The connection problem for fully nonlinear parabolic equations in one spatial dimension

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Abstract
We explicitly construct global attractors of fully nonlinear parabolic equations. The attractors are decomposed as equilibria (time independent solutions) and heteroclinic orbits (solutions that converge to distinct equilibria backwards and forwards in time). In particular, we state necessary and sufficient conditions for the occurrence of heteroclinics between hyperbolic equilibria, which is accompanied by a method that compute such conditions.

Keywords: fully nonlinear PDEs, infinite dimensional dynamical systems.

1 Main results

Consider the scalar fully nonlinear parabolic differential equation

\[ f(x, u, u_x, u_{xx}, u_t) = 0 \]

with initial data \( u(0, x) = u_0(x) \), where \( x \in (0, \pi) \) has Neumann boundary conditions, \( f \in C^2 \). Indices abbreviate partial derivatives. We assume the parabolicity condition

\[ f_q \cdot f_r < 0, \]

for every argument \( (x, u, p, q, r) := (x, u, u_x, u_{xx}, u_t) \). In particular, \( f_q, f_r \neq 0 \).

The equation (1.1) defines a semiflow denoted by \((t, u_0) \mapsto u(t)\) in a Banach space \( X^{\alpha} := C^{2\alpha+\beta}([0, \pi]) \cap \{\text{Neumann b.c.}\} \). Consider \( 2\alpha + \beta > 1 \) so that solutions are at least \( C^1 \). The appropriate functional setting is described in Section 2.1.

We suppose that the semiflow is bounded and dissipative: trajectories \( u(t) \) remain bounded, exist for all time \( t \in \mathbb{R}_+ \) and eventually enter a large ball in the phase-space \( X^{\alpha} \). Suppose also that global orbits are precompact in \( X^{\alpha} \), i.e., the closure of orbits \( \{u(t) \mid t \in \mathbb{R}_+\} \) is compact in \( X^{\alpha} \). These hypotheses guarantee the existence of a nonempty global attractor \( \mathcal{A} \subseteq X^{\alpha} \) of (1.1), which is the maximal compact invariant set. Equivalently, it is the minimal set that attracts all bounded sets of \( X^{\alpha} \), or the set of all bounded trajectories \( u(t) \) in \( X^{\alpha} \) that exist for all \( t \in \mathbb{R} \). See Figure 1.1.

The abstract setting is accurately described in the monumental work of Uraltseva, Ladyzhenskaya and Solonnikov [55], Henry [34], Lunardi [42], Babin and Vishik [3].

![Figure 1.1: The semiflow u(t) is a time t \in \mathbb{R}_+ action of the space X^\alpha, such that orbits are curves in X^\alpha parametrized by time, displaying the time evolution of an initial point u_0 \in X^\alpha. For bounded, dissipative and precompact semiflows, any initial data converges to the global attractor \mathcal{A}.](image)

We seek to decompose the attractor into smaller invariant sets, and describe how those sets are related within its internal dynamics. This conjecture to construct the global attractor for fully nonlinear equations was originally stated by Fiedler in [13].
The attractor $A \subseteq X^\alpha$ has gradient dynamics, due to the existence of a Lyapunov function in [39], which generalizes the construction of Matano [45] for quasilinear equations. See also Zelenyak [59]. More precisely, there is an energy functional $E := \int_0^\pi L(x, u, u_x)dx$ that decays according to

\begin{equation}
\frac{dE}{dt} := -\int_0^\pi L_{pp}(x, u, u_x)F^1(x, u, u_x, u_t)|u_t|^2dx \leq 0,
\end{equation}

for some Lagrange function $L$ satisfying the convexity condition $L_{pp} > 0$, where $p := u_x$, and some positive function $F^1 > 0$. Therefore, the LaSalle invariance principle holds and implies that bounded solutions converge to equilibria, and any $\omega$-limit set consists of a single equilibrium. See Figure 1.2 and [34, 45].

![Figure 1.2: Dynamical decomposition of the global attractor $A$: gradient structure induced by the one-dimensional dynamics arising from the Lyapunov function $E$.](image)

The global attractor is thereby decomposed as $A = E \cup H$, where $E$ denotes the equilibria points (time independent solutions, i.e., $u_t = 0$) and $H$ stands for the set of heteroclinic orbits, i.e., a solution $u(t) \in H$ that satisfies

\begin{equation}
\begin{array}{c}
u_- \xrightarrow{t \to -\infty} u(t) \xrightarrow{t \to \infty} u_+,
\end{array}
\end{equation}

for some $u_\pm \in E$.

The task of explicitly finding equilibria and which heteroclinics occur is often called the \textit{connection problem}. In particular, necessary and sufficient conditions are given in order to guarantee the occurrence of heteroclinics among two given equilibria, such as in (1.4). The global attractors of scalar unidimensional parabolic equations (1.1) are known as \textit{Sturm attractors}, since the connection problem can be solved by means of nodal properties firstly discovered by Sturm [54], and rediscovered by Matano [44]. This geometric description of Sturm attractors was carried out in the semilinear context with Hamiltonian reaction term $f(u)$ by Brunovský and Fiedler [6], in the more general reaction term $f(x, u, u_x)$ by Fiedler and Rocha [14], and with periodic boundary conditions by Fiedler, Rocha and Wolfrum [17, 22]. The author pursued quasilinear equations in [38] and singular diffusion in [40]. Some examples can be seen in [12], and a classification of Sturm attractors of dimension 2 and 3 can be found in the respective tryptics [18, 19, 20] and [25, 26, 27].

For the statement of the main theorem, we need a few definitions. The \textit{zero number} $z(u_*)$ denotes the number of sign changes of a continuous function $u_*(x)$. An equilibrium $u_*(x)$ is \textit{hyperbolic} if the linearization of the right hand side of (1.1) at $u_*$ has no eigenvalue in the imaginary axis; also, the number of positive eigenvalues is called the \textit{Morse index}. Lastly, we say that two different equilibria $u_-, u_+$ of (1.1) are \textit{adjacent} if there does not exist an equilibrium $u_*$ of (1.1) such that $u_*(0)$ lies between $u_-(0)$ and $u_+(0)$, i.e. $u_-(0) < u_*(0) < u_+(0)$ or $u_-(0) > u_*(0) > u_+(0)$, and

\begin{equation}
z(u_- - u_*) = z(u_- - u_+) = z(u_+ - u_*).
\end{equation}
Figure 1.3: An example of continuous functions $u_\pm(x), u_+(x)$ such that $u_\ast$ is between $u_\pm$ at $x = 0$, $z(u_- - u_+) = z(u_+ - u_\ast) = 2$, but $z(u_- - u_\ast) = 0$. Note that the zero number of the difference of solutions counts the number of intersections of the two graphs. Thus, $u_-$ and $u_\ast$ are adjacent.

Both the zero number and Morse index can be computed from a permutation of the equilibria, called the \textit{Fusco-Rocha Permutation}, as it was done in [30] and [14]. We compute such permutation abstractly in Section 2.3, and in the explicit example of the Chafee-Infante attractor in Section 3. For such, it is required that the flow of the equilibria equation of (1.1) exists for all $x \in [0, \pi]$.

**Theorem 1.1. Sturm Attractor.** Consider $f \in C^2$ satisfying the parabolicity condition (1.2) such that it generates a bounded and dissipative semiflow with precompact orbits. Suppose that all equilibria for the equation (1.1) are hyperbolic. Then,

1. the global attractor $A$ of (1.1) consists of finitely many equilibria $E$, and heteroclinic connections $H$ between them.

2. there is a heteroclinic orbit $u(t) \in H$ that converges to distinct equilibria $u_\pm \in E$ as $t \to \pm \infty$, i.e.,

   \begin{align}
   u_- \xrightarrow{t \to -\infty} u(t) \xrightarrow{t \to +\infty} u_+
   \end{align}

   if, and only if, $u_-$ and $u_\ast$ are adjacent and $i(u_-) > i(u_\ast)$.

This result meets the expectations of Fiedler [13], that mentions fully nonlinear equations should yield the same type of attractors as the semilinear ones. In order to achieve such claims, we rewrite (1.1) in two different manners, following the ideas in [39].

On one hand, we solve for the diffusion variable $q = u_{xx}$ in terms of the other variables $(x, u, u_x, u_t)$. Indeed, the parabolicity condition (1.2) implies $f_q \neq 0$, which allows us to rewrite (1.1) by the implicit function theorem as

\begin{align}
   u_{xx} = F(x, u, u_x, u_t)
\end{align}

where the parabolicity condition (1.2) becomes $F_r > 0$ at any $(x, u, p, r) = (x, u, u_x, u_t)$, since implicit differentiation implies $F_r = -f_r/f_q > 0$. We note that $F$ may not, and need not be, defined globally. We only consider $F(x, u, p, .)$ to be defined on an open interval of $r$, with limits $\pm \infty$ of $F$ at the boundaries.

Then, we split the function $F$ into two parts: one independent of $u_t$, and the other depending on $u_t$. In other words, we distinguish between the diffusion part $F^0$ related to the equilibrium ODE $u_t = 0$, and the diffusion part $F^1$ related to time changing solutions. Specifically, to account for equilibria, we define

\begin{align}
   F^0(x, u, p) := F(x, u, p, 0)
\end{align}
and suppose it is well-defined. Otherwise, let \( F^0 \equiv 0 \), artificially. To distinguish the \( u_t \) dependence, define
\[
F^1(x, u, p, r) := \begin{cases} 
\frac{F(t, u, p, r) - F^0(x, u, p)}{r} & \text{for } r \neq 0 \\
F_r(x, u, p, 0) & \text{for } r = 0.
\end{cases}
\]

The parabolic equation (1.7) can be rewritten as
\[ u_{xx} = F^0(x, u, u_x) + F^1(x, u, u_x, u_t)u_t. \]

The parabolicity condition (1.2), or \( F_r > 0 \), now reads \( F^1 > 0 \). Indeed, the monotonicity condition \( F_r > 0 \) ensures \( F^1 > 0 \) at \( r = 0 \), as well as \( \text{sign}(r) = \text{sign}(F - F^0) \) for \( r \neq 0 \). In the latter case, the numerator and denominator in (1.9) have the same sign, yielding \( F^1 > 0 \) for all \( r \).

On the other hand, we solve for the evolution variable \( r = u_t \) in terms of the others \((x, u, u_x, u_{xx})\). Again, the implicit function theorem allows us to rewrite (1.1) as
\[ u_t = \tilde{F}(x, u, u_x, u_{xx}). \]

We split \( F \) analogously to (1.10), i.e.,
\[ u_t = \tilde{F}^0(x, u, u_x) + \tilde{F}^1(x, u, u_x, u_{xx})u_{xx}, \]
where \( \tilde{F}^0, \tilde{F}^1 \) are defined similarly to (1.8), (1.9). The parabolicity condition is \( \tilde{F}^1 > 0 \).

Therefore equation (1.1) can be disguised as (1.10) or (1.12), and we use each splitting whenever it is more convenient, since they are two sides of the same coin. For instance, the former splitting (1.10) is nonlinear in \( u_t \), but linear in \( u_{xx} \), which suits better for the shooting methods in Section 2.3; whereas the latter splitting (1.12) is linear in \( u_t \), but nonlinear in \( u_{xx} \), which is more compatible with the liberalism proof in Section 2.2.

Note the following growth conditions on the nonlinearity \( \tilde{F} \) are sufficient to obtain boundedness and dissipativity of the semiflow:
\[
\begin{align*}
(1.13a) & \quad \tilde{F}_q(x, u, p, q) \geq \nu(|u|), & \text{for all } (x, u, p, q), \\
(1.13b) & \quad \tilde{F}(x, u, 0, 0) \cdot u \leq c \cdot [1 + u^2], & \text{for all } (x, u), \\
(1.13c) & \quad |\tilde{F}_x(x, u, p, 0)| \leq \mu_2(|u|)|1 + |p||, & \text{for all } (x, u, p), \\
(1.13d) & \quad \tilde{F}_u(x, u, p, 0) \leq \mu_1(|u|), & \text{for all } (x, u, p), \\
(1.13e) & \quad \tilde{F}(0, 0, p, 0) \leq 0 \leq \tilde{F}(1, 0, p, 0), & \text{for all } |p| > K \text{ and some } K \in \mathbb{R},
\end{align*}
\]
where \( \nu \) is positive and continuously decreasing; and \( \mu_1 \) is continuously increasing. See Proposition 3.5 in [42], or Chapter 6, Section 5 in [55]. Also [47, 3]. We also need that the fractional power \( \alpha > 1/2 \).

The remaining of the paper is organized as follows. We firstly introduce the necessary background in Section 2.1, including the appropriate functional setting, invariant stable/unstable manifolds, the dropping lemma and two of its consequences (a comparison of zero numbers and Morse indices within invariant manifolds, and the Morse-Smale property). In Section 2.2, we build upon the background tools in order to establish the connection problem. In Section 2.3, we describe the shooting method that unravel the information on adjacency (i.e., Morse indices and zero numbers), which is encoded in a permutation of the equilibria. In Section 3, we provide an example of fully nonlinear equations yielding the well known Chafee-Infante attractor. Lastly, we discuss the present results and future directions in Section 4.
2 Proof of main result

2.1 Background

The phase-space \( X^\alpha \) lies in the space of Hölder continuous functions \( X := C^\beta([0, \pi]) \) with Hölder coefficient \( \beta \in (0, 1) \), intersected with the Neumann boundary conditions, constructed as follows. See [43], [3]. The notation \( C^\beta \) for some \( \beta \in \mathbb{R}_+ \) indicates that \( \beta \) can be rewritten as \( [\beta] + \{\beta\} \), where the integer part \( [\beta] \in \mathbb{N} \) denotes the \( [\beta] \)-times differentiable functions whose \( [\beta] \)-derivatives is \( \{\beta\}\)-Hölder, where \( \{\beta\} \in [0, 1) \) is the fractional part of \( \beta \).

Equation (1.11) can be seen as an abstract differential equation on a Banach space,

\[
(2.1) \quad u_t = Au + g(u)
\]

where \( A : D(A) \to X \) is the linearization of the right-hand side of (1.11) at the initial data \( u_0 \), and the Nemitskii operator \( g(u) := \tilde{F}(x, u, u_x, u_{xx}) - Au \), which takes values in \( X \). The domain of \( A \) is \( D(A) := C^{2, \beta}([0, \pi]) \cap \{\text{Neumann b.c.}\} \subseteq X \), where \( \beta \in (0, 1) \). Moreover, consider the interpolation spaces \( X^\alpha := C^{2\alpha+\beta}([0, \pi]) \) between \( X \) and \( D(A) \), with \( \alpha \in (0, 1) \), such that \( A \) generates a strongly continuous semigroup in \( X^\alpha \). Therefore, solutions of the equation (1.1) defines a semiflow in \( X^\alpha \) according to the variation of constants formula. Moreover, this semiflow is dissipative due to the conditions (1.13). See Theorem 8.1.1 in Lunardi [43]. See also [36].

We suppose that \( 2\alpha + \beta > 1 \) so that solutions are in \( C^1([0, \pi]) \) and \( \alpha > 1/2 \) in order to fulfill dissipativity in (1.13). For a delicate analysis regarding the regularity theory for fully nonlinear parabolic equations, see [37, 56, 57, 10], and more recently in [9, 11].

In particular, this settles existence and uniqueness of solutions. For certain qualitative properties of solutions, such as the existence of invariant manifolds tangent to the linear eigenspaces, one needs the spectrum of \( A \). Note that \( A u = \lambda u \) is a regular Sturm-Liouville problem, since the coefficients depend only on \( x \) and are all bounded. Therefore, the spectrum \( \sigma(A) \) consists of real simple eigenvalues \( \lambda_k \) accumulating at \( -\infty \), and corresponding eigenfunctions \( \phi_k(x) \) which form an orthonormal basis of \( X \), and thereby of \( X^\alpha \). Note that \( 0 \) is not an eigenvalue, since we assumed all equilibria are hyperbolic. Moreover, there is a spectral gap between pairs of eigenvalues that allows us to obtain the following filtration of invariant manifolds.

\textbf{Proposition 2.1. Filtration of Invariant Manifolds.} [34, 46, 3, 43]. Let \( u_* \in \mathcal{E} \) be a hyperbolic equilibrium of (1.1) with Morse index \( n := i(u_*) \). Then there exists a filtration of the unstable manifold\(^1\) according to

\[
(2.2) \quad W^n_0(u_*) \subset \ldots \subset W^n_{n-1}(u_*) = W^n(u_*)
\]

where each \( W^n_k \) has dimension \( k + 1 \) and tangent space at \( u_\ast \) spanned by \( \phi_0, \ldots, \phi_k \). Analogously, there is a filtration of the stable manifold

\[
(2.3) \quad \ldots \subset W^s_{n+1}(u_*) \subset W^s_n(u_*) = W^s(u_*)
\]

where each \( W^s_k \) has codimension \( k \) and tangent space spanned by \( \phi_k, \phi_{k+1}, \ldots \).

\(^1\)Note indices are not in agreement with the dimension of each submanifold within the filtration, but with the number of zeros of the corresponding eigenfunction. For example, \( \phi_k \) has \( k \) simple zeroes, whereas the \( \dim W^u_k = k + 1 \).
The standard existence theorem of unstable manifolds only guarantees that \( W_u^u(u_s) \) is locally diffeomorphic to a ball. However, one can obtain global topological properties of unstable manifolds of equations (1.1): the topological boundary of each \( W_u^u(u_s) \) is not only homeomorphic to a sphere, but its interior is a ball, see [24]. This excludes Alexander horned spheres and lens spaces. Moreover, the unstable manifolds provide a cell decomposition of the global attractor \( \mathcal{A} \) as a regular CW-complex, see [23].

An important property is the behavior of solutions within each submanifold of the above filtration of the unstable or stable manifolds. This holds mainly due to the spectral gap of the linearized operator, and geometrically imply that solutions respect the filtration of the unstable manifold are tangent to its corresponding eigenfunction.

**Proposition 2.2. Linear Asymptotic Behavior.** [34, 1, 5]. Consider a hyperbolic equilibrium \( u_s \in \mathcal{E} \) with Morse index \( n := i(u_s) \) and a solution \( u(t) \) of (1.1).

1. If \( u(t) \in W_k^u(u_s) \setminus W_{k-1}^u(u_s) \) with \( k = 0, \ldots, i(u_s) - 1 \), where \( W_{i+1}^u(u_s) := \emptyset \). Then,

   \[
   \lim_{t \to -\infty} \frac{u(t) - u_s}{||u(t) - u_s||} = \pm \phi_k \quad \text{in } C^1.
   \]

2. If \( u(t) \) in \( W_k^s(u_s) \setminus W_{k+1}^s(u_s) \) with \( k \geq i(u_s) \). Then,

   \[
   \lim_{t \to -\infty} \frac{u(t) - u_s}{||u(t) - u_s||} = \pm \phi_k, \quad \text{in } C^1.
   \]

The conclusions of 1. and 2. also hold true by replacing the difference \( u(t) - u_s \) with the tangent vector \( u_t \). The reason this theorem works for both the tangent vector \( v := u_t \) or the difference \( v := u_1 - u_2 \) of any two solutions \( u_1, u_2 \) of the nonlinear equation (1.11) is that they satisfy a linear equation of the type

\[
(2.6) \quad v_t = a(t, x)v_{xx} + b(t, x)v_x + c(t, x)v
\]

where \( x \in (0, \pi) \) has Neumann boundary conditions, and \( a(t, x), b(t, x), c(t, x) \) are bounded for all \( (t, x) \in \mathbb{R} \times [0, \pi] \), and given by

\[
(2.7a) \quad a(t, x) := \int_1^2 \tilde{F}_q(x, u^s, u^s_x, u^s_{xx}) ds > 0,
\]

\[
(2.7b) \quad b(t, x) := \int_1^2 \tilde{F}_p(x, u^s, u^s_x, u^s_{xx}) ds,
\]

\[
(2.7c) \quad c(t, x) := \int_1^2 \tilde{F}_u(x, u^s, u^s_x, u^s_{xx}) ds,
\]

where \( q := u_{xx}, p := u_x, \) and \( u^s := (2 - s)u_1 + (s - 1)u_2 \) for \( s \in [1, 2] \).

![Figure 2.1: A two-dimensional unstable manifold, \( W_u^u(u_s) = W_t^u(u_s) \), with tangent space spanned by \( \phi_0, \phi_1 \). On the left, the unstable manifold contains the (bold) one-dimensional curve, i.e. \( W_0^u(u_s) \). Solutions in \( W_0^u(u_s) \) (resp. \( W_t^u(u_s) \)) are well-approximated by \( \phi_1 \) (resp. \( \phi_0 \)) as \( t \to -\infty \). On the right, the linear semiflow in the tangent space that locally approximates the nonlinear semiflow.](image)
A fundamental ingredient in order to solve the connection problem are the nodal properties, i.e., the zero number of certain solutions of (1.1) is nonincreasing in time \( t \), and decreases whenever a multiple zero occur. A point \((t_*, x_*) \in \mathbb{R} \times [0, \pi]\) such that \(u(t_*, x_*) = 0\) is said to be a multiple zero if \(u_\ast(t_*, x_*) = 0\). Let the zero number \(0 \leq z(u(t, .)) \leq \infty\) count the number of strict sign changes in \(x\) of a \(C^1\) function \(u(t, x) \neq 0\), for each fixed \(t\). More precisely, if \(x \rightarrow u(t, x)\) is not of constant sign, let

\[
(2.8) \quad z(u(t, .)) := \sup_k \left\{ \text{There is a partition } \{x_j\}_{j=1}^k \text{ of } [0, \pi] \text{ such that } u(t, x_j)u(t, x_{j+1}) < 0 \text{ for all } j = 0, \ldots, k - 1 \right\}.
\]

For functions which do not change sign, \(x \mapsto u(t, x) \neq 0\), and we define \(z(u) := 0\). For the trivial constant, \(x \mapsto u(t, x) \equiv 0\), we define \(z(u) \equiv 0 := -1\).

**Lemma 2.3. Dropping Lemma.** [54, 44, 2]. Consider a non-trivial solution \(v \in C^1\) of the linear equation (2.6) for \(t \in [0, T]\). Then, its zero number \(z(v(t, .))\) satisfies

1. \(z(v(t, .)) < \infty\) for any \(t \in (0, T)\).
2. \(z(v(t, .))\) is nonincreasing in time \(t\).
3. \(z(v(t, .))\) decreases at multiple zeros \((t_*, x_*)\) of \(v(t, .)\), i.e.,

\[
(2.9) \quad z(v(t_\ast - \epsilon, .)) > z(v(t_\ast + \epsilon, .)), \quad \forall \text{ sufficiently small } \epsilon > 0.
\]

Recall that both the tangent vector \(v := u_t\) and the difference \(v := u_1 - u_2\) of solutions \(u_1, u_2\) of the nonlinear equation (1.1) satisfy the linear equation (2.6), and thereby the proof of Lemma 2.3 for fully nonlinear equations is the same as in the semilinear case.

**Figure 2.2:** Example of a function \(v(t, x)\) such that its zeros are denoted in bold. Note that the number of zeros (in \(x\)) of \(v(t, x)\) is two when \(t = 0\), and this number decreases with time, since \(v(t, x)\) has no zeros for \(t > t_*\). Note that the function \(v(t_\ast, x)\) has a multiple zero at \(x_*\).

We can now combine the dropping lemma 2.3 and the asymptotic description in Proposition 2.2 in order to relate the zero number within invariant manifolds and the Morse indices of equilibria.

**Proposition 2.4. Zero number within Invariant Manifolds.** [5]. Consider a hyperbolic equilibrium \(u_* \in \mathcal{E}\) with Morse index \(i(u_*)\) and a solution \(u(t)\) of (1.1).

1. If \(u(t) \in W^u(u_*),\) then \(i(u_*) > z(u(t) - u_*)\).
2. If \(u(t) \in W^s_{\text{loc}}(u_*) \setminus \{u_*\},\) then \(z(u(t) - u_*) \geq i(u_*)\). These results also hold by replacing \(u(t) - u_*\) with the tangent vector \(u_t\). 


As a consequence of Proposition 2.4, in case of hyperbolic equilibria, the semiflow of (1.1) is Morse-Smale, i.e., the non-wandering set consists of finitely many hyperbolic equilibria, and the stable and unstable manifolds of equilibria intersect transversely.

Theorem 2.5. Transversality. [35, 1, 30]. Consider hyperbolic equilibria $u_\pm \in E$ with Morse indices $i(u_\pm)$. If $W^u(u_-)$ and $W^s(u_+)$ intersect at $u_0$, then such intersection is transverse, i.e., $X = T_{u_0} W^u(u_-) \oplus T_{u_0} W^s(u_+)$. Moreover, $W^u(u_-) \cap W^s(u_+)$ is an embedded submanifold of dimension $i(u_-) - i(u_+)$. 

![Figure 2.3: An example of a transverse heteroclinic orbit connecting two hyperbolic equilibria: from $u_-$ to $u_+$ as $t \in \mathbb{R}$ increases. The heteroclinic occur as the intersection of $W^u(u_-)$ and $W^s(u_+)$.](image)

Note that the Morse-Smale property remains true in certain cases the equilibria are not hyperbolic, as in [35]; or higher spatial dimension, generically, see [8]. We emphasize that genericity of both hyperbolicity and the Morse-Smale property has been proved for scalar unidimensional semilinear equations, which respectively imply the local and global stability of the dynamics with respect to perturbations of the system. See [4, 31, 41, 48, 7]. These results should remain true for fully nonlinear equations.

2.2 Sturm global structure

This section abstractly constructs the attractor for the equation (1.1) and prove the second part of the Theorem 1.1. Its proof is a consequence of four propositions. First, the cascading principle guarantees it is enough to construct only heteroclinics between equilibria with Morse index differing by 1. Second, on one direction, the blocking principle: certain conditions prevent the existence of a heteroclinic; whereas on the other direction, the liberalism principle: if heteroclinics are not forbidden, then they actually exist. Lastly, Wolfrum’s equivalence yield a relation between two notions of adjacencies: one that depends on a cascade between equilibria, and one that does not.

Proposition 2.6. Cascading Principle. [14]. Consider two equilibria $u_\pm$ of equation (1.1) such that $n := i(u_-) - i(u_+) > 0$. Then the following statements are equivalent:

(i) There exists a heteroclinic orbit from $u_-$ to $u_+$ in forward time, as in (1.6).

(ii) There exists a sequence (cascade) of equilibria $\{e_j\}_{j=0}^n$ with $e_0 := u_-$ and $e_n := u_+$ such that $i(e_{j+1}) = i(e_j) + 1$ and there exists a heteroclinic orbit from $e_{j+1}$ to $e_j$ in forward time for all $j = 0, \ldots, n - 1$.

The proof of Proposition 2.6 relies on nodal properties, and we refer to Lemma 1.5 of [14]. Due to the cascading principle, it suffices to construct all heteroclinic orbits between equilibria with Morse indices differing by one. The implication $(ii) \Rightarrow (i)$ is a special case of a transitivity principle that holds for Morse-Smale systems, whereas the converse implication $(i) \Rightarrow (ii)$ is not true in general and is specific of equation (1.1).
A heteroclinic from $u_-$ to $u_+$ exists if, and only if there is a cascade of consecutive heteroclinics among equilibria $\{e_j\}_{j=0}^4$ with Morse index differing by 1 such that $e_0 = u_-$ and $e_4 = u_+$.

The second result provides a condition that prevents heteroclinic orbits between equilibria with Morse indices differing by one. Before we present its content, we say that two hyperbolic equilibria $e_{j+1}$ and $e_j$ of (1.1) with $i(e_{j+1}) = i(e_j) + 1$ are blocked if one of the following conditions holds:

1. Morse blocking: $z(e_{j+1} - e_j) \neq i(e_j)$;
2. Zero number blocking: there exists an equilibria $u_*$ between $e_{j+1}$ and $e_j$ at $x = 0$ such that
   \begin{equation}
   z(e_{j+1} - u_*) = z(e_{j+1} - e_j) = z(e_j - u_*)).
   \end{equation}

The proof of the upcoming Proposition 2.7 follows from nodal properties of solutions of (1.1); see the subsequent discussion of Definition 1.6 in [14].

**Proposition 2.7. Blocking Principle.** [14]. Consider two equilibria $e_{j+1}$ and $e_j$ of (1.1) such that $i(e_{j+1}) = i(e_j) + 1$. If $e_{j+1}$ and $e_j$ are blocked, then there does not exist heteroclinic orbits from $e_{j+1}$ to $e_j$ in forward time.

The next claim is an act of liberalism: If a heteroclinic connection among two equilibria is not forbidden by the blocking law, then a connection between them does exist.

**Proposition 2.8. Liberalism Principle.** Consider hyperbolic equilibria $e_{j+1}, e_j \in \mathcal{E}$ of (1.1) such that $i(e_{j+1}) = i(e_j) + 1$. If $e_{j+1}$ and $e_j$ are not blocked, then there exists a heteroclinic orbit from $e_{j+1}$ to $e_j$ in forward time.

The proof of liberalism in Proposition 2.8 follows from an application of the Conley index theory; see [14], Lemma 1.7. However, their proof needs a modification from the original in Section 4 of [14]. Indeed, one has to replace the homotopy performed after their equation (4.15). In their case, they homotope two semilinear equations, and in our case, we homotope a semilinear equation, given by $u_t = \tilde{a}(x)u_{xx} + \tilde{f}(x, u, u_x)$ that arises from a saddle-node bifurcation, to a fully nonlinear one through

\begin{equation}
   u_t = \tilde{F}^\tau(x, u, u_x, u_{xx}),
\end{equation}

where

\begin{equation}
   \tilde{F}^\tau(x, u, u_x, u_{xx}) := (1 - \tau)(\tilde{a}u_{xx} + \tilde{f}) + \tau\hat{F} + \sum_{j=-, +} \chi_{e_j}\mu_e(\tau)[u - e_j(x)],
\end{equation}

and $\chi_{e_j}$ are cut-offs being 1 nearby $e_j$ and zero far from $e_j$, the coefficients $\mu_j(\tau)$ are zero near $\tau = 0, 1$ and shift the spectra of the linearization at $u_{\pm}$ appropriately such that uniform hyperbolicity of these equilibria is guaranteed during the homotopy.

Propositions 2.6, 2.7 and 2.8 yield the existence of heteroclinics between $u_-$ and $u_+$ if they are cascadly adjacent, i.e., if there exists a cascade of equilibria $\{e_j\}_{j=0}^n$ with $e_0 := u_-$ and $e_n := u_+$ such that for all $j = 0, \ldots, n - 1$ the following holds:
1. $i(e_{j+1}) = i(e_j) + 1$, 
2. $z(e_j - e_{j+1}) = i(e_{j+1})$, 
3. there are no equilibria $u_*$ between $e_j$ and $e_{j+1}$ at $x = 0$ satisfying (2.10).

However, Theorem 1.1 yields existence of heteroclinics relying on the notion of adjacency in (1.5), which does not involve a cascade. These notions of adjacency coincide, and this is the core of Wolfrum’s ideas in [58].

**Proposition 2.9. Wolfrum’s equivalence.** Consider equilibria $u_\pm \in \mathcal{E}$ such that $i(u_-) > i(u_+)$. The equilibria $u_\pm$ are adjacent if, and only if they are cascadly adjacent.

### 2.3 Shooting: Finding Equilibria and Computing Adjacency

The next step on our quest to construct Sturm attractors is to find all equilibria of (1.1) and compute their Morse indices and zero numbers, in order to discern which equilibria are adjacent. We will use shooting methods similar to [29, 32, 50, 51, 53].

The equilibria equation associated to (1.1) is

\[
(2.13) \quad 0 = f(x, u, u_x, u_{xx}, 0)
\]

for $x \in [0, \pi]$ with Neumann boundary conditions. Equation (2.13) can be translated as follows, according to the implicit function theorem, as in (1.7) and (1.10),

\[
(2.14) \quad u_{xx} = F^0(x, u, u_x).
\]

In turn, equation (2.14) can be reduced to a first order autonomous system,

\[
\begin{align*}
(2.15a) & \quad u' = p, \\
(2.15b) & \quad p' = F^0(x, u, p), \\
(2.15c) & \quad x' = 1
\end{align*}
\]

where the Neumann boundary condition amounts to $p = 0$ at $x = 0, \pi$ and $(\cdot)'$ denotes a derivative with respect to a new parameter $\tau := x \in [0, 1]$. We suppose solutions of (2.15) exist for all $x \in [0, \pi]$ and any initial data with $(x, u, p) = (0, u_0(0), 0)$.

Equilibria of the PDE (1.1) can be found as follows. They must be in the line

\[
(2.16) \quad L_0 := \{(x, u, p) \in \mathbb{R}^3 \mid (x, u, p) = (0, a, 0) \text{ and } a \in \mathbb{R}\},
\]

because of Neumann boundary at $x = 0$. Then, this line can be evolved under the flow of (2.15) until $x = \pi$, yielding the *shooting manifold* defined as

\[
(2.17) \quad M := \{(x, u, p) \in [0, \pi] \times \mathbb{R}^3 \mid (x, u(x, a, 0), p(x, a, 0)) \text{ and } a \in \mathbb{R}\},
\]

where $(x, u(x, a, 0), p(x, a, 0))$ is the solution of (2.15) which evolves the initial data $(0, a, 0)$. Denote by $M_x$ the cross-section of $M$ for some fixed $x \in [0, \pi]$, which is a curve parametrized by $a \in \mathbb{R}$. We call the cross-section $M_x$ by *shooting curve*, which is an object that carries all the information regarding existence, hyperbolicity and adjacency of equilibria, as in the following Lemmata. See [50, 33, 52, 28].
Figure 2.5: A shooting manifold with cross-section at \( x = \pi \) given by the shooting curve \( M_\pi \). The transverse intersections of \( M_\pi \) and \( L_\pi \) correspond to hyperbolic equilibria of (1.1).

**Lemma 2.10. Hyperbolic Equilibria by Shooting.** The set of equilibria \( E \) of (1.1) is in one-to-one correspondence with \( M_\pi \cap L_\pi \). Moreover, an equilibrium \( u_* \in E \) is hyperbolic if, and only if, \( M_\pi \) intersects \( L_\pi \) transversely at \((\pi, u_*(\pi), 0)\).

*Proof.* To prove the first part, note that a point in \( M_\pi \cap L_\pi \) satisfies the equilibria equation with Neumann boundary conditions by definition of the shooting manifolds. Conversely, any equilibrium of (1.1) satisfying Neumann boundary must be in \( M_\pi \cap L_\pi \). Due to the uniqueness of the differential equation (2.15), such correspondence above is one-to-one. The implicit function theorem guarantees these solutions solve (2.13).

To prove the second part, consider an equilibrium \( u_* \) corresponding to the boundary value \( u_*(0) = a \in \mathbb{R} \). We compare the eigenvalue problem for \( u_* \) and the differential equation satisfied by the angle of the tangent vectors of the shooting manifold.

The eigenvalue problem for \( u_* \) is the linearization of the right hand side of (1.1) at \( u_* \)

\[
\lambda u = f_q(x, u_*, p_*, q_*, 0)u_{xx} + f_p(x, u_*, p_*, q_*, 0)u_x + f_u(x, u_*, p_*, q_*, 0)u.
\]

where \( x \in [0, \pi] \) has Neumann boundary conditions. From now on, we suppress the arguments of \( f_q, f_p, f_u \). We rewrite equation (2.18) as a first order system:

\[
\begin{align*}
    u' &= p, \\
    p' &= -(f_p \cdot p + f_u \cdot u - \lambda u)/f_q, \\
    x' &= 1,
\end{align*}
\]

with Neumann boundary condition. This is well defined due to parabolicity \( f_q \neq 0 \).

On the other hand, \( M_x \) is parametrized by the initial data \( a \in \mathbb{R} \) and its tangent vector \((\partial a u(x, a), \partial a p(x, a))\) corresponding to the trajectory \( u_* \) satisfies the following linearized equation

\[
\begin{align*}
    u'_a &= p_a, \\
    p'_a &= F_p^0 \cdot p_a + F_u^0 \cdot u_a, \\
    x' &= 1,
\end{align*}
\]

with data \((u_a(0), p_a(0)) = (1, 0)\). We suppress the arguments \((u(x, a), p(x, a))\) of \( F_u^0, F_p^0 \).

Note the tangent vector equation in (2.20) is the same as the eigenvalue problem (2.19) with \( \lambda = 0 \), except the tangent equation only has Neumann boundary conditions at \( x = 0 \). Indeed, implicit differentiation of (1.1) with respect to \( u \) (resp. \( p \)), bearing (1.7) in mind, yields \( F_u = -f_u/f_q \) (resp. \( F_p = -f_p/f_q \)). To finish the proof, we compare equations (2.19) and (2.20) using well-known arguments, as in Lemma 2.7 of [38].
Thus, given a shooting curve $M_\pi$, one can find all equilibria of the PDE (1.1). Next we address the characterization of adjacency (i.e., the Morse indices and zero numbers) of equilibria by means of the shooting curve, similar to [50, 33, 52, 28].

The Fusco-Rocha permutation $\sigma$ is obtained by firstly labeling the intersection points $e_j \in M_\pi \cap L_\pi$ along the shooting curve $M_\pi$ following its parametrization, given by $\{(\pi, u(\pi, a, 0), p(\pi, a, 0))\}$ as $a \in \mathbb{R}$ increases, according to:

$$
e_1 <_{M_\pi} \cdots <_{M_\pi} e_N
$$

where $N$ denotes the number of equilibria. Secondly, label the intersection points $e_j \in M_\pi \cap L_\pi$ along $L_\pi$ as $b \in \mathbb{R}$ increases according to

$$
e_{\sigma(1)} < \cdots < e_{\sigma(N)}.
$$

Therefore $\sigma \in S_N$, the permutation group of $N$ elements. The Fusco-Rocha permutation $\sigma$ is enough to guarantee all information regarding adjacency, as in [52, 14].

**Lemma 2.11. Adjacency Through Shooting.**

1. If the equilibrium $e_j \in \mathcal{E}$ is hyperbolic, then its Morse index is given by

$$i(e_j) = 1 + \left\lfloor \frac{\theta(e_j)}{\pi} \right\rfloor,
$$

where $\theta(e_j) \in (-\pi/2, \infty)$ is the (clockwise) angle between the tangent of the curve $M_\pi$ and $L_\pi$ at their intersection $\{(\pi, e_j(\pi), 0)\}$ and $\lfloor \cdot \rfloor$ denotes the floor function.

2. Consider $e_j, e_k \in \mathcal{E}$ with boundary value $e_j(\pi) = b_j, e_k(\pi) = b_k$. Denote the vertical line at $b_j$ by $\ell_j := \{(\pi, b_j, c) \mid c \in \mathbb{R}\}$, and by $r_{jk}$ with $1 \leq j < k \leq N$ the total number of intersections of $M_\pi$ and $\ell_j$ as $b$ ranges from $b_j$ to $b_k$, taking into account the sign of the rotation of the tangent of $M_\pi$ at said intersection (i.e., +1 in case of an intersection with a clockwise rotation and −1 for counter-clockwise). Then the number of intersection points of $e_j$ and $e_k$ can be computed as

$$z(e_j - e_k) = \begin{cases} i(e_j) + r_{jk}, & \text{if } (\theta_j \mod 2\pi) \in (\pi/2, \pi) \cup (3\pi/2, 2\pi), \\ i(e_j) - 1 + r_{jk}, & \text{if } (\theta_j \mod 2\pi) \in (0, \pi/2) \cup (\pi, 3\pi/2). \end{cases}
$$

The proof of Lemma 2.11 follows Lemma 2.7 of [38] and Proposition 3 of [52], bearing in mind that the fully nonlinear shooting equation (2.13) is reduced to (2.15) with linearized equation related to the tangent of the shooting curve, as in Lemma 2.10.

**Figure 2.6:** On the left, the Fusco-Rocha permutation is obtained by consecutively labeling the equilibria along $M_\pi$ (top) and along $L_\pi$ (bottom), which yields the permutation $\sigma = (24) \in S_5$. In the middle, the angle $\theta(e_1) \in (-\pi/2, 0)$ and thus the Morse index $i(e_1) = 0$, whereas $i(e_j)$ increases by +1 (resp. −1) for each clockwise (resp. counter clockwise) $\pi$-rotation of $\theta(e_1)$; thereby $i(e_1) = 0, i(e_2) = 1, i(e_3) = 2, i(e_4) = 1, i(e_5) = 0$. On the right, the number of (clockwise) intersections $r_{j_k} = +1$ of the vertical line $\ell_j$ with the (bold) segment of $M_\pi$ ranging from $b_j$ to $b_k$. 

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3 Example: Fully nonlinear Chafee-Infante

In this section we provide the classical example of the Chafee-Infante global attractor that arises from the following fully nonlinear parabolic equation,

\[(3.1) \quad 0 = 1 - e^{uu} + u_{xx} + \lambda u[1 - u^2]\]

where \(x \in [0, \pi]\) has Neumann boundary conditions and initial data \(u_0 \in C^{2\alpha+\beta}([0, \pi])\) with \(\alpha, \beta \in (0, 1)\) such that \(\alpha \leq \min\{1/2, 2\beta + 1\}\). For the semilinear version see [33]. The equilibria equation describing the shooting curve is equal to the semilinear one,

\[(3.2a) \quad u' = p,\]
\[(3.2b) \quad p' = -\lambda u[1 - u^2],\]
\[(3.2c) \quad x' = 1.\]

This yields the Chafee-Infante shooting curves as in Figure 3.1. See [14, 40]. Thus all equilibria, their respective Morse indices and zero numbers can be computed according to Lemma 2.11. As an illustrative example, we compute the indices and intersections for the first parameter values below. This detects which equilibria are adjacent and thereby which connections occur by means of a heteroclinic orbit yielding the same structure (connection-wise) as the standard Chafee-Infante problem. See Figure 3.2.

![Figure 3.1](image)

\(\text{Figure 3.1: From left to right: the shooting curves for } \lambda \in (\lambda_0, \lambda_1), (\lambda_1, \lambda_2), \text{ and } (\lambda_2, \lambda_3).\)

For \(\lambda \in (\lambda_0, \lambda_1)\), the rotation of the tangent of \(e_0\) yields \(i(e_1) = 0, i(e_2) = 1, i(e_3) = 0\). Moreover, the intersections of vertical lines \(\ell_j\) at each equilibria \(e_j\) with the shooting curves yield \(r_{12} = r_{13} = r_{23} = 0\), and the angles of the tangent vector satisfy \((\theta(e_1), \theta(e_3)) \mod 2\pi \in (\pi, 3\pi/2)\), whereas \((\theta(e_2) \mod 2\pi \in (0, \pi/2)\), we consequently obtain \(z(e_1 - e_2) = z(e_1 - e_3) = z(e_2 - e_3) = 0\). Therefore \(e_2\) is adjacent to both \(e_1\) and \(e_0\), since there are no equilibria between \(e_2\) and either \(e_1\) or \(e_3\).

For \(\lambda \in (\lambda_1, \lambda_2)\), then \(i(e_1) = i(e_5) = 0, i(e_2) = i(e_4) = 1, i(e_3) = 2\). Moreover, the \(r_{1j} = 0\) for all \(j = 2, \ldots, 5\), \(r_{23} = r_{24} = 0, r_{25} = -1, r_{34} = 0, r_{35} = -1\), \(r_{45} = -1\). Also \((\theta(e_1), \theta(e_5) \mod 2\pi \in (\pi, 3\pi/2), (\theta(e_2), \theta(e_4) \mod 2\pi \in (\pi/2, \pi)\) and \((\theta(e_3) \mod 2\pi \in (\pi, 3\pi/2)\). Consequently,

\[(3.3a) \quad z(e_1 - e_j) = 0, \text{ for all } j = 2, \ldots, 5,\]
\[(3.3b) \quad z(e_2 - e_3) = z(e_2 - e_4) = 1, \quad z(e_2 - e_5) = 0,\]
\[(3.3c) \quad z(e_3 - e_4) = 1, \quad z(e_3 - e_5) = 0,\]
\[(3.3d) \quad z(e_4 - e_5) = 0.\]
We only compute the heteroclinic connections between equilibria with Morse index differing by one, as the remaining connections follow by transitivity. Indeed, \( e_3 \) is adjacent to \( e_2, e_4 \), as there are no equilibria between \( e_3 \) and either \( e_2 \) or \( e_4 \) at \( x = 0 \); and \( e_2, e_4 \) are adjacent to \( e_1, e_5 \) since again there are no equilibria between each pair of equilibria. By transitivity, there are connections from \( e_3 \) to \( e_1, e_5 \).

Alternatively, one could compute adjacency among pairs of equilibria according to its definition (1.5) and directly find heteroclinics between equilibria that do not necessarily have Morse indices differing by one. Indeed, \( e_3 \) is adjacent to \( e_1, e_5 \), since the only equilibrium between \( e_1 \) and \( e_3 \) at \( x = 0 \) is \( e_2 \) (resp. the only equilibrium between \( e_3 \) and \( e_5 \) at \( x = 0 \) is \( e_4 \)), yet \( z(e_1 - e_3) = z(e_1 - e_2) = 0 \) whereas \( z(e_2 - e_3) = 1 \).

For \( \lambda \in (\lambda_2, \lambda_3) \), then \( i(e_1) = i(e_7) = 0, i(e_2) = i(e_6) = 1, i(e_3) = i(e_5) = 2, i(e_4) = 3 \). Moreover, the \( r_{1j} = 0 \) for all \( j = 2, ..., 7 \), \( r_{23} = r_{24} = r_{25} = r_{26} = 0, r_{27} = -1, r_{34} = r_{35} = 0, r_{36} = -1, r_{37} = -2, r_{45} = 0, r_{46} = -1, r_{47} = -2, r_{56} = -1, r_{57} = -2, r_{67} = -1 \). Also \( (\theta(e_1), \theta(e_3), \theta(e_5), \theta(e_7) \mod 2\pi) \in (\pi, 3\pi/2), (\theta(e_2), \theta(e_6) \mod 2\pi) \in (\pi/2, \pi) \) and \( (\theta(e_4) \mod 2\pi) \in (0, \pi/2) \). Thus,

\[
\begin{align*}
(3.4a) \quad & z(e_1 - e_j) = 0, \text{ for all } j = 2, ..., 7, \\
(3.4b) \quad & z(e_2 - e_j) = 1, \text{ for all } j = 3, ..., 6, \quad z(e_2 - e_7) = 0, \\
(3.4c) \quad & z(e_3 - e_4) = z(e_3 - e_5) = 2, \quad z(e_3 - e_6) = 1, \quad z(e_3 - e_7) = 0, \\
(3.4d) \quad & z(e_4 - e_5) = 2, \quad z(e_4 - e_6) = 1, \quad z(e_4 - e_7) = 0, \\
(3.4e) \quad & z(e_5 - e_6) = 1, \quad z(e_5 - e_7) = 0, \\
(3.4f) \quad & z(e_6 - e_7) = 0.
\end{align*}
\]

This enables finding which equilibria are adjacent by analysing the equilibria \( e_\ast \) that lie between \( e_j \) and \( e_k \) at \( x = 0 \). We only compute the heteroclinic connections between equilibria with Morse index differing by one, as the remaining connections follow by transitivity. Indeed, \( e_4 \) is adjacent to \( e_3, e_5 \), as there are no equilibria between \( e_4 \) and either \( e_3 \) or \( e_5 \) at \( x = 0 \). Also, \( e_3 \) is adjacent to \( e_2 \), since there is no equilibria between \( e_3 \) and \( e_2 \) at \( x = 0 \), and \( e_3 \) is adjacent to \( e_6 \) because the equilibria between them at \( x = 0 \) are \( e_4, e_5 \), but \( z(e_3 - e_6) = 1 \) whereas \( z(e_3 - e_4) = z(e_3 - e_5) = 2 \). Similarly, \( e_5 \) is adjacent to \( e_2, e_6 \). Analogously, \( e_2 \) is adjacent to \( e_1 \) since there is no equilibria between them at \( x = 0 \), and \( e_2 \) is adjacent to \( e_7 \) since the equilibria between them at \( x = 0 \) are \( e_3, e_4, e_5, e_6 \), but \( z(e_2 - e_7) = 0 \) whereas \( z(e_2 - e_j) = 1 \) for all \( j = 3, ..., 6 \). By similar reasoning, \( e_6 \) is adjacent to \( e_1, e_7 \).

![Figure 3.2](image)

**Figure 3.2:** The Chafee-Infante attractor: dots correspond to equilibria and arrows to heteroclinics.
4 Discussion

We now provide a categorial discussion regarding the attractors of equation (1.1) towards a classification of PDEs by means of their global attractors. As a preamble, let us fix Neumann boundary conditions, since attractors do not depend on them as long as the conditions are separated, as in [13]. Moreover, we bear in mind that different PDEs (1.1) with the same Fusco-Rocha permutation yields $C^0$-orbit equivalence attractors. See [16], where this fact is conjectured to remain valid for the fully nonlinear setting. As a final note, we mention that the task of finding a nonlinearity $f$ modelling an equation (1.1) that realizes a given Fusco-Rocha permutation was proved in [15].

Denote by $\text{Sturm}(u)$ the category with objects given by the global attractors (up to orbit equivalence) of the quasilinear equation with Hamiltonian type nonlinearities,

\begin{equation}
\dot{u} = a(u)u_{xx} + f(u).
\end{equation}

There are several morphisms in the $\text{Sturm}(u)$ category. Two that come to mind are bifurcation morphisms, namely $A \mapsto p(A)$ if $p(A)$ arises from $A$ after a pitchfork bifurcation; or $A \mapsto sn(A)$ if $sn(A)$ is a result of $A$ after a saddle-node bifurcation. There are other morphisms that emerge from topological constructions that respect the Sturm structure, such as a suspension of a well-known Sturm attractors that, see [27].

In [21], it is given a characterization of the Sturm attractors of Hamiltonian type by means of the Fusco-Rocha permutation, except that the class $\text{Sturm}(u)$ consisted only of the reaction terms $f(u)$ with semilinear diffusion $a \equiv 1$. Nevertheless, the Hamiltonian quasilinear equations from (4.1) computed in [38] can be realized by semilinear ones, if one considers $u_t = u_{xx} + f(u)/a(u)$. Indeed, they have the same shooting equation, hence same permutation and same attractor (up to orbit equivalence).

Consider also the class $\text{Sturm}(u, u_{xx}, u_t)$, whose objects consist of global attractors of the following fully nonlinear equations of Hamiltonian type:

\begin{equation}
0 = f(u, u_{xx}, u_t)
\end{equation}

with parabolicity condition (1.2). Note that the global attractors of (4.2) can be realized by semilinear equations (4.1). Indeed, one can consider $F^0, F^1$ arising from $f$ through the implicit function theorem, as in (1.8) and (1.9), and the fully nonlinear equation (4.2) transvestite in $F^1(u, u_t)u_t = u_{xx} + F^0(u)$ has the same shooting equation as the usual semilinear equation $u_t = u_{xx} + F^0(u)$. Hence they possess the same Fusco-Rocha permutation and thereby same attractor (up to orbit equivalence). Thus,

\begin{equation}
\text{Sturm}(u) = \text{Sturm}(u, u_{xx}, u_t)
\end{equation}

Similarly, let $\text{Sturm}(x, u, u_x)$ be the category of attractors of (4.1) with advection dependent diffusion coefficient $a(x, u, u_x)$ and reaction $f(x, u, u_x)$. Such quasilinear attractors and the fully nonlinear attractors of (1.1), denoted by $\text{Sturm}(x, u, u_x, u_{xx}, u_t)$, can be realized by semilinear equations. Indeed, a semilinear equation with reaction $f/a$ has the same Fusco-Rocha permutation of a quasilinear equation with diffusion coefficient $a > 0$. Likewise, any fully nonlinear equation (1.1) has a Fusco-Rocha permutation that can be realized by a semilinear equation with reaction term $F^0(x, u, u_x)$ emerging from the implicit function theorem in (1.8). Therefore,

\begin{equation}
\text{Sturm}(x, u, u_x) = \text{Sturm}(x, u, u_x, u_{xx}, u_t).
\end{equation}
It is known that there are Sturm attractors for advection dependent nonlinearities which cannot be realized by Hamiltonian vector fields (see [14, 21]), i.e.,

\begin{equation}
\text{Sturm}(u) \subsetneq \text{Sturm}(x, u, u_x).
\end{equation}

In other words, despite of all the nonlinear possibilities for \( f \) in equation (1.1), the dependencies on \( u \) and \( u_x \) play crucial a role in the complexity of the attractor. Even though there are no new attractors in the class of fully nonlinear parabolic equations, we enlarge the class of models that one is able to compute the Sturm attractors, and hence the domain of the Sturm and Fusco-Rocha functors, as can be seen from the example in Section 3. It remains the question of describing a full filtration diagram of classes of Sturm attractors such as in (4.3), (4.4) and (4.5). For example, which Sturm attractors arise from the class of fully nonlinear diffusion in \( \text{Sturm}(u_{xx}, u_t) \)?

Where do the classes \( \text{Sturm}(x), \text{Sturm}(x, u), \text{Sturm}(u_x) \) and \( \text{Sturm}(u, u_x) \) fit in such a diagram? Can one give a permutation characterization to each of these classes, similar to Theorem 1 in [21] for Hamiltonian vector fields?

Next, denote by \( S(u) \) the category of objects given by the Fusco-Rocha permutations within the group of permutations \( S_n \), for all \( n \), satisfying the conditions of Theorem 1 in [21] that classify the permutations of Hamiltonian vector fields. As before, there are morphisms in this category that also arise from bifurcations, such as \( \sigma \mapsto p_*(\sigma) \) if \( p_*(\sigma) \) is obtained from \( \sigma \) after a pitchfork bifurcation and \( \sigma \mapsto s_n(\sigma) \) if \( s_n(\sigma) \) is achieved from \( \sigma \) after a saddle-node bifurcation, and morphisms that arise from topological constructions such as a suspension. Note that all previous categories are graded by the dimension \( n \) of the global attractor, for example \( S(u) = S_n(u) \times \mathbb{N}_0 \) where \( S_n(u) \) are the Fusco-Rocha permutations in the group \( S_n \) for Hamiltonian vector fields, or \( \text{Sturm}(u) = \text{Sturm}_n(u) \times \mathbb{N}_0 \) where \( \text{Sturm}_n(u) \) are the Sturm attractors of dimension \( n \). Note that the bifurcation morphisms do not preserve the graded structure, i.e., \( p_*(\sigma) \in S_{n+2} \) for \( \sigma \in S_n \).

The construction of Sturm attractors developed in the literature so far is a functor from the category of scalar parabolic equations of second order in one spatial variable, denoted by \( \text{parPDE}(2, 1) \), into the category \( \text{Sturm}(x, u, u_x, u_{xx}, u_t) \), and we call it the \textit{Sturm functor}, denoted by \( \text{Sturm} : \text{parPDE}(2, 1) \to \text{Sturm}(x, u, u_x, u_{xx}, u_t) \). The Sturm functor is not injective with respect to the objects: two different parabolic PDEs can yield the same attractor (up to \( C^0 \)-orbit equivalence) if they have the same Fusco-Rocha permutation. The Sturm functor can be factorized by the functor from \( \text{parPDE}(2, 1) \) to its Fusco-Rocha permutation in \( S(x, u, u_x, u_{xx}, u_t) \), called \textit{Fusco-Rocha functor} which we denote by \( \mathcal{FR} : \text{parPDE}(2, 1) \to S(x, u, u_x, u_{xx}, u_t) \), and the functor \( S : S(x, u, u_x, u_{xx}, u_t) \to \text{Sturm}(x, u, u_x, u_{xx}, u_t) \) which constructs the attractor from a given permutation. In other words, \( \text{Sturm} = \mathcal{FR} \circ S \).

The quest for expanding the partial differential equations we can explicitly describe its global attractor, and which kind of attractors arise from such equations, goes on.

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References

[1] S. Angenent. The Morse–Smale property for a semi-linear parabolic equation. J. Diff. Eq 62, 427 – 442, (1986).

[2] S. Angenent. The zero set of a solution of a parabolic equation. J. Reine Angew. Math. 390, 79 – 96, (1988).

[3] A.V. Babin and M.I. Vishik. Attractors of Evolution Equations. Elsevier Science, (1992).

[4] P. Brunovský and S.-N. Chow. Generic Properties of Stationary State Solutions of Reaction-Diffusion Equations. J. Diff. Eq. 53, 1–23, (1984).

[5] P. Brunovský and B. Fiedler. Numbers of Zeros on Invariant Manifolds in Reaction-diffusion Equations. Nonlinear Analysis: TMA 10, 179–193, (1986).

[6] P. Brunovský and B. Fiedler. Connecting orbits in scalar reaction diffusion equations II: The complete solution. J. Diff. Eq. 81, 106–135, (1989).

[7] P. Brunovský, R. Joly and G. Raugel. Generic Transversality of Heteroclinic and Homoclinic Orbits for Scalar Parabolic Equations. To appear in J. Dyn. Diff. Eq., (2019).

[8] P. Brunovský and P. Poláčik. The Morse-Smale Structure of a Generic Reaction-Diffusion Equation in Higher Space Dimension. J. Diff. Eq. 135, 119–181, (1997).

[9] L. Caffarelli and U. Stefanelli. A Counterexample to $C^{2,1}$ Regularity for Parabolic Fully Nonlinear Equations. Comm. PDE 33, 1216–1234, (2008).

[10] M. Crandall, M. Kocan and A. Świech. $L^p$-theory for fully nonlinear uniformly parabolic equations. Comm. PDE 25, 1997–2053, (2000).

[11] H. Dong, N.V. Krylov. On the Existence of Smooth Solutions for Fully Nonlinear Parabolic Equations with Measurable “Coefficients” without Convexity Assumptions. Comm. PDE 38, 1038–1068, (2013).

[12] B. Fiedler. Global attractors of one-dimensional parabolic equations: sixteen examples. Tatra Mountains Math. Publ. 4, 67–92, (1994).

[13] B. Fiedler. Do global attractors depend on boundary conditions? Doc. Math. J. DMV 1, 215–228, (1996).

[14] B. Fiedler and C. Rocha. Heteroclinic orbits of semilinear parabolic equations. J. Diff. Eq. 125, 239–281, (1996).

[15] B. Fiedler, C. Rocha. Realization of meander permutations by boundary value problems. J. Diff. Eq. 156, 282 – 308, (1999).

[16] B. Fiedler and C. Rocha. Orbit Equivalence of Global Attractors of Semilinear Parabolic Differential Equations. Trans. Am. Math. Soc. 352, 257–284, (2000).

[17] B. Fiedler, C. Rocha and M. Wolfrum. Heteroclinic orbits between rotating waves of semilinear parabolic equations on the circle. J. Diff. Eq. 201, 99–138, (2004).

[18] B. Fiedler and C. Rocha. Connectivity and Design of Planar Global Attractors of Sturm Type. I: Bipolar Orientations and Hamiltonian Paths. J. Reine Angew. Math. 635, 76–96, (2009).

[19] B. Fiedler and C. Rocha. Connectivity and Design of Planar Global Attractors of Sturm Type. II: Connection Graphs. J. Diff. Eq. 24, 1255–1286, (2008).
[20] B. Fiedler and C. Rocha. Connectivity and Design of Planar Global Attractors of Sturm Type. III: Small and Platonic Examples. *J. Dyn. Diff. Eq.* 2, 121–162, (2010).

[21] B. Fiedler, C. Rocha and M. Wolfrum. A permutation characterization of Sturm global attractors of Hamiltonian type. *J. Diff. Eq.* 252, 588 – 623, (2012).

[22] B. Fiedler, C. Rocha and M. Wolfrum. Sturm global attractors for $S^1$-equivariant parabolic equations. *Netw. Heterog. Media* 7, 617 – 659, (2012).

[23] B. Fiedler and C. Rocha. Nonlinear Sturm Global Attractors: Unstable Manifold Decompositions as Regular CW-Complexes. *Disc. Cont. Dyn. Sys.* 34, 5099–5122, (2014).

[24] B. Fiedler, C. Rocha. Schoenflies Spheres as Boundaries of Bounded Unstable Manifolds in Gradient Sturm Systems. *J. Dyn. Diff. Eq.* 27, 597 – 626, (2015).

[25] B. Fiedler and C. Rocha. Sturm 3-ball global attractors 1: Thom-Smale complexes and meanders. *São Paulo J. Math. Sci.* 12, 18–67, (2018).

[26] B. Fiedler and C. Rocha. Sturm 3-ball global attractors 2: Design of Thom-Smale complexes. *J. Dyn. Diff. Eq.* 31, 1549–1590, (2019).

[27] B. Fiedler and C. Rocha. Sturm 3-ball global attractors 3: Examples of Thom-Smale complexes. *Disc. Cont. Dyn. Sys.* 38, 3479–3545, (2018).

[28] B. Fiedler and C. Rocha. Meanders, zero numbers and the cell structure of Sturm global attractors. *arXiv:2002.00218*, (2021).

[29] G. Fusco and J. Hale. Stable Equilibria in a Scalar Parabolic Equation with Variable Diffusion. *SIAM J. on Math. Analysis* 16, 1152–1164, (1985).

[30] G. Fusco and C. Rocha. A permutation related to the dynamics of a scalar parabolic PDE. *J. Diff. Eq.* 91, 111–137, (1991).

[31] J. Hale, L. Magalhães, and W. Oliva. *An Introduction to Infinite Dimensional Dynamical Systems — Geometric Theory.* Springer New York, (1984).

[32] J. Hale and C. Rocha. Bifurcations in a Parabolic Equation with Variable Diffusion. *Nonlinear Analysis: TMA* 9, 479–494, (1985).

[33] J. Hale. Dynamics of a scalar parabolic equation. *Canadian App. Math. Quarterly* 12, 239–314, (1989).

[34] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag New York, (1981).

[35] D. Henry. Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations. *J. Diff. Eq.* 59, 165–205, (1985).

[36] C. Imbert and L. Silvestre. An Introduction to Fully Nonlinear Parabolic Equations. *An Introduction to the Kähler-Ricci Flow, Lecture Notes in Mathematics* 2086, Springer, eds. S. Boucksom, P. Eyssidieux, V. Guedj, 7–88, (2011).

[37] N. Krylov. Boundedly nonhomogeneous elliptic and parabolic equations. *Akad. Nauk SSSR, ser. mat.* 46, 487–523, (1982). English translation in *Math. USSR Izvestija* 20, 459–492, (1983).

[38] P. Lappicy. Sturm attractors for quasilinear parabolic equations. *J. Diff. Eq.* 265, 4642–4660, (2018).

[39] P. Lappicy and B. Fiedler. A Lyapunov function for fully nonlinear parabolic equations in one spatial variable. *São Paulo J. Math. Sc.* 13, 283–291, (2019).
[40] P. Lappicy. Sturm attractors for quasilinear parabolic equations with singular coefficients. *J. Dyn. Diff. Eq.* **32**, 359–390, (2020).

[41] K. Lu. Structural stability for scalar parabolic equations. *J. Diff. Eq.* **114** 253–271, (1994).

[42] A. Lunardi. On a Class of Fully Nonlinear Parabolic Equations. *Comm. PDE* **16** 145–172, (1991).

[43] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Springer Basel, (1995).

[44] H. Matano. Non increase of the lapnumber for a one dimensional semilinear parabolic equation. *J. Fac. Sci. Univ. Tokyo IA Math* **29**, 401–441, (1982).

[45] H. Matano. Asymptotic behavior of solutions of semilinear heat equations on $S^1$. *Nonlinear Diffusion Equations and Their Equilibrium States II*, eds. W.-M. Ni, L. A. Peletier, J. Serrin, 139–162, (1988).

[46] A. Mielke. Locally Invariant Manifolds for Quasilinear Parabolic Equations. *Rocky Mountain J. Math.* **21**, 707–714, (1991).

[47] O.A. Oleinik and S.N. Kruzhkov. Quasi-linear second-order parabolic equations with many independent variables. *Russian Math. Surveys* **16**, 105–146, (1961).

[48] W.M. Oliva. Morse-Smale semiflows. Openess and A-stability in Differential Equations and Dynamical Systems. *Proc. conference in Lisbon 2000, Fields Institute Communication* **31**, 285 – 307, (2002).

[49] J. Pimentel and C. Rocha. A permutation related to non-compact global attractors for slowly non-dissipative systems. *J. Dyn. Diff. Eq.* **28**, 1–28, (2016).

[50] C. Rocha. Generic Properties of Equilibria of Reaction-Diffusion Equations. *Proc. Roy. Soc. Edinburgh*, 45–55, (1985).

[51] C. Rocha. Examples of attractors in scalar reaction-diffusion equations. *J. Diff. Eq.* **73**, 178–195, (1988).

[52] C. Rocha. Properties of the Attractor of a Scalar Parabolic PDE. *J. Dyn. Diff. Eq.* **3**, 575–591, (1991).

[53] C. Rocha. On the Singular Problem for the Scalar Parabolic Equation with Variable Diffusion. *J. Math. Analysis and App.* **183**, 413–428, (1994).

[54] C. Sturm. Sur une classe d’équations à différences partielles. *J. Math. Pures. Appl.* **1**, 373–444, (1836).

[55] N. Uraltseva, O. Ladyzhenskaya, and V.A. Solonnikov. *Linear and Quasi-linear Equations of Parabolic Type*. American Mathematical Society, (1968).

[56] L. Wang. On the regularity theory of fully nonlinear parabolic equations I. *Comm. Pure Appl. Math.* **45**, 27–76, (1992).

[57] L. Wang. On the regularity theory of fully nonlinear parabolic equations II. *Comm. Pure Appl. Math.* **45**, 141–178, (1992).

[58] M. Wolfrum. A Sequence of Order Relations: Encoding Heteroclinic Connections in Scalar Parabolic PDE. *J. Diff. Eq.* **183**, 56–78, (2002).

[59] T. I. Zelenyak. Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable. *Differ. Uravn.* **4**, 34–45, (1968).