Research Article

New Stability Criterion for Fractional-Order Quaternion-Valued Neural Networks Involving Discrete and Leakage Delays

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Abstract

In the current work, we are devoted to the issue of uniform stability of fractional-order quaternion-valued neural networks involving discrete and leakage delays. Making use of the contracting mapping theory, we prove that the equilibrium point of the involved fractional-order quaternion-valued neural networks exists and is unique. Taking advantage of mathematical analysis strategy, a sufficient criterion involving delay to verify the global uniform stability for the considered fractional-order quaternion-valued neural networks is set up. Computer simulation figures are displayed to sustain the rationality of the established conclusions. This study generalizes and supplements the research of Xiu et al. (2020).

1. Introduction

It is public knowledge that that neural networks own broad application prospects in numerous aspects and many deals such as pattern recognition, artificial intelligence, graph manipulation, psychophysics, control engineering, bioscience, and so on [1–3]. Generally speaking, time delay usually arises in artificial neural networks and biological systems due to the time lag of signal transmission. The study shows that the time delay often brings about some unexpected dynamical phenomena such as loss of stability, periodic vibration, chaos, and so on [4]. Thus, it is an important task for us to reveal the influence of time delay on various dynamical phenomena in delayed neural networks. During the past several decades, plenty of scholars made great effort to investigate a great variety of dynamical behaviors of delayed neural networks and fruitful results have been reported. For instance, Kong et al. [5] discussed the periodic and homoclinic solutions of discontinuous delayed neural networks. In 2019, Aouiti et al. [6] obtained the sufficient condition to ensure the existence and exponential stability of piecewise pseudo almost periodic solution to neutral-type inertial neural networks involving delays and impulses. In 2019, Huang et al. [7] set up a sufficient criterion to guarantee the existence of anti-periodic solutions and exponential stability for shunting inhibitory cellular neural networks involving proportional time delays by applying Lyapunov functional, inequality skills, and some mathematical analyses. In 2020, Abdelaziz and Chérif [8] carried out the study on the piecewise asymptotic almost periodic solutions of fuzzy Cohen–Grossberg neural networks involving impulsive effect. Xu and Li [9] did a valuable and novel work on anti-periodic solution to delayed cellular neural networks involving D operator. For more detailed publications, one can refer to [10–12].

All the above works are only concerned with real-valued neural networks. Here we would like to point out that there are other types of multidimensional valued neural networks. As the extension of real-valued neural networks (RVNNs), complex-valued neural networks (CVNNs) occupy an important position in handling signal and intrinsic information of neural networks. Especially, they are often applied in different physical waves such as sound wave, elastic wave, electronic wave, optical wave, and so on. In 1843, Hamilton [13] proposed quaternion-valued neural networks (QVNNs), which are extension version of RVNNs and
CVNNs. The skew of quaternion is given by $Q = \{y = y^e + iy^j + jy^k\}$, where $y^e, y^j, y^k \in \mathbb{R}$ and $i, j, k$ obey the following operation:

\[
\begin{align*}
ij &= -ji = k, \\
jk &= -kj = i, \\
ki &= -ik = j, \\
j^2 &= k^2 = ijk = -1.
\end{align*}
\]

The investigation on QVNNs has attracted great attention from a lot of scholars since they have been found to have tremendous application in numerous areas such as color light imaging, image impression, spatial rotation, three dimension geometrical affine transformation, and so on [14–16]. At present, some fruits on the dynamics of QVNNs have been reported. For example, Lin et al. [17] dealt with the global exponential synchronization problem of inertial memristor-based QVNNs with delays. Jiang and Wang [18] studied the almost periodic solutions of delayed QVNNs. You et al. [19] made a detailed analysis on the exponential stability of discrete-time quaternion-valued neural networks with leakage delay and discrete delays. For more concrete literatures, we refer the readers to [20–24].

It is worth mentioning that all the above works on quaternion-valued neural networks mainly focus on the integer-order case and does not involve the fractional-order ones. The study on fractional-order neural networks has been keeping a very slow level due to the lack of actual background and fractional calculus theories. With the development of the research on fractional calculus, it is recognized that fractional-order differential system has greater advantages than classical integer-order one since it can give a description of the hereditary trait and memory nature for many materials and dynamic processes [14, 15]. Recently, lots of fractional-order neural networks have already aroused high attention from academic circles and a great deal of excellent fruits on fractional-order neural networks have been reported constantly. For instance, Udhayakumar et al. [25] focused on the multiple $\psi$-type stability issue for fractional-order quaternion-valued neural networks. Liu et al. [26] set up a set of sufficient conditions to guarantee the asymptotic synchronization of fractional-order neural networks involving delays. Du and Lu [27] investigated the finite-time synchronization problem for fractional-order delayed memristor-based neural networks. For more detailed studies, we refer the readers to [28–37]. However, there are few publications on fractional-order quaternion-valued neural networks. Stimulated by the discussion above, in this work, we will explore the research on the stability for fractional-order quaternion-valued neural networks involving delays. In a word, this work will mainly focus on the following issues: (a) prove the existence and uniqueness of equilibrium point of fractional-order quaternion-valued neural networks; (b) set up the sufficient criterion to ensure global uniform stability of fractional-order quaternion-valued neural networks.

In 2017, Zhang et al. [4] studied the following complex-valued neural networks:

\[
\frac{d^\alpha u_i(t)}{dt^\alpha} = -y_i u_i(t - \tau) + \sum_{h=1}^{m} \alpha_{ih} g_h(u_h(t)) + \sum_{h=1}^{m} \beta_{ih} g_h(u_h(t - \tau)) + L_i,
\]

where $i = 1, 2, \ldots, m, \rho \in (0, 1), u_i(t) \in \mathbb{C}$ ($\mathbb{C}$ denotes the set of complex numbers) stands for the state of the ith neuron at time $t$, $\alpha_{ih}, \beta_{ih} \in \mathbb{C}$ stand for the connection weight without and with time delays, respectively, $L_i \in \mathbb{C}$ denotes the external input, $\delta, \varrho \geq 0$ stand for the transmission delay and the leakage delay, respectively, and $g_h \in \mathbb{C}$ denotes the activation function. For details, one can see [4]. Making use of contraction mapping principle, a sufficient condition to ensure the existence and uniqueness of the equilibrium point for system (2) is set up. Applying mathematical analysis skills, a set of delay-dependent criteria to check the global uniform stability of system (2) is established.

In this present work, we modify system (2) as the following fractional-order quaternion-valued neural networks:

\[
\frac{d^\alpha u_i(t)}{dt^\alpha} = -y_i u_i(t - \tau) + \sum_{h=1}^{m} \alpha_{ih} g_h(u_h(t)) + \sum_{h=1}^{m} \beta_{ih} g_h(u_h(t - \tau)) + L_i,
\]

where $i = 1, 2, \ldots, m, \rho \in (0, 1), u_i(t) \in \mathbb{C}$ stands for the state of the ith neuron at time $t$, $\alpha_{ih}, \beta_{ih} \in \mathbb{C}$ stand for the connection weight without and with time delays, respectively, $L_i \in \mathbb{C}$ denotes the external input, $\delta, \varrho \geq 0$ stand for the transmission delay and the leakage delay, respectively, and $g_h \in \mathbb{C}$ denotes the activation function. For details, one can see [4]. Making use of contraction mapping principle, a sufficient condition to ensure the existence and uniqueness of the equilibrium point for system (2) is set up. Applying mathematical analysis skills, a set of delay-dependent criteria to check the global uniform stability of system (2) is established.

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\]

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This paper is organized as follows. In Section 2, the basic definitions, lemmas, and essential theories on fractional calculus and quaternion algebra are given. In Section 3, the existence and uniqueness of solution of model (3) are stated. In Section 4, a new delay-dependent criterion to check the global uniform stability of model (3) is derived. In Section 5, software simulation plots are presented to support the derived chief conclusions of this study. Section 6 ends this paper.

2. Preliminaries and Assumptions

Now we give some related notations. $\mathbb{L}^p \times \mathbb{L}^q$ stands for the set of positive integer numbers, and $i, j, k$ are imaginary units. $\mathbb{Q}^n, \mathbb{R}^{m \times n}$, $\mathbb{Q}^{m \times n}$ stand for the set of $n$-dimensional quaternion-valued vectors, $m \times n$ real-valued matrices, and quaternion-valued matrices. The norm of quaternion-valued matrices $M = (m_{ij})_{n \times n} \in \mathbb{Q}^{m \times n}$ is given by $\|M\| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |m_{ij}|^2}$. Denote $C([-T, 0], R^m)$ the Banach space of continuous $m$-real vector functions defined on $[-T, 0]$ with the norm $\|\phi(t)\| = \max_{t \in [-T, 0 \cup \{0\}]} \sup_{\|\phi(t)\|} |e^t\phi(t)|$, where $\phi \in C([-T, 0], R^m)$. For $u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^T \in C([0, +\infty), R^m)$, the norm of $u(t)$ is defined by $\|u(t)\| = \max_{t \in [1, 2, \ldots, m]} \sup_{\|\phi(t)\|} |e^t\phi(t)|$

Definition 1 (see [38]). The Caputo fractional-order derivative with order $\rho$ for the function $u(a)$ is given by

\[
\frac{d^\alpha u_i(t)}{dt^\alpha} = -y_i u_i(t - \tau) + \sum_{h=1}^{m} \alpha_{ih} g_h(u_h(t)) + \sum_{h=1}^{m} \beta_{ih} g_h(u_h(t - \tau)) + L_i,
\]
\[
\mathcal{D}^\rho u(\beta) = \frac{1}{\Gamma(l-\rho)} \int_\beta^\infty \frac{u(t)}{(t-\gamma)^{l-1}} \mathrm{d}t, \tag{4}
\]

where \(u(\beta) \in ([\beta_0, \infty), \mathbb{R}), \Gamma(s) = \int_0^\infty \beta^{s-1} e^{-\beta \delta} \mathrm{d}\beta, \beta \geq \beta_0, \Gamma(l) \equiv \int_0^\infty \beta^{l-1} e^{-\beta \delta} \mathrm{d}\beta\) and \(l \in \mathbb{Z}^+, l - 1 \leq \rho < l\). \(\mathcal{D}^\rho u(\beta)\) implies that \(\mathcal{D}^\rho u(\beta)/\mathrm{d}\beta\) is a Caputo fractional-order derivative operator.

For each \(y = y^R + iy^I + jy^J + ky^K \in \mathcal{C}\) and each activation \(g: \mathcal{C} \rightarrow \mathcal{C}\), \(g_h\) has the form:

\[
g(y) = g^R(y^R, y^I, y^J, y^K) + g^I(y^R, y^I, y^J, y^K) + g^J(y^R, y^I, y^J, y^K) + g^K(y^R, y^I, y^J, y^K). \tag{5}
\]

Thus, \(g_h(u_h)\) in system (3) has the following form:

\[
g_h(u_h) = g^R_h(y^R_h, y^I_h, y^J_h, y^K_h) + g^I_h(y^R_h, y^I_h, y^J_h, y^K_h) + g^J_h(y^R_h, y^I_h, y^J_h, y^K_h) + g^K_h(y^R_h, y^I_h, y^J_h, y^K_h). \tag{6}
\]

where \(h = 1, 2, \ldots, m\).

System (3) can be rewritten as the following equivalent form:

\[
\mathcal{D}^\rho u(t) = -\mathcal{A}u(t - \gamma) + \mathcal{B}g(u(t)) + \mathcal{C}g(u(t - \delta)) + \mathcal{L}, \tag{7}
\]

where \(u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^\top \in \mathbb{C}^m, \mathcal{A} = (a_{ij})_{m \times m} \in \mathbb{C}^{m \times m}, \mathcal{B} = (b_{ij})_{m \times m} \in \mathbb{C}^{m \times m}, \mathcal{C} = (c_{ij})_{m \times m} \in \mathbb{C}^{m \times m}, \mathcal{L} = (L_{ij})_{m \times m} \in \mathbb{C}^{m \times m}\).

The initial values of (3) can be written as follows:

\[
u_h(s) = \phi_h^R(s) + i\phi_h^I(s) + j\phi_h^J(s) + k\phi_h^K(s), \quad s \in [-\tau, 0], h = 1, 2, \ldots, m, \tag{8}
\]

where \(\tau = \max\{\gamma, \delta\}, \phi_h^R(s), \phi_h^I(s), \phi_h^J(s), \phi_h^K(s) \in C([-\tau, 0], \mathbb{R})\).

Definition 2 (see [39]). We say that the equilibrium point \(u^*\) of model (3) is stable provided that for every \(\varepsilon > 0, \exists \xi = \eta(t_0, \varepsilon) > 0\) which satisfies \(t \geq t_0 > 0, \|u(t) - u^*\| \leq \xi\) which implies \(\|u(t, t_0, \phi) - u^*\| \leq \varepsilon\) for every solution \(u(t, t_0, \phi)\) of model (1). The equilibrium point \(u^*\) of model (3) is uniformly stable provided that \(\eta\) has nothing to do with \(t_0\).

Lemma 1 (see [40]). Let \(u(t) \in C^\infty[0, \infty)\) and \(n - 1 < \rho < n \in \mathbb{Z}^+_\), then, the following equalities hold:

\[
\begin{align*}
|g^R_h(y^R_h, y^I_h, y^J_h, y^K_h) - g^R_h(x^R_h, x^I_h, x^J_h, x^K_h)| \leq & \; |g^R_h|_{y^R_h - x^R_h} + |g^R_h|_{y^I_h - x^I_h} + |g^R_h|_{y^J_h - x^J_h} + |g^R_h|_{y^K_h - x^K_h}, \\
|g^I_h(y^R_h, y^I_h, y^J_h, y^K_h) - g^I_h(x^R_h, x^I_h, x^J_h, x^K_h)| \leq & \; |g^I_h|_{y^R_h - x^R_h} + |g^I_h|_{y^I_h - x^I_h} + |g^I_h|_{y^J_h - x^J_h} + |g^I_h|_{y^K_h - x^K_h}, \\
|g^J_h(y^R_h, y^I_h, y^J_h, y^K_h) - g^J_h(x^R_h, x^I_h, x^J_h, x^K_h)| \leq & \; |g^J_h|_{y^R_h - x^R_h} + |g^J_h|_{y^I_h - x^I_h} + |g^J_h|_{y^J_h - x^J_h} + |g^J_h|_{y^K_h - x^K_h}, \\
|g^K_h(y^R_h, y^I_h, y^J_h, y^K_h) - g^K_h(x^R_h, x^I_h, x^J_h, x^K_h)| \leq & \; |g^K_h|_{y^R_h - x^R_h} + |g^K_h|_{y^I_h - x^I_h} + |g^K_h|_{y^J_h - x^J_h} + |g^K_h|_{y^K_h - x^K_h},
\end{align*}
\tag{9}
\]
for every \( x_h^i, x_h^j, x_h^k \in R \) and \( h = 1, 2, \ldots, m \).

3. Existence and Uniqueness

In this part, we investigate the existence and uniqueness of the equilibrium point for model \((3)\) (i.e., \((7)\)). Let \( u^* \in \mathcal{O}^m \) be the equilibrium point for model \((7)\); then, we have

\[
-u^* + \mathcal{B} g(u^*) + \mathcal{C} g(u^*) + \mathcal{L} = 0.
\]  

Then,

\[
\theta_{h1} = \left( \mathcal{G} \right)_h^{RR} 2 + \mathcal{G} \mathcal{G}_h^{RR} \mathcal{G} \mathcal{G}_h^{RI} + \mathcal{G} \mathcal{G}_h^{RR} \mathcal{G} \mathcal{G}_h^{KI} + \left( \mathcal{G} \right)_h^{RR} 2 + \mathcal{G} \mathcal{G}_h^{RR} \mathcal{G} \mathcal{G}_h^{KI} + \left( \mathcal{G} \right)_h^{RR} 2 + \mathcal{G} \mathcal{G}_h^{RR} \mathcal{G} \mathcal{G}_h^{KI} \right.
\]

From \((11)\), we know that if we can prove the following map: \( \Gamma: \mathcal{O}^m \rightarrow \mathcal{O}^m \),

\[
\Gamma(u) = \mathcal{A}^{-1} (\mathcal{B} + \mathcal{C}) g(u^*) + \mathcal{A}^{-1} \mathcal{L}
\]

has a unique fixed point, then we can conclude that the equilibrium point of model \((7)\) exists and is unique.

Theorem 1. If \((\mathcal{H}_1)\) and \((\mathcal{H}_2)\) hold, then model \((3)\) owns a unique equilibrium point.

Proof. Let \( u_1 = u_1^R + iu_1^I + ju_1^J + ku_1^K \) and \( u_2 = u_2^R + iu_2^I + ju_2^J + ku_2^K \). Then,

\[
g_h(u_1) - g_h(u_2) = \left| g_h(u_1) + ig_h(u_1) + jg_h(u_1) + kg_h(u_1) - (g_h(u_2) + ig_h(u_2) + jg_h(u_2) + kg_h(u_2)) \right|^2
\]

\[
= \left| g_h(u_1) - g_h(u_2) \right|^2 + \left| g_h^R(u_1) - g_h^R(u_2) \right|^2 + \left| g_h^I(u_1) - g_h^I(u_2) \right|^2 + \left| g_h^J(u_1) - g_h^J(u_2) \right|^2 + \left| g_h^K(u_1) - g_h^K(u_2) \right|^2
\]

\[
\leq \left[ \mathcal{G} \mathcal{G}_h^{RR} \left| \left| u_1^R - u_2^R \right|^2 + \mathcal{G} \mathcal{G}_h^{RI} \left| \left| u_1^R - u_2^R \right|^2 + \mathcal{G} \mathcal{G}_h^{KI} \left| \left| u_1^R - u_2^R \right|^2 \right] \right] \right.
\]
\begin{align}
&= (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{K}|^2 + (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{R}|^2 + (g_{R}^{2})^{2} |u_{1h}^{I} - u_{2h}^{I}|^2 + (g_{R}^{2})^{2} |u_{1h}^{K} - u_{2h}^{K}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{I} |u_{1h}^{R} - \nu_{l}^{I} - u_{2h}^{R} - u_{2h}^{I}|^2 + 2g_{R}^{2}R_{h}R_{l}^{R} |u_{1h}^{R} - \mu_{l}^{R} - u_{2h}^{R} - u_{2h}^{R}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{K} - u_{2h}^{K}|^2 + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \\
&\quad \left( (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{R}|^2 + (g_{R}^{2})^{2} |u_{1h}^{I} - u_{2h}^{I}|^2 + (g_{R}^{2})^{2} |u_{1h}^{K} - u_{2h}^{K}|^2 + (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{K}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{I} |u_{1h}^{R} - \nu_{l}^{I} - u_{2h}^{R} - u_{2h}^{I}|^2 + 2g_{R}^{2}R_{h}R_{l}^{R} |u_{1h}^{R} - \mu_{l}^{R} - u_{2h}^{R} - u_{2h}^{R}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{K} - u_{2h}^{K}|^2 + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \right) \\
&\quad \left( (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{R}|^2 + (g_{R}^{2})^{2} |u_{1h}^{I} - u_{2h}^{I}|^2 + (g_{R}^{2})^{2} |u_{1h}^{K} - u_{2h}^{K}|^2 + (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{K}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{I} |u_{1h}^{R} - \nu_{l}^{I} - u_{2h}^{R} - u_{2h}^{I}|^2 + 2g_{R}^{2}R_{h}R_{l}^{R} |u_{1h}^{R} - \mu_{l}^{R} - u_{2h}^{R} - u_{2h}^{R}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{K} - u_{2h}^{K}|^2 + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \\
&\quad + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 + 2g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \right) \\
&\quad \leq (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{R}|^2 + (g_{R}^{2})^{2} |u_{1h}^{I} - u_{2h}^{I}|^2 + (g_{R}^{2})^{2} |u_{1h}^{K} - u_{2h}^{K}|^2 + (g_{R}^{2})^{2} |u_{1h}^{R} - u_{2h}^{K}|^2 \\
&\quad + g_{R}^{2}R_{h}R_{l}^{I} |u_{1h}^{R} - \nu_{l}^{I} - u_{2h}^{R} - u_{2h}^{I}|^2 + g_{R}^{2}R_{h}R_{l}^{R} |u_{1h}^{R} - \mu_{l}^{R} - u_{2h}^{R} - u_{2h}^{R}|^2 \\
&\quad + g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{K} - u_{2h}^{K}|^2 + g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \\
&\quad + g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 + g_{R}^{2}R_{h}R_{l}^{K} |u_{1h}^{R} - x_{l}^{K} - u_{2h}^{R} - u_{2h}^{K}|^2 \right)
\end{align}
Then, for any \( u_1, u_2 \in \mathcal{L}^m \), one has
\[
\| \Gamma(u_1) - \Gamma(u_2) \| = \| \mathcal{U}(g(u_1) - g(u_2)) \| \leq \theta \| u_1 - u_2 \|.
\]
(15)

In view of (\( \mathcal{H}_2 \)), one can easily know that \( \Gamma(u) \) is contractive map. Thus, \( \Gamma(u) \) owns a unique fixed point, which implies that model (3) owns a unique equilibrium point. The proof finishes. \( \square \)

4. Uniform Stability

In the current part, we explore the global uniform stability issue of the equilibrium point for model (3). Let
\[
\begin{align*}
\mathcal{P}_1^0 &= 1 - \left[ e^{-\gamma} y + \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RR} + |\alpha_p h| \mathcal{G}_{h}^{JR} + |\alpha_p h| \mathcal{G}_{h}^{IK} + |\alpha_p h| \mathcal{G}_{h}^{KR} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RR} + |\beta_p h| \mathcal{G}_{h}^{IR} + |\beta_p h| \mathcal{G}_{h}^{IK} + |\beta_p h| \mathcal{G}_{h}^{KR} \right) \right],
\mathcal{P}_2^0 &= \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RI} + |\alpha_p h| \mathcal{G}_{h}^{II} + |\alpha_p h| \mathcal{G}_{h}^{II} + |\alpha_p h| \mathcal{G}_{h}^{II} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_3^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_4^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_5^0 &= 1 - \left[ e^{-\gamma} y + \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RR} + |\alpha_p h| \mathcal{G}_{h}^{JR} + |\alpha_p h| \mathcal{G}_{h}^{IK} + |\alpha_p h| \mathcal{G}_{h}^{KR} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RR} + |\beta_p h| \mathcal{G}_{h}^{IR} + |\beta_p h| \mathcal{G}_{h}^{IK} + |\beta_p h| \mathcal{G}_{h}^{KR} \right) \right],
\mathcal{P}_6^0 &= \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RI} + |\alpha_p h| \mathcal{G}_{h}^{II} + |\alpha_p h| \mathcal{G}_{h}^{II} + |\alpha_p h| \mathcal{G}_{h}^{II} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_7^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_8^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_9^0 &= 1 - \left[ e^{-\gamma} y + \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RR} + |\alpha_p h| \mathcal{G}_{h}^{JR} + |\alpha_p h| \mathcal{G}_{h}^{IK} + |\alpha_p h| \mathcal{G}_{h}^{KR} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RR} + |\beta_p h| \mathcal{G}_{h}^{IR} + |\beta_p h| \mathcal{G}_{h}^{IK} + |\beta_p h| \mathcal{G}_{h}^{KR} \right) \right],
\mathcal{P}_{10}^0 &= \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RI} + |\alpha_p h| \mathcal{G}_{h}^{II} + |\alpha_p h| \mathcal{G}_{h}^{II} + |\alpha_p h| \mathcal{G}_{h}^{II} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_{11}^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_{12}^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_{13}^0 &= e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RI} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} + |\beta_p h| \mathcal{G}_{h}^{II} \right),
\mathcal{P}_{14}^0 &= 1 - \left[ e^{-\gamma} y + \sum_{h=1}^{m} \left( |\alpha_p h| \mathcal{G}_{h}^{RR} + |\alpha_p h| \mathcal{G}_{h}^{JR} + |\alpha_p h| \mathcal{G}_{h}^{IK} + |\alpha_p h| \mathcal{G}_{h}^{KR} \right) + e^{-\gamma} \sum_{h=1}^{m} \left( |\beta_p h| \mathcal{G}_{h}^{RR} + |\beta_p h| \mathcal{G}_{h}^{IR} + |\beta_p h| \mathcal{G}_{h}^{IK} + |\beta_p h| \mathcal{G}_{h}^{KR} \right) \right].
\end{align*}
\]
\[ \mathcal{G}_2 = \sum_{j=1}^{m} \left[ a^I_{p,j} \varphi^I_h + a^I_{p,j} \varphi^I_h + a^R_{p,j} \varphi^R_h + a^K_{p,j} \varphi^K_h \right] + e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_3 = \sum_{j=1}^{m} \left[ a^I_{p,j} \varphi^I_h + a^I_{p,j} \varphi^I_h + a^R_{p,j} \varphi^R_h + a^K_{p,j} \varphi^K_h \right] + e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_4 = \sum_{j=1}^{m} \left[ a^I_{p,j} \varphi^I_h + a^I_{p,j} \varphi^I_h + a^R_{p,j} \varphi^R_h + a^K_{p,j} \varphi^K_h \right] + e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_5 = 1 + e^{-y} + e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_6 = e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_7 = e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_8 = e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right], \]

\[ \mathcal{G}_9 = 1 + e^{-y} + e^{-\frac{m}{3}} \sum_{j=1}^{m} \left[ \beta^I_{p,j} \varphi^I_h + \beta^I_{p,j} \varphi^I_h + \beta^R_{p,j} \varphi^R_h + \beta^K_{p,j} \varphi^K_h \right] \]

where \( y = \max_{i=1,\ldots,m} \gamma_i \) and \( p_1, p_2, p_3, p_4 \in \{1, 2, \ldots, m\} \).

\textbf{Theorem 2.} If \((\mathcal{H}_1), (\mathcal{H}_4)\) hold, then equilibrium point of system (3) is uniformly stable.

\textbf{Proof.} In view of Theorem 1, we know that system (3) has a unique equilibrium point \( u^* = (u_1^*, u_2^*, \ldots, u_m^*)^T \). Let \( \bar{u}(t) = u(t) - u^* \); then, system (3) becomes

\[ \frac{d^r \bar{u}(t)}{dt^r} = -\alpha \bar{u}(t - \gamma) + \mathcal{B} f(\bar{u}) + \mathcal{C} f(\bar{u}(t - \theta)), \]

where \( f(\bar{u}(t)) = g(u(t)) - g(u^*) \). The initial value

\[ \bar{u}_h(s) = \varphi_1^h(s) + \bar{\varphi}_2^h(s) + \bar{\varphi}_3^h(s) + \bar{\varphi}_4^h(s), \quad s \in [-\tau, 0], \quad h = 1, 2, \ldots, m, \]
where \( \tau = \max \{ \gamma, \delta \} \), \( \phi^R_h(s), \phi^I_h(s), \phi^K_h(s) \in C([-\tau, 0], R) \), and \( \phi^K_h(s) = \phi^R_h(s) - u^*_h, \phi^I_h(s) = \phi^I_h(s) - u^*_h, \phi^K_h(s) = \phi^K_h(s) - u^*_h \) and \( \bar{u}(t) = (\bar{u}_1(t), \bar{u}_2(t), \ldots, \bar{u}_m(t))^T \). Let \( \bar{u}(t) = \bar{u}^R(t) + j\bar{u}^I(t) + j\bar{u}^K(t) \), where

\[
\begin{align*}
\frac{d^n \bar{u}_h^R(t)}{dt^n} + i \frac{d^n \bar{u}_h^I(t)}{dt^n} + j \frac{d^n \bar{u}_h^I(t)}{dt^n} + k \frac{d^n \bar{u}_h^K(t)}{dt^n} &= -\gamma_1 \bar{u}_h^R(t - \gamma) + \sum_{h=1}^{m} \left[ \alpha_{ih}^R \bar{u}_h^R(t) - \alpha_{ih}^I \bar{u}_h^I(t) - \alpha_{ih}^K \bar{u}_h^K(t) \right] \\
&+ \sum_{h=1}^{m} \left[ \beta_{ih}^R \bar{u}_h^R(t) - \beta_{ih}^I \bar{u}_h^I(t) - \beta_{ih}^K \bar{u}_h^K(t) \right] \\
&+ i \left\{ -\gamma_1 \bar{u}_h^I(t - \gamma) + \sum_{h=1}^{m} \left[ \alpha_{ih}^I \bar{u}_h^R(t) + \alpha_{ih}^R \bar{u}_h^I(t) - \alpha_{ih}^K \bar{u}_h^K(t) \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^I \bar{u}_h^R(t) + \beta_{ih}^R \bar{u}_h^I(t) + \beta_{ih}^K \bar{u}_h^K(t) \right] \right\} \\
&+ j \left\{ -\gamma_1 \bar{u}_h^I(t - \gamma) + \sum_{h=1}^{m} \left[ \alpha_{ih}^K \bar{u}_h^R(t) + \alpha_{ih}^R \bar{u}_h^I(t) + \alpha_{ih}^I \bar{u}_h^K(t) \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^K \bar{u}_h^R(t) + \beta_{ih}^R \bar{u}_h^I(t) + \beta_{ih}^I \bar{u}_h^K(t) \right] \right\} \\
&+ k \left\{ -\gamma_1 \bar{u}_h^K(t - \gamma) + \sum_{h=1}^{m} \left[ \alpha_{ih}^K \bar{u}_h^R(t) + \alpha_{ih}^R \bar{u}_h^I(t) + \alpha_{ih}^I \bar{u}_h^K(t) \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^K \bar{u}_h^R(t) + \beta_{ih}^R \bar{u}_h^I(t) + \beta_{ih}^I \bar{u}_h^K(t) \right] \right\},
\end{align*}
\]

(22)

where

\[
\begin{align*}
g_h^R &= g_h^R(\bar{u}_h^R(t), \bar{u}_h^I(t), \bar{u}_h^K(t)), \\
g_h^I &= g_h^I(\bar{u}_h^R(t), \bar{u}_h^I(t), \bar{u}_h^K(t)), \\
g_h^K &= g_h^K(\bar{u}_h^R(t), \bar{u}_h^I(t), \bar{u}_h^K(t)), \\
g_h^R &= g_h^R(\bar{u}_h^R(t - \gamma), \bar{u}_h^I(t - \gamma), \bar{u}_h^K(t - \gamma)), \\
g_h^I &= g_h^I(\bar{u}_h^R(t - \gamma), \bar{u}_h^I(t - \gamma), \bar{u}_h^K(t - \gamma)), \\
g_h^K &= g_h^K(\bar{u}_h^R(t - \gamma), \bar{u}_h^I(t - \gamma), \bar{u}_h^K(t - \gamma)).
\end{align*}
\]

(23)

It follows from (22) that
\[
\begin{align*}
\frac{d^2 \tilde{u}_i^R(t)}{dt^2} &= -\gamma_i \tilde{u}^R_i(t - \tau) + \sum_{h=1}^{m} \left[ \alpha_{ih}^R g_h^R - \alpha_{ih}^l g_h^l - \alpha_{ih}^K g_h^K \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^R g_h^R - \beta_{ih}^l g_h^l - \beta_{ih}^K g_h^K \right], \\
\frac{d^2 \tilde{u}_i^I(t)}{dt^2} &= -\gamma_i \tilde{u}^I_i(t - \tau) + \sum_{h=1}^{m} \left[ \alpha_{ih}^I g_h^R + \alpha_{ih}^l g_h^l + \alpha_{ih}^K g_h^K \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^I g_h^R + \beta_{ih}^l g_h^l + \beta_{ih}^K g_h^K \right], \\
\frac{d^2 \tilde{u}_i^J(t)}{dt^2} &= -\gamma_i \tilde{u}^J_i(t - \tau) + \sum_{h=1}^{m} \left[ \alpha_{ih}^J g_h^R + \alpha_{ih}^l g_h^l + \alpha_{ih}^K g_h^K \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^J g_h^R + \beta_{ih}^l g_h^l + \beta_{ih}^K g_h^K \right], \\
\frac{d^2 \tilde{u}_i^K(t)}{dt^2} &= -\gamma_i \tilde{u}^K_i(t - \tau) + \sum_{h=1}^{m} \left[ \alpha_{ih}^K g_h^R - \alpha_{ih}^l g_h^l + \alpha_{ih}^K g_h^K \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^K g_h^R - \beta_{ih}^l g_h^l + \beta_{ih}^K g_h^K \right],
\end{align*}
\]

where \( i = 1, 2, \ldots, m \). In view of Lemma 1, one gets

\[
\begin{align*}
\tilde{u}_i^R(t) - \tilde{u}_i^R(0) &= \frac{d^2 \tilde{u}_i^R(t)}{dt^2} - \frac{d^2 \tilde{u}_i^R(0)}{dt^2} \\
\tilde{u}_i^I(t) - \tilde{u}_i^I(0) &= \frac{d^2 \tilde{u}_i^I(t)}{dt^2} - \frac{d^2 \tilde{u}_i^I(0)}{dt^2} \\
\tilde{u}_i^J(t) - \tilde{u}_i^J(0) &= \frac{d^2 \tilde{u}_i^J(t)}{dt^2} - \frac{d^2 \tilde{u}_i^J(0)}{dt^2} \\
\tilde{u}_i^K(t) - \tilde{u}_i^K(0) &= \frac{d^2 \tilde{u}_i^K(t)}{dt^2} - \frac{d^2 \tilde{u}_i^K(0)}{dt^2}
\end{align*}
\]

From the first equation of (25), one has

\[
\tilde{u}_i^R(t) = \tilde{u}_i^R(0) + \int_0^t (s - \theta)^{\rho - 1} \left\{ -\gamma_i \tilde{u}_i^R(t - \theta) + \sum_{h=1}^{m} \left[ \alpha_{ih}^R g_h^R - \alpha_{ih}^l g_h^l - \alpha_{ih}^K g_h^K \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^R g_h^R - \beta_{ih}^l g_h^l - \beta_{ih}^K g_h^K \right] \right\} ds,
\]

which leads to

\[
\left| \tilde{u}_i^R(t) \right| \leq \left| \tilde{u}_i^R(0) \right| + \frac{1}{\Gamma(\rho)} \int_0^t (t - s)^{\rho - 1} \left\{ -\gamma_i \tilde{u}_i^R(t - \theta) + \sum_{h=1}^{m} \left[ \alpha_{ih}^R g_h^R + \alpha_{ih}^l g_h^l + \alpha_{ih}^K g_h^K \right] + \sum_{h=1}^{m} \left[ \beta_{ih}^R g_h^R + \beta_{ih}^l g_h^l + \beta_{ih}^K g_h^K \right] \right\} ds,
\]
\[
+ \sum_{h=1}^{m} \left[ \frac{\alpha^{R}_{h}}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} |\overline{u}_{h}^{R}(s)| \, ds + \sum_{h=1}^{m} \left[ \frac{\beta^{K}_{h}}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} |\overline{u}_{h}^{K}(s)| \, ds \right] + \sum_{h=1}^{m} \left[ \frac{\beta^{L}_{h}}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} |\overline{u}_{h}^{L}(s)| \, ds \right] + \sum_{h=1}^{m} \left[ \frac{\alpha^{L}_{h}}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} |\overline{u}_{h}^{L}(s)| \, ds \right] \right] ds
\]

\[
\leq |\overline{u}_{R}^{R}(0)| + \gamma \frac{1}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} |\overline{u}_{R}^{K}(t - \gamma)| \, ds
\]

\[
+ \sum_{h=1}^{m} \left[ \frac{\alpha^{R}_{h}}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} \left[ \mathcal{G}_{h}^{R} |\overline{u}_{h}^{R}(s)| + \mathcal{G}_{h}^{K} |\overline{u}_{h}^{K}(s)| + \mathcal{G}_{h}^{L} |\overline{u}_{h}^{L}(s)| + \mathcal{G}_{h}^{K} |\overline{u}_{h}^{K}(s)| \right] ds \right]
\]

\[
+ \sum_{h=1}^{m} \left[ \frac{\beta^{R}_{h}}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} \left[ \mathcal{G}_{h}^{R} |\overline{u}_{h}^{R}(s)| + \mathcal{G}_{h}^{K} |\overline{u}_{h}^{K}(s)| + \mathcal{G}_{h}^{L} |\overline{u}_{h}^{L}(s)| + \mathcal{G}_{h}^{K} |\overline{u}_{h}^{K}(s)| \right] ds \right]
\]

\[
= |\overline{u}_{R}^{R}(0)| + \gamma \frac{1}{\Gamma (\rho)} \int_{0}^{t} (t - s)^{-\rho} |\overline{u}_{R}^{K}(t - \gamma)| \, ds
\]
Multiplying by $e^{-t}$ by (23) leads to

$$e^{-t}h^R(0) = e^{-t}h^R(0) + \gamma \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^R(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^R(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^R(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^I(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^I(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^J(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^J(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^K(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t}h^K(s) ds$$

$$= e^{-t}h^R(0) + \gamma \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^R(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^R(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^R(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^J(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^J(s) ds$$

$$+ \sum_{h=1}^m \left( |\alpha^h_R| \mathcal{G}^{RR}_h + |\alpha^h_I| \mathcal{G}^{RI}_h + |\alpha^h_J| \mathcal{G}^{JR}_h + |\alpha^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^K(s) ds$$

$$+ \sum_{h=1}^m \left( |\beta^h_R| \mathcal{G}^{RR}_h + |\beta^h_I| \mathcal{G}^{RI}_h + |\beta^h_J| \mathcal{G}^{JR}_h + |\beta^h_K| \mathcal{G}^{KR}_h \right) \frac{1}{G(\rho)} \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^K(s) ds$$

$$\times \int_0^t (t-s)^{\rho-1} e^{-t+s}e^{-\Theta}e^{-s\Theta}h^R(s) ds$$
\[\sum_{h=1}^{m} \left( |\alpha^R_h| \varphi^R_h + |\beta^I_h| \varphi^I_h + |\alpha^I_h| \varphi^R_h + |\beta^I_h| \varphi^I_h \right) \frac{1}{\Gamma(\rho)} \times \int_0^t (t-s)^{\rho-1} e^{-t-s} e^{-s} e^{-s} |u^R_h(s-\theta)| ds + \sum_{h=1}^{m} \left( |\alpha^R_h| \varphi^R_h + |\beta^I_h| \varphi^I_h + |\alpha^I_h| \varphi^R_h + |\beta^I_h| \varphi^I_h \right) \frac{1}{\Gamma(\rho)} \times \int_0^t (t-s)^{\rho-1} e^{-t-s} e^{-s} e^{-s} |u^R_h(s-\theta)| ds \]

\[+ e^{-\theta} \gamma \left[ \sup_{s \in [0, t]} \left\{ e^{-s} |u^R_h(s)| \right\} \right] \frac{1}{\Gamma(\rho)} \int_0^t \theta^{-1} e^{-s} ds \]

\[+ \sum_{h=1}^{m} \left( |\alpha^R_h| \varphi^R_h + |\beta^I_h| \varphi^I_h + |\alpha^I_h| \varphi^R_h + |\beta^I_h| \varphi^I_h \right) \left( \left| \sup_{s \in [0, t]} \left\{ e^{-s} |u^R_h(s)| \right\} \right| \right) \frac{1}{\Gamma(\rho)} \int_0^t \theta^{-1} e^{-s} ds \]

\[+ \sum_{h=1}^{m} \left( |\alpha^R_h| \varphi^R_h + |\beta^I_h| \varphi^I_h + |\alpha^I_h| \varphi^R_h + |\beta^I_h| \varphi^I_h \right) \left( \left| \sup_{s \in [0, t]} \left\{ e^{-s} |u^R_h(s)| \right\} \right| \right) \frac{1}{\Gamma(\rho)} \int_0^t \theta^{-1} e^{-s} ds \]

\[+ \sum_{h=1}^{m} \left( |\alpha^R_h| \varphi^R_h + |\beta^I_h| \varphi^I_h + |\alpha^I_h| \varphi^R_h + |\beta^I_h| \varphi^I_h \right) \left( \left| \sup_{s \in [0, t]} \left\{ e^{-s} |u^R_h(s)| \right\} \right| \right) \frac{1}{\Gamma(\rho)} \int_0^t \theta^{-1} e^{-s} ds \]
\[+ \sum_{h=1}^{m} \left( \left| \alpha^R_{ih} \mathcal{G}^{R|} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{I|} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{J|} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{K|} h \right| \right) \sup_{\sigma \leq t} \left\{ e^{-\gamma(t)} \tilde{v}_h(\sigma) \right\} \frac{1}{\Gamma(\rho)} \int_0^t v^{\rho-1} e^{-v} dv \]

\[+ \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \sup_{\sigma \leq t} \left\{ e^{-\gamma(t)} \tilde{u}_h(\sigma) \right\} \frac{1}{\Gamma(\rho)} \int_0^t v^{\rho-1} e^{-v} dv \]

\[+ e^{-\gamma \rho} \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \sup_{\sigma \leq t} \left\{ e^{-\gamma(t)} \tilde{v}_h(\sigma) \right\} \frac{1}{\Gamma(\rho)} \int_0^t v^{\rho-1} e^{-v} dv \]

\[+ e^{-\gamma \rho} \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \sup_{\sigma \leq t} \left\{ e^{-\gamma(t)} \tilde{u}_h(\sigma) \right\} \frac{1}{\Gamma(\rho)} \int_0^t v^{\rho-1} e^{-v} dv \]

\[\leq \| \hat{\phi}(t) \| + e^{-\gamma \rho} \| \hat{\phi}^{(R)}(t) \| + e^{-\gamma \rho} \| \hat{u}_h^{(R)}(s) \| \]

\[+ \sum_{h=1}^{m} \left( \left| \alpha^R_{ih} \mathcal{G}^{RR} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{RI} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{RI} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{KK} h \right| \right) \| \hat{v}_h^{(R)}(s) \| \]

\[+ \sum_{h=1}^{m} \left( \left| \alpha^R_{ih} \mathcal{G}^{RI} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{IR} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{JR} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{IK} h \right| \right) \| \hat{u}_h^{(R)}(s) \| \]

\[+ \sum_{h=1}^{m} \left( \left| \alpha^R_{ih} \mathcal{G}^{R|} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{I|} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{J|} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{K|} h \right| \right) \| \hat{v}_h^{(R)}(s) \| \]

\[+ e^{-\gamma \rho} \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \| \hat{v}_h^{(R)}(s) \| \]

\[+ e^{-\gamma \rho} \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \| \hat{u}_h^{(R)}(s) \| \]

\[+ e^{-\gamma \rho} \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \| \hat{v}_h^{(R)}(s) \| \]

\[+ e^{-\gamma \rho} \sum_{h=1}^{m} \left( \left| \beta^R_{ih} \mathcal{G}^{RR} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{RI} h \right| + \left| \beta^R_{ih} \mathcal{G}^{KK} h \right| \right) \| \hat{u}_h^{(R)}(s) \| \]

\[= \| \hat{\phi}^{(R)}(t) \| + e^{-\gamma \rho} \| \hat{\phi}^{(R)}(t) \| + e^{-\gamma \rho} \| \hat{u}_h^{(R)}(s) \| \]

\[+ \sum_{h=1}^{m} \left( \left| \alpha^R_{ih} \mathcal{G}^{RR} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{RI} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{RI} h \right| + \left| \alpha^R_{ih} \mathcal{G}^{KK} h \right| \right) \| \hat{v}_h^{(R)}(s) \| \]
It is easy to see that \( \exists p_1 \in \{1, 2, \ldots, m\} \) such that

\[
\max \left\{ \sup_t \left\{ e^{-t} |\bar{u}_R(t)| \right\} \right\} = \sup_t \left\{ e^{-t} |\bar{u}_{p_1}(t)| \right\}.
\]

Then, it follows from (28) that

\[
\|\bar{u}_R(t)\| = \max \left\{ \sup_t \left\{ e^{-t} |\bar{u}_R(t)| \right\} \right\} = \sup_t \left\{ e^{-t} |\bar{u}_{p_1}(t)| \right\}
\]

\[
\leq e^{-\gamma} y + \sum_{h=1}^m \left( |a^{R}_{p_1,h}| |\varphi^R_h| + |a^{I}_{p_1,h}| |\varphi^I_h| + |a^{K}_{p_1,h}| |\varphi^K_h| \right) + e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right)
\]

\[
\|\bar{u}^R(\sigma)\|
\]

\[
\leq \sum_{h=1}^m \left( |a^{R}_{p_1,h}| |\varphi^R_h| + |a^{I}_{p_1,h}| |\varphi^I_h| + |a^{K}_{p_1,h}| |\varphi^K_h| \right) + e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right)
\]

\[
\|\bar{u}^I(\sigma)\|
\]

\[
\leq \sum_{h=1}^m \left( |a^{R}_{p_1,h}| |\varphi^R_h| + |a^{I}_{p_1,h}| |\varphi^I_h| + |a^{K}_{p_1,h}| |\varphi^K_h| \right) + e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right)
\]

\[
\|\bar{u}^K(\sigma)\|
\]

\[
+ 1 + e^{-\gamma} y + e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right) \|\varphi^R(t)\|
\]

\[
+ e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right) \|\varphi^I(t)\|
\]

\[
+ e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right) \|\varphi^K(t)\|
\]

\[
+ e^{-\gamma} \sum_{h=1}^m \left( |\beta^{R}_{p_1,h}| |\varphi^R_h| + |\beta^{I}_{p_1,h}| |\varphi^I_h| + |\beta^{K}_{p_1,h}| |\varphi^K_h| \right) \|\varphi^R(t)\|.
\]
Then, we have

$$
\|u^p(t)\| \leq \frac{1}{\Phi_1} \left[ \alpha_1^p \|\bar{u}^p(t)\| + \alpha_2^p \|\bar{u}'(t)\| + \alpha_3^p \|\bar{u}^K(t)\| + \alpha_4^p \|\bar{\phi}^R(t)\| + \alpha_5^p \|\bar{\phi}'(t)\| + \alpha_6^p \|\bar{\phi}^K(t)\| \right].
$$

(31)

In a similar way, we can obtain

$$
\|u^R(t)\| \leq \frac{1}{\Phi_1} \left[ \beta_1^R \|\bar{u}^R(t)\| + \beta_2^R \|\bar{u}'(t)\| + \beta_3^R \|\bar{u}^K(t)\| + \beta_4^R \|\bar{\phi}^R(t)\| + \beta_5^R \|\bar{\phi}'(t)\| + \beta_6^R \|\bar{\phi}^K(t)\| \right],
$$

(32)

$$
\|u'(t)\| \leq \frac{1}{\Phi_1} \left[ \gamma_1^I \|\bar{u}^I(t)\| + \gamma_2^I \|\bar{u}'(t)\| + \gamma_3^I \|\bar{u}^I(t)\| + \gamma_4^I \|\bar{\phi}^I(t)\| + \gamma_5^I \|\bar{\phi}'(t)\| + \gamma_6^I \|\bar{\phi}^I(t)\| \right],
$$

(33)

$$
\|u^K(t)\| \leq \frac{1}{\Phi_1} \left[ \delta_1^K \|\bar{u}^K(t)\| + \delta_2^K \|\bar{u}'(t)\| + \delta_3^K \|\bar{u}^K(t)\| + \delta_4^K \|\bar{\phi}^K(t)\| + \delta_5^K \|\bar{\phi}'(t)\| + \delta_6^K \|\bar{\phi}^K(t)\| \right].
$$

(34)

By (31)–(34), we obtain

$$
\left(\alpha_1^p - \beta_1^R - \gamma_1^I - \delta_1^K\right)\|\bar{u}^p(t)\| + \left(\beta_1^R - \alpha_2^p - \gamma_2^I - \delta_2^K\right)\|\bar{u}'(t)\| + \left(\gamma_1^I - \alpha_3^p - \beta_3^R - \delta_3^K\right)\|\bar{u}^K(t)\| + \left(\delta_1^K - \alpha_4^p - \beta_4^R - \gamma_4^I\right)\|\bar{\phi}^R(t)\| + \left(\delta_2^K - \alpha_5^p - \beta_5^R - \gamma_5^I\right)\|\bar{\phi}'(t)\| + \left(\gamma_3^I - \alpha_6^p - \beta_6^R - \delta_6^K\right)\|\bar{\phi}^K(t)\| \\
\leq \left(\alpha_1^p - \beta_1^R - \gamma_1^I - \delta_1^K\right)\|\bar{u}^p(t)\| + \left(\beta_1^R - \alpha_2^p - \gamma_2^I - \delta_2^K\right)\|\bar{u}'(t)\| + \left(\gamma_1^I - \alpha_3^p - \beta_3^R - \delta_3^K\right)\|\bar{u}^K(t)\| + \left(\delta_1^K - \alpha_4^p - \beta_4^R - \gamma_4^I\right)\|\bar{\phi}^R(t)\| + \left(\delta_2^K - \alpha_5^p - \beta_5^R - \gamma_5^I\right)\|\bar{\phi}'(t)\| + \left(\gamma_3^I - \alpha_6^p - \beta_6^R - \delta_6^K\right)\|\bar{\phi}^K(t)\|.
$$

(35)

It follows from (35) that

$$
\|\bar{u}^R(t)\| + \|\bar{u}'(t)\| + \|\bar{u}^I(t)\| + \|\bar{u}^K(t)\| \leq \Pi_1\|\bar{\phi}^R(t)\| + \Pi_2\|\bar{\phi}'(t)\| + \Pi_3\|\bar{\phi}^I(t)\| + \Pi_4\|\bar{\phi}^K(t)\|,
$$

(36)

where

$$
\begin{align*}
\Pi_1 &= \frac{\alpha_1^p + \beta_1^R + \gamma_1^I + \delta_1^K}{W} > 0, \\
\Pi_2 &= \frac{\alpha_2^p + \beta_2^R + \gamma_2^I + \delta_2^K}{W} > 0, \\
\Pi_3 &= \frac{\alpha_3^p + \beta_3^R + \gamma_3^I + \delta_3^K}{W} > 0, \\
\Pi_4 &= \frac{\alpha_4^p + \beta_4^R + \gamma_4^I + \delta_4^K}{W} > 0.
\end{align*}
$$

(37)

Thus, we can come to the conclusion that the unique equilibrium point $\alpha^*$ of system (3) is uniformly stable. This ends the proof. □

Remark 1. In 2017, Zhang et al. [4] analyzed the stability issue of fractional-order complex-valued neural networks involving leakage and discrete delays. In this study, we have investigated the stability of fractional-order quaternion-valued neural networks involving discrete and leakage delays. The investigation on the stability of fractional-order quaternion-valued neural networks becomes more complex than that of Zhang et al. [4] since the fractional-order quaternion-valued neural networks have been decomposed into more multidimensional neural networks. This study replenishes the research of Zhang et al. [4].

5. Simulation Results

Consider the fractional-order quaternion-valued neural networks:
Figure 1: Continued.
Figure 1: Continued.
where $u_1(t) = u_1^R(t) + iu_1^I(t) + ju_1^J(t) + ku_1^K(t)$, $u_2(t) = u_2^R(t) + iu_2^I(t) + ju_2^J(t) + ku_2^K(t)$, and

\[
\begin{align*}
\alpha_{11} &= 0.22 + 0.25i + 0.34j + 0.53k, \\
\alpha_{12} &= 0.33 + 0.43i - 0.67j + 0.45k, \\
\alpha_{21} &= 0.19 + 0.65i - 0.67j - 0.88k, \\
\alpha_{22} &= 0.43 + 0.35i - 0.23j - 0.38k, \\
\beta_{11} &= 0.56 - 0.55i - 0.76j + 0.39k, \\
\beta_{12} &= 0.22 + 0.79i + 0.81j - 0.66k, \\
\beta_{21} &= 0.89 + 0.34i + 0.73j - 0.48k, \\
\beta_{22} &= 0.35 + 0.84i + 0.67j - 0.85k, \\
L_1 &= 0.77 + 0.33i + 0.28j - 0.36k, \\
L_2 &= 0.67 + 0.34i + 0.57j - 0.62k, \\
g_h(u_h) &= 0.3 \cos u_h^R + 0.4i \sin(u_h^I + u_h^J + u_h^K) + 0.5j \sin u_h^I + 0.2k \sin(u_h^R + u_h^I + u_h^K).
\end{align*}
\]

It is easy to obtain that model (38) owns the zero equilibrium point. Let $\mathcal{G}_h^{RR} = \mathcal{G}_h^{RI} = \mathcal{G}_h^{RJ} = \mathcal{G}_h^{RK} = \mathcal{G}_h^{I} = \mathcal{G}_h^{II} = \mathcal{G}_h^{IK} = \mathcal{G}_h^{J} = \mathcal{G}_h^{IJ} = \mathcal{G}_h^{JK} = \mathcal{G}_h^{K} = \mathcal{G}_h^{James} = 1$. Utilizing the Matlab software, we obtain that $\theta = 0.7845$ and $\|\mathcal{Y}\| = 0.6812$, $\mathcal{W} = 0.6008$. Thus, we can check that all the hypotheses of Theorem 2 are fulfilled. Thus, one can conclude that the zero equilibrium point of model (38) is uniformly stable. Figure 1 shows that the state of four parts will be close to zero gradually.

6. Conclusions

Based on earlier research studies, a class of fractional-order quaternion-valued neural networks involving discrete and
leakage delays has been set up. A set of sufficient conditions guaranteeing the existence and uniqueness of the equilibrium point for the investigated fractional-order quaternion-valued neural networks has been derived. In addition, the global uniform stability of the involved fractional-order quaternion-valued neural networks has been systematically discussed. This work generalizes the work of Zhang et al. [4]. The research idea can also be utilized to investigate many other types of fractional-order quaternion-valued neural networks. In addition, we know that Clifford analysis is more general than quaternion one [36, 37]. We will study the global uniform stability of Clifford-valued neural networks in the near future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Data Availability

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Data Availability

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