Quantum chaotic system as a model of decohering environment

JAYENDRA N. BANDYOPADHYAY (a)

Max-Planck Institute for the Physics of Complex Systems - Nöthnitzerstr. 38, D-01187 Dresden, Germany, EU

received 22 December 2008; accepted in final form 11 February 2009
published online 19 March 2009

PACS 05.45.Mt – Quantum chaos; semiclassical methods
PACS 03.65.Yz – Decoherence; open systems; quantum statistical methods
PACS 03.67.-a – Quantum information

Abstract – As a model of decohering environment, we show that a quantum chaotic system behaves equivalently as a many-body system. An approximate formula for the time evolution of the reduced density matrix of a system interacting with a quantum chaotic environment is derived. This theoretical formulation is substantiated by the numerical study of the decoherence of two qubits interacting with a quantum chaotic environment modeled by a chaotic kicked top. Like the many-body model of environment, the quantum chaotic system is an efficient decoherer, and it can generate entanglement between the two qubits which have no direct interaction.

The interaction of a quantum system with the environment creates correlations between the states of the system and of the environment. These correlations destroy the superposition of the system states — a phenomenon known as decoherence [1]. This phenomenon is believed to be responsible for the quantum-to-classical transition. Decoherence is also a major obstacle for designing a quantum computational and informational protocol [2]. Therefore, a deeper understanding of this phenomenon is required to address the fundamental questions like the quantum-classical transition and to develop a quantum computational protocol.

In general, the environment is modeled by a many-body system, e.g. infinitely many harmonic oscillators in thermal equilibrium (Feynman-Vernon or Caldeira-Leggett model) [3], the spin-boson model [4], the chaotic spin-chain [5], etc. In another approach, the random matrix model of the environment is used [6,7]. The random matrix theory has well-known connections with quantum chaotic systems. Hence, some studies have concentrated on the possibility of having quantum dissipation and decoherence due to the interaction with chaotic degrees of freedom [8]. Recently, as a model of decohering environment, a single-particle quantum chaotic system has been considered [9]. This paper shows that the kicked rotator, a well-studied model of chaotic system, can reproduce the decohering effects of a many-body environment. In comparison to complex many-body systems, this simple deterministic system is very convenient for numerical as well as analytical studies of decoherence. Hence, a single-particle quantum chaotic system warrants a special attention as a model of decohering environment.

In this letter, we establish the direct equivalence of the single-particle quantum chaotic environment and the Caldeira-Leggett-type model of many-body environment by providing a rigorous but straightforward treatment. We keep our results vis-à-vis a recent study in which decoherence in a quantum system is investigated under the influence of an environment consisting of a collection of harmonic oscillators [10]. Our derivation first assumes a weak interaction between the system and the environment. Using the interaction strength as a small parameter, we perform a perturbative-theory calculation. By exponentiating the perturbative expansion, we get an approximate formula for the non-perturbative strong-interaction effect of the environment on the system. The approximate formula is then justified by numerical evidences. In numerics, we study the decoherence of two non-interacting qubits which are individually interacting with a common quantum chaotic environment. We use a chaotic kicked top, a very well-studied model of quantum chaotic system [11], as the environment.

The most general form of the Hamiltonian of a system \( S \), interacting with an environment \( E \), is \( H = H_S + H_E + H_{SE} \), where \( H_S \) and \( H_E \) are the Hamiltonians of the system and the environment, respectively, and \( H_{SE} \) is the system-environment coupling Hamiltonian.

We assume throughout this letter that the decoherence time is much smaller than the system characteristic time.
Hence we can neglect any dynamics of the isolated system and can discard $H_S$. We consider a kicked quantum chaotic system as a model of the environment, so the general form of the time-dependent system-environment Hamiltonian is:

$$
H(t) = I_S \otimes H_E(t) + H_I(t),
$$

where $H_E(t) = H_1 + H_2 \sum_n \delta(t-n)$ and $H_I(t) = \alpha V_S \otimes V_E$; $V_S$ and $V_E$ are the coupling agents of the system and the environment, respectively. The parameter $\alpha$ determines the strength of the interaction. We now assume that $V_E$ commutes with $H_1$. This is a very natural assumption as it separates the environment dynamics from the system-environment interaction. The corresponding time-evolution operator $U$ in between two consecutive kicks is:

$$
U = U_I U_0 = U_I (I_S \otimes U_E),
$$

where the coupling part $U_I = \exp(-i \alpha V_S \otimes V_E)$, $I_S$ is a unit matrix which indicates the absence of any dynamics in the system, and $U_E = \exp(-i H_1) \exp(-i H_2)$. Furthermore, we assume that the initial system-environment joint state is an unentangled pure state of the form $|\Psi_{SE}(0)\rangle = |\psi_S(0)\rangle \otimes |\psi_E(0)\rangle$. We measure the decoherence of the system by the loss of purity $P(n) \equiv \text{Tr} \{\rho_S(n)^2\}$, which varies from $1$, for the pure state, to $1/N_S$ for the completely mixed state, $N_S$ being the Hilbert space dimension of $S$. And $\rho_S(n) \equiv \text{Tr}_E \{ |\Psi_{SE}(n)\rangle \langle \Psi_{SE}(n)| \}$ is the system reduced density matrix (RDM), where $|\Psi_{SE}(n)\rangle$ is the joint state of the system and the environment at time $n$.

We are interested in studying the entanglement between the two non-interacting qubits due to their interaction with a common chaotic environment. Following Wootters, the entanglement is measured by computing the concurrence

$$
C[\rho_S(n)] = \max (\Lambda, 0),
$$

where

$$
\Lambda \equiv \sqrt{\lambda_0} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}
$$

and $\lambda_i$'s are the eigenvalues of the matrix $R(n) = \rho_S(n) \rho_S(n)^\dagger$; $\rho_S(n) \equiv |\sigma_0 \otimes \sigma_0\rangle \langle \sigma_0 \otimes \sigma_0|$, $\rho_S(n)^\dagger$ being the complex conjugation of $\rho_S(n)$ in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The concurrence varies from $0$, for the separable state, to $1$ for the maximally entangled state.

Our perturbation theory approach is reminiscent of the method followed by Tanaka et al. in the context of the entanglement production between two coupled chaotic systems [13]. We define the interaction picture of the joint density matrix $\tilde{\rho}_{SE}(n) \equiv U_0(n)^\dagger \rho_{SE}(n) U_0(n)$, and of any arbitrary operator $O(n) \equiv U_0(n)^\dagger O U_0(n)$, where $U_0(n) \equiv (I_S \otimes U_E)^n$, and $O(n) \equiv \text{Tr}_E \{ |\Psi_{SE}(n)\rangle \langle \Psi_{SE}(n)| \}$ represents the free evolution of $O$. Thus, the time evolution of $\tilde{\rho}_{SE}(n)$ is determined by the mapping:

$$
\tilde{\rho}_{SE}(n) = U_I(n) \tilde{\rho}_{SE}(n-1) U_I(n)^\dagger.
$$

The perturbative expansion for $U_I(n)$ is

$$
U_I(n) = 1 - i \alpha V(n) - \frac{1}{2} \alpha^2 V(n)^2 + O(\alpha^3),
$$

where $V(n) \equiv V_S(n) \otimes V_E(n)$, $V_S(n)$ and $V_E(n)$ are, as usual, free evolutions of $V_S$ and $V_E$, respectively. We introduce the eigenvalues and the eigenvectors of $V_S$ as $V_S(s) = \lambda_s |s\rangle$. By tracing out the environment, we obtain the perturbative expansion of RDM of the system $S$ at time $n$ in the eigenbasis of $V_S$ up to $O(\alpha^2)$ as

$$
[\rho_S(n)]_{ss'} \simeq \left[ 1 - i \alpha (\lambda_s - \lambda_{s'}) g(n) - \frac{1}{2} \alpha^2 (\lambda_s - \lambda_{s'})^2 g(n)^2 - \alpha^2 (\lambda_s - \lambda_{s'}) f(n) - i \alpha^2 (\lambda_s^2 - \lambda_{s'}^2) \phi(n) \right] [\rho_S(0)]_{ss'},
$$

where

$$
g(n) \equiv \sum_{l=1}^{n} (V_E(l))^2, \quad f(n) \equiv \text{Re} \{ \Phi(n) \}, \quad \phi(n) \equiv \text{Im} \{ \Phi(n) \},
$$

and of any arbitrary operator $O(n) \equiv U_0(n)^\dagger O U_0(n)$, where $U_0(n) \equiv (I_S \otimes U_E)^n$, and $O(n) \equiv \text{Tr}_E \{ |\Psi_{SE}(n)\rangle \langle \Psi_{SE}(n)| \}$ represents the free evolution of $O$. Thus, the time evolution of $\tilde{\rho}_{SE}(n)$ is determined by the mapping:

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\tilde{\rho}_{SE}(n) = U_I(n) \tilde{\rho}_{SE}(n-1) U_I(n)^\dagger.
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The perturbative expansion for $U_I(n)$ is

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U_I(n) = 1 - i \alpha V(n) - \frac{1}{2} \alpha^2 V(n)^2 + O(\alpha^3),
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where $V(n) \equiv V_S(n) \otimes V_E(n)$, $V_S(n)$ and $V_E(n)$ are, as usual, free evolutions of $V_S$ and $V_E$, respectively. We introduce the eigenvalues and the eigenvectors of $V_S$ as $V_S(s) = \lambda_s |s\rangle$. By tracing out the environment, we obtain the perturbative expansion of RDM of the system $S$ at
bounded. Following ref. [13], we assume further phenomenological properties of $C_E(l,l')$: 1) In a strongly chaotic regime, the distribution function quickly becomes uniform in the phase space, hence $C_E(l,l)=(V_E(l))^2-(V_E(l'))^2$ becomes almost constant within a very short time. So we assume $C_E(l,l)\approx C_0$ for all the time interval. 2) $C_E(l,l')$ exponentially decays with the time interval $|l-l'|$ with an exponent $\gamma$, i.e., $C_E(l,l')\approx C_0 \exp(-\gamma|l-l'|)$. Substituting this in eq. (5), and performing the summations, we get [13]

$$f(n) \approx \frac{1}{2} C_0 \left[ \frac{n \coth(\frac{\gamma}{2}) - 1 - \exp(-\gamma n)}{\sinh(\gamma - 1)} \right].$$

(7)

Hence, the rate of change of $f(n)$ with time $n$ is

$$\frac{\Delta f(n)}{\Delta n} \approx \frac{1}{2} C_0 \left[ \frac{\coth(\frac{\gamma}{2}) - \gamma \exp(-\gamma n)}{\sinh(\gamma - 1)} \right].$$

(8)

At the beginning, when $n$ is very small,

$$\frac{\Delta f(n)}{\Delta n} \approx \frac{1}{2} \left( \frac{C_0 \gamma^2}{\sinh(\gamma - 1)} \right) n + \text{const},$$

(9)
i.e., $f(n)$ evolves quadratically with time ($f(n) \propto n^2$). On the other hand, when $n$ is very large,

$$\frac{\Delta f(n)}{\Delta n} \approx \frac{1}{2} C_0 \coth\left(\frac{\gamma}{2}\right).$$

(10)

This suggests a long-time linear behavior of $f(n)$. Here we identify two distinct time-dependent regimes of the decoherence function $f(n)$: a short-time quadratic evolution, and a long-time linear evolution. An identical behavior of the decoherence function is reported for the the environment consisting of a collection of harmonic oscillators [10]. Thus we finally arrive at our goal, and prove that a quantum chaotic system and a collection of harmonic oscillators behave equivalently as a model of decohering environment.

We now substantiate our results with numerics. Here we consider a chaotic kicked top as the model of environment. The total Hamiltonian is $H(t) = H_f(t) + H_E(t)$ where $H_f(t)$ is the system-environment interaction Hamiltonian, and $H_E(t)$ is the kicked-top Hamiltonian. As the environment, the following version of the kicked top model is used:

$$H_E(t) = \left( \beta J_z + \frac{k}{2j} J_j^z \right) + \frac{\pi}{2} J_y \sum_n \delta(t-n),$$

(11)

where $J_i$ is the $i$-th component of the angular-momentum operator of the top, $j$ is the size of the spin (here, $j=100$), $k$ is the parameter which decides chaoticity in the system. The most popular version of the kicked-top model does not contain the $\beta$-term. We introduce this extra term to remove the parity symmetry $RH(t)R^{-1} = H(t)$, where $R = \exp(i\pi J_y)$. The Hilbert space dimension of the kicked top is $N=2j+1=201$. We set the chaotic parameter at $k=99.0$ which corresponds to a classically strongly chaotic system, and the parameter $\beta$ at 0.47. The interaction Hamiltonian between the two qubits and the kicked top is

$$H_I(t) = \alpha h_s \otimes J_z.$$  

(12)

The parameter $\alpha$ determines the system-environment coupling strength. The system consists of two non-interacting qubits, and its coupling agent is $h_s = S_3^{(1)} \otimes I^{(2)} + I^{(1)} \otimes S_3^{(2)}$, where $S_3^{(1)} = \sigma_z^{(1)}/2$ and $\sigma_z^{(1)}$ is the third Pauli matrix. Hence, the computational basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ are also the eigenbasis of $h_s$ with eigenvalues $\{-1,0,0,1\}$. The states $|01\rangle$ and $|10\rangle$ are the degenerate eigenstates of $h_s$. Note that, according to eq. (4), the off-diagonal terms of the system state, corresponding to this degenerate subspace, will not decay with time.

The decoherence of the system is studied for two different initial states: 1) a Bell state $|\phi_s\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, and 2) a product state $|\phi_s\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. For all numerical studies, a generalized $SU(2)$ coherent state $|\phi_s\rangle$ is used as initial state for the kicked-top environment.

Figure 1 presents the result for the Bell-state case. In the computational basis, the diagonal elements of the initial density matrix $\rho_S(0)$ are $(1/2,0,0,1/2)$, and the two off-diagonal elements have non-zero values: $(00|\rho_S(0)|11) = (|11|\rho_S(0)|00) = -1/2$. These off-diagonal elements are not part of the degenerate subspace, and therefore they decay in time. So in the asymptotic limit: $\rho_S \approx \text{diag}(1/2,0,0,1/2)$, the purity $P$ and $\Lambda$ of this RDM are $1/2$ and 0, respectively. In fig. 1, we see that the purity $P$ and $\Lambda$ have reached values very close to these asymptotic values. The rate, at which these two quantities have reached the asymptotic values, is determined by the coupling strength $\alpha$. As expected, for weaker coupling...
the rate is slower than for stronger coupling. The most important fact is that our approximate formula for the RDM derived from the perturbation theory, is not only working well for the weak-coupling case but also agrees very well for the stronger-coupling cases.

Figure 2 presents the result for the product state case. Initially $\Lambda = 0$, and then it becomes negative till $n = 1142$ for $\alpha = 0.0001$, $n = 81$ for $\alpha = 0.001$, and $n = 4$ for $\alpha = 0.005$. Up to this time, the entanglement between two qubits determined by the concurrence was zero. After that, there is a time interval when $\Lambda > 0$, and the two qubits become entangled due to their common interaction with the chaotic environment. This shows that, like the many-body environment [17], the quantum chaotic environment can also generate entanglement between two non-interacting qubits which have no entanglement initially. In the asymptotic limit, the diagonal elements of $\rho_S$: $(1/4, 1/4, -1/4, -1/4)$, and the off-diagonal elements: $(01|\rho_S|10) = (10|\rho_S|01) = -1/4$ corresponding to the degenerate subspace, survive from decoherence. Therefore, in this limit, the purity $P = 3/8 = 0.375$, and $\Lambda = 0$. The figure shows again a good agreement between the numerics and the theory.

Our theoretical formulation establishes the equivalence between a quantum chaotic environment and a many-body environment by showing the equivalence between their decoherence function $f(n)$. Therefore, it is very important to investigate $f(n)$ of the kicked top carefully. In fig. 3, we plot the evolution of the decoherence function of the kicked top. Our numerical experiment suggests that the phenomenological formula for $f(n)$ given in eq. (7) is not an exact formula for all the time $n$. It describes the short-time quadratic and the long-time linear behavior well. So instead of fitting the numerics by the phenomenological formula, we interpolate it with a simple function $f(n) = an + bn^2$, a combination of a linear and a quadratic term. The interpolation (solid line of fig. 3) gives $a \approx 0.1776$ and $b \approx 1.1561 \times 10^{-3}$. We expect the parameter $a$ to be equal to the coefficient of the linear term of eq. (7), i.e. $\frac{1}{2}C_0 \coth(\gamma/2)$. For the chaotic kicked top, $C_0 = (J_1(l)^2 - (J_2(l))^2 \approx 1/3$. In addition, in the strong-chaos regime, the parameter $\gamma$ is very large which leads to $\coth(\gamma/2) \approx 1$. Hence, the coefficient of the linear term of eq. (7) is approximately equal to 1/6, which is very close to the value obtained for the parameter $a$, i.e., $a \approx 0.1776$. Thus we show that the decoherence function of the chaotic kicked top has properties which are similar to the properties of any other model of decohering environment.

In conclusion, as a model of decohering environment, we establish the equivalence between the quantum chaotic system and any other many-body systems. This is an important result, which suggests that instead of a complex many-body system one can use a simple chaotic system as a model of environment. Consequently, analytical investigations of environment-induced decoherence would be easier, and from the numerical point of view as well, this simple model requires very less computational resources.

We substantiate our analytical formulation with strong numerical evidences. These show that a quantum chaotic system is a good decoherer, and it can also create entanglement between two non-interacting particles. The later result is very important due to the recent identification of the entanglement as a resource for quantum computational and informational protocols [2].

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We thank Drs A. Tanaka, D. Cohen, and A. Lakshminarayan for useful comments and discussions.

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