Quantum states transformation under free operations plays a central role in the resource theory of coherence. In this paper, we investigate the transformation from a mixed coherent state into a pure one by using both incoherent operations and stochastic incoherent operations. We show that contrary to the strictly incoherent operations and the stochastic strictly incoherent operations, both the incoherent operations and the stochastic incoherent operations can increase the dimension of the maximal pure coherent subspace of a state. This means that the incoherent operations are generally stronger than the strictly incoherent operations when we want to transform a mixed coherent state into a pure coherent one. Our findings can also be interpreted as confirming the ability of incoherent operations to enhance the coherence of mixed states relative to certain coherence monotones under strictly incoherent operations.

I. INTRODUCTION

Quantum coherence is an important feature of quantum mechanics that is responsible for the departure between the classical and the quantum world. It is an essential component in quantum information processing [1], and it plays a central role in emergent fields, such as quantum metrology [2, 3], nanoscale thermodynamics [4–6], and quantum biology [7–10]. Recently, the quantification of coherence has attracted a growing interest due to the rapid development of quantum information science [11–15].

As a quantum resource theory, there are two fundamental ingredients: free states and free operations [16–18]. For the resource theory of coherence, the free states are quantum states which are diagonal in a prefixed reference basis. While, there are no general consensus on the set of free operations [14, 15]. Based on various physical and mathematical considerations, several free operations were presented, such as the maximally incoherent operations (MIO) [11], the incoherent operations (IOs) [13], the strictly incoherent operations (SIOs) [19, 20], and the physically incoherent operations (PIOs) [21, 22]. Here, we focus our attention on the IOs, which have the physical motivation that they cannot create coherence [13], and the SIOs, which have the physical motivation that they can neither create nor use coherence [20].

With these notions, a central topic of the resource theory of coherence is to study the states transformation under free operations. Investigations on this topic started from the deterministic transformation between pure coherent states. In Refs. [23, 24], the authors presented the necessary and sufficient conditions for the deterministic transformation between pure coherent states by using incoherent operations. Then, in Ref. [21], the author present that any pure state transformation by using strictly incoherent operations obeys the same necessary and sufficient conditions for the deterministic transformation of an incoherent operations. These results tell us that when we want to transform a pure state into another one deterministically, incoherent operations and strictly incoherent operations have the same power. After that, the probabilistic transformation between pure coherent states was studied in Refs. [24, 25]. Their results also show that the incoherent operations and strictly incoherent operations have the same power. More recently, we know that the power of incoherent operations and strictly incoherent operations is identical when performing the deterministic transformation between 2-dimensional mixed states [21, 26, 27]. Despite these results, one might still suspect that incoherent operations are more powerful than strictly incoherent operations in general.

In this paper, we show that the above conjecture is correct, i.e., we find that the incoherent operations are stronger than strictly incoherent operations when transform a mixed coherent state into a pure coherent state. Specifically, we will show that contrary to the strictly incoherent operations and the stochastic strictly incoherent operations, both the incoherent operations and the stochastic incoherent operations can increase the dimension of the maximal pure-coherent subspace of a state. Our findings imply that there is indeed an operational gap between incoherent operations and strictly incoherent operations under state transformations, which provide an answer to the open question in Ref. [21]. An interesting consequence of this results further imply there exist coherence monotones under strictly incoherent operations that can increase under incoherent operations.

This paper is organized as follows. In Sec. II, we recall some notions of the quantum resource theory of coherence, including incoherent operations, strictly incoherent operations, stochastic incoherent operations, and stochastic strictly incoherent operations. In Sec. III, we show our main results, i.e., show that the incoherent operations are generally stronger than strictly incoherent operations when we want to transform a mixed coherent state into a pure one. A concise summary of our results is presented in Sec. IV.
II. PRELIMINARIES

Let $\mathcal{H}$ be the Hilbert space of a $d$-dimensional quantum system. A particular basis of $\mathcal{H}$ is denoted as $|i\rangle$, $i = 1, 2, \ldots, d$, which is chosen according to the physical problem under consideration. Coherence of a state is then measured based on the basis chosen. Specifically, a state is said to be incoherent if it is diagonal in the basis. Any state which cannot be written as a diagonal matrix is defined as a coherent state. For a pure state $|\varphi\rangle$, we will denote $|\varphi\rangle\langle\varphi|$ as $\varphi$, i.e., $\varphi := |\varphi\rangle\langle\varphi|$ for the sake of simplicity.

An incoherent operation [13] is a completely positive trace-preserving map, expressed as

$$\Lambda(\rho) = \sum_{n=1}^{N} K_n \rho K_n^\dagger,$$

where the Kraus operators $K_n$ satisfy not only $\sum_n K_n^\dagger K_n = 1$ but also $K_n I K_n^\dagger \subset I$ for all $K_n$, i.e., each $K_n$ transforms an incoherent state into an incoherent state and such a $K_n$ is called an incoherent Kraus operator. While, a strictly incoherent operation [19, 20] is a completely positive trace-preserving map satisfying not only $\sum_n K_n^\dagger K_n = 1$ but also $K_n I K_n^\dagger \subset I$ for all $K_n$, i.e., each $K_n$ as well $K_n^\dagger$ transforms an incoherent state into an incoherent state and such a $K_n$ is called a strictly incoherent Kraus operator. Here, $I$ represents the set of incoherent states.

With the notions of the incoherent operation and the strictly incoherent operation, we can introduce the notion of the stochastic incoherent operation [28] and the stochastic strictly incoherent operation [29]. A stochastic incoherent operation is constructed by a subset of incoherent Kraus operators. Without loss of generality, we denote the subset as $\{K_1, K_2, \ldots, K_L\}$. Otherwise, we may renumber the subscripts of these Kraus operators. Then, a stochastic incoherent operation, denoted as $\Lambda_s(\rho)$, is defined by

$$\Lambda_s(\rho) = \sum_{n=1}^{L} K_n \rho K_n^\dagger \frac{\sum_{i=1}^{L} K_n^\dagger K_n}{\text{Tr}(\sum_{i=1}^{L} K_n^\dagger K_n)},$$

where $\{K_1, K_2, \ldots, K_L\}$ satisfies $\sum_{n=1}^{L} K_n^\dagger K_n \leq 1$. Clearly, the state $\Lambda_s(\rho)$ is obtained with probability $P = \text{Tr}(\sum_{n=1}^{L} K_n^\dagger K_n)$ under a stochastic incoherent operation $\Lambda_s$, while state $\Lambda(\rho)$ is fully deterministic under an incoherent operation $\Lambda$. Similarly, we can give the notion of the stochastic strictly incoherent operation by changing the incoherent Kraus operator into the strictly incoherent Kraus operator in Eq. (2).

A functional $C$ can be taken as a measure of coherence if it satisfies the four postulate [13, 20]: (C1) the coherence being zero for incoherent states; (C2) the monotonicity of coherence under incoherent operations or strictly incoherent operations; (C3) the monotonicity of coherence under selective measurements on average; and (C4) the non-increasing of coherence under mixing of quantum states. A coherence monotone satisfies (C1) and (C2), while a coherence measure satisfies (C1)-(C4). In accordance with the general criterion, several coherence measures have been put forward. Out of them, we recall the coherence rank [19], which will be considered in this paper. The coherence rank $C_r$ of a pure state (not necessarily normalized), $|\varphi\rangle = \sum_{i=1}^{R} c_i |i\rangle$ with $c_i \neq 0$, is defined as the number of nonzero terms of $c_i$ minus 1, i.e.,

$$C_r(\varphi) = R - 1.$$  \hspace{1cm} (3)

For a mixed state $\rho$, the coherence rank of it is defined as

$$C_r(\rho) = \inf_{|\varphi\rangle \langle\varphi|} \sum_i p_i C_r(\varphi_i),$$

where $p_i = \sum_i |\psi_i\rangle \langle\psi_i|$ is any decomposition of $\rho$ into pure states $\varphi_i$ with $p_i \geq 0$.

III. INCOHERENT OPERATIONS BEING STRONGER THAN STRICTLY INCOHERENT OPERATIONS

We begin our study by observing the difference between the structure of incoherent operations and that of strictly incoherent operations from their Kraus operators. This leads to the following Lemma [19, 23, 30].

Lemma 1.-- (a) For an incoherent Kraus operator, there is at most one nonzero element in each column of $K_n$;

(b) For a strictly incoherent Kraus operator, there is at most one nonzero element in each column and each row of $K_n$.

Equipped with the above Lemma, we now present the following theorem.

Theorem 1.-- Let $\rho$ be any 2-dimensional mixed states. Then for any coherent state $\varphi$, no stochastic incoherent operation can transform $\rho$ into $\varphi$ with nonzero probability.

Proof.--First, we show that if we want to judge whether there exists a stochastic incoherent operation such that

$$\Lambda(\rho) = \varphi,$$

we only need to consider the stochastic incoherent operation with the form of

$$\Lambda_s(\rho) = \frac{K \rho K^\dagger}{\text{Tr}(K \rho K^\dagger)}.$$ \hspace{1cm} (5)

To this end, we assume that we can transform a mixed state $\rho$ into a pure coherent state $\varphi$ by using some stochastic incoherent operation $\Lambda_s$, i.e.,

$$\Lambda_s(\rho) = \frac{\sum_{n=1}^{L} K_n \rho K_n^\dagger}{\text{Tr}(\sum_{n=1}^{L} K_n \rho K_n^\dagger)} = \varphi.$$ \hspace{1cm} (6)

Then, since pure states are extreme points of the set of states, there must be

$$\frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)} = \varphi$$

for all $n = 1, \ldots, L$. On the other hand, we note that

$$\Lambda_s(\rho) = \frac{K_n \rho K_n^\dagger}{\text{Tr}(K_n \rho K_n^\dagger)}$$

is also a stochastic incoherent operation. Thus, we obtain that if we can transform $\rho$ into $\varphi$ by using $\Lambda_s(\rho)$, then there exists a stochastic incoherent operation such that $\Lambda(\rho) = \varphi$. 

With the above result, to prove the theorem, it is enough to examine all the stochastic incoherent operations with the forms

$$\Lambda^{(s)}(\varphi) = \frac{K\rho K^\dagger}{\text{Tr}(K\rho K^\dagger)},$$

where $K$ being a $2 \times 2$ incoherent Kraus operators.

From the definition of incoherent operations, we can get that there are eight classes of incoherent Kraus operators with the form

$$K_1 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & 0 \\ a_3 & 0 \end{pmatrix}, \quad K_4 = \begin{pmatrix} 0 & 0 \\ 0 & a_4 \end{pmatrix},$$

$$K_5 = \begin{pmatrix} a_5 & b_5 \\ 0 & 0 \end{pmatrix}, \quad K_6 = \begin{pmatrix} a_6 & 0 \\ 0 & b_6 \end{pmatrix},$$

$$K_7 = \begin{pmatrix} 0 & 0 \\ a_7 & b_7 \end{pmatrix}, \quad K_8 = \begin{pmatrix} 0 & b_8 \\ a_8 & 0 \end{pmatrix},$$

where $a_n \neq 0$ and $b_n \neq 0$ for all $n = 1, \ldots, 8$. It is straightforward to see that $K_1, K_2, K_3, K_4, K_5, K_7$ cannot transform any mixed states into a pure coherent state with nonzero probability. Thus, we only need to examine the $\Lambda^{(s)}(\varphi) = K\rho K^\dagger$ with $n = 6, 8$. However, since $K_6$ and $K_8$ are strictly incoherent Kraus operators, by using a result in Ref. [29], which says a $d$-dimensional mixed state $\rho$ can never be transformed into a pure coherent state with its coherence rank being $d - 1$ by using a stochastic strictly incoherent operation, we immediately obtain that we cannot transform any 2-dimensional mixed state $\rho$ into a pure coherent state $\varphi$ by using stochastic incoherent operations. This completes the proof of the theorem.

With the above theorem, we immediately have the following corollary:

**Corollary 1.** Let $\rho$ be a 2-dimensional mixed state. Then, for any coherent state $|\varphi\rangle$, no incoherent operation can transform $\rho$ into $\varphi$ with probability one.

We now move onto the larger dimensional state space of 3 and show that stochastic incoherent operations and stochastic strictly incoherent operations are no longer the same. This arrive at the following theorem.

**Theorem 2.** Let $\rho$ be any 3-dimensional mixed state. If there is no incoherent projector $\mathbb{P}$ such that

$$\frac{\text{Tr}(\rho \mathbb{P} \mathbb{P})}{\text{Tr}(\rho \mathbb{P} \mathbb{P})} = \varphi,$$

with the coherence rank of $|\varphi\rangle$ being equal to or greater than 1. Then, (a) there are 3-dimensional mixed state $\rho$ and pure coherent state $\varphi$ such that the transformation from $\rho$ into $\varphi$ can be achieved by using stochastic incoherent operations; (b) it is impossible to transform $\rho$ into a pure coherent state $\varphi$ by using stochastic strictly incoherent operations.

Here, an incoherent projector is a projector which has the form $\mathbb{P}_i = \sum_{j=1}^{d} |i\rangle\langle j|$ with $1 \in \{1, \ldots, d\}$.

**Proof.** (a) Let us consider a special class of incoherent Kraus operators with the form

$$\bar{K} = k \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

where $a_1, a_2, a_3$ are all nonzero complex numbers and $k$ is some complex number such that $K^\dagger \bar{K} \leq 1$. We further choose $\rho = p_1|\varphi_1\rangle + p_2|\varphi_2\rangle$ with

$$|\varphi_1\rangle = |1\rangle + |2\rangle + |3\rangle,$$

$$|\varphi_2\rangle = |1\rangle + |2\rangle + |3\rangle.$$  

Suppose that we can transform $\rho$ into a pure state $|\varphi\rangle$ by using some stochastic incoherent operations if and only if there are $\bar{K}|\varphi_1\rangle = k_1|\varphi_1\rangle$, $\bar{K}|\varphi_2\rangle = k_2|\varphi_2\rangle$,

where $k_1$ and $k_2$ may be identical or not identical since we only consider the transformation in Eq. (9). Here we should note that this condition is independent of the ensemble decomposition of $\rho$. To see this, assume that $\rho = \sum_{i=1}^{n} \lambda_i |\lambda_i\rangle\langle \lambda_i|$ is another ensemble decomposition of $\rho$. Then, we can transform $\rho$ into a pure state $\varphi$ by using an incoherent operator $\bar{K}$ if and only if there are

$$\bar{K}|\lambda_i\rangle = k_i|\varphi\rangle,$$

for all $i = 1, \ldots, n$ with $k_i$ being complex numbers. On the other hand, $\{p_i, |\varphi_i\rangle\}$ is an ensemble for $\rho$ if and only if there exists a unitary matrix $U = (U_{ij})$ such that $[31]

$$\sqrt{p_i} |\varphi_i\rangle = \sum_j U_{ij} \sqrt{\lambda_j} |\lambda_j\rangle.$$

By using the condition in Eq. (13), we then obtain that

$$\bar{K}|\varphi_i\rangle = \frac{1}{\sqrt{p_i}} \sum_j U_{ij} \sqrt{\lambda_j} \bar{K}|\lambda_j\rangle = \sum_j U_{ij} \frac{1}{\sqrt{p_i}} \sqrt{\lambda_j} |\varphi_i\rangle

= k_i|\varphi\rangle.$$  

Now, we return to the Eq. (12) and further require a special case of $k_1$ and $k_2$ with $k_1 = k_2 = k$. Then by direct calculations, we immediately obtain

$$a_1 |\varphi_1\rangle + a_2 |\varphi_2\rangle = k \varphi_1$$

$$a_1 |\varphi_1\rangle + a_2 |\varphi_2\rangle = k \varphi_2$$

where we have used $|\varphi_1\rangle = |1\rangle + |2\rangle + |3\rangle$. Thus, any state $\rho$ with $|\varphi_1\rangle$ and $|\varphi_2\rangle$ and $\bar{K}$ satisfy the above equations can be transformed into the same pure state $|\varphi\rangle$ by using the stochastic incoherent operations

$$\Lambda^{(s)}(\rho) = \frac{\bar{K}\rho \bar{K}^\dagger}{\text{Tr}(\bar{K}\rho \bar{K}^\dagger)}.$$  

An explicit example is $\rho = \frac{1}{2} |\varphi_1\rangle\langle \varphi_1| + |\varphi_2\rangle\langle \varphi_2|$, where

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}} (\sin \frac{\pi}{12} |1\rangle + \cos \frac{\pi}{12} |2\rangle + |3\rangle)$$

$$|\varphi_2\rangle = \frac{1}{\sqrt{2}} (\cos \frac{\pi}{12} |1\rangle + \sin \frac{\pi}{12} |2\rangle + |3\rangle).$$
The corresponding incoherent Kraus operator is
\[
\tilde{K} = k \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\] (19)

In other words, we can transform the 3-dimensional mixed state
\[
\rho = \frac{1}{8} \begin{pmatrix} 2 & 1 & \sqrt{6} \\ \sqrt{6} & 2 & \sqrt{6} \\ \sqrt{6} & \sqrt{6} & 4 \end{pmatrix}
\] (20)
into the pure coherent state \( |\varphi\rangle \)
\[
\varphi = \frac{\tilde{K}\rho\tilde{K}^+}{\text{Tr}(K\rho K^+)} = \frac{1}{8} \begin{pmatrix} 3 & \sqrt{6} & 0 \\ \sqrt{6} & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\] (21)
which is a pure coherent state with its coherence rank being 1. This incoherent Kraus operator may be arrived from the incoherent operation
\[
\Lambda(\rho) = K_1\rho K_1^+ + K_2\rho K_2^+,
\] (22)
where
\[
K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\] (23)

Since there is no incoherent projector \( P \) such that, for the state in Eq. (24),
\[
\frac{P\rho P}{\text{Tr}(P\rho P)} = \varphi,
\] (24)
with the coherence rank of \( |\varphi\rangle \) being equal to or greater than 1, this completes the proof of the part (a).

(b) It is straightforward to obtain the part (b) of the theorem by using a result in Ref. [32], which says that we can transform a mixed coherent state \( \rho \) into a pure coherent state \( \varphi \) with its coherence rank \( C(\varphi) = m \leq d - 1 \) by using the stochastic strictly incoherent operations if and only if there exists an incoherent projector \( P \) such that
\[
\frac{P\rho P}{\text{Tr}(P\rho P)} = \psi,
\] (25)
with the coherence rank of \( \psi \) being \( n (\geq m) \). Since there is no such incoherent projector \( P \) for \( \rho \), thus, it is impossible to transform \( \rho \) into a pure coherent state \( \varphi \) by using stochastic strictly incoherent operations. This completes the proof of the part (b).

Theorem 2 implies that the stochastic incoherent operations are generally stronger than the stochastic strictly incoherent operations when transform a mixed coherent state into a pure coherent one by using them. In order to proceed further, let us recall the notion of the pure coherent-state subspace [33]. If there is an incoherent projector \( P \) such that \( P\rho P = \varphi \) with the coherence rank of \( \varphi \) being \( n \geq 0 \), then we say that \( \rho \) has an \( n + 1 \)-dimensional pure coherent-state subspace corresponding to \( P \). And we say that the pure coherent-state subspace with

the projector \( P \) for \( \rho \) is maximal if the pure coherent-state subspace can not be expanded to a larger one with the incoherent projector \( P' \) such that \( P'\rho P' = \varphi' \), \( \varphi' \neq \varphi \), and \( P'\varphi P' = \varphi \). From Theorem 2, we obtain that the stochastic incoherent operations can increase the dimension of the maximal pure-coherent subspace.

Next, with the above notions, let us consider the operational difference between incoherent operations and strictly incoherent operations. From corollary 1, we know that there is no operational gap between them in the 2-dimensional case. Thus, let us consider the 3-dimensional case. This leads the following theorem.

**Theorem 3**—Let \( \rho \) be any 3-dimensional mixed state with its dimension of the maximal pure-coherent subspace being 1. Then incoherent operations cannot increase the dimension of the maximal pure-coherent subspace of \( \rho \).

**Proof**—To prove the theorem, let us consider the transformation
\[
\Lambda(\rho) = \varphi,
\] (26)
where \( \Lambda(\cdot) = \sum_n K_n(\cdot)K_n^\dagger \) is some incoherent operation and \( |\varphi\rangle = (m,n,0) \) is a 3-dimensional pure coherent state with \( |m|^2 + |n|^2 = 1 \). Next, we will divide our discussion into four cases.

**Case a**—There are three linearly independent incoherent Kraus operators as
\[
K = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\] (27)
with \( a,b,c \neq 0 \), up to a permutation in \( \{K_n\} \). Without loss of generality, we assume them as
\[
K_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (28)
The relation in Eq. (26) implies that there are at least two 3-dimensional states \( |\psi_1\rangle = (x_1,x_2,x_3)^T \) and \( |\psi_2\rangle = (y_1,y_2,y_3)^T \) such that
\[
K_n|\psi_i\rangle = 0,
\] (29)
for \( n = 1,2,3 \) and \( i = 1,2 \), where \( 0 \) is the null vector. This means that there are
\[
a_1x_1 + a_2x_2 + a_3x_3 = 0,
\]
\[
b_1x_1 + b_2x_2 + b_3x_3 = 0,
\]
\[
c_1x_1 + c_2x_2 + c_3x_3 = 0.
\] (30)
Since \( K_1, K_2, K_3 \) are linearly independent, then the rank of the matrix
\[
A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}
\] (31)
is 3. We immediately obtain the uniqueness of \( |\psi\rangle = (x_1,x_2,x_3)^T \).
Case b) There are two linearly independent incoherent Kraus operators as Eq. (27) in \([K_a]\). Without loss of generality, we assume them as (up to a permutation)

\[
K_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} b_1 & b_2 & b_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

with \(a_i, b_i \neq 0\) for all \(i\) and \(j\). The relation in Eq. (26) implies that

\[
a_{1x1} + a_{2x2} + a_{3x3} = 0, \quad b_{1x1} + b_{2x2} + b_{3x3} = 0.
\]

(31)

From Eq. (31) and \(|x_1|^2 + |x_2|^2 + |x_3|^2 = 1\), by direct calculations, we immediately obtain the uniqueness of \(|\psi\rangle = (x_1, x_2, x_3)^\dagger\).

Case c) There is only one incoherent Kraus operator as Eq. (27) in \([K_a]\). Without loss of generality, we assume it as

\[
K_1 = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

with \(a_i \neq 0\) for all \(i\). Since \(\Lambda(\cdot)\) is a trace preserving map, then there must be three incoherent Kraus operators in \([K_a]\) with the following forms:

\[
K_1' = \begin{pmatrix} a_1' & a_2' & 0 \\ 0 & 0 & a_3' \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2' = \begin{pmatrix} b_1' & 0 & 0 \\ 0 & b_2' & b_3' \\ 0 & 0 & 0 \end{pmatrix}, \quad K_3' = \begin{pmatrix} c_1' & 0 & c_3' \\ 0 & c_2' & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then, \((a_1, a_2)^\dagger\) and \((a_1', a_2')\), \((a_1, a_3)^\dagger\) and \((b_1', b_2')\), and \((a_1, a_3)^\dagger\) and \((c_1', c_3')\) are all linearly independent. The relation in Eq. (26) implies that

\[
a_{1x1} + a_{2x2} + a_{3x3} = 0.
\]

(32)

and

\[
(a_1' x_1 + a_2' x_2)n = a_3' x_3 m, \\
b_1' x_1 m = (b_2' x_2 + b_3' x_3)m, \\
(c_1' x_1 + c_3' x_3)n = c_2' x_2 m.
\]

(33)

From Eqs. (32) and (33), by direct calculations, we immediately obtain the uniqueness of \(|\psi\rangle = (x_1, x_2, x_3)^\dagger\).

Case d) There is no incoherent Kraus operator as Eq. (27) in \([K_a]\). Since \(\Lambda(\cdot)\) is a trace preserving map and strictly incoherent operations cannot transform a 3-dimensional mixed states into a pure coherent states \([32]\), then there is at least a pair of incoherent Kraus operators both from one of the following three types:

\[
K_1' = \begin{pmatrix} a_1' & a_2' & 0 \\ 0 & 0 & a_3' \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2' = \begin{pmatrix} b_1' & 0 & 0 \\ 0 & b_2' & b_3' \\ 0 & 0 & 0 \end{pmatrix}, \quad K_3' = \begin{pmatrix} c_1' & 0 & c_3' \\ 0 & c_2' & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Without loss of generality, let them all have the form as \(K_1'\) with

\[
K_1 = \begin{pmatrix} a_1 & a_2 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} b_1 & b_2 & 0 \\ 0 & 0 & b_3 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Then the relations in Eq. (26) and \(|x_1|^2 + |x_2|^2 + |x_3|^2 = 1\) imply that

\[
(a_1 x_1 + a_2 x_2)n = a_3 x_3 m, \\
(b_1 x_1 + b_2 x_2)n = b_3 x_3 m.
\]

(34)

From Eq. (34), by direct calculations, we immediately obtain the uniqueness of \(|\psi\rangle = (x_1, x_2, x_3)^\dagger\).

Thus, the above four cases imply that incoherent operations cannot increase the dimension of the maximal pure-coherent subspace of a 3-dimensional \(\rho\). This completes the proof of the theorem.

Theorem 3 tells us that incoherent operations cannot increase the dimension of the maximal pure-coherent subspace of a 3-dimensional \(\rho\). Let us consider the 4-dimensional state space and show that incoherent operations can increase the dimension of the maximal pure-coherent subspace of a state.

Theorem 4.—Let \(\rho\) be a 4-dimensional mixed state with its dimension of the maximal pure-coherent subspace being 1. There is an incoherent operation such that the transformation from \(\rho\) to \(\varphi\), with its coherence rank \(\geq 1\), can be achieved with certainty.

Proof.—To this end, let us consider the incoherent operation \(\Lambda(\cdot)\) which has the form

\[
\Lambda(\cdot) = K_1(\cdot)K_1^\dagger + K_2(\cdot)K_2^\dagger
\]

with

\[
K_1 = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -b & a & 0 & 0 \\ 0 & 0 & -d & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

(36)

The relation \(K_1 K_1^\dagger + K_2 K_2^\dagger = I\) implies that \(|a|^2 + |b|^2 = 1\) and \(|c|^2 + |d|^2 = 1\). Let \(\rho = p_1|\varphi_1\rangle\langle\varphi_1| + p_2|\varphi_2\rangle\langle\varphi_2|\). Then, the deterministic transformation

\[
\Lambda(\rho) = \varphi
\]

(37)

means that, for \(i = 1, 2\), there are

\[
K_1|\varphi_1\rangle = k_1^1|\varphi_1\rangle \text{ or } 0, \\
K_2|\varphi_1\rangle = k_1^2|\varphi_1\rangle \text{ or } 0.
\]

(38)

Specifically, we assume that there are

\[
K_1|\varphi_1\rangle = k_1^1|\varphi_1\rangle \text{ and } K_1|\varphi_2\rangle = 0, \\
K_2|\varphi_1\rangle = 0 \text{ and } K_2|\varphi_2\rangle = k_2^2|\varphi_2\rangle.
\]

(39)

Equipped with these tools, we may construct an explicit example as follows. Let \(|\varphi_1\rangle = \frac{\sqrt{5}}{\sqrt{6}}(|0\rangle + |1\rangle)\), \(a = \frac{\sqrt{3}}{\sqrt{5}}\), \(b = \frac{1}{\sqrt{3}}\), \(c = \frac{1}{\sqrt{3}}\), and \(d = \frac{2}{\sqrt{3}}\). Then, from Eq. (39), by direct calculations, \(|\varphi_1\rangle\) and \(|\varphi_2\rangle\) can be chosen as

\[
|\varphi_1\rangle = \frac{1}{5\sqrt{2}}(4, 3, \sqrt{5}, 2\sqrt{5})^t
\]

(40)

\[
|\varphi_2\rangle = \frac{1}{5\sqrt{2}}(-3, 4, -2\sqrt{5}, \sqrt{5})^t.
\]

(41)

where the superscript \(t\) means transpose. Without loss of generality, we further assume that \(p_1 = p_2 = \frac{1}{2}\).
Thus, from the above discussion, the state we chosen is
\[ \rho = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4} \\
\frac{1}{4} & 0 & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & 0 & \frac{1}{4}
\end{pmatrix}, \tag{42} \]
and the incoherent operation we chosen is
\[ \Lambda(\rho) = K_1 \rho K_1^\dagger + K_2 \rho K_2^\dagger, \tag{43} \]
with its incoherent Kraus operators being
\[ K_1 = \begin{pmatrix}
\frac{2}{5} & \frac{2}{5} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad K_2 = \begin{pmatrix}
-\frac{2}{5} & -\frac{2}{5} & 0 & 0 \\
0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{44} \]

By direct calculations, from Eqs. (42) and (44), we immediately obtain that
\[ \Lambda(\rho) = \frac{1}{2} \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{45} \]

It is straightforward to examine that the dimension of the maximal pure coherent-state subspace of \( \rho \) is 1, the dimension of the maximal pure coherent-state subspace of \( \Lambda(\rho) \) is 2, and the corresponding incoherent projector is
\[ P' = |1\rangle\langle 1| + |2\rangle\langle 2|. \tag{46} \]

Thus, we have found a mixed state \( \rho \) with its dimension of the maximal pure coherent-state subspace of it being 1 while, for some incoherent operations \( \Lambda(\cdot) \), the dimension of the maximal pure coherent-state subspace of \( \Lambda(\rho) \) being \( n(>1) \). This completes the proof of the theorem.

At last, we would like to present two applications of our results.

The first application is related to the the operational gap between incoherent operations and strictly incoherent operations. As we know, although the inclusion relations between them is known as PIO \( \subset \) SIO \( \subset \) IO \( \subset \) MIO, the operational gap in terms of state transformation between them has attracted much attentions \[15, 16\]. Specifically, the operational gap of maximally incoherent operations and the incoherent operations under stochastic state transformation is presented in Refs. \[21, 34\]. The operational gap between strictly incoherent operations and physically incoherent operations under stochastic state transformation is presented in Ref. \[22\]. However, whether there is an operational gap between incoherent operations and strictly incoherent operations under deterministic state transformation is unclear and is an open question \[16\]. Here, from our Theorem 4, we see that there is indeed an operational gap between the incoherent operations and the strictly incoherent operations under deterministic state transformation.

Another application is related to the difference between coherence monotones under IO and that of SIO. Theorem 4 leads directly to the existence of coherence monotones under strictly incoherent operations that increase under incoherent operations. As we known, any deterministic transformation not achievable by strictly incoherent operations necessarily implies the increase in some coherence monotone. Thus, we arrive at the following corollary.

**Corollary 2.**—There exist coherence monotones under strictly incoherent operations that can increase under incoherent operations.

The results of Ref. \[33\] show that the dimension of the maximal pure coherent-state subspace cannot increase under strictly incoherent operations. However, from Theorem 4, we immediately obtain that the dimension of the maximal pure coherent-state subspace of a state can increase under incoherent operations.

### IV. CONCLUSIONS

In summary, we have investigated the transformation from a mixed coherent state into a pure one by using both the incoherent operations and the stochastic incoherent operations. Our results show that contrary to the strictly incoherent operations and the stochastic strictly incoherent operations, both the incoherent operations and the stochastic incoherent operations can increase the dimension of the maximal pure-coherent subspace of a state. This means that the incoherent operations are generally stronger than the strictly incoherent operations when transform a mixed coherent state into a pure coherent one by using them. Our findings confirmed that there is an operational gap between incoherent operations and strictly incoherent operations under state transformations, which provide an answer to the open question in Ref. \[21\]. As an application, we have further shown that there exist coherence monotones under strictly incoherent operations that can increase under incoherent operations. The subtle differences between the incoherent operations and the strictly incoherent operations will increase our understandings on quantum coherence transformations under free operations in the resource theory of coherence.

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