A COLOCALIZATION SPECTRAL SEQUENCE

S. SHAMIR

Abstract. Colocalization is a right adjoint to the inclusion of a subcategory. Given an S-algebra \( R \), one would like a spectral sequence which connects colocalization in the derived category of \( R \)-modules and an appropriate colocalization in the derived category of \( \pi_*R \)-modules. We show that, under suitable conditions, such a spectral sequence exists. This generalizes the local-cohomology spectral sequence of Dwyer, Greenlees and Iyengar from [5]. Some applications are presented.

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1. Introduction

Let \( \mathcal{D} \) be a category and let \( \mathcal{C} \) be a subcategory of \( \mathcal{D} \). A \( \mathcal{C} \)-colocalization of \( X \in \mathcal{D} \) is a morphism \( \eta : \mathcal{C} \to X \) in \( \mathcal{D} \) such that \( C \in \mathcal{C} \) and \( \text{hom}_{\mathcal{D}}(T, \eta) \) is an isomorphism for every \( T \in \mathcal{C} \). We will call \( C \) a \( \mathcal{C} \)-cellular approximation of \( X \) and denote it by \( \text{Cell}_{\mathcal{C}} \mathcal{D} X \).

Now consider the following setup. Let \( R \) be an \( S \)-algebra (or a dga) and let \( \mathcal{T} \) be a class of graded left \( \pi_*R \)-modules which is closed under submodules, quotient modules, coproducts and extensions. Such a class is called a hereditary torsion class (see Definition 3.1). Let \( \mathcal{D} \) be the derived category of left \( R \)-modules and let \( \mathcal{C} \) be the subcategory of \( R \)-modules \( M \) such that \( \pi_*(M) \in \mathcal{T} \). We shall use the notation \( \text{Cell}^R_{\mathcal{T}} \) for \( \text{Cell}_{\mathcal{C}}^R \mathcal{D} \). Denote by \( \mathcal{D}_\bullet \), the derived category of bounded above \( \pi_*R \)-complexes. Let \( \mathcal{C}_\bullet \) be the subcategory of \( \mathcal{D}_\bullet \) consisting of all complexes \( \mathcal{X} \) such that \( H^n(\mathcal{X}) \in \mathcal{T} \) for all \( n \). We shall use the notation \( \text{Cell}^{\pi_*R}_{\mathcal{T}} \) for \( \text{Cell}_{\mathcal{C}_\bullet}^{D^\text{b}} \). A \( \mathcal{T} \)-colocalization spectral sequence for an \( R \)-module \( M \) is a spectral sequence of the form:

\[
E^2_{p,q} = H_{p,q}(\text{Cell}^{\pi_*R}_{\mathcal{T}} \pi_*M) \Rightarrow \pi_{p+q}(\text{Cell}^R_{\mathcal{T}} M)
\]

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The Greenlees spectral sequence \([11]\) (see also \([1]\)) can be viewed as a colocalization spectral sequence, as was shown by Dwyer, Greenlees and Iyengar in \([5]\). The setup for this spectral sequence is the following. Let \(G\) be a compact Lie group, for simplicity assume that \(G\) is connected, and let \(k\) be a be a field. Denote by \(R\) the dga \(C^*(BG; k)\), i.e. singular cochains on the classifying space of \(G\). Then \(\pi_*(R)\) is the graded-commutative ring \(H^*(BG; k)\). Let \(\mathcal{T}\) be the augmentation ideal \(H^+(BG; k)\). An \(H^*(BG; k)\)-module \(M\) is \(\mathcal{I}\)-power torsion if for every \(m \in \mathcal{M}\), \(T^m 0 = 0\) for some \(n\). Let \(\mathcal{T}\) be the class of \(\mathcal{I}\)-power torsion modules. The Greenlees spectral sequence can be written as
\[
E^2_{p,q} = H_{p,q}(\text{Cell}^R_H(BG; k) H^*(BG; k)) \Rightarrow \pi_{p+q}(\text{Cell}^R_H(R))
\]

Let \(\langle k \rangle\) be the localizing subcategory of \(D(R)\) generated by \(k\), this is the minimal triangulated subcategory of \(D(R)\) which contains \(k\) and is closed under coproducts. It turns out that \(\text{Cell}^R_{\mathcal{T}} R\) is in fact \(\text{Cell}^R_{\langle k \rangle} R\). Using the machinery of \([5]\), it is easy to show that \(\text{Cell}^R_{\langle k \rangle} R \simeq \Sigma^d C_* (BG; k)\), where \(d\) is the dimension of \(G\). By \([4]\), the \(H^*(BG; k)\)-modules \(H_{p+q}(\text{Cell}^R_{\langle k \rangle} H^*(BG; k))\) are the \(\mathcal{I}\)-local cohomology groups, denoted \(H^p_{\mathcal{T}}(H^*(BG; k))\).

Thus, we can now write the Greenlees spectral sequence in its usual form:
\[
E^2_{p,q} = H^{-p}_{\mathcal{T}}(H^*(BG; k))_q \Rightarrow H_{p+q-d}(BG; k)
\]

A similar colocalization spectral sequence is presented by Greenlees and May in \([12]\). In that spectral sequence \(R\) is a commutative \(\mathcal{S}\)-algebra, \(\mathcal{I}\) is a finitely generated ideal of \(\pi_* R\) and \(\mathcal{T}\) is the class of \(\mathcal{I}\)-power torsion \(\pi_* R\)-modules.

In this paper we generalize such results. While it is not difficult to get a spectral sequence with the desired \(E^2\)-page (see Lemma \([5,3]\)), one must impose additional conditions for the spectral sequence to converge to the desired object. This is illustrated by our first result.

**Theorem 1.** Let \(R\) be an \(\mathcal{S}\)-algebra and let \(\mathcal{T}\) be a hereditary torsion class on \(\pi_* R\)-modules. For any \(R\)-module \(M\) there is a spectral sequence \(E^2_{p,q}\) whose \(E^2\)-term is
\[
E^2_{p,q} = H_{p,q}(\text{Cell}^R_{\mathcal{T}} \pi_* M).
\]

If the \(\pi_* R\)-chain complex \(\text{Cell}^R_{\mathcal{T}} \pi_* M\) is bounded below (i.e. has zero homology below a certain degree), then the spectral sequence converges strongly to \(\pi_{p+q}(\text{Cell}^R_{\mathcal{T}} M)\).

Of course, for Theorem \([1]\) to be useful, one must find cases for which the condition on the spectral sequence is satisfied. Let \(R\) be an \(\mathcal{S}\)-algebra and let \(\mathcal{I}\) be a two-sided ideal of \(\pi_* R\). We say \(\mathcal{I}\) is **almost-commutative** if \(\mathcal{I} = (x_1, ..., x_n)\) and the image of \(x_i\) in \(\pi_* R/(x_1, ..., x_{i-1})\) is central.

**Theorem 2.** Let \(R\) be an \(\mathcal{S}\)-algebra and let \(\mathcal{I}\) be an almost-commutative ideal of \(\pi_* R\). Suppose that \(\pi_* R\) is Noetherian or that \(\pi_* R\) is graded-commutative. Let \(\mathcal{T}\) be the class of \(\mathcal{I}\)-power torsion modules. Then for any \(R\)-module \(M\) there exists a strongly convergent \(\mathcal{T}\)-colocalization spectral sequence:
\[
E^2_{p,q} = H_{p,q}(\text{Cell}^R_{\mathcal{T}} \pi_* (M)) \Rightarrow \pi_{p+q}(\text{Cell}^R_{\mathcal{T}} M)
\]

Similar to Theorem \([1]\) is the following result. Note that given a ring \(k\) we use the notation \(Hk\) for the corresponding Eilenberg-Mac Lane \(\mathcal{S}\)-algebra.
Theorem 3. Let \( k \) be a field and let \( R \) be a connective \( Hk \)-algebra with augmentation \( R \to Hk \). Suppose that \( \pi_* R \) is left Noetherian. Let \( T \) be the hereditary torsion theory of \( T \)-power torsion modules. If \( M \) is a bounded-above \( R \)-module, then there is a strongly convergent \( T \)-colocalization spectral sequence for \( M \):

\[
E^2_{p,q} = H_{p,q}(\text{Cell} R^*_{T} \pi_* (M)) \Rightarrow \pi_{p+q}(\text{Cell} R^*_{T} \pi_* M)
\]

Moreover, \( \text{Cell} R^*_{T} M \) is equivalent to \( \text{Cell} R^*_{(Hk)}\pi_* M \) and \( \text{Cell} \pi_*^R_{T} \pi_* M \) is \( \text{Cell} \pi_*^R_{(k)} \pi_* M \). Hence the spectral sequence can be written as:

\[
E^2_{p,q} = H_{p,q}(\text{Cell} R^*_{(k)} \pi_* (M)) \Rightarrow \pi_{p+q}(\text{Cell} R^*_{(Hk)} \pi_* M)
\]

One source of examples for this theory is the chains of loop spaces. Let \( k \) be a commutative ring and consider the dga \( R = C_*(\Omega X; k) \) - the singular chains on the Kan loop group of a connected space \( X \). Under certain conditions Theorems 2 and 3 can be applied to the derived category of \( R \). This will be done in Theorem 7.3 and in Theorem 8.1.

Organization of this paper. We begin by presenting some background material on cellular approximations in Section 2. All of this material is well known in one form or another. Next, in Sections 3 and 4, we consider colocalization over a graded ring \( R \) with respect to a hereditary torsion theory on \( R \)-modules and recall material from [16]. The main result of [16] is recalled in Theorem 4.3. It is this theorem that enables the construction of a colocalization spectral sequence.

Once we have all the necessary background material we can construct the spectral sequences. This is done in Section 5. As noted above, the construction of a spectral sequence with the desired \( E^2 \)-page and showing the spectral sequence converges are separate matters. Thus, the proofs of Theorems 1, 2 and 3 are given in the next section, Section 6.

After proving the main theorems we give two applications of the colocalization spectral sequence. The first application, given in Section 7, can be viewed as a non-commutative analog of the Greenlees spectral sequence. In another sense, the spectral sequence of Section 7 is dual to the Greenlees spectral sequence. Thus, while the Greenlees spectral sequence relates local cohomology of the cohomology ring of a space with the space homology, the spectral sequence of Section 7 relates the colocalization of the homology ring of a certain loop space with the cohomology of that loop space.

The second application is presented in Section 8. It is known that the target of the Eilenberg-Moore spectral sequence for a fibration can sometimes be viewed as colocalization (see eg. [17]). Hence an appropriate colocalization spectral sequence would have the same target. This is made explicit in Theorem 8.1.

We have pushed all the purely algebraic computations to the appendices. Appendix A contains all the necessary results regarding colocalization over graded rings. Appendix B is a result regarding the structure of the loop space homology of certain spaces. This result is needed for the application in Section 7.

Setting, conventions and notation. We work with \( S \)-algebras in the sense of [9], where \( S \) stands for the sphere spectrum. Let \( R \) be an \( S \)-algebra, unless otherwise noted an \( R \)-module means a left \( R \)-module. The derived category of \( R \)-modules will be denoted by \( \text{D}(R) \). The homotopy groups of an \( R \)-module \( M \) are its stable homotopy groups,
denoted \( \pi_\ast M \). An \( R \)-module \( M \) is called \emph{bounded-above} if \( \pi_i M = 0 \) for all \( i > n \) for some \( n \). An \( S \)-algebra \( R \) is called \emph{connective} if \( \pi_i (R) = 0 \) for all \( i < 0 \).

For \( S \)-algebras and modules over \( S \)-algebras we follow [3] in both notation and terminology. Thus, for an \( S \)-algebra \( R \) and for \( R \)-modules \( M \) and \( N \) the notation \( M \otimes_R N \) stands for the smash product of \( M \) and \( N \) over \( R \): \( M \wedge_R N \). Similarly, \( \text{Hom}_R(M, N) \) stands for the function spectrum \( F_R(M, N) \).

As in [3], the notation \( \text{Hom}_R(−, −) \) and \( − \otimes_R − \) are used for the function spectrum and the smash product of \( R \)-modules respectively. Both the function spectrum and the smash product are taken in the derived sense. This means that we always assume to have replaced our modules by appropriate resolutions before applying the relevant functor.

Let \( X \) be a connected space and let \( k \) be a commutative \( S \)-algebra. We use \( \Omega X \) to denote the Kan loop group of \( X \), which is equivalent to the loop space of \( X \). The cochains of \( X \) with coefficients in \( k \) is the commutative \( S \)-algebra \( C^\ast\)(\( X; k \)) = \( \text{Hom}_S(\Sigma^\infty X_+, k) \). The chains of \( \Omega X \) with coefficients in \( X \) is the \( S \)-algebra \( C_\ast(\Omega X; k) = k \otimes_S \Sigma^\infty (\Omega X)_+ \).

A \emph{graded ring} means a \( \mathbb{Z} \)-graded ring. A module over a graded ring is similarly \( \mathbb{Z} \)-graded. We shall reserve calligraphic font for graded rings their modules. Let \( \mathcal{R} \) be a graded ring. As above an \( \mathcal{R} \)-module will mean a \( \mathcal{R} \)-module. An \( \mathcal{R} \)-complex is a chain complex of left \( \mathcal{R} \)-modules. Hence an \( \mathcal{R} \)-complex is in fact bigraded. To avoid confusion we denote the grading of an \( \mathcal{R} \)-module \( \mathcal{M} \) by \( \mathcal{M}(i) \).

We say that \( \mathcal{R} \) is \emph{connective} if \( \mathcal{R}(i) = 0 \) for all \( i < 0 \). Ideals of \( \mathcal{R} \) are always graded (homogeneous) ideals. Given an \( \mathcal{R} \)-module \( \mathcal{M} \), we denote by \( \Sigma^n \mathcal{M} \) the \( \mathcal{R} \)-module given by \( \Sigma^n \mathcal{M}(i) = \mathcal{M}(i-n) \). We say an \( \mathcal{R} \)-module \( \mathcal{M} \) has \emph{bounded grading} if there is some \( b \) such that \( \mathcal{M}(i) = 0 \) for all \( i > b \). We shall call such \( b \) a \emph{grading bound} of \( \mathcal{M} \).

It is taken for granted that every \( \mathcal{R} \)-module is also a \( \mathcal{R} \)-complex concentrated in degree zero. For \( \mathcal{R} \)-complexes \( \mathcal{X} \) and \( \mathcal{Y} \) we use \( \text{Hom}_R(\mathcal{X}, \mathcal{Y}) \) for the usual chain complex of morphism groups, albeit in this case the morphism groups are graded. We will automatically assume \( \text{Hom}_\mathcal{R}(−, −) \) is derived, as is the case for \( S \)-algebras. To avoid confusion, we shall denote the translation functor on \( D(\mathcal{R}) \) by \( T \). Hence, for an \( \mathcal{R} \)-module \( \mathcal{M} \) the notation \( T \mathcal{M} \) stands for the \( \mathcal{R} \)-complex with \( \mathcal{M} \) in degree 1 and zero elsewhere.

In a triangulated category \( D \) we use the notation \( \text{hom}_D(−, −) \) for the graded abelian group of homomorphisms between two objects of \( D \). A \emph{triangle} in \( D \) will always mean a distinguished (exact) triangle. Given a map \( f : X \to Y \) in \( D \) we denote by \( \text{Cone}(f) \) any object which completes \( X \xrightarrow{f} Y \) to a triangle. Note that if \( D \) is the derived category of a graded ring, then \( \text{hom}_D^\ast(−, −) \) is bigraded.

2. Cellular approximation and nullification

The key ingredient of this paper is the notion of cellular approximation. It is therefore worthwhile to set aside a section on its definition and properties. We shall follow the definitions set by Dwyer and Greenlees in [4], given here in the setting of a general triangulated category. This allows for a unified treatment of both situations we have in mind, namely the derived category of an \( S \)-algebra and the derived category of a graded ring.

Recall that a full subcategory \( C \) of a triangulated category \( D \) is called a \emph{localizing} subcategory if \( C \) is closed under completion of triangles and arbitrary coproducts. The
localizing subcategory generated by a class of objects \(a \subset D\) is the minimal localizing subcategory of \(D\) among all localizing subcategories which contain \(a\). We denote this localizing subcategory by \(\langle a \rangle\).

**Definition 2.1.** Let \(D\) be a triangulated category and let \(a\) be a class of objects in \(D\). A morphism \(f : X \to Y\) in \(D\) is called an \(a\)-equivalence if for every \(A \in a\) the morphism \(\text{hom}_D^*(A, f)\) is an isomorphism. An object \(X \in D\) is \(a\)-cellular if \(X \in \langle a \rangle\). We say \(C\) is an \(A\)-cellular approximation of \(X\), if \(C\) is \(a\)-cellular and there is an \(a\)-equivalence \(\eta : C \to X\). An \(a\)-cellular approximation of \(X\) shall be denoted by \(\text{Cell}_a X\). If \(D = D(R)\) for an \(S\)-algebra (or a ring) \(R\) then we denote an \(a\)-cellular approximation of \(X\) by \(\text{Cell}_a^R X\) whenever we need to emphasize the category we work over.

Clearly, \(a\)-cellular approximation depends only on the localizing category generated by \(a\), i.e. if \(\langle a \rangle = \langle b \rangle\) then \(\text{Cell}_a X \cong \text{Cell}_b X\) whenever such cellular approximation exists. A similar remark applies for \(a\)-equivalences. This implies that the \(a\)-equivalence \(\text{Cell}_a X \to X\) is an \(\langle a \rangle\)-colocalization of \(X\). To make the terminology a bit more convenient we have the following definition.

**Definition 2.2.** Let \(a\) be a class of objects in \(D\). We say that a map \(\eta : C \to X\) is an \(a\)-colocalization of \(X\) if \(\eta\) is an \(\langle a \rangle\)-colocalization of \(X\). Thus \(C\) is \(\text{Cell}_a X\) and \(\eta\) is an \(a\)-equivalence.

The counterpart of cellular approximation is nullification, described below.

**Definition 2.3.** An object \(X \in D\) is called \(a\)-null if \(\text{hom}_D^*(A, X) = 0\) for every \(A \in a\). An object \(N\) is an \(a\)-nullification of \(X\) if there is a morphism \(\nu : X \to N\) which is initial among maps in \(D\) from \(X\) to \(a\)-null objects. This is equivalent to saying that \(N\) is \(a\)-null and every cone of \(\nu\) is \(a\)-cellular. The morphism \(\nu\) shall be called an \(a\)-nullification map of \(X\). We use the notation \(\text{Null}_a X\) for an \(a\)-nullification of \(X\) in \(D\).

It is easy to see that for any \(X\) there is a distinguished triangle:

\[
\text{Cell}_a X \xrightarrow{\eta} X \xrightarrow{\nu} \text{Null}_a X
\]

where \(\eta\) is an \(a\)-equivalence map of \(X\) and \(\nu\) is an \(a\)-nullification map of \(X\). Hence the existence of \(a\)-cellular approximation implies the existence of \(a\)-nullification and vice versa.

In general, \(a\)-cellular approximations of objects need not exist. However, when \(\langle a \rangle\) is generated by a set of objects and \(D\) is the derived category of an \(S\)-algebra (or a graded ring) then \(a\)-cellular approximations exist for every object in \(D\). This follows from \([14]\) (see also \([7]\) for cellular approximation in a topological setting). Note that if \(\text{Cell}_a X\) can be formed for every \(X\), then an object \(C\) is \(a\)-cellular if and only if \(\text{hom}_D^*(C, N) = 0\) for every \(a\)-null object \(N\) (see for example \([4]\)).

The following proposition lists well known equivalent definitions of \(a\)-cellular approximation. Its proof is trivial and we list it simply for future reference.

**Proposition 2.4.** For a morphism \(\eta : C \to X\) in \(D\) the following are equivalent:

1. \(\eta\) is an \(a\)-equivalence and \(C\) is \(a\)-cellular,
2. the morphism \(X \to \text{Cone}(\eta)\) is an \(a\)-nullification map,
3. \(C\) is \(a\)-cellular and \(\text{Cone}(\eta)\) is \(a\)-null.
We also record a well known principle of recognizing a-cellular objects. We note this principle in both the setting of a derived category of an S-algebra (Lemma 2.5) and in that of a graded ring (Lemma 2.6). The proof of this principle, applicable for both cases, can be found, for example, in [17].

Recall that if $R$ is a connective S-algebra then one can form an Eilenberg-Mac Lane spectrum $H\pi_0(R)$ and a map of S-algebras $R \to H\pi_0(R)$ realizing the isomorphism on $\pi_0$ of both S-algebras (see [6, IV.3]).

**Lemma 2.5.** Suppose that $R$ is a connective S-algebra. Let $M$ be a bounded-above $R$-module, then $M$ is $\{H\pi_nM\}_{n\in\mathbb{Z}}$-cellular.

**Lemma 2.6.** Suppose $\mathcal{R}$ is a graded ring and $a$ is a class of $\mathcal{R}$-modules. If $C$ is an $\mathcal{R}$-complex such that

1. $C$ is bounded above and
2. $H_nC$ is an $a$-cellular $\mathcal{R}$-module for all $n$.

Then $C$ is an $a$-cellular complex.

### 3. Colocalization with respect to a hereditary torsion theory

In order to construct the spectral-sequence we require several of the results of [16], which deal with colocalization with respect to a hereditary torsion theory. Note that in [16] the setting considered was that of modules over a ring, while the setting needed here is that of graded modules over a graded ring. Nevertheless, the results (and proofs) of [16] apply also to the graded case. Throughout this section $\mathcal{R}$ denotes a graded ring.

We begin by recalling the definition and some properties of hereditary torsion theories. These do not change when passing to the setting of graded modules over a graded ring. A more comprehensive treatment of this subject can be found, for example, in [18].

**Definition 3.1.** A hereditary torsion class $\mathcal{T}$ is a class of $\mathcal{R}$-modules that is closed under submodules, quotient modules, coproducts and extensions. Closure under extensions means that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence with $M_1$ and $M_3$ in $\mathcal{T}$, then so is $M_2$. The modules in $\mathcal{T}$ will be called $\mathcal{T}$-torsion modules (or just torsion modules when the torsion theory is clear from the context). The class of torsion-free modules $\mathcal{F}$ is the class of all modules $M$ satisfying $\text{Ext}_\mathcal{R}^1(C,M) = 0$ for every $C \in \mathcal{T}$. The pair $(\mathcal{T}, \mathcal{F})$ is referred to as a hereditary torsion theory.

To every hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ there is an associated radical $t$. Given an $\mathcal{R}$-module $M$, the module $t(M)$ is the largest torsion submodule of $M$. Because $\mathcal{T}$ is hereditary, the module $M/t(M)$ is a torsion-free module.

Every hereditary torsion class $\mathcal{T}$ has an injective cogenerator (see [18, VI.3.7]). This means there exists an injective module $\mathcal{E}$ such that a module $M$ is torsion if and only if $\text{Hom}_\mathcal{R}(M, \mathcal{E}) = 0$.

The main example we have in mind is the following.

**Example 3.2.** Suppose $I \subset \mathcal{R}$ is a two-sided ideal of $\mathcal{R}$ that is finitely generated as a left $\mathcal{R}$-module. Recall an $\mathcal{R}$-module $M$ is an $I$-power torsion module if for every $m \in M$ there exists some $n$ such that $I^n m = 0$. It is not difficult to show that the class $\mathcal{T}$ of $I$-power torsion modules is a hereditary torsion class, see [18, VI.6.10]. The radical of $\mathcal{T}$
is given by $t(\mathcal{M}) = \{ m \in \mathcal{M} | I^n m = 0 \text{ for some } n \}$ and the quotient $\mathcal{M}/t(\mathcal{M})$ has no $I$-power torsion elements. One can take for an injective cogenerator of $\mathcal{I}$ the injective module $\mathcal{E} = \prod \mathcal{E}(\mathcal{R}/\mathfrak{a})$, where $\mathfrak{a}$ goes over all left ideals of $\mathcal{R}$ such that $\mathcal{R}/\mathfrak{a}$ is torsion-free and $\mathcal{E}(\mathcal{R}/\mathfrak{a})$ is the injective hull of $\mathcal{R}/\mathfrak{a}$.

It is worth noting the following results from [16].

**Proposition 3.3** ([16, Lemma 2.8]). Let $\mathcal{I}$ be a hereditary torsion class and denote by $C_\mathcal{I}$ the set of all cyclic $\mathcal{I}$-torsion modules. Then $\langle \mathcal{I} \rangle = \langle C_\mathcal{I} \rangle$ and therefore $\mathcal{I}$-colocalization and $\mathcal{I}$-nullification exists for every $\mathcal{R}$-complex.

**Proposition 3.4** ([16, Corollary 2.11]). Let $\mathcal{I}$ be a hereditary torsion class, then a bounded-above $\mathcal{R}$-complex $X$ is $\mathcal{I}$-cellular if and only if $H_n(X)$ is $\mathcal{I}$-torsion for all $n$.

We shall also require the following result.

**Lemma 3.5.** Let $\mathcal{R} \to \mathcal{S}$ be a surjection of graded rings and let $I$ be the kernel of this map. Suppose $I$ is finitely generated as a left $\mathcal{R}$-module. Let $\mathcal{I}$ be the class of $I$-power torsion modules. Then $\langle \mathcal{I} \rangle = \langle \mathcal{S} \rangle$ and therefore $\text{Cell}_\mathcal{I} \simeq \text{Cell}_\mathcal{S}$.

**Proof.** Since $\mathcal{S} \in \langle \mathcal{I} \rangle$, we need only show that $\langle \mathcal{I} \rangle \subset \langle \mathcal{S} \rangle$. As noted in Proposition 3.3, $\langle \mathcal{I} \rangle$ is generated by the cyclic $\mathcal{I}$-torsion modules. Hence it is enough to show that every $\mathcal{I}$-torsion cyclic module is in $\mathcal{S}$-cellular.

Any cyclic $\mathcal{I}$-torsion module is a cyclic $\mathcal{R}/I^n$-module for some $n$, as we will now show. Let $\mathcal{M}$ be a cyclic $\mathcal{I}$-power torsion module and let $x$ be a generator for $\mathcal{M}$. Clearly, $I^n x = 0$ for some $n$. Because $I$ is finitely generated as a left $\mathcal{R}$-module, it is easy to show that $I^n$ is also finitely generated as a left $\mathcal{R}$-module. Let $a_1, ..., a_m$ be these generators for $I^n$ and let $r$ be some element of $\mathcal{R}$. Since $a_i r \in I^n$, we can write $a_i r = \sum t_j a_j$ for some $t_1, ..., t_j \in \mathcal{R}$. Hence

$$a_i(rx) = \sum t_j a_j x = 0$$

because $a_j x = 0$ for all $j$. Therefore $I^n(rx) = 0$ for every $r \in \mathcal{R}$.

Clearly, every $\mathcal{R}/I^n$-module is $\mathcal{R}/I^n$-cellular. We see it is enough to show that for every $n$, $\mathcal{R}/I^n$ is $\mathcal{S}$-cellular. There are short exact sequences

$$0 \to I^{n-1}/I^n \to \mathcal{R}/I^n \to \mathcal{R}/I^n \to 0.$$

Noting that $I^{n-1}/I^n$ is an $\mathcal{S}$-module and hence $\mathcal{S}$-cellular, and that $\langle \mathcal{S} \rangle$ is closed under triangles, it is easy to see that an inductive argument completes the proof. □

4. **Explicit Construction of $\mathcal{I}$-nullification**

Let $\mathcal{R}$ be a graded ring and let $(\mathcal{I}, \mathcal{F})$ be a hereditary torsion theory on $\mathcal{R}$-modules with an associated radical $t$. In [16] there is described a construction for $\mathcal{I}$-nullification using torsion-free injective modules. This construction is described below, with a minor modification.

**Nullification Construction 4.1.** Given an $\mathcal{R}$-module $\mathcal{M}$, we construct a cochain complex $\mathcal{N} = \mathcal{N}^0 \xrightarrow{d_0} \mathcal{N}^1 \xrightarrow{d_1} \mathcal{N}^2 \xrightarrow{d_2} \cdots$ inductively. Let $\mathcal{M}^0 = \mathcal{M}$, let $\mathcal{G}^0 = \mathcal{M}^0/t(\mathcal{M}^0)$
and let $N^0$ be the injective hull of $G^0$. Proceed by induction, thus
\[ M^{n+1} = N^n/G^n, \]
\[ G^{n+1} = M^{n+1}/t(M^{n+1}) \]
and $N^{n+1}$ is the injective hull of $G^{n+1}$.

The differential $d^n : N^n \to N^{n+1}$ is the composition of the obvious morphisms $N^n \to M^{n+1} \to G^{n+1} \to N^{n+1}$. Note that the obvious morphism $M \to N^0$ induces a map $\nu : M \to N$ of $R$-complexes.

**Lemma 4.2 ([16, Lemma 3.2]).** The complex $N$ constructed in 4.1 is a $T$-nullification of $M$ and the map $\nu : M \to N$ is a $T$-nullification map of $M$.

**Proof.** Every $N^n$ is the injective hull of a torsion-free module and therefore $N^n$ itself is torsion-free. Hence, $N$ is a $T$-null complex. Now consider the cochain complex $C$, given by:
\[ M \to N^0 \to N^1 \to N^2 \to \cdots \]
It is easy to see that $H^n(C) = t(M^n)$ for all $n \geq 0$. This implies that $C$ is $T$-cellular (Lemma 2.6).

There is a triangle $C \to M \xrightarrow{\nu} N$. Since $N$ is $T$-null, the morphism $C \to M$ is a $T$-equivalence. By Proposition 2.4 $C$ is a $T$-cellular approximation of $M$ and $N$ is a $T$-nullification of $M$. \qed

One can use Lemma 4.2 alone to construct the desired spectral sequence. However, the following trick from [16] will make the construction of the spectral sequence immediate.

**Theorem 4.3 ([16, Theorem 2.2]).** Let $T$ be a hereditary torsion class. For every $R$-module $M$ there exists a torsion-free injective $R$-module $E$ such that the natural map of complexes
\[ M \to \text{Hom}_{\text{End}_R(E)}(\text{Hom}_R(M,E),E) \]
is a $T$-nullification map of $M$.

It will sometimes be necessary need to specify the injective module used in Theorem 4.3. For that purpose we record one more result from [16].

**Proposition 4.4 ([16, Proposition 3.5]).** Let $M$ be an $R$-module and let $E$ be an injective cogenerator of the hereditary torsion theory $T$. Denote by $Q$ the graded ring $\text{End}_R(E)$. If the $Q$-module $\text{Hom}_R(M,E)$ has a resolution composed of finitely generated projective modules in each degree, then $\text{Hom}_Q(\text{Hom}_R(M,E),E)$ is the $T$-nullification of $M$.

5. **Construction of the spectral sequences**

We will, in fact, construct two spectral sequences. First we will construct a nullification spectral sequence and from it deduce a colocalization spectral sequence. But before constructing the spectral sequences we must take a short detour and review a certain adjunction.

Let $R$ be an $S$-algebra and $E$ be an $R$-module. Denote by $E$ the $S$-algebra $\text{End}_R(E)$. The module $E$ is in fact a left $R \otimes_S E$-module. There are two functors:
\[ \text{Hom}_R(-,E) : D(R)^{op} \to D(E) \quad \text{and} \quad \text{Hom}_E(-,E) : D(E)^{op} \to D(R). \]
That $\text{Hom}_R(M, E)$ is an $E$-module comes from the extra left $E$-action on $E$ (this is implicit in [6 III.6.5]). The exact same argument applies for the functor $\text{Hom}_E(\_ , E)$.

**Definition 5.1.** For an $R$-module $M$, the $E$-*dual* of $M$ is the $E$-module $\text{Hom}_R(M, E)$. The $E$-*double dual* of $M$ is the $R$-module $\text{Hom}_E(\text{Hom}_R(M, E), E)$

**Lemma 5.2.** There is a natural transformation $1 \to \text{Hom}_E(\text{Hom}_R(\_ , E), E)$.

**Proof.** Let $U$ be an $E$-module, then:

$$\text{hom}_R(M, \text{Hom}_E(U, E)) \cong \text{hom}_{R\otimes S}(M \otimes_S U, E) \cong \text{hom}_{S\otimes R}(U \otimes_S M, E) \cong \text{hom}_E(U, \text{Hom}_R(M, E)).$$

The first and last isomorphisms are the ones described in [6 III.6.5(i)]. This leads to an adjunction:

$$D(R) \overset{G}{\underset{F}{\rightleftarrows}} D(E)^{\text{op}},$$

where $F$ is $\text{Hom}_R(\_ , E)$ and $G$ is $\text{Hom}_E(\_ , E)$. The natural transformation mentioned above is simply the unit of this adjunction. 

Note that the image of the functor $\text{Hom}_E(\_ , E) : D(E) \to D(R)$ is contained in the colocalizing subcategory generated by $E$. The colocalizing subcategory generated by $E$ is the smallest triangulated subcategory closed under retracts, isomorphisms, completion of triangles and products. Also note that if $E$ is $A$-null for some $R$-module $A$, then every object in the colocalizing subcategory generated by $E$ is $A$-null.

We are now ready to construct the spectral sequences.

**Lemma 5.3.** Let $R$ be an $S$-algebra and let $\mathcal{T}$ be a hereditary torsion theory on $\pi_*R$-modules. For every $R$-module $M$ there exists a map $M \to N$ such that $N$ is a $\mathcal{T}$-null $R$-module and there is a spectral sequence of $\pi_*R$-modules:

$$E^2_{p,q} = H_{p,q}(\text{Null}_\mathcal{T}^{\pi_*R} \pi_*M) \Rightarrow \pi_{p+q}(N)$$

This spectral sequence has conditional convergence.

**Proof.** Let $\mathcal{N}$ be the complex described in the Nullification Construction [1.1] for the $\pi_*R$-module $\pi_*M$. Let $\mathcal{E}$ be an appropriate injective $\pi_*R$-module as in Theorem [1.3]. To be specific, we choose $\mathcal{E}$ to be the product $\prod_i \mathcal{N}^i$. It follows from [16] that $\mathcal{E}$ is an appropriate injective, i.e.

$$\text{Null}_\mathcal{T}^{\pi_*R} \pi_*M \cong \text{Hom}_{\text{End}_{\pi_*R}(\mathcal{E})}(\text{Hom}_{\pi_*R}(\pi_*M, \mathcal{E}), \mathcal{E})$$

Brown representability implies that there exists an $R$-module $E$ such that there is a natural isomorphism $\pi_*\text{Hom}_R(X, E) \cong \text{Hom}_{\pi_*R}(\pi_*X, \mathcal{E})$ for every $R$-module $X$. Let $N$ be the $E$-double dual of $M$, i.e.

$$N = \text{Hom}_{\text{End}_R(E)}(\text{Hom}_R(M, E), E).$$

The $R$-module $E$ is certainly $\mathcal{T}$-null and since $N$ is in the colocalizing subcategory generated by $E$, $N$ is also $\mathcal{T}$-null.
In [6, IV.4], a spectral sequence for calculation of $\text{Ext}_{\text{End}_R(E)}(\text{Hom}_R(M, E), E)$ is presented. This spectral sequence has the form:

$$E_2^{p,q} = \text{Ext}_{\pi^*\text{End}_R(E)}^{p,q}(\pi^*\text{Hom}_R(M, E), \pi^*E) \Rightarrow \pi^{p+q}N$$

The isomorphisms:

$$\text{End}_R(E)_* \cong \text{End}_R(\mathcal{E}, \mathcal{E}), \text{Hom}_R(M, E)_* \cong \text{Hom}_R(M_*, \mathcal{E})$$

complete the proof.

**Lemma 5.4.** Let $R$, $T$, $M$ and $N$ be as in Lemma 5.3 above. Let $C$ be the homotopy fiber of the map $M \to N$, then there exists a spectral sequence of $\pi_*$-$R$-modules:

$$E_2^{p,q} = H_{p,q}(\text{Cell}^R_\pi \pi_*M) \Rightarrow \pi^{p+q}(C)$$

This spectral sequence has conditional convergence.

**Proof.** As above, $\mathcal{N}$ is the complex described in the Nullification Construction 4.11 for the $\pi_*R$-module $\pi_*M$. Let $N^p$ be the lift of $\mathcal{N}^p$ given by brown representability and let $E$ be the product $\prod N^p$.

Following the details of the construction of the Ext-spectral sequence in [6, IV.5] we see there are triangles:

$$\Sigma^{-p}N^p \xleftarrow{k^p} E^p \xleftarrow{j^p} E^{p+1} \xrightarrow{\delta^p} \Sigma^{-p-1}N^p$$

for $p \geq 0$. These fiber sequences satisfy the following properties:

1. $\pi_*(N^p) \cong N^p$.
2. $E^0 \cong \text{Hom}_E(\text{Hom}_R(M, E), E)$.
3. The composition $k^{p+1} \circ j^p$ realizes the differential $\delta^{p+1}$.
4. The homotopy limit of the tower $E^0 \leftarrow E^1 \leftarrow \cdots$ is zero.
5. The composition $M \to E^0 \to N^0$ realizes the map $\pi_* M \to \mathcal{N}^0$ in the Nullification Construction 4.11.

The only property needing explanation here is the last one. In choosing the specific module $E$ above we have ensured that $N^0$ is equivalent to its double dual, simply because $N^0$ is a retract of $E$. Now, let $f : M \to N^0$ be such that $\pi_* f : \pi_* M \to \mathcal{N}^0$ is the morphism described in the Nullification Construction 4.11. Then applying the $E$-double dual functor to $f$ yields a commutative diagram:

$$\begin{array}{ccc}
M & \xrightarrow{f} & N^0 \\
\downarrow \cong & & \downarrow \cong \\
N & \xrightarrow{\text{Hom}_E(\text{Hom}_R(N^0, E), E)} &
\end{array}$$

which proves the last property.

Let $C^0$ be the homotopy fiber of $M \to E^0$. For $p > 0$ set $C^p = \Sigma^{-1}E^{p-1}$. Thus there are triangles

$$M \xleftarrow{k^{-1}} C^0 \xleftarrow{i^{-1}} C^1 \xleftarrow{j^0} \Sigma^{-1}M \quad \text{and} \quad \Sigma^{-p-1}N^{p-1} \xleftarrow{C^p} C^{p+1} \xleftarrow{\delta^{p+1}} \Sigma^{-p}N^p$$

for $p > 0$. 

for $p > 0$. 

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Define $D^1_{p,q} = \pi_{p+q}(C^p)$, $E^1_{0,q} = \pi_q(M)$ and $E^1_{p,q} = \pi_q(E^{p-1})$ for $p < 0$. The fiber sequences above yield the following maps:

\[
i : D^1_{p,q} \to D^1_{p+1,q-1}
\]
\[
j : E^1_{p,q} \to D^1_{p-1,q}
\]
\[
k : D^1_{p,q} \to E^1_{p,q}
\]

which make $D$ and $E$ into an exact couple. It is a simple matter to see that $E^1_{p,*}$ with the map $\delta = k \circ j$ is the cochain-complex:

\[
\pi_* M \to N^0 \to N^1 \to \cdots \to N^{p-1} \overset{\delta}{\to} N^p \to \cdots
\]

This cochain complex is clearly $\text{Cell}^{\pi_* R} \pi_* M$, thus yielding the desired $E^2$-page.

To show conditional convergence we consider the following triangle of towers:

\[
\begin{array}{c}
C^0 \leftarrow C^1 \leftarrow C^2 \leftarrow \cdots \\
| \\
| \\
| \\
M \leftarrow 0 \leftarrow 0 \leftarrow \cdots \\
| \\
| \\
| \\
E^0 \leftarrow E^0 \leftarrow E^1 \leftarrow \cdots
\end{array}
\]

The homotopy limit of the bottom tower is equivalent to the homotopy limit of the tower:

\[
E^0 \leftarrow E^1 \leftarrow E^2 \leftarrow \cdots
\]

which is shown to be zero in [6, IV.5]. The homotopy limit of the middle tower is clearly zero, hence the homotopy limit of the top tower is also zero. As in [6, IV.5], this implies that:

\[
\lim(D^1_{0,*} \leftarrow D^1_{-1,*} \leftarrow \cdots) = 0
\]

and

\[
\lim^1(D^1_{0,*} \leftarrow D^1_{-1,*} \leftarrow \cdots) = 0,
\]

resulting in conditional convergence as defined in the work of Boardman [2].

6. PROOFS OF THEOREMS 1, 2 AND 3

Proof of Theorem 1. To obtain a spectral sequence $E_{p,q}^r$ with the correct $E^2$-page we need only invoke Lemma 5.1. Hence there is a triangle $C \to M \to N$, with $N$ being $T$-null and the spectral sequence on Lemma 5.1 converges conditionally to $\pi_* C$.

Now suppose the $\pi_* R$-complex $\text{Cell}_T^{\pi_* R}(\pi_* M)$ is bounded below. This immediately implies that the spectral sequence has strong convergence [2, Theorem 7.1]. It remains to show that $C$ is $\text{Cell}_T M$. By Proposition 2.4, it is enough to show that $C$ is $T$-cellular. Equivalently, we must show that $\pi_* C$ is $T$-cellular.

By Proposition 3.4 for every $p$ the $\pi_* R$-module $E^2_{p,*}$ is $T$-torsion. Hence $E^r_{p,*}$ is $T$-torsion for every $p$ and $r$, because $T$ is closed under kernels and cokernels. Since $\text{Cell}_T^{\pi_* R}(\pi_* M)$ is bounded below, the spectral sequence collapses for some $r$. We see that $\pi_* C$ has a finite filtration whose successive quotients are $T$-torsion. Because $T$ is also closed under extensions, $\pi_* C$ is $T$-torsion. 

\[\Box\]
Proof of Theorem 2. It is enough to show that the condition of Theorem 1 is satisfied. Namely that the $\pi_* R$-complex $Cell_T^R(\pi_* M)$ is bounded-below. If $\pi_* R$ is graded-commutative then this is a result of Dwyer and Greenlees [11 Proposition 6.10]. If $\pi_* R$ is left Noetherian this follows from Lemma A.3.6. □

Proof of Theorem 3. It should be pointed out that, since $\pi_* R$ is left Noetherian, $I$ is finitely generated as a left $\pi_* R$-module. This implies that the class of $I$-power torsion modules $T$ is a hereditary torsion class, see [18 VI.6.10]. It also implies that $Cell_T^R(\pi_* M) \simeq Cell_I^R(\pi_* M)$, by Lemma 3.5.

As in the proof of Theorem 1, we obtain the desired spectral sequence $E^2_{p,q}$ by invoking Lemma 5.3. Thus there is a triangle $C \to M \to N$, with $N$ being $T$-null and the spectral sequence of Lemma 5.4 converges conditionally to $\pi_* C$. By Lemma A.1.2 there exists $q_0$ such that $E^2_{p,q} = 0$ for all $q > q_0$. Strong convergence now follows from Boardman [2 Theorem 7.1].

Our next task is to show that $C$ is $Cell_T^R M$. For every $r$, the $\pi_* R$-modules $E^r_{p,*}$ are $T$-torsion. Note that $E^r_{p,q}$ is a $\pi_* R$-module that is a sub-quotient of $E^r_{p,q}$. For example $E^r_{p,q_0}$ is a submodule of $E^r_{p,*}$ and $E^r_{p,q_0-1}$ is a submodule of the quotient $E^r_{p,*}/E^r_{p,q_0}$. Hence $E^r_{p,q}$ is $T$-torsion for all $r,p$ and $q$.

Now it is clear that for every $n$ there exists $r(n)$ such that $E^{r(n)}_{p,q} = E^\infty_{p,q}$ whenever $p + q = n$. Moreover, there are only finitely many pairs $(p,q)$ such that $p + q = n$ and $E^{r(n)}_{p,q} \neq 0$. Hence, $\pi_n C$ has a finite filtration whose successive quotients are $T$-torsion.

We conclude that $\pi_n C$ is $T$-torsion for all $n$.

By Lemma 3.5, $\pi_n C$ is $k$-cellular as a $\pi_* R$-module. We claim this implies $\pi_n C$ is $k$-cellular also as a $\pi_0 R$-module. Consider the map of graded algebras $f : \pi_0 R \to \pi_* R$. We know that $\pi_n(C)$ is $k$-cellular over $\pi_* R$, this can be pulled back along $f$ to show that $\pi_n(C)$ is $k$-cellular over $\pi_0 R$.

We see that the Eilenberg-Mac Lane spectrum $H(\pi_n C)$ is an $Hk$-cellular $H\pi_0 R$-module. This can be pulled back along the map $R \to H\pi_0 R$ to show that $H(\pi_n C)$ is $Hk$-cellular as an $R$-module. Hence, by Lemma 2.3, $C$ is $k$-cellular. Since $k$ is $T$-cellular then $N$ is $k$-null and $C$ is $T$-cellular. Using Proposition 2.4 we conclude that $C$ is both $Cell_T^R M$ and $Cell_T^R N$.

7. Application: The chains of loops on an elliptic space

Elliptic spaces, defined by Felix, Halperin and Thomas in [9], have a particularly well behaved loop space homology. This enables us to apply Theorem 2 to modules over the $S$-algebra $C_*(\Omega X; \Z)$, where $X$ is an elliptic space. We begin by recalling the definition of an elliptic space from [9].

Definition 7.1. Let $k$ be a sub-ring of $\Q$. A simply connected CW complex $X$ is called $k$-elliptic if

1. $X_k$ has a finite Lusternik-Schnirelmann category, where $X_k$ is the $k$-localization of $X$.
2. Each $H_i(X; k)$ is finitely generated.
3. For $F = \Q$ or $F = \Z/p$ for prime $p$, the sequence $\dim_F H_i(X_k; F)$ grows at most polynomially.
If $X$ is $\mathbb{Z}$-elliptic we will say that $X$ is \textit{elliptic}.  

\textbf{Remark 7.2.} Note that by \cite{9}, a $k$-elliptic space $X$ has the $k$-homotopy type of a finite complex. So, for the applications we have in mind we may as well assume that $X$ is a finite complex. Moreover, by \cite[Theorem A]{9}, if $X$ is $k$-elliptic then $H^*(X; k)$ satisfies Poincaré duality and if $X$ is elliptic then $X$ has the homotopy type of a finite Poincaré complex.

\textbf{Theorem 7.3.} Let $X$ be a elliptic space and let $R$ be the $\mathbb{S}$-algebra $C_*(\Omega X; \mathbb{Z})$. Suppose that $M$ is an $R$-module such that $\pi_*M$ is a finitely generated $\pi_*R$-module. Then there is a strongly convergent colocalization spectral sequence:

$$E^2_{p,q} = H_{p,q}(\text{Cell}^{\pi_*R \pi_*M}_Z) \Rightarrow \pi_{p+q}(\text{Cell}^R_{HZ}M)$$

In addition, for $M = R$ we have:

$$E^2_{p,q} = H_{p,q}(\text{Cell}^{\pi_*R \pi_*M}_Z) \Rightarrow H^{a-p-q}(\Omega X; \mathbb{Z})$$

where $a$ is the Poincaré duality dimension of $X$.

\textbf{Proof.} Proposition \cite[3.1]{B} shows that $R$ satisfies the conditions of Theorem \cite{2}. Thus, there is a strongly convergent spectral sequence

$$E^2_{p,q} = H_{p,q}(\text{Cell}^{\pi_*R \pi_*M}_Z) \Rightarrow \pi_{p+q}(\text{Cell}^R_{T}M)$$

where $T$ is the hereditary torsion class of $I$-power torsion modules and $I = H_+ (\Omega X)$. Lemma \cite[3.3]{5} shows that $T$-colocalization is the same as $\mathbb{Z}$-colocalization. Thus, we can interpret the $E^2$-page of this spectral sequence as

$$E^2_{p,q} = H_{p,q}(\text{Cell}^{\pi_*R \pi_*M}_Z)$$

Lemma \cite[2.3]{A} shows that there exists $q_0$ such that $E^2_{p,q} = 0$ for all $q > q_0$. Therefore $\text{Cell}_T M$ is bounded-above. Since $\pi_0 R = \mathbb{Z}$, then Lemma \cite[2.5]{2} implies that $\text{Cell}_T M$ is $HZ$-cellular. Because the Eilenberg-Mac Lane spectrum $HZ$ is a $T$-cellular $R$-module, then every $T$-equivalence is also an $HZ$-equivalence. We conclude that the $T$-colocalization $\text{Cell}_T M \to M$ is also an $HZ$-colocalization. Thus we have shown the first spectral sequence.

To complete the proof we need only show that $\text{Cell}_{HZ} M \simeq \Sigma^{-a} C^*(\Omega X; \mathbb{Z})$. Now \cite[10.4 & 8.2]{5} show that

$$\text{Cell}_{HZ} R \simeq \Sigma^{-a} HZ \otimes_{C^*(X; \mathbb{Z})} HZ$$

The Eilenberg-Moore spectral sequence (see also \cite[7.6]{5}) implies that $HZ \otimes_{C^*(X; \mathbb{Z})} HZ \simeq C^*(\Omega X; \mathbb{Z})$, which completes the proof. \hfill \Box

It is a simple matter to show that Theorem \cite[7.3]{} applies also when working over a field. However, this is not very interesting as we now explain. Let $k = \mathbb{Z}/p$, let $X$ be a $k$-elliptic space and let $R = C_*(X; k)$. Then the results of Felix, Halperin and Thomas from \cite{8} immediately show there exists $p_0$ such that

$$H_{p,*}(\text{Cell}^{\pi_*R \pi_*R}_k) = \begin{cases} H^{-a-p-*}(\Omega x; k), & p = p_0; \\ 0, & \text{otherwise.} \end{cases}$$

A similar result applies when $k = \mathbb{Q}$, this again follows from \cite{8}.
Example 7.5. Let $\mathcal{R} = H_*(\Omega X; \mathbb{Z})$. It follows that $\mathcal{R} \otimes \mathbb{Q} = H_*(\Omega X; \mathbb{Q})$, let us denote $\mathcal{R} \otimes \mathbb{Q}$ by $\mathcal{Q}$. It is not difficult to show there is an equivalence of $\mathbb{Q}$-complexes

$$\mathcal{Q} \otimes \text{Cell}_G^p(\mathcal{R}) \simeq \text{Cell}_G^p(\mathcal{Q})$$

Which proves the following result:

**Proposition 7.4.** Let $X$ be an elliptic space and let $\mathcal{R}$ be the graded ring $H_*(\Omega X)$. Then there exists $p_0$ such that $H_{p,q}(\text{Cell}_G^p(\mathcal{R}) \otimes \mathbb{Q}) \neq 0$ implies $p = p_0$.

**Example 7.5.** Let $X$ be the Stiefel manifold $V_2(\mathbb{R}^5)$, i.e. the manifold of all orthonormal 2-frames in $\mathbb{R}^5$, this is a 7-dimensional Poincaré duality manifold. We shall first calculate $H_*(\Omega X)$. There is a fibration $S^3 \to X \to S^4$ and it is well known that $\pi_3(X) = \mathbb{Z}/2$. Examining the Serre spectral sequence for the fibration $\Omega X \to \Omega S^4 \to S^3$ we see there is a long exact sequence

$$\cdots \to H_{*+2}(\Omega X) \xrightarrow{\theta} H_*(\Omega X) \xrightarrow{\psi} H_*(\Omega S^4) \xrightarrow{\varphi} H_{*+1}(\Omega X) \to \cdots$$

Note that $\psi$ is a map of graded rings, $\theta$ is multiplication by the generator of $H_2(\Omega X) \cong \mathbb{Z}/2$ and $\varphi$ is a map of $H_*(\Omega X)$ modules. From these facts it is a simple exercise to show that $H_*(\Omega X) = \mathbb{Z}[u,v]/2u$ with $|u| = 2$ and $|v| = 6$.

Let $\mathcal{R} = H_*(\Omega X)$, let $\mathcal{I} = (u,v)$ and let $\mathcal{J}$ be the class of $\mathcal{I}$-power torsion modules. we shall now compute $H_*(\text{Cell}_G^2(\mathcal{R}))$. Since $\mathcal{R}$ is commutative, $H_*(\text{Cell}_G^2(\mathcal{R})) \cong H_*(\mathcal{R})$, i.e. the $\mathcal{I}$-local cohomology groups of $\mathcal{R}$ (see [4]). The calculation turns out to be a standard exercise in local-cohomology and we shall spare the reader details. Denote by $\mathbb{F}_2$ the field $\mathbb{Z}/2$, the local cohomology groups are:

$$H^0_0\mathcal{R} = 0$$
$$H^1_0\mathcal{R} = \mathbb{Z}[v,1/v]/\mathbb{Z}[v]$$
$$H^2_0\mathcal{R} = \Sigma^{-8}\text{Hom}_{\mathbb{F}_2}(\mathbb{F}_2[u,v], \mathbb{F}_2)$$
$$H^n_0\mathcal{R} = 0 \quad \text{for} \ n > 2$$

We see that the spectral sequence of Theorem 7.3 collapses at the $\mathbb{E}^2$-page and indeed yields $H^{*+7}(\Omega X; \mathbb{Z})$.

When working with coefficients in $\mathbb{Q}$ we have that $\mathcal{Q} = \mathcal{R} \otimes \mathbb{Q} = \mathbb{Q}[v]$ and the local cohomology groups are:

$$H^0_0\mathcal{Q} = 0$$
$$H^1_0\mathcal{Q} = \mathbb{Q}[v,1/v]/\mathbb{Q}[v] = \Sigma^{-6}\text{Hom}_{\mathbb{Q}}(\mathbb{Q}[V], \mathbb{Q})$$
$$H^n_0\mathcal{Q} = 0 \quad \text{for} \ n \neq 1$$

On the other hand, set $\mathcal{S} = H_*(\Omega X; \mathbb{F}_2)$ then $\mathcal{S} = \mathbb{F}_2[u,y]$ where $|y| = 3$. Let $\mathcal{J}$ be the maximal ideal of $\mathcal{S}$, then the $\mathcal{J}$-local cohomology groups are

$$H^2_\mathcal{J}\mathcal{S} = \Sigma^{-5}\text{Hom}_{\mathbb{F}_2}(\mathcal{S}, \mathbb{F}_2)$$
$$H^n_\mathcal{J}\mathcal{S} = 0 \quad \text{for} \ n \neq 2$$
8. Application: the target of the Eilenberg-Moore spectral sequence

In this section we will work solely over the field of rational number \( \mathbb{Q} \). Let \( F \to E \to B \) be a fibration sequence where the spaces \( E \) and \( B \) are connected and of finite type. The Eilenberg-Moore spectral sequence for this fibration (with coefficients in \( \mathbb{Q} \)) has the form:

\[
\mathbf{E}^2_{p,q} = \text{Tor}^{H^*(B;\mathbb{Q})}_{p,q}(H^*(E;\mathbb{Q}), \mathbb{Q}) \Rightarrow \pi^{p+q}(C^*(E;\mathbb{Q}) \otimes_{C^*(B;\mathbb{Q})} \mathbb{Q})
\]

It is standard that the space \( F \) is weakly equivalent to a \( \Omega B \)-space, thus \( C^*(F;\mathbb{Q}) \) is a \( C_* (\Omega B; \mathbb{Q}) \)-module. Now suppose that \( B \) is a finite CW complex. Then the results of \[5\] show there is an equivalence of \( C_* (\Omega B; \mathbb{Q}) \)-modules:

\[
C^*(E; \mathbb{Q}) \otimes_{C^*(B; \mathbb{Q})} \mathbb{Q} \simeq \text{Cell}^{C_*(\Omega B; \mathbb{Q})}_Q C^*(F; \mathbb{Q})
\]

Thus, when \( C^*(F; \mathbb{Q}) \) is \( \mathbb{Q} \)-cellular as a \( C_* (\Omega B; \mathbb{Q}) \)-module, the Eilenberg-Moore spectral sequence converges to the rational cohomology of the fiber \( F \). Dwyer’s strong convergence result \[3\] can be viewed in this light. In this section we shall use a colocalization spectral sequence to describe the relation between \( C^*(F; \mathbb{Q}) \) and the target of the Eilenberg-Moore spectral sequence: \( \text{Cell}^{C_*(\Omega B; \mathbb{Q})}_Q C^*(F; \mathbb{Q}) \).

**Theorem 8.1.** Let \( F \to E \to B \) be a fibration sequence where the spaces \( E \) and \( B \) are connected. Suppose that:

1. \( B \) is a finite CW complex with a finite fundamental group \( G \),
2. the universal cover \( \tilde{B} \) of \( B \) is rationally elliptic,
3. \( H_*(F; \mathbb{Q}) \) is finitely generated as a right \( H_* (\Omega B; \mathbb{Q}) \)-module.

Denote by \( \omega \) the idempotent \( 1 - \frac{1}{|G|} \sum_{g \in G} g \) in \( H_0(\Omega B; \mathbb{Q}) \) and let \( S \) be the graded algebra \( \omega H_*(\Omega B; \mathbb{Q}) \). Then there exists a strongly convergent spectral sequence:

\[
\mathbf{E}^2_{p,q} \Rightarrow \pi_{p+q}(C^*(E; \mathbb{Q}) \otimes_{C^*(B; \mathbb{Q})} \mathbb{Q})
\]

where \( \mathbf{E}^2_{p,q} \) is given by:

\[
\mathbf{E}^2_0 = H^*(F; \mathbb{Q})^G
\]

\[
\mathbf{E}^2_{-1} = \text{Ext}^0_{S^0}(H_*(F; \mathbb{Q})\omega, H_*(\Omega B; \mathbb{Q})\omega)/\mathcal{N}
\]

\[
\mathbf{E}^2_{-p} = \text{Ext}^p_S(H_*(F; \mathbb{Q})\omega, H_*(\Omega B; \mathbb{Q})\omega) \quad \text{for} \ p > 1
\]

where \( \mathcal{N} = H^*(F; \mathbb{Q})/H^*(F; \mathbb{Q})^G \).

**Remark 8.2.** The reader might wonder why we are using \( S^{op} \) in the theorem above instead of \( S \). Indeed, one can replace \( S^{op} \) with \( S \) and change the resulting right \( S \)-modules to left modules (this is possible by Remark \[A.5.5\] ). The point is that at the beginning of this section we have designated \( \text{chains}^*(F; \mathbb{Q}) \) to be a left \( C_* (\Omega B; \mathbb{Q}) \)-module, which means that \( H_*(F; \mathbb{Q}) \) is a right \( H_*(\Omega B; \mathbb{Q}) \)-module. So the statement of Theorem \[8.1\] is consistent with our choices.

**Proof of Theorem \[8.1\]** Theorem \[3\] implies there is a strongly convergent spectral sequence

\[
\mathbf{E}^2_{p,q} = H_{p,q}(\text{Cell}^{H_*(\Omega B; \mathbb{Q})}_{Q} H^*(F; \mathbb{Q})) \Rightarrow \pi_{p+q}(\text{Cell}^{C_*(\Omega B; \mathbb{Q})}_Q C^*(F; \mathbb{Q}))
\]

As noted above (see also \[17\] Lemma 5.6): \( C^*(E; \mathbb{Q}) \otimes_{C^*(B; \mathbb{Q})} \mathbb{Q} \simeq \text{Cell}^{C_*(\Omega B; \mathbb{Q})}_Q C^*(F; \mathbb{Q}) \). Thus, it remains only to describe the \( \mathbf{E}^2 \)-page of this spectral sequence. It follows from
Corollary A.5.10 that

\[ \text{Null}_{\mathbb{Q}}^{H_*(\Omega B; \mathbb{Q})} H^*(F; \mathbb{Q}) \simeq \text{Hom}_{\mathbb{Q}}(H_*(F; \mathbb{Q})\omega, H_*(\Omega B; \mathbb{Q})\omega) \]

The triangle Cell_{\mathbb{Q}} H^*(F; \mathbb{Q}) \to H^*(F; \mathbb{Q}) \to \text{Null}_{\mathbb{Q}} H^*(F; \mathbb{Q}) yields the \( \mathbf{E}^2 \)-page described in Theorem 8.1. \( \square \)

**Remark 8.3.** It is easy to see that when \( \pi_1(B) = \mathbb{Z}/2 \) then

\[ S = \omega H_*(\Omega B; \mathbb{Q})\omega \cong H_*(\Omega \tilde{B}; \mathbb{Q})^{\mathbb{Z}/2} \quad \text{and} \quad H_*(\Omega B; \mathbb{Q})\omega \cong H_*(\Omega \tilde{B}; \mathbb{Q}) \]

**Example 8.4.** It is amusing to apply Theorem 8.1 to the principal fibration

\[ \mathbb{Z}/2 \to S^{2n} \to \mathbb{R}P^{2n} \]

The algebra \( H_*(\Omega S^{2n}; \mathbb{Q}) \) is \( \mathbb{Q}[x] \) with \( |x| = 2n - 1 \) (note that this is not graded-commutative, it is simply the tensor algebra on a vector space of dimension 1 and degree \( 2n - 1 \)). Hence \( H_*(\Omega \mathbb{R}P^{2n}; \mathbb{Q}) \) is the semi-direct product \( \mathbb{Q}[x] \rtimes \mathbb{Z}/2 \). The action of \( \mathbb{Z}/2 \) on \( \pi_{2n}(S^{2n}) = \mathbb{Z} \) is the non-trivial action. So, if we denote by \( g \) is the generator of \( \mathbb{Z}/2 \), then \( g x g^{-1} = g x g = -x \). Thus:

\[ \omega H_*(\Omega \mathbb{R}P^{2n}; \mathbb{Q})\omega \cong H_*(\Omega S^{2n}; \mathbb{Q})^{\mathbb{Z}/2} \cong \mathbb{Q}[x^2] \]

\[ \omega H_*(\Omega \mathbb{R}P^{2n}; \mathbb{Q}) \cong \mathbb{Q}[x] \]

\[ \mathbb{Q}[\mathbb{Z}/2]\omega \cong \mathbb{Q} \]

Note that the isomorphism \( \mathbb{Q}[\mathbb{Z}/2]\omega \cong \mathbb{Q} \) is not an isomorphism of \( \mathbb{Q}[\mathbb{Z}/2] \)-modules, since \( \mathbb{Q}[\mathbb{Z}/2]\omega \) has a non-trivial \( \mathbb{Z}/2 \) action. Nevertheless, this is sufficient for us to compute the \( \mathbf{E}^2 \)-page of the spectral sequence of Theorem 8.1.

\[
\mathbf{E}^2_{p,q} = \begin{cases} 
\mathbb{Q}, & (p,q) = (0,0); \\
\mathbb{Q}, & (p,q) = (-1,2n-1); \\
0, & \text{otherwise.}
\end{cases}
\]

The spectral sequence then collapses in the \( \mathbf{E}^2 \)-page to reveal that

\[
\pi_n(C^*(S^{2n}; \mathbb{Q}) \otimes_{C^*(\mathbb{R}P^{2n}; \mathbb{Q})} \mathbb{Q}) \cong \begin{cases} 
\mathbb{Q}, & n = 0 \text{ or } n = -2n; \\
0, & \text{otherwise.}
\end{cases}
\]

On the other hand, \( C^*(\mathbb{R}P^{2n}; \mathbb{Q}) \cong \mathbb{Q} \) and hence \( C^*(S^{2n}; \mathbb{Q}) \otimes_{C^*(\mathbb{R}P^{2n}; \mathbb{Q})} \mathbb{Q} \cong C^*(S^{2n}; \mathbb{Q}) \). So we have just calculated the rational cohomology of \( S^{2n} \).

**APPENDIX A. COLOCALIZATION IN GRADED RINGS**

Recall that local cohomology can be viewed as a type of colocalization (see [4]). Much of the work in this section involves generalizing known results about local cohomology modules to the homology groups of various colocalization functors. Similar work has been carried out before, in various settings. The work here is done within the colocalization framework.

Throughout this section, \( \mathcal{R} \) is a graded \( k \)-algebra, where \( k \) is a commutative ring concentrated in degree zero.
A.1. Bounded grading is preserved by colocalization.

Lemma A.1.1. Let $k$ be a field and let $\mathcal{R}$ be a connective graded $k$-algebra. Suppose that $\mathcal{M}$ is an $\mathcal{R}$-module that has bounded grading with grading bound $q_0$. Then the injective hull of $\mathcal{M}$, also has grading bound $q_0$.

Proof. The module $\mathcal{R}^\vee = \text{Hom}_k(\mathcal{R}, k)$ is an injective $\mathcal{R}$-module which has grading bound 0. For every $m \in \mathcal{M}$ the is an obvious map $f : M \to \Sigma^{|m|} \mathcal{R}^\vee$ such that $f(m) \neq 0$. Hence, $\mathcal{M}$ has a monomorphism into the injective module $E = \prod_{m \in \mathcal{M}} \Sigma^{|m|} \mathcal{R}^\vee$. Since the injective hull of $\mathcal{M}$ is a submodule of $E$ and since $E$ has grading bound $q_0$, we are done. $\square$

Lemma A.1.2. Let $k$ be a field and let $\mathcal{R}$ be a connective graded $k$-algebra. Let $(\mathcal{I}, \mathcal{F})$ be a hereditary torsion theory on $\mathcal{R}$-modules with radical $t$. Suppose that $\mathcal{M}$ is an $\mathcal{R}$-module which has a grading bound $q_0$. Then for every $p$, $H_p(\text{Cell}_\mathcal{I} \mathcal{M})$ has a grading bound $q_0$.

Proof. Let $\mathcal{B}$ be an $\mathcal{R}$-module with grading bound $b$. Then $t(\mathcal{B})$, $\mathcal{B}/t(\mathcal{B})$ and the injective hull of $\mathcal{B}/t(\mathcal{B})$ all have grading bound $b$.

Now consider the complex constructed in the Nullification Construction 4.1. An easy inductive argument using Lemma A.1.1 shows that for every $p$, the module $\mathcal{N}^p$ has grading bound $q_0$. Hence, the homology groups of the complex $\mathcal{M} \to \mathcal{N}^0 \to \mathcal{N}^1 \to \cdots$ all have a grading bound $q_0$. These are the homology groups of $\text{Cell}_\mathcal{I} \mathcal{M}$. $\square$

A.2. Colocalization at a central principal ideal.

Definition A.2.1. We say that an element $x \in \mathcal{R}$ is central if $x$ is central in the graded sense, i.e. $xy = (-1)^{\deg x \deg y} yx$. For a central element $x \in \mathcal{R}$ we denoted by $\mathcal{R}[1/x]$ the colimit of the telescope $\mathcal{R} \xrightarrow{x} \mathcal{R} \xrightarrow{x} \cdots$.

Remark A.1. The reader might wonder about this definition of a central element. Indeed, we could have defined $x$ to be central if $xy = yx$ for every $y \in \mathcal{R}$, the proofs in this appendix would work equally well with this definition. The reason for Definition A.2.1 is that such cases are common in a topological settings.

Let $\mathcal{I}$ be a central principal ideal of $\mathcal{R}$. Then colocalization with respect to $\mathcal{I}$-power torsion modules is completely analogous to calculation of local cohomology. It is important to bear in mind that when the ideal $\mathcal{I}$ is finitely generated as a left $\mathcal{R}$-module, then $\mathcal{R}/\mathcal{I}$-colocalization is the same as colocalization with respect to $\mathcal{I}$-power torsion module (see Lemma 3.5).

Lemma A.2.2. Let $x$ be a central element of the graded algebra $\mathcal{R}$ and let $\mathcal{K}^\infty_x$ be the complex $\mathcal{R} \to \mathcal{R}[1/x]$, concentrated in degrees 0 and -1. The obvious map $\mathcal{K}^\infty_x \to \mathcal{R}$ induces a natural transformation $\mathcal{K}^\infty_x \otimes_{\mathcal{R}} \mathcal{K} \to \mathcal{K}$ on $D(\mathcal{R})$. This natural transformation is an $\mathcal{R}/(x)$-colocalization.
Proof. Set \( S = R/(x) \). It is easy to see that \( R[1/x] \) must be \( S \)-null, because multiplication by \( x \) yields a natural transformation on \( D_R \) which is zero on \( S \), but an isomorphism on \( R[1/x] \). It is also easy to see that both the kernel and the cokernel of the map \( R \to R[1/x] \) are \((x)\)-power torsion and hence these are \( S \)-cellular modules. Thus, by Lemma 2.6, \( K_x^\infty \) is \( S \)-cellular. Applying Proposition 2.4 to the triangle \( K_x^\infty \to R \to R[1/x] \) shows that \( K_x^\infty \) is \( \text{Cell}_S \tilde{R} \) while \( \text{Null}_{R/(x)} R \simeq R[1/x] \).

To get further in our analogy with local cohomology, we must assume \( R \) is left Noetherian.

Lemma A.2.3. Suppose \( R \) is left Noetherian and let \( x \) be a central element of \( R \). Then for every complex \( \mathcal{X} \) the natural map

\[
\text{hocolim}_n \text{Hom}_R(R/(x^n), \mathcal{X}) \to \mathcal{X}
\]

is an \( R/(x) \)-colocalization.

Proof. Let \( L_n \) be the complex \( R \xrightarrow{x^n} R \) concentrated in degrees 0 and 1. Let \( A_n = H_1(L_n) \), thus \( A_n \) is the annihilator of \( x^n \), i.e. \( A_n = \{ r \in R \mid x^n r = 0 \} \). Obviously, \( H_0(L_n) = R/(x^n) \). There are triangles \( TA_n \to L_n \to R/(x^n) \) and there are morphisms \( \psi_n : L_{n+1} \to L_n \) given by:

\[
\begin{array}{ccc}
R & \xrightarrow{x} & R \\
\downarrow{x^{n+1}} & & \downarrow{x^n} \\
R & \xrightarrow{1} & R \\
\end{array}
\]

Together, these induce morphisms of triangles:

\[
\begin{array}{ccc}
TA_{n+1} & \xrightarrow{\varphi_n} & TA_n \\
\downarrow{\psi_n} & & \downarrow{\psi_n} \\
L_{n+1} & \xrightarrow{\psi_n} & L_n \\
\downarrow{\chi_n} & & \downarrow{\chi_n} \\
R/(x^{n+1}) & \xrightarrow{\chi_n} & R/(x^n) \\
\end{array}
\]

where \( \chi_n \) is the obvious morphism and \( \varphi_n \) is given by multiplication by \( x \).

The complex \( L_n \) is in fact a complex of bimodules. Therefore \( \text{Hom}_R(L_n, \mathcal{X}) \) is again a \( R \)-complex. Upon applying the functor \( \text{Hom}_R(\_ , \mathcal{X}) \) and then taking the homotopy colimit of the resulting telescopes, one gets a triangle in \( D(R) \):

\[
(1) \quad \text{hocolim}_n \text{Hom}_R(TA_n, \mathcal{X}) \to \text{hocolim}_n \text{Hom}_R(L_n, \mathcal{X}) \to \text{hocolim}_n \text{Hom}_R(R/(x^n), \mathcal{X})
\]

It is easy to see that:

\[
\text{hocolim}_n \text{Hom}_R(L_n, \mathcal{X}) \simeq \text{hocolim}_n (K_x^n \otimes_R \mathcal{X}) \simeq K_x^\infty \otimes_R \mathcal{X}
\]
In view of the previous lemma, it will suffice to show that $\text{hocolim}_n \text{Hom}_R(TA_n, \mathcal{X}) \simeq 0$. This will be done in an indirect way by showing the complex $\text{hocolim}_n \text{Hom}_R(TA_n, \mathcal{X})$ is in fact $\mathcal{K}_x$-null.

Note that $A_n$ is a submodule of $A_{n+1}$ and for every $n$ the module $A_n$ is a two-sided ideal of $R$. Since $\mathcal{R}$ is left Noetherian, this chain of ideals stabilizes for some $n$. Let $A$ be the colimit of these ideals. Recall the map $\varphi$ by $x R$ ideal of $R$.

In view of the previous lemma, it will suffice to show that hocolim $\text{Hom}_R(TA_n, \mathcal{X})$.

It is now clear that hocolim $\text{Hom}_R(TA_n, \mathcal{X})$.

where the right hand homotopy colimit is taken over the telescope

$$\text{Hom}_R(TA_n, \mathcal{X}) \xrightarrow{x} \text{Hom}_R(TA_n, \mathcal{X}) \xrightarrow{x} \cdots$$

It is now clear that $\text{hocolim}_n \text{Hom}_R(TA_n, \mathcal{X}) \simeq \mathcal{R}[1/x] \otimes_R \text{Hom}_R(TA, \mathcal{X})$ and in particular it is $\mathcal{R}/(x)$-null.

It is easy to show that $\text{hocolim}_n \text{Hom}_R(\mathcal{R}/(x^n), \mathcal{X})$ is $\mathcal{R}/(x)$-cellular. Since two objects in the triangle $[\square]$ are $\mathcal{R}/(x)$-cellular, so is the third. We conclude that the homotopy colimit $\text{hocolim}_n \text{Hom}_R(TA_n, \mathcal{X})$ is both $\mathcal{R}/(x)$-cellular and $\mathcal{R}/(x)$-null, so it must be equivalent to 0.

A.3. Bounding the number of colocalization groups. From now on we shall assume that $\mathcal{R}$ is left Noetherian. The first property of local cohomology to be generalized is that the number of non-zero local cohomology groups is finite. We show that if $\mathcal{I}$ is an almost-central ideal of $\mathcal{R}$ then for every discrete $\mathcal{R}$-module $\mathcal{M}$, the number of non-zero homology groups $\text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M} = 0$ is finite. Note that for this result there is no need to assume $\mathcal{R}$ is graded.

**Definition A.3.1.** Recall that an element $x \in \mathcal{R}$ is central if $x$ is central in the graded sense. An $\mathcal{R}$-bimodule $\mathcal{M}$ is called $x$-centralizing if $xm = (-1)^{\deg x \deg m}mx$ for every $m \in \mathcal{M}$. Given a sequence of elements $x_1, \ldots, x_n \in \mathcal{R}$ we say that an $\mathcal{R}$-bimodule $\mathcal{M}$ is almost $x_1, \ldots, x_n$-centralizing if for every $0 \leq i \leq n$, the bimodule $\mathcal{M}/(x_1, \ldots, x_{i-1})\mathcal{M}$ is $x_i$-centralizing (where $(x_1, \ldots, x_{i-1})$ denotes the two-sided ideal generated by these elements).

A sequence of elements $x_1, \ldots, x_n \in \mathcal{R}$ will be called almost central if $\mathcal{R}$ is almost $x_1, \ldots, x_n$-centralizing. A two-sided ideal $\mathcal{I}$ of $\mathcal{R}$ will be called almost central if there is an almost central sequence of elements which generate $\mathcal{I}$ (as a two-sided ideal).

**Lemma A.3.2.** Let $\mathcal{I}$ be an almost central ideal of $\mathcal{R}$. Then $\mathcal{R}/\mathcal{I}$-colocalization is the same as colocalization with respect to localizing subcategory generated by $\mathcal{I}$-power torsion modules.

**Proof.** We learn from Lemma 3.5 that it is enough to show $\mathcal{I}$ is finitely generated as a left $\mathcal{R}$-module. Let $(x_1, \ldots, x_n)$ be an almost central sequence which generates $\mathcal{I}$, then a simple inductive argument suffices to show that $x_1, \ldots, x_n$ also generate $\mathcal{I}$ as a left module. □

**Lemma A.3.3.** Let $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots \subseteq \mathcal{M}$ be a filtration of an $\mathcal{R}$-module $\mathcal{M}$ such that $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}_i$. Suppose that $\mathcal{R}$ is left Noetherian and let $\mathcal{I}$ be an almost central ideal of $\mathcal{R}$. Then the obvious morphism

$$\text{hocolim}_n \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i \rightarrow \text{hocolim}_n \mathcal{M}_i \simeq \mathcal{M}$$
is an $\mathcal{R}/\mathcal{I}$-colocalization of $\mathcal{M}$.

Proof. The triangle

$$\bigoplus_i \mathcal{M}_i \rightarrow \bigoplus_i \mathcal{M}_i \rightarrow \hocolim_i \mathcal{M}_i \simeq \mathcal{M}$$

implies it is enough to show that

$$\phi : \bigoplus_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i \rightarrow \bigoplus_i \mathcal{M}_i$$

is an $\mathcal{R}/\mathcal{I}$-cellular approximation.

It is clear that $\bigoplus_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i$ is $\mathcal{R}/\mathcal{I}$-cellular. It remains to show that the morphism $\phi$ is an $\mathcal{R}/\mathcal{I}$-equivalence. From Lemma 4.2 we learn that $\text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i$ is bounded above. Without loss of generality we may assume that $\text{Cell}_{\mathcal{R}/\mathcal{I}}$ is a cochain complex of injective modules. Similarly, we replace $\mathcal{M}_i$ by an injective resolution $J_i$.

Because $\mathcal{R}$ is left Noetherian, a direct sum of injective modules is injective. Thus, $\bigoplus_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i$ is a cochain complex of injective modules and $\bigoplus_i J_i$ is an injective resolution of $\bigoplus_i \mathcal{M}_i$. Let us denote by $\text{Hom}_\mathcal{R}(-, -)$ the strict (i.e. not derived) internal hom of $\mathcal{R}$-chain complexes. Since $\mathcal{R}/\mathcal{I}$ is a cyclic $\mathcal{R}$-module, the natural maps

$$\bigoplus_i \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i) \rightarrow \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \bigoplus_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i)$$

$$\bigoplus_i \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, J_i) \rightarrow \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \bigoplus_i J_i)$$

are isomorphisms. Hence, there is a commutative diagram in $\mathbf{D}(k)$:

$$\begin{array}{ccc}
\bigoplus_i \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i) & \xrightarrow{\simeq} & \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \bigoplus_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{M}_i) \\
\bigoplus_i \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \mathcal{M}_i) & \xrightarrow{\simeq} & \text{Hom}_\mathcal{R}(\mathcal{R}/\mathcal{I}, \bigoplus_i \mathcal{M}_i)
\end{array}$$

which completes the proof. □

Lemma A.3.4. Let $\mathcal{X}_1 \rightarrow \mathcal{X}_2 \rightarrow \cdots$ be a telescope of $\mathcal{R}$-complexes such that all complexes are bounded-above at some fixed degree $B$. Let $\mathcal{I}$ be an almost central ideal of $\mathcal{R}$. Suppose either $\mathcal{R}$ is left Noetherian or $\langle \mathcal{R}/\mathcal{I} \rangle$ is generated by a perfect complex. Then:

$$\hocolim_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{X}_i \simeq \text{Cell}_{\mathcal{R}/\mathcal{I}} \hocolim_i \mathcal{X}_i$$

and the natural map $\hocolim_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{X}_i \rightarrow \hocolim_i \mathcal{X}_i$ is an $\mathcal{R}/\mathcal{I}$-colocalization.

Proof. If $\langle \mathcal{R}/\mathcal{I} \rangle$ is generated by a perfect complex then the results of [4] show that $\mathcal{R}/\mathcal{I}$-colocalization is smashing. The result follows.

So suppose that $\mathcal{R}$ is left Noetherian. Recall that $\hocolim_i \mathcal{X}_i$ can be written as the cone of the map

$$\bigoplus_i \mathcal{X}_i \xrightarrow{\psi_i^{-1}} \bigoplus_i \mathcal{X}_i$$

where $\psi_i$ is the map $\mathcal{X}_i \rightarrow \mathcal{X}_{i+1}$. Hence it is enough to show that

$$\bigoplus_i \text{Cell}_{\mathcal{R}/\mathcal{I}} \mathcal{X}_i \simeq \text{Cell}_{\mathcal{R}/\mathcal{I}} \bigoplus_i \mathcal{X}_i$$
Clearly, \( \oplus_i \text{Cell}_{R/I} X_i \) is \( R/I \)-cellular. So it is enough to show that the natural map 
\( \oplus_i \text{Cell}_{R/I} X_i \to \oplus_i X_i \) is a \( R/I \)-equivalence. Equivalently, it is enough to show that

\[
\bigoplus_i \text{Null}_{R/I} X_i
\]
is \( R/I \)-null.

Let \( T \) be the hereditary torsion class of \( I \)-power torsion modules. In Lemma A.3.2 we saw that \( T \)-nullification is the same as \( R/I \)-nullification. An easy generalization of the Nullification Construction 4.11 shows that if \( X \) is a bounded-above complex, then \( \text{Null}_T X \) has an injective resolution \( E \) composed only of torsion free injectives, i.e. the injective module \( E' \) is torsion-free. Moreover, we can choose this injective resolution to be bounded-above at the same degree as \( X \), i.e. if \( H_i X = 0 \) for \( i > B \) then \( E_i = 0 \) for \( i > B \). Thus, we can replace each \( \text{Null}_{R/I} X_i \) by such an injective resolution \( E(i) \).

Because \( R \) is left Noetherian, then a direct sum of injective modules is injective. Let \( b \) be a set of injective \( R \)-modules that are \( R/I \)-null. Next we show that \( \oplus_{E \in b} E \) is \( R/I \)-null. The map \( \oplus_{E \in b} E \to \prod_{E \in b} E \) is a monomorphism of injective modules, hence \( \oplus_{E \in b} E \) is a direct summand of \( \prod_{E \in b} E \). Clearly \( \prod_{E \in b} E \) is \( R/I \)-null, hence so is \( \oplus_{E \in b} E \).

The last paragraph shows that \( \oplus_i E(i) \) is a bounded-above complex which has a torsion-free injective \( R \)-module in each degree. Therefore \( \text{Ext}^*_R(R/I, \oplus_i E(i)) = 0 \).

Lemma A.3.5. Let \( x_1, ..., x_n \) be an almost central sequence in \( R \) and suppose either \( R \) is left Noetherian or \( (R/(x_1, ..., x_n)) \) is generated by a perfect complex. Set \( I = (x_1, ..., x_n) \) and \( I_j = (x_1, ..., x_j) \). Then for every \( R/I_j \)-module \( M \) the map

\[
\phi_{j,M} : \text{Cell}_{R/I_j} M \to M
\]
is an \( R/I \)-colocalization in \( D(R) \).

Proof. It is enough to show the morphism \( \phi_{j,M} \) is an \( R/I \)-equivalence. The proof will proceed by induction on \( i = n - j \). For \( i = 0 \) the proof is obvious.

For \( i > 0 \) we break the proof into two cases. In the first case, assume \( M \) is also \((x_1, ..., x_{j+1})\)-power torsion. Let \( y = x_{j+1} \) and let \( M_t \subseteq M \) be the annihilator of \( y^t \) (where \( M_0 = 0 \)). Then \( M = \text{colim}_t M_t \simeq \text{hocolim}_t M_t \). Since \( Q_t = M_t/M_{t-1} \) is an \( R/I_{j+1} \)-module, by the induction assumption the map \( \phi_{j+1,Q_t} \) is an \( R/I \)-equivalence both of \( R/I_j \)-complexes and of \( R \)-complexes.

Consider the commutative diagram where each row is a triangle:

\[
\begin{array}{ccc}
M_{t-1} & \rightarrow & M_t \\
\downarrow & & \downarrow \\
\text{Cell}_{R/I_j} R_I M_{t-1} & \rightarrow & \text{Cell}_{R/I_j} R_I M_t
\end{array}
\]

A simple inductive argument shows that \( \phi_{j,M_t} \) is an \( R/I \)-equivalence of \( R \)-modules for all \( t \). Lemma A.3.1 shows that \( \phi_{j,M_t} \simeq \text{hocolim}_t \phi_{j,M_t} \) is an \( R/I \)-colocalization both over \( R \) and over \( R/I_j \), since in both cases \( R/I \)-colocalization is given by the homotopy colimit of the telescope \( \text{Cell}_{R/I} R_I M_1 \to \text{Cell}_{R/I} R_I M_2 \to \cdots \).

Now suppose \( M \) is a general \( R/I_j \)-module. Let \( M' \) be the \( R/I_{j+1} \)-cellular approximation of \( M \) over \( R/I_j \). So, by Lemma A.2.2 \( M' \) is the complex \( M \to M[1/y] \). Since
the morphism $\mathcal{M}' \to \mathcal{M}$ is an $\mathcal{R}/I$-equivalence (over $\mathcal{R}/I_j$), the composition

$$\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/I_j} \mathcal{M}' \to \mathcal{M}' \to \mathcal{M}$$

is an $\mathcal{R}/I$-colocalization over $\mathcal{R}/I_j$.

Consider the triangle:

$$H^0(\mathcal{M}') \to \mathcal{M}' \to T^{-1}H^1(\mathcal{M}')$$

Both $H^0(\mathcal{M}')$ and $H^1(\mathcal{M}')$ are $I_{j+1}$-power torsion, so the claim is true for them. Hence $\phi_{j,M'}$ is an $\mathcal{R}/I$-colocalization also over $\mathcal{R}$ and the proof is complete. □

Lemma A.3.6. Let $I$ be an almost central ideal of a left Noetherian ring $\mathcal{R}$. Suppose $I$ has an almost central generating sequence of length $n$. Then for any $\mathcal{R}$-module $\mathcal{M}$, $H^i\text{Cell}_{\mathcal{R}/I} \mathcal{M} = 0$ for $i > n$.

Proof. Let $x_1, \ldots, x_n$ be the generating set for $I$. The proof proceeds by induction on $n$. For $n = 1$ the result follows from Lemma A.2.2.

Suppose $n > 1$. We begin with the case where $\mathcal{M}$ is an $(x_1)$-power torsion module and show that $H^i\text{Cell}_{\mathcal{R}/I} \mathcal{M} = 0$ for $i > n - 1$. As before (see the proof of Lemma A.3.5), $\mathcal{M}$ has a filtration

$$0 = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \cdots \subseteq \mathcal{M}$$

such that $\mathcal{M} = \cup_t \mathcal{M}_t$ and $\mathcal{Q}_t = \mathcal{M}_t/\mathcal{M}_{t-1}$ is an $\mathcal{R}/(x_1)$-module. The induction assumption combined with the previous lemma show that for every $t$, $H^i\text{Cell}_{\mathcal{R}/I} \mathcal{Q}_t = 0$ for $i > n - 1$. A simple inductive argument shows that also $H^i\text{Cell}_{\mathcal{R}/I} \mathcal{M}_t = 0$ for $i > n - 1$. By Lemma A.3.4, $\text{Cell}_{\mathcal{R}/I} \mathcal{M} \simeq \text{hocolim}_t \text{Cell}_{\mathcal{R}/I} \mathcal{M}_t$ and therefore also $H^i\text{Cell}_{\mathcal{R}/I} \mathcal{M} = 0$ for $i > n - 1$.

Since $\mathcal{R}/I \in \langle \mathcal{R}/(x_1) \rangle$, the map $\text{Cell}_{\mathcal{R}/(x_1)} \mathcal{M} \to \mathcal{M}$ is an $\mathcal{R}/I$-equivalence. So we might as well compute $\text{Cell}_{\mathcal{R}/I} \text{Cell}_{\mathcal{R}/(x_1)} \mathcal{M}$. Set $\mathcal{M}' = \text{Cell}_{\mathcal{R}/(x_1)} \mathcal{M}$. As above, there is a distinguished triangle

$$H^0(\mathcal{M}') \to \mathcal{M}' \to T^{-1}H^1(\mathcal{M}')$$

where both $H^0(\mathcal{M}')$ and $H^1(\mathcal{M}')$ are $(x_1)$-power torsion modules. Hence

$$H^i(\text{Cell}_{\mathcal{R}/I} H^q(\mathcal{M}')) = 0 \quad \text{for } i > n - 1 \text{ and } q = 0, 1$$

Combined with the triangle above the result follows. □

A.4. Bounding the cohomology modules of the cellular approximation. Now we recall the fact that $\mathcal{R}$ is graded. Recall that for an $\mathcal{R}$-module $\mathcal{M}$ the group of elements of $\mathcal{M}$ whose internal grading is $i$ is denoted by $\mathcal{M}(i)$ and we say that $\mathcal{M}$ has bounded grading if there is some $b$ such that $\mathcal{M}(i) = 0$ for all $i > b$. Such $b$ is called a grading bound of $\mathcal{M}$.

Lemma A.4.1. Let $\mathcal{R}$ be a connective Noetherian graded ring. Let $I$ be an almost central ideal of $\mathcal{R}$ that contains all elements of degree greater than $0$. Suppose $\mathcal{M}$ be a finitely generated $\mathcal{R}$-module, then for every $i$, $H^i\text{Cell}_{\mathcal{R}/I} \mathcal{M}$ has bounded grading.

Proof. Let $x_1, \ldots, x_n \in \mathcal{R}$ be an almost central sequence of generators for $I$. As is our custom, the proof shall proceed by induction on $n$. When $n = 0$, then $\mathcal{R} = \mathcal{R}/(0)$ and clearly every finitely generated $\mathcal{R}$-module has bounded grading.
Let $n > 0$ and set $x = x_1$. The proof shall be divided into three cases. First, suppose that the finitely generated $\mathcal{R}$-module $\mathcal{M}$ is an $(x)$-power torsion module. Construct the usual filtration on $\mathcal{M}$, namely set $\mathcal{M}_t$ to be the kernel of the map $\mathcal{M} \xrightarrow{x^t} \mathcal{M}$. Since $\mathcal{R}$ is Noetherian and $\mathcal{M}$ is finitely generated, $\mathcal{M}$ is Noetherian. This implies $\mathcal{M} = \mathcal{M}_j$ for some $j$. Set $Q_t = \mathcal{M}_t/\mathcal{M}_{t-1}$, these are finitely generated $\mathcal{R}/(x)$-modules. By the induction assumption the modules $H^i\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/(x)} Q_t$ have bounded grading. From Lemma A.3.5 we see that

$$H^i\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/(x)} Q_t = H^i\text{Cell}_{\mathcal{R}/I}^\mathcal{R} Q_t$$

A simple inductive argument shows that $H^i\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/(x)} \mathcal{M}_t$ has bounded grading, for all $t \leq j$.

Next, suppose that $x$ is regular on $\mathcal{M}$, i.e. the map $\mathcal{M} \xrightarrow{x} \mathcal{M}$ is a monomorphism. Set $\mathcal{M}' = \mathcal{M}/\mathcal{M}[1/x]$ and consider the short exact sequence $\mathcal{M} \to \mathcal{M}[1/x] \to \mathcal{M}'$. Lemma A.2.2 implies that $\mathcal{M}[1/x]$ is $\mathcal{R}/(x)$-null. Since $\mathcal{R}/I \in (\mathcal{R}/(x))$, it follows that $\mathcal{M}[1/x]$ is $\mathcal{R}/I$-null. This, in turn, implies that

$$H^i\text{Cell}_{\mathcal{R}/I} \mathcal{M} \cong H^{i-1}\text{Cell}_{\mathcal{R}/I} \mathcal{M}'$$

Note that $\mathcal{M}'$ is $\mathcal{M}/x^\infty$, i.e. $\mathcal{M}' = \text{colim}_i \mathcal{M}/x^i \mathcal{M}$.

Consider the following commutative diagram whose rows and columns are short exact sequences of $\mathcal{R}$-modules:

\[
\begin{array}{cccc}
\mathcal{M} & = & \mathcal{M} \\
\downarrow x^i & & \downarrow x^i+1 \\
\mathcal{M} & \xrightarrow{x} & \mathcal{M} & \to \mathcal{M}/x\mathcal{M} \\
\mathcal{M}/x^i\mathcal{M} & \to & \mathcal{M}/x^{i+1}\mathcal{M} & \to \mathcal{M}/x\mathcal{M}
\end{array}
\]

By the induction assumption, $H^i\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/(x)} (\mathcal{M}/x\mathcal{M})$ has bounded grading for each $i$. From the previous result, Lemma A.3.6, there is only a finite number of degrees $q$ for which $H^q\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/(x)} (\mathcal{M}/x\mathcal{M}) \neq 0$. Therefore, there is an integer $B$ such that

$$(H^q\text{Cell}_{\mathcal{R}/I}^{\mathcal{R}/(x)} \mathcal{M}/x\mathcal{M}) \langle j \rangle = 0$$

for all $q$ and for all $j \geq B$.

Using the long exact sequences arising from the triangle

$$\text{Cell}_{\mathcal{R}/I}(\mathcal{M}/x^i\mathcal{M}) \to \text{Cell}_{\mathcal{R}/I}(\mathcal{M}/x^{i+1}\mathcal{M}) \to \text{Cell}_{\mathcal{R}/I}\mathcal{M}/x\mathcal{M}$$

and an inductive argument, we see that for all $i$ and for all $q$

$$(H^q\text{Cell}_{\mathcal{R}/I}(\mathcal{M}/x^i\mathcal{M})) \langle j \rangle = 0 \text{ for } j \geq B$$

Now $\mathcal{M}' \simeq \text{hocolim}_i(\mathcal{M}/(x^i)\mathcal{M})$, so by Lemma A.3.4 there is an equivalence $\text{Cell}_{\mathcal{R}/I} \mathcal{M}' \simeq \text{hocolim}_i \text{Cell}_{\mathcal{R}/I}(\mathcal{M}/(x^i)\mathcal{M})$. It follows that for all $q$:

$$(H^q\text{Cell}_{\mathcal{R}/I} \mathcal{M}') \langle j \rangle = 0 \text{ for } j \geq B$$
Finally, let $\mathcal{M}$ be any finitely generated $\mathcal{R}$-module. Let $\mathcal{Q}$ be the largest $(x_1)$-power torsion submodule of $\mathcal{M}$ and let $\mathcal{G} = \mathcal{M}/\mathcal{Q}$. Then both $\mathcal{Q}$ and $\mathcal{G}$ are finitely generated. Since $\mathcal{Q}$ is an $(x_1)$-power torsion module, the groups $H^i\text{Cell}_{\mathcal{R}/\mathcal{I}}\mathcal{Q}$ have bounded grading. Since $\mathcal{G}$ is $(x_1)$-torsion free, it follows that $\mathcal{G}$ is $x_1$-regular. Hence the groups $H^i\text{Cell}_{\mathcal{R}/\mathcal{I}}\mathcal{G}$ also have bounded grading. The result for $\mathcal{M}$ follows from the obvious long exact sequence.

□

A.5. Colocalization over a rational semidirect product. Throughout the rest of this section $B$ is a finite CW-complex with a finite fundamental group $G$ and $\hat{B}$ is the universal cover of $B$. We assume that $\hat{B}$ is rationally elliptic. Let $\mathcal{R}$ be the rational graded algebra $H_*(\Omega\hat{B}; \mathbb{Q})$. The isomorphism $\pi_0(\Omega\hat{B}) \cong G$ gives a natural way to identify the graded algebra $\mathcal{A} = H_*(\Omega\hat{B}; \mathbb{Q})$ with $\mathcal{R} \rtimes G$ where the action of $G$ on $\mathcal{R}$ is given by conjugation. We consider $\mathcal{R}$ as a subalgebra of $\mathcal{A}$ via the obvious morphism $\mathcal{R} \to \mathcal{A}$. Note that for a right $\mathcal{A}$-module $\mathcal{M}$ we denote by $\mathcal{M}^\vee$ the left $\mathcal{A}$-module $\text{Hom}_\mathbb{Q}(\mathcal{M}, \mathbb{Q})$.

Lemma A.5.1. The $\mathcal{A}$-module $\mathcal{A}^\vee = \text{Hom}_\mathbb{Q}(\mathcal{A}, \mathbb{Q})$ is the injective hull of $\mathbb{Q}[G]$.

Proof. One need only show that $\mathbb{Q}[G] \cong \mathcal{A}^\vee(0)$ is an essential submodule of $\mathcal{A}^\vee$. Indeed it is easy to see that for every non-zero $f \in \mathcal{A}^\vee(-n)$ there exists $a \in \mathcal{A}(n)$ such that $af \in \mathcal{A}^\vee(0)$ is non-zero.

Write $\mathbb{Q}[G]$ as a direct sum $\mathbb{Q} \oplus W$, where $\mathbb{Q}$ is the trivial $G$-representation. Clearly $\mathcal{A}^\vee = \mathcal{E}(\mathbb{Q}) \oplus \mathcal{E}(W)$ where $\mathcal{E}(\mathbb{Q})$ is the injective hull of $\mathbb{Q}$ and $\mathcal{E}(W)$ is the injective hull of $W$. To easily write down these modules we need some idempotents in $\mathcal{A}(0)$, set

$$\epsilon = \frac{1}{|G|} \sum_{g \in G} g \quad \text{and} \quad \omega = 1 - \epsilon.$$ 

Lemma A.5.2. With the notation above: $\mathcal{E}(W) = (\omega \mathcal{A})^\vee$.

Proof. Note that $(\omega \mathcal{A})^\vee$ is a left $\mathcal{A}$-module via the right action of $\mathcal{A}$ on $\omega \mathcal{A}$. Since $(\omega \mathcal{A})^\vee$ is a retract of $\mathcal{A}^\vee$, it is injective. Now, every nontrivial submodule $\mathcal{M}$ of $(\omega \mathcal{A})^\vee$ is also a submodule of $\mathcal{A}^\vee$ and hence has a nontrivial intersection with $\mathcal{A}^\vee(0)$. Clearly, the intersection $\mathcal{M} \cap (\omega \mathcal{A})^\vee$ must be contained in $W$. This shows $W$ is an essential submodule of $(\omega \mathcal{A})^\vee$.

Lemma A.5.3. For any $\mathcal{A}$-module $\mathcal{M}$, $\text{Hom}_\mathcal{A}(\mathcal{M}, \mathcal{E}(W)) \cong (\omega \mathcal{M})$.

Proof. Clearly

$$\text{Hom}_\mathcal{A}(\mathcal{M}, \mathcal{E}(W)) \cong (\omega \mathcal{A} \otimes_\mathcal{A} \mathcal{M})^\vee \cong (\omega \mathcal{M})^\vee.$$ 

□

Lemma A.5.4. There is a natural isomorphism of graded rings:

$$\text{Hom}_\mathcal{A}(\mathcal{E}(W), \mathcal{E}(W)) \cong (\omega \mathcal{A})^{\text{op}}.$$ 

Proof. Lemma A.5.3 shows that

$$\text{Hom}_\mathcal{A}(\mathcal{E}(W), \mathcal{E}(W)) \cong (\omega(\mathcal{A}^\vee))^\vee$$
Since $A$ is of finite type, we have that $(\omega A)^{\vee \vee} \cong \omega A$. Clearly
$$(\omega A)^{\vee \vee} \cong (\epsilon(\omega A)^{\vee})^{\vee} \oplus (\omega A)^{\vee},$$
on the other hand
$$\omega A \cong \omega A \epsilon \oplus \omega A \omega$$
It is easy to see that under these isomorphisms we have
$$(\omega(\omega A)^{\vee})^{\vee} \cong (\epsilon(\omega A)^{\vee})^{\vee} \oplus (\omega(\omega A)^{\vee})^{\vee} \oplus (\omega A)^{\vee}$$
It is now an exercise to show that this induces a natural isomorphism of graded rings:
$$\text{Hom}_A(E(W), E(W)) \cong (\omega A)^{\text{op}}$$

From now on we shall use the notation $S$ for the graded algebra $\omega A \omega$.

**Remark A.5.5.** Note that, since $A$ is a graded Hopf algebra, then $A^{\text{op}} \cong A$. This implies $S^{\text{op}} \cong S$, since $(\omega A \omega)^{\text{op}} \cong \omega A^{\text{op}} \omega$.

**Lemma A.5.6.** The graded algebra $A$ is a left and right Noetherian ring.

**Proof.** Recall that $R$ is the enveloping algebra of a finite dimensional graded Lie algebra. Thus, $R$ is clearly Noetherian. As a left $R$-module, $A$ is a finite direct sum of copies of $R$ and therefore $A$ is a Noetherian left $R$-module. Hence $A$ is left Noetherian. Similar considerations show $A$ is also right Noetherian. □

**Lemma A.5.7.** The graded algebra $S^{\text{op}}$ is left Noetherian.

**Proof.** Let $a$ be a right ideal of $S$ and let $b$ be the right ideal of $A$ generated by $a$. By Lemma A.5.6, $b$ is finitely generated as a right $A$-module. Clearly, one can choose a finite set of generators for $b$ from the set $a$. Let $\omega x_1, \ldots, \omega x_n$ be generators of $b$ as a right $A$-module. If $\omega x \omega \in a$ then $\omega x \omega = \sum_i \omega x_i \omega r_i$ for some $r_i \in A$. Multiplying on the left and right by $\omega$ yields
$$\omega x \omega = \sum_i (\omega x_i \omega)(\omega r_i \omega)$$
Hence $a$ is finitely generated as a right $S$-module. □

**Lemma A.5.8.** If $M^{\vee}$ is a finitely generated right $A$-module, then $(\omega M)^{\vee}$ is a finitely generated left $S^{\text{op}}$-module. Moreover, in this case $(\omega M)^{\vee} \cong M^{\vee} \omega$.

**Proof.** Since $M^{\vee}$ is a finitely generated right $A$-module, and since $A$ is of finite type (i.e. $A(n)$ is finite for every $n$) then $M^{\vee}$ is of finite type. Hence $(M^{\vee})^{\vee} \cong M$. Using the isomorphism $A^{\text{op}} \cong A$, we see it is enough to prove the following claim: suppose $M$ is a finitely generated $A$-module, then $\omega M$ is a finitely generated $S$-module. The proof of this last claim is essentially the same as the proof of Lemma A.5.7 above. The isomorphism $(\omega M)^{\vee} \cong M^{\vee} \omega$ is a simple consequence of the fact that $M$ is of finite type. □

**Proposition A.5.9.** Let $M$ be an $A$-module with bounded grading such that $M^{\vee}$ is a finitely generated right $A^{\text{op}}$-module, then
$$\text{Null}_Q M \cong \text{Hom}_{\text{End}_A(E(W))}(\text{Hom}_A(M, E(W)), E(W))$$
Proof. Let $\mathcal{T}$ be the hereditary torsion class cogenerated by $\mathcal{E}(W)$. We shall start by showing that $\text{Hom}_{\text{End}_A(\mathcal{E}(W))}(\text{Hom}_A(\mathcal{M}, \mathcal{E}(W)), \mathcal{E}(W))$ is $\text{Null}_T \mathcal{M}$. Lemma A.5.7 shows that $\text{End}_A(\mathcal{E}(W))$ is a left Noetherian ring and Lemma A.5.8 shows that $\text{Hom}_A(\mathcal{M}, \mathcal{E}(W))$ is a finitely generated module over $\text{End}_A(\mathcal{E}(W))$. Hence $\text{Hom}_A(\mathcal{M}, \mathcal{E}(W))$ has a resolution by finitely generated projective $\text{End}_A(\mathcal{E}(W))$-modules. Proposition 4.4 implies that $\text{Null}_T \mathcal{M} \cong \text{Hom}_{\text{End}_A(\mathcal{E}(W))}(\text{Hom}_A(\mathcal{M}, \mathcal{E}(W)), \mathcal{E}(W))$.

Since $\text{Hom}_A(\mathbb{Q}, \mathcal{E}(W)) = 0$, we see that every $\mathbb{Q}$-cellular complex is $T$-cellular. Similarly, every $T$-null complex is $\mathbb{Q}$-null. Hence $\text{Null}_T \mathcal{M}$ is also $\mathbb{Q}$-null. It remains to show that $\text{Cell}_T \mathcal{M}$ is $\mathbb{Q}$-cellular, as that would imply by Proposition 2.3 that $\text{Null}_T \mathcal{M} \cong \text{Null}_\mathbb{Q} \mathcal{M}$.

Let $I$ be the kernel of map $A \rightarrow \mathbb{Q}$. Recall that an $A$-module is an $I$-power torsion module if and only if it is $\mathbb{Q}$-cellular. From Lemma A.1.2 we see that $H^n(\text{Cell}_T \mathcal{M})$ has bounded grading for every $n$. So it would suffice to show that every $T$-torsion module with bounded grading is an $I$-power torsion module.

Let $B$ be a $T$-torsion module with bounded grading. Since $\text{Hom}_A(B, \mathcal{E}(W)) = 0$ we conclude from Lemma A.5.3 that $\omega B = 0$. Let $J$ be the kernel of $A \epsilon \rightarrow \mathbb{Q}$, then $I \cong J \oplus A \omega$. Since $A \omega B = 0$, we are left with showing that for every $b \in B$ there exists some $n$ such that $J^n b = 0$ ($J$ is a multiplicatively closed subset of $I$). Note that $J(0) = 0$, hence $J^n (i) = 0$ for $i < n$. Because $B$ has bounded grading this implies that $J^n b = 0$ for large enough $n$. □

Corollary A.5.10. Let $\mathcal{M}$ be an $A$-module with bounded grading such that $\mathcal{M}^\vee$ is a finitely generated right $A^\text{op}$-module, then

$$\text{Null}_\mathbb{Q} \mathcal{M} \cong \text{Hom}_{S^\text{op}}(\mathcal{M}^\vee \omega, A^\vee \omega).$$

Proof. This is simply a restatement of Proposition A.5.9 using the isomorphisms of Lemma A.5.4 and Lemma A.5.8. □

Appendix B. The homology of loops on an elliptic space

This section contains the proof of the following proposition.

Proposition B.1. Let $X$ be an elliptic space and let $\mathcal{I}$ be the augmentation ideal of $H_*(\Omega X)$. Then $\mathcal{I}$ is almost commutative.

In [9], Felix, Halperin and Thomas proved that the loop space homology of an elliptic space is left and right Noetherian. The proof of proposition B.1 above follows the same lines of the proof of [9, Theorem B].

Note that for a connective graded ring $\mathcal{R}$, the augmentation ideal of $\mathcal{R}$ is the ideal of positively graded elements.

Proof of Proposition B.1. As in [9], we can assume without loss if generality that $X$ is a finite complex. Let $R$ be the sub-ring of $\mathbb{Q}$ obtained by adjoining $1/p$ to $\mathbb{Z}$ whenever $p < \text{dim} X$ or $H_*(\Omega X)$ has $p$-torsion.

By [13, Theorem A], $H_*(\Omega X; R) = UL$ - the enveloping algebra of a graded Lie algebra. The arguments in [9] show that $L$ is a free finitely generated $R$-module. Since $X$ is simply
connected, the Lie algebra $L$ is nilpotent and therefore the augmentation ideal $I_T$ of $H_*(\Omega X; R)$ is almost-commutative.

As noted in [9], the McGibbon-Wilkerson theorem [15] implies that $R = \mathbb{Z}[1/p_1, \ldots, 1/p_s]$ for primes $p_1, \ldots, p_s$. Set $R_t = \mathbb{Z}[1/p_1, \ldots, 1/p_t]$, $A_t = H_*(\Omega X; R_t)$ and $I_t = H_*(\Omega X; R_t)$. We have just shown that $I_s$ is almost-commutative and we will work inductively to show $I_t$ is almost-commutative for $t = 0, \ldots, s$. The rest of the proof is the induction step.

Set $k = R_t$, $A = A_t$, $I = I_t$ and $p = p_{t+1}$. By the induction assumption, $I[1/p]$ is an almost-commutative ideal of $A[1/p] = A_{t+1}$. Note that for any $R$-module $M$, $M[1/p] \cong M \otimes_R A[1/p]$ and that $A[1/p]$ is a flat $A$-module.

By [10], there exists $m$ such that $p^m$ annihilates all $p$-torsion in $A$. The short exact sequence

$$0 \to p^m \mathcal{I} \to \mathcal{I} \to \mathcal{I}/p^m \to 0$$

induces a short exact sequence

$$0 \to p^m \mathcal{I}[1/p] \to \mathcal{I}[1/p] \to (\mathcal{I}/p^m)[1/p] \to 0$$

Clearly, $(\mathcal{I}/p^m)[1/p] = 0$ and hence $p^m \mathcal{I}[1/p] \cong \mathcal{I}[1/p]$. By definition of $m$, the map $p^m \mathcal{I} \to \mathcal{I}[1/p]$ is an injection.

**Lemma B.2.** The ideal $p^m \mathcal{I}$ is almost-commutative.

**Proof.** Let $\mathcal{J} = \mathcal{I}[1/p]$ and let $x_1, \ldots, x_n$ be an almost-central sequence generating $\mathcal{J}$. Let $Z_i$ be the sub-ring of $A[1/p]$ generated by $x_1, \ldots, x_i$ and $R[1/p]$. Define $T_i$ to be the intersection $Z_i \cap p^m \mathcal{I}$.

We first show that the set $T_i$ is central in $A$. Let $t \in T_i$ and $x \in \mathcal{I}$, then

$$p^m(xt - tx) = (p^m x)t - t(p^m x) = 0$$

Hence $xt - tx$ is $p$-torsion, but $xt - tx \in p^m \mathcal{I}$ and $p^m \mathcal{I}$ has no $p$-torsion. So $t$ must be central in $A$. This also implies that $\mathcal{M}_1 = \mathcal{A}T_1$ is a two-sided ideal of $A$. By [9], $A$ is left Noetherian, therefore $\mathcal{M}_1$ is generated by a finite number of elements in $T_1$.

We continue by induction, thus we assume that $\mathcal{M}_{i-1} = \mathcal{A}T_{i-1}$ is an almost commutative ideal of $\mathcal{A}$ and show that the image of $T_i$ in $A/\mathcal{M}_{i-1}$ is central. Let $t \in T_i \setminus \mathcal{M}_{i-1}$ and let $x \in \mathcal{I}/\mathcal{M}_{i-1}$. As above, we see that $tx - xt$ is both $p$-torsion and an element of $p^m \mathcal{I}/\mathcal{M}_{i-1}$, hence $tx - xt = 0$. Set $\mathcal{M}_i = \mathcal{A}T_i$. Using the induction assumption and the fact that $\mathcal{A}$ is Noetherian it is easy to see that $\mathcal{M}_i$ is almost commutative ideal of $\mathcal{A}$.

It remains to show that $\mathcal{I}/p^m \mathcal{I}$ is and almost-central ideal of $A/p^m \mathcal{I}$.

**Lemma B.3.** The ideal $\mathcal{I}/p^m$ is an almost-central ideal of $A/p^m$.

**Proof.** Set $\mathcal{J}_i = \mathcal{I}/p^i$ and $\mathcal{S}_i = A/p^i$. We prove, by induction on $i$, that $\mathcal{J}_i$ is an almost-commutative ideal of $\mathcal{S}_i$, which clearly implies that $\mathcal{J}_i$ is an almost central ideal of $A/p^i$.

The ring $\mathcal{S}_1$ is a sub-ring of $H_*(\Omega X; R/p)$. The results of Felix, Halperin and Thomas from [8] show that the maximal ideal of $H_*(\Omega X; \mathbb{Z}/p)$ is almost commutative. Since $H_*(\Omega X; R/p) \cong H_*(\Omega X; \mathbb{Z}/p) \otimes_\mathbb{Z} R$, we see that the maximal ideal of $H_*(\Omega X; R/p)$ is almost commutative. Since $\mathcal{S}_1$ is left Noetherian and a sub-ring of $H_*(\Omega X; R/p)$, an argument similar to the one employed in the proof of Lemma B.2 shows that $\mathcal{J}_1$ is almost commutative.
For $i > 1$, consider the short exact sequence:

$$0 \to p^{i-1} S_i \to S_i \overset{\varphi}{\to} S_{i-1} \to 0$$

Let $b_1, ..., b_m$ be an almost central generating sequence for $\mathcal{J}_{i-1}$. For each $n = 1, ..., m$ choose $\tilde{b}_n \in \varphi^{-1}(b_i)$.

We first show that the sequence $p\tilde{b}_1, ..., p\tilde{b}_m$ is an almost central sequence in $S_i$. Let $x \in \mathcal{J}_i$, then $x\tilde{b}_1 - \tilde{b}_1 x \in p^{i-1} \mathcal{J}$ and so $xp\tilde{b}_1 - p\tilde{b}_1 x = 0$. Thus $p\tilde{b}_1$ is central in $S_i$. We continue by induction. Thus:

$$x\tilde{b}_n - \tilde{b}_n x \in p^{i-1} \mathcal{J} + S_i \{\tilde{b}_1, ..., \tilde{b}_{n-1}\}$$

and so

$$xp\tilde{b}_n - p\tilde{b}_n x \in (p\tilde{b}_1, ..., p\tilde{b}_{n-1}).$$

Next, we show that $(p\tilde{b}_1, ..., p\tilde{b}_m) = p\mathcal{J}_i$. Suppose that $x \in p\mathcal{J}_i$, then $x = py$ where $y = y' + z$ where $y' \in (\tilde{b}_1, ..., \tilde{b}_m)$ and $z \in p^{i-1} \mathcal{J}$. Hence $pz = 0$ and $x \in (p\tilde{b}_1, ..., p\tilde{b}_m)$. We conclude that $p\mathcal{J}_i$ is almost central.

It remains to show that the ideal $\mathcal{J}_i/p\mathcal{J}_i$ of $S_i/p\mathcal{J}_i$ is almost commutative. It is easy to see that it would suffice to show that the augmentation ideal of $S_i/pS_i$ is almost commutative. But $S_i/p = S_1$, and we have already shown that the augmentation ideal of $S_1$ is almost commutative.

This completes the proof of Proposition B.1.

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School of Mathematics and Statistics, Hicks Building, Sheffield S3 7RH. UK.
E-mail address: s.shamir@sheffield.ac.uk