Identities And Relations On The Hermite-based Tangent Polynomials

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Abstract

In this note, we introduce and investigate the Hermite-based Tangent numbers and polynomials, Hermite-based modified degenerate-Tangent polynomials, poly-Tangent polynomials. We give some identities and relations for these polynomials.

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1. Introduction and Notation

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler number and polynomials, Genocchi numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials, poly-Genocchi numbers and polynomials, poly-Tangent numbers and polynomials, Hermite polynomials, Hermite-based Bernoulli polynomials, Hermite-based Tangent polynomials, modified degenerate Bernoulli polynomials, modified degenerate Euler polynomials and modified degenerate Genocchi polynomials (see [1]-[20]). In this note we define the Hermite-based tangent polynomials, modified Hermite-based tangent polynomials and poly-tangent polynomials. We obtain some relations and identities for these polynomials. Throughout this paper, we always make use of the following notations: \( \mathbb{N} \) denotes the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). We recall that
the classical Stirling numbers of the first kind \( S_1(n, k) \) and second kind \( S_2(n, k) \) are defined by the relations \([15]\)

\[
(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k
\]

(1.1) respectively. Here \((x)_n = x(x-1) \cdots (x-n+1)\) denotes the falling factorial polynomials of order \(n\). The numbers \(S_2(n, m)\) also admit a representation in terms of a generating function \(\frac{(e^t - 1)^m}{m!} = \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!}\).

The Bernoulli polynomials \(B_n^{(r)}(x)\) of order \(\alpha\), the Euler polynomials \(E_n^{(r)}(x; \lambda)\) of order \(\alpha\) and the Genocchi polynomials \(G_n^{(r)}(x; \lambda)\) of order \(\alpha\) are defined as respectively:

\[
\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi,
\]

(1.3)

\[
\left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}, \quad |t| < \pi
\]

(1.4)

and

\[
\left(\frac{2t}{e^{2t} + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}, \quad |t| < \pi.
\]

(1.5)

When \(x = 0\), \(B_n^{(r)}(0) = B_n^{(r)}\), \(E_n^{(r)}(0) = E_n^{(r)}\) and \(G_n^{(r)}(0) = G_n^{(r)}\) are called Bernoulli numbers of order \(r\), Euler numbers of order \(r\) and Genocchi numbers of order \(r\), respectively.

The familiar tangent polynomials \(T_n^{(r)}(x)\) of order \(r\) are defined by the generating functions \([12, 15, 17]\)

\[
\left(\frac{2}{e^{2t} + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} T_n^{(r)}(x) \frac{t^n}{n!}, \quad |2t| < \pi.
\]

(1.6)

When \(x = 0\), \(T_n^{(r)}(0) = T_n^{(r)}\) are called the tangent numbers.

2-variable Hermite-Kampé de Fériet polynomials are defined in \([5, 11]\) as

\[
\sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} = e^{xt+yt^2}.
\]

(1.7)

Khan et al. in \([5]\) defined and studied on Hermite-based Bernoulli polynomials and Hermite-based Euler polynomials as

\[
\sum_{n=0}^{\infty} (H_B_n(x, y)) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt+yt^2}, \quad |t| < 2\pi
\]

(1.8)

and

\[
\sum_{n=0}^{\infty} (H_E_n(x, y)) \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{xt+yt^2}, \quad |t| < \pi,
\]

(1.9)

respectively.
Carlitz in [1] defined degenerate Bernoulli polynomials which are given by the generating functions to be
\[
\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_n(x | \lambda) \frac{t^n}{n!}.
\] (1.10)

When \( x = 0 \), \( B_n(\lambda) = B_n(0 | \lambda) \) are called the degenerate Bernoulli numbers.

From (1.10), we can easily derive the following equation
\[
B_n(x | \lambda) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}(\lambda) (x | \lambda)^l, \quad n \geq 0
\]
where \( (x | \lambda)^n = x(x - \lambda) \cdots (x - \lambda(n - 1)) \), \( (x | \lambda)^n = 1 \).

Dolgy et. al. [2] defined the modified degenerate Bernoulli polynomials, which are different from Carlitz’s degenerate Bernoulli polynomials as
\[
\frac{t}{(1 + \lambda t)^{1/\lambda} - 1} (1 + t\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}.
\] (1.11)

When \( x = 0 \), \( B_{n,\lambda} = B_{n,\lambda}(0) \) are called the modified degenerate Bernoulli numbers. From (1.11), we note that
\[
\lim_{\lambda \to 0} \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \lim_{\lambda \to 0} \frac{t}{(1 + \lambda t)^{1/\lambda} - 1} (1 + \lambda t)^{xt/\lambda} = t e^t - 1 = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\] (1.12)

Thus, by (1.12)
\[
\lim_{\lambda \to 0} B_{n,\lambda}(x) = B_n(x).
\]

H.-In Known et. al. [8] defined the modified degenerate Euler polynomials as
\[
\frac{2}{(1 + \lambda t)^{1/\lambda} + 1} (1 + \lambda t)^{xt/\lambda} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!}.
\] (1.13)

and T. Kim et. al. in [6] defined the modified degenerate Genocchi polynomials as
\[
\frac{2t}{(1 + \lambda t)^{1/\lambda} + 1} (1 + \lambda t)^{tx/\lambda} = \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!}.
\] (1.14)

From (1.13) and (1.14), we get
\[
\lim_{\lambda \to 0} E_{n,\lambda}(x) = E_n(x), \quad \lim_{\lambda \to 0} G_{n,\lambda}(x) = G_n(x).
\]

For \( k \in \mathbb{Z}, k > 1 \), then \( k \)-th polylogarithm is defined by Kaneko [4] as
\[
L_{ik}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.
\] (1.15)

Thus function is convergent for \(|z| < 1\), when \( k = 1 \)
\[
L_{i1}(z) = -\log(1 - z).
\] (1.16)
Kim et. al. in [7] defined the poly-Bernoulli polynomials and the poly-Genocchi polynomials as

\[ \sum_{n=0}^{\infty} \mathfrak{B}_n^{(k)}(x) \frac{t^n}{n!} = \frac{L_{ik}(1-e^{-t})}{1-e^{-t}} e^{xt} \]  \hspace{1cm} (1.17)

and

\[ \sum_{n=0}^{\infty} \mathfrak{G}_n^{(k)}(x) \frac{t^n}{n!} = \frac{2L_{ik}(1-e^{-t})}{e^t + 1} e^{xt} \]  \hspace{1cm} (1.18)

respectively.

For \( k = 1 \), by use (1.16) in (1.17) and (1.18), we get

\[ \mathfrak{B}_n^{(1)}(x) = (-1)^{n+1} B_n(x), \quad \mathfrak{G}_n^{(1)}(x) = G_n(x). \]

Hamahata [3] defined poly-Euler polynomials by

\[ \sum_{n=0}^{\infty} \mathfrak{E}_n^{(k)}(x) \frac{t^n}{n!} = \frac{2L_{ik}(1-e^{-t})}{e^t + 1} e^{xt}. \]

For \( k = 1 \), we get \( \mathfrak{E}_n^{(1)}(x) = E_n(x) \).

From (1.6), we obtain the following equalities easily

\[ T_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} T_k^{(r)} x^{n-k}, \]

\[ T_n^{(r)}(x + y) = \sum_{l=0}^{k} \binom{k}{l} T_k^{(r)}(x) y^{k-l}, \]

\[ T_n^{(r_1+r_2)}(x + y) = \sum_{k=0}^{n} \binom{n}{k} T_k^{(r_2)}(x) T_{n-k}^{(r_2)}(y) \]

and

\[ T_n^{(r)}(2(x + 1)) = 2T_n^{(r-1)}(2x). \]

2. Hermite Based Tangent Polynomials

Khan et. al. in [5] and Ozarslan [11] introduced and investigated the Hermite-based Bernoulli polynomials and Hermite-based Euler polynomials. They proved some identities and relations for these polynomials.

By this motivation, we define Hermite-based Tangent polynomials of order \( r \) as

\[ \sum_{n=0}^{\infty} \left( hT_n^{(r)}(x,y) \right) \frac{t^n}{n!} = \left( \frac{2}{e^{2t} + 1} \right)^r e^{xt+yt^2}. \]  \hspace{1cm} (2.1)
Proof. From (2.1), we replace

Let

Theorem 2. Let \( r \in \mathbb{Z}_+ \). We have

\[
HT_n^{(r)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} T_k^{(r)}(0, 0) H_{n-k}(x, y)
\]

and

\[
HT_n^{(r_1+r_2)}(x+u, y+v) = \sum_{k=0}^{n} \binom{n}{k} \left( HT_k^{(r)}(x, y) \right) H_{n-k}(u, v)
\]

\[
HT_n^{(r_1)}(x+u, y+v) = \sum_{k=0}^{n} \binom{n}{k} \left( HT_k^{(r_1)}(x, y) \right) \left( HT_n^{(r_2)}(u, v) \right).
\]

Theorem 3. There is the following implicit relation for the Hermite-based Tangent polynomials as

\[
HT_{n+m}^{(r)}(u, v) = \sum_{p=0}^{n} \binom{n}{p} \sum_{q=0}^{m} \binom{m}{q} (v-y)^{p+q} \left( HT_{n+m-p-q}^{(r)}(x, y) \right).
\]

Proof. From (2.1), we replace \( t \) by \( t+u \) and rewrite the generating function as

\[
\frac{2e^{y(t+u)^2}}{e^{2t}+1} = e^{-x(t+u)} \sum_{n=0}^{\infty} HT_n^{(r)}(x, y) \frac{t^n u^m}{n! m}.
\]

Replacing \( x \) by \( v \) in the above equation to the above equation.

We get

\[
\sum_{n,m=0}^{\infty} \left( HT_{n+m}^{(r)}(v, y) \right) \frac{t^n u^m}{n! m!} = e^{(t+u)(v-x)} \sum_{n,m=0}^{\infty} \left( HT_{n+m}^{(r)}(x, y) \right) \frac{t^n u^m}{n! m!}
\]

which on using formula \[19\] Srivastava p. 52]

\[
\sum_{N=0}^{\infty} f(N) \frac{X^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{X^n Y^m}{n! m!}
\]

in the right hand side becomes

\[
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{(v-x)^{p+q} t^p u^q}{p! q!} \sum_{n,m=0}^{\infty} \left( HT_{n+m}^{(r)}(x, y) \right) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} \left( HT_{n+m}^{(r)}(v, y) \right) \frac{t^n u^m}{n! m!}.
\]

By using Cauchy product and comparing the coefficients of both sides, we have (2.2). \( \square \)

Theorem 4. There is the following relation between the Hermite-based Tangent polynomials and the Hermite-based Bernoulli polynomials as

\[
\left( HT_n^{(r)}\left( \frac{x+u}{4}, \frac{y+v}{16} \right) \right) = 2^{r-n-k} \sum_{k=0}^{n} \binom{n}{k} \left( HT_k^{(r)}(x, y) \right) \left( HB_{n-k}^{(r)}\left( \frac{u}{2}, \frac{v}{4} \right) \right).
\]
Proof. From (2.1),
\[\sum_{n=0}^{\infty} \left( H_{B_n}^{(r)} \left( \frac{x + u}{4}, \frac{y + v}{16} \right) \right) \frac{(4t)^n}{n!} = \left( \frac{2 \times 4t}{e^{4t} - 1} \right)^{(r)} e^{(x+u)t+(y+v)t^2} \]
\[= \left( \frac{2}{e^{2t} + 1} \right)^{(r)} e^{xt^2} 2^r \left( \frac{2t}{e^{2t} - 1} \right)^{(r)} e^{yt^2} \]
\[= \sum_{n=0}^{\infty} \left( H_{T_n}^{(r)} (x, y) \right) \frac{t^n}{n!} 2^r \sum_{q=0}^{\infty} \left( H_{B_q}^{(r)} \left( \frac{u}{2}, \frac{v}{4} \right) \right) \frac{(2t)^n}{n!}. \]

By using Cauchy product and comparing the coefficients of both sides, we get (2.3). □

3. Modified Degenerate Hermite-Based Tangent Polynomials

Dolgy et. al. [2] introduced and investigated the modified degenerate Bernoulli polynomials. Known et. al. [8] defined and investigated the modified degenerate Euler polynomials. They proved some properties for these polynomials.

By these motivations, we define 2-variable modified degenerate Hermite polynomials and the modified degenerate Hermite-based Tangent polynomials of order \( r \)
\[\sum_{n=0}^{\infty} H_{n} (x, y : \lambda) \frac{t^n}{n!} = (1 + \lambda)^{\frac{xt^2 + yt^2}{x}} \] (3.1)
and
\[\sum_{n=0}^{\infty} \left( H_{T_n}^{(r)} (x, y : \lambda) \right) \frac{t^n}{n!} = \frac{2}{(1 + \lambda)^{\frac{x}{2}} + 1} \left( 1 + \lambda \right)^{\frac{xt^2 + yt^2}{x}} \] (3.2)
respectively.

From (3.1) and (3.2), we get
\[\lim_{\lambda \to 0} H_{n} (x, y : \lambda) = H_{n} (x, y), \quad \lim_{\lambda \to 0} \left( H_{T_n}^{(r)} (x, y : \lambda) \right) = \left( H_{T_n}^{(r)} (x, y) \right). \]

Similiarly, we define the modified Hermite-based Bernoulli polynomials and the modified Hermite-based Euler polynomials as
\[\sum_{n=0}^{\infty} \left( H_{B_n}^{(r)} (x, y : \lambda) \right) \frac{t^n}{n!} = \frac{t}{(1 + \lambda)^{\frac{x}{2}} - 1} \left( 1 + \lambda \right)^{\frac{xt^2 + yt^2}{x}} \] (3.3)
and
\[\sum_{n=0}^{\infty} \left( H_{E_n}^{(r)} (x, y : \lambda) \right) \frac{t^n}{n!} = \frac{2}{(1 + \lambda)^{\frac{x}{2}} + 1} \left( 1 + \lambda \right)^{\frac{xt^2 + yt^2}{x}} \] (3.4)
respectively.

From (3.2), we obtain the following relations easily
\[\left( H_{T_n}^{(r_1+r_2)} (x + u, y + v : \lambda) \right) = \sum_{k=0}^{n} \binom{n}{k} \left( H_{T_k}^{(r_1)} (x, y : \lambda) \right) \left( H_{T_{n-k}}^{(r_2)} (u, v : \lambda) \right). \]
Theorem 6.\n
\[
(HT_n^{(r)}(x, y : \lambda)) = \sum_{k=0}^{n} \binom{n}{k} \left( HT_k^{(r)}(0, 0 : \lambda) \right) \left( H_{n-k}(x, y : \lambda) \right),
\]

\[
(HT_n^{(r)}(x + 2, y : \lambda)) + (HT_n^{(r)}(x, y : \lambda)) = 2 \left( HT_n^{(r-1)}(x, y : \lambda) \right)
\]

for \( r = 1 \),

\[
(HT_n(x + 2, y : \lambda)) + (HT_n(x, y : \lambda)) = 2 \left( H_n(x, y : \lambda) \right)
\]

and

\[
(HT_n^{(r)}(x, y : \lambda)) = \sum_{k=0}^{n} \binom{n}{k} \left( HT_k^{(r)} \left( \frac{1}{2}, 0 : \lambda \right) \right) \left( H_{n-k} \left( x - \frac{1}{2}, y : \lambda \right) \right).
\]

Theorem 5. There is the following relation between the modified degenerate Bernoulli polynomials, the modified degenerate Euler polynomials and the modified degenerate Tangent polynomials as

\[
(HB_n(x, y : \lambda)) 2^{2n+1} = \sum_{q=0}^{n} \binom{n}{q} \left( HT_{n-q}(x, y : \lambda) \right) \sum_{k=0}^{q} \binom{q}{k} \left( HB_{q-k}(x, y : \lambda) \right) \left( HE_n(2x, 14y : \lambda) \right).
\]

Proof. From (3.3), (3.4) and (3.2), we write as

\[
\sum_{n=0}^{\infty} (HB_n(x, y : \lambda)) \frac{(4t)^n}{n!} = \left( \frac{4t}{(1 + \lambda)^{\frac{x}{2}} - 1} \right)^x(1 + \lambda)^{\frac{2t}{x}}
\]

\[
= \frac{1}{2} e^{\frac{2t}{x}} \left( (1 + \lambda)^{\frac{x}{2}} - 1 \right) + \frac{\lambda t}{2} e^{\frac{2t}{x}} \left( (1 + \lambda)^{\frac{x}{2}} - 1 \right) + \frac{\lambda t}{2} e^{\frac{2t}{x}} \left( (1 + \lambda)^{\frac{x}{2}} - 1 \right)
\]

By using Cauchy product and comparing the coefficient of \( \frac{t^n}{n!} \), we have (3.5).

\]

Theorem 6. \( n \in \mathbb{Z}_+ \), we have

\[
(HT_n(x + 2, y : \lambda)) + (HT_n(x, y : \lambda)) = \frac{1}{n+1} \left\{ (HB_{n+1}(x + 1, y : \lambda)) - (HB_{n+1}(x, y : \lambda)) \right\}.
\]

Proof. By using definition (3.2)

\[
\frac{2t (1 + \lambda)^{\frac{x+y}{2}}}{(1 + \lambda)^{\frac{x}{2}} + 1} \left[ (1 + \lambda)^{\frac{x}{2}} + 1 \right] = \frac{2t (1 + \lambda)^{\frac{x+y}{2}}}{(1 + \lambda)^{\frac{x}{2}} - 1} \left[ (1 + \lambda)^{\frac{x}{2}} - 1 \right]
\]

\[
\frac{2t (1 + \lambda)^{\frac{x+2+y}{2}}}{(1 + \lambda)^{\frac{x}{2}} + 1} + \frac{2t (1 + \lambda)^{\frac{x+y}{2}}}{(1 + \lambda)^{\frac{x}{2}} + 1} = \frac{2t (1 + \lambda)^{\frac{x+y}{2}}}{(1 + \lambda)^{\frac{x}{2}} - 1} - \frac{2t (1 + \lambda)^{\frac{x+y}{2}}}{(1 + \lambda)^{\frac{x}{2}} - 1}
\]
\[ t \sum_{n=0}^{\infty} \left\{ (H T_n (x + 2, y : \lambda)) + (H T_n (x, y : \lambda)) \right\} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ (H \mathfrak{B}_n (x + 1, y : \lambda)) - (H \mathfrak{B}_n (x, y : \lambda)) \right\} \frac{t^n}{n!}. \]

From the above equality, we have (3.6). \qed

4. Poly-Tangent Polynomials

In this section, we define the poly-tangent numbers and polynomials and provide some of their relevant properties.

**Definition 1.** We define the Hermite-based poly-tangent polynomials by

\[
\frac{2L_i k (1 - e^{-t})}{t (e^{2t} + 1)} e^{xt+yt^2} = \sum_{n=0}^{\infty} (H T_n^{(k)} (x, y)) \frac{t^n}{n!},
\]

(4.1)

when \( x = 0 \), \( (H T_n^{(k)}) := \left( H T_n^{(k)} (0, 0) \right) \) are called the Hermite-based poly-tangent numbers.

For \( k = 1 \) and \( L_i k (z) = -\log (1 - z) \), from (4.1)

\[
\frac{2L_i 1 (1 - e^{-t})}{t (e^{2t} + 1)} e^{xt+yt^2} = \frac{2e^{xt+yt^2}}{e^{2t} + 1} = \sum_{n=0}^{\infty} (H T_n (x, y)) \frac{t^n}{n!}.
\]

(4.2)

By (4.2), we get

\[
(H T_n^{(1)} (x, y)) = (H T_n (x, y)).
\]

**Theorem 7.** \( n, k \in \mathbb{Z}_+ \), we have

\[
(H T_n^{(k)} (x, y)) = \frac{1}{n+1} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (H T_{n+1} (x - j, y)).
\]

(4.3)

**Proof.**

\[
\sum_{n=0}^{\infty} (H T_n^{(k)} (x, y)) \frac{t^n}{n!} = 2 \sum_{m=0}^{\infty} \frac{(1 - e^{-t})^{m+1}}{(m+1)^k} \frac{e^{xt+yt^2}}{t (e^{2t} + 1)}
\]

\[
= 2 \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \frac{e^{-tj+xt+yt^2}}{t (e^{2t} + 1)}
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \frac{1}{t e^{2t} + 1} e^{t(x-j)+yt^2}
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \sum_{n=0}^{\infty} (H T_n (x - j, y)) \frac{t^n}{n!}
\]

\[
= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \sum_{n=0}^{\infty} \frac{(H T_{n+1} (x - j, y)) t^n}{n!}.
\]

Comparing the coefficients both sides, we have (4.3). \qed
**Theorem 8.** There is the following relation between poly-tangent polynomials and the Stirling numbers of the second kind and the Hermite-based Bernoulli polynomials as

\[
(H_T^{(k)}(x, y)) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{(l+r)} S_2(l + r, r) \sum_{i=0}^{n-l} \left(H_{i}^{(r)}(x, y)\right) \left(H_T^{(r)}_{n-l-i}\right). \tag{4.4}
\]

**Proof.** From (4.1), we write as

\[
\sum_{n=0}^{\infty} \left(H_T^{(k)}(x, y)\right) \frac{t^n}{n!} = \frac{2 L_{i_k} (1 - e^{-t})}{t (e^{2t} + 1)} e^{xt+yt^2}
\]

\[
= \frac{(e^t - 1)^r r!}{r!} \left(\frac{t}{e^t - 1}\right)^r e^{xt+yt^2} \frac{2 L_{i_k} (1 - e^{-t})}{t (e^{2t} + 1)}
\]

\[
= \frac{(e^t - 1)^r}{r!} \left(\sum_{n=0}^{\infty} \left(H_{i}^{(r)}(x, y)\right) \frac{t^n}{n!}\right) \left(\sum_{q=0}^{\infty} \left(H_T^{(r)}\right) \frac{t^q}{q!}\right) \frac{r!}{t^r}
\]

\[
= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{\binom{n}{l}}{(l+r)} S_2(l + r, r) \sum_{i=0}^{n-l} \left(H_{i}^{(r)}(x, y)\right) \left(H_T^{(r)}_{n-l-i}\right)\right) \frac{t^n}{n!}.
\]

Comparing the coefficients of \(\frac{t^n}{n!}\), we obtain \((4.4)\). \qed

**Theorem 9.** There is the following relation between the poly-tangent polynomials, the poly-Genocchi numbers and the Hermite-based tangent polynomials

\[
(H_T^{(k)}(x, y)) = \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} G_{n-p}^{(k)} \{(H_T^n (x + 1, y)) + (H_T^n (x, y))\}. \tag{4.5}
\]

**Proof.** From (4.1) and (1.18)

\[
\sum_{n=0}^{\infty} \left(H_T^{(k)}(x, y)\right) \frac{t^n}{n!} = \frac{2 L_{i_k} (1 - e^{-t})}{t (e^{2t} + 1)} e^{xt+yt^2}
\]

\[
= \frac{1}{2} \left(\frac{2 L_{i_k} (1 - e^{-t})}{e^t + 1}\right) \frac{2 (e^t + 1) e^{xt+yt^2}}{t (e^{2t} + 1)}
\]

\[
= \frac{1}{2} 2 L_{i_k} (1 - e^{-t}) \left(\frac{2 e^{(x+1)t+yt^2}}{t (e^{2t} + 1)} + \frac{2 e^{xt+yt^2}}{t (e^{2t} + 1)}\right)
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} G_{n-p}^{(k)} \frac{t^n}{n!} \left\{\sum_{p=0}^{\infty} \left(H_T^n (x + 1, y)) + (H_T^n (x, y))\right) \frac{t^p}{p!}\right\}.
\]

Comparing the coefficients of both sides, we have \((4.5)\). \qed

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