AN INJECTIVITY THEOREM
WITH MULTIPLIER IDEAL SHEAVES OF
SINGULAR METRICS WITH TRANSCENDENTAL SINGULARITIES

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Abstract. The purpose of this paper is to establish an injectivity theorem with multiplier ideal sheaves of singular metrics. This result is a generalization of various injectivity and vanishing theorems. To treat transcendental singularities, after regularizing a given singular metric, we study the asymptotic behavior of the harmonic forms with respect to a family of the regularized metrics. Moreover we give a method to obtain $L^2$-estimates of solutions of the $\overline{\partial}$-equation, by using the Čech complex. As an application, we obtain a Nadel type vanishing theorem.

1. Introduction

In his paper [Kol86], Kollár established the following celebrated result in the setting of algebraic geometry, which is the so-called injectivity theorem. In this paper, we consider the Kollár type injectivity theorem in the setting of complex differential geometry. Our purpose is to establish an injectivity theorem with multiplier ideal sheaves associated to singular metrics with transcendental singularities.

Theorem 1.1 ([Kol86], [EV92]). Let $F$ be a semi-ample line bundle on a smooth projective variety $X$. Then for a (non-zero) section $s$ of a positive multiple $F^m$ of the line bundle $F$, the multiplication map induced by the tensor product with $s$

$$
\Phi_s : H^q(X, K_X \otimes F) \otimes_{F} H^q(X, K_X \otimes F^{m+1})
$$

is injective for any $q$. Here $K_X$ denotes the canonical bundle of $X$.

Injectivity theorems play an important role in higher dimensional algebraic geometry and the theory of several complex variables. Therefore, according to the context and objectives, there are many contributions to such kind of injectivity theorems (for example, see [Amb12], [Eno90], [EP08], [EV92], [Fuj09], [Fuj13], [Ohs04], [Tak97], and so on).

In his paper [Eno90], Enoki gave the following injectivity theorem. Kollár’s proof of the injectivity theorem is based on the Hodge theory. On the other hand, Enoki’s proof is based on the theory of harmonic integrals, which enables us to approach the injectivity theorem from the viewpoint of complex differential geometry.

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**Theorem 1.2** ([Eno90]). Let $F$ be a semi-positive line bundle on a compact Kähler manifold $X$. Then the same conclusion as Theorem 1.1 holds.

A line bundle is said to be *semi-positive* if the line bundle admits a “smooth” metric with semi-positive curvature. A semi-ample line bundle is always semi-positive, and thus Theorem 1.2 leads to Theorem 1.1.

The above theorems can be considered as a generalization to semi-positive line bundles of the Kodaira vanishing theorem. (Indeed they lead to the Kodaira vanishing theorem.) The Kodaira vanishing theorem can be generalized to the Nadel (Kawamata-Viehweg) vanishing theorem by using “singular” metrics and their multiplier ideal sheaves. Therefore, in the same direction as this generalization, it is natural and of interest to generalize them to an injectivity theorem for line bundles equipped with singular metrics.

The following theorem is the main result of this paper, which can be seen as a generalization of the injectivity theorem and the Nadel vanishing theorem.

**Theorem 1.3** (The main theorem). Let $F$ be a line bundle on a compact Kähler manifold $X$ and $h$ be a singular metric with semi-positive curvature on $F$. Then for a (non-zero) section $s$ of a positive multiple $F^m$ satisfying $\sup_X |s|_h^m < \infty$, the multiplication map

$$\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \rightarrow H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$$

is (well-defined and) injective for any $q$. Here $\mathcal{I}(h)$ denotes the multiplier ideal sheaf associated to the singular metric $h$.

**Remark 1.4.** The multiplication map is well-defined thanks to the assumption of $\sup_X |s|_h^m < \infty$. When $h$ is a metric with minimal singularities on $F$, this assumption is always satisfied for any section $s$ of $F^m$ (see [Dem] for metrics with minimal singularities).

It is important to emphasize that a singular metric $h$ in Theorem 1.3 may have non-algebraic (transcendental) singularities. To handle singular metrics with non-algebraic singularities, we take a more transcendental approach for cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$, which gives a strong generalization of techniques of [Eno90], [Fuj12-A], [Mat13-A], [Ohs04], [Tak97]. For example, metrics with minimal singularities are important object, but they do not always have algebraic singularities. By considering them, as applications we can obtain an injectivity theorem for nef line bundles (Corollary 1.5) and Nadel type vanishing theorems (Theorem 3.21 and Corollary 1.6) as follows:

It is reasonable to expect the same conclusion as Theorem 1.1 to hold for nef line bundles, but there exist counterexamples to the injectivity theorem for nef line bundles. However, from [Kaw85, Proposition 2.1], we can show that a metric $h_{\text{min}}$ with minimal
singularities on $F$ satisfies $\mathcal{T}(h^m_{\min}) = \mathcal{O}_X$ for any $m > 0$ if $F$ is nef and abundant (that is, the numerical dimension coincides with the Kodaira dimension). Therefore Theorem 1.3 leads to the following corollary. (The same conclusion was obtained in \cite{Fuj12-A}. On projective varieties, a similar conclusion was proved in \cite{EP08} and \cite{EV92} by different methods.) It is worth pointing out that Theorem 1.2 is not sufficient to obtain Corollary 1.5. This is because, the above metric $h_{\min}$ is not smooth and does not have algebraic singularities in general even if $F$ is nef and abundant (for example, see \cite{Fuj12-B, Example 5.2}).

**Corollary 1.5.** Let $F$ be a nef and abundant line bundle on a compact Kähler manifold $X$. Then the same conclusion as Theorem 1.1 holds.

As another application, we can obtain a Nadel type vanishing (Theorem 3.21), which leads to the following corollary.

**Corollary 1.6** (cf. \cite{Cao12}, \cite{Mat13-B}). Let $F$ be a line bundle on a smooth projective variety $X$ of dimension $n$ and $h_{\min}$ be a singular metric with minimal singularities on $F$. Then

$$H^q(X, K_X \otimes F \otimes \mathcal{T}(h_{\min})) = 0 \quad \text{for any } q > n - \kappa(F).$$

Here $\kappa(F)$ denotes the Kodaira dimension of $F$.

This result is non-trivial even when the line bundle $F$ is big (that is, $\kappa(F) = n$). In his paper \cite{Cao12}, Cao proved the celebrated vanishing theorem for cohomology groups with coefficients in $K_X \otimes F \otimes \mathcal{T}(h)$. It is relatively easier to handle $\mathcal{T}(h)$ than $\mathcal{T}(h)$ (see \cite{DEL00} for the precise definition). If $h_{\min}$ has algebraic singularities, we can easily see that $\mathcal{T}_{\lambda}(h_{\min})$ agrees with $\mathcal{T}(h_{\min})$, but unfortunately $h_{\min}$ does not always have algebraic singularities. Thanks to Theorem 1.3, we can obtain Corollary 1.6 without the assumption of algebraic singularities.

**Remark 1.7.** Eight months after we finish writing our preprint, Guan and Zhou announced that they solve the strong openness conjecture in \cite{GZ13}. Although their result and Cao’s theorem leads to Corollary 1.6, we believe that it is worth displaying our techniques. This is because, our techniques are quite different from that of them and give a new viewpoint to prove the vanishing theorem via the asymptotic vanishing theorem (\cite{Mat13-A, Theorem 4.1}).

Let us briefly explain the proof of Theorem 1.3. First we recall Enoki’s proof of Theorem 1.2 which gives a proof of the special case when $h$ is smooth. In this case, the cohomology group $H^q(X, K_X \otimes F)$ can be represented by the space of the harmonic forms

$$\mathcal{H}^{n,q}(F)_h := \{ u \mid u \text{ is a smooth } F\text{-valued } (n,q)\text{-form on } X \text{ such that } \overline{\partial}u = D''_h^*u = 0 \}$$

with respect to $h$, where $D''_h^*$ is the adjoint operator of the $\overline{\partial}$-operator. For an arbitrary harmonic form $u$ in $\mathcal{H}^{n,q}(F)_h$, we can conclude that $D''_{h^{\kappa n+1}}s u = 0$, thanks to semi-positivity of the curvature of $h$. This step strongly depends on semi-positivity of the curvature of $h$. 

Then the multiplication map $\Phi_s$ induces the map from $\mathcal{H}^{n,q}(F)_h$ to $\mathcal{H}^{n,q}(F^{m+1})_{h^{m+1}}$, and thus the injectivity is obvious.

In our situation, we must consider singular metrics with non-algebraic (transcendental) singularities. It is quite difficult to directly handle transcendental singularities. For this reason, we first approximate the metric $h$ by singular metrics $\{h_{\varepsilon}\}_{\varepsilon > 0}$ that are smooth on a Zariski open set. Then we represent a given cohomology class by the associated harmonic form $u_{\varepsilon}$ with respect to $h_{\varepsilon}$ on the Zariski open set. We want to show that $su_{\varepsilon}$ is also harmonic by using the same method as Enoki’s proof. However, the same argument as Enoki’s proof fails since the curvature of $h_{\varepsilon}$ is not semi-positive. For this reason, we investigate the asymptotic behavior of the harmonic forms $u_{\varepsilon}$ with respect to a family of the regularized metrics $\{h_{\varepsilon}\}_{\varepsilon > 0}$. This asymptotic analysis contains a new ingredient, which asserts that the $L^2$-norm $\|D^{\varepsilon}_{h_{m+1}}su_{\varepsilon}\|$ converges to zero as letting $\varepsilon$ go to zero. Moreover we construct solutions $\gamma_{\varepsilon}$ of the $\overline{\partial}$-equation $\overline{\partial}\gamma_{\varepsilon} = su_{\varepsilon}$ such that the $L^2$-norm $\|\gamma_{\varepsilon}\|$ is uniformly bounded, by applying the Čech complex with the topology induced by the local $L^2$-norms. The above arguments yield

$$\|su_{\varepsilon}\|^2 = \langle \langle su_{\varepsilon}, \overline{\partial}\gamma_{\varepsilon} \rangle \rangle \leq \|D^{\varepsilon}_{h_{m+1}}su_{\varepsilon}\||\gamma_{\varepsilon}\| \to 0 \quad \text{as } \varepsilon \to 0.$$ 

From these observations, we conclude that $u_{\varepsilon}$ converges to zero in a suitable sense, which completes the proof.

This paper is organized as follows: In Section 2, we collect materials for the proof of the main result. In Section 3, we give a proof of the main result. In Section 4, we study the space of the cochains with coefficients in $K_X \otimes F \otimes \mathcal{I}(h)$.

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2. Preliminaries

In this section, we fix the notation and recall the fundamental results that are often used in this paper. For more details, refer to [Dem], [Dem-book], [GR], [Laz].

2.1. Singular metrics and multiplier ideal sheaves. Throughout this section, let $X$ be a compact Kähler manifold and $F$ be a line bundle on $X$. We first recall the definition of singular metrics and their curvature currents. Fix a smooth (hermitian) metric $g$ on $F$.

**Definition 2.1.** (Singular metrics and their curvature currents). (1) For an $L^1$-function $\varphi$ on $X$, the metric $h$ defined by

$$h := ge^{-\varphi}$$

is called a **singular metric** on $F$. Further $\varphi$ is called the **weight** of $h$ with respect to the fixed smooth metric $g$.

(2) The **curvature current** $\sqrt{-1}\Theta_h(F)$ associated to $h$ is defined by

$$\sqrt{-1}\Theta_h(F) = \sqrt{-1}\Theta_g(F) + dd^c\varphi,$$
where $\sqrt{-1}\Theta_g(F)$ is the Chern curvature of $g$.

In this paper, we simply abbreviate the singular metric (resp. the curvature current) to the metric (resp. the curvature). The Levi form $dd^c\varphi$ is taken in the sense of distributions, and thus the curvature is a $(1,1)$-current but not always a smooth $(1,1)$-form. The curvature $\sqrt{-1}\Theta_h(F)$ of $h$ is said to be positive if $\sqrt{-1}\Theta_h(F) \geq 0$ in the sense of currents. Remark that positivity of currents does not necessarily mean strictly positive.

We consider only metrics $h$ such that $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form $\gamma$ on $X$. Under this condition, the weight function $\varphi$ becomes a quasi-psh (quasi-plurisubharmonic) function. In particular $\varphi$ is upper semi-continuous and hence is bounded above. Then we can define multiplier ideal sheaves, which are coherent ideal sheaves.

**Definition 2.2.** (Multiplier ideal sheaves) Let $h$ be a metric on $F$ such that $\sqrt{-1}\Theta_h(F) \geq \gamma$ for some smooth $(1,1)$-form $\gamma$ on $X$. Then the ideal sheaf $\mathcal{I}(h)$ defined to be

$$\mathcal{I}(h)(U) := \mathcal{I}(\varphi)(U) := \{ f \in \mathcal{O}_X(U) \mid |f|e^{-\varphi} \in L^2_{\text{loc}}(U) \}$$

for every open set $U \subset X$, is called the multiplier ideal sheaf associated to $h$.

2.2. Equisingular approximations. In the proof, we apply the equisingular approximation to a given metric. In this subsection, we reformulate [DPS01, Theorem 2.3.] with our notation and give an additional property.

**Theorem 2.3.** (DPS01, Theorem 2.3.) Let $X$ be a compact Kähler manifold and $F$ be a line bundle equipped with a metric $h$ with (semi-)positive curvature. Then there exist metrics $\{h_\varepsilon\}_{1 \gg \varepsilon > 0}$ on $F$ with the following properties:

(a) $h_\varepsilon$ is smooth on $X \setminus Z_\varepsilon$, where $Z_\varepsilon$ is a subvariety on $X$.

(b) $h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h$ holds for any $0 < \varepsilon_1 < \varepsilon_2$.

(c) $\mathcal{I}(h) = \mathcal{I}(h_\varepsilon)$.

(d) $\sqrt{-1}\Theta_{h_\varepsilon}(F) \geq -\varepsilon \omega$.

Moreover, if the set $\{x \in X \mid \nu(\varphi, x) > 0\}$ is contained in a subvariety $Z$, then we can add the property that $Z_\varepsilon$ is contained in $Z$ for any $\varepsilon > 0$. Here $\nu(\varphi, x)$ denotes the Lelong number at $x$ of the weight $\varphi$ of $h$.

**Proof.** Fix a smooth metric $g$ on $F$. Then there exists an $L^1$-function $\varphi$ on $X$ with $h = ge^{-\varphi}$. By applying [DPS01, Theorem 2.3.] to $\varphi$, we obtain quasi-psh functions $\varphi_\nu$ with equisingularities. For a given $\varepsilon > 0$, by taking large $\nu = \nu(\varepsilon)$, we define $h_\varepsilon$ by $h_\varepsilon := ge^{-\varphi_\nu(\varepsilon)}$. Then the metric $h_\varepsilon$ satisfies properties (a), (b), (c), (d).

The latter conclusion follows from the proof in [DPS01]. We will see this fact shortly, by using the notation in [DPS01]. In their proof, they locally approximate $\varphi$ by $\varphi_{\varepsilon, \nu, j}$ with logarithmic pole. By inequality (2.5) in [DPS01], the Lelong number of $\varphi_{\varepsilon, \nu, j}$ is less than or equal to that of $\varphi$. Hence $\varphi_{\varepsilon, \nu, j}$ is smooth on $X \setminus Z$ since $\varphi_{\varepsilon, \nu, j}$ has a logarithmic pole. Since $\varphi_\nu$ is obtained from Richberg’s regularization of the supremum of these functions (see around (2.5) and (2.7)), we obtain the latter conclusion.
2.3. The theory of harmonic integrals. In this subsection, we collect fundamental results on the theory of harmonic integrals.

Throughout this subsection, let $Y$ be a (not necessarily compact) complex manifold with a hermitian form $\omega$ and $E$ be a line bundle on $Y$ with a smooth metric $h$. In Section 3, we need the theory of harmonic integrals on non-compact manifolds.

For $E$-valued $(p, q)$-forms $u$ and $v$, the point-wise inner product $\langle u, v \rangle_{h, \omega}$ can be defined, and the (global) inner product $\langle \langle u, v \rangle \rangle_{h, \omega}$ can also be defined by

$$\langle \langle u, v \rangle \rangle_{h, \omega} := \int_Y \langle u, v \rangle_{h, \omega} \omega^n.$$  

Then the $L^2$-space of the $E$-valued $(p, q)$-forms with respect to $h$ and $\omega$ is defined as follows:

$$L^{p,q}_{(2)}(Y, E)_{h,\omega} := \{ u \mid u \text{ is an } E\text{-valued } (p, q)\text{-form with } \|u\|_{h,\omega} < \infty. \}.$$  

The connection $D = D_{(E,h)}$ on $E$ is canonically determined by the holomorphic structure and the hermitian metric $h$ of $E$, which is called the Chern connection. The Chern connection $D$ can be written as $D = D'_h + D''_h$ with the $(1,0)$-connection $D'_h$ and the $(0,1)$-connection $D''_h$. By the definition, $D''_h$ agrees with the $\overline{\partial}$-operator. The connections $D'_h$ and $D''_h$ can be seen as closed and densely defined operators on $L^{p,q}_{(2)}(Y, E)_{h,\omega}$.

If $\omega$ is a complete form on $Y$, then the formal adjoints $D'_h$ and $D''_h$ coincide with the Hilbert space adjoints in the sense of Von Neumann (see [Dem-book], (3.2) Theorem in Chapter VIII). The following proposition can be obtained from Nakano’s identity and the density lemma (cf. [Dem82, Lemma 4.3], [Dem-book, Chapter VIII]).

**Proposition 2.4.** Assume that $\omega$ is a complete Kähler form on $Y$, and $\sqrt{-1}\Theta_h(E) \geq -C\omega$ for some positive constant $C > 0$.

Then for every $u \in L^{p,q}_{(2)}(Y, E)_{h,\omega}$ with $u \in \text{Dom} D''_h \cap \text{Dom} \overline{\partial}$, the following equality holds:

$$\|D''_h u\|_{h,\omega}^2 + \|\overline{\partial} u\|_{h,\omega}^2 = \langle \langle \sqrt{-1}\Theta_h(E)\Lambda_\omega u, u \rangle \rangle_{h,\omega} + \|D'_h u\|_{h,\omega}^2.$$  

Here $\Lambda_\omega$ denotes the adjoint of the wedge product $\omega \wedge \cdot$.

2.4. Fréchet spaces. In this subsection, we recall well-known facts on Fréchet spaces.

**Theorem 2.5** (The open mapping theorem). Let $\pi : D \to E$ be a linear map between Fréchet spaces $D$ and $E$. If $\pi$ is continuous and surjective, then $\pi$ is an open map.

This theorem leads to the following proposition. For reader’s convenience, we give a proof.

**Proposition 2.6.** Let $\pi : D \to E$ be a continuous linear map between Fréchet spaces $D$ and $E$. If the cokernel of $\pi$ is finite dimensional, then the image $\text{Im} \pi$ of $\pi$ is closed in $E$.

**Proof.** We first consider the case when $\pi : D \to E$ is injective. After taking a finite dimensional subspace $E_1$ of $E$ such that the quotient map $p : E_1 \to E/\pi(D)$ is isomorphic, we consider a continuous map $\pi_1 : D \oplus E_1 \to E$ defined to be $\pi_1(d, e) := \pi(d) + e$ for every $(d, e) \in D \oplus E_1$. Since $\pi_1$ is surjective (and injective) and continuous, the inverse map
\[ \pi_1^{-1} : E \to D \oplus E_1 \] is also continuous by the open mapping theorem. By composing \( \pi_1^{-1} \) with the second projection \( D \oplus E_1 \to E_1 \), we obtain the continuous map \( \pi_2 : E \to E_1 \). It is easy to check that the kernel of \( \pi_2 \) agrees with the image of \( \pi \), which implies that the image of \( \pi \) is closed. When \( \pi : D \to E \) is not injective, by considering the linear map \( \pi : D/\text{Ker} \pi \to E \), we can obtain the conclusion.

□

3. Proof of results

3.1. Proof of Theorem [1,3]. In this subsection, we give a proof of the main result. The proof is based on a technical combination of the theory of harmonic integrals and the \( L^2 \)-method for the \( \bar{\partial} \)-equation.

**Theorem 3.1** (=Theorem [13]). Let \( F \) be a line bundle on a compact Kähler manifold \( X \) and \( h \) be a (singular) metric with (semi-)positive curvature on \( F \). Then for a (non-zero) section \( s \) of a positive multiple \( F^m \) satisfying \( \sup_X |s|_{h^m} < \infty \), the multiplication map

\[
\Phi_s : H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \otimes s \to H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))
\]

is (well-defined and) injective for any \( q \).

**Proof of Theorem [3,4]** The proof can be divided into four steps. In Step 1, we approximate a given metric \( h \) by metrics \( \{h_\varepsilon\}_{\varepsilon > 0} \) that are smooth on a Zariski open set. In this step, we fix the notation to apply the theory of harmonic integrals and explain the sketch of the proof. For a given cohomology class in \( H^q(X, K_X \otimes F \otimes \mathcal{I}(h)) \), we take the associated harmonic form \( u_\varepsilon \) with respect to \( h_\varepsilon \). In Step 2, we study the asymptotic behavior of \( u_\varepsilon \) and \( su_\varepsilon \) as letting \( \varepsilon \) go to zero. In Step 3, we construct suitable solutions \( \gamma_\varepsilon \) of the \( \bar{\partial} \)-equation \( \bar{\partial}\gamma_\varepsilon = su_\varepsilon \), by using the Čech complex. In Step 4, we show that \( u_\varepsilon \) converges to zero in a suitable sense.

**Step 1 (The equisingular approximation of \( h \))**

Throughout the proof, we fix a Kähler form \( \omega \) on \( X \) and a smooth metric \( g \) on \( F \). For the proof, we want to apply the theory of harmonic integrals, but the metric \( h \) may not be smooth. For this reason, we approximate \( h \) by metrics \( \{h_\varepsilon\}_{\varepsilon > 0} \) that are smooth on a Zariski open set. By Theorem [2,3], we can obtain metrics \( \{h_\varepsilon\}_{\varepsilon > 0} \) on \( F \) with the following properties:

(a) \( h_\varepsilon \) is smooth on \( X \setminus Z_\varepsilon \), where \( Z_\varepsilon \) is a subvariety on \( X \).
(b) \( h_{\varepsilon_2} \leq h_{\varepsilon_1} \leq h \) holds for any \( 0 < \varepsilon_1 < \varepsilon_2 \).
(c) \( \mathcal{I}(h) = \mathcal{I}(h_\varepsilon) \).
(d) \( \sqrt{-1} \Theta_{h_\varepsilon}(F) \geq -\varepsilon \omega \).

Take the weight function \( \varphi \) (resp. \( \varphi_\varepsilon \)) of the metric \( h \) (resp. \( h_\varepsilon \)) with respect to the fixed smooth metric \( g \). The weight function \( \varphi_\varepsilon \) is bounded above on \( X \) since \( \varphi_\varepsilon \) is upper semi-continuous. Therefore, by adding constants, we may assume \( \varphi_\varepsilon \leq 0 \). (Note that we
consider only a small $\varepsilon > 0$.) In summary, we have
\[ g \leq h_{\varepsilon} = ge^{-\varphi_{\varepsilon}} \leq h = ge^{-\varphi}. \]
Since the point-wise norm $|s|_{h^{m}}$ is bounded on $X$, there exists a positive constant $C > 0$ such that $\log |s| \leq m\varphi + C$, where we locally regard $s$ as the holomorphic function under a local trivialization of $F$. It implies that the Lelong number of $m\varphi$ is less than or equal to that of $\log |s|$. In particular, the set $\{x \in X \mid \nu(\varphi, x) > 0\}$ is contained in the subvariety $Z$ defined by $Z := \{x \in X \mid s(x) = 0\}$, thus we may assume a stronger property than property (a), namely
\[(e) \quad h_{\varepsilon} \text{ is smooth on } Y := X \setminus Z, \text{ where } Z = \{x \in X \mid s(x) = 0\}.\]
Now we construct a “complete” Kähler form on $Y$ with suitable potential function. Take a quasi-psh function $\psi$ on $X$ such that $\psi$ has a logarithmic pole along $Z$ and $\psi$ is smooth on $Y$. Since a quasi-psh function is upper semi-continuous, the function $\psi$ is bounded above. Therefore we may assume $\psi \leq -e$. Then we define the $(1, 1)$-form $\tilde{\omega}$ on $Y$ by
\[ \tilde{\omega} := \ell\omega + dd^{c}\Psi, \]
where $\ell$ is a positive number and $\Psi := \frac{1}{\log(-\psi)}$. We can show that the $(1, 1)$-form $\tilde{\omega}$ satisfies the following properties for a sufficiently large $\ell > 0$:
\[ (A) \quad \tilde{\omega} \text{ is a complete Kähler form on } Y. \]
\[ (B) \quad \Psi \text{ is bounded on } X. \]
\[ (C) \quad \tilde{\omega} \geq \omega. \]
Indeed, properties (B), (C) are obvious by the definition of $\Psi$ and $\tilde{\omega}$, and property (A) follows from a straightforward computation (see \cite{Fuj12-A} Lemma 3.1 for the precise proof of property (A)). In the proof, we mainly consider harmonic forms on $Y$ with respect to $h_{\varepsilon}$ and $\tilde{\omega}$.

Let $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon}, \tilde{\omega}}$ be the space of the square integrable $F$-valued $(n, q)$-forms $\alpha$ with respect to the inner product $\| \cdot \|_{h_{\varepsilon}, \tilde{\omega}}$ defined by
\[ \| \alpha \|^2_{h_{\varepsilon}, \tilde{\omega}} := \int_{Y} |\alpha|^2_{h_{\varepsilon}, \tilde{\omega}} \tilde{\omega}^n. \]
Then we can obtain the following orthogonal decomposition:
\[ L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon}, \tilde{\omega}} = \text{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}(F)_{h_{\varepsilon}, \tilde{\omega}} \oplus \text{Im}D^{n*}_{h_{\varepsilon}}. \]
Here the operators $D^{n*}_{h_{\varepsilon}}$ and $D^{n*}_{\tilde{\omega}}$ denote the closed extensions of the formal adjoints of $D^{n}_{h_{\varepsilon}}$ and $D^{n}_{\tilde{\omega}}$ in the sense of distributions. Note that they coincide with the Hilbert space adjoints since $\tilde{\omega}$ is complete. Further $\mathcal{H}^{n,q}(F)_{h_{\varepsilon}, \tilde{\omega}}$ stands for the space of the harmonic forms with respect to $h_{\varepsilon}$ and $\tilde{\omega}$, namely
\[ \mathcal{H}^{n,q}(F)_{h_{\varepsilon}, \tilde{\omega}} = \{ \alpha \mid \alpha \text{ is an } F\text{-valued } (n, q)\text{-form such that } \overline{\partial}\alpha = D^{n*}_{h_{\varepsilon}}\alpha = 0. \}. \]
A harmonic form in $\mathcal{H}^{n,q}(F)_{h_{\varepsilon}, \tilde{\omega}}$ is smooth by the regularization theorem for elliptic operators. These results are known to specialists. The precise proof of these results can be found in \cite{Fuj12-A} Claim 1].
For every \((n, q)\)-form \(\beta\) we have \(|\beta|^2_\omega \omega^n \leq |\beta|^2_\omega \omega^n\) since the inequality \(\omega \geq \omega\) holds by property (C). Further we have \(|\beta|^2_\omega \omega^n = |\beta|^2_\omega \omega^n\) if \(\beta\) is \((n, 0)\)-form. These can be shown by a simple computation. From this inequality and property (b) of \(h_\varepsilon\), we obtain
\[
\|\alpha\|_{h_\varepsilon, \bar{\omega}} \leq \|\alpha\|_{h_\varepsilon, \omega} \leq \|\alpha\|_{h_\varepsilon, \bar{\omega}}
\]
for an \(F\)-valued \((n, q)\)-form \(\alpha\), which plays a crucial role in the proof.

Take an arbitrary cohomology class \(\{u\} \in H^q(X, K_\mathcal{X} \otimes F \otimes \mathcal{I}(h))\) represented by an \(F\)-valued \((n, q)\)-form \(u\) with \(\|u\|_{h_\varepsilon, \omega} < \infty\). In order to prove that the multiplication map \(\Phi_*\) is injective, we assume that the cohomology class of \(su\) is zero in \(H^q(X, K_\mathcal{X} \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))\). Our final goal is to show that the cohomology class of \(u\) is actually zero.

By inequality (3.1), we know \(u \in L^n_{(2)}(Y, F)_{h_\varepsilon, \bar{\omega}}\) for any \(\varepsilon > 0\). Therefore by the above orthogonal decomposition, there exist \(u_\varepsilon \in H^n_{\varepsilon, q}(F)_{h_\varepsilon, \bar{\omega}}\) and \(v_\varepsilon \in L^n_{(2)-1}(Y, F)_{h_\varepsilon, \bar{\omega}}\) such that
\[
u = u_\varepsilon + \overline{\partial} v_\varepsilon.
\]
Note that the component of \(\text{Im}D^{\varepsilon}_{h_\varepsilon}\) is zero since \(u\) is \(\overline{\partial}\)-closed.

At the end of this step, we explain the strategy of the proof. In Step 2, we show that \(\|D^{\varepsilon}_{h_\varepsilon} \ast su_\varepsilon\|_{h_\varepsilon^{m+1}, \omega}\) converges to zero as letting \(\varepsilon\) go to zero. Since the cohomology class of \(su\) is zero, there are solutions \(\gamma_\varepsilon\) of the \(\overline{\partial}\)-equation \(\overline{\partial} \gamma_\varepsilon = su_\varepsilon\). For the proof, we need to obtain \(L^2\)-estimates of them. In Step 3, we construct solutions \(\gamma_\varepsilon\) of the \(\overline{\partial}\)-equation \(\overline{\partial} \gamma_\varepsilon = su_\varepsilon\) such that the norm \(\|\gamma_\varepsilon\|_{h_\varepsilon^{m+1}, \omega}\) is uniformly bounded. Then we have
\[
\|su_\varepsilon\|_{h_\varepsilon^{m+1}, \bar{\omega}}^2 \leq \|D^{\varepsilon}_{h_\varepsilon} \ast su_\varepsilon\|_{h_\varepsilon^{m+1}, \bar{\omega}} \|\gamma_\varepsilon\|_{h_\varepsilon^{m+1}, \bar{\omega}}.
\]
By Step 2 and Step 3, we can conclude that the right hand side goes to zero as letting \(\varepsilon\) go to zero. In Step 4, from this convergence, we prove that \(u_\varepsilon\) converges to zero in a suitable sense, which implies that the cohomology class of \(u\) is zero.

**Step 2 (A generalization of Enoki’s proof of the injectivity theorem)**

The aim of this step is to prove the following proposition. The proof of the proposition can be seen as a generalization of the proof of Theorem 1.2.

**Proposition 3.2.** As letting \(\varepsilon\) go to zero, the norm \(\|D^{\varepsilon}_{h_\varepsilon} \ast su_\varepsilon\|_{h_\varepsilon^{m+1}, \bar{\omega}}\) converges to zero.

**Proof of Proposition 3.2.** We have the following inequality:
\[
\|v_\varepsilon\|_{h_\varepsilon, \bar{\omega}} \leq \|u\|_{h_\varepsilon, \bar{\omega}} \leq \|u\|_{h_\varepsilon, \omega}.
\]
The first inequality follows from the definition of \(u_\varepsilon\) and the second inequality follows from inequality (3.1). This inequality is often used in the proof. Note that \(\|u\|_{h_\varepsilon, \omega}\) does not depend on \(\varepsilon\). By applying Proposition 2.4 to \(u_\varepsilon\), we obtain
\[
0 = \langle \sqrt{-1} \Theta_{h_\varepsilon}(F) \Lambda_{\varepsilon} u_\varepsilon, u_\varepsilon \rangle_{h_\varepsilon, \bar{\omega}} + \|D^{\varepsilon}_{h_\varepsilon} u_\varepsilon\|_{h_\varepsilon, \bar{\omega}}^2.
\]
Note that the left hand side is zero since \(u_\varepsilon\) is harmonic. Let \(A_\varepsilon\) be the first term and \(B_\varepsilon\) be the second term of the right hand side of equality (3.3). First, we show that the first
term $A_\epsilon$ and the second term $B_\epsilon$ converge to zero. For simplicity, we denote the integrand of $A_\epsilon$ by $g_\epsilon$, namely

$$g_\epsilon := \langle \sqrt{-1} \Theta_{h_\epsilon}(F) \Lambda_\tilde{\omega} u_\epsilon, u_\epsilon \rangle_{h_\epsilon, \tilde{\omega}}.$$

Then there exists a positive constant $C > 0$ (independent of $\epsilon$) such that

$$g_\epsilon \geq -\epsilon C |u_\epsilon|_{h_\epsilon, \tilde{\omega}}^2.$$

It is easy to check this inequality. Indeed, let $\lambda_1^\epsilon \leq \lambda_2^\epsilon \leq \cdots \leq \lambda_n^\epsilon$ be the eigenvalues of $\sqrt{-1} \Theta_{h_\epsilon}(F)$ with respect to $\tilde{\omega}$. Then for any point $y \in Y$ there exists a local coordinate $(z_1, z_2, \ldots, z_n)$ centered at $y$ such that

$$\sqrt{-1} \Theta_{h_\epsilon}(F) = \sum_{j=1}^n \lambda_j^\epsilon dz_j \wedge d\bar{z}_j$$

and $\tilde{\omega} = \sum_{j=1}^n dz_j \wedge d\bar{z}_j$ at $y$.

When we locally write $u_\epsilon$ as $u_\epsilon = \sum_{|K|=q} f_{K_\epsilon}^\epsilon dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_K$, we have

$$g_\epsilon = \sum_{|K|=q} \left( \sum_{j \in K} \lambda_j^\epsilon \right) |f_{K_\epsilon}^\epsilon|^2_{h_\epsilon}$$

by a straightforward computation. On the other hand, from property (C) of $\tilde{\omega}$ and property (d) of $h_\epsilon$, we have $\sqrt{-1} \Theta_{h_\epsilon}(F) \geq -\epsilon \omega \geq -\epsilon \tilde{\omega}$. This implies $\lambda_j^\epsilon \geq -\epsilon$, and thus we obtain inequality (3.4).

From inequality (3.4) and equality (3.3), we obtain

$$0 \geq A_\epsilon = \int_Y g_\epsilon \tilde{\omega}^n$$

$$\geq -\epsilon C \int_Y |u_\epsilon|^2_{h_\epsilon, \tilde{\omega}} \tilde{\omega}^n$$

$$\geq -\epsilon C \|u\|_{h, \omega}^2.$$

The last inequality follows from inequality (3.2). Therefore $A_\epsilon$ converges to zero, and further we can conclude that $B_\epsilon$ also converges to zero by equality (3.3).

To apply Proposition 2.4 to $su_\epsilon$, we first check $su_\epsilon \in L^{n,q}(Y, F^{m+1})_{h_\epsilon, \tilde{\omega}}$. By the assumption the point-wise norm $|s|_{h^m}$ with respect to $h^m$ is bounded, and further we have $|s|_{h_\epsilon^m} \leq |s|_{h^m}$ from property (b) of $h_\epsilon$. They imply

$$\|su_\epsilon\|_{h_\epsilon^{m+1}, \tilde{\omega}} \leq \sup_X |s|_{h^m} \|u_\epsilon\|_{h_\epsilon, \tilde{\omega}} \leq \sup_X |s|_{h^m} \|u\|_{h, \omega} < \infty.$$

Note that the right hand side does not depend on $\epsilon$. By applying Proposition 2.4 to $su_\epsilon$, we obtain

$$\|D^{m+1}_{h_\epsilon} su_\epsilon\|_{h_\epsilon^{m+1}, \tilde{\omega}}^2 = \langle \sqrt{-1} \Theta_{h_\epsilon}^{m+1}(F^{m+1}) \Lambda_{\tilde{\omega}} su_\epsilon, su_\epsilon \rangle_{h_\epsilon^{m+1}, \tilde{\omega}} + \|D^{m+1}_{h_\epsilon} su_\epsilon\|_{h_\epsilon^{m+1}, \tilde{\omega}}^2.$$

(3.5)
Here we used $\overline{\partial} su = 0$. Now we prove that the second term of the right hand side converges to zero. Since $s$ is a holomorphic (0, 0)-form, we have $D_{h^{m+1}_e}^* su = s D_{\overline{h^{m+1}_e}}^* u$. Thus we have

$$\|D_{h^{m+1}_e}^* su\|^2_{h^{m+1}_e, \omega} \leq \sup_X |s|_{h^{m}_e}^2 \int_Y |D_{h^{m}_e}^* u|^2_{h^{m}_e, \omega} \omega^n \leq \sup_X |s|_{h^{m}_e}^2 B_e.$$ 

Since $\sup_X |s|_{h^{m}_e}^2$ is bounded and $B_e$ converges to zero, the second term $\|D_{h^{m+1}_e}^* su\|_{h^{m+1}_e, \omega}$ also converges to zero.

For the proof of the proposition, it remains to show that the first term of the right hand side of equality (3.5) converges to zero. For this purpose, we investigate $A_\varepsilon$ in details. By the definition of $A_\varepsilon$, we have

$$A_\varepsilon = \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon \bar{\omega}^n + \int_{\{g_\varepsilon \leq 0\}} g_\varepsilon \bar{\omega}^n.$$ 

Let $A_\varepsilon^+$ be the first term and $A_\varepsilon^-$ be the second term of the right hand side. Then $A_\varepsilon^+$ and $A_\varepsilon^-$ converge to zero. Indeed, a simple computation and inequality (3.4) imply

$$0 \geq A_\varepsilon^- \geq -\varepsilon C \int_{\{g_\varepsilon \leq 0\}} |u_\varepsilon|^2_{h^{m}_e, \bar{\omega}} \bar{\omega}^n \geq -\varepsilon C \int_{Y} |u_\varepsilon|^2_{h^{m}_e, \bar{\omega}} \bar{\omega}^n \geq -\varepsilon C \|u\|^2_{h^{m}_e, \omega}.$$ 

Therefore $A_\varepsilon^+$ and $A_\varepsilon^-$ converge to zero, since $A_\varepsilon = A_\varepsilon^+ + A_\varepsilon^-$ converges to zero. Now we have

$$\langle \sqrt{-1} \Theta_{h^{m+1}_e} (F^{m+1}) \Lambda \bar{\omega} s u_\varepsilon, s u_\varepsilon \rangle_{h^{m+1}_e, \omega} = (m + 1) |s|_{h^{m}_e}^2 x \langle \sqrt{-1} \Theta_{h^{m}_e} (F) \Lambda \bar{\omega} u_\varepsilon, u_\varepsilon \rangle_{h^{m}_e, \bar{\omega}}$$

$$= (m + 1) |s|_{h^{m}_e}^2 g_\varepsilon.$$ 

It implies

$$\langle \sqrt{-1} \Theta_{h^{m+1}_e} (F^{m+1}) \Lambda \bar{\omega} s u_\varepsilon, s u_\varepsilon \rangle_{h^{m+1}_e, \omega} = (m + 1) \int_Y |s|_{h^{m}_e}^2 g_\varepsilon \bar{\omega}^n$$

$$= (m + 1) \left\{ \int_{\{g_\varepsilon \geq 0\}} |s|_{h^{m}_e}^2 g_\varepsilon \bar{\omega}^n + \int_{\{g_\varepsilon \leq 0\}} |s|_{h^{m}_e}^2 g_\varepsilon \bar{\omega}^n \right\}.$$ 

Then we have

$$0 \leq \int_{\{g_\varepsilon \geq 0\}} |s|_{h^{m}_e}^2 g_\varepsilon \bar{\omega}^n \leq \sup_X |s|_{h^{m}_e}^2 \int_{\{g_\varepsilon \geq 0\}} g_\varepsilon \bar{\omega}^n \leq \sup_X |s|_{h^{m}_e}^2 A_\varepsilon^+.$$
On the other hand, we have
\[
0 \geq \int_{\{g_{\varepsilon} \leq 0\}} |s|^{2}_{h_{\varepsilon}} g_{\varepsilon} \tilde{\omega}_{\varepsilon}^{n} \geq \sup_{X} \int_{\{g_{\varepsilon} \leq 0\}} g_{\varepsilon} \tilde{\omega}_{\varepsilon}^{n} \geq \sup_{X} |s|^{2}_{h_{\varepsilon}} A_{\varepsilon}^{\varepsilon}.
\]

Therefore the right hand side of equality (3.5) converges to zero. We obtain the conclusion of Proposition 3.2. 

Step 3 (A construction of solutions of the $\overline{\partial}$-equation via the Čech complex)

The aim of this step is to prove the following proposition.

**Proposition 3.3.** There exist $F$-valued $(n, q - 1)$-forms $\alpha_{\varepsilon}$ on $Y$ with the following properties:

1. $\overline{\partial} \alpha_{\varepsilon} = u - u_{\varepsilon}$.
2. The norm $\|\alpha_{\varepsilon}\|_{h_{\varepsilon}, \tilde{\omega}}$ is uniformly bounded.

**Remark 3.4.** We have already known that there exist solutions $\alpha_{\varepsilon}$ of the $\overline{\partial}$-equation $\overline{\partial} \alpha_{\varepsilon} = u - u_{\varepsilon}$ since $u - u_{\varepsilon} \in \text{Im}\overline{\partial}$. However, for the proof of the main theorem, we need to construct solutions with uniformly bounded $L^{2}$-norm.

**Proof of Proposition 3.3.** The strategy of the proof is as follows: The main idea of the proof is to convert the $\overline{\partial}$-equation $\overline{\partial} \alpha_{\varepsilon} = u - u_{\varepsilon}$ to the equation $\delta V_{\varepsilon} = S_{\varepsilon}$ of the coboundary operator $\delta$ in the space of the cochains $C^{\bullet}(K_{X} \otimes F \otimes I(h_{\varepsilon}))$, by using the Čech complex and pursuing the De Rham-Weil isomorphism. Here the $q$-cochain $S_{\varepsilon}$ is constructed from $u - u_{\varepsilon}$. In this construction, we locally solve the $\overline{\partial}$-equation by Lemma 3.14 (Lemma 3.14 is proved at the end of this step.) The important point is that the space $C^{\bullet}(K_{X} \otimes F \otimes I(h_{\varepsilon}))$ is independent of $\varepsilon$ thanks to property (c) of $h_{\varepsilon}$ although the $L^{2}$-space $L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon}, \tilde{\omega}}$ depends on $\varepsilon$. Since $\|u - u_{\varepsilon}\|_{h_{\varepsilon}, \tilde{\omega}}$ is uniformly bounded, we can observe that $S_{\varepsilon}$ converges to some $q$-coboundary in $C^{q}(K_{X} \otimes F \otimes I(h))$ with the topology induced by the local $L^{2}$-norms with respect to $h$. Further we can observe that the coboundary operator $\delta$ is an open map. (This topology is studied in Section 4.) Then by these observations we construct solutions $V_{\varepsilon}$ of the equation $\delta V_{\varepsilon} = S_{\varepsilon}$ with uniformly bounded norm. Finally, by using a partition of unity, we conversely construct $T_{\varepsilon} \in L^{n,q}_{(2)}(Y, F)_{h_{\varepsilon}, \tilde{\omega}}$ from $S_{\varepsilon}$, which provides $\alpha_{\varepsilon}$ satisfying the properties in Proposition 3.3 thanks to the property that the norm of $V_{\varepsilon}$ is uniformly bounded. This proof gives a new method to obtain $L^{2}$-estimates of solutions of the $\overline{\partial}$-equation.

Let $U$ (resp. $U'$) be a Stein finite cover $U := \{B_{i}\}_{i \in I}$ (resp. $U' := \{B'_{i}\}_{i \in I'}$) of $X$ with $B_{i} \subseteq B'_{i}$. We may assume that there is a Stein open set $B''_{i}$ such that $B_{i} \subseteq B'_{i} \subseteq B''_{i}$.

From $U_{\varepsilon} := u - u_{\varepsilon}$, we first construct $q$-cochains $S_{\varepsilon}$ in $C^{q}(U', K_{X} \otimes F \otimes I(h_{\varepsilon}))$ calculated by $U'$, by pursuing the De Rham-Weil isomorphism. By applying Lemma 3.14 to $U_{\varepsilon}^{0} := \ldots$
\{U_{\varepsilon, i_0}\} defined by \(U_{\varepsilon, i_0} := U_\varepsilon|_{B'_0 \setminus Z}\), we obtain \(U^1_\varepsilon = \{U^1_{\varepsilon, i_0}\}\) such that \(\partial U^1_\varepsilon = U^0_\varepsilon\) and

\[
\|U^1_\varepsilon\|^2_{h, \bar{w}} := \sum_{i \in I} \int_{B'_i \setminus Z} |U^1_{\varepsilon, i}|^2_{h, \bar{w}} \bar{w}^n \leq C \int_Y |U^1_\varepsilon|^2_{h, \bar{w}} \bar{w}^n
\]

for some constant \(C\). In the proof, we denote by \(C\), different positive constants independent of \(\varepsilon\). The right hand side can be estimated by a constant independent of \(\varepsilon\) by \(\|U_\varepsilon\|_{h, w} \leq 2\|u\|_{h, w} < \infty\) (see inequality (3.2) and the definition of \(u_\varepsilon\)). By the construction of \(U^1_\varepsilon\), we have \(\partial \delta U^1_\varepsilon = \delta \partial U^1_\varepsilon = 0\). Here \(\delta\) denotes the coboundary operator of the Čech complex defined as follows: For every \((p - 1)\)-cochain \(\{f_{i_0i_1...i_{p-1}}\}\)

\[
\delta(\{f_{i_0i_1...i_{p-1}}\}) := \{\sum_{\ell=0}^{p} (-1)^\ell f_{i_0...i_\ell...i_p} |_{B'_0...i_p}\},
\]

where \(B'_0...i_p = B'_0 \cap ... \cap B'_i\). We often omit the notation of the restriction in the right hand side. By applying Lemma 3.14 again, we obtain \(U^2_\varepsilon = \{U^2_{\varepsilon, i_0i_1}\}\) such that \(\partial U^2_\varepsilon = \delta U^1_\varepsilon\) and

\[
\|U^2_\varepsilon\|^2_{h, \bar{w}} := \sum_{\{i,j\} \subseteq I} \int_{B'_i \setminus Z} |U^2_{\varepsilon, i}|^2_{h, \bar{w}} \bar{w}^n \leq C \|U^1_\varepsilon\|^2_{h, \bar{w}}
\]

for some constant \(C\). Note that the right hand side can also be estimated by a constant independent of \(\varepsilon\). By repeating this procedure, we obtain \(U^k_\varepsilon = \{U^k_{\varepsilon, i_0...i_{k-1}}\}\) for \(2 \leq k \leq q\) such that

\[
\delta(\{f_{i_0i_1...i_{k-1}}\}) \subseteq \{\sum_{\ell=0}^{k} (-1)^\ell f_{i_0...i_\ell...i_{k-1}} |_{B'_0...i_{k-1}}\},
\]

where \(S_\varepsilon\) is defined by

\[
S_\varepsilon := \delta U^q_\varepsilon := \{S_{\varepsilon, i_0...i_q}\},
\]

then \(S_\varepsilon\) determines the \(q\)-cocycle in \(C^q(U', K_X \otimes F \otimes \mathcal{I}(h_\varepsilon))\). Indeed, by the construction, there exists a positive constant \(C\) such that

\[
\int_{B'_{i_0...i_q} \setminus Z} |S_{\varepsilon, i_0...i_q}|^2_{h, \bar{w}} \bar{w}^n < C.
\]

In particular, the \(L^2\)-norm of \(S_{\varepsilon, i_0...i_q}\) with respect to the fixed \(g\) and \(\omega\) is bounded since \(S_{\varepsilon, i_0...i_q}\) is an \(F\)-valued \((n, 0)\)-form and \(g \leq h_\varepsilon\). Further the coefficients of \(S_{\varepsilon, i_0...i_q}\) are holomorphic since \(\partial S_{\varepsilon, i_0...i_q} = 0\). By the Riemann extension theorem, \(S_{\varepsilon, i_0...i_q}\) can be extended to the holomorphic \(F\)-valued \((n, 0)\)-form on \(B'_{i_0...i_q}\). The extended \(S_{\varepsilon, i_0...i_q}\) gives the \(q\)-cocycle with coefficients in \(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon)\) since \(\delta S_\varepsilon = 0\) by the definition.

From now on, we mainly handle the space \(C^p(U, K_X \otimes F \otimes \mathcal{I}(h))\) that stands for the space of the cochains with coefficients in \(K_X \otimes F \otimes \mathcal{I}(h)\) (not \(K_X \otimes F \otimes \mathcal{I}(h_\varepsilon)\)) calculated
by $\mathcal{U}$ (not $\mathcal{U}'$). Note that $S_\varepsilon$ determines the $q$-cocycle in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$. Indeed, by property (c) of $h_\varepsilon$, $S_\varepsilon$ is a $q$-cocycle in $C^q(\mathcal{U}', K_X \otimes F \otimes \mathcal{I}(h))$, and further the restriction of $S_\varepsilon$ to $\mathcal{U}$ determines the $q$-cocycle in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$. We continue to use the same notation $S_\varepsilon$ for this cocycle.

Remark 3.5. By the same argument, we can construct $q$-cocycles in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ from $F$-valued $(n, q)$-forms in $L^{n,q}_2(Y, F)_{h_\varepsilon,\omega} \cap \operatorname{Ker} \overline{\partial}$. Thus we obtain a (well-defined) homomorphism

$$\iota : L^{n,q}_2(Y, F)_{h_\varepsilon,\omega} \cap \operatorname{Ker} \overline{\partial} / \overline{\partial} \to H^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h)),$$

where $H^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ is the Čech cohomology group calculated by $\mathcal{U}$. It is well-known that the homomorphism $\iota$ is actually isomorphic (for example, see [Fuj12-A, Claim 1]).

Now we consider the topology of $C^p(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ induced by a family of the semi-norms $p_{K_{i_0...i_p}}(\cdot)$, which are defined as follows: For every $f = \{f_{i_0...i_p}\} \in C^p(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ and a relatively compact set $K_{i_0...i_p} \subset B_{i_0...i_p}$, the semi-norm $p_{K_{i_0...i_p}}(f)$ of $f$ is defined by

$$(3.10) \quad p_{K_{i_0...i_p}}(f)^2 := \int_{K_{i_0...i_p}} |f_{i_0...i_p}|^2_{h_\varepsilon,\omega} \omega^n.$$

In the proof of Proposition 3.3, this topology plays a crucial role.

Remark 3.6.

(1) The equality $|f_{i_0...i_p}|^2_{h_\varepsilon,\omega} \omega^n = |f_{i_0...i_p}|^2_0 \widetilde{\omega}^n$ holds since $f_{i_0...i_p}$ is an $F$-valued $(n, 0)$-form.

(2) Theorem 3.3 asserts that $C^p(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ is a Fréchet space (that is, complete with respect to the semi-norms $p_{K_{i_0...i_p}}(\cdot)$). See Section 4 for the proof.

In the above situation, we prove the following claim. In the proof, we use Lemma 4.2 which is proved in Section 4.

Claim 3.7. The sequence $\{S_\varepsilon\}_{\varepsilon > 0}$ has a subsequence that converges to a $q$-cochain $S$ in $C^q(\mathcal{U}, K_X \otimes F \otimes \mathcal{I}(h))$ as letting $\varepsilon$ go to zero.

Proof. Under a fixed local trivialization of $K_X \otimes F$ on $B_{i_0...i_q}'$, we regard $F$-valued holomorphic $(n, 0)$-forms as the holomorphic functions. By the construction of $S_\varepsilon$, the $L^2$-norm on $B_{i_0...i_q}'$

$$\int_{B_{i_0...i_q}'} |S_\varepsilon,i_0...i_q|^2_{h_\varepsilon,\omega} \omega^n$$

is uniformly bounded (see inequality (3.9) and Remark 3.6). Since $S_\varepsilon,i_0...i_q$ is holomorphic, the sup-norm $\sup_{K_{i_0...i_q}} |S_\varepsilon,i_0...i_q|$ is also uniformly bounded for every relatively compact set $K_{i_0...i_q} \subset B_{i_0...i_q}'$. Therefore, by Montel’s theorem, we obtain a subsequence of $\{S_\varepsilon,i_0...i_q\}_{\varepsilon > 0}$ that uniformly converges to an $F$-valued holomorphic $(n, 0)$-form $S_{i_0...i_q}$ on
bounded norm. The following coboundary operator is an open map by the open mapping theorem. Thus we can find a subsequence of \( S_{\varepsilon} \) that converges to a \( q \)-cochain \( S \) in \( C^q(U, K_X \otimes F \otimes \mathcal{I}(h)) \).

For simplicity we use the same notation \( S_{\varepsilon} \) for the subsequence obtained in Claim 3.7. Let \( Z^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \) (resp. \( B^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \)) be the space of the \( p \)-cocycles (resp. \( p \)-coboundaries). We prove the following claim.

**Claim 3.8.** \( Z^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \) and \( B^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \) are closed subspaces in \( C^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \). Moreover, \( S_{\varepsilon} \) belongs to \( B^q(U, K_X \otimes F \otimes \mathcal{I}(h)) \) for any \( \varepsilon > 0 \), and in particular the limit \( S \) of \( S_{\varepsilon} \) also belongs to \( B^q(U, K_X \otimes F \otimes \mathcal{I}(h)) \).

**Proof.** It is easy to check that the coboundary operator \( \delta \) from \( C^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \) to \( C^{p+1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \) is continuous. Therefore \( Z^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \) is a closed subspace since the kernel of \( \delta \) agrees with \( Z^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \). Now we consider the following coboundary operator:

\[
\delta : C^{p-1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \to Z^p(U, K_X \otimes F \otimes \mathcal{I}(h)).
\]

The cokernel of \( \delta \) is isomorphic to \( H^p(X, K_X \otimes F \otimes \mathcal{I}(h)) \), and thus the cokernel is a finite dimensional vector space. The open mapping theorem implies that \( B^p(U, K_X \otimes F \otimes \mathcal{I}(h)) \) is a closed subspace (see Proposition 2.6).

Let us prove the latter conclusion. Recall that the homomorphism \( \iota \) is isomorphic and \( U_{\varepsilon} \in \text{Im} \iota \subset L_{(2)}^n(Y, K_X \otimes F)_{h_{\varepsilon},w} \). Therefore \( S_{\varepsilon} \) (that is obtained from \( U_{\varepsilon} \in \text{Im} \iota \)) belongs to \( B^q(U, K_X \otimes F \otimes \mathcal{I}(h_{\varepsilon})) \) (see Remark 3.5). By property (c) of \( h_{\varepsilon} \), we have the conclusion.

By Claim 3.7 and 3.8 we construct solutions \( V_{\varepsilon} \) of the equation \( \delta V_{\varepsilon} = S_{\varepsilon} \) with uniformly bounded norm. The following coboundary operator

\[
\delta : C^{q-1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \to B^q(U, K_X \otimes F \otimes \mathcal{I}(h))
\]

is continuous and surjective linear map between Fréchet spaces, and thus this coboundary operator is an open map by the open mapping theorem.

By the latter conclusion of Claim 3.8 we can take \( V \in C^{q-1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \) with \( \delta(V) = S \). For a given family \( K := \{ K_{i_0...i_q-1} \} \) of relatively compact sets \( K_{i_0...i_q-1} \in B_{i_0...i_q-1} \), we define the open and bounded neighborhood \( \Delta_K \) of \( V \) in \( C^{q-1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \) by

\[
\Delta_K := \{ W \in C^{q-1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \mid p_{K_{i_0...i_q-1}}(W - V) < 1 \}.
\]

Then \( \delta(\Delta_K) \) is an open neighborhood of \( S \) in \( B^q(U, K_X \otimes F \otimes \mathcal{I}(h)) \) by the above observation. Therefore \( S_{\varepsilon} \) belongs to \( \delta(\Delta_K) \) for a sufficiently small \( \varepsilon > 0 \) since \( S_{\varepsilon} \) converges to \( S \). By the definition of \( \Delta_K \), we can obtain \( V_{\varepsilon} := \{ V_{\varepsilon,i_0...i_q-1} \} \) in \( C^{q-1}(U, K_X \otimes F \otimes \mathcal{I}(h)) \).
such that

\begin{equation}
\delta(V_\varepsilon) = \left\{ \sum_{\ell=0}^{q} (-1)^\ell V_{\varepsilon, i_0 \ldots \hat{i_\ell} \ldots i_q} \right\} = S_\varepsilon,
\end{equation}

\begin{equation}
p_{K_{i_0 \ldots i_q - 1}}(V_\varepsilon)^2 = \int_{K_{i_0 \ldots i_q - 1}} |V_{\varepsilon, i_0 \ldots i_q - 1}|^2 \omega^n \leq C_K
\end{equation}

for some positive constant $C_K$ (depending on $K$).

**Remark 3.9.** The above constant $C_K$ and the solution $V_\varepsilon$ depend on the choice of $K$, but not depend on $\varepsilon$.

Finally we conversely construct $F$-valued $(n, q)$-forms $T_\varepsilon \in L^{n,q}_{(2)}(Y, F)_{h, \tilde{\omega}}$ from $S_\varepsilon$. Fix a partition of unity $\{ \rho_i \}_{i \in I}$ associated to $U$ and define $T_\varepsilon^1$ by

\begin{equation}
T_\varepsilon^1 := \left\{ T_{\varepsilon, i_0 \ldots i_q - 1}^1 \right\} := \left\{ \sum_{i \in I} \rho_i S_{\varepsilon, i_0 \ldots i_q - 1} \right\}.
\end{equation}

We have $\delta T_\varepsilon^1 = S_\varepsilon$ and $\delta(\overline{\partial} T_\varepsilon^1) = \overline{\partial} S_\varepsilon = 0$ since $S_\varepsilon$ is a cocycle. In the same manner, we construct $T_\varepsilon^2$ such that $\delta T_\varepsilon^2 = \overline{\partial} T_\varepsilon^1$. By repeating this procedure, we obtain $T_\varepsilon^k$ for $2 \leq k \leq q$ such that $\delta T_\varepsilon^k = \overline{\partial} T_\varepsilon^{k-1}$. Then $\overline{\partial} T_\varepsilon^q$ determines the $F$-valued $(n, q)$-form $T_\varepsilon$ on $X$ by $\delta(\overline{\partial} T_\varepsilon^q) = 0$. By the construction, we have

\begin{equation}
T_\varepsilon := \overline{\partial} T_\varepsilon^q = \overline{\partial} \sum_{k_q \in I} \rho_{k_q} \overline{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \ldots \overline{\partial} \sum_{k_1 \in I} \rho_{k_1} S_{\varepsilon, k_1 \ldots k_q i_0}.
\end{equation}

on $B_{i_0}$. This equality is often used in the proof of Claim 3.11 and 3.13.

**Remark 3.10.** Claim 3.11 follows from properties (3.11), (3.12) of $V_\varepsilon$. On the other hand, Claim 3.13 follows from properties (3.6), (3.7).

**Claim 3.11.** There exist $F$-valued $(n, q - 1)$-forms $\beta_\varepsilon$ on $X$ satisfying the following properties:

1. $\overline{\partial} \beta_\varepsilon = T_\varepsilon$.
2. The norm $\| \beta_\varepsilon \|_{h, \tilde{\omega}}$ is uniformly bounded.

**Proof.** By equality (3.11), we have

\begin{equation}
S_{\varepsilon, k_1 \ldots k_q i_0} = V_{\varepsilon, k_2 \ldots k_q i_0} + \sum_{\ell=2}^{q} (-1)^{\ell-1} V_{\varepsilon, k_1 \ldots \hat{k_\ell} \ldots k_q i_0} + (-1)^q V_{\varepsilon, k_1 \ldots k_q}.
\end{equation}

[**Argument 1**]

Firstly we consider the first term $V_{\varepsilon, k_2 \ldots k_q i_0}$ of the right hand side. This term does not depend on $k_1$ and $V_{\varepsilon, k_2 \ldots k_q i_0}$ is holomorphic, and thus we have

\begin{equation}
\overline{\partial} \sum_{k_1 \in I} \rho_{k_1} V_{\varepsilon, k_2 \ldots k_q i_0} = \overline{\partial} V_{\varepsilon, k_2 \ldots k_q i_0} = 0.
\end{equation}

Here we used $\sum_{k_1 \in I} \rho_{k_1} = 1$. We can conclude that this term does not affect $T_\varepsilon$ from equation (3.13).
[Argument 2]
Secondly we consider the second term. For an integer $\ell$ with $2 \leq \ell \leq q$, the second term $V_{\varepsilon,k_1\ldots k_q}$ does not depend on $k_{\ell}$. It implies
\[
\overline{\partial} \sum_{k_{\ell}\in I} \rho_{k_{\ell}} \overline{\partial} \sum_{k_{\ell-1}\in I} \rho_{k_{\ell-1}} \overline{\partial} \cdots \overline{\partial} \sum_{k_1\in I} \rho_{k_1} V_{\varepsilon,k_1\ldots k_q \ldots k_{q-1}i_0} = 0.
\]
Here we used $\overline{\partial} \overline{\partial} = 0$ and $\sum_{k_{\ell}\in I} \rho_{k_{\ell}} = 1$. Therefore the second term does not affect $T_{\varepsilon}$.

[Argument 3]
Finally we consider the third term $V_{\varepsilon,k_1\ldots k_q}$. When $\beta_{\varepsilon}$ is defined by
\[
\beta_{\varepsilon} := \sum_{k_{\ell}\in I} \rho_{k_{\ell}} \overline{\partial} \sum_{k_{\ell-1}\in I} \rho_{k_{\ell-1}} \overline{\partial} \cdots \overline{\partial} \sum_{k_1\in I} \rho_{k_1} (-1)^q V_{\varepsilon,k_1\ldots k_q}.
\]
then $\beta_{\varepsilon}$ determines the $F$-valued $(n, q-1)$-form on $X$ since $V_{\varepsilon,k_1\ldots k_q}$ is independent of $i_0$. We have $T_{\varepsilon} = \overline{\partial} \beta_{\varepsilon}$ by the definition of $\beta_{\varepsilon}$ and Argument 1, 2. For the proof, it is sufficient to show that the norm $\|\beta_{\varepsilon}\|_{h,\overline{\omega}}$ is uniformly bounded. We define the $(0, q-1)$-form $\eta_{k_1\ldots k_q}$ on $X$ by
\[
\eta_{k_1\ldots k_q} := \rho_{k_1} \overline{\partial} \rho_{k_2} \wedge \overline{\partial} \rho_{k_3} \wedge \cdots \wedge \overline{\partial} \rho_{k_q}.
\]
Since $V_{\varepsilon,k_1\ldots k_q}$ is holomorphic, we can easily show
\[
\beta_{\varepsilon} = \sum_{\{k_1, \ldots, k_q\} \subseteq I} \eta_{k_1\ldots k_q} \wedge V_{\varepsilon,k_1\ldots k_q}.
\]
Since the support of $\eta_{k_1\ldots k_q}$ is relatively compact in $B_{k_1\ldots k_q}$, we can take $K := \{K_{k_1\ldots k_q}\}$ such that $\text{Supp} \ \eta_{k_1\ldots k_q} \subseteq K_{k_1\ldots k_q} \subseteq B_{k_1\ldots k_q}$. For the family $K = \{K_{k_1\ldots k_q}\}$, we may assume that the $q$-cochains $V_{\varepsilon}$ satisfy inequality (3.12). To estimate the norm $\|\beta_{\varepsilon}\|_{h,\overline{\omega}}$, we use Lemma 3.15, which is proved at the end of this step. By Lemma 3.15, there exists a positive constant $C > 0$ such that
\[
|\beta_{\varepsilon}|_{h,\overline{\omega}} \leq \sum_{\{k_1, \ldots, k_q\} \subseteq I} |\eta_{k_1\ldots k_q} \wedge V_{\varepsilon,k_1\ldots k_q}|_{h,\overline{\omega}} \leq C \sum_{\{k_1, \ldots, k_q\} \subseteq I} \chi_{K_{k_1\ldots k_q}} |V_{\varepsilon,k_1\ldots k_q}|_{h,\overline{\omega}},
\]
where $\chi_{K_{k_1\ldots k_q}}$ is the characteristic function of $K_{k_1\ldots k_q}$. Notice that $C$ depends on the choice of $\{\rho_i\}_{i \in I}$ but does not depend on $\varepsilon$. Therefore we have
\[
\|\beta_{\varepsilon}\|_{h,\overline{\omega}} \leq C \sum_{\{k_1, \ldots, k_q\} \subseteq I} p_{K_{k_1\ldots k_q}}(V_{\varepsilon})
\]
for some constant $C$. The right hand side can be estimated by a constant independent of $\varepsilon$ by inequality (3.12) and Remark 3.9. This completes the proof. \qed
The proof of the following claim is based on an argument similar to that of Claim 3.11. To avoid confusion, we use the following notation in the proof.

**Definition 3.12.** Let \( a_\varepsilon \) and \( b_\varepsilon \) be \( F \)-valued \((n,p)\)-forms on \( Y \). We write \( a_\varepsilon \equiv b_\varepsilon \), if there exist \( F \)-valued \((n,p-1)\)-forms \( \eta_\varepsilon \) on \( Y \) such that \( \overline{\partial} \eta_\varepsilon = a_\varepsilon - b_\varepsilon \) and the norm \( \| \eta_\varepsilon \|_{h_\varepsilon,\omega} \) is uniformly bounded.

**Claim 3.13.** There exist \( F \)-valued \((n,q-1)\)-forms \( \tilde{\beta}_\varepsilon \) on \( Y \) satisfying the following properties:

1. \( \overline{\partial} \tilde{\beta}_\varepsilon = U_\varepsilon - T_\varepsilon \).
2. The norm \( \| \tilde{\beta}_\varepsilon \|_{h_\varepsilon,\omega} \) is uniformly bounded.

**Proof.** By the construction of \( S_\varepsilon = \{ S_{\varepsilon,i_0...i_q} \} \) (see (3.8)), we have

\[
S_{\varepsilon,k_1...k_q} = U^q_{\varepsilon,k_2...k_q} + \sum_{\ell=2}^{q} (-1)^{\ell-1} U^q_{\varepsilon,k_1...k_\ell...k_q} + (-1)^q U^q_{\varepsilon,k_1...k_q}.
\]

[Argument 4]

Firstly we consider the second term of the right hand side of (3.14). For an integer \( \ell \) with \( 2 \leq \ell \leq q \), the second term \( U^q_{\varepsilon,k_1...k_\ell...k_q} \) is independent of \( k_\ell \). By the same reason as Argument 2 in Claim 3.11, we can conclude that this term does not affect \( T_\varepsilon \).

[Argument 5]

Secondly we consider the third term of the right hand side of (3.14). Our aim of Argument 5 is to show

\[
\overline{\partial} \sum_{k_q \in I} \rho_{k_q} \overline{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \cdots \overline{\partial} \sum_{k_2 \in I} \rho_{k_2} U^q_{\varepsilon,k_1...k_q} \equiv 0.
\]

By the Leibnitz rule, we obtain

\[
\sum_{k_q \in I} \rho_{k_q} \overline{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \wedge \left( \overline{\partial} \sum_{k_{q-2} \in I} \rho_{k_{q-2}} \cdots \overline{\partial} \sum_{k_1 \in I} \rho_{k_1} U^q_{\varepsilon,k_1...k_q} \right)
\]

\[
= \sum_{k_q \in I} \rho_{k_q} \sum_{k_{q-1} \in I} \overline{\partial} \rho_{k_{q-1}} \wedge \left( \overline{\partial} \sum_{k_{q-2} \in I} \rho_{k_{q-2}} \cdots \overline{\partial} \sum_{k_1 \in I} \rho_{k_1} U^q_{\varepsilon,k_1...k_q} \right).
\]

Here we used \( \overline{\partial} \overline{\partial} = 0 \). By repeating this procedure, we obtain

\[
\sum_{k_q \in I} \rho_{k_q} \overline{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \wedge \left( \overline{\partial} \sum_{k_{q-2} \in I} \rho_{k_{q-2}} \cdots \overline{\partial} \sum_{k_1 \in I} \rho_{k_1} U^q_{\varepsilon,k_1...k_q} \right)
\]

\[
= \sum_{k_q \in I} \rho_{k_q} \sum_{k_{q-1} \in I} \overline{\partial} \rho_{k_{q-1}} \wedge \sum_{k_{q-2} \in I} \overline{\partial} \rho_{k_{q-2}} \cdots \overline{\partial} \rho_{k_2} \wedge \left( \sum_{k_1 \in I} \overline{\partial} \rho_{k_1} \wedge \sum_{k_1 \in I} U^q_{\varepsilon,k_1...k_q} + \sum_{k_1 \in I} \rho_{k_1} \overline{\partial} U^q_{\varepsilon,k_1...k_q} \right).
\]

This is an \( F \)-valued \((n,q-1)\)-form on \( X \) since \( U^q_{\varepsilon,k_1...k_q} \) does not depend on \( i_0 \). For our aim, we need to estimate the norm of this \( F \)-valued \((n,q-1)\)-form. If we define \( \eta_{k_2...k_q} \) by

\[
\eta_{k_2...k_q} := \rho_{k_q} \overline{\partial} \rho_{k_{q-1}} \wedge \overline{\partial} \rho_{k_{q-2}} \cdots \wedge \overline{\partial} \rho_{k_2},
\]
then by Lemma 3.15 we obtain
\[
\left| \sum_{\{k_1, \ldots, k_q\} \in I} \eta_{k_2 \ldots k_q} \wedge (\bar{\partial} \rho_{k_1} \wedge U^q_{\varepsilon,k_1 \ldots k_q} + \sum_{k_1 \in I} \rho_{k_1} \bar{\partial} U^q_{\varepsilon,k_1 \ldots k_q}) \right|_{h_\varepsilon,\tilde{\omega}} \\
\leq C \left( \left| U^q_{\varepsilon,k_1 \ldots k_q} \right|_{h_\varepsilon,\tilde{\omega}} + \left| \partial U^q_{\varepsilon,k_1 \ldots k_q} \right|_{h_\varepsilon,\tilde{\omega}} \right)
\]
for some positive constant $C > 0$. The norms $\|U^q_{\varepsilon,k_1 \ldots k_q}\|_{h_\varepsilon,\tilde{\omega}}$ and $\|\partial U^q_{\varepsilon,k_1 \ldots k_q}\|_{h_\varepsilon,\tilde{\omega}}$ can be estimated by a constant independent of $\varepsilon$ by the construction (see equality (3.6) and inequality (3.7)). Therefore we have
\[
\bar{\partial} \sum_{k_q \in I} \rho_{k_q} \bar{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \cdots \bar{\partial} \sum_{k_2 \in I} \rho_{k_2} U^q_{\varepsilon,k_1 \ldots k_q} \equiv 0.
\]

[Argument 6]
Finally we consider the first term of the right hand side of (3.14). Our aim is to show
\[
\bar{\partial} \sum_{k_q \in I} \rho_{k_q} \bar{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \cdots \bar{\partial} \sum_{k_1 \in I} \rho_{k_1} U^q_{\varepsilon,k_1 \ldots k_q} \equiv U^\varepsilon.
\]
Since $U^q_{\varepsilon,k_2 \ldots k_q i_0}$ does not depend on $k_1$, we have
\[
(3.15) \quad \bar{\partial} \sum_{k_1 \in I} \rho_{k_1} U^q_{\varepsilon,k_2 \ldots k_q i_0} = \bar{\partial} U^q_{\varepsilon,k_2 \ldots k_q i_0} \\
= U^q_{\varepsilon,k_3 \ldots k_q i_0} + \sum_{\ell=3}^q (-1)^\ell U^{q-1}_{\varepsilon,k_2 \ldots \hat{k}_\ell \ldots k_q i_0} + (-1)^{q+1} U^{q-1}_{\varepsilon,k_2 \ldots k_q}.
\]
The second equality follows from equality (3.6). The second term of the right hand side of (3.15) does not effect $T^\varepsilon$, by the same reason as Argument 2 (Argument 4). Further the third term of the right hand side does not effect $T^\varepsilon$. Indeed, by the same method as Argument 5, we can show
\[
\bar{\partial} \sum_{k_q \in I} \rho_{k_q} \bar{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \cdots \bar{\partial} \sum_{k_2 \in I} \rho_{k_2} U^{q-1}_{\varepsilon,k_2 \ldots k_q} \equiv 0.
\]
In summary, we obtain
\[
\bar{\partial} \sum_{k_q \in I} \rho_{k_q} \bar{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \cdots \bar{\partial} \sum_{k_1 \in I} \rho_{k_1} U^q_{\varepsilon,k_2 \ldots k_q i_0} \equiv \bar{\partial} \sum_{k_q \in I} \rho_{k_q} \bar{\partial} \sum_{k_{q-1} \in I} \rho_{k_{q-1}} \cdots \bar{\partial} \sum_{k_2 \in I} \rho_{k_2} U^{q-1}_{\varepsilon,k_3 \ldots k_q i_0}.
\]
By repeating this procedure, we obtain
\[ \overline{\partial} \sum_{k_q \in I} \rho_{k_q} \overline{\partial} \sum_{k_q-1 \in I} \rho_{k_q-1} \cdots \overline{\partial} \sum_{k_0 \in I} \rho_{k_0} U_{\varepsilon,k_0}^q = \overline{\partial} \sum_{k_q \in I} \rho_{k_q} \overline{\partial} \sum_{k_q-1 \in I} \rho_{k_q-1} U_{\varepsilon,k_q}^2 = \overline{\partial} \sum_{k_q \in I} \rho_{k_q} U_{\varepsilon,k_q}^1 \]

The last equality follows from equality (3.6). By inequality (3.7), there exists a constant \( C \) such that
\[ \| U_{\varepsilon,k}^1 \|_{h\varepsilon,\tilde{\omega}} \leq C, \]
which implies that
\[ \overline{\partial} \sum_{k_q \in I} \rho_{k_q} U_{\varepsilon,k_q}^1 = 0. \]

On the other hand, we have
\[ \overline{\partial} \sum_{k_q \in I} \rho_{k_q} U_{\varepsilon,i_0}^1 = \overline{\partial} U_{\varepsilon,i_0}^1 = U_{i_0} = U_{\varepsilon} \]
on \( B_{i_0} \setminus Z \). This completes the proof.

From Claim 3.11 and 3.13 we can obtain the conclusion of Proposition 3.3. Indeed, if we put \( \alpha_{\varepsilon} := \beta_{\varepsilon} + \tilde{\beta}_{\varepsilon} \), then \( \alpha_{\varepsilon} \) satisfies the properties in Proposition 3.3.

At the end of this step, we prove Lemma 3.14 and 3.15, which were used in the proof of Proposition 3.3.

**Lemma 3.14.** For every \( u \in L_{n,p}^2(B' \setminus Z, F)_{h_{\varepsilon,\tilde{\omega}}} \) with \( \overline{\partial} u = 0 \), there exist \( v \in L_{n-1,p}^2(B' \setminus Z, F)_{h_{\varepsilon,\tilde{\omega}}} \) and a positive constant \( C \) (independent of \( \varepsilon \) and \( u \)) such that
\[ \overline{\partial} v = u, \]
\[ \int_{B' \setminus Z} |v|^2_{h_{\varepsilon,\tilde{\omega}}} \tilde{\omega}^n \leq C \int_{B' \setminus Z} |u|^2_{h_{\varepsilon,\tilde{\omega}}} \tilde{\omega}^n. \]

**Proof.** We can take a smooth function \( \Phi \) on \( B'' \setminus Z \) such that \( \tilde{\omega} = dd^c \Phi \) on \( B'' \setminus Z \), since we may assume that \( B'' \) is sufficiently small. When the metric \( h_{\varepsilon} \) on \( F \) is define by \( H_{\varepsilon} := h_{\varepsilon} e^{-\Phi} \), then the curvature of \( H_{\varepsilon} \) satisfies
\[ \sqrt{-1} \Theta_{H_{\varepsilon}}(F) = \sqrt{-1} \Theta_{h_{\varepsilon}}(F) + dd^c \Phi \]
\[ \geq -\varepsilon \tilde{\omega} + \tilde{\omega} \]
\[ \geq (1 - \varepsilon) \tilde{\omega}. \]
Here we used property (d) of $h_ε$ and property (C) of $\tilde{ω}$. By property (B) of $\tilde{ω}$, the function $Φ$ is a bounded function on $B'$, and thus the $L^2$-norm $\|u\|_{h_ε,\tilde{ω}}$ with respect to $H_ε$ is finite by the assumption of $\|u\|_{h_ε,\tilde{ω}} < \infty$. From the $L^2$-method for the $\tilde{\partial}$-equation (for example, see [Dem82, 4.1. Théorème]), we obtain a solution $v$ of the $\tilde{\partial}$-equation $\tilde{\partial}v = u$ such that

$$\|v\|_{H_ε,\tilde{ω}}^2 \leq \frac{C}{1 - ε} \|u\|_{H_ε,\tilde{ω}}^2$$

for some positive constant $C$. Note that $\tilde{ω}$ is not a complete form on $B' \setminus Z$, but $B' \setminus Z$ admits a complete Kähler form. Therefore we can apply [Dem82, 4.1. Théorème]. Since we consider only small $ε$, we may assume $1/2 \leq (1 - ε)$. We have $\|v\|_{H_ε,\tilde{ω}}^2 \leq C_1 \|u\|_{H_ε,\tilde{ω}}^2$ for some positive constant $C_1$ (independent of $ε$). Further there exist positive constants $C_2, C_3$ such that

$$\|v\|_{h_ε,\tilde{ω}} \leq C_2 \|v\|_{H_ε,\tilde{ω}} \quad \text{and} \quad \|u\|_{H_ε,\tilde{ω}} \leq C_3 \|u\|_{h_ε,\tilde{ω}}$$

since $Φ$ is bounded. This completes the proof.

**Lemma 3.15.** Let $W$ be an $(n, i)$-form on $Y$ and $η$ be a $(0, j)$-form on $X$. Then there exists a positive constant $C$ (depending only $i$ and $j$) such that

$$|η \wedge W|_ω \leq C |η|_ω |W|_\tilde{ω}.$$ 

**Proof.** For any point $y \in Y$, there exists a local coordinate $(z_1, z_2, \ldots, z_n)$ centered at $y$ such that

$$\tilde{ω} = \sum_{j=1}^{n} λ_j dz_j \wedge d\overline{z}_j \quad \text{and} \quad ω = \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \quad \text{at} \ y.$$ 

When we locally write $W$ and $η$ as $W = \sum_{|K|=i} W_K \ dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_K$ and $η = \sum_{|L|=j} η_L \ d\overline{z}_L$, we have

$$|η|^2_ω = \sum_{|L|=j} |η_L|^2 \quad \text{and} \quad |W|^2_\tilde{ω} = \sum_{|K|=i} |W_K|^2 \frac{1}{λ_1 \cdots λ_n} \prod_{m \in K} \frac{1}{λ_m}.$$ 

In particular for any multi-indices $L$ and $K$ with $|L| = j$ and $|K| = i$, we have

$$|η_L| \leq |η|_ω \quad \text{and} \quad |W_K| \leq \frac{1}{(λ_1 \cdots λ_n)^{1/2}} \frac{1}{\prod_{m \in K} λ_m^{1/2}} \leq |W|_\tilde{ω}.$$ 

On the other hand, an easy computation yields

$$|η \wedge W|_ω \leq \sum_{|K|=i, |L|=j} |η_L W_K dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_L \wedge d\overline{z}_K|_\tilde{ω} \leq \sum_{|K|=i, |L|=j} |η_L W_K| \frac{1}{(λ_1 \cdots λ_n)^{1/2}} \frac{1}{\prod_{m \in K \cup L} λ_m^{1/2}}.$$
Since property (C) of $\tilde{\omega}$, the eigenvalue $\lambda_m$ is larger than or equal to 1. It implies

$$|\eta \wedge W|_\tilde{\omega} \leq \sum_{|K|=i, |L|=j} |\eta_L W_K| \frac{1}{(\lambda_1 \cdots \lambda_n)^{1/2}} \prod_{m \in K} \lambda_m^{1/2}$$

$$\leq |\eta|_\omega \sum_{|K|=i, |L|=j} |W_K| \frac{1}{(\lambda_1 \cdots \lambda_n)^{1/2}} \prod_{m \in K} \lambda_m^{1/2}$$

$$\leq |\eta|_\omega \sum_{|K|=i, |L|=j} |W|_\tilde{\omega} = \left( \binom{n}{j} \binom{n}{i} \right) |\eta|_\omega |W|_\tilde{\omega}.$$  

This completes the proof. □

Step 4 (The limit of the harmonic forms)

In this step, we investigate the limit of $u_\varepsilon$ and complete the proof of Theorem 3.1. First we prove the following proposition.

**Proposition 3.16.** There exist $F^{m+1}$-valued $(n, q - 1)$-forms $\gamma_\varepsilon$ on $Y$ with the following properties:

1. $\overline{\partial} \gamma_\varepsilon = su_\varepsilon$.
2. The norm $\|\gamma_\varepsilon\|_{h^{m+1}, \tilde{\omega}}$ is uniformly bounded.

**Proof.** There exists an $F^{m+1}$-valued $(n, q-1)$-form $\gamma$ such that $\overline{\partial} \gamma = su$ and $\|\gamma\|_{h^{m+1}, \tilde{\omega}} < \infty$. (Recall that we are assuming that the cohomology class of $su$ is zero in $H^q(X, K_X \otimes F^{m+1} \otimes \mathcal{I}(h^{m+1}))$.) If we take $\alpha_\varepsilon$ satisfying the properties in Proposition 3.3 and put $\gamma_\varepsilon := -s\alpha_\varepsilon + \gamma$, then we have $\overline{\partial} \gamma_\varepsilon = su_\varepsilon$. Further an easy computation yields

$$\|\gamma_\varepsilon\|_{h^{m+1}, \tilde{\omega}} \leq \|s\alpha_\varepsilon\|_{h^{m+1}, \tilde{\omega}} + \|\gamma\|_{h^{m+1}, \tilde{\omega}}$$

$$\leq \sup_X |s|_{h^m} \|\alpha_\varepsilon\|_{h^{m}, \tilde{\omega}} + \|\gamma\|_{h^{m+1}, \tilde{\omega}}.$$  

Note that we have $\|\gamma\|_{h^{m+1}, \tilde{\omega}} \leq \|\gamma\|_{h^{m+1}, \tilde{\omega}} < \infty$. Since the norm $\|\alpha_\varepsilon\|_{h^{m+1}, \tilde{\omega}}$ is uniformly bounded, the right hand side can be estimated by a constant independent of $\varepsilon$. □

We consider the limit of the norm $\|su_\varepsilon\|_{h^{m+1}, \tilde{\omega}}$.

**Proposition 3.17.** The norm $\|su_\varepsilon\|_{h^{m+1}, \tilde{\omega}}$ converges to zero as letting $\varepsilon$ go to zero.

**Proof.** By taking $\gamma_\varepsilon \in L^{n,q-1}_{(2)}(Y, F^{m+1})_{h^{m+1}, \tilde{\omega}}$ satisfying the properties in Proposition 3.16, we obtain

$$\|su_\varepsilon\|_{h^{m+1}, \tilde{\omega}}^2 = \langle su_\varepsilon, \overline{\partial} \gamma_\varepsilon \rangle_{h^{m+1}, \tilde{\omega}}$$

$$= \langle D^{m+1}_{h^{m+1}} su_\varepsilon, \gamma_\varepsilon \rangle_{h^{m+1}, \tilde{\omega}}$$

$$\leq \|D^{m+1}_{h^{m+1}} su_\varepsilon\|_{h^{m+1}, \tilde{\omega}} \|\gamma_\varepsilon\|_{h^{m+1}, \tilde{\omega}}.$$  

By Proposition \[3.16\] the norm \(\|\gamma_{\varepsilon}\|_{h_{\varepsilon}^{m+1},\tilde{\omega}}\) is uniformly bounded. On the other hand, the norm \(\|D_{h_{\varepsilon}^{m+1},\tilde{\omega}}^n u_{\varepsilon}\|_{h_{\varepsilon}^{m+1},\tilde{\omega}}\) converges to zero by Proposition \[3.2\] Therefore the norm \(\|su_{\varepsilon}\|_{h_\varepsilon,\tilde{\omega}}\) also converges to zero.

Fix a small number \(\varepsilon_0 > 0\). Then for any positive number \(\varepsilon\) with \(0 < \varepsilon < \varepsilon_0\), by property (b) of \(h_\varepsilon\), we obtain

\[
\|u_\varepsilon\|_{h_{\varepsilon_0,\tilde{\omega}}} \leq \|u_\varepsilon\|_{h_\varepsilon,\tilde{\omega}} \leq \|u\|_{h,\omega}.
\]

It says that the norm of \(u_\varepsilon\) with respect to \(h_{\varepsilon_0}\) is uniformly bounded. Therefore there exists a subsequence of \(\{u_\varepsilon\}_{\varepsilon>0}\) that converges to \(\alpha \in L^{n,q}_2(Y,F)_{h_{\varepsilon_0},\tilde{\omega}}\) with respect to the weak \(L^2\)-topology. For simplicity, we denote this subsequence by the same notation \(u_\varepsilon\). Then we prove the following proposition.

**Proposition 3.18.** The weak limit \(\alpha\) of \(\{u_\varepsilon\}_{\varepsilon>0}\) in \(L^{n,q}_2(Y,F)_{h_{\varepsilon_0},\tilde{\omega}}\) is zero.

**Proof.** For every positive number \(\delta > 0\), we define the subset \(A_\delta\) of \(Y\) by \(A_\delta := \{x \in Y \mid |s|_{h_{\varepsilon_0}}^2 > \delta\}\). Since the weight \(\varphi_{\varepsilon_0}\) of \(h_{\varepsilon_0}\) is upper semi-continuous, \(|s|_{h_{\varepsilon_0}}^2\) is lower semi-continuous, and thus \(A_\delta\) is an open set of \(Y\). By an easy computation, we have

\[
\|su_\varepsilon\|_{h_{\varepsilon_0},\tilde{\omega}}^2 \geq \int_{A_\delta} |s|_{h_{\varepsilon_0}}^2 |u_\varepsilon|_{h_{\varepsilon_0},\tilde{\omega}}^2 \tilde{\omega}^n \geq \delta \int_{A_\delta} |u_\varepsilon|^2_{h_{\varepsilon_0},\tilde{\omega}} \tilde{\omega}^n \geq 0
\]

for any \(\delta > 0\). Since the left hand side converges to zero, the norm \(\|u_\varepsilon\|_{h_{\varepsilon_0},\tilde{\omega},A_\delta}\) on \(A_\delta\) also converges to zero. Notice that \(u_\varepsilon|_{A_\delta}\) converges to \(\alpha|_{A_\delta}\) with respect to the weak \(L^2\)-topology in \(L^{n,q}_2(A_\delta,F)_{h_{\varepsilon_0},\tilde{\omega}}\). Here \(u_\varepsilon|_{A_\delta}\) (resp. \(\alpha|_{A_\delta}\)) denotes the restriction of \(u_\varepsilon\) (resp. \(\alpha\)) to \(A_\delta\). Indeed for every \(\gamma \in L^{n,q}_2(A_\delta,F)_{h_{\varepsilon_0},\tilde{\omega}}\), the inner product \(\langle u_\varepsilon|_{A_\delta}, \gamma \rangle_{A_\delta} = \langle u_\varepsilon, \gamma \rangle_Y\) converges to \(\langle \alpha, \tilde{\gamma} \rangle_Y = \langle \alpha|_{A_\delta}, \gamma \rangle_{A_\delta}\). Here \(\tilde{\gamma}\) denotes the zero extension of \(\gamma\) to \(Y\). Since \(u_\varepsilon|_{A_\delta}\) converges to \(\alpha|_{A_\delta}\), we obtain

\[
\|\alpha|_{A_\delta\varepsilon}\|_{h_{\varepsilon_0},\tilde{\omega},A_\delta} \leq \liminf_{\varepsilon\to 0} \|u_\varepsilon|_{A_\delta}\|_{h_{\varepsilon_0},\tilde{\omega},A_\delta} = 0.
\]

(Recall the norm of the weak limit can be estimated by the limit inferior of the norms of sequences.) Therefore we have \(\alpha|_{A_\delta} = 0\) for any \(\delta > 0\). By the definition of \(A_\delta\), the union of \(\{A_\delta\}_{\delta>0}\) is equal to \(Y = X \setminus Z\), which asserts that the weak limit \(\alpha\) is zero on \(Y\). □

By using Proposition \[3.18\] we complete the proof of Theorem \[3.1\] By the definition of \(u_\varepsilon\), we have

\[
u = u_\varepsilon + \tilde{\partial}v_\varepsilon.
\]

Proposition \[3.18\] says that \(\tilde{\partial}v_\varepsilon\) converges to \(u\) with respect to the weak \(L^2\)-topology. Then it is easy to see that \(u\) is a \(\tilde{\partial}\)-exact form (that is, \(u \in \text{Im} \tilde{\partial}\)). This is because the subspace \(\text{Im} \tilde{\partial}\) is closed in \(L^{n,q}_2(Y,F)_{h_{\varepsilon_0},\tilde{\omega}}\) with respect to the weak \(L^2\)-topology. Indeed, for every
\[
\gamma = \gamma_1 + D_{h_{\xi_0}}^{n,0} \gamma_2 \in H^{n,q}(F)_{h_{\xi_0}} \oplus \text{Im} D_{h_{\xi_0}}^{n,0},
\]
we have \( \langle \langle u, \gamma \rangle \rangle = \lim_{\varepsilon \to 0} \langle \langle \bar{\vartheta}_{v, \varepsilon}, \gamma_1 + D_{h_{\xi_0}}^{n,0} \gamma_2 \rangle \rangle = 0. \)
Therefore we can conclude \( u \in \text{Im} \bar{\vartheta}. \)

In summary, we proved that \( u \) is a \( \bar{\vartheta} \)-exact form in \( L^{n,q}(Y, F)_{h_{\xi_0}, \bar{\omega}} \), which says that the cohomology class \( \{ u \} \) of \( u \) is zero in \( H^i(X, K_X \otimes F \otimes I(h_{\xi_0})) \). By property (c), we obtain the conclusion of Theorem 3.1. \( \square \)

3.2. Proof of Corollary 1.6. In this subsection, we give a proof of Theorem 3.21 which is obtained from Theorem 1.3. Theorem 3.21 leads to Corollary 1.6. First we give the following definition.

Definition 3.19. Let \( F \) be a line bundle on a compact complex manifold \( X \) and \( h \) be a (singular) metric on \( F \).

(1) We denote by \( H^0_{\text{bdd}, h}(X, F) \), the space of the sections of \( F \) with bounded norm. Namely,
\[
H^0_{\text{bdd}, h}(X, F) := \{ s \in H(X, F) \mid \sup_X |s|_h < \infty \}.
\]

(2) The generalized Kodaira dimension \( \kappa_{\text{bdd}}(F, h) \) of \( (F, h) \) is defined to be \( -\infty \) if \( H^0_{\text{bdd}, h^k}(X, F^k) = 0 \) for any \( k > 0 \). Otherwise, \( \kappa_{\text{bdd}}(F, h) \) is defined by
\[
\kappa_{\text{bdd}}(F, h) := \sup \{ m \in \mathbb{Z} \mid \limsup_{k \to \infty} \dim H^0_{\text{bdd}, h^k}(X, F^k)/k^m > 0 \}.
\]

Remark 3.20. If \( h_{\text{min}} \) is a metric with minimal singularities on \( F \), the norm \( |s|_{h_{\text{min}}} \) of every section \( s \in H^0(X, F^m) \) is bounded on \( X \). (See [Dem] or [Mat13-B] for the definition of metrics with minimal singularities.) It implies that \( H^0_{\text{bdd}, h_{\text{min}}}(X, F^m) \) is isomorphic to \( H^0(X, F^m) \) for any \( m \geq 0 \). In particular, \( \kappa_{\text{bdd}}(F, h_{\text{min}}) \) agrees with the usual Kodaira dimension \( \kappa(F) \).

At the end of this subsection, we prove the following theorem. The following theorem and the above remark lead to Corollary 1.6.

Theorem 3.21. Let \( F \) be a line bundle on a smooth projective variety \( X \) of dimension \( n \) and \( h \) be a (singular) metric with (semi)-positive curvature on \( F \). Then
\[
H^q(X, K_X \otimes F \otimes I(h)) = 0 \quad \text{for any } q > n - \kappa_{\text{bdd}}(F, h).
\]

Proof. For a contradiction, we assume that there exists a non-zero cohomology class \( \alpha \in H^q(X, K_X \otimes F \otimes I(h)) \). If sections \( \{ s_i \}_{i=1}^N \) in \( H^0_{\text{bdd}, h^m}(X, F^m) \) are linearly independent, then \( \{ s_i \alpha \}_{i=1}^N \) are also linearly independent in \( H^q(X, K_X \otimes F^{m+1} \otimes I(h^{m+1})) \). Indeed, if \( \sum_{i=1}^N c_i s_i \alpha = 0 \) for some \( c_i \in \mathbb{C} \), then we can obtain \( \sum_{i=1}^N c_i s_i = 0 \) by Theorem 1.3. Since \( \{ s_i \}_{i=1}^N \) are linearly independent, we have \( c_i = 0 \) for any \( i = 1, 2, \ldots, N \). It implies
\[
\dim H^0_{\text{bdd}, h^m}(X, F^m) \leq \dim H^q(X, K_X \otimes F^{m+1} \otimes I(h^{m+1})).
\]
On the other hand, by [Mat13-A, Theorem 4.1] we have
\[
\dim H^q(X, K_X \otimes F^m \otimes I(h^m)) = O(m^{n-q}) \quad \text{as } m \to \infty
\]
for any \( q \geq 0 \) (cf. [Dem, (6.18) Lemma]). If \( q > n - \kappa_{\text{bdd}}(F, h) \), it is a contradiction. \( \square \)
In this section, we study the space of the cochains with the topology induced by the local $L^2$-norms defined by (3.10), which played an important role in Step 3 of Section 3. In order to prove Theorem 4.3, we first recall the following result on holomorphic functions, which can be proved by the division theorem. See [GR, Section D, Chapter II] for the proof.

**Theorem 4.1.** ([GR, 2. Theorem, Section D, Chapter II]) Let $G_1, G_2, \ldots, G_N$ be holomorphic functions on an open set $B$ in $\mathbb{C}^n$ and $\mathcal{I} \subset \mathcal{O}_B$ be an ideal sheaf. Assume that holomorphic functions $\{G_i\}_{i=1}^N$ generate the stalk $\mathcal{I}_p$ of $\mathcal{I}$ at $p \in B$. Then there exist a neighborhood $L_p \Subset B$ of $p$ and a positive constant $C_p > 0$ with the following properties:

For every holomorphic function $F$ whose germ at $p$ belongs to $\mathcal{I}_p$, there exist holomorphic functions $\{h_j\}_{j=1}^N$ on $L_p$ such that

$$F = \sum_{j=1}^N h_j G_j \quad \text{and} \quad \sup_{L_p} |h_j| \leq C_p \sup_{L_p} |F|.$$  

By using this theorem, we prove the following lemma. In his paper [Cao12], Cao proved the former conclusion of the lemma when a quasi-psh function $\varphi$ has analytic singularities. For our purpose, we need a generalization of his result and the stronger conclusion (the latter conclusion of the lemma).

**Lemma 4.2.** Let $\varphi$ be a (quasi-) psh function on an open set $B$ in $\mathbb{C}^n$ and $G_1, G_2, \ldots, G_N$ be holomorphic functions on $B$ that generate the stalk of the multiplier ideal sheaf $\mathcal{I}(\varphi)$ at every point in $B$. Consider a sequence of holomorphic functions $\{f_k\}_{k=1}^\infty$ satisfying the following properties:

1. $f_k$ belongs to $H^0(B, \mathcal{I}(\varphi))$ (that is, $|f_k|e^{-\varphi}$ is locally $L^2$-integrable on $B$).
2. $\{f_k\}_{k=1}^\infty$ uniformly converges to $f$ on every relatively compact set in $B$.

Then the limit $f$ belongs to $H^0(B, \mathcal{I}(\varphi))$. Moreover, for every relatively compact set $K \Subset B$, the (local) $L^2$-norm

$$\int_K |f_k - f|^2 e^{-2\varphi}$$

converges to zero as letting $k$ go to infinity.

**Proof.** For an arbitrary point $p \in B$, there exist a neighborhood $L_p \Subset B$ of $p$ and a positive constant $C_p$ with the properties in Theorem 4.1. Since the germ of $f_k$ belongs to the stalk $\mathcal{I}(\varphi)_p$, there exist holomorphic functions $\{h_{k,j}\}_{j=1}^N$ on $L_p$ such that

$$f_k = \sum_{j=1}^N h_{k,j} G_j \quad \text{and} \quad \sup_{L_p} |h_{k,j}| \leq C_p \sup_{L_p} |f_k|.$$  

The sup-norm $\sup_{L_p} |f_k|$ on $L_p$ is uniformly bounded by property (2). The above inequality implies that the sup-norm $\sup_{L_p} |h_{n,j}|$ is also uniformly bounded, and thus by Montel’s
There exists a subsequence \( \{ h_{k,i,j} \}_{\ell=1}^{\infty} \) that uniformly converges to a holomorphic function \( h_j \) on every relatively compact set in \( L_p \). For every point \( x \) in \( L_p \) we have

\[
f(x) = \lim_{\ell \to \infty} f_{k}\ell(x)
= \lim_{\ell \to \infty} \sum_{j=1}^{N} h_{k,i,j}(x)G_j(x)
= \sum_{j=1}^{N} h_j(x)G_j(x).
\]

Therefore the germ of \( f \) belongs to \( \mathcal{I}(\varphi)_p \) since the germ of \( G_j \) belongs to \( \mathcal{I}(\varphi)_p \).

Finally, we prove the latter conclusion. We have already known that the germ of \( f_k - f \) belongs to \( \mathcal{I}(\varphi)_p \). By Theorem 4.1, there exist a relatively compact set \( L_p \subset B \) and a positive constant \( C_p \) and holomorphic functions \( \{ g_{k,j} \}_{j=1}^{N} \) on \( L_p \) such that

\[
f_k(x) - f(x) = \sum_{j=1}^{N} g_{k,j}(x) G_j(x)
\]

\[
\sup_{L_p} |g_{k,j}| \leq C_p \sup_{L_p} |f_k - f| \to 0.
\]

On the other hand, an easy computation yields

\[
\int_{L_p} |f_k - f|^2 e^{-2\varphi} \leq \int_{L_p} \left( \sum_{j=1}^{N} |g_{k,j}|^2 \right) \left( \sum_{j=1}^{N} |G_j|^2 \right) e^{-2\varphi}
\]

\[
\leq \left( \sum_{j=1}^{N} \sup_{L_p} |g_{k,j}|^2 \right) \int_{L_p} \sum_{j=1}^{N} |G_j|^2 e^{-2\varphi}.
\]

The integral of \( |G_j|^2 e^{-2\varphi} \) is finite and \( g_{k,j} \) uniformly converges to 0 on \( L_p \). Therefore the above left hand side converges to zero.

For a given relatively compact set \( K \subset B \), there are finite points \( \{ p_{\nu} \}_{\nu=1}^{m} \) and their neighborhoods \( \{ L_{p_{\nu}} \}_{\nu=1}^{m} \) that cover \( K \). Then we have

\[
\int_{K} |f_k - f|^2 e^{-2\varphi} \leq \sum_{\nu=1}^{m} \int_{L_{p_{\nu}}} |f_k - f|^2 e^{-2\varphi} \to 0.
\]

This completes the proof. \( \square \)

To prove Theorem 4.3, we recall the notation on the space of the cochains. Let \( F \) be a line bundle with a singular metric \( h \) on a compact complex manifold \( X \), and let \( \mathcal{U} := \{ B_i \}_{i \in I} \) be a Stein finite cover of \( X \) with the following properties:

1. \( F \) admits a local trivialization on \( B_i \).
2. There are holomorphic functions on \( B_i \) that generate the stalk of the multiplier ideal sheaf \( \mathcal{I}(h) \) at every point in \( B_i \).
We can take such cover since a multiplier ideal sheaf is a coherent sheaf. Let \( C^p(\mathcal{U}, F \otimes \mathcal{I}(h)) \) be the space of the \( p \)-cochains with coefficients in \( F \otimes \mathcal{I}(h) \). For a \( p \)-cochain \( f = \{ f_{i_0...i_p} \} \in C^p(\mathcal{U}, F \otimes \mathcal{I}(h)) \), we regard \( f_{i_0...i_p} \) as the holomorphic function under the trivialization of \( F \) on \( B_i \). Then the family of the semi-norms \( p_{K_{i_0...i_p}}(\cdot) \) is defined by

\[
p_{K_{i_0...i_p}}(f)^2 := \int_{K_{i_0...i_p}} |f_{i_0...i_p}|^2_h
\]

for a relatively compact set \( K_{i_0...i_p} \subseteq B_{i_0...i_p} \). The family of these semi-norms induces the topology of \( C^p(\mathcal{U}, F \otimes \mathcal{I}(h)) \). At the end of this section, we show that \( C^p(\mathcal{U}, F \otimes \mathcal{I}(h)) \) is a Fréchet space.

**Theorem 4.3.** In the above situation, the space of the \( p \)-cochains \( C^p(\mathcal{U}, F \otimes \mathcal{I}(h)) \) is a Fréchet space.

**Proof.** Let \( \{ \{ f_{k,i_0...i_p} \} \}_{k=1}^\infty \) be a Cauchy sequence in \( C^p(\mathcal{U}, F \otimes \mathcal{I}(h)) \). For simplicity, we put \( f_k := f_{k,i_0...i_p} \) and \( B := B_{i_0...i_p} \), and we regard \( f_k \) as the holomorphic function on \( B \). For the proof, it is sufficient to show that there exists a holomorphic function \( f \) on \( B \) such that

\[
\int_K |f_k - f|^2_h \to 0
\]

for every relatively compact set \( K \subseteq B \).

Take a relatively compact set \( K \subseteq B \). Since \( \{ f_k \}_{k=1}^\infty \) is a Cauchy sequence with respect to the semi-norms, the \( L^2 \)-norm \( \int_K |f_k|^2_h \) of \( f_k \) on \( K \) is uniformly bounded. Since the local weight \( \varphi \) of \( h \) is quasi-psh, \( \varphi \) is upper semi-continuous. In particular \( \varphi \) is bounded above, and thus the \( L^2 \)-norm \( \int_K |f|^2 \) is also uniformly bounded. By Montel’s theorem, there exists a subsequence \( \{ f_{k_\ell} \}_{\ell=1}^\infty \) of \( \{ f_k \}_{k=1}^\infty \) that uniformly converges to a holomorphic functions \( f \) on every relatively compact set in \( B \). This subsequence \( \{ f_{k_\ell} \}_{\ell=1}^\infty \) satisfies the assumptions of Lemma 4.2. By Lemma 4.2 we know that the limit \( f \) also belongs to \( \mathcal{I}(h) \). Moreover, we have

\[
p_K(f_{k_\ell} - f) = \int_K |f_{k_\ell} - f|^2_h \to 0
\]

for every relatively compact set \( K \subseteq B \). Since \( \{ f_k \}_{k=1}^\infty \) is a Cauchy sequence, the semi-norm \( p_K(f_k - f) \) also converges to zero. \( \Box \)

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