A RELATION BETWEEN THE KAUFFMAN AND
THE HOMFLY POLYNOMIALS FOR TORUS KNOTS

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ABSTRACT

Polynomial invariants corresponding to the fundamental representation of the
gauge group $SO(N)$ are computed for arbitrary torus knots in the framework of
Chern-Simons gauge theory making use of knot operators. As a result, a formula
which relates the Kauffman and the HOMFLY polynomials for torus knots is pre-
sented.

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1. Introduction

Knot operators [1,2] have shown to be a powerful tool in Chern-Simons gauge theory [3] to obtain general expressions for knot invariants related to torus knots and links. Computations by other methods [4,5,6,7,8,9,10] have been successful for specific knots but not to obtain general expressions for knot sequences as torus knots. Knot operators have been used in [11] where a formula for the invariants for torus knots and links carrying arbitrary representations of the gauge group $SU(2)$ has been presented. For the fundamental representation it covers the case of the Jones polynomial [12,13], while for higher dimensional representations it covers the case of the Akutsu-Wadati polynomials [14]. They have also been used in [15] where a formula for the HOMFLY polynomial [16,13] for arbitrary torus knots and links has been presented. For the case of torus knots the formula obtained in [15] for the HOMFLY polynomial coincides with the one presented by Jones in [13], and later reobtained using quantum groups by Rosso and Jones in [17].

Knot operators were constructed in [1,2] for the gauge group $SU(N)$. In this paper we will present the form of these operators for arbitrary simple compact groups. Then, we will use them to compute knot invariants for arbitrary torus knots carrying the fundamental representation of $SO(N)$. As a consequence a formula for the Kauffman polynomial [18] for this type of knots is obtained. This formula turns out to be equivalent to the one obtained in [19] using a different method. Comparing this formula for the Kauffman polynomial to the one obtained in [13,17,15] for the HOMFLY polynomial we obtain a rather simple relation between them. Denoting the HOMFLY polynomial for a torus knot \{n,m\} (n and m are coprime integers, \((n,m) = 1\)) in terms of its standard variables a and z by $P_{n,m}(a,z)$, and the Kauffman polynomial (Dubrovnik version) by $Y_{n,m}(a,z)$, we find:

\[
P_{n,m}(a,z) = \frac{1}{2}(Y_{n,m}(a,z) + Y_{n,m}(a,-z)) + \frac{z}{2(a-a^{-1})}(Y_{n,m}(a,z) - Y_{n,m}(a,-z)).
\]

(1.1)

This is the main new result presented in this paper. The existence of a formula
like (1.1) is rather remarkable. In general the Kauffman polynomial contains very many more terms than the HOMFLY polynomial. This means that an important cancellation of terms occurs in (1.1). Notice also that this formula indicates that at least for torus knots the Kauffman polynomial distinguishes more knots than the HOMFLY polynomial. Two torus knots which have the same Kauffman polynomial also have the same HOMFLY polynomial but it might happen that two torus knots have the same HOMFLY polynomial but different Kauffman polynomials. At least for torus knots one can state that the Kauffman polynomial is more fundamental than the HOMFLY polynomial.

As a byproduct of formula (1.1) it will be obtained in sect. 4 a formula for the Alexander-Conway polynomial in terms of the first derivative at $a = 1$ of the corresponding Kauffman polynomial.

The paper is organized as follows. In sect. 2 we present the generalization of the construction of knot operators based on Chern-Simons gauge theory for an arbitrary simple compact gauge group. In sect. 3 we calculate the Kauffman polynomial for torus knots obtaining a result in full agreement with a previous calculation. In sect. 4 we prove formula (1.1) and derive a formula for the Alexander-Conway polynomial for torus knots in terms of the corresponding Kauffman polynomial. In sect. 5 we add final comments and remarks on our results. The conventions used in this paper are conveniently compiled in an appendix.
2. Knot Operators for arbitrary simple gauge group

In this section we present the generalization of the operator formalism developed in [1,2] for an arbitrary simple compact gauge group and the construction of the corresponding knot operators. We begin introducing Chern-Simons gauge theory. Let $M$ be a boundaryless three-dimensional manifold and let $A$ be a connection associated to a principal $G$-bundle for some simple Lie group $G$. The action which defines Chern-Simons gauge theory has the form:

$$S_k(A) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

(2.1)

where $\text{Tr}$ is the trace over the fundamental representation of the simple gauge group $G$, and, for the moment, $k$ is an arbitrary real number. Under a gauge transformation,

$$A \rightarrow g^{-1}dg + g^{-1}Ag,$$

(2.2)

the action (2.1) transforms as:

$$S_k(A) \rightarrow S_k(A) - \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg).$$

(2.3)

The last quantity is closely related to the winding number of the map $g : M \rightarrow G$, which is defined as:

$$\Upsilon(g) = \frac{1}{48\pi} \int_M \epsilon^{\mu\nu\rho} f^{abc} \psi^2 C_\mu^a C_\nu^b C_\rho^c,$$

(2.4)

where $C_\mu^a$ is given by,

$$g^{-1}\partial_\mu g = C_\mu^a T^a,$$

(2.5)

being $T^a$, $a = 1, \ldots, \text{dim}(G)$, the generators of the simple group $G$. In (2.4) $f^{abc}$ are the structure constants corresponding to this group, and $\psi^2$ the squared length.
of the longest simple root of $G$. The quantity $\Upsilon(g)$ in (2.4) is always $2\pi$ times an integer [20]. To study its relation to the second term on the right hand side of (2.3) we must take into account that the generators can be chosen in such a way that,

$$\text{Tr}(T^a T^b) = -y \psi^2 \delta^{ab}, \quad (2.6)$$

where $y$ is the Dynkin index of the fundamental representation of the simple group $G$. It is clear from (2.6) that this index is independent of the scale chosen for the gauge group generators. From (2.4) and (2.6) follows that (2.3) can be written as:

$$S_k(A) \rightarrow S_k(A) - 2yk \Upsilon(g). \quad (2.7)$$

The values of $y$ for $SU(N)$ and $SO(N)$ are $1/2$ and $1$, respectively. For other groups $y$ is a half-integer or an integer (see the Appendix). Therefore, if $k$ is an integer the action (2.1) changes into $2\pi$ times an integer and the exponential $\exp(i S_k(M))$ is gauge invariant. Furthermore, for the case of $SO(N)$ is enough to require $k$ to be half-integer. Defining:

$$x = 2yk, \quad (2.8)$$

one has, in general, the following quantization condition:

$$x = 2yk \in \mathbb{Z}. \quad (2.9)$$

For values of $k$ satisfying the quantization condition (2.9) the partition function of the theory is defined as,

$$Z_k(M) = \int [DA]_M \exp(i S_k(A)), \quad (2.10)$$

where the functional integration is over gauge non-equivalent connections. This partition function is a topological invariant because the action $S_k(A)$ does not
depend on the metric on $M$. Other topological-invariant quantities are con-
structed introducing operators in the integrand of the functional integral present in
(2.10). These operators must be gauge-invariant and metric-independent to lead to
topological-invariant quantities. Wilson lines constitute an important class of these
operators. Let $\gamma$ be a close curve in $M$ and let $R$ be an irreducible representation
of the gauge group. The Wilson line operator associated to $\gamma$ and $R$ is:

$$W_{\gamma}^{R}(A) = \text{Tr}_{R}(P \exp \int_{\gamma} A), \quad (2.11)$$

where $P$ denotes a path-ordered product along $\gamma$. We will be interested in comput-
ing the vacuum expectation values of products of these operators, i.e., functional
integrations of the form:

$$\int [DA]_{M} (\prod_{i=1}^{n} W_{\gamma_i}^{R_i}) \exp(iS_k(A)). \quad (2.12)$$

In order to generalize the operator formalism developped in [1,2] let us assume
that there are some Wilson lines $L_i$ on the manifold $M$. We will perform a Heegaard
splitting on $M$ in such a way that no Wilson line is cut. The case in which this does
not happen has been studied in [21]. In this formalism, the vacuum expectation
values are expressed as an inner product of states in a Hilbert space. These states
are defined as functional integrals over configurations on each of the $g$-handlebodies
$M_1$ and $M_2$ which result from the Heegaard splitting. In order to construct these
states let us introduce complex local coordinates on the Riemman surface $\Sigma$ which
Corresponds to the common boundary of $M_1$ and $M_2$,

$$z = \sigma_1 + i\sigma_2, \quad \bar{z} = \sigma_1 - i\sigma_2, \quad (2.13)$$

and let us use complex components for the part of the gauge connection parallel
to the surface $\Sigma$:

$$A_z = \frac{1}{2}(A_1 - iA_2), \quad A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2). \quad (2.14)$$

Our aim is to define wave functionals which will be functional integrals over field
configurations in the \( g \)-handlebodies resulting after the splitting with the value of \( A_z \) fixed at the boundary. The inner product will be implemented as an integration over the components \( A_z \) and \( \bar{A}_z \) on the common boundary.

Following \([1,2]\) we will use the formalism of the holomorphic quantization. Wave functionals associated to the \( g \)-handle body \( M_1 \) enclosing \( p \) Wilson lines are defined as,

\[
\Psi_1[A_z] = \int [DA]_{M_1} \left( \prod_{i=1}^{p} W^{\gamma_i}_{R_i} \right) \exp \left( iS_k(A) - \frac{k}{2\pi} \int_\Sigma \text{Tr}(A_z A_{\bar{z}}) \right),
\]

(2.15)

where \([DA]_{M_1}\) represents the functional integration measure over gauge orbits such that \( A_{\bar{z}} \) is fixed at the boundary \( \Sigma \). A similar expression defines the wave functional \( \Psi_2[A_z] \) for the \( g \)-handle body \( M_2 \). The vacuum expectation value (2.12) is given by the following inner product:

\[
(\Psi_2 | \Psi_1) = \int [DA_z D\bar{A}_z]_{\Sigma} \exp \left( \frac{k}{\pi} \int_\Sigma \text{Tr}(A_z A_{\bar{z}}) \right) \Psi_2[A_{\bar{z}}] \Psi_1[A_z].
\]

(2.16)

Let us recall a few important facts related to this formalism. Boundary terms like the one in (2.15) are introduced to make the wave functional well defined, \( i.e. \), depending on \( A_{\bar{z}} \) on the boundary \( \Sigma \). Also, such a term is the one responsible for having a functional integral in (2.15) which is extremal for gauge configurations such that the field strength of \( A \) vanishes in the interior of \( M_1 \).

The commutation relations of the canonically conjugate fields \( A_z \) and \( A_{\bar{z}} \) on \( \Sigma \) can be read from the exponent of the exponential inserted in (2.16). They take the form:

\[
[A^a_z(\sigma), A^b_{\bar{z}}(\sigma')] = \frac{\pi}{2y_\psi^2k} \delta^{ab} \delta^{(2)}(\sigma - \sigma').
\]

(2.17)

Our next step is to compute explicitly the wave functionals (2.15) in order to obtain a description of the Hilbert space of the theory. To carry this out we will use standard parametrizations of the gauge fields \( A_z \) and \( A_{\bar{z}} \) on the Riemann surface \( \Sigma \). We will address the situations corresponding to genus zero and one.
2.1. Genus-zero handlebody

Let $M_1$ be a solid ball and $\Sigma = S^2$ its boundary. On $S^2$ the fields $A_z$ and $A_{\bar{z}}$ can be parametrized as:

$$ A_{\bar{z}} = u^{-1} \partial_{\bar{z}} u, \quad A_z = \bar{u}^{-1} \partial_z \bar{u}, \quad (2.18) $$

where $u$ is a single-valued map $u : S^2 \to G^c$, being $G^c$ the complexification of $G$. Since $A_{\bar{z}}^\dagger = -A_z$ one has that $u^\dagger = \bar{u}^{-1}$. The gauge transformations (2.2) take the following form for fields on the surface $S^2$:

$$ A_{\bar{z}} \to g^{-1} \partial_{\bar{z}} g + g^{-1} A_{\bar{z}} g, \quad A_z \to g^{-1} \partial_z g + g^{-1} A_z g, \quad (2.19) $$

where $g$ is map $g : S^2 \to G$. In the parametrization (2.18) these gauge transformations take the simple form $u \to ug$.

The next step is to express the measure $[\mathcal{D}A_z \mathcal{D}A_{\bar{z}}]|_{S^2}$ in (2.16) in terms of an infinite product of de Haar measures of $G^c$. This involves the computation of a Jacobian which takes the form [22,23]:

$$ [\mathcal{D}A_z \mathcal{D}A_{\bar{z}}]|_{S^2} = \exp \left( \frac{g^\vee}{2y} \Gamma(u \bar{u}^{-1}) \right) \det \partial_z \partial_{\bar{z}} |dud\bar{u}|, \quad (2.20) $$

where $g^\vee$ is the dual Coxeter number of $G$ and $\Gamma(\alpha)$ is the Wess-Zumino-Witten action [20]:

$$ \Gamma(u) = \frac{1}{2\pi} \int_{S^2} \text{Tr}(\alpha^{-1} \partial_z \alpha \alpha^{-1} \partial_{\bar{z}} \alpha) $$

$$ + \frac{i}{12\pi} \int_{M_1} \epsilon^{\mu\nu\rho} \text{Tr}(\tilde{\alpha}^{-1} \partial_\mu \tilde{\alpha} \tilde{\alpha}^{-1} \partial_\nu \tilde{\alpha} \tilde{\alpha}^{-1} \partial_\rho \tilde{\alpha}). \quad (2.21) $$

In (2.21) $\alpha$ is a map $\alpha : S^2 \to G$, and $\tilde{\alpha}$ is one of the extensions of this map to the interior of the solid ball $M_1$. The measure (2.21) does not depend on the choice of
extension of the map $\alpha$. For different choices, the resulting Wess-Zumino-Witten actions differ by $2iy$ times an integral of the form (2.4) where $M = S^2$. Therefore, since $g^\vee$ is always an integer, the measure (2.20) is well defined. It is also important to remark that this measure is gauge invariant.

In order to write wave functionals in terms of $u$ and $\bar{u}$ one would like to factor the measure (2.20) appropriately. This is however not obvious due to the Polyakov-Wiegmann condition [22] satisfied by the Wess-Zumino-Witten action:

$$\Gamma(\alpha, \beta) = \Gamma(\alpha) + \Gamma(\beta) + \langle \alpha, \beta \rangle,$$  

(2.22)

where we have introduced,

$$\langle \alpha, \beta \rangle = \frac{1}{\pi} \text{Tr}(\alpha^{-1} \partial_z \alpha \partial_z \beta \beta^{-1}).$$  

(2.23)

As in [1,2], we will solve this problem making the following choice of measure on the boundaries of $M_1$ and $M_2$: take the measure (2.20) without those factors that only depend on the gauge variables which are not being integrated over in the path integral representation of the wave functional. Working in a gauge where the radial component of $A$ on $S^2$ vanishes this amounts to choose:

$$\exp \left( \frac{g^\vee}{2y} (\Gamma(\bar{u}^{-1}) + \langle u, \bar{u}^{-1} \rangle) \right) \, d\bar{u} \quad \text{for} \quad \Psi_1,$$  

(2.24)

and,

$$\exp \left( \frac{g^\vee}{2y} (\Gamma(u) + \langle u, \bar{u}^{-1} \rangle) \right) \, du \quad \text{for} \quad \Psi_2.$$

(2.25)

In doing this an extra factor $\exp(\langle u, \bar{u}^{-1} \rangle)$ has been introduced. One must account for it in (2.16). This implies that the exponential factor in (2.16) has to be redefined
to:
\[
\exp \left( \frac{1}{\pi} \left( k + \frac{g \vee}{2y} \right) \int_{\Sigma} \text{Tr}(A_z A_{\bar{z}}) \right),
\]
(2.26)
so that the inner product (2.16) becomes:
\[
(\Psi_2 | \Psi_1) = \int d\bar{u} du | \det \partial_z \partial_{\bar{z}} | \exp \left( \frac{1}{\pi} \left( k + \frac{g \vee}{2y} \right) \int_{\Sigma} \text{Tr}(A_z A_{\bar{z}}) \right) \Psi_2 \bar{A}_{\bar{z}} \Psi_1 A_z,
\]
(2.27)
where \(A_z\) and \(A_{\bar{z}}\) are given by (2.18).

As shown in [1,2], the form of the wave functional is determined using gauge invariance. Under the gauge transformations (2.19), the wave functional (2.15) transforms as:
\[
\Psi[A_{\bar{z}}] \rightarrow \Psi[g^{-1} A_{\bar{z}} g + g^{-1} \partial_z g] = \exp \left( - \left( k + \frac{g \vee}{2y} \right) \left( \Gamma(g) + \frac{1}{\pi} \int_{\Sigma} \text{Tr}(A_{\bar{z}} \partial_z g g^{-1}) \right) \right) \Psi[A_{\bar{z}}],
\]
(2.28)
where the variation of the factor (2.24) introduced in the measure has been taken into account. Notice that in doing the gauge transformation (2.28) an extension to the interior of \(M_1\) of the map \(g\) on the boundary \(\Sigma\) has been done. The result (2.28) is independent of the choice of extension when \(k\) satisfies the quantization condition (2.9). The solution to (2.28) has the form:
\[
\Psi[A_{\bar{z}}] = \xi \exp \left( - \left( k + \frac{g \vee}{2y} \right) \Gamma(u) \right).
\]
(2.29)
It is known [3] that the Hilbert space for the case of the solid ball is one-dimensional. Independently of the form of the Wilson lines contained in the solid ball the corresponding wave functional must be proportional to (2.29). The wave functional (2.29) satisfies the Gauss law emanating from the Chern-Simons action (2.1):
\[
F_{\bar{z}z}^a \Psi[A_{\bar{z}}] = 0,
\]
(2.30)
where \(F_{\bar{z}z}^a\) are the components of the gauge field strength. To verify (2.30) one must use the commutation relations for the gauge fields \(A_z\) and \(A_{\bar{z}}\) resulting from (2.27).
2.2. Genus-one handlebody

In this section we describe the construction of the operator formalism for the case of genus one: \( \Sigma = T^2 \). The strategy is similar to the one in the previous section. The non-trivial homology structure of the torus \( T^2 \) will provide a richer framework. Let us first introduce some data to characterize the torus.

We will denote the holomorphic abelian differential of a torus \( T^2 \) with modular parameter \( \tau \) by \( \omega(z) \). Labeling the homology cycle on \( T^2 \) which is contractible in the handlebody by \( \alpha \), and the one which is not by \( \beta \), the holomorphic form \( \omega(z) \) satisfies:

\[
\int_{\alpha} \omega = 1, \quad \int_{\beta} \omega = \tau, \quad \int_{T^2} \omega \wedge \bar{\omega} = \text{Im} \tau. \tag{2.31}
\]

The gauge fields \( A_z \) and \( A_{\bar{z}} \) on \( T^2 \) can be parametrized in the following way [23]:

\[
A_{\bar{z}} = (u_a u)^{-1} \partial_{\bar{z}} (u_a u), \quad A_z = (u_a \bar{u})^{-1} \partial_z (u_a \bar{u}), \tag{2.32}
\]

where \( u \) is a single-valued map, \( u : T^2 \to G^c \), and \( u_a \) a non-single valued map, \( u_a : T^2 \to G \), which takes the form:

\[
u_a = \exp \left( i\pi \frac{z}{\text{Im} \tau} \int \bar{\omega}(z') a \cdot H - i\pi \frac{z}{\text{Im} \tau} \int \omega(z') \bar{a} \cdot H \right), \tag{2.33}\]

where,

\[
a = \sum_{i=1}^l a_i \lambda^{(i)}, \quad H = \sum_{i=1}^l H_i \lambda^{(i)}, \tag{2.34}\]

being \( \lambda^{(i)} \), \( i = 1, \ldots, l \), the fundamental weights of a simple group \( G \) of rank \( l \). A summary of the group-theoretical conventions used in this paper is contained in the Appendix. Notice that \( u_a \) is in the maximal torus of \( G \) and that \( u_a^{-1} \). As before, \( u^\dagger = \bar{u}^{-1} \), so that \( A_{\bar{z}}^\dagger = -A_z \).
The generalization of the measure (2.20) for the case of the torus has the form [23]:

\[
[\mathcal{D}A_z \mathcal{D}A_{\bar{z}}]_{T^2} = \exp \left( \frac{g^\vee}{2y} \Gamma(u\bar{u}^{-1}, C) \right) |\Pi(a, \tau)|^4 (\text{Im} \tau)^l \exp \left( - \frac{g^\vee}{y} (u_a, u_a^{-1}) \right)
\]

\[
|\det \partial_\bar{z}| \text{d}u_\bar{a} \text{d}u_a^\dagger,
\]

where \( \Gamma(g, B) \) is the gauged Wess-Zumino-Witten action [24],

\[
\Gamma(g, B) = \Gamma(g) - \frac{1}{\pi} \int_\Sigma \text{Tr} (g^{-1}B_{\bar{z}}gB_z - B_{\bar{z}}\partial_\bar{z}gg^{-1} + g^{-1}\partial_zgB_z - B_zB_{\bar{z}}),
\]

and

\[
\Pi(a, \tau) = \exp \left( \frac{g^\vee \psi^2 \pi^2}{4\text{Im} \tau} a^2 \right) \Theta_{g^\vee, \rho}(a, \tau),
\]

being \( \Theta_{g^\vee, \rho}(a, \tau) \) the Weyl antisymmetrized theta function of level \( g^\vee \) (see the Appendix), and,

\[
\rho = \sum_{i=1}^l \lambda^{(i)}.
\]

The field \( C \) in the measure (2.35) is:

\[
C_z = u_a^{-1}\partial_zu_a, \quad C_{\bar{z}} = u_a^{-1}\partial_{\bar{z}}u_a,
\]

while the measure \( \text{d}u_a \text{d}u_a^\dagger \) takes the form:

\[
\text{d}u_a \text{d}u_a^\dagger = \frac{d^la^\dagger}{(\text{Im} \tau)^l}.
\]

The measure (2.35) is invariant under the gauge transformations (2.19) which now take the form,

\[
u \to ug,
\]

which will be called of type (i); under transformations which leave the fields \( A_z \)
and $A_{\bar{z}}$ invariant,

$$u \rightarrow \hat{g}^{-1}u, \quad u_a \rightarrow u_a \hat{g},$$  \hspace{1cm} (2.42)

where $\hat{g}$ is a map from $T^2$ into the Cartan torus of $G$; and under modular transformations. This last set of transformations is described in the Appendix. The transformations (2.42), which will be denoted as type (ii), involve maps $\hat{g}$ which are labelled in the following way:

$$\hat{g}_{m,n} = \exp\left(\frac{2\pi i}{\psi^2 \text{Im} \tau} ((n + m \tau) \cdot H \int \bar{z} \omega(z') - (n + m \bar{\tau}) \cdot H \int z \omega(z))\right),$$  \hspace{1cm} (2.43)

being $n$ and $m$ elements of the lattice generated by the long roots of $G$, which will be denoted by $L_R$, i.e., $n, m \in L_R$. Notice that the maps (2.43) are not connected to the identity map.

The analogue of the Polyakov-Wiegmann condition (2.22) for the case of the Wess-Zumino-Witten action [20] takes the form:

$$\Gamma(u \bar{u}^{-1}, C) = \Gamma(u) + \langle u_a, u \rangle + \Gamma(\bar{u}^{-1}) + \langle \bar{u}^{-1}, \bar{u}_a^{-1} \rangle - \langle u_a, u_a^{-1} \rangle - \frac{1}{\pi} \int_{\Sigma} \text{Tr}(C \bar{C} \bar{u}^{-1}).$$  \hspace{1cm} (2.44)

This expression leads to similar factorization problems as the ones found from (2.22). Following [1,2] we take

$$\exp\left(\frac{g^\vee}{2y} (\Gamma(u \bar{u}^{-1}, C) - \Gamma(u) - \langle u_a, u \rangle)\right) d\bar{u}^\dagger d\bar{u}_a$$  for $\Psi(A_{\bar{z}}),$  \hspace{1cm} (2.45)

and,

$$\exp\left(\frac{g^\vee}{2y} (\Gamma(u \bar{u}^{-1}, C) - \Gamma(\bar{u}^{-1}) - \langle \bar{u}^{-1}, u_a^{-1} \rangle)\right) du u_a$$  for $\Psi(A_z).$  \hspace{1cm} (2.46)

After comparing the products of these two factors to the one in (2.35) one finds
that the inner product (2.16) now takes the form:

\[
(\Psi_2 | \Psi_1) = \int du \, d\tau \, d\Pi \, d\Pi |\Pi(a, \tau)|^4 (\text{Im} \tau)^I \exp \left( - \frac{g^\vee}{2y} \langle u_a, u_a^{-1} \rangle \right) \\
\times \exp \left( \frac{1}{\pi} (k + \frac{g^\vee}{2y}) \int_\Sigma \text{Tr}(A_z A_{\bar{z}}) \right) \frac{\psi_2[A_{\bar{z}}]}{\psi_1[A_z]} \exp \left( - \frac{g^\vee}{2y} \langle u_a, u_a \rangle \right). \tag{2.47}
\]

As in the genus-zero case, the general form of the wave-functional is obtained using arguments based on its properties under symmetry transformations. Performing a gauge transformation of type (i) (2.41) one finds:

\[
\Psi[A_{\bar{z}}] \to \exp \left( - (k + \frac{g^\vee}{2y}) (\Gamma(g) + \langle u_a u, g \rangle) \right) \Psi[A_{\bar{z}}]. \tag{2.48}
\]

Using the Polyakov–Wiegmann condition (2.22) one finds that the solution to (2.48) can be written as:

\[
\Psi[A_{\bar{z}}] = \xi \psi_{2g_k + g^\vee}(u_a) \Lambda(u_a), \tag{2.49}
\]

where \(\xi\) is a constant, \(\Lambda(u_a)\) is arbitrary, and \(\psi_{2g_k + g^\vee}(u_a)\) is a functional which satisfies:

\[
\psi_{2gr}(u_a v) = \psi_{2gr}(u_a) \exp \left( - r (\Gamma(v) + \langle u_a, v \rangle) \right), \tag{2.50}
\]

for any single-valued map \(v : T^2 \to G\).

To search for solutions to (2.50) let us perform a symmetry transformation of type (ii) (2.42). One finds,

\[
\Psi[A_{\bar{z}}] \to \exp \left( \frac{g^\vee}{2y} (\Gamma(\hat{g}) + \langle u_a, \hat{g} \rangle) \right) \Psi[A_{\bar{z}}]; \tag{2.51}
\]

which implies the following property for \(\Lambda(u_a)\) in (2.49):

\[
\Lambda(u_a \hat{g}) = \exp \left( \frac{g^\vee}{2y} (\Gamma(\hat{g}) + \langle u_a, \hat{g} \rangle) \right) \Lambda(u_a). \tag{2.52}
\]

Comparing to (2.50) it turns out that \(\Lambda(u_a)\) and \(\psi_{2gr}(u_a)\) are related in the fol-
following way:

\[ \Lambda(u_a) = \left[ \psi_g(u_a) \right]^{-1}. \quad (2.53) \]

We need now to solve for (2.50). Let us consider the situation in which the map \( v \) is a map as in (2.43) of the form \( \hat{g}_{n[i],0} \) with \( n[i] = \sum_{j=1}^l n[j] \alpha(j) \) and \( n[j] = \delta[j] \), being \( \alpha(j) \) the simple roots of the group \( G \). Equation (2.50) takes the form:

\[ \psi_{2yr}(u_a + n[i]) = \exp \left( 2yr \left( \frac{\pi}{\psi^2 \text{Im} \tau} n[i] \cdot n[i] + \frac{\pi}{\text{Im} \tau} a \cdot n[i] \right) \right) \psi_{2yr}(u_a). \quad (2.54) \]

For maps of the form \( g_{0,m[i]} \) with \( m[i] = \sum_{j=1}^l m[j] \alpha(j) \) and \( m[j] = \delta[j] \), one finds,

\[ \psi_{2yr}(u_a + m[i] \tau) = \exp \left( 2yr \left( \frac{\pi \tau \bar{\tau}}{\psi^2 \text{Im} \tau} m[i] \cdot m[i] + \frac{\pi}{\text{Im} \tau} a \cdot m[i] \bar{\tau} \right) \right) \psi_{2yr}(u_a). \quad (2.55) \]

The two types of maps under consideration generate the maps (2.43) as described in [2]. The general solution to equations (2.54) and (2.55) can be expressed in terms of theta functions of level \( r \):

\[ \psi_{2yr,p}(a, \tau) = \exp \left( \frac{yr \pi \psi^2 a^2}{2 \text{Im} \tau} \right) \Theta_{2yr,p}(a, \tau), \quad (2.56) \]

where \( p \) is an element of the weight lattice modulo \( 2yr \) times the root lattice, \( i.e., \ p \in \Lambda_W/2yr \Lambda_R \). The properties of the theta functions \( \Theta_{2yr,p}(a, \tau) \) are briefly summarized in the Appendix.

Our analysis leads to the following form for the wave functional:

\[ \Psi[A_z] = \xi \exp \left( - (k + \frac{g^\vee}{2y})(\Gamma(u) + \langle u_a, u \rangle) \right) \frac{\psi_{2yk+g^\vee}(u_a)}{\psi_{g^\vee}(u_a)}, \quad (2.57) \]

where \( \xi \) is a constant, and \( \psi_{2yk+g^\vee}(u_a) \) and \( \psi_{g^\vee}(u_a) \) represent certain linear combinations of the solutions (2.56). As shown in [1,2] the \( u \)-dependence of the wave
functional can be integrated out obtaining an effective theory. Using (2.57), the inner product (2.47) becomes:

\[
(\Psi' | \Psi) = \int du_a du_a^\dagger |\Pi(a, \tau)|^4 (\Im \tau)^{\frac{3}{2}} \exp \left( - \left( k + \frac{g^\vee}{y} \right) \langle u_a, u_a^{-1} \rangle \right)
\]

\[
\times \xi'^\xi \left[ \frac{\psi'_{2yk+g^\vee}(u_a)}{\psi_{g^\vee}(u_a)} \right] \frac{\psi_{2yk+g^\vee}(u_a)}{\psi_{g^\vee}(u_a)} \int du \bar{u} \exp \left( - \left( k + \frac{g^\vee}{2y} \right) \Gamma(u \bar{u}^{-1}, C) \right),
\]

which, after using the result [23]:

\[
\int du \bar{u} \exp \left( - \left( k + \frac{g^\vee}{2y} \right) \Gamma(u \bar{u}^{-1}, C) \right) = (\Im \tau)^{\frac{3}{2}} |\Pi(a, \tau)|^2 \exp \left( \frac{g^\vee}{2y} \langle u_a, u_a^{-1} \rangle \right),
\]

becomes:

\[
(\Psi' | \Psi) = \int du_a du_a^\dagger |\Pi(a, \tau)|^2 (\Im \tau)^{\frac{3}{2}} \exp \left( - \left( k + \frac{g^\vee}{y} \right) \langle u_a, u_a^{-1} \rangle \right)
\]

\[
\times \xi'^\xi \left[ \frac{\psi'_{2yk+g^\vee}(u_a)}{\psi_{g^\vee}(u_a)} \right] \frac{\psi_{2yk+g^\vee}(u_a)}{\psi_{g^\vee}(u_a)} .
\]

(2.60)

Weyl invariance forces to choose antisymmetric combinations of the solutions (2.56). Defining:

\[
\lambda_{2yr,p}(a, \tau) = \sum_{w \in W} \epsilon(w) \psi_{2yr,w(p)}(a, \tau),
\]

(2.61)

where \( W \) is the Weyl group and \( \epsilon(w) \) the signature of the element \( w \in W \), the effective inner product (2.60) becomes:

\[
(\lambda_{2yk+g^\vee,q} | \lambda_{2yk+g^\vee,p}) = |\xi|^2 \int \frac{d'^4a \, d^4\bar{a}}{2 \Im \tau} \left( \Im \tau \right)^{-\frac{3}{2}} \exp \left( - \left( 2yk + g^\vee \right) \frac{\pi \psi^2}{2 \Im \tau} a \cdot \bar{a} \right)
\]

\[
\times \lambda_{2yk+g^\vee,q}(a, \tau) \lambda_{2yk+g^\vee,p}(a, \tau).
\]

(2.62)

From this inner product for the effective theory one can read the commutation
relations of its basic operators:

\[ [\bar{a}^i, a_j] = \frac{2\text{Im}\tau}{\pi(2yk + g^\vee)\psi^2}\delta^i_j. \]  

(2.63)

The states \(\lambda_{2yr,p}\) of the form (2.61) which are independent in \(\Lambda_w/2yr\Lambda_r\) constitute the physical states or Hilbert space of the theory. The set of weights labeling those states constitute the fundamental chamber \(\mathcal{F}_{2yr}\).

Knot operators are associated to Wilson lines. They correspond to the form of these operators when represented in the framework of the Hilbert space which has been constructed. Let us consider a torus knot labelled by two coprime integers \(n\) and \(m\), and their corresponding Wilson line:

\[ W^{(n,m)}_{\Lambda} = \text{Tr}_\Lambda(\text{P} \exp \int_{n,m} A). \]  

(2.64)

We use the convention in which \(n\) (\(m\)) denotes the number of times that the Wilson line winds along the \(\beta\)-cycle (\(\alpha\)-cycle) on the torus.

We are interested in the form of this operator when the single valued map \(u\) in (2.32) has been integrated out. In other words, we need the expression for the Wilson line (2.64) when \(u = 1\). Using (2.33) it turns out to be:

\[ W^{(n,m)}_{\Lambda} = \text{Tr}_\Lambda \left( \exp \left( \frac{i\pi}{\text{Im}\tau} \left( (n\bar{\tau} + m)a \cdot H - (n\tau + m)\bar{a} \cdot H \right) \right) \right) \]

\[ = \sum_{\mu \in M_\Lambda} \exp\left( -\frac{\pi}{\text{Im}\tau} (n\bar{\tau} + m)a \cdot \mu + \frac{2(n\tau + m)}{(2yk + g^\vee)\psi^2}\mu \cdot \frac{\partial}{\partial a} \right), \]  

(2.65)

where in the last step we have used (2.63), and the fact that \(H\) is made out of diagonal matrices whose entries are related to the components of the set of weights \(\mu \in M_\Lambda\), being \(M_\Lambda\) the set of weights corresponding to an irreducible representation of highest weight \(\Lambda\). Using the standard properties of the theta functions which
are compiled in the Appendix one finds:

$$W_\Lambda^{(n,m)} \lambda_{2yk+g^\vee,p} = \sum_{\mu \in \mathcal{M}_\Lambda} \exp \left( \frac{2i\pi \mu^2 nm}{\psi^2(2yk + g^\vee)} + \frac{4i\pi m p \cdot \mu}{\psi^2(2yk + g^\vee)} \right) \lambda_{2yk+g^\vee,p+n\mu}. \quad (2.66)$$

These operators are called knot operators. They satisfy the following important relation:

$$W_\Lambda^{(1,0)} \rho = |\rho + \Lambda\rangle, \quad (2.67)$$

where $|\rho\rangle$ is the state corresponding to the weight (2.38). As discussed in [1,2], this relation allows to think of the operators $W_\Lambda^{(1,0)}$ as creation operators since they create the state corresponding to the highest weight $\Lambda$ when acting on the vacuum state $|\rho\rangle$.

One important ingredient in the computation of knot invariants for torus knots is the knowledge of the corresponding representation on the set of homeomorphisms on $T^2$. These homeomorphisms are generated by modular transformations $S$ and $T$ on $T^2$ which possess the following representation [25]:

$$T_{p,p'} = \delta_{p,p'} e^{2\pi i (h_p - h_{p'})},$$

$$S_{p,p'} = \frac{i^{|\Delta_+|}}{(2yk + g^\vee)^{\frac{1}{2}}} \left( \frac{\text{Vol} \mathcal{L}_R^*}{\text{Vol} \mathcal{L}_R} \right) \sum_{w \in \mathcal{W}} \epsilon(w) e^{-4\pi i \mu \cdot (w(p'))} \psi^2(2yk + g^\vee), \quad (2.68)$$

where $|\Delta_+|$ is the number of positive roots, $\mathcal{L}_R$ is the lattice of long roots and $\mathcal{L}_R^*$ its dual. In (2.68) $h_p$ and $c$ represent the conformal weight and central charge of the corresponding two-dimensional conformal field theory:

$$h_p = \frac{p^2 - \rho^2}{\psi^2(2yk + g^\vee)}, \quad c = \frac{2yk \dim(G)}{2yk + g^\vee}. \quad (2.69)$$

Knot operators provide a very useful tool to compute knot invariants in lens spaces. These spaces are boundaryless three-dimensional manifolds which can be built by joint of two tori. The gluing is carried out by an homeomorphism whose
representation in the Hilbert space which we have constructed is written in terms of the generators (2.68). If we denote this representation by \( F \), the vacuum expectation value for a Wilson line corresponding to a torus knot carrying an irreducible representation of highest weight \( \Lambda \) of a simple group \( G \) is:

\[
V_{\Lambda}^{(n,m)}|_F = \frac{\langle \rho | F W_{\Lambda}^{(n,m)} | \rho \rangle}{\langle \rho | F | \rho \rangle}.
\] (2.70)

To connect with the standard form in which polynomial invariants are written we need to correct (2.70) in three aspects. First of all in (2.70) a choice of frame for the knot and the manifold has been done. Invariants are usually expressed in the standard frames and we must correct (2.70) so that the contribution from the knot framing factor is cancelled, and that the appropriate choice of \( F \) is made. Taking the three-sphere as our choice of lens space, which will be the case of interest in this paper, the standard frame is accomplished considering \( F = S \), being \( S \) one of the two generators of modular transformations. As shown in [2] the correction relative to the frame of the knot is easily accomplished multiplying by

\[
e^{-2\pi i m h_{\rho+\Lambda}},
\] (2.71)

where \( h_{\rho+\Lambda} \) is the conformal weight given in (2.69). The second aspect leading to an additional correction for (2.70) is the fact that the orientation chosen for the torus \( T^2 \) is the opposite to the standard one. We must therefore do the following change \( m \to -m \). Finally, the third aspect is that usually knot invariants are normalized in such a way that their value for the unknot is one. We must therefore normalize (2.70) by its value for the unknot. These three aspects lead to the following proposition:

**Proposition 2.1.** The normalized knot invariant for a torus knot \( \{n, m\} \) in the standard framing, carrying a \( G \) irreducible representation of highest weight \( \Lambda \)
on $S^3$ in the standard framing, is:

$$X^{(n,m)}_{\Lambda} = e^{2\pi i n m \hbar + \Lambda} \frac{V^{(n,-m)}_{\Lambda}}{V^{(1,0)}_{\Lambda}} |_{S^3},$$

$$= e^{2\pi i n m \hbar + \Lambda} \frac{\langle \rho | SW^{(n,-m)}_{\Lambda} | \rho \rangle}{\langle \rho | SW^{(1,0)}_{\Lambda} | \rho \rangle}. \tag{2.72}$$

The structures of the knot operators (2.66) and the matrix $S_{p,p'}$ in (2.68) allow to express this invariant in terms of the variable

$$t = e^{\frac{2\pi i}{g_k + g'}},$$ \tag{2.73}

which encloses all the dependence on $k$. The main purpose of this paper is to compute (2.72) for the fundamental representation of the group $SO(N)$. This will lead to the Kauffman polynomial [18] for torus knots. The resulting formula agrees with the one given in [19]. The comparison of this formula to the corresponding known expression for the HOMFLY polynomial [13,17,15] will allow to prove (1.1).
3. Kauffman polynomial for torus knots

In this section we will make use of proposition 2.1 to compute the Kauffman polynomial for torus knots. We must evaluate (2.72) for the fundamental representation of $SO(N)$, i.e., we must make $\Lambda = \lambda^{(1)}$. The result is stated in the following theorem:

**Theorem 3.1.** The Kauffman polynomial for a torus knot $\{n, m\}$ is given by:

$$X_{\lambda^{(1)}}^{(n,m)} = \frac{[1] \lambda^{nm}}{[1] + [0; 1]} \left( \sum_{\beta, \gamma \geq 0} t^{-\frac{N}{2}(\beta - \gamma)} \lambda^{-m} (-1)^\gamma \left( \frac{1}{\lfloor \beta \rfloor} + \frac{1}{\lfloor \beta - \gamma \rfloor} \right) \right)$$

$$\times \frac{1}{\lfloor \beta \rfloor! \lfloor \gamma \rfloor!} \prod_{j=\gamma}^\beta [j; 1] + \begin{cases} 0, & n \text{ odd;} \\ 1, & n \text{ even;} \end{cases}$$

(3.1)

where:

$$[p] = t^p - t^{-p}, \quad [p; y] = t^p y - t^{-p} y, \quad \lambda = t^{N-1}, \quad t = e^{\frac{2\pi i}{2k+g^\vee}},$$

(3.2)

with $g^\vee = N - 2$.

**Proof.** The rest of this section deals with the proof of this theorem. As $SO(N)$ has two different algebras, depending on whether $N$ is odd or even, we will have to study both cases separately. We will begin with $SO(2l + 1)$, $B_l$ being the corresponding algebra. The main feature of this case is that the simple roots of $B_l$ are not all of the same length. Notice that since an $\{n, m\}$ torus knot is isotopically equivalent to the $\{-n, -m\}$ torus knot, we can restrict ourselves to torus knots with $n > 0$. Also we will consider the case in which $l > n$. Our results, however, as in the case of the HOMFLY polynomial computed in [15], are valid for arbitrary $l$. In this proof we make the following choice of normalization for the long roots:

$$\psi^2 = 2.$$  

(3.3)

Notice also that for $SO(N)$ the Dynkin index for the fundamental representation
is \( y = 1 \) and therefore (2.73) becomes:

\[
t = e^{\frac{2\pi i}{2k+g'}}. \tag{3.4}
\]

3.1. \( SO(2l + 1) \)

Let us begin working out the action of the knot operator \( W^{(n,m)}_{\lambda(1)} \) on the vacuum state. Using (2.66), (3.3), and the form of \( t \) in (3.4), we have:

\[
W^{(n,-m)}_{\lambda(1)} |\rho\rangle = \sum_{i=1}^{2l+1} t^{-\frac{1}{2}\mu_i^2 n - m \mu_i} |\rho + n \mu_i\rangle \tag{3.5}
\]

where \( \mu_i, i = 1, ..., 2l + 1 \), are the weights in \( M_{\lambda(1)} \) whose explicit expression is given in (A.20). Following the framework described in the previous section, we must find the canonical representatives in the fundamental chamber \( F_{2k+g'} \) (notice that \( 2yr = 2yk + g' \) and \( y = 1 \)) of the weights appearing in the sum. The weights present in (3.5) have the following structure:

\[
\begin{align*}
\rho + n \mu_1 &= (n + 1, 1, \ldots, 1), \\
\vdots & \\
\rho + n \mu_j &= (1, \ldots, 1 - n, 1 + n, 1, \ldots, 1), \\
\vdots & \\
\rho + n \mu_l &= (1, \ldots, 1 - n, 1 + 2n), \\
\rho + n \mu_{l+1} &= \rho, \\
\rho + n \mu_{l+2} &= (1, \ldots, 1 + n, 1 - 2n), \\
\vdots & \\
\rho + n \mu_{l+j} &= (1, \ldots, 1 + n, 1 - n, 1, \ldots, 1), \\
\vdots & \\
\rho + n \mu_{2l+1} &= (1 - n, 1, \ldots, 1).
\end{align*}
\]  

(3.6)

Every weight in the weight lattice can be written as \( w(\mu) + (2k + g')\alpha \), where \( w \) is an element of the Weyl group, \( \alpha \) a long root, and \( \mu \) is a weight whose components
are non-negative. In the Hilbert space constructed in the previous section the weights which possess one or more components which vanish are represented by null vectors. Since $2l + 1 > n$ there is no need to add terms of the form $(2k + g^\vee)\alpha$ to the weights in (3.6) to bring them to a form in which their components are non-negative. A series of Weyl reflections will be sufficient. If $n = 1$ all the weights in (3.6) except the first one and $\rho + n\mu_{l+1}$ have one vanishing component and therefore there are only these two contributions in the sum present in (3.5). If $n > 1$, notice first that the weights $\rho + n\mu_1$ and $\rho + n\mu_{l+1}$ in (3.6) are already in $F_{2k+g^\vee}$. For the rest we have the following cases:

1. Case $i = 2, \ldots, l$:
   
   a) $2 \leq i \leq n$. We perform the chain of Weyl reflections:

   $\rho + n\mu_i \xrightarrow{\sigma_1} \ldots \xrightarrow{\sigma_{i-2}\sigma_{i-1}} \nu_i = (n + 1 - i, 1, \ldots, 1, \ldots, i, 1, \ldots, 1)$, $i = 2, \ldots, l - 1$,

   (3.7)

   $\rho + n\mu_l \xrightarrow{\sigma_1} \ldots \xrightarrow{\sigma_{i-2}\sigma_{i-1}} \nu_l = (n + 1 - l, 1, \ldots, 1, 3)$.

   The weight $i = l$ will not be considered as we restrict ourselves to $n < l$.

   b) $i > n$. The chain of Weyl reflections is like the one in (3.7):

   $\rho + n\mu_i \xrightarrow{\sigma_{i+1-n}} \ldots \xrightarrow{\sigma_{i-1}} = (1, \ldots, 1, \ 0 ,1, \ldots, 2, \ldots, 1)$, $i = 1, \ldots, l - 1$,

   (3.8)

   $\rho + n\mu_l \xrightarrow{\sigma_{i+1-n}} \ldots \xrightarrow{\sigma_{i-1}} = (1, \ldots, 1, \ 0 , \ldots, 1, 3)$.

   After $n + 1$ reflections the weights get a vanishing component and therefore all these weights correspond to null vectors and do not contribute to the sum in (3.5).

   This fact is very important in this calculation because it implies that the sum (3.5) is truncated. Its upper limit turns out to be $n$ instead of $2l + 1$.

2. Case $i = l + 2, \ldots, 2l + 1$:
As \( i > n \) for the weights in this case, we would expect that all of them would achieve a vanishing component after a chain of Weyl reflections. What actually happens is that for \( n \) odd an extra weight will contribute:

\[
\rho + n\mu_{l+2} \rightarrow (1, \ldots, 1, 2-n, 2n-1) = \rho \quad \text{for } n = 1.
\]

For \( j = 2, \ldots, l \), one has the following situations:

\( n \leq j \),
\[
ho + n\mu_{l+1+j} \rightarrow \ldots \rightarrow \sigma_{j-1} \rightarrow (1, \ldots, 1, 2, \ldots, 1, \ldots, 0, \ldots, 1); 
\]

\( j < n < 2j - 1 \),
\[
ho + n\mu_{l+1+j} \rightarrow \ldots \rightarrow \sigma_{j} \rightarrow \ldots \rightarrow \sigma_{j-1} \rightarrow (1, \ldots, 1, 2, \ldots, 1, \ldots, 0, \ldots, 1); 
\]

\( n = 2j - 1 \),
\[
\rho + n\mu_{l+1+j} \rightarrow \ldots \rightarrow \sigma_{j-1} \rightarrow \ldots \rightarrow \sigma_{j} \rightarrow (1, \ldots, 1, 2j - n, n - (2j - 2) \ldots, 1) = \rho; 
\]

\( 2j - 1 < n < l \),
\[
\rho + n\mu_{l+1+j} \rightarrow \ldots \rightarrow \sigma_{j} \rightarrow \ldots \rightarrow \sigma_{j-1} \rightarrow (1, \ldots, 1, 0, \ldots, 1, \ldots, 2, \ldots, 1). 
\]

(3.9)

We see that all the vectors have a vanishing component except when \( n = 2j - 1 \), where the weight \( \nu_{l+1+j} = \rho \) belongs to \( \mathcal{F}_{2k+g^v} \). Taking into account these considerations we find that the weights contributing to the sum in (3.5) are:

\[
\nu_i = \epsilon(\omega_i) \cdot (n + 1 - i, 1 \ldots i \ldots 1), \quad i = 1, \ldots, n, 
\]

\[
\nu_{l+1} = \epsilon(\omega_{l+1}) \cdot \rho, 
\]

\[
\nu_{l+1+i} = \epsilon(\omega_{l+1+i}) \cdot \rho, \quad n = 2i - 1. 
\]

(3.10)

where \( \epsilon(\omega) \) is the signature of the Weyl chain, given by the number of Weyl reflec-
tions we have made to bring the weights to this form:

$$\omega_i = \sigma_1 \ldots \sigma_{i-1}, \quad \Rightarrow \quad \epsilon(\omega_i) = (-1)^{i-1},$$

$$\omega_{l+1} = I, \quad \Rightarrow \quad \epsilon(\omega_{l+1}) = (-1)^0 = 1,$$

$$\omega_{l+1+i} = \sigma_{l+1-i} \ldots \sigma_{l-1} \sigma_i \sigma_{l-1} \ldots \sigma_{l-i+1}, \quad \Rightarrow \quad \epsilon(\omega_{l+1+i}) = (-1)^{2i-1} = -1.$$

Using these results and the scalar products in (A.22) the sum in (3.5) becomes,

$$W^{(n,-m)}_\Lambda |\rho\rangle = \left\{ \begin{array}{ll} 0, & n \text{ odd;} \\
|\rho\rangle, & n \text{ even.} \end{array} \right. \quad (3.11)$$

This equation is valid for any $n \geq 1$ as long as $l > n$. The vacuum expectation value (2.70) which enters (2.72) takes the form:

$$V^{(n,-m)}_{\Lambda(1)} = \frac{\langle \rho | SW^{(n,-m)}_{\Lambda_1} | \rho \rangle}{\langle \rho | S | \rho \rangle} = \left\{ \begin{array}{ll} 0, & n \text{ odd;} \\
1, & n \text{ even.} \end{array} \right. \quad (3.12)$$

The weights $\nu_i$ have the general expression $\nu_i = \rho + (n-i, 0 \ldots 1 \ldots 0) = \rho + \Lambda$. If $\Lambda$ is a highest weight, the ratio $S_{\rho,\rho+\Lambda}/S_{\rho,\rho}$ can be written in terms of the character associated to $\Lambda$ with the help of the Weyl formula,

$$\frac{S_{\rho,\rho+\Lambda}}{S_{\rho,\rho}} = \frac{\sum_{w \in W} \epsilon(w) t^{\rho \cdot w(\rho+\Lambda)}}{\sum_{w \in W} \epsilon(w) t^{\rho \cdot w(\rho)}} = ch_{\Lambda}[\frac{2\pi i}{2k + g\rho^\vee}]. \quad (3.13)$$

All the weigths entering (3.12) can be thought as highest weights and therefore we can express $V^{(n,-m)}_{\Lambda(1)}$ in terms of characters:

$$V^{(n,-m)}_{\Lambda(1)} = \sum_{i=1}^{n} t^{-\frac{nm}{2} - m(2l+1-2i)} (-1)^{i-1} ch_{(n-i)\Lambda(1)+\Lambda(0)}[-\frac{2\pi i}{2k + g\rho^\vee}] + \left\{ \begin{array}{ll} 0, & n \text{ odd;} \\
1, & n \text{ even.} \end{array} \right. \quad (3.14)$$

Let us compute first $V^{(1,0)}_{\Lambda(1)}$, which is the quantity entering the denominator in (2.72). From (3.12) and (3.13) follows that one needs to compute the character for
the fundamental representation. This calculation is done very simply just summing over the weights of the representation:

\[
\text{ch}_{\lambda}^{(1)}[-\frac{2\pi i}{2k+g^\vee}\rho] = \sum_{\mu \in M_{\lambda}^{(1)}} t^{-\mu \cdot \rho} = 1 + \sum_{j=1}^{l} t^{-\mu_j \cdot \rho} + \sum_{k=1}^{l} t^{-\mu_{k+1} \cdot \rho} = 1 + \frac{t^{l} - t^{-l}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}.
\] (3.15)

Using this result, it turns out that

\[
V^{(1,0)}_{\lambda} = 1 + \frac{\lambda - \lambda^{-1}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}},
\] (3.16)

which has been written entirely in terms of the variables \(\lambda\) and \(t\) in (3.2) (notice that in this case \(N = 2l + 1\)). This result agrees with previous calculations for the unknot [7,10].

For representations different than the fundamental one, however, it is more useful to compute the character using its expression in term of a product over positive roots:

\[
\text{ch}_{\Lambda}[-\frac{2\pi i}{2k+g^\vee}\rho] = \prod_{\alpha > 0} \left[ \frac{t^{\frac{1}{2}}\alpha \cdot (\rho + \Lambda) - t^{-\frac{1}{2}}\alpha \cdot \rho}{t^{\frac{1}{2}}\alpha \cdot \rho - t^{-\frac{1}{2}}\alpha \cdot \rho} \right].
\] (3.17)

In this equation, the symbol \(\alpha > 0\) indicates that the product has to be performed over all the positive roots. For \(B_l\) these are given in the Appendix. Our next task is to compute the characters appearing in (3.14) with the help of this formula.

In order to simplify our notation, from now on we will denote \(\text{ch}_{\Lambda}[-2\pi i \rho/(2k + g^\vee)]\) simply by \(\text{ch}_{\Lambda}\). Also, we introduce the following notation regarding \(q\)-numbers and \(q\)-factorials:

\[
[p] = t^{\frac{p}{2}} - t^{-\frac{p}{2}},
\]

\[
[p]! = [p][p-1]\ldots[1],\quad [0]! = 1.
\] (3.18)

This allows us to write the character formula in the form:

\[
\text{ch}_{\Lambda} = \prod_{\alpha > 0} \frac{[\alpha \cdot (\rho + \Lambda)]}{[\alpha \cdot \rho]}.
\] (3.19)

In order to compute (3.14) we must perform the products in (3.19) for weights
of the form \((n - i)\lambda^{(1)} + \lambda^{(i)}\). Taking into account the form of the positive roots listed in (A.6), this suggests to organize the product in (3.19) splitting the set of positive roots in two groups, I and II, depending on whether the positive root contains the simple root \(\alpha^{(1)}\) or not. Another thing we have to take into account is that the metric between fundamental weights and simple roots of this algebra, for the normalization chosen for the long roots, is the following:

\[
\alpha^{(i)} \cdot \lambda^{(j)} = \text{diag}(1 \ldots 1, \frac{1}{2}), \tag{3.20}
\]

due to the fact that the simple root \(\alpha^{(l)}\) is shorter than the others. Let us carry out the computation of the character.

The products of the positive roots with the Weyl vector are:

\[
\begin{align*}
\beta^{(j)} \cdot \rho &= l - j + \frac{1}{2}, \\
\gamma^{(j,k)} \cdot \rho &= 1 + k, \\
\delta^{(j,k)} \cdot \rho &= 2l - 2j - k.
\end{align*}
\tag{3.21}
\]

and with the weights \(\nu_i = \rho + (n - i, 0 \ldots \hat{i} \ldots 0), \ i = 1, \ldots, l - 1:\

a) group I, positive roots with \(\alpha^{(1)}\):

\[
\begin{align*}
\beta^{(1)} \cdot \nu_i &= l - 1 + \frac{1}{2} + n - i + 1, \\
\gamma^{(1,k)} \cdot \nu_i &= 1 + k + n - i + \begin{cases} 
1, & i \leq k + 1; \\
0, & k \leq i - 2;
\end{cases} \\
\delta^{(1,k)} \cdot \nu_i &= 2l - 2 - k + n - i + \begin{cases} 
1, & i \leq k + 1; \\
2, & k \leq i - 2;
\end{cases}
\end{align*}
\tag{3.22}
\]
b) group II, positive roots without $\alpha_{(1)}$:

\[
\beta_{(j)} \cdot \nu_i = \begin{cases} 
    l - j + \frac{1}{2} + 1, & j \leq i; \\
    l - j + \frac{1}{2}, & j > i;
\end{cases}
\]

\[
\gamma_{(j,k)} \cdot \nu_i = \begin{cases} 
    1 + k, & j > i \text{ or } i \geq j + k + 1; \\
    2 + k, & j \leq i \leq j + k;
\end{cases}
\]

\[
\delta_{(j,k)} \cdot \nu_i = \begin{cases} 
    2l - 2j - k, & j > i; \\
    2l - 2j - k + 1, & i \leq j \leq j + k; \\
    2l - 2j - k + 2, & i \geq j + k + 1.
\end{cases}
\] (3.23)

We have these two contributions to the characters:

\[
\prod_{\alpha \in I} \frac{[\alpha \cdot \nu_i]}{[\alpha \cdot \rho]} = \frac{[l + \frac{1}{2} + n - i]}{[l - \frac{1}{2}]} \times \prod_{k=0}^{i-2} \frac{[1 + k + n - i]}{[1 + k]} \times \prod_{k=i-1}^{l-2} \frac{[2 + k + n - i]}{[1 + k]} \times \\
\prod_{k=0}^{i-2} \frac{[2l + n - i - k]}{[2l - 2 - k]} \times \prod_{k=i-1}^{l-2} \frac{[2l - 1 - k + n - i]}{[2l - 2 - k]} \times \\
\frac{1}{[n]} \frac{[n - i + l + \frac{1}{2}]}{[l - \frac{1}{2}]} \times \frac{[2l + n - i]!}{[n - i]! [2l - 2]!}. 
\] (3.24)

and,

\[
\prod_{\alpha \in \Pi} \frac{[\alpha \cdot \nu_i]}{[\alpha \cdot \rho]} = \prod_{j=2}^{i} \frac{[l - j + 1 + \frac{1}{2}]}{[l - j + \frac{1}{2}]} \times \prod_{j=2}^{i-1} \prod_{k=0}^{i-j} \frac{[2l - 2j - k + 2]}{[2l - 2j - k]} \times \\
\prod_{j=2}^{i} \prod_{k=i-j}^{i-j-1} \frac{[2l - 2j - k + 1]}{[2l - 2j - k]} \times \prod_{j=2}^{i-1} \prod_{k=i-j}^{i-j-1} \frac{[2 + k]}{[1 + k]} \times \\
\frac{[l - \frac{1}{2}]}{[l - i + \frac{1}{2}]} \times \frac{[2l - 2i + 1]}{[i - 1]!} \times \frac{[2l - 2]!}{[2l - i]!}. 
\] (3.25)

Taking into account (3.24) and (3.25) we finally obtain a formula for the character in terms of \(q\)-numbers:
\[
\text{ch}_{(n-i)\lambda^{(1)}+\lambda^{(i)}} = \prod_{\alpha > 0} \frac{[\alpha \cdot \nu_i]}{[\alpha \cdot \rho]} = \left( \frac{1}{[n]} + \frac{1}{[n + 2l - 2i + 1]} \right) \\
\times \frac{1}{[n - i]! \ [i - 1]!} \prod_{j=-(i-1)}^{n-i-1} [2l + j].
\]

From this it is straightforward to write an expression for (3.14) involving only the variables \(t\) and \(\lambda\). First we introduce the notation:

\[
[p; y] = t^\rho \lambda^y - t^{-\rho} \lambda^{-y}, \\
\beta = n - i, \\
\gamma = i - 1.
\]

Recall that \(\lambda\) is defined as \(\lambda = t^\frac{n-1}{2} = t^l\). One finds,

\[
V_{\lambda^{(i)}}^{(n,-m)} = \sum_{\gamma + \beta + 1 = n} t^{-m(\beta - \gamma)} \lambda^{-m(-1)^\gamma} \left( \frac{1}{[n]} + \frac{1}{[\beta - \gamma; 1]} \right) \\
\times \frac{1}{[\beta]! \ [\gamma]!} \prod_{j=-\gamma}^{\beta} [j; 1] + \begin{cases} 0, & \text{if } n \text{ odd;} \\ 1, & \text{if } n \text{ even.} \end{cases}
\]

It remains only to obtain the deframing phase factor. The conformal weight for the fundamental representation of \(SO(2l + 1)\) is given by (2.69):

\[
h_{\rho + \lambda^{(1)}} = \frac{(\rho + \lambda^{(1)})^2 - \rho^2}{2(2k + g^\vee)} = \frac{l}{(2k + g^\vee)^2},
\]

which gives the deframing factor:

\[
e^{2\pi i m \rho, \lambda^{(1)}} = t^{lmn} = \lambda^{nm}.
\]

From (3.28) (3.30) and (3.16) one obtains the final expression for the knot invariant (2.72), which equals the one stated in Theorem 3.1. This ends the proof for the case \(SO(2l + 1)\).
3.2. \( SO(2l) \)

As the calculation procedure is the same as in the previous case, we will simply give the main results at each step. The Lie algebra is now \( D_l \) and its main features are summarized in the Appendix.

The action of the knot operator on the vacuum state is given by:

\[
W^{(n,-m)}_{\Lambda} |\rho\rangle = \sum_{i=1}^{2l} t^{-\frac{1}{2}} \mu_i^{nr-m\mu_i} |\rho + n\mu_i\rangle,
\]

where \( \mu_i = 1,\ldots,2l \) are the weights in \( M_{\Lambda(i)} \) whose expression is in (A.25). The vectors \( \rho + n\mu_i \) have the structure:

\[
\begin{align*}
\rho + n\mu_1 &= (n+1,1,\ldots,1), \\
\vdots \\
\rho + n\mu_j &= (1,\ldots,1,1-n,1+n,1,\ldots,1), \\
\vdots \\
\rho + n\mu_{l-1} &= (1,\ldots,1,1-n,1+n,1+n), \\
\rho + n\mu_l &= (1,\ldots,1,1-n,1+n), \\
\rho + n\mu_{l+1} &= (1,\ldots,1,1+n,1-n), \\
\rho + n\mu_{l+2} &= (1,\ldots,1,1+n,1-n,1-n), \\
\vdots \\
\rho + n\mu_{l+j} &= (1,\ldots,1,1+n,1-n,1,\ldots,1), \\
\vdots \\
\rho + n\mu_{2l-1} &= (1+n,1-n,1,\ldots,1), \\
\rho + n\mu_{2l} &= (1-n,1,\ldots,1),
\end{align*}
\]
suitable chain of Weyl reflections, and the corresponding signature, are:

\[
\nu_i = (-1)^{(i-1)} \cdot (n + 1 - i, 1, \ldots, 2, \ldots, 1), \quad i = 1, \ldots, n,
\]

\[
\nu_{i+1} = \rho, \quad n = 2i.
\]

We see that, very similarly to the \(SO(2l+1)\) case, the number of weights we have to take into account is bounded by \(n\) and that there is an extra one in the case of \(n\) even. So the expression (3.31) becomes, after using the scalar products in (A.27):

\[
W^{(n,-m)}_\Lambda |\rho\rangle = \sum_{i=1}^{n} t^{-\frac{nm}{2}} m(1-i)^{-1} |\nu_i\rangle + \begin{cases} 0, & n \text{ odd;} \\ |\rho\rangle, & n \text{ even.} \end{cases}
\]

The quantity \(V^{(1,0)}_{\lambda(1)}\) is obtained from the character of the fundamental representation:

\[
V^{(1,0)}_{\lambda(1)} = \sum_{\mu \in M_{\lambda(1)}} t^{-\mu \cdot \rho} = 1 + \frac{\lambda - \lambda^{-1}}{t^2 - t^{-2}} = \left[1\right] + \left[0; 1\right] \left[1\right],
\]

where for the last equality we have use the definitions (3.18) and (3.27). To calculate \(V^{(n,-m)}_{\lambda(1)}\) we again need the characterization of the positive roots which is contained in (A.7). Then one computes the products of these roots with the Weyl vector and the weights \(\nu_i\). From these we obtain the following formula for the characters:

\[
\prod_{\alpha > 0} \frac{[\alpha \cdot \nu_i]}{[\alpha \cdot \rho]} = \left(\frac{1}{[n]} + \frac{1}{n + 2l - 2i}\right) \frac{1}{[n-i]! [i-1]!} \prod_{j=-(i-1)}^{n-i} [2l + j - 1].
\]

Using again (3.27), one gets:

\[
V^{(n,-m)}_{\lambda(1)} = \sum_{\gamma, \beta \geq 0} t^{-\frac{m(\beta-\gamma)}{2}} \lambda^{-m} (-1)^{\gamma} \left(\frac{1}{[n]} + \frac{1}{[\beta-\gamma]; 1}\right)
\]

\[
\times \frac{1}{[\beta]! [\gamma]!} \prod_{j=-\gamma}^{\beta} [j; 1] + \begin{cases} 0, & n \text{ odd;} \\ 1, & n \text{ even.} \end{cases}
\]
The framing factor for this case is given by:

$$e^{2\pi i n m h_{\rho+\lambda(1)}} = t^{\frac{nm}{2}}(2l-1) = \lambda^{nm}. \quad (3.38)$$

It is easy to see that taking into account (3.38), (3.35), and (3.37) we obtain the expression (3.1) for the knot invariant associated to the fundamental representation of $SO(2l)$. Notice also that although $\lambda$ is defined in a different way with respect to the rank of the algebra, $l$, its definition is the same for both cases in terms of the variable $N$ of $SO(N)$. This completes the proof of Theorem 3.1.

### 3.3. Natural variables of the Kauffman polynomial and Yokota’s formula

The Dubrovnik version of the Kauffman polynomial, as described in pag. 215 of [26], depends on two variables, $a$ (which is called $\alpha$ in [26]) and $z$. We will refer to these variables as the natural ones. In those variables the skein rules have the simple form shown in [26]. We will denote the Dubrovnik version of the Kauffman polynomial, normalized in such a way that for the unknot its value is one, by $Y_K(a, z)$, and will try to identify these variables in terms of ours. This can be done comparing the skein rules in [26] to the skein rules obtained from Chern–Simons theory in [5,7,8]. It turns out that,

$$a = \lambda = e^{2\pi i h_{\rho+\lambda(1)}},$$
$$z = [1] = t^{\frac{1}{2}} - t^{-\frac{1}{2}}. \quad (3.39)$$

The formula in Theorem 3.1 can therefore be stated as:

$$Y_{n,m}(\lambda, t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = X_{\lambda(1)}^{(n,m)}, \quad (3.40)$$

where $X_{\lambda(1)}^{(n,m)}$ is given in (3.1).
To compare our formula (3.40) to the one obtained by Yokota in [19] we will use (3.39) and the identification done in [19] between its variables, $q$ and $\alpha$, and the natural ones. Proceeding in this way one concludes that the relation between our variables and Yokota’s is:

\begin{align*}
q &= t^{-\frac{1}{2}}, \\
\alpha^2 &= -(q\lambda)^{-1}.
\end{align*}

(3.41)

Taking into account that Yokota uses an orientation opposite to ours, and therefore we must compare (3.40) to its formula for $Y_{n,m}(a^{-1}, -z)$, one finds complete agreement after substituting (3.41) in the formula given in [19].
4. Relation between the HOMFLY and Kauffman polynomials for torus knots

The HOMFLY [16] and Kauffman polynomials [18] have the common characteristic of being functions of two variables defined for oriented links, although their behavior under change of orientation of some of the link components is quite different. On the other hand, the skein rules that define them are also different: in the first one the relation is established among three diagrams and in the second one among four. Both are able to differentiate in many cases one knot from its mirror image, although Kauffman’s is more powerful in this sense. These two polynomials are considered as independent, in the sense that there is not a subtle change of variables taking one into the other. In [27] there are examples of knots with the same Kauffman and different HOMFLY and viceversa. We will prove that for the particular case of torus knots there is a relation between these two polynomials.

Let’s begin recalling the expression of the HOMFLY polynomial for torus knots. It was first obtained in [13], reobtained in [17] using quantum groups and in [15] from the Chern-Simons theory with gauge group $SU(N)$. The corresponding invariant has the form [15]:

$$
P_{n,m}(a, z) = P_{n,m}((\lambda t)^{\frac{1}{2}}, t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

$$= \left(1 - \frac{t}{1 - t^n}\right) \frac{\lambda^{\frac{1}{2}(m-1)(n-1)}}{\lambda t - 1} \sum_{p+i+1=n, p, i \geq 0} (-1)^i t^{mi+\frac{1}{2}p(p+1)} \frac{\prod_{j=-p}^{i} (\lambda t - t^j)}{\prod_{j=1}^{i} (t^j - 1) \prod_{j=1}^{p} (t^j - 1)}$$

$$= \frac{[1]! (\lambda t)^{\frac{1}{2}m(n-1)}}{[1; -\frac{1}{2}]} \sum_{\beta+\gamma+1=n, \beta, \gamma \geq 0} (-1)^{\gamma} t^{\frac{m}{2}(\beta-\gamma)} \prod_{\beta} \frac{1}{[\beta][\beta][\gamma]} \times \prod_{j=-\gamma}^{\beta} [j - 1; -\frac{1}{2}],$$

(4.1)

where:

$$\lambda = t^{N-1},$$

$$t = e^{2\pi i \varphi}.$$  \hspace{1cm} (4.2)

If one performs one of these two changes of variables:

$$t^{\frac{1}{2}} \rightarrow t^{-\frac{1}{2}}, \quad \text{or} \quad t^{\frac{1}{2}} \rightarrow -t^{\frac{1}{2}},$$

(4.3)
one finds that (3.1) transforms into:

\[
Y_{n,m}(a, -z) = Y_{n,m}(\lambda, t^{-\frac{1}{2}} - t^{\frac{1}{2}}) = -\frac{[1]a^{nm}}{[1] - [0; 1]} \times \left( \sum_{\gamma + \beta + 1 = n, \gamma, \beta \geq 0} t^{-\frac{m}{2}(\beta - \gamma)} \lambda^{-m} (-1)^{\gamma} \times \left( \frac{1}{[n]} - \frac{1}{[\beta - \gamma; 1]} \right) \right. \\
\left. \times \frac{1}{[\beta]! [\gamma]!} \times \prod_{j=\gamma}^{\beta} [j; 1] + \left\{ \begin{array}{ll} 0, & \text{n odd;} \\
-1, & \text{n even.} \end{array} \right. \right),
\]

(4.4)

It is worth to remark that this is exactly the formula obtained when one calculates the polynomial for torus knots associated to the fundamental representation of \( Sp(N) \) from Chern–Simons theory. This can be shown explicitly using the methods developed in the previous section, or from the form of the skein rules for the fundamental of \( Sp(N) \) obtained in [7,8] from Chern-Simons gauge theory. Let us compare (3.1), (4.1), and (4.4). The crucial point is that, using the auxiliary variable \( q \), these three expressions can be written as follows:

\[
Y_{n,m}(a, q - q^{-1}) = \\
= \frac{a^{nm}[1]_q}{[1]_q + a - a^{-1}} \times \left( \sum_{\gamma + \beta + 1 = n, \gamma, \beta \geq 0} q^{-m(\beta - \gamma)} a^{-m} (-1)^{\gamma} \times \left( \frac{1}{[n]_q} + \frac{1}{q^{\beta - \gamma} a - q^{-\gamma} a^{-1}} \right) \right. \\
\left. \times \frac{1}{[\beta]_q! [\gamma]_q!} \times \prod_{j=\gamma}^{\beta} (q^j a - q^{-1} a^{-1} + \left\{ \begin{array}{ll} 0, & \text{n odd;} \\
1, & \text{n even.} \end{array} \right. \right),
\]

(4.5)

\[
Y_{n,m}(a, -(q^{-1} - q)) = \\
= -\frac{a^{nm}[1]_q}{[1]_q - a + a^{-1}} \times \left( \sum_{\gamma + \beta + 1 = n, \gamma, \beta \geq 0} q^{-m(\beta - \gamma)} a^{-m} (-1)^{\gamma} \times \left( \frac{1}{[n]_q} - \frac{1}{q^{\beta - \gamma} a - q^{-\gamma} a^{-1}} \right) \right. \\
\left. \times \frac{1}{[\beta]_q! [\gamma]_q!} \times \prod_{j=\gamma}^{\beta} (q^j a - q^{-j} a^{-1} + \left\{ \begin{array}{ll} 0, & \text{n odd;} \\
-1, & \text{n even.} \end{array} \right. \right),
\]

(4.6)
and,

\[
P_{n,m}(a, q^{-1} - q) = \frac{a^{m(n-1)}[1]}{a - a^{-1}} \sum_{\substack{\gamma, \beta \geq 0 \\ \gamma + \beta + 1 = n}} q^{-m(\beta - \gamma)}(-1)^\gamma \frac{1}{[n]_q \beta_q! \gamma_q!} \prod_{j=-\gamma}^\beta (q^j a - q^{-j} a^{-1}),
\]

where,

\[
[n]_q = q^n - q^{-n}.
\]

The structure on the right hand side of (4.7) shows that the HOMFLY polynomial, \( P_{n,m}(a, z) \), can be expressed in terms of a linear combination of the polynomials \( Y_{n,m}(a, z) \) and \( Y_{n,m}(a, -z) \). In fact, after performing some algebra from (4.5), (4.6) and (4.7), one obtains,

\[
P_{n,m}(a, z) = \frac{1}{2} (Y_{n,m}(a, z) + Y_{n,m}(a, -z)) + \frac{z}{2(a - a^{-1})} (Y_{n,m}(a, z) - Y_{n,m}(a, -z)).
\]

This ends the proof of the relation (1.1) between the HOMFLY and Kauffman polynomials which was presented in the introduction.

For \( a = 1 \), the ordinary version of the Kauffman polynomial, \( F_K(a, z) \), becomes the unoriented polynomial invariant of ambient isotopy discovered in [28,29] which is usually denoted by \( Q_K(z) = F_K(1, z) \). Similarly, we define:

\[
\tilde{Q}_K(z) = Y_K(1, z).
\]

It turns out that for torus knots, after performing the limit \( N \to 1 \) in (3.1), which is equivalent to \( a \to 1 \), one finds:

\[
\tilde{Q}_{n,m}(z) = 1.
\]

In the case of the HOMFLY polynomial, the limit \( a \to 1 \) leads to the Alexander-
Conway polynomial, $\Delta_K(z) = P_K(1, z)$. From (4.9) and (4.11) one finds:

$$\Delta_{n,m}(z) = 1 + \frac{z}{4} \frac{\partial}{\partial a} \left( Y_{n,m}(a, z) - Y_{n,m}(a, -z) \right) \bigg|_{a=1}. \quad (4.12)$$

Notice that this expression is consistent with the fact that $\Delta_{n,m}(z)$ must be 1 plus a polynomial containing only even powers of $z$. 
5. Conclusions and prospects

In this paper we have presented the construction of the operator formalism, originally discussed in 1 and 2 for the groups $SU(2)$ and $SU(N)$ respectively, for an arbitrary simple group. The main result in this respect is the general form for knot operators presented in (2.66).

Knot operators are utilized to compute the knot invariant corresponding to the fundamental representation of the gauge group $SO(N)$. The resulting formula is presented in (3.40) and (3.1), and shown to agree with a previous expression for the Kauffman polynomial. This formula is compared to known expressions for the HOMFLY polynomial and the relation (1.1) between the Kauffman and the HOMFLY polynomials for torus knots is proved.

Our result (1.1) confirms that the Kauffman polynomial is more fundamental than the HOMFLY polynomial. The simplicity of the relation obtained suggest that it could be obtained by other methods. In this respect it would very interesting if it could be reobtained using skein rules.

It would be also worthwhile to study how our results fit in Jaeger’s expansions for the Kauffman polynomial in terms of HOMFLY polynomials (see for example pag. 219 of [26]) Finally, one should also study if there exist similar formulas for other sets of knots. In this respect one would like to start studying the situation in sets characterized by a generalization of the notion of a torus knot. A torus knot is a knot that can be placed on the surface of a standardly embedded torus in $S^3$ without self-intersection. There are knots which can be placed on a standardly embedded genus two surface without self-intersection but not on a genus one surface. One could analyze for example if there is a relation of the type (1.1) for these knots. In general one could study the problem for knots placed on a genus $g$ surface. Work in this direction will be presented elsewhere.
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APPENDIX

GROUP-THEORETICAL CONVENTIONS.

In this section of the Appendix we will summarize our group-theoretical conventions. Let $G$ be a compact simple group of rank $l$, with generators $T^a$, $a = 1, \ldots, \dim(G)$, which are chosen to be antihermitian. For the fundamental representation of $G$ they are normalized as follows:

$$\text{Tr}(T^a T^b) = -y \psi^2 \delta^{ab}$$

where $y$ is the Dynkin index of the fundamental representation and $\psi^2$ is the squared length of the longest simple root of $G$. The value of $y$ for the groups $SU(N)$, $SO(N)$, $Sp(N)$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$ are $1/2, 1, 1/2, 9, 12, 30, 6$ and $3$, respectively.

We will denote the $l$ fundamental roots of $G$ by $\alpha_i$, $i = 1, \ldots, l$. In the explicit calculations carried out in sect. 3 they have been chosen in such a way that the long roots have length $\sqrt{2}$, i.e., $\psi^2 = 2$. The Cartan matrix $g_{ij}$,

$$g_{ij} = 2 \frac{\alpha(i) \cdot \alpha(j)}{\alpha(i) \cdot \alpha(i)}$$

takes the following forms for the two Lie algebras $B_l$ ($l = (N - 1)/2$, $N$ odd) and $D_l$ ($l = N/2$, $N$ even) associated to the simple group $SO(N)$, which is the one that has been considered in this paper:

$$g_{ij}(B_l) = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & -2 & 2
\end{pmatrix}, \quad (A.3)$$

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and,

\[
g_{ij}(D_l) = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 2 & -1 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0
\end{pmatrix}.
\]  \quad (A.4)

We will denote the root lattice by \( \Lambda_R \). This \( l \)-dimensional space is generated by the fundamental roots \( \alpha_{(i)} \), which can be taken as a basis, the root basis. Any vector \( x \) in this basis has components \( x^i \) given by:

\[
x = \sum_{i=1}^{l} x^i \alpha_{(i)}.
\]  \quad (A.5)

Among all the roots in \( \Lambda_R \) there is a subset which plays an important role in the calculation performed in the paper. These are the positive roots. For \( SO(N) \) they take the form [30],

- algebra \( B_l \):

\[
\beta_{(j)} = \alpha_{(j)} + \cdots + \alpha_{(l)}, \quad j = 1, \ldots, l,
\]

\[
\gamma_{(j,k)} = \alpha_{(j)} + \cdots + \alpha_{(j+k)}, \quad i = 1, \ldots, l - 1, \quad k = 1, \ldots, l - j - 1,
\]

\[
\delta_{(j,k)} = \alpha_{(j)} + \cdots + \alpha_{(j+k)} + 2(\alpha_{(j+k+1)} + \cdots + \alpha_{(l)}),
\]

\[
j = 1, \ldots, l - 1, \quad k = 0, \ldots, l - j - 1.
\]  \quad (A.6)
- algebra $D_l$:

\[
\alpha(j), \quad j = 1, \ldots, l,
\]

\[
\beta(j) = \alpha(j) + \ldots + \alpha(l-2) + \alpha(l), \quad j = 1, \ldots, l - 2,
\]

\[
\gamma(j,k) = \alpha(j) + \ldots + \alpha(j+k), \quad j = 1, \ldots, l - 2, \quad k = 1, \ldots, l - j,
\]

\[
\delta(j,k) = \alpha(j) + \ldots + \alpha(j+k) + 2(\alpha(j+k+1) + \ldots + \alpha(l-2)) + \alpha(l-1) + \alpha(l),
\]

\[
j = 1, \ldots, l - 3, \quad k = 0, \ldots, l - 3 - j. \tag{A.7}
\]

The fundamental weights $\lambda^{(i)}$, $i = 1, \ldots, l$, satisfy:

\[
2 \frac{\alpha(i) \cdot \lambda^{(j)}}{\alpha(i) \cdot \alpha(i)} = \delta^{ij}. \tag{A.8}
\]

The fundamental weights generate over $\mathbb{Z}$ an $l$-dimensional lattice called the weight lattice which will be denoted by $\Lambda_w$. The lattices $\Lambda_R$ and $\Lambda_w$ are dual to each other and $\Lambda_R \in \Lambda_w$. The $l$-dimensional basis expanded by the fundamental weights is called the Dynkin basis. Any vector $x$ has in this basis components $x_i$ given by:

\[
x = \sum_{i=1}^{l} x_i \lambda^{(i)}. \tag{A.9}
\]

The matrix $G^{ij} = \lambda^{(i)} \cdot \lambda^{(j)}$ gives the metric in weight space, so it allows us to rise indices. Its expression for the algebras $D_l$ and $B_l$ is:

\[
G^{ij}(D_l) = \frac{1}{2} \begin{pmatrix}
2 & 2 & 2 & \cdots & 2 & 1 & 1 \\
2 & 4 & 4 & \cdots & 4 & 2 & 2 \\
2 & 4 & 6 & \cdots & 6 & 3 & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
2 & 4 & 6 & \cdots & 2(l-2) & l-2 & l-2 \\
1 & 2 & 3 & \cdots & l-2 & l/2 & (l-2)/2 \\
1 & 2 & 3 & \cdots & l-2 & (l-2)/2 & l/2
\end{pmatrix}, \tag{A.10}
\]
and,
\[
G^{ij}(B_l) = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(l-1) & l-1 \\ 1 & 2 & 3 & \cdots & l-1 & l/2 \end{pmatrix}.
\]

(A.11)

Among the weights in $\Lambda_{W}$ there is one which plays an important role in Chern-Simons theory because it can be regarded as the vacuum. This weight is denoted by $\rho$ and all its components are one:

\[
\rho = \sum_{i=1}^{l} \lambda^{(i)}.
\]

(A.12)

The irreducible representations of $G$ are characterized by highest weights $\Lambda$. Highest weights can be written uniquely as a linear combination of fundamental weights with non-negative integer coefficients $h_i$,

\[
\Lambda = \sum_{i=1}^{l} h_i \lambda^{(i)}.
\]

(A.13)

The set of weights of an irreducible representation of highest weight $\Lambda$ will be denoted as $M_\Lambda$. To build this set one may use the following rule: if a weight $\mu \in M_\Lambda$ has the $k^{th}$ Dynkin component greater than zero (i.e., $\mu_k > 0$), then the vectors obtained by subtracting $t\alpha_k$ ($t = 1, \ldots, \mu_k$) from $\mu$ are also elements of $M_\Lambda$. One can start applying this rule to $\Lambda$ and then to the successive weights obtained to build the different elements of $M_\Lambda$. The multiplicities of each weight can be obtained using Freudenthal’s formula [31].
The Weyl group is generated by $r$ reflections $\sigma_i$, $i = 1, \ldots, l$, on weight space $x \in \Lambda_W$, $\sigma_i(x) = x - \frac{2}{\alpha(i) \cdot \alpha(i)} \alpha(i) (\alpha(i) \cdot x)$. \hspace{1cm} (A.14)

It divides the weight lattice $\Lambda_W$ into domains. The fundamental domain or Weyl chamber is chosen to be the one containing all the weights $x \in \Lambda_W$ such that,

$$\alpha(i) \cdot x \geq 0. \hspace{1cm} (A.15)$$

The Weyl character for an irreducible representation of highest weight $\Lambda$ is defined as,

$$\text{ch}_\Lambda(a) = \sum_{\mu \in M_\Lambda} e^{a \cdot \mu}, \hspace{1cm} (A.16)$$

where $a = a_i \lambda^{(i)}$. The Weyl character satisfies the equation [31],

$$\text{ch}_\Lambda(a) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho) \cdot a}}{\sum_{w \in W} \epsilon(w) e^{w(\rho) \cdot a}}, \hspace{1cm} (A.17)$$

known as the Weyl character formula. When $a = -\rho$, we have an expression for the character [25] which is particularly useful:

$$\sum_{\mu \in M_\Lambda} e^{-\mu \cdot \rho} = \prod_{\alpha > 0} \frac{e^{\frac{1}{2} \alpha \cdot (\rho + \Lambda)} - e^{-\frac{1}{2} \alpha \cdot (\rho + \Lambda)}}{e^{\frac{1}{2} \alpha \cdot \rho} - e^{-\frac{1}{2} \alpha \cdot \rho}}, \hspace{1cm} (A.18)$$

where $\alpha > 0$ denotes a sum over all positive roots.

An important set of weights used in this work is the one made by Weyl-antisymmetric combinations of weights in $\Lambda_W/s\Lambda_R$ where $s$ is an arbitrary non-negative integer. This set of weights builds the fundamental chamber $F_s$.

**Fundamental representation of $SO(2l+1)$**
The fundamental representation of $B_l$ is associated to the highest weight $\Lambda = \lambda^{(1)} = (1, 0, \ldots, 0)$, and the corresponding weight space is:

$$M_{\lambda^{(1)}} = \{ \mu_i : 1 \leq i \leq 2l + 1 \},$$  \hspace{1cm} (A.19)

where:

$$\mu_1 = \lambda^{(1)} = (1, 0, \ldots, 0),$$

$$\mu_j = \mu_{j - 1} - \alpha_{(j - 1)} = (0, \ldots, -j, 1, 0, \ldots, 0), \quad j = 1 \ldots l - 1,$$

$$\mu_l = \mu_{l - 1} - \alpha_{(l - 1)} = (0, \ldots, 0, -1, 2),$$  \hspace{1cm} (A.20)

$$\mu_{l + 1} = \mu_l - \alpha_{(l)} = 0,$$

$$\mu_{l + 1 + i} = -\mu_{l + 1 - i}, \quad i = 1, \ldots, l.$$

We can write these weights as follows:

$$\mu_j = \sum_{i=1}^{l} [\delta_{j - 1, i} - \delta_{j, i}] \lambda^{(i)}, \quad j = 1, \ldots, l, \quad j \neq l - 1,$$

$$\mu_{l - 1} = [-\delta_{l - 1, i} + 2\delta_{l, i}] \lambda^{(i)},$$

$$\mu_{l + 1 + i} = -\mu_{l + 1 - i}, \quad i = 1, \ldots, l.$$  \hspace{1cm} (A.21)

We also need the scalar products $\rho \cdot \mu_i$ and $\mu_i \cdot \mu_i$. Using the form (A.20) and (A.11), we can easily find:

$$\mu_i^2 = 1, \quad i = 1, \ldots, 2l + 1, \quad i \neq l + 1,$$

$$\mu_{l + 1}^2 = 0,$$

$$\rho \cdot \mu_i = \frac{1}{2} [2i - (2i - 1)], \quad i = 1, \ldots, l,$$  \hspace{1cm} (A.22)

$$\rho \cdot \mu_{l + 1} = 0,$$

$$\rho \cdot \mu_{l + 1 + i} = -\frac{1}{2} (2i - 1), \quad i = 1, \ldots, l.$$

The action of the Weyl reflections on the fundamental weights $\lambda^{(i)}$ follows from
(A.14):

\[\sigma_1(x) = (-x_1, x_2 + x_1, x_3, \ldots, x_l),\]
\[\sigma_i(x) = (x_1, \ldots, x_{i-1} + x_i, -x_i, x_{i+1}, \ldots, x_l), \quad i = 1, \ldots, l - 2,\]
\[\sigma_{l-1}(x) = (x_1, \ldots, x_{l-3}, x_{l-2} + x_{l-1}, -x_{l-1}, x_l + 2x_{l-1}),\]
\[\sigma_l(x) = (x_1, \ldots, x_{l-2}, x_{l-1} + x_l, -x_l).\]

**Fundamental representation of SO(2l)**

In this section we present the results concerning the fundamental representation of \(D_l\). It is associated to the highest weight \(\Lambda = \lambda^{(1)} = (1, 0, \ldots, 0)\), and the corresponding weight space is:

\[M_{\lambda^{(1)}} = \{\mu_i : 1 \leq i \leq 2l\},\]  
(A.24)

where:

\[\mu_1 = \lambda^{(1)} = (1, 0, \ldots, 0),\]
\[\mu_j = \mu_{j-1} - \alpha_{(j-1)} = (0, \ldots, -\frac{j}{2}, 1, 0, \ldots, 0), \quad j = 1, \ldots, l - 2,\]
\[\mu_{l-1} = \mu_{l-2} - \alpha_{(l-2)} = (0, \ldots, 0, -1, 1, 1),\]  
(A.25)
\[\mu_l = \mu_{l-1} - \alpha_{(l-1)} = (0, \ldots, 0, -1, 1),\]
\[\mu_{l+i} = -\mu_{l+1-i}, \quad i = 1, \ldots, l.\]

We can write these weights as follows:

\[\mu_j = \sum_{i=1}^{l} \left[ -\delta_{j-1,i} + \delta_{j,i} \right] \lambda^{(i)}, \quad j = 1, \ldots, l \quad j \neq l - 1,\]  
(A.26)
\[\mu_{l-1} = \left[ -\delta_{l-2,i} + \delta_{l-1,i} + \delta_{l,i} \right] \lambda^{(i)},\]
\[\mu_{l+i} = -\mu_{l+1-i}, \quad i = 1, \ldots, l.\]

We also need the scalar products \(\rho \cdot \mu_i\) and \(\mu_i \cdot \mu_i\). Using the form (A.26) and
(A.10), we easily find:

\[ \mu_i^2 = 1, \quad i = 1, \ldots, 2l, \]
\[ \rho \cdot \mu_i = l - i, \quad i = 1, \ldots, l, \]
\[ \rho \cdot \mu_{l+i} = -(i - 1), \quad i = 1, \ldots, l. \]  

(A.27)

The action of the Weyl reflections on the fundamental weights \( \lambda_i \) follows from (A.14):

\[ \sigma_1(x) = (-x_1, x_2 + x_1, x_3, \ldots, x_l), \]
\[ \sigma_i(x) = (x_1, \ldots, x_{i-1} + x_i, -x_i, x_i + x_{i+1}, \ldots, x_l) \quad i = 1, \ldots, l - 3, \]
\[ \sigma_{l-2}(x) = (x_1, \ldots, x_{l-4}, x_{l-3} + x_{l-2}, -x_{l-2}, x_{l-1} + x_{l-2}, x_l + x_{l-2}), \]
\[ \sigma_{l-1}(x) = (x_1, \ldots, x_{l-3}, x_{l-2} + x_{l-1}, -x_{l-1}, x_l), \]
\[ \sigma_l(x) = (x_1, \ldots, x_{l-3}, x_{l-2} + x_l, x_{l-1}, -x_l). \]  

\[ \text{(A.28)} \]

**Theta functions of level } s**

The Theta functions of level } s (being \( s \) an arbitrary positive integer) play a fundamental role in the construction of the Hilbert space presented in sect. 2. They are defined as follows [25]:

\[ \Theta_{s,p}(a, \tau) = \sum_{\nu \in \mathcal{L}_R} \exp \left\{ \frac{2\pi is}{\psi^2} \left( \nu + \frac{p}{s} \right)^2 + 2\pi is(\nu + \frac{p}{s}) \cdot a \right\}, \]

\[ \text{(A.29)} \]

where \( \mathcal{L}_R \) stands for the long root lattice. These functions are well defined for \( \text{Im} \tau > 0 \), which makes the sum convergent. We will consider the case where \( p \) belongs to the weight lattice \( \Lambda_W \).

The Theta functions in (A.29) satisfy some important properties [32]. The first one, which follows trivially from its definition (A.29), is the following: a displacement of \( p \) by a vector in \( s\mathcal{L}_R \) does not change (A.29),

\[ \Theta_{s,p+s\alpha}(a, \tau) = \Theta_{s,p}(a, \tau), \quad \alpha \in \mathcal{L}_R. \]

\[ \text{(A.30)} \]

This shows that \( p \) in \( \Theta_{s,p}(a, \tau) \) lives in the domain \( p \in \frac{\Lambda_W}{s\mathcal{L}_R} \). Another important
property is the following. Consider \( m \) and \( n \) two vectors in \( \mathcal{L}_R \), \( m, n \in \mathcal{L}_R \). Then,

\[
\Theta_{s,p}(a + \frac{2(m + n\tau)}{\psi^2}) = e^{2\pi i s \frac{a}{\psi^2}} e^{-2\pi i s n \cdot a} \Theta_{s,p}(a, \tau).
\] (A.31)

Of particular interest in our analysis are the Weyl antisymmetric combinations of Theta functions of level \( s \). Let’s define them as:

\[
\Theta^A_{s,p}(a, \tau) = \sum_{w \in W} \epsilon(w) \Theta_{s,w(p)}(a, \tau),
\] (A.32)

where \( \epsilon(w) \) is the signature of the permutation corresponding the Weyl group element \( w \). These functions satisfy:

\[
\Theta^A_{s,p}(a, \tau) = \epsilon(w) \Theta^A_{s,w(p)}(a, \tau),
\] (A.33)

so they are Weyl antisymmetric. This property implies some relations between the antisymmetrized theta functions of level \( s \). Finally, we recall the behavior of the theta functions under modular transformations. The modular group is generated by the transformation \( S \),

\[
\tau \rightarrow -\frac{1}{\tau},
\]

\[
a \rightarrow a\tau,
\] (A.34)

and the transformation \( T \),

\[
\tau \rightarrow \tau + 1,
\]

\[
a \rightarrow a.
\] (A.35)

The theta functions of level \( s \) transform under them as:

\[
\Theta_{s,p}\left(\frac{a}{\tau}, -\frac{1}{\tau}\right) = \left(\frac{\text{Vol} \mathcal{L}_R}{\text{Vol} \mathcal{L}_R}\right)^{\frac{1}{2}} \left(\frac{\tau}{i s}\right)^{\frac{1}{2}} e^{i \pi s a^2 \psi^2} \sum_{q \in \frac{\Lambda W}{\mathcal{L}_R}} e^{-4\pi i \frac{a^2}{s\psi^2} q} \Theta_{s,q}(a, \tau),
\] (A.36)

and,

\[
\Theta_{s,p}(a, \tau + 1) = e^{2\pi i s \frac{a^2}{\psi^2}} \Theta_{s,p}(a, \tau).
\] (A.37)

In (A.36) \( \text{Vol} \mathcal{L}_R \) is the volume of the fundamental cell of the long root lattice \( \mathcal{L}_R \),
and Vol $\mathcal{L}_R^*$ that of its dual lattice, $\mathcal{L}_R^*$. The values of their quotient are:

\[
\left( \frac{\text{Vol} \mathcal{L}_R^*}{\text{Vol} \mathcal{L}_R} \right)^{\frac{1}{2}} = \begin{cases} 
N^{-\frac{1}{2}}, & SU(N) \\
\frac{1}{2}, & SO(N) \\
2^{-\frac{N}{4}}, & Sp(N) 
\end{cases}
\]

(A.38)
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