We present a novel framework for dynamic cut aggregation in L-shaped algorithms. The aim is to improve the parallel performance of distributed L-shaped algorithms through reduced communication latency and load imbalance. We show how optimality cuts can be aggregated into arbitrary partitions without affecting convergence of the L-shaped algorithm. Furthermore, we give a worst-case bound for L-shaped algorithms with static cut aggregation and then extend this result for dynamic aggregation. Our approach requires tunable parameters, where the optimal settings for any given problem are unknown. We devise a simple tuning procedure and show its effectiveness. We also propose a variety of aggregation schemes that fit into our framework, and evaluate them on two large-scale stochastic programming problems. In addition, we propose a fixing strategy that combines the strengths of dynamic and static cut aggregation. Major performance improvements are possible with our approach in distributed settings. Our experimental results suggest that uniform cut aggregation, as well as our fixing strategy, can yield high performance at low overhead cost. These results are supported by our worst-case bounds.

1 Introduction

The well-known L-shaped algorithm [1] was originally proposed as a single-cut algorithm. It was later extended to a multi-cut variant, with better convergence properties on many test examples [2]. Recent contributions have explored aggregation strategies that fall between a single-cut and multi-cut approach [3, 4, 5]. The aim is to preserve the convergence properties of a multi-cut algorithm, while reducing the size growth of the master problem and communication overhead in distributed implementations. In this work, we formalize this approach and also present a novel dynamic aggregation procedure based on an alternative L-shaped formulation. We show how this formulation allows us to prototype various heuristic aggregation schemes. We provide worst-case complexity results and show that large performance gains are possible in practice by solving applied problems.

2 Preliminaries

We consider finite two-stage stochastic programs of the form

$$\begin{align*}
\text{minimize} & \quad c^T x + \sum_{s=1}^{n} \pi_s q_s^T y_s \\
\text{s.t.} & \quad Ax = b \\
& \quad T_s x + W y_s = h_s, \quad s = 1, \ldots, n \\
& \quad x \geq 0, \quad y_s \geq 0, \quad s = 1, \ldots, n,
\end{align*}$$

(1)
where $A \in \mathbb{R}^{p \times n}, T_s \in \mathbb{R}^{q \times n}, s = 1, \ldots, n$ and $W \in \mathbb{R}^{q \times m}$. We use the natural decomposition into a first and second stage:

$$\begin{align*}
\text{minimize} & \quad c^T x + \sum_{s=1}^{n} \pi_s Q_s(x) \\
\text{s.t.} & \quad Ax = b \quad x \geq 0,
\end{align*}$$

where

$$Q_s(x) = \min_{y_s \in \mathbb{R}^m} q_s^T y_s$$

$$\begin{align*}
\text{s.t.} & \quad W y_s = h_s - T_s x \\
& \quad y_s \geq 0.
\end{align*}$$

We assume throughout that $Q_s(x) < \infty, \forall s = 1, \ldots, n$ for any $x \in \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. This is not a restrictive assumption since the L-shaped algorithm can handle problems where this is not true independent of the techniques we propose here. Moreover, second-stage infeasibility can often be mitigated by reassessing the model and soften constraints that can be violated for certain $x$.

### 2.1 The L-shaped algorithm

The L-shaped algorithm decomposes (1) into a master problem and $n$ subproblems. During the procedure, solution candidates $x_k$ are generated by solving a master problem:

$$\begin{align*}
\text{minimize} & \quad c^T x + \theta \\
\text{s.t.} & \quad Ax = b \quad \theta \geq \Phi(x) \quad x \geq 0,
\end{align*}$$

(2)

where $\Phi(x)$ is an outer linearization of

$$Q(x) = \sum_{s=1}^{n} \pi_s Q_s(x).$$

During the L-shaped procedure, solution iterates $x_k$ are used to parameterize subproblems of the form:

$$\begin{align*}
\text{minimize} & \quad q_s^T y_s \\
\text{s.t.} & \quad W y_s = h_s - T_s x_k \\
& \quad y_s \geq 0.
\end{align*}$$

(3)

It follows from duality theory that $\lambda_s^T (h_s - T_s x)$, where $\lambda_s$ is the dual optimizer of (3), is a valid support function for $Q_s(x)$ and hence

$$\sum_{s=1}^{n} \pi_s \lambda_s^T (h_s - T_s x)$$

is a valid support function for $Q(x)$. In the original formulation of the L-shaped algorithm [1], the above result is used in each iteration $k$ to construct optimality cuts by introducing

$$\partial Q_k = \sum_{s=1}^{n} \pi_s \lambda_s^T T_s$$

$$q_k = \sum_{s=1}^{n} \pi_s \lambda_s^T h_s.$$

If the optimality cut is not satisfied by the current master iterate, i.e., if

$$\theta_k < q_k - \partial Q_k x_k,$$
then the optimality cut is included in the master problem as follows:

$$\min_{x \in \mathbb{R}^n} \ c^T x + \theta$$

s.t. $\ Ax = b$

$$\ \partial Q_{x, k} x + \theta \geq q_k, \ \forall k$$

$x \geq 0$. \hfill (4)

If any second-stage problem is infeasible for the given $x$, feasibility cuts can be generated and included in the master problem \cite{1}. The master problem is then re-solved to generate the next iterate $x_{k+1}, \theta_{k+1}$. This is repeated until

$$\left| Q(x_k) - \theta_k \right|$$

falls within some predefined relative tolerance, or if the latest optimality cut is already satisfied by the current master iterate, upon which the algorithm terminates. The L-shaped algorithm is finitely convergent because $W$ has a finite number of bases \cite{1}.

2.2 The multi-cut L-shaped algorithm

The original L-shaped algorithm was extended in \cite{2} by including all generated optimality cuts in a disaggregate form at each iteration. In other words, optimality cuts are generated for each subproblem:

$$\partial Q_{s,k} = \pi_s \lambda_s^T h_s$$

$$q_{s,k} = \pi_s \lambda_s^T h_s,$$

and these then enter a modified master problem as follows:

$$\min_{x \in \mathbb{R}^n} \ c^T x + \sum_{s=1}^{n} \theta_s$$

s.t. $\ Ax = b$

$$\ \partial Q_{s,k} x + \theta_s \geq q_{s,k}, \ \ s = 1, \ldots, n \ \ \forall k$$

$x \geq 0$. \hfill (5)

A given disaggregate optimality cut is rejected if it is already satisfied by the current master iterate:

$$\theta_{s,k} \geq q_{s,k} - \partial Q_{s,k} x_k$$

The authors of \cite{2} show that the resulting procedure will terminate in equal or fewer iterations than the original aggregate version \cite{1} if the major iterates coincide. A simple argument in favour of a multi-cut approach is that the master problem has more available information at each iteration and is therefore able to localize the set of optimal solutions faster. However, there is no general rule that the disaggregate master problem converges in fewer iterations for all problems. Also, the size of the master problem grows faster if the cuts are not aggregated, which has a negative effect on the time to solution. As a rule of thumb, the authors suggest that the single-cut approach should be preferred when the number of scenarios is considerably larger than the number of first stage constraints, i.e., when $n \gg p$. Finally, the authors suggest that it may be advantageous to adopt a so called “hybrid approach”, where cuts are aggregated in separate clusters, but do not explore the approach further. We propose a framework around this idea, which we introduce in the following section.

2.3 The aggregated L-shaped algorithm

To formalize the hybrid approach suggested in \cite{2} consider the following definition

**Definition 1.** A partitioning scheme

$$\mathcal{S} = \{ S_1, \ldots, S_A \}$$

of $n$ scenarios is a set of partitions such that

$$S_a \subseteq \{ 1, \ldots, n \}, \quad a = 1, \ldots, A$$

$$S_a \cap S_b = \emptyset, \quad \forall a \neq b$$

$$\bigcup_{a=1}^{A} S_a = \{ 1, \ldots, n \}.$$ \hfill (6)

$$\bigcup_{a=1}^{A} S_a = \{ 1, \ldots, n \}.$$ \hfill (7)
In an aggregated L-shaped algorithm, the results of solving subproblems in the same partition $S_a$ are used to create aggregated optimality cuts

$$\partial Q_{a,k} = \sum_{s \in S_a} \pi_s \lambda_s^T T_s$$
$$q_{a,k} = \sum_{s \in S_a} \pi_s \lambda_s^T h_s,$$

which then enter the master problem as follows:

$$\begin{align*}
\text{minimize} & \quad c^T x + \sum_{a=1}^A \theta_a \\
\text{s.t.} & \quad Ax = b \\
& \quad \partial Q_{a,k} x + \theta_a \geq q_{a,k}, \quad a = 1, \ldots, A \quad \forall k \\
& \quad x \geq 0.
\end{align*}$$

(8)

A given aggregated optimality cut is rejected if it is already satisfied by the current master iterate:

$$\theta_{a,k} \geq q_{a,k} - \partial Q_{a,k} x_k$$

We give a convergence proof for a general variant of this algorithm, where the partitioning scheme can vary over iterations, in a following section. Note that the partitioning scheme $S = \{S_1\}$ with $S_1 = \{1, \ldots, n\}$ corresponds to the original single-cut algorithm, while $S = \{\{a\} \mid a = 1, \ldots, n\}$ corresponds to the multi-cut algorithm. We introduce two entities that characterize any given partitioning scheme $S$.

**Definition 2.** The **aggregation size** of the partitioning scheme $S$ is given by

$$A(S) = |S|.$$  

**Definition 3.** The **aggregation level** of the partitioning scheme $S$ is given by

$$A_L(S) = \max_{a=1,\ldots,A(S)} |S_a|.N$$

It is clear that $A(S) = 1$, $A_L(S) = n$ for single-cut L-shaped, and $A(S) = n$, $A_L(S) = 1$ for multi-cut L-shaped. Moreover, these values constitute the extremes in terms of these characteristics, i.e., $1 \leq A(S) \leq n$, $1 \leq A_L(S) \leq n$ for any partitioning scheme $S$.

We extend the worst-case complexity analysis developed in [2] to the aggregated case. Recall the following definition:

**Definition 4.** Let $b_s$ represent the maximum number of different slopes of $Q_s(x)$ in any direction parallel to one of the axes. Then, $b = \max_a b_s$ is the **slope number** of $Q(x)$.

The worst-case complexity result developed by the authors of [2] is then given by the following theorem:

**Theorem 2.1.** The maximum number of iterations required to obtain an optimal solution of (1), of the single-cut L-shaped algorithm, is given by

$$[1 + n(b - 1)]^m,$$  

(9)

while the maximum number of iterations required to obtain an optimal solution of (1), of the multi-cut L-shaped algorithm, is given by

$$1 + n(b^m - 1),$$

(10)

where $b$ is the slope number of $Q(x)$.

Using similar arguments, we postulate and prove the following extended result for the aggregated L-shaped algorithm:

**Theorem 2.2.** The maximum number of iterations required to obtain an optimal solution of (1), of an aggregated L-shaped algorithm that uses a partitioning scheme $S = \{S_1, \ldots, S_{A(S)}\}$ satisfying (7), is given by

$$1 + \sum_{a=1}^{A(S)} [1 + |S_a|(b - 1)]^m - A(S),$$

(11)

where $b$ is the slope number of $Q(x)$.
**Proof.** In the worst case, a single facet of one of the $A(S)$ aggregates is identified at each iteration, so that all facets are identified before converging. Hence, because $A(S)$ facets are identified in the first iteration, the maximum number of iterations is $1 + M - A(S)$, where $M$ is the total number of facets that can be identified in all aggregates. Consider any of the aggregates $S_a$. In the worst case, $b_a = b$ for every $Q(x)$ in that aggregate. If so, it holds that one facet of this aggregate, in every direction $j$, consists of facets from each of its $|S_a|$ constituents, for a total of $|S_a|b$ combinations. However, the facet identified in the considered aggregate at the first iteration consists of $|S_a|$ facets because $b_a$ is initially unrestricted in the master problem. In the worst case, any new facet identified in aggregate $S_a$ includes only one facet that has not been identified before. There are $b - 1$ such slopes remaining for each of the constituents, for a total number of $1 + |S_a|(b - 1)$ facets in $S_a$. Moreover, this can occur in all $m$ dual directions of the subproblems. Hence, the maximum number of iterations required to identify all facets in the given aggregate $S_a$ is given by $[1 + |S_a|(b - 1)]^m$, and hence, $M = \sum_{a=1}^{A(S)}[1 + |S_a|(b - 1)]^m$. In conclusion, the maximum number of iterations of the aggregated L-shaped is in the worst case given by

$$1 + \sum_{a=1}^{A(S)}[1 + |S_a|(b - 1)]^m - A(S).$$

Because $|S_a| \leq A_L(S)$ holds by construction, we can bound the sum in (11). This yields the following upper bound on the worst-case complexity:

**Corollary 2.3.** The maximum number of iterations of an aggregated L-shaped algorithm, using a partitioning scheme $S = \{S_1, \ldots, S_A\}$ satisfying (11), is upper bounded by

$$1 + A(S)([1 + A_L(S)(b - 1)]^m - 1),$$

where $b$ is the slope number of $Q(x)$, and $m$ is the row dimension of $W$.

Note that, the original results in Theorem 2.1 are recovered for the single-cut L-shaped algorithm ($A(S) = 1$, $A_L(S) = n$) and for the multi-cut L-shaped algorithm ($A(S) = n$, $A_L(S) = 1$). The upper bound (12) is easier to reason about than (11), but it could be pessimistic for irregular aggregation schemes where $A(S) > 1$ and $A_L(S)$ is close to $n$. Both expressions (11) and (12) can grow astronomically large already for medium-scale problems. However, the worst-case results still indicate which aggregation schemes could be more performant. We can observe that decreasing the aggregation level $A_L(S)$ decreases the worst-case complexity. In addition, we can note that the aggregated L-shaped algorithm will in general have better worst-case performance than the single-cut L-shaped algorithm for large-scale problems. For example, the worst-case complexity of a uniform partitioning scheme, where $A_L(S) = n/A(S)$, is on the order of

$$\frac{n^m(b - 1)^m}{A(S)^{m - 1}},$$

as opposed to the single-cut complexity $n^m(b - 1)^m$. The size of the master problem grows slower for the aggregated L-shaped algorithm than the multi-cut L-shaped algorithm because $A(S) \leq n$ constraints are added at each iteration as opposed to $n$ cuts. Thus, provided that the practical iteration complexity of an aggregated L-shaped variant is not far worse than the multi-cut approach, performance improvements are possible.

### 3 Review of L-shaped aggregation schemes

A comprehensive review of past contributions related to algorithmic improvements of L-shaped algorithm variants is provided in [6] and also in the dissertation [7]. We give an overview of contributions related to aggregation strategies which to the best of our knowledge could be considered the state-of-the-art.

#### 3.1 Partial cut aggregation

The first usage of an aggregation approach of type (8) was presented in [8]. The main motivation is to reduce communication overhead in the distributed setting as well as time to solution when re-solving the master problem. The $n$ subproblems are distributed uniformly on $r$ worker nodes. This topology is then used to induce a uniform partitioning scheme $S = \{S_1, \ldots, S_n\}$ where $|S_w|$ is the number of subproblems on worker $w$. This minimizes the amount of data passed from every worker at each iteration. The numerical results do not clearly favor the aggregated approach over a multi-cut approach. However, the problem sizes were only on the order of $10^4$ variables and constraints in the performed experiments.
The partial-cut approach has since been shown to be effective in various applied problems [4][5]. Moreover, the results in [8][9][10] suggest that many problems are solved more efficiently with an aggregation level somewhere between the single-cut and multi-cut, i.e., partitioning schemes \( S \) where \( 1 < A_L(S) < n \). For instance, in [10], a uniform aggregation scheme with variable aggregation level is adopted in a distributed setting to solve a large problem instance of 1000 scenarios corresponding to 2.5 million variables and 1.4 million constraints. The optimal aggregation level was found to be groups of 10 scenarios when solved on 8 worker cores. However, the beneficial effect on solution time appears problem-dependent and the optimal aggregation level \( A_L(S) \) is not known \textit{a priori}.

### 3.2 Adaptive multi-cut aggregation

A more recent aggregation approach is presented in [3]. The authors suggest an adaptive aggregation policy, where the partitioning \( \mathcal{S}_k^k = \{S_1^k, \ldots, S_{A_k}^k\} \) is allowed to vary at each iteration \( k \). The master problem is of the form (8). Hence, if \( S_k^k \neq S^{k-1} \), then the cuts generated at iteration \( k \) will not form valid supports for the second-stage objective because the master variables \( \{\theta_a\}_{a=1}^{A_k} \) adhere to a specific partitioning. This is alleviated by repartitioning the master variables to match the new partitioning \( \mathcal{S}_k^k \). Specifically, if \( S_1^{k-1}, \ldots, S_{A_k}^{k-1} \) are aggregated in \( S_k^k \), then the master variables \( \theta_1, \ldots, \theta_j \) are removed from (8) and replaced by a single new variable. Cuts from previous iterations are aggregated to adhere to the new partitioning. The authors suggest that disaggregation is also possible, but intractable in practice since it requires bookkeeping of all cuts. Consequently, \( A_L(S_k^k) \geq A_L(S^{k-1}) \) holds for the suggested adaptive aggregation scheme. The idea is therefore to initialize with no aggregation and run the adaptive aggregation scheme with the hope of eventually identifying an efficient aggregation level for the given problem.

The authors of [3] present two heuristic rules, based on a redundancy threshold and a bound on the number of aggregates, to decide how to determine the subsequent partitioning \( S_k^k \) based on \( S^{k-1} \). They mention trying other rules, based on for example cut similarity, but state that such efforts yield no significant gains in performance. The authors also perform exhaustive tests of uniform aggregation schemes of fixed size, which they refer to as \textit{static aggregation}. These results also indicate that many problems are solved faster when \( 1 < A_L(S) < n \).

### 3.3 Cut consolidation

Another aggregation technique is presented in [11]. The technique, \textit{cut consolidation}, is adopted to reduce the size of the master problem, and acts independently of the aggregation scheme used. The idea is to prune historical cuts that have become inactive, but retain their aggregation to keep some information in the master. Specifically, a consolidation scheme is used where if a set portion of cuts at a previous iteration \( k \)

\[
\partial Q_{s,k} x + \theta_s \geq q_{s,k}, \quad s = 1, \ldots, n
\]

have been inactive in the master for a set number of iterations then they are removed from the master and the special aggregate

\[
\sum_{s=1}^{n} \partial Q_{s,k} + \sum_{s=1}^{n} \theta_s \geq \sum_{s=1}^{n} q_{s,k}
\]

is added to the master. Numerical results indicate that cut consolidation can considerably reduce the time to solution, especially in combination with a partial aggregation scheme.

### 4 Dynamic cut aggregation

We propose a new aggregation procedure, which we call \textit{dynamic cut aggregation}. This procedure addresses two drawbacks of the adaptive aggregation procedure presented in [3]. First, the partitioning of the master variables \( \theta_a \), \( a = 1, \ldots, A_k \) must always match the current partitioning scheme \( S_k^k \) during the adaptive procedure. Consequently, any changes to the partitioning scheme incurs deleting and adding columns in the master problem, which can lead to significant overhead for large sparse problems. Moreover, the cuts from previous iterations have to be updated to adhere to the new partitioning. This incurs a large number of constraint replacements, which also increases the overhead in master iterations. Second, the nature of the implementation in [3] makes disaggregation of cuts non-performant. Therefore, the partitioning can only be made coarser.

The main idea of our approach is to retain the structure of the multi-cut master problem (5), while still allowing for a dynamic partitioning scheme that can vary over iterations. We build upon the constructs introduced in Section 2.3.

**Definition 5.** A dynamic partitioning scheme

\[ D = \{S_k^k\}_{k=1}^{\infty} \]  

is a sequence of partitioning schemes \( S_k^k = \{S_1^k, \ldots, S_{A_k}^k\} \), each satisfying (7).
Next, we pose an L-shaped algorithm with dynamic cut aggregation. Our reformulated master problem has the following form:

\[
\begin{align*}
\text{minimize} & \quad c^T x + \sum_{s=1}^{n} \theta_s \\
\text{s.t.} & \quad Ax = b \\
& \quad \sum_{s \in S_k^h} \partial Q_{k,s} x + \sum_{s \in S_k^h} \theta_s \geq \sum_{s \in S_k^h} q_{s,k}, \quad S^k \in \mathcal{D} \quad \forall k \\
& \quad x \geq 0.
\end{align*}
\] (14)

Again, a new cut aggregate is only added to the master problem if it not satisfied by the current master iterate, i.e., if

\[
\sum_{s \in S_a^h} \tilde{\theta}_s < \sum_{s \in S_a^h} (q_{k,s} - \partial Q_{k,s} \hat{x}).
\]

If the partitioning scheme is fixed every iteration \( S^k = \{S_1, \ldots, S_A\} \), the aggregated master problem (8) is recovered through the variable substitutions

\[
\theta_a = \sum_{s \in S_a} = 1, \ldots, A.
\]

With our reformulation (14), the number of master columns is not affected by the changes to the partitioning scheme and cuts from previous iterations remain valid. Moreover, disaggregation is possible; so, the partitioning scheme can vary between single-cut and multi-cut at each iteration. The flexibility of the formulation (14) enables us to test a large variety of aggregation schemes, which we present in a following section.

### 4.1 Convergence

We give a short proof of finite convergence for the L-shaped algorithm with dynamic cut aggregation.

**Theorem 4.1.** An L-shaped algorithm that uses dynamic cut aggregation, with a dynamic partitioning scheme \( \mathcal{D} = \{S^k\}_{k=1}^{\infty} \) for which the partitioning scheme \( S^k \) at each iteration satisfies the conditions (7), converges to an optimal solution of (1) in a finite number of iterations.

**Proof.** We assume without loss of generality that (1) has complete recourse, since we can otherwise fallback to the standard proof using a finite number of feasibility cuts. By duality,

\[
Q(x) = \sum_{s=1}^{n} \pi_s Q_s(x) = \sum_{s=1}^{n} \max_{\lambda_s \in \Lambda_s} \lambda_s^T(h_s - T_s x),
\] (15)

where \( \Lambda_s = \{\lambda \in \mathbb{R}^q \mid W^T \lambda \leq q_s\} \) and \( \bar{\Lambda}_s \) are the extreme points of \( \Lambda_s \). Hence, a full representation of (2) is given by

\[
\begin{align*}
\text{minimize} & \quad c^T x + \theta \\
\text{s.t.} & \quad Ax = b \\
& \quad \theta \geq \sum_{s=1}^{n} \pi_s \lambda_s^T(h_s - T_s x) \quad (\lambda_1, \ldots, \lambda_n) \in \bar{\Lambda}_1 \times \cdots \times \bar{\Lambda}_n \\
& \quad x \geq 0.
\end{align*}
\]

Now, for any partitioning scheme that satisfies (7), the \( A_k \) optimality cut aggregates generated during one iteration will form supports of the second-stage objective since

\[
\theta = \sum_{a=1}^{A} \sum_{s \in S_a^h} \theta_s \geq \sum_{a=1}^{A} \sum_{s \in S_a^h} \pi_s \lambda_s^T(h_s - T_s x) = \sum_{s=1}^{n} \pi_s \lambda_s^T(h_s - T_s x)
\]
for some \((\lambda_1, \ldots, \lambda_n) \in \bar{A}_1 \times \cdots \times \bar{A}_n\), which is exactly one of the facets of \(Q(x)\). Every iteration a new iterate \(\hat{x}^k\) and \(\{\theta^k_s\}_{s=1}^n\) is obtained from solving the master problems. Now, it can hold that
\[
\sum_{s \in S^k_a} \hat{\theta}_s < \sum_{s \in S^k_a} (q_{k,s} - \partial Q_{k,s}\hat{x}^k)
\]
for some, or all, of the current iteration aggregates. If so, the current set of aggregated cuts in the master do not impose
\[
\sum_{s=1}^n \theta_s \geq Q(x).
\]
Therefore, a new set of second-stage dual multipliers, not already present in the master problem, will be added through aggregated optimality cuts. Because each set of extreme points \(\bar{A}_s\) is finite, this can only occur finitely many times. Therefore, it must eventually hold that
\[
\sum_{s \in S^k_a} \hat{\theta}_s \geq \sum_{s \in S^k_a} (q_{k,s} - \partial Q_{k,s}\hat{x}^k)
\]
for all \(a = 1, \ldots, A_k\) so that
\[
\hat{\theta}^k = \sum_{a=1}^A \sum_{s \in S^k_a} \hat{\theta}_s \geq \sum_{a=1}^A \sum_{s \in S^k_a} \pi_s \lambda_s^T (h_s - T_s\hat{x}^k) = \sum_{s=1}^n \pi_s \lambda_s^T (h_s - T_s\hat{x}^k).
\]
Now, since \(\hat{\theta}^k\) is optimal and the \(\theta_s\) are free in (14) except for the cut constraints, it follows that
\[
\hat{\theta}^k = Q(\hat{x}^k) = \sum_{s=1}^n \pi_s \max_{\lambda_s \in \bar{A}_s} \lambda_s^T (h_s - T_s\hat{x}^k) \leq \sum_{s=1}^n \pi_s \max_{\lambda_s \in \bar{A}_s} \lambda_s^T (h_s - T_s\hat{x}) = Q(x).
\]
In conclusion, \(\hat{x}^k\) is an optimal solution to (1).

### 4.2 Complexity

Since the partitioning scheme used in dynamic cut aggregation can vary with iterations, the worst-case result in Theorem 2.2 does not hold and must be extended. First, we introduce some well-known combinational constructs that are required in the analysis.

**Definition 6.** A \(k\)-combination of \(n\) elements is a subset of \(1, \ldots, n\) of size \(k\). The number of \(k\)-combinations out of \(n\) elements is denoted by \(\binom{n}{k}\).

**Definition 7.** The Stirling number of the second kind is the number of ways to partition \(n\) elements into \(k\) non-empty subsets, and is denoted by \(\Lambda(n, k)\).

**Definition 8.** The \(n\)th Bell number, denoted by \(B_n\), is the number of possible partitionings of \(n\) elements. In terms of Stirling numbers it is given by
\[
B_n = \sum_{k=1}^n \binom{n}{k}
\]

We can now postulate and prove the following result for dynamic cut aggregation:

**Theorem 4.2.** The maximum number of iterations required to obtain an optimal solution of (1), of an L-shaped algorithm that uses dynamic cut aggregation with a dynamic partitioning scheme \(D = \{S^k_a\}_{k=1}^\infty\), is given by
\[
2 + \sum_{a_L=1}^n \binom{n}{a_L} [1 + a_L(b - 1)]^m - \sum_{a_L=1}^n \binom{n}{a_L} - A_0,
\]
where \(b\) is the slope number of \(Q(x)\).

**Proof.** In the worst case, a single facet of one of the \(A_k\) aggregates is identified at each iteration \(k\), so that all possible combinations of facets are identified before converging. Hence, because \(A_0\) facets are identified in the first iteration, the maximum number of iterations is \(1 + M - A_0\), where \(M\) is the total number of facets that can be identified in all possible aggregates. Consider any aggregate \(S^k_a\) at some iteration \(k\). We have already shown that the number of
facets that can be identified in this aggregate is given by \([1 + S^A_k(b - 1)]^m\). There are no assumed restrictions on the partitioning schemes in \(\mathcal{D}\). Therefore, any aggregate of the same size as \(S^A_k\) could be considered in subsequent iterations, each of which share the same number of possible facets that can identified. If the common size is denoted by \(a_L\), this number is given by \([1 + a_L(b - 1)]^m\). The number of aggregates that share the size \(a_L\) is given by the number of combinations of \(a_L\) out of \(n\). Moreover, the size of a given aggregate can vary between 1 and \(n\). Therefore, the total number of facets identifiable in all possible aggregates is given by \(\sum_{a_L=1}^{n} \binom{n}{a_L} [1 + a_L(b - 1)]^m\). When the algorithm has converged it will hold that all possible facets corresponding to some partitioning scheme \(S \in \mathcal{D}\) have been identified. Furthermore, in the worst case, there is only one facet in all other possible partitioning schemes that have not been identified before the final iteration. These facets will not be identified since the algorithm terminates; so, their total must be subtracted from the number of facets we can consider. This number is equal to the total number of possible partitioning schemes minus one due to the scheme active during the final iteration. The number of possible partitioning schemes is given exactly by the Bell number. Therefore, the maximum number of iterations required to converge is in the worst case given by:

\[
1 + \sum_{a_L=1}^{n} \binom{n}{a_L} [1 + a_L(b - 1)]^m - (B_n - 1) - A_0 = 2 + \sum_{a_L=1}^{n} \binom{n}{a_L} [1 + a_L(b - 1)]^m - \sum_{a_L=1}^{n} \binom{n}{a_L} - A_0.
\]

We can obtain a tighter bound by imposing restrictions on the dynamic partitioning scheme. For example, we can limit the size of the aggregates at each iteration, which simply removes summands in (16). The following result is obtained:

**Corollary 4.3.** The maximum number of iterations of an L-shaped algorithm with dynamic cut aggregation, where the dynamic partitioning scheme \(\mathcal{D}\) satisfies

\[
A_L(\mathcal{D}) \leq A_L(S^k) \leq \bar{A}_L(\mathcal{D}) \quad \forall S^k \in \mathcal{D}
\]

is given by

\[
2 + \sum_{a_L=A_L(\mathcal{D})}^{\bar{A}_L(\mathcal{D})} \binom{n}{a_L} [1 + a_L(b - 1)]^m - \sum_{a_L=A_L(\mathcal{D})}^{\bar{A}_L(\mathcal{D})} \binom{n}{a_L} - A_0,
\]

where \(b\) is the slope number of \(Q(x)\).

We can now again recover the original worst-case results presented in (2). The single-cut L-shaped algorithm corresponds to a dynamic aggregation scheme with \(A_L = \bar{A}_L = n\) and \(A_0 = 1\), for which we obtain

\[
2 + \binom{n}{1} [1 + n(b - 1)]^m - \binom{n}{1} - 1 = [1 + n(b - 1)]^m
\]

Likewise, the multi-cut L-shaped algorithm corresponds to a dynamic aggregation scheme with \(A_L = \bar{A}_L = 1\) and \(A_0 = n\), for which we obtain

\[
2 + \binom{n}{1} [1 + b - 1]^m - \binom{n}{1} - n = 1 + n(b^m - 1)
\]

As with static aggregation, we can improve the worst-case bound by decreasing the aggregation level of the partitioning schemes. In addition, we would expect performance improvements from any dynamic aggregation rule that limits the possible aggregate combinations. For example, we could fix the partitioning scheme after a certain number of iterations. In the worst case, we would then recover a static aggregation bound (12) plus the initial dynamic iterations. This is summarized in the following result:

**Corollary 4.4.** The maximum number of iterations of an L-shaped algorithm with dynamic cut aggregation, where the dynamic partitioning scheme \(\mathcal{D}\) satisfies

\[
S^k = S^N \quad \forall S^k \in \mathcal{D}, \quad k > N
\]

for some \(N\), is given by

\[
N + A(S^N) \left( [1 + A_L(S^N)(b - 1)]^m - 1 \right)
\]

where \(b\) is the slope number of \(Q(x)\).
The fixed scheme $S^N$ is in fact arbitrary in Corollary 4.4 but the $N$th scheme in $D$ is a natural choice. The idea is that the dynamic scheme identifies an efficient partitioning which can then be applied without the combinatorial effects of the dynamic scheme. We refer to this technique in the following as the fixing strategy.

Practical performance could of course be much better than suggested by the worst-case bounds (16), (17), and (18). In addition, any form of cut aggregation generally improves scalability in a distributed setting. Both communication latency and load imbalance among the master node and worker nodes are reduced. This holds since fewer cuts are passed from workers and the master problem does not grow as fast. Therefore, if the average iteration complexity of any aggregated L-shaped is comparable to the average multi-cut complexity in the single-core setting, then wall-clock time to solution can be greatly reduced if the aggregated L-shaped is run in parallel on distributed memory. It is not a general rule as the aggregation overhead could outweigh the gains from aggregation.

We introduce a framework for measuring empirical performance of L-shaped algorithm, to complement the worst-case analysis. Consider the following definitions.

**Definition 9.** The empirical iteration complexity of any given L-shaped algorithm $A$ applied to problem instance $P$, denoted by $N_I(A, P)$, is the number of L-shaped iterations of $A$ required to converge to an optimal solution of $P$, within some relative tolerance.

**Definition 10.** The empirical cut complexity of any given L-shaped algorithm $A$ applied to problem instance $P$, denoted by $N_C(A, P)$, is the number of optimality cuts in the master problem of $A$ after convergence to an optimal solution of $P$, within some relative tolerance.

Now, the discussion above can be made more precise. If an aggregated L-shaped algorithm has comparable empirical iteration complexity with that of the multi-cut L-shaped algorithm, but smaller empirical cut complexity, it is expected to perform better in a distributed setting. For most problems, we would expect a trade-off between these quantities. The worst-case bounds indicate that coarse aggregation schemes, with fewer cuts, require more iterations to converge. Likewise, fine aggregation schemes yield more cuts but fewer iterations to converge. The empirical complexities will not map directly to wall-clock time to solution, but we will show that low empirical cut complexity is a good indicator for when aggregation can yield better performance. We will use these quantities to benchmark the dynamic aggregation schemes introduced in the next section.

### 5 Aggregation schemes

We devise a simple abstraction for dynamic cut aggregation, and use it to extend the L-shaped algorithm implementation in LShapedSolvers.jl [10][12]. The default implementation is based on the multi-cut L-shaped formulation (5) and uses the subroutine given in Listing 1. In short, every generated optimality cut enters the master problem directly through `add_cut!`.

#### Listing 1: Subproblem subroutine in nominal L-shaped algorithm.

```julia
function resolve_subproblems!(lshaped::LShaped)
    # Update subproblems
    update_subproblems!(lshaped.subproblems, lshaped.x)
    # Solve subproblems
    for subproblem ∈ lshaped.subproblems
        cut = subproblem()
        add_cut!(lshaped, cut)
    end
    # Return current objective value
    return current_objective_value(lshaped)
end
```

We extend this implementation to include aggregation by introducing the `aggregate_cut!` and `flush!` functions, as shown in Listing 2. Cuts can be added to the master problem in either function calls. The idea is to aggregate and possibly add cuts in `aggregate_cut!` and then add any remaining aggregates in `flush!` after all cuts have been considered. We will show in the subsequent sections how this simple abstraction allows us to create a variety of aggregation schemes.
New aggregators are implemented by creating Julia objects of type `AbstractAggregation`. We recover the multi-cut implementation by having `aggregate_cut!` fall back to `add_cut!` and by having `flush!` be a no-op, as shown in Listing 3. Note, that for comparison we have also implemented partial cut aggregation in its original formulation as well as the single-cut L-shaped algorithm, using this abstraction.

5.1 Dynamic aggregation

The first proposed scheme is the dynamic aggregation. This scheme uses a fixed-length partitioning $S^k = \{S^k_1, \ldots, S^k_A\}$ where each aggregate $S^k_a$ can vary over iterations. A new optimality cut is placed in one of the aggregates based on a predefined selection rule. If the selection rule determines the chosen aggregate to be full, then the aggregated cut is added to the master problem and is then emptied. After all scenarios have been considered, any remaining non-empty aggregate is added to the master problem in the `flush!` function. The implementation in `LShapedSolvers.jl` is given in Listing 4.
Listing 4: Dynamic aggregation implementation.

```julia
struct DynamicAggregation <: AbstractAggregation
    aggregates::Vector{AggregatedOptimalityCut}
    rule::SelectionRule
end

function aggregate_cut!(lshaped, aggregation::DynamicAggregation, cut)
    (idx, full) = select(aggregation.rule, aggregation.aggregates, cut)
    aggregation.aggregates[idx] += cut
    if full
        aggregate = aggregation.aggregates[idx]
        add_cut!(lshaped, aggregate)
        aggregation.aggregates[idx] = zero(AggregatedOptimalityCut)
    end
    return nothing
end

function flush!(lshaped, aggregation::DynamicAggregation)
    for (i, aggregate) in enumerate(aggregation.aggregates)
        if !iszero(aggregate)
            add_cut!(lshaped, aggregate)
            aggregation.aggregates[i] = zero(AggregatedOptimalityCut)
        end
    end
    reset!(aggregation.rule)
    return nothing
end
```

5.1.1 Selection rules

A selection rule returns an aggregate index $1 \leq a \leq A$ based on the aggregates $S_1, \ldots, S_A$ and the cut candidate. The rule also determines if the chosen aggregate $S_a$ should be considered full and added to the master problem. Below, we list a set of selection rules we have implemented in LShapedSolvers.jl.

**SelectUniform**: Selects aggregates so that $|S_a| = T, a = 1, \ldots, A$ for some predefined $T$, with $TA \geq n$. This rule replicates partial cut aggregation, using formulation (14) instead of formulation (8). If $n$ is not divisible by $T$, then the final aggregate in the partition will consist of fewer than $T$ cuts. The worst-case bound for static aggregation (12) is recovered for this rule.

**SelectDecaying**: Functions like SelectUniform, but with $T_k = \max(T, \gamma^k T_0), \ 0 < \gamma < 1$ decaying over iterations. This rule is based on an observation that partitioning schemes with a high aggregation level are often inefficient close to the optimum. This suggests that initializing L-shaped with a high aggregation level and slowly progressing to a more disaggregate partitioning could be efficient. This would give the size conserving benefits of aggregation and fast convergence of multi-cut. Because the aggregation level decreases at each iteration, the number of aggregate combinations also decreases at each iteration. The number of iterations required to converge will therefore be much smaller than (16) when using this rule.

**SelectClosest**: Selects the aggregate that is currently closest to the considered cut. Closeness is measured by some predefined distance function. The distance functions we utilize are presented in [A]. If all aggregates are empty or no aggregate is close to the cut candidate within some relative tolerance $\tau$, then the cut candidate is placed in the next available empty aggregate. To ensure that an empty slot always exists, the rule deems an aggregate to be full if there are no empty aggregates available. The aggregation level will depend on the chosen distance tolerance and chosen distance measure. In general, the number of possible aggregate combinations can be decreased by lowering $\tau$.

**SelectClosestToReference**: Similar to the SelectClosest rule, but chooses aggregate based on the distance to some reference cut instead of calculating distances to the existing aggregates. Hence, instead of linear complexity in the number of aggregates, the rule operates in constant complexity. The idea is to reduce the aggregation overhead while still lowering L-shaped complexity. The reference cut we use is the aggregation of all cuts in the previous iteration. Consequently, the rule operates as the multi-cut L-shaped algorithm at the first iteration while the cuts are buffered for the first reference aggregate. Again, the aggregation level will depend on the chosen distance tolerance and chosen distance measure, and the number of possible aggregate combinations can be decreased by lowering $\tau$.

5.2 Cluster aggregation

The second proposed scheme is the cluster aggregation. Each iteration, the idea is to keep all new cuts in a buffer and perform aggregation with all information available. Consequently, it could be possible to determine a more effective
aggregation, albeit at the cost of larger overhead. Each incoming cut is put into a buffer. Hence, no cuts are added to the master before the flush! function is called. The flush! function adds all the new cuts, following a partitioning scheme $S^k$ determined by some predefined clustering rule. The implementation in LShapedSolvers.jl is given in Listing 5.

Listing 5: Cluster aggregation implementation.

```julia
struct ClusterAggregation <: AbstractAggregation
    buffer::Vector{OptimalityCut}
    rule::ClusterRule
end

function aggregate_cut!(lshaped, aggregation::ClusterAggregation, cut)
    push!(aggregation.buffer, cut)
    return nothing
end

function flush!(lshaped, aggregation::ClusterAggregation)
    if isempty(aggregation.buffer)
        return nothing
    end
    aggregates = cluster(aggregation.rule, aggregation.buffer)
    for aggregate in aggregates
        add_cut!(lshaped, aggregate)
    end
    empty!(aggregation.buffer)
    return nothing
end
```

5.2.1 Cluster rules

A cluster rule sorts the buffered cuts into a set of partitions $S^k = \{S^k_1, \ldots, S^k_A\}$.

ClusterByReference: Functions exactly like the dynamic selection rule with the exception that the reference cut is calculated by aggregating the cuts of the current iterate instead of the previous, because this information is now readily available. The aggregation level will depend on the chosen distance tolerance and chosen distance measure. In general, the number of possible aggregate combinations can be decreased by lowering $\tau$.

K-medoids: Sorts the cuts using k-medoids clustering [13]. K-medoids is an extension of the k-means algorithm for generalized distances. We cannot put precise bounds on the aggregation level because this will depend on the results of the k-medoid algorithm. Indirectly, the resulting clusters depend on the distance measure used. We expect that increasing $k$ will both decrease the aggregation level and reduce the possible aggregate combinations. We rely on the K-medoids algorithm implemented in the Julia package Clustering.jl[1] for the cluster calculations.

5.3 Hybrid aggregation

The final proposed scheme is the hybrid aggregation. This scheme extends the idea of the dynamic selection rule with decaying aggregation level. To recap, we have observed that aggressive aggregation is often not efficient around the optimum. Moreover, the size of the master problem may exceed memory capacity if a more disaggregate scheme is used from the beginning of the procedure. Therefore, we propose a hybrid procedure where we initially employ a scheme with heavy aggregation and then transition to a less aggregated scheme when a given condition is met. We propose to set the condition that the relative tolerance

$$\frac{|Q_k - \theta|}{|Q_k + \epsilon|}$$

decreases below some predefined threshold value. In other words, even though the transition will not be smooth like that of the decaying aggregation level rule, we gain the ability to transition between any of the schemes we have proposed so far. In the implementation, aggregate_cut! and flush! fallback to the definitions of the currently active aggregation. Inside flush!, we check if the transition condition is met and, if so, switch to the final aggregation scheme. Other transition conditions could be implemented and used instead. Although any aggregation schemes are viable here, we will typically transition from some $S_i$ to some $S_f$ where $A_L(S_f) < A_L(S_i)$ holds, cf. the discussion above. For many problems, we will probably prefer transitioning to multi-cut L-shaped. However, for large scale

[1]https://github.com/JuliaStats/Clustering.jl

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problems this may not be viable even if the optimum is almost attained, due to the addition of a large amount of cuts at every iteration. The implementation in \texttt{LShapedSolvers.jl} is given in Listing 6.

Listing 6: Hybrid aggregation implementation.

```julia
struct HybridAggregation <: AbstractAggregation
    initial::AbstractAggregation
    final::AbstractAggregation
    tolerance::Float64
end

function aggregate_cut!(lshaped, aggregation::HybridAggregation, cut)
    aggregate_cut!(lshaped, active(aggregation), cut)
    return nothing
end

function flush!(lshaped, aggregation::HybridAggregation)
    flush!(lshaped, active(aggregation))
    if gap(lshaped) <= aggregation.tolerance
        activate_final!(aggregation)
    end
    return nothing
end
```

The hybrid aggregation abstraction may also be used to implement the fixing strategy suggested in Corollary 4.4. That is, after the transition condition is met, we fix the partitioning scheme to the most recent scheme returned by the initial aggregation scheme. Note that, apart from the improved worst-case bound \((18)\), we also expect performance improvements when the initial aggregation scheme has high computational overhead.

6 Numerical experiments

We benchmark the various aggregation schemes on two applied problems. In both cases, we have the ability to sample scenarios, and hence, can create sample average approximation (SAA) \([14, 15]\) instances, on the form \((1)\), of varying sizes. Most of the proposed aggregation schemes have a set of tunable parameters, and the optimal parameter set is not known for a given problem instance. Therefore, we employ a simple tuning procedure based on the empirical complexity measures introduced in Section 4.2. For each problem and aggregation scheme, we generate relatively small SAA instances and solve them for varying parameters. The idea is to identify efficient aggregation schemes and hypothesize that they will stay efficient when the number of scenarios increases. Next, we benchmark the schemes on large-scale instances of the problems. We measure both the empirical convergence and the parallel performance in a distributed environment. In every experiment, the problem instance is solved to a relative tolerance of \(10^{-5}\).

The tuning procedure itself is also performed in parallel. Each SAA instance is distributed over 16 workers and each worker employs the considered aggregation scheme independent of the others. Therefore, we take the number of workers into consideration when scaling up the optimal parameters for larger number of scenarios. To reduce random errors, we perform each optimization 5 times and record the medians of the empirical complexities.

6.1 SSN

The first problem is the telecommunications problem SSN, first introduced in \([16]\). In brief, the aim is to provision bandwidth in a network before the precise point-to-point demands are known. We implement SSN in our stochastic programming framework \texttt{StochasticPrograms.jl} \([10]\) (SPjl). A discrete distribution is available for every demand. We sample from these distributions to generate scenarios and then create SAA instances of the SSN model. SSN has 89 decision variables in the first stage, and 706 variables and 175 constraints in the second stage. \(n = 10000\) scenarios yield a relatively tight confidence interval around the optimum \([15]\). An SAA instance of this size will be used for large-scale experiments.

6.1.1 Parameter tuning

We generate SAA instances of \(n = 10000\) scenarios to tune aggregation parameters for the SSN problem. First, we vary the size of \(T\) in the \texttt{SelectUniform} rule between 1 and \(n/16\). The results are shown in Fig. 1. There is an apparent trade-off between iteration complexity and cut complexity as \(\gamma\) is varied, where the end points effectively yield multi-cut and single-cut L-shaped. This is supported by the worst-case bound \((12)\) as iteration complexity is expected to increase with coarser aggregation. Next, we vary \(\gamma\) in the \texttt{SelectDecaying} between 0 and 1. The results are shown
Dynamic cut aggregation in L-shaped algorithms

Figure 1: Empirical complexity for \( \mathcal{P} = SSN \) with \( n = 1000 \) scenarios as a function of \( T \), when using the SelectUniform decision rule during dynamic aggregation. The best trade-off is achieved for \( T = 6 \).

Figure 2: Empirical complexity for \( \mathcal{P} = SSN \) with \( n = 1000 \) scenarios as a function of decay parameter \( \gamma \), when using the SelectDecaying decision rule during dynamic aggregation. \( T_0 = n \) and \( T = 1 \) in all experiments. The best trade-off is achieved for \( \gamma = 0.7 \).

These results are more promising because we achieve large reductions in empirical cut complexity without significantly increasing the empirical iteration complexity. The aggregation procedures on this efficient frontier are expected to perform better than multi-cut on large-scale problems.

We now test the distance based aggregation schemes. We vary the relevant distance tolerance parameter \( \tau \) and repeat the tests for all the distance measures introduced in A. The results from running the dynamic aggregators are shown in Fig. 3 and Fig. 4 and the cluster aggregators are shown in Fig. 5 and Fig. 6. Here, we also identify efficient frontiers for the smaller values of \( \tau \). The angular distance introduced in A.2 appears to be the most efficient distance measure in all of the considered selection- and cluster rules. We identify and collect the distance tolerances \( \tau \) that yield the lowest complexity for all aggregation schemes using the angular distance measure. For K-medoids clustering, we note that \( k = 27 \) gives the lowest empirical complexity.

For every aggregation scheme and selection rule, we select parameter configurations that result in low iteration and low cut complexity (see Table 1).

| Aggregation scheme       | Parameter | Distance measure                  |
|--------------------------|-----------|----------------------------------|
| SelectUniform            | \( T = 6 \) | N/A                              |
| SelectDecaying           | \( \gamma = 0.7 \) | N/A                              |
| SelectClosest            | \( \tau = 0.22 \) | with Angular Distance            |
| SelectClosestToReference | \( \tau = 0.37 \) | with Angular Distance            |
| ClusterByReference       | \( \tau = 0.29 \) | with Angular Distance            |
| K-medoids                | \( k = 27 \) | with Angular Distance            |

Table 1: Empirically optimal parameter configuration when solving \( \mathcal{P} = SSN \) with \( n = 1000 \) scenarios, for various aggregation schemes.
Dynamic cut aggregation in L-shaped algorithms

Figure 3: Empirical complexity for $\mathcal{P} = SSN$ with $n = 1000$ scenarios as a function of distance tolerance $\tau$, when using the SelectClosest decision rule during dynamic aggregation. The experiment is repeated for all distanced measures in $A$. The best trade-off is achieved for $\tau = 0.22$ with the angular distance measure.

Figure 4: Empirical complexity for $\mathcal{P} = SSN$ with $n = 1000$ scenarios as a function of distance tolerance $\tau$, when using the SelectClosestToReference decision rule during dynamic aggregation. The experiment is repeated for all distanced measures in $A$. The best trade-off is achieved for $\tau = 0.37$ with the angular distance measure.

Figure 5: Empirical complexity for $\mathcal{P} = SSN$ with $n = 1000$ scenarios as a function of distance tolerance $\tau$, when using the ClusterByReference decision rule during cluster aggregation. The experiment is repeated for all distanced measures in $A$. The best trade-off is achieved for $\tau = 0.29$ with the angular distance measure.
Figure 6: Empirical complexity for $\mathcal{P} = \text{SSN}$ with $n = 1000$ scenarios as a function of number of clusters $k$, when using K-medoids based cluster aggregation. The experiment is repeated for all distance measures in $\mathcal{A}$. The best trade-off is achieved for $k = 27$ with the angular distance measure.

In Fig. 7, we present the empirical complexities and wall-clock time to solution for these parameters together with the corresponding complexities of multi-cut and single-cut L-shaped. We note that most distance-based aggregation schemes appear on the efficient line with iteration complexity close to that of multi-cut L-shaped, but with fewer cuts. K-medoids appears slightly less efficient than the other schemes. SelectUniform seems to also fall on the efficient frontier, but with fewer cuts and more iterations. In terms of wall-clock time to solution, there is a clear performance increase for most aggregation schemes. These results are consistent with our theory that aggregation schemes with low empirical complexity will outperform multi-cut and single-cut in a distributed setting. The K-medoids scheme is not as performant. This could be attributed to the overhead from the k-medoids cluster computation. In the following, we repeat the experiments on a larger instance of SSN to observe if the performance gains from aggregation are scalable.

Figure 7: Empirical complexity and wall-clock time to solution for $\mathcal{P} = \text{SSN}$ with $n = 1000$ scenarios for multi-cut L-shaped, single-cut L-shaped, and the aggregation schemes SelectUniform, SelectDecaying, SelectClosest, SelectClosestToReference, ClusterByReference, and K-medoids, using the best parameters found so far.
6.1.2 Aggregation evaluation

Now, we solve a large-scale instance of SSN with $n = 10000$ scenarios, using the best parameter configuration found from the small-scale experiments. We conjecture that the best parameters of the distance based schemes should be invariant over scenario count because we normalize for number of scenarios in all distance measures. We also assume that the best value for SelectDecaying should stay the same when $n$ increases. The parameters for SelectUniform is scaled up by 10 to match the new amount of scenarios. To improve convergence when solving this large-scale problem, we utilize trust-region regularization. This was first suggested for L-shaped algorithms by the authors of [17], and we have observed it to be effective in our setting in [12, 10]. The results are shown in Fig. 8. We can still observe that most aggregation schemes end up on a frontier of empirical complexity. However, many rules yield relatively higher cut complexity than that in the small-scale setting. Therefore, we cannot conclude that these parameter configurations still yield the lowest empirical complexity when the problem is scaled up. We can clearly observe that low empirical complexity does not necessarily yield better run-time performance. Some aggregation schemes are outperformed by single-cut L-shaped, and SelectClosest is even outperformed by multi-cut L-shaped. We observed that the master iterations were considerably prolonged when employing SelectClosest for this problem size. The aggregation schemes SelectUniform, SelectDecaying and K-medoids are all more performant than single-cut L-shaped, with SelectUniform being the most performant of all.

We attempt to improve the performance by employing the fixing strategy suggested by Corollary [4.4]. We expect improvements to the distance based aggregation schemes because they offer little control of the combinatorial amount of partitioning schemes possible. The strategy is not used for SelectUniform and SelectDecaying because they naturally limit the combinatorial options. Using the same parameter configurations as before, we lock the partitioning scheme after 5 iterations. For clarity, we prefix the aggregation schemes that employ the fixing strategy with Hybrid. The empirical complexity results are shown in Fig. 9. We can observe improvements in empirical complexity for both HybridSelectClosest and Hybrid K-medoids. Moreover, their runtime performance is also improved significantly. In contrast, the reference-based schemes are no longer on the efficient frontier when using the fixing strategy. HybridSelectClosestToReference even performed worse in terms of wall-clock time.

![SSN - Large-scale complexity](image1)

![SSN - Large-scale performance](image2)

Figure 8: Empirical complexity and wall-clock time to solution for $P = SSN$ with $n = 10000$ scenarios for multi-cut L-shaped, single-cut L-shaped, and the aggregation schemes SelectUniform, SelectDecaying, SelectClosest, SelectClosestToReference, ClusterByReference, and K-medoids, using the best parameters found so far scaled according to $n$. Trust-region regularization is used in all experiments.
6.2 Day-ahead planning

Next, we apply the aggregated L-shaped algorithms to a large-scale energy problem. Specifically, we seek to determine optimal order strategies on the Nordic day-ahead market from the perspective of a price-taking hydropower producer. The specific day-ahead model we formulate and solve has been thoroughly introduced in [18], and we have already benchmarked various distributed L-shaped algorithms in [12, 10]. The implementation in SPJL is given in [18], so we refrain from repeating it here. The day-ahead problem, henceforth abbreviated by DA, has 1457 decision variables in the first stage, and 2765 variables and 1909 constraints in the second stage. \( n = 1000 \) scenarios yields a relatively tight confidence interval around the optimal solution [10], and we again use an SAA instance of this size for the large-scale experiment. In contrast to SSN, the large problem size stems from the size of the second stage instead of the number of scenarios.

6.2.1 Parameter tuning

We generate SAA instances of \( n = 100 \) scenarios to tune aggregation parameters for the DA problem. Next, we follow the same methodology as that in the SSN problem, varying the parameters in the selected aggregation schemes until low empirical complexity is achieved. The experiments for the \textbf{SelectUniform} rule is shown in Fig. 10. We get similar results to when \( P = SSN \) for \textbf{SelectUniform}, with the exception that the empirical complexity does not increase monotonically with \( T \). The results for \textbf{SelectDecaying} are shown in Fig. 11. Most smaller values of \( \gamma \) yield empirical complexity close to that of multi-cut L-shaped. Both of these observations can be attributed to the fact that the number of scenarios \( n \) is smaller in this tuning procedure than it was for \( P = SSN \). Hence, small values of \( \gamma \) in \textbf{SelectUniform} yield similar values of \( T \), and small values of \( \gamma \) in \textbf{SelectUniform} rapidly recover multi-cut L-shaped.

The results from running the dynamic aggregators are shown in Fig. 12 and Fig. 13 and the cluster aggregators are shown in Fig. 14 and Fig. 15. In general, the absolute distance measure appears to not be effective when solving the day-ahead problem. We also note that the complexity frontiers are not as pronounced as they were when solving...
Figure 10: Empirical complexity for $P = DA$ with $n = 100$ scenarios as a function of $\gamma = \frac{T}{n}$, when using the SelectUniform decision rule during dynamic aggregation.

Figure 11: Empirical complexity for $P = DA$ with $n = 100$ scenarios as a function of decay parameter $\gamma$, when using the SelectDecaying decision rule during dynamic aggregation. $T_0 = n$ and $T = 1$ in all experiments.

$P = SSN$ for most distance-based schemes. The only exception is the K-medoids scheme, which has a more visible frontier.

As before, we identify the parameter configurations that yield low iteration complexity and cut complexity. These parameters are presented in Table 2.

In Fig. 12 we present the empirical complexities and wall-clock time to solution for these parameters together with the corresponding results for multi-cut and single-cut L-shaped. In terms of empirical complexity, SelectClosest and

Figure 12: Empirical complexity for $P = DA$ with $n = 100$ scenarios as a function of distance tolerance $\tau$, when using the SelectClosest decision rule during dynamic aggregation. The experiment is repeated for all distanced measures in $A$. 
Figure 13: Empirical complexity for $\mathcal{P} = DA$ with $n = 100$ scenarios as a function of distance tolerance $\tau$, when using the **SelectClosestToReference** decision rule during dynamic aggregation. The experiment is repeated for all distanced measures in $\mathcal{A}$.

Figure 14: Empirical complexity for $\mathcal{P} = DA$ with $n = 100$ scenarios as a function of distance tolerance $\tau$, when using the **ClusterByReference** decision rule during cluster aggregation. The experiment is repeated for all distanced measures in $\mathcal{A}$.

Figure 15: Empirical complexity for $\mathcal{P} = DA$ with $n = 100$ scenarios as a function of number of clusters $k$, when using **K-medoids** based cluster aggregation. The experiment is repeated for all distanced measures in $\mathcal{A}$. 

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Aggregation scheme | Parameter | Distance measure
---|---|---
SelectUniform | $T = 2$ | N/A
SelectDecaying | $\gamma = 0.8$ | N/A
SelectClosest | $\tau = 0.17$ | with Angular Distance
SelectClosestToReference | $\tau = 0.43$ | with Spatioangular Distance
ClusterByReference | $\tau = 0.31$ | with Spatioangular Distance
K-medoids | $k = 60$ | with Angular Distance

Table 2: Empirically optimal parameter configuration when solving $P = DA$ with $n = 100$ scenarios, for various aggregation schemes. Note that, the optimal parameters are not the same for $P = SSN$ and $P = DA$.

Day-ahead - Small-scale complexity

Day-ahead - Small-scale performance

Figure 16: Empirical complexity and wall-clock time to solution for $P = DA$ with $n = 100$ scenarios for the aggregation schemes SelectUniform, SelectDecaying, SelectClosest, SelectClosestToReference, ClusterByReference, and K-medoids, using the best parameters found so far.

K-medoids appear the most effective. Again, we see performance improvements from using aggregation schemes, with SelectClosest being the most performant scheme.

6.2.2 Aggregation evaluation

Now, we solve a large-scale instance of DA with $n = 1000$ scenarios, using the best parameter configuration from the small-scale experiments. Again, we employ trust-region regularization in each algorithm to improve convergence. The results are shown in Fig. 17. Here, the most effective schemes are still SelectClosest and K-medoids. Again, single-cut L-shaped outperforms many of the aggregation schemes. Multi-cut L-shaped is in this case, worse than any of the tested schemes. The best performance is achieved with SelectUniform. Again, we observe that K-medoids is not as performant as one would expect from its empirical complexity.

As before, we employ the fixing strategy suggested by Corollary 4.4. The results are shown in Fig. 18. Now, we observe significant improvements in both empirical complexity and run-time performance for all considered hybrid
schemes. HybridSelectClosest and Hybrid K-medoids now outperform SelectUniform, with HybridSelectClosest being the most performant.

![Day-ahead - Large-scale complexity](image1.png)

![Day-ahead - Large-scale performance](image2.png)

Figure 17: Empirical complexity and wall-clock time to solution for $P = DA$ with $n = 1000$ scenarios for the aggregation schemes SelectUniform, SelectDecaying, SelectClosest, SelectClosestToReference, ClusterByReference, K-medoids, and HybridSelectClosest, using the best parameters found so far. Trust-region regularization is used in all experiments.

7 Discussion and conclusion

7.1 Discussion

There is no single strategy that outperforms the other in the two solved problems. The optimal parameter configurations are also not the same. This implies that the best aggregation scheme is problem-dependent. However, we can propose some rules-of-thumb based on these experiments. First, amongst the proposed distance measures in $A$, the angular distance appears to be most suited for distance-based aggregation scheme. We find that reference-based aggregation, in general, has slightly less cut complexity than that of multi-cut L-shaped and just slightly larger iteration complexity. These aggregation schemes behave like multi-cut L-shaped most iterations and aggregate almost every cut at some iterations. In brief, these schemes adequately identify non-informative iterations where most cuts are parallel and close in distance, and saves time and memory accordingly. The cluster based version is slightly more effective than the dynamic one in this regard, at the cost of larger overhead. Thus, for problems where the multi-cut L-shaped algorithm is performant, we expect performance improvements from employing the schemes based on reference comparison. Further, there may exist a reference cut that is more suitable than the full aggregate of the previous/current iteration, which could yield better performance. In general, employing advanced aggregation strategies like K-medoids clustering can improve performance, but tuning is required. In almost all experiments, we observed performance improvements from the hybrid fixing strategy.

Even though single-cut L-shaped has much worse iteration complexity than all other variants, the gain from having the lowest cut complexity in a distributed environment often makes it more performant than many of the aggregation schemes. Moreover, our approach requires separate second-stage objective variables $\theta_s$ in the master problem for each of the $n$ subproblems. This impacts the memory requirement in the master problem as the scenario count $n$ grows, while single-cut L-shaped has less memory impact thanks to both fewer variables and constraints. We also
observe numerical instability in the master problem when the number of scenarios $n$ is large. In particular, this can be observed when solving SSN with $n = 10,000$ using the SelectClosest scheme. In brief, similar to the conclusions drawn by the authors of [2], we conclude that the performance increase from aggregation diminishes when the number of scenarios grows very large. For instances when $n \gg p$, it could be beneficial to determine an initial aggregation scheme $\mathcal{S}$ and fix the master variables to $\theta_a$, $a = 1, \ldots, A(\mathcal{S})$ according to this scheme throughout the procedure. In other words, we apply a dynamic aggregation procedure as suggested in Section 4, but with fewer master variables. An immediate drawback is that subproblems which turn out to be undesirable to aggregate are grouped together in the initial partitioning. However, our static worst-case bound (12) and our experimental results from using the SelectUniform scheme and the fixing strategy indicate that this may be negligible as the L-shaped procedure progresses. We leave this suggestion as a future work.

In all experiments, we observe large performance improvements from SelectUniform and from using the fixing strategy in dynamic schemes. After tuning, we see large performance gains at low overhead cost. We can relate this observation to our worst-case results. Because the partitioning scheme is the same at each iteration when using SelectUniform, it satisfies all assumptions required for the static worst-case bound in Theorem 2.2. Moreover, the similar worst-case bound in Corollary 4.4 holds when using the fixing strategy. Therefore, although the dynamic aggregation schemes we propose could theoretically aggregate cuts in a more clever way, they could also theoretically identify more facets than static schemes before converging. This is supported by our worst-case bound on the dynamic aggregation (16) which in general is expected to be larger because of the combinatorial terms. Based on our experimental observations, and our worst-case bound, we suggest designing aggregation schemes that limits the possible combinations of aggregates.

Our derived worst-case bounds grow astronomically large quickly, but they do not give accurate estimates of average-time complexity. Instead, they allow us to reason about aggregation schemes and suggest rules of thumb. From practical experience, we would not expect the dynamical worst-case bound (16) to be attained by anything but diabolically constructed problems. An identified facet in some aggregate generally corresponds to many other facets in coarser aggregates. The worst case would therefore occur only if all facets are identified in a very specific order.

Figure 18: Empirical complexity and wall-clock time to solution for $\mathcal{P} = DA$ with $n = 1000$ scenarios for the aggregation schemes SelectUniform, SelectDecaying, HybridSelectClosest, HybridSelectClosestToReference, HybridClusterByReference, and Hybrid K-medoids, using the best parameters found so far. The partitioning scheme is fixed after 5 iterations for all schemes prefixed with Hybrid. Trust-region regularization is used in all experiments.
which is unlikely in the average case. Hence, the combinatorial explosion suggested by the worst-case bound is rarely observed in practice. Future work could involve further theoretical development around the average-time complexity of these algorithms.

7.2 Conclusion

In this work, we have presented a novel framework for dynamic cut aggregation in the L-shaped algorithm. With our approach, the optimality cuts generated at each iteration can be aggregated into arbitrary partitions which are allowed to vary at each iteration. We have given a worst-case bound for aggregated L-shaped in Theorem 2.2 that holds for any static partition scheme $\mathcal{S}$. We have also extended this worst-case result to dynamic aggregation in Theorem 4.2 and given a convergence proof for L-shaped with dynamic cut aggregation in Theorem 4.1. We have proposed three aggregation types, dynamic aggregation, cluster aggregation, and hybrid aggregation, and also introduced various decision rules that lead to a large set of aggregation schemes. We have also proposed a fixing strategy for dynamic schemes that improves the dynamic worst-case bound. The improved bound is given in Corollary 4.2.

The proposed aggregation schemes have been evaluated by solving two large-scale stochastic programs, which are both distributed over 16 worker nodes. Although the best aggregation scheme and parameter configuration are unknown for a given problem, we have shown that large performance gains are attainable through a tuning procedure. In short, the aggregation parameters are determined by solving small-scale instances. Most of our aggregation schemes converge after roughly as many iterations as that of the multi-cut L-shaped algorithm, but with fewer optimality cuts in the master problem. In most cases, this leads to performance improvements in a distributed setting. An enhanced software implementation could potentially improve the performance of aggregation schemes that yield low empirical complexity but no speed-ups in practice. It is definitely worthwhile to tune the SelectUniform scheme, or employ our fixing strategy, as this can yield large performance improvements at low cost. Our set of proposed aggregation schemes do not encompass every possible partitioning scheme and we aim to explore more strategies in the future.

A Distance measures

Many of the devised heuristics for selecting which cuts to aggregate require a measure of distance between two given optimality cuts. Let $c_s$ denote a generated optimality cut on the form

$$\partial Q_s x + \theta_s \geq q_s$$

and let $d(c_i, c_j)$ denote some distance measure between two optimality cuts of the form (19). We do not devise measures that fulfill all conditions of a metric, but we at least require that $d(c_i, c_j) \geq 0$ and that $d(c_i, c_j) = 0$ whenever $c_i = c_j$. Ideally, we want a measure so that $c_i$ and $c_j$ give similar information about the feasible region in the master problem when $d(c_i, c_j)$ is small. To this end, we borrow ideas from the following survey paper about aggregation techniques in optimization [19] when exploring measures. We stipulate and utilize the following three measures.

A.1 Absolute distance

First, we introduce the absolute distance between two optimality cuts as:

$$d(c_i, c_j) = \frac{\|c_i - c_j\|}{\max \{\|c_i\|, \|c_j\|\}}$$

where

$$\tilde{c}_s = \left[ \frac{\partial Q_s}{q_s} \right]$$

The absolute distance has the property that $d(c_i, c_j) = 0$ precisely when $c_i = c_j$. However, it will often place a heavy weight on $q_s$, since $q_s$ directly relates to the second-stage objective and it often holds that $|q_s| \gg \|\partial Q_s\|$. In many of the introduced selection rules, a cut candidate $c_s$ is often compared to an existing aggregate $c_{S_a}$ of cuts:

$$\sum_{s \in S_a} \partial Q_s x + \sum_{s \in S_a} \theta_s \geq \sum_{s \in S_a} q_s.$$  \tag{21}

Due to the summation, the distance between an aggregated cut and a single cut will generally be larger than that between two single cuts. Therefore, we normalize by the number of cuts when calculating the distance, so that

$$\tilde{c}_{S_a} = \frac{1}{|S_a|} \left[ \frac{\sum_{s \in S_a} \partial Q_s}{\sum_{s \in S_a} q_s} \right].$$
A.2 Angular distance

Next, we introduce the angular distance between two cuts as

\[ 1 - \frac{\partial Q_i \cdot \partial Q_j}{\|\partial Q_i\| \|\partial Q_j\|}. \] (22)

This distance is invariant over aggregation; so, there is no need to rescale. The maximum distance is acquired for perpendicular cuts, which are probably undesired to aggregate. The main drawback is that the distance between parallel cuts is zero.

A.3 Spatioangular distance

Finally, we introduce the spatioangular distance between two cuts as

\[ 1 - \frac{\partial Q_i \cdot \partial Q_j}{\|\partial Q_i\| \|\partial Q_j\|} + \frac{|q_i - q_j|}{\max(|q_i|, |q_j|)}. \]

This formulation alleviates the drawback of the angular distance by also measuring the distance between the bias terms \( q_i \) and \( q_j \). However, it is not as straightforward to decide at what relative tolerance the two cuts should be considered close enough for aggregation. As with the absolute distance, we again keep track of the amount of cuts included in an aggregate and rescale \( q_s \) accordingly.

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