FRACTIONAL ANALOGUE OF K-HESSIAN OPERATORS

YIJING WU

Abstract. Applying ideas of fractional analogue of Monge-Ampère operator in [1] by L. Caffarelli and F. Charro, we consider an analogue of fractional k-Hessian operators expressed as concave envelopes of fractional linear operators, and reproduce the same regularity results when \( k = 2 \).

Under the set up of global solutions prescribing data at infinity and global barriers, the key estimate is to prove that fractional 2-Hessian operator is strictly elliptic. Then we can apply nonlocal Evans-Krylov theorem [2][3] to prove such solutions are classical.

1. Introduction

Monge-Ampère operator is a special case of \( k \)-Hessian operators, which are defined by

\[
f_k(D^2u)(x) = \left( \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k} \right)^{1/k},
\]

for \( k \) integer and \( 1 \leq k \leq n \). Here \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are eigenvalues of the matrix \( D^2u(x) \), and \( f_k \) is concave and elliptic [4] [5] of \( \lambda \) when

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k.
\]

\( \Gamma_k \) is an open symmetric convex cone defined by

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n, \sigma_l(\lambda) > 0, l = 1, 2, \ldots, k \}.
\]

Here

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}
\]

is the \( k \)-th elementary symmetric polynomial. And when \( k = n \), \( \Gamma_n \) is the positive cone

\[
\Gamma_n = \{ \lambda \in \mathbb{R}^n, \lambda_i > 0, i = 1, 2, \ldots, n \}.
\]

One main ingredient of the paper [1] is the following:

The Monge-Ampère equation is a concave fully nonlinear equation. If \( u \) is a convex solution solving

\[
(\det D^2u)^{1/n}(x) = g(x),
\]

then the equation is equivalent to

\[
\inf_{M \in \mathcal{M}} L_M u(x) = g(x),
\]

Date: March 13, 2022.
where $L_M$ is a linear operator defined by

$$L_M u(x) = \text{trace}(MD^2 u(x)) = \Delta(u \circ \sqrt{M})(x),$$

and the set $\mathcal{M}$ consists of all positive symmetric matrices with determinant $n^{-n}$, independent of $x$. Moreover, the infimum is realized when $M$ is a constant multiple of the matrix of cofactor of $D^2 u(x)$.

Then we define the fractional analogue of Monge-Ampère equation as

$$F_s[u](x) = \inf_{M \in \mathcal{M}} \{-C^{-1}_{n,s}(-\Delta)^s(u \circ \sqrt{M})(x)\}.$$

Under this setting, regularity results for fractional Monge-Ampère equation are discussed in [1].

Therefore, it is natural to consider $k$-Hessian operators as concave envelopes of linear operators. We give the following definition:

**Definition 1.1.** As an analogue of definition of the Monge-Ampère operator, we define

$$f_k(D^2 u(x)) = \inf_{M \in \mathcal{M}_k} \{\text{trace}(MD^2 u(x))\}$$

$$= \inf_{M \in \mathcal{M}_k} \{\Delta(u \circ \sqrt{M})(x)\}$$

$$= \inf_{M \in \mathcal{M}_k} \{\Delta(u(\sqrt{M}x))\}.$$

Details and explanations of the set $\mathcal{M}_k$ will be further discussed in section 2. Then we are able to give a similar definition for fractional analogues of $k$-Hessian operators:

**Definition 1.2.** Define fractional $k$-Hessian operators as

$$F_{k,s}[u](x) = \inf_{M \in \mathcal{M}_k} \{-C^{-1}_{n,s}(-\Delta)^s(u \circ \sqrt{M})(x)\}$$

$$= \inf_{M \in \mathcal{M}_k} \{P.V. \int_{\mathbb{R}^n} \frac{u(\sqrt{M}x + y) - u(\sqrt{M}x)}{|\sqrt{M}^{-1}y|^{n+2s}} \det\sqrt{M}^{-1} dy\}$$

$$= \inf_{M \in \mathcal{M}_k} \{\frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u, \sqrt{M}x, y)}{|\sqrt{M}^{-1}y|^{n+2s}} \det\sqrt{M}^{-1} dy\},$$

where

$$\delta(u, x, y) = u(x + y) - 2u(x) + u(x - y).$$

The main idea of this article is to reproduce the regularity results of fractional Monge-Ampère equation in [1] to fractional $k$-Hessian equations.

In this article, our main purpose is to follow the ideas and set up of the paper [1], and to prove:

(a) On each $n - 1$ dimensional space, the fractional Laplacian is bounded from above and strictly positive. (Proposition [3.1])

(b) When $k = 2$, the operators that are close to the infimum remain strictly elliptic. (Theorem [1.2])

Here we define the strictly elliptic operator:
**Definition 1.3.** For $\epsilon_0 > 0$, we define a non-degenerate and strictly elliptic operator

$$F_{k,s}^{\epsilon_0}[u](x) = \inf_{M \in M_k} \{P.V. \int_{\mathbb{R}^n} u(\sqrt{M}x + y) - u(\sqrt{M}x)) \frac{\det \sqrt{M}^{-1} y}{|y|^{n+2s}} \frac{\lambda_{\text{min}}(M) \geq \epsilon_0} \}$$

$$= \inf_{M \in M_k} \{\frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u, \sqrt{M}x, y)}{|y|^{n+2s}} \frac{\det \sqrt{M}^{-1} y}{\lambda_{\text{min}}(M) \geq \epsilon_0} \}.$$ 

The main theorem of this article is:

**Theorem 1.4.** Consider $1/2 < s < 1$, and assume $u$ is Lipschitz continuous and semiconcave with constants $L$ and $SC$ respectively. And

(1) $(1 - s)F_{2,s}[u](x) \geq \eta_0$

for any $x \in \Omega$, in the viscosity sense for some constant $\eta_0 > 0$. Then

(2) $F_{2,s}[u](x) = F_{2,s}^{\epsilon_0}[u](x)$

for any $x \in \Omega$ in the classical sense, with

$\epsilon_0 = \epsilon_0(\eta_0, n, s, L, SC) > 0$

given by (9).

**Remark 1.5.** For simplicity, we shall assume that $0 \in \Omega$ and then prove (2) for $x = 0$. Note for the sequel that since $u$ is semiconcave, Lemma 2.2 in paper [1] implies that $F_{2,s}(x)$ is defined in the classical sense for all $x \in \Omega$ and (1) holds pointwise. And this theorem states that the infimum in the definition of $F_{2,s}[u]$ cannot be realized by matrices that are too degenerate, which proves that the fractional analogue of 2-Hessian operators are locally uniformly elliptic.

**Remark 1.6.** We can check in the proofs that $\epsilon_0$ is given by (9), that

$$\epsilon_0 = \sqrt{\frac{n}{n-1} C_4^{1/s} (\frac{\mu_0}{\mu_1})^{1/s}},$$

with $C_4 = C_4(n, s, L, SC, \eta_0)$ given by (17), $\mu_0$ given by (13) and $\mu_1$ given by (14). And this shows that Theorem 1.4 is stable as $s \to 1$, that the constant $\epsilon_0$ will not goes to 0 as $s \to 1$.

Under a framework of global solutions prescribing data at infinity and global barriers, which are set up to avoid complexity of dealing with issues from the boundary data for non-local equations, the following theories for fractional Monge-Ampère equations also work for fractional $k$-Hessian equations:

(c) Existence of solutions. (Theorem 1.7)
(d) Semiconcavity and Lipschitz continuity of solutions. (Theorem 1.8)
(e) The non-local fully nonlinear theory developed in [2] [3] applies, in particular the nonlocal Evans-Krylov theorem.
**Theorem 1.7.** There exists a unique solution of
\[
\begin{cases}
F_{k,s}[u](x) = u(x) - \phi(x) & \text{in } \mathbb{R}^n \\
(u - \phi)(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]

**Theorem 1.8.** Assume \(\phi\) is semiconcave and Lipschitz continuous, and let \(v\) be the solution of
\[
\begin{cases}
F_{k,s}[v](x) = v(x) - \phi(x) & \text{in } \mathbb{R}^n \\
(v - \phi)(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]
Then, \(v\) is Lipschitz continuous and semiconcave with the same constants as \(\phi\).

**Remark 1.9.** The difference between fractional Monge-Ampère operators and \(k\)-Hessian operators is the set of matrices \(M\) among which we take infimum of fractional linear operators. In Monge-Ampère, we consider the infimum among all positive symmetric matrices with determinant \(n^{-n}\), and in \(k\)-Hessian, we consider the infimum among all positive symmetric matrices in the set \(\mathcal{M}_k\) (which will be discussed in Section 2, Proposition 2.2). Hence, we can apply the exact same proofs of existence and \(C^{1,1}\) regularity in the fractional Monge-Ampère case, which are carefully explained in section 4.5 and 6 in [1], to prove Theorem 1.7 and Theorem 1.8 for our fractional \(k\)-Hessian equations.

Thus by what we have proved in (b), that such operators are strictly elliptic, and \(C^{1,1}\) estimates in (d), we can apply nonlocal Evans-Krylov theorem [2][3] to prove solutions of fractional 2-Hessian equations are \(C^{2s+\alpha}\), and further classical, under the framework of global solutions prescribing data at infinity and global barriers.

**Remark 1.10.** The proof for strictly ellipticity of the operator is required to improve the \(C^{1,1}\) regularity to \(C^{2s+\alpha}\) regularity. Therefore, we only care about the case \(1/2 < s < 1\) in Theorem 1.4 or there is no improvement in the regularity. We also care what would happen as \(s \to 1\), and in the Remark 1.6, we can see that Theorem 1.4 is stable as \(s \to 1\).

2. Notations and Preliminaries

In this section, we will first state some notations. And then we will discuss one important representation of Monge-Ampère operator(Proposition 2.1). Next we will derive a similar representation for \(k\)-Hessian operator(Proposition 2.2), show how we construct the set \(\mathcal{M}_k\) in Definition 1.2 and give the definition of fractional \(k\)-Hessian operator.

Given a function \(u\), we shall denote the second-order increment of \(u\) at \(x\) in the direction of \(y\) as
\[
\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x),
\]
and fractional laplacian is defined as

\[-(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x - y|^{n+2s}} dy \]

\[= \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|x - y|^{n+2s}} dy.\]

And the constant $C_{n,s}$ is a normalization constant.

For square matrices, $A > 0$ means positive definite and $A \geq 0$ positive semidefinite. We denote $\lambda_i(A)$ the eigenvalues of $A$, in particular $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ are the smallest and largest eigenvalues, respectively.

We shall denote the $n$th-dimensional ball of radius $r$ and center $x$ by $B_n^r(x) = \{y \in \mathbb{R}^n, |y - x| < r\}$, and the corresponding $(n - 1)$-dimensional sphere by $\partial B_n^r(x) = \{y \in \mathbb{R}^n, |y - x| = r\}$. $\mathcal{H}^n$ stands for the $n$-dimensional Haussdorff measure.

Let $A \subset \mathbb{R}^n$ be an open set. We say that a function $u : A \to \mathbb{R}$ is semi-concave if it is continuous in $A$ and there exists a constant $SC \geq 0$ such that

$\delta(u, x, y) \leq SC |y|^2$ for all $x, y \in \mathbb{R}^n$ such that the segment $[x - y, x + y] \subset A$.

And the constant $SC$ is called a semi-concavity constant for $u$ in $A$. Alternatively, a function $u$ is semi-concave in $A$ with constant $SC$ if $u(x) - \frac{SC}{2} |x|^2$ is concave in $A$. Geometrically, this means that the graph of $u$ can be touched from above at every point by a paraboloid of the type $a + b < x > + \frac{SC}{2} |x|^2$.

We denote the constant $C_n^k = \frac{n!}{k!(n-k)!}$ for $n, k \in \mathbb{N}$ and $n \geq k$.

We can write Monge-Ampère operator as a concave envelope of linear operators, that

**Proposition 2.1.** If $u$ is convex, then the Monge-Ampère operator $f(D^2u) = (\det D^2u)^{1/n}$ can be expressed as

$f(D^2u) = (\det D^2u)^{1/n} = \inf_{M \in \mathcal{M}} L_M u,$

where $\mathcal{M}$ is the set of all positive symmetric matrices with determinant $n^{-n}$, and the linear operator $L_M u$ is defined by

$L_M u = \text{trace}(MD^2u) = \Delta(u \circ \sqrt{M}).$

**Proof of Proposition 2.1.** Let $A = D^2u(x)$ which is positive, and we consider Monge-Ampère operator $f(A) = (\det A)^{1/n}$ as a concave envelope of linear operators, that

$f(A) = \inf_{B \in \Gamma_n} \{Df(B)(A - B) + f(B)\},$

and $Df(B)$ is a linear operator mapping $\mathbb{R}^{n\times n}$ to $\mathbb{R}$, that

$Df(B)A = \lim_{\epsilon \to 0} \frac{f(B + \epsilon A) - f(B)}{\epsilon}.$
Since $f$ is homogeneous of degree 1, that for any $t > 0$,
\[ f(tB) = tf(B), \]
and we can prove
\[ Df(B) = \lim_{\epsilon \to 0} \frac{f(B + \epsilon B) - f(B)}{\epsilon} = f(B). \]
Letting $E_{ij} \in \mathbb{R}^{n \times n}$ be the matrix with the $i,j$th entry being 1 and all other entries being 0, we can calculate
\[ Df(B)E_{ij} = \frac{1}{n}(\det B)^{\frac{1}{n}}b_{ij}^*, \]
where $b_{ij}^*$ is the $i,j$th entry of the cofactor matrix of $B$. Thus, by linearity,
\[ Df(B)A = Df(B)(a_{ij}E_{ij}) = a_{ij}\left(\frac{1}{n}(\det B)^{\frac{1}{n}}b_{ij}^*\right) = \text{trace}(AM^T), \]
where
\[ M = M(B) = Df(B) = \frac{1}{n}(\det B)^{\frac{1}{n}}b_{ij}^*. \]
And by the property of cofactor matrix $B^*$ that $B^{-1} = (\det B)^{-1}B^*$, we know
\[ \det M = n^{-n}. \]
Therefore, by the bijection between matrices and cofactor matrices, without loss of generality, we can conclude that
\[ \left(\det D^2u\right)^{1/n} = \inf_{M \in \mathcal{M}} \text{trace}(MD^2u), \]
where $\mathcal{M}$ is the set of all positive symmetric matrices with determinant $n^{-n}$. \hfill \square

Monge-Ampère operator is the n-Hessian operator. Thus, we try to find a similar way of representing the concave k-Hessian operator.

**Proposition 2.2.** If $D^2u \in \Gamma_k$, then the k-Hessian operator
\[ f_k(D^2u) = \left(\sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}\right)^{1/k} \]
is a concave envelope of linear operators, that
\[ f_k(D^2u) = \inf_{M \in \mathcal{M}_k} \{\text{trace}(MD^2u)\}. \]
And a matrix $M \in \mathcal{M}_k$ if there exists a matrix $B \in \Gamma_k$, such that the $i,j$th entry of the matrix $M$ satisfies the following conditions:
\[ M_{ii} = \frac{1}{kf_k(B)^{k-1}} \sum_{1 \leq j_1 < j_2 < \ldots < j_{k-1} \leq n} \det B_{(j_1,\ldots,j_{k-1})} \]
where
\[ 1 \leq i \leq n, \quad 1 \leq k \leq n, \quad i \neq j_{k-1}. \]
where $B_{(j_1,...,j_{k-1})}$ denotes the submatrix of $B$ formed by choosing the $j_1, j_2, ..., j_{k-1}$ th rows and columns.

When $k \geq 3$,

$$M_{ij} = -\frac{1}{k f_k(B)^{k-1}} \sum_{i \leq j_1 < j_2 < \cdots < j_{k-2} \leq n, \ j_1,...,j_{k-2} \neq i,j} \det B_{(j,j_1,...,j_{k-2})}(i,j_1,...,j_{k-2}),$$

where $B_{(j,j_1,...,j_{k-2})}(i,j_1,...,j_{k-2})$ denotes the submatrix of $B$ formed by choosing the $j, j_1, j_2, ..., j_{k-2}$ th rows and $i, j_1, j_2, ..., j_{k-2}$ th columns.

And when $k = 2$,

$$M_{ij} = -\frac{1}{2 f_2(B)} b_{ji},$$

where $b_{ji}$ denotes the $j, i$ th entry of matrix $B$.

Moreover, for each $M \in \mathcal{M}_k$, $M$ is a positive symmetric matrix.

**Proof of Proposition 2.2** Since $f_k$ is a concave function of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Gamma_k$, with $\lambda_j, j = 1, 2, ..., n$ eigenvalues of matrix $A$, we can write

$$f_k(A) = \inf_{B \in \Gamma_k} \{Df_k(B) (A - B) + f_k(B)\}.$$  

Here $Df_k(B) : \mathbb{R}^{n \times n} \to \mathbb{R}$ is an operator defined by

$$Df_k(B)A = \lim_{\epsilon \to 0} \frac{f_k(B + \epsilon A) - f_k(B)}{\epsilon}.$$  

Take a basis $\{E_{ij}\}_{i,j=1}^n$ of $\mathbb{R}^{n \times n}$, that $E_{ij}$ is a matrix with $i, j$ th entry being 1, and all other entries being 0, we can calculate that

$$Df_k(B)E_{ii} = \frac{1}{k f_k(B)^{k-1}} \sum_{i \leq j_1 < j_2 < \cdots < j_{k-1} \leq n, \ j_1,...,j_{k-1} \neq i} \det B_{(j_1,...,j_{k-1})},$$

where $B_{(j_1,...,j_{k-1})}$ denotes the submatrix of $B$ formed by choosing the $j_1, j_2, ..., j_{k-1}$ th rows and columns.

When $k \geq 3$,

$$Df_k(B)E_{ij} = -\frac{1}{k f_k(B)^{k-1}} \sum_{i \leq j_1 < j_2 < \cdots < j_{k-2} \leq n, \ j_1,...,j_{k-2} \neq i,j} \det B_{(j,j_1,...,j_{k-2})}(i,j_1,...,j_{k-2}),$$

where $B_{(j,j_1,...,j_{k-2})}(i,j_1,...,j_{k-2})$ denotes the submatrix of $B$ formed by choosing the $j, j_1, j_2, ..., j_{k-2}$ th rows and $i, j_1, j_2, ..., j_{k-2}$ th columns.

And when $k = 2$,

$$Df_2(B)E_{ij} = -\frac{1}{2 f_2(B)} b_{ji},$$

where $b_{ji}$ denotes the $j, i$ th entry of matrix $B$. 
Define a matrix \( M \in \mathbb{R}^{n \times n} \) where
\[
M_{ii} = Df_k(B)E_{ii},
\]
\[
M_{ij} = Df_k(B)E_{ij}.
\]
And we write \( M = M(B) = Df_k(B) \) to denote this relation between matrix \( B \) and \( M \). Then for any matrix \( A \in \mathbb{R}^{n \times n} \), \( A = a_{ij}E_{ij} \), by linearity,
\[
Df_k(B)A = a_{ij}Df_k(B)E_{ij} = a_{ij}M_{ij} = \text{trace}(AM^T).
\]
Moreover, since \( f_k \) is homogeneous of degree 1, so
\[
Df_k(B)B = f_k(B).
\]
And therefore,
\[
f_k(A) = \inf_{B \in \mathbb{R}^{n \times n}} \{ Df_k(B)(A - B) + f_k(B) \}
= \inf_{B \in \mathbb{R}^{n \times n}} \{ \text{trace}(AM^T), M = M(B) \}
= \inf_{M \in \mathcal{M}_k} \{ \text{trace}(AM^T) \}.
\]
We can write the set
\[
\mathcal{M}_k = \{ M \in \mathbb{R}^{n \times n}, \text{ exist } B \in \Gamma_k, M = Df_k(B) = M(B) \}.
\]
Actually, \( \mathcal{M}_k \) is the image set of all matrices in \( \Gamma_k \) under the mapping
\[
B \mapsto M = M(B) = Df_k(B),
\]
and a matrix \( M \in \mathcal{M}_k \) if there exists a matrix \( B \in \Gamma_k \) such that with entries
of \( M \) satisfying (3), (4) (when \( k \geq 3 \)) or (5) (when \( k = 2 \)).

Without loss of generality, we can assume \( M \) to be symmetric. Assume the matrix \( B \) has eigenvalues \( \lambda_1, \lambda_2, ..., \lambda_n \) and since \( f_k \) is invariant under orthonormal transformation, that \( f_k(B) = f_k(Q^T \Lambda Q) \), with \( \Lambda \) be the diagonal matrix with diagonal entries \( \lambda_1, \lambda_2, ..., \lambda_n \). Then the matrix \( M = Df_k(B) \) has same eigenvalues as \( Df_k(\Lambda) \). And since \( f_k \) is elliptic, thus the \( i \)th diagonal entry of \( Df_k(\Lambda) \) satisfies
\[
(Df_k(\Lambda))_{ii} = \lim_{\epsilon \to 0} \frac{f_k(\Lambda + \epsilon E_{ii}) - f_k(\Lambda)}{\epsilon} > 0.
\]
Therefore, if \( B \in \Gamma_k \), then \( M = Df_k(B) \) is a positive matrix. In particular, if \( B = \text{diag}\{\sigma_1, \sigma_2, ..., \sigma_n\} \) and \( f_k(B) = 1 \), then
\[
M = Df_k(B) = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\}
\]
with
\[
\lambda_i = \frac{1}{k} \left( \sum_{1 \leq i_1 < ... < i_{k-1} \leq n, i_j \neq i} \sigma_{i_1} \sigma_{i_2} ... \sigma_{i_{k-1}} \right).
\]
\( \square \)
From Proposition 2.2 we write
\[ f_k(D^2u(x)) = \inf_{M \in M_k} \{ \text{trace}(D^2u(x)MT) \} \]
\[ = \inf_{M \in M_k} \{ \text{trace}(\sqrt{M}^T D^2u(x) \sqrt{M}) \} \]
\[ = \inf_{M \in M_k} \{ \Delta(u \circ \sqrt{M})(x) \} , \]

Then it is natural to give Definition 1.2 of fractional k-Hessian operator by writing
\[ F_{k,s}[u](x) = \inf_{M \in M_k} \{-C_{n,s}^{-1}(-\Delta)^s(u \circ \sqrt{M})(x)\}. \]

3. The main mathematical results

In this section we will prove Theorem 1.4 that when \( k = 2 \), the infimum in the definition (1) of \( F_{k,s} \), cannot be realized by matrices that are too degenerate, which proves that the fractional 2-Hessian operator is locally uniformly elliptic. Then we can apply theories for uniformly elliptic non-local operators such as Evans-Krylov theorem to our fractional 2-Hessian operators, to get \( C^{2,\alpha} \) estimates for global solutions prescribing data at infinity and global barriers, and further to prove that such solutions are classical.

Our aim is to prove that as \( \epsilon \to 0 \),
\[ \inf_{M \in M_k} \{-C_{n,s}^{-1}(-\Delta)^s(u \circ \sqrt{M})(x), \lambda_{\min}(M) = \epsilon \} \to \infty. \]
And this will show that the infimum cannot be realized by matrices that are too degenerate, which is the result of Theorem 1.4. To prove this, we want to consider the integral on \( \partial B_r^n(0) \) as an average of integrals on \( \partial B_r^{n-1}(0) \). Consider a unit vector
\[ \tilde{e}(\theta) = (0, 0, \ldots, 0, \sin \theta, \cos \theta), \]
with \( \theta \in (-\pi/2, \pi/2] \). Then
\[ \text{span}\{\tilde{e}(\theta)\}^\perp = \text{span}\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{n-1}\} \]
with \( \tilde{e}_j, j = 1, 2, \ldots, n - 1 \) be the orthonormal basis of the \( n - 1 \) dimensional perpendicular space. Especially, we can consider
\[ \tilde{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0) \quad j = 1, 2, \ldots, n - 2, \]
and
\[ \tilde{e}_{n-1} = (0, 0, \ldots, 0, \cos \theta, -\sin \theta). \]
Then for any \( y \in \partial B_r^n(0) \), and \( y \perp \tilde{e}(\theta) \), we can write \( y = (y_1, y_2, \ldots, y_n) \) as
\[ y = z_1 \tilde{e}_1 + z_2 \tilde{e}_2 + \ldots + z_{n-1} \tilde{e}_{n-1}, \]
and therefore,
\[ y_j = z_j, \quad j = 1, 2, \ldots, n - 2, \]
\[ y_{n-1} = z_{n-1} \cos \theta, \]
when \( y_n = -z_{n-1} \sin \theta \).

Now let \( M \in \mathcal{M}_2 \), \( \sqrt{M}^{-1} = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), assume

\[
\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n = \epsilon^{-1/2},
\]

and write integral in \( \mathbb{R}^n \) as an average of \((n - 1)\)-dimensional subspace perpendicular to \( \tilde{e}(\theta) \), \( -\pi/2 < \theta \leq \pi/2 \), that

\[
I = \prod_{j} \lambda_j \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{(\lambda_j^2 y_1^2 + \ldots + \lambda_n^2 y_n^2)^{\frac{n+2s}{2}}} \, dy
\]

\[
= \prod_{j} \lambda_j \int_{-\pi/2}^{\pi/2} \int_{0}^{\infty} \int_{x \in B_1^{n-1}(0), x \perp \tilde{e}(\theta)} \frac{u(r(x_1 \tilde{e}_1 + \ldots + x_{n-1} \tilde{e}_{n-1})) - u(0)}{r^{1+2s}(\lambda_j^2 x_1^2 + \ldots + (\lambda_{n-1}^2 \cos^2 \theta + \lambda_n^2 \sin^2 \theta)x_{n-1}^2)^{\frac{n+2s}{2}}} \, dx \, dr \, d\theta
\]

\[
= \prod_{j} \lambda_j \int_{-\theta_0}^{\theta_0} \ldots d\theta + \prod_{j} \lambda_j \int_{\theta_0}^{\pi/2} \ldots d\theta
\]

\[
= I_1 + I_2.
\]

Our aim is to show that as \( \epsilon \to 0 \), \( I_1 \to \infty \) (Proposition 3.2), and \( I_2 \geq 0 \) (Proposition 3.3).

We need to prove the fractional laplacian of the restriction of \( u \) to any \((n - 1)\)-dimensional subspace is positive and bounded from above:

**Proposition 3.1.** Assume that \( u \) satisfies all conditions in Theorem 1.4 then

\[
0 < \mu_0 \leq (1 - s) \int_{\mathbb{R}^{n-1}} \frac{u(z_1 e_1 + z_2 e_2 + \ldots + z_{n-1} e_{n-1}) - u(0)}{|z|^{n-1+2s}} \, dz \leq \mu_1
\]

for each orthonormal basis \( \{e_j\}_{j=1}^{n-1} \) of \( \mathbb{R}^{n-1} \), where

\[
\mu_0 = \mu_0(\eta_0, n, s, L, SC)
\]

given by (13), and

\[
\mu_1 = \mu_1(n, s, L, SC)
\]

given by (14).

**Proposition 3.2.** Assume that \( u \) satisfies all conditions in Theorem 1.4. When \( M \in \mathcal{M}_2 \), \( \sqrt{M}^{-1} = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) and \( \lambda_{\min}(M) = \epsilon \), the integral

\[
I_1 = \prod_{j} \lambda_j \int_{-\theta_0}^{\theta_0} \int_{0}^{\infty} \int_{x \in B_1^{n-1}(0), x \perp \tilde{e}(\theta)} \frac{u(r(x_1 \tilde{e}_1 + \ldots + x_{n-1} \tilde{e}_{n-1})) - u(0)}{r^{1+2s}(\lambda_j^2 x_1^2 + \ldots + (\lambda_{n-1}^2 \cos^2 \theta + \lambda_n^2 \sin^2 \theta)x_{n-1}^2)^{\frac{n+2s}{2}}} \, dx \, dr \, d\theta
\]

\[
\geq C_4 \mu_0 \frac{\epsilon^{-s}}{1 - s}.
\]

Here \( C_4 = C_4(n, s, \eta_0, L, SC) \) is given by (17).
Proposition 3.3. Assume that \( u \) satisfies all conditions in Theorem 1.4. For each \( M \in M_2, \sqrt{M}^{-1} = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_n\} \), the integral
\[
\prod \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(y_1, y_2, ..., y_{n-1}, 0) - u(0)}{(\lambda_1^2 y_1^2 + ... + \lambda_{n-1}^2 y_{n-1}^2)^{s/2}} \, dy \geq 0.
\]
And this shows
\[
I_2 = \prod \lambda_j \int_{|\theta| \geq \theta_0} \int_0^\infty \int_0^\infty \frac{u(r(x_1 \bar{e}_1 + ... + x_{n-1} \bar{e}_{n-1})) - u(0)}{r^1 + 2s(\lambda_1^2 x_1^2 + ... + (\lambda_{n-1}^2 \cos^2 \theta + \lambda_n^2 \sin^2 \theta) x_{n-1}^2)^{s/2}} \, dx \, dr \, d\theta \geq 0.
\]

Proposition 3.2 and Proposition 3.3 together prove the main theorem:

Proof of Theorem 1.4. Let \( P \) be an orthogonal matrix such that
\[
P^T \sqrt{M}^{-1} P = J = \text{diag}\{\lambda_1, ..., \lambda_n\}.
\]
and \( M \in M_2, \) with \( \lambda_{\min}(M) = \epsilon \). then by Proposition 3.2 and Proposition 3.3,
\[
\int_{\mathbb{R}^n} \frac{u(y) - u(0)}{\sqrt{M}^{-1} y |y|^{n+2s}} \, dy = \prod_{j=1}^n \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(y) - u(0)}{(\lambda_1^2 y_1^2 + ... + \lambda_{n-1}^2 y_{n-1}^2)^{s/2}} \, dy = I_1 + I_2 \geq C_4 \mu_0 \epsilon^{-s} + 0 = C_4 \mu_0 \frac{\epsilon^{-s}}{1 - s}.
\]
Also, since \( I \in \Gamma_2 \), so
\[
M_0 = Df_2(I) = \sqrt{\frac{n-1}{2n}} I \in M_2.
\]
we can obtain
\[
F_{2,s}[u](0) = \inf_{M \in M_2} \{\text{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{\sqrt{M}^{-1} y |y|^{n+2s}} \, dy\} \leq \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{\sqrt{M_0^{-1} y |y|^{n+2s}}} \, dy \leq \frac{(n - 1)^{s/2}}{2n} \mu_1^{1/s} \mu_0 \frac{\mu_1}{1 - s},
\]
here the last inequality is proved by Proposition 3.1.

Therefore, when \( \epsilon \) is small enough, for instance, when
\[
\epsilon < \sqrt{\frac{2n}{n-1} C_4^{1/s} \left(\frac{\mu_0}{\mu_1}\right)^{1/s}},
\]
we can see

\[ C_4 \mu_0 \epsilon^{-s} > \left( \frac{n - 1}{2n} \right)^{s/2} \frac{\mu_1}{1 - s}. \]

Now we take

\[ \epsilon_0 = \sqrt{\frac{n}{n - 1} C_4^{1/s} \frac{\mu_0}{\mu_1}}^{1/s} < \sqrt{\frac{2n}{n - 1} C_4^{1/s} \frac{\mu_0}{\mu_1}}^{1/s}. \]

Combining (6), (7) and (8), we can obtain

\[
\inf_{M \in M_2} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u,0,y)}{\sqrt{M^{-1}} y^{n+2s}} \det \sqrt{M}^{-1} dy, \lambda_{\min}(M) \leq \epsilon_0 \right\} \\
\geq \frac{C_4 \mu_0}{1 - s} \epsilon_0^{-s} \\
> \left( \frac{n - 1}{2n} \right)^{s/2} \frac{\mu_1}{1 - s} \\
\geq F_{2,s}[u](0).
\]

Therefore,

\[
\inf_{M \in M_2} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u,0,y)}{\sqrt{M^{-1}} y^{n+2s}} \det \sqrt{M}^{-1} dy, \lambda_{\min}(M) \leq \epsilon_0 \right\} > F_{2,s}[u](0),
\]

and thus,

\[ F_{2,s}[u](0) = F_{2,s}^0[u](0), \]

with

\[ \epsilon_0 = \epsilon_0(n, s, \eta_0, S, L, C) = \sqrt{\frac{n}{n - 1} C_4^{1/s} \frac{\mu_0}{\mu_1}}^{1/s}. \]

And

\[ C_4 = C_4(n, s, \eta_0, L, SC) \]
given by (17). And this completes the proof for Theorem 1.4.

We will use the following lemmas to prove Proposition 3.1. Take a matrix \( B \in \Gamma_2 \), that

\[ B = \text{diag}\{ \frac{2}{n - 1} \epsilon, \frac{2}{n - 1} \epsilon, ..., \frac{2}{n - 1} \epsilon, h(\epsilon) \}. \]

And find \( h(\epsilon) \) such that

\[ \sigma_2(B) = 2\epsilon h(\epsilon) + \frac{2(n - 2)}{n - 1} \epsilon^2 = 1, \]

and this means

\[ h(\epsilon) = \frac{1 - \frac{2(n - 2)}{n - 1} \epsilon^2}{2\epsilon}, \]

and when \( \epsilon \) is small enough, \( h(\epsilon) \approx \frac{1}{2\epsilon} \). Then as defined,

\[ M(B) = \frac{1}{2\sigma_2(B)^{1/2}} \text{diag}\{ \frac{2(n - 2)}{n - 1} \epsilon + h(\epsilon), \frac{2(n - 2)}{n - 1} \epsilon + h(\epsilon), ..., \frac{2(n - 2)}{n - 1} \epsilon + h(\epsilon), 2\epsilon \}. \]
Here we define two constants $C_{10}$ where

$$g(\epsilon) = \left(\frac{n - 2}{n - 1} + \frac{h(\epsilon)}{2}\right)^{-1/2}.$$

And we can see that $g(\epsilon) \approx 2\sqrt{\epsilon}$ when $\epsilon$ is very small. Then, since $M \in \mathcal{M}_2$, thus by the equation (11)

$$0 < \frac{\eta_0}{1 - s} \leq \det(\sqrt{M^{-1}}) \int_{\mathbb{R}^n} \frac{u(\tilde{\gamma}, y_n)}{(g(\epsilon)^2|\tilde{\gamma}|^2 + \frac{1}{\epsilon}y_n^2)^{n+2s}} dy \leq g(\epsilon)^n e^{-1/2} \int_{\mathbb{R}^n} \frac{u(\tilde{\gamma}, y_n)}{(g(\epsilon)^2|\tilde{\gamma}|^2 + \frac{1}{\epsilon}y_n^2)^{n+2s}} dy + g(\epsilon)^{n-1} e^{-1/2} \int_{\mathbb{R}^n} \frac{u(\tilde{\gamma}, 0)}{(g(\epsilon)^2|\tilde{\gamma}|^2 + \frac{1}{\epsilon}y_n^2)^{n+2s}} dy = J_1 + J_2.$$

Lemma 3.4 will give an estimate of $J_1$ by semi-concavity and Lipschitz continuity of $u$.

**Lemma 3.4.** Assume that $u$ satisfies all conditions in Theorem 1.4. Take $\sqrt{M^{-1}} = \text{diag}\{g(\epsilon), g(\epsilon), \ldots, g(\epsilon), e^{-1/2}\}$, then

$$J_1 = g(\epsilon)^n e^{-1/2} \int_{\mathbb{R}^n} \frac{u(\tilde{\gamma}, y_n)}{(g(\epsilon)^2|\tilde{\gamma}|^2 + \frac{1}{\epsilon}y_n^2)^{n+2s}} dy \leq \epsilon^s C_1 C_2,$$

where $C_1 = C_1(s, L, SC)$ and $C_2 = C_2(n, s)$ are given by (10) and (11) respectively.

**Proof.** By Lipschitz continuity and semi-concavity of $u$,

$$J_1 \leq g(\epsilon)^n e^{-1/2} \int_{\mathbb{R}^n} \frac{\max\{2L|y_n|, SC|y_n|^2\}}{(g(\epsilon)^2|\tilde{\gamma}|^2 + \frac{1}{\epsilon}y_n^2)^{n+2s}} dy,$$

then we can do change of variables, letting

$$z_n = y_n, z_j = \frac{y_j}{|y_n|}\sqrt{\epsilon}g(\epsilon), j = 1, 2, \ldots, n - 1.$$

Then

$$dz = dy \frac{1}{|y_n|^{n-1}}(\sqrt{\epsilon}g(\epsilon))^{n-1},$$

and

$$I_1 \leq g(\epsilon)^n e^{-1/2}(\sqrt{\epsilon}g(\epsilon))^{1-n} \int_{\mathbb{R}^n} \frac{\max\{2L|z_n|, SC|z_n|^2\}}{(1 + |\tilde{\gamma}|^2)^{n+2s}|z_n|^{n+2s-n + 1} e^{-(n+2s)/2}} d\tilde{\gamma}d\tilde{z}_n \leq \epsilon^s \int_{\mathbb{R}^n} \frac{\max\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} d\tilde{z}_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |\tilde{\gamma}|^2)^{n+2s}} d\tilde{\gamma} \leq \epsilon^s C_1 C_2.$$

Here we define two constants $C_1, C_2$ by following:

$$C_1 = C_1(s, L, SC) = \int_{\mathbb{R}^n} \frac{\max\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} d\tilde{z}_n,$$

and

$$C_2 = C_2(n, s).$$

So write $\sqrt{M^{-1}} = \text{diag}\{g(\epsilon), g(\epsilon), \ldots, g(\epsilon), e^{-1/2}\}$, where
Then Lemma 3.5 gives an estimate of the integral $J_2$.

**Lemma 3.5.** Assume that $u$ satisfies all conditions in Theorem 1.4. Take $\sqrt{M}^{-1} = \text{diag}\{g(\epsilon), g(\epsilon), \ldots, g(\epsilon), \epsilon^{-1/2}\}$, then

$$J_2 = g(\epsilon)^{-2s}C_3 \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z},$$

where $C_3 = C_3(n, s)$ are given by (12).

**Proof.**

By change of variables

$$z_j = y_j, \quad j = 1, 2, \ldots, n - 1,$$

$$z_n = (\sqrt{\epsilon}g(\epsilon))^{-1}y_n,$$

we will get

$$dz = dy(\sqrt{\epsilon}g(\epsilon)|y|)^{-1},$$

and

$$J_2 = g(\epsilon)^{n-1} \epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{y}, 0) - u(0)}{(g(\epsilon)^2|\bar{y}|^2 + \frac{1}{\epsilon} y_n^2)^{n+2s}} dy.$$

Here we define a constant $C_3$ by the following:

(12) \hspace{1cm} C_3 = C_3(n, s) = \int_{\mathbb{R}} \frac{1}{(1 + z_n^2)^{\frac{n+2s}{2}}} dz_n.
Proof. From the equation, we can see

\[ 0 < \frac{\eta_0}{1 - s} \leq J_1 + J_2 \leq \epsilon^s C_1 C_2 + g(\epsilon)^{-2s} C_3 \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z}, \]

and therefore,

\[ \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \geq \frac{\eta_0}{1 - s} - \epsilon^s C_1 C_2, \]

So we only need to take \( \epsilon = \epsilon_1 \) small enough such that

\[ \eta_0 = 2(1 - s) C_1 C_2 \epsilon_1^s, \]

that

\[ \epsilon_1 = \left( \frac{\eta_0}{2(1 - s) C_1 C_2} \right)^{1/s}, \]

then

\[ \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \geq \frac{\eta_0}{2(1 - s) C_3} g(\epsilon_1)^{2s}. \]

And we have calculated that

\[ g(\epsilon) = \left( \frac{1}{4\epsilon} + \frac{n - 2}{2(n - 1)} \epsilon \right)^{-1/2}, \]

thus

\[ g(\epsilon_1)^{2s} = \left( \frac{1}{4\epsilon_1} + \frac{n - 2}{2(n - 1)} \epsilon_1 \right)^{-s}, \]

and we can define

\[ (13) \quad \mu_0 = \mu_0(n, s, \eta_0, L, SC) = \frac{\eta_0}{2(1 - s) C_3} g(\epsilon_1)^{2s}, \]

we obtain the estimates that

\[ \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \geq \mu_0 > 0. \]

And by doing any orthonormal transformation, we will be able to show if \( \{e_j\}_{j=1}^{n-1} \) are orthonormal basis of \( \mathbb{R}^{n-1} \),

\[ \int_{\mathbb{R}^{n-1}} \frac{u(z_1 e_1 + z_2 e_2 + \ldots + z_{n-1} e_{n-1}) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \geq \mu_0 > 0. \]

On the other hand, if \( u \) is Lipschitz continuous and semi-concave, then

\[ \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \leq \int_{\mathbb{R}^{n-1}} \max\{2L|\bar{z}|, SC|\bar{z}|^2\} d\bar{z} \leq \frac{\mu_1}{1 - s}, \]

with

\[ (14) \quad \mu_1 = \mu_1(n, s, L, SC) = (1 - s) \int_{\mathbb{R}^{n-1}} \max\{2L|\bar{z}|, SC|\bar{z}|^2\} d\bar{z}. \]

□
With the estimates in Proposition 3.1, now we start to prove Proposition 3.2. The main idea is that, when the smallest eigenvalue of matrix $M$ is close to 0, there will be some constraints on the eigenvalues and their square root inverse $\lambda_j$, since the matrix is in the set $\mathcal{M}_2$. We will prove that $\frac{1}{\lambda_1^{n+2s}} - \frac{1}{\lambda_{n-1}^{n+2s}}$ is very small compared with $\frac{1}{\lambda_{n-1}^{n+2s}}$. This and the lower bound in Proposition 3.1 will make it possible to prove that the integral on a (n-1)-dimensional subspace, close to \( \{x_n = 0\} \), is very large.

**Proof of Proposition 3.2.** Our aim is to show that when $\epsilon$ is very small, $I_1 \geq C_4\mu_0\epsilon^{-s}$. We take $\theta_0 = C\frac{\lambda_{n-1}}{\lambda_e}$, which is very small ($\theta_0 \leq 2C\epsilon$) and the constant $C$ depends on $\frac{\mu_0}{\mu_0}$, determined by (1.5). When $|\theta| \leq \theta_0$,

$$
\lambda_{n-1}^2 \cos^2 \theta + \lambda_n^2 \sin^2 \theta \leq (1 + C^2)\lambda_{n-1}^2
$$

and thus,

$$(1 - 4C^2\epsilon^2)\lambda_1^2 \leq \lambda_{n-1}^2 \cos^2 \theta + \lambda_n^2 \sin^2 \theta \leq (1 + C^2)\lambda_{n-1}^2$$

Let

$$
A = \int_0^\infty \int_{\{x \in \partial B_{1}^{n-1}(0), u(rx) - u(0) > 0\}} \frac{u(rx) - u(0)}{r^{1+2s}} dx dr \geq 0,
$$

and

$$
B = \int_0^\infty \int_{\{x \in \partial B_{1}^{n-1}(0), u(rx) - u(0) \leq 0\}} \frac{u(rx) - u(0)}{r^{1+2s}} dx dr \leq 0.
$$

Then by Proposition 3.1

$$
A + B \geq \frac{\mu_0}{1-s} > 0,
$$

and

$$
A \leq \frac{\mu_1}{1-s}.
$$

And we can have the following estimates

$$
(1-s)I_1 = \prod_{j} \lambda_j \int_{-\theta_0}^{\theta_0} \int_0^\infty \int_{x \in \partial B_{1}^{n-1}(0)} \frac{u(rx) - u(0)}{r^{1+2s}} \frac{1}{(\lambda_1^2 x_1^2 + \cdots (\lambda_{n-1}^2 \cos^2 \theta + \lambda_n^2 \sin^2 \theta) x_{n-1}^2)} dx dr d\theta
$$

$$
\geq 2\theta_0 \prod_{j} \lambda_j \bigg( \frac{1}{\lambda_{1}^{n+2s}} + \frac{1}{\lambda_{n-1}^{n+2s}} \bigg) \bigg( \frac{\mu_0}{\lambda_{1}^{n+2s}} + \frac{(1 + C^2)^{-(n+2s)/2}}{\lambda_{n-1}^{n+2s}} \bigg) \bigg( \frac{1}{\lambda_{1}^{n+2s}} - \frac{1}{\lambda_{n-1}^{n+2s}} \bigg)
$$

$$
\geq 2C \lambda_1 \mu_0 + \mu_1 (C_5 - 1) \lambda_1^{n+2s} + C_5 \mu_1 \bigg( \frac{\lambda_{1}^{n+2s}}{\lambda_{n-1}^{n+2s}} - \frac{1}{\lambda_{1}^{n+2s}} \bigg).
$$

Here

$$
C_5 = (1 + C^2)^{-(n+2s)/2}
$$

and take constant $C$ such that

$$
\mu_0 + \mu_1 (C_5 - 1) \geq \mu_0/2,
$$
i.e., take

\[(15) \quad C = \sqrt{\left(1 - \frac{\mu_0}{2\mu_1}\right)^{\frac{4}{n+2s}} - 1}\]

and

\[(16) \quad C_5 = 1 - \frac{\mu_0}{2\mu_1}\]

Now let’s see what constraint we will have on \(\lambda_j\) when the smallest eigenvalue of matrix \(M \in \mathcal{M}_2\) is \(\epsilon\). We want to show that the non-negative

\[\frac{1}{\lambda_1^{n+2s}} - \frac{1}{\lambda_{n-1}^{n+2s}}\]

is very small compared with \(\frac{1}{\lambda_1^{n-1}}\).

Let \(B = \text{diag}\{\sigma_1, \sigma_2, ..., \sigma_n\} \in \Gamma_2\). Assume \(\sigma_1 \leq \sigma_2 \leq ... \leq \sigma_n\), and \(\sum \sigma_i \sigma_j = 1\). Then \(M = \text{diag}\{\eta_1, \eta_2, ..., \eta_n\} = Df_2(B)\), with \(\eta_1 \geq \eta_2 \geq ... \geq \eta_n = \epsilon\), and

\[\eta_j = \frac{1}{2} \left(\sum_i \sigma_i - \sigma_j\right).\]

Then

\[\sigma_1 + \sigma_2 + ... + \sigma_{n-1} = 2\epsilon = 2\eta_n.\]

Let \(Q = \sigma_2 + \sigma_3 + ... + \sigma_{n-1}\). Then \(Q > \frac{2(n-2)}{n-1}\epsilon\). And since \(\sum \sigma_i \sigma_j = 1\), so

\[
\begin{align*}
1 &= \sigma_n \left(\sum_{i=1}^{n-1} \sigma_i\right) + \sum_{1 \leq i < j \leq n-1} \sigma_i \sigma_j \\
&= \sigma_n (2\epsilon) + \sigma_1 (Q) + \sum_{2 \leq i < j \leq n-1} \sigma_i \sigma_j \\
&\leq 2\epsilon \sigma_n + (2\epsilon - Q)Q + \frac{Q^2}{2} \\
&= 2\epsilon \sigma_n + 2\epsilon Q - \frac{Q^2}{2}.
\end{align*}
\]

Then

\[\sigma_n \geq \frac{1 + Q^2/2 - 2\epsilon Q}{2\epsilon}.\]

And therefore

\[\eta_1 = \frac{1}{2} (Q + \sigma_n) \geq \frac{1 + Q^2/2}{4\epsilon}.\]

In addition, since \(\sigma_1 = 2\epsilon - Q\), and \(\sigma_{n-1} = 2\epsilon - \sigma_1 - \sigma_2 - ... - \sigma_{n-1} \leq 2\epsilon - (n-2)\sigma_1\), so

\[0 \geq \sigma_1 - \sigma_{n-1} \geq (2n-4)\epsilon - (n-1)Q,\]

and this means

\[\eta_{n-1} - \eta_1 \geq (2n-4)\epsilon - (n-1)Q.\]

Therefore we can calculate

\[
\frac{1}{\lambda_{n-1}^{n+2s}} - \frac{1}{\lambda_1^{n+2s}} \geq \frac{n+2s}{2} \left( \frac{1}{\lambda_{n-1}^{n+2s-2}} - \frac{1}{\lambda_1^{n-1}} \right) \geq \frac{n+2s}{2} \frac{1}{\lambda_{n-1}^{n+2s-2}} ((2n-4)\epsilon - (n-1)Q).
\]
Therefore,

\[(1 - s)I_1 \geq 2C\lambda_1^2\left(\frac{\mu_0 + \mu_1(C - 1)}{\lambda_1^{n+2s}} + C\mu_1\left(\frac{1}{\lambda_1^{n+2s}} - \frac{1}{\lambda_1^{n+2s}}\right)\right)
\]

\[\geq \frac{C\mu_0}{2} + \frac{C\mu_1}{2} + C\mu_1(n + 2s)\eta_1^{s-1}(2n - 4)\epsilon - (n - 1)Q\]

\[\geq \frac{C\mu_0}{2}\epsilon^{-s} + \eta_1^{s-1}(\frac{C\mu_0}{2}\eta_1 + C\epsilon - C^7Q)\]

\[\geq \frac{C\mu_0}{2}\epsilon^{-s} + \eta_1^{s-1}(\frac{C\mu_0 1 + \frac{Q^2/2}{4\epsilon}}{2} + C\epsilon - C^7Q)\]

\[\geq \frac{C\mu_0}{2}\epsilon^{-s} + \eta_1^{s-1}\left(\frac{C\mu_0}{8\epsilon} + \sqrt{\frac{C\mu_0}{16\epsilon}Q - C^7\sqrt{\frac{4\epsilon}{C\mu_0}}}^2 + C\epsilon - C^7\frac{4\epsilon}{C\mu_0}\right)\]

\[\geq \frac{C\mu_0}{2}\epsilon^{-s} + 0\]

\[\geq C_4\mu_0\epsilon^{-s},\]

when \(\epsilon > 0\) very small, and

\[C_4 = C_4(n, s, L, SC, \eta_0) = \frac{C}{2} = \frac{1}{2} \sqrt{(1 - \frac{\mu_0}{2\mu_1})}\frac{2}{n+2s} - 1.\]

Then we want to prove Proposition 3.3 by contradiction:

**Proof of Proposition 3.3.** Assume it is not true, then for some \(M \in M_2\), there exists a positive constant \(A > 0\) such that

\[(1 - s)\prod \lambda_j \int_{\bar{y} \in \mathbb{R}^{n-1}} \frac{u(y_1, y_2, \ldots, y_{n-1}, 0) - u(0)}{(\lambda_1 y_1^2 + \ldots + \lambda_n^2 y_n^2)^{n+2s}} \bar{y} = -A < 0.\]

Then since \(M = \text{diag}\{\eta_1, \ldots, \eta_n\} \in M_2\), there exists \(B = \text{diag}\{\sigma_1, \ldots, \sigma_n\} \in \Gamma_2\). WLOG we require \(\sum \sigma_j = 1\). Then we can see \(M = Df_2(B)\) and

\[\eta_j = \frac{1}{2}(\sum_{i \neq j} \sigma_i).\]

Take another matrix \(\tilde{B} \in \Gamma_2\), that \(\tilde{B} = \text{diag}\{\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_n\}\), and let

\[\tilde{\sigma}_j = t_j\sigma_j, j = 1, 2, \ldots, n - 1\]

and

\[\tilde{\sigma}_n = g(t)\sigma_n.\]

Given any \(t > 0\) every small, first find \(n\) unknowns \(t_1, t_2, \ldots, t_{n-1}, f(t)\) such that the following \(n\) equations are satisfied:

\[\frac{\tilde{\eta}_j}{\eta_j} = \frac{\sum_{1 \leq i \leq n-1} t_i \sigma_i}{\sum \sigma_i - \sigma_j} = \frac{1}{t}, j = 1, 2, \ldots, n - 1;\]
and

$$1 = \sum_{1 \leq i < j \leq n-1} \sigma_i \sigma_j = \sum_{1 \leq i < j \leq n-1} t_i t_j \sigma_i \sigma_j + g(t) \sigma_n (t_1 \sigma_1 + t_2 \sigma_2 + \ldots + t_{n-1} \sigma_{n-1}).$$

The last equation means

$$g(t) = \frac{1 - \sum_{1 \leq i < j \leq n-1} t_i t_j \sigma_i \sigma_j}{\sigma_n (t_1 \sigma_1 + t_2 \sigma_2 + \ldots + t_{n-1} \sigma_{n-1})}.$$ 

and as $t, t_j \to 0$, $g(t) \to \infty$ if $\sigma_n > 0$. And if $\sigma_n < 0$, then $g(t) < 0$ but still we will have $\tilde{\sigma}_n = g(t) \sigma_n$ positive and goes to $\infty$. Then

$$\tilde{M} = Df_2(\bar{B}) = diag\{\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_n\}$$

with

$$\tilde{\eta}_j = \frac{1}{t} \eta_j, j = 1, 2, \ldots, n - 1;$$

$$\tilde{\eta}_n = \frac{t_1 \sigma_1 + t_2 \sigma_2 + \ldots + t_{n-1} \sigma_{n-1}}{\sigma_1 + \sigma_2 + \ldots + \sigma_{n-1}} = h(t) \eta_n$$

And as $t \to 0$,

$$h(t) \to 0.$$

Then

$$\tilde{\lambda}_j = \sqrt{t} \lambda_j, j = 1, 2, \ldots, n - 1;$$

and

$$\tilde{\lambda}_n = h(t)^{-1/2} \lambda_n.$$

Now since $\tilde{M} \in \mathcal{M}_2$ as well, therefore it satisfies the equation

$$0 < \frac{\eta_0}{1-s} \leq \prod \tilde{\lambda}_j \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{(\tilde{\lambda}_1 y_1^2 + \ldots + \tilde{\lambda}_n y_n^2)^{(n+2s)/2}} dy$$

$$\leq \prod \tilde{\lambda}_j \int_{\mathbb{R}^n} \frac{u(\bar{y}, y_n) - u(\bar{y}, 0)}{(\tilde{\lambda}_1 y_1^2 + \ldots + \tilde{\lambda}_n y_n^2)^{(n+2s)/2}} dy + \prod \tilde{\lambda}_j \int_{\mathbb{R}^n} \frac{u(\bar{y}, 0) - u(0)}{(\tilde{\lambda}_1 y_1^2 + \ldots + \tilde{\lambda}_n y_n^2)^{(n+2s)/2}} dy$$

$$= P_1 + P_2.$$

Define $\lambda = \min\{\lambda_1, \ldots, \lambda_n\} > 0$, first we can calculate $P_1$

$$P_1 \leq t^{(n-1)/2} h(t)^{-1/2} \prod \lambda_j \int_{\mathbb{R}^n} \frac{\max\{2L|y_n|, SC|y_n|^2\}}{(t(\lambda_1 y_1^2 + \ldots + \lambda_n y_n^2)^{(n+2s)/2}) dy}$$

$$\leq t^{(n-1)/2} h(t)^{-1/2} \lambda^{-n-2s} \prod \lambda_j \int_{\mathbb{R}^n} \frac{\max\{2L|y_n|, SC|y_n|^2\}}{(t(y_1^2 + \ldots + y_n^2)^{(n+2s)/2}) dy}.$$ 

Do change of variables

$$z_j = \frac{y_j}{|y_n|} \sqrt{th(t)}, j = 1, 2, \ldots, n - 1$$

and

$$z_n = y_n.$$
we can calculate
\[
P_1 \leq \lambda^{-n-2s} \prod \lambda_j h(t)^s \int_{\mathbb{R}} \frac{1}{|z_n|^{1+2s}} d\bar{z} = \int_{\mathbb{R}^n-1} \frac{1}{(1 + |\bar{z}|^2)^{(n+2s)/2}} d\bar{z}.
\]
Calculating details are similar to the proof of Proposition 3.1 and with definitions of (10) and (11) we know
\[
P_1 \leq h(t)^s C(\lambda) C_1 C_2.
\]
Then we calculate \(P_2\), that
\[
P_2 = \prod \lambda_j \int_{\mathbb{R}^n} \frac{u(\bar{y}, 0) - u(0)}{(\lambda_1^2 y_1^2 + \ldots + \lambda_n^2 y_n^2)^{(n+2s)/2}} dy
\]
\[
= t^{(n-1)/2} h(t)^{-1/2} \prod \lambda_j \int_{\mathbb{R}^n} \frac{u(\bar{y}, 0) - u(0)}{(t(\lambda_1^2 y_1^2 + \ldots + \lambda_n^2 y_n^2)^{(n+2s)/2})} dy.
\]
By change of variable,
\[
z_j = y_j, j = 1, 2, \ldots, n-1,
\]
\[
z_n = \frac{y_n}{(\lambda_1^2 y_1^2 + \ldots + \lambda_{n-1}^2 y_{n-1}^2)^{1/2}} \frac{\lambda_n}{\sqrt{th(t)}}.
\]
we can calculate this integral
\[
P_2 = t^{-s} \prod \lambda_j \int_{\mathbb{R}^n-1} \frac{u(z, 0) - u(0)}{(\lambda_1^2 z_1^2 + \ldots + \lambda_{n-1}^2 z_{n-1}^2)^{(n+2s-1)/2}} dz \int_{\mathbb{R}} \frac{1}{(1 + |z_n|^2)^{(n+2s)/2}} d\bar{z}
\]
\[
= t^{-s} C_3 \frac{1}{\lambda_n} \frac{1 - A}{1 - s},
\]
Here
\[
C_3 = C_3(n, s) = \int_{\mathbb{R}} \frac{1}{(1 + |z|^2)^{(n+2s)/2}} dz
\]
is the same as in (12). Then as \(t \to 0\), since \(A > 0\) positive, and \(h(t) \to 0\),
\[
P_1 + P_2 \leq h(t)^s C(\lambda) C_1 C_2 - t^{-s} \frac{AC_3}{(1 - s)\lambda_n} \to -\infty,
\]
and this contradicts
\[
P_1 + P_2 = \prod \lambda_j \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{(\lambda_1^2 y_1^2 + \ldots + \lambda_n^2 y_n^2)^{(n+2s)/2}} dy \geq \frac{\eta_0}{1 - s} > 0,
\]
which completes the proof of Proposition 3.3. □

**Acknowledgement**

The author would like to thank her Ph.D. advisor, Professor Luis Caffarelli, for many valuable conversations on this project. She also want to thank Professor Sun-Yung Alice Chang, who shared her ideas on this topic and pointed out the k-cone problem. She is also grateful to many colleagues and friends, especially Hui Yu, who offered many helpful comments on this paper.
References

[1] Caffarelli, Luis, and Fernando Charro. "On a Fractional Monge-Ampre Operator." Annals of PDE 1.1 (2015): 1-47.

[2] L. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math., 62(5):597638, 2009.

[3] L. Caffarelli, L. Silvestre, The Evans-Krylov theorem for nonlocal fully nonlinear equations, Ann. of Math. (2) 174 (2011), no. 2, 11631187.

[4] Wang, Xu-Jia. "The k-Hessian equation." Geometric analysis and PDEs. Springer Berlin Heidelberg, 2009. 177-252.

[5] Caffarelli, Luis, Louis Nirenberg, and Joel Spruck. "The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian." Acta Mathematica 155.1 (1985): 261-301.

Department of Mathematics, the University of Texas at Austin
E-mail address: yijingwu@math.utexas.edu