NON-NILPOTENT SUBGROUPS OF LOCALLY GRADED GROUPS

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Abstract. In this paper, we show that a locally graded group with a finite number \( m \) of non-nilpotent of class at most \( n \) subgroups is (soluble of class at most \( \lceil \log_2(n) \rceil + m + 3 \))-by-(finite of order \( \leq m! \)). Also we show that the derived length of a soluble group with a finite number \( m \) of non-nilpotent of class at most \( n \) subgroups, is at most \( \lceil \log_2(n) \rceil + m + 1 \).

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1. Introduction and results

Let \( G \) be a group. A non-nilpotent finite group whose proper subgroups are all nilpotent is well-known (called Schmidt group). In 1924, O.Yu. Schmidt studied such groups and proved that such groups are soluble [7]. Subsequently, Newman and Wiegold in [5], discussed infinite non-nilpotent groups whose proper subgroups are all nilpotent. Such groups need not be soluble in general. For example, the Tarski Monsters, which are infinite simple groups with all proper subgroups of a fixed prime order.

Following [9] we say that a group \( G \) is a \( S^m \)-group if \( G \) has exactly \( m \) non-nilpotent subgroups. More recently Zarrin in [9] generalized Schmidt’s Theorem and proved that every finite \( S^m \)-group with \( m < 22 \) is soluble. Let \( n \) be a non-negative integer. We say that a group \( G \) is an \( S^m_n \)-group, if \( G \) has exactly \( m \) non-nilpotent of class at most \( n \) subgroups. Clearly, every \( S^m_n \)-group is a \( S^r \)-group, for some \( r \leq m \). Here, we show that every locally graded group with a finite number \( m \) of non-nilpotent of class at most \( n \) subgroups, is soluble-by-finite. Recall that a group \( G \) is locally graded if every non-trivial finitely generated subgroup of \( G \) has a non-trivial finite homomorphic image. This is a rather large class of groups, containing for instance all residually finite groups and all locally(soluble-by-finite) groups.

Theorem A. Every locally graded \( S^m_n \)-group is (soluble of class at most \( \lceil \log_2(n) \rceil + m + 3 \))-by-(finite of order \( \leq m! \)).

This result suggests that the behavior of non-nilpotent of class at most \( n \) subgroups has a strong influence on the structure of the group.

Finding a upper bound for the solubility length of a soluble group is an important problem in the theory of groups, for example see [8]. It is well-known that a nilpotent group of class \( n \) (or a group without non-nilpotent of class at most \( n \) subgroups) has derived length \( \leq \lceil \log_2(n) \rceil + 1 \) (see [8], Theorem 5.1.12). Here,
we obtain a result which is of independent interest, namely, the derived length of soluble $S_m^n$-groups is bounded in terms of $m$ and $n$. (Note that every nilpotent group of class $n$ is a $S_m^n$-group with $m = 0$.)

**Theorem B.** Let $G$ be a soluble $S_m^n$-group and $d$ be the derived length of $G$. Then $d \leq \lceil \log_2(n) \rceil + m + 1$.

## 2. Proofs

If $G$ is an arbitrary group, the *norm* $B_1(G)$ of $G$ is the intersection of the normalizers of all subgroups of $G$ and $W(G)$ is the intersection of the normalizers of all subnormal subgroups of $G$. In 1934 and 1958, respectively, those concepts were considered by R. Baer and Wielandt (see also [1, 2, 3]). More recently Zarrin generalized this concept in [10]. Here we define $A_n(G)$ as the intersection of all the normalizers of non-nilpotent of class at most $n$) subgroups of $G$, i.e.,

$$A_n(G) = \bigcap_{H \in \mathcal{T}_n(G)} N_G(H),$$

where $\mathcal{T}_n(G) = \{H \mid H$ is a non-nilpotent of class at most $n$) subgroup of $G\}$ (with the stipulation that $A_n(G) = G$ if all subgroups of $G$ are nilpotent of class at most $n$). Clearly

$$B_1(G) \leq A_i(G) \leq A_{i+1}(G).$$

Moreover, in view of the proof of Theorem A, below, we can see that, for every locally graded group $G$, we have

$A_n(G)$ is a soluble normal subgroup of $G$ of class $\leq \lceil \log_2(n) \rceil + 4$.

**Proof of Theorem A.** The group $G$ acts on the set $\mathcal{T}_n(G)$ by conjugation. By assumption $|\mathcal{T}_n(G)| = m$. It is easy to see that the subgroup $A_n(G)$ is the kernel of this action and so $A_n(G)$ is normal in $G$ and $G/A_n(G)$ is embedded in $S_m$, the symmetric group of degree $m$. So

$$|G/A_n(G)| \leq m!.$$

Therefore to complete the proof it is enough to show that $H = A_n(G)$ is soluble of class at most $\lceil \log_2(n) \rceil + 4$. To see this, it is enough to show that $K = H^{(3)}$ is nilpotent of class at most $n$. Suppose on the contrary that $K$ is not nilpotent of class at most $n$. It follows that every subgroup containing $K$ is not nilpotent of class at most $n$ and so, by definition of $A_n(G)$, it is a normal subgroup of $H$. Therefore every subgroup of $H/K$ is normal. That is, $H/K$ is a Dedekind group and so, it is well-known (see [6], Theorem 5.3.7), that $H/K$ is metabelian. From which it follows that

$$H^{(2)} = H^{(3)} = K. \quad (\star)$$

We claim the following conclusions.

**Step1.** Every proper normal subgroup of $K$ is nilpotent of class at most $n$. 


Suppose, a contrary, that there exists a proper normal subgroup $M$ of $K = H^{(2)}$ such that $M$ is not nilpotent of class at most $n$. Then we can obtain, by definition of $A_n(G)$, that $H^{(2)}/M$ is a Dedekind group (so it is metabelian) and so, in view of ($\bullet$), $H^{(2)} = M$, a contradiction.

**Step 2.** The product of all proper normal subgroups of $K$, say $R$, is a proper nilpotent subgroup of $K$ of class at most $n$.

Suppose that $M_1, M_2, \ldots, M_t$ are proper normal subgroups of $H^{(2)}$. Then, by step 1, every $M_i$ is soluble and so $M_1M_2\ldots M_t$ is soluble. Now by ($\bullet$), we conclude that $H^{(2)} \neq M_1M_2\ldots M_t$. Therefore $M_1M_2\ldots M_t$ is a proper normal subgroup of $H^{(2)}$ and so, by step 1, it is nilpotent of class at most $n$. Therefore $R$ is a locally nilpotent of class at most $n$ group, and so $R$ is nilpotent of class at most $n$ (note that the class of nilpotent groups of class at most $n$ is locally closed). Also as ($\bullet$), we have $R \neq H^{(2)}$.

**Step 3.** Finishing the proof.

We note that, by definition of $A_n(G)$, every subgroup of $H^{(2)}$ which is not nilpotent of class at most $c$ is a normal subgroup of $H^{(2)}$. It follows, as $H^{(2)}/R$ is a simple group, that all proper subgroups of $H^{(2)}/R$ are nilpotent of class at most $n$. Since $H^{(2)}$ is locally graded, by the main result of [4], $H^{(2)}/R$ is locally graded. Therefore if $H^{(2)}/R$ is finitely generated then it must be finite. Thus, by Schmidt’s Theorem, $H^{(2)}/R$ is soluble, which is contrary to ($\bullet$). If $H^{(2)}/R$ is not finitely generated, then $H^{(2)}/R$ is locally nilpotent of class at most $n$ and so $H^{(2)}/R$ is nilpotent of class at most $n$, a contradiction.

Now we prove Theorem B.

**Proof of Theorem B.** Assume that a soluble group $G$ has derived length $> [\log_2 n] + 1 + m$ for some $n, m \geq 1$. Then obviously the $m+1$ derived subgroups $G, G', \ldots, G^{(m)}$ are all pairwise distinct and have solubility length $> [\log_2 n] + 1$. Therefore they cannot be nilpotent of class at most $n$. This shows that $G$ cannot be a $S^m_n$-group, a contradiction.

Finally, as every $S^m_n$-group is a $S^r_n$-group, for some $r \leq m$, and by the main result in [9], we can see that every $S^m_n$-group with $m \leq 21$ is soluble. Hence the following question arises naturally:

**Question 2.1.** Assume that $G$ is a $S^{m}_n$-group. What relations between $m, n$ guarantee that $G$ is soluble?

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