CARDINALITIES OF $k$-DISTANCE SETS IN MINKOWSKI SPACES

K. J. SWANEPOEL

Abstract. A subset of a metric space is a $k$-distance set if there are exactly $k$ non-zero distances occurring between points. We conjecture that a $k$-distance set in a $d$-dimensional Banach space (or Minkowski space), contains at most $(k+1)^d$ points, with equality iff the unit ball is a parallelotope. We solve this conjecture in the affirmative for all 2-dimensional spaces and for spaces where the unit ball is a parallelotope. For general spaces we find various weaker upper bounds for $k$-distance sets.

1. Introduction

A subset $S$ of a metric space is a $k$-distance set if there are exactly $k$ non-zero distances occurring between points of $S$. We also call a 1-distance set an equilateral set. In this paper we find upper bounds for the cardinalities of $k$-distance sets in Minkowski spaces, i.e. finite-dimensional Banach spaces (see Theorems 1 to 5), and make a conjecture concerning tight upper bounds.

In Euclidean spaces $k$-distance sets have been studied extensively; see e.g. [13, 14, 15, 12, 3, 5, 3, 9, 11, 10, 18, 24], and the books [22] and [12, sections F1 and F3].

For general $d$-dimensional Minkowski spaces it is known that the maximum cardinality of an equilateral set is $2^d$, with equality iff the unit ball of the space is a parallelotope, and that if $d \geq 3$, there always exists an equilateral set of at least 4 points [23]. It is unknown whether there always exists an equilateral set of $d + 1$ points; see [20, 21] and [22, p. 129, p. 308 problem 4.1.1]. However, Brass [7] recently proved that for each $n$ there is a $d = d(n)$ such that any $d$-dimensional Minkowski space has an equilateral set of at least $n$ points. See [17] for problems on equilateral sets in $\ell_p$ spaces.

Equilateral sets in Minkowski spaces have been used in [20] to construct energy-minimizing cones over wire-frames. See also [21].

As far as we know, $k$-distance sets for $k \geq 2$ have not been studied in spaces other than Euclidean.

Our main results are the following.

Theorem 1. If the unit ball of a $d$-dimensional Minkowski space is a parallelotope, then a $k$-distance set in $X$ has cardinality at most $(k+1)^d$. This bound is tight.

Theorem 2. Given any set $S$ of $n$ points in a $d$-dimensional Minkowski space with a parallelotope as unit ball, there exists a point in $S$ from which there are at least $\lceil n^{1/d} \rceil - 1$ distinct non-zero distances to points in $S$. This bound is tight.

Theorem 3. The cardinality of a $k$-distance set in a 2-dimensional Minkowski space is at most $(k+1)^2$, with equality iff the space has a parallelogram as unit ball.
Theorem 4. Given any set of \( n \) points in a 2-dimensional Minkowski space, there exists a point in \( S \) from which there are at least \( \lceil n^{1/2} \rceil - 1 \) distinct non-zero distances to points in \( S \).

Theorem 5. The cardinality of a \( k \)-distance set in a \( d \)-dimensional Minkowski space is at most \( \min(2^{kd}, (k + 1)^{(d - 2)/2}) \).

In the light of Theorems 1 and 3 and the results of [23], we make the following

Conjecture 1. The cardinality of a \( k \)-distance set in any \( d \)-dimensional Minkowski space is at most \( (k + 1)^d \), with equality if the unit ball is a parallelotope.

As mentioned above, [24] shows that this conjecture is true for \( k = 1 \). By Theorem 3 the conjecture is true if \( d = 2 \), and by Theorem 1 if the unit ball is a parallelotope.

In the sequel, \((\mathbb{R}^d, \|\cdot\|)\) is a \( d \)-dimensional Minkowski space with norm \( \|\cdot\| \), \( B(x, r) \) is the closed ball with centre \( x \) and radius \( r > 0 \), and \( B := B(0, 1) \) the unit ball of the space. Recall that two \( d \)-dimensional Minkowski spaces are isometric iff their unit balls are affinely equivalent (by the Mazur-Ulam Theorem; see e.g. [25 Theorem 3.1.2]). In particular, a Minkowski space has a parallelotope as unit ball iff it is isometric to \((\mathbb{R}^d, \|\cdot\|_{\infty})\), where \( \|(\lambda_1, \lambda_2, \ldots, \lambda_d)\|_{\infty} := \max_{i=1,\ldots,d} |\lambda_i| \).

We define a cone (or more precisely, an acute cone) \( P \) to be a convex set in \( \mathbb{R}^d \) that is positively homogeneous (i.e., for any \( x \in P \) and \( \lambda \geq 0 \) we have \( \lambda x \in P \)) and satisfies \( P \cap (-P) = \{0\} \). Recall that such a cone defines a partial order on \( \mathbb{R}^d \) by \( x \preceq y \iff y - x \in P \).

We denote the cardinality of a set \( S \) by \( \# S \).

For measurable \( S \subseteq \mathbb{R}^d \), let \( \text{vol}(S) \) denote the Lebesgue measure of \( S \). For later reference we state Lysternik’s version of the Brunn-Minkowski inequality (see [8 Theorem 8.1.1])

Lemma 1. If \( A, B \subseteq \mathbb{R}^d \) are compact, then
\[
\text{vol}(A + B)^{1/d} \geq \text{vol}(A)^{1/d} + \text{vol}(B)^{1/d}.
\]
If equality holds and \( \text{vol}(A), \text{vol}(B) > 0 \), then \( A \) and \( B \) are convex bodies such that
\( A = v + \lambda B \) for some \( \lambda > 0 \) and \( v \in \mathbb{R}^d \). \( \square \)

2. Proofs

Proof of Theorem 1. We may assume without loss of generality that the space is \((\mathbb{R}^d, \|\cdot\|_{\infty})\). We introduce partial orders on \( \mathbb{R}^d \) following Blokhuis and Wilbrink [6]. For each \( i = 1, \ldots, d \), let \( \leq_i \) be the partial order with cone
\[
P_i = \{(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d : \max_{j=1,\ldots,d} |\lambda_j| = \lambda_i \}.
\]

For each \( x \) in a \( k \)-distance set \( S \), let \( h_i(x) \) be the length of the longest descending \( \leq_i \)-chain starting with \( x \), i.e. \( h_i(x) \) is the largest \( h \) such that there exist \( x_1, x_2, \ldots, x_h \in S \) for which \( x >_i x_1 >_i x_2 >_i \cdots >_i x_h \).

Since \( \bigcup_{i=1}^d (P_i \cup -P_i) = \mathbb{R}^d \), for all distinct \( x, y \in \ell^\infty_2 \), there exists \( i \) such that \( x <_i y \) or \( y <_i x \). Exactly as in [6], it follows that the mapping \( x \mapsto (h_1(x), \ldots, h_d(x)) \) is injective, and thus \( \# S \leq (h + 1)^d \), where
\[
h := \max_{x \in S, i = 1, \ldots, d} h_i(x).
\]

It remains to show that \( h \leq k \). Suppose not. Then for some \( x \in S \) and some \( i \) there exist \( x_1, \ldots, x_k+1 \in S \) such that \( x >_i x_1 >_i \cdots >_i x_{k+1} \). Since \( S \) is a \( k \)-distance set, \( \|x - x_m\|_{\infty} = \|x - x_n\|_{\infty} \) for some \( 1 \leq m < n \leq k + 1 \). Also, \( x - x_m, x - x_n \in P_i \). Now note that if \( \|a\|_{\infty} = \|b\|_{\infty} \) with \( a, b \in P_i, a \neq b \), then \( a \)
and $b$ are $\leq_i$-incomparable; in particular, $b - a \notin P_i$. Therefore, $x_m - x_n \notin P_i$, a contradiction.

The set $\{0, 1, \ldots, k\}^d$ is a $k$-distance set of cardinality $(k + 1)^d$. Note that it is not difficult to see that in fact the only $k$-distance sets of cardinality $(k + 1)^d$ are of the form $S = a + \lambda[0, 1, \ldots, k]^d$ for some $a \in \mathbb{R}^d$ and $\lambda > 0$. □ □

**Proof of Theorem 4.** Consider the mapping $x \mapsto (h_1(x), \ldots, h_d(x))$ in the proof of Theorem 3. If $h$ is the length of the longest $\leq_i$-chain over all $i$, then $n \leq (h + 1)^d$. Thus there is a $\leq_i$-chain $x_0 > x_1 > \cdots > x_h$ of length $h \geq \lceil n^{1/d} \rceil - 1$. By the last paragraph of the proof of Theorem 3 the distances $\rho(x_0, x_j)$, $j = 1, \ldots, h$ are all distinct.

Lemma 2. Then a $k$-distance set in $(\mathbb{R}^d, ||||)$ has cardinality at most $(k + 1)!$. □

**Lemma 3.** The cardinality of a $k$-distance set in a metric space $(X, \rho)$ with distances $\rho_1 < \rho_2 < \cdots < \rho_k$. If $\rho_k/\rho_1 > 2^{k-1}$, then for some $i = 1, \ldots, k-1$, the relation

$x \sim_i y \iff \rho(x, y) \leq \rho_i$

is an equivalence relation.

Proof. The relation $\sim_i$ is reflexive and symmetric. If it is not transitive, there exist $x, y, z \in S$ such that $\rho(x, y), \rho(y, z) \leq \rho_i$ and $\rho(x, z) > \rho_i$. Thus $\rho_{i+1} \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z) \leq 2\rho_i$. If this holds for all $i = 1, \ldots, k-1$, we obtain $\rho_k \leq 2^{k-1} \rho_1$. □ □

**Lemma 3.** The cardinality of a $k$-distance set in a $d$-dimensional Minkowski space is at most $2^{kd}$.

Proof. Let $\{x_1, \ldots, x_m\}$ be a $k$-distance set with distances $\rho_1 < \rho_2 < \cdots < \rho_k$. Set $V := \bigcup_{i=1}^m B(x_i, \rho_i/2)$. Then we have

$$\text{vol}(V) = m(\rho_1/2)^d \text{vol}(B).$$

Also, $V - V \subseteq B(0, \rho_k + \rho_1)$, since if $x, y \in V$, there exist $i$ and $j$ such that $||x - x_i|| \leq \rho_1/2$, $||y - x_j|| \leq \rho_1/2$. Thus

$$||x - y|| \leq ||x - x_i|| + ||x_i - x_j|| + ||x_j - y|| \leq \rho_1 + \rho_k.$$

Therefore,

$$\text{vol}(V - V) \leq (\rho_1 + \rho_k)^d \text{vol}(B).$$

Substituting (3) and (4) into the Brunn-Minkowski inequality

$$\text{vol}(V - V)^{1/d} \geq \text{vol}(V)^{1/d} + \text{vol}(-V)^{1/d},$$

we obtain $\rho_1 + \rho_k \geq m^{1/d} \rho_1$, and $m \leq (1 + \rho_k/\rho_1)^d$.

If $1 + \rho_k/\rho_1 \leq 2^k$, there is nothing to prove. Otherwise, $\rho_k/\rho_1 > 2^k - 1 \geq 2^{k-1}$, and by Lemma 2, $x \sim_i y \iff \rho(x, y) \leq \rho_i$ is an equivalence relation for some $i = 1, \ldots, k-1$. By induction on $k$ we obtain that each equivalence class, being
Figure 1.

an $i$-distance set, has at most $2^{id}$ points. By choosing a representative from each equivalence class, we obtain a $(k-i)$-distance set with at most $2^{(k-i)d}$ points. Therefore, $m \leq 2^{id}2^{(k-i)d} = 2^{kd}$.

In the proof of Theorem 3, we need the following geometric lemma, which is a modification of [25, corollary 3.2.6] in 2 dimensions.

**Lemma 4.** Let $B_1$ be the convex hull of \{$(\pm 1, 0), (0, \pm 1)$\} and $B_\infty$ the square $[-1,1]^2$. For any symmetric convex disc $C$ in $\mathbb{R}^2$ there exists an invertible linear transformation taking $C$ to $C'$ such that $B_1 \subseteq C' \subseteq B_\infty$ and such that any straight-line segment contained in the boundary of $C'$ lies completely in one of the four coordinate quadrants.

**Proof.** We consider all triangles with vertices $0, x, y$, where $x$ and $y$ are on the boundary of $C$. By compactness there exist $x_0$ and $y_0$ such that the area of the triangle is a maximum. Then $\{x_0 + \lambda y_0 : \lambda \in \mathbb{R}\}$ is a support line of $C$ at $x_0$, since otherwise we can replace $x_0$ by a point on the side of the line opposite 0 to enlarge the area of the triangle. Similarly, $\{y_0 + \lambda x_0 : \lambda \in \mathbb{R}\}$ is a support line of $C$ at $y_0$. Since $C$ is symmetric, it follows that $C$ is contained in the parallelogram $\{\lambda x_0 + \mu y_0 : -1 \leq \lambda, \mu \leq 1\}$. See Figure 1.

If $x_0$ is an interior point of a straight-line segment contained in the boundary of $C$, we may shift $x_0$ to a boundary point of such a segment, without changing the area of the triangle. Thus $C$ is still contained in a parallelogram as above. A similar remark holds for $y_0$. We now apply the linear transformation sending $x_0$ and $y_0$ to the standard unit vectors $e_1$ and $e_2$, respectively (see Figure 2).

**Proof of Theorem 3** We have to find two cones $P_1$ and $P_2$ satisfying (1) and (2) of Corollary 1. By Lemma 4 we may replace the space by an isometric space $(\mathbb{R}^2, \|\cdot\|)$ such that the unit ball $B$ of $\|\cdot\|$ lies between $B_1$ and $B_\infty$, and such that any straight-line segment contained in the boundary of the unit ball lies completely in a quadrant of the plane.

We provisionally let $P_1$ be the closed first quadrant, and $P_2$ the closed second quadrant. See Figure 2. Then (1) is satisfied. The only way that (2) could fail is if there is a straight-line segment contained in the boundary of the unit ball parallel to either the x-axis or the y-axis, lying in $P_1$ or $P_2$. If there is a segment in the boundary of the unit ball in $P_1$ parallel to the x-axis, say, we remove the positive x-axis $\{(\lambda, 0) : \lambda > 0\}$ from $P_1$. If in this case there were another straight-line segment in the boundary parallel to the x-axis in $P_2$, then there would be a straight-line segment in the boundary lying in the first and second quadrants, giving a contradiction. Thus we do not have to remove the negative x-axis from $P_2$.\[\square\]
and (1) is still satisfied. We do the same thing for segments parallel to the $y$-axis, and for $P_2$. In the end, the modified $P_1$ and $P_2$ satisfy (1) and (2), and we deduce $\#S \leq (k + 1)^2$ from Corollary [1].

If equality holds, then the mapping $x \mapsto (h_1(x), h_2(x))$ in the proof of Theorem 1 [1] is a bijection from $S$ to $\{0, \ldots, k\}^2$. We now denote a point $x \in S$ by $p_{i,j}$, where $(i,j) = (h_1(x), h_2(x))$.

Suppose that two of the distances $\|p_{0,i} - p_{0,0}\|$ ($i = 1, \ldots, k$) are equal, say $\|p_{0,i} - p_{0,0}\| = \|p_{0,j} - p_{0,0}\|$ with $0 < i < j$. Then, since $p_{0,j} > p_{0,i} > p_{0,0}$, we have $p_{0,i} - p_{0,0}, p_{0,j} - p_{0,0} \in P_2$, contradicting (2).

It follows that the distances $\|p_{0,i} - p_{0,0}\|$, $i = 1, \ldots, k$ are distinct, and thus are exactly the $k$ different distances in increasing order. Similarly, the distances $\|p_{0,i} - p_{0,1}\|$, $i = 2, \ldots, k$ are in increasing order. If $\|p_{0,k} - p_{0,1}\| = \rho_k$, the three points $p_{0,0}, p_{0,1}, p_{0,k}$ again contradict (2). Thus these distances are $\rho_1, \ldots, \rho_{k-1}$ in increasing order, etc. In the end we find that $\|p_{0,i+1} - p_{0,i}\| = \rho_i$ for all $i$. Thus $\rho_k \leq k \rho_1$, by the triangle inequality. Using the Brunn-Minkowski inequality as in the proof of Lemma 3 [3] we find that equality holds in (3) and (4), implying that for $V := \bigcup_{i=1}^{\#S} B(x_i, \rho_1/2)$ we have $V - V = B(0, \rho_k + \rho_1)$, and $V - V$ and $V$ are homothetic. Thus $V$ is a ball that is perfectly packed by smaller balls. By a result of [16], this implies that the unit ball is a parallelogram. \hfill $\square$

**Proof of Theorem 4**. Follows from the proof of Theorem 3 [3] in the same way that Theorem 2 follows from Theorem 1 [1]. \hfill $\square$

**Proof of Theorem 6**. Lemma 3 already gives part of the theorem. For the remaining part we apply Corollary 1 [1]. In order for a cone $P$ to satisfy (2), it is sufficient that

$$\forall a, b \in P: \text{ if } \|a\| = \|b\| = 1, \text{ then } \|a - b\| < 1. \quad (6)$$

To see this, suppose that $P$ does not satisfy the condition in (2), i.e. there exist distinct $x, y \in P$ such that $\|x\| = \|y\|$ and $y - x \in P$. Let $a := \|x\|^{-1} x$, $b := \|y\|^{-1} y$, $c := \|y - x\|^{-1} (y - x)$, and $0 < \lambda := \|x\|/(\|y - x\| + \|x\|) < 1$. Then $a = (1 - \lambda)(a - c) + \lambda b$, and

$$1 = \|a\| \leq (1 - \lambda) \|a - c\| + \lambda \|b\| = (1 - \lambda) \|a - c\| + \lambda,$$

implying $\|a - c\| \geq 1$.

In order for (1) to be satisfied too, we need a cover of the unit sphere by sets such that, if they are extended to positive cones, are convex.

We do this with the following construction: Let $C = \{c_1, c_2, \ldots, c_m\}$ be a maximal set of unit vectors satisfying $\|c_i - c_j\|, \|c_i + c_j\| \geq \frac{1}{\delta}$ for all $1 \leq i < j \leq m$. \hfill $\square$
Then for any unit vector $x$ there exists $i$ such that $\|c_i - x\| < \frac{1}{5}$ or $\|c_i + x\| < \frac{1}{5}$. For $i = 1, \ldots, m$, let $P_i$ be the cone generated by

$$Q_i := \{ x \in \mathbb{R}^d : \|x\| = 1, \|c_i - x\| < \frac{1}{5} \},$$

i.e. $P_i := \{ \sum_j \lambda_j x_j : \lambda_j \geq 0, x_j \in Q_i \}$. Then the $P_i$’s satisfy (1) by the maximality of $C$. Each $P_i$ satisfies (6): Let $\sum_j \lambda_j x_j \in P_i$, where $\lambda_j \geq 0, \|x_j\| = 1, \|c_i - x_j\| < \frac{1}{5}$ and $\|\sum \lambda_j x_j\| = 1$. Then

$$\|c_i - \sum \lambda_j x_j\| = \|\sum \lambda_j (c_i - x_j) + (1 - \sum \lambda_j) c_i\|$$

$$< \sum j \lambda_j/5 - 1 + \sum j \lambda_j \quad \text{(since } \sum j \lambda_j \geq 1)$$

$$= \frac{2}{5} \sum j \lambda_j - 1.$$

Also, since

$$1 + \sum j \lambda_j/5 > \|\sum \lambda_j x_j\| + \sum j \|\lambda_j x_j - \lambda_j c\| \geq \sum j \lambda_j \|c\| = \sum j \lambda_j,$$

we obtain $\sum j \lambda_j < \frac{5}{2}$, and $\|c_i - \sum \lambda_j x_j\| < \frac{5}{2} \cdot \frac{5}{2} - 1 = \frac{1}{2}$.

A volume argument gives the upper bound for $\#C$: The balls $B(0, \frac{a}{11}), B(\pm c_i, \frac{1}{11})$, $i = 1, \ldots, m$ have disjoint interiors and are contained in the ball $B(0, \frac{1}{11})$. Therefore,

$$\left(\frac{a}{11}\right)^d \text{vol}(B) + 2m(\frac{1}{11})^d \text{vol}(B) \leq \left(\frac{1}{11}\right)^d \text{vol}(B),$$

giving $m \leq \frac{1}{2}(11^d - 9^d)$.

\[\square\]

\textbf{ACKNOWLEDGEMENT}

This paper is part of the author’s PhD thesis written under supervision of Prof. W. L. Fouche at the University of Pretoria. I thank the referees as well as Graham Brightwell for their suggestions on the layout of the paper.

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