SUBTLETIES OF THE MINMAX SELECTOR

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Abstract. In this note, we show that the minmax and maxmin critical values of a function quadratic nondegenerate at infinity are equal when defined in homology or cohomology with coefficients in a field. However, by an example of F. Laudenbach, this is not always true for coefficients in a ring and, even in the case of a field, the minmax-maxmin depends on the field.

1. Introduction

Given a Lagrangian submanifold $L$ in the cotangent bundle of a closed manifold $M$, obtained by Hamiltonian deformation of the zero section, the minmax selector introduced by J.-C. Sikorav provides an almost everywhere defined section $M \to L$ of the projection $T^* M \to M$ restricted to $L$. As noticed by M. Chaperon [4], this defines weak solutions of smooth Cauchy problems for Hamilton-Jacobi equations; in the classical case of a convex Hamiltonian, the minmax is a minimum and the minmax solution coincides with the viscosity solution, which is not always the case for nonconvex Hamiltonians. For a recent use of the minmax selector in weak KAM theory, see [1].

The minmax has been defined using homology or cohomology with various coefficient rings, for example $\mathbb{Z}$ in [4, 9], $\mathbb{Q}$ in [3] and $\mathbb{Z}_2$ in [8]. Also, in [9], the maxmin was mentioned as a natural analogue to the minmax. But there is no evidence showing that all these critical values coincide. G. Capitanio has given a proof [3] that the maxmin and minmax for homology with coefficients in $\mathbb{Q}$ are equal, but the criterion he uses (Proposition 2 in [3]) is not correct—see Remark 3.11 hereafter.

In this note, we investigate the maxmin and minmax for a general function quadratic at infinity, not necessarily related to Hamilton-Jacobi equations. We give both algebraic and geometric proofs that the minmax and maxmin with coefficients in a field coincide; the geometric proof, based on Barannikov’s Jordan normal form for the boundary operator of the Morse complex, improves our understanding of the problem.

A counterexample for coefficients in $\mathbb{Z}$, due to F. Laudenbach, is constructed using Morse homology; in this example, moreover, the minmax-maxmin for coefficients in $\mathbb{Z}_2$ is not the same as for coefficients in $\mathbb{Q}$. However, if the minmax and maxmin for coefficients in $\mathbb{Z}$ coincide, then all three minmax-maxmin critical values are equal.

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2. Maxmin and Minmax

Hypotheses and notation. We denote by $X$ the vector space $\mathbb{R}^n$ and by $f$ a real function on $X$, quadratic at infinity in the sense that it is continuous and there exists a nondegenerate quadratic form $Q : X \to \mathbb{R}$ such that $f$ coincides with $Q$ outside a compact subset.

Let $f^- := \{x | f(x) \leq c\}$ denote the sub-level sets of $f$. Note that for $c$ large enough, the homotopy types of $f^c$, $f^-$ do not depend on $c$, we may denote them as $f^\infty$ and $f^-$. Suppose the quadratic form $Q$ has Morse index $\lambda$, then the homology groups with coefficient ring $R$ are

$$H_*(f^\infty, f^-; R) \simeq \begin{cases} R & \text{in dimension } \lambda \\ 0 & \text{otherwise} \end{cases}$$

Consider the homomorphism of homology groups

$$i_{cs} : H_*(f^c, f^-; R) \to H_*(f^\infty, f^-; R)$$

induced by the inclusion $i_c : (f^c, f^-) \hookrightarrow (f^\infty, f^-)$.

Definition 2.1. If $\Xi$ is a generator of $H_\lambda(f^\infty, f^-; R)$, we let

$$\gamma(f, R) := \inf \{c : \Xi \in \text{Im}(i_{cs})\},$$

i.e. $\gamma(f, R) = \inf \{c : i_{cs}H_\lambda(f^c, f^-; R) = H_\lambda(f^\infty, f^-; R)\}$.

Similarly, we can consider the homology group

$$H_*(X \setminus f^-; X \setminus f\infty; R) \simeq \begin{cases} R, & \text{in dimension } n - \lambda \\ 0, & \text{otherwise} \end{cases}$$

and the homomorphism

$$j_{cs} : H_*(X \setminus f^c, X \setminus f\infty; R) \to H_*(X \setminus f^-; X \setminus f\infty; R)$$

induced by $j_c : (X \setminus f^c, X \setminus f\infty) \hookrightarrow (X \setminus f^-; X \setminus f\infty)$.

Definition 2.2. If $\Delta$ is a generator of $H_{n-\lambda}(X \setminus f^-; X \setminus f\infty; R)$, we let

$$\overline{\gamma}(f, R) := \sup \{c : \Delta \in \text{Im}(j_{cs})\} = \sup \{c : j_{cs}H_{n-\lambda}(X \setminus f^c, X \setminus f\infty; R) = H_{n-\lambda}(X \setminus f^-; X \setminus f\infty; R)\}.$$

Lemma 2.3. One has that

$$\gamma(f, R) = \inf \max f := \inf_{|\sigma| = \Xi} \max_{x \in |\sigma|} f(x)$$

$$\overline{\gamma}(f, R) = \sup \min f := \sup_{|\sigma| = \Delta} \min_{x \in |\sigma|} f(x),$$

where $\sigma$ is a relative cycle and $|\sigma|$ denotes its support. We call $\sigma$ a descending (resp. ascending) simplex if $|\sigma| = \Xi$ (resp. $|\sigma| = \Delta$).

Proof. A descending simplex $\sigma$ defines a homology class in $H_\lambda(f^c, f^-; R)$ if and only if $|\sigma| \subset f^c$, in which case one has $\max_{x \in |\sigma|} f(x) \leq c$, hence $\gamma(f, R) = \inf \max f$; choosing $c = \max_{x \in |\sigma|} f(x)$, we get equality. The case of $\overline{\gamma}$ is identical. \hfill \square
Definition 2.4. \( \gamma(f, R) \) is called a minmax of \( f \) and \( \overline{\gamma}(f, R) \), a maxmin.

Remark 2.5. As we shall see later, in view of Morse homology, the names are proper generically for Morse-excellent functions.

One can also consider cohomology instead of homology and define

\[
\underline{\alpha}(f, R) := \inf \{ c : i^*_c \neq 0 \}, \quad i^*_c : H^\lambda(f^\infty, f^{-\infty}; R) \to H^\lambda(f^c, f^{-\infty}; R)
\]

\[
\overline{\alpha}(f, R) := \sup \{ c : j^*_c \neq 0 \}, \quad j^*_c : H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^c; R) \to H^{n-\lambda}(X \setminus f^c, X \setminus f^\infty; R).
\]

Proposition 2.6 ([9], Proposition 2.4). When \( X \) is \( R \)-oriented,

\[
\overline{\alpha}(f, R) = \underline{\gamma}(f, R) \quad \text{and} \quad \underline{\alpha}(f, R) = \overline{\gamma}(f, R).
\]

Proof. We establish for example the first identity: one has the commutative diagram

\[
\begin{array}{ccc}
H_\lambda(f^c, f^{-\infty}; R) & \simeq & H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^c; R) \\
\downarrow j^*_c & & \downarrow \\
H_\lambda(f^\infty, f^{-\infty}; R) & \simeq & H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^\infty; R) \\
\downarrow & & \downarrow j^*_c \\
H_\lambda(f^\infty, f^c; R) & \simeq & H^{n-\lambda}(X \setminus f^c, X \setminus f^\infty; R)
\end{array}
\]

where the horizontal isomorphisms are given by Alexander duality ([5], section 3.3) and the columns are exact. It does follow that \( i^*_c \) is onto if and only if \( j^*_c \) is zero. \( \square \)

Definition 2.7. As long as \( X \) is finite dimensional, theClarke generalized derivative

of a locally Lipschitzian function \( f : X \to \mathbb{R} \) can be defined as follows:

\[
\partial f(x) := \text{co}\{ \lim_{x' \to x} df(x'), \ x' \in \text{dom}(df) \};
\]

where co denotes the convex envelop. A point \( x \in X \) is called a critical point of \( f \) if \( 0 \in \partial f(x) \).

Proposition 2.8. If \( f \) is \( C^2 \) then \( \underline{\gamma}(f, R) \) and \( \overline{\gamma}(f, R) \) are critical values of \( f \); they are critical values of \( f \) in the sense of Clarke when \( f \) is locally Lipschitzian.

Proof. Take \( \gamma \) for example: if \( c = \underline{\gamma}(f, R) \) is not a critical value then, for small \( \epsilon > 0 \), \( f^{c-\epsilon} \) is a deformation retract of \( f^{c+\epsilon} \) via the flow of \( -\nabla f \), hence \( \gamma(f, R) \leq c - \epsilon \), a contradiction. The same argument applies when \( f \) is only locally Lipschitzian, replacing \( \nabla f \) by a pseudo-gradient. \( \square \)

Lemma 2.9. If \( f \) is locally Lipschitzian, then

\[
\overline{\gamma}(f, R) = -\underline{\gamma}(-f, R)
\]

Proof. Using a (pseudo-)gradient of \( f \) as previously, one can see that \( X \setminus f^c \) and \( (-f)^{-c} \) have the same homotopy type when \( c \) is not a critical value of \( f \). Otherwise, choose a sequence of non-critical values \( c_n \nearrow c = \overline{\gamma}(f, R) \), then \( -c_n \geq \underline{\gamma}(-f, R) \), taking the limit, we have \( \overline{\gamma}(f, R) \leq -\underline{\gamma}(-f, R) \). Similarly, taking \( c'_n \searrow \underline{\gamma}(-f, R) \), then \( -c'_n \leq \underline{\gamma}(f, R) \), from which the limit gives us the inverse inequality \( -\underline{\gamma}(-f, R) \leq \overline{\gamma}(f, R) \). \( \square \)
The following two questions arise naturally:

1. Do we have \( \gamma(f, R) = \overline{\gamma}(f, R) \)?
2. Do \( \gamma(f, R) \) and \( \overline{\gamma}(f, R) \) depend on the coefficient ring \( R \)?

Here are two obvious elements for an answer:

**Proposition 2.10.** One has \( \gamma(f, \mathbb{Z}) \geq \overline{\gamma}(f, \mathbb{Z}) \).

**Proof.** As the intersection number of \( \Xi \) and \( \Delta \) is \( \pm 1 \), the support of any descending simplex \( \sigma \) must intersect the support of any ascending simplex \( \tau \) at some point \( \bar{x} \), hence \( \max_{x \in |\sigma|} f(x) \geq f(\bar{x}) \geq \min_{x \in |\tau|} f(x) \). □

**Proposition 2.11.** One has \( \gamma(f, \mathbb{Z}) \geq \gamma(f, R) \) and \( \overline{\gamma}(f, \mathbb{Z}) \leq \overline{\gamma}(f, R) \) for every ring \( R \).

**Proof.** A simplex \( \sigma \) whose homology class generates \( H_\lambda(f^\infty, f^{-\infty}; \mathbb{Z}) \) induces a simplex whose homology class generates \( H_\lambda(f^\infty, f^{-\infty}; R) \), hence the first inequality and, mutatis mutandis, the second one. □

**Theorem 2.12.** If \( F \) is a field, then \( \gamma(f, F) = \overline{\gamma}(f, F) \).

**Proof.** By Proposition 2.6, it is enough to prove that \( \gamma(f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F}) \).

Recall that \( \gamma(f, \mathbb{F}) \) (resp. \( \alpha(f, \mathbb{F}) \)) is the infimum of the real numbers \( c \) such that \( i_{cs} : H_\lambda(f^c, f^{-\infty}; \mathbb{F}) \to H_\lambda(f^\infty, f^{-\infty}; \mathbb{F}) \) is onto (resp. such that \( i_{cs}^* : H^\lambda(f^\infty, f^{-\infty}; \mathbb{F}) \to H^\lambda(f^c, f^{-\infty}; \mathbb{F}) \) is nonzero). Now, as \( H_\lambda(f^\infty, f^{-\infty}; \mathbb{F}) \) is a one-dimensional vector space over \( \mathbb{F} \), the linear map \( i_{cs} \) is onto if and only if it is nonzero, i.e. if and only if the transposed map \( i_{cs}^* \) is nonzero. □

**Remark.** This proof is invalid for coefficients in \( \mathbb{Z} \) since a \( \mathbb{Z} \)-linear map to \( \mathbb{Z} \), for example \( \mathbb{Z} \ni m \to km, k \in \mathbb{Z}, k > 1 \), can be nonzero without being onto; we shall see in Section 4 that Theorem 2.12 itself is not true in that case.

**Corollary 2.13.** If \( \gamma(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = \gamma \) then \( \gamma(f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F}) = \gamma \) for every field \( \mathbb{F} \).

**Proof.** This follows at once from Theorem 2.12 and Proposition 2.11. □

**Corollary 2.14.** Let \( \gamma \in \mathbb{R} \) have the following property: there exist both a descending simplex over \( \mathbb{Z} \) along which \( \gamma \) is the maximum of \( f \) and an ascending simplex over \( \mathbb{Z} \) along which \( \gamma \) is the minimum of \( f \). Then, \( \gamma(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = \gamma(f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F}) = \gamma \) for every field \( \mathbb{F} \).

**Proof.** We have \( \gamma(f, \mathbb{Z}) \leq \gamma \leq \overline{\gamma}(f; \mathbb{Z}) \) by Lemma 2.3 and \( \overline{\gamma}(f; \mathbb{Z}) \leq \gamma(f, \mathbb{Z}) \) by Proposition 2.10, hence our result by Corollary 2.13. □
3. Morse complexes and the Barannikov normal form

The previous proof of Theorem 2.12, though simple, is quite algebraic. We now
give a more geometric proof, which we find more concrete and illuminating, based on
Barannikov’s canonical form of Morse complexes. It will provide a good setting for the
counterexample in Section 4.

First, there is a continuity result for the minmax and maxmin:

**Proposition 3.1.** If \( f \) and \( g \) are two continuous functions quadratic at infinity with
the same reference quadratic form, then

\[
|\gamma(f, R) - \gamma(g, R)| \leq |f - g|_{C^0} \\
|\nu(f, R) - \nu(f, R)| \leq |f - g|_{C^0}.
\]

**Proof.** For \( f \leq g \), from Lemma 2.3, it is easy to see that \( \gamma(f) \leq \gamma(g) \). In the general
case, this implies \( \gamma(g) \leq \gamma(f) + |g - f|_{C^0} \); exchanging \( f \) and \( g \), we get
\( \gamma(f) \leq \gamma(g) + |f - g|_{C^0} \). \( \square \)

**Corollary 3.2.** To prove Theorem 2.12, it suffices to establish it for excellent Morse
functions \( f : X \to \mathbb{R} \), i.e. smooth functions having only non-degenerate critical points,
each of which corresponds to a different value of \( f \).

**Proof.** By a standard argument, given a non-degenerate quadratic form \( Q \) on \( X \), the set
of all continuous functions on \( X \) equal to \( Q \) off a compact subset contains a
\( C^0 \)-dense subset consisting of excellent Morse functions; our result follows by Proposition 3.1. \( \square \)

To prove Theorem 2.12 for excellent Morse functions, we will use Morse homology.

**Hypotheses.** We consider an excellent Morse function \( f \) on \( X \), quadratic at infinity;
for each pair of regular values \( b < c \) of \( f \), we denote by \( f_{b,c} \) the restriction of \( f \) to
\( f^c \cap (-f)^{-b} = \{ b \leq f \leq c \} \).

**Morse complexes.** Let
\[
C_k(f_{b,c}) := \{ \xi^k_\ell : 1 \leq \ell \leq m_k \}
\]
denote the set of critical points of index \( k \) of \( f_{b,c} \), ordered so that \( f(\xi^k_\ell) < f(\xi^k_m) \) for
\( \ell < m \). Given a generic gradient-like vector field \( V \) for \( f \) such that \((f, V)\) is Morse-Smale*,
the Morse complex of \((f_{b,c}, V)\) over \( R \) consists of the free \( R \)-modules
\[
M_k(f_{b,c}, R) := \{ \sum \ell a_\ell \xi^k_\ell, \quad a_\ell \in R \}
\]
together with the boundary operator \( \partial : M_k(f_{b,c}, R) \to M_{k-1}(f_{b,c}, R) \) given by
\[
\partial \xi^k_\ell := \sum_{m} \nu_{f,V}(\xi^k_\ell, \xi^{k-1}_m) \xi^{k-1}_m
\]
where, with given orientations for the stable manifolds (hence co-orientations for un-
stable manifolds), \( \nu_{f,V} \) is the intersection number of the stable manifold \( W^s(\xi^k_\ell) \) of \( \xi^k_\ell \)
\[\star\] Being Morse-Smale means that the stable and unstable manifolds of all the critical points are
transversal.
and the unstable manifold $W^u(\xi_m^{k-1})$ of $\xi_m^{k-1}$, i.e. the algebraic number of trajectories of $V$ connecting $\xi_m^k$ and $\xi_m^{k-1}$; note that

- $\nu_{f,V}(\xi_m^k, \xi_m^{k-1})$ is the same for all $b, c$ with $f(\xi_m^k)$, $f(\xi_m^{k-1})$ in $[b, c]$;
- $\nu_{f,V}(\xi_m^k, \xi_m^{k-1}) \neq 0$ implies $f(\xi_m^k) > f(\xi_m^{k-1})$: otherwise, the stable manifold of $\xi_m^{k-1}$ and the unstable manifold of $\xi_m^k$ for $V$, which cannot be transversal because of their dimensions, would intersect, contradicting the genericity of $V$.
- $\nu_{f,V}(\xi_m^k, \xi_m^{k-1}) = 0$ for two distinct critical points of the same index.

This does define a complex, i.e. $\partial \circ \partial = 0$: see for example [6, 7]. The homology $HM_s(f_{b,c}, R) := H_*(M_s(f_{b,c}, R))$ is called the Morse homology† of $f_{b,c}$.

**Lemma 3.3** (Barannikov,[2]). If $R$ is a field $\mathbb{F}$, then this boundary operator $\partial$ has a special kind of Jordan normal form as follows: each $M_k(f_{b,c}, \mathbb{F})$ has a basis

$$\Xi^k_\ell := \sum_{i < \ell} \alpha_{\ell,i}^k \xi^k_i, \quad \alpha_{\ell,i} \neq 0$$

such that either $\partial \Xi^k_\ell = 0$ or $\partial \Xi^k_\ell = \Xi^k_{\ell-1}$ for some $m$, in which case no $\ell' \neq \ell$ satisfies $\partial \Xi^{k-1}_{\ell'} = \Xi^{k-1}_{\ell'}$. If $(\Theta^k_\ell)$ is another such basis, then $\partial \Xi^k_\ell = \Xi^{k-1}_\ell$ (resp. 0) is equivalent to $\partial \Theta^k_\ell = \Theta^{k-1}_\ell$ (resp. 0); in other words, the matrix of $\partial$ in all such bases is the same.

**Proof.** We prove existence by induction. Given nonnegative integers $k, i$ with $i < m_k$, suppose that vectors $\Xi^k_q$ of the form (3.1) have been obtained for all $(p, q)$ with either $p < k$, or $p = k$ and $q \leq i$, possessing the required property that either $\partial \Xi^p_q = \Xi^{p-1}_q$ (with $j_p(q) \neq j_p(q')$ for $q \neq q'$) or $\partial \Xi^p_q = 0$. If $\partial \xi^k_{i+1} = 0$ (e.g., when $k = 0$), we take $\xi^k_{i+1} := \Xi^k_{i+1}$ and continue the induction. Otherwise, $\partial \xi^k_{i+1} = \sum \alpha_j \Xi^{k-1}_j$, $\alpha_j \in \mathbb{F}$. Moving all the terms $\Xi_{j \neq q} = \partial \Xi_q, q < i$ from the right-hand side to the left, we get

$$\partial (\xi^k_{i+1} - \sum_{q \leq i} \alpha_{j,q} \Xi^k_q) = \sum_j \beta_j \Xi^{k-1}_j.$$

Let

$$\Xi^k_{i+1} := \xi^k_{i+1} - \sum_{q \leq i} \alpha_{j,q} \Xi^k_q.$$

If $\beta_j = 0$ for all $j$, then $\partial \Xi^k_{i+1} = 0$ and the induction can go on. Otherwise,

$$\partial \Xi_{i+1} = \sum_{j \leq j_0} \beta_j \Xi^{k-1}_j =: \Xi^{k-1}_{j_0}$$

with $\beta_{j_0} \neq 0$; as $\partial \Xi^{k-1}_{j_0} = \partial \Xi^{k-1}_{i+1} = 0$, we can replace $\Xi^{k-1}_{j_0}$ by $\Xi^{k-1}_{j_0}$ and continue the induction†. \hfill $\square$

**Definition 3.4.** Under the hypotheses and with the notation of the Barannikov lemma, two critical points $\xi_m^k$ and $\xi_m^{k-1}$ of $f_{b,c}$ are coupled if $\partial \Xi^k_\ell = \Xi^{k-1}_\ell$. A critical point is free (over $\mathbb{F}$) when it is not coupled with any other critical point.

In other words, $\xi_m^k$ is free if and only if $\Xi^k_\ell$ is a cycle of $M_k(f_{b,c}, \mathbb{F})$ but not a boundary, hence the following result:

†Morse homology is defined in general for any Morse function without being excellent.

‡Note that if $\mathbb{F}$ was not a field, this would not provide a basis for noninvertible $\beta_{j_0}$. 
Corollary 3.5. For each integer \(k\), the Betti number \(\dim_{\mathbb{F}} HM_k(f_{B,C}, \mathbb{F})\) is the number of free critical points of index \(k\) of \(f_{B,C}\) over \(\mathbb{F}\).

\[\square\]

Theorem 3.6. (1) The Barannikov normal form of the Morse complex of \(f_{B,C}\) over \(\mathbb{F}\) is independent of the gradient-like vector field \(V\).

(2) So is the Morse homology \(HM_\ast(f_{B,C}, R)\); it is isomorphic to \(H_\ast(f^c, f^b; R)\).

(3) For \(b' < b < c \leq c'\), the inclusion \(i : f^c \hookrightarrow f^{c'}\), restricted to the critical set \(C_\ast(f_{B,C})\), induces a linear map \(i_* : M_\ast(f_{B,C}, R) \rightarrow M_\ast(f_{B,C}', R)\) such that \(\partial \circ i_* = i_\ast \circ \partial\) and therefore a linear map \(i_* : HM_\ast(f_{B,C}, R) \rightarrow HM_\ast(f_{B,C}', R)\), which is the usual \(i_* : H_\ast(f^c, f^b; R) \rightarrow H_\ast(f^{c'}, f^{b'}; R)\) modulo the isomorphism (ii).

Idea of the proof [6]. (1) Connecting two generic gradient-like vector fields \(V_0, V_1\) for \(f\) by a generic family, one can prove that each of the Morse complexes defined by \(f\) over \(f_{B,C}\) is the class of a cell of dimension \(\lambda\) of \(\gamma\), i.e., their algebraic number of connecting trajectories.

(2) When there is only one critical point \(a\) of \(f\) in \(\{b \leq a \leq c\}\), both \(HM_\ast(f_{B,C}, R)\) and \(H_\ast(f^c, f^b; R)\) are trivial (the flow of \(V\) defines a retraction of \(f^c\) onto \(f^b\)). When there is only one critical point \(\xi\) of \(f\) in \(\{b \leq \xi \leq c\}\), of index \(\lambda\),

\[HM_k(f_{B,C}, R) \simeq H_k(f^c, f^b; R) \simeq \begin{cases} R, & \text{if } k = \lambda, \\ 0 & \text{otherwise}. \end{cases}\]

the class of \(\xi\) obviously generates \(HM_k(f_{B,C}, R)\), whereas a generator of \(H_\ast(f^c, f^b; R)\) is the class of a cell of dimension \(\lambda\), namely the stable manifold of \(\xi\) for \(V|_{\{b \leq a \leq c\}}\); the isomorphism associates the second class to the first.

In the general case, one can consider a subdivision \(b = b_0 < \cdots < b_N = c\) consisting of regular values of \(f\) such that each \(f_{b_i, b_{i+1}}\) has precisely one critical point. One can show that the boundary operator \(\partial\) of the relative singular homology \(\partial : H_{k+1}(f_{b_{i+1}, f^b}; R) \rightarrow H_k(f^b, f^{b_{i-1}})\) can be interpreted as the intersection number of the stable manifold of the critical point in \(\{b_i \leq a \leq b_{i+1}\}\) and the unstable manifold of that in \(\{b_i-1 \leq a \leq b_i\}\), i.e., their algebraic number of connecting trajectories.

(3) The first claims are easy. The last one follows from what has just been sketched.

\[\square\]

Corollary 3.7. If \(f\) is an excellent Morse function quadratic at infinity, then it has precisely one free critical point \(\xi\) over \(\mathbb{F}\); its index \(\lambda\) is that of the reference quadratic form \(Q\) and

\[\gamma(f, \mathbb{F}) = f(\xi).\]

Proof. Clearly, the dimension of

\[HM_k(f, \mathbb{F}) = HM_k(f_{-\infty, \infty}, \mathbb{F}) \simeq H_k(f^\infty, f^{-\infty}; \mathbb{F}) = H_k(Q^\infty, Q^{-\infty}; \mathbb{F})\]

is 1 if \(k = \lambda\) and 0 otherwise. The first two assertions follow by Corollary 3.5. To prove \(\gamma(f, \mathbb{F}) = f(\xi)\), note that \(\gamma(f)\) is the infimum of the regular values \(c\) of \(f\) such that the class of \(\xi\) in \(HM_\lambda(f_{-\infty, \infty}, \mathbb{F})\) lies in the image of \(i_* : HM_\lambda(f_{-\infty, c}, \mathbb{F}) \rightarrow HM_\lambda(f_{-\infty, \infty}, \mathbb{F})\); by Theorem 3.6 (iii), which means \(c \geq f(\xi)\).

\[\square\]
Proposition 3.8. The excellent Morse function \(-f_{b,c} = (-f)_{c,-b}\) has the same free critical points over the field \(\mathbb{F}\) as \(f_{b,c}\).  

Proof. Assuming \(V\) fixed, this is essentially easy linear algebra:

- One has \(C_k(-f) = C_{n-k}(f)\) and the ordering of the corresponding critical values is reversed. Thus, the lexicographically ordered basis of \(M_*(-f)\) corresponding to \((\xi^k_\ell)_{1 \leq \ell \leq m_0, 0 \leq k \leq n}\) is \((\xi^{n-k}_{m_{n-k}+\ell+1})_{1 \leq \ell \leq m_{n-k}, 0 \leq k \leq n}\).

- The vector field \(-V\) has the same relations with \(-f\) as \(V\) has with \(f\), hence  
  \[
  \nu_{-f,-V}(\xi^{n-k}_{m_{n-k}+\ell+1}, \xi^{n-(k-1)}_{m_{n-(k-1)}-m+1}) = \nu_{V}(\xi^{n-(k-1)}_{m_{n-(k-1)}-m+1}, \xi^{n-k}_{m_{n-k}+\ell+1}).
  \]

That is, the matrix of the boundary operator of \(M_*(-f_{b,c})\) in the basis \((\xi^{n-k}_{m_{n-k}+\ell+1})\) is the matrix \(M\) obtained from the matrix \(A\) of the boundary operator of \(M_*(f_{b,c})\) in the basis \((\xi^k_\ell)\) by symmetry with respect to the second diagonal (i.e. by reversing the order of both the lines and columns of the transpose of \(A\)).

Lemma 3.3 can be rephrased as follows: there exists a block-diagonal matrix  
\[
P = \text{diag}(P_0, \ldots, P_n)
\]
where each \(P_k \in \text{GL}(m_k, \mathbb{F})\) is upper triangular, such that  
\[
P^{-1}AP = B
\]

is a Barannikov normal form, meaning the following: the entries of the column of indices \(k_\ell\) are 0 except possibly one, equal to 1, which must lie on the line of indices \(k_{m-1}\) for some \(m\) and be the only nonzero entry on this line. The normal form \(B\) is the same for every choice of \(P\) and \(V\). Clearly, \(k_\ell\) is a free critical point of \(f_{b,c}\) if and only if both the line and column of indices \(k_\ell\) of \(B\) are zero.

Equation (3.2) reads  
\[
\tilde{P}A\tilde{P}^{-1} = \tilde{B};
\]

Now, \(\tilde{P}^{-1}\) and \(\tilde{P} = (\tilde{P}^{-1})^{-1}\) are block diagonal upper triangular matrices whose \(k\)th diagonal block lies in \(\text{GL}(m_{n-k}, \mathbb{F})\); therefore, by (3.3), as \(\tilde{B}\) is a Barannikov normal form for the ordering associated to \(-f\), it is the Barannikov normal form of the boundary operator of \(M_*(-f_{b,c})\), from which our result follows at once. \[\square\]

Corollary 3.9. For any excellent Morse function \(f\) quadratic at infinity, the sole free critical point of \(-f\) over \(\mathbb{F}\) is the free critical point \(\xi\) of \(f\); hence \(\gamma(f, \mathbb{F}) = f(\xi) = -(f)(\xi) = -\gamma(-f, \xi) = -\gamma(f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F})\) by Corollary 3.7 and Lemma 2.9, which proves Theorem 2.12. \[\square\]

Before we give an example where \(\gamma(f, \mathbb{Z}) > \overline{\gamma}(f, \mathbb{Z})\), here is a situation where this cannot occur:

Proposition 3.10. Assume that \(M_*(f, \mathbb{Z})\) can be put into Barannikov normal form by a basis change (3.1) of the free \(\mathbb{Z}\)-module \(M_*(f, \mathbb{Z})\):

\[
\Xi^k_\ell := \sum_{i \leq \ell} \alpha^k_{i,\ell} \xi^k_i, \quad \alpha^k_{i,\ell} \in \mathbb{Z}, \quad \alpha^k_{i,\ell} = \pm 1.
\]

Then, \(\gamma(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = f(\xi)\), where \(\xi\) is the sole free critical point of \(f\) over \(\mathbb{Z}\).
Proof. We are in the situation of the proof of Proposition 3.8 with \( P_k \in \text{GL}(m_k, \mathbb{Z}) \), which implies that the Barannikov normal form \( B \) of the boundary operator is the same for \( Z \) as for \( Q \); it does follow that there is a unique free critical point \( \xi \) of \( f \) over \( Z \) (the same as over \( Q \)) and that it is the unique free critical point of \(-f\) over \( Z \); moreover, the proof of Corollary 3.7 shows that \( \gamma(f, Z) = \gamma(-f, Z) = f(\xi) \). We conclude as in Corollary 3.9. \( \square \)

Now that the coefficients are in \( \mathbb{Z} \), the classical method of so called sliding handles states that, under an additional condition imposed on the index of the change of basis in (3.4), namely \( 2 \leq k \leq n - 2 \), the Barannikov normal form can be realized by a gradient-like vector field for \( f \).

More precisely, let \( P : M_k(f) \to M_k(f) \) be a transformation matrix where \( P = \text{diag}(P_0, \ldots, P_n) \) with each \( P_k \in \text{GL}(m_k, \mathbb{Z}) \) such that \( P_k = \text{id} \) for \( k = 0, 1 \) or \( n - 1, n \), and \( P_k \) is upper triangular with \( \pm 1 \) in the diagonal entries for \( 2 \leq k \leq n - 2 \). Then one can construct a gradient-like vector field \( V' \) such that, if the matrix of the boundary operator for a given gradient-like vector field \( V \) is \( A \), then the matrix for \( V' \) is given by \( B = P^{-1}AP \).

Roughly speaking, one modifies \( V \), each time for one \( i \leq l \), by sliding handle of the stable sphere\(^{5}\) \( S_L(\xi^k_i) \) of \( \xi^k_i \) for \( V \) such that it sweeps across the unstable sphere \( S_R(\xi^k_i) \) of \( \xi^k_i \) with indicated intersection number. In other words, \( S'_L(\xi^k_i) \) for the resulted \( V' \) is the connected sum of \( S_L(\xi^k_i) \) and the boundary of a meridian disk of \( S_R(\xi^k_i) \) described in section 4.4 of [6]. One may refer to the Basis Theorem (Theorem 7.6) in [7] for a detailed construction of \( V' \).

Remark 3.11 (on the “proof” of Corollary 3.9 in [3]). Capitanio uses the following Criterion. A critical point \( \xi \) of \( f \) is free (over \( Q \)) if and only if, for any critical point \( \eta \) incident to \( \xi \), there is a critical point \( \xi' \), incident to \( \eta \), such that

\[
|f(\xi') - f(\eta)| < |f(\xi) - f(\eta)|.
\]

where fixing a gradient-like vector field \( V \) generic for \( f \), two critical points are called incident if their algebraic number of connecting trajectories is nonzero.

Unfortunately, this is not true: one can construct a function \( f : \mathbb{R}^{2n} \to \mathbb{R}, n \geq 2 \), quadratic at infinity with Morse index \( n \), having five critical points, two of index \( n - 1 \) and three of index \( n \), whose gradient vector field \( V \) defines the Morse complex

\[
\partial \xi^n_1 = \xi^n_2 - 1, \quad \partial \xi^n_2 = \xi^n_1 - 1, \quad \partial \xi^n_3 = 0.
\]

This complex can be reformulated into

\[
\partial \xi^n_1 = (\xi^n_2 - 1 - \xi^n_1 - 1) + \xi^n_1 - 1,
\]

\[
\partial (\xi^n_2 + \xi^n_1) = (\xi^n_2 - 1 - \xi^n_1 - 1) + 2\xi^n_1 - 1,
\]

\[
\partial (\xi^n_3 + \xi^n_2) = \xi^n_1 - 1.
\]

Hence, for a change of basis

\[
\xi^n_2 \mapsto \xi^n_2 - 1 - \xi^n_1 - 1, \quad \xi^n_2 \mapsto \xi^n_2 + \xi^n_1, \quad \xi^n_3 \mapsto \xi^n_3 + \xi^n_2.
\]

\(^{5}\)The stable and unstable sphere is defined as : \( S_L(\xi^k_i) = W^s(\xi^k_i) \cap L \) and \( S_R(\xi^k_i) = W^u(\xi^k_i) \cap L \) where \( L = f^{-1}(c) \) for some \( c \in (f(\xi^k_i), f(\xi^k_i)) \).
one can construct a gradient-like vector field \( V' \) for \( f \) by sliding handles, such that
\[
\partial \xi_1^n = \xi_2^{n-1} + \xi_1^{n-1}, \quad \partial \xi_2^n = \xi_2^{n-1} + 2\xi_1^{n-1}, \quad \partial \xi_3^n = \xi_1^{n-1}.
\]

Obviously, \( \xi_3^n \) is the only free critical point, but \( \xi_2^n \) satisfies the criterion (with incidences under \( V' \)). \( \square \)

4. An example of Laudenbach

**Proposition 4.1.** There exists an excellent Morse function \( f : \mathbb{R}^{2n} \to \mathbb{R} \) as follows:

1. it is quadratic at infinity and the reference quadratic form has index and coindex \( n > 1 \);
2. it has exactly five critical points: three of index \( n \), one of index \( n - 1 \) and one of index \( n + 1 \);
3. its Morse complex over \( \mathbb{Z} \) is given by
   \[
   \partial \xi_1^{n-1} = 0, \quad \partial \xi_1^n = \xi_1^{n-1}, \quad \partial \xi_2^n = -2\xi_1^{n-1}, \quad \partial \xi_3^n = -\xi_1^{n-1}, \quad \partial \xi_1^{n+1} = \xi_2^n - 2\xi_3^n,
   \]
   hence, for any field \( \mathbb{F}_2 \) of characteristic 2 and any field \( \mathbb{F} \) of characteristic \( \neq 2 \),
   \[
   \gamma(f, \mathbb{Z}) = \gamma(f, \mathbb{F}_2) = \gamma(f, \mathbb{F}) = f(\xi_3^n) > f(\xi_2^n) = \gamma(f, \mathbb{F}) = \gamma(f, \mathbb{Z}).
   \]

**Proof that (4.5) implies (4.6).** The Morse complex of \( f \) over \( \mathbb{F}_2 \) writes
\[
\partial \xi_1^{n-1} = 0, \quad \partial \xi_1^n = \xi_1^{n-1}, \quad \partial \xi_2^n = 0, \quad \partial(\xi_3^n + \xi_1^n) = 0, \quad \partial \xi_1^{n+1} = \xi_2^n,
\]
implying that \( \xi_3^n \) is the only free critical point, hence, by Corollary 3.7,
\[
\gamma(f, \mathbb{F}_2) = \gamma(f, \mathbb{F}) = f(\xi_3^n);
\]
as \( \gamma(f, \mathbb{Z}) \geq \gamma(f, \mathbb{F}_2) \) by Proposition 2.11 and \( \gamma(f, \mathbb{Z}) \leq f(\xi_3^n) \), we do have
\[
\gamma(f, \mathbb{Z}) = f(\xi_3^n).
\]

Similarly (keeping the numbering of the critical points defined by \( f \)) the Morse complex of \(-f\) over \( \mathbb{F} \) has the Barannikov normal form
\[
\partial(-2\xi_1^{n+1}) = 0, \quad \partial \xi_3^n = -2\xi_1^{n+1}, \quad \partial(\xi_2^n + \frac{1}{2}\xi_3^n) = 0, \quad \partial(-\xi_3^n - 2\xi_2^n + \xi_1^n) = 0, \quad \partial \xi_1^{n-1} = -\xi_3^n - 2\xi_2^n + \xi_1^n,
\]
showing that the free critical point is \( \xi_2^n \); hence, by Corollary 3.7 and Proposition 3.8,
\[
\gamma(f, \mathbb{F}) = \gamma(f, \mathbb{F}) = f(\xi_2^n);\]
finally, as we have $\gamma(f, Z) \leq \gamma(f, F)$ by Proposition 2.11, and $\gamma(f, Z) \geq f(\xi^n_1)$, we should prove $\gamma(f, Z) > f(\xi^n_1)$, which is obvious since $\xi^n_1$ and $\xi^n_{1+1}$ are boundaries in $M_s(-f, Z)$.

**How to construct such a function $f$.** It is easy to construct a function $f_0 : \mathbb{R}^{2n} \to \mathbb{R}$ with properties (1) and (2) required in the proposition and whose gradient vector field $V_0$ provides a Morse complex given by

$$\partial \xi^{n-1}_1 = 0, \quad \partial \xi^n_1 = \xi^{n-1}_1, \quad \partial \xi^n_2 = 0, \quad \partial \xi^n_3 = 0$$

$$\partial \xi_{1+1}^n = \xi^n_3.$$

For a change of basis

$$\xi^n_2 \mapsto \xi^n_2 - \xi^n_1, \quad \xi^n_3 \mapsto \xi^n_3 - 2(\xi^n_2 - \xi^n_1)$$

one can construct a gradient-like vector field $V'$ for $f_0$ by sliding handles, such that

$$\partial \xi^{n-1}_1 = 0$$

$$\partial \xi^n_1 = \xi^{n-1}_1, \quad \partial \xi^n_2 = -\xi^{n-1}_1, \quad \xi^n_3 = -2\xi^{n-1}_1$$

$$\partial \xi_{1+1}^n = -2\xi^n_2 + \xi^n_3$$

Since $(f_0, V')$ is Morse-Smale, the invariant manifolds of those critical points of the same index are disjoint, hence one can modify $f_0$ to $f$ such that

- $f$ has the same critical points of $f_0$;
- the ordering of critical points for $f$ is $f(\xi^n_2) > f(\xi^n_3) > f(\xi^n_1)$;
- $V'$ is a gradient-like vector field for $f$.

This can be realized by the preliminary rearrangement theorem (Theorem 4.1) in [7].

In other words, we have made a change of critical points $\xi^n_2 \leftrightarrow \xi^n_3$, hence obtain the required Morse complex in the proposition.

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