On the Convex Hulls of Self-Affine Fractals

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Abstract

Suppose that the set $T = \{T_1, T_2, \ldots, T_q\}$ of real $n \times n$ matrices has joint spectral radius less than 1. Then for any digit set $D = \{d_1, \ldots, d_q\} \subset \mathbb{R}^n$, there exists a unique nonempty compact set $F = \bigcup_{j=1}^q T_j(F + d_j)$, which is called a self-affine fractal. We consider an existing criterion for the convex hull of $F$ to be a polytope, which is due to Kirat and Kocyigit. In this note, we strengthen our criterion for the case $T_1 = T_2 = \cdots = T_q$. More specifically, we give an upper bound for the number of steps needed for deciding whether the convex hull of $F$ is a polytope or not. This improves our earlier result on the topic.

Keywords: Self-affine fractal, Polytope convex hulls.

1 Introduction

As usual, we use $\mathbb{R}$ for the set of real numbers. Let $M_n(\mathbb{R})$ denote the set of $n \times n$ matrices with real entries. A matrix $T \in M_n(\mathbb{R})$ is called expanding (or expansive) if all its eigenvalues have moduli $> 1$. For a set $\{T_1^{-1}, \ldots, T_q^{-1}\} \subset M_n(\mathbb{R})$ of expanding matrices, the $T_i$ may not be contractive with respect to the same norm (obviously, for each of them there is a norm, i.e. the Lind norm, depending on the matrix, with respect to which it is contractive [3]).

Let $\| \cdot \|$ denote a norm on $\mathbb{R}^n$, and $\|T\| = \sup\{\|Tx\| : x \in \mathbb{R}^n, \|x\| = 1\}$ be the induced matrix norm of a matrix $T \in M_n(\mathbb{R})$. For $\mathcal{T} = \{T_1, \cdots T_q\} \subset M_n(\mathbb{R})$, we set

$$\|\mathcal{T}\| = \max\{\|T_j\| : T_j \in \mathcal{T}\}.$$ 

Let $J_k = \{(j_1, j_2, \ldots, j_k) : 1 \leq j_i \leq q\}$, and let $j = (j_1, \cdots, j_k)$ denote a multi-index or an element in $J_k$ so that $|j| = k$ is the length of $j$. By $T_j$, we mean the product $T_{j_1} \cdots T_{j_k}$. We let $T^k = \{T_j : j \in J_k\}$. Then the number

$$\lambda(\mathcal{T}) = \lim_k \|T^k\|^{1/k} = \limsup_k \|T^k\|^{1/k} = \inf_k \|T^k\|^{1/k}$$

is called the (uniform) joint spectral radius of $\mathcal{T}$. Then we have the following.

**Proposition 1.1** [2, 1] Suppose that $\mathcal{T} = \{T_1, T_2, \ldots, T_q\} \subset M_n(\mathbb{R})$ satisfies $\lambda(\mathcal{T}) < 1$. Then for any set $D = \{d_1, \ldots, d_q\} \subset \mathbb{R}^n$, called a digit set, there exists a unique nonempty compact set $F$ satisfying

$$F = \bigcup_{j=1}^q T_j(F + d_j). \quad (1.1)$$

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We sometimes write \( F(\mathcal{T}, D) \) for \( F \) to stress the dependence on \( \mathcal{T} \) and \( D \). The compact set \( F \) in (1.1) is called a self-affine set or a self-affine fractal, and can be viewed as the invariant set or the attractor of the (affine) iterated function system (IFS) \( \{\phi_j(x) = T_j(x + d_j)\}_{j=1}^q \). Our generalization of \( F \) here includes all classical self-affine attractors. In the most classical case, \( T_j \) are assumed to be contractions.

Further, if \( \mathcal{T} \) is a set of nonsingular matrices with \( \lambda(\mathcal{T}) < 1 \), then \( \{T_1^{-1}, \ldots, T_q^{-1}\} \subseteq M_n(\mathbb{R}) \) is a set of expanding matrices. In fact, when \( \lambda(\mathcal{T}) = 1 \), there is no vector norm on \( \mathbb{R}^n \) such that the induced matrix norm satisfies \( \|T_j\| < 1 \) for all \( j \in \{1, \ldots, q\} \) (see [2] for details).

In this paper, we consider \( F = F(\mathcal{T}, D) \) in Proposition 1.1 with the assumption that \( T_1 = T_2 = \cdots = T_q = T \), and \( T \) is a nonsingular matrix. The purpose of this note is to report a strengthened criterion for the convex hull of \( F \) to be a polytope. This problem arises from the interest in the geometry of fractals.

## 2 Preliminaries

In the sequel, we will also assume that \( d_1 = 0 \). For \( j = (j_1, \ldots, j_k) \in J_k \), we let \( \phi_j = \phi_{j_1} \circ \cdots \circ \phi_{j_k} \). We set

\[
A_k = \{T_{j_1}T_{j_2}\cdots T_{j_k}d_{j_k} + \cdots + T_{j_1}T_{j_2}d_{j_2} + T_{j_1}d_{j_1} \mid d_{j_1}, \ldots, d_{j_k} \in D\}.
\]

In the Hausdorff metric, we know that

\[
A_k = \bigcup_{|j|=k} \phi_j(0), \quad k = 1, 2, \ldots,
\]

converges to \( F \) as \( k \to \infty \). The convex hull of a set \( S \subseteq \mathbb{R}^n \), denoted by \( \mathcal{C}(S) \), is the intersection of all convex sets in \( \mathbb{R}^n \) containing \( S \). It is of geometrical interest to study \( \mathcal{C}(F) \) and to determine its vertices [4]. Since \( 0 \in D \), it can be easily seen that \( A_k \subseteq A_{k'} \) and \( \mathcal{C}(A_k) \subseteq \mathcal{C}(A_{k'}) \) for \( k \leq k' \). We note that \( \mathcal{C}(F) \) is a compact set.

For brevity, a point \( x \in F \) will be denoted by a sequence \( d_{j_1}d_{j_2}\ldots d_{j_k} \ldots \). A periodic sequence is a sequence of the form

\[
d_{j_1}d_{j_2}\ldots d_{j_p},
\]

i.e., the block of digits \( d_{j_1}, d_{j_2}, \ldots, d_{j_p} \) is repeated indefinitely. An eventually periodic (e.p.) sequence \( Y \) is a sequence of the form

\[
d_{i_1}d_{i_2}\ldots d_{i_m}d_{j_1}d_{j_2}\ldots d_{j_p},
\]

where \( i_1i_2\ldots i_m \in \{1, \ldots, q\} \). For all arbitrarily large \( r \geq 2 \), a point of the form

\[
x = i_1i_2\ldots i_m(j_1j_2\ldots j_p), j_{p+1}\ldots j_{p+s}
\]

will be called a finitely eventually periodic (f.e.p.) point. For instance, suppose that \( i_1, \ldots, i_m, j_1, \ldots, j_{p+s} \in \{1, 2\} \), then \( (12)^{10}, 1(12)^{20}112, 11(12)^{40}1121, (12)^{100}21 \) are all f.e.p. points. Assume that \( W_k \subseteq V_k \), and \( W_k \) consists of f.e.p. points. Let \( W_k, \infty \) be the set of e.p. points of the form \( i_1i_2\ldots i_mj_1j_2\ldots j_p \) associated to the f.e.p. points of \( W_k \subseteq V_k \). Set

\[
W_k^c, \infty = \{x = i_1'1i_2\ldots i_mj_1j_2\ldots j_p \mid i_1, \ldots, i_mj_1j_2\ldots j_p \in W_k, 1 \leq i_1' \leq q, x \notin W_k, \infty \}.
\]

By a string of a sequence, we mean a special sequence consisting of certain consecutive terms of that sequence. For example, let \( Y = d_{j_1}d_{j_2}\ldots d_{j_k} \ldots \) be a sequence of digits, then
$d_{j_2}d_{j_3}d_{j_4}$ is a 3-string of $Y$, $d_{j_5}d_{j_6} \ldots d_{j_{k+4}}$ is a $k$-string of $Y$, and $d_{j_2}d_{j_3} \ldots$ is an $\infty$-string of $Y$.

In the literature, there are two characterization of self affine-fractals with polytope convex hulls. The first of them is due to Strichartz and Wang, and the other is due to Kirat and Kocyigit. They are as follows:

**Proposition 2.1** \[1\] Assume that $T_1 = T_2 = \cdots = T_q = T$ in $(1.1)$. Let $\{n_j\}$ be the outward unit normal vectors of the $(n - 1)$-dimensional faces of $C(D)$. Then $C(F)$ is a polytope if and only if every $n_j$ is an eigenvector of $T^*_k$ for some $k$, where $T^*$ is the classical adjoint of $A$.

**Proposition 2.2** \[2\] (i) $C(F)$ is a polytope if and only if there exists an index $k_0$ and a subset $W_{k_0} \subseteq V_{k_0}$ with f.e.p. points such that

$$W_{k_0, \infty} \cup V_{k_0} \subseteq C(W_{k_0, \infty}).$$

In such a case, $C(F) = C(W_{k_0, \infty})$.

(ii) Assume that $T_1 = T_2 = \cdots = T_q = T$ in $(1.1)$. Then $C(F)$ is a polytope if and only if $\#V_i = \#V_{i+1} = t$ for some $i$. In such a case, the points of $V(F)$ are periodic and for any $k > i$, $V_k$ consists of all $k$-strings of the points of $C(F)$.

There are two differences between Proposition 2.1 and Proposition 2.2. First, the former only deals with one matrix. Secondly, unlike the latter, the former doesn’t find the coordinates and the number of the vertices of the polytope $C(F)$. Although, Proposition 2.2 is better than Proposition 2.1 in that respect, and we can obtain many examples by using it, the second characterization cannot decide if $C(F)$ is a polytope when the conditions in Proposition 2.2 are not satisfied for $k_0$ or $i$ up to a large number.

It is our aim in this note to give an upper bound for $i$ satisfying the conditions in Proposition 2.2 in the single-matrix case. Thus we can say that $C(F)$ is not a polytope if the condition in (ii) are not met after a finite number of steps.

### 3 Statement of the Result

$C$ stands for the set of complex numbers. In the following theorem, $z_n \in C$ denotes an $n$-th root of unity.

**Theorem 3.1** Assume that $T_1 = T_2 = \cdots = T_q = T$ in $(1.1)$. Let

$$U = \{c_{n_1}z_{2n_1}, c_{n_2}z_{2n_2}, \cdots, c_{n_m}z_{2n_m}\}$$

be the set of all roots of $T^{-1}$ of the form $cz_n$, where $c \geq 0$. Assume that $U \neq \emptyset$. Set $k = 2n_1n_2 \cdots n_m$. Then if $\#V_i \neq \#V_{i+1}$ for all $i \leq k$, then $C(F)$ is not a polytope. Therefore, $C(F)$ is a polytope if and only if $\#V_i \neq \#V_{i+1}$ for some $i \leq k$.

**Remark 3.2**. Here we don’t assume that $T^{-1}$ is a similitude, it is a general expanding matrix. It follows from Proposition 2.1 that $C(F)$ is not a polytope when $U = \emptyset$. Note that $U = \emptyset$ is possible only if $F \subset \mathbb{R}^{2n}$. 

3
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