RIGHT AMENABILITY AND GROWTH OF FINITELY
RIGHT GENERATED LEFT GROUP SETS

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Abstract. We introduce right generating sets, Cayley graphs, growth functions, types and rates, and isoperimetric constants for left homogeneous spaces equipped with coordinate systems; characterise right amenable finitely right generated left homogeneous spaces with finite stabilisers as those whose isoperimetric constant is 0; and prove that finitely right generated left homogeneous spaces with finite stabilisers of sub-exponential growth are right amenable, in particular, quotient sets of groups of sub-exponential growth by finite subgroups are right amenable.

The notion of amenability for groups was introduced by John von Neumann in 1929. It generalises the notion of finiteness. A group is left or right amenable if there is a finitely additive probability measure on \( P(G) \) that is invariant under left and right multiplication respectively. Groups are left amenable if and only if they are right amenable. A group is amenable if it is left or right amenable.

The definitions of left and right amenability generalise to left and right group sets respectively. A left group set \( (M, G, \rhd) \) is left amenable if there is a finitely additive probability measure on \( P(M) \) that is invariant under \( \rhd \). There is in general no natural action on the right that is to a left group action what right multiplication is to left group multiplication. Therefore, for a left group set there is no natural notion of right amenability.

A transitive left group action \( \rhd \) of \( G \) on \( M \) induces, for each element \( m_0 \in M \) and each family \( \{g_{m_0,m}\}_{m \in M} \) of elements in \( G \) such that, for each point \( m \in M \), we have \( g_{m_0,m} \rhd m_0 = m \), a right quotient set semi-action \( \triangleleft \) of \( G/G_0 \) on \( M \) with defect \( G_0 \) given by \( m \triangleleft gg_0 = g_{m_0,mg_g^{-1}_{m_0,m}} \rhd m \), where \( G_0 \) is the stabiliser of \( m_0 \) under \( \rhd \). Each of these...
right semi-actions is to the left group action what right multiplication is to left group multiplication. They occur in the definition of global transition functions of cellular automata over left homogeneous spaces as defined in [6]. A cell space is a left group set together with choices of \( m_0 \) and \( \{g_{m_0,m}\}_{m \in M} \).

A cell space is right amenable if there is a finitely additive probability measure on \( \mathcal{P}(M) \) that is semi-invariant under \( \trianglelefteq \). For example cell spaces with finite sets of cells, abelian groups, and finitely right generated cell spaces with finite stabilisers of sub-exponential growth are right amenable, in particular, quotients of finitely generated groups of sub-exponential growth by finite subgroups acted on by left multiplication. A net of non-empty and finite subsets of \( M \) is a right Følner net if, broadly speaking, these subsets are asymptotically invariant under \( \trianglelefteq \). A finite subset \( E \) of \( G/G_0 \) and two partitions \( \{A_e\}_{e \in E} \) and \( \{B_e\}_{e \in E} \) of \( M \) constitute a right paradoxical decomposition if the map \( \_ \trianglelefteq e \) is injective on \( A_e \) and \( B_e \), and the family \( \{(A_e \trianglelefteq e) \cup (B_e \trianglelefteq e)\}_{e \in E} \) is a partition of \( M \). The Tarski-Følner theorem states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent. We prove it in [7] for cell spaces with finite stabilisers.

A cell space \( \mathcal{R} \) is finitely right generated if there is a finite subset \( S \) of \( G/G_0 \) such that, for each point \( m \in M \), there is a family \( \{s_i\}_{i \in \{1,2,\ldots,k\}} \) of elements in \( S \cup S^{-1} \) such that \( m = (((m_0 \trianglelefteq s_1) \trianglelefteq s_2) \cdots) \trianglelefteq s_k \). The finite right generating set \( S \) induces the \( S \)-Cayley graph structure on \( M \): For each point \( m \in M \) and each generator \( s \in S \), there is an edge from \( m \) to \( m \trianglelefteq s \). The length of the shortest path between two points of \( M \) yields the \( S \)-metric. The ball of radius \( \rho \in \mathbb{N}_0 \) centred at \( m \in M \), denoted by \( B_S(m,\rho) \), is the set of all points whose distance to \( m \) is less than or equal to \( \rho \). The \( S \)-growth function is the map \( \gamma_S : \mathbb{N}_0 \to \mathbb{N}_0 \), \( k \mapsto |B_S(m,k)| \); the growth type of \( \mathcal{R} \), which does not depend on \( S \), is the equivalence class \( [\gamma_S]_\sim \), where two growth functions are equivalent if they dominate each other; and the \( S \)-growth rate is the limit point of the sequence \( \left( \sqrt[k]{\gamma_S(k)} \right)_{k \in \mathbb{K}} \).
A finitely right generated cell space $\mathcal{R}$ is said to have sub-exponential growth if its growth type is not $[\exp]_\sim$, which is the case if and only if its growth rates are 1. The $S$-isoperimetric constant is a real number between 0 and 1 that measures, broadly speaking, the invariance under $\preceq|_{M \times S}$ that a finite subset of $M$ can have, where 0 means maximally and 1 minimally invariant. In the case that $G_0$ is finite, this constant is 0 if and only if $\mathcal{R}$ is right amenable, and if $\mathcal{R}$ has sub-exponential growth, then it is right amenable, and if $G$ has sub-exponential growth, then so has $\mathcal{R}$.

Cayley graphs were introduced by Arthur Cayley in his paper ‘Desiderata and suggestions: No. 2. The Theory of groups: graphical representation’[1]. The notion of growth was introduced by Vadim Arsenyevich Efremovich and Albert S. Švarc in their papers ‘The geometry of proximity’[3] and ‘A volume invariant of coverings’[5]. Mikhail Leonidovich Gromov was the first to study groups through their word metrics, see for example his paper ‘Infinite Groups as Geometric Objects’[4]. The present paper is greatly inspired by the monograph ‘Cellular Automata and Groups’[2] by Tullio Ceccherini-Silberstein and Michel Coornaert.

In Section 1 we introduce right generating sets. In Section 2 we recapitulate directed multigraphs. In Section 3 we introduce Cayley graphs induced by right generating sets. In Section 4 we introduce metrics and lengths induced by Cayley graphs. In Section 5 we consider balls and spheres induced by metrics. In Section 6 we consider interiors, closures, and boundaries of any thickness of sets. In Section 7 we recapitulate growth functions and types. In Section 8 we introduce growth functions and types of cell spaces. In Section 9 we introduce growth rates of cell spaces. In Section 10 we prove that right amenability and having isoperimetric constant 0 are equivalent, and we characterise right Følner nets. And in Section 11 we prove that having sub-exponential growth implies right amenability.

Preliminary Notions. A left group set is a triple $(M, G, \triangleright)$, where $M$ is a set, $G$ is a group, and $\triangleright$ is a map from $G \times M$ to $M$, called left group action of $G$ on $M$, such that $G \to \text{Sym}(M)$, $g \mapsto [g \triangleright \_]$, is a group
homomorphism. The action $\triangleright$ is transitive if $M$ is non-empty and for each $m \in M$ the map $\_ \triangleright m$ is surjective; and free if for each $m \in M$ the map $\_ \triangleright m$ is injective. For each $m \in M$, the set $G \triangleright m$ is the orbit of $m$, the set $G_m = (\_ \triangleright m)^{-1}(m)$ is the stabiliser of $m$, and, for each $m' \in M$, the set $G_{m,m'} = (\_ \triangleright m)^{-1}(m')$ is the transporter of $m$ to $m'$.

A left homogeneous space is a left group set $\mathcal{M} = (M, G, \triangleright)$ such that $\triangleright$ is transitive. A coordinate system for $\mathcal{M}$ is a tuple $\mathcal{K} = (m_0, \{g_{m_0,m}\}_{m \in M})$, where $m_0 \in M$ and for each $m \in M$ we have $g_{m_0,m} \triangleright m_0 = m$. The stabiliser $G_{m_0}$ is denoted by $G_0$. The tuple $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is a cell space. The map $\leq : M \times G/G_0 \rightarrow M$, $(m, gG_0) \mapsto g_{m_0,m}g^{-1}G_{m_0} \triangleright m = g_{m_0,m \triangleright g_0}G_0$ is a right semi-action of $G/G_0$ on $M$ with defect $G_0$, which means that

$$\forall m \in M : m \leq G_0 = m,$$

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall g' \in G/G_0 : m \leq g \cdot g' = (m \leq gG_0) \leq g_0 \cdot g'.$$

It is transitive, which means that the set $M$ is non-empty and for each $m \in M$ the map $m \leq \_ \leq$ is surjective; and free, which means that for each $m \in M$ the map $m \leq \_ \leq$ is injective; and semi-commutes with $\triangleright$, which means that

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall g' \in G/G_0 : (g \triangleright m) \leq g' = g \triangleright (m \leq g \cdot g').$$

The maps $\iota : M \rightarrow G/G_0$, $m \mapsto G_{m_0,m}$, and $m_0 \leq \_ \leq$ are inverse to each other. Under the identification of $M$ with $G/G_0$ by either of these maps, we have $\leq : (m, g) \mapsto g_{m_0,m} \triangleright g$.

A left homogeneous space $\mathcal{M}$ is right amenable if there is a coordinate system $\mathcal{K}$ for $\mathcal{M}$ and there is a finitely additive probability measure $\mu$ on $M$ such that

$$\forall g \in G/G_0 \forall A \subseteq M : ((\_ \leq g)|_A \text{ injective } \implies \mu(A \leq g) = \mu(A)),$$

in which case the cell space $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is called right amenable. When the stabiliser $G_0$ is finite, that is the case if and only if there is a right Følner net in $\mathcal{R}$ indexed by $(I, \leq)$, which is a net $\{F_i\}_{i \in I}$ in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$ such that

$$\forall g \in G/G_0 : \lim_{i \in I} \frac{|F_i \setminus (\_ \leq g)^{-1}(F_i)|}{|F_i|} = 0.$$
1. Right Generating Sets

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space.

In Definition 1.1 we define right generating sets of $\mathcal{R}$. And in Lemma 1.4 we show how generating sets of $G$ induce right ones of $\mathcal{R}$.

Definition 1.1. Let $S$ be a subset of $G/G_0$ such that $G_0 \cdot S \subseteq S$.

1. The set $\{g^{-1}G_0 \mid s \in S, g \in s\}$ is denoted by $S^{-1}$.
2. The set $S$ is said to right generate $\mathcal{R}$, called right generating set of $\mathcal{R}$, and each element $s \in S$ is called right generator if and only if, for each element $m \in M$, there is a non-negative integer $k \in \mathbb{N}_0$ and there is a family $\{s_i\}_{i \in \{1, 2, ..., k\}}$ of elements in $S \cup S^{-1}$ such that
   $$(((m_0 \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_k = m.$$  
3. The set $S$ is called symmetric if and only if $S^{-1} \subseteq S$.

Definition 1.2. The cell space $\mathcal{R}$ is called finitely right generated if and only if there is a right generating set of $\mathcal{R}$ that is finite.

Remark 1.3. If $S$ is a right generating set of $\mathcal{R}$, then $S \cup S^{-1}$ is a symmetric one; and, if $S$ is also finite and $G_0$ is finite, then $S \cup S^{-1}$ is finite.

Lemma 1.4. Let $T$ be a generating set of $G$. The set $S = \{g_0 \cdot tG_0 \mid g_0 \in G_0, t \in T\}$ is a right generating set of $\mathcal{R}$. And, if $T$ is symmetric, then so is $S$. And, if $T$ and $G_0$ are finite, then so is $S$.

Proof. Let $m \in M$. Then, because $\trianglelefteq$ is transitive, there is a $g \in G$ such that $m_0 \trianglelefteq gG_0 = m$. Moreover, there is a $k \in \mathbb{N}_0$ and there is a $\{t_i\}_{i \in \{1, 2, ..., k\}} \subseteq T \cup T^{-1}$ such that $t_1t_2 \cdots t_k = g$. Furthermore, there is a $\{g_i\}_{i \in \{2, 3, ..., k\}} \subseteq G_0$ such that
   $$\left(((m_0 \trianglelefteq t_1G_0) \trianglelefteq g_20t_2G_0) \trianglelefteq \cdots) \trianglelefteq g_k0t_kG_0 = m_0 \trianglelefteq t_1t_2 \cdots t_kG_0 = m.$$  

In conclusion, because $t_1G_0 \in S \cup S^{-1}$ and $\{g_i0t_iG_0\}_{i \in \{2, 3, ..., k\}} \subseteq S \cup S^{-1}$, the set $S$ is a right generating set of $\mathcal{R}$. 
Let $T$ be symmetric. Furthermore, let $s \in S$ and let $g \in s$. Then, there is a $g_0 \in G_0$, there is a $t \in T$, and there is a $g'_0 \in G_0$ such that $g_0 \cdot tG_0 = s$ and $g_0 t g'_0 = g$. Hence, because $(g'_0)^{-1} \in G_0$ and $t^{-1} \in T$,
\[
g^{-1}G_0 = (g'_0)^{-1} t^{-1} g^{-1} G_0 = (g'_0)^{-1} \cdot t^{-1} G_0 \in S.
\]
In conclusion, $S^{-1} \subseteq S$.

If $T$ and $G_0$ are finite, then so is $S$. \hfill \square

2. Directed Multigraphs

Definition 2.1. Let $V$ and $E$ be two sets, and let $\sigma$ and $\tau$ be two maps from $E$ to $V$. The quadruple $G = (V, E, \sigma, \tau)$ is called directed multigraph; each element $v \in V$ is called vertex; each element $e \in E$ is called edge from $\sigma(e)$ to $\tau(e)$; for each element $e \in E$, the vertex $\sigma(e)$ is called source of $e$ and the vertex $\tau(e)$ is called target of $e$.

Definition 2.2. Let $G = (V, E, \sigma, \tau)$ be a directed multigraph and let $e$ be an edge of $G$. The edge $e$ is called loop if and only if $\tau(e) = \sigma(e)$.

Definition 2.3. Let $G = (V, E, \sigma, \tau)$ be a directed multigraph and let $v$ be a vertex of $G$.

1. The cardinal number $\deg^+(v) = |\{e \in E \mid \sigma(e) = v\}|$ is called out-degree of $v$.
2. The cardinal number $\deg^-(v) = |\{e \in E \mid \tau(e) = v\}|$ is called in-degree of $v$.
3. The cardinal number $\deg(v) = \deg^+(v) + \deg^-(v)$ is called degree of $v$.

Definition 2.4. Let $G$ be a directed multigraph, and let $v$ and $v'$ be two vertices of $G$. The vertices $v$ and $v'$ are called adjacent if and only if there is an edge from $v$ to $v'$ or one from $v'$ to $v$.
Definition 2.5. Let $\mathcal{G} = (V, E, \sigma, \tau)$ be a directed multigraph and let $p = (e_i)_{i \in \{1, 2, \ldots, k\}}$ be a finite sequence of edges of $\mathcal{G}$. The sequence $p$ is called path from $\sigma(e_1)$ to $\tau(e_k)$ if and only if, for each index $i \in \{1, 2, \ldots, k - 1\}$, we have $\tau(e_i) = \sigma(e_{i+1})$.

Definition 2.6. Let $\mathcal{G}$ be a directed multigraph and let $p = (e_i)_{i \in \{1, 2, \ldots, k\}}$ be a path in $\mathcal{G}$. The number $|p| = k$ is called length of $p$.

Definition 2.7. Let $\mathcal{G} = (V, E, \sigma, \tau)$ be a directed multigraph. It is called

1. symmetric if and only if, for each edge $e \in E$, there is an edge $e' \in E$ such that $\sigma(e') = \tau(e)$ and $\tau(e') = \sigma(e)$;
2. strongly connected if and only if, for each vertex $v \in V$ and each vertex $v' \in V$, there is a path $p$ from $v$ to $v'$;
3. regular if and only if all vertices of $\mathcal{G}$ have the same degree and, for each vertex $v \in V$, we have $\text{deg}^-(v) = \text{deg}^+(v)$.

Definition 2.8. Let $\mathcal{G} = (V, E, \sigma, \tau)$ be a directed multigraph, let $W$ be a subset of $V$, let $F$ be the set $\{e \in E \mid \sigma(e), \tau(e) \in W\}$, let $\varsigma$ be the map $\sigma|_{F \rightarrow W}$, and let $\upsilon$ be the map $\tau|_{F \rightarrow W}$. The subgraph $\mathcal{G}[W] = (W, F, \varsigma, \upsilon)$ of $\mathcal{G}$ is called induced by $W$.

Definition 2.9. Let $\mathcal{G} = (V, E, \sigma, \tau)$ be a symmetric and strongly connected directed multigraph. The map

$$d: V \times V \rightarrow \mathbb{N}_0,$$

$$ (v, v') \mapsto \min\{|p| \mid p \text{ path from } v \text{ to } v'\},$$

is a metric on $V$ and called distance on $\mathcal{G}$.

Definition 2.10. Let $(V, E, \sigma, \tau)$ be a directed multigraph, let $\Lambda$ be a set, and let $\lambda$ be a map from $E$ to $\Lambda$. The quintuple $\mathcal{G} = (V, E, \sigma, \tau, \lambda)$ is called $\Lambda$-edge-labelled directed multigraph.

### 3. Cayley Graphs

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space and let $S$ be a right generating set of $\mathcal{R}$. 
Definition 3.1. Let \( E \) be the set \( \{(m, s, m \trianglelefteq s) \mid m \in M, s \in S\} \), and let \( \sigma: E \to M \), \( \lambda: E \to S \), and \( \tau: E \to M \) be the projections to the first, second, and third component respectively. The \( S \)-edge-labelled directed multigraph \( G = (M, E, \sigma, \tau, \lambda) \) is called \( S \)-Cayley graph of \( \mathcal{R} \).

Remark 3.2. Let \( G \) be the \( S \)-Cayley graph of \( \mathcal{R} \).

1. If \( S \) is symmetric, then \( G \) is symmetric and strongly connected.
2. The following statements are equivalent:
   a. \( G_0 \in S \);
   b. At least one vertex of \( G \) has a loop;
   c. All vertices of \( G \) have a loop.
3. Because \( \trianglelefteq \) is free, there are no multiple edges in \( G \).

Remark 3.3. Let \( G \) be the \( S \)-Cayley graph of \( \mathcal{R} \), and let \( m \) and \( m' \) be two vertices of \( G \). The vertices \( m \) and \( m' \) are adjacent if and only if there is an element \( s \in S \) such that \( m \trianglelefteq s = m' \).

Remark 3.4. Let \( G \) be the \( S \)-Cayley graph of \( \mathcal{R} \) and let \( m \) be a vertex of \( G \). The map

\[
S \to m \trianglelefteq S,
\]

\[
s \mapsto m \trianglelefteq s,
\]

is a bijection onto the out-neighbourhood of \( m \). It is injective, because \( \trianglelefteq \) is free, and it is surjective, by definition. Therefore, if \( S \) is symmetric, then the degree of \( m \) is \( 2|S| \) in cardinal arithmetic and the graph \( G \) is regular.

4. Metrics and Lengths

In this section, let \( \mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M})) \) be a cell space and let \( S \) be a symmetric right generating set of \( \mathcal{R} \).

In Definitions 4.1 and 4.6 we define the \( S \)-metric \( d_S \) and the \( S \)-length \( |\_|_S \) on \( \mathcal{R} \) induced by the \( S \)-Cayley graph. And in Lemmas 4.3 and 4.4 we show how the \( S \)-metric relates to the left group action \( \triangleright \) and the right quotient set semi-action \( \trianglelefteq \).

Definition 4.1. The distance on the \( S \)-Cayley graph of \( \mathcal{R} \) is called \( S \)-metric on \( \mathcal{R} \) and denoted by \( d_S \).
Remark 4.2. The $S$-metric on $R$ is

$$d_S: M \times M \to \mathbb{N}_0,$$

$$(m, m') \mapsto \min\{k \in \mathbb{N}_0 \mid \exists \{s_i\}_{i \in \{1, 2, \ldots, k\}} \subseteq S : ((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_k = m'\}.$$

**Lemma 4.3.** Let $m$ and $m'$ be two elements of $M$, and let $s$ be an element of $S$. Then, $d_S(m, m' \trianglelefteq s) \leq d_S(m, m') + 1$.

**Proof.** Let $k = d_S(m, m')$. Then, there is a $\{s_i\}_{i \in \{1, 2, \ldots, k\}} \subseteq S$ such that $(((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_k = m'$. Hence, $(((((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_k) \trianglelefteq s = m' \trianglelefteq s$. Therefore, $d_S(m, m' \trianglelefteq s) \leq d_S(m, m') + 1$. \(\square\)

**Lemma 4.4.** Let $m$ and $m'$ be two elements of $M$, and let $g$ be an element of $G$. Then, $d_S(g \triangleright m, g \triangleright m') = d_S(m, m')$.

**Proof.** Let $k = d_S(g \triangleright m, g \triangleright m')$. Then, there is a $\{s_i\}_{i \in \{1, 2, \ldots, k\}} \subseteq S$ such that $(((g \triangleright m) \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_k = g \triangleright m'$. Moreover, because $\triangleright$ and $\trianglelefteq$ semi-commute, for each $i \in \{1, 2, \ldots, k\}$, there is a $g_{i,0} \in G_0$, such that

$$
\left(\left(\left(g \triangleright (m \trianglelefteq g_{1,0} \cdot s_1)\right) \trianglelefteq s_2\right) \trianglelefteq \cdots \trianglelefteq s_k\right) = g \triangleright \left(\left(\left(m \trianglelefteq g_{1,0} \cdot s_1\right) \trianglelefteq g_{2,0} \cdot s_2\right) \trianglelefteq \cdots \trianglelefteq s_k\right).
$$

Hence, $(((m \trianglelefteq g_{1,0} \cdot s_1) \trianglelefteq g_{2,0} \cdot s_2) \trianglelefteq \cdots) \trianglelefteq g_{k,0} \cdot s_k = m'$. Therefore, $d_S(m, m') \leq k = d_S(g \triangleright m, g \triangleright m')$.

Taking $g \triangleright m$ for $m$, $g \triangleright m'$ for $m'$, and $g^{-1}$ for $g$ yields $d_S(g \triangleright m, g \triangleright m') \leq d_S(g^{-1} \triangleright (g \triangleright m), g^{-1} \triangleright (g \triangleright m')) = d_S(m, m')$. In conclusion, $d_S(g \triangleright m, g \triangleright m') = d_S(m, m')$. \(\square\)

**Lemma 4.5.** Let $m$ and $m'$ be two elements of $M$, let $\{s_i\}_{i \in \{1, 2, \ldots, d_S(m, m')\}}$ be a family of elements in $S$ such that $m' = (((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_{d_S(m, m')}$. Let $i$ be an element of $\{0, 1, 2, \ldots, d_S(m, m')\}$, and let $m_i = (((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_i$. Then, $d_S(m, m_i) = i$. 

Proof. By definition of $m_i$, we have $d_S(m, m_i) \leq i$ and $d_S(m_i, m') \leq d_S(m, m') - i$. Therefore, because $d_S(m, m') \leq d_S(m, m_i) + d_S(m_i, m')$, we have $d_S(m, m_i) \geq d_S(m, m') - d_S(m_i, m') \geq d_S(m, m') - (d_S(m, m') - i) = i$. In conclusion, $d_S(m, m_i) = i$. □

Definition 4.6. The map
\[ |\_|_S : M \to \mathbb{N}_0, \]
\[ m \mapsto d_S(m_0, m), \]
is called $S$-length on $\mathcal{R}$.

Remark 4.7. For each element $m \in M$, we have $|m|_S = 0$ if and only if $m = m_0$.

5. Balls and Spheres

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space and let $S$ be a symmetric right generating set of $\mathcal{R}$.

In Definition 5.1 we define balls $B_S$ and spheres $S_S$ in the $S$-metric on $\mathcal{R}$. And in the lemmata and corollaries of this section we show how balls, spheres, the left group action $\triangleright$, the right quotient set semi-action $\trianglelefteq$, and the $S$-metric relate to each other.

Definition 5.1. Let $m$ be an element of $M$ and let $\rho$ be a non-negative integer.

1. The set
\[ B_S(m, \rho) = \{m' \in M \mid d_S(m, m') \leq \rho\} \]
is called ball of radius $\rho$ centred at $m$. The ball of radius $\rho$ centred at $m_0$ is denoted by $B_S(\rho)$.

2. The set
\[ S_S(m, \rho) = \{m' \in M \mid d_S(m, m') = \rho\} \]
is called sphere of radius $\rho$ centred at $m$. The sphere of radius $\rho$ centred at $m_0$ is denoted by $S_S(\rho)$.

Remark 5.2. For each element $m \in M$, we have $S_S(m, 0) = B_S(m, 0)$, and, for each positive integer $\rho \in \mathbb{N}_+$, we have $S_S(m, \rho) = B_S(m, \rho) \setminus B_S(m, \rho - 1)$. 


Remark 5.3. For each non-negative integer \( \rho \in \mathbb{N}_0 \),
\[
\mathbb{B}_S(\rho) = \{ m \in M \mid |m|_S \leq \rho \}
\]
and
\[
\mathbb{S}_S(\rho) = \{ m \in M \mid |m|_S = \rho \}.
\]

Definition 5.4. Let \((A_k)_{k \in \mathbb{N}_0}\) be a sequence of sets.

1. The set
\[
\liminf_{k \to \infty} A_k = \bigcup_{k \in \mathbb{N}_0} \bigcap_{j \geq k} A_j
\]

is called limit inferior of \((A_k)_{k \in \mathbb{N}_0}\).

2. The set
\[
\limsup_{k \to \infty} A_k = \bigcap_{k \in \mathbb{N}_0} \bigcup_{j \geq k} A_j
\]

is called limit superior of \((A_k)_{k \in \mathbb{N}_0}\).

3. Let \(A\) be a set. The sequence \((A_k)_{k \in \mathbb{N}_0}\) is said to converge to \(A\), the set \(A\) is called limit set of \((A_k)_{k \in \mathbb{N}_0}\), and \(A\) is denoted by \(\lim_{k \to \infty} A_k\) if and only if \(\liminf_{k \to \infty} A_k = \limsup_{k \to \infty} A_k = A\).

4. The sequence \((A_k)_{k \in \mathbb{N}_0}\) is called convergent if and only if \(\liminf_{k \to \infty} A_k = \limsup_{k \to \infty} A_k\).

Lemma 5.5. Let \((A_k)_{k \in \mathbb{N}_0}\) be a non-decreasing or non-increasing sequence of sets. It converges to \(\bigcup_{k \in \mathbb{N}_0} A_k\) or \(\bigcap_{k \in \mathbb{N}_0} A_k\) respectively. \(\Box\)

Remark 5.6. For each element \(m \in M\), we have \(\mathbb{B}_S(m, 0) = \{ m \}\), and the sequence \((\mathbb{B}_S(m, \rho))_{\rho \in \mathbb{N}_0}\) is non-decreasing with respect to inclusion and converges to \(M\), and hence, for each non-negative integer \(\rho\),
\[
\bigcup_{\rho' \in \mathbb{N}_0, \rho' \geq \rho} \mathbb{B}_S(m, \rho') = M.
\]

Remark 5.7. For each element \(m \in M\) and each non-negative integer \(\rho \in \mathbb{N}_0\), in cardinal arithmetic,
\[
|\mathbb{B}_S(m, \rho)| \leq (1 + |S|)\rho,
\]
because the map
\[
(\{G_0\} \cup S)^\rho \to \mathbb{B}_S(m, \rho),
\]
\((s_1, s_2, \ldots, s_\rho) \mapsto (((m \leq s_1) \leq s_2) \leq \cdots) \leq s_\rho,\)

is surjective and \(|\{(G_0) \cup S\}^\rho \leq (1 + |S|)^\rho|.

**Lemma 5.8.** Let \(m\) be an element of \(M\), let \(\rho\) be a non-negative integer, and let \(s\) be an element of \(S\). Then, \(\mathcal{B}_S(m, \rho) \leq s \subseteq \mathcal{B}_S(m, \rho + 1)\).

**Proof.** Let \(m' \in \mathcal{B}_S(m, \rho) \leq s\). Then, there is an \(m'' \leq s = m'\). Hence, according to Lemma 4.3, we have \(d_S(m, m') = d_S(m, m'') + 1 \leq \rho + 1\). Therefore, \(m' \in \mathcal{B}_S(m, \rho + 1)\). In conclusion, \(\mathcal{B}_S(m, \rho) \leq s \subseteq \mathcal{B}_S(m, \rho + 1)\).

**Lemma 5.9.** Let \(m\) be an element of \(M\), let \(\rho\) be a non-negative integer, and let \(g\) be an element of \(G\). Then, \(g \triangleright \mathcal{B}_S(m, \rho) = \mathcal{B}_S(g \triangleright m, \rho)\).

**Proof.** First, let \(m' \in g \triangleright \mathcal{B}_S(m, \rho)\). Then, \(g^{-1} \triangleright m' \in \mathcal{B}_S(m, \rho)\) and thus \(d_S(g, g^{-1} \triangleright m') \leq \rho\). Hence, according to Lemma 4.4,

\[
d_S(g, g^{-1} \triangleright m') = d_S(g^{-1} \triangleright (g \triangleright m), g^{-1} \triangleright m') \\
= d_S(m, g^{-1} \triangleright m') \\
\leq \rho.
\]

Therefore, \(m' \in \mathcal{B}_S(g \triangleright m, \rho)\). In conclusion, \(g \triangleright \mathcal{B}_S(m, \rho) \subseteq \mathcal{B}_S(g \triangleright m, \rho)\).

Secondly, let \(m' \in \mathcal{B}_S(g \triangleright m, \rho)\). Then, \(d_S(g \triangleright m, m') \leq \rho\). Thus, according to Lemma 4.4,

\[
d_S(m, g^{-1} \triangleright m') = d_S(g \triangleright m, g \triangleright (g^{-1} \triangleright m')) \\
= d_S(g \triangleright m, m') \\
\leq \rho.
\]

Hence, \(g^{-1} \triangleright m' \in \mathcal{B}_S(m, \rho)\). Therefore, \(m' \in g \triangleright \mathcal{B}_S(m, \rho)\). In conclusion, \(\mathcal{B}_S(g \triangleright m, \rho) \subseteq g \triangleright \mathcal{B}_S(m, \rho)\).

**Corollary 5.10.** Let \(m\) be an element of \(M\), let \(\rho\) be a non-negative integer, and let \(g_m\) be an element of \(G_m\). Then, \(g_m \triangleright \mathcal{B}_S(m, \rho) = \mathcal{B}_S(m, \rho)\). In particular, \(G_m \triangleright \mathcal{B}_S(m, \rho) = \mathcal{B}_S(m, \rho)\).

**Proof.** Because \(g_m \triangleright m = m\), this is a direct consequence of Lemma 5.9.
Corollary 5.11. Let \( m \) and \( m' \) be two elements of \( M \), and let \( \rho \) be a non-negative integer. Then, \( |B_S(m, \rho)| = |B_S(m', \rho)| \).

Proof. Because there is a \( g \in G \) such that \( g \cdot m = m' \), and \( g \cdot \_ \) is injective, this is a direct consequence of Lemma 5.9. \( \square \)

Lemma 5.12. Let \( m \) and \( m' \) be two elements of \( M \) and identify \( M \) with \( G/G_0 \) by \( [m] \mapsto G_{m_0,m} \). Then, \( m \trianglelefteq m' = g_{m_0,m} \triangleright m' \).

Proof. Let \( g \in G_{m_0,m'} \). Then, \( G_{m_0,m'} = gG_0 \). Hence,
\[
\begin{align*}
m \trianglelefteq m' &= m \trianglelefteq G_{m_0,m'} \\
&= m \trianglelefteq gG_0 \\
&= g_{m_0,m} g \cdot m_0 \\
&= g_{m_0,m} \cdot (g \cdot m_0) \\
&= g_{m_0,m} \cdot m'.
\end{align*}
\] \( \square \)

Lemma 5.13. Let \( m \), \( m' \), and \( m'' \) be three elements of \( M \) and identify \( M \) with \( G/G_0 \) by \( [m] \mapsto G_{m_0,m} \). Then, there is an element \( g_0 \in G_0 \) such that \( (m \trianglelefteq m') \trianglelefteq m'' = m \trianglelefteq (m' \trianglelefteq (g_0 \triangleright m'')) \).

Proof. Because \( \trianglelefteq \) is a right semi-action, there is an element \( g_0 \in G_0 \) such that \( m \trianglelefteq g_{m_0,m'} \cdot g_0 \cdot G_{m_0,m''} = (m \trianglelefteq g_{m_0,m'} G_0) \trianglelefteq G_{m_0,m''} \). And, under the identification of \( M \) with \( G/G_0 \), we have \( G_{m_0,m''} = m'' \), \( g_{m_0,m'} G_0 = G_{m_0,m'} = m' \), and \( g_{m_0,m'} \cdot g_0 \cdot G_{m_0,m''} = m' \trianglelefteq (g_0 \triangleright m'') \). Therefore, \( m \trianglelefteq (m' \trianglelefteq (g_0 \triangleright m'')) = (m \trianglelefteq m') \trianglelefteq m'' \). \( \square \)

Corollary 5.14. Let \( m \) be an element of \( M \), let \( \rho \) be a non-negative integer, and identify \( M \) with \( G/G_0 \) by \( [m] \mapsto G_{m_0,m} \). Then, \( m \trianglelefteq B_S(\rho) = B_S(m, \rho) \).

Proof. According to Lemma 5.12 and Lemma 5.9,
\[
\begin{align*}
m \trianglelefteq B_S(\rho) &= g_{m_0,m} \triangleright B_S(\rho) \\
&= g_{m_0,m} \triangleright B_S(m_0, \rho) \\
&= B_S(g_{m_0,m} \triangleright m_0, \rho) \\
&= B_S(m, \rho). \quad \square
\end{align*}
\]
Corollary 5.15. Let \( m \) be an element of \( M \), let \( \rho \) and \( \rho' \) be two non-negative integers, and identify \( M \) with \( G/G_0 \) by \([m \mapsto G_{m_0,m}]\). Then, \( B_s(m, \rho) \trianglelefteq B_s(\rho') = B_s(m, \rho + \rho') \).

Proof. First, let \( m' \in B_s(m, \rho) \trianglelefteq B_s(\rho') \). Then, there is an \( m'' \in B_s(m, \rho) \) such that \( m'' \leq m' \trianglelefteq B_s(\rho') \). Moreover, according to Corollary 5.14, we have \( m'' \trianglelefteq B_s(\rho') = B_s(m'', \rho') \). Hence, because \( d_s \) is subadditive, \( d_s(m', m') \leq d_s(m', m'') + d_s(m'', m') \leq \rho + \rho' \). Therefore, \( m' \in B_s(m, \rho + \rho') \). In conclusion, \( B_s(m, \rho) \trianglelefteq B_s(\rho') \subseteq B_s(m, \rho + \rho') \).

Secondly, let \( m' \in B_s(m, \rho) \trianglelefteq B_s(\rho') \).

**Case 1:** \( m' \in B_s(m, \rho) \). Then, because \( m_0 \in B_s(\rho') \), we have \( m' = m' \trianglelefteq m_0 \trianglelefteq B_s(m, \rho) \trianglelefteq B_s(\rho') \).

**Case 2:** \( m' \notin B_s(m, \rho) \). Then, there is a \( j \in \{ \rho + 1, \rho + 2, \ldots, \rho + \rho' \} \) and there is a \( \{s_i\}_{i \in \{1, \ldots, j\}} \subseteq S \) such that \( ((m'' \trianglelefteq s_{\rho+1}) \trianglelefteq s_{\rho+2}) \trianglelefteq \cdots \trianglelefteq s_{\rho+\rho'} = m' \), where \( m'' = ((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_\rho \in B_s(m, \rho) \). Hence, \( m' \in B_s(m'', \rho') = m'' \trianglelefteq B_s(\rho') \subseteq B_s(m, \rho) \trianglelefteq B_s(\rho') \).

In either case, \( m' \in B_s(m, \rho) \trianglelefteq B_s(\rho') \). In conclusion, \( B_s(m, \rho + \rho') \subseteq B_s(m, \rho) \trianglelefteq B_s(\rho') \).

Definition 5.16. Let \( A \) and \( A' \) be two subsets of \( M \). The non-negative number or infinity

\[
d_s(A, A') = \min\{d_s(a, a') \mid a \in A, a' \in A'\}
\]

is called distance of \( A \) and \( A' \), where we put \( \min \emptyset = \infty \). In the case that \( A = \{a\} \), we write \( d_s(a, A') \) in place of \( d_s(\{a\}, A') \); and in the case that \( A' = \{a'\} \), we write \( d_s(A, a') \) in place of \( d_s(A, \{a'\}) \).

Lemma 5.17. Let \( m \) and \( m' \) be two elements of \( M \), and let \( \rho \) be a non-negative integer such that \( \rho \leq d_s(m, m') \). Then, \( d_s(S_s(m, \rho), m') = d_s(m, m') - \rho \).

Proof. Let \( \rho' = d_s(m, m') \).

Then, there is a \( \{s_i\}_{i \in \{1, \ldots, \rho'\}} \) such that \( ((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_{\rho'} = m' \). Let \( m'' = ((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_{\rho} \). Then, \( ((m'' \trianglelefteq s_{\rho+1}) \trianglelefteq s_{\rho+2}) \trianglelefteq \cdots \trianglelefteq s_{\rho'} = m' \). And, according to Lemma 4.5, we have \( m'' \in S_s(m, \rho) \). Thus, \( d_s(S_s(m, \rho), m') \leq d_s(m'', m') \leq \rho' - \rho \).
Suppose that \( d_S(S_S(m, \rho), m') < \rho' - \rho \). Then, there is an \( m'' \in S_S(m, \rho) \) such that \( d_S(m'', m') < \rho' - \rho \). Hence, \( d_S(m, m') \leq d_S(m, m'') + d_S(m'', m') < \rho + (\rho' - \rho) = \rho' \), which contradicts \( d_S(m, m') = \rho' \). Therefore, \( d_S(S_S(m, \rho), m') \geq \rho' - \rho \).

In conclusion, \( d_S(S_S(m, \rho), m') = \rho' - \rho = d_S(m, m') - \rho \). \( \square \)

**Corollary 5.18.** Let \( m \) be an element of \( M \), and let \( \rho \) and \( \rho' \) be two non-negative integers such that the spheres \( S_S(m, \rho) \) and \( S_S(m, \rho') \) are non-empty. Then, \( d_S(S_S(m, \rho), S_S(m, \rho')) = |\rho - \rho'| \).

**Proof.** Without loss of generality, let \( \rho \leq \rho' \). Then, for each \( m' \in S_S(m, \rho') \), according to Lemma 5.17, we have \( d_S(S_S(m, \rho), m') = \rho' - \rho \). In conclusion, \( d_S(S_S(m, \rho), S_S(m, \rho')) = \rho' - \rho = |\rho - \rho'| \). \( \square \)

**Corollary 5.19.** Let \( m \) and \( m' \) be two elements of \( M \), and let \( \rho \) be a non-negative integer. Then, \( d_S(S_S(m, \rho), m') \geq |d_S(m, m') - \rho| \).

**Proof.** If \( S_S(m, \rho) = \emptyset \), then \( d_S(S_S(m, \rho), m') = \infty \geq |d_S(m, m') - \rho| \). Otherwise, let \( \rho' = d_S(m, m') \). Then, \( S_S(m, \rho') \neq \emptyset \). Hence, according to Corollary 5.18, we have \( d_S(S_S(m, \rho), m') \geq d_S(S_S(m, \rho), S_S(m, \rho')) = |\rho - \rho'| = |d_S(m, m') - \rho| \). \( \square \)

**Lemma 5.20.** Let \( m \) and \( m' \) be two elements of \( M \), and let \( \rho \) and \( \rho' \) be two non-negative integers such that \( \rho + \rho' \leq d_S(m, m') \). Then, \( d_S(B_S(m, \rho), B_S(m', \rho')) = d_S(m, m') - (\rho + \rho') \).

**Proof.** For each \( m_\rho \in B_S(m, \rho) \) and each \( m'_\rho \in B_S(m', \rho') \), because \( d_S \) is subadditive,

\[
d_S(m, m') \leq d_S(m, m_\rho) + d_S(m_\rho, m'_\rho) + d_S(m'_\rho, m') \\
\leq \rho + d_S(m_\rho, m'_\rho) + \rho',
\]

and hence \( d_S(m_\rho, m'_\rho) \geq d_S(m, m') - (\rho + \rho') \). Therefore, \( d_S(B_S(m, \rho), B_S(m', \rho')) \geq d_S(m, m') - (\rho + \rho') \).

Moreover, there is a \( \{s_i\}_{i \in \{1, 2, \ldots, d_S(m, m')\}} \) such that \( ((m \leq s_1) \leq s_2) \cdots \leq s_{d_S(m, m')} = m' \). Let \( m_\rho = ((m \leq s_1) \leq \cdots) \leq s_\rho \) and let \( m'_\rho = ((m_\rho \leq s_{\rho + 1}) \leq \cdots) \leq s_{d_S(m, m') - \rho} \). Then, \( m_\rho \in B_S(m, \rho) \) and, because \( ((m'_\rho \leq s_{d_S(m, m') - \rho} + 1) \leq s_{d_S(m, m') - \rho + 2} \cdots) \leq s_{d_S(m, m')} = m' \), we have \( m'_\rho \in \).
The set $A - \theta = A - B_S(\theta)$ is called $\theta$-interior of $A$.

(2) The set $A + \theta = A + B_S(\theta)$ is called $\theta$-closure of $A$.

(3) The set $\partial_\theta A = A + \theta \setminus A^{-\theta}$ is called $\theta$-boundary of $A$.

(4) The set $\partial_\theta^{-1} A = A \setminus A^{-\theta}$ is called internal $\theta$-boundary of $A$.

(5) The set $\partial_\theta^{+1} A = A^{+\theta} \setminus A$ is called external $\theta$-boundary of $A$.

**Lemma 6.2.** Let $A$ be a subset of $M$, and identify $M$ with $G/G_0$ by $[m \mapsto G_{m_0,m}]$. For each non-negative integer $\theta \in \mathbb{N}_0$,

(1) $A^{-\theta} = \{m \in A \mid B_S(m,\theta) \subseteq A\}$;
\( A^{+\theta} = \bigcup_{m \in A} B_S(m, \theta) = A \trianglelefteq B_S(\theta). \)

**Proof.** Let \( \theta \in \mathbb{N}_0 \) and let \( m \in M \).

1. According to Corollary 5.14,
\[
A^{-\theta} = \{ m \in M \mid B_S(m, \theta) \subseteq A \}.
\]

Therefore, because \( m \in B_S(m, \theta) \),
\[
A^{-\theta} = \{ m \in A \mid B_S(m, \theta) \subseteq A \}.
\]

2. According to Corollary 5.14,
\[
A^{+\theta} = \{ m \in M \mid (m \trianglelefteq B_S(\theta)) \cap A \neq \emptyset \}
\]
\[
= \{ m \in M \mid \exists m' \in A : m' \in m \trianglelefteq B_S(\theta) \}
\]
\[
= \{ m \in M \mid \exists m' \in A : m' \in B_S(m, \theta) \}.
\]

Moreover, because of the symmetry of \( d_S \), for each \( m' \in A \),
\[
m' \in B_S(m, \theta) \iff d_S(m, m') \leq \theta
\]
\[
\iff m \in B_S(m', \theta).
\]

Hence, according to Corollary 5.14,
\[
A^{+\theta} = \{ m \in M \mid \exists m' \in A : m \in B_S(m', \theta) \}
\]
\[
= \bigcup_{m' \in A} B_S(m', \theta)
\]
\[
= \bigcup_{m' \in A} m' \trianglelefteq B_S(\theta)
\]
\[
= A \trianglelefteq B_S(\theta). \quad \Box
\]

**Corollary 6.3.** Let \( m \) be an element of \( M \), let \( \rho \) be a non-negative integer, and let \( \theta \) be a non-negative integer. Then,

1. \( B_S(m, \rho)^{-\theta} \supseteq B_S(m, \rho - \theta); \)
2. \( B_S(m, \rho)^{+\theta} = B_S(m, \rho + \theta); \)
3. \( \partial B_S(m, \rho) \subseteq B_S(m, \rho + \theta) \setminus B_S(m, \rho - \theta). \)

**Proof.** (1) According to Corollary 5.15, we have \( B_S(m, \rho - \theta) \trianglelefteq B_S(\theta) \subseteq B_S(m, \rho) \). Hence, according to Definition 6.1, we have \( B_S(m, \rho - \theta) \subseteq B_S(m, \rho)^{-\theta}. \)
According to Item 2 of Lemma 6.2 and Corollary 5.15, we have $B_S(m, \rho + \theta) = B_S(m, \rho) \trianglelefteq B_S(\theta) = B_S(m, \rho + \theta)$.

(3) This is a direct consequence of Items 1 and 2. □

Lemma 6.4. Let $A$ be a subset of $M$, and let $\theta$ and $\theta'$ be two non-negative integers. The following statements hold:

1. $(A^{-\theta})^{-\theta'} = A^{-(\theta + \theta')}$;
2. $\partial^- A^{-\theta} = A^{-\theta} \setminus (A^{-\theta})^{-\theta'}$;
3. $(A^{+\theta})^{+\theta'} = A^{+(\theta + \theta')}$;
4. $\partial^- A^{+\theta} = A^{+(\theta + \theta')} \setminus A^{+\theta}$;
5. Let $\theta' \leq \theta$. Then, $A^{+(\theta - \theta')} \subseteq (A^{+\theta})^{-\theta'}$ and $(A^{-\theta})^{+\theta'} \subseteq A^{-(\theta - \theta')}.$

Proof. (1) For each $m' \in A$, according to Corollary 5.15 and Lemma 6.2, we have $m' \in A^{-\theta}$ if and only if $m' \in B_S(\theta) = B_S(m, \theta) \subseteq A$. Hence, according to Corollary 5.15,

$$(A^{-\theta})^{-\theta'} = \{m' \in A \mid B_S(m', \theta') \subseteq A^{-\theta}\}$$

$= \{m' \in A \mid B_S(m', \theta') \trianglelefteq B_S(\theta) \subseteq A\}$$

$= \{m' \in A \mid B_S(m', \theta + \theta') \subseteq A\}$$

$= A^{-(\theta + \theta')}.$

(2) According to Item 1,

$$\partial^- A^{-\theta} = A^{-\theta} \setminus (A^{-\theta})^{-\theta'}$$

$$= A^{-\theta} \setminus A^{-(\theta + \theta')}.$$

(3) According to Lemma 6.2 and Corollary 5.15,

$$(A^{+\theta})^{+\theta'} = A^{+\theta} \trianglelefteq B_S(\theta')$$

$$= \left(\bigcup_{m \in A} B_S(m, \theta)\right) \trianglelefteq B_S(\theta')$$

$$= \bigcup_{m \in A} B_S(m, \theta) \trianglelefteq B_S(\theta')$$

$$= \bigcup_{m \in A} B_S(m, \theta + \theta')$$

$$= A^{+(\theta + \theta')}.$$
(4) According to Item 3,
\[ \partial_\theta^+ A^{+\theta} = (A^{+\theta})^{+\theta'} \cap A^{+\theta} \]
\[ = A^{+(\theta+\theta')} \cap A^{+\theta}. \]

(5) According to Lemma 6.2 and Item 3,
\[ A^{+(\theta-\theta')} \subseteq B_S(\theta') = (A^{+(\theta-\theta')})^{+\theta'} \]
\[ = A^{+(\theta-\theta')+\theta'} \]
\[ = A^{+\theta}. \]

Thus, for each \( m \in A^{+(\theta-\theta')} \), according to Corollary 5.15, we have
\[ B_S(m, \theta') = m \subseteq B_S(\theta') \subseteq A^{+\theta} \]
and, in particular, \( m \in A^{+\theta} \). Therefore, according to Lemma 6.2, we have
\[ A^{+(\theta-\theta')} \subseteq (A^{+\theta})^{-\theta'}. \]

According to Lemma 6.2, Item 3, and Definition 6.1,
\[ (A^{-\theta}+\theta') \subseteq B_S(\theta-\theta') = ((A^{-\theta}+\theta')+(\theta-\theta')) \]
\[ = (A^{-\theta}+\theta'+(\theta-\theta')) \]
\[ = (A^{-\theta})^{+\theta} \]
\[ = A^{-\theta} \subseteq B_S(\theta) \]
\[ \subseteq A. \]

Therefore, according to Definition 6.1, we have \( (A^{-\theta})^{+\theta'} \subseteq A^{-(\theta-\theta')} \). \( \square \)

**Lemma 6.5.** Let \( k \) be a non-negative integer, and let \( A \) and \( A' \) be two subsets of \( M \). Then, \( d_S(A, A' \setminus A^{+k}) \geq k + 1 \).

**Proof.** If \( A \) or \( A' \setminus A^{+k} \) is empty, then \( d_S(A, A' \setminus A^{+k}) = \infty \geq k + 1 \). Otherwise, let \( m' \in A' \setminus A^{+k} \). According to Item 3 of Lemma 1, we have \( A' \setminus A^{+k} = (A' \setminus A)^{-k} \). Hence, according to Lemma 6.2, we have \( B_S(m', k) \subseteq A' \setminus A \). Therefore, for each \( m \in A \), we have \( m \notin B_S(m', k) \) and hence \( d_S(m, m') \geq k + 1 \). Thus, \( d_S(A, m') \geq k + 1 \). In conclusion, \( d_S(A, A' \setminus A^{+k}) \geq k + 1 \). \( \square \)

**Corollary 6.6.** Let \( k \) be a non-negative integer, let \( k' \) be a positive integer, and let \( A \) be a subset of \( M \). Then, \( d_S(A, \partial_{k'}^+ A^{+k}) \geq k + 1 \).
Proof. Because $\partial_k^+ A^{+k} = (A^{+k})^{+k'} \setminus A^{+k}$, this is a direct consequence of Lemma 6.5. \qed

Lemma 6.7. Let $A$ be a finite subset of $M$ and let $S'$ be the set $\{G_0\} \cup S$. There is a non-negative integer $k \in \mathbb{N}_0$ such that

$$A \subseteq \{ m \in M \mid \exists \{s'_i\}_{i \in \{1,2,\ldots,k\}} \subseteq S' : (((m_0 \trianglelefteq s'_1) \trianglelefteq s'_2) \trianglerighteq \cdots) \trianglelefteq s'_k \}.$$

Proof. If $A$ is empty, then any $k \in \mathbb{N}_0$ works. Otherwise, let $k = \max_{a \in A} d_S(m_0, a)$. Because $A$ is finite, we have $k \in \mathbb{N}_0$. By the choice of $k$, we have $A \subseteq B(m_0, k)$. And, because $G_0 \in S'$ and $\_ \equiv G_0 = id_M$, we have $B(m_0, k) = \{ m \in M \mid \exists \{s'_i\}_{i \in \{1,2,\ldots,k\}} \subseteq S' : (((m_0 \trianglelefteq s'_1) \trianglelefteq s'_2) \trianglerighteq \cdots) \trianglelefteq s'_k \}$. In conclusion, the stated inclusion holds. \qed

7. Growth Functions And Types

In this section we recapitulate growth functions and types, more or less as presented in the monograph ‘Cellular Automata and Groups’[2].

Definition 7.1. Let $\gamma$ be a map from $\mathbb{N}_0$ to $\mathbb{R}_{\geq 0}$. It is called growth function if and only if it is non-decreasing, that is to say, that

$$\forall k \in \mathbb{N}_0 \forall k' \in \mathbb{N}_0 : (k \leq k' \implies \gamma(k) \leq \gamma(k')).$$

Definition 7.2. Let $\gamma$ and $\gamma'$ be two growth functions. The growth function $\gamma$ is said to dominate $\gamma'$ and we write $\gamma \succeq \gamma'$ if and only if

$$\exists \alpha \in \mathbb{N}_+ : \forall k \in \mathbb{N}_+ : \alpha \cdot \gamma(\alpha \cdot k) \geq \gamma'(k).$$

Definition 7.3. Let $\gamma$ and $\gamma'$ be two growth functions. They are called equivalent and we write $\gamma \sim \gamma'$ if and only if $\gamma \succeq \gamma'$ and $\gamma' \succeq \gamma$.

Lemma 7.4 ([2 Proposition 6.4.3]).

(1) The relation $\succeq$ is reflexive and transitive.
(2) The relation $\sim$ is an equivalence relation.
(3) If $\gamma_1 \sim \gamma_2$ and $\gamma'_1 \sim \gamma'_2$, then $\gamma_1 \succeq \gamma'_1$ implies $\gamma_2 \succeq \gamma'_2$.

Definition 7.5. Let $\gamma$ be a growth function. The equivalence class of $\gamma$ with respect to $\sim$ is denoted by $[\gamma]_\sim$ and called growth type.
Definition 7.6. Let $\Gamma$ and $\Gamma'$ be two growth types. The growth type $\Gamma$ is said to dominate $\Gamma'$ and we write $\Gamma \succeq \Gamma'$ if and only if

$$\exists \gamma \in \Gamma \exists \gamma' \in \Gamma' : \gamma \succ \gamma'.$$

Example 7.7 ([2] Examples 6.4.4)).

(1) The growth function $[k \mapsto k]$ dominates $1$ but they are not equivalent.

Proof. For each $k \in \mathbb{N}_+$, we have $k \geq 1(k)$. But, for each $\alpha \in \mathbb{N}_+$, there is a $k \in \mathbb{N}_+$, for example $k = \alpha + 1$, such that $\alpha 1(\alpha k) = \alpha < k$.\]

(2) Let $r$ and $s$ be two non-negative real numbers. Then, $[k \mapsto k^r] \succeq [k \mapsto k^s]$ if and only if $r \geq s$. And, $[k \mapsto k^r] \sim [k \mapsto k^s]$ if and only if $r = s$.

(3) Let $\gamma$ be a growth function such that it is a polynomial function of degree $d \in \mathbb{N}_0$. Then, $\gamma \sim [k \mapsto k^d]$.

(4) Let $r$ and $s$ be two elements of $\mathbb{R}_{>1}$. Then, $[k \mapsto r^k] \sim [k \mapsto s^k]$. In particular, $[k \mapsto r^k] \sim \exp$.

Proof. Without loss of generality, suppose that $r \leq s$. Then, for each $k \in \mathbb{N}_+$, we have $r^k \leq s^k$. Hence, $[k \mapsto r^k] \preceq [k \mapsto s^k]$. Moreover, let $\alpha = \lfloor \log r \log s \rfloor \in \mathbb{N}_+$. Then, for each $k \in \mathbb{N}_+$,

$$s^k = (r^\log r s^\log s)^k = r^\log r s^\log s k \leq r^\alpha k \leq \alpha r^\alpha k.$$

Hence, $[k \mapsto r^k] \succeq [k \mapsto s^k]$. In conclusion, $[k \mapsto r^k] \sim [k \mapsto s^k]$.\]

(5) Let $d$ be a non-negative integer. Then, $\exp \succeq [k \mapsto k^d]$ and $[k \mapsto k^d] \sim \exp$.

Proof. See [2] Examples 6.4.4 (d)].\]

Lemma 7.8. Let $\gamma$ be a growth function and let $d$ be a non-negative integer such that $[k \mapsto k^d] \succ \gamma$. Then, $\exp \succeq \gamma$ and $\exp \sim \gamma$.

Proof. According to Item 5 of Example 7.7, we have $\exp \succeq [k \mapsto k^d]$ and $\exp \sim [k \mapsto k^d]$. Hence, because $\succeq$ is transitive and $[k \mapsto k^d] \succ \gamma$, we have $\exp \succeq \gamma$ and $\exp \sim \gamma$.\]
8. Cell Spaces’ Growth Functions and Types

In this section, let \( R = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M})) \) be a cell space such that there is a finite and symmetric right generating set \( S \) of \( R \).

In Definition 8.1, we define the \( S \)-growth function \( \gamma_S \) of \( R \). In Lemma 8.3 and its corollaries we show that \( \gamma_S \) is dominated by \( \exp \) and that the \( \sim \)-equivalence class \([\gamma_S]_\sim\) does not depend on \( S \). In Definition 8.10 we define the growth type \( \gamma(R) \) of \( R \) as that equivalence class.

In Lemma 8.13 and its corollary we relate the inclusion-behavior of the sequence of balls to the cardinality of \( M \). And in Definition 8.17 we define the terms ‘exponential growth’, ‘sub-exponential growth’, ‘polynomial growth’, and ‘intermediate growth of \( R \)’.

**Definition 8.1.** The map
\[
\gamma_S : \mathbb{N}_0 \to \mathbb{N}_0, \\
k \mapsto |B_S(k)|,
\]
is called \( S \)-growth function of \( R \).

**Remark 8.2.** According to Remark 5.6, we have \( \gamma_S(0) = 1 \) and the sequence \((\gamma_S(k))_{k \in \mathbb{N}_0}\) is non-decreasing with respect to the partial order \( \leq \). Moreover, according to Remark 5.7, for each non-negative integer \( k \in \mathbb{N}_0 \), we have \( \gamma_S(k) \leq (1 + |S|)^k \).

**Lemma 8.3.** Let \( S' \) be a finite and symmetric right generating set of \( R \) and let \( \alpha \) be the non-negative integer \( \min\{k \in \mathbb{N}_0 \mid B_S(1) \subseteq B_{S'}(k)\} \). Then,
\[
\forall m \in M \forall m' \in M : d_{S'}(m, m') \leq \alpha \cdot d_S(m, m'),
\]
in particular,
\[
\forall m \in M : |m|_{S'} \leq \alpha \cdot |m|_S.
\]

**Proof.** For each \( m \in M \), let \( \alpha_m = \min\{k \in \mathbb{N}_0 \mid B_S(m, 1) \subseteq B_{S'}(m, k)\} \), in particular, \( \alpha_{m_0} = \alpha \).

**Proof of:** \( \forall m \in M : \alpha_m = \alpha \). Let \( m \in M \), let \( k \in \mathbb{N}_0 \), and let \( g \in G_{m_0, m} \). Then, because \( g \triangleright \_ \) is bijective, \( B_S(1) \subseteq B_{S'}(k) \) if and only if \( g \triangleright B_S(1) \subseteq g \triangleright B_{S'}(k) \). Moreover, according to Lemma 5.9, we have \( g \triangleright B_S(1) = B_S(m, 1) \) and \( g \triangleright B_{S'}(k) = B_{S'}(m, k) \). Therefore,
$\mathcal{B}_S(1) \subseteq \mathcal{B}_{S'}(k)$ if and only if $\mathcal{B}_S(m, 1) \subseteq \mathcal{B}_{S'}(m, k)$. In conclusion, $\alpha_m = \alpha$.

Proof. By induction on the distance, that is, proof by induction on $k$ of

$$\forall k \in \mathbb{N}_0 \forall m \in M \forall m' \in M : \quad (d_S(m, m') = k \implies d_{S'}(m, m') \leq \alpha \cdot k).$$

*Base Case.* Let $k = 0$. Furthermore, let $m$ and $m' \in M$ such that $d_S(m, m') = k$. Then, $m = m'$. Hence, $d_{S'}(m, m') = 0$. Therefore, $d_{S'}(m, m') \leq \alpha \cdot k$.

*Inductive Step.* Let $k \in \mathbb{N}_0$ such that

$$\forall m \in M \forall m' \in M : (d_S(m, m') = k \implies d_{S'}(m, m') \leq \alpha \cdot k).$$

Furthermore, let $m$ and $m'' \in M$ such that $d_S(m, m'') = k + 1$. Then, there is a $\{s_i\}_{i=1, \ldots, k+1} \subseteq S$ such that $m' \trianglelefteq s_{k+1} = m''$, where $m' = ((m \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots \trianglelefteq s_k$. And, according to Lemma 4.5, we have $d_S(m, m') = k$. Therefore, according to the inductive hypothesis, $d_{S'}(m, m') \leq \alpha \cdot k$. Moreover, by definition of $\alpha_{m'}$, we have $m'' = m' \trianglelefteq s_{k+1} \in \mathcal{B}_S(m', 1) \subseteq \mathcal{B}_{S'}(m', \alpha_{m'})$. Hence, because $\alpha_{m'} = \alpha$, we have $d_{S'}(m', m'') \leq \alpha_{m'} = \alpha$. In conclusion, because $d_{S'}$ is subadditive, $d_{S'}(m, m'') \leq d_{S'}(m, m') + d_{S'}(m', m'') \leq \alpha \cdot k + \alpha = \alpha \cdot (k + 1)$. \hfill $\Box$

**Corollary 8.4.** In the situation of Lemma 8.3, for each element $m \in M$ and each non-negative integer $k \in \mathbb{N}_0$, we have $\mathcal{B}_S(m, k) \subseteq \mathcal{B}_{S'}(m, \alpha \cdot k)$.

Proof. This is a direct consequence of Lemma 8.3, because for each element $m \in M$, each non-negative integer $k \in \mathbb{N}_0$, and each element $m' \in M$, if $d_S(m, m') \leq k$, then $d_{S'}(m, m') \leq \alpha \cdot k$. \hfill $\Box$

**Corollary 8.5.** In the situation of Lemma 8.3, for each non-negative integer $k \in \mathbb{N}_0$, we have $\gamma_S(k) \leq \gamma_{S'}(\alpha \cdot k)$.

Proof. This is a direct consequence of Corollary 8.4. \hfill $\Box$

**Definition 8.6.** Let $X$ be a set, and let $d$ and $d'$ be metrics on $X$. The metrics $d$ and $d'$ are called *Lipschitz equivalent* if and only if there are positive real numbers $\kappa$ and $\varkappa$ such that $\kappa \cdot d \leq d' \leq \varkappa \cdot d$. 
Corollary 8.7. Let $S'$ be a finite and symmetric right generating set of $\mathcal{R}$. The metrics $d_S$ and $d_{S'}$ are Lipschitz equivalent.

Proof. Let $\alpha = \min\{k \in \mathbb{N}_0 \mid \mathbb{B}_S(1) \subseteq \mathbb{B}_{S'}(k)\}$ and let $\alpha' = \min\{k \in \mathbb{N}_0 \mid \mathbb{B}_{S'}(1) \subseteq \mathbb{B}_S(k)\}$. If $\alpha = 0$ or $\alpha' = 0$, then $M = \{m_0\}$, hence $d_S = 0 = d_{S'}$, and therefore $d_S \leq d_{S'} \leq d_S$. Otherwise, according to Lemma 8.3, we have $\frac{1}{\alpha} \cdot d_{S'} \leq d_S \leq \alpha' \cdot d_{S'}$. \hfill $\square$

Corollary 8.8. Let $S'$ be a finite and symmetric right generating set of $\mathcal{R}$. The $S$-growth function $\gamma_S$ of $\mathcal{R}$ and the $S'$-growth function $\gamma_{S'}$ of $\mathcal{R}$ are equivalent.

Proof. According to Corollary 8.5, there is a $\alpha \in \mathbb{N}_0$ such that, for each $k \in \mathbb{N}_0$, we have $\gamma_S(k) \leq \gamma_{S'}(\alpha \cdot k)$. Hence, according to Remark 8.2, for each $k \in \mathbb{N}_0$, we have $\gamma_S(k) \leq (\alpha + 1)\gamma_{S'}((\alpha + 1) \cdot k)$. Therefore, $\gamma_S$ is dominated by $\gamma_{S'}$. Switching roles of $S$ and $S'$ yields that $\gamma_{S'}$ is dominated by $\gamma_S$. In conclusion, $\gamma_S$ and $\gamma_{S'}$ are equivalent. \hfill $\square$

Corollary 8.9. The $S$-growth function $\gamma_S$ of $\mathcal{R}$ is dominated by exp.

Proof. According to Remark 8.2, for each $k \in \mathbb{N}_0$, we have $\gamma_S(k) \leq r^k$, where $r = 1 + |S|$. Hence, $\gamma_S \preceq [k \mapsto r^k]$. Moreover, according to Item 4 of Example 7.7, we have $[k \mapsto r^k] \sim \exp$. In conclusion, $\gamma_S \preceq \exp$. \hfill $\square$

Definition 8.10. The equivalence class $\gamma(\mathcal{R}) = [\gamma_S]_\sim$ is called growth type of $\mathcal{R}$.

Lemma 8.11 ([2 Proposition 6.4.6]). Let $\gamma$ be a growth function such that $\gamma(0) > 0$. Then, $\gamma$ is equivalent to 1 if and only if $\gamma$ is bounded.

Corollary 8.12. The set $M$ is finite if and only if the growth types $\gamma(\mathcal{R})$ and $[1]_\sim$ are equal.

Proof. First, let $M$ be finite. Then, for each $k \in \mathbb{N}_0$, we have $\gamma_S(k) \leq |M| = |M| \cdot 1(|M| \cdot k)$.

Secondly, let $\gamma(\mathcal{R}) = [1]_\sim$. Then, according to Lemma 8.11, $\gamma_S$ is bounded by some $\xi \in \mathbb{R}_{>0}$. Therefore, because $M = \bigcup_{k \in \mathbb{N}_0} \mathbb{B}_S(k)$, $(\mathbb{B}_S(k))_{k \in \mathbb{N}_0}$ is non-decreasing with respect to $\subseteq$, and $(\gamma_S(k))_{k \in \mathbb{N}_0} = ((\mathbb{B}_S(k)))_{k \in \mathbb{N}_0}$, we have $|M| \leq \sup_{k \in \mathbb{N}_0} \gamma_S(k) \leq \xi$. In conclusion, $M$ is finite. \hfill $\square$
Lemma 8.13. Either the sequence $(B_S(k))_{k \in \mathbb{N}_0}$ is strictly increasing with respect to $\subseteq$ or eventually constant, that is to say, that there is a non-negative integer $k \in \mathbb{N}_0$ such that, for each non-negative integer $k' \in \mathbb{N}_0$ with $k' \geq k$, we have $B_S(k') = B_S(k)$.

Proof. According to Remark 5.6, the sequence $(B_S(k))_{k \in \mathbb{N}_0}$ is non-decreasing with respect to $\subseteq$. If it is strictly increasing with respect to $\subseteq$, it is not eventually constant. Otherwise, there is a $k \in \mathbb{N}_0$ such that $B_S(k) = B_S(k+1)$. We proof by induction on $k'$ that, for each $k' \in \mathbb{N}_0$ with $k' \geq k$, we have $B_S(k') = B_S(k)$.

Base Case. Let $k' = k$. Then, $B_S(k') = B_S(k)$. 

Inductive Step. Let $k' \in \mathbb{N}_0$ with $k' \geq k$ such that $B_S(k') = B_S(k)$. Furthermore, let $m \in B_S(k'+1)$.

Case 1: $m \in B_S(k')$. Then, according to the inductive hypothesis, $m \in B_S(k)$.

Case 2: $m \notin B_S(k')$. Then, there is a $\{s_i\}_{i \in \{1,2,\ldots,k'+1\}} \subseteq S$ such that $m' \trianglelefteq s_{k'+1} = m$, where $m' = (((m_0 \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_{k'}$. Hence, $m' \in B_S(k')$ and thus, according to the inductive hypothesis, $m' \in B_S(k)$. Therefore, according to Lemma 5.8, we have $m \in B_S(k+1)$. Thus, because $B_S(k+1) = B_S(k)$, we have $m \in B_S(k)$.

In either case, $m \in B_S(k)$. Therefore, $B_S(k'+1) \subseteq B_S(k) \subseteq B_S(k') \subseteq B_S(k'+1)$. In conclusion, $B_S(k'+1) = B_S(k)$.

In conclusion, $(B_S(k))_{k \in \mathbb{N}_0}$ is eventually constant. 

Corollary 8.14. The set $M$ is infinite if and only if the sequence $(B_S(k))_{k \in \mathbb{N}_0}$ is strictly increasing with respect to $\subseteq$.

Proof. First, let $M$ be infinite. Suppose that $(B_S(k))_{k \in \mathbb{N}_0}$ is eventually constant. Then, there is a $k \in \mathbb{N}_0$ such that, for each $k' \in \mathbb{N}_0$ with $k' \geq k$, we have $B_S(k') = B_S(k)$. Hence, according to Remark 5.6, we have $M = \bigcup_{k' \in \mathbb{N}_0, k' \geq k} B_S(k') = B_S(k)$ and therefore, according to Remark 5.7, the set $M$ is finite, which contradicts the precondition that $M$ is infinite. Thus, $(B_S(k))_{k \in \mathbb{N}_0}$ is not eventually constant. In conclusion, according to Lemma 8.13, the sequence $(B_S(k))_{k \in \mathbb{N}_0}$ is strictly increasing with respect to $\subseteq$.
Secondly, let $(B_S(k))_{k \in \mathbb{N}_0}$ be strictly increasing with respect to $\subseteq$. Then, because $M = \bigcup_{k \in \mathbb{N}_0} B_S(k)$, the set $M$ is infinite. \hfill $\Box$

**Corollary 8.15.** The set $M$ is infinite if and only if

\[(1) \quad \forall \rho \in \mathbb{N}_0 : S_S(\rho) \neq \emptyset.\]

**Proof.** We have $S_S(0) = \{m_0\} \neq \emptyset$. And, according to Remark 5.2, for each $\rho \in \mathbb{N}_+$, we have $S_S(\rho) = B_S(\rho) \setminus B_S(\rho - 1)$. Hence, $(B_S(k))_{k \in \mathbb{N}_0}$ is strictly increasing with respect to $\subseteq$ if and only if Eq. (1) holds. Therefore, according to Corollary 8.14, the set $M$ is infinite if and only if Eq. (1) holds. \hfill $\Box$

**Lemma 8.16.** The set $M$ is infinite if and only if the growth type of $R$ dominates $[k \mapsto k]_\sim$.

**Proof.** First, let $M$ be infinite. Then, according to Corollary 8.14, the sequence $(B_S(k))_{k \in \mathbb{N}_0}$ is strictly increasing with respect to $\subseteq$. Hence, because $B_S(0) = \{m_0\}$, for each $k \in \mathbb{N}_0$, we have $\gamma_S(k) = |B_S(k)| \geq k + 1$. In conclusion, $\gamma_S$ dominates $[k \mapsto k]$ and hence $\gamma(R)$ dominates $[k \mapsto k]_\sim$.

Secondly, let $M$ be finite. Then, according to Corollary 8.12, we have $R = [1]_\sim$. Hence, according to Item 1 of Example 7.7, the cell space $R$ does not dominate $[k \mapsto k]_\sim$. \hfill $\Box$

**Definition 8.17.** The cell space $R$ is said to have

1. **exponential growth** if and only if its growth type $\gamma(R)$ is equal to $[\exp]_\sim$;
2. **sub-exponential growth** if and only if it does not have exponential growth;
3. **polynomial growth** if and only if there is a non-negative integer $d \in \mathbb{N}_0$ such that $\gamma_S$ is dominated by $[k \mapsto k^d]$.
4. **intermediate growth** if and only if it has sub-exponential growth but not polynomial growth.

**Lemma 8.18.** Let $R$ have polynomial growth. The cell space $R$ has sub-exponential growth.

**Proof.** There is a $d \in \mathbb{N}_0$ such that $[k \mapsto k^d] \succcurlyeq \gamma_S$. Hence, according to Lemma 8.8, $\gamma_S \sim \exp$. In conclusion, $\gamma(R) \neq [\exp]_\sim$. \hfill $\Box$
9. Growth Rates

In this section, let \( R = ((M, G, \cdot), (m_0, \{ g_{m_0, m} \}_{m \in M})) \) be a cell space such that there is a finite and symmetric right generating set \( S \) of \( R \).

In Definition 9.2, we define the \( S \)-growth rate of \( R \). And in Lemma 9.3, we show how that growth rate and exponential growth relate to each other.

**Lemma 9.1.** The sequence \( (\sqrt[k]{\gamma_S(k)})_{k \in \mathbb{N}_0} \) converges to \( \inf_{k \in \mathbb{N}_0} \sqrt[k]{\gamma_S(k)} \in \mathbb{R}_{\geq 1} \).

**Proof.** According to Corollary 5.15,

\[
\gamma_S(k + k') = |B_S(k + k')| \\
= |B_S(k) \trianglelefteq B_S(k')| \\
\leq |B_S(k)| \cdot |B_S(k')| \\
= \gamma_S(k) \cdot \gamma_S(k').
\]

Hence, according to [2, Lemma 6.5.1], the sequence \( (\sqrt[k]{\gamma_S(k)})_{k \in \mathbb{N}_0} \) converges to \( \inf_{k \in \mathbb{N}_0} \sqrt[k]{\gamma_S(k)} \). Moreover, because, for each \( k \in \mathbb{N}_0 \), we have \( \gamma_S(k) \geq 1 \), that limit point must be in \( \mathbb{R}_{\geq 1} \). \( \square \)

**Definition 9.2.** The limit point \( \lambda_S = \lim_{k \to \infty} \sqrt[k]{\gamma_S(k)} \) is called \( S \)-growth rate of \( R \).

**Lemma 9.3.** The \( S \)-growth rate \( \lambda_S \) of \( R \) is greater than 1 if and only if the cell space \( R \) has exponential growth.

**Proof.** First, let \( \lambda_S > 1 \). According to Lemma 9.1, for each \( k \in \mathbb{N}_0 \), we have \( \sqrt[k]{\gamma_S(k)} \geq \lambda_S \) and hence \( \gamma_S(k) \geq \lambda_S^k \). Therefore, \( \gamma_S \) dominates \( \lambda_S \) and, because \( \lambda_S > 1 \), the growth function \( \lambda_S \) is equivalent to exp, and thus \( \gamma_S \) dominates exp. Moreover, according to Corollary 8.9, the growth function \( \gamma_S \) is dominated by \( \exp \). Altogether, \( \gamma_S \) is equivalent to \( \exp \). In conclusion, \( \gamma(R) = [\gamma_S]_{\sim} = [\exp]_{\sim} \).

Secondly, let \( \gamma(R) = [\exp]_{\sim} \). Then, \( \gamma_S \) and \( \exp \) are equivalent. In particular, \( \gamma_S \) dominates \( \exp \). Hence, there is a \( \alpha \in \mathbb{N}_+ \) such that, for each \( k \in \mathbb{N}_+ \), we have \( \alpha \gamma_S(\alpha k) \geq \exp(k) \). Therefore, for each \( k \in \mathbb{N}_+ \),

\[
\alpha^{\sqrt[\alpha]{\alpha}} \sqrt[\alpha]{\gamma_S(\alpha k)} = \alpha^{\sqrt[\alpha]{\alpha \gamma_S(\alpha k)}} \\
\geq \alpha^{\sqrt[\alpha]{\exp(k)}}
\]
Thus, because \((\alpha k)^{\sqrt{\alpha}}\) converges to 1 and \((\alpha k^{\gamma_s(k)})\) as subsequence of \((\sqrt{\gamma_s(k)})\), converges to \(\lambda_S\), we conclude \(\lambda_S \geq \sqrt{e} > 1\).

\[\Box\]

**Corollary 9.4.** The S-growth rate \(\lambda_S\) of \(\mathcal{R}\) is equal to 1 if and only if the cell space \(\mathcal{R}\) has sub-exponential growth.

**Proof.** This is a direct consequence of Lemma 9.3. \(\Box\)

**Corollary 9.5.** Let \(S'\) be a finite and symmetric right generating set of \(\mathcal{R}\). The S-growth rate \(\lambda_S\) of \(\mathcal{R}\) is equal to 1 or greater than 1 if and only if the \(S'\)-growth rate \(\lambda_{S'}\) of \(\mathcal{R}\) is equal to 1 or greater than 1 respectively.

**Proof.** This is a direct consequence of Corollary 9.4 and Lemma 9.3. \(\Box\)

### 10. Amenability, Følner Conditions/Nets, and Isoperimetric Constants

In this section, let \(\mathcal{R} = (\langle M, G, \triangleright = \rangle, (m_0, \{g_{m_0,m}\}_{m \in M}))\) be a finitely right generated cell space such that the stabiliser \(G_0\) is finite, and let \(S\) be a finite and symmetric right generating set of \(\mathcal{R}\).

In Definition 10.3 we define the S-isoperimetric constant of \(\mathcal{R}\), which measures, broadly speaking, the invariance under \(\ll_l \triangleright = \rr_r\) that a finite subset of \(M\) can have, where 0 means maximally and 1 minimally invariant. In Theorem 10.5 we show that \(\mathcal{R}\) is right amenable if and only if a kind of Følner condition holds, which in turn holds if and only if the S-isoperimetric constant is 0. And in Theorem 10.6 we characterise right Følner nets using \(\rho\)-boundaries.

**Remark 10.1.** Let \(g\) and \(g'\) be two elements of \(G/G_0\), and let \(A, B,\) and \(C\) be three sets. Then,

\[(\_ \ll g)^{-1}(A \setminus B) = (\_ \ll g)^{-1}(A) \setminus (\_ \ll g)^{-1}(B)\]

and

\[((\_ \ll g) \ll g')^{-1}(A) = (\_ \ll g)^{-1}((\_ \ll g')^{-1}(A)).\]
Remark 10.2. Let $A$, $B$, and $C$ be three finite sets. Then,
\[|A \setminus B| \leq |A \setminus C| + |C \setminus B|.|\]

Definition 10.3. Let $E$ be a subset of $G/G_0$ and let $\mathcal{F}$ be the set
\[\{ F \subseteq M \mid F \text{ is non-empty and finite} \}. \]
The non-negative real number
\[\iota_E(\mathcal{R}) = \inf_{F \in \mathcal{F}} \frac{|\bigcup_{e \in E} F \setminus (_{-} \trianglelefteq e)^{-1}(F)|}{|F|}\]
is called $E$-isoperimetric constant of $\mathcal{R}$.

Lemma 10.4. Let $A$ be a subset of $M$, let $g$ and $g'$ be two elements of $G/G_0$, and identify $M$ with $G/G_0$ by $[m \mapsto G_{m_0,m}]$. Then,
\[( _{-} \trianglelefteq g)^{-1}(A) \setminus ( _{-} \trianglelefteq (g \trianglelefteq g'))^{-1}(A) \subseteq \bigcup_{g_0 \in G_0} ( _{-} \trianglelefteq g)^{-1}(A \setminus ( _{-} \trianglelefteq g_0 \cdot g')^{-1}(A)). \]

Proof. Let $m \in ( _{-} \trianglelefteq g)^{-1}(A) \setminus ( _{-} \trianglelefteq (g \trianglelefteq g'))^{-1}(A)$. Then, according to Lemma 5.13, there is a $g_0 \in G_0$ such that $(m \trianglelefteq g) \trianglelefteq g_0 \cdot g' = m \trianglelefteq (g \trianglelefteq g') \notin A$. Therefore, $m \notin (( _{-} \trianglelefteq g) \trianglelefteq g_0 \cdot g')^{-1}(A)$ and hence $m \in ( _{-} \trianglelefteq g)^{-1}(A) \setminus (( _{-} \trianglelefteq g) \trianglelefteq g_0 \cdot g')^{-1}(A)$. Moreover, according to Remark 10.1, we have $(( _{-} \trianglelefteq g) \trianglelefteq g_0 \cdot g')^{-1}(A) = ( _{-} \trianglelefteq g)^{-1}(( _{-} \trianglelefteq g_0 \cdot g')^{-1}(A))$. Thus, according to Remark 10.1, we have $( _{-} \trianglelefteq g)^{-1}(A) \setminus (( _{-} \trianglelefteq g) \trianglelefteq g_0 \cdot g')^{-1}(A) = ( _{-} \trianglelefteq g)^{-1}(A \setminus ( _{-} \trianglelefteq g_0 \cdot g')^{-1}(A))$. Hence, $m \in \bigcup_{g_0 \in G_0} ( _{-} \trianglelefteq g)^{-1}(A \setminus ( _{-} \trianglelefteq g_0 \cdot g')^{-1}(A))$. \hfill \Box

Theorem 10.5. The following statements are equivalent:

1. The cell space $\mathcal{R}$ is right amenable;
2. For each positive real number $\varepsilon \in \mathbb{R}_{>0}$, there is a non-empty and finite subset $F$ of $M$ such that
   \[\forall s \in S : \frac{|F \setminus ( _{-} \trianglelefteq s)^{-1}(F)|}{|F|} < \varepsilon;\]
   (2)
3. The isoperimetric constant $\iota_S(\mathcal{R})$ is 0.

Proof. \(\square) \implies \boxed{\mathbb{1}}\) Let $\mathcal{R}$ be right amenable. Then, according to [7, Main Theorem 4], there is a right Følner net in $\mathcal{R}$. Hence, according to [7, Lemma 9], for each $\varepsilon \in \mathbb{R}_{>0}$, there is a non-empty and finite $F \subseteq M$ such that Eq. (2) holds.

\boxed{\mathbb{1}} \implies \boxed{\mathbb{2}}\) For each $\varepsilon \in \mathbb{R}_{>0}$, let there be a non-empty and finite $F \subseteq M$ such that Eq. (2) holds. Furthermore, let $\varepsilon' \in \mathbb{R}_{>0}$, let $E \subseteq G/G_0$ be
finite, and identify $M$ with $G/G_0$ by $[m \mapsto G_{m_0,m}]$. Then, according to Lemma 6.7, there is a $k \in \mathbb{N}_0$ such that

$$E \subseteq \{m \in M \mid \exists \{s'_i\}_{i \in \{1,2,\ldots,k\}} \subseteq S' : (((m_0 \not\asymp s'_1) \not\asymp s'_2) \not\asymp \cdots) \not\asymp s'_k\},$$

where $S' = \{G_0\} \cup S$. Let $\varepsilon = \varepsilon'/(|G_0|^2 \cdot k)$ and let $F \subseteq M$ be non-empty and finite such that Eq. (2) holds. Furthermore, let $e \in E$. Then, there is a $\{s'_i\}_{i \in \{1,2,\ldots,k\}} \subseteq S'$ such that $(((m_0 \not\asymp s'_1) \not\asymp s'_2) \not\asymp \cdots) \not\asymp s'_k = e$. For each $i \in \{0,1,\ldots,k\}$, let $m_i = (((m_0 \not\asymp s'_1) \not\asymp s'_2) \not\asymp \cdots) \not\asymp s'_i$ and let $F_i = (\_ \not\asymp m_i)^{-1}(F)$. Note that $m_k = e$ and that $F_0 = F$. Then, according to Remark 10.2,

$$|F \setminus F_k| = |F_0 \setminus F_k| \leq |F_0 \setminus F_1| + |F_1 \setminus F_k| \leq |F_0 \setminus F_1| + |F_1 \setminus F_2| + |F_2 \setminus F_k| \leq \ldots \leq \sum_{i=1}^{k} |F_{i-1} \setminus F_i|.$$

Let $i \in \{1,2,\ldots,k\}$. Then, because $m_i = m_{i-1} \not\asymp s'_i$, we have $F_{i-1} \setminus F_i = (\_ \not\asymp m_{i-1})^{-1}(F) \setminus (\_ \not\asymp (m_{i-1} \not\asymp s'_i))^{-1}(F)$. Hence, according to Lemma 10.4, we have $F_{i-1} \setminus F_i \subseteq \bigcup_{g_0 \in G_0} (\_ \not\asymp m_{i-1})^{-1}(F) \setminus (\_ \not\asymp g_0 \cdot s'_i)^{-1}(F))$. Therefore,

$$|F_{i-1} \setminus F_i| \leq \left| \bigcup_{g_0 \in G_0} (\_ \not\asymp m_{i-1})^{-1}(F) \setminus (\_ \not\asymp g_0 \cdot s'_i)^{-1}(F)\right| \leq \sum_{g_0 \in G_0} \left| ((\_ \not\asymp m_{i-1})^{-1}(F) \setminus (\_ \not\asymp g_0 \cdot s'_i)^{-1}(F)) \right|.$$

Thus, according to [3] Corollary 1,

$$|F_{i-1} \setminus F_i| \leq \sum_{g_0 \in G_0} |G_0| \cdot |F \setminus (\_ \not\asymp g_0 \cdot s'_i)^{-1}(F)|.$$

Hence, because $G_0 \cdot S' \subseteq S'$, $F \setminus (\_ \not\asymp G_0)^{-1}(F) = F \setminus F = \emptyset$, and Eq. (2) holds,

$$|F_{i-1} \setminus F_i| < \sum_{g_0 \in G_0} |G_0| \cdot \varepsilon \cdot |F| = |G_0|^2 \cdot \varepsilon \cdot |F|.$$
Therefore,

\[ |F \setminus F_k| \leq \sum_{i=1}^{k} |F_{i-1} \setminus F_i| < k \frac{\varepsilon'}{k} |F| = \varepsilon' |F|. \]

Thus, because \( F_k = (\_ \trianglelefteq e)^{-1}(F) \),

\[ \frac{|F \setminus (\_ \trianglelefteq e)^{-1}(F)|}{|F|} < \varepsilon'. \]

Hence, according to [7, Lemma 3.9], there is a right Følner net in \( \mathcal{R} \).

In conclusion, according to [7, Theorem 5.1], the cell space \( \mathcal{R} \) is right amenable.

Let \( \varepsilon' \in \mathbb{R}_{>0} \) and let \( \varepsilon = \varepsilon' / |S| \). Then, there is a non-empty and finite \( F \subseteq M \) such that Eq. (2) holds. Therefore,

\[ \lim_{i \in I} |\partial_\rho F_i| / |F_i| = 0. \]

In conclusion, \( \iota_S(\mathcal{R}) = 0 \).

Let \( \varepsilon \in \mathbb{R}_{>0} \). Then, because \( \iota_S(\mathcal{R}) = 0 \), there is a non-empty and finite \( F \subseteq M \) such that

\[ \frac{|\bigcup_{s \in S} F \setminus (\_ \trianglelefteq s)^{-1}(F)|}{|F|} < \varepsilon. \]

Hence, for each \( s \in S \), because \( F \setminus (\_ \trianglelefteq s)^{-1}(F) \subseteq \bigcup_{s' \in S} F \setminus (\_ \trianglelefteq s')^{-1}(F) \),

\[ \frac{|F \setminus (\_ \trianglelefteq s)^{-1}(F)|}{|F|} < \varepsilon. \]

In conclusion, Eq. (2) holds.

**Theorem 10.6.** Let \( \{F_i\}_{i \in I} \) be a net in \( \{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\} \) indexed by \((I, \leq)\). It is a right Følner net in \( \mathcal{R} \) if and only if

\[ \forall \rho \in \mathbb{N}_0 : \lim_{i \in I} \left| \frac{|\partial_\rho F_i|}{|F_i|} \right| = 0. \]
Proof. First, let \( \{F_i\}_{i \in I} \) be a right Følner net. Furthermore, let \( \rho \) be a non-negative integer. Then, for each index \( i \in I \), we have \( \partial F_i = \partial_{B(\rho)} F_i \). And, according to Remark 5.7, the ball \( B(\rho) \) is finite. Hence, according to [8, Theorem 1],
\[
\lim_{i \in I} \frac{|\partial_{B(\rho)} F_i|}{|F_i|} = 0.
\]
In conclusion, Eq. (3) holds.

Secondly, let Eq. (3) hold. Furthermore, let \( N \) be a finite subset of \( \mathbb{G}/G_0 \). Then, according to Remark 5.6, there is a non-negative integer \( \rho \) such that \( N \subseteq B(\rho) \). Hence, for each index \( i \in I \), according to [8, Item 4 of Lemma 1], we have \( \partial N F_i \subseteq \partial_{B(\rho)} F_i = \partial \rho F_i \). Therefore,
\[
\lim_{i \in I} \frac{|\partial N F_i|}{|F_i|} = 0.
\]
In conclusion, according to [8, Theorem 1], the net \( \{F_i\}_{i \in I} \) is a right Følner net. \( \square \)

11. Subexponential Growth and Amenability

In this section, let \( R = ((M, G, \cdot), (m_0, \{g_{m_0, m}\}_{m \in M})) \) be a finitely right generated cell space such that the stabiliser \( G_0 \) is finite.

In Main Theorem 11.1 we show that if \( R \) has sub-exponential growth, then it is right amenable. And in Theorem 11.3 we show that if \( G \) has sub-exponential growth, then so has \( R \).

Lemma 11.1 ([2, Lemma 6.11.1]). Let \( (r_k)_{k \in \mathbb{N}_0} \) be a sequence of positive real numbers. Then,
\[
\liminf_{k \to \infty} \frac{r_{k+1}}{r_k} \leq \liminf_{k \to \infty} \sqrt[k]{r_k}.
\]

Main Theorem 11.1. Let the cell space \( R \) have sub-exponential growth. It is right amenable.

Proof. Let \( S \) be a finite and symmetric right generating set of \( R \). According to Lemma 11.1 and Corollary 9.4,
\[
1 \leq \liminf_{k \to \infty} \frac{\gamma_S(k+1)}{\gamma_S(k)} \leq \lim_{k \to \infty} \sqrt[k]{\gamma_S(k)} = \lambda_S = 1.
\]
Therefore, \( \liminf_{k \to \infty} \frac{\gamma_S(k+1)}{\gamma_S(k)} = 1 \).
Let $\varepsilon \in \mathbb{R}_{>0}$. Then, there is a $k \in \mathbb{N}_+$ such that $\frac{\gamma_S(k)}{\gamma_S(k-1)} < 1 + \varepsilon$.
Hence, $\gamma_S(k) - \gamma_S(k-1) < \varepsilon \cdot \gamma_S(k-1)$. Furthermore, let $F = \mathbb{B}_S(k)$ and let $s \in S$. Then, according to Lemma 5.8, we have $\mathbb{B}_S(k-1) \subseteq (_{\trianglelefteq} s)^{-1}(F)$. Therefore, because $\mathbb{B}_S(k-1) \subseteq F$ and $\gamma_S(k-1) \leq \gamma_S(k)$,

$$|F \setminus (_{\trianglelefteq} s)^{-1}(F)| \leq |F \setminus \mathbb{B}_S(k-1)|$$

$$= |F| - |\mathbb{B}_S(k-1)|$$

$$= \gamma_S(k) - \gamma_S(k-1)$$

$$< \varepsilon \cdot \gamma_S(k-1)$$

$$\leq \varepsilon \cdot \gamma_S(k)$$

$$= \varepsilon \cdot |F|.$$ 

In conclusion, according to Theorem 10.5, the cell space $\mathcal{R}$ is right amenable. \hfill $\square$

**Lemma 11.2.** Let the group $G$ be finitely generated. The growth rate of $G$ dominates the one of $\mathcal{R}$.

*Proof.* There is a finite and symmetric generating set $T$ of $G$ such that $G_0 T \subseteq T$. And, according to Lemma 1.4, the set $S = \{tG_0 \mid t \in T\} = \{g_0 \cdot tG_0 \mid g_0 \in G_0, t \in T\}$ is a finite and symmetric right generating set of $\mathcal{R}$.

Let $k \in \mathbb{N}_0$ be a non-negative integer. Furthermore, let $m$ be an element of $\mathbb{B}_S^\infty(k)$. Then, there is a non-negative integer $j \in \{0, 1, 2, \ldots, k\}$ and a family $\{s_i\}_{i \in \{1, 2, \ldots, j\}}$ of elements in $S$ such that

$$m = (((m_0 \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_j.$$

And, by the definition of $S$, there is a family $\{t_i\}_{i \in \{1, 2, \ldots, j\}}$ of elements in $T$ such that $\{t_iG_0\}_{i \in \{1, 2, \ldots, j\}} = \{s_i\}_{i \in \{1, 2, \ldots, j\}}$. And, because $\trianglelefteq$ is a right semi-action, there is a family $\{g_{i,0}\}_{i \in \{1, 2, \ldots, j\}}$ of elements in $G_0$ such that

$$m = (((m_0 \trianglelefteq s_1) \trianglelefteq s_2) \trianglelefteq \cdots) \trianglelefteq s_j$$

$$= m_0 \trianglelefteq t_1 g_{2,0} t_2 g_{3,0} t_3 \cdots g_{j,0} t_j G_0$$

$$= g_{1,0} t_1 g_{2,0} t_2 g_{3,0} t_3 \cdots g_{j,0} t_j \triangleright m_0,$$
where $g_{1,0} = g_{m_0, m_0}$. And, because $G_0 T \subseteq T$, the family \( \{ g_{i,0} t_i \}_{i \in \{1, \ldots, j\}} \) is one of elements in $T$. Hence, $m \in B^G_T (j) \triangleright m_0 \subseteq B^G_T (k) \triangleright m_0$. Therefore, $B^G_S (k) \subseteq B^G_T (k) \triangleright m_0$ and thus
\[
|B^G_S (k)| \leq |B^G_T (k) \triangleright m_0| \leq |B^G_T (k)|.
\]
Hence, $\gamma^G_S (k) \leq \gamma^G_T (k)$. In conclusion, $\gamma^G_T$ dominates $\gamma^G_S$ and thus $\gamma (G)$ dominates $\gamma (\mathcal{R})$.

**Theorem 11.3.** Let the group $G$ be finitely generated and let it have sub-exponential growth. The cell space $\mathcal{R}$ has sub-exponential growth and is right amenable.

**Proof.** According to Lemma [11.2], the cell space $\mathcal{R}$ has sub-exponential growth. Hence, according to Main Theorem [11.1], it is right amenable.

\[\square\]

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