Variants of fattening and flavor symmetry restoration

Kostas Orginos and Doug Toussaint
Department of Physics, University of Arizona, Tucson, AZ 85721, USA

R.L. Sugar
Department of Physics, University of California, Santa Barbara, CA 93106, USA

The MILC Collaboration

We study the effects of different “fat link” actions for Kogut-Susskind quarks on flavor symmetry breaking. Our method is mostly empirical - we compute the pion spectrum with different valence quark actions on common sets of sample lattices. Different actions are compared, as best we can, at equivalent physical points. We find significant reductions in flavor symmetry breaking relative to the conventional or to the “link plus staple” actions, with a reasonable cost in computer time. We also develop and test a scheme for approximate unitarization of the fat links. While our tests have concentrated on the valence quark action, our results will be useful in designing simulations with dynamical quarks.
1 Introduction

As lattice QCD computations evolve towards quantitative predictions, it becomes increasingly important to make better use of computational resources by using more sophisticated discretizations of the continuum action, or “improved actions”. With Kogut-Susskind quarks the errors from the lattice discretization of the fermion action are proportional to $a^2$, where $a$ is the lattice spacing. In contrast, for the (unimproved) Wilson quark formulation, the errors are proportional to $a^4 g$. However, for the lattice spacings that are now practical for calculations with dynamical quarks, the effects of these errors in the Kogut-Susskind formulation are not small. In particular, the breaking of flavor symmetry is a large effect. The Kogut-Susskind formulation naturally describes four flavors of quarks, which is conventionally reduced to two in dynamical simulations by taking the square root of the determinant. However, only a U(1) subgroup of the original SU(4) chiral symmetry is an exact symmetry, so for nonzero lattice spacing only one of the sixteen pseudoscalar mesons has a vanishing mass when the quark mass vanishes. In principle, flavor symmetry breaking is also present for vector mesons, nucleons, etc., but these effects are generally small, and we can concentrate on the pseudoscalar mesons.

Flavor symmetry breaking can be reduced by “fattening” the links, which means that the conventional parallel transport using the link matrix is replaced by an average over paths connecting the points. This was demonstrated in Ref. [1], where the single link was replaced by an average of the simple link and three link paths, or “staples”. This modification was introduced as the simplest possible gauge invariant modification of the nearest neighbor coupling, and the relative weight of the staples and simple connection was treated as a free parameter. The improvement of flavor symmetry was found to be fairly insensitive to the value of the weighting coefficient. Lepage pointed out that this improvement could be understood as an introduction of a form factor suppressing the coupling to high momentum gluons which scatter quarks from one corner of the Brillouin zone to another [2]. Lagae and Sinclair used this understanding to construct an action where successive smearing in all four directions canceled the tree level couplings to all gluons with any momentum component equal to $\pi/a$, and showed that this further reduced flavor symmetry breaking [3]. This action includes paths to nearest neighbor points that are up to nine links in length. A fat link Kogut-Susskind fermion action, motivated by a perfect Kogut-Susskind fermion action, was tested by Bietenholz and Dilger [4] on the 2D Schwinger Model. They found that their action shows very good scaling for the pion and the eta masses. Furthermore, perfect action motivated fattening has been used by DeGrand [3] for Wilson-like fermions, showing very good scaling and renormalization factors very close to unity. In an earlier paper we investigated the single staple fattening, together with the Naik term [6] improving the dispersion relation in more detail [7]. Ref. [7] also describes an algorithm for using these actions in full QCD simulations, and extends studies of flavor symmetry breaking to the nonlocal pions, which turn out to have much larger breakings than the local non-Goldstone
pion which was used in the earlier studies.

This paper extends our studies of flavor symmetry breaking to actions involving paths longer than three links. Many of these results were briefly presented at the Lattice-98 conference\[8\]. We investigate actions that involve five and seven link paths to the nearest neighbor point, in addition to the one and three link paths. The coefficients of the various paths are chosen to minimize or to completely eliminate the tree level couplings to gluons with transverse momentum components $\pi/a$. Motivated by encouraging results using links fattened by “Ape smearing”\[9\], we also try a variant of the fattened action which makes the fattened links approximately unitary.

Recently Lepage has analyzed flavor symmetry breaking using the language of Symanzik improvement\[10\]. In addition to providing a simple construction of the seven-link action which cancels tree level couplings at momentum $\pi$, Lepage points out in this work that the fattening which reduces flavor symmetry breaking introduces an error of order $a^2$ in the low momentum physics (effectively an $a^2p^2$ error, where $p$ is a momentum), and shows how this error can be removed by an additional term in the action.

This work concentrates on the effect of the valence quark action on the flavor symmetry breaking. We have not done a study of the effects of changing the dynamical quark action; however, we have found in previous studies that improvements in flavor symmetry arising from changes in the valence quark action are indicative of improvements in full QCD simulations\[7\]. Also, we have not studied scaling, or the independence of mass ratios on the lattice spacing, although we expect that they will show improved scaling properties. Our concentration on the splittings of the pions is motivated by experience that suggests that this is the worst practical problem with Kogut-Susskind quarks. (For example, in Ref.\[7\] we found that rotational symmetry of the pion propagator was essentially restored at coarser lattice spacings than was the flavor symmetry.) Furthermore, we expect that the study of the pion spectrum provides a good guide to the quality of actions, since the low energy dynamics of QCD is approximately the dynamics of a pion gas.

2 The actions

Figure 1 illustrates the coupling to gluons with momentum components $\pm\pi$. The figure illustrates the single link couplings in $D_\mu$, connecting the central point to both the forwards and backwards nearest neighbors, as well as three link staples connecting to the same points. Each horizontal arrow represents a parallel transport by $U_\mu \approx 1 + i g a A_\mu$. The directions of the arrows in the coupling to the backwards neighbor include one minus sign because this transport involves $U_\mu^\dagger$ and another minus sign appearing explicitly in the derivative. Now consider the coupling to a $\mu$-direction gluon ($A_\mu(\pi \hat{e}_\nu)$) with momentum component $\pm\pi$ in
Figure 1: A simple link and staples in one plane, showing the relative signs of the coupling to a gluon with momentum component $\pi$ in the transverse direction (unparenthesized), and a gluon with momentum $\pi$ in the longitudinal direction (parenthesized).

the $\nu$ direction. The $\pm$ signs above the $\mu$ direction links indicate the sign of the coupling of this gluon relative to the forward simple link. Thus, if $c_1$ and $c_3$ are the weights of the one-link and three-link paths, the coupling to this gluon from the paths pictured here is $c_1 - 2c_3$. (This is not the whole story — one must also include the staples in the directions that are not shown in this figure.) Similarly, the parenthesized signs below the $\mu$ direction links show the relative coupling to a gluon with momentum $\pi$ in the $\mu$ direction ($A_\mu(\pi \hat{e}_\mu)$).

Note that this coupling is automatically cancelled between the forward and backward parts of $D_\mu$, so we don’t have to worry about longitudinal momentum $\pi$. Now one might worry that the $\nu$ direction links, required to keep the expression gauge invariant, might introduce couplings to $\nu$ direction gluons with momentum components $\pm \pi$. But for $A_\nu(\pi \hat{e}_\nu)$ gluons, the contributions of the vertical links in the center cancel, and the contribution of the links from the left and right sides cancel, since they are separated by $2a$ and traversed in opposite directions. Similarly, the coupling to $A_\nu(\pi \hat{e}_\nu)$ gluons cancels between the top and bottom halves of the figure. This argument extends to the five and seven link paths illustrated in Fig. 2, so the end result is that we can compute the couplings to gluons with momentum components equal to $\pi$ by just considering the $\mu$ direction links in $D_\mu$. Taking into account the multiplicities of the various paths and using $C_n$ for the weight of the $n$ link paths, the couplings are given below. In these expressions the coefficients explicitly show how many paths of each length contribute with positive weight and how many with negative weight. We also write explicitly $(2C_5)$ and $(6C_7)$ to indicate that in $\mathcal{D}_x$ there are two shortest paths connecting the starting point to the $\hat{x}$ direction link displaced by $+\hat{y} + \hat{z} + \hat{t}$, and six paths to the link displaced by $+\hat{y} + \hat{z} + \hat{t}$.

- Coupling to $k = (0, 0, 0, 0)$: $C_1 + 6C_3 + 12(2C_5) + 8(6C_7)$
Figure 2: The simple link, three link staple, five link staple and seven link staple used in suppressing flavor symmetry breaking. The final path is the five link path used, following Lepage, to correct the small momentum form factor.

- Coupling to $k = (0, \pi, 0, 0)$: $C_1 + (4 - 2)C_3 + (4 - 8)(2C_5) + (-8)(6C_7)$
- Coupling to $k = (0, \pi, \pi, 0)$: $C_1 + (2 - 4)C_3 + (4 - 8)(2C_5) + (+8)(6C_7)$
- Coupling to $k = (0, \pi, \pi, \pi)$: $C_1 + (-6)C_3 + (12)(2C_5) + (-8)(6C_7)$

If we are willing to use all the paths up to length seven, we can normalize the zero momentum coupling to one and set all the others to zero with $C_1 = 1/8$, $C_3 = 1/16$, $(2C_5) = 1/32$ and $(6C_7) = 1/64$. This defines our “Fat7” action and, with tadpole improved coefficients, the “Fat7tad” action. However, it is interesting to ask how helpful this complexity is — could we get by with shorter paths? If we restrict ourselves to five link and shorter paths, we can no longer satisfy all of these conditions. However, we can choose the couplings to minimize the maximum of the couplings to the high momentum gluons. This leads to $C_1 = 1/7$, $C_3 = 1/14$ and $(2C_5) = 1/28$, with all couplings to high momentum gluons reduced by a factor of seven. This defines our “Fat5” action.

Here we note that in two dimensions the equivalent of our “Fat7” action has only a 3-link staple. Our tree level formula for the relative weight of the staple would be 0.25 in two dimensions. This is very close to the relative weight of 0.238 introduced in the approximate perfect action constructed in Ref.\cite{ref4} for the 2D Schwinger Model.

All of these actions can be tadpole improved by inserting a factor of $(1/u_0)^{L-1}$ in the coefficient of each path, where $L$ is the length of the path. Using $L - 1$ instead of $L$ amounts to absorbing one power of $u_0$ into the quark mass. We use the average plaquette to define $u_0$. 
While the paths shown in Fig. 2 can be used to reduce or eliminate couplings to gluons with transverse momentum components equal to $\pi$, Lepage has pointed out that they have the undesirable effect of modifying the coupling to gluons with small nonzero transverse momentum components, essentially by introducing a second derivative coupling proportional to $a^2 p^2$. Following Ref. [10], we can correct for this by introducing a flavor conserving five link path, $+\hat{y} + \hat{y} + \hat{x} - \hat{y} - \hat{y}$, into $D_x$, giving an action with no tree level order $a^2$ corrections. While it is not clear what observable quantities are affected by this correction, it is a relatively cheap and aesthetically pleasing addition to the Fat7+Naik action. We have tested flavor symmetry breaking with this action, which we call the “Asq” action, or, when the coefficients are tadpole improved, the “Asqtad” action.

To summarize and clarify these actions, we give the coefficients of the paths in the $a^2$ improved action in a form useful for simulation. Here $c_1$ is the coefficient of the simple link, $c_3$, $c_5$ and $c_7$ are the coefficients of the three, five and seven link paths in Fig. 2, $c_N$ is the coefficient of the three link path to the third nearest neighbor (Naik term), and $c_L$ is the coefficient of the five link path implementing the correction introduced by Lepage. The origin of each term in the coefficients is identified by a subscript: “F” for flavor symmetry, “N” for the Naik correction to the quark dispersion relation, and “L” for the small momentum form factor correction. The factors of $1/2$ and $1/6$ in $c_5$ and $c_7$ compensate for the number of different paths connecting the starting point to the $\mu$ direction link parallel to the simple link. For example, the $x$ direction link displaced by $+\hat{y} + \hat{z}$ (coefficient $c_5$) is included in both $+\hat{y} + \hat{z} + \hat{x} - \hat{z} - \hat{y}$ and $+\hat{z} + \hat{y} + \hat{x} - \hat{y} - \hat{z}$, and we average over both paths.

$$
\begin{align*}
    c_1 &= (1/8)_F + (3/8)_L + (1/8)_N \\
    c_3 &= (1/16)_F \\
    c_5 &= (1/32)_F(1/2) \\
    c_7 &= (1/64)_F(1/6) \\
    c_L &= (-1/16)_L \\
    c_N &= (-1/24)_N 
\end{align*}
$$

(1)

We have also experimented with an action in which the fat links are approximately unitary. Theoretically, it is not clear why unitarity should be a concern in suppressing flavor symmetry breaking. However, results in Ref. [9] led us to consider “APE-smeared” links, where an average over the single link and nearby paths is constructed as above, and then the resulting fat link is projected back onto the nearest element of SU(3). Of course, such actions are not explicitly expressed as a sum over paths, and dynamical simulations using them would require an extension of our algorithms. However, spectrum calculations using APE-smeared links for the valence quarks are straightforward. In particular, we ran a calculation using a single iteration of APE smearing on a fairly coarse lattice. This action differs from the one link plus staple action only in the projection onto a special unitary matrix, yet, as will be seen in a later section, it produced smaller flavor symmetry breaking than the one link
plus staple plus Naik action. (The Naik term has little effect on flavor symmetry.) This motivated us to construct an action for which the fat links are approximately unitary, but are still expressed as an explicit sum over paths.

An almost unitary matrix $M$ can be expressed as a unitary matrix $U$ times a correction:

$$M = U \left( 1 + \epsilon \right),$$

where $\epsilon$ is Hermitian. Now look at $M^\dagger M$ to first order in $\epsilon$

$$M^\dagger M = (1 + \epsilon) U^\dagger U (1 + \epsilon) \approx 1 + 2 \epsilon.$$  

Now, inverting Eq. 2 to first order,

$$U = M (1 - \epsilon) = \frac{3}{2} M - \frac{1}{2} M M^\dagger M.$$  

We want to use this equation to make a fat link approximately unitary. Let $M$ be a generic fat link:

$$M = L + \alpha S$$

where $L$ is the simple link and $S$ is some sum over other paths connecting the ends of $L$. $S$ may be the sum of the staples for a simple fattening, or something more complicated involving longer paths. We will work to first order in $\alpha$. Customarily we also rescale $M$ by some factor which is $1 + b\alpha$. However, this factor cancels in the derivation, so we suppress it here. Unitarize $M$ as above to make an approximately unitary fat link $F$:

$$F = \frac{3}{2} M - \frac{1}{2} M M^\dagger M$$

$$= \frac{3}{2} (L + \alpha S) - \frac{1}{2} (L + \alpha S) \left( L^\dagger + \alpha S^\dagger \right) (L + \alpha S)$$

Expand, keeping terms up to order $\alpha$ and using $L^\dagger L = 1$, and remarkable cancellations occur

$$F = L + \frac{\alpha}{2} \left( S - LS^\dagger L \right)$$

$F$ is an approximately unitary fat link expressed as a sum over paths — a form suitable for dynamical simulations. The only change is that the sum over fattening paths, $S$, has been replaced by an average of the paths traversed in each direction, $\frac{1}{2} \left( S - S^\dagger \right)$, with $L$ inserted as necessary to maintain gauge invariance. At this point one can forget about how it was derived, and just verify from Eq. 6 that $F^\dagger F = 1$ to order $\alpha$. It is also easy to verify, by setting link matrices to $1 + iA_\mu$, that $F$ averages $A_\mu$ over position in the same way that $M$ did. The minus sign in front of the $LS^\dagger L$ term compensates for the fact that this path is now traversed in the opposite direction. This action is illustrated in Fig. 3 for the case where $S$ is the three link staple.
The two terms can be made to look more symmetric by factoring \( L \) out on the right (or left).

\[
F = \left( 1 + \frac{\alpha}{2} (SL^\dagger - LS^\dagger) \right) L
\]  

(8)

In this form, we are just following the parallel transport by \( L \) with a difference of going around closed loops in opposite directions, or a term proportional to \( F_{\mu\nu} \).

3 The simulations

Most of our spectrum calculations used two of the same sets of sample lattices used in Ref. [7]. These were lattices with two flavors of dynamical quarks, where the dynamical quarks used the "Staple+Naik" action, and a Symanzik improved gauge action at \( 10/g^{2}_{\text{imp}} = 7.3 \) and 7.5. At \( 10/g^{2}_{\text{imp}} = 7.3 \) we used \( 12^3 \times 32 \) lattices with dynamical quark masses of 0.02 and 0.04, with lattice spacings determined from \( r_0 \) [11] of about 0.15 fm. and 0.16 fm. respectively. The \( 10/g^{2}_{\text{imp}} = 7.5 \) runs used \( 16^3 \times 48 \) lattices with \( m_q = 0.015 \) and 0.030, with lattice spacings of 0.13 fm and 0.14 fm, respectively. Sample sizes ranged from 48 to 60 lattices[7], with four source time slices per lattice for spectrum calculations. In addition we computed the meson spectrum on a set of eleven large \((32^3 \times 64)\) quenched lattices with a lattice spacing around 0.07 fm., using the one plaquette gauge action at \( 6/g^{2}_{\text{conv}} = 6.15 \).

For each of the three gauge couplings we computed spectra at two values of the quark mass, which allows us to interpolate results to make fair comparisons of the actions. (For the two flavor runs, the sea quark mass was also changed.) The meson masses were interpolated assuming that the squared meson masses are linear in the quark mass. One approach to comparing actions is to interpolate to a fixed ratio of \( m_G \) to \( m_\rho \), where \( m_G \) is the Goldstone
pion mass. In so doing, we are letting each action “determine its own lattice spacing”. Since the $\rho$ mass is also dependent on the valence quark action, this results in a slightly different estimate of the lattice spacing for each action. An alternate approach is to assume that each set of lattices has a fixed lattice spacing, which could be determined either from the rho mass for a fixed choice of action, or from the static quark potential. Operationally, this results in interpolating all the valence actions on a set of lattices to the same pion mass, and we choose the Goldstone pion for this interpolation. For the runs with $10/g_{\text{imp}}^2 = 7.3$ and 7.5 we will present results with both sets of assumptions. For the fine lattices, with $6/g_{\text{conv}}^2 = 6.15$, the differences in the $\rho$ masses with different actions are insignificant, which is expected as the lattice spacing gets smaller.

As in Ref. [7], we parameterize the splitting of the pions by the dimensionless quantity

$$\delta_2 = \frac{m_{\pi}^2 - m_G^2}{m_\rho^2 - m_G^2},$$  \hspace{1cm} (9)$$

where $m_\pi$ is one of the non-Goldstone pion masses, $m_G$ is the Goldstone pion mass, and $m_\rho$ is one of the local $\rho$ masses. Since the $\rho$ masses are nearly degenerate, it makes little difference which one we use. In our analysis we used the local $\gamma_i \otimes \gamma_i$ ($\rho_2$) mass because for heavy quarks it is often estimated more accurately. We use the squared meson masses since they are approximately linear in the quark mass and we interpolate and extrapolate in quark mass. Empirically $\delta_2$ is fairly insensitive to the quark mass, and the theoretical analysis in Ref. [12] predicts that the numerator is independent of quark mass for small quark masses. The squared Goldstone pion mass is subtracted in the denominator to give more sensible behavior at large quark mass. At large $a m_q$, all of the meson masses become degenerate, but it is still possible to use the difference between the Goldstone pion mass and the rho mass as a natural scale for flavor symmetry breaking.

4 Results

Figure 4 shows the spectrum of pion masses obtained with the “Staple+Naik” action at $10/g_{\text{imp}}^2 = 7.3$ and 7.5 respectively. Here, as in all of our calculations, the pattern of near-degeneracies predicted in Ref. [12] is evident. Note that the local non-Goldstone pion, the flavor $\gamma_0\gamma_5$ pion, is one of the lightest non-Goldstone pions, so a realistic assessment of the flavor symmetry breaking requires consideration of the nonlocal pions. All to the actions we have tested produce a qualitatively similar pattern of pion masses.
Table 1: Path coefficients for the various quark actions. The first column is a name for the action. The remaining columns are the coefficients for the paths in Fig. 2 and for the connection to the site at distance three (Naik term). Notice that in the tadpole improved actions we have absorbed one power of $u_0$ into the quark mass, so $u_0$ appears to the power length $-1$.

| Action        | Link | 3-staple | 5-staple | 7-staple | Lepage | Naik  |
|---------------|------|----------|----------|----------|--------|-------|
| OneLink       | 1    | -        | -        | -        | -      | -     |
| Staple+Naik   | $\frac{9}{8}$ | $\frac{1}{8}$ | -        | -        | -      | $\frac{9}{8} - \frac{1}{27}$ |
| Fat5          | $\frac{1}{7}$ | $\frac{1}{7}$ $u_0^{-2}$ | $\frac{1}{7}$ $u_0^{-4}$ | -        | -      | -     |
| Fat5tad       | $\frac{1}{8}$ | $\frac{1}{8}$ $u_0^{-2}$ | $\frac{1}{8}$ $u_0^{-4}$ | $\frac{1}{8}$ $u_0^{-6}$ | -      | -     |
| Fat7          | $\frac{1}{8}$ | $\frac{1}{8}$ $u_0^{-2}$ | $\frac{1}{8}$ $u_0^{-4}$ | $\frac{1}{8}$ $u_0^{-6}$ | -      | -     |
| Fat7tad       | $\frac{1}{8}$ | $\frac{3}{8} + \frac{1}{8}$ | $\frac{1}{8}$ $u_0^{-2}$ | $\frac{1}{8}$ $u_0^{-4}$ | $\frac{1}{8}$ $u_0^{-6}$ | $\frac{1}{16} u_0^{-4}$ | $\frac{1}{24} u_0^{-2}$ |

Figure 5 shows $\delta_2$ from three different actions, each with three different lattice spacings. All of these results are interpolated in quark mass to the point where $m_G/m_\rho = 0.55$. The improvement of the flavor symmetry breaking is evidenced by the smaller non-Goldstone pion masses for the improved actions. In this figure we have determined the lattice spacing for each action using the $\rho$ mass evaluated with this action. Thus, even though the spectra for the different actions are evaluated on the same sets of lattices, they appear at different horizontal positions. However, as we would hope, the ambiguity in lattice spacing coming from the difference in $\rho$ mass with action becomes smaller at the smaller lattice spacings. (It is neither surprising nor upsetting that the $\rho$ mass should depend on the action. After all, in the end we expect that these actions will have better scaling behavior in all quantities, meaning that we should come closer to the continuum limit at these large lattice spacings. Among other things, this means that we expect the nucleon to rho mass ratio, a well known problem on coarse lattices, to be improved with these actions.)

In Table 4 we show $\delta_2$ for the various actions. We tabulate this for only three of the pions: the local non-Goldstone pion with flavor structure $\gamma_0\gamma_5$, and the two three-link pions, with flavor structures $\gamma_0$ and 1. The local pion is the most often studied one, and so allows comparison with other work. The flavor 1 is the worst case pion. The other three link pion is in the second worst multiplet, but it is interesting to study because its parity partner would have exotic quantum numbers, so the propagator can be fit to a simple exponential, leading to smaller errors for the mass estimates. We note in passing that the flavor 1 pion is properly called a “pion” here instead of an “eta”, since we did not compute quark-line disconnected diagrams in the propagator. In other words, one may imagine that the quark
and antiquark in this pion carry a flavor quantum number in addition to that coming from
the Kogut-Susskind quarks’ natural four flavors, and so cannot annihilate each other. Thus,
it is correct to demand that an improved action should make this pion degenerate with
the others. In order to make a fair comparison of the actions, we have interpolated (or
extrapolated) the spectrum to a fixed $m_G/m_\rho = 0.55$ point. This is done using two different
bare masses for the valence quarks. We also interpolated (or extrapolated) our spectrum
to a fixed $m_G = 545\text{MeV}$ using the heavy quark potential parameter $r_0$ in order to fix the
scale\([\text{11}]\). As we can see from the table, the computed $\delta_2$’s in both cases are in agreement
with in errors. The errors in both cases were computed with jackknife analysis. Since $\delta_2$
involves a ratio of mass differences, and the different masses are all correlated, naive error
propagation would lead to an overestimate of the errors in $\delta_2$. Therefore we used a jackknife
analysis to compute these errors. When comparing actions, we are interested in the difference
in $\delta_2$ between the two actions. Since we used the same lattices for all the actions, these $\delta_2$’s
are not independent, and one should really do a jackknife analysis in order to determine the
errors on the differences of $\delta_2$’s. We have done this and we find in general the same error as
the one computed with error propagation from the quoted $\delta_2$ errors.

Our results show that all the variants of the fat actions significantly improve flavor
symmetry, with larger improvement for the actions that suppress more couplings to gluons
with transverse momentum $\pi$. Generally, tadpole improving the tree level coefficients results
in a better action. The approximately reunitarized action with coefficient 0.25 seems to
work better than the Staple+Naik action. We have no clear understanding of why this
happens, but this may be a hint that a more careful (non-perturbative) tuning of the fattening
coefficients would result in better actions. As expected, the Fat7tad action is the best for
suppressing flavor symmetry violations. The Asqtad action is slightly worse than the Fat7tad
action in this regard. Extra $O(a^4)$ flavor symmetry violation introduced by the Naik term
and the Lepage term may be responsible for this. However, the Asqtad action has improved
rotational symmetry due to the Naik term, and has no additional $O(a^2)$ flavor conserving
errors introduced by the fattening. Thus, of all the actions we have studied, we consider this
one to be the best candidate for an improved Kogut-Susskind action.

In Ref. [11] it was shown that smearing the links by “Ape smearing”, where the link
and staples are averaged and the result in projected back onto SU(3) (i.e. replaced by the
SU(3) matrix which maximizes $\text{Tr}(U^\dagger F)$, where $F$ is the fattened link) improved the flavor
symmetry, among other nice features. We find similar results on our set of sample lattices.
It is interesting to compare the “Ape1” action in Table 4 with the “Staple-un(.25)” action,
since the difference between the two is that the Ape1 action uses links that are exactly
unitary, while the Staple-un(.25) uses the same fattening, but is only unitary to first order
in the staple weight. We see that the approximately unitary action is only slightly worse
than the Ape1 action. Also, we see as expected that that the Ape4 valence action gives
a very good suppression of flavor symmetry breaking, although using it for a dynamical
action is difficult. The approximately reunitarized actions seem to have slightly better flavor
symmetry breaking than the comparable Link+Staple action, but probably not enough to justify the extra complexity.

Another practical advantage of the fat link actions is that the fat link configurations are smoother than the original configurations, and the conjugate gradient computation of the propagators converges in fewer iterations. Quantifying this statement is tricky because, unlike the flavor symmetry breaking, where $\delta_2$ is approximately independent of quark mass, the number of conjugate gradient iterations is very sensitive to the quark mass. For example, at $10/g_{\text{imp}}^2$ at a fixed bare quark mass of 0.02 the Link+Staple action requires 21% fewer conjugate gradient iterations than the conventional action. However, after interpolation to $m_G/m_\rho = 0.55$ this advantage disappears, and both of these actions require about the same number of iterations. On the coarse lattice the approximately unitary actions do better — the Staple-un(.25) action and the Ape1 action require about 10% fewer iterations than the conventional or Link+Staple actions, and the Ape4 21% fewer. On the fine lattice, with $6/g_{\text{conv}}^2 = 6.15$, there is much less ambiguity since the various pion masses are much closer. Also, the fattening does a much better job of smoothing the configurations on the fine lattice. In this case the Link+Staple action requires 46% fewer iterations than the conventional action, and the Fat7 action 52% fewer. However, the advantage of the approximately reunitarized action has disappeared — it requires just about as many iterations as the Link+Staple. Regrettably, the Asqtad action does not do as well as the Fat7 action, only reducing the number of iterations by 37% as compared to the conventional action. This is probably because of the negative coefficient associated with the Lepage term (see Table 3), meaning that this term is actually undoing part of the smoothing accomplished by the other paths. We should note that with these complicated actions the cost of simulation with dynamical quarks is no longer completely dominated by the conjugate gradient. Except for very light quark masses, the cost of computing the fermion force and precomputing the fat links becomes comparable to the conjugate gradient cost.

5 Conclusions

In this paper we have investigated the possibility of constructing a Kogut-Susskind action with improved flavor and rotational symmetry suitable for dynamical fermion simulations. For this reason we want to keep the amount of new paths introduced into the action as small as possible. An action containing fat links with paths up to length seven was constructed. At tree level this action has no couplings of quarks to gluons with a transverse momentum component $\pi$. As a result, at tree level the flavor symmetry violating terms in the action are completely removed. In addition to flavor symmetry, the rotational symmetry is improved
Table 2: The flavor symmetry breaking measure $\delta_2$ for the various actions and lattice spacings. Results are shown for the local non-Goldstone ($\gamma_0\gamma_5$) pion and for the two three-link pions, with Kogut-Susskind flavor structures $\gamma_0$ and $\gamma_1$. The actions are those specified in Table 3, plus actions with a single staple fattening approximately unitarized as discussed in section 2. For the actions with single staple fattening we give the relative weight of each staple to the single link in parentheses (e.g. Staple-un(.25)). The “Ape1” action is the single link plus the three link staple with a relative weight of 0.25, projected back onto an element of SU(3) to numerical precision. For the “Ape4” action, this smearing and projection was repeated four times. Otherwise the coefficients of the various paths are found in Table 3. For comparing the actions we interpolate (or extrapolate) to $m_G/m_\rho = 0.55$ or alternatively to $m_G = 545\, MeV$.

| Action              | $\gamma_0\gamma_5$ | $\gamma_0$ | $1$   | $\gamma_0\gamma_5$ | $\gamma_0$ | $1$   |
|---------------------|---------------------|-----------|-------|---------------------|-----------|-------|
| 10/$g_{\text{imp}}^2 = 7.30 |                     |           |       |                     |           |       |
| OneLink             | 0.408(45)           | 0.677(62) | 0.815(73) | 0.408(66)           | 0.68(11)  | 0.83(15)   |
| Staple+Naik         | 0.171(7)            | 0.370(18) | 0.479(26) | 0.170(7)            | 0.369(18) | 0.479(25)  |
| Ape1                | 0.141(13)           | 0.326(22) | 0.424(22) | 0.141(15)           | 0.326(27) | 0.424(23)  |
| Ape4                | 0.064(10)           | 0.209(10) | 0.262(21) | 0.070(36)           | 0.272(75) | 0.296(80)  |
| Staple-un(.25)      | 0.145(9)            | 0.320(31) | 0.446(31) | 0.150(17)           | 0.319(41) | 0.457(51)  |
| Fat5                | 0.126(8)            | 0.298(14) | 0.393(16) | 0.127(8)            | 0.299(14) | 0.393(16)  |
| Fat5tad             | 0.115(5)            | 0.277(18) | 0.370(22) | 0.116(5)            | 0.279(22) | 0.375(28)  |
| Fat7                | 0.117(9)            | 0.281(13) | 0.367(14) | 0.118(8)            | 0.282(13) | 0.368(14)  |
| Fat7tad             | 0.108(5)            | 0.260(16) | 0.346(19) | 0.109(6)            | 0.263(22) | 0.353(28)  |
| Asq                 | 0.164(10)           | 0.357(21) | 0.438(20) | 0.164(12)           | 0.357(22) | 0.439(20)  |
| Asqtad              | 0.136(10)           | 0.279(21) | 0.365(22) | 0.137(10)           | 0.280(18) | 0.367(20)  |

| Action              | $\gamma_0\gamma_5$ | $\gamma_0$ | $1$   | $\gamma_0\gamma_5$ | $\gamma_0$ | $1$   |
|---------------------|---------------------|-----------|-------|---------------------|-----------|-------|
| 10/$g_{\text{imp}}^2 = 7.50 |                     |           |       |                     |           |       |
| OneLink             | 0.371(11)           | 0.569(13) | 0.594(17) | 0.363(18)           | 0.526(22) | 0.570(28)  |
| Staple+Naik         | 0.115(5)            | 0.261(7)  | 0.318(8) | 0.118(6)            | 0.266(8)  | 0.326(10)  |
| Staple-un(.25)      | 0.103(8)            | 0.236(14) | 0.304(20) | 0.106(12)           | 0.241(21) | 0.312(31)  |
| Staple-un(.40)      | 0.131(8)            | 0.235(14) | 0.288(19) | 0.131(10)           | 0.236(18) | 0.290(24)  |
| Fat5                | 0.082(5)            | 0.200(7)  | 0.245(8) | 0.086(5)            | 0.205(7)  | 0.256(8)   |
| Fat5tad             | 0.080(6)            | 0.201(10) | 0.261(13) | 0.082(9)            | 0.206(16) | 0.275(22)  |
| Fat7                | 0.074(5)            | 0.187(7)  | 0.225(8) | 0.076(5)            | 0.193(6)  | 0.236(8)   |
| Fat7tad             | 0.071(7)            | 0.184(10) | 0.236(13) | 0.072(10)           | 0.189(17) | 0.245(22)  |
| Asq                 | 0.115(5)            | 0.242(6)  | 0.288(11) | 0.116(5)            | 0.246(8)  | 0.295(12)  |
| Asqtad              | 0.101(7)            | 0.207(10) | 0.255(11) | 0.102(8)            | 0.207(12) | 0.257(15)  |

| Action              | $\gamma_0\gamma_5$ | $\gamma_0$ | $1$   | $\gamma_0\gamma_5$ | $\gamma_0$ | $1$   |
|---------------------|---------------------|-----------|-------|---------------------|-----------|-------|
| 6/$g_{\text{conv}}^2 = 6.15 |                     |           |       |                     |           |       |
| OneLink             | 0.096(38)           | 0.111(71) | 0.157(41) | —                   | —         | —     |
| Staple+Naik         | 0.014(13)           | 0.050(19) | 0.077(14) | —                   | —         | —     |
| Staple-un(.25)      | 0.026(27)           | 0.040(25) | 0.057(24) | —                   | —         | —     |
| Fat5                | 0.011(9)            | 0.043(20) | 0.057(23) | —                   | —         | —     |
| Fat7                | 0.009(8)            | 0.036(14) | 0.049(24) | —                   | —         | —     |
| Asqtad              | 0.030(20)           | 0.034(18) | 0.051(40) | —                   | —         | —     |
by introducing the Naik term. Finally, as Lepage pointed out[10], we need to introduce an extra five link staple in order to cancel errors of $O(a^2p^2)$ introduced by the fattening. The resulting action can be further improved by tadpole improvement. This action, which we call “Asqtad”, is an order $O(a^4, a^2g^2)$ accurate fermion action.

Asqtad is an action simple enough to be useful for dynamical simulations. Preliminary tests using our generic code[7] show about a factor of 4 higher cost than the code implementing the standard Kogut-Susskind action. We have found optimizations specific to the Asqtad action that could bring the cost factor down to 2-2.5. The $O(a^2)$ precision at a cost of a factor of 2-2.5 makes such an action very competitive with the other popular improved actions such as D234[13], perfect or approximately perfect actions, the Neuberger action, and domain wall fermion actions. The highly improved chiral symmetry of actions respecting (or approximately respecting) the Ginsparg-Wilson relation is not something that Asqtad can compete with. On the other hand, cost may favor the Asqtad action. The Neuberger action[14, 15], approximately perfect actions[16, 1, 4] and domain wall fermions[17] are all fairly costly to implement for dynamical fermions.

In view of the enormous price one has to pay in order to have highly improved chiral symmetry on the lattice, we think that the Asqtad action is a good candidate for a fermion action to be used in the next generation of dynamical simulations. Flavor symmetry breaking in the traditional Kogut-Suskind action at lattice spacings commonly used in high temperature QCD studies results in pions as heavy as the kaons, making it impossible to study the effects of the strange quark. Our study of the Asqtad action shows that one can achieve a good separation between the pions and the kaons at accessible lattice spacings. Thus Asqtad is an action that may prove very useful in projects in which the effects of the strange quark are to be studied.

Acknowledgements

This work was supported by the U.S. Department of Energy under contract DE-FG03-95ER-40906 and by the National Science Foundation grant number NSF–PHY97–22022. Computations were done on the Paragon at Oak Ridge National Laboratory, and the T3E’s at NERSC, NPACI and the PSC. We would like to thank Peter Lepage for helpful communications, and the members of the MILC collaboration for inspiration and many discussions.

References

[1] T. Blum et al., Phys. Rev. D 55, 1133 (1997).
[2] G.P. Lepage, proceedings of “Lattice QCD on Parallel Computers” (Tsukuba, March 1997), Nucl. Phys. (Proc. Suppl.) 60A 267 (1998).

[3] J.F. Lagae and D.K. Sinclair, Nucl. Phys. (Proc. Suppl.) 63 892 (1998); Phys. Rev. D 59:014511 (1999).

[4] W. Bietenholz and H. Dilger, eprint hep-lat/9812016

[5] T. DeGrand, eprint hep-lat/9903006

[6] S. Naik, Nucl. Phys. B316 238 (1989).

[7] K. Orginos and D. Toussaint, Phys. Rev. D 59:014501 (1999).

[8] K. Orginos and D. Toussaint, eprint hep-lat/9809148, to appear in Nucl. Phys. B (Proc. Suppl.).

[9] T. DeGrand, A. Hasenfratz and T. Kovacs, Phys. Lett. 420B, 97 (1998).

[10] G.P. Lepage, eprint hep-lat/9809157.

[11] R. Sommer, Nucl. Phys. B 411, 839 (1994).

[12] W. Lee and S. Sharpe, hep-lat/9809026, to appear in Nucl. Phys. B (Proc. Suppl.).

[13] M. Alford, T. Klassen, P. Lepage, Phys. Rev. D58 (1998) 034503

[14] R. G. Edwards, U. M. Heller, R. Narayanan, eprint hep-lat/9807017

[15] H. Neuberger, eprint hep-lat/9901003

[16] T. DeGrand, Phys.Rev. D58 (1998) 094503.

[17] T. Blum, XVI Intl. Symposium on Lattice Field Theory, forthcoming in Nucl. Phys. (Proc. Suppl.)
Figure 4: The spectrum of pion masses with the "Link+Staple+Naik" action. The two spectra are at $10/g_{imp}^2 = 7.3$ and 7.5 respectively. In both cases we have interpolated in quark mass, assuming that squared meson masses are linear in the quark mass, to the point where $m_G/m_\rho = 0.55$. The pion masses are plotted in units of the $\rho$ mass. Each pion is labeled by the gamma matrix specifying its flavor structure. The pattern of near degeneracies predicted in Ref. [12] is evident. As expected, the mass splittings among the pions are smaller for the smaller lattice spacing ($10/g_{imp}^2 = 7.5$).
Figure 5: The $\delta_2$ for three different actions versus the squared lattice spacing. From left to right, the actions are the conventional “One Link” action, the “Link+Staple+Naik” action, and the “Order $a^2$ tadpole” action. In each case, the $\rho$ mass for the respective action was used to define the length scale. The points come from two flavor runs with $10/g_{imp}^2 = 7.3$ (rightmost points in each graph), $10/g_{imp}^2 = 7.5$, and a quenched simulation with single plaquette gauge action at $6/g_{conv}^2 = 6.15$. The lines drawn in each panel are approximate slopes for the local non-Goldstone pion (lower line) and for the three link flavor 1 pion (upper line), which is the worst case pion.