Determinant Structure of the Rational Solutions for the Painlevé II Equation

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Abstract

Two types of determinant representations of the rational solutions for the Painlevé II equation are discussed by using the bilinear formalism. One of them is a representation by the Devisme polynomials, and another one is a Hankel determinant representation. They are derived from the determinant solutions of the KP hierarchy and Toda lattice, respectively.

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I. Introduction

The six Painlevé transcendents are now regarded as the nonlinear version of the special functions and hence the Painlevé equations are the most fundamental integrable systems in some sense. It is known that the Painlevé transcendents cannot be expressed by the solutions of linear equations, except for two classes of solutions, namely, special function solutions and rational solutions. The Painlevé II equation ($P_{II}$)

$$\frac{d^2}{dz^2}v = 2v^3 - 4zv + 4\alpha,$$  

(1)

is the simplest equation that admits such solutions among the Painlevé equations. In fact, it is known that it admits one parameter family of Airy function solutions for $\alpha$ being half odd integers, and only one rational solutions for each integer $\alpha$ and it has no other classical solution.

It is well known that the Painlevé equations can be derived from the similarity reduction of various soliton equations. In particular, $P_{II}$ can be reduced from the modified KdV equation. A systematic study of the rational solutions was done by Airault, who constructed the Bäcklund transformation of $P_{II}$ from the similarity reduction of the modified KdV equation. On the other hand, Okamoto revealed that the Bäcklund transformations of Painlevé equations are given by the Toda lattice equation. For the KP and Toda lattice hierarchies, the solutions are described by the Wronski determinants. The Painlevé equations are deeply connected with the KP and Toda, therefore a question naturally arising is what is the structure of the solutions of Painlevé equations. Actually for the special function type solutions of Painlevé equations, it is known that they are expressed by Wronskians whose entries are given by special functions. Such Wronskians are called the $\tau$ function. Here, we note that $\tau$ functions are originally defined for arbitrary values of parameters through the Hamiltonians of the Painlevé equations. This Wronskian structure of the solutions is quite similar to that for the soliton equations. Hence we expect that the rational solutions also have such a structure. Many studies have been done for the rational solutions, but curiously, it seems that the determinant structure of solutions itself has not been well discussed. This situation motivates us for studying the relationship of the solutions of the Painlevé equations and integrable PDE.
In this article, we present the determinant representations for the rational solutions of PII and clarify how those solutions are reduced from the \( \tau \) functions of the KP hierarchy and Toda lattice. We present two types of determinant representations. One is directly derived from the Schur polynomials, namely the algebraic solutions for the KP hierarchy, by applying a reduction procedure. Entries of the determinant are expressed by the Devisme polynomials\(^6\),\(^7\). This reduction exactly corresponds to the derivation of PII from the modified KdV equation. The bilinear form for PII is nothing but the bilinear first Bäcklund transformations of the KP hierarchy. Another one is a Hankel determinant representation which is derived from the Hankel determinant solution of the B-type Toda lattice equation\(^8\). In this case, the Toda lattice is corresponding to the Bäcklund transformation ladder of the solutions of PII.

In section II, the bilinearization of PII is presented. We give a brief review of the algebraic solutions for KP and KdV hierarchies in section III. In section IV, we give the derivation of the rational solutions for PII from the Schur polynomials. In section V, we briefly summarize the determinant solution for the B-type Toda lattice equation. The Hankel determinant representation of the rational solutions is presented in section VI. Section VII is devoted to concluding remarks.

II. Bilinear form for PII

By using the dependent variable transformation,

\[
v = \frac{d}{dz} \log \frac{g}{f},
\]

eq. (1) is decomposed into the following bilinear equations\(^9\),\(^10\),

\[
\left(D^2_z - \lambda\right) g \cdot f = 0,
\]

\[
\left(D^3_z + (4z - 3\lambda)D_z - 4\alpha\right) g \cdot f = 0,
\]

where \( D^n_z \) is the Hirota bilinear differential operator and \( \lambda \) is an arbitrary function of \( z \). Dividing eqs. (3) and (4) by \( gf \), we obtain

\[
s + v^2 = \lambda,
\]

\[
v_{zz} + 3sv + v^3 + (4z - 3\lambda)v - 4\alpha = 0,
\]
where \( s = (\log gf)_{zz} \). Eliminating \( s \) from above equations, we get \( \Pi \) (4), therefore eqs.(3) and (4) actually give the bilinear form for \( \Pi \).

Using the gauge transformation, we can take \( \lambda \) as we like. In the case of the rational solutions of \( \Pi \), taking \( \lambda \) to be 0 is convenient as is shown in the later. On the other hand, for the Airy function type solutions of \( \Pi \), \( \lambda \) is taken to be \( 2z[1] \). If we fix \( \lambda \) to be equal 0, the bilinear equations for \( \Pi \) are

\[
\begin{align*}
D_z^2 g \cdot f &= 0, \\
\left(D_z^3 + 4zD_z - 4\alpha\right) g \cdot f &= 0.
\end{align*}
\]

In this gauge, these equations allow polynomial solutions for \( f \) and \( g \) which give the rational solutions \( v \) for \( \Pi \) (4) through the variable transformation (2). In the following sections, we will show how the rational solutions are constructed from the \( \tau \) functions of KP hierarchy and Toda lattice equation.

### III. Algebraic Solutions for KP and KdV Hierarchies

We first give a brief review on the algebraic solutions of KP and KdV hierarchies [10].

**Definition 3.1** Let \( p_j(y), j = 0, 1, 2, \ldots \), be polynomials in \( y = (y_1, y_2, y_3, \ldots) \) defined by

\[
\sum_{k=0}^{\infty} p_k(y)\lambda^k = \exp \left( \sum_{n=1}^{\infty} y_n\lambda^n \right), \quad \text{and} \quad p_k(y) = 0, \quad \text{for} \quad k < 0. \tag{7}
\]

Then a set of infinitely many bilinear equations for \( \tau(x) = \tau(x_1, x_2, x_3, \ldots) \) generated by

\[
\left( \sum_{j=0}^{\infty} p_j(-2y)p_{j+1}(\bar{D}) \exp \left( \sum_{n=1}^{\infty} y_nD_{x_n} \right) \right) \tau \cdot \tau = 0, \tag{8}
\]

where

\[
\bar{D} = (D_{x_1}, \frac{1}{2}D_{x_2}, \frac{1}{3}D_{x_3}, \ldots),
\]

is called the KP hierarchy and \( \tau \) is called the \( \tau \) function.

The simplest bilinear equation included in this hierarchy is

\[
\left(D_{x_1}^4 - 4D_{x_1}D_{x_3} + 3D_{x_2}^2\right) \tau \cdot \tau = 0, \tag{9}
\]
which yields the KP equation in nonlinear form,

\[-4u_{x_3} + 6uu_{x_1} + u_{x_1x_1x_1} = 3u_{x_2} = 0, \quad (10)\]

by the dependent variable transformation,

\[u = 2(\log \tau)_{x_1x_1}. \quad (11)\]

**Proposition 3.2** The following Wronskian,

\[
\tau_{N,KP} = \begin{vmatrix}
\partial_{x_1}^{N-1} f_1 & \cdots & \partial_{x_1} f_1 & f_1 \\
\partial_{x_1}^{N-1} f_2 & \cdots & \partial_{x_1} f_2 & f_2 \\
\vdots & \ddots & \vdots & \vdots \\
\partial_{x_1}^{N-1} f_N & \cdots & \partial_{x_1} f_N & f_N
\end{vmatrix},
\]

solves the KP hierarchy, where \(f_k, k = 1, 2, \cdots, N\) are arbitrary functions in infinitely many independent variables \(x = (x_1, x_2, \cdots)\) satisfying

\[
\partial_{x_n} f_k = \partial_{x_1}^n f_k, \quad k = 1, 2, \cdots, N, \quad n = 1, 2, \cdots. \quad (12)
\]

The crucial point is that all the bilinear equations in the KP hierarchy for the \(\tau\) function (12) are reduced to the identities of determinant which are called the Plücker relations.

**Definition 3.3** A set of infinitely many bilinear equations in \(\tau(x)\) and \(\tau'(x)\) generated by

\[
\left( \sum_{j=0}^{\infty} p_j (-2y)p_{j+2}(\hat{D}) \exp \left( \sum_{n=1}^{\infty} y_n D_{x_n} \right) \right) \tau \cdot \tau' = 0, \quad (14)
\]

is called the first modified KP hierarchy.

In particular, \(\tau = \tau_{N+1,KP}, \quad \tau' = \tau_{N,KP}\) solves the first modified KP hierarchy. Hence, this is regarded as the hierarchy of the first Bäcklund transformations. Moreover, the bilinear equations in this hierarchy are regarded as the identities of \((N + 1) \times (N + 1)\) determinant and \(N \times N\) determinant, which are also the Plücker relations. First two equations of this hierarchy are given by

\[
(D_{x_1}^2 - D_{x_2}) \tau_{N+1,KP} \cdot \tau_{N,KP} = 0, \quad (15)
\]

\[
(D_{x_1}^3 - 4D_{x_3} + 3D_{x_1}D_{x_2}) \tau_{N+1,KP} \cdot \tau_{N,KP} = 0. \quad (16)
\]
In the following, we show that these two equations reduce to the bilinear form for $P_{II}$ (5) and (6) on the conditions of reduction for the algebraic solutions.

Now we discuss the algebraic solutions for the KP hierarchy. We can easily verify that the polynomials $p_k(x)$ defined by eq.(7) satisfy

$$\partial_x p_k(x) = p_{k-n}(x),$$

and hence eq.(13). Taking $f_k$ in the $\tau$ function (12) as $p_{i_k+N-k}(x)$, we have,

**Proposition 3.4** Let $Y = (i_1, i_2, \cdots, i_N)$, where $i_1 \geq i_2 \geq \cdots \geq i_N \geq 0$ be integers, be a Young diagram. Then

$$\tau_{Y,KP} = \begin{vmatrix} p_{i_1}(x) & p_{i_1+1}(x) & \cdots & p_{i_1+N-1}(x) \\ p_{i_2-1}(x) & p_{i_2}(x) & \cdots & p_{i_2+N-2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{i_N-N+1}(x) & p_{i_N-N+2}(x) & \cdots & p_{i_N}(x) \end{vmatrix},$$

(18)
gives the algebraic solution for the KP hierarchy.

The polynomial $\tau_{Y,KP}$ is called the Schur polynomial attached to the Young diagram $Y$. We note that if we define the weight of $x_n$ as $n$, then $p_k(x)$ is a polynomial with homogeneous weight $k$ and $\tau_{Y,KP}$ is also homogeneous with the weight $|Y| = i_1 + i_2 + \cdots + i_N$. This $\tau$ function gives the rational solution of eq.(10) by the dependent variable transformation (11).

Let us apply the reduction to the KdV hierarchy. This is achieved by dropping the dependence of $x_2, x_4, \cdots$, in the $\tau$ functions of KP hierarchy. In order to realize this condition, it is sufficient to choose $Y$ as $(N, N-1, \cdots, 1)$ in the algebraic solution for the KP hierarchy (18).

**Proposition 3.5**

$$\tau_{N,KdV} = \begin{vmatrix} p_N(x) & p_{N+1}(x) & \cdots & p_{2N-1}(x) \\ p_{N-2}(x) & p_{N-1}(x) & \cdots & p_{2N-3}(x) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-N+2}(x) & p_{-N+3}(x) & \cdots & p_1(x) \end{vmatrix},$$

(19)
gives the algebraic solution of the KdV hierarchy.
Proposition 3.5 can be easily verified noticing that
\[
\frac{\partial \tau_{N, \text{KdV}}}{\partial x_j} = 0, \quad j = 1, 2, 3 \cdots ,
\] (20)
which directly follows from eq. (17). From eqs. (15), (16) and (20), it is clear that the \( \tau \) function (19) satisfies the following bilinear equations,
\[
D^2_{x_1} \tau_{N+1, \text{KdV}} \ast \tau_{N, \text{KdV}} = 0 ,
\] (21)
\[
(D^3_{x_1} - 4D_{x_3}) \tau_{N+1, \text{KdV}} \ast \tau_{N, \text{KdV}} = 0 .
\] (22)
The modified KdV equation,
\[
v_{x_3} + \frac{3}{2} v^2 v_{x_1} - \frac{1}{4} v_{x_1 x_1 x_1} = 0,
\] (23)
is obtained from eqs. (21) and (22) by the dependent variable transformation,
\[
v = \frac{\partial}{\partial x_1} \log \frac{\tau_{N+1, \text{KdV}}}{\tau_{N, \text{KdV}}} .
\] (24)

IV. Rational Solutions for P\(_\text{II}\): Devisme Polynomial Representation

Now we give a determinant representation for the rational solutions of P\(_\text{II}\). We first give the definition of the Devisme polynomials.\[6],[7]\]

**Definition 4.1** The Devisme polynomials \( q_k(x_1, x_2, \cdots, x_m), \) \( k = 0, 1, 2 \cdots \), are polynomials in \( x_1, \cdots, x_m \) defined by
\[
\sum_{k=0}^{\infty} q_k(x_1, x_2, \cdots, x_m) \lambda^k = \exp \left( x_1 \lambda + x_2 \lambda^2 + \cdots + x_m \lambda^m + \frac{1}{m+1} \lambda^{m+1} \right) .
\] (25)

Then one of our main results is stated as follows.

**Theorem 4.2** Let \( q_k(z, t), \) \( k = 0, 1, 2, \cdots \), be the Devisme polynomials and \( \tau_N \) be an \( N \times N \) determinant defined by
\[
\tau_N = \begin{vmatrix}
q_N(z, t) & q_{N+1}(z, t) & \cdots & q_{N-1}(z, t) & q_{2N-1}(z, t) \\
q_{N-2}(z, t) & q_{N-1}(z, t) & \cdots & q_{2N-3}(z, t) & q_{2N-2}(z, t) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{-N+2}(z, t) & q_{-N+3}(z, t) & \cdots & q_1(z, t) & q_{-1}(z, t)
\end{vmatrix}, \quad q_k(z, t) = 0 \text{ for } k < 0 .
\] (26)
Then
\[ v = \frac{d}{dz} \log \frac{\tau_{N+1}}{\tau_N}, \]  
(27)
gives a rational solution for \( P_{11} \) (4) with \( \alpha = N + 1 \).

**Remark 4.3**

1. The \( \tau \) function (26) is derived only by putting
\[ x_1 = z, \quad x_2 = t, \quad x_3 = \frac{1}{3}, \quad x_4 = x_5 = \cdots = 0, \]  
(28)
in eq. (13). Namely, the rational solutions of \( P_{11} \) are given in terms of the special case of the Schur polynomials.

2. The \( \tau \) function (26) itself does not depend on \( t \), but we have left \( t \) dependence in the entries in order to relate the solutions with the Devisme polynomials.

Theorem 4.2 is a direct consequence of the following proposition.

**Proposition 4.4** The \( \tau \) function \( \tau_N \) (26) satisfies the bilinear equations (5) and (6) with \( \alpha = N + 1 \).

**Proof.** Putting \( x_5 = x_7 = \cdots = 0 \) in the rational solutions of KdV hierarchy (19), it is readily seen that \( \tau_{N,\text{KdV}} \) is a homogeneous weight polynomial in \( x_1 \) and \( x_3 \) with weight \( \frac{N(N+1)}{2} \). Hence, if we put
\[ f_N = \frac{1}{x_3^{N(N+1)/6}} \tau_{N,\text{KdV}}, \]  
(29)
then \( f_N \) depends only on \( t = \frac{x_1}{x_3^{1/3}} \). Thus we have
\[ \partial_{x_3} f_N = \frac{\partial t}{\partial x_3} \frac{d}{dt} f_N, \quad \partial_{x_1} f_N = \frac{\partial t}{\partial x_1} \frac{d}{dt} f_N, \]  
(30)
which yield
\[ \partial_{x_3} \tau_{N,\text{KdV}} = \frac{1}{3x_3} \left( \frac{N(N+1)}{2} \tau_{N,\text{KdV}} - x_1 \partial_{x_1} \tau_{N,\text{KdV}} \right). \]  
(31)
Substituting eq. (31) into eq. (22), we get
\[ \left( D_{x_1}^3 + \frac{4}{3x_3} x_1 D_{x_1} - \frac{4}{3x_3} (N + 1) \right) \tau_{N+1,\text{KdV}} \cdot \tau_{N,\text{KdV}} = 0. \]  
(32)
Moreover, by putting \( z = x_1 \) and \( x_3 = \frac{1}{3} \) \( \tau_{N,\text{KdV}} \) reduces to \( \tau_N \) in (26) and eqs. (3) and (5) with \( f = \tau_N, \ g = \tau_{N+1} \) and \( \alpha = N + 1 \) are obtained from eqs. (21) and (32). \( \Box \)
V. Hankel Determinant Solution for Toda Lattice

Let us consider the Toda lattice equation,
\[
\frac{d^2 u_N}{dz^2} = e^{u_{N-1}} - u_N - e^{u_{N+1}},
\]
with the symmetric lattice condition,
\[
u_N = u_{-N-1}.
\]

It is easy to see that eq. (33) is bilinearized through the dependent variable transformation,
\[
u_N = \log \frac{\tau_{N-1}}{\tau_N},
\]
from which we get
\[
D_z^2 f_N \cdot f_N = 2(f_{N+1} f_{N-1} - f_N f_N), \quad f_N = f_{-N-1}.
\]

Here we call this type of symmetric lattice as B-type Toda lattice because it concerns the BKP hierarchy\[8\],\[10\]. Using the gauge freedom, we can translate the above B-type Toda lattice equation in the following form,
\[
(D_z^2 + 2a_0)\sigma_N \cdot \sigma_N = 2\sigma_{N+1}\sigma_{N-1},
\]
where \(\sigma_N = f_N/f_0\) and \(a_0 = f_1/f_0\). It is clear that \(\sigma_N\) satisfies
\[
\sigma_N = \sigma_{-N-1},
\]
\[
\sigma_0 = 1, \quad \sigma_1 = a_0.
\]

It is possible to express the general solution of eqs. (37)-(39) in determinant form\[11\].

**Proposition 5.1** The general solution for the equations (37)-(39) for an arbitrary \(a_0\) is given in the Hankel determinant form
\[
\sigma_N = \begin{vmatrix}
a_0 & a_1 & \cdots & a_{N-1} \\
a_1 & a_2 & \cdots & a_N \\
\vdots & \vdots & \ddots & \vdots \\
a_{N-1} & a_N & \cdots & a_{2N-2}
\end{vmatrix}, \quad N \geq 0
\]
where \(a_n, \ n = 1, 2, 3, \cdots\) are recursively defined by
\[
a_{n+1} = \frac{d a_n}{dz} + \sum_{k=0}^{n-1} a_k a_{n-k-1}, \quad n \geq 0.
\]

This contains one arbitrary function \(a_0\), hence it gives the general solution for the B-type Toda lattice equation.
VI. Rational Solutions for $P_{II}$: Hankel Determinant Representation

The rational solutions for $P_{II}$ are derived only by putting

$$a_0 = z,$$

in the above $\sigma_N$.

**Theorem 6.1** Let $a_n$, $n = 0, 1, 2, \cdots$, be polynomials defined by

$$a_{n+1} = \frac{d a_n}{dz} + \sum_{k=0}^{n-1} a_k a_{n-k-1}, \quad n \geq 0, \quad a_0 = z,$$

and let $\sigma_N$ be an $N \times N$ determinant given by eq.(40). Then

$$v = \frac{d}{dz} \log \frac{\sigma_{N+1}}{\sigma_N},$$

(44)

gives a rational solution for $P_{II}$ (4) with $\alpha = N + 1$.

Similar to the previous section, theorem 6.1 is a direct consequence of the following proposition.

**Proposition 6.2** $f = \sigma_N$ and $g = \sigma_{N+1}$ satisfies the bilinear equations (3) and (4) with $\alpha = N + 1$.

To prove proposition 6.2, let us first introduce the notation $\sigma_{NY}$:

**Definition 6.3** Let $Y = (i_1, i_2, \cdots, i_h)$ be a Young diagram. Then we define an $N \times N$ determinant $\sigma_{NY}$ by

$$\sigma_{NY} = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-h-1} & a_{N-h+i_1} & \cdots & a_{N-2+i_1} & a_{N-1+i_1} \\ a_1 & a_2 & \cdots & a_{N-h} & a_{N-h+1+i_1} & \cdots & a_{N-1+i_2} & a_{N+i_1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{N-1} & a_N & \cdots & a_{2N-h-2} & a_{2N-h+1+i_1} & \cdots & a_{2N-3+i_2} & a_{2N-2+i_1} \end{vmatrix}. \quad (45)$$

We first construct the shift operators which are differential operators generating $\sigma_{NY}$ from $\sigma_N$. If entries of the determinant satisfy simple equations like (13), then construction of the shift operators is straightforward. But when we have to work on more complicated relations among the entries like (41), it is useful to apply the technique developed in [12].

We can prove the following lemma.
Lemma 6.4

\[ \sigma_{N\Omega} = \frac{d}{dz} \sigma_N. \] (46)

**Proof.** Notice that \( \sigma_{N\Omega} \) is expressed by

\[
\sigma_{N\Omega} = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \\ a_2 & a_3 & \cdots & a_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_N & a_{N+1} & \cdots & a_{2N-1} \end{pmatrix} \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix},
\] (47)

where \( \Delta_{ij} \) is the \((i, j)\)-cofactor of \( \sigma_N \) and \( A \cdot B \) denotes a standard scalar product for \( N \times N \) matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) which is defined as

\[ A \cdot B = \sum_{i,j=1}^N a_{ij} b_{ij} = \text{trace} A^t B. \] (48)

The first matrix of (47) is rewritten by using the recursion relation (11) as

\[
\begin{pmatrix}
\partial_z a_0 & \partial_z a_1 & \cdots & \partial_z a_{N-1} \\
\partial_z a_1 & \partial_z a_2 & \cdots & \partial_z a_N \\
\vdots & \vdots & \ddots & \vdots \\
\partial_z a_{N-1} & \partial_z a_N & \cdots & \partial_z a_{2N-2} \\
\end{pmatrix}
= \begin{pmatrix}
0 & a_0^2 & \cdots & \sum_{k=0}^{N-2} a_k a_{N-k-2} \\
0 & a_0 a_1 & \cdots & \sum_{k=0}^{N-1} a_k a_{N-k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=0}^{N-2} a_k a_{N-k-2} & \sum_{k=0}^{N-1} a_k a_{N-k-1} & \cdots & \sum_{k=0}^{2N-3} a_k a_{2N-k-3} \\
\end{pmatrix} + \begin{pmatrix}
0 & \sum_{k=0}^{N-2} a_k a_{N-k} \\
0 & \sum_{k=0}^{N-2} a_k a_{N-k} \\
\vdots & \vdots & \vdots \\
0 & \sum_{k=0}^{N-2} a_k a_{2N-k} \\
\end{pmatrix}.
\] (49)

The above second term is separated as

\[
\begin{pmatrix}
0 & \sum_{k=0}^{N-2} a_k a_{N-k} \\
0 & \sum_{k=0}^{N-2} a_k a_{N-k} \\
\vdots & \vdots & \vdots \\
0 & \sum_{k=0}^{N-2} a_k a_{2N-k} \\
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \cdots & 0 \\
a_0^2 & a_0 a_1 & \cdots & a_{N-1} a_0 \\
\vdots & \vdots & \ddots & \vdots \\
\sum_{k=0}^{N-2} a_k a_{N-k} & \sum_{k=1}^{N-1} a_k a_{N-k} & \cdots & \sum_{k=N-1}^{2N-3} a_k a_{2N-k} \\
\end{pmatrix}.
\] (50)
Each of these terms gives zero contribution in (17). Hence we have proved lemma 5.4. □

Next we have:

**Lemma 6.5**

\[
\sigma_{N□} + \sigma_{N□} = \left( \frac{d^2}{dz^2} + z \right) \sigma_N, \quad (51)
\]

\[
\sigma_{N□} - \sigma_{N□} = (2N - 1)z\sigma_N. \quad (52)
\]

**Proof.** We consider

\[
\sigma_{N□} + \sigma_{N□} = \left( \begin{array}{cccc}
    a_1 & a_2 & \cdots & a_{N-1} & a_{N+1} \\
    a_2 & a_3 & \cdots & a_N & a_{N+2} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    a_N & a_{N+1} & \cdots & a_{2N-2} & a_{2N}
\end{array} \right) \cdot \left( \begin{array}{cccc}
    \Delta_{□11} & \Delta_{□12} & \cdots & \Delta_{□1N} \\
    \Delta_{□21} & \Delta_{□22} & \cdots & \Delta_{□2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    \Delta_{□N1} & \Delta_{□N2} & \cdots & \Delta_{□NN}
\end{array} \right), \quad (53)
\]

where \( \Delta_{□ij} \) is \((i, j)\) cofactor of \(\sigma_{N□}\). The first matrix in the right-hand side is equal to

\[
\left( \begin{array}{cccc}
    \partial_z a_0 & \partial_z a_1 & \cdots & \partial_z a_{N-2} & \partial_z a_N \\
    \partial_z a_1 & \partial_z a_2 & \cdots & \partial_z a_{N-1} & \partial_z a_{N+1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \partial_z a_{N-1} & \partial_z a_N & \cdots & \partial_z a_{2N-3} & \partial_z a_{2N-1}
\end{array} \right)
\]

\[
+ \left( \begin{array}{cccc}
    0 & a_0 & \cdots & \sum_{k=0}^{N-3} a_k a_{N-k-3} & \sum_{k=0}^{N-1} a_k a_{N-k-1} \\
    0 & a_0 a_1 & \cdots & \sum_{k=0}^{N-3} a_k a_{N-k-2} & \sum_{k=0}^{N-1} a_k a_{N-k} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & a_0 a_{N-1} & \cdots & \sum_{k=0}^{N-3} a_k a_{2N-k-4} & \sum_{k=0}^{N-1} a_k a_{2N-k-2}
\end{array} \right)
\]

\[
+ \left( \begin{array}{cccc}
    0 & 0 & \cdots & 0 & 0 \\
    a_0^2 & a_1 a_0 & \cdots & a_{N-2} a_0 & a_N a_0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \sum_{k=0}^{N-2} a_k a_{N-k-2} & \sum_{k=1}^{N-1} a_k a_{N-k-1} & \cdots & \sum_{k=0}^{2N-4} a_k a_{2N-k-4} & \sum_{k=0}^{2N-2} a_k a_{2N-k-2}
\end{array} \right). \quad (54)
\]

Taking the scalar product, the first and second terms give \( \partial_z \sigma_{N□} \) and \( a_0 \sigma_N \), respectively, and the third term vanishes. Hence we have

\[
\sigma_{N□} + \sigma_{N□} = \left( \frac{d^2}{dz^2} + z \right) \sigma_N. \quad (55)
\]
Next we consider the following equality,

\[ \sigma_{Nn} - \sigma_{N0} = \begin{pmatrix} a_2 & a_3 & \cdots & a_{N+1} \\ a_3 & a_4 & \cdots & a_{N+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N+1} & a_{N+2} & \cdots & a_{2N} \end{pmatrix} \cdot \begin{pmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1N} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{N1} & \Delta_{N2} & \cdots & \Delta_{NN} \end{pmatrix}. \quad (56) \]

The first matrix of the right hand side of (56) is rewritten as

\[
\begin{pmatrix}
\partial_z a_1 & \partial_z a_2 & \cdots & \partial_z a_N \\
\partial_z a_2 & \partial_z a_3 & \cdots & \partial_z a_{N+1} \\
\vdots & \vdots & \ddots & \vdots \\
\partial_z a_N & \partial_z a_{N+1} & \cdots & \partial_z a_{2N-1} \\
\end{pmatrix}
\begin{pmatrix}
a_0^2 & a_0 a_1 + a_1 a_0 & \cdots & \sum_{k=0}^{N-1} a_k a_{N-k-1} \\
a_0 a_1 & a_0 a_2 + a_1 a_1 & \cdots & \sum_{k=0}^{N-1} a_k a_{N-k} \\
\vdots & \vdots & \ddots & \vdots \\
a_0 a_{N-1} & a_0 a_N + a_1 a_{N-1} & \cdots & \sum_{k=0}^{N-1} a_k a_{2N-k-2} \\
\sum_{k=1}^{N-1} a_k a_{N-k-1} & \sum_{k=2}^{N} a_k a_{N-k} & \cdots & \sum_{k=N}^{2N-2} a_k a_{2N-k-2} \\
\end{pmatrix}.
\quad (57)
\]

Here, we note that \( a_n \)'s also satisfy

\[
\partial_z a_{n+1} = 2na_{n-1},
\quad (58)
\]

which is proved by induction from eqs. (411) and (412). The first term of the right hand side of eq. (57) is rewritten by using eq. (58) as

\[
\begin{pmatrix}
0 & 2a_0 & \cdots & 2(N-1)a_{N-2} \\
0 & 2a_1 & \cdots & 2(N-1)a_{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 2a_{N-1} & \cdots & 2(N-1)a_{2N-3} \\
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \cdots & 0 \\
2a_0 & 2a_1 & \cdots & 2a_{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
2(N-1)a_{N-2} & 2(N-1)a_{N-1} & \cdots & 2(N-1)a_{2N-3} \\
\end{pmatrix}.
\quad (59)
\]

Applying the scalar product on these terms, we obtain

\[
\sigma_{Nn} - \sigma_{N0} = (2N-1)z\sigma_N.
\quad (60)
\]
Hence we have proved lemma 6.5. 

Continuing the similar argument, we get the following shift operators.

**Lemma 6.6**

\[
\sigma_{N+2} + 2\sigma_{N} + \sigma_{N+1} = \left( \frac{d^3}{dz^3} + 3z \frac{d}{dz} + 1 \right) \sigma_N, \\
\sigma_{N-1} - \sigma_N = \left( (2N - 1)z \frac{d}{dz} + (2N + 1) \right) \sigma_N, \\
\sigma_{N+1} - \sigma_N + \sigma_{N+2} = \left( 2z \frac{d}{dz} + 2(N^2 + N - 1) \right) \sigma_N.
\] (61) (62) (63)

Finally, we prove proposition 6.2. From the Plücker relations, we have

\[
\sigma_{N+1} \sigma_N - \sigma_{N+1} \sigma_{N+1} + \sigma_N \sigma_N = 0, \\
\sigma_{N+2} \sigma_N - \sigma_{N+2} \sigma_{N+1} + \sigma_{N+1} \sigma_N = 0,
\] (64) (65)

which are essentially the same as the bilinear equations (15) and (16) for \( \tau_{N,KP} \). By using lemmas 6.4, 6.5 and 6.6, we get

\[
D^2 z \sigma_{N+1} \cdot \sigma_N = 0, \\
(D^3 z + 4z \sigma_N + 4(N + 1)) \sigma_{N+1} \cdot \sigma_N = 0
\] (66) (67)

which are the desired result. Thus we have proved proposition 6.2.

**VII. Concluding Remarks**

In this article, we have presented two types of determinant representations for the rational solutions of \( P_{II} \). The Devisme polynomial representation follows from the reduction procedure of modified KdV equation and the Hankel determinant representation is obtained from the Toda lattice equation, namely the Bäcklund transformation of the solution of \( P_{II} \). These determinant structures of the rational solutions of \( P_{II} \) exactly reflect the Wronskian structure of the solution of KP hierarchy and Toda lattice equation. The relationship between those two representations is not clear yet. At least, it seems that there is no simple transformation relating the two representations.
It is known that the Airy function type solutions of P$_{II}$ are expressed as

$$v = \frac{d}{dz} \log \frac{\rho_{N+1}}{\rho_N},$$  \hspace{1cm} (68)

$$\rho_N = \begin{vmatrix} Ai & \frac{d}{dz} Ai & \cdots & \frac{d^{N-1}}{dz^{N-1}} Ai \\ \frac{d}{dz} Ai & \frac{d^2}{dz^2} Ai & \cdots & \frac{d^N}{dz^N} Ai \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{N-1}}{dz^{N-1}} Ai & \frac{d^N}{dz^N} Ai & \cdots & \frac{d^{2N-2}}{dz^{2N-2}} Ai \end{vmatrix},$$  \hspace{1cm} (69)

where $Ai$ is the Airy function satisfying

$$\frac{d^2}{dz^2} Ai = z Ai.$$  \hspace{1cm} (70)

Then $v$ satisfies P$_{II}$,

$$\frac{d^2 v}{dz^2} = 2v^3 - 2zv + (2N + 1).$$  \hspace{1cm} (71)

In [12], it was shown that the $\tau$ function (68) can be reduced from that of the KP hierarchy (12).

In the theory of KP hierarchy, an important fact is that we can introduce the $\tau$ function which is expressed in terms of determinant. Based on this fact, we can identify the solution space, and the KP hierarchy is regarded as the dynamical system on the infinite dimensional Grassmann manifold. P$_{II}$ is obtained from the similarity reduction of the modified KdV equation, but the parameter of the equation appears as the integration constant, which means that P$_{II}$ has the information of various boundary conditions of the modified KdV equation. From this observation, it looks that one cannot expect such beautiful structures in the solution space of P$_{II}$. Nevertheless, the results in this article may imply that at least for the special function type solutions and the rational solutions, such structures in the solutions of KP hierarchy may survive through the reduction. It may be an interesting problem to investigate the determinant structures for other Painlevé equations. So far, this is completely an open problem.

Recently, discrete versions of the Painlevé equations have been proposed through the singularity confinement test[13]. As for the solutions, some of them admit discrete or q-difference analog of special function type solutions expressed by determinants[14],[15]. Moreover, it was reported that the discrete Painlevé II equation admits rational solutions with determinant structure[16]. We might expect that through such determinant structures
of solutions, similarity reductions\textsuperscript{[17]} deriving the discrete Painlevé equations from discrete KP (or Toda) would become more transparent, as we have seen in the continuous $\text{P}_{II}$ case.

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