Time evolution of Wigner function in laser process derived by entangled state representation

Li-yun Hu$^{1,2*}$ and Hong-yi Fan$^2$

$^1$College of Physics & Communication Electronics, Jiangxi Normal University, Nanchang 330022, China
$^2$Department of Physics, Shanghai Jiao Tong University, Shanghai, 200030, P.R. China

*Corresponding author. E-mail addresses: hlyun2008@126.com

Evaluating the Wigner function of quantum states in the entangled state representation is introduced, based on which we present a new approach for deriving time evolution formula of Wigner function in laser process. Application of this formula to calculating time evolution of photon number is also presented, as an example, the case when the initial state is photon-added coherent state is discussed.

I. INTRODUCTION

One of the major topics in Quantum Statistical Mechanics is the evolution of pure states into mixed states [1, 2]. Such evolution usually happens when a system is immersed in a thermal environment, or a signal (a quantum state) passes through a quantum channel, and is described by a master equation. Alternately, description of evolution of density matrices $\rho$ can be replaced by its Wigner function’s evolution in phase space [3, 4]. The partial negativity of Wigner function can be considered as an indicator of nonclassicality of quantum state. On the basis of the entangled state representation and the thermo field dynamics we present a new approach for deriving time evolution formula of Wigner function in amplitude-damping channel and laser process. Application of this formula to calculating time evolution of photon number is also presented, as an example, the case when the initial state is photon-added coherent state is discussed.

II. WIGNER FUNCTION FORMULA IN THermo ENTANGLED STATE REPRESENTATION

We begin with briefly reviewing the thermo entangled state representation (TESR). On the basis of Umezawa-Takahashi thermo field dynamics (TFD) [5, 6, 7] we have constructed the TESR in doubled Fock space [8, 9],

$$|\eta\rangle = \exp \left[ -\frac{1}{2}|\eta|^2 + \eta a^\dagger - \eta^* a^\dagger + a^\dagger a^\dagger \right] |0, \bar{0}\rangle,$$

or

$$|\eta\rangle = D(\eta) |\eta = 0\rangle, \quad D(\eta) = e^{\eta a^\dagger - \eta^* a},$$

where $D(\eta)$ is the displacement operator, $\hat{a}^\dagger$ is a fictitious mode accompanying the real photon creation operator $a^\dagger$, $|0, \bar{0}\rangle = |0\rangle |\bar{0}\rangle$, and $|\bar{0}\rangle$ is annihilated by $\hat{a}$, $[\hat{a}, \hat{a}^\dagger] = 1$.

Operating $a$ and $\hat{a}$ on $|\eta\rangle$ in Eq.(1) we obtain the eigen-equations of $|\eta\rangle$,

$$(a - \hat{a}^\dagger) |\eta\rangle = \eta |\eta\rangle, \quad (a^\dagger - \hat{a}) |\eta\rangle = \eta^* |\eta\rangle,$$

$$(\eta) (a^\dagger - \hat{a}) = \eta^* (|\eta\rangle), \quad (\eta^* - \eta) |\eta\rangle = \eta (\eta^* - \eta^*).$$

Note that $[(a - \hat{a}^\dagger), (a^\dagger - \hat{a})] = 0$, thus $|\eta\rangle$ is the common eigenvector of $(a - \hat{a}^\dagger)$ and $(\hat{a} - a^\dagger)$. Using the normally ordered form of vacuum projector $|0\rangle \langle 0| = \exp(-a^\dagger \hat{a} - \hat{a}^\dagger a):$, and the technique of integration within an ordered product (IWOP) of operators [10, 11, 12], we can easily prove that $|\eta\rangle$ is complete and orthonormal,

$$\int \frac{d^2\eta}{\pi} |\eta\rangle \langle \eta| = 1, \quad \langle \eta^\prime | \eta\rangle = \pi \delta (\eta^\prime - \eta) \delta (\eta^* - \eta^*).$$

It is easily seen that $|\eta = 0\rangle$ has the properties

$$|\eta = 0\rangle = e^{a^\dagger a^\dagger} |0, \bar{0}\rangle = \sum_{n=0}^{\infty} |n, \bar{n}\rangle,$$

and

$$a |\eta = 0\rangle = \hat{a}^\dagger |\eta = 0\rangle,$$

$$(a^\dagger a)^n |\eta = 0\rangle = (\hat{a}^\dagger \hat{a})^n |\eta = 0\rangle.$$
By using \( \langle \tilde{n} | \tilde{m} \rangle = \delta_{n,m} \) and introducing \( |\rho\rangle \equiv |I\rangle \) we can reform Eq.\((7)\) as

\[
W(\alpha) = \sum_{m,n}^{\infty} \langle n, \tilde{n} | \Delta(\alpha) \rho | m, \tilde{m} \rangle \\
= \frac{1}{\pi} \langle \eta = 0 | D(2\alpha) (-1)^{a^\dagger a} | \rho \rangle \\
= \frac{1}{\pi} \langle \eta = -2\alpha | (-1)^{a^\dagger a} | \rho \rangle \\
= \frac{1}{\pi} \langle \xi = 2\alpha | \rho \rangle ,
\]

(9)

where \( |\xi\rangle \) is defined as

\[
|\xi\rangle_{\eta} = (-1)^{a^\dagger a} |\eta = -\xi\rangle \\
= \exp \left( -\frac{1}{2} \xi \xi^* + \xi a^\dagger + \xi^* a^\dagger - a^\dagger a^\dagger \right) |0, \tilde{0}\rangle \\
= D(\xi) e^{-a^\dagger a^\dagger} |0, \tilde{0}\rangle .
\]

(10)

It can be proved that

\[
\langle \eta | \xi \rangle = \frac{1}{\pi} \exp \left( \frac{\xi \eta^* - \xi^* \eta}{2} \right),
\]

(11)

a Fourier transformation kernel, so \( |\xi\rangle \) can be considered the conjugate state of \( |\eta\rangle \), which also possess orthonormal and complete properties

\[
\int \frac{d^2 \xi}{\pi} |\xi\rangle \langle \xi| = 1 , \quad \langle \xi' | \xi \rangle = \pi \delta (\xi' - \xi) \delta (\xi^* - \xi^*) .
\]

(12)

Eq.\((10)\) is just a new formula for evaluating the Wigner function of quantum states: by calculating the overlap between two “pure states” in enlarged Fock space rather than using the ensemble average in real mode space.

For example, for number state \(|n\rangle \langle n|\), noticing \(|n\rangle \langle n|I\rangle = |n, \tilde{n}\rangle\), and the generating function of two-variable Hermite polynomial \([12, 18]\) \(H_{m,n}(x,y)\),

\[
\sum_{m,n}^{\infty} \frac{t^m t^n}{m! n!} H_{m,n}(x,y) = \exp [-tt' + tx + t'y] ,
\]

(13)

we see

\[
W_{|n\rangle \langle n|}(\alpha) = \frac{1}{\pi} \langle n, \tilde{n} | \xi_{=2\alpha} \rangle = \frac{1}{\pi} e^{\frac{1}{2} \xi^2} H_{n,n}(\xi, \xi^*) \\
= (-1)^n e^{-2|\alpha|^2} L_n (4|\alpha|^2) ,
\]

(14)

in the last step in Eq.\((13)\) we have used the relation between \(H_{m,n}(x,y)\) and Laguerre polynomial \(L_m(x)\) \([17]\),

\[
L_n(xy) = \frac{(-1)^n}{n!} H_{n,n}(x,y) .
\]

(15)

Similarly, for coherent state \(|z\rangle \langle z| = \exp(-|z|^2/2 + za^\dagger)|0\rangle\) \([18, 19]\), due to \(|z\rangle \langle z|I\rangle = D(z) D(z^*)|00\rangle = |z, \tilde{z}\rangle\), we have

\[
W_{|z\rangle \langle z|}(\alpha) = \frac{1}{\pi} \langle 0, \tilde{0} | \exp (-2|\alpha|^2 + 2\alpha^* a^\dagger + 2a^\dagger a - a a^\dagger) |z, \tilde{z}\rangle \\
= \frac{1}{\pi} \exp \left[ -2|\alpha - z|^2 \right] .
\]

(16)

Further, using Eq.\((11)\) and the completeness of \(\langle \eta \rangle\) in Eq.\((4)\), we can reform Eq.\((9)\) as

\[
W(\alpha) = \int \frac{d^2 \eta}{\pi^2} \langle \xi = 2\alpha | \eta \rangle \langle \eta | \rho \rangle = \int \frac{d^2 \eta}{2\pi^2} e^{\alpha \eta^* - \alpha^* \eta} \langle \eta | \rho \rangle .
\]

(17)

Once \(\langle \eta | \rho \rangle\) is known, one can calculate the Wigner function by taking the Fourier transform of \(\langle \eta | \rho \rangle\). Eqs.\((9)\) and \((13)\) are two ways accessing to Wigner function, we can use either one to derive Wigner functions.

### III. EVOLUTION FORMULA OF WIGNER FUNCTION FOR AMPLITUDE DAMPING CHANNEL

In this section, we consider Wigner function’s time evolution in the amplitude decay channel (dissipation in a lossy cavity) described by the following master equation \([20]\)

\[
\frac{d\rho}{dt} = \kappa (2a^\dagger a \rho - a^\dagger a \rho - \rho a^\dagger a) ,
\]

(18)

where \(\kappa\) is the rate of decay. In Ref. \([21]\) we have reformed \((18)\) as

\[
\frac{d\rho}{dt} = \kappa (2a^\dagger a \rho - \rho a^\dagger a - \rho a^\dagger a) ,
\]

(19)

thus the formal solution of Eq.\((19)\) is

\[
|\rho(t)\rangle = e^{\kappa t (a^\dagger a - a a^\dagger + 1)} e^{(1-e^{2\kappa t})(a^\dagger a - a a^\dagger)/2} |\rho_0\rangle .
\]

(20)

Then projecting Eq.\((20)\) on \(\langle \eta \rangle\), and noticing \(\exp [\kappa t (a a^\dagger - a^\dagger a)]\) being the two-mode squeezing operator,

\[
\langle \eta \rangle \exp [\kappa t (a a^\dagger - a^\dagger a)] = e^{-\kappa t} \langle \eta e^{-\kappa t} \rangle ,
\]

(21)

as well as Eq.\((3)\), we obtain

\[
\langle \eta | \rho(t) \rangle = e^{-\frac{1}{2} T|\eta|^2} \langle \eta e^{-\kappa t} | \rho_0 \rangle ,
\]

(22)

where \(T = 1 - e^{-2\kappa t}\). Substituting Eq.\((22)\) into Eq.\((17)\), we derive the Wigner function at time \(t\)

\[
W(\alpha, t) = \int \frac{d^2 \eta}{2\pi^2} e^{\alpha \eta^* - \alpha^* \eta - \frac{1}{2} T|\eta|^2} \langle \eta e^{-\kappa t} | \rho_0 \rangle .
\]

(23)

Inserting the completeness relation \((12)\) into Eq.\((23)\) and noticing Eqs.\((3)\) as well as \((14)\), we can reform Eq.\((23)\)
as
\[ W(\alpha, t) = \int \frac{d^2 \xi}{\pi} \int \frac{d^2 \eta}{\pi^2} e^{-\frac{1}{2} t |\eta|^2} \langle \eta e^{-\kappa} | \xi' \rangle \langle \xi | \rho_0 \rangle \]
\[ = \int \frac{d^2 \beta d^2 \eta}{\pi^2} e^{-\frac{1}{2} t |\eta|^2 + \alpha (\alpha - \beta e^{-\kappa t})} + \eta^* (\beta e^{-\kappa t} - \alpha) \]
\[ = \frac{2}{T} \int \frac{d^2 \beta}{\pi} \exp \left[ -\frac{2}{T} |\alpha - \beta e^{-\kappa t}|^2 \right] W(\beta, 0), \]  \hspace{1cm} (24)
where \( W(\beta, 0) \) is the Wigner function at initial time, and we have used the following integral formula 17
\[ \int \frac{d^2 z}{\pi} \exp \left( \frac{1}{\zeta} (\beta + \eta z^*) \right) = \frac{1}{\zeta} e^{-\frac{\xi}{\zeta}}, \text{Re}(\zeta) < 0. \]  \hspace{1cm} (25)
Eq. (24) is the expression of time evolution of Wigner function for amplitude damping channel.

For example, for the photon-added coherent state \( C_m a^m \ket{\beta} \), where \( C_m = [m L_m (|\beta|^2)]^{-\frac{1}{2}} \) is the normalization factor, the initial Wigner function \( W(\beta, 0) \) is given by 22
\[ W(\beta, 0) = \frac{(-1)^m e^{-2|\beta - z|^2}}{\pi L_m (|\beta|^2)} L_m (|2\beta - z|^2). \]  \hspace{1cm} (26)
Substituting Eq. (26) into Eq. (24) and using Eq. (15) as well as the another generating function of \( H_{m,n}(x,y) \),
\[ H_{m,n}(x,y) = \frac{\partial^{m+n}}{\partial \tau^m \partial \tau^n} \exp [-\tau \tau' + \tau x + \tau' y] \bigg|_{\tau=\tau'=0}, \]  \hspace{1cm} (27)
we have
\[ W(\alpha, t) = \frac{2}{T \pi m!} \frac{2 \pi^2 e^{-2|\beta - z|^2}}{\partial \beta^m} e^{-\frac{1}{2} t z^* + \frac{1}{2} \frac{\alpha}{T} e^{-\kappa t} + \tau} \]
\[ \int \frac{d^2 \beta}{\pi} \exp \left[ -\frac{2}{T} |\beta|^2 + 2 \beta \left( z^* + \frac{\alpha}{T} e^{-\kappa t} + \tau \right) \right]_{\tau=\tau'=0} + 2 \beta^* \left( z + \frac{\alpha}{T} e^{-\kappa t} + \tau' \right) \]
\[ = \frac{2}{T \pi m!} \frac{2 \pi^2 e^{-2|\beta - z|^2}}{\partial \beta^m} \exp \left[ \left( 1 - 2 e^{-2\kappa t} \right) \tau \right] \]
\[ + \left[ \left( 1 - 2 e^{-2\kappa t} \right) z + 2 \alpha e^{-\kappa t} \right] \tau \]
\[ + \left[ \left( 1 - 2 e^{-2\kappa t} \right) z^* + 2 \alpha^* e^{-\kappa t} \right] \tau' \bigg|_{\tau=\tau'=0}. \]  \hspace{1cm} (28)
With use of a scaled transformation in the right-hand part of Eq. (28) we finally get
\[ W(\alpha, t) = \frac{1}{\pi L_m (|\beta|^2)} \frac{2 \pi^2 e^{-2|\alpha - z e^{-\kappa t}|^2}}{\partial \alpha^m} \]
\[ \times L_m \left[ -\frac{2 \alpha e^{-\kappa t} + z \left( 1 - 2 e^{-2\kappa t} \right)}{1 - 2 e^{-2\kappa t}} \right]^2, \]  \hspace{1cm} (29)
which is the analytical expression of the time evolution of Wigner function for any number \( m \) photon-added coherent state in photon loss channel 23. In particular, when \( t = 0 \), Eq. (29) just reduce to Eq. (26).

IV. EVOLUTION FORMULA OF WIGNER FUNCTION FOR LASER PROCESS

We now generalize the master equation to the case of laser theory. The mechanism of laser is described by the following master equation
\[ \frac{d\rho(t)}{dt} = \gamma g \left[ 2 a^\dagger \rho \left( t - a a^\dagger - \rho \right) a a^\dagger \right] \]
\[ + \kappa \left[ 2 \rho \left( a \right) - a \right] + a \left( a \right) a a^\dagger, \]  \hspace{1cm} (30)
where \( \gamma \) and \( \kappa \) are the cavity gain and the loss, respectively. Eq. (30) reduces to Eq. (15) when \( g = 0 \); while for \( g \to \kappa \) and \( \kappa \to \kappa (\tilde{n} + 1) \), Eq. (30) becomes
\[ \frac{d\rho(t)}{dt} = \kappa \left( \tilde{n} + 1 \right) \left[ 2 a a^\dagger - a - a a^\dagger \right] + \kappa \left[ 2 a \left( a \right) - a \right] + a \left( a \right) a a^\dagger \]  \hspace{1cm} (31)
which corresponds to the master equation in thermal environment 21.

Similar to the way of deriving Eq. (22), we have derived in Ref. 21
\[ \int \left| \rho(t) \right| = \exp \left[ \left( \frac{1}{2} \kappa + g \right) \left( 1 - e^{2\kappa - g t} \right) \left( a - a^\dagger \right) \right] \left| \rho_0 \right|, \]  \hspace{1cm} (32)
Thus the matrix element \( \langle \eta | \rho(t) \rangle \) is given by
\[ \langle \eta | \rho(t) \rangle = \exp \left[ -\frac{A}{2} |\eta|^2 \right] \left| e^{-g t} \right| \left| \rho_0 \right|, \]  \hspace{1cm} (33)
where
\[ A = \frac{\kappa + g}{\kappa - g} \left( 1 - e^{-2\kappa - g t} \right). \]  \hspace{1cm} (34)
According to Eq. (15), the Wigner function’s evolution for Laser process is given by
\[ W(\alpha, t) \]
\[ = \int \frac{d^2 \eta}{2 \pi^2} e^{-\frac{1}{4} |\eta|^2 + \frac{1}{4} |\eta|^2 + \eta^* \eta} \left| e^{-g t} \right| \left| \rho_0 \right| \]
\[ = \int \frac{d^2 \xi d^2 \eta}{\pi^2} e^{-\frac{1}{2} |\eta|^2 + \alpha^* \eta^*} \left| e^{-g t} \right| \left| \xi = 2 \beta \right) W(\beta, 0) \]
\[ = \int \frac{d^2 \xi d^2 \eta}{\pi^2} e^{-\frac{1}{2} |\eta|^2 + \eta^* \eta} \left| e^{-g t} \right| \left| \xi = 2 \beta \right) W(\beta, 0) \]
\[ = \frac{2}{2} \int \frac{d^2 \beta}{\pi} \exp \left[ -\frac{2}{A} |\alpha - \beta e^{-g t}|^2 \right] W(\beta, 0), \]  \hspace{1cm} (35)
where we have used Eq. (24). In particular, when \( g = 0 \), Eq. (35) reduces to Eq. (24). For \( g \to \kappa \) and \( \kappa \to \kappa (\tilde{n} + 1) \), leading to \( A = (2 \tilde{n} + 1) T \), Eq. (35) becomes
\[ W(\alpha, t) = \frac{2}{(2 \tilde{n} + 1) T} \int \frac{d^2 \beta}{\pi} W(\beta, 0) e^{-\frac{1}{2} |\alpha - \beta e^{-g t}|^2}, \]  \hspace{1cm} (36)
or
\[
W(\alpha, t) = 2 e^{2\alpha t} \int d^2 \beta W_T(\beta) W \left( e^{\alpha t}(\alpha - \sqrt{T} \beta), 0 \right),
\]
where
\[
W_T(\beta) = \frac{1}{\pi(2n+1)} e^{-\frac{2|\beta|^2}{nT}}
\]
is the Wigner function of the thermal state with mean photon number \(\bar{n}\).

Similar to the way of deriving Eq. (29), when the initial state is \(C_m a^\dagger m |z\rangle\), substituting Eq. (26) into Eq. (35) we have
\[
W(\beta, \beta^*, t) = \frac{e^{-C-2|\beta|^2}}{\pi L_m(-|z|^2)} A^n \frac{A^m}{(2nT + 1)} nT + 1 \times \frac{(n+1)^m}{2} L_m \left(-\frac{|B|^2}{A}\right), \tag{38}
\]
where
\[
A = 1 - \frac{e^{-2\alpha t} / T}{(2T\bar{n} + 1) (\bar{n} + 1)},
\]
\[
B = \frac{\sqrt{nT + 1} e^{-\kappa t} (2\beta^* - z^* e^{-\kappa t})}{(2nT + 1) \sqrt{\bar{n} + 1} T},
\]
\[
C = \frac{1}{(2T\bar{n} + 1) (\bar{n} + 1)} \left(3\bar{n} + 2\bar{n} \right) \sqrt{\bar{n} + 1} T, \tag{39}
\]
\[
+ \frac{2e^{-\kappa t} T^n + 1}{(2T\bar{n} + 1) (\bar{n} + 1)} (z^t \beta^* + \beta z^t).
\]
In particular, when \(\bar{n} = 0\), leading to
\[
A = \frac{1}{nT / (1 - 2e^{-2\alpha t})}, B = \frac{1}{\sqrt{T}} \left((1 - 2e^{-2\alpha t}) z^t + 2e^{-\kappa t} \beta^t\right), \quad -C - 2|\beta|^2 = -2|\beta - ze^{-\kappa t}|^2,
\]
thus Eq. (39) reduces to Eq. (29).

Eq. (38) manifestly shows that the Wigner function of \(C_m a^\dagger m |z\rangle\) in thermal environment is closely related to the Laguerre polynomials. In addition, due to \(L_m(-|z|^2) > 0\), so \(C_m > 0\), thus it is easily seen that when \(A > 0\), which means the condition
\[
k_T = \kappa t e^t = \frac{1}{2} \ln \frac{2(\bar{n} + 1)}{2nT + 1}, \tag{40}
\]
the Wigner function is always positive-definite. Thus we emphasize that for any values of \(m\), when the condition (40) is satisfied, the Wigner function has no chance to be negative.

\section{V. TIME EVOLUTION OF PHOTON NUMBER FOR THE LASER PROCESS}

Next we consider the photon number (PN) of density operator \(\rho\) for the laser process. According to the TFD, we can reform the PN \(p(n) = \langle n | \rho | n \rangle\) as
\[
p(n) = \langle n | \rho | n \rangle = \sum_{m=0}^{\infty} \langle n, \bar{n} | \rho | m, \bar{m} \rangle \\
= \langle n, \bar{n} | \rho | I \rangle = \langle n, \bar{n} | \rho \rangle, \tag{41}
\]
thus the PN is converted to the matrix element \(\langle n, \bar{n} | \rho \rangle\) in thermo dynamics frame. Then using the completeness of \(\langle \xi | \) and Eq. (11) as well as Eq. (14), we see
\[
p(n) = \int d^2 \xi \frac{\langle n, \bar{n} | \xi \rangle \langle \xi | \rho \rangle}{\pi} \\
= \int d^2 \xi \langle n, \bar{n} | \xi \rangle W(\alpha = \xi / 2) \\
= 4 \int d^2 \alpha W_{n, n} | \alpha \rangle W(\alpha), \tag{42}
\]
one can see this formula also in [14, 24]. Thus one can calculate the PN by combining Eq. (39) and (42).

Now we evaluate the PN of the above decoherence model in Eq. (38). Substituting Eq. (35) into Eq. (12), we see
\[
p(n) = \frac{8}{A} \int d^2 \beta W(\beta, 0) G(\beta), \tag{43}
\]
where
\[
G(\beta) = \int d^2 \alpha W_{n, n} | \alpha \rangle e^{-\frac{A}{2} |\alpha - \beta e^{-\kappa t}|^2 - 2|\alpha|^2} \\
= (-1)^n \int d^2 \alpha L_n \left(4|\alpha|^2 e^{-\frac{A}{2} |\alpha - \beta e^{-\kappa t}|^2 - 2|\alpha|^2}. \tag{44}
\]
Using Eqs. (25) and (27) we can evaluate Eq. (44) as
\[
G(\beta) = \frac{1}{n!} \frac{\partial^n + n}{\partial \tau^n \partial \kappa^n} e^{-\tau t - \frac{A}{2} |\beta|^2 e^{-2\kappa t}} \int d^2 \alpha \pi \times \exp \left[-\frac{A}{2} + 1 \right] \frac{\alpha}{A^n} \\
\times \left( \frac{A}{2} + 1 \right) \frac{\beta}{A^n} \exp \left[-\frac{1}{A^n} \right] \frac{T^n}{\alpha^n} \frac{T^n}{\beta^n} \frac{\beta e^{-\kappa t}}{A^n} + 1 \\
+ \frac{2 \beta e^{-\kappa t}}{A^n} + 1 \left( \frac{A}{2} + 1 \right) \frac{\beta e^{-\kappa t}}{A^n} \right] \tag{45}
\]
After making some scaled transformations, we finally obtain
\[
G(\beta) = \frac{A (A - 1)^n}{\tau t + 1} e^{-\frac{2e^{-2\kappa t} t - 1}{A^2}} \times \frac{\partial^n + n}{\partial \tau^n \partial \kappa^n} e^{-\tau t + \frac{2 \beta e^{-\kappa t}}{A^n} + 1} \frac{2 \beta e^{-\kappa t}}{A^n} + 1 \\
= \frac{A (A - 1)^n}{\tau t + 1} e^{-\frac{2e^{-2\kappa t} t - 1}{A^2}} L_n \left(4 |\beta|^2 e^{-\frac{2e^{-\kappa t}}{A^n} + 1} \right). \tag{46}
\]
Substituting Eq. (40) into Eq. (43) yields
\[
p(n) = \frac{4}{A + 1} \int d^2 \beta e^{-\frac{2e^{-2\kappa t} t - 1}{A^2}} \times L_n \left\{ \frac{4e^{-2\kappa t}}{1 - A^2} \right\} W(\beta, 0), \tag{47}
\]
which is a new formula for calculating the photon number distribution of the open system in environment. From Eq. (47) it is easily seen that once the Wigner function of initial state is known, one can obtain its photon number distribution by performing the integration in Eq. (47).

In particular, when \( g = 0, A = 1 - e^{-2\kappa t} = T \), Eq. (47) reduces to

\[
p(n) = \frac{4(-1)^n n e^{2\kappa t}}{(2e^{2\kappa t} - 1)^n + 1} \int d^2\beta e^{-\frac{2 e^{2\kappa t} - 1}{2 e^{2\kappa t} - 1} |\beta|^2} \times L_n \left\{ \frac{4e^{2\kappa t}}{1 - A^2} |\beta|^2 \right\} W(\beta, 0),
\]

which corresponds to the photon number of density operator in the amplitude damping quantum channel.

While for \( g \to \kappa n \) and \( \kappa \to \frac{\kappa}{n} (n + 1) \), Eq. (47) becomes to

\[
p(n) = \frac{4(A - 1)^n}{(A + 1)^{n+1}} \int d^2\beta e^{-\frac{2 e^{2\kappa t} - 1}{2 e^{2\kappa t} - 1} |\beta|^2} \times L_n \left\{ \frac{4e^{2\kappa t}}{1 - A^2} |\beta|^2 \right\} W(\beta, 0),
\]

where \( A = (2\kappa + 1) T = (2\kappa + 1) (1 - e^{-2\kappa t}) \). Eq. (49) corresponds to the photon number of system interacting with thermal bath.

For example, we still consider the photon-added coherent state field. Substituting Eq. (20) into Eq. (47) and using Eqs. (29) and (27) yields

\[
p(n) = Ne^{-2|z|^2} \int \frac{d^2\beta}{\pi} L_m(|2\beta - z|^2) L_n \left\{ \frac{4 e^{-2(\kappa - g) t}}{1 - A^2} |\beta|^2 \right\} \\
\times \exp \left\{ \frac{2}{2} (z\beta* + \beta z) - 2 \left( 1 + \frac{e^{-2(\kappa - g) t}}{A + 1} \right) |\beta|^2 \right\} \\
= Ne^{-2|z|^2} (-1)^{m+n} \frac{\partial^2 m}{\partial v^m \partial v'^m} \frac{\partial^2 n}{\partial \tau^m \partial \tau'^m} e^{-2uv' - z v'} \\
\times e^{-z v - \tau'} \int \frac{d^2\beta}{\pi} \exp \left\{ -2 \mu |\beta|^2 + 2 (\sigma \tau' + z^* + v) \beta \\
+ 2 (\sigma \tau' + z^* + v) \beta \right\} \left|_{v=v'=\tau=\tau'=0} \\
= Ne^{-2|z|^2} (-1)^{m+n} \frac{\partial^2 m}{\partial v^m \partial v'^m} \frac{\partial^2 n}{\partial \tau^m \partial \tau'^m} \\
\times \exp \left\{ \frac{2}{2} (z\beta* + \beta z) - 2 \left( 1 + \frac{e^{-2(\kappa - g) t}}{A + 1} \right) |\beta|^2 \right\} \\
\times \exp \left\{ \frac{2}{2} (z\beta* + \beta z) - 2 \left( 1 + \frac{e^{-2(\kappa - g) t}}{A + 1} \right) |\beta|^2 \right\}. 
\]

where we have set

\[
\omega = \frac{2 - \mu}{\mu}, \lambda = \frac{2\sigma}{\mu}, \sigma = \frac{e^{-2(\kappa - g) t}}{\sqrt{1 - A^2}},
\]

and

\[
N = \frac{4(A - 1)^n (-1)^m}{(A + 1)^{n+1} L_m(-|z|^2)}, m = 1 + \frac{e^{-2(\kappa - g) t}}{A + 1}. 
\]

Further expanding the exponential item \( \exp \left\{ \omega \nu' + (\lambda \sigma - 1) \tau' \right\} \), we finally obtain

\[
p(n) = \frac{N \lambda^2 \omega e^{\frac{2}{\sqrt{2} \beta} |z|^2}}{2 \mu (-\omega)^{n-m}} \sum_{l,k=0}^m m! n! \left( \omega (\lambda \sigma - 1) / \lambda \right)^l \left( \lambda \sigma + 1 - \lambda \right)^k \\
\times |H_{m-l,n-k} (i \sqrt{\omega z} i \sqrt{\omega z^*})|^2. 
\]

In particular, when \( g = 0 \), leading to \( A = \omega = T, \sigma = e^{-\kappa t}, \lambda = 1, \lambda = \sqrt{1 + T} \), thus

\[
p(n) = \frac{m!}{n!} L_m(-|z|^2) \sum_{l=0}^m \frac{\omega^m |n| e^{-2\kappa t} |z|^2}{l! \left(l - m \right)! \left| l - m \right|!} \\
\times |H_{m-l,n} (i \sqrt{\omega z} i \sqrt{\omega z^*})|^2, 
\]

which concides with Eq.(43) with idea detection efficiency in Ref. [23].

In sum, by virtue of the thermo entangled state representation that has a fictitious mode as a counterpart of the system mode, we have derived the relation between the Wigner functions at \( t \) time and the initial time when quantum system interacts with environment, such as decoherence, damping and amplification. As another quantity describing quantum system, the formula of photon number distribution has also been derived, which can be evaluated by performing an integration for the initial Wigner function. Our derivations seem more concise.

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