Minimizing Sum of Non-Convex
but Piecewise log-Lipschitz Functions using Coresets

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Abstract

We suggest a new optimization technique for minimizing the sum \( \sum_{i=1}^{n} f_i(x) \) of \( n \) non-convex real functions that satisfy a property that we call piecewise log-Lipschitz. This is by forging links between techniques in computational geometry, combinatorics and convex optimization.

Example applications include the first constant-factor approximation algorithms whose running-time is polynomial in \( n \) for the following fundamental problems:

(i) Constrained \( \ell_z \) Linear Regression: Given \( z > 0 \), \( n \) vectors \( p_1, \ldots, p_n \) on the plane, and a vector \( b \in \mathbb{R}^n \), compute a unit vector \( x \) and a permutation \( \pi : [n] \to [n] \) that minimizes \( \sum_{i=1}^{n} |p_i x - b_{\pi(i)}|^z \).

(ii) Points-to-Lines alignment: Given \( n \) lines \( \ell_1, \ldots, \ell_n \) on the plane, compute the matching \( \pi : [n] \to [n] \) and alignment (rotation matrix \( R \) and a translation vector \( t \)) that minimize the sum of Euclidean distances

\[
\sum_{i=1}^{n} \text{dist}(Rp_i - t, \ell_{\pi(i)})^z
\]

between each point to its corresponding line.

These problems are open even if \( z = 1 \) and the matching \( \pi \) is given. In this case, the running time of our algorithms reduces to \( O(n) \) using core-sets that support: streaming, dynamic, and distributed parallel computations (e.g. on the cloud) in poly-logarithmic update time. Generalizations for handling e.g. outliers or pseudo-distances such as \( M \)-estimators for these problems are also provided.

Experimental results show that our provable algorithms improve existing heuristics also in practice. A demonstration in the context of Augmented Reality show how such algorithms may be used in real-time systems.

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1 Introduction

We define below the general problem of minimizing sum of piecewise log-Lipschitz functions, and then suggest two example applications.

Minimizing sum of piecewise log-Lipschitz functions. We consider the problem of minimizing the sum \( \sum_{g \in G} g(x) \) over \( x \in \mathbb{R}^d \) of a set \( G \) of \( n \) real non-negative functions that may not be convex but satisfy the piecewise log-Lipschitz; see Definition[2]. This condition means that we can partition the range of each function \( g : \mathbb{R}^d \rightarrow [0, \infty) \) into small \( m \) subsets of \( \mathbb{R}^d \) (sub-ranges), such that \( g \) satisfies the log-Lipschitz condition on each of these sub-ranges; see Definition[1]. That is, \( g \) has a single minimum in this sub-range, and increases in a bounded ratio around its local minimum. Note that \( g \) might not be convex even in this sub-range. Formally, if the distance \( x \) from the minimum in a sub-range is multiplied by \( c > 0 \), then the value of the function increases by a factor of at most \( c^r \) for some small (usually constant) \( r > 0 \).

More generally, we wish to minimize the cost function \( f(g_1(x), \ldots, g_n(x)) \) where \( f : \mathbb{R}^n \rightarrow [0, \infty) \) is a log-Lipschitz function as explained in the previous paragraph, and \( \{g_1, \ldots, g_n\} \) are the log-Lipschitz functions in \( G \).

As an application, we reduce the following problems to minimizing such a set \( G \) of functions.

Aligning points-to-lines, known as the Perspective-n-Point (PnP) problem, is a fundamental problem in computer vision which aims to compute the position of an object (formally, a rigid body) based on its 3D model. Here, we assume that the structure of the object (its 3D model) is known. This problem is equivalent to the problem of estimating the position of a moving camera, based on a captured 2D image of a known object, which is strongly related to the very common procedure of “camera calibration” that is used to estimate the external parameters of a camera using a chessboard.

Formally, the input to the problem is an ordered set \( L = \{\ell_1, \ldots, \ell_n\} \) of \( n \) lines that intersect at the origin and an ordered set \( P = \{p_1, \ldots, p_n\} \) of \( n \) points, both in \( \mathbb{R}^3 \). Each line represents a point in the 2D image. The output is an alignment \((R, t)\) that minimizes the sum of Euclidean distances over each point (column vector) and its corresponding line, i.e.,

\[
\min_{(R,t)} \sum_{i=1}^n \text{dist}(Rp_i - t, \ell_i), \tag{1}
\]

where the minimum is over every rotation matrix \( R \in \mathbb{R}^{3 \times 3} \) and translation vector \( t \in \mathbb{R}^3 \). Here, \( \text{dist}(x, y) = \|x - y\|_2 \) is the Euclidean distance but in practice we may wish to use non-Euclidean distances, such as distances from a point to the intersection of its corresponding line with the camera’s image plane.

While dozens of heuristics were suggested over the recent decades, this problem is open even when the points and lines are on the plane, e.g. when we wish to align a set of GPS points to a map of lines (say, highways).

We tackle a variant of this problem, when both the points and lines are in \( \mathbb{R}^2 \), and the lines do not necessarily intersect at the origin.

Constrained regression is a fundamental problem in machine learning, which aims to compute a vector \( x \in \mathbb{R}^d \) such that the inner product \((p^T x)\) will predict the label or classification \( b \in \mathbb{R} \) of a point (data record) \( p \in \mathbb{R}^d \). Without loss of generality we can assume that all the entries of \( b \) are non-negative. This motivates the problem of minimizing the error on a “training dataset” of \( n \) records which are the rows of an \( n \times d \) real matrix \( A \), i.e., \( \min_{x \in \mathbb{R}^d} \|Ax - b\|_z \) with respect to \( \ell_z \)-norm where \( z > 0 \) (including the non-standard norm \( z < 1 \)). To avoid overfitting, or to decrease sparsity (number of non-zeroes in \( x \)) we may wish to keep the norm of \( x \) constant, say, 1.
This yields the constrained optimization problem
\[
\min_{\{x \in \mathbb{R}^d : ||x|| = 1\}} \|Ax - b\|.
\] (2)

Lagrange multiplier can then be used to obtain the problem
\[
\min_{\lambda \in \mathbb{R}, x \in \mathbb{R}^d} \|Ax - b\| + \lambda(\|x\| - 1).
\] (3)

Again, to our knowledge both (2) and (3) are open problems already for points on the plane \((d = 2)\), and even for the \(\ell_1\) linear regression \((z = 1)\). A possible lee-way is to calibrate \(\lambda\) manually, where large \(\lambda\) implies better sparsity and less over fitting but higher fitting error, and replace the non-convex constraint \(\|x\| = 1\) by the convex constraint \(\|x\| \leq 1\). This yields the common Lasso (least absolute shrinkage and selection operator) methods [30]
\[
\min_{\|x\| \leq 1} \|Ax - b\| + \lambda \|x\| \tag{4}
\]
for different norms (usually combinations of \(\ell_1\) and \(\ell_2\)). Due to the known value of \(\lambda\) and convex constraint on \(x\), problem (4) can usually be solved using convex optimization.

### 1.1 Generalizations

We consider more generalizations of the above problems as follows.

**Unknown matching** is the case where the matching \(\pi : [n] \to [n]\) between the \(i\)th point \(p_i\) to its line \(\ell_i\) (in the PnP problem (1)) or to the \(i\)th label \(b_i\) of \(b\) (in (3)) is unknown for every \(i \in \{1, \ldots, n\} = [n]\). E.g., in the PnP problem there are usually no labels in the observed images, and in regression \(b\) may be an (unsorted) vector of anonymous votes or classifications for all the \(n\) users.

**Non-distance functions** where for an error vector \(v = (v_1, \ldots, v_n)\) of a candidate solution, the cost that we wish to minimize over all these solutions is \(f(v)\) where \(f : \mathbb{R}^n \to [0, \infty)\), such as \(f(v) = \|v\|_\infty\) for “worst case” error, \(f(v) = \|v\|_{1/2}\) which is more robust to noisy data, or \(f(v) = \|v\|^2\) of squared distances for maximizing likelihood in the case of Gaussian noise. As in the latter two cases, the function \(f\) may not be a distance functions.

**Robustness to outliers** may be obtained by defining \(f\) to be a function that ignores or at least put less weight on very large distance, maybe based on some given threshold. Such a function \(f\) is called an \(M\)-estimator and is usually a non-convex function.

**Coreset** in this paper refers to a small representation of a set \(P\) of \(n\) input points by a weighted (scaled) subset. The approximation is \((1 + \varepsilon)\) multiplicative factor, with respect to the cost of any item (query) in a given set \(Q\). E.g. in (1) it is the union over alignments \((R, t)\), and in (2) \(Q\) is the set of unit vectors in \(\mathbb{R}^d\).

**Composable coresets** have the property that they can be merged and re-reduced; see e.g. [1, 15].

Our main motivation for designing coresets is (i) to reduce the running time of our algorithms from polynomial to near-linear in \(n\), and (ii) handle big data computation models as follows, which is straightforward using composable coresets.

**Handling big data** in our paper refers to the following computation models: (i) **Streaming** support for a single pass over possibly unbounded stream of items in \(A\) using memory and update time that is sub-linear (usually poly-logarithmic) in \(n\). (ii) **Parallel computations** on distributed data that is streamed to \(M\) machines where the running time is reduced by \(M\) and the communication between the machines to the server should be also sub-linear in its input size \(n/M\). (iii) **Dynamic data** which includes deletion of pairs. Here \(O(n)\) memory is necessary, but we still desire sub-linear update time.
1.2 Our contribution

Generic framework for defining a cost function $\text{cost}(A, q)$ for any finite input subset $A = \{a_1, \ldots, a_n\}$ from a set $X$ called ground set, and an item $q$ (called query) from a (usually infinite) set $Q$. We show that this framework enables handling the generalization in Section 1.1 such as outliers, m-estimators and non-distance functions in a straightforward way. Formally, we define

$$\text{cost}(A, q) := f(\text{lip}(D(a_1, q)), \ldots, \text{lip}(D(a_n, q))),$$  \hspace{1cm} (5)

where $f : [0, \infty)^n \rightarrow [0, \infty)$ and $\text{lip} : [0, \infty) \rightarrow [0, \infty)$ are an $O(1)$-log-Lipschitz functions, and $D : X \times Q \rightarrow [0, \infty)$; See Definition 4.

Optimization of piecewise log-Lipschitz functions. Given $x^* \in \mathbb{R}$ and piecewise $O(1)$-log-Lipschitz functions $g_1, \ldots, g_n$, we prove that one of their minima $x^* \in \mathbb{R}$ approximates their value $g_i(x^*)$ in $x$ simultaneously (for every $i \in [n]$), up to a constant factor. See Theorem 3. This yields a finite set of candidate solutions (called centroid set) that contains an approximated solution, without knowing $x^*$.

We use this result to compute $q' \in Q$ that approximates $D(a_i, q^*)$ simultaneously (for every $i \in [n]$), where $q^*$ is the query that minimizes $\text{cost}(A, q)$ over every $q \in Q$. Observation 5 proves that $\text{cost}(A, q^*)$ is the desired approximation for the optimal solution $\text{cost}(A, q^*)$ in our framework.

Simultaneous optimization and matching may be required for the special case that $A = \{(a_i, b_i)\}_{i=1}^n$ is a set of pairs, and we wish to compute the permutation $\pi^*$ and query $q^*$ that minimize $\text{cost}(A, q)$ over every $\pi^* : [n] \rightarrow [n]$ and $q^* \in Q$, where $A_{\pi} = \{(a_i, b_{\pi(i)})\}_{i=1}^n$ is the corresponding permutation of the pairs. We provide constant factor approximations for the case $f = \|\cdot\|_1$ in (5); See Theorems 8 and 13.

Constrained Regression as defined in Section 1 is our first example application of the above framework, where $A = \{(a_i, b_i)\}_{i=1}^n \subseteq \mathbb{R}^2 \times [0, \infty)$, $Q = \{x \in \mathbb{R}^2 \mid \|x\|_2 = 1\}$ is the unit circle, and $D((a_i, b_i), x) = |a_i^T x - b_i|$ for every $x \in Q$ and $i \in [n]$. We provide the first constant factor approximation for the optimal solution $\min_{x \in Q} \text{cost}(A, x)$ that takes time polynomial in $n$. Such a solution can be computed for every cost function and lip as defined in (5), e.g., $\text{lip}(x) = x^{1/2}$ where we wish to minimize the non-convex sum over $\sqrt{|a_i^T x - b_i|}$; see Theorem 7. Simultaneous optimization and matching for $f = \|\cdot\|_1$ are suggested in Theorem 8.

Approximated Points-to-Lines alignment as defined in Section 1 is our second example, where $A = \{(p_i, \ell_i)\}_{i=1}^n$ is a set of paired point $p_i$ and line $\ell_i$ on the plane, $Q = \text{ALIGNMENTS}$ is the union over every rotation matrix $R$ and a translation vector $t$, $D((p_i, \ell_i), (R, t)) = \min_{x \in \ell_i} \|R p_i - t - x\|$ for every $(R, t) \in Q$ and $i \in [n]$. We provide the first constant factor approximation for the optimal solution $\min_{(R, t) \in Q} \text{cost}(A, (R, t))$ that takes polynomial time. Such a solution can be computed for every cost function as defined in (5); see Theorem 12. Including simultaneous optimization and matching for $f = \|\cdot\|_1$; See Theorem 13.

Composable $\varepsilon$-coresets for aligning points-to-lines are suggested in Theorem 14 and 15 based on reduction to coresets for $\ell_2$ regression.

Experimental Results show that our algorithms performed better also in practice, compared to both existing heuristics and provable algorithms for related problems. A system for head tracking in the context of augmented reality shows that our algorithms can be applied in real-time using sampling and coresets. Existing solutions are either too slow or provide unstable images, as is demonstrated in the companion video [20].
2 Related Work

For the easier case of summing convex function, a framework was suggested in [11, 12]. However, for the case of summing non-convex functions as in this paper, each with more than one local minima, these techniques do not hold. This is why we had to use more involved algorithms in Section 4. Moreover, our generic framework such as handling outliers and matching can be applied also for the works in [11, 12].

The motivation was to obtain weak but faster $(\alpha, \beta)$-approximations that suffice for computing coresets. The polynomial time algorithms can then be applied on the coreset. A classic example is projective clustering where we wish to approximate $n$ points in $\mathbb{R}^d$ by a set of $k$ affine $j$-dimensional subspaces, such as $k$-means ($j = 0$) or PCA ($k = 1$). Constant factor approximations can be computed by considering every set of $k$ subspaces, each spanned by $j + 1$ input points.

Summing of non-convex but polynomial or rational functions was suggested in [31]. This is by using tools from algebraic geometry such as semi-algebraic sets and their decompositions. For high degree polynomial such techniques may be used to compute the minima in Theorem 3. In this sense, piecewise log-Lipschitz functions can be considered as generalizations of such functions, and our framework may be used to extend them for the generalizations in Section 1.1 (outliers, matching, etc.).

Aligning points to lines. The problem for aligning a set of points to a set of lines in the plane is natural e.g. in the context of GPS points [29, 27], finding sky patterns as in Fig. 1 or aligning pixels in a 2D image to an object that is pre-defined by linear segments [25], as in augmented reality applications [22, 32, 17].

The only known solutions are for the case of sum of squared distances and $d = 2$ dimensions, with no outliers, and when the matching between the points is given. In this case, the Lagrange multipliers method can be applied in order to get a set of $2$nd order polynomials. For $d = 3$ the problem is called PnP (Perspective-n-Points) and has provable solutions only for the case of exact alignment (zero fitting error) [26, 19] and numerous heuristics. When the matching is unknown ICP (Iterative closest point) is the main common technique based on greedy nearest neighbours; see references in [6]. To handle outliers RANSAC [9] is heuristically used.

Constrained regression is usually used to avoid overfitting and noise for linear regression, as explained in Section 1 and e.g. in [30, 34, 34, 33]; see references therein. The solution is usually based on relaxation to convex optimization. However, when the tradeoff parameter $\lambda$ is unknown, or when we want to ignore $k$ outliers, or use $M$-estimators, the resulting problems are non-convex.

To our knowledge, no existing provable algorithms are known for handling outliers, unknown matching, or $\ell_z$ norm for the case $z \in (0, 1)$.

Coresets have many different definitions. In this paper we use the simplest one that is based on a weighted subset of the input, which preserve properties such as sparsity of the input and numerical stability. Coresets for $\ell_p$ regression were suggested in [3] using well-conditioned matrices that we cite in Theorem 14. We improve the bounds on the coreset size using the framework from [11, 7]. We also reduce the points-to-lines aligning problem to constrained $\ell_1$ optimization in the proof of Theorem 31 which allows us to apply our algorithms on these coresets for the case of sum over point-line distances.

In most coresets papers, the main challenge is to compute the coreset. However, in this paper, the harder problem was to extract the desired constrained solution from the coresets which approximate every vector, and ignore the constraints.
\section{Optimization Framework}

In what follows, for every pair of vectors \(v = (v_1, \ldots, v_n)\) and \(u = (u_1, \ldots, u_n)\) in \(\mathbb{R}^n\) we denote \(v \leq u\) if \(v_i \leq u_i\) for every \(i \in [n]\). Similarly, \(f: \mathbb{R}^n \rightarrow [0, \infty)\) is non-decreasing if \(f(v) \leq f(u)\) for every \(v \leq u \in \mathbb{R}^d\).

The following definition is a generalization of Definition 2.1 in [14] from \(n = 1\) to \(n > 1\) dimensions, and from \(\mathbb{R}\) to \(I \subseteq \mathbb{R}^n\).

\begin{definition}[Log-Lipschitz function] Let \(r > 0\), let \(n \geq 1\) be an integer, and let \(v = (v_1, \ldots, v_n) \in \mathbb{R}^n\). Let \(I\) be a subset of \(\mathbb{R}^n\), and \(h: I \rightarrow [0, \infty)\) be a non-decreasing function. Then \(h(x)\) is \(r\)-log-Lipschitz over \(x \in I\), if for every \(c \geq 1\) and \(x \in I \cap \frac{I}{c}\), we have \(h(cx) \leq c^r h(x)\). The parameter \(r\) is referred to as the log-Lipschitz constant.
\end{definition}

Unlike previous papers, the loss fitting ("distance") function that we want to minimize in this thesis is not a log-Lipschitz function. However, it can be partitioned to a constant number of log-Lipschitz functions in the following sense.

\begin{definition}[Piecewise log-Lipschitz] Let \(g: X \rightarrow [0, \infty)\) be a continuous function over a set \(X\), and \(\text{dist}: X^2 \rightarrow [0, \infty)\) be a distance function. Let \(r \geq 0\). The function \(g\) is piecewise \(r\)-log-Lipschitz if there is a partition of \(X\) into \(m\) subsets \(X_1, \ldots, X_m\) such that for every \(i \in [m]\):

(i) \(g\) has a unique infimum \(x_i\) at \(X_i\), i.e., \(\{x_i\} = \arg\min_{x \in X_i} g(x)\).

(ii) \(h_i: [0, \max_{x \in X_i} \text{dist}(x, x_i)] \rightarrow [0, \infty)\) is an \(r\)-log-Lipschitz function; see Definition 7

(iii) \(g(x) = h_i(\text{dist}(x, x_i))\) for every \(x \in X_i\).

The set of minima is denoted by \(M(g) = \{x_1, \ldots, x_m\}\).
\end{definition}

Suppose that we have a set of piecewise \(r\)-log-Lipschitz functions, and consider the union \(\bigcup M(g)\) over every function \(g\) in this set. The following lemma states that, for every \(x \in \mathbb{R}\), this union contains a value \(x'\) such that \(g(x')\) approximates \(g(x)\) up to a multiplicative factor that depends on \(r\).

\begin{theorem}[simultaneous approximation] Let \(g_1, \ldots, g_n\) be \(n\) function, where \(g_i: \mathbb{R} \rightarrow [0, \infty)\) is a piecewise \(r\)-log-Lipschitz function for every \(i \in [n]\), and let \(M(g_i)\) denote the minima of \(g_i\) as in Definition 2. Let \(x \in \mathbb{R}\). Then there is \(x' \in \bigcup_{i \in [n]} M(g_i)\) such that for every \(i \in [n]\),

\[ g_i(x') \leq 2^r g_i(x) \]  \hspace{1cm} (6)
\end{theorem}

\begin{definition}[Optimization framework] Let \(X\) be a set called ground set, let \(A = \{a_1, \ldots, a_n\} \subseteq X\) be a finite input set and let \(Q\) be a set of queries. Let \(D: X \times Q \rightarrow [0, \infty)\) be a function. Let \(\text{lip}: [0, \infty) \rightarrow [0, \infty)\) be an \(s\)-log-Lipschitz function \(f: [0, \infty)^n \rightarrow [0, \infty)\) be an \(s\)-log-Lipschitz function. Let \(q \in Q\). We define

\[ \text{cost}(A, q) = f(\text{lip}(D(a_1, q)), \ldots, \text{lip}(D(a_n, q))). \]

The following observation states that if we find a query \(q \in Q\) that approximates the function \(D\) for every input element, then it also approximates the function cost as defined in Definition 4.

\begin{observation} Let \(\text{cost}(A, q) = f(\text{lip}(D(a_1, q)), \ldots, \text{lip}(D(a_n, q)))\) be defined as in Definition 4. Let \(q^*, q' \in Q\) and let \(c \geq 1\). If \(D(a_i, q') \leq c \cdot D(a_i, q^*)\) for every \(i \in [n]\), then \(\text{cost}(A, q') \leq c^{s} \cdot \text{cost}(A, q^*)\).
\end{observation}
4 Algorithms for Aligning Points to Lines

In this section, we introduce our notations, describe our algorithms, and give an overview for each algorithm. See Sections D.1, D.2 and D.3 for an intuition of the algorithms presented in this section.

Notation. Let $\mathbb{R}^{n \times d}$ be the set of $n \times d$ real matrices. We denote by $\|p\| = \|p\|_2 = \sqrt{p_1^2 + \ldots + p_d^2}$ the length of a point $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$, by $\text{dist}(p, \ell) = \min_{x \in X} \|p - x\|_2$ the Euclidean distance from $p$ to an line $\ell$ in $\mathbb{R}^d$ for every line $\ell$, by $\text{proj}(p, X)$ its projection on $X$, i.e., $\text{proj}(p, X) = \arg\min_{x \in X} \text{dist}(p, x)$, and by $\text{sp}\{p\} = \{kp \mid k \in \mathbb{R}\}$ we denote the linear span of $p$. For a matrix $V \in \mathbb{R}^{d \times m}$ we denote by $V^\perp \in \mathbb{R}^{d \times (d-m)}$ an arbitrary matrix whose columns are mutually orthogonal unit vectors, and also orthogonal to every vector in $V$. Hence, $[V \mid V^\perp]$ is an orthogonal matrix. If $p \in \mathbb{R}^2$, then $p^\perp := q$ such that $q^T p = 0$ and $q_x > 0$. We denote $\{n\} = \{1, \ldots, n\}$ for every integer $n \geq 1$.

In this paper, every vector is a column vector, unless stated otherwise. A matrix $R \in \mathbb{R}^{2 \times 2}$ is called a rotation matrix if it is orthogonal and its determinant is 1, i.e., $R^TR = I$ and $\det(R) = 1$. For $t \in \mathbb{R}^2$ that is called a translation vector, the pair $(R, t)$ is called an alignment. We define ALIGNMENTS to be the union of all possible alignments in 2-dimensional space.

For $\pi : [n] \rightarrow [n]$ and a set $A = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ of $n$ pairs of elements, $A_\pi$ is defined as $A_\pi = \{(a_1, b_{\pi(1)}), \ldots, (a_n, b_{\pi(n)})\}$.

Algorithms. We now present algorithms that compute a constant factor approximation for the problem of aligning points to lines, when the matching is either known or unknown. Algorithm 2 handles the case when the matching is given, i.e. given an ordered set $P = \{p_1, \ldots, p_n\}$ of $n$ points, and a corresponding set $L = \{\ell_1, \ldots, \ell_m\}$ of $n$ lines, both in $\mathbb{R}^2$, we wish to find an alignment that minimizes, for example, the sum of distances between each point in $P$ and it’s corresponding line in $L$.

Formally, let $A = \{(p_i, \ell_i)\}_{i=1}^n$ be a set of $n \geq 3$ point-line pairs, $z \geq 1$, and $D_z : A \times \text{ALIGNMENTS} \rightarrow [0, \infty)$ such that $\text{cost}(A, (R, t)) = \min_{q \in \ell} \|Rp - t - q\|_z$ is the $\ell_z$ distance between $R(p - t)$ and $\ell$ for every $(p, \ell) \in A$ and $(R, t) \in \text{ALIGNMENTS}$. Let $\text{cost}, s, r$ be as defined in Definition 4 for $D = D_2$. Then Algorithm 2 outputs a set of alignments that is guaranteed to contain an alignment which approximates $\min_{(R,t) \in \text{ALIGNMENTS}} \text{cost}(A, (R, t))$ up to a constant factor; See Theorem 12.

Algorithm 3 handles the case when the matching is unknown, i.e. given unordered sets $P$ and $L$ consisting of $n$ points and $n$ lines respectively, we wish to find a matching function $\pi : [n] \rightarrow [n]$ and an alignment $(R, t)$ that minimize, for example, the sum of distances between each point $p_i \in P$ and its corresponding line $\ell_{\pi(i)} \in L$.

Formally, let $\text{cost}$ be as defined above but with $f = \|\cdot\|_1$. Then Algorithm 5 outputs a set of alignments that is guaranteed to contain an alignment which approximates $\min_{(R,t,\pi)} \text{cost}(A_\pi, (R, t))$ up to a constant factor, where the minimum is over every alignment $(R, t)$ and matching function $\pi$; See Theorem 13.
4.1 Algorithm 1: Z-CONFIGURATIONS

Algorithm 1: Z-CONFIGURATIONS $(v, p, q, z)$

**Input:** A unit vector $v = (v_x, v_y)^T$ such that $|v_y| > 0$, and $p, q, z \in \mathbb{R}^2$ such that $p \neq q$.

**Output:** A tuple of matrices $P, Q, Z \in \mathbb{R}^{2 \times 2}$ that satisfy Lemma 9.

1. Set $r_1 \leftarrow \|p - q\|$, $r_2 \leftarrow \|p - z\|$, $r_3 \leftarrow \|q - z\|$.
2. Set $d_1 \leftarrow \frac{r_1^2 + r_2^2 - r_3^2}{2r_1}$, $d_2 \leftarrow \sqrt{r_2^2 - d_1}$.
3. Set $s \leftarrow \frac{v_x}{v_y}$; $R \leftarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $P \leftarrow r_1 \begin{pmatrix} s & 1 \\ 0 & 0 \end{pmatrix}$.
4. Set $Q \leftarrow P^T$.
5. Set $b \leftarrow \begin{cases} 1, & (z - p)^T(q - p) > 0 \\ 0, & \text{otherwise} \end{cases}$.
6. Set $Z \leftarrow P + \frac{d_1}{r_1} (Q - P) + b \cdot \frac{d_2}{r_1} R(Q - P)$.
7. return $(P, Q, Z)$.

**Overview of Algorithm 1**  
Algorithm 1 takes as input a unit vector $v$ and three points $p, q, z \in \mathbb{R}^2$. The vector $v$ represents a direction of a line $\ell$ that intersects the origin. It computes 3 matrices $P, Q, Z$ that satisfy Lemma 9. See Section D.1 for intuition and interpretation of those matrices. In Lines 1-3 we define constants. In Line 3 we define a rotation matrix that rotates the coordinates system by $\pi/2$ radians counter clockwise around the origin. In Lines 3 and 4 we compute the output matrices $P$ and $Q$ respectively.

In Line 5, $b = 1$ if $z$ is in the halfplane to the left of the vector $q - p$, and $b = 0$ otherwise. In Line 6 we compute the output matrix $Z$; See Fig. 2 for an illustration.

4.2 Algorithm 2: ALIGN

In this section we present the main algorithm, called ALIGN; See Algorithm 2. The output of this algorithm satisfies Lemma 10. The algorithm uses our main observations and general technique for minimizing the sum of distances between the point-line pairs.

4.2.1 Overview of Algorithm 2

The input for Algorithm 2 is a set of $n$ pairs, each consists of a point and a line on the plane. The algorithm runs exhaustive search on all the possible tuples and outputs candidate set $C$ of $O(n^3)$ alignments. Alignment consists of a rotation matrix $R$ and a translation vector $t$. Theorem 10 proves that one of these alignments is the desired approximation. See Section D.2 for intuition.

Line 2 identifies each line $\ell_i$ by its direction (unit vector) $v_i$ and distance $b_i > 0$ from the origin. Lines 3-16 iterates over every triple $(j, k, l)$ of input pairs such that $j \neq k$, and turns it into a constant number of alignments $C_1$. In Lines 6-13 we handle the case where the lines $\ell_j$ and $\ell_k$ are not parallel. In Line 15 we handle the case where $\ell_j$ and $\ell_k$ are parallel.

The case where $\ell_j$ and $\ell_k$ are not parallel. Lines 6-7 compute a rotation matrix $R_{v_j}$ that rotates $v_j$ to the $x$-axis. Line 8 calls the sub-procedure Algorithm 1 for computing three matrices $P', Q'$ and $Z'$. In
Algorithm 2: ALIGN($A$)

**Input:** A set $A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\}$ of $n$ pairs, where for every $i \in [n] = \{1, \ldots, n\}$, we have that $p_i$ is a point and $\ell_i$ is a line, both on the plane.

**Output:** A set $C \subseteq \text{ALIGNMENTS}$ of alignments that satisfies Lemma 10.

1. Set $C \leftarrow \emptyset$.
2. Set $v_i \in \mathbb{R}^2$ and $b_i \geq 0$ such that $\ell_i = \{q \in \mathbb{R}^2 \mid v_i^T q = b_i\}$ for every $i \in [n]$.
3. **for every** $j, k, l \in [n]$ s.t. $j \neq k$ **do**
   4. Set $C_1, C_2 \leftarrow \emptyset$.
   5. **if** $|v_j^Tv_k| \neq 1$ **then**
     6. Set $v_j^\perp \leftarrow$ a unit vector in $\mathbb{R}^2$ that is orthogonal to $v_j$.
     7. Set $R_{v_j} \leftarrow \begin{pmatrix} -v_j^T & -v_j^T \end{pmatrix}$. /* $R_{v_j}$ aligns $v_j$ with the $x$-axis. */
     8. Set $(P', Q', Z') \leftarrow \text{Z-CONFIGURATIONS}(R_{v_j}v_k, p_j, p_k, p_l)$. // See Algorithm 1
     9. Set $P \leftarrow R_{v_j}^TP', Q \leftarrow R_{v_j}^TQ', Z \leftarrow R_{v_j}^TZ'$.
    10. Set $s \leftarrow \ell_j \cap \ell_k$.
        // $\ell_j \cap \ell_k$ contains one point since $\ell_j$ and $\ell_k$ are not parallel.
    11. Set $c_l \leftarrow \text{dist}(s, \ell_l)$.
    12. Set $X \leftarrow \arg \min_{x \in \mathbb{R}^2 : \|x\| = 1} |v_i^T Z x - c_l|$. // The set of unit vectors that minimize the distance between $p_i$ and $\ell_l$ while maintaining $p_j \in \ell_j$ and $p_k \in \ell_k$.
    13. Set $C_1 \leftarrow \{(R, t) \in \text{ALIGNMENTS} \mid Rp_j - t = P x$ and $Rp_k - t = Q x$ and $Rp_l - t = Z x$ for every $x \in X\}$. /* The set of alignments that align the points $(p_j, p_k, p_l)$ with points $(P x, Q x, Z x)$ for every $x \in X$. */
    14. **else**
    15. Set $C_2 \leftarrow \{(R, t)\}$ such that $(R, t) \in \text{ALIGNMENTS}$, $Rp_j - t \in \ell_j$ and $(R, t) \in \arg \min \text{dist}(Rp_k - t, \ell_k)$.
    16. Set $C \leftarrow C \cup C_1 \cup C_2$.
17. **return** $C$. 

Line 3 we revert the effect of the rotation matrix $R_{ij}$. Lines 10–11 compute the distance between $l_t$ and the intersection between $l_j$ and $l_k$ since we assumed this intersection point is the origin in Algorithm 1. The matrix $Z$ and the line $l_t$ are used to compute a set $X$ of $O(1)$ unit vectors in Line 12. Every $x \in X$ defines a possible positioning for the triplet. In Line 13, we define an alignment $(R, t)$ for each $x \in X$. The union of the alignments in $C_t$ is then added to the output set $C$ in Line 16.

The case where $l_j$ and $l_k$ are parallel. In the case, we place $p_j \in l_j$, and place $p_k$ as close as possible to $l_k$. If there are more than one alignment that satisfies those conditions, then we pick an arbitrary one. This is done in Line 13.

4.3 Algorithm 3: ALIGN+MATCH

**Algorithm 3:** ALIGN+MATCH$(A, \text{cost})$

**Input:** A set $A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\}$ and a cost function as in Theorem 13

**Output:** An element $(\hat{R}, \hat{t}, \hat{\pi})$ that satisfies Theorem 13

1. Set $X \leftarrow \emptyset$.
2. for every $i_1, i_2, i_3, j_1, j_2, j_3 \in [n]$ do
3.   Set $X' \leftarrow \text{ALIGN}\{(p_{i_1}, \ell_{j_1}), (p_{i_2}, \ell_{j_2}), (p_{i_3}, \ell_{j_3})\}$.
   // See Algorithm 2
4.   Set $X \leftarrow X \cup X'$.
5. Set $S \leftarrow \{(R, t, \hat{\pi}(A, (R, t), \text{cost})) \mid (R, t) \in X\}$.
   /* see Definition 11 */
6. Set $(\hat{R}, \hat{t}, \hat{\pi}) \in \arg\min_{(R', t', \pi') \in S} \text{cost} (A_{x'}, (R', t'))$.
7. return $(\hat{R}, \hat{t}, \hat{\pi})$

Overview of Algorithm 3. Algorithm 3 takes as input a set of $n$ points and lines in $\mathbb{R}^2$, and a cost function as defined in Theorem 13. The algorithm computes an alignment $(\hat{R}, \hat{t}) \in \text{ALIGNMENTS}$ and a matching function $\hat{\pi}$ that approximate the minimal value of the given cost function; See Theorem 13.

In Line 2, we iterate over every $i_1, i_2, i_3, j_1, j_2, j_3 \in [n]$. In Lines 3–4, we match $p_{i_1}$ to $\ell_{j_1}$, $p_{i_2}$ to $\ell_{j_2}$ and $p_{i_3}$ to $\ell_{j_3}$, compute their corresponding set of alignments $X'$ by a call to Algorithm 2, and then add $X'$ to the set $X$. Finally, in Lines 5–6, we compute the optimal matching for every alignment in $X$, and pick the alignment and corresponding matching that minimize the given cost function.

5 Statements of Main Results

5.1 Constrained regression

The following lemma proves that for every two paired sets $a_1, \ldots, a_n \subseteq \mathbb{R}^2$ and $b_1, \ldots, b_n \geq 0$ and unit vector $x \in \mathbb{R}^2$, there exists $x' \in \arg\min_{\|y\|=1} |a_k^T y - b_k|$ for some $k \in [n]$ that approximates $|a_i^T x - b_i|$ for every $i \in [n]$.

**Lemma 6.** Let $a_1, \ldots, a_n \subseteq \mathbb{R}^2$ and $b_1, \ldots, b_n \geq 0$. Then there is a set $C$ of $|C| \in O(n)$ unit vectors that can be computed in $O(n)$ time such that (i) and (ii) hold as follows:
Let \( t \) be defined in Definition 4.

\[ |a_i^T x' - b_i| \leq 4 \cdot |a_i^T x - b_i|. \quad (7) \]

(ii) There is \( k \in [n] \) such that \( x' \in \arg \min_{|y|=1} |a_i^T y - b_k| \). The following theorem generalizes the approximation obtained in Lemma 6 to the family of cost functions defined in Definition 4.

**Theorem 7.** Let \( A = \{(a_1, b_1), \ldots, (a_n, b_n)\} \) be a set of \( n \geq 2 \) pairs, where for every \( i \in [n] \), we have that \( a_i \in \mathbb{R}^2 \) and \( b_i \geq 0 \). Let \( D(a_i, b_i, x) = |a_i^T x - b_i| \) for every \( i \in [n] \) and unit vector \( x \in \mathbb{R}^2 \). Let \( \text{cost}, s, r \) be as defined in Definition 4. Then in \( n^{O(1)} \) time we can compute a unit vector \( x' \in \mathbb{R}^2 \) such that

\[ \text{cost}(A, x') \leq 4^{rs} \cdot \min_{x \in \mathbb{R}^2 : \|x\|=1} \text{cost}(A, x). \]

Recall that for \( \pi : [n] \to [n] \) and a set \( A = \{(a_1, b_1), \ldots, (a_n, b_n)\} \) of \( n \) pairs of elements, \( A_\pi \) is defined as \( A_\pi = \{(a_1, b_\pi(1)), \ldots, (a_n, b_\pi(n))\} \). Lemma 6 proves that for every set \( A = \{(a_1, b_1), \ldots, (a_n, b_n)\} \subset \mathbb{R}^2 \times \mathbb{R} \) and unit vector \( x \in \mathbb{R}^2 \), there is \( (a_k, b_k) \in A \) such that the \( x' \in \arg \min_{|y|=1} |a_i^T y - b_k| \) approximates \( |a_i x - b_j| \) for every \( i \in [n] \). Hence, by computing the union \( C \) of minimizers of \( |a_i x - b_j| \) for every \( i, j \in [n] \), we are guaranteed that for any permutation \( \pi : [n] \to [n] \) and unit vector \( x \), one of the vectors in \( C \) will approximate \( |a_i x - b_{\pi(i)}| \) for every \( i \in [n] \).

**Theorem 8.** Let \( A = \{(a_1, b_1), \ldots, (a_n, b_n)\} \) be a set of \( n \geq 2 \) pairs, where for every \( i \in [n] \), we have that \( a_i \in \mathbb{R}^2 \) and \( b_i \geq 0 \). Let \( D(a, b, x) = |a^T x - b| \) for every \( a \in \mathbb{R}^2 \), \( b \geq 0 \) and unit vector \( x \in \mathbb{R}^2 \). Let \( \text{cost}, r \) be as defined in Definition 4 for \( D \) and \( f(v) = \|v\|_1 \). Then in \( n^{O(1)} \) we can compute a unit vector \( x' \in \mathbb{R}^2 \) and \( \pi' : [n] \to [n] \) that satisfy the following

\[ \text{cost}(A_{\pi'}, x') \leq 4^r \cdot \min_{x, \pi} \text{cost}(A_\pi, x), \]

where the minimum is over every unit vector \( x \in \mathbb{R}^2 \) and \( \pi : [n] \to [n] \).

### 5.2 Aligning Points-To-Lines

Table 1 summarizes the important results that we obtained for the problem of aligning points-to-lines.

The following Lemma proves that the matrices computed in Algorithm 1 satisfy some set of properties.

**Lemma 9.** Let \( v \) be a unit vector and \( \ell = \text{sp} \{v\} \) be the line in this direction. Let \( p, q, z \in \mathbb{R}^2 \) be the vertices of a triangle such that \( \|p - q\| > 0 \). Let \( P, Q, Z \in \mathbb{R}^{2 \times 2} \) be the output of a call to Z-CONFIGURATIONS(\( v, p, q, z \)); see Algorithm 1. Then the following hold:

(i) \( p \in \ell \)-axis and \( q \in \ell \) iff there is a unit vector \( x \in \mathbb{R}^2 \) such that \( p = Px \) and \( q = Qx \).

(ii) For every unit vector \( x \in \mathbb{R}^2 \), we have that \( z = Zx \) if \( p = Px \) and \( q = Qx \).

What follows is the main Lemma of ALIGN; See Algorithm 2. The proof of this lemma is divided into 3 steps that correspond to the steps discussed in the intuition for Algorithm 2 in Section D.2.
Table 1: Main results of this paper. Let $P$ be a set of $n$ points and $L$ be a set of $n$ lines, both in $\mathbb{R}^2$. Let $\pi: [n] \rightarrow [n]$ be a matching function and let $A_\pi = \{(p_1, \ell_{\pi(1)}), \ldots, (p_n, \ell_{\pi(n)})\}$. In this table we assume $x, v \in [0, \infty)$ and $n \in [0, \infty)^n$. For every point-line pair $(p, \ell) \in A$ and $(R, t) \in \text{ALIGNMENTS}$, let $D_z((p, \ell), (R, t)) = \min_{x' \in \ell} \|Rp - t - x'\|_z$ for $z \geq 1$. Let cost be as defined in Definition 4 for $D = D_z$, $A = A_\pi$ and $s = 1$. The approximation factor is relative to the minimal value of the cost function. Rows marked with a $\ast$ have that the minimum of the function cost is computed both over every $(R, t) \in \text{ALIGNMENTS}$ and matching function $\pi$.

| Function Name | $f(v)$ | lip$(x)$ | $z$ | $r$ | Time | Approximation factor | Ref. | Related Work |
|---------------|--------|----------|-----|-----|------|----------------------|-----|--------------|
| Sum of Squared Euclidean distances | $\|v\|_1$ | $x^2$ | 2  | 2  | $O(n^3)$ | $16^2$ | 12 | See Section 2 |
| Sum of Euclidean distances | $\|v\|_1$ | $x$ | 2  | 1  | $O(n^3)$ | 16 | 12 | None |
| Sum of $\ell_z$ distances to the power of $r$ | $\|v\|_1$ | $x^r$ | $z \geq 1$ | $r \geq 1$ | $O(n^3)$ | $(\sqrt{2} \cdot 16)^r$ | 12 | None |
| Sum of Euclidean distances with $k$ outliers | $\|v\|_{1,n-k}$ | $x$ | $z \geq 1$ | 1  | $O(n^3)$ | 16 | 12 | RANSAC [9] |
| Sum of Euclidean distances with unknown matching | $\|v\|_1$ | $x$ | 2  | 1  | $O(n^9)$ | $16^\ast$ | 13 | ICP [6] |
| Sum of $\ell_z$ distances to the power of $r$ with unknown matching | $\|v\|_1$ | $x^r$ | $z \geq 1$ | $r \geq 1$ | $O(n^9)$ | $(\sqrt{2} \cdot 16)^r^\ast$ | 13 | None |

Lemma 10. Let $A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\}$ be set of $n \geq 3$ pairs, where for every $i \in [n]$, we have that $p_i$ is a point and $\ell_i$ is a line, both on the plane. Let $C \subseteq \text{ALIGNMENTS}$ be an output of a call to $\text{ALIGN}(A)$; see Algorithm 2. Then for every alignment $(R^*, t^*) \in \text{ALIGNMENTS}$ there exists an alignment $(R, t) \in C$ such that for every $i \in [n],$

$$\text{dist}(Rp_i - t, \ell_i) \leq 16 \cdot \text{dist}(R^*p_i - t^*, \ell_i). \quad (8)$$

Moreover, $|C| \leq O(n^3)$ and can be computed in $O(n^3)$ time.

We now define an optimal matching for a given input set of pairs $A$, a query $q$ and a cost function.

Definition 11 (Optimal matching). Let $n \geq 1$ be an integer and $\text{Perms}(n)$ denote the union over every permutation (bijection functions) $\pi: [n] \rightarrow [n]$. Let $A = \{a_1, \ldots, a_n\}$ be an input set, where $a_i = (p_i, \ell_i)$ is a pair of elements for every $i \in [n]$, and let $Q$ be a set of queries. Consider a function cost as defined in Definition 4 for $f(v) = \|v\|_1$. Let $q \in Q$. A permutation $\pi$ is called an optimal matching for $(A, q, \text{cost})$ if it satisfies that

$$\pi(A, q, \text{cost}) \in \arg\min_{\pi \in \text{Perms}(n)} \text{cost}(A_\pi, q).$$

The following theorem generalizes the approximation obtained in Lemma 10 to the family of cost functions defined in Definition 4.
Theorem 12. Let \( A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\} \) be a set of \( n \geq 3 \) pairs, where for every \( i \in [n] \), we have that \( p_i \) is a point and \( \ell_i \) is a line, both on the plane. Let \( z \geq 1 \) and let \( D_z : A \times \text{ALIGNMENTS} \to [0, \infty) \) such that \( D_z((p, \ell), (R, t)) = \min_{q \in \ell} \|Rp - t - q\|_z \) is the \( z \)-distance between \( Rp - t \) and \( \ell \). Let cost, \( s, r \) be as defined in Definition 4 for \( D = D_z \). Let \( w = 1 \) if \( z = 2 \) and \( w = \sqrt{2} \) otherwise. Let \( C \) be the output of a call to \( \text{ALIGN}(A) \); see Algorithm 2. Then there exists \((R', t') \in C\) such that

\[
\text{cost}(A, (R', t')) \leq (w \cdot 16)^r s \cdot \min_{(R, t) \in \text{ALIGNMENTS}} \text{cost}(A, (R, t)).
\]

Furthermore, \( C \) and \((R', t')\) can be computed in \( n^{O(1)} \) time.

Recall that for \( \pi : [n] \to [n] \) and a set \( A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\} \) of \( n \) pairs of elements, \( A_\pi \) is defined as \( A_\pi = \{(p_{\pi(1)}, \ell_{\pi(1)}), \ldots, (p_{\pi(n)}, \ell_{\pi(n)})\} \).

Theorem 13. Let \( A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\} \) be a set of \( n \geq 3 \) pairs, where for every \( i \in [n] \) we have that \( p_i \) is a point and \( \ell_i \) is a line, both on the plane. Let \( z \geq 1 \) and \( D_z((p, \ell), (R, t)) = \min_{q \in \ell} \|Rp - t - q\|_z \) for every point \( p \) and line \( \ell \) on the plane and alignment \((R, t)\). Consider cost, \( r \) to be a function as defined in Definition 4 for \( D = D_z \) and \( f(v) = \|v\|_1 \). Let \( w = 1 \) if \( z = 2 \) and \( w = \sqrt{2} \) otherwise. Let \((\tilde{R}, \tilde{t}, \tilde{\pi})\) be the output alignment \((\tilde{R}, \tilde{t})\) and permutation \( \tilde{\pi} \) of a call to \( \text{ALIGN+MATCH}(A, \text{cost}) \); see Algorithm 3. Then

\[
\text{cost}(A_{\tilde{\pi}}, (\tilde{R}, \tilde{t})) \leq (w \cdot 16)^r \min_{(R, t, \pi)} \text{cost}(A_{\pi}, (R, t)),
\]

where the minimum is over every alignment \((R, t)\) and permutation \( \pi \).

Moreover, \((\tilde{R}, \tilde{t}, \tilde{\pi})\) can be computed in \( n^{O(1)} \) time.

### 5.3 Coresets for Big Data

In this section we assume the set \( \text{ALIGNMENTS} \) contains the union of every pair \((R, t)\) of \( d \)-dimensional rotation matrix and translation vector, respectively.

Theorem 14 (coreset for points-to-lines alignment). Let \( d \geq 2 \) be an integer. Let \( A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\} \) be a set of \( n \) pairs, where for every \( i \in [n] \), \( p_i \) is a point and \( \ell_i \) is a line, both in \( \mathbb{R}^d \), and let \( w = (w_1, \ldots, w_n) \in [0, \infty)^n \). Let \( \varepsilon, \delta \in (0, 1) \). Then in \( nd^{O(1)} \log n \) time we can compute a weights vector \( u = (u_1, \ldots, u_n) \in [0, \infty)^n \) that satisfies the following pair of properties.

(i) With probability at least \( 1 - \delta \), for every \((R, t) \in \text{ALIGNMENTS} \) it holds that

\[
(1 - \varepsilon) \sum_{i \in [n]} w_i \cdot \text{dist}(Rp_i - t, \ell_i) \leq \sum_{i \in [n]} u_i \cdot \text{dist}(Rp_i - t, \ell_i) \leq (1 + \varepsilon) \sum_{i \in [n]} w_i \cdot \text{dist}(Rp_i - t, \ell_i).
\]

(ii) The weights vector \( u \) has \( \frac{d^{O(1)}}{\varepsilon^2} \log \frac{1}{\delta} \) non-zero entries.

Corollary 15 (streaming, distributed, dynamic data). Let \( A = \{(p_1, \ell_1), (p_2, \ell_2), \ldots\} \) be a (possibly infinite) stream of pairs, where for every \( i \in [n] \), \( p_i \) is a point and \( \ell_i \) is a line, both in the plane. Let \( \varepsilon, \delta \in (0, 1) \). Then, for every integer \( n > 1 \) we can compute with probability at least \( 1 - \delta \) an alignment \((R^*, t^*)\) that satisfies

\[
\sum_{i=1}^{n} \text{dist}(R^*p_i - t^*, \ell_i) \in O(1) \cdot \min_{(R, t) \in \text{ALIGNMENTS}} \sum_{i=1}^{n} \text{dist}(Rp_i - t, \ell_i),
\]

for the \( n \) points seen so far in the stream, using \( (\log(n/\delta)/\varepsilon)^{O(1)} \) memory and update time per a new pair. Using \( M \) machines the update time can be reduced by a factor of \( M \).
6 Conclusion and Open Problems

We described a general framework for approximating functions under different constraint, and used it for obtaining generic algorithms for minimizing a finite set of piecewise log-Lipschitz functions. The generic result reduces the many optimization problems to a problem of computing a query that approximates simultaneously every input item. We solve this problem for two examples: constrained regression and points-to-lines alignment. Coresets for these problems allowed us to turn our polynomial time algorithms into near linear time, and support big data models, as well as practical real-time implementations. Unlike other papers, since the non-convex constraints can be ignored by the coreset, the main challenge was to compute the solution in polynomial time, even on the small coreset.

Open problems include generalization of our results to higher dimensions, as some of our suggested coresets. We generalize our algorithms for the case when no matching permutation $\pi$ is given between the points and lines (or regression labels). However, the results are less general than our results for the known matching case, and we do not have coresets for these cases, which seems challenging. On the contrary, reducing the constant factor approximations in this work to $1 + \varepsilon$ approximations seems easier using existing techniques. We leave this to future papers.

References

[1] P. Agarwal, S. Har-Peled, and K. Varadarajan. Approximating extent measures of points. *Journal of the ACM*, 51(4):606–635, 2004.

[2] Martin Anthony and Peter L Bartlett. *Neural network learning: Theoretical foundations*. Cambridge University Press, 2009.

[3] Artem Barger and Dan Feldman. k-means for streaming and distributed big sparse data. In *Proceedings of the 2016 SIAM International Conference on Data Mining*, pages 342–350. SIAM, 2016.

[4] Herbert Bay, Tinne Tuytelaars, and Luc Van Gool. Surf: Speeded up robust features. In *European conference on computer vision*, pages 404–417. Springer, 2006.

[5] J. L. Bentley and J. B. Saxe. Decomposable searching problems i. static-to-dynamic transformation. *Journal of Algorithms*, 1(4):301–358, 1980.

[6] Paul J Besl and Neil D McKay. Method for registration of 3-d shapes. In *Sensor Fusion IV: Control Paradigms and Data Structures*, volume 1611, pages 586–607. International Society for Optics and Photonics, 1992.

[7] Vladimir Braverman, Dan Feldman, and Harry Lang. New frameworks for offline and streaming coreset constructions. *arXiv preprint arXiv:1612.00889*, 2016.

[8] Anirban Dasgupta, Petros Drineas, Boulos Harb, Ravi Kumar, and Michael W Mahoney. Sampling algorithms and coresets for $\ell_p$ regression. *SIAM Journal on Computing*, 38(5):2060–2078, 2009.

[9] Konstantinos G Derpanis. Overview of the ransac algorithm. *Image Rochester NY*, 4(1):2–3, 2010.

[10] Richard O Duda and Peter E Hart. Use of the hough transformation to detect lines and curves in pictures. *Communications of the ACM*, 15(1):11–15, 1972.
[11] D. Feldman and M. Langberg. A unified framework for approximating and clustering data. In Proceedings of the 43rd ACM Symposium on the Theory of Computing (STOC), pages 569–578, 2011. See http://arxiv.org/abs/1106.1379 for fuller version.

[12] Dan Feldman, Amos Fiat, Micha Sharir, and Danny Segev. Bi-criteria linear-time approximations for generalized k-mean/median/center. In Proceedings of the twenty-third annual symposium on Computational geometry, pages 19–26. ACM, 2007.

[13] Dan Feldman, Melanie Schmidt, and Christian Sohler. Turning Big Data into Tiny Data: Constant-size Coresets for k-means, PCA and Projective Clustering. In Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1434 – 1453, 2013.

[14] Dan Feldman and Leonard J Schulman. Data reduction for weighted and outlier-resistant clustering. In Proceedings of the twenty-third annual ACM-SIAM symposium on Discrete Algorithms, pages 1343–1354. SIAM, 2012.

[15] Dan Feldman and Tamir Tassa. More constraints, smaller coresets: Constrained matrix approximation of sparse big data. In Proceedings of the 21th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, pages 249–258. ACM, 2015.

[16] Martin A Fischler and Robert C Bolles. Random sample consensus: a paradigm for model fitting with applications to image analysis and automated cartography. Communications of the ACM, 24(6):381–395, 1981.

[17] Kevin A Geisner, Brian J Mount, Stephen G Latta, Daniel J McCulloch, Kyungsuk David Lee, Ben J Sugden, Jeffrey N Margolis, Kathryn Stone Perez, Sheridan Martin Small, Mark J Finocchio, et al. Realistic occlusion for a head mounted augmented reality display, September 1 2015. US Patent 9,122,053.

[18] Bert M Haralick, Chung-Nan Lee, Karsten Ottenberg, and Michael Nölle. Review and analysis of solutions of the three point perspective pose estimation problem. International journal of computer vision, 13(3):331–356, 1994.

[19] Shunsuke Hijikata, Kenji Terabayashi, and Kazunori Umeda. A simple indoor self-localization system using infrared leds. In Networked Sensing Systems (INSS), 2009 Sixth International Conference on, pages 1–7. IEEE, 2009.

[20] Ibrahim Jubran and Dan Feldman. Demonstration of our algorithms in a real-time system. Available at url = https://drive.google.com/open?id=19I6Jd6F8ET9386yahKx3Dgr6gBTuzYhJ.

[21] Harold W Kuhn. The hungarian method for the assignment problem. Naval Research Logistics (NRL), 2(1-2):83–97, 1955.

[22] Bor-Woei Kuo, Hsun-Hao Chang, Yung-Chang Chen, and Shi-Yu Huang. A light-and-fast SLAM algorithm for robots in indoor environments using line segment map. J. Robotics, 2011:257852:1–257852:12, 2011.

[23] Bruce D Lucas, Takeo Kanade, et al. An iterative image registration technique with an application to stereo vision. 1981.
[24] Eric Marchand, Hideaki Uchiyama, and Fabien Spindler. Pose estimation for augmented reality: a hands-on survey. *IEEE transactions on visualization and computer graphics*, 22(12):2633–2651, 2016.

[25] Gian Luca Mariottini and Domenico Prattichizzo. Uncalibrated video compass for mobile robots from paracatadioptric line images. In *2007 IEEE/RSJ International Conference on Intelligent Robots and Systems, October 29 - November 2, 2007, Sheraton Hotel and Marina, San Diego, California, USA*, pages 226–231, 2007.

[26] Raul Mur-Artal, Jose Maria Martinez Montiel, and Juan D Tardos. Orb-slam: a versatile and accurate monocular slam system. *IEEE Transactions on Robotics*, 31(5):1147–1163, 2015.

[27] Rohan Paul, Dan Feldman, Daniela Rus, and Paul Newman. Visual precis generation using coresets. In *2014 IEEE International Conference on Robotics and Automation, ICRA 2014, Hong Kong, China, May 31 - June 7, 2014*, pages 1304–1311, 2014.

[28] Rahul Raguram, Jan-Michael Frahm, and Marc Pollefeys. A comparative analysis of ransac techniques leading to adaptive real-time random sample consensus. In *European Conference on Computer Vision*, pages 500–513. Springer, 2008.

[29] Cynthia R. Sung, Dan Feldman, and Daniela Rus. Trajectory clustering for motion prediction. In *2012 IEEE/RSJ International Conference on Intelligent Robots and Systems, IROS 2012, Vilamoura, Algarve, Portugal, October 7-12, 2012*, pages 1547–1552, 2012.

[30] Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 267–288, 1996.

[31] Antoine Vigneron. Geometric optimization and sums of algebraic functions. *ACM Transactions on Algorithms (TALG)*, 10(1):4, 2014.

[32] John Wang and Edwin Olson. Apriltag 2: Efficient and robust fiducial detection. In *2016 IEEE/RSJ International Conference on Intelligent Robots and Systems, IROS 2016, Daejeon, South Korea, October 9-14, 2016*, pages 4193–4198, 2016.

[33] Jiyan Yang, Yin-Lam Chow, Christopher Ré, and Michael W Mahoney. Weighted sgd for lp regression with randomized preconditioning. In *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, pages 558–569, 2016.

[34] Hui Zou. The adaptive lasso and its oracle properties. *Journal of the American statistical association*, 101(476):1418–1429, 2006.


\section{Optimization Framework}

\begin{theorem}[Theorem 3]
Let \(g_1, \ldots, g_n\) be \(n\) function, where \(g_i : \mathbb{R} \rightarrow [0, \infty)\) is a piecewise \(r\)-log-Lipschitz function for every \(i \in [n]\), and let \(M(g_i)\) denote the minima of \(g_i\) as in Definition 2. Let \(x \in \mathbb{R}\). Then there is \(x' \in \bigcup_{i \in [n]} M(g_i)\) such that for every \(i \in [n]\),

\[ g_i(x') \leq 2^r g_i(x). \tag{10} \]

\textit{Proof.} Let \(x \in \mathbb{R}\). Let \(x' \in \bigcup_{i \in [n]} M(g_i)\) be the closest item to \(x\), i.e., that minimizes \(\text{dist}(x, x')\). Ties broken arbitrarily. Put \(i \in [n]\). We have that \(g_i\) is piecewise \(r\)-log-Lipschitz function. Let \(M(g_i) = \{x_1, \ldots, x_m\}\), let \(X_1, \ldots, X_m\) be a partition of \(\mathbb{R}\), and let \(h_1, \ldots, h_m\) such that properties (i)-(iii) in Definition 2 hold for \(g_i\). Let \(j \in [m]\) such that \(x \in X_j\). The rest of the proof is by the following case analysis: (i) \(x' \in X_j\), (ii) \(x' \in X_{j+1}\) and \(j \leq m - 1\), and (iii) \(x' \in X_{j-1}\) and \(j \geq 2\). If \(x' \in X_{j-2}\) or \(x' \in X_{j+2}\), it will imply that \(\text{dist}(x_{j-2}, x) < \text{dist}(x', x)\) or \(\text{dist}(x_{j+2}, x) < \text{dist}(x', x)\) respectively, which contradicts the definition of \(x'\). Hence, there are no more cases.

\textbf{Case (i) \(x' \in X_j\).} We prove this case for any \(y \in X_j\) that satisfies \(\text{dist}(x, y) \leq \text{dist}(x, x_j)\). Then this case will trivially hold for \(y = x'\) since \(\text{dist}(x, x') \leq \text{dist}(x, x_j)\). Let \(y \in X_j\) such that \(\text{dist}(x, y) \leq \text{dist}(x, x_j)\). Then it holds that

\[ \text{dist}(x, y) \leq \text{dist}(x, x) + \text{dist}(x, y) \leq 2\text{dist}(x, x) \tag{11} \]

by the definition of \(y\) and the triangle inequality. We then have that

\[ g_i(y) = h_j(\text{dist}(x, y)) \leq h_j(2\text{dist}(x, x)) \leq 2^r h_j(\text{dist}(x, x)) = 2^r g_i(x), \]

where the first and last equalities hold by the definition of \(h_j\), the first inequality holds by combining that \(h_j\) is a monotonic increasing function with (11), and the second inequality holds since \(h_j\) is \(r\)-log-Lipschitz in \(X_j\) by property (ii) of Definition 2.

\textbf{Case (ii) \(x' \in X_{j+1}\) and \(j \leq m - 1\).} In this case \(x' < x_{j+1}\) by its definition and the fact that \(x \in X_j\). Hence, \(x' \in (x_j, x_{j+1})\). Let \(y \in (x_j, x_{j+1})\) such that \(g_i(y) = \sup_{y' \in (x_j, x_{j+1})} g_i(y')\). Combining the definition of \(y\) with the assumption that \(g_i\) is continuous and non-decreasing in \((x_j, x_{j+1}) \cap X_j\), and is non-increasing in \((x_j, x_{j+1}) \cap X_{j+1}\), we have that

\[ \lim_{z \rightarrow y^+} g_i(z) = \lim_{z \rightarrow y^-} g_i(z). \tag{12} \]

Hence,

\[ g_i(x') \leq \lim_{z \rightarrow y^+} g_i(z) = \lim_{z \rightarrow y^-} g_i(z) \leq 2^r g_i(x), \]

where the first derivation holds by comibing that \(g_i\) is non-increasing in \((x_j, x_{j+1}) \cap X_{j+1}\) and the definition of \(y\), the second derivation is by (12), and since \(\text{dist}(x, z) \leq \text{dist}(x, x') \leq \text{dist}(x, x_j)\) the last derivation holds by substituting \(y = z\) in Case (i).

\textbf{Case (iii) \(x' \in X_{j-1}\) and \(j \geq 2\).} The proof for this case is symmetric to Case (ii).

Hence, for all cases there is \(x' \in \bigcup_{i \in [n]} M(g_i)\) such that for every \(i \in [n]\),

\[ g_i(x') \leq 2^r g_i(x). \]

\end{proof}

\begin{observation}[Observation 5]
Let \(\text{cost}(A, q) = f(\text{lip}(D(a_1, q)), \ldots, \text{lip}(D(a_n, q)))\) be defined as in Definition 2. Let \(q^*, q' \in Q\) and let \(c \geq 1\). If \(D(a_i, q') \leq c \cdot D(a_i, q^*)\) for every \(i \in [n]\), then

\[ \text{cost}(A, q') \leq c^{r^*} \cdot \text{cost}(A, q^*). \]

\end{observation}

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Proof. We have that
\[
\text{cost} (A, q') = f \left( \text{lip} \left( D (a_1, q') \right), \cdots, \text{lip} \left( D (a_n, q') \right) \right) \tag{13}
\]
\[
\leq f \left( \text{lip} \left( c \cdot D (a_1, q^*) \right), \cdots, \text{lip} \left( c \cdot D (a_n, q^*) \right) \right) \tag{14}
\]
\[
\leq c^s \cdot f \left( \text{lip} \left( D (a_1, q^*) \right), \cdots, \text{lip} \left( D (a_n, q^*) \right) \right) \tag{15}
\]
\[
= c^s \cdot \text{cost} (A, q^*), \tag{16}
\]
where [13] holds by the definition of \text{cost}, [14] holds by the assumption in Observation 5, [15] holds since \text{lip} is \( r \)-log-Lipschitz, [16] holds since \( f \) is \( s \)-log-Lipschitz, and [17] holds by the definition of \text{cost}. \qed

B Proofs of Main Results

In this section we presents the main results and their proofs.

B.1 Constrained regression

The following lemma states that if a function \( f \) is concave in an interval \( X \) that contains 0, then \( f \) is 1-log-Lipschitz.

Lemma 18. Let \( X \subset \mathbb{R} \) be an interval that contains 0, and let \( f : X \rightarrow [0, \infty) \) such that the second derivative of \( f \) is defined and satisfies \( f''(x) \leq 0 \) for every \( x \in X \). Then \( f \) is log-Lipschitz for every \( x \in X \), i.e., for every \( c \geq 1 \) and \( x \in X \cap \frac{X}{c} \), it holds that
\[
f(cx) \leq cf(x). \tag{18}
\]

Proof. Since \( f''(x) \leq 0 \) for every \( x \in X \), we have that \( f \) is concave in the interval \( X \). Therefore, for every \( a, b \in X \), it holds that
\[
f(b) \leq f(a) + f'(a)(b - a). \tag{18}
\]
By substituting \( a = x \) and \( b = 0 \) in (18), we have
\[
f(0) \leq f(x) + f'(x)(0 - x) = f(x) - xf'(x).
\]
Rearranging terms yields
\[
xf'(x) \leq f(x) - f(0) \leq f(x), \tag{19}
\]
where the second derivation is since \( f(0) \geq 0 \) by the definition of \( f \). Hence, it holds that
\[
f(cx) \leq f(x) + f'(x)(c - 1) \leq f(x) + f(x)(c - 1) = cf(x),
\]
where the first inequality holds by substituting \( a = x \) and \( b = cx \) in (18), and the second inequality holds by (19). \qed

Lemma 19 (Lemma 6). Let \( a_1, \cdots, a_n \subseteq \mathbb{R}^2 \) and \( b_1, \cdots, b_n \geq 0 \). Then there is a set \( C \) of \( |C| \in O(n) \) unit vectors that can be computed in \( O(n) \) time such that (i) and (ii) hold as follows:
(i) For every unit vector \( x \in \mathbb{R}^2 \) there is a vector \( x' \in C \) such that for every \( i \in [n] \),
\[
|a_i^T x' - b_i| \leq 4 \cdot |a_i^T x - b_i|.
\]
(20)

(ii) There is \( k \in [n] \) such that \( x' \in \arg \min_{\| y \| = 1} |a_k^T y - b_k| \)

Proof. Proof of (i): We prove the claim independently for every \( i \in [n] \). By Lemma 18, it suffices to prove that \( |a_i^T x - b_i| \) is piecewise \( r \)-log-Lipschitz over every unit vector \( x \), and then define \( C \) to include all the minima of \( |a_i^T x - b_i| \), over every \( i \in [n] \). The proof that \( |a_i^T x - b_i| \) is piecewise \( r \)-log-Lipschitz is based on the case analysis of whether \( b_i \geq \|a_i\| \) or \( b_i < \|a_i\| \) in Claims 19.1 and 19.2 respectively. The proofs of these cases use Lemma 18.

Indeed, let \( i \in [n] \), \( \alpha_i \in [0, 2\pi) \) such that \( a_i/\|a_i\| = (\cos \alpha_i, \sin \alpha_i) \), and let \( I = [\alpha_i, \alpha_i + \pi) \) be an interval. Here we assume \( \|a_i\| \neq 0 \), otherwise (20) trivially holds for \( i \). For every \( t \in \mathbb{R} \), let
\[
g_i(t) = \begin{cases} \|a_i\| \cdot |\sin(t - \alpha_i) - \frac{b_i}{\|a_i\|}|, & t \in I \\ 0, & \text{otherwise} \end{cases}
\]
and let \( x(t) = (\sin t, -\cos t) \). Hence, for every \( t \in I \),
\[
g_i(t) = \|a_i\| \cdot |\sin(t - \alpha_i) - \frac{b_i}{\|a_i\|}| = \|a_i\| \cdot |\sin(t) \cos(\alpha_i) - \cos(t) \sin(\alpha_i) - \frac{b_i}{\|a_i\|}| = \|a_i\| \cdot \frac{a_i^T}{\|a_i\|} x(t) - \frac{b_i}{\|a_i\|} = \|a_i\| \cdot \frac{a_i^T}{\|a_i\|} x(t) - \frac{b_i}{\|a_i\|} = \|a_i\| \cdot \frac{a_i^T}{\|a_i\|} x(t) - b_i.
\]
(21)

Claim 19.1. If \( \frac{b_i}{\|a_i\|} \geq 1 \) then \( g_i(t) \) is piecewise 2-log-Lipschitz for every \( t \in I \).

Proof. Suppose that indeed \( \frac{b_i}{\|a_i\|} \geq 1 \). In this case, \( \frac{b_i}{\|a_i\|} \geq 1 \geq \sin(t - \alpha_i) \), so the absolute value in \( g_i \) can be removed, i.e., for every \( t \in \mathbb{R} \),
\[
g_i(t) = \begin{cases} \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - \sin(t - \alpha_i) \right), & t \in I \\ 0, & \text{otherwise} \end{cases}
\]

Let \( I' = [0, \pi/2] \) and let \( h : I' \to [0, \infty) \) such that \( h(x) = \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - \sin(\pi/2 + x) \right) \). Since \( h(x) = h(-x) = g_i(\pi/2 + \alpha_i + x) \) for every \( x \in I' \), we have
\[
g_i(t) = \begin{cases} h(\pi/2 + \alpha_i - t), & t \in I \\ 0, & \text{otherwise}. \end{cases}
\]
(22)

We now prove that \( h(x) \) is 2-log-Lipschitz for every \( x \in I' \). For every \( x \in I' \) it holds that
\[
h(x) = \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - \sin(\pi/2 + x) \right) = \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - \left( 1 - 2 \sin^2 \frac{x}{2} \right) \right) = \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - 1 + 2 \sin^2 \frac{x}{2} \right),
\]
(23)

where the first equality holds by the definition of \( h \) and the second equality holds since \( \sin(\pi/2 + x) = \cos(x) = (1 - 2 \sin^2 \left( \frac{x}{2} \right) ) \) for every \( x \in \mathbb{R} \).
Let \( c \geq 1 \), \( X = [0, \pi/2] \), and \( f : X \to [0, \infty) \) such that \( f(x) = \sin(x) \). Since \( f''(x) = -\sin(x) \leq 0 \) for every \( x \in X \), we have by Lemma 18 that
\[
\sin(cx) = f(cx) \leq c \cdot f(x) = c \cdot \sin(x)
\]
for every \( x \in X \cap \frac{X}{c} \).

By taking the square of the last inequality, it holds that for every \( x \in X \cap \frac{X}{c} \),
\[
\sin^2(cx) \leq c^2 \cdot \sin^2(x).
\] (24)

Thus, for every \( x \in I' \cap \frac{I'}{c} \), it holds that
\[

h(cx) = \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - 1 + 2 \sin^2 \frac{cx}{2} \right) \leq \|a_i\| \cdot \left( \left( \frac{b_i}{\|a_i\|} - 1 \right) + 2c^2 \sin^2 \frac{x}{2} \right)
\]
\[
\leq c^2 \cdot \|a_i\| \cdot \left( \frac{b_i}{\|a_i\|} - 1 + 2 \sin^2 \frac{x}{2} \right) = c^2 \cdot h(x),
\] (25)

where the first and last equalities are by (23), the first inequality is by combining \( I' = X \) and (24), and the second inequality holds since \( \frac{b_i}{\|a_i\|} - 1 \geq 0 \). By Definition 1, it follows that \( h(x) \) is 2-log-Lipschitz for every \( x \in I' \).

By substituting \( g = g_i \), \( m = 1 \), \( X = X_1 = I \) and \( \text{dist}(a, b) = |a - b| \), Properties (i)-(iii) of Definition 2 hold for \( g_i \) since
(i) It is clear that \( g_i \) has a unique infimum \( x_1 = \pi/2 + \alpha_i \) in \( X_1 \).
(ii) \([0, \max_{x \in I} |x - x_1|] = [0, \pi/2] = I' \) and by (25) \( h : I' \to [0, \infty) \) is 2-log-Lipschitz.
(iii) \( g_i(x) = h(|\pi/2 + \alpha_i - t|) = h(\text{dist}(\pi/2 + \alpha_i, t)) \) by (22).

Hence, \( g_i(t) \) is a piecewise 2-log-Lipschitz function for every \( t \in I \). \( \Box \)

Claim 19.2. If \( \frac{b_i}{\|a_i\|} < 1 \), it then holds that \( g_i(t) \) is 2-piecewise Lipschitz for every \( t \in [-\frac{\pi}{2} + \alpha_i, \frac{3\pi}{2} + \alpha_i] \).
**Proof.** Let

\[
\alpha = \arcsin\left(\frac{b_i}{\|a_i\|}\right) \in [0, \pi/2)
\]

\[
\alpha_{i1} = -\frac{\pi}{2} + \alpha_i, \alpha_{i2} = \alpha + \alpha_i, \alpha_{i3} = \frac{\pi}{2} + \alpha_i, \alpha_{i4} = \pi - \alpha + \alpha_i, \alpha_{i5} = \frac{3\pi}{2} + \alpha_i,
\]

\[
X_1 = [\alpha_{i1}, \alpha_{i2}), I_1 = [0, \alpha_{i2} - \alpha_{i1}) = [0, \pi/2 + \alpha),
\]

\[
X_2 = [\alpha_{i2}, \alpha_{i3}), I_2 = [0, \alpha_{i3} - \alpha_{i2}) = [0, \pi/2 - \alpha),
\]

\[
X_3 = [\alpha_{i3}, \alpha_{i4}), I_3 = [0, \alpha_{i4} - \alpha_{i3}) = [0, \pi/2 - \alpha),
\]

\[
X_4 = [\alpha_{i4}, \alpha_{i5}), I_4 = [0, \alpha_{i5} - \alpha_{i4}) = [0, \pi/2 + \alpha),
\]

\[
h_1(t)/\|a_i\| = \frac{b_i}{\|a_i\|} - \sin(\alpha_{i2} - t - \alpha_i) = \frac{b_i}{\|a_i\|} - \sin(\alpha - t) \text{ for every } t \in I_1,
\]

\[
h_2(t)/\|a_i\| = \sin(\alpha_{i2} + t - \alpha_i) - \frac{b_i}{\|a_i\|} = \sin(\alpha + t) - \frac{b_i}{\|a_i\|} \text{ for every } t \in I_2,
\]

\[
h_3(t)/\|a_i\| = \sin(\alpha_{i4} - t - \alpha_i) - \frac{b_i}{\|a_i\|} = \sin(\pi - \alpha - t) - \frac{b_i}{\|a_i\|} = \sin(\alpha + t) - \frac{b_i}{\|a_i\|} \text{ for every } t \in I_3,
\]

\[
h_4(t)/\|a_i\| = \frac{b_i}{\|a_i\|} - \sin(\alpha_{i4} + t - \alpha_i) = \frac{b_i}{\|a_i\|} - \sin(\pi - \alpha + t) = \frac{b_i}{\|a_i\|} - \sin(\alpha - t) \text{ for every } t \in I_4.
\]

First, observe that \( I = [\alpha_i, \alpha_i + \pi) \subseteq [-\frac{\pi}{2} + \alpha_i, \frac{3\pi}{2} + \alpha_i] = X_1 \cup X_2 \cup X_3 \cup X_4. \) Second, for every \( t \in \mathbb{R}, \) it holds that,

\[
g_i(t) = \begin{cases} 
  h_1(\alpha_{i2} - t), & t \in X_1 \\
  h_2(t - \alpha_{i2}), & t \in X_2 \\
  h_3(\alpha_{i4} - t), & t \in X_3 \\
  h_4(t - \alpha_{i4}), & t \in X_4 \\
  0, & \text{otherwise}.
\end{cases}
\]

Let \( c \geq 1. \) We now prove that \( g_i(t) \) is piecewise 2-log-Lipschitz for every \( t \in [-\frac{\pi}{2} + \alpha_i, \frac{3\pi}{2} + \alpha_i] \) by proving that

(i) \( h_1(t) \) is 2-log-Lipschitz in \( I_1, \) i.e., \( h_1(ct) \leq c^2 \cdot h_1(t) \) for every \( t \in I_1 \cap \frac{I_1}{c} = I_1/c. \)

(ii) \( h_2(t) \) is 1-log-Lipschitz in \( I_2, \) i.e., \( h_2(ct) \leq c \cdot h_2(t) \) for every \( t \in I_2 \cap \frac{I_2}{c} = I_2/c. \)

By combining (ii), \( I_2 = I_3 \) and \( h_2(t) = h_3(t) \) for every \( t \in I_3, \) we get that \( h_3(t) \) is 1-log-Lipschitz in \( I_3, \)

i.e., \( h_3(ct) \leq c \cdot h_3(t) \) for every \( t \in I_3 \cap \frac{I_3}{c}. \)

By combining (i), \( I_1 = I_4 \) and \( h_1(t) = h_4(t) \) for every \( t \in I_4, \) we get that \( h_4(t) \) is 2-log-Lipschitz in \( I_4, \)

i.e., \( h_4(ct) \leq c^2 \cdot h_4(t) \) for every \( t \in I_4 \cap \frac{I_4}{c}. \)

(i): Clearly, if \( t = 0, \) Claim [19.2] trivially holds as

\[
h_1(ct) = h_1(t) \leq c \cdot h_1(t).
\]

Let \( d_1 : [I_1 \cap \frac{I_1}{c} \setminus \{0\}] :\Rightarrow [0, \infty) \) such that \( d_1(t) = \frac{h_1(ct)}{h_1(t)}. \) The denominator is positive since \( h_1(t) > 0 \) in the range of \( d_1. \)
We now prove that \( d_1 \) does not get its maximum at point \( t' = \frac{\pi/2 + \alpha}{c} \). Observe that

\[
c \cdot h_1'(ct') \cdot (h_1(t')) - h_1(ct') \cdot (h_1'(t')) = c \cdot \cos(\alpha - ct') \cdot \left( \frac{b_i}{\|a_i\|} - \sin(\alpha - t') \right) - \left( \frac{b_i}{\|a_i\|} - \sin(\alpha - ct') \right) \cdot \cos(\alpha - t')
\]

\[
= \left( \frac{b_i}{\|a_i\|} + 1 \right) \cdot \left( \cos \left( \alpha - \frac{\pi/2 + \alpha}{c} \right) \right) < 0,
\]

where the third derivation holds since \( \alpha - ct' = -\pi/2 \), and the last derivation holds since \( \alpha - \frac{\pi/2 + \alpha}{c} \in (-\pi/2, \pi/2) \).

Since \( d_1'(t') = \frac{c \cdot h_1'(ct') \cdot (h_1(t')) - h_1(ct') \cdot (h_1'(t'))}{h_1''(t')} \), we have that the sign of \( d_1'(t') \) is equal to the sign of \( c \cdot h_1'(ct') \cdot (h_1(t')) - h_1(ct') \cdot (h_1'(t')) \), which is negative by \( 27 \). Hence, \( d_1 \) doesn’t get its maximum at points \( t' \).

We now prove that that \( d_1(t) \leq c^2 \) for \( t \in (0, \frac{\pi/2 + \alpha}{c}) \). Suppose that \( t^* \) is a number that maximizes \( d_1(t) \) over the open interval \( (0, \frac{\pi/2 + \alpha}{c}) \). Since \( d_1 \) is continuous in an open interval, the derivation of \( d_1 \) is zero, i.e.,

\[
0 = d_1'(t^*) = c \cdot h_1'(ct^*) \cdot (h_1(t^*)) - h_1(ct^*) \cdot (h_1'(t^*)).
\]

We also have that \( h_1'(t^*)/\|a_i\| = \cos(\alpha - t^*) > 0 \) since

\[
\alpha - t^* \in (\alpha - (\pi/2 + \alpha), \alpha] \subseteq (-\pi/2, \pi/2).
\]

Hence, it holds that

\[
d_1(t^*) = \frac{h_1(ct^*)}{h_1(t^*)} = \frac{e \cdot h_1'(ct^*)}{h_1'(t^*)} = \frac{e \cdot \|a_i\| \cdot \cos(\alpha^*_2 - ct^* - \alpha_i)}{\|a_i\| \cdot \cos(\alpha^*_2 - t^* - \alpha_i)} = e \cdot \frac{\cos(\alpha - ct^*)}{\cos(\alpha - t^*)}. \tag{28}
\]

Let \( f(x) = \cos(\alpha-x) = \cos(x-\alpha) \) and observe that \( f''(x) = -\cos(x-\alpha) \leq 0 \) for every \( x \in I_1 \cap \frac{I_1}{c} = I_1 \).

Substituting \( f \) and \( X = I_1 \) in Lemma \( 18 \) yields

\[
\cos(\alpha - cx) = f(cx) \leq c \cdot f(x) = c \cdot \cos(\alpha - x). \tag{29}
\]

Hence, for every \( t \in \frac{I_1}{c} \) it follows that

\[
d_1(t) \leq d_1(t^*) = c \cdot \frac{\cos(\alpha - ct)}{\cos(\alpha - t)} \leq c^2, \tag{30}
\]

where the first equality holds by \( 28 \) and the last inequality is by substituting \( x = t \) in \( 29 \).

We also have that

\[
\lim_{t \to 0} d_1(t) = \lim_{t \to 0} \frac{h_1(ct)}{h_1(t)} = \lim_{t \to 0} \frac{c \cdot h_1'(ct)}{h_1'(t)} = e \cdot \lim_{t \to 0} \frac{\|a_i\| \cdot \cos(\alpha - ct)}{\|a_i\| \cdot \cos(\alpha - t)} = e \cdot \frac{\cos(\alpha)}{\cos(\alpha)} = e, \tag{31}
\]

where the second derivation holds by L’Hospital’s rule since \( h_1(ct) = h_1(t) = 0 \) for \( t = 0 \).
By combining (27), (30) and (31), we get that $d_1(t) = \frac{h_1(ct)}{h_1(t)} \leq c^2$ for every $t \in I_1 \cap I_1/c \setminus \{0\}$. By combining (26) with the last inequality, (i) holds as $h_1(ct) \leq c^2 h_1(t)$ for every $t \in I_1 \cap I_1/c$.

(ii): We have that $h_2(t) = \|a_i\| \cdot (\sin(t + \alpha) - \frac{b_i}{\|a_i\|})$ for $t \in I_2$. Since $h''_2(t) = -\|a_i\| \cdot \sin(t + \alpha) \leq 0$ for every $t \in I_2/c$, by substituting $f(x) = h_2(x)$ and $X = I_2$ in Lemma 18 (ii) holds as 

$$h_2(ct) = f(ct) \leq c \cdot f(t) = c \cdot h_2(t)$$

(32)

for every $t \in I_2 \cap I_2/c$.

Observe that $\alpha_{i2}$ and $\alpha_{i4}$ are the minima of $g_i(t)$ over $t \in [-\pi/2 + \alpha_i, \alpha_i + 3\pi/2]$ when $\frac{b_i}{\|a_i\|} < 1$ and that $\{X_1, \cdots, X_4\}$ is a partition of $[-\frac{\pi}{2} + \alpha_i, \frac{3\pi}{2} + \alpha_i]$. By Definition 2 we have that $g_i(t)$ is piecewise 2-log-Lipschitz function in $[-\frac{\pi}{2} + \alpha_i, \frac{3\pi}{2} + \alpha_i]$ since

(i) $g_i$ has a unique infimum $x_1 = \alpha_{i2} = \alpha + \alpha_i$ in $X_1$, $h_1 : [0, \max_{x \in X_1} |x_1 - x|] \to [0, \infty)$ is 2-log-Lipschitz in $I_1$, and $g_i(t) = h_1(\alpha_{i2} - t) = h_1(|\alpha_{i2} - t|)$ for every $t \in X_1$.

(ii) $g_i$ has a unique minimum $x_2 = \alpha_{i2} = \alpha + \alpha_i$ in $X_2$, $h_2 : [0, \max_{x \in X_2} |x_2 - x|] \to [0, \infty)$ is 1-log-Lipschitz in $I_2$, and $g_i(t) = h_2(t - \alpha_{i2}) = h_2(|t - \alpha_{i2}|)$ for every $t \in X_2$.

(iii) $g_i$ has a unique infimum $x_3 = \alpha_{i4} = \pi - \alpha + \alpha_i$ in $X_3$, $h_3 : [0, \max_{x \in X_3} |x_3 - x|] \to [0, \infty)$ is 1-log-Lipschitz in $I_3$, and $g_i(t) = h_3(\alpha_{i4} - t) = h_3(|\alpha_{i4} - t|)$ for every $t \in X_3$.

(iv) $g_i$ has a unique minimum $x_4 = \alpha_{i4} = \pi - \alpha + \alpha_i$ in $X_4$, $h_4 : [0, \max_{x \in X_4} |x_4 - x|] \to [0, \infty)$ is 2-log-Lipschitz in $I_4$, and $g_i(t) = h_4(t - \alpha_{i4}) = h_4(|t - \alpha_{i4}|)$ for every $t \in X_4$.

Since $I \subset [-\frac{\pi}{2} + \alpha_i, \frac{3\pi}{2} + \alpha_i]$, by Claims 19.1 and 19.2 it holds that $g_i$ is piecewise 2-log-Lipschitz function for every $t \in I$.

Hence, by substituting $g(p, \cdot)$ in Lemma 3 with $g_i(\cdot)$, and $r = 2$, there exists $t' \in \bigcup_{i \in [n]} M(g_i(\cdot))$ such that for every $t \in I$, it holds that 

$$g_i(t') \leq 2^r \cdot g_i(t) = 4g_i(t).$$

(33)

Let $x \in \mathbb{R}^2$ be a unit vector, let $t \in \mathbb{R}^2$ such that $x = x(t)$, let $x' = (\sin t', -\cos t')$, and let $k \in [n]$ be the index such that $t' \in M(g_k(\cdot))$. It holds that 

$$|a_i^T x' - b_i| = |a_i^T x(t') - b_i| = g_i(t') \leq 4 \cdot g_i(t) = 4 \cdot |a_i^T x(t) - b_i| = |a_i^T x - b_i|,$$

where the first equality holds by the definition of $x'$, the second equality holds by (21), the third equality holds by (33), the fourth holds by (21), and the last equality holds since $x = x(t)$.

**Proof of (ii):** The proof follows immediately by the definition of $k$ in (i).

Observe that it takes $O(1)$ time to compute $M(g_i(\cdot))$ for some $i \in [n]$. Hence, Lemma 19 holds by letting $C = \left\{ x(t) \mid t \in \bigcup_{i \in [n]} M(g_i(\cdot)) \right\}$, since it takes $O(n)$ time to compute $C$.

**Theorem 20 (Theorem 7).** Let $A = \{ (a_1, b_1), \cdots, (a_n, b_n) \}$ be a set of $n \geq 1$ pairs, where for every $i \in [n]$, we have that $a_i \in \mathbb{R}^2$ and $b_i \geq 0$. Let $D((a_i, b_i), x) = |a_i^T x - b_i|$ for every $i \in [n]$ and unit vector $x \in \mathbb{R}^2$. Let $\text{cost}, s, r$ be as defined in Definition 4. Then in $n^{O(1)}$ time we can compute a unit vector $x' \in \mathbb{R}^2$ such that 

$$\text{cost}(A, x') \leq 4^r s \cdot \min_{x \in \mathbb{R}^2 : |x| = 1} \text{cost}(A, x).$$

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Proof. Put \(i \in [n]\) and let \(x^* \in \arg\min_{x \in \mathbb{R}^2 : \|x\| = 1} \text{cost}(A, x)\). By Lemma 6 (i) there is a set \(C\) of \(|C| \in O(n)\) unit vectors that can be computed in \(O(n)\) time, and \(x' \in C\) such that the following holds

\[
|a_i^T x' - b_i| \leq 4 \cdot |a_i^T x^* - b_i|.
\]

By substituting \(q' = x'\) and \(q^* = x^*\) in Observation 5, Theorem 20 holds as

\[
\text{cost}(A, x') \leq 4^r s \cdot \text{cost}(A, x^*) = 4^r s \cdot \min_{x \in \mathbb{R}^2 : \|x\| = 1} \text{cost}(A, x).
\]

Furthermore, the vector \(x' \in C\) can be computed in \(O(n^2)\) time by computing the vector \(x \in C\) that minimizes \(\text{cost}(A, x)\) over every \(x \in C\). \(\square\)

B.2 Aligning Points-To-Lines

Figure 1: Localization using sky patterns. (Left:) An observed image of the stars in Ursa Major. Each star represents a point and is marked with a blue circle. (Right:) A known (initial) model, represented as a set of linear segments.

Lemma 21 (Lemma 9). Let \(v\) be a unit vector and \(\ell = \text{sp} \{v\}\) be the line in this direction. Let \(p, q, z \in \mathbb{R}^2\) be the vertices of a triangle such that \(\|p - q\| > 0\). Let \(P, Q, Z \in \mathbb{R}^{2 \times 2}\) be the output of a call to \(Z\text{-Configurations}(v, p, q, z)\); see Algorithm 1. Then the following hold:

(i) \(p \in x\)-axis and \(q \in \ell\) iff there is a unit vector \(x \in \mathbb{R}^2\) such that \(p = Px\) and \(q = Qx\).

(ii) For every unit vector \(x \in \mathbb{R}^2\), we have that \(z = Zx\) if \(p = Px\) and \(q = Qx\).

Proof. We use the variables that are defined in Algorithm 1.

(i): \(\Leftarrow\) Let \(\alpha \in [0, 2\pi)\) and \(x = (\sin \alpha, \cos \alpha)^T\) such that \(p = Px\) and \(q = Qx\). We need to prove that \(p \in x\)-axis and \(q \in \ell\). It holds that

\[
\text{dist}(p, x\text{-axis}) = |(0, 1)p| = |(0, 1)Px| = 0,
\]

\[
\text{dist}(q, \ell) = |(0, 0)q| = |(0, 0)Qx| = 0.
\]

\[\Rightarrow\]
where the first equality holds since the vector \((0, 1)\) is orthogonal to the \(x\)-axis and the last equality holds since \((0, 1)P = (0, 0)\) by the definition of \(P\) in Line 3 of Algorithm 1. Hence, \(p \in x\)-axis. It also holds that

\[
\text{dist}(q, \ell) = |v^T q| = |v^T Qx|
\]

\[
= r_1 \left| \begin{pmatrix} v_y \\ -v_x \end{pmatrix}^T \begin{pmatrix} s & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} \right| = r_1 \left| \begin{pmatrix} 0 & 0 \\ \sin \alpha & \cos \alpha \end{pmatrix} \right| = 0,
\]

where the third equality holds by the definition of \(v^\perp = (v_y, -v_x)^T\), and the fourth equality holds since \(s = \frac{v_x}{v_y}\). Hence, \(q \in \ell\).

(i): \(\Rightarrow\) Let \(p \in x\)-axis, \(q \in \ell\) and let \(r_1 = ||p - q||\). We need to prove that there exists a unit vector \(x \in \mathbb{R}^2\) such that \(p = Px\) and \(q = Qx\). Let \(t_p, t_q \in \mathbb{R}\) such that \(p = (t_p, 0)^T, q = t_q(v_x, v_y)^T\) and \(\Delta(p, q, o)\) be a triangle, where \(o\) is the origin of \(\mathbb{R}^2\). Let \(\alpha \in [0, \pi)\) be the interior angle at vertex \(p\) and let \(\beta \in [0, \pi)\) be the angle at vertex \(q\) in triangle \(\Delta(p, q, o)\). Let \(x = (\sin \alpha, \cos \alpha)^T\). By simple trigonometric identities in triangle \(\Delta(p, q, o)\), it holds that

\[
\sin \beta = v_y \cos \alpha + v_x \sin \alpha. \tag{34}
\]

We continue with the following case analysis: (i): \(\alpha, \beta \neq 0\), (ii): \(\alpha = 0\) and (iii): \(\beta = 0\).

**Case (i) \(\alpha, \beta \neq 0\):** By the law of sines we have that \(||q - o|| / \sin(\alpha) = ||p - o|| / \sin \beta\). Hence, respectively,

\[
\frac{t_q}{\sin \alpha} = r_1 \frac{1}{v_y}, \tag{35}
\]

so, \(\sin \alpha = \frac{t_q v_y}{r_1}\), and

\[
\frac{t_p}{\sin \beta} = r_1 \frac{1}{v_y}, \tag{36}
\]

so, \(t_p = r_1 \frac{\sin \beta}{v_y}\).

Then it holds that

\[
Qx = r_1 \begin{pmatrix} s & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} = r_1 \begin{pmatrix} s \sin \alpha \\ \sin \alpha \end{pmatrix} = r_1 \begin{pmatrix} \frac{v_x}{v_y} \sin \alpha \\ \frac{v_x}{v_y} \sin \alpha \end{pmatrix} = t_q \begin{pmatrix} v_x \\ v_y \end{pmatrix} = t_q \cdot v = q, \tag{37}
\]

where the third equality holds by combining (35) with the definition \(s = \frac{v_x}{v_y}\), and the last equality holds by the definition of \(t_q\). It also holds that

\[
P = r_1 \begin{pmatrix} s & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} = r_1 \begin{pmatrix} s \sin \alpha + \cos \alpha \\ 0 \end{pmatrix} = r_1 \begin{pmatrix} \frac{v_x}{v_y} \sin \alpha + \cos \alpha \\ 0 \end{pmatrix} = \begin{pmatrix} t_p \\ 0 \end{pmatrix} = p
\]

where the third equality holds by the definition \(s = \frac{v_x}{v_y}\), the fourth equality holds by (34), the fifth equality holds by (36), and the last equality holds by the definition of \(t_p\). Hence, the vector \(x = (\sin \alpha, \cos \alpha)^T\) satisfies that \(p = Px\) and \(q = Qx\).
Case (ii) $\alpha = 0$: In this case, since $\alpha = 0$, it holds that $q$ intersects the origin $o$, i.e., $q = (0,0)^T$, and $p = \pm (r_1, 0)$ since $\|p - q\| = r_1$. Then by setting $x = \pm (0, 1)^T$ respectively, it holds that, $P x = \pm (r_1, 0)^T = p$ and $Q x = (0, 0)^T = q$. Hence, there exists a unit vector $x \in \mathbb{R}^2$ such that $p = P x$ and $q = Q x$.

Case (iii) $\beta = 0$: In this case, since $\beta = 0$, it holds that $p$ intersects the origin $o$, i.e., $p = (0,0)^T$. Similarly to (37), it holds that $Q x = q$ for $x = (\sin \alpha, \cos \alpha)^T$. It also holds that

$$P x = r_1 \begin{pmatrix} s & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} = r_1 \begin{pmatrix} s \sin \alpha + \cos \alpha \\ 0 \end{pmatrix} = p$$

(38)

where the fourth equality holds by combining (34) and $\beta = 0$. Hence, there exists a unit vector $x \in \mathbb{R}^2$ such that $q = Q x$ and $p = P x$.

(ii): Suppose that $p = P x$ and $q = Q x$ for some unit vector $x \in \mathbb{R}^2$. By (i) we have that $p \in x$-axis and $q \in \ell$. We need to prove that $z = Z x$. Consider the triangle $\Delta(p, q, z)$. Vertex $z$ lies at one of the intersection points of the pair of circles circles $c_1, c_2$ whose radii are $r_2 = \|p - z\|$, $r_3 = \|q - z\|$, centered at $p$ and $q$, respectively; See Fig. 2. The position of $z$ is given by

$$z = p + d_1 \cdot \frac{q - p}{\|q - p\|} + b \cdot d_2 \left( \frac{q - p}{\|q - p\|} \right)^\perp$$

$$= p + \frac{d_1}{r_1} \cdot (q - p) + b \cdot \frac{d_2}{r_1} \cdot (q - p)^\perp$$

(39)

where $b$ is 1 or $-1$ if $z$ lies respectively in the halfplane to the left or to the right of the vector $(q - p)$. If $z$ lies exactly on the line spanned by the vector $(q - p)$, then $b$ can take either 1 or $-1$ since, in that case, $d_2 = 0$; See illustration in Fig 2. The first equality in (39) holds since the vector $q - p$ and $(q - p)^\perp$ are an orthogonal basis of $\mathbb{R}^2$, and the last equality holds since $R$ is a $\pi/2$ radian counter clockwise rotation matrix, i.e., for every vector $v \in \mathbb{R}^2$ it satisfies that $v^\perp = R v$. Substituting $p = P x$ and $q = Q x$ in (39) proves property (ii) as

$$z = P x + \frac{d_1}{r_1} \cdot (Q x - P x) + b \cdot \frac{d_2}{r_1} \cdot R (Q x - P x) = \left( P + \frac{d_1}{r_1} \cdot (Q - P) + b \cdot \frac{d_2}{r_1} \cdot R \right) (Q - P) x = Z x.$$  

\[\square\]

**Corollary 22.** Let $\ell$ be a line on the plane that intersects the origin and is spanned by the unit direction vector $v \in \mathbb{R}^2$. Let $p, q \in \mathbb{R}^2$ such that $p$ is on the $x$-axis, $q$ is on line $\ell$ and $\|p - q\| > 0$. For every $z \in \mathbb{R}^2$, let $(P, Q, Z(z))$ denote the output of a call to Z-CONFIGURATIONS($v, p, q, z$); see Algorithm[?]. Then

(i). There is a unit vector $x \in \mathbb{R}^2$ such that for every $z \in \mathbb{R}^2$ we have $z = Z(z) x$.

(ii). For every unit vector $x \in \mathbb{R}^2$, there is a corresponding alignment $(R, t)$, such that $R z - t = Z(z) x$ for every $z \in \mathbb{R}^2$. 

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Proof. (i): By Lemma 21(i), since \( p \in x\)-axis and \( q \in \ell \), there exists a unit vector \( x \in \mathbb{R}^2 \) such that \( p = Px \) and \( q = Qx \). By Lemma 21(ii), since \( p = Px \) and \( q = Qx \), it holds that \( z = Z(z)x \) for every \( z \in \mathbb{R}^2 \).

(ii): Let \( x \in \mathbb{R}^2 \) be a unit vector and let \((R, t)\) be the alignment that aligns the points \( p \) and \( q \) with the points \( Px \) and \( Qx \), i.e., \( Rp - t = Px \) and \( Rq - t = Qx \). Such an alignment always exists since \( \|Pp - Qq\| = \|p - q\| \) for every unit vector \( x \in \mathbb{R}^2 \). Let \( z \in \mathbb{R}^2 \). Consider the triangle \( \Delta(p', q', z') \) after applying the alignment \((R, t)\) to triangle \( \Delta(p, q, z) \). It holds that \( p' = Rp - t, q' = Rq - t \) and \( z' = Rz - t \), respectively. By the definition of \((R, t)\), it also holds that \( p' = Px \) and \( q' = Qx \). By Lemma 21(ii), the unit vector \( x \) satisfies that \( z' = Z(z)x \). Hence, \( z' = Rz - t = Z(z)x \). \(\square\)

Lemma 23 (Lemma \[10\]). Let \( A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\} \) be set of \( n \geq 3 \) pairs, where for every \( i \in [n] \), we have that \( p_i \) is a point and \( \ell_i \) is a line, both on the plane. Let \( C \subseteq \text{ALIGNMENTS} \) be an output of a call to \( \text{ALIGN} (A) \); see Algorithm \[2\]. Then for every alignment \((R^*, t^*) \in \text{ALIGNMENTS} \) there exists an alignment \((R, t) \in C \) such that for every \( i \in [n] \),

\[
\text{dist}(Rp_i - t, \ell_i) \leq 16 \cdot \text{dist}(R^*p_i - t^*, \ell_i).
\]

Moreover, \( |C| \in O(n^3) \) and can be computed in \( O(n^3) \) time.

Proof. Let \((R^*, t^*) \in \text{ALIGNMENTS} \). Without loss of generality, assume that the set is already aligned by \((R^*, t^*)\), i.e., \( R^* = I \) and \( t^* = 0 \).

Step 1. Put \( i \in [n] \). We first prove that there is \( j \in [n] \), such that translating \( p_j \) until it intersects \( \ell_j \) does not increase the distance \( \text{dist}(p_i, \ell_i) \) by more than a multiplicative factor of 2. See Fig. 2.
Let $t_i = p_i - \text{proj}(p_i, \ell_i)$ and let $j \in \arg\min_{j' \in [n]} \|t'_j\|$. It follows that,
\begin{align*}
dist(p_i - t_j, \ell_i) &\leq \dist(p_i - t_j, p_i) + \dist(p_i, \ell_i) \\
&= \|t_j\| + \dist(p_i, \ell_i) \\
&\leq \|t_i\| + \dist(p_i, \ell_i) \\
&= 2 \cdot \dist(p_i, \ell_i),
\end{align*}
where (41) holds due to the triangle inequality, (42) holds due to the definition of $j$ and (43) holds since $\|t_i\| = \dist(p_i, \ell_i)$. For every $i \in [n]$, let $p'_i = p_i - t_j$ and without loss of generality assume that $p'_j$ is the origin, otherwise translate the coordinate system.

Put $i \in [n]$. Let $v_i \in \mathbb{R}^2$ be a unit vector and $b_i \geq 0$ such that $\ell_i = \{ q \in \mathbb{R}^2 \mid v_i^T q = b_i \}$.

In what follows we assume that $\ell_1, \cdots, \ell_n$ cannot all be parallel, i.e., there exists at least two lines that intersect. We address the case where all the lines are parallel in Claim 23.3.

**Step 2.** We now prove that there is $k \in [n]$ such that translating $p'_k$ in the direction of $\ell_j$, until it gets as close as possible to $\ell_k$ will not increase the distance $\dist(p'_i, \ell_i)$ by more than a multiplicative factor of $2$. See Fig. 4.

Let $\lambda_i \in \arg\min_{\lambda \in \mathbb{R}} \dist(p'_i - \lambda \cdot v_j, \ell_i)$. If there is a finite number of such minimizers set $\lambda_i$ to be the smallest one, and $\infty$ otherwise, i.e., $\lambda_i = \infty$ if and only if $\ell_j$ and $\ell_i$ are parallel lines. Observe that there is at least one index $i \in [n]$ such that $\lambda_i \neq \infty$ since the lines are not all parallel.

**Claim 23.1.** There exists $k \in [n] \setminus \{j\}$ and a corresponding translation vector $t' = \lambda_k \cdot v_j$, such that for every $i \in [n]$,
\[
\dist(p'_i - t', \ell_i) \leq 2 \cdot \dist(p'_i, \ell_i).
\]
Figure 4: Illustration of Step 2 in the proof of Lemma \ref{lem:step2} (Left) The pair \((p_3', \ell_3)\) is the pair with minimal distance in the direction of \(\ell_4\) among the 4 pairs \((p_1', \ell_1), (p_2', \ell_2), (p_3', \ell_3), (p_5', \ell_5)\). (Right) The set of points is translated by the vector \(t' = \lambda_3 \cdot v_4\), where \(v_4\) is the direction vector of \(\ell_4\) and \(\lambda_3\) is the minimal translation magnitude required in order for point \(p_3'\) to get as close as possible to \(\ell_3\).

Proof. Let \(k = \arg \min_{i \in [n]} |\lambda_i|\). Let \(\ell_i' = \{p_i' - \lambda v_j \mid \lambda \in \mathbb{R}\}\). Observe that \(\ell_i'\) is a line in \(\mathbb{R}^2\). If \(\ell_i'\) is parallel to \(\ell_i\), then since \(p_i' - \lambda_k \cdot v_j, p_i' \in \ell_i'\), the claim trivially holds as

\[
\text{dist}(p_i' - t', \ell_i) = \text{dist}(p_i' - \lambda_k \cdot v_j, \ell_i) = \text{dist}(p_i', \ell_i) \leq 2 \cdot \text{dist}(p_i', \ell_i).
\]

We now assume that \(\ell_i'\) and \(\ell_i\) are not parallel. Let \(\alpha\) be the angle between \(\ell_i\) and \(\ell_i'\), and let \(p = \ell_i' \cap \ell_i\). Observe that

\[
\lambda_i = \|p_i' - p\|, \tag{44}
\]

and that for every \(q \in \ell_i'\),

\[
\text{dist}(q, \ell_i) = \sin(\alpha) \cdot \|q - p\|. \tag{45}
\]

Then the claim holds as

\[
\text{dist}(p_i' - t', \ell_i) = \text{dist}(p_i' - \lambda_k \cdot v_j, \ell_i)
= \sin(\alpha) \cdot \|p_i' - \lambda_k \cdot v_j - p\|
\leq \sin(\alpha) \cdot (\|p_i' - p\| + \|\lambda_k \cdot v_j\|) \tag{46}
\leq \sin(\alpha) \cdot (\|p_i' - p\| + |\lambda_k|) \tag{47}
\leq \sin(\alpha) \cdot 2 \|p_i' - p\| \tag{48}
= 2 \sin(\alpha) \cdot \|p_i' - p\|
= 2 \cdot \text{dist}(p_i', \ell_i), \tag{50}
\]

where (46) holds by substituting \(q = p_i' - \lambda_k \cdot v_j\) in (45), (47) holds by the triangle inequality, (48) since \(\|v_j\| = 1\), (49) holds by combining the definition of \(k\) with (44), and (50) holds by substituting \(q = p_i'\) in (45). \(\square\)
Let \( k = \arg \min_{i \in [n]} |\lambda_i| \) and \( t' = \lambda_k \cdot v_j \) be the index and the translation vector computed in Claim 23.1 respectively. For every \( i \in [n] \), let \( p_i'' \) denote the point \( p_i \) after translation by \( t_j + t' \), i.e., \( p_i'' = p_i - t_j - t' \).

Observe that by the definition of \( k \), \( \lambda_k \), and \( t' \), we know that \( t_j \) and \( \ell_k \) are not parallel and that \( p_i'' \in \ell_i \).

**Step 3.** In the following claim we prove that there exists an alignment \((R'', t'')\) \( \in \) ALIGNMENTS that satisfies the following:

(i) \( R''p_i'' - t'' \in \ell_i \) and \( R''p_i'' - t'' \in \ell_k \),

(ii) \( (R'', t'') \) minimizes the distance \( \dist(p_i'', \ell_i) \) over some index \( i \in [n] \) and

(iii) \( (R'', t'') \) satisfies \( \dist(R''p_i'' - t'', \ell_i) \leq 4 \cdot \dist(p_i'', \ell_i) \) for every \( i \in [n] \).

**Claim 23.2.** Let \( \text{ALIGNMENTS}_1 \) be the union over all alignments \((R, t) \in \) ALIGNMENTS such that \( Rp_j'' - t \) lies on \( \ell_j \) and \( Rp_k'' - t \) lies on \( \ell_k \), i.e., \( \text{ALIGNMENTS}_1 = \{ (R, t) \in \text{ALIGNMENTS} \mid Rp_j'' - t \in \ell_j \text{ and } Rp_k'' - t \in \ell_k \} \). Then there exists \( l \in [n] \setminus \{ j, k \} \) and an alignment \((R'', t'')\) \( \in \) \( \arg \min_{(R, t) \in \text{ALIGNMENTS}_1} \dist(Rp_i - t, \ell_i) \) such that for every \( i \in [n] \),

\[
\dist(R''p_i'' - t'', \ell_i) \leq 4 \cdot \dist(p_i'', \ell_i).
\]

**Proof.** Put \( i \in [n] \). We have that \( p_i'' \in \ell_j \) since \( p_j'' \) is the origin, and \( p_k'' \in \ell_k \) by the definition \( t' \). Let \( R_{v_j} \in \mathbb{R}^{2 \times 2} \) be a rotation matrix that aligns \( v_j \) with the \( x \)-axis, i.e., \( R_{v_j} v_j = (1, 0)^T \). By substituting \( p = p_j'', q = p_k'', z = p_i'' \) and \( v = R_{v_j} v_k \) in Lemma 9(i) and (ii), it holds that there is a unit vector \( x \in \mathbb{R}^2 \) and matrices \( Z_j = R_{v_j}^T P, Z_k = R_{v_j}^T Q \) and \( Z_i = R_{v_j}^T Z \) such that \( p_j'' = Z_j x, p_k'' = Z_k x \) and \( p_i'' = Z_i x \).

Observe that

\[
\dist(p_i'', \ell_i) = |v_i^T Z_i x - b_i| = |v_i^T Z_i y - b_i|,
\]

where the second equality holds since \( p_i'' = Z_i x \). By substituting \( a_i = v_i^T Z_i \) in Lemma 19(i), there is a set \( D \) of \( O(n) \) unit vectors, \( y \in D \) and an index \( l \in [n] \), such that for every unit vector \( x \in \mathbb{R}^2 \) it holds that

\[
|v_i^T Z_i y - b_i| \leq 4 \cdot |v_i^T Z_i x - b_i|.
\]

Substituting \( p = p_j'' \) and \( q = p_k'' \) in Corollary 22(ii), there is an alignment \((R'', t'')\) \( \in \) \( \text{ALIGNMENTS}_1 \) that aligns the set \( \{ Z_i x \mid i \in [n] \} \) with the set \( \{ Z_i y \mid i \in [n] \} \), i.e. \( R''p_i'' - t = R''Z_i x - t'' = Z_i y \). Hence, the following holds

\[
\dist(R''p_i'' - t'', \ell_i) = |v_i^T (R''p_i'' - t'') - b_i| = |v_i^T (R''Z_i x - t'') - b_i| = |v_i^T Z_i y - b_i| \leq 4 \cdot |v_i^T Z_i x - b_i| = 4 \cdot \dist(p_i'', \ell_i),
\]

where (53) holds since \( p_i'' = Z_i x \), (54) holds since \( R''Z_i x - t'' = Z_i y \), (55) holds by (52) and (56) holds by (51).

It follows from the definition of \( y \) and \( l \) that the vector \( y \) minimizes \( |v_i^T Z_i y - b_i| \). Similarly to (54), it holds that \( \dist(R''p_i'' - t'', \ell_i) = |v_i^T Z_i y - b_i| \), so \((R'', t'') \) \( \in \) \( \arg \min_{(R, t) \in \text{ALIGNMENTS}_1} \dist(Rp_l - t, \ell_l) \). Hence, \( l \) and \((R'', t'')\) satisfy the requirements of this Claim. \( \square \)

Hence, By the proof in Step 1, there exists an alignment that aligns a point \( p_j \) with \( \ell_j \) for some \( j \in [n] \), and does not increase the distances of the other pairs by more than a multiplicative factor of 2. This is
the reason that all the output alignments of Algorithm 2 satisfy that one of the input points intersects its corresponding line.

By the proof in Step 2, there exists an alignment that aligns a point \( p_k \) with \( \ell_k \) for some \( k \in [n] \), and does not increase the distances of the other pairs by more than a multiplicative factor of 2. This is the reason that all the output alignments of Algorithm 2 satisfy that two of the input points intersect their corresponding lines, if the input lines are not all parallel.

By the proof in Step 3, there exists an alignment that minimizes the distance between \( p_l \) and \( \ell_l \) for some \( l \in [n] \), and maintains that \( p_j \in \ell_j \) and \( p_k \in \ell_k \). This alignment provably guarantees that the distance of the other pairs does not increase by more than a multiplicative factor of 2.

Hence, if the input lines are not all parallel, we proved there are \( j, k, l \in [n] \) where \( j \neq k \), and \((R, t) \in \text{ALIGNMENTS}\), such that the following holds

\[
Rp_j - t \in \ell_j, Rp_k - t \in \ell_k, \text{dist}(Rp_l - t, \ell_l) \text{ is minimized over every alignment that satisfies}
\]

\[\text{dist}(Rp_l - t, \ell_l) \leq 16 \cdot \text{dist}(p_i, \ell_i) \text{ for every } i \in [n].\]  

(57)

If \( \ell_1, \cdots, \ell_n \) are parallel. Recall that \( p'_i = p_i - (p_j - \text{proj}(p_j, \ell_j)). \) By the proof in Step 1, we have that \( \text{dist}(p'_i, \ell_i) \leq 2 \cdot \text{dist}(p_i, \ell_i). \) To handle the case where all the lines \( \ell_1, \cdots, \ell_n \) are parallel, we now prove that rotating the points around \( p'_j \) until the distance \( \text{dist}(p_h, \ell_h) \) for some \( h \in [n] \) is minimized will not increase the distance \( \text{dist}(p'_i, \ell_i) \) for every \( i \in [n] \) by more than a multiplicative factor of 4.

**Claim 23.3.** There exists \( h \in [n] \setminus \{j\} \) and a rotation matrix \( R' \in \arg \min_R \text{dist}(Rp'_h, \ell_h) \), where the minimum is over every rotation matrix \( R \in \mathbb{R}^{2 \times 2} \), such that for every \( i \in [n] \),

\[
\text{dist}(R'p'_i, \ell_i) \leq 4 \cdot \text{dist}(p'_i, \ell_i).
\]

**Proof.** Let \( x \in \mathbb{R}^2 \) be a unit vector. Let \( R_i \in \mathbb{R}^{2 \times 2} \) be a rotation matrix that rotates \( x \) to the direction of \( p'_i \), i.e., \( p'_i = \|p'_i\| \cdot R_i x \), and define \( a_i = \|p'_i\| \cdot R_i^T v_i \). We then have that

\[
\text{dist}(p'_i, \ell_i) = |v_i^T p'_i - b_i| = \|p'_i\| \cdot v_i^T R_i x - b_i| = |a_i^T x - b_i|.
\]

(58)

By Lemma 19(i), there is a set \( C \) of \( O(n) \) unit vectors and \( x' \in C \), such that for every unit vector \( x \in \mathbb{R}^2 \) it holds that

\[
|a_i^T x' - b_i| \leq 4 \cdot |a_i^T x - b_i|.
\]

(59)

Let \( h \in [n] \) be the index from Lemma 6(ii). Define \( R' \in \mathbb{R}^{2 \times 2} \) to be the rotation matrix that satisfies \( R' x = x' \). Hence, it holds that

\[
\text{dist}(R'p'_i, \ell_i) = |v_i^T R' p'_i - b_i|
\]

(60)

\[
= \|p'_i\| \cdot v_i^T R' R_i x - b_i| \]

(61)

\[
= \|p'_i\| \cdot v_i^T R_i R' x - b_i| \]

(62)

\[
= |a_i^T R' x - b_i| \]

(63)

\[
\leq 4 \cdot |a_i^T x - b_i| \]

(64)

\[
= 4 \cdot \text{dist}(p'_i, \ell_i),
\]

(65)
that satisfy (57) are computed in Lines 6-13 and stored in $C$.

By the definition of $a_i$, (63) holds by the definition of $R'$, (64) holds by (59) and (65) holds by (58).

By the definition of $h$ and $x'$, it follows that $x'$ minimizes $|a_h^T x' - b_h|$. Similarly to (63), it holds that $\text{dist}(R'p'_h, \ell_h) = |a_h^T x' - b_h|$, so $R' \in \arg \min_R \text{dist}(R'p'_h, \ell_h)$. Hence, $h$ and $R'$ satisfy the requirements of this Claim.

Hence, by Step 1 and Claim 23.3 if all the lines are parallel, there are $j, h \in [n]$, where $j \neq h$, and $(R, t) \in \text{ALIGNMENTS}$, such that the following holds

\[ Rp_j - t \in \ell_j, \]
\[ \text{dist}(Rp_h - t, \ell_h) \text{ is minimized over every alignment that satisfies } Rp_j - t \in \ell_j, \] and
\[ \text{dist}(R\ell_i - t, \ell_i) \leq 8 \cdot \text{dist}(p_i, \ell_i) \text{ for every } i \in [n]. \] (66)

Algorithm 2 iterates over every triplet $(j, k, l) \in [n]$. There are $O(n^3)$ such triplets. The alignments that satisfy (57) are computed in Lines 6-13 and stored in $C_1$. The alignments that satisfy (66) are computed in Line 15 and stored in $C_2$. The sets $C_1$ and $C_2$ computed at every iteration are of constant size, i.e., $|C_1|, |C_2| \in O(1)$. We output $C$, the union of all those alignments. Hence, $C$ will contain an alignment $(R, t)$ that satisfies (40), and $|C| \in O(n^3)$. Furthermore, the running time of Algorithm 2 is $O(n^3)$ since Lines 4-16 take $O(1)$ time to compute, and are executed $O(n^3)$ times.

**Theorem 24 (Theorem 12).** Let $A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\}$ be set of $n \geq 3$ pairs, where for every $i \in [n]$, we have that $p_i$ is a point and $\ell_i$ is a line, both on the plane. Let $z \geq 1$ and let $D_z : A \times \text{ALIGNMENTS} \to [0, \infty)$ such that $D_z((p, \ell), (R, t)) = \min_{q \in \ell} \|Rp - t - q\|_z$ is the $\ell_z$ distance between $Rp - t$ and $\ell$. Let
cost, s, r be as defined in Definition 4 for $D = D_z$. Let $w = 1$ if $z = 2$ and $w = \sqrt{2}$ otherwise. Let $C$ be the output of a call to ALIGN($A$); see Algorithm 2. Then there exists $(R', t') \in C$ such that

$$\text{cost}(A, (R', t')) \leq (w \cdot 16)^r s \cdot \min_{(R, t) \in \text{ALIGNMENTS}} \text{cost}(A, (R, t)).$$

Furthermore, $C$ and $(R', t')$ can be computed in $n^{O(1)}$ time.

**Proof.** Put $i \in [n]$ and let $(R^*, t^*) \in \arg \min_{(R, t) \in \text{ALIGNMENTS}} \text{cost}(A, (R, t))$. By Lemma 23, $|C| = O(n^3)$ and can be computed in $O(n^3)$ time. Furthermore, there exists $(R', t') \in C$ that satisfies

$$D_z((p_i, \ell_i), (R', t')) = \text{dist}(R' \ell_i - t', \ell_i) \leq 16 \cdot \text{dist}(R^* p_i - t^*, \ell_i) = 16 \cdot D_z((p_i, \ell_i), (R^*, t^*)). \quad (67)$$

By (67) and since the $\ell_2$-norm of every vector in $\mathbb{R}^2$ is approximated up to $w$ by its $\ell_2$-norm, we obtain that

$$D_z((p_i, \ell_i), (R', t')) \leq w \cdot 16 \cdot D_z((p_i, \ell_i), (R^*, t^*)).$$

By substituting $q' = (R', t'), q^* = (R^*, t^*)$ and $D = D_z$ in Observation 17, Theorem 24 holds as

$$\text{cost}(A, (R', t')) \leq (w \cdot 16)^r s \cdot \text{cost}(A, (R^*, t^*)) = (w \cdot 16)^r s \cdot \min_{(R, t) \in \text{ALIGNMENTS}} \text{cost}(A, (R, t)),$$

where the last derivation holds by the definition of $(R^*, t^*)$.

Furthermore, it takes $O(n \cdot |C|) = n^{O(1)}$ time to compute $(R', t')$. \qed

**Theorem 25 (Theorem 13).** Let $A = \{(p_1, \ell_1), \ldots, (p_n, \ell_n)\}$ be a set of $n \geq 3$ pairs, where for every $i \in [n]$ we have that $p_i$ is a point and $\ell_i$ is a line, both on the plane. Let $z \geq 1$ and $D_z((p, \ell), (R, t)) = \min_{q \in \ell} \|Rp - t - q\|_z$ for every point $p$ and line $\ell$ on the plane and alignment $(R, t)$. Consider cost, $r$ to be a function as defined in Definition 4 for $D = D_z$ and $f(v) = \|v\|_1$. Let $w = 1$ if $z = 2$ and $w = \sqrt{2}$ otherwise. Let $(\tilde{R}, \tilde{t}, \tilde{\pi})$ be the output alignment $(\tilde{R}, \tilde{t})$ and permutation $\tilde{\pi}$ of a call to ALIGN+MATCH$(A, \text{cost})$; see Algorithm 3. Then

$$\text{cost}(A_{\tilde{\pi}}, (\tilde{R}, \tilde{t})) \leq (w \cdot 16)^r s \cdot \min_{(R, t, \pi)} \text{cost}(A_{\pi}, (R, t)), \quad (68)$$

where the minimum is over every alignment $(R, t, \pi)$ and permutation $\pi$. Moreover, $(\tilde{R}, \tilde{t}, \tilde{\pi})$ can be computed in $n^{O(1)}$ time.

**Proof.** Let $(R^*, t^*, \pi^*) \in \arg \min_{(R, t, \pi)} \text{cost}(A_{\pi}, (R, t))$. By Theorem 24, we can compute a set $C \subseteq \text{ALIGNMENTS}$ such that there exists an alignment $(R, t) \in C$ that satisfies

$$\text{cost}(A_{\pi^*}, (R, t)) \leq (w \cdot 16)^r s \cdot \text{cost}(A_{\pi^*}, (R^*, t^*)) \quad (69)$$

where $C$ is computed by a call to ALIGN($A_{\pi^*}$); see Algorithm 2.

In Line 3 of Algorithm 2, we iterate over every triplet of indices $j, k, l \in [n]$. Each such index corresponds to a point-line matched pair. Hence, $j, k$ and $l$ correspond to a triplet of matched point-line pairs. In Lines 16 we add $O(1)$ alignments to the set $C$ that correspond to that triplet of point-line pairs. Therefore, every alignments $(R', t') \in C$ correspond to some 3 matched point-line pairs $(p_j, \ell_{\pi^*(j)}), (p_k, \ell_{\pi^*(k)}), (p_l, \ell_{\pi^*(l)}) \in A$. Let $(p_{i1}, \ell_{\pi^*(i1)}), (p_{i2}, \ell_{\pi^*(i2)}), (p_{i3}, \ell_{\pi^*(i3)})$ be the triplet of matched pairs that corresponds to the alignment $(R, t)$. Hence, it holds that

$$(R, t) \in \text{ALIGN}\left(\{(p_{i1}, \ell_{\pi^*(i1)}), (p_{i2}, \ell_{\pi^*(i2)}), (p_{i3}, \ell_{\pi^*(i3)})\}\right).$$

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By iterating over every \( j_1, j_2, j_3 \in [n] \), when \( j_1 = \pi^*(i_1), j_2 = \pi^*(i_2), j_3 = \pi^*(i_3) \), it holds that \( (R, t) \in \text{ALIGN} \left( \{(p_{i_1}, \ell_{j_1}), (p_{i_2}, \ell_{j_2}), (p_{i_3}, \ell_{j_3})\} \right) \).

In Lines 2 of Algorithm 3 we iterate over every tuple of 6 indices \( i_1, i_2, i_3, j_1, j_2, j_3 \in [n] \), and compute, using Algorithm 2, the set of alignments \( X' \) that corresponds to the triplet of pairs \( (p_{i_1}, \ell_{j_1}), (p_{i_2}, \ell_{j_2}), (p_{i_3}, \ell_{j_3}) \). We then add those alignments to the set \( X \). Hence, it is guaranteed that the alignment \( (R, t) \in C \) that satisfies (69) is also in \( X \).

Kuhn and Harold suggested in [21] an algorithm that given the pairwise distances (fitting loss) between two sets of \( n \) elements \( P \) and \( L \), it finds an assignment for every \( p \in P \) to an element \( \ell \in L \) that minimizes the sum of distances between every assigned pair. This algorithm takes \( O(n^3) \) time. We use this algorithm to compute the optimal matching function \( \tilde{\pi}(A, (R', t'), \text{cost}) \) for every \( (R', t') \in X \) in Line 5 of Algorithm 3. Let \( \pi = \tilde{\pi}(A, (R, t), \text{cost}) \). Since \( \pi \) is an optimal matching function for the pairs of \( A \), the alignment \( (R, t) \), and the function \( \text{cost} \), it satisfies that

\[
\text{cost} \left( A_{\pi}, (R, t) \right) \leq \text{cost} \left( A_{\pi^*}, (R, t) \right)
\] (70)

Since \( (R, t) \in X \) and \( \pi = \tilde{\pi}(A, (R, t), \text{cost}) \), we obtain that \( (R, t, \pi) \in S \), where \( S \) is the set defined in Line 5 of Algorithm 3. Combining \( (R, t, \pi) \in S \) with the definition of \( (\tilde{R}, \tilde{t}, \tilde{\pi}) \) in Line 6, it holds that

\[
\text{cost} \left( A_{\tilde{\pi}}, (\tilde{R}, \tilde{t}) \right) \leq \text{cost} \left( A_{\pi}, (R, t) \right).
\] (71)

Hence, the following holds

\[
\text{cost} \left( A_{\tilde{\pi}}, (\tilde{R}, \tilde{t}) \right) \leq \text{cost} \left( A_{\pi}, (R, t) \right) \leq \text{cost} \left( A_{\pi^*}, (R, t) \right) \leq (w \cdot 16)^r \cdot \text{cost} \left( A_{\pi^*}, (R^*, t^*) \right),
\]

where the first derivation holds by (71), the second derivation holds by (70) and the third derivation is by (69). Furthermore, the running time of the algorithm is \( O(n^9) = n^{O(1)} \).

C Coresets for Big Data

The following theorem states Theorem 4 from [8] for the case where \( p = 1 \).

**Theorem 26 (Theorem 4 in [8]).** Let \( A \) be an \( n \times d \) matrix of rank \( r \). Then there exists an \( n \times r \) matrix \( U \) that satisfies the following

(i) \( U \) is a basis for the column pace of \( A \), i.e., there exists a \( r \times d \) matrix \( G \) such that \( A = UG \).

(ii) \( \|U\|_1 \leq r^{1.5} \).

(iii) For every \( z \in \mathbb{R}^r \), \( \|z\|_\infty \leq \|Uz\|_1 \).

(iv) \( U \) can be computed in \( \text{O}(nrd + nr^5 \log n) \) time.

**Lemma 27.** Let \( P = [p_1 | \ldots | p_n]^T \in \mathbb{R}^{d \times n} \), where \( p_i \in \mathbb{R}^d \) for every \( i \in [n] \). Then a function \( s : P \to [0, \infty) \) can be computed in \( \text{O}(nd^5 \log n) \) time such that

(i) For every \( i \in [n] \),

\[
\sup_{x \in \mathbb{R}^d, \|x\| > 0} \frac{|p_i^T x|}{\|Px\|_1} \leq s(p_i).
\]

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\[(ii) \quad \sum_{i=1}^{n} s(p_i) \in d^{O(1)}\]

**Proof.** Let \(r \leq d\) denote the rank of \(A\).

(i) By substituting \(A = P\) in Theorem \((26)\), we get that there exists a matrix \(U \in \mathbb{R}^{n \times r}\) and a matrix \(G \in \mathbb{R}^{r \times d}\) with the following properties:

\[
(1) \quad P = UG, \quad (2) \quad \|U\|_1 \leq r^{1.5} \leq d^{1.5}, \quad (3) \quad \text{for every } z \in \mathbb{R}^r, \quad \|z\|_1/r \leq \|z\|_\infty \leq \|Uz\|_1,
\]

and (4) \(U\) and \(G\) can be computed in \(O(ndr + nr^5 \log n) = O(nd^5 \log n)\) time.

Let \(u_i^T \in \mathbb{R}^r\) be the \(i\)th row of \(U\) for \(i \in [n]\).

Observe that for every \(u, y \in \mathbb{R}^r\) such that \(\|y\| \neq 0\), it holds that

\[
\left| \frac{u^T y}{\|y\|_1} \right| = \left| \frac{u^T y}{\|y\|_2} \right| \cdot \frac{\|y\|_2}{\|y\|_1} \leq \|u\|_2 \cdot \frac{\|y\|_2}{\|y\|_1} \leq \|u\|_2,
\]

where the second derivation holds since \(\frac{\|y\|_1}{\|y\|_2} \) is a unit vector, and the last derivation holds since \(\|y\|_2 \leq \|y\|_1\).

Let \(x \in \mathbb{R}^d\) such that \(\|x\| > 0\). Then it follows that

\[
\frac{|p_i^T x|}{\|Px\|_1} = \frac{|u_i^T Gx|}{\|UGx\|_1} \leq r \cdot \frac{|u_i^T Gx|}{\|Gx\|_1} = r \cdot \frac{|u_i^T Gx|}{\|Gx\|_1} \leq r \cdot \|u_i\|_2 \leq r \cdot \|u_i\|_1,
\]

where the first derivation holds by combining Property (1) in \((72)\) and the definition of \(u_i\), the second derivation holds by substituting \(z = Gx\) in property (3) in \((72)\), and the fourth derivation holds by substituting \(u = u_i\) and \(y = Gx\) in \((73)\). Let \(G^+\) be the pseudo inverse of \(G\). It thus holds that \(GG^+\) is the \(r\) dimensional identity matrix. Hence, the following holds.

\[
\|u_i\|_1 = \sum_{j=1}^{r} |u_i^T e_j| = \sum_{j=1}^{r} |u_i^T G^+ e_j| = \sum_{j=1}^{r} |p_i^T G^+ e_j|,
\]

where \(e_j\) is the \(j\)th standard vector of \(\mathbb{R}^r\).

By combining \((74)\) and \((75)\), we have

\[
\frac{|p_i^T x|}{\|Px\|_1} \leq r \cdot \sum_{j=1}^{r} |p_i^T G^+ e_j|.
\]

By combining the definition of \(s(p_i)\) and that \((76)\) holds for every \(x \in \mathbb{R}^d\), Property (i) of Lemma \((27)\) holds as

\[
\max_{x \in \mathbb{R}^d: \|x\| > 0} \frac{|p_i^T x|}{\|Px\|_1} \leq r \cdot \sum_{j=1}^{r} |p_i^T G^+ e_j| \leq d \cdot \sum_{j=1}^{r} |p_i^T G^+ e_j| = s(p_i).
\]

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(ii): We have that
\[
\sum_{i=1}^{n} s(p_i) = \sum_{i=1}^{n} \left( d \cdot \sum_{j=1}^{r} |p_i^T G^+ e_j| \right) = d \cdot \sum_{j=1}^{r} \sum_{i=1}^{n} |p_i^T G^+ e_j| = d \cdot \sum_{j=1}^{r} \|PG^+ e_j\|_1
\]
\[
= d \cdot \sum_{j=1}^{r} \|UGG^+ e_j\|_1 = d \cdot \sum_{j=1}^{r} \|U e_j\|_1 \leq d^{2.5} \cdot \sum_{j=1}^{r} \|e_j\|_1 = d^{3.5},
\]
where the first inequality holds since \(\|U\|_1 \leq r^{1.5}\) by Property (2) in [72]. Therefore, Property (ii) of Lemma \[27\] holds as
\[
\sum_{i=1}^{n} s(p_i) \leq d^{3.5} \in O(1).
\]
Furthermore, the time needed to compute \(s\) is dominated by the computation time of \(U\) and \(G\), which is bounded by \(O(d^5 \log n)\) due to Property (4) in \(72\).

**Definition 28 (Definition 4.2 in \[7\]).** Let \(P\) be a finite set, and let \(w : P \to [0, \infty)\). Let \(Q\) be a function that maps every set \(S \subseteq P\) to a corresponding set \(Q(S)\), such that \(Q(T) \subseteq Q(S)\) for every \(T \subseteq S\). Let \(f : P \times Q(P) \to \mathbb{R}\) be a cost function. The tuple \((P, w, Q, f)\) is called a query space.

**Definition 29 (VC-dimension).** For a query space \((P, w, Q, f)\), \(S \subseteq P\), \(q \in Q(S)\) and \(r \in [0, \infty)\) we define
\[
\text{range}(q, r) = \{p \in P \mid w(p) \cdot f(p, q) \leq r\}.
\]
The VC-dimension of \((P, w, Q, f)\) is the smallest integer \(d_{VC}\) such that for every \(S \subseteq P\) we have
\[
|\{\text{range}(q, r) \mid q \in Q(S), r \in [0, \infty)\}| \leq |S|^{d_{VC}}.
\]

**Theorem 30 (Theorem 5.5 in \[7\]).** Let \((P, w, Q, f)\) be a query space; see Definition \[28\]. Let \(s : P \to [0, \infty)\) such that
\[
\sup_{q} \frac{w(p) f(p, q)}{\sum_{p \in P} w(p) f(p, q)} \leq s(p),
\]
for every \(p \in P\) and \(q \in Q(P)\) such that the denominator is non-zero. Let \(t = \sum_{p \in P} s(p)\) and Let \(d_{VC}\) be the VC-dimension of query space \((P, w, Q, f)\). Let \(c \geq 1\) be a sufficiently large constant and let \(\varepsilon, \delta \in (0, 1)\). Let \(S\) be a random sample of
\[
|S| \geq \frac{ct}{\varepsilon^2} \left( d_{VC} \log t + \log \frac{1}{\delta} \right)
\]
points from \(P\), such that \(p\) is sampled with probability \(s(p)/t\) for every \(p \in P\). Let \(u(p) = \frac{t \cdot w(p) s(p)}{|S|}\) for every \(p \in S\). Then, with probability at least \(1 - \delta\), for every \(q \in Q\) it holds that
\[
(1 - \varepsilon) \sum_{p \in P} w(p) \cdot f(p, q) \leq \sum_{p \in S} u(p) \cdot f(p, q) \leq (1 + \varepsilon) \sum_{p \in P} w(p) \cdot f(p, q).
\]

**Theorem 31 (Theorem 14).** Let \(d \geq 2\) be an integer. Let \(A = \{(p_1, \ell_1), \cdots, (p_n, \ell_n)\}\) be set of \(n\) pairs, where for every \(i \in [n]\), \(p_i\) is a point and \(\ell_i\) is a line, both in \(\mathbb{R}^d\), and let \(w = (w_1, \cdots, w_n) \in [0, \infty)^n\). Let \(\varepsilon, \delta \in (0, 1)\). Then in \(O(1) \log n\) time we can compute a weights vector \(u = (u_1, \cdots, u_n) \in [0, \infty)^n\) that satisfies the following pair of properties.
(i) With probability at least $1 - \delta$, for every $(R, t) \in \text{ALIGNMENTS}$ it holds that
\[
(1 - \varepsilon) \cdot \sum_{i \in [n]} w_i \cdot \text{dist}(R p_i - t, \ell_i) \leq \sum_{i \in [n]} u_i \cdot \text{dist}(R p_i - t, \ell_i) \leq (1 + \varepsilon) \cdot \sum_{i \in [n]} w_i \cdot \text{dist}(R p_i - t, \ell_i).
\]

(ii) The weights vector $u$ has $\frac{d \log(1)}{2\varepsilon^2}$ non-zero entries.

**Proof.** Put $(R, t) \in \text{ALIGNMENTS}$. We represent a line $\ell$ by a basis of its orthogonal complement $V \in \mathbb{R}^{(d-1) \times d}$ and its translation $b$ from the origin. Formally, let $V \in \mathbb{R}^{(d-1) \times d}$ be a matrix whose rows are mutually orthogonal unit vectors, and $b \in \mathbb{R}^{d-1}$ such that
\[
\ell = \left\{ q \in \mathbb{R}^d \mid \| V q - b \| = 0 \right\}.
\]
Let
\[
x(R, t) = (R_{1*}, \ldots, R_{d*}, -t^T, -1)^T.
\]
Let $p \in \mathbb{R}^d$, $\omega \geq 0$ and for every $k \in [d - 1]$, let
\[
s_k = \omega \cdot (V_{(k,1)} p^T, \ldots, V_{(k,d)} p^T, V_{k*}, b_{(k)})^T.
\]
Then it holds that
\[
\omega \cdot \text{dist}(R p - t, \ell) = \omega \cdot \| V(R p - t) - b \|
\]
\[
= \omega \cdot \| V(R_{1*} p, \ldots, R_{d*} p)^T - V t - b \|
\]
\[
= \omega \cdot \left\| \begin{pmatrix} V_{1*}(R_{1*} p, \ldots, R_{d*} p)^T - V_{1*} t - b_{(1)} \\ \vdots \\ V_{(d-1)*}(R_{1*} p, \ldots, R_{d*} p)^T - V_{(d-1)*} t - b_{(d-1)} \end{pmatrix} \right\|
\]
\[
= \omega \cdot \left\| \begin{pmatrix} V_{1*} : R_{1*} p + \cdots + V_{1*} : R_{d*} p - V_{1*} t - b_{(1)} \\ \vdots \\ V_{(d-1)*} : R_{1*} p + \cdots + V_{(d-1)*} : R_{d*} p - V_{(d-1)*} t - b_{(d-1)} \end{pmatrix} \right\|
\]
\[
= \omega \cdot \left\| \begin{pmatrix} x(R, t)^T (V_{(1,1)} p^T, \ldots, V_{(1,d)} p^T, V_{1*}, b_{(1)})^T \\ \vdots \\ x(R, t)^T (V_{(d-1,1)} p^T, \ldots, V_{(d-1,d)} p^T, V_{(d-1)*}, b_{(d-1)})^T \end{pmatrix} \right\|
\]
\[
= \omega \cdot \left\| \begin{pmatrix} (R, t)^T s_1 \\ \vdots \\ (R, t)^T s_{d-1} \end{pmatrix} \right\|,
\]
where (78) holds by the definition of $x(R, t)$ and (79) holds by the definition of $s_k$ for every $k \in [d - 1]$. Hence,
\[
\omega \cdot \text{dist}(R p - t, \ell) = \gamma \cdot \left\| \begin{pmatrix} x(R, t)^T s_1 \\ \vdots \\ x(R, t)^T s_{d-1} \end{pmatrix} \right\|.
\]
We have $\frac{\|v\|_1}{\sqrt{d}-1} \leq \|v\|_2 \leq \|v\|_1$ for every $v \in \mathbb{R}^{d-1}$. Plugging this in (80) yields

$$\frac{1}{\sqrt{d}} \left\| \begin{pmatrix} x(R, t)^T s_1 \\ \vdots \\ x(R, t)^T s_{d-1} \end{pmatrix} \right\|_1 \leq \omega \cdot \text{dist}(Rp - t, \ell) \leq \left\| \begin{pmatrix} x(R, t)^T s_1 \\ \vdots \\ x(R, t)^T s_{d-1} \end{pmatrix} \right\|_1.$$  \hspace{1cm} (81)

For every $i \in [n]$ and $k \in [d-1]$, let $V_i \in \mathbb{R}^{(d-1) \times d}$ and $b_i \in \mathbb{R}^{d-1}$ such that $\ell_i = \{ q \in \mathbb{R}^d \mid \|V_i q - b_i\| = 0 \}$, and let

$$s_{i,k} = w_i \cdot (V_i(k,1)p_1^T, \ldots, V_i(k,d)p_1^T, V_i(k,d), b_i(k))^T.$$  

Similarly to (81) for every $i \in [n]$ it holds that

$$\frac{1}{\sqrt{d}} \left\| \begin{pmatrix} x(R, t)^T s_{i,1} \\ \vdots \\ x(R, t)^T s_{i,d-1} \end{pmatrix} \right\|_1 \leq w_i \cdot \text{dist}(Rp_i - t, \ell_i) \leq \left\| \begin{pmatrix} x(R, t)^T s_{i,1} \\ \vdots \\ x(R, t)^T s_{i,d-1} \end{pmatrix} \right\|_1.$$  \hspace{1cm} (82)

Let $S_i = [s_{i,1} | \ldots | s_{i,d-1}] \in \mathbb{R}^{(d^2+d+1) \times (d-1)}$, and let $S = [S_1 | \ldots | S_n] \in \mathbb{R}^{(d^2+d+1) \times (n(d-1))}$. Then the following holds

$$\sup_{(R,t)} \frac{\sum_{j \in [n]} w_j \cdot \text{dist}(Rp_j - t, \ell_j)}{\sum_{j \in [n]} \frac{1}{\sqrt{d}} \left\| x(R, t)^T s_{j,1} \right\|_1} \leq \sup_{(R,t)} \frac{\sum_{k \in [d-1]} |x(R, t)^T s_{i,k}|}{\|x(R, t)^T S\|_1} \leq \sqrt{d} \cdot \frac{\sum_{k \in [d-1]} |x(R, t)^T s_{i,k}|}{\|x(R, t)^T S\|_1} \leq \sqrt{d} \cdot \frac{\left( \sup_{(R,t)} \frac{|x(R, t)^T s_{i,1}|}{\|x(R, t)^T S\|_1} + \cdots + \sup_{(R,t)} \frac{|x(R, t)^T s_{i,d-1}|}{\|x(R, t)^T S\|_1} \right)}{\|x(R, t)^T S\|_1} \leq \sqrt{d} \cdot \left( \sup_{(R,t)} \frac{|x(R, t)^T s_{i,1}|}{\|x(R, t)^T S\|_1} + \cdots + \sup_{(R,t)} \frac{|x(R, t)^T s_{i,d-1}|}{\|x(R, t)^T S\|_1} \right) \leq \sqrt{d} \cdot \sum_{k \in [d-1]} \sup_{(R,t)} \frac{|x(R, t)^T s_{i,k}|}{\|x(R, t)^T S\|_1} \leq \sqrt{d} \cdot \sum_{k \in [d-1]} \sup_{x \in \mathbb{R}^{d^2+d+1} : \|x\|_1 = 0} \frac{|x^T s_{i,k}|}{\|x^T S\|_1}.$$  \hspace{1cm} (83) to (87)

where the supremum is over every alignment $(R,t)$ such that $\sum_{j \in [n]} w_j \cdot \text{dist}(Rp_j - t, \ell_j) \neq 0$, and where (83) holds by (82), (84) holds by the definition of $S$, (85) holds since the maximum of a sum is at most the sum of maxima, and (87) holds since $\{x(R, t) \mid (R,t) \in \text{ALIGNMENTS}\} \subseteq \mathbb{R}^{d^2+d+1} \setminus 0$.  

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Therefore,
\[
\sup_{(R,t)} \sum_{j \in [n]} w_j \cdot \text{dist}(Rp_j - t, \ell_j) \leq \sqrt{d} \cdot \sum_{k \in [d-1]} \sup_{x \in \mathbb{R}^{d^2 + d + 1} : \|x\| > 0} \frac{|x^T s_i, k|}{\|x^T S\|_1}.
\]
(88)

By substituting \( P = S^T \) in Lemma 27 we get that in \( nd^{O(1)} \log n \) we can compute a function \( s \) that maps every column \( s_i, k \) of \( S \) to a non-negative real such that the following two properties hold for every \( i \in [n] \) and \( k \in [d-1] \):
\[
\sup_{x \in \mathbb{R}^{d^2 + d + 1} : \|x\| > 0} \frac{|x^T s_i, k|}{\|x^T S\|_1} \leq s(s_i, k), \quad \text{and}
\]
\[
\sum_{i \in [n], k \in [d-1]} s(s_i, k) \in d^{O(1)}.
\]
(89)

Let \( s' : A \to [0, \infty) \) such that \( s'((p_i, \ell_i)) = \sqrt{d} \cdot \sum_{k \in [d-1]} s(s_i, k) \) for every \( i \in [n] \). Then we have
\[
\sup_{(R,t)} \sum_{j \in [n]} w_j \cdot \text{dist}(Rp_j - t, \ell_j) \leq \sqrt{d} \cdot \sum_{k \in [d-1]} \sup_{x \in \mathbb{R}^{d^2 + d + 1} : \|x\| > 0} \frac{|x^T s_i, k|}{\|x^T S\|_1} \leq \sqrt{d} \cdot \sum_{k \in [d-1]} s(s_i, k) = s'((p_i, \ell_i)),
\]
(90)

where the first derivation holds by (89), and the last is by the definition of \( s' \).

We also have that
\[
\sum_{i \in [n]} s'((p_i, \ell_i)) = \sum_{i \in [n]} \left( \sqrt{d} \cdot \sum_{k \in [d-1]} s(s_i, k) \right) = \sqrt{d} \cdot \sum_{i \in [n], k \in [d-1]} s(s_i, k) \in \sqrt{d} \cdot d^{O(1)} = d^{O(1)},
\]
(92)

where the first derivation is by the definition of \( s' \) and the third derivation is by (90).

The VC-dimension of the corresponding query space is bounded by \( d_{VC} \in O(d^2) \) by 2.

Let \( t = \sum_{i \in [n]} s'((p_i, \ell_i)) \). Let \( Z \subseteq A \) be a random sample of
\[
|Z| \in \frac{d^{O(1)}}{\varepsilon^2} \left( d^2 \log d + \log \frac{1}{\delta} \right) = \frac{d^{O(1)}}{\varepsilon^2} \log \frac{1}{\delta}
\]
pairs from \( A \), where \((p, \ell) \in A \) is sampled with probability \( s'((p, \ell)) / t \) and let \( u = (u_1, \ldots, u_n) \) where
\[
u_i = \begin{cases} 
\frac{t \cdot w_i}{s'((p_i, \ell_i)) |Z|}, & \text{if } (p_i, \ell_i) \in Z, \\
0, & \text{otherwise}
\end{cases}
\]
for every \( i \in [n] \). By substituting \( P = A, Q(\cdot) \equiv \text{ALIGNMENTS}, f((p_i, \ell_i), (R, t)) = w_i \cdot \text{dist}(Rp_i - t, \ell_i) \) for every \( i \in [n] \), \( s = s', t \in d^{O(1)} \) and \( d_{VC} = O(d^2) \), in Theorem 30 Property (i) of Theorem 14 holds as
\[
(1 - \varepsilon) \cdot \sum_{i \in [n]} w_i \cdot \text{dist}(Rp_i - t, \ell_i) \leq \sum_{i \in [n]} u_i \cdot \text{dist}(Rp_i - t, \ell_i) \leq (1 + \varepsilon) \cdot \sum_{i \in [n]} w_i \cdot \text{dist}(Rp_i - t, \ell_i).
\]

Furthermore, Property (i) of Theorem 31 holds since the number of non-zero entries of \( u \) is equal to \( |Z| \in \frac{d^{O(1)}}{\varepsilon^2} \log \frac{1}{\delta} \).

The time needed to compute \( u \) is bounded by the computation time of \( s \), which is bounded by \( nd^{O(1)} \log n \). □
Corollary 32 (Corollary 15). Let \( A = \{(p_1, \ell_1), (p_2, \ell_2), \ldots\} \) be a (possibly infinite) stream of pairs, where for every \( i \in [n] \), \( p_i \) is a point and \( \ell_i \) is a line, both in the plane. Let \( \varepsilon, \delta \in (0, 1) \). Then, for every integer \( n > 1 \) we can compute with probability at least \( 1 - \delta \) an alignment \((R^*, t^*)\) that satisfies
\[
\sum_{i=1}^{n} \text{dist}(R^*p_i - t^*, \ell_i) \in O(1) \cdot \min_{(R,t) \in \text{ALIGNMENTS}} \sum_{i=1}^{n} \text{dist}(Rp_i - t, \ell_i),
\]
for the \( n \) points seen so far in the stream, using \((\log(n/\delta) / \varepsilon)O(1)\) memory and update time per a new pair. Using \( M \) machines the update time can be reduced by a factor of \( M \).

Proof. The coreset from Theorem 25 is composable by its definition (see Section 1.1). The claim then follows directly for the traditional merge-and-reduce tree technique as explained in many coreset papers, e.g. [1, 13, 5, 3, 15].

D Intuition

In this section we give an intuition for the algorithms presented in Section 4.

D.1 Intuition behind Algorithm 1

The algorithm’s input is a triangle, and a line \( \ell \) that intersects the origin. The triangle is represented by its three vertices \( p, q, z \in \mathbb{R}^2 \) and denoted by \( \Delta(p, q, z) \). The line is defined by its direction (unit vector) \( v \).

The usage of this algorithm in the main algorithm (Algorithm 2) is to compute the union over every feasible configuration \( \Delta(p', q', z') \), which is a rotation and a translation of \( \Delta(p, q, z) \) such that \( p' \) is on the \( x \)-axis, and \( q' \) is on the input line (simultaneously).

To this end, the output of the Algorithm 1 is a tuple of three \( 2 \times 2 \) matrices \( P, Q \) and \( Z \) such that the union of \( (P, Q, Z) \) over every unit vector \( x \) is the desired set. That is, for every feasible configuration \( \Delta(p', q', z') \) there is a unit vector \( x \in \mathbb{R}^2 \) such that \((p', q', z') = (Px, Qx, Zx)\), and vice versa.

Geometric interpretation. For every matrix \( Z \in \mathbb{R}^{2 \times 2} \), the set \( \{Zx \mid x \in \mathbb{R}^2, ||x|| = 1\} \) defines the boundary of an ellipse in \( \mathbb{R}^2 \). Hence, the shape formed by all possible locations of vertex \( z \in \mathbb{R}^2 \), assuming \( p \in x\)-axis and \( q \in \ell \), is an ellipse. Furthermore, this ellipse is centered around the intersection point of the \( x\)-axis and \( \ell \) (the origin, in this case). See Fig. 6.

D.2 Intuition behind Algorithm 2

The idea behind the algorithm consists of three steps. At each step we reduce the set of feasible alignments by adding an increasing number of constraints. Each constraint typically increases our approximation factor by another constant.

Consider any alignment of the input points to lines, and suppose that \((p, \ell_1)\) is the closest pair after applying this alignment. By the triangle inequality, translating the set \( P \) of points so that \( p \) intersects \( \ell_1 \) will increase the distance between every other pair by a factor of at most 2. Hence, minimizing \((1)\) under the constraint that \( p \in \ell_1 \) would yield a 2-approximation to the original (non-constrained) problem.

Similarly, we can then translate \( P \) in the direction of \( \ell_1 \) (while maintaining \( p \in \ell_1 \)) until the closest pair, say \((q, \ell_2)\), intersects. The result is an 4-approximation to the initial alignment by considering all the possible alignments of \( P \) such that \( p \in \ell_1 \) and \( q \in \ell_2 \). There are still infinite such alignments which satisfy the last constraint.
Figure 6: Illustration for Algorithm 1. A triangle $\Delta(p, q, z)$ whose vertex $p$ intersects the $x$-axis, and its vertex $q$ is on the line $sp\{v\}$. There are such infinitely many triangles, and the union over every possible solution $z$ are the points on the green ellipse. If $(P, Q, Z)$ is the output of Algorithm 1 then $(p, q, z) = (Px, Qx, Zx)$ for some unit vector $x$.

Hence, we add a third step. Let $(z, \ell_3)$ be the pair that requires the minimal rotation of the vector $q - p$ in order to minimize $\text{dist}(z, \ell_3)$ under the constraints that $p \in \ell_1$ and $q \in \ell_2$. We now rotate the vector $q - p$ and translate the system to maintain the previous constraints until $\text{dist}(z, \ell_3)$ is minimized.

The result is a 16-approximation to (1) by considering all the possible alignments of $P$ such that: $z$ is closest to $\ell_3$ among all alignments that satisfy $p \in \ell_1$ and $q \in \ell_2$. Unlike the previous steps, there are only a finite number of such alignments, namely $\binom{n}{3}$.

D.3 Intuition behind Algorithm 3

In Algorithm 3, we go over every triplet of points and triplet of lines from the input set $A$. Each such tuple of 3 points and 3 lines define a set of $O(1)$ alignments using Algorithm 2. For each such alignment, we compute the optimal matching function for the given cost function, using naive optimal matching algorithms. We then return the alignment and matching that yield the smallest cost.

E Experimental Results

To demonstrate the correctness and robustness of our algorithms from Section 4, we conducted the following experiments on synthetic and real data. We then compare the results to state-of-the-art solutions.

Hardware: All the following tests were conducted using MATLAB R2016b on a Lenovo Y700 laptop with an Intel i7-6700HQ CPU and 16GB RAM.
E.1 Synthetic Data

The input data-set was generated by first computing a set $L = \{\ell_1, \cdots, \ell_n\}$ of $n$ random lines in the plane. Then, a set $P_0$ of $n$ random points was generated, each in the range $[0, 100]$. A set $P_1$ was then computed by projecting the $i$th point of the points in $P_0$ onto its corresponding line $\ell_i$ in $L$. The set $P_1$ was then translated and rotated, respectively, by a random rotation matrix $R \in \mathbb{R}^{2 \times 2}$ and a translation vector $t \in [0, 10]^{2}$ to obtain a set $P_2$. Then, a random noise $N_i \in [0, 1]^2$ was added for every point $p_i \in P_2$, and additional noise $N_i \in [0, r]^2, r = 200$ was added to a subset $S \subseteq P_2$ of $|S| = k \cdot |P_2|$ points (outliers), where $k \in [0, 1]$ is the percentage of outliers. The resulting set was denoted by $P = \{p_1, \cdots, p_n\}$.

The randomness in the previous paragraph is defined as follows. Let $N(\mu, \sigma)$ denote a Gaussian distribution with mean $\mu$ and standard deviation $\sigma$. Similarly, let $U(r)$ denote uniform distribution over $[0, r]$. For every $i \in [n]$ and $j \in [2]$, the line $\ell_i = \{x \in \mathbb{R}^2 \mid a_i^T x = b_i\}$ was constructed such that $a_{i,j} \in N(1/2, 1/2), \|a_i\| = 1$ and $b_i \in U(10)$. For every $i \in [2], j \in [2]$, we choose $t_j \in U(10), N_{i,j} \in N(1/2, 1/2)$, and $N_{i,j} \in N(r/2, r/2)$. The rotation matrix was chosen such that $R_{1,1} \in U(1)$ and $R^T R = I$.

Experiment. The goal of each algorithm in the experiment was to compute $\tilde{R}$ and $\tilde{t}$, approximated rotation $R$ and translation $t$, respectively, using only the set $L$ of lines and the corresponding noisy set $P$ of points. Two tests were conducted, one with a constant number of input points $n$, with increasing percentage $k$ of outliers, and another test with a constant value for $k$, while increasing the number $n$ of input points. For the first test we compare the threshold M-estimator of the sum of fitting distances

$$\text{cost}(P, L, \tilde{R}, \tilde{t}) = \sum_{i=1}^{n} \min\{\text{dist}(\tilde{R}p_i - \tilde{t}, \ell_i), th\}. \quad (93)$$

For the second test we compare the running time (in second) of the suggested methods.

Algorithms. We apply each of the following algorithms that gets the pair of sets $P$ and $L$ as input.

Our FAST-APPROX-ALIGNMENT samples a set of $\sqrt{n}$ points $P'$ from $P$. Let $L' \subseteq L$ be lines in $L$ corresponding to the points in $P'$. It then runs Algorithm 2 with the set of corresponding point-line pairs from $P'$ and $L'$. The algorithm returns a set $C$ of alignments. We then chose the alignment $(\tilde{R}, \tilde{t}) \in C$ that minimizes Eq. (93) using $th = 10$.

Algorithm LMS returns an alignment $(\tilde{R}, \tilde{t})$ that minimizes the sum of squared fitting distances (Least Mean Squared), i.e., $(\tilde{R}, \tilde{t}) \in \arg\min_{R,t} \sum_{i=1}^{n} \text{dist}^2(Rp_i - t, \ell_i)$, where the minimum is over every valid alignment $(R, t)$. This is done by solving the set of polynomials induced by the sum of squared fitting distances function using the Lagrange Multipliers method.

Adaptive RANSAC + Algorithm LMS is an iterative method that uses $m$ iterations, proportional to the (unknown) percentage of outliers. See [28]. At the $j^{th}$ iteration it samples 3 pairs of corresponding points and lines from $P$ and $L$, respectively, it then computes their optimal alignment $(R_j, t_j)$ using Algorithm LMS, and then updates the percentage of inliers found based on the current alignment. A point $p_i$ is defined as an inlier in the $j^{th}$ iteration if $\text{dist}(R_j p_i - t_j, \ell_i) < th$, where $th = 10$. It then picks the alignment $(\tilde{R}, \tilde{t})$ that has the maximum number of inliers over every alignment in $\{(R_1, t_1), \cdots, (R_m, t_m)\}$.

Adaptive RANSAC + our FAST-APPROX-ALIGNMENT similar to the previous Adaptive RANSAC Algorithm, but instead of using Algorithm LMS, it uses our FAST-APPROX-ALIGNMENT to compute the alignment of the triplet of pairs at each iteration.

Each test was conducted 10 times. The average sum of fitting distances, along with the standard deviation are shown in Fig. 7a, Fig. 7b and Fig. 7c. The average running times, along with the standard deviation, are shown in Fig. 8a, Fig. 8b and Fig. 8c.
We have conducted a test with real-world data to emphasize the potential use of our algorithm in real-world applications. Video link: https://drive.google.com/open?id=19I6Jd6F8ET9386yahKx3Dgr6gBTuzYhJ.

**Experiment: Potential application for Augmented Reality.** A small camera was mounted on a pair of glasses. The glasses were worn by a person (me), while observing the scene in front of him, as shown in Fig 9. The goal was to insert 2D virtual objects into the video of the observed scene, while keeping them aligned with the original objects in the scene.

The experiment. At the first frame of the video we detect a set \( P \) of “interesting points” (features) using a SURF feature detector [4], and draw virtual objects on top of the image. We track the set \( P \) throughout the video using the KLT algorithm [23]. Let \( Q \) denote the observed set of points in a specific frame. In every new frame we use Hough Transform [10] to detect a set \( L \) of lines, and match them naively to the set \( P \): every point \( p \in P \) is matched to its closest line \( \ell \in L \) among the yet unmatched lines in \( L \). We then apply the following algorithms. In practice, we noticed that the approach of detecting a set of lines in the currently observed image rather than detecting a set of interest points, is more robust to noise, since a line
in the image is much more stable than a point / corner.

**Algorithm LMS** begins similar to Algorithm LMS from the previous test. We then apply the output alignment of this algorithm to the initial virtual object, in order to estimate its location in the current frame.

**Adaptive RANSAC -Homography** gets the paired sets $P$ and $Q$, and computes a Homography mapping [24] represented as a matrix $H \in \mathbb{R}^{3 \times 3}$. This is an iterative method that uses $m$ iterations, proportional to the (unknown) percentage of outliers. At the $j^{th}$ iteration it samples 4 pairs of corresponding points from $P$ and $Q$, respectively, it then computes a Homography mapping $H_j$ using those 4 pairs, and then updates the percentage of inliers found based on the current alignment. The $i^{th}$ pair $(p_i, q_i)$ for every $i \in [n]$ is defined as an inlier in the $j^{th}$ iteration if $\|p_i' - q_i\| < 1$, where $p_i'$ is the result of applying $H_j$ to the point $p_i$. It then picks the Homography $\tilde{H}$ that has the maximum number of inliers over every Homography in $\{H_1, \cdots, H_m\}$. We estimate the location of the virtual object in the current frame by applying $\tilde{H}$ to the initial virtual object.

We estimate the location of the virtual object in the current frame using $H$.

**Our FAST-APPROX-ALIGNMENT** begins similar to our FAST-APPROX-ALIGNMENT from the previous test. We then apply the output alignment of this algorithm to the initial virtual object, in order to estimate it’s location in the current frame.