Isolation of the cuspidal spectrum: The function field case

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Abstract Isolating cuspidal automorphic representations from the whole automorphic spectrum is a basic problem in the trace formula approach. For example, matrix coefficients of supercuspidal representations can be used as test functions for this. However, they kill a large class of interesting cuspidal automorphic representations. For the case of number fields, multipliers of the Schwartz algebra are used in the recent work (see Beuzart-Plessis et al. (2021)) to isolate all the cuspidal spectrum. In particular, they are suitable for the comparison of orbital integrals. These multipliers are then applied to the proof of the Gan-Gross-Prasad conjecture for unitary groups (see Beuzart-Plessis et al. (2021, 2022)). In this article, we prove the similar result on isolating the cuspidal spectrum in Beuzart-Plessis et al. (2021) for the function field case.

Keywords cuspidal spectrum, Hecke algebra, trace formula, automorphic representation

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1 Isolation of the cuspidal spectrum

The spectral expansion is a central but difficult problem in the study of (relative) trace formulae. Simple trace formulae can largely simplify this problem. The traditional simple (relative) trace formulae use matrix coefficients of supercuspidal representations as (local components of) test functions, which exclude many important cases. In [7], Lindenstrauss and Venkatesh introduced a new type of simple trace formula to prove the Weyl’s law for spherical cusp forms on locally symmetric spaces associated with a split adjoint semisimple group $G$ over $\mathbb{Q}$. Their approach is based on the observation that there are strong relations between the spectrum of the Eisenstein series at different places.

Recently, in [3], Beuzart-Plessis et al. developed a new technique for isolating components on the spectral side of the trace formula. Precisely, they introduced an analogue of the Bernstein center at Archimedean places, and constructed enough multipliers preserving matching of test functions by considering Schwartz functions, instead of smooth functions with compact supports. Using these multipliers, one can isolate the cuspidal spectrum without the full spectral decomposition, and establish the refined

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Gan-Gross-Prasad conjecture for a large class of representations (see also [2]), which is important to the work [8] on the Beilinson-Bloch-Kato conjecture for certain Rankin-Selberg motives.

The goal of this article is to give a proof of the result on isolating the cuspidal spectrum (see [3, Theorem 1.1]) for the function field case. Similar to the number field case, the result here is expected to be applied to the general situation of the trace formula approach over function fields. This work came to be through an effort to understand the paper [3], and will be the starting point of our project on the trace formula approach for the arithmetic problems over function fields.

Let $F$ be the function field of a smooth projective and geometrically connected curve over the finite field $F_q$. Denote by $\mathcal{A} = \mathcal{A}_F$ the Adele ring of $F$. Let $G$ be a connected reductive group over $F$ and $Z$ be the center of $G$. Let $S_G$ be the set of all the primes of $F$ such that $G(F_p)$ is ramified. We fix a maximal compact subgroup $K_0$ of $G(\mathcal{A})$, and a Haar measure $dg = \prod_v dg_v$ on $G(\mathcal{A})$ such that $K_0, v$ is hyperspecial maximal with volume one under $dg_v$ for every place $v$ not in $S_G$.

Take a unitary automorphic character $\omega : Z(F)\backslash Z(\mathcal{A}) \to \mathbb{C}^\times$. We define $L^2(G(F)\backslash G(\mathcal{A}))_\omega$ to be the $L^2$-completion of the space of smooth functions $\varphi$ on $G(\mathcal{A})$ satisfying

- $\varphi(\gamma g) = \omega(\gamma)\varphi(g)$ for every $\gamma \in Z(F)$ and $g \in G(\mathcal{A})$;
- $|\varphi|^2$ is integrable on $Z(\mathcal{A})G(\mathcal{A})/G(\mathcal{A})$.

Denote by $L^2_0(G(F)\backslash G(\mathcal{A}))_\omega$ the subspace of $L^2(G(F)\backslash G(\mathcal{A}))_\omega$ consisting of functions $\varphi$ which are cuspidal, i.e., the constant term

$$\varphi_P(g) = \int_{U(F)\backslash U(\mathcal{A})} \varphi(ug) du$$

is zero for all the proper parabolic subgroups $P$ of $G$, where $U$ is the unipotent radical of $P$. The group $G(\mathcal{A})$ acts on $L^2_0(G(F)\backslash G(\mathcal{A}))_\omega$ via the right regular action $R$, and $L^2_0(G(F)\backslash G(\mathcal{A}))_\omega$ is closed under this action. Denote by $C_c^\infty(G(\mathcal{A}))$ the algebra (without an identity element) of smooth functions on $G(\mathcal{A})$ with compact supports. Then the action $R$ induces an action of $C_c^\infty(G(\mathcal{A}))$ on $L^2_0(G(F)\backslash G(\mathcal{A}))_\omega$ by

$$R(f) = \int_{G(\mathcal{A})} f(g)R(g)dg \quad (f \in C_c^\infty(G(\mathcal{A}))).$$

Let $S$ be a set of places of $F$ containing $S_G$. Let $K \subset K_0$ be an open compact group of the form $K = \prod_{v \in S} K_v \times \prod_{v \not\in S} K_{0,v} = K_S \times K_0^S$. Assume that the character $\omega$ is invariant under the action of $K \cap Z(\mathcal{A})$. Denote by $L^2(G(F)\backslash G(\mathcal{A})/K)_\omega$ the subspace of $L^2(G(F)\backslash G(\mathcal{A}))_\omega$ invariant under the action of $K$ via $R$. Similarly, we have the space $L^2_0(G(F)\backslash G(\mathcal{A})/K)_\omega$ consisting of cuspidal functions. Denote by $C_c^\infty(K\backslash G(\mathcal{A})/K)$ the algebra of bi-$K$-invariant functions in $C_c^\infty(G(\mathcal{A}))$. Then $C_c^\infty(K\backslash G(\mathcal{A})/K)$ acts on both $L^2(G(F)\backslash G(\mathcal{A})/K)_\omega$ and $L^2_0(G(F)\backslash G(\mathcal{A})/K)_\omega$ via $R$.

For every place $v \not\in S_G$, let $H_v = C_c^\infty(K_{0,v}\backslash (F_v)\backslash K_{0,v})$ be the spherical Hecke algebra of $G_v$ with respect to $K_{0,v}$. Let $\mathcal{H}^{(S)}$ be the restricted tensor product of $H_v$ for places $v \not\in S$. Then $\mathcal{H}^{(S)}$ can be viewed as a subalgebra of $C_c^\infty(K\backslash G(\mathcal{A})/K)$ by the embedding $f^{(S)} \mapsto 1_{K_S} \otimes f^{(S)}$, where $f^{(S)} \in \mathcal{H}^{(S)}$ and $1_{K_S}$ is the characteristic function of $K_S$. In particular, the Hecke algebra $\mathcal{H}^{(S)}$ acts on both $L^2(G(F)\backslash G(\mathcal{A})/K)_\omega$ and $L^2_0(G(F)\backslash G(\mathcal{A})/K)_\omega$ via $R$.

Let $\pi = \bigotimes_v \pi_v$ be an irreducible admissible representation of $G(\mathcal{A})$. Then the algebra $C_c^\infty(G(\mathcal{A}))$ also acts on $\pi$ by

$$\pi(f) = \int_{G(\mathcal{A})} f(g)\pi(g)dg \quad (f \in C_c^\infty(G(\mathcal{A}))).$$

Denote by $\pi^K$ the invariant subspace of $\pi$ under $K$. Then $C_c^\infty(K\backslash G(\mathcal{A})/K)$ acts on $\pi^K$.

Assume that $\pi^K$ is non-zero. In particular, $\pi_v$ is unramified for all $v \not\in S$. We call such a representation $\pi$ is $(G, S)$-CAP if there is a proper parabolic subgroup $P$ of $G$ and a cuspidal automorphic representation $\sigma$ of $M(\mathcal{A})$, where $M$ is the Levi part of $P$ such that for all but finitely many places $v$ of $F$ not in $S$, the unramified representation $\pi_v$ is a constituent of $I_P^G(\sigma_v)$. Here, $I_P^G(\sigma_v)$ denotes the normalized parabolic induction of $\sigma_v$.

**Theorem 1.1.** Suppose that $\pi$ is an irreducible admissible representation of $G(\mathcal{A})$ which is not $(G, S)$-CAP. Then there exists a Hecke algebra element $\mu \in \mathcal{H}^{(S)}$ such that
Theorem 1.1 for $G = \text{PGL}_2$ as an example. The main ingredients for the proof are the following:

- **Spectral decomposition along the cuspidal supports**: we have the spectral decomposition of unitary $G(\mathbb{A})$-modules

$$L^2(G(F) \backslash G(\mathbb{A})) = \left( \bigoplus_{\pi} L^2_{\pi} \right) \oplus \left( \bigoplus_{[\sigma]} L^2_{[\sigma]} \right).$$

Here,

- $\pi$ runs over cuspidal automorphic representations of $G(\mathbb{A})$, and $L^2_{\pi}$ is its $L^2$-completion.
- $[\sigma]$ runs over the equivalent classes of unitary automorphic characters $\sigma : F^\times \backslash \mathbb{A}^\times \rightarrow \mathbb{C}^\times$ under the action $\sigma \mapsto \sigma^{-1}$ of the non-trivial element in the Weyl group $W$, and the action $\sigma \mapsto \sigma_\lambda = \sigma \cdot |\lambda|$
(\lambda \in \mathbb{C}/([\frac{2\pi i}{\log q}]\mathbb{Z}) of unramified characters. The space \( L^2_\sigma \) consists of Eisenstein series associated with the induced representations \( I^G_\sigma(\sigma_\lambda) \) with \( \lambda \in \mathbb{C}/([\frac{2\pi i}{\log q}]\mathbb{Z}) \). Here, \( P \subset \text{PGL}_2 \) is the parabolic subgroup consisting of upper-triangular matrices, and we view the character \( \sigma_\lambda \) as a representation on the Levi subgroup of \( P \), i.e., the subgroup of diagonal matrices (see Section 3 for the precise definition of \( L^2_\sigma \) in the general situation).

- **Harder’s theorem on the finiteness of cuspidal representations**: for any open compact subgroup \( K \) of \( G(\mathbb{A}) \), the space \( L^2_\sigma(G(F) \backslash G(\mathbb{A})/K) \) is of finite dimension (see Corollary 3.4 in Section 3).

- **The Satake isomorphism**: for any place \( v \not\in S \) (here, \( S \) is the set of places of \( F \) given in Section 1), consider the trace map

\[ \mathcal{H}_v \to C(\widehat{G}(F_v)_{\text{un}}), \quad f \mapsto (\pi \mapsto \text{tr}(\pi(f))), \]

where \( \mathcal{H}_v \) is the spherical Hecke algebra at \( v \), \( \widehat{G}(F_v)_{\text{un}} \) is the set of equivalent classes of unramified representations of \( G(F_v) \) with Fell topology, and \( C(\widehat{G}(F_v)_{\text{un}}) \) is the space of continuous functions on \( \widehat{G}(F_v)_{\text{un}} \). The trace map factors through the Satake isomorphism (see Section 4)

\[ S : \mathcal{H}_v \to \mathbb{C}[T, T^{-1}]^W, \]

so that if \( \pi = I^G_\sigma([\cdot | v_\lambda]) \), one has

\[ \text{tr}(\pi(f)) = (Sf)(q_v^\lambda, q_v^{-\lambda}) \]

for any \( f \in \mathcal{H}_v \). Here, \( q_v \) is the cardinality of the residue field of \( F_v \) and \([\cdot | v_\lambda] \) is the normalized abstract value on \( F_v \) which maps uniformizers to \( q_v^{-1} \).

**Step 1** (Killing the continuous spectrum). We now apply the trick of Lindenstrauss and Venkatesh [7] in the function field case, which is based on the strong relation of an Eisenstein series at two different places.

Let \( \pi = \bigotimes_v \pi_v \) be an irreducible admissible representation of \( G(\mathbb{A}) \). Let \( K \) be an open compact subgroup of \( G(\mathbb{A}) \) such that \( \pi^K \) is nonzero. Let \( S \) be a finite place of \( F \) such that \( K \) is maximal outside \( S \). Assume that \( \pi \) is not \((G, S)\)-CAP. If \( \pi \) is cuspidal automorphic, then \( \pi \) is not Eisenstein.

There are only finitely many classes of characters \([\sigma_i] \), so that we may also assume that these \([\sigma_i]\)’s are all unramified outside \( S \). Here, the finiteness comes from the finiteness of the divisor class number of \( F \), i.e., the cardinality \((F^\times \backslash \mathbb{A}^1/\mathcal{O}^\times)\). For higher rank groups, we need Harder’s theorem on finiteness of cuspidal representations (see Theorem 3.3).

Fix a place \( v_\infty \) outside \( S \) with \( q_\infty \) being its cardinality of residue field. Take \( \alpha_\infty \in \mathbb{C} \) such that

\[ \pi_{v_\infty} = I^G_\sigma([\cdot | v_\infty^{\alpha_\infty}]), \]

and fix a class of characters \([\sigma] \). By twisting an unramified character at \( v_\infty \), we may assume \( \sigma_{v_\infty} = 1 \). Let \( v_1 \) and \( v_2 \) be two distinct places outside \( S \). Let \( S_W = \{v_1, v_2\} \), and denote by \( q_1 \) and \( q_2 \) the cardinalities of the residue fields at \( v_1 \) and \( v_2 \), respectively. We may assume that \( q_1 \) and \( q_2 \) are both powers of \( q_\infty \). By the assumption that \( \pi \) is not \((G, S)\)-CAP, we can choose places \( v_1 \) and \( v_2 \) such that

\[ \pi_{v_1} \neq I^G_\sigma([\sigma_{v_1}]_{v_1}), \quad \pi_{v_2} \neq I^G_\sigma([\sigma_{v_2}]_{v_2}). \quad (2.1) \]

Let \( T_1 = T_{v_1} \in \mathcal{H}_{v_1} \) and \( T_2 = T_{v_2} \in \mathcal{H}_{v_2} \) be the usual generators of \( \mathcal{H}_{v_1} \) and \( \mathcal{H}_{v_2} \) so that if we define \( \beta_1 = \sigma_{v_1}(w_{v_1}) \) and \( \beta_2 = \sigma_{v_2}(w_{v_2}) \), then for any \( \lambda \in \mathbb{C} \) and \( i = 1 \) or \( i = 2 \) one has

\[ \text{tr}(I^G_\sigma([\sigma_\lambda]_{v_i}(T_i))) = \beta_1 q_i^{-\lambda} + \beta_1^{-1} q_i^\lambda. \]

Consider the following two elements in \( \mathcal{H}_{v_1} \otimes \mathbb{C}[q_\infty^\lambda, q_\infty^{-\lambda}] \):

\[ T_{1,1} = T_1 - (\beta_1 q_1^{-\lambda} + \beta_1^{-1} q_1^\lambda), \quad T_{1,w} = T_1 - (\beta_1 q_1^\lambda + \beta_1^{-1} q_1^{-\lambda}), \]

and the following two elements in \( \mathcal{H}_{v_2} \otimes \mathbb{C}[q_\infty^\lambda, q_\infty^{-\lambda}] \):

\[ T_{w,1} = T_2 - (\beta_2 q_2^{-\lambda} + \beta_2^{-1} q_2^\lambda), \quad T_{w,w} = T_2 - (\beta_2 q_2^\lambda + \beta_2^{-1} q_2^{-\lambda}). \]
Here, the indices \{1, w\} are referred to the elements in the Weyl group \(W\). By the condition (2.1), we have
\[
T_{1,1}(\pi_{v_1}, \alpha_{\infty}) = \text{tr}(\pi_{v_1}(T_1)) - (\beta_1 q_1^{-\alpha_{\infty}} + \beta_1^{-1} q_1^{\alpha_{\infty}}) = \text{tr}(\pi_{v_1}(T_1)) - \text{tr}(I^G_{\pi}(\sigma_{\infty})_{v_1}(T_1)) \neq 0,
\]
and similarly,
\[
T_{w,w}(\pi_{v_2}, \alpha_{\infty}) \neq 0.
\]
In particular, there exist two constants \(C_1\) and \(C_w\) such that
\[
T_1(\pi_{S_{w}}, \alpha_{\infty}) \neq 0 \quad \text{and} \quad T_w(\pi_{S_{w}}, \alpha_{\infty}) \neq 0
\]
for
\[
T_1 = C_1T_{1,1} + C_wT_{w,1} \quad \text{and} \quad T_w = C_1T_{1,w} + C_wT_{w,w}.
\]
They are both elements of \(\mathcal{H}_{S_{w}} \otimes \mathbb{C}[q_{\infty}^\lambda, q_{\infty}^{-\lambda}]\).

Finally, consider
\[
T = T_1 \cdot T_w \in \mathcal{H}_{S_{w}} \otimes \mathbb{C}[q_{\infty}^\lambda, q_{\infty}^{-\lambda}]^W \cong \mathcal{H}(v_1, v_2, v_\infty).
\]
By our construction, the action of \(T\) on \(\pi\) is non-zero. On the other hand, note that for any \(\lambda \in \mathbb{C}\),
\[
T_{1,1}(I^G_{\lambda}(\sigma_{\lambda})_{v_1}, \lambda) = T_{w,1}(I^G_{\lambda}(\sigma_{\lambda})_{v_2}, \lambda) = 0.
\]
In particular, \(T\) annihilates \(I^G_{\lambda}(\sigma_{\lambda})\) for any \(\lambda\), and hence \(T\) kills the continuous spectrum \(L_2^2(\sigma)\). As there are finitely many \([\sigma]\)'s, a finite product of such Hecke elements \(T\) kills the space orthogonal to \(L_2^2(G(F)\backslash G(\mathbb{A})/K)\) in \(L^2(G(F)\backslash G(\mathbb{A})/K)\), but does not kill \(\pi\).

**Step 2** (Isolating \(\pi\)). By Harder’s theorem, there are only finitely many cuspidal representations in the cuspidal spectrum \(L_2^2(G(F)\backslash G(\mathbb{A})/K)\). Denote by \(\pi_1, \ldots, \pi_n\) the cuspidal representations which are not nearly equivalent to \(\pi\). In particular, for \(\pi_1\), there is a place \(v_1\) of \(F\) outside the union of \(S\) and \(\bigcup_{[\sigma]} S_{\sigma}\), such that \(\pi_{1,v_1} \neq \pi_{v_1}\). Here, \(S_{\sigma}\) is a finite set of places such that the Hecke algebra element used to kill the Eisenstein part \(L_2^2(\sigma_{\sigma})\) above lies in \(\mathcal{H}_{S_{\sigma}}\), and \([\sigma]\) runs over all such equivalence classes. Equivalently, we have \(\text{tr}(\pi_{1,v_1}) \neq \text{tr}(\pi_{v_1})\), and hence the Hecke algebra element \([T_{v_1} - \text{tr}(\pi_{1,v_1}(T_{v_1}))]\) kills \(\pi_1\), but does not kill \(\pi\). Continuing this procedure for \(\pi_2, \ldots, \pi_n\), we can construct a Hecke algebra element which kills all the cuspidal representations not nearly equivalent to \(\pi\) in the spectrum, but does not kill \(\pi\). This finishes the proof.

**Remark 2.1.** In [11], Yun and Zhang killed the continuous part in the case where \(G = \text{PGL}_2\) and \(K\) is maximal, by employing the so-called *Eisenstein ideal* in the Hecke algebra. Take \(S = \emptyset\), and let \(\mathcal{H} = \bigotimes_{v} \mathcal{H}_v\) be the spherical Hecke algebra of \(G\) with respect to \(K\). Consider the ring homomorphisms (see [11, (4.1)]))
\[
a_{\text{Eis}} : \mathcal{H} \xrightarrow{S} \mathbb{C}[A(\mathbb{A})/A(\mathbb{O})] \rightarrow \mathbb{C}[A(F)\backslash A(\mathbb{A})/A(\mathbb{O})],
\]
where \(A\) is the diagonal subgroup of \(G\), and \(\mathcal{O} = \prod_v \mathcal{O}_v\). The image of \(a_{\text{Eis}}\) can be described clearly, which is the subspace of \(\mathbb{C}[A(F)\backslash A(\mathbb{A})/A(\mathbb{O})]\) invariant under the involution from the Weyl group \(W\) of \(G\) (see [11, Lemma 4.2(2)]). The Eisenstein ideal \(I_{\text{Eis}}\) of \(\mathcal{H}\) is then defined to be the kernel of \(a_{\text{Eis}}\). By the spectral decomposition of \(L^2(G(F)\backslash G(\mathbb{A})/K)\), and the fact that the characters of Eisenstein series factor through \(a_{\text{Eis}}\) (i.e., \(\text{tr}(I^G_{\lambda}(\chi)) = \chi \circ a_{\text{Eis}}\) for any unramified Hecke character \(\chi\) on \(A(\mathbb{A}))\), any element in \(I_{\text{Eis}}\) kills the continuous spectrum, and vice versa. On the other hand, the ideal \(I_{\text{Eis}}\) is large enough in the sense that for any cuspidal automorphic representation \(\pi\) of \(G(\mathbb{A})\) which is unramified everywhere, there exists an element \(f \in I_{\text{Eis}}\) such that \(\text{tr}(\pi(f)) \neq 0\). In fact, if the statement is not true, \(I_{\text{Eis}}\) will be contained in the kernel of \(\text{tr}\pi\) so that \(\text{tr}\pi\) factors through \(a_{\text{Eis}}\). Since the image of \(a_{\text{Eis}}\) is the \(W\)-invariant subspace of \(\mathbb{C}[A(F)\backslash A(\mathbb{A})/A(\mathbb{O})]\), \(\text{tr}\pi\) is given by a \((W, \emptyset)\)-CAP. Moreover, based on the above, one can obtain an element \(\mu \in I_{\text{Eis}}\) satisfying the conditions in Theorem 1.1, by applying Harder’s theorem.

Comparing with the strategy above, for a given cuspidal automorphic representation \(\pi\), in this section we construct an explicit element \(\mu_\pi \in I_{\text{Eis}}\), which isolates \(\pi\) from \(L^2(G(F)\backslash G(\mathbb{A})/K)\). In particular, \(\mu_\pi\) depends on \(\pi\), while \(I_{\text{Eis}}\) does not.
Remark 2.2. We discuss a possible generalization of the above Eisenstein ideal for a general reductive group $G$ and any level $K \subset G(\mathbb{A})$. Let $S$ be a finite set of places such that $K^{(S)}$ is maximal. For each standard Levi subgroup $M$ of $G$ (after fixing a Borel subgroup of $G$), denote by $\mathcal{H}^{(S)}_M = \bigotimes_{v \notin S} \mathcal{H}_{M,v}$ the spherical Hecke algebra of $M$ outside $S$. Similar to the Satake transform (for the minimal Levi), for each $v \notin S$, consider the map

$$\mathcal{S}_{M,v} : \mathcal{H}_v \to \mathcal{H}_{M,v}, \quad f_v \mapsto \left( m \mapsto \delta_{P_0}(m)^{1/2} \int_{N(F_v)} f_v(mn)dn \right).$$

Then for each irreducible unramified representation $I^G_v(\sigma_v)$ of $G(F_v)$, one has

$$\text{tr}(I^G_v(\sigma_v)(f_v)) = \text{tr}(\sigma_v(\mathcal{S}_{M,v}(f_v))) \quad (f_v \in \mathcal{H}_v).$$

Let $\mathcal{S}_M = \bigotimes_{v \notin S} \mathcal{S}_{M,v}$, and consider the following map:

$$a_M : \mathcal{H}^{(S)} \xrightarrow{\mathcal{S}_M} \mathcal{H}^{(S)}_M \to \text{End}_{\mathcal{H}^{(S)}_M}(\mathcal{A}_{\text{cusp}}(M(\mathbb{A})/K \cap M(\mathbb{A}))).$$

Here, $\mathcal{A}_{\text{cusp}}(M(\mathbb{A})/K \cap M(\mathbb{A}))$ is the space of cusp forms on $M(\mathbb{A})$ with level $K \cap M(\mathbb{A})$. Denote by $I_M$ the kernel of $a_M$. Consider the following ideal of $\mathcal{H}^{(S)}$:

$$I_{\text{Eis}} = \bigcap_M I_M.$$

Then by the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))/K$, any element in $I_{\text{Eis}}$ kills the continuous spectrum, and vice versa. On the other hand, one needs to know that the ideal $I_{\text{Eis}}$ is large enough in the sense that for any cuspidal automorphic representation $\pi$ on $G(\mathbb{A})$ with $\pi^K \neq 0$, there exists an element $f \in I_{\text{Eis}}$ such that $\text{tr}(\pi(f)) \neq 0$. One may prove this by studying the image of $a_M$ for each $M$ as in Remark 2.1, but it seems more involved. However, the property that $I_{\text{Eis}}$ is large enough will follow from Theorem 1.1 immediately, which ensures that there is a $\mu \in I_{\text{Eis}}$ such that $\text{tr}(\pi^{(S)}(\mu)) = 1$.

3 Spectral decomposition along the cuspidal data

In this section, we recall the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A}))_\omega$ along the cuspidal supports in the case of function field, following [9].

For convenience, we list some notation first, which will be used in the remaining parts of this paper.

- Let $P_0$ be a fixed minimal parabolic subgroup of $G$ defined over $F$ with the Levi subgroup $M_0$. A subgroup $M$ of $G$ is called a standard Levi subgroup if there exists a parabolic subgroup of $G$ containing $P_0$, of which $M$ is the unique Levi subgroup containing $M_0$.
- Let $T_0$ be the maximal split torus in the center of $M_0$. For any standard Levi subgroup of $G$, let $T_M$ be the maximal split torus in the center of $M$, which is contained in $T_0$.
- Fix a maximal compact subgroup $K_0 \subset G(\mathbb{A})$ such that
  (i) $G(\mathbb{A}) = P_0(\mathbb{A})K_0$;
  (ii) for every standard parabolic subgroup $P = MU$, $P(\mathbb{A}) \cap K_0 = (M(\mathbb{A}) \cap K_0)(U(\mathbb{A}) \cap K_0)$ and $M(\mathbb{A}) \cap K_0$ is a maximal compact subgroup of $M(\mathbb{A})$.

The choice of $K_0$ fixes a choice of the maximal compact subgroup of $M(\mathbb{A})$ for every standard Levi $M$.

- Denote by $\text{Rat}(M)$ the group of rational characters of $M$. Then define $\text{Re}(\mathfrak{a}_M) = \text{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_M^0 = \text{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{C}$.
- For $\chi \in \text{Rat}(M)$, denote by $|\chi|$ the continuous character on $M(\mathbb{A})$ given by

$$|\chi|(m) = \prod_v |\chi_v(m_v)|_v \quad (m = (m_v)_v \in M(\mathbb{A})),$$

where $\chi_v : M(F_v) \to F_v^\times$ is the algebraic character induced by $\chi$. Then define

$$M(\mathbb{A})^1 = \bigcap_{\chi \in \text{Rat}(M)} \text{Ker}|\chi|.$$
Denote by $X_M$ the group of characters on $M(\mathbb{A})^1\backslash M(\mathbb{A})$, which can be realized as a quotient of $\mathfrak{a}_M^*$. In fact, let $\chi_1, \ldots, \chi_r$ be a $\mathbb{Z}$-basis of $\text{Rat}(M)$. The map
\[ j : M^1(\mathbb{A}) \backslash M(\mathbb{A}) \rightarrow (q^{\mathbb{Z}})^r, \quad m \mapsto (|\chi_1|(m), \ldots, |\chi_r|(m)) \]
defines a topological group isomorphism onto its image, which is a subgroup of $(q^{\mathbb{Z}})^r$ with finite index. Then
\[ \kappa : \mathfrak{a}_M^* \rightarrow X_M, \quad \chi_i \mapsto |\chi_i| \]
is a surjective morphism of groups, and the kernel of $\kappa$ is of the form $(\frac{2\pi i}{\log q})L$, where $L$ is a lattice of $\text{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$. We also define $\text{Re}(X_M) = \kappa(\text{Re}(\mathfrak{a}_M^*)))$, and $\kappa$ induces an isomorphism $\text{Re}(\mathfrak{a}_M^*) \simeq \text{Re}(X_M)$.

Denote by $X_M^G$ the subgroup of $X_M$ with characters trivial on $Z(\mathbb{A})$ (recall that $Z = Z_G$, the center of $G$). In particular, there is a perfect pairing
\[ X_M^G \times M(\mathbb{A})^1Z(\mathbb{A}) \backslash M(\mathbb{A}) \rightarrow \mathbb{C}^\times. \tag{3.1} \]

For standard Levi subgroups $M \subset M'$ of $G$, denote by $\text{Re}(\mathfrak{a}_{M'}^*)$ the real vector subspace of $\text{Re}(\mathfrak{a}_{M}^*)$ generated by $\text{Re}(X_M)$, the set of roots (see [9, Subsection I.1.6]) of $M'$ relative to $T_M$. Identifying $\text{Re}(\mathfrak{a}_{M'}^*)$ with a real vector subspace of $\text{Re}(\mathfrak{a}_{M}^*)$ by restriction, we have
\[ \text{Re}(\mathfrak{a}_{M}^*) = \text{Re}(\mathfrak{a}_{M'}^*) \oplus \text{Re}(\mathfrak{a}_{M'}^*)^*. \]

Moreover, the elements of $\text{Re}(\mathfrak{a}_{M'}^*)^*$ can be identified with the elements of $\text{Re}(X_M)$, which are trivial on the center of $M'(\mathbb{A})$. After the tensor product by $\mathbb{C}$, one also has the decomposition
\[ \mathfrak{a}_{M}^* = \mathfrak{a}_{M'}^* \oplus (\mathfrak{a}_{M'}^*)^*, \tag{3.2} \]

where $(\mathfrak{a}_{M'}^*)^* = \text{Re}(\mathfrak{a}_{M'}^*) \otimes_{\mathbb{R}} \mathbb{C}$.

For a compact open subgroup $K \subset G(\mathbb{A})$ such that $G(\mathbb{A}) = P(\mathbb{A})K$, one defines a map
\[ m_P : G(\mathbb{A}) \rightarrow M^1(\mathbb{A}) \backslash M(\mathbb{A}) \]
by $m_P(g) = M^1m$ if $g = muk$ with $u \in U(\mathbb{A})$, $m \in M(\mathbb{A})$ and $k \in K$.

We recall some notions on automorphic forms and automorphic representations. Let $P = MU$ be a standard parabolic subgroup. We call a smooth (locally constant) function
\[ \varphi : U(\mathbb{A})M(F) \backslash G(\mathbb{A}) \rightarrow \mathbb{C} \tag{3.3} \]
an automorphic form if
\begin{enumerate}
\item[(i)] $\varphi$ is of moderate growth;
\item[(ii)] $\varphi$ is $K_0$-finite;
\item[(iii)] $\varphi$ is $3(G(F_v))$-finite for any place $v$ of $F$.
\end{enumerate}
Here, $3(G(F_v))$ is the Bernstein center (see [1]) of $G(F_v)$. We denote the space of all such automorphic forms by $\mathcal{A}(U(\mathbb{A})M(F) \backslash G(\mathbb{A}))$. For a unitary automorphic character $\omega : Z(F) \backslash Z(\mathbb{A}) \rightarrow \mathbb{C}^\times$, we also denote by $\mathcal{A}(U(\mathbb{A})M(F) \backslash G(\mathbb{A}))_{\omega}$ the automorphic forms $\varphi$ with the central character $\omega$, i.e., $\varphi(zg) = \omega(z)\varphi(g)$ for all $z \in Z(\mathbb{A})$. We say that $\varphi$ is cuspidal if for all $z \in Z(\mathbb{A})$, it has constant term along $P'$ is zero. The space of cuspidal automorphic forms on $U(\mathbb{A})M(F) \backslash G(\mathbb{A})$ is denoted by $\mathcal{A}_0(U(\mathbb{A})M(F) \backslash G(\mathbb{A}))$.

Moreover, for any $k \in K_0$, we define $\varphi_k : M(F) \backslash M(\mathbb{A}) \rightarrow \mathbb{C}$ by
\[ \varphi_k(m) = m^{-\rho_P} \varphi(mk), \]
where $\rho_P$ is the half-sum of roots of $M$ in the Lie algebra of $U$. Then a smooth function (3.3) is an automorphic form if it is $K_0$-finite and for all $k \in K_0$, $\varphi_k$ is an automorphic form on $M(F) \backslash M(\mathbb{A})$ ([9, Subsection I.2.17]).

The spectral decomposition is given by Eisenstein series associated with different cuspidal data. We set some more notation:
• Denote by $\Pi_0(M(\mathbb{A}))$ the set of cuspidal automorphic representations $\sigma$ of $M(\mathbb{A})$, i.e., equivalence classes of irreducible subquotients of the space of cusp forms $A_0(M(F) \backslash M(\mathbb{A}))$.

• For any unitary automorphic character $\omega : Z(F) \backslash Z(\mathbb{A}) \to \mathbb{C}^\times$, let $\Omega_M(\omega)$ be the set of unitary automorphic characters $\omega_M : Z_M(F) \backslash Z_M(\mathbb{A}) \to \mathbb{C}^\times$ such that $\omega_M|_{Z(\mathbb{A})} = \omega$.

• Denote by $\Pi_0(M(\mathbb{A}))_{\omega}$ the subspace of $\Pi_0(M(\mathbb{A}))$ consisting of cuspidal automorphic representations with the central character $\omega_M \in \Omega_M(\omega)$.

• For $\sigma \in \Pi_0(M(\mathbb{A}))_{\omega}$, denote by $\mathcal{A}(M, \sigma)$ the subspace consisting of

\[ \varphi \in \mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\omega} \]

such that $\varphi_k \in \mathcal{A}(M(F)\backslash M(\mathbb{A}))_{\sigma}$ for all $k \in K_0$, where $\mathcal{A}(M(F)\backslash M(\mathbb{A}))_{\sigma}$ is the isotypic submodule of $\sigma$ in $\mathcal{A}(M(F)\backslash M(\mathbb{A}))$. The group $X^G_M$ acts on the space $\Pi_0(M(\mathbb{A}))_{\omega}$ via

\[ \sigma \mapsto \sigma_\lambda = \sigma \circ \lambda \]

with $\lambda \in X^G_M$ and $\sigma \in \Pi_0(M(\mathbb{A}))_{\omega}$. We say that $\sigma$ is equivalent to $\sigma'$ if there exists $\lambda \in X^G_M$ such that $\sigma_\lambda \simeq \sigma'$, and denote such an equivalent class by $\mathcal{P}$. A cuspidal datum (of central character $\omega$) is a pair $(M, \mathcal{P})$, where $M$ is a standard Levi of $G$, and $\mathcal{P}$ is an equivalence class of $\sigma \in \Pi_0(M(\mathbb{A}))_{\omega}$ as above. Two cuspidal data $(M, \mathcal{P})$ and $(M', \mathcal{P'})$ are called equivalent if there exists some $w \in G(F)$ such that $w \cdot M = M'$ and $w \cdot \mathcal{P} = \mathcal{P'}$. By the Bruhat decomposition, if such an $w$ exists, we can suppose it lies in the Weyl group of $G$.

The group $X^G_M$ also acts on the space $\mathcal{A}(U(\mathbb{A})M(F)\backslash G(\mathbb{A}))_{\omega}$ via

\[ \varphi \mapsto \varphi_\lambda := \varphi \cdot (\lambda \circ m_p). \]

Then any $\lambda \in X^G_M$ induces an isomorphism

\[ \lambda : \mathcal{A}(M, \sigma) \xrightarrow{\sim} \mathcal{A}(M, \sigma_\lambda). \]

For any $\varphi \in \mathcal{A}(M, \sigma)$, the Eisenstein series on $G(F) \backslash G(\mathbb{A})$ associated with $\varphi$ is defined by

\[ E(\varphi, \sigma)(g) = \sum_{\gamma \in P(F) \backslash G(F)} \varphi(\gamma g), \]

whenever the sum converges.

**Proposition 3.1** (See [9, Subsection II.1.5, Proposition]). There exists an open cone $C^G_M$ in $X^G_M$ such that for any $\varphi \in \sigma$, if $\lambda \in C^G_M$, then the summation defining $E(\varphi_\lambda, \sigma_\lambda)(g)$ converges absolutely and uniformly when $g$ varies in a compact set. Moreover, one also has

\[ E(\varphi, \sigma) \in \mathcal{A}(G(F)\backslash G(\mathbb{A}))_{\omega} \]

if it is convergent.

Let $P(X^G_M)$ be the set of Paley-Wiener functions on $X^G_M$, i.e., the image of the Fourier transform (recall (3.1))

\[ f \mapsto \hat{f}(\lambda) = \int_{M(\mathbb{A})^1Z(\mathbb{A})\backslash M(\mathbb{A})} f(m)\lambda(m)dm \]

on the space $C^\infty_c(M(\mathbb{A})^1Z(\mathbb{A})\backslash M(\mathbb{A}))$. A section $\Phi : X^G_M \to \mathcal{A}(M, \sigma)$ is called a Paley-Wiener section if $\Phi$ is a sum of sections of the form

\[ X^G_M \ni \lambda \mapsto \hat{f}(\lambda) \cdot \varphi \]

for some $\hat{f} \in P(X^G_M)$ and $\varphi \in \mathcal{A}(M, \sigma)$. Denote by $P(M, \sigma)$ the space of all the Paley-Wiener sections on $\mathcal{A}(M, \sigma)$. For any $\Phi \in P(M, \sigma)$, consider the pseudo-Eisenstein series (see [9, Subsections II.1.11 and II.1.12])

\[ \theta_\Phi(g) = \int_{\lambda \in X^G_M, \text{Re}(\lambda) = \lambda_0} E(\Phi(\lambda)\lambda, \sigma_\lambda) d\lambda, \]
where $\lambda_0$ is an arbitrary element in $\text{Re}(X_M^G)$ which is positive enough.

Let $L^2(G(F)\backslash G(\mathbb{A}))_\omega$ be the space of functions on $G(F)\backslash G(\mathbb{A})$ with the central character $\omega$ and square-integrable modulo the center $Z(\mathbb{A})$. By computing the inner product of two pseudo-Eisenstein series, one obtains the following spectral decomposition result.

**Theorem 3.2** (Spectral decomposition along cuspidal data [9, Subsection II.2.4, Proposition]). Let $\mathbb{X}$ be an equivalence class of cuspidal data. Denote by $L^2(G(F)\backslash G(\mathbb{A}))_{\mathbb{X}}$ the closed subspace of $L^2(G(F)\backslash G(\mathbb{A}))_\omega$ spanned by $\theta_{\Phi}$ with $\Phi \in P(M, \sigma)$, where $(M, \sigma)$ is an arbitrary representative of $\mathbb{X}$. Then

$$L^2(G(F)\backslash G(\mathbb{A}))_\omega = \bigoplus_{\mathbb{X}} L^2(G(F)\backslash G(\mathbb{A}))_{\mathbb{X}}.$$  

We also need some finiteness properties for the spectral decomposition in our proof later. The following theorem is due to Harder:

**Theorem 3.3** (See [6, Corollary 1.2.3]). Let $G$ be a reductive group over $F$ and $\omega$ be a unitary character of $Z(F)\backslash Z(\mathbb{A})$. Then for any open compact subgroup $K$ of $G(\mathbb{A})$, the vector space $L^2_0(G(F)\backslash G(\mathbb{A})/K)_\omega$ is of finite dimension.

**Corollary 3.4.** Let $G$ be a reductive group over $F$ and $\omega$ be a unitary character of $Z(F)\backslash Z(\mathbb{A})$. Let $K$ be an open compact subgroup of $G(\mathbb{A})$. Then there are only finitely many cuspidal data occurring in the spectral decomposition of $L^2(G(F)\backslash G(\mathbb{A})/K)_\omega$.

**Proof.** Let $\mathbb{X}$ be an equivalence class of cuspidal data. Assume that $L^2(G(F)\backslash G(\mathbb{A})/K)_\mathbb{X} \neq 0$. Then for any $(M, \sigma) \in \mathbb{X}$, we have $(I_M^G \sigma)^K \neq 0$. Using [10, Subsection III.2.2, Lemma], one sees that there exists an open compact subgroup $K_M$ of $M(\mathbb{A})$ depending on $K$ such that $\sigma^{K_M} \neq 0$ for any $\sigma$ with $(I_M^G \sigma)^K \neq 0$. We claim that by modifying $\sigma$ to $\sigma_\lambda$ with $\lambda \in X_M^G$, the central character of $\sigma$ belongs to a finite set. Hence, by Harder’s theorem (see Theorem 3.3), there are only finitely many such $\sigma$’s.

To prove the claim, consider the set $\Sigma$ of characters $\omega_M : Z_M(F)\backslash Z_M(\mathbb{A})/K_{Z_M} \rightarrow \mathbb{C}^*$ with $\omega_M|_{Z(\mathbb{A})} = \omega$, where $K_{Z_M}$ is a fixed open compact subgroup of $Z_M(\mathbb{A})$. The group of characters of the quotient $Z_G(\mathbb{A})Z_M(\mathbb{A})^{1}\backslash Z_M(\mathbb{A})$ acts on $\Sigma$. The claim is then equivalent to saying that the number of the orbits of $\Sigma$ under this action is finite. To see the finiteness, let $T$ be the torus $Z_M/Z_G$ over $F$ and $K_T$ be the image of $K_{Z_M}$ in $T(\mathbb{A})$. Consider the exact sequence

$$1 \rightarrow Z_G(F)\backslash Z_G(\mathbb{A})/(K_{Z_M} \cap Z_G(\mathbb{A})) \rightarrow Z_M(F)\backslash Z_M(\mathbb{A})/K_{Z_M} \rightarrow T(F)/T(\mathbb{A})K_T \rightarrow 1.$$  

We may write $T = T_s \times T_0$ with $T_s \cap T_0$ being finite, where $T_s$ is a split torus and $T_0$ is an anisotropic torus. For the anisotropic part, the quotient $T_0(F)/T_0(\mathbb{A})$ is compact. For the split part, if we denote its rank by $d$, then $T_s(F)/T_s(\mathbb{A}) \cong (F^\times/|\cdot|^d) \times \mathbb{Z}^d$ with $(F^\times/|\cdot|^d)^d$ compact. Note that we may modify $\omega_M$ by a character on $Z_G(\mathbb{A})Z_M(\mathbb{A})^{1}\backslash Z_M(\mathbb{A})$ such that $\omega_M$ is trivial on $\mathbb{Z}^d$. Therefore, there must be finite number of such orbits by the exact sequence above.

\[ \Box \]

## 4 Proof of Theorem 1.1

We prove Theorem 1.1 in the general case in this section. We recall some basics on unramified representations and the Satake isomorphism at first, and the basic reference is [4].

Keep the notation used in Sections 1 and 3. Recall that $T_0$ is the maximal split torus of $M_0$. We denote by $d$ the rank of $T_0$, and fix a basis of $\text{Rat}(T_0)$, i.e., $\chi_1, \ldots, \chi_d$. Let $v$ be a place of $F$ outside $S$. Denote by $M_0(F_v)_{\text{un}}$ the group of unramified characters on $M_0(F_v)$. Then we have an isomorphism

$$\mathbb{C}/\left( \frac{2\pi i}{\log q_v} \right)^d \cong M_0(F_v)_{\text{un}}, \quad \lambda_1, \ldots, \lambda_d \mapsto \lambda_1^{\chi_1} \cdots \lambda_d^{\chi_d},$$

so that we may view $M_0(F_v)_{\text{un}}$ as a torus over $\mathbb{C}$. Denote by $\mathbb{C}[M_0(F_v)_{\text{un}}]$ the ring of regular functions on $M_0(F_v)_{\text{un}}$. Under the above isomorphism, one has

$$\mathbb{C}[M_0(F_v)_{\text{un}}] \cong \mathbb{C}[^{\chi_1}\lambda_1, ^{\chi_2}\lambda_2, \ldots, ^{\chi_d}\lambda_d].$$  

(4.1)
Let $v_1$ be another place of $F$ such that $q_{v_1} = q_0^k$ for some integer $k$. Then by taking $q_{v_1}^{\pm\lambda_i} \mapsto (q_0^\pm)^k$ ($i = 1, \ldots, d$), we have an injection

$$
\mathbb{C}[\tilde{M}_0(F_{v_1})_{\text{un}}] \hookrightarrow \mathbb{C}[\tilde{M}_0(F_v)_{\text{un}}]
$$

from (4.1).

There is a perfect pairing

$$
\tilde{M}_0(F_v)_{\text{un}} \times \tilde{M}_0(F_v)/\tilde{M}_0(\mathcal{O}_v) \rightarrow \mathbb{C}^\times.
$$

For each $f \in C_c^\infty(\tilde{M}_0(F_v)/\tilde{M}_0(\mathcal{O}_v))$, one considers its Fourier transform

$$
\hat{f}(\chi) = \int_{\tilde{M}_0(F_v)/\tilde{M}_0(\mathcal{O}_v)} f(m)\chi(m)dm \quad (\chi \in \tilde{M}_0(F_v)_{\text{un}}),
$$

which gives an isomorphism

$$
\sim : C_c^\infty(\tilde{M}_0(F_v)/\tilde{M}_0(\mathcal{O}_v)) \sim \mathbb{C}[\tilde{M}_0(F_v)_{\text{un}}].
$$

Denote by $\tilde{G}(F_v)_{\text{un}}$ the set of irreducible unramified representations of $G(F_v)$, i.e., the irreducible smooth representations $\pi_v$ of $G(F_v)$ with the non-zero invariant subspace $\pi_v^{K_{0,v}}$. For any $\chi_v \in \tilde{M}_0(F_v)_{\text{un}}$, there is a unique subquotient of $I_{\tilde{\beta}_0}^0(\chi_v)$ which is an irreducible unramified representation of $G(F_v)$. This in fact gives an isomorphism

$$
\tilde{M}_0(F_v)_{\text{un}}/W \sim \tilde{G}(F_v)_{\text{un}},
$$

where $W = N_{\tilde{G}(F_v)}(\tilde{M}_0(F))/\tilde{M}_0(F)$ is the Weyl group of $G$. Conversely, for an irreducible unramified representation $\pi_v \in \tilde{G}(F_v)_{\text{un}}$, we denote by $\chi_{\pi_v} \in \tilde{M}_0(F_v)_{\text{un}}/W$ the $W$-orbit of the unramified character corresponding to $\pi_v$ as above.

For each $\pi_v \in \tilde{G}(F_v)_{\text{un}}$, the spherical Hecke algebra $\mathcal{H}_v = C_c^\infty(K_{0,v}\backslash G(F_v)/K_{0,v})$ acts on the spherical line $\pi_v^{K_{0,v}}$ of $\pi_v$, which gives a map

$$
\text{tr} : \mathcal{H}_v \rightarrow C(\tilde{G}(F_v)_{\text{un}}), \quad f \mapsto (\pi_v \mapsto \text{tr}(\pi_v(f))).
$$

Recall that $C(\tilde{G}(F_v)_{\text{un}})$ is the space of continuous functions on $\tilde{G}(F_v)_{\text{un}}$. Consider the Satake isomorphism

$$
S : \mathcal{H}_v \sim \mathbb{C}[\tilde{M}_0(F_v)/\tilde{M}_0(\mathcal{O}_v)]^W
$$

given by

$$
(Sf)(m) = \delta_{P_0}(m)^{1/2} \int_{U_0(F_v)} f(mn)dn \quad (f \in \mathcal{H}_v).
$$

Then the composition map

$$
\mathcal{H}_v \xrightarrow{\mathcal{U}} C(\tilde{G}(F_v)_{\text{un}}) \sim C(\tilde{M}_0(F_v)_{\text{un}}/W)
$$

factors through the isomorphism

$$
\mathcal{H}_v \xrightarrow{S} C_c^\infty(\tilde{M}_0(F_v)/\tilde{M}_0(\mathcal{O}_v))^W \rightarrow \mathbb{C}[\tilde{M}_0(F_v)_{\text{un}}]^W.
$$

In particular, we will view elements in $\mathcal{H}_v$ as functions on $\tilde{M}_0(F_v)_{\text{un}}$ in the following.

Let $\pi$ be an irreducible admissible representation of $G(\mathbb{A})$ with the central character $\omega$. Let $K = K_S \times K_0^{(S)}$ be an open compact subgroup of $G(\mathbb{A})$ such that $\pi^K \neq 0$. In particular, $\pi$ is unramified outside $S$. Assume that $\pi$ is not $(G,S)$-CAP. Let $X = [[M,\sigma]]$ be an equivalence class of the cuspidal datum with $M \neq G$ such that $L^2(G(F)/G(\mathbb{A}))/K_X \neq 0$. In particular, $\sigma$ is also unramified outside $S$. In the following, we want to construct a Hecke algebra element $\mu_\sigma \in \mathcal{H}(S)$ such that

(1) $R(\mu_\sigma)$ acts on $L^2(G(F)\backslash G(\mathbb{A}))/K_X$ by zero;
(2) $\pi(\mu_\sigma) = 1$. 
**Step 1** (Killing the continuous spectrum). Note that the restriction map $\mathfrak{a}^*_M \hookrightarrow \mathfrak{a}^*_M$ is injective, and we fix a splitting of this injection

$$\ell : \mathfrak{a}^*_M \to \mathfrak{a}^*_M.$$  

For any place $v$ of $F$ outside $S$, we also have a surjective map

$$\mathfrak{a}^*_M \to \overline{M_0(F_v)_{\text{un}}} \to \overline{M_0(F_v)_{\text{un}}}/W. \quad (4.2)$$

In the following we fix a place $v_\infty$ of $F$ outside $S$, and fix $\alpha_\pi \in \mathfrak{a}^*_M$ (resp. $\alpha_\sigma \in \mathfrak{a}^*_M$) such that its image under the surjection (4.2) is $\chi_{\pi v_\infty}$ (resp. $\chi_{\alpha v_\infty}$).

As $\pi$ is not $(G, S)$-CAP, for each $w \in W$, there is a place $v[w] \not\subseteq S \cup \{v_\infty\}$ such that the following hold:

- $\pi_{v[w]}$ is not a subquotient of $I_{P^M_f}(\sigma_{v[w], \ell_{(w\alpha_\pi)}}, \ell_{(\alpha_\pi)})$. Here, $\ell_{(w\alpha_\pi)} - \ell_{(\alpha_\pi)}$ lie in the subspace $(\mathfrak{a}^*_M)^*$ of $\mathfrak{a}^*_M$ (see (3.2)), and we realize $\ell_{(w\alpha_\pi)} - \ell_{(\alpha_\pi)}$ as its image in $\overline{M_0(F_v[w])_{\text{un}}}$ via (4.2). We will keep this convention in the following.

- $q_{v[w]}$ is a power of $q_{v_\infty}$.

Let $h_w \in \mathcal{H}_{v[w]}$ such that

$$h_w(\chi_{\pi v[w]}) \neq h_w(\chi_{\mathfrak{r}_P^g \sigma_{v[w], \ell_{(w\alpha_\pi)}}, \ell_{(\alpha_\pi)})}. \quad (4.3)$$

Write $\mathfrak{h}^* = \overline{M_0(F_{v_\infty})_{\text{un}}}$ for short. Since $q_{v[w]}$ is a power of $q_{v_\infty}$, the following map:

$$\mathfrak{a}^*_M \xrightarrow{\delta} \mathfrak{a}^*_M \to \overline{M_0(F_v[w])_{\text{un}}} \quad (4.4)$$

factors through $\mathfrak{h}^*$, and hence we have

$$h_w(\chi_{\mathfrak{r}_P^g \sigma_{v[w], \ell_{(w\alpha_\pi)}}, \ell_{(\alpha_\pi)})} \in \mathbb{C}[\mathfrak{h}^*].$$

Then (4.3) gives a map

$$\mathfrak{a}^*_M \to \overline{\mathcal{C}[\mathfrak{h}^*]}. \quad (4.4)$$

For any $\lambda \in \mathfrak{a}^*_M$, define

$$h_{w, w'}(\lambda, \cdot) := h_w(\chi_{\mathfrak{r}_P^g \sigma_{v[w], \ell_{(w\alpha_\pi)}}, \ell_{(\alpha_\pi)})} \in \mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{H}_{v[w]}.$$

As $h_{w, w}(\alpha_\pi, \chi_{\pi v[w]}) \neq 0$ for any $w \in W$, there exist constants $C_w$ such that for any $w' \in W$,

$$\sum_{w \in W} C_w \cdot h_{w, w'}(\alpha_\pi, \chi_{\pi v[w]}) \neq 0.$$ 

Define $S_W = \{v[w]\}_{w \in W}$ and

$$\mu_{w'} = \sum_{w \in W} C_w \cdot h_{w, w'} \in \mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{H}_{S_W}.$$ 

Then

$$\mu_{\sigma} = \prod_{w \in W} \mu_{w'} \in \mathbb{C}[\mathfrak{h}^*] \otimes \mathcal{H}_{S_W} = \mathcal{H}_{S_W \cup \{v_\infty\}}$$

satisfying $\pi(\mu_{\sigma}) \neq 0$. Finally, for any $\lambda \in \mathfrak{a}^*_M$, also denoting by $\lambda$ its image in $\mathfrak{a}^*_M$, we have

$$\mu_{\lambda}(\alpha_\sigma + \lambda, \chi_{\mathfrak{r}_P^g \sigma_{v[w_\infty]}}) = \sum_{w \in W} C_w [h_w(\chi_{\mathfrak{r}_p^g \sigma_{v[w], \lambda}}) - h_w(\chi_{\mathfrak{r}_P^g \sigma_{v[w], \ell_{(\alpha_\pi)}}, \ell_{(\alpha_\pi)})] = 0.$$
Therefore, $R(\mu_\sigma)$ annihilates $L^2(G(F)\backslash G(\mathcal{A})/K)_\chi$. As there are only finitely many $\mathcal{X} = [(M, \sigma)]$ with $M \neq G$, a finite product of such $\mu_\sigma$’s kills the orthogonal space of the cuspidal spectrum $L^2_0(G(F)\backslash G(\mathcal{A})/K)_\omega$ in $L^2(G(F)\backslash G(\mathcal{A})/K)_\omega$, but does not kill $\pi$.

**Step 2** (Isolating $\pi$). Recall that (see Corollary 3.4) there are only finitely many (equivalence classes of) cuspidal representations in the cuspidal spectrum $L^2_0(G(F)\backslash G(\mathcal{A})/K)$. Denote by $\pi_1, \ldots, \pi_n$ the cuspidal representations which are not nearly equivalent to $\pi$. In particular, for $\pi_1$, there is a place $v_1$ of $F$ outside the union of $S$ and $\bigcup_{[(M, \sigma)]} S_\sigma$ such that $\pi_{1,v_1} \not\sim \pi_{1,v_1}$. Here, $S_\sigma$ is a finite set of places such that $\mu_\sigma \in \mathcal{H}_{S_\sigma}$, and $[(M, \sigma)]$ runs over all the equivalence classes of cuspidal data. It follows that we may find $T_{v_1} \in \mathcal{H}_{v_2}$ such that

$$T_{v_1}(\chi_{\pi_{1,v_1}}) \neq T_{v_1}(\chi_{\pi_{1,v_1}}).$$

In particular, the Hecke element $T_{v_1} - T_{v_1}(\chi_{\pi_{1,v_1}}) \in \mathcal{H}_{v_1}$ kills $\pi_1$, but does not kill $\pi$. Continuing this procedure for $\pi_2, \ldots, \pi_n$, we can construct a Hecke algebra element $\mu_0$ which kills all the cuspidal representations not nearly equivalent to $\pi$ in the spectrum, but does not kill $\pi$. Consider the finite product

$$\mu' = \mu_0 \cdot \prod_{[(M, \sigma)]} \mu_\sigma \in \mathcal{H}^{(S)},$$

where each $\mu_\sigma$ is constructed in Step 1 to kill $L^2(G(F)\backslash G(\mathcal{A})/K)_{[(M, \sigma)]}$. Then $\mu'$ satisfies the first condition in Theorem 1.1 which acts on $\pi^K$ by a non-zero constant. Finally, $\mu = \pi(\mu')^{-1} \mu'$ is a Hecke algebra element required in Theorem 1.1.

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