Research Article

Dynamical Analysis of Long Fiber-Reinforced Laminated Plates with Elastically Restrained Edges

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Received 30 December 2010; Revised 7 August 2011; Accepted 14 September 2011

Academic Editor: Kok Keong Choong

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This paper presents a variational formulation for the free vibration analysis of unsymmetrically laminated composite plates with elastically restrained edges. The study includes a micromechanics approach that allows starting the study considering each layer as constituted by long unidirectional fibers in a continuous matrix. The Mori–Tanaka method is used to predict the mechanical properties of each lamina as a function of the elastic properties of the components and of the fiber volume fraction. The resulting mechanical properties for each lamina are included in a general Ritz formulation developed to analyze the free vibration response of thick laminated anisotropic plates resting on elastic supports. Comprehensive numerical examples are computed to validate the present method, and the effects of the different mechanical and geometrical parameters on the dynamical behavior of different laminated plates are shown. New results for general unsymmetrical laminates with elastically restrained edges are also presented. The analytical approximate solution obtained in this paper can also be useful as a basis to deal with optimization problems under, for instance, frequency constraints.

1. Introduction

Fiber-reinforced composite laminated plates are extensively used in many engineering applications. The free vibration analysis of these plates plays a very important role in the design of civil, aerospace, mechanical, and marine structures. In addition to the favorable high specific strength and high specific stiffness, fiber-reinforced composite laminates offer the possibility of optimal design through the variation of stacking pattern, angle of fiber orientation, fiber content, and so forth, known as composite tailoring. All these mechanical and geometrical characteristics, as well as the various coupling effects that take place, must be considered in the prediction of the laminates dynamical response to assure that this is reliable, accurate, and adequate to the design requirements.

It is well known that laminated composite plates have relatively low transverse shear stiffness, playing the shear deformation an important role in the global and local behavior of these structures. Among the numerous theories used for laminated plates that include the transverse shear strain, the first-order shear deformation theory (FSDT) [1, 2] is adequate for the computation of global responses (such as natural frequencies) and simultaneously has some advantages due to its simplicity and low computational cost. Many investigations have been reported for free vibration analysis of moderately thick composite laminates using the FSDT kinematics (see for instance [3–13]). However, the results are, in most cases, limited to certain lamination schemes and boundary conditions. As far as the study of thick plates with elastically restrained edges is concerned, most of the previous works are limited to isotropic ones ([14–19] among others). But, limited information is found for the case of thick anisotropic laminated plates resting on elastic supports. For instance, Setoodeh and Karami [20] implemented a layer-wise laminated plate theory linked with three-dimensional elasticity approach for vibration and buckling of symmetric and antisymmetric fiber-reinforced composite plates having elastically restraint edges support and results for cross-ply laminates are presented, whereas Karami et al. [21] applied the differential quadrature method for the free vibration
analysis of moderately thick symmetric laminated plates with elastically restrained edges. For the same boundary
conditions, semianalytical solutions for the free vibration
of angle-ply symmetrically laminated plates were presented
by Ashour [22]. Nallim and Grossi [23] also studied the
vibration of symmetric laminated plates resting on elastic
support employing the Ritz method and beam orthogonal
polynomials as approximated functions. These kind of
approximate functions (in one or two variables) have been
used by many authors to the free vibration analysis of, both
homogeneous and nonhomogeneous, plates (Chakraverty
et al. [24–26] and Chow et al. [27], among others).
In this paper, a general Ritz formulation for the free
vibration analysis of anisotropic laminated plates is de-
veloped. All kind of boundary conditions including elastically
restrained edges are considered enhancing the study. This
feature allows for more realistic analysis of some structural
problems. The analysis includes a micromechanical approach
(according to the classification of Altenbach et al. [28]),
where the average mechanical properties of each anisotropic
lamina are estimated from the known characteristics of the
fibers and the matrix materials taking into account the
fiber volume ratio and the fiber-packing arrangement. At
structural level, the dynamic response of the unsymmetrical
laminate is calculated by Mori-Tanaka [17] along with the
first-order shear deformation theory and the Ritz
method with beam orthogonal polynomials as coordinate
functions. Approximate analytical solution developed
here is very useful to understand, both qualitatively and
quantitatively, the behavior of complex laminated plates.

2. Formulation

2.1. Effective Elastic Moduli of Long Fiber-Reinforced Laminae.

The micromechanics-based Mori-Tanaka method [29] is
used in this section to predict the elastic mechanical prop-
erties of the orthotropic unidirectional laminae. This method
may be viewed as the simplest mean field approach for
inhomogeneous materials that encompass the full physical
range of phase volume fraction.

Eshelby’s results [30] show that if an elastic homogeneous
ellipsoidal inclusion in an infinite linear elastic matrix is
subjected to an eigenstrain $\varepsilon^f$, uniform strain states $\varepsilon^c$ is
induced, and it is related to the eigenstrain by the expression

\[
\varepsilon^c = S^f : \varepsilon^f,
\]

where $S^f$ is the Eshelby tensor, which depends on the rein-
fforcement dimensions and the Poisson ratio of the matrix
$\nu_m$. The components of this tensor for a circular, cylindrical
inclusion with an infinite length-to-diameter ratio parallel to
the 1-axis (parallel to the fiber direction, Figure 1) are

\[
S_{1111} = S_{1133} = S_{1222} = 0, \quad S_{3333} = S_{2222} = \frac{5 - 4\nu_m}{8(1 - \nu_m)},
\]

\[
S_{3322} = \frac{4\nu_m - 1}{8(1 - \nu_m)}, \quad S_{3311} = S_{2211} = \frac{2\nu_m}{2(1 - \nu_m)},
\]

\[
S_{3232} = \frac{3 - 4\nu_m}{8(1 - \nu_m)}, \quad S_{3131} = S_{1212} = \frac{1}{4}.
\]

The transformations strains are obtained considering the
equivalent homogeneous inclusion for inhomogeneous
inclusions developed by Eshelby [33] together with the inter-
action effects of Mori-Tanaka [29]. These transformations
strains are used to equate the total stresses in the inhomo-
genities and their equivalent inclusions, as described in the
following equation:

\[
C_f : (\varepsilon^f + \varepsilon^m + \varepsilon^c) = C_m : (\varepsilon^f + \varepsilon^m + \varepsilon^c - \varepsilon^f) \quad (3)
\]

where $C_f$ and $C_m$ are the stiffness tensors of fiber and matrix,
respectively, $\varepsilon^f$ is the uniform far field strain applied to the
domain at infinity, and $\varepsilon^m$ is the average elastic strain defined
by Mori-Tanaka which is given by

\[
e^{int} = -k_f (\varepsilon^c - \varepsilon^f), \quad (4)
\]

where $k_f$ is the fiber volume fraction.

\[
C = C_m \times \left\{ I - k_f \left[ (C_f - C_m) \left( S^f - k_f (S^f - I) + C_m \right) \right]^{-1} \right. \\
\times \left. (C_f - C_m) \right\}^{-1}
\]

Finally, the stiffness tensor $C$ for different unidirectional
laminae can be obtained from energy considerations [34]
and (1) to (4) as

where $I$ is the fourth order identity tensor.

Using this method the mechanical properties of unidirectional
carbon/epoxy laminae are found considering various
fiber volume fractions, and they are depicted in Table 1. These
properties, for each unidirectional lamina, are then
used in the next section to obtain the reduced constitutive
matrix.

2.2. General Laminated Plate Resting on Elastic Supports.

Let us consider a rectangular fiber-reinforced composite
laminated plate, of dimension $a \times b$ and total thickness $h$
Table 1: Mechanical properties of unidirectional laminae (AS4-3501-6), obtained using Mori-Tanaka method. Fiber and matrix properties $E_{11} = 225$ GPa, $E_{22} = 15$ GPa, $G_{12} = 15$ GPa, $G_{13} = 7$ GPa, $v_{12} = 0.20; E_m = 4.2$ GPa, $v_m = 0.34$ ([30]).

| $k_f$   | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|
| $E_1$   | 26.29 | 48.38 | 70.46 | 92.54 | 114.62 | 136.70 | 158.78 | 180.86 |
| $E_2$   | 5.11  | 5.69 | 6.31 | 7.01 | 7.81 | 8.76 | 9.87 | 11.23 |
| $G_{12}$ | 1.84 | 2.17 | 2.57 | 3.07 | 3.70 | 4.54 | 5.68 | 7.35  |
| $G_{23}$ | 1.75 | 1.97 | 2.22 | 2.52 | 2.88 | 3.32 | 3.88 | 4.61  |

(35)

$Q_{ij}$ are the components of the plane-stress reduced constitutive matrix [35] which are function of the elastic constant determined in Section 2.1 and the ply angle $\beta_k$.

2.3. Energy Functional Components. Taking into account (7) and (9), the strain energy due to the laminated plate deflection can be written as

$$U_p = \frac{1}{2} \int_R \left[ \{\epsilon_0\} [A] \{\epsilon_0\}^T + \{\epsilon_0\} [B] \{\epsilon_1\}^T + \{\epsilon_1\} [B] \{\epsilon_0\}^T + \{\epsilon_1\} [D] \{\epsilon_1\}^T + \{\epsilon_0^*\} [A^*] \{\epsilon_0^*\}^T \right] dx dy,$$

where $R$ is the mid-surface area (Figure 1) and the stiffness coefficients [35, 36] are given by $(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij} \phi dz$ $(i, j = 1, 2, 6), A_{ij} = \int_{-h/2}^{h/2} Q_{ij} \phi dz$ $(i, j = 4, 5), k_{ij}$ being the shear correction factors.

The strain energy corresponding to the elastic edge restraints is given by

$$U_t = \frac{1}{2} \int_0^b \left[ t_{ix}^0 \left( \frac{\partial u}{\partial x} \right)^2 + t_{iy}^0 \left( \frac{\partial v}{\partial y} \right)^2 + t_{iz}^0 \left( \frac{\partial w}{\partial z} \right)^2 + t_{ixx}^0 \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + t_{iyy}^0 \left( \frac{\partial^2 v}{\partial y^2} \right)^2 + t_{izz}^0 \left( \frac{\partial^2 w}{\partial z^2} \right)^2 \right] dz,$$

$$+ \frac{1}{2} \int_0^b \left[ t_{iyy}^\prime \left( \frac{\partial \phi_y}{\partial y} \right)^2 + t_{iyy}^\prime \left( \frac{\partial \phi_y}{\partial y} \right)^2 \right] dz,$$

$$+ \frac{1}{2} \int_0^b \left[ t_{iyy}^\prime \left( \frac{\partial \phi_y}{\partial y} \right)^2 + t_{iyy}^\prime \left( \frac{\partial \phi_y}{\partial y} \right)^2 \right] dy,$$

(11)

where $t_i^0 (i = 1, \ldots, 4)$ and $t_i^\prime (i = 1, \ldots, 4)$ are the elastic translational and rotational coefficients, respectively.

The kinetic energy is expressed as

$$T = \frac{1}{2} \int_R \left[ I_0 \left( \frac{\partial \dot{u}}{\partial x} \right)^2 + I_0 \left( \frac{\partial \dot{v}}{\partial y} \right)^2 + 2I_1 \left( \frac{\partial \phi_x}{\partial x} \dot{u} + \frac{\partial \phi_y}{\partial y} \dot{v} + \frac{\partial \phi_z}{\partial z} \dot{w} \right) \right] dx dy,$$

(12)

and

$$I_i = \int_{-h/2}^{h/2} \rho^{(k)} \phi^2 dz, \quad (i = 0, 1, 2),$$

(13)

where $\rho^{(k)}$ is the material density of the $k$-th layer.

3. Application of the Ritz Method for the Free Vibration Analysis

The Ritz method is applied to determine analytical approximate solutions for dynamical behavior of arbitrarily laminated plates resting on elastic supports. During free vibration, the displacements components are assumed split in
...the translational restraint parameter and the rotational restraint parameter in time; that is, the spatial and temporal parts, being the last one periodic in time; that is,

\[ u_0(x, y, t) = U(x, y) \sin \omega t, \]
\[ v_0(x, y, t) = V(x, y) \sin \omega t, \]
\[ w_0(x, y, t) = W(x, y) \sin \omega t, \]
\[ \phi_x(x, y, t) = \Phi_x(x, y) \sin \omega t, \]
\[ \phi_y(x, y, t) = \Phi_y(x, y) \sin \omega t, \]

where \( \omega \) is the natural frequency in radian.

Putting these displacements into the energy functional components (10) to (12)) the maximum values of the kinetic energy (\( T_{\text{max}} \)) and the strain energies (\( U_{p,\text{max}}, U_{l,\text{max}} \)) are derived. Then, the energy functional for free vibration of the laminated plate is given by

\[ \Pi = U_{p,\text{max}} + U_{l,\text{max}} - T_{\text{max}}, \]

which is to be minimized according to the Ritz principle.

3.1. Boundary Conditions and Approximating Functions.

There are some options when choosing the unknown functions of displacement components to apply the Ritz method. Particularly, the use of orthogonal polynomials as coordinate...
functions has important advantages related to numerical stability and fast convergence as has been demonstrated in previous works [23, 37, 38], even for plates with complicated boundary conditions and high degree of anisotropy. For these reasons, in this work, the displacement components are expressed by sets of beam characteristic orthogonal polynomials \( \{p_i(x)\}, \{q_j(y)\}\), \((i = u, v, w, \phi_x, \phi_y)\), resulting in

\[
\begin{align*}
U(x,y) &\approx U_{MN}(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} p_i(x) q_j(y), \\
V(x,y) &\approx V_{MN}(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(v)} p_i(x) q_j^{(v)}(y), \\
W(x,y) &\approx W_{MN}(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(w)} p_i(x) q_j^{(w)}(y), \\
\Phi_x &\approx \Phi_{x_{MN}}(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(\phi_x)} p_i(x) q_j^{(\phi_x)}(y), \\
\Phi_y &\approx \Phi_{y_{MN}}(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij}^{(\phi_y)} p_i(x) q_j^{(\phi_y)}(y),
\end{align*}
\]

where \( c_{ij}, c_{ij}^{(v)}, c_{ij}^{(w)}, c_{ij}^{(\phi_x)}, c_{ij}^{(\phi_y)} \) are the unknown coefficients, and \( M, N \) are the numbers of polynomials in each coordinate.

The procedure for the construction of the orthogonal polynomials has been developed by Bhat [39]. The first members of the sets, \( p_i^{(u,v)}(x) \) and \( q_j^{(u,v)}(y) \) of \( u, v, w, \phi_x, \phi_y \), are obtained as the simplest polynomials that satisfy all

the geometrical boundary conditions of the plate in their respective \( x \) and \( y \) directions. The higher members of each set are constructed by employing the Gram-Schmidt orthogonalization procedure. The coefficients of the polynomials are chosen in such a way as to make the polynomials orthonormal. However, the functions \( p_i^{(u,v)}(x) \) and \( q_j^{(u,v)}(y) \) for \( (\bullet) = \phi_x, \phi_y \) are obtained from relative rotation conditions starting from polynomials of an order lower than the chosen for the transversal displacements and then applying the sequence of Gram-Schmidt orthogonalization procedure. This particular choice is made to avoid the overestimation of the rate of elastic energy due to the shear respect to the rate due to the bending. This concept has been applied by Auciello and Ercolano [40], to Timoshenko beams, to avoid

Figure 6: Variation of the fundamental frequency coefficient \( \bar{\omega} \) with the translational restraint parameter \( T_k^R, T_k^r = T_k^r = R_i = 0 \) \((i = 1, \ldots, 4)\) for different aspect ratios \( [0^\circ/45^\circ] \) carbon-epoxy (Table 1), with \( k_f = 0.6 \).

Figure 7: Effect of the fiber orientation on the first vibration frequency coefficient \( \bar{\omega} \), for two different fiber volume fraction \( k_f \) with \( C_1C_1C_1C_1 \) boundary condition and \( a/h = 10 \).

Figure 8: Effect of the fiber orientation on the first vibration frequency coefficient \( \bar{\omega} \), for two different fiber volume fraction \( k_f \) with \( S_1S_1S_1 \) boundary condition and \( a/h = 10 \).
Table 2: Notations for various combinations of boundary conditions, in which \( n \) and \( s \) indicate the directions normal and tangential to the respective plate edges.

| Boundary Condition                  | In-plane constraints | Transverse constraints |
|------------------------------------|----------------------|------------------------|
| Clamped: \( w = 0; \phi_n = 0 \)   | \( u_n = 0, u_s = 0 \) | \( N_n = 0, C_2 = 0 \) |
| Simply supported: \( w = 0; M_n = 0; \phi_s = 0 \) | \( u_n = 0, N_s = 0 \) | \( S_1 = 0, S_2 = 0 \) |
| Free: \( M_n = 0; M_s = 0; Q_n = 0 \) | \( N_n = 0, N_s = 0 \) | \( C_4 = 0, S_4 = 0 \) |

Table 3: Convergence study of frequencies \( \omega_i^* = \omega_i a^2 \sqrt{\rho/(E_2 h^2)} \) for a two-layered \([0^\circ/45^\circ]\) square plate. \( E_1/E_2 = 25, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, \nu_{12} = 0.25 \).

| \( a/h \) | \( M, N \) \((M = N)\) | Mode sequence number |
|-----------|----------------|-------------------|
| \( a/h \) | \( M, N \) \((M = N)\) | Mode sequence number |
| 4         | 15.487 23.582 | 30.182 35.243 42.217 49.098 |
| 4         | 19.746 30.947 | 42.338 50.728 71.617 81.996 |
| 5         | 19.222 30.415 | 41.756 50.225 70.270 80.750 |
| 6         | 19.221 30.279 | 41.586 49.434 53.531 62.069 |
| 7         | 19.219 30.271 | 41.572 49.047 53.090 61.032 |
| 8         | 19.219 30.268 | 41.569 49.016 53.047 60.655 |
| 9         | 19.218 30.267 | 41.567 49.008 53.033 60.600 |
| 10        | 19.218 30.267 | 41.567 49.007 53.030 60.587 |
| Shi et al. [5] | 15.504 23.399 | 29.991 33.170 37.740 42.973 |

Table 4: Comparison of fundamental frequency coefficient \( \omega_i^* = \omega_i a^2 \sqrt{\rho/(E_2 h^2)} \) for a four-layered \([45^\circ/−45^\circ/45^\circ/−45^\circ]\) plate with different aspect ratios \( a/b \). \( E_1/E_2 = 40, G_{12} = 0.6E_1, G_{13} = G_{23} = 0.5E_2, \nu_{12} = 0.25 \).

| \( a/h \) | 0.2 | 0.6 | 0.8 | 1   | 1.2  | 1.6  | 2   |
|-----------|-----|-----|-----|-----|-----|-----|-----|
| Present   | 9.013 | 13.02 | 15.74 | 18.62 | 21.59 | 27.66 | 34.57 |
| Alibeigloo et al. [31] | 8.559 | 12.565 | 15.187 | 17.983 | 20.895 | 27.031 | 33.634 |
| Redy [32] | 8.724 | 12.965 | 15.712 | 18.609 | 21.567 | 27.736 | 34.247 |
| Present   | 9.965 | 15.409 | 19.293 | 23.638 | 28.377 | 39.062 | 51.480 |
| Alibeigloo et al. [31] | 9.420 | 14.790 | 18.487 | 22.637 | 27.200 | 37.534 | 49.499 |
| Redy [32] | 9.667 | 15.385 | 19.304 | 23.676 | 28.381 | 38.940 | 51.132 |
| Present   | 10.056 | 15.666 | 19.700 | 24.25 | 29.25 | 40.70 | 54.25 |
| Alibeigloo et al. [31] | 9.5016 | 15.0261 | 18.8586 | 23.195 | 28.003 | 39.05 | 52.686 |
| Redy [32] | 9.816 | 15.689 | 19.759 | 24.343 | 29.321 | 40.653 | 53.989 |
Table 5: (a) Frequency parameters $\omega_i$ for $[0°/45°]$ and $[0°/90°]$ carbon-epoxy AS4-3501-6 (Table 1), with different translational restraint parameter $T^{tr}_w$, $T^{tr}_u = T^{tr}_v = R_i = 0$ ($i = 1, \ldots, 4$), and $a/h = 10$. (b) Frequency parameters $\omega_i$ for $[0°/45°]$ and $[0°/90°]$ carbon-epoxy AS4-3501-6 (Table 1), with different translational restraint parameter $T^{tr}_w$, $T^{tr}_u = T^{tr}_v = R_i = 0$ ($i = 1, \ldots, 4$), and $a/h = 100$.

(a)

| $k_f$ | Mode | Translational restraint parameter $T^{tr}_w$ |
|-------|------|------------------------------------------|
|       |      | 0.1 | 1 | 10 | 100 | 1000 | 10000 | 1.00E + 10 |
|       |      |     |   |    |    |     |       |
| [0°/45°] |     |     |   |    |    |     |       |   |
| 0.2    | 1   | 0.060 | 0.187 | 0.520 | 0.969 | 1.161 | 1.191 | 1.195 |
| 2   | 0.085 | 0.266 | 0.803 | 1.827 | 2.380 | 2.470 | 2.481 |
| 3   | 0.085 | 0.267 | 0.822 | 2.079 | 2.930 | 3.063 | 3.078 |
| 4   | 0.754 | 0.810 | 1.224 | 2.609 | 3.776 | 4.004 | 4.032 |
| 1   | 0.083 | 0.256 | 0.689 | 1.219 | 1.419 | 1.448 | 1.452 |
| 2   | 0.117 | 0.367 | 1.083 | 2.284 | 2.829 | 2.912 | 2.922 |
| 3   | 0.117 | 0.368 | 1.125 | 2.723 | 3.638 | 3.767 | 3.782 |
| 4   | 0.890 | 0.978 | 1.590 | 3.325 | 4.530 | 4.735 | 4.759 |
| 0.4    | 1   | 0.101 | 0.310 | 0.824 | 1.443 | 1.671 | 1.705 | 1.709 |
| 2   | 0.142 | 0.445 | 1.303 | 2.685 | 3.290 | 3.382 | 3.393 |
| 3   | 0.142 | 0.447 | 1.364 | 3.274 | 4.342 | 4.492 | 4.509 |
| 4   | 1.040 | 1.151 | 1.903 | 3.936 | 5.292 | 5.517 | 5.543 |
| [0°/90°] |     |     |   |    |    |     |       |   |
| 0.2    | 1   | 0.060 | 0.187 | 0.528 | 0.942 | 1.084 | 1.104 | 1.106 |
| 2   | 0.085 | 0.267 | 0.814 | 1.970 | 2.657 | 2.756 | 2.768 |
| 3   | 0.085 | 0.267 | 0.814 | 2.070 | 2.930 | 3.063 | 3.078 |
| 4   | 0.621 | 0.977 | 1.164 | 2.642 | 3.708 | 3.888 | 3.909 |
| 1   | 0.083 | 0.258 | 0.705 | 1.131 | 1.321 | 1.340 | 1.342 |
| 2   | 0.117 | 0.368 | 1.110 | 2.546 | 3.251 | 3.344 | 3.355 |
| 3   | 0.117 | 0.368 | 1.110 | 2.546 | 3.251 | 3.344 | 3.355 |
| 4   | 0.750 | 0.858 | 1.540 | 3.424 | 4.519 | 4.675 | 4.675 |
| 0.4    | 1   | 0.101 | 0.313 | 0.847 | 1.395 | 1.553 | 1.574 | 1.577 |
| 2   | 0.142 | 0.446 | 1.343 | 3.035 | 3.829 | 3.932 | 3.944 |
| 3   | 0.142 | 0.446 | 1.343 | 3.035 | 3.829 | 3.932 | 3.944 |
| 4   | 0.910 | 1.040 | 1.864 | 4.089 | 5.329 | 5.514 | 5.536 |

(b)

| $k_f$ | Mode | Translational restraint parameter $T^{tr}_w$ |
|-------|------|------------------------------------------|
|       |      | 0.1 | 1 | 10 | 100 | 1000 | 10000 | 1.00E + 10 |
|       |      |     |   |    |    |     |       |   |
| [0°/45°] |     |     |   |    |    |     |       |   |
| 0.2    | 1   | 0.060 | 0.187 | 0.525 | 1.014 | 1.276 | 1.329 | 1.340 |
| 2   | 0.085 | 0.268 | 0.813 | 1.928 | 2.708 | 2.881 | 2.908 |
| 3   | 0.085 | 0.269 | 0.832 | 2.214 | 3.489 | 3.771 | 3.815 |
| 4   | 0.804 | 0.858 | 1.266 | 2.774 | 4.519 | 5.027 | 5.098 |
| 1   | 0.083 | 0.256 | 0.697 | 1.292 | 1.577 | 1.633 | 1.646 |
| 2   | 0.117 | 0.369 | 1.099 | 2.445 | 3.269 | 3.440 | 3.468 |
| 3   | 0.117 | 0.370 | 1.142 | 2.952 | 4.444 | 4.746 | 4.794 |
| 4   | 0.947 | 1.032 | 1.642 | 3.588 | 5.531 | 6.011 | 6.073 |
| 0.4    | 1   | 0.101 | 0.310 | 0.835 | 1.531 | 1.859 | 1.925 | 1.940 |
| 2   | 0.142 | 0.447 | 1.324 | 2.880 | 3.804 | 3.995 | 4.028 |
| 3   | 0.143 | 0.450 | 1.385 | 3.555 | 5.304 | 5.654 | 5.711 |
| 4   | 1.104 | 1.210 | 1.962 | 4.255 | 6.457 | 6.980 | 7.046 |
the shear locking effect and is extended here for laminated plates.

The classical boundary conditions considered in this study are depicted in Table 2. By keeping in mind that in the Ritz method only the geometric boundary conditions need to be satisfied, it is possible to work with any sets of required edge boundary condition and also is very simple the consideration of elastically restrained edges where there are no essential boundary conditions to satisfy. Upon inserting the displacement forms (16) into the energy functional of the system (15), the minimization with respect to the coefficients of the displacement functions is given by

\[
\frac{\partial \Pi}{\partial c^{(a)}_{ij}} = 0, \quad \frac{\partial \Pi}{\partial c^{(w)}_{ij}} = 0, \quad \frac{\partial \Pi}{\partial c^{(u)}_{ij}} = 0, \quad \frac{\partial \Pi}{\partial c^{(v)}_{ij}} = 0, \quad \frac{\partial \Pi}{\partial c^{(w)}_{ij}} = 0, \quad \frac{\partial \Pi}{\partial c^{(w)}_{ij}} = 0. \tag{17}
\]

From (17) a set of algebraic simultaneous equations is obtained. The number of these equations becomes \(5 \times M \times N\). The algebraic equations obtained are given as follows, in the form of the generalized eigenvalue problem:

\[
(K - \omega^2 M) \{C\} = \{0\}, \tag{18}
\]

where \(K\) and \(M\) are stiffness and inertia matrices, respectively (their expressions are given in the Appendix, \(\{C\}\) contains the unknown coefficients of (16). For a nontrivial solution, the eigenvalues which make the determinant equal to zero, correspond to the free vibration frequencies.

### 4. Verification of the Formulation and Numerical Applications

#### 4.1. General Description

The variational algorithm developed in this paper was programmed in Fortran language and is used for the free vibration analysis of generally laminated thin and moderately thick laminated plates having different geometric parameters, stacking sequences, material properties, fiber volume fractions, and boundary conditions. The examples considered in this study are confined to laminates with layers of equal thickness, even though the procedure was formulated for plies with arbitrary thickness. In all cases, the shear correction factor was taken as 5/6.

Let us introduce the terminology to be used throughout the remainder of the paper for describing the boundary conditions of the considered plates. The designation \(C_iS_iF_i\), for example, identifies a plate with edges (1) clamped, (2) simply supported, (3) free, and (4) simply supported (see Figure 1). This notation also applies to out-of-plane constraints according to Table 2. When the edges are elastically restrained against rotation or translation, the following nondimensional restraint parameters are used

\[
T_i = \frac{a_i^3 R_i}{D_0}, \quad (i = 1, \ldots, 4 \text{ and } \bullet = u, v, w), \tag{19}
\]

where \(D_0 = E_s h^3/12(1 - \nu_{12} \nu_{23})\).

The main purposes of the numerical applications presented in this section are twofold. One is to demonstrate the accuracy, the flexibility, and the efficiency of the proposed method and the other is to produce some results which may be regarded as benchmark solutions for other academic research workers and design engineers.

#### 4.2. Validation and Convergence Studies

The accuracy and reliability of the results obtained with the present approach are next demonstrated by comparing them with some selected values published by Shi et al. [5] for moderately thick \((a/h = 10, 20)\) and thin \((a/h = 100)\) arbitrarily clamped laminated plates. The comparison presented in Table 3 authenticates the validity of the present method for arbitrarily laminated plates. Very close agreement for the first

### Table 3

| \(k_j\) | Mode | Translational restraint parameter \(T_i^M\) |
|---|---|---|
| \([0^\circ/90^\circ]\) | 0.1 | 1 | 10 | 100 | 1000 | 10000 | \(1.00E + 10\) |
| 1 | 0.060 | 0.188 | 0.533 | 0.981 | 1.161 | 1.190 | 1.194 |
| 2 | 0.085 | 0.268 | 0.823 | 2.086 | 3.043 | 3.212 | 3.233 |
| 3 | 0.085 | 0.268 | 0.823 | 2.086 | 3.043 | 3.212 | 3.233 |
| 4 | 0.666 | 0.732 | 1.197 | 2.805 | 4.365 | 4.732 | 4.782 |
| 2 | 0.083 | 0.258 | 0.714 | 1.239 | 1.422 | 1.449 | 1.453 |
| 3 | 0.117 | 0.370 | 1.126 | 2.738 | 3.783 | 3.948 | 3.969 |
| 4 | 0.979 | 0.901 | 1.583 | 3.699 | 5.418 | 5.774 | 5.821 |
| 2 | 0.101 | 0.313 | 0.857 | 1.464 | 1.674 | 1.706 | 1.710 |
| 3 | 0.117 | 0.370 | 1.126 | 2.738 | 3.783 | 3.948 | 3.969 |
| 4 | 0.143 | 0.449 | 1.363 | 3.271 | 4.453 | 4.638 | 4.661 |
| 2 | 0.143 | 0.449 | 1.363 | 3.271 | 4.453 | 4.638 | 4.661 |
| 3 | 0.967 | 1.093 | 1.917 | 4.427 | 6.392 | 6.797 | 6.851 |

#### Table 2

| \(k_f\) | Mode | Edge boundary conditions |
|---|---|---|
| 1 | 0 | \(n^\circ/90^\circ\) |
| 2 | 0.060 | \(0^\circ/90^\circ\) |
| 3 | 0.085 | \(0^\circ/90^\circ\) |
| 4 | 0.666 | \(0^\circ/90^\circ\) |
| 2 | 0.083 | \(0^\circ/90^\circ\) |
| 3 | 0.117 | \(0^\circ/90^\circ\) |
| 4 | 0.979 | \(0^\circ/90^\circ\) |
| 2 | 0.101 | \(0^\circ/90^\circ\) |
| 3 | 0.117 | \(0^\circ/90^\circ\) |
| 4 | 0.143 | \(0^\circ/90^\circ\) |
| 2 | 0.143 | \(0^\circ/90^\circ\) |
| 3 | 0.967 | \(0^\circ/90^\circ\) |

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Table 6: (a) Frequency parameters \( \bar{\omega}_i \) for \([0^\circ/45^\circ]\) and \([0^\circ/90^\circ]\) carbon-epoxy AS4-3501-6 (Table 1), with different rotational restraint parameter \( R_i, T_i^T = T_i^U = T_i^V = \infty \) \((i = 1, \ldots, 4)\), and \(a/h = 10\). (b) Frequency parameters \( \bar{\omega}_i \) for \([0^\circ/45^\circ]\) and \([0^\circ/90^\circ]\) carbon-epoxy AS4-3501-6 (Table 1), with different rotational restraint parameter \( R_i, T_i^T = T_i^U = T_i^V = \infty \) \((i = 1, \ldots, 4)\), and \(a/h = 100\).

(a)

| \( k_f \) | Mode | Rotational restraint parameter \( R_i \) |
| --- | --- | --- |
| | | 0.1 | 1 | 10 | 100 | 1000 | 10000 | 1.00E + 10 |
| [0°/45°] | | | | | | | | |
| 0.2 | 1 | 1.461 | 1.623 | 1.943 | 2.056 | 2.071 | 2.072 | 2.072 |
| 2 | 2.709 | 2.903 | 3.268 | 3.390 | 3.406 | 3.407 | 3.407 |
| 3 | 3.350 | 3.497 | 3.861 | 4.010 | 4.030 | 4.032 | 4.032 |
| 4 | 4.322 | 4.477 | 4.818 | 4.951 | 4.969 | 4.971 | 4.971 |
| 0.4 | 1 | 1.837 | 2.029 | 2.371 | 2.483 | 2.497 | 2.498 | 2.498 |
| 2 | 3.249 | 3.493 | 3.875 | 3.988 | 4.002 | 4.003 | 4.003 |
| 3 | 4.112 | 4.302 | 4.709 | 4.854 | 4.872 | 4.874 | 4.874 |
| 4 | 5.129 | 5.333 | 5.700 | 5.823 | 5.838 | 5.840 | 5.840 |
| 0.6 | 1 | 1.386 | 1.562 | 1.916 | 2.143 | 2.048 | 2.050 | 2.050 |
| 2 | 2.901 | 3.110 | 3.563 | 3.719 | 3.738 | 3.740 | 3.740 |
| 3 | 3.901 | 3.563 | 3.719 | 3.738 | 3.740 | 3.741 | 3.741 |
| 4 | 4.052 | 4.687 | 4.764 | 4.970 | 4.994 | 4.996 | 4.996 |
| [0°/90°] | | | | | | | | |
| 0.2 | 1 | 1.598 | 1.814 | 2.318 | 2.534 | 2.564 | 2.567 | 2.567 |
| 2 | 3.108 | 3.410 | 4.115 | 4.415 | 4.457 | 4.461 | 4.462 |
| 3 | 4.161 | 4.421 | 5.275 | 5.759 | 5.831 | 5.839 | 5.840 |
| 4 | 5.366 | 5.689 | 6.611 | 7.100 | 7.173 | 7.181 | 7.181 |
| 0.4 | 1 | 2.044 | 2.316 | 2.901 | 3.135 | 3.166 | 3.169 | 3.170 |
| 2 | 3.783 | 4.185 | 4.997 | 5.311 | 5.353 | 5.358 | 5.358 |
| 3 | 5.265 | 5.627 | 6.699 | 7.240 | 7.317 | 7.325 | 7.326 |
| 4 | 6.450 | 6.911 | 8.018 | 8.523 | 8.594 | 8.601 | 8.601 |
| 0.6 | 1 | 2.428 | 2.748 | 3.423 | 3.689 | 3.725 | 3.729 | 3.729 |
| 2 | 4.418 | 4.896 | 5.821 | 6.171 | 6.217 | 6.222 | 6.223 |
| 3 | 6.250 | 6.692 | 7.967 | 8.593 | 8.682 | 8.691 | 8.692 |
| 4 | 7.498 | 8.058 | 9.337 | 9.897 | 9.974 | 9.983 | 9.983 |

(b)
sixth nondimensional frequencies $\omega_i^2 = \omega_i^2 a^2 \sqrt{\rho/(E_i h^2)}$ is obtained for all cases and display monotonic convergence tendency to constant values. For thick plates, as shown in Table 3, as number of $N$ and $M$ is increased from 7 to 10, the frequency parameter decreases merely 0.002% for the first mode and 0.26% for the sixth. For thin plates the relative decreases of the frequency parameters are 0.004% for the first mode and 1.88% for the sixth as the numbers of polynomials $N, M$ are increased from 7 to 10, exhibiting slower convergence rate than that of moderately thick plates. Consequently the number of beam characteristic polynomials used in the following computations for thin and thick plates is chosen as $N = 7$ and $M = 10$.

The validation of the proposed methodology for different aspect ratios $(a/b)$ is presented in Table 4, showing a good agreement with Alibeigloo et al. [31] and Reddy [32].

4.3. Numerical Results and Discussion. Several examples including new results for arbitrarily laminated plates with elastically restrained edges are presented in this section. The elastic properties of the composite materials used here are those shown in Table 1. The influence of different values of fiber volume ratios $(k_f)$ is analyzed in several figures and tables.

Values of the first four frequency parameters $\overline{\omega}_i = \omega_i a^2 \sqrt{\rho/E_i}$ for square thick $(a/h = 10)$ and thin $(a/h = 100)$ unsymmetric laminated plates are shown for increasing values of the translational restraint parameter $T_{11}^\infty$, in Tables 5(a) and 5(b). Moreover, the influence of rotational restraint parameter $R_i$ in the free vibration frequency coefficients is shown in Tables 6(a) and 6(b).

In Figures 2–4 the fundamental frequency coefficients $\overline{\omega}$ corresponding to two laminated square plates are plotted against the restraint parameters $R_i$ and $T_{11}^\infty$. Figure 2 shows the variation of $\overline{\omega}$ for various values of the rotational restraint $R_i$, while Figure 3 shows the variation of $\overline{\omega}$ for various values of the translational restraint $T_{11}^\infty$. A major increase of frequency occurs when the elastic restraint values are in the interval $0.1–50$. Figure 4 shows the variation of $\overline{\omega}$ for various values of the rotational and translational restraint parameters: (a) $R_i = 0$, $T_{11}^\infty = 0$; (b) $R_i = S$, $T_{11}^\infty = \infty$, and (c) $R_i = T_{11}^\infty = S$. The obtained curves illustrate the restraint parameters intervals for which the frequency coefficient is sensitive to $R_i$ and $T_{11}^\infty$.

To assess the influence of the aspect ratio $a/b$ in the laminated plate response, values of the first four frequency parameters $\overline{\omega}_i = \omega_i a^2 \sqrt{\rho/E_i}$ for rectangular thick $(a/h = 10)$ unsymmetric laminated plates are shown, for increasing values of the translational restraint parameter $T_{11}^\infty$ (Table 7(a)) and the rotational restraint parameter $R_i$ (Table 7(b)), considering $a/b = 1/3$ and $a/b = 1$.

Figure 5 shows the variation of $\overline{\omega}$ for various values of the rotational restraint $R_i$, while Figure 6 shows the variation of $\overline{\omega}$ for various values of the translational restraint $T_{11}^\infty$ for rectangular laminated plates.

To evaluate the effect of different fiber orientation angles $(\beta)$ and fiber volume fraction on the dynamic properties of the laminates, the variation of the first free vibration coefficient $\overline{\omega}_i$ is plotted in Figures 7 and 8, considering two lamination stacking sequences, $[\beta/–\beta]$ and $[0/\beta]$. Two boundary conditions have been included, $C_1 C_1 C_1 C_1$ in Figure 7 and $S_1 S_1 S_1 S_1$ in Figure 6. It is observed that the $[\beta/–\beta]$ laminate is more sensitive to the fiber orientation angle than $[0/\beta]$ lamination scheme. The adimensional frequency parameter is noticeable higher as the fiber volume fraction $k_f$ increases and as the boundary conditions become clamped.

Finally, the first four free vibration coefficients are presented in Table 8 to illustrate the influence of various fiber volume fractions and boundary conditions on the dynamical behavior of an unsymmetric $[0°/45°]$ laminated plate.

5. Concluding Remarks

A Ritz approach for free vibration analysis of general laminated plates with edges elastically restrained against translation and rotation is presented in this work. The study...
Table 7: (a) Frequency parameters $\omega_i$ for $[0^\circ/45^\circ]$ carbon-epoxy AS4-3501-6 (Table 1), with different translational restraint parameter $T_{wi}$, $T_{wi} = T_{ui} = R_i = 0$ ($i = 1, \ldots, 4$), and $a/h = 10$. (b) Frequency parameters $\omega_i$ for $[0^\circ/45^\circ]$ carbon-epoxy AS4-3501-6 (Table 1), with different rotational restraint parameter $R_i$, $T_{wi} = T_{ui} = T_{vi} = \infty$ ($i = 1, \ldots, 4$), and $a/h = 10$.

### (a)

| $k_f$ | Mode | Translational restraint parameter $T_{wi}$ |
|-------|------|------------------------------------------|
|       |      | 0.1 | 1 | 10 | 100 | 1000 | 10000 | 1.00E+10 |
|       |      |     |   |    |    |     |      |        |
| 0.1   | 1    | 0.158 | 0.464 | 1.043 | 2.045 | 2.778 | 2.903 | 2.917 |
| 0.2   | 2    | 0.184 | 0.574 | 1.625 | 3.592 | 4.520 | 4.704 | 4.726 |
|       | 3    | 0.255 | 0.793 | 2.167 | 3.948 | 6.915 | 7.139 | 7.149 |
|       | 4    | 1.652 | 1.823 | 2.838 | 5.127 | 7.177 | 7.419 | 7.449 |
| 0.4   | 1    | 0.199 | 0.578 | 1.268 | 2.392 | 3.046 | 3.144 | 3.156 |
|       | 2    | 0.233 | 0.723 | 1.999 | 4.175 | 5.024 | 5.169 | 5.186 |
|       | 3    | 0.322 | 0.999 | 2.649 | 4.774 | 7.759 | 8.125 | 8.148 |
|       | 4    | 1.832 | 2.071 | 3.374 | 6.021 | 7.973 | 8.173 | 8.196 |
| 0.6   | 1    | 0.235 | 0.681 | 1.487 | 2.757 | 3.455 | 3.557 | 3.569 |
|       | 2    | 0.275 | 0.853 | 2.347 | 4.837 | 5.763 | 5.915 | 5.933 |
|       | 3    | 0.380 | 1.179 | 3.115 | 5.589 | 7.759 | 8.125 | 8.148 |
|       | 4    | 2.104 | 2.394 | 3.945 | 6.992 | 9.186 | 9.408 | 9.434 |
|       |      |     |   |    |    |     |      |        |
|       |      |     |   |    |    |     |      |        |
| 0.1   | 1    | 0.111 | 0.341 | 0.904 | 1.724 | 2.239 | 2.345 | 2.364 |
| 0.2   | 2    | 0.141 | 0.443 | 1.335 | 3.326 | 4.739 | 5.023 | 5.067 |
|       | 3    | 0.171 | 0.539 | 1.627 | 3.705 | 5.946 | 6.580 | 6.673 |
|       | 4    | 1.423 | 1.542 | 2.285 | 4.768 | 7.832 | 8.845 | 8.980 |
| 0.4   | 1    | 0.140 | 0.429 | 1.108 | 2.012 | 2.501 | 2.598 | 2.617 |
|       | 2    | 0.178 | 0.559 | 1.664 | 3.964 | 5.396 | 5.649 | 5.686 |
|       | 3    | 0.216 | 0.681 | 2.037 | 4.480 | 6.688 | 7.229 | 7.313 |
|       | 4    | 1.587 | 1.731 | 2.729 | 5.680 | 8.857 | 9.749 | 9.869 |
| 0.6   | 1    | 0.165 | 0.506 | 1.296 | 2.316 | 2.847 | 2.954 | 2.976 |
|       | 2    | 0.210 | 0.660 | 1.957 | 4.587 | 6.194 | 6.469 | 6.509 |
|       | 3    | 0.256 | 0.804 | 2.400 | 5.243 | 7.625 | 8.198 | 8.292 |
|       | 4    | 1.815 | 1.989 | 3.182 | 6.582 | 10.085 | 11.034 | 11.163 |

### (b)

| $k_f$ | Mode | Rotational restraint parameter $R_{\phi x, \phi y}$ |
|-------|------|-----------------------------------------------|
|       |      | a/b = 2 |
| 0.1   | 1    | 3.444 | 3.858 | 4.588 | 4.806 | 4.833 | 4.835 | 4.836 |
| 0.2   | 2    | 5.467 | 5.742 | 6.278 | 6.450 | 6.472 | 6.474 | 6.474 |
|       | 3    | 8.161 | 8.349 | 8.750 | 8.890 | 8.908 | 8.909 | 8.910 |
|       | 4    | 8.357 | 8.741 | 9.485 | 9.722 | 9.751 | 9.754 | 9.754 |
| 0.4   | 1    | 3.854 | 4.314 | 4.989 | 5.183 | 5.183 | 5.183 | 5.186 |
|       | 2    | 6.061 | 6.362 | 6.855 | 7.009 | 7.009 | 7.011 | 7.011 |
|       | 3    | 8.971 | 9.187 | 9.552 | 9.678 | 9.678 | 9.678 | 9.678 |
|       | 4    | 8.987 | 9.437 | 10.142 | 10.37 | 10.347 | 10.349 | 10.349 |
| 0.6   | 1    | 4.421 | 4.946 | 5.682 | 5.866 | 5.887 | 5.889 | 5.889 |
|       | 2    | 6.961 | 7.309 | 7.858 | 8.006 | 8.024 | 8.026 | 8.026 |
|       | 3    | 10.160 | 10.584 | 10.999 | 11.124 | 11.139 | 11.140 | 11.141 |
|       | 4    | 10.352 | 10.720 | 11.512 | 11.709 | 11.732 | 11.734 | 11.735 |
includes the effective elastic moduli of each lamina obtained using the Mori-Tanaka mean field theory, which allows taking into account the influence of the fiber volume ratios and the elastic properties of the components (fiber and matrix) into the vibration behavior. The formulation is based on the first-order shear deformation theory, and the generalized displacements are approximate using sets of characteristic orthogonal polynomials generated by the Gram-Schmidt procedure. The consideration of all possible rotational and translational restraints allows generating any classical
boundary condition, only approaching the corresponding spring parameter to zero or infinity. The algorithm is computationally efficient, and the solutions are stables and convergent. Close agreement with existing results in the literature is shown and new results are presented in tables and figures which could be useful for design and optimization problems of general long fiber-reinforced laminated plates.

Appendix

The matrices $K$ and $M$ in (18) are given by

$$
[K] = \begin{bmatrix}
K_{i,j,k,h}^{uu} & K_{i,j,k,h}^{uv} & K_{i,j,k,h}^{uw} & K_{i,j,k,h}^{uh} \\
K_{i,j,k,h}^{vu} & K_{i,j,k,h}^{vv} & K_{i,j,k,h}^{vw} & K_{i,j,k,h}^{vh} \\
K_{i,j,k,h}^{wu} & K_{i,j,k,h}^{wv} & K_{i,j,k,h}^{ww} & K_{i,j,k,h}^{wh} \\
K_{i,j,k,h}^{uh} & K_{i,j,k,h}^{vh} & K_{i,j,k,h}^{wh} & K_{i,j,k,h}^{kh}
\end{bmatrix},
$$

(A.1)

where

$$
3K_{i,j,k,h}^{uu} = A_{11} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ A_{16} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ A_{66} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ A_{26} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy,
$$

$$
K_{i,j,k,h}^{uv} = A_{12} \int \int p_i^{(v)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ A_{16} \int \int p_i^{(v)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ A_{66} \int \int p_i^{(v)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ A_{26} \int \int p_i^{(v)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy,
$$

$$
K_{i,j,k,h}^{vw} = 0,\quad K_{i,j,k,h}^{uw} = 0,
$$

$$
K_{i,j,k,h}^{uh} = B_{11} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ B_{16} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ B_{66} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy
+ B_{26} \int \int p_i^{(u)} p_k^{(u)} q_j^{(u)} q_h^{(u)} \, dx \, dy,
$$

$$
K_{i,j,k,h}^{vh} = B_{12} \int \int p_i^{(v)} p_k^{(v)} q_j^{(v)} q_h^{(v)} \, dx \, dy
+ B_{16} \int \int p_i^{(v)} p_k^{(v)} q_j^{(v)} q_h^{(v)} \, dx \, dy
+ B_{66} \int \int p_i^{(v)} p_k^{(v)} q_j^{(v)} q_h^{(v)} \, dx \, dy
+ B_{26} \int \int p_i^{(v)} p_k^{(v)} q_j^{(v)} q_h^{(v)} \, dx \, dy,
$$

$$
K_{i,j,k,h}^{wh} = B_{22} \int \int p_i^{(w)} p_k^{(w)} q_j^{(w)} q_h^{(w)} \, dx \, dy
+ B_{26} \int \int p_i^{(w)} p_k^{(w)} q_j^{(w)} q_h^{(w)} \, dx \, dy
+ B_{66} \int \int p_i^{(w)} p_k^{(w)} q_j^{(w)} q_h^{(w)} \, dx \, dy
+ B_{26} \int \int p_i^{(w)} p_k^{(w)} q_j^{(w)} q_h^{(w)} \, dx \, dy,
$$

$$
K_{i,j,k,h}^{kh} = 0.
$$
\[
K_{ijkh}^{\phi\phi} = K A_{44} \int_{R} \left[ p_i^{(w)} p_k^{(w)} q_j^{(w)} q_h^{(w)} \right] dx dy \\
+ D_{22} \int_{R} \left[ p_i^{(\phi)} p_k^{(\phi)} q_j^{(\phi)} q_h^{(\phi)} \right] dx dy \\
+ D_{26} \int_{R} \left[ p_i^{(\phi)} p_k^{(\phi)} q_j^{(\phi)} q_h^{(\phi)} \right] dx dy \\
+ D_{66} \int_{R} \left[ p_i^{(\phi)} p_k^{(\phi)} q_j^{(\phi)} q_h^{(\phi)} \right] dx dy \\
+ \frac{r_1}{2} \int_{0}^{a} \left[ p_i^{(\phi)} p_k^{(\phi)} q_j^{(\phi)} q_h^{(\phi)} \right] y=0 dx \\
+ \frac{r_3}{2} \int_{0}^{a} \left[ p_i^{(\phi)} p_k^{(\phi)} q_j^{(\phi)} q_h^{(\phi)} \right] y=b dx.
\]

(A.2)

with \( K = 5/6 \)

\[
[M] = \begin{bmatrix}
M_{ijkh}^{uu} & 0 & 0 & 0 \\
0 & M_{ijkh}^{\phi\phi} & 0 & 0 \\
0 & 0 & M_{ijkh}^{ww} & 0 \\
sym & M_{ijkh}^{\psi\psi} & M_{ijkh}^{\psi\phi} & M_{ijkh}^{\psi\psi}
\end{bmatrix}
\]

(A.3)

Acknowledgment

The authors wish to thank the economic support of CONICET (PIP no. 0105/2010) and CIUNSa.

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