Appendix B

The bulk of this appendix is the output from a Maple worksheet that contains many of the “gritty” details of our second order Schwarzschild calculation. A Maple worksheet that reproduces the entire calculation is available for download at the GRTensor website.

\[ \texttt{restart:} \]
\[ \texttt{grtw():} \]
\[ \texttt{GRTensorII Version 1.70 (R5)} \]
\[ 31 \text{ May 1998} \]
\[ \textit{Developed by Peter Musgrave, Denis Pollney and Kayll Lake} \]
\[ \textit{Copyright 1994 – 1998 by the authors.} \]
\[ \texttt{Latest version available from: \url{http://astro.queensu.ca/~grtensor/}} \]
\[ \texttt{mine():} \]
\[ \texttt{read ‘myutils.mpl’;} \]
\[ \texttt{qload(qschw):} \]
\[ \textit{Default spacetime = qschw} \]
\[ \textit{For the qschw spacetime:} \]
\[ \textit{Coordinates} \]
\[ x^{a} = [r, \theta, \phi, t] \]
\[ \textit{Line element} \]
\[ ds^{2} = \frac{(1 + \frac{1}{2} (\varepsilon h_{2}(r, t) + \varepsilon^{2} H_{2}(r, t))) \%1) \, d\theta^{2} + r^{2} (1 + \frac{1}{2} (\varepsilon k(r, t) + \varepsilon^{2} K(r, t))) \%1) \, d\phi^{2}}{1 - 2 \frac{m}{r}} + (\varepsilon h_{1}(r, t) + \varepsilon^{2} H_{1}(r, t)) \%1 \, d\theta \, d\phi \]
\[ - (1 - 2 \frac{m}{r}) (1 - \frac{1}{2} (\varepsilon h_{0}(r, t) + \varepsilon^{2} H_{0}(r, t))) \%1) \, dt^{2} \]
\[ \%1 := 3 \cos(\theta)^{2} - 1 \]
\[ \textit{Constraints} = [\varepsilon^{3} = 0, \varepsilon^{4} = 0, \varepsilon^{5} = 0, \varepsilon^{6} = 0, \varepsilon^{7} = 0, \varepsilon^{8} = 0, \varepsilon^{9} = 0, \varepsilon^{10} = 0] \]

As can be seen from the form of the metric, the lower case functions are the first order perturbations, and the upper case functions are the second order perturbations. To begin our analysis we calculate the exact contravariant metric tensor, which is then quadratically perturbed using the \texttt{quadpert()} routine.

\[ \texttt{grcalc(g(up,up));} \]
\[ \textit{CPU Time} = .394 \]
> gralter(g(up,up),quadpert,simplify,factor);

Component simplification of a GRTensorII object:

Applying routine quadpert to object g(up,up)
Applying routine simplify to object g(up,up)
Applying routine factor to object g(up,up)

\[ \text{CPU Time } = .305 \]

> grdisplay(g(up,up));

For the qschw spacetime:

Contravariant metric tensor
\[ g(up, up) \]

\[ g^{rr} = \frac{1}{4}((r - 2m)(-4 + 6h_2(r, t)e\cos(\theta)^2 - 2e h_2(r, t) - e^2 h_2(r, t)^2 + 6e^2 h_2(r, t)^2 \cos(\theta)^2 - 9e^2 h_2(r, t)^2 \cos(\theta)^4 - 6e^2 h_1(r, t)^2 \cos(\theta)^2 + 9e^2 h_1(r, t)^2 \cos(\theta)^4 + 6e^2 H_2(r, t) \cos(\theta)^2 - 2e^2 H_2(r, t) + e^2 h_1(r, t)^2))/r \]

\[ g^{rt} = -\frac{1}{4}e((3 \cos(\theta)^2 - 1)h_1(r, t)h_2(r, t)e\cos(\theta)^2 - 3e h_1(r, t)h_0(r, t)e\cos(\theta)^2 - 2e H_1(r, t) + e h_1(r, t)h_0(r, t) - e h_1(r, t)h_2(r, t) - 2h_1(r, t)) \]

\[ g^{\theta \theta} = -\frac{1}{4}((4 + 6e k(r, t)e\cos(\theta)^2 - 2e k(r, t) - 9e^2 e\cos(\theta)^4 k(r, t)^2 + 6e^2 k(r, t)^2 e\cos(\theta)^2 + 6e^2 K(r, t) e\cos(\theta)^2 - e^2 k(r, t)^2 - 2e^2 K(r, t))/r^2 \]

\[ g^{\phi \phi} = \frac{1}{4}((4 + 6e k(r, t)e\cos(\theta)^2 - 2e k(r, t) - 9e^2 e\cos(\theta)^4 k(r, t)^2 + 6e^2 k(r, t)^2 e\cos(\theta)^2 + 6e^2 K(r, t) e\cos(\theta)^2 - e^2 k(r, t)^2 - 2e^2 K(r, t))/r^2(1 + \cos(\theta))(\cos(\theta) + 1)) \]

\[ g^{tt} = \frac{1}{4}((4 + 6e h_0(r, t) e\cos(\theta)^2 - 2e h_0(r, t) - 6e^2 h_0(r, t)^2 e\cos(\theta)^2 + 9e^2 e\cos(\theta)^4 h_0(r, t)^2 + 6e^2 h_1(r, t)^2 e\cos(\theta)^2 - 9e^2 h_1(r, t)^2 e\cos(\theta)^4 - e^2 h_1(r, t)^2 - 2e^2 H_1(r, t))/(-r + 2m) \]

With these quadratically perturbed contra/co-variant components of the metric we can now calculate the perturbed Ricci tensor to second order.

> grcalcaltor(R(dn,dn),13);

Simplification will be applied during calculation.
Applying routine 'Apply constraints repeatedly' to object Chr(dn,dn,dn)

Applying routine 'Apply constraints repeatedly' to object Chr(dn,dn,up)

Applying routine 'Apply constraints repeatedly' to object R(dn, dn)

\[ CPU Time = 5.811 \]

\[ > \text{gralter}(\text{R}(\text{dn}, \text{dn}), \text{expand}); \]

Component simplification of a GRTensorII object:

Applying routine expand to object R(dn, dn)

\[ CPU Time = .211 \]

\[ > \text{grdisplay}(\text{R}(\text{dn}, \text{dn})); \]

For the qschw spacetime:

\[
\text{Covariant Ricci}
\]

\[
R_{rr} = 86717 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{r\theta} = 28579 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{rt} = 53004 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{\theta\theta} = 87057 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{\theta t} = 20438 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{\phi\phi} = 72053 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{tt} = 82672 \text{ words. Exceeds grOptionDisplayLimit}
\]

For completeness, the definitions of the Legendre function, \( P_2(\cos(\theta)) \), that we will need to calculate the projections listed in Table 1, and the standard Zerilli substitutions, that we will need to eliminate \( H_1 \), and, \( K \), appear below.

\[
> p2 := \frac{5}{2} \times (3 \times \cos(\theta)^2 - 2 - 1) / 2;
\]

\[
p2 := \frac{15}{4} \cos(\theta)^2 - \frac{5}{4}
\]

\[
> \text{newH}(r, t) :=
\]

\[
(2 \times r^2 - 6 \times m \times r - 3 \times m^2)/(r - 2 \times m)/(2 \times r + 3 \times m) \times \text{chi}(r, t) + r^2/(r - 2 \times m) \times \text{eta}(r, t);
\]

\[
\text{newH}(r, t) := \frac{(2 r^2 - 6 m r - 3 m^2) \chi(r, t) + r^2 \eta(r, t)}{(r - 2 m)(2 r + 3 m)}
\]

\[
> \text{Kdot}(r, t) := 6 \times (r^2 + m \times r + m^2) / r^2 / (2 \times r + 3 \times m) \times \text{chi}(r, t) + \text{eta}(r, t);
\]

\[
\text{Kdot}(r, t) := 6 \times \frac{r^2 + m r + m^2}{r^2 (2 r + 3 m)} \chi(r, t) + \eta(r, t)
\]
Following the prescription set out in Table 1, we can now extract the linear contributions to the seven non-trivial Ricci components. The resulting PDEs are called \textit{leqn}_1...7.

\begin{verbatim}
> a1:=coeff(collect(grcomponent(R(dn,dn),[theta,theta]),epsilon),epsilon,1):
> g1:=coeff(collect(grcomponent(R(dn,dn),[phi,phi]),epsilon),epsilon,1):
> ag1:=int(expand((a1-g1/sin(theta)^2)*(2*cot(theta)*diff(p2,theta)+2*(2+1)*p2)*sin(theta)),theta=0..Pi);
> leqn[1]:={h[2](r,t) = solve(ag1,h[2](r,t))};
> a2:=subs(leqn[1],a1):
> g2:=subs(leqn[1],g1):
> grmap(R(dn,dn),subs,leqn[1],'x');
> b1:=coeff(collect(grcomponent(R(dn,dn),[theta,r]),epsilon),epsilon,1):
> leqn[2]:=termsimp(collect(int(b1*diff(p2,theta)*sin(theta),theta=0..Pi),lperts));
> c1:=coeff(collect(grcomponent(R(dn,dn),[theta,t]),epsilon),epsilon,1):
> leqn[3]:=termsimp(collect(int(c1*diff(p2,theta)*sin(theta),theta=0..Pi),lperts));
> d1:=coeff(collect(grcomponent(R(dn,dn),[t,r]),epsilon),epsilon,1):
> leqn[4]:=termsimp(collect(int(d1*p2*sin(theta),theta=0..Pi),lperts));
\end{verbatim}

Applying routine \texttt{subs} to \texttt{R(dn,dn)}

Applying routine \texttt{simplify} to \texttt{R(dn,dn)}, \texttt{expand}.

Component simplification of a \texttt{GRTensorII} object:

\begin{verbatim}
treating routine \texttt{simplify} to object \texttt{R(dn,dn)}
CPU Time = 14.021
\end{verbatim}

\begin{verbatim}
> leqn[2] := -2 m h_0(r,t) \left[ \frac{1}{-r + 2 m} \frac{\partial}{\partial r} h_0(r,t) \right] - \left( \frac{\partial}{\partial r} k(r,t) \right) + \left( \frac{\partial}{\partial t} h_0(r,t) \right) \left[ \frac{r}{-r + 2 m} \right] \left( \frac{\partial}{\partial t} k(r,t) \right)\right.
\end{verbatim}

\begin{verbatim}
> leqn[3] := \frac{2}{r^2} \frac{h_1(r,t)}{m} \left( \frac{\partial}{\partial t} h_0(r,t) \right) - \left( \frac{\partial}{\partial t} k(r,t) \right) \left( \frac{\partial}{\partial r} h_0(r,t) \right) - \left( \frac{\partial}{\partial t} k(r,t) \right) - \left( \frac{\partial}{\partial t} h_0(r,t) \right) \left( \frac{\partial}{\partial r} k(r,t) \right)\right.
\end{verbatim}

\begin{verbatim}
> d1 := \frac{3}{r} \frac{h_1(r,t)}{m} \left( \frac{\partial^2}{\partial t \partial r} k(r,t) \right) + \left( \frac{\partial}{\partial t} h_0(r,t) \right) \left( \frac{\partial}{\partial r} h_0(r,t) \right) - \left( \frac{\partial}{\partial t} h_0(r,t) \right) \left( \frac{\partial}{\partial r} k(r,t) \right)\right.
\end{verbatim}

24
> e1:=coeff(collect(grcomponent(R(dn,dn),[r,r]),epsilon),epsilon,1):
> leqn[5]:=termsimp(collect(int(e1*p2*sin(theta),theta=0..Pi),lperts));

\[
\begin{align*}
\text{leqn}_5 & := -3 \frac{h_0(r, t)}{(-r + 2 m) r} + \frac{1}{2} \left( \frac{\partial^2}{\partial t^2} h_0(r, t) \right) - \left( \frac{\partial^2}{\partial r^2} k(r, t) \right) + \frac{r}{-r + 2 m} \frac{\partial^2 h_1(r, t)}{-r + 2 m} + \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} h_0(r, t) \right)
- \frac{\partial^2}{\partial t^2} h_0(r, t) - \frac{m (-r + 2 m) \left( \frac{\partial}{\partial r} k(r, t) \right)}{r^3} - m \frac{\partial^2 h_1(r, t)}{r^2}
\end{align*}
\]

> f1:=coeff(collect(grcomponent(R(dn,dn),[t,t]),epsilon),epsilon,1):
> leqn[6]:=termsimp(collect(int(f1*p2*sin(theta),theta=0..Pi),lperts));

\[
\begin{align*}
\text{leqn}_6 & := -3 \frac{(-r + 2 m) h_0(r, t)}{r^3} - \frac{1}{2} (-r + 2 m) \left( \frac{\partial^2}{\partial t^2} h_0(r, t) \right) - \left( \frac{\partial^2}{\partial r^2} k(r, t) \right)
- \frac{(-r + 2 m) \left( \frac{\partial}{\partial r} h_1(r, t) \right)}{r^3} - \frac{1}{2} \left( \frac{\partial^2}{\partial t^2} h_0(r, t) \right) + \frac{(-r + 2 m) \left( \frac{\partial}{\partial r} h_0(r, t) \right)}{r^2}
- m (-r + 2 m) \left( \frac{\partial}{\partial r} k(r, t) \right) - \frac{(3 m - 2 r) \left( \frac{\partial}{\partial r} h_1(r, t) \right)}{r^2}
\end{align*}
\]

> leqn[7]:=termsimp(collect(int((a2+g2*sin(theta)^2)*p2*sin(theta),theta=0..Pi),lperts));

\[
\text{leqn}_7 := 4 k(r, t) + 2 h_0(r, t) + \left( \frac{\partial^2}{\partial t^2} k(r, t) \right) (-r + 2 m) r - \frac{r^3 \left( \frac{\partial^2}{\partial r^2} k(r, t) \right)}{-r + 2 m} - 2 \left( \frac{\partial}{\partial r} h_0(r, t) \right) (-r + 2 m)
+ 2 \left( \frac{\partial}{\partial r} k(r, t) \right) (3 m - 2 r) - 2 \left( \frac{\partial}{\partial r} h_1(r, t) \right) r
\]

While this completes our formal analysis of the linear problem, these seven equations, \(\text{leqn}_{1..7}\), must now be combined in some unobvious ways in order to facilitate the reduction of the source terms in the second order calculations. These details can be found in the “complete” worksheet at the GRTensor web-site.

In order to determine the second order PDEs \(\text{geqn}_{1..7}\) we simply repeat the linear analysis, but now selecting only the \(\epsilon^2\) components of the Ricci tensor. The second order perturbation functions will, necessarily, appear in these PDEs in exactly the same manner as the linear perturbation functions appear in the \(\text{leqn}_i\) equations.

> a1:=simplify(coeff(collect(
> grcomponent(R(dn,dn),[theta,theta]),epsilon),epsilon,2),trig):
> gg1:=simplify(expand(1/sin(theta)^2)*coeff(collect(
> grcomponent(R(dn,dn),[phi,phi]),epsilon),epsilon,2),trig):
> zz1:=simplify(expand(gg1-a1));

\[
\text{zz1} := \frac{3}{2} H_2(r, t) - \frac{3}{2} H_0(r, t) - \frac{3}{2} h_1(r, t)^2 + \frac{3}{2} h_0(r, t)^2 + \frac{21}{2} h_1(r, t)^2 \cos(\theta)^2 - \frac{21}{2} \cos(\theta)^2 h_0(r, t)^2
+ 9 \cos(\theta)^4 h_0(r, t)^2 + \frac{3}{2} H_0(r, t) \cos(\theta)^2 - \frac{3}{2} H_2(r, t) \cos(\theta)^2
- 9 h_1(r, t)^2 \cos(\theta)^4
\]
zz2:=int(zz1*(2*cot(theta)*diff(p2,theta)+2*(2+1)*p2)*sin(theta), theta=0..Pi);

zz2 := -12 H_2(r, t) - \frac{1}{7} h_0(r, t)^2 + \frac{1}{7} h_1(r, t)^2 + 12 H_0(r, t)

qeqn1:= \{ H[2](r, t) = solve(zz2, H[2](r, t)) \};

qeqn1 := \{ H_2(r, t) = -\frac{1}{7} h_0(r, t)^2 + \frac{1}{7} h_1(r, t)^2 + H_0(r, t) \}

qeqn[1]:=qeqn1;

qeqn[1] := \{ H_2(r, t) = -\frac{1}{7} h_0(r, t)^2 + \frac{1}{7} h_1(r, t)^2 + H_0(r, t) \}

Applying routine subs to R(dn,dn)

aa1:=simplify(coeff(collect(grcomponent(R(dn,dn),[theta,theta]),epsilon),epsilon,2),trig):

gg1:=simplify(expand(1/sin(theta)^2*(coeff(collect(grcomponent(R(dn,dn),[phi,phi]),epsilon),epsilon,2))),trig):

qeqn[2]:=termsimp(collect(int((gg1+aa1)*p2*sin(theta),theta=0..Pi),qperts)):

bb1:=simplify(coeff(collect(grcomponent(R(dn,dn),[theta,r]),epsilon),epsilon,2),trig):

qeqn[3]:=termsimp(collect(int(bb1*diff(p2,theta)*sin(theta),theta=0..Pi),qperts)):

ccl1:=simplify(coeff(collect(grcomponent(R(dn,dn),[t,theta]),epsilon),epsilon,2),trig):

qeqn[4]:=termsimp(collect(1/3*int(ccl1*diff(p2,theta)*sin(theta),theta=0..Pi),qperts)):

dd1:=simplify(coeff(collect(grcomponent(R(dn,dn),[r,t]),epsilon),epsilon,2),trig):

qeqn[5]:=termsimp(collect(int(dd1*p2*diff(H[0](r,t),t)*sin(theta),theta=0..Pi),qperts)):

eel1:=simplify(coeff(collect(grcomponent(R(dn,dn),[r,r]),epsilon),epsilon,2),trig):

qeqn[6]:=termsimp(collect(expand(int(eel1*p2*sin(theta),theta=0..Pi)),qperts)):

qsub1:= \{ \text{diff}(H[0](r,t),t) = solve(qeqn[4],\text{diff}(H[0](r,t),t)) \};

qsub1 := \{ \text{diff}(H_0(r,t),t) = \frac{1}{7} (14 H_1(r, t) m + \frac{14}{\partial r} H_1(r, t) m r - 7 \frac{\partial}{\partial t} K(r, t))^2 + 3 h_0(r, t) \frac{\partial}{\partial r} h_0(r, t)^2 - 3 h_1(r, t) \frac{\partial}{\partial r} h_1(r, t)^2 + 2 k(r, t) (\frac{\partial}{\partial t} k(r, t))^2)/r^2 \}

d := simplify(coeff(collect(grcomponent(R(dn,dn),[theta,r]),epsilon),epsilon,2),trig):

qeqn[1];

\{ H_2(r, t) = -\frac{1}{7} h_0(r, t)^2 + \frac{1}{7} h_1(r, t)^2 + H_0(r, t) \}
The derivation of the first Zerilli equation begins with the elimination of $\dot{H}_0$, using $qeqn_4$, in the $R^{(2)}_{\text{rt}}$ component of the Ricci tensor. We then make the previously defined Zerilli substitutions, and make various substitutions involving the $leqn_i$ equations. The resulting expression is called etaeqn in our Maple worksheet. To simplify this rather awkward expression, we employ a new routine, mcollect(), that collects specified terms that are multiplied together. This routine could, for example, collect together terms that have a factor of, $ab$, but not, $a^2$, or $b^2$. A more complete description of this routine appears in Appendix A. (Readers who are familiar with the computer algebra environment will, no doubt, already see the utility of such a procedure.)
It is now a simple matter to put these results together to arrive at the first Zerilli equation for the second order calculation:

\[
\eta_{\text{final}} := \eta(r, t) = \left( \frac{\partial}{\partial r} \chi(r, t) \right) \left( -r + 2m \right) \frac{1}{r} - \frac{1}{7} \left( r - 2m \right) \left( -h_0(r, t) \left( \frac{\partial}{\partial t} h_0(r, t) \right) \right.
\]

\[
+ 2k(r, t) \left( \frac{\partial}{\partial t} h_0(r, t) \right) + r \left( \frac{\partial}{\partial r} k(r, t) \right) \left( \frac{\partial}{\partial t} h_0(r, t) \right) + 2k(r, t) \left( \frac{\partial}{\partial t} k(r, t) \right) + 4 \frac{h_1(r, t)}{r} \frac{k(r, t)}{r} \left( r - 2m \right) \frac{1}{r^2} h_0(r, t)
\]

\[
- 2 - 2m \left( \frac{\partial}{\partial r} k(r, t) \right) \frac{1}{r} h_0(r, t) + h_1(r, t) \left( \frac{\partial}{\partial t} h_1(r, t) \right) - 2 \left( -r + 2m \right) \frac{h_1(r, t)}{r^2} \frac{h_0(r, t)}{r^2} \right)
\]

This agrees with the result found in [1].

The development of the second order Zerilli “wave equation” follows a similar procedure to that of the eta_{\text{final}} result. We first take the time derivative of eqn_2, eliminate \dot{H}_0 and \dot{H}'_0 using eqn_4, make the Zerilli substitutions, and substitute for \eta from our first Zerilli result. We then repeat this process for eqn_3 and add the resulting equations such that the \ddot{\chi} term vanishes. The result of these operations, which is called chieqn in our worksheet, is essentially the raw form of the result we are seeking. This expression is, however, rather large.

\[
\text{atmp1} := \text{diff(eqns[2], t)};
\]
\[
\text{atmp2} := \text{termsimp(collect(expand(}
\]
\[
\text{subs(qsub1, diff(qsub1, r), atmp1)), } \{ \text{diff(K(r, t), t, t, t)} \})
\]
\[
\text{atmp3} := \text{termsimp(collect(expand(eval(subs(H[1](r, t)=newH(r, t),}
\]
\[
\text{diff(K(r, t), t)=Kdot(r, t), diff(K(r, t), t,}
\]
\[
\text{r)=diff(Kdot(r, t), r), diff(K(r, t), t, r, r)=diff(Kdot(r, t), r, r, atmp2)), zfuncs)}
\]
\begin{verbatim}
> atmp4:=termsimp(collect(expand(eval(subs(etafinal,atmp3))),zfuncs)):
> coefa:=simplify(coeff(atmp4,diff(chi(r,t),t,r,t),1));
\[
coefa := -r^2
\]
> btmp1:=termsimp(collect(expand(
    subs(qsub1,diff(qsub1,r),diff(qeqn[3],t))),
    {diff(H[1](r,t),t,t),
     diff(K(r,t),t,t,t)})):
> btmp2:=termsimp(collect(expand(eval(
    subs(H[1](r,t)=newH(r,t),diff(K(r,t),t)=Kdot(r,t),
    diff(K(r,t),r,t)=diff(Kdot(r,t),r),btmp1))),zfuncs)):
> btmp3:=termsimp(collect(expand(eval(subs(etafinal,btmp2))),zfuncs)):
> coefb:=simplify(coeff(btmp3,diff(chi(r,t),r,t,t),1));
\[
coefb := 3 r^2 - r + 2 m
\]
> ftmp1:=termsimp(collect(expand(atmp4/coefa-btmp3/coefb),
    {chi(r,t),diff(chi(r,t),r),diff(chi(r,t),r,r),diff(chi(r,t),t,t),
     diff(chi(r,t),r,r,r)})):
> ftmp2:=termsimp(collect(expand(eval(
    subs(k(r,t)=0,h[1](r,t)=0,h[0](r,t)=0,ftmp1)))),
    {chi(r,t),
     diff(chi(r,t),r),diff(chi(r,t),r,r),diff(chi(r,t),t,t),
     diff(chi(r,t),r,r,r)})):
> chi1:=termsimp(collect(
    diff(chi(r,t),t,t)-solve(ftmp1,diff(chi(r,t),t,t)),zfuncs)):
> chieqn:=ftmp1:
> nops(chieqn);
\end{verbatim}

Although, at this point we can at least verify that it reproduces the correct linear Zerilli equation when the source terms are set to zero.

\begin{verbatim}
> termsimp(collect(expand(eval(subs(k(r,t)=0,h[1](r,t)=0,h[0](r,t)=0,
    chieqn/coef(chieqn,diff(chieqn,diff(chi(r,t),t,t),1))),
    {chi(r,t),diff(chi(r,t),r),
    diff(chi(r,t),r,r),diff(chi(r,t),t,t),
    diff(chi(r,t),r,r,r)});
\end{verbatim}

\begin{align*}
- \frac{(\frac{\partial^2}{\partial r^2} \chi(r,t)) (-r + 2 m)^2}{r^2} + \frac{(\frac{\partial^2}{\partial t^2} \chi(r,t))}{r^2} - 6 \frac{(-r + 2 m) (3 m^3 + 6 r m^2 + 4 r^2 m + 4 r^3) \chi(r,t)}{r^4 (2 r + 3 m)^2} + 2 \frac{(\frac{\partial}{\partial r} \chi(r,t)) (-r + 2 m) m}{r^3}
\end{align*}
This is just the linear Zerilli wave equation with the quadrapole potential, as required.

In order to make contact with the results presented in [1], we now work with the “renormalized” function $\zeta$.

\[
\chi(r,t) = \zeta(r,t) + 2/7 (r^2/(2r+3m) k(r,t) \frac{\partial}{\partial t} k(r,t) + k(r,t)^2)
\]

This substitution has considerably reduced the number of terms in the expression. We now eliminate $h_0$ in favor of $\psi$ using:

\[
h_0(r,t) = 3 \frac{(\psi(r,t) - 1/3 r k(r,t))(2r+3m)}{r(r-2m)} + r(\frac{\partial}{\partial r} k(r,t))
\]

After making this substitution, and a fairly extensive simplification of the source terms using the linear perturbation equations\footnote{While we have omitted these details here, they can be found in the full Maple worksheet at the GRTensor website.}, we arrive at the unsimplified answer $\zeta_{eqn}$.

Here again we make use of the `mcollect()` routine to simplify our result.
We now make one final set of substitutions, $\mu = (r - 2m)$, and $\lambda = (2r + 3m)$, and extract the source terms from the second order Zerilli wave equation:

$$zeta_{source} := \text{subs}\left( (r-2m)=\mu, (2m-r)=-\mu, (2r+3m)=\lambda, \right)$$

$$k\text{factor}(\text{eval}(\text{subs}(\zeta(r,t)=0,-\text{final17})), 12/7*(r-2m)^3/(2r+3m));$$

$$zeta_{source} := \frac{12}{7} \mu^3 \left( -\frac{1}{3} (7m-3r) \left( \frac{\partial}{\partial t} \psi(r,t) \right) \left( \frac{\partial^2}{\partial r^2} \psi(r,t) \right) \right)$$

$$+ \frac{(8r^2 + 12mr + 7m^2) \psi(r,t)}{\mu r^4 \lambda}$$

$$- \frac{(2r^3 + 4r^2m + 9r \lambda + 6m^2)}{r^6 \lambda} \psi(r,t) \%1$$

$$- \frac{12}{\mu^2 r^6 \lambda} \left( (2r^5 - 9r^4m - 6r^3m^2 + 2r^2m^3 + 15rm^4 + 15m^5) \psi(r,t)^2 \right)$$

$$+ \frac{112r^5 + 480r^4m + 692r^3m^2 + 762r^2m^3 + 441rm^4 + 144m^5}{\mu^2 r^6 \lambda} \psi(r,t) \frac{\partial^2}{\partial r^2} \psi(r,t)$$

$$- \frac{4}{\mu^2 r^3} \left( r^2 + mr + m^2 \right) \left( \frac{\partial}{\partial r} \psi(r,t) \right) \left( \frac{\partial^2}{\partial r^2} \psi(r,t) \right)$$

$$+ \frac{1}{3} \frac{\mu \%1^2 \lambda}{r^4}$$

$$+ \frac{1}{3} \left( 12r^3 + 36r^2 + 59rm^2 + 90m^3 \right) \left( \frac{\partial}{\partial r} \psi(r,t) \right)^2$$

$$- \frac{2}{\mu r^5} \left( 32r^5 + 88r^4m + 296r^3m^2 + 310r^2m^3 + 561rm^4 + 270m^5 \right) \psi(r,t) \left( \frac{\partial}{\partial r} \psi(r,t) \right)$$

$$- \frac{1}{3} \frac{\mu \%1 \lambda}{r^4}$$

$$+ \frac{1}{3} \left( \frac{\partial}{\partial r} \psi(r,t) \right) \left( \frac{\partial^2}{\partial r^2} \psi(r,t) \right)$$

$$- \frac{m}{3} \frac{\mu \%1 \lambda}{r^3}$$

$$+ \frac{1}{3} \left( \frac{\partial}{\partial r} \psi(r,t) \right) \left( \frac{\partial^2}{\partial r^2} \psi(r,t) \right)$$

$$- \frac{12}{\mu r^5} \left( r^2 + mr + m^2 \right) \left( \frac{\partial}{\partial r} \psi(r,t) \right)^2 \frac{\partial}{\partial r} \psi(r,t)$$

$$+ \frac{1}{3} \frac{\mu \%1 \lambda}{r^4}$$

$$- \frac{12}{\mu r^5} \left( r^2 + mr + m^2 \right) \left( \frac{\partial}{\partial r} \psi(r,t) \right)^2 \frac{\partial}{\partial r} \psi(r,t)$$

$$+ \frac{1}{3} \frac{\mu \%1 \lambda}{r^4}$$

$$- \frac{12}{\mu r^5} \left( r^2 + mr + m^2 \right) \left( \frac{\partial}{\partial r} \psi(r,t) \right)^2 \frac{\partial}{\partial r} \psi(r,t)$$

$$- \frac{1}{3} \frac{\mu \%1 \lambda}{r^4}$$

We now make one final set of substitutions, $\mu = (r - 2m)$, and $\lambda = (2r + 3m)$, and extract the source terms from the second order Zerilli wave equation:
\%1 := \frac{\partial^2}{\partial r^2} \psi(r, t)

This is the effective source term for the second order Zerilli equation.
Abstract

This article outlines our derivation of the second order perturbations to a Schwarzschild black hole, highlighting our use of, and necessary reliance on, computer algebra. The particular perturbation scenario that is presented here is the case of the linear quadrupole seeding the second order quadrupole. This problem amounts to finding the second order Zerilli wave equation, and in particular the effective source term due to the linear quadrupole. With one minor exception, our calculations confirm the earlier findings of Gleiser, et. al. On route to these results we also illustrate that, with the aid of computer algebra, the linear Schwarzschild problem can be solved in a very direct manner (i.e., without resorting to the usual function transformations), and it is this “direct method” that drives the higher order perturbation analysis. The calculations were performed using the GRTensorII computer algebra package, running on the Maple V platform, along with several new Maple routines that we have written specifically for these types of problems. Although we have chosen to consider only the “quadrupole-quadrupole” calculation in this article, the GRTensor environment, with the inclusion of these new routines, would allow this analysis to be repeated for a far more general problem. These routines, along with Maple worksheets that reproduce our calculations, are publicly available at the GRTensor website: www.astro.queensu.ca/~grtensor. The interested reader is invited to download and use them to reproduce our results and experiment.

Introduction and Overview

The use of perturbative methods within general relativity to investigate the stability of black holes has, at the linear level, a long and successful history. The need for such analysis is obvious. While these objects are very mathematically rich and interesting, we need to know if they are indeed physical objects. If it were found that black holes were unstable to small perturbations we certainly would not be justified in expecting them to be the ultimate end state of runaway gravitational collapse. Of course, the linear analysis has shown that black holes are in fact stable within the context of first order perturbation theory. Regge and Wheeler were the first to successfully perform a linear black hole analysis in the late 1950’s with their ground-breaking study of the (spherically symmetric) Schwarzschild metric. Their results were later clarified and refined by Zerilli in 1970, and it is this work that has become the standard for the linear analysis of the Schwarzschild problem. It is important to note that all of these calculations were carried out using the “intuitive” metric perturbation (MP) method, i.e., by analyzing perturbations of the metric tensor itself. This procedure will be described in greater detail below, but for now we simply note that this stands in stark contrast to the only successful linear analysis of the (rotating, axis symmetric) Kerr black hole. It is now not difficult to show by direct calculation that when one linearizes the Einstein equations about the Kerr vacuum solution the “regular” angular modes, i.e., spherical harmonics, do not decouple. This is nothing less than a disaster for the metric perturbation analysis. Here one can no longer uniquely project out the contributions from each of the infinite number of multipoles required to construct a general perturbation.

The successful analysis of the linearized Kerr problem, first performed by Teukolsky in the
early 1970’s, avoids this problem of angular mode coupling entirely by recasting the problem in terms of the Neuman-Penrose (NP) formalism. (The NP formalism is an example of the more general tetrad approach to general relativity, that is distinguished by the fact that all of the tetrad vectors are chosen to be null. See for example, Chandrasekhar [5].) Much of the power that the NP approach brings to linear perturbation problems is a direct consequence of its formulation of the Einstein equations. Here six of the Einstein equations are linear(!) at the outset. An immediate result of this rather remarkable fact is that any quantities that are zero in the background solution automatically appear as linearized perturbations. The essence of this approach then, is that one does not have to linearize the Einstein equations at all. It should be noted that performing a “useful” linearization of the Einstein equations about a given metric, even with the aid of modern computer algebra packages, can be a non-trivial task. While the NP and MP methods can in principle be related to each other via the perturbed tetrad vectors, which must combine to form the perturbed metric functions, the two approaches can appear to have surprisingly little in common. We shall return to this point in our concluding remarks. As a final comment on the linearized Kerr analysis we note that while the NP scheme does make the problem tractable, the analysis remains rather complex, and is, in the words of Chandrasekhar [5], “...prolixious in its complexity.”.

The ultimate goal of this article is to present our calculations of second order perturbations to the Schwarzschild metric, with an emphasis on the computer algebra methods, packages, and techniques that we employ. To properly set the stage for this end result we must first outline the details of the linear perturbation theory and its connection to the second, and higher, order perturbations. This material appears in the following section. The next section will then give an example of the linear theory at work. Here we will follow through the linear Schwarzschild calculation and see that with the aid of modern computer algebra this analysis can be carried out in a thoroughly transparent fashion. Here we will not derive the usual Zerilli results, but rather illustrate, and find the prescription for, the decoupling of the linear perturbation modes. With this material covered we will then move on to illustrate the second order calculations. The main goal of this section will be to find the form of the effective potential in the Zerilli wave equation. This is where we will see the absolute necessity of employing computer algebraic methods to perform these types of calculations. We will also see that with the addition of a few new simplification routines the analysis can be made much simpler. The final section will, of course, summarize our results and attempt to view them in the broader context of more general problems. The current direction of our ongoing research will also be discussed.

Perturbation Theory: A Short Review

At the outset it is important that we clearly define what we mean by a perturbation, and its relative order. Equally important is that we come to fully understand the linear problem. Indeed, once the linear problem has been completely solved, we will see that the second order problem is, in principle, “just” a matter of computational volume. In what follows we will limit our discussion to the MP method, and give only a brief description of the methodology. (A discussion of some of the subtler points of the associated questions of gauge can be found in [1].) We start by writing the metric as a known, background, vacuum solution with the addition of higher order terms that mimic an expansion in a “small” perturbation parameter, $\epsilon$,

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + \epsilon g_{\alpha\beta}^{(1)} + \epsilon^2 g_{\alpha\beta}^{(2)} + \cdots,$$

and substitute this “expansion” into the vacuum Einstein equations,

$$R_{\alpha\beta} = 0.$$

To produce an $n^{th}$ order perturbation we simply truncate all terms with powers higher than $O(\epsilon^n)$ in the resulting equations; so that a first order perturbation involves keeping only up to $O(\epsilon)$ terms.
This, along with our restriction to vacuum background solutions, allows us to express the linearly perturbed Einstein equations as a homogeneous linear operator, that depends on the background metric, acting on the linear perturbations:

$$\epsilon \mathcal{L}_{\,g_{\alpha\beta}}(g_{\alpha\beta}^{(1)}) = 0.$$ 

The next step in the MP process is to perform the “usual” mode expansion of the perturbation functions,

$$g^{(1)} = \sum_{i=0}^{\infty} h_i(r, t) u_i(\theta, \phi),$$

where $g^{(1)}$ represents all of the perturbed metric functions, and the $u_i(\theta, \phi)$ are typically spherical harmonics or Legendre functions. If all goes well, when these expansions are put through the linear operator the angular modes will decouple. The mathematical statement of this is,

$$\left\langle f_1(u_j) \right| \mathcal{L} \left( \sum_{i=0}^{\infty} h_i u_i \right) = f_2(h_j),$$

where $f_1$ and $f_2$ are some functions of their arguments and their derivatives. Essentially, we need a prescription for picking out the contributions that a particular $h_j$ makes to the linearized equations. As one might guess, such a unique decomposition is not always possible.

The nature of any decoupling will of course depend on the symmetries of the background metric, which determines the functional form of $\mathcal{L}$. Thus, one might also correctly guess that the spherical symmetry of the background Schwarzschild metric will allow for the separation of spherical harmonic modes, while the background Kerr metric, which obeys the far more restrictive axial symmetry, will not. As a preview of the next section we note that in the linear Schwarzschild analysis, three of the perturbed Ricci components take the form,

$$\mathcal{L}_{Schw} \left( \sum_{i=0}^{\infty} h_i u_i \right) = \sum_{i=0}^{\infty} f_2(h_i) u_i = 0,$$

so that a simple projection of $\langle u_j |$ will yield $f_2(h_j) = 0$, which is one of the desired PDEs governing the metric perturbations. For the simpler case of decoupling modes then, the angular functions are known and the problem is then to determine the differential equations that the $h_i$ obey. Whereas, in the more general case the angular eigenfunctions must themselves be determined as part of the solution. For the case of the linearized Kerr metric, for instance, these angular functions can be shown to form a general Sturm-Liouville system, but they must be determined numerically, see e.g., [4].

Assuming that we have completely solved the linearized problem, i.e., once the angular eigenfunctions and the PDEs relating the $h_i$, are known, we can begin to quantitatively examine the second order perturbations. Consider truncating the main perturbation equation at $O(\epsilon^3)$. It is not hard to convince oneself that the $O(\epsilon^2)$ terms occurring in the perturbed Einstein equations now yield the inhomogeneous system,

$$\epsilon^2 \mathcal{L}_{\,g_{\alpha\beta}}(g_{\alpha\beta}^{(2)}) = \mathcal{S} \left( \epsilon^2 \left( g_{\alpha\beta}^{(1)} \right)^2 \right),$$

where $\mathcal{L}$ is exactly the same linear operator found in the first order analysis, and $\mathcal{S}$ is just the collection of quadratically occurring linear perturbations (assumed known from the linear analysis).

---

1 This is really just the statement that the linearly perturbed Einstein equations reduce to, $R_{\alpha\beta}^{(1)} = 0$, in vacuum.

2 Recall, for instance, that for Legendre functions, $\langle u_i | u_j \rangle = \int_0^\pi d\theta \sin(\theta) u_i u_j \propto \delta_{ij}$. 

---
that are produced by keeping all of the $O(\epsilon^2)$ terms that appear in the perturbed Einstein equations. The physical significance of this inhomogeneous relationship, and the need for a clear understanding of the linear analysis, is now apparent. The quadratic, first order perturbation, terms act as an effective source term for the second order perturbations, allowing us to quantify the back reaction of the linear perturbations on the system. The determination of the PDEs for the second order perturbations now follows exactly the same procedure as in the linear analysis, but here the projections of the angular eigenfunctions will pick out components of the source terms from $\mathcal{S}$. Note that without a clear analysis of the linear problem one would not know how to calculate what the projected components of the source terms are. While this may seem obvious, much of the linear Schwarzschild problem can be solved by simply assuming that the angular modes decouple, and setting various coefficients of $\cos^n(\theta)$ equal to zero. If one chooses this, albeit simpler, solution technique for the linear problem the second order analysis is doomed.

This completes our general discussion of MP methods, and we now move on to examine the particular case of perturbations to the Schwarzschild spacetime. The next section will demonstrate, with the aid of computer algebra, the decoupling of the linear Schwarzschild modes. As we shall see, the use of computer algebra allows the linear analysis to be examined in a thoroughly transparent manner. With the computer doing all of the work, we do not have to resort to invoking the standard function transformations, which mix the original perturbation functions, to simplify the calculations. The final result of the analysis presented in the next section will be a simple procedure for extracting the decoupled perturbation modes, of arbitrary order, for a Schwarzschild black hole.

The Linear Schwarzschild Analysis

The bulk of this section is the input/output from a GRTensorII session run on the Maple V platform. (The Maple worksheet itself will be made available for download from the GRTensor website.) While reading this section one should bear in mind what our objective is. Rather than actually deriving the standard linear Schwarzschild results, i.e. the Zerilli wave equation [3], we simply want to show that the angular modes of the linearized Schwarzschild metric will decouple. In so doing we will develop a procedure to consistently extract a particular multipole from the perturbed Ricci tensor. Note that in what follows we consider only a single $\omega_i^{th}$ term from the general perturbation expansion presented in the last section, with its index suppressed. This is done for simplicity and clarity, and with no loss of generality due to the linearity of $\mathcal{L}$. (The concerned reader can mentally place a summed index on all of the perturbation terms.)

Our general approach to this problem begins by constructing the standard form of the covariant, linearly perturbed, Schwarzschild metric,

$$g_{\alpha\beta} \simeq g^{(0)}_{\alpha\beta} + \epsilon g^{(1)}_{\alpha\beta},$$

where,

$$g^{(0)}_{\alpha\beta} = \text{diag}(g_{rr}, g_{\theta\theta}, g_{\phi\phi}, g_{tt}) = \text{diag}\left((1 - 2M/r)^{-1}, r^2, r^2 \sin^2(\theta), -(1 - 2M/r)\right).$$

For a discussion of the particular form of the perturbed metric that we use, which will be seen shortly, see e.g. [1]. The next step in this process is to calculate the exact contravariant metric tensor, $g^{\alpha\beta}$, that is associated with the linearly perturbed covariant metric, $g_{\alpha\beta}$, and then to linearize it w.r.t. $\epsilon$. The resulting object is the linearly perturbed contravariant metric tensor,

$$g^{(0)}{}^{\alpha\beta} + \epsilon g^{(1)}{}^{\alpha\beta} = g^{(0)}{}^{\alpha\beta} + \left(\frac{\partial g^{\alpha\beta}}{\partial \epsilon} \bigg|_{\epsilon=0}\right) \epsilon .$$

These linearized co/contra-variant forms of the metric tensor are the fundamental tools of the linear theory: With them one can calculate the linear perturbation to any tensor quantity by simply
following through the normal, unperturbed, calculation while dropping all of the non-linear $\epsilon$ terms that appear. While this procedure can certainly produce calculations that would be intractable "by hand", it is a very simple matter to instruct the Maple computer engine to follow this algorithmic procedure. To arrive at the linearized Einstein vacuum equations then, we simply instruct GRTensor to calculate the Ricci tensor from the linearly perturbed metric, while truncating the non-linear $\epsilon$ terms.

$$R_{\alpha\beta} \left( g^{(0)}_{\gamma\delta} + \epsilon g^{(1)}_{\gamma\delta} + \epsilon g^{(1)}_{\gamma\delta} \right) = R_{\alpha\beta}^{(0)} + \epsilon R_{\alpha\beta}^{(1)} + O(\epsilon^2) = 0.$$ 

Of course, $R_{\alpha\beta}^{(0)} = 0$, for the Schwarzschild spacetime, so that truncation of the non-linear $\epsilon$ term yields the linearized Einstein equations, $R_{\alpha\beta}^{(1)} = 0$.

To begin the Maple session we load the GRTensorII libraries, our new simplification/perturbation routines\(^3\), and the linearly perturbed metric.

```maple
> restart:
> readlib(grii):
> grtensor()

GRTensorII Version 1.64 (R3)
4 November 1997
Developed by Peter Musgrave, Denis Pollney and Kayll Lake
Copyright 1994−1997 by the authors.

Latest version available from: http://astro.queensu.ca/~grtensor/
To initiate help type ?grtensor

> mine():
> read ‘myutils.mpl’;
> qload(lpschw):
Calculating ds for lpschw ... Done. (0.000000 sec.)

Default spacetime = lpschw

For the lpschw spacetime:

Coordinates

$x(\text{up})$

$$x^a = [ r \theta \phi t ]$$

Line element

$$ds^2 = \frac{(1 + \epsilon H_2(r,t) u(\theta))}{1 - 2 \frac{m}{r}} dr^2 + 2 \epsilon H_1(r,t) u(\theta) dt^2$$

$$+ r^2 (1 + \epsilon K(r,t) u(\theta)) d\theta^2 + r^2 \sin(\theta)^2 (1 + \epsilon K(r,t) u(\theta)) d\phi^2$$

$$- \left(1 - 2 \frac{m}{r}\right) (1 - \epsilon H_0(r,t) u(\theta)) dt^2$$

Constraints = $[\epsilon^2 = 0, \epsilon^3 = 0, \epsilon^4 = 0, \epsilon^5 = 0, \epsilon^6 = 0, \epsilon^7 = 0, \epsilon^8 = 0, \epsilon^9 = 0, \epsilon^{10} = 0]$
```

\(^3\)These routines are contained in the "myutils.mpl" file and are detailed in Appendix A.
These constraints will allow us to truncate the non-linear perturbation terms during calculation, which greatly reduces both the CPU and memory requirements. Note that the function $u(\theta)$ appearing in the metric represents an arbitrary Legendre function. (Although, it will actually remain an arbitrary function of $\theta$ until we eliminate $u''$ in terms of $u'$ and $u$, using the Legendre equation, later in the calculation.)

Our analysis begins with the calculation of the exact $g^{\alpha\beta}$ corresponding to the input metric, which we must then linearize in $\epsilon$. The linearization is accomplished using the linpert() routine, the details of which can be found in Appendix A. We are then left with, $g^{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon g^{(1)}_{\alpha\beta}$, and, $g^{\alpha\beta} = g^{(0)}_{\alpha\beta} + \epsilon g^{(1)}_{\alpha\beta}$, as the co/contra-variant components of the metric tensor$^4$.

> grcalc(g(up,up));
Calculating detg for lpschw ... Done. (0.017000 sec.)
Calculating g(up,up) for lpschw ... Done. (0.067000 sec.)

    CPU Time = .067

> gralter(g(up,up),linpert,simplify,factor);
Component simplification of a GRTensorII object:
Applying routine linpert to object g(up,up)
Applying routine simplify to object g(up,up)
Applying routine factor to object g(up,up)

    CPU Time = .133

> grdisplay(g(up,up));
For the lpschw spacetime :
Contravariant metric tensor

\[
g^{a\ b} = \begin{bmatrix}
r \frac{- (1 + \epsilon H_2(r,t) u(\theta)) (r - 2 m)}{0, \frac{1 + \epsilon H_1(r,t) u(\theta)}{0}} \\
0, \frac{-1 + \epsilon K(r,t) u(\theta)}{r^2}, 0, 0 \\
0, 0, \frac{-1 + \epsilon K(r,t) u(\theta)}{r^2 (\cos(\theta) - 1) (\cos(\theta) + 1)}, 0 \\
\frac{\epsilon H_1(r,t) u(\theta)}{0, 0}, -r \frac{(1 + \epsilon H_0(r,t) u(\theta))}{r - 2 m}
\end{bmatrix}
\]

With the linearized co/contra-variant forms of the metric tensor in hand, it is a simple matter to calculate the perturbed Ricci tensor, or any other first order (tensor) quantities of interest for that matter. We now calculate the Ricci tensor, which amounts to calculating the perturbed Einstein tensor, applying the constraints as we go to kill off the higher order terms.

> grcalcalter(R(dn,dn),13);
Simplification will be applied during calculation.

Applying routine Apply constraints repeatedly to object g(dn,dn,pdn)
Calculating g(dn,dn,pdn) for lpschw ... Done. (0.033000 sec.)
Applying routine Apply constraints repeatedly to object Chr(dn,dn,dn)
Calculating Chr(dn,dn,dn) for lpschw ... Done. (0.050000 sec.)
Applying routine Apply constraints repeatedly to object Chr(dn,dn,up)

$^4$Direct calculation shows that, $g^{\alpha\gamma}g_{\gamma\beta} = \delta^\alpha_\beta + O(\epsilon^2)$, as required for consistency at $O(\epsilon)$. 


Calculating Chr(dn,dn,up) for lpschw ... Done. (0.117000 sec.)
Applying routine Apply constraints repeatedly to object R(dn,dn)
Calculating R(dn,dn) for lpschw ... Done. (0.733000 sec.)

CPU Time = .933

Up to this point the angular function \( u(\theta) \) has been completely arbitrary, except that one would like it to be a member of a complete set. We now fix \( u \) as a Legendre function by making the following substitution in \( R_{\alpha \beta} \): \[ u'' = -\cot(\theta)u' - n(n+1)u = 0. \] (For simplicity we set \( j = n(n+1) \).)

\[
\text{grmap}(R(dn,dn),\text{subs},\text{diff}(u(\theta),\theta)^2=\text{-cos}(\theta)/\text{sin}(\theta)\times\text{diff}(u(\theta),\theta)-j*u(\theta),'x');
\]

We can now examine the seven non-trivial components of the perturbed Ricci tensor: \( R^{(1)}_{r\theta} \), \( R^{(1)}_{rt} \), \( R^{(1)}_{\theta t} \), and the four diagonal components. The \( R^{(1)}_{r\theta} \) component is\(^5\)

\[
b1 := hcollect(expand(coef(collect(grcomponent(R(dn,dn),[r,theta]),epsilon),epsilon,1)), ufuncs,lperts);
\]

\[
b1 := \left( \frac{1}{2} \frac{(-m + r) H_2(r,t)}{(r - 2 m)} - \frac{1}{2} \frac{(-3 m + r) H_0(r,t)}{(r - 2 m)} + \frac{1}{2} \left( \frac{\partial}{\partial r} H_0(r,t) \right) - \frac{1}{2} \frac{\partial}{\partial r} K(r,t) \right)
\]

\[ - \frac{1}{2} r \frac{\partial}{r-2m} H_1(r,t) \left( \frac{\partial}{\partial \theta} u(\theta) \right) \]

(The \texttt{hcollect()} routine, described in Appendix A, is a hierarchical collection procedure, “ufuncs” is the set of \( u \) and its derivatives, and “lperts” is the set of linear MP and their derivatives up to second order.) If we considered the full series perturbation mentioned above, we would have an infinite sum of these expressions, each with index \( i \). Noting that the \( u'_i(\theta) \) form an orthogonal set however, we could uniquely select out any desired term in the imagined summation by taking the appropriate projection on the linearized expression (and any potential source term). Examining \( R^{(1)}_{t\theta} \) we similarly find:

\[
c1 := hcollect(coef(collect(grcomponent(R(dn,dn),[t,theta]),epsilon),epsilon,1)), ufuncs,lperts);
\]

\[
c1 := \left( \frac{H_1(r,t) m}{r^2} + \frac{1}{2} \frac{(r - 2 m)}{r} H_1(r,t) \right) - \frac{1}{2} \left( \frac{\partial}{\partial t} H_0(r,t) \right) - \frac{1}{2} \frac{\partial}{\partial t} K(r,t) \right) \left( \frac{\partial}{\partial \theta} u(\theta) \right)
\]

Here again we simply need to project with \( u'_i(\theta) \) in order to extract the \( "i^{th}" \) term from any summed expression. We next find that \( R^{(1)}_{rt} \), \( R^{(1)}_{rr} \), and \( R^{(1)}_{tt} \) can all be expressed as: \( f(h_i)u_i(\theta) = 0 \). For these three components we need only project with the \( u_i \) function itself.

\[
d1 := hcollect(expand(coef(collect(grcomponent(R(dn,dn),[t,r]),epsilon),epsilon,1)), ufuncs,lperts);
\]

\[
d1 := \left( \frac{1}{2} \frac{H_1(r,t)}{r^2} \right) - \left( \frac{\partial}{\partial t} \frac{\partial}{\partial r} K(r,t) \right) + \frac{\partial}{\partial t} \frac{H_2(r,t)}{r} - \frac{(-3 m + r)}{(r - 2 m)} \frac{\partial}{\partial r} \frac{K(r,t)}{r} \right) \left( \frac{\partial}{\partial \theta} u(\theta) \right)
\]

\(^5\)Although \( R^{(0)}_{\alpha \beta} = 0 \), we will still take the time to collect only the linear \( \epsilon \) terms as a matter of good practice.
So far all of the perturbed Ricci components have had a very simple angular dependence in terms of either, $u(\theta)$, or its derivative. The last two non-trivial components of $R^{(1)}_{\alpha\beta}$ do not, however, exhibit this simplicity.

Here we seem to run into trouble; our expression has both $u_i$ and $u'_i$ terms. The projection $\langle u_i|u'_i \rangle$ will in general be non-zero. Thus we could not decouple the modes if this expression were summed over. But when we combine this expression with $R^{(1)}_{\phi\phi}$ we will find a, seemingly fortuitous, resolution.

$$g1 := (\cos(\theta) + 1)(\cos(\theta) - 1) - \frac{1}{2} \left(2H_2(r,t) r - 4H_2(r,t) m - 2K(r,t) r + 4K(r,t) m\right)$$
from which one obtains seven PDEs that govern the MP functions. Here

projection functions. that relates the seven linear combinations of the perturbed Ricci components to their associated first derivative. We can summarize the details of this decoupling procedure in the following table

\[
\begin{array}{l}
G, \\
\end{array}
\]

linear combination of its arguments, and

projections, these also form a complete orthogonal set, we can, therefore, uniquely select out any term in such a series by projecting with:

By adding these two expressions we have eliminated the \( u'_{1} \) terms, and can therefore select any \( u'_{i} \) term by simply projecting with \( u_{i} \). Similarly:

\[
\frac{1}{2} \frac{H_{2}(r,t) - H_{0}(r,t)}{\sin(\theta)} \left( j u(\theta) \sin(\theta) + 2 \cos(\theta) \left( \frac{\partial}{\partial \theta} u(\theta) \right) \right)
\]

Here again we have the potential for mode coupling from the presence of both \( u'_{1} \) and \( u_{i} \) terms. If, however, we add/subtract these last two components a remarkable thing happens:

\[
\begin{align*}
&> \text{hcollect}(a1+g1/sin(theta)\^2, ufuncs, lperts); \\
&\left( \frac{1}{2} (j + 4) H_{2}(r,t) + (-2 + j) K(r,t) + r^{3} \left( \frac{\partial}{\partial r} K(r,t) \right) - \left( \frac{\partial}{\partial r} K(r,t) \right) \right) (r - 2m) r \\
&- \frac{1}{2} H_{0}(r,t) j + \left( \frac{\partial}{\partial r} H_{0}(r,t) \right) (r - 2m) + \left( \frac{\partial}{\partial r} H_{2}(r,t) \right) (r - 2m) \\
&- 2 (-3m + 2r) \left( \frac{\partial}{\partial r} K(r,t) \right) - 2 \left( \frac{\partial}{\partial \theta} H_{1}(r,t) \right) r u(\theta)
\end{align*}
\]

While this subtraction seems to again leave us with the potential for mode coupling, we notice that:

\[
i(i + 1)u_{i} + 2 \cos(\theta)u'_{i}(\theta) = -(1 - x^{2})d^{2}u_{i}/dx^{2},
\]

where \( x = \cos(\theta) \). Recognizing this as the \( P_{i}^{2}(x) \) associated Legendre function, and recalling that these also form a complete orthogonal set, we can, therefore, uniquely select out any term in such a series by projecting with:

\[
i(i + 1)u_{i} + 2 \cos(\theta)du_{i}/d\theta.
\]

The decoupling of the linear problem is thus reduced to a prescription for performing seven projections,

\[
\left\langle F_{i} \left( R_{\alpha\beta}^{(1)} \right) | G_{i} (P_{1}) \right\rangle = 0,
\]

from which one obtains seven PDEs that govern the MP functions. Here \( i = 1..7, \) \( F_{i} \) produces a linear combination of its arguments, and \( G_{i} \) is at most a function of the given argument and its first derivative. We can summarize the details of this decoupling procedure in the following table that relates the seven linear combinations of the perturbed Ricci components to their associated projection functions.
Perturbed Ricci components | l-mode Projection function
\[\begin{array}{|c|c|}
\hline
\delta R_{\theta\theta}, \delta R_{\theta t} & dP_1/d\theta \\
\delta R_{rr}, \delta R_{rt}, \delta R_{tt} & P_1 \\
\delta R_{\theta\theta} + \delta R_{\phi\phi} / \sin^2(\theta) & P_1 \\
\delta R_{\theta t} - \delta R_{\phi\phi} / \sin^2(\theta) & (l+1)P_1 + 2\cot(\theta)dP_1/d\theta \\
\hline
\end{array}\]

Table 1.

This table is the central result of this section, and is a kind of Rosetta stone for the Schwarzschild perturbation analysis. It allows one to consistently and algorithmically decouple the angular perturbations, regardless of the order of the perturbation expansion. Thus the \(\delta R_{\alpha\beta}\) can be any order of the perturbed Ricci tensor, \(R^{(n)}_{\alpha\beta}\), or even the total perturbation, \(\sum_n \epsilon^n R^{(n)}_{\alpha\beta}\).

While we have managed to decouple the perturbation modes, we should note that there is a great redundancy in the system of seven PDEs that are generated by these projections. The remainder of the solution to the linear problem is, essentially, the elimination of this excess information. As a simple example of this redundancy, and as a prelude to the results of the next section, we note that the \(\theta^\theta - \phi^\phi\) projection in the linear analysis gives,

\[\int_0^\pi d\theta \sin(\theta) P^2_l(\theta) \left( \delta R_{\theta\theta} - \delta R_{\phi\phi} / \sin^2(\theta) \right) = 0 \rightarrow H_0(r,t) = H_2(r,t).\]

At the linear level then, \(H_0\) and \(H_2\) are degenerate.

The reduction of the system is typically accomplished by making the Zerilli transformation,

\[H_1(r,t) = \frac{(2r^2 - 6m - 3m^2)}{(r - 2m)(2r + 3m)} \chi(r,t) + \frac{r^2 \eta(r,t)}{r - 2m}, \quad \frac{\partial K(r,t)}{\partial t} = 6 \frac{r^2 + rm + m^2}{r^2(2r + 3m)} \chi(r,t) + \eta(r,t).\]

With these substitutions the redundancy can now be expressed as: \(\eta(r,t) = (1 - 2m/r) \partial_r \chi\). Using this result to eliminate \(\eta\) throughout the system one then finds a “wave equation” in the single new function \(\chi(r,t)\). This is the celebrated Zerilli “wave equation”:

\[\frac{\partial^2 \chi(r,t)}{\partial t^2} - \frac{\partial^2 \chi(r,t)}{\partial r^2} + V_1(r)\chi(r,t) = 0.\]

Here, \(r^* = r + 2M \log(r/2M - 1)\), and the effective potential, \(V_1\), has an explicit dependence on the angular mode number, \(l\). Considering our discussion of perturbation theory from the last section, we expect that the second order Zerilli wave equation will be the same as that of the linear case, but with the addition of a new source term. Finding the form of this source term, which is not unique\(^7\), is the central problem of the second order calculation. Calculating the source term will, ultimately, be the goal of the next section.

As the final comment of this section, we invite the interested reader to download the Maple worksheet that produced the linearized results presented here. One is then free to reproduce these results and experiment with the tools at hand. In particular one should try to replicate the linearized equations without the aid of the \texttt{termsimp()} and \texttt{hcollect()} routines.

The Quadratic Schwarzschild Analysis

We found in the last section that the non-trivial components of the linearly perturbed Ricci tensor grouped into five components, and two linear combinations of components, each having an associated projection function. Making use of this linear analysis we can now solve the second order

---

\(^6\) This is a direct result of the fact that higher order perturbations will simply produce the same seven equations, now in terms of the higher order functions, along with possible source terms.

\(^7\) We will find that this degeneracy is broken at the second order level by the presence of “source terms”.

\(^8\) See, for example, [1], for a discussion of the gauge dependence of the source term.
Schwarzschild problem. Here the second order perturbations appear exactly as the linear ones have, but with the addition of “source” terms that are quadratic in the (assumed known) linear perturbations. The key here is that while the quadratic source terms will in general be a combination of an arbitrary number of multipoles, we now have a unique prescription for determining their contribution to any desired second order multipole perturbation.

From the point of view of the second order analysis, the form of the linear perturbation functions are, apart from consistency, simply a matter of choice. The recent work that has appeared in the literature assumes that the linear perturbations have only a quadrupole component, see, e.g., \([1]\). This restriction can be justified on physical grounds by arguing that most of the spectral power will be concentrated in this mode and that the full source could be reconstructed by simply summing over the actual multipole modes of the linear problem. But perhaps the best rational for this restriction is that there exists a particular class of interesting problems, see, e.g., \([1]\), for which the only non-zero multipole is, at the linear level, the quadrupole. For this case then, the treatment is exact, and it is this scenario that we will study. The remainder of this section will outline the calculation of the second order quadrupole results, i.e., \(g_{\alpha\beta}^{(1)}\) and \(g_{\alpha\beta}^{(2)}\), will both be pure quadrupole moments.

In solving this particular problem then, we are examining how the first order quadrupole moment seeds the second order quadrupole moment. We will return to discuss this point in our conclusion, but for now one should simply notice that the following analysis could easily be repeated with any two given multipoles. Further details of the calculation can be found in Appendix B, and as in the last section, the complete Maple worksheet that produced these results can be downloaded from the GRTensor website.

The quadratically perturbed covariant metric for the “quadrupole-quadrupole” perturbation is a simple extension of the linear case.

\[
\begin{align*}
 ds^2 &= \left(1 + \frac{1}{2} \left(\varepsilon h_2(r,t) + \varepsilon^2 H_2(r,t)\right) \left(3\cos(\theta)^2 - 1\right)\right) dr^2 \\
 & \quad + \left(\varepsilon h_1(r,t) + \varepsilon^2 H_1(r,t)\right) \left(3\cos(\theta)^2 - 1\right) d r \ dt \\
 & \quad + r^2 \left(1 + \frac{1}{2} \left(\varepsilon k(r,t) + \varepsilon^2 K(r,t)\right) \left(3\cos(\theta)^2 - 1\right)\right) d \theta^2 \\
 & \quad + r^2 \sin(\theta)^2 \left(1 + \frac{1}{2} \left(\varepsilon k(r,t) + \varepsilon^2 K(r,t)\right) \left(3\cos(\theta)^2 - 1\right)\right) d \phi^2 \\
 & \quad - \left(1 - \frac{2}{r} \right) \left(1 - \frac{1}{2} \left(\varepsilon h_0(r,t) + \varepsilon^2 H_0(r,t)\right) \left(3\cos(\theta)^2 - 1\right)\right) dt^2
\end{align*}
\]

Constraints = \([\varepsilon^4 = 0, \varepsilon^5 = 0, \varepsilon^6 = 0, \varepsilon^7 = 0, \varepsilon^8 = 0, \varepsilon^9 = 0, \varepsilon^{10} = 0, \varepsilon^3 = 0]\)

As can be seen from the form of the metric, the lower case functions are the first order perturbations, and the upper case functions are the second order perturbations.

The calculation now proceeds exactly as in the linear case, except that we now expand the contravariant metric to second order and truncate higher than \(O(\varepsilon^2)\) terms during the calculation of the Ricci tensor. One can gain an immediate appreciation for the increase in complexity of this problem over the linear case by examining the size of the second order Ricci tensor.

\[
R_{rr} = 86717 \text{ words. Exceeds grOptionDisplayLimit}
\]

\[
R_{r\theta} = 28579 \text{ words. Exceeds grOptionDisplayLimit}
\]
As one might expect, the quadratically perturbed Ricci tensor is substantially larger than its linear counterpart, and it is here that our new simplification routines will become indispensable. Of course, what we have really calculated here is,

\[ R_{\alpha\beta} \simeq R^{(0)}_{\alpha\beta} + \epsilon R^{(1)}_{\alpha\beta} + \epsilon^2 R^{(2)}_{\alpha\beta}, \]

where \( R^{(0)}_{\alpha\beta} = 0 \). Given that the Ricci tensor now contains terms that can be either linear or quadratic in \( \epsilon \), care must be taken when extracting a particular order of perturbation.

In order to determine the PDEs that govern the second order perturbations we simply follow the procedure set out in the linear analysis. Here we select only the \( \epsilon^2 \) components of the Ricci tensor, and we perform the \( l = 2 \) projections. This projects the squared quadrupole of the linear perturbation onto the second order quadrupole\(^9\). The second order perturbation functions will, necessarily, appear in these PDEs in exactly the same manner as the linear perturbation functions appear in the last section. The only new feature, which lies at the heart of the second order theory, is the manner in which the quadratically occurring linear perturbations appear. We will not show the raw expressions for most of these PDEs as many of their source terms are quite large. Instead, we give the detailed expressions for two of the simpler equations, both to illustrate how these equations can differ from their linear counterparts and to make contact with the results presented in [1]. The results of the “θθ” and “tt” projections yield,

\[ H_2(r, t) = H_0(r, t) - \frac{1}{7} h_0(r, t)^2 + \frac{1}{7} h_1(r, t)^2, \]

and,

\[
2 \frac{m H_1(r, t)}{r^2} + \frac{(r - 2m)}{r} \left( \frac{\partial}{\partial r} H_1(r, t) \right) - \left( \frac{\partial}{\partial t} H_0(r, t) \right) - \left( \frac{\partial}{\partial t} K(r, t) \right) + \frac{3}{14} \left( \frac{\partial}{\partial t} h_0(r, t)^2 \right) \\
+ \frac{1}{7} \left( \frac{\partial}{\partial t} k(r, t)^2 \right) - \frac{3}{14} \left( \frac{\partial}{\partial t} h_1(r, t)^2 \right) = 0,
\]

respectively. The degeneracy between \( H_0 \) and \( H_2 \) is therefore broken at second order by the appearance of “source” terms. Of course, one can still eliminate \( H_2 \) from the analysis using this first of these expressions. As with the linearized case, the second order perturbations form a very redundant system. The effective source terms, of course, inherit all of the redundancy of the linear problem as well. The completion of our second order calculation will be the elimination of this redundancy, à la Zerilli, in a manner that will allow us to reproduce the results of [1].

The derivation of the first of the Zerilli equations proceeds by eliminating \( \dot{H}_0 \) from \( R^{(2)}_{tt} \) using the \( R^{(2)}_{rt} \) component of the Ricci tensor. One must then eliminate the redundant source terms, a procedure that is not unique. Here we choose to preferentially eliminate certain source terms to reproduce the results in [1]. The details of this process are not very enlightening and are relegated to the Maple worksheet that can be found at the GRTensor website. The result of these manipulations is essentially the raw result for the \( \eta(r, t) \) expression which can be found in Appendix B. One should note that the quadratic nature of the linear perturbations that appear in these types of expressions make them almost impossible to simplify with the standard Maple routines. To overcome this...
problem we use our new \texttt{mcollect()} routine (see Appendix A), after successive application of which, we find,

\[
\eta(r,t) = \frac{(r - 2m)}{r} \frac{\partial}{\partial r} \chi(r,t) - \frac{1}{7}(r - 2m) \left( - h_0(r,t) \frac{\partial}{\partial t} h_0(r,t) \right) \\
+ 2 \left( \frac{\partial}{\partial t} h_0(r,t) \right) k(r,t) + r \left( \frac{\partial}{\partial r} k(r,t) \right) \left( \frac{\partial}{\partial t} h_0(r,t) \right) + 2 k(r,t) \left( \frac{\partial}{\partial t} k(r,t) \right) \\
+ 4 \frac{h_1(r,t) k(r,t)}{r} - 2 m \frac{h_1(r,t)}{r} \left( \frac{\partial}{\partial r} k(r,t) \right) + h_1(r,t) \left( \frac{\partial}{\partial t} h_1(r,t) \right) \\
+ 2 \frac{(r - 2m) h_1(r,t) h_0(r,t)}{2 r^2} - \frac{r^2}{r} \left( \frac{\partial}{\partial r} k(r,t) \right) \left( \frac{\partial}{\partial t} h_1(r,t) \right) \\
+ 2 \left( \frac{\partial}{\partial r} k(r,t) \right) \frac{h_0(r,t) m}{r - 2m} / (2r + 3m).
\]

This result, which appears here exactly as outputed from the Maple session, agrees with the result found in [1]. The new source terms that appear in this second order result add a considerable degree of complexity to the problem, as one must use this result to eliminate all of the \( \eta \) dependence, which includes up to third order derivatives, from the analysis. One should also note that without the use of our new simplification routines this analysis would not have been as seamless: Ideally, when performing computer algebra calculations, one should never have to resort to a “by hand” calculation as this increases the chance of error by a very large factor. Our new simplification routines were designed precisely to avoid the necessity of any “by hand” calculations.

The development of the second order Zerilli “wave equation” follows a similar procedure to this last result. We first take the time derivative of \( R^{(2)}_{t\phi} \), eliminate \( H_0 \) and \( H'_0 \) using \( R^{(2)}_{rt} \), make the Zerilli substitutions, and substitute for \( \eta \) from our first Zerilli result. We then repeat this process for \( R^{(2)}_{t\theta} \) and add the resulting equations such that the \( \chi' \) term vanishes. The result of these operations is essentially the raw form of the result we are seeking. Having 421 terms, this expression is, however, rather large.

In order to make contact with the results presented in [1], we now work with the “renormalized” perturbation function, \( \zeta \), defined by,

\[
\chi(r,t) = \zeta(r,t) + \frac{2}{7} \frac{r^2 k(r,t)}{2r + 3m} \left( \frac{\partial}{\partial r} k(r,t) \right) + \frac{2}{7} k(r,t)^2
\]

This choice of transformation reduces the number of terms by over a factor of three. The next problem we must tackle is to eliminate all of the redundant information contained in the source terms for \( \zeta \). In principle this should be relatively simple, and certainly straight forward; in practice it is neither. The basic necessity is to use all of the PDEs found in the linear analysis to eliminate as many of the “higher order” derivatives as is possible, and to preferentially eliminate any \( h_1 \) dependence. The rational for this is that, to linear order, all of the radiation information is determined by the Zerilli function, \( \psi \), which depends only on \( k \) and \( h_0 \). One therefore expects that the source terms can be expressed solely in terms of these two functions, and ultimately, solely in terms of \( \psi \). Our approach is thus to express \( h_0 \) in terms of \( \psi \) and \( k \) once the \( h_1 \) dependence has been eliminated from the source terms.

\[
h_0(r,t) = 3 \frac{\left( \psi(r,t) - \frac{1}{3} r k(r,t) \right)}{r (r - 2m)} \left( 2r + 3m \right) + \left( \frac{\partial}{\partial r} k(r,t) \right) r
\]
After performing all of these eliminations\textsuperscript{10}, making the above substitution for $h_0$, and, further simplifying the result to eliminate the $k$ dependence, we arrive at the unsimplified form of our final result, which is still rather large at 154 terms. To simplify this expression we once again use a repeated application of \texttt{mcollect()}. Here we must collect and simplify the 17 different types of quadratic $\psi$ terms that occur. Making one last set of substitutions, $\mu = (r-2m)$, and, $\lambda = (2r+3m)$, and extracting the source terms from the resulting expression yields:

$$
\frac{12}{7} \mu^3 \left( -12 \frac{(r^2 + m r + m^2)^2}{r^4 \mu^3 \lambda} \left( \frac{\partial}{\partial t} \psi(r, t) \right)^2
\right.

- 4 \frac{(2 r^3 + 4 r^2 m + 9 r m^2 + 6 m^3) \psi(r, t) \%2}{r^6 \lambda}

+ \left( 112 r^5 + 480 r^4 m + 692 r^3 m^2 + 762 r^2 m^3 + 441 r m^4 + 144 m^5 \right) \psi(r, t) \left( \frac{\partial}{\partial t} \psi(r, t) \right)

\left/ \left( r^5 \mu^2 \lambda^3 \right) - \frac{1}{3} \left( \frac{\partial}{\partial t} \psi(r, t) \right) \left( \frac{\partial^3}{\partial r^3 \partial t} \psi(r, t) \right) \right.

+ \frac{1}{3} \frac{(18 r^3 - 4 r^2 m - 33 r m^2 - 48 m^3) \left( \frac{\partial}{\partial r} \psi(r, t) \right) \left( \frac{\partial}{\partial r} \psi(r, t) \right)}{r^4 \mu^2 \lambda}

+ \frac{1}{3} \frac{(12 r^3 + 36 r^2 m + 59 r m^2 + 90 m^3) \left( \frac{\partial}{\partial r} \psi(r, t) \right)}{r^6 \mu}

+ 12 \frac{(2 r^5 + 9 r^4 m + 6 r^3 m^2 - 2 r^2 m^3 - 15 r m^4 - 15 m^5) \psi(r, t)}{r^8 \mu^2 \lambda}

- 4 \frac{(r^2 + m r + m^2) \left( \frac{\partial}{\partial t} \psi(r, t) \right) \%1}{r^3 \mu^2}

- 2

\left( 32 r^5 + 88 r^4 m + 296 r^3 m^2 + 510 r^2 m^3 + 561 r m^4 + 270 m^5 \right) \psi(r, t) \left( \frac{\partial}{\partial r} \psi(r, t) \right)

\left/ \left( r^7 \mu \lambda^2 \right) + \frac{1}{3} \left( \frac{\partial}{\partial t} \psi(r, t) \right) \left( \frac{\partial^3}{\partial r^3 \partial t} \psi(r, t) \right) \right.

- \left( 2 r^2 - m^2 \right) \left( \frac{\partial}{\partial r} \psi(r, t) \right) \%2

+ \frac{8 r^2 + 12 m r + 7 m^2) \psi(r, t) \%1}{r^3 \mu \lambda} + \frac{1}{3} \frac{(3 r - 7 m) \left( \frac{\partial}{\partial r} \psi(r, t) \right)}{r^3 \mu}

- \frac{m \psi(r, t) \left( \frac{\partial^3}{\partial r^3 \partial t} \psi(r, t) \right)}{r^3 \lambda} + \frac{4 \left( 3 r^2 + 5 m r + 6 m^2 \right) \left( \frac{\partial}{\partial r} \psi(r, t) \right) \%2}{r^5} + \frac{1}{3} \frac{\mu \lambda \%2}{r^4}

- \frac{1}{3} \frac{\lambda \%1^2}{r^2 \mu}\right) / \lambda

\%1 := \frac{\partial^2}{\partial t \partial r} \psi(r, t)

\%2 := \frac{\partial^2}{\partial r^2} \psi(r, t)

This is our final result, and also appears exactly as outputed from the Maple worksheet. We have derived the effective Zerilli source term that dictates how the linear quadrupole perturbation seeds the second order quadrupole perturbation for a Schwarzschild black hole. This expression is almost identical to that presented in [1]. The difference between the above expression and the previously published result is that the $\psi^2$ term that appears in [1] only has factor of $1/\mu^2$, while we have found

\textsuperscript{10}These details can be found in the worksheet at the GRTensor website.
that it has a $1/\mu^3$ dependence.

To conclude this section we again invite the interested reader to download the full Maple worksheet from the GRTensor website. One can then reproduce these calculations and, of course, experiment. One could, for instance, alter the worksheet to calculate the source term for any particular pair of linear-quadratic multipole perturbations.

**Conclusion and Discussion**

This article has presented a detailed account of our calculation of second order perturbations to a Schwarzschild black hole. Our methodology was chosen to illustrate both the utility, and necessity, of employing computer algebraic methods to examine these types of problems. We began our analysis with an explicit demonstration that the linear Schwarzschild perturbations decouple, a result that is well known but that is not always clearly presented. We then used these linear results to examine the second order pure quadrapole case. These second order results confirm, with one minor exception\(^\text{11}\), the earlier work by Gleiser, et. al., in [1].

As mentioned earlier, there are interesting problems for which the quadrapole perturbation provides an exact description of the system to first order. Given this, one may then wish to consider how the linear quadrapole seeds an arbitrary second order multipole perturbation. This more general result has been presented by Pullin [6]. (This result contains a source term that remains inconsistent with our results.) With our general method it would take relatively little effort to examine this for any second order multipole\(^\text{12}\) or even to reproduce the general result from [6]. Our methods, of course, are not limited to examining perturbations that have only a linear quadrapole contribution. Indeed, given the computational speed with which our simple “pure quadrapole” case was solved, there is no reason to believe that more realistic perturbations could not be examined. One could, for example, study how the inclusion of the first few multipoles at the linear level affect any given second order multipole. One might even consider higher order calculations. Given the minimal computational requirements of our second order calculations it is likely that third order Schwarzschild perturbations would be manageable. While the utility of a third order calculation is certainly questionable, the potential for its investigation does exist.

The largest obstacle to such calculations, and even to our present calculation, is finding the correct strategy for eliminating the redundancy in the linear perturbation equations. With some forethought, however, one could create a Maple routine that would examine a system of redundant PDEs, such as the ones we have encountered, and automatically generate a list of simplifying equations. One of these expressions might, for example, allow one to eliminate a second order derivative in terms of a linear combination of first order derivatives. Such a routine would certainly go a long way towards making these kinds of calculations much easier.

With the methodology of Schwarzschild perturbations on such firm ground one is quite naturally led to carrying this analysis over to the calculation of perturbations to the Kerr metric. As mentioned in the introduction, this approach fails almost immediately when applied to the Kerr spacetime. The reason for this is simple: The background Kerr metric is not spherically symmetric, which results in perturbation equations that are not separable and that can therefore not be decoupled. One is free to try and find a coordinate system in which the equations become separable, but one should be forewarned that this is not a rewarding pursuit. None of the common coordinate systems of the

\(^{11}\)One, out of seventeen, of our Zerilli source terms differs from the previously published results.

\(^{12}\)Gleiser, [7], for example, has considered the linear quadrapole in conjunction with the second order monopole perturbation. This is essentially an investigation of how the second order mass perturbation is affected by the first order gravitational radiation.

\(^{13}\)The calculations presented in Appendix B were performed within a Maple worksheet running on a 600 MHz Alpha. The total CPU time was just under two minutes.
Kerr spacetime produce separable perturbation equations. Unless one can guess, or derive, such a coordinate system, this approach to the Kerr problem is a dead end. It thus appears that one must appeal to the tetrad solution of the linear Kerr problem for guidance.

The central result of the tetrad analysis is the Teukolsky equation (see, e.g., [4]), $\mathcal{L}_T \psi = 0$, where $\mathcal{L}_T$ is a complex linear operator and the $\psi_i$ are the $\psi_0$ and $\psi_4/\rho^4$ scalars from the NP formalism. All of the perturbation information is therefore encoded in just two complex scalars, so that one expects that there are only four independent MP functions. Our approach to this problem is to try and reverse engineer the NP solution and express the metric perturbations in terms of the perturbed NP quantities. If this can be done we will be assured that the resulting linear perturbation equations will decouple. By examining how the combinations of the separable eigenfunctions appear in the resulting linearized Ricci tensor one could then develop an algorithm, similar to our Schwarzschild result, to decouple the linear perturbations of the Kerr spacetime. If this linear problem can be solved with “reasonable” computing resources then one should be able to repeat the analysis for second order Kerr perturbations, just as we have done for the Schwarzschild case.

Our preliminary calculations in this effort indicate that even the linear Kerr problem can generate extremely large and complex expressions. Although much of this complexity disappears after extensive simplification. Despite the size of some of these intermediate objects, we are confident that the linear Kerr problem can be solved by this “reverse engineering” method. Our work is now proceeding in this direction, and we expect to have definite Kerr results shortly. Our findings here will be the subject of Paper II. If the linear Kerr calculation requires a large percentage of our full computing resources then it is likely that our findings for the second order Kerr case will simply be that a full solution to the problem will have to wait for the next generation of computer/computing engine. As our final comment we wish to note that the difficulty and complexity of the linear Kerr problem can be truly surprising.

Acknowledgments

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Appendix A

This appendix contains some of the details of our new Maple routines that were used in our calculations. Most of these routines are fairly small, and did not require a significant amount of coding. (There are, however, some notable exceptions.) Our examples will make use of the following two polynomials that are similar to those found in our calculations:

\[ p_1 := \frac{(r^2+M r+M^2)^2 (r-2 M)^2}{r^2 (2 r+3 M)^2} \]
\[ p_2 := \frac{(2 r^2-6 M r-3 M^2)^2 (r-2 M)^2}{r (2 r+3 M)} \]

• termsimp(expr): The termsimp() routine applies factor(simplify()) to each term in a type + or * expression individually, and then reconstructs the final expression. This can be very useful if one needs to simplify an expression after having applied the collect() routine, as Maple’s simplify() and collect() tend to be inverse operations.

e.g.
\[ > \text{temp} := \text{expand}(p1)*H[0] + \text{expand}(p2)*H[1]; \]
\[ \text{temp} := \left( \frac{r^4}{(2 r+3 M)^2} - 2 \frac{r^3 M}{(2 r+3 M)^2} - \frac{r^2 M^2}{(2 r+3 M)^2} - 2 \frac{r M^3}{(2 r+3 M)^2} + 5 \frac{M^4}{(2 r+3 M)^2} \right) \frac{H_0}{r} + \left( \frac{4 r^3}{2 r+3 M} - 24 \frac{r^2 M}{2 r+3 M} + 24 \frac{r M^2}{2 r+3 M} + 36 \frac{M^3}{2 r+3 M} + 9 \frac{M^4}{r (2 r+3 M)} \right) \frac{H_1}{r^2 (2 r+3 M)^2} \]

We can now compare this with the result of applying the Maple simplify() routine.

\[ > \text{simplify(temp);} \]
\[ \left( H_0 r^6 - 2 H_0 r^5 M - H_0 r^4 M^2 - 2 H_0 r^3 M^3 + 5 H_0 r^2 M^4 + 4 H_0 M^5 r + 4 H_0 M^6 + 8 H_1 r^6 - 36 H_1 r^5 M - 24 H_1 r^4 M^2 + 144 H_1 r^3 M^3 + 126 H_1 r^2 M^4 + 27 H_1 r M^5 \right) \left( \frac{r^2}{(2 r+3 M)^2} \right) \]

This rather cumbersome expression is a result of the fact that simplify() does not “know” that the \( H_i \)’s should be treated as the primary objects. The effect of using collect() (which would produce an expression similar to temp) and then termsimp() essentially tells Maple that the “collected” objects are the primary ones.

14 While these examples may seem somewhat artificial to the uninitiated, one should bear in mind that the result of a large algebraic calculation is typically an almost fully expanded expression that must be simplified.
• hcollect(expr, list_one, list_two): The hcollect() routine is a hierarchical collection routine. The collect() routine is first applied to the expression with list_one as an argument. The resulting expression, which has been factored w.r.t. the elements of list_one, is then collected w.r.t. the elements of list_two. The utility of this function is that it is designed not to undo any of the first collection, which is the typical result of applying successive collect() calls with different lists. This routine automatically calls termsimp() when reconstructing the final expression.

\[\text{e.g.}\]
> temp:=expand(p1*(u(r)+(r-2*M)/r*diff(u(r),r))*H[0]+p2*(u(r)+(2*r+3*M)*diff(u(r),r))*H[1]);

\[
\begin{align*}
temp := & \frac{r^4 H_0 u(r)}{(2 r + 3 M)^2} + \frac{r^4 H_0 \left( \frac{\partial}{\partial r} u(r) \right)}{(2 r + 3 M)^2} - \frac{4 r^3 H_0 \left( \frac{\partial}{\partial r} u(r) \right) M}{(2 r + 3 M)^2} - \frac{2 r^3 H_0 M u(r)}{(2 r + 3 M)^2} \\
& + 3 \frac{r^2 H_0 M^2 \left( \frac{\partial}{\partial r} u(r) \right)}{(2 r + 3 M)^2} - \frac{r^2 H_0 M^2 u(r)}{(2 r + 3 M)^2} - \frac{2 r H_0 M^3 u(r)}{(2 r + 3 M)^2} + 9 \frac{H_0 M^4 \left( \frac{\partial}{\partial r} u(r) \right)}{(2 r + 3 M)^2} \\
& + 5 \frac{H_0 M^4 u(r)}{(2 r + 3 M)^2} - 6 \frac{H_0 M^5 \left( \frac{\partial}{\partial r} u(r) \right)}{r (2 r + 3 M)^2} + \frac{4 H_0 M^5 u(r)}{r (2 r + 3 M)^2} - \frac{4 H_0 M^6 \left( \frac{\partial}{\partial r} u(r) \right)}{r^2 (2 r + 3 M)^2} \\
& + 4 \frac{H_0 M^6 u(r)}{r^2 (2 r + 3 M)^2} - 8 \frac{H_0 M^7 \left( \frac{\partial}{\partial r} u(r) \right)}{r^3 (2 r + 3 M)^2} + \frac{4 H_1 u(r)}{r^2 (2 r + 3 M)^2} + 8 \frac{r^4 H_1 \left( \frac{\partial}{\partial r} u(r) \right)}{r^2 (2 r + 3 M)^2} \\
& - 36 \frac{r^3 H_1 \left( \frac{\partial}{\partial r} u(r) \right)}{2 r + 3 M} - 24 \frac{r^2 H_1 M u(r)}{2 r + 3 M} - 24 \frac{r^2 H_1 M \left( \frac{\partial}{\partial r} u(r) \right)}{2 r + 3 M} + 24 \frac{H_1 M^2 u(r)}{2 r + 3 M} \\
& + 144 \frac{r H_1 M^3 \left( \frac{\partial}{\partial r} u(r) \right)}{2 r + 3 M} + 36 \frac{H_1 M^3 u(r)}{2 r + 3 M} + 126 \frac{H_1 M^4 \left( \frac{\partial}{\partial r} u(r) \right)}{2 r + 3 M} + 9 \frac{H_1 M^4 u(r)}{2 r + 3 M} \\
& + 27 \frac{H_1 M^5 \left( \frac{\partial}{\partial r} u(r) \right)}{r (2 r + 3 M)} \\
& > \text{hcollect(temp,\{u(r),diff(u(r),r)\},\{H[0],H[1]\})} ;
\]

\[
\begin{align*}
& \left( \frac{\left( r^2 + M r + M^2 \right)^2 (r - 2 M)^3 H_0}{r^3 (2 r + 3 M)^2} + \frac{\left( r^2 - 6 M r - 3 M^2 \right) H_1}{r} \right) \left( \frac{\partial}{\partial r} u(r) \right) \\
& + \left( \frac{(r - 2 M)^2 \left( r^2 + M r + M^2 \right)^2 H_0}{r^2 (2 r + 3 M)^2} + \frac{\left( r^2 - 6 M r - 3 M^2 \right) H_1}{r (2 r + 3 M)} \right) u(r)
\end{align*}
\]

As with any “free code”, one should always perform some random consistency checks:

> simplify(hcollect(temp,\{u(r),diff(u(r),r)\},\{H[0],H[1]\})-temp);

• mcollect(expr, \{arg\_one [, arg\_two ]\}): The mcollect() routine will collect the terms of an expression w.r.t. the product (arg\_one arg\_two). If arg\_two is not supplied the collection is done w.r.t. (arg\_one)^2. (This routine required a relatively large amount of coding. In particular, one has to very careful, from a coding standpoint, when the arguments can be derivatives of a function.) This routine automatically calls termsimp() when reconstructing the final expression.

\[\text{e.g.}\]
> temp:=expand(p1*(u(r)+(r-2*M)/r*diff(u(r),r))*u(r)+(r-2*M)/r*diff(u(r),r));

\[
\begin{align*}
temp := & \frac{2 r^4 u(r) \left( \frac{\partial}{\partial r} u(r) \right)}{(2 r + 3 M)^2} + 11 \frac{r^2 \left( \frac{\partial}{\partial r} u(r) \right)^2 M^2}{(2 r + 3 M)^2} - 2 \frac{r^3 M u(r)^2}{(2 r + 3 M)^2} - 6 \frac{r M^3 \left( \frac{\partial}{\partial r} u(r) \right)^2}{(2 r + 3 M)^2}
\end{align*}
\]

\[\text{e.g.}\]
> temp:=expand(p1*(u(r)+(r-2*M)/r*diff(u(r),r))*u(r)+(r-2*M)/r*diff(u(r),r));
\[- \frac{r^2 M^2 u(r)^2}{(2 r + 3 M)^2} + 9 \frac{M^4 \left( \frac{\partial}{\partial r} u(r) \right)^2}{(2 r + 3 M)^2} - 2 \frac{r M^3 u(r)^2}{(2 r + 3 M)^2} - 24 \frac{M^5 \left( \frac{\partial}{\partial r} u(r) \right)^2}{r (2 r + 3 M)^2} \]
\[+ 5 \frac{M^4 u(r)^2}{(2 r + 3 M)^2} + 8 \frac{M^6 \left( \frac{\partial}{\partial r} u(r) \right)^2}{r^2 (2 r + 3 M)^2} + 4 \frac{M^5 u(r)^2}{r (2 r + 3 M)^2} + 16 \frac{M^8 \left( \frac{\partial}{\partial r} u(r) \right)^2}{r^4 (2 r + 3 M)^2} \]
\[+ r^4 u(r)^2 \left( \frac{\partial}{\partial r} u(r) \right)^2 + 4 \frac{M^6 u(r)^2}{r^2 (2 r + 3 M)^2} - 8 \frac{r^3 u(r) \left( \frac{\partial}{\partial r} u(r) \right) M}{(2 r + 3 M)^2} \]
\[- 0 \frac{r^3 \left( \frac{\partial}{\partial r} u(r) \right)^2}{(2 r + 3 M)^2} + 6 \frac{r^2 M^2 u(r) \left( \frac{\partial}{\partial r} u(r) \right)^2}{(2 r + 3 M)^2} + 18 \frac{M^4 u(r) \left( \frac{\partial}{\partial r} u(r) \right)}{(2 r + 3 M)^2} \]
\[-12 \frac{M^5 u(r) \left( \frac{\partial}{\partial r} u(r) \right)}{r (2 r + 3 M)^2} - 8 \frac{M^6 u(r) \left( \frac{\partial}{\partial r} u(r) \right)}{r^2 (2 r + 3 M)^2} - 16 \frac{M^7 u(r) \left( \frac{\partial}{\partial r} u(r) \right)}{r^3 (2 r + 3 M)^2} \]
\[mtemp := \frac{(r^2 + M r + M^2)^2 (r - 2 M)^2 u(r)^2}{r^2 (2 r + 3 M)^2} + 2 \frac{(r^2 + M r + M^2)^2 (r - 2 M)^3 u(r) \left( \frac{\partial}{\partial r} u(r) \right)}{r^3 (2 r + 3 M)^2} \]
\[+ \frac{(r^2 + M r + M^2)^2 (r - 2 M)^4 \left( \frac{\partial}{\partial r} u(r) \right)^2}{r^4 (2 r + 3 M)^2} \]

And just to be sure:

\[> \text{simplify(mtemp-temp);} \]

\[0 \]

- **kfactor**(*expr, fctr*): The `kfactor()` routine simply forces Maple to pull the factor `fctr` out of the expression. The chief utility of this is largely cosmetic, although it can be useful when comparing expressions to published results.

  *e.g.*

  \[> \text{temp:=p1*H[0]+p2*H[1];} \]

  \[\text{temp} := \frac{(r^2 + M r + M^2)^2 (r - 2 M)^2 H_0}{r^2 (2 r + 3 M)^2} + \frac{(2 r^2 - 6 M r - 3 M^2)^2 H_1}{r (2 r + 3 M)} \]

  \[> \text{kfactor(temp,(r^2+M*r+M^2)^2*(r-2*M)^2)}; \]

  \[(r^2 + M r + M^2)^2 (r - 2 M)^2 \left( \frac{H_0}{r^2 (2 r + 3 M)^2} + \frac{(2 r^2 - 6 M r - 3 M^2)^2 H_1}{r (2 r + 3 M) (r^2 + M r + M^2)^2 (r - 2 M)^2} \right) \]

- **linpert**(*expr*): The `linpert()` routine returns the first order Taylor expansion in \( \epsilon \), about \( \epsilon = 0 \), of the supplied expression. While this routine could be made far more elaborate in terms of options and parameters, it was specifically designed as a single argument function so that it could be used within GRTensorII in a seamless manner.

  *e.g.*

  \[> \text{temp:=1/sqrt(1+epsilon*f(r,theta,phi,t));} \]

  \[\text{temp} := \frac{1}{\sqrt{1 + \epsilon f(r, \theta, \phi, t)}} \]
\[ > \text{linpert(temp);} \]
\[ 1 - \frac{1}{2} \varepsilon f(r, \theta, \phi, t) \]

- \textbf{quadpert(expr):} The \textit{quadpert()} routine is similar to the \textit{linpert()} routine, except that it returns the second order Taylor expansion of the expression in \( \varepsilon \).

\textit{e.g.}
\[ > \text{quadpert(temp);} \]
\[ 1 - \frac{1}{2} \varepsilon f(r, \theta, \phi, t) + \frac{3}{8} f(r, \theta, \phi, t)^2 \varepsilon^2 \]