Chain of point-like potentials in $\mathbb{R}^3$ and infiniteness of the number of bound states

A.A.Boitsev, I.Yu.Popov and O.V.Sokolov

St. Petersburg National Research University of Information Technologies, Mechanics and Optics, 49 Kronverskiy, St. Petersburg, 197101, Russia

E-mail: boitsevanton@gmail.com, popov1955@gmail.com

Abstract. Infinite chain of point-like potentials having the Hamiltonian with infinite number of eigenvalues below the continuous spectrum is constructed. The background of the model is the theory of self-adjoint extensions of symmetric operators in the Hilbert space. The analogous example of the Hamiltonian is obtained for the system of three-dimensional waveguides coupled through point-like windows.

1. Introduction

The problem of existence of infinite number of eigenvalues of the Schrödinger operator. It appears in different physical situations. Particularly, in three-particle problem it is known as the Efimov effect [1], [2], [3]. Strict mathematical justification for the existence of infinitely many bound states below the continuous spectrum for short-range perturbations of a much larger class of spinorbit Hamiltonians, which includes, in particular, the above Rashba Hamiltonian was obtained in [4]. The authors uses min-max principle in the spirit of [5]. In the present paper we construct an operator having infinitely many different eigenvalues below the continuous spectrum. The result can be interested from the point of view of functional model due to close relation between the properties of the characteristic functions and the operator spectrum [6]. More precisely, we construct a chain of zero-range potentials with specific intensities which gives us the operator in question. This type of potential has useful advantage- it allows us to construct the spectral equation in an explicit form, correspondingly, the spectral analysis of the operator is simpler. Point-like interaction is introduced by a conventional way [8], [7]. The construction is based on the theory of self-adjoint extensions of symmetric operators. It is widely used for the description of various quantum systems.

Let us describe briefly the model operator construction. Consider a chain of $n$ delta-potentials in $\mathbb{R}^3$. The positions of the potentials are $x_m$. First, we restrict the Laplace operator onto the set of smooth functions vanishing at the points $x_m$. The obtained operator is symmetric non-self-adjoint. To construct a self-adjoint extension, it is more convenient to deal with the corresponding restriction of the adjoint operator instead of the extension of the symmetric operator. There are several ways of extensions descriptions, e.g., boundary triplets method ([9], von Neumann formulas ([11]), Krein resolvent formula ([10]). We will use here a variant of the second approach which allows one in the case of semi-boundedness of the Hamiltonian to present...
an element from the domain of the adjoint operator in the following form:

$$\psi(x) = \psi_0(x) + \sum_{m=1}^{n} \tilde{a}_m \cdot \frac{e^{-ik_0|x-x_m|}}{4\pi|x-x_m|},$$

where $\psi_0$ belongs to the domain of the Friedrichs extension of the initial symmetric operator, $\tilde{a}_m$ is some constant, $k_0 = \sqrt{\lambda_0}$, $\lambda_0$ is a regular value of the spectral parameter (we choose real negative value of $\lambda_0$, $\Re k_0 > 0$). Let $\psi_0(x_m) = b_m, m = 1, 2, \ldots, n$. We will not describe all possible self-adjoint extensions, but will consider a particular extension determined by the following correlation between $n$-vectors:

$$b_i = \alpha_i \cdot \bar{a}_i, \quad i = 1, \ldots, n.$$  

Here the matrix parameterizing the extension is chosen in the diagonal form, $\alpha_j, j = 1, \ldots, n$, are some real parameters. The physical meaning of the parameters are the potential intensities.

2. Main result

The main result of the paper is the following theorem.

**Theorem.** The Laplace operator in $L_2(\mathbb{R}^3)$ perturbed by infinite chain of zero-range potentials posed at the points $(0,0,0), \ldots, x_m, \ldots$,

$$x_1 = (0,0,0) \quad x_m = x_{m-1} + (m^m \cdot 2^{2m^2},0,0), m = 2,3,\ldots,$$

of intensities, correspondingly, $\alpha_m$,

$$\alpha_m = \frac{1}{2^m} + \frac{1}{4\pi} - \frac{i k_0}{4\pi},$$

has infinite number of eigenvalues below the continuous spectrum.

Note that the analogous consideration can be performed for another system. Consider two half spaces of $\mathbb{R}^3$ and let us deal with the Laplace operator in $L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3)$ with the Neumann boundary condition at the plane. Let the set of points $(0,0,0), \ldots, (x_m,0,0), \ldots$, belongs to the plane. Let us restrict the operator on the set of smooth functions vanishing near the chosen points. It is symmetric operator with infinite deficiency indices. Elements from the domain of the adjoint operators have the form:

$$\begin{pmatrix} \psi_{+}(x) \\ \psi_{-}(x) \end{pmatrix} = \begin{pmatrix} \psi_{0,+}(x) \\ \psi_{0,-}(x) \end{pmatrix} + \begin{pmatrix} \sum_{m=1}^{n} \tilde{a}_{m+} \cdot \frac{e^{-ik_0|x-x_m|}}{2\pi|x-x_m|} \\ \sum_{m=1}^{n} \tilde{a}_{m-} \cdot \frac{e^{-ik_0|x-x_m|}}{2\pi|x-x_m|} \end{pmatrix},$$

To construct a selfadjoint extension it is necessary to determine a linear relation between the coefficients $a_m$ and $b_m$ (analogously to the procedure for point-like potential described above). This procedure gives us a model of point-like windows in the plane (see [12], [13]). The type of the chosen self-adjoint extension is related with the type of physical coupling through the windows. It is easy to see that in this case one can obtain the analogous theorem. Moreover, one can obtain similar theorem for the case when the half spaces are replaced by straight waveguides coupled through a system of point-like windows [14]. In this situation it is necessary to replace

$$\frac{e^{-ik_0|x-x_m|}}{2\pi|x-x_m|}$$

by the Green functions for the waveguides. The theorem is obtained due to the coincidence of the singularities of the Green functions for the waveguides and for the free space. We will not formulate the theorem for this case because the changes in respect to the above written theorem is rather evident.
3. Main steps of the proof

Our goal is to construct such infinite chain that gives us infinite number of eigenvalues. To ensure it we consider a limiting procedure \((n \to \infty)\) and make a special choice of potentials positions \(x_m\) and potentials intensities \(\alpha_m\). More precisely, we construct the chain step by step. At each stage we add one center to the chain and look for the behavior of the eigenvalues. The addition of the zero-range potential leads to the appearance of new eigenvalue and to the shift of the eigenvalues obtained at the previous stage. We choose the position and the intensity of the added zero-range potential in a proper way to avoid gluing of the eigenvalues. The resulting Hamiltonian with infinite chain of zero-range potentials is obtained as a strong resolvent limit of the constructed sequence of operators.

Let us choose the following positions of the centers:

\[
x_1 = (0, 0, 0) \quad x_m = x_{m-1} + (m^m \cdot 2^{2m^2}, 0, 0), m = 2, 3, \ldots,
\]

In accordance with the description of the operator domain given above we seek the eigenfunction corresponding to the eigenvalue \(\lambda\) in the form:

\[
\sum_{m=1}^{n} a_m \cdot e^{-ik|x-x_m|} / 4\pi |x-x_m|,
\]

\(k = \sqrt{\lambda}\). Let us determine parameters \(\tilde{a}_m, b_m\) from the definition (1). Due to the fact that the singularities of \(e^{-ik|x-x_m|} / 4\pi |x-x_m|\) and \(e^{-ik_0|x-x_m|} / 4\pi |x-x_m|\) are identical, one gets \(\tilde{a}_m = a_m\). To get the coefficient \(b_m\) one should subtract the singular term \(e^{-ik_0|x-x_m|} / 4\pi |x-x_m|\).

Let us introduce the notations

\[
t = \frac{ik}{4\pi} \quad t \in \mathbb{R}, \quad t > 0, \quad \Delta e_{i,j} = \frac{e^{-4\pi t|x_i-x_j|}}{4\pi |x_i-x_j|}
\]

Due to (4) the relations for \(b_i\) are as follows

\[
b_i = \left(t + \frac{1}{4\pi}\right) a_i + \sum_{j=1, j \neq i}^{n} \Delta e_{i,j} \cdot a_j.
\]

In accordance with our choice of the extension (2) one has

\[
b_i = \alpha_i \cdot a_i, \quad i = 1, \ldots, n
\]

Let us choose the extension parameters (potential intensities \(\alpha_j\)) by a special way

\[
\alpha_j = \frac{1}{2j} + \frac{1}{4\pi} - \frac{i k_0}{4\pi}
\]

Subtracting (6) from (5), one obtains that the system of equations has non-trivial solutions if and only if

\[
\tilde{f}_n(t) = \begin{vmatrix}
t - \beta_1 & \Delta e_{1,2} & \Delta e_{1,3} & \ldots & \Delta e_{1,n} \\
\Delta e_{1,2} & t - \beta_2 & \Delta e_{2,3} & \ldots & \Delta e_{2,n} \\
\Delta e_{1,3} & \Delta e_{2,3} & t - \beta_3 & \ldots & \Delta e_{3,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Delta e_{1,n} & \Delta e_{2,n} & \Delta e_{3,n} & \ldots & t - \beta_n
\end{vmatrix} = 0
\]
This equation (7) doesn’t have more than \(n\) roots. Our goal is to show that the equations has exactly \(n\) roots.

Below we assume that \(t \in (0; 1)\).

Expanding the determinant \(\tilde{f}_n(t)\) along the last column, one gets

\[
\tilde{f}_n(t) = \tilde{f}_{n-1}(t) \cdot (t - \beta_n) + \epsilon_n(t).
\]  

(8)

Making algebraic transformations and using the Hadamard inequality, we come to the following estimation

\[
|\epsilon_n(t)| < \frac{1}{2^{2n^2}}.
\]  

(9)

Let

\[
d(n) = \frac{1}{2^{2n^2}}
\]  

(10)

**Remark.** If \(\tilde{f}_n(t)\) has \(n\) roots in the case when \(\epsilon_n(t) = d(n)\) then it has \(n\) roots when \(|\epsilon_n(t)| < d(n)\).

Consider the case when \(\epsilon_n(t) = d(n)\). One can see that

\[
\tilde{f}_1(t) = t - \frac{1}{2},
\]

\[
\tilde{f}_n(t) = \tilde{f}_{n-1}(t) \cdot \left(t - \frac{1}{2^n}\right) + d(n).
\]

Hence,

\[
\tilde{f}_n(t) = \left(\cdots \left(\left(t - \frac{1}{2} \right) \left(t - \frac{1}{2^2}\right) + d(2) \right) \left(t - \frac{1}{2^n}\right) + d(3)\right) \left(t - \frac{1}{2^n}\right) + \cdots \left(t - \frac{1}{2^n}\right) + d(n).
\]

Simple transformation gives us

\[
\tilde{f}_n(t) = \prod_{k=1}^{n} \left(t - \frac{1}{2^k}\right) + \sum_{i=2}^{n-1} d(i) \cdot \prod_{k=i+1}^{n} \left(t - \frac{1}{2^k}\right) + d(n).
\]  

(11)

Let us introduce the notation

\[
f_n(t) = \prod_{k=1}^{n} \left(t - \frac{1}{2^k}\right),
\]  

(12)

\[
D_n(t) = \sum_{i=2}^{n-1} d(i) \cdot \prod_{k=i+1}^{n} \left(t - \frac{1}{2^k}\right) + d(n),
\]  

(13)

correspondingly,

\[
\tilde{f}_n(t) = f_n(t) + D_n(t).
\]  

(14)

The function \(f_n(t)\) has \(n\) zeroes in (0; 1). To show that \(\tilde{f}_n(t)\) also has \(n\) zeroes in this interval, we prove that for any interval of the form \(\left(\frac{1}{2^m}, \frac{1}{2^{m-1}}\right)\) there exists \(t \in \left(\frac{1}{2^m}, \frac{1}{2^{m-1}}\right)\) such that \(f_n(t) \cdot \tilde{f}_n(t) > 0\). Here \(m = 2, 3, ... n\).

Consider the intervals where \(f_n(t) > 0\). They are as follows \(\left(\frac{1}{2^{2m+1}}, \frac{1}{2^{2m}}\right)\), \(m = 1,...\lfloor n/2 \rfloor - 1\). It can be shown that \(\tilde{f}_n(t) > 0\) for these intervals.
Consider the intervals where \( f_n(t) < 0 \). They are \( \left( \frac{1}{2^{m-1}}, \frac{1}{2^{m-1}} \right) \), \( m = 1, \ldots, \lfloor n/2 \rfloor \). Fix \( m \), so we have \( t \in \left( \frac{1}{2^m}, \frac{1}{2^{m-1}} \right) \). Let \( t_0 = \frac{1}{2^m} + \frac{1}{2^{m+1}} \) be the middle of the interval. It can be shown that \( \tilde{f}_n(t_0) < 0 \).

As a conclusion, we have proved that any interval of the form \( \left( \frac{1}{2^m}, \frac{1}{2^{m-1}} \right) \) contains a point such that \( f_n(t) \cdot \tilde{f}_n(t) > 0 \), where \( m = 2, 3, \ldots, n \). Then, from the fact that a continuous on \([a, b]\) function \( f(x) \) with the condition \( f(a) \cdot f(b) < 0 \) has at least one root on \((a, b)\) we obtain the fact that \( \tilde{f}_n(t) \) has exactly \( n \) roots.

4. Final steps
As a result, we come to the conclusion, that the constructed Hamiltonians (with \( n \) zero-range potentials) has \( n \) different eigenvalues below the continuous spectrum for any fixed \( n \), and the limiting procedure \( (n \to \infty) \) does not lead to the gluing of the eigenvalues (due to the fact that they are localized in non-intersecting intervals). The required Hamiltonian is obtained as a strong resolvent limit of this operator sequence. Namely, let

\[
X = \{ x_j \in \mathbb{R}^3, j \in \mathbb{N} \}, \quad X_n = \{ x_j \in \mathbb{R}^3, j \in \mathbb{N}, j \leq n \},
\]

\( \alpha : X \to \mathbb{R} \).

Let \(-\Delta_{X, \alpha}, -\Delta_{X_n, \alpha}\) be the Laplace operators with potentials posed at the points from \( X \) (correspondingly, \( X_n \)) of intensities (extension parameters \( \alpha_j \), see (2)) determined by \( \alpha \) (see above). We denote the corresponding resolvent by \( R(\lambda), \quad R_n(\lambda) \).

**Lemma 2.** The resolvent \( R(\lambda) \) is a strong limit of the sequence of operators \( R_n(\lambda) \):

\[
R(\lambda) = s - \lim_{n \to \infty} R_n(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

**Proof.** The statement of the lemma follows from the theorem 3.1.1.1 [8] due to the fact that in our case (for our choice of \( X \) and \( \alpha \))

\[
\inf_{j \neq j'} \inf_{j, j' \in \mathbb{N}} |x_j - x_{j'}| = d > 0
\]

and \( \alpha \) is bounded.

Hence, the Hamiltonian with the constructed infinite chain has infinite number of different eigenvalues.

Thus, the proof of the theorem is completed.

**Acknowledgements**
This work was partially financially supported by the Government of Russian Federation (grant 074-U01), by Ministry of Science and Education of the Russian Federation (GOSZADANIE 2014/190, Project 14.Z50.31.0031), by grant of Russian Foundation for Basic Researches and grants of the President of Russia (state contracts 14.124.13.2045-MK and 14.124.13.1493-MK).

**References**
[1] Efimov V.N. Bound states of three resonances of interacting particles. Nuclear Phys. 12 (5) (1970) 1080-1091.
[2] Yafaev D.R. On the theory of the discrete spectrum of the three-particle Schrodinger operator. Mat. Sborhik.94 (4) (1974) 567-593.
[3] Tamura H. The Efimov effect of three-body Schrodinger operators. J. Funct. Anal. 95 (2) (1991) 433-459.
[4] Bruning J., Geyler V., Pankrashkin K. On the number of bound states for weak perturbations of spinorbit Hamiltonians. J. Phys. A: Math. Theor. 40 F113 - F117 (2007).
[5] Yang K., de Llano M. Simple variational proof that any two-dimensional potential well supports at least one bound state. Am. J. Phys. 57, 85-86 (1989).
[6] Sz-Nagy B., Foias C., Bercovici H., Kerchy L. Harmonic Analysis of Operators on Hilbert Space. Springer, Berlin, 2010.
[7] Pavlov B.S. Extensions theory and explicitly solvable models. Uspekhi Mat.Nauk 42 (6), 99-131 (1987).
[8] Albeverio S., Gesztesy F., Hoegh-Krohn R. and Holden H. with an appendix by Exner P., Solvable Models in Quantum Mechanics: Second Edition. (AMS Chelsea Publishing, Providence, R.I. 2005).
[9] Berndt J., Malamud M. M., Neidhardt H. Scattering matrices and Weyl functions. Proc. London Math. Soc. 97 (3), P. 568–598 (2008).
[10] Alonso A., Simon B. The Birman-Krein-Vishik theory of selfadjoint extensions of semi-bounded operators. J. Oper. Theory. 4, 251-270 (1980).
[11] Birman M. S., Solomyak M. Z. Spectral Theory of Self-Adjoint Operators in Hilbert Space (D. Reidel Publishing Company, Dordrecht-Boston-Lancaster, 1987).
[12] Popov I.Yu. Extensions theory and localization of resonances for domains of trap type. Math. USSR-Sbornik 71 (1), 209-234 (1992)
[13] Popov I.Yu. The resonator with narrow slit and the model based on the operator extensions theory. J. Math. Phys. 33 (11), 3794-3801 (1992)
[14] Popov I.Yu. Asymptotics of bound states and bands for laterally coupled waveguides and layers. J. Math. Phys. 43 (1), 215-234 (2002).