GLOBAL FUKAYA CATEGORY AND THE SPACE OF $A_\infty$ CATEGORIES II

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1. INTRODUCTION

In part I we have constructed a “classifying” space $\mathcal{A}_\infty^K$ of unital $A_\infty$ categories, over a field $K$ which arose as the maximal Kan subcomplex of a certain natural quasi-category $A_\infty - \text{Cat}_K^\omega$ with vertices unital $A_\infty$ categories, and morphisms (essentially) just $A_\infty$ bimodules. For further motivation of why one may be interested in having such a space see part I. It is natural to wonder if and why this is the “right” space of $A_\infty$ categories. The most telling point is that a Hamiltonian $M$ fibration $P$ over $X$, induces a classifying map

$$f_P : X \to \mathcal{A}_\infty^K.$$ (1.1)

We shall further illustrate here that $\mathcal{A}_\infty^K$ is very closely intertwined with symplectic geometry and the theory of Fukaya categories, so it does seem to be “right” geometrically. What about algebraically? Recall the conjecture in Part I: (after adjusting the grading, see the section on conventions)

Conjecture 1.1.

$$\pi_i(\mathcal{A}_\infty^K, C) \simeq \text{HH}^{i-2}(C), i > 2,$$

$$\pi_2(\mathcal{A}_\infty^K, C) \to \text{HH}^0(C)^*, \quad * \text{ denoting the subgroup of invertible elements.}$$

If this was true then one can argue that at least on one level $A_\infty$ is algebraically right, as this statement is a direct analogue of a theorem due to Toen [14] for the classifying space of dg-categories. We shall explore this in the case $C = \text{Fuk}(S^2)$. Specifically using the classifying maps (1.1) corresponding to Hamiltonian $S^2$ fibrations $P$ over $S^4$ we prove:

Theorem 1.2. There is an injection of $\mathbb{Z}$ into $\pi_4(\mathcal{A}_\infty^Q, \text{Fuk}(S^2))$, with $\text{Fuk}(S^2)$ considered as an $A_\infty$ category over $Q$.

On the other hand $\text{HH}^2(S^2) \simeq \mathbb{Q}\text{H}^2(S^2) \simeq \mathbb{Q}$, which is a case of the Kontsevich conjecture [5]. Although there may still be no proof of this in the literature, (as far as I know) this is widely expected to hold, at least for monotone $(M, \omega)$ satisfying the generation criterion of Abouzaid [1], which of course holds for $S^2$. So in this case the above conjecture is reduced to the statement that $\pi_4(\mathcal{A}_\infty^Q, \text{Fuk}(S^2))$ has one generator over $\mathbb{Q}$.

To prove homotopical non triviality of the classifying map $f_P$ for a non-trivial Hamiltonian $S^2$ fibration $P$ as above, we show that the “global Fukaya category” associated to $P$ is non-trivial as a (co)-Cartesian fibration over $S^4$. We do this by first extracting a Kan fibration over $S^4$, and then attacking it by elementary techniques of homotopy theory. The calculation in particular will imply that $\pi_3$ of the $A_\infty$ nerve of $\text{Fuk}(S^2)$ admits a $\mathbb{Z}$ injection, (but by itself this is not enough
to recover the statement on $\pi_4(A^\infty_\infty, \text{Fuk}(S^2))$. Thus the calculation is in a sense simple and natural, reducing what are often complicated homological algebra manipulations to “simple” topology and this is one of the rewards of the categorical formulations in part I. It is however only “algebra” that becomes simpler, there still remains a difficult chain level analytical calculation that must be performed, and for this we use a technique based on Hofer geometry.

Other than the topological/algebraic application above the calculation also yields an application in Hofer geometry, in section 10. On the way in Section 8 we also construct a higher dimensional version of the relative Seidel morphism $[\mu]$ in the monotone context, and show its non triviality in Section 9. The Sections 10 and 8 are logically independent of the $\infty$-categorical and even the $A_\infty$ setup and maybe read independently.

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3. Conventions and notations

We use notation $\Delta^n$ to denote the standard topological $n$-simplex. For the standard representable $n$-simplex as a simplicial set we mostly use the notation $\Delta^n$, and in general the under-bullet notation implies we are dealing with a simplicial set, (we supressed this in the introduction). For a topological space $X$ and singular simplex $\Sigma : \Delta^n \to X$ we may denote its image just by $\Sigma$.

Our Fukaya categories here follow homological grading conventions, although it will be implicit. So in particular differential is in degree $-1$, and multiplication is degree $-n$. As we will be in the Morse-Bott setup and it is simpler dealing with chains. However in the Conjecture 1.1, to be consistent with Toen, the differential on $C$ should in degree +1, and multiplication graded (in degree 0). So to obtain a special case of the conjecture as discussed in the introduction we have to first dualize our Fukaya categories to the co-chain setup, we shall not make this explicit. Although we follow Fukaya-Oh-Ono-Ohta for some things we use Seidel’s notation $\mu^k$ for composition operations in the $A_\infty$ categories as opposed to $m_k$. Mostly because the letter $m$ seems better used for naming morphisms in our quasi-categories.

We shall often denote Hamiltonian connections by a caligraphic letter such as $\mathcal{A}$. Not to be confused with the space of $A_\infty$ categories denoted by $\mathcal{A}_\infty$.

4. Preliminaries on coupling forms

We refer the reader to [6, Chapter 6] for more details on what follows. A Hamiltonian fibration is a smooth fiber bundle

$$M \hookrightarrow P \rightarrow X,$$

with structure group $\text{Ham}(M, \omega)$. A coupling form, originally appearing in [3], for a Hamiltonian fibration $M \hookrightarrow P \overset{\pi}{\to} X$, is a closed 2-form $\tilde{\Omega}$ whose restriction to fibers coincides with $\omega$ and which has the property:

$$\int_M \tilde{\Omega}^{n+1} = 0 \in \Omega^2(X).$$

Such a 2-form determines a Hamiltonian connection $\mathcal{A}_{\tilde{\Omega}}$, by declaring horizontal spaces to be $\tilde{\Omega}$ orthogonal spaces to the vertical tangent spaces. A coupling form generating a given connection $\mathcal{A}$ is unique. A Hamiltonian connection $\mathcal{A}$ in turn determines a coupling form $\tilde{\Omega}_\mathcal{A}$ as follows. First we ask that $\tilde{\Omega}_\mathcal{A}$ generates the connection $\mathcal{A}$ as above. This determines $\tilde{\Omega}_\mathcal{A}$, up to values on $\mathcal{A}$ horizontal lifts $\tilde{v}, \tilde{w} \in T_pP$ of $v, w \in T_xX$. We specify these values by the formula

$$\tilde{\Omega}_\mathcal{A}(\tilde{v}, \tilde{w}) = R_\mathcal{A}(v, w)(p),$$

where $R_\mathcal{A}|_x$ is the curvature 2-form with values in $C^\infty_{\text{norm}}(p^{-1}(x))$.

5. A Morse-Bott version of the construction in Part I

Although we do not absolutely need the construction in this section for the calculation in section 6, it will make some geometry much simpler, and help with exposition. Let us briefly recall from part I the construction of an $A_\infty$ category $F_P(\Sigma)$ associated to a given non-degenerate smooth $n$-simplex $\Sigma : \Delta^n \to X$, for $M \hookrightarrow P \to X$ a Hamiltonian fibration over $X$. This discussion will still be ungraded over $\mathbb{K} = \mathbb{F}_2$. 


Given
\[ \Sigma : \Delta^n \to X, \]
a collection of Lagrangian submanifolds \( \{ L_{\rho(k)} \}_{k=0}^{d} \),
\[ \rho : \{0, \ldots, d\} \to \{ x_i \}_{i=0}^{n}, \]
for \( x_i \) the vertices of \( \Delta^n \), (later on we shall often denote these by \( i \) alone) with each \( L_{\rho(k)} \in \text{Fuk}(P_{\Sigma^0(\rho(k))}) \) we constructed structure maps
\[ \mu^d_{\Sigma} : \text{hom}(L_{\rho(0)}, L_{\rho(1)}) \otimes \cdots \otimes \text{hom}(L_{\rho(d-1)}, L_{\rho(d)}) \to \text{hom}(L_{\rho(0)}, L_{\rho(d)}), \]
for the \( A_\infty \) category \( F_P(\Sigma) \). These structure maps are defined by:
\[ \langle \mu^d_{\Sigma}(\gamma_1, \ldots, \gamma_d), \gamma_0 \rangle = \sum_{A} \int_{M(\gamma_k), \Sigma, \mathcal{F}, \mathcal{A}} 1, \]
(the integral just means count). Where the moduli space
\[ \overline{M}(\{ \gamma_k \}, \Sigma, \mathcal{F}, \mathcal{A}) \]
is defined as follows. First one takes a certain natural system \( \mathcal{U} \), specifying maps
\[ u(m_1, \ldots, m_d) : \mathcal{S} \to \Delta^n, \]
where \( m_k : \Delta^1 \to \Delta^n \) are edges of \( \Delta^n \), from \( \rho(k-1) \) to \( \rho(k) \). Where \( \mathcal{S} \) is the “universal curve” over the compactified moduli space \( \overline{M} \) of complex structures on the disk \( D^2 \) with \( d+1 \) punctures on the boundary. The fiber of \( \mathcal{S} \) over \( r \) is denoted by \( \mathcal{S}_r \) or just by \( \mathcal{S}_r \) to simplify notation. The ends of \( \mathcal{S} \) are labeled by \( \{ e_i \} \), with subscripts \( i \) respecting the natural cyclic order.

Then very briefly: the above moduli space consists of pairs \( (\sigma, r) \) for \( \sigma \) a (pseudo)-holomorphic, with certain Lagrangian boundary conditions, class \( A \) (stable)-section of
\[ \mathcal{S}_r = (\Sigma \circ u(m_1, \ldots, m_d, n)|_{\mathcal{S}_r})^* P, \]
asymptotic to some collection of generators
\[ \{ \gamma_k \in \text{hom}_{\Sigma(m_k)}(L_{\rho(k-1)}, L_{\rho(k)}), \gamma_0 \in \text{hom}_{\Sigma(m_k)}(L_{\rho(0)}, L_{\rho(d)}) \}, \]
at the respective ends \( e_k \), whenever the expected dimension of this moduli space is 0. Here \( \gamma_k \) is an isolated flat section for an auxiliary connection \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \) on \( \Sigma(m_k)^* P \). What pseudo-holomorphic means is determined by a chosen natural system of Hamiltonian connections \( \mathcal{F} \), compatible with \( \mathcal{U} \). The Lagrangians \( L_i \) are assumed here to be monotone and unobstructed. Let us set
\[ \Sigma \circ m_i = \overline{m}_i. \]

Suppose now that all \( L_{\rho(k)} \) are Hamiltonian isotopic so that there is a Hamiltonian connection \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \) on \( \overline{m}_k P \) whose parallel transport map over the interval takes \( L_{\rho(k-1)} \) to \( L_{\rho(k)} \) by a diffeomorphism. We will now adjust the definition of morphism spaces and the construction of the structure maps to use a Morse-Bott version of the construction, which is essentially a direct translation of the beautiful Morse-Bott construction used in \([2]\) in the context of the usual Fukaya category. Let \( \mathcal{S}(L_{\rho(k-1)}, L_{\rho(k)}, \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)})) \) denote the space of \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \) flat sections with boundary on \( L_{\rho(k-1)}, L_{\rho(k)} \). For \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \) as above
\[ \mathcal{S}(L_{\rho(k-1)}, L_{\rho(k)}, \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)})) \simeq L_{\rho(k-1)} \simeq L_{\rho(k)}. \]
Then our complex

\[ \text{hom}_{\text{Diff}^k}(L_{\rho(k-1)}, L_{\rho(k)}; \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)})) \]

is freely generated over \( F_2 \) by the set \( C(S(L_{\rho(k-1)}, L_{\rho(k)})) \) of smooth singular simplices in \( S(L_{\rho(k-1)}, L_{\rho(k)}) \).

For \( c \in \text{hom}_{\text{Diff}^k}(L_{\rho(k-1)}, L_{\rho(k)}; \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)})) \) the differential is

\[ (5.1) \quad \partial c = \sum_A \partial e v_A(\overline{\mathcal{M}}^h(c, A(L_{\rho(k-1)}, L_{\rho(k)}), A)) \]

where

\[ (5.2) \quad \overline{\mathcal{M}}^h(c, A(L_{\rho(k-1)}, L_{\rho(k)}), A) \]

is the (possibly virtual) fundamental chain of the moduli space of stable, total class \( A \) finite energy \( J(\overline{\mathcal{A}}(L_{\rho(k-1)}, L_{\rho(k)}))-\)holomorphic sections \( \text{see of} \)

\[ \overline{m}_k \mathcal{P} \times \mathbb{R} \rightarrow [0, 1] \times \mathbb{R}, \]

with boundary on the constant \( \overline{\mathcal{A}}(L_{\rho(k-1)}, L_{\rho(k)}) \) invariant Lagrangian sub-bundle

\[ L_{\rho(k-1)} \times \mathbb{R} \sqcup L_{\rho(k)} \times \mathbb{R} \]

of \( \overline{m}_k \mathcal{P} \), asymptotic to a flat section in the image of \( c \) at \( \infty \). And where

\[ \overline{\mathcal{A}}(L_{\rho(k-1)}, L_{\rho(k)}) \]

is the \( \mathbb{R} \) translation invariant extension of \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \) to \( \overline{m}_k \mathcal{P} \). The sum is finite over \( A \) for monotonicity reasons. Energy here can be taken to be \( L_2 \) energy, with respect to the almost Kahler metric \( g_j \) of the projection of \( \text{see} \) to \( (M, \omega, j) \). The projection satisfies a version of the classical Floer equation, and this energy coincides with the topological energy

\[ (5.3) \quad \langle \langle \text{sec}, -\tilde{\Omega}\mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \rangle \rangle, \]

for \( \tilde{\Omega}\mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \) the coupling form, by classical arguments. Let us make better sense of (5.1) by splitting of the classical differential as follows.

There is a distinguished class in \( \pi_0(\mathcal{P}(L_{\rho(k-1)}, L_{\rho(k)})) \), corresponding to the constant class under the canonical isomorphism

\[ \pi_0(\mathcal{P}(L_{\rho(k-1)}, L_{\rho(k)})) \cong \pi_0(\mathcal{P}(L_{\rho(k-1)}, L_{\rho(k)})) \cong \pi_0(\mathcal{P}(L_{\rho(k)}, L_{\rho(k)})), \]

induced by the connection \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \). Let \( \mathcal{P}(L_{\rho(k-1)}, L_{\rho(k)})_0 \) denote the component of this class. Then we clearly have \( S(L_{\rho(k-1)}, L_{\rho(k)}) \subset \mathcal{P}(L_{\rho(k-1)}, L_{\rho(k)})_0 \).

In particular an element of \( \mathcal{P}(L_{\rho(k-1)}, L_{\rho(k)})_0 \) determines a well defined class in

\[ \pi_2(M, L_{\rho(k)} \mathbb{R}) \equiv \pi_2(M, L_{\rho(k)}), \]

where the isomorphism is induced by \( \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \). The class \( A \) in (5.2) which we previously just thought of as a relative class then can be thought of as an absolute class as above. Note next that if an element \( \text{see} \in \overline{\mathcal{M}}(c, \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}), A) \) is non-constant then the energy and hence

\[ \langle \langle \mathcal{P}, -\tilde{\Omega}\mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}) \rangle \rangle \]

is positive, which immediately implies that \( A \neq 0 \) (in the absolute sense as described above). So \( \overline{\mathcal{M}}^h(c, \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}), A = 0) \) coincides with the space of constant sections, and

\[ \partial \overline{\mathcal{M}}^h(c, \mathcal{A}(L_{\rho(k-1)}, L_{\rho(k)}), A = 0) = \partial_{d}c, \]
for $\partial_d$ the classical boundary operator on chains.

The maps $\mu^d$ are then defined in terms of a certain moduli space
\begin{equation}
\overline{\mathcal{M}}^{MB}(\{\gamma_k\}_{k=1}^{k=d}; \mathcal{F}, \Sigma^n, A),
\end{equation}
where $\gamma_k$ is the name for a morphism corresponding to a generator
\[c_{\gamma_k} \in C(S(L_{\rho(k-1)}, L_{\rho(k)})].
\]

**Notation 5.1.** The possibly slightly awkward notation is meant to fix the use of letters $\gamma$ for generators in hom complexes of our $A_\infty$ categories universally, while the underlying geometric objects are either flat sections (which by abuse of notation are also denoted by $\gamma$) or a singular simplex in the space of flat sections. When we shall want to emphasize the underlying singular simplex we shall write $c_{\gamma}$.

In the above notation $\mathcal{F}$ denotes a natural system of perturbations (i.e. connections), compatible with a natural system $\mathcal{U}$ of maps
\[u^d(m_1, \ldots, m_d, n) : \overline{S}_d \to \Delta^n,
\]
from the “universal curve” $\overline{S}_d$ over the moduli space $\mathcal{K}_d$ of complex structures on a disk with $d + 1$ punctures on the boundary. Natural means that these satisfy certain axioms, see part I. One can readily extend the construction of such natural systems to the present Morse-Bott setting. Suppose then that we have objects $L_{\rho(0)}, \ldots, L_{\rho(d)} \in \mathcal{F}(\Sigma)$. Then a smooth element $(\sigma, r)$ of (5.4) consists of:

- A class $A \mathcal{F}(\{L_{\rho(k)}\}; \Sigma^n, r)$-holomorphic section $\sigma$ of
  \[\overline{S}_r = (\Sigma \circ u^d(m_1, \ldots, m_d, d)|_{\Sigma_r})^* P.
\]
  Where for each $r$, $\mathcal{F}(\{L_{\rho(k)}\}; \Sigma^n, r)$ is a (part of a natural system $\mathcal{F}$) Hamiltonian connection on \[\overline{S}_r \to \Sigma_r.
\]

- The boundary of $\sigma$ is in a certain subfibration $\mathcal{L}_r$ of $\overline{S}_r$ over the boundary of $\Sigma_r$, with fibers Lagrangian submanifolds of the respective fibers of $\overline{S}_r$. Over connected components of the boundary of $\Sigma_r$ by construction of the maps $u_d : \overline{S}_r$ is naturally trivialized, and $\mathcal{L}_r$ coincides with the constant subfibration with fiber $L_{\rho(k)}$ over the boundary component between ends $e_{k-1}$, $e_k$. This determines $\mathcal{L}_r$ uniquely.

- At the $e_k$ end $\sigma$ is asymptotic to a flat section in the image of $c_{\gamma_k}$.

We won’t describe the “stable map” compactification of this moduli space as it is analogous to what is described in [2]. Regularity is obtained in [2] using Kuranishi structures. Although in this paper we shall be exclusively within unobstructed monotone setup, and so virtual techniques should not be strictly necessary, we still have to construct explicit fundamental chains, and this seems conceptually easier with Kuranishi structures approach as opposed to non abstract regularization. However this is more or less a formal point, as we shall explicitly regularize all relevant moduli spaces that we encounter in our specific example via certain Hamiltonian perturbations.

We define $\mu^d(\gamma_1, \ldots, \gamma_d)$ by:
\[\mu^d(\gamma_1, \ldots, \gamma_d) = \sum_A ev_{A*},
\]
where
\[ ev_A : \mathcal{M}^{MB} (\gamma_i, F, \Sigma^n, A) \rightarrow S(L_{\rho(0)}, L_{\rho(d)}), \]
is the “evaluation map” taking a finite energy section to its asymptotic constraint at the \( L_{\rho(0)}, L_{\rho(d)} \) end, and \( ev_{A*} \) denotes the push-forward of the fundamental chain.

The proof of the \( A_\infty \) associativity equations is analogous to what was done previously in part I.

5.1. Exact Hamiltonian connections. It will be helpful for certain calculations with the Atiyah-Singer index theorem, and necessary for certain invariance arguments, to slightly restrict the class of Hamiltonian connections \( F(\{L_{\rho(k)}\}, \Sigma^n, r) \) on \( \mathcal{S}_r \) for each \( r \). Specifically by their properties over any sufficiently “large” (for example in the sense of a complete Riemannian metric on \( \mathcal{S}_r \)) domain \( D \subset \mathcal{S}_r \) diffeomorphic to \( D^2 \), the holonomy path of \( F(\{L_{\rho(k)}\}, \Sigma^n, r) \) over \( \partial D \) stabilizes i.e. is independent of such \( D \). We require that for each such connection \( A \) this limiting holonomy path \( p_A \) is a loop and we shall call such an \( A \) exact.

6. Setup

A Hamiltonian \( S^2 \) fibration over \( S^4 \) is classified by an element \([g] \in \pi_3(\text{Ham}(S^2))\). Such an element determines a fibration \( P_g \) over \( S^4 \) via the clutching construction:
\[ P_g = S^2 \times D^4_+ \sqcup S^2 \times D^4_- / \sim, \]
with \( D^4_- \), \( D^4_+ \) being 2 different names for the standard 4-ball \( D^4 \), and the equivalence relation \( \sim \) is \((x, d) \sim \tilde{g}(x, d), \tilde{g}(x, d) = (g^{-1}x, d) \), for \( d \in \partial D^4 \). From now on \( P_g \) will denote such a fibration for a non-trivial class \([g] \).

A bit of possibly non-standard terminology: we say that \( A \) is a model for \( B \) if there is a morphism \( \text{mod} : A \rightarrow B \) which is a weak equivalence, in the context of some model category. The map \( \text{mod} \) will be called a modelling map. Thus a replacement (fibrant-cofibrant) is a very special kind of model. In our context the modeling map \( \text{mod} \) always turns out to be a monomorphism, but this is not always essential.

6.1. A model for the maximal Kan subcomplex of \( N(\text{Fuk}(S^2)) \). Let us first consider a simpler quasi-isomorphic model \( \text{Fuk}^{eq}(S^2) \) for (our version of) the Fukaya category of \( S^2 \). The category \( \text{Fuk}^{eq}(S^2) \) is constructed using the same geometric setup from part I, but we restrict the objects to oriented great circles, with a choice of a relative spin structure, and we use the Morse-Bott version of the construction, as outlined in the previous section, using \( SO(3) \)-connections to construct \( \text{hom} \) complexes. The associated Donaldson-Fukaya category is isomorphic as a linear category over \( \mathbb{Q} \) to \( \text{FH}(L_0, L_0) \) (considered as a linear category with one object) for \( L_0 \in \text{Fuk}^{eq}(S^2) \) A morphism (1-edge) \( f \) is an isomorphism in \( N\text{Fuk}^{eq}(S^2) \) if and only if it is the nerve of a morphism in \( \text{Fuk}^{eq}(S^2) \) which projects to an isomorphism in the Donaldson-Fukaya category \( D\text{Fuk}^{eq}(S^2) \). Such a morphism will be called a \( c \)-isomorphism.

Consequently the maximal Kan subcomplex of the \( A_\infty \) nerve of \( \text{Fuk}^{eq}(S^2) \), is characterized as the maximal subcomplex of this nerve with 1-simplices corresponding to \( c \)-isomorphisms in \( \text{Fuk}^{eq}(S^2) \). Let us reduce notational complexity by denoting the maximal Kan sub-complex of \( N(\text{Fuk}^{eq}(S^2)) \) by \( K(S^2) \).
Remark 6.1. It would be most interesting to identify the geometric realization of $K(S^2)$ as a space, up to homotopy. As we are working over $\mathbb{Q}$ we speculate that this nothing more than rationalization of $S^2$, see [13]. Although it seems we can also work over $\mathbb{Z}$ in this particular instance. Indeed it is fairly easy to see a non-trivial element in $\pi_2(K(S^2))$, as well as that $\pi_4(K(S^2)) = 0$. Our argument will show that there is $\mathbb{Z}$ injection into $\pi_3(K(S^2)) \neq 0$. It is probably not too difficult to extend the arguments to verify the expectation above but we leave this for future work.

6.2. A model for the maximal Kan-subcomplex of $Fuk^\infty(P_g)$. The strategy is then as follows. First we construct a similar maximal Kan subcomplex $K(P_g)$ for a quasi-category $Fuk^\infty_{eq}(P_g)$, itself modeling the full (as in with all objects and morphisms) quasi-category $Fuk^\infty(P_g)$, starting with very special perturbation data, and show that the resulting Kan fibration

$$K(P_g) \to S^4_{\bullet},$$

is non-trivial. To clarify, as we shall use this shorthand often, our categories arise from some analytic data, and when we say perturbation data for an $A_\infty$ or by extension quasi-category, in this context we mean the perturbation data necessary for the analytic construction of the various structure maps. The process of taking maximal Kan subcomplex is functorial and so it will follow that $Fuk^\infty(P_g)$ is also non-trivial as a (co)-Cartesian fibration. The construction of $Fuk^\infty_{eq}(P_g)$ proceeds exactly as before by starting with an analogous analytic pre-$\infty$-functor

$$F^{eq} : \text{Simp}(S^4) \to A_{\infty - \text{Cat}},$$

but taking fewer objects as we do for $Fuk^\infty(S^2)$, (eq stands for equator). More specifically the objects of $F^{eq}(\Sigma)$, for $\Sigma : \Delta^n \to S^4$ are great circles in the fibers over the images of vertices of $\Delta^n$. What great circle means is unambiguous since the structure group of $P_g$ is $SU(2)$. However we shall require rather special perturbation data.

We shall take a certain model for the smooth singular set of $S^4$, which recall is a Kan complex with $n$-simplices smooth maps

$$\Sigma : \Delta^n \to S^4.$$

Restricting to a model for the singular set of $S^4$ is justified by Proposition 4.5 of Part I.

First we model $D^4_\bullet$ as follows. Take the standard representable 3-simplex $\Delta^4_\bullet$, and the standard representable 0-simplex $\Delta^0_\bullet$. Then identify all faces of $\Delta^3_\bullet$ (by faces here we mean morphisms $[n] \to [3]$, which are not surjective: so $n$ need not be 2) with with the (degenerate) faces of $\Delta^0_\bullet$. This obviously gives a Kan complex $S^3_{\bullet,\text{mod}}$ modelling the singular set of $S^3$. Now take the cone on $S^3_{\bullet,\text{mod}}$, denoted by $C(S^3_{\bullet,\text{mod}})$, the resulting Kan complex $D^4_{\bullet,\text{mod}}$ is our model for $D^4_\bullet$. We then model $S^4_\bullet$ by taking a pair of copies $D^4_{\bullet,\pm}$ of $D^4_{\bullet,\text{mod}}$ and identifying them along $S^3_{\bullet,\text{mod}}$. Note that the modelling map $\text{mod} : S^4_{\bullet,\text{mod}} \to S^4_\bullet$ in this case can be given by an embedding. We shall then from now on identify simplicial sets $S^4_{\bullet,\text{mod}}, D^4_{\bullet,\pm}$ as subsets of $S^4_\bullet$ without referring to the modeling map.

We have the natural projection

$$\text{pr} : (P_+ = S^2 \times D^4_\bullet) \to S^2.$$
Given $\Sigma^1: \Delta^1 \to D^4_+ \text{ in } D^{4,\text{mod}}_0$, and $L^0_0 \in P_+, \Sigma(0)$, $L^1_0 \in P_+, \Sigma(1)$ identified via $pr$ as Lagrangian submanifolds of $S^2$ with a particular great circle $L_0$, we define

$$\text{hom}_{Fuk}(\Sigma)(L^0_0, L^1_0)$$

using Morse-Bott complex for the trivial connection (with respect to the natural trivialization of $P_+ = S^2 \times D^4_+$). For future use fix a particular fundamental chain $\gamma_{[L_0]}$ representing the fundamental class in $\text{hom}_{Fuk}(\Sigma)(L^0_0, L^1_0)$, for every $L^0_0, L^1_0$ as above. Let us denote by $m_{L^0_0, L^1_0}$ the corresponding invertible 1-edge in $Fuk(P|_{D^{4,\text{mod}}_+})$.

Fix a particular point $x_0 \in D^4_+$ and choose perturbation data for the Morse-Bott version of the construction of $Fuk^0(x_0)$, as for $Fuk^c\omega(S^2)$. Then using the trivial connection on $S^2 \times D^4_+$, we may identify great circle Lagrangians in other fibers with those over $x_0$, and consequently also get induced perturbation data for the construction of $Fuk^0(\Sigma)$, for any $\Sigma : \Delta^n \to D^4_+$. The global Fukaya category restricted over $D^{4,\text{mod}}_0$, with respect to this perturbation data can then be naturally identified with the product $D^{4,\text{mod}}_+ \times N(Fuk^0(x_0))$, so that the 1-edges of the form $e \times \{L_0\}, e \in D^{4,\text{mod}}_+$ correspond under the identification to the edges $\{m_{L^0_0, L^1_0} \}$. Let $\text{const}_{L_0}$ denote the constant section of this product, corresponding to some object $L_0 \in Fuk^c\omega(S^2)$. For $\Sigma : \Delta^n \to D^4_+$, in $\partial D^{4,\text{mod}}_0$, we define the perturbation data for $Fuk^0(\Sigma)$ by pulling it back by

$$\tilde{\gamma} : S^2 \times (\partial D^4_+ \simeq S^3) \to S^2 \times (\partial D^4_+ \simeq S^3).$$

For the moment extend this perturbation data over $D^4_+$ in any way. This determines a transition map

$$Fuk(P|_{\partial D^{4,\text{mod}}_+}) \to Fuk(P|_{\partial D^{4,\text{mod}}_+}).$$

We shall denote this map by $\tilde{\gamma}$ to avoid over complicating notation, but the reader should be wary that it is now acting on simplicial sets.

Set

$$\text{sec} = \tilde{\gamma}^{-1} \circ \text{const}_{L_0}|_{\partial D^{4,\text{mod}}_+}.$$

**Theorem 6.2.** Suppose that $g : S^3 \to \text{Ham}(S^2, \omega)$ represents a non-trivial class in $\pi_3$. Then the class $[\text{sec}] = \tilde{\gamma}^{-1}[\text{const}_{L_0}|_{\partial D^{4,\text{mod}}_+}]$ is non-vanishing in

$$\pi_3(K(P_g)|_{D^{4,\text{mod}}_+}) \simeq \pi_3(K(S^2)),$$

where $K(P_g)$ denotes the maximal Kan sub-fibration of $NFuk^c\omega(P_g) \to S^{4,\text{mod}}$.

In particular $K(P_g)$ is a non-trivial Kan fibration over $S^4_\bullet$, and so $Fuk^\infty(P_g)$ is a non-trivial (co-)Cartesian fibration over $S^4_\bullet$.

For $g$ the generator this class $[\text{sec}]$ in $\pi_3(K(S^2))$ can be thought of as a “quantum” analogue of the class of the classical Hopf map, cf Remark 6.1.

**Proof of Theorem 1.2.** By the main theorem of Part I. We have the universal classifying map

$$f_U : B\text{Ham}(S^2, \omega)_\bullet \to A^0_\infty,$$

taking a point to $Fuk(S^2)$, where the subscript $\bullet$ denotes that we take the singular set. In particular there is an induced map on $\pi_4(B\text{Ham}(S^2, \omega)_\bullet) \simeq \mathbb{Z}$. If

$$c_l_g : S^4 \to B\text{Ham}(S^2)$$
denotes the classifying map for \( P_g \), then \( \text{Fuk}^{\infty}(P_g) \) is isomorphic as a (co)-Cartesian fibration to the pull-back of the universal (co)-Cartesian fibration

\[
P \to A^\otimes_{\infty}
\]

by \( f_U \circ cl_g \) by construction. Then by the theorem above if \([g]\) is non-trivial the class of \( f_U[cl_g] \) is non-vanishing in \( \pi_4 \).

\[\square\]

7. Proof of Theorem 6.2 Part I

As indicated 2 sections will be dedicated to the argument, so that we may better subdivide it.

7.1. Outline of the argument. In our simplicial set \( D^{4,\text{mod}} \) we have a single natural non-degenerate 4-simplex \( \Sigma^4 \). It is the image of the natural non-degenerate 4-simplex of \( C(\Delta^3) \simeq \Delta^4 \) for the natural composition

\[C(\Delta^3) \to C(\Delta^4) \to D^{4,\text{mod}}.\]

From now on \( \Sigma^4 \) always refers to this simplex. Likewise we have a pair of vertices

\[\Sigma^4(0) = \Sigma^0_0, \Sigma^4(i) = \Sigma^0_1, i \neq 0\]

in \( D^{4,\text{mod}} \) with \( 0 \in \Delta^4 \) corresponding to the cone vertex of \( C(\Delta^3) \). The induced 1-edge between \( \Sigma^0_0 \) will be denoted by \( \Sigma^1 \). When we write \( \Sigma^0_0 \) we mean the simplicial subset of \( D^{4,\text{mod}} \) corresponding to \( \Sigma^0_0 \) and likewise elsewhere.

If \( \text{sec} \) is null-homotopic in our Kan complex \( K(P_g|D^{4,\text{mod}}) \), then by definition of homotopy groups of a Kan complex there would be a diagram:

\[
\begin{array}{ccc}
\Delta^3 \times I & \xrightarrow{H} & K(P_g|D^{4,\text{mod}}) \\
\downarrow^{i_0} \downarrow \downarrow^{i_1} & & \downarrow^{\text{null}} \\
\Delta^3 & \longrightarrow & \Delta^3
\end{array}
\]

Here \( \partial \Delta^3 \times I \) maps into \( L_{0,\bullet} \subset K(P_g|D^{4,\text{mod}}) \), with the latter being our name for the lift of \( \Sigma^0_0 \) corresponding to the object \( L_0 \). The map \( \text{null} \) is the constant map to \( L_{0,\bullet} \). It will be convenient to deform the maps, so that in the above diagram \( \partial \Delta^3 \times I \) maps into \( m_{L_{0}^0,L_{1}^0,\bullet} \) lying over \( \Sigma^1 \), with \( m_{L_{0}^0,L_{1}^0,\bullet} \) defined as before, and so that \( \text{null} \) is the constant map to \( \Sigma^0_0 \). This can be done since we are dealing with a Kan complex, (in particular the classical homotopy extension theorem has an analogue.) Note that the base point is now varying. The deformed map will still be denoted by \( H \).

Assuming that this is done, it follows that \( H \) factors as

\[
\Delta^3 \times I \to C(\Delta^4) \simeq \Delta^4 \xrightarrow{T} K(P_g|D^{4,\text{mod}}),
\]

for a certain induced \( T \). We may suppose without loss of generality that \( T \) lies over \( \Sigma^4 \) in \( D^{4,\text{mod}} \). This can be seen as follows. \( H \) lies over a null-homotopy of \( S^3,\text{mod} \),
in $D_{\bullet-\mod}$. Any pair of such null-homotopies are themselves homotopic, as we know the corresponding relative homotopy groups:

$$\pi_4(D_{\bullet-\mod}, \partial D_{\bullet-\mod}) \simeq \mathbb{Z}.$$ 

Using Kan lifting property of the fibration we may perturb $H$ itself so that it lies over the (un-pointed) null-homotopy of $S^3$, inducing (likewise to (7.1)) the 4-simplex $\Sigma^4$.

Clearly $T$ also induces a natural 4-simplex in $K(P_{g|D_{\bullet-\mod}})$, which for simplicity we also call $T$. One of the faces of $T$ is $sec$ and all the other faces are degenerate. By discussion above, the simplex $T$ is in the image of the universal map

$$K(F^{eq}(\Sigma^4)) \rightarrow K(P_{g|D_{\bullet-\mod}}),$$

where $K(F^{eq}(\Sigma^4))$ denotes the maximal Kan replacement of the $A_\infty$ nerve of $F^{eq}(\Sigma^4)$. For $i \neq 0$, the edges $e_i$ of $T$ from $T(i-1)$ to $T(i)$ are

$$\tilde{g}^{-1} m_{L_i^{-1}, L_0} \cdot = m_{L_i^{-1}, L_0}$$

for analogously defined $L_0^{-1}, L_i$ and $m_{L_0^{-1}, L_0}$, since $\tilde{g}^{-1}$ is identity over $\Sigma_{\bullet}^0$.

The edge $e_0$ from $T(0)$ to $T(4)$, is by construction $m_{L_0^{-1}, L_4}$, and similarly with other edges, $e_i, e_i'$ from $T(i)$ to $T(i')$. Let us call by $\{\gamma_i, \gamma_i', \gamma_0\}$ the associated morphisms in $F^{eq}(\Sigma^4)$, abbreviating $\gamma_{i-1, i}$ by $\gamma_i$, and $\gamma_{i, 4}$ by $\gamma_0$. We shall now setup up our perturbation data and show that $T$ cannot exist for this data.

**Lemma 7.1.** There exists perturbation data so that the simplex $T$ exists if and only if

$$\langle \mu_{\Sigma^4}(\gamma_1 \otimes \ldots \otimes \gamma_4, \gamma_0) = 0. $$

This is proved further below. We shall show that this correlator does not vanish, this will be a contradiction and will finish the argument. However the calculation of the correlator will require significant setup.

7.2. Preliminary perturbation data. Let $A_{\partial}$ be the connection on $S^2 \times S^3$ obtained by pull-back of the trivial connection on $S^2 \times S^3$ by $\tilde{g}$. Let $\Omega_{\partial}$ denote the associated coupling form on $S^2 \times S^3$. With $rad \in [0, 1]$ denoting the radial coordinate of $D^4$, and $\{rad\} \times S^3$ the radius $rad$ spheres in $D^4$, we extend $\tilde{\Omega}_{\partial}$ to a coupling form $\tilde{\Omega}$ on $S^2 \times D^4$ by

(7.2) $$\tilde{\Omega}(v, w) = \eta(rad) \cdot \tilde{\Omega}_{\partial}(pr_* v, pr_* w),$$

for $v, w \in T_z(S^2 \times D^4)$, with $z$ having radial coordinate $rad$, and where $\eta : [0, 1] \rightarrow [0, 1]$ is a smooth function satisfying

$$0 \leq \eta'(rad),$$

and

(7.3) $$\eta(rad) = \begin{cases} 
1 & \text{if } 1 - \delta \leq \text{rad} \leq 1, \\
\text{rad}^2 & \text{if } \text{rad} \leq 1 - 2\delta,
\end{cases}$$

for a small $\delta > 0$. At this point we fix our system of natural maps $U$, (this is done for the sake of simplicity). We do this so that for $d = 3, d = 4$ the maps $u^d(m_1, \ldots, m_d, \Sigma^4)$ are the particular maps constructed as in the proof of Proposition 3.4 in Part I, i.e.

$$u^d(m_1, \ldots, m_d, \Sigma^4, r) = f_r \circ \text{ret}_r.$$
in the notation of the proof of that Proposition. We then set the connections in the (preliminary) perturbation data to be simply the (pull-back) of the connection \( A \), by the maps

\[
\mathcal{S}_d \xrightarrow{u^d(m_1,\ldots,m_d,\Sigma^4)} \Delta^d \xrightarrow{\Sigma^4} D^4.
\]

The resulting system of connections \( \mathcal{F} \) is automatically a natural system in the sense of Part I. Moreover the family of connections \( \{ \mathcal{F}(m_1,\ldots,m_d,\Sigma^4,r) \} \) for \( 2 \leq d < 4 \) are the trivial connections on the trivial bundles

\[ S^2 \hookrightarrow \mathcal{S}_r \to \mathcal{S}_r. \]

And for the induced almost complex structures the moduli spaces

\[ \mathcal{M}^{MB}(\gamma_i, r, \Sigma^4, \{ \mathcal{F}(m_1,\ldots,m_d,\Sigma^4, r) \}, A), 2 \leq d < 4, \]

are regular after a suitable choice of the family \( \{ j_x \} \) of almost complex structures on the fibers of \( S^2 \times D^4, x \in D^4 \). To see this let us just take \( j_x = j_{st} \) for \( j_{st} \) the standard integrable almost complex structure on \( S^2 \). Then we have:

**Lemma 7.2.** Whenever \( A \) is such that

\[ \mathcal{M}^{MB}(\gamma_i, r, \Sigma^4, \{ \mathcal{F}(m_1,\ldots,m_d,\Sigma^4, r) \}, A), 2 \leq d < 4, \]

has virtual dimension \( \leq 1 \), the moduli space is empty unless \( A \) is the class of the constant section, \( d = 2 \) and in this case the moduli space consists purely of constant sections of \( \mathcal{S}_r = S^2 \times \mathcal{S}_r \), where \( \{ \gamma_i \} \) is any chain of morphisms, with \( s(\gamma_{i+1}) = t(\gamma_i) \), (source/target).

**Proof.** The sub-bundle \( \mathcal{L}_r \) of \( \mathcal{S}_r = S^2 \times \mathcal{S}_r \) over the boundary of \( \mathcal{S}_r \) in this case is just the constant sub-bundle with fiber \( L_0 \). Let us denote by \( \Xi_r \) the sub-bundle of the vertical tangent bundle of \( \mathcal{S}_r \), corresponding to the vertical tangent bundle of \( \mathcal{L}_r \). By Riemann-Roch (Appendix B) we get that the expected dimension of the moduli space of \( A \) class holomorphic sections of \( \mathcal{S}_r \), with boundary in \( \mathcal{L}_r \), and constraints \( \gamma_1' , \ldots , \gamma_d' \) is

\[ 1 + \text{Maslov}^{vert}(A) + 1 - \deg \gamma_1' + \ldots + 1 - \deg \gamma_d' \]

where \( \text{Maslov}^{vert}(A) \) is the Maslov number of

\[ (\sigma^* T^{vert}(\mathcal{S}_r), \sigma^* \Xi_r, \{ \sigma^* \Xi_{r,i} \}), \sigma \text{ a class } A \text{ section}, \]

for \( \Xi_{r,i} \) totally real subbundle of \( T^{vert} \mathcal{S}_r \) over the \( e_i \) end of \( \mathcal{S}_r \), induced via parallel transport for the connection \( \mathcal{F}(m_1,\ldots,m_d,\Sigma^4, r) \) over the arcs

\[ [0,1] \times \{ \tau \} \subset [0,1] \times \mathbb{R}_{\geq 0} \]

in the strip coordinate charts at the ends, of the fibers of \( \mathcal{L}_r \). In other words in our particular case \( \Xi_{r,i} \) is the constant subbundle of \( T^{vert} \mathcal{S}_r \) over the \( e_i \) end with fiber the tangent bundle of \( L_0 \), (with respect to trivialization at the end).

Consequently the expected dimension of

\[ \mathcal{M}^{MB}(\{ \gamma_i \}, r, \Sigma^4, \{ \mathcal{F}(m_1,\ldots,m_d,\Sigma^4, r) \}, A) \]

is:

\[ \geq 1 + \text{Maslov}^{vert}(A) + (\dim \mathcal{R}_d = d - 2). \]

So either \( d = 3 \), and \( \text{Maslov}^{vert}(A) \leq -2 \) or \( d = 2 \) and \( \text{Maslov}^{vert}(A) \leq 0 \). However negative Maslov number sections are impossible since the projection \( \mathcal{S}_r \to S^2 \) is holomorphic, and \( L_0 \) is monotone with positive monotonicity constant.
The leaves the Maslov number 0 case, and again since the projection to $S^2$ is holomorphic the projections of such sections are constant.

Regularity of $\{F(m_1, \ldots, m_d, \Sigma^4, r)\}, A$ (for $2 \leq d < 4$) is then immediate, and moreover it immediately follows that
\begin{align}
\mu^2_{\Sigma^4}(\gamma_{i-1}, \gamma_i) &= \gamma_{i-2}, \quad 0 < i \leq 4 \\
\mu^3_{\Sigma^4}(\gamma'_1, \gamma'_2, \gamma'_3) &= 0,
\end{align}
for this perturbation data, where
$$\{\gamma'_k\}_{k=1}^3 \subset \{\gamma_i, i' \mid i < i', 0 \leq i \leq 4, 0 < i' \leq 4\}$$
is any possible triple of $\mu^3$-composable morphisms.

To keep notation a bit simpler let us abbreviate $F(m_1, \ldots, m_4, \Sigma^4, r)$ as $A_r$. We now need to understand the moduli spaces $\mathcal{M}^{MB}(\gamma_1, \ldots, \gamma_4, \Sigma^4, \{A_r\}, A)$.

By the dimension formula above since we need the expected dimension of this moduli space to be 1, the class $A$ must have Maslov number $-2$, see Appendix B.

**Notation 7.3.** From now on $A$ refers to this class.

Unfortunately the above moduli space is not regular for such an $A$, in fact there are nodal elements of the above moduli space with codimension 2 breaking, (that is Fredholm index -2 nodal curves.) We want to find a regular family $\{A'^{\text{reg}}_r\}$, which restricts on the boundary of $\mathcal{R}_4$ to the old family determined by $F$. Let us call such a family admissible. Now for any (not necessarily regular) admissible family $\{A'_r\}$ class $A$ curves must stay away from the boundary of $\mathcal{R}_4$. We could deduce this from Lemma 7.2 and Gromov compactness, but it is much simpler to appeal to the following lemma: (which we shall state in more generality later):

**Lemma 7.4.** A stable $J_{A'_r}$-holomorphic class $A$ section $\sigma$ of $S^2 \times S_r$, with boundary in $\mathcal{L}_r$, gives a lower bound
$$-\langle [\omega], A \rangle = 1/2 \cdot \text{area}(S^2) = - \int_{S_r} \sigma^* \tilde{\Omega}_{A'_r} \leq \text{area}(A'_r),$$
where
$$\text{area}(A'_r) = \inf_\alpha \left\{ \int_{S_r} \alpha \left( \tilde{\Omega}_{A'_r} + \pi^*(\alpha) \right) \text{ is nearly symplectic} \right\},$$
where $\alpha$ is a 2-form on $S_r$, $\tilde{\Omega}_{A'_r}$ the coupling form and nearly symplectic means that
$$(\tilde{\Omega}_{A'_r} + \pi^*(\alpha))(\tilde{v}, jv) \geq 0,$$for $\tilde{v}, jv$ horizontal lifts with respect to $\tilde{\Omega}_{A'_r}$, of $v, jv \in T_z(S_r)$, for all $z \in S_r$.

**Proof.** This follows by the classical energy identity for holomorphic curves, proof is omitted. \qed

It follows that a given regularizing perturbation of $\{A_r\}$ may be assumed to vanish near the boundary $\mathcal{R}_4$. And so for the resulting system $F^{\text{reg}}$,
$$\{F^{\text{reg}}(m_1, \ldots, m_4, \Sigma^4, r)\}$$
will be admissible.
Proof of Lemma 7.1. We use the system $\mathcal{F}_{reg}$. The following argument will be over $\mathbb{F}_2$ as opposed to $\mathbb{Q}$ as the signs will not matter. Recall that all codimension faces of $T$ are determined. Let $\{f_j\}, j : [n_j] \to [4]$, a monomorphism, (equivalently cardinality $n_{j+1}$ subset of $[n]=\{0, \ldots, n\}$) be as in the definition of the $A_\infty$ nerve in Part I, corresponding to the various (arbitrary codimension) faces of $T$. If the 4-simplex $T$ exists then there is an $f_{[4]} \in \text{hom}_{\mathcal{F}_{reg}(\Sigma^4)}(L_0^4, L_0^4)$ so that

\begin{equation}
\mu_{\Sigma^4}^4 f_{[4]} = \sum_{1 \leq j < 4} f_{[4]-i} + \sum_{s} \sum_{(j_1, \ldots, j_s) \in \text{decomp}_{s}} \mu_{\Sigma^4}^s(j_1, \ldots, j_s).
\end{equation}

By (7.4), (7.5) we must have $f_{j'} = 0$ whenever $n_{j'} = 2$, and $f_{[4]-i} = 0, 0 \leq i \leq 4$. ($\leq$ is intended.) Given this (7.6) holds if and only if $\mu_{\Sigma^4}^4(\gamma_1, \ldots, \gamma_4) = 0$. \hfill $\square$

On the other hand because $A$ class curves always stay away from boundary of $\mathcal{R}_4$ and because $L_0$ is unobstructed, $\mu_{\Sigma^4, \mathcal{F}_{reg}}^4(\gamma_1, \ldots, \gamma_4)$ is well defined in homology $HF(L_0^0, L_0^4)$ for any choice of system $\mathcal{F}_{reg}$ satisfying our admissibility condition near boundary of $\mathcal{R}_4$. We shall compute this class by relating it to the higher Seidel morphism.

7.3. Connection between $\mu_{\Sigma^4, \mathcal{F}_{reg}}^4(\gamma_1, \ldots, \gamma_4)$ and the higher Seidel morphism. For each $r \in \mathcal{R}_4$, we may close the open ends $\{e_i\}, i \neq 0$ of $S_r$, first by cutting of $[0, 1] \times \mathbb{R}_{>0}$ in the strip coordinate charts at the ends $\{e_i\}$ and then attaching half disks

$$HD^2 = \{z \in \mathbb{C} \mid |z| \leq 1, \text{Im} \ z \geq 0\}.$$

Let us denote the closed off surface by $S_r^* \simeq D^2 - z_0$, where $z_0 \in \partial D^2$ is a point corresponding to the end $e_0$. Since $S_r^*$ is naturally trivialized at the ends, we may similarly close off $S_r$ by gluing bundles $S^2 \times HD^2$ at the ends obtaining an $S^2$ bundle $S_r^*$ over $S_r^*$. If we put a trivial Hamiltonian connection on the trivial bundle $S^2 \times HD^2$ then for any admissible system $\mathcal{F}$ as above, $\mathcal{F}(m_1, \ldots, m_4, \Sigma^4, r)$ would naturally induce a Hamiltonian connection $A_r^*$ on $S_r^*$. Likewise the Lagrangian subbundle $L_r$ of $S_r$ over the boundary of $S_r$ would be naturally closed off to a subbundle $L_r^*$ over the boundary of $S_r^*$, and $A_r^*$ preserves this subbundle. As constructed $\{A_r^*\}$ is also small near boundary in the sense that

$$\text{area}(A_r^*) < 1/2 \cdot \text{area}(S^2)$$

for $r$ near the boundary of $\mathcal{R}_4$, and consequently there are no elements $(\sigma, r)$ of the moduli space

\begin{equation}
\mathcal{M}(\{S_r^*\}, \{L_r^*\}, A, \{A_r^*\}),
\end{equation}

for $r$ near the boundary of $\mathcal{R}_4$, (by the discussion above). With the above moduli space consisting of pairs $(\sigma, r)$ with $\sigma$ a $J(A_r^*)$-holomorphic (stable) section of $S_r^*$ with boundary on $L_r^*$.

We have an “evaluation” map

$$ev : \mathcal{M}(\{S_r^*\}, \{L_r^*\}, A, \{A_r^*\}) \to S(L_0^0, L_0^4) \simeq S^2,$$

sending a (stable) section to its asymptotic constraint at the $e_0$ end, i.e. $(L_0^0, L_0^4)$ end. We shall relate it to our correlator $\mu^4$, but before we do that let us formalize conditions for invariance of the homology class of $ev_*$, which denotes the push forward of the fundamental chain.
Definition 7.5. Let \( \{ \tilde{S}_{r,i}, S_{r,i}, \mathcal{L}_{r,i}, A_{r,i} \} \), \( i = 1, 2, r \in \mathcal{K} \) for some parameter manifold \( \mathcal{K} \), be a pair of 4-tuples where \( \tilde{S}_{r,i} \) is a Hamiltonian \( S^2 \)-fibration over a Riemann surface with boundary and one end: \( S_{r,i} \), and where \( \mathcal{L}_{r,i} \) is a subbundle of \( \tilde{S}_{r,i} \) over the boundary, with fiber a Lagrangian submanifold of the respective fiber, and \( A_{r,i} \) a Hamiltonian connection on \( B_{r,i} \) preserving \( \mathcal{L}_{r,i} \), and exact as in Section 5.1. We say that such a pair are \textit{admissibly concordant} if there is a 4-tuple
\[
\{ \tilde{T}_r, T_r, \mathcal{L}'_r, A'_r \},
\]
with
- \( \tilde{T}_r \) a Hamiltonian \( S^2 \)-bundle over \( T_r \), which is itself a fibration
  \( \text{fib} : T_r \to [0, 1] \)
with fiber a Riemann surface (i.e. with complex structure) with boundary.
- \( \mathcal{L}'_r \) as before is a Lagrangian subbundle over \( \partial \text{fib}^{-1}(0) - \text{fib}^{-1}(1) \).
- Where \( A'_r \) is an exact Hamiltonian connection preserving \( \mathcal{L}'_r \), small near boundary of \( K \).
- There is a diffeomorphism (in the natural sense, preserving all structure)
  \( \sqcup_i \{ \tilde{S}_{r,i}, S_{r,i}, \mathcal{L}_{r,i} \} \to \{ \tilde{T}_r |_{\text{fib}^{-1}(0, 1)}, \text{fib}^{-1}(0, 1), \mathcal{L}'_r |_{\text{fib}^{-1}(0, 1)} \} \),
which pulls back \( A'_r \) to \( A_{r,1}, A_{r,2} \) over the respective components.

Lemma 7.6. \([ev_*] \) is a cycle, its homology class depends only on the admissible concordance class of the family \( \{ \tilde{S}^*_r, S^*_r, \mathcal{L}^*_r, A^*_r \} \), and this homology class coincides with the homology class of
\[
\mu_{\Sigma^4, r=0}(\gamma_1, \ldots, \gamma_4).
\]
Proof. The fact that \([ev_*] \) is a cycle follows by the fact that \( L_0 \) is unobstructed and monotone. The second point follows by the following. An admissible concordance will induce a chain part of which boundary will be the pair of evaluation cycles corresponding to the data of the boundary of the concordance. There is no other boundary by \( L_0 \) being unobstructed (so that contributions from disk bubbling cancel away), and by the fact that for an admissible \( \{ A^*_r \} \), \( A \) class curves stay away from boundary of \( \overline{\mathcal{R}}_4 \), the last part is immediate from standard gluing arguments.

The map \( ev \) is closely related to a relative form of the higher Seidel morphism, which in its most basic form is a group homomorphism:
\[
\Psi : \pi_{k-1} \Omega_{L_0} \text{Lag}(M) \simeq \pi_k(\text{Lag}(M, L_0)) \to F H(L_0, L_0) \quad k > 1
\]
with \( \text{Lag}(M) \) denoting the space whose components are Hamiltonian isotopic Lagrangian submanifolds of \( (M, \omega) \). This generalizes the idea behind relative Seidel morphism. We shall present this construction in the next section, (the reader may also safely read that section first) and it will be immediate that:
\[
\Psi([\text{lag}]) = [ev_*],
\]
for a certain map
\[
\text{lag} : S^2 \to \Omega_{L_0} \text{Lag}(S^2).
\]
7.3.1. Constructing lag. Let $\operatorname{dom}_r$ be the maximal sub-domain of $[0, 4] \times [0, 1]$, whose interior is mapped by $\tilde{g}_r$ into the complement of $\partial D^4 \cup \Sigma^1$, inside $D^4$. By construction this domain is homeomorphic to a disk. A segment $\operatorname{bound}_r$ of its boundary is mapped by $\Sigma^4$ to a loop in $\partial D^4$ based at $\Sigma^0_1$ and that we call $c_r$, and the rest of the boundary $\operatorname{bound}_r^{\text{reg}}$ is mapped into $\Sigma^1$. Let
\[ L'_r \to \tilde{g}_r P |_{\partial \operatorname{dom}_r}, \]
be the sub-bundle determined by $A$ and $L_0$. The latter means that $L'_r$ is the $A$ invariant sub-bundle, which coincides with the constant sub-bundle $L_0$ over the boundary component $\operatorname{bound}_r^{\text{reg}}$. After fixing a parametrization of the boundary of $\operatorname{dom}_r$, (which we can do in a continuous in $r$ way) $L'_r$ determines a loop $\gamma_r$ in the space $\operatorname{Lag}^{\text{reg}}(S^2) \simeq S^2$ of great circles in $S^2$. So we get a map
\[ \operatorname{lag} : \mathcal{R}_4 \to \Omega_{L_0} \operatorname{Lag}^{\text{reg}}(S^2). \]

The boundary of $\mathcal{R}_4 \simeq D^2$, (with $\simeq$ denoting homeomorphism) is mapped by $\operatorname{lag}$ to the point given by the constant loop at $L_0$.

**Lemma 7.7.** $\{ \tilde{S}_r^\ast, S_r^\ast, L_r^\ast, A_r^\ast \}$ is admissibly concordant to $\{ S^2 \times \mathcal{D}, \mathcal{D}, L_{\gamma_r}, A_{\gamma_r} \}$, where $L_{\gamma_r}$ is the Lagrangian sub-bundle determined by the loop $\operatorname{lag}(r)$ over the boundary of a Riemann surface $\mathcal{D}$ biholomorphic to $D^2 - z_0$, $z_0 \in \partial D^2$, and $A_{\gamma_r}$ is any exact Hamiltonian connection on $M \times \mathcal{D}$ small near boundary of $\mathcal{R}_4$ and preserving $\mathcal{L}_{\gamma_r}$.

**Proof.** By construction each tuple $\{ \tilde{S}_r^\ast, S_r^\ast, L_r^\ast, A_r^\ast \}$ corresponds to the pullback of the bundle $P$ and the connection $A$ by some map
\[ S_r^\ast \to D_r^4, \]

taking the ends of the boundary $\partial S_r^\ast$ into $\partial \Sigma^1_0 \subset \partial D^4$. Also by construction there is a deformation retraction
\[ \text{deform} : S_r^\ast \times I \to S_r^\ast, \]
relative to the end $e_0$, onto a submanifold $D^2_r$ diffeomorphic to $D^2 - z_0$, $z_0 \in \partial D^2$, with $z_0$ corresponding to $e_0$ end, and is such that the holonomy path of $A$ over $\partial D^2_r$ is $\gamma_r$ (up to parametrization). The family of such retraction can be clearly made smooth in $r$. Our admissible concordance is obtained by the pullback of $P, A$ by this family of retractions. We may then finally use smooth Riemann mapping theorem to identify each $(D^2_r, j_r)$ with $(\tilde{D}, j_{st})$, smoothly in $r$, where $j_r$ is the induced almost complex structure.

To conclude (7.8), we only need to note that for reasons of dimension $A$ is the only class that could contribute to $\Psi([\operatorname{lag}])$. Aside from the connection $A$ the map $\operatorname{lag}$ also naturally depends on the choice of the modeling map $D^4_{\ast, \text{mod}} \to D^4_{\ast}$, (specifically we need to specify to which 4-simplex $\Sigma^4 : \Delta^4 \to D^4_{\ast}$, we identify the non-degenerate 4-simplex of $D^4_{\ast, \text{mod}}$).

**Lemma 7.8.** The class $[\operatorname{lag}] \in \pi_2(\Omega_{L_0} \operatorname{Lag}^{\text{reg}}(S^2), L_0) \simeq \mathbb{Z}$ is independent of all choices, and coincides with $[\text{hopf}_r(g)]$, for $\text{hopf} : S^3 \to S^2$ the Hopf map.

**Proof.** Independence of all choices is obvious. The second assertion is some simple topology. For concreteness let us specify our $\Sigma^4 : \Delta^4 \to D^4_{\ast}$. We consider $D^4$ as the
standard 4-ball in $\mathbb{R}^4$, with $r$ the radial coordinate, $r \in [0, 1]$. Let $h : \Delta^4 \to [0, 1]$ be the re-normalized linear height function

$$h : \Delta^4 \to [0, 1],$$

s.t. $h(x_0) = 0$ and $h(d_0(\Delta^4)) = 1$, where $d_0(\Delta^4)$ is the face not containing the vertex $x_0$. Then $\Sigma^4$ takes the $r$ level sets for $h$ to the $r$-spheres in $D^4$ by the map $\text{map}_r$. To define $\text{map}_r$ first define $\text{comp}_r$ by naturally identifying the $r$ level sets for $h$ with polytope subspaces $\xi_r$ of $\mathbb{R}^4$, with center of mass at the origin and following up with the inverse stereographic projection $\text{stereo}^{-1} : \xi_r \to S^4 \subset \mathbb{R}^5$, with $S^4 \subset \mathbb{R}^5$ isometrically embedded so that it intersects the plane $x_5 = 0$ only at the origin, with $0 \in S^4$ going to the origin. There is a homotopy $\partial \text{Homot}$ of $\text{comp}_r|_{\partial \xi_r}$, defined by postcomposing with time $t \in [0, \epsilon]$ flow map, for the height reparametrized gradient flow on $S^4$ for the standard height function, induced by the above embedding $S^4 \subset \mathbb{R}^5$ (so that the maximizer corresponds to infinity under stereographic projection), and with $\epsilon$ large enough that the image of $\text{comp}_r|_{\partial \xi_r}$ is taken by the flow map to the maximum. Take a neighborhood $N_{\tau_0}$ of $\partial \xi_r$ consisting of points $\tau \cdot x, \tau \in [0, \tau_0], x \in \partial \xi_r$. (With \cdot the canonical action of $\mathbb{R}$ on $\mathbb{R}^4$.) There is then a canonical deformation retraction of $N_{\tau_0}$ to $\partial \xi_r$; use this to extend $\partial \text{Homot}$ to a map

$$\text{Homot} : \xi_r \times [0, \epsilon] \to S^4,$$

then $\text{Homot}|_{\xi_r \times \{\epsilon\}}$ is our map $\text{map}_r$. Given this explicit data it is not too difficult to verify that if $g$ was the generator then the family of loops $r \mapsto c_r \in \Omega_{\infty} S^3$ for $r \in \mathcal{R}_4 \simeq D^2$ represents the generator $\text{gen}$ of $\pi_2(\Omega \Sigma^q S^3) \simeq \pi_3(S^3)$. To do this it may help to first consider a simpler manifestation of this with $D^3$ playing the role of $D^4$. To briefly sketch this, by naturality axioms in part I, for $\Delta^3$ the map $\nu^3(m_1, \ldots, m_3, r) : \overline{\mathcal{R}_r} \to \Delta^3, r \in \mathcal{R}_3 \simeq [0, 1]$, for $r = 1$ will be surjective onto the $1, 2, 3$ face (face containing vertices $1, 2, 3$), and for $r = 0$ it will only hit the 1, 2, 3 face in the 1, 2 edge. For our particularly constructed maps $\nu^3(m_1, \ldots, m_3)$ we see that from $r = 0$ to $r = 1$ we sweep out the 1, 2, 3 face, by curves $c_r$, with end points on the vertices $1, 3$, with $c_r$ a boundary of the intersection of the image of $\nu^3(m_1, \ldots, m_3, r)$ with the 1, 2, 3 face.

On the other hand our class $[\text{lag}]$ is the image of the generator of $\pi_2(\Omega \Sigma^q S^3)$ by the map induced by the Hopf map

$$\text{hopf} : S^3 \to \text{Lag}^{eq}(S^2) \simeq S^2$$

$$s \mapsto g(s) \cdot L_0,$$

with \cdot denoting the natural action. So the assertion follows. $\Box$

8. Higher relative Seidel morphism

The relative Seidel morphism appears in Seidel’s [12] in the exact case and further developed in [4] in the monotone case. Let $\text{Lag}(M)$ denote the space whose components are Hamiltonian isotopic Lagrangian submanifolds of $M$, we may also denote the component of $L$ by $\text{Lag}(M, L)$. Then the relative Seidel morphism is a homomorphism

$$S : \pi_1(\text{Lag}(M), L_0) \to FH(L_0, L_0),$$

defined for $L_0$ monotone and unobstructed. Here $FH(L_0, L_0)$ denotes the ungraded Morse-Bott Floer homology of $L_0$ over $\mathbb{F}_2$, generated by smooth singular chains on $L_0$. The notation is the one used by Fukaya-Oh-Ono-Ohta who we are
following throughout this paper, the group $F H(L_0, L_0)$ is canonically isomorphic to $F H(L_0, L_0, A^r(L_0, L_0))$ with our previous notation, for $A^r(L_0, L_0)$ the trivial connection.

To a loop $\gamma$ in $Lag(M)$ based at $L_0$ we may associate in the obvious way a smooth Lagrangian sub-bundle $L_\gamma$ of $M \times D^2$ over the boundary $\partial D^2$, which means as before that the fibers of $L_\gamma$ are Lagrangian submanifolds of the respective fibers of $M \times D^2$. The fiber over $z_0 \in \partial D^2$ of $L_\gamma$ is $L_0$ while the fiber over $z_\theta$ is $\gamma(\theta)$.

Pick a Hamiltonian connection on $M \times D^2 \to \partial D^2$, whose holonomy over the boundary $\partial D^2$ preserves $L$. We shall call such an $A L_\gamma$-admissible. We have the moduli space $M(A, A)$ whose smooth elements are class $A, J_A$-holomorphic sections of $M \times D^2$ with boundary on $L$. Here $J_A$ is induced as previously by $A$ after fixing a choice of an almost complex structure on $M$. We shall assume that $A$ is suitably regular. We have an evaluation map $ev_A : M(A, A) \to L_0$, taking a section to its value at $z_0$. Then

$$S([\gamma]) = \sum_A [ev_A],$$

where $ev_A$ is push forward map for the fundamental chain, the sum is over all section classes, and is finite by monotonicity.

There is a natural extension of $S$ to a (graded if one wishes, with an $S^1$ equivariant extension to the free loop space if one wishes) ring homomorphism

$$\Psi : H_*(\Omega_{L_0}Lag(M), \mathbb{Q}) \to F H(L_0, L_0),$$

much like the author’s extension [9] of the Seidel homomorphism, the algebra structure on the left is the Pontryagin ring structure and the ring structure on the right is with respect to quantum multiplication. The map $\Psi$ is defined as follows. To a smooth cycle

$$f : B \to \Omega_{L_0}Lag(M),$$

for $B$ a smooth closed oriented manifold, we have an associated family

$$\{M \times D^2, \mathcal{L}_b\},$$

$b \in B$, $\mathcal{L}_b$ a Lagrangian subbundle of $M \times D^2$ over $\partial D^2$ associated as before to the loop $f(b)$. Given a suitably regular family $\{A_b\}$ of Hamiltonian connections on $M \times D^2$ with $A_b$ $f_b$-admissible. Define:

$$\mathcal{M}(\{A_b\}, B),$$

to be the space whose smooth elements are pairs: $(\sigma, b)$ for $\sigma$ a $J(A_b)$-holomorphic class $B$ section of $M \times D^2$ with boundary on $\mathcal{L}_b$. We have a map:

$$ev_B : \mathcal{M}(\{A_b\}, B) \to L_0,$$

given by evaluation of a section at $z_0 \in \partial D^2$. We may then define as previously:

$$\Psi([f]) = \sum_B [ev_B].$$
9. Proof of Theorem 6.2 Part II (computation of the higher Seidel element)

9.1. Computation of $[ev_*]$ via Morse theory for the Hofer length functional. We shall now compute $\Psi([\text{lag}])$ and so $[ev_*]$ by constructing a very large but geometrically special perturbation of $\{A_{\gamma(r)}\}$. Since $\Psi$ is a group homomorphism we may restrict for simplicity to the case where $g : S^3 \to S^3$ and so $\text{lag}$ represent generators of the respective fundamental groups.

Under certain conditions the spaces of perturbation data for certain problems in Gromov-Witten theory admit a Hofer like functional. Although these spaces of perturbations are usually contractible, there maybe a gauge group in the background that we have to respect, the reader may think of the situation in classical Yang-Mills theory. The basic idea of regularization that we now do consists of pushing the perturbation data as far down as possible (in the sense of the functional) to obtain a mini-max (for the functional) data, which turns out to be especially nice and amenable to calculation. We define the (restricted) positive Hofer length functional

$$L^+ : P\text{Lag}^{eq}(S^2) \to \mathbb{R},$$

$$L^+(\gamma) = \inf_{H^\gamma} \int_0^1 \max H^\gamma_t dt,$$

where $H^\gamma : S^2 \times [0,1] \to \mathbb{R}$ is the function normalized to have zero mean at each moment, generating a lift of $\gamma$ to $SO(3)$ starting at $id$. (That is $H^\gamma$ generates a path in $SO(3)$, which moves $L_0$ along $\gamma$.) And where $P\text{Lag}^{eq}(S^2)$ denotes the path space with some fixed end points. (Which we may later prescribe.) It is elementary to verify that the infimum over such $H^\gamma$ is uniquely attained, and we denote the corresponding function by

$$(9.1) H^\gamma_0.$$

Note that $\text{Lag}^{eq}(S^2)$ is naturally diffeomorphic to $S^2$ and moreover it is easy to check that the functional $L^+$ is proportional to the Riemannian length functional $L_{\text{met}}$ on the path space of $S^2$, with its standard round metric $\text{met}$. The idea of the computation is then this: perturb $\text{lag}$ to be transverse to the (infinite dimensional) stable manifolds for the energy functional $\Omega_{L_0}\text{Lag}^{eq}(S^2)$, push it down by the “infinite time” negative gradient flow for the energy functional on $\Omega_{L_0}\text{Lag}^{eq}(S^2)$, and use the resulting representative to compute $[ev_*]$. Unfortunately the energy functional is degenerate on the based path space giving rise to an unnecessarily complicated picture for the limiting representative. Also we shall arrange details so as to (mostly) avoid dealing with infinite dimensional differential topology.

9.1.1. The “energy” minimizing perturbation data. Let us fix a pair of non conjugate points $L_0, L'_0 \in S^2$. The class $[\text{lag}]$ induces a class

$$[\text{lag}] \in \pi_2(P_{L_0,L'_0}(S^2),\text{geod}),$$

where $\text{geod}$ is the minimizing geodesic from $L_0$ to $L'_0$, (which is unique by the non-conjugacy assumption). Then classical Morse theory [8] tells us that the functional $L_g$ on $P_{L_0,L'_0}(S^2)$ is Morse non-degenerate with a single critical point in each degree. Consequently $[\text{lag}]$ has a representative in the 2-skeleton of $P_{L_0,L'_0}(S^2)$, for the Morse cell decomposition induced by $L_g$. Furthermore since $\pi_2(S^1) = 0$ such a representative cannot entirely lie in the 1-skeleton. It follows by elementary Morse
theory that there is a representative $\lag' : S^2 \to \mathcal{P}_{L_0, L_0'}(S^2)$, for $[\lag]$ s.t. the function $\lag'' L_0$ is Morse with a unique maximizer max, (necessarily of index 2), and s.t. $\gamma_0 = \lag'(\text{max})$ is the index 2 geodesic. (In principle there maybe more than one such maximizer, but recall that we assumed that $\lag$ represents the generator, in which case by further deformation we may insure that there is only one as the degree of $\lag'$ is the intersection number of $\lag'$ with the stable manifold of the geodesic $\gamma_0$). Such a representative can easily be constructed by hand. We shall also suppose that each path $\lag'(r)$ is constant for time near 0, 1.

We shall now use this to compute $\Psi([\lag])$ via a neck-stretching argument. Let $\{\mathcal{A}_{\tau}^-\}, r \in \mathbb{R}_4$ be a family of connections on $S^2 \times \mathcal{D}$, s.t. $\mathcal{A}_{\tau}^-$ is trivial and preserves $\mathcal{L}_{\lag'(r)}$, over the end $[0, 1] \times \mathbb{R}_{\geq 0}$, (under identification with the strip coordinate chart at the end). Where $\mathcal{L}_{\lag'(r)}$ is the Lagrangian sub-bundle over the boundary of $\mathcal{D}$, which over the end $[0, 1] \times \mathbb{R}_{\geq 0} \subset \mathcal{D}$ is the constant subbundle with fiber $L_0$ over $0 \times \mathbb{R}_{\geq 0} \subset \partial [0, 1] \times \mathbb{R}_{\geq 0}$ and fiber $L'_0$ over $1 \times \mathbb{R}_{\geq 0} \subset \partial [0, 1] \times \mathbb{R}_{\geq 0}$.

And $\mathcal{A}_{\tau}^-$ is such that that the (counter-clockwise) holonomy path of $\mathcal{A}_{\tau}^-$ over the path in the boundary of $\mathcal{D}^2$ from $z_0 \in \{0\} \times \mathbb{R}_{\geq 0}$ to $z_1 \in \{1\} \times \mathbb{R}_{\geq 0}$ (under identification) is the path $\lag'(r)$ up to parametrization. We shall call such an $\mathcal{A}$: $\lag'(r)$ compatible.

Let $\mathcal{A}^+$ be the Hamiltonian connection on $S^2 \times [0, 1] \times \mathbb{R}$, with the following coupling form:

$$\tilde{\Omega}^+ = \omega - d(\eta(\tau) \cdot H_{0}^{\text{geod}} d\theta),$$

for $\eta : \mathbb{R} \to [0, 1]$ the canonical smooth extension of the function $\eta$ as in (7.3), $\theta$ the coordinate on $[0, 1]$, and $H_{0}^{\text{geod}}$ as in (9.1). We suppose that $\{\mathcal{A}_{\tau}^-\}$ is such that the natural glued family $\{\mathcal{A}_{\tau}^- \# \mathcal{A}^+\}$ is small for $r$ near the boundary of $\mathcal{R}_4$. We shall call such a pair $(\mathcal{A}_{\tau}^-, \mathcal{A}^+)$ compatible with $\lag'(r), \text{geod}$. Given a pair $\{\mathcal{A}_{\tau}^-, \mathcal{A}^+\}$ we then have an evaluation map

$$ev^\pm : \overline{\mathcal{M}}(\{\mathcal{A}_{\tau}^-, \mathcal{A}^+\}, \mathcal{A}) \to \mathcal{S}(L_0, L_0', \mathcal{A}^{\text{geod}}(L_0, L_0')),$$

where

$$\overline{\mathcal{M}}(\{\mathcal{A}_{\tau}^-, \mathcal{A}^+\}, \mathcal{A})$$

is the moduli space whose smooth elements are triples $(\sigma^-, \sigma^+, r)$ for $\sigma^-$ a finite energy $J(\mathcal{A}_{\tau}^-)$-holomorphic section of $S^2 \times \mathcal{D}$ with $\mathcal{L}_{\lag'(r)}$ boundary condition, $\sigma^+$ a finite energy $J(\mathcal{A}^+)$-holomorphic section of $S^2 \times [0, 1] \times \mathbb{R}$, with $\mathcal{L}_{\text{geod}}$ boundary condition, s.t. the asymptotic constraint of $\sigma^+$ at the $-\infty$ end coincides with the asymptotic constraint of $\sigma^-$, and such that the glued section has Maslov number $-2$ i.e. is in class $A$. While $\mathcal{A}^{\text{geod}}(L_0, L_0')$ denotes the Hamiltonian connection on $S^2 \times [0, 1] \to [0, 1]$, whose holonomy path is generated by $H_{0}^{\text{geod}}$. We denote the
push-forward of the (virtual) fundamental chain by $ev^\pm_*$. By gluing the homology class of $ev^\pm_*$ coincides with $\Psi([\text{lag}])$.

9.1.2. Construction of a particularly suitable $\{A_r^-\}$. Set

$$H^r : S^2 \times [0, 1] \to \mathbb{R}$$

where $\cdot$ denotes concatenation (see (9.1)). For each $r \in \mathcal{R}_4$ we define the coupling form $\tilde{\Omega}_r$ on $S^2 \times D_-$:

$$\tilde{\Omega}_r = \omega - d(\eta(r) \cdot H^r d\theta),$$

for $(\text{rad}, \theta)$ the modified angular coordinates on $D$, $\theta \in [0, 1]$ and $\eta$ as in (7.2). The induced connection is our $A_r^-$. 

9.1.3. The properties of $\{A_r^-\}$. Let $C(L_0, L'_0)$ be the space of coupling forms $\tilde{\Omega}$ on $M \times D$ s.t. for each such $\tilde{\Omega}$ the associated connection is compatible with $p_{\tilde{\Omega}}$, for some

$$p_{\tilde{\Omega}} \in \mathcal{P}_{L_0, L'_0} \text{Lag}(S^2),$$

with the latter denoting path space as usual. Define

$$\text{area} : C(L_0, L'_0) \to \mathbb{R}$$

$$\text{area}(\tilde{\Omega}) = \inf_\alpha \int_D \alpha(\tilde{\Omega} + \pi^*(\alpha) \text{ is nearly symplectic}),$$

where $\alpha$ is a 2-form on $D^2$, and nearly symplectic means that

$$\text{(9.2)} \quad \tilde{\Omega} + \pi^*(\alpha)\tilde{v}, \tilde{j}v \geq 0,$$

for $\tilde{v}, \tilde{j}v$ horizontal lifts with respect to $\tilde{\Omega}$, of $v, jv \in T_z([0, 4] \times [0, 1])$, for all $z \in [0, 4] \times [0, 1]$. Then by elementary calculation we have:

$$\text{(9.3)} \quad \text{area}(\tilde{\Omega}_r^-) = L^+(\text{lag}'(r)).$$

To verify (9.3) first show that the infinum is attained on a uniquely defined 2-form $\alpha_{\tilde{\Omega}}$:

$$\text{(9.4)} \quad \alpha_{\tilde{\Omega}}(v, w) = \max_D R_{\tilde{\Omega}}(v, w),$$

where $R_{\tilde{\Omega}}$ is the Lie algebra valued curvature 2-form of (the connection induced by) $\tilde{\Omega}$, and we are using the isomorphism $\text{lieHam}(S^2, \omega) \simeq C_\text{norm}^\infty(S^2)$. The following then readily follows.

**Lemma 9.1.** The function $r \mapsto \text{area}(\tilde{\Omega}_r^-)$ has a unique maximizer, coinciding with the maximizer $\max$ of $\text{lag}' L_g$ and area is Morse at $\max$ with index 2.

The previous discussion will allow us to show that the moduli space

$$\overline{\mathcal{M}}(\{A_r^-, A^+\}, A)$$

will localize over $\max \in \mathcal{R}_4$. Let us now find specific holomorphic sections for the data.
9.1.4. Finding class $A$ elements for the data. Define

$$\sigma_{\text{max}}^{-} : \mathcal{D} \to S^2 \times \mathcal{D},$$

to be the constant section

$$z \mapsto x_{\text{max}}^{-}$$

for $x_{\text{max}}$, the maximizer of $H_{\text{max}}^{-}$. Then $\sigma_{\text{max}}^{-}$ is a flat section for $A_{\text{max}}^{-}$, with boundary on $\mathcal{L}_{\text{lag}}^{(\text{max})}$, and consequently holomorphic. Likewise define

$$\sigma_{\text{max}}^{+} : [0, 1] \times \mathbb{R} \to S^2 \times [0, 1] \times \mathbb{R},$$

to be the constant section

$$z \mapsto x_{\text{max}}^{+}$$

for $x_{\text{max}}^{+}$, the maximizer of $H_{0}^{\text{geod}}$.

**Lemma 9.2.**

$$x_{\text{max}}^{-} = x_{\text{max}}^{+}$$

*Proof.* This follows by the fact that $L_0, L'_0$ are non conjugate in $\text{Lag}^{eq}(S^2) \simeq S^2$ and so $\text{lag}^{(\text{max})}$ and $\text{geod}$ are both arcs of the same great geodesic. In particular the generating function $H_{0}^{\text{geod}}$ of $\text{geod}$ is a positive multiple of $-H_{\text{max}}^{-}$, which is a height function on $S^2$ and in particular these functions have the same maximizer. \qed

In particular $(\sigma_{\text{max}}^{-}, \sigma_{\text{max}}^{+}, \text{max})$ is an actual element of

$$\overline{\mathcal{M}}(\{A_{\text{max}}^{-}, A_{\text{max}}^{+}\}, A),$$

if it is indeed in class $A$. However this latter fact is an elementary calculation by the fact that $\text{lag}^{(\text{max})}$ is an index 2 geodesic. Specifically after gluing and compactifying the end as in Appendix B, we get the induced loop $l$ in

$$\text{Lag}(T_{x_{\text{max}}}S^2) \simeq \text{Lag}(\mathbb{R}^2),$$

corresponding to the pull-back by the glued section of the boundary Lagrangian subbundle. By our conventions for the Hamiltonian vector field:

$$\omega(X_H, \cdot) = -dH(\cdot)$$

and since $\text{lag}^{(\text{max})}$ is index 2, $l$ is homotopic to a degree 1 loop in $\text{Lag}(\mathbb{R}^2)$ going counter-clockwise, so by homotopy invariance and normalization the Maslov number of $l$ is $-2$.

Let us denote by

$$\overline{\mathcal{M}}(S^2 \times [0, 1] \times \mathbb{R}, \mathcal{L}_{\text{geod}}, A_{\text{max}}^{+}, \gamma_{[L_0]}),$$

the space whose smooth elements are $J(A_{\text{max}}^{+})$-holomorphic sections of $S^2 \times [0, 1] \times \mathbb{R} \to [0, 1] \times \mathbb{R}$, with boundary on $\mathcal{L}_{\text{geod}}$, asymptotic to $\gamma_{[L_0]}$ at the $-\infty$ end, and are in the class of $\sigma_{\text{max}}^{+}$. We have an evaluation map:

$$ev^{+} : \overline{\mathcal{M}}^q(S^2 \times [0, 1] \times \mathbb{R}, \mathcal{L}_{\text{geod}}, A_{\text{max}}^{+}, \gamma_{[L_0]}) \to S(L_0, L'_0, A_{\text{geod}}^{\text{geod}}(L_0, L'_0)) \simeq S^2.$$

**Lemma 9.3.** For a generic $A_{\text{max}}^{+}$, $ev^{+}$ represents the fundamental class of $S^2$.

*Proof.* Note first that $[ev^{+}]$ is the contribution to

$$PSS(\gamma_{[L_0]}) \in FH(L_0, L'_0, A_{\text{geod}}^{\text{geod}}(L_0, L'_0))$$

in a particular class: $h$, which satisfies:

$$\langle [\tilde{\Omega}^{+}], h \rangle = -L^{+}(\text{geod}).$$
Where
\[ PSS : FH(L_0, L_0', A^{tr}(L_0, L_0')) \to FH(L_0, L_0', A^{geo}(L_0, L_0')) \]
is the continuation map in Floer homology, for \( A^{tr}(L_0, L_0') \) the trivial connection. Any other class that could contribute is of the form
\[ h + A, \]
for \( A \neq 0 \) a class in \( H_2(S^2, L_0) \). By dimension formula \( A \) must have negative Maslov number (otherwise dimension would be too high), and so by monotonicity
\[ -\langle [\omega], A \rangle \geq 1/2 \cdot \text{area}(S^2). \]
\[ -\langle [\tilde{\Omega}], h + A \rangle \geq L^+(\text{geo}) + 1/2 \cdot \text{area}(S^2). \]
But by (analogue of) Lemma 9.5, this is impossible. Since \( PSS \) is an isomorphism the conclusion follows. \( \square \)

**Lemma 9.4.** The only elements of the moduli space\[ \mathcal{M}(S^2 \times D, \{ A^{+}_{\tau}, A^{+} \}, A) \]
are of the form \( (\sigma_{\max}, \sigma^{+,\cdot}, \max) \)
\[ \sigma^{+,\cdot} \in \mathcal{M}(S^2 \times [0, 1] \times \mathbb{R}, L_{geo}, A^{+, \gamma[L_0]}) \]
Proof. We need the following lemma:

**Lemma 9.5.** A stable, finite energy, (meaning suitable \( L^2 \) energy as usual) \( J_A \)-holomorphic section \( \sigma \) of \( M \times D \), with boundary in a \( A \)-invariant Lagrangian sub-bundle \( L \) over \( \partial D \), gives a lower bound
\[ -\int_{D} \sigma^{*}\tilde{\Omega}_A \leq \text{area}(\tilde{\Omega}_A), \]
where \( \tilde{\Omega}_A \) is the coupling form of \( A \). Here \( A \) is assumed to be translation invariant and hence flat in some chosen strip coordinate system \( M \times [0, 1] \times \mathbb{R}_{\geq 0} \) at the end of \( D \). This insures that a finite energy holomorphic section is asymptotic to a flat section at the end, and both sides will be obviously finite.

Proof. As \( L \) is \( A \) invariant by an elementary calculation \( \tilde{\Omega}_A \) vanishes on \( L \). In other words for any symplectic form on \( M \times D \) of the kind:
\[ \tilde{\Omega}_A + \pi^{*}\alpha, \]
for \( \alpha \) a 2-form on \( D \) \( L \) is a Lagrangian submanifold of \( M \times D \). The lemma is then essentially immediate by energy positivity for \( J \)-holomorphic curves and further details are omitted. \( \square \)

Let’s call an \( A \) preserving \( L \): \( L \)-admissible. The pairing
\[ -\int_{D} \sigma^{*}\tilde{\Omega}_A, \]
is invariant with respect to isotopy of the pair \( (L, A) \), for \( L \)-admissible \( A \) fixing the pair at the end, and under variation of \( \sigma \) in its relative homology class, (relative to \( L \) and relative to asymptotic boundary constraint at the end, which exists by finite energy assumption). The assumption on fixing the pair at the end is necessary to make sense of the relative class of \( \sigma \). We can prove invariance of the pairing
directly. Let \( \{(\mathcal{L}_t, \mathcal{A}_t)\}, \ t \in [0,1] \) be a smooth variation. Then we get an \( \tilde{\mathcal{L}} \) admissible Hamiltonian connection \( \tilde{\mathcal{A}} \) on

\[
(\tilde{M} = M \times D \times [0,1]) \to D \times [0,1],
\]

where \( \tilde{\mathcal{L}} \) is the natural Lagrangian sub-bundle of \( \tilde{M} \), over \( \partial D \times [0,1] \) induced by \( \{\mathcal{L}_t\} \). There is then an induced closed coupling form \( \tilde{\Omega}_{\tilde{\mathcal{A}}} \) on \( \tilde{M} \) extending \( \tilde{\mathcal{A}} \) over \( M \times D \times \{i\} \), for \( i = 0 \) or \( i = 1 \). Invariance then readily follows by a relative form of Stokes theorem. We may denote the above pairing by integration by:

\[
\langle [\tilde{\Omega}_{\tilde{\mathcal{A}}}], [\sigma] \rangle.
\]

Let us go back to our argument. By direct calculation:

\[
\langle [\tilde{\Omega}_{\tilde{\mathcal{A}}}], [\sigma]\rangle = -\int_{D} (\sigma^{-})^{*}\tilde{\Omega}_{\tilde{\mathcal{A}}} = L^{+}(\text{lag}(\text{max})).
\]

So by previous pair of points we have:

\[
L^{+}(\text{lag}'(\text{max})) \leq \text{area}(\tilde{\Omega}_{\tilde{\mathcal{A}}}) = L^{+}(\text{lag}'(r)),
\]

whenever there is an element

\[
(\sigma, r) \in \overline{\mathcal{M}}(\{S^{2} \times D, D, \mathcal{L}_{\text{lag}'(r)}, \mathcal{A}_{\text{max}}^{-}, \gamma_{[L_{0}]}\}),
\]

where \( \gamma_{[L]} \) denotes the asymptotic constraint of (i.e. flat section in \( S(L_{0}, L'_{0}) \)) of \( \sigma_{\text{max}}^{-} \), and the sections \( \sigma \) are required to have the same constraint and be in the class of \( \sigma_{\text{max}}^{-} \).

But clearly this is impossible unless \( r = \text{max} \), as

\[
\text{area}(\tilde{\Omega}_{\tilde{\mathcal{A}}}) = L^{+}(\text{lag}'(\text{max})),
\]

by direct calculation, and \( L^{+}(\text{lag}'(r)) < L^{+}(\text{lag}'(\text{max})) \) for \( r \neq \text{max} \).

We now show that there are no elements \( (\sigma^{-}, r) \) other than \( (\sigma_{\text{max}}^{-}, \text{max}) \) of the moduli space

\[
\overline{\mathcal{M}}(S^{2} \times D, D, \mathcal{A}_{\text{max}}^{-}, \gamma_{[L]}).
\]

We have by (9.4)

\[
0 = \langle [\tilde{\Omega}_{\tilde{\mathcal{A}}}] + \alpha_{\tilde{\Omega}_{\tilde{\mathcal{A}}}}^{-}, [\sigma]\rangle,
\]

and so given another element \( (\sigma^{-}, \text{max}) \) by invariance we have:

\[
0 = \langle [\tilde{\Omega}_{\tilde{\mathcal{A}}}] + \alpha_{\tilde{\Omega}_{\tilde{\mathcal{A}}}}^{-}, [\sigma] \rangle.
\]

It follows that \( \sigma^{-} \) is necessarily \( \tilde{\Omega}_{\tilde{\mathcal{A}}}^{-} \)-horizontal, since

\[
(\tilde{\Omega}_{\tilde{\mathcal{A}}}^{-} + \alpha_{\tilde{\Omega}_{\tilde{\mathcal{A}}}}^{-})(v, J_{\tilde{\Omega}_{\tilde{\mathcal{A}}}}^{-} w) \geq 0,
\]

and is strictly positive for \( v \) in the vertical tangent bundle of

\[
S^{2} \hookrightarrow S^{2} \times D \to D
\]

and so otherwise right hand side would be positive. The claim then follows, if every element \( (\sigma^{-}, \sigma^{+}, r) \) in \( \overline{\mathcal{M}}(S^{2} \times D, D, \{\mathcal{A}_{\text{max}}^{-}, \mathcal{A}_{\text{max}}^{+}\}, A) \), was breaking along \( \gamma_{[L_{0}]} \), i.e. if \( \sigma^{-} \) has \( \gamma_{[L_{0}]} \) as an asymptote at the end. However this is the only possibility. This readily follows by the following observation: since \( ev^{+} \) is part of the PSS isomorphism, the only classes \( A^{+} \) for which the homology classes of \( (ev_{A}^{-})_{*} \):

\[
ev^{+} : \overline{\mathcal{M}}(S^{2} \times [0,1] \times \mathbb{R}, \mathcal{L}_{\text{grad}}, \mathcal{A}_{\text{max}}^{+}, 0, \gamma_{[pt]}, A^{+}) \to S(L_{0}, L'_{0}, A_{\text{grad}}^{+}(L_{0}, L'_{0}))
\]
are non trivial, are such that \((ev^+_A)_*\) is degree 0. Which implies that \(ev^±\) on the components of the moduli space with elements \((σ^−, σ^+, r)\) for which \([σ^+] = A^+\), for \(A^+\) as above must be homologically trivial. □

9.1.5. Regularity. Assembling everything together it will follow that

\[Ψ([lag]) = [ev^±_*] = [L_0],\]

if we knew that \((σ^-_{\text{max}}, \text{max})\) was a regular element of

\[\mathcal{M}(S^2 × D, D, \{A^-\}, γ_{L_0}),\]

where the latter denotes the moduli space whose smooth elements are pairs \((σ^−, r)\) for \(σ^-\) a \(J(A^-)\)-holomorphic section with asymptotic constraint \(γ_{L_0}\), in the relative class of \(σ^-_{\text{max}}\), with boundary on \(E_{log}(r)\). How did we get a positive sign for \([L_0]\)?

The sign of \([L_0]\) depends on the choice of the relative spin structure on \(L_0\), we simply assume for convinience that it was chosen so that this sign is positive. We won’t answer directly if \((σ^-_{\text{max}}, \text{max})\) is regular, it likely is, although the argument (which I imagine) involves some algebraic geometry. But it is regular after a suitably small Hamiltonian perturbation of the family \(\{A^-\}\) vanishing at \(A^-_{\text{max}}\). This is the essential regularity mentioned earlier.

**Lemma 9.6.** For any family \(\{A'_r\}\) sufficiently \(C^\infty\) close to \(\{A^-\}\) and s.t.

\[A'^+_{\text{max}} = A^-_{\text{max}},\]

\((σ^-_{\text{max}}, \text{max})\) is the only element of the corresponding moduli space

\[\mathcal{M}(S^2 × D, D, \{A'_r\}, γ_{L_0}),\]

Moreover there is such a family \(\{A'^{\text{reg}}_r\}\) such that

\[\mathcal{M}(S^2 × D, D, \{A'^{\text{reg}}_r\}, γ_{L_0}),\]

is regular.

**Proof.** Given Lemma 9.1 the proof of this is completely analogous to the proof [11, Theorem 1.20]. □

This finishes the section and the proof of the theorem.

10. Application to Hofer geometry

Here is one application to Hofer geometry. This is a relative analogue of the theorem given in the authors thesis [10]. The geometrically interesting fact here is the existence of a lower bound on the minimax below, as at the moment we have extremely poor understanding of “Hofer small” balls in the group of Hamiltonian symplectomorphisms and the spaces of Lagrangians, for any symplectic manifold.

**Theorem 10.1.** Let \((M, ω) = (S^2, ω)\) and \(L_0 \subset M\) the equator. And let \(f : S^2 → \Omega_{L_0}Lag(S^2)\) represent the generator of \(π_2\). Then

\[Ψ([f]) = [L_0].\]

Moreover

\[\min_{f' ∈ [f]} \max_{s ∈ S^2} L^+(f'(s)) = 1/2 \cdot \text{area}(S^2, ω).\]
Proof: The first part of the statement follows immediately from our calculation \([ev^\pm] = [L_0]\), from the previous section, upon noting that only the class \(B\) with Maslov number \(-2\), (i.e. \(A\) in the previous notation) can contribute to the invariant by dimension count. To see the second point first note that by 9.5, and by invariance of the associated pairing (see discussion following that lemma) we get that any element \((\sigma, s) \in \overline{\mathcal{M}}(\{A_s\}, A)\), \(s \in S^2\) gives a lower bound:

\[
\frac{1}{2} \text{area}(S^2, \omega) \leq L^+(f(s)),
\]

and since \(\overline{\mathcal{M}}(\{A_s\}, A)\) is not empty we get:

\[
\min_{f' \in \{f\}} \max_{s \in S^2} L^+(f'(s)) \geq \frac{1}{2} \cdot \text{area}(S^2, \omega).
\]

On the other hand the lower bound is sharp by the explicit construction in Section 9.1.1.

\[\square\]

Appendix A. Homotopy groups of Kan complexes

Given a pointed Kan complex \((X_\bullet, x)\) and \(n \geq 1\) the \(n\)th simplicial homotopy group of \((X_\bullet, x)\): \(\pi_n(X_\bullet, x)\) is defined to be the set of equivalence classes of morphisms

\[
\Sigma : \Delta^n \to X_\bullet,
\]

for \(\Delta^n \cong \text{hom}_\Delta([k], [n])\), for \(\Delta\) the simplicial category. Such that \(\Sigma\) takes \(\partial \Delta^n\) to \(x\). Since for us \(X_\bullet\) is often the singular set associated to a topological space \(X\), we note that such morphisms are in complete correspondence with maps:

\[
\Sigma : \Delta^n \to X,
\]

taking the topological boundary of \(\Delta^n\) to \(x\).

Two such maps are equivalent if there is a diagram (simplicial homotopy):

\[
\begin{array}{ccc}
\Delta^n & \xrightarrow{\Sigma_1} & X_\bullet \\
| & \downarrow \sigma_0 \downarrow \sigma_1 & \\
\Delta^n \times I_\bullet & \xrightarrow{H} & X_\bullet \\
| & \downarrow \eta_1 & \\
\Delta^n & \xrightarrow{\Sigma_2} & X_\bullet
\end{array}
\]

such that \(\partial \Delta^n \times I_\bullet\) is taken by \(H\) to \(x\). The simplicial homotopy groups of a Kan complex \((X_\bullet, x)\) coincide with the classical homotopy groups of the geometric realization \(\left|X_\bullet, x\right|\). But the power of the above definition is that if we know our Kan complex well, (like in the example of the present paper) the simplicial homotopy groups are very computable since they are completely combinatorial in nature.
Appendix B. On the Maslov number

Let $S$ be obtained from a compact Riemann surface with boundary, by removing a finite number of points from the boundary. Let $V \to S$ be a rank $r$ complex vector bundle, trivialized at the open ends, with respect to fixed strip coordinate charts at the ends $\{e_i\}$. We denote by
$$b_{i-1,i} : \mathbb{R} \to \partial S$$
the component of the boundary going from $e_{i-1}$ end to the $e_i$ end. Let
$$\Xi : \partial S \subset S$$
be a totally real rank $r$ subbundle of $V$, which is constant in the coordinates
$$\mathbb{C}^r \times [0,1] \times \mathbb{R}_{\geq 0},$$
at the ends. And let $\Xi_i$ be likewise totally real rank $r$ subbundles of $V|_{e_i}$, s.t. in the coordinates $\mathbb{C}^r \times [0,1] \times \mathbb{R}_{\geq 0}$ the restrictions $\Xi_i|_{[0,1] \times \{\tau\}}$ have a smooth limit $\Xi_i^\infty$ as $\tau \to \infty$. Finally we shall require that the real vector space $\Xi_i^\infty|_{0 \in [0,1]}$ coincides with
$$\lim_{\tau \to -\infty} \Xi|_{b_{i-1,i}((\tau))},$$
and the real vector space $\Xi_i^\infty|_{1 \in [0,1]}$ coincides with
$$\lim_{\tau \to \infty} \Xi|_{b_{i,i+1}((\tau))},$$
again with respect to the trivialization of $V$ at that end.

Then there is a Maslov number $\operatorname{Maslov}(V, \Xi, \{\Xi_i\})$ associated to this data coinciding with the boundary Maslov index in the sense of [7, Appendix C3], for the modified pair $(V/\Xi, \Xi/\Xi_i)$ obtained from $(V, \Xi, \{\Xi_i\})$ by closing each $e_i'$ end of $V \to S$ via gluing with
$$(C^r \times (D^2 - z_0), \bar{\Xi}_{i'}^\infty, \bar{\Xi}_{i',0}^\infty).$$
Where $\bar{\Xi}_{i'}^\infty$ is a totally real rank $r$ subbundle of $C^r \times (D^2 - z_0)$, $z_0 \in \partial D^2$ over the boundary arc, which is constant in the coordinates $\mathbb{C}^r \times [0,1] \times \mathbb{R}_{\geq 0}$ at end, and such that identifying the arc of the boundary in the compliment of the chart, with $[0,1]$ we have
$$\bar{\Xi}_{i'}^\infty|_{\rho} = \Xi_{i'}^\infty|_{\rho},$$
for $\rho \in [0,1]$. And where likewise $\bar{\Xi}_{i',0}^\infty$ is a totally real rank $r$ subbundle of
$$C^r \times (D^2 - z_0),$$
over the open end s.t. in the coordinates $\mathbb{C}^n \times [0,1] \times \mathbb{R}_{\geq 0}$ we likewise have
$$\bar{\Xi}_{i',0}^\infty|_{[0,1] \times \tau} = \Xi_i^\infty.$$
For a real linear Cauchy-Riemann operator on $V$, the Fredholm index is given by:
$$r \cdot \chi(V, \Xi, \{\Xi_i\}) + \operatorname{Maslov}(V, \Xi, \{\Xi_i\}).$$
The proof of this is analogous to [7, Appendix C], we can also reduce it the statement in [7, Appendix C] via a gluing argument, together with computation of the Fredholm index of a real linear Cauchy-Riemann operator on $(C^r \times (D^2 - z_0), \bar{\Xi}_{i'}^\infty, \bar{\Xi}_{i',0}^\infty)$. 
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