Symmetries of the Bosonic String S-Matrix

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The bracket operation on mutually local BRST classes may be combined with Lorentz invariance and analyticity to write an infinite set of finite difference relations on string scattering amplitudes. When combined with some simple physical criteria these relations uniquely determine the genus zero string S-matrix for $N \leq 26$-particle scattering in $\mathbb{R}^{25,1}$ in terms of a single parameter, $\kappa$, the string coupling. We propose that the high-energy limit of the relations are the Ward identities for the high-energy symmetries of string theory.

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1. Introduction

String theory is a generalization of gauge theory. Historically, understanding the importance of general covariance and local isotopic-spin invariance were crucial steps in the formulation of general relativity and Yang-Mills theory. One might therefore suspect that a complete formulation of string theory will be predicated upon a deeper understanding of the symmetries in the theory.

In [1] D. Gross proposed that one useful tool for discovering fundamental string symmetries is the analysis of the high energy behavior of string scattering amplitudes. In field theories of spontaneously broken gauge invariance this technique works very well. Using the saddle-point analysis of [2] Gross derived an infinite set of linear relations between bosonic string 4-particle $S$-matrix amplitudes. Usually linear relations between scattering amplitudes are derived from an underlying symmetry, and it was suggested in [1] that such a symmetry must explain these linear relations. Some work has subsequently been done with a view towards understanding this mysterious symmetry in [3] [4] [5].

In [6] it was suggested that the infinite dimensional hyperbolic symmetries which arise upon toroidal compactification of time might be the source of the high energy symmetries of string theory. The original program of [6] is probably misguided, for reasons explained in appendix B below. Nevertheless, as we show in the present paper, the basic idea that generalized Kac-Moody algebras are high-energy symmetries of string theory is correct. In fact, more is true. We can replace high energy Ward identities, which relate amplitudes at the same values of $s, t$ by finite difference relations for the exact amplitudes. These finite difference relations put strong constraints on the genus zero string $S$-matrix. We will show that they determine scattering amplitudes at all mass levels in terms of tachyon scattering. Moreover, when supplemented with some mild analyticity requirements (e.g. Regge-like behavior at $s \to \infty$) the finite difference relations even determine the tachyon amplitude itself. A technical point discussed in section 3.3 limits our discussion to $N$-particle scattering for $N \leq 26$.

In essence the answer to the problem posed in [1] is very simple: the underlying symmetries are the bracket algebras defined by on-shell mutually local BRST invariant chiral vertex operators.

2. Review of Bosonic String Scattering

It is convenient to summarize some standard facts [7].
2.1. On-Shell States

The states of the open bosonic string are defined in terms of the BRST cohomology \( H^* \). We focus on the chiral ghost number 1 cohomology, \( H = H^{g=1} \). The space \( H \) is graded by level number \( n \) and momentum \( p \in \mathbb{R}^{25,1} \):

\[
H = \bigoplus_{n \in \mathbb{Z}^+} \int_{\mathbb{R}^{25,1}} dp \ H[p,n]
\]

(2.1)

where \( H[p,n] = 0 \) unless \( p^2 = 2 - 2n \) and \( \dim H[p,n] = p_{24}(n) \). \( H \) is an induced representation of the Poincaré group in \( \mathbb{R}^{25,1} \).

Cohomology classes have representatives of the form \( cV \) where \( V \) is a chiral vertex operator satisfying the physical state conditions. These conditions state that \( V \) is a dimension one Virasoro primary. \( ^1 \) Such operators have the form \( V = \mathcal{P} e^{ipX} \) where \( \mathcal{P} \) is a polynomial in \( \partial^* X \) of dimension \( n \). For example:

- \( n = 0 \): \( \mathcal{P} = 1 \)
- \( n = 1 \): \( \mathcal{P} = i\zeta \cdot \partial X \quad \zeta \in T\mathbb{R}^{25,1} \)
  \[ \zeta \cdot p = 0 \quad \zeta \sim \zeta + \lambda p \]
- \( n = 2 \): \( \mathcal{P} = ip \cdot \zeta \cdot \partial^2 X + \partial X \cdot \zeta \cdot \partial X \quad \zeta \in (T\mathbb{R}^{25,1})^\otimes 2 \)
  \[ \text{tr}(\zeta) - 2p \cdot \zeta \cdot p = 0 \quad \zeta \sim \zeta + \frac{1}{2} [p \otimes \chi + \chi \otimes p - \frac{1}{3}(p \cdot \chi)\eta] \]

(2.2)

The fields \( X^\mu \) are always normalized to have the correlator

\[
\partial X^\mu(z) \partial X^\nu(w) \sim -\frac{\eta^{\mu\nu}}{(z-w)^2}.
\]

We use units where \( \alpha' = \frac{1}{2} \) for open string amplitudes. Polarization tensors, or multiplets of polarization tensors are generically denoted by \( \zeta \).

2.2. S-Matrix Amplitudes

S-matrix amplitudes are multilinear functions \( A : H^\otimes n \to \mathbb{C} \) constructed as follows. The operator formalism associates a measure \( \Omega(V_1, \ldots, V_n) \) on the moduli space \( F \) of ordered points on the boundary of the unit disk. We define \( A = \int_F \Omega \). By Mobius invariance of \( \Omega \), \( A \) is invariant under cyclic permutations.

\(^1\) The ghosts will not play an essential role in this paper so we will often identify a class with \( V \).
The definition of \( \int F \Omega \) requires some care. Using Mobius invariance of \( \Omega \) we transform the disk to the upper half plane and write:

\[
\mathcal{A}(V_1, \ldots, V_n) = \kappa^{n-2} \int_{F_{n-3}([0,1])} dy_i \langle 0 | V_n(y_n) V_{n-1}(y_{n-1}) \cdots V_1(y_1) | 0 \rangle \tag{2.3}
\]

where \( y_n = \infty, y_{n-1} = 1, y_1 = 0 \) and \( F_{n-3}([0,1]) \) is the moduli space of \( n-3 \) ordered points on the interval. The integral (2.3) is given meaning as follows. The integrand is a function of the \( y_i \) and of the relativistic invariants \( s_{ij} \equiv p_i \cdot p_j \). It follows from the o.p.e. that there is a domain where \( \text{Re}(s_{ij}) \) is sufficiently large and positive, or negative, (depending on \( ij \)) so that the integral is absolutely convergent and defines a holomorphic function of the \( s_{ij} \). The analytic continuation from this domain defines an amplitude which is a meromorphic function of the \( s_{ij} \).

It will be important to specify clearly the independent relativistic invariants that \( \mathcal{A} \) depends upon. These invariants are formed out of momenta \( p_i \) and polarization tensors \( \zeta_i \). Relativistic invariants formed from the \( p_i \) alone parametrize the different orbits of \( SO(1,d-1) \) on the space of \( n-1 \) independent momenta. Taking into consideration the relevant little group we see that the number of independent relativistic invariants for \( n \) particle scattering in \( d \) dimensions is

\[
nd - \frac{1}{2}d(d+1) \quad \text{for} \quad n \geq d+1
\]

\[
\frac{1}{2}n(n-1) \quad \text{for} \quad n \leq d+1
\]

For \( n \leq d \) a set of algebraically independent invariants can be chosen to be an appropriate collection of the \( s_{ij} \). For example, we may choose \( s_{ij} \) for \( 1 \leq i < j \leq n-1 \). (We make a different choice below.) For \( n \geq d+1 \) the story is more complicated. By classical invariant theory the ring of polynomial invariants is generated by the \( s_{ij} \) and by \( [p^{i_1}, \ldots, p^{i_d}] = \epsilon^{i_1 \cdots i_d \mu_1 \cdots \mu_d} p_{\mu_1}^{i_1} \cdots p_{\mu_d}^{i_d} \). The relations are generated by \( [p^{i_1}, \ldots, p^{i_d}] [p^{j_1}, \ldots, p^{j_d}] = \det_{s,t} p_{s}^{i_1} \cdots p_{s}^{i_d} . p_{t}^{j_1} \cdots p_{t}^{j_d} \), \( \det \Delta = 0 \) where \( \Delta \) is any \( (d+1) \)-dimensional minor of the matrix \( (s_{ij}) \), and \( \sum_{\sigma} \pm [p^{i_{\sigma(1)}}, \ldots, p^{i_{\sigma(d)}}] p^{j_{\sigma(d+1)}} \cdot p^{j} = 0 \) \([11]\). Thus for \( n \geq d+1 \) a maximal algebraically independent set of invariants can be taken to be an appropriate set of \( s_{ij} \) together with \([p^{1}, \ldots, p^{d}]\).

We separate the independent relativistic invariants into three types:

- Levels of the particles: \( p_i^2 = 2 - 2n_i \), \( i = 1, \ldots n \)
- Scalar product invariants. According to the above discussion for \( n \leq d \) these can be taken to be the \( s_{ij} \), \( 1 \leq i < j \leq n-2 \), together with \( 1 \leq i \leq n-3, j = n-1 \). For \( n \geq d+1 \) we must choose an appropriate set of \( n(d-1) - \frac{1}{2}d(d+1) - 1 \) \( s_{ij} \)’s.
• Polarization invariants. These are formed from contractions of the \( \zeta \)'s with themselves or the \( \zeta \)'s with \( p \)'s. The only way the relativistic invariants \( [p_1^i, \ldots, p_4^i] \) can enter a (bosonic) string amplitude is through the polarization invariants.

**Example:** The case \( n = 4 \) is of particular importance. This is the function \( \mathcal{A} : \mathcal{H}^4 \rightarrow \mathfrak{C} \) given by:

\[
\mathcal{A}(V_1, V_2, V_3, V_4) = \kappa^2 \int_0^1 dz \langle 0 | V_4(\infty) V_3(1) V_2(z) V_1(0) | 0 \rangle
\]  

(2.5)

The kinematic invariants are \( s = p_1 \cdot p_2, t = p_2 \cdot p_3 \). We let

\[
\left\{ \begin{array}{cccc}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
p_1 & p_2 & p_3 & p_4
\end{array} \right\}
\]  

(2.6)

stand for an ordered set of independent polarization invariants. Thus, the amplitudes are functions with independent arguments:

\[
\mathcal{A}_{n_1, n_2, n_3, n_4} \left( \left\{ \begin{array}{cccc}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
p_1 & p_2 & p_3 & p_4
\end{array} \right\} | s, t \right)
\]  

(2.7)

2.3. Analyticity Properties

As an analytic function \( \mathcal{A} \) is a polynomial in the polarization invariants. The coefficients of this polynomial are meromorphic functions of the \( s_{ij} \). One of the goals of this paper is to replace the complicated formula (2.3) by a symmetry principle. We will need to take the following three analyticity properties as axiomatic:

**AP1:** Location of poles. The dual diagrams of the genus zero open string amplitudes are binary trees rooted at the center of the unit disk with ordered terminal vertices on the boundary of the disk as in:

\[ \text{Diagram} \]

---

\[ ^2 \text{We choose a lexicographic ordering, with } \zeta \text{ in front of } p. \]
Only poles at $2s_{12} \in -p_1^2 - p_2^2 + \{2, 0, -2, \ldots\}$

Each terminal vertex carries an ingoing momentum $p_i$. Using momentum conservation we associate a momentum $p_I$ with each internal edge $I$. We assume that the amplitudes can only have poles when $p_I^2 \in \{2, 0, -2, -4, \ldots\}$ for some $I$ in some dual diagram. Explicitly, this is the condition that $(p_i + \cdots + p_{i+k})^2 \in \{2, 0, -2, -4, \ldots\}$ for some $i, k$, where indices are understood modulo $n$.

The proof that the $n$-point amplitudes (2.3) satisfy the axiom AP1 makes use of the operator product expansion and Mobius invariance of $\Omega$.

**AP2**: Growth at infinity. We assume that the amplitude has at most power law growth as any $s_{ij}$ tends to infinity, holding all other independent kinematic invariants fixed. This is a generalization of Regge behavior.

The axiom AP2 can be motivated by considering the $n$-tachyon scattering amplitude:

$$A = \kappa^{n-2} \int_0^1 \prod_{i=2}^{n-2} dy_i \theta(y_i - y_{i-1}) \prod_{1 \leq i < j \leq n-1} (y_j - y_i)^{s_{ij}}$$  \hspace{1cm} (2.8)

where $y_{n-1} = 1, y_1 = 0$. The standard argument proceeds as follows. If $Re(s_{ij}) \to +\infty$ then since all the factors in the product in (2.8) are $\leq 1$ the integration is dominated by the region where those factors raised to a power $s_{ij}$ are near one. Making an exponential change of variables and isolating this region proves the claim. As the simplest example
of this argument, suppose we take the $s_{ij}$ appearing in (2.8) as independent variables ($s_{1,n-1}$ will be considered dependent). Consider the limit $Re(s_{12}) \to +\infty$ holding all other independent $s_{ij}$ fixed. We write $y_2 = e^{-x_2}$ and the integral is dominated by the region $x_2 \sim 0$. Making this approximation in the rest of the integrand we find the asymptotics $s_{12}^{-n_{12}}$. In general we find

$$A \sim \alpha s_{ij}^\beta$$

where $\beta$ is a combination of the other $s_{kl}$'s. This argument extends to any correlation function and justifies the axiom AP2.

Finally we need an axiom that relates the amplitudes for different values of $n$. These are the well-known and standard tree-level unitarity equations:

**AP3**: Factorization. If we cut a dual diagram on some internal edge $I$ we decompose it into two dual diagrams $I_1, I_2$. We assume that when $p_{I_1}^2 \to 2 - 2n_I$ and other momenta are in general position the residue at the pole is

$$\sum_{a,b} A(V_i, \ldots, V_{i+k}, V_a) G^{ab} A(V_b, V_{i+k+1}, \ldots, V_{i-1})$$

where $\{V_a\}$ is a basis for $H[p_I, n_I]$ and $G^{ab}$ is the inverse of the positive definite metric on $H[p_I, n_I]$ whose existence is assured by the no-ghost theorem.

The proof that property AP3 is satisfied again makes use of the operator product expansion.

**Remark**: In stating the factorization axiom we have not made any use of a complex structure or of local coordinates on the disk. One could state a stronger factorization axiom valid for all values of the momenta. This requires the introduction of the full off-shell BRST chain complex and the introduction of local coordinates. We will not need that in the present paper.

### 2.4. High Energy-Fixed Angle Scattering

We define $s = p_1 \cdot p_2 \equiv -2E^2$, and $t = p_2 \cdot p_3 \equiv 2E^2 \sin^2 \frac{1}{2} \theta$. In the limit of high energy scattering, where all masses are effectively zero, $E$ and $\theta$ have the physical interpretation of center of mass energy and scattering angle. In the plane of scattering the spatial momenta look like:
The high energy limit is defined to be the limit where \( E^2 \to \infty \) along any ray other than the positive or negative real axis, holding \( \theta \) and all other independent relativistic invariants fixed. \( ^3 \)

Gross and Mende studied the asymptotic behavior of the amplitudes in the high-energy limit \[2\]. The amplitudes are dominated by a saddle point. For open strings: \( ^4 \)

\[
\mathcal{A} \sim \mathcal{A}^{s.p.} \left[ 1 + \mathcal{O}\left(1/s, 1/t, 1/(s + t)\right) \right]
\]

\[
\mathcal{A}^{s.p.} \equiv \sqrt{2\pi} \sqrt{\frac{st}{(s + t)^3}} \langle 0|V_1(\infty)V_2(z_0)V_3(1)V_4(\infty)|0 \rangle
\]

where \( z_0 \equiv s/(s + t) = 1/\cos^2 \frac{1}{2}\theta \).

The proof of (2.10) may be obtained by combining Stirling’s formula with the analytic structure of the amplitudes described above. Alternatively, one may use a saddle point analysis of either the integral over moduli space or of the path integral. The saddle point typically lies outside the domain of integration, and for good reason. The true high energy asymptotics obtained by taking \( E^2 \to \infty \) along the positive real axis must encounter an infinite set of poles when any intermediate state goes onshell. These poles are manifestly absent from (2.10). The saddle point analysis applies to the asymptotics of analytically continued \( S \)-matrix elements. This is important to bear in mind when considering generalizations to \( n \geq 5 \) point functions and to higher orders of string perturbation theory, in which case one probably must consider asymptotics only for \( E^2 \to \pm i\infty \).

\(^3\) Warning: this differs slightly from the \( \alpha' \to \infty \) limit of \[1\], since we hold expressions of the form \( \zeta \cdot p^k \) constant, even for polarizations with longitudinal components.

\(^4\) The result is BRST invariant despite appearances. The choice of the point \( z_0 \) magically cancels all differences arising from different choices of representative \( \mathcal{P} \) for the cohomology class.
3. Bracket Relations

3.1. Bracket

Two years ago it was shown that the BRST cohomology of 2D string theory has a rich algebraic structure related to BV quantization [12] [13] [14]. Subsequently, Lian and Zuckerman studied the algebraic structures of the BRST cohomology based on an arbitrary chiral algebra (=vertex operator algebra [15]) in [16] [17]. As shown by Lian and Zuckerman, there is an operation \( \{ \cdot, \cdot \} \) \( \mathcal{H}^{g_1} \times \mathcal{H}^{g_2} \to \mathcal{H}^{g_1+g_2-1} \) which they called the “Gerstenhaber bracket.” The bracket is very fundamental: it exists for arbitrary bosonic string cohomology and plays a role related to the BV anti-bracket for the on-shell BV structure of string theory. In the present case we may identify ghost number 1 classes with chiral physical state operators and the bracket is simply the standard “commutator” of dimension one currents:

\[
J \otimes V \to \{ J, V \}(z) \equiv \oint_z dw J(w) V(z) \quad (3.1)
\]

In general, the ghost number one cohomology based on a chiral algebra is a Lie algebra.

In our case \( \mathcal{H} \) in (2.1) is not based on a chiral algebra because the fields have monodromy when considered as chiral vertex operators. Nevertheless, when \( J, V \) are mutually local (3.1) still makes sense. In particular, given two momenta \( p, q \) with \( q^2 = 2 - 2n_1 \), \( p^2 = 2 - 2n_2 \), \( (p+q)^2 = 2 - 2n_3 \). we have a map:

\[
\{ \cdot, \cdot \} : \mathcal{H}[q, n_1] \otimes \mathcal{H}[p, n_2] \to \mathcal{H}[p+q, n_3]
\]

We may extend the bracket to all of \( \mathcal{H} \):

\[
\{ \cdot, \cdot \} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}
\]

by defining it to be zero on pairs not mutually local. In this paper “mutually local” means the o.p.e. has only integral powers of \( z - w \). Equivalently, the monodromy is trivial (although the braiding matrix could be \((-1))

6 Alternatively, one can take the point of view that the bracket is only defined on some pairs of states. That is, it is like multiplication in a groupoid, or composition of morphisms in a category.

\[
\text{B1. If } p_1, p_2 \text{ are the momenta of } V_1, V_2 \text{ then:}
\]

\[
\{ V_1, V_2 \} = -(-1)^{p_1 \cdot p_2} \{ V_2, V_1 \}
\]
B2. If \( V_1, V_2, V_3 \) are all pairwise mutually local then:

\[
(-1)^{p_1:p_3} \{V_1, \{V_2, V_3\}\} + (-1)^{p_3:p_2} \{V_3, \{V_1, V_2\}\} + (-1)^{p_2:p_1} \{V_2, \{V_3, V_1\}\} = 0 \quad (3.2)
\]

Unfortunately (3.2) does not hold for all triples in \( \mathcal{H} \).

The bracket on ghost number one cohomology generalizes the Lie algebra of the Gerstenhaber bracket in two ways. First, property B1 shows that the bracket is “vectorially-graded.” Second, the vectorially-graded Jacobi relation only holds for mutually local triples.

A table of useful structure constants for \( \{\cdot, \cdot\} \) can be found in appendix A.

3.2. Relations

Let \( J \) be any chiral physical state operator of momentum \( q \) and let \( V_i, i = 1, 2, 3, 4 \) be chiral physical state operators of momenta \( p_i \) such that \( q + \sum p_i = 0 \). Assume that \( q \cdot p_i \) are integral so that \( J \) and \( V_i \) are mutually local. In this case we can regard the integrand of (2.3) as a correlator of chiral vertex operators for conformal field theory on the plane. Using standard contour deformation arguments we derive the identity

\[
0 = \langle 0| V_4(\infty)V_3(1)V_2(z)\{J, V_1\}(0)|0 \rangle + (-1)^{q:p_2} \langle 0| V_4(\infty)V_3(1)\{J, V_2\}(z)V_1(0)|0 \rangle + (-1)^{q:p_2+q:p_3} \langle 0| V_4(\infty)\{J, V_3\}(1)V_2(z)V_1(0)|0 \rangle + (-1)^{q:p_2+q:p_3+q:p_4} \langle 0| \{J, V_4\}(\infty)V_3(1)V_2(z)V_1(0)|0 \rangle \quad (3.3)
\]

Now we simply integrate \( z \) from 0 to 1, assuming that \( Re(s) \) and \( Re(t) \) are sufficiently positive that the integral is absolutely convergent. This gives the finite difference relations:

\[
0 = A_{n_1,n_2,n_3,n_4} \left( \begin{array}{cccc}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
p_1 + q & p_2 & p_3 & p_4
\end{array} \right) \begin{array}{c}s + q \cdot p_2, \ t \end{array} + (-1)^{q:p_2} A_{n_1,n_2,n_3,n_4} \left( \begin{array}{cccc}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
p_2 + q & p_3 & p_4
\end{array} \right) \begin{array}{c}s + q \cdot p_1, \ t + q \cdot p_3 \end{array} + (-1)^{q:p_2+q:p_3} A_{n_1,n_2,n_3,n_4} \left( \begin{array}{cccc}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
p_1 & p_2 + q & p_3 & p_4
\end{array} \right) \begin{array}{c}s, \ t + q \cdot p_2 \end{array} + (-1)^{q:p_2+q:p_3+q:p_4} A_{n_1,n_2,n_3,n_4} \left( \begin{array}{cccc}
\zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\
p_1 & p_2 & p_3 + q & p_4
\end{array} \right) \begin{array}{c}s, \ t \end{array} \quad (3.4)
\]

\footnote{We do not assume that the \( V_i \) are mutually local.}
Here \( \tilde{n}_i, \tilde{\zeta} \) refer to the transformed operator under the bracket.

The relations (3.4) are finite difference equations relating scattering amplitudes for particles at different mass levels. The essential content of these identities is displayed by a matrix of the levels of the states involved. Each row of the matrix encodes the level-number of the states in an amplitude. The general Ward identity is associated with a matrix of the form:

\[
\begin{pmatrix}
\tilde{n}_1 & n_2 & n_3 & n_4 \\
n_1 & \tilde{n}_2 & n_3 & n_4 \\
n_1 & n_2 & \tilde{n}_3 & n_4 \\
n_1 & n_2 & n_3 & \tilde{n}_4
\end{pmatrix}
\] (3.5)

Here integers \( n_i \geq 0 \), are levels of the untransformed particles, while the integers \( \tilde{n}_i \) are the levels of the transformed particles. The level \( n_J \) of the symmetry current \( J \) is easily extracted from the sum of the diagonal minus the anti-diagonal:

\[
\sum \tilde{n}_i - \sum n_i = -q^2 = 2n_J - 2
\] (3.6)

If \( \tilde{n}_i \) is negative then \( \{ J, V \} = 0 \) and the corresponding amplitude simply vanishes.

This procedure generalizes to \( N \)-particle scattering. Whenever \( J \) of momentum \( q \) is mutually local w.r.t all the \( p_i \) and \( q + \sum p_i = 0 \) we have an identity

\[
\sum_i (-1)^{q \cdot p_2 + \cdots + q \cdot p_i} A(V_1, \ldots, \{ J, V_i \}, \ldots V_N) = 0
\] (3.7)

These relations are subject to an important technical restriction discussed in the next section.

**Remark:** Although we have used currents \( J \) to derive relations on amplitudes, they are not unbroken symmetry currents in the theory since there are many operators in the BRST cohomology which are not mutually local w.r.t. any given \( J \). Nevertheless, using Lorentz invariance and analyticity we are still able to derive relations on amplitudes.

### 3.3. Existence of required momenta

The identities (3.4) would be of little use if one could rarely choose momenta in the way we have indicated. In this section we show that we can always find momenta corresponding to any prescription of the levels and kinematic invariants (at least, for \( N \leq d = 26 \)). In doing so it is necessary to allow the momenta to be complex. We may do this since the BRST conditions and the vertex operator calculus make perfect sense when momenta are complex.
Lemma. Suppose $N \leq d$. For any complex numbers $n_i, \tilde{n}_i, i = 1, \ldots, N$, and $\frac{1}{2}N(N - 3)$ complex numbers $z_{ij}, 1 \leq i < j \leq N - 2$, $z_{1,N-1}, \ldots, z_{N-3,N-1}$, there exist momenta $p_1, \ldots, p_N \in \mathbb{C}^d$ such that, if we define $q \equiv -\sum p_i$ then

$$p_i^2 = 2 - 2n_i \quad (3.8a)$$

$$(p_i + q)^2 = \left(\sum_{j \neq i} p_j\right)^2 = 2 - 2\tilde{n}_i \quad (3.8b)$$

$$p_i \cdot p_j = z_{ij} \quad (3.8c)$$

Proof: For $N \leq d$ the invariants $s_{ij}$ made from $N$ momenta $p_i$ are algebraically independent. It is straightforward to solve (3.8) as a linear system of equations to find the $p_i \cdot p_j$ in terms of linear combinations of the $n_i, \tilde{n}_i, z_{ij}$. We can regard the equations for $p_i \cdot p_j$ as equations for the intersection of $\frac{1}{2}N(N + 1)$ quadrics in $\mathbb{P}^{Nd}$. These intersect in a variety of codimension at most $\frac{1}{2}N(N + 1)$. Even if this variety lies at infinity we can use the solution - we simply must take a momentum $\to \infty$ limit in the amplitudes. ♠

The different sets of polarization invariants in (3.4) are polynomially related. Therefore, using the analyticity of $\mathcal{A}$ we conclude that (3.4) holds for all values of $s, t$ and all polarization tensors satisfying these polynomial relations. Similarly, provided $n_i, n_j$ are positive integers and $\tilde{n}_i$ are integers, we can find momenta for arbitrary kinematic invariants for which the identities (3.7) can be written.

Unfortunately one cannot generalize the lemma to the case $N > d$. For $N \geq d + 1$ the $s_{ij}$ are not algebraically independent, hence we cannot specify arbitrarily the $n_i, \tilde{n}_i$ and independent $s_{ij}, i \neq j$ (for $N$-particle scattering).

We are therefore stuck with the rather distasteful limitation to $N$-particle scattering for $N \leq d$.

3.4. High Energy Limit

Combining (2.10) with (3.4) we obtain the high energy identities:

$$0 = z_0^{p_2 \cdot q} \mathcal{A}^{s,p}(\{J, V_1\}, V_2, V_3, V_4)$$

$$+ (-1)^{q \cdot p_2} z_0^{p_1 \cdot q} (1 - z_0)^{p_3 \cdot q} \mathcal{A}^{s,p}(V_1, \{J, V_2\}, V_3, V_4)$$

$$+ (-1)^{q \cdot p_2 + q \cdot p_3} (1 - z_0)^{p_2 \cdot q} \mathcal{A}^{s,p}(V_1, V_2, \{J, V_3\}, V_4)$$

$$+ (-1)^{q \cdot p_1} \mathcal{A}^{s,p}(V_1, V_2, V_3, \{J, V_4\}) \quad (3.9)$$

The momenta are allowed to take values in the extended complex plane $\hat{\mathbb{C}}$. 

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where \( z_0 = s/(s + t) = 1/\cos^2 \frac{1}{2} \theta, 1 - z_0 = t/(s + t) = -\tan^2 \frac{1}{2} \theta, \) and each amplitude in (3.9) is evaluated at the same value of \( s, t. \) We take \( s, t \to \infty \) holding \( q \cdot p_i \) fixed. The amplitude \( \mathcal{A}^{s-p} \) in (2.10) has the form \( RU \) where \( R \) is a rational function of \( s, t \) and

\[
U \equiv \sqrt{\frac{st}{(s + t)^3}} \exp \left\{ t \log t + s \log |s| - (s + t) \log |s + t| \right\} = \frac{\cos \frac{1}{2} \theta}{\sqrt{2E \sin^3 \frac{1}{2} \theta}} \exp \left\{ 2E^2 \left( \sin^2 \frac{1}{2} \theta \log \sin^2 \frac{1}{2} \theta + \cos^2 \frac{1}{2} \theta \log \cos^2 \frac{1}{2} \theta \right) \right\}
\]

(3.10)

Shifts such as \( s, t \to s + q \cdot p_2, t + q \cdot p_3 \) lead to an order one change in the amplitude from the exponential factor in (3.10), leading to the “extra” powers of \( z_0, 1 - z_0 \) in (3.9). The change \( \delta z_0 \) in the position of the saddle-point in moduli space is \( O(1/s, 1/t, 1/(s + t)) \).

Similarly, the shifts in \( s, t \) change the rational function \( R \) by terms of the same order. Hence (3.9) holds up to factors \( 1 + O(1/s, 1/t, 1/(s + t)) \).

4. Six Examples

**Example 1:** The simplest example of (3.4) has level matrix:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2
\end{pmatrix}
\]

(4.1)

From the entries we read off that this is a “tachyonic identity,” i.e. \( q^2 = 2. \) We also can read off:

\[ p_1^2 = 2 \quad p_1 \cdot q = p_2 \cdot q = p_3 \cdot q = -1 \quad p_4 \cdot q = +1 \]

Using (A.1) of appendix A we see that (3.4) implies

\[ \mathcal{A}(s - 1, t) + \mathcal{A}(s, t - 1) = \mathcal{A}(s - 1, t - 1) \]

(4.2)

where \( \mathcal{A} \) is the basic Veneziano amplitude \( \mathcal{A}_{0000} = \mathcal{A}(s, t) \) for scattering of four states at level 0.

**Example 2:** Consider the level matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

(4.3)
This encodes a lightlike Ward identity with \( J = \frac{i}{\lambda^2} \partial X e^{i q \cdot X} \) and \( q^2 = 0 \). It relates \( \gamma T T T \) scattering to \( T T T T \) scattering, where \( \gamma \) refers to the level one photon and \( T \) refers to the level zero tachyon. For the \( \gamma T T \) amplitude \((2.7)\) simplifies to:

\[
A_{1000}(\zeta_1 \cdot p_2, \zeta_1 \cdot p_3|s,t)
\]

From \((4.3)\) we read off

\[
\begin{align*}
p_1^2 &= p_2^2 = p_3^2 = p_4^2 = 2 \\
p_1 \cdot q &= -1 \quad p_2 \cdot q = p_3 \cdot q = 0 \quad p_4 \cdot q = +1
\end{align*}
\]

Using \((A.3)\) we have:

\[
A_{1000}(\zeta_1 \cdot p_2, \zeta_1 \cdot p_3|s,t) = -\zeta_1 \cdot p_2 A(s - 1, t) - \zeta_1 \cdot p_3 A(s, t) \quad (4.4)
\]

From \((3.9)\) we see that at high energies \( \gamma T T T \) scattering is related to tachyon scattering via

\[
0 = A_{1000}^s(p)(\zeta_1 \cdot p_2, \zeta_1 \cdot p_3|s,t) + (1 + t/s)\zeta_1 \cdot p_2 A^{s,s}(s, t) + \zeta_1 \cdot p_3 A^{s,s}(s, t) \quad (4.5)
\]

Similarly, one easily derives relations on similar amplitudes. For example:

\[
A_{0100}(\zeta_2 \cdot p_1, \zeta_2 \cdot p_3|s,t) = \zeta_2 \cdot p_1 A(s - 1, t) - \zeta_2 \cdot p_3 A(s, t - 1) \quad (4.6)
\]

**Example 3:** We now consider the level matrix

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & -1
\end{pmatrix} \quad (4.7)
\]

again giving a lightlike Ward identity. This relates \( \gamma \gamma T T \) to \( \gamma T T T \) scattering.

We consider the \( \gamma \gamma T T \) amplitude to be a function of seven arguments:

\[
A_{1100}(\zeta_1 \cdot \zeta_2, \zeta_1 \cdot p_2, \zeta_1 \cdot p_3, \zeta_2 \cdot p_1, \zeta_2 \cdot p_3|s,t)
\]

Using \((A.3)\) and \((A.5)\) \((3.4)\) becomes (after shifting arguments and setting \( \zeta_2 \cdot q = 0 \)):

\[
A_{1100}(x_1, x_2, x_3, x_4, x_5|s,t) = -A_{0100}(x_2x_4 - x_1, x_2x_5|s - 1, t) \\
- x_3 A_{0100}(x_4, x_5, s, t) \\
= (x_1 - x_2x_4)A(s - 2, t) + x_2x_5 A(s - 1, t - 1) - x_3x_4 A(s - 1, t) \\
+ x_3x_5 A(s, t - 1) \quad (4.8)
\]
Which has the high energy limit:

\[ A_{1100}^{s,p}(x_1, x_2, x_3, x_4, x_5|s,t) = - (1 + t/s) A_{0100}^{s,p}(x_2 x_4 - x_1, x_2 x_5|s,t) - x_3 A_{0100}^{s,p}(x_4, x_5|s,t) \] (4.9)

Notice that if we were restricted to using real momenta and polarizations then \( p^2 = q^2 = p \cdot q = 0 \Rightarrow p \parallel q \) and we would not derive relations on the most general amplitude (e.g. \( s \) would be fixed). This is not true if we use complex momenta. After we derive the Ward identity we can specialize to physical values of the relativistic invariants, and, as in (4.9), solve for the amplitude in terms of previously known amplitudes.

**Example 4**: We now consider

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & -1
\end{pmatrix}
\] (4.10)

which will express \( A_{1110} \) in terms of amplitudes at lower total level. The ordered set of relativistic invariants in \( A_{1110} \) is:

\[
\left\{ \zeta_1 \zeta_2 \zeta_3 \zeta_4, p_1, p_2, p_3, p_4 \right\} = \{ \zeta_1 \cdot \zeta_2, \zeta_1 \cdot \zeta_3, \zeta_2 \cdot \zeta_3, \zeta_1 \cdot p_2, \zeta_1 \cdot p_3, \zeta_2 \cdot p_1, \zeta_2 \cdot p_3, \zeta_3 \cdot p_1, \zeta_3 \cdot p_2 \} \] (4.11)

Following the procedure of the previous two examples we get:

\[
A_{1110}(x_1, \ldots, x_9|s,t) = A_{1100}(x_1 x_8, x_4 x_8, x_5 x_8 - x_2, x_6, x_7|s,t) + A_{1100}(x_1 x_9, x_4, x_5, x_6 x_9, x_7 x_9 - x_3|s,t - 1) \] (4.12)

which has the high energy limit:

\[
A_{1110}^{s,p}(x_1, \ldots, x_9|s,t) = A_{1100}^{s,p}(x_1 x_8, x_4 x_8, x_5 x_8 - x_2, x_6, x_7|s,t) + (1 + s/t) A_{1100}^{s,p}(x_1 x_9, x_4, x_5, x_6 x_9, x_7 x_9 - x_3|s,t) \] (4.13)

**Example 5**: We take

\[
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
\] (4.14)

which will relate the 4-photon amplitude to the tachyon amplitude. We now need the bracket \((A.4)\)
The ordered set of relativistic invariants occurring in $A_{1111}$ is:

$$\left\{ \zeta_1, \zeta_2, \zeta_3, \zeta_4 \right\} = \left\{ \zeta_1 \cdot \zeta_2, \zeta_1 \cdot \zeta_3, \zeta_1 \cdot \zeta_4, \zeta_2 \cdot \zeta_3, \zeta_2 \cdot \zeta_4, \zeta_3 \cdot \zeta_4 \right\}$$

(4.15)

Hence

$$\zeta_1 \cdot p_2, \zeta_1 \cdot p_3, \zeta_2 \cdot p_1, \zeta_2 \cdot p_3, \zeta_3 \cdot p_1, \zeta_3 \cdot p_2, \zeta_4 \cdot p_1, \zeta_4 \cdot p_2$$

So the four-photon amplitude is a function of 16 arguments, and (3.4) becomes

$$A_{1111}(x_1, \ldots, x_{14}|s, t)$$

$$= -A_{1110}(x_1 x_{13}, x_2 x_{13}, x_4, x_7 x_{13}, x_8 x_{13} + x_3, x_9, x_{10}, x_{11}, x_{12}|s, t)$$

$$- A_{1110}(x_{14}, x_2, x_4 x_{14}, x_7, x_8, x_9 x_{14}, x_{10} x_{14} + x_5, x_{11}, x_{12}|s, t + 1)$$

$$+ x_6 A_{1100}(x_1, x_7, x_8, x_9, x_{10}|s, t)$$

(4.16)

with high energy limit:

$$A_{1111}^{s, p_1}(x_1, \ldots, x_{14}|s, t)$$

$$= -A_{1110}^{s, p_1}(x_1 x_{13}, x_2 x_{13}, x_4, x_7 x_{13}, x_8 x_{13} + x_3, x_9, x_{10}, x_{11}, x_{12}|s, t)$$

$$- \frac{t}{(s + t)} A_{1110}^{s, p_1}(x_{14}, x_2, x_4 x_{14}, x_7, x_8, x_9 x_{14}, x_{10} x_{14} + x_5, x_{11}, x_{12}|s, t)$$

$$+ x_6 A_{1100}^{s, p_1}(x_1, x_7, x_8, x_9, x_{10}|s, t)$$

(4.17)

**Example 6:** As a final example we look at a timelike identity with level matrix:

$$\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

(4.18)

Hence $q^2 = -2$ and

$$J = V_{\zeta, q} \equiv [i q \cdot \zeta \cdot \partial^2 X + \partial X \cdot \zeta \cdot \partial X]e^{iqX}$$

$$\text{tr}(\zeta) - 2q \cdot \zeta \cdot q = 0$$

(4.19)

We parametrize the scattering of 1 level 2 on 3 level 0 states by the function of 8 variables:

$$A_{2000}(p_1 \cdot \zeta, p_1 \cdot \zeta, p_1 \cdot \zeta, p_1 \cdot \zeta, p_2 \cdot \zeta, p_2 \cdot \zeta, p_2 \cdot \zeta, p_2 \cdot \zeta, p_3 \cdot \zeta, p_3 \cdot \zeta, p_3 \cdot \zeta, p_3 \cdot \zeta)|s, t)$$

(4.20)

Now we use the bracket in (4.6). After shifting arguments a little one finds the bracket relation:

$$A_{2000}(p_1 \cdot \zeta, p_1 \cdot \zeta, \ldots, p_3 \cdot \zeta, p_3 |s, t) = -(p_2 \cdot \zeta, p_2 + p_1 \cdot \zeta, p_2, p_2 \cdot \zeta, p_2, p_2 \cdot \zeta, p_3, p_3 \cdot \zeta, p_3 |s, t)$$

(4.21)
with high energy limit:

\[
A^p_{2000}(p_1 \cdot \zeta \cdot p_1, \ldots p_3 \cdot \zeta \cdot p_3|s, t) = -\frac{t}{s}(1 + \frac{t}{s})(p_2 \cdot \zeta \cdot p_2 + p_1 \cdot \zeta \cdot p_2)A^p_{s.p.}(s, t)
\]

\[
+ \frac{t}{s}(p_3 \cdot \zeta \cdot p_3 + p_1 \cdot \zeta \cdot p_3)A^p_{s.p.}(s, t)
\]

\[
-(1 + \frac{t}{s})(p_3 \cdot \zeta \cdot p_3 + p_1 \cdot \zeta \cdot p_3 + p_2 \cdot \zeta \cdot p_2 + p_1 \cdot \zeta \cdot p_2 + 2p_2 \cdot \zeta \cdot p_3)A^p_{s.p.}(s, t)
\]

As an exercise the reader may care to work out some futuristic identities for \(A_{2001}\).

5. Determination of Tachyon Amplitudes

In the previous section we saw that the relations (3.4) lead to a host of interlevel amplitude identities. In the present section we discuss finite difference relations on the tachyon amplitudes themselves. We show that relations (3.4) (3.7) together with the analyticity properties \(AP_1, AP_2\) determine the \(N\)-tachyon scattering amplitudes up to an overall constant \(c_N\).

5.1. Derivation of the Veneziano formula

Example one of the previous section has already produced one identity on the tachyon scattering amplitude \(A(s, t)\). The relation (4.2) by itself is not sufficiently strong to determine the function \(A\). However, we can combine it with (4.6) using the decoupling of BRST trivial states. Decoupling of the longitudinal photon implies that \(A_{0100}(s,t|s,t) = 0\). Combining this with (4.6) we get

\[
sA(s - 1, t) = tA(s, t - 1)
\]

Now the recursion relations (4.2) and (5.1) determine the value of \(A\) for \(s, t \in \mathbb{Z}_+\) to be given by the Veneziano formula:

\[
A_{0000} = A(s, t) = c_4 \frac{\Gamma(s + 1) \Gamma(t + 1)}{\Gamma(s + t + 2)}
\]

where \(c_4 = A(0, 0)\) is assumed nonzero.

In order to obtain the amplitude for all \(s, t\) we must “analytically continue from the integers,” an idea familiar from studies of the S-matrix in \(D = 2\) spacetime dimensions. (For reviews see [19]. ) It is at this point that we must invoke the analyticity properties \(AP_1, AP_2\) of section 2.3.
Let us define
\[ A(s, t) = c_4 \frac{\Gamma(s + 1)\Gamma(t + 1)}{\Gamma(s + t + 2)} \tilde{A}(s, t) \] (5.3)

The functional equations for \( A \) imply that \( \tilde{A}(s, t) \) is periodic of period one in both \( s, t \). Moreover, by \textbf{AP1} it is an entire function. By \textbf{AP2}, the entire function \( \tilde{A} \) is of exponential type [20]. A periodic entire function of exponential type must be a trigonometric polynomial [20], that is:
\[ \tilde{A}(s, t) = \sum_{n, m} c_{n, m} e^{2\pi i(ns + mt)} \]
where the sum is finite. Now applying \textbf{AP2} again we see that \( \tilde{A} \) must in fact be constant, so \( \tilde{A} = 1 \).

In conclusion, with a mild analyticity assumption we see that the Veneziano amplitude is fixed by symmetry.

5.2. \( n \)-particle scattering, \( n \leq 26 \)

The above procedure can be extended to higher point functions, although the amount of work involved goes up rapidly with \( n \). We take the independent kinematic variables to be \( s_{ij} \) for \( 1 \leq i < j \leq n - 2 \) and \( s_{i,n-1} \) for \( 1 \leq i \leq n - 3 \). The equation
\[ 2 - n = \sum_{1 \leq i < j \leq n-1} s_{ij} \] (5.4)
expresses \( s_{n-2,n-1} \) in terms of the other invariants. It is often useful to think of the variables as an upper triangular matrix:
\[
\begin{pmatrix}
* & s_{12} & s_{13} & \cdots & s_{1,n-1} \\
0 & * & s_{23} & \cdots & s_{2,n-1} \\
0 & \cdots & * & s_{n-3,n-2} & s_{n-3,n-1} \\
0 & \cdots & 0 & * & s_{n-2,n-1} \\
0 & \cdots & 0 & 0 & *
\end{pmatrix}
\] (5.5)
where \( s_{n-2,n-1} \) is not an independent variable but is fixed by (5.4).

The tachyon amplitudes are functions on the space of upper triangular matrices (5.3) defined by (2.8).

The generalization of (1.2) is
\[
\mathcal{A}(..., s_{ab}, ..., s_{ac}, ..., s_{bc}, ...) = \mathcal{A}(..., (s_{ab} - 1), ..., (s_{ac} + 1), ..., s_{bc}, ...)
- \mathcal{A}(..., (s_{ab} - 1), ..., (s_{ac}), ..., (s_{bc} + 1), ...)
\] (5.6)
which holds \( \forall a, b, c \) such that \( 1 \leq a < b < c \leq n - 1 \) where all other variables in the ellipsis are held fixed. We call these “triangle relations” since they relate a triangle of variables in (5.3).

Using the triangle relations one easily reduces an arbitrary tachyon amplitude for \( s_{ij} \in \mathbb{Z}_+, 1 \leq i < j \leq n - 2 \) to the case where:

\[
s = \begin{pmatrix}
* & 0 & 0 & \cdots & 0 & s_{1,n-1} \\
0 & * & 0 & \cdots & 0 & s_{2,n-1} \\
& \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & * & s_{n-3,n-1} \\
0 & \cdots & 0 & 0 & s_{n-2,n-1} \\
0 & \cdots & 0 & 0 & 0 & \star
\end{pmatrix}
\]

We denote a tachyon amplitude evaluated for such a set of invariants by the function \( F(s_{1,n-1}, \ldots, s_{n-3,n-1}) \).

Next, one may write lightlike relations connecting \( A_{0...010...0} \) to tachyon scattering. As above, one can put \( A_{0...010...0} \) to zero by evaluating at special polarization invariants corresponding to longitudinal photons. In this way we obtain a set of \( n - 1 \) relations for the function \( F \). These equations are:

\[
0 = \sum_{1 \leq j \leq n-2; j \neq a} \left[ \sum_{\epsilon \in \mathbb{Z}_2^{n-2}: \epsilon_j = \epsilon_a = 0} (-1)^{|\epsilon|} F\left( \vec{s} + (n - 4 - |\epsilon|) \hat{e}_a + \vec{\epsilon} \right) \right] + s_{a,n-1} \sum_{\epsilon \in \mathbb{Z}_2^{n-3}: \epsilon_a = 0} (-1)^{|\epsilon|} F\left( \vec{s} + (n - 4 - |\epsilon|) \hat{e}_a + \vec{\epsilon} \right)
\]

which holds for \( 1 \leq a \leq n - 3 \) and where \( \hat{e}_a \) is a unit vector in the \( a \) direction, \( |\epsilon| \) is the sum of the nonzero entries, and the last entry of an \( n - 2 \) vector is dropped in the argument of \( F \). We have two additional identities for \( a = n - 2 \):

\[
(6 - 2n - \sum_{1 \leq j \leq n-3} s_{i,n-1}) \sum_{\epsilon \in \mathbb{Z}_2^{n-3}} (-1)^{|\epsilon|} F\left( \vec{s} + \vec{\epsilon} \right) = \sum_{1 \leq j \leq n-3} \left[ \sum_{\epsilon \in \mathbb{Z}_2^{n-3}: \epsilon_j = 0} (-1)^{|\epsilon|} F\left( \vec{s} + \vec{\epsilon} \right) \right]
\]

and for \( a = n - 1 \):

\[
\sum_{1 \leq j \leq n-3} s_{j,n-1} F(\vec{s} - \hat{e}_j) = (n - 3 + \sum_{1 \leq j \leq n-3} s_{i,n-1}) F(\vec{s})
\]
As we will show below, these equations are sufficient to determine the functional dependence of $F$ for $s_{ij} \in \mathbb{Z}$, up to a finite set of arbitrary constants. The remaining undetermined constants can be fixed (up to one overall scale) by use of the axiom AP1. We illustrate this by giving the general solution to the functional equations (5.7)(5.8)(5.9) for the case of the 5-particle function. We then give the general argument.

5.3. 5-particle function

The functional equations become

\[
x \left[ F(x + 1, y) - F(x, y) - F(x, y + 1) + F(x - 1, y + 1) \right] +
+ 2F(x + 1, y) - F(x, y) - F(x + 1, y + 1) = 0
\]

\[
y \left[ F(x + 1, y) - F(x + 1, y - 1) - F(x, y + 1) + F(x, y) \right] +
+ F(x + 1, y) + F(x, y) - 2F(x, y + 1) = 0
\]

\[
(x + y + 4) \left[ F(x + 1, y + 1) - F(x + 1, y) - F(x, y + 1) + F(x, y) \right] +
F(x + 1, y) + F(x, y + 1) - 2F(x, y) = 0
\]

\[
(x + y + 2)F(x, y) = xF(x - 1, y) + yF(x, y - 1)
\] (5.10)

The equations must be evaluated in the quadrant $x = -2 - a, y = -2 - b, a, b \geq 0$, otherwise the amplitude might have poles. The general solution to (5.10) in this quadrant is easily found to be

\[
F(x, y) = \frac{2 \left[ 4\beta + (3\alpha - 4\beta)x + (8\beta - 3\alpha)y \right]}{(x + 1)(y + 1)(x + y + 2)}
\] (5.11)

where $\alpha, \beta$ are undetermined. By AP1 we only allow poles in $x$ or in $x + y$ (since $y$ itself is not related to the squared momentum in any channel). This requirement fixes $3\alpha = 4\beta$ so that

\[
F(x, y) = \frac{6\alpha}{(x + 1)(x + y + 2)}
\] (5.12)

It is now a simple matter to use the triangle relations together with the relation

\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{1}{n - j + x} = -\frac{\Gamma(-n - x)\Gamma(n + 1)}{\Gamma(1 - x)}
\]
to obtain the complete five-particle amplitude in terms of a generalized hypergeometric function:

\[
A(s_{12}, s_{13}, s_{14}, s_{23}, s_{24}) = 6\alpha\frac{\Gamma(s_{12} + 1)\Gamma(s_{23} + 1)\Gamma(1 + s_{34})\Gamma(-1 - s_{12} - s_{13} - s_{14})}{\Gamma(2 + s_{23} + s_{34})\Gamma(-s_{13} - s_{14})} \binom{3}{2}s_{34}\left(-s_{13}, -1 - s_{12} - s_{13} - s_{14}, 1 + s_{34}; -s_{13} - s_{14}, 2 + s_{23} + s_{34}\mid 1\right)
\] 

(5.13)

where \(s_{34}\) is defined by (5.4).

Equation (5.13) is derived for integral values of \(s_{ij}\). In order to “continue” to all values of \(s_{ij}\) we must combine results on entire functions with the analyticity property \(\text{AP2}\) of section 2.3, as in the previous section. This is straightforward when \(s_{13} = 0\). In the general case we must use a Mellin-Barnes representation for \(3F_2\) to establish appropriate asymptotics.

The formula (5.13) can actually be derived directly from the integral representation (2.8) using formulae in, e.g., [21]. Similar formulae have appeared in a different context in [22].

5.4. \(6 \leq n \leq 26\)

The above discussion generalizes to \(n\)-particle scattering for \(6 \leq n \leq 26\). The analyticity property \(\text{AP1}\) combined with the functional equation (5.9) is sufficiently strong to obtain the general formula.

We now prove this. To begin we use (5.9) with \(s_2 = s_3 = \cdots = 0\):

\[
s_1F(s_1 - 1, 0) = (n - 3 + s_1)F(s_1, 0)
\] 

(5.14)

which implies

\[
F(s_1, 0) = \frac{c}{(s_1 + 1)\cdots(s_1 + n - 3)}
\] 

(5.15)

where \(c\) is a constant. Now given \(F(s_1, 0)\) we can again use (5.9) to derive

\[
F(s_1, 1, 0) = \frac{c_1s_1 + c_2}{(s_1 + 1)\cdots(s_1 + n - 2)}
\] 

(5.16)

where \(c_1, c_2\) are constants. We can carry on in this way and easily establish by induction that if \(s_2, s_3, \ldots\) are nonnegative integers then

\[
F(s_1, s_2, \ldots, s_{n-3}) = \frac{P_{s_2, \ldots, s_{n-3}}(s_1)}{(s_1 + 1)\cdots(s_1 + s_2 + \cdots + s_{n-3} + n - 3)}
\] 

(5.17)
where \( P^{s_2,\ldots,s_{n-3}}(s_1) \) is a polynomial of degree \( s_2 + \cdots + s_{n-3} \).

Now, as with the 5pt function we can fix the coefficients of the polynomial by invoking AP1. By AP1 the function \( F(\vec{s}) \) can only have poles when

\[
s_{1,n-1} \in \{-1, 0, 1, 2, \ldots\} \\
s_{1,n-1} + s_{2,n-1} \in \{-2, -1, 0, 1, 2, \ldots\} \\
s_{1,n-1} + s_{2,n-1} + s_{3,n-1} \in \{-3, -2, -1, 0, 1, \ldots\} \\
\vdots \\
s_{1,n-1} + \cdots + s_{n-3,n-1} \in \{-(n-3), -(n-4), \ldots\}
\]

Therefore, the polynomial in the numerator of (5.17) must cancel the poles:

\[
s_{1,n-1} \in \{-s_2 - 1, \ldots, -2\} \\
s_{1,n-1} \in \{-s_2 - 3, \ldots, -s_2 - s_3 - 2\} \\
\vdots \\
s_{1,n-1} \in \{-s_2 - \cdots - s_{n-4} - (n-3), \ldots, -s_2 - \cdots - s_{n-3} - (n-4)\}
\]

This fixes all the constants in \( P \) up to an overall scale and we obtain the result

\[
F(\vec{s}) = c_n \prod_{j=1}^{n-3} \frac{-1}{j + \sum_{\ell=1}^{n-1} s_{\ell,n-1}}
\]

which can also be checked directly from the integral representation (2.8).

One can proceed from here to evaluate the general tachyon amplitude using the triangle relations and AP2. The result after “putting back” \( s_{12}, s_{23}, \ldots s_{n-3,n-2} \) is still a product of gamma functions. The general result is a multiple hypergeometric function of \( \frac{1}{2}(n-3)(n-4) \) arguments. For example, the six-point function turns out to be:

\[
\sum_{j_1,j_2,j_3 \geq 0} \frac{(-s_{14})_{j_1}(-s_{24})_{j_2}(-s_{13})_{j_3} \Gamma(-s_{12} - s_{13} - s_{14} - s_{15} + j_1 + j_3)}{j_1!j_2!j_3! \Gamma(1 - s_{13} - s_{14} - s_{15} + j_1 + j_3) \Gamma(4 - s_{34} - s_{35} - s_{45} + j_1 + j_2 + j_3)} \times \frac{\Gamma(1 + s_{12})\Gamma(1 + s_{23})\Gamma(1 + s_{34})}{\Gamma(5 - s_{23} - s_{34} - s_{35} - s_{45} + j_1 + j_2 + j_3) \Gamma(5 - s_{23} - s_{34} - s_{35} - s_{45} + j_1 + j_2)}
\]

Multiple hypergeometric functions have been studied to some extent in the literature, see, e.g., [21]. In order to apply AP2 we must give a Mellin-Barnes representation to series like (5.21) to establish the appropriate asymptotics in \( s_{ij} \). We have not carried out this procedure in complete detail, but fully expect that it can be done.
6. S is unique

We now argue that the solution to the bracket relations is essentially unique.

6.1. 4-particle S-matrix

We begin by showing that the identities (3.4) completely fix the 4-particle S-matrix for all particles in terms of the level zero S-matrix $A(s,t)$. Indeed, this can already be done simply by using the lightlike Ward identities. The proof of this assertion is a simple application of the no-ghost theorem and DDF operators \[7\]. If $k_0$ is a lightlike vector, $p_0^2 = 2$, and $p_0 \cdot k_0 = 1$ then the bracket

$$\{i\zeta \cdot \partial X e^{-i\tau k_0 X}, \cdot\} : \mathcal{H}[p_0 - (n - \ell)k_0, n - \ell] \to \mathcal{H}[p_0 - nk_0, n]$$

is equivalent to applying DDF operators $\zeta \cdot A_{-\ell}$ to $\mathcal{H}[p_0 - (n - \ell)k_0, n - \ell]$. Therefore, the no-ghost theorem implies that

$$\{,\} : \oplus_{1 \leq \ell \leq n} \mathcal{H}[-\ell k_0, 1] \otimes \mathcal{H}[p_0 - (n - \ell)k_0, n - \ell] \to \mathcal{H}[p_0 - nk_0, n] \quad (6.1)$$

is a surjective map.

We would like to proceed as in the examples of section four using the Ward identities to reduce the level-numbers of various states in the amplitude. We use induction on $n_T = \sum n_i$, the sum of the level numbers of the states in an amplitude. Consider the lightlike Ward identities of the form

$$\begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ n_1 - \ell & n_2 - \ell & n_3 & n_4 \\ n_1 - \ell & n_2 & n_3 & n_4 \\ n_1 - \ell & n_2 & n_3 & n_4 \end{pmatrix} \quad (6.2)$$

for $1 \leq \ell \leq n_1$. These identities relate the amplitude encoded by the first row to amplitudes with smaller values of $n_T$. Since (6.1) is surjective, we can map an arbitrary amplitude, that is, an amplitude where the level $n_1$ state has an arbitrary polarization, to amplitudes with smaller total level number.
6.2. N-particle scattering

The above argument generalizes easily to \(N\)-particle scattering, using the level matrix

\[
\begin{pmatrix}
  n_1 & n_2 & n_3 & \ldots & n_N \\
  n_1 - \ell & n_2 - \ell & n_3 & \ldots & n_N \\
  \vdots & \ddots & \vdots \\
  n_1 - \ell & n_2 & n_3 & \ldots & n_N \\
  n_1 - \ell & n_2 & n_3 & \ldots & n_N \\
\end{pmatrix}
\]  \tag{6.3}

as long as we can specify the levels and kinematic invariants arbitrarily. From the discussion of section 3.3 we see that this is possible for \(N \leq 26\) and hence \(N\)-particle scattering for arbitrary levels can be expressed in terms of tachyon scattering.

We can complete the argument for uniqueness by using the results of section 5, where we showed that the bracket relations determine the \(N\)-tachyon amplitude up to a constant \(c_N\) (see (5.20)).

6.3. Summary

We have established:

**Theorem 1**: A multilinear function \(A_N : \mathcal{H}^{\otimes N} \rightarrow \mathbb{C} \), \(N \leq 26\), which satisfies

1. Poincaré invariance
2. The bracket relations (3.7).
3. The analyticity properties \(\text{AP1 and AP2}\)

is uniquely determined up to an overall constant, \(c_N\).

Moreover, an easy argument gives:

**Theorem 2**: A set of multilinear functions \(\{A_N\}_{N \leq 26}\) which satisfy 1,2,3 above as well as \(\text{AP3}\) are uniquely specified up to one parameter \(\kappa\) by \(c_N = \kappa^{N-2}\).

Theorem 2 follows in the standard way by examining the residue of a tachyon amplitude at a tachyon pole. By \(\text{AP3}\) this must be a product of tachyon amplitudes. Hence

\(c_{N_1+1}c_{N_2+1} = c_{N_1+N_2}\) so \(c_N = \kappa^{N-2}\), where \(\kappa\) is the string coupling.

**Remark**: We offer one speculation on how the above results can be generalized to \(N > 26\).

In establishing theorem 1 we only used the bracket relations for currents \(J\) at levels 0,1. In order to extend the results to \(N > 26\) it will be necessary to use higher level currents. In the notation of section 3.3, by fixing the \(n_i, \tilde{n}_i\) we are only free to choose momenta to fix independent invariants \(s_{ij} (i \neq j)\), on a codimension \(N\) subvariety \(\mathcal{I}(n_i, \tilde{n}_i)\) of the variety \(\mathcal{I}(n_i)\) of all invariants \(s_{ij}\) at fixed \(n_i\). Nonetheless, it is possible that the amplitude
can still be uniquely determined by generalizing the idea of “analytic continuation from the integers” used previously. By varying the $\tilde{n}_i$ at fixed $n_i$ one can possibly determine the amplitude $A_{n_1,...,n_N}$ on “enough” subvarieties $I(n_i,\tilde{n}_i) \subset I(n_i)$ that, when combined with the axioms AP1, AP2, the amplitude is fixed on the entire variety $I(n_i)$.

7. Closed Strings

The closed string cohomology for ghost number $(1,1)$ is

$$\mathcal{H}_c = H_c^{1,1} = \bigoplus_{n \in \mathbb{Z}} \int_{\mathbb{R}^{25,1}} dp \ H[p,n] \otimes \bar{H}[p,n]$$

(7.1)

where the superscript refers to left and right ghost number and the bar always stands for “right-mover,” and not “complex conjugate.” S-matrix amplitudes are multilinear functions $A_c : \mathcal{H}_c^{\otimes n} \to \Phi$ constructed as follows. The operator formalism associates a measure $\Omega(V_1 \otimes \bar{V}_1, \ldots, V_n \otimes \bar{V}_n)$ on the moduli space of the n-punctured Riemann sphere, $\mathcal{M}_{0,n}$. We integrate $\Omega$ over $\mathcal{M}_{0,n}$ for an appropriate domain of $s_{ij}$ and continue analytically from there. There are two ways we may try to extend the above results to closed strings.

7.1. Factorization of Amplitudes

A beautiful result of Kawai, Lewellen, and Tye [23] states that the closed and open four-particle amplitudes are related by

$$A_c(V_1 \otimes \bar{V}_1, V_2 \otimes \bar{V}_2, V_3 \otimes \bar{V}_3, V_4 \otimes \bar{V}_4) = -\sin(\pi t)A(V_1, V_2, V_3, V_4)A(\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4)$$

(7.2)

At high energy we have

$$A_c(V_1 \otimes \bar{V}_1, V_2 \otimes \bar{V}_2, V_3 \otimes \bar{V}_3, V_4 \otimes \bar{V}_4) \sim A^{s,p}(V_1, V_2, V_3, V_4)A^{s,p}(\bar{V}_1, \bar{V}_2, \bar{V}_3, \bar{V}_4)$$

(7.3)

Similar, but more complicated remarks hold for $n$-particle scattering. We may combine (7.2) with previous results to relate scattering of massive closed string states to closed string tachyonic scattering.

9 We take $\alpha' = 2$ for closed string amplitudes.
7.2. Algebraic structures for closed strings

It is possible to extend the definition of the bracket to the closed string, as shown in \cite{16 \cite{17}. Composing this bracket with $b^+ = b_0 + \bar{b}_0$ we obtain a map of the physical states to themselves: $H^{1,1}_c \otimes H^{1,1}_c \rightarrow H^{1,1}_c$. This map is essentially a tensor product of open string brackets and does not turn the physical states into a Lie algebra. Also, one cannot justify the analog of (3.7). Thus, the most straightforward generalization of the previous discussion does not work.

Instead what one can do is consider $\mathcal{H}_c$ to be the “diagonal” subspace of the larger space

$$\tilde{\mathcal{H}}_c = \mathcal{H}_{\text{open}} \otimes \mathcal{H}_{\text{open}}$$

defined by equality of left and right momenta: $p = \bar{p}$. Amplitudes on the larger space $\tilde{\mathcal{H}}_c$ are only defined for $n$-tuples which are pairwise mutually local. The physical amplitudes are a subset of this expanded set. It is straightforward to write bracket relations for this expanded set of amplitudes. Suppose $J \otimes \bar{J}$ has momentum $(q, \bar{q})$, and suppose further that $q + \sum p_i = 0$, $q \cdot p_i \in \mathbb{Z}$, $\bar{q} + \sum \bar{p}_i = 0$, $\bar{q} \cdot \bar{p}_i \in \mathbb{Z}$. Then

$$\sum_{i, \bar{i}} (-1)^{q \cdot p_2 + \cdots + q \cdot p_i} (-1)^{\bar{q} \cdot \bar{p}_2 + \cdots + \bar{q} \cdot \bar{p}_i} A_c (V_1 \otimes \bar{V}_1, \ldots, \{J, V_i\} \otimes \bar{V}_i, \ldots, V_i \otimes \{\bar{J}, \bar{V}_i\}, \ldots, V_n \otimes \bar{V}_n) = 0$$

(7.4)

Evidently, the previous sections apply to the left- and right- degrees of freedom separately and fully determine the expanded set of amplitudes, hence \textit{a fortiori}, the physical ones, at least for $n \leq 26$.

One implication of these remarks is that the $\alpha' \rightarrow 0$ limit of scattering amplitudes of gravitons are in principle completely determined by the bracket relations. This raises the interesting issue of the relation of bracket algebras and algebras of vector fields.

8. Conclusions

8.1. What we did

We have shown that the bracket operation on mutually local BRST classes may be combined with Lorentz invariance and analyticity to write an infinite set of functional relations on string scattering amplitudes. These relations are rather restrictive. The central results of this paper are theorems 1 and 2 of section 6.3 which state that the bracket
relations together with the analyticity axioms \( \text{AP1, AP2, AP3} \) of section 2.3 uniquely determine the genus zero open string \( S \)-matrix in terms of a single free parameter, \( \kappa \), the string coupling constant, at least for \( N \leq 26 \)-particle scattering.

Some readers will be puzzled by our emphasis on high-energy limits. This limit might seem irrelevant if the bracket relations already determine the amplitude for all energies. The finite-difference relations are not, properly speaking, Ward identities since they relate amplitudes at different energies. Our chief concern has been understanding the symmetries of string theory and how they are realized in different backgrounds. Since high energy limits of the finite-difference relations look more like Ward-identities they deserve special attention.

Some readers will object that this paper contains nothing new. After all, the structure constants of the bracket are just the on-shell three point functions. By factorization, a knowledge of all the three-point functions in principle determines the full \( S \)-matrix. If one were mainly interested in explicit formulae for amplitudes the factorization approach would be impracticable, whereas our approach could be made quite efficient. The important point is, however, that the present discussion clarifies the fundamental role of the underlying algebraic structure of the bracket.

8.2. What we should do

There is plenty of room for further work.

Clearly the restriction on \( N \leq 26 \) for \( N \)-particle scattering is extremely unsatisfactory. It is possible that the result of this paper can be extended to \( N > 26 \) by combining the procedure of “analytic continuation from the integers” with bracket relations for states other than lightlike states. Indeed, in arriving at our result we have used only a small subset of the entire set of bracket relations.

There are several possible generalizations of the present study. These include:

1. String perturbation theory. Unfortunately, it is far from clear how to extend the above results to quantum perturbation theory. The problem is that a chiral BRST class \( J \) is not mutually local with respect to all states. That means \( \{ J, \cdot \} \) does not commute with an insertion of the identity operator \( \sum |I\rangle\langle I| \) or, more geometrically, contour integrals of \( J \) cannot be pulled around handles. This problem disappears for total closed string compactification, but, curiously, such theories have other difficulties with loop amplitudes [6]. For the same reason extension to mixed open-closed string scattering is not trivial.
Our attitude towards this problem is that in this paper we have managed to understand better the classical symmetries of string theory. It remains to be seen how the symmetries are realized quantum-mechanically. We are pursuing some ideas in this direction.

2. Fermionic strings. It should be interesting to generalize these results to superstring amplitudes. We intend to return to this in a future work.

3. Other backgrounds. We have restricted attention to the background of 26-dimensional Minkowski space. Bracket relations should exist for amplitudes in any background with an uncompactified Minkowski space component. If the monodromy of the internal parts of vertex operators is abelian one should be able to adjust spacetime momenta to obtain mutual locality in some situations. In such situations the technical lemma of section 3.3 shows that one could write bracket relations for \( N \leq d \) -particle scattering where \( d \) is the number of uncompactified dimensions. The symmetry algebra of mutually local BRST classes will depend on the background. It is not clear how effective the relations will be in other backgrounds.

The bracket relations are very reminiscent of the “\( W_\infty \) Ward identities” which have been used to obtain amplitudes for string scattering in 1 + 1 dimensional spacetime \cite{13, 24–26}. These techniques made essential use of the existence of a nontrivial ghost number zero cohomology, something which is absent in the critical bosonic string. We hope that the bracket relations will play a role in general backgrounds analogous to the “\( W_\infty \) Ward identities” of 1 + 1 dimensional string theory.

Finally, finite difference relations for correlation functions are known to arise in certain exactly solvable quantum field theories as well as in studies of \( q \)-deformed affine algebras. It would be very interesting to discover an underlying quantum group symmetry in the critical bosonic string \( S \)-matrix.

8.3. What are we doing?

Beyond these questions of generalization there is the much larger question of exactly what role the bracket should play in the formulation of string theory.

We believe that the bracket relations are a stringy expression of spontaneous symmetry breaking. On physical grounds one expects that symmetries which connect scattering of states at different mass levels must be spontaneously broken. We will say that a current \( J(z) \) (or, more generally, a ghost number 1 BRST class) is “broken” if it is not mutually local with respect to some on-shell state. In the open string the on-shell condition is simply \( Q \psi = 0 \) so the only unbroken currents are \( \partial X^\mu \). In the closed string statespace - viewed
as a subspace of $H_{\text{open}} \otimes \bar{H}_{\text{open}}$ - the on-shell condition requires furthermore that $p_L = p_R$. According to this terminology all holomorphic BRST classes are off-shell and broken with the exception of $c\partial X^\mu$ and its conjugate. Nevertheless, through the bracket relations these holomorphic classes constrain the couplings of on-shell particles.

From the above point of view, the uncompactified string contains infinitely many broken symmetry algebras. These are the sets $\mathcal{L} = \{V_i\}$ of mutually local BRST invariant states in $H_{\text{open}}$ which are closed under the bracket. We have seen in section 3.1 that such sets $\mathcal{L}$ can be given the structure of a vectorially-graded Lie algebra. We can further justify the name “broken symmetry algebra” for such sets $\mathcal{L}$ by noting that $\mathcal{L}$ can also be given a Lie algebra structure, and that this Lie algebra is an algebra of unbroken symmetries of some closed string toroidal compactification. The reason for this is that, if $\{p_i\}$ is the set of momenta associated to $\{V_i\}$ then $\Gamma = \langle p_i \rangle$ is an even integral lattice. (\$Gamma$ may be Euclidean or Lorentzian.) The lattice is even since $p_i^2 = 2 - 2n_i$ and integral by mutual locality. By the Frenkel-Kac construction $[27]$ $[28]$, we can introduce cocycle operators to turn the vectorially-graded bracket algebra of the $\{V_i\}$ into a true Lie algebra structure on

$$L_\Gamma = \oplus_{n \geq 0, p \in \Gamma} H[p, n].$$

To see that this is the unbroken symmetry of a toroidal compactification recall that

$$(\Gamma; 0) \oplus (0; \Gamma) \hookrightarrow \{(p_L; p_R) | p_L, p_R \in \Gamma^*, p_L - p_R \in \Gamma\} \cong II^{D,D}$$

and the unbroken symmetry associated with this background is $L_\Gamma \oplus L_\Gamma$. It is further natural to consider maximal sets of mutually local BRST classes $\{V_i\}$. The associated lattices have rank 26 and are necessarily hyperbolic. The associated Lie algebras are unbroken symmetries of totally compactified backgrounds. Among these backgrounds there is a distinguished compactification, namely, compactification on the torus defined by $\Gamma_* = II^{25,1}$ in the open case or on the Narain compactification for $\Gamma_* = (II^{25,1}; 0) \oplus (0; II^{25,1})$ in the closed case. Here $II^{25,1}$ is the unique even self-dual lattice in $\mathbb{R}^{25,1}$. The corresponding algebra $L_* = L_{\Gamma_*}$ has been dubbed the “fake Monster Lie algebra” by Borcherds $[29]$ $[30]$. Based on the analogy with Euclidean compactifications, which duly reproduces the Higgs mechanism in the $\alpha' \rightarrow 0$ limit $[7]$, we may say that the uncompactified string is a spontaneously broken gauge theory with the gauge algebra $L_* \cong \mathbb{R}^{26}$ and $L_* \oplus L_* \cong \mathbb{R}^{26} \oplus \mathbb{R}^{26}$ respectively.
At high energies the bracket relations become Ward identities. This should be understood as some kind of stringy high-energy symmetry restoration which replaces the analogous notion in spontaneously broken gauge theory. (As shown in appendix B, the standard field-theoretic approach does not generalize straightforwardly.) In this sense the above hyperbolic Lie algebras of toroidal compactification “explain” the linear relations on high energy amplitudes discussed in [1].

Remark: The uniqueness of $S$ is a generalization of Proposition 14 of [3], with the fake Monster Lie algebra replaced by the “vectorially-graded Lie algebroid of physical states.” It is interesting to note that the structure constants of these two algebraic objects are closely related. The structure constants of appendix A are, essentially, analytic continuations of the structure constants of $L_*$. One could say that the bosonic string $S$-matrix is made out of the structure constants of $L_*$.  

8.4. What we dream of doing

It is clear from a study of unbroken symmetries in toroidal compactification that the Lie algebras $L_*, L_* \oplus L_*$, while distinguished, are not the full story. As explained in [3] $L_* \oplus L_*$ is not a universal symmetry for closed string toroidal compactification. We hope that there is some kind of “universal” algebraic structure in string theory which will replace compact Lie algebras in the complete formulation of string theory as a generalization of gauge theory.

In nonabelian gauge theories like the standard model the on-shell asymptotic states form representations of the symmetry group. Since the representations do not just involve the adjoint there is, in general, no algebraic structure on the space of on-shell states. String theories are a very interesting class of theories in which the on-shell states themselves form an algebra. String theory might some day be regarded as the theory of symmetry in its purest form: a single symmetry principle fixes entirely the particle content and the interactions.

Note added: The earlier version of this paper erroneously claimed that the lemma of section 3.3 applied to $N$-particle scattering for all $N$, not just $N \leq 26$, and hence that bracket relations fixed all $N$-particle amplitudes. I thank E. Witten for pointing out this error. I also thank H. Verlinde for insisting on the point when I didn’t listen.

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Appendix A. Table of some structure constants for the bracket

- \( \{ p_1^2 = 2, p_2^2 = 2 \} \):

\[
\{ e^{ip_1 X}, e^{ip_2 X} \} = \begin{cases} 0 & \text{for } p_1 \cdot p_2 \geq 0 \\ e^{i(p_1+p_2)X} & \text{for } p_1 \cdot p_2 = -1 \\ S_n(p_1 \cdot X(z)) e^{i(p_1+p_2)X} & \text{for } p_1 \cdot p_2 = -(n+1) \end{cases}
\]

(A.1)

where \( S_n[f(z)] \) is a differential polynomial in \( f(z) \) defined by Taylor expansion:

\[
e^{i(f(w)-f(z))} \equiv \sum_{n \in \mathbb{Z}} (w-z)^n S_n[f(z)]
\]

The \( S_n \) are essentially Schur polynomials in the derivatives of \( f \). The first few nonvanishing ones are

\[
S_0 = 1 \\
S_1 = i \partial Y \\
S_2 = \frac{i}{2} \partial^2 Y - \frac{1}{2}(\partial Y)^2
\]

(A.2)

- \( \{ p_1^2 = 0, p_2^2 = 2 \} \):

\[
\{ i \zeta_1 \partial X e^{ip_1 X}, e^{ip_2 X} \} = \zeta_1 \cdot p_2 e^{i(p_1+p_2)X} \\
\quad = i(\zeta_1 + (p_2 \cdot \zeta_1)p_1) \cdot \partial X e^{i(p_1+p_2)X} \quad p_1 \cdot p_2 = 0
\]

(A.3)

- \( \{ p_1^2 = 0, p_2^2 = 0 \} \):

\[
\{ i \zeta_1 \cdot \partial X e^{ip_1 X}, i \zeta_2 \cdot \partial X e^{ip_2 X} \} = [\zeta_1 \cdot \zeta_2 - (\zeta_1 \cdot p_2)(\zeta_2 \cdot p_1)] e^{i(p_1+p_2)X} \quad p_1 \cdot p_2 = +1
\]

(A.4)

\[
\{ i \zeta_1 \cdot \partial X e^{ip_1 X}, i \zeta_2 \cdot \partial X e^{ip_2 X} \} = i \zeta' \cdot \partial X e^{i(p_2+p_1)X} \quad p_1 \cdot p_2 = 0
\]

\[
\zeta' = (\zeta_1 \cdot \zeta_2 - (\zeta_2 \cdot p_1)(\zeta_1 \cdot p_2))p_1 + (\zeta_1 \cdot p_2)\zeta_2 - (\zeta_2 \cdot p_1)\zeta_1
\]

(A.5)
• $\{p_1^2 = -2, p_2^2 = 2\}$:

$$\{V_{\zeta,p_1}, e^{ip_2 X}\} = -(p_2 + p_1) \cdot \zeta \cdot p_2 e^{i(p_2 + p_1) \cdot X} \quad p_1 \cdot p_2 = +1$$

$$= V_{\zeta', p_2 + p_1} \quad p_1 \cdot p_2 = -1$$

$$\zeta' = \zeta' + (p_1 \otimes p_2 \cdot \zeta + \zeta \cdot p_2 \otimes p_1) + \frac{1}{2} [(p_2 + p_1) \cdot \zeta \cdot p_2] p_1 \otimes p_1$$

(A.6)

Appendix B. High energy symmetries and spontaneous symmetry breaking

In [6] we proposed that the infinite dimensional symmetries arising upon time-compactification - of which $L_+ \oplus L_-$ is a spectacular example - are the high-energy symmetries of [1]. The idea was that Ward identities derived in a symmetric compactification could be “parallel transported” to the decompactified background. The decompactification forces momenta to infinity, hence the connection with high energy scattering. This proposal was based on a field theoretic intuition: in field theory the effects of spontaneous symmetry breaking disappear at high energies. It turns out that this intuition is not very good in string theory. Closed string scattering at high energy is sensitive to the breaking of toroidal symmetries. In this appendix we demonstrate that surprising fact with a few simple examples.

B.1. Recovery of Ward Identities in Field Theory

We consider enhanced symmetries and symmetry breaking in the framework of toroidal compactification. (We only consider compactification of spacelike dimensions here.) Suppose we have an unbroken symmetry current $J_q = e^{iq \cdot Y}$ at a Narain lattice $\Gamma$ which has a vector $(q; 0)$ with $q^2 = 2$. Consider a theory at $\Gamma' = g(\lambda) \cdot \Gamma$ where $g(\lambda)$ is a one-parameter family of $O(d,d)$ rotations taking $(q; 0)$ to $(q_L; q_R)$ with $q_R \neq 0$. The symmetry associated with $J$ is spontaneously broken in spacetime leading to massive gauge bosons with mass $m_2^2 / m_{pl}^2 = q_R^2(\lambda)$ and $\alpha' \sim 1/m_{pl}^2$ is related to the Planck mass.

Consider the scattering amplitudes $A(V_1, \ldots V_n)$ for (massless and massive) gauge bosons. The field-theory limit is obtained by letting $q^2_R, p_i \cdot p_j \to 0$ and expanding in these variables. The amplitudes have an expansion $A = \sum \epsilon^n A^{(n)}$ in powers of $\epsilon = 1/m_{pl}$, where each $A^{(n)}$ is a rational function of its arguments.

If we have a Ward identity in the unbroken phase $\sum A(V_1, \ldots \delta V_i, \ldots V_m) = 0$ then we may ask what happens to the corresponding amplitudes in the broken phase. The precise definition of “corresponding amplitudes” requires a transport operator $T^\lambda$ on the BRST
cohomology. We will simply rotate momenta by $O(d, d)$ transformations: this will suffice for our examples. \(^{10}\) For each $n$ we have

$$\lim_{\lambda \to 0} \sum_i A^{(n)}(T^\lambda V_1, \ldots, T^\lambda \delta V_i, \ldots, T^\lambda V_m) = 0 \quad (B.1)$$

and since $A^{(n)}$ are \textit{rational} functions of $p_i \cdot p_j$ and of the symmetry-breaking masses $q_R^2(\lambda)$ we conclude that the sum of the terms is proportional to the symmetry breaking masses. Therefore, for dimensional reasons, at each order in the expansion in $1/m_{pl}$ the sum of the terms grows more slowly in energy than the individual terms. In particular, for the leading terms $\sim m_{pl}^n$ where $n$ is a nonnegative power, the sum of the terms vanishes. This is the recovery of the Ward identity at high energy.

**Example.** To be explicit, consider a circular compactification of a single dimension $Y$. Let $R = e^\lambda$. At $\lambda = 0$ we have an unbroken $SU(2) \times SU(2)$ gauge theory which is broken to $U(1) \times U(1)$ for $\lambda > 0$. Introduce the notation:

$$e(j, \bar{j}, R) \equiv e^{\sqrt{2}(j + \bar{j}R + (j - \bar{j})R)}e^{i \sqrt{2}(j + \bar{j}R - (j - \bar{j})R)}Y(z)\bar{Y}(\bar{z}) \quad (B.2)$$

Focus on a single factor $SU(2) \to U(1)$. In the broken phase we have massive gauge bosons, call them $W^{\pm}(\epsilon)$, with corresponding vertex operators:

$$W^{\pm}(\epsilon, p) = e(\pm 1, 0, R)\epsilon \cdot \partial X e^{i p \cdot X} \quad (B.3)$$

where $X, p$ refer to uncompactified dimensions. \(^{(B.3)}\) are on shell for $p \cdot \epsilon = 0, p^2 + 2 \sinh^2 \lambda = 0$. There are also massless gauge bosons for the unbroken $U(1)$ generators, call them $Z(\epsilon)$ described by vertex operators:

$$Z(\epsilon, p) = \cosh \lambda \partial Y \epsilon \cdot \bar{\partial} X e^{i p \cdot X} \quad (B.4)$$

which are on-shell for $p \cdot \epsilon = 0, p^2 = 0$. (The factor of $\cosh \lambda$ in \(^{(B.4)}\) is motivated by the transport in \(\text{[?]})$. It can be dropped without changing our conclusions.)

At the $su(2)$ point $\lambda = 0$ we have the $su(2)$ Ward identity:

$$0 = A[W^+(\epsilon_1), Z(\epsilon_2), W^-(\epsilon_3), Z(\epsilon_4)](s, u)$$

$$+ A[W^+(\epsilon_1), W^-(\epsilon_2), Z(\epsilon_3), Z(\epsilon_4)](s, u)$$

$$+ A[W^+(\epsilon_1), W^-(\epsilon_2), W^-(\epsilon_3), W^+(\epsilon_4)](s, u) \quad (B.5)$$

\(^{10}\) See \(\text{[?]}\) for a detailed discussion of a parallel transport on the entire CFT statespace.
What happens at $\lambda > 0$? For simplicity assume that all polarizations $\epsilon_i$ are transverse to the plane of scattering and that $\epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4 = \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 = 0$, $\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 = 1$. In the field theory limit we have

$$A[W^+, Z, W^-, Z](s, u) = \frac{2(u - m^2)}{s} + \frac{1}{m_{pl}^2} \left(2m^2 u/s + u - s - 2m^4/s - m^2\right) + \cdots$$

$$A[W^+, W^-, Z, Z](s, u) = 2 + \frac{1}{m_{pl}^2} \frac{(2s - u)(s - m^2) - u^2}{s - m^2} + \cdots \tag{B.6}$$

$$A[W^+, W^-, W^-, W^+](s, u) = 2 \frac{(2m^2 - s - u)}{s - m^2} + \frac{1}{m_{pl}^2} \frac{(u - s)(s + u - 2m^2)}{s - m^2} + \cdots$$

where $s \equiv p_1 \cdot p_2, u \equiv p_1 \cdot p_3, m^2 \equiv 2 \sinh^2 \lambda$. The sum of the three amplitudes is

$$2 \frac{m^2 m^2 - u}{s} \frac{m^2 - u}{s - m^2} + \frac{1}{m_{pl}^2} \frac{m^4 2m^2 - s - 2u}{s} \frac{s - m^2}{s - m^2} + \cdots$$

The high-energy field-theoretic limit is now obtained by letting $s \to -\infty, u \to \infty$ holding $m^2$ fixed. As promised, at each order in $1/m_{pl}$ the sum vanishes more rapidly than each of the separate amplitudes, and, in particular, the leading term of order $O(m_{pl}^0)$ vanishes.

B.2. String theory Ward identities are not recovered at high energy

We illustrate this with two simple examples.

**Example 1:**

We now take the high energy limit of the three amplitudes in (B.5). The high energy asymptotics of the amplitudes are:

$$A[W^+(\epsilon_1), Z(\epsilon_2), W^-(\epsilon_3), Z(\epsilon_4)] \sim -U^2 (1 + \sinh^2 \lambda) e^{4 \sinh^2 \lambda \log(-s/u-1)}$$

$$\frac{(s + u)^3}{s^3 u^2} \left[s^2 + su + 2u^2 + 2 \sinh^2 \lambda u^2\right](1 + O(1/s, 1/u, 1/(s + u)))$$

$$A[W^+(\epsilon_1), W^-(\epsilon_2), Z(\epsilon_3), Z(\epsilon_4)] \sim -U^2 (1 + \sinh^2 \lambda) e^{4 \sinh^2 \lambda \log(u/s + 1)}$$

$$\frac{(s + u)^3}{s^3 u^2} \left[u^2 + us + 2s^2 + 2 \sinh^2 \lambda s^2\right](1 + O(1/s, 1/u, 1/(s + u))) \tag{B.7}$$

$$A[W^+(\epsilon_1), W^-(\epsilon_2), W^-(\epsilon_3), W^+(\epsilon_4)] \sim U^2 e^{4 \sinh^2 \lambda \log((u/s + 1)(-s/u-1))}$$

$$\frac{(s + u)^6}{s^4 u^2} (1 + O(1/s, 1/u, 1/(s + u)))$$
where $U$ is defined in (3.10).

While the sum of the three terms in (B.7) vanishes at $\lambda = 0$ this is not the case for any $\lambda \neq 0$: the sum is not any smaller in magnitude than any of the three terms in the sum.

**Example 2:**

At the $SU(2)$ point the vertex operators $\phi^{\pm\pm} = e(\pm \frac{1}{2}, \pm \frac{1}{2}, R = 1)e^{i\pi \cdot X}$ describe a scalar field multiplet in the $(2, 2)$ of $SU(2) \times SU(2)$. At the $SU(2)$ point we have the Ward identity (again for $J^+$):

$$A(\phi^{++}, \phi^{+-}, \phi^{+\pm}) + A(\phi^{-+}, \phi^{++}, \phi^{+-}, \phi^{-\pm}) + A(\phi^{--}, \phi^{--}, \phi^{+-}, \phi^{++}) = 0 \quad (B.8)$$

In the broken phase $\lambda > 0$ and in the high energy limit the sum of the three terms becomes:

$$\left[ \frac{1}{|z_0|^2} |z_0 - 1|^2 + \frac{1}{|z_0|^{1/R^2} - 1} \right] U^2 = \left[ (\sin^2 \frac{\theta}{2}) R^2 + (\cos^2 \frac{\theta}{2}) 1/R^2 - 1 \right] U^2 \quad (B.9)$$

Again we see that the sum of terms is no smaller than the individual terms.

### B.3. Physical interpretation

We believe that the failure to recover the Ward identities in the high energy limit is related to the notorious delocalization of strings at ultrahigh energies [31]. This delocalization makes the string amplitudes once again sensitive to relatively long-distance physics, like the specific nature of the compactification scheme.
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