(2 + 1)-dimensional DKP oscillator in an external magnetic field

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Abstract

In this work we investigate the (2 + 1)-dimensional DKP oscillator in an external magnetic field. By choosing conveniently the six components of the DKP field we obtain several cases in an unified way, including the previously reported simplified DKP oscillator (SDKPO) as well as the non-occurrence of the Zitterbewegung frequency in the two-dimensional DKP oscillator. The main novelty we found is the splitting in the frequency of the DKP oscillator according to the spin projection that arises as an interplay between the oscillator, the external field and the vectorial sector. Thereby, our study manifests the versatility of the non trivial representations in order to characterize properties of the spin (scalar and vectorial) sectors in DKP oscillators.

Keywords: DKP equation; vectorial coupling; DKPO; splitting
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1 Introduction

The Duffin-Kemmer-Petiau (DKP) equation is a relativistic first-order wave equation which describes scalar and vector fields through a unified formalism \cite{[1],[2],[3]}. The reader can find a historical review of this formalism in \cite{[4]}. Using its representations \cite{[5],[6]} or projectors \cite{[7],[8]} as well as a rich variety of couplings \cite{[9]} for the spin zero and spin one sectors, this theory has been applied to the study of many different problems like the scattering of $K^+$ nucleus \cite{[10]}, quantum chromodynamics \cite{[11]}, studies on the S-matrix \cite{[12]}, studies on causality of the DKP theory \cite{[13]}, minimal and non-minimal coupling in a general representation of DKP matrices \cite{[14]}, Bose-Einstein condensation \cite{[15]}, breaking of Lorentz symmetry \cite{[16]}, curved spacetime \cite{[17]}, Aharonov-Bohm potential \cite{[18],[19]}, studies of the phase in Aharonov-Casher effect \cite{[20]}, Galilean five-dimensional formalism \cite{[21]-[24]} and several others applications \cite{[25]-[37]}.

In the work \cite{[38]} the authors have showed that the (2+1)-dimensional Dirac oscillator, with frequency $\omega$, in an external magnetic field taken along the $z$-direction and represented by the vector potential $A = \frac{B}{2} (-y, x)$, can be mapped onto the Dirac oscillator without magnetic field but with reduced angular frequency given by $\omega - \bar{\omega}$, where $\bar{\omega} = \frac{eB}{2mc}$ and $e$ and $m$ are the charge and the mass of the electron. This result has been used, for example, to study atomic transitions in a radiation field \cite{[38]} as well as its corrections through the generalized uncertainty principle \cite{[39]}. Following the same approach the (2+1)-dimensional DKP oscillator (DKPO) has been studied under an external magnetic field \cite{[40]}, where the authors have calculated the eigensolutions of massive spin-0 and spin-1 particles both in commutative and non-commutative space.

In this letter we revisit this problem in order to show some more properties in the spin one sector which can be found with other conditions for components of the DKP field. The work is organized as follows. In Section 2 we begin by considering a scalar $4 \times 4$ and a $6 \times 6$ representations for the DKP field in the presence of an external magnetic field, that allows to study several cases of interest in the literature. Then, we analyze the scalar and vectorial cases as well as some special ones, thus obtaining the simplified DKPO and the two-dimensional DKPO along with their energies and eigenfunctions. In the vectorial sector $s = 1$ of the DKP representation we found a relationship (splitting) between the frequency of the DKP oscillator and the projection spin of the particles. Finally, in Section 3 we draw the conclusions and some
perspectives are outlined.

2 DKP equation and some representations

The DKP equation is written by the form

\[(i\hbar \beta^\mu \partial_\mu - mc) \Psi = 0,\]  

where \(\Psi\) is a DKP field of mass \(m\) and \(\beta\)-matrices satisfy the DKP algebra

\[\beta^\mu \beta^\lambda \beta^\nu + \beta^\nu \beta^\lambda \beta^\mu = \beta^\mu \eta^\lambda \nu + \beta^\nu \eta^\lambda \mu,\]  

with the Lorentz metric \(g^{\mu\nu}\) given by \(g^{00} = -g^{11} = -g^{22} = 1\). As predicted by [5] and showed in [6], the DKP equation in (2+1)-dimensional space-time has two representations: a \(4 \times 4\) for the scalar sector and a \(6 \times 6\) for the vector sector. The DKP representation for the scalar sector is given by matrices

\[\beta^0 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \beta^1 = \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{pmatrix}.\]  

The DKP representation for the vector sector in (2+1)-dimensions is given by the \(6 \times 6\) matrices

\[\beta^0 = \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \beta^1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},\]

\[\beta^2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.\]
The DKPO in the presence of an external magnetic field $A = \frac{B}{2} (-y, x)$ is performed by the coupling

$$\mathbf{p} \rightarrow \mathbf{p} - \frac{q}{c} \mathbf{A} - im\omega \eta^0 \mathbf{r}$$

(5)

where $\omega$ is the angular frequency of the oscillation, $q$ is the charge of the boson and $\eta^0 = 2 (\beta^0)^2 - 1$. The coupled DKP equation is written by

$$\left[ -\beta^0 E + c \beta \cdot \left( \mathbf{p} - \frac{q}{c} \mathbf{A} - im\omega \eta^0 \mathbf{r} \right) + mc^2 \right] \Psi = 0.$$  

(6)

where we have used $-i\hbar \partial_\mu = p_\mu = (-\frac{E}{c}, \mathbf{p})$.

### 2.1 Scalar DKPO in a magnetic field

Using a four components $\Psi^T = (\Psi_0, \Psi_1, \Psi_2, \Psi_3, ...)$ and $\tilde{\omega} = \frac{qB}{2mc}$, the coupled equation \[6\] provides the following four equations

$$mc^2 \Psi_0 = E \Psi_3$$

(7)

$$-mc^2 \Psi_1 = c(p_x + m\tilde{\omega}y - im\omega x) \Psi_3$$

(8)

$$-mc^2 \Psi_2 = c(p_y - m\tilde{\omega}x - im\omega y) \Psi_3$$

(9)

$$-mc^2 \Psi_3 = -E\Psi_0 - c(p_x + m\tilde{\omega}y + im\omega x) \Psi_1 - c(p_y - m\tilde{\omega}x + im\omega y) \Psi_2$$

(10)

Performing the combination of these equations we have

$$\left( E^2 - m^2 c^4 \right) \Psi_3 = c^2 \left[ \mathbf{p}^2 + m^2 \omega_0^2 \mathbf{r}^2 - 2m\hbar \omega - 2m\tilde{\omega} L_z \right] \Psi_3$$

(11)

where $\omega_0 = \sqrt{\omega^2 + \tilde{\omega}^2}$, $r^2 = x^2 + y^2$ and $L_z = xp_y - yp_x$ is the $z$-component of the angular momentum. This result is in agreement with \[10\]. Thus, following the standard approach, the energy spectrum is given by

$$E_{nl} = \pm mc^2 \sqrt{1 - \frac{2\hbar (|l|\tilde{\omega} + \omega)}{mc^2} + \frac{\hbar\omega_0}{mc^2} (4n + 2(|l| + 1))}$$

(12)

where $n = 0, 1, 2, 3,...$ is the quantum number associated to the energy and $l = 0, \pm 1, \pm 2, \pm 3...$ is the angular momentum quantum number. The wave function $\Psi_3$ is calculated using the polar coordinates $(r, \theta)$ using the standard approach

$$\left( \Psi_3 \right)_{nl} = \left( \frac{m\omega_0}{\hbar} \right)^{|l|/2} e^{-\frac{m\omega_0}{2\hbar} r^2} L_n \left( \frac{m\omega_0}{\hbar} r^2 \right)$$

(13)
whose non-relativistic limit is calculated taking $E = \epsilon + mc^2$ with $\epsilon << mc^2$ in order to obtain $(E^2 - m^2c^4) / 2mc^2 \approx \epsilon$. The others components of DKP wave function, $\Psi$, can be easily calculated using the expressions (7)-(9).

### 2.2 Vector DKPO in a magnetic field: splitting in the frequency

Using the equation (B3) of the reference [14] and taking $R^0\Psi = 0$ (make the scalar component equal to zero) its equation (49) is simplified canceling the term proportional to $1/m^2$, obtaining thus a simplified equation for the $(3 + 1)$-dimensional vector DKP oscillator (SDKPO), which is composed by the usual three-dimensional oscillator added to a spin-orbit coupling. Following this reasoning in $(2 + 1)$-dimensional case we choose conveniently the six components of the DKP field $\Psi^T = (a, b, d, 0)$ where $a = (a_1, a_2)$ and $d = (d_1, d_2)$. The null component was chosen in order to obtain the $(2 + 1)$-dimensional SDKPO. Using $\Psi$, the representation (4) as well as the coupling (6) in the (1), we obtain the six equations

\[ mc^2 a_1 = Ed_1 \]
\[ mc^2 a_2 = Ed_2 \]
\[ -mc^2 b = c (p_x + m\tilde{\omega} y - im\omega x) d_2 - c (p_y - m\tilde{\omega} x + im\omega y) d_1 \]
\[ -mc^2 d_1 = - Ea_1 + c (p_y - m\tilde{\omega} x + im\omega y) b \]
\[ -mc^2 d_2 = - Ea_2 - c (p_x + im\tilde{\omega} y + im\omega x) b \]
\[ 0 = -c (p_x + m\tilde{\omega} y - im\omega x) a_1 - c (p_y - m\tilde{\omega} x - im\omega y) a_2 \]

Performing the substitution of two first equations into the others we have the equations for the components $b, d_1$ and $d_2$ given by

\[ (E^2 - m^2c^4) b = c^2 [(p_x + m\tilde{\omega} y - im\omega x) (p_x + im\tilde{\omega} y + im\omega x) + (p_y - m\tilde{\omega} x - im\omega y) (p_y - m\tilde{\omega} x + im\omega y)] b \]
\[ (E^2 - m^2c^4) d_1 = -c^2 (p_y - m\tilde{\omega} x + im\omega y) [(p_x + m\tilde{\omega} y - im\omega x) d_2 - (p_y - m\tilde{\omega} x - im\omega y) d_1] \]
\[ (E^2 - m^2c^4) d_2 = c^2 (p_x + m\tilde{\omega} y + im\omega x) [(p_x + m\tilde{\omega} y - im\omega x) d_2 - (p_y - m\tilde{\omega} x + im\omega y) d_1] \]
\[ 0 = [(p_x + m\tilde{\omega} y - im\omega x) d_1 + (p_y - m\tilde{\omega} x - im\omega y) d_2] \]
Performing the substitution of (15) and (16) in (14) it follows that

\[
(E^2 - m^2 c^4) b = c^2 \left[ p^2 + m^2 \omega_0^2 r^2 + 2m\hbar\omega - 2m\tilde{\omega} L_z \right] b
\]  

(18)

where \( \omega_0 = \sqrt{\omega^2 + \tilde{\omega}^2} \). The expression above show us that the component \( b \), like (11), behaves as a scalar DKPO in a magnetic field. The energy spectrum for (18) is obtained by the exchange \( \omega \rightarrow -\omega \) in (12), while its wave function is equal to (13). In other hand, if we apply the operator \( c^2 (p_x + m\tilde{\omega} y + i\hbar\omega) \) to (17) and add the result to (15) as well as apply the operator \( c^2 (p_y - m\tilde{\omega} x + i\hbar\omega) \) to (17) and to add the result to (16) we have

\[
(E^2 - m^2 c^4) d_1 = c^2 \left[ p^2 + m^2 (\omega^2 + \tilde{\omega}^2) r^2 - 2m\hbar\omega - 2m\tilde{\omega} L_z \right] d_1
+ ic^2 \left[ 2m\hbar\tilde{\omega} + 2m\omega L_z - 2m^2\omega\tilde{\omega} r^2 \right] d_2
\]  

(19)

\[
(E^2 - m^2 c^4) d_2 = c^2 \left[ p^2 + m^2 (\omega^2 + \tilde{\omega}^2) r^2 - 2m\hbar\omega - 2m\tilde{\omega} L_z \right] d_2
- ic^2 \left[ 2m\hbar\tilde{\omega} + 2m\omega L_z - 2m^2\omega\tilde{\omega} r^2 \right] d_1
\]  

(20)

Thus, defining \( d_1 \) and \( d_2 \) by

\[
d_1 = \frac{\varphi_2 + \varphi_1}{2}
\]  

(21)

\[
d_2 = \frac{\varphi_2 - \varphi_1}{2i}
\]  

(22)

it follows that the equations (19)-(20) can be rewritten by the form

\[
(E^2 - m^2 c^4) \varphi_1 = c^2 \left[ p^2 + m^2 (\omega + \tilde{\omega})^2 r^2 - 2m\hbar (\omega + \tilde{\omega}) - 2m (\omega + \tilde{\omega}) L_z \right] \varphi_1
\]  

(23)

and

\[
(E^2 - m^2 c^4) \varphi_2 = c^2 \left[ p^2 + m^2 (\omega - \tilde{\omega})^2 r^2 - 2m\hbar (\omega - \tilde{\omega}) + 2m (\omega - \tilde{\omega}) L_z \right] \varphi_2
\]  

(24)

The equations above still can be rewritten by a more compact form, that is

\[
(E^2 - m^2 c^4) \varphi_i = c^2 \left[ p^2 + m^2 \omega_i^2 r^2 - 2m\hbar\omega_i - 2m\omega_i L_z s_i \right] \varphi_i
\]  

(25)
where $i = 1, 2$ with $\omega_1 = \omega + \tilde{\omega}$, $\omega_2 = \omega - \tilde{\omega}$ and $s_1 = -s_2 = 1$. The last term in the right side of the equation above plays the role of a spin-orbit term even if $2\tilde{\omega}L_z$ be a angular momentum projection on $z$-direction. Following this reasoning we can interpret the functions $\varphi_i$ as components of a vector $(\varphi_1, \varphi_2)$ whose spin projections along the $z$-direction are $+1$ and $-1$, respectively. Thus, in the presence of an external magnetic field, the spin one DKPO presents a scalar component, $b$, and two vector components, $\varphi_1$ and $\varphi_2$, which oscillate with the angular frequencies $\omega_0$, $\omega_1$ and $\omega_2$, respectively.

Figure 1: Schematic representation of the frequency splitting of the DKPO (Eq. (25)) in the presence of an external magnetic field, according to the spin projection $s$.

The expression (25) has the form of the equation for the (2+1)-dimensional SDKPO whose energy spectrum and eigensolution are:

$$E_{nl}^{i} = \pm mc^2 \sqrt{1 + \frac{\hbar \omega_i}{mc^2} (4n + 2|l| (1 - s_i) + 2s_i)} \quad (26)$$

where $n = 0, 1, 2, 3, \ldots$, $l = 0, \pm 1, \pm 2, \pm 3\ldots$ and $E_{nm}^1$ and $E_{nm}^2$ represents the eigenvalues for the energy in the cases associated to the angular frequencies $\omega_1$ and $\omega_2$, respectively. The wave functions $\varphi_i$ are expressed by

$$(\varphi_i)_{nl} = \left(\frac{m \omega_i}{\hbar} r^2\right)^{|l|/2} e^{i\theta} e^{-\frac{m \omega_i}{2\hbar} r^2} L_n \left(\frac{m \omega_i}{\hbar} r^2\right) \quad (27)$$

whose non-relativistic limit is calculated by the same way pointed at the end of the previous Section. The others components of the spin one DKP wave
function, $\Psi$, can be easily calculated using the expressions (7)-(9), whose non-relativistic limit is obtained taking $E = \epsilon + m^2 c^4$ with $\epsilon \ll mc^2$. Below we explore the special cases associated to the situations where $d_1 = 0, d_2 = 0, d_1 = \pm d_2, \omega = \pm \tilde{\omega}$.

2.3 Specials Cases

2.3.1 Cases $d_1 = 0$ or $d_2 = 0$

In these cases we consider the equations (19)-(20) in order to obtain: $\varphi_1 = -\varphi_2$ for $d_1 = 0$ or $\varphi_1 = \varphi_2$ for $d_2 = 0$. Hence, for both cases we have

$$\left(E^2 - m^2 c^4\right) \varphi_{1,2} = c^2 \left[p^2 + m^2 \omega_0^2 r^2 - 2\hbar \omega - 2\tilde{\omega} L_z\right] \varphi_{1,2} \quad (28)$$

These cases show us the each component $\varphi_{1,2}$ of the wave function behaves as scalar DKPO in the presence of the magnetic field, with angular frequency $\omega_0 = \sqrt{\omega^2 + \tilde{\omega}^2}$ and, consequently, presenting the same results as pointed in (12)-(13) including the non-occurrence of the Zitterbewegung frequency. This result is in complete agreement with [40].

2.3.2 Case $d_1 = id_2$

In this case we consider (19)-(20) in order to obtain $\varphi_1 = 0$ and

$$\left(E^2 - m^2 c^4\right) \varphi_2 = c^2 \left[p^2 + m^2 \omega_2^2 r^2 - 2\hbar \omega_2 - 2\omega_2 L_z S_-\right] \varphi_2 \quad (29)$$

Thus, the problem is mapped onto DKPO without magnetic field but with reduced frequency $\omega_2 = \omega - \tilde{\omega}$ and spin projection $S_- = -1$.

2.3.3 Case $d_1 = -id_2$

in this case we consider (19)-(20) in order to obtain $\varphi_2 = 0$ and

$$\left(E^2 - m^2 c^4\right) \varphi_1 = \left[p^2 + m^2 \omega_1^2 r^2 - 2\hbar \omega_1 - 2\omega_1 L_z S_+\right] \varphi_1 \quad (30)$$

Thus, the problem is mapped onto DKP oscillator without magnetic field but with increased frequency $\omega_1 = \omega + \tilde{\omega}$ and spin projection $S_+ = +1$.  

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2.3.4 Case $\omega = \tilde{\omega}$

In this case we just need make $\omega = \tilde{\omega}$ in the (25), obtaining

$$\begin{align*}
(E^2 - m^2 c^4) \varphi_1 &= c^2 \left[ p^2 + m^2 (2\omega)^2 r^2 - 2\hbar(2\omega) - 2(2\omega)L_z S_+ \right] \varphi_1, \quad (31) \\
(E^2 - m^2 c^4) \varphi_2 &= c^2 \left[ p^2 \right] \varphi_2,
\end{align*}$$

Hence, the problem is mapped onto DKP oscillator without magnetic field but with duplicated frequency $2\omega$ and spin projection $S_+ = +1$ for the component $\varphi_1$ while the component $\varphi_2$ stops the oscillation and behaves as a free particle.

2.3.5 Case $\omega = -\tilde{\omega}$

In this case we just need make $\omega = -\tilde{\omega}$ in the (25), obtaining

$$\begin{align*}
(E^2 - m^2 c^4) \varphi_2 &= \left[ p^2 + m^2 (2\omega)^2 r^2 - 2\hbar(2\omega) - 2(2\omega)L_z S_- \right] \varphi_2, \quad (33) \\
(E^2 - m^2 c^4) \varphi_1 &= \left[ p^2 \right] \varphi_1, \quad (34)
\end{align*}$$

Hence, the problem is mapped onto DKP oscillator without magnetic field but with duplicated frequency $2\omega$ and spin projection $S_- = -1$ for the component $\varphi_2$ while the component $\varphi_1$ stops the oscillation and behaves as a free particle.

3 Conclusions

We have reported the $(2 + 1)$-dimensional DKP oscillator in an external magnetic field from a scalar $4 \times 4$ and a $6 \times 6$ representations which allow to study several cases investigated in the literature, as well as calculating their energies and eigenfunctions, in a unified way. Thus, the energy spectrum and wave function for the scalar DKPO under a magnetic field are shown in the equations (12)-(13), respectively, where the angular frequency $\omega$ is exchanged to $\omega_0$. On the other side, the vector DKPO interacting with a magnetic field presents a splitting $\omega \rightarrow (\omega_1, \omega_0, \omega_2)$ in the frequency of the oscillation (Fig. 1) that corresponds to the spin projections $s = +1, 0, -1$ of the vector DKP field. The energy spectrum and wave function of this oscillator are presented in (26)-(27). Some important situations were studied and two critical cases occurs when $\omega = \pm \tilde{\omega} \rightarrow B = \pm \frac{2m\omega}{q}$ for which the component $\varphi_1$ or $\varphi_2$
stops oscillating. We consider that the present work opens new perspectives in the study of the properties of the spin (scalar and vectorial) sectors in DKP oscillators, by means of non trivial representations. We hope this can be investigated in future researches with more examples.

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