Binary black hole system at equilibrium

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An exact and analytical solution of four dimensional vacuum General Relativity representing a system of two static black holes at equilibrium is presented. The metric is completely regular outside the event horizons, both from curvature and conical singularities. The balance between the two Schwarzschild sources is granted by an external gravitational field, without the need of extra matter fields besides gravity, nor strings or struts. The geometry of the solution is analysed. The Smarr law, the first and the second law of black hole thermodynamics are discussed.

I. INTRODUCTION

Black holes are fundamental objects of our Universe. Nevertheless there are scarce exact and analytical solutions that model black holes within the context of General Relativity, which represents the standard framework of gravitational interactions. The requirement of asymptotic flatness further restricts the number of available solutions, basically due to the existence of uniqueness theorems [1].

The recent remarkable observational results from gravitational waves detection reveal that the major source of gravitational waves is provided by the interaction between massive black holes [2]. Unfortunately, there are no exact analytical solutions describing well-founded multi-source black holes in pure General Relativity. The few attempts built so far need singular matter to sustain the gravitational attraction between the sources, as can be seen from double Kerr attempts, see [3,4]. This matter is often interpreted as struts or cosmological strings, however these objects not only have stability issues and no experimental plausibility at the moment; they also raise some fundamental theoretical issues related to the singular behaviour of the spacetime, which therefore can hardly be thought as a proper manifold. The $\delta$-like stress-energy tensor associated to the strut, located between the black holes, violates all the reasonable energy conditions because, in order to prevent the gravitational sources from the collapse, it has to mimic a repulsive gravitational force generated by anti-gravitational matter. Alternatively, the black holes can be pulled by axial strings of proper matter, i.e. of positive energy density, which however are divergent quantities that extend to infinity. Both scenarios are clearly out of current empirical experience.

On the other hand, in the presence of electromagnetic fields, it is possible to gain the equilibrium between two black holes thanks to the electromagnetic repulsive force that balances the gravitational attraction. In fact some solutions of these kind are known, basically the Majumdar–Papapetrou black holes [5,7] and the Emperan dihole [8]. While these solutions are regular outside the event horizons of the sources, they need to be composed of charged matter, which is improbable from the astrophysical point of view. Furthermore they are extremal and their physical parameters are very fine tuned to maintain the equilibrium, therefore they are likely to be unstable under perturbations.

The purpose of this Letter is to explore the possibility to build analytical metrics modelling two black hole sources at equilibrium, aiming at the most phenomenologically realistic scenario as possible. For this reason, we will discard any source of singularities outside the event horizons, not only Dirac or cosmic strings, but also Misner strings. In fact, the gravimagnetic parameter (NUT) is known to bring in physical issues, e.g. closed timelike curves.

The strategy of our approach is based on an Ernst’s
insight, who managed to regularise the nodal singularity of the accelerating Schwarzschild black hole in [9]. Similarly we examine the possibility of balancing the gravitational attraction of the two (or more) sources by an external gravitational field, which serves as a setting for embedding our multi-source solution. Thus the spacetime asymptotic will not be restricted to the flat one; instead we prefer to consider an axisymmetric multi-polar expansion [10] able to model a generic gravitational background. The multipolar expansion is the key ingredient that allows to circumvent the no-go theorem about the non-existence of static configurations of many-body systems, with a suitable separation condition.

These kind of solutions are know in the literature, in the single source case, as distorted black holes: they have been introduced by Doroshkevich, Zel’dovich and Novikov [13], and lately Chandrasekhar [15] studied their equilibrium conditions. Geroch and Hartle [16] enlightened then some of their general properties. Distorted black holes are especially valuable when the black hole is not isolated but embedded in a strong field regime, such as in the center of galaxies, where approximate, perturbative or numerical methods loose accuracy. Distorted metrics are generally considered local in the sense that they are thought to be at large distances from the black holes, and in fact they do not contribute with a stress-energy tensor [7]. A full solution should also include these sources, as done up to the 6th order in the multipole expansion in [17] for a thin disk of matter. In [18] it is shown that the inclusion of the acceleration parameter interposes a Rindler horizon between the black sources and the spatial infinity.

While the number of gravitational multipoles of the background can in principle be left arbitrary [19], in this Letter we focus only on the dipole and quadrupole terms since it represents the simplest setting in which the black holes distinctive parameters remain unconstrained. We stress that our regularisation method relies only on the presence of (at least) two multipole momenta, therefore it can be achieved for any couple on non-null poles.

In practice we superimpose a double black hole solution to the external field background by means of the Belinski–Zakharov “inverse scattering method” [20–22]. The details of the inverse scattering construction in the context of external gravitational fields are left for a future work [19].

II. METRIC AND REGULARISATION

Consider the static and axisymmetric metric, solution of the Einstein equations in vacuum $R_{\mu \nu} = 0$, in cylindrical Weyl coordinates $(t, \rho, z, \phi)$

$$\begin{align*}
\text{ds}^2 &= -V(\rho, z)dt^2 + \rho^2 V^{-1}(\rho, z)d\phi^2 \\
&
+ f(\rho, z)(d\rho^2 + dz^2),
\end{align*}$$

where

$$V = \frac{\mu_1 \mu_3}{\mu_2 \mu_4} \exp \left[2b_1 \rho + 2b_2 \left(z^2 - \frac{\rho^2}{2}\right)\right],$$

$$f = 16C_f \frac{\rho^4}{W_{11}W_{22}W_{33}W_{44}W_{12}W_{14}Y_{12}Y_{23}Y_{34}} \exp \left[-b_1^2 \rho^2 + \frac{b_2^2}{2} (\rho^2 - 8z^2) \rho^2 - 4b_1 b_2 z \rho^2 + 2b_1 (-z + \mu_1 - \mu_2 + \mu_3 - \mu_4) + b_2 (-2z^2 + \rho^2 + 4z(\mu_1 - \mu_2) + \mu_1^2 + \mu_2^2 + (\mu_3 - \mu_4)(4z + \mu_3 + \mu_4))\right],$$

and $W_{ij} = \rho^2 + \mu_i \mu_j$, $Y_{ij} = (\mu_i - \mu_j)^2$. The functions $\mu_i = \sqrt{\rho^2 + (z - w_i)^2} - (z - w_i)$ contain the constants $w_i$, that are chosen with ordering $w_1 < w_2 < w_3 < w_4$ and with values

$$\begin{align*}
w_1 &= z_1 - m_1, \\
w_2 &= z_1 + m_1, \\
w_3 &= z_2 - m_2, \\
w_4 &= z_2 + m_2.
\end{align*}$$

The parameters $m_i$ and $z_i$ refer to the mass and to the position of the $i$-th black hole, respectively, while $b_1$ and $b_2$ are the dipole and quadrupole momenta of the external gravitational field polar expansion. The constant $C_f$ is a gauge parameter.
The metric \( \text{[1]} \) represents two Schwarzschild black holes immersed in an external back-reacting gravitational field \(^3\). The two event horizons extend in the regions \( w_1 < z < w_2 \) and \( w_3 < z < w_4 \) when \( \rho = 0 \), while the curvature singularities are covered by the horizons. This can be understood thanks to a criterion given in \([23]\): the kernel of the \((t, \phi)\) part of the metric can be 2-dimensional only inside the horizon. By removing the external field \((b_1 = b_2 = 0)\), one straightforwardly recovers the double-Schwarzschild solution (also known as Bach–Weyl metric) \([24,26]\).

A physical solution should be regular on the \( z \)-axis, i.e. it must not be affected by conical singularities. These angular defects are present when the ratio between the length and the radius of small circles around the \( z \)-axis is different from 2\( \pi \). A small circle around the \( z \)-axis has radius \( R = \sqrt{\rho^2 \rho^0} \) and length \( L = 2\pi \sqrt{\rho^0 \rho} \) (cf. \([27]\)), then the regularity condition corresponds to \( L/(2\pi R) \to 1 \) as \( \rho \to 0 \). It is easy to prove that, for the diagonal metric \( \text{[1]} \), the above condition is equivalent to \( fV \to 1 \) as \( \rho \to 0 \). The latter must be satisfied everywhere, outside the black holes, to avoid conical singularities. We thus require

\[
fV = 1,
\]

over the intervals \(-\infty < z < w_1, w_2 < z < w_3 \) and \( w_4 < z < +\infty \) as \( \rho \to 0 \). Assuming a 2\( \pi \)-periodic azimuthal angle \( \phi \), we have to fix three out of seven parameters in order to satisfy \( \text{[5]} \). We choose the gauge parameter \( C_f \) and the external field parameters \( b_1, b_2 \) as follows:

\[
C_f = 256 m_1^2 m_2^2 (m_1 + m_2 + z_1 - z_2)^2 (m_1 + m_2 - z_1 + z_2)^2, \\
b_1 = \frac{(m_1 + m_2 (z_1 - z_2))}{4 m_1 m_2 (z_1 - z_2)} \log \left[ \frac{(m_1 - m_2 + z_1 - z_2)(m_1 - m_2 - z_1 + z_2)}{(m_1 + m_2 + z_1 - z_2)(m_1 + m_2 - z_1 + z_2)} \right], \\
b_2 = \frac{(m_1 + m_2)}{8 m_1 m_2 (z_1 - z_2)} \log \left[ \frac{(m_1 - m_2 + z_1 - z_2)(m_1 - m_2 - z_1 + z_2)}{(m_1 + m_2 + z_1 - z_2)(m_1 + m_2 - z_1 + z_2)} \right].
\]

It is understood that \( C_f, b_1, b_2 \) will assume the above values, from now on. Note that the physical parameters characterising the black holes were left unconstrained: this means that the physical properties of the binary black hole system are completely generic. This fact guarantees a wide flexibility in a possible phenomenological scenario \(^4\). Of course one could alternatively keep the \( b_i \) generic, to model an arbitrary external gravitational background: in that case the intrinsic black holes parameters would adjust to fit the given background.

One can infer the order of magnitude of the multipole momenta \([7,8]\) for an experimental setting. We make an explicit example by considering the data for the black holes masses and separation provided by the GW150914 event \([2]\): we take \( m_1 = 29 M_\odot, m_2 = 36 M_\odot, [z_1 - z_2] = 520 M_\odot \), and we find \( b_1 \sim 10^{-4}/M_\odot \) and \( b_2 \sim 10^{-7}/M_\odot^2 \). These are the values that ensure the equilibrium, therefore in the case of a merging process they represent only an upper bound. The relevance of the contributions of \( b_1 \) and \( b_2 \) in the context of the Geroch–Hansen multipoles \([28]\) will be discussed in \([19]\).

We notice that it is possible, within our scheme, to build an array of collinear black holes immersed in an external gravitational field with complete multipolar expansion \([19]\), thus generalising the Israel–Kahn solution \([25]\). In that case, the regularisation would require the complete multipole expansion \([\text{[7]}]\) of the external field to fix the conical singularities, without constraining the physical parameters of the black holes, such as their masses or positions on the \( z \)-axis.

To ensure phenomenological significance of the solution \([1]\), it is essential that the separation of the two sources is finite. The proper distance between the two

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\(^3\) A Mathematica notebook containing this metric can be found at https://sites.google.com/site/marcoastorino/papers/2104-07086 and as an ancillary file on the arXiv webpage.

\(^4\) In fact the presented solution is not the minimal one which can be regularised. It is sufficient to consider only one multipole term for the external gravitational field to remove all the singularities from the double black hole configuration, at the price of fixing the position or the mass of a black hole.

\(^5\) Actually an arbitrary multipole expansion can be useful to model any generic stationary and axisymmetric external gravitational field.
FIG. 1. Embedding diagram in $\mathbb{E}^3$ of the surfaces of two black hole event horizons for the parametric values $z_1 = 5$, $z_2 = 15$, $m_1 = 4$, $m_2 = 4$, in Solar mass units $M_\odot$. This picture shows the deformation of the horizons due to both the external gravitational field and the mutual interaction between the sources. The horizon surface is smooth because of the absence of any conical singularities.

black hole horizons for $\rho = 0$ and fixed $\phi$ is given by

$$\ell = \int_{w_2}^{w_3} dz \sqrt{g_{zz}}|_{\rho=0} \propto \int_{w_2}^{w_3} dz \sqrt{\frac{(w_4 - z)(z - w_1)}{(w_3 - z)(z - w_2)}} e^{2b_1(2w_3 - 2w_4 - z) + 2b_2(2w_3^2 - 2w_4^2 - z^2)}. \tag{9}$$

It can be shown that the integral converges, and hence the equilibrium can be achieved for finite proper distance.

III. NEAR-HORIZON LIMIT

We consider the near-horizon limit of the metric (1), in order to show that it contains two distorted Schwarzschild black holes. We zoom in to the first black hole horizon by performing the change of coordinates

$$\rho = \sqrt{r(r - 2m_1)} \sin \theta, \quad z = z_1 + (r - m_1) \cos \theta, \tag{10}$$

and by taking the limit $r \to 2m_1$, by which the metric (1) boils down to

$$ds^2 \simeq h(\theta) \left[ - \left( 1 - \frac{2m_1}{r} \right) e^{F_1(\theta)} dt^2 + \frac{D^2 e^{F_2(\theta)}}{1 - 2m_1/r} dz^2 \right] + (2m_1)^2 \left[ D^2 h(\theta) e^{F_2(\theta)} d\theta^2 + \frac{\sin^2 \theta}{h(\theta)} e^{-F_1(\theta)} d\phi^2 \right], \tag{11}$$

where

$$h(\theta) = \frac{m_1 \cos \theta + m_2 + z_1 - z_2}{m_1 \cos \theta - m_2 + z_1 - z_2}, \tag{12}$$

$$F_1(\theta) = 2\left[ b_1 + b_2(z_1 + m_1 \cos \theta) \right] (z_1 + m_1 \cos \theta), \tag{13}$$

$$F_2(\theta) = 2b_1(m_1 \cos \theta - 2m_1 - 4m_2 - z_1) + 2b_2(m_1^2 \cos^2 \theta + 2m_1 z_1 \cos \theta - 2m_1^2 - z_1^2 - 4m_1 z_1 - 8m_2 z_2), \tag{14}$$

and

$$D = \frac{m_1 + m_2 - z_1 + z_2}{m_1 - m_2 - z_1 + z_2}. \tag{15}$$

One clearly recognises in (11) the structure of a distorted Schwarzschild black hole [16]. Actually the first black hole horizon is deformed by the presence of both the external field and the second black hole. Indeed, when the external field and the second black hole vanish, one recovers the standard Schwarzschild metric. Obviously, a similar description holds for the second black hole as well.

A pictorial representation of the deformation that the two horizons undergo is given in the embedding diagram of Fig. 1.

Another element about the geometry of the black hole horizon can be clarified by computing the length of the equatorial and polar circles in the near-horizon geome-
try (11). The equator ($\theta = \pi/2$) length is given by

$$L_{\text{equator}} = 2m_1 \int_0^{2\pi} e^{-F_1(\pi/2)/2} \sqrt{\hat{h}(\pi/2)} \, d\phi$$

$$= 4\pi m_1 \sqrt{\frac{z_1 - z_2 - m_2}{z_1 - z_2 + m_2}} e^{-2(b_1 + b_2 z_1)z_1},$$

while the polar length is

$$L_{\text{polar}} = 4m_1 D \int_0^\pi \sqrt{\hat{h}(\theta)} e^{F_2(\theta)/2} \, d\theta.$$  \hspace{1cm} (17)

It is not possible to analytically perform the latter integral, nevertheless we can consider a numerical comparison between (16), (17) and $L_{\text{Schwarzschild}} = 4\pi m_1$. The result is

$$L_{\text{equator}} > L_{\text{Schwarzschild}}, \hspace{1cm} (18a)$$

$$L_{\text{polar}} > L_{\text{Schwarzschild}}. \hspace{1cm} (18b)$$

Thus, not only a deformation along the $z$-axis occurs, but there is also an enlargement of both the equatorial and the polar circle with respect to the Schwarzschild one. This is consistent with the behaviour of the black hole temperature, as we will see below.

### IV. SMARR LAW AND THERMODYNAMICS

Going back to the exact metric (1), we now compute some physical quantities for that spacetime, in order to discuss the thermodynamics of the system.

The mass of the two black holes is found by means of the Komar–Tomimatsu integral [29, 30]. The formula for the conserved mass, in the case of the static spacetime (1), takes the form

$$M = \alpha \int_{w_{21}}^{w_{21}} \int_{w_{21-1}} dz \rho \sqrt{g_{tt}} \frac{\partial}{\partial z} g_{tt} \bigg|_{\rho = 0}, \hspace{1cm} (19)$$

where $\alpha$ is a constant that takes into account the normalisation of the timelike Killing vector $\xi \equiv \alpha \partial_t$, which generates the stationary symmetry. It is well known that, in the absence of asymptotic flatness, $\alpha$ is not necessarily equal to one, as happens for black holes in AdS [31] or Melvin [32, 33] backgrounds. The integration is performed over the intervals $w_1 < z < w_2$ and $w_3 < z < w_4$, which correspond to the black holes, and the result is

$$M_1 = \alpha m_1, \hspace{1cm} M_2 = \alpha m_2. \hspace{1cm} (20)$$

The horizons areas are found integrating over the horizon surfaces, i.e.

$$A = \int_0^{2\pi} d\phi \int_{w_{21-1}}^{w_{21}} dz \sqrt{g_{zz} g_{\phi\phi}} \bigg|_{\rho = 0}, \hspace{1cm} (21)$$

thus

$$A_1 = 16\pi m_1^2 \frac{m_1 + m_2 - z_1 + z_2}{m_1 - m_2 - z_1 + z_2} e^{-2b_1(2m_2 + m_1 + z_1) - 2b_2((m_1 + z_1)^2 + 4m_2 z_2)}, \hspace{1cm} (22)$$

$$A_2 = 16\pi m_2^2 \frac{m_2 + m_1 - z_1 + z_2}{m_2 - m_1 - z_1 + z_2} e^{-2b_1(m_2 + z_2) - 2b_2(m_2 + z_2)^2}. \hspace{1cm} (23)$$

The temperature is obtained from the surface gravity as $T = \kappa/(2\pi)$. Recalling that $\kappa^2 = -\frac{1}{2} \langle \nabla \xi \rangle^2$, the metric (1) gives rise to

$$\kappa^2 = -\frac{\alpha^2 \langle \partial_z V \rangle^2 + \langle \partial_\rho V \rangle^2}{4 F/V} \bigg|_{\rho = 0}. \hspace{1cm} (24)$$

The temperatures are then

$$T_1 = \frac{\alpha \cdot m_1 - m_2 - z_1 + z_2}{8\pi m_1} \frac{m_1 + m_2 - z_1 + z_2}{m_1 - m_2 - z_1 + z_2} e^{2b_1(2m_2 + m_1 + z_1) + 2b_1((m_1 + z_1)^2 + 4m_2 z_2)}, \hspace{1cm} (25)$$

$$T_2 = \frac{\alpha \cdot m_2 - m_1 - z_1 + z_2}{8\pi m_2} \frac{m_2 + m_1 - z_1 + z_2}{m_2 - m_1 - z_1 + z_2} e^{2b_1(m_2 + z_2) + 2b_2(m_2 + z_2)^2}. \hspace{1cm} (26)$$

One can verify that the same results are found via the Euclidean method [34]. We notice that the presence of the external field lowers the black holes temperature, with respect to the Schwarzschild one, and hence the surface gravity. A lower surface gravity means a lower gravitational “pressure” on the horizon, which then can swell up. This feature is in agreement with (18) and with the related observations. Moreover, it explains how the external gravitational field acts, providing an external pressure in the region of the holes, to sustain the mutual collapse of the binary system.

Defining the entropy as $S = A/4$, the above quantities satisfy the Smarr law [35] both for the individual black holes $M_i = 2T_i S_i$ ($i = 1, 2$) and for the double configuration

$$M_1 + M_2 = 2T_1 S_1 + 2T_2 S_2. \hspace{1cm} (27)$$

This result holds regardless of the value of the constant $\alpha$. Nevertheless, a choice for $\alpha$ must be done in order to study the thermodynamics of the system.

We are interested in the first law of thermodynamics from a local point of view: the involved quantities are evaluated on the horizons, therefore the sources at infinity (which generate the external field) are not accessible to local observers near the black holes. Hence we will discard work terms, in the first law, due to the variation of the parameters $b_i$ [16].
We consider the system at thermal equilibrium from now on, i.e. $T_1 = T_2 \equiv \bar{T}$. This condition is satisfied by imposing $m_1 = m_2$. We furthermore choose

$$\alpha = \sqrt{\frac{z_1 - z_2 - 2m_1}{z_1 - z_2}} \times e^{-(b_1 + b_2(2m_1 + z_2))(m_1 + z_2)}, \tag{28}$$

in order to fulfill a Christodoulou–Ruffini mass formula\[^30\], as it happens for regular metrics in which the asymptotic symmetry is different from the flat one\[^33\] \[^37\] \[^38\]. For black hole configurations endowed with $N$ disconnected horizons the best proposal is an additive generalisation such that

$$\sum_{i=1}^{N} M_i = \sum_{i=1}^{N} \sqrt{\frac{A_i}{16\pi}}. \tag{29}$$

It is worth noting that $m_1 = m_2$ is not the only possibility for thermal equilibrium, but it is clearly the simplest one. Moreover, these choice guarantees the integrability of the masses\[^{20}\].

Defining the total mass $\bar{M} = M_1 + M_2$ and the total entropy $\bar{S} = S_1 + S_2$, the first law of thermodynamics

$$\delta \bar{M} = \bar{T} \delta \bar{S}, \tag{30}$$

is verified. We notice that the variation in\[^{30}\] is taken with respect to the free parameters $m_1$, $z_1$, $z_2$, in which $\alpha$ depends: thus the presence of $\alpha$ is crucial to the first law.

We now turn to the verification of the second law of thermodynamics. At this scope we consider a process in which the initial state is described by two black holes at finite distance with total mass $\bar{M}$ and entropy $\bar{S}$, while the final state is modeled by a single black hole of mass

$$M_0 = m_0 e^{-\frac{1}{2}(b_1 + (m_0 + z_0)b_2)(m_0 + z_0)} \tag{31},$$

and entropy

$$S_0 = 4\pi m_0^2 e^{2(b_1 + (m_0 + z_0)b_2)(m_0 + z_0)}. \tag{32}$$

Here $m_0$ and $z_0$ are the mass parameter and the position of the single black hole, respectively. $M_0$ and $S_0$ are computed from metric\[^1\] with a single black hole.

Note that, contrary to\[^{10}\] \[^{11}\], we do not assume $z_0 = 0 \ a \ priori$. The presence of the external gravitational field breaks the translation invariance along the $z$-axis, since the field acts differently on different points of the axis. This fact is relevant, because it allows to regularise the single black hole metric with $b_1 \neq 0$. In the single black hole case, the regularisation condition is $b_1 = -2b_2 z_0$.

In order to have a meaningful comparison we need to compare the two states at equal energy\[^{1}\] and also their background must coincide. Being $b_2$ unconstrained in the single black hole configuration, we just have to equate the values for $b_1$, which fixes the position of the single source at

$$z_0 = \frac{z_1 + z_2}{2}. \tag{33}$$

Then we require $M_0 = \bar{M}$, that fixes $m_0$ and $z_2$ as

$$m_0 = 2m_1 \sqrt{\frac{2m_1 - z_1 + z_2}{2z_2 - z_1}}, \tag{34} \quad z_2 = z_1 + (\sqrt{17} - 1)m_1. \tag{35}$$

The entropy variation is given by $\Delta S = S_0 - \bar{S}$, where the value of the parameters found above must be substituted into the expression. The result, which is a function of $m_1$ and $z_1$, is quite involved, but it can be plotted as in Fig.\[^2\]. It is clear from the plot that the function is always positive or null, i.e.

$$\Delta S \geq 0, \tag{36}$$

which verifies the second law of thermodynamics. This is one of the few cases in which the second law for a binary system can be verified analytically; it has been done, e.g., for the Majumdar–Papapetrou solution\[^{39}\].

\[^6\] We are assuming that there is no energy loss due to emitted radiation.
V. CONCLUSIONS

In this Letter we built the first analytical solution describing two black holes at equilibrium in General Relativity. Its feasibility relies on two novel fundamental features for this kind of model: (i) the spacetime is completely regular outside the black hole horizons and (ii) the metric, thanks to its multipolar expansion, can be naturally embedded in phenomenological settings, such as the center of galaxies. Thus the presented solution can be considered of some realistic astrophysical interest, or even as a reference for Numerical Relativity. In fact it is possible to transform perturbatively our binary model to a rotating frame (relative to the center of mass of the sources) [10]. Obviously, it would be necessary a solution generating technique that deals with a metric depending on three coordinates to exactly model the gravitational coalescence of two sources, which is an intrinsically dynamical phenomenon; unfortunately such a method is not available at the moment. However the system described in this work could be at least interpreted, in the above co-rotating frame and at first approximation, as a stationary phase of a binary black hole interaction during a merging process.

Generalisation to an infinite array of sources can be achieved in a straightforward manner: to this end, the complete multipolar expansion of the external gravitational field will be necessary to obtain a regular solution. Also the inclusion of electromagnetic charge, NUT and angular momentum represents a natural generalisation of the spacetime studied here [19].

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