SMOOTH COMPACTNESS FOR SPACES OF ASYMPTOTICALLY CONICAL SELF-EXPANDERS OF MEAN CURVATURE FLOW

JACOB BERNSTEIN AND LU WANG

ABSTRACT. We show compactness in the locally smooth topology for certain natural families of asymptotically conical self-expanding solutions of mean curvature flow. Specifically, we show such compactness for the set of all two-dimensional self-expanders of a fixed topological type and, in all dimensions, for the set self-expanders of low entropy and for the set mean convex self-expanders with strictly mean convex asymptotic cones. From this we deduce that the natural projection map from the space of parameterizations of asymptotically conical self-expanders to the space of parameterizations of the asymptotic cones is proper for these classes.

1. INTRODUCTION

A hypersurface, i.e., a properly embedded codimension-one submanifold, \( \Sigma \subset \mathbb{R}^{n+1} \), is a self-expander if

\[
H_\Sigma - \frac{x^\perp}{2} = 0.
\]

Here

\[
H_\Sigma = \Delta \Sigma x = -H_\Sigma n_\Sigma = -\text{div}(n_\Sigma)n_\Sigma
\]

is the mean curvature vector, \( n_\Sigma \) is the unit normal, and \( x^\perp \) is the normal component of the position vector. Self-expanders arise naturally in the study of mean curvature flow. Indeed, \( \Sigma \) is a self-expander if and only if the family of homothetic hypersurfaces

\[
\{\Sigma_t\}_{t>0} = \left\{\sqrt{t}\Sigma\right\}_{t>0}
\]

is a mean curvature flow (MCF), that is, a solution to the flow

\[
\left(\frac{\partial x}{\partial t}\right)^\perp = H_{\Sigma_t}.
\]

Self-expanders are expected to model the behavior of a MCF as it emerges from a conical singularity [1]. They are also expected to model the long time behavior of the flow [2].

Given an integer \( k \geq 2 \) and \( \alpha \in (0, 1) \) let \( \Gamma \) be a \( C^{k,\alpha}_\infty \)-asymptotically conical \( C^{k,\alpha}_\infty \)-hypersurface in \( \mathbb{R}^{n+1} \) and let \( L(\Gamma) \) be the link of the asymptotic cone of \( \Gamma \). For instance, if \( \lim_{\rho \to 0^+} \rho L(\Gamma) = \mathcal{C} \) in \( C^{k,\alpha}_\text{loc}(R^{n+1} \setminus \{0\}) \), where \( \mathcal{C} \) is a cone, then \( \Gamma \) is \( C^{k,\alpha}_\infty \)-asymptotically conical with asymptotic cone \( \mathcal{C} \). For technical reasons, the actual definition is slightly weaker – see Section 3 of [2] for the details. We denote the space of \( C^{k,\alpha}_\infty \)-asymptotically conical \( C^{k,\alpha}_\infty \)-hypersurfaces in \( \mathbb{R}^{n+1} \) by \( \mathcal{ACH}^{k,\alpha}_n \).

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We now introduce the classes we will consider. First for \( g, e \geq 0 \) and \( k \geq 2 \) let

\[
\mathcal{E}_{\text{top}}^{k,\alpha}(g, e) = \left\{ \Gamma \in \mathcal{ACH}_2^{k,\alpha} : \Gamma \text{ satisfies (1.1) and } \Gamma \text{ is of genus } g \text{ with } e \text{ ends} \right\},
\]

be the space of \( C^{k,\alpha} \)-asymptotically conical self-expanders in \( \mathbb{R}^3 \) with genus \( g \) and \( e \) ends. Similarly, for any \( n, k \geq 2 \) and \( h_0 > 0 \) let

\[
\mathcal{E}_{n,\text{mc}}^{k,\alpha}(h_0) = \left\{ \Gamma \in \mathcal{ACH}_n^{k,\alpha} : \Gamma \text{ satisfies (1.1), } H_\Gamma > 0, H_{\mathcal{L}(\Gamma)} \geq h_0 \right\},
\]

be the space of \( C^{k,\alpha} \)-asymptotically conical self-expanders in \( \mathbb{R}^{n+1} \) which are strictly mean convex and have uniformly strictly mean convex asymptotic cones. Finally, for \( n, k \geq 2 \) and \( 1 < \Lambda_0 < 2 \) let

\[
\mathcal{E}_{n,\text{ent}}^{k,\alpha}(\Lambda_0) = \left\{ \Gamma \in \mathcal{ACH}_n^{k,\alpha} : \Gamma \text{ satisfies (1.1) and } \lambda[\Gamma] \leq \Lambda_0 \right\},
\]

be the space of \( C^{k,\alpha} \)-asymptotically conical self-expanders in \( \mathbb{R}^{n+1} \) which have entropy less than or equal to \( \Lambda_0 \). See Section 1.2 for the definition of entropy.

We prove the following smooth compactness result for the spaces \( \mathcal{E}_{\text{top}}^{k,\alpha}(g, e), \mathcal{E}_{n,\text{mc}}^{k,\alpha}(h_0) \) and, under suitable hypotheses on \( \Lambda_0, \mathcal{E}_{n,\text{ent}}^{k,\alpha}(\Lambda_0) \):

**Theorem 1.1.** Fix \( k \geq 2 \) and \( \alpha \in (0, 1) \). The following holds:

1. If \( \Sigma_i \in \mathcal{E}_{\text{top}}^{k,\alpha}(g, e) \) and \( \mathcal{L}(\Sigma_i) \to \sigma \in C^{k,\alpha}(\mathbb{S}^2) \), then there is a \( \Sigma \in \mathcal{E}^{k,\alpha}(g, e) \) with \( \mathcal{L}(\Sigma) = \sigma \) so that, up to passing to a subsequence, \( \Sigma_i \to \Sigma \) in \( C^{\infty}_{\text{loc}}(\mathbb{R}^3) \).
2. If \( \Sigma_i \in \mathcal{E}_{n,\text{mc}}^{k,\alpha}(h_0) \) and \( \mathcal{L}(\Sigma_i) \to \sigma \in C^{k,\alpha}(\mathbb{S}^n) \), then there is a \( \Sigma \in \mathcal{E}_{n,\text{mc}}^{k,\alpha}(h_0) \) with \( \mathcal{L}(\Sigma) = \sigma \) so that, up to passing to a subsequence, \( \Sigma_i \to \Sigma \) in \( C^{\infty}_{\text{loc}}(\mathbb{R}^n) \).
3. If Assumption (\( \Sigma_n, \Lambda_0 \)) of Section 1.2 holds, \( \Sigma_i \in \mathcal{E}_{n,\text{ent}}^{k,\alpha}(\Lambda_0) \) for \( \Lambda_0 < \Lambda < 2 \) and \( \mathcal{L}(\Sigma_i) \to \sigma \in C^{k,\alpha}(\mathbb{S}^n) \), then there is a \( \Sigma \in \mathcal{E}_{n,\text{ent}}^{k,\alpha}(\Lambda_0) \) with \( \mathcal{L}(\Sigma) = \sigma \) so that, up to passing to a subsequence, \( \Sigma_i \to \Sigma \) in \( C^{\infty}_{\text{loc}}(\mathbb{R}^{n+1}) \).

In [2], the authors showed that the space \( \mathcal{ACE}_n^{k,\alpha}(\Gamma) \) – see [24] below – of asymptotically conical parameterizations of self-expanders modeled on \( \Gamma \) (modulo reparameterizations fixing the parameterization of the asymptotic cone) possesses a natural Banach manifold structure modeled on \( C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1}) \). They further showed that the map

\[
\Pi : \mathcal{ACE}_n^{k,\alpha}(\Gamma) \to C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})
\]

given by \( \Pi([f]) = \text{tr}_{\infty}^{\Sigma}[f] \) is smooth and Fredholm of index 0. As such, by work of Smale [23], as long as \( \Pi \) is proper it possesses a well-defined mod 2 degree. These results are all analogs of work of White [25] who proved such results for a large class of variational problems for parameterizations from compact manifolds – see also [27].

In general, the map \( \Pi : \mathcal{ACE}_n^{k,\alpha}(\Gamma) \to C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1}) \) is not proper. However, using Theorem 1.1 we give several natural subsets of \( \mathcal{ACE}_n^{k,\alpha}(\Gamma) \) on which the restriction of \( \Pi \) is proper. This should be compared to [26]. As a first step, it is necessary to shrink the range of \( \Pi \). To that end, for any \( \Gamma \in \mathcal{ACH}_n^{k,\alpha} \), let

\[
\lambda_{\text{emb}}^{k,\alpha}(\Gamma) = \left\{ \varphi \in C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1}) : \delta^H_1[\varphi] \text{ is an embedding} \right\},
\]

be the space of parameterizations of embedded cones. Here \( \delta^H_1[\varphi] \) is the homogeneous degree one extension of \( \varphi \). This is readily seen to be an open subset of \( C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1}) \). It also follows from the definition of \( \mathcal{ACE}_n^{k,\alpha}(\Gamma) \) that \( \Pi : \mathcal{ACE}_n^{k,\alpha}(\Gamma) \to \lambda_{\text{emb}}^{k,\alpha}(\Gamma) \).

**Theorem 1.2.** For any \( \Gamma \in \mathcal{ACH}_2^{k,\alpha} \), \( \Pi : \mathcal{ACE}_2^{k,\alpha}(\Gamma) \to \lambda_{\text{emb}}^{k,\alpha}(\Gamma) \) is proper.
Theorem 1.3. For $\Gamma \in ACH^{k,\alpha}_n$ and $\Lambda > 1$, let
\[ V_{\text{ent}}(\Gamma, \Lambda) = \left\{ \varphi \in V_{\text{emb}}^{k,\alpha}(\Gamma) : \lambda[\varphi](\mathcal{L}(\Gamma)) < \Lambda \right\} \]
and
\[ U_{\text{ent}}(\Gamma, \Lambda) = \left\{ [f] \in ACH^{k,\alpha}_n(\Gamma) : \lambda[f](\Gamma) < \Lambda \right\} . \]
The following is true:
1. $U_{\text{ent}}(\Gamma, \Lambda)$ is an open subset of $ACH^{k,\alpha}_n(\Gamma)$.
2. $V_{\text{ent}}(\Gamma, \Lambda)$ is an open subset of $C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$.
3. If $(\ast_{n,\Lambda})$ holds for $\Lambda < 2$, then $\Pi|_{U_{\text{ent}}(\Gamma, \Lambda)} : U_{\text{ent}}(\Gamma, \Lambda) \to V_{\text{ent}}(\Gamma, \Lambda)$ is proper.

Theorem 1.4. For $\Gamma \in ACH^{k,\alpha}_n$, let
\[ V_{\text{mc}}(\Gamma) = \left\{ \varphi \in V_{\text{emb}}^{k,\alpha}(\Gamma) : H_{\sigma} > 0 \text{ where } \sigma = \mathcal{L}[\varphi](\mathcal{L}(\Gamma)) \right\} \]
and
\[ U_{\text{mc}}(\Gamma) = \left\{ [f] \in ACH^{k,\alpha}_n(\Gamma) : H_f(\Gamma) > 0, \Pi([f]) \in V_{\text{mc}}(\Gamma) \right\} . \]
The following is true:
1. $U_{\text{mc}}(\Gamma)$ is an open subset of $ACH^{k,\alpha}_n(\Gamma)$.
2. $V_{\text{mc}}(\Gamma)$ is an open subset of $C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$.
3. $\Pi|_{U_{\text{mc}}(\Gamma)} : U_{\text{mc}}(\Gamma) \to V_{\text{mc}}(\Gamma)$ is a local diffeomorphism.
4. $\Pi|_{U_{\text{mc}}(\Gamma)} : U_{\text{mc}}(\Gamma) \to V_{\text{mc}}(\Gamma)$ is proper.

In particular, for each component $V'$ of $V_{\text{mc}}(\Gamma)$, there is an integer $l' \geq 0$ so $U' = \Pi^{-1}(V') \cap U_{\text{mc}}(\Gamma)$ has $l'$ components and for each component $U''$ of $U'$, $\Pi|_{U''} : U'' \to V'$ is a (finite) covering map.

Finally, as an application of Theorem 1.4 and a result of Huisken [12], we have the following existence and uniqueness result for self-expanders of a given topological type asymptotic to cones that satisfy a natural pinching condition.

Corollary 1.5. For $n \geq 2$, $k \geq 2$ and $\alpha \in (0, 1)$ let $\sigma \subset S^n$ be a connected, strictly mean convex, $C^{k,\alpha}$ hypersurface. In addition, if $n \geq 3$, suppose that $\sigma$ satisfies
\[
|A_{\sigma}|^2 < \frac{1}{n-2}H^2_{\sigma} + 2, \quad n \geq 4; \]
\[
|A_{\sigma}|^2 < \frac{3}{4}H^2_{\sigma} + \frac{4}{3}, \quad n = 3.
\]
There exists a smooth self-expander $\Sigma \in ACH^{k,\alpha}_n$ with $\mathcal{L}(\Sigma) = \sigma$, $H_{\Sigma} > 0$ and so $\Sigma$ is diffeomorphic to $\mathbb{R}^n$. Moreover, if $\pi_0(\text{Diff}^+(S^{n-1})) = 0$, i.e., the group of orientation-preserving diffeomorphisms of $S^{n-1}$ is path-connected, then $\Sigma$ is the unique self-expander with these properties.

Remark 1.6. Hypothesis (1.2) is required only so that classical mean curvature flow can be used to show the space of admissible $\sigma$ is path-connected.

Remark 1.7. By work of Cerf [3, 4] and Smale [21, 22] it is known, for $n \in \{2, 3, 4, 6\}$, that $\pi_0(\text{Diff}^+(S^{n-1})) = 0$.

Remark 1.8. Using only the mean convexity condition, a variational argument due to Ilmanen that is sketched in [16] and carried out by Ding in [6] gives the existence of a self-expanding solution with link $\sigma$. However, this method cannot directly say anything about the topology of the constructed self-expanders.
2. Notation and Background

In this section we fix notation and also recall the main definitions from \([2]\) we need. The interested reader should consult Sections 2 and 3 of \([2]\) for specifics and further details.

2.1. Basic notions. Denote a (open) ball in \(\mathbb{R}^n\) of radius \(R\) and center \(x\) by \(B_R^n(x)\) and the closed ball by \(\overline{B}_R^n(x)\). We often omit the superscript \(n\) when its value is clear from context. We also omit the center when it is the origin.

For an open set \(U \subset \mathbb{R}^{n+1}\), a hypersurface in \(U\), \(\Gamma\), is a smooth, properly embedded, codimension-one submanifold of \(U\). We also consider hypersurfaces of lower regularity and given an integer \(k \geq 2\) and \(\alpha \in (0, 1)\) we define a \(C^{k, \alpha}\)-hypersurface in \(U\) to be a properly embedded, codimension-one \(C^{k, \alpha}\) submanifold of \(U\). When needed, we distinguish between a point \(p \in \Gamma\) and its position vector \(x(p)\).

Consider the hypersurface \(S^n \subset \mathbb{R}^{n+1}\), the unit \(n\)-sphere in \(\mathbb{R}^{n+1}\). A hypersurface in \(S^n\), \(\sigma\), is a closed, embedded, codimension-one smooth submanifold of \(S^n\) and \(C^{k, \alpha}\)-hypersurfaces in \(S^n\) are defined likewise. Observe, that \(\sigma\) is a closed codimension-two submanifold of \(\mathbb{R}^{n+1}\) and so we may associate to each point \(p \in \sigma\) its position vector \(x(p)\). Clearly, \(|x(p)| = 1\).

A cone is a set \(C \subset \mathbb{R}^{n+1} \setminus \{0\}\) that is dilation invariant around the origin. That is, \(\rho C = C\) for all \(\rho > 0\). The link of the cone is the set \(\mathcal{L}[C] = C \cap S^n\). The cone is regular if its link is a smooth hypersurface in \(S^n\) and \(C^{k, \alpha}\)-regular if its link is a \(C^{k, \alpha}\)-hypersurface in \(S^n\). For any hypersurface \(\sigma \subset S^n\) the cone over \(\sigma\), \(\mathcal{C}[\sigma]\), is the cone defined by

\[
\mathcal{C}[\sigma] = \{ pp; p \in \sigma, \rho > 0 \} \subset \mathbb{R}^{n+1} \setminus \{0\}.
\]

Clearly, \(\mathcal{L}[\mathcal{C}[\sigma]] = \sigma\).

2.2. Function spaces. Let \(\Gamma\) be a properly embedded, \(C^{k, \alpha}\) submanifold of an open set \(U \subset \mathbb{R}^{n+1}\). There is a natural Riemannian metric, \(g_T\), on \(\Gamma\) of class \(C^{k-1, \alpha}\) induced from the Euclidean one. As we always take \(k \geq 2\), the Christoffel symbols of this metric, in appropriate coordinates, are well-defined and of regularity \(C^{k-2, \alpha}\). Let \(\nabla_T\) be the covariant derivative on \(\Gamma\). Denote by \(d_T\) the geodesic distance on \(\Gamma\) and by \(B_R^T(p)\) the (open) geodesic ball in \(\Gamma\) of radius \(R\) and center \(p \in \Gamma\). For \(R\) small enough so that \(B_R^T(p)\) is strictly geodesically convex and \(q \in B_R^T(p)\), denote by \(\tau_{p,q}^\Gamma\) the parallel transport along the unique minimizing geodesic in \(B_R^T(p)\) from \(p\) to \(q\).

Throughout the rest of this subsection, let \(\Omega\) be a domain in \(\Gamma\), \(l\) an integer in \([0, k]\), \(\beta \in (0, 1)\) and \(d \in \mathbb{R}\). Suppose \(l + \beta \leq k + \alpha\). We first consider the following norm for functions on \(\Omega\):

\[
\|f\|_{l, \Omega} = \sum_{i=0}^{l} \sup_{\Omega} |\nabla_T^i f|.
\]

We then let

\[
C^l(\Omega) = \{ f \in C^l_{loc}(\Omega); \|f\|_{l, \Omega} < \infty \}.
\]

We next define the Hölder semi-norms for functions \(f\) and tensor fields \(T\) on \(\Omega\):

\[
[f]_{l,\Omega} = \sup_{p,q \in \Omega} \frac{|f(p) - f(q)|}{d_T(p,q)^\beta}, \quad [T]_{l,\Omega} = \sup_{p,q \in \Omega} \frac{|T(p) - (\tau_{p,q}^\Gamma)^* T(q)|}{d_T(p,q)^\beta},
\]

where \(\delta = \delta(\Gamma, \Omega) > 0\) so that for all \(p \in \Omega\), \(B_\delta^\Gamma(p)\) is strictly geodesically convex. We further define the norm for functions on \(\Omega\):

\[
\|f\|_{l,\Omega} = \|f\|_{l,\Omega} + [\nabla_T f]_{l,\Omega}.
\]
and let
\[ C^{l,\beta} (\Omega) = \left\{ f \in C^{l,\beta}_{loc}(\Omega) : \| f \|^{(d)}_{l,\beta;\Omega} < \infty \right\} . \]

We also define the following weighted norms for functions on \( \Omega \):
\[
\| f \|^{(d)}_{l,\beta;\Omega} = \sum_{i=0}^{l} \sup_{p \in \Omega} (|x(p)| + 1)^{-d+i} |\nabla^i f(p)|.
\]

We then let
\[ C^\beta_d (\Omega) = \left\{ f \in C^\beta_{loc}(\Omega) : \| f \|^{(d)}_{1,\beta;\Omega} < \infty \right\} . \]

We further define the following weighted Hölder semi-norms for functions \( f \) and tensor fields \( T \) on \( \Omega \):
\[
[f]^{(d)}_{\beta;\Omega} = \sup_{p,q \in \Omega} \left( |\nabla f(p)| + 1 \right)^{-d+\beta} \| f(p) - f(q) \| \frac{d_{\Gamma}(p,q)}{d_{\Gamma}(p,q)^{\beta}}, \quad \text{and,}
\]
\[
[T]^{(d)}_{\beta;\Omega} = \sup_{p,q \in \Omega} \left( |\nabla T(p)| + 1 \right)^{-d+\beta} \| T(p) - T(q) \| \frac{d_{\Gamma}(p,q)}{d_{\Gamma}(p,q)^{\beta}},
\]
where \( \eta = \eta(\Omega, \Gamma) \in (0, \frac{1}{4}) \) so that for any \( p \in \Gamma \), letting \( \delta_p = \eta(|x(p)| + 1) \), \( B_{\delta_p}^\Gamma(p) \) is strictly geodesically convex. Finally, we define the norm for functions on \( \Omega \):
\[
\| f \|^{(d)}_{l,\beta;\Omega} = \| f \|^{(d)}_{l,\beta;\Omega} + \| \nabla^l f \|^{(d-l)}_{\beta;\Omega},
\]
and we let
\[ C^{l,\beta}_{\Omega} (\Omega) = \left\{ f \in C^{l,\beta}_{loc}(\Omega) : \| f \|^{(d)}_{l,\beta;\Omega} < \infty \right\} . \]

We follow the convention that \( C^{l,0}_{\Omega} = C^{l}_{\Omega} \), \( C^{0,0} = C^0 \) and \( C^{l,0}_d = C^l_d \) and that \( C^{0,\beta}_d = C^{\beta}_d \), \( C^{0,\beta}_d = C^{\beta}_d \), \( C^{0,\beta}_d = C^{\beta}_d \). The notation for the corresponding norms is abbreviated in the same fashion.

2.3. Homogeneous functions and homogeneity at infinity. Fix a \( C^{k,\alpha} \)-regular cone \( \mathcal{C} \) with its link \( \mathcal{L} \). By our definition \( \mathcal{C} \) is a \( C^{k,\alpha} \)-hypersurface in \( \mathbb{R}^{n+1} \setminus \{0\} \). For \( R > 0 \) let \( \mathcal{C}_R = \mathcal{C} \setminus B_R \). There is an \( \eta = \eta(\mathcal{L}, R) \) so that for any \( p \in \mathcal{C}_R, B_{\delta_p}^\mathcal{L}(p) \) is strictly geodesically convex, where \( \delta_p = \eta(|x(p)| + 1) \). We also fix an integer \( l \in [0, k] \) and \( \beta \in [0, 1] \) with \( l + \beta \leq k + \alpha \).

A map \( f \in C^{l,\beta}_{loc}(\mathcal{C}; \mathbb{R}^M) \) is homogeneous of degree \( d \) if \( f(rp) = \rho^d f(p) \) for all \( \rho \in \mathbb{C} \) and \( \rho > 0 \). Given a map \( \varphi \in C^{l,\beta}(\mathbb{L}; \mathbb{R}^M) \), the homogeneous extension of degree \( d \) of \( \varphi \) is the map \( \varphi^H_d \in C^{l,\beta}_{loc}(\mathcal{C}; \mathbb{R}^M) \) defined by
\[
\varphi^H_d \left( \frac{1}{\rho^d} \right) = \varphi \left( \frac{1}{\rho^d} \right) \left( \frac{1}{\rho^d} \right).
\]

Conversely, given a homogeneous \( \mathbb{R}^M \)-valued map of degree \( d \), \( f \in C^{l,\beta}_{loc}(\mathcal{C}; \mathbb{R}^M) \), let \( \varphi = \text{tr}[f] \in C^{l,\beta}(\mathbb{L}; \mathbb{R}^M) \), the trace of \( f \), be the restriction of \( f \) to \( \mathbb{L} \). Clearly, \( f \) is the homogeneous extension of degree \( d \) of \( \varphi \).

A map \( g \in C^{l,\beta}_{loc}(\mathbb{L}; \mathbb{R}^M) \) is asymptotically homogeneous of degree \( d \) if
\[
\lim_{\rho \to 0^+} \rho^d g \left( \frac{1}{\rho^d} p \right) = f(p) \in C^{l,\beta}_{loc}(\mathcal{C}; \mathbb{R}^M) \]
for some \( f \in C^{\ell,\beta}_{\text{loc}}(C; \mathbb{R}^M) \) that is homogeneous of degree \( d \). For such a \( g \) we define the trace at infinity of \( g \) by \( \text{tr}_\infty^d [g] = \text{tr}[f] \). We define

\[
C^{\ell,\beta}_{d,H}(C; \mathbb{R}^M) = \left\{ g \in C^{\ell,\beta}_{d}(C; \mathbb{R}^M) : g \text{ is asymptotically homogeneous of degree } d \right\}.
\]

It is straightforward to verify that \( C^{\ell,\beta}_{d,H}(C; \mathbb{R}^M) \) is a closed subspace of \( C^{\ell,\beta}_{d}(C; \mathbb{R}^M) \) and that

\[
\text{tr}_\infty^d : C^{\ell,\beta}_{d,H}(C; \mathbb{R}^M) \to C^{\ell,\beta}(\mathcal{L}; \mathbb{R}^M)
\]

is a bounded linear map. Finally, \( x|_{CR} \in C^{k,\alpha}_{1,H}(C_R; \mathbb{R}^{n+1}) \) and \( \text{tr}_\infty^1 [x|_{CR}] = x|_{\mathcal{L}} \).

2.4. Asymptotically conical hypersurfaces. A \( C^{k,\alpha} \)-hypersurface, \( \Gamma \subset \mathbb{R}^{n+1} \), is \( C^{k,\alpha} \)-asymptotically conical if there is a \( C^{k,\alpha} \)-regular cone, \( C \subset \mathbb{R}^{n+1} \), and a homogeneous transverse section, \( \nu \), on \( C \) such that \( \Gamma \) outside some compact set is given by the \( \nu \)-graph of a function in \( C^{1,\alpha}_k \cap C^{1,0}_0(C_R) \) for some \( R > 1 \). Here a transverse section is a regularized version of the unit normal – see Section 2.4 of \([2]\) for the precise definition. Observe, that by the Arzelà-Ascoli theorem one has that, for every \( \beta \in [0, \alpha) \),

\[
\lim_{\rho \to 0^+} \rho \Gamma = C \text{ in } C^{k,\beta}_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\}).
\]

Clearly, the asymptotic cone, \( C \), is uniquely determined by \( \Gamma \) and so we denote it by \( C(\Gamma) \). Let \( \mathcal{L}(\Gamma) \) denote the link of \( C(\Gamma) \) and, for \( R > 0 \), let \( C_R(\Gamma) = C(\Gamma) \setminus B_R \). Denote the space of \( C^{k,\alpha} \)-asymptotically conical \( C^{k,\alpha} \)-hypersurfaces in \( \mathbb{R}^{n+1} \) by \( \mathcal{AHC}^{k,\alpha}_n \).

Finally, let \( K \) be a compact set of \( \Gamma \) and denote by \( \Gamma' = \Gamma \setminus K. \) By definition, we may choose \( K \) large enough so \( \pi_v \) – the projection of a neighborhood of \( C(\Gamma) \) along \( \nu \) – restricts to a \( C^{k,\alpha} \) diffeomorphism of \( \Gamma' \) onto \( C_R(\Gamma) \). Denote its inverse by \( \theta_{\nu;\Gamma'} \).

2.5. Traces at infinity. Fix an element \( \Gamma \in \mathcal{AHC}^{k,\alpha}_n \). Let \( l \) be an integer in \([0, k]\) and \( \beta \in [0, 1) \) such that \( l + \beta < k + \alpha. \) A map \( f \in C^{l,\beta}_{\text{loc}}(\Gamma; \mathbb{R}^M) \) is asymptotically homogeneous of degree \( d \) if \( f \circ \theta_{\nu;\Gamma'} \in C^{l,\beta}_{d,H}(C_R(\Gamma); \mathbb{R}^M) \) where \( \nu \) is a transverse section and \( \Gamma', \theta_{\nu;\Gamma'} \) are introduced in the previous subsection. The trace at infinity of \( f \) is then

\[
\text{tr}_\infty^d [f] = \text{tr}_\infty^d [f \circ \theta_{\nu;\Gamma'}] \in C^{l,\beta}(\mathcal{L}(\Gamma); \mathbb{R}^M).
\]

Whether \( f \) is asymptotically homogeneous of degree \( d \) and the definition of \( \text{tr}_\infty^d \) are independent of the choice of homogeneous transverse sections on \( C(\Gamma) \). Clearly, \( x|_{\Gamma} \) is asymptotically homogeneous of degree \( 1 \) and \( \text{tr}_\infty^1 [x|_{\Gamma}] = x|_{\mathcal{L}(\Gamma)} \).

We next define the space

\[
C^{l,\beta}_{d,H}(\Gamma; \mathbb{R}^M) = \left\{ f \in C^{l,\beta}_{d}(\Gamma; \mathbb{R}^M) : f \text{ is asymptotically homogeneous of degree } d \right\}.
\]

One can check that \( C^{l,\beta}_{d,H}(\Gamma; \mathbb{R}^M) \) is a closed subspace of \( C^{l,\beta}_{d}(\Gamma; \mathbb{R}^M) \), and the map

\[
\text{tr}_\infty^d : C^{l,\beta}_{d,H}(\Gamma; \mathbb{R}^M) \to C^{l,\beta}(\mathcal{L}(\Gamma); \mathbb{R}^M)
\]

is a bounded linear map. We further define the set \( C^{l,\beta}_{d,0}(\Gamma; \mathbb{R}^M) \subset C^{l,\beta}_{d,H}(\Gamma; \mathbb{R}^M) \) to be the kernel of \( \text{tr}_\infty^d \).
2.6. Asymptotically conical embeddings. Fix an element $\Gamma \in ACH_{n}^{k,\alpha}$. We define the space of $C_{k,\alpha}^{k,\alpha}$-asymptotically conical embeddings of $\Gamma$ into $\mathbb{R}^{n+1}$ to be

$$AHC^{k,\alpha}_{n}(\Gamma) = \left\{ f \in C^{k,\alpha}_{1} \cap C^{k}_{1,H}(\Gamma;\mathbb{R}^{n+1}) : f \text{ and } \phi_{t}^{H} \circ \text{tr}_{\infty}^{1}[f] \text{ are embeddings} \right\}.$$ 

Clearly, $AHC^{k,\alpha}_{n}(\Gamma)$ is an open set of the Banach space $C^{k,\alpha}_{1} \cap C^{k}_{1,H}(\Gamma;\mathbb{R}^{n+1})$ with the $\| \cdot \|_{k,\alpha}$ norm. The hypotheses on $f$, $\text{tr}_{\infty}^{1}[f] \in C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$ ensure

$$C[f] = \phi^{H} \circ \text{tr}_{\infty}^{1}[f] : C(\Gamma) \to \mathbb{R}^{n+1} \setminus \{0\}$$

is a $C^{k,\alpha}$ embedding. As this map is homogeneous of degree one, it parameterizes the $C^{k,\alpha}$-regular cone $C(f(\Gamma))$ -- see [2] Proposition 3.3.

Finally, we introduce a natural equivalence relation on $AHC^{k,\alpha}_{n}(\Gamma)$. First, say a $C^{k,\alpha}$ diffeomorphism $\phi : \Gamma \to \Gamma$ fixes infinity if $\{x\} \circ \phi \in AHC^{k,\alpha}_{n}(\Gamma)$ and $\text{tr}_{\infty}^{1}[x|_{\Gamma} \circ \phi] = x|_{\mathcal{L}(\Gamma)}$.

Two elements $f, g \in AHC^{k,\alpha}_{n}(\Gamma)$ are equivalent, written $f \sim g$, provided there is a $C^{k,\alpha}$ diffeomorphism $\phi : \Gamma \to \Gamma$ that fixes infinity so that $f \circ \phi = g$. Given $f \in AHC^{k,\alpha}_{n}(\Gamma)$ let $[f]$ be the equivalence class of $f$. Following [2] we define the space

$$(2.1) \quad AEE_{n}^{k,\alpha}(\Gamma) = \left\{ [f] : f \in AHC^{k,\alpha}_{n}(\Gamma) \text{ and } f(\Gamma) \text{ satisfies (1.1)} \right\}.$$

3. Smooth Compactness

In this section we prove Theorem 3.1. We first prove compactness in the asymptotic region and then treat the three special cases separately.

3.1. Asymptotic regularity of self-expanders. Fix a unit vector $e$, a point $x_{0} \in \mathbb{R}^{n+1}$ and $r, h > 0$. Let

$$C_{e}(x_{0}, r, h) = \left\{ x \in \mathbb{R}^{n+1} : \|x - x_{0}\| < h, \|x - x_{0}\|^{2} < r^{2} + \|x - x_{0}\| \cdot e^{2} \right\}$$

be the solid open cylinder with axis $e$ centered at $x_{0}$ and of radius $r$ and height $2h$.

**Definition 3.1.** Suppose that $l \geq 0$ is an integer and $\beta \in [0, 1)$. A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a $C^{l,\beta}$ $e$-graph of size $\delta$ on scale $r$ at $x_{0}$ if there is a function $f : B_{r}^{e} \subset P_{e} \to \mathbb{R}$ with

$$\sum_{j=0}^{l} r^{-1+j} \|\nabla^{j} f\|_{0} + r^{-1+l+\beta} \|\nabla^{l} f\|_{\beta} < \delta,$$

where $P_{e}$ is the $n$-dimensional subspace of $\mathbb{R}^{n+1}$ normal to $e$ and the last term on the left hand side will be dropped if $\beta = 0$, so that

$$\Sigma \cap C_{e}(x_{0}, r, \delta r) = \{ x_{0} + x(x) + f(x)e : x \in B_{r}^{e} \}.$$

Moreover, a hypersurface $\sigma \subset \mathbb{S}^{n}$ is a $C^{l,\beta}$ $e$-graph of size $\delta$ on scale $r$ at $x_{0}$ if $C[\sigma]$ is so. We omit $C^{l,\beta}$ in the above definitions when the hypersurface is of $C^{l,\beta}$ class or when it is clear from context.

Let us summarize some elementary properties of this concept.

**Proposition 3.2.** Let $l \geq 2$ be an integer and $\beta \in [0, 1)$. The following is true:

1. If $\Sigma$ is a $C^{l,\beta}$ hypersurface in $\mathbb{R}^{n+1}$, then for every $\delta > 0$ and $p \in \Sigma$, there is an $r = r(\Sigma, p, \delta) > 0$ so that $\Sigma$ is an $e_{p}(\Sigma, p, \delta)$-graph of size $\delta$ on scale $r$ at $p$.
2. If $\sigma \subset \mathbb{S}^{n}$ is an $e$-graph of size $\delta$ on scale $r$ at $x_{0}$ and $p > 0$, then $C[\sigma]$ is an $e$-graph of size $\delta$ on scale $p r$ at $p x_{0}$.
(3) Given a $C^{l,\beta}$-hypersurface $\sigma \subset \mathbb{S}^n$ and $\delta > 0$, there is an $r = r(\sigma, \delta) > 0$ so that $\sigma$ is an $n\mathcal{L}[\sigma](p)$-graph of size $\delta$ on scale $r$ at every $p \in \sigma$.

(4) Suppose $\sigma_i \subset \mathbb{S}^n$ converges in $C^{l,\beta}(\mathbb{S}^n)$ to $\sigma$. If $p_i \in \sigma_i \rightarrow p \in \sigma$ and $\sigma$ is an $n\mathcal{L}[\sigma](p)$-graph of size $\delta$ on scale $2r$ at $p$, then there is an $i_0$ so that for $i \geq i_0$, $\sigma_i$ is an $n\mathcal{L}[\sigma_i(p_i)]$-graph of size $2\delta$ on scale $r$ at $p_i$.

The pseudo-locality property of mean curvature flow gives certain asymptotic regularity for self-expanders that are weakly asymptotic to a cone.

**Proposition 3.3.** Let $l \geq 2$ be an integer and $\beta \in [0, 1)$. For each $\delta > 0$ and $r > 0$ there exist constants $R, M, \gamma, \eta > 0$ depending only on $n, l, \beta, \delta$ and $r$ so that if $\Sigma$ is a self-expander in $\mathbb{R}^{n+1}$ with

$$\lim_{\rho \rightarrow 0^+} \mathcal{H}^n \cap (\rho \Sigma) = \mathcal{H}^n \cap \mathcal{C}[\sigma]$$

for $\sigma$ a $C^{l,\beta}$-hypersurface in $\mathbb{S}^n$, and $\Sigma$ is a $n\mathcal{L}[\sigma](p)$-graph of size $\delta$ on scale $r$ at every $p \in \sigma$, then

1. $\Sigma$ is a $C^{l,\beta}$ $n\mathcal{L}[\sigma](p)$-graph of size $\gamma$ on scale $\eta |x(p)|$ at every $p \in \mathcal{C}[\sigma] \setminus \bar{B}_R$.
2. There is a function $f : \mathcal{C}(\Sigma) \setminus \bar{B}_R \rightarrow \mathbb{R}$ satisfying

$$|f(p)| + |\nabla_{\mathcal{C}[\sigma]} f(p)| \leq M |x(p)|^{-1}$$

and so

$$\Sigma \setminus \bar{B}_{2R} = \{x(p) + f(p)n_{\mathcal{C}[\sigma]}(p) : p \in \mathcal{C}[\sigma] \setminus \bar{B}_R\} \setminus \bar{B}_{2R}.$$ 

**Proof.** For simplicity we set $\mathcal{C} = \mathcal{C}[\sigma]$. Consider the mean curvature flow (thought of as a space-time track)

$$\mathcal{S} = \bigcup_{t > 0} \left( \sqrt{t} \Sigma \right) \times \{t\}.$$ 

Let $\bar{\mathcal{S}} = \mathcal{S} \cup (\mathcal{C} \times \{0\})$ so, by our hypothesis, $\bar{\mathcal{S}}$ is an integer Brakke flow (cf. [15] §2 and [14] §6). Denote by $\bar{\mathcal{S}}_t$ the time $t$ slice of $\bar{\mathcal{S}}$. For Item (1), it is sufficient to prove that there are constants $\tau, \gamma, \eta > 0$ depending only on $n, l, \beta, \delta$ and $r$ so that for all $t \in [0, \tau]$, $\bar{\mathcal{S}}_t$ is a $C^{l,\beta}$ $n\mathcal{L}[\sigma](p)$-graph of size $\gamma$ on scale $\eta$ at every $p \in \sigma$.

By the pseudo-locality property for mean curvature flow (cf. [17] Theorem 1.5 and Remarks 1.6)) there is an $r \in (0, 1)$ depending on $n, \delta$ and $r$ so that for every $t \in [0, 16 \epsilon^2]$ and $p \in \sigma$, $\bar{\mathcal{S}}_t \cap C_n(p, 4\epsilon, 4\epsilon)$ is the graph of a function $\psi_p(t, x)$ over $T_p \mathcal{C}$ with

$$(4\epsilon)^{-1} \|\psi_p(t, \cdot)\|_0 + \|D_x \psi_p(t, \cdot)\|_0 \leq 1.$$ 

Moreover, as $\psi_p(t, x)$ satisfies

$$\frac{\partial \psi_p}{\partial t} = \sqrt{1 + |D_x \psi_p|^2} \text{div} \left( \frac{D_x \psi_p}{\sqrt{1 + |D_x \psi_p|^2}} \right),$$

it follows from the Hölder estimate for quasi-parabolic equations (cf. [18] Theorem 1.1 of Chapter 6)) that for every $\alpha' \in (0, 1),$

$$\sup_{t \in [0,4\epsilon^2]} |D_x \psi_p(t, \cdot)|_{\alpha', B^\circ_2} + \sup_{x \in \bar{B}_2^\circ} |D_x \psi_p(\cdot, x)|_{\alpha', [0,4\epsilon^2]} \leq C(n, \alpha', \epsilon).$$

Furthermore, we appeal to the estimates of fundamental solutions and the Schauder theory (cf. [18] (13.1) and Theorem 5.1)) to get that

$$\sup_{t \in [0,4\epsilon^2]} \|\psi_p(t, \cdot)\|_{l, B^\circ_2} \leq C'(n, l, \beta, \epsilon).$$
Using the equation of $\psi_p$ and the fact that $\psi_p(0,0) = |D_x \psi_p(0,0)| = 0$, it follows from (3.1) that

$$|\psi_p(t,x)| \leq \tilde{C} \left( |x|^2 + t \right)$$

and $|D_x \psi_p(t,x)| \leq \tilde{C} \left( |x| + \sqrt{t} \right)$,

where $\tilde{C} = \tilde{C}(n, C, C') > C'$. In particular, for $\rho \in (0, 1)$ and for every $t \in [0, \rho^2 e^2]$, $$(\rho e)^{-1} \|\psi_p(t,\cdot)\|_{0, B_{\rho e}^n} + \|D_x \psi_p(t,\cdot)\|_{0, B_{\rho e}^n} \leq 4\tilde{C}\rho e.$$ This together with (3.1) further gives

$$\sum_{j=0}^{l} (\rho e)^{j-1} \|D_x^j \psi_p(t,\cdot)\|_{0, B_{\rho e}^n} + (\rho e)^{l+\beta-1} \|D_x^\beta \psi_p(t,\cdot)\|_{\beta, B_{\rho e}^n} \leq 5\tilde{C}\rho e.$$

Now we choose $\rho = (5\tilde{C}e)^{-1/2}$ so $5\tilde{C}\rho e < 4\rho^{-1}$. Hence, for each $t \in [0, \rho^2 e^2]$, $\bar{S}_t$ is a $C^{l,\beta}$ $n_C(p)$-graph of size $4\rho^{-1}$ on scale $\rho e$ at every $p \in \sigma$. The claim follows immediately for $\tau = \rho^2 e^2$, $\gamma = 4\rho^{-1}$ and $\eta = \rho e$.

As $\bar{S}$ is a mean curvature flow away from $(0,0)$, by comparing with shrinking spheres, one observes that $\Sigma \setminus B_{2R} \subset \bigcup_{p \in C \setminus B_R} C_{\delta,\rho}(p, \eta|x(p)|, \gamma \eta|x(p)|)$ for some $R > \tau^{-1/2}$ depending on $n, l, \beta, \delta$ and $r$. Thus, invoking estimate (3.2) gives that for every $q \in \Sigma \setminus B_{2R}$,

$$|x(q) - x(\pi(q))| + |n_{\Sigma}(q) - n(C(\pi(q)))| \leq \tilde{C}(n, \tilde{C})|x(q)|^{-1}.$$ Here $\pi$ is the nearest point projection from $\Sigma \setminus B_{2R}$ onto $C$. Hence, Item 3 follows easily from this estimate and the implicit function theorem. \hfill $\Box$

**Corollary 3.4.** If $\Sigma_i \in \mathcal{ACH}_{n, \alpha}^k$ are self-expanders and there is a $C^{k,\alpha}$-hypersurface in $\Sigma^* \setminus \sigma$, so $\mathcal{L}(\Sigma_i) \to \sigma$ in $C^2(\Sigma^*)$, then there is an $R' = R'(n, k, \alpha, \sigma) > 0$ and a $C^{k,\alpha}$-asymptotically conical self-expanding end $\Sigma$ in $\mathbb{R}^{n+1} \setminus B_{R'}$ with $\mathcal{L}(\Sigma) = \sigma$ so that, up to passing to a subsequence,$$
\Sigma_i \to \Sigma \text{ in } C^\infty_{loc}(\mathbb{R}^{n+1} \setminus B_{R'})\).$$

**Proof.** By Items (3) and (4) of Proposition 3.2 there are $\delta, r > 0$ and $i_0$ so that for $i \geq i_0$ each $\mathcal{L}(\Sigma_i)$ is an $n_C(p)$-graph of size $\delta$ on scale $r$ for all $p \in \mathcal{L}(\Sigma_i)$. In particular, we may apply Item (2) of Proposition 3.2 using these constants to obtain an $\mathcal{R}$ so that one has uniform graphical estimates for the $\Sigma_i$ in $\mathbb{R}^{n+1} \setminus B_R$. It then follows from the Arzelà-Ascoli theorem and standard elliptic estimates (see [10] Theorems 6.17 and 8.24)) that there is a self-expanding end, $\Sigma$, in $\mathbb{R}^{n+1} \setminus B_{R'}$ so that up to passing to a subsequence $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1} \setminus B_{R'})$. In fact, by Item (2) of Proposition 3.2 applied to the $\Sigma_i$ together with the Arzelà-Ascoli theorem, $\Sigma$ is $C^{(\alpha)}_i$-asymptotic to $C[\sigma]$. Combining this fact with Item (1) of Proposition 3.2 applied to $\Sigma$ gives that $\Sigma$ is actually $C^{k,\alpha}_i$-asymptotic to $C[\sigma]$ which completes the proof. \hfill $\Box$  

### 3.2. Entropy and smooth compactness of $C_{n,\alpha}^{k,\alpha}(\Lambda_0)$

We recall the notion of entropy introduced by Colding and Minicozzi [5] and use it to prove Item (5) of Theorem 1.1 as well as introduce several auxiliary results needed in other parts of the article.

First of all, for a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ the Gaussian surface area of $\Sigma$ is

$$F[\Sigma] = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}} d\mathcal{H}^n.$$
Colding and Minicozzi [5] introduced the following notion of entropy of a hypersurface:

\[ \lambda[\Sigma] = \sup_{\rho > 0, y \in \mathbb{R}^{n+1}} F[\rho \Sigma + y]. \]

By identifying \( \Sigma \) with \( \mathcal{H}^n[\Sigma] \), one extends \( F \) and \( \lambda \) in an obvious manner to Radon measures on \( \mathbb{R}^{n+1} \). We first record a number of simple observations about the entropy of asymptotically conical self-expanders.

**Lemma 3.5.** If \( \Sigma \in \mathcal{ACH}^{k,\alpha}_n \) is a self-expander, then

\[ \lambda[\Sigma] = \lambda[C(\Sigma)]. \]

**Proof.** On the one hand, for \( \Sigma \in \mathcal{ACH}^{k,\alpha}_n \),

\[ \lim_{\rho \to 0^+} \rho \Sigma = C(\Sigma) \text{ in } C^k_{loc}(\mathbb{R}^{n+1} \setminus \{0\}). \]

In particular,

\[ \lim_{\rho \to 0^+} \mathcal{H}^n[\rho \Sigma] = \mathcal{H}^n[C(\Sigma)]. \]

By the lower semicontinuity and scaling invariance of entropy

\[ \lambda[\Sigma] \geq \lambda[C(\Sigma)]. \]

On the other hand, we define \( \mathcal{M} = \{\mu_t\}_{t \geq 0} \) to be a family of Radon measures on \( \mathbb{R}^{n+1} \) given by

\[ \mu_t = \begin{cases} \mathcal{H}^n[\sqrt{t} \Sigma] & \text{if } t > 0, \\ \mathcal{H}^n[C(\Sigma)] & \text{if } t = 0, \end{cases} \]

so \( \mathcal{M} \) is an integer Brakke flow (see [15, §2] and [14, §6]). Thus, the Huisken monotonicity formula [13] (see also [15, Lemma 7]) implies

\[ \lambda[\Sigma] \leq \lambda[C(\Sigma)], \]

which finishes the proof. \( \square \)

**Lemma 3.6.** There is a constant \( \tilde{M} = \tilde{M}(n) \) so that if \( \Sigma \in \mathcal{ACH}^{k,\alpha}_n \) is a self-expander, then for any \( R > 0 \),

\[ \mathcal{H}^n(\Sigma \cap B_R) \leq \tilde{M}\lambda[C(\Sigma)]R^n. \]

**Proof.** One computes

\[ R^{-n}\mathcal{H}^n(\Sigma \cap B_R) = \mathcal{H}^n((R^{-1}\Sigma) \cap B_1) \leq \tilde{M}(n)F[R^{-1}\Sigma] \leq \tilde{M}(n)\lambda[\Sigma] \]

and so the claim follows from Lemma 3.5. \( \square \)

**Lemma 3.7.** Fix any \( \epsilon > 0 \). If \( \mathcal{C} \subset \mathbb{R}^{n+1} \) is a \( C^2 \)-regular cone and \( \mathcal{L}[\mathcal{C}] \) is an \( n_C(p) \)-graph of size \( \delta \) on scale \( r \) at every \( p \in \mathcal{L}[\mathcal{C}] \), then there is an \( \mathcal{R} = \mathcal{R}(n, \epsilon, \delta, r) \) so that either

\[ \lambda[\mathcal{C}] \leq 1 + \epsilon \text{ or } \lambda[\mathcal{C}] = F[\mathcal{C} + x_0] \text{ for some } x_0 \in \bar{B}_R. \]

**Proof.** First observe, that as \( \mathcal{C} \) is invariant by homotheties one has

\[ \lambda[\mathcal{C}] = \sup_{x \in \mathbb{R}^{n+1}} F[\mathcal{C} + x]. \]

Next observe that an elementary covering argument gives an \( A = A(n, \delta, r) \) so that

\[ \mathcal{H}^{n-1}(\mathcal{L}[\mathcal{C}]) \leq A. \]

Hence, there is an \( A' = A'(n, \delta, r) > 0 \) so for all \( R > 0 \) and \( x \in \mathbb{R}^{n+1}, \nabla \mathcal{H}^n(\mathcal{C} \cap B_R(x)) \leq A'R^n. \]
A consequence of this is that there is an \( \tilde{R} = \tilde{R}(n, \epsilon, \delta, r) \) so that for all \( x \notin \bar{B}_R \),
\[
F[C + x] \leq 1 + \epsilon.
\]
The claim follows from this. \( \square \)

**Lemma 3.8.** Let \( \sigma_i \) and \( \sigma \) be \( C^2 \)-hypersurfaces in \( \mathbb{S}^n \). If \( \sigma_i \to \sigma \) in \( C^2(\mathbb{S}^n) \), then
\[
\lambda[\mathcal{C}[\sigma_i]] \to \lambda[\mathcal{C}[\sigma]].
\]
**Proof.** As \( \sigma_i \to \sigma \) in \( C^2(\mathbb{S}^n) \),
\[
\mathcal{H}^n[\mathcal{C}[\sigma_i]] \to \mathcal{H}^n[\mathcal{C}[\sigma]].
\]
By the lower semicontinuity of entropy
\[
\liminf_{i \to \infty} \lambda[\mathcal{C}[\sigma_i]] \geq \lambda[\mathcal{C}[\sigma]].
\]
By Lemma 3.7, given \( \epsilon > 0 \) there is an \( R = R(n, \epsilon, \sigma) > 0 \) so that for \( i \) sufficiently large, either \( \lambda[\mathcal{C}[\sigma_i]] \leq 1 + \epsilon \) or \( \lambda[\mathcal{C}[\sigma_i]] = F[\mathcal{C}[\sigma_i] + x] \) for some \( x_i \in \bar{B}_R \). Observe, that there is an \( A = A(n, \sigma) \) so that for \( i \) sufficiently large, and for every \( r > 0 \) and \( x \in \mathbb{R}^{n+1} \),
\[
\mathcal{H}^n(\mathcal{C}[\sigma_i] \cap B_r(x)) \leq Ar^n.
\]
Hence, we get
\[
\limsup_{i \to \infty} \lambda[\mathcal{C}[\sigma_i]] \leq \max \{1 + \epsilon, \lambda[\mathcal{C}[\sigma]]\}.
\]
Passing \( \epsilon \) to 0, as \( \lambda[\mathcal{C}[\sigma]] \geq 1 \), it follows that
\[
\limsup_{i \to \infty} \lambda[\mathcal{C}[\sigma_i]] \leq \lambda[\mathcal{C}[\sigma]],
\]
completing the proof. \( \square \)

We are now ready to prove that \( \mathcal{E}_{n,\text{ent}}^{k,\alpha}(\Lambda_0) \) is compact. In order to do so we introduce a necessary hypothesis about the entropy of minimal cones. Let \( \mathcal{RM}_{C_n} \) denote the space of regular minimal cones in \( \mathbb{R}^{n+1} \), that is \( C \in \mathcal{RM}_{C_n} \) if and only if it is a proper subset of \( \mathbb{R}^{n+1} \) and \( C \setminus \{0\} \) is a hypersurface in \( \mathbb{R}^{n+1} \) that is invariant under dilation about 0 and with vanishing mean curvature. Let \( \mathcal{RM}_{C_n}^{*} \) denote the set of non-flat elements of \( \mathcal{RM}_{C_n} \) i.e., cones with non-zero curvature somewhere. For any \( \Lambda > 0 \), let
\[
\mathcal{RM}_{C_n}(\Lambda) = \{ C \in \mathcal{RM}_{C_n} : \lambda[C] < \Lambda \} \quad \text{and} \quad \mathcal{RM}_{C_n}^{*}(\Lambda) = \mathcal{RM}_{C_n}^{*} \cap \mathcal{RM}_{C_n}(\Lambda).
\]
Now fix a dimension \( n \geq 3 \) and a value \( \Lambda > 1 \). Consider the following hypothesis:

\((\ast_{n, \Lambda})\)

For all \( 3 \leq l \leq n \), \( \mathcal{RM}_{C_l}(\Lambda) = \emptyset \).

Observe that all regular minimal cones in \( \mathbb{R}^2 \) consist of unions of rays and so \( \mathcal{RM}_{C_2}^{*} = \emptyset \). Likewise, as great circles are the only geodesics in \( \mathbb{S}^2 \), \( \mathcal{RM}_{C_2}^{*} = \emptyset \). As a consequence of Allard’s regularity theorem and a dimension reduction argument, there is always some \( \Lambda_n > 1 \) so that \((\ast_{n, \Lambda_n})\) holds.

**Proof of Item \((5)\) of Theorem 1.1.** First, by Corollary 3.4 there is an \( R > 0 \) so that, up to passing to a subsequence, \( \Sigma_i \setminus B_R \) converges in \( C_{L^1}^{\infty}(\mathbb{R}^{n+1} \setminus \\bar{B}_R) \) to some hypersurface \( \Sigma' \) in \( \mathbb{R}^{n+1} \setminus \bar{B}_R \). Moreover, \( \Sigma' \) is a \( C_{k,\alpha}^{k,\alpha} \)-asymptotically conical self-expander in \( \mathbb{R}^{n+1} \setminus \bar{B}_R \) and \( \mathcal{C}(\Sigma') = \sigma \).

As \( \lambda[\Sigma_i] \leq \Lambda_0 < \Lambda \), Lemma 3.6 and the standard compactness results, imply that, up to passing to a further subsequence, \( \Sigma_i \cap B_{2R} \) converges in the sense of measures to some integral varifold, \( V \), in \( B_{2R} \). As such, there is an integral varifold, \( \Sigma \), in \( \mathbb{R}^{n+1} \) that agrees with \( V \) in \( B_{2R} \) and \( \Sigma \) outside \( B_R \). In particular, \( \Sigma \) is smooth and properly embedded in \( \mathbb{R}^{n+1} \setminus \bar{B}_R \). The lower semicontinuity of entropy gives \( \lambda[\Sigma] \leq \Lambda_0 < \Lambda \), and so it follows
from $\mathcal{E}_{n,\Lambda}$, a dimension reduction theorem \cite{28} Theorem 4 and Allard’s regularity theorem (e.g., \cite{20} Theorem 24.2) that $\Sigma$ is actually smooth and properly embedded in $\mathbb{R}^{n+1}$. That is, $\Sigma \in \mathcal{E}_{n,\text{mc}}^k(A_0)$. Finally, a further consequence of Allard’s regularity theorem \cite{29} is that $\Sigma_i \to \Sigma$ in $C^\infty_{\text{loc}}(\mathbb{R}^{n+1})$, finishing the proof. \hfill \Box

3.3. Smooth compactness of $\mathcal{E}_{k,\alpha}^\top(g,e)$. Combining the asymptotic compactness property, the area estimates from Lemma 3.6 and a result of White \cite{24} Theorem 3 (4) we can easily prove Item 1 of Theorem 1.1.

Proof of Item 1 of Theorem 1.1. By Corollary 3.4 there is an $R > 0$ so that, up to passing to a subsequence, $\Sigma_i \setminus \bar{B}_R \to \Sigma'$ in $C^\infty_{\text{loc}}(\mathbb{R}^3 \setminus \bar{B}_R)$, where $\Sigma'$ is a $C^k,\alpha$-asymptotically conical self-expander in $\mathbb{R}^3 \setminus \bar{B}_R$, $\Sigma'$ consists of $e$ annuli and $\mathcal{L}(\Sigma') = \sigma$. In particular, for $r > R$ sufficiently large, $\partial B_r$ meets $\Sigma$ transversally and also meets each $\Sigma_i$ transversally. As such, $\Sigma_i \cap B_r$ has genus $g$ and $\Sigma_i \cap \partial B_r$ has $e$ components. Thus, it follows that there is a constant $C = C(g, e)$ so

$$\int_{\partial B_r \cap \Sigma_i} \kappa < C$$

where $\kappa$ denotes the geodesic curvature of the boundary curve. Moreover, by Lemma 3.8 for $\epsilon$ sufficiently large, $\lambda(\mathcal{C}(\Sigma_i)) \leq \lambda(\mathcal{C}(\sigma)) + 1$ and so, by Lemma 3.6 there is a $C'$ so

$$H^2(\Sigma_i \cap B_r) \leq C'.$$

Hence, it follows from \cite{24} Theorem 3 (4) that, up to passing to a further subsequence, $\Sigma_i \cap B_r \to \Sigma''$ in $C^\infty_{\text{loc}}(B_r)$ where $\Sigma''$ is a self-expander in $B_r$ and the convergence is with multiplicity one. It is clear that $\Sigma' = \Sigma''$ in $B_r \setminus \bar{B}_R$. Therefore the result follows with $\Sigma = \Sigma' \cup \Sigma''$. \hfill \Box

3.4. Smooth compactness of $\mathcal{E}_{n,\text{mc}}^k(h_0)$. We combine Lemma 3.6 with a curvature estimate for mean convex self-expanders in order to prove Item 2 of Theorem 1.1.

First we show a curvature estimate for strictly mean convex asymptotically conical self-expanders with strictly mean convex link. Our argument uses the maximum principle and is completely analogous to the one used in \cite{5} for mean convex self-shrinkers.

Lemma 3.9. Let $\Sigma \subset \mathbb{R}^{n+1}$, be an asymptotically conical self-expander. If $\Sigma$ is strictly mean convex and whose asymptotic cone, $\mathcal{C}(\Sigma)$ is $C^2$-regular and strictly mean convex, then for $p \in \Sigma$,

$$|A_{\Sigma}(p)|^2 \leq K|x(p)|^2$$

where

$$K = \frac{1}{4} \sup_{\mathcal{C}(\Sigma)} \frac{|A_{\mathcal{C}(\Sigma)}|^2}{H_{\mathcal{C}(\Sigma)}^2} = \frac{1}{4} \sup_{\mathcal{C}(\Sigma)} \frac{|A_{\mathcal{L}(\Sigma)}|^2}{H_{\mathcal{L}(\Sigma)}^2} < \infty.$$

Proof. By definition of asymptotic cones and scaling invariance

$$\lim_{R \to \infty} \sup_{\partial B_R \cap \Sigma} \frac{|A_{\Sigma}|^2}{H_{\Sigma}^2} = 4K.$$

Hence, for every $\epsilon > 0$, there is an $R_\epsilon$ so if $R > R_\epsilon$, then

$$\sup_{\partial B_R \cap \Sigma} \frac{|A_{\Sigma}|^2}{H_{\Sigma}^2} \leq 4K + \epsilon.$$
Computing as in [5] one has

\[ L_\Sigma H_\Sigma = \left( \Delta_\Sigma + \frac{x}{2} \cdot \nabla_\Sigma + |A_\Sigma|^2 - \frac{1}{2} \right) H_\Sigma = -H_\Sigma \]

which follows from a variant on Simons’ identity, for the rough drift Laplacian,

\[ (L_\Sigma A_\Sigma)_{ij} = -(A_\Sigma)_{ij}. \]

For details see the computations in (10.10)-(10.12) and (5.7)-(5.8) of [5]. These are carried out for self-shrinkers, however the formulas for self-expanders follow with a simple sign change.

Now consider the new operator

\[ \mathcal{L}_{H_\Sigma^2} = \Delta_\Sigma + \frac{x}{2} \cdot \nabla_\Sigma + 2(\nabla_\Sigma \log H_\Sigma) \cdot \nabla_\Sigma \]

We compute that

\[ (\mathcal{L}_{H_\Sigma^2} A_\Sigma)_{ij} = 0 \]

and so

\[ \mathcal{L}_{H_\Sigma^2} \frac{|A_\Sigma|^2}{H_\Sigma^2} = 2 \left| \nabla_\Sigma \frac{A_\Sigma}{H_\Sigma} \right|^2 \geq 0. \]

Hence, by the maximum principle, for any \( R > R_* \),

\[ \sup_{B_R \cap \Sigma} \frac{|A_\Sigma|^2}{H_\Sigma^2} \leq 4K + \epsilon. \]

Hence, letting \( \epsilon \to 0 \), gives

\[ \sup_{\Sigma} \frac{|A_\Sigma|^2}{H_\Sigma^2} \leq 4K. \]

That is, for any \( p \in \Sigma \),

\[ |A_\Sigma(p)|^2 \leq 4K H_\Sigma^2(p) \leq K(|x(p) \cdot \nu_\Sigma(p)|)^2 \leq K|x(p)|^2. \]

This proves the claim. \( \square \)

We are now ready to complete the proof of Theorem 1.1.

**Proof of Item 2 of Theorem 1.1** By Corollary 3.4 there is an \( R > 0 \) so that, up to passing to a subsequence,

\[ \Sigma_i \setminus \bar{B}_R \to \Sigma' \text{ in } C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \bar{B}_R), \]

where \( \Sigma' \) is a \( C^{k,\alpha} \)-asymptotically conical self-expander in \( \mathbb{R}^{n+1} \setminus \bar{B}_R \) and \( \mathcal{L}(\Sigma') = \sigma \). The nature of the convergence, ensures that \( H_\sigma \geq h_0 \). Moreover, by Lemma 3.8 for \( i \) sufficiently large, \( \lambda[C(\Sigma_i)] \leq \lambda[C[\sigma]] + 1 \). Thus, by Lemma 3.6 there is a uniform \( C' \) so

\[ \mathcal{H}^n(\Sigma_i \cap B_r) \leq C'. \]

As \( \sigma \) is strictly mean convex, for \( i \) sufficiently large each \( \mathcal{L}(\Sigma_i) \) is strictly mean convex. Indeed, setting

\[ K = \sup_{\sigma} \frac{|A_\sigma|^2}{H_\sigma^2} \in (0, \infty) \]

one has, after possibly throwing out a finite sequence of the \( \Sigma_i \), that

\[ \sup_{\mathcal{L}(\Sigma_i)} \frac{|A_{\mathcal{L}(\Sigma_i)}|^2}{H_{\mathcal{L}(\Sigma_i)}^2} \leq 4K. \]
Hence, by Lemma 3.9
$$| A_{\Sigma_i}(p) |^2 \leq K |x(p)|^2.$$  
That is,
$$\sup_{\Sigma_i \cap B_{2R}} | A_{\Sigma_i} |^2 \leq 4KR^2.$$  
Combining this with the area bound and the Arzelà-Ascoli theorem, gives that, up to passing to a subsequence, the $\Sigma_i$ converge, possibly with multiplicities, in $C_{\text{loc}}^{\infty}(B_{2R})$ to a limit $\Sigma''$. As $\Sigma''$ is a smooth solution to (1.1) and there are no closed self-expanders, each component of $\Sigma''$ meets $\Sigma'$. In particular, as the $\Sigma_i$ converge with multiplicity one to $\Sigma'$ in $B_{2R} \setminus B_R$, the $\Sigma_i$ converge to $\Sigma''$ with multiplicity one in $B_{2R}$. Hence, setting $\Sigma = \Sigma' \cup \Sigma''$ one obtains a smooth asymptotically conical self-expander with $\Sigma_i \to \Sigma$ in $C_{\text{loc}}^{\infty}(\mathbb{R}^{n+1})$. As each $\Sigma_i$ has positive mean curvature, $\Sigma$ has non-negative mean curvature. However, as $\sigma = \mathcal{L}(\Sigma)$ has $H_{\Sigma(\Sigma)} \geq h_0 > 0$, and $L_\Sigma H_\Sigma = -H_\Sigma \leq 0$, the strong maximum principle implies $\Sigma$ has positive mean curvature completing the proof. \hfill \Box

4. Properness of Map II

Before proving Theorems 1.2, 1.3 and 1.4 we need the following auxiliary proposition that relates sequential compactness in $ACE_{k,\alpha}^n(\Gamma)$ to locally smooth compactness in $\mathbb{R}^{n+1}$.

**Proposition 4.1.** For $\Gamma \in \mathcal{A}\mathcal{C}\mathcal{H}_{n,\alpha}^{k,\alpha}$, if $\varphi \in \mathcal{V}_{\text{emb}}^{k,\alpha}(\Gamma)$, and $[f_i] \in ACE_{k,\alpha}^n(\Gamma)$ satisfy:

1. $\text{tr}^1_{\infty}[f_i] = \varphi_i \to \varphi$ in $C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$;
2. $f_i(\Gamma) = \Sigma_i \to \Sigma$ in $C_{\text{loc}}^{\infty}(\mathbb{R}^{n+1})$ for some hypersurface $\Sigma$,

then $\Sigma \in \mathcal{A}\mathcal{C}\mathcal{H}_{n,\alpha}^{k,\alpha}$ and is a self-expander. Moreover, there is a parameterization $f : \Gamma \to \Sigma \subset \mathbb{R}^{n+1}$ so that

1. $[f] \in ACE_{n,\alpha}^{k,\alpha}(\Gamma)$;
2. $\text{tr}^1_{\infty}[f] = \varphi$; and;
3. $[f_i] \to [f]$ in the topology of $ACE_{n,\alpha}^{k,\alpha}(\Gamma)$.

**Proof.** First observe that as each $\Sigma_i$ satisfies (1.1), the nature of the convergence ensures that $\Sigma$ does as well. Let
$$C_i = \mathcal{E}_1^{\text{H}}[\varphi_i](\mathcal{C}(\Gamma)) = \mathcal{C}(\Sigma_i)$$  
and $C = \mathcal{E}_1^{\text{H}}[\varphi](\mathcal{C}(\Gamma))$. Then $C_i$ and $C$ are $C^{k,\alpha}$-regular cones. By our hypothesis (I),
$$C_i \to C$$  
in $C_{\text{loc}}^{k,\alpha}(\mathbb{R}^{n+1} \setminus \{0\})$, which further implies
$$\mathcal{L}(\Sigma_i) \to \mathcal{L}[C]$$  
in $C^{k,\alpha}(\mathbb{S}^n)$.  
Thus, by Corollary 3.4 and our hypothesis (2), we have $\Sigma \in \mathcal{A}\mathcal{C}\mathcal{H}_{n,\alpha}^{k,\alpha}$ with $\mathcal{C}(\Sigma) = C$.  
As $\varphi \in \mathcal{V}_{\text{emb}}^{k,\alpha}(\Gamma)$, the hypothesis (II) ensures that
$$\mathcal{E}_1^{\text{H}}[\varphi_i] \circ \mathcal{E}_1^{\text{H}}[\varphi]^{-1} \to x|_C$$  
in $C_{\text{loc}}^{k,\alpha}(C; \mathbb{R}^{n+1})$.

Observe, that $\mathcal{E}_1^{\text{H}}[\varphi_i] \circ \mathcal{E}_1^{\text{H}}[\varphi]^{-1}$ are homogeneous of degree one, so we denote their traces at infinity by $\tilde{\varphi}_i$. Thus, letting $\mathcal{C}$ be the link of $C$,
$$\tilde{\varphi}_i \to x|_\mathcal{C}$$  
in $C^{k,\alpha}(\mathcal{L}; \mathbb{R}^{n+1})$.

Let $h_i \in C_{1,\alpha}^{k,\alpha} \cap C_{1,\alpha}^{k}(\Sigma; \mathbb{R}^{n+1})$ be chosen so that
$$\mathcal{L}_\Sigma h_i = \Delta h_i + \frac{1}{2} x \cdot \nabla_\Sigma h_i - \frac{1}{2} h_i = 0$$  
and $\text{tr}^1_{\infty}[h_i] = \tilde{\varphi}_i - x|_\mathcal{C}$.  

By [2] Corollary 5.8 there is a unique such \( h_i \) and it satisfies the estimate

\[
\|h_i\|^{(1)}_{k,\alpha} \leq C' \|\tilde{\varphi}_i - x\|_{k,\alpha}
\]

for some constant \( C' = C'(\Sigma, n, k, \alpha) \). We let

\[
g_i = x|\Sigma| + h_i \text{ and } \Upsilon_i = g_i(\Sigma).
\]

It is clear that for \( i \) sufficiently large, \( g_i \in ACH_{n}^{k,\alpha}(\Sigma) \) and \( \text{tr}^1[g_i] = \tilde{\varphi}_i \). Thus, by [2] Proposition 3.3, \( \Upsilon_i \in ACH_{n}^{k,\alpha} \) and \( C(\Upsilon_i) = C_i \).

Now pick a transverse section \( v \) on \( \Sigma \) that, outside a compact set, equals \( \sigma_w[w] = w \circ \pi_w|\Sigma \) for \( w \) a chosen homogeneous transverse section on \( C \). For \( i \) sufficiently large, \( v_i = v \circ g_i^{-1} \) is an asymptotically homogeneous transverse section on \( \Upsilon_i \). By Proposition 3.3 for \( i \) large, \( \Sigma_i \) lies in a \( v_i \)-regular neighborhood of \( \Upsilon_i \) and is transverse to \( \sigma_{v_i}[v_i] = v_i \circ \pi_{v_i} \). In particular, \( \pi_{v_i}|\Sigma_i : \Sigma_i \to \Upsilon_i \) is an element of \( ACH_{n}^{k,\alpha}(\Sigma_i) \). Thus, for \( i \) large, there is a unique \( u_i \in C_{1}^{k,\alpha} \cap C_{1,0}^{k,\alpha}(\Sigma) \) so that \( \Sigma_i \) can be parametrized by the map

\[
\tilde{f}_i = (\pi_{v_i}|\Sigma_i)^{-1} \circ g_i = g_i + u_i v
\]

which is an element of \( ACH_{n}^{k,\alpha}(\Sigma) \) by [2] Proposition 3.3. Moreover, \( \|u_i\|^{(1)}_{k,\alpha} \) is uniformly bounded and \( \|u_i\|^{(1)}_{k,\alpha} \to 0 \).

Observe, that there is a \( \delta > 0 \) (independent of \( i \)) so that for \( i \) sufficiently large

\[
\inf_{p \in \Sigma} |v \cdot (n_{\Sigma_i} \circ \tilde{f}_i)| > \delta.
\]

As \( \Sigma_i \) is a self-expander, it follows from [2] Lemma 7.2] and direct calculations that

\[
\mathcal{L}_\Sigma u_i = - \frac{\mathbf{n}_{\Sigma_i} \circ \tilde{f}_i}{\mathbf{v} \cdot (n_{\Sigma_i} \circ \tilde{f}_i)} \left( 2\nabla_\Sigma u_i \cdot \nabla_\Sigma v + u_i \left( \mathcal{L}_\Sigma + \frac{1}{2} \right) v + (g^{-1}_i - g^{-1}_\Sigma)^{ij} (\nabla^j_{\Sigma} \tilde{f}_i)^{ji} \right)
\]

where \( g_{\tilde{f}} \) and \( g_{\Sigma} \) are the pull-back metrics of Euclidean one by \( \tilde{f}_i \) and \( x|\Sigma_i \), respectively.

One further uses [2] Proposition 3.1 to see that, for \( i \) large, the right hand side are elements of \( C_{n}^{k,\alpha}(\Sigma) \) with uniformly bounded \( \|\cdot\|_{k,\alpha} \) norm. Hence, by [2] Theorem 5.7 and Corollary 5.8, \( u_i \in D^{k,\alpha}(\Sigma) \) and \( \|u_i\|_{k,\alpha} \) is uniformly bounded. Here

\[
D^{k,\alpha}(\Sigma) = \left\{ u \in C_{1}^{k,\alpha} \cap C_{0}^{k,\alpha} \cap C_{-1}^{k,\alpha}(\Sigma) : \nabla_\Sigma u \in C_{-1}^{k,\alpha}(\Sigma) \right\}
\]

is a Banach space with norm

\[
\|u\|_{k,\alpha} = \|u\|^{(1)}_{k,\alpha} + \sum_{k-1 \leq i \leq k} \|\nabla^i_\Sigma u\|_{\alpha}^{(1-k)} + \|x \cdot \nabla_\Sigma u\|_{k-2,\alpha}^{(1-k)}.
\]

As \( D^{k,\alpha}(\Sigma) \) is compactly embedded in \( C_{-1}^{k,\alpha}(\Sigma) \), we have \( \|u_i\|^{(1)}_{k,\alpha} \to 0 \). Thus it follows from [2] Lemma 7.5 that \( \|u_i\|^{(1)}_{k-1,\alpha} \to 0 \) and so \( \|\tilde{f}_i - x|\Sigma_i\|^{(1)}_{k,\alpha} \to 0 \).

We pick a large integer \( I \) so that \( \tilde{f}_I \) is well-defined as above. Choose a representative \( f_I \) of \( \{f_I\} \). We define \( f = f_{I}^{-1} \circ f_I \), and it is clear that \( f(\Gamma) = \Sigma \). Moreover, by [2] Proposition 3.3, \( f \in ACH_{n}^{k,\alpha}(\Gamma) \) and

\[
C[f] = C[\tilde{f}_I]^{-1} \circ C[f_I] = (\sigma^H_1[\varphi_I] \circ \sigma^H_1[\varphi_I]^{-1} \circ \sigma^H_1[\varphi_I] = \sigma^H_1[\varphi].
\]

Thus, \( [f] \) represents a class in \( AC^{k,\alpha}_n(\Gamma) \) which has \( \Pi([f]) = \text{tr}^1_{\infty}[f] = \varphi \).
Hence, to complete the proof it remains only to show that \([f_i] \to [f]\) in the topology of \(ACE^{k,\alpha}_n(\Gamma)\). Observe that \(\tilde{f}_i \circ f(\Gamma) = \Sigma_i\), and invoking [2] Proposition 3.3 again, \(\tilde{f}_i \circ f \in ACE^{k,\alpha}_n(\Gamma)\) and

\[
C[\tilde{f}_i \circ f] = C[\tilde{f}_i] \circ C[f] = (\delta_1^{H}[\varphi_i] \circ \delta_1^{H}[\varphi]^{-1}) \circ \delta_1^{H}[\varphi] = \delta_1^{H}[\varphi_i].
\]

This gives that \(\tilde{f}_i \circ f\) is an element of \([f_i]\). Moreover, by [2] Proposition 3.1, \(\tilde{f}_i \circ f \to f\) in \(C^{k,\alpha}(\Gamma;\mathbb{R}^{n+1})\). Therefore, \([f_i] \to [f]\) in the topology of \(ACE^{k,\alpha}_n(\Gamma)\). \(\square\)

The proofs of properness of \(\Pi\) now follow easily.

**Proof of Theorem 1.2** The result follows directly from Item (1) of Theorem 1.1 Proposition 4.1 and an elementary topology fact, Lemma A.1. \(\square\)

**Proof of Theorem 1.3** First, by Lemma 3.8 \(V_{ext}(\Gamma, \Lambda)\) is open in \(C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})\). Next, by Lemma 3.8

\[
\lambda([f(\Gamma)]) = \lambda([C[\sigma]]) \text{ where } \sigma = \Pi([f])|L(\Gamma)).
\]

Thus, the continuity of \(\Pi\) and Lemma 3.8 imply that \(U_{ext}(\Gamma, \Lambda)\) is open in \(ACE^{k,\alpha}_n(\Gamma)\). If \(Z \subset V(\Gamma, \Lambda)\) is compact, then it follows from Lemma 3.8 that there is a \(\Lambda_0 < \Lambda\) so that

\[
\lambda(\delta^{H}_1[\varphi]|C(\Gamma)) \leq \Lambda_0
\]

for all \(\varphi \in Z\). Hence, if \(\tau_{n,\Lambda}\) holds for some \(\Lambda < 2\), then the last claim follows from Item (3) of Theorem 1.1 Proposition 4.1 and Lemma A.1. \(\square\)

**Proof of Theorem 1.4** Items (1) and (2) are straightforward. For \([f] \in U_{mc}(\Gamma)\) let \(\Sigma = f(\Gamma)\). As \(H_{\Sigma} \geq 0\) and \(L_{\Sigma}H_{\Sigma} = -H_{\Sigma} \leq 0\), \(L_{\Sigma}\) has a positive super-solution. Hence, it follows from the trick of Fischer-Colbrie and Schoen [9] that \(\Sigma\) is strictly stable. In particular, \(\Sigma\) admits no non-trivial Jacobi fields and so Item (3) follows from [2] Theorem 1.1 (4)]. Furthermore, if \(Z \subset V_{mc}(\Gamma)\) is compact, then there is an \(h_0 > 0\) so \(H_{\varphi} \geq h_0\), where \(\varphi\) is the link of the cone \(\delta^{H}_1[\varphi]|C(\Gamma))\), for all \(\varphi \in Z\). As such, Item (4) follows from Item (2) of Theorem 1.1 Proposition 4.1 and Lemma A.1.

It remains only to show the final remark. Observe, that by Item (4) and that \(V_{mc}(\Gamma)\) is a compactly generated Hausdorff space, the map \(\Pi|_{U_{mc}(\Gamma)}\) is a closed map. Hence, following the arguments in [9] Proposition 4.46, the final remark is an immediate consequence of Items (3) and (4). \(\square\)

5. Existence of mean convex asymptotically conical self-expanders

We conclude by proving Corollary 1.5. We first show existence and uniqueness of mean convex self-expanders asymptotic to rotationally symmetric cones. This result is not new (see [1][2]), but we include a proof for the sake of completeness.

**Proposition 5.1.** For \(n \geq 2\) let \(C \subset \mathbb{R}^{n+1}\) be a connected non-flat rotationally symmetric cone. There is a unique smooth self-expander \(\Sigma\) that is smoothly asymptotic to \(C\). Moreover, \(\Sigma\) satisfies \(H_{\Sigma} > 0\) and is an entire graph and so is diffeomorphic to \(\mathbb{R}^n\).

**Proof.** Without loss of generality we assume that \(e_{n+1}\) is the axis of symmetry of \(C\) and \(C\) lies in the half-space \(\{x_{n+1} \geq 0\}\). First, we show the existence of a strictly mean convex self-expanders asymptotic to \(C\). Clearly, there is a \(\tau > 0\) so \(C\) is the graph of a Lipschitz function \(u_0: \mathbb{R}^n \to \mathbb{R}\) given by \(u_0(\tau|x|) = \tau|x|\). Let \(u_\tau(x) = \tau \sqrt{i^{n-1} + |x|^2}\) so \(u_\tau: \mathbb{R}^n \to \mathbb{R}\) is a sequence of smooth functions which are strictly convex, have Lipschitz constant at most \(\tau\) and are asymptotic to \(u_0\). Moreover, the \(u_\tau\) converge to \(u_0\) uniformly. Let \(\Sigma^i \subset \mathbb{R}^{n+1}\) be the graphs of the \(u_\tau\). These are smooth convex hypersurfaces that are asymptotic to \(C\). Thus, Ecker-Huisken [7] and the strong maximum principle applied to
the evolution equation for mean curvature \[8\] imply that there is a unique strictly mean convex, graphical solution, \(\mathcal{M}^i = \{\Sigma^i_t\}_{t \geq 0}\) to mean curvature flow starting from \(\Sigma^i\). Moreover, if we denote by \(u^i(t, \cdot)\) the functions whose graphs are the \(\Sigma^i_t\), then the \(u^i(t, \cdot)\) have uniformly bounded Lipschitz constant.

Hence, by Brakke’s compactness (cf. \[14, \S 7\]), the \(\mathcal{M}^i\) converges as Brakke flows to \(\mathcal{M} = \{\Sigma_t\}_{t \geq 0}\) with \(\Sigma_0 = C\). Furthermore, the interior estimates for graphical mean curvature flow \[8\] and the Arzelà-Ascoli theorem imply that for \(t > 0\), \(\Sigma_t\) has nonnegative mean curvature and is given by the graph of a smooth function \(u(t, \cdot)\) on \(\mathbb{R}^n\) with a uniform Lipschitz constant. Moreover, as \(t \to 0^+\), \(\Sigma_t \to C\) in \(C^\infty_{\text{loc}}(\mathbb{R}^{n+1} \setminus \{0\})\). By the parabolic maximum principle, such a solution \(u(t, \cdot)\) for \(t > 0\) is unique in the class of functions with at most linear growth. A consequence of this is that \(u = \sigma\) of Grayson \[11\] and Zhu \[30, Corollary 4.2\] give that \(u\) with at most linear growth. A consequence of this is that \(\Sigma = \Sigma_0\) such, the maximum principle implies that \(\sigma\) to a contradiction with the strong maximum principle. By similar arguments, \(\Sigma^i\) is another smooth self-expander smoothly asymptotic to \(C\). Let \(\Sigma^i = \Sigma\) proving the uniqueness.

**Proof of Corollary 1.5.** Let \(\{\sigma_i\}_{t \in [0, T]}\) be the solution of mean curvature flow in \(\mathbb{R}^n\) with \(\sigma_0 = \sigma\) and \(T\) the maximal time of existence. Pick a parameterization \(\tilde{\varphi}_0: \tilde{\sigma} \to \sigma\) and \(\varphi_1: \sigma \to \sigma_1\) a corresponding evolution of parameterizations. As \(\sigma_0\) is mean convex, the maximum principle implies that \(\sigma_1\) is as well for all \(t \in (0, T)\). When \(n = 2\), results of Grayson \[11\] and Zhu \[30, Corollary 4.2\] give that \(\sigma_t\) smoothly shrinks to a round point in finite time. When \(n \geq 3\), by \[12\], the condition \(\{12\}\) is preserved, which, together with the mean convexity and connectedness, implies that \(T < \infty\) and the flow disappears in a round point at time \(T\).

Choose a rotationally symmetric hypersurface \(\tilde{\sigma} \subset \mathbb{S}^n\) and a \(T_0\) sufficiently close to \(T\), so \(\sigma_{T_0}\) is a small normal graph over \(\tilde{\sigma}\). Thus, there is a path of smooth, strictly mean convex embeddings, \(\psi_t: \tilde{\sigma} \to \mathbb{S}^n, t \in [0, 1]\), so \(\psi_0(\tilde{\sigma}) = \sigma_{T_0}\) and \(\psi_1 = x|_{\tilde{\sigma}}\). Thus, setting

\[
\tilde{\varphi}_t = \begin{cases} 
\varphi_t \circ \varphi_{T_0}^{-1} \circ \psi_0 
& 0 \leq t \leq T_0 \\
\psi_{t-T_0} 
& T_0 < t \leq T_0 + 1
\end{cases}
\]

one obtains a path of strictly mean convex \(C^{k, \alpha}\)-embeddings of \(\tilde{\sigma}\) into \(\mathbb{S}^n\) connecting \(\tilde{\varphi}_{T_0+1} = x|_{\tilde{\sigma}}\) to \(\tilde{\varphi}_0: \tilde{\sigma} \to \sigma\). By Proposition 5.1 there is a unique smooth self-expander \(\Gamma\) that is smoothly asymptotic to \(C|_{\tilde{\sigma}}\) and \(H_{\Gamma} > 0\). Hence, Theorem 4.1 implies there is an \(f_0 \in \mathcal{U}_{\text{inc}}(\Gamma)\) with \(\Pi(f_0) = \tilde{\varphi}_0\) and so \(\Sigma = f_0(\Gamma)\) is the desired element.

In what follows we restrict our discussion to those \(n\) such that \(\pi_0(\text{Diff}^+(\mathbb{S}^{n-1})) = 0\). For \(n \geq 3\) we let

\[\mathcal{V} = \{ \varphi \in \mathcal{V}_{\text{inc}}(\Gamma) : \mathcal{L}[\varphi |_{\text{inc}}(\Gamma)] \text{ satisfies } \{12\} \},\]

which is an open subset of \(\mathcal{V}_{\text{inc}}(\Gamma)\); for \(n = 2\) let \(\mathcal{V} = \mathcal{V}_{\text{inc}}(\Gamma)\). By the previous discussions and the hypothesis that \(\text{Diff}^+(\mathbb{S}^{n-1}) \simeq \text{Diff}^+(\mathbb{S}^{n-1})\) is path-connected, we observe \(\mathcal{V}\) has exactly two components. Let \(\mathcal{V}_+\) be the component of \(\mathcal{V}\) that contains \(x|_{\mathcal{L}(\Gamma)}\) and
let $\mathcal{V} = \{U_i\}$ be the component containing $x|_{\mathcal{L}(\Gamma)} \circ I$ where $I \in \text{Diff}(\mathcal{L}(\Gamma))$ is an orientation-reversing involution. If $\mathcal{U}_k = \mathcal{U}_{\text{misc}}(\Gamma) \cap \Pi^{-1}(\mathcal{V})$, then it follows from Theorem 1.4 and the uniqueness of $\Gamma$ that $\Pi|_{\mathcal{U}_k}: \mathcal{U}_k \to \mathcal{V}_k$ is a diffeomorphism.

Now suppose $\varphi, \psi \in \mathcal{V}_+$ so that $\varphi^H(\varphi)(\mathcal{C}(\Gamma)) = \psi^H(\psi)(\mathcal{C}(\Gamma))$. If $[f], [g] \in \mathcal{U}_+$ with $\Pi([f]) = \varphi$ and $\Pi([g]) = \psi$, then we will show $f(\Gamma) = g(\Gamma)$. Observe that there is a path of homogeneous $\Phi_\lambda \in \text{Diff}^1(\mathcal{C}(\Gamma))$ with $\Phi_0 = \varphi^H(\varphi)^{-1} \circ \varphi^H(\psi)$ and $\Phi_1 = x|_{\mathcal{C}(\Gamma)}$. Thus, Theorem 1.4 and the uniqueness of $\Gamma$ imply that there is a path of $[h_\lambda] \in \mathcal{AC}^{\infty, \alpha}(\Gamma)$ so that $h_\lambda(\Gamma) = \Gamma$ and $\Pi([h_\lambda]) = \text{tr}_{1}\Phi_\lambda$. In particular, $[f \circ h_0] \in \mathcal{U}_+$ and $\Pi([f \circ h_0]) = \psi$. Thus, $\Pi|_{\mathcal{U}_+}$ is a diffeomorphism, $[g] = [f \circ h_0]$ and so $g(\Gamma) = f(\Gamma)$. By symmetries, the same claim holds on $\mathcal{V}_-$. Finally, observe that as $\Gamma$ is diffeomorphic to $\mathbb{R}^n$ there is an orientation-reversing diffeomorphism $I \in \mathcal{AC}^{\infty, \alpha}(\Gamma)$ so that $\text{tr}_{1}\Phi_\lambda[I] = I$. Moreover, $[f] \in \mathcal{U}_+$ if and only if $[f \circ I] \in \mathcal{U}_+$. This concludes the proof of uniqueness.

\section{Appendix A.}

\textbf{Lemma A.1.} Let $X$ be a topological space. Suppose $X$ has a countable cover, $\{A_i\}_{i \in \mathbb{N}}$, of closed subsets so that each $A_i$ is metrizable. If $K$ is a sequentially compact subspace of $X$, then it is compact.

\textbf{Proof.} Let $\{U_\alpha\}$ be an arbitrary collection of open sets of $X$ that covers $K$. As $K$ is sequentially compact, so is every $K \cap A_i$. Since $A_i$ is metrizable, there is a finite subcollection of $\{U_\alpha\}$ that covers $K \cap A_i$. Thus, there is a countable subcollection $\{U_{\beta}\}_{\beta \in \mathbb{N}} \subset \{U_\alpha\}$ that covers $K$. We show there is a finite subcollection of $\{U_{\alpha}\}_{\alpha \in \mathbb{N}}$ that covers $K$, implying the compactness of $K$. We argue by contradiction. Suppose not, then pick $x_j \in K \setminus \bigcup_{\alpha \in \mathbb{N}} U_{\alpha}$. Thus, up to passing to a subsequence, $x_j \to x \in K$. However, $x \in U_{\alpha_{i_0}}$ for some $i_0$, so for $j$ sufficiently large $x_j \in U_{\alpha_{i_0}}$, which leads to a contradiction. \hfill $\Box$

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