Mutually unbiased bases: tomography of spin states and the star-product scheme

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Abstract
Mutually unbiased bases (MUBs) are considered within the framework of a generic star-product scheme. We rederive that a full set of MUBs is adequate for a spin tomography, i.e. knowledge of all probabilities to find a system in each MUB-state is enough for a state reconstruction. Extending the ideas of the tomographic-probability representation and the star-product scheme to MUB tomography, dequantizer and quantizer operators for MUB symbols of spin states and operators are introduced, ordinary and dual star-product kernels are found. Since MUB projectors are to obey specific rules of the star-product scheme, we reveal the Lie algebraic structure of MUB projectors and derive new relations on triple- and four-products of MUB projectors. An example of qubits is considered in detail. MUB tomography by means of the Stern–Gerlach apparatus is discussed.

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1. Introduction

Since the early days of quantum mechanics, much attention has been paid to the problem of a good description of quantum states. The notions of the wave function $\psi$ and the density matrix $\rho$ are very widely known and used. Nevertheless, these notions give rise to the problem of interpretation, especially in the case of measuring a quantum state. Outcomes of quantum observables are known to be probabilistic. In view of this, quasiprobability distribution functions such as the Wigner $W$-function [1], the Sudarshan–Glauber $P$-function [2, 3] and the Husimi $Q$-function [4] are often used in quantum optics along with the wave function and density matrix formalism. The main drawback of $W$, $P$- and $Q$-functions is that they cannot be measured experimentally. The problem of measuring quantum states resulted in the development of quantum tomography and then in a tomographic-probability representation of quantum mechanics (the historical background is given in the review [5]). According to such a representation, any quantum state of light is described by measurable tomograms: optical, symplectic and photon-number ones (see e.g. the review [6]). As far as a finite-dimensional Hilbert space is concerned, one can alternatively utilize a spin tomogram [7, 8] and a spin tomogram with a finite number of rotations [9]. Apart from being appropriate for reconstructing the density matrix, quantum tomograms are themselves notions of quantum states. Within the framework of the tomographic-probability representation, operators are described by tomographic symbols satisfying rules of the corresponding star-product scheme. We cannot help mentioning some of the probability-based approaches to quantum mechanics, namely the expectation-value representation [10, 11] and the Bayesian interpretation [12, 13] utilizing a symmetric informationally complete positive operator-valued measure (SIC-POVMs; discussed, e.g., in [14–16]).

The aim of this paper is to develop the star-product [17] quantization scheme based on mutually unbiased bases (MUBs) [18, 19]. MUBs themselves represent a highly symmetrical structure and have many interesting properties (see e.g. [20] and references therein). For example, a full set of MUBs is known to exist whenever the dimension of Hilbert space is a prime number or the power of a prime. We consider neither the problem of existence of MUBs in a given Hilbert space nor the problem of how many MUBs there exist. We assume that the full set of MUBs is known for the space involved. In this paper, we combine MUBs with the tomographic-probability representation. As a result, MUB tomography of spin states is introduced, MUB symbols of
quantum operators are considered within the framework of the star-product scheme, the Lie algebraic structure of MUBs is pointed out and new properties of MUB projectors are derived. Special attention is given to qubits.

The paper is organized as follows. In section 2, MUBs are briefly reviewed. In section 3, MUB-based tomography is considered, and scanning and reconstruction procedures are presented. In section 4, we follow the ideas of a generic star-product scheme [21, 22] and analyze the star product of MUB symbols. In section 5, an example of qubits is considered in detail. In section 6, practical realization of MUB tomography by means of the Stern–Gerlach apparatus is discussed. In section 7, conclusions and prospects are presented.

2. MUBs

Let us consider a d-dimensional Hilbert space $\mathcal{H}_d$ endowed with a full set of MUBs. If this is the case, MUBs consist of $d+1$ bases $\{|a\alpha\rangle\}_{a=0}^{d-1}$, where $a = 0, \ldots, d$ is responsible for the basis number, and index $\alpha = 0, \ldots, d - 1$ refers to one of the basis states belonging to the particular basis $a$. MUBs are to satisfy the following property:

$$\langle a\alpha | b\beta \rangle^2 = \frac{1}{d} (1 - \delta_{a,b}) + \delta_{a,b} \delta_{a,\beta}, \quad (1)$$

where $\delta_{a,b}$ is a Kronecker delta symbol. Equation (1) implies that each basis is orthonormal, and arbitrary two states belonging to different bases are equiangular, i.e., $\langle a\alpha | b\beta \rangle^2 = \frac{1}{d}$ if $a \neq b$.

Let us now consider rank-1 MUB projectors $\hat{\Pi}_{a\alpha} = |a\alpha\rangle \langle a\alpha|$. Obviously, operators $\hat{\Pi}_{a\alpha}$ are semi-positive and satisfy the trace relation $\text{Tr}[\hat{\Pi}_{a\alpha} \hat{\Pi}_{b\beta}] = \frac{1}{d} (1 - \delta_{a,b}) + \delta_{a,b} \delta_{a,\beta}$. An immediate consequence of orthonormality is

$$\sum_{a=0}^{d-1} \hat{\Pi}_{a\alpha} = \hat{I} \quad \text{for all } a = 0, \ldots, d, \quad (2)$$

$$\sum_{a=0}^{d-1} \sum_{a'=0}^{d-1} \hat{\Pi}_{a\alpha} = (d+1) \hat{I}, \quad (3)$$

where $\hat{I}$ is the identity operator. Since the relation (2) is valid for all $a = 0, \ldots, d$, the total number of linearly independent operators $\hat{\Pi}_{a\alpha}$ equals $1 + (d+1)(d-1) = d^2$. In fact, for each fixed $a$ operators $\{\hat{\Pi}_{a\alpha}\}_{\alpha=0}^{d-1}$ are orthogonal in the sense that $\text{Tr}[\hat{\Pi}_{a\alpha} \hat{\Pi}_{b\beta}] = \delta_{a,b}$. This implies a linear independence of $\{\hat{\Pi}_{a\alpha}\}_{a=0}^{d-1}$ and, equivalently, the linear independence of the set of operators comprising $\hat{I}$ and $\{\hat{\Pi}_{a\alpha} = \frac{1}{d+1} \hat{I}_{a=0}\}$, which is a $(d-1)$-dimensional subspace of traceless operators acting on $\mathcal{H}_d$. From (1) it follows that two subspaces $\mathcal{S}_a$ and $\mathcal{S}_{a'}$, $a \neq a'$, are orthogonal (in trace sense) and do not have common nonzero elements. Hence, the space of operators acting on $\mathcal{H}_d$ is $\{\hat{I}\} \oplus \mathcal{S}_{a=0} \oplus \cdots \oplus \mathcal{S}_{a=d}$. This means that the identity operator $\hat{I}$ together with $d^2 - 1$ operators $\{\hat{\Pi}_{a\alpha}\}$, $a = 0, \ldots, d$, $\alpha = 0, \ldots, d - 2$ form a basis of operators acting on $d$-dimensional Hilbert space. As a result, any operator including the density operator $\hat{\rho}$ of a quantum state can be resolved through these basis operators. Indeed,

$$\hat{\rho} = c_1 \hat{I} + \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} c_{b\beta} \hat{\Pi}_{b\beta}, \quad (4)$$

where $c_1$ and $c_{b\beta}$ are real parameters because both $\hat{\rho}$ and $\hat{\Pi}_{b\beta}$ are Hermitian. Applying the trace operation to both sides of (4) and utilizing $\text{Tr}[\hat{\rho}] = \text{Tr}[\hat{\Pi}_{b\beta}] = 1$, we readily obtain

$$c_1 = \frac{1}{d} \left[ 1 - \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} c_{b\beta} \right], \quad (5)$$

$$\hat{\rho} = \frac{1}{d} \hat{I} + \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} c_{b\beta} \left( \hat{\Pi}_{b\beta} - \frac{1}{d} \hat{I} \right). \quad (6)$$

It is worth mentioning that the latter expansion has a form similar to an expansion through mutually orthogonal (in trace sense) unitary operators called generators of the MUB [23].

Taking into account the non-negativity of operators $\hat{\Pi}_{a\alpha}$ and the sum rule (3), it is not hard to see that operators $\hat{E}_{a\alpha} = (d+1)^{-1} \hat{\Pi}_{a\alpha}$ altogether form a POVM. We will refer to such a POVM as a MUB-POVM.

3. MUB tomography

In an experiment, probabilities of measurement outcomes are only accessible. Tomography is a procedure allowing one to reconstruct the density operator $\hat{\rho}$ with the help of measured probabilities. We will consider projective (von Neumann) measurements associated with MUBs. In other words, we assume that the probabilities

$$p_{a\alpha} = \langle a\alpha | \hat{\rho} | a\alpha \rangle = \text{Tr}[\hat{\rho} \hat{\Pi}_{a\alpha}] \quad (7)$$

are known for all $a = 0, \ldots, d$, $\alpha = 0, \ldots, d - 1$. As a consequence of expressions (2)–(3), we obtain the following normalization conditions:

$$\sum_{a=0}^{d-1} p_{a\alpha} = 1, \quad \sum_{a=0}^{d-1} \sum_{\alpha=0}^{d-1} p_{a\alpha} = d + 1. \quad (8)$$

The physical meaning of $p_{a\alpha}$ is the probability to find a system in the state $|a\alpha\rangle$ that is itself an element of MUBs. The problem is to express the density operator $\hat{\rho}$ through probabilities $p_{a\alpha}$. This problem is solved in [24]. For the sake of the subsequent consideration, we rewrite the result and present it in a slightly different manner.

**Proposition** A reconstruction procedure of MUB tomography reads

$$\hat{\rho} = \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} p_{b\beta} \left( \hat{\Pi}_{b\beta} - \frac{1}{d+1} \hat{I} \right). \quad (9)$$

**Proof.** Multiplying both sides of equation (6) by $\hat{\Pi}_{b\beta}$ and applying the trace operation, we obtain

$$p_{b\beta} = \frac{1}{d} + \sum_{a=0}^{d-1} \sum_{\alpha=0}^{d-2} M_{a\alpha, b\beta} c_{b\beta}. \quad (10)$$
The right-hand equality in (23) from which it is not hard to find an explicit solution of system (10) is nothing else but a linear system of equations with respect to $c_k$, and can be rewritten in matrix form as follows:

$$
\begin{pmatrix}
 p_0 - 1/d \\
 \vdots \\
 p_{d^2+d-1} - 1/d 
\end{pmatrix} = \mathbf{M} \begin{pmatrix}
 c_0 \\
 \vdots \\
 c_{d^2+d-1} 
\end{pmatrix},
\tag{11}
$$

where $\mathbf{M}$ is a $(d^2 - 1) \times (d^2 - 1)$ block-diagonal matrix of the form

$$
\mathbf{M} = \begin{pmatrix}
 \mathbf{M} & \mathbf{0} \\
 \mathbf{0} & \mathbf{M} 
\end{pmatrix},
$$

with each $(d - 1) \times (d - 1)$ block $\mathbf{M}$ being equal to

$$
\mathbf{M} = \frac{1}{d} \begin{pmatrix}
 d - 1 & -1 & \cdots & -1 \\
 -1 & d - 1 & \cdots & -1 \\
 \vdots & \vdots & \ddots & \vdots \\
 -1 & -1 & \cdots & d - 1
\end{pmatrix}.
\tag{13}
$$

A direct calculation of the inverse matrix yields

$$
\mathbf{M}^{-1} = \begin{pmatrix}
 2 & 1 & 1 & \cdots & 1 \\
 1 & 2 & 1 & \cdots & 1 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 1 & 1 & 2 & \cdots & 1
\end{pmatrix},
$$

from which it is not hard to find an explicit solution of system (10). The result is

$$
c_{b^d} = p_{b^d} - \frac{1}{d} \sum_{\beta=0}^{d-2} \left( p_{b^d} - \frac{1}{d} \right) = p_{b^d} - p_{b,d-1}.
\tag{15}
$$

The right-hand equality in (15) is due to the normalization condition (8). Substituting the obtained value of $c_{b^d}$ in (6) and making use of (3) and (8), we finally have

$$
\hat{\rho} = \frac{1}{d(d+1)} \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-1} p_{b^d} \hat{\rho}^b + \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} \left( p_{b^d} - p_{b,d-1} \right) \left( \hat{\rho}^b \right) \left( 1 - \hat{\rho}^b \right) 
$$

$$
= \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} p_{b^d} \left( \hat{\rho}^b - \frac{1}{d+1} \hat{\rho}^b \right) - \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} \left( p_{b^d} - p_{b,d-1} \right) \left( 1 - \hat{\rho}^b \right) 
$$

$$
= \sum_{b=0}^{d-1} p_{b,d-1} \left( \hat{\rho}^b - \frac{1}{d+1} \hat{\rho}^b \right) 
$$

$$
= \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} p_{b^d} \left( \hat{\rho}^b - \frac{1}{d+1} \hat{\rho}^b \right) 
$$

$$
= \sum_{b=0}^{d-1} \sum_{\beta=0}^{d-2} \left( p_{b^d} - p_{b,d-1} \right) \left( 1 - \hat{\rho}^b \right) 
\tag{16}
$$

This completes the proof of the proposition.

Note that reconstruction formula (9) is not unique and many alternative expressions can be found, but we will use this formula in view of its symmetry. Also, an immediate consequence of formula (9) is that if MUBs exist then any qudit state can be represented by a single probability distribution $(d+1)^{-1} p_{0}$.

From this point of view, MUB-based representation of qudit states is a partial case of an inverse spin-$s$ portrait method [9] with an extra requirement on the symmetry.

### 4. Star-product scheme

Any operator $\hat{A}$ acting on Hilbert space of quantum states can be alternatively described by a symbol $f_A(x)$, which is a function of a particular set of variables $x$. The relation between $\hat{A}$ and $f_A(x)$ is defined through

$$
f_A(x) = \text{Tr}[\hat{A} \hat{U}(x)],
$$

where $\hat{U}(x)$ and $\hat{D}(x)$ are dequantizer and quantizer operators, respectively; an explicit form of the sign of integration $\int dx$ depends on the scheme used. It is worth noting that substituting the second equality (17) for $\hat{A}$ in the definition of symbol of the operator, we readily find that a function $D(x_1, x) = \text{Tr}[\hat{D}(x_1) \hat{U}(x)]$ has the sense of a delta-function on symbols, i.e.

$$
\int dx_1 \int dx_2 f_A(x_1) f_B(x_2) K(x_1, x_2, x),
$$

where the star denotes a star product of symbols defined through

$$
(f_A \star f_B)(x) = \int dx_1 dx_2 f_A(x_1) f_B(x_2) K(x_1, x_2, x),
$$

The term $K(x_1, x_2, x)$ is usually referred to as star-product kernel [21, 22]. As the star product is associative, from the relation $f_A \star f_B \star f_C = f_A \star (f_B \star f_C) = (f_A \star f_B) \star f_C$ we readily obtain a star-product kernel of three symbols $K^{(3)}(x_1, x_2, x_3, x)$ as well as an additional requirement on the star-product kernel

$$
K^{(3)}(x_1, x_2, x_3, x) = \int dy K(x_1, x_2, y) K(y, x_3, x),
$$

which is valid for all sets of variables $x_1, x_2, x_3$ and $x$. 


4.1. Dual star-product scheme

The star-product scheme of the form

$$f_\alpha^d (x) = \text{Tr}[\hat{A} \hat{D}(x)], \quad \hat{A} = \int \text{d}x f_\alpha^d (x) \hat{U}(x)$$

is called dual with respect to (17). Apparently, the dual star-product kernel reads

$$K^d (x_1, x_2) = \text{Tr}[\hat{U}(x_1) \hat{U}(x_2) \hat{D}(x)]$$

and satisfies the relation (23).

4.2. Relation between tomographic schemes

Suppose that we are given two star-product schemes: (i) \(\hat{U}(x), \hat{D}(x)\) and (ii) \(\hat{U}(\xi), \hat{D}(\xi)\). The relation between the corresponding symbols is

$$f_\alpha (x) = \int \text{d}x f_\alpha (\xi) K_{\alpha \rightarrow \alpha} (x, \xi), \quad f_\alpha (\xi) = \int \text{d}x f_\alpha (x) K_{\alpha \rightarrow \alpha} (x, \xi),$$

where intertwining kernels \(K_{\alpha \rightarrow \alpha} (x, \xi)\) and \(K_{\alpha \rightarrow \alpha} (\xi, x)\) read

$$K_{\alpha \rightarrow \alpha} (x, \xi) = \text{Tr}[\hat{D}(\xi) \hat{U}(x)], \quad K_{\alpha \rightarrow \alpha} (\xi, x) = \text{Tr}[\hat{D}(x) \hat{U}(\xi)].$$

The relation between MUB symbols and symbols of a SIC-POVM will be considered for qubits in section 5.

4.3. MUB star-product scheme

Comparing the MUB-scanning procedure (7) and the reconstruction procedure (9) with the scheme (17), it is not hard to see that MUB tomography can be treated as a star-product scheme with the following dequantizer and quantizer operators:

$$\hat{U}_{aa} = \hat{\Pi}_{aa}, \quad \hat{D}_{aa} = \hat{\Pi}_{aa} - \frac{1}{d+1} \hat{I},$$

where \(x = [a, a] \quad a, \ldots, d \quad a, \ldots, d - 1 \quad \text{and} \quad \int \text{d}x = \sum_{a=0}^d \sum_{a'=0}^{d-1} \), i.e. \(x\) is a set of discrete variables and integration \(\int \text{d}x\) implies summation. The MUB-tomographic symbol of any operator \(\hat{A}\) acting on \(d\)-dimensional Hilbert space is

$$f_\alpha (a, \alpha) = \text{Tr}[\hat{A} \hat{\Pi}_{aa}],$$

$$\hat{A} = \sum_{a=0}^d \sum_{a'=0}^{d-1} f_\alpha (a, \alpha) \left( \hat{\Pi}_{aa} - \frac{1}{d+1} \hat{I} \right).$$

The delta-function on MUB-tomographic symbols is

$$\delta (a, \alpha; b, \beta) = \text{Tr}[\hat{D}_{aa} \hat{U}_{bb}] = \text{Tr}[\hat{\Pi}_{aa} \hat{\Pi}_{bb}] - \frac{1}{d+1} = \frac{1}{d(d+1)} + \delta_{a,b} \left( \delta_{\alpha,\beta} - \frac{1}{d} \right).$$

Note that the obtained delta-function contains extra terms in addition to the Kronecker delta symbol \(\delta_{a,b} \delta_{\alpha,\beta}\). There is no contradiction here and it can easily be checked that the residual part always gives zero on summation with any MUB symbol.

The MUB star-product kernel is expressed through the MUB triple product \(T_{ab,bf,cy} = \text{Tr}[\hat{\Pi}_{aa} \hat{\Pi}_{bb} \hat{\Pi}_{cc}]\) as follows:

$$K(a, \alpha; b, \beta; c, \gamma) = \text{Tr}[\hat{D}_{aa} \hat{D}_{bb} \hat{U}_{cc}]$$

$$= T_{ab,bf,cy} + \delta_{a,c} + \delta_{b,c} - \frac{d+2}{d+1} = \frac{d+2}{d(d+1)^2}. \quad (31)$$

Star-product kernel \(K(a, \alpha; b, \beta; c, \gamma)\) necessarily meets the condition (23), from which we derive a new relation on the MUB-triple product

$$\sum_{c=0}^d \sum_{\gamma=0}^{d-1} \left( T_{ab,bf,cy} T_{cc,kx,\lambda} - T_{aa,cc,\lambda} T_{bb,kx,\lambda} \right)$$

and find an expression that relates the four-product

$$\text{Tr}[\hat{\Pi}_{aa} \hat{\Pi}_{bb} \hat{\Pi}_{cc} \hat{\Pi}_{\lambda}]$$

and the triple-product

$$\text{Tr}[\hat{\Pi}_{aa} \hat{\Pi}_{bb} \hat{\Pi}_{cc}] = \sum_{c=0}^d \sum_{\gamma=0}^{d-1} \left( T_{ab,bf,cy} T_{cc,kx,\lambda} - T_{aa,cc,\lambda} T_{bb,kx,\lambda} \right)$$

$$= \frac{1}{d+1} \left( \frac{1}{d} - \frac{d}{d+1} \right). \quad (33)$$

It is worth mentioning that the same result can be alternatively obtained by using the dual MUB star-product kernel of the form

$$K^d (a, \alpha; b, \beta; c, \gamma) = \text{Tr}[\hat{D}_{aa} \hat{D}_{bb} \hat{U}_{cc}]$$

$$= T_{ab,bf,cy} - \frac{1}{d+1} \left( \frac{1}{d} - \delta_{a,b} \right). \quad (34)$$

4.4. Lie algebraic structure of MUB-POVM

The developed MUB star-product scheme enables us to reveal the Lie algebraic structure of MUB projectors. In fact, following the ideas of [16], let us consider a commutator

$$\hat{C} = [\hat{\Pi}_{aa}, \hat{\Pi}_{bb}] = \hat{\Pi}_{aa} \hat{\Pi}_{bb} - \hat{\Pi}_{bb} \hat{\Pi}_{aa}.$$ Since MUB-projectors are Hermitian, we obtain \(\hat{C}^+ = -\hat{C}\). This means that the MUB symbol of such a commutator is purely imaginary, that is,

$$f_E(c, \gamma) = \text{Tr}[\hat{\Pi}_{cc} \hat{U}_{cc}] = T_{ab,bf,cy} - T_{bb,a,cc} = i J_{ab,bf,cy}. \quad (35)$$

where \(J_{ab,bf,cy}\) is real and satisfies the condition

$$\sum_{\gamma=0}^{d-1} J_{ab,bf,cy} = 0.$$ Using this condition and reconstruction formula (29), we readily obtain

$$\hat{C} = -\sum_{c=0}^d \sum_{\gamma=0}^{d-1} i J_{ab,bf,cy} \left( \hat{\Pi}_{cc} - \frac{1}{d+1} \hat{I} \right). \quad (36)$$
\[ [\hat{\Pi}_{a\alpha}, \hat{\Pi}_{b\beta}] = \sum_{e=0}^{d-1} \sum_{y=0}^{d-y-1} ij_{a\alpha,b\beta,e,y} \hat{\Pi}_{c\gamma}, \quad (37) \]

The latter equation means that MUB-projectors form the Lie algebra \( gl(d, \mathbb{C}) \), with \( ij_{a\alpha,b\beta,e,y} \) being structure constants. Evidently, MUB-POVM effects \( [\hat{E}_{a\alpha}, \hat{E}_{b\beta}] \) satisfy
\[ [\hat{E}_{a\alpha}, \hat{E}_{b\beta}] = (d+1)^{-1} \sum_{e=0}^{d-1} \sum_{y=0}^{d-y-1} ij_{a\alpha,b\beta,e,y} \hat{E}_{c\gamma}. \]

5. MUB star-product scheme for qubits

MUB-projectors in two-dimensional Hilbert space can be chosen as follows:
\[ \hat{\Pi}_{a=0,a=0} = \frac{1}{2}(I + \hat{\sigma}_z), \quad \hat{\Pi}_{a=0,a=1} = \frac{1}{2}(I - \hat{\sigma}_z), \quad (38) \]
\[ \hat{\Pi}_{a=1,a=0} = \frac{1}{2}(I + \hat{\sigma}_y), \quad \hat{\Pi}_{a=1,a=1} = \frac{1}{2}(I - \hat{\sigma}_y), \quad (39) \]
\[ \hat{\Pi}_{a=2,a=0} = \frac{1}{2}(I + \hat{\sigma}_z), \quad \hat{\Pi}_{a=2,a=1} = \frac{1}{2}(I - \hat{\sigma}_z), \quad (40) \]

where \( \hat{\sigma}_z = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \) is a set of Pauli operators.

The delta-function on MUB symbols for qubits is
\[ \mathcal{D}(a, \alpha; b, \beta) = \frac{1}{2} + \delta_{a,b}\delta_{\alpha,\beta} - \frac{1}{4}. \]

A direct calculation shows that the triple product of MUB projectors for qubits reads
\[ T_{a\alpha,b\beta,c\gamma} = \frac{1}{4}[1 + 2(\delta_{a,b}\delta_{\alpha,\beta} + \delta_{a,\beta}\delta_{\beta,\gamma} + \delta_{c,a}\delta_{\gamma,\alpha}) - (\delta_{a,b} + \delta_{a,c} + \delta_{c,a})] + i\epsilon_{abc}(\delta_{a,0} - \delta_{a,1})(\delta_{\beta,0} - \delta_{\beta,1})(\delta_{\gamma,0} - \delta_{\gamma,1}), \quad (41) \]

where \( \epsilon_{abc} \) is the Levi–Civita symbol. Substituting the obtained triple product into formulae (31), (34) and (33), it is easy to calculate the ordinary and dual MUB star-product kernels as well as the four-product for qubits.

5.1. Relation to SIC star-product scheme

A star-product scheme based on SIC-POVM is considered in [25]. A self-dual star-product scheme that is very similar to the SIC star-product scheme is considered in [26]. In the case of qubits, SIC projectors read \( \check{P}_k = \frac{1}{2}(I + (\hat{\sigma} \cdot \mathbf{n}_k)) \), \( k = 1, \ldots, 4 \), where \( \mathbf{n}_1 = \frac{1}{\sqrt{3}}(1, 1, 1) \), \( \mathbf{n}_2 = \frac{1}{\sqrt{3}}(1, -1, -1) \), \( \mathbf{n}_3 = \frac{1}{\sqrt{3}}(-1, 1, -1) \) and \( \mathbf{n}_4 = \frac{1}{\sqrt{3}}(-1, -1, 1) \). The dequantizer is \( \check{u}_a = \frac{1}{2}\check{P}_2 \) and the quantizer is \( \check{D}_2 = 3\check{P}_2 - I \). Calculation of intertwining kernels (26) between MUB and SIC star-product schemes yields
\[ K_{\text{SIC} \to \text{MUB}}(k; a, \alpha) = \frac{1}{2}(1 + \sqrt{3}S(k; a, \alpha)), \quad (42) \]
\[ K_{\text{MUB} \to \text{SIC}}(a, \alpha; k) = \frac{1}{12}(1 + \sqrt{3}S(k; a, \alpha)), \quad (43) \]

where \( S(k; a, \alpha) \) is a sign function taking values \( \pm 1 \) in accordance with table 1. We hope that analogous simple relations between MUB and SIC quantization schemes exist in all prime dimensions (MUBs and SIC-POVMs are also compared in [27, 28]).

### 6. MUB-tomography and Stern–Gerlach measurements

On passing a beam of spin-\( j \) particles in a state \( \rho \) through the Stern–Gerlach apparatus oriented along the z-axis, we are able to measure probabilities to find particles in each split beam, i.e. in the state \( |jm\rangle \), where \( m = -j, \ldots, j \) is the spin projection on the z-axis. States \( |jm\rangle \) form the first basis in \( d \)-dimensional Hilbert space with \( d = 2j + 1 \).

Suppose a magnetic field \( \mathbf{B}_a \) is applied to spin particles before they are passed through the Stern–Gerlach magnetic field gradient. This results in a unitary transformation of the initial state \( \rho \rightarrow \check{u}_a^d \rho \check{u}_a \). The probabilities of outcomes read
\[ p_a(m) = \langle jm|\check{u}_a^d \rho \check{u}_a|m\rangle. \]

On the other hand, \( p_a(m) = \langle \alpha\alpha|\rho|\alpha\alpha \rangle \), where \( |\alpha\alpha \rangle = \check{u}_a|jm, m = \alpha - j \rangle, \alpha = 0, \ldots, 2J \).

States \( |\alpha\alpha\rangle \) form a new basis in Hilbert space, with parameter \( a \) being a label of this basis. Thus, applying different magnetic fields \( \mathbf{B}_a, a = 0, \ldots, 2j + 1 \), we obtain a set of \( 2j + 2 \) bases \( |\alpha\alpha\rangle \) as mentioned in [29, 30]. Analogous problems can be studied in beam physics, where neutron beams can be split according to the suggested mechanism. Actually, using constant and radio-frequency oscillating fields, interferometry experiments with neutrons can be realized in the same manner as experiments with polarizations of photons [33].

However, the application of constant magnetic fields \( |\mathbf{B}_a\rangle_{a=0}^{2j+1} \) gives rise to unitary transformations \( |\check{u}_a\rangle_{a=0}^{2j+1} \) of the group \( SU(2) \). Conversely, the MUB-condition (1) can only be met when \( u_a \in SU(N) \) with \( N = 2j + 1 \). As a result, MUBs can be constructed via the conventional Stern–Gerlach technique for qubits \( j = \frac{1}{2} \) only. For higher spins, the minimal necessary number of unitary transformations \( \check{u}_a \in SU(2) \) is known to be \( 4j + 1 \) [9], which is greater than \( 2j + 2 \). The possible solution of this problem is to exploit sequential Stern–Gerlach apparatuses, each being properly adjusted for a corresponding previously split beam.

### 7. Conclusions

Starting from peculiarities of MUBs, we have shown that whenever MUBs exist they can be used in quantum state tomography. Then we developed the MUB-tomographic-probability representation of quantum states by considering MUB-projectors within the framework of a star-product scheme. The dequantizer and the quantizer

| \( k \) | 00 | 01 | 10 | 11 | 20 | 21 |
|---|---|---|---|---|---|---|
| \( \alpha\alpha \) | + | − | + | − | + | − |
| 2 | + | − | + | − | + | − |
| 3 | − | + | − | − | + | + |
| 4 | − | + | − | − | + | − |
of the MUB star-product scheme were shown to be easily expressed through MUB projectors. This takes place due to a high symmetry of MUBs. For the constructed MUB quantization scheme, ordinary and dual star-product kernels were calculated and expressed through the triple product of MUB projectors. Employing the specific rules of the star-product kernel, we derived a new relation on triple-and four-products. Applying the MUB star product scheme to a commutator of MUB projectors, we revealed the Lie algebraic structure of MUB projectors and found structure constants. The obtained results can be used both in searching for and in the classification of MUBs in higher dimensions.

An example of qubits was considered in detail. In particular, an explicit form of all star-product characteristics was obtained. The intertwining kernels between MUB and SIC star-product schemes were found. These kernels can be used in order to find SIC-POVMs whenever MUBs exist, for instance in all prime dimensions. Finally, an implementation of MUB tomography via the Stern–Gerlach apparatus was discussed. The conventional experiment turns out to be appropriate for MUB tomography of qubits.

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