MATCHED PAIRS OF COURANT ALGEBROIDS

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Abstract. We introduce the notion of matched pairs of Courant algebroids and give several examples arising naturally from complex manifolds, holomorphic Courant algebroids, and certain regular Courant algebroids. We consider the matched sum of two Dirac subbundles, one in each of two Courant algebroids forming a matched pair.

1. Introduction

Matched pairs of algebraic structures occur naturally in several contexts of mathematics. For instance, a matched pair of groupoids, introduced by Mackenzie in [10] while studying double (Lie) groupoids, are two groupoids $G \to M$ and $H \to M$ over the same base manifold $M$ together with a representation of $G$ on $H$ and a representation of $H$ on $G$ compatible such that their product $G \bowtie H$ is again a groupoid. The infinitesimal version is a matched pair of Lie algebroids, which were introduced by Lu in [9] and studied by Mokri in [12]. They consist of two Lie algebroids $A_1$ and $A_2$ over the same base manifold $M$, together with an $A_1$-module structure on $A_2$ and an $A_2$-module structure on $A_1$, such that their direct sum $A_1 \bowtie A_2$ is again a Lie algebroid. Further examples arise from the study of holomorphic Poisson structures and holomorphic Lie algebroids [7].

The main goal of this note is to study matched pairs of Courant algebroids. More precisely, we investigate the question under which conditions the direct sum of two Courant algebroids over the same base manifold is still a Courant algebroid. We derive conditions which are similar to those of Lu and Mokri [9, 12]. However, there is a significant difference between matched pairs of Courant algebroids and matched pairs of Lie algebroids. It turns out, unlike Lie algebroids, that each component of the direct sum Courant algebroid of a matched pair is no longer a Courant subalgebroid.

Examples of matched pairs of Courant algebroids have appeared in literature. In connection with the study of port-Hamiltonian systems, Merker considered the Courant algebroid $TM \oplus T^*M \oplus E \oplus E^*$, where $E \to M$ be a vector bundle endowed with a flat connection $\nabla$ [11]. This is indeed a very simple example of matched pairs of Courant algebroids.

Another class of matched pairs of Courant algebroids arise when studying holomorphic Courant algebroids along a similar line as in the study of holomorphic Lie algebroids [7]. In particular, we prove that a holomorphic Courant algebroid over a complex manifold $X$ is equivalent to a matched pair of (smooth) Courant algebroids satisfying certain special properties, one of which is the standard Courant algebroid $T^{0,1}_X \oplus (T^{0,1}_X)^*$. A third class of examples arise from the construction of regular Courant algebroids, which were recently classified by one of the authors in a joint work [1].

The paper is organized as follows. In Section 2 we review the notion of Courant algebroids and matched pairs of Lie algebroids. In Section 3 we introduce the definition of matched pairs...
of Courant algebroids. In Section 4 we give four classes of examples: Courant algebroids with flat connections, complex manifolds, holomorphic Courant algebroids, and flat regular Courant algebroids. In Section 5 we give a definition of matched pairs of Dirac structures and show that a matched pair of Dirac structures is a matched pair of Lie algebroids. A supergeometric description of matched pairs of Courant algebroids will be discussed elsewhere.

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2. Preliminaries

We recall the definition of Courant algebroids based on the Dorfman bracket originally introduced in [4, 3]. For a comparison to the bracket introduced by Courant [8], see [13, 2].

Definition 2.1. A (real) Courant algebroid is a real vector bundle \( E \to M \) endowed with a symmetric non-degenerate \( \mathbb{R} \)-bilinear form \( \langle \cdot, \cdot \rangle \) on \( E \) with values in \( \mathbb{R} \), an \( \mathbb{R} \)-bilinear product \( \diamond \) on the space of sections \( \Gamma(E) \) called Dorfman bracket and a bundle map \( \rho : E \to TM \) (over the identity) called anchor map satisfying

\[
\phi \diamond (\phi_1 \diamond \phi_2) = \phi_1 \diamond (\phi \diamond \phi_2) + (\phi \diamond \phi_1) \diamond \phi_2 \tag{1}
\]

\[
\phi \diamond (f \phi') = (\rho(\phi)f)\phi' + f(\phi \diamond \phi') \tag{2}
\]

\[
\phi \diamond \phi = \frac{1}{2} D \langle \phi, \phi \rangle \tag{3}
\]

\[
\rho(\phi) \langle \phi', \phi' \rangle = 2 \langle \phi \diamond \phi', \phi' \rangle, \tag{4}
\]

where \( \phi, \phi_1, \phi_2, \phi' \in \Gamma(E) \), \( f \in C^\infty(M) \) and \( D : C^\infty(M) \to \Gamma(E) \) is defined by the relation \( \langle Df, \phi \rangle = \rho^*(df)\phi \).

For any \( \phi, \psi \in \Gamma(E) \), we have [14]

\[
\rho(\phi \diamond \psi) = [\rho(\phi), \rho(\psi)]. \tag{5}
\]

Moreover \( (Df) \diamond \phi = 0 \).

Remark 2.2. Complex Courant algebroids are defined similarly except that the pairing is \( \mathbb{C} \)-valued, the anchor is \( TM \otimes \mathbb{C} \)-valued, and \( \mathbb{C} \)-linearity replaces \( \mathbb{R} \)-linearity.

In one of his letters to Alan Weinstein, Pavol Ševera described the following example:

Example 2.3. To each closed 3-form \( H \) on \( M \) is associated a Courant algebroid structure on \( TM \oplus T^*M \) with inner product

\[
\langle X \oplus \alpha, Y \oplus \beta \rangle = \alpha(X) + \beta(Y), \tag{6}
\]

anchor map

\[
\rho(X \oplus \alpha) = X, \tag{7}
\]

and Dorfman bracket

\[
(X \oplus \alpha) \diamond (Y \oplus \beta) = [X, Y] \oplus \left( L_X \beta - i_Y d\alpha + i_{X \wedge Y} H \right), \tag{8}
\]

where \( X, Y \in \mathfrak{X}(M) \) and \( \alpha, \beta \in \Omega^1(M) \). This Dorfman bracket is called standard if \( H = 0 \) and \( H \)-twisted if \( H \neq 0 \).
**Definition 2.4.** Given an anchored vector bundle $E \xrightarrow{\rho} TM$ and a vector bundle $V$ over a smooth manifold $M$, an $E$-connection on $V$ is a bilinear operator $\nabla : \Gamma(E) \otimes \Gamma(V) \to \Gamma(V)$ fulfilling

\begin{align}
\nabla_{f\psi} v &= f\nabla_{\psi} v, \\
\nabla_{\psi} (fv) &= (\rho(\psi)f)v + f\nabla_{\psi} v
\end{align}

for all $f \in C^\infty(M)$, $\psi \in \Gamma(E)$, and $v \in \Gamma(V)$.

**Definition 2.5 ([12]).** Two Lie algebroids $A$ and $A'$ form a matched pair when a flat $A$-connection $\tilde{\nabla}$ on $A'$ and a flat $A'$-connection $\bar{\nabla}$ on $A$ satisfying

\begin{align}
\tilde{\nabla}_{[b, c]} &= \tilde{\nabla}_b \circ \tilde{\nabla}_c + [\tilde{\nabla}_b, \tilde{\nabla}_c] + \tilde{\nabla}_{\tilde{\nabla}_b} c - \tilde{\nabla}_{\tilde{\nabla}_c} b \\
\bar{\nabla}_{a} [\beta, \gamma] &= \bar{\nabla}_a \beta + \bar{\nabla}_a \gamma + \bar{\nabla}_{\gamma} a \beta - \bar{\nabla}_{\beta} a \gamma
\end{align}

(for all $\alpha, \beta, \gamma \in \Gamma(A')$ and $a, b, c \in \Gamma(A)$) are specified.

The following theorem is due to Mokri.

**Theorem 2.6 ([12]).** Let $A$ and $A'$ be a pair of Lie algebroids (with anchors $\rho_A$ and $\rho_{A'}$, and Lie brackets $[,]_A$ and $[,]_{A'}$, resp.). If a pair of connections $\tilde{\nabla}$ and $\bar{\nabla}$ makes $(A, A')$ into a matched pair of Lie algebroids, then the vector bundle $A \oplus A'$ is a Lie algebroid when endowed with the anchor map $\rho_A + \rho_{A'}$ and the bracket

$$[a + \alpha, b + \beta] = ([a, b]_A + \tilde{\nabla}_a b - \bar{\nabla}_b a) + ([\alpha, \beta]_{A'} + \bar{\nabla}_a \beta - \tilde{\nabla}_b \alpha).$$

Conversely, given a Lie algebroid $L$ and two Lie subalgebroid $A$ and $A'$ such that $L = A \oplus A'$ as vector bundles, then $(A, A')$ is a matched pair of Lie algebroids whose pair of connections is determined by the following relation:

$$[a, \beta] = -\tilde{\nabla}_\beta a + \bar{\nabla}_a \beta.$$

### 3. Matched pairs

Let $(E, \langle , \rangle, \rho, \phi)$ be a Courant algebroid. Assume we are given two subbundles $E_1, E_2$ of $E$ such that $E = E_1 \oplus E_2$ and $E_1^\perp = E_2$. Let $\text{pr}_1$ (resp. $\text{pr}_2$) denote the projection of $E$ onto $E_1$ (resp. $E_2$) and $i_1$ (resp. $i_2$) denote the inclusion of $E_1$ (resp. $E_2$) into $E$, respectively. Assume that $E_k$ is itself a Courant algebroid with anchor $\rho_k = \rho \circ i_k$, inner product $\langle a, b \rangle_k = \langle i_k a, i_k b \rangle$, and Dorfman bracket $a \circ_k b = \text{pr}_k (i_k a \circ i_k b)$. A natural question is how to recover the Courant algebroid structure on $E$ from $E_1$ and $E_2$.

**Proposition 3.1.** The inner product and the anchor map of $E$ are uniquely determined by their restrictions to the subbundles $E_1$ and $E_2$. Indeed, for $a, b \in \Gamma(E_1)$ and $\alpha, \beta \in \Gamma(E_2)$, we have

$$\langle a \oplus \alpha, b \oplus \beta \rangle = \langle a, b \rangle_1 + \langle \alpha, \beta \rangle_2,$$

$$\rho(a \oplus \alpha) = \rho_1(a) + \rho_2(\alpha),$$

where $a \oplus \alpha$ is shorthand for $i_1(a) + i_2(\alpha)$.
Proposition 3.2. The bracket on $E$ induces an $E_1$-connection on $E_2$:
\[
\vec{\nabla}_a \beta = \text{pr}_2 \left( (i_1 a) \diamond (i_2 \beta) \right)
\] (15)

and an $E_2$-connection on $E_1$:
\[
\vec{\nabla}_\beta a = \text{pr}_1 \left( (i_2 \beta) \diamond (i_1 a) \right).
\] (16)

These connections preserve the inner products on $E_1$ and $E_2$:
\[
\rho(a) \langle a, b \rangle_1 = \langle \vec{\nabla}_a a, b \rangle_1 + \langle a, \vec{\nabla}_a b \rangle_1
\] (17)
\[
\rho(a) \langle \alpha, \beta \rangle_2 = \langle \vec{\nabla}_a \alpha, \beta \rangle_2 + \langle \alpha, \vec{\nabla}_a \beta \rangle_2
\] (18)

Moreover, we have
\[
(a \oplus 0) \diamond (0 \oplus \beta) = -\vec{\nabla}_\beta a \oplus \vec{\nabla}_a \beta
\] (19)

for $a \in \Gamma(E_1)$ and $\beta \in \Gamma(E_2)$.

Proof. The first two equations follow from Leibniz rule (2). The next two equations follow from ad-invariance (4). The last equation uses in addition axiom (3). \qed

Proposition 3.3. (a) For all $a, b \in \Gamma(E_1)$, we have
\[
(a \oplus 0) \diamond (b \oplus 0) = (a \diamond_1 b) \oplus \left( \frac{1}{2} \mathcal{D}_2 \langle a, b \rangle_1 + \Omega(a, b) \right).
\] (20)

where $\Omega: \wedge^2 \Gamma(E_1) \to \Gamma(E_2)$ is defined by the relation
\[
\Omega(a, b) = \frac{1}{2} \text{pr}_2 (i_1 a \diamond i_1 b - i_1 b \diamond i_1 a).
\] (21)

In fact $\Omega$ is entirely determined by the connection $\vec{\nabla} : \Gamma(E_2) \otimes \Gamma(E_1) \to \Gamma(E_1)$ through the relation
\[
\langle \gamma, \Omega(a, b) \rangle_2 = \frac{1}{2} \left( \langle \vec{\nabla}_\gamma a, b \rangle_1 - \langle a, \vec{\nabla}_\gamma b \rangle_1 \right)
\] (22)

(b) For all $\alpha, \beta \in \Gamma(E_2)$, we have
\[
(0 \oplus \alpha) \diamond (0 \oplus \beta) = \left( \frac{1}{2} \mathcal{D}_1 \langle \alpha, \beta \rangle_2 + \mathcal{U}(\alpha, \beta) \right) \oplus (\alpha \diamond_2 \beta),
\] (23)

where $\mathcal{U}: \wedge^2 \Gamma(E_2) \to \Gamma(E_1)$ is defined by the relation
\[
\mathcal{U}(\alpha, \beta) = \frac{1}{2} \text{pr}_1 (i_2 \alpha \diamond i_2 \beta - i_2 \beta \diamond i_2 \alpha).
\] (24)

In fact, $\mathcal{U}$ is entirely determined by the connection $\vec{\nabla} : \Gamma(E_1) \otimes \Gamma(E_2) \to \Gamma(E_2)$ through the relation
\[
\langle c, \mathcal{U}(\alpha, \beta) \rangle_1 = \frac{1}{2} \left( \langle \vec{\nabla}_c \alpha, \beta \rangle_2 - \langle \alpha, \vec{\nabla}_c \beta \rangle_2 \right).
\] (25)

Proof. From axiom (3), we conclude that the symmetric part of the bracket is given by
\[
(a \oplus 0) \diamond (a \oplus 0) = \frac{1}{2} \mathcal{D}_1 \langle a, a \rangle_1 = \frac{1}{2} \mathcal{D}_1 \langle a, a \rangle_1 + \frac{1}{2} \mathcal{D}_2 \langle a, a \rangle_1.
\]

Moreover using (4) we get
\[
\langle 0 \oplus \gamma, (a \oplus 0) \diamond (b \oplus 0) \rangle = \rho(a \oplus 0) \langle 0 \oplus \gamma, b \oplus 0 \rangle - \langle (a \oplus 0) \diamond (0 \oplus \gamma), (b \oplus 0) \rangle = \langle \vec{\nabla}_a a, b \rangle_1
\]

The formula for $\mathcal{U}$ occurs under analog considerations for $0 \oplus \alpha$ and $0 \oplus \beta$. \qed
As a consequence, we obtain the formula

\[(a \oplus \alpha) \diamond (b \oplus \beta) = \left( a \circ_1 b + \overline{\nabla}_a b - \overline{\nabla}_b a + \mathcal{U}(\alpha, \beta) + \frac{1}{2}D_1 \langle \alpha, \beta \rangle_2 \right) \oplus \left( \alpha \circ_2 \beta + \overline{\nabla}_a \beta - \overline{\nabla}_b \alpha + \Omega(a, b) + \frac{1}{2}D_2 \langle a, b \rangle_1 \right),\]  

which shows that the Dorfman bracket on \(\Gamma(E)\) can be recovered from the Courant algebroid structures on \(E_1\) and \(E_2\) together with the connections \(\overline{\nabla}\) and \(\overline{\nabla}\).

**Lemma 3.4.** For any \(f \in C^\infty(M)\), \(b \in \Gamma(E_1)\), and \(\beta \in \Gamma(E_2)\), we have \(\overline{\nabla}_{D_1f} \beta = 0\) and \(\overline{\nabla}_{D_2f} b = 0\).

**Proof.** We have \(\overline{\nabla}_{D_1f} \beta = \text{pr}_2((D_1f \oplus 0) \diamond (0 \oplus \beta)) = \text{pr}_2(Df \diamond (0 \oplus \beta)) = 0\). \(\square\)

Set

\[\overline{R}(a, b)\alpha := \overline{\nabla}_a \overline{\nabla}_b \alpha - \overline{\nabla}_b \overline{\nabla}_a \alpha - \overline{\nabla}_{a_{ab}} \alpha, \quad (27)\]

\[\overline{R}(\alpha, \beta)a := \overline{\nabla}_\alpha \overline{\nabla}_\beta a - \overline{\nabla}_\beta \overline{\nabla}_\alpha a - \overline{\nabla}_{\alpha_{ab}} a . \quad (28)\]

**Lemma 3.5.** The curvatures \(\overline{R}\) and \(\overline{\nabla}\) are sections of \(\wedge^2 E^*_1 \otimes \mathfrak{o}(E_2) \cong \wedge^2 E^*_1 \otimes \wedge^2 E^*_2\), where \(\mathfrak{o}(E_2)\) is the bundle of skew-symmetric endomorphisms of \((E_2, \langle \cdot, \cdot \rangle_2)\).

**Proof.** The \(C^\infty(M)\)-linearity of \(\overline{R}(a, b)\alpha\) in \(\alpha\) follows from \((5)\). The curvature \(\overline{R}\) is also \(C^\infty(M)\)-linear in \(b\). In view of Lemma 3.4, it is also skew-symmetric with respect to \(a\) and \(b\). \(\square\)

**Theorem 3.6.** Assume we are given two Courant algebroids \(E_1\) and \(E_2\) over the same manifold \(M\) and two connections \(\overline{\nabla}: \Gamma(E_1) \otimes \Gamma(E_2) \to \Gamma(E_2)\) and \(\overline{\nabla}: \Gamma(E_2) \otimes \Gamma(E_1) \to \Gamma(E_1)\) preserving the fiberwise metrics and satisfying \(\overline{\nabla}_{D_1f} \beta = 0\) and \(\overline{\nabla}_{D_2f} b = 0\) for all \(f \in C^\infty(M)\), \(b \in \Gamma(E_1)\), and \(\beta \in \Gamma(E_2)\). Then the inner product \((13)\), the anchor map \((14)\), and the bracket \((15)\) on the direct sum \(E = E_1 \oplus E_2\) satisfy \((2)\), \((3)\), and \((4)\). Moreover, the Jacobi identity \((1)\) is fully equivalent to the following group of properties:

\[\overline{\nabla}_a (a_1 \circ_1 a_2) - (\overline{\nabla}_a a_1) \circ_1 a_2 - a_1 \circ_1 (\overline{\nabla}_a a_2) - \overline{\nabla}_{a_{a_2}} a_1 + \overline{\nabla}_{a_{a_1}} a_2 \]

\[= -\mathcal{U}(\alpha, \Omega(a_1, a_2)) + \frac{1}{2}D_2 \langle a_1, a_2 \rangle - \frac{1}{2}D_2 \langle \alpha, \Omega(a_1, a_2) + \frac{1}{2}D_2 \langle a_1, a_2 \rangle \rangle \]

\[\overline{\nabla}_a (a_1 \circ_2 a_2) - (\overline{\nabla}_a a_1) \circ_2 a_2 - a_1 \circ_2 (\overline{\nabla}_a a_2) - \overline{\nabla}_{a_{a_2}} a_1 + \overline{\nabla}_{a_{a_1}} a_2 \]

\[= -\Omega(a, \mathcal{U}(a_1, a_2)) + \frac{1}{2}D_1 \langle a_1, a_2 \rangle - \frac{1}{2}D_2 \langle a, \mathcal{U}(a_1, a_2) + \frac{1}{2}D_1 \langle a_1, a_2 \rangle \rangle \]

\[\overline{\nabla}_a \overline{\nabla}_b \alpha = \overline{\nabla}_b \overline{\nabla}_a \alpha = \overline{\nabla}_{a_b} \alpha = 0 \quad (29)\]

\[\overline{\nabla}_{\alpha_{a_1 a_2}} a_3 + \text{c.p.} = 0 \quad (30)\]

\[\overline{\nabla}_{\alpha_{a_1 a_2}} \mathcal{U} = 0 \quad (31)\]

**Proof.** Axioms \((2)\), \((3)\), and \((4)\) are straightforward computations from the definition \((26)\) using \((22)\) and \((25)\). To check the Jacobi identity, it is helpful to compute for triples of sections of \(E_1\) and \(E_2\), respectively, which gives 8 cases. Equations \((31)\), \((32)\), and \((33)\) arise when pairing the Jacobiator with an arbitrary section of \(E_1 \oplus E_2\). \(\square\)

We are now ready to introduce the main notion of the paper.
Definition 3.7. Two Courant algebroids $E_1$ and $E_2$ over the same manifold $M$ together with a pair of connections satisfying the properties listed in Theorem 3.6 are said to form a matched pair. The induced Courant algebroid structure on the direct sum vector bundle $E_1 \oplus E_2$ is called the matched sum of the pair $(E_1, E_2)$.

4. Examples

4.1. Courant algebroids with a flat connection. The first example goes back to Merker in [11]. Start with a Courant algebroid $(E_1, \phi_1, \rho_1)$ and assume that $\nabla$ is a metric connection on a pseudo Euclidean vector bundle $(V, \langle , \rangle)$ over the same manifold. If, in addition, $\nabla_{DS} s = 0$ for all smooth functions $f \in C^\infty(M)$, then the curvature $R$, defined as

$$
R(\psi_1, \psi_2)v = \nabla_{\psi_1} \nabla_{\psi_2} v - \nabla_{\psi_2} \nabla_{\psi_1} v - \nabla_{[\psi_1, \psi_2]} v
$$

is an element of $\Gamma(\wedge^2 E_1 \otimes \mathfrak{o}(V))$. We require this to vanish. We can endow $E_2 = V$ with the trivial Courant bracket and trivial anchor. Furthermore, we assume that the $E_2$-connection on $E_1$ is trivial. Then,

$$(\psi \oplus v) \circ (\psi' \oplus v') = \left( (\psi \circ_1 \psi') + \frac{1}{2} D(v, v') + \Omega(v, v') \right) \circ \left( \nabla_{\psi'} v' - \nabla_{\psi} v \right). \quad (34)$$

This Courant algebroid plays an important role in the study of port-Hamiltonian systems [11]. Let $E \to M$ be a vector bundle endowed with a flat connection $\nabla$. The port-Hamiltonian system can be described by a Dirac structure $D \subset CM \oplus V$, where $CM := TM \oplus T^*M$ is the standard Courant algebroid, $V := E \oplus E^*$ the trivial Courant algebroid with vanishing bracket and anchor, and $CM \oplus V$ their matched pair as explained above. An interesting family of Dirac structures that does not-necessarily project to Dirac structures on $CM$ or $V$ arises from a 2-form $\omega \in \Omega^2(M)$ and a so-called port map, which is a bundle map $A : TM \to E$. The Dirac structure is now the graph of the bundle map

$$
TM \oplus E^* \xrightarrow{\begin{pmatrix} \omega^\# & -(A \circ \omega^\#)^* \\ 0 & 0 \end{pmatrix}} T^*M \oplus E.
$$

The integrability conditions are $d\omega = 0$ and $d_\phi (A \circ \omega^\#) = 0$, where $d_\phi$ is the Lie algebroid differential of the sum Lie algebroid $TM \oplus E$ of the matched pair of Lie algebroids $(TM, E)$.

Conversely, given a bivector field $\pi \in \Gamma(\wedge^2 TM)$ and a port map $A : T^*M \to E$, we can consider the graph of the bundle map $\begin{pmatrix} \pi & -A^* \\ A & 0 \end{pmatrix} : T^*M \oplus E \to T^*M \oplus E$. This is always isotropic. It is integrable iff $[\pi, \pi] = 0$, $[\pi, A]_\oplus = 0$, and $[A, A]_\oplus = 0$, where $[ , , ]_\oplus$ is the Schouten bracket of the sum Lie algebroid of the matched pair $(TM, E)$.

4.2. Complex manifolds. Let $X$ be a complex manifold. Its tangent bundle $T_X$ is a holomorphic vector bundle. Consider the almost complex structure $j : T_X \to T_X$. Since $j^2 = -\text{id}$, the complexified tangent bundle $T_X \otimes \mathbb{C}$ decomposes as the direct sum of $T_X^{0,1}$ and $T_X^{1,0}$. Let $\text{pr}^{0,1} : T_X \otimes \mathbb{C} \to T_X^{0,1}$ and $\text{pr}^{1,0} : T_X \otimes \mathbb{C} \to T_X^{1,0}$ denote the canonical projections. The complex vector bundles $T_X$ and $T_X^{1,0}$ are canonically identified with one another by the bundle map $\frac{1}{2} (\text{id} - \sqrt{-1} j) : T_X \to T_X^{1,0}$.

Definition 4.1. A complex Courant algebroid is a complex vector bundle $E \to M$ endowed with a symmetric nondegenerate $\mathbb{C}$-bilinear form $\langle , \rangle$ on the fibers of $E$ with values in $\mathbb{C}$, a $\mathbb{C}$-bilinear Dorfman bracket $\diamond$ on the space of sections $\Gamma(E)$ and an anchor map $\rho : E \to TM \otimes \mathbb{C}$ satisfying
relations (1), (2), (3), and (4), where \( \mathcal{D} : C^\infty(M; \mathbb{C}) \to \Gamma(E) \) is defined by \( \langle \mathcal{D} f, \phi \rangle = \rho^* (df) \phi, \forall \phi \in \Gamma(E), f \in C^\infty(M; \mathbb{C}) \).

The complexified tangent bundle \( T_X \otimes \mathbb{C} \) of a complex manifold \( X \) is a smooth complex Lie algebroid. As a vector bundle, it is the direct sum of \( T_X^{0,1} \) and \( T_X^{1,0} \), which are both Lie subalgebroids.

The Lie bracket of (complex) vector fields induces a \( T_X^{1,0} \)-module structure on \( T_X^{0,1} \):
\[
\nabla^0 Y = \text{pr}^{0,1} [X, Y] \quad (X \in T_X^{1,0}, Y \in T_X^{0,1}),
\]
and a \( T_X^{0,1} \)-module structure on \( T_X^{1,0} \):
\[
\nabla^0 Y = \text{pr}^{0,1} [Y, X] \quad (X \in T_X^{1,0}, Y \in T_X^{0,1}).
\]
The flatness of these connections is a byproduct of the integrability of \( j \). We use the same symbol \( \nabla^0 \) to denote the induced connections on the dual spaces:
\[
\nabla^0 \beta = \text{pr}^{1,0} (\mathcal{L}_X \beta) \quad (X \in T_X^{1,0}, \beta \in \Omega_X^{0,1}),
\]
\[
\nabla^0 \alpha = \text{pr}^{1,0} (\mathcal{L}_Y \alpha) \quad (Y \in T_X^{0,1}, \alpha \in \Omega_X^{1,0}).
\]

As in Example 2.3, associated to each \( H^{3,0} \in \Omega^{3,0}(X) \) such that \( \partial H^{3,0} = 0 \), there is a complex twisted Courant algebroid structure on \( C_X^{1,0} = T_X^{1,0} \oplus (T_X^{1,0})^* \). Moreover, if two \( (3, 0) \)-forms \( H^{3,0} \) and \( H'^{3,0} \) are \( \partial \)-cohomologous, then the associated twisted Courant algebroid structures on \( C_X^{1,0} \) are isomorphic. Similarly, associated to each \( H^{0,3} \in \Omega^{0,3}(X) \) such that \( \partial H^{0,3} = 0 \), there is a complex twisted Courant algebroid structure on \( C_X^{0,1} = T_X^{0,1} \oplus (T_X^{0,1})^* \).

**Proposition 4.2.** Let \( H = H^{3,0} + H^{2,1} + H^{1,2} + H^{0,3} \in \Omega^3(X) \otimes \mathbb{C} \), where \( H^{i,j} \in \Omega^{i,j}(X) \), be a closed 3-form. Let \( (C_X^{1,0})_{H^{3,0}} \) be the complex Courant algebroid structure on \( C_X^{1,0} = T_X^{1,0} \oplus (T_X^{1,0})^* \) twisted by \( H^{3,0} \), and \( (C_X^{0,1})_{H^{0,3}} \) be the complex Courant algebroid structure on \( C_X^{0,1} = T_X^{0,1} \oplus (T_X^{0,1})^* \) twisted by \( H^{0,3} \). Then \( (C_X^{1,0})_{H^{3,0}} \) and \( (C_X^{0,1})_{H^{0,3}} \) form a matched pair of Courant algebroids, with connections given by
\[
\nabla^0 Y = \nabla_X Y \oplus \nabla^0 \beta + H^{1,2}(X, Y, \cdot) \quad (X, Y, \beta) \in T_X^{1,0} \oplus T_X^{0,1} \oplus (T_X^{1,0})^* \oplus (T_X^{0,1})^*
\]
and
\[
\nabla^0 \alpha = \nabla_Y X \oplus \nabla^0 \alpha + H^{2,1}(Y, X, \cdot) \quad (Y, X, \alpha) \in T_X^{0,1} \oplus T_X^{1,0} \oplus (T_X^{1,0})^* \oplus (T_X^{0,1})^*
\]
for all \( X \in T_X^{1,0}, Y \in T_X^{0,1}, \alpha \in \Omega_X^{0,1}, \beta \in \Omega_X^{1,0} \). The resulting matched sum Courant algebroid is isomorphic to the standard complex Courant algebroid \( (T_X \oplus T_X^*) \otimes \mathbb{C} \) twisted by \( H \).

**Proof.** The Courant algebroid \( (C_X^{1,0})_{H^{3,0}} \) is the twisted complex Courant algebroid \( (T_X^{1,0} \oplus T_X^{0,1})^* \) twisted by \( H^{3,0} \). Therefore the 3-form \( H^{3,0} \) must be closed under \( \partial \). This is true since the \( \Omega^{(4,0)} \) component of \( H \) vanishes. The analog considerations are true for \( (C_X^{0,1})_{H^{0,3}} \).

It is clear that \( (T_X \oplus T_X^*) \otimes \mathbb{C} \) can be twisted by \( H \). Straightforward computations show that this induces the twisted standard brackets on \( C_X^{1,0} \) and \( C_X^{0,1} \) respectively. Also the connections are induced by this Courant bracket. \( \square \)

4.3. **Holomorphic Courant algebroids.** Let \( X \) be a complex manifold. We denote by \( C_X^{1,0} = T_X^{1,0} \oplus (T_X^{1,0})^* \) and \( C_X^{0,1} = T_X^{0,1} \oplus (T_X^{0,1})^* \) the standard Courant algebroid.

**Definition 4.3.** A holomorphic Courant algebroid consists of a holomorphic vector bundle \( E \) over a complex manifold \( X \), with sheaf of holomorphic sections \( \mathcal{E} \), endowed with a fiberwise \( \mathbb{C} \)-valued inner product \( \langle \cdot , \cdot \rangle \) inducing a homomorphism of sheaves of \( \mathcal{O}_X \)-modules \( \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \) and \( \mathcal{O}_X \to \mathcal{E} \), a bundle map \( E \to T_X \) inducing a homomorphism of sheaves of \( \mathcal{O}_X \)-modules \( \mathcal{E} \to \mathcal{O}_X \).
where \( \Theta_X \) denotes the sheaf of holomorphic sections of \( T_X \), and a homomorphism of sheaves of \( \mathbb{C} \)-modules \( \mathcal{E} \otimes_{\mathcal{O}} \mathcal{E} \to \mathcal{E} \) satisfying relations \([1],[2],[3],\) and \([4]\), where \( D = \delta \circ \partial : \mathcal{O}_X \to \mathcal{E} \) and \( \phi, \phi_1, \phi_2, \phi' \in \mathcal{E}, f \in \mathcal{O}_X \).

Holomorphic Courant algebroids were also studied by Gualtieri, and we refer to [5] for more details.

**Lemma 4.4** (Theorem 2.6.26 in [6]). Let \( E \) be a complex vector bundle over a complex manifold \( X \), and let \( \mathcal{E} \) be a sheaf of \( \mathcal{O}_X \)-modules of sections of \( E \to X \) such that, for each \( x \in X \), there exists an open neighborhood \( U \subset X \) with \( \Gamma(U; E) = C^\infty(U, \mathbb{C}) \cdot \mathcal{E}(U) \). Then the following assertions are equivalent:

(a) The vector bundle \( E \) is holomorphic with sheaf of holomorphic sections \( \mathcal{E} \).
(b) There exists a (unique) flat \( T^{0,1}_X \)-connection \( \nabla \) on \( E \to X \) such that

\[
\mathcal{E}(U) = \{ \sigma \in \Gamma(U; E) \text{ s.t. } \nabla_Y \sigma = 0, \forall Y \in \mathfrak{X}^{0,1}(X) \}.
\]

**Lemma 4.5.** Let \( E \to X \) be a holomorphic vector bundle with \( \mathcal{E} \) as sheaf of holomorphic sections, and \( \nabla \) the corresponding flat \( T^{0,1}_X \)-connection on \( E \). Let \( \langle , \rangle \) be a smoothly varying \( \mathbb{C} \)-valued fiberwise symmetric nondegenerate \( \mathbb{C} \)-bilinear form on \( E \). The following assertions are equivalent:

(a) The inner product \( \langle \cdot , \cdot \rangle \) induces a homomorphism of sheaves of \( \mathcal{O}_X \)-modules \( \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \Theta_X \).
(b) For all \( \phi, \psi \in \Gamma(E) \) and \( Y \in \mathfrak{X}^{0,1}(X) \), we have

\[
Y \langle \phi, \psi \rangle = \langle \nabla_Y \phi, \psi \rangle + \langle \phi, \nabla_Y \psi \rangle.
\]

**Proposition 4.6.** Let \( E \to X \) be a holomorphic vector bundle with \( \mathcal{E} \) as the sheaf of holomorphic sections, and \( \nabla \) the corresponding flat \( T^{0,1}_X \)-connection on \( E \). Let \( \rho \) be a homomorphism of (complex) vector bundles from \( E \) to \( T_X \). The following assertions are equivalent:

(a) The homomorphism \( \rho \) induces a homomorphism of sheaves of \( \mathcal{O}_X \)-modules \( \mathcal{E} \to \Theta_X \).
(b) The homomorphism \( \rho^{1,0} : E \to T^{1,0}_X \) obtained from \( \rho \) by identifying \( T_X \) to \( T^{1,0}_X \) satisfies the relation

\[
\rho^{1,0}(\nabla_Y \phi) = \text{pr}^{1,0}[Y, \rho^{1,0} \phi], \quad \forall Y \in \Gamma(T^{0,1}_X), \phi \in \Gamma(E^{1,0}). \quad (41)
\]

**Lemma 4.7.** Let \( (E, \langle \cdot , \cdot \rangle, \rho, \cdot \diamond \cdot) \) be a holomorphic Courant algebroid over a complex manifold \( X \). Denote the sheaf of holomorphic sections of the underlying holomorphic vector bundle by \( \mathcal{E} \) and the corresponding \( T^{0,1}_X \)-connection on \( E \) by \( \nabla \). Then, there exists a unique complex Courant algebroid structure on \( E \to X \) with inner product \( \langle \cdot , \cdot \rangle \) and anchor map \( \rho^{1,0} = \frac{1}{2}(\text{id} - ij) \circ \rho : E \to T^{1,0}_X \subset T_X \otimes \mathbb{C} \), the restriction of whose Dorfman bracket to \( \mathcal{E} \) coincides with \( \diamond \). Such a complex Courant algebroid is denoted by \( E^{1,0} \).

**Proof.** To prove the uniqueness, assume that there are two complex Courant algebroid structures on the smooth vector bundle \( E \to X \) whose anchor map and inner products coincide. Moreover the Dorfman brackets coincide on \( \mathcal{E} \), the sheaf of holomorphic sections. Then the Dorfman brackets coincide on all of \( \Gamma(E) \), because the holomorphic sections over a coordinate neighborhood of \( X \) are dense in the set of smooth sections.

To argue for the existence, note that we want the Leibniz rules

\[
\phi \circ (g \cdot \psi) = \rho^{1,0}(\phi)[g] \cdot \psi + g \cdot (\phi \circ \psi)
\]

\[
(f \cdot \phi) \circ \psi = -\rho^{1,0}(\psi)[f] \cdot \phi + f(\phi \circ \psi) + \langle \phi, \psi \rangle \cdot \rho^{1,0} \circ df
\]

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for $\phi, \psi \in \Gamma(E)$ and $f, g \in C^\infty(X)$. Therefore we define the Dorfman bracket of the smooth sections $f \cdot \phi$ and $g \cdot \psi$, $\forall \phi, \psi \in \mathcal{E}$, as

$$ (f \cdot \phi) \diamond (g \cdot \psi) = f \rho^{1,0}(\phi)[g] \cdot \psi + f g \cdot (\phi \diamond \psi) - g \rho^{1,0}(\psi)[f] \cdot \phi + \langle \phi, \psi \rangle g \cdot \rho^{(1,0)*}\partial f. $$

First we should argue that this extension is consistent with the bracket defined for holomorphic sections. Assume therefore that $f \cdot \phi$ is again holomorphic, where $f \in C^\infty(X)$ and $\phi \in \mathcal{E}$. But this implies that $f$ is holomorphic on the open set where $\phi$ is not zero. Therefore on this open set the extension says

$$ f \rho^{1,0}(\phi)[g] \cdot \psi + f g \cdot (\phi \diamond \psi) - g \rho^{1,0}(\psi)[f] \cdot \phi + \langle \phi, \psi \rangle g \cdot \rho^{(1,0)*}\partial f $$

where we have used the fact that the two anchor maps coincide for holomorphic sections and $\rho^{(1,0)*}\partial f = \rho^{*}\partial f$. Therefore the term is just $(f \cdot \phi) \diamond (g \cdot \psi)$.

Since the first derivatives of $f$ and $g$ are continuous, the above relation also holds on the closure of the open domain where both $\phi$ and $\psi$ do not vanish. But on the complement, which is open, this term is just 0, because at least one of the terms $\phi$ or $\psi$ as well as the bracket $\phi \diamond \psi$ is also 0.

It follows from a straightforward but tedious computation that this bracket fulfills the four axioms (3)–(4). As an example, we show the proof of the axiom (3):

$$ (f \cdot \phi) \diamond (f \cdot \phi) = f^2 \cdot (\phi \diamond \phi) + \langle \phi, \phi \rangle f \cdot \rho^{(1,0)*}\partial f $$

where, in the last step, we used again the fact that $\rho^{*}\partial = \rho^{(1,0)*}\partial$ when applied to holomorphic functions. \hfill \square

Define a flat $C^{0,1}_X$-connection $\nabla$ on $E$ by

$$ \nabla_{Y \oplus \eta} e = \nabla_Y e, $$

and a flat $E$-connection $\tilde{\nabla}$ on $C^{0,1}_X$ by

$$ \tilde{\nabla}_e (Y \oplus \eta) = \nabla_{\rho^{1,0}(e)}^{0,1} (Y \oplus \eta) = \text{pr}^{0,1}(\mathcal{L}_{\rho^{1,0}(e)} Y \oplus \eta) = \text{pr}^{0,1}([\rho^{1,0}(e), Y] \oplus \mathcal{L}_{\rho^{1,0}(e)} \eta), $$

where $\text{pr}^{0,1}$ denotes the projection of $(T_X \oplus T_X^*) \otimes \mathbb{C}$ onto $T^{0,1}_X \oplus (T^{0,1}_X)^*$. We can write the identity (43) as

$$ \rho(\nabla_Y \phi) - \tilde{\nabla}_\phi Y = [Y, \rho(\phi)]. $$

Thus we have the following

**Proposition 4.8.** Let $(E, \langle, \rangle, \rho, \diamond)$ be a holomorphic Courant algebroid over a complex manifold $X$. Denote the sheaf of holomorphic sections by $\mathcal{E}$ and the corresponding flat $T^{0,1}_X$-connection on $E$ by $\nabla$. Then the two complex Courant algebroids $C^{0,1}_X$ and $E^{1,0}$, together with the flat connections $\nabla$ and $\tilde{\nabla}$ given by (42) and (43), constitute a matched pair of complex Courant algebroids, which we call the companion matched pair of the holomorphic Courant algebroid $E$.  


Proof. We check that $\mathcal{C}^{0,1}_X \oplus E^{1,0}$ is a Courant algebroid. The axioms (2)–(4) are straightforward computations by using the fact that the connections preserve the inner product on each Courant algebroid.

It remains to check that the Jacobi identity holds. By Proposition 3.6 it suffices to check that the five properties (29)–(33) hold. All equations are trivially satisfied when $\alpha_i \in \mathcal{E}$ and $\alpha_i \in \tilde{\Theta} \oplus \tilde{\Omega}_X \subset \Gamma(C^{0,1}_X)$. So we are left to check that multiplying the $\alpha_i$ (or the $\alpha_i$ respectively) with smooth functions we get the same additional terms on each side of the equations. This will be demonstrated for (29). The left hand side of (29) is

$$\text{LHS}(a_2) := \llbracket_{\alpha} (a_1 \circ a_2) - (\llbracket_{\alpha} a_1) \circ a_2 - a_1 \circ (\llbracket_{\alpha} a_2) - \llbracket_{\alpha_{a_2}} a_1 + \llbracket_{\alpha_{a_1}} a_2.$$  

Then

$$\text{LHS}(f \cdot a_2) = f \cdot \text{LHS}(a_2) + \left( [\rho(\alpha), \rho(a_1)] + \rho(\llbracket_{\alpha_1} \alpha) - \rho(\llbracket_{\alpha} a_1) \right) [f] \cdot a_2$$

and the second term vanishes due to (44). For the right hand side

$$\text{RHS}(a_2) := -\mathcal{U}(\alpha, \Omega(a_1, a_2) + \frac{1}{2} \langle a_1, a_2 \rangle) - \frac{1}{2} \mathcal{D}(\alpha, \Omega(a_1, a_2) + \frac{1}{2} \langle a_1, a_2 \rangle),$$

we have

$$\text{RHS}(f \cdot a_2) = f \cdot \text{RHS}(a_2).$$

Thus the right hand side is also $C^\infty(X)$-linear in $a_2$. Analog considerations result in coinciding terms for both sides of (29) when multiplying $a_1$ or $\alpha$ by a smooth function.

In fact, the converse is also true.

**Proposition 4.9.** Let $X$ be a complex manifold. Assume that $(\mathcal{C}^{0,1}_X, B)$ is a matched pair of complex Courant algebroids such that the anchor of $B$ takes values in $T^{1,0}_X$, both connections $\llbracket_{\alpha}$ and $\llbracket_{\alpha}$ are flat with the $B$-connection $\llbracket_{\alpha}$ on $\mathcal{C}^{0,1}_X$ being given by $\llbracket_{\alpha}(Y \oplus \eta) = \nabla^\rho_{\rho(e)}(Y \oplus \eta)$. Then there is a unique holomorphic Courant algebroid $E$ such that $B = E^{1,0}$.

**Proof.** The flat $\mathcal{C}^{0,1}$-connection induces a flat $T^{0,1}$-connection on $B$. Hence there is a holomorphic vector bundle $E \to X$ such that $E^{1,0} = B$, according to Lemma 4.4. Since the connection $\llbracket_{\alpha}$ preserves the inner product on $B$ and is compatible with the anchor map (44), $E$ inherits a holomorphic inner product and a holomorphic anchor map. It remains to check that the induced Dorfman bracket is holomorphic as well. If $a_1, a_2 \in \mathcal{E}$ are two holomorphic sections of $E$ and $\alpha \in \Theta_X \oplus \tilde{\Omega}_X$ is an anti-holomorphic section of $\mathcal{C}^{0,1}_X$, then all terms of Equation (29) except the first one, vanish. But then the first one $\llbracket_{\alpha} (a_1 \circ a_2)$ also has to vanish. This shows that the Dorfman bracket of two holomorphic sections is itself holomorphic.

4.4. Flat regular Courant algebroid. A Courant algebroid $E$ is said to be regular if $F := \rho(E)$ has constant rank, in which case $\rho(E)$ is an integrable distribution on the base manifold $M$ and $\ker \rho/(\ker \rho)^\perp$ is a bundle of quadratic Lie algebras over $M$. It was proved by Chen et al. [1] that the vector bundle underlying a regular Courant algebroid $E$ is isomorphic to $F^* \oplus \mathcal{G} \oplus F$, where $F$ is the integrable subbundle $\rho(E)$ of $TM$ and $\mathcal{G}$ the bundle $\ker \rho/(\ker \rho)^\perp$ of quadratic Lie algebras over $M$. Thus we can confine ourselves to those Courant algebroid structures on $F^* \oplus \mathcal{G} \oplus F$ whose anchor map is

$$\rho(\xi_1 + r_1 + x_1) = x_1,$$

whose pseudo-metric is

$$\langle \xi_1 + r_1 + x_1, \xi_2 + r_2 + x_2 \rangle = \langle \xi_1 | x_2 \rangle + \langle \xi_2 | x_1 \rangle + \langle r_1, r_2 \rangle \mathcal{G},$$
and whose Dorfman bracket satisfies
\[ Pr_G(r_1 \circ r_2) = [r_1, r_2]^G, \]  
where \( \xi_1, \xi_2 \in F^* \), \( r_1, r_2 \in G \), and \( x_1, x_2 \in F \). We call them standard Courant algebroid structures on \( F^* \oplus G \oplus F \).

**Theorem 4.10** ([11]). A Courant algebroid structure on \( F^* \oplus G \oplus F \), with pseudo-metric \([46] \) and anchor map \([15] \), and satisfying \([17] \), is completely determined by an \( F \)-connection \( \nabla \) on \( G \), a bundle map \( R : \wedge^2 F \to G \), and a 3-form \( H \in \Gamma(\wedge^3 F^*) \) satisfying the compatibility conditions

\[
L_x(r, s)^G = (\nabla_x r, s)^G + (r, \nabla_x s)^G, \\
\nabla_x [r, s]^G = [\nabla_x r, s]^G + [r, \nabla_x s]^G, \\
(\nabla_x R(y, z) - R([x, y], z)) + c.p. = 0, \]

\[
\nabla_x \nabla_y r - \nabla_y \nabla_x r - \nabla_{[x, y]} r = [R(x, y), r]^G, \\
d^FRH = \langle R \wedge R \rangle^G
\]

for all \( x, y, z \in \Gamma(F) \) and \( r, s \in \Gamma(G) \). Here \( \langle R \wedge R \rangle^G \) denotes the 4-form on \( F \) given by
\[
\langle R \wedge R \rangle^G(x_1, x_2, x_3, x_4) = 1_4 \sum_{\sigma \in S_4} \text{sgn}(\sigma)(R(x_{\sigma(1)}, x_{\sigma(2)}), R(x_{\sigma(3)}, x_{\sigma(4)}))^G,
\]

where \( x_1, x_2, x_3, x_4 \in F \).

The Dorfman bracket on \( E = F^* \oplus G \oplus F \) is then given by

\[
x_1 \circ x_2 = H(x_1, x_2, _) + R(x_1, x_2) + [x_1, x_2], \\
r_1 \circ r_2 = P(r_1, r_2) + [r_1, r_2]^G, \\
\xi_1 \circ \xi_2 = r_1 \circ \xi_2 = \xi_1 \circ \xi_2 = 0, \\
x_1 \circ \xi_2 = L_{x_1} \xi_2, \\
\xi_1 \circ x_2 = -L_{x_2} \xi_1 + d^E(\xi_1 | x_2), \]

\[
x_1 \circ r_2 = -r_2 \circ x_1 = -2Q(x_1, r_2) + \nabla_x r_2,
\]

for all \( \xi_1, \xi_2 \in \Gamma(F^*), r_1, r_2 \in \Gamma(G), x_1, x_2 \in \Gamma(F) \). Here \( d^E : C^\infty(M) \to \Gamma(F^*) \) denotes the leafwise de Rham differential. The maps \( P : \Gamma(G) \otimes \Gamma(G) \to \Gamma(F^*) \) and \( Q : \Gamma(F) \otimes \Gamma(G) \to \Gamma(F^*) \) are defined, respectively, by the relation

\[
\langle P(r_1, r_2) | y \rangle = 2(r_2, \nabla_y r_1)^G
\]

and

\[
\langle Q(x, r) | y \rangle = (r, R(x, y))^G.
\]

**Definition 4.11.** A standard Courant algebroid \( E = F^* \oplus G \oplus F \) is said to be flat if \( \langle R \wedge R \rangle^G \) vanishes.

**Proposition 4.12.** Let \( E = F^* \oplus G \oplus F \) be a flat standard Courant algebroid. Then \( (F \oplus F^*)_H \) and \( G \) form a matched pair of Courant algebroids, and \( E \) is isomorphic to their matched sum, where \( (F \oplus F^*)_H \) denotes the twisted Courant algebroid on \( F \oplus F^* \) by the 3-form \( H \). Here the \( (F \oplus F^*)_H \) connection on \( G \) is given by

\[
\nabla_{x+\xi} r = \nabla_x r,
\]
while the $G$-connection on $(F \oplus F^*)_H$ is given by
\[
\tilde{\nabla}_r (X \oplus \xi) = 2Q(X, r) \oplus 0 .
\] (62)

Proof. It follows from the flatness of the Courant algebroid that the 3-form $H \in \Omega^3_M(F)$ is $d_F$ closed. Therefore we can construct the twisted standard Courant algebroid $(F \oplus F^*)_H$. By comparing each component of the Dorfman bracket of the sum Courant algebroid $G \oplus (F \oplus F^*)_H$ with that of the bracket (53)–(58), we see that, for our choice (61) and (62), both coincide. \( \square \)

5. Matched pairs of Dirac structures

Recall that a Dirac structure $D$ in a Courant algebroid $(E, \langle.,.\rangle, \varpi, \rho)$ with split signature is a maximal isotropic and integrable subbundle.

Proposition 5.1. Given a matched pair $(E_1, E_2)$ of Courant algebroids of split signature and Dirac structures $D_1 \subset E_1$ and $D_2 \subset E_2$, the direct sum $D_1 \oplus D_2$ is a Dirac structure in the Courant algebroid $E_1 \oplus E_2$ iff $\tilde{\nabla}_{\alpha} a \in \Gamma(D_1)$ and $\tilde{\nabla}_{\alpha} \alpha \in \Gamma(D_2)$ for all $\alpha \in \Gamma(D_2)$ and $a \in \Gamma(D_1)$.

Proof. It is obvious that $D_1 \oplus D_2$ is maximal isotropic. It remains to check that the $E_2$ ($E_1$)-component of the bracket of any two sections of $D_1$ is automatically in $D_2$ ($D_1$). Indeed, we have
\[
\langle 0 \oplus \alpha, (a \oplus 0) \circ (b \oplus 0) \rangle = \langle \tilde{\nabla}_{\alpha} a, b \rangle_1 .
\]
Since $D_1$ is isotropic, the RHS vanishes. It thus follows from the maximal isotropy of $D_2$ that $(a \oplus 0) \circ (b \oplus 0)$ is in $D_2$.

Definition 5.2. Let $(E_1, E_2)$ be a matched pair of Courant algebroids. A Dirac structure $D_1$ in $E_1$ and a Dirac structure $D_2$ in $E_2$ are said to form a matched pair of Dirac structures if their direct sum $D_1 \oplus D_2$ is a Dirac structure in the matched sum $E_1 \oplus E_2$.

Corollary 5.3. Let $D_1$ (resp. $D_2$) be a Dirac structure in a Courant algebroid $E_1$ (resp. $E_2$). If $(E_1, E_2)$ is a matched pair of Courant algebroids and $(D_1, D_2)$ is a matched pair of Dirac structures, then $(D_1, D_2)$ is a matched pair of Lie algebroids.

Example 5.4. Let $CM := TM \oplus T^*M$ be the standard Courant algebroid, $V \to M$ a vector bundle with a flat connection $\nabla$. Endowing $V^*$ with the dual connection, $CM$ and $V \oplus V^*$ are matched pair of Courant algebroids.

Let $\omega \in \Omega^2(M)$ and $L \in \Gamma(\wedge^2 V^*)$. Graph $\omega \subset CM$ is a Dirac structure in $CM$ iff $d\omega = 0$. On the other hand, Graph $L^* \subset V \oplus V^*$ is automatically a Dirac structure. Then (Graph $\omega$, Graph $L^*$) is a matched pair of Dirac structure iff $[\nabla_X, L^*] = 0$ for all $X \in \Gamma(TM)$. In this case the direct sum Dirac structure is the graph of bundle map
\[
\begin{pmatrix} \omega^\# & 0 \\ 0 & L^* \end{pmatrix}: TM \oplus V \to T^*M \oplus V^.*
\]

On the other hand, we can consider the Dirac structure on $CM$ given by the graph of a Poisson bivector $\pi$ on $M$, and the Dirac structure on $V \oplus V^*$ given by the graph of $\Lambda \in \Gamma(\wedge^2 V)$. They form a matched pair of Dirac structures if and only if $[\pi, \Lambda]_{\oplus} = 0$.
References

[1] Zhuo Chen, Mathieu Stiénon, and Ping Xu, On regular Courant algebroids, arXiv:0909.0319 (2009).
[2] Theodore James Courant, Dirac manifolds, Transactions of the American Mathematical Society 319 (1990), no. 2, 631–661.
[3] Irene Dorfman, Dirac structures and integrability of nonlinear evolution equations, Nonlinear Science: Theory and Applications, John Wiley & Sons Ltd., Chichester, 1993.
[4] Irene Ya. Dorfman, Dirac structures of integrable evolution equations, Physics Letters. A 125 (1987), no. 5, 240–246.
[5] Marco Gualtieri, Generalized Kähler geometry, arXiv:1007.3485 (2010).
[6] Daniel Huybrechts, Complex geometry, Universitext, Springer-Verlag, Berlin, 2005.
[7] Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu, Holomorphic poisson manifolds and holomorphic Lie algebroids, International Mathematics Research Notices. IMRN (2008), Art. ID rnn 088, 46.
[8] Zhang-Ju Liu, Alan Weinstein, and Ping Xu, Manin triples for Lie bialgebroids, Journal of Differential Geometry 45 (1997), no. 3, 547–574.
[9] Jiang-Hua Lu, Poisson homogeneous spaces and Lie algebroids associated to Poisson actions, Duke Mathematical Journal 86 (1997), no. 2, 261–304.
[10] Kirill C. H. Mackenzie, Double Lie algebroids and second-order geometry. I, Advances in Mathematics 94 (1992), no. 2, 180–239.
[11] Jochen Merker, On the geometric structure of Hamiltonian systems with ports, Journal of Nonlinear Science 19 (2009), no. 6, 717–738.
[12] Tahar Mokri, Matched pairs of Lie algebroids, Glasgow Mathematical Journal 39 (1997), no. 2, 167–181.
[13] Dmitry Roytenberg, Courant algebroids, derived brackets and even symplectic supermanifolds, arXiv:math/9910078 (1999).
[14] Kyousuke Uchino, Remarks on the definition of a Courant algebroid, Letters in Mathematical Physics 60 (2002), no. 2, 171–175.

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