Compacton Solutions in a Class of Generalized Fifth Order Korteweg-de Vries Equations

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We study generalized Korteweg-de Vries (KdV) equations derivable from the Lagrangian:

\[ L(p, m, n, l) = \int dx \left\{ \frac{1}{2} \phi_x \phi_t + \alpha (\phi_x)^{p+2} - \frac{\beta (\phi_x)^m (\phi_{xx})^2}{(p+1)(p+2)} + \frac{\gamma}{2} \phi_x \phi_t \phi_{xxx} \phi_{xxxx} \right\}. \]

The usual field \( u(x, t) \) of the generalized KdV equation is defined by \( u(x, t) = \phi_x(x, t) \).

The equation of motion derived from this Lagrangian has solitary wave solutions of both the usual (non-compact) and compact variety ("compactons"). For the particular case that \( p = m = n + l \), the solitary wave solutions have compact support and the feature that their width is independent of the amplitude. We discuss the Hamiltonian structure of these theories and find that mass, momentum, and energy are conserved. We find in general that these are not completely integrable systems. Numerical simulations show that an arbitrary compact initial wave packet whose width is wider than that of a compacton breaks up into several compactons all having the same width. Upon scattering the compactons found here behave similarly to those found in the equations of Rosenau and Hyman, the scattering is almost elastic, with the left over wake eventually turning into compacton-anticompacton pairs. Often there are two different compacton solutions for a single set of parameters having the same generic form \( A \cos^n (d \xi) \), where \( \xi = x - ct \) and \(-\pi/2 \leq d \xi \leq \pi/2 \). When this is the case, the wider solution is stable, and this solution is a minimum of the Hamiltonian.

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I. INTRODUCTION

The observed stationary and dynamical patterns in nature are usually finite in extent. However most equations that admit solitary wave solutions yield solutions that are infinite in extent, although of a localized nature. Therefore, the recently discovered solitary waves with compact support (compactons) found by Rosenau and Hyman represents a welcome development. Rosenau and Hyman found in numerical experiments that the compact solitary waves on collision reemerge as compactons, with a tiny amount of energy going into a zero mass ripple which eventually reemerges into a compacton-anticompacton pair of small (< 5%) amplitude. The original equation studied by Rosenau and Hyman was:

\[ K(n, m) : u_t + (u^n)_x + (u^m)_{xxx} = 0, \quad (1.1) \]

For \( 1 \leq m = n \leq 3 \) this equation had solitary wave solutions of the form \( \cos(a \xi)^{2/(m-1)} \), where \( \xi = x - ct \) and \(-\pi/2 \leq a \xi \leq \pi/2 \). For \( m=2,3 \) they obtained:

\[ K(2, 2) : u_c = \frac{4c}{3} \cos^2(\xi/4), \]

\[ K(3, 3) : u_c = \left( \frac{3c}{2} \right)^{1/2} \cos(\xi/3). \quad (1.2) \]

Unlike the ordinary KdV equation the above equation is not derivable from a first order Lagrangian except for \( n = 1 \), and did not possess the usual conservation laws of energy, momentum, and mass that the KdV equation possessed. Because of this, we considered earlier a different generalization of the KdV equation based on a first order Lagrangian formulation. That is, we considered:

\[ L(l, p) = \int \left( \frac{1}{2} \phi_x \phi_t + \frac{(\phi_x)^l}{l(l-1)} - \alpha (\phi_x)^p (\phi_{xx})^2 \right) dx. \quad (1.3) \]

This Lagrangian leads to a generalized sequence of KdV equations of the form:

\[ K^*(l, p) : u_t + u^{l-2} u_x + \alpha (2u^p u_{xxx} + 4pu^{p-1} u_x u_{xx} + p(p-1)u^{p-2} u_x^3) = 0, \quad (1.4) \]

where

\[ u(x, t) = \phi_x(x, t). \quad (1.5) \]

These equations have the same terms as the equations considered by Rosenau and Hyman, but the relative weights of the terms are different. In that study we
showed that for $0 \leq p \leq 2$ and $l = p + 2$, these models admit compacton solutions for which the width is independent of the amplitude. Since higher order KdV equations also admit soliton solutions, it is interesting to enquire whether higher order generalized KdV equations also admit compacton solutions, and whether there are cases where the width is independent of the amplitude. This is the first motivation for the present study. The second motivation is to study the scattering properties of the compactons numerically to see if they behave similarly to those found by Rosenau and Hyman. The generalization which we will study in this paper is described by the Lagrangian:

$$L(p, m, n, l) = \int dx \left[ \frac{1}{2} \varphi_x \varphi_t + \alpha \frac{(\varphi_x)^{p+2}}{(p+1)(p+2)} - \beta (\varphi_x)^m (\varphi_{xx})^2 + \frac{\gamma}{2} \varphi_x \varphi_x \varphi_{xxx} \right].$$  \hspace{1cm} (1.6)

which includes one extra term with higher derivatives. This Lagrangian Eq. (1.6) generalizes both the usual KdV Lagrangian and our previous generalized KdV Lagrangian, and shares with both equations that because of the invariance of the action under time, and space translations as well as the shift of the field by a constant ($\varphi \rightarrow \varphi + \alpha$) that energy, momentum and mass are conserved by Noether’s theorem.

The rest of the paper is organized follows: In section 2 we will discuss the Hamiltonian structure of these classes of theories and determine the equation for the travelling wave solutions to Eq. (2.1). In section 3 we give arguments based on a variational approach (see [2]) as well as dimensional reasons for there being solutions with the width of the solitary wave being independent of the amplitude and we determine the connection between the energy and momentum of these solitary waves: namely $E \propto P_c$. In particular we find that the condition for having solitary waves with width independent of amplitude is $\alpha = m - p$. In section 4 we obtain exact compacton solutions of the form $A \cos \gamma d(x - ct)$ and verify that the relations among the global variables obtained using the variational approach are exact. In section 5 we give numerical results for the scattering of two compactons, as well as for the breakup of an arbitrary wave packet into several compactons. In an appendix we show that most of the exact solutions as well as the criteria for their stability could be obtained by assuming the exact compact ansatz and minimizing the action on that class of functions. The unstable solutions are maxima of the effective Hamiltonian.

II. GENERALIZED KDV EQUATION AND PROPERTIES

From the Lagrangian Eq. (1.6) we obtain the generalized KdV equation:

$$u_t + \frac{\alpha}{p+1} \partial_x u^{p+1} - \beta m \partial_x (u^{m-1} u_x^2) + 2 \beta \partial_x x (u^m u_x)$$

$$+ \frac{\gamma n}{2} \partial_x (u^{n-1} u_x^2) - \frac{\gamma l}{2} \partial_x x (u^n u_x^{l-1} u_x^2)$$

$$+ \gamma \partial_{xxx} (u^n u_x^l u_{xx}) = 0,$$  \hspace{1cm} (2.1)

as well as the conserved Hamiltonian:

$$H = \int dx \left[ -\alpha \frac{u^{p+2}}{(p+1)(p+2)} + \beta u^m u_x^2 - \frac{\gamma}{2} u^n u_x^l u_{xx}^2 \right],$$  \hspace{1cm} (2.2)

where $u(x, t) = \varphi(x, t)$. We notice that the Lagrangian given by Eq. (1.1) is invariant under the transformations (i) $\varphi(x, t) \rightarrow \varphi(x, t) + c_1$; (ii) $x \rightarrow x + c_2$ and (iii) $t \rightarrow t + c_3$, where $c_1$, $c_2$ and $c_3$ are constants. By a direct application of Noether’s theorem this leads to the three conservation laws of mass $M$, momentum $P$ and energy $E$, where $E$ is given by Eq. (2.2) and $M$ and $P$ are given by

$$M = \int u(x,t) dx; \hspace{1cm} P = \frac{1}{2} \int u^2(x,t) dx.$$  \hspace{1cm} (2.3)

The equation of motion Eq. (2.1) is also invariant under the transformations as given by $
abla \rightarrow ku; \hspace{0.5cm} x \rightarrow k^n x; \hspace{0.5cm} t \rightarrow k^b t$

provided that

$$\frac{n + l - p}{l + 4} = a = \frac{m - p}{2}; \hspace{0.5cm} b = a - p.$$  \hspace{1cm} (2.4)

However, as seen below for the interesting compacton case where $m = p = n + l$, these equations are invariant under the more restricted transformations:

$$u \rightarrow ku; \hspace{0.5cm} x \rightarrow x; \hspace{0.5cm} t \rightarrow k^{-p} t.$$  \hspace{1cm} (2.4)

We shall see later that this condition is necessary so as to have compactons whose width is independent of the amplitude. As an aside we would like to point out that all the compactons so far discovered with width independent of amplitude have the property that the field equations are invariant under Eq. (2.4).

The canonical structure of these theories is similar to that found in [2] and by postulating that the $u(x)$ satisfy the Poisson bracket structure [3]$$\{u(x), u(y)\} = \partial_x \delta(x - y).$$  \hspace{1cm} (2.5)

we obtain that

$$u_t = \partial_x \frac{\delta H}{\delta u} = \{u, H\}$$  \hspace{1cm} (2.6)

with $H$ being given by Eq. (2.2). We also find that with our definition of $P$ given by Eq. (2.3), that $P$ is indeed the generator of the space translations:

$$\{u(x, t), P\} = \frac{\partial u}{\partial x}.$$  \hspace{1cm} (2.7)
Since our equation is a generalization of the equation discussed in [3], one is also not able to show the existence of a bi-Hamiltonian structure using the conserved momentum as a possible second Hamiltonian as was done for the ordinary KdV equation. So on these grounds one expects that our general Lagrangian does not correspond to an exactly integrable system, except for the original KdV equation case. This will be verified by our numerical results on scattering where there is some energy going into compacton pair production following scattering.

A. Equation for solitary waves

If we assume a solution to (2.1) in the form of a travelling wave:

\[ u(x, t) = f(x - ct) \equiv f(y), \] (2.8)

one obtains for \( f(y) \):

\[
\begin{align*}
Cf_y &= \frac{\alpha}{p + 1} \partial_y f^{p+1} - \beta m \partial_y (f^{m-1} f_y^2) + 2 \beta \partial_y (f^m f_y) \\
&\quad + \frac{\gamma n}{2} \partial_y (f^{n-1} f_y f_y') - \frac{\gamma l}{2} \partial_y (f^n f_y^{-1} f_y'') \\
&\quad + \gamma \partial_{yy} (f^n f_y f_y'),
\end{align*}
\] (2.9)

Integrating once we obtain:

\[
\begin{align*}
f(y) &= \frac{\alpha}{p + 1} f^{p+1} - \beta m f^{m-1} f_y^2 + 2 \beta f^m f_y \\
&\quad + \frac{\gamma n}{2} (f^{n-1} f_y f_y') - \frac{\gamma l}{2} f^n f_y^{-1} f_y'' \\
&\quad + \gamma \partial_{yy} (f^n f_y f_y') + c_1,
\end{align*}
\] (2.10)

where \( c_1 \) is a constant of integration. This equation needs several more integrations before a solution in terms of quadrature is obtained, unlike the previous equation we studied where two integrations were sufficient. Thus an explicit solution in terms of quadratures is not available and one must use “educated” guess ansatze to find exact solutions. It is therefore quite useful to look at a simple variational approach to the solitary wave problem to understand some of the global features of the solitary waves. Previous experience with variational methods have shown that many of the global relations among conserved quantities as well as conditions for the width to be independent of the amplitude can be obtained from a simple variational ansatz. Also whether the solitary wave is expected to be stable or not can be inferred from the fact that the variational solution is a minimum of the action or just a saddle point of the action. Before addressing these questions, we just remind the reader that at \( \gamma = 0 \) the Lagrangian we are studying reduces to the previous generalized KdV problem we studied [2] so that these equations include all the KdV solitons and compactons that we discussed earlier as a special case. We will find exact solutions of Eq. (2.10) by inserting the ansatz:

\[ f(y) = A \cos'(dy) \] (2.11)

and determining the parameters \( A, \), \( r, \) and \( d \) by a consistency argument.

III. VARIATIONAL APPROACH

Our time-dependent variational approach for studying solitary waves is based on the principle of least Action. In previous work [2] [6] [4], we introduced a post-Gaussian variational approximation, a continuous family of trial variational functions more general than Gaussians, which can still be treated analytically. We assumed a variational ansatz of the form

\[ u_v(x, t) = A(t) \exp[-b(t) |x - q(t)|^{2a}] \]

The variational parameters have a simple interpretation in terms of expectation values with respect to the “probability” \( P \)

\[ P(x, t) = \frac{|u_v(x, t)|^2}{2P}, \] (3.1)

where the conserved momentum \( P \) is defined as above.

\[ P \equiv \frac{1}{2} \int |u_v(x, t)|^2 \, dx. \] (3.2)

We have that

\[ q(t) = \langle x \rangle \]

\[ G_{2a} \equiv \langle |x - q(t)|^{2a} \rangle = \frac{1}{4sb}. \] (3.3)

and

\[ A(t) = \frac{P^{1/2}(2b)^{1/4a}}{[\Gamma(1/2a)]^{1/2b}}. \] (3.4)

Extremizing the effective action for the trial wave function \( u_v \) leads to Lagrange’s equations for the variational parameters. We find that for all values of the parameters \((l, m, n, p)\), the dynamics of the variational parameters lead to solitary waves moving with constant velocity and constant width. However the solutions found are often maxima or saddle points of the action. We will find that when this happens the exact solitary wave solution is unstable. For the special case of \( p = m = n + l \) we will show below that the width of the soliton is independent of the amplitude and velocity. For that case we also obtain the relationship:

\[ E = -\frac{2c}{p + 2} P \]
We will find that the exact solitary wave solutions satisfy this relationship as long as the integration constant \( c_1 \) of Eq. (2.10) is zero.

The starting point for the variational calculation is the action

\[
\Gamma = \int L dt, \tag{3.5}
\]

where \( L \) is given by (3.3).

Inserting the trial wave function \( u_v \) we obtain:

\[
\Gamma(q, \beta, P, s) = \int dt [-P\dot{q} - H_{eff}]; \tag{3.6}
\]

where \( H_{eff} \) is the Hamiltonian evaluated using the variational wave function \( u_v \). We find

\[
H_{eff} = C_1(s) b^{(p+2)/4 s} P^{(p+2)/2} + C_2(s) b^{(m+4)/4 s} P^{(m+2)/2} + C_3(s) b^{(n+3l+8)/4 s} P^{(n+l+2)/2}, \tag{3.7}
\]

where \( C_1, C_2 \) and \( C_3 \) are functions of \( s, \alpha, \beta, \gamma \) but independent of \( P, b \). We notice that there is no momentum conjugate to \( b \). We eliminate the variable of constraint \( b \) (using \( \delta \Gamma/\delta b = 0 \)) and obtain the equation:

\[
0 = \frac{b}{4s} C_1(s) d^p P^{(p+2)/2} + \frac{m+4}{4s} C_2(s) d^{(m+4)/2} P^{(m+2)/2} + \frac{(n+3l+8)}{4s} C_3(s) d^{(n+3l+8)/2} P^{(n+l+2)/2}, \tag{3.8}
\]

where

\[
d = b^{\frac{m}{2}}. \tag{3.9}
\]

From Eq. (3.8) we see that when

\[
p = m = n + l \tag{3.10}
\]

the momentum \( P \) factors out of the equation, so that the width of the soliton which depends on \( b \) does not depend on \( P \) and thus is independent of the amplitude or velocity. This special case is precisely the case when the exact solution is a compacton whose width is independent of amplitude as we shall show below. From Lagrange’s equation \( \delta \Gamma/\delta P = 0 \) we have that

\[
-\frac{\delta H_{eff}}{\delta P} = \dot{q} = c. \tag{3.11}
\]

When \( p = m = n + l \) this yields the relationship:

\[
E = -\frac{2c}{p+2} P, \tag{3.12}
\]

which we will verify is true for the exact compacton solutions when \( c_1 = 0 \). For this special case of \( p = m = n + l \) the functions \( C_i \) are given by:

\[
C_1 = -N\alpha \Gamma(\frac{1}{p+2}) (p+1)(p+2)s(p+2) \tag{3.13}
\]

\[
C_2 = 4N\beta s \Gamma(2 - \frac{1}{2s})(p+2)\frac{1}{2s} - 2 \tag{3.14}
\]

\[
C_3 = -(2s)^{l+1}(-1)^lN\gamma(p+2)^{\frac{l+3}{2s} - 4 - 1} \Gamma(l+2 - \frac{l+3}{2s}) \tag{3.15}
\]

\[
(2s - 1)[p^2(2s - 1) + 2p(1 + l - 2ls) + l^2(2s - 1) + 2l(s - 1) + 4s - 1] \tag{3.16}
\]

where \( N = 2\frac{1}{\Gamma(1 + \frac{1}{2s})} \)

For the special case that \( p = m = n + l \) our procedure for determining the optimal trial wave function is very straightforward. The effective Hamiltonian simplifies to (from now on we drop the subscript on \( H_{eff} \)):

\[
H(d, s) = P^4\int d^p C_1(s) + d^4 C_2(s) + d^{2l+8} C_3(s) \tag{3.17}
\]

Minimizing with respect to the width parameter \( d \), we obtain:

\[
pC_1 + (p + 4)d^4 C_2 + (p + 2l + 8)d^{2l+8} C_3 = 0 \tag{3.18}
\]

For \( l = 0 \), one can analytically solve this for \( d(s) \) and one usually finds that there are 2 real solution in the vicinity of \( s = 1 \) which is the Gaussian trial wave function. Inserting these 2 solutions \( (d_i(s)) \) into \( H \) we obtain an analytic expression of the form

\[
H_i(s) = H(d_i(s), s) \tag{3.19}
\]

Imposing \( dH_i/ds = 0 \), we find that the two solutions \( \{d_i(s_i), s_i\} \) have the property that one is a saddle of \( H \) in the \( \{d, s\} \) plane and one is a minimum. We find in our numerical simulation that only the solutions that is a minimum is stable. In general it is easy to find the extrema of \( H \) in the \( \{d, s\} \) plane graphically by making a 3 dimensional plot of \( H \). What is convenient about the class of trial wave functions discussed here, is that one can explicitly determine the effective action for all values of \( p, l, m, n \) and get the general features of the results for all values of the parameters. In the appendix we will use the form of the exact compacton wave function in the variational approach. By doing so we will recover most of the exact compacton solutions that we find in the next section. We also obtain the result that when there are multiple solutions the stable one is a minimum of the effective Hamiltonian for the exact shape function.

**IV. EXACT COMPACTON SOLUTIONS**

Motivated by our variational results we will concentrate on the particular case where \( p = m = n + l \neq 0 \), which corresponds to Compactons whose width is independent of amplitude. Assuming a solution of the form

\[
u(x, t) = A \cos^p[2\pi(x - ct)], -\frac{\pi}{2} \leq x - ct \leq \frac{\pi}{2} \tag{4.1}
\]

\[
u(x, t) = 0; \quad |x - ct| > \frac{\pi}{2}, \tag{4.2}
\]

\[C_2 = 4N\beta s \Gamma(2 - \frac{1}{2s})(p+2)\frac{1}{2s} - 2 \tag{4.3}
\]

\[C_3 = -(2s)^{l+1}(-1)^lN\gamma(p+2)^{\frac{l+3}{2s} - 4 - 1} \Gamma(l+2 - \frac{l+3}{2s}) \tag{4.4}
\]

\[(2s - 1)[p^2(2s - 1) + 2p(1 + l - 2ls) + l^2(2s - 1) + 2l(s - 1) + 4s - 1] \tag{4.5}
\]

where \( N = 2\frac{1}{\Gamma(1 + \frac{1}{2s})} \)
we look for consistent solutions for $A$, $r,c$ and $d$ in terms of $\alpha$, $\beta$ and $\gamma$ and $c_1$ which are real. We check whether there is an analogous variational solution of the post-Gaussian variety corresponding to a minimum and not just a saddle point or maximum of the action. We also show in the appendix that if we a assume a trial wave function of the exact type:

$$u(x, t) = A \cos^r[d(t)(x - q(t))]$$

the exact solutions are either maxima or minima of the effective Hamiltonian as a function of $d$. We find that when there is a unique solution, then it is a minimum of the Energy. When there are two solution, the wider one is stable and is also a minimum of the Energy. The narrower one is unstable and is a maximum of the Energy.

Now let us look at some special cases.

**A. $p = m = n$ and $l = 0$ case**

For the case $p = m = n$ and $l = 0$ it is possible to find a general class of solutions for arbitrary $p$. Inserting a trial solution of the form:

$$u(y) = A \cos^r(dy) \quad (4.2)$$

into Equation (2.11) we obtain the consistency equation:

$$0 = -3c_1 + 2Acr^2
+ A^1\rho x^{-4+r+rp} (1 - r) r d^4 \gamma \times
(12 - 10r + 2r^2 - 11r^p + 5r^2 p + 2r^2 p^3)
+ 2A^1\rho x^{-2+2+rp} r d^2 (2 - 2 + r + rp) \times
(-\beta + 2d^2 \gamma - 2rd^2 \gamma + r^2 d^2 \gamma - r^2 d^2 \gamma p + 2r^2 d^2 \gamma p)
+ A^1\rho x^{2+2} (2 + p) (2 - r^2 d^2 \gamma - 2r^2 d^2 \gamma p) x^{rp+rp}
- \frac{2}{1 + p} A^1\rho A^{rp} x^{rp+rp} \quad (4.3)$$

Here $x = \cos(dy)$. All the powers of $x$ must have zero coefficient for the trial solution to be an actual solution. This leads to various conditions depending on the values of $r$ and $p$. If $r(1+p) = 4$ then there can also be solutions with $c_1$ being nonzero. First let us consider the case when $c_1 = 0$. In that case for consistency we need either $rp = 2$ or $rp = 4$.

1. $rp = 2$

When $rp = 2$ Eq. (4.3) reduces to the conditions:

$$0 = \gamma (-1 + r)^2 r (1 + r),$$

$$A^{2/r} = -\alpha + 4 \beta d^2 + 6 \beta \beta d^2 + 2 \beta^2 \beta d^2 - 8 r d^4 \gamma
- \frac{14}{5} d^4 \gamma - 7 d^4 \gamma - r^4 d^4 \gamma,$$

$$0 = -\alpha + 4 \beta d^2 + 6 \beta \beta d^2 - 8 d^4 \gamma r + 2 \beta d^2 \gamma r^2
- \frac{14}{5} d^4 \gamma r - 7 d^4 \gamma r - d^4 \gamma r^4. \quad (4.4)$$

The first condition tells us that either $\gamma = 0$ or $r = 1$.

When $\gamma = 0$, we get the solution we found in our earlier work [3]: Namely:

$$d^2 = \frac{\alpha p^2}{4 \beta (1 + p) (2 + p)}$$

$$A^p = \frac{c (1 + p) (2 + p)}{2 \alpha}$$

When $\gamma \neq 0$ we instead get the solution $r = 1 \ (p = 2)$ and

$$A^2 = \frac{c}{2 \beta d^2 - 6 d^4 \gamma}$$

and two possible solutions for the width:

$$d^2 = \frac{12 \beta \pm \sqrt{144 \beta^2 - 120 \alpha \gamma}}{60 \gamma}$$

Which means we also can write the equation for $A^2$ as

$$c = \frac{A^2}{5} (\alpha - 2 \beta d^2)$$

Thus when $\gamma \neq 0$ one only gets a solution when $p = 2$. A particular case of this solution is: $\alpha = 6, \gamma = 3$ and $\beta = 4$. Then there are two solutions having $d^2 = 1/3$ or $d^2 = 1/5$. The first is

$$u = \sqrt{3c/2} \cos \left(\frac{x - ct}{\sqrt{3}}\right) \quad (4.5)$$

The conserved quantities for this solution are

$$M = 3\sqrt{2c}; P = \frac{3}{8} \sqrt{3c\pi}; E = -\frac{3}{16} \sqrt{3c^2\pi}$$

Thus we find that the relationship:

$$\frac{E}{P} = -\frac{2c}{p + 2} = -\frac{c}{2} \quad (4.6)$$

that we derived from our variational approach is exact here.

The second solution :

$$u = \sqrt{\frac{25c}{2 \pi}} \cos \frac{y}{\sqrt{5}} \quad (4.7)$$

has as its conserved quantities:

$$M = 5\sqrt{\frac{10c}{11}}; P = \frac{25\sqrt{5\pi}}{88}; E = -\frac{25\sqrt{3c^2\pi}}{176}$$

Thus again we find that the relationship:

$$\frac{E}{P} = -\frac{2c}{p + 2} = -\frac{c}{2} \quad (4.8)$$

is exact.
Thus the solution is relations yields
grove where the relationship

\[ \gamma \frac{\partial H}{\partial \dot{\gamma}} = -\dot{q} \]  

pertains. (See the appendix for details) If we insert this into the action, we obtain for the reduced Hamiltonian (where \( L = -P\dot{q} - H[P] \)), using the above values of \( \alpha, \beta \) and \( \gamma \):

\[ H = \frac{p^2}{\pi} (-9d^6 + 8d^3 - 3d) \]  

From Lagrange's equations \( \frac{\partial H}{\partial \dot{q}} = -\dot{q} \) we obtain a \( 2E/P \) gives back the exact result that \( d^2 = 1/5 \) which is a minimum of the energy and \( d^2 = 1/3 \) which is a maximum of the energy. This suggests that the narrower compacton with \( d^2 = 1/3 \) is unstable. We have confirmed this numerically. We have also shown numerically (see below) that if we start with initial compact data which is wider than the compacton with \( d^2 = 1/5 \) it breaks up into a number of these compactons.

2. \( rp = 4 \) case

Let us now consider the case \( r = \frac{4}{p} \), with \( c_1 = 0 \). When \( \gamma = 0 \) one of the consistency conditions \( Ac = 0 \) can only be satisfied for static solutions. When \( \gamma \neq 0 \) the consistency conditions lead to:

\[ d^2 = \frac{\beta}{\gamma(r^2 + 6r - 2)} \]  
\[ A^2 = \frac{c}{(r - 1)^2 (5 + r) d^4 \gamma} \]  

with the parameters also obeying the constraint:

\[ \alpha = \frac{(2 + r) (4 + r) (-4 + 4r + r^2) \beta^2}{(-2 + 6r + r^2)^2 \gamma} \]  

Let us look at 2 examples from this class of solutions:

For the case

(i) \( p = m = n = 1; l = 0 \)

the solution is of the form:

\[ u = A \cos^4(dy) \]

Choosing for example \( \gamma = 1/38 \) and \( \beta = 1 \), the above relations yields \( d = 1, A = 19c/216 \) and \( \alpha = 672/19 \).

Thus the solution is

\[ u = \frac{9c}{216} \cos^4(x - ct) \]  

(4.13)

For this solution we have that the global quantities are:

\[ M = \frac{19c\pi}{576}; \quad P = \frac{12635c^2\pi}{11943936}; \quad E = \frac{12635c^3\pi}{17915904} \]

So that \( E = -\frac{c^2}{6}P \). We show in the appendix that this solution can also be found by minimizing the effective action using the compacton ansatz. In that case one also gets another solution with \( d^2 = \frac{7}{12} \), which is the maximum of the Hamiltonian, with \( d^2 = 1 \) being a minimum as a function of \( d \). The Hamiltonian as a function of the width \( d \) for the above choice of parameters is given by:

\[ H = \frac{192\sqrt{d} (-77 + 19 d^2 - 2 d^4) P^2}{95 \sqrt{35} \pi}. \]

For the case

(ii) \( p = m = n = 2, l = 0 \)

In this case we obtain the conditions:

\[ r = 2; \quad d^2 = \frac{\beta}{14\gamma}; \quad A^2 = \frac{c}{28d^4\gamma} \]  

(4.14)

Choosing \( \beta = 1, \gamma = \frac{1}{14}, \alpha = \frac{96}{\pi} \), we obtain \( c = 2A^2 \) and the solution is

\[ u = \sqrt{\frac{c}{2}} \cos^2(x - ct). \]

(4.15)

For this choice of parameters we obtain for the conserved quantities:

\[ M = \frac{\pi \sqrt{c}}{2\sqrt{2}}; \quad P = \frac{3\pi c}{32}; \quad E = \frac{3\pi c^2}{64} \]

We notice that we obtain the relationship: \( E/P = -2c/(p+2) \) which we obtained by the variational method. For these parameters, if we use the compacton form as the trial wave function, and use the relation:

\[ A^2 = \frac{16dP}{3\pi} \]

then the effective Hamiltonian for the variational parameter \( d \) is

\[ H = \frac{8P^2}{9\pi} (10d - 5d^3 + d^5) \]

(4.16)

This Hamiltonian has 2 stationary points as a function of \( d \), \( d = 1 \) (minimum) and \( d = \sqrt{2} \) (maximum) The second solution is not a solution to the equation of motion.

Next, let us consider a particular special solution for the case \( c_1 \neq 0; \quad p = m = n = 1, l = 0 \).

For this case we have \( r = 2 \). Assuming a solution of the form

\[ u(y) = A \cos^2(dy) \]


we get the consistency equations:
\[ c_1 = -6A^2d^4\gamma. \]  
\[ d^2 = \frac{1}{12\gamma}[\beta \pm (\beta^2 - \alpha\gamma)^{1/2}] \]  
\[ A = \frac{c}{8d^2(\beta - 8\gamma d^2)}. \]

For the special choice of \( \alpha = 5, \beta = 3 \) and \( \gamma = 1 \), there are two real solutions for \( d^2 \), corresponding to \( d^2 = 1/12 \) and \( d^2 = 5/12 \). The first solution which is stable and which we will discuss further in our section on numerical simulations is:
\[ u = \frac{9c}{14} \cos^2\left(\frac{x - ct}{\sqrt{12}}\right). \]

This solution has for its constants of motion:
\[ M = \frac{9\sqrt{3}\pi c}{14}; \quad P = \frac{243\pi c^2\sqrt{3}}{1568}; \quad E = \frac{2349c^3\pi}{21952}. \]

Thus we obtain:
\[ \frac{E}{P} = \frac{29c}{42}. \]

This shows a failure of the relationship 3.12.

The second solution which we found to be numerically unstable is
\[ u = -\frac{9c}{10} \cos^2\left(\frac{5\sqrt{3}}{12}(x - ct)\right); \quad c > 0. \]

The conserved quantities \( E, M, P \) are given by
\[ M = -\frac{9\pi c\sqrt{3}}{10\sqrt{5}}; \quad P = \frac{243\pi c^2\sqrt{3}}{800\sqrt{5}}; \quad E = -\frac{81\pi c^3\sqrt{3}}{1600\sqrt{5}}. \]

Now we find that \( E/P = -c/6 \neq -2c/(p + 2) \).

The special solutions with \( c_1 \neq 0 \) are not obtainable from a variational calculation but these are a very restricted class of solutions.

**B. Case \( p = m = l, n = 0 \)**

For this case inserting a trial solution of the form \( u = A\cos^p dy \) usually leads to an overdetermined set of equations.

For example for \( p = 1, r = 2 \) we obtain the conditions \((x = \cos dy)\):
\[ 0 = -c_1 - 48A^2d^5\gamma \sin(dy)x + (Ac - 8A^2\beta d^2)x^2 \]
\[ + 96A^2d^5\gamma \sin(dy)x^3 + \left(-\frac{1}{2}A^2\alpha + 12A^2\beta d^2\right)x^4. \]

which has only a trivial solution.

For \( p = 2 \) the situation is simpler and one obtains when \( r = 1 \) the two relations:
\[ A^2 = \frac{c}{2\beta d^2 + 5d^6\gamma} \]
and
\[ 0 = 18d^6\gamma + 12d^2\beta - \alpha. \]

These equations can have 2 or even 3 positive solutions for \( d^2 \). One particular case is \( \alpha = 216, \beta = 21, \gamma = -2 \). In that case we get two positive solutions for \( d^2 \) namely \( d = 1 \) leading to: (i)
\[ u = \frac{\sqrt{c/2}}{4} \cos(x - ct) \]
where the constants of motion are:
\[ M = \frac{1}{2}\sqrt{c/2}; \quad P = \frac{c\pi}{128}; \quad E = -\frac{\pi c^2}{256} \]
and \( d^2 = 2 \) giving the solution: (ii)
\[ u = \frac{\sqrt{c}}{2} \cos\sqrt{3}(x - ct) \]
and we obtain for the conserved quantities:
\[ M = \sqrt{c/2}; \quad P = \frac{c\pi}{16\sqrt{2}}; \quad E = -\frac{\pi c^2}{32\sqrt{2}}. \]

Thus both these solutions again obey the relation: \( E/P = -2c/(p + 2) \). The second solution with \( d^2 = 2 \) turns out to be numerically unstable.

For \( p = 2 \) and \( r = 2 \) we get relations:
\[ c_1 = 8A^3d^6\gamma \]
\[ c = 208A^2\beta d^6\gamma \]
\[ -\alpha + 48\beta d^2 + 1152d^6\gamma = 0 \]
as well as one constraints among the parameters
\[ \beta = -48d^4\gamma \]

Eliminating the constraint, we obtain for the width:
\[ d^6 = -\frac{\alpha}{1152\gamma} \]

As an example if we choose \( \gamma = -3 \) and \( \alpha = 3456 \) then we have \( d = 1 \) and for our solution:
\[ u = \frac{1}{4}(\frac{c}{39})^{1/2}\cos^2(x - ct) \]

We have not exhausted all possible solutions for this case, but the method for finding them should be clear to the reader by now.
C. Some Other More General Cases

When \( l + n = p = m, \) we instead have one parameter family of solutions depending on the velocity \( c. \) That is for fixed \( \alpha, \beta \) and \( \gamma \) there is a solution of different amplitude for different velocities \( c. \) In some special cases there is the possibility for two different solutions with the same value of \( c. \) However, in general for a given \( \alpha, \beta \) and \( \gamma \) there is only one solution with a fixed velocity \( c. \) To illustrate this fact, let us consider the case

\[ p = n; \ m = l = 0 \]

Inserting a trial solution of the form

\[ u = A \cos^{2/3} \theta [d(x - ct)] \]

into Equation (2.10), we obtain for example if \( \gamma = 1, \beta = 1, \) and \( \alpha = -1/2 \) the conditions:

\[ d = \frac{p}{2} [(p + 1)(2p^2 + 4p + 3)]^{-1/4} \]

\[ A^p = \frac{8\beta}{(4-p^2)} [(p + 1)(2p^2 + 4p + 3)]^{1/2} \]

\[ c = \frac{2\beta(p^2 + 8p + 4)}{(4-p^2)(p + 1)(2p^2 + 4p + 3)}. \] (4.29)

Choosing for example \( p = n = 1, l = m = 0, \gamma = 1 \) and \( \alpha = -1/2 \) we get the single solution:

\[ u = 8\sqrt{2} \cos^2 \left( \frac{x - \frac{11\sqrt{2}}{3} t}{2(18)^{1/4}} \right) \] (4.30)

The constants of motion for this case are:

\[ M = 8 \pi (2)^{3/4} \sqrt{3}; \ P = 48 \pi (2)^{1/4} \sqrt{3}; \ E = \frac{400 \pi (2)^{3/4}}{3 \sqrt{3}} \]

So that \( E/P = -\frac{25\pi}{39} \) For the special case when \( p = n = 2, \) one obtains

\[ u = A \cos \theta dy \]

When \( m = l = 0 \) we find

\[ d^4 = -\frac{\alpha}{30\gamma}; \ A^2 = \frac{150\gamma}{\alpha^2} \left( c + \frac{\alpha \beta}{15\gamma} \right). \]

Choosing further \( \beta = \alpha = 1; \gamma = -1/30, \) we obtain

\[ u = \sqrt{10 - c} \cos y \]

So again for this special case we get a continuous family of solutions as long as \( c < 10. \) The constants of motion are now:

\[ M = 2 \sqrt{10 - c}; \ P = \frac{\pi}{4} (10 - c); \ E = \frac{\pi}{40} (10 - c)(30 - c). \]

V. NUMERICAL STUDY OF THE GENERALIZED KDV EQUATION

A. Numerical Method

In our calculations, we approximated the spatial derivatives with a pseudo-spectral method using the discrete Fourier transform (DFT). The equations were integrated in time with a variable order, variable timestep Adams-Bashford- Moulton method. The numerical errors were monitored by varying the number of discrete Fourier modes between 128 and 512 and varying the estimated time error per unit step between \( 10^{-6} \) and \( 10^{-9} \) to insure that the solutions were well converged. Mass and momentum were conserved to an accuracy of at least \( 10^{-6} \) and the Hamiltonian was conserved to an accuracy of better than \( 10^{-2} \) in all of the calculations.

The numerical approximation must respect the delicate balance between the nonlinear numerical dispersion terms in the equation. For example, when the third term in (2.1) is expanded, it has a diffusion-like term \( 2\beta m(u^{n-1}u_x)u_{xx}. \) On the trailing edge of the compacton \( u_x > 0 \) and this term acts like a destabilizing backward diffusion operator. The solution could be unstable if it were not for the stabilizing nonlinear dispersion. This balance is easily lost in numerical approximation if the aliasing, due to the nonlinearities, is not handled carefully. To identify numerical artifacts due to aliasing and other discrete effects, we solved the equations with the nonlinear terms expanded in different formulations. For example, we compared the solutions of the equations when they were differenced in conservation form and non conservation form to identify possible numerical inaccuracies.

Also, the lack of smoothness at the edge of the compacton reduces the spectral method to first order near the edge and introduces dispersive errors into the calculation. To reduce these errors and the errors due to aliasing, we filtered the time derivatives by explicitly adding an artificial dissipation term to the equations. This term was defined in Fourier space to approximate the effects of linear second order dissipation \( (\Delta x u_{xx}) \) on the top 1/3 of the Fourier modes, have no effect on the lower 1/3 of the modes and used a linear transition between the two regions.

Most of the calculations shown here solved the conservative form of the equations with 128 DFT modes and a time error of \( 10^{-8} \) per unit time.

B. Numerical Investigations

For the original Rosenau and Hyman compactons, numerical investigations showed some remarkable properties— namely that whatever initial compact data was given, it eventually evolved into compactons. When two compactons scattered any
energy not in the original pair of compactons emerged as compacton-anticompacton pairs. We will find that the compactons of this fifth order generalized KdV equation have similar properties to those previously found in the studies of Rosenau and Hyman on their third order generalized KdV equation.

The first generic feature of these equations is that arbitrary initial compact data, as long as the width of the packet is larger than that of the compacton evolves into several compactons with the number depending on the initial energy. We show this for two different cases. The first case is related to the compacton of Eq. (4.20). We start of with an initial pulse which is four times the width of the compacton and watch it evolve. This is shown in Fig. 1.

In figures 2 and 3 we show the same phenomena for the compacton system described by Eq. (4.7), again starting from initial data wide compared to the compacton solution.

Fig. 1. Pulse with an initial width four times that of the compacton of Eq. (4.20) pertaining to the parameters $p = m = n = 1, l = 0$ and $\alpha = 5, \beta = 3, \gamma = 1$ namely: $u_0 = \frac{9}{122} \cos^2\left(\frac{x-30}{4\sqrt{12}}\right)$. The initial wide pulse breaks into compactons that collide elastically. Note the phase shift of the slower pulse after colliding with a faster, higher compacton.

In figures 2 and 3 we show the same phenomena for the compacton system described by Eq. (4.7), again starting from initial data wide compared to the compacton solution.

Fig. 2. Similar situation as in Fig. 1 for the parameters $p = m = n = 2, l = 0$ and $\alpha = 6, \beta = 4, \gamma = 3$ relevant to Eq. (4.7). An initial compact wave (solid line) $u_0 = \sqrt{\frac{25}{22}} \cos\left(\frac{x-30}{6}\right)$ wider than a compacton with breaks into a string of compactons with the shape $A\cos\left(\frac{x-ct}{\sqrt{\gamma}}\right)$.

Fig. 3. Same compact wave break up as in Fig.2 displayed differently.

In figures 4, and 5 we show similar features of the breakup of a compact wave for the compacton Eq. (4.13).
Fig. 4 Break up of a compact wave with four times the width of the compacton: $p = m = n = 1, l = 0$ and $\alpha = \frac{672}{19}, \beta = 1, \gamma = \frac{1}{19}$. An initial compact wave (solid line) $u_0 = 2\frac{19}{216}\cos^4\left(x - \frac{7\pi}{4}\right)$ breaks into a string of compactons with the shape $A\cos^4(x - ct)$ by time $t = 10$ (dashed line).

Fig. 5 Gray scale contour plot of the evolution of the compactons in Figs. 4.

Then next generic feature is what happens when two compactons of different speeds collide. The compactons remain coherent and experience a phase shift. This is shown in Fig. 6 for the compactons described by Eq. (4.20).

Unlike the solitons of the original integrable KdV equation, these compactons after scattering also can cause “pair” production of a compacton and anti compacton. This is shown in Fig. 7 in the resolving of the wake left behind during the collision shown in Fig. 6.

Fig. 7. A ripple (solid line) is created when the compactons first collide in Fig. 6. This ripple was extracted from the solution at $t = 25$ and used as an initial condition. By time $t = 500$, the ripple (dashed line) has separated into compactons traveling in opposite directions. These compactons have a shape proportional to $A\cos^2\left(x\sqrt{\frac{c}{3}}\right)$.

When a compacton and an anti-compacton collide, as well as when one starts with initial data which is narrower than the width of the stable compacton one finds numerically blowup at late times. Whether this is an numerical artifact is not absolutely certain. This effect is shown in Fig. 8 for the same compactons as in Figs. 6 and 7.
Fig. 8. Possible blowup after a compacton anti-compacton collision. \( p = m = n = 1, l = 0 \) and \( \alpha = 5, \beta = 3, \gamma = 1 \). Two compactons described by Eq. (4.20) with speed \( c = +1 \) and \( c = -1 \) collide. The numerical solution breaks down slightly after \( t = 15 \). It is not clear if the break down is due to the steep gradients in the solution, or because there is a true singularity that develops in the equations.

APPENDIX - EXACT VARIATIONAL ANSATZ

In this section we determine to what extent we could recover from the variational ansatz the exact solitary wave solutions we have discovered earlier by trial and error. That is if we assumed solutions of the form

\[
A(t) \cos \left[ b(t)(x - q(t)) \right], \quad -\frac{\pi}{2} \leq b(x - q(t)) \leq \frac{\pi}{2}
\]

for compact solitary waves and

\[
A(t) \text{sech}^2 \left[ b(t)(x - q(t)) \right]
\]

for ordinary solitary waves would we recover all the exact solutions?

First let us show that for the KdV equation and for the generalized KdV equation that we investigated earlier, we indeed obtain the exact solution. The Lagrangian for the KdV equation is

\[
L = \int dx \left[ \frac{1}{2} \varphi_x \varphi_t - \frac{1}{2} (\varphi_x)^3 - \frac{1}{2} (\varphi_{xx})^2 \right].
\]

The conserved Hamiltonian is given by

\[
H = \int dx \left[ (\varphi_x)^3 + \frac{1}{2} (\varphi_{xx})^2 \right].
\]

Assuming the trial wave function:

\[
\varphi_x = u(x,t) = A(t) \text{sech}^2 \left[ b(t)(x - q(t)) \right]
\]

we find that the reduced action is

\[
\Gamma = -P \dot{q} - H[A(P), b],
\]

where

\[
P = \int \frac{1}{2} u^2 dx = \frac{2A^2(t)}{3b}
\]

and

\[
H = \frac{8}{15} A^2 \left( \frac{2A}{b} + b \right)
\]

We can rewrite \( H \) in terms of \( A \) as follows:

\[
H = \frac{4}{5} P(\pm (6bP)^{1/2} + b^2)
\]

where we have used the two possible solutions:

\[
A = \pm \left( \frac{3bP}{2} \right)^{1/2}
\]

Since the Hamiltonian is independent of \( q \), we have that \( P \) is conserved. \( b \) is a variable of constraint and is eliminated by the equation:

\[
\frac{\partial H}{\partial b} = 0 = \pm (6PbP)^{1/2} + 2b
\]

Only the negative choice for \( A \) in terms of \( P \) yields a positive solution for \( b \) namely:

\[
b = \frac{(6P)^{1/3}}{4^{2/3}}
\]

Eliminating \( b \) the reduced action is

\[
\Gamma = -P \dot{q} + \frac{(3P)^{5/3}}{5}
\]

Varying the action we find the velocity is a constant:

\[
\dot{q} = \frac{(3P)^{2/3}}{2} = c
\]

Thus \( A = -\frac{c}{2} \) and we get the usual exact answer:

\[
u(x,t) = -\frac{c}{2} \text{sech}^2 \left[ \frac{c^{1/2}}{2} (x - ct) \right]
\]

We also find that

\[
H = \frac{4}{5} P(-6bP^{1/2} + b^2)
\]

has a minimum at the exact value of \( b \) for fixed \( P \).

Next we consider the class of exact compact solitary waves that we found for the generalized KdV equation of ref. \( \ref{3} \). In this case the Lagrangian is

\[
L = \int dx \left[ \frac{1}{2} \varphi_x \varphi_t + \alpha \frac{1}{p(p+1)} (\varphi_x)^{p+2} - \beta \varphi_x (\varphi_{xx})^2 \right],
\]

and the Hamiltonian is:

\[
\Gamma = -P \dot{q} - H[A(P), b],
\]

where

\[
P = \int \frac{1}{2} u^2 dx = \frac{2A^2(t)}{3b}
\]

and

\[
H = \frac{8}{15} A^2 \left( \frac{2A}{b} + b \right)
\]

We can rewrite \( H \) in terms of \( A \) as follows:

\[
H = \frac{4}{5} P(\pm (6bP)^{1/2} + b^2)
\]

where we have used the two possible solutions:

\[
A = \pm \left( \frac{3bP}{2} \right)^{1/2}
\]

Since the Hamiltonian is independent of \( q \), we have that \( P \) is conserved. \( b \) is a variable of constraint and is eliminated by the equation:

\[
\frac{\partial H}{\partial b} = 0 = \pm (6PbP)^{1/2} + 2b
\]

Only the negative choice for \( A \) in terms of \( P \) yields a positive solution for \( b \) namely:

\[
b = \frac{(6P)^{1/3}}{4^{2/3}}
\]

Eliminating \( b \) the reduced action is

\[
\Gamma = -P \dot{q} + \frac{(3P)^{5/3}}{5}
\]

Varying the action we find the velocity is a constant:

\[
\dot{q} = \frac{(3P)^{2/3}}{2} = c
\]

Thus \( A = -\frac{c}{2} \) and we get the usual exact answer:

\[
u(x,t) = -\frac{c}{2} \text{sech}^2 \left[ \frac{c^{1/2}}{2} (x - ct) \right]
\]

We also find that

\[
H = \frac{4}{5} P(-6bP^{1/2} + b^2)
\]

has a minimum at the exact value of \( b \) for fixed \( P \).
\[ H = \int dx \frac{-\alpha}{p(p+1)} (\varphi_x)^{p+2} + \beta \varphi_x^p (\varphi_{xx})^2. \]

Now we assume a solution of the form \( r = 2/p \)
\[
\varphi_x = u(x,t) = A \cos^{2/p} [d(t)(x - q(t))] \quad (5.11)
\]

We obtain for the reduced action
\[
\Gamma = -P \dot{q} - H[A,d] \quad (5.12)
\]

where now
\[
P = \frac{A^2 \sqrt{\pi} \Gamma(1/2 + 2/p)}{2d(t) \Gamma(1/2 + 2/p)} \quad (5.13)
\]

Replacing \( A \) by
\[
\frac{2dP \Gamma(1+2/p)}{\sqrt{\pi} \Gamma(1/2 + 2/p)}
\]

we obtain:
\[
H = (2d)^2 P^{1+2/p} \frac{4\beta d^2(2 + 3p + p^2) - \alpha(4p + p^2)}{(2p + 5p^2 + 4p^3 + p^4) \pi^{\frac{3}{2}} \Gamma^\frac{3}{2}(p + \frac{2}{p})} \quad (5.15)
\]

We determine the constraint variable \( d \) by \( \frac{\partial H}{\partial d} = 0 \), we obtain
\[
d^2 = \frac{\alpha p^2}{4\beta(p+1)(p+2)} \quad (5.16)
\]

Lagrange’s equations gives:
\[
\dot{q} = c = -\frac{\partial H}{\partial P} = -\frac{p+2}{2} \left\{ \frac{H}{P} \right\} \quad (5.17)
\]

We then have that
\[
A^p = \frac{c(p+1)(p+2)}{2\alpha}
\]

and recover our previous exact result [3]:
\[
u(x,t) = \left[ \frac{c(p+1)(p+2)}{2\alpha} \right]^{1/p} \cos^{2/p} \left\{ \frac{p(x-ct)}{4\alpha(p+1)(p+2)} \right\} \quad (5.18)
\]

As a function of \( d \) for fixed \( P \), \( H \) is a minimum at the constraint equation value of \( d \). As an example, when \( p = 1, P = 1 \) and \( \beta = 1/2, \alpha = 1 \) one obtains for \( H[d] \):
\[
H = \frac{2}{9} \left( \frac{d}{3\pi} \right)^{1/2} (-5 + 12d^2)
\]

which has a minimum at \( d^2 = 1/12 \).

Now let us look at our generalized equation when \( \gamma \neq 0 \). For the special case \( p = m = n; l = 0 \) considered in this paper we have that the Lagrangian is:
\[
L(p = m = n; l = 0) = \int dx \left[ \frac{1}{2} \varphi_x \dot{\varphi}_x + \alpha \frac{(\varphi_x)^{p+2}}{(p+1)(p+2)} - \beta (\varphi_x)^p (\varphi_{xx})^2 + \frac{\gamma}{2} \varphi_x^2 \varphi_{xxx} \right] \quad (5.19)
\]

Introducing a trial variational function of the form:
\[
u = A \cos^r d(t)(x - q(t)) \quad (5.20)
\]

with \( r = 4/p \) and the constraint:
\[
\alpha = \frac{(2 + r)(4 + r)(-4 + 4r + r^2)\beta^2}{\gamma(-2 + r^2 + 6r^2)} \quad (5.21)
\]

Using the fact that
\[
P = A^2 \sqrt{\pi} \Gamma(1/2 + r) \frac{1}{2d(t) \Gamma(1 + r)}
\]

to eliminate \( A \) in favor of \( P \), we again find we can write the reduced action as
\[
\int dt \left\{ -P \dot{q} - H[P,d] \right\} \quad (5.22)
\]

where
\[
H = (2d)^2/r P^{1+2/r} \Gamma[1+ r]^{1+2/r} \times
\]
\[
(12\beta^2 + 8\beta^2 \gamma - 32d^2 \gamma^2 + 20\beta^2 r - 32\beta d^2 \gamma r + 176d^4 \gamma^2 r - 19\beta^2 r^2 - 32\beta d^2 \gamma r^2 - 172d^4 \gamma^2 r^2 - 24d^3 \gamma^3) \times
\]
\[
+152\beta d^2 \gamma r^3 - 152d^4 \gamma^2 r^3 - 4\beta^2 r^4 + 50\beta d^2 \gamma r^4 - 152d^4 \gamma^2 r^4 + 4\beta^2 d^2 \gamma r^5 - 40d^4 \gamma^2 r^5 - 3d^3 \gamma^3 r^6) \times
\]
\[
(4\gamma^{1/r}(-2 + 6r + r^2)^2 \Gamma[3 + r] \Gamma[1/2 + r]^{2/r})^{-1} \quad (5.23)
\]

From the equation that eliminates the constraint variable \( d \):
\[
\frac{\partial H}{\partial d} = 0
\]

we find there are two solutions for \( d^2 \). One solution,
\[
d^2 = \frac{\beta}{\gamma(r^2 + 6r - 2)} \quad (5.24)
\]

is a minimum of \( H[d] \) for fixed \( P \) and is an exact solution of the generalized KdV equation. The other solution for
\[
d^2 = \frac{\beta}{\gamma(-2 + 6r + r^2)} \frac{-4 + 4r + r^2}{(8 + 4r + 3r^2)} \quad (5.25)
\]

is a maximum of the energy \( H[d] \) for fixed \( P \) and is not a solution of the equation of motion. An example discussed
earlier is the case \( p = r = 2 \) with \( \beta = 1, \gamma = 1/14, \) and \( \alpha = 96/7. \) In that case we have

\[
H[d] = -\frac{8P^2}{9\pi}(10d - 5d^3 + d^5), \quad (5.26)
\]

with two extrema: \( d = 1 \) which is a minimum of \( H \) and yields an exact solution, \( u = \sqrt{c/2}\cos^2(x - ct), \) and \( d = \sqrt{2} \) which is a maximum and leads to \( u = \sqrt{3c/2}\cos^2\sqrt{2}(x - ct) \) which is not a solution of the original generalized KdV equation.

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