Cut-and-join operators and Macdonald polynomials from the 3-Schur functions

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ABSTRACT

Schur polynomials of infinitely many time-variables are among the most important special functions of modern mathematical physics. They are directly associated with the characters of linear and symmetric group and are therefore labeled by Young diagrams. They possess a somewhat mysterious deformation to Macdonald and Kerov polynomials, which no longer has group-theory interpretation, still preserves most of the nice properties of Schur functions. The family of Schur-Macdonald function, however, is not big enough – needed for various applications are counterparts of the Schur functions, labeled by plane (3d) partitions. Recently a very concrete suggestion was made on how this generalization can be done – and miraculous coincidences on this way can serve as a support to the idea, which, however, needs a lot of work to become a reliable and efficient theory. In particular, one can expect that Macdonald and even entire Kerov deformations should appear in this theory on equal right with the ordinary 2-Schur functions. In this paper we demonstrate in some detail how this works for Macdonald polynomials and how they emerge from the 3-Schur functions when the vector time-variables, associated with plane-partitions, are projected onto the ordinary scalar times under non-vanishing angles, which depend on $q$ and $t$. We also explain how the cut-and-join operators smoothly interpolate between different cases. Most of consideration is restricted to level two.

1 Introduction

One of the tasks of modern theoretical physics is to make quantum field and string theories calculable beyond perturbation expansions. A good understanding of non-perturbative phenomena is achieved by development of instanton calculus in supersymmetric theories, where perturbative effects can be almost eliminated. Then one can look not only on the space-time dependencies, but on the far more interesting structures in the space of theories, their coupling constants and boundary conditions. These properties are best described in terms of integrable systems [1,2], and efficiently handled by the formalism of matrix, tensor and network models. However, while the simplest matrix models are already well investigated, this is not yet the case with their generalizations. One of the main obstacles in these studies is the need for more general special functions with distinguished hidden-symmetry properties, which can be used to express the universal (common) features of many different models and theories and can provide a language to express the answers in quantitative form. There are many reasons to expect that the relevant new functions should include the generalization of $\tau$-functions in one particular direction: from Young diagrams to plane (3d) partitions. Then, since for the ordinary $\tau$-functions the starting point was the theory of Schur polynomials, one can ask for their generalization to 3-Schur functions, which would depend on the properly extended set of time-variables.

To be more concrete, 2d Virasoro conformal blocks [3] and Nekrasov functions [4], encoding the instanton effects in 4d SYM theories, involve sums over Young diagrams (partitions of integers) – and AGT equivalence [5] between these two types of quantities is best described by peculiar ("conformal" or "Dotsenko-Fateev") matrix models [6], efficiently handled by the theory of Schur functions. Increasing space-time dimension leads to $q$ and $t$-deformations, substituting Schur by Macdonald polynomials [7]. However, they are not enough for description of generic 6d case, where the single-loop Virasoro is lifted to a double-loop DIM algebra [8]. Young diagrams are substituted by plane (3d) partitions, and matrix models are promoted to network models [9], defined on a rich variety of graphs. Needed in this case are the new "3-Schur" functions, depending on additional time variables – they were recently introduced in [10]. We assume familiarity with that paper and further elaborate in one of the many directions which is opened by discovery of this new class of special functions. Namely, we provide more details on embedding of the theory of Macdonald polynomials into that of the 3-Schur functions. We also begin developing the formalism of the cut-and-join operators which in the case of 3-Schurs has a non-abelian extension, not well studied even in the case of ordinary 2-Schur functions.

1 For alternative attempt on the lines of [11] see [12]. Perhaps, 3-Schurs are not yet the end of the story: further generalization to 8d involves solid (4d) partitions, and Nekrasov functions seem to dramatically simplify [13] – still underlying symmetries and even the relevant set of time variables, nothing to say about the 4-Schur functions, are not understood.
We now remind just a few basic things about 2-Schur functions and the way they are lifted to the 3-Schur case. Young diagrams are partitions of integers, i.e. they are ordered sequences of positive numbers, \( R = [r_1 \geq r_2 \geq \cdots \geq r_R > 0] \), often depicted as diagrams on the plane with \( l_R \) lines of boxes (squares), of lengths \( r_1, \ldots, r_R \). We often call the total number of boxes \( |R| = r_1 + \cdots + r_R \) the level. A generating function for all Young diagrams depends on infinitely many "time-variables" \( p_k \) with diagram \( R \) one associates a monomial \( p_R = \prod_{i=1}^{l_R} p_{s_i} \). The ordinary Schur (2-Schur) functions \( S_R \{ p_k \} \) are functions of these time-variables, which are the common eigenfunctions of the infinitely many commuting "cut-and-join" operators \( W_\Delta \), also labeled by Young diagrams:

\[
W_\Delta S_R \{ p \} = \psi_R(\Delta)S_R \{ p \}
\]

with eigenvalues \( \psi_R(\Delta) \) being the characters of symmetric group \( S_R \) (the name "cut-and-join" is inherited from the simplest of these operators \( W_{[2]} \), which has a clear interpretation of this kind and appears in a whole variety of applications). Furthermore, the coefficients of Schur functions are again the symmetric-group characters:

\[
S_R \{ p \} = \sum_{\Delta} \frac{\psi_R(\Delta)p_\Delta}{z_\Delta}
\]

with the standard combinatorial factors \( z_\Delta \), and orthogonality of characters imply orthogonality of Schur functions in appropriate scalar product

\[
\langle p_\Delta | p_{\Delta'} \rangle = z_\Delta \cdot \delta_{\Delta,\Delta'}
\]

what can be used as their alternative defintion (as orthogonalization of a monomial basis in the space of symmetric polynomials). Schur functions form a ring with multiplication, describing that of representations of linear group \( sl_N \), and they reduce to the characters of these representations at the Miwa locus \( p_k = tr X^k \) for \( N \times N \) matrix \( X \). Among the most important properties of Schur functions is Cauchy summation formula

\[
\sum_R S_R \{ p \} S_R \{ p' \} = \exp \left( \sum_k \frac{p_k p'_k}{k} \right)
\]

Other well known facts, like determinantal representations and Plucker/Hirota relations are peculiar for Schur functions and more difficult to generalize – in neither of the three directions: to Macdonald-Kerov \( [15] \), generalized Macdonald \( [11] \) and 3-Schur \( [10] \) functions.

Coming straight to 3-Schur functions, they are labeled by plane partitions \( \pi \), which are the two-indexed sets of non-increasing integers, \( \pi_{i,j} \geq \pi_{i+1,j} \) and \( \pi_{i,j} \geq \pi_{i,j+1} \). They can be depicted as piles of cubes lying in the corner in 3 dimensions – thus they are sometime called 3-partitions. The generating functions of such partitions require an extended set of time variables \( \tilde{p}_k = p_k^{(a)} \) with additional index \( a = 1, \ldots, k, \) i.e. dimension of the vector \( \tilde{p}_k \) is equal to its "level" (grading) \( k \). The 3-Schur are functions of these time-variables, and this means that \( S_{\pi} \{ \tilde{p}_k \} \) at level \( m = |\pi| \) are described by the rank-\( m \) tensors, what relates this theory to another rapidly developing new branch of theoretical physics – the theory of tensor models \( [18] \). Both stories are particular parts of the underestimated, still successful non-linear algebra that this enter into the discussion of the underestimated, still successful non-linear algebra theory. We will also continue studying the other properties of 3-Schur functions, in particular, provide an evidence that they satisfy the Cauchy summation formula. This adds to the mysteries of 3-Schur functions (like resolvability of overdefined system of orthogonality relations or linear dependencies between emerging \( \tilde{p} \)-vectors, which are crucial for the entire program to work, see \( [10] \) and the present paper). Like in \( [10] \), we use the simplest examples at level \( |\pi| = 2 \) to reveal the emerging phenomena and then demonstrate that they survive in transition to at least the next level 3. Extension to higher levels will be worked out elsewhere.
2 On the choice of time-variables for Macdonald polynomials

We begin with a simple general comment about Macdonald polynomials [7]. As usual in modern theory
and its applications [1], we consider them not just as symmetric polynomials of the Miwa variables $x_a$, but as
functions of infinitely-many time-variables $p_k$, which reduce to $\{x_a\}$ on the Miwa locus $p_k = \text{tr } X^k = \sum x_a^k$.
Level of sophistication depends on the choice of time-variables, and for different purposes different choices are
convenient. We will need and use a rather unconventional one, which we now describe.

After a simple rescaling of time variables

\[ p_k \rightarrow \left( \frac{1 - t}{1 - q} \right)^k p_k, \]

\[ M_{[n]} \rightarrow \left( \frac{1 - q}{1 - t} \right)^n \prod_{i=1}^n \left( \frac{1 - q^{i-1}t}{1 - q^i} \right) M_{[n]} \] (5)

symmetric and antisymmetric Macdonald polynomials acquire a simple form:

\[ M_{[n]} = \text{Schur}_{[n]} \left\{ \frac{(1 - q)^k}{1 - q^k} p_k \right\} \]

\[ M_{[1^n]} = \text{Schur}_{[1^n]} \left\{ \frac{(1 - t^{-1})^k}{1 - t^{-k}} p_k \right\} \] (6)

Other polynomials, however, remain more sophisticated.

In what follows we will apply another rescaling, which does not change time-variables at $t = q$, i.e. for
ordinary Schur functions, but makes their scalar product and Cauchy formula independent of $q$ and $t$:

\[ \langle p_k | p_l \rangle = k \delta_{ik} \] (7)

Then $\langle M_R | M_{R'} \rangle \sim \delta_{R,R'}$ and

\[ \sum_R M_R \{p\} M_R \{p'\} = \exp \left( \sum_k p_k p'_k \right) \] (8)

This makes particular expressions more complicated, see (65) for the first example. However, this is the most
convenient choice to study relation to 3-Schur functions.

3 Cut-and-join operators for 2-Schur and Macdonald functions

Another piece of the standard theory which we need is that of the cut-and-join operators [14]. The name
comes from the simplest example,

\[ \hat{W}_{[2]} = \sum_{k,l} \left( (k + l)p_k p_l \frac{\partial}{\partial p_{k+l}} + k l p_k + l \frac{\partial^2}{\partial p_k \partial p_l} \right) = \frac{1}{2} : \text{tr } (X \partial X)^2 : \] (9)

which already shows up in different branches of science. In fact such operators exist for arbitrary Young diagram
$\Delta = [\ldots, 3^{m_3}, 2^{m_2}, 1^{m_1}]$:

\[ \hat{W}_\Delta \sim : \text{tr } \prod_i (X \partial X)^{m_i} : \sim \prod_i \hat{W}_{[m_i]} + \ldots \] (10)

though its expression through the time-variables is more involved. Combinatorial factor, mentioned in the
Introduction, is $z_\Delta = \prod_i a^{m_i} \cdot m_i!$.

One of the many properties of these operators is that they all commute and thus have the common eigen-
functions – which are exactly the Schur functions:

\[ \hat{W}_\Delta S_R = \psi_R(\Delta) S_R \] (11)

with eigenvalues, which are the characters $\psi_R(\Delta)$ of the universal symmetric group $S_\infty$ (available through the
command $\text{Chi}$ in MAPLE{combinat} – unfortunately this is no so for the operators themselves). At particular
level $|R|$ only the first $|R|$ time-variables, i.e. $p_k$ with $k \leq |R|$, are operative, and operators $\hat{W}_\Delta$ are accordingly
and self-consistently reduced. Moreover, only \(|\Delta| \leq |R|\) are relevant – other operators reduce to zero. In addition to this sub-ring of commuting and hermitian operators \(\hat{W}_\Delta\) (which we naturally call Cartanian), there are operators which convert one Schur function into another and form a non-abelian structure – not yet attracting attention, which it deserves.

In the simplest case of level two, which will be at the center of our consideration in this paper, there are just two Schur functions

\[
S_{[2]} = \frac{p_2 + p_1^2}{2} = S_{[2]}^+ \quad \text{and} \quad S_{[1,1]} = \frac{-p_2 + p_1^2}{2} = S_{[2]}^-
\]

(12)

where at the r.h.s. we introduced a notation, which will be consistent with lifting to plane partitions. These two polynomials are the common eigenfunctions of dilatation operator and cut-and-join operators:

\[
\hat{W}_{[1]} = \sum_k p_k \partial_k = p_1 \partial_1 + 2p_2 \partial_2 + \ldots, \quad \hat{W}_{[1]}S_{[2]} = 2S_{[2]}, \quad \hat{W}_{[1]}S_{[1,1]} = 2S_{[1,1]},
\]

\[
\hat{W}_{[2]} = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 + \ldots, \quad \hat{W}_{[2]}S_{[2]} = S_{[2]}, \quad \hat{W}_{[2]}S_{[1,1]} = -S_{[1,1]},
\]

(13)

\[
\hat{W}_{[1]} = \frac{1}{2} \hat{W}_{[2]}(\hat{W}_{[1]} - 1) = p_2 \partial_2 + \frac{1}{2} p_1 \partial_1^2 + 2p_2 p_1 \partial_2 \partial_1 + 2p_2^2 \partial_2^2 + \ldots, \quad \hat{W}_{[1]}S_{[2]} = S_{[2]}, \quad \hat{W}_{[1]}S_{[1,1]} = S_{[1,1]}
\]

where dots denote terms with \(p_k \geq 3\), unobservable at the level 2. The raising and lowering operators at this level are

\[
\hat{W}_{2}^+ = \frac{p_2 + p_1^2}{4} \cdot (\partial_1^2 - 2 \partial_2) + \ldots, \quad \hat{W}_{2}^+ S_{[2]} = 0, \quad \hat{W}_{2}^+ S_{[1,1]} = S_{[2]}
\]

(14)

and its hermitian conjugate

\[
\hat{W}_{2}^- = -\frac{p_2 + p_1^2}{4} \cdot (\partial_1^2 + 2 \partial_2) + \ldots, \quad \hat{W}_{2}^- S_{[2]} = S_{[1,1]}, \quad \hat{W}_{2}^- S_{[1,1]} = 0
\]

(15)

In this particular example these operators look obvious, but this is not the case in general – and especially after the deformations, which will be our main concern.

For Macdonald polynomials one often considers difference operators, but in fact the differential cut-and-join operators also exist – and these are more relevant for our consideration at this stage. After one more rescaling of \(p_2\) in the level-two polynomials

\[
M_{[2]} \sim M_{[2]}^+ = \frac{1}{2} \left( \sqrt{\frac{(1-q)(1+t)}{(1+q)(1-t)}} \cdot \hat{W}_{[2]}^+ \right), \quad M_{[1,1]} = M_{[2]}^- = \frac{1}{2} \left( -\sqrt{\frac{(1+q)(1-t)}{(1-q)(1+t)}} \cdot \hat{W}_{[2]}^- \right)
\]

(16)

the simplest of these operators acquires the form

\[
\hat{W}M_{[2]} = \frac{p_2 \partial_1^2 + p_1 \partial_2 \partial_2 - 2\alpha p_2 \partial_2}{\sqrt{(1-q^2)(1-t^2)}}, \quad \sigma = \sqrt{\frac{q-t}{(1-q)(1-t^2)}} = \sqrt{\frac{q^t - t^q}{(q-q^{-1})(t-t^{-1})}}
\]

(17)

Its two eigenvalues in the space of the level-2 operators are

\[
\sqrt{\frac{(1-q)(1+t)}{(1+q)(1-t)}} = i \sqrt{\frac{(1-q)(1+t^{-1})}{(1+q)(1-t^{-1})}} \quad \text{and} \quad -\sqrt{\frac{(1+q)(1-t)}{(1-q)(1+t)}} = i \sqrt{\frac{(1+q)(1-t^{-1})}{(1-q)(1+t^{-1})}}
\]

(18)

Not surprisingly, the theory of cut-and-join operators can be lifted to the 3-Schur level. Like 3-Schur functions themselves, they will be enlarged to a set of commuting (Cartanian) operators, labeled by plane partitions – and 3-Schurs will be their common eigenfunctions. Non-abelian extension also survives and the corresponding non-hermitian operators convert 3-Schur functions into other 3-Schurs. Moreover, now there is no distinguished ordering between partitions of a given size (dualism is substituted by trialism), ”highest and lowest weights” like \(S_{[r]}\) and \(S_{[1,r]}\) get more abundant, and the number of such operators increase, together with increase of the Cartanian sub-ring.

4
4 Differential cut-and-join operators. Level 2

4.1 3-Schur functions

As already mentioned, we assume familiarity with ref. [10] and do not repeat the arguments and definitions from that paper. To simplify the formulas we will use $p_2, \tilde{p}_2$ instead of $p_2^{(1)}$ and $p_2^{(2)}$.

At level 2 there are three plane partitions and three 3-Schur functions

\[
S^0_{[2]} = \frac{\sqrt{2} p_2 + \tilde{p}_2^2}{2}, \quad S^\pm_{[2]} = \pm \frac{\sqrt{2} p_2 - \frac{1}{\sqrt{2}} \tilde{p}_2 + p_2^2}{2}
\]

They are the eigenfunctions of the two linear-independent (and, more symmetrically, three – but linear dependent) cut-and-join operators

\[
\hat{W}^0_{[2]} = -\hat{W}^{0'}_{[2]} - \hat{W}^{0''}_{[2]} = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 - \frac{1}{\sqrt{2}} (\tilde{p}_2 \partial_2 + p_2 \partial_2) + \ldots
\]

and

\[
\hat{W}^{0\pm}_{[2]} = \frac{1}{\sqrt{3}} (\hat{W}^{0''}_{[2]} - \hat{W}^{0'}_{[2]} - \hat{W}^{0'}_{[2]} = \frac{\tilde{p}_2}{2} \partial_1^2 + p_1^2 \partial_2 - \frac{1}{\sqrt{2}} (p_2 \partial_2 + \tilde{p}_2 \partial_2) + \ldots
\]

The dots, to be omitted below, stand for the terms with $p_n$, $n \geq 3$, which are irrelevant at level 2. Superscript 0 means that these are "Cartanian" generators, which leave the 3-Schur functions intact:

\[
\hat{W}^{00}_{[2]} S^0_{[2]} = 0, \quad \hat{W}^{00}_{[2]} S^\pm_{[2]} = \pm \frac{\sqrt{3}}{2} S^\pm_{[2]},
\]

\[
\hat{W}^{0\pm}_{[2]} S^0_{[2]} = \sqrt{2}, \quad \hat{W}^{0\pm}_{[2]} S^\pm_{[2]} = -\frac{1}{\sqrt{2}} S^\pm_{[2]} \]

There are also additional pairs/triples of $W$ with superscripts $\pm$, which act as raising and lowering operators, for example

\[
\hat{W}^\pm_{[2]} = p_1^2 \left( -\frac{1}{2} \frac{\partial}{\partial p_2} + \sqrt{\frac{3}{2}} \frac{\partial}{\partial \tilde{p}_2} \right) - \frac{p_2}{4} \frac{\partial^2}{\partial p_1^2} - \frac{1}{\sqrt{2}} (\tilde{p}_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial \tilde{p}_2}) + \ldots
\]

act as raising and lowering generators in the chain (+0−):

\[
\hat{W}^+_{[2]} S^+_{[2]} = 0, \quad \hat{W}^+_{[2]} S^0_{[2]} = -\sqrt{\frac{3}{2}} S^+_{[2]}, \quad \hat{W}^+_{[2]} S^0_{[2]} = \sqrt{\frac{3}{2}} S^0_{[2]}
\]

\[
\hat{W}^-_{[2]} S^-_{[2]} = -\sqrt{\frac{3}{2}} S^0_{[2]}, \quad \hat{W}^-_{[2]} S^0_{[2]} = \sqrt{\frac{3}{2}} S^0_{[2]}, \quad \hat{W}^-_{[2]} S^-_{[2]} = 0
\]

There are two more pairs of operators – for the chains (0 −) and (− + 0) (i.e. the corresponding raising operators nullify respectively $S^0_{[2]}$ and $S^-_{[2]}$). They can be obtained by $\pm \frac{2\pi}{3}$ rotations in the plane ($p_2, \tilde{p}_2$). Thus, together with the triple of Cartanian $\hat{W}^0_{[2]}$ acting at level 2 is the non-abelian set of 1 + 9 operators (not all linear independent), 1 stands for the dilatation

\[
\hat{W}_{[1]} = \sum_k k \tilde{p}_k \partial_k = \sum_k \sum_{a=1}^k k p_k^{(a)} \frac{\partial}{\partial p_k^{(a)}}
\]

which was observable already at level one.

4.2 Relation/reduction of 3-Schur to ordinary 2-Schur functions

Interpolation from $\hat{W}^0_{[2]}$ in (20) to the ordinary (2-Schur) operator (13),

\[
\hat{W}_{[2]} = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2
\]
is straightforward:

\[ \mathcal{W}^0_{[3]}(h) = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 - h(\hat{\partial}_2 \partial_2 + p_2 \hat{\partial}_2) \]  

(27)

The three interpolating eigenfunctions are

\[ S^0_{[2]}(h) \sim (1 - \lambda^2)\hat{\partial}_2 + h\lambda p_2 + hp_1^2 \]  

(28)

with eigenvalues \( \lambda \) which are the three roots of the characteristic equation

\[ \lambda(\lambda^2 - 1 - h^2) = 0 \]  

(29)

i.e. the three functions are

\[ S^0_{[2]}(h) \sim h p_1^2 + \hat{\partial}_2, \quad S^+_0_{[2]}(h) \sim -h \hat{\partial}_2 + \sqrt{1 + h^2} p_2 + p_1^2 \]  

(30)

and in the 2-Schur limit the first of them depends only on the "foreign" time-variable \( \hat{\partial}_2 \).

The \( h \)-deformation of the second operator (21) is

\[ \mathcal{W}^{0, \pm}_{[2]}(h) = h \left( \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 \right) - h^2 p_2 \partial_2 + (1 - h^2) \hat{\partial}_2 \partial_2 \]  

(31)

at \( h \to 0 \) turns into \( \hat{\partial}_2 \partial_2 \), i.e. acts only on the "hidden" coordinate \( \hat{\partial}_2 \) and "disappears" (decouples) from the world of the 2-Schur functions: it has eigenvalues \( -h^2 \) for \( S^+_0_{[2]}(h) \), which survive in this limit, and 1 for the decoupling \( S^0_{[2]}(h) \).

Note that all these level-two cut-and-join operators consist of three independent combinations \( \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2, \) \( p_2 \hat{\partial}_2 + \hat{\partial}_2 \partial_2 \) and \( p_2 \partial_2 - \hat{\partial}_2 \partial_2 \), which are hermitian (self-conjugate) in the 3-Schur generalization

\[ < p^{(a)}_k | p^{(b)}_l > = k \delta_{k,l} \]  

(32)

of the metric \( \hat{g} \). In this metric \( \hat{p}_k^l = k \hat{\partial}_k \) and \( \hat{\partial}_k^l = \frac{1}{k} \hat{p}_k \). The forth hermitian combination \( p_2 \partial_2 + \hat{\partial}_2 \partial_2 \) is a part of the dilatation operator

\[ \mathcal{W}^0_{[1]} = \sum_k k \hat{p}_k \hat{\partial}_k \]  

(33)

which acts already at the level one.

### 4.3 Rotation in the \( p_2 \)-plane

Clearly, the structures of the last (underlined) terms in \( \mathcal{W}^0_{[2]} \) in (17) and \( \mathcal{W}^0_{[2]} \) in (20) are different, so there can be no direct interpolation between them. The rescue comes from existence of two other cut-and-join operators for 3d-Schurs,

\[ \mathcal{W}^{0, \prime}_{[2]} = p_1^2 \left( -\frac{1}{2} \frac{\partial}{\partial p_2} - \frac{\sqrt{3}}{2} \frac{\partial}{\partial p_2} \right) - \frac{p_2 + \sqrt{3} p_2}{4} \frac{\partial^2}{\partial p_1^2} + \frac{1}{2 \sqrt{2}} \left( \frac{\partial}{\partial p_2} + \frac{\sqrt{3}}{2} \frac{\partial}{\partial p_2} \right) \]  

(34)

of which we can take a linear combination:

\[ -u \cdot \mathcal{W}^{0, \prime}_{[2]} - v \cdot \mathcal{W}^{0, \prime \prime}_{[2]} = u \cdot \left( \frac{p_2 + \sqrt{3} p_2}{4} \left( \frac{\sqrt{3} \partial_2 + \hat{\partial}_2}{\sqrt{2}} + \partial_1^2 \right) + \frac{\sqrt{3} p_2 + \hat{\partial}_2}{2 \sqrt{2}} \right) \]  

(35)
Here we introduced the rotation parameters are \( c = \frac{\sqrt{(u-v)} - \cos \theta}{2} \) and \( s = \frac{\sqrt{(u-v)} - \sin \theta}{2} \). For \( c = 1 \) and \( s = 0 \) we return back to \( \mathcal{W}_{[2]}^0 \). The 3-Schur functions are still the eigenfunctions of this operator, but the eigenvalues are now

\[
\lambda^0 = s \sqrt{2} \quad \text{for} \quad \mathcal{S}^0_{[2]} \quad \text{and} \quad \lambda^\pm = \frac{-s \pm c \sqrt{3}}{1} \quad \text{for} \quad \mathcal{S}^\pm_{[2]}
\]

We can now switch to the new coordinates in the space \((\hat{p}_2, \hat{\theta})\):

\[
P_2 = c p_2 + s \hat{p}_2, \quad \hat{P}_2 = -s p_2 + c \hat{p}_2
\]

\[
D_2 = c \theta_2 + s \hat{\theta}_2, \quad \hat{D}_2 = -s \theta_2 + c \hat{\theta}_2
\]

and obtain:

\[
\hat{\mathcal{W}}_{[2]}^\theta := \frac{P_2}{2} \partial_1^2 + p_2^2 D_2 - \frac{\cos(3\theta)}{\sqrt{2}} (\hat{P}_2 D_2 + P_\hat{2} \hat{D}_2) - \frac{\sin(3\theta)}{\sqrt{2}} (P_2 D_2 - \hat{P}_2 \hat{D}_2)
\]

where \( \cos(3\theta) = (c^2 - 3s^2)c = (1 - 4s^2)c \) and \( \sin(3\theta) = (3c^2 - s^2)s = (3 - 4s^2)c \). Orthogonal operator is

\[
\hat{\mathcal{W}}_{[2]}^{\theta \perp} = \frac{\hat{P}_2}{2} \partial_1^2 + p_2^2 \hat{D}_2 + \frac{\sin(3\theta)}{\sqrt{2}} (\hat{P}_2 D_2 + P_\hat{2} \hat{D}_2) - \frac{\cos(3\theta)}{\sqrt{2}} (P_2 D_2 - \hat{P}_2 \hat{D}_2)
\]

and eigenvalues are equal to \( \lambda_{\perp} = \sqrt{2} \cdot \frac{1 - \frac{\sin(3\theta)}{\cos(3\theta)}}{\lambda^2} \):

\[
\lambda^0 = \sqrt{2} \sin \theta, \quad \lambda^\pm = \sqrt{2} \sin \left( \theta \pm \frac{\pi}{3} \right)
\]

\[
\lambda^0_{\perp} = \sqrt{2} \cos \theta, \quad \lambda^\pm_{\perp} = \sqrt{2} \cos \left( \theta \pm \frac{\pi}{3} \right)
\]

Note that two eigenvalues \( \lambda \) coincide when, say, \( \theta = \frac{\pi}{6} \), but the corresponding \( \lambda_{\perp} \) do not: denominator in the mapping \( \lambda \mapsto \lambda_{\perp} \) vanishes at such points, and the Hopital's resolution provides different \( \lambda_{\perp} \), according to [10].

Another remark is that the mapping \( \theta \mapsto 3\theta \) is exactly the one, which solves generic cubic equation:

\[
x^3 - bx - c = 0 \implies \quad x = \sqrt[3]{\frac{4b^3}{3} \sin \left( \theta + \frac{2k\pi}{3} \right)}, \quad k = 0, \pm 1 \quad \text{provided} \quad \sqrt[3]{\frac{27c^2}{4b^3}} = \sin(3\theta)
\]

Thus it is not surprising that eigenvalues are nicely described at level \( n = 2 \), when the number of 3-Schur functions is three. However, at higher levels \( n > 2 \), the number of functions and the order of characteristic equation for the eigenvalues \( \lambda \) increase, then the existence of trigonometric solutions requires it to be of a special type.

### 4.4 Interpolation/reduction from 3-Schur to Macdonald polynomials

Now we possess the structure, which appears in the last term in [11], and can interpolate to Macdonald polynomials. It is enough to multiply \( P_2 \) and \( D_2 \) by \( h \sqrt{2} \) and adjust the rotation angle so that \( \sin(3\theta) = 2\sqrt{2}\sigma \), i.e.

\[
\sin(3\theta) = 2\sqrt{2}\sigma = 2\sqrt{2} \frac{(q-t)}{\sqrt{(1-q^2)(1-t^2)}} = 2\sqrt{2}i \frac{1 - qt^{-1}}{\sqrt{(1-q^2)(1-t^{-2})}} = 2\sqrt{2} \frac{\sqrt{q/t - \sqrt{t/q}}}{\sqrt{(q^{-1})(1-t^{-1})}}
\]

In other words

\[
\hat{\mathcal{W}}_{[2]}^\theta(h) = \frac{P_2}{2} \partial_1^2 + p_2^2 D_2 - h \cos(3\theta)(\hat{P}_2 D_2 + P_\hat{2} \hat{D}_2) - \frac{\sin(3\theta)}{\sqrt{2}} (P_2 D_2 - \hat{P}_2 \hat{D}_2)
\]

interpolates between

\[
\hat{\mathcal{W}}_{[2]}^\theta = \frac{P_2}{2} \partial_1^2 + p_2^2 D_2 - \frac{\cos(3\theta)}{\sqrt{2}} (\hat{P}_2 D_2 + P_\hat{2} \hat{D}_2) - \frac{\sin(3\theta)}{\sqrt{2}} (P_2 D_2 - \hat{P}_2 \hat{D}_2)
\]

at \( h = \frac{1}{\sqrt{2}} \) and

\[
\hat{\mathcal{W}}_{[2]}^\theta = \frac{P_2}{2} \partial_1^2 + p_2^2 D_2 - \frac{\sin(3\theta)}{\sqrt{2}} P_2 D_2
\]
at $h = 0$. Their eigenfunctions

$$S^{λ}_{[2]}(θ, h) = h \cos(3θ) \left( λP_2 + p_1^2 \right) + \left( 1 - \frac{\sin(3θ)}{\sqrt{2}} λ - λ^2 \right) \tilde{P}_2 = h \cos(3θ) \left( λP_2 + p_1^2 \right) - \frac{λh^2 \cos^2(3θ)}{λ - h^2 \sqrt{2} \sin(3θ)} \tilde{P}_2 =$$

$$= h \cos(3θ)p_1^2 + \left( λh \cos(3θ) \cos θ + \frac{λh^2 \cos^2(3θ) \sin θ}{λ - h^2 \sqrt{2} \sin(3θ)} \right) p_2 + \left( hλ \cos(3θ) \sin θ - \frac{λh^2 \cos^2(3θ) \cos θ}{λ - h^2 \sqrt{2} \sin(3θ)} \right) \tilde{P}_2 \quad (46)$$

are the $θ$-deformations of $S^{2}_{[2]}$, and the eigenvalues $λ$ are now defined as the three roots of the $(h, θ)$-deformed characteristic equation

$$\left( λ - h^2 \sqrt{2} \sin(3θ) \right) \left( λ^2 + \frac{\sin(3θ)}{\sqrt{2}} λ - 1 \right) = λh^2 \cos^2(3θ) \quad (47)$$

$$\uparrow$$

$$\sqrt{2}λ(λ^2 - 1 - h^2) + \sin(3θ)(λ^2(1 - 2h^2) + 2h^2) = 0$$

$$h = \frac{1}{\sqrt{2}} \quad (3 - \text{Schurs}) \quad \upharpoonright \quad h = 0 \quad (2 - \text{Schurs}) \quad \uparrow$$

$$\sqrt{2}λ(2λ^2 - 3) = -2 \sin(3θ) = 2s(4s^2 - 3) \quad \sqrt{2}λ(λ^2 + \frac{\sin(3θ)}{\sqrt{2}} λ - 1) = \sqrt{2}λ(λ - λ_+)(λ - λ_-) = 0$$

$$λ = \sqrt{2}λ = \sqrt{2} \sin θ, \quad -\frac{1}{\sqrt{2}} s + \sqrt{\frac{3}{2} c} = \sqrt{2} \sin \left( θ \pm \frac{2π}{3} \right) \quad λ = 0, \quad λ_+ = \sqrt{\frac{1 + q_1(1 + r)}{1 - q_1(1 + r)}}, \quad λ_- = -\sqrt{\frac{1 + q_1(1 + r)}{1 - q_1(1 + r)}},$$

Eigenfunctions (46) are $θ$-deformations of $S^{2}_{[2]}$ and interpolate between the triple of 3-Schur functions at $h = \frac{1}{\sqrt{2}}$

$$\frac{1}{2h \cos(3θ)}S^{λ}_{[2]}(θ, h = \frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}}(\sin θ P_2 + \cos θ \tilde{P}_2) + \frac{1}{2}p_1^2 = \frac{\sqrt{2}p_2 + p_1^2}{2} = S^{0}_{[2]}, \quad (48)$$

$$\frac{1}{2h \cos(3θ)}S^{λ}_{[2]}(θ, h = \frac{1}{\sqrt{2}}) = -s ± \frac{c \sqrt{2}}{2} P_2 - \frac{c ± s \sqrt{2}}{2} \tilde{P}_2 + \frac{1}{2}p_1^2 p_2 = ± \frac{\sqrt{2}}{2} p_2 - \frac{1}{\sqrt{2}} \tilde{P}_2 + p_1^2 = S^{±}_{[2]}$$

and the pair of MacDonald polynomials (65) at $h = 0$:

$$\lim_{h \to 0} \frac{1}{2h \cos(3θ)}S^{λ}_{[2]}(θ, h = 0) = \frac{1}{2} (λ_± P_2 + p_1^2) = M^{±}_{[2]} \quad (49)$$

In the latter case the third function becomes independent of $P_2$ and $p_1$

$$S^{λ}_{[2]}(θ, h = 0) = \tilde{P}_2 \quad (50)$$

and "decouples".

### 4.5 More on deformed cut-and-join operators

A function of the form (46) with arbitrary (unrelated) $λ$, $h$ and $θ$ is annihilated by three operators:

$$P_2 \partial^2 + p_1^4 P_2 + (λ - λ^{-1})P_2D_2 + λ\tilde{P}_2 \tilde{D}_2 - λ,$$

$$\tilde{P}_2 \partial^2 + p_1^4 \tilde{P}_2 + μλP_2D_2 + \left( μλ - (μλ)^{-1} \right) \tilde{P}_2 \tilde{D}_2 - μλ \quad (51)$$

and

$$P_2 \tilde{D}_2 + \tilde{P}_2 D_2 - μP_2D_2 - μ^{-1} \tilde{P}_2 \tilde{D}_2 \quad (52)$$

with $μ = \frac{1 - \sin(3θ)}{hλ \cos(3θ)}$. However, we want an operator which annihilates three such functions with three different $λ$ at once – or, to be more precise, have them as eigenfunctions. This means that we can add, say the third
operator with some λ-independent coefficient \( w \) to the first one, and all the terms, except for the \( p \)-independent one, should not depend on \( \lambda \). There are two such terms, \( uP_2D_2 + vP_2\tilde{D}_2 \), with

\[
u = \lambda - \frac{1}{\lambda} - w\mu, \quad v = \lambda - \frac{w}{\mu} \quad \implies \quad (\lambda - v)(\lambda^2 - u\lambda - 1) = \lambda w^2 \tag{53}\]

Comparing this with the cubic equation \( \lambda^3 - (1 - 2h^2)(1 + h^2\sin^2(\theta))\lambda^2 + h^2(h^2 - 2)\cos^2(3\theta)\lambda - h^4\cos^4(3\theta) = 0 \) for \( \lambda \), \( \mu\lambda \) also satisfies a cubic equation - what is indeed the case, because its inverse \( \frac{1}{\mu\lambda} = -\frac{\lambda - h^2\sqrt{2}\sin(3\theta)}{\lambda h\cos(3\theta)} \) is a linear function of \( \lambda^{-1} \), which satisfies a cubic equation. It remains to write it down explicitly:

\[
\mu = \frac{1}{\mu\lambda} \frac{\sin(3\theta)}{h\sqrt{2}} \lambda - \frac{h\cos(3\theta)}{\lambda - h^2\sqrt{2}\cos(3\theta)} \quad \implies \quad \mu\lambda + h\cos(3\theta) \left( \mu^2\lambda^2 - \frac{1 - h^2 - 2h^4\sin^2(3\theta)}{h\cos(3\theta)} \cdot \mu\lambda - 1 \right) = 2h^4\sin^2(3\theta)\mu\lambda \tag{56}\]

which implies that \( u = -h\cos(3\theta), \quad v = \frac{1 - h^2 - 2h^4\sin^2(3\theta)}{h\cos(3\theta)}, \quad w = \sqrt{2}h^2\sin(3\theta), \) and the second operator, supplementing \( \mu\lambda \) and \( \lambda \), is

\[
\mathcal{V}(\theta)(h) = \frac{P_2}{2} \partial_1^2 + p_1^2D_2 - h\cos(3\theta)(P_2D_2 + P_2\tilde{D}_2) - \frac{\sin(3\theta)}{\sqrt{2}}(P_2D_2 - 2h^2\tilde{P}_2\tilde{D}_2) \tag{54}\]

As usual, it gets independent of \( P_2 \) and \( p_1 \) in the limit \( h = 0 \), and is unseen at the level of Macdonald polynomials.

The eigenvalues of this operator are

\[
\lambda_\perp = h\cos(3\theta), \quad \lambda_\parallel = 1 - \frac{\sin(3\theta)}{\sqrt{2}} \lambda - \lambda^2 \tag{57}\]

and solve the cubic equation:

\[
\lambda_\perp^3 - (1 - 2h^2)(1 + h^2\sin^2(3\theta))\lambda_\perp^2 + h^2(h^2 - 2)\cos^2(3\theta)\lambda_\perp - h^4\cos^4(3\theta) = 0 \tag{58}\]

which, after substitution of \( \lambda = h\cos(3\theta) \), is, of course, equivalent to \( \lambda^3 \).

At symmetric (i.e. the 3-Schur) point \( h = \frac{1}{\sqrt{2}} \), the entire operator is divisible by \( \cos(3\theta) \) (thus in \( \lambda = h\cos(3\theta) \), \( \lambda_\parallel = \frac{1 - \sin(3\theta)}{\sqrt{2}} \lambda - \lambda^2 \) and \( \lambda_\perp = \) the entire operator is divisible by \( \cos(3\theta) \). This means that the entire operator is divisible by \( \cos(3\theta) \). This means that the entire operator is divisible by \( \cos(3\theta) \).

\[
\lambda^\parallel = \sin(\theta), \quad \lambda^\parallel = \sin(\theta \pm \frac{2\pi}{3}) \tag{59}\]

\[
\frac{\lambda^\parallel}{\cos(3\theta)} = \frac{(1 - \sin^2(\theta))(1 - 4\sin^2(\theta))}{\cos(3\theta)} = \cos(\theta), \quad \frac{\lambda^\perp}{\cos(3\theta)} = \frac{(1 - 4\sin^2(\theta))\cos(\theta)}{\cos(3\theta)} = \cos(\theta \pm \frac{2\pi}{3}) \tag{59}\]

4.6 \( h \)-evolution in pictures

To visualize the situation, we can use the following picture:
Black dots denote triples of eigenvalues of the two Cartanian operators at different values of deformation parameters \( h \) (horizontal axis) and \( \theta \) (two pictures). At \( h = 0 \) on the vertical axis we get eigenvalues for Schur (at \( \theta = 0 \)) and Macdonald (at generic \( \theta \)) functions. At \( h = \frac{1}{\sqrt{2}} \) these are the eigenvalues of the 3-Schur functions in original coordinates \( \tilde{p}_2 \) and \( \theta \)-rotated \( \tilde{F}_2 \). Rotation is actually along a circle in the \((p_2, \tilde{p}_2)\) plane, which can be associated with a circle in the vertical plane of \( W \) eigenvalues. Similarly, at \( h = 0 \) the \( p_2 \) axis, relevant for description of ordinary 2-Schur and Macdonald functions can be associated with a vertical line.

Interpolation, which we described above, connects the black dots at fully-3d \( \mathbb{Z}_3 \)-symmetric point \( h = \frac{1}{\sqrt{2}} \) with those at the 2d-point \( h = 0 \) in a straightforward way. However, while this can seem natural at \( \theta = 0 \), when \( \theta \) increases towards \( \frac{\pi}{2} \), it is natural to expect that twisting will result in a jump to another branch of interpolating function – what is not yet immediately seen in our formulas.

Some information is provided by discriminant of \([47]\)

\[
\text{disc}_\lambda \left\{ \lambda(\lambda^2 - 1 - h^2) + \frac{\sin(3\theta)}{\sqrt{2}} \left( \lambda^2(1 - 2h^2) + 2h^2 \right) \right\} = \]

\[
= 4(1 + h^2)^3 + \frac{\sin^2(3\theta)}{2} (1 - 38h^2 - 75h^4 + 76h^6 + 4h^8) - 2h^2(1 - 2h^2)^3 \sin^4(3\theta)
\]

(60)

but it is limited, because vanishing of this discriminant means only that the projections on the vertical line of the "trajectories" \( \lambda_0(h) \) intersect – and this is necessary, but not sufficient for a true intersection. For example, as obvious from the picture, for \( \sin(3\theta) = 1 \), say, \( \theta = \frac{\pi}{6} \), this discriminant vanishes already at the symmetric (3d) point \( h = \frac{1}{\sqrt{2}} \):

\[
\text{disc}_\lambda [47]_{\sin(3\theta) = 1} \sim (1 + h^2)^2(1 - 2h^2)^2
\]

(61)

Indeed, vertical projections obviously coincide at this point, but the points themselves remain different – coincident are the two eigenvalues of \( W_{[2]}^0 \), but not of \( W_{[2]}^0 \), and no reshuffling takes place.

Technically, it is more practical to plot \( h^2 \) as a function of \( \lambda \) and then rotate the picture:

\[
\lambda(\lambda^2 - 1 - h^2) + \frac{\sin(3\theta)}{\sqrt{2}} \left( \lambda^2(1 - 2h^2) + 2h^2 \right) = 0 \quad \Longrightarrow \quad h^2 = \frac{\lambda(\lambda - \lambda_+)(\lambda - \lambda_-)}{\lambda + \sqrt{2}(\lambda^2 - 1) \sin^2(3\theta)}
\]

(62)

where \( \lambda_- = -\lambda_+^{-1} \) and \( \sin(3\theta) = -\sqrt{2}(\lambda_+ + \lambda_-) \). Then we easily get the following pattern from the evolution of \( \lambda \) with \( h^2 \) for \( \frac{1}{\sqrt{2}} < |\lambda_+| < \sqrt{2} \), i.e. \( |\sin(3\theta)| < 1 \):
This particular picture is for positive $1 < \rho < \sqrt{2}$, i.e. for negative $\sin(3\theta)$, the switch to positive $\sin(3\theta)$ via the changes $\rho \rightarrow \rho^{-1}$ or $\rho \rightarrow -\rho$ is upside-down reflection:

The region $h^2 < 0$ is "unphysical", but it is there that the behaviour of evolution lines is defined. This particular picture is for positive $1 < \lambda_+ < \sqrt{2}$, i.e. for negative $\sin(3\theta)$, the switch to positive $\sin(3\theta)$ via the changes $\lambda_+ \rightarrow \lambda_+^{-1}$ or $\lambda_+ \rightarrow -\lambda_+$ is upside-down reflection:

$$0 < \sin(3\theta) < 1$$

We added also a plot for the $h$-evolution of $\frac{\lambda_+}{\cos(3\theta)}$ from [22], which does not depend on the sign of $\theta$. Division by $\cos(3\theta)$ makes the picture more informative, by resolving the zero of $\lambda_+$ at $\sin(3\theta) = \pm 1$, which is common for all branches.

The slope of the asymptotic line is $\sqrt{2}\sin(3\theta)$, at vanishing $\theta$ this line becomes horizontal axis $\lambda = 0$. The two horizontal asymptotic lines are at the roots of $\lambda + \sqrt{2}(\lambda^2 - 1)\sin(3\theta)$, i.e. at

$$\lambda_{\pm}^\infty = -\frac{1 \pm \sqrt{1 + 8\sin^2(3\theta)}}{2\sqrt{2}\sin(3\theta)}$$ \hspace{1cm} (63)

For vanishing $\theta$, i.e. when $t = q$, these turn into zero and infinity, while the eigenvalues at $h = 0$ become $0, \pm 1$, associated with the ordinary 2-Schur functions [22], and at $h = \frac{1}{\sqrt{2}}$ they become $0, \pm \sqrt{3}$, associated with original (non-rotated) 3-Schurs [19]. Note in passing, that nicely expressed through $q$ and $t$ is a surprisingly similar, still different combination

$$\sqrt{1 + \frac{\sin^2(3\theta)}{8}} \hspace{1cm} \frac{1 - qt}{\sqrt{(1 - q^2)(1 - t^2)}}$$ \hspace{1cm} (64)

The turning points are the roots of the equation $\frac{\partial h}{\partial \lambda} = 0$, i.e. of $(\lambda^4 - \frac{3}{2}\lambda^2 + 1)\sin(3\theta) + \sqrt{2}\lambda (\lambda^2 - \sin^2(3\theta))$. 
When $\sin(3\theta)$ approaches $-1$, the curve tends to the asymptotes, moreover, the crossing, marked by a white circle, tends to $h^2 = \frac{1}{2}$ (since $\lambda_\infty \rightarrow \frac{1-3\sqrt{2}}{2\sqrt{2}} = -\sqrt{2}, \frac{1}{2}$) so that the two negative eigenvalues $\lambda$'s at the 3-Schur point $h^2 = \frac{1}{2}$ coincide: these are $\sqrt{2}\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{2}}$ and $\sqrt{2}\sin\left(-\frac{\pi}{6} - \frac{2\pi}{3}\right) = \sqrt{2}\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{2}}$. 

In the last picture for $\lambda_\perp$ the turning point between the nearly vertical branches is actually much higher than schematically shown, it tends to infinity when $\sin(3\theta) = 1$. Also, as already mentioned, at this point all the branches of the true $\lambda_\perp$ actually vanish at $h^2 = \frac{1}{2}$ — we eliminated this semi-artificial zero by division over $\cos(3\theta)$. Still inclusion of $\cos(3\theta)$ seems important for smooth interpolation, and this singularity at the codimension-two point $(\sin(3\theta) = 1, \ h^2 = \frac{1}{2})$ in the moduli space of 3-Schur deformations does not look easily unavoidable.

Nothing special happens at $\sin(3\theta) = 1$ at the Macdonald point $h = 0$, still further increase of $\lambda_\perp$ beyond $\sqrt{2}$ requires analytical continuation to $|\sin(3\theta)| > 1$, when eigenvalues at $h^2 = \frac{1}{2}$ are no longer real (alternatively one can consider complex-valued eigenvalues at $h = 0$, i.e. complex-valued $q$ and $t$).

This last picture corresponds to "unphysical" situation when $|\sin(3\theta)| > 1$.

Coming back to the switch between branches at the points $\sin(3\theta) = 0$ and $\sin(3\theta) = \pm 1$, it is useful to redraw the pictures in still another projection:
Coordinates on the plane are \( \lambda \) and \( \lambda_\perp \). Thick lines show the evolution from \( h^2 = \frac{1}{3} \) (black dots) to \( h^2 = 0 \) (white dots). Thin lines describe evolution in the region \( h^2 > \frac{1}{3} \), where some of the lines/branches actually merge. Evolution in "unphysical" region \( h^2 < 0 \) is more involved and is schematically shown in just one of the pictures. The series of pictures clarifies how reshuflings at \( \sin(3\theta) = 0 \) and \( |\sin(3\theta)| = 1 \) take place. In the former case the rightmost white and black dots merge with no essential reshuffling of evolution lines. In the latter case this white dot runs to \( +\infty \) and re-appears at \( -\infty \), with an abrupt switch of roles between thick and thin segments of evolution lines. Small circles show the positions of eigenvalues at \( \theta = 0 \). Those on the vertical line are \( \lambda^\pm_1 = \pm 1 \) and maximal deviation from them are to \( |\lambda_\perp| = 2^{\pm 2} \), what corresponds to \( t^{\pm 1} = \frac{1+3q}{3+q} \) – beyond this region one should consider complex angles \( \theta \). This is also needed is \( t \) and/or \( q \) are complex.

### 4.7 In the search for triality

In the limit \( h \to 0 \) the three eigenvalues \( \lambda \) tend to 0 and to the two values \( \sqrt{\frac{(1-q)(1+t)}{(1+q)(1-t)}} = i \sqrt{\frac{(1-q)(1+t^{-1})}{(1+q)(1-t^{-1})}} \) and \( -\sqrt{\frac{(1+q)(1-t)}{(1-q)(1+t)}} = i \sqrt{\frac{(1+q)(1-t^{-1})}{(1-q)(1+t^{-1})}} \) which are obviously related by the symmetry \( q \leftrightarrow t^{-1} \). The same symmetry relates the two Macdonald polynomials if we write them as

\[
M^+_2 = \frac{1}{2} \left( i \sqrt{\frac{(1-q)(1+t^{-1})}{(1+q)(1-t^{-1})}} \cdot P_2 + p^2_2 \right), \quad M^-_2 = \frac{1}{2} \left( i \sqrt{\frac{(1+q)(1-t^{-1})}{(1-q)(1+t^{-1})}} \cdot P_2 + p^2_1 \right)
\]

(65)

Note that the sign difference in the coefficients of \( p_2 \) is an automatic consequence of this symmetry – results from a transition from symmetric formulation with \( t^{-1} \) to asymmetric, but more practical formulation with \( t \). The third Schur tends to a function of the "hidden" variable \( \bar{P}_2 \) and decouples from the world of 2-Schurs, which depend on \( P_2 \) and \( p_1 \) (remaining orthogonal to them).

A natural question is what happens to the third 3-Schur functions in this limit and if there is any sign of triality – an expected symmetry between \( q_1 = q \), \( q_2 = t^{-1} \) and \( q_3 = tq^{-1} \). Unfortunately, direct association with three "original" values of \( \theta \), is not simple: characteristic equations involve

\[
\sin(3\theta) = -4 \sin \theta \cdot \sin \left( \theta + \frac{2\pi}{3} \right) \cdot \sin \left( \theta - \frac{2\pi}{3} \right) = 2\sqrt{2} \frac{q-t}{\sqrt{(1-q^2)(1-t^2)}}
\]

(66)

and are fully \( Z_3 \)-invariant. Resolution of triality mystery remains among the challenges for the future work.

One of the inspirations here can be that the third parameter \( tq^{-1} \) naturally appears in the cut-and-join operator for generalized Macdonald functions of [11] – and this can be right direction to look for triality.
4.8 What happened: a summary

We now summarize what actually happened in this simple level-two example. We started from a $Z_3$-symmetric triple of 3-Schur functions, and rescaled with the help of $h$-deformation one of the two dimensions in the $(p_2, \tilde{p}_2)$ plane — this led to the pair of ordinary 2-Schur functions, while the third dimension in $Z_3 \subset SO(2)$, at other levels they change — but one “decoupled”. If we do the same, but first $\theta$-rotated the $(p_2, \tilde{p}_2)$ plane, then the same procedure leads to a pair of Macdonald functions. The reason why deviation of $\theta$ from zero affects the answer is that the symmetry in the $(p_2, \tilde{p}_2)$ plane was just a $Z_3 \subset SO(2)$, not the full rotation group: thus rescaling “along” one of the preferred directions (associated with one of the special angles $\theta = 0, \pm \frac{2\pi}{3}$ and with particular 3-Schur function) is different from that in arbitrary direction. As a remnant of original $Z_3$-symmetry we get triality — when parameters $q_{1,2,3}$ are changed under of a simple rotation.

Underlined in the above paragraph are numbers, peculiar to the level two, at other levels they change — but the whole scheme remains the same.

5 Level 3

5.1 Macdonald polynomials

In coordinates, which respect $\mathbb{Z}_3$, the three Macdonald polynomials at level 3, complementing \([63]\),

$$M_{[2]}^+ = M_{[2]} = \frac{1}{2} \left( \sqrt{\frac{(1-q)(1+t)}{(1+q)(1-t)} \cdot p_2 + p_1^2} \right), \quad M_{[2]}^- = M_{[11]} = \frac{1}{2} \left( -\sqrt{\frac{(1-q)(1-t)}{(1+q)(1+t)} \cdot p_2 + p_1^2} \right) \quad (67)$$

are:

$$M_{[3]}^+ = M_{[3]} = \frac{1-q}{1+q+q^2} \cdot \frac{1}{3} \left( \frac{1-q}{1+q+q^2} \cdot p_3 \right), \quad M_{[3]}^- = M_{[111]} = \frac{1-q}{1+q+q^2} \cdot \frac{1}{3} \left( \frac{1-q}{1+q+q^2} \cdot p_3 \right) \quad (68)$$

The squared norms of these polynomials in metric $\mathbb{Z}_3$ are:

$$||M_{[1]}||^2 = 1, \quad ||M_{[2]}||^2 = \frac{1-q}{1+q+q^2} \cdot \frac{1}{3} \left( \frac{1-q}{1+q+q^2} \cdot p_3 \right), \quad ||M_{[3]}||^2 = \frac{1-q}{1+q+q^2} \cdot \frac{1}{3} \left( \frac{1-q}{1+q+q^2} \cdot p_3 \right) \quad (69)$$

Extension of operator \([64]\) is

$$\tilde{W}M_{[2]} = \frac{p_2}{2} \partial_1^2 + p_1^2 \partial_2 - 2\sigma_5 \partial_5 + u \cdot (2p_3 \partial_5 \partial_3 + 3\partial_1 + \partial_3) + v \cdot p_3 \partial_3 + w \cdot p_2 \partial_5 \partial_1 + \ldots, \quad (70)$$

where

$$u = \frac{2\sqrt{(1+q+q^2)(1+t+t^2)}}{2 + q + t + 2qt} \quad (71)$$

describes the $q,t$-deformation of a term, present in the ordinary (2-Schur) cut-and-join operator, while the three other parameters are peculiar for Macdonald polynomials (and vanish for 2-Schurs):

$$\sigma = \frac{q-t}{\sqrt{(1-q^2)(1-t^2)}}, \quad v = -\frac{9(q-t)}{2 + q + t + 2qt} \cdot \frac{(1+q)(1+t)}{(1-q)(1-t)}, \quad w = -\frac{2(q-t)}{2 + q + t + 2qt} \cdot \frac{(1-q)(1+t)}{(1+q)(1+t)} \quad (72)$$

In addition to this $\tilde{W}M_{[2]}$ at level 3 we can observe two higher $W$-operators $\tilde{W}M_{[3]}$ and $\tilde{W}M_{[21]}$, which were vanishing at level two.
5.2 3-Schur functions

According to [10] the six Schur functions associated with six plane partitions at level three (i.e. made from three cubes)

\[ S^0_{[3]} = \frac{2\tilde{p}_3 + \sqrt{6}\tilde{p}_3}{3} + \frac{\tilde{p}_2 p_1 + p_1^3}{6}, \quad S^\pm_{[3]} = \frac{4\tilde{p}_3 - \sqrt{6}\tilde{p}_3 + 3\sqrt{2}p_3}{6} + \frac{(-\tilde{p}_2 + \sqrt{3}p_2)p_1}{2\sqrt{2}} + \frac{p_3^3}{6}, \]

\[ S^0_{[21]} = -\frac{\tilde{p}_3 + \sqrt{6}\tilde{p}_3}{6} - \frac{\tilde{p}_2 p_1 + p_1^3}{3}, \quad S^\pm_{[21]} = -\frac{2\tilde{p}_3 - \sqrt{6}\tilde{p}_3 + 3\sqrt{2}p_3}{12} + \frac{(-\tilde{p}_2 + \sqrt{3}p_2)p_1}{2\sqrt{2}} + \frac{p_3^3}{3} \]  

(73)

Time variables \( \tilde{p}_2 = (p_2, \tilde{p}_2) \) are the same as at level two. Now they are complemented by a three-dimensional vector \( \tilde{p}_3 = (p_3, \tilde{p}_3, \tilde{\tilde{p}}_3) \). The Cauchy formula persists:

\[ \sum_n S_n[p] S_n[p'] < S_n[S_n] > = \exp \left( \sum_n \frac{\tilde{p}_n \tilde{p}_n'}{n} \right) = \exp \left( p_1 p_1' + \frac{p_2 p_2' + \tilde{p}_2 \tilde{p}_2'}{2} + \frac{p_3 p_3' + \tilde{p}_3 \tilde{p}_3'}{3} + \ldots \right) \]  

(74)

The simplest cut-and-join operator is extended to

\[ \hat{\mathcal{W}}^0_{[2]} = \frac{p_2^3}{2} \frac{\partial}{\partial p_1^2} + p_1^2 \frac{\partial}{\partial p_2} + \frac{1}{\sqrt{2}} \left( p_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_2} \right) + \frac{3(\sqrt{3}p_2 + p_2 p_1)}{2} \frac{\partial}{\partial p_3} - \frac{3(p_3 + \sqrt{3}p_2 p_1)}{2\sqrt{2}} \frac{\partial}{\partial p_3} + \]  

\[ + \frac{3}{2} \left( \sqrt{3}	ilde{p}_3 - \frac{\tilde{p}_3 + \sqrt{3}p_2 p_1}{\sqrt{2}} \right) \frac{\partial}{\partial p_3} + \frac{\sqrt{3}p_3 + p_2 p_1}{\sqrt{2}} \frac{\partial}{\partial p_2 p_1} + \left( \tilde{p}_3 + \sqrt{3}p_3 + p_2 p_1 \right) \frac{\partial^2}{\partial p_2 p_1} + \ldots \]  

(75)

(as usual, dots stand for terms which do not act at level three and will be omitted) and it now acquires six new eigenfunctions [23]:

\[ \hat{\mathcal{W}}^0_{[2]} S^0_{[3]} = 0, \quad \hat{\mathcal{W}}^0_{[2]} S^\pm_{[3]} = \pm 3 \sqrt{\frac{3}{2}} S^\pm_{[3]}, \quad \hat{\mathcal{W}}^0_{[2]} S^0_{[21]} = 0, \quad \hat{\mathcal{W}}^0_{[2]} S^\pm_{[21]} = \frac{3}{2} \sqrt{\frac{3}{2}, S^\pm_{[21]} } \]  

(76)

Also extended is [24]:

\[ \hat{\mathcal{W}}^{0,\perp}_{[2]} = \frac{p_2^3}{2} \frac{\partial}{\partial p_1} + p_2^3 \frac{\partial}{\partial p_2} + \frac{1}{\sqrt{2}} \left( p_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_2} \right) \left( p_1 \frac{\partial}{\partial p_1} - 1 \right) + \frac{3}{2\sqrt{2}} \left( \tilde{p}_3 \frac{\partial}{\partial p_3} - p_3 \frac{\partial}{\partial p_3} \right) + \frac{3}{2\sqrt{3}} \left( \tilde{\tilde{p}}_3 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + \tilde{p}_3 \frac{\partial}{\partial \tilde{\tilde{p}}_3} \right) + \]  

\[ + \frac{1}{2} \left( 3\tilde{p}_2 p_1 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + 2\tilde{\tilde{p}}_3 \frac{\partial^2}{\partial \tilde{\tilde{p}}_3 p_1} \right) + \frac{\sqrt{3}}{2\sqrt{2}} \left( \left( 3\tilde{\tilde{p}}_2 p_1 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + 2\tilde{\tilde{p}}_3 \frac{\partial^2}{\partial \tilde{\tilde{p}}_3 p_1} \right) - \left( 3\tilde{\tilde{p}}_2 p_1 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + 2\tilde{\tilde{p}}_3 \frac{\partial^2}{\partial \tilde{\tilde{p}}_3 p_1} \right) \right) + \ldots \]  

(77)

with

\[ \hat{\mathcal{W}}^{0,\perp}_{[2]} S^0_{[3]} = 3\sqrt{2} S^0_{[3]}, \quad \hat{\mathcal{W}}^{0,\perp}_{[2]} S^\pm_{[3]} = -\frac{3}{\sqrt{2}} S^\pm_{[3]}, \quad \hat{\mathcal{W}}^{0,\perp}_{[2]} S^0_{[21]} = -\frac{3}{2\sqrt{2}} S^0_{[21]}, \quad \hat{\mathcal{W}}^{0,\perp}_{[2]} S^\pm_{[21]} = -\frac{3}{2\sqrt{2}} S^\pm_{[21]} \]  

(78)

as well as the raising and lowering operators \( \hat{\mathcal{W}}^{\pm}_{[2]} \) and \( \hat{\mathcal{W}}^{\pm,\perp}_{[2]} \).

5.3 Level-three cut-and-join operators

Also emerging at level three is the whole new family \( \hat{\mathcal{W}}_{[3]} \), which were not seen (acted by zero) at lower levels. In particular there is a triple (since the \( \tilde{p}_3 \) is now three-dimensional) of linear independent ”Cartanian” operators:

\[ \hat{\mathcal{W}}^0_{[3]} = \left( p_3 \frac{\partial^3}{\partial p_3} + 3p_1 \frac{\partial^3}{\partial p_3} \right) - 3 \sqrt{2} \left( p_2 p_1 \frac{\partial^2}{\partial p_2 p_1} + \tilde{p}_2 p_1 \frac{\partial^2}{\partial p_2 p_1} \right) + \frac{15}{2} \left( \tilde{p}_3 \frac{\partial}{\partial p_3} + p_3 \frac{\partial}{\partial p_3} \right) - \frac{9}{2} \sqrt{2} \left( \tilde{\tilde{p}}_3 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + \tilde{\tilde{p}}_3 \frac{\partial}{\partial \tilde{\tilde{p}}_3} \right) + \]  

\[ + \frac{3\sqrt{3}}{2} \left( 3\tilde{p}_2 p_1 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + 2\tilde{\tilde{p}}_3 \frac{\partial^2}{\partial \tilde{\tilde{p}}_3 p_1} \right) - \frac{3}{2\sqrt{2}} \left( 3\tilde{\tilde{p}}_2 p_1 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + 2\tilde{\tilde{p}}_3 \frac{\partial^2}{\partial \tilde{\tilde{p}}_3 p_1} \right) - \frac{3}{2\sqrt{2}} \left( 3\tilde{\tilde{p}}_2 p_1 \frac{\partial}{\partial \tilde{\tilde{p}}_3} + 2\tilde{\tilde{p}}_3 \frac{\partial^2}{\partial \tilde{\tilde{p}}_3 p_1} \right) \]  

(79)

with

\[ \hat{\mathcal{W}}^0_{[3]} S^0_{[3]} = 0, \quad \hat{\mathcal{W}}^0_{[3]} S^\pm_{[3]} = \pm 9\sqrt{2} : S^\pm_{[3]}, \quad \hat{\mathcal{W}}^0_{[3]} S^0_{[21]} = 0, \quad \hat{\mathcal{W}}^0_{[3]} S^\pm_{[21]} = \pm \frac{9}{2\sqrt{2}} S^\pm_{[21]} \]  

(80)
then

\[ \dot{W}_{[3]}^{0,\perp} = \left( \hat{p}_3 \frac{\partial^3}{\partial p^3} + 3 \hat{p}_1 \frac{\partial}{\partial \hat{p}_3} \right) + 3 \sqrt{3} \left( \hat{p}_2 \hat{p}_1 \frac{\partial^2}{\partial \hat{p}_2 \partial p_1} - p_2 p_1 \frac{\partial^2}{\partial p_2 \partial p_1} \right) + \frac{15}{2} \left( \hat{p}_3 \frac{\partial}{\partial \hat{p}_3} + \hat{p}_3 \frac{\partial}{\partial \hat{p}_3} \right) + \frac{9}{2} \sqrt{3} \left( \hat{p}_3 \frac{\partial}{\partial \hat{p}_3} - p_3 \frac{\partial}{\partial \hat{p}_3} \right) + \frac{3 \sqrt{3}}{2} \left( 2 \hat{p}_2 \hat{p}_1 \frac{\partial}{\partial \hat{p}_3} + 2 \hat{p}_2 \frac{\partial}{\partial \hat{p}_3} + 2 \hat{p}_2 \frac{\partial}{\partial \hat{p}_3} \right) + \frac{3 \sqrt{3}}{2} \left( 2 \hat{p}_2 \hat{p}_1 \frac{\partial}{\partial \hat{p}_3} + 2 \hat{p}_2 \frac{\partial}{\partial \hat{p}_3} + 2 \hat{p}_2 \frac{\partial}{\partial \hat{p}_3} \right) \right) \]

with

\[ \dot{W}_{[3]}^{0,\perp} S_{[3]}^0 = 6 \sqrt{6} \cdot S_{[3]}^0, \quad \dot{W}_{[3]}^{0,\perp} S_{[3]}^\perp = -3 \sqrt{6} \cdot S_{[3]}^\perp, \quad \dot{W}_{[3]}^{0,\perp} S_{[21]}^0 = 3 \sqrt{\frac{3}{2}} \cdot S_{[21]}^0, \quad \dot{W}_{[3]}^{0,\perp} S_{[21]}^\perp = -\frac{3}{2} \sqrt{\frac{3}{2}} \cdot S_{[21]}^\perp \]

and

\[ \dot{W}_{[3]}^{0,\perp} S_{[3]}^0 = 12 S_{[3]}^0, \quad \dot{W}_{[3]}^{0,\perp} S_{[3]}^\perp = 12 S_{[3]}^\perp, \quad \dot{W}_{[3]}^{0,\perp} S_{[21]}^0 = - \frac{3}{2} S_{[21]}^0, \quad \dot{W}_{[3]}^{0,\perp} S_{[21]}^\perp = - \frac{3}{2} S_{[21]}^\perp \]

Note that there is no full symmetry between the three times \( p_3, \hat{p}_3, \) and \( \hat{p}_3 \) – like there was no between \( p_2, \hat{p}_2 \) and \( \hat{p}_2 \) already at level 2. Still in the latter case the symmetry was just \( Z_3 \), while for \( \hat{p}_3 \) it is a little more involved.

Like the linear independent pair \( \dot{W}_{[2]}^0, \dot{W}_{[2]}^{0,\bot} \) was treated in [10] as an explicitly symmetric (but not independent) triple \( \dot{W}_{[2]}^0, \dot{W}_{[2]}^m, \dot{W}_{[2]}^{0,m} \), one can do something similar with \( W_{[3]} \) family. An option is to label such overfull basis of Cartanian operators by plane partitions. The study of these structures, as well as of additional ”raising” and ”lowering” \( W \)-operators, which can make it truly ”rigid”, is beyond the scope of the present paper.

### 5.4 Interpolation from 3-Schurs to 2-Schurs

Interpolation to the ordinary 2-Schurs at level three is rather straightforward. First of all, we assume that evolution affects not the scalar product \( \langle 32 \rangle \), but the shape of functions and operators of the given time-variables – perhaps, by making \( h \)-dependent rotations and rescalings. This means that evolution preserves hermiticity: self-conjugate combinations remain self-conjugate. Thus it is convenient to rewrite \( \dot{W}_{[3]}^0 \), putting together the self-conjugate combinations (as we already did for more complicated \( W \)-operators in the previous subsection):

\[ \dot{W}_{[2]}^0 = \left( \frac{p_2}{2} \frac{\partial^2}{\partial p_2^2} + p_1 \frac{\partial}{\partial p_2} \right) + \frac{1}{2} \left( 2 \hat{p}_3 \frac{\partial^2}{\partial \hat{p}_2 \partial p_1} + 3 \hat{p}_2 p_1 \frac{\partial^2}{\partial p_2 \partial p_1} \right) + \frac{9}{2} \sqrt{3} \left( 2 \hat{p}_3 \frac{\partial}{\partial \hat{p}_3} + 3 \hat{p}_2 p_1 \frac{\partial}{\partial \hat{p}_3} \right) + \frac{3 \sqrt{3}}{2} \left( 2 \hat{p}_2 p_1 \frac{\partial}{\partial \hat{p}_3} + 2 \hat{p}_2 \frac{\partial}{\partial \hat{p}_3} + 2 \hat{p}_2 \frac{\partial}{\partial \hat{p}_3} \right) \]

and compare this with the specialization of \( \dot{W}_{[2]} \) to \( t = q \).

\[ \dot{W}_{[2]} = \frac{p_2}{2} \frac{\partial^2}{\partial p_2^2} + p_1 \frac{\partial}{\partial p_2} + \left( 2 \hat{p}_3 \frac{\partial^2}{\partial \hat{p}_2 \partial p_1} + 3 \hat{p}_2 p_1 \frac{\partial}{\partial \hat{p}_3} \right) + \ldots \]
ideal choice of coordinates, suitable for the h-evolution. The choice in \([10]\) was h-independent, orthogonal and properly normalized

\[
\mathbf{p}_3 := \sqrt{\frac{3}{5}} \hat{p}_3 - \sqrt{\frac{2}{5}} \hat{p}_3, \quad \mathbf{p}_3^\perp := \sqrt{\frac{2}{5}} \hat{p}_3 + \sqrt{\frac{3}{5}} \hat{p}_3
\]  

(87)

The price to pay is that in these variables the operators \(\hat{W}(h)\) at \(h \neq 0\) will have an admixture of \(\mathbf{p}_3^\perp\). Instead separation of variables is nice in Schur functions: at \(h^2 = \frac{1}{2}\)

\[
S^{0}_{[3]} = \frac{4\hat{p}_3 - \sqrt{6} \hat{p}_3 \pm 3 \sqrt{2} \hat{p}_3}{6} - \frac{(-\hat{p}_3 \pm \sqrt{3} \hat{p}_3 \hat{p}_1)}{2 \sqrt{2}} + \frac{p_3^3}{6} \hspace{1cm} = \frac{1}{6} \sqrt{\frac{3}{2}} p_3^\perp + \frac{1}{2} \sqrt{\frac{3}{2}} p_3 + \frac{1}{2} \sqrt{\frac{3}{2}} \hat{p}_3 - \frac{1}{2} \sqrt{\frac{3}{2}} \hat{p}_3 p_1 + \frac{p_3^3}{6}
\]

\[
S^{\pm}_{[3]} = \frac{2\hat{p}_3 - \sqrt{6} \hat{p}_3 \pm 3 \sqrt{2} \hat{p}_3}{6} + \frac{(-\hat{p}_3 \pm \sqrt{3} \hat{p}_3 \hat{p}_1)}{2 \sqrt{2}} + \frac{p_3^3}{6} \hspace{1cm} = \frac{1}{6} \sqrt{\frac{3}{2}} p_3^\perp - \frac{1}{2} \sqrt{\frac{3}{2}} p_3 - \frac{1}{2} \sqrt{\frac{3}{2}} \hat{p}_3 + \frac{p_3^3}{6}
\]

\[
S^{0}_{[2]} = \frac{-\hat{p}_3 + \sqrt{6} \hat{p}_3}{6} - \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{3} \hspace{1cm} = \frac{1}{3} \sqrt{\frac{2}{3}} p_3^\perp - \frac{1}{6} \sqrt{\frac{2}{3}} p_3 - \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{6}
\]

\[
S^{\pm}_{[2]} = \frac{-2\hat{p}_3 - \sqrt{6} \hat{p}_3 + 3 \sqrt{2} \hat{p}_3}{12} - \frac{(-\hat{p}_3 \pm \sqrt{3} \hat{p}_3 \hat{p}_1)}{2 \sqrt{2}} + \frac{p_3^3}{3} \hspace{1cm} = \frac{1}{6} \sqrt{\frac{3}{2}} p_3^\perp + \frac{1}{6} \sqrt{\frac{3}{2}} p_3 + \frac{1}{2} \sqrt{\frac{3}{2}} \hat{p}_3 p_1 + \frac{p_3^3}{6}
\]

and deformation to generic \(h\) is:

\[
3 - \text{Schur point } h^2 = \frac{1}{2} \hspace{1cm} \text{generic } h
\]

\[
S^{0}_{[3]} = \frac{2}{3} \sqrt{\frac{3}{2}} p_3^\perp + \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{6} \hspace{1cm} \rightarrow \hspace{1cm} \frac{1}{h} \left( \frac{3 - h^2}{3 h^2 - 1} \right) \cdot \left( \frac{2}{3} \sqrt{\frac{3}{2}} p_3^\perp + \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{6} \right)
\]

\[
S^{\pm}_{[3]} = \frac{1}{3} \sqrt{\frac{2}{3}} p_3^\perp + \sqrt{\frac{3}{2}} p_3 \pm \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} \pm \frac{1}{2} \sqrt{\frac{3}{2}} p_2 p_1 + \frac{p_3^3}{6} \hspace{1cm} \rightarrow \hspace{1cm} \frac{1 + h^2}{\sqrt{3(1 - h^2)(3 - h^2)}} \cdot \frac{2}{3} \sqrt{\frac{3}{2}} p_3 + \frac{1 + h^2}{2} \cdot \frac{p_2 p_1 + p_3^3}{6} + \frac{h}{\sqrt{3(1 - h^2)(3 - h^2)}} \cdot \left( \frac{1 + h^2}{3 - h^2} \right) \cdot \frac{p_3^3}{3} - \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{6}
\]

\[
S^{0}_{[2]} = \frac{\hat{p}_3 + \sqrt{6} \hat{p}_3}{6} + \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{3} \hspace{1cm} \rightarrow \hspace{1cm} \frac{1}{6} \sqrt{\frac{3}{2}} p_3^\perp - \frac{1}{6} \sqrt{\frac{3}{2}} p_3 - \frac{\hat{p}_3 \hat{p}_1}{\sqrt{2}} + \frac{p_3^3}{6}
\]

\[
S^{\pm}_{[2]} = \frac{-\hat{p}_3 - \sqrt{6} \hat{p}_3 + 3 \sqrt{2} \hat{p}_3}{12} - \frac{\hat{p}_3 \hat{p}_1}{2 \sqrt{2}} + \frac{p_3^3}{3} \hspace{1cm} \rightarrow \hspace{1cm} \frac{1}{6} \sqrt{\frac{3}{2}} p_3^\perp + \frac{1}{2} \sqrt{\frac{3}{2}} p_3 + \frac{1}{2} \sqrt{\frac{3}{2}} \hat{p}_3 p_1 + \frac{p_3^3}{6}
\]

Surviving in the 2-Schur limit, i.e. when \(h = 0\) are the expressions in boxes. As necessary, \(S^{\pm}_{[2]}\), and \(S^{0}_{[2]}\) are finite and reproduce the standard 2-Schur functions, while \(S^{0}_{[3]}\), and \(S^{\pm}_{[3]}\) get singular, but the leading pieces are independent of \(\mathbf{p}_3, \mathbf{p}_2\) and \(\mathbf{p}_1\) – only on the "decoupling" time-variables \(\mathbf{p}_3^\perp, \mathbf{p}_3\) and \(\hat{\mathbf{p}}_2\). Actually, in this limit the coefficient in front of \(\mathbf{p}_3^\perp\) becomes pure imaginary, but this does not seem to cause any problem. The \(h\)-dependence in the \(\hat{\mathbf{p}}_3\) sector can look somewhat sophisticated, but it is unambiguously defined (deduced) from that in the \(\hat{\mathbf{p}}_2\) sector and orthogonality, once one fixes recursion rules in a nearly obvious way, see \([10]\).

The \(h\)-deformation of operator \([55]\) is

\[
\hat{\mathcal{W}}_0^{0}_{[2]}(h) = \left( \frac{p_2}{2} \frac{\partial^2}{\partial p_1^2} + p_1 \frac{\partial}{\partial p_2} \right) + \sqrt{\frac{3(1 - h^2)}{3 - h^2}} \left( \frac{2p_4}{h^2} \frac{\partial^2}{\partial p_2 \partial p_1} + 3p_2 p_1 \frac{\partial}{\partial p_3} \right) - \frac{h}{2} \sqrt{\frac{3h^2 - 1}{3 - h^2}} \left( 2p_3^\perp \frac{\partial^2}{\partial p_2 \partial p_1} + 3p_2 p_1 \frac{\partial}{\partial p_3} \right) + 
\]

\[
+ h \left( \frac{p_2}{2} \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial p_2} \right) \left( p_1 \frac{\partial}{\partial p_1} - 1 \right) + \frac{6h}{\sqrt{1 + h^2}} \left( p_3 \frac{\partial}{\partial p_3} + p_3 \frac{\partial}{\partial p_3} \right) + 
\]

\[
+ \frac{p_3^2}{1 + h^2} + \frac{h^2}{1 + h^2} \left( p_3 \frac{\partial}{\partial p_3} + p_3 \frac{\partial}{\partial p_3} \right) + 
\]
+ 3 \sqrt{3(1 - h^2)(h^2 - 1)} \left( \frac{\partial}{\partial p_3} + p_3 \right)^3 - \frac{\sqrt{3}}{2} \left( \frac{\partial^2}{\partial p_2 \partial p_1} + 3 \hat{p}_2 p_1 \frac{\partial}{\partial p_3} \right) \right) \tag{88}

At \( h = 0 \) the first line turns into (86), as expected and the second line vanishes. However, the third line remains finite – but it vanishes on functions of the variables \( p_1, p_2, p_3 \), and in this sense the operator is properly reduced to the ordinary 2-Schur one. Note also, that underlined term is occasionally zero at level 3 (where there are just two monomial on which it could act, \( \hat{p}_2 p_1 \) and \( p_3 p_1 \), which are both of degree one in \( p_1 \)), but it acts non-trivially at level 2. The eigenvalues of (88) for the \( h \)-deformed 3-Schur functions are:

\[
\begin{align*}
\hat{W}_{[2]}^0(h) S_{[2]}^0(h) &= 0, \\
\hat{W}_{[2]}^0(h) S_{[2]}^\pm(h) &= \pm \sqrt{1 + h^2} S_{[2]}^\pm(h), \\
\hat{W}_{[2]}^0(h) S_{[3]}^\pm(h) &= \pm 3 \sqrt{1 + h^2} S_{[3]}^\pm(h), \\
\hat{W}_{[2]}^0(h) S_{[21]}^\pm(h) &= \pm \frac{3}{2} S_{[21]}^\pm(h)
\end{align*}
\tag{89}
\]

\( h \)-deformations of \( W_{[2]}^\pm \) and \( W_{[3]} \) are also easy to deduce.

6 Conclusion

In this paper we elaborated a little further on the suggestion of [10] to define the analogue of Schur functions for plane partitions (nicknamed 3-Schurs) through the recursion procedure. Concretely, we provided detailed description of cut-and-join operators, for which the 3-Schurs are the common eigenfunctions, and used them to better explain the reduction procedures, converting 3-Schur to the ordinary 2-Schur and Macdonald polynomials, which depend only on the ordinary partitions (Young diagrams). Most formulas are worked out for the first non-trivial case of "level two", when there are three plane and two ordinary partitions, with two cubes and two boxes respectively. The Schur part of the story is explicitly lifted to level three, but we do not discuss Macdonald part at this level – on one hand it rather straightforwardly follows the path, explained at level two, on another hand it involves tedious calculations, which would make more sense after the Schur part is independently checked and approved. Moreover, at level three it can happen that 3-Schur functions are naturally describing the entire multi-parametric set of Kerov functions [13], not only their Macdonal locus (at level two the difference is not seen: generic Kerov in this case is exactly Macdonal). We also did not reveal the relation to triple Macdonald polynomials of [12], which are directly applicable to network model studies [9] – still do not yet allow to formulate the \( \sim \) character relation [18], what can actually require building a systematic first-principle theory of 3-Schur functions, which is the target of [10] and of the present paper. This, however, can take time and quite some effort. As to more local problems, the main one at the moment is the lack of clear view on \( \text{triviality} \) between \( q, t^{-1}, tq^{-1} \), which is expected to naturally occur if reduction from 3-Schur to Macdonalds is properly described – but so far it does not. All these open questions point to the obvious directions for further research in this promising field.

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