Mean convergence of Fourier-Dunkl series

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Abstract

In the context of the Dunkl transform a complete orthogonal system arises in a very natural way. This paper studies the weighted norm convergence of the Fourier series expansion associated to this system. We establish conditions on the weights, in terms of the $A_p$ classes of Muckenhoupt, which ensure the convergence. Necessary conditions are also proved, which for a wide class of weights coincide with the sufficient conditions.

Keywords: Dunkl transform, Fourier-Dunkl series, orthogonal system, mean convergence

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1. Introduction

For $\alpha > -1$, let $J_\alpha$ denote the Bessel function of order $\alpha$:

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n(x/2)^{2n}}{n! \Gamma(\alpha + n + 1)}$$

(a classical reference on Bessel functions is [17]). Throughout this paper, by $\frac{J_\alpha(z)}{2^\alpha}$ we denote the even function

$$\frac{1}{2^\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n(z/2)^{2n}}{n! \Gamma(\alpha + n + 1)}, \quad z \in \mathbb{C}. \quad (1)$$

In this way, for complex values of $z$, let

$$I_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n! \Gamma(\alpha + n + 1)};$$

the function $I_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order $\alpha$, usually denoted by $I_\alpha$. Also, let us take

$$E_\alpha(z) = I_\alpha(z) + \frac{z}{2(\alpha + 1)} I_{\alpha+1}(z), \quad z \in \mathbb{C}.$$ 

These functions are related with the so-called Dunkl transform on the real line (see [6] and [7] for details), which is a generalization of the Fourier transform. In particular, $E_{-1/2}(x) = e^x$ and the Dunkl transform

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of order $\alpha = -1/2$ becomes the Fourier transform. Very recently, many authors have been investigating the
behaviour of the Dunkl transform with respect to several problems already studied for the Fourier transform;
for instance, Paley-Wiener theorems [1], multipliers [4], uncertainty [16], Cowling-Price’s theorem [11],
transplantation [14], Riesz transforms [15], and so on. The aim of this paper is to pose and analyse in this
new context the weighted $L^p$ convergence of the associated Fourier series in the spirit of the classical scheme
which, for the trigonometric Fourier series, can be seen in Hunt, Muckenhoupt and Wheeden’s paper [10].

The function $\mathcal{I}_\alpha$ is even, and $E_\alpha(ix)$ can be expressed as

$$E_\alpha(ix) = 2^\alpha \Gamma(\alpha + 1) \left( \frac{J_\alpha(x)}{x^{\alpha+1}} + \frac{J_{\alpha+1}(x)}{x^{\alpha+1}} \right).$$

Let $\{s_j\}_{j \geq 1}$ be the increasing sequence of positive zeros of $J_{\alpha+1}$. The real-valued function
$\text{Im} E_\alpha(ix) = \frac{2^\alpha}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(ix)$ is odd and its zeros are $\{s_j\}_{j \in \mathbb{Z}}$ where
$s_{-j} = -s_j$ and $s_0 = 0$. In connection with the Dunkl transform on the real line, two of the authors introduced
the functions $e_j, j \in \mathbb{Z},$ as follows:

$$e_0(x) = 2^{(\alpha+1)/2} \Gamma(\alpha + 2)^{1/2},$$

$$e_j(x) = \frac{2^{\alpha/2} \Gamma(\alpha + 1)^{1/2}}{[\mathcal{I}_\alpha(is_j)]} E_\alpha(is_j), \quad j \in \mathbb{Z} \setminus \{0\}.$$ 

The case $\alpha = -1/2$ corresponds to the classical trigonometric Fourier setting: $\mathcal{I}_{-1/2}(z) = \cos(iz),$ $\mathcal{I}_{1/2}(z) = \frac{e^{iz}}{iz},$ $s_j = \pi j,$ $E_{-1/2}(is_j) = e^{\pi j},$ and $\{e_j\}_{j \in \mathbb{Z}}$ is the trigonometric system with the appropriate
multiplicative constant so that it is orthonormal on $(-1, 1)$ with respect to the normalized Lebesgue measure
$(2\pi)^{-1/2} dx$.

For all values of $\alpha > -1$, in $\mathbb{R}$ the sequence $\{e_j\}_{j \in \mathbb{Z}}$ was proved to be a complete orthonormal system
in $L^2((-1, 1), d\mu_\alpha), \ d\mu_\alpha(x) = (2^{\alpha+1} \Gamma(\alpha + 1))^{-1/2} |x|^{2\alpha+1} dx.$ That is to say

$$\int_{-1}^{1} e_j(x) e_k(x) d\mu_\alpha(x) = \delta_{jk}$$

and for each $f \in L^2((-1, 1), d\mu_\alpha)$ the series

$$\sum_{j=-\infty}^{\infty} \left( \int_{-1}^{1} f(y) e_j(y) d\mu_\alpha(y) \right) e_j(x),$$

which we will refer to as Fourier-Dunkl series, converges to $f$ in the norm of $L^2((-1, 1), d\mu_\alpha)$. The next step
is to ask for which $p \in C(1, \infty), \ p \neq 2,$ the convergence holds in $L^p((-1, 1), d\mu_\alpha)$. The problem is equivalent,
by the Banach-Steinhaus theorem, to the uniform boundedness on $L^p((-1, 1), d\mu_\alpha)$ of the partial sum
operators $S_n f$ given by

$$S_n f(x) = \int_{-1}^{1} f(y) K_n(x, y) d\mu_\alpha(y),$$

where $K_n(x, y) = \sum_{j=-n}^{n} e_j(x) e_j(y)$. We are interested in weighted norm estimates of the form

$$\|S_n(f)U\|_{L^p((-1, 1), d\mu_\alpha)} \leq C \|fV\|_{L^p((-1, 1), d\mu_\alpha)},$$

where $C$ is a constant independent of $n$ and $f$, and $U, V$ are nonnegative functions on $(-1, 1)$.

Before stating our results, let us fix some notation. The conjugate exponent of $p \in (1, \infty)$ is denoted by $p'$. That is,

$$\frac{1}{p} + \frac{1}{p'} = 1,$$ 

or

$$p' = \frac{p}{p-1}.$$
For an interval \((a, b) \subseteq \mathbb{R}\), the Muckenhoupt class \(A_p(a, b)\) consists of those pairs of nonnegative functions \((u, v)\) on \((a, b)\) such that
\[
\left( \frac{1}{|I|} \int_I u(x) \, dx \right) \left( \frac{1}{|I|} \int_I v(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq C,
\]
for every interval \(I \subseteq (a, b)\), with some constant \(C > 0\) independent of \(I\). The smallest constant satisfying this property is called the \(A_p\) constant of the pair \((u, v)\).

We say that \((u, v) \in A^\delta_p(a, b)\) (where \(\delta > 1\)) if \((u^\delta, v^\delta) \in A_p(a, b)\). It follows from Hölder’s inequality that \(A^\delta_p(a, b) \subseteq A_p(a, b)\).

If \(u \equiv 0\) or \(v \equiv \infty\), it is trivial that \((u, v) \in A_p(a, b)\) for any interval \((a, b)\). Otherwise, for a bounded interval \((a, b)\), if \((u, v) \in A_p(a, b)\) then the functions \(u\) and \(v^{-\frac{1}{\delta-1}}\) are integrable on \((a, b)\).

Throughout this paper, \(C\) denotes a positive constant which may be different in each occurrence.

2. Main results

We state here some \(A_p\) conditions which ensure the weighted \(L^p\) boundedness of these Fourier-Dunkl orthogonal expansions. For simplicity, we separate the general result corresponding to arbitrary weights in two theorems, the first one for \(\alpha \geq -1/2\) and the second one for \(-1 < \alpha < -1/2\).

**Theorem 1.** Let \(\alpha \geq -1/2\) and \(1 < p < \infty\). Let \(U, V\) be weights on \((-1, 1)\). Assume that
\[
\left( U(x)^p |x|^{(\alpha + \frac{1}{2})(2-\delta)} , V(x)^p |x|^{(\alpha + \frac{1}{2})(2-\delta)} \right) \in A^\delta_p(-1, 1)
\]
for some \(\delta > 1\) (or \(\delta = 1\) if \(U = V\)). Then there exists a constant \(C\) independent of \(n\) and \(f\) such that
\[
\|S_n(f)U\|_{L^p((-1, 1), d\mu_\alpha)} \leq C\|fV\|_{L^p((-1, 1), d\mu_\alpha)}.
\]

**Theorem 2.** Let \(-1 < \alpha < -1/2\) and \(1 < p < \infty\). Let \(U, V\) be weights on \((-1, 1)\). Let us suppose that \(U, V\) satisfy the conditions
\[
\left( U(x)^p |x|^{(2\alpha + 1)(1-\delta)} , V(x)^p |x|^{(2\alpha + 1)(1-\delta)} \right) \in A^\delta_p(-1, 1),
\]
\[
\left( U(x)^p |x|^{2\alpha+1} , V(x)^p |x|^{3\alpha+1} \right) \in A^\delta_p(-1, 1)
\]
for some \(\delta > 1\) (or \(\delta = 1\) if \(U = V\)). Then there exists a constant \(C\) independent of \(n\) and \(f\) such that
\[
\|S_n(f)U\|_{L^p((-1, 1), d\mu_\alpha)} \leq C\|fV\|_{L^p((-1, 1), d\mu_\alpha)}.
\]

As we mentioned in the introduction, the case \(\alpha = -1/2\) corresponds to the classical trigonometric case. Accordingly, \(2\) reduces then to \((U^p, V^p) \in A^\delta_p(-1, 1)\). It should be noted also that taking real and imaginary parts in these Fourier-Dunkl series we would obtain the so-called Fourier-Bessel series on \((0, 1)\) (see [18, 2, 3, 14]), but the known results for Fourier-Bessel series do not give a proof of the above theorems. Also in connection with Fourier-Bessel series on \((0, 1)\), Lemma \(3\) below can be used to improve some results of \(2\).

Theorems \(4\) and \(2\) establish some sufficient conditions for the \(L^p\) boundedness. Our next result presents some necessary conditions. To avoid unnecessary subtleties, we exclude the trivial cases \(U \equiv 0\) and \(V \equiv \infty\).

**Theorem 3.** Let \(-1 < \alpha, 1 < p < \infty, \text{ and } U, V \text{ weights on } (-1, 1), \text{ neither } U \equiv 0 \text{ nor } V \equiv \infty. \text{ If there exists some constant } C \text{ such that, for every } n \text{ and every } f,
\]
\[
\|S_n(f)U\|_{L^p((-1, 1), d\mu_\alpha)} \leq C\|fV\|_{L^p((-1, 1), d\mu_\alpha)},
\]
then $U \leq CV$ almost everywhere on $(-1, 1)$, and
\[
U(x)p^{|x|^{(\alpha+\frac{1}{2})(2-p)}} \in L^1((-1, 1), dx),
\]
\[
\left(V(x)p^{|x|^{(\alpha+\frac{1}{2})(2-p)}}\right)^{-\frac{1}{(2-p)}} = V(x)^{-p'}|x|^{(\alpha+\frac{1}{2})(2-p')} \in L^1((-1, 1), dx),
\]
\[
U(x)p^{|x|^{2\alpha+1}} \in L^1((-1, 1), dx),
\]
\[
\left(V(x)p^{|x|^{(2\alpha+1)(1-p)}}\right)^{-\frac{1}{(1-p)}} = V(x)^{-p'}|x|^{2\alpha+1} \in L^1((-1, 1), dx).
\]

Notice that the first two integrability conditions imply the other two if $\alpha \geq -1/2$, while the last two imply the other if $-1 < \alpha < -1/2$.

When $U$, $V$ are power-like weights, it is easy to check that the conditions of Theorem 3 are equivalent to the $A_p$ conditions 2, 3, 11. By power-like weights we mean finite products of the form $|x-t|^\gamma$, for some constants $t$, $\gamma$. For these weights, therefore, Theorems 1, 2 and 3 characterize the boundedness of the Fourier-Bessel expansions. For instance, we have the following particular case:

**Corollary.** Let $b, A, B \in \mathbb{R}$, $1 < p < \infty$, and
\[
U(x) = |x|^b(1-x)^A(1+x)^B.
\]

Then, there exists some constant $C$ such that
\[
\|US_n f\|_{L^p((-1,1),d\mu_\alpha)} \leq C\|f\|_{L^p((-1,1),d\mu_\alpha)}
\]
for every $f$ and $n$ if and only if $-1 < Ap < p - 1$, $-1 < Bp < p - 1$ and
\[
-1 + p\left(\alpha + \frac{1}{2}\right)_+ < bp + 2\alpha + 1 < p - 1 + p(2\alpha + 1) - p\left(\alpha + \frac{1}{2}\right)_+,
\]
where $(\alpha + \frac{1}{2})_+ = \max\{\alpha + \frac{1}{2}, 0\}$.

In the unweighted case ($U = V = 1$) the boundedness of the partial sum operators $S_n$, or in other words the convergence of the Fourier-Dunkl series, holds if and only if
\[
\frac{4(\alpha + 1)}{2\alpha + 3} < p < \frac{4(\alpha + 1)}{2\alpha + 1}
\]
in the case $\alpha \geq -1/2$, and for the whole range $1 < p < \infty$ in the case $-1 < \alpha < -1/2$.

**Remark.** These conditions for the unweighted case are exactly the same as in the Fourier-Bessel case when the orthonormal functions are $2^{1/2}|J_{\alpha+1}(s_n)|^{-1}J_{\alpha}(s_n x) x^{-\alpha}$ and the orthogonality measure is $x^{2\alpha+1} dx$ on the interval $(0, 1)$.

Other variants of Bessel orthogonal systems exist in the literature, see [2, 3, 18]. For instance, one can take the functions $2^{1/2}|J_{\alpha+1}(s_n)|^{-1}J_{\alpha}(s_n x)$, which are orthonormal with respect to the measure $x dx$ on the interval $(0, 1)$. The conditions for the boundedness of these Fourier-Bessel series, as can be seen in [2], correspond to taking $A = B = 0$ and $b = \alpha - \frac{2\alpha+1}{p}$ in our corollary. Another usual case is to take the functions $(2x)^{1/2}|J_{\alpha+1}(s_n)|^{-1}J_{\alpha}(s_n x)$, which are orthonormal with respect to the measure $dx$ on $(0, 1)$. Passing from one orthogonality to another consists basically in changing the weights. Then, from the weighted $L^p$ boundedness of any of these systems we easily deduce a corresponding weighted $L^p$ boundedness for any of the other systems.

In the case of the Fourier-Dunkl series on $(-1, 1)$ we feel, however, that the natural setting is to start from $J_{\alpha}(z) z^{-\alpha}$, since these functions, defined by [11], are holomorphic on $\mathbb{C}$; in particular, they are well defined on the interval $(-1, 1)$. 4
3. Auxiliary results

We will need to control some basic operator in weighted $L^p$ spaces on $(-1,1)$. For a function $g : (0,2) \to \mathbb{R}$, the Calderón operator is defined by

$$Ag(x) = \frac{1}{x} \int_0^x |g(y)| \, dy + \int_x^2 \frac{|g(y)|}{y} \, dy,$$

that is, the sum of the Hardy operator and its adjoint. The weighted norm inequality

$$\|Ag\|_{L^p((0,2),u)} \leq C\|g\|_{L^p((0,2),v)}$$

holds for every $g \in L^p((0,2),v)$, provided that $(u,v) \in A_0^\delta((0,2))$ for some $\delta > 1$, and $\delta = 1$ is enough if $u = v$ (see [12, 13]). Let us consider now the operator $J$ defined by

$$Jf(x) = \int_{-1}^1 \frac{f(y)}{2-x-y} \, dy$$

for $x \in (-1,1)$ and suitable functions $f$. With the notation $f_1(t) = f(1-t)$, we have

$$|Jf(x)| = \left| \int_0^2 \frac{f(1-t)}{1-x+t} \, dt \right| \leq A(f_1)(1-x)$$

and a simple change of variables proves that the weighted norm inequality

$$\|Jf\|_{L^p((-1,1),u)} \leq C\|f\|_{L^p((-1,1),v)}$$

holds for every $f \in L^p((-1,1),v)$, provided that $(u,v) \in A_0^\delta((-1,1))$ for some $\delta > 1$ (or $\delta = 1$ if $u = v$).

The Hilbert transform on the interval $(-1,1)$ is defined as

$$Hg(x) = \int_{-1}^1 \frac{g(y)}{x-y} \, dy.$$

The above weighted norm inequality holds also for the Hilbert transform with the same $A_0^\delta((-1,1),1)$ condition (see [12, 13]). In both cases, the norm inequalities hold with a constant $C$ depending only on the $A_0^\delta$ constant of the pair $(u,v)$.

Our first objective is to obtain a suitable estimate for the kernel $K_n(x,y)$. With this aim, we will use some well-known properties of Bessel (and related) functions, that can be found on [17]. For the Bessel functions we have the asymptotics

$$J_\nu(z) = \frac{z^\nu}{2^{\nu+1} \Gamma(\nu+1)} + O(z^{\nu+2}), \quad (5)$$

if $|z| < 1$, $|\arg(z)| \leq \pi$; and

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[ \cos \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + O(e^{-\text{Im}(z)} z^{-1}) \right], \quad (6)$$

if $|z| \geq 1$, $|\arg(z)| \leq \pi - \theta$. The Hankel function of the first kind, denoted by $H^{(1)}_\nu$, is defined as

$$H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z),$$

where $Y_\nu$ denotes the Weber function, given by

$$Y_\nu(z) = \frac{J_\nu(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}, \quad \text{if } \nu \notin \mathbb{Z},$$

$$Y_n(z) = \lim_{\nu \to n} \frac{J_\nu(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}, \quad \text{if } n \in \mathbb{Z}.$$
From these definitions, we have

\[ H_{\nu}^{(1)}(z) = \frac{J_{\nu}(z) - e^{-\nu i \pi} J_{\nu}(z)}{i \sin \nu \pi}, \quad \text{if} \ \nu \notin \mathbb{Z}, \]

\[ H_{n}^{(1)}(z) = \lim_{\nu \to n} \frac{J_{\nu}(z) - e^{-\nu i \pi} J_{\nu}(z)}{i \sin \nu \pi}, \quad \text{if} \ n \in \mathbb{Z}. \]

For the function \( H_{\nu}^{(1)} \), the asymptotic

\[ H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{(z-\nu^2/2-\nu/4)} [C + O(z^{-1})] \tag{7} \]

holds for \(|z| > 1, -\pi < \arg(z) < 2\pi\), with some constant \( C \).

As usual for the \( L^p \) convergence of orthogonal expansions, the results are consequences of suitable estimates for the kernel \( K_n(x, y) \). The next lemma contains an estimate for the difference between the kernel \( K_n(x, y) \) and an integral containing the product of two \( E_n \) functions. This integral can be evaluated using Lemma 1 in [5]. Next, to obtain the estimate we consider an appropriate function in the complex plane having poles in the points \( s_j \) and integrate this function along a suitable path.

**Lemma 1.** Let \( \alpha > -1 \). Then, there exists some constant \( C > 0 \) such that for each \( n \geq 1 \) and \( x, y \in (-1, 1) \),

\[ |K_n(x, y) - \int_{-M_n}^{M_n} E_\alpha(izx) E_\alpha(izy) \, d\mu_\alpha(z)| \leq C \left( \frac{|xy|^{(\alpha+1)/2}}{2 - x - y} + 1 \right), \]

where \( M_n = (s_n + s_{n+1})/2 \).

**Proof.** Using elementary algebraic manipulations, the kernel \( K_n(x, y) \) can be written as

\[ K_n(x, y) = 2^{\alpha+1} \Gamma(\alpha + 2) + \frac{2^{\alpha+1} \Gamma(\alpha + 1)}{(xy)^{\alpha}} \sum_{j=1}^{n} J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y). \tag{8} \]

Let us find a function whose residues at the points \( s_j \) are the terms in the series, so that this series can be expressed as an integral. The identities

\[ -J'_{\alpha+1}(z) H_{\alpha+1}^{(1)}(z) + J_{\alpha+1}(z) (H_{\alpha+1}^{(1)})'(z) = \frac{2i}{\pi z} \]

(see [19, p. 76]), and

\[ z J'_{\alpha+1}(z) + (\alpha + 1) J_{\alpha+1}(z) = -z J_{\alpha}(z), \]

give

\[ -J'_{\alpha+1}(s_j) H_{\alpha+1}^{(1)}(s_j) = \frac{2i}{\pi s_j} \]

and

\[ J_{\alpha+1}(s_j) = -J_{\alpha}(s_j) \]

for every \( j \in \mathbb{N} \). Then,

\[
-\frac{2i}{\pi} |xy|^{1/2} \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2} \\
= -\frac{2i}{\pi} |xy|^{1/2} \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J_{\alpha+1}(s_j)^2} \\
= |xy|^{1/2} s_j H_{\alpha+1}^{(1)}(s_j) \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J'_{\alpha+1}(s_j)} \\
= \lim_{z \to s_j} (z - s_j) H_{x,y}(z) = \text{Res}(H_{x,y}, s_j),
\]
where we define

\[ H_{x,y}(z) = |xy|^{1/2} z H^{(1)}_{α+1}(z) \frac{J_α(zx)J_α(zy) + J_α+1(zx)J_α+1(zy)}{J_α+1(z)} \]

(the factor $|xy|^{1/2}$ is taken for convenience). The fact that $J_α(-z) = e^{πi}J_α(z)$ gives $\text{Res}(H_{x,y}, s_j) = \text{Res}(H_{x,y}, -s_j)$.

Since the definition of $H^{(1)}_{α+1}(z)$ differs in case $α ∈ Z$, for the rest of the proof we will assume that $α ∉ Z$; the other case can be deduced by considering the limit.

The function $H_{x,y}(z)$ is analytic in $C \setminus \{(-∞, -M_n] ∪ [M_n, ∞) ∪ \{±s_j : j = 1, 2, \ldots\}\}$. Moreover, the points $±s_j$ are simple poles. So, we have

\[ \int_{S_α} H_{x,y}(z) \, dz = 0, \]

where $S(ε)$ is the interval $[-M_n, M_n]$ warped with upper half circles of radius $ε$ centered in $±s_j$, with $j = 1, \ldots, n$ and $S$ is the path of integration given by the interval $M_n + i[0, ∞)$ in the direction of increasing imaginary part and the interval $-M_n + i[0, ∞)$ in the opposite direction. The existence of the integral is clear for the path $S(ε)$; for this fact can be checked by using (4), (6) and (7). Indeed, on $S$ one has

\[ |xy|^{1/2} z J_α(zx)J_α(zy) \leq Ce^{π(α+2)}e^{-2Im(z)}H^{(α)}_{x,y}(|z|) \]

where

\[ h^{(α)}_{x,y}(|z|) = \max\{|xz|^{α+1}/2, 1\} \max\{|yz|^{α+1}/2, 1\} \]

for $-1 < α < -1/2$, and

\[ h^{(α)}_{x,y}(|z|) = 1 \]

for $α \geq -1/2$. Thus

\[ |H_{x,y}(z)| \leq C \left( h^{(α)}_{x,y}(|z|) + h^{(α+1)}_{x,y}(|z|) \right) e^{-π(2-x-y)}, \]

and the integral on $S$ is well defined.

From the definition of $H_{x,y}(z)$, we have

\[ \int_{S(ε)} H_{x,y}(z) \, dz = \int_{S(ε)} \frac{|xy|^{1/2} z J_{α-1}(z) J_α(zx)J_α(zy) + J_α+1(zx)J_α+1(zy)}{i sin(α+1)π} \, dz \]

\[ - |xy|^{1/2} e^{-i(α+1)π} \int_{S(ε)} z (J_α(zx)J_α(zy) + J_α+1(zx)J_α+1(zy)) \, dz. \]

The function in the first integral is odd, and the function in the second integral has no poles at the points $s_j$.

Then, the first integral equals the integral over the symmetric path $-S(ε) = \{z : -z ∈ S(ε)\}$. Putting $|z-s_j| = ε$ for the positively oriented circle, this gives

\[ \lim_{ε→0} \int_{S(ε)} H_{x,y}(z) \, dz = \lim_{ε→0} \frac{1}{2} \sum_{|s_j|<M_n} \int_{|z-s_j| = ε} \frac{|xy|^{1/2} z J_{α-1}(z) J_α(zx)J_α(zy) + J_α+1(zx)J_α+1(zy)}{i sin(α+1)π} \, dz \]

\[ - |xy|^{1/2} e^{-i(α+1)π} \int_{-M_n}^{M_n} z (J_α(zx)J_α(zy) + J_α+1(zx)J_α+1(zy)) \, dz \]

\[ = -πi \sum_{|s_j| < M_n} \text{Res}(H_{x,y}, s_j) \]

\[ - |xy|^{1/2} e^{-i(α+1)π}(1 - e^{2πiα}) \int_0^{M_n} z (J_α(zx)J_α(zy) + J_α+1(zx)J_α+1(zy)) \, dz \]
\[
\begin{align*}
&= -4|xy|^{1/2} \sum_{j=1}^{n} \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2} \\
&\quad + 2|xy|^{1/2} \int_0^{M_n} z \left( J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy) \right) dz.
\end{align*}
\]

This, together with (10), gives
\[
\sum_{j=1}^{n} \frac{J_\alpha(s_j x) J_\alpha(s_j y) + J_{\alpha+1}(s_j x) J_{\alpha+1}(s_j y)}{J_\alpha(s_j)^2} = \frac{1}{4|xy|^{1/2}} \int_\mathbf{S} H_{x,y}(z) \, dz + \frac{1}{2} \int_0^{M_n} z \left( J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy) \right) dz.
\]

Then, it follows from (8) that
\[
K_n(x, y) = 2^{\alpha+1} \Gamma(\alpha + 2) + \frac{2^{\alpha+1} \Gamma(\alpha + 1)}{(xy)^\alpha |xy|^{1/2}} \int_\mathbf{S} H_{x,y}(z) \, dz + 2^{\alpha+1} \Gamma(\alpha + 1) \frac{M_n}{(xy)^\alpha} | \int_0^{M_n} z \left( J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy) \right) dz |.
\]

Now, it is easy to check the identity
\[
\frac{2^\alpha \Gamma(\alpha + 1)}{(xy)^\alpha} \int_0^{M_n} z \left( J_\alpha(zx) J_\alpha(zy) + J_{\alpha+1}(zx) J_{\alpha+1}(zy) \right) dz = \int_{-M_n}^{M_n} E_\alpha(ixz) E_\alpha(izy) d\mu(z),
\]
so that
\[
\left| K_n(x, y) - \int_{-M_n}^{M_n} E_\alpha(ixz) E_\alpha(izy) d\mu(z) \right| \leq 2^{\alpha+1} \Gamma(\alpha + 2) + \frac{2^{\alpha+1} \Gamma(\alpha + 1)}{|xy|^{\alpha+1/2}} \int_\mathbf{S} H_{x,y}(z) \, dz.
\]

We conclude showing that
\[
\left| \int_\mathbf{S} H_{x,y}(z) \, dz \right| \leq C \left( \frac{1}{2 - x - y} + |xy|^{\alpha+1/2} \right), \tag{11}
\]
for \(-1 < x, y < 1\). For \(\alpha \geq -1/2\), the bound (11) follows from (10). Indeed, in this case
\[
\left| \int_\mathbf{S} H_{x,y}(z) \, dz \right| \leq C \int_0^{\infty} e^{-t(2-x-y)} \, dt = \frac{C}{2 - x - y}.
\]

For \(-1 < \alpha < -1/2\), we have \(|H_{x,y}(z)| \leq C|x|^{\alpha+1/2} e^{-1/2-y(z)(2-x-y)}\) if \(z \in \mathbf{S}\). With this inequality we obtain (11) as follows:
\[
\left| \int_\mathbf{S} H_{x,y}(z) \, dz \right| \leq C|x|^{\alpha+1/2} \int_0^{\infty} e^{-t(2-x-y)} \, dt = C \frac{|xy|^{\alpha+1/2}}{2 - x - y} \leq C \left( |xy|^{\alpha+1/2} + \frac{1}{2 - x - y} \right).
\]

From the previous lemma and the identity (see [5])
\[
\int_{-1}^{1} E_\alpha(ixz) E_\alpha(izy) d\mu(z) = \frac{1}{2^{\alpha+1} \Gamma(\alpha + 2)} \frac{x\mathcal{I}_{\alpha+1}(ix)\mathcal{I}_{\alpha}(iy) - y\mathcal{I}_{\alpha+1}(iy)\mathcal{I}_{\alpha}(ix)}{x - y},
\]
which holds for \(\alpha > -1, x, y \in \mathbb{C},\) and \(x \neq y\), we obtain that
\[
|K_n(x, y) - B(M_n, x, y) - B(M_n, y, x)| \leq C \left( \frac{|xy|^{-(\alpha+1/2)}}{2 - x - y} + 1 \right) \quad \tag{12}
\]
with
\[ B(M_n, x, y) = \frac{M_n^{2(\alpha+1)}}{2^{\alpha+1}\Gamma(\alpha+2)} \frac{xT_{\alpha+1}(iM_n x)J_{\alpha}(iM_n y)}{x-y} \]
or, by the definition of \( T_{\alpha} \) and the fact that \( \frac{J_{\alpha}(z)}{z^{\alpha+n}} \) is even,
\[ B(M_n, x, y) = 2^\alpha \Gamma(\alpha+1) \frac{M_n xJ_{\alpha+1}(M_n|x|)J_{\alpha}(M_n|y|)}{|x|^{\alpha+1}|y|^\alpha (x-y)}. \]

4. Proof of Theorem 1

We can split the partial sum operator \( S_n \) into three terms suitable to apply (12):
\[
S_n f(x) = \int_{-1}^{1} f(y)B(M_n, x, y)\,d\mu_{\alpha}(y) + \int_{-1}^{1} f(y)B(M_n, y, x)\,d\mu_{\alpha}(y) + \int_{-1}^{1} f(y)\left[ K_n(x, y) - B(M_n, x, y) - B(M_n, y, x) \right] \,d\mu_{\alpha}(y)
\]
\[ =: T_{1,n} f(x) + T_{2,n} f(x) + T_{3,n} f(x), \tag{13} \]

With this decomposition, the theorem will be proved if we see that
\[
\|UT_{j,n} f\|_{L^p((-1,1),d\mu_{\alpha})}^p \leq C\|V f\|_{L^p((-1,1),d\mu_{\alpha})}^p, \quad j = 1, 2, 3,
\]
for a constant \( C \) independent of \( n \) and \( f \).

4.1. The first term

We have
\[
T_{1,n} f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{-1}^{1} f(y)B(M_n, x, y)|y|^{2\alpha+1} \,dy
\]
\[ = \frac{M_n^{1/2} xJ_{\alpha+1}(M_n|x|)}{2|x|^{\alpha+1}} \int_{-1}^{1} \frac{f(y)M_n^{1/2} J_{\alpha}(M_n|y|)|y|^\alpha}{x-y} \,dy. \]

According to (5) and (6) and the assumption that \( \alpha \geq -1/2 \), we have
\[ |J_{\alpha}(z)| \leq Cz^{-1/2}, \quad |J_{\alpha+1}(z)| \leq Cz^{-1/2}, \]
for every \( z > 0 \). Using these inequalities and the boundedness of the Hilbert transform under the \( A_p \) condition (2) gives
\[
\|UT_{1,n} f\|_{L^p((-1,1),d\mu_{\alpha})}^p = C \int_{-1}^{1} \int_{-1}^{1} \frac{f(y)M_n^{1/2} J_{\alpha}(M_n|y|)|y|^\alpha}{x-y} \,dy \,dx
\]
\[ \leq C \int_{-1}^{1} \int_{-1}^{1} \frac{f(y)M_n^{1/2} J_{\alpha}(M_n|y|)|y|^\alpha}{x-y} \,dy \,dx
\]
\[ \leq C \int_{-1}^{1} \int_{-1}^{1} f(x)|x|^{\alpha+1} \left| V(x) \right|^{\alpha+1} \,dx \]
\[ \leq C \int_{-1}^{1} |f(x)|^p V(x)^p |x|^{2\alpha+1} \,dx = C\|V f\|_{L^p((-1,1),d\mu_{\alpha})}^p.
\]
4.2. The second term

This term is given by

\[ T_{2,n}f(x) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} \int_{-1}^{1} f(y) B(M_n, y, x) |y|^{2\alpha+1} dy \]

\[ = \frac{M_n^{1/2} J_\alpha(M_n|x|)}{2|x|^\alpha} \int_{-1}^{1} f(y) y M_n^{1/2} J_{\alpha+1}(M_n|y|) |y|^\alpha dy \]

and everything goes as with the first term.

4.3. The third term

According to (12),

\[ |T_{3,n}f(x)| \leq C |x|^{-(\alpha+1/2)} \int_{-1}^{1} \frac{f(y)|y|^{\alpha+1/2}}{2 - x - y} dy + C \int_{-1}^{1} |f(y)| |y|^{2\alpha+1} dy \]

so it is enough to have both

\[ \int_{-1}^{1} \left| \int_{-1}^{1} f(y) dy \right|^p \frac{|x|^{2\alpha+1-p(\alpha+1/2)}}{U(x)^p} dx \]

and

\[ \left| \int_{-1}^{1} f(x) dx \right| |x|^{2\alpha+1} \int_{-1}^{1} U(x)^p |x|^{2\alpha+1} dx \]

bounded by

\[ C \int_{-1}^{1} |f(x)|^p V(x)^p |x|^{2\alpha+1} dx. \]

For the boundedness of (14) it suffices to impose

\[ \left( U(x)^p |x|^{2\alpha+1-p(\alpha+1/2)}, V(x)^p |x|^{2\alpha+1-p(\alpha+1/2)} \right) \in A_p^f(-1, 1), \]

but this is exactly (2). By duality, the boundedness of (15) is equivalent to

\[ \left( \int_{-1}^{1} U(x)^p |x|^{2\alpha+1} dx \right)^{-\frac{1}{p-1}} \left( \int_{-1}^{1} V(x)^{-p/(p-1)} |x|^{2\alpha+1} dx \right)^{p-1} < \infty. \]

Now, it is easy to check that

\[ \left( \int_{-1}^{1} U(x)^p |x|^{2\alpha+1} dx \right) \left( \int_{-1}^{1} V(x)^{-p/(p-1)} |x|^{2\alpha+1} dx \right)^{p-1} \]

\[ \leq \left( \int_{-1}^{1} U(x)^p |x|^\alpha dx \right) \left( \int_{-1}^{1} \left( V(x)^p |x|^\beta \right)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C, \]

the last inequality following from the \( A_p \) condition (2).
5. Proof of Theorem 2

We begin with a simple lemma on $A_p$ weights.

Lemma 2. Let $1 < p < \infty$, $(u, v) \in A_p(-1, 1)$, $(u_1, v_1) \in A_p(-1, 1)$. Let $w$, $\zeta$ be weights on $(-1, 1)$ such that either

$$w \leq C(u + u_1) \quad \text{and} \quad \zeta \geq C_1(v + v_1)$$

or

$$w^{-1} \geq C(u^{-1} + u_1^{-1}) \quad \text{and} \quad \zeta^{-1} \leq C_1(v^{-1} + v_1^{-1})$$

for some constants $C, C_1$. Then $(w, \zeta) \in A_p(-1, 1)$ with a constant depending only on $C, C_1$ and the $A_p$ constants of $(u, v)$ and $(u_1, v_1)$.

Proof. Assume that $w \leq C(u + u_1)$ and $\zeta \geq C_1(v + v_1)$. For any interval $I \subseteq (-1, 1)$,

$$\left( \frac{1}{|I|} \int_I \zeta^{-\frac{1}{p-1}} \right)^{p-1} \leq \frac{1}{C_1} \min \left\{ \left( \frac{1}{|I|} \int_I v^{-\frac{1}{p-1}} \right)^{p-1}, \left( \frac{1}{|I|} \int_I v_1^{-\frac{1}{p-1}} \right)^{p-1} \right\}.$$

Therefore,

$$\left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I \zeta^{-\frac{1}{p-1}} \right)^{p-1} \leq \frac{C}{C_1} \left( \frac{1}{|I|} \int_I u \right) \left( \frac{1}{|I|} \int_I v^{-\frac{1}{p-1}} \right)^{p-1} + \frac{C}{C_1} \left( \frac{1}{|I|} \int_I u_1 \right) \left( \frac{1}{|I|} \int_I v_1^{-\frac{1}{p-1}} \right)^{p-1}.$$

This proves that $(w, \zeta) \in A_p(-1, 1)$ with a constant depending on $C, C_1$ and the $A_p$ constants of $(u, v)$ and $(u_1, v_1)$.

Assume now that $w^{-1} \geq C(u^{-1} + u_1^{-1})$ and $\zeta^{-1} \leq C_1(v^{-1} + v_1^{-1})$. Then

$$\frac{1}{|I|} \int_I w \leq \frac{1}{C_1} \min \left\{ \frac{1}{|I|} \int_I u, \frac{1}{|I|} \int_I u_1 \right\} \quad \text{(16)}$$

for any interval $I \subseteq (-1, 1)$. On the other hand, the inequality

$$\frac{1}{2}(a^\lambda + b^\lambda) \leq (a + b)^\lambda \leq 2^\lambda (a^\lambda + b^\lambda), \quad a, b \geq 0, \quad \lambda > 0 \quad \text{(17)}$$

gives

$$\zeta^{-\frac{1}{p-1}} \leq C_1^{-1} (v^{-1} + v_1^{-1})^{\frac{1}{p-1}} \leq C_1^{\frac{1}{p-1}} 2^{\frac{1}{p-1}} (v^{-\frac{1}{p-1}} + v_1^{-\frac{1}{p-1}}),$$

and

$$\left( \frac{1}{|I|} \int_I \zeta^{-\frac{1}{p-1}} \right)^{p-1} \leq 2^p C_1 \left( \frac{1}{|I|} \int_I v^{-\frac{1}{p-1}} \right)^{p-1} + 2^p C_1 \left( \frac{1}{|I|} \int_I v_1^{-\frac{1}{p-1}} \right)^{p-1}.$$

This, together with (16), proves that $(w, \zeta) \in A_p(-1, 1)$ with a constant depending on $C, C_1$ and the $A_p$ constants of $(u, v)$ and $(u_1, v_1)$. \qed

Now, we use the following estimate for the Bessel functions, which is a consequence of (5), (6) and $-1 < \alpha < -1/2$:

$$|z^{1/2} J_{\alpha}(z)| \leq C(1 + z^{\alpha + 1/2}), \quad z \geq 0,$$

and

$$|z^{1/2} J_{\alpha+1}(z)| \leq C(1 + z^{\alpha + 1/2})^{-1}, \quad z \geq 0.$$

In particular, there exists a constant $C$ such that, for $x \in (-1, 1)$ and $n \geq 0$, we have

$$M_n^{1/2} |J_{\alpha}(M_n|x|)| \leq C|x|^{-1/2}(1 + |M_nx|^{\alpha + 1/2}).$$
and
\[ M_{n}^{1/2}|J_{n+1}(M_{n}|x|)| \leq C \frac{|x|^{-1/2}}{1 + |M_{n}x|^{\alpha+1/2}}. \]

Moreover, the inequality (17) gives
\[ 2^{\alpha+1/2}|x|^{\alpha+1/2}(|x| + M_{n}^{-1})^{-(\alpha+1/2)} \leq 1 + |M_{n}x|^{\alpha+1/2} \leq 2|x|^{\alpha+1/2}(|x| + M_{n}^{-1})^{-(\alpha+1/2)} \]
so that we get
\[ M_{n}^{1/2}|J_{n}(M_{n}|x|)| \leq C|x|^{-\alpha}(|x| + M_{n}^{-1})^{-(\alpha+1/2)} \tag{18} \]
and
\[ M_{n}^{1/2}|J_{n+1}(M_{n}|x|)| \leq C|x|^{-\alpha+1}(|x| + M_{n}^{-1})^{\alpha+1/2}. \tag{19} \]

To handle these expressions, the following result will be useful:

**Lemma 3.** Let \( 1 < p < \infty \), a sequence \( \{M_{n}\} \) of positive numbers that tends to infinity, two nonnegative functions \( U \) and \( V \) defined on the interval \((-1, 1)\), \(-1 < \alpha < -1/2\) and \( \delta > 1 \) (\( \delta = 1 \) if \( U = V \)). If \( (3) \) and \( (4) \) are satisfied, then
\[
(U(x)^p(|x| + M_{n}^{-1})^{p(\alpha+1/2)}|x|^{2\alpha+1}(1-p), V(x)^p(|x| + M_{n}^{-1})^{p(\alpha+1/2)}|x|^{2\alpha+1}(1-p)) \in A_{p}^\delta(-1, 1), \tag{20}
\]
\[
(U(x)^p(|x| + M_{n}^{-1})^{-p(\alpha+1/2)}|x|^{2\alpha+1}, V(x)^p(|x| + M_{n}^{-1})^{-p(\alpha+1/2)}|x|^{2\alpha+1}) \in A_{p}^\delta(-1, 1), \tag{21}
\]
“uniformly”, i.e., with \( A_{p}^\delta \) constants independent of \( n \).

**Proof.** As a first step, let us observe that \( (3) \) and \( (4) \) imply
\[
(U(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)}, V(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)}) \in A_{p}^\delta(-1, 1).
\]
To prove this, just put
\[
U(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)} = \left( U(x)^p|x|^{(2\alpha+1)(1-p)} \right)^{1/2} \left( U(x)^p|x|^{(2\alpha+1)} \right)^{1/2}
\]
(the same with \( V \)) and check the \( A_{p}^\delta \) condition using the Cauchy-Schwarz inequality and \( (3), (4) \).

Now, \( (17) \) yields
\[
\left[ U(x)^p(|x| + M_{n}^{-1})^{p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)} \right]^{-\delta} \leq \frac{1}{2} \left[ U(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)} \right]^{-\delta} + \frac{1}{2} \left[ V(x)^pM_{n}^{-p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)} \right]^{-\delta}
\]
and
\[
\left[ V(x)^p(|x| + M_{n}^{-1})^{p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)} \right]^{-\delta} \leq 2^{-p\delta(\alpha+\frac{1}{2})} \left[ V(x)^p|x|^{(2\alpha+1)(1-\frac{1}{2}p)} \right]^{-\delta} + 2^{-p\delta(\alpha+\frac{1}{2})} \left[ V(x)^pM_{n}^{-p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)} \right]^{-\delta}.
\]
Thus, Lemma \( 2 \) gives \( (20) \) with an \( A_{p}^\delta \) constant independent of \( n \), since the \( A_{p}^\delta \) constant of the pair
\[
(U(x)^pM_{n}^{-p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)}, V(x)^pM_{n}^{-p(\alpha+1/2)}|x|^{(2\alpha+1)(1-p)})
\]
is the same constant of the pair
\[
(U(x)^p|x|^{(2\alpha+1)(1-p)}, V(x)^p|x|^{(2\alpha+1)(1-p)})
\]
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i.e., it does not depend on $n$. The proof of (21) follows the same argument, since

\[ \left[ U(x)^p (|x| + M_n^{-1})^{-p(\alpha + \frac{3}{2})} |x|^{2\alpha + 1} \right]^\delta \]

\[ \leq 2^{-\delta(\alpha + \frac{3}{2})} \left[ U(x)^p |x|^{(2\alpha + 1)(1 - \frac{\delta}{2})} \right]^\delta + 2^{-\delta(\alpha + \frac{3}{2})} \left[ U(x)^p M_n^{p(\alpha + \frac{1}{2})} |x|^{2\alpha + 1} \right]^\delta \]

and

\[ \left[ V(x)^p (|x| + M_n^{-1})^{-p(\alpha + \frac{3}{2})} |x|^{2\alpha + 1} \right]^\delta \geq \frac{1}{2} \left[ V(x)^p |x|^{(2\alpha + 1)(1 - \frac{\delta}{2})} \right]^\delta + \frac{1}{2} \left[ V(x)^p M_n^{p(\alpha + \frac{1}{2})} |x|^{2\alpha + 1} \right]^\delta. \]

We already have all the ingredients to start with the proof of Theorem 2. Let us take the same decomposition $S_n f = T_{1,n} f + T_{2,n} + T_{3,n} f$ as in (13) in the previous section and consider each term separately.

### 5.1. The first term

As in the proof of Theorem 1, by using (19) we have

\[ \| U T_{1,n} f \|_{L^p((-1,1), d\mu_n)} = \int_{-1}^{1} \left[ \int_{-1}^{1} \frac{f(y) M_n^{1/2} J_\alpha(M_n |y|) |y|^{\alpha + 1}}{x - y} dy \right]^p U(x)^p M_n^{p/2} |J_{\alpha + 1}(M_n |x|)|^p dx \]

\[ \leq C \int_{-1}^{1} \left[ \int_{-1}^{1} \frac{f(y) M_n^{1/2} J_\alpha(M_n |y|) |y|^{\alpha + 1}}{x - y} dy \right]^p U(x)^p (|x| + M_n^{-1})^{p(\alpha + 1/2)} |x|^{(2\alpha + 1)(1 - p)} dx. \]

Now, by the $A_p$ condition (20), this is bounded by

\[ C \int_{-1}^{1} f(x)^p V(x)^p |x|^{2\alpha + 1} dx = C \| V f \|_{L^p((-1,1), d\mu_n)}^p. \]

### 5.2. The second term

The definition of $T_{2,n}$ and (18) yield

\[ \| U T_{2,n} f \|_{L^p((-1,1), d\mu_n)} = \int_{-1}^{1} \left[ \int_{-1}^{1} \frac{f(y) y M_n^{1/2} J_{\alpha + 1}(M_n |y|) |y|^{\alpha}}{y - x} dy \right]^p U(x)^p M_n^{p/2} |J_\alpha(M_n |x|)|^p |x|^{2\alpha + 1 - \alpha p} dx \]

\[ \leq C \int_{-1}^{1} \left[ \int_{-1}^{1} \frac{f(y) y M_n^{1/2} J_{\alpha + 1}(M_n |y|) |y|^{\alpha}}{y - x} dy \right]^p U(x)^p (|x| + M_n^{-1})^{-p(\alpha + 1/2)} |x|^{2\alpha + 1} dx. \]

Now, by the $A_p$ condition (21), this is bounded by

\[ C \int_{-1}^{1} f(x) x M_n^{1/2} J_{\alpha + 1}(M_n |x|) |x|^\alpha \left[ V(x)^p (|x| + M_n^{-1})^{-p(\alpha + 1/2)} |x|^{2\alpha + 1} \right] dx, \]

which, by (19) is in turn bounded by

\[ C \int_{-1}^{1} f(x)^p V(x)^p |x|^{2\alpha + 1} dx = C \| V f \|_{L^p((-1,1), d\mu_n)}^p. \]
5.3. The third term

Taking limits when \( n \to \infty \) in Eq. (20), we get (22), so the proof of the boundedness of the third summand in Theorem 1 is still valid for Theorem 2.

6. Proof of Theorem 3

The following lemma is a small variant of a result proved in [8]. We give here a proof for the sake of completeness.

**Lemma 4.** Let \( \nu > -1 \). Let \( h \) be a Lebesgue measurable nonnegative function on \([0, 1]\), \( \{\rho_n\} \) a positive sequence such that \( \lim_{n \to \infty} \rho_n = +\infty \) and \( 1 \leq p < \infty \). Then

\[
\lim_{n \to \infty} \int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x) \, dx \geq M \int_0^1 h(x)x^{-p/2} \, dx
\]

(in particular, that limit exists), where \( M \) is a positive constant independent of \( h \) and \( \{\rho_n\} \).

**Proof.** We can assume that \( h(x)x^{-p} \) is integrable on \((0, \delta)\) for some \( \delta \in (0, 1) \), since otherwise

\[
\int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x) \, dx = \infty
\]

for each \( n \), as follows from Eq. (5), and (22) is trivial. Assume also for the moment that \( h(x)x^{-p/2} \) is integrable on \((0, 1)\). For each \( x \in (0, 1) \) and \( n \), let us put

\[
\varphi(x, n) = (\rho_n x)^{1/2} J_\nu(\rho_n x) - \sqrt{\frac{2}{\pi}} \cos \left( \frac{\nu \pi}{2} - \frac{\pi}{4} \right).
\]

The estimate (6) gives

\[
\lim_{n \to \infty} \varphi(x, n) = 0
\]

for each \( x \in (0, 1) \). Moreover, in case \( \rho_n x \geq 1 \) the same estimate gives

\[
|\varphi(x, n)| \leq \frac{C}{\rho_n x} \leq C
\]

with a constant \( C \) independent of \( n \) and \( x \), while for \( \rho_n x \leq 1 \) it follows from (5) that

\[
|\varphi(x, n)| \leq C \left( (\rho_n x)^{\nu+1/2} + 1 \right).
\]

Without loss of generality we can assume that \( \rho_0 \geq 1 \). Then, (23) and (24) give \( |\varphi(x, n)| \leq C(x^{\nu+1/2} + 1) \) with a constant \( C \) independent of \( x \) and \( n \), so that, by the dominate convergence theorem,

\[
\lim_{n \to \infty} \int_0^1 (\rho_n x)^{1/2} J_\nu(\rho_n x) - \sqrt{\frac{2}{\pi}} \cos \left( \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \left| h(x)x^{-p/2} \right. dx = 0.
\]

Therefore,

\[
\lim_{n \to \infty} \int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x) \, dx = \lim_{n \to \infty} \int_0^1 \left| \sqrt{\frac{2}{\pi}} \cos \left( \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \right|^p h(x)x^{-p/2} \, dx.
\]

Now we use Fejér’s lemma: if \( f \in L^1(0, 2\pi) \), and \( g \) is a continuous, \( 2\pi \)-periodic function, then

\[
\lim_{\lambda \to \infty} \frac{1}{2\pi} \int_0^{2\pi} g(\lambda t) f(t) \, dt = \hat{g}(0) \hat{f}(0) = \frac{1}{2\pi} \int_0^{\pi} g(t) \, dt \frac{1}{2\pi} \int_0^{\pi} f(t) \, dt
\]

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where $\hat{f}$, $\hat{g}$ denote the Fourier transforms of $f$, $g$. After a change of variables, Fejér’s lemma applied to the right hand side of (26) gives

$$
\lim_{n \to \infty} \int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x) \, dx = M \int_0^1 h(x)x^{-p/2} \, dx
$$

for some constant $M$, thus proving (22).

Finally, in case $h(x)x^{-p/2}$ is not integrable on $(0, 1)$, let us take the sequence of increasing measurable sets

$$
K_j = \{x \in (0, 1) : h(x)x^{-p/2} \leq j\}, \quad j \in \mathbb{N},
$$

and define $h_j = h$ on $K_j$ and $h_j = 0$ on $(0, 1) \setminus K_j$. Applying (22) to each $h_j$ and then the monotone convergence theorem proves that

$$
\lim_{n \to \infty} \int_0^1 |\rho_n^{1/2} J_\nu(\rho_n x)|^p h(x) \, dx = \infty,
$$

which is (22). \qed

We can now prove Theorem 3.

**Proof of Theorem 3.** The first partial sum of the Fourier expansion is

$$
S_0 f = e_0 \int_{-1}^1 f(x) \, d\mu_\alpha = (\alpha + 1) \int_{-1}^1 f(x)|x|^{2\alpha+1} \, dx,
$$

so that the inequality $\|S_0(f)U\|_{L^p((-1,1),d\mu_\alpha)} \leq C\|fV\|_{L^p((-1,1),d\mu_\alpha)}$ gives, by duality,

$$
U(x)^p|x|^{2\alpha+1} \in L^1((-1,1), dx), \quad V(x)^{-p'}|x|^{2\alpha+1} \in L^1((-1,1), dx).
$$

In fact, this is needed just to ensure that the partial sums of the Fourier expansions of all functions in $L^p(U^p \, d\mu_\alpha)$ are well defined and belong to $L^p(U^p \, d\mu_\alpha)$. These are the last two integrability conditions of Theorem 3.

Now, if

$$
\|S_n(f)U\|_{L^p((-1,1),d\mu_\alpha)} \leq C\|fV\|_{L^p((-1,1),d\mu_\alpha)}
$$

then the difference

$$
S_n f - S_{n-1} f = e_n \int_{-1}^1 f \, d\mu_\alpha - e_{n-1} \int_{-1}^1 f \, d\mu_\alpha
$$

$$
= e_n \int_{-1}^1 f \, d\mu_\alpha + e_n \int_{-1}^1 f \, d\mu_\alpha
$$

is bounded in the same way. Taking even and odd functions, and using that Re $e_n$ is even and Im $e_n$ is odd, gives

$$
\|U \text{Re } e_n\|_{L^p((-1,1),d\mu_\alpha)} \leq C \|fV\|_{L^p((-1,1),d\mu_\alpha)}
$$

and the same inequality with Im $e_n$. Recall that

$$
\text{Re } e_n(x) = 2^{\alpha/2} \Gamma(\alpha + 1)^{1/2} \frac{|s_n|^\alpha}{|J_\alpha(s_n)|} \frac{J_\nu(s_n x)}{(s_n x)^\alpha}.
$$

Taking into account that $|J_\nu(x)|$ is an even function (recall that $J_\alpha(z)/z^\alpha$ is taken as an even function) and $|J_\alpha(s_n)| \leq C s_n^{-1/2}$ (this follows from (5)), Lemma 11 gives

$$
\lim_{n \to \infty} \inf \int_{-1}^1 \left| \frac{1}{J_\alpha(s_n)} J_\nu(s_n x) \right|^p h(x) \, dx \geq C \int_{-1}^1 h(x)|x|^{-p/2} \, dx
$$
for every measurable nonnegative function $h$. Therefore,

$$\liminf_{n \to \infty} |U \operatorname{Re} e_n|_{L^p((-1,1), d\mu_\alpha)} \geq C \left( \int_{-1}^1 U(x)^p |x|^{-p\alpha - \frac{d}{2} + 2\alpha + 1} dx \right)^{\frac{1}{p}}$$

and the corresponding lower bound for $\liminf_n \|V^{-1} \operatorname{Re} e_n\|_{L^{p'}((0,1), d\mu_\alpha)}$ holds. The same bounds hold for $\operatorname{Im} e_n$. Thus, (27) implies

$$\left( \int_{-1}^1 U(x)^p |x|^{-p\alpha - \frac{d}{2} + 2\alpha + 1} dx \right)^{\frac{1}{p}} \left( \int_{-1}^1 V(x)^{p'} |x|^{-p'\alpha - \frac{d}{2} + 2\alpha + 1} dx \right)^{\frac{1}{p'}} \leq C$$

or, in other words, the first two integrability conditions of Theorem 3.

Take now $f = U/(1 + V + UV)$ and any measurable set $E \subseteq (-1,1)$. Then $f \in L^2(d\mu_\alpha)$ by Hölder’s inequality, the obvious inequality $|f| \leq UV^{-1}$ and the integrability conditions $U \in L^p(d\mu_\alpha)$, $V^{-1} \in L^{p'}(d\mu_\alpha)$, already proved. Since $(e_j)_{j \in \mathbb{Z}}$ is a complete orthonormal system in $L^2((-1,1), d\mu_\alpha)$, we have $S_n(f|_E) \to f|_E$ in the $L^2(d\mu_\alpha)$ norm. Therefore, there exists some subsequence $S_{n_j}(f|_E)$ converging to $f|_E$ almost everywhere. Fatou’s lemma then gives

$$\int_{-1}^1 |f|_E|^p U^p d\mu_\alpha \leq \liminf_{j \to \infty} \int_{-1}^1 |S_{n_j}(f|_E)|^p U^p d\mu_\alpha.$$  

Under the hypothesis of Theorem 3 each of the integrals on the right hand side is bounded by

$$C^p \int_{-1}^1 |f|_E|^p U^p d\mu_\alpha$$

(observe, by the way, that $fV \in L^p(d\mu_\alpha)$, since $|fV| \leq 1$). Thus,

$$\int_{-1}^1 |f|_E|^p U^p d\mu_\alpha \leq C^p \int_{-1}^1 |f|_E|^p U^p d\mu_\alpha$$

for every measurable set $E \subseteq (-1,1)$. This gives $fU \leq CfV$ almost everywhere, and $U \leq CV$.  

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