On the Cauchy problem of 2D viscous shallow water system in Besov spaces

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Abstract

In this paper we consider the Cauchy problem for 2D viscous shallow water system in Besov spaces. We firstly prove the local well-posedness of this problem in $B^{s}_{p,r}(\mathbb{R}^{2})$, $s > \frac{2}{p} + 1$ by using the Littlewood-Paley theory, the Bony decomposition and the theories of transport equations and transport diffusion equations. Then we give a blow-up criterion of solutions to the system in $B^{s}_{p,r}$, $s > \frac{2}{p} + 1$. Moreover, by this blow-up criterion, we can prove the global existence of the system with small enough initial data in $B^{s}_{p,r}(\mathbb{R}^{2})$, $p \leq 2$ and $s > 1 + \frac{2}{p}$. Our obtained results generalize and cover the recent results in \cite{12}.

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Keywords: Viscous shallow water system; Littlewood-Paley theory; Besov spaces; Blow up criterion; global existence

Contents

1 Introduction

2 Preliminaries

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1 Introduction

We consider the following Cauchy problems for 2D viscous shallow water equations

\[
\begin{aligned}
&h(u_t + (u \cdot \nabla)u) - \nu \nabla \cdot (h\nabla u) + h\nabla h = 0, \\
h_t + \text{div}(hu) = 0, \\
u|_{t=0} = u_0, \\
h|_{t=0} = h_0,
\end{aligned}
\]

(1.1)

where \(h(x, t)\) is the height of fluid surface, \(u(x, t) = (u^1(x, t), u^2(x, t))\) is the horizontal velocity field, \(x = (x_1, x_2) \in \mathbb{R}^2\), and \(0 < \nu < 1\) is the viscous coefficients. For the initial data \(h_0(x)\), we suppose that it is a small perturbation of some positive constant \(\bar{h}_0\). We study the Cauchy problems (1.1) in Besov space \(B^s_{p,r}(\mathbb{R}^2)\), \(s > \frac{2}{p} + 1\). For the sake of convenience, we let the notation \(B^s_{p,r}\) stand for \(B^s_{p,r}(\mathbb{R}^2)\) in the following text, and also let the notations \(L^p\) and \(H^s\) stand for \(L^p(\mathbb{R}^2)\) and \(H^s(\mathbb{R}^2)\), respectively.

Recently, Bresch et al. [3, 4] have systematically introduced the viscous shallow water equations. Bui in [5] proved the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for the shallow water equations with initial data in \(C^{2+\alpha}\) by using Lagrangian coordinates and Hölder space estimates. Kloeden in [8] and Sundbye in [10] independently showed the global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem using Sobolev space estimates by following the energy method of Matsumura and Nishida [9]. Sundbye in [11] proved the existence and uniqueness of classical solutions to the Cauchy problem using the method of [9].
Wang and Xu in [12] obtained local solutions for any initial data and global solutions for small initial data \( h_0 - \bar{h}_0, u_0 \in H^s, s > 2 \). Haspot got global existence in time for small initial data \( h_0, u_0 \in B^0_{2,1} \cap \dot{B}^1_{2,1} \) as a special case in [7], and Chen, Miao and Zhang in [6] to prove the local existence in time for general initial data and the global existence in time for small initial data where \( h_0 - \bar{h}_0 \in \dot{B}^0_{2,1} \cap \dot{B}^1_{2,1} \) and \( u_0 \in \dot{B}^0_{2,1} \) with additional conditions that \( h \geq h_0 \) and \( h_0 \) is a strictly positive constant.

In the paper, we mainly use the Littlewood-Paley theory, the Bony decomposition and the Besov space theories for transport equations and transport-diffusion equations to obtain the local existence and uniqueness of solutions for any initial data in \( B^s_{p,r}(\mathbb{R}^2) \), \( s > \frac{2}{p} + 1 \), and a blow-up criterion in \( B^s_{p,r}(\mathbb{R}^2) \), \( s > \frac{2}{p} + 1 \). Moreover, by the blow-up criterion, we can prove the global existence of the system with small enough initial data in \( B^s_{p,r}(\mathbb{R}^2) \), \( p \leq 2 \) and \( s > 1 + \frac{2}{p} \).

The main results of this paper are as follows:

**Theorem 1.1.** Let \( u_0, h_0 - \bar{h}_0 \in B^s_{p,r}, s > \frac{2}{p} + 1, \|h_0 - \bar{h}_0\|_{B^s_{p,r}} < \bar{h}_0 \). Then there exists a positive time \( T \), a unique solution \((u, h)\) of the Cauchy problem (1.1) such that

\[
\begin{align*}
  u, h - \bar{h}_0 &\in \dot{L}^\infty([0, T]; B^s_{p,r}) \cap C([0, T]; B^s_{p,r}), \\
u, h &\in \dot{L}^2([0, T]; B^{s+1}_{p,r}).
\end{align*}
\]

**Theorem 1.2.** Let \( u_0, h_0 \in B^s_{p,r} \times B^s_{p,r}, s > 1 + \frac{2}{p} \), and let \( u, h \) be the corresponding solution of the Cauchy problem (1.1) in \( B^s_{p,r} \times B^s_{p,r} \). Assume \( T^* \) is the maximal existence time of solution. If \( T^* \) is finite, then we have

\[
\int_0^{T^*} \|\nabla u\|_{L^\infty}^t + \|h\|_{L^\infty}^t + \|\nabla (\ln(1 + h))\|_{L^\infty}^t dt = \infty,
\]

where \( r_1 = \max\{r', 2\} \).

**Theorem 1.3.** Let \( u_0, h_0 \in B^s_{p,r}, p \leq 2, s > 1 + \frac{2}{p} \). If there exists an \( \varepsilon \) small enough such that \( \|u_0\|_{B^s_{p,r}} + \|h_0\|_{B^s_{p,r}} < \varepsilon \), then the corresponding solution of the Cauchy problem (1.1) in \( B^s_{p,r} \) is global in time.
2 Preliminaries

First of all, we transform the system (1.1). For a sake of convenience, we take $h_0 = 1$. Substituting $h$ by $1 + h$ in (1.1), we have

\[
\begin{aligned}
&u_t + (u \cdot \nabla)u - \nu \Delta u - \nu \nabla(\ln(1 + h)) \nabla u + \nabla h = 0, \\
h_t + \text{div}u + \text{div}(hu) = 0,
\end{aligned}
\]

(2.1)

\[u|_{t=0} = u_0, \quad h|_{t=0} = h_0,
\]

here $h_0 \in B^s_{p,r}$, and $\|h_0\|_{B^s_{p,r}} \leq \frac{1}{C_0 C^s_{p}}$, $C_0, C^s_{p}$ is determined below.

Then we introduce the Littlewood-Paley decomposition briefly.

**Proposition 2.1.** \[\text{Littlewood-Paley Decomposition:}\]

Let $B = \{\xi \in \mathbb{R}^2, |\xi| \leq \frac{4}{3}\}$ be a ball, and $C = \{\xi \in \mathbb{R}^2, \frac{2}{3} \leq |\xi| \leq \frac{8}{3}\}$ be an annulus. There exist two radial functions $\chi$ and $\varphi$ valued in the interval $[0, 1]$, belonging respectively to $\mathcal{D}(B)$ and $\mathcal{D}(C)$, such that

\[
\begin{aligned}
&\forall \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \\
&\forall \xi \in \mathbb{R}^2 \backslash \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \\
&|j - j'| \geq 2 \Rightarrow \text{Supp}\varphi(2^j \cdot) \cap \text{Supp}\varphi(2^{j'} \cdot) = \emptyset, \\
&j \geq 2 \Rightarrow \text{Supp}\chi \cap \text{Supp}\varphi(2^j \cdot) = \emptyset,
\end{aligned}
\]

the set $\tilde{C} \overset{\text{def}}{=} B(0, 2/3) + C$ is an annulus, and we have

\[
|j - j'| \geq 5 \Rightarrow 2^j \tilde{C} \cap 2^{j'} C = \emptyset.
\]

Further, we have

\[
\begin{aligned}
&\forall \xi \in \mathbb{R}^2, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1, \\
&\forall \xi \in \mathbb{R}^2 \backslash \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1.
\end{aligned}
\]
Now we can define the nonhomogeneous dyadic blocks $\Delta_j$ and the nonhomogeneous low-frequency cut-off operator $S_j$ as follows:

$$\Delta_j u = 0, \text{ if } j \leq -2,$$

$$\Delta_{-1} u = \chi(D)u = \int_{\mathbb{R}^2} \hat{h}(y)u(x-y)dy,$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^2} h(2^jy)u(x-y)dy, \text{ if } j \geq 0.$$

and

$$S_j u = \sum_{j' \leq j-1} \Delta_{j'} u.$$

Where $h = \mathcal{F}^{-1}\varphi$ and $\hat{h} = \mathcal{F}^{-1}\chi$.

Next we define the Besov spaces:

**Definition 2.2.** Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^s_{p, r}$ consists of all tempered distribution $u$ such that:

$$\left(\sum_{j \geq -1} (2^{js} \|\Delta_j u\|_{L^p})\right)^{\frac{1}{r}} < \infty,$$

and naturally the Besov norm is defined as follows

$$\|u\|_{B^s_{p, r}} = \left(\sum_{j \geq -1} (2^{js} \|\Delta_j u\|_{L^p})\right)^{\frac{1}{r}}.$$

**Definition 2.3.** The Bony decomposition: The nonhomogeneous paraproduct of $v$ by $u$ is defined by

$$T_u v = \sum_j S_{j-1} u \Delta_j v.$$

The nonhomogeneous remainder of $u$ by $v$ is defined by

$$R(u, v) = \sum_{|k-j| \leq 1} \Delta_k u \Delta_j v.$$

The operators $T$ and $R$ are bilinear, and we have the following Bony decomposition

$$uv = T_u v + T_v u + R(u, v).$$

**Lemma 2.4.** Bernstein-Type inequalities:

Let $C$ be an annulus and $B$ a ball. A constant $C$ exists such that for any nonnegative integer $k$, any couple $(p, q)$ in $[1, \infty]^2$ with $q \geq p \geq 1$, and any function $u$ of $L^p$, we have

$$\text{Supp} \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p},$$

$$\text{Supp} \hat{u} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^k \|u\|_{L^p}.$$
Then we give some properties of the Besov spaces which will be used in this paper.

Lemma 2.5. Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any $s \in \mathbb{R}$, the space $B_{p_1,r_1}^s$ is continuously embedded in $B_{p_2,r_2}^{s-d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}$. Obviously, we also have that the space $B_{p,r}^{s_1}$ is continuously embedded in $B_{p,r}^{s_2}$ and $B_{p,\infty}^{s_2}$ is continuously embedded in $B_{p,r}^{s_1}$ if $s_1 < s_2$.

Lemma 2.6. If $u \in B_{p,r}^s$, then $\nabla u \in B_{p,r}^{s-1}$, and we have

$$\|\nabla u\|_{B_{p,r}^{s-1}} \leq C\|u\|_{B_{p,r}^s}.$$ 

Lemma 2.7. The set $B_{p,r}^s$ is a Banach space and satisfies the Fatou property, namely, if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence of $B_{p,r}^s$, then an element $u$ of $B_{p,r}^s$ and a subsequence $u_{\psi(n)}$ exist such that:

$$\lim_{n \to \infty} u_{\psi(n)} = u \text{ in } S', \quad \|u\|_{B_{p,r}^s} \leq C \liminf_{n \to \infty} \|u_{\psi(n)}\|_{B_{p,r}^s}.$$ 

Lemma 2.8. If $s > \frac{d}{p}$ or $s = \frac{d}{p}$, $r = 1$, then the $B_{p,r}^s$ space is continuously embedded in $L^\infty$, i.e.

$$\|u\|_{L^\infty} \leq C_{s,p}\|u\|_{B_{p,r}^s}.$$ 

Lemma 2.9. Let $f$ be a smooth function, $f(0) = 0$, $s > 0$, $(p, r) \in [1, \infty]^2$. If $u \in B_{p,r}^s \cap L^\infty$, then so does $f \circ u$, and we have

$$\|f \circ u\|_{B_{p,r}^s} \leq C(s, f', \|u\|_{L^\infty}) \|u\|_{B_{p,r}^s}.$$ 

Lemma 2.10. A constant $C$ exists which satisfies the following inequalities for any couple of real numbers $(s, t)$ with $t$ negative and any $(p, r_1, r_2)$ in $[1, \infty]^3$:

$$\|T\|_{L(L^\infty \times B_{p,1}; B_{p,\infty}^s)} \leq C^{s+1},$$

$$\|T\|_{L(B_{p,\infty}^s \times B_{p,\infty}^s; B_{p,\infty}^{s+t})} \leq \frac{C^{s+t+1}}{-t} \quad \text{with} \quad \frac{1}{p} \overset{\text{def}}{=} \min\{1, \frac{1}{r_1} + \frac{1}{r_2}\}.$$ 

Lemma 2.11. A constant $C$ exists which satisfies the following inequalities. Let $(s_1, s_2)$ be in $\mathbb{R}^2$ and $(p_1, p_2, r_1, r_2)$ be in $[1, \infty]^4$. Assume that

$$\frac{1}{p} \overset{\text{def}}{=} \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \quad \text{and} \quad \frac{1}{r} \overset{\text{def}}{=} \frac{1}{r_1} + \frac{1}{r_2} \leq 1.$$ 

If $s_1 + s_2 > 0$, then we have, for any $(u, v)$ in $B_{p_1,r_1}^{s_1} \times B_{p_2,r_2}^{s_2}$,

$$\|R(u, v)\|_{B_{p,r}^{s_1+s_2}} \leq C^{s_1+s_2+1} s_1 + s_2 \|u\|_{B_{p_1,r_1}^{s_1}} \|v\|_{B_{p_2,r_2}^{s_2}}.$$ 

If $r = 1$ and $s_1 + s_2 = 0$, then we have, for any $(u, v)$ in $B_{p_1,r_1}^{s_1} \times B_{p_2,r_2}^{s_2}$,

$$\|R(u, v)\|_{B_{p,\infty}^{s_1+s_2}} \leq C^{s_1+s_2+1} \|u\|_{B_{p_1,r_1}^{s_1}} \|v\|_{B_{p_2,r_2}^{s_2}}.$$
Lemma 2.12. [2] For any $s > 0$ and $(p, r) \in [1, \infty]^2$, the space $B^s_{p,r} \cap L^\infty$ is an algebra, and a constant exists such that:

$$\|uv\|_{B^s_{p,r}} \leq \frac{C^{s+1}}{s}\left(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}\right).$$

Moreover, if $s > \frac{d}{p}$ or $s = \frac{d}{p}, r = 1$, we have

$$\|uv\|_{B^s_{p,r}} \leq \frac{C^{s+1}}{s}\|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}}.$$

For the transport equations

$$\begin{align*}
\partial_t f + v \cdot \nabla f &= g \\
|f|_{t=0} &= f_0,
\end{align*}$$

we have

Lemma 2.13. [2] Let $1 \leq p \leq p_1 \leq \infty$, $1 \leq r \leq \infty$. Assume that

$$s \geq -d \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \text{or} \quad s \geq -1 - d \min\left(\frac{1}{p_1}, \frac{1}{p'}\right) \quad \text{if} \ \text{div} \ v = 0$$

with strict inequality if $r < \infty$. There exists a constant $C$, depending only on $d, p, p_1, r$ and $s$, such that for all solutions $f \in L^\infty([0, T]; B^s_{p,r})$ of (2.9), initial data $f_0$ in $B^s_{p,r}$, and $g$ in $L^1([0, T]; B^s_{p,r})$, we have, for a.e. $t \in [0, T]$,

$$\|f\|_{L^\infty(B^s_{p,r})} \leq \left(\|f_0\|_{B^s_{p,r}} + \int_0^t \exp(-CV_{p_1}(t'))\|g(t')\|_{B^s_{p,r}} \, dt'\right) \exp(CV_{p_1}(t))$$

with, if the inequality is strict in (2.10),

$$V'_{p_1}(t) = \begin{cases} 
\|\nabla v(t)\|_{B^{s-1}_{p_1,1}}, & \text{if} \ s > 1 + \frac{d}{p_1} \ \text{or} \ s = 1 + \frac{d}{p_1}, \ r = 1, \\
\|\nabla v(t)\|_{B^{s-1}_{p_1,\infty} \cap L^\infty}, & \text{if} \ s < 1 + \frac{d}{p_1}
\end{cases}$$

and, if equality holds in (2.10) and $r = \infty$,

$$V'_{p_1} = \|\nabla v(t)\|_{B^{d}_{p_1,1}}.$$

If $f = v$, then for all $s > 0(s > -1, \text{if} \ \text{div} \ u = 0)$, the estimate holds with

$$V'_{p_1}(t) = \|\nabla u\|_{L^\infty},$$

where $\|u\|_{L^\infty(B^s_{p,r})}$ is defined in Lemma 2.15.
For the transport diffusion equations
\begin{align}
\left\{ \begin{array}{l}
\partial_t f + v \cdot \nabla f - \nu \Delta f = g \\
f|_{t=0} = f_0,
\end{array} \right.
\end{align}
\tag{2.12}
we have the following lemma.

**Lemma 2.14.** \[2\] Let $1 \leq p_1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $s \in \mathbb{R}$ satisfy (2.10), and $V_{p_1}$ be defined as in Lemma 2.8.

There exists a constant $C$ which depends only on $d, r, s$ and $s - 1 - \frac{d}{p_1}$, and is such that for any smooth solution of (11) and $1 \leq \rho_1 \leq \rho \leq \infty$, we have
\[
\nu \frac{1}{2} \|f\|_{L^{p_1+2\rho}_{p,r}(B_{p,r}^s)} \leq C \left( (1 + \nu T)^{\frac{1}{2} s} \|f_0\|_{B_{p,r}^s} + (1 + \nu T)^{1 + \frac{1}{2} r - \frac{1}{2} \frac{1}{\rho_1} - 1} \|g\|_{L^{p_1}_{p,2\rho}(B_{p,r}^{s-2+2\rho})} \right).
\]

For the space $\tilde{L}_{p,T}^p(B_{p,r}^s)$, we have the following properties:

**Lemma 2.15.** \[2\] For all $T > 0$, $s \in \mathbb{R}$, and $1 \leq r, \rho \leq \infty$, we set
\[
\|u\|_{\tilde{L}_{p,T}^p(B_{p,r}^s)} \overset{def}{=} \|2^{is} \Delta_j u\|_{L^p_t(L^r)}(Z).
\]
We can then define the space $\tilde{L}_{p,T}^p(B_{p,r}^s)$ as the set of tempered distributions $u$ over $(0, T) \times \mathbb{R}^d$ such that $\|u\|_{\tilde{L}_{p,T}^p(B_{p,r}^s)} < \infty$. By the Minkowski inequality, we have
\[
\|u\|_{\tilde{L}_{p,T}^p(B_{p,r}^s)} \leq \|u\|_{L^p_t(L^r)}(Z) \quad \text{if } r \geq \rho,
\]
\[
\|u\|_{\tilde{L}_{p,T}^p(B_{p,r}^s)} \leq \|u\|_{L^p_t(L^r)}(Z) \quad \text{if } r \leq \rho.
\]
The general principle is that all the properties of continuity for the product, composition, remainder, and paraproduct remain true in those space.

Moreover when $s > 0$, $1 \leq p \leq \infty$, $1 \leq \rho_1, \rho_2, \rho_3, \rho_4 \leq \infty$, and
\[
\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\rho_3} + \frac{1}{\rho_4},
\]
we have
\[
\|uv\|_{\tilde{L}_{p,T}^p(B_{p,r}^s)} \leq C \left( \|u\|_{L^{p_1}_{p,T}(L^{\infty})} \|v\|_{L^{p_2}_{p,T}(B_{p,r}^s)} + \|v\|_{L^{p_3}_{p,T}(L^{\infty})} \|u\|_{L^{p_4}_{p,T}(B_{p,r}^s)} \right).
\]

**Lemma 2.16.** \[2\] The time-space estimate for heat equation:

Let $C$ be an annulus and $\lambda$ a positive real number. Let $u_0$, $f$ satisfy $\text{Supp} \hat{u}_0, \text{Supp} \hat{f}(t) \subset \lambda C$ for all $t \in [0, T]$. Consider $u$, a solution of
\[
\partial_t u - \nu \Delta u = f \quad \text{and} \quad u|_{t=0} = u_0.
\]
Then there exists a positive constant $C$, depending only on $C$, such that for any $1 < a < b < \infty$ and $1 \leq p \leq q \leq \infty$, we have

$$\|u\|_{L^q_t(L^b)} \leq C(\nu \lambda^2)^{-\frac{1}{q}} \lambda^{d\left(\frac{1}{b} - \frac{1}{a}\right)} \|u_0\|_{L^a} + C(\nu \lambda^2)^{-1 + \frac{1}{p} - \frac{1}{q}} \lambda^{(\frac{1}{b} - \frac{1}{a})} \|f\|_{L^p_t(L^a)}.$$  

Lemma 2.17. [2] The commutator estimate:

Let $s \in \mathbb{R}$, $1 < r < \infty$, and $1 \leq p \leq p_1 < \infty$. Let $v$ be a vector field over $\mathbb{R}^d$. Assume that $s > -d \min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\}$ or $s > -1 - d \min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\}$ if $\text{div} \ v = 0$.

Define $R_j(v, f) \overset{d}{=} [v \cdot \nabla, \Delta_j]f$ or $R_j(v, f) \overset{d}{=} \text{div} (|v| \Delta_j) f$, if $\text{div} v = 0$. There exists a constant $C$, depending continuously on $p$, $p_1$, $s$ and $d$, such that

$$\|2^j s \| R_j \|_{L^p} \| \leq C \|\nabla v\|_{B_{p_1, \infty}^{s \infty}} \|f\|_{B_{p, r}^s} \text{ if } s < 1 + \frac{d}{p_1}.$$  

Further, if $s > 0$ (or $s > -1$ if $\text{div} v = 0$) and $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_2}$, then

$$\|2^j s \| R_j \|_{L^p} \| \leq C \left(\|\nabla v\|_{L^{p_1}} \|f\|_{B_{p_1, r}^{s_1}} + \|\nabla f\|_{L^{p_1}} \|
abla v\|_{B_{p_1, r}^{s_1}}\right).$$  

Lemma 2.18. [12] Let $s > 2$, $u_0, h_0 \in H^s$. Then there exist a positive time $T$, a unique solution $(u, h)$ of the Cauchy problem (2.1) such that

$$u, h \in L^\infty([0, T], H^s), \nabla u \in L^2([0, T]; H^s).$$  

Furthermore, there exists a constant $c$ such that if $\|u_0\|_{H^s} + \|h_0\|_{H^s} \leq c$, then $T = \infty$.

Remark 2.19. For the sake of convenient, for the fixed $s, p, r$, we let $C_0(>1)$ be the maximum constant of Lemmas 2.6-2.18.

3 The local existence

In order to study the local existence of solution, we define the function set $(u, h) \in \chi(0, T, s, p, r, E_1, E_2)$, if $(u, h) \in L^\infty([0, T]; B_{p, r}^s)$, and

$$\|u\|_{L^\infty([0, T]; B_{p, r}^s)} \leq E_1, \|h\|_{L^\infty([0, T]; B_{p, r}^s)} \leq E_2,$$

where

$$E_1 = 4C_0 \|u_0\|_{B_{p, r}^s}, E_2 = 4C_0 \|h_0\|_{B_{p, r}^s}.$$
Next, we will prove Theorem 1.1 by the method of successive approximations. Let us define the sequence \((u_n, h_n)\) by the following linear system:

\[
\begin{aligned}
(u_1, h_1) &= S_2(u_0, h_0), \\
\partial_t u_{n+1} + (u_n \cdot \nabla) u_{n+1} - \nu \Delta u_n &= \frac{\nu}{1 + h_n} \nabla h_n \nabla u_n + \nabla h_n, \\
\partial_t h_{n+1} + (u_n \cdot \nabla) h_{n+1} &= -\text{div} u_n - h \text{div} u_n, \\
(u_{n+1}, h_{n+1})|_{t=0} &= S_{n+2}(u_0, h_0).
\end{aligned}
\]

(3.1)

Since \(S_q\) are smooth operators, the initial date \(S_{n+2}(u_0, h_0)\) are smooth functions. If \((u_n, h_n)\) are smooth, then we have that for any \(t \in [0, T]\),

\[
\|h_n\|_{L^\infty} \leq C_{s,p} \|h_n\|_{B^s_{p,r}} \leq C_{s,p} E_2 = \frac{4C_0 C_{s,p}}{8C_0 C_{s,p}} = \frac{1}{2}.
\]

Thus \(\frac{\nu}{1 + h_n} \nabla h_n \nabla u_n + \nabla h_n\) and \(-\text{div} u_n - h \text{div} u_n\) are also smooth functions. Note that the first equation in (3.1) is a transport diffusion equation for \(u_{n+1}\), and the second equation is a transport equation for \(h_{n+1}\). Then the local existence of the smooth function for the Cauchy problem (3.1) is obvious.

We split the proof of Theorem 1.1 into two steps: Estimation for big norms and Convergence for small norms.

### 3.1. Estimation for big norms

In this subsection, we want to prove the following proposition.

**Proposition 3.1.** Suppose that \((u_0, h_0) \in B^s_{p,r} \times B^s_{p,r}, s > 1 + \frac{2}{p}\) and \(\|h_0\|_{B^s_{p,r}} \leq \frac{1}{\nu C_{s,p}}\), then there exists a positive time \(T_1\), such that for any \(n \in \mathbb{N}, (u_n, h_n) \in \chi([0, T_1], s, p, r, E_1, E_2)\).

**Proof:** Let \(T(\geq T_1)\) satisfy

\[
T \leq 1, \; e^{C_0 E_1 T} \leq 2, \; e^{2C_0 E_1 T} \leq 2, \; (1 + \nu T)^{\frac{1}{2}} \leq 2.
\]

Then we prove the proposition by induction. Firstly let \((u_1, h_1) = S_2(u_0, h_0)\). Thus we have

\[
\|u_1\|_{L^\infty_{T_1}(B^s_{p,r})} \leq \|u_0\|_{B^s_{p,r}} \leq E_1, \; \|h_1\|_{L^\infty_{T_1}(B^s_{p,r})} \leq \|h_0\|_{B^s_{p,r}} \leq E_2.
\]

If

\[
\|u_n\|_{L^\infty_{T_1}(B^s_{p,r})} \leq E_1, \; \|h_n\|_{L^\infty_{T_1}(B^s_{p,r})} \leq E_2,
\]

then we have

\[
\|u_{n+1}\|_{L^\infty_{T_1}(B^s_{p,r})} \leq E_1, \; \|h_{n+1}\|_{L^\infty_{T_1}(B^s_{p,r})} \leq E_2.
\]

Thus, by induction, we prove that \((u_n, h_n) \in \chi([0, T], s, p, r, E_1, E_2)\) for all \(T \leq 1\) and \(n \in \mathbb{N}\).
then for $h_{n+1}$, in the view of Lemma \textbf{2.8}, \textbf{Lemma 2.13} and \textbf{Lemma 2.15}, for all $t < T_1$, we have

\[ ||h_{n+1}||_{L^\infty_t(B_{p,r}^t)} \leq \left( ||S_{n+2}h_0||_{B_{p,r}^t} + ||\text{div}u_n||_{L^1_t(B_{p,r}^t)} + ||h_n\text{div}u_n||_{L^1_t(B_{p,r}^t)} \right) \]

\[ \exp(C_0 \int_0^t ||\nabla u_n(t')||_{B_{p,r}^{t'}} dt') \]

\[ \leq 2\left( \frac{E_2}{2} + t^\frac{1}{2} \||\text{div}u_n||_{L^2_t(B_{p,r}^t)} + \|h_n\|_{L^2_t(B_{p,r}^t)} \right) \]

\[ + ||\text{div}u_n||_{L^2_t(B_{p,r}^t)} \|h_n\|_{L^2_t(L^\infty)} \]

\[ \leq 2\left( \frac{E_2}{2} + t^\frac{1}{2} \||\text{div}u_n||_{L^2_t(B_{p,r}^t)} + Ct \|h_n\|_{L^\infty_t(B_{p,r}^t)} \|u_n\|_{L^\infty_t(B_{p,r}^t)} \]

\[ + Ct^\frac{1}{2} ||\text{div}u_n||_{L^2_t(B_{p,r}^t)} \|h_n\|_{L^\infty_t(B_{p,r}^t)} \]

\[ \leq \frac{E_2}{2} + CtE_1E_2 + C(1 + E_2)t^\frac{1}{2} \|u_n\|_{L^2_t(B_{p,r}^t)}. \]

Now, we estimate $\|u_n\|_{L^2_t(B_{p,r}^t)}$. By Lemmas \textbf{2.8}, \textbf{2.14} and Lemmas \textbf{2.11}, \textbf{2.15} we get

\[ ||u_n||_{L^2_t(B_{p,r}^t)} \leq \nu^{-\frac{1}{2}}C_0 \left( (1 + \nu t)^{-\frac{1}{2}} \left( \int_0^t \|\nabla u_{n-1}||_{B_{p,r}^{t'}} dt' \right) \right) \]

\[ + C \left( \|u_0\|_{B_{p,r}^t} + E_2 + \|\nabla \ln(1 + h_{n-1})\|_{L^\infty_t(B_{p,r}^t)} \|u_{n-1}||_{L^\infty_t(B_{p,r}^t)} \right) \]

\[ \leq C \left( \|u_0\|_{B_{p,r}^t} + E_2 + (\|\nabla u_{n-1}||_{L^\infty_t(B_{p,r}^t)} + \|\nabla \ln(1 + h_{n-1})\|_{L^\infty_t(L^\infty)} \right) \]

\[ + \|\nabla \ln(1 + h_{n-1})\|_{L^\infty_t(B_{p,r}^t)} \|u_{n-1}||_{L^\infty_t(L^\infty)} \right) \]

\[ \leq C(E_1 + E_2 + E_1E_2). \]

Thus letting

\[ T'_1 = \min\{T, (4CE_1)^{-1}, \ (4C^2(1 + E_2)(E_1 + E_2 + E_1E_2))^{-2}E_2^2 \}, \] we can get that, for any $t \leq T_1 \leq T'_1$,

\[ \|h_{n+1}\|_{L^\infty_t(B_{p,r}^t)} \leq E_1. \]
For $u_{n+1}$, from Lemmas 2.8, 2.9 and Lemmas 2.14, 2.15, we obtain

\begin{equation}
\|u_{n+1}\|_{L_t^\infty(B_{p,r}^s)} \leq C_0 e^{C_0 \int_0^T \|\nabla u_n\|_{B_{p,r}^{s-1}}} \times \\
\left(\|u_0\|_{B_{p,r}^s} + (1 + \nu t)^{\frac{1}{2}} \nu^{-\frac{1}{2}} \|\nabla h_n\|_{L_t^\infty(B_{p,r}^{s-1})} + \nabla h_n\|_{L_t^2(B_{p,r}^{s-1})}\right) \\
\leq 2C_0 \left(\|u_0\|_{B_{p,r}^s} + \left(\frac{1 + \nu t}{\nu}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \left(\|\nabla h_n\|_{L_t^\infty(B_{p,r}^{s-1})} + \|\nabla (1 + h_n)\|_{L_t^\infty(B_{p,r}^{s-1})}\right)\right) \\
\leq 2C_0 \left\{\|u_0\|_{B_{p,r}^s} + \left(\frac{1 + \nu t}{\nu}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \left(\|\nabla h_n\|_{L_t^\infty(B_{p,r}^{s-1})} + \nu C \left(\|\nabla u_n\|_{L_t^\infty(B_{p,r}^{s-1})} \|\nabla (1 + h_n)\|_{L_t^\infty(B_{p,r}^{s-1})}\right)\right)\right\} \\
\leq 2C_0 \left(\|\nabla u_0\|_{B_{p,r}^s} + \nu C t^{\frac{1}{2}} (E_2 + E_1 E_2)\right) \\
\leq \frac{\nu E_2}{2} + Ct^{\frac{1}{2}} (E_2 + E_1 E_2).
\end{equation}

Then, let

$$T_1 = \min\{T'_1, (2C(E_2 + E_1 E_2))^{-2} E_1^2\}.$$

Thus, for all $t \leq T_1$, we have

$$\|u_{n+1}\|_{L_t^\infty(B_{p,r}^s)} \leq E_1.$$

This completes the proof of Proposition 3.1.

**Remark 3.2.** From the proof of Proposition 3.1, we can see that $\|u_{n+1}\|_{L_t^\infty(B_{p,r}^{s-1})}$ is bounded by $E_1$ uniformly.

### 3.2. Convergence of small norms

**Proposition 3.3.** Suppose that $(u_0, h_0) \in B_{p,r}^s \times B_{p,r}^r$, $s > 1 + \frac{2}{p}$ and $\|h_0\|_{B_{p,r}^r} \leq \frac{1}{C_{s,r,p}}$, then there exists a positive time $T_2(\leq T_1)$, such that $(u_n, h_n)$ is a Cauchy sequence in $\chi([0, T_2], s-1, p, r, E_1, E_2)$.

Proof: From the equations in (3.1), we have

\begin{equation}
\begin{cases}
\partial_t(u_{n+1} - u_n) + (u_n \cdot \nabla)(u_{n+1} - u_n) - \nu \Delta (u_{n+1} - u_n) = \sum_{j=1}^5 F_j \\
\partial_t((h_{n+1} - h_n) + (u_n \cdot \nabla)(h_{n+1} - h_n) = \sum_{j=1}^4 J_j, \\
(u_{n+1} - u_n, h_{n+1} - h_n)|_{t=0} = \Delta_{n+1}(u_0, h_0),
\end{cases}
\end{equation}
where

$$\sum_{j=1}^{5} F_j = (u_n - u_{n-1}) \cdot \nabla u_n + \nabla (h_n - h_{n-1}) + \frac{\nu}{1+h_n} \nabla h_n \nabla (u_n - u_{n-1})$$
$$+ \frac{\nu}{1+h_n} \nabla u_{n-1} \nabla (h_n - h_{n-1}) + \nu(\frac{1}{1+h_n} - \frac{1}{1+h_{n-1}}) \nabla h_{n-1} \nabla u_{n-1},$$
(3.6)

$$\sum_{j=1}^{4} J_j = (u_n - u_{n-1}) \cdot \nabla h_n + \text{div} (u_n - u_{n-1}) + h_n \text{div} (u_n - u_{n-1})$$
$$+ (h_n - h_{n-1}) \text{div} u_{n-1}.$$

Then we estimate the Besov norm of $u_{n+1} - u_n$ and $h_{n+1} - h_n$. For any $t \leq T_2 \leq T_1$, by Lemma 2.14 we have

$$\|u_{n+1} - u_n\|_{L_t^\infty(B_{p,r}^{s-1})} \leq C_0 e^\int_0^t \|\nabla u_n\|_{B_{p,r}^{s-1}} \, dt' \times$$
$$\left(\|S_{n+2} u_0 - S_{n+1} u_0\|_{B_{p,r}^{s-1}} + \left(\frac{1}{\nu p} + \frac{1}{\nu r}\right) \sum_{j=1}^{5} F_j \|L_t^2(B_{p,r}^{s-1})\right)$$
$$\leq 2C_0 \left(\|\Delta_{n+1} u_0\|_{B_{p,r}^{s-1}} + \|\sum_{j=1}^{5} F_j \|_{L_t^2(B_{p,r}^{s-2})}\right),$$
(3.7)

here

$$\|\sum_{j=1}^{5} F_j \|_{L_t^2(B_{p,r}^{s-2})} \leq \|(u_n - u_{n-1}) \cdot \nabla u_n\|_{L_t^2(B_{p,r}^{s-2})}$$
$$+ \|\nabla (h_n - h_{n-1})\|_{L_t^2(B_{p,r}^{s-2})} + \|\nu \frac{h_n}{1+h_n} \nabla (u_n - u_{n-1})\|_{L_t^2(B_{p,r}^{s-2})}$$
$$+ \|\nu \frac{h_n - h_{n-1}}{1+h_n} \nabla u_{n-1}\|_{L_t^2(B_{p,r}^{s-2})} + \|\nu \frac{h_n - h_{n-1}}{(1+h_n)(1+h_{n-1})} \nabla h_{n-1} \nabla u_{n-1}\|_{L_t^2(B_{p,r}^{s-2})}$$
$$= I_1 + I_2 + I_3 + I_4 + I_5,$$
(3.8)

Next, we deal with $I_j$, $j = 1, 2, 3, 4, 5$ term by term.

By Lemmas 2.3, 2.6, Lemma 2.8 and Lemma 2.13 we have

$$I_1 \leq t^{\frac{s}{2}} \|(u_n - u_{n-1}) \cdot \nabla u_n\|_{L_t^\infty(B_{p,r}^{s-2})}$$
$$\leq Ct^{\frac{s}{2}} \|(u_n - u_{n-1}) \cdot \nabla u_n\|_{L_t^\infty(B_{p,r}^{s-1})}$$
$$\leq Ct^{\frac{s}{2}} \left(\|u_n - u_{n-1}\|_{L_t^\infty(B_{p,r}^{s-1})} \|\nabla u_n\|_{L_t^\infty(B_{p,r}^{s-1})} + \|\nabla u_n\|_{L_t^\infty(B_{p,r}^{s-1})} \|u_n - u_{n-1}\|_{L_t^\infty(B_{p,r}^{s-1})}\right)$$
$$\leq Ct^{\frac{s}{2}} \|u_n - u_{n-1}\|_{L_t^\infty(B_{p,r}^{s-1})} \|u_n\|_{L_t^\infty(B_{p,r}^{s-1})}$$
$$\leq CE_2 t^{\frac{s}{2}} \|u_n - u_{n-1}\|_{L_t^\infty(B_{p,r}^{s-1})},$$
(3.9)

From Lemma 2.8 it’s easy to see that

$$I_2 \leq t^{\frac{s}{2}} \|\nabla (h_n - h_{n-1})\|_{L_t^\infty(B_{p,r}^{s-2})} \leq Ct^{\frac{s}{2}} \|h_n - h_{n-1}\|_{L_t^\infty(B_{p,r}^{s-1})}.$$  
(3.10)
In view of Lemmas 2.8-2.9 and Lemma 2.15 we get

\begin{equation}
I_3 = \|\nu T\nabla \ln(1+h)\|_{L^\infty(B_{p,r}^{-2})} \|\nabla(u_n - u_{n-1})\|_{L^\infty(B_{p,r}^{-2})}
\end{equation}

By Lemma 2.10, we have

\begin{equation}
I_3 = \|\nu \nabla \ln(1 + h_n)\|_{L^\infty(B_{p,r}^{-2})} \|\nabla(u_n - u_{n-1})\|_{L^\infty(B_{p,r}^{-2})}
\end{equation}

(3.11)

\begin{equation}
I_3 = \|\nu \nabla \ln(1 + h_n)\|_{L^\infty(B_{p,r}^{-2})} \|\nabla(u_n - u_{n-1})\|_{L^\infty(B_{p,r}^{-2})}
\end{equation}

(3.12)

\begin{equation}
I_3 = \|\nu \nabla \ln(1 + h_n)\|_{L^\infty(B_{p,r}^{-2})} \|\nabla(u_n - u_{n-1})\|_{L^\infty(B_{p,r}^{-2})}
\end{equation}

(3.13)

where \( \varepsilon \) is a positive real number and small enough, and it equals 0 when \( s = 1 + \frac{2}{p} \). When \( s > 2 + \frac{2}{p} \), we also have

\begin{equation}
I_3 \leq C t^{\frac{1}{2}} \|\nabla(u_n - u_{n-1})\|_{L^\infty(B_{p,r}^{-2})} \|\nabla \ln(1 + h_n)\|_{L^\infty(B_{p,r}^{-2})}
\end{equation}

(3.14)

For \( I_{33} \), when \( s - 2 + \frac{2}{p} > 0 \), from Lemma 2.11 we have

\begin{equation}
I_{33} \leq C t^{\frac{1}{2}} \|R(\nabla(u_n - u_{n-1}), \nabla \ln(1 + h_n))\|_{L^\infty(B_{p,r}^{-2})}
\end{equation}

(3.15)
otherwise, we have $1 < s \leq 2$, then we get
\[
I_{33} \leq C \| R(\nabla(u_n - u_{n-1}), \nabla(\text{ln}(1 + h_n))) \|_{\dot{L}^p_t(B^{s-2+\delta}_{2,r})} \\
\leq C \| R(\nabla(u_n - u_{n-1}), \nabla(\text{ln}(1 + h_n))) \|_{L^p_t(B^{s-3+\delta}_{2,r})} \\
\leq C \| \nabla(u_n - u_{n-1}) \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \| \nabla(\text{ln}(1 + h_n)) \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
\leq Ct^{s-3+\delta} \| h_n \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \| u_n - u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
\leq CE_2 t^{s-3+\delta} \| u_n - u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \| u_n - u_{n-1} \|_{L^\infty(B^{s-1}_{p,r})}. 
\]

Then we deal with $I_4$ by the similar method.
\[
I_4 = \| \nu \frac{\nabla(h_n - h_{n-1})}{1 + h_n} \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
= \nu \| (1 - \frac{h_n}{1 + h_n}) \nabla(h_n - h_{n-1}) \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
\leq \nu \| \nabla(h_n - h_{n-1}) \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} + \nu \| \frac{h_n}{1 + h_n} \nabla(h_n - h_{n-1}) \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
\leq \nu \| \nabla(h_n - h_{n-1}) \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} + \nu \| T_{\frac{h_n}{1 + h_n}} \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
+ \nu \| T_{\nabla(h_n - h_{n-1})} \frac{h_n}{1 + h_n} \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} + \nu \| R(\nabla(h_n - h_{n-1}), \frac{h_n}{1 + h_n} \nabla u_{n-1}) \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
= I_{41} + I_{42} + I_{43} + I_{44}. 
\]

Set $t^s = \max\{t_1^s, t_2^{s-3}\}$. Similar to the argument in the proof of $I_3$, we obtain
\[
I_{41} \leq CE_1 t^s \| h_n - h_{n-1} \|_{L^\infty_t(B^{s-1}_{p,r})}. 
\]

Following the procedure of $I_{31} - I_{33}$ respectively, we have
\[
I_{42} \leq CE_1 E_2 t^s \| h_n - h_{n-1} \|_{L^\infty_t(B^{s-1}_{p,r})}, 
\]
\[
I_{43} \leq CE_1 E_2 t^s \| h_n - h_{n-1} \|_{L^\infty_t(B^{s-1}_{p,r})}, 
\]
\[
I_{44} \leq CE_1 E_2 t^s \| h_n - h_{n-1} \|_{L^\infty_t(B^{s-1}_{p,r})}. 
\]

Similarly as $I_4$, we have
\[
I_5 = \| \nu \frac{h_n - h_{n-1}}{(1 + h_n)(1 + h_{n-1})} \nabla h_n - h_{n-1} \nabla u_{n-1} \|_{\dot{L}^p_t(B^{s-2}_{p,r})} \\
\leq CE_1 E_2(1 + E_2)^2 t^s \| h_n - h_{n-1} \|_{L^\infty_t(B^{s-1}_{p,r})}. 
\]

We also have
\[
\| \Delta_{n+1} u_0 \|_{B^{s-1}_{p,r}} \leq 2^{-(n+1)} \| \Delta_{n+1} u_0 \|_{B^s_{p,r}} \leq 2^{-(n+1)} \| u_0 \|_{B^s_{p,r}}. 
\]
For $h_{n+1} - h_n$, we have

$$\|h_{n+1} - h_n\|_{L_t^\infty(B_{p,r}^{-1})} \leq \exp(C_0 \int_0^t \|\nabla u_n\|_{B_{p,r}^{-1}}) \times$$

(3.24)

$$\left(\|\Delta_{n+1}h_0\|_{B_{p,r}^{-1}} + \sum_{j=1}^{4} J_j \|L_j^1(B_{p,r}^{-1})\right) \leq 2(\|\Delta_{n+1}h_0\|_{B_{p,r}^{-1}} + \|J_1 + J_4\|_{L_t^\infty(B_{p,r}^{-1})} + t^{\gamma} \|J_2 + J_3\|_{L_t^\infty(B_{p,r}^{-1})}).$$

From Lemmas 2.5, 2.6, Lemma 2.8 and Lemma 2.13 we have

$$\|J_1\|_{L_t^\infty(B_{p,r}^{-1})} = \|(u_n - u_{n-1}) \cdot \nabla h_n\|_{L_t^\infty(B_{p,r}^{-1})} \leq \|u_n - u_{n-1}\|_{L_t^\infty(L^\infty)} \|\nabla h_n\|_{L_t^\infty(L^\infty)} + \|\nabla h_n\|_{L_t^\infty(B_{p,r}^{-1})} \|u_n - u_{n-1}\|_{L_t^\infty(L^\infty)} \leq C\|u_n - u_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})}.$$

(3.25)

In view of Lemma 2.6 we get

$$\|J_2\|_{L_t^2(B_{p,r}^{-1})} = \|div (u_n - u_{n-1})\|_{L_t^2(B_{p,r}^{-1})} \leq C\|u_n - u_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})}.$$

(3.26)

By Lemmas 2.5, 2.6, Lemma 2.8 and Lemma 2.13 we obtain

$$\|J_3\|_{L_t^2(B_{p,r}^{-1})} = \|h_n \cdot div (u_n - u_{n-1})\|_{L_t^2(B_{p,r}^{-1})} \leq \|h_n\|_{L_t^\infty(L^\infty)} \|div (u_n - u_{n-1})\|_{L_t^2(B_{p,r}^{-1})} + \|div (u_n - u_{n-1})\|_{L_t^\infty(L^\infty)} \|h_n\|_{L_t^\infty(B_{p,r}^{-1})} \leq C\|h_n\|_{L_t^\infty(B_{p,r}^{-1})} \|u_n - u_{n-1}\|_{L_t^2(B_{p,r}^{-1})} \leq CE_2\|u_n - u_{n-1}\|_{L_t^2(B_{p,r}^{-1})}.$$

(3.27)

Similarly as $J_1$, we have

$$\|J_4\|_{L_t^\infty(B_{p,r}^{-1})} = \|(h_n - h_{n-1}) \cdot div u_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})} \leq \|h_n - h_{n-1}\|_{L_t^\infty(L^\infty)} \|div u_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})} + \|div u_{n-1}\|_{L_t^\infty(L^\infty)} \|h_n - h_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})} \leq C\|h_n - h_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})} \|u_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})} \leq CE_1\|h_n - h_{n-1}\|_{L_t^\infty(B_{p,r}^{-1})}.$$

(3.28)

We also have

$$\|\Delta_{n+1}h_0\|_{B_{p,r}^{-1}} \leq 2^{-(n+1)}\|\Delta_{n+1}h_0\|_{B_{p,r}} \leq 2^{-(n+1)}\|h_0\|_{B_{p,r}}.$$

(3.29)

Moreover, by Lemma 2.14, we have

$$\|u_{n+1} - u_n\|_{L_t^2(B_{p,r}^{-1})} \leq \nu^{-\frac{\gamma}{2}} C_0 (1 + \nu t)^{\frac{\gamma}{2}} \int_0^t \|\nabla u_n\|_{B_{p,r}^{-1}} dt' \times$$

(3.30)

$$\left(1 + \nu t\right)^{\frac{\gamma}{2}} \|\Delta_{n+1}u_0\|_{B_{p,r}^{-1}} + (1 + \nu t)^{\frac{\gamma}{2}} \sum_{j=1}^{5} F_j \|L_j^2(B_{p,r}^{-2})\| \leq 4\nu^{-\frac{\gamma}{2}} C_0 \|\Delta_{n+1}u_0\|_{B_{p,r}^{-1}} + C \sum_{j=1}^{5} F_j \|L_j^2(B_{p,r}^{-2})\|.$$
Combining (3.7)-(3.30), we get

\[
\|u_{n+1} - u_n\|_{L^\infty_t(B_{p,r}^{s+1})} \leq C_0 2^{-n} \|u_0\|_{B_{p,r}} + C T' \left( E_2 \|u_{n+1} - u_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + E_2 \|u_n - u_{n-1}\|_{L^2_t(B_{p,r})} \right) + (1 + E_1 (1 + E_2)) \|h_n - h_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})},
\]

(3.31)

\[
\|u_{n+1} - u_n\|_{L^2_t(B_{p,r}^{s+1})} \leq 2^{-n} C_0 2^{-n} \|u_0\|_{B_{p,r}} + C T' \left( E_2 \|u_{n+1} - u_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + E_2 \|u_n - u_{n-1}\|_{L^2_t(B_{p,r})} \right) + (1 + E_1 (1 + E_2)) \|h_n - h_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})},
\]

(3.32)

and

\[
\|h_{n+1} - h_n\|_{L^\infty_t(B_{p,r}^{s+1})} \leq 2^{-n} \|h_0\|_{B_{p,r}} + C T' \left( E_2 \|u_{n+1} - u_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + E_1 \|h_n - h_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} \right) + C T' (1 + E_2) \|u_n - u_{n-1}\|_{L^2_t(B_{p,r})},
\]

(3.33)

Choose a suitable \( T_2 \leq T_1 \) such that:

\[
\begin{cases}
CE_2 T'^2 \leq \frac{1}{T_2}, \\
C (1 + E_1 (1 + E_2) + E_1 E_2 (1 + E_2)^2) T'^2 \leq \frac{1}{T_2}, \\
CE_3 T_2 \leq \frac{1}{T_2}, \quad CE_2 T_2 \leq \frac{1}{T_2}, \\
C (1 + E_2) T_2^2 \leq \frac{1}{T_2}.
\end{cases}
\]

(3.34)

Thus, for any \( t \leq T_2 \), we can obtain

\[
\|u_{n+1} - u_n\|_{L^\infty_t(B_{p,r}^{s+1})} \leq \frac{1}{4} 2^{-n} E_1 + \frac{1}{T_2} \|u_n - u_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + \frac{1}{T_2} \|\tilde{u}_n - \tilde{u}_{n-1}\|_{L^2_t(B_{p,r})} + \frac{1}{T_2} \|h_n - h_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})},
\]

(3.35)

\[
\|u_{n+1} - u_n\|_{L^2_t(B_{p,r}^{s+1})} \leq \frac{1}{4} 2^{-n} E_1 + \frac{1}{T_2} \|u_n - u_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + \frac{1}{T_2} \|\tilde{u}_n - \tilde{u}_{n-1}\|_{L^2_t(B_{p,r})} + \frac{1}{T_2} \|h_n - h_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})},
\]

(3.36)

and

\[
\|h_{n+1} - h_n\|_{L^\infty_t(B_{p,r}^{s+1})} \leq \frac{1}{4} 2^{-n} E_2 + \frac{1}{T_2} \|u_n - u_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + \frac{1}{T_2} \|\tilde{u}_n - \tilde{u}_{n-1}\|_{L^2_t(B_{p,r})} + \frac{1}{T_2} \|h_n - h_{n-1}\|_{L^\infty_t(B_{p,r}^{s+1})},
\]

(3.37)

We will temporarily assume that, for any \( k \leq n \)

\[
\|u_k - u_{k-1}\|_{L^\infty_t(B_{p,r}^{s+1})} + \|u_k - u_{k-1}\|_{L^2_t(B_{p,r})} + \|h_k - h_{k-1}\|_{L^\infty_t(B_{p,r}^{s+1})} \leq 4 \nu^{-1} 2^{-k} (E_1 + E_2),
\]

(3.38)
Then
\[
\|u_{n+1} - u_n\|_{L^\infty_t(B^r_{p,r})} + \|u_{n+1} - u_n\|_{L^2_t(B^r_{p,r})} + \|h_{n+1} - h_n\|_{L^\infty_t(B^r_{p,r})} \\
\leq \frac{1}{2}2^{-n}(E_1 + E_2) + \frac{1}{2}\nu^{-1}2^{-n}E_1 \\
+ \frac{1}{4}\|u_n - u_{n-1}\|_{L^\infty_t(B^r_{p,r})} + \|u_n - u_{n-1}\|_{L^2_t(B^r_{p,r})} + \|h_n - h_{n-1}\|_{L^\infty_t(B^r_{p,r})} \\
\leq 4\nu^{-1}2^{-n-1}(E_1 + E_2).
\]
(3.39)

In order to complete the proof of Proposition 3.3, we only need justly the inequalities (3.38) hold for \(k = 1\). It is obvious that
\[
\|u_1 - u_0\|_{L^\infty_t(B^r_{p,r})} + \|u_1 - u_0\|_{L^2_t(B^r_{p,r})} + \|h_1 - h_0\|_{L^\infty_t(B^r_{p,r})} \\
\leq 4(\|u_0\|_{B^r_{p,r}} + \|h_0\|_{B^r_{p,r}}) \\
\leq 2\nu^{-1}(E_1 + E_2).
\]
(3.40)

This complete the proof of Proposition 3.3

3.3. Existence and uniqueness of local solution

In this subsection, we investigate the uniqueness of the local solution to the system (3.1). By Proposition 3.3, the approximative sequence \((u_n, h_n)\) of the problem (2.1) is a Cauchy sequence in \(\chi([0,T], s - 1, p, r, E_1, E_2)\) with \(s > \frac{d}{p} + 2\), or \(s = \frac{d}{p} + 2, r = 1\). So the limit \((u,h)\) is a solution of the Cauchy problem (2.1). From Proposition 3.3 we obtain that this sequence is bounded in \(\chi([0,T], s, p, r, E_1, E_2)\). So it’s also the Cauchy sequence in \(\chi([0,T], s', p, r, E_1, E_2)\) for all \(s' < s\) by interpolation, and the limit is in \(\chi([0,T], s, p, r, E_1, E_2)\). Thus we have proved local existence result in Theorem 1.1.

For the uniqueness result in Theorem 1.1 let \((u, h)\) and \((v, g)\) satisfy the problem (2.1) with the initial data \((u_0, h_0)\), \((v_0, h_0)\) in \(B^s_{p,r} \times B^s_{p,r}\). Then we have
\[
\begin{align*}
\partial_t(u - v) + u \cdot \nabla(u - v) - \nu \Delta(u - v) &= G_1(u, h) - G_1(v, g), \\
\partial_t(h - g) + u \cdot \nabla(h - g) &= (u - v)\nabla g + G_2(u, h) - G_2(v, g), \\
(u - v)|_{t=0} &= 0, (h - g)|_{t=0} = 0.
\end{align*}
\]
(3.41)

Using Lemmas 2.13, 2.14 we can get
\[
\|u - v\|_{L^\infty_t(B^r_{p,r})} + \|h - g\|_{L^\infty_t(B^r_{p,r})} \\
\leq C e^{Ct} (\|u_0 - v_0\|_{B^r_{p,r}} + \|h_0 - g_0\|_{B^r_{p,r}}) + C_1 t \|u - v\|_{L^\infty_t(B^r_{p,r})} + C_2 t \|h - g\|_{L^\infty_t(B^r_{p,r})}.
\]
(3.42)

This gives the uniqueness of Theorem 1.1.
3.4. Continuity

In this subsection, we will prove that \( u, h \in C([0, T]; B^s_{p, r}) \). From the equations we can get \( u, h \in C([0, T]; B^s_{p, r}) \). Then we have that \( \Delta_j u, \Delta_j h \in C([0, T]; B^s_{p, r}) \) for any \( j \geq -1 \), from which it follows that \( S_j u, S_j h \in C([0, T]; B^s_{p, r}) \) for all \( j \in \mathbb{N} \). We claim that the sequence of continuous \( B^s_{p, r} \)-valued functions \( \{S_j f\}_{j \in \mathbb{N}} \) converges uniformly on \([0, T]\). Indeed, by Proposition 2.17 we have

\[
\Delta_{j'}(u - S_j u) = \sum_{|j' - j''| \leq 1, j'' \geq j} \Delta_j' \Delta_j'' u, \quad \Delta_{j'}(h - S_j h) = \sum_{|j' - j''| \leq 1, j'' \geq j} \Delta_j' \Delta_j'' h,
\]

from which it follows that

\[
\|u - S_j u\|_{L^\infty(B^s_{p, r})} \leq C\left( \sum_{j' \geq j - 1} 2^{j' sr}\|\Delta_j' u\|_{L^\infty(L^p)}\right)^{\frac{1}{r}},
\]

\[
\|h - S_j h\|_{L^\infty(B^s_{p, r})} \leq C\left( \sum_{j' \geq j - 1} 2^{j' sr}\|\Delta_j' h\|_{L^\infty(L^p)}\right)^{\frac{1}{r}}.
\]

Applying the operator \( \Delta_{j'} \) in (2.1), we get

\[
\left\{
\begin{array}{l}
\partial_t \Delta_j' u + \Delta_j'((u \cdot \nabla)u) - \nu \Delta_j'((\Delta u) - \nu \Delta_j'(\nu\nabla(\ln(1+h)\nabla u)) + \Delta_j'(\nabla h) = 0,
\partial_t \Delta_j' h + \Delta_j'(div u) + \Delta_j'(h \, div u) + \Delta_j'(u \cdot \nabla h) = 0,
\Delta_j' u|_{t=0} = \Delta_j' u_0, \quad \Delta_j' h|_{t=0} = \Delta_j' h_0.
\end{array}
\right.
\]

When \( j \geq 1 \), we have the Fourier transform of \( \Delta_j' u_0 \) and \( \Delta_j' f \) is supported in an annulus \( 2^{j'} C \), by Lemma 2.17 for \( \rho > 2 \), we have

\[
\|\Delta_j' u\|_{L^\infty(L^p)} \leq C\|\Delta_j' u_0\|_{L^p} + 2^{2j'(-1+\frac{1}{r})}\|\Delta_j' f\|_{L^p(L^p)},
\]

where \( f = -\nabla h - u \cdot \nabla u + \nu \nabla(\ln(1+h)) \nabla u \). By Lemma 2.15 we have

\[
\|f\|_{L^p(B^s_{p, r})} \leq \|\nabla h\|_{L^p(B^s_{p, r})} + \|u \cdot \nabla u\|_{L^p(B^s_{p, r})} + \nu \|\nabla(\ln(1+h)) \nabla u\|_{L^p(B^s_{p, r})} \leq C\|h\|_{L^p(B^s_{p, r})} + C\|u\|_{L^\infty(B^s_{p, r})} \|\nabla u\|_{L^p(B^s_{p, r})} + C\|\nabla u\|_{L^p(B^{s+1}_p)} \|\nabla(\ln(1+h))\|_{L^p(B^{s+1}_p)} \leq C\|h\|_{L^p(B^s_{p, r})} + C\|h\|_{L^p(B^{s+1}_p)} \|\nabla u\|_{L^p(B^{s+1}_p)},
\]

Thus, by (3.45) and (3.46), for \( \rho > 2 \), we have

\[
\|u - S_j u\|_{L^p(B^s_{p, r})} \leq C\sum_{j' \geq j - 1} 2^{j' sr}\|\Delta_j' u\|_{L^p(L^p)} \leq \sum_{j' \geq j - 1} 2^{j' sr}\left(\|\Delta_j' u_0\|_{L^p} + 2^{2j'(-1+\frac{1}{r})}\|\Delta_j' f\|_{L^p(L^p)}\right)^{\frac{1}{r}} 
\]

\[
\leq C\sum_{j' \geq j - 1} 2^{j' sr}\|\Delta_j' u_0\|_{L^p} + C\sum_{j' \geq j - 1} \left(2^{j'(s-2+\frac{1}{r})}\|\Delta_j' f\|_{L^p(L^p)}\right)^{\frac{1}{r}} 
\]

\[
\leq C\sum_{j' \geq j - 1} 2^{j' sr}\|\Delta_j' u_0\|_{L^p} + C\sum_{j' \geq j - 1} \left(2^{j'(s-1-\frac{1}{r})}\|\Delta_j' f\|_{L^p(dt)\frac{1}{r}}\right)^{\frac{1}{r}} 
\]

\[
\leq C\sum_{j' \geq j - 1} 2^{j' sr}\|\Delta_j' u_0\|_{L^p} + C\|f\|_{L^p(B^s_{p, r})} \sum_{j' \geq j - 1} \left(2^{j'(2-\rho)}\int_0^T d_j'(t) dt\right)^{\frac{1}{r}},
\]

19
where \(d_{j'}(t) \in \ell^r\) and \(\|d_{j'}(t)\|_{\ell^r} = 1\).

The first term clearly tends to 0 when \(j\) goes to \(\infty\). For the second term, when \(\rho \leq r\), we have

\[
\sum_{j' \geq j-1} (2^{j'(2-\rho)} \int_0^T d_{j'}(t) dt) \right)^{\frac{\rho}{2}} \leq C \sum_{j' \geq j-1} (T \sum_{j' \geq j-1} (\int_0^T d_{j'}(t) dt) \right)^{\frac{\rho}{2}} \leq CT \sum_{j' \geq j-1} (\int_0^T d_{j'}(t) dt).
\]

By virtue of Lebesgue’s dominated convergence theorem and \(\rho \leq r\), the second term tends 0 when \(j\) goes to \(\infty\). As regards \(\rho > r\), by Yong’s inequality, we have

\[
\sum_{j' \geq j-1} (2^{j'(2-\rho)} \int_0^T d_{j'}(t) dt) \right)^{\frac{\rho}{2}} \leq C \sum_{j' \geq j-1} \left(2^{j'(2-\rho)} \int_0^T d_{j'}(t) dt \right)^{\frac{\rho}{2}} \leq C \sum_{j' \geq j-1} 2^{j'(2-\rho)} + C \int_0^T \sum_{j' \geq j-1} d_{j'}(t) dt \leq C \sum_{j' \geq j-1} 2^{j'(2-\rho)} + C \int_0^T \sum_{j' \geq j-1} d_{j'}(t) dt.
\]

By virtue of Lebesgue’s dominated convergence theorem and \(\rho > 2\), for \(\rho > r\), the second term tends 0 when \(j\) goes to \(\infty\) as well. This completes the proof of continuity for \(u\).

As regards \(h\), multiplying the second equation of (3.44) by \(\Delta_{j'} h |\Delta_{j'} h|^{p-2}\), integrating over \(\mathbb{R}^2\) yields

\[
\frac{1}{p} \partial_t \|\Delta_{j'} h\|_{L^p}^p \leq \int_{\mathbb{R}^2} \Delta_{j'}(div u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx
\]

\[
+ \int_{\mathbb{R}^2} \Delta_{j'}(h \cdot \nabla u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx + \int_{\mathbb{R}^2} \Delta_{j'}((u \cdot \nabla) h) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx
\]

\[
= H_1 + H_2 + H_3.
\]

Next, we deal with \(H_1 - H_3\) term by term. For \(H_1\), it’s easy to check that

\[
H_1 = \int_{\mathbb{R}^2} \Delta_{j'}(div u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx
\]

\[
\leq C \|\Delta_{j'} h \|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1}.
\]

Then we divide \(H_2\) to three terms

\[
H_2 = \int_{\mathbb{R}^2} \Delta_{j'}(h \cdot \nabla u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx
\]

\[
= \int_{\mathbb{R}^2} \Delta_{j'}(T_h \nabla u + T_{div_u} h + R(div u, h)) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx
\]

\[
= H_{21} + H_{22} + H_{23}.
\]

For \(H_{21}\), by a simple computation, we get

\[
H_{21} = \sum_{|j' - q| \leq 4} \int_{\mathbb{R}^2} \Delta_{j'}(S_q h \cdot \nabla u) \Delta_{j'} h |\Delta_{j'} h|^{p-2} dx
\]

\[
\leq C \|h\|_{L^\infty} \sum_{|j' - q| \leq 4} \|\Delta_q (div u)\|_{L^p} \|\Delta_{j'} h\|_{L^p}^{p-1}.
\]
For $H_{22}$ and $H_{23}$, by a discrete Young's inequality, we have

$$H_{22} = \sum_{|j' - q| \leq 4} \int_{R^2} \Delta_j' (S_q(div u) \Delta_q h) \Delta_j h |\Delta_j' h|^p dx$$

(3.54)

$$\leq \|div u\|_{L^\infty} \sum_{|j' - q| \leq 4} \|\Delta_q h\|_{L^p} \|\Delta_j h\|_{L^{p-1}}$$

$$\leq \|div u\|_{L^\infty} \sum_{|j' - q| \leq 4} d_q 2^{-q} \|h\|_{B^{p'}_{p,r}} \|\Delta_j h\|_{L^{p-1}}$$

$$\leq C \|div u\|_{L^\infty} d_j(t) 2^{-j'} \|h\|_{B^{p'}_{p,r}} \|\Delta_j h\|_{L^{p-1}},$$

and

$$H_{23} = \sum_{q \geq j' - N, |q - q'| \leq 1} \int_{R^2} \Delta_j' (\Delta'_q(div u) \Delta_q h) \Delta_j h |\Delta_j' h|^p dx$$

(3.55)

$$\leq \|div u\|_{L^\infty} \sum_{q \geq j' - N} \|\Delta_q h\|_{L^p} \|\Delta_j h\|_{L^{p-1}}$$

$$\leq \|div u\|_{L^\infty} \sum_{q \geq j' - N} d_q 2^{-q} \|h\|_{B^{p'}_{p,r}} \|\Delta_j h\|_{L^{p-1}}$$

$$\leq C \|div u\|_{L^\infty} d_j(t) 2^{-j'} \|h\|_{B^{p'}_{p,r}} \|\Delta_j h\|_{L^{p-1}}.$$

For $H_3$, we can divide it into two parts

$$H_3 = \int_{R^2} \Delta_j' ((u \cdot \nabla) h) \Delta_j h |\Delta_j' h|^p dx$$

(3.56)

$$= \int_{R^2} (u \cdot \nabla) \Delta_j h \Delta_j h |\Delta_j' h|^p dx - \int_{R^2} R_j' (u, h) \Delta_j h |\Delta_j' h|^p dx$$

$$= H_{31} + H_{32},$$

where the definition of $R_j'(u, h)$ is the same as that in Lemma 2.17. For $H_{31}$, we obtain that

$$H_{31} = \int_{R^2} (u \cdot \nabla) \Delta_j h \Delta_j h |\Delta_j' h|^p dx$$

(3.57)

$$= \frac{1}{p} \int_{R^2} u \cdot \nabla |\Delta_j h|^p dx$$

$$= -\frac{1}{p} \int_{R^2} div u |\Delta_j h|^p dx$$

$$\leq C \|\nabla u\|_{L^\infty} \|\Delta_j h\|_{L^p}.$$

For $H_{32}$, by the second of Lemma 2.17 with $p_2 = \infty, p_1 = p$, we have

$$H_{32} = \int_{R^2} R_j'(u, h) \Delta_j h |\Delta_j' h|^p dx$$

(3.58)

$$\leq C \|R_j'(u, h)\|_{L^p} \|\Delta_j h\|_{L^{p-1}}$$

$$\leq C d_j(t) 2^{-j'} \left( \|\nabla u\|_{L^\infty} \|h\|_{B^{p'}_{p,r}} + \|\nabla h\|_{L^\infty} \|u\|_{B^{p'}_{p,r}} \right) \|\Delta_j h\|_{L^{p-1}}.$$

Combining (3.50)-(3.58), we get

$$\partial_t \|\Delta_j h\|_{L^p} \leq C \left( \|\Delta_j' div u\|_{L^p} \|\Delta_j h\|_{L^{p-1}} + \|h\|_{L^\infty} \sum_{|j' - q| \leq 4} \|\Delta_q div u\|_{L^p} \|\Delta_j h\|_{L^{p-1}} \right.$$  

(3.59)  

$$+ \|div u\|_{L^\infty} d_j(t) 2^{-j'} \|h\|_{B^{p'}_{p,r}} \|\Delta_j h\|_{L^{p-1}} + \|\nabla u\|_{L^\infty} \|\Delta_j h\|_{L^p}$$  

$$+ 2^{-j'} d_j(t) \left( \|\nabla u\|_{L^\infty} \|h\|_{B^{p'}_{p,r}} + \|\nabla h\|_{L^\infty} \|u\|_{B^{p'}_{p,r}} \right) \|\Delta_j h\|_{L^{p-1}} \right),$$
where $d_j'(t) \in \ell^r$, and $\|d_j'(t)\|_{\ell^r} = 1$.

Then it follows

\[
\partial_t \|\Delta_j h\|_{L^p} \leq C \left( \|\Delta_j \text{div } u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \text{div } u\|_{L^p} \right. \\
+ \|\text{div } u\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \text{div } u\|_{L^p} \\
\left. + 2^{-j^*} d_j'(t) \left(\|\nabla u\|_{L^\infty} \|h\|_{B^p_{p,r}} + \|\nabla h\|_{L^\infty} \|u\|_{B^p_{p,r}}\right)\right) \\
(3.60) \\
\leq C \left( \|\Delta_j \text{div } u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \text{div } u\|_{L^p} \right. \\
\left. + 2^{-j^*} d_j'(t) \left(\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}\right) \left(\|u\|_{B^p_{p,r}} + \|h\|_{B^p_{p,r}}\right)\right).
\]

For any $t \in (0,T]$, integrating (3.60) from 0 to $t$, and taking $r$ power, we get

\[
(3.61) \\
\|\Delta_j h(t)\|_{L^p} \leq C \|\Delta_j h_0\|_{L^p} + C \left( \int_0^t 2^{-j^*} d_j'(t') \left(\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}\right) \left(\|u\|_{B^p_{p,r}} + \|h\|_{B^p_{p,r}}\right) dt'\right)^r \\
+ C \left( \int_0^t \left(\|\Delta_j \text{div } u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \text{div } u\|_{L^p}\right) dt'\right)^r.
\]

By the Minkowski inequality, it follows that

\[
\|h - S_{j^*} h\|_{L^\infty(\mathbb{R}^d)} \leq C \left( \sum_{j' \geq j - 1} 2^{j^* r} \|\Delta_j h\|_{L^p}(\mathbb{R}^d)\right)^{\frac{1}{r}} \\
\leq C \left( \sum_{j' \geq j - 1} 2^{j^* r} \|\Delta_j h_0\|_{L^p}\right)^{\frac{1}{r}} + \\
C \left( \sum_{j' \geq j - 1} \left( \int_0^T d_j'(t) \left(\|\nabla u\|_{L^\infty} + \|\nabla h\|_{L^\infty}\right) \left(\|u\|_{B^p_{p,r}} + \|h\|_{B^p_{p,r}}\right) dt'\right)^{\frac{1}{r}} \right)^{\frac{1}{r}} \\
+ C \left( \int_0^T \left(\|\Delta_j \text{div } u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \text{div } u\|_{L^p}\right) dt\right)^{\frac{1}{r}} \\
(3.62) \\
\leq C \left( \sum_{j' \geq j - 1} 2^{j^* r} \|\Delta_j h_0\|_{L^p}\right)^{\frac{1}{r}} + \\
C \left( \int_0^T \left(\|\Delta_j \text{div } u\|_{L^p} + \|h\|_{L^\infty} \sum_{|j'-q| \leq 4} \|\Delta_q \text{div } u\|_{L^p}\right) dt\right)^{\frac{1}{r}}
\]

The first term and the second term clearly tends to 0 when $j$ goes to $\infty$. For the third term, by the Young inequality, we only need to deal with

\[
C \left( \sum_{j' \geq j - 1} \left( 2^{j^*} \int_0^T \|\Delta_j \text{div } u\|_{L^p} dt \right)^{\frac{1}{r}} \right)^{\frac{1}{2}},
\]

it tends to 0 when $j$ goes to $\infty$ provided $\|\text{div } u\|_{L^p(\mathbb{R}^d)}$ is bounded. Actually, by Lemma 2.4.3 we
have
\[
\|u\|_{L^2_t(B_{p,r}^{s+1})} \leq C\nu^{-1}\exp(C(1 + \nu T) \int_0^T \|\nabla u\|_{L^\infty} dt) \times \\
\left( (1 + \nu T) \|u_0\|_{B_{p,r}^{-1}} + (1 + \nu T) \|\nabla h + \nabla u\nabla (\ln(1 + h))\|_{L^1_t(B_{p,r}^{-1})} \right)
\]
\[
(3.63)
\leq C_T \left( \|u_0\|_{B_{p,r}^s} + \|h\|_{L^2_t(B_{p,r}^s)} + \|\nabla u\|_{L^\infty_t(L^\infty_x)} \|\nabla (\ln(1 + h))\|_{L^1_t(B_{p,r}^{-1})} + \|\nabla (\ln(1 + h))\|_{L^\infty_t(L^\infty_x)} \|\nabla u\|_{L^1_t(B_{p,r}^{-1})} \right)
\]
\[
\leq C_T (\|u_0\|_{B_{p,r}^s} + T \|h\|_{L^2_t(B_{p,r}^s)} + T \|u\|_{L^\infty_t(L^\infty_x)} \|h\|_{L^\infty_t(B_{p,r}^s)}).
\]

This completes the proof of continuity for $h$.

## 4 Blow-up criteria and Global existence

In this section, we will present a blow-up criterion in $B_{p,r}^s$, $s > \frac{2}{p} + 1$ and get the global existence of the system with small enough initial data in $B_{p,r}^s(\mathbb{R}^2)$, $p \leq 2$ and $s > 1 + \frac{2}{p}$. Next we divide the section into three subsections.

### 4.1 A priori estimate

In this subsection, we give a priori estimate as follows:

**Proposition 4.1.** Let $u_0, h_0 \in B_{p,r}^{s+\varepsilon} \times B_{p,r}^{s+\varepsilon}$, where $\varepsilon$ is a small enough positive real number when $r > 2$, and $\varepsilon = 0$ when $r \leq 2$, and let $(u, h)$ be the corresponding solution of Cauchy problem (2.1) in $B_{p,r}^{s+\varepsilon} \times B_{p,r}^{s+\varepsilon}$. Assume that $T^*$ is the maximal existence time of the solution, and

\[
\int_0^{T^*} \|\nabla u\|_{L^\infty}^2 + \|h\|_{L^\infty}^2 + \|\nabla (\ln(1 + h))\|_{L^\infty}^2 dt < C_{T^*}.
\]

Then we have

\[
\|u\|_{L^\infty_t(B_{p,r}^s)} + \|h\|_{L^\infty_t(B_{p,r}^s)} \leq C(s, p, r, \nu, C_{T^*}, \|u_0\|_{B_{p,r}^{s+\varepsilon}}, \|h_0\|_{B_{p,r}^{s+\varepsilon}}).
\]

**Proof:** By Lemma 2.13 for any $T < T^*$ we have

\[
\|u\|_{L^\infty_t(B_{p,r}^s)} \leq C e^{\int_0^T \|\nabla u\|_{L^\infty} dt} (\|u_0\|_{B_{p,r}^s} + (\frac{1 + \nu T}{p})^\frac{1}{2} \|\nabla h + \nabla u\nabla (\ln(1 + h))\|_{L^1_t(B_{p,r}^{-1})})
\]
\[
\leq C_T \left( \|u_0\|_{B_{p,r}^s} + \|h\|_{L^2_t(B_{p,r}^s)} + \|\nabla u\nabla (\ln(1 + h))\|_{L^1_t(B_{p,r}^{-1})} \right)
\]
\[
\leq C_T \left( \|u_0\|_{B_{p,r}^s} + \|h\|_{L^2_t(B_{p,r}^s)} + \|\nabla u\|_{L^2_t(B_{p,r}^s)} + \|\nabla (\ln(1 + h))\|_{L^\infty_t(B_{p,r}^{-1})} \right)
\]
\[
(4.1)
\]
\[
\left( \int_0^T \|\nabla u\|_{L^\infty}^2 + \|\nabla (\ln(1 + h))\|_{L^\infty}^2 dt \right)^\frac{1}{2} \left( \|u\|_{B_{p,r}^s}^2 + \|h\|_{B_{p,r}^s}^2 \right).
\]
From (3.66), we have

\[ \partial_t \| \Delta_j h \|_{L^p}^2 \leq C \left( \| \Delta_j \text{div } u \|_{L^p} \| \Delta_j h \|_{L^p} + \| h \|_{L^\infty} \sum_{|j-q| \leq 4} \| \Delta_q \text{div } u \|_{L^p} \| \Delta_j h \|_{L^p} \right. \]

\[ \left. + \| \text{div } u \|_{L^\infty} d_j(t) 2^{-js} \| h \|_{B^p_{p,2}} \| \Delta_j h \|_{L^p} + \| \nabla u \|_{L^\infty} \| \Delta_j h \|_{L^p} \right) \]

\[ + 2^{-js} d_j(t) \left( \| \nabla u \|_{L^\infty} \| h \|_{B^p_{p,2}} + \| \nabla h \|_{L^\infty} \| u \|_{B^p_{p,2}} \right) \| \Delta_j h \|_{L^p} \]

\[ \leq C \left( 1 + \| h \|_{L^\infty} \right) \| \Delta_j h \|_{L^p}^2 + C \| \Delta_j \text{div } u \|_{L^p}^2 + C \left( \sum_{|j-q| \leq 4} \| \Delta_q \text{div } u \|_{L^p} \right)^2 \]

\[ + C \left( \| \text{div } u \|_{L^\infty} \| \Delta_j h \|_{L^p} + \| \nabla u \|_{L^\infty} \| \Delta_j h \|_{L^p} \right) \]

\[ + 2^{-js} d_j(t) \left( \| \nabla u \|_{L^\infty} \| h \|_{B^p_{p,2}} + \| \nabla h \|_{L^\infty} \| u \|_{B^p_{p,2}} \right) \| \Delta_j h \|_{L^p} \]

\[ \leq C \left( 1 + \| h \|_{L^\infty} + \| \nabla h \|_{L^\infty}^2 + \| \nabla u \|_{L^\infty}^2 \right) 2^{-2js} d_j^2(t) \left( \| h \|_{B^p_{p,2}}^2 + \| h \|_{B^p_{p,2}}^2 \right) \]

\[ + C \| \Delta_j \text{div } u \|_{L^p}^2 + C \left( \sum_{|j-q| \leq 4} \| \Delta_q \text{div } u \|_{L^p} \right)^2. \]

Then it follows that

\[ \| \Delta_j h \|_{L^p}^2 \leq \| \Delta_j h_0 \|_{L^p}^2 + C \int_0^t \left( \| \Delta_j \text{div } u \|_{L^p}^2 + \left( \sum_{|j-q| \leq 4} \| \Delta_q \text{div } u \|_{L^p} \right)^2 \right) \]

\[ + C \int_0^t \left( 1 + \| h \|_{L^\infty} + \| \nabla h \|_{L^\infty}^2 + \| \nabla u \|_{L^\infty}^2 \right) \| \Delta_j h \|_{L^p}^2 \]

\[ + \| h \|_{L^\infty}^2 \| \Delta_j \text{div } u \|_{L^p}^2 + C \int_0^t \left( \| h \|_{B^p_{p,2}}^2 + \| h \|_{B^p_{p,2}}^2 \right) dt'. \]

Multiplying by $2^{2js}$ and taking the sum over $j$ from $-1$ to $\infty$ thus give

\[ \| h \|_{L^p_{\infty}(B^p_{p,2})}^2 \leq \| h_0 \|_{B^p_{p,2}}^2 + C \sum_{j \geq -1} 2^{2js} \int_0^t \left( \| \Delta_j \text{div } u \|_{L^p}^2 + \left( \sum_{|j-q| \leq 4} \| \Delta_q \text{div } u \|_{L^p} \right)^2 \right) \]

\[ + C \sum_{j \geq -1} \int_0^t d_j^2(t') \left( 1 + \| h \|_{L^\infty}^2 + \| \nabla u \|_{L^\infty}^2 + \| \nabla h \|_{L^\infty}^2 \right) \left( \| h \|_{B^p_{p,2}}^2 + \| h \|_{B^p_{p,2}}^2 \right) dt' \]

\[ \leq \| h_0 \|_{B^p_{p,2}}^2 + C \| \text{div } u \|_{L^p_{\infty}(B^p_{p,2})}^2 \]

\[ + C \int_0^T \left( 1 + \| h \|_{L^\infty}^2 + \| \nabla u \|_{L^\infty}^2 + \| \nabla h \|_{L^\infty}^2 \right) \left( \| u \|_{B^p_{p,2}}^2 + \| h \|_{B^p_{p,2}}^2 \right) dt'. \]

By Lemma 2.13 again, we have

\[ \| \text{div } u \|_{L^p_{\infty}(B^p_{p,2})} \leq C \nu^{-\frac{1}{2}} e^{C(1 + \nu T)^{\frac{1}{2}}} \int_0^T \| \nabla u \|_{L^\infty} \times \]

\[ \left( (1 + \nu T)^{\frac{1}{2}} \| u_0 \|_{B^p_{p,2}} + (1 + \nu T)^{\frac{1}{2}} \nu^{-\frac{1}{2}} \| \nabla h + \nabla \left( \ln(1 + h) \right) \| u \|_{L^p_{\infty}(B^p_{p,2})} \right. \]

\[ \leq C_T \left( \| u_0 \|_{B^p_{p,2}} + \| h \|_{L^p_{\infty}(B^p_{p,2})} + \| \nabla u \|_{L^p_{\infty}(B^p_{p,2})} \right) \]

\[ \leq C_T \left( \| u_0 \|_{B^p_{p,2}} + \| h \|_{L^p_{\infty}(B^p_{p,2})} + \left( \int_0^T \| \nabla u \|_{L^\infty}^2 + \| \nabla \left( \ln(1 + h) \right) \|_{L^\infty}^2 \right) \left( \| u \|_{B^p_{p,2}}^2 + \| h \|_{B^p_{p,2}}^2 \right) dt \right)^{\frac{1}{2}}. \]

Combining (4.1), (4.4) and (4.5), and noting that

\[ \| \nabla h \|_{L^\infty} \leq (1 + \| h \|_{L^\infty}) \| \nabla \left( \ln(1 + h) \right) \|_{L^\infty}, \]

24
we get

\begin{equation}
\|u\|^2_{L^\infty_t(B_{p,r}^s)} + \|h\|^2_{L^\infty_t(B_{p,r}^s)} \leq CT(\|u_0\|^2_{B_{p,r}^s} + \|h_0\|^2_{B_{p,r}^s}) + C\int_0^{T^*} (1 + \|h\|^2_{L^\infty} + \|\nabla(1+h)\|^2_{L^\infty} + \|\nabla u\|^2_{L^2_t(B_{p,r}^s)})(\|u\|^2_{L^\infty_t(B_{p,r}^s)} + \|h\|^2_{L^\infty_t(B_{p,r}^s)})dt.
\end{equation}

By the virtue of Gronwall’s inequality, we can obtain

\begin{equation}
\|u\|^2_{L^\infty_t(B_{p,r}^s)} + \|h\|^2_{L^\infty_t(B_{p,r}^s)} \leq CT(\|u_0\|^2_{B_{p,r}^s} + \|h_0\|^2_{B_{p,r}^s})e^{CT\int_0^{T^*} (1 + \|h\|^2_{L^\infty} + \|\nabla(1+h)\|^2_{L^\infty} + \|\nabla u\|^2_{L^2_t(B_{p,r}^s)})dt}.
\end{equation}

4.2. Blow-up criterion in $B_{p,r}^s$.

In this subsection, we establish a blow-up criterion in common $B_{p,r}^s$ with $s > 1 + \frac{2}{p}$.

**Proposition 4.2.** Let $u_0, h_0 \in B_{p,r}^s \times B_{p,r}^s$, $s > 1 + \frac{2}{p}$, and let $(u, h)$ be the corresponding solution of the Cauchy problem (2.1) in $B_{p,r}^s \times B_{p,r}^s$. Assume that $T^*$ is the maximal existence time of solution. If $T^*$ is finite, then we have,

\[\int_0^{T^*} \|\nabla u\|_{L^\infty_t}^\prime \|\nabla h\|_{L^\infty_t}^\prime + \|\nabla(1+h)\|_{L^\infty_t}^\prime dt' = \infty,\]

where $r_1 = \max\{r', 2\}$.

Proof: If

\[\int_0^{T^*} \|\nabla u\|_{L^\infty_t}^\prime \|\nabla h\|_{L^\infty_t}^\prime + \|\nabla(1+h)\|_{L^\infty_t}^\prime dt' < \infty,\]

in view of Proposition 4.1, then we have

\[\|u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + \|h\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} < C T^*,\]

here $\varepsilon = 0$ when $r \leq 2$ and $\varepsilon$ is a small enough positive real number when $r > 2$. Then we take a suitable $\rho \geq 2$ such that $B_{p,r}^{s+1+\frac{2}{p}}$ is continuously embedded $B_{p,r}^{s-\varepsilon}$, thus by Lemma 2.14-2.15 for any $t < T^*$, we have

\begin{align*}
\|u\|_{L^\infty_t(B_{p,r}^{s+1+\frac{2}{p}})} &\leq C e^{\int_0^t \|\nabla u\|_{L^\infty} \left(\|u_0\|_{B_{p,r}^s} + (1 + \nu T)^{-\frac{2}{p}} \|\nabla h\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + \frac{2}{p} \right) dt} \\
&\leq C e^{\int_0^t \|\nabla u\|_{L^\infty} \left(\|u_0\|_{B_{p,r}^s} + (1 + \nu t)^{-\frac{2}{p}} \|\nabla h\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + \frac{2}{p} \right) dt} \\
&\leq C_1 + C_2 t^\frac{2}{p} \|h\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + C_2 \|\nabla(1+h)\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \|\nabla u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \\
&\leq C_1 + C_2 t^\frac{2}{p} \|h\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + C_2 \|\nabla(1+h)\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \|\nabla u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \\
&+ C_2 \|\nabla(1+h)\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \|\nabla u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + \|\nabla u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \|\nabla(1+h)\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \\
&\leq C_3 + C_4 \|u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} + \|u\|_{L^\infty_t(B_{p,r}^{s-\varepsilon})} \\
&< C(s, p, r, \nu, T^*, \|u_0\|_{B_{p,r}^s}).
\end{align*}
For $h$, in the case of $r \geq 2$, by Lemma 2.13 we have, for any $t < T^*$

$$
\|h\|_{L^\infty_t(B_{p,r}^*)} \leq \exp(C \int_0^t \|\nabla u\|_{B_{p,r}^*} \, dt') (\|h_0\|_{B_{p,r}^*} + \|\nabla u\|_{L^2_t(B_{p,r}^*)} + \|h \, \text{div} \, u\|_{L^2_t(B_{p,r}^*)})
$$

(4.9)

$$
\leq \exp(C \int_0^t \|u\|_{B_{p,r}^*} \, dt') (\|h_0\|_{B_{p,r}^*} + T^{\frac{1}{2}} \|\text{div} \, u\|_{L^2_t(B_{p,r}^*)} + \|h\|_{L^2_t(L^\infty)} \|\nabla u\|_{L^2_t(B_{p,r}^*)})
$$

In view of Lemma 2.13 we obtain that

(4.10) $$
\|\text{div} \, u\|_{L^2_t(B_{p,r}^*)} \leq C \|u\|_{L^2_t(B_{p,r}^*)} + C (1 + \|\nabla (1 + h)\|_{L^2_t(B_{p,r}^*)})
$$

$$
\leq C \|h\|_{L^2_t(B_{p,r}^*)} + \|\nabla (1 + h)\|_{L^2_t(B_{p,r}^*)}
$$

Combining (4.9), we have

(4.11) $$
\|h\|_{L^\infty_t(B_{p,r}^*)} \leq C(1 + \|h\|_{L^2_t(B_{p,r}^*)}).
$$

Then it follows

(4.12) $$
\|h\|_{L^\infty_t(B_{p,r}^*)}^2 \leq C + C \int_0^t \|h(t)\|_{B_{p,r}^*}^2 \, dt'
$$

where $C$ in (4.9)-(4.12) only depends on $s, p, r, \nu, T^*, \|h_0\|_{B_{p,r}^*}, \|u\|_{L^\infty_t(B_{p,r}^*)}, \|h\|_{L^2_t(L^\infty)}, \|\text{div} \, u\|_{L^2_t(L^\infty)}$.

By the virtue of the Gronwall inequality, we have

(4.13) $$
\|h\|_{L^\infty_t(B_{p,r}^*)}^2 \leq C e^{CT}.
$$
Reagrd as $r < 2$, we rewrite (4.9) as follows

\[
\|h\|_{L_t^\infty(B_{p,r}^\nu)} \leq \exp(C \int_0^t \|\nabla u\|_{B_{p,r}^{s,1}} dt') \left(\|h_0\|_{B_{p,r}^s} + \|\text{div } u\|_{L_t^1(B_{p,r}^s)} + \|h\|_{L_t^0(B_{p,r}^s)} \right)
\]

(4.14)

\[
\|\text{div } u\|_{L_t^1(B_{p,r}^s)} + \|h\|_{L_t^1(B_{p,r}^s)} < (1 + v(t)) \|\nabla(\ln(1 + h))\|_{L_t^1(B_{p,r}^s)} \leq C(1 + \|h\|_{L_t^1(B_{p,r}^s)}).
\]

(4.15)

In view of Lemma [2.13], we obtain that

(4.16)

\[
\|h\|_{L_t^\infty(B_{p,r}^\nu)} \leq C(1 + \|h\|_{L_t^1(B_{p,r}^s)}).
\]

Then it follows

(4.17)

\[
\|h\|_{L_t^\infty(B_{p,r}^\nu)} \leq C + C \int_0^t \|h(t)\|_{L_t^\infty(B_{p,r}^s)} dt' \leq C + C \int_0^t \|h(t)\|_{L_t^\infty(B_{p,r}^s)} dt',
\]

where $C$ in (4.14)-(4.17) only depends on $s, p, r, u, T^*, \|h_0\|_{B_{p,r}^s}, \|u\|_{L_t^\infty(B_{p,r}^s)}, \|h\|_{L_t^\infty(B_{p,r}^s)}$, $\|\text{div } u\|_{L_t^\infty(B_{p,r}^s)}$.

By the virtue of the Gronwall inequality, we have

(4.18)

\[
\|h\|_{L_t^\infty(B_{p,r}^\nu)} \leq Ce^{C'T}.
\]
Combining Thorem 1.1 and the continuity of \( u, h \) in \( B_{p,r}^s \) completes the proof of Proposition 4.2.

4.3. Global existence

At last, we give some corollaries about the global existence.

**Corollary 4.3.** Let \( u_0, h_0 \in B_{p_1,r_1}^{s_1} \cap B_{p_2,r_2}^{s_2} \) and \( (s_1, p_1, r_1) \) and \( (s_2, p_2, r_2) \) satisfy the same conditions in Thorem 1.2, and let \( T_1, T_2 \) be the maximal existence time of the Cauchy problem (2.1) in \( B_{p_1,r_1}^{s_1} \) and \( B_{p_2,r_2}^{s_2} \) respectively, then we have \( T_1 = T_2 \).

Proof: Assume that \( T_1 (\leq \infty) < T_2 \). By Proposition 4.2, we have

\[
\int_0^{T_1} \| \nabla u \|_{L_{\infty}}^{r_1} + \| h \|_{L_{\infty}}^{r_1} + \| \nabla (\ln(1 + h)) \|_{L_{\infty}}^{r_1} \, dt = \infty.
\]

On the other hand, by \( T_1 < T_2 \), we have

\[
\| u \|_{L_{\infty}^T(B_{p_2,r_2}^{s_2})} + \| h \|_{L_{\infty}^T(B_{p_2,r_2}^{s_2})} < \infty.
\]

In view of \( s_2 > 1 + \frac{2}{p_2} \), we get

\[
\| u(t) \|_{L_{\infty}} + \| h(t) \|_{L_{\infty}} + \| \nabla u(t) \|_{L_{\infty}} + \| \nabla \ln(1 + h(t)) \|_{L_{\infty}} \leq C \big( \| u(t) \|_{B_{p_2,r_2}^{s_2}} + \| u(t) \|_{B_{p_2,r_2}^{s_2}} \big).
\]

Then we obtain

\[
\int_0^{T_1} \| \nabla u \|_{L_{\infty}}^{r_1} + \| h \|_{L_{\infty}}^{r_1} + \| \nabla (\ln(1 + h)) \|_{L_{\infty}}^{r_1} \, dt \leq C \int_0^{T_1} \big( \| u \|_{L_{\infty}^T(B_{p_2,r_2}^{s_2})} + \| h \|_{L_{\infty}^T(B_{p_2,r_2}^{s_2})} \big) \, dt'
\]

\[
(4.19) < \infty,
\]

which leads to a contradiction. So we have \( T_1 \geq T_2 \). Of course, we also have \( T_2 \geq T_1 \) by the same argument.

**Corollary 4.4.** Let \( u_0, h_0 \in B_{p,r}^s \cap H^{s_1} \), \( (s, p, r) \) satisfy the same condition in Corollary 1.2, \( s_1 > 2 \). If there exists an \( \varepsilon \) small enough, such that \( \| u_0 \|_{H^{s_1}} + \| h_0 \|_{H^{s_1}} < \varepsilon \). Then the corresponding solution of the Cauchy problem (2.1) in \( B_{p,r}^s \) is global in time.

Proof: It’s an obvious conclusion of Corollary 4.3 and Lemma 2.18.

**Proof of Theorem 1.3:** By \( s > 1 + \frac{2}{p} \), there exists a real number \( s' \) such that \( s > s' > 2 \). Then the space \( B_{p,r}^s \) is continuous embedding in \( H^s \), and for any \( u, h \in B_{p,r}^s \), we have

\[
\| u \|_{H^{s'}} + \| h \|_{H^{s'}} \leq C \big( \| u \|_{B_{p,r}^s} + \| h \|_{B_{p,r}^s} \big).
\]
Thus by Corollary 4.3 and Lemma 2.18 we obtain the desired result in Theorem 1.3. This completes the proof.

**Remark 4.1** Note that Wang and Xu in [12] established the local well-posedness of the system (1.1) and got the global solutions to the system (1.1) for small initial data in Sobolev spaces $H^s$, $s > 2$. Our obtained results in Theorem 1.1 with $p = 2$ and $r = 2$ cover the recent results in [12].

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