Thermodynamics of the apparent horizon in the generalized energy-momentum-squared cosmology

Prabir Rudra, a Behnam Pourhassanb,c
aDepartment of Mathematics, Asutosh College, Kolkata-700 026, India.
bSchool of Physics, Damghan University, Damghan, 3671641167, Iran.
cCanadian Quantum Research Center 204-3002 32 Ave Vernon, BC V1T 2L7 Canada.
E-mail: prudra.math@gmail.com, rudra@associates.iucaa.in, b.pourhassan@du.ac.ir, b.pourhassan@candqrc.ca

ABSTRACT: In this note, we explore the thermodynamic properties of the universe in the background of the generalized energy-momentum-squared gravity. We derive the energy density of matter from the non-standard continuity equation and use it in our analysis. We consider two types of models depending on the nature of coupling between curvature and matter and perform thermodynamic analysis on them using the cosmic apparent horizon. The models are kept as generic as possible from the mathematical point of view in order to gain a wide applicability of the work. In this work we have considered power law and exponential form of models. All the thermodynamic parameters are expressed in terms of the cosmic apparent horizon radius and its time derivatives and their time evolution are studied. By using temperature, heat capacity analysis and the evolution trend of Helmholtz free energy the conditions for thermodynamic stability of the models are derived. It is seen that our stability analysis considerably constrain the parameter space of the model.

KEYWORDS: Modified gravity, Thermodynamics, Cosmic apparent horizon, Energy-Momentum, Stability.
1 Introduction

During the last two decades, the study of cosmology have been centered around the fact that the rate of expansion of the universe is actually accelerating [1, 2]. Since the accelerated expansion of the universe is not an expected phenomenon, the community is bent on finding a proper reason for this. From the research that have been already performed, it is seen that there can be a dual possibility regarding the explanation of this acceleration phenomenon. The first possibility is the concept of dark energy (DE), which aims at modifying the matter content of the universe by introducing exotic components with negative pressure. It is expected that such components will violate the energy conditions of the universe. A review on dark energy may be found in the Ref. [3]. The other possibility, which has been presented from time to time is the concept of modified gravity. Here we actually introduce suitable modifications to the Einstein’s gravity so that the modified theory incorporates the accelerated expansion of the universe. The reader is suggested to refer to [4–6] for detailed information on modified gravity theories. It should be stated over here that both the theories aim at modifying the Einstein’s equation of general relativity (GR) in their own ways.
Moreover, dark energy and modified gravity can be shown to be equivalent to each other via proper fine tuning.

The simplest and the most popular way of modifying the Einstein’s gravity is by replacing the gravity Lagrangian $\mathcal{L}_{EH} = R$ of the Einstein-Hilbert action by an analytical function of the Ricci scalar, $\mathcal{L}_{f(R)} = f(R)$, thus giving rise to $f(R)$ gravity. Using $f(R)$ modifications to Einstein’s gravity, we can explore the non-linear effects of the curvature of spacetime. Extensive reviews on $f(R)$ gravity can be found in Refs. [7, 8]. Further modifications can be affected in the gravity Lagrangian by introducing an analytical function of Ricci scalar $R$ and the matter Lagrangian $L_m$, giving rise to $f(R, L_m)$ gravity [9]. From such modifications, the contributions coming from the matter part of the universe are also taken into account along with the higher order corrections of the spacetime curvature. A specialty of these theories is that the particles experience an extra force in the gravity well, in the direction orthogonal to the four-velocity. Moreover, due to this extra force the particles undergo a non-geodesic motion. Further developments in $f(R, L_m)$ theories can be found in Refs. [10–12].

Narrowing down on these classes of theories Harko et al in [13] proposed the $f(R, T)$ theory where the matter Lagrangian is given by the scalar $T$ which represents the trace of the energy-momentum tensor $T_{\mu\nu}$. Now replacing ordinary matter by a scalar field we get $f(R, T^\phi)$ theories [13] where $T^\phi$ is the trace of the energy momentum tensor of a scalar field $\phi$. Further attempts to generalize such theories resulted in the development of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theories [14], where a coupling between curvature and matter is called into play. From the Lagrangian it is seen that contributions from contraction between the curvature tensor and the energy-momentum tensor also comes into play. So the basic idea is to create new scalar invariants via contraction of tensors and explore their effects on the dynamics of the universe. Following this path Katirici and Kavuk [15] proposed the Energy-momentum-squared gravity (EMSG) where the gravity Lagrangian is given by an analytic function $f(R, T_{\mu\nu}T^{\mu\nu})$ containing the Ricci scalar and the contraction between energy momentum tensors. This is a covariant generalization to GR where we allow a term proportional to $T_{\mu\nu}T^{\mu\nu}$ to be present in the gravity Lagrangian.

Since its induction, the theory has received very good response and a fair amount of research has been performed on EMSG. Cosmology in EMSG theory was studied in [16, 17]. Cosmological bouncing scenario to avoid the singularity was studied by Roshan and Shojai in [18] using a particular model $f(R, T^2) = R + \eta T^2$, where $\eta$ is a constant representing the coupling parameter. A dynamical system analysis in the background of EMSG was studied by Bahamonde et. al in [19]. A generalization of EMSG has been achieved via the energy-momentum-powered gravity (EMPG) by the authors of the Refs. [16, 20]. In this theory they have introduced the model $f(R, T^2) = R + \eta(T^2)^n$ where $\eta$ is the coupling parameter and $n$ is the power parameter. Observational data from neutron stars have been used to constrain model parameters of EMSG by Akarsu et al in [21]. Matter wormhole solutions have been explored in the background of EMSG in [22]. Other important studies in EMSG gravity can be found in Ref. [23, 24]. Continuing the journey of modifications Akarsu et al. in [25] proposed the energy-momentum-Log gravity (EMLG), where a specific form of logarithmic function $f(T_{\mu\nu}T^{\mu\nu}) = \alpha \ln(\lambda T_{\mu\nu}T^{\mu\nu})$ is considered, $\alpha$ and $\lambda$ being constants.

It was as late as the 1970s that scientists began to understand that there is a deep underlying connection between thermodynamics and gravitation. The initial breakthrough in this topic was achieved via black holes (BH). It was found that the area of the horizon of a BH is connected with the entropy of the system. This presented a direct link between the geometry of the system with thermodynamics. Motivated from this the initial form of thermodynamic studies were limited to black hole thermodynamics [26]. It was shown in Ref. [27] that the surface gravity of a BH is related to the temperature, and it was also shown that these quantities satisfy the first law of thermodynamics (FLT), $\delta Q = Tds$. Using this FLT and the properties of BH entropy the
authors of [28] derived the Einstein’s equation of GR. Friedmann equations were derived from the thermodynamic point of view by Cai and Kim in [29]. The first law of thermodynamics at the cosmic apparent horizon using the modified Friedmann equations have been studied for general relativity, Gauss-Bonnet, and Lovelock gravity [29, 30], then extended to the scalar-tensor gravity [31], braneworld universe [32–34], Horava-Lilshitz gravity [35, 36], \( f(R) \) gravity [37], \( f(R, \mathcal{L}) \) theory [12] generic \( f(R, \phi, \partial \phi) \) gravity [38]. From the physical point of view, apparent horizons are related to the observable boundary of the universe. On the other hand, from mathematical point of view, there are two kind of apparent horizons. One of them is related to the black holes, while the other is situated near the expanding boundary of universe. The latter is called the cosmic apparent horizon which is not a null surface. Using the Palatini formalism the laws of thermodynamics were studied in the background of \( f(R) \) gravity in [39]. Thermodynamic prescription of cosmological horizons in the background of \( f(T) \) gravity was studied in Ref. [40]. Thermodynamics in the background of \( f(T, \mathcal{L}) \) gravity theory using the FLRW spacetime was investigated in [41]. The authors of Ref. [12] studied thermodynamics in the background of \( f(R, \mathcal{L}) \) theories, where \( \mathcal{L} \) represents the matter Lagrangian density. Charged black hole perturbations in this theory was studied by the authors in [42]. The effects of coupling between matter and geometry components of the universe is a very important aspect of modern cosmology and quite expectedly thermodynamics is reasonably sensitive to such couplings. There is a clear indication of this in the Refs. [12, 41].

In this work we would like to further develop the EMSG theory by performing a thermodynamic study of universe in its background. A thermodynamic analysis of any cosmological model is very important as far as its viability as a successful model is concerned. Precisely any model which has an aim to become a successful model of universe must satisfy the thermodynamic conditions of the universe. We would also like to investigate various forms of coupling effects on the thermodynamic properties of the model. So the motivation of the work is very straightforward and moreover this is probably the first attempt towards studying the thermodynamics of the universe in EMSG theory. The paper is organized as follows: In section II we have listed the basic equations of EMSG theory. Section III has been dedicated to selection of specific models. In section IV a detailed thermodynamic study is performed. Finally the paper ends with a discussion and conclusion in section V.

2 Basic equations of Energy-momentum squared cosmology

The action of the energy-momentum-squared gravity model is written as [15, 16, 20]

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R, T^2) + S_m, \tag{2.1}
\]

where \( f \) is a function depending on the square of the energy-momentum tensor \( T^2 = T^{\mu\nu} T_{\mu\nu} \) and the scalar curvature \( R \). Here, \( \kappa^2 = 8\pi G \) and \( S_m \) represents the action corresponding to the matter component.

On varying the action (2.1) with respect to the metric \( g_{\mu\nu} \), we arrive at the following field equations

\[
R_{\mu\nu} f_R + g_{\mu\nu} \Box f_R - \nabla_\mu \nabla_\nu f_R - \frac{1}{2} g_{\mu\nu} f = \kappa^2 T_{\mu\nu} - f T^2 \Theta_{\mu\nu}, \tag{2.2}
\]

where \( \Box = \nabla_\mu \nabla^\mu \), \( f_R = \partial f / \partial R \), \( f T^2 = \partial f / \partial T^2 \) and

\[
\Theta_{\mu\nu} = \frac{\delta (T^2)}{\delta g^{\mu\nu}} = \frac{\delta (T^{\alpha\beta} T_{\alpha\beta})}{\delta g^{\mu\nu}} = -2 L_m \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) - T T^2_{\mu\nu} + 2 T^{\alpha\beta} T_{\nu\alpha} - 4 T^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g_{\alpha\beta}}, \tag{2.3}
\]
where $T$ is the trace of the energy-momentum tensor. By taking covariant derivatives in the field equation (2.2), one finds the following conservation equation

$$
\kappa^2 \nabla^\mu T_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} \nabla^\mu f + \nabla^\mu (f T^2 \Theta_{\mu\nu}).
$$

(2.4)

As one can see from the above equation that the standard conservation equation $\dot{\rho} + 3H (\rho + p) = 0$ does not hold for this theory. Instead we will get a non-standard continuity equation from the above equation which will govern the properties of matter and the matter energy density of this system. It should be noted here that this is a unique feature of this theory and occurs due to the curvature-matter (squared form) coupling in the gravitational Lagrangian.

In the following, we will concentrate on the flat FLRW cosmology for this model whose metric is described by

$$
ds^2 = -dt^2 + a^2(t) \delta_{ik} dx^i dx^k,
$$

(2.5)

where $\delta_{ik}$ is the Kronecker delta and $a(t)$ the scale factor representing the expansion of the universe. Here we will consider that the matter content is described by a standard perfect fluid with the energy momentum tensor given by, $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}$ with $u_\mu$ being the 4-velocity and $\rho$ and $p$ are the energy density and the pressure of the fluid respectively. Using this energy-momentum tensor we get $T^2 = T_{\mu\nu} T^{\mu\nu} = \rho^2 + 3p^2$. Further, let us assume $L_m = p$ which allows us to rewrite $\Theta_{\mu\nu}$ defined in equation (2.3) as a quantity which does not depend on the function $f$, as given below [15, 16]

$$
\Theta_{\mu\nu} = -\left(\rho^2 + 4\rho p + 3p^2\right) u_\mu u_\nu.
$$

(2.6)

The modified FLRW equations which corresponds to this particular action are given by

$$
-3f_R \left(\dot{H} + H^2\right) + \frac{f}{2} + 3H \dot{f}_R = \kappa^2 \left(\rho + \frac{1}{\kappa^2} f T^2 \Theta^2\right),
$$

(2.7)

$$
-4f_R (\dot{H} + 3H^2) + \frac{1}{2} f + \dot{f}_R + 2H \dot{f}_R = -\kappa^2 \rho,
$$

(2.8)

where dots denote differentiation with respect to the cosmic time $t$, $H = \dot{a}/a$ is the Hubble parameter and using Eq. (2.6) the expression for $\Theta^2$ is calculated as,

$$
\Theta^2 \equiv \Theta_{\mu\nu} \Theta^{\mu\nu} = \rho^2 + 4\rho p + 3p^2
$$

(2.9)

The conservation equation (2.4) can be written as follows

$$
\kappa^2 [\dot{\rho} + 3H (\rho + p)] = -\Theta^2 f_{T^2} - f_{T^2} \left[3H \Theta^2 + \frac{d}{dt} \left(2\rho p + \frac{1}{2} \Theta^2\right)\right].
$$

(2.10)

Clearly as stated above the standard conservation equation does not hold in $f(R, T^2)$ cosmology for an arbitrary function. The covariant divergence of the field equations produces non-zero terms on the right hand side, thus leading to the modified continuity equation given above. If one chooses $f(R, T^2) = f(R)$, all the terms on the RHS of the above equation are zero and the standard conservation equation is recovered.

We can rewrite the modified FLRW equations in the standard form as,

$$
3H^2 = \kappa^2 \rho_{c.f.} = \kappa^2 (\rho + \rho_{\text{modified}}),
$$

(2.11)

$$
3H^2 + 2\dot{H} = -\kappa^2 p_{c.f.} = -\kappa^2 (p + p_{\text{modified}}),
$$

(2.12)

where we have defined the energy density and pressure for the EMSG modifications as

$$
\rho_{\text{modified}} = \frac{1}{f_R} \left[\rho + \frac{1}{\kappa^2} \left(f_{T^2} \left(\rho^2 + 4\rho p + 3p^2\right) - \frac{f}{2} - 3H \dot{f}_R + 3H \dot{f}_R\right)\right] - \rho
$$

(2.13)

$$
p_{\text{modified}} = -\left[\frac{1}{f_R} \left(p + \frac{1}{\kappa^2} \left(\frac{f}{2} + \dot{f}_R + 2H \dot{f}_R\right)\right) + \frac{\dot{H}}{\kappa^2}\right] - p
$$

(2.14)
\( \rho \) and \( p \) are respectively the energy density and pressure of matter. It should be noted here that the energy density and pressure contributions from the modified gravity can be considered equivalent to the contributions from a dark energy component and so our aim is to consider non-exotic components in the matter sector. This will facilitate a better understanding of the exotic nature of the modified gravity. In the following, a standard barotropic equation of state will be assumed for the matter fluid as given by,

\[
p = w \rho \tag{2.15}
\]

where \( w \) is the equation of state (EoS) parameter. Using this relation in equation (2.9) one gets,

\[
\Theta^2 = (1 + 4w + 3w^2) \rho^2 \tag{2.16}
\]

Also the conservation equation (2.10) becomes

\[
\dot{\rho} + 3H(w + 1)\rho = -f T^2 \left[ 3 (3w^2 + 4w + 1) H \rho^2 + (3w^2 + 8w + 1) \rho \dot{\rho} \right] - (3w^2 + 4w + 1) \rho^2 \dot{f} T^2 \tag{2.17}
\]

Finally we can define the effective equation of state (EoS) as,

\[
w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{w \rho + p_{\text{modified}}}{\rho + p_{\text{modified}}} \tag{2.18}
\]

In order to realize the late cosmic acceleration we should have \( w_{\text{eff}} < -1/3 \), which corresponds to dark energy. Since we aim to consider the matter EoS, \( w \geq -1/3 \) (non-exotic), then the role of the EoS of modified gravity \( w_{\text{modified}} \) is so much more significant for realizing the accelerated expansion of the universe, where we have considered \( w_{\text{modified}} = \frac{p_{\text{modified}}}{\rho_{\text{modified}}} \).

Since the continuity equation given by equation (2.17) is non-standard consisting of unorthodox terms in the RHS, it is not a trivial task to integrate it for this model. The obvious reason being the non-linear terms of \( \rho \) in the RHS of the equation, which makes it a non-linear differential equation. The non-zero term on the right hand side of the modified continuity equation (2.17) poses a real mathematical challenge for this operation. Here we would like to solve the continuity equation in a model independent way and express the energy density parameter \( \rho \) in terms of the redshift parameter \( z \). We see that the conservation equation (2.17) is not integrable for any arbitrary value of \( w \) by the known mathematical methods. This means that we will have to input numerical values for \( w \) in the equation and check for solutions. From the work of Board et al., in [16] we see that the equation is integrable for only \( w = -1/3 \) and \( w = -1 \). We know that \( w = -1 \) corresponds to the \( \Lambda \)CDM cosmology and \( w < -1/3 \) indicates the boundary between the exotic and non-exotic sectors. So from our perspective the solution corresponding to \( w = -1/3 \) is crucial. On solving equation (2.17) for \( w = -1 \), we get two real solutions for the density parameter \( \rho \) given by [43],

\[
\rho = \frac{1}{f T^2} \quad \text{and} \quad \rho = C_0 \tag{2.19}
\]

where \( C_0 \) is a constant.

For \( w = -1/3 \) we get only one real value for \( \rho \) given by [43],

\[
\rho = -\frac{3W \left[ \frac{4}{3} \left\{ -e^{-C_1(f T^2)^3(z + 1)^6} \right\}^{1/3} \right]}{4f T^2} \tag{2.20}
\]

where \( W [y] \) is the Lambert \( W \) function (see appendix), \( z \) is the redshift parameter and \( C_1 \) is the constant of integration. For the reader’s convenience we would like to provide a short mathematical
definition of the Lambert $W$ function. Although the expressions of $\rho$ for $w = -1$ are too trivial yet we will use the values of $\rho$ obtained for both $w = -1, -1/3$ for our further analysis. But it is to be noted that the solution generated for $w = -1/3$ will be the one that we will expect to give us more interesting results. The reason is straightforward and has already been discussed above. We should state here that due to the narrow range of solution obtained for $\rho$ our thermodynamic analysis may be constrained significantly. But this is a property of the model and needs to be accepted given the limitations of our mathematical capabilities. But we should state here that as far as cosmological implication is concerned our solution is quite fine and ready to produce interesting results.

3 Selection of Model

In literature some models of EMSG can be found which have shown promising results till now. In Ref. [16] Board and Barrow considered a fairly generic model that gave promising results as far as cosmology is concerned. The model is given by,

$$f(R, T^2) = R + \eta(T^2)^n$$  \hspace{1cm} (3.1)

where $\eta$ and $n$ are constant parameters. This is a generalized form of EMSG known as energy-momentum powered gravity (EMPG) [16]. Some solutions of this model with $n = 1/2$ and $n = 1/4$ have been discussed in Ref. [16]. In the EMPG model $n > 1/2$ corresponds to high energy densities and thus compatible with early universe. $n < 1/2$ correspond to low energy densities and thus suit the late universe. For $n = 1$ this reduces to the following special case used in Refs. [15, 18]

$$f(R, T^2) = R + \eta T^2$$  \hspace{1cm} (3.2)

Using the same functional form a more generic model of EMSG was considered in Ref. [19] as given below,

$$f(R, T^2) = \alpha R^m + \beta(T^2)^m$$  \hspace{1cm} (3.3)

where $\alpha$, $\beta$, $n$ and $m$ are all constants. In [19] another model of a different form was considered given by,

$$f(R, T^2) = f_0 R^n(T^2)^m$$  \hspace{1cm} (3.4)

where $f_0$, $n$ and $m$ are constants. In Ref. [25] the authors have studied a special class of EMSG models called the energy-momentum-log gravity (EMLG) which was characterized by the form $f(T^\mu_\nu T^\nu_\mu) = \alpha \ln(\lambda T^\mu_\nu T^\nu_\mu)$. Here $\alpha$ is a constant and $\lambda$ has dimensions inverse energy density squared so that $\lambda T^\mu_\nu T^\nu_\mu$ is dimensionless. This form has some specific advantageous features as discussed in Ref. [25].

Motivated by all the above mentioned models, we proceed to consider some generic models for the present study. Our idea is to consider some generic mathematical functions that will help us explore the effects of the the scalar invariants $R$ and $T^2$ and their coupling on the thermodynamic properties. We will basically consider two different types of models and believe that all other models will be some sub-classes of either of the two forms considered in the present study. In that sense this work will have a far wider range compared to the other works.

3.1 Minimal coupling between $R$ and $T^2$

Here we will consider the models of the form: $f(R, T^2) = f_1(R) + f_2(T^2)$, where $f_1(R)$ and $f_2(T^2)$ are analytic functions of $R$ and $T^2$ respectively. We can see that here the curvature $R$ and the matter component $T^2$ are coupled minimally in the additive sense. Now we may generate various toy models by considering various functional forms for $f_1(R)$ and $f_2(T^2)$ along with coupling constants.
3.1.1 Power Law models
We consider, \( f_1(R) = \alpha_1 R^n \), \( f_2(T^2) = \alpha_2 (T^2)^m \). So the model is given by,

\[
f(R, T^2) = \alpha_1 R^n + \alpha_2 (T^2)^m \tag{3.5}
\]

where \( \alpha_1, \alpha_2, n \) and \( m \) are constants. Here \( \alpha_1 \) and \( \alpha_2 \) act as coupling constants between the geometric and matter sectors. This model is identical with the model given in equation (3.3) [19]. This model comprises of two power law forms, one each on \( R \) and \( T^2 \) combined together additively. We call this model energy-momentum doubly powered gravity (EMDPG). This model will help us explore the non-linear effects of the scalar invariants on the thermodynamic properties. For \( \alpha_1 = n = 1 \), we get the model given in equation (3.1). Moreover for \( \alpha_1 = n = m = 1 \), we get the model given in equation (3.2). The model reduces to GR for \( \alpha_1 = n = 1 \) and \( \alpha_2 = 0 \).

3.1.2 Exponential models
Here we consider \( f_1(R) = g_1 \exp (\beta_1 R) \), \( f_2(T^2) = g_2 \exp (\beta_2 T^2) \). So the model becomes,

\[
f(R, T^2) = g_1 \exp (\beta_1 R) + g_2 \exp (\beta_2 T^2) \tag{3.6}
\]

where \( g_1, g_2, \beta_1 \) and \( \beta_2 \) are constants. Here \( g_1 \) and \( g_2 \) act as coupling constants. As can be seen from the model, here we have considered exponential forms for both \( R \) and \( T^2 \). We name this model energy-momentum doubly exponential gravity (EMDEG). For \( g_1 = 1, g_2 = 0 \) and retaining the linear terms from the Taylor series expansion of the first exponential we can realize GR from this model.

3.2 Non-minimal coupling between \( R \) and \( T^2 \)
Here the model is characterized by \( f(R, T^2) = f_1(R) + f_2(R)f_3(T^2) \), where \( f_1(R) \), \( f_2(R) \) are analytic functions of \( R \) and \( f_3(T^2) \) is an analytic function of \( T^2 \). This is our second form of the models which will be considered in this study. In this form, two scalar invariants \( R \) and \( T^2 \) are non-minimally coupled (NMC) to each other (multiplicative sense). Now we may construct various toy models by considering specific functional forms for the analytic functions. Here we will consider one toy model for our analysis as given below.

3.2.1 Power law models
We consider \( f_1(R) = \alpha_1 R^n \), \( f_2(R) = \alpha_2 R^m \), \( f_3(T^2) = (T^2)^l \). So the model becomes,

\[
f(R, T^2) = \alpha_1 R^n + \alpha_2 R^m (T^2)^l \tag{3.7}
\]

where \( \alpha_1, n, \alpha_2, m \) and \( l \) are constants. Here \( \alpha_2 \) is the coupling parameter. For \( \alpha_1 = 0 \) we get the model discussed in equation (3.4). We name this model energy-momentum-triply-powered-gravity (EMTPG). The model reduces to GR for \( \alpha_1 = n = 1 \) and \( \alpha_2 = 0 \). Investigating this model we will try to understand the effects of non-minimal curvature matter coupling on the thermodynamics of the universe.

4 Thermodynamics in EMSG
In this section we will study the thermodynamics of the universe in the background of EMSG. Our idea is to investigate various thermodynamic parameters for the toy models of EMSG as discussed above. The basic aim of the study will be to check the thermodynamic stability of the models and thus constrain the parameter space in such an attempt. In order to study the thermodynamics of
the model we should first express the Ricci scalar in terms of the Hubble expansion parameter $H$ as given below [12],
\[ R = 6(\dot{H} + 2H^2). \tag{4.1} \]
The Hubble expansion parameter can be written in terms of the cosmic apparent horizon as,
\[ H = r_A^{-1} \tag{4.2} \]
where $r_A$ is the radius of the cosmic apparent horizon. Therefore equation (4.1) is reduced to the following relation,
\[ R = \frac{6}{r_A^2}[2 - \dot{r}_A]. \tag{4.3} \]

As we know, in GR, the horizon entropy is given by the Bekenstein-Hawking (B-H) formula, which is equal to the apparent horizon area divided by $4G$, where $G$ is gravitational coupling [44–46]. However, in modified gravitational theories we have the Wald entropy [47] which is obtained using the replacement of $G$ by the effective gravitational coupling ($G \to G_{eff} = G'/f_R$, where $G' = \mathcal{F} = 1 + f_{R^2}/8\pi G$ [48], so at leading order, one can write $G' \approx G$, which is our case of interest). Hence the Wald entropy of generalized energy-momentum-squared gravity can be expressed as following,
\[ S = \frac{A f_R}{4G}, \tag{4.4} \]
where
\[ A = 4\pi r_A^2 \tag{4.5} \]
is the cosmic apparent horizon area. Similarly the thermodynamic volume of the system may be given by,
\[ V = \frac{4}{3}\pi r_A^3 \tag{4.6} \]
Moreover, using the surface gravity ($\kappa_s$) at the cosmic apparent horizon and the equation (4.2) one can obtain the temperature of the cosmic apparent horizon as [29],
\[ \bar{T} = \frac{\kappa_s}{2\pi} = \frac{|1 - \dot{r}_A^2|}{2\pi r_A}. \tag{4.7} \]
Then, one can obtain specific heat at constant volume by using the following general formula,
\[ C_V = \bar{T} \left( \frac{\partial S}{\partial \bar{T}} \right)_{V}. \tag{4.8} \]
$C_V$ is a very important quantity to study the model stability from thermodynamic point of view. We know that if it is negative, then the system is said to be in an unstable phase, whereas positivity of $C_V$ shows the stability of the system. Obviously this means that by keeping the volume constant, we are invariably talking about a closed system. On the other hand, if we think about the horizon of the expanding universe it is not a closed system. But it must be stated here that all the thermodynamic parameters are calculated taking into consideration that the time is fixed. In such a scenario the changing scale factor or the expansion does not have any role to play and considering a constant volume is possible.

Along with this we need to have a positive temperature $T > 0$. Our main goal of this paper is to find time dependent cosmic apparent horizon by using the thermodynamic rules. Then, we can use it to determine Hubble expansion parameter and hence scale factor. Having scale factor we can study all cosmological consequences. An important cosmological parameter is the deceleration parameter given by,
\[ q = -(1 + \frac{\dot{H}}{H^2}). \tag{4.9} \]
Exploring its evolution for the interesting models described above will be a straightforward task and by properly fine tuning the parameters we can realize the accelerated expansion of the universe. The internal energy could be expressed as \[ U = V \rho_{\text{eff}} \] (4.10)

and thermodynamical work can be given by,
\[ W = \frac{\rho_{\text{eff}} - p_{\text{eff}}}{2}. \] (4.11)

Therefore, one can write the first law of thermodynamics as
\[ \dot{T} dS = dU - W dV. \] (4.12)

Finally, the Helmholtz free energy is given by,
\[ F = U - \bar{T} S. \] (4.13)

This is another parameter which is closely linked to the thermodynamic stability of a system. It is basically a thermodynamic potential that measures the useful work obtained from a closed thermodynamic system at constant temperature. Here we consider a fixed time to realize this constant temperature and closed system by eliminating the effects of expanding universe. At constant temperature \( F \) is minimized under equilibrium condition.

Above we have reviewed the basic thermodynamic parameters that will be used in this study. Now we will proceed to investigate the evolution of the above discussed parameters in the various EMSG models separately.

### 4.1 Thermodynamics in EMDPG model

For this model, using equation (3.5) we have,
\[ f_R = n \alpha_1 R^{n-1} \] (4.14)

and
\[ f_{T^2} = m \alpha_2 (T^2)^{m-1}. \] (4.15)

Using equation (4.14) in the relation (4.4) one can obtain the entropy of the system as,
\[ S = \frac{n \pi \alpha_1}{G r_A^{2/3}} \left[ \frac{2|2 - \dot{r}_A|}{r_A^{2/3}} \right]^{n-1}. \] (4.16)

Then, using the equations (4.7), (4.8) and (4.16) one can obtain the specific heat as,
\[ C_V = \frac{n \pi \alpha_1}{G} \left[ \frac{2|2 - \dot{r}_A|}{r_A^{2/3}} \right]^{n-1} \frac{(n - 1)r_A\dot{r}_A + 2(n - 2)\dot{r}_A|2 - \dot{r}_A|}{r_A\dot{r}_A - \dot{r}_A^2 + 2\ddot{r}_A}. \] (4.17)

As we know that in order to have a stable model we should have \( C_V \geq 0 \). It will be the case if the following conditions are satisfied simultaneously,
\[ (n - 1)r_A\dot{r}_A + 2(n - 2)\dot{r}_A|2 - \dot{r}_A| \geq 0 \]
\[ r_A\dot{r}_A - \dot{r}_A^2 + 2\ddot{r}_A > 0. \] (4.18)

From the second condition we get, \( r_A < Ae^{\gamma t} - \frac{2}{\gamma} \) where \( A \) and \( \gamma \) are arbitrary constants. Without any loss of generality we can consider \( A \) and \( \gamma \) as positive constants. Therefore, we introduce a small constant \( \varepsilon (0 < \varepsilon < 1) \) and choose the following solution from the above obtained condition,
\[ r_A = \varepsilon (Ae^{\gamma t} - \frac{2}{\gamma}) \] (4.19)
In that case, the denominator of (4.17) is positive. But, numerator of (4.17) i.e., the first condition of (4.18) yields,

\[(n - 1)\varepsilon(A\gamma e^{\gamma t} - 2) + 2(n - 2)|2 - \varepsilon A\gamma e^{\gamma t}| \geq 0.\]  

(4.20)

This equation tells that relatively smaller values of \(n\) (\(n < 2.5\) with our selected model parameters) yields partly stable model. In fact, stability is exhibited at the late time and the model is unstable initially. Initial instability may be attributed to the particle creation process and phase transition era. It means that the EMDPG model is initially in an unstable phase which transits to the stable phase in late time. Obviously these time limits depend on the values of the model parameters \(\gamma\) and \(\varepsilon\), and by proper fine tuning we may alter these limits as desired keeping an eye on the observational data. Finally, we find that in our scale for the case of \(n \geq 2.5\), the model is completely stable at the late time where both \(C_V > 0\) and \(T > 0\). These are illustrated in Fig. 1(a), where the typical behavior of the specific heat at constant volume is represented. Also, solid red line of Fig. 1 show the temperature which is independent of \(n\). We see that beyond \(t = 6\) (late time), both temperature and specific heat remain in the positive region.

There is also another possibility for the positivity of \(C_V\) from equation (4.17) given as below,

\[(n - 1)r_A\ddot{r}_A + 2(n - 2)\dot{r}_A|2 - \dot{r}_A| \leq 0\]

\[r_A\ddot{r}_A - \dot{r}_A^2 + 2\dot{r}_A < 0\]  

(4.21)

the result of which is represented in Fig.1(b). It is obtained if both the conditions of (4.18) be negative. The result is again similar to (4.19) with \(\varepsilon > 1\) in this case. From Fig.1(b) we can see that, this yields an unstable model at the late time (present epoch) where the specific heat shoots towards the negative region for almost all values of \(n\). So we are justified in ignoring such a possibility. In order to get a clear idea of temperature we have plotted it along with the specific heat in a zoomed range in the other two plots of Fig. 1. In those figures we can clearly see that the temperature remains in the positive level thus indicating model stability.

Therefore, by using the equations (4.2) and (4.19) we can obtain the cosmological scale factor as,

\[a(t) = a_0 (A\gamma - 2e^{-\gamma t})^{\frac{1}{2}}.\]  

(4.22)

Hence, by using the equation (4.9) we can obtain,

\[q = -1 - \varepsilon A\gamma e^{\gamma t}.\]  

(4.23)

In order to continue our thermodynamic analysis we need to include the expressions for energy density \(\rho_{\text{eff}}\) and pressure \(p_{\text{eff}}\) respectively from (2.13) and (2.14). We can see from section 2.1 that we have two different cases of \(w = -1\) and \(w = -\frac{1}{3}\) where we may find values for energy densities and pressure for the EMSG model. We will study the two cases separately.

4.1.1 \(\Lambda CDM\)

Using the relation (4.15) in the first solution given by (2.19) confirms the second one which is a constant as follow,

\[\rho = \left(\frac{m_0\alpha}{\sqrt{1 - m}}\right) \equiv C_{01} \equiv C_0\]  

(4.24)

and hence,

\[p = -C_{01} \equiv -C_0\]  

(4.25)
Figure 1. Specific heat of the EMDPG model in terms of time $t$ in unit of $G$. The initial conditions are taken as $A = 1$, $\alpha_1 = 1$, and $\gamma = 0.4$. (a) is for $0 < \varepsilon < 1$ from (4.18); (b) is for $\varepsilon > 1$ from (4.21). Temperature has also been shown in both the plots. The two other plots below show a clear picture of Temperature $T$ in a zoomed range.

Our numerical analysis show that the first law of thermodynamics for this model is satisfied at the initial times if we choose small values for $A$ and $C_{101}$. In the case of $A = 0$ the first law of thermodynamics is satisfied completely. We find that $w_{eff}$ yields a negative unity at the late time corresponding to $\Lambda CDM$ era. By using the equation (4.13) we can obtain Helmholtz free energy as following,

$$F = -\frac{A \alpha_1 \gamma (A \gamma e^{\gamma t} - 2) \varrho_1^2}{32 \pi G} - \frac{4 \pi \gamma^3 \varepsilon^3 (A \gamma e^{\gamma t} - 2)^3 \varrho_{11}}{3 n \alpha_1 \varepsilon_{11}^{n-1}}, \quad (4.26)$$

where $\varrho_1$ and $\varrho_{11}$ defined in the appendix section. In the Fig. 2 we can see typical behavior of the Helmholtz free energy for the EMDPG model with $w = -1$. We find that the cases of $m = 0.5$ and $m = 2$ have similar behavior. We can see a local minimum at the late time which may correspond to the model stability at the late time. However, in this epoch the first law of thermodynamics is not satisfied and hence there is a confusion regarding stability in this phase. So we are motivated to explore other models, but before that we consider briefly the case with $w = -\frac{1}{3}$ for this model.
Figure 2. Helmholtz free energy of the EMDPG model with respect to time \( t \) for with \( w = -1 \) in unit of \( G \). The initial conditions are taken as \( \alpha_1 = \alpha_2 = 1, A = 0.1, \gamma = 0.4 \) and \( \varepsilon = 0.5 \).

### 4.1.2 \( w = -\frac{1}{4} \)

Using the relation (4.15) in the solution given by (2.20) one can obtain,

\[
a = \left[ \rho^3 e^{3m\alpha_2(\frac{1}{4})^m \rho^{2m-1} - C_1} \right]^{-\frac{1}{4}}
\]

or

\[
z = \left[ \rho^3 e^{3m\alpha_2(\frac{1}{4})^m \rho^{2m-1} - C_1} \right]^{\frac{1}{8}} - 1
\]

It yields the following equation,

\[
H = \frac{1}{\rho A} = \frac{(3m\alpha_2(\frac{1}{4})^m (m - \frac{1}{2}) \rho^{2m-1} - \frac{3}{4}) \dot{\rho}}{3\rho}
\]

Hence, we have the cosmic apparent horizon radius in terms of \( \rho \) and its derivative. In order to have analytical solutions we consider special case of \( m = \frac{1}{2} \), (which was physically identical to the case of \( m = 2 \) in the previous case), following which we can write,

\[
\rho = \rho_0 \frac{a^2}{\dot{a}^2}
\]

where

\[
\rho_0 = \left[ e^{\frac{3}{8} \alpha_2 \sqrt{-\frac{3}{4} - C_1}} \right]^{-\frac{1}{4}}
\]

is a constant. It is clear that increasing time, increases scale factor and decreases density. Then, using (4.22) we can obtain time dependent energy density.

By using the equation (4.13) we can obtain Helmholtz free energy as,

\[
F = -\frac{A_1 \alpha_1^2 e^t}{32\pi G} - \frac{4\pi \gamma_2^2 e^t (A_1^2 e^t - 2)^2}{3m \alpha_1 \dot{q}_1}
\]

where \( \dot{q}_1 \) and \( \dot{q}_{12} \) are defined in appendix. In the Fig.3 we have obtained the typical behavior of the Helmholtz free energy and see the occurrence of a minimum which may be a sign of model stability.

We find that this model satisfies the first law of thermodynamics at both the early and late times. But there is an intermediate era where the first law of thermodynamics is violated for a short
Figure 3. Typical behavior of the Helmholtz free energy for the EMDPG model with $w = -\frac{1}{3}$ in unit of $G$ against time $t$. The parameters are considered as $m = 2$, $\alpha_1 = \alpha_2 = 1$, $A = 0.1$, $\gamma = 0.4$, $\varepsilon = 0.5$ and unit value for other constant.

period of time. It may correspond to the time of structure formation. In order to demonstrate it by numerical analysis, we rewrite (4.12) as follows,

$$X \equiv \bar{T}dS - dU + WdV. \quad (4.33)$$

Then we see the evolution of $X$ versus time $t$ as represented in Fig.4. From our construction of equation (4.33) we see that when $X = 0$ then the first law of thermodynamics is satisfied, otherwise it is violated. From the plot we can see that there is an intermediate phase of violation of the first law as stated above. In these thermodynamic studies of the EMSG models we indeed need effective energy density and effective pressure which we have provided in appendix for the reader’s convenience.

Figure 4. The first law of thermodynamics of the EMDPG model with $w = -\frac{1}{3}$ in unit of $G$ versus time. The initial conditions are $m = 2$, $\alpha_1 = \alpha_2 = 1$, $A = 0.1$, $\gamma = 0.4$, $\varepsilon = 0.5$ and unit values for other constants.
4.2 Thermodynamics in EMDEG model

In this model, by using the equation (3.6) we have,

\[ f_R = g_1 \beta_1 e^{\beta_1 R} \]  

and

\[ f_T = g_2 \beta_2 e^{\beta_2 T^2} \]  

Using the equations (4.34) and (4.3) in the relation (4.4) one can obtain the entropy as follows,

\[ S = \frac{\pi g_1 \beta_1}{G} r_A^2 \exp \left( \frac{2|2 - \dot{r}_A|}{r_A^2} \right) \]  

Then, using the equations (4.7), (4.8) and (4.36) one can obtain the specific heat as,

\[ C_V = \frac{2\pi g_1 \beta_1}{G} \frac{\beta_1 r_A \ddot{r}_A + 4\beta_1 \dot{r}_A - 2\beta_1 \dot{r}_A^2 - \ddot{r}_A r_A^2}{r_A \ddot{r}_A - \dot{r}_A^2 + 2\dot{r}_A} \exp \left( \frac{2|2 - \dot{r}_A|}{r_A^2} \right) \]  

As before, we should have \( C_V \geq 0 \) for stability. It will be realized if the following conditions are satisfied simultaneously,

\[ \beta_1 r_A \ddot{r}_A + 4\beta_1 \dot{r}_A - 2\beta_1 \dot{r}_A^2 - \ddot{r}_A r_A^2 \geq 0 \]
\[ r_A \ddot{r}_A - \dot{r}_A^2 + 2\dot{r}_A \geq 0 \]  

Although, we can also assume both equations as negative valued which yields to the similar result as the previous model with a change in the range of the parameter \( \epsilon \). We can see that the second condition is the same as previous case and satisfied with the same solution \( r_A \leq A e^{\gamma t} - \frac{2}{\gamma} \), where as before \( A \) and \( \gamma \) are some constants. Therefore, we can consider \( A \) and \( \gamma \) as positive constants without any loss of generality. However the first condition is different from the previous case and we find that both conditions are satisfied with the following solution,

\[ r_A = A e^{\gamma t} - \frac{2}{\gamma} + \epsilon \]  

where \( \epsilon \) is a positive constant. Using the constraint of temperature being a positive quantity, we find the lower bound of this parameter as \( \epsilon \geq \frac{2}{\gamma} \). Without the loss of generality we can choose \( \epsilon = \frac{2}{\gamma} \) to find,

\[ r_A = A e^{\gamma t} \]  

Using this we find,

\[ T = \frac{|2 - \gamma r_A|}{4\pi r_A} \]  

which is a positive quantity. In that case the first condition of (4.38) is reduced to the following equation,

\[ \beta_1 r_A (\gamma r_A - 4) + \gamma r_A^3 \geq 0 \]  

It satisfied for infinitesimal \( \beta_1 \) or \( \gamma A \geq 4 \). Hence, we choose \( A = \frac{4}{\gamma} \) and consider the following solution from equation (4.40),

\[ r_A = \frac{4}{\gamma} e^{\gamma t} \]  

In that case the specific heat is a completely positive quantity as follows,

\[ C_V = \frac{8\pi g_1 \beta_1 |1 - 2e^{\gamma t}|(\beta_1 \gamma^2 (e^{\gamma t} - 1) + 4e^{2\gamma t})e^{\frac{2\gamma^2 t + 2\gamma^2 t^2}{4e^{2\gamma t}}}}{\gamma^2 G} \]  


In the Fig. 5 we can see the typical behavior of specific heat which is completely positive, indicating that this model is more stable than the previous one. The plot on the right shows the temperature in the zoomed range, where we see that it remains perfectly in the positive region showing model stability.

**Figure 5.** Specific heat of the EMDEG model in unit of $G$ versus time for $\gamma = g_1 = 1$. Temperature has also been shown in the plot. The plot on the right shows temperature in the zoomed range.

Therefore, by using the equations (4.2) and (4.43) we can obtain scale factor as,

\[
a(t) = a_0 e^{-\frac{\gamma}{4}t}
\]  

(4.45)

Then, by using the equation (4.9) we can obtain the deceleration parameter as,

\[
q = -1 + 4e^{\gamma t}
\]  

(4.46)

Now we use the energy densities and the pressure of matter for the two cases separately.

**4.2.1 $\Lambda$CDM**

Using the relation (4.35) in the first solution given by (2.19), confirms the second one as given below

\[
\rho = \frac{1}{g_2 \beta_2 e^{\frac{1}{2} W[y_2^2]}^2} \equiv C_{02} \equiv C_0
\]  

(4.47)

where $W[y]$ is the Lambert W function, and hence we have,

\[
p = -C_{02} \equiv C_0
\]  

(4.48)

where $C_{02}$ denotes a constant of the second model. Similar to the previous case, we find that the first law of thermodynamics is satisfied initially while being violated at the late time. We find that larger values of $\gamma$ yields a more stable period. By using the equation (4.13) we can obtain Helmholtz free energy as following,

\[
F = -\frac{2\beta_1}{\gamma} |2e^{\gamma t} - 1| \varrho_2 - \frac{256\pi e^{3\gamma t}}{3g_1 \beta_1 \gamma^3 \varrho_2} \varrho_{21}
\]  

(4.49)

where $\varrho_2$ and $\varrho_{21}$ are defined in appendix. Helmholtz free energy of this model is an increasing function of time.
4.2.2 \( w = -\frac{1}{3} \)

Using the relation (4.35) in the solution given by (2.20), one can obtain scale factor in terms of energy density as,

\[
a = \left[ \rho^3 \left( e^{-\frac{4}{3} \rho g \beta e^{\frac{4}{3} \rho^2}} \right)^3 e^{C_2} \right]^{-\frac{1}{2}}
\]

which is used to obtain the following redshift,

\[
z = \left[ \rho^3 e^{-4 \rho g \beta e^{\frac{4}{3} \rho^2} + C_1} \right]^{\frac{1}{2}} - 1
\]

where \( C_1 \) and \( C_2 \) are some integration constants. It yields the following equation for the Hubble expansion parameter,

\[
H = \frac{1}{r_A} = \frac{16 \left( g_2 \beta \rho (\frac{4}{3} + \rho^2 e^{\frac{4}{3} \rho^2} - \frac{4}{27}) \right)}{9 \rho}
\]

Hence, we have the cosmic apparent horizon in terms of \( \rho \) and its derivative. In order to have an analytical relation we consider special cases of the early and the late times.

At the early time, we assume \( \rho \gg 1 \) and find,

\[
\rho \approx \frac{3(\gamma_2 - e^{-\gamma t})}{8 q_2 \beta e^{\frac{4}{3} W[\frac{4(\gamma_2 - e^{-\gamma t})}{8 q_2 \beta}]} - \frac{4}{27}}
\]

where \( W[y] \) is the Lambert W function, and \( \gamma_2 = C_2 \gamma \) with \( C_2 \) is an integration constant. It yields the following Helmholtz free energy,

\[
F = -\frac{2 \beta_1 e^{\gamma t} - 2 e^{\gamma t} - 1}{\gamma_2} - \frac{256 \pi e^{-3 \gamma t}}{3 g_1 \beta_1 \gamma_2} \bar{\rho}_2
\]

and see appendix for definition of \( \bar{\rho}_2 \) and \( \bar{\rho}_{22} \). On the other hand, at the late time we assume \( \rho \ll 1 \) and find,

\[
\rho \approx C_2 e^{\frac{4}{3} e^{-\gamma t}}
\]

It yields the following expression for the Helmholtz free energy,

\[
F = -\frac{2 \beta_1 e^{\gamma t} - 2 e^{\gamma t} - 1}{\gamma_2} - \frac{256 \pi e^{-3 \gamma t}}{3 g_1 \beta_1 \gamma_2} \bar{\rho}_{23}
\]

and see appendix for definition of \( \bar{\rho}_2 \) and \( \bar{\rho}_{23} \). In the plots of the Fig. 6 we can see typical behavior of Helmholtz free energy at the early and the late time. We can see that Helmholtz free energy is increasing function of time. Regarding the first law of thermodynamics we find similar result with the previous model. It means that the first law of thermodynamics is satisfied both at the late and the early times. Analyzing the entropy we see that the second law of thermodynamics is satisfied too which means that the model entropy (4.36) is an increasing function of time.

4.3 Thermodynamics in EMTPG model

In this model, by using the equation (3.7) we have,

\[
f_R = n \alpha_1 R^{n-1} + m \alpha_2 R^{m-1} (T^2)^l,
\]

and

\[
f_T^2 = l \alpha_2 R^m (T^2)^{l-1}
\]
Figure 6. Typical behavior of the Helmholtz free energy of the EMDEG model with $w = -\frac{1}{3}$ in unit of $G$ versus time for $g_1 = g_2 = 1$, $\beta_1 = \beta_2 = 1$, $\gamma = 0.4$ and unit value for other constants.

Hence, if we use the equation (4.57) in the entropy expression (4.4) we get,

$$S = \frac{\pi}{G} r_A^2 \left( n\alpha_1 \left[ \frac{2|2 - \dot{r}_A|}{r_A^2} \right]^{n-1} + m\alpha_2 \left[ \frac{2|2 - \dot{r}_A|}{r_A^2} \right]^{m-1} \right) \left( 1 + 3w^2 \right)^{2l+1}.$$  (4.59)

We can see that, unlike the previous models, here the entropy is dependent on the energy density and hence we need explicit form of the energy density, which is dependent on the cosmological era (value of $w$). So we proceed to study the cases as before.

4.3.1 $\Lambda$CDM

In this case, by using the first solution of (2.19) one can obtain,

$$\rho = \frac{1}{\left( 4^{l-1}l\alpha_2 R^m \right)^{\frac{1}{l-1}}}.$$  (4.60)

Therefore, using the equations (4.60) and (4.3) in the equation (4.59) we can write,

$$S = \frac{\pi}{G} r_A^2 \left( n\alpha_1 \left[ \frac{2|2 - \dot{r}_A|}{r_A^2} \right]^{n-1} + m\alpha_2 \left[ \frac{2|2 - \dot{r}_A|}{r_A^2} \right]^{m-1} \right) \left( \frac{1}{4^{l-1}l\alpha_2} \right)^{\frac{n}{l-1}}.$$  (4.61)

It can be simplified as,

$$S = r_A^2 \left( N_0 R^N + M_0 R^M \right)$$  (4.62)

where $N = n - 1$, $M = \frac{2l(n-1) - 3m+1}{2l-1}$, $N_0 = \frac{2}{\alpha_2} n\alpha_1$ and $M_0 = \frac{2}{\alpha_2} m\alpha_2$ are constants. In order to satisfy the second law of thermodynamics we should have

$$\frac{dS}{dt} \geq 0.$$  (4.63)

We see that the equation (4.63) is satisfied if we choose,

$$r_A = r_{01} t + \frac{r_{02}}{t}.$$  (4.64)

where $r_{01}$ and $r_{02}$ are arbitrary constants. In this case, suitable values of $M$ and $N$ can yield a stable model. For example, in the Fig. 7, we can see typical behavior of the specific heat for
We find that larger values of these parameters also yield positive specific heat. Also, from dashed blue line of Fig. 7 we can see that temperature is positive in this model. To get a better idea about this we have plotted temperature in a zoomed range in the figure on the right. From the figure it is clearly evident that temperature is positive showing the stability of the model. The entropy at the early time \( t \ll 1 \) may be written in the following form,

\[
S \approx \frac{X_0}{t^2}
\]  
(4.65)

where

\[
X_0 = N_0 e^{N \ln 2} + M_0 e^{M \ln 2}
\]  
(4.66)

is a constant. Moreover the specific heat of the early time \( t \ll 1 \) may be written in the following form,

\[
C_V \approx X_0 + \frac{X_1}{t^2}
\]  
(4.67)

where the constant \( X_0 \) is given by the equation (4.66) and \( X_1 \) is a constant depending on the model parameters. In this model the scale factor is obtained as,

\[
a = a_0 \left( r_01 t^2 + r_02 \right)^{\frac{1}{r_01}}
\]  
(4.68)

Then, the deceleration parameter may be given by,

\[
q = -1 + r_01 - \frac{r_02}{t^2}
\]  
(4.69)

![Figure 7](image_url)

**Figure 7.** Specific heat of the EMTPG model with \( w = -1 \) in unit of \( G \) versus time for \( M = N = 2 \) and unit value of other parameters. Temperature has also been plotted in the figure. The figure on the right shows temperature in the zoomed range.

### 4.3.2 \( w = -\frac{1}{3} \)

Using the relation (4.58) in the solution given by (2.20), one can obtain,

\[
z + 1 = \frac{1}{a} \left[ \rho_3 \exp \left( C_1 - \rho_3^{2t-13} (\frac{4}{3})^4 3 \alpha_2 R^m \right) \right]^{\frac{1}{4}}
\]  
(4.70)
Motivated by the previous subsection, we assume the cosmic apparent horizon radius as given by the equation (4.64). In this case, it is clear that \( R > 0 \) as well as \( T > 0 \). We know that energy density is a decreasing function of the cosmic time and hence we assume,

\[
\rho \propto \frac{1}{t} \tag{4.71}
\]

In that case we are able to study thermodynamics of the model numerically. We will show that this model is stable and the first law of thermodynamics is satisfied for a suitable choice of \( n, m \) and \( l \).

In order to check validity of the first law of thermodynamics we use the equation (4.33). According to Fig. 8 we find that the first law is violated at the early time while satisfied at the late time, if we choose suitable values for \( n, m \) and \( l \). This is represented by the left plot of the Fig. 8.

![Figure 8](image)

**Figure 8.** The first law of thermodynamics of the EMTPG model with \( w = -\frac{1}{3} \) in unit of \( G \) versus time for \( \alpha_1 = \alpha_2 = 1 \) and unit value for other constants.

Also, our numerical study on the Helmholtz free energy and internal energy indicated that the thermodynamic potentials have a maximum with negative value (for selected values of \( l, m \) and \( n \)) which satisfy the first law of thermodynamics) which is a sign of the model stability. We can confirm this point by analyzing the specific heat. We show in the Fig. 9 that the specific heat and temperature are positive, and hence the model may be stable. Temperature has also been shown in a zoomed scale to get a better idea about its nature in the figure on the right. The specific heat has initially a higher value, which decays to an infinitesimal constant value at the late time.

In Fig. 10 we have plotted the deceleration parameter against redshift for the different models. Plot for the \( \Lambda \)CDM model has also been shown so that a comparison can be made considering it as a reference. It is seen that the trajectories for the deceleration parameter \( q \) enter the negative region in the late time \((z < 0.6)\), showing accelerated expansion of the universe. The shape of the trajectories are similar, but there is a difference in phase in each model from the standard \( \Lambda \)CDM model. These deviations are expected due to the modifications of gravity.

5 Conclusion

In this work we have explored the thermodynamic properties of universe in the background of the energy-momentum-squared gravity. We reviewed the field equations of the EMSG gravity theory and solved the non-standard continuity equation to get the expression for the energy density \( \rho \) of matter. It is found that the continuity equation is generically integrable for only two values
Figure 9. Specific heat of the EMTPG model with $w = -\frac{1}{3}$ in unit of $G$ versus time for $\alpha_1 = \alpha_2 = 1$ and unit value for other constants. Temperature has also been shown in the figure. The figure on the right shows temperature in the zoomed range.

Figure 10. The deceleration parameter has been plotted against redshift for the different models.

of the equation of state $w = -1, -1/3$. We obtained reasonably non-trivial expression for the energy density corresponding to $w = -1/3$ and relatively trivial expressions for $w = -1$. However we have conducted our thermodynamic analysis using both the values. Then we selected our model by considering various forms of coupling between matter and curvature. Two different types of models were considered based on two different types of coupling between $R$ and $T^2$, namely minimal and non-minimal coupling. Various functional forms (power law and exponential) were considered and different toy models were constructed. Thermodynamic studies were undertaken for each of these toy models separately and the obtained results were discussed in detail. In the thermodynamic study, the basic thermodynamic parameters like the entropy, specific heat, Helmholtz free energy, etc. were determined in terms of the cosmic apparent horizon radius, $r_A$ and its time derivatives. The conditions for the stability of the model have been found using the conditions of positivity of the specific heat $C_V$, temperature $T$ and the existence of a local minima in the evolution of Helmholtz free energy $F$. Since we have two different expressions of energy density
for $w = -1/3, -1$, we have performed the thermodynamic analysis for both the cases separately for all the models. This gave us idea about the thermodynamic properties of the universe in EMSG gravity for different cosmological eras. We know that $w = -1$ corresponds to the $\Lambda$CDM scenario, whereas $w = -1/3$ actually represents the thin boundary between the exotic and non-exotic matter. This scenario is cosmologically really interesting in the sense that it corresponds to the era where the transition from ordinary matter to dark energy takes place. It is expected that the thermodynamic properties of the system at this juncture would be really fascinating and may reveal some important information about the universe. In all the three models that we have studied we have seen that by proper fine tuning of the parameters, stability of the model can be achieved. Obviously various parameters needed to be constrained considerably to attain this. This is obviously because the laws of thermodynamics had to be fulfilled and the other stability conditions needed to be satisfied, which was not possible over a large part of the domain. There have been various works where the model parameters of EMSG were constrained using observational data sets [17, 21, 25, 49]. In this work we were able to considerably constrain the parameter space from the thermodynamic point of view of the system. We think that the correct choice of the parameter space could be made by taking into consideration both these types of analysis, which takes us one step closer towards finding the correct model. In this regard this work is a significant development to the EMSG theory of gravity.

Acknowledgments

P.R. acknowledges the Inter University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for granting visiting associateship. We thank the referee for his/her valuable comments that helped us to improve the quality of the paper significantly.

6 Appendix

6.1 Definition of Lambert $W$ function

Lambert $W$ function returns the value $x$ that solves the equation

$$y = xe^x$$

(6.1)

Also known as omega function it is a multi-valued function, namely the branches of the inverse relation of the function $f(u) = ue^u$, where $u$ is any complex number and $e^u$ is the exponential function.

Here we provide some calculations and expressions for some parameters which were not provided in the body of the paper for the convenience of the reader. The expressions for effective energy density and pressure of each model is presented below. We report them here because these expressions are somewhat large in size and by doing so we preserve a good presentation of the paper.

6.2 EMDPG

For $w = -1$:

$$\rho_{eff} = -\frac{g_{11}}{n\alpha_1 g_{1}^{-\frac{1}{1-w}}},$$

(6.2)
where

\[ g_1 = \frac{2 \gamma^2 |\varepsilon A \gamma e^{-\gamma t} - 2|}{(\varepsilon A \gamma e^{-\gamma t} - 2 \varepsilon)^2}, \quad (6.3) \]

and

\[
g_{11} = C_01 - \frac{\alpha_1}{2} g_1^n - \frac{4 n C_02 n \alpha_2}{2} - \frac{3 A \gamma^3 n \alpha_1 e^{-\gamma t} g_{12}^{n-1}}{\varepsilon (A \gamma e^{-\gamma t} - 2)^2},
\]

\[
- \frac{3(n-1) n \alpha_1 \gamma g_1^{n-2}}{\varepsilon (A \gamma e^{-\gamma t} - 2)} \left[ \frac{2 \chi A \gamma^4 e^{\gamma t}}{\varepsilon (A \gamma e^{-\gamma t} - 2)^2} - 2 A \gamma^2 e^{\gamma t} g_1 \right],
\quad (6.4)
\]

where \( C_01 \) given by the equation (4.24). Also, \( \chi = 0 \) if \( t < 0 \), \( \chi = 1 \) if \( t > \frac{\ln \frac{\gamma}{\kappa}}{\gamma} \), and \( \chi = -1 \) if \( t < \frac{\ln \frac{\gamma}{\kappa}}{\gamma} \).

\[
p_{\text{eff}} = \frac{A \gamma^3 e^{\gamma t}}{\kappa^2 \varepsilon (A \gamma e^{-\gamma t} - 2)^2} + \frac{C_01 - \frac{1}{\eta} (p_{11} - (n-1) n \alpha_1 p_{12})}{n \alpha_1 g_1^{n-1}},
\quad (6.5)
\]

where

\[
p_{11} = \frac{\alpha_1}{2} g_1^n + \frac{4 n C_02 n \alpha_2}{2} + \frac{3(n-1) n \alpha_1 \gamma g_1^{n-2}}{\varepsilon (A \gamma e^{-\gamma t} - 2)} \left[ \frac{2 \chi A \gamma^4 e^{\gamma t}}{\varepsilon (A \gamma e^{-\gamma t} - 2)^2} - 2 A \gamma^2 e^{\gamma t} g_1 \right],
\quad (6.6)
\]

and

\[
p_{12} = \frac{g_1^{n-2}}{2} \left( \frac{2 \chi A \gamma^4 e^{\gamma t}}{A \gamma e^{-\gamma t} - 2} - \frac{3 A \gamma^3 e^{\gamma t}}{A \gamma e^{-\gamma t} - 2} - 1 \right) + \frac{2 \chi A \gamma^4 e^{\gamma t}}{A \gamma e^{-\gamma t} - 2} \left( 1 - \frac{4 A \gamma^2 e^{\gamma t}}{A \gamma e^{-\gamma t} - 2} \right)
\]

\[
+ \frac{n-2}{2} g_1^{n-1} \left[ \frac{2 \chi A \gamma^4 e^{\gamma t}}{A \gamma e^{-\gamma t} - 2} - 2 A \gamma^2 e^{\gamma t} g_1 \right],
\quad (6.7)
\]

For \( w = -\frac{1}{3} \):

\[
p_{\text{eff}} = -\frac{g_{12}}{n \alpha_1 g_1^{n-1}},
\quad (6.8)
\]

where \( g_1 \) is given by equation (6.3), while

\[
g_{12} = \frac{\delta}{\sigma^2} \left( \frac{1}{A \gamma e^{-\gamma t} - 2} \right)^{-\frac{1}{2}} - \frac{\alpha_1}{2} g_1^n - \frac{\alpha_2}{2} \left( \frac{2 \delta e^{\gamma t}}{\sigma^2 (A \gamma e^{-\gamma t} - 2)} \right)^2 - \frac{3 A \gamma^3 n \alpha_1 e^{\gamma t} g_{12}^{n-1}}{\varepsilon (A \gamma e^{-\gamma t} - 2)^2},
\quad (6.9)
\]

and

\[
p_{\text{eff}} = \frac{A \gamma^3 e^{\gamma t}}{\kappa^2 \varepsilon (A \gamma e^{-\gamma t} - 2)^2} + \frac{\delta (p_{13} - (n-1) n \alpha_1 p_{12})}{n \alpha_1 g_1^{n-1}},
\quad (6.10)
\]

where

\[
p_{13} = \frac{\alpha_1}{2} g_1^n + \frac{\alpha_2}{2} \left( \frac{2 \delta e^{\gamma t}}{\sigma^2 (A \gamma e^{-\gamma t} - 2)} \right)^2 + \frac{2(n-1) n \alpha_1 \gamma g_1^{n-2}}{\varepsilon (A \gamma e^{-\gamma t} - 2)} \left[ \frac{2 \chi A \gamma^4 e^{\gamma t}}{A \gamma e^{-\gamma t} - 2} - 2 A \gamma^2 e^{\gamma t} g_1 \right],
\quad (6.11)
\]

and \( p_{12} \) given by equation (6.7).
6.3 EMDEG

For \( w = -1 \):

\[
\rho_{\text{eff}} = -\frac{\varrho_{21}}{g_1 \beta_1 \varrho_2},
\]

where

\[
\varrho_2 = e^{\frac{\beta_2 (4\gamma t - 2\varrho_2)}{8e^{-\gamma t}}},
\]

and

\[
\varrho_{21} = C_{02} - \frac{g_1}{2} \varrho_2 - \frac{1}{2} g_2 e^{4\beta_2 C_{02}} - \frac{3}{4} g_1 \beta_1 \gamma e^{-\gamma t} \varrho_2
\]

\[-\frac{3}{8} g_1 \beta_1^2 \gamma^4 e^{-2\gamma t} \varrho_2 \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{-\gamma t}} \right],
\]

where \( C_{02} \) given by the equation (4.47).

\[
\rho_{\text{eff}} = \frac{\gamma^2}{4\kappa^2} e^{-\gamma t} + \frac{C_{02} - \varrho_2 (p_{21} + g_1 \beta_1 \varrho_2 p_{22})}{g_1 \beta_1 \varrho_2},
\]

where

\[
p_{21} = \frac{1}{2} g_1 \beta_1 \varrho_2 + \frac{1}{2} g_2 e^{4\beta_2 C_{02}} + \frac{1}{4} g_1 \beta_1^2 \gamma e^{-2\gamma t} \varrho_2 \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{-\gamma t}} \right],
\]

and

\[
p_{22} = \frac{1}{4} \beta_1 \gamma^6 e^{-2\gamma t} \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{-\gamma t}} \right]^2
\]

\[+ \frac{\gamma^4}{2} |4e^{\gamma t - 2}| e^{-2\gamma t} - \frac{3}{2} \gamma^4 e^{-\gamma t}.
\]

For \( w = -\frac{1}{3} \):

Early time:

\[
\rho_{\text{eff}} = -\frac{\varrho_{22}}{g_1 \beta_1 \varrho_2},
\]

where \( \varrho_2 \) given by (6.13) and

\[
\varrho_{22} = -\frac{x_2 g_2 \beta_2}{e^{LW(x_2)/2}} - \frac{g_1}{2} \varrho_2 - \frac{1}{2} g_2 e^{4\beta_2 x_2 LW(x_2)} - \frac{3}{4} g_1 \beta_1 \gamma e^{-\gamma t} \varrho_2
\]

\[-\frac{3}{8} g_1 \beta_1^2 \gamma^4 e^{-2\gamma t} \varrho_2 \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{-\gamma t}} \right],
\]

where \( LW(x_2) \) is Lambert W function and,

\[
x_2 = -\frac{3}{8} \gamma - e^{-\gamma t}.
\]

\[
p_{\text{eff}} = \frac{\gamma^2}{4\kappa^2} e^{-\gamma t} - \frac{x_2 g_2 \beta_2}{e^{LW(x_2)/2}} + \frac{1}{g_1 \beta_1 \varrho_2} (p_{23} + g_1 \beta_1 \varrho_2 p_{22}),
\]

- 23 -
where \( p_{22} \) given by (6.17), while
\[
p_{23} = \frac{1}{2} g_1 \beta_1 g_2 + \frac{1}{2} g_2 e^{\frac{4 \beta_2}{3} t} + \frac{1}{4} \gamma^4 g_1 \beta_1^2 e^{-2 \gamma t} g_2 \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{\gamma t}} \right]. \tag{6.22}
\]

Late time:

\[
\rho_{\text{eff}} = -\frac{\rho_{23}}{g_1 \beta_1 g_2}, \tag{6.23}
\]
where \( g_2 \) given by (6.13) and
\[
g_{23} = \frac{c_{23} e^{-\gamma t}}{2} - \frac{g_1}{2} g_2 - \frac{1}{2} g_2 e^{\frac{4 \beta_2}{3} t} - \frac{3}{4} g_1 \beta_1^2 e^{-\gamma t} g_2
- \frac{3}{8} g_1 \beta_1^2 \gamma e^{-2 \gamma t} g_2 \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{\gamma t}} \right], \tag{6.24}
\]
where \( c_{23} \) is an integration constant.

\[
p_{\text{eff}} = \frac{\gamma^2}{4 \kappa^2 e^{-\gamma t}} + \frac{c_{23} e^{-\gamma t}}{2} - \frac{1}{4} g_2 \left( p_{24} + g_1 \beta_1^2 g_2 p_{22} \right), \tag{6.25}
\]
where \( p_{22} \) given by (6.17), while
\[
p_{24} = \frac{1}{2} g_1 \beta_1 g_2 + \frac{1}{2} g_2 \exp \left( 4 \beta_2 c_{23} e^{-\gamma t} \right) + \frac{1}{4} \gamma^4 g_1 \beta_1^2 e^{-2 \gamma t} g_2 \left[ 1 - \frac{|4e^{\gamma t - 2}|}{2e^{\gamma t}} \right]. \tag{6.26}
\]
Finally, effective energy density and pressure of EMTPG model is obtained in a similar way.

References

[1] S. Perlmutter et. al. :- Astrophys. J. **517** 565 (1999).
[2] A. G. Riess et al. :- Astron. J. **116** 1009 (1998).
[3] P. Brax :- Rep. Prog. Phys. **81** 016902 (2018).
[4] S. Nojiri, S. D. Odintsov, V. K. Oikonomou :- Phys. Rep. **692** 1 (2017).
[5] S. Nojiri and S. D. Odintsov :- Int. J. Geom. Methods Mod. Phys. **04** 115 (2007).
[6] S. Capozziello, R. D’Agostino, O. Luongo :- Int. J. Mod. Phys. D **28** 1930016 (2019).
[7] T. P. Sotiriou, V. Faraoni :- Rev. Mod. Phys. **82** 451 (2010).
[8] A. De Felice, S. Tsujikawa :- Living Rev. Relativity **13** 3 (2010).
[9] T. Harko, F. S. N. Lobo :- Eur. Phys. J. C. **70** 373 (2010).
[10] R. Ribeiro, J. Pramos :- Phys. Rev. D **90** 124065 (2014).
[11] R. P. L. Azevedo, J. Pramos :- Phys. Rev. D **94** 064036 (2016).
[12] B. Pourhassan, P. Rudra :- Phys. Rev. D **101** 084057 (2020).
[13] T. Harko, F. S. N. Lobo, S. Nojiri, S. D. Odintsov :- Phys. Rev. D. **84** 024020 (2011).
[14] Z. Haghani, T. Harko, F. S. N. Lobo, H. R. Sepangi, S. Shahidi :- Phys. Rev. D. **88** 4 (2013)
[15] N. Katirci, M. Kavuk :- Eur. Phys. J. Plus **129** 163 (2014)
[16] C. V. R. Board, J. D. Barrow :- Phys. Rev. D. **96**, 12 (2017)
[17] O. Akarsu, N. Katirci, S. Kumar, R. C. Nunes, M. Sami :- Phys. Rev. D **98** 6 (2018)
