A multiplicative property of quantum flag minors

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Abstract

We study the multiplicative properties of the quantum dual canonical basis $B^*$ associated to a semi-simple complex Lie group $G$. We provide a subset $D$ of $B^*$ such that the following property holds: if two elements $b, b'$ in $B^*$ $q$-commute and if one of these elements is in $D$, then the product $bb'$ is in $B^*$ up to a power of $q$, where $q$ the quantum parameter. If $G$ is $SL_n$, then $D$ is the set of so-called quantum flag minors and we obtain a generalization of a result of Leclerc-Nazarezw-Thibon, [11].

0 Introduction

0.1 Let $G$ be a semisimple complex Lie group and fix a maximal unipotent subgroup $U^-$ of $G$. Let $g$ and $n^-$ be respectively the Lie algebras of $G$ and $U^-$. G. Lusztig and M. Kashiwara have constructed the so-called canonical basis $B$ of the enveloping algebra $U(n^-)$ of $n^-$, which has properties of compatibility with standard filtrations.

Let $\mathbb{C}[U^-]$ be the $\mathbb{C}$-algebra of regular functions on $U^-$. Then, the action of $U^-$ on itself by left multiplication provides an action of $U(n^-)$ on $\mathbb{C}[U^-]$ by differential operators. Now, consider the pairing $U(n^-) \times \mathbb{C}[U^-] \rightarrow \mathbb{C}, \delta \times f \mapsto \delta(f(e))$, where $e$ is the identity of $U^-$. Then, this pairing provides the so-called dual canonical basis $B^*$ of $\mathbb{C}[U^-]$. This article is concerned with some multiplicative properties of this basis.

0.2 Let $q$ be an indeterminate and let $U_q(n^-), \mathbb{C}_q[U^-]$ be respectively the quantum analogue of the classical objects $U(n^-)$ and $\mathbb{C}[U^-]$. We still note $B, B^*$, the canonical basis, resp. the dual canonical basis, of $U_q(n^-)$, resp. $\mathbb{C}_q[U^-]$. We say that two elements $b$ and $b'$ of $B^*$ $q$-commute if $bb' = q^{mn^b}b'$, for an integer $m$. We say that they are multiplicative if $q^{n^b}b' \in B^*$ for an integer $n$. It is known, see [19], that if two elements are multiplicative, then they $q$-commute. The converse was believed to be true until Bernard Leclerc found counter examples.

Suppose that two elements $b$ and $b'$ of the dual canonical basis $q$-commute. Let’s discuss now in which cases they are known to be multiplicative.  
1) for all $b, b'$, if $g$ is of type $A_n, n \leq 3, B_2, [1]$, see also [4].
2) if $g$ is of type $A_n$ and $b$ is a small quantum minor, [19].
3) if $g$ is of type $A_n$ and $b, b'$ are quantum flag minors, [11].

0.3 Let’s present the results of this article. Let $W$ be the Weyl group of $g$ and let $w_0$ be its longest element. For each reduced decomposition $\tilde{w}_0$ of $w_0$, we have constructed in [4] a subalgebra $A_{\tilde{w}_0}$ of $\mathbb{C}_q[U^-]$ such that:
1) The space $A_{\tilde{w}_0}$ is generated by a part of $B^*$,
2) pair of elements in $B^* \cap A_{\tilde{w}_0}$ are multiplicative,
3) the algebras $A_{\tilde{w}_0}$ and $\mathbb{C}_q[U^-]$ are equal up to localization.
They are called adapted algebras associated to a reduced decomposition of $w_0$. They are connected to the more general theory of cluster algebras, [8], see 5.2.

The main result of the article is the following:

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Theorem. Let $\tilde{w}_0$ be a reduced decomposition corresponding to an orientation of the Coxeter graph of $\mathfrak{g}$. Let $b$ and $b'$ be two $q$-commuting elements of the dual canonical basis. If $b$ is in $A_{\tilde{w}_0}$ then $b$ and $b'$ are multiplicative.

As a particular case, we obtain:

Corollary. Let $\mathfrak{g}$ be of type $A_n$. Let $b$ be a quantum flag minor and let $b'$ be any element of $\mathcal{B}^*$ which $q$-commutes with $b$, then $b$ and $b'$ are multiplicative.

We refer to [11] for motivations of this result. Actually, it provides a criterion of irreducibility for module on the affine Hecke algebra of type $A$ which are induced by the so-called evaluation modules.

Let’s sketch the proof of the theorem. If two elements $b$ and $b'$ of the dual canonical basis $q$-commute, then, the only property we have to obtain in order to prove that $b$ and $b'$ are multiplicative is

$$q^nb'b' \in b'^q + qL^*,$$

where $n$ is an integer, $b''$ is in $B^*$ and where $L^*$ is the $\mathbb{Z}[q]$-lattice generated by $B^*$. The natural question is: how to control the powers of $q$ in the multiplications of elements of the dual canonical basis? The control of these powers are based on two main ideas:

1) Kashiwara proved that bases of integrable modules of the quantized enveloping algebra $U_q(\mathfrak{g})$ cristallizes at $q = 0$, with compatibility with the tensor product. To be more precise, let $P^+$ be the semigroup of integral dominant weights and let $\overline{b}, \overline{b}'$ be the corresponding elements in the crystal bases $B(\lambda), B(\lambda')$, of the integrable modules of highest weight, respectively, $\lambda$ and $\lambda'$ in $P^+$. We can assert that if $\overline{b} \otimes \overline{b}'$ belongs to the connected component of the crystal $B(\lambda) \otimes B(\lambda')$ corresponding to $B(\lambda + \lambda')$, then (0.3.1) holds. The property $\overline{b} \otimes \overline{b}' \in B(\lambda + \lambda')$ can be checked easily via the Littelmann’s path model of the crystal basis, [13], [14], by comparing chains of elements of the Weyl group for the Bruhat ordering.

2) The quiver approach of the algebra $\mathbb{C}[U^{-}]$ enables to interpret powers of $q$ which appear in multiplications in terms of $\dim \text{Hom}(M, N)$ and $\dim \text{Ext}^1(M, N)$, where $M$ and $N$ are representations of a quiver. An important tool is that the map $\dim \text{Hom}(M, ?)$ is increasing for the so-called degeneration ordering, see [3].

1 Notations.

1.1 Let $\mathfrak{g}$ be a semi-simple Lie $\mathbb{C}$-algebra of rank $n$ with Cartan matrix $A = (a_{ij})$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the triangular decomposition and set $N := \dim \mathfrak{n}$. Let $\{\alpha_i\}_i$ be a basis of the root system $R$ resulting from this decomposition and let $R^+$ be the set of positive roots. Let $P$ be the weight $\mathbb{Z}$-lattice generated by the fundamental weights $\omega_i, i \in I := \mathbb{Z} \cap [1, n]$ and set $P^+ := \sum_i \mathbb{Z}_{\geq 0} \omega_i$. Let $W$ be the Weyl group, generated by the reflections corresponding to the simple roots $s_i := s_{\alpha_i}$, with longest element $w_0$. We note $<, >$ the $W$-invariant form on $P$.

1.2 In this section we define the quantized enveloping algebra of $\mathfrak{g}$ and the properties of its Poincaré-Birkhoff-Witt basis. We refer to [6] for precisions and proofs.

Let $q$ be an indeterminate. Let $U_q(\mathfrak{g})$ be the quantized enveloping Hopf $\mathbb{Q}(q)$-algebra as defined in [6]. Let $U_q(\mathfrak{n})$, resp $U_q(\mathfrak{n}^-)$, be the upper, resp lower, “nilpotent” subalgebra of $U_q(\mathfrak{g})$. The algebra $U_q(\mathfrak{n})$, resp $U_q(\mathfrak{n}^-)$, is generated by $E_i$, resp $F_i$, $1 \leq i \leq n$ with quantum Serre relations. For all $\lambda$ in $Q := \oplus \mathbb{Z}_{\geq 0} \omega_i$, let $K_\lambda$ be the corresponding element in the algebra $U_q^0 = \mathbb{Q}(q)[Q]$ of the torus of $U_q(\mathfrak{g})$.

For all $\mu \in Q$, let $U_q(\mathfrak{n})_\mu$ be the subspace of $U_q(\mathfrak{n})$ generated by the products $E_1^{m_1} \cdots E_n^{m_n}$ such that $\sum_i m_i \omega_i = \mu$. An element $X$ of $U_q(\mathfrak{n})_\mu$ will be called (homogeneous) element of weight $\mu$. We set $\text{wt}(X) := \mu$.

Recall the triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q^0 \otimes U_q(\mathfrak{n})$. We define the following subalgebras of $U_q(\mathfrak{g})$:

$$U_q(b) = U_q(\mathfrak{n}) \otimes U_q^0,$$

$$U_q(b^-) = U_q(\mathfrak{n}^-) \otimes U_q^0.$$
As in [22], [16], we introduce Lusztig's automorphisms $T_i$, $1 \leq i \leq n$, which define a braid action on $U_q(\mathfrak{g})$ by

\begin{equation}
T_i(E_i) = -F_iK_{\alpha_i}, \quad T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^{-a_{ij} - s} q^{a_{ij} + s} E_i^{(s)} E_j^{(-a_{ij} - s)}, \quad 1 \leq i, j \leq n, \ i \neq j
\end{equation}

\begin{equation}
T_i(F_i) = -K_{-\alpha_i}E_i, \quad T_i(F_j) = (-1)^{-a_{ij} - s} \sum_{s=0}^{-a_{ij}} q^{a_{ij} - s} F_i^{(-a_{ij} - s)} F_j^{(s)}, \quad 1 \leq i, j \leq n, \ i \neq j
\end{equation}

where $E_i^{(k)} = \frac{1}{[k]_{q^a_i}!} E_i^k$, $[k]_{q^a_i} = [k]_{q^a_i} [k-1]_{q^a_i} \ldots [1]_{q^a_i}$, $\beta_k := s_{i_1} \ldots s_{i_k-1}(\alpha_k)$. It is well known, [17], that the elements $\{\beta_k, 1 \leq k \leq N\}$ is the set of positive roots and that

$$\beta_1 < \beta_2 < \ldots < \beta_N$$

defines a so-called convex ordering on $R^+$. This ordering will identify the semigroup $\mathbb{Z}^+_{\geq 0}$ with the semigroup $\mathbb{Z}_2^N$. In the sequel, we note $\{\epsilon_k, 1 \leq k \leq N\}$ the natural basis of this semigroup.

For all $k$, define $E_{\beta_k}^{\tilde{w}_0} = E_{\beta_k} = T_1 \ldots T_{i_k-1}(E_{\alpha_{i_k}})$. For all $m = (m_i) \in \mathbb{Z}_2^N$, set $E_{\tilde{w}_0}(m) = E(m) := E_{\beta_1}^{(m_1)} \ldots E_{\beta_N}^{(m_N)}$. It is known that $\{E(m), m \in \mathbb{Z}_2^N\}$ is a basis of $U_q(\mathfrak{g})$ called the Poincaré-Birkhoff-Witt basis, in short PBW-basis, associated to the reduced decomposition $\tilde{w}_0$. In the same way, we can define the PBW-basis $\{F(m), m \in \mathbb{Z}_2^N\}$ of $U_q(\mathfrak{g})$.

In the sequel, we call a (left) factor of $\tilde{w}_0$ a reduced decomposition $\tilde{w} = s_{i_1} \ldots s_{i_k}$, $1 \leq k \leq N$. Conversely, we say that $\tilde{w}_0$ is a (right) completion of $\tilde{w}$.

Let $w$ be in the Weyl group and let $\tilde{w} = s_{i_1} \ldots s_{i_k}$ be a reduced decomposition of $w$ which is completed to $\tilde{w}_0 = s_{i_1} \ldots s_{i_k}$ in $U_q(\mathfrak{g})$. Let $U_q(n_0)$ be the subalgebra of $U_q(\mathfrak{g})$ which depends only on $w$ and not on the reduced decomposition $\tilde{w}$. In the sequel, we shall note it simply $U_q(n_0)$.

Recall the following theorem, [6, 1.7]:

**Theorem.** Fix a reduced decomposition $\tilde{w}_0$ of $w_0$ and set

$$F_{\alpha}^{\tilde{w}_0}(U_q(n)) = \oplus_{m < n} Q(q)E_{\tilde{w}_0}(m), \quad m \in \mathbb{Z}_2^N$$

where $<$ is the right lexicographical ordering of $\mathbb{Z}_2^N$. Then, the spaces $F_{\alpha}^{\tilde{w}_0}(U_q(n))$, $m \in \mathbb{Z}_2^N$, define a $\mathbb{Z}_2^N$-filtration of $U_q(n)$. The associated graded algebra $Gr^{\tilde{w}_0}(U_q(n))$ is generated by $Gr^\alpha(E_\alpha)$, $\alpha \in R^+$ with relations (up to a sign):

$$Gr^\alpha(E_\alpha)Gr^\beta(E_\beta) = q^{-\alpha, \beta} Gr^\beta(E_\beta)Gr^\alpha(E_\alpha), \quad \alpha < \beta.$$

As in [11, 4.2], we define the bilinear forms $d^{\tilde{w}_0} = d$ and $c^{\tilde{w}_0} = c$ on $\mathbb{Z}_2^N \times \mathbb{Z}_2^N$ such that

$$Gr(E(m))Gr(E(n)) \in q^{-d(m,n)}(\pm 1 + q\mathbb{Z}[q])Gr(E(m + n))$$

$$Gr(E(m))Gr(E(n)) = q^{c(m,n)}Gr(E(n))Gr(E(m)),$$

up to a sign. To be more precise, we have,

$$d(m,n) = \sum_{i,j} \min_{i,j} (\beta_i, \beta_j) > m_i n_j + \frac{1}{2} \sum_{i} \min_{\beta_i} > m_i n_i.$$

We define the $\mathbb{Z}[q]$-lattice $\mathcal{L}$ generated by the $\{E^{\tilde{w}_0}(m), m = (m_i) \in \mathbb{Z}_2^N\}$. From [15, Proposition 2.3], this lattice does not depend on the choice of $\tilde{w}_0$. 

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1.3 There exists, see [23], a unique non degenerate Hopf pairing \((,\) on \(U_q(b) \times U_q(b^-)\) such that

\[
(u^+, u^-_1, u^-_2) = (\Delta(u^+), u^-_1 \otimes u^-_2), \quad u^+ \in U_q(b) ; u^-_1, u^-_2 \in U_q(b^-)
\]

\[
(u^+_1, u^+_2, u^-) = (u^+_2 \otimes u^+_1, \Delta(u^-)), \quad u^- \in U_q(b^-) ; u^+_1, u^+_2 \in U_q(b)
\]

where \(\Delta\) is the comultiplication of the Hopf algebra \(U_q(g)\).

Let \(\{E(m)^*, m \in \mathbb{Z}^N_{\geq 0}\}\) be the dual basis of \(\{F(m), m \in \mathbb{Z}^N_{\geq 0}\}\) for this pairing. From [12], we claim that:

**Claim.** For all \(m\) in \(\mathbb{Z}^N_{\geq 0}\) there exists a function \(f_m(q)\) in \(\mathbb{Q}(q)\) such that \(E(m)^* = f_m(q)E(m)\) and \(f_m(0) = 1\). Moreover, this function is an eigenvector for the \(\mathbb{Q}\)-automorphism of \(\mathbb{Q}(q)\) defined by \(q \mapsto q^{-1}\).

We set

\[
\mathcal{L}^* = \oplus_{m \in \mathbb{Z}^N_{\geq 0}} \mathbb{Z}[q] E(m)^*.
\]

1.4 In this section, we give some results about Lusztig’s canonical basis \(B\) and its dual \(B^*\).

First of all, for \(\lambda\) in \(P^+\), let \(V_q(\lambda)\) be the simple \(U_q(g)\)-module with highest weight \(\lambda\). Choose a highest weight vector \(v_\lambda\). It is known that \(V_q(\lambda)\) verifies the Weyl character formula. For all \(w\) in \(W\), let \(v_{w,\lambda}\) be a (non zero) extremal vector of weight \(w\lambda\) and let \(V_q(w,\lambda) := U_q(n).v_{w,\lambda}\) be the Demazure module.

Let \(B\) be the Lusztig’s canonical basis of \(U_q(n^-)\), [15], which coincides with Kashiwara’s global basis, [9]. It verifies the following property:

**Theorem.** Fix \(\lambda\) in \(P^+\), and let \(B(\lambda) := \{b \in B ; bv_\lambda \neq 0\}\). Then, the set \(B(\lambda).v_\lambda\) is a basis of \(V_q(\lambda)\). Moreover, for all \(w\) in \(W\) there exists a unique subset \(B_w\) of \(B\), which does not depend on \(\lambda\) and such that \((B(\lambda) \cap B_w).v_\lambda\) is a basis of \(V_q(w,\lambda)\).

**Remark.** In the sequel, if no confusion occurs, we shall identify \(B(\lambda)\) with \(B(\lambda), v_\lambda\).

Let \(B^* \subset U_q(n)\) be the dual basis in \(U_q(n)\), i.e. \((b^*, b') = \delta_{b,b'}\). Remark that this basis is not really canonical since it depends on the choice of a Hopf pairing \((,\) ). Nethertheless, we shall call it the dual canonical basis.

Let \(\eta\) be the \(\mathbb{Q}\)-automorphism of \(U_q(g)\) such that \(\eta(E_i) = E_i, \eta(F_i) = F_i,\) and \(\eta(q) = q^{-1}\). Let \(\sigma\) be the \(\mathbb{Q}(q)\)-antiautomorphism of \(U_q(g)\) such that \(\overline{E}_i = E_i, \overline{F}_i = F_i\). We can now give a characterization of \(B^*\), see [11, Proposition 16]. This characterization will give rise to the Lusztig’s parametrization of the dual canonical basis \(B^*\). It depends on the choice of a reduced decomposition of \(w_0\).

**Proposition.** Fix a reduced decomposition \(\tilde{w}_0\) of \(w_0\). Then, for each \(m\) in \(\mathbb{Z}^N_{\geq 0}\), there exists a unique homogeneous element \(X := B^{\tilde{w}_0}(m)^* = B(m)^*\) in \(U_q(n)\) such that

\[
\sigma \eta(X) = (-1)^{tr(X)} q^{<w(X), wt(X)>/2} q_X X, \quad X \in E(m)^* + q \mathcal{L}^*,
\]

where \(wt(X) = \sum_i k_i \alpha_i\), is the weight of \(X\), \(q_X = \prod_i q_i^{k_i}\), and \(tr(X) = \sum_i k_i\).

**Remark.** First remark that the proposition implies that \(\mathcal{L}^*\), resp. \(\mathcal{L}^*\), is the \(\mathbb{Z}[q]\)-lattice generated by \(B\), resp. \(B^*\). Remark also that the eigenvalue of \(X = B(m)^*\) for \(\sigma \eta\) only depends on the weight of \(X\). Now, it can be easily seen that the first condition in the proposition can be replaced by "\(X\) is an eigenvector for \(\sigma \eta\)".

We note \(B^{\tilde{w}_0}(m) = B(m)\) the corresponding element in the canonical basis \(B\). For \(w\) in \(W\) and \(\lambda\) in \(P^+\), we set \(B_w := \{b^* \in B^*, b \in B_w\}\), and \(B(\lambda)^* := \{b^* \in B^*, b \in B(\lambda)\}\).
2 Preliminary results on the dual canonical basis.

2.1 In this section, we obtain a property of triangularity for the decomposition matrix of the dual canonical basis in a general PBW basis.

Proposition. Let \( \tilde{w}_0 \) be a reduced decomposition of \( w_0 \). Then, for all \( \alpha \) in \( \mathbb{R}^+ \), the element \( (E_\alpha^{\tilde{w}_0})^* \) of the dual PBW-basis belongs to \( B^* \).

Proof. : By Proposition 1.4 and Remark 1.4, it is enough to prove that \( (E_\alpha^{\tilde{w}_0})^* \) is an eigenvector for \( \sigma_\eta \). By Claim 1.3, it is sufficient to prove that this is true for \( E_\alpha^{\tilde{w}_o} \).

Set \( \tilde{w}_0 = s_{i_1}\ldots s_{i_N} \). We have \( E_\alpha^{\tilde{w}_0} = T_{i_1} \ldots T_{i_{p-1}}(E_{\tilde{p}}) \) for \( p \) such that \( \alpha = \beta_p \). By (1.2.1)-(1.2.3), we obtain that for all homogeneous element \( X \) in \( U_q(n) \) of weight \( \mu \), we have \( \sigma_\eta T_i(X) = (-q_\alpha)^{(s_{i_k} \mu - \mu)} T_i(\sigma_\eta(X)) \). Remark that \( E_{\tilde{p}} \) is an homogeneous eigenvector for \( \sigma_\eta \). By induction, this is also true for \( E_\alpha^{\tilde{w}_0} \).

The following corollary generalizes [15, Theorem 9.13 (a)].

Corollary. Let \( \tilde{w}_0 \) be a reduced decomposition of \( w_0 \). Then, for all \( m \) in \( \mathbb{Z}^N_{\geq 0} \), we have the following homogeneous decompositions.

(i) \( \sigma_\eta(E^{\tilde{w}_0}(m)) = e_m^\eta E^{\tilde{w}_0}(m) + \sum_{n \prec m} e_n^\eta E^{\tilde{w}_0}(n), \quad e_m^\eta \in \mathbb{Z}[q, q^{-1}] \).

(ii) \( B^{\tilde{w}_0}(m) = F^{\tilde{w}_0}(m) + \sum_{n \prec m} d_m^n F^{\tilde{w}_0}(n), \quad d_m^n \in q\mathbb{Z}[q] \).

(iii) \( B^{\tilde{w}_0}(m)^* = E^{\tilde{w}_0}(m)^* + \sum_{n \prec m} c_m^n E^{\tilde{w}_0}(n)^*, \quad c_m^n \in q\mathbb{Z}[q] \).

Proof. Let \( m \) be in \( \mathbb{Z}^N_{\geq 0} \) and set \( \sigma_\eta(E(m)) = \sum_n e_n^\eta E(n), \quad e_n^\eta \in \mathbb{Z}[q, q^{-1}] \). Up to a multiplicative scalar, we have from the previous proposition: \( \sigma_\eta(E(m)) = \sigma_\eta(E^{(m_1)} \ldots E^{(m_N)}) \).

By duality, it is sufficient to prove (iii). The coefficients \( c_m^n \) are in \( q\mathbb{Z}[q] \) by Proposition 1.3. Now, the property of triangularity comes from Proposition 1.4 and (i).

Remark. In the previous corollary, the lexicographical ordering \( \prec \) can be replaced by a coarser ordering: the one generated by \( m \leq \epsilon_k + \epsilon_{k'} \), \( 1 \leq k < k' \leq N \) \( E(m) \) is a term of the PBW decomposition of \( E_{\beta_k}E_{\beta_{k'}} - q^{<\beta_{k'},\beta_k>}E_{\beta_{k'}}E_{\beta_k} \).

2.2 The previous corollary implies the compatibility of the dual canonical basis with the space \( U_q(n_w), \quad w \in W \).

Proposition. Let \( w \) be in \( W \). Then, \( U_q(n_w) \) is generated as a space by a part of \( B^* \).

Proof. Let \( \tilde{w} \) be a reduced decomposition of \( w \) and let \( \tilde{w}_0 \) be a reduced decomposition of \( w_0 \) which completes \( \tilde{w} \). Fix an element of the PBW-basis which belongs to \( U_q(n_w) \). Then, by 1.2, all smaller elements of the PBW-basis, with the same weight, belong to \( U_q(n_w) \). By Corollary 2.1, this implies the proposition.

3 Quantum flag minors.

3.1 For all \( \lambda \) in \( P^+ \), the weights of \( B(\lambda)^* \) are the \( \lambda - \mu \), where \( \mu \) runs over the weights of \( V_q(\lambda) \) (with multiplicity). The quantum flag minors, see [10], [2, 4.2], are elements of \( B(\pi_i)^* \), \( 1 \leq i \leq n \), which correspond to extremal vectors. To be more precise, let \( w \) be in \( W \) and let \( \tilde{w} = s_{i_1}\ldots s_{i_\ell} \) be a reduced decomposition of \( w \). There exists a unique element in \( B(\pi_{i_k})^* \) with weight \( (Id - w)(\pi_{i_k}) \). Note \( \Delta_w^* \) this element. \( \Delta_w^* \) is a quantum flag minor and each quantum flag minor can be written in this way.

The following proposition generalizes some properties of the \( q \)-center of \( U_q(n) = U_q(n_{w_0}) \) proved in [4, Proposition 3.2] to \( U_q(n_w) \).

Proposition. Fix an element \( w \) in \( W \) and let \( \tilde{w} = s_{i_1}\ldots s_{i_k} \) be a reduced decomposition of \( w \). We have

(i) \( X_\mu \) is an element of weight \( \mu \) in \( U_q(n_w) \), then \( \Delta_{\tilde{w}}^* X_\mu = q^{<(Id + w)(\pi_{i_k}, \mu)} X_\mu \Delta_{\tilde{w}}^* \).
We can now study the general case. Let $m \in \mathbb{Z}$, such that $\tilde{\Delta}^* \text{Bruhat ordering}$. Moreover, the chain associated to $\Delta^*$ is compared for the Bruhat ordering. This implies that $\Delta^* \tilde{\Delta}$ and $k' < k$. Hence

$$\Delta^* \tilde{\Delta} = q^{<(Id-w)\varpi_{ik'}, w \varpi_{ik'} = 0, by W-invariance. Now, (i) comes from the fact that the division ring generated by the $\Delta^*$, $1 \leq k' \leq k$, is the division ring of $U_q(n_w)$, [5, corollary 3.2].

The proof of (ii) is a straightforward generalization of [4, Proposition 3.2]. Let’s sketch the proof. Let $b_\mu$ and $\Delta^*$ be the elements in $\mathcal{B}$ corresponding respectively to $b_\mu^*$ and $\Delta^*_\tilde{\Delta}$ and suppose $b_\mu \in \mathcal{B}(\lambda)$, $\lambda \in P^+$. Then, $\Delta^* b_\mu \in \mathcal{B}^*(\varpi_{ik'}) \otimes \mathcal{B}(\lambda)$ and this element corresponds to an element of the crystal basis at $q = 0$. We know that $b_\mu$ is in $\mathcal{B}_{\tilde{\Delta}}$. Hence, by Littelmann’s path model, we can associate to $b_\mu$ a chain of elements in $W$ which are lower than $w$ for the Bruhat ordering. Moreover, the chain associated to $\Delta^*$ is reduced to $w$. So, both chains can be compared for the Bruhat ordering. This implies that $\Delta^* \otimes b_\mu \in \mathcal{B}(\varpi_{ik'} + \lambda)$ at the crystal level. This implies (ii) by [11, Proposition 33].

3.2 By [7, Theorem 3.2] and Proposition 2.2, we have:

**Lemma.** Let $\tilde{w}_0 = s_{i_1} \ldots s_{i_N}$ be a reduced decomposition of $w_0$ and let $w = s_{i_1} \ldots s_{i_k}$, $1 \leq k \leq N$. Let $m \in \mathbb{Z}^N_{\geq 0}$ such that $m_{k'} = 0$ for $k' > k$. Then, $B^{\tilde{w}_0}(m)$ is in $\mathcal{B}_w$.

Remark that this lemma can be directly proved by applying the Berenstein-Zelevinsky formula, [2, Theorem 3.7], for the transition map between the Lusztig parametrization and the string parametrization of the dual canonical basis. Indeed, by [9, Theorem 12.4], the elements of $\mathcal{B}_{w}^*$ are characterized by a string parametrization.

We can now prove the key proposition:

**Proposition.** Let $\tilde{w}_0 = s_{i_1} \ldots s_{i_N}$ be a reduced decomposition of $w_0$ and let $w = s_{i_1} \ldots s_{i_k}$, $1 \leq k \leq N$. Fix $m \in \mathbb{Z}^N_{\geq 0}$. Then,

$$q^{d(n_{\tilde{w}}, m)} \Delta^*_w E^{\tilde{w}_0}(m)^* \in E^{\tilde{w}_0}(n_{\tilde{w}} + m)^* + q \mathcal{L}^*,$$

where $n_{\tilde{w}}$ is such that $\Delta^*_w = B^{\tilde{w}_0}(n_{\tilde{w}})^*$.

**Proof.** We first suppose that $m$ is in $\mathbb{Z}^N_{\geq 0} \times \{0\}^{N-k}$. Then, by Proposition 3.1 and the previous lemma, we have

$$q^{<\nu - \varpi_{ik}, \mu>} \Delta^*_w B^{\tilde{w}_0}(m)^* \in \mathcal{B}^* + q \mathcal{L}^*,$$

where $\nu$ and $\mu$ are respectively the weights of $\Delta^*_w$ and $B^{\tilde{w}_0}(m)^*$.

Recall the notations of 1.2. By [11, 4.2], we have

$$d(n_{\tilde{w}}, m) + d(m, n_{\tilde{w}}) = <\nu, \mu>, \quad d(n_{\tilde{w}}, m) - d(m, n_{\tilde{w}}) = c(n_{\tilde{w}}, m).$$

Using Proposition 3.1, we obtain $d(n_{\tilde{w}}, m) = <\nu - \varpi_{ik}, \mu>$. Hence, there exists $m'$ in $\mathbb{Z}^N_{\geq 0}$ such that

$$q^{d(n_{\tilde{w}}, m)} \Delta^*_w B^{\tilde{w}_0}(m)^* \in E^{\tilde{w}_0}(m')^* + \sum_{n} q\mathbb{Z}[q]E^{\tilde{w}_0}(n)^*.$$

Using Theorem 1.2 and Claim 1.3, we obtain $m' = n_{\tilde{w}} + m$. By Proposition 2.2, this implies

$$q^{d(n_{\tilde{w}}, m)} \Delta^*_w E^{\tilde{w}_0}(m)^* \in E^{\tilde{w}_0}(n_{\tilde{w}} + m)^* + q \mathcal{L}^*.$$

We can now study the general case. Let $m$ be in $\mathbb{Z}^N_{\geq 0}$ and decompose $m = n + p$, with $n \in \mathbb{Z}^k_{\geq 0} \times \{0\}^{N-k}$ and $p = 0$ for $l \leq k$. We have $q^{d(n_{\tilde{w}}, m)} \Delta^*_w E(m)^* = q^{d(n_{\tilde{w}}, m)} \Delta^*_w E(n)^* E(p)^* \in (E(n_{\tilde{w}} + n)^* + \sum q\mathbb{Z}[q]E(r)^* E(p)^*)$, where $r$ runs over $\mathbb{Z}^k_{\geq 0} \times \{0\}^{N-k}$. Hence, $q^{d(n_{\tilde{w}}, m)} \Delta^*_w E(m)^* \in E(n_{\tilde{w}} + n)^* + \sum q\mathbb{Z}[q]E(r + p)^* \in E(\tilde{w}_0 + n + m)^* + q \mathcal{L}^*$. This ends the proof.
4 Quiver orientations and quantum flag minors

According to [4], Proposition 3.2 is almost what we need if we want to test the Berenstein-Zelevinsky conjecture when one element is a quantum flag minor. In fact, we have to prove an analogue of this proposition where \( E^{\omega_0}(\mathfrak{m}) \) is replaced by \( B^{\omega_0}(\mathfrak{m}) \). This can be obtained if we prove some increasing property of the linear form \( d^{\omega_0}(n_\omega, ?) \). By results of Reineke, [19], and Bongartz, [3], it is possible to prove those properties by the quiver approach, [20], of quantum groups.

4.1 This section refers to [21] for notations and definitions. Define the graph \( \Gamma \) as follows: if \( g \) is simply laced, i.e. of type A-D-E, \( \Gamma \) is the Coxeter graph of \( g \), if \( g \) is not simply laced then \( \Gamma \) is the A-D-E graph such that the graph of \( g \) is a quotient of this graph, see [16, par. 14], [18, par. 7]. Fix an orientation \( \Gamma^0 \) of \( \Gamma \) and let \( \text{Mod} \Gamma^0 \) be the category of \( k \)-representations of \( \Gamma^0 \), where \( k \) is an algebraically closed field.

Let \( \text{Ind} \Gamma^0 \) be the set of indecomposable modules of \( \text{Mod} \Gamma^0 \). Let \( \tau: \text{Ind} \Gamma^0 \to \text{Ind} \Gamma^0 \cup \{0\} \) be the Auslander-Reiten translation, with the convention that \( \tau(M) = 0 \) if \( M \) is a projective module. Recall, [21], that

\[
\zeta(M, N) = \epsilon(N, \tau(M)), \quad M, N \in \text{Ind} \Gamma^0.
\]

By the work of Ringel, [20], the algebra \( U_q(n) \) can be realized as a deformation of the Hall algebra associated to \( \Gamma^0 \). In particular, for a special reduced decomposition \( \tilde{w}_0(\Gamma^0) \) of \( w_0 \), the PBW-basis of \( U_q(n) \) can be naturally parametrized by \( \text{Mod} \Gamma^0 \). The following proposition is a recollection of results which can be found in [19], [15, par. 4]. It gives informations on the link between the \( \text{Mod} \Gamma^0 \)-parametrization and the Lusztig parametrization associated to \( \tilde{w}_0(\Gamma^0) \).

**Proposition.** There exists a reduced decomposition \( \tilde{w}_0(\Gamma^0) = s_{i_1}s_{i_2} \ldots s_{i_N} \), defined up to commutations, and a \( \mathbb{Z}_{\geq 0} \)-linear bijection \( \iota: \text{Mod} \Gamma^0 \to \mathbb{Z}_{\geq 0}^N \) such that:

(i) if \( M \) is indecomposable non projective and if \( i(M) = e_k \), then \( i(\tau(M)) = e_{k'} \), where \( k' < k \) is maximal such that \( i_k = i_{k'} \),

(ii) \( d(i(M), i(N)) = \epsilon(N, M) - \zeta(M, N) \).

Moreover, let \( i \) be a sink of \( \Gamma^0 \) then, up to commutations, \( i_k = i \). Let \( \Gamma^0_i \) be the oriented graph obtained from \( \Gamma^0 \) by reversing the arrows which go to \( i \), we have \( \tilde{w}_0(\Gamma^0_i) = s_{i_2} \ldots s_{i_N} s_{i_1} \), where \( i^* \) is such that \( w_0(\alpha_i) = -\alpha_i^* \).

4.2 In the following sections, we shall replace the lexicographical ordering \( \prec \) by the coarser ordering discussed in Remark 2.1. We still note it \( \prec \). If the reduced decomposition \( \tilde{w}_0 \) is associated to a graph orientation, then this ordering is the Ext-ordering on \( \text{Mod} \Gamma^0 \simeq \mathbb{Z}_{\geq 0}^N \), [3], which is also the degeneration ordering, [loc. cit., Corollary 4.2, Proposition 3.2]. We are now ready to prove the expected “increasing property” of \( d^{\omega_0}(n_\omega, ?) \).

**Proposition.** Let \( \tilde{w}_0(\Gamma^0) = s_{i_1}s_{i_2} \ldots s_{i_N} \) be the reduced decomposition of \( w_0 \) associated to \( \Gamma^0 \) and let \( \tilde{w} = s_{i_1}s_{i_2} \ldots s_{i_k} \) be a subword of \( \tilde{w}_0(\Gamma^0) \). Let \( n_\omega \) be as in 3.1. Then, the form \( d(n_\omega, ?) \) is increasing for \( \prec \).

**Proof.** By [5, 3.2.2], we have \( n_\omega = \sum e_l \) where \( l \) runs over the set of integers of \([1, k]\) such that \( i_l = i_k \). From Proposition 4.1, we obtain that

\[
d(n_\omega, ?) = \sum_{i \geq 0} \epsilon(?, \tau^i(M_k)) - \sum_{i \geq 0} \zeta(\tau^i(M_k), ?),
\]

where \( M_k \) is the indecomposable module corresponding to \( e_k \). By (4.1.1), we obtain after elimination that \( d(n_\omega, ?) = \epsilon(?, M_k) \), which is increasing by [3, Proposition 3.2].
4.3 Now, the natural question is to know if all quantum flag minor can be associated to a graph decomposition $\overrightarrow{\Gamma}$. In clear, fix $u$ in $W$ and a reduced decomposition $\tilde{u}$ of $u$. Is there an orientation $\overrightarrow{\Gamma}$ of $\Gamma$ and a subword $\tilde{w}$ of $\tilde{w}_0 = \tilde{w}_0(\overrightarrow{\Gamma})$ such that $\Delta_{\tilde{w}} = \Delta_{\tilde{w}_0}$. This is true for the $A_n$ case, but not in general:

**Claim.** Let $\mathfrak{g}$ be of type $A_n$. Then, all quantum flag minors can be realized as $B^{\tilde{w}_0}(\mathfrak{n}_\mathfrak{g})^*$ where $\tilde{w}_0$ is associated to an orientation of $\Gamma$.

**Proof.** It is known that for all $k$, $1 \leq k \leq n$, the set of quantum minors in $B(\varpi_k)^*$ is naturally indexed by a set of lines $I = \{i_1 < i_2 < \ldots < i_k\}$. We generally note it $\Delta_{\tilde{w}}(I,J)$ where $J = \{1, \ldots, k\}$ is the set of its columns. This (flag) quantum minor corresponds to an extremal vector in $V_q(\varpi_k)$ associated to

$$w = s_{i_1-1} \ldots s_{i_2-1} \ldots s_{i_k-1} \ldots s_1 W^k,$$

where $W^k := \{u \in W, u\varpi_k = \varpi_k\}$.

We claim that this reduced decomposition $\tilde{w} = s_{i_1-1} \ldots s_{i_2-1} \ldots s_{i_k-1} \ldots s_1$ can be completed to a reduced decomposition of $w_0$ which is associated to an orientation of $\Gamma$. Indeed, this follows from the last assertion of Proposition 4.1 and the well known fact: $\tilde{w}_0 := s_1 s_2 s_1 s_2 \ldots s_n$ is a reduced decomposition associated to the orientation $\overrightarrow{\Gamma}$ of $A_n$ such that all arrows are oriented to the left. Indeed, with the notations of Proposition 4.1, $\tilde{w}$ can be completed to the reduced decomposition of $w_0$ associated to the quiver $\overrightarrow{\Gamma}_1$, where $i$ is the following sequence of reversing arrows

$$i = 121 \ldots (n-k) \ldots 1(n-k+1) \ldots i_1(n-k+2) \ldots i_2 \ldots n \ldots i_k.$$

**Remark.** The claim is not true if we take $\mathfrak{g}$ of type $D_4$. With the standard notations, the quantum flag minor associated to the reduced decomposition $\tilde{w} = s_2 s_1 s_3 s_2$ can not be realized from a quiver orientation.

5 Adapted algebras associated to a quiver orientation.

5.1 We first recall some basic facts on adapted algebras associated to a reduced decomposition of $w_0$, see [4]. Fix a reduced decomposition $\tilde{w}_0$ of $w_0$ and let $W$ be the set of left subwords of $\tilde{w}_0$. Then, the quantum flag minors $\Delta_{\tilde{w}_0}, \tilde{w} \in W$, $q$-commute. Moreover, they generate a $Q(q)$-algebra $A_{\tilde{w}_0}$ such that

1) $A_{\tilde{w}_0}$ is a $q$-polynomial algebra of GK-dimension $N = \text{Card } \tilde{W}$,
2) As a space, $A_{\tilde{w}_0}$ is generated by a part of the dual canonical basis, namely the monomials in the $\Delta_{\tilde{w}_0}, \tilde{w} \in W$.

Now, we reach the main theorem:

**Theorem.** Fix an orientation $\overrightarrow{\Gamma}$ of $\Gamma$ and set $\tilde{w}_0 = \tilde{w}_0(\overrightarrow{\Gamma})$. Suppose that $b^*$ in $A_{\tilde{w}_0} \cup B^*$ and $(b')^*$ in $B^*$ $q$-commute, then they are multiplicative, i.e. their product is an element of the dual canonical basis up to a power of $q$.

**Proof.** Set $\tilde{w}_0 = s_{i_1} s_{i_2} \ldots s_{i_N}$. The elements $b^* = B^{\tilde{w}_0}(\mathfrak{m})^*$ and $(b')^* = B^{\tilde{w}_0}(\mathfrak{m'})^*$ $q$-commute. Hence, in order to prove that they are multiplicative, it is sufficient, see [11, 5.1], to prove that

$$q d^{\tilde{w}_0}(\mathfrak{m}, \mathfrak{m'}) B^{\tilde{w}_0}(\mathfrak{m})^* B^{\tilde{w}_0}(\mathfrak{m'})^* \in B^{\tilde{w}_0}(\mathfrak{m} + \mathfrak{m'})^* + q\mathcal{L}^*.$$

Recall that, by Corollary 2.2,

$$B^{\tilde{w}_0}(\mathfrak{m'})^* = E^{\tilde{w}_0}(\mathfrak{m'}) + \sum_{n < m'} c^m_{m'} E^{\tilde{w}_0}(\mathfrak{n})^*,$$

with $c^m_{m'} \in q\mathbb{Z}[q]$. Moreover, $B^{\tilde{w}_0}(\mathfrak{m})^*$ is a product of $\Delta_{\tilde{w}}$, where $\tilde{w}$ runs over a subset $\tilde{W}$ of $W$. This implies that $d^{\tilde{w}_0}(\mathfrak{m}, ?) = \sum d^{\tilde{w}_0}(\mathfrak{n}_\mathfrak{g}, ?)$, where $\tilde{w}$ runs over $\tilde{W}$ with multiplicity. By Proposition 4.2, this asserts that the form $d^{\tilde{w}_0}(\mathfrak{m}, ?)$ is increasing for $\prec$. Hence, (5.1.1) is a consequence of Proposition 3.2 and (5.1.2).
Remark. It is likely that the theorem works for all reduced decomposition of $w_0$. In fact, all we have to prove is a generalization of Proposition 4.2 for general $\tilde{w}_0$.

By Claim 4.3, we deduce the following corollary which generalizes a result of [11]:

**Corollary.** Let $\mathfrak{g}$ be of type $A_n$. Let $b^*$ and $(b')^*$ be $q$-commuting elements in $B^*$. If $b^*$ is a quantum flag minor, then $b^*$ and $(b')^*$ are multiplicative.

5.2 In order to understand a “multiplicative” description of the dual canonical basis, S. Fomin and A. Zelevinsky have defined the so-called cluster algebras, [8]. Roughly speaking, cluster algebras are algebras equipped with a distinguished set of generators called cluster variables and this set is divided into a union of subsets called clusters. The cluster variables verify the so-called exchange relations. For each symmetrizable Cartan datum, Fomin and Zelevinsky associate a cluster algebra.

As a conclusion, we would like to enlight the link between the results of 5.1 and the theory of cluster algebras. This follows some discussions with A. Zelevinsky. One of the most amazing fact in the Fomin-Zelevinsky theory is that many algebras, as $\mathbb{C}[U^-], \mathbb{C}[G]$, encountered in the representation theory of semi-simple Lie groups seem to be realized in the algebraic framework of cluster algebras. In particular the subalgebras $A_{\tilde{w}_0}$ should specialize at $q = 1$ onto a subalgebra of $\mathbb{C}[U^-]$ generated by a cluster. The connection with our problem is the following: it is reasonable to think that if $b$ and $b'$ are $q$-commuting elements of the dual canonical basis, and if $b$ is specialized at $q = 1$ onto a cluster variable, then $b$ and $b'$ are multiplicative. What we proved in 5.1 is a particular case of this conjecture when the cluster is associated to some reduced word.

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