Quasinormal modes and late-time tails in the background of Schwarzschild black hole pierced by a cosmic string: scalar, electromagnetic and gravitational perturbations

Songbai Chen
Department of Physics, Fudan University, Shanghai 200433, P. R. China
Institute of Physics and Department of Physics,
Hunan Normal University, Changsha, Hunan 410081, P. R. China

Bin Wang
Department of Physics, Fudan University, Shanghai 200433, P. R. China

Rukeng Su
China Center of Advanced Science and Technology (World Laboratory),
P.B.Box 8730, Beijing 100080, Peoples Republic of China
Department of Physics, Fudan University, Shanghai 200433, P. R. China

Abstract

We have studied the quasinormal modes and the late-time tail behaviors of scalar, electromagnetic and gravitational perturbations in the Schwarzschild black hole pierced by a cosmic string. Although the metric is locally identical to that of the Schwarzschild black hole so that the presence of the string will not imprint in the motion of test particles, we found that quasinormal modes and the late-time tails can reflect physical signatures of the cosmic string. Compared with the scalar and electromagnetic fields, the gravitational perturbation decays slower, which could be more interesting to disclose the string effect in this background.

PACS numbers: 04.30.Nk, 04.70.Bw

*Electronic address: chsb@fudan.edu.cn
†Electronic address: wangb@fudan.edu.cn
‡Electronic address: rksu@fudan.ac.cn
I. INTRODUCTION

Quasinormal mode (QNM) is believed as a characteristic sound of black holes, which describes the damped oscillations under perturbations in the surrounding geometry of a black hole with frequencies and damping times of the oscillations entirely fixed by the black hole parameters. The study of QNM has been an intriguing subject of discussions for the last few decades. Despite its potential astrophysical interest which could lead to the direct identification of the black hole existence through gravitational wave observation to be realized in the near future, the QNM has also been argued as a testing ground for fundamental physics. Motivated by the discovery of the AdS/CFT correspondence, the investigation of QNM in anti-de Sitter (AdS) spacetimes became appealing in the past several years. It was argued that the QNMs of AdS black holes have direct interpretation in term of the dual conformal field theory (CFT). Attempts of using QNMs to investigate the dS/CFT correspondence has also been given. Recently QNMs in asymptotically flat spaces have acquired further attention, since the possible connection between the classical vibrations of a black hole spacetime and various quantum aspects was proposed by relating the real part of the QNM frequencies to the Barbero-Immirzi (BI) parameter, a factor introduced by hand in order that loop quantum gravity reproduces correctly the black hole entropy. The extension has been done in the dS background, however in the AdS black hole spacetime, the direct relation has not been found. The charged situation in both dS and AdS cases were fully addressed.

The motivation of the present paper is to examine whether the QNM can tell us the signature of the string. We are going to consider curved spacetimes containing strings. An example is a black hole with a straight string passing through it, whose metric is described as

\[
\begin{align*}
    ds^2 &= (1 - \frac{2M}{br})dt^2 - (1 - \frac{2M}{br})^{-1}dr^2 - r^2d\theta^2 - b^2r^2\sin^2\theta d\phi^2,
\end{align*}
\]  

where \(M\) is the parameter which relates to the black hole energy. \(b\) is related to the linear mass density \(\rho_s\) of the string by \(b = 1 - 4\rho_s\) and is justified in the range \(0 < b \leq 1\) since \(\rho_s \ll 1\). The metric is locally identical to the Schwarzschild metric, and the motion of test particles is locally unchanged by the presence of the string. Thus it would be difficult to observe the string influence through its effect on the geodesics. It is of interest to ask what physical phenomenon will present due to the presence of the string. In this work we will examine the QNM in the background of Schwarzschild black hole pierced by a cosmic string and will disclose phenomenons...
in the QNM in terms of the cosmic string.

We will extend the study of the wave dynamics outside the black hole to the late time scale. At late times, quasinormal oscillations are swamped by the relaxation process. This relaxation is the requirement of the black hole no-hair theorem \[17\]. Price \[18\] first studied the late-time behaviors of the massless scalar, gravitational and electromagnetic perturbations in the Schwarzschild black hole spacetime and found that for a fixed \(r\) the late-time tails are commonly dominated by the factor \(t^{-(2l+3)}\). This late-time behavior has also been confirmed in other static spacetimes for massless scalar perturbations \[19\]. For the massive scalar field, the late-time behaviors of perturbations persist quite different properties. For example, the intermediate late-time behavior is dominated by the oscillatory inverse power-law form \(t^{-(l+3/2)} \sin \mu t\) and the asymptotic one is by the \(t^{-5/6} \sin \mu t\) in the static spacetimes \[20\]. Similar properties of the late-time behaviors have also been found in the rotating black hole backgrounds \[21\]. For a black hole with a straight string passing through it, the black hole may have hairs in the form of strings \[16\]. It would be interesting to investigate whether the late-time tail contains the information of the string.

The organization of this paper is as follows. In Sec.II we will derive the wave equation of the scalar, electromagnetic and gravitational perturbations in the Schwarzschild black hole pierced by a cosmic string. In Sec.III we will calculate the fundamental quasinormal frequencies of the massless scalar, electromagnetic and gravitational fields by using the third-order WKB approximation. In Sec.IV we will extend our discussion to the late-time tail behaviors of the massless scalar, electromagnetic and gravitational perturbations. For the massive late-time tail behaviors, we just consider the scalar perturbation. Our conclusions and discussions will be presented in the last section.

II. THE WAVE EQUATION OF THE SCALAR, ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS IN THE SCHWARZSCHILD BLACK HOLE PIERCED BY A COSMIC STRING

For the Schwarzschild black hole pierced by a cosmic string, we can choose the null tetrad as follows

\[
\begin{align*}
  l^\mu &= \left( \frac{r^2}{\Delta r}, 1, 0, 0 \right), \\
  n^\mu &= \frac{1}{2} \left( 1, -\frac{\Delta r}{r^2}, 0, 0 \right), \\
  m^\mu &= \frac{1}{\sqrt{2}r} \left( 0, 0, 1, \frac{i}{b \sin \theta} \right),
\end{align*}
\]
where \( \Delta_r = r(r - 2M/b) \), and we find that the non-vanishing spin-coefficients have the values

\[
\rho = -\frac{1}{r}, \quad \beta = -\alpha = \frac{1}{2\sqrt{2r}} \cot \theta, \quad \mu = \frac{\Delta_r}{2r^3}, \quad \gamma = \frac{M}{2br^2}.
\]  

(3)

The Teukolsky’s master equations \[22\] for massless scalar \((s = 0)\), electromagnetic \((s = 1)\) and gravitational \((s = 2)\) perturbations in Newman-Penrose formalism can be written as \[23\]

\[
\begin{align*}
&\left\{ [D - (2s - 1)\varepsilon + \varepsilon^* - 2s\rho - \rho^*] (\Delta - 2s\gamma + \mu) \\
&\quad - [\delta - (2s - 1)\beta - \alpha^* - 2s\tau + \pi^*] (\bar{\delta} - 2s\alpha + \pi) - (s-1)(2s-1)\Psi_2 \right\} \Phi_s = 0,
\end{align*}
\]

and

\[
\begin{align*}
&\left\{ [\Delta + (2s - 1)\gamma - \gamma^* + 2s\mu + \mu^*] (D + 2s\varepsilon - \rho) \\
&\quad - [\bar{\delta} + (2s - 1)\alpha + \beta^* + 2s\pi - \tau^*] (\bar{\delta} + 2s\beta - \tau) - (s-1)(2s-1)\Psi_2 \right\} \Phi_{-s} = 0.
\end{align*}
\]

(4)

(5)

Assuming that the azimuthal and time dependence of the wave-functions in equations (4) and (5) have the form \( e^{-i(\omega t - m\varphi)} \), we find that the derivative operators are

\[
D \equiv l^\mu \partial_\mu = D_0, \quad \Delta \equiv n^\mu \partial_\mu = -\frac{\Delta_r}{2r^2} D_0^\dagger, \quad \delta \equiv m^\mu \partial_\mu = \frac{1}{\sqrt{2r}} L_0, \quad \bar{\delta} \equiv m^\mu \partial_\mu = \frac{1}{\sqrt{2r}} L_0^\dagger.
\]

(6)

where

\[
\begin{align*}
\mathcal{D}_n &= \frac{\partial}{\partial r} - \frac{iK}{\Delta_r} + \frac{n}{\Delta_r} \frac{d\Delta_r}{dr}, \\
D_0^\dagger &= \frac{\partial}{\partial r} + \frac{iK}{\Delta_r} + \frac{n}{\Delta_r} \frac{d\Delta_r}{dr}, \\
\mathcal{L}_n &= \frac{\partial}{\partial \theta} + \frac{m}{b \sin \theta} + n \cot \theta, \\
L_0^\dagger &= \frac{\partial}{\partial \theta} - \frac{m}{b \sin \theta} + n \cot \theta, \\
K &= r^2 \omega.
\end{align*}
\]

(7)

Adopting the Newman-Penrose formalism, we can easily obtain the separated equations\[23\] for massless scalar, electromagnetic and gravitational perturbations around a Schwarzschild black hole pierced by a cosmic string

\[
[\Delta_r D_1 - s D_0^\dagger + 2(2s - 1)i\omega r] \Delta^s R_s = \lambda \Delta^s R_s,
\]

\[
\mathcal{L}_1^s \mathcal{L}_s S_s = -\lambda S_s,
\]

(8)

with the spin number \( s = 0, +1 \) and \(+2\), where \( \lambda \) is a separation constant, \( R_s \) and \( S_s \) are only functions of \( r \) and \( \theta \), respectively.

Using the transformation theory, we can obtain the radial equation for the massless scalar, electromagnetic and gravitational perturbations in this black hole spacetime through tedious calculations, which leads to the form

\[
\frac{d^2 P_s}{dr_s^2} + [\omega^2 - V_s] P_s = 0,
\]

(9)
where \( r_* \) is the tortoise coordinate (which is defined by \( dr_* = \frac{r^2}{M} dr \)) and the effective potential \( V_s \) reads

\[
V_s = \begin{cases} 
(1 - \frac{2M}{r_*})(\frac{\lambda}{r_*} + \frac{2M}{r_*}) & s = 0, \\
(1 - \frac{2M}{r_*})\frac{\lambda}{r_*} & s = 1, \\
(1 - \frac{2M}{r_*})(\frac{\lambda r^2}{r_*} - \frac{6M}{r_*}) & s = 2.
\end{cases}
\] (10)

From the second equation in (8), we can obtain directly the angular equation for the perturbations

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{(m/b + s \cos \theta)^2}{\sin^2 \theta} + s \right] S_s = -\lambda S_s,
\] (11)

where \( m \) is the angular quantum number. For the parameter \( b = 1 \) (i.e., the linear mass density of the cosmic string \( \rho_s = 0 \)), the angular equation (11) reduces to that in the usual spherically symmetric case. The solution can be expressed as the expansion in the spin-weighted associated Legendre polynomial \( sP_l^m(\cos \theta) \) with the eigenvalue \( \lambda = (l + s)(l - s + 1) \) which is independent of the angular number \( m \). However when \( b \neq 1 \), the spherical symmetry is broken and then \( \lambda \) will depend not only on the angular number \( l \) but also on \( m \).

To investigate the quasinormal modes and late-time tail behaviors of the external perturbations in the black hole spacetime, we must first determine \( \lambda \) appeared in the above equations. We restrict ourselves to \( m > 0 \) and rewrite equation (11) as

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s \right] S_s = -\lambda S_s,
\] (12)

Considering that the deviation of parameter \( b \) from the unity is very small, which is physically justified for a cosmic string with \( \rho_s \ll 1 \), the last two terms in the left-hand-side of the above equation can be regarded as a perturbative term. Using the general perturbation theory, we have

\[
S_s = sP_l^m(\cos \theta) + \gamma \zeta_l^m(\cos \theta) + O(\gamma^2),
\]

\[
\lambda = \lambda_0 + \gamma \lambda_1 + O(\gamma^2),
\] (13)

where \( \gamma \) is a dimensionless parameter denoting the perturbative scale. For convenience, we set \( \gamma = 1 \) throughout our paper. Substituting the variables (13) into the angular equation (12), we can obtain

\[
(D_0 + \lambda_0) sP_l^m(\cos \theta) = 0,
\] (14)

\[
(D_0 + \lambda_0) \zeta_l^m(\cos \theta) + (D_1 + \lambda_1) sP_l^m(\cos \theta) = 0,
\] (15)

where

\[
D_0 = \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{(m/b + s \cos \theta)^2}{\sin^2 \theta} + s,
\]

\[
D_1 = \frac{(b^2 - 1)m^2}{b^2 \sin^2 \theta} + \frac{2ms(b - 1) \cos \theta}{b \sin^2 \theta}.
\] (16)
From the zeroth order equation (14), we have

$$\lambda_0 = (l + s)(l - s + 1).$$  \hspace{1cm} (17)

Multiplying equation (15) by \(sP_{lm}(\cos \theta)\) from the left side and integrating it over \(\theta\), we obtain

$$\lambda_1 = \begin{cases} 
\frac{m(2l+1)(1-b)}{2(m^2-s^2)b^2}[(1+b)m^2 - 2bs^2] \quad & |m| > |s|, \\
\frac{sm^2(2l+1)(1-b)^2}{2(s^2-m^2)b^2} \quad & |m| < |s|.
\end{cases}$$  \hspace{1cm} (18)

As we discussed above, when \(b \neq 1\) the eigenvalue \(\lambda\) of the angular equation (11) depends not only on the multiple moment \(l\), but also on the parameter \(b\), angular number \(m\) and the spin \(s\). Obviously, the eigenvalue \(\lambda\) increases with the increase of the parameters \(m\) and \(\rho_s\). At last, it must be noted that the equation (18) is not valid as \(|m| = |s|\).

### III. QUASINORMAL MODES OF SCALAR, ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS IN THE SCHWARZSCHILD BLACK HOLE PIERCED BY A COSMIC STRING

In this section, we will apply the third-order WKB approximation to evaluate the fundamental quasinormal modes \((n = 0)\) of massless scalar, electromagnetic and gravitational perturbations in the Schwarzschild black hole with a cosmic string passing through it. We expect to see what effects of cosmic string can be reflected in the QNMs behavior. The formula for the complex quasinormal frequencies \(\omega\) in this approximation is given by [25]

$$\omega^2 = [V_0 + (-2V''_0)^{1/2}\Lambda] - i(n + \frac{1}{2})(-2V''_0)^{1/2}(1 + \Omega),$$  \hspace{1cm} (19)

where

$$\Lambda = \frac{1}{(-2V''_0)^{1/2}} \left\{ \frac{1}{8} \left( \frac{V_0^{(4)}}{V_0'} \right)^2 \left( \frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left( \frac{V_0'''}{V_0'} \right)^2 (7 + 60\alpha^2) \right\},$$

$$\Omega = \frac{1}{(-2V''_0)^{1/2}} \left\{ \frac{5}{6912} \left( \frac{V_0'''}{V_0'} \right)^4 (77 + 188\alpha^2) - \frac{1}{384} \left( \frac{V_0'''}{V_0'} \right)^2 (51 + 100\alpha^2) + \frac{1}{2304} \left( \frac{V_0^{(4)}}{V_0'} \right)^2 (67 + 68\alpha^2) + \frac{1}{288} \left( \frac{V_0^{(5)}}{V_0''} \right)^2 (19 + 28\alpha^2) - \frac{1}{288} \left( \frac{V_0^{(6)}}{V_0''} \right)^2 (5 + 4\alpha^2) \right\},$$  \hspace{1cm} (20)

and

$$\alpha = n + \frac{1}{2}, \quad V_0^{(s)} = \left. \frac{dbV}{dr_s^*} \right|_{r_s=r_s(r_p)}.$$
\( n \) is overtone number and \( r_p \) is the value of polar coordinate \( r \) corresponding to the peak of the effective potential.\(^{10}\)

Setting \( M = 1 \) and substituting the effective potential\(^{11}\) into the formula above, we can obtain the quasinormal frequencies of scalar, electromagnetic and gravitational perturbations in the Schwarzschild black hole pierced by a cosmic string.

\[
\begin{array}{c|c|c|c}
 b & \omega (m=0) & \omega (m=1) & \omega (m=2) \\
0.90 & 0.953613-0.086705i & 0.973696-0.086701i & 0.993374-0.086697i \\
0.92 & 0.974805-0.088632i & 0.990725-0.088628i & 1.006393-0.088625i \\
0.94 & 0.995996-0.090559i & 1.007830-0.090556i & 1.019526-0.090554i \\
0.96 & 1.017188-0.092485i & 1.025008-0.092484i & 1.032770-0.092482i \\
0.98 & 1.038379-0.094412i & 1.042256-0.094411i & 1.046119-0.094410i \\
1.00 & 1.059570-0.096339i & 1.059570-0.096339i & 1.059570-0.096339i \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 b & \omega (m=3) & \omega (m=4) & \omega (m=5) \\
0.90 & 1.012668-0.086693i & 1.031602-0.086690i & 1.050194-0.086687i \\
0.92 & 1.021820-0.088622i & 1.037019-0.088619i & 1.051997-0.088617i \\
0.94 & 1.031090-0.090551i & 1.042525-0.090549i & 1.053837-0.090547i \\
0.96 & 1.040473-0.092480i & 1.048120-0.092479i & 1.055712-0.092477i \\
0.98 & 1.049968-0.094410i & 1.053803-0.094409i & 1.057623-0.094408i \\
1.00 & 1.059570-0.096339i & 1.059570-0.096339i & 1.059570-0.096339i \\
\end{array}
\]

**TABLE I:** The fundamental \((n=0)\) quasinormal frequencies of scalar field \((s=0)\) in the Schwarzschild black hole spacetime with a cosmic string for \(l = 5\).

\[
\begin{array}{c|c|c|c}
 b & \omega (m=0) & \omega (m=2) & \omega (m=3) \\
0.90 & 0.943084-0.086386i & 0.983959-0.086403i & 1.003138-0.086410i \\
0.92 & 0.964041-0.088305i & 0.996406-0.088319i & 1.011798-0.088325i \\
0.94 & 0.984999-0.090225i & 1.009027-0.090235i & 1.020604-0.090240i \\
0.96 & 1.005956-0.092145i & 1.021815-0.092152i & 1.029555-0.092155i \\
0.98 & 1.026913-0.094064i & 1.034765-0.094068i & 1.038645-0.094069i \\
1.00 & 1.047871-0.095984i & 1.047871-0.095984i & 1.047871-0.095984i \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 b & \omega (m=4) & \omega (m=5) \\
0.90 & 1.022140-0.086417i & 1.040844-0.086424i \\
0.92 & 1.027075-0.088331i & 1.042160-0.088337i \\
0.94 & 1.032117-0.090245i & 1.043520-0.090249i \\
0.96 & 1.037256-0.092158i & 1.044926-0.092161i \\
0.98 & 1.042517-0.094071i & 1.046376-0.094073i \\
1.00 & 1.047871-0.095984i & 1.047871-0.095984i \\
\end{array}
\]

**TABLE II:** The fundamental \((n=0)\) quasinormal frequencies of electromagnetic field \((s=1)\) in the Schwarzschild black hole spacetime with a cosmic string for \(l = 5\).

Since the properties of the fundamental QNM frequencies for different \(l\) are similar in this background, we only list the QNM frequencies for \(l = 5\) in Table 1. From figures (1), (2) and (3), we clearly see that quasinormal frequencies for different spin \(s\) fields depend not only on the parameter \(b\) but also on the angular number \(m\), which is quite different from the QNMs disclosed in the general spherically symmetric backgrounds. For the
TABLE III: The fundamental \((n = 0)\) quasinormal frequencies of gravitational field \((s=2)\) in the Schwarzschild black hole spacetime with a cosmic string for \(l = 5\).

| \(b\) | \(\omega\) \(m = 0\) | \(\omega\) \(m = 1\) | \(\omega\) \(m = 3\) |
|------|-----------------|-----------------|-----------------|
| 0.90 | 0.911027-0.085386i | 0.911771-0.085388i | 0.975143-0.085534i |
| 0.92 | 0.931272-0.087283i | 0.931738-0.087403i | 0.981941-0.087403i |
| 0.94 | 0.951517-0.089181i | 0.951774-0.089182i | 0.989058-0.089271i |
| 0.96 | 0.971762-0.091078i | 0.971874-0.091079i | 0.996487-0.091139i |
| 0.98 | 0.992007-0.092976i | 0.992034-0.092976i | 1.004220-0.093006i |
| 1.00 | 1.012252-0.094873i | 1.012252-0.094873i | 1.012252-0.094873i |

| \(b\) | \(\omega\) \(m = 4\) | \(\omega\) \(m = 5\) |
|------|-----------------|-----------------|
| 0.90 | 0.993689-0.085571i | 1.012586-0.085608i |
| 0.92 | 0.997042-0.087435i | 1.012359-0.087466i |
| 0.94 | 1.000576-0.089297i | 1.012212-0.089322i |
| 0.96 | 1.004291-0.091157i | 1.012145-0.091175i |
| 0.98 | 1.008183-0.093016i | 1.012158-0.093025i |
| 1.00 | 1.012252-0.094873i | 1.012252-0.094873i |

fixed \(l, m\) and \(s\), both the real part and the absolute value of the imaginary part of quasinormal frequencies are almost linear functions of the parameter \(b\). This is not surprising because in terms of the Pöschl-Teller potential approximation it was found that the real and imaginary parts of quasinormal frequencies of the scalar, electromagnetic and gravitational fields for large multiple number \(l\) in the black hole with a cosmic string can be well approximated as \(\omega_R \sim \frac{b}{3\sqrt{3M}} \sqrt{(l+1) + \lambda}\) and \(\omega_I \sim -i\frac{b}{3\sqrt{3M}}(n + \frac{1}{2})\). We have also examined the QNMs dependence on the angular number \(m\) for fixed \(l\) and chosen \(b\). Results are shown in Fig.3, where we observed that with the increase of \(m\), both the real parts and absolute value of the imaginary parts of the QNM frequencies for electromagnetic and gravitational fields increase, while for scalar perturbation, the real part increases and the absolute value of the imaginary part decreases. The dependence on the angular quantum number \(m\) has not been observed for spherical cases before, it is similar to those in the Kerr black hole spacetime \[26\]. Compared with the scalar and electromagnetic perturbations, the gravitational perturbation has both smaller real part and smaller absolute value of the imaginary part of quasinormal frequencies. The gravitational perturbation can thus last longer than other perturbations, thus gravitational perturbation could be of more interesting to detect the string effect in this background.

**IV. LATE-TIME TAILS OF SCALAR, ELECTROMAGNETIC AND GRAVITATIONAL PERTURBATIONS IN THE SCHWARZSCHILD BLACK HOLE PIERCED BY A COSMIC STRING**

In this section, we will adopt the technique of spectral decomposition \[27\] to investigate the late-time tails of scalar, electromagnetic and gravitational fields in the background of the Schwarzschild black hole with a
FIG. 1: Dependence of the real part of quasinormal frequencies ($n = 0$) for scalar (the left), electromagnetic (the middle) and gravitational (the right) perturbations on the parameter $b$ for fixed $l = 5$ and chosen $m$.

cosmic string passing through it.

In terms of this method, the retarded Green’s function $G(r_s, r'_s; t)$, which determines the time evolution of a scalar field in the background spacetime, can be expressed as

$$G(r_s, r'_s; t) = \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} \tilde{G}(r_s, r'_s; \omega) e^{-i\omega t} d\omega,$$  \hfill (21)
FIG. 2: Dependence of the imaginary part of quasinormal frequencies ($n = 0$) for scalar (the left), electromagnetic (the middle) and gravitational (the right) perturbations on the parameter $b$ for fixed $l = 5$ and chosen $m$.

where $c$ is a positive constant and $\tilde{G}(r_*, r'_*; \omega)$ is defined by

$$
\tilde{G}(r_*, r'_*; \omega) = -\frac{1}{W(\omega)} \left[ \tilde{\Psi}_1(r_*, \omega) \tilde{\Psi}_2(r'_*, \omega), \quad r_* < r'_*; \right. \\
\left. \tilde{\Psi}_1(r'_*, \omega) \tilde{\Psi}_2(r_*, \omega), \quad r_* > r'_*, \right]
$$

(22)

with

$$
W(\omega) = W(\tilde{\Psi}_1, \tilde{\Psi}_2) = \tilde{\Psi}_1 \tilde{\Psi}_{2,r_\ast} - \tilde{\Psi}_2 \tilde{\Psi}_{1,r_\ast},
$$

(23)
FIG. 3: Dependence of the real part (the left) and the imaginary part (the right) of quasinormal frequencies \((n = 0)\) on the angular number \(m\) for fixed \(l = 5\) and \(b = 0.9\).

\(\tilde{\Psi}_1(r_*, \omega)\) and \(\tilde{\Psi}_2(r_*, \omega)\) are linearly independent solutions to the homogeneous equation

\[
\left[ \frac{d^2}{dr_*^2} + \omega^2 - V(r) \right] \tilde{\Psi}_i(r_*, \omega) = 0, \quad i = 1, 2.
\]  

(24)

It is by now well known that there exists a branch cut in \(\tilde{\Psi}_2\) placed along the negative imaginary \(\omega\)-axis and the contribution of the Green function \(G(r_*, r_*'; t)\) to late-time tail comes from the integral of \(\tilde{G}(r_*, r_*'; \omega)\) around this branch cut which is denoted by \(G_C(r_*, r_*'; t)\). Thus, in the study of late-time evolution of an
external field, we just need to consider $G^C(r_*, r'_*; t)$.

### A. Late-time behavior of the massless perturbations

Following Ref. [19], we can adopt the low-frequency approximation to study the asymptotic late-time behavior of the massless perturbational fields in the Schwarzschild black hole pierced by a cosmic string. Neglecting terms of the order $O(M^2/r^2)$ and the higher order terms, we can expend the wave equation (24) for the massless perturbational fields as a power series in $M/r$

\[
\left( \frac{d^2}{dr^2} + \omega^2 + \frac{4M'\omega^2}{r} - \frac{\rho^2 - \frac{1}{4}}{r^2} \right) \zeta(r, \omega) = 0,
\]

(25)

where $\zeta(r, \omega) = \sqrt{1 - \frac{2M}{br}} \tilde{\Psi}$, $M' = M/b$ and $\rho = \sqrt{(l + \frac{1}{2})^2 + \lambda_1}$. We can directly obtain that the asymptotic behavior of the massless scalar, electromagnetic and gravitational fields at timelike infinity in the background of the Schwarzschild black hole pierced by a cosmic string

\[
G^C(r_*, r'_*; t) = \frac{2^{2\rho}M[\Gamma(\rho + \frac{1}{2})]^2(-1)^{2\rho+1}\Gamma(2\rho+2)}{\pi b[\Gamma(2\rho+1)]^2}(r'_* r_*)^{\rho+1/2}t^{-2\rho-2}.
\]

(26)

It is easy to find that different from the usual spherical black hole, the late-time tails of the massless scalar, electromagnetic and gravitational fields in this background are related not only to the multiple moment $l$, but also to the parameter $b$, which has the connection to the linear mass energy of the cosmic string, the angular number $m$ and the spin $s$ of perturbational fields. From figure 4, we can obtain that for the fixed $m$, the larger the parameter $b$ is, the more slowly the perturbation decays. Moreover, figure 5 also tells us that for the fixed $b$, the larger the parameter $m$ is, the more quickly the perturbation dies out. From figure 6, we also find that the larger the spin $s$ is, the more slowly the perturbation decays, which means the gravitational field decays slower, which is consistent with the discussion above in the QNM investigation. These results are different from that in the usual Schwarzschild case with $b = 1$, where the damping exponent of the perturbational fields are independent of $m$ and $s$. 

FIG. 4: Graphs of $\ln G^C(r_s, r'_s; t)$ (the left $s = 0$, the middle $s = 1$, the right $s = 2$) versus $t$ for the change of $b$, with $l = 5$ and $m = 5$. Here, $M = 1, r_s = 10$ and $r'_s = 100$. The parameter $b$ makes the perturbations decay slowly.

FIG. 5: Graphs of $\ln G^C(r_s, r'_s; t)$ (the left $s = 0$, the middle $s = 1$, the right $s = 2$) versus $t$ for the change of $m$, with $l = 5$ and $b = 0.9$. Here, $M = 1, r_s = 10$ and $r'_s = 100$. The parameter $m$ makes the perturbations decay quickly.

FIG. 6: Graphs of $\ln G^C(r_s, r'_s; t)$ (the left $b = 0.9$, the middle $b = 0.93$, the right $b = 0.95$) versus $t$ for the change of $s$, with $l = 5$ and $m = 3$. Here, $M = 1, r_s = 10$ and $r'_s = 100$. The spin $s$ makes the perturbations decay slowly.
B. Late-time behavior of the massive perturbational field

Here we just consider the late-time tails of the massive scalar field because in modern physics the electromagnetic and gravitational fields are massless. As in the massless case, for the massive scalar field we will also evaluate the contribution of $G^c(r_*, r'_*; t)$ to the late-time tail. The only difference in the latter is the integral of the Green’s function $G(r_*, r'_*; t)$ around the branch cut performs in the interval $-\mu \leq \omega \leq \mu$ \cite{20} ($\mu$ is the mass of the scalar particle) rather than along the total negative imaginary $\omega$-axis. Assuming that both the observer and the initial data are situated far away from the black hole so that $r \gg M$, we can expand the wave equation \cite{21} for the massive scalar fields as a power series in $M/r$ and obtain (neglecting terms of the order $O(M^2/r^2)$ and the higher orders)

$$\left[ \frac{d^2}{dr^2} + \omega^2 - \mu^2 + \frac{4M'\omega^2 - 2M^2\mu^2}{r} - \frac{\rho^2 + \frac{1}{r^2}}{r^2} \right] \zeta(r, \omega) = 0. \quad (27)$$

For the massive scalar field in this background, the intermediate late-time tail (which is the tail in the range $M_b \ll r \ll t \ll bM/\mu^2$) has the behavior

$$G^c(r_*, r'_*; t) = \frac{(1 + e^{(2\rho+1)i\pi}) \Gamma(-2\rho)\Gamma(\frac{1}{2} + \rho)\Gamma(1 + \rho)\mu^\rho}{\pi\rho^2 - 3\rho - 2} \frac{\Gamma(2\rho)\Gamma(\frac{1}{2} - \rho)}{\Gamma(\frac{1}{2} - \rho)} \left( r'_* r_* \right)^{\frac{1}{2} + \rho} t^{-\rho - 1} \cos[\mu t - \frac{\pi(\rho + 1)}{2}]. \quad (28)$$

Comparing equation \cite{28} with equation \cite{20}, we find that the power-law tail of the massive scalar field at a fixed radius in the intermediate late-time depends not only on the structure parameter $b$ of the background metric, but also on the multiple number $l$ and the angular quantum number $m$ of the wave modes. Moreover, the intermediate late-time behavior of the massive scalar field is dominated by an oscillatory inverse power-law tail which decays slower than that of the massless case. Furthermore, figure 7 also tells us that with the

FIG. 7: The left and the right graphs show $\ln|G^c(r_*, r'_*; t)|$ versus $t$ for different $b$ and $m$, respectively. Here, $M = 1$, $r_* = 10$ and $r'_* = 100$. 
increase of $b$, the massive scalar field dies out slower. For the case $b \neq 1$, the perturbation field decays faster when $m$ increases.

We also obtain directly that the asymptotic late-time tail (which is caused by a resonance backscattering of spacetime curvature at very late-time $\mu t \gg b^2/(\mu M)^2$) can be described by

$$G^C(r_*, \rho_*; t) \simeq \frac{\mu}{2\sqrt{3}} (2\pi)^{-5/6} (M \mu / b)^{1/3} (\mu t)^{-5/6} \times \sin \left\{ \mu t - \frac{3}{2} (2\pi M / b)^{2/3} (\mu t)^{1/3} - \varphi_s(\omega_0) - \frac{\pi}{4} \right\}, \quad (29)$$

where $\omega_0 \sim \mu \sqrt{1 - [2\pi M / (tb)]^{2/3}}$ is the frequency of wave at the saddle-point. This equation tells us that the decay rate of the asymptotic late-time tail of the massive scalar field is still in the form $t^{-5/6}$, which is the same as that in the usual black hole spacetimes and can be regarded as a quite general feature for the late-time decay of massive scalar field. Moreover, the oscillation has the period $2\pi/\mu$ which is modulated by two types of long-term phase shift. The first term $2\pi (M / b)^{2/3} (\mu t)^{1/3}$ represents a monotonously increasing phase shift and the second one $\varphi_s(\omega_0)$ denotes a periodic phase shift. Both of them depend on the parameter $b$, which means that in this background the phase shifts also depend on the the mass per length of the string.

\[V. \ \text{CONCLUSIONS AND DISCUSSIONS}\]

The metric of the Schwarzschild black hole with a straight string passing through it is locally identical to the Schwarzschild metric, so that the presence of the string will not imprint in the motion of test particles since locally the geodesics of test particles will not be changed.

In this paper we have studied QNMs and the late-time tail behaviors of scalar, electromagnetic and gravitational perturbations in the Schwarzschild black hole spacetime pierced by a cosmic string. We have observed physical signatures by the presence of the cosmic string. With the increase of the mass per length of the string, we observed that both the real part and the absolute value of the imaginary part of quasinormal frequencies decrease. The print of the cosmic string in the QNMs can be read if we compare with that of the usual Schwarzschild black hole. Furthermore we observed that with a cosmic string passing through the Schwarzschild black hole background, quasinormal frequencies depend also on the angular quantum number $m$, which has not been observed in the usual spherical black hole background before. With the increase of the angular quantum number $m$, both the real parts and the magnitude of the imaginary parts of the QNM frequencies for electromagnetic and gravitational fields increase, however for the scalar perturbation, the real part increases while the magnitude of the imaginary part decreases. Compared with other perturbations, the gravitational perturbation can last longer which could be more interesting to observation.
We have extended our study to the late-time tail behaviors of scalar, electromagnetic and gravitational perturbations in the Schwarzschild black hole pierced by a cosmic string. We also observed the physical prints in terms of the cosmic string. Although the decay rate for the asymptotic late-time tail of the massive scalar field is still $t^{-5/6}$, which looks the same as that in the usual black hole cases, the influence due to the presence of the cosmic string appears in the late-time tail in the massless perturbations and the intermediate late-time tail for the massive scalar field. We found that the damping exponents of the late-time tail for the massless perturbations and the intermediate late-time tail for the massive scalar field increase with the increase of the linear mass density of the cosmic string $\rho_s$ and the angular quantum numbers $m$. Moreover, we find that due to the presence of linear mass density, the decay rates of the massless perturbations also depend on the spin $s$ of fields. Compared with other perturbational fields, the gravitational field decays slower at timelike infinity in this background.

It would be of great interest to get more insights in the observational signatures of the cosmic string. This would depend a lot on the preciseness of the gravitational wave observation in the future if we expect the QNM or the late time tail to tell us the its signature. There are some important questions waiting to be addressed, such as what the constraints on the mass density of the string are, how close to one $b$ has to be observationally consistent etc. Answers to these interesting questions are called for.

Acknowledgments

This work was partially supported by NNSF of China, Ministry of Education of China and Shanghai Educational Commission. S. B. Chen’s work was partially supported by the National Basic Research Program of China under Grant No. 2003CB716300 and the Scientific Research Fund of Hunan Normal University under Grant No.22040639.

[1] H. P. Nollert, Class. Quant. Grav. 16, R159 (1999).
[2] K. D. Kokkotas and B. G. Schmidt, Living Rev. Rel. 2, 2 (1999).
[3] B. Wang, Braz. J. Phys. 35, 1029 (2005).
[4] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D 62, 024027 (2000).
[5] B. Wang, C.Y. Lin, and E. Abdalla, Phys. Lett. B 481, 79 (2000); B. Wang, C. Molina, and E. Abdalla, Phys. Rev. D 63, 084001 (2001); J.M. Zhu, B. Wang, and E. Abdalla, Phys. Rev. D 63, 124004 (2001).
[6] V. Cardoso and J. P. S. Lemos, Phys. Rev. D 63, 124015 (2001); V. Cardoso and J. P. S. Lemos, Phys. Rev. D 64, 084017(2001); E. Berti and K. D. Kokkotas, Phys. Rev. D 67, 064020 (2003); V. Cardoso and J. P. S. Lemos, Class. Quant. Grav. 18, 5257 (2001); E. Winstanley, Phys. Rev. D 64, 104010 (2001); J. Cristomoto, S. Lepe and J. Saavedra, Class. Quant. Grav. 21, 2801-2810 (2004) ; S. Lepe, F. Mendez, J. Saavedra, L. Vergara, Class. Quant. Grav. 20 2417-2428 (2003).
[7] D. Birmingham, I. Sachs, S. N. Solodukhin, Phys. Rev. Lett. 88 (2002) 151301; D. Birmingham, Phys. Rev. D 64, 064024 (2001).
[8] B. Wang, E. Abdalla and R. B. Mann, Phys. Rev. D 65, 084006 (2002); J. S. F. Chan and R. B. Mann, Phys. Rev. D 59, 064025 (1999).
[9] S. Musiri, G. Siopsis, Phys. Lett. B 576, 300-313 (2003); R. Aros, C. Martinez, R. Troncoso, J. Zanelli, Phys. Rev. D 67, 044014 (2003); A. Nunez, A. O. Starinets, Phys. Rev. D 67, 124013 (2003).
[10] E. Abdalla, B. Wang, A. Lima-Santos and W. G. Qiu, Phys. Lett. B 538, 435 (2002); E. Abdalla, K. H. Castello-Branco and A. Lima-Santos, Phys. Rev. D 66, 104018 (2002).
[11] S. Hod, Phys. Rev. Lett. 81, 4293 (1998); A. Corichi, Phys. Rev. D 67, 087502 (2003); L. Motl, Adv. Theor. Math. Phys. 6, 1135-1162 (2003); L. Motl and A. Neitzke, Adv. Theor. Math. Phys. 7, 307-330 (2003); A. Maassen van den Brink, J. Math. Phys. 45, 327 (2004); O. Dreyer, Phys. Rev. Lett. 90, 08130 (2003); G. Kunstatter, Phys. Rev. Lett. 90, 161301 (2003); N. Andersson and C. J. Howls, Class. Quant. Grav. 21, 1623-1642 (2004); V. Cardoso, J. Natario and R. Schiappa, J. Math. Phys. 45, 4698-4713 (2004).
[12] V. Cardoso and J. P. S. Lemos, Phys. Rev. D 67, 084020 (2003); K. H. C. Castello-Branco and E. Abdalla, gr-qc/0309090; Jose Natario and Ricardo Schiappa, Adv. Theor. Math. Phys. 8, 1001-1131 (2004).
[13] B. Wang, C. Y. Lin and C. Molina, Phys. Rev. D 70, 064025 (2004).
[14] J. Natario, R. Schiappa, Adv. Theor. Math. Phys. 8 (2004) 1001.
[15] B. Linet, gr-qc/9904044, Class. Quant. Grav. 16, 2947 (1999).
[16] M. Aryal, L. H. Ford and A. Vilenkin, Phys. Rev. D 67, 084014 (1998).
[17] R. Ruffini and J. A. Wheeler, Phys. Today 24(1), 30 (1971); C. W. Misner, K. S. Thorne, and J. A. Wheeler, *gravitation* (Freeman, San Francisco, 1973).
[18] R. H. Price, Phys. Rev. D 5, 2419 (1972).
[19] S. Hod and T. Piran, Phys. Rev. D 58, 024017 (1998); L. Barack, Phys. Rev. D 61, 024026 (2000); L. M. Burko and G. Khanna, Phys. Rev. D 67, 084014 (2003); E. S. C. Ching, P. T. Leung, W. M. Suen, and K. Young, Phys. Rev. D 52, 2118 (1995); P. R. Brady, S. Droz, and S. M. Morsink Phys. Rev. D 58, 084034 (1998); C. Gundlach, R. H. Price, and J. Pullin, Phys. Rev. D 49, 883 (1994); L. M. Burko and G. Khanna, Phys. Rev. D 70, 044018 (2004); V. Cardoso, S. Yoshida, O. J. C. Dias, and J. P. S. Lemos, Phys. Rev. D 68, 061503(R) (2003).
[20] H. Koyama and A. Tomimatsu, Phys. Rev. D 63, 064032 (2001); Phys. Rev. D 64, 044014 (2001); R. Moderski and M. Rogatko, Phys. Rev. D 64, 044024 (2001); Phys. Rev. D 63, 084014 (2001); Phys. Rev. D 72, 044027 (2005); S. Hod and T. Piran, Phys. Rev. D 58, 044018 (1998).
[21] S. Hod, Phys. Rev. D 58, 104022 (1998); L. Barack and A. Ori, Phys. Rev. Lett. 82, 4388 (1999); W. krivan, Phys. Rev. D 60, 101501(R) (1999); Q. Y. Pan and J. L. Jing, Chin. Phys. Lett. 21, 1873 (2004).
[22] S. A. Teukolsky, Phys. Rev. Lett. 29, 1114 (1972); Astrophys. J. 185,635 (1973); S. A. Teukolsky and W. H. Press, Astrophys. J. 193, 443 (1974).
[23] G. F. Torres del Castillo, J. Math. Phys. 29, 2078 (1988); J. Math. Phys. 30, 446 (1989); U. Khanal, Phys. Rev. D 28, 1291 (1983).
[24] R. A. Breuer, M. P. Ryan and S. Waller, Proc. R. Soc. Lond. A. 358, 71-86 (1977); E. Newman and R. Penrose, J. Math. Phys. 3, 566 (1966); J. N. Goldberg, elf, J. Math. Phys. 8, 2155 (1967).
[25] B. F. Schutz and C. M. Will, Astrophys. J. Lett. Ed. 291, L33 (1985); S. Iyer and C. M. Will, Phys. Rev. D 35, 3621 (1987); S. Iyer, Phys. Rev. D 35, 3632 (1987).
[26] E. Berti and V. Cardoso, Phys. Rev. D 74, 104020 (2006).
[27] E. W. Leaver, Proc. R. Soc. Lond. A. 402 285 (1985) ; Phys. Rev. D 34 384 (1986).