**LENGTH AND DEPTH**

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**Abstract.** We prove that one can construct a sequence \( \langle B_i : i < \kappa \rangle \) of Boolean algebras such that \( \prod_{i<\kappa} \text{Depth}(B_i)/D < \text{Depth}(\prod_{i<\kappa} B_i/D) \) and even \( \prod_{i<\kappa} \text{Length}(B_i)/D < \text{Depth}(\prod_{i<\kappa} B_i/D) \). The proof is carried out in ZFC and thus answers Problem 30 from [7].

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0. Introduction

The purpose of this paper is to solve an open problem about Depth of Boolean algebras, from [7]. We refer to this monograph for a comprehensive account of Depth (and Length), and here we only spell out the pertinent definitions.

**Definition 0.1.** Depth and Length of Boolean algebras. Let $B$ be a Boolean Algebra.

1. $\text{Length}(B) = \sup \{ \theta : \exists A \subseteq B, |A| = \theta, A \text{ is linearly-ordered by } <_B \}.$
2. $\text{Length}^+(B) = \sup \{ \theta^+ : \exists A \subseteq B, |A| = \theta, A \text{ such that } A \text{ is linearly-ordered by } <_B \}.$
3. $\text{Depth}(B) = \sup \{ \theta : \exists A \subseteq B, |A| = \theta, A \text{ is well-ordered by } <_B \}.$
4. $\text{Depth}^+(B) = \sup \{ \theta^+ : \exists A \subseteq B, |A| = \theta, A \text{ is well-ordered by } <_B \}.$

We shall focus on an ultraproduct construction for Length and Depth, but the question of ultraproducts is more general. Let $I$ be any invariant of Boolean algebras. Suppose we are given a sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras and a uniform ultrafilter $D$ over $\kappa$. One can assign a set-theoretic value to these objects in the following two alternating ways. In the first way one computes $I(B_i)$ for every $i < \kappa$, yielding a sequence of cardinals. Then one computes the product $\prod_{i<\kappa} I(B_i)/D$. In the second way, one creates a new Boolean algebra $B = \prod_{i<\kappa} B_i/D$, called the product algebra. Then one computes $I(B)$.

These two ways are similar in the sense that both of them are based on an ultraproduct construction and on application of the invariant $I$, the only difference being the order of the operations. In the first way, we apply $I$ to each Boolean algebra and then take ultraproduct. In the second way we begin with the ultraproduct and only then apply $I$.

However, there is a metamathematical distinction between these ways. The first way is the set theoretical line of thinking, in which we first of all get rid of the algebraic structure by translating each algebra into a cardinality. The rest of the game becomes purely set theoretical, namely a computation of the size of an ultraproduct of cardinals. The second way reflects an algebraic attitude. As a first step we create a new algebraic structure, namely the product algebra. Having a new algebra we ask about its properties, namely we compute the value of $I$ over this algebra. So these two courses reflect different attitudes, and a natural question is the comparison of the outcome of these alternatives.

The comparison question is examined in Monk, [7] with respect to every cardinal invariant. For many invariants, it is consistent that $\prod_{i<\kappa} I(B_i)/D \leq I(\prod_{i<\kappa} B_i/D)$ whenever $D$ is a uniform ultrafilter over $\kappa$. Hence the interesting question will be the possibility to produce a strict inequality of the form $\prod_{i<\kappa} I(B_i)/D < I(\prod_{i<\kappa} B_i/D)$. This question is sharpened if one wishes to prove such inequality in ZFC.
Question 0.2. [7] Problem 30].
Is an example with \( \text{Depth}(\prod_{i \in I} A_i/F) < \prod_{i \in I} \text{Depth}(A_i)/F \) possible in ZFC?

In the main result of this paper we have a singular cardinal \( \mu \) such that \( \text{Length}(B_i) \leq \mu \) (and hence \( \text{Depth}(B_i) \leq \mu \)) for every \( i < \kappa \). We require that \( \kappa < \theta = \text{cf}(\mu) < \mu \), and \( \mu^\kappa = \mu \) (so necessarily \( \mu \) is of uncountable cofinality). If \( D \) is a uniform ultrafilter over \( \kappa \) then \( \prod_{i < \kappa} \text{Depth}(B_i)/D \leq \prod_{i < \kappa} \text{Length}(B_i)/D \leq \mu^\kappa = \mu \). On the other hand, \( \text{Depth}(\prod_{i < \kappa} B_i/D) = \mu^+ \), thus the answer to the above problem is positive.

Notice that the gap here is one cardinality, and in ZFC this is the best possible gap, as exemplified by the constructible universe. Likewise, [4] shows that no ZFC example is available when \( \mu > \text{cf}(\mu) = \omega \). A ZFC example for Length appeared in [9], and an opposite inequality of the form \( \text{Length}(\prod_{i < \kappa} B_i/D) < \prod_{i < \kappa} \text{Length}(B_i)/D \) appeared in [6] using large cardinals. In another direction, inequalities for \( \text{Depth}^+ \) and \( \text{Length}^+ \) were proved in ZFC, in [3]. We indicate that larger gaps for Depth and Length can be forced, and we hope to show this in a subsequent work.

For a general background in Boolean algebras we refer to the excellent monograph [5]. Here we quote only the following version of Sikorski’s extension theorem which will be used several times:

Theorem 0.3. If \( \mathcal{B}_0 \) is a Boolean algebra freely generated from \( \{x_\gamma : \gamma \in \lambda\} \) except the inequalities mentioned in \( \Gamma \subseteq \{(x_\alpha \leq x_\beta) : \alpha, \beta \in \lambda\} \), \( f : \{x_\gamma : \gamma \in \lambda\} \to \mathcal{B}_1 \) is homomorphic and \( (x_\alpha \leq x_\beta) \in \Gamma \Rightarrow f(x_\alpha) \leq_{\mathcal{B}_1} f(x_\beta) \) then \( f \) has a homomorphic extension \( \hat{f} : \mathcal{B}_0 \to \mathcal{B}_1 \).

The rest of the paper contains two additional sections. The first one is purely combinatorial and some background in pcf theory is introduced. The second is devoted to the algebraic construction and contains the main result.
1. Pcf theory

For a general survey of pcf theory we suggest [1]. For advanced theorems we refer to [8]. We include here the basic definitions and facts to be used throughout the paper. A set of cardinals $a$ is called progressive if it composed of regular cardinals and $|a| < \min(a)$. Pcf theory analyzes the spectrum of cofinalities of the form $tcf(\prod a/J)$ when $J$ is an ideal over $a$. By $J_a^{bd}$ we denote the ideal of bounded subsets of $a$. The shorthand tcf stands for true cofinality. A central notion in pcf theory is the collection of all true cofinalities for a specific $a$. Assume $a$ is a progressive set, and let $pcf(a) = \{\lambda : (\exists J)(tcf(\prod a, < J) = \lambda)\}$. Occasionally we require that $\lim_1(a) = \mu$, which means that $a \cap \partial \in I$ whenever $\partial < \mu$.

If $a$ is progressive then $pcf(a)$ has a last member, denoted by $\max pcf(a)$. We call $a$ an interval if every regular cardinal between $\min(a)$ and $\sup(a)$ belongs to $a$. The following is a fundamental theorem of pcf theory:

**Theorem 1.1.** The no-holes theorem.
If $a$ is a progressive interval then $pcf(a)$ is an interval as well.

Another central notion which we need is the pseudo power (abbreviated as pp). The name indicates that this notion may serve as a parallel to the classical notion of the power operation with respect to singular cardinals. If $a$ is a progressive set and $\kappa < |a|$ then $pcf_\kappa(a) = \bigcup\{pcf(b) : b \subseteq a$ and $b$ is unbounded in $\kappa\}$. If $\lambda > cf(\lambda) = \kappa$ then $pp_\kappa(\lambda) = \sup\{tcf(\prod a, J) : \lambda = \sup(a), |a| \leq \kappa, J \supseteq J_a^{bd}\}$. By $pp_\kappa^+(\lambda)$ we denote the first regular cardinal which has no representation as the true cofinality of $\prod a/J$ for some ideal $J$ over $a$.

The celebrated paradox of Achilles and the tortoise comes from the school of Parmenides. One point is recurrent in modern treatment to this paradox, and it is connected to the separation between infinitude and boundedness. The paradox describes an infinite sum of intervals in which Achilles tries to take over the tortoise and fails. The Greeks concluded that he will never take over the tortoise, hence the paradox. However, infinitude need not imply this conclusion. The sum of intervals in which Achilles cannot take over the tortoise is bounded, despite its infinite description. If one takes a broader look at the race, beyond the bounded point of sum of intervals then Achilles takes over the tortoise (but see [2]).

A progressive set is a tortoise set if $\partial \in a$ implies $\max pcf(a \cap \partial) < \partial$. The progress of Achilles is mirrored here in $pcf(a \cap \partial)$ while $\max pcf(a \cap \partial)$ reflects the position of the tortoise. The *Achilles and the tortoise* theorem says that if $a$ is progressive then there is no tortoise set $b \subseteq pcf(a)$ of size $|a|^+$. As in the original paradox, if one takes a broader look at $pcf(a)$ then eventually Achilles takes over.

A progressive set $a$ has the tortoise property when $a$ is a tortoise set. Though the size of a tortoise set within a given set of the form $pcf(a)$ is limited by $|a|^+$, one can create, in advance, a tortoise set of any size. We shall
use a theorem which ensures the tortoise property (under some assumptions on the pseudo power, known to be consistent):

**Claim 1.2. pp and the tortoise property.**
Assume that:
(a) \( \theta = \text{cf}(\mu) < \mu < \lambda = \text{cf}(\lambda) \).
(b) \( \text{pp}^+(\mu) > \lambda \).
(c) \( \alpha < \mu \Rightarrow \alpha^\theta < \mu \).

Then there are a progressive set \( a \) and an ideal \( J \) which satisfy the tortoise property.

**Proof.**
Choose an ideal \( J_1 \) on \( \theta \), and a sequence \( \bar{\lambda} = \langle \lambda_\varepsilon : \varepsilon < \theta \rangle \) of regular cardinals so that \( \lim_{\bar{\lambda}}(\lambda_\varepsilon) = \mu \) and \( \text{tcf}(\prod \lambda_\varepsilon, <, J_1) = \lambda \). This can be done since \( \text{pp}^+(\mu) > \lambda \) (actually, it gives \( \text{tcf}(\prod \lambda_\varepsilon, <, J_1) \geq \lambda \), but the no-holes theorem provides an ideal which gives exactly \( \lambda \)).

By [8], Ch. VIII (Theorem 1.1) we may assume that \( J_1 = \mathcal{J}^\text{bd} \) and \( \lambda \) is an increasing sequence such that \( \text{max \ pcf}(\lambda_j : j < \varepsilon) < \lambda_\varepsilon \) for every \( \varepsilon < \theta \). Define \( a = \{ \lambda_\varepsilon : \varepsilon < \theta \} \), and conclude the proof.

The fact that \( a \) is a tortoise set is needed, actually, for the small cardinality of \( T \) in the following claim:

**Claim 1.3. Assume \( a \) satisfies the tortoise property with \( J \), and (a) – (c) from Claim 1.2 hold.**
There is a sequence \( \bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \prod a \) which is \( J \)-increasing and cofinal in \( (\prod a, <, J) \), such that \( T_{\bar{f}} = \{ f_\alpha \upharpoonright (a \cap \theta) : \alpha < \lambda \} \) has less than \( \partial \)-many members.

The proof of this claim can be found in [8], Chapter II. The last concept that we need is \( \text{Pr}_1 \). It is a combinatorial principle which asserts that there is a very variegated coloring in some sense. Formally, \( \text{Pr}_1(\lambda, \mu, \kappa, \theta) \) says that there exists a coloring \( c : [\lambda]^2 \rightarrow \kappa \) such that for every sequence \( \langle t_{\alpha, \xi} : \alpha \in \mu, \xi < \xi \rangle \) of ordinals of \( \lambda \) where \( \xi < \theta \) and \( \alpha \neq \beta \Rightarrow t_{\alpha} \cap t_{\beta} = \emptyset \) and for every color \( \gamma \in \kappa \) there are \( \alpha < \beta < \lambda \) so that \( c \upharpoonright (t_{\alpha} \times t_{\beta}) = \{ \gamma \} \). One can think of \( \text{Pr}_1 \) as the polarized version of the negative square brackets relation.

Suppose that \( \theta = \text{cf}(\mu) < \mu \), \( a = \{ \lambda_i : i < \theta \} \subseteq \text{Reg} \) and \( \text{lim}_J(a) = \mu \) for some ideal \( J \) over \( \theta \). Suppose further that \( \kappa < \theta \) and \( \{ a_j : j \in \kappa \} \) is a disjoint partition of \( a \) into \( \kappa \)-many \( J \)-positive sets. Let \( \bar{f} = \langle f_\alpha : \alpha \in \lambda \rangle \) be a scale which exemplifies the fact that \( \lambda = \text{tcf}(\prod a, J) \) and notice that \( \lambda > \mu \). We define the derived coloring \( c : [\lambda]^2 \rightarrow \kappa \) as follows. Given \( \alpha < \beta < \lambda \) let \( i_{\alpha, \beta} \) be the first ordinal \( i < \theta \) such that \( f_\alpha(\lambda_j) < f_\beta(\lambda_j) \) for every \( j \in [i, \theta) \). Define \( c(\alpha, \beta) = \ell \) iff \( \lambda_{i_{\alpha, \beta}} \in a_\ell \). We shall make use of the following theorem:
Theorem 1.4. Assume that:

(a) $\kappa < \theta = \text{cf}(\mu) < \mu$.
(b) $\mu^+ = \text{tcf}(\prod a, J)$, $J$ is the ideal of bounded subsets of $\theta$.
(c) $\{a_j : j \in \kappa\}$ is a disjoint partition of $a$ into $\kappa$-many $J$-positive sets.

Then there exists a scale $\bar{f} = (f_\alpha : \alpha \in \mu^+)$ such that:

(8) $\bar{f}$ exemplifies $\mu^+ = \text{tcf}(\prod a, J)$.
(9) $\bar{f}$ satisfies the statement of Claim 1.3.
(10) The coloring $c$ derived from $\bar{f}$ exemplifies $\text{Pr}_1(\mu^+, \mu^+, \kappa, \aleph_0)$.

Proof.

A scale which satisfies the first two requirements is known to exist, hence it suffices to show that every scale satisfies the third requirement as well. Actually, one can prove the stronger statement $\text{Pr}_1(\lambda, \lambda, \theta, \theta)$ where $\lambda = \text{tcf}(\prod a, J)$. For this end, we decompose $a$ into $\theta$ many disjoint sets $a_i$, each of which is unbounded in $\sup(a)$. We define a function $h : a \to \theta$ by:

$$h(\partial) = i \iff \partial \subseteq a_i.$$

For any pair of ordinals $\alpha < \beta < \lambda$ we let $t(\alpha, \beta) = \sup \{\partial \in a : f_\alpha(\partial) \geq f_\beta(\partial)\}$ and $t^+(\alpha, \beta) = \min \{\partial \in a : \partial \supseteq t(\alpha, \beta)\}$.

We define now a symmetric coloring over the pairs of $\lambda$ with $\theta$-many colors. Given $\alpha < \beta < \lambda$ we let:

$$c(\alpha, \beta) = h(t^+(\alpha, \beta)).$$

We claim that $c$ exemplifies the property $\text{Pr}_1(\lambda, \lambda, \theta, \theta)$. For proving this fact, suppose that $\xi < \theta$. Suppose, further, that $\langle \alpha_\zeta : \zeta < \xi \rangle$ is an increasing sequence of ordinals of $\lambda$ for every $\beta < \lambda$, and these sequences are disjoint. Fix a color $i(*) < \theta$. We shall try to find two ordinals $\beta_0 < \beta_1 < \lambda$ such that $\forall \zeta, \eta < \xi, c(\alpha_{\beta_0 \zeta}, \alpha_{\beta_1 \eta}) = i(*)$. This means that we must find some $\partial \in a_{i(*)}$ such that $f_{\alpha_{\beta_1 \zeta}}$ dominates $f_{\alpha_{\beta_0 \zeta}}$ from $\partial$ onwards, whenever $\zeta, \eta < \xi$.

Let $\chi$ be a sufficiently large regular cardinal. Choose an elementary submodel $M < \mathcal{H}(\chi)$ such that:

(a) $\lambda, \alpha, \xi \in M$ as well as any other relevant object like the scale and the array of sequences.
(b) $\xi \subseteq M$ and $a \subseteq M$.
(c) $|M| < \mu$.

We may assume, without loss of generality, that $\beta \leq \alpha_{\beta_0 \zeta}$ for every $\beta < \lambda$ and each $\zeta < \xi$. For every $\partial \in a$ let $g(\partial) = \sup(M \cap \partial)$ if $\sup(M \cap \partial) < \partial$ and $g(\partial) = 0$ otherwise. Notice that $g \in \prod a$, and $g(\partial) > 0$ for almost every $\partial \in a$ (apart from a bounded set) since $|M| < \mu$.

Choose an ordinal $\beta_1 \in \lambda$ such that $\beta_1 > \sup(M \cap \lambda)$ and $g <_J f_{\beta_1}$. Since $\beta_1 \leq \alpha_{\beta_0 \zeta}$ for every $\zeta < \xi$ and $(f_\alpha : \alpha \in \lambda)$ is $<_J$-increasing, $g <_J f_{\beta_1 \zeta}$ for every $\zeta < \xi$. Choose $\partial_\zeta \in a$ such that $\partial_\zeta \subseteq \partial \in a \Rightarrow g(\partial) < f_{\beta_1 \zeta}(\partial)$, for each $\zeta < \xi$. Fix an element $\partial(0) \in a$ such that $\forall \zeta < \xi, \partial_\zeta < \partial(0)$. This can be done since $\xi < \theta = \text{cf}(\theta)$. Recall that each $a_i$ is unbounded in $a$, and
this holds in particular at $a_\iota(*)$. Hence one can choose $\partial(1) \in a_\iota(*)$ such that $\partial(0) < \partial(1)$ and $\sup(M \cap \partial) < \partial$ for every $\partial \in a$ such that $\partial \geq \partial(1)$.

For every $\beta \in \lambda$ we define a function $f^*_\beta \in \prod_a$ such that $f^*_\beta \leq_J f^*_\beta$ as follows. Given $\partial \in a$ let $f^*_\beta(\partial) = \min\{f_{\alpha, \beta, \xi}(\partial) : \xi < \xi\}$. The fact that $f^*_\beta \leq_J f^*_\beta$ comes from the inequality $\beta \leq \alpha_{\beta, \xi}$ which holds at every $\xi < \xi$ and the assumption that $\xi < \theta = cf(\theta)$.

Let $b = \{\partial \in a : \sup\{f^*_\beta(\partial) : \beta \in \lambda\} = \partial\}$. It is easy to see that $b = J_a$. In particular, $\partial(1) \in a_\iota(*) \subseteq a = b$. This means that $\sup\{f^*_\beta(\partial(1)) : \beta \in \lambda\} = \partial(1)$, so we can choose $\gamma \in \lambda$ such that $\sup\{f_{\alpha_{\beta, \gamma}}(\partial(1)) : \gamma < \xi\} < f^*_\gamma(\partial(1))$. Denote $f^*_\gamma(\partial(1))$ by $\delta$.

Let $N = Sk^{H(\lambda)}(M \cup \{\delta\})$. We claim that there exists an ordinal $\beta_0 \in N \cap \lambda$ such that $N \models f^*_\beta(\partial(1)) = \delta$. Indeed, $H(\lambda)$ knows that there is such an ordinal, since $f^*_\gamma(\partial(1)) = \delta$, so by elementarity we can choose $\beta_0$ for which $N$ satisfies the same statement. We claim that the pair $(\beta_0, \beta_1)$ does the job and confirms $Pr_1(\lambda, \lambda, \theta, \theta)$. For proving this fact, assume that $\zeta, \eta < \xi$. By the above considerations, $f_{\alpha_{\beta_1, \eta}}(\partial(1)) < f^*_\beta(\partial(1)) = \delta$. Likewise, $\delta = f^*_\beta(\partial(1)) \leq f_{\alpha_{\beta_0, \zeta}}(\partial(1))$, the last inequality follows from the definition of $f_{\beta_0}^*$. Summing up, $f_{\alpha_{\beta_1, \eta}}(\partial(1)) < f_{\alpha_{\beta_0, \zeta}}(\partial(1))$ and hence $\partial(1) \leq t(\alpha_{\beta_0, \zeta}, \alpha_{\beta_1, \eta})$.

Assume now that $\partial \in a, \partial > \partial(1)$. Notice that $\sup(M \cap \partial) = \sup(N \cap \partial)$ in this case. Since $\beta_1 > \sup(M \cap \lambda)$ we see that $\beta_1 > \sup(N \cap \lambda)$ and in particular $\beta_1 > \beta_0$. Remark that $\alpha_{\beta_0, \zeta} \in N \cap \lambda$ since $\beta_0, \zeta \in N \cap \lambda$ and the sequence of the sequences $\langle \alpha_{\beta, \gamma} : \zeta < \xi\rangle$ is an element of $M < N$. Hence $\alpha_{\beta_0, \zeta} < \beta_1 \leq \alpha_{\beta_1, \eta}$.

Applying this to $\partial > \partial(1)$ we see that $f_{\alpha_{\beta, \gamma}}(\partial) \in N$, so $f_{\alpha_{\beta, \gamma}}(\partial) \in N \cap \partial = M \cap \partial < \sup(M \cap \partial) = g(\partial)$. But we have seen already that $g(\partial) < f_{\alpha_{\beta_1, \eta}}(\partial)$, as $\partial > \partial(1) > \partial(0)$ and by the choice of $\partial(0)$. Summing up, $f_{\alpha_{\beta, \gamma}}(\partial) < f_{\alpha_{\beta_1, \eta}}(\partial)$ which means that $\partial(1) = t(\alpha_{\beta_0, \zeta}, \alpha_{\beta_1, \eta}) = t^*(\alpha_{\beta_0, \zeta}, \alpha_{\beta_1, \eta})$ (here $\sup = \max$ is exemplified by $\partial(1)$). By the very definition of our coloring we see that $c(\alpha_{\beta_0, \zeta}, \alpha_{\beta_1, \eta}) = h(t^*(\alpha_{\beta_0, \zeta}, \alpha_{\beta_1, \eta})) = h(\partial(1)) = i(*)$. Since $\zeta, \eta < \xi$ were arbitrary, the proof is accomplished.
2. Length, Depth and other animals

There is a connection between the existence of large linearly ordered subsets of a Boolean algebra and the structure of automorphisms of this algebra. If $A \subseteq B$ is linearly ordered and $x, y \in A$ then there is no automorphism $f : B \rightarrow B$ such that $f(x) = y \land f(y) = x$. Hence large linearly ordered subsets imply many restrictions on automorphisms and homomorphisms defined over a given Boolean algebra. Similarly, if we have a variety of automorphisms then the size of linearly ordered subsets decreases.

If one wishes to have many homomorphisms then a free Boolean algebra (or any other algebraic structure) is the best way to ensure it. If $B$ is a free Boolean algebra then $\text{Length}(B) \leq \aleph_0$ no matter how large is the size of $B$. In this case there are many homomorphisms and hence only countable linearly ordered sets. One can control the number of homomorphisms and the size of linearly ordered sets by imposing some constraints on the freeness of the algebra. Our algebras will be free, but we also wish to limit the possible homomorphisms, so we define a set of desired inequalities and we take algebras which are freely generated except some prescribed list of inequalities.

The purpose of these inequalities is to make sure that there are no large linearly ordered sets in each algebra. On the other hand, we need a large linearly ordered set in the product algebra. We define partial orders which determine the order at each algebra, and set our inequalities. Along the definition, many of these partial orders are satisfied at every index (this will make sure that a $D$-positive set of indices will always obey these partial orders, when $D$ is a uniform ultrafilter), and hence in the product algebra there will be a large linearly ordered set. On the other hand, at each specific index we will keep a constraint on the pertinent partial order. This will make sure that there is no very large linearly ordered set at each algebra. Moreover, linearly ordered can be replaced by well-ordered.

The idea has been used in [3] for Depth$^+$ and Length$^+$ but in a different manner. The basic theorem to be proved below is about Length$^+$ as well. It is phrased in such a way that enables us to derive an immediate consequence about Length and Depth.

**Theorem 2.1.** Assume that:

(8) $\kappa < \theta = \text{cf}(\mu) < \mu$.

(2) $\alpha < \mu \Rightarrow \alpha^+ < \mu$.

(3) $\text{pp}^+(\mu) > \lambda = \text{cf}(\lambda)$.

(7) $\lambda$ can be represented as true cofinality with $J_0^{\text{bd}}$.

Then there is a sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras, such that:

(a) For every $i < \kappa$, $\text{Length}^+(B_i) \leq \lambda$.

(b) $\text{Depth}^+(B) > \lambda$ where $B = \prod_{i<\kappa} B_i/D$ is the product algebra, $D$ being any uniform ultrafilter over $\kappa$. 

Proof.
Since $\lambda < \text{pp}^+(\mu)$ one can find a sequence $\lambda = \langle \lambda_i : i < \theta \rangle$ and an ideal $I$ over $\theta$ such that:

(a) $\lambda_i = \text{cf}(\lambda_i) < \mu$ for every $i < \theta$.
(b) $\lim_I(\lambda) = \mu$.
(c) $\text{tcf}(\prod_{i<\theta} \lambda_i; I) \geq \lambda$.

We may assume that $\text{tcf}(\prod_{i<\theta} \lambda_i; I) = \lambda$ by virtue of Theorem 1.1. Without loss of generality, $\lambda$ is increasing, $I = J_0^{\text{bd}}$ and max $\text{pcf}\{\lambda_j : j < i\} < \lambda_i$ for every $i < \theta$.

Let $a = \{\lambda_i : i < \theta\}$, and decompose $a$ into $(a_i : i < \kappa)$ such that each $a_i$ is $I$-positive and $i < \kappa \Rightarrow \{j \in \theta : \lambda_j \in a_i\}$ is a stationary subset of $\theta$. Define $J = \{b \subseteq a : \langle j < \theta : \lambda_j \in b \rangle \in I\}$. So $J$ is an ideal over $a$ and $J \supseteq J_0^{\text{bd}}$.

Further, $\text{tcf}(\prod_{i<\theta} \lambda_i, I) = \text{tcf}(\prod a, J) = \lambda$. Using Claim 2, we choose a sequence $f = (f_\alpha : \alpha \in \lambda) \subseteq \prod a$, increasing and cofinal in $(\prod a, J)$, such that $|T_\alpha| < \partial$ for every $\alpha \in a$, where $T_\alpha = \{f_\alpha \upharpoonright (a \cap \partial) : \alpha \in \lambda\}$.

Fix any uniform ultrafilter $D$ over $\kappa$. For $\alpha < \beta < \lambda$ let $\partial_{\alpha \beta} = \min\{\partial \in a : \partial' > \partial \Rightarrow f_\alpha(\partial') < f_\beta(\partial')\}$. We define a binary relation $<_i$ over $\lambda + 1 \cup \{-1\}$, for each $i < \kappa$, as follows:

$$\alpha <_i \beta \Leftrightarrow$$

$$(\alpha < \beta \land \partial_{\alpha \beta} \in \bigcup \{a_j : j < i\}) \lor$$

$$(\alpha = -1 \land \beta \in \lambda + 1) \lor$$

$$(\alpha \in \lambda \cup \{-1\} \land \beta = \lambda)$$

It is routine to verify that each $<_i$ is a partial order. Likewise, if $\alpha < \beta$ then $\{i \in \kappa : \alpha <_i \beta\} \in D$, since it is an end-segment of $\kappa$ and $D$ is uniform.

Let $B_i$ be the Boolean algebra generated freely from $\{x_\alpha : \alpha \in \lambda + 1 \cup \{-1\}\}$ except the inequalities mentioned in $\Gamma_i$, where $\Gamma_i = \{(x_\alpha \leq x_\beta) : \alpha \leq_i \beta\} \cup \{x_{-1} = 0, x_1 = 1\}$. Put another way, $B_i \models x_\alpha < x_\beta \Leftrightarrow \alpha <_i \beta$. Let $B$ be the product algebra $\prod_{i<\kappa} B_i / D$.

It is easy to see that $\text{Depth}^+(B) > \lambda$. For this, we point to a well-ordered subset of $B$ of size $\lambda$. For every $\alpha \in \lambda$ let $c_\alpha \in \prod_{i<\kappa} B_i$ be the constant function with value $x_\alpha$, and let $c_\alpha / D \in B$ be its $D$-equivalence class. We may assume that $\alpha \neq \beta \Rightarrow c_\alpha / D \neq c_\beta / D$, hence the set $A = \{c_\alpha / D : \alpha \in \lambda\} \subseteq B$ is of size $\lambda$. Now if $\alpha < \beta$ then $\{i < \kappa : B_i \models x_\alpha < x_\beta\} = \{i < \kappa : \alpha <_i \beta\} \in D$, being an end-segment of $\kappa$. Consequently, $\alpha < \beta \Rightarrow c_\alpha / D <_B c_\beta / D$. It follows that $A$ is a well ordered subset of $B$ and hence $\text{Depth}(B) \geq \lambda$, so $\text{Depth}^+(B) > \lambda$.

The other side of the coin is that $\text{Length}^+(B_i) \leq \lambda$ for every $i < \kappa$. For proving this, fix an ordinal $i \in \kappa$ and assume toward a contradiction that $\lambda < \text{Length}^+(B_i)$ for some $i < \kappa$. Fix a linearly ordered set $\{a_\alpha : \alpha < \lambda\} \subseteq B_i$ which exemplifies this statement.

Each $a_\alpha$ is an element of the Boolean algebra $B_i$ and hence expressible as $\sigma_\alpha(x_{\gamma(\alpha,0)}, \ldots, x_{\gamma(\alpha,\text{ord}(\alpha)-1)})$, where $\sigma_\alpha$ is a Boolean term and $\gamma(\alpha, \ell) \in$
\[ \lambda + 1 \cup \{-1\} \]. But \( \lambda = \text{cf}(\lambda) > \aleph_0 \), and the number of Boolean terms is \( \aleph_0 \). So without loss of generality \( \sigma_\alpha = \sigma \) and \( n(\alpha) = n \) for every \( \alpha \in \lambda \), where \( \sigma \) is a fixed Boolean term and \( n \) is a fixed natural number. We can write \( a_\alpha = \sigma(x_{\gamma(\alpha,0)}, \ldots, x_{\gamma(\alpha,n-1)}) \) for every \( \alpha \in \lambda \), and we may assume that \( \langle \gamma(\alpha, \ell) : \ell < n \rangle \) is always an increasing sequence of ordinals.

For every \( \alpha \in \lambda \) there is a set \( p_\alpha \subseteq n \times n \) such that \( (k, \ell) \in p_\alpha \) iff \( \gamma(\alpha, k) <_i \gamma(\alpha, \ell) \). Being a subset of \( n \times n \), it is a finite set. Hence for \( \lambda \) many ordinals we have \( p_\alpha = p \) for some fixed \( p \subseteq n \times n \). By concentrating on these ordinals we may assume that \( \alpha < \beta < \lambda \Rightarrow p_\alpha = p_\beta \). This means that \( \gamma(\alpha, k) <_i \gamma(\alpha, \ell) \) iff \( \gamma(\beta, k) <_i \gamma(\beta, \ell) \) for every \( \alpha, \beta \in \lambda \).

Since \( f \) is an increasing sequence in \( (\prod a, J) \), for every \( \alpha \in \lambda \) there exists \( \partial_\alpha \in a \) such that the elements of \( \langle f(\gamma(\alpha, \ell)) \mid (a \cap \partial_\alpha) : \ell < n \rangle \) are pairwise distinct. Since \( |a| = \mu < \text{cf}(\lambda) = \lambda \) we may assume without loss of generality that \( \partial_\alpha = \partial \) for some fixed \( \partial \in a \) and each \( \alpha \in \lambda \).

We apply the Delta-system lemma to the collection of \( \lambda \) finite sequences \( \langle \gamma(\alpha, \ell) : \ell < n \rangle \). We can assume now that there exists \( m < n \) such that \( \ell < m \Rightarrow \gamma(\alpha, \ell) = \gamma(\ell) \) and if \( \alpha < \beta \) then \( \gamma(\alpha, n-1) < \gamma(\beta, m) \). Recall that \( i \in \kappa \) is fixed, and choose \( j \in (i, \kappa] \) baring in mind that \( a_j \in J^+ \). From Theorem 1.4 one can find \( \alpha < \beta < \lambda \) such that for every \( k, \ell \in [m, n) \) it is true that \( \partial_{\gamma(\alpha,k)\gamma(\beta,\ell)} \in a_j \).

Define \( S = \{ \gamma(\alpha, \ell), \gamma(\beta, \ell) : \ell < n \} \cup \{-1, 1\} \), so \( S \) is a finite subset of \( \lambda + 1 \cup \{-1\} \). Observe that \( (S, <_i) \) is a finite partial order with \(-1, \lambda \in S \) and \( s \in S \Rightarrow -1, s \leq_i \lambda \). Hence any partial ordering \( P \supseteq (S, <_i) \) projects onto \( (S, <_i) \). In particular, there exists a function \( h : \lambda + 1 \cup \{-1\} \rightarrow S \) such that \( h \upharpoonright S = \text{id}_S \) and \( \gamma_0 \leq_i \gamma_1 \Rightarrow h(\gamma_0) \leq h(\gamma_1) \).

Let \( B_S \) be the (finite) Boolean algebra generated freely from \( \{ x_\gamma : \gamma \in S \} \) except the relations in \( \Gamma_S = \{ (x_\alpha \leq x_\beta) : \alpha \leq_i \beta, \alpha, \beta \in S \} \). Define a mapping \( f : \{ x_\gamma : \gamma \in \lambda \} \rightarrow B_S \) by \( f(x_\gamma) = x_{h(\gamma)} \). By Theorem 0.3 there is a homomorphism \( \hat{f} : B_i \rightarrow B_S \) such that \( f \subseteq \hat{f} \) and hence \( \hat{f}(x_\gamma) = x_\gamma \) whenever \( \gamma \in S \).

Now define a function \( g : S \rightarrow S \) by letting, for every \( \ell < n \), \( g(\gamma(\alpha, \ell)) = \gamma(\beta, \ell) \) and \( g(\gamma(\beta, \ell)) = \gamma(\alpha, \ell) \). Notice that \( g \) is a permutation of \( S, g \circ g = \text{id}_S \) and \( g \) maps \( \Gamma_S \) onto itself, namely \( (x_\alpha \leq x_\beta) \in \Gamma_S \Rightarrow (g(x_\alpha) \leq g(x_\beta)) \in \Gamma_S \). This follows from the fact that \( \gamma(\alpha, k) <_i \gamma(\alpha, \ell) \) iff \( \gamma(\beta, k) <_i \gamma(\beta, \ell) \) and the fact that \( \gamma(\alpha, k) \) and \( \gamma(\beta, \ell) \) are \( <_i \)-incomparable for every \( k, \ell \).

By another application of Theorem 0.3 there exists an automorphism \( \hat{g} : B_S \rightarrow B_S \) which satisfies \( \hat{g}(x_\gamma) = x_{g(\gamma)} \). Observe that \( \hat{g} \) respects Boolean terms in the following sense:

\[
\hat{g}(\sigma(\ldots, x_{\gamma(\alpha, \ell)}, \ldots)) = \sigma(\ldots, x_{\gamma(\beta, \ell)}, \ldots).
\]

Back to the chosen pair of ordinals \( \alpha < \beta \), since \( \{ a_\gamma : \gamma \in \lambda \} \) is \( <_{B_1} \)-linearly ordered we know that \( (a_\alpha <_{B_1} a_\beta) \lor (a_\beta <_{B_1} a_\alpha) \). Without loss of generality \( a_\alpha <_{B_1} a_\beta \), so \( B_1 \models \sigma(\ldots, x_{\gamma(\alpha, \ell)}, \ldots) < \sigma(\ldots, x_{\gamma(\beta, \ell)}, \ldots) \).
Since \( \hat{f} : B_i \to B_S \) is homomorphic, we see that:
\[ B_S \models \sigma(\ldots, x_{\gamma(\alpha, \ell)}, \ldots) < \sigma(\ldots, x_{\gamma(\beta, \ell)}, \ldots). \]
Likewise, \( \hat{g} : B_S \to B_S \) is homomorphic and hence we also have:
\[ B_S \models \sigma(\ldots, x_{\gamma(\beta, \ell)}, \ldots) < \sigma(\ldots, x_{\gamma(\alpha, \ell)}, \ldots). \]
This contradiction accomplishes the proof of the theorem.

The above theorem deals with \( \text{Length}^+ \) and \( \text{Depth}^+ \), but we can deduce the following result about \( \text{Depth} \) and \( \text{Length} \):

**Theorem 2.2.** Assume that:

1. \( \kappa < \theta = \text{cf}(\mu) < \mu \).
2. \( \alpha < \mu \Rightarrow \alpha^+ < \mu \).
3. \( \lambda = \mu^+ \).

Then there exists a sequence of Boolean algebras \( \langle B_i : i < \kappa \rangle \) such that:

1. \( |B_i| = \lambda \) for every \( i < \kappa \).
2. \( \text{Depth}(B_i) \leq \text{Length}(B_i) \leq \mu \) for every \( i < \kappa \).
3. If \( D \) is a uniform ultrafilter over \( \kappa \) and \( B = \prod_{i < \kappa} B_i / D \) is the product algebra then \( \text{Depth}(B) \geq \lambda \).
4. \( \prod_{i < \kappa} \text{Length}(B_i) / D = \mu \), so \( \prod_{i < \kappa} \text{Length}(B_i) / D < \text{Depth}(B) \).

**Proof.**

Since \( \lambda = \mu^+ \), \( pp_\mu^+(\mu) > \lambda \) and hence Theorem 2.1 applies. Let \( \langle B_i : i < \kappa \rangle \) be a sequence of Boolean algebras as asserted in Theorem 2.1. For every \( i < \kappa \) we have \( \text{Length}^+(B_i) \leq \lambda \), hence (recalling that \( \lambda = \mu^+ \)) \( \text{Length}(B_i) \leq \mu \) and a fortiori \( \text{Depth}(B_i) \leq \mu \).

On the other hand, \( \text{Depth}^+(B) > \lambda \) so \( \text{Depth}(B) \geq \lambda \). It follows that \( \prod_{i < \kappa} \text{Length}(B_i) / D \leq \mu^\kappa = \mu \) (the equality \( \mu^\kappa = \mu \) is due to (2)), and \( \mu < \lambda = \text{Depth}(B) \) so the proof is accomplished.

A detailed examination of the construction yields accurate values of \( \text{Length} \) and \( \text{Depth} \) for \( B \) and each \( B_i \) in the above theorems. This is expressed by the following two lemmata.

**Lemma 2.3.** In the construction of Theorem 2.1, if \( \lambda = \text{cf}(\lambda) < \mu^{+\omega} \) then \( \text{Depth}(B) = \text{Length}(B) = \lambda \).

**Proof.**

We have seen already that the set \( A = \{ c_\alpha / D : \alpha \in \lambda \} \subseteq B \) is of size \( \lambda \) and well ordered by \( \leq_B \), so \( \text{Depth}(B) \geq \lambda \). We claim that if \( \lambda < \mu^{+\omega} \) then \( |B| = \lambda \) and hence \( \text{Length}(B) \leq \lambda \) as well. For this notice that \( \mu^\kappa = \mu \) and hence by induction on \( \lambda \in (\mu, \mu^{+\omega}) \) we have \( \lambda^\kappa = \lambda \). Since \( |B_i| = \lambda \) for every \( i < \kappa \) we see that \( \lambda \leq |B| = \prod_{i < \kappa} |B_i / D| \leq \lambda^\kappa = \lambda \), so we are done.

On the other hand:
Lemma 2.4. In the construction of Theorem 2.2, \(\text{Length}(B_i) = \mu\) (and even \(\text{Depth}(B_i) = \mu\)) for every \(i < \kappa\).

Proof. Fix an ordinal \(i \in \kappa\) and define the following set:
\[
A = \{\partial \in a : \sup\{f_\alpha(\partial) : \alpha \in \lambda\} = \partial\}.
\]
We may assume that \(A = a \mod J\), as this requirement can be added to the choice of \(\bar f\). In particular, \(\sup(A) = \mu\).

Fix any \(\partial \in A\) such that \(\partial \in \bigcup\{a_j : j < i\}\), and notice that these elements are unbounded in \(\mu\). Choose a sequence of ordinals \(\langle \alpha_\varepsilon : \varepsilon \in \partial\rangle\) so that \(\langle f_{\alpha_\varepsilon}(\partial) : \varepsilon \in \partial\rangle\) is an increasing sequence of ordinals in \(\partial\). Recall that \(|\{f_\alpha \upharpoonright (a \cap \partial) : \alpha < \lambda\}| \leq \max\ pcf(a \cap \partial) < \partial\), and \(\partial = \text{cf}(\partial)\). Hence without loss of generality there is some fixed \(g \in \prod(a \cap \partial)\) such that \(\varepsilon < \partial \Rightarrow f_{\alpha_\varepsilon} \upharpoonright (a \cap \partial) = g\).

Now if \(\varepsilon < \zeta < \partial\) and \(\partial \in \bigcup\{a_j : j < i\}\) then, by the choice of \(g\), \(\partial\) is the first point in which \(f_{\alpha_\varepsilon}(\partial) \neq f_{\alpha_\zeta}(\partial)\). Moreover, \(f_{\alpha_\varepsilon}(\partial) < f_{\alpha_\zeta}(\partial)\) by the choice of \(\langle f_{\alpha_\varepsilon}(\partial) : \varepsilon \in \partial\rangle\) and hence \(\alpha_\varepsilon < \alpha_\zeta\). By the definition of the order in \(B_i\) we see that \(x_{\alpha_\varepsilon} \leq_{B_i} x_{\alpha_\zeta}\) so \(\langle x_{\alpha_\varepsilon} : \varepsilon \in \partial\rangle\) implies that \(\text{Length}(B_i) \geq \partial\) and even \(\text{Depth}(B_i) \geq \partial\).

As \(\sup(A) = \mu\) we conclude that \(\text{Length}(B_i) \geq \mu\) (for Depth, ditto). On the other hand, in Theorem 2.2 we have seen that \(\text{Length}^+(B_i) \leq \lambda = \mu^+\). It follows that any linearly ordered subset of \(B_i\) is of size at most \(\mu\), so \(\text{Depth}(B_i) \leq \text{Length}(B_i) \leq \mu\), and we are done.
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