Cramér asymptotics for finite time first passage probabilities of general Lévy processes

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Abstract

We derive the exact asymptotics of \(P(\sup_{u \leq t} X(u) > x)\) if \(x\) and \(t\) tend to infinity with \(x/t\) constant, for a general Lévy process \(X\) that admits exponential moments. The proof is based on a renewal argument and a two-dimensional renewal theorem of Höglund [9].

1 Introduction

The study of boundary crossing probabilities of Lévy processes has applications in many fields, including ruin theory (see e.g. Rolski et al. [13] and Asmussen [2]), queueing theory (see e.g. Borovkov [6] and Prabhu [11]), statistics (see e.g. Siegmund [15]) and mathematical finance (see e.g. Roberts and Shortland [12]).

As in many cases closed form expressions for (finite time) first passage probabilities are either not available or intractable, a good deal of the literature has been devoted to logarithmic or exact asymptotics for first passage probabilities, using different techniques. Martin-Löf [10] and Collamore [7] derived large deviation results for first passage probabilities of a general class of processes. Employing two-dimensional renewal theory and asymptotic properties of ladder processes, respectively, Höglund [9] and von Bahr [3] obtained exact asymptotics for ruin probabilities of the classical risk process (see also Asmussen [2]). Bertoin and Doney [5] generalised the classical Cramér-Lundberg approximation (of the perpetual ruin probability of a classical risk process) to general Lévy processes.

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In this paper we obtain the exact asymptotics of the finite time ruin probability \( P(\tau(x) \leq t) \), where \( \tau(x) = \inf\{t \geq 0 : X(t) > x\} \), for a general Lévy process \( X(t) (X(0) = 0) \), if \( x \) and \( t \) jointly tend to infinity in fixed proportion, generalising Arfwedson [1] and Höglund [9] who treated the case of a classical risk process. The proof is based on an embedding of the ladder process of \( X \) and a two-dimensional renewal theorem of Höglund [9].

The remainder of the paper is organized as follows. In Section 2 the main result is presented, and its proof is given in Section 3.

## 2 Main result

Let \( X \) be a Lévy process with non-monotone paths that satisfies

\[
E[e^{\alpha_0 X(1)}] < \infty \quad \text{for some } \alpha_0 > 0,
\]

and denote by \( \tau(x) = \inf\{t \geq 0 : X(t) > x\} \) the first crossing time of \( x \). We exclude the case that \( X \) is a compound Poisson process with non-positive infinitesimal drift, as this corresponds to the random walk case which has already been treated in the literature.

The law of \( X \) is determined by its Laplace exponent \( \psi(\theta) = \log E[e^{\theta X(1)}] \) that is well defined on the maximal domain \( \Theta = \{\theta \in \mathbb{R} : \psi(\theta) < \infty\} \). Restricted to the interior \( \Theta^0 \), the map \( \theta \mapsto \psi(\theta) \) is convex and differentiable, with derivative \( \psi'(\theta) \). Moreover, \( \psi'(0+) = E[X(1)] \) if \( E[|X(1)|] < \infty \). By the strict convexity of \( \psi \), it follows that \( \psi' \) is strictly increasing on \( (0, \infty) \) and we denote by \( \Gamma : \psi'(0, \infty) \to (0, \infty) \) its right-inverse function.

Associated to the measure \( P \) is the exponential family of measures \( \{P(c) : c \in \Theta\} \) defined by their Radon-Nikodym derivatives

\[
\left. \frac{dP(c)}{dP} \right|_{\mathcal{F}_t} = \exp(cX(t) - \psi(c)t).
\]

It is well known that under this change of measure \( X \) is still a Lévy process and its new Laplace exponent satisfies

\[
\psi^{(c)}(\alpha) = \psi(\alpha + c) - \psi(c).
\]

Related to \( X \) and its running supremum are the local time \( L \) of \( X \) at its supremum, its right-continuous inverse \( L^{-1} \) and the upcrossing ladder

\footnote{For \( \theta \in \Theta \setminus \Theta^0 \), \( \psi'(\theta) \) is understood to be \( \lim_{\eta \to \theta, \eta \in \Theta^0} \psi'(\eta) \).}
process \( H \) respectively. The Laplace exponent \( \kappa \) of the bivariate (possibly killed) subordinator \( (L^{-1}, H) \),

\[
e^{-\kappa(\alpha,\beta)t} = E[e^{-\alpha L^{-1} t - \beta H t} 1_{(L^{-1} t < \infty)}],
\]

is related to \( \psi \) via the Wiener-Hopf factorisation identity

\[
u - \psi(\theta) = k\kappa(u,-\theta)\hat{\kappa}(u,\theta), \quad u \geq 0, \theta \in \Theta^o,
\]

for some constant \( k > 0 \) where \( \hat{\kappa} \) is the Laplace exponent of the dual ladder process. Refer to Bertoin [4, Ch. VI] for further background on the fluctuation theory of Lévy processes.

Bertoin and Doney [5] showed that, if the Cramér condition holds, that is \( \gamma > 0 \), where

\[
\gamma := \sup\{\theta \in \Theta : \psi(\theta) = 0\},
\]

the Cramér-Lundberg approximation remains valid for a general Lévy process:

\[
\lim_{x \to \infty} e^{\gamma x} P[\tau(x) < \infty] = C_\gamma,
\]

where \( C_\gamma \geq 0 \) is positive if and only if \( E[e^{\gamma X(1)}|X(1)|] < \infty \) and is then given by \( C_\gamma = \beta_\gamma/|\gamma m_\gamma| \), where

\[
\beta_\gamma = -\log P[H_1 < \infty], \quad m_\gamma = E[e^{\gamma H_1} H_1 1_{(H_1 < \infty)}].
\]

Further, Doob’s optional stopping theorem implies the following bound:

\[
e^{\gamma x} P(\tau(x) < \infty) = E^{(\gamma)} [e^{-\gamma (X(\tau(x)) - x)} 1_{(\tau(x) < \infty)}] \leq 1.
\]

The result below concerns the asymptotics of the finite time ruin probability \( P(\tau(x) \leq t) \) when \( x, t \) jointly tend to infinity in fixed proportion. For a given proportion \( v \) the rate of decay is either equal to \( \gamma vt \) or to \( \psi^*(v)t \), where \( \psi^* \) is the convex conjugate of \( \psi \):

\[
\psi^*(u) = \sup_{\alpha \in \mathbb{R}} (\alpha u - \psi(\alpha)).
\]

We restrict ourselves to Lévy processes satisfying the following condition

\[
\sigma > 0 \text{ or the Lévy measure is non-lattice,} \quad (H)
\]

where \( \sigma \) denotes the Gaussian coefficient of \( X \). Recall that a measure is called non-lattice if its support is not contained in a set of the form \( \{a + bh, h \in \mathbb{Z}\} \), for some \( a, b > 0 \). Note that \( (H) \) is satisfied by any Lévy process whose Lévy measure has infinite mass.

We write \( f \sim g \) if \( \lim_{x,t \to \infty, x=vt+o(t^{1/2})} f(x,t)/g(x,t) = 1. \)
Theorem 1 Assume that \((H)\) holds. Suppose that \(0 < \psi'(\gamma) < \infty\) and that there exists a \(\Gamma(v) \in \Theta^0\) such that \(\psi'(\Gamma(v)) = v\). If \(x\) and \(t\) tend to infinity such that \(x = vt + o(t^{1/2})\) then

\[
P(\tau(x) \leq t) \sim \begin{cases} C_\gamma e^{-cx}, & \text{if } 0 < v < \psi'(\gamma), \\ D_v t^{-1/2} e^{-\psi^*(v)t}, & \text{if } v > \psi'(\gamma), \end{cases}
\]

with \(C_0 = 1\) and \(D_v\) given by

\[
D_v = \frac{-v \log E[e^{-\eta_vL^{-1}}\delta_{(L^{-1} < \infty)}]}{\eta_v E[e^{\tilde{\Phi}(v)H_1 - \eta_vL^{-1}H_1\delta_{(L^{-1} < \infty)}}]} \times \frac{1}{\Gamma(v)\sqrt{2\pi \eta_v'}},
\]

where \(\eta_v = \psi(\Gamma(v))\).

Remark 1 (a) For a spectrally negative Lévy process the joint exponent of the ladder process is given by \(\kappa(\alpha, \beta) = \beta + \tilde{\Phi}(\alpha)\) \((\alpha, \beta \geq 0)\), where \(\tilde{\Phi}(\alpha)\) is the largest root of \(\psi(-\theta) = \alpha\), and thus

\[
D_v = \overline{D}_v := \frac{v}{\psi(\Gamma(v)) \sqrt{2\pi \eta_v'}}\big|_{\eta_v = \psi(\Gamma(v))}, \quad C_0 = 1.
\]

Indeed,

\[
D_v = \overline{D}_v \times \frac{\kappa(\eta_v, 0)}{\Gamma(v)\beta \frac{\partial}{\partial \beta} \kappa(\eta_v, \beta)|_{\beta = -\Gamma(v)} \exp\{-\kappa(\eta_v, -\Gamma(v))\}} = \overline{D}_v \times \frac{1}{\exp\{-\Phi(\eta_v) + \Gamma(v)\}} = \overline{D}_v
\]

since \(\Phi(\eta_v) = \Gamma(v)\).

(b) If \(X\) is spectrally positive, \(\kappa(\alpha, \beta) = [\alpha - \psi(-\beta)]/\tilde{\Phi}(\alpha) - \beta\) (see e.g. [4, Thm VII.4]), where \(\tilde{\Phi}(\alpha)\) is the largest root of \(\psi(-\theta) = \alpha\) and we find that

\[
D_v = \frac{\Gamma(v) + \tilde{\Gamma}(v)}{\Gamma(v)\Gamma(v)} \frac{1}{\sqrt{2\pi \eta_v'}}\big|_{\eta_v = \psi(\Gamma(v))}, \quad C_0 = \frac{\psi'(0)}{\psi'(\gamma)},
\]

where \(\tilde{\Gamma}(v) = \sup\{\theta : \psi(-\theta) = \psi(\Gamma(v))\}\), recovering formulas that can be found in Arfwedson [1] and Feller [8] respectively, for the case of a classical risk process.
Remark 2 Heuristically, in the case $v > \psi'(\gamma)$, the asymptotics in Thm. 1 can be regarded as a consequence of the central limit theorem, that is, under the tilted measure $P^{\Gamma(v)}$, asymptotically
\begin{align*}
\frac{\tau(x) - x/v}{\omega \sqrt{x}}
\end{align*}
follows a standard normal distribution, where by (2.3) and choice of $\Gamma(v)$,
\begin{align*}
\omega^2 &= \frac{\operatorname{Var}(\Gamma(v))[X_1]}{(E(\Gamma(v))[X_1])^3} = \frac{\psi'(\Gamma(v))''(0)}{(\psi'(\Gamma(v))')(0) \psi''(\Gamma(v))} = \frac{\psi''(\Gamma(v))}{\psi''(\Gamma(v)) v^3}.
\end{align*}
This explains why the asymptotics remain valid if $x$ deviates $o(x^{1/2}) = o(t^{1/2})$ from the line $vt$.

In the boundary case $v = \psi'(\gamma)$, in which case $E(\Gamma(v)[\tau(x)]) = t$, the exact asymptotics of $P(\tau(x) \leq t)$ may depend on the way in which $x/t$ tends to $v$. Note that this case is excluded from Theorem 1.

Remark 3 In the case $0 < v < \psi'(\gamma)$, the asymptotics in Theorem 1 are a consequence of the law of large numbers. To see why this is the case, note that $e^{\gamma x} P(\tau(x) \leq t) = e^{\gamma x} P(\tau(x) < \infty) - e^{\gamma x} P(t < \tau(x) < \infty)$, where the first term tends to $C_\gamma$ in view of (2.7), while for the second term the Markov property and (2.8) imply that
\begin{align*}
& e^{\gamma x} P(t < \tau(x) < \infty) \\
&= \int_{-\infty}^{\infty} P(\tau(x) > t, X(t) \in dy) e^{\gamma y} e^{\gamma (x-y)} P(\tau(x - y) < \infty) \\
&\leq \int_{-\infty}^{\infty} P(X(t) \in dy) e^{\gamma y} = P^{(\gamma)}(X(t) \leq x),
\end{align*}
which tends to 0 as $t$ tends to infinity in view of the law of large numbers since $E^{(\gamma)}[X(t)] = tv'(\gamma) > x$. The proof below deals with the case that $v > \psi'(\gamma)$.

3 Proof of Theorem 1

The idea of the proof is to lift asymptotic results that have been established for random walks by Höglund [9] and Arfwedson [11] to the setting of Lévy processes by considering suitable random walks embedded in the Lévy process (more precisely, in its ladder process). We first briefly recall these results following the Höglund [9] formulation.
3.1 Review of Höglund’s random walk asymptotics

Let \((S, R) = \{(S_i, R_i), i = 1, 2, \ldots\}\) be a (possibly killed) random walk starting from \((0, 0)\) whose components \(S\) and \(R\) have non-negative increments, and consider the crossing probabilities

\[
G_{a,b}(x, y) = P(N(x) < \infty, S_{N(x)} > x + a, R_{N(x)} \leq y + b),
\]

\[
K_{a,b}(x, y) = P(N(x) < \infty, S_{N(x)} > x + a, R_{N(x)} \geq y + b),
\]

where \(a \geq 0, b \in \mathbb{R}\) and \(N(x) = \min\{n : S_n > x\}\). Let \(F\) denote the (possibly defective) distribution function of the increments of the random walk with joint Laplace transform \(\phi\) and set \(F_{(u,v)}(dx, dy) = e^{-ux-uy} F(dx, dy) / \phi(u,v)\).

Let

\[
V(\zeta) = E_{\zeta}[(R_1 E_{\zeta}[S_1] - S_1 E_{\zeta}[R_1])^2] / E_{\zeta}[S_1]^3
\]

for \(\zeta = (\xi, \eta)\) where \(E_{\zeta}\) denotes the expectation w.r.t. \(F_{\zeta}\).

For our purposes it will suffice to consider random walks that satisfy the following non-lattice assumption (the analogue of the non-lattice assumption in one dimension):

The additive group spanned by the support of \(F\) contains \(\mathbb{R}^2_+\). \(\text{(G)}\)

Specialised to our setting Prop. 3.2 in Höglund (1990) jointly with the remark given on p. 380 therein read as follows:

**Proposition 1** Assume that \((G)\) holds, and that there exists a \(\zeta = (\xi, \eta)\) with \(\phi(\zeta) = 1\) such that \(v = E_{\zeta}[S_1] / E_{\zeta}[R_1]\), where \(\phi\) is finite in a neighbourhood of \(\zeta\) and \((0, \eta)\). If \(x, y\) tend to infinity such that \(x = vy + o(y^{1/2}) > 0\) then it holds that

\[
G_{a,b}(x, y) \sim D(a, b)x^{-1/2}e^{x\xi+y\eta} \quad \text{if} \quad \eta > 0,
\]

\[
K_{a,b}(x, y) \sim D(a, b)x^{-1/2}e^{x\xi+y\eta} \quad \text{if} \quad \eta < 0,
\]

for \(a \geq 0, b \in \mathbb{R}\), where \(D(a, b) = C(a, b) \cdot (2\pi V(\zeta))^{-1/2}\), with \(V(\zeta) > 0\) and

\[
C(a, b) = \frac{1}{|\eta| E_{\zeta}[S_1]} e^{bn} \int_a^\infty P_{\zeta}(S_1 \geq x) e^{\xi x} dx.
\]

3.2 Embedded random walk

Denote by \(e_1, e_2, \ldots\) a sequence of independent \(\exp(q)\) distributed random variables and by \(\sigma_n = \sum_{i=1}^n e_i, \text{ with } \sigma_0 = 0\), the corresponding partial sums,
and consider the two-dimensional (killed) random walk \(\{(S_i, R_i), i = 1, 2, \ldots\}\) starting from \((0, 0)\) with step-sizes distributed according to

\[ F(q)(dt, dx) = P(H_{\sigma_1} \in dx, L_{\sigma_1}^{-1} \in dt), \]

and write \(G^{(q)}\) for the corresponding crossing probability

\[ G^{(q)}(x, y) = G_{0,0}(x, y) = P(N(x) < \infty, R_{N(x)} \leq y). \]

Note that \(F(q)\) is a probability measure that is defective precisely if \(X\) drifts to \(-\infty\), with Laplace transform \(\phi\) given by

\[ \phi(u, v) = \int \int e^{-ut-vx} F(q)(dt, dx) = \frac{q}{q - \kappa(u, v)}. \]

The key step in the proof is to derive bounds for \(P(\tau(x) \leq t)\) in terms of crossing probabilities involving the random walk \((S, R)\):

**Lemma 1** Let \(M, q > 0\). For \(x, t > 0\) it holds that

\[ G^{(q)}(x, t) \leq P(\tau(x) \leq t) \leq G^{(q)}(x, t + M)/h(0-, M), \]

where \(h(0-, M) = \lim_{x \uparrow 0} h(x, M)\), with \(h(x, t) \equiv P(H_{\sigma_1} > x, L_{\sigma_1}^{-1} \leq t)\).

**Proof:** Let \(T(x) = \inf\{t \geq 0 : H_t > x\}\) and note that \(\tau(x) = L_{T(x)}^{-1}\). By applying the Markov property it follows that

\[ P(\tau(x) \leq t) = P(T(x) < \infty, L_{T(x)}^{-1} \leq t) \]

\[ = \sum_{n=1}^{\infty} P(\sigma_{n-1} \leq T(x) < \sigma_n, L_{T(x)}^{-1} \leq t) \]

\[ = \sum_{n=1}^{\infty} P(H_{\sigma_{n-1}} \leq x, H_{\sigma_n} > x, L_{T(x)}^{-1} \leq t) \]

\[ = \sum_{n=1}^{\infty} \int \int P(H_{\sigma_{n-1}} \in dy, L_{\sigma_{n-1}}^{-1} \in ds) \]

\[ \times P(H_{\sigma_1} > x - y, L_{T(x-y)}^{-1} \leq t - s) \]

\[ = \sum_{n=0}^{\infty} F^{(q)^n} \ast f(x, t) = (U \ast f)(x, t), \]

where \(U = \sum_{n=0}^{\infty} F^{(q)^n}\), \(f(x, t) = P(H_{\sigma_1} > x, L_{T(x)}^{-1} \leq t)\) and \(\ast\) denotes convolution. Following a similar reasoning it can be checked that

\[ G^{(q)}(x, t) = U \ast h(x, t). \]

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In view of (3.4) and (3.5), the lower bound in (3.1) follows since

$$f(x,t) \geq h(x,t),$$

taking note of the fact that $H_{\sigma_1} > x$ precisely if $T(x) < \sigma_1$, while the upper bound in (3.1) follows by observing that for fixed $M > 0$,

$$h(x,t + M) \geq P(H_{\sigma_1} > x, L_{T(x)}^{-1} \leq t)P(L_{T(x)}^{-1} \leq M) = f(x,t)h(0-, M),$$

where we used the strong Markov property of $L^{-1}$ and the lack of memory property of $\sigma_1$.

\[\square\]

Applying Höglund’s asymptotics in Proposition 1 yields the following result:

**Lemma 2** Let the assumptions of Proposition 1 hold true. If $x, t \to \infty$ such that for $v > \psi'(\gamma)$ we have $x = vt + o(t^{1/2})$ then

$$G^q(x, t + M) \sim D_{q,M}t^{-1/2}e^{-\psi^*(v)t}, \quad M \geq 0,$$

where $D_{q,M} = \sqrt{2\pi\psi''(\Gamma(v))}C_{q,M}$ with

$$C_{q,M} = e^{\psi(\Gamma(v))M} \frac{\kappa(\psi(\Gamma(v)), 0)}{c_v\psi(\Gamma(v))\Gamma(v) q + \kappa(\psi(\Gamma(v)), 0)^{\frac{1}{2}}},$$

where $c_v = E[e^{\Gamma(v)H_1 - \psi(\Gamma(v))L_1^{-1}}H_11_{L_1^{-1} < \infty}]$.

Lemma 2 is a consequence of the following auxiliary identities:

**Lemma 3** Let $u > \gamma$, $u \in \Theta^\circ$.

$$\phi(z, -u) = 1 \iff \kappa(z, -u) = 0 \iff \psi(u) = z \quad (3.6)$$

$$\psi'(u) = E^{(u)}[X(1)] = E^{(u)}[H_{\sigma_1}] \cdot (E^{(u)}[L_{\sigma_1}^{-1}])^{-1} \quad (3.7)$$

$$\psi''(u) = E^{(u)}[(H_{\sigma_1} - \psi'(u)L_{\sigma_1}^{-1})^2] \cdot (E^{(u)}[L_{\sigma_1}^{-1}])^{-1}$$

$$= \psi'(u)E^{(u)}[(H_{\sigma_1} - \psi'(u)L_{\sigma_1}^{-1})^2] \cdot (E^{(u)}[H_{\sigma_1}])^{-1} \quad (3.8)$$

$$\psi^*(v) = v\Gamma(v) - \psi(\Gamma(v)) \quad \text{for } v > 0 \text{ with } \Gamma(v) \in \Theta^\circ. \quad (3.9)$$

\textbf{Proof:} Eq (3.6): Note that for $u, z > 0$ it holds that $\hat{\kappa}(z, u) > 0$. In view of the identity (2.5) the statement follows.
Eq (3.7): Note that if \( u > \gamma \) then by the fact that \( \psi(0) = \psi(\gamma) = 0 \) and the strict convexity of \( \psi \) it follows that \( \psi(u) > 0 \). In view of (2.5) it follows then that \( \kappa(\psi(u), -u) = 0 \) for \( u \in \Theta^0, u > \gamma \). Differentiating with respect to \( u \) shows that

\[
\psi'(u) = \partial_2 \kappa(\psi(u), -u)(\partial_1 \kappa(\psi(u), -u))^{-1}.
\]

Also, note that \( E^{(u)}[H_{\sigma_1}] = q^{-1} E^{(u)}[H_1], E^{(u)}[L_{\sigma_1}^{-1}] = q^{-1} E^{(u)}[L_1^{-1}] \) and

\[
E^{(u)}[H_1] = \partial_2 \kappa(\psi(u), -u), \quad E^{(u)}[L_1^{-1}] = \partial_1 \kappa(\psi(u), -u).
\]

Eq (3.8) follows as a matter of calculus, by differentiation of (3.10) with respect to \( u \). Finally, Eq. (3.9) follows from the definition of \( \psi^* \). 

Proof of Lemma 2: The proof follows by an application of Prop. 1 to \( G^{(q)}(x, t + M) \) with

\[
(S_1, R_1) = (H_{\sigma_1}, L_{\sigma_1}^{-1}) \quad \text{and} \quad \zeta = (-\Gamma(v), \eta_v).
\]

Note that, by (3.6) with \( u = \Gamma(v) \), \( \phi(\zeta) = 1 \), and that \( \eta_v = \psi(\Gamma(v)) > 0 \) if \( v > \psi'(\gamma) \). For this choice of the parameters, \( E^v[S_1] = E^{(\Gamma(v))}[H_{\sigma_1}] = c_v/q \), and Eqs. (3.9), (3.7), (3.8) imply that \( \xi x + \eta t = -\psi^*(v)t \) and

\[
V(\zeta) = \psi''(\Gamma(v))/\psi'(\Gamma(v)) = \psi''(\Gamma(v))/v.
\]

To complete the proof we are left to verify the form of the constants. The calculation of the \( C_{q,M} = C(0,0)\psi^{\gamma M} \) goes as follows:

\[
C_{q,M} = \frac{q e^{\psi(\Gamma(v)) M}}{\psi(\Gamma(v)) c_v} \left( \int_0^{\infty} e^{-\Gamma(v)x} E^{(\Gamma(v))[H_{\sigma_1} - \psi(\Gamma(v))L_{\sigma_1}^{-1}]}(x \in H_{\sigma_1} < \infty) dx \right)
\]

\[
= \frac{q e^{\psi(\Gamma(v)) M}}{\psi(\Gamma(v)) c_v} \left( 1 - E^{[e^{-\psi(\Gamma(v))L_{\sigma_1}^{-1}}]}(L_{\sigma_1}^{-1} < \infty) \right)
\]

\[
= \frac{q e^{\psi(\Gamma(v)) M}}{\psi(\Gamma(v)) c_v} \left( 1 - \frac{q}{q + \kappa(\psi(\Gamma(v)), 0)} \right)
\]

\[
= \frac{q e^{\psi(\Gamma(v)) M}}{\psi(\Gamma(v)) c_v} \kappa(\psi(\Gamma(v)), 0)
\]

in view of the definition (2.1) of \( \kappa \). Combining all results completes the proof.

As final preparation for the proof of Theorem 1 we show that the non-lattice condition holds:

**Lemma 4** Suppose that (H) holds true. Then \( F^{(q)} \) satisfies (G).
Proof: The assertion is a consequence of the following identity between measures on $(0,\infty)^2$ (which is itself a consequence of the Wiener-Hopf factorisation, see e.g. Bertoin [4, Cor VI.10])

$$P(X_t \in dx)dt = t \int_0^\infty P(L_{u-1}^{-1} \in dt, H_u \in dx)u^{-1}du. \quad (3.11)$$

Fix $(y,v) \in (0,\infty)^2$ in the support of $\mu_X(dx,dt) = P(X_t \in dx)dt$ and let $B$ be an arbitrary open ball around $(y,v)$. Then $\mu_X(B) > 0$; in view of the identity (3.11) it follows that there exists a set $A$ with positive Lebesgue measure such that

$$P((L_{u-1}^{-1}, H_u) \in B) > 0$$

for all $u \in A$ and thus $P((L_{\sigma_1^{-1}}, H_{\sigma_1}) \in B) > 0$. Since $B$ was arbitrary we conclude that $(y,v)$ lies in the support of $F^{(q)}$. To complete the proof we next verify that if a Lévy process $X$ satisfies (H) then $\mu_X$ satisfies (G). To this end, let $X$ satisfy (H). Suppose first that its Lévy measure $\nu$ has infinite mass or $\sigma > 0$. Then $P(X_t = x) = 0$ for any $t > 0$ and $x \in \mathbb{R}$, according to Sato [14, Thm. 27.4]. Thus, the support of $P(X_t \in dx)$ is uncountable for any $t > 0$, so that $\mu_X$ satisfies (G). If $\nu$ has finite mass then it is straightforward to verify that $P(X_t \in dx)$ is non-lattice for any $t > 0$ if $\nu$ is, and that then $\mu_X$ satisfies (G). \qed

Proof of Theorem 1: Suppose that $v > \psi'(\gamma)$ (the case $v < \psi'(\gamma)$ was shown in Remark 3). Writing $l(t,x) = t^{1/2}e^{\psi'(\nu)t}P(\tau(x) \leq t)$, Lemmas 1, 2 and 3 imply that

$$s = \limsup_{x,t \to \infty, x = tv + o(t^{1/2})} l(t,x) \leq D_{q,M}/h(0-, M),$$

$$i = \liminf_{x,t \to \infty, x = tv + o(t^{1/2})} l(t,x) \geq D_{q,0}.$$

By definition of $h$ and $D_{q,M}$ it directly follows that, as $q \to \infty$,

$$D_{q,0} \to D_v, \quad D_{q,M} \to D_v e^{\psi'(\nu)M} \quad \text{and} \quad h(0-, M) = P(L_{\sigma_1^{-1}} \leq M) \to 1.$$

Letting $M \downarrow 0$ yields that $s = i = D_v$, and the proof is complete. \qed

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