Abstract

We present a systematic method to expand the quantum complexity of interacting theory in series of coupling constant. The complexity is evaluated by the operator approach in which the transformation matrix between the second quantization operators of reference state and the target state defines the quantum gate. We start with two coupled oscillators and perturbatively evaluate the geodesic length of the associated group manifold of gate matrix. Next, we generalize the analysis to \( N \) coupled oscillators which describes the lattice \( \lambda\phi^4 \) theory. Especially, we introduce simple diagrams to represent the perturbative series and construct simple rules to efficiently calculate the complexity. General formulae are obtained for the higher-order complexity of excited states. We present several diagrams to illuminate the properties of complexity and show that the interaction correction to complexity may be positive or negative depending on the magnitude of reference-state frequency.
1 Introduction

Achieving a better understanding of physics behind a black-hole horizon is important if one wants to precisely describe the bulk geometry in terms of the information of boundary CFT \[1,2,3,4,5,6,7\]. In the context of the eternal AdS-Schwarzchild black hole, for instance, a related question concerns the size of a wormhole growing linearly with time; this phenomenon has been conjectured to dual to the growth of “complexity” of the dual CFT \[8\]. In the complexity=volume (CV) conjecture \[8\], the complexity is dual to the volume of an extremal codimension-one bulk surface anchored to a time slice of the boundary. In the complexity=action (CA) conjecture \[9,10,11,12\], one identifies the complexity with a gravitational action evaluated on the Wheeler-DeWitt (WDW) patch, anchored also on a time slice of the boundary.

Several efforts were made to provide a definition of the complexity in the field theory \[13,14,15,16,17\]. The complexity in there is defined as the number of operations \(\{\mathcal{O}_I\}\) needed to transform a reference state \(|\psi_R\rangle\) to a target state \(|\psi_T\rangle\). These operators are also called as quantum gates: the more gates one needs, the more complex the target state is. One can define an affine parameter “s” associated to an unitary operator \(U(s)\) and use a set of function, \(Y^I(s)\), to character the quantum circuit. The unitary operation connecting the reference state and target state is

\[ U(s) = \mathcal{P} e^{-\int_0^s Y^I(s) \mathcal{O}_I}, \quad |\psi_R\rangle = U(0)|\psi_R\rangle, \quad |\psi_T\rangle = U(1)|\psi_R\rangle, \quad (1.1) \]
where $\vec{P}$ is a time ordering along $s$. The complexity $C$ and circuit depth $D[U]$ (cost function) are \[ C = \min \{ Y^I \}, \quad D[U] = \int_0^1 ds \sum_I |Y^I(s)|^2. \] (1.2)

Above definitions were shown to be consistent with a gravitational computation [13]. The initial studies in field theory considered the Gaussian ground state wavefunctions in reference state and target state [13, 15, 16]. The theories studied so far are the free field theory or exponential type wavefunction in interacting model [17]. The operator approach had also been used in [14, 16] to study the complexity of fermion theory.

In our previous paper [18] we adopt the operator approach, in which the transformation matrix between the second quantization operators of reference state and target state is regarded as the quantum gate, to evaluate the complexity in free scalar field theory. Since that in the operator approach we need not to use the explicit form of the wave function we can study the complexity in the excited states. We first examined the system in which the reference state is two oscillators with same frequency $\omega_f$ while the target state is two oscillators with frequency $\omega_1$ and $\omega_2$. We explicitly calculated the complexity in several excited states and proved that the square of geodesic length in the general state $|N_1, N_2\rangle$ is

\[ D^2_{(N_1, N_2)} = (N_1 + 1) \left( \ln \sqrt{\frac{\omega_1}{\omega_f}} \right)^2 + (N_2 + 1) \left( \ln \sqrt{\frac{\omega_2}{\omega_f}} \right)^2 \] (1.3)

The results was furthermore extended to the $N$ couple harmonic oscillators which correspond to the lattice version of free scalar field, see sec.5 of [18].

In this paper we extend [18] by including interactions to further study the complexity using the operator approach. We present a systematic method to evaluate the complexity of the $\lambda\phi^4$ field theory by the perturbation of small coupling constant. An outline of the paper is as follows.

In section II, as that in [13] we describes the lattice scalar field as coupled oscillators. In section III we consider two coupled oscillators and find that, to the $\lambda^n$ order the square distance of excited state between target and reference state is

\[ D^{(n)}_{(N_1, N_2)} = (N_1 + 1) \left( \ln \sqrt{R_1^{(n)}} \right)^2 + (N_2 + 1) \left( \ln \sqrt{R_2^{(n)}} \right)^2 \] (1.4)

in which $R_1^{(n)}$ and $R_2^{(n)}$ are described in (3.24). In section IV we generalize it to the case of $N$ coupled oscillators which correspond to the lattice version of $\lambda\phi^4$ theory. We use new kind of simple diagrams, figures 3, 4 and 5, to represent the perturbative series and construct simple rules, figures 1 and 2, to calculate the complexity therein. We find that the diagrams are classified into three classes : odd $N$, odd $\frac{N}{2}$, and even $\frac{N}{2}$. We explicitly calculate the complexity in the cases of $N=2,3,4, 5$ to any order of $\lambda$. Using these experiences we then in section V

\footnote{Note that the excited-state wavefunction of harmonic oscillation is not pure exponential form and the wavefunction approach is hard to work.}
derive the general formulas of complexity in (5.5), (5.12), and (5.17). Then, we present several diagrams to illuminate the properties of complexity and find that the interaction correction to complexity may be positive or negative depending on the magnitude of reference-state frequency. We conclude in Sec. 6.

2 Interacting Scalar Field and Coupled Oscillators

The $d$-dimensional massive scalar Hamiltonian with a \( \hat{\lambda}\phi^4 \) interaction is

\[
H = \frac{1}{2} \int d^{d-1}x \left[ \pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2 + \frac{\hat{\lambda}}{12}\phi(x)^4 \right].
\]

(2.1)

Placing the theory on a square lattice with lattice spacing \( \delta \), one has

\[
H = \frac{1}{2} \sum_{\vec{n}} \left\{ \frac{p(\vec{n})^2}{\delta^{d-1}} + \delta^{d-1} \sum_i \frac{1}{\delta^2} \left[ (\phi(\vec{n}) - \phi(\vec{n} - \hat{a}_i))^2 + m^2\phi(\vec{n})^2 + \frac{\hat{\lambda}}{12}\phi(\vec{n})^4 \right] \right\},
\]

(2.2)

where \( \hat{a}_i \) are unit vectors pointing toward the spatial directions of the lattice. By redefining

\[
X(\vec{n}) = \delta^{d/2}\phi(\vec{n}), \quad P(\vec{n}) = \frac{p(\vec{n})}{\delta^{d/2}}, \quad M = \frac{1}{\delta}, \quad \omega = m, \quad \Omega = \frac{1}{\delta}, \quad \lambda = \frac{\hat{\lambda}}{24\delta^4},
\]

(2.3)

the Hamiltonian becomes

\[
H = \sum_{\vec{n}} \left\{ \frac{P(\vec{n})^2}{2M} + \frac{1}{2} M \left[ \omega^2X(\vec{n})^2 + \Omega^2\sum_i \left( X(\vec{n}) - X(\vec{n} - \hat{a}_i) \right)^2 + 2\lambda X(\vec{n})^4 \right] \right\},
\]

(2.4)

When \( \vec{n} \) is an one dimensional vector the Hamiltonian describes an infinite family of coupled one dimensional oscillators. We will extensively study the one dimensional oscillators in this paper while the extension to higher dim is just to replace the site index “ \( i \)” to “ \( \vec{i} \)”, as that described in [13].

3 Two Coupled Oscillators

First we consider a simple case of two coupled oscillators (\( M = 1 \)):

\[
H = \frac{1}{2} \left[ \vec{p}_1^2 + \vec{p}_2^2 + \omega^2(x_1^2 + x_2^2) + \Omega^2(x_1 - x_2)^2 + 2\lambda(x_1^4 + x_2^4) \right]
\]

(3.1)

Defining

\[
\tilde{x}_{1,2} = \frac{1}{\sqrt{2}}(x_1 \pm x_2), \quad \tilde{p}_{1,2} = \frac{1}{\sqrt{2}}(p_1 \pm p_2), \quad \omega_1^2 = \omega^2, \quad \omega_2^2 = \omega^2 + 2\Omega^2,
\]

(3.2)

the Hamiltonian is

\[
H = \frac{1}{2} \left( \tilde{p}_1^2 + \omega_1^2\tilde{x}_{1}^2 + \tilde{p}_2^2 + \omega_2^2\tilde{x}_2^2 \right) + \frac{\lambda}{4} \left( (x_1 + x_2)^4 + (x_1 - x_2)^4 \right) = K + V
\]

(3.3)
In the second quantization, we define
\[ a_1^\dagger = \sqrt{\frac{\omega_1}{2}} x_1 + i \frac{1}{\sqrt{2 \omega_1}} p_1, \quad a_2^\dagger = \sqrt{\frac{\omega_2}{2}} x_2 + i \frac{1}{\sqrt{2 \omega_2}} p_2, \quad [a_{1,2}, a_{1,2}^\dagger] = 1 \] (3.4)
\[ x_{1,2} = \frac{1}{2 \omega_{1,2}} (a_{1,2}^\dagger + a_{1,2}), \quad p_{1,2} = i \frac{\omega_{1,2}}{2} (a_{1,2}^\dagger - a_{1,2}). \] (3.5)
The state wavefunction is \( \psi(x_1, x_2) = (x_1, x_2 | a_1^\dagger a_2^\dagger | 0) \).

### 3.1 Kinetic Term of Two Coupled Oscillators

The kinetic term has a diagonal form:
\[ K^{(\text{tar})} = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + \frac{1}{2} (\omega_1 + \omega_2) \] (3.6)
where the constant terms are irrelevant to the following discussions. We choose the reference state with the associated kinetic term given by \[13\]
\[ K^{(\text{ref})} = \omega_f (a_1^{(\text{ref})})^\dagger a_1^{(\text{ref})} + \omega_f (a_2^{(\text{ref})})^\dagger a_2^{(\text{ref})}. \] (3.7)
Note that one can obtain \( K^{(\text{tar})} \) from \( K^{(\text{ref})} \) via the replacement
\[ a_{1,2}^{(\text{ref})} \rightarrow \sqrt{\frac{\omega_{1,2}}{\omega_f}} a_{1,2} \implies K^{(\text{ref})} \rightarrow K^{(\text{tar})}. \] (3.8)

In the operator approach, the gate matrices defined in (1.1) for operators \( \{a_1, a_1^\dagger, a_2, a_2^\dagger\} \) are \( e^{Y_i} = \{ \sqrt{\frac{\omega_i}{\omega_f}}, \sqrt{\frac{\omega_i}{\omega_f}}, \sqrt{\frac{\omega_i}{\omega_f}}, \sqrt{\frac{\omega_i}{\omega_f}} \} \), which are simply the U(1) group elements. Using (1.2), the square distance between target and reference state for the gate matrix \( Y_i \) is \( D_i^2 = (Y_i)^2 \).

As the ground state is annihilated by \( a_1, a_2 \), i.e. \( a_1 a_2 |0, 0\rangle = 0 \) for target state, and \( a_1^{(\text{ref})}, a_2^{(\text{ref})} |0, 0\rangle^{\text{ref}} = 0 \) for reference state, there are two gate matrices that can be read from the transformations \( a_1^{(\text{ref})} \rightarrow \sqrt{\frac{\omega_1}{\omega_f}} a_1 \) and \( a_2^{(\text{ref})} \rightarrow \sqrt{\frac{\omega_2}{\omega_f}} a_2 \). The squared distance between target and reference state calculated from the two gate matrices is given by
\[ D^2_{(0,0)} = \left( \ln \left( \sqrt{\frac{\omega_1}{\omega_f}} \right) \right)^2 + \left( \ln \left( \sqrt{\frac{\omega_2}{\omega_f}} \right) \right)^2. \] (3.9)

For excited state, \( a_1^{N_1} a_2^{N_2} |N_1, N_2\rangle = 0 \), or \( |N_1, N_2\rangle = \frac{(a_1^\dagger)^{N_1} (a_2^\dagger)^{N_2}}{\sqrt{N_1! N_2!}} |0, 0\rangle \), the square distance between target and reference state is
\[ D^2_{(N_1, N_2)} = (N_1 + 1) \left( \ln \left( \sqrt{\frac{\omega_1}{\omega_f}} \right) \right)^2 + (N_2 + 1) \left( \ln \left( \sqrt{\frac{\omega_2}{\omega_f}} \right) \right)^2. \] (3.10)
This matches with the result obtained earlier in [18]. Recall that the state wavefunction is described by \( \Psi_n(x) = \frac{1}{\sqrt{m!}} (x | a_1^\dagger)^n |0 \) the gate matrix of excited-state wavefunction, \( \Psi_n(x) \), is thus related to the gate matrix of field operators, \( (a_1^\dagger)^n \).
3.2 Interacting Term of Two Coupled Oscillators

We next study the correction to the complexity due to the interaction term:

\[ V = \frac{\lambda}{4} ((x_1 + x_2)^4 + (x_1 - x_2)^4) \] (3.11)

\[ = \frac{\lambda}{4 \cdot 2^2} \left[ \left( \sqrt{\frac{1}{\omega_1}}(a_1^\dagger + a_1) + \sqrt{\frac{1}{\omega_2}}(a_2^\dagger + a_2) \right)^4 \right] \] (3.12)

We will consider \( \langle N_1, N_2 | V | N_1, N_2 \rangle \) for the excited state \( |N_1, N_2\rangle \) with fixed \( N_1 \) and \( N_2 \). In this way, only the terms that have the same power of \( a_i \) and \( a_i^\dagger \) are relevant. Therefore we only need to consider

\[ (a_1^\dagger + a_1)^4 = \left( (a_1^\dagger)^2 + a_1^\dagger a_1 + a_1 a_1^\dagger + (a_1)^2 \right) \left( (a_1^\dagger)^2 + a_1^\dagger a_1 + a_1 a_1^\dagger + (a_1)^2 \right) \]

\[ = 6a_1^\dagger a_1 a_1^\dagger a_1 + 6a_1^\dagger a_1 + 3 + \text{irrelevant terms} \] (3.13)

We obtain, after dropping irrelevant terms,

\[ H = \left( \omega_1 + \frac{3\lambda}{2} \left( 1 + \frac{N_1}{\omega_1 \omega_2} \right) \right) a_1^\dagger a_1 + \left( \omega_2 + \frac{3\lambda}{2} \left( \frac{1}{\omega_2} \frac{N_1}{\omega_1} + \frac{1 + N_2}{2\omega_2^2} \right) \right) a_2^\dagger a_2. \] (3.14)

The associated Hamiltonian of the reference state can be chosen as

\[ H^{(\text{ref})} = \left( \omega_f + \frac{3\lambda}{2} \left( \frac{1 + N_1}{\omega_f^2} + \frac{1 + N_2}{\omega_f^2} \right) \right) a_1^{(\text{ref})} a_1^{(\text{ref})} + \left( \omega_f + \frac{3\lambda}{2} \left( \frac{1 + N_1}{\omega_f^2} + \frac{1 + N_2}{2\omega_f^2} \right) \right) a_2^{(\text{ref})} a_2^{(\text{ref})}. \] (3.15)

In the case of zero-order of \( \lambda \),

\[ K^{(\text{ref})} = \omega_f (a_1^{(\text{ref})})^\dagger a_1^{(\text{ref})} + \omega_f (a_2^{(\text{ref})})^\dagger a_2^{(\text{ref})}, \quad K^{(\text{tar})} = \omega_1 a_1^\dagger a_1 + \omega_1 a_2^\dagger a_2 \] (3.16)

which implies transformations

\[ a_1^{(\text{ref})} \rightarrow \sqrt{\frac{\omega_1}{\omega_f}} a_1^{(\text{ref})}, \quad a_2^{(\text{ref})} \rightarrow \sqrt{\frac{\omega_2}{\omega_f}} a_2^{(\text{ref})} \] (3.17)

or

\[ N^{(\text{ref})}_{(1,2)} \rightarrow R^{(0)}_{(1,2)} N^{(\text{ref})}_{(1,2)} \] (3.18)

where

\[ R^{(0)}_{(i)} = \frac{\omega_i}{\omega_f}. \] (3.19)

The quantum gate are described by two \( 1 \times 1 \) matrices, \( \exp\left(\sqrt{R^{(0)}_{1}}\right) \) and \( \exp\left(\sqrt{R^{(0)}_{2}}\right) \). This is the case of purely kinetic term, i.e. a free theory.
Now consider a perturbation to the complexity for the two coupled oscillators. At the first order of $\lambda$, we have transformations

\[
\begin{align*}
&\begin{cases}
\left(\omega_1 + \frac{3\lambda}{2} \left(\frac{1+N_1 R_1^{(0)}}{2\omega_f^2} + \frac{1+N_2 R_2^{(0)}}{\omega_1 \omega_2}\right)\right) a_1^{\dagger} a_1 \rightarrow \left(\omega_f + \frac{3\lambda}{2} \left(\frac{1+N_1}{2\omega_f^3} + \frac{1+N_2}{\omega_f^3}\right)\right) (a_1^{\text{ref}})^{\dagger} a_1^{\text{ref}} \\
\left(\omega_2 + \frac{3\lambda}{2} \left(\frac{1+N_1}{\omega_2 \omega_1} + \frac{1+N_2}{2\omega_f^2}\right)\right) a_2^{\dagger} a_2 \rightarrow \left(\omega_f + \frac{3\lambda}{2} \left(\frac{1+N_1}{\omega_2^3} + \frac{1+N_2}{2\omega_f^3}\right)\right) (a_2^{\text{ref}})^{\dagger} a_2^{\text{ref}}
\end{cases}
\end{align*}
\]

(3.20)

The factors $N_{(1,2)}$ are within the coupling term, i.e. $\frac{3\lambda}{2}$ and we only need to consider their zero-order transform. Recall (3.18), we have to multiple them by $R_{(1,2)}^{(0)}$ factors. Therefore the first-order transformations are

\[
R_1^{(1)} = \frac{\omega_1 + \frac{3\lambda}{2} \left(\frac{1+N_1 R_1^{(0)}}{2\omega_f^2} + \frac{1+N_2 R_2^{(0)}}{\omega_1 \omega_2}\right)}{\omega_f + \frac{3\lambda}{2} \left(\frac{1+N_1}{2\omega_f^3} + \frac{1+N_2}{\omega_f^3}\right)}, \quad R_2^{(1)} = \frac{\omega_2 + \frac{3\lambda}{2} \left(\frac{1+N_1 R_1^{(0)}}{\omega_2 \omega_1} + \frac{1+N_2 R_2^{(0)}}{2\omega_f^2}\right)}{\omega_f + \frac{3\lambda}{2} \left(\frac{1+N_1}{\omega_2^3} + \frac{1+N_2}{2\omega_f^3}\right)}
\]

(3.21)

and the square distance is

\[
D_{(0,0)}^{(1,2)} = \left(\ln \left(\sqrt{R_1^{(1)}}\right)\right)^2 + \left(\ln \left(\sqrt{R_2^{(1)}}\right)\right)^2.
\]

(3.22)

For excited states, the first-order square distance is

\[
D_{(N_1,N_2)}^{2} = (N_1 + 1) \left(\ln \left(\sqrt{R_1^{(1)}}\right)\right)^2 + (N_2 + 1) \left(\ln \left(\sqrt{R_2^{(1)}}\right)\right)^2.
\]

(3.23)

Extending to higher-order interactions is straightforward. The recursion relations are

\[
R_1^{(n)} = \frac{\omega_1 + \frac{3\lambda}{2} \left(\frac{1+N_1 R_1^{(n-1)}}{2\omega_f^2} + \frac{1+N_2 R_2^{(n-1)}}{\omega_1 \omega_2}\right)}{\omega_f + \frac{3\lambda}{2} \left(\frac{1+N_1}{2\omega_f^3} + \frac{1+N_2}{\omega_f^3}\right)}, \quad R_2^{(n)} = \frac{\omega_2 + \frac{3\lambda}{2} \left(\frac{1+N_1 R_1^{(n-1)}}{\omega_2 \omega_1} + \frac{1+N_2 R_2^{(n-1)}}{2\omega_f^2}\right)}{\omega_f + \frac{3\lambda}{2} \left(\frac{1+N_1}{\omega_2^3} + \frac{1+N_2}{2\omega_f^3}\right)}
\]

(3.24)

with initial values $R_{(1,2)}^{(0)}$ defined in (3.19). For excited states, the $n$-order square distance is

\[
D_{(N_1,N_2)}^{(n)} = (N_1 + 1) \left(\ln \left(\sqrt{R_1^{(n)}}\right)\right)^2 + (N_2 + 1) \left(\ln \left(\sqrt{R_2^{(n)}}\right)\right)^2,
\]

(3.25)

which is the $n$-order complexity of two coupled oscillators.

Note that original relations (3.24) can be expanded as

\[
R_1^{(n)} \approx \frac{\omega_1}{\omega_f} + \frac{3\lambda}{2\omega_f} \left(\frac{1+N_1 R_1^{(n-1)}}{2\omega_f^2} + \frac{1+N_2 R_2^{(n-1)}}{\omega_1 \omega_2} - \frac{1+N_1}{2\omega_f^3} - \frac{1+N_2}{\omega_f^3}\right)
\]

(3.26)

\[
R_2^{(n)} \approx \frac{\omega_2}{\omega_f} + \frac{3\lambda}{2\omega_f} \left(\frac{1+N_1 R_1^{(n-1)}}{\omega_2 \omega_1} + \frac{1+N_2 R_2^{(n-1)}}{2\omega_f^2} - \frac{1+N_1}{\omega_2^3} - \frac{1+N_2}{2\omega_f^3}\right)
\]

(3.27)

In this way, the perturbative series of $R_i^{(n)}$ in coupling constant $\lambda$ is explicitly showing up. However, to save the space, in following sections we will not expand the original relations, likes as (3.24), to the relations, likes as (3.26) or (3.27).
4 N Coupled Oscillators

4.1 Kinetic Term of N Coupled Oscillators

For $N$ coupled oscillators,
\[ H = \frac{1}{2} \sum_{k=1}^{N} \left[ \frac{\mathcal{p}_k^2}{2} + \omega_k^2 \frac{\mathcal{x}_k^2}{2} + \Omega^2 (\mathcal{x}_k - \mathcal{x}_{k+1})^2 + 2\lambda \mathcal{x}_k^4 \right]. \] (4.1)

We impose a periodic boundary condition $\mathcal{x}_{k+N+1} = \mathcal{x}_k$. The normal coordinates are chosen to be
\[ \mathcal{x}_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp \left( \frac{2\pi i k}{N} j \right) \tilde{\mathcal{x}}_j, \quad p_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp \left( \frac{-2\pi i k}{N} j \right) \tilde{\mathcal{p}}_j \] (4.2)

Note that the relative sign between the Fourier series of $x_k$ and $p_k$ is important to have standard commutation relation $[x_k, p_k] = \delta_{k1, k2}$ [13]. The Hamiltonian now becomes
\[ H = \frac{1}{2} \sum_{k=1}^{N} \left( \mathcal{p}_k^2 + \omega_k^2 \mathcal{x}_k^2 \right) + V, \quad \omega_k^2 = \omega^2 + 4\Omega^2 \sin^2 \frac{\pi k}{N} \] (4.3)

Defining
\[ x_k = \frac{1}{\sqrt{2\omega_k}} (a_k + a_k^{\dagger}), \quad p_k = i \sqrt{\frac{\omega_k}{2}} (a_k^{\dagger} - a_{-k}), \quad [a_k, a_k^{\dagger}] = 1, \] (4.4)

the kinetic term can be written as
\[ \sum_k K_k = \frac{1}{2} \sum_k \left( \mathcal{p}_k^2 + \omega_k^2 \mathcal{x}_k^2 \right) = \sum_k \omega_k a_k^{\dagger} a_k, \] (4.5)

up to an irrelevant constant.

The states in $N$ oscillators can be defined by the creation operators $a_1^{\dagger}, a_2^{\dagger}, ..., a_N^{\dagger}$, such that $\psi(x_1, x_2, ...) = \langle x_1, x_2, ... | a_1^{\dagger} a_2^{\dagger} ... a_N^{\dagger} | 0 \rangle$. As before, to find the complexity of such state we choose a reference state with the associated kinetic term given by
\[ K^{(\text{ref})} = \sum_k \omega_f (a_k^{\dagger}(\text{ref}))^{\dagger} a_k^{\dagger}(\text{ref}) \] (4.6)

The square distance for the $n_k$-th excited state is
\[ D^2_{\{N_1, N_2, ..., N_k, ..., N_N\}} = (N_1 + 1) \left( \ln \left( \sqrt{\frac{\omega_1}{\omega_f}} \right) \right)^2 + (N_2 + 1) \left( \ln \left( \sqrt{\frac{\omega_2}{\omega_f}} \right) \right)^2 + \ldots \]
\[ = \sum_{k=1}^{N} (N_k + 1) \left( \ln \left( \sqrt{\frac{\omega_k}{\omega_f}} \right) \right)^2 \] (4.7)

where $\omega_k$ is defined in (4.3).
4.2 Interacting Term of N Coupled Oscillators: Perturbative Algorithm

We adopt the following steps to systematically study a perturbation theory of the complexity:

(I) We express potential $V$ in terms of $a$, $a^\dagger$:

$$V = \lambda \sum_{k=1}^{N} \tilde{x}_k^4 = \lambda \sum_{k=1}^{N} \left( \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp \left( \frac{-2\pi ik}{N} j \right) x_j \right)^4$$

$$= \frac{\lambda}{4N^2} \sum_{k=1}^{N} \left( \sum_{j=1}^{N} \exp \left( \frac{-2\pi ik}{N} j \right) \left( \frac{1}{\sqrt{\omega_j}} (a_j + a_j^\dagger) \right) \right)^4 \quad (4.8)$$

(II) Define

$$A(j) = \frac{1}{\sqrt{\omega_j}} (a_j + a_j^\dagger). \quad (4.9)$$

Then, as calculated in (3.12) and (3.13),

$$A(j)^4 = \frac{6}{\omega_j} (a_j^\dagger a_j a_j^\dagger a_j + a_j^\dagger a_j) + \text{irrelevant terms} \quad (4.10)$$

$$A(j)^2 = \frac{1}{\omega_j} (2a_j a_j + 1) + \text{irrelevant terms} \quad (4.11)$$

which lead to two relations that will be extensively used in later calculations:

$$A(j)^4 = 6 \times \left[ \frac{N_j + 1}{\omega_j^2} a_j a_j^\dagger \right] \quad (4.12)$$

$$6 A(i)^2 A(j)^2 = 6 \times \frac{1}{\omega_i \omega_j} \left( 4a_i^\dagger a_i a_j^\dagger a_j + 2a_i^\dagger a_i + 2a_j^\dagger a_j \right)$$

$$= 6 \times \left[ \frac{2N_i + 2}{\omega_i \omega_j} a_j a_j^\dagger + \frac{2N_j + 2}{\omega_i \omega_j} a_i^\dagger a_i \right], \quad i \neq j \quad (4.13)$$

The term $a_j^\dagger a_j a_j^\dagger a_j$ in (4.10) is written as $N_j a_j^\dagger a_j$ in (4.12), as we did in sec. 3.2. Sec.3.2 also tells us that we will let $N_j \to R_j^{(n-1)} N_j$ in calculating the complexity at the n'th order of $\lambda$.

(III) Adopting the series expansion (4.8), we can develop diagrammatic rules based on two basic elements, “circle” and “pair”, which appear in (4.12) and (4.13). We plot them in figure 1 and 2:

\[ \text{Figure 1: The “circle” element} \]

\[ \text{Figure 2: The “pair” element} \]

\[ \text{Figure 3:} A(j)^4 \]

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2The reason of using $6 A(i)^2 A(j)^2$ instead of $A(i)^2 A(j)^2$ is because that in the series expansion of the potential it always appears the combination factor $6 A(i)^2 A(j)^2$, as can be seen in several examples in next subsection.
The diagrams for the potential $V$ then can be classified into three classes: odd $N$, odd $\frac{N}{2}$, and even $\frac{N}{2}$. We discuss corresponding rules in the following.

- **Odd $N$**: We write numbers $1,2,...,N$ on a horizon and assign “circle” on $N$. Then we assign “pair” on $(1,N-1),...,(i,N-i),..,(\frac{N-1}{2},\frac{N+1}{2})$.

  An N=9 example is plotted in figure 3.

- **Odd $\frac{N}{2}$**: We write numbers $1,2,...,N$ on a horizon and assign “circle” on $\frac{N}{2}$ and on $N$. We also assign a “pair” on $(\frac{N}{2},N)$, “pair” on $(1,N-1),...,(i,N-i),..,(\frac{N}{2}-1,\frac{N}{2}+1)$, and assign “pair” on $(1,\frac{N}{2}-1),...,(i,\frac{N}{2}-i),..,(\frac{N}{4}-\frac{1}{2},\frac{N}{4}+\frac{1}{2})$. Finally, we assign “pair” on $(\frac{N}{2}+1,N-1),...,(i,\frac{3N}{2}-i),..,(\frac{3N}{4}-\frac{1}{2},\frac{3N}{4}+\frac{1}{2})$.

  An N=10 example is plotted in figure 4.

- **Even $\frac{N}{2}$**: We write numbers $1,2,...,N$ on a horizon and assign “circle” on $N$, $\frac{N}{4}$, $\frac{N}{2}$, and $\frac{3N}{4}$. Also assign “pair” on $(\frac{N}{2},N)$ and $(\frac{N}{4},\frac{3N}{4})$. Assign “pair” on $(1,N-1),...,(i,N-i),..,(\frac{N}{2}-1,\frac{N}{2}+1)$ and assign “pair” on $(1,\frac{N}{2}-1),...,(i,\frac{N}{2}-i),..,(\frac{N}{4}-1,\frac{N}{4}+1)$. Finally, we assign “pair” on $(\frac{N}{2}+1,N-1),...,(i,\frac{3N}{2}-i),..,(\frac{3N}{4}-1,\frac{3N}{4}+1)$.

  An N=12 example is plotted in figure 5.

From figures 3, 4, and 5, we can see a pairing property: Assign the “circle” element $A(j)^4$
pairing with “j” once and assign the “pair” element $A(i)^2A(j)^2$ pairing with each “i” and “j” once, then the odd $N$ diagrams have pairings in each “j” once while the even $N$ diagrams have pairings for each “j” twice.

### 4.3 Interacting Term of N Coupled Oscillators: Some Calculations

We now take several values of $N$ as examples to plot the diagrams and use (4.12) and (4.13) to calculate the associated complexity. General formulae will be presented in the next section.

- **N=2:**

  ![Figure 6: N=2 diagram](image)

  As shown in figure 6 the series expansion (4.8) is

  $$V_{N=2} = \frac{\lambda}{\omega_1^2 \cdot 2} \left[ A(1)^4 + A(2)^4 + 6A(1)^2A(2)^2 \right]$$

  $$= \frac{6\lambda}{\omega_1^2} \left[ \frac{1}{\omega_1^2} (N_1 + 1)a_1^\dagger a_1 + \frac{1}{\omega_2^2} (N_2 + 1)a_2^\dagger a_2 + \frac{2}{\omega_1\omega_2} ((N_2 + 1)a_1^\dagger a_1 + (N_1 + 1)a_2^\dagger a_2) \right]$$

  $$= \frac{6\lambda}{\omega_1^2} \left[ \frac{1 + N_1}{\omega_1^2} + \frac{2 + 2N_2}{\omega_1\omega_2} \right]a_1^\dagger a_1 + \left( \frac{1 + N_2}{\omega_2^2} + \frac{2 + 2N_1}{\omega_1\omega_2} \right)a_2^\dagger a_2. \quad (4.14)$$

  We have used (4.12) and (4.13). The above result matches with (3.14).

  The associated complexity can be evaluated to any order in $\lambda$:

  $$R_1^{(n)} = \frac{\omega_1 + \frac{6\lambda}{\omega_1^2} \left( \frac{1 + N_1 R_1^{(n-1)}}{\omega_1^2} + \frac{2 + 2N_2 R_2^{(n-1)}}{\omega_1\omega_2} \right)}{\omega_f + \frac{6\lambda}{\omega_1^2} \left( \frac{1 + N_1 R_1^{(n-1)}}{\omega_1^2} + \frac{2 + 2N_2 R_2^{(n-1)}}{\omega_1\omega_2} \right)}, \quad R_2^{(n)} = \frac{\omega_2 + \frac{6\lambda}{\omega_2^2} \left( \frac{1 + N_2 R_2^{(n-1)}}{\omega_2^2} + \frac{2 + 2N_1 R_1^{(n-1)}}{\omega_1\omega_2} \right)}{\omega_f + \frac{6\lambda}{\omega_2^2} \left( \frac{1 + N_2 R_2^{(n-1)}}{\omega_2^2} + \frac{2 + 2N_1 R_1^{(n-1)}}{\omega_1\omega_2} \right)} \quad (4.15)$$

  with initial values $R_1^{(1,2)}$ defined in (3.19). For excited states, the $n$-order squared distance is

  $$D_{(N_1, N_2)}^{(n)^2} = \sum_{i=1}^{2} (N_i + 1) \left( \ln \left( \sqrt{R_i^{(n)}} \right) \right)^2,$$

  which is the $n$-order complexity of 2 coupled oscillators. While above results exactly match (3.24) we have expressed them in the new form that helps us to identify rules for computing a general $N$ result.

- **N=3:**

  ![Figure 7: N=3 diagram](image)
As shown in figure 7 the series expansion \([4.8]\) is

\[
V_{N=3} = \frac{\lambda}{4 \cdot 3} \left[ A(3)^4 + 6A(1)^2A(2)^2 \right] = \frac{6\lambda}{4 \cdot 3} \left[ 1 + \frac{N_3}{\omega_3^2} a_3^\dagger a_3 + \frac{2 + 2N_2}{\omega_1\omega_2} a_1^\dagger a_1 + \frac{2 + 2N_1}{\omega_1\omega_2} a_2^\dagger a_2 \right] (4.16)
\]

We have recurrent relations

\[
R_1^{(n)} = \frac{\omega_1 + \frac{6\lambda}{4} \left( \frac{2+2N_2 R_2^{(n-1)}}{\omega_1 \omega_2} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{2+2N_2}{\omega_f^2} \right)}, \quad R_2^{(n)} = \frac{\omega_2 + \frac{6\lambda}{4} \left( \frac{2+2N_1 R_1^{(n-1)}}{\omega_1 \omega_2} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{2+2N_1}{\omega_f^2} \right)}
\]

\[
R_3^{(n)} = \frac{\omega_3 + \frac{6\lambda}{4} \left( \frac{1+N_3 R_3^{(n-1)}}{\omega_3^2} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{1+N_4}{\omega_f^2} \right)}.
\]

(4.17)

For excited states, \(D_{(N_1,N_2,N_3)}^{(n)} = \sum_{i=1}^{3} (N_i + 1) \left( \ln \left( \sqrt{R_i^{(n)}} \right) \right)^2\), which is the \(n\)-order complexity of 3 coupled oscillators.

- \(N=4\):

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\end{array} \quad \begin{array}{c}
4 \\
\end{array}
\]

Figure 8: N=4 diagram

As shown in figure 8 the series expansion \([4.8]\) is

\[
V_{N=4} = \frac{\lambda}{4 \cdot 4} \left[ A(1)^4 + A(2)^4 + A(3)^4 + A(4)^4 + 6A(1)^2A(3)^2 + 6A(2)^2A(4)^2 \right] = \frac{6\lambda}{4 \cdot 4} \left[ \left( \frac{1+N_1}{\omega_1^2} + \frac{2 + 2N_3}{\omega_1\omega_3} \right) a_1^\dagger a_1 + \left( \frac{1+N_2}{\omega_2^2} + \frac{2 + 2N_4}{\omega_2\omega_4} \right) a_2^\dagger a_2 \\
+ \left( \frac{1+N_3}{\omega_3^2} + \frac{2 + 2N_1}{\omega_1\omega_3} \right) a_3^\dagger a_3 + \left( \frac{1+N_4}{\omega_4^2} + \frac{2 + 2N_2}{\omega_2\omega_4} \right) a_4^\dagger a_4 \right], \quad (4.18)
\]

We have recurrent relations

\[
R_1^{(n)} = \frac{\omega_1 + \frac{6\lambda}{4} \left( \frac{1+N_1 R_1^{(n-1)}}{\omega_1^2} + \frac{2+2N_3 R_3^{(n-1)}}{\omega_1\omega_3} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{1+N_1}{\omega_f^2} + \frac{2+2N_3}{\omega_f^2} \right)}, \quad R_2^{(n)} = \frac{\omega_2 + \frac{6\lambda}{4} \left( \frac{1+N_2 R_2^{(n-1)}}{\omega_2^2} + \frac{2+2N_4 R_4^{(n-1)}}{\omega_2\omega_4} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{1+N_2}{\omega_f^2} + \frac{2+2N_4}{\omega_f^2} \right)}
\]

\[
R_3^{(n)} = \frac{\omega_3 + \frac{6\lambda}{4} \left( \frac{1+N_3 R_3^{(n-1)}}{\omega_3^2} + \frac{2+2N_1 R_1^{(n-1)}}{\omega_1\omega_3} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{1+N_3}{\omega_f^2} + \frac{2+2N_1}{\omega_f^2} \right)}, \quad R_4^{(n)} = \frac{\omega_4 + \frac{6\lambda}{4} \left( \frac{1+N_4 R_4^{(n-1)}}{\omega_4^2} + \frac{2+2N_2 R_2^{(n-1)}}{\omega_2\omega_4} \right)}{\omega_f + \frac{6\lambda}{4} \left( \frac{1+N_4}{\omega_f^2} + \frac{2+2N_2}{\omega_f^2} \right)}.
\]

(4.19)
For excited state, $D_{(N_1,N_2,N_3,N_4)}^{(n)2} = \sum_{i=1}^{4} (N_i + 1) \left( \ln \left( \sqrt{R_i^{(n)}} \right) \right)^2$, which is the $n$-order complexity of 4 coupled oscillators.

- **N=5:**

  \[ \sum_{i=1}^{5} (N_i + 1) \left( \ln \left( \sqrt{R_i^{(n)}} \right) \right)^2, \]

  Figure 9: N=5 diagram

  As shown in figure 9 the series expansion (4.8) is

  \[ V_{N=5} = \frac{\lambda}{4 \cdot 5} \left[ 6A(1)^2A(4)^2 + 6A(2)^2A(3)^2 + A(5)^4 \right] \]

  \[ = \frac{6\lambda}{4 \cdot 3} \left[ \frac{1 + N_5}{\omega_5^2} a_5^\dagger a_5 + \frac{2 + 2N_4}{\omega_1\omega_4} a_1^\dagger a_1 + \frac{2 + 2N_3}{\omega_3\omega_2} a_2^\dagger a_2 + \frac{2 + 2N_2}{\omega_3\omega_2} a_3^\dagger a_3 + \frac{2 + 2N_1}{\omega_1\omega_4} a_4^\dagger a_4 \right] \]

  We have recurrent relations

  \[ R_1^{(n)} = \frac{\omega_1 + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_4}{\omega_1\omega_4} \right)}{\omega_f + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_4}{\omega_f^2} \right)}, \quad R_2^{(n)} = \frac{\omega_2 + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_3}{\omega_3\omega_2} \right)}{\omega_f + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_3}{\omega_f^2} \right)} \]

  \[ R_3^{(n)} = \frac{\omega_3 + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_2}{\omega_3\omega_2} \right)}{\omega_f + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_2}{\omega_f^2} \right)}, \quad R_4^{(n)} = \frac{\omega_4 + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_1}{\omega_1\omega_4} \right)}{\omega_f + \frac{6\lambda}{4 \cdot 3} \left( \frac{2+2N_1}{\omega_f^2} \right)} \]

  \[ R_5^{(n)} = \frac{\omega_5 + \frac{6\lambda}{4 \cdot 3} \left( \frac{1+2N_5}{\omega_5^2} \right)}{\omega_f + \frac{6\lambda}{4 \cdot 3} \left( \frac{1+2N_5}{\omega_f^2} \right)}. \]

  Figure 10: N=6 diagram

  As shown in figure 10 the series expansion (4.8) is

  \[ V_{N=6} = \frac{\lambda}{4 \cdot 6} \left[ A(3)^4 + A(6)^4 + 6A(1)^2A(5)^2 + 6A(2)^2A(4)^2 + 6A(1)^2A(2)^2 \right. \]

  \[ \left. + 6A(3)^2A(6)^2 + 6A(4)^2A(5)^2 \right] \]

  (4.23)
• N=7:

As shown in figure 11 the series expansion (4.8) is

\[ V_{N=7} = \frac{\lambda}{4 \cdot 7} \left[ 6A(1)^2A(6)^2 + 6A(2)^2A(5)^2 + 6A(3)^2A(4)^2 + A(7)^4 \right] \] (4.24)

• N=8:

As shown in figure 12 the series expansion (4.8) is

\[ V_{N=8} = \frac{\lambda}{4 \cdot 8} \left[ A(2)^4 + A(4)^4 + A(6)^4 + A(8)^4 + 6A(1)^2A(7)^2 + 6A(2)^2A(6)^2 + 6A(3)^2A(5)^2 
+ 6A(1)^2A(3)^2 + 6A(4)^2A(8)^2 + 6A(5)^2A(7)^2 \right] \] (4.25)

With these experiences we will in the next section derive general formulae of the complexity for any \( N \) to any order in \( \lambda \).

5 Complexity of N Coupled Oscillators

5.1 Complexity of N Coupled Oscillators : General Formulae

From the above analysis and relations (4.12) and (4.13), we find

\[ V_N = \frac{\lambda}{4 \cdot N} \left[ \sum_{\text{"circle"}_j} A(j)^4 + 6 \sum_{\text{"pair"}_{(i,j)}} A(i)^2A(j)^2 \right] \]

\[ = \frac{6\lambda}{4 \cdot N} \left[ \sum_{\text{"circle"}_i} \frac{1 + N_j}{\omega_j^2} a_j^\dagger a_j + \sum_{\text{"pair"}_{(i,j)}} \frac{2 + 2N_i}{\omega_i\omega_j} a_j^\dagger a_j + \frac{2 + 2N_j}{\omega_i\omega_j} a_i^\dagger a_i \right] \] (5.1)

where “circle” and “pair” can be read from diagrams; see figures 3, 4, and 5.

• Odd \( N \) : We recall, from Sec 4.2, the odd \( N \) case is simplest as it only has one “circle” located at \( N \), and each “pair” is independent to each other (figure 3). Eq(5.1) becomes

\[ V_{\text{odd}\ N} = \frac{6\lambda}{4 \cdot N} \left[ \frac{1 + N_N}{\omega_N^2} a_N^\dagger a_N + \sum_{i=1}^{N-1} \frac{2 + 2N_{N-i}}{\omega_i\omega_{N-i}} a_i^\dagger a_i \right] \] (5.2)
By adding the kinematic term (4.5) and defining the recursion relations

\[ R_{N,\text{odd}}^{(n)} = \frac{\omega_N + 6\lambda \frac{1+N_N R_{N-1}^{(n-1)}}{4N}}{\omega_f + 6\lambda \frac{1+N_N}{4N} - \omega_f^2} \quad \text{(5.3)} \]

\[ R_{i,\text{odd}}^{(n)} = \frac{\omega_i + 6\lambda \frac{2+2N_{-i} R_{N-i}^{(n-1)}}{4N}}{\omega_f + 6\lambda \frac{2+2N_{-i}}{4N} - \omega_f^2}, \quad 1 \leq i \leq N-1 \quad \text{(5.4)} \]

the n-order complexity is

\[ D_{(N_1,\ldots,N_N)}^{(n)2} = (N_N + 1) \left( \ln \left( \sqrt[R_{N,\text{odd}}^{(n)}]{N_N} \right) \right)^2 + \sum_{i=1}^{N-1} (N_i + 1) \left( \ln \left( \sqrt[R_{i,\text{odd}}^{(n)}]{N_i} \right) \right)^2 \quad \text{(5.5)} \]

where \( R_i^{(0)} \) is defined in (3.19).

- Odd \( \frac{N}{2} \): These cases have two “circle” located at \( \frac{N}{2} \) and \( N \), pairing with each other (figure 4). The potential is

\[ V_{\text{odd \( \frac{N}{2} \)}}^{\text{"circle"}} = \frac{6\lambda}{4N} \left[ \left( \frac{1+N_N}{\omega_N^2} + \frac{2+2N_{-i}}{\omega_N^2 \omega_{N-i}} \right) a_i^\dagger a_i + \left( \frac{1+N_N}{\omega_N^2} + \frac{2+2N_{-i}}{\omega_N^2 \omega_{N-i}} \right) a_i^\dagger a_i \right] \quad \text{(5.6)} \]

The remaining contributions are those from pure “pairing” sites. Recalling the figure 2 and the relation (4.13) we can evaluate the corresponding potential. The result is

\[ V_{\text{odd \( \frac{N}{2} \)}}^{\text{"pair"}} = \frac{6\lambda}{4N} \left[ \sum_{i=1}^{N-1} \left( \frac{2+2N_{-i}}{\omega_i \omega_{N-i}} + \frac{2+2N_{-i}}{\omega_i \omega_{N-i}} \right) a_i^\dagger a_i + \sum_{i=\frac{N}{2}+1}^{N-1} \left( \frac{2+2N_{-i}}{\omega_i \omega_{N-i}} + \frac{2+2N_{-i}}{\omega_i \omega_{N-i}} \right) a_i^\dagger a_i \right] \quad \text{(5.7)} \]

By adding the kinematic term (4.5) and defining the recursion relations

\[ R_{\frac{N}{2},\text{even}}^{(n)} = \frac{\omega_N + 6\lambda \frac{1+N_N R_{N-1}^{(n-1)}}{4N}}{\omega_f + 6\lambda \frac{1+N_N}{4N} - \omega_f^2} \quad \text{(5.8)} \]
the potential of the two sites. We find

\[
V^{\text{circle}}_{\text{even } \frac{N}{2}} = 6\lambda \left[ \frac{1 + N_N}{\omega_N^2} + \frac{2 + 2N_N}{\omega_N \omega_N} \right] a_N^\dagger a_N + \left( \frac{1 + N_N}{\omega_N^2} + \frac{2 + 2N_N}{\omega_N \omega_N} \right) a_N^\dagger a_N + \left( \frac{1 + N_N}{\omega_N^2} + \frac{2 + 2N_N}{\omega_N \omega_N} \right) a_N^\dagger a_N
\]

(5.13)

Again, the remaining contributions are those from pure “pairing” sites. We find

\[
V^{\text{pair*}}_{\text{even } \frac{N}{2}} = \frac{6\lambda}{4 \cdot N} \left[ \sum_{i=1, \neq \frac{N}{4}}^{\frac{N}{4} - 1} \left( \frac{2 + 2N_{N-i}}{\omega_i \omega_{N-i}} + \frac{2 + 2N_{N-i}}{\omega_i \omega_{N-i}} \right) a_i^\dagger a_i + \sum_{i=\frac{N}{4}+1, \neq \frac{3N}{4}}^{\frac{N}{4} - 1} \left( \frac{2 + 2N_{N-i}}{\omega_i \omega_{N-i}} + \frac{2 + 2N_{N-i}}{\omega_i \omega_{N-i}} \right) a_i^\dagger a_i \right]
\]

(5.14)

The above result is the same as the odd \( \frac{N}{4} \), i.e. \( \ref{5.7} \), but drop the “circle” at \( \frac{N}{4} \) and \( \frac{3N}{4} \) since the potential of the two “circle” has been considered in \( V^{\text{circle}}_{\text{even } \frac{N}{2}} \).
By adding the kinematic term (4.5) and defining the recursion relations

\[
R_{N,\text{even}}^{(n)} = \frac{\omega N}{4} + \frac{6\lambda}{4N} \left( \frac{1 + N N}{\omega \frac{1}{4}} + \frac{2 + 2N N}{\omega N + \omega \frac{1}{4}} \right),
\]

\[
R_{3N,\text{even}}^{(n)} = \frac{\omega 3N}{4} + \frac{6\lambda}{4N} \left( \frac{1 + N N}{\omega \frac{1}{4}} + \frac{2 + 2N N}{\omega N + \omega \frac{1}{4}} \right),
\]

the n-order complexity is

\[
D_{(N_1,\ldots,N_N)}^{(n)} = (N_\frac{n}{2} + 1) \left( \ln \left( \sqrt{R_{N,\text{even}}^{(n)}} \right) \right)^2 + (N_N + 1) \left( \ln \left( \sqrt{R_{N,N,\text{even}}^{(n)}} \right) \right)^2 + \sum_{i=1,\neq \frac{N}{2}}^{N-1} (N_i + 1) \left( \ln \left( \sqrt{R_{i,\text{even}}^{(n)}} \right) \right)^2 + \sum_{i=\frac{N}{2}+1,\neq \frac{3N}{4}}^{N-1} (N_i + 1) \left( \ln \left( \sqrt{R_{i,\text{even}}^{(n)}} \right) \right)^2 \]

These general formulae allow one to obtain higher-order complexity for excited states at any

5.2 Complexity of N Coupled Oscillators : Numerical Results

We now use above formulas to perform numerical calculations and plot several diagrams to il-

luminate the properties of complexity.

(1) We plot in figure 13 the complexity for various lattice site number N.

![Figure 13: Complexity v.s. lattice site number N](image)

It shows that the complexity increases with site number N, as that in free theory. This consists
with the relation : complexity=volume (CV) conjecture since in one dimension the volume
is proportional to site number N. To plot figure 13 (and following figures) we choose the scale
of $\omega = 1$ and use the following values: $N_i = 1$, $\omega_f = 1$, $\lambda = 0.1$. The dependence of complexity on $N_i$, $\omega_f$, or $\lambda$ is illuminated in the following figures.

(2) We plot in figure 14 the complexity for various excited state $N_i$.

![Figure 14: Complexity v.s. excited state $N_i$](image1)

It shows that the complexity becomes larger in higher excited state, as that in free theory.

(3) We plot in figure 15 the complexity for various reference state frequency $\omega_f$.

![Figure 15: Complexity v.s. reference state frequency $\omega_f$](image2)

It shows that the complexity becomes larger for large $\omega_f$, as that in free theory.

(4) We plot in figure 16 the complexity for various coupling constant $\lambda$ in lattice $\lambda\phi^4$ theory.

![Figure 16: Complexity v.s. coupling constant $\lambda$](image3)
Figure 16: Complexity v.s. coupling constant $\lambda$. Left-hand diagram is that with $\omega = 10$, $\omega_f = 0.01$. Right-hand diagram is that with $\omega = \omega_f = 1$.

It shows that the complexity may increase or decrease while increasing coupling constant $\lambda$. The property of how the complexity depends on $\lambda$ can be seen, for example, from eq.(3.26) and eq.(3.27). The interaction correction to complexity in the two relations is proportional the coefficient of $\frac{3\lambda}{\omega_f}$, which is negative for large $\omega_f$ and become positive for small $\omega_f$. Figure 16 is consistent with this argument.

6 Concluding Remarks

We adopt operator approach to compute the complexity of the lattice $\lambda\phi^4$ scalar theory. A perturbation algorithm has been developed for computing the complexity to obtain the general formulae (5.5), (5.12), and (5.17) which can be used to obtain higher-order complexity of excited states for any $N$ lattice sites. The interaction correction to complexity may be positive or negative depending on the magnitude of reference-state frequency.

We conclude the paper by the remark: Our algorithm is based on a simple relation

$$\lambda a_j^\dagger a_j a_j^\dagger a_j \rightarrow \lambda N_j a_j^\dagger a_j \rightarrow \lambda N_j R_j^{(n-1)} a_j^\dagger a_j \quad (6.1)$$

in which the first arrow is due to the perturbation property while the second one is use to calculate the complexity. The relation is explained in sec.3.2. The similar relation could be found in many other systems. For examples:

- It is easily to see that our method could be used in interacting Fermion theory.
- For the theory which has two different field operators $a_j$ and $b_j$ and associated interaction is $\lambda \phi^2 \xi^2$ the relation will become

$$\lambda a_j^\dagger a_j b_j^\dagger b_j \rightarrow \lambda \frac{1}{2} N_j^{(b)} a_j^\dagger a_j + \lambda N_j^{(a)} b_j^\dagger b_j \rightarrow \lambda \frac{1}{2} N_j^{(b)} R_j^{(b)(n-1)} a_j^\dagger a_j + \lambda N_j^{(a)} R_j^{(a)(n-1)} b_j^\dagger b_j \quad (6.2)$$

in which the fields $\phi$ and $\xi$ could be Boson or Fermion field.
- For the $\lambda\phi^6$ theory the relation will become

$$\lambda a_j^\dagger a_j a_j^\dagger a_j \rightarrow \lambda N_j^{(a)} a_j^\dagger a_j \rightarrow \lambda (N_j R_j^{(n-1)})^2 a_j^\dagger a_j \quad (6.3)$$

Of course, the associated diagrams and basic rules in each case shall be slightly modified.

In this way, our algorithm can be applied to many quantum field theories and several many-body models in condense matter. We will study the problem in the next series of paper.

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