A van der Corput lemma and weak mixing over groups

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2005-8-17

Abstract. We study weak mixing of all orders for weakly mixing measure preserving dynamical systems, where the dynamics is given by the action of an abelian second countable topological group which has an invariant measure under the group operation. One of the main technical tools we use is a van der Corput lemma for Hilbert space valued functions on a second countable topological group.
1 Introduction

Weak mixing is an important notion in ergodic theory, introduced by Koopman and von Neumann [9] in 1932 for actions of the group $\mathbb{R}$. Furstenberg [5, 6] studied weakly mixing $\mathbb{Z}$-actions in order to give an ergodic theoretic proof of Szemerédi’s theorem in combinatorial number theory, and in the process he proved that weakly mixing systems are weakly mixing of all orders. In the case of $\mathbb{Z}$, a measure preserving transformation $T$ of a probability space $(X, \Sigma, \nu)$, namely a set $X$ with $\sigma$-algebra $\Sigma$ on which a measure $\nu$ with $\nu(X) = 1$ is defined, is called weakly mixing if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \nu(A \cap T^{-n}(B)) - \nu(A)\nu(B) \right| = 0$$

(1.1)

for all $A, B \in \Sigma$. We call this system weakly mixing of all orders if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left| \nu(A_{0} \cap T^{-m_{1}n}(A_{1}) \cap \ldots \cap T^{-m_{k}n}(A_{k})) - \nu(A_{0})\nu(A_{1})\ldots\nu(A_{k}) \right| = 0$$

(1.2)

for all $A_{0}, \ldots, A_{k} \in \Sigma$, all $m_{1}, \ldots, m_{k} \in \mathbb{N}$ with $m_{1} < m_{2} < \ldots < m_{k}$, and all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

However, weak mixing has also been studied for more general group (and semigroup) actions, notably in [4] and [2] (but also see references therein), in terms of invariant means [11] on certain spaces of functions, instead of in terms of the explicit form $\frac{1}{N} \sum_{n=1}^{N}$ above. In particular, various characterizations of weak mixing over the groups $\mathbb{Z}$ and $\mathbb{R}$ were extended to more general groups.

In this paper we study weak mixing of all orders for more general group actions than $\mathbb{Z}$ and $\mathbb{R}$. One of the technical tools we use, is a so-called van der Corput lemma which we discuss in Section 2. This type of lemma and related inequalities, inspired by the classical van der Corput difference theorem and van der Corput inequality, have been used by Bergelson [1], Furstenberg [7], Niculescu, Ströh, and Zsidó [10], and others, to study polynomial ergodic theorems, nonconventional ergodic averages, and noncommutative recurrence, for example. In Section 2 we extend the van der Corput lemma to groups more general than $\mathbb{Z}$. The main results of this section are given by Theorems 2.7 and 2.7'. Instead of working with an invariant mean on spaces of functions, we generalize the $\frac{1}{N} \sum_{n=1}^{N}$ form more directly for groups with an invariant measure, since this seems convenient in our proof of the van der Corput lemma. Because of this, we also study weak mixing by generalizing the $\frac{1}{N} \sum_{n=1}^{N}$ form directly, rather than using the invariant mean approach of
[2]. The groups over which we work, need to have an invariant measure, and a space-filling sequence (defined in Section 2). After some preliminaries on weak mixing in Section 3, we devote Section 4 to showing how weak mixing implies weak mixing of all orders, for actions of abelian second countable topological groups of this type. The form of weak mixing of all orders we prove, involves replacing the multiplication with $m_1, \ldots, m_k$ in (1.2), by homomorphisms of the group over which we work. The main result is Theorem 4.4.

2 A van der Corput lemma

This section is devoted to proving a van der Corput lemma, stated in two versions in Theorems 2.7 and 2.7'. Our proof of the van der Corput lemma will roughly follow that of [7] over the group $\mathbb{Z}$. In this section we will work over a second countable topological group (i.e. a second countable topological space which is also a group with continuous product and inverse), since for second countable topological spaces $X, Y$, and their Borel $\sigma$-algebras $S, T$, the product $\sigma$-algebra obtained from $S, T$ is the same as the Borel $\sigma$-algebra of the topological space $X \times Y$. This is needed in order to apply Fubini’s theorem, which requires measurability in the product $\sigma$-algebra. The groups that we will consider in this paper, will only be required to be abelian from Definition 4.2 and onwards, in Section 4.

For $(Y, \mu)$ a measure space and $\mathcal{H}$ a Hilbert space, consider a bounded $f : \Lambda \to \mathcal{H}$ with $\Lambda \subset Y$ measurable and $\mu(\Lambda) < \infty$, and $\langle f(\cdot), x \rangle$ measurable for every $x \in \mathcal{H}$. Define $\int_{\Lambda} f d\mu$ by requiring

$$\left\langle \int_{\Lambda} f d\mu, x \right\rangle := \int_{\Lambda} \langle f(y), x \rangle \, d\mu(y)$$

for all $x \in \mathcal{H}$. We will often use the notation $\int_{\Lambda} f(y) \, d\mu = \int_{\Lambda} f \, d\mu$, since there will be no ambiguity in the measure being used. Iterated integrals (when they exist) will be written as $\int_B \int_A f(y, z) \, dy \, dz$, which of course simply means $\int_B \left[ \int_A f(y, z) \, dy \right] \, dz$, and similarly for triple integrals.

For a group $G$ we call $K \subset G$ a subsemigroup if $ab \in K$ for all $a, b \in K$. A right invariant measure on a topological group $G$, is a positive measure $\mu$ on the Borel $\sigma$-algebra of $G$, with $\mu(\Lambda g) = \mu(\Lambda)$ for all Borel $\Lambda \subset G$ and $g \in G$. Similarly for a left invariant measure. If the measure is both right and left invariant, we simply call it invariant. We define such measures for topological semigroups in the same way.

When we say that a net $\{\Lambda_\alpha\}$ has some property for $\alpha$ “large enough”, then we mean that there is a $\beta$ in the directed set such that the property
holds for all $\alpha \geq \beta$.

**Definition 2.1.** Consider a Borel measurable subsemigroup $K$ of a topological group $G$ with right invariant measure $\mu$. A net $\{\Lambda_\alpha\}$ of Borel subsets of $K$ is called a space-filling net in $K$ (or a Følner net in $K$) if $\mu(\Lambda_\alpha) < \infty$, $\mu(\Lambda_\beta) > 0$ for $\beta$ large enough, and

$$\lim_{\beta} \frac{1}{\mu(\Lambda_\beta)} \mu(\Lambda_\beta \Delta (\Lambda_\beta g)) = 0$$

for all $g \in K$. This net $\{\Lambda_\alpha\}$ is called uniformly space-filling if in addition

$$\lim_{\beta} \frac{1}{\mu(\Lambda_\beta)} \sup_{g \in \Lambda_\alpha} \mu(\Lambda_\beta \Delta (\Lambda_\beta g)) = 0$$

for all $\alpha$ in the directed set of the net.

At the end of Section 4, we briefly consider simple examples of such nets.

**Proposition 2.2.** Let $G$ be a second countable topological group with right invariant measure $\mu$. Let $\{\Lambda_\alpha\}$ be a uniformly space-filling net in a Borel measurable subsemigroup $K$ of $G$. Consider a bounded $f : K \to \mathcal{H}$ with $\mathcal{H}$ a Hilbert space, such that $\langle f(\cdot), x \rangle$ is Borel measurable for every $x \in \mathcal{H}$. Then

$$\lim_{\beta} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} f \, d\mu - \frac{1}{\mu(\Lambda_\beta)} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\beta} \int_{\Lambda_\alpha} f(gh) \, dh \, dg \right| = 0$$

for every $\alpha$ in the directed set of the net.

**Proof.** By Fubini’s theorem

$$\int_{\Lambda_\beta} \int_{\Lambda_\alpha} \langle f(gh), x \rangle \, dh \, dg = \int_{\Lambda_\beta \times \Lambda_\alpha} \langle f(gh), x \rangle \, d(g, h)$$

$$= \int_{\Lambda_\beta} \int_{\Lambda_\alpha} \langle f(gh), x \rangle \, dg \, dh$$

which by definition means that

$$\int_{\Lambda_\beta} \int_{\Lambda_\alpha} f(gh) \, dh \, dg = \int_{\Lambda_\alpha} \int_{\Lambda_\beta} f(gh) \, dg \, dh$$

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and in particular these iterated integrals exists. From this and the fact that 
$\mu$ is a right invariant measure, we have

\[
\left\| \frac{1}{\mu(\Lambda_{\beta})} \int_{\Lambda_{\beta}} f d\mu - \frac{1}{\mu(\Lambda_{\beta})} \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\beta}} \int_{\Lambda_{\alpha}} f(gh) dhdg \right\| \\
= \left\| \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} \left[ \frac{1}{\mu(\Lambda_{\beta})} \int_{\Lambda_{\beta}} f(g) dg \right] dh - \frac{1}{\mu(\Lambda_{\alpha})} \frac{1}{\mu(\Lambda_{\beta})} \int_{\Lambda_{\alpha}} \left[ \int_{\Lambda_{\beta}} f(gh) dg \right] dh \right\| \\
= \left\| \frac{1}{\mu(\Lambda_{\alpha})} \frac{1}{\mu(\Lambda_{\beta})} \int_{\Lambda_{\alpha}} \left[ \int_{\Lambda_{\beta}} f(g) dg - \int_{\Lambda_{\beta}} f(gh) dg \right] dh \right\| \\
= \left\| \frac{1}{\mu(\Lambda_{\alpha})} \frac{1}{\mu(\Lambda_{\beta})} \int_{\Lambda_{\alpha}} \left[ \int_{\Lambda_{\beta} \setminus (\Lambda_{\beta} \cap \Lambda_{\beta}h)} f(g) dg - \int_{(\Lambda_{\beta}h) \setminus (\Lambda_{\beta} \cap \Lambda_{\beta}h)} f(g) dg \right] dh \right\| .
\]

But if $b \in \mathbb{R}$ is an upper bound for $\|f(K)\|$, we have

\[
\left\| \int_{\Lambda_{\beta} \setminus (\Lambda_{\beta} \cap \Lambda_{\beta}h)} f(g) dg - \int_{(\Lambda_{\beta}h) \setminus (\Lambda_{\beta} \cap \Lambda_{\beta}h)} f(g) dg \right\| \\
\leq b \mu (\Lambda_{\beta} \setminus (\Lambda_{\beta} \cap \Lambda_{\beta}h)) + b \mu ((\Lambda_{\beta}h) \setminus (\Lambda_{\beta} \cap \Lambda_{\beta}h))) \\
= b \mu (\Lambda_{\beta} \Delta (\Lambda_{\beta}h)) \\
\leq b \sup_{h \in \Lambda_{\alpha}} \mu (\Lambda_{\beta} \Delta (\Lambda_{\beta}h))
\]

therefore

\[
\left\| \frac{1}{\mu(\Lambda_{\beta})} \int_{\Lambda_{\beta}} f d\mu - \frac{1}{\mu(\Lambda_{\beta})} \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\beta}} \int_{\Lambda_{\alpha}} f(gh) dhdg \right\| \\
\leq \frac{1}{\mu(\Lambda_{\beta})} b \sup_{h \in \Lambda_{\alpha}} \mu (\Lambda_{\beta} \Delta (\Lambda_{\beta}h)) \\
\to 0
\]
in the $\beta$ limit. □

**Lemma 2.3.** Let $\mathfrak{H}$ be a Hilbert space, $(Y, \mu)$ a measure space, and $\Lambda \subset Y$ a measurable set with $\mu(\Lambda) < \infty$. Consider an $f : \Lambda \to \mathfrak{H}$ with $\|f(\cdot)\|$ measurable, and $(f(\cdot), x)$ measurable for every $x \in \mathfrak{H}$, and with $\int_{\Lambda} \|f(y)\| dy < \infty$ (which means $\int_{\Lambda} f d\mu$ exists). Then

\[
\left\| \int_{\Lambda} f d\mu \right\|^2 \leq \mu(\Lambda) \int_{\Lambda} \|f(y)\|^2 dy .
\]
Proof. By definition of $\int_A fd\mu$,
\[
\left\| \int_A fd\mu \right\|^2 = \left\langle \int_A fd\mu, \int_A fd\mu \right\rangle = \int_A \left\langle f(y), \int_A f d\mu \right\rangle dy \\
= \int_A \left[ \int_A \langle f(y), f(z) \rangle dz \right] dy.
\]
For any $a, b \in \mathfrak{A}$ we have $2 \text{Re} \langle a, b \rangle \leq ||a||^2 + ||b||^2$, and since the object above is real, we have
\[
\left\| \int_A fd\mu \right\|^2 = \int_A \left[ \int_A \text{Re} \langle f(y), f(z) \rangle dz \right] dy \\
\leq \frac{1}{2} \int_A \left[ \int_A (||f(y)||^2 + ||f(z)||^2) dz \right] dy \\
= \mu(A) \int_A ||f(y)||^2 dy . \quad \Box
\]

Proposition 2.4. Consider the situation in Proposition 2.2, except that we don’t need the net. Assume furthermore that $F : K \times K \to \mathbb{C} : (g, h) \mapsto \langle f(g), f(h) \rangle$ is Borel measurable, and that $\Lambda_1, \Lambda_2 \subset K$ are Borel sets with $\mu(\Lambda_j) < \infty$. Then
\[
\left\| \int_{\Lambda_2} \int_{\Lambda_1} f(gh) dh dg \right\|^2 \\
\leq \mu(\Lambda_2) \int_{\Lambda_1} \int_{\Lambda_1} \int_{\Lambda_2} \langle f(gh_1), f(gh_2) \rangle dgdh_1dh_2
\]
and in particular these integrals exist.

Proof. The double integral exists as in Proposition 2.2’s proof. Let’s now consider the triple integral. Since $F$ is Borel measurable and $G$’s product is continuous, $(g, h_1) \mapsto \langle f(gh_1), f(gh_2) \rangle$ is Borel measurable on $K \times K = K^2$ and hence measurable in the product $\sigma$-algebra on $K^2$. By Fubini’s theorem we have
\[
\int_{\Lambda_1} \int_{\Lambda_2} \langle f(gh_1), f(gh_2) \rangle dgdh_1 = \int_{\Lambda_1 \times \Lambda_2} \langle f(gh_1), f(gh_2) \rangle d(h_1, g)
\]
and in particular the iterated integral exists. Furthermore, $K \times K^2 \to K^2 : (h_2, h_1, g) \mapsto (gh_1, gh_2)$ is continuous, so $K \times K^2 \to \mathbb{C} : (h_2, h_1, g) \mapsto \langle f(gh_1), f(gh_2) \rangle$ is Borel measurable.
\[ \langle f(gh_1), f(gh_2) \rangle \] is measurable in the product \( \sigma \)-algebra of \( K \) and \( K^2 \). Hence by Fubini’s theorem
\[
\int_{\Lambda_1} \int_{\Lambda_2} \langle f(gh_1), f(gh_2) \rangle \, d(h_1, g) \, d_h = \int_{\Lambda_1 \times \Lambda_2} \langle f(gh_1), f(gh_2) \rangle \, d(h_1, h_2, g)
\]
and in particular, the triple integral exists, and we can do the three integrals in any order. By Lemma 2.3 it follows that
\[
\left\| \int_{\Lambda_2} \int_{\Lambda_1} f(gh) \, dhdg \right\|^2 \\
\leq \mu(\Lambda_2) \int_{\Lambda_2} \left\| \int_{\Lambda_1} f(gh) \, dh \right\|^2 \, d g \\
= \mu(\Lambda_2) \int_{\Lambda_2} \left( \int_{\Lambda_1} f(gh_1) \, dh_1, \int_{\Lambda_1} f(gh_2) \, dh_2 \right) \, d g \\
= \mu(\Lambda_2) \int_{\Lambda_2} \int_{\Lambda_1} \left( \int_{\Lambda_1} f(gh_1) \, dh_1, \int_{\Lambda_1} f(gh_2) \, dh_2 \right) \, d h_1 d g \\
= \mu(\Lambda_2) \int_{\Lambda_2} \int_{\Lambda_1} \int_{\Lambda_1} \langle f(gh_1), f(gh_2) \rangle \, d h_2 d h_1 d g \\
= \mu(\Lambda_2) \int_{\Lambda_2} \int_{\Lambda_1} \int_{\Lambda_2} \langle f(gh_1), f(gh_2) \rangle \, d g d h_1 d h_2
\]
and note in particular that the part of this argument after the inequality proves that \( g \mapsto \left\| \int_{\Lambda_1} f(gh) \, dh \right\|^2 \) is measurable (and therefore its square root too), which means that Lemma 2.3 does indeed apply to this situation. \( \square \)

For the next three results we give two versions of each, one set of results for nets, and one for sequences but with other assumptions a bit weaker. The weaker assumptions in case of sequences are possible, since in this case we can apply Lebesgue’s dominated convergence theorem (see the proof of Proposition 2.6’). The case of sequences will be used in Section 4 when we study weak mixing of all orders.

**Lemma 2.5.** Consider the situation in Proposition 2.2, but assume that \( f : G \to \mathcal{F} \) is bounded and \( \langle f(\cdot), x \rangle \) measurable for all \( x \notin \mathcal{F} \). The net is still uniformly space-filling only in \( K \), though. Assume that \( F : G^2 \to \mathbb{C} : (g, h) \mapsto \langle f(g), f(h) \rangle \) is Borel measurable. Then \( \int_{\Lambda} \langle f(g), f(gh) \rangle \, d g \) exists for all measurable \( \Lambda \subset G \) with \( \mu(\Lambda) < \infty \), and all \( h \in G \). Assume that
\[
\gamma_h := \lim_{\beta} \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(g), f(gh) \rangle \, d g
\]

exists for all \( h \in G \), and that
\[
\lim \sup_{\beta} \sup_{h \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(g), f(gh) \rangle dg - \gamma_h \right| = 0 \tag{2.5.1}
\]
for all \( \alpha \). Then
\[
\lim \sup_{\beta} \sup_{h_1, h_2 \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle dg - \gamma_{h_1^{-1}h_2} \right| = 0 \tag{2.5.2}
\]
for all \( \alpha \). In particular \( \Lambda_\alpha \times \Lambda_\alpha \ni (h_1, h_2) \mapsto \gamma_{h_1^{-1}h_2} \) is bounded for every \( \alpha \).

**Proof.** Since \( F \) is Borel, and \( G \to G^2 : g \mapsto (g, gh) \) is continuous, the map \( G \to \mathbb{C} : g \mapsto \langle f(g), f(gh) \rangle \) is Borel for every \( h \in G \), hence the integrals are defined. Now,

\[
\sup_{h_1, h_2 \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle dg - \gamma_{h_1^{-1}h_2} \right|
\]

\[
\leq \sup_{h_1, h_2 \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle dg - \int_{\Lambda_\beta} \langle f(g), f(gh_1^{-1}h_2) \rangle dg \right|
\]

\[
+ \sup_{h_1, h_2 \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(g), f(gh_1^{-1}h_2) \rangle dg - \gamma_{h_1^{-1}h_2} \right|
\]

but since \( \mu \) is a right invariant measure
\[
\frac{1}{\mu(\Lambda_\beta)} \left| \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle dg - \int_{\Lambda_\beta} \langle f(g), f(gh_1^{-1}h_2) \rangle dg \right|
\]

\[
= \frac{1}{\mu(\Lambda_\beta)} \left| \int_{\Lambda_\beta} \langle f(g), f(gh_1^{-1}h_2) \rangle dg - \int_{\Lambda_\beta} \langle f(g), f(gh_1^{-1}h_2) \rangle dg \right|
\]

\[
= \frac{1}{\mu(\Lambda_\beta)} \left| \int_{\Lambda_\beta \setminus \Lambda_{\Delta h_1}} \langle f(g), f(gh_1^{-1}h_2) \rangle dg \right|
\]

\[
\leq \frac{1}{\mu(\Lambda_\beta)} \left| \int_{\Lambda_\beta \setminus \Lambda_{\Delta h_1}} \langle f(g), f(gh_1^{-1}h_2) \rangle dg \right|
\]

\[
+ \int_{\Lambda_\beta \setminus \Lambda_{\Delta h_1}} \left| \langle f(g), f(gh_1^{-1}h_2) \rangle \right| dg
\]

\[
= \frac{1}{\mu(\Lambda_\beta)} \left| \int_{\Lambda_\beta \setminus \Lambda_{\Delta h_1}} \langle f(g), f(gh_1^{-1}h_2) \rangle \right| dg
\]

\[
\leq \frac{\mu(\Lambda_\beta \setminus \Lambda_{\Delta h_1})}{\mu(\Lambda_\beta)} b
\]
for all \( h_1 \in K \), where \( b \) is an upper bound for \((g, h_1, h_2) \mapsto \left| \langle f(g), f(gh_1^{-1}h_2) \rangle \right|\), which exists since \( f \) is bounded. This proves (2.5.2). Since \( f \) is bounded, so is \((h_1, h_2) \mapsto \langle f(gh_1), f(gh_2) \rangle \) and its integral with respect to \( g \) over \( \Lambda_\beta \). Hence (2.5.2) implies that \( \Lambda_\alpha \times \Lambda_\alpha \ni (h_1, h_2) \mapsto \gamma h_1^{-1}h_2 \) is bounded. \( \square \)

**Lemma 2.5'.** Consider the situation in Lemma 2.5, except that \( \{\Lambda_\alpha\} \) need not be uniform. Assume

\[
\gamma_h := \lim_\beta \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(g), f(gh) \rangle \, dg
\]

exists for all \( h \in G \). (We need not assume 2.5.1.) Then

\[
\lim_\beta \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle \, dg = \gamma h_1^{-1}h_2
\]

for all \( h_1 \in K \) and \( h_2 \in G \).

**Proof.** Simply repeat Lemma 2.5’s proof without the sup’s. \( \square \)

**Proposition 2.6.** Consider the situation in Lemma 2.5. Assuming that \( K^2 \to \mathbb{C} : (h_1, h_2) \mapsto \gamma h_1^{-1}h_2 \) is Borel measurable, we have

\[
\lim_\beta \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle \, dg \, dh_1 \, dh_2 = \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \gamma h_1^{-1}h_2 \, dh_1 \, dh_2
\]

for all \( \alpha \).

**Proof.** The triple integral exists by Proposition 2.4. The double integral exists by Fubini’s theorem, since \((h_1, h_2) \mapsto \gamma h_1^{-1}h_2\) is bounded on \( \Lambda_\alpha \times \Lambda_\alpha \) for every \( \alpha \) by Lemma 2.5. Then

\[
\left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle \, dg \, dh_1 \, dh_2 - \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \gamma h_1^{-1}h_2 \, dh_1 \, dh_2 \right|
\]

\[
\leq \mu(\Lambda_\alpha)^2 \sup_{h_1, h_2 \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle \, dg \, dh_1 \, dh_2 - \gamma h_1^{-1}h_2 \right|
\]

\[\to 0\]

in the \( \beta \) limit by Lemma 2.5. \( \square \)

**Proposition 2.6’.** Consider the situation in Lemma 2.5’, but assume the space-filling net in \( K \) is in fact a sequence \( \{\Lambda_n\}_{n \in \mathbb{N}} \). Then

\[
\lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_m} \int_{\Lambda_m} \int_{\Lambda_n} \langle f(gh_1), f(gh_2) \rangle \, dg \, dh_1 \, dh_2 = \int_{\Lambda_m} \int_{\Lambda_m} \gamma h_1^{-1}h_2 \, dh_1 \, dh_2
\]
for all \( m \), and in particular these integrals exist.

**Proof.** The triple integral exists by Proposition 2.4. Let \( b \) be an upper bound for \( (g, h_1, h_2) \mapsto |\langle f(gh_1), f(gh_2) \rangle| \), which exists since \( f \) is bounded. Fix any \( m \in \mathbb{N} \), and set

\[
A_n(h_1, h_2) := \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle f(gh_1), f(gh_2) \rangle \, dg
\]

for all \( h_1, h_2 \in \Lambda_m \) and all \( n \). Note that \( A_n(h_1, h_2) \) exists and is a measurable function of \( h_1 \) because of the existence of the triple integral. Then \( |A_n(h_1, h_2)| \leq \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} b \, dg = b \), which implies that the sequence \( A_n(\cdot, h_2) \) with \( h_2 \) fixed, is dominated by \( B : \Lambda_m \to \mathbb{R} : h \mapsto b \). But \( B \in L^1(\Lambda_m, \mu) \), namely \( \int_{\Lambda_m} |B| \, d\mu = b \mu(\Lambda_m) < \infty \), hence \( \Lambda_m \ni h_1 \mapsto \gamma_{h_1^{-1}h_2} \) is in \( L^1(\Lambda_m, \mu) \) and

\[
\lim_{n \to \infty} \int_{\Lambda_m} A_n(h_1, h_2) \, dh_1 = \int_{\Lambda_m} \gamma_{h_1^{-1}h_2} \, dh_1
\]

by Lebesgue’s dominated convergence theorem and Lemma 2.5’. Note in particular that the last integral exists. Now set

\[
C_n(h_2) := \int_{\Lambda_m} A_n(h_1, h_2) \, dh_1
\]

for all \( h_2 \in \Lambda_m \), and keep in mind that this exists and is a measurable function of \( h_2 \) by Proposition 2.4, as for \( A_n(\cdot, h_2) \) earlier. Then \( |C_n(h_2)| \leq \int_{\Lambda_m} b \, dh_1 \leq \mu(\Lambda_m) b \), so the sequence \( C_n \) is dominated by \( D : \Lambda_m \to \mathbb{R} : h \mapsto \mu(\Lambda_m) b \), and \( D \in L^1(\Lambda_m, \mu) \). Hence by Lebesgue’s dominated convergence theorem, the function

\[
\Lambda_m \ni h_2 \mapsto \int_{\Lambda_m} \gamma_{h_1^{-1}h_2} \, dh_1
\]

is in \( L^1(\Lambda_m, \mu) \), and

\[
\lim_{n \to \infty} \int_{\Lambda_m} C_n(h_2) \, dh_2 = \int_{\Lambda_m} \int_{\Lambda_m} \gamma_{h_1^{-1}h_2} \, dh_1 \, dh_2
\]

as required. \( \square \)

Now we can finally state a van der Corput lemma:

**Theorem 2.7.** Let \( G \) be a second countable topological group with right invariant measure \( \mu \). Let \( \{\Lambda_\alpha\} \) be a uniformly space-filling net in a Borel measurable subsemigroup \( K \) of \( G \). Consider a bounded \( f : G \to \mathcal{H} \), with
a Hilbert space, such that \( \langle f(\cdot), x \rangle \) and \( \langle f(\cdot), f(\cdot) \rangle : G^2 \to \mathbb{C} \) are Borel measurable (for all \( x \in \mathcal{H} \)). Assume that

\[
\gamma_h := \lim_{\beta} \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(g), f(gh) \rangle \, dg
\]

exists for all \( h \in G \), and that

\[
\lim_{\beta} \sup_{h \in \Lambda_\alpha} \left| \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} \langle f(g), f(gh) \rangle \, dg - \gamma_h \right| = 0
\]

for all \( \alpha \). Assume furthermore \( G \to \mathbb{C} : h \mapsto \gamma_h \) is Borel measurable and that

\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)^2} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \gamma_{h^{-1} h_2} dh_1 dh_2 = 0 \quad .
\] (2.7.1)

Then

\[
\lim_{\beta} \frac{1}{\mu(\Lambda_\beta)} \int_{\Lambda_\beta} f \, d\mu = 0 \quad .
\]

Proof. Note that since \( G^2 \to G : (h_1, h_2) \mapsto h_1^{-1} h_2 \) is continuous, its composition with \( h \mapsto \gamma_h \), namely \( (h_1, h_2) \mapsto \gamma_{h^{-1} h_2} \), is Borel. By Proposition 2.2 and Proposition 2.4 we just have to show that for any \( \varepsilon > 0 \) there is an \( \alpha \) and \( \beta_0 \) such that \( |A_{\alpha,\beta}| < \varepsilon \) for all \( \beta > \beta_0 \) where

\[
A_{\alpha,\beta} := \frac{1}{\mu(\Lambda_\beta)} \frac{1}{\mu(\Lambda_\alpha)^2} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \int_{\Lambda_\beta} \langle f(gh_1), f(gh_2) \rangle \, dg \, dh_1 \, dh_2
\]

But this follows from Proposition 2.6 and our assumptions, namely

\[
\limsup_{\alpha} \lim_{\beta} A_{\alpha,\beta} = \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)^2} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \gamma_{h^{-1} h_2} dh_1 dh_2 = 0 \quad . \square
\]

Next we give a van der Corput lemma for a space-filling sequence instead of a net:

**Theorem 2.7′.** Let \( G \) be a second countable topological group with right invariant measure \( \mu \). Let \( \{\Lambda_n\} \) be a uniformly space-filling sequence in a Borel measurable subsemigroup \( K \) of \( G \). Consider a bounded \( f : G \to \mathcal{H} \), with \( \mathcal{H} \) a Hilbert space, such that \( \langle f(\cdot), x \rangle \) and \( \langle f(\cdot), f(\cdot) \rangle : G^2 \to \mathbb{C} \) are Borel measurable (for all \( x \in \mathcal{H} \)). Assume

\[
\gamma_h := \lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle f(g), f(gh) \rangle \, dg
\]
exists for all \( h \in G \). Also assume that

\[
\lim_{m \to \infty} \frac{1}{\mu(\Lambda_m)^2} \int_{\Lambda_m} \int_{\Lambda_m} \gamma_{h_1^{-1}h_2}dh_1dh_2 = 0 \tag{2.7.1}
\]

(note that the integral exists by Proposition 2.6'). Then

\[
\lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} f d\mu = 0 .
\]

**Proof.** Just as for Theorem 2.7, but using Proposition 2.6' instead of 2.6, and therefore without the need to show that \((h_1, h_2) \mapsto \gamma_{h_1^{-1}h_2}\) is Borel. \(\square\)

Theorem 2.7' is the version of the van der Corput lemma that we will apply in Section 4 to prove that weak mixing implies weak mixing of all orders. However, we still need a few refinements regarding conditions (2.7.1) and (2.7'.1):

**Lemma 2.8.** Let \( G \) be a second countable topological group with left invariant measure \( \mu \). Let \( \Lambda \subset G \) be Borel and \( \mu(\Lambda) < \infty \), and \( S \subset G \) Borel such that \( \Lambda^{-1}\Lambda := \{ h_1^{-1}h_2 : h_1, h_2 \in \Lambda \} \subset S \). For a Borel \( f : G \to \mathbb{R}^+ \) we then have

\[
\int_{\Lambda} \int_{\Lambda} f(h_1^{-1}h_2)dh_1dh_2 \leq \mu(\Lambda) \int_{S} f d\mu .
\]

**Proof.** Let \( \chi \) denote characteristic functions, and set \( \varphi : \Lambda \times \Lambda \to G : (h_1, h_2) \mapsto h_1^{-1}h_2 \). Then \( f \circ \varphi \) is Borel on \( \Lambda \times \Lambda \), and therefore measurable in the product \( \sigma \)-algebra on \( \Lambda \times \Lambda \) obtained from \( \Lambda \)'s Borel \( \sigma \)-algebra, since \( \varphi \) is continuous. Let \( Y \subset \Lambda^{-1}\Lambda \) be Borel in \( G \). For \( W \subset G \times G \), let \( W_g := \{ h : (g, h) \in W \} \). Then, since \( \varphi^{-1}(Y) \) is Borel in \( \Lambda \times \Lambda \) and hence Borel in \( G \times G \), it follows that \( \varphi^{-1}(Y) \) is in the product \( \sigma \)-algebra on \( G \times G \), hence we can consider \( (\mu \times \mu)(\varphi^{-1}(Y)) = \int_{\Lambda} \mu((\varphi^{-1}(Y))_{g})dg \). Now

\[
\varphi^{-1}(Y) = \{(g, gh) : h \in Y, g \in \Lambda \cap (Ah^{-1})\} \subset \{(g, gh) : h \in Y, g \in \Lambda\} =: V
\]

but \( V_g = gY \), therefore \( \mu(\varphi^{-1}(Y)) \leq \mu(V) = \mu(gY) = \mu(Y) \), since \( \mu \) is a left invariant. Hence

\[
\int_{\Lambda \times \Lambda} \chi_{Y} \circ \varphi d(\mu \times \mu) = (\mu \times \mu)(\varphi^{-1}(Y)) \leq \mu(\Lambda)\mu(Y) = \mu(\Lambda) \int_{S} \chi_{Y}d\mu
\]

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There is an increasing sequence \( f_n : S \to \mathbb{R}^+ \) of simple functions converging pointwise to \( f \). From the above we know that
\[
\int_{\Lambda \times \Lambda} f_n \circ \varphi d(\mu \times \mu) \leq \mu(\Lambda) \int_S f_n d\mu
\]
and by applying Lebesgue’s monotone convergence first on the right and then of the left of this inequality, we obtain
\[
\int_{\Lambda} \int_{\Lambda} f (h_1^{-1} h_2) dh_1 dh_2 = \int_{\Lambda \times \Lambda} f \circ \varphi d(\mu \times \mu) \leq \mu(\Lambda) \int_S f d\mu
\]
as required, where we have used Fubini’s theorem, which holds in this case, since \( f \) is non-negative. □

**Proposition 2.9.** Let \( G \) be a second countable topological group with left invariant measure \( \mu \). Let \( \{\Lambda_\alpha\} \) be a uniformly space-filling net in a Borel measurable subsemigroup \( K \) of \( G \). Consider a Borel measurable function \( \gamma_h : G \to \mathbb{C} \). Also assume that each \( \Lambda_\alpha \) is open, and that
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha^{-1} \Lambda_\alpha} |\gamma_h| dh = 0 .
\]

Then
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)^2} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \gamma_{h_1^{-1} h_2} dh_1 dh_2 = 0
\]
if the iterated integral exists for all \( \alpha \geq \alpha_0 \) for some \( \alpha_0 \).

**Proof.** Since \( \Lambda_\alpha \) is open, \( \Lambda_\alpha^{-1} \Lambda_\alpha \) is Borel, and so
\[
\left| \frac{1}{\mu(\Lambda_\alpha)^2} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \gamma_{h_1^{-1} h_2} dh_1 dh_2 \right| \leq \frac{1}{\mu(\Lambda_\alpha)^2} \int_{\Lambda_\alpha} \int_{\Lambda_\alpha} \left| \gamma_{h_1^{-1} h_2} \right| dh_1 dh_2 \leq \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha^{-1} \Lambda_\alpha} |\gamma_h| dh
\]
by Lemma 2.8. □

As opposed to Theorems 2.7 and 2.7’, the measure in this proposition has to be left invariant, hence when it is applied in tandem with Theorem 2.7 or 2.7’, the measure will have to be invariant. Clearly Proposition 2.9 would also work if \( \Lambda_\alpha \) wasn’t necessarily open, but we had \( \Lambda_\alpha^{-1} \Lambda_\alpha \subset S_\alpha \) with \( S_\alpha \) measurable and \( \lim_{\alpha} \int_{S_\alpha} |\gamma_h| dh = 0 \).
3 Weak mixing

In this section we define weak mixing, and study some of its characterizations using simple tools like density limits. This sets the stage for our study of weak mixing of all orders in the next section. The discussion here is in a fairly abstract setting, which for the most part does not require the net \( \{ \Lambda_\alpha \} \) to be space-filling. As we will see, the net is only required to be space-filling in order for the definition of weak mixing to be independent of the net being used, and in the next section in the final step of the proof of weak mixing to all orders, where the van der Corput lemma is used.

**Definition 3.1. Dynamical system, measure preserving dynamical system.** Let \((X, \Sigma, \nu)\) be a probability space. Let \(K\) be any semigroup. For each \(g \in K\) let \(T_g : X \to X\) be such that \(T_g \circ T_h = T_{gh}\) for all \(g, h \in K\). Denote \(g \mapsto T_g\) by \(T\). If \(T^{-1}_g(\Sigma) \subset \Sigma\) for all \(g \in K\), then \((X, \Sigma, \nu, T, K)\) is called a dynamical system (over \(K\); at times it will be convenient to explicitly state the semigroup). If, additionally, \(\nu(T^{-1}_g(A)) = \nu(A)\) for all \(A \in \Sigma\) and \(g \in K\), then \((X, \Sigma, \nu, T, K)\) is called a measure preserving dynamical system.

For a group (respectively semigroup) \(G\), let \(\mathcal{M}_G\) denote the set of all group (respectively semigroup) homomorphisms \(G \to G\).

**Definition 3.2. Weak mixing and ergodicity.** Let \(K\) be a semigroup with a \(\sigma\)-algebra and a measure \(\mu\). Let \(\{ \Lambda_\alpha \}\) be a net of measurable subsets of \(K\), such that \(\mu(\Lambda_\alpha) > 0\) for \(\alpha\) large enough, and with \(\mu(\Lambda_\alpha) < \infty\) for every \(\alpha\). Let \(M \subset \mathcal{M}_K\). Assume that \((X, \Sigma, \nu, T, K)\) is a dynamical system and that \(g \mapsto \nu(A_0 \cap T^{-1}_g(\varphi(A_1)))\) is measurable for all \(A_0, A_1 \in \Sigma\) and all \(\varphi \in M\).

(i) \((X, \Sigma, \nu, T, K)\) is said to be \(M\)-weakly mixing relative to \(\{ \Lambda_\alpha \}\), if

\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \left| \nu(A_0 \cap T^{-1}_g(\varphi(A_1))) - \nu(A_0)\nu(A_1) \right| dg = 0
\]

for all \(A_0, A_1 \in \Sigma\), and for all \(\varphi \in M\).

(ii) \((X, \Sigma, \nu, T, K)\) is said to be \(M\)-ergodic relative to \(\{ \Lambda_\alpha \}\), if

\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \nu(A_0 \cap T^{-1}_g(\varphi(A_1))) dg = \nu(A_0)\nu(A_1)
\]

for all \(A_0, A_1 \in \Sigma\), and for all \(\varphi \in M\).
Remarks on Definition 3.2. In the case of $K = \mathbb{N}$, $\Lambda_n = \{1, \ldots, n\}$ and with $M = \{id_n\}$, Definition 3.2(i) corresponds to the usual definition of weak mixing for an action of the semigroup $\mathbb{N}$, as given in (1.1). Since all homomorphisms of $\mathbb{N}$ are of the form $n \mapsto kn$ for some $k \in \mathbb{N}$, one can then easily show that $\{id_n\}$-weak mixing implies $M$-weak mixing.

For general $K$ our definition of weak mixing is quite abstract. We don’t assume the dynamical system to be measure preserving, or the net $\{\Lambda_\alpha\}$ to be space-filling in $K$, simply because these assumptions are unnecessary in many of the results that follow, though they are required when proving $M$-weak mixing of all orders. In “practical” cases that one usually studies in ergodic theory, one would expect these assumptions to hold, for example $\Lambda_n$ mentioned above is space-filling in $\mathbb{N}$. Under these assumptions, we will see in Corollary 3.10 that the definition of weak mixing is independent of the space-filling net we use, i.e. if a measure preserving dynamical system is $M$-weakly mixing relative to one space-filling net, then it is $M$-weakly mixing relative to all space-filling nets in $K$. The proof of Corollary 3.10, as well as the parts of Propositions 3.8 and 3.9 which are used in this proof, are the only places in this paper where we will use ergodicity.

In general the assumption that a dynamical system is $M$-weakly mixing, is a restriction on $M$, since for example one would not expect $g \mapsto \nu(A_0 \cap T_{\varphi(g)}^{-1}(A_1))$ to even be measurable for all homomorphisms $\varphi : K \to K$.

As a last remark, note that if $K$ has an identity $e$, and the homomorphism given by $\varphi_0(g) = e$ for all $g \in G$ was in $M$, then the system wouldn’t be $M$-weakly mixing, hence we wouldn’t want $\varphi_0$ to be in $M$. We mention this simply because $\varphi_0$ does appear in the theory to follow, but not as an element of $M$.

We now turn to a few technical tools which we will need in Section 4.

**Definition 3.3. Density zero, density limit.** Let $(G, \mu)$ be a measure space (with $G$ not necessarily a group or semigroup) and $\{\Lambda_\alpha\}$ a net of measurable subsets of $G$. Assume that $\mu(\Lambda_\alpha) > 0$ for $\alpha$ large enough, and that $\mu(\Lambda_\alpha) < \infty$ for every $\alpha$.

(i) A set $R \subset G$ is said to have *density zero relative to* $\{\Lambda_\alpha\}$, and we write $D_{\{\Lambda_\alpha\}}(R) = 0$ if and only if there exists a measurable set $S \subset G$, with $R \subset S$ such that

$$
\lim_{\alpha} \frac{\mu(\Lambda_\alpha \cap S)}{\mu(\Lambda_\alpha)} = 0.
$$

(ii) We say that $f : G \to L$, with $L$ a real or complex normed space, has *density limit* $a \in L$ relative to $\{\Lambda_\alpha\}$, if and only if for each $\varepsilon > 0$,
\[ D_{\{\Lambda_\alpha\}}(S_\varepsilon) = 0, \text{ where} \]
\[ S_\varepsilon := \{ h \in G : \| f(h) - a \| \geq \varepsilon \} , \]
and we write it as
\[ D_{\{\Lambda_\alpha\}}^{-} \lim f = D_{\{\Lambda_\alpha\}}^{-} \lim_h f(h) = a . \]

Note that if \( R \) and \( S \) have density zero relative to \( \{\Lambda_\alpha\} \) and \( V \subset S \), then \( R \cap S, R \cup S \) and \( V \) also have density zero relative to \( \{\Lambda_\alpha\} \).

**Proposition 3.4.** Let \( f, g : G \to L \) with \( (G, \mu) \) and \( L \) as in Definition 3.3, and assume that
\[ D_{\{\Lambda_\alpha\}}^{-} \lim f = a \quad \text{and} \quad D_{\{\Lambda_\alpha\}}^{-} \lim g = b . \]
Then
\[ D_{\{\Lambda_\alpha\}}^{-} \lim (f + g) = a + b \]
and
\[ D_{\{\Lambda_\alpha\}}^{-} \lim (\beta f) = \beta a \]
for any \( \beta \in \mathbb{C} \). Furthermore, if \( f, g \) are real-valued functions and \( f(h) \leq g(h) \) for all \( h \in G \), then \( a \leq b \).

**Proof.** For each \( \varepsilon > 0 \), let
\[ R_\varepsilon := \{ h \in G : \| f(h) - a \| \geq \varepsilon \} \quad \text{and} \quad S_\varepsilon := \{ h \in G : \| g(h) - b \| \geq \varepsilon \} . \]
By definition, \( R_\varepsilon \) and \( S_\varepsilon \) have density zero relative to \( \{\Lambda_\alpha\} \). Let
\[ V_\varepsilon := \{ h \in G : \| (f + g)(h) - (a + b) \| \geq \varepsilon \} \]
and
\[ V'_\varepsilon := \{ h \in G : \| f(h) - a \| + \| g(h) - b \| \geq \varepsilon \} . \]
Since \( \| (f + g)(h) - (a + b) \| \leq \| f(h) - a \| + \| g(h) - b \| \), it is clear that \( V_\varepsilon \subset V'_\varepsilon \). Also, clearly \( V'_\varepsilon \subset R_\varepsilon \cup S_\varepsilon \). But \( R_\varepsilon \cup S_\varepsilon \) has density zero relative to \( \{\Lambda_\alpha\} \), and hence the same holds for \( V'_\varepsilon \) and then \( V_\varepsilon \). Hence
\[ D_{\{\Lambda_\alpha\}}^{-} \lim (f + g) = a + b . \]

Letting \( W_\varepsilon := \{ h \in G : \| (\beta f)(h) - \beta a \| \geq \varepsilon \} \), it is easily seen that \( W_\varepsilon \) has density zero relative to \( \{\Lambda_\alpha\} \), hence
\[ D_{\{\Lambda_\alpha\}}^{-} \lim (\beta f) = \beta a . \]
Finally, suppose that $f, g$ are real-valued functions, i.e. $L = \mathbb{R}$, and $f(h) \leq g(h)$ for all $h \in G$. From the previous two results in this proposition, we have that

$$D\{\Lambda_\alpha\}^{-}\lim (g - f) = b - a .$$

Hence for any $\varepsilon > 0$, the set

$$W_\varepsilon := \{h \in G : |(g - f)(h) - (b - a)| \geq \varepsilon\}$$

has density zero relative to $\{\Lambda_\alpha\}$. Suppose now that $b - a =: \rho < 0$. Since $(g - f)(h) \geq 0$ for all $h \in G$, we must have that the set $W_{|\rho|/2}$ consists of all of $G$. Hence

$$\frac{\mu(\Lambda_\alpha \cap W_{|\rho|/2})}{\mu(\Lambda_\alpha)} = \frac{\mu(\Lambda_\alpha)}{\mu(\Lambda_\alpha)} = 1 ,$$

contradicting the stated fact that $W_{|\rho|/2}$ has density zero relative to $\{\Lambda_\alpha\}$. Therefore $b - a \geq 0$. □

We now give a Koopman-von Neumann type lemma:

**Lemma 3.5.** Let $(G, \mu)$ be a measure space, and let $\{\Lambda_\alpha\}$ be a net of measurable subsets of $G$. Assume that $\mu(\Lambda_\alpha) > 0$ for $\alpha$ large enough, and that $\mu(\Lambda_\alpha) < \infty$ for every $\alpha$. Let $f : G \to [0, \infty)$ be bounded and measurable. Then the following are equivalent:

1. $D\{\Lambda_\alpha\}^{-}\lim f = 0$
2. $\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} f \, d\mu = 0$

**Proof.** For every $\varepsilon > 0$, let $S_\varepsilon := \{h \in G : f(h) \geq \varepsilon\}$, which is a measurable set, since $f$ is measurable.

$(1) \Rightarrow (2)$: From (1) we have that each $S_\varepsilon$ has density zero relative to $\{\Lambda_\alpha\}$. Given any $\varepsilon > 0$ and index $\alpha$, consider the term

$$\frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} f \, d\mu = \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha \cap S_\varepsilon} f \, d\mu + \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha \cap S_\varepsilon^c} f \, d\mu .$$

Since $S_\varepsilon$ has density zero relative to $\{\Lambda_\alpha\}$

$$0 \leq \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha \cap S_\varepsilon} f \, d\mu \leq \frac{\mu(\Lambda_\alpha \cap S_\varepsilon)}{\mu(\Lambda_\alpha)} \sup f(G) \to 0$$

in the $\alpha$ limit. Also,

$$0 \leq \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha \cap S_\varepsilon^c} f \, d\mu \leq \frac{\mu(\Lambda_\alpha \cap S_\varepsilon^c)}{\mu(\Lambda_\alpha)} \varepsilon \leq \varepsilon$$

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hence
\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} f d\mu = 0. \]

(2) ⇒ (1): Clearly \( \varepsilon \chi_{S_\varepsilon} \leq f \). Also note that \( D_{\{\Lambda_\alpha\}}(S_\varepsilon) = 0 \), since \( S_\varepsilon \) is measurable and
\[ \varepsilon \frac{\mu(\Lambda_\alpha \cap S_\varepsilon)}{\mu(\Lambda_\alpha)} \leq \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} f d\mu \]
which tends to zero in the \( \alpha \) limit. □

**Corollary 3.6.** Consider the situation in Lemma 3.5, except that we use \( f : G \to \mathbb{R} \), assumed to be bounded and measurable. Then
\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} |f|^2 dh = 0 \]
is and only if
\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} |f(h)| dh = 0. \]

**Proof.** Given any \( \varepsilon > 0 \). Let
\[ S_\varepsilon := \{ h \in G : |f(h)|^2 \geq \varepsilon^2 \} = \{ h \in G : |f(h)| \geq \varepsilon \}. \]
Suppose that \( \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} |f|^2 dh = 0 \), i.e. \( D_{\{\Lambda_\alpha\}}\lim_{h} |f(h)|^2 = 0 \) by Lemma 3.5. By the definition of the density limit we have \( D_{\{\Lambda_\alpha\}}(S_\varepsilon) = 0 \). Since \( \varepsilon > 0 \) is arbitrary, we conclude that \( D_{\{\Lambda_\alpha\}}\lim |f| = 0 \), and hence \( \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} |f(h)| dh = 0 \) by Lemma 3.5.

The converse follows similarly. □

As a result, the \( |\cdot| \) in Definition 3.2(i) of weak mixing, can be replaced by \( |\cdot|^2 \).

**Lemma 3.7.** Consider the situation in Lemma 3.5, except that we use \( f : G \to \mathbb{C} \), assumed to be bounded and measurable. Let \( \beta \in \mathbb{C} \).

If
\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} f(h) dh = \beta \]
and
\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} |f(h)|^2 dh = \beta^2, \]
then
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} [f(h) - \beta]^2 \, dh = 0.
\]

Proof. This follows immediately if we note that
\[
\frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} [f(h) - \beta]^2 \, dh
= \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} (|f(h)|^2 - 2\beta f(h) + \beta^2) \, dh
\to 0
\]
in the \(\alpha\) limit. □

Next we consider standard characterizations of weak mixing, that we will need. The first proposition does not require the system to be measure preserving, but the second does.

**Proposition 3.8.** Let \(K\) be a semigroup with a \(\sigma\)-algebra and a measure \(\mu\), and let \(\{\Lambda_\alpha\}\) be a net of measurable subsets of \(K\). Assume that \(\mu(\Lambda_\alpha) > 0\) for \(\alpha\) large enough, and that \(\mu(\Lambda_\alpha) < \infty\) for every \(\alpha\). Let \(M \subset \mathcal{M}_K\). Let \((X, \Sigma, \nu, T, K)\) be a dynamical system. Set \((T \times T)_h = T_h \times T_h\) for all \(h \in K\), where \((T_h \times T_h)(x_1, x_2) = (T_h(x_1), T_h(x_2))\) for all \((x_1, x_2) \in X \times X\). Consider the following statements:

1. \((X, \Sigma, \nu, T, K)\) is \(M\)-weakly mixing relative to \(\{\Lambda_\alpha\}\).
2. \((X \times X, \Sigma \times \Sigma, \nu \times \nu, T \times T, K)\) is \(M\)-weakly mixing relative to \(\{\Lambda_\alpha\}\).
3. \((X \times X, \Sigma \times \Sigma, \nu \times \nu, T \times T, K)\) is \(M\)-ergodic relative to \(\{\Lambda_\alpha\}\).
4. \(D_{\{\Lambda_\alpha\}}\lim \nu(A_0 \cap T_{\phi(h)}^{-1}(A_1)) = \nu(A_0)\nu(A_1)\) for all \(A_0, A_1 \in \Sigma\) and for each \(\phi \in M\).

Then (1) and (4) are equivalent. Also, (2) implies (3), which in turn implies (1).

Proof. (1) \(\Leftrightarrow\) (4): Given any \(\phi \in M\), let \(f(h) := \left| \nu(A_0 \cap T_{\phi(h)}^{-1}(A_1)) - \nu(A_0)\nu(A_1) \right|\), and apply Lemma 3.5.

(2) \(\Rightarrow\) (3): Follows immediately from Definition 3.2.
(3) ⇒ (1): Let $A_0, A_1 \in \Sigma$ and $\varphi \in M$. We have
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \nu(A_0 \cap T_{\varphi(g)}^{-1}(A_1)) dg
= \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} (\nu \times \nu)((A_0 \times X) \cap (T_{\varphi(g)} \times T_{\varphi(g)})^{-1}(A_1 \times X)) dg
= (\nu \times \nu)(A_0 \times X)(\nu \times \nu)(A_1 \times X)
= \nu(A_0)\nu(A_1),
\]
and also
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \nu(A_0 \cap T_{\varphi(g)}^{-1}(A_1))^2 dg
= \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} (\nu \times \nu)((A_0 \times X) \cap (T_{\varphi(g)} \times T_{\varphi(g)})^{-1}(A_1 \times A_1)) dg
= (\nu \times \nu)(A_0 \times A_0)(\nu \times \nu)(A_1 \times A_1)
= \nu(A_0)^2\nu(A_1)^2.
\]
Therefore by Lemma 3.7 we have that
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} (\nu(A_0 \cap T_{\varphi(g)}^{-1}(A_1)) - \nu(A_0)\nu(A_1))^2 dg = 0,
\]
and it follows from Corollary 3.6 that $(X, \Sigma, \nu, T, K)$ is $M$-weakly mixing relative to $\{\Lambda_\alpha\}$. □

**Proposition 3.9.** Consider the situation in Proposition 3.8, but also assume that the dynamical system $(X, \Sigma, \nu, T, K)$ is measure preserving. Then the following are equivalent:

(1) $(X, \Sigma, \nu, T, K)$ is $M$-weakly mixing relative to $\{\Lambda_\alpha\}$.

(2) $(X \times X, \Sigma \times \Sigma, \nu \times \nu, T \times T, K)$ is $M$-weakly mixing relative to $\{\Lambda_\alpha\}$.

(3) $(X \times X, \Sigma \times \Sigma, \nu \times \nu, T \times T, K)$ is $M$-ergodic relative to $\{\Lambda_\alpha\}$.

(4) \[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \left| \langle f_1, f_2 \circ T_{\varphi(h)} \rangle - \langle f_1, 1 \rangle \langle 1, f_2 \rangle \right| dh = 0 \quad \text{and} \quad h \mapsto \langle f_1, f_2 \circ T_{\varphi(h)} \rangle \text{ is measurable for all } f_1, f_2 \in L^2(\nu) \text{ and for each } \varphi \in M.
\]

**Proof.** By Proposition 3.8, we already have (2) ⇒ (3) ⇒ (1). Now for the rest:
(1) ⇒ (2): Given any \( \varphi \in M \) and \( A, B, C, D \in \Sigma \), we have
\[
|\nu \times \nu((A \times C) \cap (T \times T)^{-1}_\varphi(B \times D)) - \nu \times \nu(A \times C)\nu \times \nu(B \times D)|
= |\nu(A \cap T^{-1}_\varphi(B))\nu(C \cap T^{-1}_\varphi(D)) - \nu(A)\nu(B)\nu(C)\nu(D)|
\leq \nu(A)\nu(D)|\nu(A \cap T^{-1}_\varphi(B)) - \nu(A)\nu(B)|
+ \nu(C)\nu(D)|\nu(C \cap T^{-1}_\varphi(D)) - \nu(C)\nu(D)|
\leq \nu(A)|\nu(C \cap T^{-1}_\varphi(D)) - \nu(C)\nu(D)|
+ \nu(C)\nu(D)|\nu(A \cap T^{-1}_\varphi(B)) - \nu(A)\nu(B)|.
\]
Hence by 3.8(1 and 4) and Proposition 3.4,
\[
D_{\{\Lambda_n\}}\lim_h|\nu \times \nu((A \times C) \cap (T \times T)^{-1}_\varphi(B \times D)) - \nu \times \nu(A \times C)\nu \times \nu(B \times D)| = 0.
\]
So again by 3.8(1 and 4), and since the system is measure preserving and the rectangles form a semi-algebra that generates \( \Sigma \times \Sigma \), the proof follows in a standard way (see e.g. [12]).

(1) ⇒ (4): This is true if \( f_1, f_2 \) are characteristic functions of measurable sets and given any \( \varphi \in M \). The desired result is obtained by forming linear combinations and approximating in a standard way (see e.g. [12]).

(4) ⇒ (1): This follows by taking \( f_1 \) and \( f_2 \) to be characteristic functions of measurable sets, given any \( \varphi \in M \). □

We can now show that the definition of \( M \)-weak mixing relative to a space-filling net, is independent of the space-filling net being used:

**Corollary 3.10.** If a measure preserving dynamical system \((X, \Sigma, \nu, T, K)\) is \( M \)-weakly mixing relative to some space-filling net in \( K \), then it is \( M \)-weakly mixing relative to every space-filling net in \( K \).

**Proof.** In [3] it is shown that the ergodicity of a measure-preserving dynamical system is independent of the space-filling net being used, and the proof holds for \( M \)-ergodicity, as we defined it here, as well. Hence \( M \)-weak mixing is also independent of the space-filling net, by the equivalence in Proposition 3.9(1 and 3). □

## 4 Weak mixing of all orders

In this section we show that weak mixing implies weak mixing of all orders. Our approach is strongly influenced by that of [8] for the case of the group \( \mathbb{Z} \). The proof is by induction, two steps of which are given by the following:
Proposition 4.1. Let $K$ be a semigroup with a $\sigma$-algebra and measure $\mu$, and assume that $K$ has an identity element $e$. Let $\{\Lambda_\alpha\}$ be a net of measurable subsets of $K$ such that $\mu(\Lambda_\alpha) > 0$ for $\alpha$ large enough, and with $\mu(\Lambda_\alpha) < \infty$ for every $\alpha$. Let $M \subset \mathcal{M}_K$. Now we will use the following notation: $(X, \Sigma, \nu, T, K)$ will denote any measure preserving dynamical system, but with $K$ fixed, and $id$ will denote the identity mapping $X \to X$. Let $\omega := \int_X f d\nu$ for all $f \in L^\infty(\nu)$, and let $(\omega \otimes \omega)(f) := \int_{X \times X} f d(\nu \times \nu)$ for all $f \in L^\infty(\nu \times \nu)$. Given $k \in \mathbb{N}$, let $\varphi_1, \ldots, \varphi_k$ denote elements of $M$, and let $f_0, \ldots, f_k$ denote real-valued elements of $L^\infty(\nu)$. Let $\varphi_0(h) = e$ for all $h \in K$.

Consider the following statements (where the existence of the integrals contained in each statement form part of that statement):

1$[k]$: The integral $\int_{\Lambda_\alpha} \omega \left( \prod_{j=0}^k f_j \circ T_{\varphi_j(g)} \right) dg$ exists for all $\alpha \geq \alpha_0$ for some $\alpha_0$, and

$$\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \left( \omega \left( \prod_{j=0}^k f_j \circ T_{\varphi_j(g)} \right) - \prod_{j=0}^k \omega(f_j) \right)^2 dg = 0 .$$

2$[k]$: $\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega \left( \prod_{j=0}^k f_j \circ T_{\varphi_j(g)} \right) dg = \prod_{j=0}^k \omega(f_j) .

3$[k]$: For $\kappa := \prod_{j=1}^k \omega(f_j)$, we have

$$\lim_{\alpha} \left\| \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \prod_{j=1}^k f_j \circ T_{\varphi_j(g)} dg - \kappa \right\|_{L^2(\nu)} = 0 .$$

Then

(i) 1$[k]$ implies 2$[k]$.

(ii) If 3$[k]$ holds for all measure preserving dynamical systems over $K$ with $T_e = id$ which are $M$-weakly mixing relative to the given net $\{\Lambda_\alpha\}$, and all $f_1, \ldots, f_k$ and all $\varphi_1, \ldots, \varphi_k$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$, then 1$[k]$ also holds for all measure preserving dynamical systems over $K$ with $T_e = id$ which are $M$-weakly mixing relative to $\{\Lambda_\alpha\}$ and all $f_0, \ldots, f_k$ and all $\varphi_1, \ldots, \varphi_k$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$. 

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Proof. (i) Use Corollary 3.6.

(ii) The strong convergence in $3[k]$ implies weak convergence, i.e.

\[
\lim_{\alpha} \left\langle \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)} \, dg, \ f_0 \right\rangle = \langle \kappa \cdot 1, \ f_0 \rangle
\]

\[
= \omega(\kappa f_0)
\]

\[
= \prod_{j=0}^{k} \omega(f_j).
\]

Furthermore, by the definition of the integral, and from the assumption that $T_{\varphi_0(h)} = T_e = id$, we have that

\[
\lim_{\alpha} \left\langle \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)} \, dg, \ f_0 \right\rangle
\]

\[
= \lim_{\alpha} \frac{1}{\mu(\Lambda_{\alpha})} \left\langle \int_{\Lambda_{\alpha}} \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)} \, dg, \ f_0 \right\rangle
\]

\[
= \lim_{\alpha} \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} \left\langle \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)}), \ f_0 \right\rangle \, dg
\]

\[
= \lim_{\alpha} \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} \omega \left( \prod_{j=0}^{k} f_j \circ T_{\varphi_j(g)} \right) \, dg ,
\]

hence

\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} \omega \left( \prod_{j=0}^{k} f_j \circ T_{\varphi_j(g)} \right) \, dg = \prod_{j=0}^{k} \omega(f_j) \quad (4.1.1)
\]

and in particular the integral on the left exists for all $\alpha \geq \alpha_0$ for some $\alpha_0$.

Since by Proposition 3.9(1 and 2) the product system $(X \times X, \Sigma \times \Sigma, \nu \times \nu, T \times T, K)$ is an $M$-weak mixing dynamical system relative to $\{\Lambda_{\alpha}\}$, and the product system is measure preserving with $(T \times T)_e = id \times id$, we can apply (4.1.1) to the product system to obtain

\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_{\alpha})} \int_{\Lambda_{\alpha}} (\omega \otimes \omega) \left( \prod_{j=0}^{k} (f_j \otimes f_j) \circ (T \times T)_{\varphi_j(g)} \right) \, dg
\]

\[
= \prod_{j=0}^{k} (\omega \otimes \omega)(f_j \otimes f_j).
\]
where for every $f_1, f_2 \in L^\infty(\nu)$ we define $f_1 \otimes f_2 : X \times X \to \mathbb{R}$ by $(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1)f_2(x_2)$ for all $(x_1, x_2) \in X \times X$. By Fubini's theorem, namely $(\omega \otimes \omega)(f_1 \otimes f_2) = \omega(f_1)\omega(f_2)$ for all $f_1, f_2 \in L^\infty(\nu)$, we have
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega \left( \left( \prod_{j=0}^{k} f_j \circ T_{\varphi_j(g)} \right)^2 \right) dg = \prod_{j=0}^{k} \omega(f_j)^2 ,
\]
proving 1[\text{k}] by Lemma 3.7. □

Note that the only property of weak mixing which is used in Proposition 4.1’s proof, is that if a dynamical system is $M$-weakly mixing and measure preserving, then so is its product with itself. This is the only reason that the systems in Proposition 4.1 are required to be measure preserving, otherwise Proposition 3.9(1 and 2) would not apply. Proposition 4.1 would still hold if we considered dynamical systems with some abstract property, call it $E$, instead of “$M$-weak mixing and measure preserving”, as long as the product of an $E$ dynamical system with itself is again an $E$ dynamical system. In particular, even though Proposition 4.1 is expressed in terms of functions instead of sets, we did not need the characterization of $M$-weak mixing in terms of functions, given by Proposition 3.9(4).

In order to complete the induction argument, we need 1[\text{l}], and that if 2[\text{k}–1] holds for all measure preserving dynamical systems over $K$ with $T_\varepsilon = \text{id}$ which are $M$-weakly mixing relative to $\{\Lambda_\alpha\}$, then the same is true for 3[\text{k}]. The latter requires some more work, and we will need to specialize the $M$ that we will allow. Firstly note that for an abelian group $G$ and any homomorphisms $\varphi_1$ and $\varphi_2$ of $G$, the function $\varphi' : G \to G$ defined by
\[
\varphi'(g) := \varphi_2(g)\varphi_1(g)^{-1}
\]
is also a homomorphism of $G$. Even though from now on we will use only abelian groups, we will continue to use multiplicative notation, as in (4.1).

**Definition 4.2.** Let $G$ be an abelian group and let $M \subset \mathcal{M}_G$. We call $M$ translational if for all $\varphi_1, \varphi_2 \in M$ with $\varphi_1 \neq \varphi_2$, the homomorphism $\varphi'$ defined by (4.1) is also in $M$.

**Proposition 4.3.** Let $G$ be an abelian group with a $\sigma$-algebra and measure $\mu$, and let $M \subset \mathcal{M}_G$ be translational. Let $(X, \Sigma, \nu, T, G)$ be a measure preserving dynamical system. Let $\omega(f) := \int_X f d\nu$ for all $f \in L^\infty(\nu)$. Let $\{\Lambda_\alpha\}$ be a net of measurable subsets of $K$ with $\mu(\Lambda_\alpha) < \infty$ for all $\alpha$, and $\mu(\Lambda_\beta) > 0$ for $\beta$ large enough. Set $\varphi_0(g) = e$ for all $g \in G$. Assume that for some $k \in \mathbb{N}$
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega \left( \left( \prod_{j=0}^{k-1} f_j \circ T_{\varphi_j(g)} \right) \right) dg = \prod_{j=0}^{k-1} \omega(f_j)
\]

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for all real-valued \( f_0, \ldots, f_{k-1} \in L^\infty(\nu) \) and all \( \varphi_1, \ldots, \varphi_{k-1} \in M \) with \( \varphi_j \neq \varphi_l \) when \( j \neq l \) for \( j, l \in \{1, \ldots, k-1\} \), and in particular the existence of the integral over \( \Lambda_\alpha \) and the limit is assumed. Now set

\[
\gamma_h := \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \langle u_g, u_{gh} \rangle \, dg
\]

exists (where \( \langle \cdot, \cdot \rangle \) is taken in \( L^2(\nu) \); \( L^\infty(\nu) \subset L^2(\nu) \) since \( \nu(X) < \infty \)), and

\[
\gamma_h = \prod_{j=1}^k \omega \left( f_j \circ T_{\varphi_j(h)} \right) - \kappa^2
\]

for all \( h \in K \).

**Proof.** We have

\[
\langle u_g, u_{gh} \rangle = \int_X \left( \prod_{j=1}^k \left( f_j \circ T_{\varphi_j(g)} - \kappa \right) \prod_{l=1}^k \left( f_l \circ T_{\varphi_l(gh)} - \kappa \right) \right) \, d\nu
\]

\[
= \int_X \left[ \prod_{j=1}^k \left( f_j \circ T_{\varphi_j(g)} \right) \left( f_j \circ T_{\varphi_j(gh)} \right) - \kappa \prod_{j=1}^k f_j \circ T_{\varphi_j(g)} - \kappa \prod_{j=1}^k f_j \circ T_{\varphi_j(gh)} \right] \, d\nu + \kappa^2
\]

and note that all three these last integrals exist, since \( f_j \circ T_{\varphi_j(g)} \) and products of such functions are in \( L^\infty(\nu) \subset L^1(\nu) \). We now consider the three integrals in turn:
(a) Since $G$ is abelian and $T$ is measure preserving,

\[ \int_X \left[ \prod_{j=1}^k (f_j \circ T_{\varphi_j(g)}) (f_j \circ T_{\varphi_j(gh)}) \right] d\nu = \int_X \left[ \prod_{j=1}^k (f_j \circ T_{\varphi_j(h)}) (f_j \circ T_{\varphi_j(gh)} \circ T_{\varphi_j(g)}) \right] d\nu \]

\[ = \int_X \left\{ \prod_{j=1}^k \left[ f_j \left( f_j \circ T_{\varphi_j(h)} \right) \right] \circ T_{\varphi_j(g) \varphi_1(g)^{-1}} \right\} \circ T_{\varphi_1(g)} d\nu \]

\[ = \int_X \left\{ \prod_{j=1}^k \left[ f_j \left( f_j \circ T_{\varphi_j(h)} \right) \right] \circ T_{\varphi_j(g) \varphi_1(g)^{-1}} \right\} d\left( \nu \circ T_{\varphi_1(g)}^{-1} \right) \]

\[ = \int_X \left\{ \prod_{j=1}^k \left[ f_j \left( f_j \circ T_{\varphi_j(h)} \right) \right] \circ T_{\varphi_j(g) \varphi_1(g)^{-1}} \right\} d\nu \]

\[ = \omega \left( \prod_{j=0}^{k-1} \left[ f_{j+1} \left( f_{j+1} \circ T_{\varphi_{j+1}(h)} \right) \right] \circ T_{\varphi_j(g)} \right) \]

where $\varphi_j'(g) := \varphi_{j+1}(g) \varphi_1(g)^{-1}$ for all $g \in G$ and $j = 0, \ldots, k - 1$, so $\varphi_j' \in M$ for $j = 1, \ldots, k - 1$ since $M$ is translational, $\varphi_j' \neq \varphi_l'$ when $j \neq l$ for $j, l \in \{1, \ldots, k - 1\}$, and $\varphi_0'(g) = e$ for all $g \in G$. Hence

\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \int_X \left[ \prod_{j=1}^k (f_j \circ T_{\varphi_j(g)}) (f_j \circ T_{\varphi_j(gh)}) \right] d\nu dg = \prod_{j=0}^{k-1} \omega \left( f_{j+1} \left( f_{j+1} \circ T_{\varphi_{j+1}(h)} \right) \right) \]

by assumption.

(b) For the second integral, again using the fact that $T$ is measure preserving, it follows as in (a) that

\[ \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \int_X \left[ \prod_{j=1}^k f_j \circ T_{\varphi_j(g)} \right] d\nu dg = \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega \left( \prod_{j=0}^{k-1} f_{j+1} \circ T_{\varphi_j(g)} \right) dg \]

\[ = \prod_{j=0}^{k-1} \omega(f_{j+1}) \]

\[ = \kappa \]

by assumption.
Lastly, again since $G$ is abelian and $T$ is measure preserving,
\[
\lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \int_X \left[ \prod_{j=1}^{k} f_j \circ T_{\varphi_j(gh)} \right] \ d\nu \ d g = \lim_{\alpha} \frac{1}{\mu(\Lambda_\alpha)} \int_{\Lambda_\alpha} \omega \left( \prod_{j=0}^{k-1} (f_{j+1} \circ T_{\varphi_{j+1}(h)}) \circ T_{\varphi_j'(g)} \right) \ d g = \prod_{j=0}^{k-1} \omega(f_{j+1}) = \kappa
\]
by assumption.

From (a)-(c)
\[
\gamma_h = \prod_{j=1}^{k} \omega \left( f_j \left( f_j \circ T_{\varphi_j(h)} \right) \right) - \kappa^2
\]
and in particular $\gamma_h$ exists. $\square$

We now state and prove our final result, namely that weak mixing implies weak mixing of all orders. This is where our van der Corput lemma is finally applied, along with Propositions 4.1 and 4.3, and the characterization of $M$-weak mixing given by Proposition 3.9(1 and 4) which so far we have not used.

**Theorem 4.4.** Let $(X, \Sigma, \nu, T, G)$ be a measure preserving dynamical system for an abelian second countable topological group $G$ with invariant measure $\mu$, and with $T = \text{id}$. Let $M \subset \mathcal{M}_G$ be translational. Assume that $(X, \Sigma, \nu, T, G)$ is $M$-weakly mixing relative to a uniformly space-filling sequence of open sets $\{\Lambda_n\}$ in $G$, and that $(X, \Sigma, \nu, T, G)$ is $M$-weakly mixing relative to the sequence $\{\Lambda_n^{-1}\}$, so in particular we require $\mu(\Lambda_n^{-1}) > 0$ for $n$ large enough and $\mu(\Lambda_n^{-1}) < \infty$ for every $n$, and where we assume that $\mu(\Lambda_n^{-1}) \leq c \mu(\Lambda_n)$ for $n$ large enough and some strictly positive real number $c$. Assume furthermore that $G \to L^\infty(\nu) : g \mapsto f \circ T_{\varphi(g)}$ is continuous in the $L^\infty$-norm topology on $L^\infty(\nu)$ for all $\varphi \in M$. Then
\[
\lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \left( \omega \left( \prod_{j=0}^{k} f_j \circ T_{\varphi_j(\varphi)} \right) - \prod_{j=0}^{k} \omega(f_j) \right)^2 \ d g = 0
\]
for any real-valued \( f_j \in L^\infty(\nu) \) and any \( \varphi_1, ..., \varphi_k \in M \) with \( \varphi_j \neq \varphi_l \) when \( j \neq l \) for \( j, l \in \{1, ..., k\} \), and with \( \varphi_0(g) = e \) for all \( g \in G \), where \( \omega(f) := \int_X f d\nu \).

**Proof.** We need to complete the induction argument started in Proposition 4.1, and we will continue using its notation, but with \( K = G \). Since \( G \to L^\infty(\nu) : g \mapsto f \circ T_{\varphi(g)} \) is continuous, so is \( F : G \to L^\infty(\nu) : g \mapsto \prod_{j=0}^{k} f_j \circ T_{\varphi_j(g)} \) in the \( L^\infty \)-topology. Since \( \nu(X) = 1 \), we have \( ||f||_2 \leq ||f||_\infty \) for all \( f \in L^\infty(\nu) \), so the \( L^\infty \)-topology is finer that the \( L^2 \)-topology, hence \( F \) is continuous in the \( L^2 \)-topology on \( L^\infty(\nu) \) as well. It follows that

\[
G \times G \to \mathbb{R} : (g, h) \mapsto \left\langle \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)}, \prod_{j=1}^{k} f_j \circ T_{\varphi_j(h)} \right\rangle
\]

is continuous. Keep in mind that \( \omega\left( \left( \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)} \right) \left( \prod_{j=1}^{k} f_j \circ T_{\varphi_j(h)} \right) \right) = \left\langle \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)}, \prod_{j=1}^{k} f_j \circ T_{\varphi_j(h)} \right\rangle \). Now we write

\[
u_u := \prod_{j=1}^{k} f_j \circ T_{\varphi_j(g)} - \kappa
\]

for all \( g \in G \), where \( \kappa := \prod_{j=1}^{k} \omega(f_j) \). It follows that \( G \times G \to \mathbb{C} : (g, h) \mapsto \left\langle \nu_u, \nu_u \right\rangle \) is continuous and therefore Borel measurable. Note that \( g \mapsto \left\langle u_g, x \right\rangle \) is also Borel measurable for all \( x \in L^2(\nu) \). Furthermore, \( G \to L^2(\nu) : g \mapsto \nu_u \) is bounded, since each \( f_j \) is essentially bounded and \( \nu(X) = 1 \). (We need these properties, since we will be applying Theorem 2.7 to the function \( g \mapsto u_g \).) Since \( \mu(\Lambda_n^{-1}\Lambda_n) \leq c\mu(\Lambda_n) \), and we have \( M \)-weak mixing relative to \( \{\Lambda_n^{-1}\Lambda_n\} \), it follows from Proposition 3.9(1 and 4) that

\[
\lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n^{-1}\Lambda_n} |\omega(f_0 \circ T_{\varphi(g)}) - \omega(f_0)\omega(f_1)| \, dg = 0 \quad (4.4.1)
\]

for all \( \varphi \in M \). By Proposition 4.3, assuming \( 2[k - 1] \) for all measure preserving dynamical systems over \( G \) with \( T_e = id \), which are \( M \)-weakly mixing relative to \( \{\Lambda_n\} \), and of course for all \( f_0, ..., f_{k-1} \) and all \( \varphi_1, ..., \varphi_{k-1} \in M \) with \( \varphi_j \neq \varphi_l \) when \( j \neq l \) for \( j, l \in \{1, ..., k-1\} \), we have

\[
\gamma_n := \lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} \langle u_g, u_{gh} \rangle \, dg = \prod_{j=1}^{k} \omega(f_j \circ T_{\varphi_j(n)}) - \prod_{j=1}^{k} \omega(f_j)^2
\]
for any $f_1, \ldots, f_k$ and all $\varphi_1, \ldots, \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$, for all $h \in G$. Using the identity $\prod_{j=1}^{k} a_j - \prod_{j=1}^{k} b_j = \sum_{j=1}^{k} \left( \prod_{i=1}^{j-1} a_i \right) (a_j - b_j) \left( \prod_{i=j+1}^{k} b_i \right)$ it follows that

$$\int_{\Lambda_m \Lambda_n} |\gamma_h| \, dh \leq \sum_{j=1}^{k} A_j \prod_{i=j+1}^{k} \omega(f_i) \int_{\Lambda_m \Lambda_n} \left| \omega \left( f_j \circ T_{\varphi_j(h)} \right) - \omega(f_j)^2 \right| \, dh$$

where $A_j := \sup_{h \in G} \left| \prod_{i=1}^{j-1} \omega \left( f_i \circ T_{\varphi_i(h)} \right) \right|$ which exists in $\mathbb{R}$, since the $f_j$'s are essentially bounded. Note that $\int_{\Lambda_m \Lambda_n} |\gamma_h| \, dh$ exists, since the integrand is continuous. Hence

$$\lim_{m \to \infty} \frac{1}{\mu(\Lambda_m)} \int_{\Lambda_m} |\gamma_h| \, dh = 0$$

by (4.4.1). From Proposition 2.9 and Theorem 2.7' we then have

$$\lim_{n \to \infty} \frac{1}{\mu(\Lambda_n)} \int_{\Lambda_n} u_g \, dg = 0$$

where the limit is taken in the $L^2$-norm, i.e. $3[k]$ holds for all measure preserving dynamical systems over $G$ with $T_e = id$, which are $M$-weakly mixing relative to $\{\Lambda_\alpha\}$, and all $f_1, \ldots, f_k$ and all $\varphi_1, \ldots, \varphi_k \in M$ with $\varphi_j \neq \varphi_l$ when $j \neq l$ for $j, l \in \{1, \ldots, k\}$. But $1[1]$ holds for all $f_0, f_1 \in L^\infty(\nu)$ and all $\varphi \in M$ for all measure preserving dynamical systems over $G$ with $T_e = id$, which are $M$-weakly mixing relative to $\{\Lambda_\alpha\}$, because of Proposition 3.9(1 and 4) and Corollary 3.6, completing the induction argument started in Proposition 4.1, and proving $1[k]$ for all $k \in \mathbb{N}$. □

By Corollary 3.6, the $[\cdot]^2$ in the integrand in Theorem 4.4, can be replaced by $|\cdot|$, to have the same form as Definition 3.2(i) of weak mixing.

Note that if $\{\Lambda_n^{-1} \Lambda_n\}$ is also space-filling in $G$, then the assumption that the system be $M$-weakly mixing relative to $\{\Lambda_n^{-1} \Lambda_n\}$ can be dropped because of Corollary 3.10. If $\{\Lambda_n^{-1} \Lambda_n\}$ does not have the properties required in Theorem 4.4, for example if the system is not $M$-weak mixing relative to $\{\Lambda_n^{-1} \Lambda_n\}$, but there is some other uniformly space-filling sequence $\{\Lambda'_n\}$ such that $\{\Lambda_n^{-1} \Lambda'_n\}$ does have the required properties, then we can replace $\{\Lambda_n\}$ by $\{\Lambda'_n\}$ because of Corollary 3.10, to get weak mixing of all orders relative to $\{\Lambda'_n\}$. We now briefly consider examples of space-filling sequences with the required properties.

In the simple case where $G = \mathbb{Z}$ with the counting measure $\mu$, and $\Lambda_n = \{-n, \ldots, n\}$ which is uniformly space-filling in $\mathbb{Z}$, we have $\Lambda_n^{-1} \Lambda_n =$
\{-2n, \ldots, 2n\}, so \(\mu(\Lambda_n) \leq \mu(\Lambda_n^{-1}\Lambda_n) \leq 2\mu(\Lambda_n)\) for \(n \geq 1\), and if the dynamical system is weak mixing relative to \(\{\Lambda_n\}\), then it is also weak mixing relative to \(\Lambda_n^{-1}\Lambda_n = \Lambda_{2n}\). Hence the conditions of Theorem 4.4 are satisfied. Furthermore, if the system is only weak mixing relative to \(\{0, \ldots, n\}\), so we are working over the semigroup \(\mathbb{N} \cup \{0\}\), and \(T\) is injective, then it is easily seen that it is also weak mixing relative to \(\Lambda_n\). This implies the usual version of weak mixing of all orders when working on the semigroup \(\mathbb{N} \cup \{0\}\), for an injective \(T\).

As another example of a sequence with the properties in Theorem 4.4, let \(\Lambda_m\) be the open ball of radius \(m\) in \(\mathbb{R}^q\) for any positive integer \(q\). Note that \(\{\Lambda_m\}\) is a uniformly space-filling sequence in \(\mathbb{R}^q\). Then \(\Lambda_m^{-1}\Lambda_m = \Lambda_{2m}\), which means that \(M\)-weak mixing relative to \(\{\Lambda_m\}\), implies \(M\)-weak mixing relative to \(\{\Lambda_m^{-1}\Lambda_m\}\), while \(\mu(\Lambda_m^{-1}\Lambda_m) = 2^q\mu(\Lambda_m)\), as is required in Theorem 4.4.

Concerning the assumption that \(M\) is translational, a simple example would be of the following type: Use the group \(G = \mathbb{R}^q\). Let \(M\) be all \(q \times q\) non-zero diagonal real matrices acting as linear operators on \(\mathbb{R}^q\). (We exclude the zero matrix simply because this would make \(M\)-weak mixing impossible.) Then \(M\) is a translational set of homomorphisms of \(\mathbb{R}^q\). The same is true if we drop the condition that the matrices be diagonal. Similarly if we work with \(\mathbb{Z}^q\) instead of \(\mathbb{R}^q\) and use matrices over the integers.

Acknowledgment. We thank Richard de Beer, Willem Fouché, Johan Swart and Gusti van Zyl for useful discussions.

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