Bifurcations of phase portraits of spherical pendulum with vibrating suspension point

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June 28, 2023

Abstract

We consider a spherical pendulum whose suspension point performs high-frequency spatial vibrations. Dynamics of this pendulum can be described by averaging of its Hamiltonian over phases of vibrations. We impose conditions on vibrations such that the averaged Hamiltonian has a rotational symmetry. Under these conditions we present a bifurcation diagram for the phase portraits of the averaged system. We show numerical simulations of different examples of vibrations. Bifurcation of phase portraits of spherical physical pendulum with vibrating suspension point are considered as well.

1 Introduction

Small high frequency excitation has a considerable effect of dynamics of a mechanical system. This was first demonstrated in classical works [1, 2, 3] for a simple pendulum with vertical vibrating suspension point. N.N. Bogolyubov [2] developed nonlinear theory by using the averaging method, and P.L. Kapitsa developed a method of separation of slow and fast variables for this [3, 12] (see also [13]). Dynamics of a simple pendulum with vibrating suspension point was later considered in many publications, see [4, 5, 6] and references therein. It is demonstrated in many works, e.g., in [11] that the problem can be simplified by using averaging in Hamiltonian form. The case of arbitrary planar vibrations of the suspension point of a planer simple pendulum is considered in [7], who used the Hamiltonian approach of [14] to construct the averaged system and gave a complete description of bifurcations of its phase portraits. Generalisations to double-link and multiple-link pendulums are contained in [15, 16, 17].

A spherical pendulum (see, e.g., [13]) is a classical model problem in mechanics along with a simple pendulum. Dynamics of a spherical pendulum with high frequency vertical harmonic vibration of the suspension point was considered in [9]. It was shown that the averaged over the fast vibration system has either one or three equilibria in dependence of system’s parameters.

We consider a spherical pendulum whose suspension point performs high-frequency arbitrary spatial vibrations. We construct averaged Hamiltonian and impose conditions on vibrations such that this Hamiltonian has a rotational symmetry. We use a set of parameters different from that in [9]. This allows to obtain simple parametric form of the bifurcation curve in the plane of parameters. We present a complete description of bifurcation diagram of its phase portraits of the averaged system. Numerical examples demonstrate a very good agreement of dynamics of the exact and the averaged system. Bifurcations of phase portraits of spherical physical pendulum (rigid rod case) with high-frequency arbitrary vibrating suspension point are discussed as well.

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2 Hamiltonian of spherical pendulum with vibrating suspension point

We consider a spherical pendulum, Fig. 7, whose suspension point performs high-frequency periodic vibrations. Let \( l, m \) be length of the massless rod and mass of the bob for this pendulum. Let \( \xi(t), \eta(t) \) and \( \tau(t) \) be, respectively, the horizontal and the vertical Cartesian coordinates of the suspension point. It is assumed that \( \xi(t), \eta(t) \) and \( \tau(t) \) are given periodic fast oscillating functions of time. Owing to the spherical geometry of the problem, spherical coordinates in Fig. 7 are used to describe the position of pendulum’s bob:

\[
\begin{align*}
  x &= l \sin \varphi \cos \alpha + \xi(t), \\
  y &= l \sin \varphi \sin \alpha + \eta(t), \\
  z &= -l \cos \varphi + \tau(t).
\end{align*}
\]

Figure 1: Spherical pendulum

Then the kinetic and potential energies of the bob are

\[
T = \frac{1}{2}m \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) = \frac{1}{2}ml^2 \left( \dot{\varphi}^2 + \sin^2 \varphi \cdot \dot{\alpha}^2 \right) + ml \left( \cos \varphi \cos \alpha \cdot \dot{\varphi} - \sin \varphi \sin \alpha \cdot \dot{\alpha} \right) \cdot \dot{\xi}(t) \\
+ ml \left( \cos \varphi \sin \alpha \cdot \dot{\varphi} + \sin \varphi \cos \alpha \cdot \dot{\alpha} \right) \cdot \dot{\eta}(t) + ml \sin \varphi \cdot \dot{\varphi} \cdot \dot{\tau}(t) + \frac{1}{2}m \left( \ddot{\xi}^2(t) + \ddot{\eta}^2(t) + \ddot{\tau}^2(t) \right),
\]

\[
V = -mgl \cos \varphi + mg \cdot \tau(t).
\]

The Lagrangian is

\[
L = T - V.
\]

The generalised momenta conjugate to \( \varphi \) and \( \alpha \) are

\[
p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = ml^2 \dot{\varphi} + ml \left( \cos \varphi \cos \alpha \cdot \dot{\xi}(t) + \cos \varphi \sin \alpha \cdot \dot{\eta}(t) + \sin \varphi \cdot \dot{\tau}(t) \right),
\]

\[
p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = ml^2 \sin^2 \varphi \cdot \dot{\alpha} + ml \left( -\sin \varphi \sin \alpha \cdot \dot{\xi}(t) + \sin \varphi \cos \alpha \cdot \dot{\eta}(t) \right).
\]

Thus

\[
\dot{\varphi} = \frac{p_\varphi - ml \left( \cos \varphi \cos \alpha \cdot \dot{\xi}(t) + \cos \varphi \sin \alpha \cdot \dot{\eta}(t) + \sin \varphi \cdot \dot{\tau}(t) \right)}{ml^2},
\]

\[
\dot{\alpha} = \frac{p_\alpha - ml \left( -\sin \varphi \sin \alpha \cdot \dot{\xi}(t) + \sin \varphi \cos \alpha \cdot \dot{\eta}(t) \right)}{ml^2 \sin^2 \varphi}.
\]
Then we get the Hamiltonian
\[
H = \frac{1}{2} m l^2 \left( p_\varphi - m l \left( \cos \varphi \cos \alpha \cdot \xi(t) + \cos \varphi \sin \alpha \cdot \eta(t) + \sin \varphi \cdot \tau(t) \right) \right)^2 \\
+ \frac{1}{2} m l^2 \sin^2 \varphi \cdot \left( p_\alpha - m l \left( -\sin \varphi \sin \alpha \cdot \dot{\xi}(t) + \sin \varphi \cos \alpha \cdot \dot{\eta}(t) \right) \right)^2 \\
- \frac{1}{2} m \left( \dot{\xi}^2(t) + \dot{\eta}^2(t) + \dot{\tau}^2(t) \right) - m g l \cos \varphi + m g \cdot \tau(t).
\]

3 Averaged Hamiltonian

Assume that \( \xi = \varepsilon \xi(\omega t/\varepsilon), \eta = \varepsilon \eta(\omega t/\varepsilon), \tau = \varepsilon \tau(\omega t/\varepsilon) \), where \( \varepsilon \) is a small parameter, \( \dot{\xi}, \dot{\eta}, \dot{\tau} \) are 2\( \pi \)-periodic functions of the argument \( \omega t/\varepsilon \) with zero average. Dynamics can be approximately described by averaging of the Hamiltonian over phases of vibrations \([10]\). The averaged Hamiltonian is
\[
\bar{H} = \frac{1}{2 m l^2} \left( p_\varphi^2 + \frac{p_\alpha^2}{\sin^2 \varphi} \right) + \frac{1}{2} m \left( \cos^2 \varphi \cos^2 \alpha + \sin^2 \alpha \right) \cdot \bar{\xi}^2(t) \\
+ \frac{1}{2} m \left( \cos^2 \varphi \sin^2 \alpha + \cos^2 \alpha \right) \cdot \bar{\eta}^2(t) + \frac{1}{2} m \sin^2 \varphi \cdot \bar{\tau}^2(t) \\
+ m \left( \cos^2 \varphi \cos \alpha \sin \alpha - \cos \alpha \sin \alpha \right) \cdot \bar{\xi}(t) \bar{\eta}(t) + m \cos \varphi \cos \alpha \sin \varphi \cdot \bar{\tau}(t) \bar{\xi}(t) \\
+ m \cos \varphi \sin \alpha \sin \varphi \cdot \bar{\eta}(t) \bar{\tau}(t) - m g l \cos \varphi.
\]

Here “bars” in the right hand side denote averaging with respect to time.

4 The Hamiltonian under symmetry conditions

In what follows we consider vibrations such that the averaged Hamiltonian does not depend on \( \alpha \). The conditions for this are
\[
\begin{align*}
\bar{\xi}^2(t) &= \bar{\eta}^2(t), \\
\bar{\xi}(t) \bar{\eta}(t) &= \bar{\eta}(t) \bar{\tau}(t) = \bar{\tau}(t) \bar{\xi}(t) = 0.
\end{align*}
\]

Then \( \alpha \) is a cyclic coordinate, and \( p_\alpha \) is a first integral of the averaged system. The averaged Hamiltonian simplifies to
\[
\bar{H} = \frac{p_\varphi^2}{2 m l^2} + \bar{V},
\]
where
\[
\bar{V} = \frac{p_\alpha^2}{2 m l^2 \sin^2 \varphi} + \frac{1}{2} m \cos^2 \varphi \cdot \bar{\eta}^2(t) + \frac{1}{2} m \bar{\eta}^2(t) + \frac{1}{2} m \sin^2 \varphi \cdot \bar{\tau}^2(t) - m g l \cos \varphi.
\]

The third term \( \frac{1}{2} m \bar{\eta}^2(t) \) is not important here, and we omit it. Denote \( A = \bar{\tau}^2(t), B = p_\alpha^2 \) and \( C = \bar{\eta}^2(t) \). For convenience, we suppose that \( m = 1, l = 1, g = 1 \), and we still use the notation \( \bar{V} \). We have up to a constant term
\[
\bar{V} = \frac{B}{2 \sin^2 \varphi} + \frac{1}{2} \sin^2 \varphi \cdot (A - C) - \cos \varphi.
\]

Dynamics of variables \( \varphi, p_\varphi \) is described by the Hamiltonian system with one degree of freedom; \( \bar{H} \) is the Hamiltonian of this system. Behaviour of the variable \( \alpha \) is described by the equation
\[
\dot{\alpha} = \frac{p_\alpha}{\sin^2 \varphi}.
\]
5 Bifurcations of phase portraits

We will divide the parameter plane of the problem into two domains corresponding to different types of phase portraits of the averaged system. Boundary between these domains is a critical curve corresponding to degenerate equilibria: first and second derivatives of $\bar{V}$ vanish for parameters on these curves. The number of equilibria changes at crossing such a curve in the plane of parameters.

The critical curve is defined by the equations

$$\frac{\partial \bar{V}}{\partial \varphi} = -\frac{B \cos(\varphi)}{(\sin(\varphi))^3} + \frac{1}{2} (A-C) \sin(2\varphi) + \sin(\varphi) = 0,$$
$$\frac{\partial^2 \bar{V}}{\partial \varphi^2} = 3 \frac{B \cos(\varphi)^2}{(\sin(\varphi))^4} + \frac{B}{(\sin(\varphi))^2} + (A-C) \cos(2\varphi) + \cos(\varphi) = 0. \tag{6}$$

Equations (6) is a system of linear non-homogeneous equations with respect to $A-C$ and $B$. Solving it, we get a parametric representation of the bifurcation curve:

$$\begin{cases}
A-C = -\frac{3\cos^2(\varphi) + 1}{4\cos^3(\varphi)}, \\
B = -\frac{1}{4} \cdot \frac{\sin^6\varphi}{\cos^3\varphi}.
\end{cases} \tag{7}$$

As we defined $B$ to be positive, we should consider $\varphi \in [\pi/2, \pi]$. Therefore there are no bifurcations with $\varphi \in [0, \pi/2]$. The relations (7) give a parametric representation with the parameter $\varphi$ of the critical curve $\Gamma$ corresponding to degenerate equilibria. This curve is shown in Fig.8. It divides the plane $A-C,B$ into domains I and II. The pendulum has one equilibrium, if parameters are in domain I, and three equilibria, if parameters are in domain II. Phase portraits of the system for parameters in domains I and II are shown in Fig.3. The line $B = 0$ corresponds to a simple pendulum (cf. [7]).

6 Numerical examples

In this Section we provide different numerical examples with suspension point vibrations.

Clearly, symmetry conditions (3) are satisfied for these examples of vibrations. We take $g = 1$, $l = 1$, $m = 1$ and $\varepsilon = 0.001$. Then examples of vibrations 1 and 2 correspond to points $P(17.5,1)$ in domain II and $Q(1.5,1)$ in domain I in the parameter plane respectively with initial value $p_a(0) = 1$, examples of vibrations 3 and 4 correspond to points $P(17.5,4)$ in domain II and $Q(1.5,4)$ in
domain I in the parameter plane respectively with initial value \( p_{\alpha}(0) = 2 \). We calculate numerically trajectories of the system with Hamiltonian \( H \) of exact system (1) with initial conditions \( p_\phi(0) = 1 \), \( \phi(0) = \pi/2 \) and \( \alpha(0) = 0 \) for examples of vibrations 1, 2 and 3, 4 respectively.

Behaviour of \( p_\alpha \) and of the averaged Hamiltonian \( \bar{H} \) along trajectories of the exact system with examples of vibrations 1, 2 and 3, 4 are shown in Fig. 4 and Fig. 5 respectively. The time-step of the plot is 0.001. One can see that values \( p_\alpha \) and \( \bar{H} \) are subject to only small oscillations. Projection onto the plane \( \phi, p_\phi \) of these trajectories are shown in cyan colour in Fig. 6. The trajectories of the averaged system with the same initial conditions are shown in red colour in the same figure. One can see that the corresponded trajectories agree very well.

### 7 Spherical physical pendulum case

Physical pendulum is more realistic than simple pendulum in practice. Similar to the spherical pendulum we discussed above, we consider a spherical physical pendulum (rigid rod case) with uniform density, Fig. 7, whose suspension point performs high-frequency periodic vibrations in arbitrary direction. Let \( l_p, m_p \) be total length and mass of the pendulum, \( \lambda \) be the length from the suspension point. We still let \( \xi(t) \), \( \eta(t) \) and \( \tau(t) \) be, respectively, the horizontal and the vertical Cartesian coordinates of the suspension point. A new system of spherical coordinates in Fig. 7 are used to describe the position of pendulum:

\[
\begin{align*}
    x_\lambda &= \lambda \sin \varphi \cos \alpha + \xi(t), \\
    y_\lambda &= \lambda \sin \varphi \sin \alpha + \eta(t), \\
    z_\lambda &= -\lambda \cos \varphi + \tau(t).
\end{align*}
\]
Figure 4: Behaviour of $p_\alpha$ vs time.
Figure 5: Value of $\bar{H}$ vs time.
Figure 6: Trajectories of the exact (cyan) and the averaged (red) systems for examples 1 to 4.

Figure 7: Spherical physical pendulum
Then the kinetic and potential energies of the rigid rod are

\[
T_P = \int_0^1 \frac{1}{2} m \left( \dot{x}_\lambda^2 + y_\lambda^2 + z_\lambda^2 \right) d\lambda
\]

\[
= \frac{1}{6} ml^2 (\dot{\phi}^2 + \sin^2 \varphi \cdot \dot{\alpha}^2) + \frac{1}{2} ml (\cos \varphi \cos \alpha \cdot \dot{\phi} - \sin \varphi \sin \alpha \cdot \dot{\alpha}) \cdot \dot{\xi} (t)
\]

\[
+ \frac{1}{2} ml (\cos \varphi \sin \alpha \cdot \dot{\phi} + \sin \varphi \cos \alpha \cdot \dot{\alpha}) \cdot \eta (t) + \frac{1}{2} ml \sin \varphi \cdot \dot{\varphi} \cdot \dot{\tau} (t)
\]

\[
+ \frac{1}{2} m \left( \dot{\varphi}^2 (t) + \dot{\eta}^2 (t) + \dot{\xi}^2 (t) \right),
\]

\[
V_P = \int_0^1 mgz_\lambda d\lambda = -\frac{1}{2} mgl \cos \varphi + mg \cdot \tau (t).
\]

The Lagrangian is

\[
L_P = T_P - V_P.
\]

Denote the generalised momenta conjugate to \( \varphi \) and \( \alpha \) as \( p_\varphi \) and \( p_\alpha \). Using the same definition of high-frequency vibrations \( \xi, \eta, \tau \), dynamics of physical pendulum can also be approximately described by averaging of the Hamiltonian over phases of vibrations \([10]\). The averaged Hamiltonian is

\[
\bar{H}_P = \frac{1}{6ml^2} \left( p_\varphi^2 + \frac{p_\alpha^2}{\sin^2 \varphi} \right) + \frac{1}{12} m \left( \cos^2 \varphi \cos^2 \alpha + \sin^2 \alpha \right) \cdot \dot{\xi}^2 (t)
\]

\[
+ \frac{1}{12} m \left( \cos^2 \varphi \sin^2 \alpha + \cos^2 \alpha \right) \cdot \eta^2 (t) + \frac{1}{12} m \sin^2 \varphi \cdot \dot{\tau}^2 (t)
\]

\[
+ \frac{1}{6} m \cos \varphi \cos \alpha \sin \alpha \cdot \xi (t) \eta (t) + \frac{1}{6} m \cos \varphi \cos \alpha \sin \varphi \cdot \dot{\tau} (t) \xi (t)
\]

\[
+ \frac{1}{6} m \cos \varphi \sin \alpha \sin \varphi \cdot \eta (t) \dot{\tau} (t) - \frac{1}{2} mgl \cos \varphi.
\]

Applying the assumption in \([3]\) again, the averaged Hamiltonian simplifies to

\[
\bar{H}_P = \frac{p_\varphi^2}{6ml^2} + \bar{V}_P,
\]

where

\[
\bar{V}_P = \frac{B}{6 \sin^2 \varphi} + \frac{1}{12} \sin^2 \varphi \cdot (A - C) - \frac{1}{2} \cos \varphi
\]

up to a constant. Here \( \bar{H}_P \) is of the same structure with \( \bar{H} \). Thus we will have a similar bifurcation curve to which in spherical pendulum case. we get a parametric representation of the bifurcation curve:

\[
\begin{cases}
A - C = -\frac{9 \cos^2 (\varphi) + 3}{4 \cos^3 (\varphi)}, \\
B = \frac{3}{8} \frac{\sin^6 \varphi}{\cos^3 \varphi}.
\end{cases}
\]

The relations \([11]\) give a parametric representation with the parameter \( \varphi \) of the critical curve \( \Gamma_P \) corresponding to degenerate equilibria. This curve is shown in Fig. 8(left). It divides the plane \( (A - C) - \bar{O} - \bar{B} \) into domains \( I_P \) and \( II_P \). If parameters are in domain \( I_P \), the pendulum has one equilibrium, and if parameters are in domain \( II_P \), the pendulum has three equilibria. Compare with the spherical pendulum, the number of equilibria changes at a larger vertical vibration or smaller horizontal vibration for spherical physical pendulum in rigid rod case, see Fig. 8(right). Phase portraits of the system for parameters in domains \( I_P \) and \( II_P \) are shown in Fig. 9.
Figure 8: Critical curves of spherical physical pendulum.

Figure 9: Phase portrait for domain I (left) and II (right).
8 Relation to exact problem

In the 4-dimensional phase space of the averaged system, the motion occurs on invariant surfaces \( p_\alpha = \text{const}, \bar{H}(P) = \text{const} \). These surfaces are 2-dimensional tori, or closed trajectories, or separatrix surfaces. We consider here a compact part of the phase space and assume that vibrations are described by smooth enough functions, so, KAM theory is applicable.

According to KAM theory (see, e.g., [11]), 5-dimensional extended phase space of the original system with coordinates \( \varphi, \alpha, p_\varphi, p_\alpha, t \) is filled by 3-dimensional invariant tori close to tori \( p_\alpha = \text{const}, \bar{H}(P) = \text{const} \) up to a reminder of a measure which is small when \( \varepsilon \) is small. This is the case of a proper degeneration in KAM theory. To see this we can introduce a new time \( \tilde{t} = t/\varepsilon \), then we obtain the new system

\[
\tilde{H}(P) = \frac{1}{\varepsilon} \tilde{P}_t + H(P) \left( p_\varphi, \varphi, p_\alpha, \alpha, \tilde{t} \right)
\]

with Hamiltonian \( \varepsilon H(P) \), in which the variables \( \varphi, \alpha, p_\varphi, p_\alpha \) are slow and the variable \( t \) is fast, \( \tilde{P}_t \) and \( \tilde{t} \) are conjugate variables. We get a system with 3 degrees of freedom in the case of proper degeneration.

If symmetry conditions (3) are satisfied only approximately, with some accuracy \( \delta \), then KAM theory ensure that the extended phase space of the original system is filled by 3-dimensional invariant tori close to tori \( p_\alpha = \text{const}, \bar{H}(P) = \text{const} \) up to a reminder of a measure which is small when \( \delta \) and \( \varepsilon \) are small.

Thus, our study of phase portraits of the averaged system provides considerable information about dynamics of the exact (not averaged) problem.

9 Conclusion

We considered a spherical (physical) pendulum whose suspension point performs high-frequency spatial vibrations. This system has 2\( \frac{1}{2} \) degrees of freedom with canonical phase variables \( \varphi, \alpha, p_\varphi, p_\alpha \) described in Section 2. After averaging with respect to phases of fast vibrations we got a system with 2 degrees of freedom. Conditions (3) imply that the averaged system has a rotational symmetry with the cyclic coordinate \( \alpha \). Value \( p_\alpha \) is a first integral of the averaged system. Dynamics of \( \varphi, p_\varphi \) is described by a Hamiltonian system with one degree of freedom and can be studied in a phase plane. We described bifurcations of phase portraits of this system. The function \( \bar{H}(P) \) is its Hamiltonian. Our study of phase portraits of the averaged system provides considerable information about dynamics of the exact (not averaged) problem.

Acknowledgement

We are glad to express our gratitude to Prof. Anatoly Neishtadt for suggestion of the topic of this work and for discussions.
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