TRANSFER OPERATORS, INDUCED PROBABILITY SPACES, AND RANDOM WALK MODELS

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Abstract. We study a family of discrete-time random-walk models. The starting point is a fixed generalized transfer operator $R$ subject to a set of axioms, and a given endomorphism in a compact Hausdorff space $X$. Our setup includes a host of models from applied dynamical systems, and it leads to general path-space probability realizations of the initial transfer operator. The analytic data in our construction is a pair $(h, \lambda)$, where $h$ is an $R$-harmonic function on $X$, and $\lambda$ is a given positive measure on $X$ subject to a certain invariance condition defined from $R$. With this we show that there are then discrete-time random-walk realizations in explicit path-space models; each associated to a probability measures $\mathbb{P}$ on path-space, in such a way that the initial data allows for spectral characterization: The initial endomorphism in $X$ lifts to an automorphism in path-space with the probability measure $\mathbb{P}$ quasi-invariant with respect to a shift automorphism. The latter takes the form of explicit multi-resolutions in $L^2$ of $\mathbb{P}$ in the sense of Lax-Phillips scattering theory.

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1. Introduction

We study a family of stochastic processes indexed by a discrete time index. Our results encompass the more traditional random walk models, but our study here goes beyond that. The processes considered are generated by a single positive operator, say $R$, defined on $C(X)$ where $X$ is a given compact Hausdorff space. From a given positive operator $R$ we then derive an associated system of generalized transition probabilities, and an induced probability space; the induction realized as a probability space of infinite paths having $X$ as a base space. In order

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for us to pin down the probability space, i.e., the induced path-space measure $\mathbb{P}$, two more ingredients will be needed, one is a prescribed endomorphism $\sigma$ in $X$, consistent with $R$; and, the other, is a generalized harmonic function $h$ on $X$, i.e., $R(h) = h$. We further explore the interplay between the harmonic functions $h$ and the associated path-space measures $\mathbb{P}$. To do this we note that the given endomorphism $\sigma$ in $X$ induces an automorphism in the path-space. We show that the path-space measure $\mathbb{P}$ is quasi-invariant, and we compute the corresponding Radon-Nikodym derivative. Our motivation derives from the need to realize multiresolution models in a general setting of dynamical systems as they arise in a host of applications: in symbolic dynamics, e.g., [BJO04, BJKR02], in generalized multiresolution model, e.g., [DJ09]; in dynamics arising from an iteration of substitutions, e.g., [Bea91]; in geometric measure theory, and for Iterated Function Systems (IFS), e.g., [Hut81, Urb09]; or in stochastic analysis, e.g., [AJ12, Hut81, MNB16, GF16, ZXL16, TSI+15, Pes13, KLTMV12].

2. The Setting

Organization of the paper: The setup starts with a fixed and given compact Hausdorff space $X$, and positive operator $R$ (a generalized transfer operator), defined on the function algebra $C(X)$, and subject to two simple axioms. Candidates for $X$ will include compact Bratteli diagrams, see e.g., [Mat06, Mat04, Dan01, HPS92]. Given $R$, in principle one is then able to derive a system of generalized transition probabilities for discrete time processes starting from points in $X$; see details in the present section. However, in order to build path-space probability spaces this way, more considerations are required, and this will be explored in detail in Sections 4 and 5 below. Section 3 deals with a subfamily of systems where the generalized transfer operator $R$ is associated to an Iterated Function system (IFS). In Section 6, we derive some conclusions from the main theorems in the paper.

Let $X$ be a compact Hausdorff space, and $\mathcal{M}(X)$ be the space of all measurable functions on $X$. Let $R: C(X) \rightarrow \mathcal{M}(X)$ be a positive linear mapping, i.e., $f \geq 0 \Rightarrow Rf \geq 0$.

Definition 2.1. Let $\mathcal{L}(R)$ be the set of all positive Borel measures $\lambda$ on $X$ s.t.

\begin{align*}
R(C(X)) &\subset L^1(\lambda), \\
\lambda \cdot R &\ll \lambda \text{ (absolutely continuous).}
\end{align*}

Note 2.2. Let $R$ be as above, and set

$$
\mu = \lambda \cdot R, \quad \text{and} \quad W = \frac{d\mu}{d\lambda} = \text{Radon-Nikodym derivative},
$$

then

\begin{equation}
\lambda(Rf) = \text{defn. } \int_X RF d\lambda = \int_X fW d\lambda, \quad \forall f \in C(X).
\end{equation}

$W$ depends on both $R$ and $\lambda$.

Definition 2.3. For all $x \in X$, let

$$
\mu_x = P(\cdot | x)
$$

be the conditional measure, where

$$
\mu_x(f) := (Rf)(x) = \int_X f(y) d\mu_x(y).
$$
Lemma 2.4. Let $X$, and $R$ (positive in $C(X)$) be as before, then there is a system of measures $P(\cdot \mid x)$ such that

$$
(Rf)(x) = \int_X f(y) P(dy \mid x), \quad \forall f \in C(X).
$$

(2.6)

Proof. Immediate from Riesz’ theorem applied to the positive linear functional,

$$
C(X) \ni f \rightarrow R(f)(x), \quad \forall x \in X.
$$

\[ \square \]

Corollary 2.5. $C(X) \ni f \rightarrow R(f)(x)$ extends to $F \in \mathcal{M}(X)$, measurable functions on $X$, s.t. the extended operator $\tilde{R}$ is as follows:

$$
\tilde{R}(F)(x) = \int_X F(y) P(dy \mid x), \quad F \in \mathcal{M}(X).
$$

(2.7)

We will write $R$ also for the extension $\tilde{R}$.

Remark 2.6. Let $X$, and $R$ be as specified in Definition 2.1. Set

$$
\mathcal{L}_1(R) = \{ \lambda \in \mathcal{L}(R) \mid \lambda(X) = 1 \}.
$$

Clearly, $\mathcal{L}_1(R)$ is convex. In this generality, we address two questions:

Q1. We show that $\mathcal{L}_1(R)$ is non-empty.

Q2. What are the extreme points in $\mathcal{L}_1(R)$?

Lemma 2.7. Let $\mu_x = P(\cdot \mid x)$ be as above. Let $\lambda \in \mathcal{L}(R)$, and let $W$ be the Radon-Nikodym derivative from (2.3). Then

$$
\int_X P(\cdot \mid x) d\lambda(x) = W(\cdot) d\lambda(\cdot).
$$

(2.8)

Proof. Immediate from the definition. Indeed, for all Borel subset $E \subset X$, the following are equivalent ($f = \chi_E$):

\[
\begin{align*}
\int Rf \, d\lambda &= \int fW \, d\lambda \\
&\upharpoonright \\
\int f(y) P(dy \mid x) \, d\lambda(x) &= \int f(x) W(x) \, d\lambda(x) \\
&\upharpoonright \\
\int P(E \mid x) \, d\lambda(x) &= \int_E W(y) \, d\lambda(y) \\
&\upharpoonright \\
\int_X P(\cdot \mid x) \, d\lambda(x) &= W(\cdot) \, d\lambda(\cdot)
\end{align*}
\]

\[ \square \]
Remark 2.8. In general, \( \lambda \neq P(\cdot \mid x_0), \) \( x_0 \in X. \) Note that \( \lambda = P(\cdot \mid x_0) \in \mathcal{L}(R) \) iff

\[
\int_y P(\cdot \mid y) P(dy \mid x_0) = W(\cdot) P(\cdot \mid x_0), \quad \text{i.e.,} \\
\int_y P(dz \mid y) P(dy \mid x_0) = W(z) P(dz \mid x_0)
\] (2.9)

However, condition (2.9) is very restrictive, and it is not satisfied in many cases. See Example 2.9 below.

Example 2.9 (Iterated Function System (IFS); see e.g., [Jor12, DJ09]). Let \( X = [0,1] = \mathbb{R}/\mathbb{Z}, \) and \( \lambda = \) Lebesgue measure. Fix \( v > 0, \) a positive function on \([0,1]\), and set

\[
(Rf)(x) = v\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) + v\left(\frac{x+1}{2}\right) f\left(-\frac{x+1}{2}\right)
\]

Then

\[
P(\cdot \mid x) = v\left(\frac{x}{2}\right) \delta_{\frac{x}{2}} + v\left(\frac{x+1}{2}\right) \delta_{\frac{x+1}{2}} \not\ll \lambda.
\]

Assumption (Additional axiom on \( R \)). Let \( R \) be the positive mapping in Definition 2.1. Assume there exists \( \sigma : X \rightarrow X, \) measurable and onto, such that

\[
R((f \circ \sigma) g) = fRg, \quad \forall f, g \in C(X).
\] (2.10)

\( R \) in (2.10) is a generalized conditional expectation.

Lemma 2.10. Let \( R \) satisfy (2.10) and let \( \{P(\cdot \mid x)\}_{x \in X} \) be the family of conditional measures in Definition 2.3. Then,

\[
P(E \mid x) = \int_E \frac{f(\sigma(y))}{f(x)} P(dy \mid x)
\] (2.11)

for all \( f \in C(X), \) and all \( E \in \mathcal{B}(X); \) where \( \mathcal{B}(X) \) denotes all Borel subsets of \( X. \)

Proof. We have

\[
R((f \circ \sigma) g)(x) = f(x) R(g)(x)
\]

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\[
\int f(\sigma(y)) g(y) P(dy \mid x) = f(x) \int g(y) P(dy \mid x), \quad \forall f, g \in C(X), \forall x \in X.
\]

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\[
E \int f(\sigma(y)) P(dy \mid x) = f(x) E P(E \mid x), \quad \forall f \in C(X), \forall E \in \mathcal{B}(X),
\]

and the assertion follows. \( \square \)

Lemma 2.11. Suppose (2.10) holds and \( \lambda \in \mathcal{L}(R). \) Set \( W = \) the Radon-Nikodym derivative, then the operator \( S : f \rightarrow Wf \circ \sigma \) is well-defined and linear in \( L^2(\lambda) \) with \( C(X) \) as dense domain. In general \( S \) is unbounded. Moreover,

\[
S \subset R^*, \text{ containment of unbounded operators},
\] (2.12)
where $R^*$ denotes the adjoint operator to $R$, i.e.,
\[
\int_X (Wf \circ \sigma) g \, d\lambda = \int_X f R(g) \, d\lambda; \tag{2.13}
\]
holds for all $f, g \in C(X)$. That is,
\[
R^* f = Wf \circ \sigma, \quad \forall f \in C(X), \tag{2.14}
\]
as a weighted composition operator.

Further, the selfadjoint operator $RR^*$ is the multiplication operator:
\[
RR^* f = R(W)f, \quad \forall f \in C(X); \tag{2.15}
\]
i.e., multiplication by the function $R(W)$.

**Proof.** For all $f, g \in C(X)$, we have
\[
\int_X f R(g) \, d\lambda = \int_X R((f \circ \sigma) g) \, d\lambda = \int_X \underbrace{(W(f \circ \sigma))g}_{= S(f)} \, d\lambda,
\]
and so (2.12)-(2.13) follow. Also,
\[
RR^* f = R(W) f = m f,
\]
where $m = R(W)$. The assertion (2.15) follows from this. \hfill \square

**Corollary 2.12.** $S$ is isometric in $L^2(\lambda) \iff R(W) = 1$.

**Corollary 2.13.** $R$ defines a bounded operator on $L^2(\lambda)$, i.e., $L^2(\lambda) \overset{R}{\longrightarrow} L^2(\lambda)$ is bounded $\iff R(W) \in L^{\infty}(\lambda)$.

**Proof.** Immediate from (2.15) since
\[
||RR^*||_{2 \rightarrow 2} = ||R||^2_{2 \rightarrow 2} = ||R^*||^2_{2 \rightarrow 2}. \tag{2.16}
\]

**Remark 2.14.** Let $\lambda \in \mathcal{L}(R), \mu = \lambda \cdot R$, and $W = d\mu/d\lambda$ as before. Even if $W \in L^1(\lambda)$, the following two operators are still well-defined:
\[
L^2(\lambda) \supset \left\{ \begin{array}{ccc} C(X) & \ni f & \overset{R}{\rightarrow} Rf \in L^\infty(\lambda) \subset L^2(\lambda) \\ C(X) & \ni f & \overset{S}{\rightarrow} W(f \circ \sigma) \in L^2(\lambda) \end{array} \right\},
\]
and
\[
\langle Sf, g \rangle_{L^2} = \langle f, Rg \rangle_{L^2}, \quad \forall f, g \in C(X).
\]

**Corollary 2.15.** Assume $\frac{d\lambda}{W}$ is well-defined. Then $\lambda \circ \sigma^{-1} \ll \lambda$, and
\[
\frac{d\lambda \circ \sigma^{-1}}{d\lambda} = R\left(\frac{1}{W}\right),
\]
where $R\left(\frac{1}{W}\right)$ is defined as in (2.7) of Corollary 2.5.
Proof. Recall the pull-back measure $\lambda \circ \sigma^{-1}$, where $\sigma^{-1}(E) = \{ z \in X \mid \sigma(z) \in E \}$, for all Borel sets $E \subset X$. One checks that
\[
\int f \, d\lambda \circ \sigma^{-1} = \int f \circ \sigma \, d\lambda = \int \frac{1}{W} W f \circ \sigma \, d\lambda = \int R \left( \frac{1}{W} \right) f \, d\lambda, \quad \forall f \in C(X);
\]
and the assertion follows. □

Corollary 2.16. Let $R$, $\lambda$, $W$ be as above, and assume that $\|R(W]\|_\infty \leq 1$. Let $h$ be a function on $X$ solving the equation
\[
Rh = h, \quad h \in L^2(\lambda), \quad (R\text{-harmonic}) \quad (2.17)
\]
then the following implication holds:
\[
h(x) \neq 0 \implies R(W)(x) = 1. \quad (2.18)
\]
Proof. By (2.16), $R$ is contractive, i.e., $\|R\|_{2 \rightarrow 2} = \|R^*\|_{2 \rightarrow 2} \leq 1$, $\|Rf\|_{L^2(\lambda)} \leq \|f\|_{L^2(\lambda)}$; and so $R^*h = h$; and, by (2.15),
\[
h = hR(W), \quad \text{pointwise}, \quad (2.19)
\]
i.e., $h(x) = h(x)R(W)(x)$, for all $x \in X$, and (2.18) follows. □

Corollary 2.17. Suppose $\lambda \in \mathcal{L}(R)$ with $R$, $\sigma$, $W = d\mu/d\lambda$ satisfying the usual axioms, then $\lambda$ is $\sigma$-invariant, i.e.,
\[
\int f \circ \sigma \, d\lambda = \int f \, d\lambda, \quad \forall f \in C(X) \quad (2.20)
\]
\[
\Downarrow
\]
\[
\frac{1}{W} \exists! \text{ exists, and } R \left( \frac{1}{W} \right) = 1 \text{ on the support of } \lambda. \quad (2.21)
\]

Proof. $(2.20) \implies (2.21)$ follows from Corollary 2.15. (Also see Corollary 2.5.)

Assume (2.21), then
\[
\text{LHS}_{(2.20)} = \int \frac{1}{W} W f \circ \sigma \, d\lambda = \int R \left( \frac{1}{W} \right) f \, d\lambda = \int f \, d\lambda, \quad \forall f \in C(X). 
\]
\[
\Box
\]

Corollary 2.18. Let $X$, $R$, $\lambda$, $W$, $\sigma$ be as specified above. Recall that $f \geq 0 \implies Rf \geq 0$, and $R((f \circ \sigma)g) = fR(g)$, $\forall f, g \in C(X)$. Assume further that $W \in L^2(\lambda)$, then
\[
\int_X |W|^2 f \circ \sigma \, d\lambda = 0, \text{ for some } f \in C(X) \quad (2.22)
\]
\[
\Downarrow
\]
\[
\int_X f R(W) \, d\lambda = 0. \quad (2.23)
\]

Proof. Use that $L^2(\lambda) \ni f \xrightarrow{R^*} W f \circ \sigma \in L^2(\lambda)$, we conclude that
\[
\int_X |W|^2 f \circ \sigma \, d\lambda = \int_X W f \circ \sigma W \, d\lambda = \langle R^*f, W \rangle_{L^2(\lambda)} \quad (2.24)
\]
\[
= \langle f, R(W) \rangle_{L^2(\lambda)} = \int_X f(x) R(W)(x) \, d\lambda(x). 
\]
Corollary 2.19. Let $X$, $R$, $\lambda$, $W$, $\sigma$ be as above, and let $E \subset X$ be a Borel set; then
\[
\int_{\sigma^{-1}(E)} |W|^2 \, d\lambda = \int_E R(W) \, d\lambda,
\]  
and so in particular, $R(W) \geq 0$ a.e. on $X$ w.r.t. $\lambda$.

Proof. Approximate $\chi_E$ with $f \in C(X)$ and use (2.24), we have
\[
\int |W|^2 \circ \sigma \, d\lambda = \int R(W) \, f \, d\lambda,
\]  
which is (2.25).

Example 2.20. Let $X = [0,1] = \mathbb{R}/\mathbb{Z}$, and $d\lambda = dx =$ Lebesgue measure. Fix $W > 0$, a positive function over $[0,1]$, and set
\[
(Rh)(x) = \frac{1}{2} \left( W \left( \frac{x}{2} \right) h \left( \frac{x}{2} \right) + W \left( \frac{x+1}{2} \right) h \left( \frac{x+1}{2} \right) \right). 
\]  
Let $\sigma(x) = 2x \mod 1$, $x \in X$, then
\[
\int_0^1 g(x) (Rh)(x) \, dx = \int_0^1 W(x) g(\sigma(x)) h(x) \, dx, \quad \forall f, g \in C(X). 
\]  
Proof. We introduce the mappings $\tau_0$ and $\tau_1$, as in Fig 2.1-2.2, so that $\sigma(\tau_i(x)) = x$, for all $x \in X$, $i = 0, 1$. One checks that
\[
R((g \circ \sigma) h)(x) = g(x) (Rh)(x). 
\]  
Note that $\lambda \in L'(R)$. Indeed, we have
\[
\lambda(Rh) = \int_0^1 (Rh)(x) \, dx 
\]  
by (2.26),
\[
= \int_0^1 \frac{1}{2} \left( Wh \left( \frac{x}{2} \right) + Wh \left( \frac{x+1}{2} \right) \right) \, dx 
\]  
and so $\frac{d\mu}{dx}(x) = W(x)$, where $\mu = \lambda \cdot R$.

Example 2.21. Let $R$ be as in (2.26), and let $h$ be an $R$-harmonic function, i.e.,
\[
(Rh)(x) = \frac{1}{2} \left( Wh \left( \frac{x}{2} \right) + Wh \left( \frac{x+1}{2} \right) \right) = h(x), \quad x \in X = [0,1]. 
\]  
Setting $\widehat{h}(n) = \int_0^1 e(nx) h(x) \, dx$, with $e(nx) := e^{i2\pi nx}$, it follows from (2.29) that
\[
\widehat{h}(n) = \int_0^1 e(nx) (Rh)(x) \, dx 
\]  
= $\int_0^1 W(x) e(2nx) h(x) \, dx = (Wh)^\wedge(2n), \quad \forall n \in \mathbb{Z}$.
An iteration gives

\[ \hat{h}(n) = \int_0^1 W(x)e^{(2nx)(Rh)(x)}\,dx \]

\[ = \int_0^1 W(x)W(2x)e^{(2^{2nx})h(x)}\,dx \]

\[ \cdots \]

\[ = \int_0^1 W(x)W(2x)\cdots W(2^{k-1}x)e^{(2^{k}\cdot nx)}h(x)\,dx, \]

and so

\[ \hat{h}(n) = (W_k h)^\wedge (2^k n), \quad \forall n \in \mathbb{Z}, \forall k = 0, 1, 2, \cdots; \]

where \( W_k(x) := W(x)W(2x)\cdots W(2^{k-1}x) \).

**Figure 2.1.** \( \sigma(x) = 2x \mod 1 \)

**Figure 2.2.** \( \tau_0(x) = x/2, \tau_1(x) = (x+1)/2 \)
3. Iterated Function Systems: The General Case

In this section we discuss a subfamily of systems where the generalized transfer operator $R$ is associated with an Iterated Function system (IFS).

Let $X$ be a compact Hausdorff space, $n \in \mathbb{N}$, and let

$$\tau_i : X \rightarrow X, \quad 1 \leq i \leq n$$

be a system of endomorphisms. Let

$$p_i > 0, \text{ s.t. } \sum_{i=1}^{n} p_i = 1.$$ \hfill (3.2)

Following [Hut81, Jor12, FH09, Urb09, DABJ09, DJ09], we say that (3.1)-(3.2) is an Iterated Function System (IFS) if there is a Borel probability measure $\lambda$ on $X$ such that

$$\sum_{i=1}^{n} p_i \int_X f(\tau_i(x)) \, d\lambda(x) = \int_X f(x) \, d\lambda(x)$$ \hfill (3.3)

holds for all $f \in C(X)$. Note that (3.3) may also be expressed as follows:

$$\sum_{i=1}^{n} p_i \lambda \circ \tau_i^{-1} = \lambda.$$ \hfill (3.4)

The measure $\lambda$ is called an IFS measure.

Let $W \in L^1(\lambda)$, $W \geq 0$, and set

$$(R_W f)(x) = \sum_{i=1}^{n} p_i (Wf)(\tau_i(x)), \quad x \in X, f \in C(X),$$ \hfill (3.5)

where $(Wf)(\tau_i(x)) := W(\tau_i(x)) f(\tau_i(x))$.

**Lemma 3.1.** If $W$ is as above, and if $\lambda$ is an IFS measure, then $\lambda \in L^1(R_W)$, see Remark 2.6.

**Proof.** We establish the conclusion by verifying that, under the assumptions, we have

$$\int_X (R_W f)(x) \, d\lambda(x) = \int_X W(x) f(x) \, d\lambda(x), \quad \forall f \in C(X),$$ \hfill (3.6)

i.e., $W$ is the Radon-Nikodym derivative, $d\mu_W / d\lambda = W$, where $\mu_W = \lambda \cdot R_W$. Indeed,

$$\text{LHS}_{(3.6)} = \text{by (3.5)} \sum_{i=1}^{n} p_i \int_X (Wf)(\tau_i(x)) \, d\lambda(x)$$

$$= \text{by (3.3)} \int_X (Wf)(x) \, d\lambda(x) = \text{RHS}_{(3.6)}.$$

\[ \square \]

**Remark 3.2.** The setting of Example 2.21, we have an IFS corresponding to the two mappings in Figure 2.2, and, in this setting, the corresponding IFS measure $\lambda$ on the unit interval $X = [0,1]$ can then easily be checked to be the restriction to $[0,1]$ of the standard Lebesgue measure. It is important to mention that there is a rich literature on IFS measures, see e.g., [Hut81, Jor12, FH09, Urb09, DABJ09, DJ09], and the variety of IFS measures associated to function systems includes explicit classes measures of fractal dimension.
4. The Set $\mathcal{L}_1(R)$ from a Quadratic Estimate

In order to build a path-space probability space from a given generalized transfer operator $R$, a certain spectral property for $R$ must be satisfied, and we discuss this below; see Theorem 4.1. The statement of the problem requires the introduction of a Hilbert space of sigma functions, also called square densities.

Let $X$ be a locally compact Hausdorff space, and let $R : C(X) \rightarrow \mathcal{M}(X)$ be given, subject to the conditions in Definition 2.1 and Remark 2.6.

For every probability measure $\lambda$ on $X$, we apply the Radon-Nikodym decomposition (see [Rud87]) to the measure $\lambda R$, getting

$$\lambda R = \mu_{abs} + \mu_{sing}$$

(4.1)

where the two terms on the RHS in (4.1) are absolutely continuous w.r.t $\lambda$, respectively, with $\mu_{sing}$ and $\lambda$ mutually singular. Hence there is a positive $W_{\lambda} \in L^1(\lambda)$ such that

$$\mu_{abs} = W_{\lambda} \sqrt{d\lambda}.$$  

When $\lambda$ is fixed, set

$$\tau = \frac{1}{2} (\mu_{abs} + \lambda R).$$

(4.2)

We have the following:

**Theorem 4.1.** Let $(X,R)$ be as described above, and let $\text{Prob}(X)$ be the convex set of all probability measures on $X$. For $\lambda \in \text{Prob}(X)$, let $\tau$ be the corresponding measure given by (4.2). Then $\mathcal{L}_1(R) \neq 0$ if and only if

$$\inf_{\lambda \in \text{Prob}(X)} \int_X \left| \sqrt{d(\lambda R)} - W_{\lambda} \sqrt{d\lambda} \right|^2 d\tau = 0.$$  

(4.3)

**Proof.** To carry out the proof details, we shall make use of the Hilbert space $\text{Sig}(X)$ of sigma-functions on $X$. While it has been used in, for example [Nel69, KM46, Hid80, Jor11], we shall introduce the basic facts which will be needed.

Elements in $\text{Sig}(X)$ are equivalence classes of pairs $(f,\mu)$, where $f \in L^2(\mu)$, and $\mu$ is a positive finite measure on $X$; we say that $(f,\mu) \sim (g,\nu)$ for two such pairs iff

$$f \sqrt{\frac{d\mu}{d\tau}} = g \sqrt{\frac{d\nu}{d\tau}}$$

a.e. on $X$ w.r.t. $\tau$. (4.4)

If $\text{class}(f_i,\mu_i)$, $i = 1, 2$, are two equivalence classes, then the operations in $\text{Sig}(X)$ are as follows: First set $\tau_s = \frac{1}{2} (\mu_1 + \mu_2)$, then the inner product in $\text{Sig}(X)$ is

$$\int_X f_1 \sqrt{\frac{d\mu_1}{d\tau_s}} f_2 \sqrt{\frac{d\mu_2}{d\tau_s}} d\tau_s,$$

and the sum is

$$\text{class} \left( f_1 \sqrt{\frac{d\mu_1}{d\tau_s}} + f_2 \sqrt{\frac{d\mu_2}{d\tau_s}}, \tau_s \right).$$

It is known that these definitions pass to equivalence classes; and that $\text{Sig}(X)$ is a Hilbert space; in particular, it is complete.

In order to complete the proof of the theorem, we shall need the following facts about the Hilbert space $\text{Sig}(X)$; see e.g., [Nel69]: First some notation; we set

$$f \sqrt{d\mu} = \text{class}(f,\mu) \in \text{Sig}(X);$$

(4.5)
and when \( \mu \) is fixed, we set \( M_2(\mu) \) to be the closed subspace in \( \text{Sig}(X) \) spanned by 
\[
\left\{ f \sqrt{d\mu} \mid f \in L^2(\mu) \right\}.
\]

We then have:
\[
\left\| f \sqrt{d\mu} \right\|^2_{\text{Sig}(X)} = \left\| f \right\|^2_{L^2(\mu)} = \int_X |f|^2 d\mu; \tag{4.6}
\]
and so, in particular,
\[
L^2(\mu) \ni f \mapsto f \sqrt{d\mu} \in \text{Sig}(X) \tag{4.7}
\]
defines an isometry with range \( M_2(\mu) \). We shall abbreviate \( \sqrt{d\mu} \) as \( \sqrt{\mu} \).

For two measures \( \mu \) and \( \nu \), the following three facts holds:
\[
\begin{align*}
[\mu \ll \nu] & \iff M_2(\mu) \subseteq M_2(\nu), \\
[\mu \approx \nu] & \iff M_2(\mu) = M_2(\nu), \text{ and} \\
[\mu \text{ and } \nu \text{ are mutually singular}] & \iff M_2(\mu) \perp M_2(\nu). 
\end{align*} \tag{4.8}
\]

As a result, we note that therefore, the decomposition in (4.1) is orthogonal in \( \text{Sig}(X) \), and further that a fixed \( \lambda \in \text{Prob}(X) \) is in \( L^1(\mathbb{R}) \) if and only if 
\[
M_2(\lambda R) \subseteq M_2(\lambda) \tag{4.9}
\]

\[
\inf_{\lambda \in \text{Prob}(X)} \left\| \sqrt{\lambda R} - W\sqrt{\lambda} \right\|^2_{\text{Sig}(X)} = 0 \tag{4.10}
\]

Moreover, (4.9) is a restatement of (4.3).

We now turn to the conclusions in the theorem: One implication is clear. If now the infimum in (4.10) is zero, then there is a sequence \( \{\lambda_n\} \subset \text{Prob}(X) \) such that 
\[
\lim_{n} \left\| \sqrt{\lambda_n R} - W_{\lambda_n} \sqrt{\lambda_n} \right\|^2_{\text{Sig}(X)} = 0. \tag{4.11}
\]
Combining (4.10) and (4.11), and possibly passing to a subsequence, we conclude that there is a sequence \( W_{\lambda_n} \sqrt{\lambda_n} \) which is convergent in \( \text{Sig}(X) \). Let the limit be \( W_{\lambda_0} \sqrt{\lambda_0} \), and it follows that \( \lambda_0 \in \mathcal{L}_1(\mathbb{R}) \).

\[ \square \]

**Remark 4.2.** Since \( \text{Sig}(X) \) is a Hilbert space, we conclude that the sequence \( \{W_{\lambda_n} \sqrt{\lambda_n}\}_n \) in \( \text{Sig}(X) \) satisfies 
\[
\lim_n \left\| \sqrt{\lambda_0 R} - W_{\lambda_n} \sqrt{\lambda_n} \right\|^2_{\text{Sig}(X)} = 0;
\]
where the desired measure \( \lambda_0 \in \mathcal{L}_1(\mathbb{R}) \) may be taken to be
\[
d\lambda_0(\cdot) = \sum_{n=1}^{\infty} \frac{1}{2^n} d\lambda_n(\cdot). \]
5. From Endomorphism to Automorphism

In this section (Theorem 5.2), we build a path-space probability space from a given generalized transfer operator \( R \) assumed to satisfy the spectral property from above.

There is a generalized family of multi-resolution measures on solenoids, and we shall need the following facts (see e.g., [Hut81, Jor12, FH09, Urb09, DABJ09, DJ09]):

Let \( X \) be a compact Hausdorff space, and let \( \sigma : X \to X \) be a continuous endomorphism onto \( X \). Let

\[
\Omega := \prod_{0}^{\infty} X = X \times X \times \cdots
\]  

be the infinite Cartesian product with coordinate mappings \( Z_n : \Omega \to X \),

\[
Z_n (x_0, x_1, x_2, \cdots) = x_n \in X, \quad n \in 0, 1, 2, \cdots.
\]  

The associated solenoid \( \text{Sol}_{\sigma} (X) \) is defined as follows:

\[
\text{Sol}_{\sigma} (X) = \{ (x_n)_{n=0}^{\infty} \in \Omega \mid \sigma (x_{n+1}) = x_n, \quad n = 0, 1, 2, \cdots \};
\]  

and set

\[
\tilde{\sigma} (x_0, x_1, x_2, \cdots) := (\sigma (x_0), x_0, x_1, x_2, \cdots).
\]  

We give \( \text{Sol}_{\sigma} (X) \) its relative projective topology, and note that the restricted random variable \((Z_n)_{n=0}^{\infty}\) from (5.2) are then continuous. Moreover \( \tilde{\sigma}, \) in (5.4), is invertible with

\[
\tilde{\sigma}^{-1} (x_0, x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, \cdots),
\]  

\[
\tilde{\sigma}\tilde{\sigma}^{-1} = \tilde{\sigma}^{-1} \tilde{\sigma} = \text{Id}_{\text{Sol}_{\sigma} (X)}.
\]  

Let \((X, R, \lambda, W)\) be as specified in Section 2. In particular, \( R \) is positive, i.e., \( f \in C (X), \ f \geq 0 \implies R (f) \geq 0 \), and

\[
R ((f \circ \sigma) g) = fR (g), \quad \forall f, g \in C (X).
\]  

Moreover, \( W \) is the Radon-Nikodym derivative of the measure \( f \mapsto \lambda (R (f)) \) w.r.t. \( \lambda \), i.e.,

\[
\int_{X} R (f) d\lambda = \int_{X} fW d\lambda, \quad \forall f \in C (X).
\]  

Let \( h \in L^{\infty} (\lambda), \ h \geq 0, \) satisfying

\[
Rh = h, \quad \text{and} \quad \int_{X} h d\lambda = 1.
\]  

Remark 5.1. In view of equations (2.8) and (5.9), it is natural to think of these conditions as a generalized Perron-Frobenius property for \( R \).

**Theorem 5.2.** With the assumptions (5.7)-(5.9), we have the following conclusions:

1. For every \( x \in X \), there is a unique Borel probability measure \( \mathbb{P}_x \) on \( \text{Sol}_{\sigma} (X) \) such that for all \( n \) and all \( f_0, f_1, \cdots, f_n \in C (X), \)

\[
\int_{Z_n^{-1}(x)} (f_0 \circ Z_0) \cdots (f_n \circ Z_n) d\mathbb{P}_x
\]  

\[
= f_0 (x) R (f_1 R (f_2 R (\cdots R (f_n h) \cdots))) (x). \tag{5.10}
\]
Moreover, setting
\[ P = \int_X P_x \, d\lambda (x), \]  
we get that \( P \) is a probability measure on \( \text{Sol}_\sigma (X) \) such that
\[ E_P (\cdots | Z_0 = x) = P_x \]  
where the LHS in (5.12) is the conditional measure, and the RHS is the measure from (5.10).

We have the following Radon-Nikodym derivative:
\[ \frac{dP \circ \tilde{\sigma}}{dP} = W \circ Z_0, \]  
as an identity of the two functions specified in (5.13). Equivalently, setting
\[ U \psi = \left( \sqrt{W \circ Z_0} \right) \psi \circ \tilde{\sigma}, \quad \psi \in L^2 (\text{Sol}_\sigma (X), P), \]  
then \( U \) is a unitary operator in \( L^2 (\text{Sol}_\sigma (X), P) \).

Proof. This is the basic Kolmogorov inductive limit construction. We note that, by Stone-Weierstrass, the space of cylinder-functions
\[ (f_0 \circ Z_0 ) (f_1 \circ Z_1 ) \cdots (f_n \circ Z_n) \]  
is dense in \( C (\text{Sol}_\sigma (X)) \). Fix \( x \in X \), and start with \( Z_0^{-1} (x) \), set
\[ L^x_n (f_1, f_2, \cdots, f_n) = R (f_1 R (f_2 \cdots R (f_n h) \cdots )) (x). \]  
We get the desired consistency:
\[ L^x_{n+1} (f_1, f_2, \cdots, f_n, \mathbb{1}) = L^x_n (f_1, f_2, \cdots, f_n) \]  
where \( \mathbb{1} \) denotes the constant function 1 on \( X \). Indeed,
\[ R (f_{n-1} R (f_n R (\mathbb{1} h))) (x) = R (f_{n-1} R (f_n R (h))) (x) = R (f_{n-1} R (f_n h)) (x), \]  
(by 5.9)
as claimed in (5.16).

Lemma 5.3. Let \( R, X, P (\cdot | x) \) be as above. Assume \( h \geq 0 \) on \( X \), and \( Rh = h \). Then
\[ |R (f h) (x)| \leq \|f\|_\infty h (x) \]  
where \( \|f\|_\infty \) is the \( P (\cdot | x) \) \( L^\infty \)-norm on functions on \( X \).

Proof. We may apply Cauchy-Schwarz to \( P (\cdot | x) \) in a sequence of steps as follows:
\[ |R (f h) (x)| = \left| R (fh^{\frac{1}{2}} h^{\frac{1}{2}}) (x) \right| \]  
\[ \leq \left( R (|f|^2 h) (x) \right)^{\frac{1}{2}} \left( R (h) (x) \right)^{\frac{1}{2}} \]  
by Schwarz
\[ = \left( R (|f|^2 h) (x) \right)^{\frac{1}{2}} h (x)^{\frac{1}{2}} \]  
since \( Rh = h \)
\[ \leq \left[ R (|f|^p h) (x) \right]^{\frac{1}{p}} h (x)^{\frac{1}{2} + \frac{1}{2p} + \cdots + \frac{1}{2p}} \]  
by induction, and let \( p \to \infty \)
An elementary result in measure theory (see [Rud87]) shows that
\[ \lim_{p \to \infty} R(|f|^p) (x) \frac{1}{p} = \|f\|_\infty; \]  
(5.19)
and so the desired estimate (5.17) holds.

**Corollary 5.4.** Let \( R, h, \sigma, W, \lambda, h \in \mathcal{L}(R) \) be as above, where \( \mu = \lambda \cdot R \), and \( W = \frac{d\mu}{dx} \). Assume \( h > 0 \) on \( X \), and \( Rh = h \). Let \( \mathbb{P} \) and \( \mathbb{P}_x \) be the measures on \( \text{Sol}_\sigma(X) \) as in Theorem 5.2, where \( \mathbb{P}_x \) is determined by
\[ \int_{Z_0^{-1}(x)} (f_1 \circ Z_1) \cdots (f_n \circ Z_n) \, d\mathbb{P}_x = L^\lambda_n(f_1, \ldots, f_n) \]
and so the desired estimate (5.25) holds. □

**Corollary 5.5.** Let \( X, R, \sigma, \lambda, h \) be as described above; in particular, \( Rh = h \) is assumed. Let \( \{\mathbb{P}_x\}_{x \in X} \) be the corresponding measures from Corollary 5.4. Then
\[ h(x) = \mathbb{P}_x(Z_0^{-1}(x)) \text{ for all } x \in X. \]

**Corollary 5.6.** Let \( R, h, \sigma, W, \lambda \) be as above, and let \( \mathbb{P} \) and \( \mathbb{P}_x \) be the measures on \( \text{Sol}_\sigma(X) \), then \( \mathbb{V}_0 \) is isometric, where
\[ \mathbb{V}_0 : L^2(X, h \, d\lambda) \to L^2(\text{Sol}_\sigma(X), \mathbb{P}) \]
is given by
\[ \mathbb{V}_0 g = g \circ Z_0, \quad g \in L^2(X, h d\lambda), \]  
(5.23)
and
\[ (V^* \psi)(x) = \frac{\mathbb{E}(\psi \mid x)}{h(x)}, \quad \forall x \in X, \forall \psi \in L^2(\text{Sol}_\sigma(X), \mathbb{P}). \]  
(5.24)

**Proof.** Since \( \mathbb{P} = \int_X \mathbb{P}_x d\lambda(x) \), it follows that \( \mathbb{V}_0 \) in (5.23) is isometric, i.e.,
\[ \|\mathbb{V}_0 g\|_{L^2(\mathbb{P})}^2 = \int_X |g|^2 h \, d\lambda = \|g\|_{L^2(h \, d\lambda)}^2, \quad \forall g \in C(X). \]

To prove (5.24), we must establish
\[ \int_{\text{Sol}_\sigma(X)} (g \circ Z_0) \psi \, d\mathbb{P} = \int_X g(x) \mathbb{E}(\psi \mid x) \, d\lambda(x). \]  
(5.25)
Since the space of the cylinder functions \( \psi = (f_0 \circ Z_0)(f_1 \circ Z_1) \cdots (f_n \circ Z_n) \) is dense in \( C(\text{Sol}_\sigma(X)) \), it suffices to prove (5.25) for \( \psi \). But then
\[ (g \circ Z_0) \psi = (g f_0) \circ Z_0 (f_1 \circ Z_1) \cdots (f_n \circ Z_n), \]
and so (5.25) follows from (5.20). □
Corollary 5.7. Fix $x \in X$, and set
\[
\mathbb{E}(\psi \mid x) = \int_{Z_0^{-1}(x)} \psi \, d\mathbb{P}_x, \quad \psi \in L^2(\mathbb{P}).
\] (5.26)
For all $n \in \mathbb{N}$, if $A_i \subset X$, $i = 1, \cdots, n$, are Borel sets, then
\[
\mathbb{P}_x(Z_1 \in A_1, Z_2 \in A_2, \cdots, Z_n \in A_n) = \int_{A_1} \int_{A_2} \cdots \int_{A_n} h(y_n) P(dy_n \mid y_{n-1}) \cdots P(dy_2 \mid y_1) P(dy_1 \mid x).
\] (5.27)

Proof. Recall that $\chi_A \circ Z_i = \chi_{Z_i^{-1}(A)}$, if $A \subset X$ is a Borel set; and
\[
Z_i^{-1}(A) = \{x \in \text{Sol}_\sigma(X) \mid Z_i(x) \in A\},
\] (5.28)
where $Z_i(x_0, x_1, x_2, \cdots) = x_i$ is the coordinate mapping. Also, $P(\cdot \mid x)$ satisfies
\[
R(f)(x) = \int_X f(y) P(dy \mid x), \quad \forall x \in X.
\] (5.29)

Now set $f_i = \chi_{A_i}$, with $A_i \subset X$ Borel sets, and apply the mapping
\[
f_i \mapsto R(f_1 R(f_2 \cdots R(f_n h) \cdots))(x).
\]
If we specialize (5.27) to individual transition probabilities, we get, $x \in X$, $A \subset X$ a Borel set, and
\[
\mathbb{P}(Z_1 \in A \mid Z_0 = x) = \int_A h(y) P(dy \mid x),
\]
\[
\mathbb{P}(Z_2 \in B, Z_1 \in A \mid Z_0 = x) = \int_A \int_B h(y_2) P(dy_2 \mid y_1) P(dy_1 \mid x), \quad y_1 \in A, y_2 \in B.
\]
Note that, fix $n > 1$, then
\[
\mathbb{P}(Z_n \in A \mid Z_0 = x) = \mathbb{P}_x(Z_n \in A) = R^n(\chi_A h)(x), \quad \text{and}
\]
\[
\mathbb{P}_x(Z_{n+1} \in B, Z_n \in A) = R^n(\chi_A R(\chi_B h))(x) \neq \mathbb{P}_x(Z_2 \in B, Z_1 \in A),
\]
so it is not Markov. \hfill \Box

Hence the transition from $n$ to $n+1$ gets more “flat” as $n$ increases, the transition probability even out with time.

5.1. Multi-Resolutions. Let $X, \sigma, R, h, \text{and } \lambda$ be as in the setting of Theorem 5.2 above. In particular, we are assuming that:

(i) $R((f \circ \sigma) g) = f R(g), \forall f, g \in C(X),$

(ii) $Rh = h$, $h \geq 0$,

(iii) $\int_X R(f) d\lambda = \int_X f W d\lambda, \forall f \in C(X)$, and

(iv) $\int_X h(x) d\lambda(x) = 1$.

We pass to the probability space $(\text{Sol}_\sigma(X), \mathbb{P}_x, \mathbb{P})$ from the conclusion in Theorem 5.2.

Definition 5.8. Let $\mathcal{H}$ be a Hilbert space, and $\{\mathcal{H}_n\}_{n \in \mathbb{N}_0}$ a given system of closed subspaces such that $\mathcal{H}_n \subset \mathcal{H}_{n+1}$, for all $n$.

We further assume that $\cup_n \mathcal{H}_n$ is dense in $\mathcal{H}$, and that a unitary operator $U$ in $\mathcal{H}$ satisfying $U(\mathcal{H}_n) \subset \mathcal{H}_{n-1}$, for all $n \in \mathbb{N}$. Then we say that $((\mathcal{H}_n)_{n \in \mathbb{N}_0}, U)$ is a multi-resolution for the Hilbert space $\mathcal{H}$.
Theorem 5.9. Let $\mathcal{H} = L^2(Sol_\sigma(X), \mathbb{P})$ be the Hilbert space from the construction in Theorem 5.2, and let $\mathcal{H}_n$ be the closed subspaces defined from the random walk process $(Z_n)_{n \in \mathbb{N}}$. Finally, let $U$ be the operator in part (3) of Theorem 5.2. Then this constitutes a multi-resolution.

Proof. As indicated above, the setting is specified in Theorem 5.2, and we set

$$\mathcal{H} := L^2(Sol_\sigma(X), \mathbb{P});$$

and, for each $n \in \mathbb{N}$, let $\mathcal{H}_n \subset \mathcal{H}$, be the closed subspace spanned by

$$\{ f \circ Z_n \mid f \in C(X) \}. \quad (5.30)$$

Since

$$f \circ Z_n = (f \circ \sigma) \circ Z_{n+1} \quad (5.31)$$

it follows that $\mathcal{H}_n \subseteq \mathcal{H}_{n+1}$. It further follows from Theorem 5.2 that $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$ is dense in $\mathcal{H}$. And, finally, the unitary operator $U$ from part (3) of Theorem 5.2 satisfies

$$U(\mathcal{H}_n) \subset \mathcal{H}_{n-1}, \quad \forall n \in \mathbb{N}. \quad (5.32)$$

$\square$

Corollary 5.10. Let $X, \sigma, R, h, \lambda, \mathbb{P}$ be as stated above; and let $((\mathcal{H}_n), U)$ be the corresponding multi-resolution from Theorem 5.9.

Then $\mathcal{H}_0 \simeq L^2(X, h \, d\lambda)$, and $\cap_{n \geq 0} U^n \mathcal{H}_m = \mathcal{H}_0$ holds for all $m \in \mathbb{N}$. Finally, $U$ restricts to a unitary operator in $\mathcal{H} \ominus \mathcal{H}_0$; and the spectrum of this restriction is pure Lebesgue spectrum, i.e., there is a Hilbert space $\mathcal{K}$ (the multiplicity space) such that $U \mid_{\mathcal{H} \ominus \mathcal{H}_0}$ is unitarily equivalent to a subshift of the bilateral shift $S$ in $L^2(T, \text{Leb}; \mathcal{K})$, where $T$ is the circle group $\{ z \in \mathbb{C} \mid |z| = 1 \}$, and the bilateral shift is then given on functions $\psi \in L^2(T, \text{Leb}; \mathcal{K})$ by

$$(S\psi)(z) = z\psi(z), \quad \psi : T \rightarrow \mathcal{K}, \quad z \in T, \quad \text{multiplication by } z.$$

Proof. The conclusion follows from an application of the Stone-von Neumann uniqueness theorem [Sum01] combined with the present theorems in Sections 4 and 5 above. (For more details on the spectral representation for operators with multi-resolution, see also [LP67].) $\square$

6. Harmonic Functions from Functional Measures

Let $R : C(X) \rightarrow \mathcal{M}(X)$ be as specified in (2.1); and let the measure system $\{ P(\cdot \mid x) \}_{x \in X}$ be as specified in (2.3).

Let $(\Omega, \mathcal{F})$ be a measure space; i.e., $\mathcal{F}$ is a specified sigma-algebra of events in a given sample space $\Omega$, and let $Z_0$ be an $X$-valued random variable, i.e., it is assumed that $Z_0^{-1}(A) \in \mathcal{F}$ for every Borel set $A \subset X$. For recent applications, we refer to [AJ12, JP12, JKS12, JPT15, CJ15].

Theorem 6.1. Let $R, X, \Omega, \mathcal{F}$, and $Z_0$ be as specified above. Suppose $\{ P_x \}_{x \in X}$ is a system of positive measures on $\Omega$ indexed by $X$, and set

$$h(x) = P_x(Z_0^{-1}(x)), \quad x \in X. \quad (6.1)$$

Assume that

$$\int_X P_y(\cdot) P(dy \mid x) = P_x(\cdot), \quad (6.2)$$

then $h$ in (6.1) is harmonic for $R$, i.e., we have

$$R(h) = h, \quad \text{pointwise on } X. \quad (6.3)$$
Proof. Using (2.5) in Definition 2.3, we get the following:

\[
(Rh)(x) = \int_X h(y) P(dy | x)
\]
\[
= \int_X \mathbb{P}_y (Z_0^{-1}(y)) P(dy | x)
\]
\[
= \mathbb{P}_x (Z_0^{-1}(x)) = h(x), \quad x \in X.
\]

Corollary 6.2. Let \( R \) be a transfer operator. Then if \( \lambda \in \mathcal{L}_1(R) \), then there is a solution \( h \geq 0 \), on \( X \), to \( Rh = h \), and \( \int_X h(x) d\lambda(x) = 1 \).

Proof. This is a conclusion of Corollary 5.5 and Theorem 6.1. Indeed, given \( \lambda \in \mathcal{L}_1(X) \), let \( \{\mathbb{P}_x\}_{x \in X} \) be the system from Theorem 5.2, then \( h(x) = \mathbb{P}_x (Z_0^{-1}(x)) \) is the desired solution.

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