On transmission line resonances in high $T_C$ dc SQUIDs

U Sinha$^1$, A Sinha$^2$ and E J Tarte$^3$

$^1$ Institute for Quantum Computing, University of Waterloo, 200 University Avenue West, Waterloo, ON, N2L 3G1, Canada
$^2$ Perimeter Institute for Theoretical Physics, 31 Caroline Street North, Waterloo, ON, N2L 2Y5, Canada
$^3$ Department of Electrical and Electronic Engineering, University of Birmingham, Birmingham, UK

E-mail: usinha@iqc.ca, asinha@perimeterinstitute.ca and e.tarte@bham.ac.uk

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Abstract
In this paper, we study transmission line resonances in high $T_C$ dc SQUIDs (superconducting quantum interference devices). These resonances are exhibited in the characteristics of SQUIDs which are fabricated on substrates with high dielectric constant, such as strontium titanate. The power balance equation is analytically derived both for symmetric and for asymmetric SQUIDs. Using this, we investigate SQUID current–voltage $I(V)$, voltage–flux $V(\Phi)$ and voltage modulation $\Delta V$ characteristics.

1. Introduction
Superconducting quantum interference devices (SQUIDs) are the most sensitive magnetic flux sensors known today. Studying and improving SQUID properties has been a major research field in the last few decades [1, 2]. Most high $T_C$ SQUIDs are fabricated on strontium titanate (STO) substrates. It has been seen that the large dielectric constant of STO strongly affects the performance of high $T_C$ dc SQUIDs by causing transmission line resonances to appear in their current–voltage characteristics [3–5]. This was pointed out for instance, by Lee et al who experimentally showed that their hairpin shaped SQUID loop could be treated as a quarter wave microwave resonator in order to predict the voltage at which the resonance appeared in their measurements [6]. Enpuku et al used a theoretical approach to predict the effect of the resonances based upon a transformation of the distributed resonator structure of the SQUID loop into a series of lumped element resonators [3]. This enabled them to calculate the current–voltage characteristics of the SQUID and later, the voltage–flux curves for other devices. However, it was necessary for them to solve the resulting differential equations numerically. In this paper, we develop an analytic approach to investigate the effects of transmission line resonances on dc SQUID resonance characteristics. We are able to obtain good agreement between experiment and theory for devices in the literature [6]. We find that the resonance positions are mainly controlled by the product of the junction critical current $I_0$ and junction normal state resistance $R_S(I_0 \times R_S)$ as well as the dielectric constant of the substrate $\epsilon_R$. We also investigate the effects of introducing asymmetry in junction parameters. We find that the resonance positions are not very sensitive to the asymmetry. The actual magnitude of the current–voltage curve and $\Delta V$ curves are however sensitive to the degree of asymmetry introduced. We ignore the effects of thermal noise in performing our calculations. Noise analysis can be performed perturbatively as in [7]. In such calculations, the lowest order result is the noise free case where our analysis will be useful.

We begin with a brief outline of relevant previous work in the study of the effect of transmission line resonances on SQUID characteristics in section 2. In section 3, we derive the power balance relation in terms of SQUID parameters, which involves deriving an expression for the circulating current through the SQUID inductance. Section 4 discusses current–voltage $I(V)$ curves for symmetric SQUIDs whereas in section 5, effects of introducing asymmetry on $I(V)$, voltage–flux $V(\Phi)$ curves and SQUID modulation–current $\Delta V(I)$ curves are discussed. The paper ends with a discussion of results obtained in section 6.

2. SQUID circuit equations
The geometry of the SQUID washer that we have used is shown in figure 1(a). This is a geometry commonly used to...
Figure 1. (a) Geometry of the SQUID washer. Here \( l \) is the length of the slit, \( w \) is the width of the electrode and \( s \) is the width of the slit. (b) Equivalent circuit of the SQUID when its parasitic capacitance distributing along the slit of the washer is taken into account. The circuit consists of the SQUID coupled to a series of LCR circuits. \( R_d \) is a damping resistance in parallel to the SQUID.

manufacture SQUIDs [6, 8, 9] and is the geometry Enpuku et al used to numerically investigate the effects of large dielectric constant of strontium titanate (STO) on the characteristics of high \( T_c \) dc SQUIDs [3]. The slit of the SQUID washer makes up the SQUID inductance where \( l, s, w \) and \( d \) denote the slit length, slit width, electrode width and thickness of electrode respectively. For this geometry, the inductance per unit length of the slit \( \bar{L} \) and parasitic capacitance per unit length \( \bar{C} \) are given by [3, 10]:

\[
\bar{L} = \bar{L}_M + \bar{L}_K
\]

(1)

where \( \bar{L}_M \) is the magnetic inductance per unit length and \( \bar{L}_K \) is the kinetic inductance per unit length of the SQUID slit given by [11]:

\[
\bar{L}_M = \frac{\mu_0 K(k)}{K(k')}
\]

\[
\bar{L}_K = \frac{2\mu_0 \lambda^2}{dwk^2K^2(k')} \left[ \frac{w}{s} \ln \left( \frac{4ws}{d(w+s)} \right) \right]
\]

(2)

\[
+ \frac{w}{2w+s} \ln \left( \frac{4w(2w+s)}{d(w+s)} \right)
\]

(3)

and

\[
\bar{C} = \frac{\varepsilon R + 1}{2\varepsilon^2 \bar{L}_M}
\]

(4)

where \( \mu_0 \) is the permeability of free space, \( K(k) \) is the complete elliptic integral of the first kind [12] with a modulus \( k = \frac{\sqrt{1-k^2}}{1+k^2} \), \( k' = (1-k^2)^{1/2} \), \( \lambda \) is the penetration depth of the film, \( \varepsilon R \) is the dielectric constant of the STO substrate and \( c \) is the velocity of light in vacuum. The SQUID slit behaves as a transmission line with distributed inductance \( \bar{L} \) and distributed capacitance \( \bar{C} \). The impedance \( Z_{AB} \) of the slit seen from terminals A and B is given by [3]:

\[
Z_{AB} = i\bar{Z}_0 \tan(\Omega\sqrt{\bar{L}\bar{C}}) + i\Omega L_{PR}
\]

(5)

where \( \bar{Z}_0 = \sqrt{\bar{L}/\bar{C}} \) is the characteristic impedance of the slit transmission line, \( \Omega \) is the angular frequency of measurement and \( L_{PR} \) is the junction parasitic inductance. The first term arises because the hairpin shaped slit can be treated as a shorted transmission line of length \( l \).

Now, using the formula [3]

\[
\tan \left( \frac{\pi x}{2} \right) = -\frac{4x}{\pi} \sum_{n=1}^{\infty} \frac{1}{\pi^2 - (2n - 1)^2},
\]

equation (5) can be expanded as (in the lossless case with \( L_{PR} = 0 \)):

\[
Z_{AB} = \sum_{n=1}^{\infty} \frac{1}{\Omega C_n + 1/i\Omega L_n},
\]

(7)

with \( C_n = \bar{C}l/2 \) and \( L_n = 8\bar{L}l/\pi^2(2n - 1)^2 \). This transformation to an equivalent circuit has the advantage that it allows us to consider loss in the transmission line. In the lossy case, the rf-loss \( R_n \) is added to equation (7) leading to:

\[
Z_{AB} = \sum_{n=1}^{\infty} \frac{1}{\Omega C_n + 1/i\Omega L_n + 1/R_n},
\]

(8)

where \( R_n = Q\sqrt{L_n/C_n} \) and \( Q \) is a quality factor. Here \( l \) is the SQUID slit length. This expression allows us to express the impedance \( Z_{AB} \) by the series of \( L-C-R \) resonant circuits as shown in figure 1(b).

We will now show derivations of various quantities in the lossless limit and then quote the results obtained for the lossy case which can be derived analogously. The circuit equations can be easily derived by the application of Kirchhoff’s laws. In figure 1(b), current entering point C should equal current leaving point C. We are assuming the most general case in which the SQUID is made up of junctions which are asymmetric [13, 14]. Let the average junction critical current be \( I_0 \), the average junction normal state resistance be \( R_S \) and the average junction capacitance be \( C_J \). Let the asymmetry parameter in \( I_0 \) be \( \kappa \), that in \( R_S \) be \( \rho \) and that in \( C_J \) be \( \chi \). So specifically, let us split [13] \( 2I_0 = (1 + \kappa)I_0 + (1 - \kappa)I_0 \), \( 2/R_S = (1 + \rho)/R_S + (1 - \rho)/R_S \) and \( 2C_J = (1 + \chi)C_J + (1 - \chi)C_J \). Then, this gives for the first junction:

\[
(1 + \kappa)I_0 \sin(\theta_1) + \frac{V_1(1 + \rho)}{R_S} + (1 + \chi)C_J \frac{dV_1}{dt} = \frac{I_0}{2} - J = \frac{V_2 - V_1}{R_D},
\]

(9)

and writing an analogous equation for point D gives for junction 2:

\[
(1 - \kappa)I_0 \sin(\theta_2) + \frac{V_2(1 - \rho)}{R_S} + (1 - \chi)C_J \frac{dV_2}{dt} = \frac{I_0}{2} + J = \frac{V_1 - V_2}{R_D}.
\]

(10)

Here \( J \) is the circulating current through the SQUID inductance, \( I_0 \) is the SQUID bias current, \( V_1 \) and \( V_2 \) are voltages across junctions 1 and 2, \( \theta_1 \) and \( \theta_2 \) are the phases of junctions 2 and 1 and \( R_D \) is a damping resistance in parallel to the SQUID inductance. The circulating current in Fourier

\[\text{it is possible to consider different quality factors for each } n \text{ but we will not do so here.}\]
space is given by:

\[ j = \frac{\bar{V}_2 - \bar{V}_1 + \Phi}{Z_{AB}}, \]

where \( \Phi \) is the Dirac delta function and \( \Phi \) is the total flux applied to the SQUID loop.

We will follow [3] and use normalized circuit equations. We normalize currents by \( I_0 \), voltage by \( V_0 R_S \) and time by \( \Phi_0 / 2\pi I_0 R_S \). The ac Josephson relation gives \( v = d\theta / dt = V / I_0 R_S \), where \( v \) is the normalized voltage and \( \tau \) is the normalized time. Here we have neglected random noise currents. The normalized circuit equations are:

\[ (1 + \chi)\beta_C \dot{\theta}_1 = \frac{1}{2}(i_n + j) - (1 + \kappa)\dot{\theta}_1, \]

\[ - (1 + \kappa)\sin(\theta_1) - \gamma(\dot{\theta}_1 - \dot{\theta}_2), \]

\[ (1 - \chi)\beta_C \dot{\theta}_2 = \frac{1}{2}(i_n - j) - (1 - \rho)\dot{\theta}_2, \]

\[ - (1 - \kappa)\sin(\theta_2) + \gamma(\dot{\theta}_1 - \dot{\theta}_2), \]

where \( \beta_C = 2\pi I_0 C_1 R_S^2 / \Phi_0 \) is the SQUID McCumber parameter with \( \Phi_0 \) being the flux quantum and \( \gamma = R_S / R_0 \). In principle, these equations can be solved numerically using the dielectric constant \( \varepsilon_r \) as a parameter having a finite number of resonant circuits. Enpuku et al present numerical solutions of equations (12) and (13) in their paper. As shown in appendix A, they however consider a finite number 'n' of resonant circuits and the expression they use for the circulating current \( j \) becomes indeterminate in the continuum limit. The analytic handle will enable closed form expressions for \( I_0 (V) \), which can subsequently be used to simulate graphs quickly and conveniently.

Normalizing equation (11) and the expressions for impedance in the lossless limit, equation (7) and lossy limit, equation (8), the expression for the normalized circulating current \( j \) in Fourier space is given by:

\[ j = \left( \frac{1}{2} i_n + j \right) + \frac{8\pi \delta(\omega)\phi}{\beta} A(\omega), \]

where \( \omega = \Omega \Phi_0 / 2\pi I_0 R_S \) and dimensionless. Here \( \phi \) is the externally applied flux normalized to \( \Phi_0 \) and \( \beta = 2\pi I_0 \Phi_0 \) is the SQUID inductance parameter. In the lossless case with \( L_{PR} = 0 \)

\[ A(\omega) = \frac{2}{\pi} \frac{\Phi_0}{I_0} \frac{(2\pi \omega L_{PR} \sqrt{LC} / \Phi_0)}{\tan(2\pi \omega L_{PR} \sqrt{LC} / \Phi_0)}, \]

and in the lossy case, performing the sum in equation (8):

\[ A(\omega) = 2i \frac{\nu}{\mu} [\Psi(\xi - i\xi) + \Psi(\xi + i\xi)], \]

where \( \xi = 1/2 + |\mu| L C / 2\pi C, \xi = |\mu| L C / 2\pi C, \mu = L \sqrt{C} (1 - 4Q^2) / 2\pi Q, \nu = -Q \Phi_0 / 2|I_0| L C, |\mu| = L \omega \sqrt{C} (1 - 4Q^2) / 2\pi Q \) and \( \Psi(x) = \theta_1 \log(G(x)) \) is the di-gamma function with \( \Gamma(x) \) being the standard gamma function [12]. If \( L_{PR} \neq 0 \) then \( A(\omega) \) should be replaced by \( A(\omega) / (1 + \pi I_0 A(\omega)L_{PR} / \Phi_0) \) in the calculations.

The expression for circulating current can be written explicitly in the frequency domain in terms of \( V_1 \) and \( V_2 \) which are obtained by solving equations (9) and (10). The equivalent expression for \( j \) in the time domain (equation (B.22)) can only be explicitly obtained after solving the equations and taking into account the properties of the SQUID loop. This is presented in appendix B.

3. Power balance equation

The calculation details for the SQUID power balance equation have been included in appendix B. After some straightforward but tedious algebra, the power balance relation so derived is as follows:

\[ i_0 = 2v_{dc} + \frac{I^2 v_{dc}}{2|d|} \left( (1 + \kappa^2)[dd + r\bar{r}] + 2\kappa [d\bar{r} - d\bar{r}] \right) \]

\[ + (1 - \kappa^2)(-dd - d\bar{r} + d\bar{r}) \cdot \left( 2\pi \phi + \sin(2\pi \phi) \right) \]

\[ + \frac{I^2 v_{dc}}{2\beta^2|ds - d\bar{r}|^2} \left( 1 + 2\gamma - 1 - \frac{A - \bar{A}}{2v_{dc}} \right) \]

\[ \times (1 + \kappa^2)[\beta^2 s\bar{s} + r\bar{r}] - 2\kappa \beta [s\bar{r} + s\bar{r}] \]

\[ - (1 - \kappa^2)[\beta^2 s\bar{s} - r\bar{r}] \cdot \cos(2\pi \phi) \]

\[ - i\beta [s\bar{r} - s\bar{r}] \sin(2\pi \phi)). \]

Here \( d = d(v_{dc}), s = s(v_{dc}), r = r(v_{dc}), A = A(v_{dc}) \) and \( \bar{x} \) denotes the complex conjugate of \( x \). The phase difference between the two junctions have been set to \( 2\pi \phi \).

\[ I = \sqrt{2} v_{dc}(1 + v_{dc}^2)^{1/2} - v_{dc} \]

\[ s(\omega) = (\omega)^2 \beta_C + i\omega), \]

\[ d(\omega) = A(\omega) - (\omega)^2 \beta_C + i\omega + 2i\omega \gamma), \]

\[ r(\omega) = i\omega \psi + \chi \beta_C \omega^2. \]

Note that the power balance relation has the correct symmetries. For instance interchanging the two junctions leads to \( \kappa \rightarrow -\kappa, \rho \rightarrow -\rho, \chi \rightarrow -\chi \) and \( \psi_1 \leftrightarrow \psi_2 \). The expression is symmetric under this transformation. Another interesting observation is the fact that the \( \rho \) and \( \chi \) dependence are both packaged in the function \( r(\omega) \).

4. Symmetric SQUIDs

For symmetric SQUIDs, i.e. \( \kappa = \rho = \chi = 0 \) and in the lossless case where \( A = \bar{A} \), we have

\[ i_0 = 2v_{dc} + \frac{I^2 v_{dc}}{2|d|} \left( 1 + \cos(2\pi \phi) \right) \]

\[ + \frac{I^2 v_{dc}}{2|d|} (1 + 2\gamma - 1 - \cos(2\pi \phi)). \]

where \( s, d \) are given by equation (19) and \( A \) is given by equation (15). In [18], Enpuku et al derive the SQUID power balance equation in the absence of transmission line resonances. By allowing the slit length to tend to zero, hence treating the SQUID inductance as a lumped inductance instead of a distributed parameter, we retrieve Enpuku’s expression for the power balance relation:

\[ i_0 v_{dc} = 2v_{dc} I^2 (1 + \cos(2\pi \phi)) \]

\[ + \frac{I^2 (1 - \cos(2\pi \phi))(1 + 2\gamma)\beta^2 v_{dc}^3}{2(1 - \beta^2 \beta_C v_{dc}^2)^2 + \beta^2 \beta_C(1 + 2\gamma)^2). \]

Here we have neglected the capacitance in parallel to the shunt resistance, which they had included in their circuit as our analysis is based on a circuit, which does not involve such
respectively. Enpuku et al’s formula does not lead to any broad resonances. We have chosen ε = 1. i.e. for Φ_{ext} = nΦ_{0}, the resonances have no effect on the current–voltage characteristics, whereas for \cos 2nφ = \pm 1, Φ_{ext} = \pm (2n + 1)Φ_{0}, the resonances have maximum effect.

The maxima occur at:

\[ v_{dc} = \frac{mΦ_{0}}{2 l I_{0} R_{S}} \sqrt{LC} \quad (23) \]

The minima occur at:

\[ v_{dc} = \frac{(2n + 1)Φ_{0}}{4 l I_{0} R_{S}} \sqrt{LC} \quad (24) \]

If we approximate \sqrt{LC} by \sqrt{\frac{c}{2ε}}, where c is the speed of light in free space, then equation (24) is the one used by Lee et al to predict the voltage at which the resonance occurred for their device. This approximation is only true if the kinetic inductance of the SQUID loop is negligible.

5. Introducing asymmetry

5.1. Current–voltage characteristics

In this section we wish to consider the effects of introducing asymmetry to the SQUID. We have introduced current and resistance asymmetry systematically and seen the effect on the I(V) curve both for the \( \phi = 0 \) and \( \phi = \pi \) cases. Figure 3(a) shows the comparison between Lee’s experimental I(V) and our theoretical simulations for the \( \phi = 0 \) case and figure 3(b) shows the same for the \( \phi = \pi \) case. The comparison between experiment and theory in the \( \phi = 0 \) case is not very good at large voltages. In the \( \phi = \pi \) case, introducing current asymmetry shifts the origin to a point higher than zero as expected. Furthermore, in the \( \phi = \pi \) case, the curve with no asymmetry seems to be a good fit to the experimental data. When both current and resistance asymmetries are considered, it leads to a general flattening of the curves, leading to very broad resonances. We have chosen \( \kappa = 0.5 \) and \( \rho = 0.2 \) as typical asymmetry parameters. We find that increasing \( \rho \) to a value higher than 0.2 leads to extremely flattened \( i_{dc}(v_{dc}) \) curves and also multiple roots in the \( v_{dc}(v_{dc}) \) curves (next section) which indicates that 0.2 is probably the highest resistance asymmetry we can consider in this case. We have also chosen to demonstrate trends for \( \kappa = 0.5 \) as we feel that for practical SQUIDs this represents one of the highest asymmetries.

5.2. Voltage–flux characteristics

Equation (17) can also be used to simulate the voltage–flux \( v_{dc}–\phi \) characteristics of a SQUID. We have again used Lee et al’s SQUID parameter values to simulate \( v_{dc}–\phi \) curves for various cases namely in the absence of any asymmetry in SQUID parameters, in the presence of just current asymmetry, in the presence of just resistance asymmetry and also in the presence of both current and resistance asymmetries.

Figure 4(a) shows the \( v_{dc}–\phi \) curve in the absence of asymmetries. Figure 4(b) shows the \( v_{dc}–\phi \) curve in the case when a current asymmetry \( \kappa = 0.5 \) is considered.

Figure 5(a) shows the \( v_{dc}–\phi \) curve in the case when a normal state resistance asymmetry \( \rho = 0.2 \) is considered. Figure 5(b) shows the \( v_{dc}–\phi \) curve in the case when a current
asymmetry $\kappa = 0.5$ and resistance asymmetry $\rho = 0.2$ are considered. In each of the four cases, normalized bias current $i_b$ ranges from 2.4 to 3.6.

From figure 4, we find that in the absence of any asymmetry we have perfectly symmetric $v_{dc}(\phi)$ curves. The difference between the maximum and minimum $v_{dc}$ decreases i.e. $\Delta v_{dc}$ decreases for increasing values of bias current, $\Delta v_{dc}$ goes to zero for $i_b = 3.3$. Then it again increases for higher bias current values. The complete flattening of the $v_{dc}(\phi)$ curve or $\Delta v_{dc}$ going to zero indicates a resonance position. With the introduction of current asymmetry (figure 4), the voltage values are less than the corresponding ones in the absence of asymmetry for a particular bias current; for instance, for $i_b = 3.6$, maximum $v_{dc}$ in the absence of asymmetry is around 1.62 whereas for the same bias current the maximum voltage decreases to around 1.45 in the presence of current asymmetry. Both figure 5(a), where only resistance asymmetry is considered and figure 5(b), where both resistance and current asymmetries are considered exhibit skewed $v_{dc}(\phi)$ curves. Beyer et al [15] found that $v_{dc}(\phi)$ curves got skewed on introduction of asymmetry in SQUID parameters. We find that complete flattening of $v_{dc}(\phi)$ curves is absent when resistance asymmetry is introduced i.e. the minima in the curves do not go to zero. From Lee’s experimental data (for example see figures 3(a) and (b)) we find, that at $i_b = 3.3$, the voltage values for $\phi = 0$ and 1/2 are almost the same. This indicates a resonance position. Thus we expect flattening of the voltage–flux curves at this position. This indicates that even though some current asymmetry may be present in the SQUID (see figure 4(b)), resistance asymmetry is probably absent.

5.3. Voltage modulation $\Delta V$

Figure 6 shows the comparison between voltage modulation $\Delta v_{dc}(i_b)$ where both $\Delta v_{dc}$ and $i_b$ are in normalized units in cases where no asymmetry is considered and when various asymmetries are introduced.

We find that the resonance position is almost unaffected when just current asymmetry is introduced. However, the magnitude of $\Delta V$ decreases by around 20%. On introduction of just resistance asymmetry on the other hand, the magnitude of $\Delta V$ increases, by almost 100% for the first peak and by around 20% for the next couple of peaks. The resonance position also seems to shift towards the left (towards smaller current values). On introduction of both current and resistance asymmetries, the resonances get rather flattened and there is a general decrease in magnitude of $\Delta V$ in keeping with the observation for just current asymmetry and the resonance position, especially the first peak shifts a little leftward in keeping with the observation for just resistance asymmetry. The other peaks do not show any significant shift.
6. Summary of results

In this paper, we analytically investigated the effects of transmission line resonances on dc SQUID characteristics. We obtain a closed form expression for the $i_b(v_{dc})$ characteristics of the SQUID. We use SQUID parameters used by Lee et al. [6] and compare our theoretical simulations with their experimental data. We find that the resonance positions are mainly controlled by $I_0 R_S$ as well as the dielectric constant of the substrate $\epsilon_R$. We also investigated the effects of introducing asymmetry in junction parameters. We find that the resonance positions are not very sensitive to the asymmetry. The actual magnitude of the current–voltage curves are however sensitive to the degree of asymmetry introduced. Our analytical solution may provide us with a method of theoretically determining the value of the dielectric constant of the substrate if we use that as a fit parameter between experimental and theoretical graphs. In an earlier work, we have determined $\epsilon_R$ of STO thin films using a Josephson junction based technique called Josephson Broadband Spectroscopy [16, 17].

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Appendix A

Enpuku et al.'s expression for the circulating current $J$ (equation (12) of [3]) is given by:

$$J = j_0 \left( \theta_2 - \theta_1 + \sum_{n=1}^{\infty} m_n \theta_{r,n} \right) + \frac{4}{\beta} \phi_{ex}$$  \hspace{1cm} (A.1)

where $m_n = M_n / L_n$, $M_n$ is the mutual inductance between SQUID and $n$th resonant circuit, $L_n$ is the inductance of the $n$th resonant circuit, $\theta_{r,n}$ is the phase corresponding to the voltage $V_{r,n}$ across $L_n$, $\phi_{ex}$ is the external applied flux and $j_0 = \frac{2}{\pi} \left( \frac{1}{1 - \sum_{n=1}^{\infty} \alpha_n^2} \right)$ where $\beta$ is the SQUID inductance parameter and $\alpha_n$ is the coupling constant between the $n$th resonant circuit and the lumped SQUID loop given by $\alpha_n = \frac{\sqrt{2}}{\pi \sqrt{n-1}}$. A drawback in the expression of $J$ is immediately apparent. Substituting the expression for $\alpha_n$, we get for $J$:

$$J = \frac{1}{(1 - \sum_{n=1}^{\infty} \alpha_n^2)} \frac{1}{\pi \beta} \left( \theta_2 - \theta_1 + \sum_{n=1}^{\infty} m_n \theta_{r,n} \right) + \frac{4 \phi_{ex}}{\beta}$$ \hspace{1cm} (A.2)

There is a problem with this expression in the continuum limit when $n \rightarrow \infty$. In this limit,

$$\theta_2 - \theta_1 = - \sum_{n=1}^{\infty} m_n \theta_{r,n}$$ \hspace{1cm} (A.3)

$$\Rightarrow J = \frac{0}{\beta} + \frac{4 \phi_{ex}}{\beta},$$ which is undefined.

Appendix B

In the following derivation of the SQUID power balance relation, random noise currents have been neglected. Multiplying equation (12) by $\dot{\theta}_1$ and equation (13) by $\dot{\theta}_2$, adding the two and noting that the voltage across the SQUID averaged over a time $T$ is given by:

$$\frac{1}{T} \int_0^T \dot{\theta}_1 \, dt = v_{dc} = \frac{1}{T} \int_0^T \dot{\theta}_2 \, dt$$  \hspace{1cm} (B.1)
we obtain:

$$v_{ik}i_B = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \left( \dot{\theta}_1^2 + \dot{\theta}_2^2 + \gamma (\dot{\theta}_1 - \dot{\theta}_2)^2 - \frac{1}{2} J (\dot{\theta}_1 - \dot{\theta}_2) \right).$$ (B.2)

Equation (B.2) can be rewritten as:

$$i_B v_{ik} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1}{2} \left( \dot{\delta}^2 + (1 + 2 \gamma \dot{\delta})^2 - j(t) \dot{\delta} \right) dt$$ (B.3)

where $S = \theta_1 + \theta_2$ and $D = \theta_1 - \theta_2$. The left-hand side of equation (B.2) represents the input power from the bias current $i_B$. The first two terms on the right-hand side of equation (B.2) represent the power dissipated in the two SQUID shunt resistances, the third term represents the power dissipated in the damping resistance while the fourth term represents the power dissipated in the resonant circuit. The time averaged voltage as a function of bias current and applied flux $V(i_B, \Phi)$ for the SQUID can be obtained if the voltage waveforms for the two junctions $v_1$ and $v_2$ are known. When a dc voltage $v_{dc}$ appears across the SQUID, the normalized Josephson currents, $\sin(\theta_1)$ and $\sin(\theta_2)$ oscillate with time since the relation

$$\sin(\theta_i) = \sin \left( \int v_i dt \right)$$

holds, where $i = 1, 2$. Moreover, by averaging over the period of a Josephson oscillation, it is simple to show that the fundamental frequency of the Josephson current may be given by $v_{dc}/2\pi$. Therefore, we can express the Josephson current in the presence of $v_{dc}$ as:

$$\sin(\theta_i) = \sum_{n=1}^{\infty} i_{i,n} \cos(n v_{dc} t + \psi_{i,n}) \quad i = 1, 2$$ (B.4)

where $i_{i,n}$ and $\psi_{i,n}$ represent the amplitude and phase of the $n$th Fourier mode respectively. The above relation means that the Josephson current can be regarded as an ac current generator in the presence of $v_{dc}$. The amplitude of the current generator is given by [18]:

$$I = i_1 = i_2 = \sqrt{2 v_{dc}(1 + v_{dc}^2)^{1/2} - v_{dc}}.$$ (B.5)

The phases of the two current generators are different and the phase difference is given by the normalized applied flux as $\psi_1 - \psi_2 = 2 \pi \phi$. In the present calculation, only the first harmonic of the Fourier expansion has been considered i.e. $\sin(\theta_i) = i_i \cos(v_{dc} t + \psi_i)$ has been used. We have found that this approximation is in good agreement with numerical simulations as well over the Josephson frequency range. In this paper, the following definition has been used for the Fourier transform:

$$\tilde{X}(\omega) = \int_{-\infty}^{\infty} X(t) e^{-i \omega t} dt$$ (B.6)

and

$$X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{X}(\omega) e^{i \omega t} d\omega.$$ (B.7)

Taking Fourier transform of equation (12) and (13) gives:

$$-(1 + \chi) \omega^2 \beta C \tilde{\theta}_1 = \frac{i}{2} \dot{\beta} \delta(\omega) + \frac{i}{2} j + i \omega (1 + \rho) \dot{\theta}_1 - (1 + \kappa) \sin(\theta_1) + i \omega \gamma (\dot{\theta}_1 - \dot{\theta}_2).$$ (B.8)

and

$$-(1 - \chi) \omega^2 \beta C \tilde{\theta}_2 = \frac{i}{2} \dot{\beta} \delta(\omega) - \frac{i}{2} j + i \omega (1 - \rho) \dot{\theta}_2 - (1 - \kappa) \sin(\theta_1) - i \omega \gamma (\dot{\theta}_1 - \dot{\theta}_2).$$ (B.9)

Denoting $\tilde{S} = \dot{\theta}_1 + \dot{\theta}_2$ and $\tilde{D} = \dot{\theta}_1 - \dot{\theta}_2$, we get:

$$\tilde{S} = a(\omega) \kappa + b(\omega) - (i \alpha \omega + \omega^2 \beta C) \tilde{D} - \frac{i}{\beta} \delta(\omega)$$ (B.10)

and

$$\tilde{D} = a(\omega) + \kappa b(\omega) - (i \alpha \omega + \omega^2 \beta C) \tilde{S} - j$$ (B.11)

where

$$a(\omega) = I_1(\omega) - I_2(\omega)$$

and

$$b(\omega) = I_1(\omega) + I_2(\omega)$$

with

$$I_1(\omega) = I \pi (\delta(-v_{dc} + \omega) e^{-i\psi} + \delta(v_{dc} + \omega) e^{i\psi}),$$ (B.12)

and

$$I_2(\omega) = I \pi (\delta(-v_{dc} + \omega) e^{-i\psi} + \delta(v_{dc} + \omega) e^{i\psi}).$$ (B.13)

Using the notation

$$s(\omega) = (\omega^2 \beta C + i \omega), \quad d(\omega) = A(\omega) - (\omega^2 \beta C + i \omega + 2i \omega \gamma),$$ (B.14)

$$r(\omega) = i \alpha \omega + \beta C \omega^2,$$ (B.15)

$$f_1 = \kappa \alpha(\omega) + b(\omega) - \delta(\omega) i_1,$$ (B.16)

$$f_2 = \beta(a(\omega) + \kappa b(\omega)) - 8 \pi \alpha(\omega) \delta(\omega),$$ (B.17)

we get

$$\tilde{S} = -\frac{\beta d(\omega) f_1 + r(\omega) f_2}{\beta[d(\omega) s(\omega) - r(\omega) r(-\omega)]},$$ (B.18)

$$\tilde{D} = -\frac{s(\omega) f_2 + \beta r(\omega) \omega f_1}{\beta[d(\omega) s(\omega) - r(\omega) r(-\omega)]}.$$ (B.19)

The inverse Fourier transform leads to

$$S(t) = 2 v_{dc} t - \frac{4 \rho \phi}{\beta} e^{-i v_{dc} t}$$

$$+ \frac{I}{2} [d(v_{dc}) s(\omega) - r(\omega) r(-\omega)]$$

$$\times [(1 - \kappa) d(v_{dc}) - r(\omega) e^{-i \omega t}]$$

$$+ (1 + \kappa) [d(v_{dc}) + r(\omega) e^{i \omega t}] + c.c.,$$ (B.20)

$$D(t) = \frac{4 \rho \phi}{\beta} - \frac{\pi i \alpha \beta}{4}$$

$$+ \frac{I}{2} [d(v_{dc}) s(\omega) - r(\omega) r(-\omega)]$$

$$\times [(1 - \kappa)(\beta s(v_{dc}) + r(\omega)) e^{i \omega t}]$$

$$+ (1 + \kappa) [\beta s(v_{dc}) - r(\omega) e^{i \omega t}] + c.c.,$$ (B.21)

$$j(t) = \text{const} - \frac{I}{2} [d(v_{dc}) s(\omega) - r(\omega) r(-\omega)]$$

$$\times [(1 - \kappa) (\beta s(v_{dc}) + r(\omega)) e^{i \omega t}]$$

$$+ (1 + \kappa) (\beta s(v_{dc}) - r(\omega)) e^{i \omega t}] + c.c.$$ (B.22)

The constant term in $j(t)$ is given by $A(0)(8 \pi /\beta - \pi i \alpha \beta /4)$ but plays no role in the following. After some tedious algebra it can be shown that (B.3) leads to equation (17).
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