Complexity reduction in resonant open quantum system Tavis-Cummings model with quantum circuit mapping

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Tavis-Cummings (TC) cavity quantum electrodynamical effects, describing the interaction of \( N \) atoms with an optical resonator, are at the core of atomic, optical and solid state physics. The full numerical simulation of TC dynamics scales exponentially with the number of atoms. By restricting the open quantum system to a single excitation, typical of experimental realizations in quantum optics, we analytically solve the TC model with an arbitrary number of atoms achieving reduced modeling complexity. This solution allows us to devise the Quantum Mapping Algorithm of Resonator Interaction with \( N \) Atoms (Q-MARINA), an intuitive TC mapping to a quantum circuit with linear space and time scaling, whose \( N + 1 \) qubits represent atoms and a lossy cavity, while the dynamics is encoded through \( 2N \) entangling gates. Finally, we benchmark the robustness of the algorithm on a quantum simulator and superconducting quantum processors against the quantum master equation solution on a classical computer.

Introduction. The Tavis-Cummings (TC) model \[1\], which describes interaction of \( N \) atoms with an optical cavity, has been a cornerstone in the studies of quantum optical systems relevant for the development of quantum networks \[2\]–\[4\], all-photonic quantum simulators \[5\], quantum memories \[6\]–\[7\], quantum transport \[8\] and entanglement sharing protocols \[9\]–\[11\]. Rapid progress in experimental development in the field of nanophotonics, renders the impracticality and scarceness of theoretical approaches unsatisfactory, especially in the open quantum system setting where the cavity interacts with the environment. Although recent results demonstrate that TC model is integrable and can be solved using a variant of quantum inverse methods (QIM) \[12\]–\[13\], solutions obtained in this way pose difficulties in extracting physical quantities and capturing dynamical correlations in the system. On the other hand, numerical solutions obtained through the quantum master equation \[14\] are limited by the exponential runtime complexity in Hilbert space size, and have thus far been performed for a single digit number of atoms. Due to the impracticality of analytical approaches based on QIM and exponentially rising cost of numerical solutions of the quantum master equations for such systems, theoretical verifications of experimental results are constrained to low number of atoms. Increasing the size of the Hilbert space has been pursued via approximate methods, such as the effective Hamiltonian \[15\], scattering matrix \[4\] and quantum trajectories \[16\] approaches.

The availability of the Noisy Intermediate Scale Quantum (NISQ) devices has attracted interest for simulating open quantum systems. To date, two prevailing directions have emerged, the first using operator sum representation, where Sz.-Nagy theorem is used to relate Kraus operators with unitary dilatation matrices \[17\]–\[18\] that can then be directly implemented on a quantum circuit \[2\]. The alternative approach is starting directly from the Lindblad master equation, and has been applied so far to both Markovian and non-Markovian open quantum systems consisting of 1 or 2 qubits \[19\]. Both approaches have recently been verified on a canonical model of light matter interaction systems: the Jaynes-Cummings model \[2, 19\]. The approach based on the operator-sum representation has been further generalised and applied to quantum simulate the complex open quantum system, governed by the Fenna-Matthews-Olson Dynamics modelling the quantum theory of electron transfer in biological systems \[20\]. Cavity quantum electrodynamical models that involve multiple emitters, such as the TC model, have not yet been considered. However, this in particular is the area where classical methods quickly saturate numerical resources and quantum devices may be able to expand the Hilbert size of systems studied in quantum communication, memories and simulators. Moreover, studying a quantum system on purely quantum hardware may provide representations that are intuitive in nature, as both emitters and qubits are two-level systems.

In this Letter, we study a resonant open TC model with arbitrary number of atoms and first provide an analytical solution for the singly-excited system with reduced simulation complexity. We then design the Quantum Mapping Algorithm of Resonator Interaction with \( N \) Atoms (Q-MARINA) which maps the open TC system with \( N \) atoms to a gate-based quantum circuit with only \( N+1 \) qubits. We simulate the system on a superconducting quantum computer available through IBM Quantum program \[21\], and provide comparison of the linear quantum simulation scaling with the exponential scaling of the
Here, we use the collective system operators $S_i$ while their interaction is described by the Tavis-Cummings separated by 2

lorentzian profile with linewidth $\kappa$ and a cavity resonantly coupled to $N$ atoms (solid orange line) featuring two polariton peaks separated by $2g\sqrt{N}$.

The model. We consider $N$ two-level systems, modeling an ensemble of atoms (or spins), coupling to the environment of discrete bosonic modes. The system and the environment Hamiltonians $H_S$ and $H_E$ are:

$$H_S = \omega_S S_z,$$

$$H_E = \sum_k \omega_k b_k^\dagger b_k,$$  \hspace{1cm} (1)

while their interaction is described by the Tavis-Cummings Hamiltonian $H_I$:

$$H_I = \sum_k g_k b_k S^+ + g_k^* b_k^\dagger S^-.$$  \hspace{1cm} (2)

Here, we use the collective system operators $S_z = \sum_{j=1}^N \frac{1}{2} \sigma_j^z$ and $S^\pm = \sum_{j=1}^N \sigma_j^\pm = \sum_{j=1}^N \frac{1}{2} (\sigma_j^+ \pm i \sigma_j^-)$, with commutation relations $[\sigma_j, \sigma_k] = 2i\epsilon_{j,k} \sigma_l$ and $[S_z, S^\pm] = \pm S^\pm$.

To solve the model analytically, we aim to obtain the time dependent Hamiltonian in the interaction picture in the form of:

$$H_I(t) = \sum_k g_k b_k(t) S^+ + g_k^* b_k^\dagger(t) S^-.$$  \hspace{1cm} (3)

Here, the form of $b_k(t) = b_k e^{-i\omega_k t}$ is easily derived, however, finding an elegant expression for $S^\pm(t)$ requires closer consideration.

We first note that, in the Hilbert space of the system, the operator $S_z = \frac{1}{2} \sum_{n=1}^N (\otimes^{n-1} I \otimes \sigma_z \otimes^{N-n} I) = \text{diag}(\{e_{xp}\})$ is diagonal in terms we will call $x_p, 1 \leq p \leq 2^N$. We find that, $x_p$ is a function of the Hamming weight $W(p-1)$, i.e. the digit sum of the binary representation of the number $p-1$, as:

$$x_p = \frac{N}{2} - W(p-1).$$

Therefore, the term $e^{iH_E t} = \text{diag}(\{e^{i\omega_{xp} t}\})$ must too be diagonal, which allows us to obtain a closed form solution:

$$S^\pm(t) = S^\pm e^{+i\omega_x t},$$  \hspace{1cm} (4)

thus completing the Eq. \ref{3} for the time-dependent interaction Hamiltonian of the TC model with $N$ identical two-level atoms.

Reduced density matrix. The wavefunction of an $N$-atom Tavis Cummings system in the low-excitation regime is given by the superposition of the vacuum state $|g0\rangle$, single excitations of the $n$-th atom $|e_n0\rangle$ and the single excitations of the $k$-th bosonic mode $|g1_k\rangle$

$$|\Psi_N(t)\rangle = c_0|g0\rangle + \sum_{n=1}^N c_{sn}(t)|e_n0\rangle + \sum_k c_k(t)|g1_k\rangle.$$  \hspace{1cm} (5)

The Schrödinger equation with Hamiltonian given in Eq. \ref{3} yields a system of differential equations:

$$\dot{c}_{sn} = -i \sum_k g_k e^{i(x_2N-2N-n-x_2)\omega_s - \omega_k t} c_k(t),$$  \hspace{1cm} (6)

$$\dot{c}_k = -ig_k^* \sum_{n=1}^N e^{i[\omega_k + \omega_s(x_2N-2N-n)] t} c_{sn}(t).$$  \hspace{1cm} (7)

We next note that $W(2^N-1) = N$ and $W(2^N-2^N-n-1) = N-1$ therefore $x_2N = -\frac{N}{2}$ and $x_2N-2^N-n = 1 - \frac{N}{2}$.

The system of differential equations transforms to

$$\dot{c}_{sn} = -i \sum_k g_k e^{i(\omega_s - \omega_k) t} c_k(t),$$  \hspace{1cm} (8)

$$\dot{c}_k = -ig_k^* \sum_{n=1}^N e^{i(\omega_k - \omega_s) t} c_{sn}(t).$$  \hspace{1cm} (9)

It follows that the $k$-th cavity mode and the $n$-th atom amplitude can be expressed as

$$c_k(t) = -i \int_0^t dt' g_k e^{i(\omega_k - \omega_s) t'} \sum_{n=1}^N c_{sn}(t'),$$  \hspace{1cm} (10)

$$\dot{c}_{sn}(t) = - \int d\omega J(\omega) \int_0^t dt' e^{i(\omega - \omega_s) (t-t')} \sum_{m=1}^N c_{sm}(t'),$$  \hspace{1cm} (11)

where we approximate the environment coupling terms with Lorentzian density of states modeling the cavity dynamics $\sum_k |g_k|^2 = \int d\omega J(\omega)$. The term

$$J(\omega) = \frac{g^2}{2\pi} \frac{\kappa}{(\omega_s - \omega)^2 + (\kappa/2)^2}$$  \hspace{1cm} (12)

describes an optical resonator with loss rate $\kappa$ coupled to an atom at interaction rate $g$, and represents the channel through which the system interacts with the environment. For a closed system, the cavity would respond to
1) 1-dimensional density matrix as:

\[ \rho_s(t) = \langle 0 | \Psi_N(t) | \Psi_N(t) \rangle |0 \rangle + \sum_k |1_k| \langle \Psi_N(t) | \Psi_N(t) |1_k \rangle. \]  

From here, we express the diagonal elements of the \((N+1)\)-dimensional density matrix as:

\[ \rho_{s,n}^*(t) = |c_{s,n}(t)|^2, 1 \leq n \leq N \]  

\[ \rho_{s,N+1,N+1}(t) = 1 - \sum_{n=1}^{N} |c_{s,n}(t)|^2 \]  

In the following, we show that this dynamics can be mapped onto a quantum circuit with \(N+1\) qubit, thus enabling quantum modeling of the Tavis-Cummings open quantum system on a gate-based quantum computer.

Quantum circuit. Here, we devise the Quantum Mapping Algorithm of Resonator Interaction with \(N\) Atoms (Q-MARINA), an \((N+1)\)-qubit quantum circuit that evolves an open quantum system of \(N\) atoms and a resonant cavity in the single-excitation regime. The quantum circuit consists of \(N\) system qubits \(Q_{Sn}\) and one environment qubit \(Q_E\). The initial state is the excited state of one of the atoms, here \(Q_{S1}\) which is subject to an X-gate. Subsequent application of CU3 and CNOT gates between \(Q_{S1}\) and \(Q_E\) entangles the first atom and the environment, and then \(N-1\) sequences of CU3 and CNOT entangling gates are applied to each of the qubits \(Q_{S2},...,Q_{SN}\) paired with \(Q_E\), in the opposite direction than for the \(Q_{S1}\). The corresponding quantum circuit is shown in Fig. 2. Here, the parameters of the CU3 gates, \(\text{CU3}(2\theta_n)\) are selected to implement the Lorentzian density of states of the cavity open to the environment into the circuit:

\[ \theta_1 = \arccos (c_{s1}(t)), \]  

\[ \theta_n = \arcsin \left( \frac{c_{s,n}(t)}{\sin \theta_1 \prod_{m=2}^{N-1} \cos \theta_m} \right), \]  

thus resulting in excited state measurement probabilities of the system qubits \(Q_{Sn}\) equal to \(|c_{s,n}(t)|^2\).
The quantum circuit requires star connectivity as all system qubits $Q_{Sn}$ interact with the environment qubit $Q_E$, therefore we selected devices that can support that layout in a 3- and 4-qubit circuits within the computers’ heavy-hexagon topology. Our quantum device results demonstrate the agreement with the previously obtained benchmarks on the QASM simulator and numerical QME solutions, as shown in Fig. 4. These results demonstrate that the open quantum system Tavis-Cummings physics can be simulated on the existing quantum hardware with an intuitive mapping between atoms and qubits and a substantial reduction in complexity implemented through the entangling gates with a single environment qubit.

**Summary and outlook.** In this work, we have explored quantum circuit mapping of the dynamics of $N$ two-level atoms in a a lossy optical cavity. By restricting the open quantum system to a single excitation, typical of experimental realizations in quantum optics, we have analytically solved the TC model with an arbitrary number of atoms achieving reduced modeling complexity. This solution enabled us to devise the Quantum Mapping Algorithm of Resonator Interaction with $N$ Atoms (Q-MARINA), an intuitive TC mapping to a quantum circuit with linear space and time scaling. The close agreement between the quantum master equation with Q-MARINA executed on QASM simulator and IBM Q quantum devices shows that NISQ era quantum computers can be used to simulate open quantum system dynamics of highly dimensional models.

This constitutes a first step towards using superconducting NISQ processors to design new optical quantum devices. The results obtained on existing quantum devices are limited by the quantum computer size and the corresponding topology which provides the desired star-connectivity to up to 4 qubits. Alternative quantum platforms which provide all-to-all connectivity, such as those based on trapped ions or atoms, may provide options to scale the problem size by at least an order of magnitude. Once the number of qubits is scaled, the number of entangling gates relative to the qubit coherence time will be the measure of the performance, as the circuit depth scales linearly with the number of atoms.

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5

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