Taste breaking in staggered fermions from random matrix theory

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We discuss the construction of a chiral random matrix model for staggered fermions. This model includes $O(a^2)$ corrections to the continuum limit of staggered fermions and is related to the zero momentum limit of the Lee-Sharpe Lagrangian for staggered fermions. The naive construction based on a specific expansion in lattice spacing ($a$) of the Dirac matrix produces the term which gives the dominant contribution to the observed taste splitting in the pion masses. A more careful analysis can include extra terms which are also consistent with the symmetries of staggered fermions. Lastly I will mention possible uses of the model including studies of topology and fractional powers of the fermion determinant.

Staggered fermions are a popular way to include quarks on the lattice. A single staggered fermion determinant in the continuum limit actually describes four tastes of quarks, so named to distinguish them from physical flavors. At finite lattice spacing the tastes interact and the continuum $SU(4)$ taste symmetry is broken. The size of the taste breaking can be reduced by using improved actions, but for current lattice spacings the effects are clearly present in chiral fits [1].

The form of the taste breaking terms for a single flavor (four tastes) of staggered fermions was worked out as corrections to the chiral Lagrangian by Lee and Sharpe [2]. This was later extended to arbitrary flavor content (most notably 2+1 flavors) by Bernard and Aubin [3].

Here we will outline how to include taste breaking in a chiral random matrix model. This model should be equivalent to low energy lattice QCD with staggered fermions and can be used to study properties of the lattice theory which are sensitive to the smallest eigenvalues of the Dirac matrix. A more detailed study of the staggered matrix model and its applications will appear elsewhere.

1. Random Matrix Theory

Chiral random matrix theory has been used extensively to model the low energy eigenmodes of the Dirac operator. Its construction is based solely on the symmetries of the Dirac operator. In a chiral basis the standard Dirac operator for a single quark flavor takes the form

$$D = \left( \begin{array}{cc} m \mathbb{I}_N & iW \\ iW^\dagger & m \mathbb{I}_N \end{array} \right)$$

(1)

where $W$ is a complex $N \times N$ matrix, $\mathbb{I}_N$ is the $N \times N$ identity matrix and $m$ is the quark mass. Here we will only consider the case of zero topological charge and three or more colors which corresponds to the Dirac matrix given above. Due to universality the partition function near the chiral limit is not sensitive to the exact form for the distribution of the various matrix elements. One can therefore make the simplest choice of an independent Gaussian measure for each matrix element

$$\mu(W) = \exp\left( -\alpha N \text{Tr} W^\dagger W \right).$$

(2)

The parameter $\alpha$ is related to the chiral limit of the chiral condensate ($\Sigma$) and the volume ($V$) by $\sqrt{\alpha} = V\Sigma/2N$. For a recent review of chiral random matrix models see [4] and its references.

In order to extend chiral random matrix theory to include taste breaking we need to examine the form of the taste breaking interactions. The original motivation for this work came from examining the classical expansion of the staggered Dirac operator near the continuum limit given by Kluberg-Stern et. al. [5]. Their expansion starts

$$(\gamma_\mu \otimes \mathbb{I}_4) D_\mu + (\mathbb{I}_4 \otimes \mathbb{I}_4) m + a (\gamma_5 \otimes \xi_5) D_5^2.$$  

(3)

The notation $(S \otimes T)$ has the standard meaning of the outer product of a $4 \times 4$ spin matrix $S$ with
a $4 \times 4$ taste matrix $T$ with $\xi_{\nu 5} = \xi_\nu \xi_5 = \gamma_\nu^5 \gamma_5$. The first part is just the standard Dirac operator for four identical flavors of mass $m$. The remaining term is suppressed by a factor of the lattice spacing and breaks the $SU(4)$ taste symmetry. There is also an $O(a^2)$ term which for our purposes is similar to the included taste breaking term and will be ignored. The remaining corrections are at least $O(a^2)$. This form of the expansion is specific to the particular basis chosen for the continuum Dirac fermions. In fact Luo \cite{3} has shown that with an improved basis the $O(a)$ terms can be removed giving corrections only at $O(a^2)$. This ambiguity in basis to use for the continuum expansion can be avoided by instead starting with the same four-fermion operators in the continuum effective Lagrangian used by Lee and Sharpe \cite{2}. Using a Hubbard-Stratonovitch transformation these terms can split into fermion bilinears which can be used to construct a matrix theory with the correct symmetries. We do not present the details of this construction here but will refer to it later in the paper.

For simplicity here we will continue to construct the matrix model based on the expansion \cite{4} that was given. The taste breaking term in the chiral basis has the form

\begin{equation}
T = (A_\nu \otimes \sigma_3) \otimes \xi_{\nu 5} \tag{4}
\end{equation}

where $A_\nu$ is an $N \times N$ Hermitian matrix and $\sigma_3$ a Pauli matrix. We again make the simplest choice for the integration measure over the matrices $A_\nu$ which is that of independent Gaussian matrix elements given by

\begin{equation}
\mu(A_\nu) = \exp(-\beta N \text{ Tr } A_\nu^2) \tag{5}
\end{equation}

with some unknown parameter $\beta$. The partition function is then

\begin{equation}
Z = \int \mu(W) \prod_\nu \mu(A_\nu) \text{ d}A_\nu \text{ Det}(M) \tag{6}
\end{equation}

with

\begin{equation}
M = D \otimes 1_4 + a T . \tag{7}
\end{equation}

We now will relate this model to the staggered chiral Lagrangian given by Lee and Sharpe.

2. Chiral Lagrangian

The random matrix model \cite{8} can be transformed into a nonlinear sigma model using standard methods. In the large $N$ limit the sigma model can be expanded around its saddle point which gives

\begin{equation}
\int_{U(4)} dU \exp(-m V \Sigma \text{ Re Tr } U - a^2 \mathcal{V}) \tag{8}
\end{equation}

with

\begin{equation}
\mathcal{V} = N \left( \frac{\alpha}{\beta} \sum_\nu \text{ Re Tr } (U \xi_{\nu 5} U \xi_{5 \nu}) \right). \tag{9}
\end{equation}

This is equivalent to the zero momentum limit of an effective chiral theory. Notice that the taste breaking is $O(a^2)$ in the chiral Lagrangian even though we started with a term of $O(a)$ in the Dirac operator. The form of $\mathcal{V}$ is identical to the term with coefficient $C_4$ in the Lee-Sharpe Lagrangian if we set $N\alpha/\beta = VC_4$. It is interesting to note that this term was previously found to be the dominant contributor to the mass splittings between pions with different taste structures. In Figure 1 we show the difference between the squares of various pion masses and the square of the Goldstone pion ($\xi_5$) mass in units of the lattice spacing. On the $x$-axis are the taste generators used with the corresponding contribution to the splitting from the $C_4$ term written below. The data are from simulations of improved “$a^2$-tad” staggered fermions at $\beta_{imp} = 7.3$ and $am = 0.02$ tabulated in \cite{7}. The results agree very well with the linear rise predicted from the $C_4$ term.

We have so far considered only the taste breaking term given in \cite{8}. A more general construction alluded to earlier gives the following additional terms for the matrix model

\begin{align}
(A_\nu^{(1)} \otimes \sigma_{0,3}) \otimes i \xi_\nu & \rightarrow C_3 \\
(A_\nu^{(2)} \otimes \sigma_1) \otimes i I_4 & \rightarrow \text{ const.} \\
(A_\nu^{(3)} \otimes \sigma_1) \otimes i \xi_5 & \rightarrow C_1 \\
(A_\nu^{(4)} \otimes \sigma_1) \otimes \xi_{\mu \nu} & \rightarrow C_6 . \tag{10}
\end{align}

The $\sigma_{0,3}$ can be either $\sigma_0 = I_2$ or $\sigma_3$. Also the $\sigma_3$ in \cite{8} can be replaced with $\sigma_0$ to produce another valid term. The arrow shows which term from the chiral Lagrangian the matrix model term
Figure 1. Pion mass splittings for different taste content relative to the Goldstone pion. The contribution from the $C_4$ term of the staggered chiral Lagrangian is given at the bottom.

produces in the large $N$ limit. The corresponding terms in the chiral Lagrangian are

$$
\begin{align*}
C_1 & \text{ Tr } (\xi_5 U \xi_5 U^\dagger) \\
C_3 & \sum_\mu \text{ Re Tr } (\xi_\mu U \xi_\mu U) \\
C_6 & \sum_{\mu < \nu} \text{ Tr } (\xi_{\mu\nu} U \xi_{\nu\mu} U^\dagger).
\end{align*}
$$

The two remaining terms from [2] are

$$
\begin{align*}
C_2 & \text{ Re } \left[ \text{ Tr } (U^2) - \text{ Tr } (\xi_5 U \xi_5 U) \right] \\
C_5 & \frac{1}{2} \sum_\mu \text{ Tr } (\xi_\mu U \xi_\mu U^\dagger) - \text{ Tr } (\xi_{\mu\mu} U \xi_{\mu\mu} U^\dagger).
\end{align*}
$$

They can also be produced from the matrix model using terms similar to those in [10], however some of the terms would have to be Hermitian (note the terms in [10] are all anti-Hermitian) which would make them not suitable for modeling the Dirac matrix. This point requires further study.

3. Conclusions and Future Applications

We have presented a chiral random matrix model for staggered fermions. In its simplest form [6] it includes just the term that is known to be dominant in the pion mass splittings. Additional terms consistent with the staggered fermion symmetries can also be included, but the construction of the most general form of the matrix model still requires further study.

This model can also be extended to include additional flavors and non-zero topological charge. Both of these extensions will be covered in a future publication. Since topology is related to the lowest eigenvalues of the Dirac matrix, this should prove to be a useful model for studying the topology of staggered fermions.

Current lattice simulations using staggered fermions typically take the square root or fourth root of the fermion determinant to reduce the number of tastes from four to produce one or two physical flavors in the continuum limit. The effects of the root on the low energy spectrum are not well understood. One could study its effects in the matrix theory where analytic results are easier to produce. If the distribution of Dirac eigenvalues in the matrix model agrees with lattice calculations, then the effect of the root should be modeled properly by the matrix theory. This is currently perhaps the most interesting application of the staggered random matrix model.

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