Exploring Fermionic T-duality

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Abstract

The fermionic T-duality transformation developed by Berkovits and Maldacena is applied to D-brane and the pp-wave solutions of type IIB supergravity. The pp-wave is found to be self-dual under the combination of dualities. We explore the consequences of applying the transformation and discuss various properties of the new transformed solutions.

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T-duality is one of the remarkable features of string theory [1]. It is a map between different string backgrounds that leaves the partition function of the string sigma model invariant [2]. From the point of view of the world sheet it is an abelian two-dimensional S-duality. From the spacetime viewpoint it is somewhat mysterious since it provides an equivalence between completely different geometries. A key application of the duality is to use this symmetry as a solution generating mechanism in supergravity [3] where one begins with a particular solution and then through application of the T-duality rules produces a new set of solutions. This technique has proved particularly useful in constructing solutions deformed by NS flux such as for the gravity duals of noncommutative theories [4], beta deformed Yang-Mills [5] and so-called dipole deformed theories [6] (similar techniques have also been used for deformation of M-theory geometries[7]).
T-duality is also crucial in establishing the connection between the different branes of type II string theory and has been a central pillar in string duality for many years. It is surprising then that it was not until 2008 that fermionic T-duality was developed [8, 9]. Usual bosonic T-duality relies on using an isometry of the background to generate the T-duality transformation. Fermionic T-duality can be viewed as extending this idea to isometries of the fermionic directions in superspace. From the viewpoint of supergravity component fields these are just the supersymmetry transformations, thus instead of using isometries to generate the T-duality transformations one uses the supersymmetries. The details of the transformation will be reviewed later in the paper.

Let us note various aspects of this transformation. Firstly, it is not a full symmetry of string theory like bosonic T-duality since it is broken at one loop in $g_s$. (This is because of the presence of fermionic zero modes in the path integral beyond tree level that make the path integral vanish. It is interesting to consider if one could extend the duality beyond tree level by soaking up these zero modes and making sense of such a path integral including the fermionic insertion).

Secondly, apart from a shift in the dilaton the NSNS sector of the theory is left invariant. Fermionic T-duality is a transformation primarily of the RR fields. (This really explains the delay in the study of fermionic T-duality; deriving the RR transformations in bosonic T-duality from the string world sheet was only done recently and required using the Berkovits formulation [10]).

Thirdly, because of the requirement that we deal with commuting supersymmetries (just as one deals with commuting isometries in ordinary T-duality) it is necessary that we deal with complexified Killing spinors and in turn complexified RR-fluxes. Thus the transformed background will be a solution of complexified supergravity. One open and indeed crucial question is to determine when these transformations map back to a real supergravity solution. In fact, one need not map directly to a purely real solution since if there exists a time-like isometry (which is almost certain for a supersymmetric solution) then one can do bosonic T-duality in the timelike direction [11]. This transformation has the effect (amongst other things) of multiplying the RR-fluxes by an imaginary unit. Thus it can make purely imaginary fluxes real. This was precisely the case for the fermionic dual of $AdS_5 \times S^5$ described by Berkovits and Maldacena where after eight fermionic T-dualities there remained some imaginary RR flux. This was then made real by application of timelike T-duality.

In any case, perhaps we should be interested in complexified supergravity in its own right. In quantum field theory (such as Yang-Mills) there has been a great deal of progress made by complexifying the theory and then using the power of complex analysis. This
was the origin of the S-matrix programme which has now seen something of a revival [12] with recent works on amplitude physics again relying on an implicit complexification of the theory to achieve results. In fact, the motivation for studying fermionic T-duality [8, 9] was to derive the duality between certain amplitudes and Wilson lines in Yang-Mills theory\footnote{DSB is grateful to various participants of the FPUK meeting in Cambridge for discussions on this issue.}. Whether we can learn really more about string theory per se through complexification of backgrounds has yet to be seen but ideas along these lines have appeared before (see for example the discussion in [11]).

Related work on fermionic T-duality has appeared in [13, 14, 15].

## 2 Fermionic T-duality

Here we review in detail the fermionic T-duality transformation procedure derived in [8]. The type II supersymmetry transformations are parameterized by the Killing spinors of the solution. These Killing spinors will determine the transformed solution as follows.

Take, $\epsilon$, a Killing spinor that parameterizes an unbroken supersymmetry. It is a Majorana-Weyl spinor of $(1+9)$-dimensional spacetime, that is, real with sixteen components. Since type II supergravity is an $\mathcal{N} = 2$ theory, there is also another Killing spinor, which is denoted by $\hat{\epsilon}$. A pair $e = (\epsilon, \hat{\epsilon})$ generates one supersymmetry transformation. However, the two spinors within this pair are not independent – they are related by the Killing spinor equations (see section 3 below), and furthermore by the constraint

$$\epsilon \gamma_\mu \epsilon + \hat{\epsilon} \gamma_\mu \hat{\epsilon} = 0 \quad (2.1)$$

for all $\mu \in \{0, \ldots, 9\}$. Here $\gamma_\mu$ are blocks comprising the 10-dimensional gamma-matrices in the Weyl representation (see appendix B). This constraint arises from insisting that the supersymmetry with which one carries out the T-duality transformation is commuting just as one requires that the Killing vectors in bosonic T-duality commute. Since $\gamma_0$ is a unit matrix, the above relation cannot hold for real spinors, and they must be artificially complexified. This is a characteristic property of fermionic T-duality, which then leads to complex RR fluxes after the transformation.

After the choice of the Killing spinors satisfying (2.1) has been made, one calculates an auxiliary scalar field $C$ defined by the following differential equation:

$$\partial_\mu C = i \epsilon \gamma_\mu \epsilon - i \hat{\epsilon} \gamma_\mu \hat{\epsilon}. \quad (2.2)$$
By using the constraint (2.1) we can simplify this to be:

$$\partial_\mu C = 2i\epsilon\gamma_\mu\epsilon. \quad (2.3)$$

The transformation of the dilaton is given by

$$\phi' = \phi + \frac{1}{2} \log C, \quad (2.4)$$

and the RR forms transformation can be written succinctly in terms of the bispinor $F^{\alpha\beta}$:

$$\frac{i}{16} e^{\phi'} F' = \frac{i}{16} e^{\phi} F - \epsilon \otimes \hat{\epsilon} C. \quad (2.5)$$

The RR field strength bispinor incorporates all RR forms of IIB supergravity:

$$F^{\alpha\beta} = (\gamma^\mu)^{\alpha\beta} F_\mu + \frac{1}{3!} (\gamma^\mu_1 \mu_2 \mu_3)^{\alpha\beta} F_{\mu_1 \mu_2 \mu_3} + \frac{1}{2 \times 5!} (\gamma^\mu_1 \ldots \mu_5)^{\alpha\beta} F_{\mu_1 \ldots \mu_5}. \quad (2.6)$$

We have a factor of $+16$ as compared to $-4$ of [8] in the transformation law (2.5) because of a different normalisation of RR fields, which is implied by the action (A.2) that we use. In fact, the formula (2.6) is only correct for backgrounds with trivial NS two-form (which is the case for the D-brane and pp-wave backgrounds). If there is a nontrivial $B$-field, then instead of just the RR field strengths one should use the modified RR field strengths that are invariant under the supergravity gauge transformations as given in equation (A.6). This correction is beyond the first order in component fields and thus was omitted from the original derivation\textsuperscript{4}.

In the case when the fermionic T-duality is performed with respect to several supersymmetries, parameterized by the Killing spinors $e_i = (\epsilon_i, \hat{\epsilon}_i), i \in \{1, \ldots, n\}$, the formulae (2.3), (2.4), and (2.5) are generalized to

$$\partial_\mu C_{ij} = 2i\epsilon_i \gamma_\mu \epsilon_j, \quad (2.7a)$$

$$\phi' = \phi + \frac{1}{2} \sum_{i=1}^{n} (\log C)_{ii}, \quad (2.7b)$$

$$\frac{i}{16} e^{\phi'} F' = \frac{i}{16} e^{\phi} F - \sum_{i,j=1}^{n} (\epsilon_i \otimes \hat{\epsilon}_j) (C^{-1})_{ij}. \quad (2.7c)$$

\textsuperscript{4}We are grateful to Giuseppe Policastro and Nathan Berkovits for clarifying this point.
The set of the Killing spinors must obey

\[ \epsilon_i \gamma_\mu \epsilon_j + \hat{\epsilon}_i \gamma_\mu \hat{\epsilon}_j = 0 \]  \hspace{1cm} (2.8)

for all \( i, j \in \{1, \ldots, n\} \).

In summary, the recipe to perform fermionic T-duality on a given solution is as follows:

1. Find the Killing spinors of the solution. In IIB supergravity these are represented by pairs \( e = (\epsilon, \hat{\epsilon}) \) of 16-component real spinors of the same chirality.

2. Choose a complex linear combination of the Killing spinors \( e' = (\epsilon', \hat{\epsilon}') \) that satisfies the condition (2.1). These Killing spinors describe the supersymmetry that we are dualising with respect to.

3. Calculate \( C \) from (2.3). To do this consistently, one should work in world indices (i.e. one should integrate \( \partial_\mu C = 2i\epsilon' (e''_\nu \gamma_\nu) \epsilon' \), where \( e''_\nu \) is the vielbein, and world indices are underlined to distinguish them from flat ones).

4. If there are any RR fields in the original background, substitute them into (2.6) to calculate the matrix \( F^{\alpha\beta} \).

5. Use \( F^{\alpha\beta}, \epsilon^\alpha, \hat{\epsilon}^\beta, \) and \( C \) to calculate the transformed RR background \( F'^{\alpha\beta} \) via (2.5).

6. Use (2.6) again, this time to find the contributions of \( F_1, F_3, \) and \( F_5 \) to \( F'^{\alpha\beta} \) separately.

7. Check that the transformed background is a solution to the field equations.

Since the above recipe of doing fermionic T-duality involves a great deal of 16 by 16 matrix manipulations, it is the easiest to implement it using a simple Mathematica program to perform steps 2, 4, 5, and 6. (A copy of the program is available via an email to the authors). The only nontrivial step in such program is number 6, where one starts with a 16 by 16 matrix \( F' \), and one needs to find the corresponding 1-, 3-, and 5-form components. This calculation is done by separating the matrices in equation (2.6) into their symmetric and antisymmetric parts. On the left-hand side of the equation we have a matrix \( F' \), which has been calculated from (2.5). This should be split into symmetric and antisymmetric parts by brute force. As to the right-hand side of (2.6), it is naturally separated into symmetric and antisymmetric parts. Namely, a single \( \gamma \)-matrix is symmetric, as well as a product of five \( \gamma \)-matrices, whereas a triple product is antisymmetric. This can be verified explicitly by using the matrix representation given in appendix B.
3 Fermionic T-duals of the D1-brane

We begin with the D1-brane solution in IIB. This background has nonzero dilaton, metric and RR 2-form potential. These are given by the following [16]:

\[ e^{2\phi} = 1 + \frac{Q}{(\delta_{mn}x^m x^n)^6}; \]
\[ g_{\mu\nu} = (e^{-\phi}\eta_{ij}, e^\phi\delta_{mn}), \]
\[ g^{\mu\nu} = (e^\phi\eta^{ij}, e^{-\phi}\delta^{mn}), \]
\[ \sqrt{|g|} = e^{3\phi}, \]
\[ (C_2)_{01} = e^{-2\phi} - 1; \quad (F_3)_{01m} = -2e^{-2\phi}\partial_m\phi, \]

and all the other fields (\( B_2, C_0, C_4 \)) are zero. The notation is

\[ \eta_{ij} = \text{diag}(-1, 1), \quad i, j \in \{0, 1\}, \]
\[ \delta_{mn} = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1), \quad m, n \in \{2, \ldots, 9\}. \]

All components of \( C_2 \) and \( F_3 \), other than specified in (3.5), are zero. The indices in (3.5) are world indices.

The form of the transformed solution depends on the choice of the Killing spinor used for the transformation. So a few words about D-brane Killing spinors are in order.

Dp-branes are invariant under the supersymmetry transformations parameterized by the spinors that satisfy the following projection condition:

\[ (1 \pm \Gamma^{0\ldots p}\mathcal{O})\bar{\epsilon} = 0, \]

where \( \mathcal{O} \) is an operator that depends on the supergravity type and on the dimensionality of the brane in question. Thus a generic type II D-brane in ten dimensions has sixteen unbroken supersymmetries generated by the Killing spinors that satisfy the above constraint.

Confining our attention to the case of D1-brane we have

\[ \bar{\epsilon} = \begin{pmatrix} \epsilon \\ \hat{\epsilon} \end{pmatrix}, \]

where \( \epsilon \) and \( \hat{\epsilon} \) are the two chiral Majorana-Weyl spinors that are the supersymmetry
parameters of type IIB supergravity. This is written in the two-component formalism, so that $e$ is just a two-component column vector, not a 32-component 10d spinor. The operator $O$ is given by the Pauli matrix $\sigma_1$, so that the Killing spinor constraint takes the form

$$(1 \pm \Gamma^{01}\sigma_1)\epsilon = \left( \begin{array}{c} \epsilon \\ \hat{\epsilon} \end{array} \right) \pm \left( \begin{array}{c} \Gamma^{01}\hat{\epsilon} \\ \Gamma^{01}\epsilon \end{array} \right) = 0,$$

(3.8)

Taking the minus sign for definiteness we see that, for example, we can take $\epsilon$ to be arbitrary 16-component MW spinor, in which case $\hat{\epsilon} = \Gamma^{01}\epsilon$.

Technically, the above algebraic constraint on the Killing spinor can be thought of as arising from the requirement that the supersymmetry variation of dilatino vanishes. There is also an analogous requirement for gravitino, which is the second fermionic doublet in type II supergravity. Since the variation of gravitino contains derivatives of the supersymmetry parameter, this second constraint leads to a differential equation on $\epsilon$. Solving this equation introduces coordinate dependence into the Killing spinor (note that so far our $\epsilon$ and $\hat{\epsilon}$ were constant). Thus, it turns out that

$$\epsilon = e^{-\phi}\epsilon_0$$

(3.9)

for an arbitrary constant $\epsilon_0$, and $\hat{\epsilon} = \Gamma^{01}\epsilon$, as before. The function $e^\phi$ has been defined in (3.1).

Using explicit realisation of the gamma-matrices, we see that corresponding to an arbitrary

$$\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{16})^T,$$  

(3.10)

is

$$\hat{\epsilon} = (\epsilon_{16}, -\epsilon_{15}, -\epsilon_{14}, \epsilon_{13}, -\epsilon_{12}, \epsilon_{11}, \epsilon_{10}, -\epsilon_{9}, -\epsilon_8, \epsilon_7, \epsilon_6, -\epsilon_5, \epsilon_4, -\epsilon_3, -\epsilon_2, \epsilon_1)^T,$$

(3.11)

where the factors of $e^{-\phi}$ have been omitted for simplicity ($^T$ means transpose, so that $\epsilon$ and $\hat{\epsilon}$ are columns). Setting all $\epsilon_i$ but $\epsilon_1$ to zero, we get the first basis element (which we call $e_1$). It includes both $\epsilon$ and $\hat{\epsilon}$. Repeating this process for all of the sixteen parameters, we end up with the sixteen basis elements $e_i$.

The next step in our programme is to pick a particular linear combination of the Killing spinors, so that it satisfies the condition (2.1). As mentioned earlier, this constraint cannot be satisfied by real Killing spinors. We consider the simplest possible linear combinations, i.e. those of the form

$$e' = e_a + ie_b; a, b \in \{1, \ldots, 16\}.$$  

(3.12)
Using the explicit form of gamma-matrices it is easy to check that (2.1) is satisfied by any such combination, apart from those of the form $e_a + ie_{17-a}$.

The result of the fermionic T-duality transformation can be of two types depending on the values of $a$ and $b$ in (3.12):

- If $(a \leq 8$ and $b \leq 8)$, or $(a \geq 9$ and $b \geq 9)$, then the result is of the ‘simple’ type. This is characterized by vanishing $\epsilon \gamma_\mu \epsilon$, which means that $C$ in (2.4) and (2.5) is just a constant. The dilaton is shifted by a constant, the RR field components that were present in the original background (3.5) are multiplied by a constant, and several new components of $F_3$ and $F_5$ emerge.

- If $(a \leq 8$ and $b \geq 9)$, or $(a \geq 9$ and $b \leq 8)$, then the result is of the ‘complicated’ type. Despite $\epsilon \gamma_\mu \epsilon + \hat{\epsilon} \gamma_\mu \hat{\epsilon}$ is still zero, as required by (2.1), $\epsilon \gamma_\mu \epsilon$ is nonzero in this case. This means that $C$ is not a constant (in our examples $C$ will be a linear complex-valued function of the coordinates transverse to the brane, see below). The dilaton is shifted by a logarithm of this function, the RR fields are scaled by a power of it, and some new components of $F_3$ and $F_5$ appear again, but also the components that were present in the original solution (3.5) get additive terms.

Let us give some explicit examples. As a representative of the ‘simple’ group of transformed D1-branes we will consider the result of the duality with a Killing spinor parameter $e_1 + ie_2$. For a ‘complicated’ class of backgrounds we will use $e_1 + ie_9$. In both cases we have the same metric (3.2) and $B$-field (zero), as in the original D1-brane solution – this is a general property of fermionic T-duality. In the particular case of D1-brane and for Killing spinor combinations of the form (3.12) it turns out that RR scalar is also the same before and after the transformation (zero). Shown below are transformed dilaton and the new RR fields.

### 3.1 ‘Simple’ case

Taking a Killing spinor parameter of the transformation to be

$$e_1 + ie_2 = \left\{ \begin{array}{c} \{1, i, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \} \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -i, 1 \} \end{array} \right\},$$

we get from (2.3)

$$\partial_\mu C = 0, \forall \mu \quad \Rightarrow \quad C = \text{const.}$$

(3.14)
Thus, the dilaton dependence after the duality is
\[ e^{2\phi'} = C \left( 1 + \frac{Q}{(\delta_{mn}x^m x^n)^6} \right). \]  \hspace{1cm} (3.15)

The R-R 3-form has the components (world indices are used everywhere)
\[ (F_3)_{0lm} = -2C^{-1/2}e^{-2\phi} \partial_m \phi \]  \hspace{1cm} (3.16)

(compare to (3.5)) and eight new constant components
\[ F_{236} = i, \quad F_{268} = -1, \quad F_{356} = 1, \quad F_{568} = -i, \]
\[ F_{237} = 1, \quad F_{278} = i, \quad F_{357} = -i, \quad F_{578} = -1. \]  \hspace{1cm} (3.17)

There also appear 16 constant components of the 5-form:
\[ F_{02369} = -i, \quad F_{02689} = 1, \quad F_{03569} = -1, \quad F_{05689} = i, \]
\[ F_{02379} = -1, \quad F_{02789} = -i, \quad F_{03579} = i, \quad F_{05789} = 1, \]  \hspace{1cm} (3.18a)
\[ F_{14578} = -i, \quad F_{13457} = -1, \quad F_{12478} = 1, \quad F_{12347} = i, \]
\[ F_{14568} = 1, \quad F_{13456} = -i, \quad F_{12468} = i, \quad F_{12346} = -1. \]  \hspace{1cm} (3.18b)

Note that the indices in (3.18a) result from appending 0 and 9 to the indices of the 3-form components in (3.17). The components in (3.18b) are required by the self-duality. All the values given in (3.17) and (3.18) must be additionally multiplied by \( 2C^{-3/2} \).

### 3.2 ‘Complicated’ case

As an example of this type of a transformed background let’s take the following linear combination of Killing spinors:
\[ e_1 + i e_9 = \left\{ \begin{array}{l}
\{1, 0, 0, 0, 0, 0, 0, 0, i, 0, 0, 0, 0, 0, 0, 0, 0\} \\
\{0, 0, 0, 0, 0, 0, 0, -i, 0, 0, 0, 0, 0, 0, 0, 1\}
\end{array} \right\}, \]  \hspace{1cm} (3.19)

from which it follows that
\[ \partial_{0,...,7} C = 0, \quad \partial_8 C = -4, \quad \partial_9 C = 4i \quad \Rightarrow \quad C = 4i(x^9 + ix^8). \]  \hspace{1cm} (3.20)
We see that the dilaton is now complex-valued:

\[ e^{2\phi'} = Ce^{2\phi} = 4i(x^9 + ix^8) \left( 1 + \frac{Q}{(\delta_{mn}x^m x^n)^6} \right). \]  

(3.21)

The RR fields transform similarly to the simple case with one important difference: of the eight newly appearing components of the 3-form only six have truly new indices:

\[
F_{278} = i, \quad F_{348} = -i, \quad F_{568} = -i, \\
F_{279} = 1, \quad F_{349} = -1, \quad F_{569} = -1,
\]  

(3.22)

whereas the lacking two appear as additive contributions to the (01\(m\)) components that were present before the transformation:

\[
F_{012} = -2C^{-1/2}e^{-2\phi}\partial_2\phi, \quad \ldots, \quad F_{017} = -2C^{-1/2}e^{-2\phi}\partial_7\phi, \\
F_{018} = -2C^{-1/2}e^{-2\phi}(\partial_8\phi - C^{-1}), \quad F_{019} = -2C^{-1/2}e^{-2\phi}(\partial_9\phi + iC^{-1}).
\]  

(3.23)

Again there are sixteen components of self-dual 5-form field strength. These components, as well as those of the 3-form in (3.22), should be multiplied by \(2C^{-3/2}\):

\[
F_{02368} = 1, \quad F_{02458} = 1, \quad F_{03578} = -1, \quad F_{04678} = 1, \\
F_{02369} = -i, \quad F_{02459} = -i, \quad F_{03579} = i, \quad F_{04679} = -i, \\
F_{14579} = -1, \quad F_{13679} = -1, \quad F_{12469} = 1, \quad F_{12359} = -1, \\
F_{14578} = -i, \quad F_{13678} = -i, \quad F_{12468} = i, \quad F_{12358} = -i.
\]  

(3.24)

### 3.3 Solution checking the fermionic T-dual

We have verified that the transformed backgrounds are indeed solutions to type IIB supergravity equations of motion.

In the so-called, ‘simple’ case, all the equations are trivial apart from the Einstein equation (A.21) which is satisfied by the transformed solution because the RR fields’ energy-momentum tensors change trivially under the transformation – being quadratic in RR field strengths that scale as \(C^{-1/2}\) (3.16), they simply get multiplied by \(C^{-1}\), which is cancelled by the transformation of the dilaton:

\[
\left[ \frac{e^{2\phi} - 1}{2} \left[ T^\mu_\nu^{(1)} + T^\mu_\nu^{(3)} + \frac{1}{2} T^\mu_\nu^{(5)} \right] \right] = \frac{Ce^{2\phi}}{2} \left[ \frac{1}{C} T^\mu_\nu^{(1)} + \frac{1}{C} T^\mu_\nu^{(3)} + \frac{1}{C} \frac{1}{2} T^\mu_\nu^{(5)} \right],
\]  

(3.25)

so that the right-hand side of (A.21) does not change (the left-hand side does not change trivially because the dilaton is shifted by a constant and because the curvature is not...
An interesting question, however, is how it so happens that the new components of the 3- and 5-form do not contribute to the energy-momentum tensor. The reason is an accurate balance of real and imaginary units, scattered around (3.17) and (3.18).

In the so called, ‘complicated’ case, the auxiliary field $C$ in the transformation is no longer constant. As a result the function $C = 4i(x^9 + ix^8)$ (3.20) enters into the expressions for the transformed fields and the verification of most equations is nontrivial.

To gain a flavour of the cancellations involved we will give an example of solving the dilaton field equation (A.16). Using

$$\phi' = \phi + \frac{1}{2} \log C,$$ (3.26)

we calculate

$$\nabla^2 \phi' = \frac{1}{\sqrt{|g|}} \partial_m \left( \sqrt{|g|} g^{mn} \partial_n \phi' \right) = -\frac{e^{-\phi}}{2C^2} \delta^{mn} \left( \partial_m C \partial_n C - 2C \partial_m \phi \partial_n C \right),$$ (3.27)

$$\left( \partial \phi' \right)^2 = e^{-\phi} \delta^{mn} \left( \partial_m \phi \partial_n \phi + \frac{1}{C} \partial_m \phi \partial_n C + \frac{1}{4C^2} \partial_m C \partial_n C \right),$$ (3.28)

where we have taken into account that for the dilaton in the D1-brane background

$$\delta^{mn} \left( \partial_m \partial_n \phi + 2 \partial_m \phi \partial_n \phi \right) \equiv 0,$$ (3.29)

and that the second derivatives of $C$ vanish.

For the function $C = 4i(x^9 + ix^8)$ we get

$$\delta^{mn} \partial_m C \partial_n C = (\partial_8 C)^2 + (\partial_9 C)^2 = 0,$$ (3.30)

$$\delta^{mn} \partial_m \phi \partial_n C = -4(\partial_8 \phi - i \partial_9 \phi),$$ (3.31)

and substituting this into the dilaton field equation (A.16) yields

$$R + 4 \nabla^2 \phi' - 4(\partial \phi')^2 = -5e^{-\phi} \delta^{mn} \partial_m \partial_n \phi - \frac{16e^{-\phi}}{C} (\partial_8 \phi - i \partial_9 \phi)$$

$$\quad -10e^{-\phi} \delta^{mn} \partial_m \phi \partial_n \phi + \frac{16e^{-\phi}}{C} (\partial_8 \phi - i \partial_9 \phi) = 0.$$ (3.32)

All other field equations have been checked and involve many complicated cancellations. Carrying out these checks one obtains a healthy respect for the nontriviality of this duality from the point of view of the supergravity equations of motion.
4 pp-wave

Another type IIB background that is interesting to consider is the pp-wave solution [18]. This is a maximally supersymmetric solution, and so by dualizing it with respect to any of its Killing spinors we can get another maximally supersymmetric background of (complexified) type IIB supergravity.

In our conventions the pp-wave background is given by

\[ ds^2 = 2dx^+dx^- - \lambda^2 \delta_{\mu\nu}x^\mu x^\nu dx^+ dx^+ + \delta_{\mu\nu}dx^\mu dx^\nu, \]  

(4.1a)

\[ F_{+1234} = 4\lambda = F_{+5678} \]  

(4.1b)

(in this section we use the lightcone coordinates \( x^\pm = \frac{1}{\sqrt{2}}(x^9 \pm x^0) \), and \( x^\mu = \{x^1, \ldots, x^8\} \)). This solves the field equations for any constant \( \lambda \): the dilaton equation is \( R = 0 \), which holds for the above metric, and the only nontrivial Einstein equation is \( R_{++} = \frac{1}{4}T_{++} \), which also holds with \( R_{++} = 8\lambda^2 \). All the other equations are trivial due to the vanishing of almost all of the type IIB fields.

The Killing spinors of this background have been derived in [18] and in our notation are given by

\[ \epsilon = (1 - ix^{+4678}) \left( \cos \frac{\lambda x^+}{2} - i \sin \frac{\lambda x^+}{2} \right) \left( \cos \frac{\lambda x^+}{2} - i \sin \frac{\lambda x^+}{2} \right)^\dagger \epsilon_0, \]  

(4.2)

for an arbitrary \( \epsilon_0 \), where \( 1 \) is a 32 \( \times \) 32 unit matrix, \( I = \Gamma_1\Gamma_2\Gamma_3\Gamma_4, J = \Gamma_5\Gamma_6\Gamma_7\Gamma_8 \), and

\[ \mathbb{A}_\mu = \begin{cases} 8\lambda \Gamma_- \Gamma_\mu, & \mu = 1, 2, 3, 4, \\ 8\lambda \Gamma_- J \Gamma_\mu, & \mu = 5, 6, 7, 8. \end{cases} \]  

(4.3)

The formula (4.2) is written in the complex notation for the supersymmetry transformations, see appendix B. Both \( \epsilon \) and \( \epsilon_0 \) are Weyl spinors, i.e. complex, 16-component. Since full 32 by 32 gamma-matrices \( \Gamma_\mu \) are used here, half of the components of \( \epsilon \) and \( \epsilon_0 \) are zero.

In order to get the 32 basis elements \( \{\epsilon_k = (\epsilon_k, \hat{\epsilon}_k)\} \) we first substitute arbitrary complex constants as the components of \( \epsilon_0 \):

\[ (\epsilon_0)_k = \alpha_k + i\beta_k, \quad k \in \{1, \ldots, 16\}, \quad \alpha_k, \beta_k \in \mathbb{R}, \]  

(4.4)

the rest 16 components of \( \epsilon_0 \) being zero. Next we evaluate (4.2) and get 16 complex components of \( \epsilon \). Now, the real and imaginary parts of this Weyl spinor are our Killing
spinors $(\epsilon, \hat{\epsilon})$ in real notation. There are 32 independent pairs $e = (\epsilon, \hat{\epsilon})$, corresponding to the thirty-two real parameters $\alpha_k, \beta_k$.

The basis Killing spinor pairs then fall into two groups, those that depend on $x^+$ only (‘group $A$’), and those that depend on the transverse coordinates $x^1, \ldots, x^8$ (‘group $B$’). We get 16 group $A$ Killing spinors by keeping any of $\alpha_1, \ldots, \alpha_8$ (which we refer to as ‘group $A1$’) or $\beta_1, \ldots, \beta_8$ (‘group $A2$’), while setting all other parameters to zero. Spinors that comprise group $B$ result from keeping any of $\alpha_9, \ldots, \alpha_{16}$ (‘group $B1$’) or $\beta_9, \ldots, \beta_{16}$ (‘group $B2$’).

Not all of these Killing spinors satisfy the constraint (2.1) (or its generalisation (2.8), if one wants to perform multiple fermionic T-dualities). If we pick a pair to construct a complex linear combination $f = e_a + ie_b$ so that $e_a$ and $e_b$ belong to different groups ($A$ and $B$), then the condition (2.1) cannot be satisfied. Thus, necessarily $e_a, e_b \in A$ or $e_a, e_b \in B$. According to the subdivision into subgroups $A1, A2, B1$, and $B2$, there are four quite distinct fermionic T-dual backgrounds:

- $e_a, e_b \in A1$ or $e_a, e_b \in A2$;
- $e_a, e_b \in B1$ or $e_a, e_b \in B2$;
- $e_a \in B1$, $e_b \in B2$, or the other way round;
- $e_a \in A1$, $e_b \in A2$, or the other way round.

The first case is much like the ‘simple’ case of the transformed D1-brane discussed in the section 3.1 above. Namely, $C$ is just a constant, dilaton is shifted by its logarithm and RR 5-form is scaled by its power. Twenty-four new RR field components appear, eight in $F_3$ and sixteen more in $F_5$. These look much like those given in (3.17) and (3.18) multiplied additionally by a sine or a cosine of $2\lambda x^+$. Crucially, these new RR fluxes do not contribute to the stress-energy, precisely as in a D-brane case.

In the second case the transformed background is more complex. It also has constant $C$, and therefore a constant dilaton and a constant scaling factor for the 5-form components. New in this case is that there are four nonvanishing components of RR 1-form, thirty-two components of the 3-form and fifty-six components of the 5-form. All of these look like $\text{const} \cdot (x^\mu + ix^\nu)$ for some $\mu, \nu \in \{1, \ldots, 8\}$. Again, their stress-energy vanishes, so that no modification of the Einstein equations occurs.

The third case is interesting, the defining equation for $C$ is nontrivial. We can proceed however forgetting about the factors of $C$ in all the RR form components. Three points are characteristic of a dual background in this case: there is no 3-form, but all the 1-form
and the 5-form components are nonzero; all of these are either first or (more often) second order polynomials in the transverse coordinates; and they have nonvanishing stress-energy. The Einstein equations are still satisfied due to the nontrivial spacetime dependence of the dilaton, which is proportional to \( \log C \).

We will look in detail at the fourth case. This can be also characterized by nontrivial contribution of the new components to the stress-energy tensor, and a spacetime-dependent dilaton.

## 4.1 Transformed pp-wave

The linear combination of the Killing spinors that we will use is \( f = e_1 + ie_9 \), where \( e_1 \) is what results from keeping only \( \alpha_1 = 1 \) in (4.4) while setting all the other parameters to zero (so this is a group \( A1 \) element), and \( e_9 \) corresponds to \( \beta_1 = 1 \) (group \( A2 \)). Explicitly this has the following form:

\[
f = \begin{cases} 
\{ \cos \lambda x^+, 0, 0, i \sin \lambda x^+, 0, 0, 0, 0, 0, 0, 0, 0, 0 \} \\
\{ i \cos \lambda x^+, 0, 0, -\sin \lambda x^+, 0, 0, 0, 0, 0, 0, 0, 0 \}
\end{cases},
\]

where the first line is \( \epsilon \) and the second line is \( \hat{\epsilon} \). This Killing spinor manifestly satisfies the constraint (2.1), since in this case \( \hat{\epsilon} = i\epsilon \), and thus

\[
\epsilon \gamma_\mu \epsilon + \hat{\epsilon} \gamma_\mu \hat{\epsilon} = \epsilon \gamma_\mu \epsilon - \epsilon \gamma_\mu \epsilon \equiv 0.
\]

The defining equation for \( C \) (2.3) takes the form

\[
\partial_+ C = 2\sqrt{2} i \cos 2\lambda x^+, \quad \Rightarrow \quad C = \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+.
\]

The dilaton now depends on \( x^+ \):

\[
\phi' = \frac{1}{2} \log \left( \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+ \right).
\]

The RR 5-form components that were nonzero in the original background (4.1b) take the values

\[
F_{+1234} = F_{+5678} = 3\lambda \left( \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+ \right)^{-1/2}.
\]
The transformed background also has nonzero RR 1-form

\[ F_+ = -\cos 2\lambda x^+ \left( \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+ \right)^{-3/2} \]  

(4.10)

and the following new components of the 5-form:

\[ F_{+1256} = F_{+1368} = F_{+1458} = F_{+2367} = F_{+2457} = F_{+3478} \]  

(4.11a)

\[ F_{+1236} = F_{+1245} = F_{+3678} = F_{+2567} = F_{+2457} = F_{+3478} \]  

(4.11b)

\[ F_{+1348} = F_{+1568} = F_{+2347} = F_{+2567} = F_{+2457} = F_{+3478} \]  

(4.11c)

\[ F_{+1278} = F_{+1467} = F_{+2358} = F_{+3456} = \cos 2\lambda x^+ \left( \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+ \right)^{-3/2} \]  

(4.11d)

\[ F_{+1357} = F_{+2468} = -\cos 2\lambda x^+ \left( \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+ \right)^{-3/2}. \]  

(4.11e)

The only nonvanishing component of the energy-momentum tensors of these RR fields is the (++) component, and this is readily calculated to give

\[ T^{(1)}_{++} = \frac{i\lambda^3 \cos^2 2\lambda x^+}{\sqrt{2} \sin^3 2\lambda x^+}, \]  

(4.12a)

\[ T^{(5)}_{++} = 15\sqrt{2}i\lambda^3 \cos^2 2\lambda x^+ \sin^3 2\lambda x^+ - 8\sqrt{2}i\lambda^3 \frac{1}{\sin^3 2\lambda x^+}. \]  

(4.12b)

The combination that enters the Einstein equations (A.21) is

\[ \frac{e^{2\phi}}{2} \left( T^{(1)}_{++} + \frac{1}{2} T^{(5)}_{++} \right) = -8\lambda^2 \cos^2 2\lambda x^+ \sin^2 2\lambda x^+ + 4\lambda^2 \frac{1}{\sin^2 2\lambda x^+}. \]  

(4.13)

Recalling that \( R_{++} = 8\lambda^2 \) and calculating the second derivative of the dilaton to be

\[ \nabla_+ \nabla_+ \phi = \partial_+ \partial_+ \phi = -2\lambda^2 \frac{1}{\sin^2 2\lambda x^+}, \]  

(4.14)
we see that the Einstein equation (A.21) is satisfied by the transformed background:

\[ 8\lambda^2 - 4\lambda^2 \frac{1}{\sin^2 2\lambda x^+} + 8\lambda^2 \cos^2 2\lambda x^+ \frac{1}{\sin^2 2\lambda x^+} - 4\lambda^2 \frac{1}{\sin^2 2\lambda x^+} \equiv 0. \quad (4.15) \]

All the other field equations are satisfied trivially.

### 4.2 Purely imaginary fermionic T-dual background

In the previous sections the transformed solutions were all complex. Here we give an example of a solution that one can potentially make sense of within non-complexified supergravity. This is produced by carrying out two independent fermionic T-dualities on the pp-wave. The result of the transformation has purely imaginary RR forms, so that timelike bosonic T-duality [11] will make it real.

We begin by picking a second Killing spinor alongside with the one that has been used in the previous subsection:

\[ f_1 = \begin{cases} \{ \cos \lambda x^+, 0, 0, i \sin \lambda x^+, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \} \\ \{ i \cos \lambda x^+, 0, 0, -\sin \lambda x^+, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \} \end{cases}, \]

\[ f_2 = \begin{cases} \{ i \sin \lambda x^+, 0, 0, \cos \lambda x^+, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \} \\ \{ -\sin \lambda x^+, 0, 0, i \cos \lambda x^+, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \} \end{cases}. \quad (4.16a, 4.16b) \]

The additional Killing spinor is a sum \( f_2 = e_4 + ie_{12} \), where \( e_4 \) is a group A1 Killing spinor defined by \( \alpha_4 = 1 \) in (4.4) while setting all the other parameters to zero, and \( e_{12} \) corresponds to \( \beta_4 = 1 \) (group A2). The pair \( (f_1, f_2) \) can be checked to satisfy (2.8).

The auxiliary function \( C \) is a two by two matrix, defined by (2.7a):

\[ C_{ij} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad (4.17) \]

where

\[ a = \frac{i\sqrt{2}}{\lambda} \sin 2\lambda x^+, \quad (4.18) \]

\[ b = \frac{\sqrt{2}}{\lambda} \cos 2\lambda x^+. \quad (4.19) \]

The matrices \( \log C \) and \( C^{-1} \), which are needed in order to implement the formulae (2.7), have the same structure, but with different values for \( a \) and \( b \). Namely, we have...
for the inverse of $C$

\[ a' = -\frac{i\lambda}{\sqrt{2}} \sin 2\lambda x^+, \quad (4.20) \]

\[ b' = \frac{\lambda}{\sqrt{2}} \cos 2\lambda x^+, \quad (4.21) \]

and for $\log C$:

\[ a'' = \frac{i\pi}{2} + \log \frac{\sqrt{2}}{\lambda}, \quad (4.22) \]

\[ b'' = -\frac{i\pi}{2} + i 2\lambda x^+. \quad (4.23) \]

Using $\log C$ we can calculate the transformed dilaton:

\[ \phi' = \frac{1}{2} \Tr \log C = a'' = \frac{i\pi}{2} + \log \frac{\sqrt{2}}{\lambda}, \quad e^{\phi'} = i\frac{\sqrt{2}}{\lambda}. \quad (4.24) \]

Thus the string coupling is purely imaginary in this background. From this we can already predict, that the transformed background will necessarily have purely imaginary RR flux, so that the sign of the combination $e^{2\phi} F^2$ is invariant.

In order to derive this explicitly we calculate the contribution of the Killing spinors to the RR field strength bispinor, which is represented by the last term in (2.7c):

\[
(\epsilon_i \otimes \hat{\epsilon}_j) (C^{-1})_{ij} = -\frac{i\lambda}{\sqrt{2}} \sin 2\lambda x^+ [\epsilon_1 \otimes \hat{\epsilon}_1 + \epsilon_2 \otimes \hat{\epsilon}_2] + \frac{\lambda}{\sqrt{2}} \cos 2\lambda x^+ [\epsilon_1 \otimes \hat{\epsilon}_2 + \epsilon_2 \otimes \hat{\epsilon}_1],
\]

(4.25)

where $\epsilon_i$ and $\hat{\epsilon}_i$ are explicit components of $f_i = (\epsilon_i, \hat{\epsilon}_i)$. Substituting the values of the Killing spinors as given in (4.16), we arrive at the following background, which is indeed purely imaginary:

\[ F_{+1234} = F_{+5678} = -i\lambda^2 \sqrt{2} \]

\[ F_{+1256} = F_{+1368} = F_{+1458} = F_{+2367} = F_{+2457} = F_{+3478} = i\lambda^2 \sqrt{2}. \quad (4.26a) \]

\[ F_{+1256} = F_{+1368} = F_{+1458} = F_{+2367} = F_{+2457} = F_{+3478} = i\lambda^2 \sqrt{2}. \quad (4.26b) \]

All other components of RR forms vanish. This background clearly satisfies Einstein equations, because

\[ R_{++} + 2\nabla_+ \nabla_+ \phi' - \frac{e^{2\phi'}}{4} T^{(5)}_{++} = 8\lambda^2 - \frac{1}{4} \left( -\frac{2}{\lambda^2} \right) \left( -16\lambda^4 \right) \equiv 0. \quad (4.27) \]
4.3 Self-duality of pp-wave

We shall now show that the pp-wave background is self-dual under the fermionic T-duality with respect to eight supersymmetries that we denote by \( \{ f_1, \ldots, f_8 \} \). Corresponding Killing spinors are all of the same form as those used to demonstrate how a single or double T-duality is done in the two previous subsections. Namely, recapitulating the discussion after (4.4), we pick sixteen real Killing spinors \( \{ e_1, \ldots, e_8 \} \in A_1, \{ e_9, \ldots, e_{16} \} \in A_2 \). Then the eight complex Killing spinors, satisfying (2.8), are given by

\[
 f_i = e_i + i e_{i+8}, \quad i \in \{ 1, \ldots, 8 \}. \tag{4.28}
\]

In particular, \( f_1 \) is exactly the same as \( f \) that was used in section 4.1 and was given by (4.5).

With this choice of supersymmetries we get the following matrix \( C \):

\[
 C = \begin{pmatrix}
 a & 0 & 0 & b \\
 0 & a & -b & 0 \\
 0 & -b & a & 0 \\
 b & 0 & 0 & a
\end{pmatrix}, \tag{4.29}
\]

where \( a \) and \( b \) are the same as in the previous subsection:

\[
 a = \frac{i \sqrt{2}}{\lambda} \sin 2\lambda x^+, \tag{4.30}
\]

\[
 b = \frac{\sqrt{2}}{\lambda} \cos 2\lambda x^+. \tag{4.31}
\]

The matrices \( \log C \) and \( C^{-1} \) again have the same structure, but with different values for \( a \) and \( b \), which coincide with those given in the previous subsection, see eqs. (4.20) to (4.23).

The transformed dilaton is then evaluated to be

\[
 \phi' = 4a'' = 2\pi i + 4 \log \frac{\sqrt{2}}{\lambda}, \quad e^{\phi'} = \frac{4}{\lambda^4}. \tag{4.32}
\]
\[ (\epsilon_i \otimes \hat{\epsilon}_j) (C^{-1})_{ij} = -\frac{i\lambda}{\sqrt{2}} \sin 2\lambda x^+ [\epsilon_1 \otimes \hat{\epsilon}_1 + \ldots + \epsilon_8 \otimes \hat{\epsilon}_8] \]
\[ + \frac{\lambda}{\sqrt{2}} \cos 2\lambda x^+ \left[ \epsilon_1 \otimes \hat{\epsilon}_4 + \epsilon_4 \otimes \hat{\epsilon}_1 - \epsilon_2 \otimes \hat{\epsilon}_3 - \epsilon_3 \otimes \hat{\epsilon}_2 - \epsilon_5 \otimes \hat{\epsilon}_8 - \epsilon_8 \otimes \hat{\epsilon}_5 + \epsilon_6 \otimes \hat{\epsilon}_7 + \epsilon_7 \otimes \hat{\epsilon}_6 \right], \quad (4.33) \]

where \((\epsilon_i, \hat{\epsilon}_i) = f_i\).

An important feature of this matrix, which becomes obvious only after explicit substitution of the Killing spinors, is that it is proportional to the first term on the right-hand side of (2.7c). This leads to the RR field bispinor after the transformation being proportional to itself before the transformation. More precisely, we have for the transformed RR background

\[ F_{+1234} = -\lambda^5 = F_{+5678}, \quad (4.34) \]

with all other components vanishing. This is just the original flux that was supporting the pp-wave geometry before we have done fermionic T-duality, but multiplied by a constant \(-\lambda^4\). Since this constant is equal to \(-e^{-\phi'}\) (4.32), the Einstein equations hold for the new background because they involve a product \(e^{2\phi'} T^{(5)}_{\mu\nu}\):

\[ R_{++} + 2\nabla^+ \nabla_+ \phi' - \frac{e^{2\phi'}}{4} T^{(5)}_{++} = 8\lambda^2 - \frac{1}{4} \left( \frac{4}{\lambda^4} \right)^2 (\lambda^{10} + \lambda^{10}) \equiv 0. \quad (4.35) \]

This transformation clearly leaves the string spectrum invariant since it is just a field redefinition of the Ramond-Ramond field strength.

Interestingly, if one splits the eight supersymmetries that were used in this section into two groups \(\{f_1, \ldots, f_4\}\) and \(\{f_5, \ldots, f_8\}\) and performs fermionic T-dualities of the original pp-wave background with respect to each of these groups independently, then the resulting background has the dilaton \(e^{\phi'} = \frac{2}{\lambda^2}\) in both cases, and the RR forms in the two cases are given by

\[ F_{+1458} = F_{+2367} = \pm 2\lambda^3. \quad (4.36) \]

Thus each group of four fermionic T-dualities also results in a pp-wave background that has undergone a certain rotation in transverse directions as compared to the original pp-wave.
5 Discussion

Fermionic T-duality has many interesting properties, some of them are quite unexpected. First, one should note that fermionic T-duality does not commute with bosonic T-duality. This is easily seen with the D1-brane case where new Ramond-Ramond fields are produced that break the $SO(1, 1) \times SO(8)$ symmetry of the original D1-brane solution. In retrospect this should not be a surprise since it is known that supersymmetries and isometries do not commute either. One can also think of examples where T-duality breaks supersymmetry (at the level of supergravity).

We have also checked to see whether fermionic T-duality is nilpotent and we see that it is not always so. In the examples carried out above the transformation is only nilpotent up to a root of unity. This is undoubtedly a consequence of the fermionic nature of the transformation.

One of the main goals of this paper was to find transformations to real solutions. This has been successful in that we have shown that the pp-wave can be transformed to produce real solutions but in that case the transformed solution is again the pp-wave up to some field redefinitions or rotations.

It seems somewhat distant at this point to be able to know when a real solution is possible and what the new solutions will be. We intend to pursue this question in further work.

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A Type IIB supergravity action and equations of motion

We will give the relevant action and equations of motion for IIB supergravity so that all our conventions are transparent. Our metric signature is mostly plus, $(- + \ldots +)$; antisymmetric Levi-Civita tensor is defined with $\epsilon_{0 \ldots 9} = 1$. Apart from the metric, which
is represented by \( g_{\mu\nu} \), the bosonic field content of type IIB supergravity is given by two real scalars, dilaton \( \phi \) and RR scalar \( C_0 \), two real antisymmetric second-rank tensors \( B \) and \( C_2 \) and a fourth-rank real tensor \( C_4 \), whose field strength \( F_5 = dC_4 \) is self-dual:

\[
F_{\mu_1...\mu_5} = \frac{1}{5!} \varepsilon_{\mu_1...\mu_5\nu_1...\nu_5} F^{\nu_1...\nu_5}.
\]  

From string theory point of view, the fields \( C_0, C_2, \) and \( C_4 \) are potentials of the RR fields \( F_{n+1} = dC_n \). Three remaining fields \( g, B, \) and \( \phi \) belong to the NSNS sector of type IIB superstring.

The action of type IIB supergravity in the string frame is a sum of three terms

\[
S = S_{\text{NSNS}} + S_{\text{RR}} + S_{\text{CS}},
\]

where

\[
S_{\text{NSNS}} = \frac{1}{2\kappa^2} \int d^{10} x \sqrt{|g|} e^{-2\phi} \left[ R + 4(\partial\phi)^2 - \frac{1}{2\cdot 3!} H_3^2 \right],
\]

\[
S_{\text{RR}} = -\frac{1}{4\kappa^2} \int d^{10} x \sqrt{|g|} \left[ F_3^2 + \frac{1}{3!} \tilde{F}_3^2 + \frac{1}{2\cdot 5!} \tilde{F}_5^2 \right],
\]

\[
S_{\text{CS}} = -\frac{1}{4\kappa^2} \int C_4 \wedge H_3 \wedge F_3.
\]

Here \( H_3 = dB_2 \) is the field strength of the NSNS antisymmetric tensor field, and we use a common notation \( F_n^2 = F_{\mu_1...\mu_n} F^{\nu_1...\nu_n} g^{\mu_1\nu_1} ... g^{\mu_n\nu_n} \). Modified field strengths \( \tilde{F}_n \) are used in \( S_{\text{RR}} \), and only there:

\[
\tilde{F}_3 = F_3 - C_0 H_3,
\]

\[
\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3.
\]

Note that these reduce to ordinary \( F_n \) if the \( B \)-field is zero.

The equations of motion of the two scalars in the theory (A.2) are the simplest. The dilaton equation reads

\[
R = 4(\partial\phi)^2 - 4\nabla^2 \phi + \frac{1}{2} \frac{H_3^2}{2},
\]

and the RR scalar field equation is

\[
\nabla^2 C_0 + \frac{1}{3!} H_3 \tilde{F}_3 = 0.
\]  

The equations for \( B_2, C_2, \) and \( C_4 \) are respectively (note that the first two equations
have been simplified somewhat by substitution of the third one):

\[
\nabla_\mu \left[ e^{-2\phi} H - C_9 \tilde{F} \right]^{\alpha\beta\mu} = \frac{1}{2} \frac{1}{3!} \tilde{F}^{\alpha\beta\mu\nu\lambda} F_{\mu\nu\lambda} - \frac{1}{2} \frac{1}{5!} \frac{1}{3!} \epsilon^{\alpha\beta\mu_1...\mu_5\nu_1...\nu_3} \tilde{F}_{\mu_1...\mu_5} F_{\nu_1...\nu_3}; \quad (A.9)
\]

\[
\nabla_\mu \tilde{F}^{\alpha\beta\mu} = - \frac{1}{2} \frac{1}{3!} \tilde{F}^{\alpha\beta\mu\nu\lambda} H_{\mu\nu\lambda} + \frac{1}{2} \frac{1}{5!} \frac{1}{3!} \epsilon^{\alpha\beta\mu_1...\mu_5\nu_1...\nu_3} \tilde{F}_{\mu_1...\mu_5} H_{\nu_1...\nu_3}; \quad (A.10)
\]

\[
\nabla_\mu \tilde{F}^{\mu\nu_1...\nu_4} = \frac{1}{\sqrt{|g|}} \frac{1}{3!} \frac{1}{3!} \epsilon^{\nu_1...\nu_4\lambda_1...\lambda_3\rho_1...\rho_3} H_{\lambda_1...\lambda_3} F_{\rho_1...\rho_3}. \quad (A.11)
\]

Finally the Einstein equations, after simplifying by substitution of the Ricci scalar as given by the dilaton equation (A.7) are:

\[
R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi = \frac{1}{4} H_{\mu\alpha\beta} H_\nu^{\alpha\beta} + \frac{e^{2\phi}}{2} \left[ T^{(1)}_{\mu\nu} + T^{(3)}_{\mu\nu} + \frac{1}{2} T^{(5)}_{\mu\nu} \right], \quad (A.12)
\]

where

\[
T^{(1)}_{\mu\nu} = \partial_\mu C \partial_\nu C - \frac{1}{2} g_{\mu\nu} (\partial C)^2, \quad (A.13)
\]

\[
T^{(3)}_{\mu\nu} = \frac{1}{2} \tilde{F}_{\mu\alpha\beta} F^{\alpha\beta}_\nu - \frac{1}{2} g_{\mu\nu} \frac{1}{3!} \tilde{F}_3^2, \quad (A.14)
\]

\[
T^{(5)}_{\mu\nu} = \frac{1}{4!} \tilde{F}_{\mu\alpha_1...\alpha_4} F^{\alpha_1...\alpha_4}_\nu. \quad (A.15)
\]

(the \(\tilde{F}_5^2\) term in the 5-form energy-momentum is identically zero since \(\tilde{F}_5 = \star \tilde{F}_5\)).

The supergravity field equations, which we have derived here, simplify considerably in the case of zero \(B\)-field, as is relevant for D-brane solutions. For the dilaton, RR scalar,
\( B_2, C_2, C_4, \) and \( g \) we have correspondingly

\[
R = 4(\partial \phi)^2 - 4\nabla^2 \phi, \tag{A.16}
\]
\[
\nabla^2 C_0 = 0, \tag{A.17}
\]
\[
\nabla_\mu (C_0 F)^{\alpha \beta \mu} = -\frac{1}{12} \epsilon^{\alpha \beta \mu \nu \mu_1 \nu_1 \nu_3} F_{\mu_1 \mu_2} F_{\nu_1 \nu_3}, \tag{A.18}
\]
\[
\nabla_\mu F^{\alpha \beta \mu} = 0, \tag{A.19}
\]
\[
\nabla_\mu F^{\mu \nu_1 \nu_4} = 0, \tag{A.20}
\]
\[
R_{\mu \nu} + 2\nabla_\mu \nabla_\nu \phi = \frac{e^{2\phi}}{2} \left[ T^{(1)}_{\mu \nu} + T^{(3)}_{\mu \nu} + \frac{1}{2} T^{(5)}_{\mu \nu} \right]. \tag{A.21}
\]

\section*{B Gamma-matrices and supersymmetry transformations}

We work with the real 32 by 32 representation for the gamma-matrices of \((9 + 1)\)-dimensional spacetime, that exist due to the isomorphism \( Cl(9, 1) \cong \text{Mat}(\mathbb{R}, 32) \). It is convenient to exploit the periodicity property of the Clifford algebras

\[
Cl(9, 1) \cong Cl(1, 1) \otimes Cl(8, 0) \tag{B.1}
\]

and construct the gamma-matrices as tensor products of \( \{\sigma_1, i\sigma_2\} \), which are the gammatrices of \( Cl(1, 1) \) with the following symmetric \( \{\Sigma_1, \ldots, \Sigma_8\} \), which are the gamma-matrices of 8-dimensional Euclidean space:

\[
\Sigma^1 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2,
\]
\[
\Sigma^2 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_2,
\]
\[
\Sigma^3 = \sigma_2 \otimes 1 \otimes \sigma_3 \otimes \sigma_2,
\]
\[
\Sigma^4 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes 1,
\]
\[
\Sigma^5 = \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes 1,
\]
\[
\Sigma^6 = \sigma_2 \otimes \sigma_2 \otimes 1 \otimes \sigma_1,
\]
\[
\Sigma^7 = \sigma_2 \otimes 1 \otimes 1 \otimes \sigma_3,
\]
\[
\Sigma^8 = \sigma_1 \otimes 1 \otimes 1 \otimes 1. \tag{B.2}
\]
\[ \Sigma^9 = \Sigma^1 \cdots \Sigma^8 = \sigma_3 \otimes 1 \otimes 1 \otimes 1, \] which is a chirality operator in 8D. In particular, the representation we use is:

\[
\Gamma^0 = i\sigma_2 \otimes 1_{16} = \begin{pmatrix} 0 & 1_{16} \\ -1_{16} & 0 \end{pmatrix}, \quad (\Gamma^0)^2 = -1; \\
\Gamma^i = \sigma_1 \otimes \Sigma^i = \begin{pmatrix} 0 & \Sigma^i \\ \Sigma^i & 0 \end{pmatrix}, \quad (\Gamma^i)^2 = 1.
\] (B.3)

The 10-dimensional chirality operator is \( \Gamma^{10} = \Gamma^0 \cdots \Gamma^9 = \sigma_3 \otimes 1_{16} \). Spinors of definite chirality are defined as usual, \( \Gamma^{10} \psi^\pm = \pm \psi^\pm \); they provide two inequivalent real 16-dimensional representations of \( \text{Spin}(9,1) \), \( S_+ \) and \( S_- \). These are Majorana-Weyl spinors; we can also define \( S_+ \oplus S_- \), which is a Majorana spinor (real 32 component) and \( S_+ \otimes \mathbb{C} \ (S_- \otimes \mathbb{C}) \), which are Weyl spinors (complex 16 component) of positive (negative) chirality.

The \( \gamma^\mu \) matrices, which are used throughout the paper, are defined as off-diagonal 16 by 16 blocks of the \( \Gamma^\mu \) matrices:

\[
\Gamma^\mu = \begin{pmatrix} 0 & \gamma^{\mu\alpha\beta} \\ \gamma_{\alpha\beta} & 0 \end{pmatrix},
\] (B.4)

so that they are analogs of Pauli matrices in 4D. Explicitly

\[
\gamma^{\mu\alpha\beta} = (1, \Sigma^i), \\
\gamma_{\alpha\beta} = (-1, \Sigma^i).
\] (B.5a, B.5b)

The \( \gamma^\mu \) matrices are symmetric and they satisfy a condition

\[
\gamma^\mu_{\alpha\beta} \gamma^{\nu\beta\gamma} + \gamma^\nu_{\alpha\beta} \gamma^{\mu\beta\gamma} = 2\eta^{\mu\nu} \delta^\gamma_\alpha.
\] (B.6)

Position of the spinor indices reflects the convention to denote the positive chirality spinors with \( \psi^\alpha \) and the negative chirality spinors with \( \chi_\alpha \). For example, action of a gamma-matrix on a Majorana spinor is given by

\[
\Gamma^\mu \Psi = \begin{pmatrix} 0 & \gamma^{\mu\alpha\beta} \\ \gamma_{\alpha\beta} & 0 \end{pmatrix} \begin{pmatrix} \psi^\beta \\ \chi_\beta \end{pmatrix} = \begin{pmatrix} (\gamma^\mu \chi)^\alpha \\ (\gamma^\mu \psi)_\alpha \end{pmatrix},
\] (B.7)

and action on chiral (Majorana-Weyl or Weyl) spinors can be written by setting \( \psi \) or \( \chi \) to zero.
Since a charge conjugation matrix in this representation can be taken to be $C = \Gamma^0$:

$$CT^i C^{-1} = -\Gamma^T,$$  \hspace{1cm} \text{(B.8)}

the Lorentz-covariant bilinear takes the form (using Majorana conjugation $\bar{\Psi} = \Psi^T C$):

$$\bar{\Psi} \Gamma^\mu \Phi = \left( \begin{array}{cc} \psi^\alpha & \chi_\alpha \\ 0 & -1^{\alpha \beta} \\ \end{array} \right) \left( \begin{array}{cc} 0 & \gamma_\beta \gamma_\gamma \\ \gamma^{\mu \beta \gamma} & 0 \\ \end{array} \right) \left( \begin{array}{c} \phi_\gamma \\ \varphi_\gamma \\ \end{array} \right) =$$

$$= \psi^\alpha \gamma^{\mu \alpha \beta} \phi_\beta - \chi_\alpha \gamma^{\mu \alpha \beta} \varphi_\beta.$$  \hspace{1cm} \text{(B.9)}

For chiral spinors, such as the supersymmetry parameters of IIB supergravity, this bilinear reduces to $\psi^\alpha \gamma^{\mu \alpha \beta} \phi_\beta$ (in the case of positive chirality). This type of 16-component spinor bilinear is used, e.g. in the formula (2.1).

Killing spinor equations result from requiring that the supersymmetry variations of the fermions vanish. The fermions in type IIB supergravity are the doublets of gravitini and dilatini, which have opposite chirality. We take the dilatini $\lambda, \hat{\lambda}$ to have negative chirality. The supersymmetry parameters $\epsilon, \hat{\epsilon}$ are of the same (positive) chirality as the gravitini $\psi_{\mu}, \hat{\psi}_{\mu}$. Supersymmetry variations in the two-component formalism are:

$$\delta \psi_{\mu} = \nabla_{\mu} \epsilon - \frac{1}{4} \bar{H}_{\mu} \epsilon - \frac{e^\phi}{8} \left( \bar{F}_1 + \bar{F}_3 + \frac{1}{2} \bar{F}_5 \right) \Gamma_{\mu} \hat{\epsilon},$$  \hspace{1cm} \text{(B.10)}

$$\delta \hat{\psi}_{\mu} = \nabla_{\mu} \hat{\epsilon} + \frac{1}{4} \bar{H}_{\mu} \hat{\epsilon} + \frac{e^\phi}{8} \left( \bar{F}_1 - \bar{F}_3 + \frac{1}{2} \bar{F}_5 \right) \Gamma_{\mu} \epsilon,$$  \hspace{1cm} \text{(B.11)}

$$\delta \lambda_{\mu} = \phi \phi \epsilon - \frac{1}{2} \bar{H} \epsilon + \frac{e^\phi}{2} \left( 2 \bar{F}_1 + \bar{F}_3 \right) \hat{\epsilon},$$  \hspace{1cm} \text{(B.12)}

$$\delta \hat{\lambda}_{\mu} = \phi \phi \hat{\epsilon} + \frac{1}{2} \bar{H} \hat{\epsilon} - \frac{e^\phi}{2} \left( 2 \bar{F}_1 - \bar{F}_3 \right) \epsilon,$$  \hspace{1cm} \text{(B.13)}

where

$$\bar{F}_n = \frac{1}{n!} F_{\mu_1 \ldots \mu_n} \Gamma^{\mu_1 \ldots \mu_n},$$  \hspace{1cm} \text{(B.14)}

$$\bar{H}_{\mu} = \frac{1}{2} H_{\mu \nu \rho} \Gamma^{\nu \rho}.$$  \hspace{1cm} \text{(B.15)}

Sometimes it is more convenient to derive and solve the Killing spinor equations in terms of the single complex gravitino, dilatino and supersymmetry parameter, defined as

$$\Psi_{\mu} = \psi_{m} + i \hat{\psi}_{\mu}, \quad \Lambda = \lambda + i \hat{\lambda}, \quad \epsilon = \epsilon + i \hat{\epsilon}.$$  \hspace{1cm} \text{(B.16)}
The above transformations can be rewritten in the complex notation as

\[
\delta \Psi_\mu = \nabla_\mu \varepsilon - \frac{1}{4} \hat{F}_{\mu} \varepsilon^* + \frac{i e^\phi}{8} \left( \hat{F}_1 + \frac{1}{2} \hat{F}_5 \right) \Gamma_\mu \varepsilon - \frac{i e^\phi}{8} \hat{F}_3 \Gamma_\mu \varepsilon^*, \tag{B.17}
\]

\[
\delta \Lambda = \bar{\phi} \phi \varepsilon - \frac{1}{2} \hat{F}_1 \varepsilon^* - i e^\phi \hat{F}_1 \varepsilon + \frac{i e^\phi}{2} \hat{F}_3 \varepsilon^*. \tag{B.18}
\]

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