A CDH APPROACH TO ZERO-CYCLES ON SINGULAR VARIETIES

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Abstract. We study the Chow group of zero-cycles on singular varieties from the perspective of the cdh topology and $KH$-theory. We define the cdh versions of the Chow groups of zero-cycles and albanese maps and formulate conjectures about the Roitman torsion theorem and “finite-dimensionality” of the Chow group in this setup. We prove these conjectures for surfaces. We compare the cdh analogue of the Chow group of zero-cycles on a projective variety with the Chow group of its desingularization. This is used to prove a finite-dimensionality result for this version of the Chow group for a normal projective threefold.

In order to apply the cdh techniques to study the known Chow group of zero-cycles $CH^d(X)$ on a singular scheme $X$ of dimension $d$, we show under certain cohomological conditions that there is a natural injection $CH^d(X) \hookrightarrow F^dKH_0(X)$ up to torsion, if $X$ has only isolated singularities. This reduces the finite-dimensionality problem for $CH^d(X)$ to its cdh analogue.

As a byproduct of this, we prove some finite-dimensionality result for the Chow group of zero-cycles $CH^3(X)$ on a threefold $X$ with isolated singularities, assuming the same for smooth threefolds.

1. Introduction

The purpose of this work is to understand the Chow groups of zero-dimensional algebraic cycles on singular varieties from the perspective of cdh topology on schemes, which was developed by Voevodsky [52] in his study of the triangulated category of motives and motivic cohomology. The Chow groups of algebraic cycles on smooth varieties have been studied for a long time and they were first used in a nontrivial way by Grothendieck, when he proved the Riemann-Roch theorem for the $K$-group of algebraic vector bundles on such varieties. Ever since, these groups of algebraic cycles on smooth varieties and the Abel-Jacobi maps from them into the intermediate Jacobians have have gone through a phenomenal developments which one hopes, would culminate in the proof of the famous general Hodge conjecture regarding them. One also knows now that the Chow groups of algebraic cycles and their generalization by Bloch [6] in terms of the higher Chow groups provide a theory of motivic cohomology, which describes the algebraic $K$-theory of vector bundles on smooth varieties.

On the other hand, almost nothing is known about the existence of such a theory of the Chow groups of algebraic cycles on singular varieties. It still looks a difficult question whether there is a motivic cohomology theory which could describe the algebraic $K$-theory on the Grothendieck group of vector bundles on a singular scheme $X$ of dimension $d$, Levine and Weibel [42] invented the Chow group of zero cycles, often denoted by $CH^d(X)$.

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This group has since been extensively studied by Levine, Srinivas and many others. However, except for the case of curves (cf. [42]) and normal surfaces (cf. [37]), this Chow group is still far from being fully understood, although many properties, such as the singular analogue of the Roitman’s torsion theorem and Mumford’s infinite-dimensionality theorem have been now been proven (cf. [41], [5], [18], [19], [21]). In higher dimensions, a formula for Chow groups of zero cycles on a variety $X$ with only Cohen-Macaulay isolated singularities in terms of the analogous group for a resolution of singularities of $X$ was recently obtained in [35] (see also [33]). The main motivation behind the lookout for such a formula is the hope that many questions about the Chow groups of zero cycles on such singular varieties could be reduced to answering the similar questions about the zero cycles on smooth varieties, which are supposedly better understood.

One such question, which is the prime focus of this paper on the application side, is the singular analogue of the generalized Bloch conjecture. To state this, recall that if $X$ is a smooth projective variety of dimension $d$ over an algebraically closed field $k$, then the natural map from the variety to its albanese variety defines a surjective group homomorphism

$$c_{0,X} : CH^d(X)_{\text{deg}0} \to J^d(X),$$

where $J^d(X)$ is the albanese variety of $X$. In fact, one knows that $J^d(X)$ is a universal regular quotient of $CH^d(X)_{\text{deg}0}$. One says that $CH^d(X)$ is finite-dimensional, if the map $c_{0,X}$ is an isomorphism, which means that the “connected component” of $CH^d(X)$ is parameterized by a projective variety. It was conjectured by Bloch [7] that a smooth projective surface $X$ with $H^2_{\text{Zar}}(X, O_X) = 0$, is finite-dimensional. This conjecture is open till date though it has been proven for surfaces which are not of general type [9]. The following is the generalized form of Bloch’s conjecture in higher dimension.

**Conjecture 1.1. (GBC)** Let $X$ be a smooth projective variety of dimension $d$ over $k$ such that $H^i_{\text{Zar}}(X, \Omega^i_X) = 0$ for $0 \leq i \leq d - 2$. Then $CH^d(X)$ is finite-dimensional.

We refer the reader to [54] for more detail on this conjecture and its relations with several outstanding conjectures on algebraic cycles. We shall say that $GBC(X)$ holds if the generalized Bloch conjecture holds for the variety $X$. For $d \geq 2$, we shall say that $GBC(d)$ holds, if this conjecture holds for all smooth projective varieties of dimension up to $d$.

Now suppose $X$ is a singular projective variety. It is shown in [21] that there is a smooth commutative algebraic group $J^d_*(X)$ over $k$ and a surjective group homomorphism

$$c_{0,X} : CH^d(X)_{\text{deg}0} \to J^d_*(X),$$

which is a universal regular quotient in the category of regular maps from $CH^d(X)_{\text{deg}0}$ to smooth commutative algebraic groups. The map $c_{0,X}$ is called the albanese map. As in the case of smooth varieties, one says that $CH^d(X)$ is finite-dimensional if the albanese map is an isomorphism. One has then the following singular analogue of the generalized Bloch conjecture.

**Conjecture 1.2.** Let $X$ be a projective variety of dimension $d$ over $k$ such that $H^i_{\text{Zar}}(X, \Omega^i_X) = 0$ for $0 \leq i \leq d - 2$. Then $CH^d(X)$ is finite-dimensional.

In the current work, we propose a new direction for studying the Chow group of zero-cycles on singular varieties. This involves approaching the problem from the perspective of the cdh topology on singular schemes. This topology was invented by Voevodsky [52] in his study of motives over a field of characteristic zero.
This topology although does not do anything extraordinary on smooth schemes, it has proven to be of immense help in solving many $K$-theoretic problems about singular schemes which are otherwise known for smooth schemes. This has been ubiquitously used for example, in [11], [12] and [15], to solve some very important problems on the algebraic $K$-theory of singular schemes.

This work is an attempt to study the questions about algebraic cycles on singular schemes by relating them with the homotopy invariant $K$-theory and the cdh cohomology of the $K$-theory sheaves. One of the main tools for making this program work is the recent result [12, Theorem 1.6], which roughly speaking, estimates the difference between the algebraic $K$-theory and the homotopy $K$-theory of singular schemes in terms of their Hochschild and cyclic homology. Needless to say, these Hochschild and cyclic homology are much simpler objects to deal with. The hope is that this, together with many known results about the $KH$-theory, will give a new direction in constructing the motivic cohomology of singular schemes.

We now describe the various results of this paper in some detail. Our results are based on exploiting the relation between the Zariski and the cdh cohomology of quasi-coherent and $K$-sheaves. Towards this, we describe the Zariski cohomology of the Kähler differentials on normal crossing schemes in terms of their cdh cohomology in Section 3. As mentioned above, the difference between the algebraic and the homotopy $K$-groups can be measured in terms of the cyclic homology of schemes. In Section 4, we describe the relevant part of the cyclic homology of schemes with isolated singularities, in terms of the cdh cohomology of the Kähler differentials. This is done by using the resolution of singularities, the cyclic homology of smooth schemes and the computations of these groups for normal crossing schemes.

In Section 5, we study various filtrations on the algebraic $K$-theory of singular schemes. In order to understand the behavior of these filtrations under the push-forward map on the algebraic $K$-theory, we establish a Riemann-Roch theorem (cf. Theorem 5.3) for the local complete intersection morphism of singular schemes. This is an extension of the similar result of Grothendieck et al. for $K_0$ to higher $K$-theory. The proof involves exploiting the axiomatic approach of Fulton and Lang [22] to the Grothendieck’s Riemann-Roch theorem, and the study of filtrations on higher $K$-theory by Soulé [50]. This is then used to show in Corollary 5.7 that on a scheme with only isolated singularities, the Chow group of zero-cycles coincides with the lowest level of the gamma filtration on $K_0$. The main result here is Theorem 5.8, where we show that under certain vanishing of the cohomology of differential forms, the Chow group of zero-cycles can be embedded inside the lowest level of the Brown filtration on $KH_0$. This brings the study of the known Chow group of zero-cycles to the cdh world.

The Chow group of algebraic cycles on schemes are often studied via the Abel-Jacobi maps from these groups to the various intermediate Jacobians. In order to facilitate this technique to work through in the cdh approach, we study these objects in Section 6. We interpret the Hodge filtration on the analytic cohomology of singular complex varieties in terms of the cdh cohomology of differential forms. This follows from a cdh-descent theorem for the Du Bois complex on these schemes.

In Section 11, we construct the theory of Chern classes on the $KH$-theory with values in the Deligne cohomology. These Chern classes are compatible with the similar Chern classes on the algebraic $K$-theory. This construction is achieved by comparing the $KH$-theory with the descent $K$-theory of [45]. We use these Chern classes to define the cdh version of the albanese maps of (12). This facilitates the formulation of two conjectures about the cdh analogues of the Roitman torsion theorem and the finite-dimensionality problem for the Chow group of zero-cycles.

We prove our main result about the cdh version of the Chow group of zero-cycles in Theorem 11.2. As an application, we prove Conjectures 8.9 and 8.10 for surfaces in Section 12. As further applications, we prove the finite-dimensionality results
Theorem [13.2] and Corollary [13.3] for the Chow groups of zero-cycles on normal projective three-folds in Section [13].

We end this section with few comments. As said before, this work is an attempt to study the Chow group of algebraic cycles on singular schemes using the cdh topology on such schemes. In most of the proofs, we have put extra conditions on the nature of the singularity, which are seemingly needless. The results here lead one to many other related questions on the Chow groups of singular schemes. The applications in this paper are mostly focussed on the finite-dimensionality question for the Chow group of zero-cycles and its cdh analogue. In the sequel [36], we exploit and refine the techniques of this paper to obtain generalizations of Theorem [5.8] for varieties with arbitrary singularities. This in turn will be used to obtain formulas for the known Chow group of zero-cycles in terms of the Chow group of a resolution of singularities.

2. Recollections and preliminary results

In this paper, we fix once and for all, a ground field $k$ of characteristic zero. We shall assume $k$ to be algebraically closed although many of the results do not need this extra assumption, as the reader will observe. We make this assumption mainly because our applications will be over the field $\mathbb{C}$ of complex numbers. Let $\text{Sch}/k$ denote the category of all separated schemes of finite type over $k$ and let $\text{Sm}/k$ denote the category of smooth schemes of finite type over $k$. For $X \in \text{Sch}/k$, $X^N$ will often denote the normalization of $X_{\text{red}}$. Recall from [28] that an abstract blow-up is a Cartesian square

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{i} & X
\end{array}
\]

in $\text{Sch}/k$ where $i$ is a closed immersion, $f$ is proper and $(X' – Y')_{\text{red}} \to (X – Y)_{\text{red}}$ is an isomorphism. An elementary Nisnevich square is a Cartesian square (2.1) where $f$ is étale, $i$ is an open immersion and $(X' – Y') \to (X – Y)$ is an isomorphism. The cdh topology on $\text{Sch}/k$ is generated by covers \{ $X' \to X$ \} given by the above abstract blow-ups and elementary Nisnevich squares.

For a presheaf of spectra $\mathcal{E}$ on $\text{Sch}/k$, let $\mathbb{H}_{\text{cdh}}(-, \mathcal{E})$ denote its fibrant replacement in the cdh topology. We shall denote the nonconnective $K$-theory spectra and its homotopy invariant counterpart (cf. [55]) on $\text{Sch}/k$ by $K$ and $KH$ respectively. For a subfield $F \subseteq k$, let $\mathcal{H}H(/F)$ and $\mathcal{H}C(/F)$ denote the presheaves of Eilenberg-Mac Lane spectra of Hochschild and cyclic homology respectively on $\text{Sch}/k$ (cf. [11] Section 2) over $F$. If the field $F$ is not indicated, we shall assume these homology to be taken over the field $\mathbb{Q}$ of rational numbers. For any $X$ in $\text{Sch}/k$, we let $HH_*(X/F)$ to be $\pi_0 \mathcal{H}H(X/F)$. One defines the cyclic homology of schemes in a similar way. We shall also write $\pi_i K(X)$ (resp $\pi_i KH(X)$) as $K_i(X)$ (resp $KH_i(X)$). We recall from [12] Proposition 2.1 that the Hochschild, cyclic homology and their cdh-fibrant replacements have a gamma filtrations (Hodge filtration) which give the canonical $\lambda$-decompositions

\[
\begin{align*}
HH_n(X/F) & \cong \bigoplus_i HH_n^{(i)}(X/F), \\
HC_n(X/F) & \cong \bigoplus_i HC_n^{(i)}(X/F), \\
\mathbb{H}_{\text{cdh}}^n(X, \mathcal{H}H)^n & \cong \bigoplus_i \mathbb{H}_{\text{cdh}}^n(X, \mathcal{H}H^{(i)})
\end{align*}
\]
\[ \mathbb{H}_{cdh}^n(X, \mathcal{H}) \cong \bigoplus_i \mathbb{H}_{cdh}^n(X, \mathcal{H}_i) . \]

Moreover, the natural map \( \mathcal{H} \to \mathbb{H}_{cdh}(-, \mathcal{H}) \) preserves these decompositions. The same holds for the cyclic homology and negative cyclic homology.

For a presheaf of spectra \( \mathcal{E} \), let \( \tilde{\mathcal{E}} \) denote the homotopy fiber of the natural map of spectra \( \mathcal{E} \to \mathbb{H}_{cdh}(-, \mathcal{E}) \).

**Theorem 2.1.** ([12, Theorem 1.6]) For any scheme \( X \) in \( \text{Sch}/k \), the Chern character map \( \mathcal{K} \to \tilde{\mathcal{H}}^n \) from the K-theory to the negative cyclic homology spectra induces a natural weak equivalence
\[ \tilde{\mathcal{K}}(X) \cong \tilde{\mathcal{H}}^n(X) \cong \Omega^{-1} \tilde{\mathcal{H}}(X) . \]

**Corollary 2.2.** There is a natural fibration sequence of spectra
\[ \Omega^{-1} \tilde{\mathcal{H}}(X) \to \mathcal{K}(X) \to \mathcal{K}(X) . \]

**Proof.** This follows from Theorem 2.1 and the weak equivalence \( \mathbb{H}_{cdh}(-, \mathcal{K}) \cong \mathcal{K} \) as shown in [28, Theorem 6.4]. \[ \square \]

We shall write \( a : (\text{Sch}/k)_{cdh} \to (\text{Sch}/k)_{\text{Zar}} \) for the natural morphism from the cdh to the Zariski site of \( \text{Sch}/k \). For a Zariski sheaf \( \mathcal{F} \), let \( \mathcal{F}_{cdh} \) denote its associated cdh-sheaf \( a^* \mathcal{F} \) and let \( H^*_{cdh}(X, \mathcal{F}) \) denote the cdh-cohomology of \( \mathcal{F}_{cdh} \). For any subfield \( F \subseteq k \) and for \( i \geq 0 \), let \( \Omega^i_{\mathcal{F}} \) denote the presheaf \( X \mapsto \Omega^i_{\mathcal{F}}(X) \) of Kähler differentials. Let \( \Omega^{\geq i}_{X/F} \) denote the brutal truncations of the algebraic de Rham complex \( \Omega^\bullet_{X/F} \) of \( X \). It is known (cf. [52, 1 Exp. VI, 2.11, 5.2]) that the cdh-cohomology commutes the filtered direct limits of sheaves. In particular, for a sheaf of \( k \)-modules \( \mathcal{F} \) and a \( k \)-vector space \( V \), one has \( H^*_{cdh}(X, \mathcal{F} \otimes_k V) \cong H^*_{cdh}(X, \mathcal{F}) \otimes_k V \).

We shall often use this fact for the sheaves of the form \( \Omega^i_{k/Q} \otimes_k \Omega^{\geq i}_{X/k} \).

We also recollect here a few preliminary results that will be used often. The case \( i = 0 \) of the following result was proven in [11, Lemma 6.5]. The general case is a simple consequence of [12, Theorem 2.4, Lemma 2.8]. We shall use such a result for comparing the Zariski and the cdh-cohomology of sheaves of differential forms.

**Lemma 2.3.** For a \( k \)-scheme (i.e., a scheme in \( \text{Sch}/k \)), the complex of Zariski sheaves \( R\alpha_* a^* \Omega^\bullet_{X/k} \) (resp \( R\alpha_* a^* \Omega^\bullet_{X/k} \)) has quasi-coherent (resp coherent) cohomology sheaves for any subfield \( F \subseteq k \) and any \( i \geq 0 \).

**Lemma 2.4.** Let \( F \to G \) be a morphism of Zariski sheaves on a \( k \)-scheme \( X \) such that \( \phi \) is an isomorphism outside a closed subscheme \( Z \hookrightarrow X \) of dimension \( d \). Then \( H^{d+1}(X, \mathcal{F}) \to H^{d+1}(X, \mathcal{G}) \) and \( H^i(X, \mathcal{F}) \cong H^i(X, \mathcal{G}) \) for \( i \geq d + 2 \).

**Proof.** It suffices using the standard long exact cohomology sequences, to show that if \( \mathcal{F} \) is a Zariski sheaf on \( X \) such that \( \mathcal{F}|_U = 0 \), where \( U \to X \) is the complement of \( Z \), then \( H^i(X, \mathcal{F}) = 0 \) for \( i \geq d + 1 \). We have the short exact sequence (cf. [29, Exercise 1.19, Chapter II])
\[ 0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* (\mathcal{F}|_Z) \to 0 . \]

Our assumption then implies that \( H^i(X, \mathcal{F}) \cong H^i(X, i_* (\mathcal{F}|_Z)) \) for all \( i \). But the last term is same as \( H^i(Z, \mathcal{F}|_Z) \) by [29, Lemma III.2.10]. The proof now follows from the Grothendieck vanishing theorem on \( Z \). \[ \square \]
Lemma 2.5. Let $X$ be a $k$-scheme of dimension $d$. Then for either of Zariski and cdh-sites, the following hold for any subfield $F \subseteq k$ and any $j \geq 0$.

(i) $H^d(X, \Omega^{-j}_{X/F}) \to H^d(X, \Omega^j_{X/F}) \to \mathbb{H}^{d+j}(X, \Omega^j_{X/F}) \to 0$

is exact.

(ii) $\mathbb{H}^i(X, \Omega^j_{X/F}) = 0$ for $i \geq j + d + 1$.

Proof. This is a simple exercise in the hypercohomology using the fact that the Zariski or the cdh cohomological dimension of $X$ is $d$ (cf. [52]). We give a sketch. For $j = 0$, this follows from the above fact. So assume that the result holds for $0 \leq j' \leq j - 1$ and $j \geq 1$. The long exact cohomology sequence for the exact sequence

$$0 \to \Omega^{-j}_{X/F} \to \Omega^j_{X/F} \to \Omega^{j-1}_{X/F} \to 0$$

gives a diagram

$$
\begin{array}{ccc}
H^d(X, \Omega^{-j}_{X/F}) & \longrightarrow & \mathbb{H}^{d-j-1}(X, \Omega^{j-1}_{X/F}) \\
\downarrow & & \downarrow \\
\mathbb{H}^{d+j}(X, \Omega^{j}_{X/F}) & \longrightarrow & \mathbb{H}^{d+j}(X, \Omega^{j}_{X/F}) \to \mathbb{H}^{d+j}(X, \Omega^{j-1}_{X/F})
\end{array}
$$

The vertical map is surjective and the last term on the bottom exact sequence is zero by induction. This proves the first exact sequence of the lemma. Finally, the exact sequence

$$H^{i-j}(X, \Omega^j_{X/F}) \to \mathbb{H}^i(X, \Omega^j_{X/F}) \to \mathbb{H}^i(X, \Omega^{j-1}_{X/F})$$

for $i \geq d + j + 1$ and the induction prove the second part of the lemma. \qed

3. Zariski and cdh cohomology on normal crossing schemes

In this section, we compare the Zariski and the cdh cohomology of sheaves of differential forms on normal crossing schemes. We begin with the following local result.

For a finite morphism $f : Y \to X$ of $k$-schemes and for a subfield $F \subseteq k$, let

$$\Omega^i_{(X,Y)/F} := \text{Ker}(\Omega^i_{X/F} \to f_* (\Omega^i_{Y/F})) \text{ for } i \geq 0.$$ 

If $X = \text{Spec}(A)$ and $Y$ is the closed subscheme of $X$ defined by an ideal $I$, we shall often write the above as $\Omega^i_{(A,I)/F}$.

Proposition 3.1. Let $A$ be a reduced and essentially of finite type $k$-algebra. Assume that the normalization map $f : A \to B$ of $A$ is unramified. Let $I \subset A$ be a conducting ideal for the normalization. Then the natural map $\Omega^i_{(A,I)/F} \to \Omega^i_{(B,I)/F}$ is surjective for any $i \geq 0$ and any subfield $F \subseteq k$.

Proof. Let $A' = A/I$ and $B' = B/I$. We prove the proposition by the induction on $i \geq 0$. For $i = 0$, we have $\Omega^0_{(A,I)/F} = I = \Omega^0_{(B,I)/F}$. For $i = 1$, we use the
commmutative diagram of exact sequences

\[
\begin{array}{ccc}
HH_0^F(A, I) & \longrightarrow & \Omega_{A/F}^1 \\
\downarrow & & \downarrow \\
HH_0^F(B, I) & \longrightarrow & \Omega_{B/F}^1 \\
\downarrow & & \downarrow \\
HH_0^F(A, B, I),
\end{array}
\]

where \( HH_0^F(A, B, I) \) is the double relative Hochschild homology (cf. [32]). Since
\( HH_0^F(A, B, I) \cong I \otimes_A \Omega_{B/A}^1 \) by [10] Theorem 3.4] and since \( A \to B \) is unramified,
we see that \( HH_0^F(A, B, I) = 0 \) and in particular, the top vertical map on the
extreme left in the above diagram is surjective. A diagram chase shows that
\( \Omega_{B/A}^1 \) is unramified, the first fundamental exact sequence of Kähler differentials implies that
\[
\Omega_{B/A}^1 = B \Omega_{A/F}^1.
\]
Moreover, \( \Omega_{(A,I)/F}^i \subseteq \Omega_{A/F}^i \) is generated as an \( A \)-module by the exterior forms of
the type \( a_0 da_1 \wedge \cdots \wedge da_i \), where \( a_p \in A \) for all \( p \) and \( a_p \in I \) for some \( p \). This
\[
\begin{align*}
\Omega_{(A,I)/F}^i & = \sum_{\text{some } a_p \in I} A a_0 da_1 \wedge \cdots \wedge da_i, \\
\Omega_{(B,I)/F}^i & = \sum_{\text{some } b_p \in I} B b_0 db_1 \wedge \cdots \wedge db_i \\
& = \sum_{\text{some } b_p \in I} A b_0 db_1 \wedge \cdots \wedge db_i.
\end{align*}
\]
Hence, it suffices to show that
\[
\begin{align*}
\beta & = b_0 db_1 \wedge \cdots \wedge db_i \in \text{Image } \left( \Omega_{(A,I)/F}^i \to \Omega_{(B,I)/F}^i \right). \tag{3.3}
\end{align*}
\]
By permuting the orders of differentials (which only changes the sign), can assume that \( b_p \in I \) for some \( p \leq i - 1 \). Then we have \( \beta' = b_0 db_1 \wedge \cdots \wedge db_{i-1} \in \Omega_{(B,I)/F}^{i-1} \) by
\[
\begin{align*}
\beta & = \beta' \wedge db_i.
\end{align*}
\]
By induction, we see that \( \beta' \in \text{Image } \left( \Omega_{(A,I)/F}^{i-1} \to \Omega_{(B,I)/F}^{i-1} \right) \). This implies from
\[
\begin{align*}
\beta' & = \sum_{\text{some } a_p \in I} a_0 da_1 \wedge \cdots \wedge da_{i-1},
\end{align*}
\]
with \( a_p \in A \) for \( 0 \leq p \leq i - 1 \). So can assume that \( \beta' = a_0 da_1 \wedge \cdots \wedge da_{i-1} \). We
then have
\[
\begin{align*}
\beta & = \beta' \wedge db_i = a_0 da_1 \wedge \cdots \wedge da_{i-1} \wedge db_i.
\end{align*}
\]
If \( a_0 \in I \), then the case \( i = 1 \) implies that \( a_0 db_i \in \Omega_{(A,I)/F}^1 \) and hence \( \beta \in \Omega_{(A,I)/F}^i \).
So we suppose that \( a_p \in I \) for some \( 1 \leq p \leq i - 1 \). We can again assume that \( a_1 \in I \). It follows from (3.1) that \( da_1 \wedge db_i \) is of the form \( \sum b'_p da_1 \wedge d\alpha_p \), where \( \alpha_p \in A \) and \( b'_p \in B \) for all \( p \). Hence we can assume that
\[
\beta = (bda_1 \wedge d\alpha_1) \wedge (a_0 da_2 \wedge \cdots \wedge da_{i-1}),
\]
where \( \alpha_i \in A \) and \( b \in B \). In particular, we have \( bda_1 \in \Omega^1_{(B,I)/F} \) and hence in the image of \( \Omega^1_{(A,I)/F} \) by the case \( i = 1 \). This in turn implies that up to sign
\[
\beta = bda_1 \wedge (da_2 \wedge \cdots \wedge da_{i-1} \wedge d\alpha_i)
\in \Omega^1_{(A,I)/F} \wedge \Omega^1_{A/F} = \Omega^1_{(A,I)/F}.
\]
This proves (3.3) and finishes the proof of the proposition.

\[\square\]

**Lemma 3.2.** Let \( E \) be a reduced seminormal curve which is affine and essentially of finite type over \( k \). Then the following hold for any subfield \( \bar{F} \subseteq \bar{k} \) and any \( i \geq 0 \).

\[
H^0_{\text{Zar}}(E, \mathcal{O}_E) \cong H^0_{\text{cdh}}(E, \mathcal{O}_E)
\]

\[
H^0_{\text{Zar}}(X, \Omega^i_{E/F}) \rightarrow H^0_{\text{cdh}}(X, \Omega^i_{E/F}) \text{ and } H^0_{\text{cdh}}(E, \Omega^i_{E/F}) = 0 \text{ for } j > 0.
\]

**Proof.** Let \( \overline{E} \overset{f}{\rightarrow} E \) denote the normalization of \( E \). Let \( S = E_{\text{sing}} \) be the reduced singular locus of \( E \) and let \( \overline{S} = f^{-1}(S) \). Since \( E \) is a seminormal curve, the map \( f \) is unramified and hence the proof of Proposition 3.1 implies that there is an exact sequence

\[
(3.4) \quad H^0_{\text{Zar}}(E, \Omega^i_{E/F}) \rightarrow H^0_{\text{Zar}}(\overline{E}, \Omega^i_{\overline{E}/F}) \oplus H^0_{\text{Zar}}(S, \Omega^i_{S/F}) \rightarrow H^0_{\text{Zar}}(\overline{S}, \Omega^i_{\overline{S}/F}) \rightarrow 0.
\]

for \( 0 \leq i \leq 1 \) and the first arrow from the left is injective for \( i = 0 \). We thus have the following commutative diagram of exact sequences for \( 0 \leq i \leq 1 \).

\[
(3.5) \quad \xymatrix{ H^0_{\text{Zar}}(E, \Omega^i_{E/F}) \ar[r] & H^0_{\text{Zar}}(\overline{E}, \Omega^i_{\overline{E}/F}) \ar[r] & H^0_{\text{Zar}}(S, \Omega^i_{S/F}) \ar[r] & H^0_{\text{Zar}}(\overline{S}, \Omega^i_{\overline{S}/F}) \ar[r] & 0 \\
0 \ar[r] & H^0_{\text{cdh}}(E, \Omega^i_{E/F}) \ar[r] & H^0_{\text{cdh}}(\overline{E}, \Omega^i_{\overline{E}/F}) \ar[r] & H^0_{\text{cdh}}(S, \Omega^i_{S/F}) \ar[r] & H^0_{\text{cdh}}(\overline{S}, \Omega^i_{\overline{S}/F}) \ar[r] & H^1_{\text{cdh}}(E, \Omega^i_{E/F}).}
\]

The top sequence is exact by (3.4) and the bottom sequence is the Mayer-Vietoris exact sequence for the cdh cohomology (cf. [12, Theorem 2.7]). The smoothness of \( \overline{E}, S \) and \( \overline{S} \) and [12, Corollary 2.5] imply that the middle and the right vertical maps are isomorphisms. For the same reason, the assertion of the lemma holds for these schemes. A diagram chase now proves the lemma for \( 0 \leq i \leq 1 \). Furthermore, it also follows that for \( i \geq 2 \),

\[
\Omega^i_{k/F} \otimes_k H^0_{\text{cdh}}(\overline{E}, \mathcal{O}_{\overline{E}/F}) \rightarrow \Omega^i_{k/F} \otimes_k H^0_{\text{cdh}}(\overline{S}, \mathcal{O}_{\overline{S}/F}) \cong H^0_{\text{cdh}}(\overline{S}, \Omega^i_{\overline{S}/F}).
\]

In particular, \( H^2_{\text{cdh}}(E, \Omega^i_{E/F}) = 0 \) for \( i \geq 0 \) and \( j \geq 1 \).

We now only need to show that

\[
(3.6) \quad H^0_{\text{Zar}}(E, \Omega^i_{E/F}) \rightarrow H^0_{\text{cdh}}(E, \Omega^i_{E/F}) \text{ for } i \geq 2.
\]
For this, we first assume that $F = k$. Then $\Omega^1_{S/k}$ and $\Omega^1_{\overline{S}/k}$ vanish and hence the top exact sequence in the above diagram shows that $\Omega^1_{E/k} \rightarrow \Omega^1_{\overline{E}/k}$. This in turn gives a commutative diagram

$$
\begin{array}{ccc}
\Omega^i_{E/k} & \rightarrow & \Omega^i_{\overline{E}/k} \\
\downarrow & & \downarrow_{\cong} \\
H^0_{\text{cdh}}(E, \Omega^i_{E/k}) & \cong & H^0_{\text{cdh}}(\overline{E}, \Omega^i_{\overline{E}/k}),
\end{array}
$$

where the the isomorphism of the right vertical map and the bottom horizontal map follows from the smoothness of $E, S$ and $\overline{S}$ and the Mayer-Vietoris exact sequence for the $\text{cdh}$ cohomology. This shows the desired surjectivity of the left vertical map. For any $F \subseteq k$, we note that $\Omega^i_{E/F}$ has a finite decreasing filtration $\{F^j\Omega^i_{E/F}\}_{0 \leq j \leq i}$ such that there is a surjection

$$
\Omega^i_{k/F} \otimes_k \Omega^i_{E/k} \rightarrow \frac{F^j\Omega^i_{E/F}}{F^{j+1}\Omega^i_{E/F}}
$$

which is an isomorphism on the smooth locus of $E$. Now (3.6) follows by an easy induction on $i$ and $j$. $\square$

The above lemma is a special case of the following more general result.

**Proposition 3.3.** Let $E \hookrightarrow X$ be a strict normal crossing divisor on a smooth $k$-scheme $X$ such that $X$ is affine and essentially of finite type over $k$ and its dimension is $d + 1$. Then the following hold for any subfield $F \subseteq k$ and any $i \geq 0$.

$$
\begin{align*}
H^0_{\text{Zar}}(E, \mathcal{O}_E) & \cong H^0_{\text{cdh}}(E, \mathcal{O}_E) \\
H^0_{\text{Zar}}(X, \Omega^i_{E/F}) & \rightarrow H^0_{\text{cdh}}(X, \Omega^i_{E/F}) \quad \text{and} \\
H^j_{\text{cdh}}(E, \Omega^i_{E/F}) & = 0 \quad \text{for } j > 0.
\end{align*}
$$

**Proof.** Let $A$ be the coordinate ring of $E$ and let $A \xrightarrow{f} B$ be the normalization of $A$. Let $I \subseteq A$ be the reduced conducting ideal for the normalization. In particular, $I$ is the ideal of the closed subscheme $E_{\text{sing}}$. We prove the proposition by induction on $d$. The case $d = 1$ is shown in Lemma 3.2. So we assume $d \geq 2$ and that the result holds for the normal crossing divisors of dimension up to $d - 1$.

We observe that since $E$ is a strict normal crossing divisor on $X$ which is smooth, the normalization is simply the disjoint union of the irreducible components of $E$. Moreover, the map $f$ is unramified, as can be easily checked by local calculations. That is,

$$
\Omega^1_{\overline{E}/E} = 0.
$$

If $E$ is irreducible, then it is smooth and the result is known (cf. [12, Corollary 2.5]). So we assume that $E$ is not irreducible and prove the assertion by induction on the number of components of $E$. Let $E = E_1 \cup \cdots \cup E_r$ be the union of all its irreducible components. Let $E' = E_2 \cup \cdots \cup E_r$ and $D = E' \cap E_1$. Then $E'$ is also a strict normal crossing divisor on $X$ and $D$ is a strict normal crossing divisor on $E_1$. We first claim that the sequence

$$
\Omega^i_{E/F} \rightarrow \Omega^i_{E_1/F} \oplus \Omega^i_{E'/F} \rightarrow \Omega^i_{D/F} \rightarrow 0
$$

(3.8)
is exact. To show this, it suffices to prove that the map \( \Omega_{(E,E')/F}^i \rightarrow \Omega_{(E_1,D)/F}^i \) is surjective.

Let \( I' \) denote the ideal of \( E' \) as a closed subscheme of \( E \). Then the fact that \( E \) is a snc, implies that \( J = I'O_{E_1} \) is the ideal of \( D \) as a closed subscheme of \( E_1 \).

In other words, one has \( I' \rightarrow J \). The desired surjectivity now follows from the presentations of \( \Omega_{(E,E')/F}^i \) and \( \Omega_{(E_1,D)/F}^i \) in (3.2). This proves the claim.

It is easy to see from the definitions of \( E' \) and \( D \) that the diagram

\[
\begin{array}{ccc}
D^N & \rightarrow & E'^N \\
\downarrow & & \downarrow \\
D & \rightarrow & E'
\end{array}
\]

is Cartesian. We now claim that the map

\[
(3.9) \quad \Omega_{(E,E')/F}^i \rightarrow \Omega_{(D,D^N)/F}^i
\]

To prove this claim, we can work locally on \( X \) at points of \( E \). Thus we can let \( A = \hat{R}/(a) \), where \( R = (R,m,k) \) is the regular local ring of a closed point on \( X \) with maximal ideal \( \mathfrak{m} = (x_1, \ldots, x_{d+1}) \) and residue field \( k \), and \( a = x_{i_1} \cdots x_{i_r} \) with \( 1 \leq i_1 < \cdots < i_r \leq d+1 \). We prove the claim in the case when \( r = d+1 \). The case \( r < d+1 \) is simpler and follows in the same way.

In this case, \( E_1, E' \) and \( D \) can be described by the local rings \( A_1 = R/(x_1) \), \( A' = R/(b) \), and \( S = A_1/(b) \) respectively, where \( b = x_2 \cdots x_{d+1} \). In particular, we have \( A'^N = \prod_{2 \leq i \leq d+1} R/x_i \) and \( S^N = \prod_{2 \leq i \leq d+1} A_1/x_i \). Since \( \Omega_{(A',A'^N)/F}^i \rightarrow \Omega_{(S,S^N)/F}^i \) is \( A' \)-linear, it is enough to show that the map \( \Omega_{(A',A'^N)/F}^i \rightarrow \hat{A}' \rightarrow \Omega_{(S,S^N)/F}^i \rightarrow \hat{A}' \) is surjective. Hence we can replace \( R, A' \) and \( S \) by their completions, and assume that \( A' = k[[x_1, \ldots, x_{d+1}]]/(b) \) and \( S = k[[x_2, \ldots, x_{d+1}]]/(b) \). In particular, we have \( A' \cong S[[x_1]] \) and hence the maps \( A' \rightarrow S \) and \( A'^N \rightarrow S^N \) have sections and hence the map \( \Omega_{(A',A'^N)/F}^i \rightarrow \Omega_{(S,S^N)/F}^i \) is surjective. This proves (3.9).

To prove the proposition for \( E \), we consider the diagram

\[
\begin{array}{cccccc}
\Omega_{E/F}^i & \rightarrow & \Omega_{E'/F}^i + \Omega_{E_1/F}^i & \rightarrow & \Omega_{D/F}^i & \rightarrow & 0 \\
\downarrow & & \downarrow \alpha & & \downarrow \beta & & \\
0 & \rightarrow & H^0_{cdh}(E, \Omega_{E/F}^i) & \rightarrow & \begin{cases} H^0_{cdh}(E', \Omega_{E'/F}^i) \\ \oplus H^0_{cdh}(E_1, \Omega_{E_1/F}^i) \end{cases} & \rightarrow & H^0_{cdh}(D, \Omega_{D/F}^i) \rightarrow H^1_{cdh}(E, \Omega_{E/F}^i),
\end{array}
\]

where the top sequence is exact by (3.8) and the bottom sequence is the Mayer-Vietoris exact sequence for the cdh cohomology. The middle and the right vertical maps are surjective by induction on the dimension and number of components. The last map on the bottom row is surjective as the next term in the long exact sequence is zero by induction again. This shows in particular that \( H^j_{cdh}(E, \Omega_{E/F}^i) = 0 \) for \( j \geq 1 \). We now only need to show that \( \text{Ker}(\alpha) \rightarrow \text{Ker}(\beta) \) is surjective to finish the proof of the proposition. However, since \( E'^N, E_1 \) and \( D^N \) are smooth, \( \text{Ker}(\alpha) \) and \( \text{Ker}(\beta) \) are same as \( \Omega_{(E',E'^N)/F}^i \) and \( \Omega_{(D,D^N)/F}^i \) respectively by [14, Remark 5.6.1]. The required surjectivity now follows from (3.9). \( \square \)
Corollary 3.4. Let $E \to X$ be a strict normal crossing divisor on a smooth $k$-scheme $X$ of dimension $d + 1$. Then the natural map

$$H^j_{\text{Zar}}(E, \Omega^i_{E/F}) \to H^j_{\text{cdh}}(E, \Omega^i_{E/F})$$

is surjective for $j = d - 1$ and isomorphism for $j = d$ for any subfield $F \subseteq k$ and any $i \geq 0$.

Proof. For the morphism $a : E_{\text{cdh}} \to E_{\text{Zar}}$ of sites, it follows from Proposition 3.3 that $R^ia_*a^*\Omega^i_{E/F} = 0$ for $j \geq 1$ and $i \geq 0$. The Leray spectral sequence implies that $Ra_*a^*\Omega^i_{E/F} \cong a_*a^*\Omega^i_{E/F}$. On the other hand, it also follows from Lemma 2.3 and Proposition 3.3 that $\Omega^i_{E/F} \to a_*a^*\Omega^i_{E/F}$ is a surjective map of quasi-coherent sheaves with kernel supported on $E_{\text{sing}}$. The corollary now follows from Lemma 2.4.

Corollary 3.5. Let $\tilde{X} \to X$ be a resolution of singularities of a normal quasiprojective $k$-variety $X$ of dimension $d + 1$ with the reduced strict normal crossing exceptional divisor $E = f^{-1}(X_{\text{sing}})$. Then the map $\mathbb{H}^{2d-1}_{\text{Zar}}(E, \Omega^{<d}_{E/F}) \to \mathbb{H}^{2d-1}_{\text{cdh}}(E, \Omega^{<d}_{E/F})$ is an isomorphism. Furthermore, the map $\mathbb{H}^{2d-1}(\tilde{X}, \Omega^{<d}_{\tilde{X}/F}) \to \mathbb{H}^{2d-1}(E, \Omega^{<d}_{E/F})$ is surjective in the Zariski and the cdh topology if the map $H^{d+1}_{\text{cdh}}(X, \Omega^{d-1}_{X/F}) \to H^{d+1}_{\text{cdh}}(\tilde{X}, \Omega^{d-1}_{\tilde{X}/F})$ is an isomorphism.

Proof. The first isomorphism follows at once from Lemma 2.5 and Corollary 3.4. For the other assertion, we can use the first assertion to reduce to the case of cdh topology. Now, the surjectivity $H^d_{\text{cdh}}(E, \Omega^{d-1}_{E/F}) \to \mathbb{H}^{2d-1}_{\text{cdh}}(E, \Omega^{<d}_{E/F})$ (cf. Lemma 2.5) implies that we only have to show that the map $H^d_{\text{cdh}}(\tilde{X}, \Omega^{d-1}_{\tilde{X}/F}) \to H^d_{\text{cdh}}(E, \Omega^{d-1}_{E/F})$ is surjective. But this follows directly from our assumption and the Mayer-Vietoris exact sequence for the cdh cohomology of $\Omega^{d-1}_{E/F}$ for the resolution map once we observe that $H^d_{\text{cdh}}(X_{\text{sing}}, \Omega^{d-1}_{X_{\text{sing}}/F})$ vanishes as $X$ is normal.

The following result about the Zariski and the cdh cohomology of the Kähler differentials on seminormal varieties is of independent interest. This is a weaker form of the previous result.

Proposition 3.6. Let $X$ be a reduced seminormal $k$-scheme of dimension $d$. Then the natural map

$$H^d_{\text{Zar}}(X, \Omega^i_{X/F}) \to H^d_{\text{Zar}}(X, a_*a^*\Omega^i_{X/F})$$

is an isomorphism for any subfield $F \subseteq k$ and any $i \geq 0$.

Proof. It follows from Lemma 2.3 that the map $\Omega^i_{X/F} \to a_*a^*\Omega^i_{X/F}$ is a morphism of quasi-coherent sheaves. In particular, the cokernel sheaf $\mathcal{F}^i_{X/F}$ is also quasi-coherent.

Claim. $\mathcal{F}^i_{X/F}$ is supported on a closed subscheme $Z$ which has codimension at least two in $X$.

Proof of the claim. Let $\tilde{X} \to X$ be the normalization map. Let $S \to X$ be the
reduced conductor subscheme and let \( \bar{S} = S \times_X \tilde{X} \). Let \( \bar{S} \hookrightarrow \tilde{X} \) be the inclusion map. Let \( \mathcal{H}^i_{(X, \bar{X}, S)/F} \) denote the cokernel of the map \( \Omega^i_{(X, \bar{X}, S)/F} \rightarrow \pi_* \left( \Omega^i_{(\tilde{X}, \bar{S})/F} \right) \). The seminormality of \( X \) implies that \( \pi \) is unramified in codimension one. Hence we conclude from Proposition \( \ref{proposition:semi-normality} \) that \( \text{support} \left( \mathcal{H}^i_{(X, \bar{X}, S)/F} \right) = Z_1 \), where \( Z_1 \) is a closed subscheme of \( X \) of codimension at least two. In particular, the sequence

\[
\Omega^i_{X/F} \rightarrow \pi_* \Omega^i_{\tilde{X}/F} \oplus \Omega^i_{\bar{S}/F} \rightarrow \pi_* \Omega^i_{\bar{S}/F} \rightarrow 0
\]

is exact on the open subscheme \( U_1 = X - Z_1 \). Thus we get a commutative diagram

\[
\begin{array}{ccc}
\Omega^i_{X/F} & \rightarrow & \Omega^i_{\tilde{X}/F} \oplus \Omega^i_{\bar{S}/F} \\
\downarrow \theta^i & & \downarrow \\
0 & \rightarrow & a_* a^* \Omega^i_{X/F} \oplus a_* a^* \Omega^i_{\tilde{X}/F} \rightarrow a_* a^* \Omega^i_{\bar{S}/F} \rightarrow \cdots,
\end{array}
\]

where the bottom sequence is the Mayer-Vietoris exact sequence for \( cdh \)-cohomology because \( S \) and \( \bar{S} \) are reduced.

Since \( X \) is seminormal, it follows from \([38, \text{Theorems 3.5, 3.8}]\) that there is a dense open subscheme \( U_2 \subset X \) such that for \( Z_2 = X - U_2 \), we have

(i) \( \text{Codim}_X (Z_2) \geq 2 \) and

(ii) \( \tilde{X} \cap U_2, S \cap U_2 \) and \( \bar{S} \cap \pi^{-1}(U_2) \) are all smooth.

In particular, the restriction of \( R^i a_* a^* \Omega^i_{\bar{S}/F} \) on \( U_2 \) is zero. Moreover, the middle as well as the right vertical maps are isomorphisms by \([12, \text{Corollary 2.5}]\) on \( U_2 \).

A diagram chase in (3.12) shows that \( \theta^i \) is surjective on \( U = U_1 \cap U_2 \) and the codimension of \( Z = X - U \) is at least two. This proves the claim.

It also follows from \([12, \text{Corollary 2.5}]\) that the kernel of \( \theta^i \) is a quasi-coherent sheaf supported on \( X_{\text{sing}} \). The proof of Lemma \( \ref{lemma:theta} \) now proves the desired isomorphism of the top cohomology groups.

\[\square\]

4. Zariski and cdh cohomology of some singular schemes

Our aim in this section is to compare some Zariski and \( cdh \) cohomology of projective varieties with isolated singularities using the results of Section \( 3 \) and the resolution of singularities. Let \( X \) be a normal projective \( k \)-variety of dimension \( d + 1 \geq 2 \) with only isolated singularities. Let \( S = X_{\text{sing}} \) denote the singular locus of \( X \) with the reduced induced structure. Let \( f : \tilde{X} \rightarrow X \) be a resolution of singularities of \( X \) such that the reduced exceptional divisor \( E \hookrightarrow \tilde{X} \) has smooth components with strict normal crossings. For \( n \geq 1 \), let \( nE \) denote the \( n \)th infinitesimal thickening of \( E \) defined by the sheaf of ideals \( \mathcal{I}^n \) on \( \tilde{X} \), where \( \mathcal{I} \) is the sheaf of ideals defining \( E \). Let \( \overline{E} \rightarrow E \) be the normalization map of \( E \). Note that \( \overline{E} \) is simply the disjoint union of irreducible components of \( E \).

**Lemma 4.1.** The natural map

\[
H^d_{\text{Zar}} \left( E, \frac{\Omega^d_{E/k}}{\Omega^{d-1}_{E/k}} \right) \rightarrow H^d_{\text{Zar}} \left( \overline{E}, \frac{\Omega^d_{\overline{E}/k}}{\Omega^{d-1}_{\overline{E}/k}} \right)
\]

is an isomorphism.
Proof. It suffices to show by Lemma 2.4 that the map
\[ \frac{\Omega^d_{E/k}}{\Omega^{d-1}_{E/k}} \to \frac{\Omega^d_{E/k}}{\Omega^{d-1}_{E/k}} \]
is surjective. For this, it is enough to show that the map \( \Omega^d_{E/k} \to \Omega^d_{E/k} \) is surjective. One can now check by an easy local calculation (cf. proof of Lemma 4.2) that, in fact the composite map \( \tilde{\Omega}^d_{\tilde{X}/k} \to \Omega^d_{E/k} \) is surjective. This completes the proof. \( \square \)

Let \( \omega_{\tilde{X}/k} \) denote the canonical line bundle on \( \tilde{X} \). For \( n \geq 0 \), the differential map \( \Omega^d_{(n+1)E/k} \to \Omega^{d+1}_{(n+1)E/k} \) induces the map
\[ \frac{\Omega^{d+1}_{(n+1)E/k}}{\Omega^{d+1}_{(n+1)E/k}} \to \frac{\omega_{\tilde{X}/k}}{\Omega^d_{E/k}} \]
Also, we have the following natural surjections.
\[ \omega_{\tilde{X}/k} \otimes_{\mathcal{O}_X} \mathcal{O}_n \]

Lemma 4.2. For any \( n \geq 1 \), the above maps induce the surjective maps
\[ \frac{\Omega^{d+1}_{(n+1)E/k}}{\Omega^{d+1}_{(n+1)E/k}} \to \frac{\omega_{\tilde{X}/k}}{\Omega^d_{E/k}} \] which are isomorphisms on the smooth locus of \( E \).

Proof. This is a local calculation and can be checked at the local ring of closed points of \( \tilde{X} \). So as in the proof of Proposition 2.3, let \( R = (R, m, k) \) be the regular local ring of a closed point on \( X \) with maximal ideal \( m = (x_1, \ldots, x_{d+1}) \) and residue field \( k \). For \( n \geq 1 \), let \( A_n \) denote the local ring of \( nE \) at the chosen point.

We first consider the case when \( A = R/x_1 \) is smooth. In that case, \( A_{n+1} \) is in fact of the form \( A[x]/(x^{n+1}) \). On can then explicitly calculate that
\[ \Omega^d_{(A_{n+1}, A)/k} = \left( \Omega^d_{A/k} \otimes A_x A_{n+1} \right) \oplus \left( A_{n+1}/x^n \right) dx, \]
\[ \Omega^i_{(A_{n+1}, A)/k} = \left( \Omega^i_{A/k} \otimes A_x A_{n+1} \right) \oplus \left( \Omega^{i-1}_{A/k} \otimes A \left( A_{n+1}/x^n \right) dx \right) \] for \( 2 \leq i \leq d \) and
\[ \Omega^d_{A_{n+1}/k} = \Omega^d_{A/k} \otimes A \left( A_{n+1}/x^n \right) dx. \]
The desired isomorphism \[ \frac{\Omega^d_{(A_{n+1}, A)/k}}{\Omega^{d+1}_{(A_{n+1}, A)/k}} \cong \Omega^d_{A_{n+1}/k} \cong \frac{\Omega^d_{(A_{n+1})/k}}{\Omega^{d+1}_{(A_{n+1})/k}} \] can be now directly checked.

We now assume that \( A \) is not smooth. Let \( A_n = R/(a^n) \), where \( a = x_1 \cdots x_r \) with \( 1 \leq i_1 < \cdots < i_r \leq d+1 \). We prove the assertion in the case when \( r = d+1 \). The case \( r < d+1 \) is simpler and follows in the same way. The normalization of \( A \) in this case is the ring \( A_N = \prod_{1 \leq i \leq d+1} R/x_i \). We describe the various Kähler differentials in terms of generators and relations. We fix a few notations.
For $1 \leq i \leq d+1$, let
\[ a_i = \prod_{j \neq i} x_j \quad \text{and} \quad dX_i = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{d+1}. \]
For $1 \leq i < j \leq d+1$, we let
\[ dY_{ij} = dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{d+1}. \]
Finally, we put $dw = dx_1 \wedge \cdots \wedge dx_{d+1}$. We then have
\[ \Omega^{1}_{R/k} = \bigoplus_{1 \leq i \leq d+1} Rdx_i \]
\[ \Omega^{1}_{A_{n+1}/k} = \bigoplus_{1 \leq i \leq d+1} Rdx_i \bigg( \bigoplus_{1 \leq i \leq d+1} a^{n+1}Rdx_i, \sum_{1 \leq i \leq d+1} a^n a_i dx_i \bigg) \]
\[ \Omega^{n}_{A_{n}/k} = \bigoplus_{1 \leq i \leq d+1} \left( \bigoplus_{j \neq i} (R/x_i) dx_j \right). \]
Taking the various exterior powers, we get
\[ \Omega^{d}_{A_{n+1}/k} = \bigoplus_{1 \leq i \leq d+1} RdX_i \bigg( \bigoplus_{1 \leq i \leq d+1} a^{n+1}RdX_i, \sum_{1 \leq i \leq d+1} a^n a_i dx_i \wedge dY_{ij} \bigg) \]
\[ \Omega^{d}_{A_{n}/k} = \bigoplus_{1 \leq i \leq d+1} R/x_i dx_i \quad \text{and} \]
\[ \Omega^{d+1}_{A_{n+1}/k} = \frac{Rdw}{a^nRdw} \bigg( a^{n+1}Rdw, \bigoplus_{1 \leq i \leq d+1} Ra^n a_i dw \bigg). \]
It can now be directly checked from the above description that the differential map $\partial: \Omega^{d}_{(A_{n+1}, A_{n})/k} \to \Omega^{d+1}_{A_{n+1}/k}$ is surjective and there is a surjection $\Omega^{d+1}_{A_{n+1}/k} \to Rdw/a^nRdw = \frac{\gamma_{R/k}^{d+1}}{a^n\Omega^{n}_{R/k}}$. We omit more details. This finishes the proof of the lemma. \(\square\)

**Corollary 4.3.** For every $n \geq 1$, the natural map
\[ \frac{H^{d}_{\text{Zar}}(nE, \Omega^{d}_{nE/k})}{H^{d}_{\text{Zar}}(nE, \Omega^{d-1}_{nE/k})} \to H^{d}_{\text{Zar}}(E, \frac{\Omega^{d}_{E/k}}{\Omega^{d-1}_{E/k}}) \]
is an isomorphism.

**Proof.** Using the exact sequence
\[ H^{d}_{\text{Zar}}(nE, \Omega^{d-1}_{nE/k}) \to H^{d}_{\text{Zar}}(nE, \Omega^{d}_{nE/k}) \to H^{d}_{\text{Zar}}(nE, \frac{\Omega^{d}_{nE/k}}{\Omega^{d-1}_{nE/k}}) \to 0, \]
it suffices to show that the map
\[ H^d_{\text{Zar}} \left( nE, \frac{\Omega^d_{nE/k}}{\Omega^{d-1}_{nE/k}} \right) \to H^d_{\text{Zar}} \left( E, \frac{\Omega^d_{E/k}}{\Omega^{d-1}_{E/k}} \right) \]
is an isomorphism.

By Lemma 4.1, we can replace \( E \) by \( \overline{E} \). It is easily seen from the above calculations that the map \( \Omega^d_{nE/k} \to \Omega^d_{\overline{E}/k} \) is surjective. Thus we have an exact sequence of sheaves
\[ \frac{\Omega^d_{(nE,\overline{E})/k}}{\Omega^{d-1}_{(nE,\overline{E})/k}} \to \frac{\Omega^d_{nE/k}}{\Omega^{d-1}_{nE/k}} \to 0. \]
Hence, we only need to show that \( H^d_{\text{Zar}} \left( \overline{X}, \omega_{\overline{X}/k} \otimes \mathcal{O}_{nE} \right) = 0 \) for all \( n \geq 1 \).

Since \( \left\{ H^d_{\text{Zar}} \left( \overline{X}, \omega_{\overline{X}/k} \otimes \mathcal{O}_{nE} \right) \right\}_{n \geq 1} \) is an inverse system of surjective maps of finite-dimensional \( k \)-vector spaces, the map
\[ \lim_{\leftarrow} H^d_{\text{Zar}} \left( \overline{X}, \omega_{\overline{X}/k} \otimes \mathcal{O}_{nE} \right) \to H^d_{\text{Zar}} \left( \overline{X}, \omega_{\overline{X}/k} \otimes \mathcal{O}_{nE} \right) \]
is surjective for each \( n \geq 1 \). Hence, it suffices to show that the inverse limit vanishes. However, this inverse limit is isomorphic to \( H^0_{\text{Zar}} \left( X, R^d f_* \omega_{\overline{X}/k} \right) \) by the formal function theorem. On the other hand, \( R^d f_* \omega_{\overline{X}/k} \) is zero by the Grauert-Riemenschneider vanishing theorem (cf. [20, p. 59]). This completes the proof of the corollary.

**Lemma 4.4.** Let \( f : \overline{X} \to X \) be a resolution of singularities of a normal projective \( k \)-variety of dimension \( d + 1 \geq 2 \) with only isolated singularities as above, with reduced exceptional divisor \( E \). Then the natural map
\[ \begin{array}{ccc}
    H^d_{\text{Zar}} \left( X, R^d f_* \Omega^d_{\overline{X}/k} \right) & \to & H^d_{\text{Zar}} \left( E, \Omega^d_{E/k} \right) \\
    H^d_{\text{Zar}} \left( X, R^d f_* \Omega^{d-1}_{\overline{X}/k} \right) & \to & H^d_{\text{Zar}} \left( E, \Omega^{d-1}_{E/k} \right)
\end{array} \]
is an isomorphism.

**Proof.** By the formal function theorem, it suffices to show that
\[ \lim_{\leftarrow} H^d_{\text{Zar}} \left( \overline{X}, \Omega^d_{X/k} \otimes \mathcal{O}_{nE} \right) \to H^d_{\text{Zar}} \left( E, \frac{\Omega^d_{E/k}}{\Omega^{d-1}_{E/k}} \right) \]
is an isomorphism.

By Corollary 4.3 we only need to show that the map
\[ \begin{array}{c}
    \lim_{\leftarrow} H^d_{\text{Zar}} \left( \overline{X}, \Omega^d_{X/k} \otimes \mathcal{O}_{nE} \right) \\
    \lim_{\leftarrow} H^d_{\text{Zar}} \left( \overline{X}, \Omega^{d-1}_{X/k} \otimes \mathcal{O}_{nE} \right) \to \lim_{\leftarrow} H^d_{\text{Zar}} \left( nE, \Omega^{d-1}_{nE/k} \right)
\end{array} \]
is an isomorphism. We consider the following diagram of exact sequences.

\[
\begin{align*}
H^d_{\text{Zar}} \left( \tilde{X}, \frac{\mathcal{I}_{nE} \Omega^{d-1}_{X/k}}{\mathcal{I}_{2nE}} \right) & \oplus H^d_{\text{Zar}} \left( \tilde{X}, \frac{\mathcal{I}_{2nE} \otimes \Omega^{d-2}_{nE/k}}{\mathcal{I}_{2nE}} \right) \to H^d_{\text{Zar}} \left( \tilde{X}, \frac{\Omega^{d-1}_{X/k}}{\mathcal{I}_{2nE}} \right) \to H^d_{\text{Zar}} \left( nE, \Omega^{d-1}_{nE/k} \right) \to 0 \\
H^d_{\text{Zar}} \left( \tilde{X}, \frac{\mathcal{I}_{nE} \Omega^{d-1}_{X/k}}{\mathcal{I}_{2nE}} \right) & \to H^d_{\text{Zar}} \left( \tilde{X}, \frac{\Omega^{d}_{X/k}}{\mathcal{I}_{nE}} \right) \to H^d_{\text{Zar}} \left( nE, \Omega^{d}_{nE/k} \right) \to 0.
\end{align*}
\]

The exactness of these sequences follows from the right exactness of corresponding sequences of sheaves and the fact that all the underlying sheaves are supported on $E$. The left vertical map from the first factor of the direct sum is the quotient map and hence surjective. Taking the inverse limit over $n$, using the Mittag-Leffler property of these cohomology groups, and using the isomorphism

\[\lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d-1}_{X/k} \otimes \mathcal{O}_{2nE} \right) \cong \lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d-1}_{X/k} \otimes \mathcal{O}_{nE} \right),\]

we obtain

\[
\begin{align*}
\lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d}_{X/k} \otimes \mathcal{O}_{nE} \right) & \cong \lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d}_{X/k} \otimes \mathcal{O}_{2nE} \right) \\
\lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d-1}_{X/k} \otimes \mathcal{O}_{nE} \right) & \cong \lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d-1}_{X/k} \otimes \mathcal{O}_{2nE} \right) \\
& \cong \lim_n H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d-1}_{X/k} \otimes \mathcal{O}_{nE} \right) \\
& \cong \lim_n H^d_{\text{Zar}} \left( nE, \Omega^{d-1}_{nE/k} \right).
\end{align*}
\]

This completes the proof. \(\square\)

**Lemma 4.5.** Let $f : \tilde{X} \to X$ be as in Lemma 4.4. Assume that $H^d_{\text{Zar}} \left( X, \Omega^{d-1}_{X/k} \right) \cong H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d-1}_{\tilde{X}/k} \right)$. Then the natural map $H^d_{\text{Zar}} \left( X, \Omega^{d}_{X/k} \right) \to H^d_{\text{cdh}} \left( X, \Omega^{d}_{X/k} \right)$ is an isomorphism.

**Proof.** We first observe that the map

\[H^d_{\text{Zar}} \left( X, \Omega^{i}_{X/F} \right) \to H^d_{\text{cdh}} \left( X, \Omega^{i}_{X/F} \right)
\]

is surjective for all $i \geq 0$ and all subfields $F \subseteq k$ by [12 Proposition 2.6]. In particular, $H^d_{\text{cdh}} \left( X, \Omega^{d-1}_{X/k} \right) \cong H^d_{\text{cdh}} \left( \tilde{X}, \Omega^{d-1}_{\tilde{X}/k} \right)$. The Leray spectral sequence for $f$, applied to the sheaves $\Omega^{i}_{X/k}$ for the Zariski site, gives the exact sequences

\[H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{i}_{\tilde{X}/k} \right) \to H^0_{\text{Zar}} \left( X, R^d f_* \Omega^{i}_{\tilde{X}/k} \right) \to H^{d+1}_{\text{Zar}} \left( X, \Omega^{i}_{X/k} \right) \to H^{d+1}_{\text{Zar}} \left( \tilde{X}, \Omega^{i}_{\tilde{X}/k} \right) \to 0
\]

for all $i \geq 0$. Applying this for $i \geq d - 1$ and using our assumption, we then get an exact sequence

\[H^d_{\text{Zar}} \left( \tilde{X}, \Omega^{d}_{\tilde{X}/k} \right) \to H^0_{\text{Zar}} \left( X, R^d f_* \Omega^{d}_{\tilde{X}/k} \right) \to H^{d+1}_{\text{Zar}} \left( X, \Omega^{d}_{X/k} \right) \to H^{d+1}_{\text{Zar}} \left( \tilde{X}, \Omega^{d}_{\tilde{X}/k} \right) \to 0.
\]
We now compare this with the similar Mayer-Vietoris exact sequence for the cdh cohomology to get a commutative diagram of exact sequences

\[
\begin{align*}
H^d_{\text{Zar}}\left(\tilde{X}, \Omega^d_{\tilde{X}/k}\right) &\to \frac{H^d_{\text{Zar}}\left(X, R^df_*\Omega^d_{\tilde{X}/k}\right)}{H^d_{\text{Zar}}\left(X, R^df_*\Omega^d_{\tilde{X}/k}\right)} \to H^{d+1}_{\text{Zar}}\left(X, \Omega^d_{\tilde{X}/k}\right) \to \frac{H^{d+1}_{\text{Zar}}\left(X, \Omega^d_{\tilde{X}/k}\right)}{H^{d+1}_{\text{Zar}}\left(X, \Omega^d_{\tilde{X}/k}\right)}
\end{align*}
\]

The left and the right vertical maps on both ends are isomorphisms by [12, Corollary 2.5]. The second vertical map from the left is an isomorphism by Lemma 4.4 and Corollary 3.4. Hence the remaining vertical map is also an isomorphism by 5-lemma.

For the remaining part of this section, recall our convention that a Kähler differential (or Hochschild and cyclic homology) without the mention of the coefficient field means that the underlying field is taken to be \(\mathbb{Q}\).

**Corollary 4.6.** Let \(X\) be as in Lemma 4.4 such that \(H^{d+1}_{\text{Zar}}\left(X, \Omega^i_{X/k}\right) = 0\) for \(0 \leq i \leq d - 1\). Then \(H^{d+1}\left(X, \Omega^i_X\right) = 0\) for the Zariski or the cdh-site and for any \(i \leq d - 1\). Moreover, the map \(H^{d+1}_{\text{Zar}}\left(X, \Omega^i_X\right) \to H^{d+1}_{\text{cdh}}\left(X, \Omega^i_X\right)\) is an isomorphism.

**Proof.** For the first assertion, we only need to show the vanishing of the Zariski cohomology by (4.2). The case \(i \geq 0\) is part of the assumption. For any \(i \geq 1\), there is a filtration \(\{F^j\Omega^i_X\}_{0 \leq j \leq i}\) such that there is a surjection

\[
\Omega^j_{k/Q} \otimes_k \Omega^{i-j}_{X/k} \to \frac{F^j\Omega^i_X}{F^{j+1}\Omega^i_X}
\]

which is an isomorphism on \(X_{\text{smooth}}\) and this latter set is finite. Hence the first assertion follows from our assumption and an easy induction on \(1 \leq i \leq d - 1\) and \(0 \leq j \leq i\). Furthermore, this also implies that the map \(H^{d+1}\left(X, \Omega^i_X\right) \to H^{d+1}\left(X, \Omega^i_{X/k}\right)\) is an isomorphism for both Zariski and cdh-sites. The second assertion of the corollary now follows from Lemma 4.5.

The following is the main result of this section.

**Proposition 4.7.** Let \(X\) be a normal projective \(k\)-variety of dimension \(d + 1 \geq 2\) with only isolated singularities such that \(H^{d+1}_{\text{Zar}}\left(X, \Omega^i_{X/k}\right) = 0\) for \(0 \leq i \leq d - 1\). Then the natural maps

\[
HC_0^{(d+1)}(X) \to \mathbb{H}^0_{\text{cdh}}\left(X, \mathcal{H}C^{(d+1)}\right) \quad \text{and} \\
HC_{-1}^{(d)}(X) \to \mathbb{H}^1_{\text{cdh}}\left(X, \mathcal{H}C^{(d)}\right)
\]

are respectively surjective and isomorphism.

**Proof.** One knows that \(HC_i^{(i)}(A) = \Omega^i_A/\Omega^{i-1}_A\) and \(HC_j^{(i)}(A) = 0\) for \(j > i\) for any local ring of \(X\) (cf. [14, Theorems 4.6.7, 4.6.8]). Hence the spectral sequence

\[
E_2^{p,q} = H^p\left(X, \mathcal{H}C_{X,q}\right) \Rightarrow HC_{-p-q}(X)
\]
Lemma 2.5 now implies that there are commutative diagrams and exact sequences

\[ H^0_{cdh}(X, \mathcal{HC}(d+1)) \cong H_{Zar}^{2d+2}(X, \Omega^{d+1}_X) \] and

\[ H^0_{cdh}(X, \mathcal{HC}(d+1)) \cong H_{Zar}^{2d+1}(X, \mathcal{HC}(d+1)) \].

On the other hand, it follows from [12, Theorem 2.2] that

\[ H^0_{cdh}(X, \mathcal{HC}(d+1)) \cong H_{Zar}^{2d+2}(X, \Omega^{d+1}_X) \] and

\[ H^0_{cdh}(X, \mathcal{HC}(d)) \cong H_{Zar}^{2d+1}(X, \Omega^d_X) \].

Lemma 2.5 now implies that there are commutative diagrams and exact sequences

\[ H^d_{Zar}(X, \Omega^{d+1}_X) \longrightarrow H^0_{cdh}(X, \mathcal{HC}(d+1)) \]

\[ H^d_{Zar}(X, \Omega^{d+1}_X) \longrightarrow H^d_{Zar}(X, \Omega^d_X) \longrightarrow H^d_{cdh}(X, \mathcal{HC}(d+1)) \]

\[ H^d_{Zar}(X, \Omega^{d+1}_X) \longrightarrow H^d_{Zar}(X, \Omega^d_X) \longrightarrow H^d_{cdh}(X, \mathcal{HC}(d+1)) \rightarrow 0. \]

The left vertical map in the top square is surjective by [12]. This proves the first assertion of the proposition. Observe that we have not used the conditions of the proposition until now. In particular, the first surjectivity always holds.

Now assume the given conditions. In this case, the terms on the left ends of both the rows in the bottom diagram are zero by Corollary 4.6. The middle vertical map is an isomorphism again by Corollary 4.6. Hence the right vertical map is also an isomorphism, proving the second assertion. \( \square \)

5. \( K \)-theory and \( KH \)-theory of some singular schemes

Recall from Theorem 2.1 that the Chern character from the algebraic \( K \)-theory to the negative cyclic homology induces a natural weak equivalence \( \tilde{K}(X) \cong \Omega^{-1}\mathcal{HC}(X) \) for any \( k \)-scheme \( X \). In particular, there is a homotopy fibration sequence

\[ \tilde{K}(X) \rightarrow \mathcal{HC}(X)[-1] \xrightarrow{\phi_{\mathcal{HC}}} \mathbb{H}_{cdh}^1(X, \mathcal{HC})[-1]. \]

In particular, the homotopy groups of \( \tilde{K}(X) \) have \( \lambda \)-decomposition such that the above gives a long exact sequence of homotopy groups which preserves this decomposition (cf. [13]). Thus we have \( \tilde{K}_n(X) := \pi_n\tilde{K}(X) = \bigoplus_\lambda \tilde{K}_{n,\lambda}(X) \).

Lemma 5.1. Let \( X \) be a \( k \)-scheme of dimension \( d + 1 \). Then there is an exact sequence

\[ 0 \rightarrow \tilde{K}^{(d+1)}_0(X) \rightarrow H^d_{Zar}(X, \Omega^d_X) \rightarrow H^d_{cdh}(X, \Omega^d_X) \rightarrow 0. \]

Proof. We have the following long exact sequence coming from the above fibration sequence.

\[ H^0_{cdh}(X, \mathcal{HC}(d+1)) \rightarrow \tilde{K}_0^{(d+1)}(X) \rightarrow H^0_{cdh}(X, \mathcal{HC}(d+1)) \to H^0_{cdh}(X, \mathcal{HC}(d+1)) \to \tilde{K}_0^{(d+1)}(X) \]

The lemma now follows from the identification of various terms in this exact sequence in (4.3) and (4.4). Lemma 2.5 and [12, Theorem 2.6] (cf. proof of Proposition 4.4). \( \square \)
Proposition 5.2. Let \( X \) be a normal projective \( k \)-variety of dimension \( d + 1 \geq 2 \) with only isolated singularities such that \( H^{d+1}_{Zar}(X, \Omega^i_{X/k}) = 0 \) for \( 0 \leq i \leq d - 1 \). Then \( \tilde{K}_0^{(d+1)}(X) = 0 \).

Proof. Follows directly from Lemma 5.1 and Proposition 4.7. \( \square \)

5.1. Gamma filtration of \( K_*(X) \). Recall from [50] that for any \( k \)-scheme \( X \) of dimension \( d \), there are natural \( \gamma \)-operations \( \gamma^j \) on \( K_*(X) \) which naturally define the Adams operations \( \psi^j : K_n(X) \to K_n(X) \) for each \( n \) and for \( j \in \mathbb{Z} \). All these operations commute with the pull-back maps on \( K \)-groups of schemes. These \( \gamma \)-operations define a natural decreasing filtration

\[
0 = F_{\gamma}^n K_n(X) \subseteq F_{\gamma}^{n+d} K_n(X) \subseteq \cdots \subseteq F_{\gamma}^0 K_n(X) = K_n(X)
\]

such that \( F_{\gamma}^1 K_0(X) \) is the subgroup of \( K_0(X) \) generated by vector bundles of virtual rank zero. Our purpose here is to describe \( F_{\gamma}^{n+d} K_n(X) \) in terms of algebraic cycles for certain singular schemes. This description will be later used in this work to study the Chow group of zero-cycles on such schemes.

We further recall that \( K_{n,q}(X) := K_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \) has a canonical decomposition \( K_{n,q}(X) = \bigoplus_i K_{n,q}^{(i)}(X) \) in terms of the eigenspaces of the Adams operators \( \psi^j \) (which does not depend on \( j \)). This Adams decomposition is related to the \( \gamma \)-filtration by the natural isomorphism

\[
\frac{F_{\gamma}^i K_{n,q}(X)}{F_{\gamma}^{i+1} K_{n,q}(X)} \cong K_{n,q}^{(i)}(X).
\]

In particular, one has

(5.1) \( F_{\gamma}^{n+d} K_{n,q}(X) \cong K_{n,q}^{(n+d)}(X) \).

The following is the generalization of the Grothendieck Adams-Riemann-Roch theorem for \( K_0 \) (cf. [26]) to the higher \( K \)-theory of singular schemes.

Theorem 5.3. Let \( Y \xrightarrow{f} X \) be a regular embedding of quasi-projective \( k \)-varieties of codimension \( d \geq 1 \). Let \( N_{Y/X} \) denote the normal bundle of \( Y \) in \( X \). Then for any \( n \geq 0 \) and \( j \in \mathbb{Z} \), the diagram

\[
\begin{array}{ccc}
K_n(Y) & \xrightarrow{\theta^j(N_{Y/X})} & K_n(Y) \\
\downarrow f_* & & \downarrow f_* \\
K_n(X) & \xrightarrow{\psi^j} & K_n(X)
\end{array}
\]

is commutative, where \( \theta^j \)'s are the cannibalistic operators (cf. [50, Section 4.5]).

Before we prove this theorem, we prove the following result on the deformation to the normal cone of the regular embedding \( Y \xrightarrow{f} X \). Let \( X' = \mathbb{P}(N_{Y/X} \oplus \mathcal{O}_Y) \), \( M = Bl_{Y \times \{\infty}\}}(X \times \mathbb{P}^1) \) and consider the following deformation to the normal cone
In this diagram, all the vertical arrows are the closed regular embeddings, $i_0$ and $i_\infty$ are the obvious inclusions of $Y$ in $Y \times \mathbb{P}^1$ along the specified points, $i$ and $j$ are inclusions of the inverse images of $\mathbb{A}^1$ and $\infty$ respectively under the map $\pi$, $u$ and $f'$ are zero section embeddings and $p_Y$ is the projection map. The map $\phi$ is the composite $M \to X \times \mathbb{P}^1 \to X$. In particular, one has $p_Y \circ i_0 = p_Y \circ i_\infty = id_Y$ and $\phi \circ h = id_X$.

**Lemma 5.4.** Consider the diagram (5.2) and let $y \in K_n(Y)$. Then there exists $z \in K_n(M)$ such that $f_*(y) = h^*(z)$ and $f'_*(y) = i^*(z)$.

**Proof.** Put $\tilde{y} = p_Y^*(y)$ and $z = F_*(\tilde{y})$. Then

\[
\begin{align*}
f_*(y) &= f_*((p_Y \circ i_0)^*(x)) \\
&= f_* \circ i_0^* \circ p_Y^*(y) = f_* \circ i_0^*(\tilde{y}) \\
&= f_* \circ u^* \circ j^*(\tilde{y}) \\
&= u^* \circ F_*(j^*(\tilde{y})) \\
&= u^* \circ j^* \circ F_*(\tilde{y}) \quad \text{(since $j$ is an open immersion)} \\
&= h^* \circ F_*(\tilde{y}) = h^*(z).
\end{align*}
\]

Similarly,

\[
\begin{align*}
f'_*(y) &= f'_*((p_Y \circ i_\infty)^*(x)) \\
&= f'_* \circ i_\infty^* \circ p_Y^*(y) = f'_* \circ i_\infty^*(\tilde{y}) \\
&= i^* \circ F_*(\tilde{y}) \\
&= i^*(z).
\end{align*}
\]

Here, the fifth equality in the first array and the fourth equality in the second follow from [46, Proposition 2.11].

**Proof of Theorem 5.3.** This theorem for $K_0$ was proven in [26] and was also proven in an axiomatic way in [22, Theorem 6.3] using the method of the deformation to the normal cone. The proof for the higher $K$-theory can also be given using this method. We only give a brief sketch and leave the details to the readers. If $f$ is the zero-section embedding for a vector bundle $X \to Y$, then this is proven by Soulé ([50, Theorem 3]). Actually, Soulé assumes $X$ and $Y$ to be smooth, but the proof of this particular case of the zero-section embedding goes through even in the singular case.

Following the axiomatic approach in [22, Chapter II, Theorem 1.2] for $K_0$, the general case is deduced from the above using the above deformation to the normal cone diagram. The reader can check in loc. cit. that the general case for the higher $K$-theory follows directly from (i) the general case for $K_0$,
(ii) the zero-section embedding case for higher $K$-groups and
(iii) Lemma 5.4
once we have the projection formula for the push-forward and pull-back maps on
the higher $K$-theory for a local complete intersection morphism. But this is already shown in [16]. We refer to [22] Theorem II.1.2, Lemmas V.6.1, 6.2] for more detail.

**Corollary 5.5.** Let $Y \xrightarrow{f} X$ be as in Theorem 5.3. Then for any $n, i \geq 0$, one has

$$f_*(F^d_\gamma K_n,\mathbb{Q}(Y)) \subseteq F^{i+d}_\gamma K_n,\mathbb{Q}(X).$$

**Proof.** This follows directly from Theorem 5.3 exactly in the same way as is proven for $K_0$ in [22] Proposition V.6.4).

For more applications, recall that (cf. [28]) for a $k$-scheme $X$, there are Brown-Gersten spectral sequences

$$E_2^{p,q} = H^p_{\text{Zar}}(X,\mathcal{K}_q) \Rightarrow K_{-p-q}(X).$$

There are similar spectral sequences (cf. [28], Theorem 1]

$$E_2^{p,q} = H^p_{\text{cdh}}(X,\mathcal{K}_q) \Rightarrow KH_{-p-q}(X)$$

for the $KH$-theory, and there is a natural morphism from the first spectral sequence to the second. This induces the associated Brown filtration (as called so in [25]) $F_B^*$ on these $K$-theories.

Let $X$ be a quasi-projective $k$-variety of dimension $d$ and let $x \in X$ be a smooth closed point of $X$. Then $\{x\} \xrightarrow{i} X$ is a regular embedding and hence there is a natural map $i_* : \mathbb{Z} = K_0(\{x\}) \to K_0(X)$. Let $F^d K_0(X)$ denote the subgroup of $K_0(X)$ generated by the images of the classes of smooth points via these maps.

**Corollary 5.6.** Let $X$ be a quasi-projective $k$-variety of dimension $d$. Then

$$CH^d(X) \to F^d K_0(X) \hookrightarrow F^d_\gamma K_0(X) \to F^d_B K_0(X).$$

up to torsion.

**Proof.** The first surjectivity is already known and can be easily proved from the definitions of the terms. The second inclusion follows from Corollary 5.5 and third inclusion follows from [25] (26), p.138].

**Corollary 5.7.** Let $X$ be a normal quasi-projective $k$-variety of dimension $d$ with only isolated singularities. Then there are isomorphisms

$$H^d_{\text{Zar}}(X,\mathcal{K}_{X,d}) \cong CH^d(X) \cong F^d K_0(X) \cong F^d_\gamma K_0(X) \cong F^d_B K_0(X)$$

up to torsion.

**Proof.** The first two isomorphisms are already known (cf. [2] and [10]). The Brown-Gersten spectral sequence implies that there is a surjection $H^d_{\text{Zar}}(X,\mathcal{K}_{X,d}) \to F^d_B K_0(X)$ and hence must be an isomorphism because of the first two isomorphisms. The remaining isomorphisms now follow from Corollary 5.6.

**Theorem 5.8.** Let $X$ be a normal projective $k$-variety of dimension $d$ with only isolated singularities such that $H^i_{\text{Zar}}(X,\Omega^i_{X/k}) = 0$ for $i \leq d-2$. Then the natural map $K_0(X) \to KH_0(X)$ induces the inclusion

$$CH^d(X) \xrightarrow{\sim} F^d_B K_0(X) \hookrightarrow F^d_B KH_0(X).$$

up to torsion.
Proof. If \( d = 1 \), this is well known, so we assume \( d \geq 2 \). The natural map from the first to the second spectral sequence above gives the map \( F^d_B K_0(X) \to F^d_B K H_0(X) \). To prove its inclusion, we can replace \( F^d_B K_0(X) \) by \( F^{d-\gamma} K_0(X) \) by Corollary 5.7. This group can in turn be replaced by \( K_0^{(d)}(X) \) by (5.1). We have seen earlier that there are \( \lambda \)-decompositions on \( K_0(X) \) and \( \tilde{K}_0(X) \), and there is a fibration sequence of spectra \( \tilde{K}(X) \to K(X) \to K H(X) \) by Corollary 2.2. This fibration sequence in particular yields an exact sequence of eigenspaces

\[
\tilde{K}_0^{(d)}(X) \to K_0^{(d)}(X) \to F^d_B K H_0(X).
\]

Thus we only need to show that \( \tilde{K}_0^{(d)}(X) = 0 \). But this is proven in Proposition 5.2. \( \square \)

6. \textit{cdh cohomology and Hodge theory of singular schemes}

We shall assume the ground field \( k \) to be the field of complex numbers \( \mathbb{C} \) for the rest of this paper. For a \( \mathbb{C} \)-scheme \( X \), we shall denote its associated analytic space by \( X_{\text{an}} \). For simplicity of presentation, we shall assume all \( \mathbb{C} \)-schemes to be quasi-projective for the rest of this work. If \( A \) is an abelian group, then the analytic singular cohomology \( H^*(X_{\text{an}}, A) \) will be simply written as \( H^*(X, A) \). For a chain complex \( F^\bullet \) of presheaves of abelian groups on the analytic, Zariski or the \( \text{cdh} \)-site, we shall consider \( F^\bullet \) also as a presheaf of Eilenberg-Mac Lane spectra and write \( R\Gamma(\cdot, F^\bullet) \) as \( \mathbb{H}(\cdot, F^\bullet) \).

The Hodge theory of singular schemes was invented by Deligne in [16] and [17], where he showed using Hironaka’s resolution of singularities that for every \( \mathbb{C} \)-scheme \( X \), the analytic cohomology \( H^*(X, \mathbb{Z}) \) has a mixed Hodge structure and hence is equipped with a natural weight and Hodge filtration. This was achieved by showing the descent property of the singular cohomology: if \( X_{\bullet} \twoheadrightarrow X \) is a smooth and proper simplicial hypercovering, then the map \( \mathbb{Z}_X \to R\pi_*(\mathbb{Z}_{X_{\bullet}}) \) is a weak equivalence. Our purpose here is to interpret the Hodge theory and the Hodge cohomology of \( X \) in terms of the \( \text{cdh} \) cohomology of the algebraic Kühlé differentials. We show in particular that for a singular projective \( \mathbb{C} \)-scheme \( X \), there is a natural isomorphism \( \mathbb{H}^*_\text{cdh}(X, \Omega^\bullet_{X/\mathbb{C}}) \cong H^*(X, \mathbb{C}) \) such that the Hodge filtration corresponds to the Betti filtration on the \( \text{cdh} \) cohomology. We deduce several consequences which are later used to study the Chow groups of zero-cycles on such varieties in terms of \( \text{cdh} \) cohomology of differential forms.

6.1. \textit{cdh-descent for Du Bois complex}. Let \( X \) be a \( \mathbb{C} \)-scheme and let \( X_{\bullet} \twoheadrightarrow X \) be a smooth proper hypercovering of \( X \). Let

\[
\Omega^\bullet_X := R\pi_*(\Omega^\bullet_{X_{\bullet}/\mathbb{C}}).
\]

This complex was invented by Du Bois in [18]. He showed that \( \Omega^\bullet_X \) is a filtered complex of sheaves with quasi-coherent cohomology sheaves such that

\[
F^i\Omega^\bullet_X = R\pi_*(\Omega_{X_{\bullet}/\mathbb{C}}^\leq i)[-i]
\]

and

\[
Gr_F^i(\Omega^\bullet_X) = \frac{F^i\Omega^\bullet_X}{F^{i+1}\Omega^\bullet_X} \cong R\pi_*(\Omega_{X_{\bullet}/\mathbb{C}}^i)[-i].
\]

In particular, \( \Omega^\bullet_X \in D(qc/X) \), where \( D(qc/X) \) is the derived category of quasi-coherent sheaves on \( X \). If \( X \) is projective, then \( \Omega^\bullet_X \) has coherent cohomology.
sheaves. The exact triangle
\[ R\pi_* \left( \Omega^i_{X/\mathbb{C}}[-i] \right) \rightarrow R\pi_* \left( \Omega^i_{X/\mathbb{C}} \right) \rightarrow R\pi_* \left( \Omega^{i-1}_{X/\mathbb{C}} \right) \]
now shows that
\[ \frac{\Omega^i_X}{F^i \Omega^i_X} \cong R\pi_* \left( \Omega^{i-1}_{X/\mathbb{C}} \right). \]

We let
\[ \Omega^i_X := \text{Gr}_F (\Omega^i_X) \]
Lemma 6.1. Let \( X \) be a \( \mathbb{C} \)-scheme of dimension \( d \). Then
\[ F^d \Omega^i_X \cong Rf_* \Omega^i_{X/\mathbb{C}} \text{ and } F^d \Omega^i_X = 0 \text{ in } D(qc/X) \text{ for } i \geq d+1, \]
where \( \tilde{X} \xrightarrow{\sim} X \) is any resolution of singularities of \( X_{\text{red}} \).

Proof. cf. [18] Proposition 4.1. \( \square \)

Lemma 6.2. For any \( \mathbb{C} \)-scheme \( X \) and \( i \geq 0 \), the natural map
\[ H_\text{Zar} (X, F^i \Omega^i_X) \rightarrow H_{\text{cdh}} (X, F^i \Omega^i_X) \]
is a weak equivalence.

Proof. We prove by a descending induction on \( i \geq 0 \). We know from Lemma 6.1 that \( F^{d+1} \Omega^i_X = 0 \), where \( d \) is the dimension of \( X \). In particular, \( F^d \Omega^i_X \cong \text{Gr}_F F^d \Omega^i_X = \Omega^i_X[-d] \). On the other hand, we have for any \( i \geq 0 \),
\begin{align*}
H_{\text{cdh}} (X, \Omega_X^i) & = H_{\text{cdh}} \left( X, R\pi_* \Omega^i_{X/\mathbb{C}} \right) \\
& = H_{\text{cdh}} \left( X, \Omega^i_{X,\mathbb{C}} \right) \\
& = H_{\text{Zar}} \left( X, \Omega^i_{X,\mathbb{C}} \right) \\
& = H_{\text{Zar}} \left( X, R\pi_* \Omega^i_{X,\mathbb{C}} \right) \\
& = H_{\text{Zar}} \left( X, \Omega^i_X \right),
\end{align*}
where the third equality follows from [12] Corollary 2.5 since \( X_\bullet \) is smooth. This in particular proves the result for \( F^{d+1} \Omega^i_X \). Let us now assume that the lemma holds for \( F^{d+1} \Omega^i_X \) and consider the commutative diagram of exact triangles
\[ H_{\text{Zar}} (X, F^{i+1} \Omega^i_X) \rightarrow H_{\text{Zar}} (X, F^i \Omega^i_X) \rightarrow H_{\text{Zar}} (X, \Omega^i_X)[i] \]
\[ H_{\text{cdh}} (X, F^{i+1} \Omega^i_X) \rightarrow H_{\text{cdh}} (X, F^i \Omega^i_X) \rightarrow H_{\text{cdh}} (X, \Omega^i_X)[i]. \]
The left vertical map is a weak equivalence by induction and we have just shown that the right vertical map is also a weak equivalence. Hence so is the middle vertical map. \( \square \)

For a \( \mathbb{C} \)-scheme \( X \), let \( \Omega^\bullet_{X/\mathbb{C}} \) denote the filtered de Rham complex of \( X \) with the Betti filtration \( F^i \Omega^\bullet_{X/\mathbb{C}} = \Omega^\bullet_{X/\mathbb{C}}[-i] \) and \( \text{Gr}_F \Omega^\bullet_{X/\mathbb{C}} = \Omega^\bullet_{X/\mathbb{C}}[-i] \). Note that this is a finite filtration as \( X \) is quasi-projective. The morphism \( X_\bullet \xrightarrow{\pi} X \) induces the natural map of filtered complexes
\[ (\Omega^\bullet_{X/\mathbb{C}}, F) \rightarrow (\Omega^\bullet_X, F), \]
which gives rise to the map of filtered complexes

\[(a^*\Omega^\bullet_{X/\mathbb{C}}, F) \xrightarrow{\theta_X} (a^*\Omega^\bullet_X, F).\]

These maps are known (cf. [18]) to be weak equivalences if \(X\) is smooth. For singular schemes, they are related by the following.

**Proposition 6.3.** (cf. [14] Lemma 7.1) The above map induces the weak equivalence

\[H_{\text{cdh}}(X, F^i\Omega^\bullet_{X/\mathbb{C}}) \xrightarrow{\sim} H_{\text{Zar}}(X, F^i\Omega^\bullet_X)\]

for every \(i \geq 0\).

**Proof.** By Lemma 6.2, we can replace the right hand side by the corresponding \(\text{cdh}\) hypercohomology. As in Lemma 6.1, we prove by a descending induction on \(i\). We first show the statement of the proposition for the graded pieces. Since we are now working with the \(\text{cdh}\) cohomology, we can assume that \(X\) is reduced. We use an induction on the dimension \(d\) of \(X\). If \(d = 0\), then \(X\) is smooth and hence the statement is obvious. So suppose that \(d \geq 1\). Let \(\tilde{X} \xrightarrow{f} X\) be a resolution of singularities of \(X\). Let \(S \hookrightarrow X\) denote the singular locus of \(X\) and let \(\tilde{S} \hookrightarrow \tilde{X}\) be its inverse image. Then descent property of the \(\text{cdh}\) cohomology gives a commutative diagram of exact triangles

\[
\begin{align*}
H_{\text{cdh}}(\tilde{S}, \Omega^i_{\tilde{S}/\mathbb{C}})[-1] &\to H_{\text{cdh}}(X, \Omega^i_{X/\mathbb{C}}) \to H_{\text{cdh}}(\tilde{X}, \Omega^i_{\tilde{X}/\mathbb{C}}) \oplus H_{\text{cdh}}(S, \Omega^i_S) \\
&\downarrow \downarrow \\
H_{\text{cdh}}(\tilde{S}, \Omega^i_{\tilde{S}/\mathbb{C}})[-1] &\to H_{\text{cdh}}(X, \Omega^i_X) \to H_{\text{cdh}}(\tilde{X}, \Omega^i_{\tilde{X}}) \oplus H_{\text{cdh}}(S, \Omega^i_S),
\end{align*}
\]

where the left and the right vertical maps are weak equivalence by induction on dimension and by the smoothness of \(\tilde{X}\). Hence the middle one is also a weak equivalence. Now suppose the proposition is proven for \(F^{\geq i+1}\) and use the same argument as in the proof of Lemma 6.2 to get the proof for \(F^i\). \(\square\)

We now collect some important consequences of the above comparison results.

**Corollary 6.4.** For any \(\mathbb{C}\)-scheme \(X\) and any \(n \geq 0\), there is a natural isomorphism

\[H^n_{\text{cdh}}(X, \Omega^\bullet_{X/\mathbb{C}}) \xrightarrow{\sim} H^n(X, \mathbb{C}).\]

If \(X\) is projective, there is a spectral sequence

\[E_1^{p,q} = H_{\text{cdh}}^p(X, \Omega^q_{X/\mathbb{C}}) \Rightarrow H^{p+q}(X, \mathbb{C}).\]

Moreover, this spectral sequence degenerates and the induced filtration on \(H^*(X, \mathbb{C})\) coincides with its Hodge filtration.

**Proof.** It follows directly from Proposition 6.3 and [18] Theorem 4.5. \(\square\)

**Corollary 6.5.** For a projective \(\mathbb{C}\)-scheme \(X\), the Hodge filtration on \(H^n(X, \mathbb{C})\) is given by

\[H^n(X, \mathbb{C}) = F^0H^n(X, \mathbb{C}) \supseteq \cdots \supseteq F^nH^n(X, \mathbb{C}) \supseteq F^{n+1}H^n(X, \mathbb{C}) = 0,\]

where

\[\frac{F^iH^n(X, \mathbb{C})}{F^{i+1}H^n(X, \mathbb{C})} \cong H^i_{\text{cdh}}(X, \Omega^i_{X/\mathbb{C}}).\]
Moreover, there is a natural isomorphism

$$H^n_{cdh} \left(X, \Omega^{\geq i}_{X/\mathbb{C}} \right) \xrightarrow{\sim} H^i_{cdh} \left(X, F^i \Omega^\bullet_X \right) \xrightarrow{\sim} F^i H^n \left(X, \mathbb{C} \right)$$

for all $n, i \geq 0$.

**Proof.** The first assertion follows at once from Corollary 6.4. For the second, we use the isomorphism $F^i \Omega^\bullet_X \cong R\pi_* \left( \Omega^{\geq i}_{X/\mathbb{C}} \right)$ from (6.2), which gives

$$H^i_{cdh} \left(X, \Omega^{\geq i}_{X/\mathbb{C}} \right) \cong H^i_{Zar} \left(X, F^i \Omega^\bullet_X \right) \cong H^i_{Zar} \left(X, R\pi_* \left( \Omega^{\geq i}_{X/\mathbb{C}} \right) \right) \cong H^i_{Zar} \left(X, \Omega^{\geq i}_{X/\mathbb{C}} \right) \cong F^i H^n \left(X, \mathbb{C} \right),$$

where the first isomorphism follows from Proposition 6.3 and the last isomorphism follows from [3, (1),(3)]. This proves the second assertion. \[\square\]

Another consequence of Proposition 6.3 is the following simple criterion for the Du Bois singularities of a variety.

**Corollary 6.6.** Let $X$ be a quasi-projective variety over a field $k$ of characteristic zero. Then $X$ has Du Bois singularities if and only if

$$H^i_{Zar} \left(X, \mathcal{O}_X \right) \xrightarrow{\sim} H^i_{cdh} \left(X, \mathcal{O}_X \right)$$

for all $i \geq 0$.

**Proof.** Recall from [51] that $X$ is said to have Du Bois singularity if the natural map $\mathcal{O}_X \to \Omega^0_X$ is a quasi-isomorphism. By the Lefschetz pencil argument (cf. [47]), we can assume the base field to be $\mathbb{C}$. But it is then an immediate consequence of Proposition 6.3. \[\square\]

The following well known fact is a simple consequence of Proposition 3.3 and Corollary 6.6.

**Corollary 6.7.** A normal crossing singularity is Du Bois.

**Remark 6.8.** There have been a lot of work by various people on Du Bois singularities, partially due to the complicated nature of the Du Bois complex $\Omega^0_X$. One hopes that the above criterion of such a singularity in terms of the cdh cohomology of the structure sheaf might play a useful role in the study of Du Bois singularity. For example, it was conjectured by Kollar (proven now by Kollar and Kovács [31]) that the log canonical singularity is Du Bois. One could ask if the above criterion would help in simplifying the proof of [31].

6.2. **Deligne complexes.** Recall from [19] and [3, Section 1] that for a projective $\mathbb{C}$-scheme $X$ and for the morphism of analytic sites $X_{an} \xrightarrow{\pi} X_{an}$, the Deligne complex of $X$ is defined as the complex $\mathbb{Z}_D(\pi) = R\pi_* \left( \mathbb{Z}_{D,X}(\pi) \right)$, where

$$(6.7) \quad \mathbb{Z}_{D,X}(\pi) := \left( \mathbb{Z}(\pi) \xrightarrow{(2\pi i)^n} \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{q-1}_X \right)$$

is the Deligne complex of the smooth simplicial scheme $X_\bullet$. In particular, there is an exact triangle

$$(6.8) \quad R\pi_* \left( \Omega^{\leq q}_X \right) [-1] \to \mathbb{Z}_D(\pi) \to \mathbb{Z}(\pi).$$
It follows at once (using GAGA) from (6.4) and Proposition 6.3 that there is an exact triangle

\[(6.9) \quad \left( R\alpha_* \Omega_X^{<q} \right)[{-1}] \to Rb_* \mathcal{D}(q) \to Rb_* \mathcal{Z}(q), \]

where \(X_{an} \to X_{Zar}\) is the usual morphism of sites. The Deligne cohomology groups of \(X\) are defined by \(H^n_D(X, \mathbb{Z}(q)) := H^n(X_{an}, \mathcal{D}(q))\).

For singular \(\mathbb{C}\)-schemes, Levine [41] had introduced a modified version of the classical Deligne cohomology. It turns out that there are Chern class maps from the algebraic \(K\)-theory to this modified Deligne cohomology which detect more nontrivial elements in the \(K\)-groups of the singular schemes than the above classical Deligne cohomology. We refer to [34] for some applications of the Chern classes into the modified Deligne cohomology. For a projective \(\mathbb{C}\)-scheme \(X\), the \textit{modified Deligne cohomology groups} \(H^n_D(X, \mathbb{Z}(q))\) are defined as the analytic hypercohomology of the truncated complex

\[(6.10) \quad \mathbb{Z}_{\mathcal{D}^*}(q) := \left( \mathbb{Z}(q) \xrightarrow{(2\pi i)^q} \mathcal{O}_X \to \Omega_X^{1} \to \cdots \to \Omega_X^{q-1} \right)\]

For a morphism \(Y \to X\) of schemes, the relative modified Deligne cohomology groups \(H^n_{D^*}(X, \mathbb{Z}(q))\) are defined as the hypercohomology of the complex \(\text{Cone}(\mathbb{Z}_{D^*}(q) \to Rf_* \mathbb{Z}_{D^*}(Y(q))[-1]).\)

It is clear from the definition that the modified Deligne cohomology agrees with the classical one for smooth projective varieties. In particular, a smooth proper hypercovering \(X_{\bullet} \to X\) defines a natural map \(\mathbb{Z}_{\mathcal{D}^*}(q) \to \mathbb{Z}_{\mathcal{D}}(q)\). Moreover, it follows from (6.9) that there is a commutative diagram of exact triangles

\[(6.11) \quad \Omega_X^{<q}[{-1}] \longrightarrow Rb_* \mathbb{Z}_{\mathcal{D}^*}(q) \longrightarrow Rb_* \mathbb{Z}(q) \]

\[\alpha \downarrow \quad \alpha \downarrow \quad \alpha \]

\[\left( R\alpha_* \Omega_X^{<q} \right)[{-1}] \to Rb_* \mathcal{D}(q) \to Rb_* \mathcal{Z}(q). \]

**Lemma 6.9.** There is a commutative diagram of long exact sequences

\[\cdots \to \mathbb{H}^n_{\text{Zar}} \left( X, \Omega_X^{<q} \right) \to \mathbb{H}^n_{\mathcal{D}}(X, \mathbb{Z}(q)) \to H^{n+1}(X, \mathbb{Z}) \to H^*_{\text{Zar}} \left( X, \Omega_X^{<q} \right) \cdots \]

\[\alpha_q \downarrow \quad \cdots \to \mathbb{H}^n(X, \mathbb{C}) \to H^1_{\mathcal{D}}(X, \mathbb{Z}) \to H^{n+1}(X, \mathbb{Z}) \to \mathbb{H}^{n+1}(X, \mathbb{C}) \cdots \]

where the \(\alpha_q\) are all surjective.

**Proof.** The exact sequences and commutative diagram follow from (6.11) and Corollary 6.5. To prove the required surjectivity, we consider the commutative diagram

\[\mathbb{H}^n_{\text{Zar}} \left( X, \Omega_X^{\bullet} \right) \to \mathbb{H}^n_{\text{Zar}} \left( X, \Omega_X^{<q} \right) \]

\[\alpha_q \downarrow \quad \cdots \to \mathbb{F}^q H^n(X, \mathbb{C}) \to H^n(X, \mathbb{C}) \to \mathbb{H}^n(X, \mathbb{C}) \cdots \]
The bottom sequence is exact and \( \frac{H^p(X, \mathbb{C})}{F_pH^p(X, \mathbb{C})} \simeq H^p_{cdh} \left( X, \Omega^q_{X/\mathbb{C}} \right) \) by Corollary 6.5. On the other hand, the composite map \( C_{X_{an}} \to \Omega^*_{X_{an}} \to R\pi_*\Omega^*_{X_{an}} \) is an isomorphism by the cohomological descent of analytic cohomology and the de Rham theorem. In particular, the left vertical map is split surjective. Hence the right vertical map is surjective too.

\[ J^p(\mathcal{X}) = \frac{H^{2p-1}(X, \mathbb{C}(p))}{F_pH^{2p-1}(X, \mathbb{C}(p)) + H^{2p-1}(X, \mathbb{Z}(p))} \cong \frac{H^{2p-1}_{cdh}(X, \Omega^q_{X/\mathbb{C}})}{H^{2p-1}(X, \mathbb{Z}(p))}, \]

where the second isomorphism follows from Corollary 6.5. If \( X \) is smooth, the intermediate Jacobian can also be written as \( J^p(\mathcal{X}) = \frac{H^{2p-1}(X, \mathbb{R})}{H^{2p-1}(X, \mathbb{Z})} \). In particular, this is a real torus. In fact, one knows that this has a complex structure which is smooth, the intermediate Jacobian can be written as \( J^p(\mathcal{X}) = \frac{H^{2p-1}(X, \mathbb{R})}{H^{2p-1}(X, \mathbb{Z})} \). In particular, this is a real torus. In fact, one knows that this has a complex structure which makes it a complex torus. In general, \( J^p(\mathcal{X}) \) does not admit a polarization, i.e., it is not an abelian variety. However, in case \( H^i(X) = 0 \) for \( |i - j| > 1 \) and \( i + j = 2p - 1 \), then, \( J^p(\mathcal{X}) \) is indeed an abelian variety (cf. [43, p.171, 172]). We conclude in particular that \( J^{d-1}(\mathcal{X}) \) is an abelian variety if \( H^d(X, \Omega^*_X) = 0 \) for \( 0 \leq i \leq d - 2 \).

For a smooth and projective \( \mathbb{C} \)-scheme \( X \) of dimension \( d \) and \( p \geq 0 \), let \( A^p(X) = CH^p(X)_{alg} \) be the subgroup of \( CH^p(X) \) consisting of algebraic cycles which are algebraically equivalent to zero. Let \( CH^p(X)_{hom} \) be the subgroup of homologically trivial cycles in \( CH^p(X) \), i.e., this is simply the kernel of the topological cycle class map \( CH^p(X) \to H^{2p}(X, \mathbb{Z}) \). One knows that \( A^p(X) \subset CH^p(X)_{hom} \subset CH^p(X) \). There are Abel-Jacobi maps

\[ CH^p(X)_{hom} \xrightarrow{A^p} J^p(\mathcal{X}). \]

Let \( J^p_a(X) \) denote the image of \( A^p(X) \) under the Abel-Jacobi map. Then \( J^p_a(X) \) is an abelian subvariety of \( J^p(X) \). Recall that the famous Hodge conjecture says that the topological cycle class map \( CH^p(X)_{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^p(X, \Omega^q_X) \) is surjective. This conjecture is known for \( p \in \{0, 1, d - 1, d\} \). We now recall the general Hodge conjecture.

For any \( l \geq 1 \) and \( p \geq 0 \), let

\[ F^p_lH^l(X, \mathbb{Q}) = \bigcup_{\text{codim}_X(Y) \geq p} \text{Ker} \left( H^l(X, \mathbb{Q}) \to H^l(X - Y, \mathbb{Q}) \right). \]

It is known that \( F^p_lH^l(X, \mathbb{Q}) \) has a Hodge structure and

\[ F^p_lH^l(X, \mathbb{Q}) \subset F^pH^l(X, \mathbb{C}) \cap H^l(X, \mathbb{Q}). \]

The general Hodge conjecture says the following.

**Conjecture 6.10.** \((\text{GHC}(p, l, X))\) \( F^p_lH^l(X, \mathbb{Q}) \) is the largest sub-Hodge structure of \( F^pH^l(X, \mathbb{C}) \cap H^l(X, \mathbb{Q}). \)

There are few cases when this conjecture is known. On case which interests us is when \( X \) is a hypersurfaces in \( \mathbb{P}^d \) of degree \( \leq 4 \). In this case, \( \text{GHC}(1, 3, X) \) is known (cf. [43, Chapter 7.IX]). It is also known for any smooth projective variety \( X \) that \( \text{GHC}(p - 1, 2p - 1, X) \) is equivalent to saying that \( J^p_a(X) \) is the largest abelian subvariety of \( J^p(X) \). We conclude:
Corollary 6.11. Let $X$ be a smooth projective variety of dimension $d$ such that $H^d(X, \Omega^i_X) = 0$ for $0 \leq i \leq d - 2$. Then

$$GHC(d - 2, 2d - 3, X) \Leftrightarrow J^{d-1}_d(X) = J^{d-1}(X).$$

Now we come back to the general case of singular projective schemes. Suppose $X$ is a singular projective $\mathbb{C}$-scheme of dimension $d$. If $\tilde{X} \to X$ is a resolution of singularities of $X$, then the isomorphism $H^{2d}(X, \mathbb{Z}(d)) \xrightarrow{\cong} H^{2d}(\tilde{X}, \mathbb{Z}(d))$ and Lemma 6.9 yields an exact sequence

$(6.14) \quad 0 \to J^d(X) \to H^{2d}_D(X, \mathbb{Z}(q)) \to H^{2d}(X, \mathbb{Z}(d)) \to 0.$

We define the $p$th generalized intermediate Jacobian as

$(6.15) \quad J^p_*(X) = \frac{H^{2p-1}_{\text{Zar}}(X, \Omega^d_X)}{H^{2p-1}(X, \mathbb{Z}(p))}.$

It follows from Lemma 6.9 that the natural map $J^p_*(X) \to J^p(X)$ is surjective. $J^p_*(X)$ and $J^p(X)$ are also called the generalized albanese and the albanese varieties of $X$. In fact, one knows from [21, Theorem 2] that $J^d_*(X)$ is a commutative algebraic group over $\mathbb{C}$ and $J^d(X)$ is its universal semi-abelian quotient variety. Recall from [21] that the Chow group of zero-cycles $CH^d(X)$ is defined as the free abelian group of smooth closed points of $X$ modulo the subgroup generated by the cycles defined by the rational functions on all the Cartier curves on $X$. We refer to loc. cit. for the complete definition. Let $A^d(X) = CH^d(X)_{\text{deg}0}$ denote the kernel of the map $CH^d(X) \xrightarrow{\text{deg}} H^{2d}(X, \mathbb{Z})$. We have seen in (1.2) that there a regular map $A^d(X) \xrightarrow{\text{deg}} J^d(X)$ which is a universal regular quotient.

7. Chern Classes on $K$-Theory

Recall from [4] (see also [24], [3]) that for a $\mathbb{C}$-scheme $X$, there are natural Chern classes

$(7.1) \quad c_{q,p} : K_{2q-p}(X) \to \mathbb{H}^p_D(X, \mathbb{Z}(q)),$

which have all the functorial properties with respect to the pull-back maps on $K$-theory and Deligne cohomology. These functorial properties also define such Chern class maps from the relative $K$-groups to the relative Deligne cohomology. Gillet [24] has constructed universal Chern classes into generalized cohomology theories of which the above is a special case. It was shown in [11] that the modified Deligne cohomology also satisfies Gillet’s conditions for being a generalized cohomology theory and hence there are functorially defined Chern classes

$(7.2) \quad c^*_{q,p} : K_{2q-p}(X) \to \mathbb{H}^p_D(X, \mathbb{Z}(q))$

such that the diagram

$(7.3) \quad K_{2q-p}(X) \xrightarrow{c^*_{q,p}} \mathbb{H}^p_D(X, \mathbb{Z}(q)) \xrightarrow{\iota} \mathbb{H}^p_D(X, \mathbb{Z}(q)) \xrightarrow{c_{q,p}} H^p(X, \mathbb{Z}(q))$

commutes, where $c^*_{q,p}$ is the topological Chern class.
Proposition 7.1. For any $\mathbb{C}$-scheme $X$, there are Chern class maps
\[ c^h_{q,p} : KH_{2q-p}(X) \to \mathbb{H}^p_D(X, \mathbb{Z}(q)) \]
such that the composite $K_{2q-p}(X) \to KH_{2q-p}(X) \to \mathbb{H}^p_D(X, \mathbb{Z}(q))$ is the Chern class map of \((7.1)\). Furthermore, these Chern classes are functorial with the pull-back maps of (relative) $KH$-theory and the Deligne cohomology.

Proof. The Chern classes from the homotopy invariant $K$-theory are obtained by comparing it with the descent $K$-theory of \([45]\). It was shown in \([45, \text{Theorem 4.1}]\) that the algebraic $K$-theory functor $K : (Sm/\mathbb{C})^{op} \to HoSp$ from the category of smooth schemes over $\mathbb{C}$ to the homotopy category of spectra is a functor to a descent category in the sense of Guillén-Navarro \([27]\). Hence it uniquely extends to a functor $KD : (Sch/\mathbb{C})^{op} \to HoSp$ which satisfies the descent property in the sense that if $X_\bullet \rightarrow X$ is a Guillén-Navarro smooth proper hypercubical resolution of $X$, then $KD(X_\bullet) \cong K(X_\bullet)$. One defines $KD_i(X) = \pi_i(KD(X))$.

Using the cdh-descent property of $KH$-theory (cf. \([28]\)) and the uniqueness of the descent $K$-theory, it was shown in loc. cit. that there is a weak equivalence $KH(X) \cong KD(X)$ of spectra. To see that the Chern class maps of \((7.1)\) descend to $KD(X)$, we choose a proper smooth hypercubical resolution $X_\bullet \rightarrow X$ as above. This gives natural maps
\[
\begin{array}{cccc}
K D_{2q-p}(X) & \rightarrow & K D_{2q-p}(X_\bullet) & \rightarrow & K_{2q-p}(X_\bullet) \\
\downarrow c^h_{2q-p} & & & & \downarrow c_{q,p} \\
\mathbb{H}^p_D(X, \mathbb{Z}(q)) & \cong & \mathbb{H}^p_D(X_\bullet, \mathbb{Z}(q)) & \cong & \mathbb{H}^p_D(X_\bullet, \mathbb{Z}(q)) \\
\end{array}
\]
giving the desired factorization. The functoriality of $c^h$ now easily follows from the above using similar properties the usual Chern classes of smooth schemes. \qed

Recall that the descent spectral sequence \((5.4)\) induces a functorial Brown filtration $F^*_B$ on the $KH$-theory. Since we shall be considering only this filtration on $KH_*(X)$, we shall drop the subscript and simply write $F^*KH_*(X)$.

7.1. Milnor and Quillen $K$-theory. Recall that for a $\mathbb{C}$-algebra $A$, the Milnor $K$-theory $K^M_*(A)$ is the quotient of the tensor algebra $T(A^*)$ of units in $A$ over $\mathbb{Z}$ by the two-sided ideal generated by homogeneous elements $\{a \otimes (1-a)|a, 1-a \in A^*\}$. For any variety $X$ over $\mathbb{C}$, let $K^M_{m,x}$ denote the sheaf of Milnor $K$-groups on $X$. This is the sheaf associated to the presheaf which on affine open subsets of $X$ is given by above. In particular, its stalk at any point $x$ of $X$ is the Milnor $K$-group of the local ring $O_{X,x}$. For any closed embedding $i : Y \hookrightarrow X$, let $K^M_{m,(X,Y)}$ be the sheaf of relative Milnor $K$-groups defined so that the sequence of sheaves
\[(7.4)\]
\[ 0 \to K^M_{m,(X,Y)} \to K^M_{m,X} \to i_*(K^M_{m,Y}) \to 0 \]
is exact. Note that the map $K^M_{m,X} \to i_*(K^M_{m,Y})$ is always surjective.

There is a natural map of $K$-theory sheaves $K^M_{*,X} \to K_{*,X}$ for any variety $X$, and it is known (cf. \([50]\)) that this map is injective up to torsion and the Milnor $K$-sheaves are the smallest piece of the gamma filtration on the corresponding Quillen $K$-sheaves. It is well known (loc. cit.) that the Chow groups of algebraic cycles on smooth varieties can also be described as the cohomology of Milnor $K^M$-sheaves. The results of \([33]\) and \([35]\) suggest that similar identifications should be valid for
singular varieties as well. In fact, we strongly suspect the following which seems to be believed by experts.

**Conjecture 7.2.** For a quasi-projective scheme $X$ of dimension $d$ over an algebraically closed field $k$ of characteristic zero, there is an isomorphism

$$CH^d(X) \cong H^d_{\text{Zar}}(X, \mathcal{K}^M_d).$$

One also believes that $H^d_{\text{Zar}}(X, \mathcal{K}^M_d)$ is in general smaller than $H^d_{\text{Zar}}(X, \mathcal{K}_d)$. However, there is one case where they coincide. We refer to [33, Corollary 4.2] for a proof.

**Proposition 7.3.** Let $X$ be an affine or projective variety of dimension $d$ over $\mathbb{C}$ with only isolated singularities. Then

$$CH^d(X) \cong H^d_{\text{Zar}}(X, \mathcal{K}^M_d) \cong H^d_{\text{Zar}}(X, \mathcal{K}_d).$$

We shall also need the following comparison result for the cohomology of Milnor and Quillen $K$-sheaves on smooth schemes.

**Lemma 7.4.** Let $X$ be a smooth $\mathbb{C}$-scheme of dimension $d$. Then for any $i \geq 0$,

$$H^i_{\text{Zar}}(X, \mathcal{K}^M_i) \cong H^i_{\text{Zar}}(X, \mathcal{K}_i) \quad \text{for} \quad j \geq i - 1.$$

**Proof.** This follows directly by comparing the Gersten resolutions for appropriate $\mathcal{K}^M$ and $\mathcal{K}$-sheaves and using the fact that $K^M_i(R) \cong K_i(R)$ for any local ring $R$ over $\mathbb{C}$ and for $0 \leq i \leq 2$. The Gersten resolution for the Milnor $K$-sheaves is proven in [30]. \qed

For a quasi-projective scheme $X$ of dimension $d$ over $\mathbb{C}$ and $i \geq 0$, let

$$F^dK^M_i(X) := \text{Image} \left( H^d_{\text{Zar}}(X, \mathcal{K}^M_{d+i}) \to K_i(X) \right),$$

and

$$F^dKH^M_i(X) := \text{Image} \left( H^d_{\text{cdh}}(X, \mathcal{K}^M_{d+i}) \to KH_i(X) \right).$$

Note that these maps are induced by the natural maps from Milnor sheaves to Quillen sheaves followed by the maps induced by the Brown-Gersten spectral sequences.

For the rest of this paper, we choose and fix the following resolution of singularities diagram for a projective $\mathbb{C}$-scheme $X$ of dimension $d$.

$$E \xrightarrow{\gamma} \tilde{X} \xrightarrow{f} X,$$

where $S = X_{\text{sing}}$ and $E = f^{-1}(S)$ is the reduced exceptional divisor which is assumed to be strict normal crossing. We recall the following Mayer-Vietoris property of the Deligne cohomology from [3, Variant 3.2].

**Lemma 7.5.** ([3, Variant 3.2]) The above resolution diagram induces the following long exact sequence of Deligne cohomology.

$$\cdots \to H^i_{D}(X, \mathbb{Z}(q)) \to H^i_{D}(\tilde{X}, \mathbb{Z}(q)) \oplus H^i_{D}(S, \mathbb{Z}(q)) \to H^i_{D}(E, \mathbb{Z}(q)) \to H^{i+1}_{D}(X, \mathbb{Z}(q)).$$

Such a Mayer-Vietoris exact sequence also holds for the singular cohomology. The functoriality property of the Chern classes with the pull-back maps of (relative) $K$-theory and Deligne cohomology gives such maps between the Mayer-Vietoris sequences of $KH$-theory and Deligne cohomology.
8. Chern classes for normal crossing schemes

In this section, we prove some results about the cdh cohomology of $K$-sheaves and the Chern classes from them into the Deligne cohomology of normal crossing schemes.

Lemma 8.1. Let $E$ be a strict normal crossing divisor of dimension $d$ on a smooth scheme. Then the cup product map

$$H^2_d (E, \mathbb{Z}(d)) \otimes H^1_D (E, \mathbb{Z}(1)) \to H^2_{d+1} (E, \mathbb{Z}(d+1))$$

is surjective.

Proof. We can assume that $E$ is connected. It is easy to see that $H^1_D (E, \mathbb{Z}(1)) \cong H^0 (E, \mathbb{C}^*) \cong \mathbb{C}^*$ (cf. [3, Section 1]). It follows from Lemma 6.9 that

$$H^2_{d+1} (E, \mathbb{Z}(d+1)) = H^2_d (E, \mathbb{C}^*) / F^{d+1} H^2_d (E, \mathbb{C}) \cong H^2_d (E, \mathbb{Z}) \otimes \mathbb{C}^* / F^{d+1} H^2_d (E, \mathbb{C}).$$

On the other hand, if $E^N \to E$ is the smooth normalization of $E$, then the map $H^2_d (E, \mathbb{Z}) \to H^2_d (E^N, \mathbb{Z})$ is an isomorphism of Hodge structures. Moreover, $F^{d+1} H^2_d (E^N, \mathbb{C}) = 0$ by Hodge theory. In particular, we get $F^{d+1} H^2_d (E, \mathbb{C}) = 0$. We conclude that

$$H^2_d (E, \mathbb{Z}(d)) \otimes \mathbb{C}^* \cong H^2_d (E, \mathbb{Z})(d+1)).$$

In particular, we get

$$H^2_d (E, \mathbb{Z}(d)) \otimes H^1_D (E, \mathbb{Z}(1)) \cong H^2_d (E, \mathbb{Z}(d)) \otimes \mathbb{C}^* \cong H^2_{d+1} (E, \mathbb{Z}(d+1)).$$

which shows that the left vertical arrow is surjective.

We next recall the following result which signifies the importance of using Milnor $K$-sheaves in place of Quillen $K$-sheaves to study algebraic cycles on singular varieties. We refer to [35, Proposition 8.2] for a proof. It is not clear if such a result is true for the cohomology of Quillen $K$-sheaves.

Proposition 8.2. Let $E$ be as in Lemma 8.1. Then the natural cup product map

$$H^d (E, \mathcal{K}^M_{i,E}) \otimes \mathbb{C}^* \to H^d (E, \mathcal{K}^M_{i+1,E})$$

is surjective for all $i \geq d$ in either of Zariski and cdh topology.

We remark here that the proof of the above proposition in [35, Proposition 8.2] is given for the Zariski cohomology, but the same (in fact easier) proof also works for the cdh cohomology.

Corollary 8.3. Let $E$ be as in Lemma 8.1. Then the cup product maps in $K$-theory induce the following diagram:

$$H^d_{cdh} (E, \mathcal{K}^M_d) \otimes \mathbb{C}^* \to H^d_{cdh} (E, \mathcal{K}^M_{d+1}) \to F^d KH^M_0 (E) \otimes \mathbb{C}^* \to F^d KH^M_1 (E),$$
where all the arrows are surjective.

Proof. Follows immediately from Proposition 8.2 and (7.3).

Lemma 8.4. Let $E$ be as in Lemma 8.1. Then the Chern class maps

$$H^d_{cdh}(E, K^M_d) \to \mathbb{H}^{2d}_D(E, \mathbb{Z}(d))$$

$$H^d_{cdh}(E, K^M_{d+1}) \to \mathbb{H}^{2d+1}_D(E, \mathbb{Z}(d+1))$$

are surjective.

Proof. We prove the first assertion by induction on dimension of $E$. If $E$ is smooth, then we can replace $K^M$ by $K$ using Lemma 7.4. The assertion is then standard. If $E$ has dimension zero, then it is smooth. So assume that the result holds for all strict normal crossing divisors of dimension less than $d$ which is at least one. Let $E_N \rightarrow E$ be the normalization map and let $\tilde{S} = f^{-1}(S = E_{\text{sing}})$. Then we have the commutative diagram of exact sequences

$$H^{d-1}_{cdh}(\tilde{S}, K^M_d) \to H^d_{cdh}(E, K^M_d) \to H^d_{cdh}(E_N, K^M_d) \to 0$$

$$\mathbb{H}^{2d-1}_D(\tilde{S}, \mathbb{Z}(d)) \to \mathbb{H}^{2d}_D(E, \mathbb{Z}(d)) \to \mathbb{H}^{2d}_D(E_N, \mathbb{Z}(d)) \to 0.$$

Since $\tilde{S}$ is a strict normal crossing divisor on $E_N$ (which is smooth), the map $H^{d-1}_{cdh}(\tilde{S}, K^M_{d-1}) \to \mathbb{H}^{2d-2}_D(\tilde{S}, \mathbb{Z}(d-1))$ is surjective by induction. Now the commutative diagram

$$H^{d-1}_{cdh}(\tilde{S}, K^M_{d-1}) \otimes \mathbb{C}^* \to H^{d-1}_{cdh}(\tilde{S}, K^M_d)$$

$$\mathbb{H}^{2d-2}_D(\tilde{S}, \mathbb{Z}(d-1)) \otimes \mathbb{C}^* \to \mathbb{H}^{2d-1}_D(\tilde{S}, \mathbb{Z}(d))$$

and Lemma 8.1 imply that the left vertical map in (8.3) is surjective. Hence so is the middle vertical map. This proves the first surjectivity of the lemma. The second surjectivity now follows from the first, Lemma 8.1 and the commutative diagram (8.4) for $E$.

It follows from [21, Theorem 2] and [34, Lemma 2.2] that the Chern class map in 7.2 from the algebraic $K$-theory to the modified Deligne cohomology gives rise to a commutative diagram of exact sequences

$$0 \to F^dK_0(X)_{\text{deg}0} \to F^dK_0(X) \xrightarrow{c^0} H^{2d}(X, \mathbb{Z}) \to 0$$

$$0 \to J^d_*(X) \to H^{2d}_D(X, \mathbb{Z}(d)) \to H^{2d}(X, \mathbb{Z}) \to 0.$$

In fact, one knows from [21, Theorem 2] that $J^d_*(X)$ is a smooth and commutative algebraic group which is a universal regular quotient of $A^d(X) = F^dK_0(X)_{\text{deg}0}$ in the category of smooth commutative group algebraic groups over $\mathbb{C}$. The following is the $cdh$ analogue of this albanese map.
For a smooth and projective $\mathbb{C}$-scheme $X$ of dimension $d$ and $i \geq 0$, let
\begin{equation}
H^i_{cdh} (X, \mathcal{K}_M^d)_{\text{hom}} = \text{Ker} \left( H^i_{cdh} (X, \mathcal{K}_i^M) \to H^{2i} (X, \mathbb{Z}(i)) \right).
\end{equation}

Note that this group is same as the subgroup of $\text{CH}^i (X)_{\text{hom}}$ by Lemma 7.4. We shall also write $H^d_{cdh} (X, \mathcal{K}_d^d)_{\text{deg0}}$ as $H^d_{cdh} (X, \mathcal{K}_d^M)_{\text{deg0}}$. This makes sense even if $X$ is singular.

**Proposition 8.5.** For a projective $\mathbb{C}$-scheme $X$ of dimension $d$, the Chern class maps induce the following commutative diagram with exact rows.

\begin{equation}
0 \to H^d_{cdh} (X, \mathcal{K}_d^M)_{\text{deg0}} \to H^d_{cdh} (X, \mathcal{K}_d^M) \to H^{2d} (X, \mathbb{Z}) \to 0
\end{equation}

Remark 8.6. The reader should be warned that this proposition can not be deduced from the diagram (8.5) because there is a priori no map from $F^d \text{KH}_0 (X)$ (or from $\text{CH}^d (X)$) to $F^d \text{KH}_0 (X)$ which factors the classical albanese map $F^d \text{KH}_0 (X) \to H^d_D (X, \mathbb{Z}(d))$. This is because of the lack of the current knowledge on the question whether $F^d \text{KH}_0 (X)$ is isomorphic to $F^d \text{KH}_0 (X)$.

**Proof.** The bottom row is exact by (6.14). To show that $c^\text{top}_0$ is surjective, it suffices to show that the composite map $H^d_{cdh} (X, \mathcal{K}_d^M) \to F^d \text{KH}_0 (X) \to H^{2d} (X, \mathbb{Z})$ is surjective. But this follows from the commutative diagram

\begin{equation}
H^d_{cdh} (X, \mathcal{K}_d^M) \to H^d_{cdh} \left( \tilde{X}, \mathcal{K}_d^M \right) \to H^{2d} (X, \mathbb{Z}) \to H^{2d} (\tilde{X}, \mathbb{Z}),
\end{equation}

where the top horizontal arrow is surjective by Mayer-Vietoris, the right vertical arrow is surjective by the smoothness of $\tilde{X}$ plus Lemma 7.4 and the bottom horizontal arrow is an isomorphism, as can be checked by the Mayer-Vietoris sequence for the singular cohomology. $H^d_{cdh} (X, \mathcal{K}_d^M)_{\text{deg0}}$ and $F^d \text{KH}_0 (X)_{\text{deg0}}$ are defined to make the top and the middle rows exact. This makes the above diagram commutative. We only need to show that the middle vertical arrow is surjective to complete the proof.

We first observe that the functorial property of the descent spectral sequence (5.3) and the Mayer-Vietoris property of the cdh cohomology gives the map of spectral sequences $E_2^{p,q} (E) = H^p_{cdh} (E, \mathcal{K}_q) \to H^{p+q+1} (X, \mathcal{K}_q) = E_{2+1}^{p+1,q}$. In particular, this induces the natural map $F^{d-1} \text{KH}_1 (E) = E_{\infty}^{d-1,d} (E) \to E_{\infty}^{d,d} (X) = F^d \text{KH}_0 (X)$. This restricts to the map $F^{d-1} \text{KH}_1^M (E) \to F^d \text{KH}_0^M (X)$. We get a commutative...
Corollary 8.7. Let $\square$Lemma 8.4. vertical map on the left is surjective to finish the proof. But this is shown in using the isomorphism in (8.1), we get a commutative diagram

$$H^d_{cdh} (E, \mathcal{K}^M_d) \rightarrow H^d_{cdh} (X, \mathcal{K}^M_d) \rightarrow H^d_{cdh} (\bar{X}, \mathcal{K}^M_d) \rightarrow 0$$

$$F^{d-1}KH^M_1(E) \rightarrow F^dKH^M_0(X) \rightarrow F^dKH^M_0(\bar{X}) \rightarrow 0$$

$$\overset{c_1,E}{\longrightarrow} \overset{c_0,X}{\longrightarrow} \overset{c_0,\bar{X}}{\longrightarrow} \overset{0}{H^{2d-1}_d (E, \mathbb{Z}(d)) \rightarrow H^{2d}_d (X, \mathbb{Z}(d)) \rightarrow H^{2d}_d (\bar{X}, \mathbb{Z}(q)) \rightarrow 0.}$$

The top and the bottom rows are exact. The left and the middle vertical maps on the top are surjective by (7.5). The top right vertical map is isomorphism by the Bloch's formula for smooth varieties and Lemma 7.4. This shows in particular that the middle row is also exact. The right vertical map on the bottom is known to be surjective as $\bar{X}$ is smooth. We now only have to show that the composite vertical map on the left is surjective to finish the proof. But this is shown in Lemma 8.3.

Corollary 8.7. Let $E$ be as in Lemma 8.7. Then the Chern class $c_1 = c_{d+1,2d+1} : KH_1(E) \rightarrow \mathbb{H}^{2d+1}_D (E, \mathbb{Z}(d + 1))$ induces the exact sequences

$$F^dKH^M_0(X)_{d \geq 0} \otimes \mathbb{C}^* \rightarrow F^dKH^M_1(E) \overset{c_1}{\rightarrow} \mathbb{H}^{2d+1}_D (E, \mathbb{Z}(d + 1)) \rightarrow 0. \quad (8.10)$$

$$H^d_{cdh} (E, \mathcal{K}^M_d) \otimes \mathbb{C}^* \rightarrow H^d_{cdh} (E, \mathcal{K}^M_{d+1}) \overset{c_1}{\rightarrow} \mathbb{H}^{2d+1}_D (E, \mathbb{Z}(d + 1)) \rightarrow 0. \quad (8.11)$$

Proof. Tensoring the diagram (8.7) of Proposition 8.5 (for $F = E$) with $\mathbb{C}^*$ and using the isomorphism in (8.1), we get a commutative diagram

$$0 \rightarrow F^dKH^M_0(X)_{d \geq 0} \otimes \mathbb{C}^* \rightarrow F^dKH^M_0(X) \otimes \mathbb{C}^* \rightarrow \mathbb{H}^{2d+1}_D (E, \mathbb{Z}(d + 1)) \rightarrow 0$$

$$0 \longrightarrow \text{Ker}(c_1) \longrightarrow F^dKH^M_0(E) \overset{c_1}{\longrightarrow} \mathbb{H}^{2d+1}_D (E, \mathbb{Z}(d + 1)) \rightarrow 0,$$

where the first map on the top row is injective because $H^{2d}(E, \mathbb{Z})$ is a free abelian group of finite rank. The corollary now follows from Corollary 8.3. The second exact sequence follows exactly in the same way using Corollary 8.3 again.

Definition 8.8. Let $X$ be a projective $\mathbb{C}$-scheme of dimension $d$ over $\mathbb{C}$. We say that $F^dKH^M_0(X)$ is finite-dimensional, if the map $F^dKH^M_0(X) \rightarrow H^{2d}_D (X, \mathbb{Z}(d))$ is an isomorphism.

One could now formulate the following cdh version of the Roitman torsion theorem and the finite-dimensionality problem for the Chow group of zero-cycles on singular projective schemes.

Conjecture 8.9. For a projective $\mathbb{C}$-scheme $X$ of dimension $d$, the map

$$F^dKH^M_0(X) \rightarrow H^{2d}_D (X, \mathbb{Z}(d))$$

is isomorphism on torsion subgroups.

Conjecture 8.10. Let $X$ be projective $\mathbb{C}$-scheme of dimension $d$. Then $F^dKH^M_0(X)$ is finite-dimensional if and only if $H^i_{cdh} (X, \Omega^i_{X/\mathbb{C}}) = 0$ for $0 \leq i \leq d - 2$. 
Proposition 9.3. Let $\varphi : H \to K$ be a (possibly nonreduced) curve over $\mathbb{C}$. Then the map $H^1_{\text{Zar}}(E, \mathcal{O}_E) \to H^1_{\text{cdh}}(E, \mathcal{O}_E)$ is surjective. This map is an isomorphism if $E$ is seminormal.

Proof. Since $H^1_{\text{Zar}}(E, \mathcal{O}_E) \to H^1_{\text{Zar}}(E_{\text{red}}, \mathcal{O}_{E_{\text{red}}})$ and is an isomorphism in the cdh topology, we can assume that $E$ is reduced. This result is well known when $E$ is smooth. In general, let $E^N \to E$ be the normalization of $E$ and let $S \to E$ be a conducting subscheme for the normalization. Put $\~S = S \times_E E^N$. The smoothness of $E^N$ and the Leray spectral sequence imply that $H^*_{\text{cdh}}(E^N, \mathcal{O}^*) \cong H^*_{\text{cdh}}(E, \mathcal{O}^*)$. Now, the exact sequence

$$0 \to \mathcal{O}_E^* \to \mathcal{O}_{E^N}^* \to \mathcal{O}_{\~S}^* \to 0$$

gives the commutative diagram of exact sequences

$$\mathcal{O}^*(E^N) \oplus \mathcal{O}^*(S) \to \mathcal{O}^*(\~S) \to H^1_{\text{Zar}}(E, \mathcal{O}^*) \to H^1_{\text{Zar}}(E^N, \mathcal{O}^*) \to 0$$

$$\mathcal{O}^*_{\text{cdh}}(E^N) \oplus \mathcal{O}^*_{\text{cdh}}(S_{\text{red}}) \to \mathcal{O}^*_{\text{cdh}}(\~S_{\text{red}}) \to H^1_{\text{cdh}}(E, \mathcal{O}^*) \to H^1_{\text{cdh}}(E^N, \mathcal{O}^*) \to 0.$$

Now we observe that $S_{\text{red}}$ and $\~S_{\text{red}}$ are zero-dimensional and hence smooth. In particular, $\mathcal{O}^*(S) \to \mathcal{O}^*_{\text{cdh}}(S)$ and is an isomorphism when $S$ is reduced, which is the case when $E$ is seminormal. The same holds for $\~S$. We always have $\mathcal{O}^*(E^N) \cong \mathcal{O}^*_{\text{cdh}}(E^N)$. A diagram chase now proves the result. □

Lemma 9.2. For any curve $E$ over $\mathbb{C}$, there is an exact sequence

$$\text{Pic}^0(E) \otimes \mathbb{C}^* \to H^1_{\text{cdh}}(E, \mathcal{K}_2) \to H^3_{\text{B}}(E, \mathbb{Z}(2)) \to 0.$$

Proof. By Lemma 9.1 it suffices to prove (8.11) for $E$. We can assume that $E$ is reduced. As in the proof of Lemma 9.1 we consider the commutative diagram of exact sequences

$$\mathcal{O}^*_{\text{cdh}}(\~S_{\text{red}}) \otimes \mathbb{C}^* \to H^1_{\text{cdh}}(E, \mathcal{O}^*)_{\text{deg0}} \otimes \mathbb{C}^* \to H^1_{\text{cdh}}(E^N, \mathcal{O}^*)_{\text{deg0}} \otimes \mathbb{C}^* \to 0$$

Since $\~S$ is zero-dimensional, the left vertical map is surjective using the isomorphism $K_2^M(\mathbb{C}) \cong K_2(\mathbb{C})$. The lemma now follows easily by a diagram chase, Corollary 8.7 for $E^N$ and the isomorphism $H^3_{\text{B}}(E, \mathbb{Z}(2)) \cong H^3_{\text{B}}(E^N, \mathbb{Z}(2))$. □

Proposition 9.3. Let $E$ be a curve over $\mathbb{C}$. Then the map $H^1_{\text{Zar}}(E, \mathcal{K}_2) \to H^1_{\text{cdh}}(E, \mathcal{K}_2)$ is surjective. This map is an isomorphism if $E$ is seminormal.
Proof. We can assume $E$ to be reduced. For smooth $E$, there is nothing to prove.

We now assume that $E$ is seminormal. Let $E^N \xrightarrow{f} E$ be the normalization of $E$ as above.

We consider the commutative diagram of exact sequences

$$\begin{array}{ccc}
\mathcal{K}_2(E,S) & \xrightarrow{f_*} & \mathcal{K}_2(E) \\
\downarrow & & \downarrow \\
\mathcal{K}_2(E^N,S) & \xrightarrow{f_*} & \mathcal{K}_2(E^N)
\end{array}$$

The double relative $K$-theory exact sequence tells us that the cokernel of the left vertical map is contained in $\mathcal{K}_{1(E,E^N,S)}$, which in turn is isomorphic to $\mathcal{I}_S \otimes \Omega^1_{E^N/E}$ by the main result of Geller-Weibel [23]. But this last term is zero because $E$ is seminormal. In particular, the left vertical map in the above diagram is surjective. Hence we get exact sequence

$$K_2(E) \to f_* (K_2(E^N)) \oplus K_2(S) \to f_* (\mathcal{K}_2(S)) \to 0.$$

Taking the associated long exact sequences of the Zariski and cdh cohomology, we get the following commutative diagram.

$$\begin{array}{c}
\left\{ \begin{array}{c}
H^0_{\text{Zar}}(E^N,\mathcal{K}_2) \\
H^0_{\text{Zar}}(S,\mathcal{K}_2)
\end{array} \right\} \to H^0_{\text{Zar}}(\tilde{S},\mathcal{K}_2) \to H^1_{\text{Zar}}(E,\mathcal{K}_2) \to H^1_{\text{Zar}}(E^N,\mathcal{K}_2) \to 0 \\
\downarrow \\
\left\{ \begin{array}{c}
H^0_{\text{cdh}}(E^N,\mathcal{K}_2) \\
H^0_{\text{cdh}}(S,\mathcal{K}_2)
\end{array} \right\} \to H^0_{\text{cdh}}(\tilde{S},\mathcal{K}_2) \to H^1_{\text{cdh}}(E,\mathcal{K}_2) \to H^1_{\text{cdh}}(E^N,\mathcal{K}_2) \to 0.
\end{array}$$

The smoothness of $E$, $S$ and $\tilde{S}$ implies that the first two vertical maps from the left and the last vertical map on the right are isomorphisms. Hence the remaining vertical map is also an isomorphism. This proves the case of seminormal curves. For a general reduced curve, we compare the commutative diagram

$$\begin{array}{c}
\text{Pic}^0(E) \otimes \mathbb{C}^* \to H^1_{\text{Zar}}(E,\mathcal{K}_2) \supseteq H^3_D(E,\mathbb{Z}(2)) \to 0 \\
\downarrow \\
\text{Pic}^0(E) \otimes \mathbb{C}^* \to H^1_{\text{cdh}}(E,\mathcal{K}_2) \supseteq H^3_D(E,\mathbb{Z}(2)) \to 0,
\end{array}$$

where the top row is exact by [34, Lemma 3.1] and the bottom row is exact by Lemma [9.2]. This proves the result. \hfill \Box

Corollary 9.4. Let $E$ be a curve over $\mathbb{C}$. Then

(i) The map $H^1_{\text{cdh}}(E,\mathcal{K}_2)_{\text{tors}} \to H^1_{\text{D}}(E,\mathbb{Z}(2))_{\text{tors}}$ is split surjective and is an isomorphism if $E$ is seminormal.

(ii) $H^1_{\text{cdh}}(E,\mathcal{K}_2) \otimes \mathbb{Q}/\mathbb{Z} = 0$.

Proof. The first part follows directly from Proposition [9.3] and [3, Theorem 5.3]. The second part follows from Proposition [9.3] and [3, Proposition 8.4]. \hfill \Box
10. Chern classes for smooth schemes

Let $X$ be a smooth projective $\mathbb{C}$-scheme of dimension $d$. In this section, we prove some results about the Chern classes from the $K$-theory of $X$ into its Deligne cohomology. We shall assume all abelian groups in this section to be tensored with $\mathbb{Q}$, i.e., the abelian group $A$ will actually mean $A \otimes \mathbb{Z} \mathbb{Q}$. Let $\mathcal{H}_D(q)$ denote the sheaf on $X_{\text{Zar}}$ associated to the presheaf $U \mapsto \mathbb{H}_D^q(U_{an}, \mathbb{Q}(q))$, where

$$Q_{D,U}(q) := \left( \mathbb{Q}(q) \rightarrow \mathcal{O}_U \rightarrow \cdots \rightarrow \Omega_U^{q-1} \right).$$

Lemma 10.1. Let $X$ be a smooth projective $\mathbb{C}$-scheme of dimension $d$. Then $H^d_{\text{Zar}}(X, \mathcal{H}^{d-1}_D(1)) = 0$.

Proof. Let $U$ be an affine neighborhood of a closed point on $X$. Then the map $\mathbb{C} \rightarrow \left( \mathcal{O}_{U_{an}} \rightarrow \cdots \rightarrow \Omega_{U_{an}}^d \right)$ is a quasi-isomorphism of complexes by the Poincaré lemma. In other words, there is exact sequence of complexes on $U_{an}$:

$$0 \rightarrow \Omega_{U_{an}}^d \rightarrow \mathbb{C}/\mathbb{Q}(d)[-1] \rightarrow \mathbb{Q}(d) \rightarrow 0,$$

which gives the isomorphism $H^{d-2}(U_{an}, \mathcal{C}/\mathbb{Q}(d)) \cong \mathbb{H}_D^{d-1}(U, \mathbb{Q}(d))$. Since the map $H^*(\mathcal{O}_{an}, \mathbb{Q}) \rightarrow H^*(\mathcal{O}_{an}, \mathbb{C})$ is injective, we get exact sequence of Zariski sheaves

$$0 \rightarrow \mathcal{H}^{d-2} \rightarrow \mathcal{H}^{d-2}_{\mathcal{C},U} \rightarrow \mathcal{H}^{d-1}_D(d) \rightarrow 0.$$

This in particular gives a surjection $H^d_{\text{Zar}}(X, \mathcal{H}^{d-2}_{\mathcal{C},U}) \twoheadrightarrow H^d_{\text{Zar}}(X, \mathcal{H}^{d-1}_D(1))$. But the Bloch-Ogus-Gersten sequence (cf. [8, (0.3)]) implies that $H^p_{\text{Zar}}(X, \mathcal{H}^q_{\mathcal{C},U}) = 0$ for $p > q$. \hfill $\square$

Corollary 10.2. Let $X$ be as in Lemma 10.1. Then the Chern class map $K_1(X) \xrightarrow{c_1} H^{d-1}_D(X, \mathbb{Q}(d))$ gives rise to the Chern class map

$$H^{d-1}_D(X, \mathcal{K}_d) \xrightarrow{F^d K_1(X)} H^d_D(X, \mathbb{Q}(d)).$$

Proof. We can replace $H^{d-1}_D(X, \mathcal{K}_d)$ with $H^{d-1}_{\text{Zar}}(X, \mathcal{K}_d)$ by Lemma 7.4. The first surjection then follows at once from the Brown-Gersten spectral sequence (5.3). This spectral sequence also implies that there is a surjection $H^d_{\text{Zar}}(X, \mathcal{K}_{d+1}) \twoheadrightarrow F^d K_1(X)$. Thus we only need to show that the composite $H^d_{\text{Zar}}(X, \mathcal{K}_{d+1}) \twoheadrightarrow F^d K_1(X) \rightarrow H^{2d-1}_D(X, \mathbb{Q}(d))$ is zero. Now, the functoriality of the Chern classes gives a commutative diagram

$$\begin{array}{ccc}
H^d_{\text{Zar}}(X, \mathcal{K}_{d+1}) & \xrightarrow{c_{d+1}} & F^d K_1(X) \\
\downarrow & & \downarrow c_1 \\
H^d_{\text{Zar}}(X, \mathcal{H}^{d-1}_D) & \twoheadrightarrow & H^{2d-1}_D(X, \mathbb{Q}(d)),
\end{array}$$

where the bottom horizontal arrow is the edge map in the spectral sequence

$$E_2^{i,j} = H^{i+j}_{\text{Zar}}(X, \mathcal{H}^j_D(q)) \Rightarrow H^{i+j}_D(X, \mathbb{Q}(d)).$$

The corollary now follows immediately from Lemma 10.1. \hfill $\square$

Lemma 10.3. Let $X$ be as in Lemma 10.1. Then $H^{2d-1}(X, \mathbb{Q}) \cap F^d H^{2d-1}(X, \mathbb{C}) = 0$ and hence

$$H^{2d-1}(X, \mathbb{Q}) \hookrightarrow \frac{H^{2d-1}(X, \mathbb{C})}{F^d H^{2d-1}(X, \mathbb{C})}.$$
Proof. We observe that \( H^{2d-1}(X, \mathbb{Q}) \) is invariant under the complex conjugation on \( H^{2d-1}(X, \mathbb{C}) \). On the other hand,
\[
F^d H^{2d-1}(X, \mathbb{C}) \cap \overline{F^d H^{2d-1}(X, \mathbb{C})} = 0
\]
by the Hodge theory. The lemma now follows.

Let \( X \) be a smooth and projective \( \mathbb{C} \)-scheme of dimension \( d \). For any \( i \geq 0 \), let
\[
H^{2i}(X, \mathbb{Q})_{\text{alg}} = H^{2i}_{\text{Zar}}(X, \Omega^i_{X/\mathbb{C}}) \cap H^{2i}(X, \mathbb{Q}).
\]
Note that
\[
H^{2i}(X, \mathbb{Q}) \hookrightarrow H^{2i}(X, \mathbb{Q}) \otimes \mathbb{C} \cong H^{2i}(X, \mathbb{C})
\]
and the Hodge decomposition shows that \( H^{2i}(X, \mathbb{Q})_{\text{alg}} \otimes \mathbb{C} \cong H^i_{\text{Zar}}(X, \Omega^i_{X/\mathbb{C}}) \) under this isomorphism.

Denote the image of the map \( H^{2d-1}_{\text{Zar}}(X, \Omega^{d-1}_{X/\mathbb{C}}) \to \mathbb{H}^{2d-1}(X, \mathbb{Q}(d)) \) by the subgroup \( \mathbb{H}^{2d-1}_{\text{D}}(X, \mathbb{Q}(d))_{\text{alg}} \).

Proposition 10.4. Let \( X \) be as in Lemma \ref{lemma}. Then
\[
\mathbb{H}^{2d-1}_{\text{D}}(X, \mathbb{Q}(d))_{\text{alg}} \subseteq \text{Image} \left( H^{2d-1}_{\text{D}}(X, \mathcal{K}^M_d) \xrightarrow{c_1} \mathbb{H}^{2d-1}_{\text{D}}(X, \mathbb{Q}(d)) \right).
\]

Proof. We can replace \( H^{2d-1}_{\text{Zar}}(X, \mathcal{K}^M_d) \) with \( H^{2d-1}_{\text{Zar}}(X, \mathcal{K}_d) \) by Lemma \ref{lemma}. The injectivity of maps \( H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{C}) \) and \ref{eq:2} together imply that
\[
H^{2d-2}(X, \mathbb{Q})_{\text{alg}} \otimes \mathbb{C} \cong \frac{H^{2d-2}(X, \mathbb{Q})_{\text{alg}} \otimes \mathbb{C} \otimes \mathbb{C}}{H^{2d-2}(X, \mathbb{Q})_{\text{alg}}} \cong \frac{H^{2d-1}_{\text{Zar}}(X, \Omega^{d-1}_{X/\mathbb{C}})}{H^{2d-2}(X, \mathbb{Q})_{\text{alg}}}.
\]

The validity of the Hodge conjecture for codimension \( (d - 1) \) cycles on \( X \) (cf. \cite[p. 91]{cites}) implies that \( H^{d-1}_{\text{Zar}}(X, \mathcal{K}_{d-1}) \to H^{2d-2}(X, \mathbb{Q})_{\text{alg}} \) under the topological Chern class maps
\[
H^{d-1}_{\text{Zar}}(X, \mathcal{K}_{d-1}) \xrightarrow{c_1} CH^{d-1}(X) \to \mathbb{H}^{2d-2}_{\text{D}}(X, \mathbb{Q}(d - 1)) \xrightarrow{c_1} H^{2d-2}(X, \mathbb{C}) \leftarrow H^{2d-2}(X, \mathbb{Q}).
\]

The proposition now follows by using the commutative diagram
\[
\begin{array}{ccc}
H^{d-1}_{\text{Zar}}(X, \mathcal{K}_{d-1}) \otimes \mathbb{C} & \longrightarrow & H^{d-1}_{\text{Zar}}(X, \mathcal{K}_d) \\
\downarrow & & \downarrow c_1 \\
\frac{H^{d-1}_{\text{Zar}}(X, \Omega^{d-1}_{X/\mathbb{C}})}{H^{2d-2}(X, \mathbb{Q})_{\text{alg}}} & \cong & \frac{H^{2d-2}(X, \mathbb{Q})_{\text{alg}} \otimes \mathbb{C} \otimes \mathbb{C}}{H^{2d-2}(X, \mathbb{Q})_{\text{alg}}} \to \mathbb{H}^{2d-1}_{\text{D}}(X, \mathbb{Q}(d)),
\end{array}
\]
where the first map on the bottom row is the isomorphism of \ref{eq:3}.

11. Zero-cycles on singular varieties and their resolutions

In this section, we prove our main result about the comparison between the cdh analogue of the Chow group of zero-cycles on a normal projective variety \( X \) and that of a resolution of singularities of \( X \). We need the following intermediate result.
Lemma 11.1. Consider the resolution diagram \([7,6]\) for a projective (not necessarily normal) \(\mathbb{C}\)-scheme \(X\) of dimension \(d\). Then the map

\[
H^{d-1}_{\text{Zar}}(\widetilde{X}, \mathcal{K}_{d}) \xrightarrow{\sim} H^{d-1}_{D}(\widetilde{X}, \mathbb{Q}(d))
\]

is surjective.

Proof. It follows from Lemmas \([6,9]\) and \([10,3]\) that

\[
\mathbb{H}^{2d-2}_{\text{Zar}}\left(\widetilde{X}, \Omega^{d-1}_{\widetilde{X}/\mathbb{C}}\right) \cong H^{2d-1}_{D}(\widetilde{X}, \mathbb{Q}(d)).
\]

Using the exact sequence

\[
0 \to H^{d-1}_{\text{Zar}}(X, \Omega^{d-1}_{X/\mathbb{C}}) \to \mathbb{H}^{2d-2}_{\text{Zar}}\left(\widetilde{X}, \Omega^{d-1}_{\widetilde{X}/\mathbb{C}}\right) \to H^{2d-2}_{\text{Zar}}(\widetilde{X}, \mathbb{Q}) \to 0
\]

and Proposition \([10,4]\) it suffices to show that

\[
\mathbb{H}^{2d-2}_{\text{cdh}}(X, \Omega^{d-1}_{X/\mathbb{C}}) \to \mathbb{H}^{2d-2}_{\text{cdh}}\left(\widetilde{X}, \Omega^{d-1}_{\widetilde{X}/\mathbb{C}}\right).
\]

But this follows from the Mayer-Vietoris exact sequence

\[
\mathbb{H}^{2d-2}_{\text{cdh}}(X, \Omega^{d-1}_{X/\mathbb{C}}) \to \mathbb{H}^{2d-2}_{\text{cdh}}\left(\widetilde{X}, \Omega^{d-1}_{\widetilde{X}/\mathbb{C}}\right) \oplus \mathbb{H}^{2d-2}_{\text{cdh}}(S, \Omega^{d-1}_{S/\mathbb{C}}) \to \mathbb{H}^{2d-2}_{\text{cdh}}\left(E, \Omega^{d-1}_{E/\mathbb{C}}\right)
\]

together with the fact that \(\mathbb{H}^{2d-2}_{\text{cdh}}(S, \Omega^{d-1}_{S/\mathbb{C}}) = 0 = \mathbb{H}^{2d-2}_{\text{cdh}}(E, \Omega^{d-1}_{E/\mathbb{C}})\), which follows from Lemma \([2,5]\). \(\square\)

The following is our main result about the cdh version of the Chow group of zero-cycles on singular schemes.

Theorem 11.2. Let \(X\) be a normal and projective \(\mathbb{C}\)-scheme of dimension \(d \geq 2\). Suppose that for a resolution diagram \([7,6]\) for \(X\),

\[
H^{d-1}_{\text{cdh}}\left(\widetilde{X}, \mathcal{K}_{d-1}^M\right)_{\text{hom}} \otimes \mathbb{C}^* \to H^{d-1}_{\text{cdh}}(E, \mathcal{K}_{d-1}^M)_{\text{hom}} \otimes \mathbb{C}^*.
\]

Then the map

\[
F^dKH^M_0(X) \to F^dKH^M_0(\widetilde{X}) \cong CH^d(\widetilde{X})
\]

is an isomorphism.

Proof. Since \(H^d_{\text{cdh}}(\widetilde{X}, \mathcal{K}_d^M) \cong F^dKH^M_0(\widetilde{X}) \cong F^dK_0(\widetilde{X}) \cong CH^d(\widetilde{X})\) and

\[
H^d_{\text{cdh}}(X, \mathcal{K}_d^M) \to F^dKH^M_0(X),
\]

it suffices to show the stronger assertion that

\[
H^d_{\text{cdh}}(X, \mathcal{K}_d^M) \xrightarrow{\sim} H^d_{\text{cdh}}\left(\widetilde{X}, \mathcal{K}_d^M\right).
\]
Let us write \( H_{d-1} \mathcal{D}(\bar{X}, \mathcal{Q}(d)) \) as \( H_{d-1} \mathcal{D}(\bar{X}, \mathcal{Q}(d)) \) and consider the following commutative diagram of Mayer-Vietoris exact sequences.

\[
\begin{array}{ccc}
H_{d-1} \mathcal{D}(\bar{X}, \mathcal{K}_d^M) \otimes \mathbb{C}^* & \rightarrow & H_{d-1} \mathcal{D}(E, \mathcal{K}_d^M) \otimes \mathbb{C}^* \\
\downarrow & & \downarrow \\
H_{d-1} \mathcal{D}(\bar{X}, \mathcal{Q}(d)) & \rightarrow & H_{d-1} \mathcal{D}(E, \mathcal{Q}(d)) \\
\downarrow & & \downarrow \\
\mathbb{H}_{2d-1} \mathcal{D}(\bar{X}, \mathcal{Q}(d)) & \rightarrow & \mathbb{H}_{2d-1} \mathcal{D}(E, \mathcal{Q}(d)) \\
\end{array}
\]

The bottom row is exact by Lemma 7.5 since the normality of \( X \) implies that \( \dim(X_{\text{sing}}) \leq d - 2 \) and hence \( \mathbb{H}_i \mathcal{D}(X_{\text{sing}}, \mathcal{Q}(d)) = 0 \) for \( i \geq 2d - 1 \). The middle row is exact also because of the normality of \( X \). Furthermore, the universal property of the albanese variety of \( X \) (cf. [21]) shows that \( J^d(X) \cong J^d(\bar{X}) \) and hence it follows from (6.14) that \( \mathbb{H}_{2d} \mathcal{D}(X, \mathcal{Q}(d)) \cong \mathbb{H}_{2d} \mathcal{D}(\bar{X}, \mathcal{Q}(d)) \). In particular, the map \( \gamma \) is surjective.

The left lower vertical map is surjective by Lemma 11.1. The second lower vertical map from the left is surjective by Lemma 8.4. The second column from the left is exact by Corollary 8.7. A diagram chase shows that we only need to show that \( \text{Image}(\alpha) \subseteq \text{Image}(\beta) \). But this follows from the assumption of the theorem.

12. Chow groups of zero-cycles on surfaces

In this section, we deduce some consequences of Theorem 11.2 for the Chow group of zero-cycles on surfaces. In particular, we prove the cdh version of the Roitman torsion theorem and compare the Chow group of a surface with arbitrary singularity with the Chow group of a resolution of singularities.

12.1. Roitman torsion for surfaces. The following is a version of the Roitman torsion theorem for singular surfaces in the cdh topology. It proves Conjecture 8.9 for surfaces.

**Theorem 12.1.** Let \( X \) be a projective surface over \( \mathbb{C} \). Then the Chern class map \( H^2_{\text{cdh}}(X, \mathcal{K}_2) \xrightarrow{c_{0,X}} \mathbb{H}^4_{\text{cdh}}(X, \mathbb{Z}(2)) \) is isomorphism on torsion subgroups.

**Proof.** This follows by imitating the proof of the Zariski version of the Roitman torsion theorem in [3]. We consider the resolution diagram (7.6) for \( X \) and use the
following diagram.

\[
\begin{array}{c}
\begin{array}{c}
H^1_{\text{cdh}} \left( \bar{X}, \mathcal{K}_2 \right)_{\text{tors}} \\
\oplus \\
H^1_{\text{cdh}} (S, \mathcal{K}_2)_{\text{tors}}
\end{array}
\end{array}
\xrightarrow{\text{Corollary 9.4}}
\begin{array}{c}
H^1_{\text{cdh}} (E, \mathcal{K}_2)_{\text{tors}} \\
\rightarrow \\
H^2_{\text{cdh}} (X, \mathcal{K}_2)_{\text{tors}} \\
\rightarrow \\
H^2_{\text{cdh}} \left( \bar{X}, \mathcal{K}_2 \right)_{\text{tors}} \rightarrow 0
\end{array}
\]

The exactness of the bottom row is already shown in loc. cit. The top row is exact without taking the torsion part by the Mayer-Vietoris property of the cdh cohomology. Corollary 9.4 now shows that the top row is exact except at \( c_{0, X} \). The first vertical map from left is surjective by Corollary 9.4 and [loc. cit., Proposition 8.4]. The second vertical map from the left is an isomorphism by Corollary 9.4 since \( E \) is seminormal. The last vertical map on the right is an isomorphism by the Roitman torsion for smooth surfaces. A diagram chase shows that \( c_{0, X} \) is an isomorphism on the torsion subgroups. \( \square \)

12.2. Chow group of singular surfaces. The following is our main result for the cdh analogue of the Chow group zero-cycles on singular surfaces.

**Theorem 12.2.** Let \( X \) be a projective surface over \( \mathbb{C} \) and let \( \bar{X} \rightarrow X \) be a resolution of singularities of \( X \) as in (7.6). Then

1. \( H^2_{\text{cdh}} (X, \mathcal{O}_X) \xrightarrow{\sim} H^2_{\text{cdh}} (\bar{X}, \mathcal{O}_{\bar{X}}) \Rightarrow \ker(c_{0, X}) \xrightarrow{\sim} \ker(c_{0, \bar{X}}). \)
2. \( H^2_{\text{cdh}} \left( X, \Omega^i_{X/\mathbb{C}} \right) \xrightarrow{\sim} H^2_{\text{cdh}} \left( \bar{X}, \Omega^i_{\bar{X}/\mathbb{C}} \right) \) for \( i \leq 1 \Rightarrow F^2K \mathcal{H}_0(X) \xrightarrow{\sim} CH^2(\bar{X}). \)

**Proof.** Since \( \mathcal{K}^M_{2, X} \cong \mathcal{K}_{2, X} \), we can use \( F^2K \mathcal{H}^M_0(X) \) and \( F^2K \mathcal{H}_0(X) \) interchangeably.

Assume first that \( H^2_{\text{cdh}} (X, \mathcal{O}_X) \xrightarrow{\sim} H^2_{\text{cdh}} (\bar{X}, \mathcal{O}_{\bar{X}}). \) The exact sequence

\[
H^1_{\text{cdh}} (\bar{X}, \mathcal{O}) \oplus H^1_{\text{cdh}} (S, \mathcal{O}) \rightarrow H^1_{\text{cdh}} (E, \mathcal{O}) \rightarrow H^2_{\text{cdh}} (X, \mathcal{O}) \rightarrow H^2_{\text{cdh}} (\bar{X}, \mathcal{O}) \rightarrow 0,
\]

Corollary 3.4 for \( E, \) [12, Corollary 2.5, Proposition 2.6] for \( \bar{X} \) and \( S \), and our assumption together imply that

\[
(12.1) \quad H^1_{Zar}(\bar{X}, \mathcal{O}) \oplus H^1_{Zar}(S, \mathcal{O}) \rightarrow H^1_{Zar}(E, \mathcal{O}).
\]

Since these groups are the Lie algebras of the associated Picard varieties, we conclude that

\[
(12.2) \quad \text{Pic}^0(\bar{X}) \oplus \text{Pic}^0(S) \rightarrow \text{Pic}^0(E).
\]
We now consider the following commutative diagram of exact sequences (12.3)

\[
\begin{array}{cccc}
H^1_{cdh}(\tilde{X}, K_1) \otimes \mathbb{C}^* & \xrightarrow{\alpha} & H^1_{cdh}(E, K_1) \otimes \mathbb{C}^* & \xrightarrow{\beta} & H^2_{cdh}(\tilde{X}, K_2) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{cdh}(\tilde{X}, K_2) & \xrightarrow{\beta} & H^1_{cdh}(E, K_2) & \rightarrow & H^2_{cdh}(X, K_2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^3_D(\tilde{X}, \mathbb{Z}(2)) & \xrightarrow{\gamma} & \mathbb{H}^3_D(E, \mathbb{Z}(2)) & \rightarrow & \mathbb{H}^4_D(X, \mathbb{Z}(2)) \\
\end{array}
\]

In this diagram, the second column from the left is exact as it is the quotient of the exact sequences given by Lemma 6.2. This already implies that \( \ker(c_{0, \chi}) \rightarrow \ker(c_{0, \tilde{X}}) \). To show the injectivity, it suffices now to show the same with rational coefficients because of Theorem 12.1. In this case, the left lower vertical map is surjective, which follows directly from (12.2). This proves the first assertion.

To prove the second part of the theorem, it suffices to show that \( c_{0, X} \) is an isomorphism. Under the given assumption, it follows from Lemma 2.5 that \( H_3(X, \mathbb{Z}) \rightarrow H^3(\tilde{X}, \mathbb{Z}) \) and \( H^4(X, \mathbb{Z}) \rightarrow H^4(\tilde{X}, \mathbb{Z}) \). It follows now from (6.12) that \( \mathbb{H}^3_D(X, \mathbb{Z}(2)) \xrightarrow{\cong} \mathbb{H}^4_D(\tilde{X}, \mathbb{Z}(2)) \). In particular, the map \( \gamma \) is surjective. This also shows using Theorem 12.1 that we only have to show the isomorphism with rational coefficients. Now, using a simple diagram chase and arguing as in the proof of Theorem 11.2, we only have to show that the top horizontal map is surjective, which follows directly from (12.2). \( \square \)

The following recovers [37, Theorem 1.3] and [34, Theorem 1.3] by a different and more conceptual approach.

**Corollary 12.3.** Let \( X \) be a projective surface over \( \mathbb{C} \) with a resolution of singularities \( \tilde{X} \) such that \( H^2_{zar}(X, O_X) = 0 \). Then the finite-dimensionality of \( CH^2(\tilde{X}) \) implies the same for \( CH^2(X) \).

**Proof.** By the Roitman torsion theorem (cf. [3], [3]), it suffices to prove the result rationally. So we assume all the groups to be tensored with \( \mathbb{Q} \). The surjectivity \( H^2_{zar}(X, O_X) \rightarrow H^2_{zar}(\tilde{X}, O_{\tilde{X}}) \) and our assumption imply from Theorem 12.2 that \( F^2KH_0(X) \xrightarrow{\cong} \mathbb{H}^4_D(X, \mathbb{Z}(2)) \). Thus we need to show that

\[
\ker(CH^2(X) \rightarrow F^2KH_0(X)) \xrightarrow{\cong} \ker(\mathbb{H}^4_D(X, \mathbb{Z}(2)) \rightarrow \mathbb{H}^4_D(X, \Omega^1_X))
\]

where the second isomorphism follows from our assumption, (6.12), (6.14) and (6.15).

However, the Bloch’s formula \( H^2_{zar}(X, K_2) \cong F^2K_0(X) \cong CH^2(X) \) of [39], Corollary 5.5 and 5.1 imply that \( \ker(CH^2(X) \rightarrow F^2KH_0(X)) \subseteq \tilde{K}_0(X) \).

The corollary now follows from the exact sequence

\[
\Omega^1_\mathbb{C} \otimes H^2(X, O_X) \rightarrow H^2(X, \Omega^1_X) \rightarrow H^2(X, \Omega^1_{X/c}) \rightarrow 0,
\]
in the Zariski and the cdh topology, the surjection \( H^2_{cdh}(X, \mathcal{O}_X) \to H^2_{cdh}(X, \mathcal{O}_X) \) and Lemma 5.1.

**Corollary 12.4.** Let \( X \) be a strict normal crossing divisor on a smooth projective threefold such that \( H^2_{cdh}(X, \mathcal{O}_X) = 0 \). Then the finite-dimensionality of \( CH^2(\tilde{X}) \) implies that the maps

\[
\begin{align*}
H^2_{Zar}(X, \mathcal{K}_2) & \to CH^2(X) \to \mathbb{H}^4_{dR}(X, \mathbb{Z}(2)) \\
H^2_{cdh}(X, \mathcal{K}_2) & \to F^dKH_0(X) \to \mathbb{H}^4_{dR}(X, \mathbb{Z}(2))
\end{align*}
\]

are all isomorphisms.

**Proof.** The first horizontal map on the top row is an isomorphism by the main result of [39]. Since \( X \) is a strict normal crossing divisor, it follows from our assumption and Corollary 3.4 that \( H^2_{Zar}(X, \mathcal{O}_X) = H^2_{cdh}(X, \mathcal{O}_X) = 0 \). Hence, the second horizontal map on the top row is an isomorphism by [34, Theorem 1.3]. It follows from Theorem 12.2 that both the horizontal maps on the bottom row are isomorphisms. Since \( X \) is seminormal, the right vertical map is an isomorphism by [34, Corollary 6.2]. Hence all other vertical maps are also isomorphisms. \( \square \)

### 13. Finite-dimensionality for normal threefolds

In this section, we use Theorem 11.2 to deduce a conditional result on the finite-dimensionality of the Chow group of zero-cycles on normal threefolds. So let \( X \) be a normal and projective threefold over \( \mathbb{C} \) and consider a resolution of singularities diagram for \( X \) as in (7.6). Since \( E \) is a surface, there is an isomorphism \( H^2_{Zar}(E, \mathcal{K}_{2,X}) \cong CH^2(X) \) by the main result of [39]. Since such an isomorphism also holds for \( \tilde{X} \), there is a natural map \( CH^2(\tilde{X}) \to CH^2(E) \) which in turn induces the restriction map

\[
A^2(\tilde{X}) = CH^2(\tilde{X})_{alg} \to CH^2(\tilde{X})_{hom} \to A^2(E) = CH^2(E)_{deg0}.
\]

**Lemma 13.1.** Assume that \( GHC(1, 3, \tilde{X}) \) and \( GBC(E^N) \) hold. Suppose that \( H^3_{cdh}(X, \Omega^1_{X/\mathbb{C}}) = 0 \) for \( i \leq 1 \). Then the map \( A^2(\tilde{X}) \to A^2(E) \) is surjective.

**Proof.** As in the proof of Corollary 3.4, it follows from our assumption and Corollary 3.4 that the map \( H^2_{Zar}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^2_{Zar}(E, \mathcal{O}_E) \). On the other hand, the surjectivity \( H^2_{Zar}(\tilde{X}, \Omega^1_{\tilde{X}/\mathbb{C}}) \to H^3_{Zar}(\tilde{X}, \Omega^1_{\tilde{X}/\mathbb{C}}) \) and the Hodge theory imply that \( H^2_{Zar}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0 \). We conclude that \( H^2_{Zar}(E, \mathcal{O}_E) = 0 \). Now, the Bloch’s conjecture \( GBC(E^N) \) and [34, Theorem 1.3] imply that the albanese map \( A^2(E) \to J^2(E) \) is an isomorphism. Since \( E \) is a strict normal crossing divisor, it follows from [34, Corollary 6.5] that \( J^2(E) \) is a semi-abelian variety and hence \( J^2(E) \cong J^2(E) \).

In particular, we get \( A^2(E) \cong J^2(E) \).

It also follows from (6.12) and Corollary 3.5 that the morphism of complex algebraic groups \( J^2(\tilde{X}) \to J^2(E) \) is surjective. The lemma now follows from Corollary 6.11. \( \square \)
Theorem 13.2. Let $X$ be a normal and projective threefold over $\mathbb{C}$ such that $H^3_{cdh}(X, \Omega^i_{X/\mathbb{C}}) = 0$ for $0 \leq i \leq 1$. Assume GBC(2) and GHC(1, $3$, $\tilde{X}$) for some resolution of singularities $\tilde{X}$ as in (14.1). Then $F^3KH_0^M(X) \cong CH^3(\tilde{X})_Q$.

Proof. In this proof, we assume all groups as tensored with $\mathbb{Q}$ without mentioning it explicitly. Since $H^3_{cdh}(X, \mathcal{K}^M_3) \rightarrow F^3KH_0^M(X)$, we need to show that

$$H^3_{cdh}(X, \mathcal{K}^M_3) \cong H^3_{cdh}(\tilde{X}, \mathcal{K}^M_3).$$

We only need to show that the hypothesis of Theorem 11.2 is satisfied. Our assumption and the Mayer-Vietoris exact sequence show that $H^2_{cdh}(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^2_{cdh}(E, \mathcal{O}_E)$. On the other hand, our assumption $H^3_{cdh}(X, \Omega^1_{X/\mathbb{C}}) = 0$ implies that the same holds for $\tilde{X}$. The Hodge theory now implies that $H^2_{cdh}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. We conclude that $H^2_{cdh}(E, \mathcal{O}_E) = 0$.

Since $E$ is a strict normal crossing divisor, and since $CH^2(E^N)$ is finite-dimensional by our assumption, we can now apply Corollary 12.4 to reduce to proving that $H^2_{Zar}(\tilde{X}, \mathcal{K}^M_2)_{hom} \otimes \mathbb{C}^* \rightarrow H^2_{Zar}(E, \mathcal{K}^M_2)_{hom} \otimes \mathbb{C}^*$. For this, it suffices to show that $A^2(\tilde{X}) \rightarrow A^2(E)$, using the isomorphism $H^2_{Zar}(\tilde{X}, \mathcal{K}^M_2) \cong CH^2(\tilde{X})$. But this is shown in Lemma 13.1. □

Corollary 13.3. Let $X$ be a normal and projective threefold over $\mathbb{C}$ with only isolated singularities such that $H^3_{Zar}(X, \Omega^i_{X/\mathbb{C}}) = 0$ for $0 \leq i \leq 1$. Assume GBC(2) and GHC(1, $3$, $\tilde{X}$) for some resolution of singularities $\tilde{X}$ as in (7.6). Then $CH^3(X) \cong CH^3(\tilde{X})$.

Proof. As in the proof of Corollary 12.3, it suffices to prove the result with the rational coefficients. It follows from our assumption and [11] Proposition 2.6] that $H^3_{cdh}(X, \Omega^i_{X/\mathbb{C}}) = 0$ for $0 \leq i \leq 1$. In particular, Theorem 13.2 applies. The corollary now follows from Theorems 5.8 and 13.2. □

14. Zero-cycles in arbitrary dimension

In this last section, we give a weaker form of Theorem 13.2 in arbitrary dimension using Theorem 11.2. This situation particularly applies in case of projective cones over smooth projective varieties.

Theorem 14.1. Let $X$ be a normal and projective variety of dimension $d$ over $\mathbb{C}$ such that $H^j_{cdh}(X, \Omega^i_{X/\mathbb{C}}) = 0$ for $0 \leq i \leq d - 2$ and $j \geq d - 1$. Let $\tilde{X} \rightarrow X$ be a resolution of singularities such that the reduced exceptional divisor is smooth. Assume that GBC($d-1$) and GHC($d-2, 2d-3, \tilde{X}$) hold. Then $F^dKH_0^M(X) \cong CH^d(\tilde{X})_Q$.

Proof. Following the proof of Theorem 13.2 we can use Theorem 11.2 and Lemma 7.4 to reduce to showing that the map

$$CH^{d-1}(\tilde{X})_{hom} \otimes \mathbb{C}^* \rightarrow CH^{d-1}(E)_{hom} \otimes \mathbb{C}^*$$

(14.1)
is surjective, where \( E \) is the reduced exceptional divisor on \( \tilde{X} \). The \( cdh \) cohomology exact sequence (which uses the normality of \( X \))

\[
H^{d-1}_{cdh} \left( \tilde{X}, \Omega^i_{\tilde{X}/\mathbb{C}} \right) \to H^{d-1}_{cdh} \left( E, \Omega^i_{E/\mathbb{C}} \right) \to H^d_{cdh} \left( X, \Omega^i_{X/\mathbb{C}} \right)
\]

shows that \( H^{d-1}_{cdh} \left( E, \Omega^i_{E/\mathbb{C}} \right) = 0 \) for \( 0 \leq i \leq d - 3 \). Together with our assumption, this implies that the map

\begin{equation}
(14.2) \quad CH^{d-1}(E)_{hom} \xrightarrow{\cong} J^{d-1}(E).
\end{equation}

On the other hand, the vanishing assumption on the cohomology of \( X \) and its normality imply that \( \mathbb{H}^{2d-2}_{cdh} \left( X, \Omega^{d-1}_{X/\mathbb{C}} \right) = 0 = \mathbb{H}^{2d-3}_{cdh} \left( X_{\text{sing}}, \Omega^{d-1}_{X_{\text{sing}}/\mathbb{C}} \right) \). This in turn implies from (6.12) that the map \( J^{d-1}(\tilde{X}) \to J^{d-1}(E) \) is surjective. The required surjectivity in (14.1) now follows from Corollary 6.11.

**Remark 14.2.** A conscious reader would observe that one does not need the assumption \( H^{d-1}_{cdh}(X, \mathcal{O}_X) = 0 \). This condition is needed only for \( \tilde{X} \), which follows from the vanishing assumption on the top cohomology groups and Hodge theory.

**Corollary 14.3.** Let \( X \) be a normal and projective variety of dimension \( d \) over \( \mathbb{C} \) with only isolated singularities such that \( H^i_{\text{Zar}} \left( X, \Omega^i_{X/\mathbb{C}} \right) = 0 \) for \( 0 \leq i \leq d - 2 \). Let \( \tilde{X} \to X \) be a resolution of singularities such that the reduced exceptional divisor is smooth and \( H^i_{\text{Zar}} \left( \tilde{X}, \Omega^i_{\tilde{X}/\mathbb{C}} \right) = 0 \) for \( 1 \leq i \leq d - 2 \). Assume that \( GBC(d-1) \) and \( GHC(d-2, 2d-3, \tilde{X}) \) hold. Then \( CH^d(X) \xrightarrow{\cong} CH^d(\tilde{X}) \).

**Proof.** Follows from Theorems 14.1, 5.8 and Remark 14.2 as in the proof of Corollary 13.3.

The following improves [35, Theorem 1.5] for the Chow group of zero-cycles on projective cones.

**Corollary 14.4.** Let \( Y \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \) be a smooth projective variety of dimension \( d \) and let \( X \) be the projective cone over \( Y \). Assume that \( H^i_{\text{Zar}} \left( X, \Omega^i_{X/\mathbb{C}} \right) = 0 \) for \( 0 \leq i \leq d - 1 \). Then \( CH^{d+1}(X) \) is finite-dimensional if \( CH^d(Y) \) is so.

**Proof.** Let \( \tilde{X} \xrightarrow{\ell} X \) be the blow-up of \( X \) at the vertex. Then \( \tilde{X} \xrightarrow{p} Y \) is a \( \mathbb{P}^1 \)-bundle, and \( f \) is a resolution of singularities of \( X \). Moreover, \( E \hookrightarrow \tilde{X} \) maps isomorphically to \( Y \) under the map \( p \). It is also easy to see that \( CH^d(Y) \cong CH^{d+1}(\tilde{X}) \) and \( J^d(Y) \cong J^{d+1}(\tilde{X}) \). Hence we only need to show that \( CH^{d+1}(X) \xrightarrow{\cong} CH^{d+1}(\tilde{X}) \). We only have to show the injectivity as the surjectivity is already known. But this follows from the exact sequence

\[
H^d_{cdh} \left( \tilde{X}, \mathcal{K}_{d+1} \right) \to H^d_{cdh} \left( E, \mathcal{K}_{d+1} \right) \to H^{d+1}_{cdh} \left( X, \mathcal{K}_{d+1} \right) \to H^{d+1}_{cdh} \left( \tilde{X}, \mathcal{K}_{d+1} \right) \to 0,
\]

the fact that \( E \hookrightarrow \tilde{X} \) has a section, Corollary 5.7 and Theorem 5.8.

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