Maximal regularity of the time-periodic Navier-Stokes system

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Time-periodic solutions to the linearized Navier-Stokes system in the $n$-dimensional whole-space are investigated. For time-periodic data in $L^q$-spaces, maximal regularity and corresponding a priori estimates for the associated time-periodic solutions are established. More specifically, a Banach space of time-periodic vector fields is identified with the property that the linearized Navier-Stokes operator maps this space homeomorphically onto the $L^q$-space of time-periodic data.

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1 Introduction

We investigate the time-periodic, linearized Navier-Stokes system in the $n$-dimensional whole-space with $n \geq 2$. More specifically, we consider the linearized Navier-Stokes system

$$\begin{cases}
\partial_t u - \Delta u - \lambda \partial_1 u + \nabla p = f & \text{in } \mathbb{R}^n \times \mathbb{R}, \\
\text{div } u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}
\end{cases}$$

(1.1)

for an Eulerian velocity field $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and pressure term $p : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as well as data $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ that are all $T$-time-periodic, that is,

$$\forall (x,t) \in \mathbb{R}^n \times \mathbb{R} : \quad u(x,t) = u(x,t + T) \quad \text{and} \quad p(x,t) = p(x,t + T)$$

(1.2)

and

$$\forall (x,t) \in \mathbb{R}^n \times \mathbb{R} : \quad f(x,t) = f(x,t + T).$$

(1.3)
The time period $T > 0$ remains fixed. The constant $\lambda \in \mathbb{R}$ determines the type of linearization. If $\lambda = 0$ the system (1.1) is a Stokes system, while $\lambda \neq 0$ turns (1.1) into an Oseen system. These are the two possible ways to linearize the Navier-Stokes system.

The main goal in this paper is to identify a Banach space $X^q$ with the property that for any vector field $f \in L^q(\mathbb{R}^n \times (0, T))^n$ satisfying (1.3) there is a unique solution $(u, p)$ in $X^q$ to (1.1)–(1.2) such that

$$\|(u, p)\|_{X^q} \leq C \|f\|_q,$$

with a constant $C$ depending only on $\lambda, T, q$, and $n$. Note that any $f \in L^q(\mathbb{R}^n \times (0, T))^n$ has a trivial extension to a time-periodic vector field satisfying (1.3). More precisely, we wish to establish a functional analytic setting in which (1.1)–(1.3) can be written on an operator form

$$A(u, p) = f$$

such that the operator

$$A : X^q \to L^q(\mathbb{R}^n \times (0, T))^n$$

is a homeomorphism. We say that a function space $X^q$ with this property establishes maximal regularity in the $L^q$-setting for the linearized, time-periodic Navier-Stokes system. In this context, regularity refers not only to the order of differentiability of $(u, p)$, but also, and in particular, to the summability of $u$ and $p$.

In order to identify the space $X^q$, we shall employ the theory of Fourier multipliers. This may not seem surprising, as similar results for both the corresponding steady-state and initial-value problem are traditionally established using Fourier multipliers. However, it is not directly clear how to employ the Fourier transform on the space-time domain $\mathbb{R}^n \times (0, T)$. The main idea behind the approach presented in the following is to identify $\mathbb{R}^n \times (0, T)$ with the group $G := \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z}$. Equipped with the canonical topology, $G$ is a locally compact abelian group. As such, there is a naturally defined Fourier transform $\mathcal{F}_G$ associated to $G$. Moreover, as $G$ inherits a differentiable structure from $\mathbb{R}^n \times \mathbb{R}$ in a canonical way, we may view (1.1) as a system of differential equations on $G$. Employing the Fourier transform $\mathcal{F}_G$, we then obtain a representation of the solution in terms of a Fourier multiplier defined on the dual group $\hat{G}$. Based on this representation, the space $X^q$ and corresponding a priori estimate (1.4) will be established. Since multiplier theorems like the theorems of Mihlin, Lizorkin or Marcinkiewicz are only available in an Euclidean setting $\mathbb{R}^m$, and not in the general setting of group multipliers, we shall employ a so-called transference principle. More specifically, we shall use a theorem, which in its original form is due to de Leeuw [2], that enables us to study the properties of a multiplier defined on $\hat{G}$ in terms of a corresponding multiplier defined on $\mathbb{R}^{n+1}$.

2 Statement of main result

In order to state the main result, we first introduce an appropriate decomposition of the problem. For this purpose, we express the data $f$ as a sum of a time-independent
part $f_s$ and a time-periodic part $f_{tp}$ with vanishing time-average over the period, that is, $f = f_s + f_{tp}$ with $f_s : \mathbb{R}^n \to \mathbb{R}^n$ and $f_{tp} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfying (1.3) and $\int_0^T f_{tp}(x, t) \, dt = 0$. Based on this decomposition of $f$, we split the investigation of (1.1) into two parts: A steady-state problem with data $f_s$ corresponding to time-independent unknowns $(u_s, p_s)$, and a time-periodic problem with data $f_{tp}$ corresponding to time-periodic unknowns $(u_{tp}, p_{tp})$ with vanishing time-average over the period. In this way, we obtain a solution $(u, p) = (u_s + u_{tp}, p_s + p_{tp})$ to (1.1). We shall identify function spaces $X^q_s$ and $X^q_{tp}$ that establish maximal regularity for $u_s$ and $u_{tp}$ separately. We thus obtain $X^q = (X^q_s \oplus X^q_{tp}) \times X^q_{\text{pres}}$, where $X^q_{\text{pres}}$ is the function space that establishes maximal regularity for the pressure term $p$. It turns out that the regularity, more specifically the summability, of the velocity fields in $X^q_s$ and $X^q_{tp}$ are substantially different.

We denote points in $\mathbb{R}^n \times \mathbb{R}$ by $(x, t)$, and refer to $x$ as the spatial and $t$ as the time variable. Our main result is formulated in terms of the function spaces:

\[
C^\infty_{\text{per}}(\mathbb{R}^n \times \mathbb{R}) := \{ U \in C^\infty(\mathbb{R}^n \times \mathbb{R}) \mid \forall t \in \mathbb{R} : U(\cdot, t + T) = U(\cdot, t) \},
\]

\[
C^\infty_{0, \text{per}}(\mathbb{R}^n \times [0, T]) := \{ u \in C^\infty(\mathbb{R}^n \times [0, T]) \mid \exists U \in C^\infty(\mathbb{R}^n \times \mathbb{R}) : u = U|_{[0, T]} \},
\]

\[
C^\infty_{0, \text{per}, \sigma}(\mathbb{R}^n \times [0, T]) := \{ u \in C^\infty_{0, \text{per}}(\mathbb{R}^n \times [0, T]) \mid \text{div}_x u = 0 \},
\]

for $q \in (1, \infty)$ the Lebesgue space of solenoidal vector fields

\[
L^q_x(\mathbb{R}^n \times (0, T)) := \left\{ u \in C^\infty_{0, \text{per}}(\mathbb{R}^n \times [0, T]) \mid \text{div}_x u = 0 \right\},
\]

\[
\|u\|_q := \|u\|_{L^q(\mathbb{R}^n \times (0, T))}
\]

and the Sobolev space of time-periodic, solenoidal, vector fields

\[
W^{2,1,q}_{\sigma, \text{per}}(\mathbb{R}^n \times (0, T)) := C^\infty_{0, \text{per}}(\mathbb{R}^n \times [0, T])\big|^{2,1,q},
\]

\[
\|u\|_{2,1,q} := \left( \sum_{|\alpha| \leq 2, |\beta| \leq 1} \|\partial_\alpha^\beta u\|_{L^q_x(\mathbb{R}^n \times (0, T))}^q + \|\partial_t^\alpha u\|_{L^q_x(\mathbb{R}^n \times (0, T))}^q \right)^{1/q}
\]

In order to incorporate the decomposition described above on the level of function spaces, we introduce the projection

\[
\mathcal{P} : C^\infty_{0, \text{per}}(\mathbb{R}^n \times [0, T]) \to C^\infty_{0, \text{per}}(\mathbb{R}^n \times [0, T]), \quad \mathcal{P}u(x, t) := \frac{1}{T} \int_0^T u(x, s) \, ds
\]

(2.1)

and put $\mathcal{P}_\perp := \text{Id} - \mathcal{P}$. Observe that $\mathcal{P}$ and $\mathcal{P}_\perp$ decompose a time-periodic vector field $u$ into a time-independent part $\mathcal{P}u$ and a time-periodic part $\mathcal{P}_\perp u$ with vanishing time-average over the period. Both projections extend by continuity to bounded operators on $L^q_x(\mathbb{R}^n \times (0, T))$ and $W^{2,1,q}_{\sigma, \text{per}}(\mathbb{R}^n \times (0, T))$. Finally, we put

\[
W^{2,1,q}_{\sigma, \text{per}, \perp}(\mathbb{R}^n \times (0, T)) := \mathcal{P}_\perp W^{2,1,q}_{\sigma, \text{per}}(\mathbb{R}^n \times (0, T)),
\]

\[
L^q_x(\mathbb{R}^n \times (0, T)) := \mathcal{P}_\perp L^q_x(\mathbb{R}^n \times (0, T)).
\]

Our first main theorem states that $X^q_{tp} = W^{2,1,q}_{\sigma, \text{per}, \perp}(\mathbb{R}^n \times (0, T))$. We state the theorem in a setting of solenoidal vector-fields and thereby eliminate the need to characterize the pressure term. More specifically, we show:

3
Theorem 2.1. Let \( q \in (1, \infty) \). For each \( f \in L^q_{\sigma,+}(\mathbb{R}^n \times (0, T)) \) there is a unique solution \( u \in W^{2,1,q}_{\sigma,\text{per},+}(\mathbb{R}^n \times (0, T)) \) to (1.1), that is, a solution in the sense that the trivial extensions of \( u \) and \( f \) to time-periodic vector fields on \( \mathbb{R}^n \times \mathbb{R} \) together with \( p = 0 \) constitutes a solution in the standard sense of distributions. Moreover,
\[
\|u\|_{2,1,q} \leq C_1 \|f\|_q,
\]
where \( C_1 = C_1(\lambda, \mathcal{T}, n, q) \).

The other maximal regularity space \( X^q_{\sigma} \) is already well-known from the theory of steady-state, linearized Navier-Stokes systems. In order to characterize it, one has to distinguish between \( \lambda = 0 \) (Stokes case) and \( \lambda \neq 0 \) (Oseen case). Moreover, the case \( n = 2 \) has to be treated separately. For \( n \geq 3, \lambda = 0, q \in (1, \frac{q}{2}) \) we put
\[
X^q_{\sigma,\text{Stokes}}(\mathbb{R}^n) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^n)^n \mid \text{div } v = 0, \|v\|_{q,\text{Stokes}} < \infty \right\},
\]
\[
\|v\|_{q,\text{Stokes}} := \|v\|_{n-q} + \|\nabla v\|_{n-q} + \|\nabla^2 v\|_q.
\]
For \( n \geq 3, \lambda \neq 0, q \in (1, \frac{n+1}{2}) \) we put
\[
X^q_{\sigma,\text{Oseen}}(\mathbb{R}^n) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^n)^n \mid \text{div } v = 0, \|v\|_{q,\lambda,\text{Oseen}} < \infty \right\},
\]
\[
\|v\|_{q,\lambda,\text{Oseen}} := \|\lambda\|_{n+1}^{\frac{2}{n+1}} \|v\|_{n+1,2,\mathbb{R}^n} + \|\lambda\|_{n+1}^{\frac{2}{n+1}} \|\nabla v\|_{n+1,2,\mathbb{R}^n} + \|\lambda\| \|\partial_1 v\|_q + \|\nabla^2 v\|_q.
\]
For \( n = 2, \lambda \neq 0, q \in (1, \frac{3}{2}) \) we put
\[
X^q_{\sigma,\text{Oseen2D}}(\mathbb{R}^2) := \left\{ v \in L^1_{\text{loc}}(\mathbb{R}^2)^2 \mid \text{div } v = 0, \|v\|_{q,\lambda,\text{Oseen2D}} < \infty \right\},
\]
\[
\|v\|_{q,\lambda,\text{Oseen2D}} := \|\lambda\|_{2}^{\frac{1}{2}} \|v\|_{2,\mathbb{R}^2} + \|\lambda\|_{2}^{\frac{1}{2}} \|\nabla v\|_{2,\mathbb{R}^2} + \|\lambda\| \|\partial_1 v\|_2 + \|\nabla^2 v\|_2.
\]

We shall not treat the case \( n = 2, \lambda = 0 \) explicitly; see however remark 2.3 below. To characterize the maximal regularity for the pressure \( p \), we put
\[
X^q_{\text{pres}}(\mathbb{R}^n \times (0, T)) := \{ p \in L^1_{\text{loc}}(\mathbb{R}^n \times (0, T)) \mid \|p\|_{X^q_{\text{pres}}} < \infty \},
\]
\[
\|p\|_{X^q_{\text{pres}}} := \left( \frac{1}{T} \int_0^T \|p(\cdot, t)\|_{a}^q + \|\nabla p(\cdot, t)\|_{a}^q dt \right)^{1/q}.
\]

We are now in a position to state the theorem that establishes maximal regularity in the general \( L^q \)-setting.

Theorem 2.2 (Maximal regularity in \( L^q \)-setting). Let
\[
X^q_{\sigma}(\mathbb{R}^n) := X^q_{\sigma,\text{Stokes}}(\mathbb{R}^n) \quad \text{if } n \geq 3, \lambda = 0, q \in (1, \frac{n}{2}),
\]
\[
X^q_{\sigma}(\mathbb{R}^n) := X^q_{\sigma,\text{Oseen}}(\mathbb{R}^n) \quad \text{if } n \geq 3, \lambda \neq 0, q \in (1, \frac{n+1}{2}),
\]
\[
X^q_{\sigma}(\mathbb{R}^n) := X^q_{\sigma,\text{Oseen2D}}(\mathbb{R}^2) \quad \text{if } n = 2, \lambda \neq 0, q \in (1, \frac{3}{2}).
\]
For every \( f \in L^q(\mathbb{R}^n \times (0, T))^n \) there is unique solution\(^1\)

\[
(u, p) = (v \oplus w, p) \in \left( X^q_\sigma(\mathbb{R}^n) \oplus W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^n \times (0, T)) \right) \times X^q_{\text{pres}}(\mathbb{R}^n \times (0, T))
\]

to (1.1) in the sense that the trivial extensions of \((u, p)\) and \(f\) to time-periodic vector fields on \(\mathbb{R}^n \times \mathbb{R}\) that satisfy (1.2)–(1.3) constitute a solution in the standard sense of distributions. Moreover,

\[
\|v\|_{X^q_\sigma} + \|w\|_{2,1,q} + \|p\|_{X^q_{\text{pres}}} \leq C_2 \|f\|_q,
\]

where \(C_2 = C_2(\lambda, T, n, q)\)

Remark 2.3. The projection \(P\) induces the decomposition

\[
L^q_\sigma(\mathbb{R}^n \times (0, T)) = L^q_\sigma(\mathbb{R}^n) \oplus L^q_{\sigma,\perp}(\mathbb{R}^n \times (0, T)).
\]

Theorem 2.1 states that

\[
\partial_t - \Delta - \lambda \partial_1 : W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^n \times (0, T)) \to L^q_{\sigma,\perp}(\mathbb{R}^n \times (0, T))
\]

is a homeomorphism. Theorem 2.2 is therefore basically a consequence of Theorem 2.1 combined with the well-known fact that in the steady-state setting the same operator maps \(X^q_\sigma(\mathbb{R}^n)\), defined in (2.3)–(2.5), homeomorphically onto \(L^q_\sigma(\mathbb{R}^n)\). Generally, if one can identify steady-state function spaces \(\mathcal{X}_\sigma(\mathbb{R}^n)\) and \(\mathcal{Y}_\sigma(\mathbb{R}^n)\) of solenoidal vector fields such that \(-\Delta - \lambda \partial_1 : \mathcal{X}_\sigma(\mathbb{R}^n) \to \mathcal{Y}_\sigma(\mathbb{R}^n)\) is a homeomorphism, Theorem 2.1 can be employed to show that

\[
\partial_t - \Delta - \lambda \partial_1 : \mathcal{X}_\sigma(\mathbb{R}^n) \oplus W^{2,1,q}_{\sigma,\text{per},\perp}(\mathbb{R}^n \times (0, T)) \to \mathcal{Y}_\sigma(\mathbb{R}^n) \oplus L^q_{\sigma,\perp}(\mathbb{R}^n \times (0, T))
\]

is a homeomorphism. If one for example wishes to investigate a case in which the parameters \(n, \lambda, q\) are not covered by (2.3)–(2.5), one must simply identify such function spaces \(\mathcal{X}_\sigma(\mathbb{R}^n)\) and \(\mathcal{Y}_\sigma(\mathbb{R}^n)\) for this particular choice of parameters. In Theorem 2.1 we have covered only the cases where these spaces are well-known by what can be considered standard theory.

3 Notation

Points in \(\mathbb{R}^n \times \mathbb{R}\) are denoted by \((x, t)\) with \(x \in \mathbb{R}^n\) and \(t \in \mathbb{R}\). We refer to \(x\) as the spatial and to \(t\) as the time variable.

For a sufficiently regular function \(u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}\), we put \(\partial_t u := \partial_x u\). For any multiindex \(\alpha \in \mathbb{N}_0^n\), we let \(\partial^{\alpha}_x u := \sum_{j=1}^n \partial_j^{\alpha_j} u\) and put \(|\alpha| := \sum_{j=1}^n \alpha_j\). Moreover, for \(x \in \mathbb{R}^n\) we let \(x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}\). Differential operators act only in the spatial variable unless otherwise indicated. For example, we denote by \(\Delta u\) the Laplacian of \(u\) with respect

\(^1\)We use \(\oplus\) to denote either the sum of two subspaces of \(L^1_{\text{loc}}(\mathbb{R}^n \times (0, T))\) whose intersection contains only 0, or the sum of two functions from such subspaces.
to the spatial variable, that is, $\Delta u := \sum_{j=1}^{n} \partial_{j}^{2} u$. For a vector field $u : \mathbb{R}^{n} \times \mathbb{R} \to \mathbb{R}^{n}$ we let $\text{div} u := \sum_{j=1}^{n} \partial_{j} u_{j}$ denote the divergence of $u$.

For two vectors $a, b \in \mathbb{R}^{n}$ we let $a \otimes b \in \mathbb{R}^{n} \times \mathbb{R}^{n}$ denote the tensor with $(a \otimes b)_{ij} := a_{i}b_{j}$. We denote by $I$ the identity tensor $I \in \mathbb{R}^{n \times n}$.

For a vector space $X$ and $A, B \subset X$, we write $X = A \oplus B$ iff $A$ and $B$ are subspaces of $X$ with $A \cap B = \{0\}$ and $X = A + B$. We also write $a \oplus b$ for elements of $A \oplus B$.

Constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

4 Reformulation in a group setting

In the following, we let $G$ denote the group

$$ G := \mathbb{R}^{n} \times \mathbb{R}/\mathbb{T}\mathbb{Z} $$

(4.1)

with addition as the group operation. We shall reformulate (1.1)–(1.3) and the main theorems in a setting of functions defined on $G$. For this purpose, we must first introduce a topology and an appropriate differentiable structure on $G$. It is then possible to define Lebesgue and Sobolev spaces on $G$.

4.1 Differentiable structure, distributions and Fourier transform

The topology and differentiable structure on $G$ is inherited from $\mathbb{R}^{n} \times \mathbb{R}$. More precisely, we equip $G$ with the quotient topology induced by the canonical quotient mapping

$$ \pi : \mathbb{R}^{n} \times \mathbb{R} \to \mathbb{R}^{n} \times \mathbb{R}/\mathbb{T}\mathbb{Z}, \quad \pi(x, t) := (x, [t]). $$

(4.2)

Equipped with the quotient topology, $G$ becomes a locally compact abelian group. We shall use the restriction

$$ \Pi : \mathbb{R}^{n} \times [0, T) \to G, \quad \Pi := \pi|_{\mathbb{R}^{n} \times [0, T)} $$

to identify $G$ with the domain $\mathbb{R}^{n} \times [0, T)$, as $\Pi$ is clearly a (continuous) bijection.

Via $\Pi$, one can identify the Haar measure $dg$ on $G$ as the product of the Lebesgue measure on $\mathbb{R}^{n}$ and the Lebesgue measure on $[0, T)$. The Haar measure is unique up-to a normalization factor, which we choose such that

$$ \forall u \in C_{0}(G) : \int_{G} u(g) \, dg = \frac{1}{T} \int_{0}^{T} \int_{\mathbb{R}^{n}} u \circ \Pi(x, t) \, dx \, dt, $$

where $C_{0}(G)$ denotes the space of continuous functions of compact support. For the sake of convenience, we will omit the $\Pi$ in integrals of $G$-defined functions with respect to $dx \, dt$, that is, instead of $\frac{1}{T} \int_{0}^{T} \int_{\mathbb{R}^{n}} u \circ \Pi(x, t) \, dx \, dt$ we simply write $\frac{1}{T} \int_{0}^{T} \int_{\mathbb{R}^{n}} u(x, t) \, dx \, dt$.

Next, we define by

$$ C^{\infty}(G) := \{ u : G \to \mathbb{R} \mid u \circ \pi \in C^{\infty}(\mathbb{R}^{n} \times \mathbb{R}) \} $$

(4.3)
the space of smooth functions on $G$. For $u \in C^\infty(G)$ we define derivatives

$$
\forall (\alpha, \beta) \in N_0^n \times N_0^n : \quad \partial_t^\alpha \partial_x^\beta u := \left[ \partial_t^\alpha \partial_x^\beta (u \circ \pi) \right] \circ \Pi^{-1}.
\leqno{(4.4)}
$$

It is easy to verify for $u \in C^\infty(G)$ that also $\partial_t^\alpha \partial_x^\beta u \in C^\infty(G)$. We further introduce the subspace

$$
C^\infty_0(G) := \{ u \in C^\infty(G) \mid \text{supp } u \text{ is compact} \}
$$

of compactly supported smooth functions. Clearly, $C^\infty_0(G) \subset C^\infty_0(G)$.

With a differentiable structure defined on $G$ via (4.3), we can introduce the space of tempered distributions on $G$. For this purpose, we first recall the Schwartz-Bruhat space of generalized Schwartz functions; see for example [1]. More precisely, we define for $u \in C^\infty(G)$ the semi-norms

$$
\forall (\alpha, \beta, \gamma) \in N_0^n \times N_0^n \times N_0^n : \quad \rho_{\alpha, \beta, \gamma}(u) := \sup_{(x,t) \in G} |x^\gamma \partial_t^\alpha \partial_x^\beta u(x,t)|,
$$

and put

$$
\mathcal{S}(G) := \{ u \in C^\infty(G) \mid \forall (\alpha, \beta, \gamma) \in N_0^n \times N_0^n \times N_0^n : \rho_{\alpha, \beta, \gamma}(u) < \infty \}.
$$

Clearly, $\mathcal{S}(G)$ is a vector space, and $\rho_{\alpha, \beta, \gamma}$ a semi-norm on $\mathcal{S}(G)$. We endow $\mathcal{S}(G)$ with the semi-norm topology induced by the family $\{ \rho_{\alpha, \beta, \gamma} \mid (\alpha, \beta, \gamma) \in N_0^n \times N_0^n \times N_0^n \}$. The topological dual space $\mathcal{S}'(G)$ of $\mathcal{S}(G)$ is then well-defined. We equip $\mathcal{S}'(G)$ with the weak* topology and refer to it as the space of tempered distributions on $G$. Observe that both $\mathcal{S}(G)$ and $\mathcal{S}'(G)$ remain closed under multiplication by smooth functions that have at most polynomial growth with respect to the spatial variables.

For a tempered distribution $u \in \mathcal{S}'(G)$, distributional derivatives $\partial_t^\alpha \partial_x^\beta u \in \mathcal{S}'(G)$ are defined by duality in the usual manner:

$$
\forall \psi \in \mathcal{S}(G) : \quad \langle \partial_t^\alpha \partial_x^\beta u, \psi \rangle := \langle u, (-1)^{|(\alpha, \beta)|} \partial_t^\alpha \partial_x^\beta \psi \rangle.
$$

It is easy to verify that $\partial_t^\alpha \partial_x^\beta u$ is well-defined as an element in $\mathcal{S}'(G)$. For tempered distributions on $G$, we keep the convention that differential operators like $\Delta$ and div act only in the spatial variable $x$ unless otherwise indicated.

We shall also introduce tempered distributions on $G$’s dual group $\hat{G}$. We associate each $(\xi, k) \in \mathbb{R}^n \times \mathbb{Z}$ with the character $\chi : G \to \mathbb{C}$, $\chi(x, t) := e^{ix \cdot \xi + i k t}$ on $G$. It is standard to verify that all characters are of this form, and we can thus identify $\hat{G} = \mathbb{R}^n \times \mathbb{Z}$. By default, $\hat{G}$ is equipped with the compact-open topology, which in this case coincides with the product of the Euclidean topology on $\mathbb{R}^n$ and the discrete topology on $\mathbb{Z}$. The Haar measure on $\hat{G}$ is simply the product of the Lebesgue measure on $\mathbb{R}^n$ and the counting measure on $\mathbb{Z}$.

A differentiable structure on $\hat{G}$ is obtained by introduction of the space

$$
C^\infty(\hat{G}) := \{ w \in C(\hat{G}) \mid \forall k \in \mathbb{Z} : \text{ } w(\cdot, k) \in C^\infty(\mathbb{R}^n) \}.
$$
To define the generalized Schwartz-Bruhat space on the dual group \( \hat{G} \), we further introduce the semi-norms
\[
\forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0 \times \mathbb{N}_0^n : \ \hat{\rho}_{\alpha, \beta, \gamma}(w) := \sup_{(\xi, k) \in G} |k^\beta \xi^\alpha \partial^\gamma_\xi w(\xi, k)|.
\]
We then put
\[
\mathcal{I}(\hat{G}) := \{ w \in C^\infty(\hat{G}) \mid \forall (\alpha, \beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0 \times \mathbb{N}_0^n : \ \hat{\rho}_{\alpha, \beta, \gamma}(w) < \infty \}.
\]
We endow the vector space \( \mathcal{I}(\hat{G}) \) with the semi-norm topology induced by the family of semi-norms \( \{ \hat{\rho}_{\alpha, \beta, \gamma} \mid (\alpha, \beta, \gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0 \times \mathbb{N}_0^n \} \). The topological dual space of \( \mathcal{I}(\hat{G}) \) is denoted by \( \mathcal{I}'(\hat{G}) \). We equip \( \mathcal{I}'(\hat{G}) \) with the weak* topology and refer to it as the space of tempered distributions on \( \hat{G} \).

All function spaces have so far been defined as real vector spaces of real functions. Clearly, we can define them analogously as complex vector spaces of complex functions. When a function space is used in context with the Fourier transform, which we shall introduce below, we consider it as a complex vector space.

The Fourier transform \( \mathcal{F}_G \) on \( G \) is given by
\[
\mathcal{F}_G : L^1(G) \to C(\hat{G}), \quad \mathcal{F}_G(u)(\xi, k) := \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} u(x, t) e^{-ix \cdot \xi - ik \cdot t} \, dx \, dt.
\]
If no confusion can arise, we simply write \( \mathcal{F} \) instead of \( \mathcal{F}_G \). The inverse Fourier transform is formally defined by
\[
\mathcal{F}^{-1} : L^1(\hat{G}) \to C(G), \quad \mathcal{F}^{-1}(w)(x, t) := w^\vee(x, t) := \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} w(\xi, k) e^{ix \cdot \xi + ik \cdot t} \, d\xi.
\]
It is standard to verify that \( \mathcal{F} : \mathcal{I}(G) \to \mathcal{I}(\hat{G}) \) is a homeomorphism with \( \mathcal{F}^{-1} \) as the actual inverse, provided the Lebesgue measure \( d\xi \) is normalized appropriately. By duality, \( \mathcal{F} \) extends to a mapping \( \mathcal{I}'(G) \to \mathcal{I}'(\hat{G}) \). More precisely, we define
\[
\mathcal{F} : \mathcal{I}'(G) \to \mathcal{I}'(\hat{G}), \quad \forall \psi \in \mathcal{I}(\hat{G}) : \langle \mathcal{F}(u), \psi \rangle := \langle u, \mathcal{F}(\psi) \rangle.
\]
Similarly, we define
\[
\mathcal{F}^{-1} : \mathcal{I}'(\hat{G}) \to \mathcal{I}'(G), \quad \forall \psi \in \mathcal{I}(G) : \langle \mathcal{F}^{-1}(u), \psi \rangle := \langle u, \mathcal{F}^{-1}(\psi) \rangle.
\]
Clearly \( \mathcal{F} : \mathcal{I}'(G) \to \mathcal{I}'(\hat{G}) \) is a homeomorphism with \( \mathcal{F}^{-1} \) as the actual inverse.

The Fourier transform in the setting above provides us with a calculus between the differential operators on \( G \) and the polynomials on \( \hat{G} \). As one easily verifies, for \( u \in \mathcal{I}'(G) \) and \( \alpha \in \mathbb{N}_0^n, \ l \in \mathbb{N}_0 \) we have
\[
\mathcal{F} \left( \partial^\alpha_t \partial_x^\alpha u \right) = i^{l+|\alpha|} \left( \frac{2\pi}{T} \right)^l k^l \xi^\alpha \mathcal{F}(u)
\]
as identity in \( \mathcal{I}'(\hat{G}) \).
4.2 Sobolev spaces

We let $L^q(G)$ denote the usual Lebesgue space with respect to the Haar measure $dg$. A standard mollifier argument shows that $C_0^\infty(G)$ is a dense subset of $L^q(G)$. It is standard to verify that $L^q(G) \subset \mathcal{S}'(G)$.

We define by

$$W^{2,1,q}(G) := \{ u \in L^q(G) \mid \|u\|_{2,1,q} < \infty \},$$

$$\|u\|_{2,1,q} := \left( \sum_{|\alpha| \leq 2, |\beta| \leq 1} \|\partial^\alpha_x u\|_q^q + \|\partial^\alpha_t u\|_q^q \right)^{1/q}.$$

a Sobolev space. The functions $\partial^\alpha_x u$ and $\partial^\alpha_t u$ above are well-defined at the outset as tempered distributions. The condition $\|u\|_{2,1,q} < \infty$ expresses that these derivatives belong to $L^q(G)$. Standard mollifier arguments yield that $C_0^\infty(G)$ is dense in $W^{2,1,q}(G)$.

We further define the following subspaces of solenoidal vector fields:

$$C_{0,\sigma}^\infty(G) := \{ u \in C_0^\infty(G)^n \mid \text{div } u = 0 \}, \quad (4.5)$$

$$L^q_{\sigma}(G) := C_{0,\sigma}^\infty(G)^{1/q}, \quad (4.6)$$

$$W^{2,1,q}_{\sigma}(G) := C_{0,\sigma}^\infty(G)^{1/2,1,q}. \quad (4.7)$$

We have the following characterization of the spaces above:

**Lemma 4.1.** For any $q \in (1, \infty)$:

$$L^q_{\sigma}(G) = \{ u \in L^q(G)^n \mid \text{div } u = 0 \}, \quad (4.8)$$

$$W^{2,1,q}_{\sigma}(G) = \{ u \in W^{2,1,q}(G)^n \mid \text{div } u = 0 \}. \quad (4.9)$$

The above identities are well-known if the underlying domain of the function spaces is, for example, $\mathbb{R}^n$. A proof can be found in [4, Chapter III.4]. Simple modifications to this proof (see [7, Lemma 3.2.1]) suffice to show the identities in the case where $\mathbb{R}^n$ is replaced with $G$.

Next, we shall define the Helmholtz projection on the Lebesgue space $L^q(G)^n$ of vector fields defined on $G$. For this purpose we employ the Fourier transform $\mathcal{F}_G$ and define the Helmholtz projection as a Fourier multiplier:

**Definition 4.2.** For $f \in L^2(G)^n$ we define by

$$\mathcal{P}_Hf := \mathcal{F}_G^{-1}\left[ \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{f} \right] \quad (4.10)$$

the Helmholtz projection.

It is not immediately clear from the definition of the Helmholtz projection via the Fourier multiplier in (4.10) that $\mathcal{P}_Hf$ is a real function if $f$ is real. This, however, is a simple consequence of the fact that the multiplier in question is even.
Since the multiplier on the right-hand side in (4.10) is bounded, it is natural to initially define \( \mathcal{P}_H \) on \( L^2(G)^n \). If namely \( f \in L^2(G)^n \), by Plancherel’s theorem also \( \mathcal{P}_H f \in L^2(G)^n \). We state in the following lemma that \( \mathcal{P}_H \) can be extended to \( L^q(G)^n \), and that it is a projection with the desired properties of a Helmholtz projection.

**Lemma 4.3.** Let \( q \in (1, \infty) \). Then \( \mathcal{P}_H \) extends uniquely to a continuous projection \( \mathcal{P}_H : L^q(G)^n \to L^q(G)^n \). Moreover, \( \mathcal{P}_H L^q(G)^n = L^q_0(G) \).

**Proof.** It is easy to see that we can define on \( G := \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \) the partial Fourier transforms \( \mathcal{F}_{\mathbb{R}^n} : \mathcal{S}(G) \to \mathcal{S}(G) \) and \( \mathcal{F}_{\mathbb{R}/T\mathbb{Z}} : \mathcal{S}(G) \to \mathcal{S}(G) \) in the canonical way, and that \( \mathcal{F}_G = \mathcal{F}_{\mathbb{R}^n} \circ \mathcal{F}_{\mathbb{R}/T\mathbb{Z}} \). Consequently, for \( f \in L^2(G)^n \cap L^q(G)^n \)

\[
\mathcal{F}_G^{-1} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{f} = \mathcal{F}_G^{-1} \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \mathcal{F}_{\mathbb{R}^n}(f).
\]

From the boundedness of the classical Helmholtz projection on \( L^q(\mathbb{R}^n)^n \), which one recognizes on the right-hand side above, it therefore follows that \( \|\mathcal{P}_H f\|_q \leq \|f\|_q \). Thus, \( \mathcal{P}_H \) extends uniquely to a continuous map \( \mathcal{P}_H : L^q(G)^n \to L^q(G)^n \). One readily verifies that \( \mathcal{P}_H \) is a projection, and that \( \text{div} \mathcal{P}_H f = 0 \). By Lemma 4.1, \( \mathcal{P}_H L^q(G)^n \subset L^q_0(G) \) follows. On the other hand, since \( \text{div} f = 0 \) implies \( \xi_j \hat{f}_j = 0 \), we have \( \mathcal{P}_H f = f \) for all \( f \in L^q_0(G) \). Hence we conclude \( \mathcal{P}_H L^q(G)^n = L^q_0(G) \). \( \square \)

Since \( \mathcal{P}_H : L^q(G)^n \to L^q(G)^n \) is a continuous projection, it decomposes \( L^q(G)^n \) into a direct sum

\[
L^q(G) = L^q_0(G) \oplus \mathcal{G}^q(G) \quad (4.11)
\]
of closed subspaces with

\[
\mathcal{G}^q(G) := (\text{Id} - \mathcal{P}_H)L^q(G)^n. \quad (4.12)
\]

### 4.3 Time-averaging

We shall now introduce the time-averaging projection (2.1) into the setting of vector fields defined on the group \( G \).

**Definition 4.4.** We let

\[
\mathcal{P} : C_{0,s}^\infty(G) \to C_{0,s}^\infty(G), \quad \mathcal{P} f(x, t) := \frac{1}{T} \int_0^T f(x, s) \, ds,
\]

\[
\mathcal{P}_\perp : C_{0,s}^\infty(G) \to C_{0,s}^\infty(G), \quad \mathcal{P}_\perp := \text{Id} - \mathcal{P}.
\]

**Lemma 4.5.** Let \( q \in (1, \infty) \). The operator \( \mathcal{P} \) extends uniquely to a continuous projection \( \mathcal{P} : L^q_0(G) \to L^q_0(G) \) and \( \mathcal{P} : W^{2,1,q}_\sigma(G) \to W^{2,1,q}_\sigma(G) \). The same is true for \( \mathcal{P}_\perp \).
Proof. Clearly, $\mathcal{P}$ is a projection. Employing first Minkowski’s integral inequality and then Hölder’s inequality, we estimate
\[
\|\mathcal{P} f\|_{L^q} = \left( \frac{1}{T} \int_0^T \left[ \int_0^T f(x, s) \, ds \right]^q \, dx \right)^{1/q} 
\leq \frac{1}{T} \int_0^T \left( \int_0^T |f(x, s)|^q \, dx \right)^{1/q} \, ds \leq \|f\|_{L^q}.
\]
Thus, by density $\mathcal{P}$ extends uniquely to a continuous projection $\mathcal{P} : L^q(G) \to L^q(G)$. Estimating derivatives of $\mathcal{P} f$ in the same manner, we find that $\mathcal{P}$ is also bounded with respect to the $W^{2,1,q}(G)$-norm. Consequently, $\mathcal{P}$ extends uniquely to a continuous projection $\mathcal{P} : W^{2,1,q}(G) \to W^{2,1,q}(G)$. It follows trivially that the same is true for $\mathcal{P}_\perp$. 

We use $\mathcal{P}$ and $\mathcal{P}_\perp$ to decompose $L^q(G)$ and $W^{2,1,q}(G)$ into direct sums of functions that are time-independent, i.e., steady states, and functions that have vanishing time-average. Put
\[
L^q_{\sigma,\perp}(G) := \mathcal{P}_\perp L^q_{\sigma}(G), 
\]
\[
W^{2,1,q}_{\sigma,\perp}(G) := \mathcal{P}_\perp W^{2,1,q}_{\sigma}(G). 
\]
Identifying $\mathbb{R}^n$ as a subdomain of $G$, we observe:

**Lemma 4.6.** Let $q \in (1, \infty)$. The projection $\mathcal{P}$ induces the decompositions
\[
L^q_{\sigma}(G) = L^q_{\sigma}(\mathbb{R}^n) \oplus L^q_{\sigma,\perp}(G), 
\]
\[
W^{2,1,q}_{\sigma}(G) = W^{2,q}_{\sigma}(\mathbb{R}^n) \oplus W^{2,1,q}_{\sigma,\perp}(G). 
\]

**Proof.** Since, by Lemma 4.5, $\mathcal{P} : L^q(G) \to L^q(G)$ is a (continuous) projection, it follows that $L^q(G) = \mathcal{P} L^q(G) \oplus \mathcal{P}_\perp L^q(G)$. Consequently, to show (4.15) we only need to verify that $\mathcal{P} L^q_{\sigma}(G) = L^q_{\sigma}(\mathbb{R}^n)$. This, however, is an easy consequence of the fact that $\mathcal{P} f$ is independent on $t$ and thus $\|\mathcal{P} f\|_{L^q(\mathbb{R}^n)} = \|\mathcal{P} f\|_{L^q(G)}$. The decomposition (4.16) follows analogously.

Next, we compute the symbols of the projections $\mathcal{P}$ and $\mathcal{P}_\perp$ with respect to the Fourier transform on $G$.

**Lemma 4.7.** For $f \in \mathcal{S}(G)$
\[
\mathcal{P} f = \mathcal{F}_G^{-1}[\kappa_0 \cdot \hat{f}], 
\]
\[
\mathcal{P}_\perp f = \mathcal{F}_G^{-1}[(1 - \kappa_0) \cdot \hat{f}] 
\]
with
\[
\kappa_0 : \mathcal{G} \to \mathbb{C}, \quad \kappa_0(\xi, k) := \begin{cases} 
1 & \text{if } k = 0, \\
0 & \text{if } k \neq 0.
\end{cases}
\]
Proof. We simply observe that
\[
\mathcal{F}_G[\mathcal{P} f](\xi, k) = \frac{1}{T} \int_0^T \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T f(x, s) \, ds \, e^{-ix\cdot\xi - i\frac{2\pi}{T}kt} \, dx \, dt
\]
\[
= \kappa_0(\xi, k) \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T f(x, s) \, ds \, e^{-ix\cdot\xi} \, dx
\]
\[
= \kappa_0(\xi, k) \hat{f}(\xi, 0) = \kappa_0(\xi, k) \hat{f}(\xi, k).
\]

\[\square\]

4.4 Reformulation

Since the topology and differentiable structure on \(G\) is inherited from \(\mathbb{R}^n \times \mathbb{R}\), we obtain the following equivalent formulation of the time-periodic problem (1.1), including the periodicity conditions (1.2)–(1.3), as a system over \(G\)-defined vector fields:

\[
\begin{aligned}
\partial_t u - \Delta u - \lambda \partial_1 u + \nabla p &= f \quad \text{in } G, \\
\text{div } u &= 0 \quad \text{in } G
\end{aligned}
\]

with unknowns \(u : G \to \mathbb{R}^n\) and \(p : G \to \mathbb{R}\), and data \(f : G \to \mathbb{R}^n\). Observe that in this formulation the periodicity conditions are not needed anymore. Indeed, all functions defined on \(G\) are by definition \(T\)-time-periodic.

Based on the new formulation above, we also obtain the following new formulation of Theorem 2.1:

**Theorem 4.8.** Let \(q \in (1, \infty)\). Put \(A_{TP} := \partial_t - \Delta - \lambda \partial_1\). Then

\[
A_{TP} : W^{2,1,q}_r(G) \to L^q_{r,\perp}(G)
\]

homeomorphically. Moreover

\[
\|A_{TP}^{-1}\| \leq C_3 P(\lambda, T),
\]

where \(C_3 = C_3(q, n)\) and \(P(\lambda, T)\) is a polynomial in \(\lambda\) and \(T\).

The main challenge is now to prove Theorem 4.8. This will be done in the next section. At this point we just emphasize the crucial advantage obtained by formulating the problem in a group setting, which is the ability by means of the Fourier transform \(\mathcal{F}_G\) to express the solution \(u\) to (4.19) in terms of a Fourier multiplier. If we namely apply the Fourier transform \(\mathcal{F}_G\) on both sides of the equations in (4.19), we obtain the equivalent system\(^2\)

\[
\begin{aligned}
(i \frac{2\pi}{T} k) \hat{u} + |\xi|^2 \hat{u} - \lambda i \xi_1 \hat{u} + i \hat{p} \xi &= \hat{f} \quad \text{in } \hat{G}, \\
\xi_j \hat{u}_j &= 0 \quad \text{in } \hat{G}
\end{aligned}
\]

\(^2\)We make use of the Einstein summation convention and implicitly sum over all repeated indices.
Dot-multiplying the first equation with $\xi$, we obtain the relation $i\hat{p}|\xi|^2 = \xi_j\hat{f}_j$ and thus
\[
\left(\frac{2\pi}{T}k + |\xi|^2 - \lambda i\xi_1\right)\hat{u} = \left(I - \frac{\xi \otimes \xi}{|\xi|^2}\right)\hat{f}.
\]
Formally at least, we can therefore deduce
\[
u = \mathcal{F}_G^{-1}\left[\frac{1}{\left(\frac{2\pi}{T}k + |\xi|^2 - \lambda i\xi_1\right)}\mathcal{P}_Hf\right]. \quad (4.21)
\]
This representation formula for the solution $\nu$ shall play a central role in the proof of Theorem 4.8 presented in the next section.

5 Proof of main theorems

The main tool in the proof of Theorem 4.8 is the following theorem on transference of Fourier multipliers, which enables us to “transfer” multipliers from one group setting into another. The theorem is originally due to de Leeuw [2], who established the transference principle between the torus group and $\mathbb{R}$. The more general version below is due to Edwards and Gaudry [3, Theorem B.2.1].

**Theorem 5.1.** Let $G$ and $H$ be locally compact abelian groups. Moreover, let
\[
\Phi : \hat{G} \to \hat{H}
\]
be a continuous homomorphism and $q \in [1, \infty]$. Assume that $m \in L^\infty(\hat{H}; \mathbb{C})$ is a continuous $L^q$-multiplier, that is, there is a constant $C_4$ such that
\[
\forall f \in L^2(H) \cap L^q(H) : \|\mathcal{F}_H^{-1}[m \cdot \mathcal{F}_H(f)]\|_q \leq C_4\|f\|_q.
\]
Then $m \circ \Phi \in L^\infty(\hat{G}; \mathbb{C})$ is also an $L^q$-multiplier with
\[
\forall f \in L^2(G) \cap L^q(G) : \|\mathcal{F}_G^{-1}[m \circ \Phi \cdot \mathcal{F}_G(f)]\|_q \leq C_4\|f\|_q.
\]

**Remark 5.2.** We shall employ Theorem 5.1 with $H := \mathbb{R}^n \times \mathbb{R}$ and $G := \mathbb{R}^n \times \mathbb{R} / T\mathbb{Z}$. A proof of the theorem for this particular choice of groups can be found in [7, Theorem 3.4.5].

We are now in a position to prove Theorem 4.8.

**Proof of Theorem 4.8.** By construction, $\Lambda_{TP}$ is a bounded mapping from $W^{2,1,q}_G(G)$ into $L^q_\sigma(G)$. As one may easily verify, the diagram
\[
\begin{array}{ccc}
W^{2,1,q}_G(G) & \xrightarrow{\Lambda_{TP}} & L^q_\sigma(G) \\
\mathcal{P}_\perp & \downarrow & \mathcal{P}_\perp \\
\mathcal{P}_\perp & \xrightarrow{\Lambda_{TP}} & \mathcal{P}_\perp \\
W^{2,1,q}_G(G) \quad & \quad \Lambda_{TP} & \quad \mathcal{P}_\perp \quad \mathcal{P}_\perp \\
\end{array}
\]
commutes. Thus also
\[ A_{TP} : W^{2,1,q}_{\sigma,\perp}(G) \rightarrow L^q_{\sigma,\perp}(G) \]  
(5.1)
is a bounded map.

We shall now show that the mapping (5.1) is onto. To this end, consider first a vector field \( f \in \mathcal{P}^\perp \mathcal{C}_0^\infty(G) \). Clearly, \( f \in \mathcal{S}(G) \). In view of (4.21), we put
\[ u := \mathcal{F}^{-1}_G \left[ \frac{1}{i^{\frac{2\pi}{T}} k + |\xi|^2 - \lambda i \xi_1} \hat{f} \right]. \]
At the outset, it is not clear that \( u \) is well-defined. However, since \( f = \mathcal{P}^\perp f \) we recall (4.18) to deduce \( \hat{f} = (1 - \kappa_0) \hat{f} \) and thus
\[ u = \mathcal{F}^{-1}_G \left[ \frac{1 - \kappa_0(\xi, k)}{i^{\frac{2\pi}{T}} k + |\xi|^2 - \lambda i \xi_1} \hat{f} \right] = \mathcal{F}^{-1}_G \left[ M(\xi, k) \cdot \hat{f} \right] \]  
(5.2)
with
\[ M : \hat{G} \rightarrow \mathbb{C}, \quad M(\xi, k) := \frac{1 - \kappa_0(\xi, k)}{|\xi|^2 + i(\frac{2\pi}{T} k - \lambda \xi_1)}. \]  
(5.3)
Observe that the denominator \(|\xi|^2 + i(\frac{2\pi}{T} k - \lambda \xi_1)\) in the definition (5.3) of \( M \) vanishes only at \((\xi, k) = (0, 0)\). Since the numerator \( 1 - \kappa_0(\xi, k) \) in (5.3) vanishes in a neighborhood around \((0, 0)\), we see that \( M \in L^\infty(\hat{G}; \mathbb{C}) \). It therefore follows from (5.2) that \( u \) is well-defined as an element of \( \mathcal{S}'(G) \). It follows directly from the definition of \( u \) that \( A_{TP} u = f \). The challenge is now to show that \( u \in W^{2,1,q}_{\sigma,\perp}(G) \) and establish the estimate
\[ \|u\|_{2,1,q} \leq c \|f\|_q \]  
(5.4)
for some constant \( c \). We shall use the transference principle for multipliers, that is, Theorem 5.1, to establish (5.4). For this purpose, let \( \chi \) be a “cut-off” function with
\[ \chi \in \mathcal{C}_0^\infty(\mathbb{R}; \mathbb{R}), \quad \chi(\eta) = 1 \text{ for } |\eta| \leq \frac{1}{2}, \quad \chi(\eta) = 0 \text{ for } |\eta| \geq 1. \]
We then define
\[ m : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}, \quad m(\xi, \eta) := \frac{1 - \chi(\frac{2\pi}{T} \eta)}{|\xi|^2 + i(\eta - \lambda \xi_1)}. \]  
(5.5)
We can consider \( \mathbb{R}^n \times \mathbb{R} \) as a group \( H \) with addition as group operation. Endowed with the Euclidean topological, \( H \) becomes a locally compact abelian group. It is well-known that the dual group \( \hat{H} \) can also be identified with \( \mathbb{R}^n \times \mathbb{R} \) equipped with the Euclidean topology. We can thus consider \( m \) as mapping \( m : \hat{H} \rightarrow \mathbb{C} \). In order to employ Theorem 5.1, we define \( \Phi : \hat{G} \rightarrow \hat{H}, \Phi(\xi, k) := (\xi, \frac{2\pi}{T} k) \). Clearly, \( \Phi \) is a continuous homomorphism. Moreover, \( M = m \circ \Phi \). Consequently, if we can show that \( m \) is a
continuous $L^q(\mathbb{R}^n \times \mathbb{R})$-multiplier we may conclude from Theorem 5.1 that $M$ is an $L^q(G)$-multiplier. We first observe that the only zero of the denominator $|\xi|^2 + i(\eta - \lambda \xi_1)$ in definition (5.5) of $m$ is $(\xi, \eta) = (0, 0)$. Since the numerator $1 - \chi(\frac{\xi}{2\pi})$ in (5.5) vanishes in a neighborhood of $(0, 0)$, we see that $m$ is continuous; in fact $m$ is smooth. We shall now apply Marcinkiewicz’s multiplier theorem, see for example [5, Corollary 5.2.5] or [8, Chapter IV, §6], to show that $m$ is an $L^q(\mathbb{R}^n \times \mathbb{R})$-multiplier. Note that Marcinkiewicz’s multiplier theorem can be employed at this point since $m$ is a Fourier multiplier in the Euclidean $\mathbb{R}^n \times \mathbb{R}$ setting. To employ Marcinkiewicz’s multiplier theorem, we must verify that

$$
\sup_{\varepsilon \in \{0,1\}^{n+1}} \sup_{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}} |\xi_1^\varepsilon_1 \cdots \xi_n^\varepsilon_n \eta^\varepsilon_{n+1} \partial_1^\varepsilon_1 \cdots \partial_n^\varepsilon_n \partial_\eta^\varepsilon_{n+1} m(\xi, \eta)| \leq c_1. \tag{5.6}
$$

Since $m$ is smooth, (5.6) follows if we can show that all functions of type

$$(\xi, \eta) \rightarrow \xi_1^\varepsilon_1 \cdots \xi_n^\varepsilon_n \eta^\varepsilon_{n+1} \partial_1^\varepsilon_1 \cdots \partial_n^\varepsilon_n \partial_\eta^\varepsilon_{n+1} m(\xi, \eta)$$

stay bounded as $|\xi(\xi, \eta)| \rightarrow \infty$. Since $m$ is a rational function with non-vanishing denominator away from $(0, 0)$, this is easy to verify. Consequently, we conclude (5.6). If we analyze the bound on the functions more carefully, we find that $m_1$ can be chosen such that $c_1 = P_1(\lambda, \mathcal{T})$ with $P_1(\lambda, \mathcal{T})$ a polynomial in $\lambda$ and $\mathcal{T}$. By Marcinkiewicz’s multiplier theorem, see for example [5, Corollary 5.2.5] or [8, Chapter IV, §6], $m$ is an $L^q(\mathbb{R}^n \times \mathbb{R})$-multiplier. We now recall Theorem 5.1 and conclude that $M = m \circ \Phi$ is an $L^q(G)$-multiplier. Since $u = \mathcal{F}_G^{-1}[M(\xi, \eta) \cdot \hat{f}]$, we thus obtain

$$
\|u\|_q \leq c_2 P_1(\lambda, \mathcal{T}) \|f\|_q, \tag{5.7}
$$

with $c_2 = c_2(q, n)$. Differentiating $u$ with respect to time and space, we further obtain from the equation $u = \mathcal{F}_G^{-1}[M(\xi, k) \cdot \hat{f}]$ the identities

$$
\partial_\xi u = \mathcal{F}_G^{-1}[(i\frac{2\pi}{\mathcal{T}} k) M(\xi, k) \cdot \hat{f}],
$$

and

$$
\partial^\alpha_\xi u = \mathcal{F}_G^{-1}[(i\xi)^\alpha M(\xi, k) \cdot \hat{f}],
$$

respectively. We can now repeat the argument above with $(i\frac{2\pi}{\mathcal{T}} k) M(\xi, k)$ in the role of the multiplier $M$, and $(i\frac{2\pi}{\mathcal{T}} \eta) m(\xi, \eta)$ in the role of $m$, to conclude

$$
\|\partial_\xi u\|_q \leq c_3 P_2(\lambda, \mathcal{T}) \|f\|_q, \tag{5.8}
$$

with $c_3 = c_3(q, n)$. Similarly, for $|\alpha| \leq 2$ we repeat the argument above with $(i\xi)^\alpha M(\xi, k)$ in the role of $M$, and $(i\xi)^\alpha m(\xi, \eta)$ in the role of $m$, and obtain

$$
\|\partial^\alpha_\xi u\|_q \leq c_4 P_3(\lambda, \mathcal{T}) \|f\|_q. \tag{5.9}
$$

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Collecting (5.7), (5.8) and (5.9), we thus conclude
\[ \|u\|_{2,1,q} \leq c_5 P_4(\lambda, T)\|f\|_q, \]
with $c_5 = c_5(q, n)$. Since $\mathcal{P}_H f = f$, we see directly from (5.2) that $\mathcal{P}_H u = u$ and thus $\text{div } u = 0$. Clearly, also $\mathcal{P}_\perp u = u$. Recalling (4.9) and (4.14), it follows that $u \in W_{\sigma, \perp}^{2,1,q}(G)$. Consequently, we have constructed for arbitrary $f \in \mathcal{P}_\perp C_{0, \sigma}(G)$ a vector field $u \in W_{\sigma, \perp}^{2,1,q}(G)$ such that $A T_F u = f$ and for which (5.10) holds. Since $C_{0, \sigma}(G)$ is a dense subset of $L^2_{\sigma, \perp}(G)$, it follows that $\mathcal{P}_\perp C_{0, \sigma}(G)$ is dense in $L^2_{\sigma, \perp}(G)$. Thus, by a standard density argument we can find for any $f \in L^2_{\sigma, \perp}(G)$ a vector field $u \in W_{\sigma, \perp}^{2,1,q}(G)$ that satisfies $A T_F u = f$ and (5.10). In particular, we have verified that the mapping (5.1) is onto.

Finally, we must verify that the mapping (5.1) is injective. Consider therefore a vector field $u \in W_{\sigma, \perp}^{2,1,q}(G)$ with $A T_F u = 0$. Employing the Fourier transform $\mathcal{F}_G$, we then deduce $(i 2\pi k + |\xi|^2 - \lambda\xi_1) \hat{u} = 0$. Since the polynomial $|\xi|^2 + i(2\pi k - \lambda\xi_1)$ vanishes only at $(\xi, k) = (0, 0)$, we conclude that $\text{supp } \hat{u} \subset \{(0,0)\}$. However, since $\mathcal{P} u = 0$ we have $\kappa_0 \hat{u} = 0$, whence $(\xi, 0) \notin \text{supp } \hat{u}$ for all $\xi \in \mathbb{R}^n$. Consequently, $\text{supp } \hat{u} = \emptyset$. It follows that $\hat{u} = 0$ and thus $u = 0$. We conclude that the mapping (5.1) is injective.

Since the mapping (5.1) is bounded and bijective, it is a homeomorphism by the open mapping theorem. The bound (4.20) follows from (5.10).

\[ \square \]

Remark 5.3. In the proof above, it is crucial that the numerator $1 - \chi(\frac{T}{2\pi} \eta)$ in the fraction that defines $m$ in (5.5) vanishes in a neighborhood of the only zero of the denominator $|\xi|^2 + i \eta - \lambda \xi_1$. Consequently, $m$ is a bounded and even smooth multiplier. The key reason $m$ can be chosen with this structure is that the data $f \in L^2_{\sigma, \perp}(G)$ is $\mathcal{P}_\perp$-invariant, that is, $\mathcal{P}_\perp f = f$. In other words, we would not be able to carry out the argument for a general $f \in L^2_{\sigma, \perp}(G)$. This observation illustrates the necessity of the decomposition induced by the projections $\mathcal{P}$ and $\mathcal{P}_\perp$.

Proof of Theorem 2.1. The statements in Theorem 2.1 were established in Theorem 4.8 for the mapping (4.19). To prove Theorem 2.1, we therefore only need to verify that (4.19) is fully equivalent to (1.1)–(1.3). The verification can be reduced to three simple observations. We first observe that $\Pi$ induces, by lifting, an isometric isomorphism between $W_{\sigma, \perp}^{2,1,q}(G)$ and $W_{\sigma, \perp}^{2,1,q}(\mathbb{R}^n \times (0, T))$. We next observe that the trivial extension of a function $F : \mathbb{R}^n \times (0, T) \to \mathbb{R}$ to a time-periodic function on $\mathbb{R}^n \times \mathbb{R}$ is given by $F \circ (\Pi^{-1} \circ \pi)$. Finally, we observe that $\pi$ induces, by lifting, an embedding of $W_{\sigma, \perp}^{2,1,q}(G)$ into the subspace
\[ \{ u \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \mid \partial_\beta \partial_\alpha u \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \text{ for } |\beta| \leq 1, |\alpha| \leq 2 \} \]
of distributions $\mathcal{D}'(\mathbb{R}^n \times \mathbb{R})$. Consequently, recalling (4.4), we deduce for any element $u \in W_{\sigma, \perp}^{2,1,q}(G)$ that $\partial_\beta \partial_\alpha u = [\partial_\beta \partial_\alpha (u \circ \Pi)] \circ \Pi^{-1}$ for $|\alpha| \leq 2, |\beta| \leq 1$ with the derivatives $\partial_\beta \partial_\alpha (u \circ \Pi)$ being understood in the sense of distributions and thereby, by the embedding above, as functions in $L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$. 

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Now consider a vector field \( f \in L^q_{\sigma,\perp}(\mathbb{R}^n \times (0, T)) \). Clearly, \( f \circ \Pi^{-1} \in L^q_{\sigma,\perp}(G) \). Recall Theorem 4.8 and put \( \tilde{u} := A_{\Pi}^{-1}(f \circ \Pi^{-1}) \in W^{2,1,q}_{\sigma,\perp}(G) \). Then \( u := \tilde{u} \circ \Pi \in W^{2,1,q}_{\sigma,\perp}(\mathbb{R}^n \times (0, T)) \) with

\[
\|u\|_{2,1,q} = \|\tilde{u}\|_{2,1,q} \leq \|A_{\Pi}^{-1}\|f \circ \Pi^{-1}\|_q = \|A_{\Pi}^{-1}\|f\|_q.
\]

Recalling (4.20), we see that \( u \) satisfies (2.2). Moreover, since \( A_{\Pi} \tilde{u} = f \circ \Pi^{-1} \) it follows that \( [A_{\Pi}(\tilde{u} \circ \pi)] \circ \Pi^{-1} = f \circ \Pi^{-1} \), from which we can easily deduce \( A_{\Pi}(u \circ (\Pi^{-1} \circ \pi)) = f \circ (\Pi^{-1} \circ \pi) \). The latter identity shows that the trivial extensions of \( u \) and \( f \) to time-periodic vector fields on \( \mathbb{R}^n \times \mathbb{R} \) satisfy (1.1) with \( p = 0 \).

It remains to verify uniqueness of the solution \( u \). If, however, \( \overline{u} \in W^{2,1,q}_{\sigma,\perp}(\mathbb{R}^n \times (0, T)) \) is another solution, then \( (u - \overline{u}) \circ \Pi^{-1} \in W^{2,1,q}_{\sigma,\perp}(G) \) with \( A_{\Pi}[(u - \overline{u}) \circ \Pi^{-1}] = 0 \). Hence \( u - \overline{u} = 0 \) by the injectivity of \( A_{\Pi} \) established in Theorem 4.8.

We proceed with the proof of Theorem 2.2. In order to characterize the pressure term in (1.1), we need the following lemma:

**Lemma 5.4.** Let \( q \in (1, n) \). Put

\[
X^q_{\text{pres}}(G) := \{ p \in \mathcal{F}'(G) \cap L^1_{\text{loc}}(G) \mid \|p\|_{X^q_{\text{pres}}} < \infty \},
\]

\[
\|p\|_{X^q_{\text{pres}}} := \left( \frac{1}{T} \int_0^T \|p(c, t)^{q-nq} dt \right)^{1/q}.
\]

Then

\[
\text{grad} : X^q_{\text{pres}}(G) \to \mathcal{F}^q(G), \quad \text{grad} p := \nabla p
\]

is a homeomorphism. Moreover, \( \|\text{grad}^{-1}\| \) depends only on \( n \) and \( q \).

**Proof.** Clearly, \( \text{grad} \) is bounded. Consider \( p \in \ker \text{grad} \). Then \( \nabla p = 0 \) and it thus follows by standard arguments that \( p(x, t) = c(t) \). Since \( \|p\|_{X^q_{\text{pres}}} < \infty \), we must have \( p = 0 \). Consequently, \( \text{grad} \) is injective. To show that \( \text{grad} \) is onto, we consider the mapping

\[
\mathcal{I} : \mathcal{F}(G)^n \to \mathcal{F}(G), \quad \mathcal{I}(f) := \mathcal{F}_{\mathbb{R}^n}^{-1} \left[ \frac{\xi_j}{|\xi|^2} \cdot \mathcal{F}_{\mathbb{R}^n}(f_j) \right],
\]

where \( \mathcal{F}_{\mathbb{R}^n} : \mathcal{F}(G) \to \mathcal{F}(G) \) denotes the partial Fourier transform. Observe that

\[
\nabla \mathcal{I}(f) = (\text{Id} - \mathcal{P}_H)f.
\]

Since \( \mathcal{I} \) can expressed as a Riesz potential composed with a sum of Riesz operators, well-known properties of the Riesz potential (see for example [6, Theorem 6.1.3]) and the Riesz operators (see for example [5, Corollary 4.2.8]) yield

\[
\frac{1}{T} \int_0^T \|\mathcal{I}(f)(c, t)^{q-nq} dt \leq c_1 \frac{1}{T} \int_0^T \|f(c, t)^{q} dt = c_1 \|f\|_q^q
\]

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with \( c_1 = c_1(n, q) \). In combination with (5.13), (5.14) implies \( \|\mathcal{I}(f)\|_{X^q_{\text{pres}}} \leq c_2 \|f\|_q \). By a density argument, we can extend \( \mathcal{I} \) uniquely to a bounded map \( \mathcal{I} : L^q(G)^n \to X^q_{\text{pres}}(G) \) that satisfies (5.13) for all \( f \in L^q(G)^n \). We can now show that \( \text{grad} \) is onto. If namely \( f \in \mathcal{G}^q(G) \), then \( \nabla \mathcal{I}(f) = f \). We conclude by the open mapping theorem that \( \text{grad} \) is a homeomorphism. In fact, the inverse is given by \( \mathcal{I} \), whence by (5.14) \( \|\text{grad}^{-1}\| \) depends only on \( n \) and \( q \).

Proof of Theorem 2.2. Let \( f \in L^q(\mathbb{R}^n \times (0, T))^n \). We put \( \tilde{f} := f \circ \Pi^{-1} \in L^q(G)^n \) and employ the Helmholtz projection to decompose \( \tilde{f} = \mathcal{P}_H \tilde{f} + (\text{Id} - \mathcal{P}_H) \tilde{f} \). By Lemma 4.3, \( \mathcal{P}_H \tilde{f} \in L^q_\sigma(G) \). We further decompose \( \mathcal{P}_H \tilde{f} = \mathcal{P}_H f + \mathcal{L} \mathcal{P}_H f \) and recall from Lemma 4.6 that \( \mathcal{L} \mathcal{P}_H f \in L^q_{\perp, \perp}(G) \).

Well-known theory for the linearized, steady-state Navier-Stokes system, see for example [4, Theorem IV.2.1] and [4, Theorem VII.4.1], implies existence of a unique solution \( v \in X^q_{\text{per}}(\mathbb{R}^n) \) to \( -\Delta v - \lambda \partial_t v = \mathcal{P}_H f \) that satisfies

\[
\|v\|_{X^q_{\text{per}}} \leq c_1 \mathcal{P} \mathcal{P}_H f \leq c_1 \|f\|_q, \tag{5.15}
\]

with \( c_1 = c_1(n, q) \). Here, \( X^q_{\text{per}}(\mathbb{R}^n) \) is defined by (2.3), (2.4) or (2.5) depending on the choice of \( n, \lambda, q \).

Recalling Theorem 4.8, we put \( \tilde{w} := A^{-1}_{\text{TP}} \mathcal{P}_H \tilde{f} \in W^{2,1, q}_{\perp, \perp}(G) \). As in the proof of Theorem 2.1, we then observe that \( w := \tilde{w} \circ \Pi \in W^{2,1, q}_{\perp, \perp}(\mathbb{R}^n \times (0, T)) \) and that the trivial extension of \( w \) to a time-periodic vector field on \( \mathbb{R}^n \times \mathbb{R} \) is given by \( w \circ (\Pi^{-1} \circ \pi) \) and satisfies \( A_{\text{TP}}[w \circ (\Pi^{-1} \circ \pi)] = [\mathcal{P}_H \mathcal{P}_H \tilde{f}] \circ \pi \) in the sense of distributions. Moreover, using (4.20) we can estimate

\[
\|w\|_{2,1, q} = \|\tilde{w}\|_{2,1, q} \leq \|A^{-1}_{\text{TP}}\| \|\mathcal{P}_H \tilde{f}\|_q \leq c_2 \|f\|_q \tag{5.16}
\]

with \( c_2 = c_2(\lambda, T, q, n) \).

We now put \( u := v + w \). We then have \( u \in X^q_{\text{per}}(\mathbb{R}^n) \oplus W^{2,1, q}_{\perp, \perp}(\mathbb{R}^n \times (0, T)) \) and \( A_{\text{TP}}[u \circ (\Pi^{-1} \circ \pi)] = [\mathcal{P}_H \tilde{f}] \circ \pi \) in the sense of distributions.

Finally, we recall Lemma 5.4 and put \( \tilde{p} := \text{grad}^{-1}[\text{Id} - \mathcal{P}_H] \tilde{f} \in X^q_{\text{pres}}(G) \). One readily verifies that \( \Pi \) induces, by lifting, an isometric isomorphism between \( X^q_{\text{pres}}(G) \) and \( X^q_{\text{pres}}(\mathbb{R}^n \times (0, T)) \). Thus \( p := \tilde{p} \circ \Pi \in X^q_{\text{pres}}(\mathbb{R}^n \times (0, T)) \). Since \( \nabla \tilde{p} = (\text{Id} - \mathcal{P}_H) \tilde{f} \), it follows by the same observations as those made in the proof of Theorem 2.1 that \( \nabla [p \circ (\Pi^{-1} \circ \pi)] = [(\text{Id} - \mathcal{P}_H) \tilde{f}] \circ \pi \) in the sense of distributions. Moreover, we can estimate

\[
\|p\|_{X^q_{\text{pres}}} = \|\tilde{p}\|_{X^q_{\text{pres}}} \leq \|\text{grad}^{-1}\| \|(\text{Id} - \mathcal{P}_H) \tilde{f}\|_q \leq c_3 \|f\|_q \tag{5.17}
\]

with \( c_3 = c_3(q, n) \).

We conclude that \( A_{\text{TP}}[u \circ (\Pi^{-1} \circ \pi)] + \nabla [p \circ (\Pi^{-1} \circ \pi)] = f \circ (\Pi^{-1} \circ \pi) \) in the sense of distributions. Since both \( v \) and \( w \) are solenoidal vector fields, we see that also \( \text{div} [u \circ (\Pi^{-1} \circ \pi)] = 0 \). We have thus shown that the trivial extension of \( (u, p) \) and \( f \) to time-periodic vector fields on \( \mathbb{R}^n \times \mathbb{R} \) is a solution to (1.3)–(1.3). Furthermore, by (5.15), (5.16) and (5.17) we have established (2.6).
It remains to show uniqueness of the solution. Assume 

$$(\overline{u}, \overline{p}) \in \left( X^q_s(\mathbb{R}^n) \oplus W^{2,1,q}_{\sigma,\text{per,\perp}}(\mathbb{R}^n \times (0, T)) \right) \times X^q_{\text{pres}}(\mathbb{R}^n \times (0, T))$$

is another solution. Then 0 = $\mathcal{P}_\perp \mathcal{P}_H \left[ A_{TP}(\overline{u} - u) \circ \Pi^{-1} \right]$. The injectivity of $A_{TP}$ established in Theorem 4.8 thus implies $\mathcal{P}_\perp (\overline{u} - u) \circ \Pi^{-1} = 0$. Similarly, we see that $0 = \mathcal{P}_H \left[ A_{TP}(\overline{u} - u) \circ \Pi^{-1} \right] = (\Delta - \lambda \partial_t) \left[ \mathcal{P}(\overline{u} - u) \circ \Pi^{-1} \right]$. By well-known theory for the linearized, steady-state Navier-Stokes system, see again [4, Theorem IV.2.1] and [4, Theorem VII.4.1], $(\Delta - \lambda \partial_t) : X^q_s(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ is a homeomorphism, whence $\mathcal{P}(\overline{u} - u) \circ \Pi^{-1} = 0$ follows. We can now conclude that $\overline{u} - u = 0$. As a consequence hereof we then have $\nabla (\overline{p} - p) = 0$, from which, by the injectivity of grad established in Lemma 5.4, we deduce $\overline{p} - p = 0$. Hence $(\overline{u}, \overline{p}) = (u, p)$. 

\[ \square \]

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