Обобщенная предельная теорема для периодической дзета-функции Гурвица

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Аннотация

С времен Бора и Йессена (1910–1935) в теории дзета-функций применяются вероятностные методы. В 1930 г. они доказали первую теорему для дзета-функции Римана \( \zeta(s) \), \( s = \sigma + it \), которая является прототипом современных предельных теорем, характеризующих поведение дзета-функции при помощи слабой сходимости вероятностей мер. Более точно, они получили, что при \( \sigma > 1 \) существует предел

\[
\lim_{T \to \infty} \frac{1}{T} \int_{[0,T]} \{ t : \log \zeta(\sigma + it) \in R \},
\]

где \( R \) — прямоугольник на комплексной плоскости со сторонами, параллельными осям, а \( JA \) обозначает меру Жордана множества \( A \subset \mathbb{R} \). Два года спустя они распространили приведенный результат на полуплоскость \( \sigma > \frac{1}{2} \).

Идеи Бора и Йессена были развиты в работах Винтнера, Борщсениуса, Йессена, Сельберга и других известных математиков. Современные версии теорем Бора-Йессена для широкого класса дзета-функций были получены в работах К. Матсумото.

В основном теория Бора-Йессена применялась для дзета-функций, имеющих эйлерово произведение по простым числам. В настоящей статье доказывается предельная теорема для дзета-функций, не имеющих эйлерова произведения и являющихся обобщением классической дзета-функции Гурвица. Пусть \( \alpha, 0 < \alpha \leq 1 \), фиксированный параметр, а

\[ a = \{ a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \] — периодическая последовательность комплексных чисел. Тогда периодическая дзета-функция Гурвица \( \zeta(s, \alpha; a) \) в полуплоскости \( \sigma > 1 \) определяется рядом Дирихле

\[
\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s},
\]

и мероморфно продолжается на всю комплексную плоскость. Пусть \( B(\mathbb{C}) \) — борелевское \( \sigma \)-поле комплексной плоскости, \( \text{meas} A \) — мера Лебега измеримого множества \( A \subset \mathbb{R} \), а функция \( \varphi(t) \) при \( t \geq T_0 \) имеет монотонную положительную производную \( \varphi'(t) \), при \( t \to \infty \) удовлетворяющую оценкам \( (\varphi'(t))^{-1} = o(t) \) и \( \varphi(2t) \max_{t \leq u \leq 2t} (\varphi'(u))^{-1} \ll t \). Тогда в статье получено, что при \( \sigma > \frac{1}{2} \)

\[
\frac{1}{T} \text{meas} \{ t \in [0,T] : \zeta(\sigma + i\varphi(t), \alpha; a) \in A \}, \quad A \in B(\mathbb{C}),
\]

при \( T \to \infty \) слабо сходится к некоторой в явном виде заданной вероятностной мере на \( (\mathbb{C}, B(\mathbb{C})) \).

Ключевые слова: дзета-функция Гурвица, мера Хаара, периодическая дзета-функция Гурвица, предельная теорема, слабая сходимость.

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A generalized limit theorem for the periodic Hurwitz zeta-function

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Abstract

Probabilistic methods are used in the theory of zeta-functions since Bohr and Jessen time (1910–1935). In 1930, they proved the first theorem for the Riemann zeta-function \( \zeta(s) \), \( s = \sigma + it \), which is a prototype of modern limit theorems characterizing the behavior of \( \zeta(s) \) by weakly convergent probability measures. More precisely, they obtained that, for \( \sigma > \frac{1}{2} \), there exists the limit

\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{J} \{ t \in [0, T] : \log \zeta(\sigma + it) \in R \},
\]

where \( R \) is a rectangle on the complex plane with edges parallel to the axes, and \( \mathbb{J}A \) denotes the Jordan measure of a set \( A \subset \mathbb{R} \). Two years latter, they extended the above result to the half-plane \( \sigma > \frac{1}{2} \).

Ideas of Bohr and Jessen were developed by Wintner, Borchsenius, Jessen, Selberg and other famous mathematicians. Modern versions of the Bohr-Jessen theorems, for a wide class of zeta-functions, were obtained in the works of K. Matsumoto.

The theory of Bohr and Jessen is applicable, in general, for zeta-functions having Euler’s product over primes. In the present paper, a limit theorem for a zeta-function without Euler’s product is proved. This zeta-function is a generalization of the classical Hurwitz zeta-function. Let \( \alpha, 0 < \alpha \leq 1 \), be a fixed parameter, and \( a = \{ a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \) be a periodic sequence of complex numbers. The periodic Hurwitz zeta-function \( \zeta(s, \alpha; a) \) is defined, for \( \sigma > 1 \), by the Dirichlet series

\[
\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},
\]

and is meromorphically continued to the whole complex plane. Let \( B(\mathbb{C}) \) denote the Borel \( \sigma \)-field of the set of complex numbers, \( \text{meas A} \) be the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \), and let the function \( \varphi(t) \) for \( t \geq T_0 \) have the monotone positive derivative \( \varphi'(t) \) such that \( (\varphi'(t))^{-1} = o(t) \) and \( \varphi(2t) \max_{1 \leq u \leq 2t} (\varphi'(u))^{-1} \ll t \). Then it is obtained in the paper that, for \( \sigma > \frac{1}{2} \),

\[
\frac{1}{T} \text{meas} \{ t \in [0, T] : \zeta(\sigma + i\varphi(t), \alpha; a) \in A \}, \quad A \in B(\mathbb{C}),
\]

converges weakly to a certain explicitly given probability measure on \( (\mathbb{C}, B(\mathbb{C})) \) as \( T \to \infty \).

Keywords: Haar measure, Hurwitz zeta-function, limit theorem, periodic Hurwitz zeta-function, weak convergence.

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1. Introduction

The idea of application of probabilistic methods in the theory of zeta-functions is due to Bohr and Jessen. In [2], they proved a theorem for the Riemann zeta-function

\[ \zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad s = \sigma + it, \quad \sigma > 1, \]

which is a prototype of a modern limit theorems on weakly convergent probability measures. Denote by \( J_A \) the Jordan measure of a measurable set \( A \subset \mathbb{R} \), and let \( R \) be a rectangle on the complex plane with edges parallel to the axis. Then they proved that, for \( \sigma > 1 \), there exists the limit

\[ \lim_{T \to \infty} \frac{1}{T} \int [t \in [0, T] : \log \zeta(\sigma + it) \in R]. \]

Two years later, Bohr and Jessen extended [3] the above result to the half-plane \( \sigma > \frac{1}{2} \). In this case, a problem arises because of possible zeros of \( \zeta(s) \). Therefore, they defined the set

\[ G = \left\{ s \in \mathbb{C} : \sigma > \frac{1}{2} \right\} \setminus \bigcup_{s_j=\sigma+it_j} \left\{ s = \sigma + it_j : \frac{1}{2} < \sigma < \sigma_j \right\}, \]

where \( s_j \) runs over all zeros of \( \zeta(s) \) in the region \( \left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \right\} \), and proved that there exists the limit

\[ \lim_{T \to \infty} \frac{1}{T} \int [t \in [0, T] : \sigma + it \in G, \log \zeta(\sigma + it) \in R]. \]

In the sixth decade of the last century, the theory of weak convergence of probability measures was created. Therefore, it became possible to state Bohr-Jessen type theorems in the sense of weakly convergent probability measures, for results, see [6] and [8].

The present note is devoted to limit theorems for the periodic Hurwitz zeta-function. Let \( \alpha, 0 < \alpha \leq 1 \) be a fixed parameter, and let \( a = \{a_m : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \) be a periodic sequence of complex numbers with minimal period \( q \in \mathbb{N} \). The periodic Hurwitz zeta-function \( \zeta(s, \alpha; a) \) was introduced in [7], and is defined, for \( \sigma > 1 \), by the Dirichlet series

\[ \zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}. \]

If \( a_m \equiv 1 \), then \( \zeta(s, \alpha; a) \) becomes the classical Hurwitz zeta-function

\[ \zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}, \quad \sigma > 1, \]

which has a meromorphic continuation to the whole complex plane with the unique simple pole at the point \( s = 1 \) with residue 1. The periodicity of the sequence \( a \) implies, for \( \sigma > 1 \), the equality

\[ \zeta(s, \alpha; a) = \frac{1}{q^s} \sum_{l=0}^{q-1} a_l \zeta \left( s, \frac{l + \alpha}{q} \right). \]
Therefore, the function \( \zeta(s, \alpha; a) \) also can be continued meromorphically to the whole complex plane with the unique simple pole at the point \( s = 1 \) with residue

\[
a = \sum_{l=0}^{q-1} a_l.
\]

If \( a = 0 \), then the periodic Hurwitz zeta-function is entire.

In [4], [9] and [11], limit theorems on weakly convergent probability measures on the complex plane for the function \( \zeta(s, \alpha; a) \) were proved. Denote by \( \mathcal{B}(X) \) the Borel \( \sigma \)-field of the space \( X \). Then, for example, it was obtained in [10] that if the parameter \( \alpha \) is transcendental and \( \sigma > \frac{1}{2} \) is fixed, then, on \( (\mathbb{C}, \mathcal{B}(\mathbb{C})) \), there exists a probability measure \( P_\sigma \) such that

\[
\frac{1}{T} \text{meas}\{ t \in [0, T] : \zeta(\sigma + it, \alpha; a) \in A \}, \quad A \in \mathcal{B}(\mathbb{C}),
\]

converges weakly to \( P_\sigma \) as \( T \to \infty \). Moreover, the measure \( P_\sigma \) is given explicitly.

The aim of this note is a generalization of the above theorem for

\[
P_{T, \sigma, \alpha; a}(A) \overset{\text{def}}{=} \frac{1}{T - T_0} \text{meas}\{ t \in [T_0, T] : \zeta(\sigma + i\varphi(t), \alpha; a) \in A \}, \quad A \in \mathcal{B}(\mathbb{C}),
\]

for certain functions \( \varphi(t) \) and \( T_0 > 0 \). For its statement, we need some notation and definitions.

Let \( \gamma \) be the unit circle on the complex plane, and

\[
\Omega = \prod_{m=0}^{\infty} \gamma_m,
\]

where \( \gamma_m = \gamma \) for all \( m \in \mathbb{N}_0 \). With the product topology and pointwise multiplication, the torus \( \Omega \) is a compact topological Abelian group. Therefore, on \( (\Omega, \mathcal{B}(\Omega)) \), the probability Haar measure \( m_H \) can be defined. This gives the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \). Denote by \( \omega(m) \) the \( m \)-th component, \( m \in \mathbb{N}_0 \), of an element \( \omega \in \Omega \), and, on the probability space \( (\Omega, \mathcal{B}(\Omega), m_H) \) define, for \( \sigma > \frac{1}{2} \), the complex-valued random element \( \zeta(\sigma, \alpha; a) \)

\[
\zeta(\sigma, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m \omega(m)}{(m + \alpha)^\sigma}.
\]

Let \( P_{\zeta, \sigma} \) be the distribution of the random element \( \zeta(\sigma, \alpha; a) \), i.e.,

\[
P_{\zeta, \sigma, \alpha; a}(A) = m_H \{ \omega \in \Omega : \zeta(\sigma, \alpha; a) \in A \}, \quad A \in \mathcal{B}(\mathbb{C}).
\]

Now, define the class of functions. We say that \( \varphi \in L(T_0) \) if \( \varphi \) is a real differentiable function for \( t \geq T_0 > 0 \) such that \( \varphi'(t) \) is monotonic positive, \( \frac{1}{\varphi(t)} = o(t) \) and \( \varphi(2t) \max_{t \leq u \leq 2t} \frac{1}{\varphi(u)} \ll t \) as \( t \to \infty \). For example, the function \( \varphi(t) = t^4 + 2t^3 + t^2 \) is an element of the class \( L(1) \).

The main result of this note is the following theorem.

**Theorem 1.** Suppose that the parameter \( \alpha \) is transcendental, \( \sigma > \frac{1}{2} \) is fixed and \( \varphi \in L(T_0) \). Then \( P_{T, \sigma, \alpha; a} \) converges weakly to the measure \( P_{\zeta, \sigma, \alpha; a} \) as \( T \to \infty \).

2. **Lemmas**

We start with a limit theorem for probability measures on \( (\Omega, \mathcal{B}(\Omega)) \). For \( A \in \mathcal{B}(\Omega) \), let

\[
Q_{T, \alpha}(A) = \frac{1}{T - T_0} \text{meas}\{ t \in [T_0, T] : (m + \alpha)^{-i\varphi(t)} : m \in \mathbb{N}_0 \} \in A \}.
\]
Lemma 1. Suppose that \( \varphi \in L(T_0) \). Then \( Q_{T,\alpha} \) converges weakly to the Haar measure \( m_H \) as \( T \to \infty \).

Proof. We apply the Fourier transform method. Let the sign “′” mean that only a finite number of integers \( k_m \) are distinct from zero. Denote by \( g_T(k) = (k_m : k_m \in \mathbb{Z}, m \in \mathbb{N}_0) \) the Fourier transform of \( Q_{T,\alpha} \). Then the definition of \( Q_{T,\alpha} \) implies that

\[
g_{T,\alpha}(k) = \int_{\Omega} \left( \prod_{m=0}^{\infty} \omega^{k_m}(m) \right) dQ_{T,\alpha} = \frac{1}{T - T_0} \int_{T_0}^{T} \prod_{m=0}^{\infty} (m + \alpha)^{-ik_m \varphi(t)} dt
\]

\[
= \frac{1}{T - T_0} \int_{T_0}^{T} \exp\{-i\varphi(t)\sum_{m=0}^{\infty} k_m \log(m + \alpha)\} dt. \tag{1}
\]

Clearly,

\[
g_{T,\alpha}(0) = 1. \tag{2}
\]

Since \( \alpha \) is transcendental, the set \( \{ \log(m + \alpha) : m \in \mathbb{N}_0 \} \) is linearly independent over the field of rational numbers, thus the finite sum

\[
r \overset{\text{def}}{=} \sum_{m=0}^{\infty} k_m \log(m + \alpha) \neq 0
\]

for \( k \neq 0 \). Obviously,

\[
\int_{T_0}^{T} \exp\{-ir\varphi(t)\} dt = \int_{T_0}^{T} \cos(r\varphi(t)) dt - i \int_{T_0}^{T} \sin(r\varphi(t)) dt. \tag{3}
\]

If the function \( \varphi'(t) \) is decreasing, then \( (\varphi'(t))^{-1} \) is increasing. Thus, by the mean value theorem for integrals,

\[
\int_{T_0}^{T} \cos(r\varphi(t)) dt = \frac{1}{r} \int_{T_0}^{T} \frac{r\varphi'(t) \cos(r\varphi(t))}{\varphi'(t)} dt = \frac{1}{r\varphi'(T)} \int_{\xi}^{T} \varphi'(t) \cos(r\varphi(t)) dt
\]

\[
= \frac{1}{r\varphi'(T)} \int_{\xi}^{T} d\sin(r\varphi(t)) = o(T), \tag{4}
\]

as \( T \to \infty \), where \( T_0 \leq \xi \leq T \). Similarly, we find that

\[
\int_{T_0}^{T} \sin(r\varphi(t)) dt = o(T), \quad T \to \infty. \tag{5}
\]

If the function \( \varphi'(t) \) is increasing, then \( (\varphi'(t))^{-1} \) is decreasing, and we obtain by similar arguments that

\[
\int_{T_0}^{T} \exp\{-ir\varphi(t)\} dt = O\left( \frac{1}{r\varphi'(T_0)} \right). \tag{6}
\]

Now, the estimates (4)–(6), and equalities (3) and (1) show that

\[
\lim_{T \to \infty} g_{T,\alpha}(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}
\]

The right-hand side of the latter equality is the Fourier transform of the Haar measure \( m_H \). This and a continuity theorem for probability measures on compact groups prove the lemma. \( \square \)
Now, we will deal with absolutely convergent Dirichlet series. Let $\theta > \frac{1}{2}$ be a fixed number, and

$$v_n(m, \alpha) = \exp \left\{ - \left( \frac{m + \alpha}{n + \alpha} \right)^\theta \right\}.$$  

Define the functions

$$\zeta_n(s, \alpha; a) = \sum_{m=0}^\infty \frac{a_m v_n(m, \alpha)}{(m + \alpha)^s}$$  

and

$$\zeta_n(s, \alpha; \omega; a) = \sum_{m=0}^\infty \frac{a_m \omega(m) v_n(m, \alpha)}{(m + \alpha)^s}.$$  

We note that the above series are absolutely convergent for $\sigma > \frac{1}{2}$ [5]. Consider the function

$$u_{n, \sigma, \alpha; a} : \Omega \to \mathbb{C}$$  

given by the formula

$$u_{n, \sigma, \alpha; a}(\omega) = \zeta_n(\sigma, \alpha; \omega; a), \quad \sigma > \frac{1}{2}.$$  

Then the function $u_{n, \sigma, \alpha; a}$ is continuous. Moreover,

$$P_{r, n, \sigma, \alpha; a} = Q_{T, a} u_{n, \sigma, \alpha; a}^{-1}.$$  

This observation together with Theorem 5.1 of [1] gives the following assertion.

**Lemma 2.** Suppose that $\varphi \in L(T_0)$. Then, for $\sigma > \frac{1}{2},$

$$P_{r, n, \sigma, \alpha; a}(A) \overset{\text{def}}{=} \frac{1}{T - T_0} \text{meas}\{ t \in [T_0, T] : \zeta_n(\sigma + i\varphi(t), \alpha; a) \in A \}, \quad A \in \mathcal{B}(\mathbb{C}),$$  

converges weakly to measure $P_{n, \sigma, \alpha; a} = m_H u_{n, \sigma, \alpha; a}^{-1}$ as $T \to \infty$.

Now we will approximate $\zeta(\sigma, \alpha; a)$ by $\zeta_n(s, \alpha; a)$. For this, we need a mean square estimate.

**Lemma 3.** Suppose that $\varphi \in L(T_0)$ and $\sigma > \frac{1}{2}$ is fixed. Then, for $\tau \in \mathbb{R},$

$$\int_{T_0}^T |\zeta(\sigma + i\tau + i\varphi(t), \alpha; a)|^2 dt \ll_{\sigma, \alpha, a} T (1 + |\tau|).$$  

**Proof.** Suppose that $T \geq T_0$. Then

$$\int_T^{2T} |\zeta(\sigma + i\tau + i\varphi(t), \alpha; a)|^2 dt = \int_T^{2T} \frac{1}{\varphi'(t)} |\zeta(\sigma + i\tau + i\varphi(t), \alpha; a)|^2 d\varphi(t)$$

$$\ll \max_{T \leq t \leq 2T} \frac{1}{\varphi'(t)} \int_T^{2T} d \left( \int_T^{t+\varphi(t)} |\zeta(\sigma + iu, \alpha; a)|^2 du \right)$$

$$\ll \max_{T \leq t \leq 2T} \frac{1}{\varphi'(t)} \left( \int_T^{t+\varphi(t)} |\zeta(\sigma + iu, \alpha; a)|^2 du \right) \int_T^{2T}. \quad (7)$$  

For $\sigma > \frac{1}{2}$, the estimate

$$\int_{T_0}^T |\zeta(\sigma + iu, \alpha; a)|^2 du \ll_{\sigma, \alpha, a} T$$
is true [5]. Therefore,

\[
\left( \int_T^{T + \varphi(t)} |\zeta(\sigma + iu, \alpha; a)|^2 \, du \right)^{2T} \lesssim_{\sigma, \alpha, a} |t| + \varphi(2T).
\]

This together with hypothesis that \( \varphi(2T) \max_{T \leq t \leq 2T} \frac{1}{\varphi(t)} \ll T \) and (7) gives

\[
\int_T^{2T} |\zeta(\sigma + i\tau + i\varphi(t), \alpha; a)|^2 \, dt \lesssim_{\sigma, \alpha, a} |\tau| + \varphi(2T) \max_{T \leq t \leq 2T} \frac{1}{\varphi'(t)} \lesssim_{\sigma, \alpha, a} T (1 + |\tau|).
\]

Taking \( 2^{-k-1} T \) in place of \( T \) and summing over \( k \in \mathbb{N} \), gives the estimate of the lemma. \( \square \)

**Lemma 4.** Suppose that \( \varphi \in L(T_0) \) and \( \sigma > \frac{1}{2} \). Then

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - T_0} \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; a) - \zeta_n(\sigma + i\varphi(t), \alpha; a)| \, dt = 0.
\]

**Proof.** Define the function

\[
l_n(s, \alpha) = \frac{s}{\theta} \Gamma\left(\frac{s}{\theta}\right) (m + \alpha)^s,
\]

where \( \Gamma(s) \) is the Euler gamma-function, and the number \( \theta \) comes from the definition of \( v_n(m, \alpha) \). Then the function \( \zeta(s, \alpha; a) \) has the integral representation [5]

\[
\zeta_n(s, \alpha; a) = \frac{1}{2\pi i} \int_{\theta - i\infty}^{\theta + i\infty} \zeta(s + z, \alpha; a) \frac{l_n(z, \alpha)}{z} \, dz.
\]

Then, using the residue theorem and properties of the gamma-function, we obtain that

\[
\frac{1}{T - T_0} \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; a) - \zeta_n(\sigma + i\varphi(t), \alpha; a)| \, dt
\]

\[
\lesssim_{\sigma, \alpha; a} \int_{-\infty}^{\infty} |l_n(\sigma_1 + i\tau, \alpha)| \left( \frac{1}{T - T_0} \int_{T_0}^{T} |\zeta(\sigma_2 + i\tau + i\varphi(t), \alpha; a)| \, dt \right) \, d\tau + o(1)
\]

as \( T \to \infty \), where \( \sigma_1 < 0 \) and \( \sigma_2 > \frac{1}{2} \). Hence, in view of Lemma 3,

\[
\frac{1}{T - T_0} \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; a) - \zeta_n(\sigma + i\varphi(t), \alpha; a)| \, dt
\]

\[
\lesssim_{\sigma, \alpha; a} \int_{-\infty}^{\infty} |l_n(\sigma_1 + i\tau, \alpha)| (1 + |\tau|) \, d\tau + o(1)
\]

as \( T \to \infty \). Thus, by the properties of \( l_n(s, \alpha) \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - T_0} \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; a) - \zeta_n(\sigma + i\varphi(t), \alpha; a)| \, dt = 0.
\]

\( \square \)

We recall that \( P_{n, \sigma, \alpha; a} \) is the limit measure in Lemma 2.

**Lemma 5.** The sequence \( \{ P_{n, \sigma, \alpha; a} : n \in \mathbb{N} \} \) is tight, i.e., for every \( \varepsilon > 0 \), there exists a compact set \( K = K(\varepsilon) \subset \mathbb{C} \) such that

\[
P_{n, \sigma, \alpha; a}(K) > 1 - \varepsilon
\]

for all \( n \in \mathbb{N} \).
Proof. Let $\xi$ be a random variable defined on a certain probability space with measure $\mathbb{P}$, and uniformly distributed on $[0,1]$. Define the complex-valued random element $X_{T,n,\sigma,a} = X_{T,n,\sigma,a}(\sigma)$ by

$$X_{T,n,\sigma,a} = \zeta_n(\sigma + i\varphi(\xi T), \alpha; a).$$

Then the assertion of Lemma 2 is equivalent to the relation

$$X_{T,n,\sigma,a} \xrightarrow{\mathcal{D}} T \rightarrow \infty X_{n,\sigma,a}, \quad (8)$$

where $X_{n,\sigma,a}(\sigma)$ is the complex-valued random element having the distribution $P_{n,\sigma,a}$. By Lemma 3 with $\tau = 0$, for $\sigma > \frac{1}{2}$,

$$\int_{T_0}^T |\zeta(\sigma + i\varphi(t), \alpha; a)|^2 dt \ll_{\sigma,a} T.$$

Hence, the Cauchy inequality implies

$$\int_{T_0}^T |\zeta(\sigma + i\varphi(t), \alpha; a)|^2 dt \ll \left( (T - T_0) \int_{T_0}^T |\zeta(\sigma + i\varphi(t), \alpha; a)|^2 dt \right)^{1/2} \ll_{\sigma,a} T.$$

Therefore, using Lemma 4, we obtain that, for $\sigma > \frac{1}{2}$,

$$\sup \limsup_{n \in \mathbb{N}} \frac{1}{T - T_0} \int_{T_0}^T |\zeta_n(\sigma + i\varphi(t), \alpha; a)| dt \leq C_{\sigma,a} < \infty. \quad (9)$$

Let $\varepsilon > 0$ be an arbitrary fixed number, and $M = M_{\sigma,a}(\varepsilon) = C_{\sigma,a}\varepsilon^{-1}$. Then, by (9),

$$\sup \limsup_{n \in \mathbb{N}} \mathbb{P}(|X_{T,n,\sigma,a}| > M) \leq \sup \limsup_{n \in \mathbb{N}} \frac{1}{T - T_0} \text{meas} \{ t \in [T_0, T] : |\zeta_n(\sigma + i\varphi(t), \alpha; a)| > M \}$$

$$\leq \sup \limsup_{n \in \mathbb{N}} \frac{1}{T - T_0 M} \int_{T_0}^T |\zeta_n(\sigma + i\varphi(t), \alpha; a)| dt \leq \varepsilon.$$

This together with (8) shows that

$$\mathbb{P}(|X_{n,\sigma,a}| > M) \leq \varepsilon \quad (10)$$

for all $n \in \mathbb{N}$. The set $K = K(\varepsilon) = \{ s \in \mathbb{C} : |s| \leq M \}$ is compact, and, by (10),

$$\mathbb{P}(X_{n,\sigma,a} \in K) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$, or equivalently,

$$P_{n,\sigma,a}(K) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{P_{n,\sigma,a} : n \in \mathbb{N}\}$ is tight. $\square$

3. Proof of Theorem 1

The existence of the limit measure for $P_{T,\sigma,a}$ as $T \rightarrow \infty$ easily follows from Lemmas 4 and 5, relation (8) and Theorem 4.2 of [1].

Proof. [Proof of Theorem 1] By the Prokhorov theorem [1, Theorem 6.1], and Lemma 5, the sequence $\{P_{n,\sigma,a} : n \in \mathbb{N}\}$ is relatively compact, i.e., every subsequence $\{P_{n_k,\sigma,a} : n_k \in \mathbb{N}\} \subset \{P_{n,\sigma,a}\}$ contains a weakly convergent subsequence. Thus, there exists a subsequence $\{P_{n_r,\sigma,a}\}$ such that $P_{n_r,\sigma,a}$ converges weakly to a certain probability measure $P_{\sigma,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $r \rightarrow \infty$. In other words,

$$X_{n_r,\sigma,a}(\sigma) \xrightarrow{\mathcal{D}} r \rightarrow \infty P_{\sigma,a}. \quad (11)$$
Define one more complex-valued random element

\[ X_{T,\alpha;a} = X_{T,\alpha;a}(\sigma) = \zeta(\sigma + i\varphi(\xi T), \alpha; a). \]

Then Lemma 4 shows that, for every \( \varepsilon > 0 \) and \( \sigma > \frac{1}{2} \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P} \left( |X_{T,\alpha;a}(\sigma) - X_{T,n,\alpha;a}(\sigma)| \geq \varepsilon \right) = \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - T_0} \text{meas} \left\{ t \in [T_0, T] : |\zeta(\sigma + i\varphi(t), \alpha; a) - \zeta_n(\sigma + i\varphi(t), \alpha; a)| \geq \varepsilon \right\} \\
\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T - T_0} \varepsilon \int_{T_0}^{T} |\zeta(\sigma + i\varphi(t), \alpha; a) - \zeta_n(\sigma + i\varphi(t), \alpha; a)| dt = 0.
\]

The later equality, relations (8) and (11), and Theorem 4.2 of [1] prove that

\[
X_{T,\sigma,\alpha;a}(\sigma) \xrightarrow{D} P_{\sigma,\alpha;a}.
\]

in other words, \( P_{T,\sigma,\alpha;a} \) converges weakly to \( P_{\sigma,\alpha;a} \) as \( T \to \infty \). Moreover, relation (12) shows that the measure \( P_{\sigma,\alpha;a} \) does not depend of the subsequence \( P_{n,\sigma,\alpha;a} \). Therefore, we have the relation

\[
X_{n,\alpha;a}(\sigma) \xrightarrow{n \to \infty} P_{\sigma,\alpha;a}.
\]

This relation allows to identify the measure \( P_{\sigma,\alpha;a} \). Namely, in [5], it was proved that, for \( \sigma > \frac{1}{2} \),

\[
\frac{1}{T} \text{meas} \left\{ t \in [0, T] : \zeta(\sigma + i t, \alpha; a) \in A \right\}, \quad A \in \mathcal{B}(\mathbb{C}),
\]

as \( T \to \infty \), also converges weakly to the limit measure \( P_{\sigma,\alpha;a} \) of \( P_{n,\sigma,\alpha;a} \) as \( n \to \infty \), and that \( P_{\sigma,\alpha;a} \) coincides with \( P_{\zeta,\sigma,\alpha;a} \). Therefore, \( P_{T,\sigma,\alpha;a} \) converges weakly to \( P_{\zeta,\sigma,\alpha;a} \) as \( T \to \infty \). The theorem is proved. \( \Box \)

4. Conclusions

In the paper, a generalized limit theorem for the periodic Hurwitz zeta-function

\[
\zeta(s, \alpha; a) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}, \quad \text{Res} = \sigma > 1,
\]

where \( 0 < \alpha \leq 1 \) is a fixed transcendental parameter and \( a = \{a_m : m \in \mathbb{N}_0\} \) is a periodic sequence of complex numbers, is obtained. More precisely, it is proved that, for \( \sigma > \frac{1}{2} \),

\[
\frac{1}{T - T_0} \text{meas} \left\{ t \in [T_0, T] : \zeta(\sigma + i\varphi(t), \alpha; a) \in A \right\}, \quad A \in \mathcal{B}(\mathbb{C}),
\]

converges weakly to the explicitly given probability measure on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) as \( T \to \infty \). Here the function \( \varphi(t) \) for \( t \geq T_0 \) has a monotone positive derivative \( \varphi'(t) \) satisfying the estimates \( (\varphi'(t))^{-1} = o(t) \) and \( \varphi(2t) \max_{t \leq u \leq 2t} \frac{1}{\varphi'(u)} \ll t \) as \( t \to \infty \). The theorem obtained generalized previous author’s results with \( \varphi(t) = t \). Moreover, it can be extended to a collection of periodic Hurwitz zeta-functions. Also, the case of rational \( \alpha \) can be considered.
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