The symmetries of the Fokker - Planck equation in two dimensions

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ABSTRACT

We calculate all point symmetries of the Fokker - Planck equation in two-dimensional Euclidean space. General expression of symmetry group action on arbitrary solution of Fokker - Planck equation is presented.

1. The symmetries of the Fokker - Planck equation in two dimensions

The object of our considerations is a special case of Fokker - Planck equation, which describes evolution of 2D continuum of non-interacting particles imbedded in a dense medium without outer forces. The interaction between particles and medium causes combined diffusion in physical space and velocities space. The only force, which acts on particles, is damping force proportional to velocity.

The 3D variant of this equation was investigated in our work [1]. In this work fundamental solution of 3D equation was obtained by means of Fourier transform.

The 1D variant of this equation was investigated in our work [2]. All point symmetries of the Fokker - Planck equation in one-dimensional Euclidean space were calculated.

In present work we continue this investigation for more complex 2D equation.

The Fokker - Planck equation in two dimensions is

\[
\frac{\partial n}{\partial t} + u \frac{\partial n}{\partial x} + v \frac{\partial n}{\partial y} - au \frac{\partial n}{\partial u} - av \frac{\partial n}{\partial v} - 2an - k \left( \frac{\partial^2 n}{\partial u^2} + \frac{\partial^2 n}{\partial v^2} \right) = 0; \tag{1}
\]

where
\( n = n(t, x, y, u, v) \) - density;
\( t \) - time variable;
\( x, y \) - space coordinates;
\( u, v \) - velocity;
\( a \) - coefficient of damping;
\( k \) - coefficient of diffusion.

The list of symmetries of the Fokker - Planck equation in one dimension follows. The calculations of symmetries are rather awkward. They are carried out to APPENDIX 1.

Instead of classic "\( \xi - \phi \)" notation we use another ("\( \delta \)") notation. This notation was presented in our work [2].

Addition of arbitrary solution

\[
v_1 = A \frac{\partial}{\partial n}; \tag{2}
\]
where $A$ is arbitrary solution of the (1) equation.

Scaling of density

\[ v_2 = n \frac{\partial}{\partial n}; \]  

(3)

The reason of symmetries (2-3) existence is linearity of PDE (1).

Time shift

\[ v_3 = \frac{\partial}{\partial t}; \]  

(4)

Space translations

\[ v_4 = \frac{\partial}{\partial x}; \quad v_5 = \frac{\partial}{\partial y}. \]  

(5)

Space rotation

\[ v_6 = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v} + y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \]  

(6)

Transformations (5) and (6) build two-dimensional Euclidean movements group.

Extended Galilean transformations, which besides time and space coordinates affect the density

\[ v_7 = \frac{\partial}{\partial u} + t \frac{\partial}{\partial x} - \frac{an}{2k} (ax + u) \frac{\partial}{\partial n}; \quad v_8 = \frac{\partial}{\partial v} + t \frac{\partial}{\partial y} - \frac{an}{2k} (ax + v) \frac{\partial}{\partial n}. \]  

(7)

Negative exponent transformations - they affect time and space coordinates, contain time-dependent common multiplier (negative exponent). They do not affect density.

\[ v_9 = e^{-at} \left( -a \frac{\partial}{\partial u} + \frac{\partial}{\partial x} \right); \quad v_{10} = e^{-at} \left( -a \frac{\partial}{\partial v} + \frac{\partial}{\partial y} \right). \]  

(8)

Positive exponent transformations - they affect time, space and density, contain time-dependent common multiplier (positive exponent).

\[ v_{11} = e^{at} \left( a \frac{\partial}{\partial u} + \frac{\partial}{\partial x} - \frac{a^2}{2} n u \frac{\partial}{\partial n} \right); \quad v_{12} = e^{at} \left( a \frac{\partial}{\partial v} + \frac{\partial}{\partial y} - \frac{a^2}{2} n v \frac{\partial}{\partial n} \right). \]  

(9)

One-parameter groups, generated by vector fields $v_1 - v_{12}$, are enumerated in the following list. The list contains images of the point $(n, t, x, y, u, v)$ by transformation $exp(\varepsilon v_i)$

$G_1$: $(n + \varepsilon A, t, x, y, u, v)$;

$G_2$: $(e^\varepsilon n, t, x, y, u, v)$;

$G_3$: $(n, t + \varepsilon, x, y, u, v)$;

$G_4$: $(n, t, x + \varepsilon, y, u, v)$;

$G_5$: $(n, t, x, y + \varepsilon, u, v)$;

$G_6$: $(n, t, \cos(\varepsilon)x + \sin(\varepsilon)y, -\sin(\varepsilon)x + \cos(\varepsilon)y, \cos(\varepsilon)u + \sin(\varepsilon)v, -\sin(\varepsilon)u + \cos(\varepsilon)v)$;

(10)

$G_7$: $\left( \exp \left[ -\frac{a}{2k} (\varepsilon (ax + u) + \frac{1}{2} \varepsilon^2 (at + 1) \right] n, t, x + \varepsilon t, y, u + \varepsilon, v). \right.$
For relatively nontrivial integration of \( G_7, G_8, G_{11}, G_{12} \) we refer to [2].

The fact, that \( G_i \) are symmetries of PDE (1) means, that if \( f(t, x, y, u, v) \) is arbitrary solution of (1), the functions

\[
\begin{align*}
\text{u}^{(1)} & : f(t, x, y, u, v) + \varepsilon A(t, x, y, u, v); \\
\text{u}^{(2)} & : \varepsilon f(t, x, y, u, v); \\
\text{u}^{(3)} & : f(t - \varepsilon, x, y, u, v); \\
\text{u}^{(4)} & : f(t, x - \varepsilon, y, u, v); \\
\text{u}^{(5)} & : f(t, x, y - \varepsilon, u, v); \\
\text{u}^{(6)} & : f(t, \cos(\varepsilon)x - \sin(\varepsilon)y, \sin(\varepsilon)x + \cos(\varepsilon)y, \cos(\varepsilon)u - \sin(\varepsilon)v, \sin(\varepsilon)u + \cos(\varepsilon)v); \\
\text{u}^{(7)} & : \exp \left[ -\frac{a}{2k} \left( \varepsilon(ax + u) - \frac{1}{2} \varepsilon^2(at + 1) \right) \right] f(t, x - \varepsilon t, y, u - \varepsilon, v); \\
\text{u}^{(8)} & : \exp \left[ -\frac{a}{2k} \left( \varepsilon(ay + v) - \frac{1}{2} \varepsilon^2(at + 1) \right) \right] f(t, x, y - \varepsilon t, u, v - \varepsilon); \\
\text{u}^{(9)} & : f(t, x - \varepsilon e^{-at}, y, u + \varepsilon a e^{-at}, v); \\
\text{u}^{(10)} & : f(t, x, y - \varepsilon e^{-at}, u, v + \varepsilon \varepsilon a e^{-at}); \\
\text{u}^{(11)} & : \exp \left[ \frac{a^2}{k} e^{at} \left( \varepsilon u - \frac{1}{2} \varepsilon^2 a e^{at} \right) \right] f(t, x - \varepsilon e^{at}, y, u - \varepsilon a e^{at}, v); \\
\text{u}^{(12)} & : \exp \left[ -\frac{a^2}{k} e^{at} \left( \varepsilon v - \frac{1}{2} \varepsilon^2 a e^{at} \right) \right] f(t, x, y - \varepsilon e^{at}, u, v - \varepsilon a e^{at}),
\end{align*}
\]

where \( \varepsilon - \) arbitrary real number, also are solutions of (1). Here \( A \) is another arbitrary solution of (1).

We systematically replaced "old coordinates" by their expressions through "new coordinates". Note, that due to these replacements terms with \( \varepsilon^2 \) in \( u^{(7)}, u^{(8)}, u^{(11)} \) and \( u^{(12)} \) change their signs.
We have trivial solution \( n = e^{2at} \) at our disposal. If we act on this solution by transformations (10), we obtain 4 new solutions:

\[
\begin{align*}
n &= \exp \left[ 2at - \frac{a}{2k} \left( e(ax + u) - \frac{1}{2} e^2 (at + 1) \right) \right]; \\
n &= \exp \left[ 2at - \frac{a}{2k} \left( e(ay + v) - \frac{1}{2} e^2 (at + 1) \right) \right]. \\
n &= \exp \left[ 2at - \frac{a^2}{k} e^{at} \left( eu - \frac{1}{2} e^2 a e^{at} \right) \right]; \\
n &= \exp \left[ 2at - \frac{a^2}{k} e^{at} \left( ev - \frac{1}{2} e^2 a e^{at} \right) \right].
\end{align*}
\]

General expression is

\[
U = e^{\xi_3} \exp \left[ -\frac{a}{2k} \left( \xi_7(a \bar{x} + \bar{u}) - \frac{1}{2} \xi_5^2 (at + 1) \right) \right] \exp \left[ -\frac{a}{2k} \left( \xi_8(a \bar{y} + \bar{v}) - \frac{1}{2} \xi_6^2 (at + 1) \right) \right] \times
\]

\[
\times \exp \left[ -\frac{a^2}{k} e^{at} \left( \xi_{11} (\bar{u} - \xi_7) - \frac{1}{2} \xi_{11}^2 a e^{at} \right) \right] \exp \left[ -\frac{a^2}{k} e^{at} \left( \xi_{12} (\bar{v} - \xi_8) - \frac{1}{2} \xi_{12}^2 a e^{at} \right) \right] \times
\]

\[
\times f(t - \xi_3, \bar{x} - \xi_4 - \xi_7t - \xi_9 e^{-at} - \xi_{11} e^{at}, \bar{y} - \xi_5 - \xi_8t - \xi_{10} e^{-at} - \xi_{12} e^{at}, \bar{u} - \xi_7 + \xi_9 a e^{-at} - \xi_{11} a e^{at}, \bar{v} - \xi_8 + \xi_{10} a e^{-at} - \xi_{12} a e^{at}) +
\]

\[
+ \xi_1 A(t, x, y, u, v);
\]

where

\[
\begin{align*}
\bar{x} &= \cos(\xi_6)x - \sin(\xi_6)y; \\
\bar{y} &= \sin(\xi_6)x + \cos(\xi_6)y; \\
\bar{u} &= \cos(\xi_6)u - \sin(\xi_6)v; \\
\bar{v} &= \sin(\xi_6)u + \cos(\xi_6)v.
\end{align*}
\]
DISCUSSION

Looking at the list of all point symmetries of the Fokker-Planck equation in two-dimensional Euclidean space, we see that there is no simple way to get, for example, fundamental solution of PDE, using these symmetries. We have not at our disposal such an instrument, as scaling of independent variables $t, x, y, u, v$. The result (12-15) of action of symmetry group on trivial solution is not very interesting from physical point of view.

Indirect way of use of Galilean transformations (7) was demonstrated in [1]. The transformation was used for generalisation of solution, which was obtained in the form of exponent of quadratic form of space coordinates and velocities with time dependent coefficients.

There is need of further investigations of Fokker-Planck equation and its set of symmetries, which may lead to another physically interesting results. We can follow the scheme of [7]: to consider invariant solutions for some one-parameter group, thus reduce the independent variables number. To find for obtained in such a way equation all point symmetries - and so long.

In the work [7] this scheme was represented for equations of elasticity and plasticity.

ACKNOWLEDGMENTS

We wish to thank Jos A. M. Vermaseren from NIKHEF (the Dutch Institute for Nuclear and High-Energy Physics), for he made his symbolic computations program FORM release 3.1 available for download for non-commercial purposes (see [8]). This wonderful program makes difficult task of symmetries search more accessible.

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APPENDIX 1

The infinitesimal invariance criteria for PDE (1) is

\[
\begin{align*}
\delta \frac{\partial n}{\partial t} + \delta u \frac{\partial n}{\partial x} + u \delta \left( \frac{\partial n}{\partial x} \right) + \delta v \frac{\partial n}{\partial y} + v \delta \left( \frac{\partial n}{\partial y} \right) - \\
-a \delta u \frac{\partial n}{\partial u} - au \delta \left( \frac{\partial n}{\partial u} \right) - a \delta v \frac{\partial n}{\partial v} - av \delta \left( \frac{\partial n}{\partial v} \right) - 2a \delta n - k(\delta \frac{\partial^2 n}{\partial u^2} + \delta \frac{\partial^2 n}{\partial v^2}) = 0.
\end{align*}
\]

(A1-1)

According to [2] (APPENDIX 1, eq. (A1-8) and (A1-18)), we have for variations of derivatives following expression:

\[
\begin{align*}
\delta \frac{\partial n}{\partial x} &= \frac{\partial}{\partial x} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial x} - \frac{\partial n}{\partial x} \left( \frac{\partial}{\partial x} (\delta x) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial x} \right) \right) - \\
- \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial x} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial x} \right) \right) - \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial x} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial x} \right) \right) - \\
- \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial x} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial x} \right) \right) - \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial x} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial x} \right) \right) - \\
\delta \frac{\partial n}{\partial y} &= \frac{\partial}{\partial y} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial y} - \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial y} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \\
- \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial y} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial y} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \\
- \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial y} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \\
\delta \frac{\partial n}{\partial u} &= \frac{\partial}{\partial u} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial u} - \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial u} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \\
- \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial u} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial u} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \\
- \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial u} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \\
\delta \frac{\partial n}{\partial v} &= \frac{\partial}{\partial v} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial v} - \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial v} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \\
- \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial v} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial v} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \\
- \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial v} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \\
\delta \frac{\partial n}{\partial t} &= \frac{\partial}{\partial t} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial t} - \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial t} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial t} \right) \right) - \\
- \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial t} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial t} \right) \right) - \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial t} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial t} \right) \right) - \\
- \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial t} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial t} \right) \right) - \\
\delta \frac{\partial n}{\partial y} &= \frac{\partial}{\partial y} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial y} - \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial y} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \\
- \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial y} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial y} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \\
- \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial y} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial y} \right) \right) - \\
\delta \frac{\partial n}{\partial u} &= \frac{\partial}{\partial u} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial u} - \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial u} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \\
- \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial u} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial u} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \\
- \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial u} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial u} \right) \right) - \\
\delta \frac{\partial n}{\partial v} &= \frac{\partial}{\partial v} (\delta n) + \frac{\partial}{\partial n} (\delta n) \frac{\partial n}{\partial v} - \frac{\partial n}{\partial v} \left( \frac{\partial}{\partial v} (\delta v) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \\
- \frac{\partial n}{\partial y} \left( \frac{\partial}{\partial v} (\delta y) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \frac{\partial n}{\partial u} \left( \frac{\partial}{\partial v} (\delta u) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right) - \\
- \frac{\partial n}{\partial t} \left( \frac{\partial}{\partial v} (\delta t) + \frac{\partial}{\partial n} \left( \frac{\partial n}{\partial v} \right) \right). \tag{A1-6}
\end{align*}
\]
$$-\frac{\partial n}{\partial t} \left( \frac{\partial^2}{\partial t^2} (\delta t) + \frac{\partial^2}{\partial n^2} (\delta t) \right) - \frac{\partial n}{\partial x} \frac{\partial n}{\partial v} (\delta x) - \frac{\partial n}{\partial x} \frac{\partial n}{\partial v} (\delta x) + \frac{\partial^2}{\partial n^2} (\delta x) \frac{\partial n}{\partial v} - \frac{\partial n}{\partial y} \frac{\partial n}{\partial v} (\delta y) + \frac{\partial^2}{\partial n^2} (\delta y) \frac{\partial n}{\partial v}$$

$$-\frac{\partial n}{\partial v} \frac{\partial n}{\partial v} (\delta v) + \frac{\partial^2}{\partial n^2} (\delta v) \frac{\partial n}{\partial v} - \frac{\partial n}{\partial u} \frac{\partial n}{\partial v} (\delta u) + \frac{\partial^2}{\partial n^2} (\delta u) \frac{\partial n}{\partial v}$$

$$-\frac{\partial^2 n}{\partial v dx} (\delta x) - \frac{\partial n}{\partial n} \frac{\partial n}{\partial v} (\delta u) - \frac{\partial^2 n}{\partial v^2} (\delta v) + \frac{\partial n}{\partial n} \frac{\partial n}{\partial v} (\delta v) - \frac{\partial^2 n}{\partial t^2} (\delta t) + \frac{\partial n}{\partial n} \frac{\partial n}{\partial v} (\delta t) \frac{\partial n}{\partial v};$$

We eliminate $\frac{\partial n}{\partial t}$ in (A1-1) using original equation

$$\frac{\partial n}{\partial t} \left( \frac{\partial^2}{\partial t^2} (\delta t) + \frac{\partial^2}{\partial n^2} (\delta t) \right) = 0;$$

Collecting similar terms, we obtain following equations:

$$-2ku \frac{\partial^2}{\partial n^2} (\delta t) + 2k \frac{\partial^2}{\partial n^2} (\delta x) = 0;$$

$$-ku \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta x) = 0;$$

$$-2ku \frac{\partial^2}{\partial n^2} (\delta t) + 2k \frac{\partial^2}{\partial n^2} (\delta x) = 0;$$

$$-ku \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta x) = 0;$$

$$-auv \frac{\partial}{\partial v} (\delta t) + 2auv \frac{\partial}{\partial n} (\delta t) + au \frac{\partial}{\partial u} (\delta x) - au \frac{\partial}{\partial u} (\delta t) + av \frac{\partial}{\partial v} (\delta x) -$$

$$-2au \frac{\partial}{\partial n} (\delta x) - ku \frac{\partial^2}{\partial u^2} (\delta t) - ku \frac{\partial^2}{\partial v^2} (\delta t) + k \frac{\partial^2}{\partial u^2} (\delta x) + k \frac{\partial^2}{\partial v^2} (\delta x) +$$

$$+uv \frac{\partial}{\partial y} (\delta t) - u \frac{\partial}{\partial x} (\delta t) + u \frac{\partial}{\partial t} (\delta t) + u^2 \frac{\partial}{\partial x} (\delta x) - v \frac{\partial}{\partial y} (\delta t) + \delta u - \frac{\partial}{\partial \tilde{t}} (\delta x) = 0;$$
\[
\frac{\partial}{\partial y} \frac{\partial}{\partial u} n
\]

\[-2kv \frac{\partial^2}{\partial n \partial u} (\delta t) + 2k \frac{\partial^2}{\partial n \partial u} (\delta y) = 0; \quad (A1-15)\]

\[
\frac{\partial}{\partial y} \frac{\partial}{\partial u^2} n
\]

\[-kv \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta y) = 0; \quad (A1-16)\]

\[
\frac{\partial}{\partial y} \frac{\partial}{\partial v} n
\]

\[-2kv \frac{\partial^2}{\partial n \partial v} (\delta t) + 2k \frac{\partial^2}{\partial n \partial v} (\delta y) = 0; \quad (A1-17)\]

\[
\frac{\partial}{\partial y^2} n
\]

\[-kv \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta y) = 0; \quad (A1-18)\]

\[
\frac{\partial}{\partial y} u
\]

\[-auv \frac{\partial}{\partial u} (\delta t) + au \frac{\partial}{\partial u} (\delta y) + 2avn \frac{\partial}{\partial n} (\delta t) + av \frac{\partial}{\partial v} (\delta y) - av^2 \frac{\partial}{\partial v} (\delta t) - \]

\[-2an \frac{\partial}{\partial n} (\delta y) - kv \frac{\partial^2}{\partial u^2} (\delta t) - kv \frac{\partial^2}{\partial u^2} (\delta y) - k \frac{\partial^2}{\partial u^2} (\delta y) + \]

\[+u^2 \frac{\partial}{\partial x} (\delta t) - u \frac{\partial}{\partial x} (\delta y) - uv \frac{\partial}{\partial y} (\delta y) + \frac{\partial}{\partial y} (\delta t) + v^2 \frac{\partial}{\partial y} (\delta t) + \delta v - \frac{\partial}{\partial t} (\delta y) = 0; \quad (A1-19)\]

\[
\frac{\partial}{\partial u} \frac{\partial}{\partial v} n
\]

\[2aku \frac{\partial^2}{\partial n \partial v} (\delta t) + 2akv \frac{\partial^2}{\partial n \partial u} (\delta t) + 2k \frac{\partial^2}{\partial n \partial u} (\delta v) + 2k \frac{\partial^2}{\partial n \partial v} (\delta u) = 0; \quad (A1-20)\]

\[
\frac{\partial^2}{\partial u \partial v} n
\]

\[aku \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta u) = 0; \quad (A1-21)\]

\[
\frac{\partial^2}{\partial u^2} n
\]

\[2k \frac{\partial}{\partial n} (\delta u) + 2k^2 \frac{\partial^2}{\partial n ^2} (\delta t) = 0; \quad (A1-22)\]

\[
\frac{\partial^2}{\partial v^2} n
\]

\[2k^2 \frac{\partial^2}{\partial n^2} (\delta t) = 0; \quad (A1-23)\]

\[
\frac{\partial^2}{\partial u \partial v} n
\]
\[ 2k \frac{\partial}{\partial n} (\delta v) = 0; \quad (A1-24) \]

\[ \frac{\partial n}{\partial u} \frac{\partial^2 n}{\partial u \partial x} \]

\[ 2k \frac{\partial}{\partial n} (\delta x) = 0; \quad (A1-25) \]

\[ \frac{\partial n}{\partial u} \frac{\partial^2 n}{\partial u \partial y} \]

\[ 2k \frac{\partial}{\partial n} (\delta y) = 0; \quad (A1-26) \]

\[ \frac{\partial n}{\partial u} \frac{\partial^2 n}{\partial u \partial t} \]

\[ 2k \frac{\partial}{\partial n} (\delta t) = 0; \quad (A1-27) \]

\[ \frac{\partial n}{\partial u} \]

\[ aku \frac{\partial^2}{\partial u^2} (\delta t) + aku \frac{\partial^2}{\partial v^2} (\delta t) + 4akn \frac{\partial^2}{\partial n \partial u} (\delta t) - auv \frac{\partial}{\partial v} (\delta t) + \]

\[ + au \frac{\partial}{\partial u} (\delta u) - au \frac{\partial}{\partial t} (\delta t) - au^2 \frac{\partial}{\partial x} (\delta t) + av \frac{\partial}{\partial v} (\delta u) - 2an \frac{\partial}{\partial n} (\delta u) - a\delta u + a^2 uv \frac{\partial}{\partial v} (\delta t) - \]

\[ - 2a^2 un \frac{\partial}{\partial n} (\delta t) + a^2 u^2 \frac{\partial}{\partial u} (\delta t) - 2k \frac{\partial^2}{\partial n^2} (\delta n) + k \frac{\partial^2}{\partial u^2} (\delta u) + 2k \frac{\partial^2}{\partial v^2} (\delta u) - u \frac{\partial}{\partial x} (\delta u) - v \frac{\partial}{\partial y} (\delta u) - \frac{\partial}{\partial t} (\delta u) = 0; \quad (A1-28) \]

\[ \frac{\partial n}{\partial u} \frac{\partial n}{\partial u^2} \frac{\partial n}{\partial v^2} \]

\[ akv \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta v) = 0; \quad (A1-29) \]

\[ \frac{\partial n}{\partial u^2} \frac{\partial n}{\partial u^2} \frac{\partial n}{\partial v^2} \]

\[ k^2 \frac{\partial^2}{\partial n^2} (\delta t) = 0; \quad (A1-30) \]

\[ \frac{\partial n}{\partial u^2} \frac{\partial n}{\partial u^2} \frac{\partial n}{\partial v^2} \]

\[ k^2 \frac{\partial^2}{\partial n^2} (\delta t) = 0; \quad (A1-31) \]

\[ \frac{\partial n}{\partial u^3} \]

\[ 2aku \frac{\partial^2}{\partial n \partial u} (\delta t) + 2akn \frac{\partial^2}{\partial n^2} (\delta t) - k \frac{\partial^2}{\partial n^2} (\delta n) + 2k \frac{\partial^2}{\partial n \partial u} (\delta u) = 0; \quad (A1-32) \]

\[ \frac{\partial n}{\partial u^3} \]

\[ aku \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta u) = 0; \quad (A1-33) \]
\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial u^2} \]
\[ k^2 \frac{\partial^2}{\partial n \partial v} (\delta t) + k^2 \frac{\partial^2}{\partial n \partial v} (\delta t) = 0; \quad (A1-34) \]
\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial u^2} \]
\[ 2k \frac{\partial}{\partial n} (\delta v) + 2k^2 \frac{\partial^2}{\partial n \partial v} (\delta t) = 0; \quad (A1-35) \]
\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial u^2} \]
\[ 2k \frac{\partial}{\partial n} (\delta u) = 0; \quad (A1-36) \]
\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial v \partial x} \]
\[ 2k \frac{\partial}{\partial n} (\delta x) = 0; \quad (A1-37) \]
\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial v \partial y} \]
\[ 2k \frac{\partial}{\partial n} (\delta y) = 0; \quad (A1-38) \]
\[ \frac{\partial n}{\partial v} \frac{\partial^2 n}{\partial n^2} \]
\[ akv \frac{\partial^2}{\partial u^2} (\delta t) + akv \frac{\partial^2}{\partial v^2} (\delta t) + 4akn \frac{\partial^2}{\partial n \partial v} (\delta t) - \\
-av^2 \frac{\partial}{\partial y} (\delta t) - 2an \frac{\partial}{\partial n} (\delta v) - a\delta v + av \frac{\partial}{\partial u} (\delta t) - 2a^2 vn \frac{\partial}{\partial n} (\delta t) + \\
+av^2 \frac{\partial}{\partial y} (\delta t) + k \frac{\partial^2}{\partial u^2} (\delta v) - 2k \frac{\partial^2}{\partial n \partial v} (\delta n) + k \frac{\partial^2}{\partial v^2} (\delta v) - u \frac{\partial}{\partial x} (\delta v) - v \frac{\partial}{\partial y} (\delta v) - \frac{\partial}{\partial t} (\delta v) = 0; \quad (A1-40) \]
\[ \frac{\partial n}{\partial v^2} \frac{\partial^2 n}{\partial u^2} \]
\[ k^2 \frac{\partial^2}{\partial n^2} (\delta t) = 0; \quad (A1-41) \]
\[ \frac{\partial n}{\partial v^2} \frac{\partial^2 n}{\partial v^2} \]
\[ k^2 \frac{\partial^2}{\partial n^2} (\delta t) = 0; \quad (A1-42) \]
\[
\frac{\partial n}{\partial v^2} \quad 2akv \frac{\partial^2}{\partial n \partial v} (\delta t) + 2akn \frac{\partial^2}{\partial n^2} (\delta t) - k \frac{\partial^2}{\partial n^2} (\delta n) + 2k \frac{\partial^2}{\partial n \partial v} (\delta v) = 0; \tag{A1-43}
\]

\[
\frac{\partial n}{\partial v^3} \quad akv \frac{\partial^2}{\partial n^2} (\delta t) + k \frac{\partial^2}{\partial n^2} (\delta v) = 0; \tag{A1-44}
\]

\[
\frac{\partial^2 n}{\partial u^2} \quad aku \frac{\partial}{\partial u} (\delta t) + akv \frac{\partial}{\partial v} (\delta t) - 2akn \frac{\partial}{\partial n} (\delta t) - ku \frac{\partial}{\partial x} (\delta t) - kv \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial u} (\delta u) - k \frac{\partial}{\partial t} (\delta t) + k^2 \frac{\partial^2}{\partial u^2} (\delta t) + k^2 \frac{\partial^2}{\partial v^2} (\delta t) = 0; \tag{A1-45}
\]

\[
\frac{\partial^2 n}{\partial v^2} \quad aku \frac{\partial}{\partial u} (\delta t) + akv \frac{\partial}{\partial v} (\delta t) - 2akn \frac{\partial}{\partial n} (\delta t) - ku \frac{\partial}{\partial x} (\delta t) - kv \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial v} (\delta v) - k \frac{\partial}{\partial t} (\delta t) + k^2 \frac{\partial^2}{\partial u^2} (\delta t) + k^2 \frac{\partial^2}{\partial v^2} (\delta t) = 0; \tag{A1-46}
\]

\[
\frac{\partial^2 n}{\partial u \partial v} \quad 2k \frac{\partial}{\partial u} (\delta u) + 2k \frac{\partial}{\partial v} (\delta v) = 0; \tag{A1-47}
\]

\[
\frac{\partial^2 n}{\partial u \partial x} \quad 2k \frac{\partial}{\partial u} (\delta x) = 0; \tag{A1-48}
\]

\[
\frac{\partial^2 n}{\partial u \partial y} \quad 2k \frac{\partial}{\partial u} (\delta y) = 0; \tag{A1-49}
\]

\[
\frac{\partial^2 n}{\partial v \partial x} \quad 2k \frac{\partial}{\partial v} (\delta x) = 0; \tag{A1-50}
\]

\[
\frac{\partial^2 n}{\partial v \partial y} \quad 2k \frac{\partial}{\partial v} (\delta y) = 0; \tag{A1-51}
\]

\[
\frac{\partial^2 n}{\partial t \partial u}
\]
From (A1-37 - A1-39), (A1-48 - A1-51) we see, that \(\delta x = \delta x(x, y, t);\) \(\delta y = \delta y(x, y, t);\) \(\delta t = \delta t(x, y, t).\) Using these expressions, we simplify the rest of the equations (A1-10 - A1-54).

\[
2k \frac{\partial}{\partial u} (\delta t) = 0; \quad (A1-52)
\]

\[
\frac{\partial^2 n}{\partial t \partial v} = 2k \frac{\partial}{\partial v} (\delta t) = 0; \quad (A1-53)
\]

\[
2a kn \frac{\partial^2 (\delta t)}{\partial u^2} + 2a kn \frac{\partial^2 (\delta t)}{\partial t^2} - 2a un \frac{\partial}{\partial x} (\delta t) - au \frac{\partial}{\partial u} (\delta n) - \quad (A1-54)
\]

\[
-2a vn \frac{\partial}{\partial y} (\delta t) - av \frac{\partial}{\partial v} (\delta n) + 2an \frac{\partial}{\partial n} (\delta n) - 2an \frac{\partial}{\partial t} (\delta t) - \quad (A1-54)
\]

\[
-2a \delta n + 2a^2 un \frac{\partial}{\partial u} (\delta t) + 2a^2 vn \frac{\partial}{\partial v} (\delta t) - 4a^2 n^2 \frac{\partial}{\partial n} (\delta t) - k \frac{\partial^2}{\partial u^2} (\delta n) - \quad (A1-54)
\]

\[
-k \frac{\partial^2}{\partial v^2} (\delta n) + u \frac{\partial}{\partial x} (\delta n) + v \frac{\partial}{\partial y} (\delta n) + \frac{\partial}{\partial t} (\delta n) = 0.
\]

\[
\frac{uv}{\partial y} (\delta t) - u \frac{\partial}{\partial x} (\delta x) + u \frac{\partial}{\partial t} (\delta t) + u^2 \frac{\partial}{\partial x} (\delta t) - v \frac{\partial}{\partial y} (\delta x) + \frac{\partial}{\partial t} (\delta x) = 0; \quad (A1-55)
\]

\[
\frac{uv}{\partial x} (\delta t) - u \frac{\partial}{\partial x} (\delta y) - v \frac{\partial}{\partial y} (\delta t) + v^2 \frac{\partial}{\partial y} (\delta t) + \frac{\partial}{\partial t} (\delta y) = 0; \quad (A1-56)
\]

\[
-av \frac{\partial}{\partial y} (\delta t) + au \frac{\partial}{\partial u} (\delta u) - au \frac{\partial}{\partial t} (\delta t) - au^2 \frac{\partial}{\partial x} (\delta t) + \quad (A1-57)
\]

\[
+av \frac{\partial}{\partial v} (\delta u) - a \delta u - 2k \frac{\partial^2}{\partial u n} (\delta n) + k \frac{\partial^2}{\partial u^2} (\delta n) + \quad (A1-58)
\]

\[
+k \frac{\partial^2}{\partial v^2} (\delta u) - u \frac{\partial}{\partial x} (\delta u) - v \frac{\partial}{\partial y} (\delta u) - \frac{\partial}{\partial t} (\delta u) = 0;
\]

\[
-k \frac{\partial^2}{\partial n^2} (\delta n) = 0; \quad (A1-58)
\]

\[
-av \frac{\partial}{\partial x} (\delta t) + au \frac{\partial}{\partial u} (\delta v) + av \frac{\partial}{\partial v} (\delta v) - av \frac{\partial}{\partial t} (\delta t) - \quad (A1-59)
\]

\[
-av^2 \frac{\partial}{\partial y} (\delta t) - a \delta v + k \frac{\partial^2}{\partial u^2} (\delta v) - 2k \frac{\partial^2}{\partial u n} (\delta n) + \quad (A1-59)
\]

\[
+k \frac{\partial^2}{\partial v^2} (\delta v) - u \frac{\partial}{\partial x} (\delta v) - v \frac{\partial}{\partial y} (\delta v) - \frac{\partial}{\partial t} (\delta v) = 0;
\]

\[
-k \frac{\partial^2}{\partial n^2} (\delta n) = 0; \quad (A1-60)
\]
\[-k u \frac{\partial}{\partial x} (\delta t) - k v \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial u} (\delta u) - k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-61)\]

\[-k u \frac{\partial}{\partial x} (\delta t) - k v \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial v} (\delta v) - k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-62)\]

\[2k \frac{\partial}{\partial v} (\delta u) + 2k \frac{\partial}{\partial u} (\delta v) = 0; \quad (A1-63)\]

\[-2a u n \frac{\partial}{\partial x} (\delta t) - a u \frac{\partial}{\partial u} (\delta n) - 2a v n \frac{\partial}{\partial y} (\delta t) - a v \frac{\partial}{\partial v} (\delta n) + \quad (A1-64)\]

\[+2a n \frac{\partial}{\partial n} (\delta t) - 2a n \frac{\partial}{\partial t} (\delta t) - 2a \delta n - k \frac{\partial^2}{\partial u^2} (\delta n) - \quad (A1-65)\]

\[-k \frac{\partial^2}{\partial v^2} (\delta n) + u \frac{\partial}{\partial x} (\delta n) + v \frac{\partial}{\partial y} (\delta n) + \frac{\partial}{\partial t} (\delta n) = 0. \quad (A1-66)\]

From (A1-58) and (A1-60) we conclude, that

\[\delta n = A + nB; \quad (A1-67)\]

where \(A = A(x, y, u, v, t), B = B(x, y, u, v, t).\) Using this expression, we simplify the rest of equations (A1-55 - A1-64).

\[uv \frac{\partial}{\partial y} (\delta t) - u \frac{\partial}{\partial x} (\delta x) + u \frac{\partial}{\partial t} (\delta t) + u^2 \frac{\partial^2}{\partial x^2} (\delta t) - v \frac{\partial}{\partial y} (\delta x) + \delta u - \frac{\partial}{\partial t} (\delta x) = 0; \quad (A1-68)\]

\[uv \frac{\partial}{\partial x} (\delta t) - u \frac{\partial}{\partial x} (\delta y) - v \frac{\partial}{\partial y} (\delta y) + v \frac{\partial}{\partial t} (\delta t) + v^2 \frac{\partial^2}{\partial y^2} (\delta t) + \delta v - \frac{\partial}{\partial t} (\delta y) = 0; \quad (A1-69)\]

\[\delta v = \frac{\partial}{\partial y} (\delta t) + \frac{\partial}{\partial u} (\delta u) - \frac{\partial}{\partial t} (\delta t) - au \frac{\partial}{\partial x} (\delta t) + \quad (A1-70)\]

\[+av \frac{\partial}{\partial v} (\delta u) + a u + k \frac{\partial^2}{\partial u^2} (\delta u) + k \frac{\partial^2}{\partial v^2} (\delta v) - 2k \frac{\partial B}{\partial u} - u \frac{\partial}{\partial x} (\delta u) - v \frac{\partial}{\partial y} (\delta u) - \frac{\partial}{\partial t} (\delta u) = 0; \quad (A1-71)\]

\[-au v \frac{\partial}{\partial x} (\delta t) + au \frac{\partial}{\partial u} (\delta v) + av \frac{\partial}{\partial v} (\delta v) - av \frac{\partial}{\partial t} (\delta t) - \quad (A1-72)\]

\[-av^2 \frac{\partial^2}{\partial y^2} (\delta t) - a \delta v + k \frac{\partial^2}{\partial u^2} (\delta v) + k \frac{\partial^2}{\partial v^2} (\delta v) - 2k \frac{\partial B}{\partial v} - u \frac{\partial}{\partial x} (\delta v) - v \frac{\partial}{\partial y} (\delta v) - \frac{\partial}{\partial t} (\delta v) = 0; \quad (A1-73)\]

\[-ku \frac{\partial}{\partial x} (\delta t) - kv \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial u} (\delta u) - k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-74)\]

\[-ku \frac{\partial}{\partial x} (\delta t) - kv \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial v} (\delta v) - k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-75)\]

\[2k \frac{\partial}{\partial v} (\delta u) + 2k \frac{\partial}{\partial u} (\delta v) = 0; \quad (A1-76)\]

\[-2au \frac{\partial}{\partial x} (\delta t) - au \frac{\partial B}{\partial u} - 2av \frac{\partial}{\partial y} (\delta t) - av \frac{\partial B}{\partial v} - 2a \frac{\partial}{\partial t} (\delta t) - \quad (A1-77)\]

\[-k \frac{\partial^2 B}{\partial u^2} - k \frac{\partial^2 B}{\partial v^2} + u \frac{\partial B}{\partial x} + v \frac{\partial B}{\partial y} + \frac{\partial B}{\partial t} = 0; \quad (A1-78)\]
\[-au \frac{\partial A}{\partial u} - av \frac{\partial A}{\partial v} - 2au\delta - k \frac{\partial^2 A}{\partial u^2} - k \frac{\partial^2 A}{\partial v^2} + u \frac{\partial A}{\partial x} + v \frac{\partial A}{\partial y} + \frac{\partial A}{\partial t} = 0. \] (A1-74)

Equation (A1-74) is simply Fokker - Planck equation for $A$.

We solve (A1-66 - A1-67) and find $\delta u, \delta v$

\[
\delta u = -(uv \frac{\partial}{\partial y} (\delta t) - u \frac{\partial}{\partial x} (\delta x) + u \frac{\partial}{\partial t} (\delta t) + u^2 \frac{\partial^2}{\partial x^2} (\delta t) - v \frac{\partial}{\partial y} (\delta x) - \frac{\partial}{\partial t} (\delta x)); \] (A1-76)

\[
\delta v = -(uv \frac{\partial}{\partial x} (\delta t) - u \frac{\partial}{\partial y} (\delta y) - v \frac{\partial}{\partial y} (\delta y) + v \frac{\partial}{\partial t} (\delta t) + v^2 \frac{\partial^2}{\partial y^2} (\delta t) - \frac{\partial}{\partial t} (\delta y)). \] (A1-77)

This gives for (A1-67 - A1-73)

\[
\begin{align*}
-2auv \frac{\partial}{\partial y} (\delta t) - au \frac{\partial}{\partial t} (\delta t) - 2au^2 \frac{\partial}{\partial x} (\delta t) - a \frac{\partial}{\partial y} (\delta x) - 2k \frac{\partial}{\partial x} (\delta t) - & \\
-2k \frac{\partial B}{\partial u} + 2uv \frac{\partial^2}{\partial xy} (\delta t) - 2uv \frac{\partial^2}{\partial x^2} (\delta t) + u \frac{\partial}{\partial y} (\delta x) + u^2 \frac{\partial^2}{\partial x^2} (\delta t) + u \frac{\partial}{\partial t} (\delta t) - 2u \frac{\partial^2}{\partial t^2} (\delta x) + & \\
+2u^2 v \frac{\partial^2}{\partial x \partial y} (\delta t) + 2u^2 \frac{\partial^2}{\partial t \partial x} (\delta t) - u^2 \frac{\partial^2}{\partial x^2} (\delta t) & = 0;
\end{align*} \] (A1-78)

\[
\begin{align*}
-2auv \frac{\partial}{\partial x} (\delta t) - au \frac{\partial}{\partial y} (\delta y) - 2au^2 \frac{\partial}{\partial y} (\delta y) - & \\
-2auu \frac{\partial}{\partial x} (\delta t) - av \frac{\partial}{\partial t} (\delta t) - 2au^2 \frac{\partial}{\partial y} (\delta y) - & \\
-2k \frac{\partial B}{\partial v} + 2uv \frac{\partial^2}{\partial t \partial x} (\delta t) - 2uv \frac{\partial^2}{\partial x \partial y} (\delta y) + 2uv^2 \frac{\partial^2}{\partial x \partial y} (\delta t) - & \\
-2u \frac{\partial^2}{\partial y \partial x} (\delta y) + & \\
+u^2 v \frac{\partial^2}{\partial x^2} (\delta t) - u^2 \frac{\partial^2}{\partial y^2} (\delta y) + v \frac{\partial^2}{\partial t^2} (\delta t) - 2v \frac{\partial^2}{\partial y \partial x} (\delta y) + 2u^2 \frac{\partial^2}{\partial y \partial x} (\delta t) - & \\
-v^2 \frac{\partial^2}{\partial y^2} (\delta y) + v^3 \frac{\partial^2}{\partial y^2} (\delta t) - & \\
-\frac{\partial^2}{\partial t^2} (\delta y) = 0;\end{align*} \] (A1-79)

\[
\begin{align*}
-5ku \frac{\partial}{\partial x} (\delta t) - 3kv \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial x} (\delta x) - 3k \frac{\partial}{\partial t} (\delta t) - & \\
-3ku \frac{\partial}{\partial x} (\delta t) - 5kv \frac{\partial}{\partial y} (\delta t) + 2k \frac{\partial}{\partial x} (\delta y) - 3k \frac{\partial}{\partial t} (\delta t) - & \\
-2ku \frac{\partial}{\partial y} (\delta t) - 2kv \frac{\partial}{\partial x} (\delta t) + 2k \frac{\partial}{\partial y} (\delta x) + 2k \frac{\partial}{\partial x} (\delta y) = 0;\end{align*} \] (A1-80)

\[
\begin{align*}
-2au \frac{\partial}{\partial x} (\delta t) - au \frac{\partial}{\partial u} (\delta t) - 2av \frac{\partial}{\partial y} (\delta t) - & \\
-2au \frac{\partial}{\partial x} (\delta t) - au \frac{\partial}{\partial u} (\delta t) - & \\
-2a \frac{\partial}{\partial t} (\delta t) - & \\
-k \frac{\partial^2 B}{\partial u^2} - & \\
-2k \frac{\partial^2 B}{\partial v^2} + u \frac{\partial B}{\partial x} + v \frac{\partial B}{\partial y} + \frac{\partial B}{\partial t} = 0.\end{align*} \] (A1-81)

Now we can collect similar terms in (A1-80 - A1-82) and so split them into nine equations:

\[-5k \frac{\partial}{\partial x} (\delta t) = 0; \] (A1-84)

\[-3k \frac{\partial}{\partial y} (\delta t) = 0; \] (A1-85)
\[
2k \frac{\partial}{\partial x} (\delta x) - 3k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-86)
\]
\[
-3k \frac{\partial}{\partial x} (\delta t) = 0; \quad (A1-87)
\]
\[
-5k \frac{\partial}{\partial y} (\delta t) = 0; \quad (A1-88)
\]
\[
2k \frac{\partial}{\partial y} (\delta y) - 3k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-89)
\]
\[
-2k \frac{\partial}{\partial y} (\delta t) = 0; \quad (A1-90)
\]
\[
-2k \frac{\partial}{\partial x} (\delta t) = 0; \quad (A1-91)
\]
\[
2k \frac{\partial}{\partial y} (\delta x) + 2k \frac{\partial}{\partial x} (\delta y) = 0. \quad (A1-92)
\]

From (A1-84 - A1-85), (A1-87 - A1-88), (A1-90 - A1-91) we see, that \( \delta t = \delta t(t) \), which results in further simplifications

\[
-au \frac{\partial}{\partial t} (\delta t) - a \frac{\partial}{\partial t} (\delta x) - 2k \frac{\partial B}{\partial u} - 2u \frac{\partial B}{\partial x} (\delta x) + u \frac{\partial^2 B}{\partial t^2} - 2u \frac{\partial^2 B}{\partial t \partial x} (\delta x) - \quad (A1-93)
\]
\[
-2u \frac{\partial^2 B}{\partial x^2} (\delta x) - 2uv \frac{\partial^2 B}{\partial x \partial y} (\delta x) - \frac{\partial^2 B}{\partial y^2} = 0;
\]
\[
-av \frac{\partial}{\partial t} (\delta t) - a \frac{\partial}{\partial t} (\delta y) - 2k \frac{\partial B}{\partial v} - 2uv \frac{\partial B}{\partial x \partial y} (\delta y) - 2u \frac{\partial^2 B}{\partial t \partial x} (\delta y) - \quad (A1-94)
\]
\[
-u^2 \frac{\partial^2 B}{\partial x^2} (\delta y) + v \frac{\partial^2 B}{\partial y^2} (\delta y) - \frac{\partial^2 B}{\partial y^2} (\delta y) - \frac{\partial^2 B}{\partial t^2} (\delta y) = 0;
\]
\[
2k \frac{\partial}{\partial x} (\delta x) - 3k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-95)
\]
\[
2k \frac{\partial}{\partial y} (\delta y) - 3k \frac{\partial}{\partial t} (\delta t) = 0; \quad (A1-96)
\]
\[
2k \frac{\partial}{\partial y} (\delta x) + 2k \frac{\partial}{\partial x} (\delta y) = 0; \quad (A1-97)
\]

\[
-uu \frac{\partial B}{\partial u} - av \frac{\partial B}{\partial v} - 2u \frac{\partial B}{\partial t} (\delta t) - k \frac{\partial^2 B}{\partial u^2} - k \frac{\partial^2 B}{\partial v^2} + u \frac{\partial B}{\partial x} + v \frac{\partial B}{\partial y} + \frac{\partial B}{\partial t} = 0. \quad (A1-98)
\]

We integrate (A1-95 - A1-96) and find

\[
\delta x = C + 3/2x \frac{\partial}{\partial t} (\delta t); \quad (A1-99)
\]
\[
\delta y = D + 3/2y \frac{\partial}{\partial t} (\delta t); \quad (A1-100)
\]
where \( C = C(y,t), D = D(x,t) \). We substitute these expressions to (A1-93 - A1-94), (A1-97 - A1-98) and obtain

\[
-3/2ax \frac{\partial^2}{\partial t^2} (\delta t) - au \frac{\partial}{\partial t} (\delta t) - a \frac{\partial C}{\partial t} - 2k \frac{\partial B}{\partial u} - 3/2x \frac{\partial^3}{\partial t^3} (\delta t) - \quad (A1-101)
\]

\[
-2u \frac{\partial^2}{\partial t^2} (\delta t) - 2v \frac{\partial^2 C}{\partial t \partial y} - v^2 \frac{\partial^2 C}{\partial y^2} - \frac{\partial^2 C}{\partial t^2} = 0;
\]

\[
-3/2ay \frac{\partial^2}{\partial t^2} (\delta t) - av \frac{\partial}{\partial t} (\delta t) - a \frac{\partial D}{\partial t} - 2k \frac{\partial B}{\partial y} - 3/2y \frac{\partial^3}{\partial t^3} (\delta t) - \quad (A1-102)
\]

\[
-2u \frac{\partial^2}{\partial t \partial x} (\delta t) - u^2 \frac{\partial^2 D}{\partial x^2} - 2v \frac{\partial^2}{\partial t^2} (\delta t) - \frac{\partial^2 D}{\partial t^2} = 0;
\]

\[
2k \frac{\partial C}{\partial y} + 2k \frac{\partial D}{\partial x} = 0; \quad (A1-103)
\]

\[
-\frac{\partial B}{\partial u} - \frac{\partial B}{\partial v} - 2a \frac{\partial}{\partial t} (\delta t) - k \frac{\partial^2 B}{\partial u^2} - k \frac{\partial^2 B}{\partial y^2} + u \frac{\partial B}{\partial x} + v \frac{\partial B}{\partial y} + \frac{\partial B}{\partial t} = 0. \quad (A1-104)
\]

We find \( \frac{\partial B}{\partial u} \) from (A1-101) and \( \frac{\partial B}{\partial v} \) from (A1-102):

\[
\frac{\partial B}{\partial u} = \frac{1}{k} (-3/4ax \frac{\partial^2}{\partial t^2} (\delta t) - 1/2au \frac{\partial}{\partial t} (\delta t) - 1/2a \frac{\partial C}{\partial t} - 3/4x \frac{\partial^3}{\partial t^3} (\delta t) - \quad (A1-105)
\]

\[
-\frac{\partial C}{\partial y} - v \frac{\partial^2 C}{\partial t \partial y} - 1/2v^2 \frac{\partial^2 C}{\partial y^2} - 1/2 \frac{\partial^2 C}{\partial t^2};
\]

\[
\frac{\partial B}{\partial v} = \frac{1}{k} (-3/4ay \frac{\partial^2}{\partial t^2} (\delta t) - 1/2av \frac{\partial}{\partial t} (\delta t) - 1/2a \frac{\partial D}{\partial t} - 3/4y \frac{\partial^3}{\partial t^3} (\delta t) - \quad (A1-106)
\]

\[
-\frac{\partial B}{\partial t \partial x} - 1/2u^2 \frac{\partial^2 D}{\partial x^2} - v \frac{\partial^2 D}{\partial t^2} (\delta t) - 1/2 \frac{\partial^2 D}{\partial t^2}.
\]

Differentiating (A1-105) by \( v \) we have

\[
\frac{\partial^2 B}{\partial u \partial v} = \frac{1}{k} (-v \frac{\partial^2 C}{\partial y^2} - \frac{\partial^2 C}{\partial t \partial y}); \quad (A1-107)
\]

Differentiating (A1-106) by \( u \) we have

\[
\frac{\partial^2 B}{\partial u \partial v} = \frac{1}{k} (-u \frac{\partial^2 D}{\partial x^2} - \frac{\partial^2 D}{\partial t \partial x}). \quad (A1-108)
\]

We know, that \( C = C(y,t), D = D(x,t) \) and so we conclude from (A1-107 - A1-108)

\[
C = C_1 y + E; \quad (A1-109)
\]

\[
D = C_2 x + F; \quad (A1-110)
\]

where \( E = E(t), F = F(t), C_1 + C_2 = 0. \)

We find derivatives of \( B \).

\[
\frac{\partial B}{\partial u} = \frac{1}{2k} (3/2ax \frac{\partial^2}{\partial t^2} (\delta t) + au \frac{\partial}{\partial t} (\delta t) + a \frac{\partial E}{\partial t} - 3/2x \frac{\partial^3}{\partial t^3} (\delta t) - 2u \frac{\partial^2 E}{\partial t^2} (\delta t) - \frac{\partial^2 E}{\partial t^2}); \quad (A1-111)
\]
\[
\frac{\partial B}{\partial v} = \frac{1}{2k} \left( 3/2a^2 + \frac{\partial^2}{\partial t^2} (\delta t) + av \frac{\partial}{\partial t} (\delta t) + a \frac{\partial F}{\partial t} - 3/2y \frac{\partial^3}{\partial t^3} (\delta t) - 2v \frac{\partial^2}{\partial t^2} (\delta t) - \frac{\partial^2 F}{\partial t^2} \right); \tag{A1-112}
\]

\[
\frac{\partial^2 B}{\partial v \partial t} = \frac{\partial^2 B}{\partial v \partial u} = 0. \tag{A1-113}
\]

Integration of (A1-111 - A1-112) gives

\[
B = G + \frac{1}{2k} \left( 3/2uax \frac{\partial^2}{\partial t^2} (\delta t) + 1/2au^2 \frac{\partial}{\partial t} (\delta t) + uu \frac{\partial E}{\partial t} - 3/2ux \frac{\partial^3}{\partial t^3} (\delta t) - u^2 \frac{\partial^2}{\partial t^2} (\delta t) - u \frac{\partial^2 E}{\partial t^2} \right) \tag{A1-114} + \frac{1}{2k} \left( 3/2vy \frac{\partial^2}{\partial t^2} (\delta t) + 1/2av^2 \frac{\partial}{\partial t} (\delta t) + va \frac{\partial F}{\partial t} - 3/2vy \frac{\partial^3}{\partial t^3} (\delta t) - v^2 \frac{\partial^2}{\partial t^2} (\delta t) - v \frac{\partial^2 F}{\partial t^2} \right);
\]

where \( G = G(x, y, t) \).

Substitution of (A1-114) to (A1-98), collecting and equating to zero similar terms by \( u, v \) gives

\[
3/4a^2k^{-1}x \frac{\partial^2}{\partial t^2} (\delta t) + 1/2a^2k^{-1} \frac{\partial E}{\partial t} - 3/4k^{-1}x \frac{\partial^4}{\partial t^4} (\delta t) - 1/2k^{-1} \frac{\partial^3 E}{\partial t^3} + \frac{\partial G}{\partial x} = 0; \tag{A1-115}
\]

\[
1/2a^2k^{-1} \frac{\partial}{\partial t} (\delta t) - 5/4k^{-1} \frac{\partial^3}{\partial t^3} (\delta t) = 0; \tag{A1-116}
\]

\[
3/4a^2k^{-1}y \frac{\partial^2}{\partial t^2} (\delta t) + 1/2a^2k^{-1} \frac{\partial F}{\partial t} - 3/4k^{-1}y \frac{\partial^4}{\partial t^4} (\delta t) - 1/2k^{-1} \frac{\partial^3 F}{\partial t^3} + \frac{\partial G}{\partial y} = 0; \tag{A1-117}
\]

\[
1/2a^2k^{-1} \frac{\partial}{\partial t} (\delta t) - 5/4k^{-1} \frac{\partial^3}{\partial t^3} (\delta t) = 0; \tag{A1-118}
\]

\[
-a \frac{\partial}{\partial t} (\delta t) + 2 \frac{\partial^2}{\partial t^2} (\delta t) + \frac{\partial G}{\partial t} = 0; \tag{A1-119}
\]

Integrate (A1-115), (A1-117) and obtain following expression \((H = H(t))\):

\[
G = H - \frac{1}{k} \left( 3/8a^2x^2 \frac{\partial^2}{\partial t^2} (\delta t) + 1/2ax^2 \frac{\partial E}{\partial t} - 3/8x^2 \frac{\partial^4}{\partial t^4} (\delta t) - 1/2x \frac{\partial^3 E}{\partial t^3} \right) - \frac{1}{k} \left( 3/8a^2y^2 \frac{\partial^2}{\partial t^2} (\delta t) + 1/2ay^2 \frac{\partial F}{\partial t} - 3/8y^2 \frac{\partial^4}{\partial t^4} (\delta t) - 1/2y \frac{\partial^3 F}{\partial t^3} \right) \tag{A1-120}
\]

Substitution of (A1-120) to (A1-119), collecting and equating to zero terms by \( x, y \) gives

\[
-1/2a^2k^{-1} \frac{\partial^2 E}{\partial t^2} + 1/2k^{-1} \frac{\partial^2 E}{\partial t^3} = 0; \tag{A1-121}
\]

\[
-3/8a^2k^{-1} \frac{\partial^3}{\partial t^3} (\delta t) + 3/8k^{-1} \frac{\partial^4}{\partial t^4} (\delta t) = 0; \tag{A1-122}
\]

\[
-1/2a^2k^{-1} \frac{\partial^2 F}{\partial t^2} + 1/2k^{-1} \frac{\partial^2 F}{\partial t^3} = 0; \tag{A1-123}
\]

\[
-3/8a^2k^{-1} \frac{\partial^3}{\partial t^3} (\delta t) + 3/8k^{-1} \frac{\partial^4}{\partial t^4} (\delta t) = 0; \tag{A1-124}
\]

\[
-a \frac{\partial}{\partial t} (\delta t) + 2 \frac{\partial^2}{\partial t^2} (\delta t) + \frac{\partial H}{\partial t} = 0. \tag{A1-125}
\]
From (A1-116), (A1-118), (A1-122), (A1-124) we conclude, that
\[ \delta t = \text{const} = C_3. \]  
(A1-126)

From (A1-121), (A1-123) we conclude, that
\[ E = C_4 + C_5 t + C_6 e^{-at} + C_7 e^{at}; \]  
(A1-127)
\[ F = C_8 + C_9 t + C_{10} e^{-at} + C_{11} e^{at}; \]  
(A1-128)

From (A1-125) and (A1-126) we see, that
\[ H = C_{12}. \]  
(A1-129)

We obtain using (A1-126 - A1-129) and backward substitution final expressions for variations:
\[ \delta n = -1/2ak^{-1}unC_5 - 1/2ak^{-1}vnC_9 - 1/2a^2k^{-1}xnC_5 - \]  
\[ -1/2a^2k^{-1}ynC_9 - a^2k^{-1}unC_7 e^{at} - a^2k^{-1}vnC_{11} e^{at} + nC_{12} + A; \]  
\[ \delta x = yC_1 + tC_3 + C_4 + C_6 e^{-at} + C_7 e^{at}; \]  
(A1-131)
\[ \delta y = xC_2 + tC_9 + C_8 + C_{10} e^{-at} + C_{11} e^{at}; \]  
(A1-132)
\[ \delta u = -aC_6 e^{-at} + aC_7 e^{at} + vC_1 + C_5; \]  
(A1-133)
\[ \delta v = -aC_{10} e^{-at} + aC_{11} e^{at} + uC_2 + C_9; \]  
(A1-134)
\[ \delta t = C_3. \]  
(A1-135)

This ends calculations.