Hochschild homology and cohomology for involutive $A_\infty$-algebras

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Abstract

We present a study of the homological algebra of bimodules over $A_\infty$-algebras endowed with an involution. Furthermore we introduce a derived description of Hochschild homology and cohomology for involutive $A_\infty$-algebras.

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1 Introduction

Hochschild homology and cohomology are homology and cohomology theories developed for associative algebras which appears naturally when one studies its deformation theory. Furthermore, Hochschild homology plays a central role in topological field theory in order to describe the closed states part of a topological field theory.

An involutive version of Hochschild homology and cohomology was developed by Braun in [Bra14] by considering associative and $A_\infty$-algebras endowed with an involution and morphisms which commute with the involution.

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This paper pretends to take a step further with regards to [FVG15]. Whilst in the latter paper we develop the homological algebra required to give a derived version of Braun's involutive Hochschild homology and cohomology for involutive associative algebras, this research is devoted to develop the machinery required to give a derived description of involutive Hochschild homology and cohomology for $A_\infty$-algebras endowed with an involution.

As in [FVG15], this research has been driven by the author’s research on Costello’s classification of topological conformal field theories [Cos07], where he proves that an open 2-dimensional theory is equivalent to a Calabi-Yau $A_\infty$-category. In [FV15], the author extends the picture to unoriented topological conformal field theories, where open theories now correspond to involutive Calabi-Yau $A_\infty$-categories, and the closed state space of the universal open-closed extension turns out to be the involutive Hochschild chain complex of the open state algebra.

2 Basic concepts

2.1 Coalgebras and bicomodules

An involutive graded coalgebra over a field $K$ is a graded $K$-module $C$ endowed with a coproduct $\Delta : C \to C \otimes_K C$ of degree zero together with an involution $\star : C \to C$ such that:

1. The map $\Delta$ makes the following diagram commute

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_K C \\
\downarrow \Delta & & \downarrow \Delta \otimes \text{Id}_C \\
C \otimes_C C & \xrightarrow{\text{Id}_C \otimes \Delta} & C \otimes_K \tilde{C} \otimes_K C
\end{array}
$$

2. The involution and $\Delta$ are compatible: $\Delta(c^\star) = (\Delta(c))^\star$, for $c \in C$, where the involution on $C \otimes_K C$ is defined as: $(c_1 \otimes c_2)^\star = c_2^\star \otimes c_1^\star$, for $c_1, c_2 \in C$.

An involutive coderivation on an involutive coalgebra $C$ is a map $L : C \to C$ preserving involutions and making the following diagram commutative:

$$
\begin{array}{ccc}
C & \xrightarrow{L} & C \\
\downarrow \Delta & & \downarrow \Delta \\
C \otimes_K C & \xrightarrow{L \otimes \text{Id}_C + \text{Id}_C \otimes L} & C \otimes_K C
\end{array}
$$

Denote with $\text{iCoder}(-)$ the spaces of coderivations of involutive coalgebras. Observe that $\text{iCoder}(-)$ are Lie subalgebras.

An involutive differential graded coalgebra is an involutive coalgebra $C$ equipped with an involutive coderivation $b : C \to C$ of degree $-1$ such that $b^2 = b \circ b = 0$. 

2
A morphism between two involutive coalgebras \( C \) and \( D \) is a graded map \( C \xrightarrow{f} D \) compatible with the involutions which makes the following diagram commutative:

\[
\begin{array}{ccc}
C & f & D \\
\Delta_C & \sim & \Delta_D \\
C \otimes_K C & \sim & D \otimes_K D
\end{array}
\]

Example 2.1. The cotensor coalgebra of an involutive graded \( K \)-bimodule \( A \) is defined as \( \hat{T}A = \bigoplus_{n \geq 0} A \otimes_K n \). We define an involution in \( A \otimes_K n \) by stating:

\[
(a_1 \otimes \cdots \otimes a_n)^* := (a_n^* \otimes \cdots \otimes a_1^*).
\]

The coproduct on \( \hat{T}A \) is given by:

\[
\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n).
\]

Observe that \( \Delta \) commutes with the involution.

Proposition 2.2. There is a canonical isomorphism of complexes:

\[
\text{iCoder}(TSA) \cong \text{Hom}_{A^{-} \text{\text{-}bimod}}(\text{Bar}(A), A).
\]

Proof. The proof follows the arguments in Proposition 4.1.1 [FVG15], where we show the result for the non-involutive setting in order to restrict to the involutive one.

Since \( \text{Bar}(A) = A \otimes_K TSA \otimes_K A \), the degree \(-n\) part of \( \text{Hom}_{A^{-} \text{\text{-}bimod}}(\text{Bar}(A), A) \) is the space of degree \(-n\) linear maps \( TSA \to A \), which is isomorphic to the space of degree \((-n - 1)\) linear maps \( TSA \to SA \). By the universal property of the tensor coalgebra, there is a bijection between degree \((-n - 1)\) linear maps \( TSA \to SA \) and degree \((-n - 1)\) coderivations on \( TSA \). Hence the degree \( n \) part of \( \text{Hom}_{A^{-} \text{\text{-}bimod}}(\text{Bar}(A), A) \) is isomorphic to the degree \( n \) part of \( \text{Coder}(TSA) \). One checks directly that this isomorphism restricts to an isomorphism of graded vector spaces

\[
\text{Hom}_{A^{-} \text{\text{-}bimod}}(\text{Bar}(A), A) \cong \text{iCoder}(TSA).
\]

Finally, one can check that the differentials coincide under the above isomorphism, cf. Section 12.2.4 [LV12].

Remark 2.3. Proposition 2.2 allows us to think of a coderivation as a map \( \hat{T}A \to A \). Such a map \( f : \hat{T}A \to A \) can be described as a collection of maps \( \{f_n : A \otimes^n \to A\} \) which will be called the components of \( f \).
If $b$ is a coderivation of degree $-1$ on $\widehat{TA}$ with $b_n : A^\otimes_k S^n \to A$, then $b^2$ becomes a linear map of degree $-2$ with

$$b_n^2 = \sum_{i+j=n+1}^{n-1} \sum_{k=0}^{n-1} b_i \circ \left( \text{Id}^\otimes_k \circ b_j \circ \text{Id}^\otimes_k \right).$$

The coderivation $b$ will be a differential for $\widehat{TA}$ if, and only if, all the components $b_n^2$ vanish.

Given a (involutive) graded $\mathbb{K}$-bimodule $A$, we denote the suspension of $A$ by $SA$ and define it as the graded (involutive) $\mathbb{K}$-bimodule with $SA_i = A_{i-1}$. Given such a bimodule $A$, we define the following morphism of degree $-1$ induced by the identity $s : A \to SA$ by $s(a) = a$.

**Lemma 2.4 (cf. Lemma 1.3 [GJ90]).** If $b_k : (SA)^\otimes_k \to SA$ is an involutive linear map of degree $-1$, we define $m_k : A^\otimes_k \to A$ as $m_k = s^{-1} \circ b_k \circ s^\otimes_k$. Under these conditions:

$$b_k(a_1 \otimes \cdots \otimes a_k) = \sigma m_k(a_1 \otimes \cdots \otimes a_k),$$

where $\sigma := (-1)^{\left(\sum_{i=1}^{n} \left|a_{1}\right| + \cdots + \left|a_{i} \right| - i \right)}$.

**Proof.** The proof follows the arguments of Lemma 1.3 [GJ90]. We only need to observe that the involutions are preserved as all the maps involved in the proof are assumed to be involutive. □

Let $\overline{m}_k := \sigma m_k$, then we have $b_k(a_1 \otimes \cdots \otimes a_k) = \overline{m}_k(a_1 \otimes \cdots \otimes a_k)$.

**Proposition 2.5.** Given an involutive graded $\mathbb{K}$-bimodule $A$, let $e_i = |a_1| + \cdots + |a_i| - i$ for $a_i \in A$ and $1 \leq i \leq n$. A boundary map $b$ on $\widehat{TA}$ is given in terms of the maps $\overline{m}_k$ by the following formula:

$$b_n(a_1 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} \sum_{i=1}^{n-k+1} (-1)^{e_{i-1}} (a_1 \otimes \cdots \otimes a_{i-1} \otimes \overline{m}_k(a_i \otimes \cdots \otimes a_{i+k-1}) \otimes \cdots \otimes a_n).$$

**Proof.** This proof follows the arguments of Proposition 1.4 [GJ90]. The only detail that must be checked is that $b_n$ preserves involutions:

$$b_n((a_1 \otimes \cdots \otimes a_n)^*) = \sum_{j,k} \pm (a_n^* \otimes \cdots \otimes a_j^* \otimes \overline{m}_k(a_{j-1}^* \otimes \cdots \otimes a_{k+1}^*) \otimes \cdots \otimes a_1^*))$$

$$= \sum_{j,k} \pm (a_1^* \otimes \cdots \otimes \overline{m}_k(a_{j-1}^* \otimes \cdots \otimes a_{j-1}^*) \otimes a_j \otimes \cdots \otimes a_n)^*$$

$$= (b_n(a_1 \otimes \cdots \otimes a_n))^*.$$ □

Given an involutive coalgebra $C$ with coproduct $\rho$ and counit $\varepsilon$, for an involutive graded vector space $P$, a left coaction is a linear map $\Delta^L : P \to C \otimes_k P$ such that

1. $(\text{Id} \otimes \rho) \circ \Delta^L = (\rho \otimes \text{Id}) \circ \Delta^L$;
2. $(\text{Id} \otimes \varepsilon) \circ \Delta^L = \text{Id}.$
Analagously we introduce the concept of right coaction.

Given an involutive coalgebra \((C, \rho, \varepsilon)\) with involution \(*\) we define an involutive \(C\)-bicomodule as an involutive graded vector space \(P\) with involution \(†\), a left coaction \(\Delta^L : P \to C \otimes \mathbb{K} P\) and a right coaction \(\Delta^R : P \to P \otimes \mathbb{K} C\) which are compatible with the involutions, that is the diagrams below commute:

\[
\begin{array}{c}
P \xrightarrow{(-)^*} P \\
\xrightarrow{\Delta^L} C \otimes \mathbb{K} P \\
\end{array} \quad \begin{array}{c}
P \xrightarrow{\Delta^R} C \otimes \mathbb{K} P \\
\xrightarrow{\Delta^L \otimes \text{Id}_C} C \otimes \mathbb{K} P \otimes \mathbb{K} C \\
\end{array}
\]

Where

\[
\begin{array}{c}
(-, -)^* : C \otimes \mathbb{K} P \to P \otimes \mathbb{K} C \\
c \otimes p \mapsto p^* \otimes c^* \\
\end{array}
\]

For two involutive \(C\)-bicomodules \((P_1, \Delta^L_1)\) and \((P_2, \Delta^L_2)\), a morphism \(P_1 \overset{f}{\to} P_2\) is defined as an involutive morphism making diagrams below commute:

\[
\begin{array}{c}
P_1 \overset{\Delta^L_1}{\to} C \otimes \mathbb{K} P_1 \\
\xrightarrow{\text{Id}_C \otimes f} C \otimes \mathbb{K} P_2 \\
\end{array} \quad \begin{array}{c}
P_1 \overset{\Delta^R_1}{\to} P_1 \otimes \mathbb{K} C \\
\xrightarrow{\text{Id}_C \otimes f} P_2 \otimes \mathbb{K} C \\
\end{array}
\]

\[
\begin{array}{c}
P_1 \overset{\Delta^L_1}{\to} C \otimes \mathbb{K} P_2 \\
\xrightarrow{\Delta^L_1 \otimes \text{Id}_C} C \otimes \mathbb{K} P_1 \otimes \mathbb{K} C \\
\end{array} \quad \begin{array}{c}
P_1 \overset{\Delta^R_1}{\to} C \otimes \mathbb{K} P_2 \otimes \mathbb{K} C \\
\xrightarrow{\Delta^L_1 \otimes \text{Id}_C} C \otimes \mathbb{K} P_1 \otimes \mathbb{K} C \\
\end{array}
\]

\[\text{(4)}\quad \text{(5)}\]

### 2.2 \(A_{\infty}\)-algebras and \(A_{\infty}\)-quasi-isomorphisms

An involutive \(\mathbb{K}\)-algebra is an algebra \(A\) over a field \(\mathbb{K}\) endowed with a \(\mathbb{K}\)-linear map (an involution) \(* : A \to A\) satisfying:

1. \(0^* = 0\) and \(1^* = 1\);
2. \((a^*)^* = a\) for each \(a \in A\);
3. \((a_1a_2)^* = a_2^*a_1^*\) for every \(a_1, a_2 \in A\).

**Example 2.6.**

1. Any commutative algebra \(A\) becomes an involutive algebra if we endow it with the identity as involution.

2. Let \(V\) an involutive vector space. The tensor algebra \(\bigoplus_n V^\otimes_n\) becomes an involutive algebra if we endow it with the following involution: \((v_1, \ldots, v_n)^* = (v_n^*, \ldots, v_1^*)\). This example is particularly important and we will come back to it later on.

3. For a discrete group \(G\), the group ring \(\mathbb{K}[G]\) is an involutive \(\mathbb{K}\)-algebra with involution given by inversion \(g^* = g^{-1}\).

Given an involutive algebra \(A\), an involutive \(A\)-bimodule \(M\) is an \(A\)-bimodule endowed with an involution satisfying \((a_1ma_2)^* = a_2^*m^*a_1^*\).
Given two involutive $A$-bimodules $M$ and $N$, a *involutive morphism* between them is a morphism of $A$-bimodules $f : M \to N$ compatible with the involutions.

**Lemma 2.7.** The composition of involutive morphisms is an involutive morphism.

**Proof.** Given $f : M \to N$ and $g : N \to P$ two involutive morphisms:

$$(f \circ g)(m^*) = f((g(m))^*) = f((g(m))^*) = f(g(m))^* \quad \square$$

Involutive $A$-bimodules and involutive morphisms form the category $A\text{-}iBimod$.

Given a (involutive) graded $\mathbb{K}$-module $A$, we denote the suspension of $A$ by $SA$ and define it as the graded (involutive) $\mathbb{K}$-module with $SA_i = A_{i-1}$. An *involutive $A_\infty$-algebra* is an involutive graded vector space $A$ endowed with involutive morphisms $b_n : (SA)^{\otimes n} \to SA$, $n \geq 1$,

$$b_n : (SA)^{\otimes n} \to SA, \; n \geq 1, \quad (6)$$

of degree $n - 2$ such that the identity below holds:

$$\sum_{i+j+l=n} (-1)^{i+j+l+|b|} b_{i+1+j+1} \circ (\text{Id}^{\otimes j} \otimes b_j \otimes \text{Id}^{\otimes l}) = 0, \; \forall n \geq 1. \quad (7)$$

**Remark 2.8.** Condition (7) says, in particular, that $b_2^2 = 0$.

**Example 2.9.**

1. The concept $A_\infty$-algebra is a generalization for that of a differential graded algebra. Indeed, if the maps $b_n = 0$ for $n \geq 3$ then $A$ is a differential $\mathbb{Z}$-graded algebra and conversely an $A_\infty$-algebra $A$ yields a differential graded algebra if we require $b_n = 0$ for $n \geq 3$.

2. The definition of $A_\infty$-algebra was introduced by Stasheff whose motivation was the study of the graded abelian group of singular chains on the based loop space of a topological space.

For an involutive $A_\infty$-algebra $(A, b_n)$, the involutive bar complex is the involutive differential graded coalgebra $\text{Bar}(A) = \hat{T}SA$, where we endow $\text{Bar}(A)$ with a coderivation defined by $b_i = s^{-1} \circ b_i \circ s^{\otimes i}$ (cf. Definition 1.2.2.3 [LH03]).

Given two involutive $A_\infty$-algebras $C$ and $D$, a *morphism* of $A_\infty$-algebras $f : C \to D$ is an involutive morphism of degree 0 between the associated involutive differential graded coalgebras $\text{Bar}(C) \to \text{Bar}(D)$.

It follows from Proposition 2.2 that the definition of an involutive $A_\infty$-algebra can be summarized by saying that it is an involutive graded $\mathbb{K}$-module $A$ equipped with an involutive coderivation on $\text{Bar}(A)$ of degree $-1$.

**Remark 2.10.** From [Bra14, Definition 2.8], we have that a morphism of involutive $A_\infty$-algebras $f : C \to D$ can be given by an involutive morphism of differential graded coalgebras $\text{Bar}(C) \to \text{Bar}(D)$, that is, a series of involutive homogeneous maps of degree zero

$$f_n : (SC)^{\otimes n} \to SD, \; n \geq 1,$$
such that
\[ \sum_{i+j+k=n} f_{i+j+k} \otimes \left( \text{Id}_{SC} \otimes b_j \otimes \text{Id}_{SC} \right) = \sum_{i_1+\cdots+i_s=n} b_k \circ (f_{i_1} \otimes \cdots \otimes f_{i_s}). \] (8)

The composition \( f \circ g \) of two morphisms of involutive \( \mathcal{A}_{\infty} \)-algebras is given by
\[ (f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_s \circ (g_{i_1} \otimes \cdots \otimes g_{i_s}); \]
the identity on \( SC \) is defined as \( f_1 = \text{Id}_{SC} \) and \( f_n = 0 \) for \( n \geq 2 \).

For an involutive \( \mathcal{A}_{\infty} \)-algebra \( A \), we define its associated homology algebra \( H_\bullet(A) \) as the homology of the differential \( b_1 \) on \( A \): \( H_\bullet(A) = H_\bullet(A, b_1) \).

**Remark 2.11.** Endowed with \( b_2 \) as multiplication, the homology of an \( \mathcal{A}_{\infty} \)-algebra \( A \) is an associative graded algebra, whereas \( A \) is not usually associative.

Let \( f : A_1 \to A_2 \) a morphism of involutive \( \mathcal{A}_{\infty} \)-algebras with components \( f_n \); for \( n = 1 \), \( f_1 \) induces a morphism of algebras \( H_\bullet(A_1) \to H_\bullet(A_2) \). We say that \( f : A_1 \to A_2 \) is an \( \mathcal{A}_{\infty} \)-quasi-isomorphism if \( f_1 \) is a quasi-isomorphism.

### 2.3 \( \mathcal{A}_{\infty} \)-bimodules

Let \( (A, b^A) \) be an involutive \( \mathcal{A}_{\infty} \)-algebra. An **involutive \( \mathcal{A}_{\infty} \)-bimodule** is a pair \((M, b^M)\) where \( M \) is a graded involutive \( \mathbb{K} \)-module and \( b^M \) is an involutive differential on the Bar \((A)\)-bicomodule
\[
\text{Bar}(M) := \text{Bar}(A) \otimes_{\mathbb{K}} SM \otimes_{\mathbb{K}} \text{Bar}(A).
\]

Let \((M, b^M)\) and \((N, b^N)\) be two involutive \( \mathcal{A}_{\infty} \)-bimodules. We define a **morphism of involutive \( \mathcal{A}_{\infty} \)-bimodules** \( f : M \to N \) as a morphism of \( \text{Bar}(A) \)-bicomodules
\[
F : \text{Bar}(M) \to \text{Bar}(N)
\]
such that \( b^N \circ F = F \circ b^M \).

**Proposition 2.12.** If \( f : A_1 \to A_2 \) is a morphism of involutive \( \mathcal{A}_{\infty} \)-algebras, then \( A_2 \) becomes an involutive bimodule over \( A_1 \).

**Proof.** As we are assuming that both \( A_1 \) and \( A_2 \) are involutive \( \mathcal{A}_{\infty} \)-algebras and that \( f \) is involutive, we do not need to care about involutions. When it comes to the bimodule structure, this result holds as \( \text{Bar}(A_2) \) is made into a bicomodule of \( \text{Bar}(A_1) \) by the homomorphism of involutive coalgebras \( f : \text{Bar}(A_1) \to \text{Bar}(A_2) \), see Proposition 3.4 \cite{GJ90}.

**Remark 2.13 (Section 5.1 \cite{KS09}).** Let \( i\text{Vect} \) be the category of involutive \( \mathbb{Z} \)-graded vector spaces and involutive morphisms. For an involutive \( \mathcal{A}_{\infty} \)-algebra \( A \), involutive \( A \)-bimodules and their respective morphisms form a differential graded category. Indeed, following \cite{KS09}, Definition 5.1.5: let \( A \) be an
involutive $A_\infty$-algebra and let us define the category $\text{A-\textit{iBimod}}$ whose class of objects are involutive $A$-bimodules and where $\text{Hom}_{\text{A-\textit{iBimod} }}(M, N)$ is:

$$\text{Hom}_{i\text{Vect}}^{\star}(\text{Bar}(A) \otimes_K SM \otimes_K \text{Bar}(A), \text{Bar}(A) \otimes_K SN \otimes_K \text{Bar}(A)),$$

Let us recall that

$$\text{Hom}_{i\text{Vect}}^{\star}(\text{Bar}(A) \otimes_K SM \otimes_K \text{Bar}(A), \text{Bar}(A) \otimes_K SN \otimes_K \text{Bar}(A))$$

is by definition

$$\prod_{i \in Z} \text{Hom}_{i\text{Vect}}((\text{Bar}(A) \otimes_K SM \otimes_K \text{Bar}(A))^i, (\text{Bar}(A) \otimes_K SN \otimes_K \text{Bar}(A))^{i+n}).$$

The morphism

$$\text{Hom}_{i\text{Vect}}^{n+1}(\text{Bar}(A) \otimes_K SM \otimes_K \text{Bar}(A), \text{Bar}(A) \otimes_K SN \otimes_K \text{Bar}(A)) \to \text{Hom}_{i\text{Vect}}^{n+1}(\text{Bar}(A) \otimes_K SM \otimes_K \text{Bar}(A), \text{Bar}(A) \otimes_K SN \otimes_K \text{Bar}(A))$$

sends a family $\{f_i\} \in Z$ to a family $\{b^N \circ f_i - (-1)^n f_{i+1} \circ b^M\} \in Z$. Observe that the zero cycles in $\text{Hom}_{i\text{Vect}}^{\star}(\text{Bar}(A) \otimes_K M \otimes_K \text{Bar}(A), \text{Bar}(A) \otimes_K N \otimes_K \text{Bar}(A))$ are precisely the morphisms of involutive $A$-bimodules. This morphism defines a differential, indeed: for fixed indices $i, n \in Z$ we have

$$d^2(f_i) = d(b^N f_i - (-1)^n f_{i+1} b^M)$$

$$= b^N (b^N f_i - (-1)^n f_{i+1} b^M) - (-1)^n (b^N f_i - (-1)^n f_{i+1} b^M) b^M$$

$$= (-1)^n b^N f_{i+1} b^M - (-1)^{n+1} b^N f_{i+1} b^M = 0,$$

where (!) points out the fact that $b^N \circ b^N = 0 = b^M \circ b^M$.

For a morphism $\phi \in \text{Hom}_{i\text{Vect}}^{\star}(\text{Bar}(A) \otimes_K M \otimes_K \text{Bar}(A), \text{Bar}(A) \otimes_K N \otimes_K \text{Bar}(A))$ and an element $x \in \text{Bar}(A) \otimes_K M \otimes_K \text{Bar}(A)$, $\text{Hom}_{\text{A-\textit{iBimod} }}(M, N)$ becomes an involutive complex if we endowed it with the involution $\phi^\ast(x) = \phi(x^\ast)$.

The functor $\text{Hom}_{\text{A-\textit{iBimod} }}(M, \_)$ pairs an involutive $A$-bimodule $F$ with the involutive $K$-vector space $\text{Hom}_{\text{A-\textit{iBimod} }}(M, F)$ of involutive homomorphisms. Given a homomorphism $f : F \to G$, for $F, G \in \text{Obj } (\text{A-\textit{iBimod})}$, $\text{Hom}_{\text{A-\textit{iBimod} }}(M, \_)$ pairs $f$ with the involutive map:

$$f_* : \text{Hom}_{\text{A-\textit{iBimod} }}(M, F) \to \text{Hom}_{\text{A-\textit{iBimod} }}(M, G).$$

We prove that $f_*$ preserves involutions:

$$(f_* \phi^\ast)(x) = (f \circ \phi^\ast)(x) = f(\phi(x^\ast)) = f((\phi(x))^\ast) = ((f(\phi(x)))^\ast) = (f_* \phi)(x^\ast).$$

We define the functor $\text{Hom}_{\text{A-\textit{iBimod} }}(\_, M)$, which sends an involutive homomorphism $f : F \to G$, for $F, G \in \text{Obj } (\text{A-\textit{iBimod})}$, to

$$\varphi : \text{Hom}_{\text{A-\textit{iBimod} }}(G, M) \to \text{Hom}_{\text{A-\textit{iBimod} }}(F, M).$$

We prove that $\varphi$ preserves involutions:

$$(\varphi \phi^\ast)(x) = (\phi \circ \varphi^\ast)(x) = \phi((\varphi(x))^\ast) = \phi((\phi(x)))^\ast) = (\varphi \phi)(x^\ast).$$
Let us check that the involution is preserved:

\[ \varphi(\phi^*)(x) = (\phi^* \circ f)(x) = \varphi(f(x)^*) = \varphi(f(x^*)) = \varphi(\phi)(x^*) = (\varphi(\phi))^*(x) \]

Let \( A \) be an involutive \( A_\infty \)-algebra and let \( (M, b^M) \) and \( (N, b^N) \) be involutive \( A \)-bimodules. For \( f, g : M \to N \) morphisms of \( A \)-bimodules, an \( A_\infty \)-homotopy between \( f \) and \( g \) is a morphism \( h : M \to N \) of \( A \)-bimodules satisfying

\[ f - g = b^N \circ h + h \circ b^M. \]

We say that two morphisms \( u : M \to N \) and \( v : N \to M \) of involutive \( A \)-bimodules are homotopy equivalent if \( u \circ v \sim \text{Id}_N \) and \( v \circ u \sim \text{Id}_M \).

### 3 The involutive tensor product

For an involutive \( A_\infty \)-algebra \( A \) and involutive \( A \)-bimodules \( M \) and \( N \), the involutive tensor product \( M \widetilde{\otimes}_\infty N \) is the following object in \( i\text{Vect}_K \):

\[
M \widetilde{\otimes}_\infty N := \frac{M \otimes_K \text{Bar}(A) \otimes_K N}{(m^* \otimes a_1 \otimes \cdots \otimes a_k \otimes n - m \otimes a_1 \otimes \cdots \otimes a_k \otimes n^*)}
\]

Observe that, for an element of \( M \widetilde{\otimes}_\infty N \) of the form \( m \otimes a_1 \otimes \cdots \otimes a_k \otimes n \), we have: \((m \otimes a_1 \otimes \cdots \otimes a_k \otimes n)^* = m^* \otimes a_1 \otimes \cdots \otimes a_k \otimes n = m \otimes a_1 \otimes \cdots \otimes a_k \otimes n^*\).

**Proposition 3.1.** For an involutive \( A_\infty \)-algebra \( A \) and involutive \( A \)-bimodules \( M, N \) and \( L \), the following isomorphism holds:

\[
\tau : \text{Hom}_{i\text{Vect}} \left( M \widetilde{\otimes}_\infty N, L \right) \cong \text{Hom}_{i\text{Vect}} \left( \frac{M \otimes_K \text{Bar}(A)}{\sim}, \text{Hom}_{A^{-\text{bimod}}}(N, L) \right),
\]

where in \( M \otimes_K \text{Bar}(A) : (m \otimes a_1 \otimes \cdots \otimes a_k)^* = m^* \otimes a_1 \otimes \cdots \otimes a_k, \sim \) denotes the relation \( m \otimes a_1 \otimes \cdots \otimes a_k = m^* \otimes a_1 \otimes \cdots \otimes a_k \) and \( \frac{M \otimes_K \text{Bar}(A)}{\sim} \) has the identity map as involution.

**Proof.** Let \( f : M \widetilde{\otimes}_\infty N \to L \) be an involutive map. We define:

\[
\tau(f) := \tau_f \in \text{Hom}_{i\text{Vect}} \left( \frac{M \otimes_K \text{Bar}(A)}{\sim}, \text{Hom}_{A^{-\text{bimod}}}(N, L) \right),
\]

where \( \tau_f \) of \( M \otimes a_1 \otimes \cdots \otimes a_k \) := \( \tau_f [m \otimes a_1 \otimes \cdots \otimes a_k] \in \text{Hom}_{A^{-\text{bimod}}}(N, L) \). Finally, for \( n \in N \) we define:

\[
\tau_f [m \otimes a_1 \otimes \cdots \otimes a_k](n) := f( m \otimes a_1 \otimes \cdots \otimes a_k \otimes n).
\]

We need to check that \( \tau \) preserves the involutions, indeed:

\[
\tau_f^*[m \otimes a_1 \otimes \cdots \otimes a_k](n) = f^*(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n) =
\]

\[
= (f( m \otimes a_1 \otimes \cdots \otimes a_k \otimes n))^* = (\tau_f)^*[m \otimes a_1 \otimes \cdots \otimes a_k](n).
\]

In order to see that \( \tau \) is an isomorphism, we build an inverse. Let us consider an involutive map

\[
g_1 : \frac{M \otimes_K \text{Bar}(A)}{\sim} \to \text{Hom}_{A^{-\text{bimod}}}(N, L)
\]

\[
m \otimes a_1 \otimes \cdots \otimes a_k \mapsto g_1[m \otimes a_1 \otimes \cdots \otimes a_k]
\]
and define a map
\[
g_2 : M \boxtimes_{\infty} N \to L
\]
\[
m \otimes a_1 \otimes \cdots \otimes a_k \otimes n \mapsto g_1[m \otimes a_1 \otimes \cdots \otimes a_k](n)
\]
We check that \(g_2\) is involutive:
\[
g_2((m \otimes a_1 \otimes \cdots \otimes a_k \otimes n)^*) = g_2(m^* \otimes a_1 \otimes \cdots \otimes a_k \otimes n) =
\]
\[
= g_1[m^* \otimes a_1 \otimes \cdots \otimes a_k](n) = (g_1[m \otimes a_1 \otimes \cdots \otimes a_k])^*(n)
\]
\[
= (g_1[m \otimes a_1 \otimes \cdots \otimes a_k](n))^* = (g_2(m \otimes a_1 \otimes \cdots \otimes a_k \otimes n))^*.
\]
The rest of the proof is standard and follows the steps of Theorem 2.75 [Rot09] or Proposition 2.6.3 [Wei94].

For an \(A\)-bimodule \(M\), let us define \((-)\boxtimes_{\infty} M\) as the covariant functor
\[
\begin{array}{c}
A-\text{iBimod} \\
B \xrightarrow{\sim} B\boxtimes_{\infty} M
\end{array}
\]
This functor sends a map \(B_1 \xrightarrow{f} B_2\) to \(B_1 \boxtimes_{\infty} M \xrightarrow{f \boxtimes_{\infty} \text{Id}_M} B_2 \boxtimes_{\infty} M\).

The functor \((-)\boxtimes_{\infty} M\) is involutive: let us consider an involutive map \(f : B_1 \to B_2\) and its image under the tensor product functor, \(g = f \boxtimes_{\infty} \text{Id}_M\). Hence:
\[
g((b,a)^*) = g(b^*,a) = (f(b^*),a) = (f(b),a)^* = (g(b,a))^*.
\]
Given an involutive \(A_{\infty}\)-algebra \(A\), we say that an involutive \(A\)-bimodule \(F\) is \textit{flat} if the tensor product functor \((-)\boxtimes_{\infty} F : A-\text{iBimod} \to A-\text{iBimod}\) is exact, that is: it takes quasi-isomorphisms to quasi-isomorphisms. From now on, we will assume that all the involutive \(A\)-bimodules are flat.

**Lemma 3.2.** If \(P\) and \(Q\) are homotopy equivalent as involutive \(A_{\infty}\)-bimodules then, for every involutive \(A_{\infty}\)-bimodule \(M\), the following quasi-isomorphism in the category of involutive \(A_{\infty}\)-bimodules holds:
\[
P \boxtimes_{\infty} M \simeq Q \boxtimes_{\infty} M.
\]
**Proof.** Let \(f : P \simeq Q : g\) be a homotopy equivalence. It is clear that
\[
h \sim k \Rightarrow h \boxtimes_{\infty} \text{Id}_M \sim k \boxtimes_{\infty} \text{Id}_M.
\]
Therefore, we have:
\[
P \boxtimes_{\infty} M \to Q \boxtimes_{\infty} M \to P \boxtimes_{\infty} M
\]
\[
p \boxtimes a \mapsto f(p) \boxtimes a \mapsto g(f(p)) \boxtimes a
\]
and
\[
Q \boxtimes_{\infty} M \to P \boxtimes_{\infty} M \to Q \boxtimes_{\infty} M
\]
\[
q \boxtimes a \mapsto g(q) \boxtimes a \mapsto f(g(q)) \boxtimes a
\]
the result follows since \(f \circ g \sim \text{Id}_Q\) and \(g \circ f \sim \text{Id}_P\).  

Lemma 3.3. Let $A$ be an involutive $A_{\infty}$-algebra. If $P$ and $Q$ are homotopy equivalent as involutive $A$-bimodules then, for every involutive $A$-bimodule $M$, the following quasi-isomorphism holds:

$$\text{Hom}_{A_{\text{bimod}}} (P, M) \simeq \text{Hom}_{A_{\text{bimod}}} (Q, M).$$

Proof. Consider $f : P \to Q$ a homotopy equivalence and let $g : Q \to P$ be its homotopy inverse. If $[-,-]$ denotes the homotopy classes of morphisms, then both $f$ and $g$ induce the following maps:

$$f_* : [P, M] \to [Q, M], \quad \alpha \mapsto \alpha \circ g$$
$$g_* : [Q, M] \to [P, M], \quad \beta \mapsto \beta \circ f$$

Now we have:

$$f_* \circ g_* \circ \beta = f_* \circ \beta \circ f = \beta \circ g \circ f \sim \beta;$$
$$g_* \circ f_* \circ \alpha = g_* \circ \alpha \circ g = \alpha \circ f \circ g \sim \alpha. \quad \Box$$

4 Involutive Hochschild homology and cohomology

4.1 Hochschild homology for involutive $A_{\infty}$-algebras

We define the involutive Hochschild chain complex of an involutive $A_{\infty}$-algebra $A$ with coefficients in an $A$-bimodule $M$ as follows:

$$C^{\text{inv}}_\bullet (M, A) = M \check{\otimes} \infty \text{Bar}(A).$$

The differential is the same given in Section 7.2.4 [KS09]. The involutive Hochschild homology of $A$ with coefficients in $M$ is

$$\text{HH}_n (M, A) = H C^{\text{inv}}_n (M, A).$$

Lemma 4.1. For an involutive $A_{\infty}$-algebra $A$ and a flat $A$-bimodule $M$, the following quasi-isomorphism holds:

$$C^{\text{inv}}_\bullet (M, A) \simeq M \check{\otimes} \infty A.$$

Proof. The result follows from:

$$M \check{\otimes} \infty A \simeq M \check{\otimes} \infty \text{Bar}(A) = C^{\text{inv}}_\bullet (M, A).$$

Observe that we are using that $M$ is flat and that there is a quasi-isomorphism between $\text{Bar}(A)$ and $A$ (Proposition 2, Section 2.3.1 [Fer12]). \quad \Box
4.2 Hochschild cohomology for involutive $A_{\infty}$-algebras

The involutive Hochschild cochain complex of an involutive $A_{\infty}$-algebra $A$ with coefficients on an $A$-bimodule $M$ is defined as the $\mathbb{K}$-vector space

$$C^*_{inv}(A, M) := \text{Hom}_{A^{-}\text{Bimod}}(\text{Bar}(A), M),$$

with the differential defined in section 7.1 of [KS09].

**Proposition 4.2.** For an involutive $A_{\infty}$-algebra $A$ and an $A$-bimodule $M$, we have the following quasi-isomorphism: $C^*_{inv}(A, M) \simeq \text{Hom}_{A^{-}\text{Bimod}}(A, M)$.

**Proof.** The result follows from:

$$C^*_{inv}(A, M) = \text{Hom}_{A^{-}\text{Bimod}}(\text{Bar}(A), M) :=$$

$$\text{Hom}^{\text{inv}}_i(\text{Bar}(A) \otimes_\mathbb{K} S \text{Bar}(A) \otimes_\mathbb{K} \text{Bar}(A), \text{Bar}(A) \otimes_\mathbb{K} SM \otimes_\mathbb{K} \text{Bar}(A)) \simeq$$

$$\text{Hom}^{\text{inv}}_i(\text{Bar}(A) \otimes_\mathbb{K} SA \otimes_\mathbb{K} \text{Bar}(A), \text{Bar}(A) \otimes_\mathbb{K} SM \otimes_\mathbb{K} \text{Bar}(A)) =$$

$$\text{Hom}_{A^{-}\text{Bimod}}(A, M).$$

Here (!) points out the fact that $S \text{Bar}(A)$ is a projective resolution of $SA$ in $i\text{Vect}$ and hence we have the quasi-isomorphism $S \text{Bar}(A) \simeq SA$. Observe that $S \text{Bar}(A)$ is projective in $i\text{Vect}$, therefore the involved functors in the proof are exact and preserve quasi-isomorphisms. $\square$

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