On Some Sequences of Polynomials Generating the Genocchi Numbers

A. K. Svinin

1Matrosov Institute for System Dynamics and Control Theory of Siberian Branch of Russian Academy of Sciences, 134 Lermontova str., Irkutsk, 664033 Russia

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Abstract—We consider sequences of Genocchi numbers of the first and the second kind. For these numbers, an approach based on their representation using sequences of polynomials is developed. Based on this approach, for these numbers some identities generalizing the known identities are constructed.

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1. INTRODUCTION

The classic Genocchi numbers \((G_{2n})_{n \geq 1} = (1, 1, 3, 17, 155, \ldots)\) or, in other words, the Genocchi numbers of the first order are defined in the most simple way by the exponential generating function (see, for instance, [1], [2])

\[
\frac{2t}{e^t + 1} = t + \sum_{n \geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.
\]

(1)

These numbers are widely applied in different mathematical areas in generation of various objects. The most known application of these numbers is, similarly to the Euler numbers, enumeration of permutations of the given kind [3]. Some papers by Dumont et al. are dedicated to this direction (see, for instance, [4]–[6]), as well as the later papers by another authors.

The generating function (1) implies that the numbers \(G_{2n}\) are connected to the Bernoulli numbers by relation \(G_{2n} = (-1)^{n+1} 2(4^n - 1)B_{2n}\), these numbers are defined as

\[
\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n \geq 1} B_{2n} \frac{t^{2n}}{(2n)!}.
\]

The Genocchi numbers are contained in some number triangles. As a classic example one can consider Seidel triangle consisting of numbers \(g_{n,j}\), where \(j \geq 1, 1 \leq n \leq (j+1)/2\), which are defined in the following way. Let \(g_{1,1} = 1\) and the rest of the numbers in the triangle be defined by the formulae

\[
g_{n,2j} = \sum_{q \geq n} g_{q,2j-1}, \quad g_{n,2j+1} = \sum_{q \leq n} g_{q,2j}.
\]

In Seidel triangle, the Genocchi numbers are defined as \(G_{2n} = g_{n,2n-1}\). In addition, in Seidel triangle one can find the so-called median Genocchi numbers or the Genocchi numbers of the second kind \((H_{2n-1})_{n \geq 1} = (1, 2, 8, 56, \ldots)\), namely, \(H_{2n-1} = g_{1,2n}\). It is known that these numbers have a useful application in enumerating of various objects. D. Barsky [7] and D. Dumont [4] proved that number \(H_{2n+1}\) is divisible by \(2^n\) for any \(n \geq 0\), and thus, number \(h_n = H_{2n+1}/2^n\) is integer for

*E-mail: svinin@icc.ru
any $n \geq 0$. The numbers $(h_n)_{n \geq 0}$ are called normalized median Genocchi numbers. One can find a good description of the combinatorial sense of these numbers in [8].

When speaking about the Genocchi numbers, we have to mention classic identities with these numbers. The most known identity is the implicit recurrent Seidel relation

$$\left\lfloor \frac{n}{2} \right\rfloor \sum_{j=0}^{\min(n,2)} (-1)^j \binom{n}{2j} G_{2n-2j} = 0 \ \forall n \geq 2.$$ 

It is known that the Genocchi numbers of the first and the second kind are connected by the relation

$$H_{2n-1} = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \binom{n}{2j+1} G_{2n-2j} \ \forall n \geq 1.$$ 

One of the ways to generate the Genocchi numbers, both of the first and the second kind, is connected to the fact that they can be given as values of a sequence of polynomials defined by a recurrent relation for some value of an argument. The objective of the paper is the development and application of this approach. In construction of these polynomials, numbers of Stirling type are used, thus in the next section, we briefly discuss the approach to construct these numbers. In Section 3, we give two examples of a sequence of polynomials defined by the respective numbers of Stirling type and show how they participate in construction of some identities including the Genocchi numbers. Thus, one of the first results of the paper is the construction of new identities with the Genocchi numbers, which generalize the known identities. In Section 4, we introduce a countable class of sequences of polynomials, which uniformly and naturally define a countable class of integer sequences. The first two sequences in this class are the Genocchi numbers of the first and the second kind respectively. We call the rest of the integer sequences belonging to this class the numbers of Genocchi type.

2. THE NUMBERS OF STIRLING TYPE

First, we discuss the general approach to the construction of the numbers of Stirling type [9]. In this approach, two-parametric sets of numbers are defined which are of interest to us. By an arbitrary number sequence $(a_n)_{n \geq 1}$ we define two two-parametric sets of polynomials in variables $(a_1, a_2, \ldots)$ by the relations

$$t(n+1,j) = t(n,j-1) - a_n t(n,j), \quad (2)$$

$$T(n+1,j) = T(n,j-1) + a_j T(n,j) \quad (3)$$

with the condition $T(n,0) = t(n,0) = \delta_{n,0}$, $t(n,j) = T(n,j) = 0$ for $n < j$. it happens that $T(n,j)$ and $t(n,j)$ are symmetric polynomials. One can define them more accurately by means of the generating functions

$$t(t-a_1) \cdots (t-a_{n-1}) = \sum_{j=0}^n t(n,j) t^j, \quad \sum_{n \geq j} T(n,j) t^{n-j} = \frac{1}{1-a_1 t} \cdots \frac{1}{1-a_j t}.$$ 

Using an appropriate number sequence $(a_n)_{n \geq 1}$, we obtain some numbers of Stirling type of the first and the second kind respectively. For instance, the choice $a_n = n$ corresponds to the classic Stirling numbers $s(n,j)$ and $S(n,j)$. In the case of $a_n = n^2$ we obtain the so-called central factorial numbers [2], [10], which we will denote, as in the general case, by the symbols $t(n,j)$, $T(n,j)$. In its turn, if $a_n = n(n+1)$, we obtain the so-called Legendre–Stirling numbers [11], which we denote as $Ls(n,j)$, $LS(n,j)$. 

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3. SEQUENCES OF POLYNOMIALS GENERATING THE GENOCCHI NUMBERS

3.1. Gandhi polynomials. The common property of the Genocchi numbers of the first and the second kind is that they can be obtained as values of the respective polynomials \( F_n(z) \) for some value of the argument. These polynomials are defined by the recurrent relation of the form

\[
F_{n+1}(z) = f(z)F_n(z + 1) - g(z)F_n(z),
\]

starting with \( F_1(z) = 1 \), where \( f(z) \) and \( g(z) \) are some appropriate polynomials of the second degree. For instance, numbers \( G_{2n} \) are defined by Gandhi polynomials [12]–[14], which are given by the recurrent relation

\[
F_{n+1}(z) = (z + 1)^2F_n(z + 1) - z^2F_n(z).
\]  (4)

The first of them have the form

\[
F_2(z) = 2z + 1, \quad F_3(z) = 6z^2 + 8z + 3, \quad F_4(z) = 24z^3 + 60z^2 + 54z + 17,
\]

\[
F_5(z) = 120z^4 + 480z^3 + 762z^2 + 556z + 155, \ldots.
\]

In [13], [14], it was proved that Gandhi polynomials are connected with the Genocchi numbers of the first kind by relations \( F_n(0) = G_{2n} \), \( F_n(1) = G_{2n+2} \) for all \( n \geq 1 \).

Remark 1. In fact, in [12], Gandhi defined polynomials \( A_n(z) \):

\[
A_{n+1}(z) = z^2A_n(z + 1) - (z - 1)^2A_n(z),
\]

starting with \( A_0(z) = 1 \), which are connected with \( F_n(z) \) by relation \( F_n(z) = A_{n-1}(z + 1) \). Polynomials \( F_n(z) \), defined by relation (4), were used, for instance, in [6].

It is straightforward to show [15] that Gandhi polynomials can be represented in the form

\[
zF_n(z) = \sum_{j=0}^{n}(-1)^{n+j}T(n, j)j!(z)^j,
\]  (5)

where \((z)^0 := 1\) and \((z)^n := z(z + 1)\cdots(z + n - 1)\ \forall n \geq 1\), and \( T(n, j) \) are central factorial numbers of the second kind which were mentioned beforehand. To prove the representation (5), it is more convenient to consider polynomials \( P_n(z) := zF_n(z) \) which, due to (4), satisfy the recurrent relation

\[
P_{n+1}(z) = z(z + 1)P_n(z + 1) - z^2P_n(z),
\]

starting with \( P_1(z) = z \), which can be rewritten in the operator form \( P_{n+1}(z) = R(P_n(z)) \), where \( R := z(z + 1)\Lambda - z^2\) and \( \Lambda \) is a shift operator acting by the rule \( \Lambda(f(z)) = f(z + 1) \). To prove the relation (5), one can use the easily justified relation

\[
R((z)^n) = -n^2(z)^n + (n + 1)(z)^{n+1} \ \forall n \geq 0
\]

and the recurrent relation for central factorial numbers of the second kind.

In its turn, using \( F_n(1) = G_{2n+2} \), it is easy to deduce from (5) the known identity [2], [4], [15]

\[
G_{2n+2} = \sum_{j=0}^{n}(-1)^{n+j}T(n, j)(j!)^2 \ \forall n \geq 1.
\]  (6)

Now, we will deduce an identity for numbers \( G_{2n} \), which generalizes (6).

Proposition 1. The values of Gandhi polynomials for integer non-negative values of the argument are defined by the relation

\[
F_n(m)(m!)^2 = \sum_{j=0}^{m}(-1)^{m+j}t(m, j)G_{2n+2j},
\]  (7)

where \( t(n, j) \) are central factorial numbers of the first kind.
Here, we do not prove this proposition since it will be proved later for a more general case. In its turn, (5) implies

$$F_n(m) = \frac{1}{m!} \sum_{j=0}^{n} (-1)^{n+j} T(n,j) j!(j + m - 1)!.$$  (8)

Comparing (7) with (8), we obtain

**Proposition 2.** For all $m \geq 0$, $n \geq 1$, we have the identity

$$\sum_{j=0}^{m} (-1)^{m+j} t(m,j) G_{2n+2j} = m! \sum_{j=0}^{n} (-1)^{n+j} T(n,j) j!(j + m - 1)! \forall m \geq 0.$$  (9)

**Remark 2.** At the first sight, an undefined symbol $(-1)!$ arises in formula (9) for $m = 0$. However, since $T(n,0) = 0$ for all $n \geq 1$, in this case the formula is correct, too, and it can be written as

$$G_{2n} = \sum_{j=1}^{n} (-1)^{n+j} T(n,j)(j - 1)! j! \forall n \geq 1.$$  

### 3.2. Polynomials which generate the Genocchi numbers of the second kind.

In [6] (see also [16]), a sequence $(C_n(z))_{n \geq 1}$ was defined by the relation

$$C_{n+1}(z) = (z + 1)^2 C_n(z + 1) - z(z + 1) C_n(z),$$

starting with $C_1(z) = 1$, and it was shown that $C_n(1) = H_{2n+1} \forall n \geq 1$. It is straightforward to show that there exist polynomials $\tilde{F}_n(z)$ which are connected with $C_n(z)$ by the relation $C_n(z) = (z + 1) \tilde{F}_{n-1}(z + 1)$ for all $n \geq 2$. Clearly, these polynomials are defined by the recurrent relation

$$\tilde{F}_{n+1}(z) = z(z + 1) \tilde{F}_n(z + 1) - (z - 1) z \tilde{F}_n(z),$$  (10)

starting with $\tilde{F}_1(z) = 1$. Some of the first polynomials have the form

$$\tilde{F}_2(z) = 2z, \quad \tilde{F}_3(z) = 6z^2 + 2z, \quad \tilde{F}_4(z) = 24z^3 + 24z^2 + 8z, \quad \tilde{F}_5(z) = 120z^4 + 240z^3 + 192z^2 + 56z, \ldots.$$  

It follows easily from (10) that $\tilde{F}_n(0) = 0 \forall n \geq 2$. If $C_n(1) = H_{2n+1}$, then, obviously, $\tilde{F}_n(2) = H_{2n+1}/2$. In its turn, using the recurrent relation (10), one can calculate the values of polynomials $\tilde{F}_n(z)$ for any natural value of the argument $z$. For example, we can write

$$\tilde{F}_n(1) = \frac{H_{2n-1}}{0!}, \quad \tilde{F}_n(2) = \frac{H_{2n+1}}{1!}, \quad \tilde{F}_n(3) = \frac{2H_{2n+1} + H_{2n+3}}{2!3!},$$

$$\tilde{F}_n(4) = \frac{12H_{2n+1} + 8H_{2n+3} + H_{2n+5}}{3!4!}, \ldots$$

for all $n \geq 1$. We can give a general formula for the values of polynomials defined by the recurrent relation (10).

**Proposition 3.** The values of polynomials $\tilde{F}_n(z)$ for integer non-negative values of the argument are defined by the relation

$$\sum_{j=0}^{m} (-1)^{m+j} L_s(m,j) H_{2n+2j-1} = \tilde{F}_n(m + 1)m!(m + 1)!,$$  (11)

where $L_s(n,j)$ are the Legendre–Stirling numbers of the first kind.
We will not prove this proposition since it is a partial case of a more general statement which will be proved here later.

3.3. Representation of the Genocchi numbers of the second kind in terms of the Legendre–Stirling numbers. In [16], seemingly, the relation

$$H_{2n-1} = \sum_{j=0}^{n} (-1)^{n+j} LS(n, j) (j!)^2 \forall n \geq 0,$$  \hspace{1cm} (12)

was obtained, which is, clearly, similar to representation (6) for the Genocchi numbers of the first kind with the difference that numbers $T(n, j)$ are replaced here by the Legendre–Stirling numbers. Since (6) is deduced from the respective representation for Gandhi polynomials (5), it is natural to suppose that a similar representation exists also for polynomials $\tilde{F}_n(z)$, and this is actually true.

**Proposition 4.** Polynomials $\tilde{F}_n(z)$ are represented in the form

$$z\tilde{F}_n(z) = \sum_{j=0}^{n} (-1)^{n+j} LS(n, j) j! (z)^j.$$  \hspace{1cm} (13)

The proof of (13) is similar to the proof of (5). It is more convenient to use polynomials $\tilde{P}_n(z) := z\tilde{F}_n(z)$, which, clearly, satisfy the recurrent relation

$$\tilde{P}_{n+1}(z) = z^2 \tilde{P}_n(z+1) - (z-1)z \tilde{P}_n(z)$$  \hspace{1cm} (14)

with $\tilde{P}_1(z) = z$. We rewrite this relation in the operator from $\tilde{P}_{n+1}(z) = R(\tilde{P}_n(z))$ with the operator $R = z^2 \Delta - (z-1)z$. It is straightforward to check that

$$R((z)^n) = -(n+1)(z)^n + (n+1)(z)^{n+1} \forall n \geq 0.$$  \hspace{1cm} \Box

In its turn, it follows from (13) that

$$\tilde{F}_n(m+1) = \frac{1}{(m+1)!} \sum_{j=0}^{n} (-1)^{n+j} LS(n, j) j! (j+m)!.$$  \hspace{1cm}

Comparing this quantity with (11), we obtain

**Proposition 5.** For all $m \geq 0$ and $n \geq 1$, we have the identity

$$\sum_{j=0}^{m} (-1)^{m+j} LS(m, j) H_{2n+2j-1} = m! \sum_{j=0}^{n} (-1)^{n+j} LS(n, j) j! (j+m)!.$$  \hspace{1cm} (15)

Thus, we obtained the identity which generalizes (12).

4. GENERAL APPROACH

4.1. Polynomials defined by the numbers of Stirling type. Presently, we have two substantive examples of sequences of polynomials $F_n(z)$ defined by the formula

$$zF_n(z) = \sum_{j=0}^{n} (-1)^{n+j} T(n, j) j! (z)^j,$$  \hspace{1cm} (16)

where $T(n, j)$ are the numbers of Stirling type of the second kind which correspond to some sequence $(a_n)_{n \geq 1}$ and satisfy (3). Note that for these two examples, the sequence of polynomials satisfies a recurrent relation of the form

$$F_{n+1}(z) = h(z)F_n(z+1) - g(z)F_n(z),$$  \hspace{1cm} (17)
starting with $F_1(z) = 1$, where $h(z)$ and $g(z)$ are some polynomials of the second degree, however in a general case a sequence of polynomials of the form (16) may not satisfy any such relation. Taking into account that

$$(z)^n = \sum_{j=0}^{n} (-1)^{n+j} s(n, j) z^j,$$

where $s(n, j)$ are classic Stirling numbers of the first kind, we obtain

$$F_n(z) = \sum_{j=1}^{n} (-1)^{n+j} \left( \sum_{q=j}^{n} q! s(q, j) T(n, q) \right) z^{j-1}. \quad (18)$$

For instance,

$$F_1(z) = 1, \quad F_2(z) = 2z - (T(2, 1) - 2), \quad F_3(z) = 6z^2 - (2T(3, 2) - 18) z + (T(3, 1) - 2T(3, 2) + 12),$$

$$F_4(z) = 24z^3 - (6T(4, 3) - 144) z^2 + (2T(4, 2) - 18T(4, 3) + 264) z$$

$$- (T(4, 1) - 2T(4, 2) + 12T(4, 3) + 144), \ldots .$$

Here we use the fact that $T(n, n) = 1 \forall n \geq 1$.

**4.2. Representation for the Genocchi numbers of the second kind.** Let us make a detour and notice that one can obtain the following representation for the Genocchi numbers of the second kind with the help of (18).

**Proposition 6.** For all $n \geq 2$, we have the identity

$$H_{2n-3} = (-1)^n \sum_{j=2}^{n} LS(n, j) j! s(j, 2) - \delta_{n, 2}.$$

**Proof.** Consider a sequence of polynomials $\tilde{F}_n(z)$ corresponding to the number sequence $a_n = n(n+1)$. As we know, these polynomials satisfy (10). From this relation we obtain

$$\tilde{F}'_{n+1}(z) = (2z + 1) \tilde{F}_n(z + 1) + z(\tilde{F}'_n(z + 1) - (2z - 1) \tilde{F}_n(z) - (z - 1)z \tilde{F}'_n(z)).$$

Hence, in particular, it follows that

$$\tilde{F}_n(0) = \tilde{F}_{n-1}(1) + \tilde{F}_{n-1}(0) = H_{2n-3} + \delta_{n, 2}.$$

In its turn, we have from (18)

$$\tilde{F}_n(0) = \sum_{j=2}^{n} LS(n, j) j! s(j, 2).$$

\[\square\]

**4.3. A recurrent relation.** Consider the operator $R := f(z) \Lambda - g(z)$, where $f(z), g(z)$ is a pair of polynomials of the second degree.

**Proposition 7.** Let $(a_n)_{n \geq 1}$ be an arbitrary number sequence. The relation

$$R ((z)^n) = -a_n(z)^n + (n + 1)(z)^{n+1} \quad \forall n \geq 1 \quad (19)$$

is valid if and only if

$$f(z) = z(z + 4 + a_1 - a_2), \quad g(z) = z^2 + (3 + a_1 - a_2)z + 2 + 2a_1 - a_2, \quad (20)$$

$$a_n = -a_1(n - 2) + a_2(n - 1) + (n - 2)(n - 1) \quad \forall n \geq 3. \quad (21)$$
Proof. On one hand, it is easy to check that the substitution of (20) and (21) into relation (19) results in the identity for any \( n \geq 1 \). On the other hand, let \( f(z) = f_2 z^2 + f_1 z + f_0 \), \( g(z) = g_2 z^2 + g_1 z + g_0 \), where \((f_2, f_1, f_0, g_2, g_1, g_0)\) are six undetermined coefficients. Comparing coefficients at the same degrees of \( z \) in (19) for \( n = 1, 2 \), we obtain a system of linear equations for this coefficients, whence we uniquely determine \( f_2 = 1, f_1 = 4 + a_1 - a_2, f_0 = 0 \) and \( g_2 = 1, g_1 = 3 + a_1 - a_2, g_0 = 2 + 2a_1 - a_2 \). Obviously, relation (19) for \( n \geq 3 \) uniquely defines \( a_n \) for \( n \geq 3 \), and we know that these values are given by expression (21).

By direct calculations, it is straightforward to check that, due to (19), the sequence of polynomials \( P_n(z) = zF_n(z) \) defined by formula (16) satisfies the recurrent relation

\[
P_{n+1}(z) = f(z)P_n(z + 1) - g(z)P_n(z),
\]

starting with \( P_1(z) = z \), where polynomials \( f(z) \) and \( g(z) \) are defined by the formula (20). The case of \((a_1, a_2) = (1, 4)\) corresponds to Gandhi polynomials. In its turn, if \((a_1, a_2) = (2, 6)\), we obtain a sequence of polynomials defined in (10). Note that in both cases, we have the relation \( a_2 = 2a_1 + 2 \), and hence, \( f(z) - g(z) = z \). In this case, the estimation of polynomials \( P_n(z) \) with the help of the recurrent relation (22) can be made starting with \( P_0(z) = 1 \).

4.4. The numbers of Genocchi type. Now, consider a countable class of number sequences \( a_n = n(n + v) \) for integer \( v \geq 0 \). Using formulae (2), (3), for each of such sequence we calculate the respective numbers of Stirling type \( t^{(v)}(m, j), T^{(v)}(m, j) \), and then with the help of them we define a sequence of polynomials \( (F_n^{(v)}(z))_{n \geq 1} \):

\[
zF_n^{(v)}(z) = \sum_{j=0}^{n} (-1)^{n+j} T^{(v)}(n, j) j!(z)^j.
\]

It is clear that, due to Proposition 7, for polynomials \( F_n^{(v)}(z) \) we have the recurrent relation

\[
F_{n+1}^{(v)}(z) = (z + 1)(z - v + 1) F_n^{(v)}(z + 1) - z(z - v) F_n^{(v)}(z),
\]

starting with \( F_1^{(v)}(z) \). In the matter of fact, these are polynomials in two variables \( v \) and \( z \), which can take any complex values. Several first of them have the form

\[
F_2^{(v)}(z) = 2z - v + 1, \quad F_3^{(v)}(z) = 6z^2 - (6v - 8)z + v^2 - 4v + 3,
\]

\[
F_4^{(v)}(z) = 24z^3 - (36v - 60)z^2 + (14v^2 - 60v + 54)z - v^3 + 11v^2 - 27v + 17, \ldots.
\]

Remark 3. It is clear that polynomials \( F_n^{(v)}(z) = zF_n^{(v)}(z) \) for \( n \geq 1 \) satisfy the recurrent relation

\[
P_{n+1}^{(v)}(z) = z(z - v + 1) P_n^{(v)}(z + 1) - z(z - v) P_n^{(v)}(z),
\]

starting with \( P_1^{(v)}(z) = z \). Let \( v \) be an arbitrary complex number. It is easy to check that the sequence \( a_n = n(n + v) \) presents the general case of a sequence of the form (21) with condition \( a_2 = 2a_1 + 2 \). In this case \( f(z) - g(z) = z \), and hence, these polynomials can be esteemed starting with \( P_0^{(v)}(z) = 1 \).

We draw attention to the similarity of formulae expressing identities connected with the Genocchi numbers and the respective polynomials. Note that formulae (7), (11) can be rewritten in the form

\[
\sum_{j=0}^{m} (-1)^{m+j} t^{(v)}(m, j) g_{n+j}^{(v)} = F_n^{(v)}(m + v) \frac{m!(m + v)!}{v!}
\]

for \( v = 0, 1 \). Here \( g_0^{(v)} = G_{2n} \), \( g_1^{(v)} = H_{2n-1} \) for \( n \geq 1 \). First, we note that, if we assume the validity of (25), for any fixed \( v \geq 0 \) numbers \( g_n^{(v)} \) are defined as

\[
g_n^{(v)} = F_n^{(v)}(v) \quad \forall n \geq 1, v \geq 0.
\]
Proposition 8. For any integer \( v \geq 0 \) and numbers defined by the relation (26), formula (25) is valid for all \( m \geq 0 \).

**Proof.** For any fixed \( n \geq 1 \), numbers \( g_n^{(v)} \) can be determined as values of the respective polynomial \( g_n(v) := F_n^{(v)}(v) \) for integer values of the argument \( v \geq 0 \). We need to prove that, for the numbers defined in this way, the relation (25) is valid also for the rest of \( m \geq 1 \). Denote

\[
\xi_{m,n}^{(v)} = \sum_{j=0}^{m} (-1)^{m+j} t^{(v)}(m,j) g_{n+j}^{(v)}.
\]

Numbers \( \xi_{m,n}^{(v)} \) satisfy the equality

\[
\xi_{m+1,n}^{(v)} = \xi_{m,n+1}^{(v)} + m (m + v) \xi_{m,n}^{(v)}
\]

with condition \( \xi_{0,n}^{(v)} = g_{n}^{(v)} \). In particular, from the latter relation, we obtain

\[
\xi_{1,n}^{(v)} = g_{n+1}^{(v)}, \quad \xi_{2,n}^{(v)} = g_{n+2}^{(v)} + (v+1) g_{n+1}^{(v)}, \ldots
\]

Note that (27) is defined only by the fact that the numbers of Stirling type of the first kind \( t^{(v)}(n,j) \), by definition, satisfy the recurrent relation

\[
t^{(v)}(n+1,j) = t^{(v)}(n,j) - n(n+v) t^{(v)}(n,j).
\]

Thus, we need to prove that

\[
\xi_{m,n}^{(v)} = F_n^{(v)}(m + v) \frac{m!(m+v)!}{v!}.
\]

Substituting this expression into (27), we obtain

\[
F_n^{(v)}(m + v + 1)(m + 1)! (m + v + 1) = F_{n+1}^{(v)}(m + v)! (m + v)! + m(m + v) F_n^{(v)}(m + v)! (m + v)!.
\]

Clearly, the latter relation is valid due to (24). \( \square \)

It follows from (23) that

\[
F_n^{(v)}(m + v) = \frac{1}{(m + v)!} \sum_{j=0}^{n} (-1)^{n+j} T^{(v)}(n,j) j! (j + m + v - 1)!
\]

Comparing this formula with (25), we obtain

**Proposition 9.** The relation

\[
\sum_{j=0}^{m} (-1)^{m+j} t^{(v)}(m,j) g_{n+j}^{(v)} = \frac{m!}{v!} \sum_{j=0}^{n} (-1)^{n+j} T^{(v)}(n,j) j! (j + m + v - 1)!
\]

is an identity for any \( m, v \geq 0 \) and \( n \geq 1 \).

Obviously, formula (28) generalizes identities (9), (15) onto the Genocchi numbers, and we can consider a countable class of integer valued sequences \( g_n^{(v)} \). Unfortunately, the combinatorial sense of these numbers for integer \( v \geq 2 \) is not known yet.

**Remark 4.** Note that Propositions 8, 9 do not state themselves that \( g_n^{(0)}, g_n^{(1)} \) are the Genocchi numbers. However, this is true and can be obtained from what is proved in [6], [13], [14].
In the following table, we give some of the numbers $g_n^{(v)}$ for small values of $n$ and $v$:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|
| $g_n^{(0)}$ | 1 | 1 | 3 | 17 | 2073 | 38227 |
| $g_n^{(1)}$ | 1 | 2 | 8 | 56 | 608 | 9440 | 198272 |
| $g_n^{(2)}$ | 1 | 3 | 15 | 123 | 26691 | 623391 |
| $g_n^{(3)}$ | 1 | 4 | 24 | 224 | 59904 | 1531264 |
| $g_n^{(4)}$ | 1 | 5 | 35 | 365 | 116765 | 3229475 |
| $g_n^{(5)}$ | 1 | 6 | 48 | 552 | 206688 | 6131712 |
| $g_n^{(6)}$ | 1 | 7 | 63 | 791 | 340935 | 10774687 |

Some of the first polynomials $g_n^{(v)}$ have the form

$g_1(v) = 1$, $g_2(v) = v + 1$, $g_3(v) = (v + 1)(v + 3)$, $g_4(v) = (v + 1)(v^2 + 10v + 17)$,

$g_5(v) = (v + 1)(v^3 + 25v^2 + 123v + 155)$, $g_6(v) = (v + 1)(v^4 + 56v^3 + 590v^2 + 2000v + 2073)$,

$g_7(v) = (v + 1)(v^5 + 119v^4 + 2362v^3 + 15942v^2 + 42485v + 38227)$, \ldots.

**Proposition 10.** Polynomials $g_n^{(v)}$, starting with $n = 2$, are divisible by $v + 1$.

**Proof.** Substituting $z = v$ into the recurrent relation (24), we obtain

$$g_{n+1}^{(v)} = (v + 1)F_n^{(v)}(v + 1) \forall n \geq 1.$$ 

Since $F_n^{(v)}(v + 1)$ is a polynomial in $v$, this finishes the proof. \qed

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