Optimal unbiased functional filtering in the frequency domain

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(Received 14 November 2013; final version received 1 March 2014)

The functional filtering problem is solved in the frequency domain. From the recently obtained time-domain results, a simple frequency optimal filter is obtained and its design is solved by the spectral factorization. Continuous and -time cases are considered. A numerical example is given to illustrate the presented approach.

Keywords: unbiased filter; algebraic Riccati equation; spectral factorization; frequency domain; innovation sequence

1. Introduction

The problem of optimal linear estimation has received a considerable interest since the publication of the Kalman filter (see references Kailath, Sayed, & Hassibi, 2000; Kwakernaak & Sivan, 1972; Lewis, 1986; Middleton & Goodwin, 1990). This problem is concerned with the reconstruction or the estimation of the states of the system using the input and output measurements, it is of great importance in the optimal control design and fault diagnosis. It is well known from the Kalman–Bucy filter theory that the order of the filter is the same as the order of the system. However, in many practical situations, one may be interested only by a partial state estimation or functional estimation (Oreilly, 1983). Recently, much attention has been given to the reduced-order filter design. The case where the measurements are partially noise free was reported in Haas (1984), Soroka and Shaked (1988), Gessing (1992), and references therein.

The frequency domain method for full-order filtering with noisy or noise-free measurements has been considered in Shaked (1976), Shaked and Soroka (1987), and Bekir (1988). This approach has been shown to be very efficient, it is based on the power spectral density matrix of the noisy system output signal, this can be derived in practice from the output measurements.

In this paper we consider the reduced-order filtering problem presented and treated in Nagpal, Helmick, and Sims (1987), Nakamizo (1997), and recently in Darouach (2000), in the time domain. Starting from the recent results on the optimal unbiased functional filtering (Darouach, 2000), we investigate the frequency domain approach. Using the spectral factorization on the algebraic Riccati equation (ARE) in the s plane for the continuous time case and in the z plane for the -time case, we derive the transfer function of the filter and the properties of its associate innovation sequence. From the results of Bekir (1988), we show that the standard full-order case can be deduced directly from the obtained results.

The main reason of formulating the results of the time domain in the frequency one is the advantages that it presents for the observer-based control design (Hippe & Deutscher, 2009). In fact, in this case, the compensator is driven by the input and the output of the system. So only the input–output behavior of the compensator (characterized by its transfer function) influences the properties of the closed-loop system. The additional degrees of freedom given by the frequency approach can then be used for robustness purpose for example (Hippe & Deutscher, 2009).

2. Continuous-time unbiased functional filter in frequency domain

In this section we shall present a new method for the determination of the transfer function matrix for continuous-time systems. It is based on the spectral factorization of a partial output.

Consider the following time invariant continuous-time system:

\[
\begin{align*}
\dot{x} &= Ax + w, \quad (1a) \\
y &= Cx + v, \quad (1b) \\
Z &= Lx, \quad (1c)
\end{align*}
\]

where \(x \in \mathbb{R}^n\) is the state vector, \(y \in \mathbb{R}^p\) is the measurement output and \(Z \in \mathbb{R}^r\) is the vector to be estimated, with \(r \leq n\). The matrices \(A, C,\) and \(L\) are constant and of appropriate dimensions, without loss of generality, it is assumed that rank \(C = p\) and rank \(L = r\). Let the initial state \(x_0\) be a

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random vector with zero mean and given covariance matrix \( P_0 \). The plant noise and the measurement noise, \( w \) and \( v \), are zero mean white processes with: \( E[w(t)w(\tau)^T] = Q\delta(t-\tau) \) and \( E[v(t)v(\tau)^T] = R\delta(t-\tau) \), where \( \delta \) is the Dirac function. \( Q \) and \( R \) are symmetric positive-semidefinite matrices. We assume that \( x_0, w, \) and \( v \) are mutually uncorrelated.

As in reference Darouach (2000), define the following matrices: \( L_1 = I - L^+L, A_1 = LAL_1, \Pi = (CL_1)^+, T = \alpha(I - (CL_1)(CL_1)^+), Q = LQL^T + A_1R_1A_1^T, S = -A_1RT^T, \Theta = TRT^T, \bar{A} = LAL^+ - A_1, \bar{C} = TCL^+, \bar{A}_1 = \bar{A} - S\Theta^{-1}C, \bar{Q}_s = \bar{Q} - S\Theta^{-1}ST^T, \) and \( A_{1s} = A_1 + S\Theta^{-1}T \), where \( L^+ \) is any generalized inverse of \( L \) satisfying \( LL^+L = L \) and \( \alpha \) is a full row rank matrix such that \( \begin{bmatrix} \bar{Q} \\ \bar{A} \end{bmatrix} \) is of full column rank. In Remark 1 we shall explain how to determine this matrix.

In the sequel we shall make the following assumptions (Darouach, 2000):

(A1) The existence condition rank \( \begin{bmatrix} L_\lambda A \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ L \end{bmatrix} \) and \( \Theta > 0 \) are satisfied,

(A2) The pair \((\bar{C}, \bar{A})\) is detectable or equivalently

\[
\text{rank} \begin{bmatrix} \lambda L - LA \\ C \end{bmatrix} = \text{rank} \begin{bmatrix} LA \\ C \\ L \end{bmatrix},
\]

\( \forall \lambda \in \mathbb{C}, \text{Re}(\lambda) \geq 0. \)

(A3) The pair \((\bar{A}_1, Q_1^{1/2})\) has no unreachable mode on the stability boundary, or equivalently

\[
\text{rank} \begin{bmatrix} AL^+ - j\omega L^+ \\ 0 \\ -CL^+ \\ -CL^+ \end{bmatrix} = n + p,
\]

\( \forall \omega \in \mathbb{R}. \)

These assumptions are necessary to guarantee the convergence and stability of the functional filter.

The optimal unbiased filter which estimate the functional \( Z(t) \), based on the measurement \( y(t), 0 \leq \tau \leq t \), is given by Darouach (2000):

\[
\dot{\hat{Z}} = \bar{A}_s\hat{Z} + K_s(Ty - \bar{C}\hat{Z}) + A_1sv
\]

and the error covariance propagates as the following Riccati differential equation (RDE):

\[
\dot{P} = (\bar{A}_s - K_s\bar{C})P + P(\bar{A}_s - K_s\bar{C})^T + Q_s + K_s\Theta K_s^T
\]

and its associate ARE is given by

\[
0 = (\bar{A}_s - K_s\bar{C})P + P(\bar{A}_s - K_s\bar{C})^T + Q_s + K_s\Theta K_s^T
\]

with the optimal gain matrix \( K_s = P\bar{C}^T\Theta^{-1}. \) Under assumptions A1–A2–A3, the convergence and stability of the filter \( (2) \) are guaranteed and the above ARE has an unique stabilizing solution satisfying

\[
0 = \bar{A}_sP + P\bar{A}_s^T - P\bar{C}^T\Theta^{-1}\bar{C}P + Q_s.
\]

We can make the following remarks concerning the above time-domain functional filtering.

Remark 1 The determination of the parameter \( \alpha \) can be made as follows. Let \( CL_1 = U\begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \) be the singular value decomposition of \( CL_1 \), then we have \((CL_1)^+ = V^T\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix}U^T\)

\[
\begin{bmatrix} \Pi \\ T \end{bmatrix} = (CL_1)^+ = \begin{bmatrix} V^T \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & I \end{bmatrix}U^T \end{bmatrix}^{1/2}
\]

From the last matrix, one can see that one choice of the parameter matrix \( \alpha = [0 I]^TU^T \). In this case the covariance matrix \( \Theta \) which is assumed to be positive definite is given by \( \Theta = [0 I]^TU^TRU \).

Remark 2 In this remark we shall present a geometric interpretation of the functional filtering. In fact, consider system \((1)\), it is easy to see that \( x = L^+Z + L_1d \), where \( d \) is an arbitrary vector, is a general solution to Equation \((1c)\). This is also a division of the state space into a space of dimension \( r \) corresponding to the functional \( Z \) to be estimated and its complement of dimension \( (n-r) \) in terms of projection operators, \( L^+L \) and \( L_1 \), respectively. Inserting the above value of \( x \) in Equations \((1a)\) and \((1b)\) we obtain the following model:

\[
\dot{\hat{Z}} = LAL^+Z + LAL_1d + Lw,
\]

\[
y = CL^+Z + CL_1d + v.
\]

Now, by pre-multiplying Equation \((5b)\) by the full column rank matrix \( \begin{bmatrix} P \end{bmatrix} \) and by using the relation \( TCL_1 = 0 \), we obtain

\[
\dot{\hat{Z}} = LAL^+Z + LAL_1d + Lw,
\]

\[
\Pi y = \Pi CL^+Z + \Pi CL_1d + \Pi v,
\]

\[
Ty = \bar{C}Z + Tv.
\]
\[ \rank \left[ \frac{L_A}{\xi} \right] = \rank \left[ \xi \right], \text{ see Darouach, 2000}, \text{ then we obtain} \]

\[
\begin{align*}
\dot{Z} &= \bar{A}_1 Z + A_{1v} y + A_{1w} v + L w, \\
\Pi v &= \Pi C L^T Z + \Pi C L d + \Pi v, \\
T_y &= \bar{C} Z + T_v.
\end{align*}
\tag{7a,7b,7c}
\]

From Equations (7a) and (7c), one can obtain the presented functional filter (2). However, Equation (7b) is dependent on the unknown parameter vector \( d \), so it can be used to estimate this parameter.

When matrix \( CL_1 \) is of full row rank we have \((CL_1)(CL_1)^T = I\), in this case Equations (6a) and (6b), and the filter equation (2) becomes \( \dot{Z} = \bar{A} \dot{Z} + A_{1v} y \), which leads to the transfer function from \( y \) to \( \dot{Z}, \hat{Z}(s) = H(s)y(s) \), where \( H(s) = (sI - \bar{A})^{-1} A_1 \).

Before providing the method of the determination of the above filter transfer function, we provide the definition of the s-spectrum of the output (7c).

**Definition 1** The s-spectrum of the partial output process \( \{Ty\} \) for system (7) is given by

\[
S_{Ty}(s) = \left[ \frac{\bar{C}}{(sI - \bar{A})} \right] \begin{bmatrix} O_s & 0 \\ 0 & \Theta(sI - \bar{A}_s)^{-1} \bar{C}^T \end{bmatrix}.
\]

Also we have the following definition for the spectral factorization.

**Definition 2** Let \( S(s) \) be the spectral density or the s-spectrum of the signal \( x(t) \), then we can write \( S(s) = W(s) W^T(-s) \) as the spectral factorization of \( S(s) \).

Now, from Equation (2), the transfer function from \( y \) to \( \dot{Z} \) is given by

\[
H(s) = (sI - \bar{A}_s + K_s \bar{C})^{-1} (K_s T + A_{1s}).
\]

The following theorem gives this transfer function from the partial output s-spectrum factorization.

**Theorem 1** Under assumptions A1–A2–A3, the transfer function \( H(s) \) is given by

\[
H(s) = (sI - \bar{A}_s)^{-1} [F \nabla(-s) (T - \bar{C}(sI - \bar{A}_s)^{-1} A_{1s}) + A_{1s}] + A_{1s},
\]

where \( \nabla(s) \) is the square and invertible matrix and its inverse is analytic in the right half of the plane \( s \) with \( \nabla(s) = \bar{C}(sI - \bar{A}_s)^{-1} F + I \), where \( F = P \bar{C}^T \Theta^{-1} \), and \( \nabla(s) \) is the standard transfer function associated to the full-order Kalman filter (Bekir, 1988).

The different steps to compute the transfer function \( H(s) \) are given by the following remark:

**Remark 3** One can see that if \( L = I \), we obtain \( L_1 = 0, T = \alpha = I, \Theta = R, \bar{C} = \bar{C}, A_{1s} = 0, A_1 = 0, S = 0, \bar{A} = A, F = P \bar{C}^T R^{-1} \), and the transfer function of the filter becomes \( H(s) = (sI - A)^{-1} F \nabla(-s) \), with \( \nabla(s) = C(sI - A)^{-1} F + I \), which is the standard transfer function associated to the full-order Kalman filter (Bekir, 1988).

**Remark 4** The procedure of the determination of the transfer function \( H(s) \) can be done as follows:

1. Solve the factorization equation (8) to obtain \( \nabla(s) \).
(2) Solve equation \( \nabla(s) = \tilde{C}(sI - \mathcal{A}_s)^{-1}F + I \), to obtain \( \mathcal{F} \).

(3) Calculate \( H(s) \) from the expression given in Theorem 1.

The innovation process \( v_r(t) \) is defined as the quantity carrying the new information contained in the current observation. From Equation (2) or (7) we can define the quantity \( v_r(t) = Ty(t) - \tilde{C}\mathcal{Z}(t) \), and we have the following theorem.

**Theorem 2** Under assumptions A1, A2, and A3, the transfer function \( H_v(s) \) from \( y \) to \( v_r \) is given by

\[
H_v(s) = \nabla^{-1}(s)[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]
\]  
(9)

and the spectrum of \( v_r(t) \) defined by \( \Phi_{v_r}(j\omega) \) is given by

\[
\Phi_{v_r}(j\omega) = \Theta.
\]  
(10)

**Proof** We have \( v_r(s) = Ty(s) - \tilde{C}\mathcal{Z}(s) = [T - \tilde{C}H(s)]y(s) \), and from the above results it is easy to see that

\[
[T - \tilde{C}H(s)] = T - \tilde{C}(sI - \mathcal{A}_s)^{-1}I \times [\mathcal{F}\nabla^{-1}(s)(sI - \mathcal{A}_s)^{-1}A_{1s} + A_{1s}]
\]

\[
= [T - \tilde{C}(sI - \mathcal{A}_s)^{-1}\mathcal{F}\nabla^{-1}(s)]
\]

\[
\times [T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]
\]

\[
= \nabla^{-1}(s)[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]
\]

which gives

\[
v_r(s) = \nabla^{-1}(s)[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]y(s).
\]

Now, the s-spectrum of the process \( \{v_r(t)\} \) is given by

\[
S_{v_r}(s) = \nabla^{-1}(s)[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]S_y(s)
\]

\[
\times [T^T - A_{1s}^T(-sI - \mathcal{A}_s)^{-1}C^T]\nabla^{-T}(-s),
\]  
(11)

where \( S_y(s) \) is the s-spectrum of the output process \( \{y(t)\} \) given by

\[
S_y(s) = C(sI - A)^{-1}Q(-sI - A)^{-1}CT + R.
\]  
(12)

On the other hand, we have \( TCL_1 = 0 \) or equivalently \( TC = TCL^+L = \tilde{C}L \), then

\[
[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]C = TC - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}
\]

\[
= \tilde{C}L - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}
\]

\[
= (sI - \mathcal{A}_s)^{-1}(sL - \mathcal{A}_sL - A_{1s}C)
\]

and by using the value of \( A_{1s} \) and the fact that \( LAL_1^T = L^T \), we obtain

\[
[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]C = \tilde{C}(sI - \mathcal{A}_s)^{-1}L(sI - A)
\]

and from the fact that \( A_{1s}RA_{1s}^T = A_RA_{1s}^T - S\Theta^{-1}ST \), and \( A_{1s}RT^T = 0 \), we obtain

\[
[T - \tilde{C}(sI - \mathcal{A}_s)^{-1}A_{1s}]R[T^T - A_{1s}^T(-sI - \mathcal{A}_s)^{-1}\tilde{C}^T]
\]

\[
= \Theta + \tilde{C}(sI - \mathcal{A}_s)^{-1}(A_RA_{1s}^T - S\Theta^{-1}ST)
\]

\[
\times (-sI - \mathcal{A}_s)^{-1}\tilde{C}^T.
\]

By using these results and substituting Equation (12) in Equation (11) we obtain \( S_{v_r}(s) = \Theta \), which proves the theorem.

### 3. Discrete-time unbiased functional filter in frequency domain

In this section we present the extension of the above results to the discrete-time systems. The system considered is in the following form:

\[
x(t + 1) = Ax(t) + w(t),
\]  
(13a)

\[
y(t) = Cx(t) + v(t),
\]  
(13b)

\[
Z(t) = Lx(t),
\]  
(13c)

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( y(t) \in \mathbb{R}^p \) is the measurement output and \( Z(t) \in \mathbb{R}^r \) is the vector to be estimated, with \( r \leq n \). The matrices \( A, C, \) and \( L \) are real and of appropriate dimensions. Without loss of generality, it is assumed that \( rank C = p \) and \( rank L = r \). The processes \( w(t) \) and \( v(t) \) are zero-mean and white, uncorrelated with each other and with the initial state of the system \( x_0 \), with:

\[
E\{w(i)w(j)^T\} = Q\delta_{ij} \text{ and } E\{v(i)v(j)^T\} = R\delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker \( \delta \) function (identically zero except when \( i = j \)).

We shall consider the predictor and the estimator filters cases.

#### 3.1. One step predictor filter

This case can be obtained from the continuous case, we shall use the same notations as in the continuous case.

In the sequel we shall make the following adapted assumptions to the discrete-time case:

(A/2) The pair \((\tilde{C}, \mathcal{A}_s)\) is detectable or equivalently

\[
\text{rank } \begin{bmatrix} \mathcal{A}_s - \tilde{C}L \end{bmatrix} = \text{rank } \begin{bmatrix} \mathcal{L} \end{bmatrix}, \forall \lambda \in \mathbb{C}, |\lambda| \geq 1.
\]

(A/3) The pair \((\mathcal{A}_s, Q^{1/2})\) has no unreachable mode on the stability boundary, or

\[
\text{rank } \begin{bmatrix} A\mathcal{L}^+ - \exp(i\omega)L^+ & Q^{1/2} & 0 \\ -\exp(i\omega)CL^+ & 0 & R^{1/2} \end{bmatrix} = n + p,
\]

\( \forall \omega \in [0, 2\pi] \).
Then the optimal unbiased estimate of $Z(t + 1)$, based on the measurement $y(\tau)$, $0 \leq \tau \leq t$, is given by

$$\hat{Z}(t + 1) = \overline{A}_s \hat{Z}(t) + K_s(T)y(t) - \bar{C}\hat{Z}(t) + \overline{A}_s y(t).$$

(14)

The ARE associated with this estimate is

$$P = (\overline{A}_s - K_s\bar{C})P(\overline{A}_s - K_s\bar{C})^T + Q_s + K_s\Theta K_s^T.$$

(15)

Under assumptions A1–A2–A3, the convergence and stability of the filter (14) are guaranteed and the above ARE has an unique stabilizing solution. In this case we obtain $K_s = \overline{A}_sP\bar{C}^{-1}\Psi^{-1}$ with $\Psi = (\bar{C}P\bar{C}^T + \Theta)$. Inserting this value in Equation (15) leads to

$$P = \overline{A}_sP\overline{A}_s^T - \overline{A}_sP\bar{C}^{-1}\Psi^{-1}\bar{C}P\overline{A}_s^T + Q_s.$$

(16)

It is easy to see that the transfer function from $y$ to $\hat{Z}$ is given by $\hat{H}(z) = \overline{H}(z)v(z)$, where

$$H(z) = (zI - \overline{A}_s + K_s\bar{C})^{-1}(K_sT + A_{1s}).$$

As in the continuous case we can give the definition of the z-spectrum of the -time output counter part of Equation (7c).

**Definition 3** The z-spectrum of the partial output process $\{Ty\}$ for the -time system associated to Equation (7) is given by

$$S_{Ty}(z) = \begin{bmatrix} \hat{C}(zI - \overline{A}_s)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (z^{-1}I - \overline{A}_s^{-1})^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

The following theorem gives the transfer function $H(z)$ from the spectral factorization of the z-spectrum of the partial output process $\{Ty\}$.

**Theorem 3** Under assumptions A1–A2–A3, the transfer function $H(z)$ is given by $H(z) = [(zI - \overline{A}_s)^{-1}\overline{G}\Delta^{-1}(z)\Delta^{-1}(z)](T - (zI - \bar{C})^{-1}A_{1s}) + A_{1s}$ where $\Delta(z)$ is a square and invertible matrix and its inverse is analytic outside the unit circle, with $\Delta(z)\Psi\Delta^{-1}(z^{-1}) = S_{Ty}(z)$.

**Proof** First, we have the following equality:

$$P - \overline{A}_sP\overline{A}_s^T = (zI - \overline{A}_s)P(z^{-1}I - \overline{A}_s^T) + (zI - \overline{A}_s)P\overline{A}_s^T + \overline{A}_sP(z^{-1}I - \overline{A}_s^T).$$

Inserting this value in Equation (15), and premultiplying by $\bar{C}(zI - \overline{A}_s)^{-1}$ and postmultiplying by $(z^{-1}I - \overline{A}_s^T)^{-1}\bar{C}^T$ and adding $\Theta$ to the two sides of the obtained equality, and after some algebraic manipulations we obtain the following spectral factorization:

$$\Delta(z)\Psi\Delta^{-1}(z^{-1}) = \Theta + \hat{C}(zI - \overline{A}_s)^{-1}Q_s(z^{-1}I - \overline{A}_s^{-1})^{-1}\hat{C}^T,$$

where $\Delta(z)$ is a square and invertible matrix and its inverse is analytic outside the unit circle, with

$$\Delta(z) = \hat{C}(zI - \overline{A}_s)^{-1}\mathcal{G}\overline{A}_sP\bar{C}^T\Psi^{-1} + I,$$

where $\mathcal{G} = \overline{A}_sP\bar{C}^T\Psi^{-1}$. As in the continuous case, $\Delta(z)$ represents the return difference matrix of the filter (14).

Now the transfer function $H(z)$ can be written as

$$H(z) = [(zI - \overline{A}_s)K_sT + (zI - \overline{A}_s + K_s\bar{C})^{-1}A_{1s}].$$

After some algebraic manipulation we obtain

$$H(z) = [(zI - \overline{A}_s)^{-1}$$

$$\times [\mathcal{G}\Delta^{-1}(z)(T - \bar{C}(zI - \overline{A}_s)^{-1}A_{1s}) + A_{1s}].$$

From Equation (2), we can define the innovation process $v_r = Ty - \hat{C}\hat{Z}$, and we have the following theorem.

**Theorem 4** Under assumptions A1–A2–A3, the transfer function $H_{vu}(z)$ from $y$ to $v_r$ is given by

$$H_{vu}(z) = \Delta^{-1}(z)[T - \hat{C}(zI - \overline{A}_s)^{-1}A_{1s}]$$

and the spectrum of $v_r(t)$ defined by $\Phi_v(e^{j\omega})$ is given by

$$\Phi_{v_r}(e^{j\omega}) = \Psi.$$  

(18)

**Proof** We have $v_r(z) = Ty(z) - \hat{C}\hat{Z}(z) = [T - \hat{C}(zI - \overline{A}_s)^{-1}A_{1s}]\nu(z)$. As in the continuous case we have $v_r(z) = \Delta^{-1}(z)[T - \hat{C}(zI - \overline{A}_s)^{-1}A_{1s}]\nu(z)$. The z-spectrum of the process $v_r(t)$ is given by Kailath et al. (2000)

$$S_{v_r}(z) = \Delta^{-1}(z)[T - \hat{C}(zI - \overline{A}_s)^{-1}A_{1s}]\mathcal{S}_{\nu}(z)$$

$$\times [T^T - A_{1s}^T(z^{-1}I - \overline{A}_s^{-1})^{-1}\hat{C}^T]\Delta^{-T}(z^{-1}),$$

(19)

where $S_{\nu}(z)$ is the z-spectrum of the output $\nu(t)$ given by

$$S_{\nu}(z) = C(zI - A)^{-1}Q(z^{-1}I - A^T)^{-1}C^T + R.$$  

(20)

Now using the same ideas as in Theorem 3 we obtain

$$[T - \hat{C}(zI - \overline{A}_s)^{-1}A_{1s}]\mathcal{R}[T^T - A_{1s}^T(z^{-1}I - \overline{A}_s^{-1})^{-1}\hat{C}^T]$$

$$= \Theta + \hat{C}(zI - \overline{A}_s)^{-1}(A_{1s}R_{1s}^T - S\Theta^{-1}S^T)$$

$$\times (z^{-1}I - \overline{A}_s^{-1})^{-1}\hat{C}^T.$$

Using Equations (20) and (19) we obtain $S_{v_r}(z) = \Psi$, which proves the theorem. ■
Remark 5 If \( L = I \), we obtain \( G = APC^T(PCA_T + R)^{-1} \), and the transfer function of the filter becomes \( H(z) = (zI - A)^{-1}G\Delta^{-1}(z) \), with \( \Delta(z) = C(zI - A)^{-1}G + I \), which is the standard transfer function associated to the full-order Kalman filter.

Remark 6 As in Bekir (1988), we can deduce the singular case from the above results, in fact if \( R = 0 \), then \( \Psi = \hat{C}P\hat{C}^T \), and in this case the spectral factorization becomes \( \Delta(z)\Delta^T(z^{-1}) = \hat{C}(zI - \overline{A})^{-1}Q_s(z^{-1}I - \overline{A}^T)^{-1}\hat{C}^T \), where \( \Delta(z) \) is given by \( \Delta(z) = C(zI - A)^{-1}G + \hat{C} \), and \( G \) is a matrix which can be determined as shown in Bekir (1988).

Remark 7 The procedure of the determination of the transfer function \( H(z) \) can be done as in Remark 3.

3.2. Filter

Define the following matrices Darouach (2000), \( L_1 = I - L^+L, A_{1r} = A_1 + s\Omega^{-1}T, \overline{A} = LAL^+ - A_1CAL^+, \hat{C} = TCAL^+, A_1 = LAL_1P, \Pi = (CAL_1)^+, \) and \( T = \alpha(I - (CAL_1)(CAL_1)^+) \), \( \overline{A}_s = \hat{A} - s\Omega^{-1}C, K_s = K - s\Omega^{-1}, \) and \( Q_s = \overline{Q} - s\Omega^{-1}ST, \) with \( \overline{Q} = (L - A_1C)Q(L - A_1C)^T + A_1RAT, S = [(L - A_1C)QCT - A_1R]^T \) and \( \Omega = T(CQC^T + R)^T \).

We shall make the following assumptions:

- \( A'1 \)-The existence condition rank \( \left[ \frac{C}{L} \right] = \) rank \( \left[ \frac{C}{L} \right] \) and \( \Omega > 0 \) are satisfied.

- \( A'2 \) and \( A'3 \) are also satisfied.

Then the optimal unbiased estimate of \( Z(t + 1) \), based on the measurement \( y(t) \), \( 0 \leq t \leq t + 1 \), is given by Darouach (2000)

\[
\hat{Z}(t + 1) = \overline{A}_s\hat{Z}(t) + \hat{K}_s(Ty(t + 1) - \hat{C}\hat{Z}(t)) + A_{1r}y(t + 1).
\]

The error covariance propagates as the following (Riccati difference equation) RDE:

\[
P(t + 1) = (\overline{A}_s - \hat{K}_s\hat{C})P(t)(\overline{A}_s - \hat{K}_s\hat{C})^T + Q_s + K_s\Omega K_s^T.
\]

We define the reduced innovation process as Darouach (2000)

\[
v_r(t + 1) = Ty(t + 1) - \hat{C}\hat{Z}(t).
\]

The ARE associated with the above RDE is

\[
P = (\overline{A}_s - \hat{K}_s\hat{C})P(\overline{A}_s - \hat{K}_s\hat{C})^T + Q_s + K_s\Omega K_s^T.
\]

Under assumptions \( A'1 \), \( A'2 \), and \( A'3 \), the convergence and stability of the filter (21) are guaranteed and the above ARE has an unique stabilizing solution Darouach, 2000. In this case we obtain

\[
K_s = \overline{A}_sP\overline{C}^T \hat{Y}^{-1}
\]

with \( \hat{Y} = (\hat{C}P\hat{C}^T + \Omega) \). Inserting this value in Equation (24) leads to

\[
P = \overline{A}_sP\overline{C}^T - \overline{A}_sP\overline{C}^T \hat{Y}^{-1} \hat{C}P\overline{C}^T + Q_s.
\]

Remark 8 As in Remark 2, we shall present an intuitive explanation of the estimator filter. In fact, consider system (13), we have

\[
x(t + 1) = Ax(t) + w(t), \quad \text{(26a)}
\]

\[
y(t + 1) = Cx(t) + Cw(t) + v(t + 1), \quad \text{(26b)}
\]

\[
Z(t) = Lx(t). \quad \text{(26c)}
\]

Following the same approach as in Remark 2 and by using the relation \( LAL_1 - LAL_1\Pi CAL_1 = 0 \), from Darouach (2000), this is equivalent to rank \( \left[ \frac{L_A}{A_L} \right] \), we obtain the following model:

\[
Z(t + 1) = \overline{A}_sZ + A_{1r}y(t + 1) + w_s, \quad \text{(27a)}
\]

\[
\Pi y(t + 1) = \Pi CAL_1 Z(t) + \Pi CAL_1 w(t) + \Pi Cw(t) + \Pi v(t + 1), \quad \text{(27b)}
\]

\[
 Ty(t + 1) = \tilde{C}Z(t + 1) + \tilde{v}, \quad \text{(27c)}
\]

where \( w_s = -A_{1r}v(t + 1) + (L - A_{1r}C)w(t) \) and \( \tilde{v} = TCw(t) + TV(t + 1) \) are zero-mean and white processes and uncorrelated with each other, with \( E\{w_s(\cdot)|w_s(\cdot)^T\} = Q_s\delta_{ij} \) and \( E\{\tilde{v}(\cdot)|\tilde{v}(\cdot)^T\} = \Omega_{\tilde{v}}\). From the above results, we can see that the presented functional filter (21) can be obtained directly from Equations (27a) and (27c).

It is easy to see that the transfer function from \( y \) to \( \hat{Z} \) is given by \( \hat{Z}(z) = H(z)y(z) \), where

\[
H(z) = (zI - A_s + K_s\tilde{C})^{-1}(K_sT + A_{1r})z.
\]

Let the z-spectrum of the partial output process \( \{TY\} \) for system (27) be given by

\[
S_{TY}(z) = \tilde{C}(zI - A_s)^{-1}I \left[ \begin{array}{cc} Q_s & 0 \\ 0 & \Omega \end{array} \right] \left[ \begin{array}{c} (z^{-1}I - \overline{A}^T) \tilde{C}^T \\ I \end{array} \right].
\]

Then the following theorem gives the transfer function \( H(z) \).

Theorem 5 Under assumptions \( A'1 \), \( A'2 \), and \( A'3 \), the transfer function \( H(z) \) is given by

\[
H(z) = z[I - \hat{A}_s]^{-1} \times [H\Delta_{A_s}^{-1}(z)T - (zI - A_s)^{-1}A_{1r} + A_{1r}].
\]

where \( \Delta_{A_s}(z) \) is a square and invertible matrix and its inverse is analytic outside the unit circle with
\[ \Delta_1(z) = \hat{C}(zI - \bar{A}_s)^{-1}H + I, \text{ where } H = \bar{A}_sP\hat{C}^TY^{-1}, \text{ and } \Delta_1(z)Y \Delta^{-1}_1(z^{-1}) = S_{\bar{T}_y}(z). \]

**Proof** Can be obtained from the proof of Theorem 3 by replacing \( \Theta \) by \( \Omega \).

Before giving the properties of the innovation process \( v_r(t+1) = Ty(t+1) - \hat{C}\hat{Z}(t) \), we can give the following lemma.

**LEMMA 1** Under assumption \( A'1 \), we have

\[ \hat{C}(zI - \bar{A}_s)^{-1}[L(zI - A) + A_{1s}CA] = \hat{C}L = TCA. \] (28)

**Proof** In fact \( L(zI - A) + A_{1s}CA = zL - LA + A_{1s}CA + S\Omega^{-1}TCA \) and \(-LA + A_{1s}CA + S\Omega^{-1}TCA = -LA + LAL_1\Pi CA + S\Omega^{-1}\hat{C}L \) and from assumption \( A'1 \), we have (Darouach, 2000) \( LAL_1\Pi CA = LAL_1 + LAL_1\Pi CA L \), by using the value of \( \bar{A}_s \), we obtain \(-LA + LAL_1\Pi CA + S\Omega^{-1}\hat{C}L = \bar{A}_sL \) and \( L(zI - A) + A_{1s}CA = (zI - \bar{A}_s) L \).

For the innovation process defined above, we have the following theorem.

**THEOREM 6** Under assumptions \( A'1 \), \( A'2 \), and \( A'3 \), the transfer function \( H_{v_r}(z) \) from \( \gamma \) to \( v_r \) is given by

\[ H_{v_r}(z) = \Delta^{-1}_1(z)[T - \hat{C}(zI - \bar{A}_s)^{-1}A_{1s}] \] (29)

and the spectrum of \( v_r(t) \) defined by \( \Phi_{v_r}(e^{j\omega}) \) is given by

\[ \Phi_{v_r}(e^{j\omega}) = \Upsilon. \] (30)

**Proof** We have \( v_r(z) = (Tz)\gamma(z) - \hat{C}\hat{Z}(z) = (Tz - \hat{C}H(z))\Gamma(z), \) as in Theorem 4 we have \( v_r(z) = \Delta^{-1}_1(z)[T - \hat{C}(zI - \bar{A}_s)^{-1}A_{1s}] \).

Now the z-spectrum of the process \( \{v_r(t)\} \) is given by

\[ S_{v_r}(z) = z\Delta^{-1}_1(z)[T - \hat{C}(zI - \bar{A}_s)^{-1}A_{1s}] S_{\gamma}(z) \]

\[ \times [T^T - A_{1s}^T(z^{-1}I - \bar{A}_s^{-1})\hat{C}^TY^{-1}(z^{-1})], \] (31)

where \( S_{\gamma}(z) \) is the z-spectrum of the output \( y(t) \) given by Equation (22), and from Lemma 1 we have \( z[T - \hat{C}(zI - \bar{A}_s)^{-1}A_{1s}]C = B(z)(zI - A) \), where \( B(z) = \hat{C}(zI - \bar{A}_s)^{-1}[(L - A_{1s}C) - S\Omega^{-1}\hat{C}] + TC \), this leads to \( \Delta_1(z)S_{\gamma}(z)\Delta^{-1}_1(z^{-1}) = B(z)Q^{BT}(z^{-1}) + [\hat{C}(zI - \bar{A}_s)^{-1}A_{1s} - T][R[A_{1s}^T(zI - \bar{A}_s)^{-1}\hat{C}^T - T]], \) which gives after some algebraic manipulations \( \Delta_1(z)S_{\gamma}(z)\Delta^{-1}_1(z^{-1}) = S_{\bar{T}_y}(z) \), and from Theorem 5 we obtain \( S_{v_r}(z) = \Upsilon. \)

4. **A numerical example**

We illustrate the results of the previous sections with the following simple example. Consider the continuous-time system in form of (1) with \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, L = [1 \ 0], Q = I, \) and \( R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \)

From the results of Section 2, we obtain \( L^+ = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, L_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix}, \) \( \Pi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \) \( \alpha = [0 \ 1], T = [-0.5 \ 0.5], \) \( \Theta = 0.25, \) \( \bar{A} = \bar{A}_s = 1, A_{1s} = [0 \ 0], \) \( \hat{Q} = Q, = 2, \) \( S = 0, \) and \( \hat{C} = -0.5. \) Matrix \( P \) can be obtained from Equation (4) which can be written as \( P^2 - 2P - 2 = 0 \) and since \( P > 0, \) we have \( P = 1 + \sqrt{3}, \) then the gain matrix \( K_s = -2P = -2(1 + \sqrt{3}), \) the functional filter (2) is then given by

\[ \hat{Z} = -\sqrt{3}\hat{Z} - (1 + \sqrt{3})(-1 \ 1)\nu. \] (32)

From this equation we can deduce the transfer function \( H(s) \) of the filter

\[ H(s) = \frac{-1 + \sqrt{3}}{s + \sqrt{3}}[\begin{bmatrix} -1 & 1 \end{bmatrix}\nu]. \] (33)

Now, we shall use the procedure of Remark 4. First, from Equation (8) we have \( \nabla(s)\Theta \nabla^T(-s) = S_{\bar{T}_y}(s), \) with

\[ S_{\bar{T}_y}(s) = [\hat{C}(sI - \bar{A}_s)^{-1}I] \begin{bmatrix} Q_0 \\ 0 \end{bmatrix} \begin{bmatrix} -sI - \hat{A}_s^{\top} \end{bmatrix}^{-1}\hat{C}^T \]

\[ = \begin{bmatrix} 0.25(s^2 - 3) \\ s^2 - 1 \end{bmatrix} \]

\[ = \nabla(s)\Theta \nabla^T(-s). \]

Which gives, since \( \Theta = 0.25, \) \( \nabla(s) = (s + \sqrt{3})/(s - 1). \) By solving \( \nabla(s) = \hat{C}(sI - \bar{A}_s)^{-1}F + I, \) to obtain \( F, \) we obtain \( F = K_s = 2(1 + \sqrt{3}), \) from Theorem 1 we obtain the transfer function of the filter

\[ H(s) = \frac{(s - 1)^{-1} -2(1 + \sqrt{3})(s - 1)T}{s + \sqrt{3}} \]

\[ = \frac{-(1 + \sqrt{3})}{s + \sqrt{3}}[\begin{bmatrix} -1 & 1 \end{bmatrix}\nu]. \]

This example shows how we can determine, the transfer function of the filter, directly from the spectral factorization of the partial output \( T_y. \)

5. **Conclusion**

In this paper, the optimal unbiased functional filtering problem in the frequency domain has been studied. Starting from the obtained results on the optimal unbiased functional
filtering in the time domain (Darouach, 2000), a simple frequency optimal filter is presented and its design is solved by the spectral factorization. It generalizes the existing results on the determination of the transfer function of the Kalman filter. Continuous and -time cases are considered.

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