Estimating the Average of a Lipschitz-Continuous Function from One Sample

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Abstract

We study the problem of estimating the average of a Lipschitz continuous function \( f \) defined over a metric space, by querying \( f \) at only a single point. More specifically, we explore the role of randomness in drawing this sample. Our goal is to find a distribution minimizing the expected estimation error against an adversarially chosen Lipschitz continuous function. Our work falls into the broad class of estimating aggregate statistics of a function from a small number of carefully chosen samples. The general problem has a wide range of practical applications in areas as diverse as sensor networks, social sciences and numerical analysis. However, traditional work in numerical analysis has focused on asymptotic bounds, whereas we are interested in the best algorithm. For arbitrary discrete metric spaces of bounded doubling dimension, we obtain a PTAS for this problem. In the special case when the points lie on a line, the running time improves to an FPTAS. Both algorithms are based on approximately solving a linear program with an infinite set of constraints, by using an approximate separation oracle. For Lipschitz-continuous functions over \([0, 1]\), we calculate the precise achievable error as \( 1 - \frac{\sqrt{3}}{2} \approx 0.134 \), which improves upon the \( \frac{1}{4} \) which is best possible for deterministic algorithms.

1 Introduction

One of the most fundamental problems in data-driven sciences is to estimate some aggregate statistic of a real-valued function \( f \) by sampling \( f \) in few places. Frequently, obtaining samples incurs a cost in terms of human labor, computation, energy or time. Thus, researchers face an inherent tradeoff between the accuracy of estimating the aggregate statistic and the number of samples required. With samples a scarce resource, it becomes an important computational problem to determine where to sample \( f \), and how to post-process the samples.

Naturally, there are many mathematical formulations of this estimation problem, depending on the aggregate statistic that we wish to estimate (such as the average, median or maximum value), the error objective that we wish to minimize (such as worst-case absolute error, average-case squared error, etc.), and on the conditions imposed on the function. In this paper, we study algorithms optimizing a worst-case error objective, i.e., we assume that \( f \) is chosen adversarially. Motivated by the applications described below, we use Lipschitz-continuity to impose a “smoothness” condition.

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on $f$. (Note that without any smoothness conditions on $f$, we cannot hope to approximate any aggregate function in an adversarial setting without learning all function values.) That is, we assume that the domain of $f$ is a metric space, and that $f$ is Lipschitz-continuous over its domain. Thus, nearby points are guaranteed to have similar function values.

Here, we focus on perhaps the simplest aggregation function: the average $\bar{f}$. Despite its simplicity, it has many natural applications, such as

1. In sensor networks covering a geographical area, the average of a natural phenomenon (such as temperature or pressure) is frequently one of the most interesting quantities. Here, nearby locations tend to yield similar measurements. Since energy is a scarce resource, it is desirable to sample only a few of the deployed sensors.

2. In population surveys, researchers are frequently interested in the average of quantities such as income or education level. A metric on the population may be based on job similarity, which would have strong predictive value for these quantities. Interviewing a subject is time-consuming, and thus sample sizes tend to be much smaller than the entire population.

3. In numerical analysis, one of the most fundamental problems is numerical integration of a function. If the domain is continuous, this corresponds precisely to computing the average. If the function to be integrated is costly to evaluate, then again, it is desirable to sample a small number of points.

If $f$ is to be evaluated at $k$ points, chosen deterministically and non-adaptively, then previous work [4] shows that the optimum sampling locations for estimating the average of $f$ form a $k$-median of the metric space. However, the problem becomes significantly more complex when the algorithm gets to randomize its choices of sampling locations. In fact, even the seemingly trivial case of $k = 1$ turns out to be highly non-trivial, and is the focus of this paper. Addressing this case is an important step toward the ultimate goal of understanding the tradeoffs between the number of samples and the estimation error.

Formally, we thus study the following question: Given a metric space $\mathcal{M}$, a randomized sampling algorithm is described by (1) a method for sampling a location $x \in \mathcal{M}$ from a distribution $p$; (2) a function $g$ for predicting the average $\bar{f}$ of the function $f$ over $\mathcal{M}$, using the sample $(x, f(x))$. The expected estimation error is then $E(p, g, f) = \sum_{x \in \mathcal{M}} p_x \cdot |g(x, f(x)) - \bar{f}|$. (The sum is replaced by an integral, and $p$ by a density, if $\mathcal{M}$ is continuous.) The worst-case error is $E_w(p, g) = \sup_{f \in L} E(p, g, f)$, where $L$ is the set of all 1-Lipschitz continuous functions defined on $\mathcal{M}$. Our goal is to find a randomized sampling algorithm (i.e., a distribution $p$ and function $g$, computable in polynomial time) that (approximately) minimizes $E_w(p, g)$.

In this paper, we provide a PTAS for this problem of minimizing $E_w(p, g)$, for any discrete metric space $\mathcal{M}$ with constant doubling dimension. (This includes constant-dimensional Euclidean metric spaces.) For discrete metric spaces $\mathcal{M}$ embedded on a line, we improve this result to an FPTAS. Both of these algorithms are based on a linear program with infinitely many constraints, for which an approximate separation oracle is obtained.

We next study the perhaps simplest variant of this problem, in which the metric space is the interval $[0, 1]$. While the worst-case error of any deterministic algorithm is obviously $\frac{1}{4}$ in this case, we show that for a randomized algorithm, the bound improves to $1 - \sqrt{2}$. We prove this by providing an explicit distribution, and obtaining a matching lower bound using Yao’s Minimax Principle. Our
result can also be interpreted as showing how “close” a collection of Lipschitz-continuous functions on \([0,1]\) must be.

1.1 Related Work

Estimating the integral of a smooth function \(f\) using its values at a discrete set of points is one of the core problems in numerical analysis. The tradeoffs between the number of samples needed and the estimation error bounds have been investigated in detail under the name of Information Based Complexity (IBC) \([10, 11]\). More generally, IBC studies the problem of computing approximations to an operator \(S(f)\) on functions \(f\) from a set \(F\) (with certain “smoothness” properties) using a finite set of samples \(N(f) = [L_1(f), L_2(f), \ldots, L_n(f)]\). The \(L_i\) are functionals. For a given algorithm \(U\), its error is \(E(U) = \sup_{f \in F} \|S(f) - U(f)\|\). The goal in IBC is to find an \(\epsilon\)-approximation \(U\) (i.e., ensuring that \(E(U) \leq \epsilon\)) with least information cost \(c(U) = n\).

One of the common problems in IBC is multivariate integration of real-valued functions with a smoothness parameter \(r\) over \(d\)-dimensional unit balls. For such problems, Bakhvalov \([2]\) designed a randomized algorithm providing an \(\epsilon\)-approximation with cost \(\Theta(\frac{1}{\epsilon^2 d/(d + 2r)})\). Bakhvalov \([2]\) and Novak \([9]\) also show that this cost is asymptotically optimal. The papers by Novak \([9]\) and Mathe \([7]\) show that if \(r = 0\), then simple Monte-Carlo integration algorithms (which sample from the uniform distribution) have an asymptotically optimal cost of \(\frac{1}{\epsilon^2}\).

In \([13, 14]\), Wozniakowski studied the average case complexity of linear multivariate IBC problems, and derived conditions under which the problems are tractable, i.e., have cost polynomial in \(\frac{1}{\epsilon}\) and \(d\). Wojtaszczyk \([12]\) proved that the multivariate integration problem is not strongly tractable (polynomial in \(\frac{1}{\epsilon}\) and independent of \(d\)).

In \([3]\), Baran et al. study the IBC problem in the univariate integration model for Lipschitz continuous functions, and formulate approximation bounds in an adaptive setting. That is, the sampling strategy can change adaptively based on the previously sampled values. They provide deterministic and randomized \(\epsilon\)-approximation algorithms, which, for any problem instance \(P\), use \(O(\log(\frac{1}{\epsilon \text{OPT}}) \cdot \text{OPT})\) samples for the deterministic case and \(O(\text{OPT}^{4/3} + \text{OPT} \cdot \log(\frac{1}{\epsilon}))\) samples for the randomized case. Here, \(\text{OPT}\) is the optimal number of samples for the problem instance \(P\). They prove that their algorithms are asymptotically optimal, compared to any other adaptive algorithm.

There are two main differences between the results in IBC and our work: first, IBC treats the target approximation as given and the number of samples as the quantity to be minimized. Our goal is to minimize the expected worst-case error with a fixed number of samples (one). More importantly, results in IBC are traditionally asymptotic, ignoring constants. For a single sample, this would trivialize the problem: it is implicit in our proofs that sampling at the metric space’s median is a constant-factor approximation to the best randomized algorithm.

The deterministic version of our problem was studied previously in \([4]\). There, it was shown that the best sampling locations for reading \(k\) values non-adaptively constitute the optimal \(k\)-median of the metric space. Thus, the algorithm of Arya et al. \([1]\) gives a polynomial-time \((3 + \epsilon)\)-approximation algorithm to identify the best \(k\) values to read.
2 Preliminaries

We are interested in real-valued Lipschitz-continuous functions over metric spaces of constant doubling dimension (e.g., [6]). Let \((\mathcal{M}, d)\) be a compact metric space with distances \(d(x, y)\) between pairs of points. W.l.o.g., we assume that \(\max_{x,y \in \mathcal{M}} d(x, y) = 1\). We require \((\mathcal{M}, d)\) to have constant doubling dimension \(\beta\), i.e., for every \(\delta\), each ball of diameter \(\delta\) can be covered by at most \(c^\beta\) balls of diameter \(\delta/c\), for any \(c \geq 2\).

A real-valued function \(f\) is Lipschitz-continuous (with constant 1) if \(|f(x) - f(y)| \leq d(x, y)\) for all points \(x, y\). We define \(L\) to be the set of all such Lipschitz-continuous functions \(f\), i.e., \(L = \{f \mid |f(x) - f(y)| \leq d(x, y)\}\) for all \(x, y\)\}. Since we will frequently want to bound the function values, we also define \(L_c = \{f \in L \mid \int x f(x) dx \leq c\}\). Notice that \(L_c\) is a compact set.

We wish to predict the average \(\overline{f} = \int x f(x) dx\) of all the function values. When \(\mathcal{M}\) is finite of size \(n\), then the average is of course \(\overline{f} = \frac{1}{n} \sum x f(x)\) instead. The algorithm first gets to choose a single point \(x\) according to a (polynomial-time computable) density function \(p\); it then learns the value \(f(x)\), and may post-process it with a prediction function \(g(x, f(x))\) to produce its estimate of the average \(\overline{f}\). The goal is to minimize the expected estimation error of the average, assuming \(f\) is chosen adversarially from \(L\) with knowledge of the algorithm, but not its random choices. Formally, the goal is to minimize \(E_w(p, g) = \sup_{f \in L} (\int x p_x \cdot |\overline{f} - g(x, f(x))| dx)\). If \(\mathcal{M}\) is finite, then \(p\) will be a probability distribution instead of a density, and the error can now be written as \(E_w(p, g) = \sup_{f \in L} (\int x p_x \cdot |\overline{f} - g(x, f(x))| dx)\).

Formally, we consider an algorithm to be the pair \((p, g)\) of the distribution and prediction function. Let \(\mathcal{A}\) denote the set of all such pairs, and \(\mathcal{D}\) the set of all deterministic algorithms, i.e., algorithms for which \(p\) has all its density on a single point. Our analysis will make heavy use of Yao’s Minimax Principle [8]. To state it, we define \(\mathcal{L}\) to be the set of all probability distributions over \(L\). We also define the estimation error \(\Delta(f, A) = \int x p_x \cdot |\overline{f} - g(x, f(x))| dx\), where \(A\) corresponds to the pair \((p, g)\).

Theorem 2.1 (Yao’s Minimax Principle [8])

\[
\sup_{g \in \mathcal{L}} \inf_{A \in \mathcal{D}} E_{f \sim q} |\Delta(f, A)| = \inf_{A \in \mathcal{A}} \sup_{f \in L} \Delta(f, A).
\]

We first show that without loss of generality, we can focus on algorithms whose post-processing is just to output the observed value, i.e., algorithms \((p, id)\) with \(id(x, y) = y\), for all \(x, y\). When \(g\) is the identity function, we simply write \(\Delta(f, p) = \int x p_x \cdot |\overline{f} - f(x)| dx\) for the error incurred by using the distribution \(p\).

Theorem 2.2 Let \(A^* = (p^*, g^*)\) be the optimum randomized algorithm. Then, for every \(\epsilon > 0\), there is a randomized algorithm \(A = (p, id)\), such that \(E_w(A) \leq E_w(A^*) + \epsilon\).

Proof. Let \(\mathcal{A}^l\) denote the set of all (randomized) algorithms using the identity function for post-processing, i.e., \(\mathcal{A}^l = \{A = (p, id) \mid p\) is a distribution over \(\mathcal{M}\}\}.

For the analysis, we are interested in equivalence classes of functions; we say that \(f, f'\) are equivalent if either (1) there exists a constant \(c\) such that \(f(x) = c + f'(x)\) for all \(x\), or (2) there is a constant \(c\) such that \(f(x) = c - f'(x)\) for all \(x\). Let \(\hat{f}\) denote the equivalence class of \(f\), i.e., the set of all vertical translations of \(f\) and its vertical reflection.
Given $\epsilon$, we let $r$ be a large enough constant defined below. A distribution $q$ over $L_r$ is called equivalence-uniform if the distribution, restricted to any equivalence class, is uniform. That is, for any $f' \in \tilde{f}$ with $f', f \in L_r$, we have $q(f') = q(f)$. Let $U_r$ denote the set of all equivalence-uniform distributions over $L_r$. We will show two facts:

1. If $q \in U_r$, then for any deterministic algorithm $A \in D$, there is a deterministic algorithm $A' \in D \cap A^I$ which outputs simply the value it sees, such that

$$E_{f \sim q}[\Delta(f, A')] \leq E_{f \sim q}[\Delta(f, A)] + \epsilon/2.$$  

2. For any distribution $q \in L$ of Lipschitz-continuous functions, there is an equivalence-uniform distribution $q' \in U_r$ (where $r$ may depend on $q$) such that

$$\inf_{A \in D \cap A^I} E_{f \sim q}[\Delta(f, A)] \leq \inf_{A \in D \cap A^I} E_{f \sim q'}[\Delta(f, A)] + \epsilon/2.$$  

Using these two inequalities, and applying Yao’s Minimax Theorem twice then completes the proof as follows:

$$\inf_{A \in A'} \sup_{f \in L} \Delta(f, A) = \sup_{q \in L} \inf_{A \in D \cap A^I} E_{f \sim q}[\Delta(f, A)] \leq \sup_{q \in U_r} \inf_{A \in D \cap A^I} E_{f \sim q}[\Delta(f, A)] + \epsilon/2 \\ \leq \sup_{q \in U_r} \inf_{A \in D} E_{f \sim q}[\Delta(f, A)] + \epsilon \\ \leq \sup_{q \in L} \inf_{A \in D} E_{f \sim q}[\Delta(f, A)] + \epsilon \\ = \inf_{A \in A'} \sup_{f \in L} \Delta(f, A) + \epsilon.$$

It thus remains to show the two inequalities. We begin with Inequality (1). Let $x$ be the point at which $A$ samples the function. For any function $f \in L_r$ let $f$ be the “flipped” function around $x$, defined by $\tilde{f}(y) = 2f(x) - f(y)$ for all $y$. Let $L$ be the set of all functions $f$ such that both $f$ and $\tilde{f}$ are in $L_r$. Because $\tilde{f} = 2f(x) - f$ and $|f(x) - \tilde{f}| \leq \frac{1}{2}$ for all $x$, we obtain that $L \supseteq L_{r-1}$. Also, because $\tilde{f} \in \tilde{f}$ and $q \in U_r$, we have $q(f) = q(\tilde{f})$ whenever $f \in L$. We thus obtain that

$$E_{f \sim q}[\Delta(f, A)] = \int_{L} |g(x, f(x)) - \tilde{f}|q(f)df + \int_{L} |g(x, f(x)) - \tilde{f}|q(f)df \\ \geq \frac{1}{2} \int_{L} (|g(x, f(x)) - \tilde{f}| + |g(x, \tilde{f}(x)) - \tilde{f}|)q(f)df \\ = \frac{1}{2} \int_{L} (|g(x, f(x)) - \tilde{f}| + |g(x, f(x)) - \tilde{f}|)q(f)df \\ \geq \frac{1}{2} \int_{L} |\tilde{f} - \tilde{f}|q(f)df.$$  

1 Unfortunately, this definition does not extend to $L$, since $\tilde{f}$ is not bounded, and a uniform distribution is thus not defined. This issue causes the $\epsilon$ terms in the theorem.
For the first inequality, we dropped the second integral, and used the symmetry of the distribution to write the first integral twice and then regroup. The second step used that by definition, \( f(x) = f(x) \), and the third the inverse triangle inequality. By definition, \( f(x) \) lies between \( \bar{f} \) and \( \underline{f} \); therefore, \( |\underline{f} - \bar{f}| = |\underline{f} - f(x)| + |f(x) - \bar{f}| \), and we can further bound

\[
\mathbb{E}_{f \sim q}[\Delta(f, A)] \geq \frac{1}{2} \int_L |\underline{f} - f(x)| + |f(x) - \bar{f}| q(f) df \\
= \int_L |\underline{f} - f(x)| q(f) df \\
\geq \int_{L_r} |\underline{f} - f(x)| q(f) df - \epsilon/2,
\]

because symmetry of \( q \) implies that \( \text{Prob}[f \notin L] \leq 1/r \leq \epsilon/2 \) (we will set \( r \geq 2/\epsilon \)), and Lipschitz continuity implies that \( |\underline{f} - f(x)| \leq 1 \) for all \( x \).

Next, we prove Inequality (2). Let \( q \) be an arbitrary distribution, and \( r \) large enough such that \( \text{Prob}_{f \sim q}[f \notin L_r] \leq \epsilon/2 \). First, we truncate \( q \) to a distribution \( q'' \) over \( L_r \): we set \( q''(f) = 0 \) for all \( f \notin L_r \), and renormalize by setting \( q''(f) = \frac{1}{\text{Prob}_{f \sim q}[f \in L_r]} \cdot q(f) \) for \( f \in L_r \). Next, we define a distribution \( \bar{q} \) over equivalence classes \( \bar{f} \) as \( \bar{q}(\bar{f}) = \int_{f' \in \bar{f}} q''(f') df' \); finally, let \( q' \) be defined by choosing an equivalence class \( \bar{f} \) according to \( \bar{q} \), and subsequently choosing a member of \( \bar{f} \cap L_r \) uniformly at random; clearly, \( q' \) is equivalence-uniform. Let \( A \in \arg\min_{A \in \mathcal{D} \cap A'} \mathbb{E}_{f \sim q}[\Delta(f, A)] \) be a deterministic algorithm with identity post-processing function minimizing the expected estimation error for \( q' \); let \( x \) be the point at which \( A \) samples the function.

Since the algorithm always samples at \( x \) and outputs \( f'(x) \), the estimation error \( |f'(x) - \bar{f}| \) is the same for all \( f' \in \bar{f} \), because all these \( f' \) are simply shifted or mirrored from each other. If used instead on the initial distribution \( q \), \( A \) has expected estimation error

\[
\int_L |f(x) - \bar{f}| q(f) df \leq \text{Prob}_{f \sim q}[f \notin L_r] + \int_{L_r} |f(x) - \bar{f}| q''(f) df \\
\leq \epsilon/2 + \int_{f' \in \bar{f}} |f'(x) - \bar{f}| q(f') df' d\bar{f} \\
= \epsilon/2 + \int |f(x) - \bar{f}| \bar{q}(\bar{f}) d\bar{f} \\
= \epsilon/2 + \int_{f' \in \bar{f}} |f'(x) - \bar{f}| q'(f') df' d\bar{f} \\
= \epsilon/2 + \int_{L_r} |f(x) - \bar{f}| q'(f) df.
\]

The inequality in the first step came from upper-bounding the estimation error outside \( L_r \) by 1, and using that \( q''(f) \geq q(f) \).

### 3 Discrete Metric Spaces

In this section, we focus on finite metric spaces, consisting of \( n \) points. Thus, instead of integrals and densities, we will be considering sums and probability distributions. The characterization of
using the identity function for post-processing from Theorem 2.2 holds in this case as well; hence, without loss of generality, we assume that all algorithms simply output the value they observe. The problem of finding the best probability distribution for a single sample can be expressed as a linear program, with variables $p_x$ for the sampling probabilities at each of the $n$ points $x$, and a variable $Z$ for the estimation error.

Minimize $Z \
subject to (i) \sum_x p_x = 1 
(ii) \sum_x p_x \cdot |f - f(x)| \leq Z \quad \text{for all } f \in L 
(iii) 0 \leq p_x \leq 1 \quad \text{for all points } x 

Since this LP (which we refer to as the “exact LP”) has infinitely many constraints, our approach is to replace the set $L$ in the second constraint with a set $Q_\delta$. We will choose $Q_\delta$ carefully to ensure that it “approximates” $L$ well, and such that the resulting LP below (which we refer to as the “discretized LP”) can be solved efficiently.

Minimize $\hat{Z} 
subject to (i) \sum_x p_x = 1 
(ii) \sum_x p_x \cdot |f - f(x)| \leq \hat{Z} \quad \text{for all } f \in Q_\delta 
(iii) 0 \leq p_x \leq 1 \quad \text{for all points } x 

To define the notion of approximation formally, let $o$ be a 1-median of the metric space, i.e., a point minimizing $\sum_x d(o, x)$. Let $m = \frac{1}{n} \sum_x d(o, x)$ be the average distance of all points from $o$.

Because we assumed w.l.o.g. that $\max_{x, y \in M} d(x, y) = 1$, at least one point has distance at least $\frac{1}{2}$ from $o$, and therefore, $m \geq \frac{1}{2n}$. The median value $m$ forms a lower bound for randomized algorithms in the following sense.

Lemma 3.1 The worst-case expected error for any randomized algorithm is at least $\frac{1}{4 \cdot 6^\beta} \cdot m$, where $\beta$ is the doubling dimension of the metric space.

Proof. Consider any randomized algorithm with probability distribution $p$; w.l.o.g., the algorithm outputs the value it observes. Let $R = \{x | \frac{m}{2} \leq d(o, x) \leq \frac{3m}{2}\}$ be the ring of points at distance between $\frac{m}{2}$ and $\frac{3m}{2}$ from $o$. We distinguish two cases:

1. If $\sum_{x \in R} p_x \leq \frac{1}{2}$, then consider the Lipschitz-continuous function defined by $f(x) = d(x, o)$. This function has average $\overline{f} = m$. With probability at least $\frac{1}{2}$, the algorithm samples a point outside $R$, and thus outputs a value outside the interval $[\frac{m}{2}, \frac{3m}{2}]$, which incurs error at least $\frac{m}{2}$. Thus, the expected error is at least $\frac{m}{4}$.

2. If $\sum_{x \in R} p_x > \frac{1}{2}$, then consider a collection of balls $B_1, \ldots, B_k$ of diameter $\frac{m}{2}$ covering all points in $R$. Because $R$ is contained in a ball of diameter $3m$, the doubling constraint implies that $k \leq 6^\beta$ balls are sufficient. At least one of these balls — say, $B_1$ — has $\sum_{x \in B_1} p_x \geq \frac{1}{2k}$. Fix an arbitrary point $y \in B_1$, and define the Lipschitz-continuous function $f$ as $f(x) = d(x, y)$. Because $o$ was a 1-median, we get that $\overline{f} \geq m$. With probability at least $\frac{1}{2k}$, the algorithm will choose a point inside $B_1$ and output a value of at most $\frac{m}{2}$, thus incurring an error of at least $\frac{m}{2}$. Hence, the expected error is at least $\frac{1}{2k} \cdot \frac{m}{2} \geq \frac{1}{4 \cdot 6^\beta} \cdot m$. 


We now formalize our notion for a set of functions $Q_\delta$ to be a good approximation.

**Definition 3.2 (δ-approximating function classes)** For any sampling distribution $p$, define $E_1(p) = \max_{f \in L} \Delta(f, p)$ and $E_2(p) = \max_{f \in Q_\delta} \Delta(f, p)$ to be the maximum error of sampling according to $p$ against a worst-case function from $L$ and $Q_\delta$, respectively. The class $Q_\delta$ is said to δ-approximate $L$ if the following two conditions hold:

1. For each $f \in L$, there is a function $f' \in Q_\delta$ such that $|\Delta(f', p) - \Delta(f, p)| \leq \frac{\delta}{2} \cdot E_1(p)$, for all distributions $p$.

2. For each $f \in Q_\delta$, there is a function $f' \in L$ such that $|\Delta(f', p) - \Delta(f, p)| \leq \frac{\delta}{2} \cdot E_1(p)$, for all distributions $p$.

**Theorem 3.3** Assume that for every $\delta$, $Q_\delta$ is a class of functions δ-approximating $L$, such that the following problem can be solved in polynomial time (for fixed $\delta$): Given $p$, find a function $f \in Q_\delta$ maximizing $\Delta(f, p)$.

Then, solving the discretized LP [4] instead of the exact LP [3] gives a PTAS for the problem of finding a sampling distribution that minimizes the worst-case expected error.

**Proof.** First, an algorithm to find a function $f \in Q_\delta$ maximizing $\sum_x p_x \cdot |\overline{f} - f(x)|$ gives a separation oracle for the discretized LP. Thus, using the Ellipsoid Method (e.g., [5]), an optimal solution to the discretized LP can be found in polynomial time, for any fixed $\delta$.

Let $p$, $q$ be optimal solutions to the exact and discretized LPs, respectively. Let $f_1 \in L$ maximize $\sum_x q_x \cdot |\overline{f} - f(x)|$ over $f \in L$, and $f_2 \in Q_\delta$ maximize $\sum_x p_x \cdot |\overline{f} - f(x)|$ over $f \in Q_\delta$. Thus, $\Delta(f_1, q) = E_1(q)$ and $\Delta(f_2, p) = E_2(p)$.

Now, applying the first property from Definition 3.2 to $f_1 \in L$ gives us a function $f'_1 \in Q_\delta$ such that $|\Delta(f'_1, q) - E_1(q)| \leq \frac{\delta}{2} E_1(q)$. Since $E_2(q) \geq \Delta(f'_1, q)$, we obtain that $E_2(q) \geq E_1(q)(1 - \frac{\delta}{4})$.

Similarly, applying the second property from Definition 3.2 to $f_2 \in Q_\delta$, gives us a function $f'_2 \in L$ with $|\Delta(f'_2, p) - E_2(p)| \leq \frac{\delta}{2} E_1(p)$. Since $E_1(p) \geq \Delta(f'_2, p)$, we have that $E_1(p) \geq E_2(p) - \frac{\delta}{4} E_1(p)$, or $E_1(p) \geq \frac{E_2(p)}{1 + \frac{\delta}{4}}$. Also, by optimality of $q$ in $Q_\delta$, $E_2(q) \leq E_2(p)$. Thus, we obtain that $E_1(q) \leq \frac{E_2(q)}{1 - \frac{\delta}{2}} \leq \frac{E_2(p)}{1 - \frac{\delta}{2}} \leq \frac{E_2(p)(1 + \frac{\delta}{4})}{1 - \frac{\delta}{2}} \leq E_1(p)(1 + 2\delta)$.

In light of Theorem 3.3, it suffices to exhibit classes of functions δ-approximating $L$ for which the corresponding optimization problem can be solved efficiently. We do so for metric spaces of bounded doubling dimension and metric spaces that are contained on the line.

### 3.1 A PTAS for Arbitrary Metric Spaces

We first observe that since the error for any translation of a function $f$ is the same as for $f$, we can assume w.l.o.g. that $f(o) = 0$ for all functions $f$ considered in this section. Thus, in this section, we implicitly restrict $L$ to functions with $f(o) = 0$.

We next describe a set $Q_\delta$ of functions which δ-approximate $L$. Roughly, we will discretize function values to different multiples of $\gamma$, and consider distance scales that are different multiples of $\gamma$. We later set $\gamma = \frac{\delta}{256 \cdot 6 + 6}$. We then show in Lemma 3.4 that $Q_\delta$ has size $n^{\log(2(1+\gamma)/\gamma)(2/\gamma)^3} = n^{O(1)}$ for constant $\delta$; the discretized LP can therefore be solved in time $O(poly(n) \cdot n^{\log(2(1+\gamma)/\gamma)(2/\gamma)^3})$
(using exhaustive search for the separation oracle), and we obtain a PTAS for finding the optimum distribution for arbitrary metric spaces.

We let \( k = \log_2 \frac{1}{2m} \), and define a sequence of \( k \) rings of exponentially decreasing diameter around \( o \), that divide the space into \( k + 1 \) regions \( R_1, \ldots, R_{k+1} \). Specifically, \( R_{k+1} = \{ x \mid d(x, o) \leq 2m \} \), and \( R_i = \{ x \mid 2^{-i} < d(x, o) \leq 2^{-i+1} \} \) for \( i = 1, \ldots, k \). Notice that because \( m \geq \frac{1}{2n} \), we have that \( k \leq \log n \) suffices to obtain a disjoint cover.

Since the metric space has doubling dimension \( \beta \), each region \( R_i \) can be covered with at most \((2/\gamma)\beta\) balls of diameter \( 2\gamma \cdot 2^{-i} \). Let \( B_{i,j} \) denote the \( j \)th ball from the cover of \( R_i \); without loss of generality, each \( B_{i,j} \) is non-empty and contained in \( R_i \) (otherwise, consider its intersection with \( R_i \) instead). We call \( B_{i,j} \) the \( j \)th grid ball for region \( i \). Thus, the grid balls cover all points, and there are at most \((2/\gamma)^\beta \cdot \log n\) grid balls. See Figure 1 for an illustration of this cover.

For each grid ball \( B_{i,j} \), let \( o_{i,j} \in B_{i,j} \) be an arbitrary, but fixed, representative of \( B_{i,j} \). The exception is that for the grid ball containing \( o \), \( o \) must be chosen as the representative. We now define the class \( Q_\delta \) of functions \( f \) as follows:

1. For each \( i, j \), \( f(o_{i,j}) \) is a multiple of \( \gamma \cdot 2^{-i} \).

2. For all \( (i, j), (i', j') \), the function values satisfy the following relaxed Lipschitz-condition:
   \[ |f(o_{i,j}) - f(o_{i',j'})| \leq d(o_{i,j}, o_{i',j'}) + \gamma \cdot (2^{-i} + 2^{-i'}) \]

3. All points in \( B_{i,j} \) have the same function value, i.e., \( f(x) = f(o_{i,j}) \) for all \( x \in B_{i,j} \).

![Figure 1: Covering with grid balls](image)

We first show that the size of \( Q_\delta \) is polynomial in \( n \).

**Lemma 3.4** The size of \( Q_\delta \) is at most \( n^{\log(2(1+\gamma)/\gamma)(2/\gamma)^\beta} \).
Proof. Because of the first and second constraint in the definition of $Q_δ$, each point $o_{i,j}$ can take on at most $\frac{d(o_{i,j},o)+2\cdot 2^{-i}}{\gamma^2 2^{-i}} \leq \frac{2(1+\gamma)2^{-i}}{\gamma^2 2^{-i}} = \frac{2(1+\gamma)}{\gamma}$ distinct values. Setting the values for all $o_{i,j}$ uniquely determines the function $f$; the relaxed Lipschitz condition will result in some of these functions not being in $Q_δ$, and thus only decreases the number of possible functions. Because there are at most $(2/\gamma)^{\beta} \cdot \log n$ grid balls, there are at most $(2(1+\gamma)/\gamma)(2/\gamma)^{\beta} \cdot \log n = n \log(2(1+\gamma)/\gamma)(2/\gamma)^{\beta}$ functions in $Q_δ$.

We need to prove that $Q_δ$ approximates $L$ well, by verifying that for each function $f \in L$, there is a “close” function in $Q_δ$, and vice versa. We first show that for any function satisfying the relaxed Lipschitz condition, we can change the function values slightly and obtain a Lipschitz continuous function.

Lemma 3.5 For each $x \in \mathcal{M}$, let $s_x$ be some non-negative number. Assume that $f$ satisfies the “relaxed Lipschitz condition” $|f(x) - f(y)| \leq d(x,y) + s_x + s_y$ for all $x, y$. Then, there is a Lipschitz continuous function $f' \in L$ such that $|f(x) - f'(x)| \leq s_x$ for all $x$.

Proof. We describe an algorithm which runs in iterations $\ell$, and sets the value of one point $x$ per iteration. $S_\ell$ denotes the set of $x$ such that $f'(x)$ has been set. We maintain the following two invariants after the $\ell$th iteration:

1. $f'$ satisfies the Lipschitz condition for all pairs of points in $S_\ell$, and $|f'(x) - f(x)| \leq s_x$ for all $x \in S_\ell$.

2. For any function $f''$ satisfying the previous condition, $f'(x) \leq f''(x)$ for all $x \in S_\ell$.

Initially, this clearly holds for $S_0 = \emptyset$. And clearly, if it holds after iteration $n$, the function $f'$ satisfies the claim of the lemma.

We now describe iteration $\ell$. For each $x \notin S_{\ell-1}$, let $t_x = \max_{y \in S_{\ell-1}} (f'(y) - d(x,y))$. We show below that for all $x$, we have $t_x \leq f(x) + s_x$. Let $x \notin S_{\ell-1}$ be a point maximizing $\max_{x \in S_{\ell-1}} (f(x) - s_x, t_x)$, and set $f'(x) = \max(f(x) - s_x, t_x)$. It is easy to verify that this definition satisfies both parts of the invariant.

It remains to show that $t_x \leq f(x) + s_x$ for all points $x \notin S_{\ell-1}$. Assume that $t_x > f(x) + s_x$ for some point $x$. Let $x_1$ be the point in $S_{\ell-1}$ for which $t_x = f'(x_1) = d(x,x_1)$. By definition, either $f'(x_1) = f(x_1) - s_{x_1}$, or there is an $x_2$ such that $f'(x_1) = f'(x_2) = d(x_1, x_2)$. In this way, we obtain a chain $x_1, \ldots, x_r$ such that $f'(x_i) = f'(x_{i+1}) - d(x_i, x_{i+1})$ for all $i < r$, and $f'(x_r) = f(x_r) - s_{x_r}$. Rearranging as $f'(x_{i+1}) - f'(x_i) = d(x_i, x_{i+1})$, and adding all these equalities for $i = 1, \ldots, r$ gives us that $f(x_r) - f'(x_1) = s_{x_r} + \sum_{i=1}^{r-1} d(x_i, x_{i+1})$. By assumption, we have $f'(x_1) - d(x,x_1) = t_x > f(x) + s_x$. Substituting the previous equality, rearranging, and applying the triangle inequality gives us that

$$f(x_r) - f(x) > s_x + s_{x_r} + d(x,x_1) + \sum_{i=1}^{r-1} d(x_i, x_{i+1}) \geq s_x + s_{x_r} + d(x, x_r),$$

which contradicts the relaxed Lipschitz condition for the pair $x, x_r$.

We now use Lemma 3.5 to obtain, for any given $f \in Q_δ$, a function $f' \in L$ close to $f$.

Lemma 3.6 Let $f \in Q_δ$. There exists an $f' \in L$ such that $|\Delta(f, p) - \Delta(f', p)| \leq \frac{\delta}{2} \cdot E_1(p)$, for all distributions $p$. 

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Proof. Because $f \in Q_\delta$, it must satisfy, for all $(i, j)$ and $(i', j')$, the relaxed Lipschitz condition $|f(o_{ij}) - f(o_{i'j'})| \leq d(o_{ij}, o_{i'j'}) + \gamma (2^{-i} + 2^{-i'})$. Now, we apply Lemma 3.1 with $s_{o_{ij}} = \gamma \cdot 2^{-i}$ to get function values $f'(o_{ij})$ for all $i, j$ that satisfy the Lipschitz condition and the condition that $|f'(o_{ij}) - f(o_{ij})| \leq \gamma \cdot 2^{-i}$. For any other point $x$, let $L_{max}(x, f) = \min_{i, j}(f'(o_{ij}) + d(x, o_{ij}))$ and $L_{min}(x, f) = \max_{i, j}(f'(o_{ij}) - d(x, o_{ij}))$, and set $f'(x) = \frac{1}{2}(L_{max}(x, f) + L_{min}(x, f))$. It is easy to see that $L_{min}(x, f) \leq L_{max}(x, f)$ for all $x$, and that this definition gives a Lipschitz continuous function $f'$. For a point $x \in B_{i, j}$, triangle inequality, the above construction, and the fact that $B_{i, j}$ has diameter at most $2\gamma \cdot 2^{-i}$ imply that

$$
|f'(x) - f(x)| \leq |f'(x) - f'(o_{ij})| + |f'(o_{ij}) - f(o_{ij})| + |f(o_{ij}) - f(x)|
$$

$$
\leq 2\gamma \cdot 2^{-i} + \gamma \cdot 2^{-i} + 0
$$

$$
= 3\gamma \cdot 2^{-i}.
$$

For each point $x$, let $i(x)$ be the index of the region $i$ such that $x \in R_i$. Now, using the triangle inequality and Lemma 3.1, we can bound

$$
|\overline{f'} - \overline{f}| \leq \frac{1}{n} \cdot \sum_{x} |f'(x) - f(x)|
$$

$$
\leq \frac{1}{n} \cdot \sum_{x} 3\gamma \cdot 2^{-i(x)}
$$

$$
\leq \frac{1}{n} \cdot (\sum_{x \notin R_{k+1}} 3\gamma \cdot d(x, o) \cdot \sum_{x \in R_{k+1}} 3\gamma \cdot m)
$$

$$
\leq \frac{1}{n} \cdot (3\gamma nm + 3\gamma nm)
$$

$$
\leq 24 \cdot 6^\beta \cdot \gamma \cdot E_1(p).
$$

Similarly, we can bound

$$
\sum_{x} p_x \cdot |f'(x) - f(x)| \leq 3\gamma \cdot (\sum_{x \notin R_{k+1}} p_x \cdot d(x, o) \cdot \sum_{x \in R_{k+1}} p_x \cdot m)
$$

$$
\leq 3\gamma \cdot (m + \sum_{x \notin R_{k+1}} p_x \cdot d(x, o)).
$$

Let $f''$ be defined as $f''(x) = d(x, o)$. Clearly, $f'' \in L$, $\overline{f''} = m$, and the estimation error for $p$ when the input is $f''$ is

$$
\Delta(f'', p) = \sum_{x} p_x \cdot |f''(x) - m| \geq \sum_{x \notin R_{k+1}} p_x \cdot |d(x, o) - m| \geq (\sum_{x \notin R_{k+1}} p_x \cdot d(x, o)) - m.
$$

Combining these observations, and using Lemma 3.1 and the fact that $\Delta(f'', p) \leq E_1(p)$, we get

$$
\sum_{x} p_x \cdot |f'(x) - f(x)| \leq 6\gamma \cdot m + 3\gamma \Delta(f'', p) \leq (8 \cdot 6^\beta + 1) \cdot 3\gamma \cdot E_1(p).
$$

Now, by using the fact that $|\Delta(f, p) - \Delta(f', p)| \leq |\overline{f'} - \overline{f}| + \sum_{x} p_x \cdot |f'(x) - f(x)|$, and setting $\gamma = \frac{\delta}{48 \cdot 6^\beta + 6}$, we obtain the desired bound.

Finally, we need to analyze the converse direction.

Lemma 3.7 Let $f \in L$. There exists an $f' \in Q_\delta$ such that $|\Delta(f, p) - \Delta(f', p)| \leq \frac{\delta}{2} \cdot E_1(p)$, for all distributions $p$. 

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Proof. The proof is similar to that of Lemma 3.6. First, for each grid ball representative \( o_{i,j} \), we let \( f'(o_{i,j}) \) be \( f(o_{i,j}) \), rounded down (up for negative numbers) to the nearest multiple of \( \gamma \cdot 2^{-i} \). Then, for all points \( x \in B_{i,j} \), we set \( f'(x) = f'(o_{i,j}) \). Clearly, the resulting function \( f' \) is in \( Q_\delta \).

By a similar argument as before, \( |f'(x) - f(x)| \leq 3\gamma \cdot 2^{-i(x)} \), for all points \( x \). Thus, we immediately get \( |f' - f| \leq 24 \cdot 6^2 \cdot \gamma \cdot E_1(p) \) as well.

Define the function \( f'' \) exactly as in the proof of Lemma 3.6. Then, exactly the same bounds as in that proof apply, and give us the claim.

### 3.2 An FPTAS for points on a line

In this section, we show that if the metric consists of a discrete point set on the line, then the general PTAS of the previous section can be improved to an FPTAS.

Since we assumed the maximum distance to be 1, we can assume w.l.o.g. that the points are \( 0 = x_1 \leq x_2 \leq \cdots \leq x_n = 1 \). Also, because w.l.o.g. the post-processing is the identity function, we only need to consider functions \( f \in L_0 \), i.e., such that \( \sum_i f(x_i) = 0 \). We define \( \gamma = \frac{\delta}{14n} \), and the class \( Q_\delta \) to contain the following functions \( f \):

1. For each \( i \), \( f(x_i) \) is a multiple of \( \gamma \).
2. The function values satisfy the relaxed Lipschitz-condition \( |f(x_i) - f(x_j)| \leq d(x_i, x_j) + \gamma \) for all \( i, j \).
3. The sum is “close to 0”, in the sense that \( \sum_i f(x_i) \leq n\gamma \).

We first establish that, given a probability distribution \( p \), a function \( f \in Q_\delta \) maximizing \( \sum_i p_{x_i} \cdot |f(x_i)| \) can be found in polynomial time using Dynamic Programming. To set up the recurrence, let \( a[j, t, s] \) be the maximum expected error that can be achieved with function values at \( x_1, \ldots, x_j \), under the constraints that \( f(x_j) = t \) and \( \sum_{i \leq j} f(x_i) = s \). Then, we obtain the recurrence

\[
a[1, t, s] = \begin{cases} \frac{p_{x_1} \cdot t}{\gamma} & \text{if } s = t \\ -\infty & \text{otherwise}\end{cases}
\]

\[
a[j+1, t, s] = \max_{y \in [-\gamma n \gamma, \gamma n \gamma]} \max_{s, t \in [-1, 1]} |p_{x_{j+1}}| \max_{a[n, t, s]}\left( a[n, t, s] \right) + a[j, y, s-t] \]

The maximizing value is then \( \max_{s \in [-\gamma n \gamma, \gamma n \gamma]} \max_{a[n, t, s]}\left( a[n, t, s] \right) \). The total number of entries is \( O(n \cdot \frac{1}{\gamma}) \), and each entry requires time \( O(\min(1, \frac{1}{\gamma})) \) to compute. The overall running time is thus \( O(n \cdot \frac{1}{\gamma}) = O(n \frac{1}{\gamma}) \), giving us an FPTAS.

All we need now is to show that \( Q_\delta \) \( \delta \)-approximates \( L \). We use the following lemma:

#### Lemma 3.8
For each \( f \in L \), there is a function \( f' \in Q_\delta \) such that \( |\Delta(f, p) - \Delta(f', p)| \leq 2\gamma \) for all distributions \( p \). Also, for each \( f \in Q_\delta \), there is a function \( f' \in L \) such that \( |\Delta(f, p) - \Delta(f', p)| \leq 6\gamma \) for all distributions \( p \).

**Proof.** For the first part, define \( f' \) by rounding each \( f(x_i) \) down (toward 0 for negative values) to the nearest multiple of \( \gamma \). Clearly, \( f' \in Q_\delta \). Furthermore, the average changes by at most \( \gamma \), and \( \sum_x p_x f'(x) - f(x) \leq \sum_x p_x \gamma = \gamma \).

For the second part, first create a Lipschitz continuous function \( f'' \) from \( f \) according to Lemma 3.5, then define \( f'(x) = f''(x) - f'' \) for all \( x \). The first step changed each function value by at most \( \gamma \), and because \( f'' - f'' \leq f'' + f'' \leq 2\gamma \). Thus, \( f'(x) - f(x) \leq 3\gamma \) for all \( x \). Thus, \( |f' - f| + \sum_x p_x \cdot |f'(x) - f(x)| \leq 6\gamma \).
By Lemma 3.1 applied with $\beta = 1$, any randomized algorithm must have expected error at least $\frac{1}{12n}$. In particular, substituting the definition of $\gamma = \frac{\delta}{144n}$ gives us that $6\gamma \leq \frac{1}{2} \cdot E_1(p)$ for all distributions $p$. Thus, $Q_\delta$ approximates $L$ well.

4 Sampling in the Interval $[0, 1]$

In this section, we focus on what is probably the most basic version of the problem: the metric space is the interval $[0, 1]$. In this continuous case, we can explicitly characterize the optimum sampling distribution and estimation error. It is easy to see (and follows from a more general result in [4]) that the best deterministic algorithm samples at $\frac{1}{2}$ and outputs the value read. The worst-case error of this algorithm is $\frac{1}{4}$. We prove that randomization can lead to the following improvement.

**Theorem 4.1** An optimal distribution that minimizes the worst-case expected estimation error is to sample uniformly from the interval $[2 - \sqrt{3}, \sqrt{3} - 1]$. This sampling gives a worst-case error of $1 - \frac{\sqrt{3}}{2} \approx 0.134$.

Following the discussion in Section 2, we restrict our analysis w.l.o.g. to functions $f \in L_0$, i.e., we assume that $\int_0^1 f(x)dx = 0$. Then, the expected error of a distribution $p$ against input $f$ is $\Delta(f, p) = \int_0^1 p_x |f(x)|dx$. The key part of the proof of Theorem 4.1 is to show that when the algorithm samples uniformly over an interval $[c, 1 - c]$, then with loss of only an arbitrarily small $\epsilon$, we can focus on functions consisting of just two line segments.

**Theorem 4.2** For any $b$, define $f_b(x) = \frac{1}{2} + b^2 - b - |b - x|$. If $p$ is uniform over $[c, 1 - c]$ where $c = 2 - \sqrt{3}$, then for every $\epsilon > 0$, there exists some $b = b(\epsilon)$ such that for all functions $f \in L_0$, we have $\Delta(f_b, p) \geq \Delta(f, p) - \epsilon$.

All of Section 4.1 is devoted to the proof of Theorem 4.2. Here, we show how to use Theorem 4.2 to prove the upper bound from Theorem 4.1.

Let $c = 2 - \sqrt{3}$, so that the algorithm samples uniformly from $[c, 1 - c]$. Let $\epsilon > 0$ be arbitrary; we later let $\epsilon \to 0$. Let $b = b(\epsilon)$ be the value whose existence is guaranteed by Theorem 4.2. We distinguish two cases:

1. If $b \leq c$, then

\[
\Delta(f_b, p) = \frac{1}{1 - 2c} \cdot \int_c^{1-c} |\frac{1}{2} + b^2 - b - |b - x||dx
\]

\[
= \frac{1}{1 - 2c} \cdot \left(\frac{1}{2}(b^2 + \frac{1}{2} - c)^2 + \frac{1}{2}(1 - c - b^2)^2\right)
\]

\[
= \frac{1}{1 - 2c} \cdot (b^4 + (\frac{1}{2} - c)^2).
\]

2. If $b \geq c$, then

\[
\Delta(f_b, p) = \frac{1}{1 - 2c} \cdot \int_c^{1-c} |\frac{1}{2} + b^2 - b - |b - x||dx
\]
The first formula is increasing in $b$, and thus maximized at $b = c$; at $b = c$, the value equals that of the second formula, so the maximization must happen for $b \geq c$. A derivative test shows that it is maximized for $b = \frac{\sqrt{3} - 1}{2}$, giving an error of $1 - \sqrt{\frac{3}{2}}$. By Theorem 4.2, for any function $f$, the error is at most $1 - \sqrt{\frac{3}{2}} + \epsilon$, and letting $\epsilon \to 0$ now proves an upper bound of $1 - \sqrt{\frac{3}{2}}$ on the error of the given distribution.

Next, we prove optimality of the uniform distribution over $[2 - \sqrt{3}, \sqrt{3} - 1]$, by providing a lower bound on all randomized sampling distributions. Again, by Theorem 2.2, we focus only on algorithms which output the value $f(x)$ after sampling at $x$, by incurring an error $\epsilon > 0$ that can be made arbitrarily small. Our proof is based on Yao’s Minimax principle: we explicitly prescribe a distribution $q$ over $L_0$ such that for any deterministic algorithm using the identity function, the expected estimation error is at least $1 - \sqrt{\frac{3}{2}}$. Since a deterministic algorithm is characterized completely by its sampling location $x$, this is equivalent to showing that $E_{f \sim q}[|f(x)|] \geq 1 - \sqrt{\frac{3}{2}}$ for all $x$.

We let $b = \frac{\sqrt{3} - 1}{2}$, and define two functions $f, f'$ as $f(x) = \frac{1}{2} + b^2 - b - |x - b|$ and $f'(x) = f(1 - x)$. The distribution $q$ is then simply to choose each of $f$ and $f'$ with probability $\frac{1}{2}$. Fix a sampling location $x$; by symmetry, we can restrict ourselves to $x \leq \frac{1}{2}$. Because $f = f'$, the expected estimation error is

$$\frac{1}{2}(|f(x)| + |f'(x)|) = \frac{1}{2}(|\frac{1}{2} + b^2 - b - |x - b|| + |\frac{1}{2} + b^2 - b - |1 - x - b||)$$

$$= \begin{cases} \frac{1}{2} - b, & \text{if } x \leq b \\ \frac{1}{2} - x, & \text{if } b \leq x \leq \frac{1}{2} - b^2 \\ b^2, & \text{if } \frac{1}{2} - b^2 \leq x \leq \frac{1}{2}. \end{cases}$$

This function is clearly non-increasing in $x$, and thus minimized at $x = \frac{1}{2}$, where its value is $b^2 = 1 - \sqrt{\frac{3}{2}}$. Thus, even at the best sampling location $x = \frac{1}{2}$, the error cannot be less than $1 - \sqrt{\frac{3}{2}}$. This completes the proof of Theorem 4.1.

Notice that the proof of Theorem 4.1 has an interesting alternative interpretation. For a (finite) multiset $S \subseteq L_0$ of Lipschitz continuous functions $f$ with $\int_S f(x)dx = 0$, we say that $S$ is $\delta$-close if there exist $x, y$ such that $\frac{1}{n} \sum_{f \in S} |f(x) - y| \leq \delta$. In other words, the average distance of the functions from a carefully chosen reference point is at most $\delta$. Then, the proof of Theorem 4.1 implies:

**Theorem 4.3** Every set $S \subseteq L_0$ is $(1 - \sqrt{\frac{3}{2}})$-close, and this is tight.

### 4.1 Proof of Theorem 4.2

We begin with the following lemma which guarantees that we can focus on functions $f$ with finitely many zeroes.
Lemma 4.4 For any $\epsilon > 0$ and any function $f$, there exists a function $f'$ such that there are at most $O(1/\epsilon)$ points $x$ with $f'(x) = 0$, and $\Delta(f', p) \geq \Delta(f, p) - \epsilon$, for all distributions $p$.

Proof. Let $\epsilon > 0$ be arbitrary. We prove the lemma by modifying $f$ to ensure that it meets the requirements, and showing that its estimation error decreases by at most $\epsilon$ in the process.

We replace $f$ with a function $f'$ with the following properties: (1) $f'$ is Lipschitz continuous, (2) $\int_0^1 \! f(x) \, dx = \int_0^1 \! f'(x) \, dx$, (3) $|f(x) - f'(x)| \leq \epsilon$ for all $x$, and (4) for each $j = 1, \ldots, 1/\epsilon$, the set $Z_j^f = \{ x \in [(j-1)\epsilon, j\epsilon] | f'(x) = 0 \}$ contains at most three points. The error can change by at most $\epsilon$ due to the third condition, and the fourth condition ensures the bound on the number of zeroes.

To describe the construction, first focus on one interval $[(j-1)\epsilon, j\epsilon]$, and define $x^- = \inf Z_j^f$, $x^+ = \sup Z_j^f$, and $\delta = x^+ - x^-$. Now let $\alpha = \frac{\delta^2 + 4f_{\epsilon^+}(x)dx}{45}$, and define the function $f'$ such that

$$f'(x) = \begin{cases} 
\alpha - |x^- + \alpha - x|, & \text{if } x \in [x^-, x^- + 2\alpha] \\
\alpha - \delta/2 + |x^+ + \alpha - \delta/2 - x|, & \text{if } x \in [x^- + 2\alpha, x^+] \\
f(x), & \text{if } x \in [(j-1)\epsilon, j\epsilon] \setminus [x^-, x^+].
\end{cases}$$

Intuitively, this replaces the function on the interval by a zigzag shape with the same integral that has the same leftmost and rightmost zero.

Do this for each $j$. By the careful choice of $\alpha$, the integral remains unchanged. Because each function value changes by at most $\delta \leq \epsilon$, the third condition is satisfied; the fourth condition is directly by construction, and Lipschitz continuity is obvious.

Next, we show a series of lemmas restricting the functions $f$ under consideration. When we say that $f$ has a certain property without loss of generality, we mean that changing $f$ to $f'$ with that property can be accomplished while ensuring that $\Delta(f', p) \geq \Delta(f, p)$ for all uniform distributions $p$ over intervals $[c, 1-c]$. Since our goal is to characterize the functions that make the algorithm’s error large, this restriction is indeed without loss of generality.

We focus on points $x \in (c, 1-c)$ with $f(x) = 0$. Let $c \leq z_1 \leq \ldots \leq z_k \leq 1-c$ be all such points. For ease of notation, we write $z_0 = c$ and $z_{k+1} = 1-c$. By continuity, $f(x)$ has the same sign for all $x \in (z_i, z_{i+1})$, for $i = 0, \ldots, k$. We show that w.l.o.g., $f$ is as large as possible over areas of the same sign.

Lemma 4.5 Assume w.l.o.g. that $f(x) \geq 0$ for all $x \in [z_i, z_j]$, with $j > i$. Then, w.l.o.g., $f$ maximizes the area over $[z_i, z_j]$ subject to the Lipschitz constraint and the function values at $z_i$ and $z_j$. More formally, w.l.o.g., $f$ satisfies,

1. If $1 \leq i < j \leq k$, then $f(x) = \min(x - z_i, z_j - x)$ for all $x \in [z_i, z_j]$.

2. If $i = 0$, then $f(x) = \min(f(c) + (x-c), z_1 - x)$ for all $x \in [c, z_1]$, and if $i = k$, then $f(x) = \min(f(1-c) + (1-c) - x, x - z_k)$ for all $x \in [z_k, 1-c]$.

Proof. We prove the first part here (the proof of the second part is analogous). Define a function $f'$ as $f'(x) = \min(x - z_i, z_j - x)$ for $x \in [z_i, z_j]$, and $f'(x) = f(x)$ otherwise. Let $f'' = f' - f'$, so
that \( f'' \) is renormalized to have integral 0. Since \( f'(x) \geq f(x) \) for all \( x \), and \( \overline{f} = 0 \), we have that \( \overline{f} \geq 0 \). Then

\[
\int_c^{1-c} |f''(x)| - |f(x)| \, dx \\
= \int_{z_1}^{z_2} |f''(x)| - |f(x)| \, dx + \int_c^{z_1} |f(x) - \overline{f}| - |f(x)| \, dx + \int_{z_j}^{1-c} |f(x) - \overline{f}| - |f(x)| \, dx \\
\geq \int_{z_1}^{z_2} (|f'(x) - \overline{f}| - |f'(x)|) + (|f'(x)| - |f(x)|) \, dx - (1 - 2c - (z_j - z_i)) \overline{f} \\
\geq \int_{z_1}^{z_2} f'(x) - f(x) \, dx - (1 - 2c) \overline{f} \\
= 2c \cdot \overline{f} \\
\geq 0.
\]

Thus, the estimation error of \( f'' \) is at least as large as the one for \( f \), so w.l.o.g., \( f \) satisfies the statement of the lemma.

\[ \text{Lemma 4.6} \quad \text{w.l.o.g., } k \leq 2, \text{ i.e., there are at most two points } x \in (c, 1 - c) \text{ such that } f(x) = 0. \]

\[ \text{Proof.} \quad \text{Assume that } f(z_1) = f(z_2) = f(z_3) = 0. \text{ Consider mirroring the function on the interval } [z_1, z_3]. \text{ Formally, we define } f'(x) = f(z_3 - (x - z_1)) \text{ if } x \in [z_1, z_3], \text{ and } f'(x) = f(x) \text{ otherwise.} \]

Clearly, \( f' \) is Lipschitz continuous and has the same average and same expected estimation error as \( f \). However, the signs of \( f' \) on the intervals \( [c, z_1] \) and \( [z_1, z_1 + z_3 - z_2] \) are now the same; similarly for the intervals \( [z_1 + z_3 - z_2, z_3] \) and \( [z_3, 1 - c] \). Thus, applying Lemma 4.5, we can further reduce the number of points \( x \) with \( f(x) = 0 \), without decreasing the estimation error.

Hence, it suffices to focus on functions \( f \) that have at most two points \( z \in (c, 1 - c) \) with \( f(z) = 0 \). We distinguish three cases accordingly:

1. If there is no point \( z \in (c, 1 - c) \) with \( f(z) = 0 \), then \( f(c) \) and \( f(1 - c) \) have the same sign; without loss of generality, \( f \) is negative over \( (c, 1 - c) \). Then, the expected error is maximized when \( \int_0^c f(x) \, dx \) and \( \int_{1-c}^1 f(x) \, dx \) are as positive as possible, subject to the Lipschitz condition and the constraint that \( \int_0^1 f(x) \, dx = 0 \). Otherwise, we could increase the value of \( \int_0^c f(x) \, dx \) and \( \int_{1-c}^1 f(x) \, dx \), and then lower the function to restore the integral to 0. By doing this, the expected estimation error cannot decrease. Thus, by Lemma 4.5, \( f \) is of the form \( f(x) = |x - b| + f(b) \), where \( b = \arg\min_{x \in (c, 1-c)} f(x) \).

2. If there is exactly one point \( z \in (c, 1 - c) \) with \( f(z) = 0 \), then \( f(c) \) and \( f(1 - c) \) have opposite signs. Without loss of generality, assume that \( f(c) > 0 > f(1 - c) \) and that \( z \leq \frac{1}{2} \). (Otherwise, we could consider \( f'(x) = f(1 - x) \) instead.)

The expected error is maximized when \( f(c) \) is as large as possible, and \( \int_z^{1-c} f(x) \, dx \) is as negative as possible, subject to the Lipschitz condition and the constraint that \( \int_0^1 f(x) \, dx = 0 \). Because \( z \leq \frac{1}{2} \) and the integral of the function \( f'(x) = z - x \) is thus negative, by starting from
$f'$, then raising the function in the interval $[1 - c, 1]$ and, if necessary, increasing $f'(1 - c)$, it is always possible to ensure that $f(x) = z - x$ for all $x \in [0, z]$. Then, $\int_{1-c}^{1} f(x)dx$ is as negative as possible if, for some value $b$, $f$ is of the following form: $f(x) = -(x - z)$ for $x \leq b$, and $f(x) = -(b - z) + (x - b) = z + x - 2b$ for $x \geq b$. Thus, $f$ overall is of the form $f(x) = |x - b| - (b - z)$.

3. If there are two points $z_1 < z_2 \in (c, 1 - c)$ with $f(z_1) = f(z_2) = 0$, then $f(c)$ and $f(1 - c)$ have the same sign; w.l.o.g., they are both positive. We distinguish two subcases:

- If $z_2 - z_1 \geq (1 - c - z_2) + (z_1 - c)$, then the expected error is maximized when $\int_{0}^{z_1} f(x)dx$ and $\int_{z_2}^{1} f(x)dx$ are as positive as possible, subject to the Lipschitz condition and the constraint that $\int_{0}^{1} f(x)dx = 0$. Otherwise, we could increase the value of $\int_{0}^{z_1} f(x)dx$ and $\int_{z_2}^{1} f(x)dx$, and then lower the function by some small resulting $\delta$ to restore the integral to 0. If the function is thus lowered by $\delta$, then for the interval $(z_1, z_2)$, the error increases by $\delta$, while for the intervals $[c, z_1]$ and $[z_2, 1 - c]$, it at most decreases by $\delta$. By the condition $z_2 - z_1 \geq (1 - c - z_2) + (z_1 - c)$, the lowering would overall increase the error. Now, applying Lemma 4.5 gives us that w.l.o.g., $f(x) = |x - \frac{z_1 + z_2}{2}| - \frac{z_2 - z_1}{2}$.

- If $z_2 - z_1 < (1 - c - z_2) + (z_1 - c)$, then the expected error is maximized when $\int_{0}^{z_1} f(x)dx$ and $\int_{1-c}^{1} f(x)dx$ are as negative as possible, subject to the Lipschitz condition and the constraint that $\int_{0}^{1} f(x)dx = 0$. Otherwise, we could decrease the value of $\int_{0}^{z_1} f(x)dx$ and $\int_{1-c}^{1} f(x)dx$, and then raise the function to restore the integral to 0. An argument just as in the previous case shows that the error cannot decrease. Hence, w.l.o.g., $f(x) = f(c) - (c - x) for x \in [0, c]$ and $f(x) = f(1 - c) - (x - (1 - c))$ for $x \in [1 - c, 1]$. We next claim that there must be at least one point $z \in [0, c] \cup (1 - c, 1]$ such that $f(z) = 0$. For contradiction, assume that $f$ is positive in $[0, c] \cup (1 - c, 1]$. Then, $f(0), f(1) > 0$, and therefore, $f(c), f(1 - c) > c$. Because $f(z_1) = f(z_2) = 0$, this implies that $z_1 > 2c$ and $z_2 < 1 - 2c$. But with our choice of $c = 2 - \sqrt{3}$, this implies that $z_2 < z_1$, a contradiction.

Without loss of generality, assume that the interval $(1 - c, 1]$ contains such a point $z$; define $z_3 = \min\{z \in (1 - c, 1] | f(z) = 0\}$. Further, assume that we have applied Lemma 4.5 to $f$, such that $f$ maximizes the area in the intervals $[c, z_1], [z_1, z_2]$ and $[z_2, 1 - c]$. Consider mirroring the function on the interval $[z_2, z_3]$. Formally, we define $f'(x) = f(z_3 - (x - z_1)) if x \in [z_1, z_3]$, and $f'(x) = f(x)$ otherwise. Clearly, $f'$ is Lipschitz continuous and has the same integral (namely, zero) as $f$.

Next, we define a new function $f''$ by modifying $f'$ so that it is as negative as possible in the interval $[z_1 + z_3 - z_2, 1]$. Formally, we define $f''(x) = z_1 + z_3 - z_2 - x if x \in [z_1 + z_3 - z_2, 1]$, and $f''(x) = f'(x)$ otherwise. (See Figure 2 for an illustration of this mirroring, and the resulting shapes of $f'$ and $f''$)

Notice that $f''$ is not normalized to have an integral of 0, since $\int_{z_1 + z_3 - z_2}^{1} f''(x) < \int_{z_1 + z_3 - z_2}^{1} f'(x)$. However, since (by assumption on the current case) $z_2 - z_1 < (1 - c - z_2) + (z_1 - c)$, raising $f''$ to restore the integral to 0 can only increase the resulting estimation error, by an argument similar to the previous case. The remainder of the proof for this case is as follows: We will first prove that the estimation error of $f''$ is at least as large as the estimation error of $f$. This implies that even after normalizing $f''$,
its estimation error remains at least as large as that of \(f\). Finally we can use Lemma 4.5 on the normalized version of \(f''\) to reduce the number of points \(x \in (c, 1 - c)\) with \(f''(x) = 0\) down to either one or zero, without decreasing the estimation error, and thus reduce this subcase to one of the previous two cases.

We now compare the estimation error of \(f\) against that of \(f''\). Simply by definition of \(f''\), we have that
\[
\int_{z_2}^{1-c} |f(x)|dx = \int_{z_1+1-c-z_2}^{z_1} |f''(x)|dx, \quad \text{and} \quad \int_{c}^{z_1} |f(x)|dx = \int_{c}^{z_1} |f''(x)|dx.
\]
Furthermore, we have that
\[
\int_{z_1}^{z_2} |f(x)|dx \leq \int_{z_1+(1-c-z_2)}^{1-c} |f''(x)|dx.
\]
This follows, since for any values \(p, q\) such that \(0 < p < q\), we have
\[
\int_{0}^{q} |\frac{q}{2} - |x - \frac{q}{2}||dx \leq \int_{0}^{p} |p - x|dx.
\]
Hence,
\[
\int_{c}^{1-c} |f''(x)|dx \geq \int_{c}^{1-c} |f(x)|dx.
\]

In all three cases, we have thus shown that w.l.o.g., \(f(x) = |x - b| - t\), for some values \(b, t\). Finally, the normalization \(\int_{0}^{1} f(x)dx = 0\) implies that \(t = \frac{1}{2} + b^2 - b\), completing the proof of Theorem 4.2.

5 Future Work

Our work is a first step toward obtaining optimal (as opposed to asymptotically optimal) randomized algorithms for choosing \(k\) sample locations to estimate an aggregate quantity of a function \(f\). The most obvious extension is to extend our results to the case of estimating the average using \(k\) samples. It would be interesting whether approximation guarantees for the \(k\)-median problem (the deterministic counterpart) can be exceeded using a randomized strategy.

Also, our precise characterization of the optimal sampling distribution for functions on the \([0, 1]\)
interval should be extended to higher-dimensional continuous metric spaces. Another natural direction is to consider other aggregation goals, such as predicting the function’s maximum, minimum, or median. For predicting the maximum from $k$ deterministic samples, a 2-approximation algorithm was given in [4], which is is best possible unless P$\neq$NP. However, it is not clear if equally good approximations can be achieved for the randomized case. For the median, even the deterministic case is open.

On a technical note, it would be interesting whether finding the best sampling distribution for the single sample case is NP-hard. While we presented a PTAS in this paper, no hardness result is currently known.

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