A flux-conservative formalism for convective and dissipative multi-fluid systems, with application to Newtonian superfluid neutron stars

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Abstract
We develop a flux-conservative formalism for a Newtonian multi-fluid system, including dissipation and entrainment (i.e. allowing the momentum of one fluid to be a linear combination of the velocities of all fluids). Maximum use is made of mass, energy and linear and angular momentum conservation to specify the equations of motion. Also used extensively are insights gleaned from a convective variational action principle, the key being the distinction between each velocity and its canonically conjugate momentum (which is modified because of entrainment). Dissipation is incorporated to second order in the ‘thermodynamic forces’ via the approach pioneered by Onsager, which makes it transparent how to guarantee the law of increase of entropy. An immediate goal of the investigation is to understand better the number, and form, of independent dissipation terms required for a consistent set of equations of motion in the multi-fluid context. A significant, but seemingly innocuous detail is that one must be careful to isolate ‘forces’ that can be written as total gradients, otherwise errors can be made in relating the net internal force to the net externally applied force. Our long-range aim is to provide a formalism that can be used to model dynamical multi-fluid systems both perturbatively and via fully nonlinear 3D numerical evolutions. To elucidate the formalism we consider the standard model for a heat-conducting, superfluid neutron star, which is believed to be dominated by superfluid neutrons, superconducting protons and a highly degenerate, ultra-relativistic gas of normal fluid electrons. We determine that in this case there are, in principle, 19 dissipation coefficients in the final set of equations. A final reduction of the system is made by neglecting heat conduction. This leads to an extension of the standard two-fluid model for neutron star cores, which has been used in a number of
previous applications, and illustrates how mutual friction is represented in our formalism.

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1. Introduction

The chemistry community has great experience in dealing with multi-fluid systems. Perhaps indicative of the cultural divide between research areas is the development of a multi-fluid literature for superfluid neutron stars which largely ignores the chemistry successes. Of particular note is the Onsager formulation of dissipative, multi-fluid systems [1]. On the other hand, action-based derivations of the equations of motion for multi-fluid systems [2–5] have, as far as we can tell, not crossed over in the other direction. In this work, we attempt to span the divide by utilizing fully mass, energy, and linear and angular momentum conservation, the Onsager formulation and action-based constructs to build a general dissipative model for multi-fluid systems. We facilitate our use of the conservation laws by borrowing from the engineering community the idea of ‘control’ volumes. Even though we restrict the discussion to the Newtonian regime, we do not hesitate to use ideas that have been developed for general relativity. In fact, the analysis is often much simplified if the true covariant nature of the physics is retained.

Although much of our analysis is general, and can be used in any multi-constituent fluid context—for example, superfluid helium and bubbly liquid/gas mixtures (see [6, 7] for discussions of multicomponent fluids, and a series of papers by Geurst [8–10] for interesting discussions), our immediate goal is to produce a formalism that can be applied to Newtonian superfluid neutron stars, in order to model their detailed dynamics. The importance of this target is enhanced by the fact that the first generation of large-scale, ground-based, gravitational-wave detectors now operates near their design sensitivity [11]. Mature neutron stars can radiate gravitationally in a number of ways, many of which depend crucially on dissipation in the superfluid interior. An obvious example is the potentially strong damping of gravitational-wave driven mode instabilities (e.g. of the so-called r-modes) due to either superfluid mutual friction [12, 13] or a viscous boundary layer at the neutron star core–crust interface [14, 15]. Other astrophysical contexts requiring a detailed model for superfluid dissipation include the pulsar glitches [16], the standard model which is based on the transfer of angular momentum between components rotating relative to each other and possible neutron star free precession [17], the damping of which depends crucially on the interior viscosity.

In the standard superfluid neutron star model, the neutrons in the inner crust and the outer core are believed to be superfluid, and the protons in the outer core are thought to be superconducting. Because of overall charge neutrality, the neutrons and protons will coexist with a normal, but highly degenerate fluid of electrons (and possibly muons in the more dense regions). At the core temperature expected in a mature neutron star (likely four orders of magnitude below the relevant Fermi temperatures), the superfluid and superconducting nature, respectively, of the neutrons and protons implies that they function as dynamically independent, inter-penetrating fluids. The conventional wisdom is that viscous damping of the oscillations of these objects is a result of electron–electron scattering. However, this begs the following question: how is it that damping of the electron motion on the one hand leads to damping, on the other hand, of the superfluid neutrons, which make up the bulk of the...
matter in the star? Despite these kinds of problems having been studied for decades [18, 19], the answer is not at all obvious from the extant literature (although we have recently clarified some aspects in [20, 21]). Indeed, the conventional wisdom seems to have been guided by the classic discussion of Cutler and Lindblom [22], who noted that ‘neutron star matter becomes more viscous in the superfluid state than it was in the normal state’. However, while they included the effect of superfluidity in their determination of the viscosity coefficients, they did not consider the effect on the actual fluid dynamics. Obviously aware of this fact, they remark that ‘The superfluidity of neutron star matter, will, of course, have drastic effects both on the dissipation coefficients and on the dynamics of this material.’ The implications of the first statement on the possibility of detectable gravitational radiation from neutron stars are potentially far ranging, leading most directly to the conclusion that gravitational-wave driven instabilities are ineffective in mature neutron stars [12], provided, of course, that superfluid dynamics can be ignored. However, subsequent work on the role of superfluidity in neutron stars reveals that the effects on the dynamics can be significant, and sometimes unanticipated (see [23–26] for recent discussions).

To date, most studies of superfluid neutron stars have drawn upon Landau’s original two-fluid model for He4 [27]. In the case of neutron stars, both neutrons and protons are expected to form Cooper pairs, which then form condensates which at the fluid level can be described by the standard equations. The main distinction in the neutron star case is that the two fluids no longer represent the superfluid and normal parts of a single-particle species. Instead, the two degrees of freedom describe the neutrons and a conglomerate of all charged components which are expected to be electromagnetically coupled on a short timescale. Following the traditional route, Mendell [19, 28] extended the non-dissipative zero-temperature equations to account for the main dissipation mechanisms. The result is a set of equations which include the mutual friction coupling due to electrons scattering off of rotational vortices in the condensates (see also [21]). These equations were later used to study the damping of unstable modes of oscillation in a spinning neutron star. The conclusions drawn from this work are that the mutual friction is a key dissipation mechanism which may, in fact, suppress rotational instabilities entirely [13, 29].

As the work of Mendell shows, the mathematical description of the problem becomes considerably more complicated once the so-called entrainment effect [30], which accounts for the fact that the flow of one fluid may impart momentum in the other fluid, is included. This complexity is partly due to a common confusion concerning Landau’s description of superfluid hydrodynamics. At the heart of the problem lies a failure to distinguish clearly between transport velocities and momenta. This distinction is made clear in recent variational formulations of the problem [2–5], work which draws heavily on the fully covariant relativistic description of Carter and collaborators [31–37], and which incorporates entrainment in a natural way. This leads to a mathematical description which is in many ways less complex than the standard one.

Our present analysis differs somewhat from the recent variational derivations [2–5] by emphasizing the conservation laws for mass, linear and angular momentum, and energy. We find this approach appealing because it remains close to our physical intuition. Furthermore, we believe that our discussion may serve as a useful ‘introduction’ to the more technically involved work described in [2–5]. Of course, we rely heavily on the variational methods for rigorous mathematical support of our results in the non-dissipative case. Until very recently there had been no serious discussion of the dissipative problem within the variational framework. The discussion by Carter and Chamel [3–5] provided a useful first step by translating (and extending somewhat) the covariant relativistic model devised by Carter [31] and Carter and Khalatnikov [38], yet the discussion remains somewhat abstract. A key aim of the present
discussion is to make contact with applications and prepare the ground for building realistic models of astrophysical superfluids.

The present work has two main aims. First of all, we want to consider the multi-fluid dissipation problem in ‘complete’ generality. In doing this we rely on the key conservation laws and ask what the most general form for the dissipation and interaction forces may be. The dissipative terms are then constrained by ensuring that the second law of thermodynamics is satisfied. To make progress in this direction we make use of Onsager’s celebrated symmetry principle [1]. Our derivation follows the guidelines set out in work on superfluid He⁴ (see [39] and [40] for nice descriptions) and is basically an extension of the general analysis championed, for instance, by Landau and Lifshitz [27] for one-fluid systems. We extend previous work in two important ways: we include the entrainment effect in a (hopefully) clear way, and we also allow for the presence of mutual interaction terms which have traditionally not been considered (e.g. in [3–5]). The latter turns out to be crucial if we want to be able to represent the mutual friction forces which arise due to the presence of vortices. Up to this point, i.e. in sections 2 and 3, the discussion is general and should be relevant for any multi-fluid system. Our second objective is met in section 4, where we specify the equations to the conditions that should prevail in the outer core of a mature neutron star. We consider a mixture of superfluid neutrons, superconducting protons, normal electrons and entropy. This model is instructive in a number of ways. We show how the dynamical degrees of freedom are reduced in a natural way if we assume that the electrons and protons are coupled electromagnetically. (It should be noted that we do not account for detailed electromagnetic effects in this analysis [41].) We also deduce that this system, in general, requires the specification of no less than 19 distinct dissipation coefficients. This discussion is followed, and the paper concluded, by a final reduction to the dissipative extension of the standard two-fluid model which has been used in recent work on neutron star dynamics. This model, which is obtained by neglecting heat conduction, provides the simplest system that should be considered in ‘realistic’ applications. We discuss the interpretation of the various dissipative terms, some of which appear to be new. In particular, we show how the standard form for the mutual friction force [21] is represented within our formalism.

2. Conservation laws for multi-fluid systems

We begin our analysis by discussing the conservation laws that must be satisfied by any multi-fluid system. This discussion serves two important purposes. First of all, it establishes our notation in the comfortable context of the well-known balance laws of physics. Secondly, it allows us to highlight points where our analysis differs from the standard literature, e.g. [6, 7]. Although the reader should not expect to find any true revelations here, a number of points are worth emphasizing. We treat the entropy as a distinct (massless) fluid. We allow the individual constituent stress tensors to be asymmetric (which is the generic case), and finally we use the results from recent variational derivations to introduce the entrainment effect (which is analogous to the ‘virtual mass effect’ in the standard theory of mixtures [6, 7]) in a transparent way. The discussion is intended to make our analysis self-contained.

Any general fluid formalism must respect the three laws concerning

- conservation of mass;
- conservation of linear and angular momentum;
- conservation of energy.

Although entropy is not conserved in general, the law of increase of entropy is also a vital property of any fluid. In imposing these conservation laws for a multi-fluid system, we
A flux-conservative formalism for convective and dissipative multi-fluid systems

Consider some ‘control volume’ $V$ with boundary $\partial V$, letting $\eta_i$ denote the unit normal form to $\partial V$. The control volume is to be contrasted with the notion of a fluid element, which is a region of the fluid small enough to be considered ‘infinitesimal’ with respect to the whole fluid and yet contain enough particles that a (local) thermodynamic treatment is warranted.

We consider a general multi-fluid system, the dynamical description of which is based on determining the space and time dependence of each particle number density $n_x$ and the corresponding particle number density current $n'_x$. These are natural variables to use in describing the system. It is assumed that the different fluids are inter-penetrating, i.e. the different constituents do not have to be separated by interfaces. We let $x, y, z$ be particle constituent indices that range over the whole set of fluids in the system. Repeated constituent indices will not satisfy a summation convention, but vector indices $i, j, k$ do satisfy such a convention (as usual). Ostensibly, all formulae will be written in a general coordinate basis, with $g_{ij} = g_{ji}$ representing the (flat-space) metric, except for certain integrals written below, where a Cartesian basis is implicit (since tensor components are being integrated). The metric is used to raise and lower indices, e.g. for a vector $v_i = g_{ij}v^j$ and $v^i = g^{ij}v_j$, where the inverse metric $g^{ij}$ satisfies $g_{jk}g^{ki} = \delta^i_j$ and $\delta^i_j$ is the Kronecker-delta tensor. When integrating over volumes, the measure is given by $dV = \sqrt{g}d^3x$, where $g$ is the determinant of the metric. Finally, all spatial derivatives are given by the covariant derivative $\nabla_i$ that is compatible with the metric (i.e. $\nabla_ig_{jk} = 0$).

2.1. Conservation of mass

Let $m^x$ denote the ‘particle’ mass of the $x$th constituent, so that $\rho_x = m^xn_x$ is the mass density of that same constituent. The total (local) mass density of the system, $\rho$, is thus

$$\rho = \sum_x \rho_x = \sum_x m^xn_x. \quad (1)$$

The total mass–density current $\rho'$ is likewise given by

$$\rho' = \sum_x m^xn'_x. \quad (2)$$

When $\rho$ is integrated over the control volume $V$, we obviously determine the mass of fluid $M$ contained in $V$, i.e.

$$M = \int_V \rho \, dV. \quad (3)$$

When $\rho'$ is integrated over $\partial V$, we have a representation of the amount of mass that leaves (or enters, depending on the situation) $V$. Overall conservation of mass thus means

$$\frac{d}{dt} \int_V \rho \, dV = - \int_{\partial V} \rho' \eta_i \, dA = - \int_V \nabla_i \rho' \, dV. \quad (4)$$

This implies the local condition

$$\partial_t \rho + \nabla_i \rho' = 0. \quad (5)$$

If $\Gamma_x$ represents the particle number creation rate per unit volume of the $x$th constituent, then we have

$$\partial_t n_x + \nabla_i n'_x = \Gamma_x. \quad (6)$$

Multiplying this by $m^x$, summing over all $x$ and using the conservation of mass, equation (5), one immediately sees that

$$\sum_x m^x \Gamma_x = 0. \quad (7)$$
Worth noting at this point is that we will consider the entropy to be a fluid, the main distinction being that its mass is zero. Its 'number' density is simply the entropy per unit volume, and it is assumed to satisfy a creation rule like equation (6).

2.2. Conservation of linear momentum

Let $p_i^x$ represent the (local) linear momentum per particle (which will later be understood to be canonically conjugate to $n_i^x$) of the $x$th constituent. The total (local) linear momentum of the fluid $p_i$ will then obviously be given by

$$p_i = \sum_x p_i^x. \tag{8}$$

Likewise, let $\pi_i$ denote the total linear momentum density, so that

$$\pi_i = \sum_x n_i^x p_i^x = \sum_x \pi_i^x. \tag{9}$$

The total (global) linear momentum $P_i$ of the fluid contained in the control volume will be given by

$$P_i = \int_V \pi_i \, dV. \tag{10}$$

Finally, let $T_{ij}$ represent the $i$th component of the total linear momentum that flows (per unit time and unit area) in a direction perpendicular to the $j$th direction, and $f_i$ be the net external force density acting on the control volume. Overall conservation of linear momentum in $V$ is then given by

$$\frac{dP_i}{dt} = \frac{d}{dt} \int_V \pi_i \, dV = -\int_{\partial V} T_{ij} n_j \, dA + \int_V f_i \, dV = -\int_V \left( \nabla_j T_{ij} - f_i \right) \, dV, \tag{11}$$

which implies that

$$\partial_t \pi_i + \nabla_j T_{ij} = f_i. \tag{12}$$

Here we could have added a term that corresponds to the net internal forces. However, the weak law of action and reaction implies that this contribution is zero [42]. There are also the well-known difficulties for action and reaction when charged particles are present (cf [42]).

The individual fluid components will not satisfy exactly the same conservation law, since momentum and energy can be exchanged between them. However, it is worthwhile to assume that the $x$th constituent satisfies an equation of the form

$$\partial_t \pi_i^x + \nabla_j T_{xji} = \bar{f}_i, \tag{13}$$

noting that this is still completely general, as nothing has been proposed about either $T_{xji}$ or $\bar{f}_i$. That being said, one should not be too surprised by such a choice, since it represents the limiting form familiar from single-fluid studies. Consistency with overall linear momentum conservation now demands that

$$f_i - \sum_x \bar{f}_i = \nabla_j \left( T_{ij} - \sum_x T_{xji} \right). \tag{14}$$

Although it may seem natural to suppose that the left- and right-hand sides should vanish independently, we will establish below that this is not correct.
2.3. Conservation of energy

Let $U$ denote the total energy per unit volume in $V$ and $U$ be the total energy of the control volume, i.e.,

$$ U = \int_V U \, dV. \quad (15) $$

Also, let $Q^i$ denote the flow of energy (per unit time and unit area) perpendicular to the $i$th direction and $\epsilon^{\text{ext}}$ the energy creation rate per unit volume due to external sources. Conservation of energy in $V$ is then naturally expressed as

$$ \frac{dU}{dt} = \int_V \left( \epsilon^{\text{ext}} - \nabla_i Q^i \right) \, dV, \quad (16) $$

or

$$ \partial_t U + \nabla_i Q^i = \epsilon^{\text{ext}}. \quad (17) $$

Here it should be noted that we must specify a relation $U = U(n_x, n_i)$ to close the system of equations. That is, we need to provide an equation of state for the matter.

2.4. Conservation of angular momentum

In order to get an appropriate definition of the total angular momentum $L^i$ in the control volume, let us first reconsider the total linear momentum $P_i$ in the context of Newton’s second law. We need to determine those internal and external contributions that act within the fluid to change $P_i$, so as to properly define the net torque.

Referring back to equation (10), we can use $P_i$ in Newton’s second law to determine the net ‘force’ $F_i$ acting on the control volume:

$$ F_i \equiv \frac{dP_i}{dt} = \int_V \partial_t \pi_i \, dV = \int_V \left( f_i - \nabla_j T^j_i \right) \, dV. \quad (18) $$

This suggests that the appropriate definition of the total torque $T^i$ acting on the control volume (in Cartesian coordinates) is

$$ T^i = \frac{1}{2} \epsilon^{ijk} \int_V \left( x_j \left[ f_k - \nabla_l T^l_j \right] - x_k \left[ f_j - \nabla_l T^l_j \right] \right) \, dV, \quad (19) $$

where the volume form $\epsilon^{ijk} = \sqrt{g} \begin{vmatrix} ijk \end{vmatrix}$ and $\begin{vmatrix} ijk \end{vmatrix}$ is completely antisymmetric with $[123] = 1$. To see that this definition is consistent, we consider the time rate of change of $L^i$ and show that it is equal to $T^i$.

The total angular momentum of the control volume is defined to be

$$ L^i = \frac{1}{2} \epsilon^{ijk} \int_V \left( x_j \pi_k - x_k \pi_j \right) \, dV, \quad (20) $$

and its time rate of change is equal to

$$ \frac{dL^i}{dt} = \frac{1}{2} \epsilon^{ijk} \int_V \left( x_j \partial_t \pi_k - x_k \partial_t \pi_j \right) \, dV $$

$$ = \frac{1}{2} \epsilon^{ijk} \int_V \left( x_j \left[ f_k - \nabla_l T^l_j \right] - x_k \left[ f_j - \nabla_l T^l_j \right] \right) \, dV $$

$$ = T^i, \quad (21) $$

which is the desired result. Of course, equation (21) implies that the total angular momentum contained within the control volume is conserved when the net torque vanishes. We can use
this to our advantage now by showing that conservation results only if the symmetry $T_{ij} = T_{ji}$ holds.

When a system is completely isolated, and the control volume is such that it completely contains the system and has a boundary taken well outside the system, we should find that $L_i$ is conserved, i.e. constant in time, if in addition the net force $f_i$ acting is zero. Equation (21) can be rewritten as

$$\frac{dL_i}{dt} = \frac{1}{2} \epsilon^{ijk} \left\{ \int_V (x_j f_k - x_k f_j) \, dV - \int_V (T_{jk} - T_{kj}) \, dV - \int_{\partial V} (x_j T^l_k - x_k T^l_j) \, \eta_l \, dA \right\}. \tag{22}$$

The first term on the right-hand side vanishes because the net force $f_i$ is zero. The last term vanishes because we have taken the boundary of the control volume well outside the system. Hence, if angular momentum is to be conserved, the middle term must also vanish, which implies $T_{ij} = T_{ji}$.

Because of the inherent linearity in our definitions when summing over the constituent indices, we immediately see that an appropriate definition for the control volume force $F^i_x$ acting on the $x$th constituent is

$$F^i_x = \int_V (f^i_x - \nabla_j T^{xj}_i) \, dV, \tag{23}$$

while for the control volume angular momentum we have

$$L^i_x = \frac{1}{2} \epsilon^{ijk} \int_V (x_j \pi^k_i - x_k \pi^j_i) \, dV, \tag{24}$$

where the corresponding torque is

$$T^{i}_x = \frac{1}{2} \epsilon^{ijk} \int_V (x_j \left[ T^k_j - \nabla_l T^{xj}_l \right] - x_k \left[ T^j_k - \nabla_l T^{xj}_l \right]) \, dV. \tag{25}$$

This naturally implies

$$\frac{dL^i_x}{dt} = \frac{1}{2} \epsilon^{ijk} \left\{ \int_V (x_j \overline{T}^k_j - x_k \overline{T}^j_i) \, dV - \int_V (T^{i}^j - T^{k}^i) \, dV - \int_{\partial V} (x_j T^{xj}_i - x_k T^{xj}_i) \, \eta_l \, dA \right\}. \tag{26}$$

At this point we make an important observation: in contrast to the comments following equation (22), we argue that it is a mistake to infer that $T^{xj}_i = T^{xj}_i$. This assumption would be too restrictive. In particular, we will show later that it would not allow us to represent the standard form for the vortex-mediated mutual friction in a superfluid system [21].

2.5. Lagrange’s generalized action principle and equations of motion

Further progress can be made in specifying the unknown quantities $U$, $Q^i$, $\overline{T}^i_j$, $T^{xj}_i$ and $T^{i}^j$, assuming that $f_i$ and $\epsilon^{\text{ext}}$ are prescribed. Consider that there is a Lagrangian density $\mathcal{L}$ whose variation leads to the multi-fluid equations in the fully conservative, non-dissipative case. It consists of the total kinetic energy density minus the ‘potential’ energy density, which we take to be the total internal energy density $\mathcal{E}(n_s, n'_s)$,

$$\mathcal{L} = \sum_x \frac{1}{2} m^x g_{ij} n^i_x n^j_x / n_x - \mathcal{E}(n_s, n'_s). \tag{27}$$

Minimizing the action associated with $\mathcal{L}$ in the usual way will necessarily lead to the conservative Euler–Lagrange equations. To incorporate dissipation, we use Lagrange’s
generalized approach (which is championed by Carter, see for example [31]), which is to obtain the Euler–Lagrange equations in the usual way, but instead of equating them to zero we put them equal to an appropriate sum of generalized forces that may include dissipation. The net effect is that the Lagrangian serves to define the momenta, whose identification is perhaps the most crucial element of the multi-fluid construction that fully incorporates entrainment.

The first step towards identifying the unknown quantities is to write the total energy density \( U \) (to be distinguished from the total internal energy density \( E \)) in terms of the fundamental variables of the action. Recall that a Hamiltonian density \( H \) can be generated as a Legendre transformation on \( L \), i.e.

\[
\mathcal{H} \equiv \sum_x n_x p_x^i - L. \tag{28}
\]

The momenta are defined via

\[
p_x^i \equiv \left. \frac{\partial L}{\partial n_x^i} \right|_{n_x^j}, \tag{29}
\]

where \( v_x^i = n_x^i/n_x \) is the transport velocity of the \( x \)th constituent. The simplest, and most natural way to obtain the total energy density in terms of the fundamental variables is to set \( U = H \).

Accepting that, we now take a time derivative of \( H \), and use equations (6) and (13), to find

\[
\partial_t U = -\nabla_i \left\{ \sum_x \left( \left[ \bar{p}_x^i + \frac{1}{2} m^x v_x^2 - v_x^i p_x^j \right] n_x^i + v_x^j T_x^{ij} \right) + \sum_x \left( v_x^j T_x^i + T_x^{ij} \nabla_j v_x^i + n_x^j \nabla_i \left[ \bar{p}_x^i + \frac{1}{2} m^x v_x^2 - v_x^i p_x^j \right] \right) + \left[ \bar{p}_x^i + \frac{1}{2} m^x v_x^2 - v_x^i p_x^i \right] \right\} \tag{30},
\]

where \( v_x^2 = g_{ij} v_x^i v_x^j \) and

\[
\bar{p}^x \equiv \left. \frac{\partial E}{\partial n_x^i} \right|_{n_x^j}. \tag{31}
\]

Comparing with equation (17) we can identify

\[
Q^i = \sum_x \left( \left[ \bar{p}_x^i + \frac{1}{2} m^x v_x^2 - v_x^i p_x^j \right] n_x^i + v_x^j T_x^{ij} \right), \tag{32}
\]

\[
\epsilon^{ext} = \sum_x \left( v_x^j T_x^i + T_x^{ij} \nabla_j v_x^i + n_x^j \nabla_i \left[ \bar{p}_x^i + \frac{1}{2} m^x v_x^2 - v_x^i p_x^j \right] + \left[ \bar{p}_x^i + \frac{1}{2} m^x v_x^2 - v_x^i p_x^i \right] \right) \tag{33}.
\]

It is useful at this point to note that there is a more complete Legendre transformation on \( \mathcal{L} \) that can be made by using the functions \( p_0^i \) defined as

\[
p_0^i \equiv \left. \frac{\partial L}{\partial n_x^i} \right|_{n_x^j} = -\left( \bar{p}_x^i + \frac{1}{2} m^x v_x^2 \right). \tag{34}
\]

This leads to the new function \( \Psi \) given by

\[
\Psi = -\left( \sum_x \left[ n_x p_0^i + n_x^i p_x^i \right] - \mathcal{L} \right). \tag{35}
\]
We will now establish that $\Psi$ enters the total momentum conservation equation as a generalized pressure. To do this we need to analyse more closely the stress-tensor $T_{ji}$.

Widely discussed in the general relativity literature (for instance, [43]) is the fact that the stress–energy–momentum tensor can be obtained from a variation of the Lagrangian with respect to the metric. It is natural that metric variations are associated with stresses, since stress leads to deformations and the metric is what gives distances and volumes. In appendix A we provide a non-rigorous account (using energy-conservation arguments) of how to obtain the conservative piece of $T_{ji}$, to be denoted $C_{ji}$, by varying $L$ with respect to $g_{ij}$ (see [2] for a rigorous derivation). Our variation is to be understood as a displacement, in a small amount of time, of each fluid element within a fixed control volume, keeping the individual particle numbers and their associated velocities fixed.

The discussion in appendix A prompts us to consider a less general form of the internal energy, i.e.

$$E = \mathcal{E}(n_x, w^2_{xy}), \quad (36)$$

where

$$w_{xy}^i = v_{xy}^i - v_{xy}^j, \quad w^2_{xy} = g_{ij}w_{xy}^i w_{xy}^j. \quad (37)$$

This is manifestly Galilean invariant and locally isotropic. It is worth pointing out that this is not the most general form for the energy that we can allow. For example, it would be straightforward to include an explicit dependence on the vorticity, e.g. in terms of the ‘vorticity’ vector $W_{js}$ (defined below by equation (60)) we could allow the energy to depend on $W^2_{js} = g_{ij}W_{js}^i W_{js}^j$. This would be analogous to the superfluid helium model discussed by Bekarevich and Khalatnikov [44] where the circulation around a vortex contributes to the energy budget, and would lead to terms that could be interpreted as the ‘vortex tension’. Although one can think of other such possibilities we prefer to work with equation (36) here. Once the complete framework has been developed for this case, it should be relatively easy to extend it to include more general cases.

From appendix A we find that $C_{ji}$ is given simply as

$$C_{ji} = \Psi \delta_{ji} + \sum_x n_x^j p_x^i. \quad (38)$$

Thus, $\Psi$ behaves as the total pressure. We are naturally led to write the full stress tensor as a sum of $C_{ji}$ and a dissipation piece $D_{ji}$, i.e.

$$T_{ji} = C_{ji} + D_{ji}. \quad (39)$$

At this point it is important to note that, since the pressure $\Psi$ is not generally separable there is, in general, no way to write $T_{ji}$ as just a sum of the $T^{\perp}_{ji}$. In other words, there is in general no useful notion of partial pressures, a point which is examined in more detail in appendix B. Basically, it is more natural to work with the individual chemical potentials. Then the final equations retain the natural symmetry between different chemical constituents.

The ramifications of a lack of partial pressures can be extracted by returning to equation (14), and inserting equation (39). We then get

$$f_i = \sum_x f_x^i = \nabla_j \left( D_{ji} - \sum_x \left[ T^{\perp}_{ji} - n_x^j p_x^i \right] \right), \quad (40)$$

where we have defined

$$\overline{T}_{ji} = f_x^i + n_x \nabla_i p_x^i + n_x^j \nabla_i p_x^j, \quad (41)$$
and used the fact that
\[ \nabla_i \psi = - \sum_n \left( n_i \nabla_i p_0^n + n_i^n \nabla_i p_i^n \right). \] (42)

Letting \( T^{xj}_i \) be a sum of a conservative and dissipative piece, i.e.
\[ T^{xj}_i = C^{xj}_i + D^{xj}_i, \] (43)
the consistency relation equation (14) becomes
\[ f_i - \sum_n f^n_i = \nabla_j \left( D^j_i - \sum_n D^{xj}_i \right) - \sum_n \nabla_j \left( C^{xj}_i - n_i^n p_i^n \right). \] (44)

The key point is that we have now isolated terms in such a way that consistency of equations (12) and (13) is guaranteed by taking
\[ f_i = \sum_n f^n_i, \quad D^i_j = \sum_n D^{xj}_i, \quad C^{xj}_i = n_i^n p_i^n. \] (45)

Here we recall the discussion of section 2.4 in which it was demonstrated that the total stress tensor \( T^{xj}_i \) is symmetric. Since the conservative piece \( C^{xj}_i \) can be shown to be symmetric, by using equations (29) and (36) in equation (38), this means that we must require \( D^j_i \) to be symmetric. However, as we already stated following equation (26), this does not mean that we should assume that the individual \( D^{xj}_i \) are symmetric.

Finally, the equation of motion for the \( x \)th constituent takes the form
\[ \partial_t \pi^x_i + \nabla_j \left( p_i^n \pi^{n x} + D^{xj}_i \right) - \left( n_i^n \nabla_i p_0^n + n_i^n \nabla_i \pi^x_i \right) = f^x_i, \] (46)
while the total external energy creation rate per unit volume becomes, cf equation (33),
\[ \epsilon^{\text{ext}} = \sum_n \left( \pi^x_n f^n_x + D^{xj}_i \nabla_j \pi^x_i - \left[ p_0^x + \pi^x_i p^x_i \right] \Gamma^x \right). \] (47)

One can verify that our equations are consistent with those of earlier formulations, with the exception that our formulation has led to a balance of forces, equation (45), that does not include the various dissipation stress-tensors \( D^{xj}_i \). Thus, to complete the system we can specify each \( D^{xj}_i \), and each \( f^x_i \) independently, with only the latter having to add to the total external force density \( f_i \) acting on the whole system. We will now show how to limit further the possible forms for the \( f^x_i \) and \( D^{xj}_i \), by employing Onsager’s formulation [1] for multi-fluid systems.

3. The Onsager formulation for dissipative multi-fluids

Because we have many independent fluids, the number of potential dissipation coefficients can be quite large. However, Onsager [1] demonstrated long ago that microscopic reversibility implies certain equalities among ‘off-diagonal’ pieces of the entropy creation rate. Since this argument plays a key role in our analysis it is worth outlining the main ideas. A more detailed, pedagogical, description can be found in [45].

We begin by noting that the entropy, here represented by the number density \( n_s = s \), is maximal for a system in equilibrium. This means that any perturbation away from equilibrium must be represented by quadratic deviations. Specifically, in the ‘thermal frame’ associated with the entropy velocity \( v^s_i \), the conservation law equation (6) implies
\[ \frac{\Delta s}{\Delta t} \approx \frac{s - s_{\text{eq}}}{\Delta t} \approx \Gamma^s. \] (48)
Comparing this to the anticipated expansion, see Jaynes [46] for an elegant exposition of this, near an equilibrium

\[ s \approx s_{eq} - \frac{\Delta t}{2T} \sum_{a,b} X_a L^{ab} X_b, \]  

(49)

we can identify [47]

\[ T \Gamma_s = -\frac{1}{2} \sum_{a,b} X_a L^{ab} X_b = \sum_{a=1}^N J^a X_a, \]

(50)

where the \( N \) individual \( X_a \) are known as ‘thermodynamic forces’ and the \( J^a \) as ‘fluxes.’ The thermodynamic forces represent a measure of the departure from global equilibrium in the system, with the fluxes arising in response. The Onsager symmetry principle simply states that \( L^{ab} = L^{ba} \).

Let us now see how we can apply this idea to our formalism. Our entropy creation rate is obtained from equation (47), by solving for \( \Gamma_s \). Noting that the chemical potentials \( \mu_x \), which are obtained from

\[ \mu_x \equiv \frac{\partial E}{\partial n_x} \bigg|_{w^2} \]

(51)

are related to the \( \overline{\mu}^x \) via

\[ \overline{\mu}^x = \mu^x - m^x v_x^2 + v_x^i p_x^i, \]

(52)

and that the temperature is given by \( \mu^s = T \), we can write

\[ \epsilon^\text{ext} = f_i v_i^j + D^i_j \nabla_j v_i^j + T \Gamma_s + \sum_{x \neq s} \left( \Gamma_x \mu^x + \overline{f}^x_{i} w_{x,i} + D^x_{ij} \nabla_{w} w_{x,j} \right). \]

(53)

We recall that \( f_i \) and \( D^i_j \) are the total force and dissipation tensors, respectively. We have assumed the locally isotropic, manifestly Galilean-invariant form for \( E \) (i.e. equation (36)), and we have defined

\[ \overline{f}^x_{i} = f^x_{i} - \frac{1}{2} m^x \Gamma_{x} g_{ij} \left( v_{x,j}^i + v_{s,j}^i \right). \]

(54)

We will take the total system to be closed, which means that \( \epsilon^\text{ext} = 0 \) and \( f_i = 0 \). In order to apply (50) we also need to work in the entropy frame, which means all velocities will appear as the difference \( w_{x,i} = v_{x,i} - v_{s,i} \).

The ‘thermodynamic forces’, which drive the system to equilibrium in the sense of equation (49), are seen from equation (53) to be (for \( x \neq s \))

\[ X_a = \begin{cases} \{ X^x \equiv \mu^x \}, & a = 1 \\ \{ X^i \equiv w_{x,i} \}, & a = 2 \\ \{ X^{ij} \equiv \nabla_i w_{x,j} \}, & a = 3, \end{cases} \]

(55)

implying that the corresponding ‘fluxes’ are (again for \( x \neq s \))

\[ J^a = \begin{cases} \{ J_x \equiv -\Gamma_{x} \}, & a = 1 \\ \{ J^i \equiv -\overline{f}^x_{i} \}, & a = 2 \\ \{ J^{ij} \equiv -D^x_{ij} \}, & a = 3. \end{cases} \]

(56)
The Onsager formulation for our system is thus

\[
\sum_{y \neq s} (L_{xy} \mu_y^{x} + \tilde{L}_{xy} \mu_y^{x} w_{y}^{i} + \tilde{L}_{xy} \nabla w_{y}^{i}) = 0,
\]

(57)

The tensorial aspects of the ‘\(L\)’ and ‘\(\tilde{L}\)’ coefficients are handled by assuming that they can only be constructed from combinations of the thermodynamic forces, \(\mu_x\), \(w_i^x\) and \(\nabla w_i^x\), and the background geometry terms, i.e. the metric \(g_{ij}\) and the volume form \(\epsilon_{ijk}\). Moreover, in keeping with the spirit of the Onsager expansion and due to the fact that we are supposedly close to equilibrium, we consider only those terms that lead to quadratic combinations of the thermodynamic forces. With these restrictions it is clear one would not expect coupling between forces with different tensorial nature. We readily find that \(\tilde{L}_{xyi} = \tilde{L}_{yxi} = 0\) and that we can rule out the use of \(w_i^x\) and \(\nabla w_i^x\) in any of the coefficients. The most general coefficients within our assumptions that can be written are

\[
L_{xy} = \gamma_{xy} = \gamma_{yx},
\]

\[
\tilde{L}_{xy}^{ij} = \tau_{xy}^{ij} = \tau_{yx}^{ij},
\]

\[
L_{xy}^{ijkl} = \zeta_{xy}^{ijkl} + \eta_{xy}^{ijkl}(g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{3} g_{ij} g_{kl},
\]

\[
\tilde{L}_{xy}^{ijkl} = \zeta_{yx}^{ijkl} + \eta_{yx}^{ijkl}(g_{ik} g_{jl} + g_{il} g_{jk}) - \frac{1}{3} g_{ij} g_{kl},
\]

(58)

Clearly, each coefficient inherits for its spatial indices the symmetries of either \(g_{ij}\) or \(\epsilon_{ijk}\).

There has also been a lot of thought put into the coefficient \(L_{xy}^{ijkl}\). The key point is that the most general four-index object that can be constructed is a linear combination of the three terms \(g_{ik} g_{jl}\), \(g_{il} g_{jk}\) and \(g_{ik} g_{jl} - g_{il} g_{jk}\).

Our discussion will be greatly facilitated by introducing the trace-free shear \(\Theta_{ij}^{xs}\), vorticity \(\omega_{ij}^{xs}\) and expansion \(\Theta_{xs}\):

\[
\Theta_{ij}^{xs} = \frac{1}{2} \left( \nabla w_i^x + \nabla w_j^x - \frac{2}{3} g_{ij} (\Theta_{xs}) \right) = \Theta_{ij}^{xs},
\]

\[
\Theta_{xs} = \nabla w_i^x.
\]

(59)

We will also need the vector that is dual to the vorticity, i.e.

\[
W_i^{xs} = \frac{1}{2} \epsilon_{ij}^k \omega_{ij}^{xs} \quad \Leftrightarrow \quad \omega_{ij}^{xs} = \epsilon_{ij}^k W_k^{xs}.
\]

(60)

This allows us to utilize the well-known result that

\[
\nabla w_i^x = \Theta_{ij}^{xs} + \epsilon_{ij}^k W_k^{xs} + \frac{1}{3} g_{ij} (\Theta_{xs}).
\]

(61)

Here we should emphasize that the proper definition of vorticity, for which one recovers the Kelvin–Helmholtz conservation theorem, is in terms of the momentum [2, 21]. In the absence of entrainment the quantity we use and the conserved vorticity are identical; but when entrainment is present, the difference is crucial. In terms of the analysis presented here, our introduction of the ‘vorticity’ \(\omega_{ij}^{xs}\) is purely a matter of convenience. It is not associated with a conservation law, but it is useful in separating purely symmetric (in the spatial indices) from antisymmetric objects.

The fluxes are now of the form

\[
\Gamma_x = - \sum_{y \neq s} (\gamma_{xy} \mu_y^{x} + \tau_{x}^{y} (\Theta_{ys})),
\]

(62)
\[ \dot{j}_s = -2g_{ij} \sum_{y \neq s} (R^{s} w^j_{ys} + S^{s} W^j_{ys}), \]  
(63) 

\[ D^{s}_{ij} = - \sum_{y \neq s} (g_{ij} \left[ \tau^{s}_{ys} + \zeta^{s}_{ys} \Theta_{ys} \right] + 2\eta^{s}_{ij} g_{ik} g_{jl} \Theta^{ij}_{ys} + \epsilon_{ijk} \left[ S^{s}_{ys} w^k_{ys} + \sigma^{s}_{ys} W^k_{ys} \right]), \]  
(64) 

and we see that the entropy creation rate (in the thermal frame) is 
\[ T \Gamma_s = \sum_{s, y \neq s} \left\{ \gamma^{s}_{ys} \mu^{s}_y + 2\tau^{s}_{ys} \mu^{s}_y \Theta^{s}_{ys} + \zeta^{s}_{ys} \Theta^{s}_{ys} + 2\eta^{s}_{ij} g_{ik} g_{jl} \Theta^{ij}_{ys} + \right. \]  
\[ \left. + 2g_{ij} \left[ R^{s} w^j_{ys} w^i_{ys} + 2S^{s} w^j_{ys} W^i_{ys} + \sigma^{s} W^i_{ys} W^j_{ys} \right] \right\}. \]  
(65) 

It is easy to show, given the explicit transformations in [2] that the chemical potentials are Galilean invariant. Since all other terms on the right-hand side of equation (65) depend only on velocity differences we see that the entire expression is invariant. This is, of course, as it should be. After all, the difference between reversible \((T / \Gamma_s = 0)\) and irreversible \((T / \Gamma_s > 0)\) processes cannot depend on the observer.

There are several different strategies to adopt in demonstrating the positive-definite nature of \(\Gamma_s\). Common to all is the simple notion of producing terms that look like \(V^T M V\), where \(M\) is a \(d \times d\) symmetric matrix, and \(V\) is a \(d\)-dimensional vector. At that point basic theorems from linear algebra (cf [48]) can be employed that will yield the constraints under which the matrix coefficients lead to positive-definite quantities. In our case, the most natural way to gather together terms is to introduce two new vectors (in this linear algebra sense): 
\[ V_1 = \left[ \begin{array}{c} \{\mu^{s}\} \\ \{\Theta^{s}_{ys}\} \end{array} \right], \quad V_2 = \left[ \begin{array}{c} \{w^i_{ys}\} \\ \{W^i_{ys}\} \end{array} \right], \]  
(66) 

whose dimension is \(d = 2(N - 1)\). Likewise, the matrices that define the quadratic terms are 
\[ M_1 = \left[ \begin{array}{c} \{y^{s}_{ys}\} \\ \{\tau^{s}_{ys}\} \end{array} \right], \quad M_2 = \left[ \begin{array}{c} \{S^{s}\} \\ \{\sigma^{s}\} \end{array} \right]. \]  
(67) 

The term containing \(\Theta^{ij}_{ys} \Theta^{kl}_{ys}\) is obviously already of the appropriate form. Hence, once the necessary constraints from linear algebra are obtained on the matrix coefficients, i.e. the dissipation coefficients \(\gamma^{s}, \zeta^{s}, \text{etc.}, \Gamma_s \geq 0\) can be guaranteed in the entropy frame.

Finally, we return to the inertial frame, in which the entropy velocity can appear independently of the other velocities. If we consider the system to be closed, then the entropy creation rate just picks up the additional term proportional to \(D^{s}_{ij}\). As \(D^{s}_{ij}\) still remains undefined, we can just as well work with \(D_{ij}\). Because \(D_{ij}\) is contracted with \(\nabla^i v^j\), and it must be symmetric, the only proposal consistent with our main assumptions is 
\[ D_{ij} = - \left( \zeta^{\text{tot}} \Theta_g g_{ij} + 2\eta^{\text{tot}} g_{ik} g_{jl} \Theta^{ij}_{s} \right), \]  
(68) 

where \(\Theta_g\) and \(\Theta^{ij}_s\) are defined as in equation (59) but replacing \(w^i_{ys}\) with \(v^i_s\). Using the previous, linear algebra-based line of reasoning, we can determine, in principle, the constraints on the dissipation coefficients \(\zeta^{\text{tot}}\) and \(\eta^{\text{tot}}\) that will guarantee the positivity of \(D_{ij} \nabla^i v^j\), without spoiling the positive-definite nature of the other terms.

4. The neutron, proton, electron and entropy system

4.1. The problem of neutron star dissipation

At zero temperature, the outer core of a neutron star consists of three inter-penetrating fluids: superfluid neutrons, superconducting protons and a highly degenerate gas of normal fluid
electrons. Because of electromagnetic coupling the charged components lock together on a timescale that is much shorter than, e.g., the timescale of stellar rotation or oscillation. Hence, the problem usually reduces to a two-fluid system. At a finite temperature, the situation is considerably more complex. Most obviously, we need to also account for the entropy. Provided that the system is far below the various superfluid transition temperatures, the entropy is associated with the electrons. However, there will also be contributions from excitations of the neutron and proton condensates. Accounting for these requires more thought. Based on studies of the analogous problem for superfluid helium there would seem to be (at least) two possible strategies. The difference between them is nicely described by Geurst [10].

In the first approach, championed by Carter and his collaborators (see for example [2]), each superfluid component is made up of the condensate and a massless gas of excitations. In our formalism this would amount to associating an entropy component with both the neutrons and the protons. It is then not difficult to show that the ‘normal fluid density’ associated with the \( x \)th constituent is directly proportional to the entrainment coefficient \( \alpha^{xs} \) (cf [2]). In principle, both the neutrons and the protons will thus contribute to the normal fluid. This effect becomes particularly important near the superfluid transition temperatures. Far below the transition temperature, the temperature is essentially zero and one would expect both \( \alpha^{ns} \) and \( \alpha^{ps} \) to be effectively zero. Anyway, in order to account for a ‘normal’ part of each superfluid we need to begin with a four-fluid system. Otherwise there would be no explicit distinction between the entrainment coefficients, e.g. between those that couple the fluids to the entropy, and those (i.e. \( \alpha^{ns} \), \( \alpha^{ce} \) and \( \alpha^{pe} \)) that couple the species of particles to each other. As the electrons are not superconducting, we can assume that \( \alpha^{es} = 0 \).

The second possibility is perhaps closer to the orthodox approach to superfluids [10, 27], in which one introduces an ad hoc separation of the mass density into a ‘superfluid’ density and a ‘normal’ fluid density. In our framework, this would involve dividing each density \( n_x \) into a piece associated with the condensate and a piece corresponding to the quasiparticle excitations. This philosophy was used in the recent calculation of entrainment parameters by Gusakov and Haensel [49].

At the end of the day, the separation of a fluid constituent into two pieces is purely formal. One could not conceivably separate the superfluid condensate from the excitations in a real physical system. The mathematical formalism that we have developed should be flexible enough to allow us to represent both approaches to the finite-temperature problem. As we will discuss elsewhere, the main issue concerns the interpretation of the various variables (especially the entrainment coefficients). This is an important problem, the solution of which impacts on the values of the various dissipation coefficients discussed here.

It should also be noted that, despite neutron stars being self-gravitating bodies, we will not couple gravity to our model at this point. In the Newtonian context this is easily done by introducing an external force [2]. We do not include this force here as our main focus is on the fluid aspects of the problem.

Let us proceed by assuming that there are four independent densities, e.g. the number densities for the neutrons, protons and electrons, as well as the entropy density. These are denoted by \( (n_n, n_p, n_e, s) \). \textit{A priori}, the system will consist of four independent velocities, e.g. the neutron, proton, electron and entropy velocities which are denoted by \( (v_n^s, v_p^s, v_e^s, v_s^s) \). The entropy fluid is massless. We consider this four-fluid system first, in order to identify all the entrainment coefficients, before reducing to the case where all ‘normal’ fluids flow together. We believe that this strategy is more natural than that which imposes the existence of only two distinct transport velocities from the beginning.
For the four-fluid system, the ‘first law of thermodynamics’ is given by the expression
\[
d\mathcal{E} = \mu_n \, d n_n + \mu_p \, d n_p + \mu_e \, d n_e + T \, ds + \alpha_{np} \, d u_{np}^2 + \alpha_{ne} \, d u_{ne}^2
\]
\[+ \alpha_{ep} \, d u_{ep}^2 + \alpha_{ns} \, d u_{ns}^2 + \alpha_{es} \, d u_{es}^2, \tag{69}\]
where we are again assuming the Galilean invariant, locally isotropic form for \( \mathcal{E} \). Because of overall charge neutrality we have
\[
n_p = n_e. \tag{70}\]
This, in conjunction with equations (6) and (7), implies
\[
\Gamma_p = \Gamma_e = -\Gamma_n \tag{71}\]
and
\[
m^\alpha = m^p + m^e \equiv m, \tag{72}\]
where \( m^\alpha \), \( m^p \) and \( m^e \) are the neutron, proton and electron masses, respectively. The electrons should also closely track the protons because of the electromagnetic attraction \([18, 28]\). Thus, it is a good approximation to simply set
\[
v'_p = v'_e \equiv v'_c. \tag{73}\]
In a later work, we will consider electromagnetic coupling in more detail in order to fully justify this constraint (see also \([50]\)).

In the absence of heat conduction \( v'_s = v'_c \). Because of this the entropy flux is often written as
\[
s v'_{sc} = s v'_c + \frac{1}{T} q'_s \Rightarrow q'_s = s T w'_{sc}, \tag{74}\]
where \( q'_s \) is the heat-flux vector. However, by introducing the heat-flux vector we break the natural symmetry of our formalism since \( v'_c \) is put on a special footing. In general, \( q'_s \) is dynamically independent of the other two velocities. A standard result is to argue that any deviations of \( v'_s \) from co-motion with \( v'_c \) will be due to gradients of the temperature. Because of the definition of temperature, cf equation (69), the entropy flux still represents a dynamically independent variable, since \( T \) depends on the relative velocities \( w'_{xy} \), which include \( v'_s \). This kind of velocity dependence is manifested in the momenta \( p'_i \), for instance, via the entrainment coefficients \( \alpha_{xy}^{\alpha} \).

It is convenient to define
\[
\alpha^{nc} = \alpha^{np} + \alpha^{ne}, \quad \alpha^{cs} = \alpha^{ps} + \alpha^{es}, \tag{75}\]
and
\[
\mu_c = \mu_p + \mu_e. \tag{76}\]
Now the three independent fluid momentum densities can be written
\[
\pi^{n}_i = g_{ij} \left( m_{n} v_{n}^j - 2 \left[ (\alpha^{nc} + \alpha^{as}) w_{nc}^j - \alpha^{ns} w_{sc}^j \right] \right), \tag{77}\]
\[
\pi^{c}_i = n_p p_{c}^p + n_e p_{c}^e = g_{ij} \left( m_{n} v_{n}^j + 2 \left[ \alpha^{nc} w_{nc}^j + \alpha^{es} w_{sc}^j \right] \right), \tag{78}\]
\[
\pi^{s}_i = 2 g_{ij} \left( \alpha^{ns} w_{nc}^j - [\alpha^{ns} + \alpha^{cs}] w_{sc}^j \right). \tag{79}\]
Defining
\[
D^{c}_{i} = D^{p}_{i} + D^{e}_{i}, \tag{80}\]
we find for the three force densities, cf (46),

$$f_i^n = \partial_t \pi_i^n + \nabla_j \left( v_j^n \pi_i^n + D^{n j}_i \right) + n_n \nabla_i \left( \mu_n - \frac{1}{2} m v_i^n \right) + \pi_j^n \nabla_i v_j^n, \quad (81)$$

$$f_i^c = \partial_t \pi_i^c + \nabla_j \left( v_j^c \pi_i^c + D^{c j}_i \right) + n_c \nabla_i \left( \mu_c - \frac{1}{2} m v_i^c \right) + \pi_j^c \nabla_i v_j^c, \quad (82)$$

$$f_i^s = \partial_t \pi_i^s + \nabla_j \left( v_j^s \pi_i^s + D^{s j}_i \right) + s \nabla_i T + \pi_j^s \nabla_i v_j^s. \quad (83)$$

We can put the finishing touches on our ‘neutron star model’ by building the thermodynamic ‘forces’ and ‘fluxes.’ Key to this will be the assertion that the thermodynamic forces is to be equal to zero, then the coefficients multiplying each force must vanish. This assumption does not seem unreasonable. We must first consider the constraints that result from the equalities among the particle creation rates $\Gamma_n, \Gamma_p$ and $\Gamma_e$.

By setting $\Gamma_e = \Gamma_p = -\Gamma_n$, and imposing linear independence, we determine that

$$\gamma_n = -\gamma_p = -\gamma_e = \gamma_{pe} = \gamma_{ee} \quad (84)$$

and

$$\tau^n_n = -\tau^n_p = -\tau^n_e, \quad (85)$$

$$2\tau^n_j = \tau^b_p + \tau^p_v, \quad (86)$$

$$\tau^p_p = \tau^e_v. \quad (87)$$

If we define

$$\chi^{nn} = \chi^{np} + \chi^{ne}, \quad (88)$$

$$\chi^{cc} = \chi^{bp} + 2\chi^{pc} + \chi^{ce},$$

then the fluid forces are

$$f_i^n = -2g_{ij} \left( R^{nn} w_{ns}^j + R^{nc} w_{cs}^j + S^{nn} \omega_{ns}^j + S^{nc} \omega_{cs}^j \right), \quad (89)$$

$$f_i^c = -2g_{ij} \left( R^{cc} w_{ns}^j + R^{cs} w_{cs}^j + S^{cc} \omega_{ns}^j + S^{cs} \omega_{cs}^j \right), \quad (90)$$

and using the fact that we are considering a closed system, we also have

$$f_i^n = -f_i^s = -\left( f_i^n + f_i^s + \frac{1}{2} m \Gamma_n g_{ij} w_{nc}^j \right). \quad (91)$$

Finally, the independent dissipation coefficients become, cf equation (64),

$$D^{nn}_{ij} = -g_{ij} \left[ \tau^n_n (\mu^n - \mu^n) + \zeta^{nn} \Theta_{ns} + \zeta^{nc} \Theta_{cs} \right] + 2g_{jk} g_{jl} \left[ \eta^{nn} \Theta_{ns}^{kl} + \eta^{nc} \Theta_{cs}^{kl} \right] + \epsilon_{ijk} \left[ (S^{nn} w_{ns}^k + S^{nc} \omega_{cs}^k + \sigma^{nn} \omega_{ns}^k + \sigma^{nc} \omega_{cs}^k) \right], \quad (92)$$

$$D^{cc}_{ij} = -g_{ij} \left[ 2\tau^n_n (\mu^n - \mu^n) + \zeta^{cc} \Theta_{ns} + \zeta^{cs} \Theta_{cs} \right] + 2g_{jk} g_{jl} \left[ \eta^{cc} \Theta_{ns}^{kl} + \eta^{cs} \Theta_{cs}^{kl} \right] + \epsilon_{ijk} \left[ (S^{cc} w_{ns}^k + S^{cs} \omega_{cs}^k + \sigma^{cc} \omega_{ns}^k + \sigma^{cs} \omega_{cs}^k) \right]. \quad (93)$$

We also get

$$D^{s s}_{ij} = D_{ij} = D^{nn}_{ij} - D^{cc}_{ij} = -g_{ij} \left[ \zeta^{nn} \Theta_{ns} - \tau^n_n (\mu^n - \mu^n) - (\zeta^{nn} + \zeta^{nc}) \Theta_{ns} \right] - \frac{1}{2} m g_{ij} \left[ \eta^{nn} \Theta_{ns}^{kl} - (\eta^{nn} + \eta^{cc}) \Theta_{ns}^{kl} - (\eta^{cc} + \eta^{nc}) \Theta_{cs}^{kl} \right] - \epsilon_{ijk} \left[ (S^{nn} + S^{cc}) w_{ns}^k + (S^{cc} + S^{cs}) \omega_{cs}^k + (\sigma^{nn} + \sigma^{cc}) \omega_{ns}^k + (\sigma^{cc} + \sigma^{cs}) \omega_{cs}^k) \right]. \quad (94)$$

It is worth noting that this construction ensures that $D_{ij}$ is symmetric, as required for a closed system.
The final step is to use the positivity of the entropy to set constraints on the dissipation coefficients. It follows from equations (65) and (68), and some algebra, that the entropy creation rate is

$$T \Gamma_s = \zeta^{tot} + 2 \eta^{tot} g_{ij} g_{jk} \Theta^i \Theta^j + 2 g_{ij} g_{kl} \left[ \Theta^i \Theta^j \Theta^k \Theta^l \right] \begin{bmatrix} \eta^{nn} & \eta^{nc} \\ \eta^{cn} & \eta^{cc} \end{bmatrix} \begin{bmatrix} \Theta^{kl} \\ \Theta^{kl} \end{bmatrix}$$

$$+ \left[ \mu^e - \mu^h \Theta_{ns} \Theta_{cs} \right] \begin{bmatrix} \gamma_{nn} & -\tau_{nn} \\ -\tau_{nn} & \xi_{nn} \end{bmatrix} \begin{bmatrix} 2 \tau_{nn} \\ \zeta_{nc} \end{bmatrix} \begin{bmatrix} \Theta_{ns} \\ \Theta_{cs} \end{bmatrix}$$

$$+ 2 g_{ij} \left[ w_{ni}^i w_{ci}^i w_{ni}^i w_{ci}^i \right] \begin{bmatrix} R_{nn} & R_{nc} \\ R_{cn} & R_{cc} \end{bmatrix} \begin{bmatrix} S_{nn} \\ S_{nc} \end{bmatrix} \begin{bmatrix} \sigma_{nn} \\ \sigma_{nc} \end{bmatrix} \begin{bmatrix} W_{ns} \\ W_{cs} \end{bmatrix}.$$ (95)

This expression shows that the problem is extremely rich. Our system has, in general, 19 independent dissipation coefficients. This can be compared to the general case for superfluid helium which, according to Putterman [40], requires 13 independent coefficients. It may be appropriate to ask whether the problem is, in fact, so complex that we have no chance of making real progress on solving it. One can certainly make this argument. However, we believe that this would be quite wrong. The key point is that we need to be able to describe all significant physical phenomena that our system may exhibit. The general equations that we have derived should allow us to do this. They certainly highlight the many ways in which a dissipative multi-fluid system differs from the standard Navier–Stokes equations. Of course, it is also important not to make the description impossible to comprehend. To avoid falling into this trap, it would seem sensible to reduce the problem further and focus attention on one feature at a time. This way one can hope to develop a better intuition of the interpretation of the different dissipation coefficients. It should also be remembered that, even though they may be mathematically allowed, many of the terms in equation (95) may vanish in a real physical system. Especially if some of the constituents are superfluid.

In principle, it is now a straightforward matter to determine the constraints on the dissipation coefficients, since each defining matrix is symmetric and real, and thus linear-algebra theorems can be employed to determine when each contribution is positive definite, specifically, if all the eigenvalues of the defining matrix are positive then the quadratic form will be positive definite. For example, it is not difficult to show that if both $\eta^{nn}$ and $\eta^{cc}$ are negative then there are no values for the coefficients that will lead to a positive-definite form. On the other hand, if $\eta^{nn} \geq 0$ and $\eta^{cc} \geq 0$, then the quadratic form is positive if and only if

$$\eta^{nn} \eta^{cc} > (\eta^{nc})^2.$$ (96)

A similar analysis for the second and third matrices would be tractable analytically, since the characteristic equations for the eigenvalues will be third and fourth order, respectively. However, it is not clear that working out the detailed conditions would add any further insight at the present time. After all, many of the coefficients in our general expression are new, and it may be more productive to begin by seeing if we can interpret their physical meaning.

4.2. Final reduction: the two-fluid equations

In order to connect with previous work on superfluid neutron stars we conclude our discussion by making one further reduction. We will neglect heat conduction and let the entropy flow
with the charged fluid components. This simply means that we take \( v_i = v_i^c \) in all equations in the previous section. Then we only have two independent velocities in the problem, and hence only need two equations of motion. Given this, it is natural to combine the momenta of the entropy and the charged components and consider

\[
\tilde{\pi}_i^c = \pi_i^c + \pi_i^s = g_{ij} \left[ m n_c v_i^c + 2(\alpha^{nc} + \alpha^{ns}) w_{nc}^i \right] = m n_c g_{ij} \left( v_i^c + \varepsilon_c w_{nc}^i \right)
\]

where we have defined the entrainment parameter

\[
\varepsilon_c = \frac{2(\alpha^{nc} + \alpha^{ns})}{m n_c}.
\]

Next, combining equations (82) and (83) we see that

\[
\begin{align*}
\partial_t \tilde{\pi}_i^c + \nabla_j \left( v_i^c \tilde{\pi}_j^c \right) + \nabla_i \left( \varepsilon_c m n_c v_i^c \right) + s \nabla_i T &= -\hat{f}_n^i - \frac{1}{2} m (1 - 2 \varepsilon_n) \Gamma_n w_{nc}^i - \nabla_j \left( D^{nj} - D^{ni} \right). \\
\end{align*}
\]

Rewriting this equation in terms of the explicit transport velocities, as in [2, 51], we obtain

\[
\begin{align*}
m n_c \left( \partial_t + \varepsilon_c \nabla_i \right) \left( v_i^c + \varepsilon_c w_{nc}^i \right) + n_c \nabla_i \left( \mu_c - \frac{1}{2} m v_i^c \right) &\quad = -\hat{f}_n^i - \frac{1}{2} m (1 - 2 \varepsilon_n) \Gamma_n w_{nc}^i - \nabla_j \left( D^{nj} - D^{ni} \right),
\end{align*}
\]

where we have used the fact that the Lie derivative (which measures the rate at which a quantity changes relative to its motion) of a covector \( a_i \) follows from

\[
\varepsilon_c = \frac{2(\alpha^{nc} + \alpha^{ns})}{m n_c}.
\]

The two equations of motion, equations (100) and (102), can be directly compared to equations (176) and (177) in [2]. If we introduce

\[
\hat{f}_i^\text{mut} = \hat{f}_n^i + \frac{1}{2} m (1 - 2 \varepsilon_n) \Gamma_n w_{nc}^i - \nabla_j D^{nj},
\]

we find that the two results agree perfectly in the zero-temperature \((T = 0)\), non-dissipative \((D^{nj} = D^{nj} = 0)\) limit. Thus our representation reduces to the familiar result in the relevant limit. In the more general case that we have considered, several new aspects appear naturally. Finite temperature effects are explicitly represented via the presence of the \( s \nabla_i T \) term in the equation for the charged fluid/entropy equation. Implicitly, we are also accounting for the possibility of a normal part of the neutrons via the entrainment coefficient \( \alpha^{ns} \). The two fluids are coupled in a number of ways. In particular, the mutual coupling via \( \hat{f}_n^i \) and \( D^{nj} \), which accords with Newton’s third law, should be noted. The presence of the ‘total’ dissipation in terms of \( D_{nj} \) is also relevant.

It is instructive to consider the above equations of motion in the context of what we already know about dissipative fluids. In particular, we would like to understand which of our many coefficients can be considered as known, and which represent potentially new aspects of the problem. To carry out this exercise, let us begin on familiar territory and consider the total dissipation part. From our definitions, we have

\[
-\nabla^j D_{ij} = \nabla^j \left( \Theta^\text{tot} g_{ij} + 2 \eta^\text{tot} g_{ik} g_{jl} \Theta^{kl} \right)
\]
which should be compared to the standard right-hand side of the Navier–Stokes equations [27]
\[ \nabla^j \left[ \eta \left( \nabla_j v_i + \nabla_i v_j - \frac{2}{3} g_{ij} \nabla^l v^l \right) \right] + \nabla_l (\zeta \nabla^l v^i) = \nabla^j \left( 2\eta \Theta_{ij} \right) + g_{ij} \nabla^j (\zeta \Theta). \]  
(106)
Comparing the two expressions we can readily identify the coefficients of the standard shear and bulk viscosities as
\[ \eta = \eta^{\text{tot}} \quad \text{and} \quad \zeta = \zeta^{\text{tot}}. \]  
(107)
In addition to the mutual friction coefficients discussed below, these are the only two dissipation coefficients that have previously been discussed for neutron stars. We have recently summarized the current understanding of the shear viscosity in [20]. There is no similar summary for the bulk viscosity but the reader may find [12], where the bulk viscosity damping of the unstable r-modes is discussed, useful. Recent work on hyperon bulk viscosity, and the superfluid suppression thereof, is also highly relevant [52, 53].

Not surprisingly, the interior coupling terms are not as straightforward to explain. This is quite obvious since we have tried to write a more or less general expression for the permissible terms in a multi-fluid context where previous studies have been limited. The problem is exacerbated by the fact that the bulk of the extant work was carried out in the ‘orthodox’ framework [27]. To interpret the result, we consider the combination
\[ \hat{f}_{ni} = \nabla_j D^{nji} = -2R^{nn} u^{nc}_{ic} + \epsilon_{ijk} \left( \nabla_j S^{nn} \right) w^{nc}_{ic} + \nabla_j \left[ g_{ij} (\tau^{nn} (\mu_n - \mu_c) - \zeta^{nn} \Theta_{nc}) + 2\eta^{nn} \Theta^{nc} + \sigma^{nn} \omega^{nc} \right], \]  
(108)
which follows from the various definitions and some straightforward algebra. Given this expression we can interpret the individual terms in the following way: since it is proportional to the relative velocity, the term involving $R_{nn}$ is associated with a force analogous to standard resistivity. Meanwhile, $S^{nn}$ relates to a Magnus-type force which acts orthogonally to the relative flow [21]. The two terms involving $\Theta_{nc}$ and $\Theta^{nc}_{ij}$ are simply analogues of the standard bulk and shear viscosities, although here it is the expansion and shear of the relative flow which is important. The coefficient $\tau^{nn}$ accounts for effects due to the fluids being driven out of chemical equilibrium by the flow. In recent work, Carter and collaborators have referred to this type of interaction as ‘transfusion’. The final term in equation (108), related to $\sigma^{nn}$, appears to be new.

To conclude our discussion, let us consider the particular form of the mutual friction force which arises because of a balance between (i) the Magnus force acting on quantised superfluid neutron vortices and (ii) scattering of electrons off of the vortices. In the case when the vortices are straight, this results in a force [21]
\[ f_{ni}^{mf} = B' \rho_n \epsilon_{ij} \kappa^i w^{nc}_{ic} + B \rho_n \epsilon_{ijk} \epsilon_{lm} \hat{\kappa}^{j} \hat{\kappa}^{l} w^{nc}_{im}, \]  
(109)
acting on the neutrons. (Correcting an overall sign error resulting from writing the constituent indices in the wrong order in the corresponding formula in [21].) Here $\rho_n$ is the vortex surface density and $\kappa^i$ is a vector which is aligned with the vortices and has magnitude $h/2m_n$ (a hat denotes a unit vector); see [21] for further details.

Comparing this expression to equation (108) we see that only the first two terms in the latter are relevant. In order to incorporate equation (109) in our framework, we need to be able to identify the coefficients $R_{nn}$ and $\nabla_i S^{nn}$. We easily see that we must have
\[ \nabla_i S^{nn} = B' \rho_n \epsilon_{ij} \kappa^i. \]  
(110)
The corresponding expression for $R_{nn}$ follows from
\[ B\rho_n \epsilon_{ijk} \epsilon_{lm} \hat{\kappa}^{j} \hat{\kappa}^{l} w^{nc}_{im} = B\rho_n \epsilon_{ijk} (\hat{\kappa}^i \hat{\kappa}^j - g_{ij}) w^{nc}_{im} = 2R_{nn} w^{nc}_{im}. \]  
(111)
Multiplying each side by $w^i_{en}$ we find that (formally)

$$R^{nn} = \frac{1}{2} \rho_{en} v_\kappa B\left[1 - \left(\tilde{u}^i_{en} \tilde{k}_i\right)^2\right]. \quad (112)$$

The above expressions can in principle be used, given the explicit estimates for the two coefficients $B$ and $B'$ obtained in [21], to incorporate mutual friction in our dissipative neutron star model. This will allow us to extend our previous studies of superfluid neutron star oscillations and instabilities in a number of exciting directions.

Finally, it is also worth mentioning an alternative form of the mutual friction force, originally proposed by Gorter and Mellink [54], see also [55]. Representing the mutual interaction in the case of superfluid turbulence, where the vortices form a complicated tangle, the corresponding form for $f_{mf}$ is proportional to the cube of the relative velocity. This type of force was considered for neutron stars for the first time in the recent work by Peralta et al [56]. In order to include it in the present framework we would simply have the coefficient $R^{nn}$ proportional to the square of the relative velocity. It should be noted, however, that even though the form of this force is reasonably well established for superfluid helium the exact coefficient remains to be derived in the neutron star case. This is an important problem that deserves attention in the near future.

5. Concluding remarks

We have described a general formalism which can be used to model dissipative multi-fluid systems. Our discussion was based on the key conservation laws for mass, energy and linear and angular momentum, together with recent progress in deriving the equations of motion for the non-dissipative problem from a variational principle [2–5], and the classic Onsager symmetry principle which provides a strategy for including dissipative terms in such a way that the second law of thermodynamics is satisfied. Following this approach, we derived a set of multi-fluid equations which has a number of advantages over the ‘standard’ formalism used to model, in particular, superfluid He$^4$. The most important difference is that we have made due distinction between velocities and momenta, which means that the entrainment effect is accounted for in a natural way.

As an application of the general formalism, we considered two models for a neutron star core. The first model accounts for three distinct fluids, the superfluid neutrons, a conglomerate of charged particles (protons and electrons) and the entropy. We have shown that this model, in principle, requires 19 distinct dissipation coefficients to be determined. Detailed microscopic analysis is needed to understand the relevance of the majority of these coefficients. We also discussed how the normal fluid fraction of neutrons that should be present at finite temperatures can be associated with the entrainment parameter $a^{nu}$ which encodes how the equation of state depends on the relative motion between the neutrons and the entropy. Our second model is obtained by neglecting heat conductivity. This couples the entropy to the charged fluid and reduces the number of independent degrees of freedom to 2. We discussed the form of the various dissipative terms in this two-fluid model and interpreted their physical meaning. We also showed how the mutual friction, which is mediated by the superfluid vortices, can be accounted for within our model.

One can envisage a number of interesting applications of the framework that we have developed in this paper. In the case of neutron stars, the possibilities range from the viscous damping of neutron star oscillations due to electron–electron scattering [20] to intricate issues concerning the effect of entrainment on the mutual friction force [21]. Since our model allows for much more complicated multi-fluid dynamics than has so far been considered in the literature, one would have to be somewhat suspicious about any statements about the damping
of superfluid neutron star oscillations which were made without consideration of the various
degrees of freedom. It would certainly be interesting to revisit the problem of oscillation
modes driven unstable by gravitational radiation [12] and investigate the role of the true multi-
fluid degrees of freedom. Our hope is this work will inspire a significant improvement of
our understanding of superfluid neutron star dynamics, and perhaps generate input from other
communities (e.g. chemistry) that have much experience with multi-fluid systems.

Finally, it is worth emphasizing that our formalism is quite general, and one would expect
it to be useful also in other problem areas. Applying it to other multi-phase flow problems,
as discussed in for example [6–9], seems relatively straightforward. There are also more
exotic possibilities. In particular, we believe that our formalism could be quite relevant for
problems involving fluid analogues of various curved spacetimes, such as black holes and
big-bang cosmologies, that can be potentially observable in laboratory systems (see [57] for a
discussion of ‘sonic’ horizons in superfluids, and [58] for a recent review). For this kind of
application, it would be particularly important to establish whether the ‘spacetime’ effects are
likely to be rendered experimentally undetectable by various forms of dissipation.

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Appendix A

Here we will motivate why the conservative part of the total stress tensor $C^j_i$ can be obtained
from a variation of the action with respect to the metric. The analysis is by no means rigorous,
but it does lead to the result that has been obtained rigorously in [2]. Our aim is to give the
reader not familiar with all aspects of action-based field theory some basic physical intuition,
using simple energy-conservation arguments.

Let us consider a displacement $\delta \xi^i$, in a small amount of time $\delta t = t_2 - t_1$ of a
closed, isolated system such that the individual constituent particle numbers as well as their
corresponding velocities are kept fixed. We will also suppose that each constituent undergoes
the same displacement so that their respective velocities can be written as

$$v^i_x \approx \frac{\delta \xi^i}{\delta t}.$$  \hspace{1cm} (A.1)

Because the system is closed and isolated, the total energy will be conserved. From equations
(15) and (28) the change in total energy can be written as

$$\delta U = U(t_2) - U(t_1)$$

$$= \int_{t_1}^{t_2} \frac{dt}{d\tau} \int_V \left( \sum_x n^i_x p^{i}_x - L \right) dV$$

$$= \sum_x \int_{t_1}^{t_2} dt \int_V \frac{\partial}{\partial t} (v^i_x \pi^{i}_x) - \delta \int_V L dV, \hspace{1cm} (A.2)$$

where the last term represents the difference in the integral before (at time $t_1$) and after (at
time $t_2$) the displacement is put in place. If we now impose that the total energy does not
change $\delta U = 0$, and that the velocities remain fixed, i.e. $\partial v_i / \partial t = 0$, then upon insertion of equation (13) we find
\[
\delta \int_V \mathcal{L} \, dV = \sum_x \int_{t_1}^{t_2} \frac{d}{dt} \int_V v_i \left( \mathcal{T}_{ij} - \nabla_j C_{ij} \right) \, dV = \int_V \delta \left( f_i - \nabla_j C_{ij} \right) \, dV, \tag{A.3}
\]
where the second equality follows because the displacement is the same for each constituent, we have used equation (14), and we have kept only terms linear in $\delta \xi^i$. Since the system is isolated, the net force $f_i = 0$. Finally, after integrating by parts, we can write
\[
\delta \int_V \mathcal{L} \, dV \approx \frac{1}{2} \int_V C_{ij} \left( \nabla_i \delta \xi_j + \nabla_j \delta \xi_i \right) \, dV - \int_{\partial V} C_{ij} n_j \delta \xi^i \, dA. \tag{A.4}
\]
The last term will vanish if we impose suitable boundary conditions on the control volume (or assume that the boundary is well outside the fluids).

We now need to establish that the deformation of the system set up by the displacement $\delta \xi^i$ induces a change in the metric, which in fact will define the $\delta$ variation in equation (A.4). Consider an active coordinate transformation where the metric is pushed (at time $t_2$) to the new points $\mathbf{x}' = \mathbf{x} + \delta \xi$. Denoting the transformed metric as $\overline{g}_{ij}$, ordinary tensor analysis implies
\[
\overline{g}_{ij}(\mathbf{x}') = \frac{\partial x^k}{\partial x^i} \frac{\partial x^l}{\partial x^j} g_{kl}(x^m + \delta x^m). \tag{A.5}
\]
Next we define
\[
\delta g_{ij} = \overline{g}_{ij}(x') - g_{ij}(x'), \tag{A.6}
\]
and, keeping terms up to linear order in the displacement, we find
\[
\delta g_{ij} = \nabla_i \delta \xi_j + \nabla_j \delta \xi_i. \tag{A.7}
\]
Thus in equation (A.4) we can rewrite the displacement on the right-hand side in terms of $\delta g_{ij}$. Next we must determine how to vary the fundamental variables $n_x$ and $n'_{ix}$ under the displacement. Recall that we are assuming that the number of $x$-th-constituent particles $N_x$ remains fixed. This implies
\[
\delta N_x = \delta \int_V n_x \, dV = 0 \implies \delta n_x = -\frac{1}{2} \frac{\delta \sqrt{g}}{\sqrt{g}}, \tag{A.8}
\]
where matrix theory (cf [48]) can be used to show
\[
\delta \sqrt{g} = \frac{1}{2} \sqrt{g} g^{ij} \delta g_{ij}. \tag{A.9}
\]
Thus, using also equation (A.7), we know how the displacement changes $n_x$. Furthermore, we also know how the particle number density currents $n'_{ix}$ change since $n'_{ix} = n_x v_i$, but the velocities remain fixed, and so $\delta n'_{ix} = v_i' \delta n_x$. Therefore, we have
\[
\delta n_x = -\frac{1}{2} n_x g^{ij} \delta g_{ij}, \quad \delta n'_{ix} = -\frac{1}{2} n'_{ix} g^{ij} \delta g_{ij}. \tag{A.10}
\]
The small variation indicated on the left-hand side of equation (A.4) is the difference between the integral before the displacement and the integral after the displacement. But as the integral is a function only of $n_x$, $n'_{ix}$ and $g_{ij}$, and the variations $\delta n_x$ and $\delta n'_{ix}$ induced by the displacement are equivalent to a variation of the metric, we see that $\delta$ on the left-hand side of equation (A.4) can be equally understood as a variation of the metric. Thus, equation (A.4) can be reduced to
\[
\int_V \left( C_{ij} - \frac{2}{\sqrt{g}} \frac{\partial (\sqrt{g} \mathcal{L})}{\partial g_{ij}} \right) \delta g_{ij} \, dV = 0, \tag{A.11}
\]
which implies
\[ C^{ij} = \frac{2}{\sqrt{g}} \frac{\partial (\sqrt{g} L)}{\partial g_{ij}}. \] (A.12)

This result should be compared to the standard formula used in general relativity (see, for example, [43]).

Finally, using the less-general internal energy given in equation (36) we find
\[ C_{j}^{i} = \Psi \delta_{j}^{i} + \sum_{x} n_{x}^{j} p_{x}^{i}, \] (A.13)

which is equation (38) in the main body of the paper.

Appendix B

Consider the simplest application, where each particle number is conserved independently and there are no forces acting on a constituent, i.e. \( f_{x}^{i} = 0 \). Then
\[ \partial_{t} n_{x} + \nabla i n_{x}^{i} = 0 \] (B.1)

and
\[ \frac{\partial}{\partial t} (n_{x} p_{x}^{i}) + \nabla j (p_{x}^{j} n_{x}^{i}) = n_{x} \nabla i p_{x}^{i} + n_{x}^{i} \nabla j p_{x}^{j}. \] (B.2)

Recall that
\[ \nabla i \Psi = -\sum_{x} \left( n_{x} \nabla i p_{x}^{0} + n_{x}^{i} \nabla j p_{x}^{j} \right). \] (B.3)

We see that the term appearing in the summation is precisely the term on the right-hand side of the above ‘Euler’ equation. Hence, unless \( \Psi \) takes the particular form of
\[ \Psi = \psi^{1}(p_{0}^{1}, p_{1}^{1}) + \psi^{2}(p_{0}^{2}, p_{1}^{2}) + \cdots + \psi^{N}(p_{0}^{N}, p_{1}^{N}) = \sum_{\sigma=1}^{N} \psi^{\sigma}(p_{0}^{\sigma}, p_{1}^{\sigma}), \] (B.4)

which results in the exact differential
\[ \nabla i \psi^{\sigma} = -\left( n_{\sigma} \nabla i p_{0}^{\sigma} + n_{\sigma}^{i} \nabla j p_{1}^{\sigma} \right) \] (B.5)

for each constituent, such that
\[ n_{\sigma} = -\frac{\partial \psi^{\sigma}}{\partial p_{0}^{\sigma}}, \quad n_{\sigma}^{i} = -\frac{\partial \psi^{\sigma}}{\partial p_{1}^{\sigma}}, \] (B.6)

the right-hand side of the ‘Euler’ equation will not be an exact differential, and so cannot be written as a total divergence.

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A flux-conservative formalism for convective and dissipative multi-fluid systems

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