Teitelbaum’s exceptional zero conjecture in the function field case

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1
Introduction

Let $E$ be an elliptic curve over $\mathbb{Q}$ of conductor $N$ and $p$ a prime number. Since $E$ is modular, it corresponds to a cusp form of weight 2, i.e. an analytic function $f(z)$ on the complex upper half plane $\mathcal{H}$ such that $f\left(\frac{az + b}{cz + d}\right) = (cz + d)^2 f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and it fulfills suitable growth conditions at the points in $\mathbb{P}^1(\mathbb{Q})$ (the so-called cusps). Mazur, Tate and Teitelbaum construct in [MTT] a $p$-adic $L$-function $L_p(E, s)$ using a $p$-adic measure associated to $f(z)$ via the modular symbols $\Psi_E: \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \rightarrow \mathbb{Q}$ associated to $E$.

Furthermore, they propose $p$-adic versions of the Birch and Swinnerton-Dyer conjecture. It turns out that $L_p(E, s)$ always vanishes at the central point 1 if $E$ has split multiplicative reduction at $p$. In this case they conjecture (the so-called exceptional zero conjecture):

$$L'_p(E, 1) = \frac{\log_p(q_E)}{\text{ord}_p(q_E)} L_{\text{norm}}(E, 1),$$

where $q_E$ is the Tate period of $E$ at $p$, $\log_p$ the $p$-adic logarithm with $\log_p(p) = 1$ and $L_{\text{norm}}(E, s)$ a suitable normalisation of the complex $L$-function. This was proved by Greenberg and Stevens [GS]. The proof is very technical and makes use of Hida’s theory of $p$-adic $L$-functions.

In their paper [MTT], Mazur and Tate state slightly stronger, refined versions of their conjectures, avoiding the construction of a $p$-adic $L$-function. In particular, they give an exponentiated version of the exceptional zero conjecture:

$$\prod_{a \mod p^n \atop (a, p) = 1} a^{\text{ord}_p(q_E) \Psi_E((a/p^n)-(i\infty))} \equiv \left(\frac{q_E}{p^{\text{ord}_p(q_E)}}\right)^{\Psi_E((0)-(i\infty))} \text{ in } (\mathbb{Z}/p^n\mathbb{Z})^*$$

for every $n \geq 1$ (proved by de Shalit [dS] under certain assumptions).

Now let $E$ be an elliptic curve over $F = \mathbb{F}_q(T)$ of conductor $n\infty$ with split multiplicative reduction at the place $\infty = \frac{1}{T}$ and $F_\infty$ the completion of $F$ at $\infty$. By Drinfeld’s work [Dr], $E$ has a uniformisation

$$\overline{M}_0(n) \rightarrow E$$
by a Drinfeld modular curve and it corresponds to a Drinfeld cusp form on 
\( \Omega_\infty = \mathbb{P}^1(C_\infty) - \mathbb{P}^1(F_\infty) \) (\( C_\infty \) a completion of an algebraic closure of \( F_\infty \)) or, equivalently, to a cuspidal harmonic cochain \( c_\infty \) on the Bruhat-Tits tree \( T_\infty \) (with values in \( \mathbb{Q} \)). Teitelbaum ([T2]) defines the group of modular symbols to be

\[ M := \text{Div}^0(\mathbb{P}^1(F)) \]

and denotes the divisor \((r) - (s)\) by \([r, s]\). He defines a map

\[ M \rightarrow \mathbb{Q} \]

\[ [r, s] \mapsto [r, s] \cdot c_\infty, \]

where \([r, s] \cdot c_\infty\) is the sum of the values of \( c_\infty \) along the axis connecting the ends corresponding to \( r \) and \( s \) on \( T_\infty \).

If \( E \) has split multiplicative reduction at a further place \( p \) different from \( \infty \), he defines a measure on \( \mathcal{O}_p \) analogous to the classical case. (Here \( \mathcal{O}_p \) is the ring of integers of the completion of \( F \) at \( p \).) However, the absence of a logarithm obstructs the definition of a \( p \)-adic \( L \)-function analogous to the number field case. But it is still possible to formulate the refined exceptional zero conjecture:

\[
\left( \frac{q_p}{\pi^\text{ord}_p(q_p)} \right)^{[0,\infty] \cdot c_\infty} = \lim_{n \to \infty} \prod_{a \mod \pi_p^n, (a, \pi_p) = 1} \text{ord}_p(q_p)
\]

where \( q_p \) is the Tate period associated to \( E \) at \( p \). This is carried out in [T2], where also some numerical evidence can be found.

In this paper we prove Teitelbaum’s conjecture up to a root of unity. Therefore we can regard our result as an analogue of the Greenberg-Stevens formula. We achieve this by developing an analogue of Darmon’s integration theory on \( \mathcal{H}_p \times \mathcal{H} \), where \( \mathcal{H}_p = \mathbb{P}^1(C_p) - \mathbb{P}^1(\mathbb{Q}_p) \) is the \( p \)-adic upper half plane [Da]. This is done in sections 2.1 and 2.2 for arbitrary global function fields and choices of \( p \) and \( \infty \), i.e., in a more general setting than in [T2].

The function field situation is considerably easier than the number field case. Since both places are non-archimedean, we can work with certain harmonic cochains on the product of the Bruhat-Tits trees at \( p \) and \( \infty \):

\[ \mathcal{T} = T_p \times T_\infty. \]

By this we mean the set product of the vertices/edges. The symmetry of this situation will prove very useful for later calculations.
Since $E$ has split multiplicative reduction at $p$ and $\infty$, it corresponds to two harmonic cochains $c_p$ and $c_\infty$ on $T_p$ and $T_\infty$, respectively. In section 2.3 we compare them by relating them to a certain space of automorphic newforms.

Darmon defines a period $I_\psi \in \mathbb{C}^*_p$ (for details compare [Da]). The order at $p$ and the $p$-adic logarithm of this period are closely related to special values of a certain partial $L$-function and the first derivative of the $p$-adic $L$-function attached to $E/\mathbb{Q}$, respectively. He obtains the following reformulation of the result of Greenberg and Stevens:

$$\log_p(I_\psi) = \frac{\log_p(q_E)}{\ord_p(q_E)} \ord_p(I_\psi).$$

We define an analogous period in section 3.1. Furthermore, we show an exponentiated version of the above formula (theorem 3.2):

$$I_\psi = \zeta \cdot q_p^{\ord_p(I_\psi)}. $$

where $\zeta$ is a root of unity.

In the final section we assume the setting in [T2] and show that Teitelbaum’s conjecture is equivalent (up to a root of unity) to the above theorem.

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Notations

$F$ is the function field of a geometrically connected smooth projective algebraic curve $C$ over the finite field $\mathbb{F}_q$ of characteristic $p$. For a closed point $p \in C$ denote by

- $A_p = \Gamma(C - \{p\}, \mathcal{O}_C)$ the ring of regular functions on $C - \{p\}$ and more generally, $A_S = \Gamma(C - S, \mathcal{O}_C)$ the ring of regular functions on $C - S$ for a finite set $S$ of closed points,
- $\nu_p$ the corresponding valuation of $F$ at $p$,
- $\pi_p$ a uniformiser,
- $F_p$ the completion of $F$ at $p$ with valuation ring $\mathcal{O}_p$ and maximal ideal $m_p$,
- $C_p$ the completion of an algebraic closure of $F_p$,
- $\Omega_p = \mathbb{P}^1(C_p) - \mathbb{P}^1(F_p)$ Drinfeld’s upper half plane and $\Omega_p^* = \Omega_p \cup \mathbb{P}^1(F)$.
- $\mathcal{A}$ is the adele ring of $F$ with ring of integers $\mathcal{O}$. These decompose into a $p$-part and a ”finite” part with respect to $p$:

$$\mathcal{A} = A_{f,p} \times F_p \quad \text{and} \quad \mathcal{O} = \mathcal{O}_{f,p} \times \mathcal{O}_p.$$ 

Elements of $\mathcal{A}$ are denoted by $x = (x_q)_q$.

A subgroup $\Gamma$ of $GL_2(F)$ is arithmetic (w.r.t. $p$) if it is commensurable with $GL_2(A_p)$. For such a group, we write

$$\overline{\Gamma} := \Gamma / \Gamma \cap \text{Center}(GL_2(F))$$

and

$$\overline{\Gamma} := \overline{\Gamma}^{ab/\text{torsion}}.$$ 

Sometimes we use $G$ and $Z$ to denote $GL_2$ and its centre.

For an ideal $n$ of $A_p$, we define the Hecke congruence subgroup associated to $n$ (w.r.t. $p$):

$$\Gamma_0^p(n) := \left\{ \gamma \in GL_2(A_p) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod n \right\}.$$
1 Preliminaries

1.1 The Bruhat-Tits Tree for $\text{PGL}_2(F_p)$

The vertices of the tree $\mathcal{T}_p$ are defined to be homothety classes of $\mathcal{O}_p$-lattices (i.e. free $\mathcal{O}_p$-modules of rank 2) in $F_p^2$. Two such classes $[L], [L']$ are connected by an edge if there exists $L'' \in [L]$ such that $L'' \subseteq L$ and $L/L''$ has length 1 as $\mathcal{O}_p$-module, i.e. $L/L'' \cong \mathcal{O}_p/\pi_p \mathcal{O}_p$.

We denote by $\mathcal{E}(\mathcal{T}_p), \vec{\mathcal{E}}(\mathcal{T}_p), \mathcal{V}(\mathcal{T}_p)$ respectively the set of edges, oriented edges and vertices of $\mathcal{T}_p$. We consider the following action of $\text{GL}_2(F_p)$ from the left on $\mathcal{T}_p$:

$$\gamma_*[L] := [L\gamma^{-1}]$$

We note that this action is different from the one considered in [Se]. It induces the following identifications:

$$\text{GL}_2(F_p)/\text{GL}_2(\mathcal{O}_p) : Z(F_p) \xrightarrow{\gamma} \mathcal{V}(\mathcal{T}_p)$$

$$\xrightarrow{\gamma_*v_0} \gamma_*v_0$$

and

$$\text{GL}_2(F_p)/\mathcal{I}_p : Z(F_p) \xrightarrow{\gamma} \vec{\mathcal{E}}(\mathcal{T}_p)$$

$$\xrightarrow{\gamma_*e_0} \gamma_*e_0,$$

where $Z$ is the centre of $\text{GL}_2$.

$$\mathcal{I}_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_p) : c \equiv 0 \mod p \right\}$$

the Iwahori group, $v_0 = [\mathcal{O}_p^2]$, $v_{-1} = [\pi_p \mathcal{O}_p \oplus \mathcal{O}_p]$, $e_0$ the edge with origin $o(e_0) = v_{-1}$ and terminus $t(e_0) = v_0$, and $e_0$ is the edge opposite to $e_0$. We also note that $\text{GL}_2(F_p)$ acts canonically from the left on $\text{GL}_2(F_p)/\text{GL}_2(\mathcal{O}_p) : Z(F_p)$ and $\text{GL}_2(F_p)/\mathcal{I}_p : Z(F_p)$.

**Lemma 1.1.** Each edge $e$ of $\mathcal{T}_p$ can uniquely be written as a product $\gamma e_0$, where $\gamma$ is an element of the following disjoint union of sets:

$$\left\{ \begin{pmatrix} \pi_p^n & u \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}, u \in F_p/\pi_p^n \mathcal{O}_p \right\}$$

$$\cup \left\{ \begin{pmatrix} \pi_p^n & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \pi_p & 0 \end{pmatrix} : n \in \mathbb{Z}, u \in F_p/\pi_p^n \mathcal{O}_p \right\}.$$

**Proof.** This follows easily from the above identification (see [Gek1]).
The ends of \( T_p \) (= infinite half-lines, equivalent if coinciding up to a finite number of edges) correspond to \( \partial \Omega_p = \mathbb{P}^1(F_p) \). We label the path \( A(\infty, 0) \) from \( \infty \) to 0 by \((v_i)_{-\infty < i < \infty}\) resp. \((e_i)_{i \in \mathbb{Z}}\), where \( t(e_i) = v_i \). In the following, we will call \( v_0 \) the base vertex, \( e_0 \) the base edge and \( A(\infty, 0) \) the base axis of \( T_p \). In particular, \( e_i \) (and \( v_i \)) is represented by \( \left( \pi^i_p, 0 \right) \) in lemma 1.1. (For more detail see \[GR, 1.3\] and \[Se\].)

Let \( M \) be an abelian group. A harmonic cochain with values in \( M \) is a map \( \varphi : \hat{E}(T_p) \to M \) that is alternating, i.e. \( \varphi(\overline{e}) = -\varphi(e) \) for all \( e \in \hat{E}(T_p) \) and harmonic, i.e.

\[
\sum_{t(e) = v} \varphi(e) = 0
\]

for all \( v \in V(T_p) \). We denote the group of \( M \)-valued harmonic cochains by \( H(T_p, M) \).

The identification of \( \partial \Omega_p \) with the ends of \( T_p \) induces a bijection between \( H(T_p, M) \) and \( \text{Meas}_0(\partial \Omega_p, M) \), the set of \( M \)-valued measures (i.e., finitely additive functions on compact open subsets) of total mass 0 on \( \partial \Omega_p \). If \( \varphi \) is a harmonic cochain, then \( \mu(U(e)) := -\varphi(e) \) defines a measure of total mass 0, where the set \( U(e) \) consists of all ends going through \( e \in \hat{E}(T_p) \).

Let \( \Gamma \) be an arithmetic subgroup of \( \text{GL}_2(F) \). Then the group of \( M \)-valued cuspidal harmonic cochains for \( \Gamma \) is the following:

\[
H(T_p, M)^\Gamma := \left\{ \varphi \in H(T_p, M) : \varphi \text{ is } \Gamma\text{-invariant and of finite support modulo } \Gamma \right\}.
\]

The group \( \text{GL}_2(F_p) \) acts on \( H(T_p, M)^\Gamma \) by

\[
\gamma \ast \varphi := \varphi \circ \gamma^{-1}.
\]

This action is compatible with its action on \( T_p \).

There is an injection

\[
j : \hat{T} \longrightarrow H(T_p, \mathbb{Z})^\Gamma,
\]

with finite cokernel, such that for a chosen vertex \( v \in V(T_p) \), \( j(\alpha)(e) \) counts the number of \( \gamma e \) lying in \( A(v, \alpha v) \) as \( \gamma \) varies in \( \hat{T} \) \( \text{[GR 3.3.3]} \). It is proved to be bijective if \( A_p \) is a polynomial ring and \( \Gamma \) is a Hecke congruence group.
Although unknown to be surjective in general, no case is known where this map fails to be bijective.

If \( M \) is a subgroup of the complex numbers \( \mathbb{C} \), there is a pairing \( <, \rangle_\Gamma \) on \( \mathbb{H}_1(\mathcal{T}_p, M) \), given by

\[
< \varphi_1, \varphi_2 >_\Gamma := \sum_{e \in \mathcal{E}(\Gamma \setminus \mathcal{T}_p)} \varphi_1(e) \varphi_2(e) \frac{1}{\# \Gamma_e},
\]

where

\[
\Gamma_e := \text{Stab}_\Gamma(e)
\]

is the (finite) stabiliser of \( e \) in \( \Gamma \).

**Definition 1.2.** Let \( \Gamma' \) be a subgroup of \( \Gamma \) of finite index and \( \{ \gamma_1, \ldots, \gamma_r \} \) be a set of representatives for \( \Gamma' \setminus \Gamma \). We define the following trace map:

\[
\text{Tr}_{\Gamma'} : \mathbb{H}_1(\mathcal{T}_p, M)^{\Gamma'} \rightarrow \mathbb{H}_1(\mathcal{T}_p, M)^{\Gamma} \varphi \mapsto \sum_{i=1}^{r} \gamma_i^{-1} * \varphi.
\]

**Lemma 1.3.** The pairing above is compatible with the trace map, i.e.:

\[
< \varphi, \psi >_{\Gamma'} = < \text{Tr}_{\Gamma'} \varphi, \psi >_{\Gamma},
\]

where \( \varphi \in \mathbb{H}_1(\mathcal{T}_p, M)^{\Gamma'} \) and \( \psi \in \mathbb{H}_1(\mathcal{T}_p, M)^{\Gamma} \).

**Proof.** By definition,

\[
< \text{Tr}_{\Gamma'} \varphi, \psi >_{\Gamma} = \sum_{i=1}^{r} < \gamma_i^{-1} * \varphi, \psi > = \sum_{i=1}^{r} \sum_{e \in \mathcal{E}(\Gamma \setminus \mathcal{T}_p)} \varphi(\gamma_i e) \varphi(e) \frac{1}{\# \Gamma_e}.
\]

The \( \Gamma \)-orbit of \( e \) decomposes into \( \Gamma' \)-orbits as follows:

\[
\Gamma e = \Gamma' \gamma_1 e \cup \ldots \cup \Gamma' \gamma_r e.
\]

This decomposition is not necessarily disjoint. We observe that for all \( i \), the number of \( j \)'s such that \( \Gamma' \gamma_i e = \Gamma' \gamma_j e \), is equal to

\[
\frac{\# \Gamma_e}{\# \Gamma'_{\gamma_i e}}.
\]

This concludes the proof. \( \square \)
1.2 Automorphic forms

Let $\mathcal{K} \subseteq G(\mathcal{O})$ be an open subgroup and $L$ a subfield of the complex numbers. Later we will be interested in groups of the form

$$\mathcal{K}_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}) : c \equiv 0 \mod n \right\}$$

for an effective divisor $n$ of $F$.

**Definition 1.4.** The space of automorphic cusp forms $W_p(\mathcal{K}, L)$ at $p$ consists of functions

$$f : G(\mathbb{A}) \to L$$

such that

(i) for all $\gamma \in G(F)$, $g \in G(\mathbb{A})$ and $k \in \mathcal{K}Z(F_p)$,

$$f(\gamma g k) = f(g)$$

and

(ii) for all $g \in G(\mathbb{A})$,

$$\int_{F \backslash \mathbb{A}} f\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0,$$

where $dx$ is a Haar measure on $F \backslash \mathbb{A}$.

In particular, an automorphic cusp form is a function on

$$Y_p(\mathcal{K}) := G(F) \backslash G(\mathbb{A}) / \mathcal{K} \cdot Z(F_p).$$

There is a natural $G(F_p)$-action (from the right) on the space

$$V_{sp,p}(L) := \{ f : \mathbb{P}^1(F_p) \to L : f \text{ locally constant} \}/L.$$

This is called the $(L$-valued$)$ special representation $\varrho_{sp,p}$ of $G(F_p)$. An automorphic cusp form $f \in W_p(\mathcal{K}, L)$ transforms like $\varrho_{sp,p}$ if the $L$-vector space generated by its right $G(F_p)$-translates is isomorphic to a finite number of copies of $\varrho_{sp,p}$. We denote the space of such forms by $W_{sp,p}(\mathcal{K}, L)$.

Assume that $\mathcal{K}$ decomposes as a product

$$\mathcal{K} = \mathcal{K}_{f,p} \times \mathcal{I}_p,$$
where $\mathcal{I}_p$ is the Iwahori group. Then we can choose a system $R_p$ of representatives for the finite set $G(F) \backslash G(\mathcal{A}_{f,p})/\mathcal{K}_{f,p}$ and we define
\[ \Gamma_{\mathcal{I}} := G(F) \cap \mathcal{I} \mathcal{K}_{f,p} \mathcal{I}^{-1} \]
to be the intersection in $G(\mathcal{A}_{f,p})$ for $\mathcal{I} \in R_p$. Every element $g \in G(\mathcal{A})$ can be written as a product
\[ g = \gamma (x \times 1_p) (k \times 1_p) (1_{f,p} \times g_p), \]
for some $\gamma \in G(F) \subseteq G(\mathcal{A})$, $k \in \mathcal{K}_{f,p}$, $g_p \in G(F_p)$, $1_p \in G(F_p)$ and $1_{f,p} \in G(\mathcal{A}_{f,p})$ the respective units, and a uniquely determined $x \in R_p$. This leads to the following identification ([GR, 4.5.4]):
\[ \Phi_p : Y_p(\mathcal{K}) \xrightarrow{\cong} \bigsqcup_{x \in R_p} \Gamma_{\mathcal{I}} \backslash G(F_p) / Z(F_p) \cdot \mathcal{I}_p \]
\[ [g] \longmapsto [g_p], \]
where $[\cdot]$ denotes the double class of an element. Of course, the group on the right is the same as $\bigsqcup_{x \in R_p} \mathcal{E}(\Gamma_{\mathcal{I}} \backslash \mathcal{T}_p)$ (see sect. 1.1). The following important theorem can e.g. be found in [GR, 4.7.6]:

**Theorem 1.5 (Drinfeld).** Assume $\mathcal{K} = \mathcal{K}_{f,p} \times \mathcal{I}_p$ with an open subgroup $\mathcal{K}_{f,p} \subseteq G(\mathcal{O}_{f,p})$. Under the identification above, the following spaces are isomorphic:
\[ W_{sp,p}(\mathcal{K}, L) \cong \bigoplus_{x \in R_p} H^1(\mathcal{T}_p, L)^{\Gamma_{\mathcal{I}}}. \]

One defines a theory of Hecke operators on cuspidal harmonic cochains which is compatible with the Hecke algebra on automorphic forms. Using the Petersson product, it is therefore possible to define new- and oldforms (see below).

**Remark.** Let $F = \mathbb{F}_q(T)$, $\deg(p) = 1$ and $\mathcal{K} = \mathcal{K}_0(\mathfrak{p} \mathfrak{n}) = \mathcal{K}_{f,p} \times \mathcal{I}_p$, where $\mathfrak{n}$ is a positive divisor such that $\mathfrak{p} \not| \mathfrak{n}$. Then $R_p$ consists only of one element and
\[ W_{sp,p}(\mathcal{K}_0(\mathfrak{p} \mathfrak{n}), L) \cong H^1(\mathcal{T}_p, L)^{\Gamma_0}. \]

The space of automorphic cusp forms admits a non-degenerate pairing, the so-called Petersson product, given by
\[ (f_1, f_2)_{\mu} := \int_{Y_p(\mathcal{K})} f_1(g) \overline{f_2(g)} d\mu(g), \]
where $\mu$ is a suitably normalised Haar measure on $G(A)/Z(G(F_\infty))$.

With the identifications of theorem 1.5, the Haar measure becomes

$$\mu(e) = \frac{1}{\#(\Gamma_{\mathbb{Q}})_e},$$

and the pairing in section 1.1 is seen to be the Petersson product. (More details can be found in [GR, 4.8].)

1.3 Theta functions and Tate curves

Let $E/F$ be an elliptic curve of conductor $p^n$ with split multiplicative reduction at $p$ and $\Gamma$ an arithmetic group. Then $E$ is a Tate curve at $p$, i.e.

$$E(F_p) \cong F_p^\times / q_{E,p} \mathbb{Z}_E,$$

where $q_{E,p} \in F_p^\times$ is the Tate period. This isomorphism can be constructed explicitly by means of theta functions. The contents of this section are a review of [GR, Sections 5, 7 and 9].

**Theorem 1.6 (Gekeler-Reversat).** Let $\omega \in \Omega_p$ be a randomly chosen point. For $\alpha \in \Gamma$, one defines

$$u_\alpha : \begin{cases} \Omega_p^* & \longrightarrow C_p^\times \\ z & \mapsto \prod_{\varepsilon \in \Gamma} \frac{z - \varepsilon \omega}{z - \varepsilon \alpha \omega}. \end{cases}$$

This product converges to a $C_p^\times$-valued holomorphic theta function, i.e. for all $\beta \in \Gamma$, $u_\alpha$ satisfies a functional equation

$$u_\alpha(\beta z) = c_\alpha(\beta) u_\alpha(z).$$

The definition of $u_\alpha$ is independent of the chosen $\omega$ and only depends on the class of $\alpha$ in $\Gamma$. This induces a group homomorphism

$$c : \begin{cases} \Gamma & \longrightarrow \text{Hom}(\Gamma, F_p^\times) \\ \alpha & \mapsto (\beta \mapsto c_\alpha(\beta)) \end{cases}.$$

Furthermore, the map

$$\bar{\cdot} : \begin{cases} \Gamma \times \Gamma & \longrightarrow F_p^\times \\ (\alpha, \beta) & \mapsto c_\alpha(\beta) \end{cases}$$

defines a symmetric bilinear pairing.
We observe that by definition, \( u_\gamma(\infty) = 1 \) for all \( \gamma \in \Gamma \).

The elliptic curve \( E \) has a uniformisation

\[ \overline{M}_0(n) \to E \]

by a Drinfeld modular curve, defined over \( F \). The affine algebraic curve \( M_0(n) \) is the moduli scheme representing the functor of Drinfeld modules of rank 2 with level \( n \)-structure. The rigid analytic variety \( M^\text{an}_0(n) \) associated to \( M_0(n) \otimes_F C_p \) decomposes as:

\[ M^\text{an}_0(n) = \coprod_{\bar{\gamma} \in R_p} \Gamma_{\bar{\gamma}} \backslash \Omega_p. \]

This curve can be compactified to a projective curve \( \overline{M}_0(n) \) by adding a finite number of points (the so-called cusps), which can be seen as coming from \( \mathbb{P}^1(F) \subset \partial \Omega_p \). As above, we get a decomposition:

\[ \overline{M}^\text{an}_0(n) = \coprod_{\bar{\gamma} \in R_p} \overline{M}_{\Gamma_{\bar{\gamma}}}(C_p) = \coprod_{\bar{\gamma} \in R_p} \Gamma_{\bar{\gamma}} \backslash \Omega^*_p, \]

and its Jacobian decomposes accordingly. It is enough to work on a chosen component \([\text{GR (9.6)}]\).

Assume that \( E \) is the strong Weil curve in its isogeny class. It corresponds to a primitive Hecke eigenform

\[ c_p = (c_{p,\bar{\gamma}})_{\bar{\gamma} \in R_p} \in \bigoplus_{\bar{\gamma} \in R_p} H^\text{new}_2(\mathcal{T}_p, \mathbb{Z})^{\Gamma_{\bar{\gamma}}}, \]

with rational eigenvalues. I.e. \( c_p \) is normalised such that for all \( \bar{\gamma} \in R_p \), \( c_{p,\bar{\gamma}} \in j(\Gamma_{\bar{\gamma}}) \) but \( c_{p,\bar{\gamma}} \notin nj(\Gamma_{\bar{\gamma}}) \) for \( n > 1 \) \([\text{GR (9.1)}]\). We now choose a component \( \Gamma_{\bar{\gamma}} \backslash \Omega^*_p \), as above, and consider the corresponding newform \( c_p = c_{p,\bar{\gamma}} \). Let \( \gamma_p \in \Gamma_{\bar{\gamma}} \) be its preimage under \( j \). Then the analytic uniformisation of \( E \) at \( p \) is given by the diagram:

\[
\begin{array}{cccccc}
1 & \to & \Gamma_{\bar{\gamma}} & \overset{\pi}{\to} & \text{Hom}(\Gamma_{\bar{\gamma}}, C_p^\times) & \to & \text{Jac}(\overline{M}_{\Gamma_{\bar{\gamma}}}(C_p)) & \to & 0 \\
& & \downarrow{\text{ev}_p} & & \downarrow{\text{pr}_p} & & \\
1 & \to & \Lambda_p & \to & C_p^\times & \to & E(C_p) & \to & 0,
\end{array}
\]

where \( \text{ev}_p \) is the evaluation map at \( \gamma_p \), \( \Lambda_p \) is the image of \( \Gamma_{\bar{\gamma}} \) under \( \text{ev}_p \) and \( \text{pr}_p \) is the map induced by \( \text{ev}_p \). It is explicitly given by:

\[ \text{pr}_p : \text{Jac}(\overline{M}_{\Gamma_{\bar{\gamma}}}(C_p)) \to E(C_p) \]

\[ (a) \to \frac{u_{\gamma_p}(a)}{u_{\gamma_p}(b)} \]

where \( a, b \in \Omega^*_p \).
There exists a divisor $d$ of $q^{\deg(p)} - 1$ and $t \in F_p^\times$ with $|t|_p < 1$ (both dependent on $c_p$) such that $\Lambda_p = \mu_d \times t^Z$ ($\mu_d$ the $d$-th roots of unity in $F_p^\times$). The Tate period is given by $q_{E,p} = t^d$.

In the case that $A_p$ is isomorphic to a polynomial ring, Gekeler shows in [Gek2] that $d = 1$. In particular,

$$\Lambda_p = q_{E,p}^Z$$

if $F = \mathbb{F}_q(T)$ and $p$ is of degree 1.

An analogue of the well known Manin-Drinfeld theorem was established in [Gek3]:

**Theorem 1.7 (Gekeler).** For each congruence subgroup $\Gamma_0^p(n)$ of $GL_2(F)$, the cuspidal divisor group

$$C_\Gamma := \text{Div}_0(M_0(n)(C_p) - M_0(n)(C_p)) \mod \text{(principal divisors)}$$

$$= \text{Div}_0(\Gamma_0^p(n) \backslash \mathbb{P}^1(F)) \mod \text{(principal divisors)}$$

is finite.

### 1.4 Multiplicative integrals

Based on one of Bertolini’s ideas, the following multiplicative integrals are considered in [L]. Let $X$ be a topological space such that a basis for its topology is given by its open compact subsets. Given a continuous function $f : X \to C_p^\times$ and a measure $\mu \in \text{Meas}(X, \mathbb{Z})$, i.e. a finitely additive function on the compact open subsets of $X$, we define:

$$\int_X f(t) d\mu(t) := \lim_{\alpha \to \infty} \prod_{U \in C_\alpha} f(t)^{\mu(U)},$$

where $\{C_\alpha\}$ is the direct system of finite covers of $X$ by compact open subsets, and $t \in U$ is chosen arbitrarily. This integral exists and is independent of the choices of the $t$’s. Furthermore, it induces a continuous homomorphism from the group of continuous functions $C(X, C_p^\times)$ to $C_p^\times$.

For future reference, we state one important (but obvious) property:

**Lemma 1.8.** Let $X$ and $\mu$ be as above. Then

$$\int_X c d\mu(t) = c^{\mu(X)},$$

for all $c \in C_p^\times$. 
If we choose $X$ to be the boundary $\partial \Omega_p$, the computation of a multiplicative integral can be accomplished as follows. Choose a vertex $v \in \vec{E}(\mathcal{T}_p)$ and, for $e \in \vec{E}(\mathcal{T}_p)$ pointing away from $v$, define $\text{dist}_v(e)$ to be the distance between $o(e)$ and $v$ (i.e. the number of edges of a geodesic). Then, for $f \in C(\partial \Omega_p, C^\infty_p)$,

$$\int_{\partial \Omega_p} f(t) d\mu(t) = \lim_{n \to \infty} \prod_{\text{dist}_v(e) = n} f(t_e) \mu(U(e)),$$

where $t_e \in U(e)$ is chosen arbitrarily. This is independent of the choice of $v$, which is normally chosen to be the base vertex $v_0$ (see sect. 1.1).

Another important property is the following:

**Lemma 1.9.** Let $U$ be an open subset of $\partial \Omega_p$ and $\gamma \in GL_2(F_p)$. Then

$$\int_{\gamma U} f \gamma d\mu_\gamma = \int_{U} (f \circ \gamma) d\mu,$$

where $\gamma \mu(V) = \mu(\gamma^{-1}V)$.

It is possible to obtain a multiplicative version of Teitelbaum’s Poisson inversion formula ([T1]):

**Proposition 1.10.** Let $\alpha \in \Gamma_p(\infty)$. Then

$$\frac{u_\alpha(z_2)}{u_\alpha(z_1)} = \int_{\partial \Omega_p} \frac{t-z_2}{t-z_1} d\mu_{j(\alpha)}(t),$$

for all $z_1, z_2 \in \Omega_p^*$. 

**Proof.** This is a combination of propositions 8 and 24 in [L].  \qed
2 Integration on $\mathcal{T}_p \times \mathcal{T}_\infty/\Gamma$

2.1 Harmonic cochains

We choose a further place $\infty$ of $F$, different of $p$, and set

$$S := \{p, \infty\}.$$ 

We define the following subgroup of $GL_2(\mathbb{A}_S)$:

$$\Gamma := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{A}_S) : \det(\gamma) \in \mathbb{F}_q \text{ and } c \in \mathfrak{n} \right\}$$

where $\mathfrak{n} \leq \mathbb{A}_S$ is an ideal corresponding to an effective divisor of $F$, which we will also call $\mathfrak{n}$, s.t. $p, \infty \not|\mathfrak{n}$. We will use the same notation for the corresponding ideals in $\mathbb{A}_p$ and $\mathbb{A}_\infty$.

**Lemma 2.1.** Let $q \in S$. Then the group $\Gamma \cap SL_2(\mathbb{F}_q)$ is dense in $SL_2(\mathbb{F}_q)$.

**Proof.** Let $\Gamma^c$ be the closure of $\Gamma \cap SL_2(\mathbb{F}_q)$ in $SL_2(\mathbb{F}_q)$. By the strong approximation theorem, $\mathbb{A}_S$ and $\mathfrak{n}$ are dense in $\mathbb{F}_q$. Therefore, since $\Gamma$ contains the subgroups

$$\begin{pmatrix} 1 & 0 \\ \mathfrak{n} & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \mathbb{A}_S \\ 0 & 1 \end{pmatrix},$$

we see that $\Gamma^c$ contains the full triangular subgroups

$$\begin{pmatrix} 1 & 0 \\ \mathbb{F}_q & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \mathbb{F}_q \\ 0 & 1 \end{pmatrix}.$$ 

Let $\alpha \in SL_2(\mathbb{F}_q)$ and put $a := \alpha(\infty)$, $b := \alpha(0)$. The matrix

$$\beta := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ (a - b)^{-1} \\ 0 \\ 1 \end{pmatrix} \in \Gamma^c$$

also sends $\infty$ to $a$ and 0 to $b$. It follows that $\beta^{-1}\alpha$ is a diagonal matrix in $SL_2(\mathbb{F}_q)$, i.e., of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}.$$ 

But all such diagonal matrices are contained in $\Gamma^c$, since

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^{-1} - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x - 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}$$

(this is a special case of Whitehead’s lemma). \qed
In the following we use the letters $w$ and $s$ to denote vertices and edges on $T_\infty$, whereas we keep $v$ and $e$ for $T_p$.

**Corollary 2.2.** The group $\Gamma$ acts transitively on the sets of unoriented edges of $T_p$ and $T_\infty$. Its action on oriented edges has two orbits:

$$\vec{E}(T_p) = \Gamma e_0 \coprod \Gamma \overline{e}_0 \quad \text{and} \quad \vec{E}(T_\infty) = \Gamma s_0 \coprod \Gamma \overline{s}_0.$$  

**Proof.** We observe that $\vec{E}(T_p) = \text{SL}_2(F_p)e_0 \coprod \text{SL}_2(F_p)\overline{e}_0$. Let $e$ be in the $\text{SL}_2(F_p)$-orbit of $e_0$. The set

$$T_e := \{ \gamma \in \text{SL}_2(F_p) : \gamma e_0 = e \}$$

is open in $\text{SL}_2(F_p)$, since it is a coset of the stabiliser of $e_0$. Lemma 2.1 implies that

$$T_e \cap \Gamma \neq \emptyset,$$

i.e., $e$ is in the $\Gamma$-orbit of $e_0$. We can apply the same argument for the orbit of $\overline{e}_0$. The statement for $T_\infty$ follows analogously. \qed 

We want to study harmonic cochains on the product $\mathcal{T} = T_p \times T_\infty$. Let $\vec{E}(\mathcal{T})$ (resp. $\mathcal{V}(\mathcal{T})$) denote the product of the corresponding sets of (directed) edges (resp. vertices). We observe that for the base vertex $v_0 \in \mathcal{V}(T_p),

$$\Gamma_{v_0} := \Gamma \cap \text{Stab}(v_0) = \Gamma_0^\infty(n)$$

and for the base edge $e_0 \in \mathcal{V}(T_p),

$$\Gamma_{e_0} := \Gamma \cap \text{Stab}(e_0) = \Gamma_0^\infty(pn).$$

The stabilisers of the other vertices in $T_p$ are $\Gamma$-conjugate to $\Gamma_{v_0}$. For an edge $e \in \vec{E}(T_p)$, either the stabiliser of $e$ or $\overline{e}$ is conjugate to $\Gamma_{e_0}$, depending on whether $e$ is in the $\Gamma$-orbit of $e_0$ or $\overline{e}_0$. The same holds for the base vertex/edge $w_0, s_0$ on $T_\infty$ with the corresponding subgroups of $\text{GL}_2(A_p)$.

**Definition 2.3.** Let $M$ be an abelian group. The space of $\Gamma$-invariant harmonic cochains $\mathbb{H}_\Gamma(\mathcal{T}, M)^\Gamma$ with values in $M$, cuspidal with respect to $S$, consists of functions $c : \vec{E}(\mathcal{T}) \to M$ such that:

(i) $\forall \gamma \in \Gamma \forall (e, s) \in \vec{E}(\mathcal{T}) : c(\gamma e, \gamma s) = c(e, s)$

and $c$ has compact support modulo $\Gamma$,

(ii) $\forall e \in \vec{E}(T_p) : c(e, \cdot) \in \mathbb{H}(T_\infty, M)$
2.1 Harmonic cochains

(iii) \( \forall s \in \mathcal{E}(\mathcal{T}_\infty) : c(\cdot, s) \in H(\mathcal{T}_p, M) \).

**Remark.** These conditions imply that

\[
\forall e \in \mathcal{E}(\mathcal{T}_p) : c(e, \cdot) \in H(\mathcal{T}_\infty, M)^{\Gamma_e} \quad \text{and} \quad \forall s \in \mathcal{E}(\mathcal{T}_\infty) : c(\cdot, s) \in H(\mathcal{T}_p, M)^{\Gamma_s}.
\]

**Definition 2.4.** Let \( L \) be a subgroup of the complex numbers. The subspace of \( p \)-newforms \( H^{p\text{-new}}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(\mathfrak{p} n)} \) of \( H(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(\mathfrak{p} n)} \) is defined to be the orthogonal complement of the images of the natural inclusions of \( H(\mathcal{T}_\infty, M)^{\Gamma_{v_0}} \) and \( H(\mathcal{T}_\infty, M)^{\Gamma_{v^{-1}}} \) into \( H(\mathcal{T}_\infty, M)^{\Gamma_0} \), induced by the equality

\[
\Gamma_{e_0} = \Gamma_{v_0} \cap \Gamma_{v^{-1}},
\]

with respect to the Petersson product.

The following is an analogue of [Da, Lemma 1.3,3.]:

**Proposition 2.5.** The map

\[
H_{p\text{-new}}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(\mathfrak{p} n)} \quad \rightarrow \quad H(\mathcal{T}_p, L)^{\Gamma_e} \quad c \quad \mapsto \quad c_\infty(\cdot) := c(e_0, \cdot)
\]

is a well-defined isomorphism.

**Proof.** Since \( \mathcal{E}(\mathcal{T}_p) = \Gamma_{e_0} \sqcup \cdots \sqcup \Gamma_{e_0} \), \( c \) is completely determined by \( c(e_0, \cdot) \). On the other hand, given \( c_\infty \in H_{p\text{-new}}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(\mathfrak{p} n)} \), we can define a cochain \( c \in H(\mathcal{T}_p, L)^{\Gamma_e} \) by \( c(\gamma e_0, s) := c_\infty(\gamma^{-1} s) \) and \( c(\gamma e_0, s) := -c_\infty(\gamma^{-1} s) \) for \( \gamma \in \Gamma \), respectively, provided the condition of harmonicity at the vertices \( v_0 \) and \( v^{-1} \) is satisfied (observe that there are also only two \( \Gamma \)-orbits for vertices on \( \mathcal{T}_p \)). Let \( \{ \gamma_1, \ldots, \gamma_r \} (r = q^{\deg(p)} + 1) \) be a set of representatives of the quotient \( \Gamma_0^\infty(\mathfrak{n})/\Gamma_0^\infty(\mathfrak{n}) \). Then the map

\[
\gamma_i \mapsto \gamma_i v^{-1}
\]

defines a bijection between \( \Gamma_0^\infty(\mathfrak{n})/\Gamma_0^\infty(\mathfrak{n}) \) and the \( \Gamma_{v_0} \)-orbit of \( v^{-1} \), i.e. the set of edges with origin \( v_0 \). Condition (iii) above yields

\[
0 = \sum_{o(\mathfrak{e}) = v_0} c(\mathfrak{e}, s) = \sum_{i=1}^r c(\gamma_i e_0, s) = -\sum_{i=1}^r c_\infty(\gamma_i^{-1} s).
\]

I.e., \( c_\infty \) is in the kernel of the trace map

\[
\Gamma_0^\infty(\mathfrak{n}) : H(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(\mathfrak{n})} \quad \rightarrow \quad H(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(\mathfrak{n})} \quad \psi \quad \mapsto \quad \sum \psi \circ \gamma_i^{-1} = \sum \gamma_i \ast \psi
\]
By lemma \ref{lemma1.3},

\[ <\varphi, \psi >_{\Gamma^\infty_0(n)} = < \text{Tr}_{\Gamma^\infty_0(n)} \varphi, \psi >_{\Gamma^\infty_0(n)} \]

for \( \varphi \in H_0(\mathcal{T}_\infty, L)^{\Gamma^\infty_0(n)} \) and \( \psi \in H_0(\mathcal{T}_\infty, L)^{\Gamma^\infty_0(n)} \).

The edges exiting from \( v_{-1} \) are the orbit of \( e_0 \) in \( \Gamma_{v_{-1}} \). Similar reasoning as above implies that

\[ Tr_{\Gamma_{v_{-1}}}^{\Gamma^\infty_0(pn)}(c_\infty) = 0. \]

Hence \( c_\infty \) is orthogonal to \( H_1(\mathcal{T}_\infty, L)^{\Gamma_{v_{-1}}} \).

Now we consider the following group:

\[ \tilde{\Gamma} := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(A_S) : c \in n \right\}. \]

We observe that

\[ \tilde{\Gamma}_{v_0} \cap \text{GL}_2(A_\infty) = \Gamma^\infty_0(n) \]

and

\[ \tilde{\Gamma}_{e_0} \cap \text{GL}_2(A_\infty) = \Gamma^\infty_0(pn), \]

where \( \tilde{\Gamma} := \text{Stab}_{\tilde{\Gamma}}(\cdot) \).

We define the space of \( \tilde{\Gamma} \)-invariant, \( S \)-cuspidal harmonic cochains

\[ H_1(\mathcal{T}, M)^{\tilde{\Gamma}} \]

with values in an abelian group \( M \) as in definition \ref{def2.3}. This is a subspace of \( H_1(\mathcal{T}, M)^{\Gamma} \).

By the \( S \)-unit theorem, there exists \( u \in F \) such that

\[ A_S^\times = \mathbb{F}_q^\times \times u^\mathbb{Z} \]

and

\[ \text{div}(u) = h_p \cdot [p] - h_\infty \cdot [\infty]. \]

The group \( \tilde{\Gamma} \) is generated by \( \Gamma \) and the matrix

\[ \gamma = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in \tilde{\Gamma} - \Gamma. \]

In the case that \( h_p \) is odd, \( \gamma e_0 \) is in the \( \Gamma \)-orbit of \( \overline{e}_0 \) and there exists \( \beta \in \Gamma \) such that \( \gamma e_0 = \beta \overline{e}_0 \). Alternatively, if \( h_p \) is even, there exists \( \beta \in \Gamma \) such that \( \gamma e_0 = \beta e_0 \). In either case we define

\[ \alpha_p := \beta^{-1} \gamma \in \tilde{\Gamma} - \Gamma. \]
Then the Atkin-Lehner involution at $p$ on $\mathbb{H}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(pn)}$ is given by:

$$W_p c_\infty := \alpha_p * c_\infty.$$ 

An easy calculation shows that the image of $\mathbb{H}(\mathcal{T}, L)^{\Gamma}$ in $\mathbb{H}_{p\text{-new}}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(pn)}$ (via the isomorphism in proposition 2.5) is

$$\mathbb{H}_{p\text{-new, }-}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(pn)},$$

the space of those $p$-newforms which are eigenforms under $W_p$ of eigenvalue $(-1)^{h_p}$. This shows:

**Corollary 2.6.** The following diagram (with horizontal isomorphisms) is commutative:

\[
\begin{array}{ccc}
\mathbb{H}_*(\mathcal{T}, M)^{\Gamma} & \xrightarrow{\cong} & \mathbb{H}_{p\text{-new, }-}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(pn)} \\
\bigcap & \bigcap & \\
\mathbb{H}_*(\mathcal{T}, M)^{\Gamma} & \xrightarrow{\cong} & \mathbb{H}_{p\text{-new}}(\mathcal{T}_\infty, L)^{\Gamma_0^\infty(pn)}.
\end{array}
\]

**Remark.** Similarly, by interchanging $p$ and $\infty$ and exchanging $e_0$ with $s_0$ we get:

\[
\begin{array}{ccc}
\mathbb{H}_*(\mathcal{T}, L)^{\Gamma} & \xrightarrow{\cong} & \mathbb{H}_{\infty\text{-new, }-}(\mathcal{T}_p, L)^{\Gamma_0(\infty)} \\
\bigcap & \bigcap & \\
c & \bigcap & c_p(\cdot).
\end{array}
\]

## 2.2 Double integrals

Let $x, y \in \mathbb{P}^1(F)$. We denote the geodesic joining the corresponding ends on $\mathcal{T}_\infty$ by $A(x, y)$. For any $c \in \mathbb{H}_*(\mathcal{T}, M)^{\Gamma}$, it follows from the definitions that

$$c\{x \to y\} : e \mapsto \sum_{s \in A(x, y)} c(e, s)$$

is a $M$-valued harmonic cochain on $\mathcal{T}_p$. (Note that since $c$ is cuspidal, this is a finite sum for all $e$.) Therefore it corresponds to a measure

$$\mu_c\{x \to y\}$$

on $\mathbb{P}^1(F_p)$ of total mass 0.
If we choose $\gamma \in \tilde{\Gamma}$ such that $e = \gamma e_0$, we see that
\[
c_{x \to y}(e) = \sum_{s \in A(x,y)} c_{\infty}(s)
\]
\[
= \sum_{s \in A(x,y)} \gamma \ast c_{\infty}(s)
\]
(compare cor. 2.6). Or equivalently,
\[
\mu_c(x \to y)(U(e)) = - \sum_{s \in A(x,y)} \gamma \ast c_{\infty}(s).
\]

This implies the following:

**Lemma 2.7.** For all $c \in \mathbb{H}(T, M)^{\tilde{\Gamma}}$, the map $(x, y) \mapsto \mu_c(x \to y)$ induces a $\tilde{\Gamma}$-module homomorphism
\[
\mathcal{M} \rightarrow \text{Meas}_0(\partial \Omega_p, M)
\]
from the group of modular symbols $\mathcal{M} = \text{Div}^0(\mathbb{P}^1(F))$ to measures on $\partial \Omega_p$.

**Remark.** The analogous statement for $\Gamma$-invariant cochains is also true.

For $c \in \mathbb{H}(T, \mathbb{Z})^{\tilde{\Gamma}}$, we define the following double (multiplicative) integral:
\[
\int_{z_1}^{z_2} \int_{x_1}^{x_2} \omega := \int_{\mathbb{P}^1(F_p)} \left(\frac{t - z_2}{t - z_1}\right) d\mu_c(x \to y) \in \mathbb{C}_p^\times
\]
for $z_1, z_2 \in \Omega_p$.

**Remark:** Firstly, we observe that this definition depends on the choice of $c$. Since later on we will work with an especially chosen $c_E$, we dropped it from the notation. Secondly, although $\omega$ itself is not defined, this definition should be regarded as a period for a "rigid analytic modular form of weight $(2, 2)$ on $(\Omega_p \times \Omega_\infty)/\Gamma"$ (compare [Da]).

**Lemma 2.8.** The double integrals above have the following properties:
\[
\int_{z_1}^{z_3} \int_{x_1}^{x_2} \omega = \int_{z_1}^{z_2} \int_{x_1}^{x_2} \omega \times \int_{z_2}^{z_3} \int_{x_1}^{x_2} \omega,
\]
\[
\int_{z_1}^{z_2} \int_{x_1}^{x_3} \omega = \int_{z_1}^{z_2} \int_{x_1}^{x_2} \omega \times \int_{z_2}^{z_3} \int_{x_2}^{x_3} \omega,
\]
\[
\int_{\gamma z_1}^{\gamma z_2} \int_{\gamma x_1}^{\gamma x_2} \omega = \int_{z_1}^{z_2} \int_{x_1}^{x_2} \omega,
\]
for all $z_1, z_2, z_3 \in \Omega_p$, $x_1, x_2, x_3 \in \mathbb{P}^1(F)$ and $\gamma \in \tilde{\Gamma}$. 
Proof. The first two properties follow directly from the definitions, and the third property from:

\[
\begin{align*}
\int_{\mathbb{P}^1(F_p)} & \frac{t - \gamma z_2}{t - \gamma z_1} d\mu_c \{ \gamma x_1 \to \gamma x_2 \}(t) \\
= & \int_{\gamma(\mathbb{P}^1(F_p))} \frac{t - \gamma z_2}{t - \gamma z_1} d\gamma * \gamma^{-1} * \mu_c \{ \gamma x_1 \to \gamma x_2 \}(t) \\
= & \int_{\mathbb{P}^1(F_p)} \frac{\gamma t - \gamma z_2}{\gamma t - \gamma z_1} * \mu_c \{ \gamma x_1 \to \gamma x_2 \}(t) \\
= & \int_{\mathbb{P}^1(F_p)} \frac{cz_1 + d}{cz_2 + d} \frac{t - z_2}{t - z_1} d\mu_c \{ x_1 \to x_2 \}(t) \\
= & \int_{\mathbb{P}^1(F_p)} \frac{t - z_2}{t - z_1} d\mu_c \{ x_1 \to x_2 \}(t),
\end{align*}
\]

where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\Gamma}. \)

\[\square\]

2.3 Automorphic cusp forms

For an open subgroup \( K \) of \( G(O) \) and a field \( L \) of characteristic zero, we denote the intersection of the spaces of automorphic cusp forms at \( p \) and at \( \infty \) in the following way:

\[ W_{S}(K, L) := W_p(K, L) \cap W_{\infty}(K, L) \]

and similarly for \( W_{sp} \) and newforms.

Assume that \( K \) decomposes as

\[ K = K_{f,S} \times I_p \times I_{\infty}. \]

Since \( K_{f,S} \) is of finite index in \( O_{f,S} \) and \( G(F) \backslash G(A_{f,S})/G(O_{f,S}) \) is finite (of cardinality equal to the class number \( \text{cl}(A_S) \) of \( A_S \), compare [GR (4.1.4)]), the set

\[ G(F) \backslash G(A_{f,S})/K_{f,S} \]

is finite. Let \( R_S \) be a system of representatives and

\[ \Gamma_{\mathfrak{p}} := G(F) \cap \mathfrak{p}K_{f,S} \mathfrak{p}^{-1} \subseteq G(A_{f,S}), \]

for \( \mathfrak{p} \in R_S \). Every element \( g \in G(A) \) can be written as

\[ g = \gamma(\mathfrak{p} \times 1_p \times 1_{\infty})(\mathfrak{k} \times 1_p \times 1_{\infty})(\mathbf{1}_{f,S} \times g_p \times g_{\infty}), \]
for some $\gamma \in G(F)$, $k \in \mathcal{K}_{f,S}$, $g_p \in G(F_p)$, $g_\infty \in G(F_\infty)$ and a uniquely determined $\underline{x} \in R_S$. We define

$$Y_S(\mathcal{K}) := G(F) \backslash G(\mathcal{A}) / G(\mathcal{F} \times G(\mathcal{F}_p \times F_\infty))$$

and get the following generalisation of [GR, 4.5.4].

**Lemma 2.9.** The following map is a well-defined isomorphism:

$$\Phi_S : Y_S(\mathcal{K}) \xrightarrow{\cong} \prod_{\underline{x} \in R_S} \Gamma_{\underline{x}} \backslash G(F_p \times F_\infty) / Z(F_p \times F_\infty) \cdot (\mathcal{I}_p \times \mathcal{I}_\infty)$$

$$[g] \mapsto [g_p \times g_\infty].$$

**Proof.** Straightforward. \qed

Again, we observe that the right hand side is equal to

$$\prod_{\underline{x} \in R_S} \tilde{\varepsilon}(\Gamma_{\underline{x}} \backslash \mathcal{T})$$

(as in section 1.2).

**Remark.** Since the class groups of $A_p$ and $A_\infty$ surject onto the class group of $A_S$, the class number $\text{cl}(A_S)$ fulfils:

$$\text{cl}(A_S) \mid \gcd(\text{cl}(A_p), \text{cl}(A_\infty)).$$

In other words,

$$\#R_S \mid \gcd(\#R_p, \#R_\infty),$$

where $R_p$ and $R_\infty$ are the corresponding systems of representatives at $p$ and $\infty$. E.g., $R_S$ consists only of one element if $F$ is the rational function field and one of the places is of degree one.

**CONVENTION:** As already mentioned in section 1.3, the results in [GR] are obtained by restricting, without loss of generality, to a single $\underline{x} \in R_p$. The same method applies in our case. To keep our notations simple, we will assume that $R_S$ consists only of one element. To obtain the general case, one has to slightly generalise the definitions and results in the preceding sections.

If we choose $\mathcal{K} = \mathcal{K}_0(\mathcal{F}_p \mathcal{F}_\infty)$, then it decomposes as above (see remark after thm. 1.5). Furthermore, we can easily check that

$$G(F) \cap \mathcal{K}_{f,S} = \tilde{\Gamma}.$$
2.3 Automorphic cusp forms

The image of the natural inclusion $W_{sp,S} \hookrightarrow W_{sp,p}$ can easily be identified as

$$W_{sp,p}^{\infty-\text{new,-}}(K_0(pn)),$$

the space corresponding to $H_{\infty-\text{new,-}}(T_p, L)^{\Gamma_0^p(n)}$ via the isomorphism in theorem 1.5 (Again, the analogue statement is true if we interchange $p$ and $\infty$.) This leads us to the following generalisation of Drinfeld’s theorem:

**Theorem 2.10.** The following spaces are isomorphic:

$$W_{sp,S}(K_0(pn), L) \cong H_p(T, L)^{\tilde{\Gamma}}.$$

Furthermore, the following diagram, which consists entirely of isomorphisms, is commutative:

$$\begin{array}{ccc}
W_{sp,p}^{\infty-\text{new,-}}(K_0(pn)) & \cong & H_{\infty-\text{new,-}}(T_p, L)^{\Gamma_0^p(n)} \\
W_{sp,S}(K_0(pn), L) & \cong & H_p(T, L)^{\tilde{\Gamma}} \\
W_{sp,\infty}^{\infty-\text{new,-}}(K_0(pn)) & \cong & H_{\infty}^{\text{new,-}}(T_\infty, L)^{\Gamma_{\infty}^\infty(pn)}.
\end{array}$$

**Proof.** The corresponding commutative diagram of the underlying spaces is the following:

$$\begin{array}{ccc}
Y_p(K) & \xrightarrow{\Phi_p} & \tilde{E}(\Gamma_0^p(n) \backslash T_p) \\
Y_S(K) & \xrightarrow{\Phi_S} & \tilde{E}(\Gamma \backslash T) \\
Y_\infty(K) & \xrightarrow{\Phi_\infty} & \tilde{E}(\Gamma_0^\infty(pn) \backslash T_\infty)
\end{array}$$

(the vertical maps are surjections). The theorem follows from theorem 1.5 and corollary 2.6 (resp. its analogue in the succeeding remark).

**Remark.** Note that this is a slightly simplified formulation of the theorem, due to our assumption that $R_S$ contains only one element. In general, we have to consider direct sums of spaces of harmonic cochains on the right hand side in the diagram. E.g.,

$$W_{sp,S}(K_0(pn), L) \cong \bigoplus_{x \in R_S} H_p(T, L)^{\Gamma_p}.$$
3 Teitelbaum’s conjecture

3.1 Darmon’s period

Let $K = F \times F$ and choose an $F$-algebra embedding $\psi : K \to M_2(F)$. This induces an action of $\psi(K^\times)$ on $\Omega_p^\ast$ with two fixed points $x_\psi, y_\psi \in \mathbb{P}^1(F)$. The image of $\psi(K^\times) \cap \tilde{\Gamma}$ in $\text{PGL}_2(F)$ is of rank 1. We choose a generator $\gamma_\psi$ of the free part. Assume that $x_\psi$ is repulsive and $y_\psi$ attractive w.r.t. $\gamma_\psi$. To such an embedding and $c \in H^1(T, \mathbb{Z})$, we can associate a period

$$I_\psi = I_{\psi,c} := \int_z \gamma_\psi^z \int_{x_\psi} y_\psi \omega \in C_p^\times,$$

where we choose $z \in \Omega_p$ arbitrarily.

**Lemma 3.1.** The period $I_\psi$ is independent of the choice of $z \in \Omega_p$.

**Proof.**

$$\int_{x_\psi} \gamma_\psi^z \int_{x_\psi} y_\psi \omega \div \int_{x_\psi} y_\psi \omega = \int_{x_\psi} \gamma_\psi^z \int_{x_\psi} y_\psi \omega \div \int_{x_\psi} \gamma_\psi^z \int_{x_\psi} y_\psi \omega = \int_{x_\psi} \int_{x_\psi} y_\psi \omega \div \int_{x_\psi} y_\psi \omega = 1,$$

where the second equality follows from the third property in lemma 2.8. □

**Remark.** Note that, unlike in Darmon’s setting, $I_\psi$ depends on the choice of $\gamma_\psi$. This is due to the fact that the group of roots of unity in $A_S^\times$ is nontrivial (see the remark after Prop. 3.3).

To simplify calculations, we consider only the case where $\psi$ is the diagonal embedding. The fixed points of $\psi$ are $\{\infty, 0\}$. As already mentioned in section 2.1, by the $S$-unit theorem there exists $u \in F$ such that

$$A_S^\times = \mathbb{P}_q^\times \times u^\mathbb{Z}$$

and

$$\text{div}(u) = h_p \cdot [p] - h_\infty \cdot [\infty].$$

We make the following choice:

$$\gamma_\psi = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in \tilde{\Gamma}.$$
It acts as shift by $h_p$ along the axis $A(\infty, 0)$ on $\mathcal{T}_p$ and as shift by $h_\infty$ along $A(0, \infty)$ on $\mathcal{T}_\infty$.

Following Darmon, we define the winding element attached to $\psi$ as

$$W_\psi := \sum_{e \in A(v, \gamma \psi v)} \mu_c\{\infty \to 0\}(U(e)),$$

where $v \in \mathcal{V}(\mathcal{T}_p)$ is chosen arbitrarily. By observing that $c\{\infty \to 0\}$ is invariant under $\gamma_v \in \tilde{\Gamma}$, it is easy to see that this is independent of the choice of $v$ ([Da, Lemma 2.10]). In particular,

$$W_\psi = - \sum_{i=0}^{h_p-1} c\{\infty \to 0\}(e_i).$$

Let $E$ be an elliptic curve over $F$ of conductor $pn\infty$ with split multiplicative reduction at $p$ and at $\infty$. Assume that $E$ is the strong Weil curve in its isogeny class. By the function field version of the Shimura-Taniyama-Weil conjecture [GR, Section 8], there is a newform

$$c = c_E \in W_{sp,S}^{new}(\mathcal{K}_0(pn\infty), \mathbb{Z})$$

which corresponds to $E$. It is uniquely determined up to constants, such that

(i) $c$ is an eigenform under the Hecke algebra with rational eigenvalues and

(ii) for all idèle class characters $\chi$ of $F$,

$$L(c, \chi, s) = L(E, \chi, s).$$

We denote its images in the spaces of harmonic cochains on $\mathcal{T}_p$, $\mathcal{T}_\infty$ and $\mathcal{T}$ by $c_p$, $c_\infty$ and $c$, respectively. We choose $c$ such that $c_\infty$ is primitive, i.e., is normalised such that $c_\infty \in j(\Gamma_0^\infty(pn))$ but $c_\infty \notin nj(\Gamma_0^\infty(pn))$ for $n > 1$. This determines $c$ up to sign (see section 1.3 and [GR 9.1]).

**Theorem 3.2.** Put $r = \frac{W_\psi}{\nu_p(q_{E,p})} \in \mathbb{Q}$. Then

$$I_\psi = \zeta \cdot q_{E,p}^r,$$

where $\zeta$ is a root of unity and $q_{E,p}$ the Tate period of $E$ at $p$. 
Before we can prove theorem 3.2, we need to establish a few tools. A fundamental domain for the action of $\gamma_\psi$ on $\mathbb{P}^1(F_p) - \{\infty, 0\}$ is given by

$$\mathcal{F}_\psi = \bigcup_{i=0}^{h_p-1} \pi_p O_p^i \cup \bigcup_{i=0}^{h_p-1} U(v_i),$$

where $\pi_p$ denotes the matrix $\left( \begin{array}{cc} \pi_p & 0 \\ 0 & 1 \end{array} \right)$, and for all $i$,

$$U(v_i) = \{ t \in \mathbb{P}^1(F_p) - \{\infty, 0\} : \nu_p(t) = i \}.$$ 

Each of these sets corresponds to all ends of $T_p$ which have origin in $v_i$ and no edge in common with $A(\infty, 0)$.

We define

$$m_0 := -c(\infty \to 0)(e_0) = -c(\infty \to 0)(e_{h_p})$$

(since $c(\infty \to 0)$ is invariant under the action of $\gamma_\psi$).

**Proposition 3.3.** For all $z \in \Omega_p$,

$$I_\psi = u^{m_0} \cdot \int_{\mathcal{F}_\psi} t \, d\mu_c(\infty \to 0)(t) \in F_p^\times.$$

**Proof.** The proof proceeds exactly as in [Da, Prop. 2.7] (or [L, Lemma 16]). 

**Remark.** Proposition 3.3 shows that the definition of $I_\psi$ depends on the choice of $\gamma_\psi$. E.g., if we define a period $I'_\psi$ using

$$\gamma'_\psi = \left( \begin{array}{cc} \xi u & 0 \\ 0 & 1 \end{array} \right),$$

where $\xi$ is a root of unity, we obtain $I'_\psi = \xi^{m_0} \cdot I_\psi$.

We define $\mathcal{L} = \{s_1, \ldots, s_{h_\infty}\}$ to be the geodesic connecting the vertices $w_0$ and $w_{h_\infty}$ on $T_\infty$. Let $\alpha_1, \ldots, \alpha_{h_\infty} \in \tilde{\Gamma}$ such that $s_i = \alpha_i s_0$ for $i = 1, \ldots, h_\infty$. Furthermore, let

$$\lambda := \sum_{i=1}^{h_\infty} \alpha_i \in \mathbb{Z}[\tilde{\Gamma}]$$
and

\[ \mu_L := \lambda \ast \mu_{E,p} \]
\[ = \sum_{i=1}^{h_\infty} \alpha_i \ast \mu_{E,p}, \]

where \( \mu_{E,p} \) is the measure on \( \partial \Omega_p \) corresponding to \( c_p \).

**Lemma 3.4.**

\[ \mu_c\{\infty \to 0\} = \sum_{n \in \mathbb{Z}} \gamma_n^\psi \ast \mu_L. \]

**Proof.** Since \( A(\infty,0) = \bigcup_{n \in \mathbb{Z}} \gamma_n^\psi \mathcal{L} \subseteq \tilde{\mathcal{T}}(T_\infty) \), and using the \( \tilde{\Gamma} \)-invariance of \( c \), we conclude:

\[ \mu_c\{\infty \to 0\}(U(e)) \overset{\text{def.}}{=} - \sum_{s \in A(\infty,0)} c(e,s) \]
\[ = - \sum_{n \in \mathbb{Z}} \sum_{i=1}^{h_\infty} c(e, \gamma_n^\psi \alpha_i s_0) \]
\[ = - \sum_{n \in \mathbb{Z}} \sum_{i=1}^{h_\infty} c((\gamma_n^\psi \alpha_i)^{-1}, s_0) \]
\[ = \left( \sum_{n \in \mathbb{Z}} \gamma_n^\psi \ast \mu_L \right)(U(e)). \]

\[ \square \]

**Proposition 3.5.** For \( i = 1, \ldots, h_\infty \), let \( P_i := \alpha_i^{-1}(0) \) and \( Q_i := \alpha_i^{-1}(\infty) \).

Then

\[ u_0 \int_{\mathcal{F}_\psi} t \, d\mu_c\{\infty \to 0\}(t) = \prod_{i=1}^{h_\infty} \frac{u_{E,p}(P_i)}{u_{E,p}(Q_i)}, \]

where \( u_{E,p} \) is the theta function associated to \( E \) at \( p \).

**Proof.** Similar reasoning as in [L] Prop. 25] shows:

\[ u_0 \int_{\mathcal{F}_\psi} t \, d\mu_c\{\infty \to 0\}(t) = \int_{\partial \Omega_p} t \, d\mu_L \]
\[ \overset{1.9}{=} \prod_{i=1}^{h_\infty} \int_{\partial \Omega_p} \alpha_i(t) d\mu_{E,p}(t) \]
\[ = \prod_{i=1}^{h_\infty} \int_{\partial \Omega_p} \frac{t - P_i}{t - Q_i} d\mu_{E,p}(t). \]
The statement follows from Teitelbaum’s formula (Prop. 1.10).

Combining propositions 3.3 and 3.5 we get

\[ I_\psi = h_\infty \prod_{i=1}^{u} \frac{u_E,\psi(P_i)}{u_E,\psi(Q_i)}. \]

Since \( P_i \) and \( Q_i \) are cusps for all \( i \), Theorem 1.7 shows that the fraction on the right is (up to a root of unity) a rational power of the Tate period. Theorem 3.2 now follows from the next lemma.

**Lemma 3.6.** \( \nu_p(I_\psi) = W_\psi \).

**Proof.** Since \( \nu_p(u) = h_p \),

\[
\nu_p(I_\psi) = h_p m_0 + \nu_p \left( \int_{\mathcal{F}_\psi} td\mu_c\{ \infty \to 0 \}(t) \right) \\
= h_p m_0 + \sum_{i=0}^{h_p-1} \mu_c\{ \infty \to 0 \}(\pi_i^* \mathcal{O}_p) \\
= h_p c\{ \infty \to 0 \}(e_0) + \sum_{i=0}^{h_p-1} i(c\{ \infty \to 0 \}(e_i) - c\{ \infty \to 0 \}(e_{i+1})) \\
= W_\psi,
\]

where the last equality follows from the \( \gamma_\psi \)-invariance of \( c\{ \infty \to 0 \} \).

**Remark:** Under the assumption that \( E/\mathbb{Q} \) is unique in its \( \mathbb{Q} \)-isogeny class, Darmon ([Da]) conjectures that

\[ I_\psi \in q\mathbb{Z}. \]

His assumption implies especially that \( E(\mathbb{Q}) \) is torsion free. Since in our case \( I_\psi \) is an \( F \)-rational torsion point, theorem 3.2 implies:

**Corollary 3.7 (Darmon’s conjecture).** Assume that \( E(F)_{\text{tor}} = 0 \). Then

\[ I_\psi \in q\mathbb{Z}_{E,p}. \]
3.2 The main result

Finally, we take a look at Teitelbaum’s function field version of the exceptional zero conjecture ([12]). He considers the rational function field \( F = \mathbb{F}_q(T) \) and the usual place at infinity \( \infty \) with uniformiser \( \pi_\infty = \frac{1}{T} \). We note that in this case \( h_p = 1 \) and \( W_\psi = m_0 \). We choose \( \pi_p \) to be an element of \( A_\infty = \mathbb{F}_q[T] \). Since \( A_S = \mathbb{F}_q[T, \pi_p^{-1}] \), \( u \) can chosen to be \( u = \pi_p \).

Teitelbaum defines a measure on \( \mathcal{O}_p \) by
\[
\mu_{\text{Teit}}(a + \pi_p^n \mathcal{O}_p) := \left[ \frac{a}{\pi_p^n}, \infty \right] \cdot c_\infty,
\]
where
\[
\left[ \frac{a}{\pi_p^n}, \infty \right] \cdot c_\infty = \sum_{s \in A(\frac{a}{\pi_p^n}, \infty)} c_\infty(s).
\]

Given the elements
\[
m_p = \nu_p(q_{E,p}), \quad \tilde{q} = \frac{q_{E,p}}{\pi_p^{m_p}},
\]
\[
q(c_\infty) = \lim_{n \to \infty} \prod_{a \mod \pi_p^n, (a, \pi_p) = 1} a^{[\frac{a}{\pi_p^n}, \infty] \cdot c_\infty} = \int_{\mathcal{O}_p^\times} t d\mu_{\text{Teit}}(t),
\]
he formulates:

**Conjecture 3.8 (Teitelbaum).**
\[
\tilde{q}^{[0, \infty] \cdot c_\infty} = q(c_\infty)^{m_p}.
\]

Our result towards this conjecture is the following:

**Theorem 3.9.** There exists a root of unity \( \xi \in \mathbb{F}_p^{\times} \) such that
\[
\xi \cdot \tilde{q}^{[0, \infty] \cdot c_\infty} = q(c_\infty)^{m_p}.
\]

**Proof of theorem 3.9** By theorem 3.2 and proposition 3.3
\[
\zeta \cdot (q_{E,p})^{\frac{W_\psi}{m_p}} = \pi_p^{W_\psi} \int_{\mathcal{O}_p^\times} t d\mu_c(\infty \to 0)(t)
\]
for some root of unity \( \zeta \in \mathbb{C}_p^\times \). We set \( \xi := \zeta^{m_p} \) and raise to the power \( m_p \):
\[
\xi \cdot q_{E,p}^{W_\psi} = \pi_p^{W_\psi \cdot m_p} \left( \int_{\mathcal{O}_p^\times} t d\mu_c(\infty \to 0)(t) \right)^{m_p} \in \mathbb{F}_p^\times.
\]
Also,

\[
W_\psi = - \sum_{s \in A(\infty, 0)} c_\infty(s) = \sum_{s \in A(0, \infty)} c_\infty(s) = [0, \infty] \cdot c_\infty \in \mathbb{Z}.
\]

This implies that \( \xi \in \mu(F^\times_p) = \mathbb{R}^\times_{q_{\log(p)}} \), since \( q_{E, p}^{W_\psi} \in F_p^\times \). Dividing by \( \pi^{W_\psi - m_p} \), the theorem follows from:

**Lemma 3.10.**

\[ \mu_c \{ \infty \to 0 \} |_{E_p} = \mu_{\text{Teit}}. \]

**Proof.** Since \( A_\infty \) is dense in \( \mathcal{O}_p \), we only have to consider compact open sets of the form

\[ a + \pi_p^n \mathcal{O}_p = U(\begin{pmatrix} \pi_p^n & a \\ 0 & 1 \end{pmatrix} e_0), \]

where \( a \in A_\infty \).

By Lemma 2.7, \( \mu_c \{ \infty \to 0 \} \) is invariant under the action of \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \). In particular,

\[ \mu_c \{ \infty \to 0 \}(a + \pi_p^n \mathcal{O}_p) = \mu_c \{ \infty \to 0 \}(-a + \pi_p^n \mathcal{O}_p). \]

Furthermore,

\[
\mu_c \{ \infty \to 0 \}(-a + \pi_p^n \mathcal{O}_p) = - \sum_{s \in A(\infty, 0)} c(\begin{pmatrix} \pi_p^n & -a \\ 0 & 1 \end{pmatrix} e_0, s) \\
\overset{(*)}{=} - \sum_{s \in A(\infty, 0)} c(e_0, \begin{pmatrix} \pi_p^{-n} & a \\ 0 & 1 \end{pmatrix} s) \\
= - \sum_{s \in A(\infty, \frac{a}{\pi_p^n})} c_\infty(s) \\
= [\frac{a}{\pi_p^n}, \infty] \cdot c_\infty \\
= \mu_{\text{Teit}}(a + \pi_p^n \mathcal{O}_p).
\]

In (\( * \)) we used the \( \tilde{\Gamma} \)-invariance of \( c \) and the fact that \( \begin{pmatrix} \pi_p^n & -a \\ 0 & 1 \end{pmatrix} \in \tilde{\Gamma}. \)
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