ON LOW DIMENSIONAL KC-SPACES

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Abstract. The KC property, a separation axiom between weakly Hausdorff and Hausdorff, requires compact subsets to be closed. Various assumptions involving local conditions, dimension, connectivity, and homotopy show certain KC-spaces are in fact Hausdorff. Several low dimensional examples of compact, connected, non-Hausdorff KC-spaces are exhibited in which the nested intersection of compact connected subsets fails to be connected.

1. Introduction

What are the strongest properties permitted of a space $X$, if the nested intersection of compact connected subsets of $X$ can fail to be connected? $X$ must be non-Hausdorff, but examples otherwise rich in structure are not so obvious since for example, standard texts such as [11] devote little attention to such spaces.

To weaken slightly the Hausdorff condition, we turn to KC-spaces, spaces in which compact subspaces are closed. We construct a variety of non-Hausdorff KC-spaces in which that nested intersection of closed compact connected subspaces fails to be connected. We establish two theorems which eliminate certain classes of non-Hausdorff $KC$ examples, and finally we observe that the paper’s content applies to $WH$, the category of weakly Hausdorff spaces.

For historical background, Example 99 [14] shows why KC-spaces can be non-Hausdorff. More generally a technique for constructing KC-spaces is implicit in the 1967 paper by Wilansky [18] which shows the Alexandroff compactification of a $k+KC$-space is again a KC-space. KC-spaces are also called maximal compact spaces, and in certain contexts such spaces are guaranteed to be Hausdorff. For example the 1985 paper of A.H. Stone [15] shows that 1st countable maximal compact spaces are in fact $T_2$. The general theory of KC-spaces has continued to develop over the last decade [13, 16] and remains an active area of research. Recent advances include a proof of a long standing conjecture that minimal $KC$-spaces are compact $[2]$, and several questions posed in [12] are settled in [1].

In the paper at hand Theorem 1 shows every simply connected, locally path connected, 1-dimensional KC-space is $T_2$. Theorem 2 shows the Hausdorff property is also guaranteed in KC-spaces which are locally connected by continua and in which compact connected subsets always have connected intersection. Corollaries 1 and 2 yield new criteria by which dendrites and certain dendroids can be recognized.

The remainder of the paper demonstrates, via example, the futility of weakening the hypotheses of Theorems 1 and 2. We exhibit a series of non-Hausdorff examples typically constructed as the Alexandroff compactification $W \cup \{\infty\}$ of a $T_2$ space $W$.

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such that \( W \) fails to be locally compact at precisely one point. Subsections 4.2, 4.3, 4.4 and 4.5 exhibit 1-dimensional counterexamples. Subsections 4.6 and 4.7 exhibit non-Hausdorff 2-dimensional KC-spaces, the latter of which is both contractible and locally contractible.

The general relevance of KC-spaces is bolstered by the following observations. Every KC-space \( X \) is weakly Hausdorff, (i.e. maps from Hausdorff compacta into \( X \) have closed image in \( X \)). In various contexts, weakly Hausdorff spaces are better behaved than Hausdorff spaces [17]. For example, as noted ([9] (p. 485)) in reference to Peter May’s book “The geometry of iterated loops spaces”[9] (p. 485) “The weak Hausdorff rather than the Hausdorff property should be required...in order to validate some of the limit arguments used in [8].” In particular the Hausdorff property fails in general to be preserved (with the direct limit topology) by the union of closed Hausdorff subspaces [6]. On the other hand, by definition, the KC property is preserved under unions (with the direct limit topology) of closed KC-subspaces.

This paper suggests two open questions. Is the \( n \)-connected example from subsection 4.6 contractible? Does there exist a 1-dimensional contractible non-Hausdorff KC-space?

2. Definitions

A continuum is a compact connected Hausdorff (\( T_2 \)) space, and in particular we allow that a continuum is not metrizable.

A Peano continuum is a compact, connected, locally path connected metrizable space.

The continuum \( Y \) is hereditarily unicoherent if \( C \cap D \) is connected whenever \( C \) and \( D \) are subcontinua of \( Y \).

The space \( Y \) is generalized hereditarily unicoherent if \( C \cap D \) is connected whenever \( C \) and \( D \) are closed compact connected subspaces of \( Y \).

A space \( X \) is a KC-space if each compact subspace of \( X \) is closed in \( X \).

A space \( X \) is a \( k \)-space if \( A \) is closed in \( X \) whenever \( A \cap K \) is closed for all compact closed sets \( K \subset X \).

An arc is a space homeomorphic to \([0,1]\).

A dendroid is a hereditarily unicoherent continuum. (Note every dendroid is \( T_2 \) but not necessarily metrizable).

A dendrite is a locally path connected continuum which contains no simple closed curve. (Note every dendrite is \( T_2 \) but not necessarily metrizable).

If \((X, T_X) \) and \( \infty \notin X \) and if \( Y = X \cup \{\infty\} \) then the Alexandroff compactification of \( X \) is the space \((Y, T_Y) \) such \( T_Y \) is union of \( T_X \) and sets \( V \) such that \( V \subset Y \) and \( Y \setminus V \) is compact and closed in \( X \).

The space \( X \) is locally compact if for each \( x \in X \) there exists an open set \( U \) such that \( x \in U \) and \( \overline{U} \) is compact.

The space \( X \) has (covering) dimension \( n \), if \( n \) is minimal such that for each open covering \( \mathcal{G} \) of \( X \) there exists an open covering \( \mathcal{G}_1 \) of \( X \) such that if \( U \in \mathcal{G}_1 \) there exists \( V \in \mathcal{G} \) such that \( U \subset V \), and such that each element \( x \in X \) belongs to at most \( n + 1 \) distinct sets of the collection \( \mathcal{G}_1 \).

The space \( X \) is connected by continua if for each pair \( \{x, y\} \subset X \) there exists a continuum \( Z \) and a map \( f: Z \rightarrow X \) such that \( \{x, y\} \subset im(f) \). The space \( X \) is
locally connected by continua (denoted lcc) if for each \( x \in X \) there exists an open set \( U \) such that \( x \in U \) and \( U \) is connected by continua.

**Remark 1.** Familiar examples (such as the Alexandroff compactification of the rationals discussed in subsection 4.1) show that ‘connected’ is strictly weaker than ‘connected by continua’. The property ‘locally connected by continua’ is at least as strong as ‘locally connected’. However if \( X \) is locally connected, it need not be the case that maps of ordered continua are adequate to connect distinct points of \( X \) [4] [7].

3. Promoting KC-spaces to Hausdorff

The main results of this section are Theorems 1 and 2, Corollaries 1 and 2 which establish conditions under which certain KC-spaces are necessarily Hausdorff.

**Lemma 1.** Suppose \( Y \) is a KC-space. If \( Z \) is a compact \( T_2 \) space and \( f : Z \to Y \) is continuous then \( \text{im}(f) \) is a \( T_2 \) subspace of \( Y \). Each path connected subspace of \( Y \) is arcwise connected.

**Proof.** Suppose \( f : Z \to Y \) is a map and \( Z \) is a compact \( T_2 \) space. Suppose \( \{a,b\} \subset \text{im}(f) \) and \( a \neq b \). Let \( A = f^{-1}(\{a\}) \) and \( B = f^{-1}(\{b\}) \). Since \( Z \) is compact and \( T_2 \), \( Z \) is normal. Apply normality of \( Z \) to obtain disjoint open sets \( U \) and \( V \) in \( Z \) such that \( A \subset U \), \( B \subset V \) and \( U \cap V = \emptyset \). Let \( K = Z \setminus U \) and \( C = Z \setminus V \). Then \( Z = K \cup C \) and thus \( \text{im}(f) = f(K) \cup f(C) \). Moreover, since each of \( f(K) \) and \( f(C) \) is compact, and since \( Y \) is a KC-space, each of \( f(K) \) and \( f(C) \) is closed in \( Y \). Thus \( \text{im}(f) \setminus f(K) \) and \( \text{im}(f) \setminus f(C) \) establish the \( T_2 \) property of \( \text{im}(f) \).

Suppose \( A \subset Y \) and \( A \) is path connected and \( \{a,b\} \subset A \) and \( a \neq b \). Obtain a path \( \alpha : [0,1] \to A \) such that \( \alpha(0) = a \) and \( \alpha(1) = b \). Since \( [0,1] \) is a compact \( T_2 \) space, \( \text{im}(\alpha) \) is a path connected \( T_2 \) subspace of \( A \) and hence \( \text{im}(\alpha) \) is arcwise connected. Thus \( A \) is arcwise connected. \( \Box \)

**Lemma 2.** Suppose \( X \) is a 1-dimensional KC-space. Then \( X \) is aspherical, and moreover \( X \) is simply connected if and only if \( X \) contains no simple closed curve.

**Proof.** To see that \( X \) is aspherical suppose \( f : S^n \to X \) is a map and \( n \geq 2 \). Then \( \text{im}(f) \) is \( T_2 \) by Lemma 1. By the Hahn Mazurkiewicz theorem \( \text{im}(f) \) is a 1 dimensional Peano continuum, and hence \( \text{im}(f) \) is aspherical (Cor. p578 [4]). Thus \( X \) is aspherical.

If \( X \) is simply connected then, to obtain a contradiction, suppose \( X \) contains a simple closed curve \( S \subset X \). Since \( X \) is simply connected there exists a map \( f : D^2 \to X \) such that \( f_{\partial D^2} \) is an embedding onto \( S \). By the Hahn Mazurkiewicz theorem and lemma 1 \( \text{im}(f) \) is a one dimensional Peano continuum. Hence \( S \) is a retract of \( \text{im}(f) \) (Thm 3.1 [3]), and hence the loop \( S \) is both essential and inessential in \( \text{im}(f) \) and we have a contradiction.

Conversely suppose \( X \) contains no simple closed curve and \( f : \partial D^2 \to X \) is any map. By the Hahn Mazurkiewicz theorem and lemma 1 \( \text{im}(f) \) is a 1 dimensional Peano continuum which contains no simple closed curve and thus \( \text{im}(f) \) is contractible (Thm. p. 578 [5]) and in particular \( f \) is inessential in \( X \). Thus \( X \) is simply connected. \( \Box \)

**Theorem 1.** Suppose \( X \) is a locally path connected KC-space and suppose \( X \) contains no simple closed curve. Then \( X \) is Hausdorff.
Proof: Suppose \( \{a, b\} \subset X \) and \( a \neq b \). Since \( X \) is locally path connected the components of \( X \) are open and thus if \( a \) and \( b \) belong to distinct components \( Y_a \) and \( Y_b \), the open sets \( Y_a \) and \( Y_b \) separate \( a \) and \( b \).

For the remaining case suppose \( Y \) is a component of \( X \) and \( \{a, b\} \subset Y \). Then \( Y \) is arcwise connected since \( Y \) is connected and locally path connected. Thus \( Y \) is arcwise connected by Lemma 1 and let the arc \( \alpha \subset Y \) have endpoints \( \{a, b\} \). Let \( k \in \alpha \setminus \{a, b\} \). We claim \( a \) and \( b \) belong to distinct components \( U \) and \( V \) of \( Y \setminus \{k\} \).

To obtain a contradiction suppose \( U \) is a component of \( Y \setminus \{k\} \) and \( \{a, b\} \subset U \). Since \( U \) is open, \( U \) is locally path connected. Thus \( U \) is path connected since \( U \) is connected. By Lemma 1 there exists an arc \( \beta \subset U \) with endpoints \( \{a, b\} \). Note \( k \notin \beta \). Let \( J \) denote the component of \( \alpha \setminus \beta \) such that \( k \in J \). Let \( \{x, y\} \) denote the endpoints of \( J \) in \( \alpha \). Let \( I \) denote the component of \( \beta \setminus \{x, y\} \) with endpoints \( \{x, y\} \). Observe \( J \cup I \cup \{x, y\} \) is a simple closed curve and we have a contradiction.

Let \( U \) and \( V \) denote the components of \( Y \setminus \{k\} \) such that \( a \in U \) and \( b \in V \). Recall \( Y \) is open in \( X \) and \( \{k\} \) is closed in \( X \). Thus \( U \) and \( V \) are open in \( X \) and this proves \( X \) is \( T_2 \).

Combining Theorem 1 and Lemma 2 we obtain the following.

**Corollary 1.** If \( X \) is a locally path connected, simply connected, 1-dimensional KC-space then \( X \) is \( T_2 \), hence if \( X \) is also compact and connected then \( X \) is a dendrite.

By definition, every hereditarily unicoherent continuum contains no simple closed curve, and every locally path connected space is lcc. Pairwise replacement of the corresponding notions in the hypothesis of Theorem 1 yields the following theorem.

**Theorem 2.** Suppose the KC-space \( X \) is generalized hereditarily unicoherent and suppose \( X \) is locally connected by continua. Then \( X \) is Hausdorff.

Proof: Suppose \( \{a, b\} \subset X \) and \( a \neq b \). Since \( X \) is lcc, Lemma 2 ensures the components of \( Y \) are open. Thus if \( A \) and \( B \) belong to distinct components \( Y_a \) and \( Y_b \), then the open sets \( Y_a \) and \( Y_b \) separate \( a \) and \( b \).

Suppose \( \{a, b\} \) belong to some component \( Y \subset X \). Then since \( Y \) is connected and lcc, \( Y \) is connected by continua by Lemma 5. Obtain a continuum \( Z \) and a map \( f : Z \to Y \) such that \( \{a, b\} \subset im(f) \). Since \( Y \) is a KC-space and \( Z \) is a continuum \( im(f) \) is a continuum by Lemma 1.

Let \( A \) denote the collection of all compact connected sets in \( Y \) which contain \( \{a, b\} \) and let \( B \) denote the collection of all subcontinua of \( im(f) \) which contain \( \{a, b\} \).

Let \( \alpha \) denote the intersection of all sets in \( A \) and let \( \beta \) denote the intersection of all sets in \( B \). Then \( \alpha \subset \beta \) since \( B \subset A \). On the other hand if \( \gamma \in A \) then \( im(f) \cap \gamma \in A \cap B \) and hence \( \beta \subset \alpha \). Thus \( \alpha = \beta \). Note \( \alpha \) is a continuum by Lemma 3.

Since \( Y \) is \( T_1 \), \( \{a, b\} \) is not connected and there exists \( k \in \beta \setminus \{a, b\} \). Now we claim \( a \) and \( b \) belong to distinct components \( U \) and \( V \) of \( Y \setminus \{k\} \). To obtain a contradiction suppose there exists a component \( U \subset Y \setminus \{k\} \) such that \( \{a, b\} \subset U \). Then \( U \) is open since \( Y \) is \( T_1 \).

Since \( U \) is connected and lcc, \( U \) is connected by continua (by Lemma 3). In particular there exists a compact connected set \( \gamma \subset U \) such that \( \{a, b\} \subset \gamma \). Note \( \gamma \in A \) and hence \( \alpha \subset \gamma \). On the other hand \( k \in \alpha \setminus \gamma \) and we have a contradiction.
Thus \(a\) and \(b\) belong to distinct components \(U\) and \(V\) of \(Y\backslash \{k\}\). Since \(Y\) is open in \(X\) and since \(\{k\}\) is closed in \(X\) it follows that \(U\) and \(V\) are open in \(X\) and this proves \(X\) is Hausdorff. 

This yields immediately alternate criteria for dendroid recognition.

**Corollary 2.** Suppose the compact KC-space \(X\) is connected, generalized hereditarily unicoherent and suppose \(X\) is locally connected by continua. Then \(X\) is a dendroid.

4. **Examples of connected, compact, non-Hausdorff KC-spaces**

The examples in the following subsections are compact, connected, non-Hausdorff KC-spaces, and each contains a nested sequence of compact connected subspaces \(...A_3 \subset A_2 \subset A_1\) such that \(\bigcap_{n=1}^{\infty} A_n\) is not connected.

To construct such examples we begin with a 1st countable, Hausdorff space \(X\), such that \(X\) fails to be locally compact, and then manufacture the Alexandroff compactification \(X \cup \{b\}\). Since \(X\) is both a KC-space and a \(k\) space, Theorem 5 of [18] ensures that \(Y\) is a connected, non-Hausdorff, compact KC-space. See Lemma 8 for an alternate argument.

The examples also serve to illustrate how slight weakening of the hypotheses in Theorems 1 and 2 can destroy the guarantee of the \(T_2\) condition.

4.1. **The Alexandroff compactification of the rationals.** To reinforce the relevance of the remaining examples in this paper, we begin with a discussion of a well studied space which fails to enjoy most of the properties of interest elsewhere in the paper at hand. Let \(Y = Q \cup \{\infty\}\) denote the Alexandroff compactification of the rational numbers \(Q\) (see also [14]). Lemma 8 ensures \(Y\) is a compact non-Hausdorff KC-space. The space \(Y\) is not locally connected, not connected by continua, not generalized hereditarily unicoherent, and \(\dim(Y) = 1\), argued as follows.

Since \(Q\) is not connected Lemma 8 does not guarantee that \(Y\) is connected. To see why \(Y\) is connected, observe if \(A \subset Q\) and \(A\) is compact, then \(Q \backslash A = Y\) and in particular \(Y\) cannot be the disjoint union of two nonempty compact subspaces.

For similar reasons, if \(U \subset Q\) and \(U\) is open, then \(\overline{U}\) is connected in \(Y\), since if \(A \subset U\) and \(A\) is compact, then \(\overline{U \backslash A} = \overline{U}\). Thus if \(Q_+\) and \(Q_-\) denote the positive and negative rationals then \(\overline{Q_+} \cap \overline{Q_-} = \{0, \infty\}\) and thus \(Y\) is not generalized hereditarily unicoherent.

Every nontrivial continuum is uncountable and hence has constant image in \(Y\). Thus \(Y\) is not connected by continua.

To see that \(\dim(Y) = 1\), given an open covering \(G\) of \(Y\) let \(\infty \in V\) and note \(C = Y \backslash V\) is a compact zero dimensional set of real numbers. Thus there exists a covering of \(C\) by pairwise disjoint open sets \(\{U_1, .., U_n\}\) subordinate to \(G \backslash V\). Each point of \(Y\) belongs to at most 2 sets in \(\{V, U_1, .., U_n\}\) and hence \(\dim(Y) \leq 1\) (and \(\dim(Y) > 0\) since \(Y\) is not \(T_2\)).

Notice if \(A_n = (0, \frac{1}{n}) \cap Q\) then \(A_n\) is compact and connected but \(\bigcap_{n=1}^{\infty} A_n = \{0, \infty\}\), and the latter set is not connected.

4.2. **1-dimensional and n-connected for all \(n\).** In the previous example \(Q \cup \{\infty\}\) is not path connected. For the current example \(Z\) is 1-dimensional, compact,
non-Hausdorff, connected and \( n \)-connected for all \( n \). Corollary \([\square]\) demands that \( Z \) is not locally path connected and in this example \( Z \) is not locally connected.

Consider the following planar set \( T \subset \mathbb{R}^2 \) such that \( T = ((-\infty, \infty) \times \{0\}) \cup ((0) \times [0, \infty)) \). Note \( T \) is a metrizable space and in particular \( T \) is Hausdorff and 1st countable. Let \( \mathcal{A} \) denote the following collection of open subsets of \( T \). For each positive integer \( n \) consider the open subspace \( A_n = ((n, \infty) \times \{0\}) \cup \{0\} \times \cup_{k=n}^{\infty}(2k, 2k+1) \).

Let \( \mathcal{A} = \{A_1, A_2, \ldots\} \) and note \( \mathcal{A} \) is countable, \( A_{n+m} \cap A_n = A_{n+m} \) and if \( x \in T \) there exist open sets \( U \) and \( A_n \) such that \( x \in U \) and \( A_n \cap U = \emptyset \). Thus if \( T_\mathcal{A} \) denotes the open sets of \( T \) and if \( \mathcal{S}_n \) denotes the sets of the form \( \{a\} \cup A \) for \( A \in \mathcal{A} \), and if \( Y = X \cup \{a\} \) is the space with topology generated by \( T_{\mathcal{A}} \cup \mathcal{S}_n \), then Lemma \( [\square] \) ensures that \( Y \) is a connected 1st countable \( T_2 \) space.

To check that \( Y \) is not locally compact at \( a \), consider the closure of a basic open set \( C = \{a\} \cup A_n \) and note \( \{2n, 2n+2, \ldots\} \subset C \) and this sequence has no subsequential limit in \( Y \).

Thus if \( Z = Y \cup \{b\} \) denotes the Alexandroff compactification of \( Y \), then Lemma \( \blacklozenge \) ensures \( Z \) is a compact, connected, non-Hausdorff KC-space.

Note \( T \) is path connected. Let \([-\infty, \infty]\) denote the two point compactification of \((\infty, \infty)\). For \( N \geq 0 \) define \( j_N : ([\infty, \infty) \times \{0\}) \cup ([0, N]) \rightarrow Z \) via \( j(-\infty, 0) = b \), \( j(\infty, 0) = c \) and \( j(x) = x \) otherwise. By construction \( j_N \) is continuous and one to one, hence by Lemma \( [\Box] \) \( j_N \) is an embedding. Hence \( im(j_N) \) is contractible. Note \( \{a, b\} \subset im(j_N) \) and \( (0, 0) \in T \cap im(j_N) \). Thus \( Z \) is path connected.

Observe if \( \alpha : [0, 1] \rightarrow Z \) is a map such that \( \alpha(0) \neq a \) and \( \alpha(1) = a \) then there exists \( N \) such that \( im(\alpha) \subset \{a, b\} \cup (\infty, \infty) \times \{0\}) \cup ([0, N]) \). To see that \( Z \) is \( n \)-connected, suppose \( f : S^n \rightarrow Z \) is a map. Since \( S^n \) is a Peano continuum obtain a surjective map \( \beta : [0, 1] \rightarrow S^n \). Let \( \alpha = f \beta \) and note \( im(f) = im(\alpha) \). Thus \( im(f) \) is contained in the contractible subspace \( \{a, b\} \cup (\infty, \infty) \times \{0\}) \cup ([0, N]) \).

Hence \( f \) is inessential.

To check that \( \dim(Y) = 1 \), take an open covering \( G \) of \( Y \) with \( a \in U \) and \( b \in V \). Replace \( U \) and \( V \) by basic open sets \( U_1 \subset U \) and \( V_1 \subset V \) such that \( Y \setminus (U_1 \cup V_1) = ([\infty, N] \times \{0\}) \cup ([0, \infty) \times \{0\}) \) and such that \( \{(N, 1) \cup (0, 0)\} \subset U_1 \setminus V_1 \) and \( (N, 1) \in V_1 \setminus U_1 \) and manufacture a covering \( \mathcal{G}_1 \) of \( Y \) subordinate to \( \mathcal{G} \) such that \( \{U_1, V_1\} \subset \mathcal{G}_1 \) and each element of \( Y \) is contained in at most two elements of the covering.

The local basis \( \mathcal{S}_n \) shows \( Z \) is not locally connected at \( a \) (in fact \( Z \) is not locally connected at \( b \) either).

Let \( A_n = \{a, b\} \cup ([0, \infty)) \) and note \( A_n \) is compact and connected but \( \cap_{n=1}^{\infty} A_n = \{a, b\} \).

### 4.3. 1-dimensional and generalized hereditarily unicoherent

Neither of the previous examples are generalized hereditarily unicoherent. Theorem \([\Box]\) ensures if \( D \) is a 1-dimensional, non-Hausdorff, generalized hereditarily unicoherent space, then \( D \) is not locally connected by continua and the example at hand is not locally connected.

Recall the previous example and the discussion of the subspace \( D \subset C \) such that \( D = \{a, b\} \cup ([0, \infty)) \). Thus \( D \) is a connected, 1-dimensional, compact non-Hausdorff KC-space, \( D \) is generalized hereditarily unicoherent, but \( D \) is not path connected or locally connected.
4.4. 1-dimensional and locally contractible. To construct a 1-dimensional, non-Hausdorff KC compactum \( Y \) such that \( Y \) is locally contractible, we glue together countably many closed rays at the common minimal point, and then apply Alexandroff compactification to obtain \( Y \). Corollary \( \text{I} \) demands that \( Y \) cannot be simply connected.

Let \( \{e_1, e_2, \ldots\} \) denote the standard unit vectors \( e_n = (0, \ldots, 0, 1, 0, 0, \ldots) \) in familiar \( l_2 \) Hilbert space (the space of square summable sequences), and let \( X \) denote the subspace of \( l_2 \) consisting of points of the form \( \alpha e_n \) with \( \alpha \in [0, \infty) \).

Notice \( X \) is a connected 1-dimensional metric space and \( X \) fails to be locally compact at precisely the point \( (0, 0, 0, \ldots) \). Thus by Lemma \( \text{II} \) if \( Y = X \cup \{ \infty \} \) denotes the Alexandroff compactification of \( X \) then \( Y \) is a non-Hausdorff, compact, connected KC-space.

Note \( X \) is locally contractible. To check local contractibility at \( \infty \), we manufacture a homotopy of \( Y \setminus (0, 0, 0, \ldots) \), shrinking basic open neighborhoods \( U \) of \( \infty \) within themselves to \( \infty \), as follows.

Define \( H : Y \setminus (0, 0, 0, \ldots) \times [0, \infty] \to Y \setminus (0, 0, 0, \ldots) \) so that \( H((\alpha e_n, t)) = (\alpha + t)e_n \) if \( t < \infty \) and \( H(x, t) = \infty \) otherwise.

To check that \( H \) is continuous suppose \( U \) is a subbasic open set in \( Y \setminus (0, 0, 0, \ldots) \).

If \( \infty \notin U \) and \( U = (0e_n, \alpha e_n) \) then \( H^{-1}(U) = (0e_n, \alpha e_n) \) which is open.

If \( \infty \notin U \) and \( U = (\alpha e_n, \infty e_n) \) then \( H^{-1}(U) = (0e_n, \infty e_n) \) which is open.

If \( \infty \in U \) we can assume \( Y \setminus U \) is connected and \( (0, 0, 0, \ldots) \notin U \) and note \( H^{-1}(U) = U \) which is open. Thus \( H \) is continuous.

To see informally why \( Y \) is 1-dimensional, given an open covering \( \mathcal{G} \) of \( Y \), observe there exist respective subordinate basis elements \( U \) and \( V \) of the special points \( \{(0, 0, 0, \ldots, \infty)\} \) such that \( Y \setminus (U \cup V) \) is the countable union of disjoint closed line segments \( \beta_1, \beta_2, \ldots \) such that the respective endpoints of \( \beta \beta_n \) are respective limit points of \( U \) and \( V \). All the segments \( \beta_n \) can be simultaneously lengthened slightly to pairwise disjoint open arcs \( \gamma_1, \gamma_2, \ldots \) such that the ends of \( \gamma_n \) are contained respectively in \( U \) and \( V \). In particular extending the standard construction that \( \dim(\beta_1 \cup \beta_2) = 1 \) relative to the covering \( \mathcal{G} \) yields the desired covering \( \mathcal{G}_1 \).

Let \( J_n = [0e_n, \infty e_n] \cup \{ \infty \} \) denote the loop containing \( e_n \) with endpoints \( (0, 0, 0, \ldots) \) and \( \infty \). Let \( A_N = \bigcup_{n=N}^{\infty} J_n \) and note \( A_N \) is compact and connected but the two point set \( \cap_{N=1}^{\infty} A_N = \{(0, 0, 0, \ldots), \infty\} \) is not connected.

4.5. 1-dimensional, lcc, and simply connected. In similar manner to the example from subsection 4.4 to construct a 1-dimensional, simply connected, non-Hausdorff KC compactum \( Y \) such that \( Y \) is locally connected by continua we glue together countably many ‘long lines’ at the common minimal point, and then apply Alexandroff compactification to obtain \( Y \).

Let \( L \) denote the (noncompact) long line (i.e. \( L \) is obtained by attaching open intervals between consecutive points of the minimal uncountable well ordered set \( S_0 \), to obtain a connected 1-dimensional nonseparable space \( L \) such that each point of \( L \) has a neighborhood homeomorphic to \( (0, 1) \)).
Let $z$ denote the minimal point of $L$ and let $X$ denote the quotient space of $\{1, 2, 3, \ldots \} \times L$ obtained by identifying $(m, z)$ and $(n, z)$. Thus we are gluing countably many copies of $L$ together at the minimal point. Then $X$ is a 1st countable $T_2$ space which fails to be locally compact. Thus if $Y$ is the Alexandroff compactification of $X$ then Lemma 8 ensures $Y$ is a connected, compact, non-Hausdorff KC-space.

To see why $\dim(Y) = 1$, we can apply essentially the same argument as in subsection 4.4.

To see that $Y$ is simply connected note by construction $Y$ is 1 dimensional and contains no simple closed curves and hence by Lemma 2 $Y$ is simply connected.

Let $A_N$ denote the union of the closed arcs $L_N, L_{N+1}, \ldots$ and note $A_N$ is compact and connected but $\bigcap_{N=1}^{\infty} A_N$ is not connected.

4.6. 2-dimensional, locally contractible, and n-connected for all $n$. Theorem 1 forbids the existence of a 1-dimensional, non-Hausdorff, locally contractible, $n$-connected, KC-space. Our construction of such a space of dimension 2 is equivalent to taking a closed disk, deleting a half open interval from the boundary, and then taking the Alexandroff compactification of the remaining space. For convenient coordinates, we begin with the closed first quadrant $X \subset R^2$ and first attach a point $a$ to create a $T_2$ space which is not locally compact at $a$. We then take the Alexandroff compactification of $X \cup \{a\}$ to obtain the desired space $Y = X \cup \{a\} \cup \{b\}$. The idea behind the example is similar to the totally disconnected space constructed in example 99 [14].

Intuition suggests $Y$ is not contractible but we do not settle this question. However if $Y$ fails to be contractible, then $Y$ would serve to highlight a potential difference between KC-spaces and the familiar theory of absolute retracts, since every finite dimensional, $n$-connected, compact, locally contractible metric space is necessarily contractible (Theorem 4.2.33 [?]).

Let $X = [0, \infty) \times [0, \infty)$ with the standard topology (i.e. $X$ is the closed 1st quadrant in the Euclidean plane). We will attach two points $a$ and $b$ to $X$.

Let $A$ denote the collection of open sets of $X$ of the form $[0, \infty) \times (n, \infty)$ for $n \in \{1, 2, 3, \ldots \}$. Note $A$ is countable, closed under finite intersections, and given $(x, y) \in X$ there exists an open set $U \subset X$ and $A \in A$ such that $U \cap A = \emptyset$. Let $a \notin X$. If $S_a$ denotes the collection of sets of the form $\{a\} \cup A$ for $A \in S_a$, Lemma 7 ensures that $X \cup \{a\}$ is a $T_2$ 1st countable space. Observe $\{a\} \cup ([0, \infty) \times (n, \infty))$ fails to have compact closure in $X \cup \{a\}$ since the sequence $((0, n+1), (1, n+1), \ldots)$ has no subsequential limit in $X \cup \{a\}$. Thus $X \cup \{a\}$ is not locally compact at $a$.

Let $b \notin X \cup \{a\}$ and let $Y = X \cup \{a\} \cup \{b\}$ denote the Alexandroff compactification of $X \cup \{a\}$. Lemma 8 ensures $Y$ is a compact, connected, non-Hausdorff KC-space.

To obtain a particular local basis $S_0$ at $b$, let $M$ denote the collection of non-decreasing maps $f : [0, \infty) \to [1, \infty)$. For each $f \in M$ let $U_f = \{(x, y) | f(y) < x\}$. Note $U_f$ is open in $X$. Let $B$ denote the collection of sets of the form $U_f$ and let $S_b$ denote the collection of sets of the form $\{b\} \cup U_f$ for $U_f \in B$. To check that $S_b$ is a collection of open sets in $Y$, let $\{b\} \cup U_f \in S_b$ and consider the complement $C = Y \setminus \{\{b\} \cup U_f\}$. To check that $C$ is compact, given a covering $G$ of $C$ by basic open sets in $Y$, obtain $U \in G$ such that $a \in U$. Notice $C \setminus U$ is compact in $X$, since $C \setminus U$ is a topological disk whose simple closed curve boundary is the concatenation of 3 arcs: a line segment $\alpha \subset \partial U$, an arc $\beta$ contained in $\partial U_f$, and a third arc $\gamma$ contained in the union of the $x$ and $y$ axes in $X$. Thus we can cover
$C \setminus U$ by a finite collection of the open sets in $G \setminus \{U\}$. Thus $S_b$ is a collection of open sets in $Y$ each of which contains $b$. To check that $S_b$ is a local basis suppose we have a compact set $C \subset Y \setminus \{b\}$. Observe for each $n > 0$ there exists $m$ such that $(m, n) \times [0, a] \cap C = \emptyset$, (since otherwise there would exist a sequence $\{(x_n, y_n)\} \subset C$ such that $\{x_n\}$ is bounded and $y_n \to \infty$ and $\{(x_n, y_n)\}$ will be a noncompact closed subspace of the compact space $C$). Hence we can manufacture $f \in M$ and $U_f \in B$ such that $C \subset Y \setminus U_f$ and, thus $S_b$ is a basic local basis at $b$.

To check that $\pi_n(Z) = 0$ for all $n \geq 0$ it suffices to show if $Z$ is a compact $T_2$ space, then each map $f : Z \to Y$ is homotopic to a constant, and the strategy is to show that both subspaces $X \cup \{a\}$ and $X \cup \{b\}$ are contractible and in particular $X \cup \{a\}$ admits strong deformation retracts onto large compact spaces of $X \cup \{a\}$. Normality of $Z$ combined with the standard pasting Lemma from general topology will allow us to push $f$ into $X \cup \{b\}$, and then we will homotop $f$ to the constant map $b$, by contracting $X \cup \{b\}$ to the point $b$.

To see that $X \cup \{a\}$ is homeomorphic to a closed topological disk with a half open interval deleted from the boundary, let $h : [0, \infty) \to [0, 1)$ be any homeomorphism and thus we can consider $X$ as the space $[0, 1) \times [0, 1]$. Now consider the following operations on the familiar closed unit square $[0, 1] \times [0, 1]$. Take the quotient space by identifying $[0, 1) \times \{1\}$ to a point and note the quotient space $X_1$ is still a closed topological disk. Now delete from $X_1$ the side $\{1\} \times [0, 1)$ to obtain the space $X_2$. Then $X_2$ is homeomorphic to $X \cup \{a\}$ (and $a$ corresponds to the top side $[0, 1) \times \{1\}$ of $X_2$).

Now suppose $Z$ is any compact $T_2$ space and $f : Z \to Y$ is any map. Our first task is to homotop $f$ to a map $g$ such that $im(g) \subset X \cup \{b\}$.

If $im(f) \subset X \cup \{b\}$ let $g = f$. Otherwise let $A = f^{-1}(\{a\})$ and $B = f^{-1}(\{b\})$ and apply normality of $Z$ to obtain an open set $U \subset Z$ such that $A \subset U$ and $\overline{U} \cap B = \emptyset$. Let $\partial U = \overline{U} \setminus U$ and note $f(\partial U) \subset X$. Obtain a compact topological disk $D \subset X$ such that $f(\partial U) \subset D$ and obtain a strong deformation retract $H_t : X \cup a \to X \cup a$ onto $D$. Let $J_t : X \to X \cup b$ denote the constant homotopy. Observe $(\overline{U} \times [0, 1]) \cap ((Z \setminus U) \times [0, 1]) \subset \partial U \times [0, 1]$. Now apply the pasting lemma [11], gluing together the union of the restricted homotopies $H_1(f_{\overline{U}}) \cup J_b(f_{Z \setminus U})$, and obtain a homotopy of $f$ to a map $g = H_1(f_{\overline{U}}) \cup f_{Z \setminus U}$ such that $g(Z) \subset X \cup \{b\}$.

Thus we have shown that $Y \setminus \{b\}$ is locally contractible, and (for $n \geq 0$) any map $S^n : Z \to Y$ is homotopic in $Y$ to a map $g : S^n \to Y \setminus \{a\}$. To complete the proof that $Y$ is $n$ connected and locally contractible it suffices to show $Y \setminus \{a\}$ is contractible and locally contractible. To accomplish this we will manufacture a global contraction from $X \cup \{b\}$ to the $b$ whose restrictions shrink basic open sets $\{b\} \cup U_f$ within themselves to $b$.

Let $[0, \infty)$ denote the one point compactification of $[0, \infty)$ and define a function $H : (X \cup \{b\}) \times [0, \infty) \to X \cup \{b\}$ so that $H(x, y, t) = (x + t, y)$ if $(x, y) \in X$ and $t < \infty$, $H(b, t) = b$ for all $t \in [0, \infty]$, and $H(x, y, \infty) = b$. To check that $H$ is continuous it suffices to check that the preimage under $H$ of subbasic sets is open.

Suppose $U \subset X \cup \{b\}$ is a subbasic open set. Let $J$ be open in $[0, \infty)$. If $b \notin U$ note if $U = [0, x) \times J$ then $H^{-1}(U) = [0, x) \times J$ is open, and if $U = (x, \infty] \times J$ then $H^{-1}(U) = [0, \infty) \times J$ is open. If $b \in U$ then let $U = \{b\} \cup B$ for $B \in B$ and $B = U_f$ with $f \in M$, and observe $H^{-1}(U) = U \times [0, \infty)$. 
To see informally why \( \dim(Y) = 2 \) suppose \( \mathcal{G} \) is an open covering of \( Y \). Obtain subordinate basis elements \( \{a\} \cup U \) and \( \{b\} \cup U_f \) such that \( U = [0, \infty) \times (n, \infty) \) and \( U_f = \{(x,y) | f(y) < x\} \) with \( f \in \mathcal{M} \). Let \( D = Y \setminus (U \cup V) \). Notice \( D \) is a closed topological disk (as described in the earlier paragraph) and the boundary point \( z = (f(y),y) \) poses the ‘greatest risk’ of belonging to too many open sets in the cover \( \mathcal{G}_1 \) currently under construction, since \( (f(y),y) \) is a (unique) point of \( D \) which is a limit point of both sets \( U \) and \( V \). So now cover \( z \) by a tiny round open disk \( V \) subordinate to the original cover \( \mathcal{G} \), and then proceed (as in a standard proof that the topological disk \( D \setminus U \) is 2 dimensional) to build the desired cover \( \mathcal{G}_1 \) subordinate to \( \mathcal{G} \).

Let \( A_N \) denote the complement in \( Y \) of \( [0, n) \times [0, n) \). Note \( A_N \) is compact and connected but \( \{a, b\} = \bigcap_{n=1}^{\infty} A_N \).

4.7. 2-dimensional, compact, non-Hausdorff, KC, contractible, and locally contractible. To create a 2-dimensional non-Hausdorff KC compactum \( Y \) such that \( Y \) is both contractible and locally contractible, apply directly Lemma 9 to the locally contractible example from subsection 4.4. Recall \( J_n = [0, e_n, \infty) \cup \{\infty\} \) denotes the loop containing \( e_n \) with endpoints \( (0,0,0,\ldots) \) and \( \infty \). Let \( A_N = (\bigcup_{n=1}^{\infty} J_n) \times [0,1] \) and note \( A_N \) is compact and connected but the two point set \( \bigcap_{n=1}^{\infty} A_N \) is not connected.

5. Supplemental Lemmas

Justification of the theorems and examples throughout this paper relies on various well known or elementary results concerning KC-spaces or basic general topology. For convenience we include proofs.

The nested intersection of subcontinua indexed by an arbitrary ordered set \( I \) is connected, and this yields a proof (in conjunction with Zorn’s Lemma) that any two points of a hereditarily unicoherent continuum are contained within a canonical subcontinuum (Lemma 3).

Familiar facts about locally path connected spaces (such spaces have open components and such spaces are connected if they are path connected), have counterparts with the notion of ‘path connected’ replaced by ‘connected by a continua’, and this is the content of Lemmas 4 and 5.

Lemmas 7 and 8 validate the basic properties of the main examples of the paper at hand, and are essentially a special case of Theorem 5.

Lemma 9 yields a method to construct a contractible KC-space with straightforward justification.

**Lemma 3.** Suppose the continuum \( X \) is a hereditarily unicoherent continuum and \( \{a,b\} \subset X \). Let \( Y \) denote the intersection of all subcontinua containing \( \{a,b\} \). Then \( Y \) is connected.

**Proof.** To see that \( Y \) is connected apply Zorn’s Lemma as follows. Let \( S \) denote the collection of subcontinua of \( X \) that contain \( \{a,b\} \), partially ordered by reverse inclusion such that \( A \leq B \) if \( B \subset A \). Since \( Y \) is a \( T_2 \) space, the nested intersection of any linearly ordered collection of subcontinua is connected, and thus every chain in \( S \) has an upper bound. By Zorn’s Lemma let \( M \) be a maximal element in \( S \). To see that \( M = Y \) note if \( A \subset S \) then \( M \cap A \subset S \) (since \( X \) is hereditarily unicoherent) and moreover \( A \cap M \subset M \). Thus \( M = M \cap A \) since \( M \) is maximal. Thus \( M \subset Y \). Conversely, since \( M \subset S \) it follows that \( Y \subset M \). \( \square \)
Lemma 4. Suppose $X$ and $Y$ are disjoint continua and $a \in X$ and $b \in Y$ and $Z$ is the quotient of $X \cup Y$ identifying $a$ and $b$. Then $Z$ is a continuum. Suppose $f : X \to W$ and $g : Y \to W$ are maps such that $f(a) = g(b)$. Then $f \cup g : Z \to W$ is a continuous function.

Proof. Let $q : X \cup Y \to Z$ denote the quotient map. Since by definition $q$ is onto and continuous, $Z$ is compact and connected. To check $Z$ is $T_2$ suppose $\{c, d\} \subset Z$. If $c \neq \{a, b\}$ and $d \neq \{a, b\}$ then apply directly the $T_2$ property of $X \cup Y$ to obtain disjoint open sets separating $c$ and $d$.

Suppose $x \in X \backslash \{a\}$. Apply the $T_2$ property of $X$ to obtain open sets $U$ and $V_x$ in $X$ such that $U \cap V_x = \emptyset$ and $x \in U$ and $a \in V_x$. Then $q^{-1}(U)$ and $q^{-1}(V_x \cup Y)$ separate $x$ and $\{a, b\}$.

By a symmetric argument if $y \in Y \backslash \{a, b\}$ then $y$ and $\{a, b\}$ can be separated in $Z$.

Notice $f \cup g$ is well defined. Recall $q$ is a quotient map and hence $f \cup g$ is continuous since if $z \in Z$ and $B = q^{-1}(z)$ then $(f \cup g)|_B$ is constant.

Lemma 5. Suppose the nonempty connected space $X$ is locally connected by continua. Then $X$ is connected by continua.

Proof. Fix $x \in X$ and let $A$ denote the set of $y$ in $X$ such that there exists a continuum $Z_y$ and a map $f : Z_y \to X$ such that $\{x, y\} \subset \text{im}(f)$.

To see that $A$ is open, given $y \in A$ obtain a continuum $Z_y$ and a map $f : Z_y \to X$ such that $\{x, y\} \subset \text{im}(f)$. Apply the lcc property of $X$ at $y$ and obtain an open set $U$ such $y \in U$ and $U$ is connected by continua. Given $z$ in $U$ obtain a continuum $Z_z$ and a map $g : Z_z \to U$ such that $\{y, z\} \subset \text{im}(g)$. Obtain $a \in Z_y$ and $b \in Z_z$ such that $f(a) = y = g(b)$. Let $Z$ denote space obtained from $Z_y \cup Z_z$ by identifying $a$ and $b$. Apply Lemma 4 to conclude $Z$ is a continuum and the map $f \cup g : Z \to X$ satisfies $\{x, z\} \subset \text{im}(f \cup g)$. Thus $A$ is open.

To see that $A$ is closed suppose $z$ is a limit point of $A$. Obtain an open set $U_z$ such that $z \in U_z$ and $U_z$ is connected by continua. Obtain $y \in A \cap U_z$. Obtain disjoint continua $Z_y$ and $Z_z$ and maps $f : Z_y \to X$ and $g : Z_z \to X$ such that $\{x, y\} \subset \text{im}(f)$ and $\{y, z\} \subset \text{im}(g)$. Obtain $a \in Z_y$ and $b \in Z_z$ such that $f(a) = y = g(b)$. Let $Z$ denote space obtained from $Z_y \cup Z_z$ by identifying $a$ and $b$. Apply Lemma 4 to conclude $Z$ is a continuum and the map $f \cup g : Z \to X$ satisfies $\{x, z\} \subset \text{im}(f \cup g)$. Thus $A$ is closed.

Since $X$ is connected and $A$ is both open and closed, $A = X$. Thus $X$ is connected by continua.

Lemma 6. Suppose the space $X$ is lcc and $Y$ is a component of $X$. Then $Y$ is an open subspace of $X$ and $Y$ is connected by continua.

Proof. Suppose $y \in Y$. Since $X$ is lcc obtain an open set $U$ such that $y \in U$ and such that $U$ is connected by continua. Then $U$ is connected (since $U$ is the union of images of continua, each of which contains the common point $y$) and hence $U \subset Y$. Thus $Y$ is open and the open set $U$ also shows $Y$ is lcc. Hence by Lemma 5 $Y$ is connected by continua.

Lemma 7. Suppose $(X, \mathcal{T}_X)$ is a connected $T_2$ space and $A \subset \mathcal{T}_X$ such that $A \neq \emptyset$, and such that if $\{A_1, A_2\} \subset A$ then $A_1 \cap A_2 \in A$, and suppose $a \notin X$ and $Y = X \cup \{a\}$. Suppose $\mathcal{S}_a$ denotes the collection of sets of the form $\{a\} \cup A$ for $A \in A$. Then $\mathcal{T}_X \cup \mathcal{S}_a$ is a basis for a topology $(Y, \mathcal{T}_Y)$ on $Y$, $(Y, \mathcal{T}_Y)$ is connected if $\emptyset \notin A$, 

and \((Y, \mathcal{T}_Y)\) is \(T_2\) if for each \(x \in X\) there exists \(U \in \mathcal{T}_X\) and \(A \in \mathcal{A}\) such that \(U \cap A = \emptyset\). If \(X\) is first countable and \(\mathcal{A}\) is countable then \(Y\) is first countable.

**Proof.** To check \(\mathcal{T}_X \cup \mathcal{S}_a\) is a basis observe \(\mathcal{T}_X \cup \mathcal{S}_a\) covers \(Y\) since \(\mathcal{A} \neq \emptyset\) and hence \(\mathcal{T}_X \cup \mathcal{S}_a\) is a subbasis. Thus sets of the form \(U_1 \cap U_2 \cap \ldots \cap U_n\) form a basis for a topology on \(Y\). Observe \(\mathcal{T}_X \cup \mathcal{S}_a\) is closed under the operation of finite intersections and thus \(\mathcal{T}_X \cup \mathcal{S}_a\) is a basis for \((Y, \mathcal{T}_Y)\). If \(\emptyset \notin \mathcal{A}\) then \(A = X \cap (\{a\} \cup \mathcal{A}) \neq \emptyset\) for each basic open set \(\{a\} \cup A \in \mathcal{S}_a\) and thus \(X\) is dense in \(Y\) and hence \(Y\) is connected since \(X\) is connected. Since \(\mathcal{S}_a\) is a local basis at \((Y, \mathcal{T}_Y)\) is \(T_2\) if for each \(x \in X\) there exists \(U \in \mathcal{T}_X\) and \(A \in \mathcal{A}\) such that \(U \cap A = \emptyset\) and by also by definition \(Y\) is first countable at \(a\) if \(\mathcal{S}_a\) is countable. □

**Lemma 8.** Suppose \((X, \mathcal{T}_X)\) is a 1st countable \(T_2\) space and \(Y = X \cup \{b\}\) is the Alexandroff compactification of \(X\). Then \(Y\) is a compact KC-space. If \(X\) is not locally compact then \(Y\) is not Hausdorff. If \(X\) is connected and not compact, then \(Y\) is connected.

**Proof.** Since \(X\) is \(T_2\) compact sets in \(X\) are closed in \(X\) and hence \(X\) is a KC.

To check that \(Y\) is a KC-space suppose \(C \subset Y\) and \(C\) is not closed in \(Y\). If \(b \in \overline{C}\) then \(C\) is not a compact subspace of \(X\) (since otherwise \(Y \setminus C\) shows \(b\) is not a limit point of \(C\)). Since open sets in \(X\) are also open in \(Y\), it follows that \(C\) is not a compact subspace of \(Y\). If \(x \in X\) and \(x \in \overline{C}\), since \(X\) is 1st countable, obtain a countable collection of open sets \(\{U_n\} \subset \mathcal{T}_X\) such that \(\cup U_1 \subset U_2 \subset U_1\) and if \(V\) is an open in \(X\) and \(x \in V\) there exists \(U_N\) such that \(x \in U_N \subset V\). Since \(x\) is a limit point of \(C\) select \(c_n \in C \cap U_n\) and note \(c_n \to x\) and the set \(A = \{x, c_1, c_2, \ldots\}\) is compact in \(X\). Thus \(\{c_1, c_2, \ldots\}\) is not a closed subspace of \(X\) and hence since \(X\) is a KC-space \(\{c_1, c_2, \ldots\}\) is not compact in \(X\). Obtain an open cover \(G \subset \mathcal{T}_X\) of \(\{c_1, c_2, \ldots\}\) such that no finite subcover of \(G\) covers \(\{c_1, c_2, \ldots\}\). Observe \(G \cup \{Y \setminus A\}\) covers \(C\) and has no finite subcover. Thus \(C\) is not compact in \(Y\) and hence \(Y\) is a KC-space.

If \(X\) is not locally compact obtain \(a \in X\) such that \(X\) is not locally compact at \(a\). To see that \(Y\) is not Hausdorff suppose \(a \in U\) and \(b \in V\) and each of \(U\) and \(V\) are open in \(Y\). To obtain a contradiction suppose \(U \cap V = \emptyset\). Then \(b \notin U\) and hence \(U \subset X\). By definition of \(Y\), \(U\) is open in \(X\). Hence \(\overline{U} \cap X\) is not compact since \(X\) is not locally compact at \(a\). Thus, since \(X\) is a KC-space, \(\overline{U} \cap X\) is not closed in \(Y\). Hence \(b\) is a limit point of \(\overline{U} \cap X\) and \(b \in \overline{U} \cap U\). Since \(b\) is a limit point of \(U\) we obtain the contradiction \(V \cap U = \emptyset\). This shows \(X\) is not \(T_2\).

If \(X\) is connected and not locally compact, then \(X\) not compact, and thus \(b\) is not an isolated point of \(Y\), and hence the connected subspace \(X\) is dense in \(X \cup \{b\}\) and hence \(Y\) is connected. □

**Lemma 9.** If \(X\) is metrizable and \(Y = X \cup \{\infty\}\) denotes the Alexandroff compactification of \(X\), then \(Y \times [0, 1]\) is a KC-space. If \(Y \times [0, 1]\) is a KC-space and \(Z\) denotes the quotient of \(Y\) such that \((x, 1) \sim (y, 1)\) then \(Z\) is a contractible KC-space and \(Z\) is locally contractible at the point determined by \(Y \times \{1\}\).

**Proof.** To check that \(Y \times [0, 1]\) is a KC-space suppose \(C \subset Y \times [0, 1]\) and suppose \(C\) is not closed in \(Y\) and suppose \(z \in \overline{C}\). If \(z \in X \times [0, 1]\) then, since \(X \times [0, 1]\) is an open metrizable subspace of \(X \times [0, 1]\), name a sequence \(z_n \to z\) such that \(z_n \in C\). Note \(\{z_1, z_2, \ldots\}\) is a noncompact closed subspace of the space \(C\) and conclude \(C\) cannot be compact. We have reduced to the case that \(C \cap X\) is closed in \(X\) and
z ∈ {∞} × [0, 1]. If there exists zn ∈ ( {∞} × [0, 1]) ∩ C such that zn → z then C is not compact since C ∩ {z1, z2, ...} is closed in C and not compact. Thus to check the final case we assume z is isolated in C ∩ {∞} × [0, 1]. Obtain a closed interval [c, d] such that {z} = ( {∞} × [c, d]) ∩ C and {z} ∈ ( {∞} × (c, d)). Let D = C ∩ (X × [c, d]) and observe D is a closed subspace of C. Let E denote the image of D under the natural projection Π : D → X. To see why D cannot be compact, note if D were compact then E is compact in X, and if U = Y \ E then the open set U × (c, d) would show z is not a limit point of C, a contradiction. Thus since D is a closed noncompact subspace of C, we conclude C is not compact and hence Y × [0, 1] is a KC-space.

To check that Z is a KC-space suppose C ⊂ Z and suppose C is not closed. If C ∩ (Y × [0, 1]) is not closed in Y × [0, 1], then C is not compact since Y × [0, 1] is a KC-space. We have reduced to the case that C is closed in Y × [0, 1]. By definition of the quotient topology, the natural quotient map q : Y × [0, 1] → Z shows C is not closed in Y × [0, 1] and Thus C is not compact, and hence q(C) = C is not compact in Z. Thus Z is a KC-space.

Ignoring specific properties of the spaces Y and Z, since q is a quotient map, the natural strong deformation retraction from Y × [0, 1] onto Y × {1} induces a contraction of Z and the contraction shows that Z is locally contractible at the special point.

6. Conclusions

Theorems 1 and 2 show that every locally path connected KC-space which contains no simple closed curve is T2, and every generalized hereditarily unicoherent KC-space which is locally connected by continua is T2.

It follows directly from Corollary 1 that every contractible, locally contractible, 1-dimensional KC-space is T2. However subsection 4.7 shows there exists a 2-dimensional non-Hausdorff KC compactum Y such that Y is both locally contractible and contractible.

The example Y in subsection 4.7 is compact, 2-dimensional, locally contractible, and n-connected for all n but Y has not been shown to be Y contractible. If Y fails to be contractible then Y would amplify another potential contrast between finite dimensional KC-spaces and finite dimensional metric spaces (since every locally contractible n-connected compact metric space is contractible).

For each non-Hausdorff KC-space Y constructed in this paper there exists a sequence of compact connected subspaces ...A3 ⊂ A2 ⊂ A1 ⊂ Y such that \bigcap_{n=1}^{∞} A_n is not connected.

Examples are constructed in subsections 4.2, 4.3, 4.4, and 4.5 of 1-dimensional non-Hausdorff KC-spaces in order to show that slight weakening of the hypotheses of Theorems 1 and 2 can cause the T2 conclusion to fail. The theorems and examples lead naturally to the unsettled question of whether or not every contractible 1-dimensional KC-space is Hausdorff.

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