Model study of dissipation in quantum phase transitions

Subhasis Sinha and Sushanta Dattagupta

Indian Institute of Science Education and Research, Mohanpur, Nadia 741252, India.

We consider a prototypical system of an infinite range transverse field Ising model coupled to a bosonic bath. By integrating out the bosonic degrees, an effective anisotropic Heisenberg model is obtained for the spin system. The phase diagram of the latter is calculated as a function of coupling to the heat bath and the transverse magnetic field. Collective excitations at low temperatures are assessed within a spin-wave like analysis that exhibits a vanishing energy gap at the quantum critical point. We also consider another limit where the system reduces to a generalized spin-boson model of two interacting spins. By increasing the coupling strength with the heat bath, the two-spin wavefunction changes from an entangled state to a factorized state of two spins which are aligned along the transverse field. We also discuss the possible realization and application of the model to different physical systems.

PACS numbers: 05.30.Rt, 64.70.Tg, 64.60.De, 03.65.Yz

I. INTRODUCTION

The twin (and apparently disjoint) topics of quantum phase transition (QPT) and quantum dissipation (QD) have seen a great upsurge of activity in recent years. The QPT is of significance in many areas of contemporary interest in the Condensed Matter, such as metal-insulator transition, quantum magnetism, ferroelectricity, superconductivity and in the general question of coherence and quantum computation. On the other hand, QD is ubiquitous present because of environmental influences on otherwise unitary evolution of quantum systems. Dissipation, though considered a pest, needs to be understood (and tamed), in order to tackle decoherence effects in quantum many body systems. Our aim in this paper is to analyze the combined presence of these two seemingly disparate phenomena of QPT and QD via simple model systems. Our hope is to elucidate on the irreversible effects near a quantum critical point (QCP) because of dissipative interactions with the environment.

The simplest model of a QPT is an Ising system of coupled spins (which are taken to be polarized in one direction only, say the z-axis), subjected to an additional (‘magnetic’) field $\Gamma$ along the z-axis. The latter couples to the x-component of the spins which, because of its non-commutativity with the z-component, triggers quantum dynamics in the system. The resultant 'Transverse Ising Model (TIM)' is a prototype for analyzing the conflicting presence of order along z-axis induced by the Ising coupling and 'disorder', caused by the tilting of the spins to the x-axis by $\Gamma$. The net result is the occurrence of a QCP at temperature $T = 0$ in the $\Gamma - T$ phase diagram as $\Gamma$ is increased to a critical value $\Gamma_c$, when the system transits from a ferromagnetic to a paramagnetic phase (see Fig.1). We will be interested in analyzing questions such as what is the analogue of ‘phase ordering’ in classical systems (see) when the quantum mechanical system of Fig.1 is subjected to a sudden ‘quenching’ along the $\Gamma$-axis across $\Gamma_c$, maintaining the temperature at $T = 0$. Because a change in the value of $\Gamma$ (from above $\Gamma_c$ to below) will inevitably lead to an irreversible transition from one equilibrium configuration to another, dissipation needs to be dovetailed to the discussion. Again, a straightforward and widely studied model of quantum dissipation, popularized in recent years by Ford et al, and Caldeira and Leggett is the one in which the quantum system at hand is linearly coupled to a bosonic bath. TIM is an effective model which has application from solid state materials of rare earth magnets to a coupled Josephson array, where dissipation arises in a natural way. Our model Hamiltonian can then be written,

$$H = -\frac{1}{2} \sum_{i\neq j} J_{ij} s_i^z s_j^z - \Gamma \sum_i s_i^x + \frac{1}{\sqrt{N}} \sum_{i,k} s_i^z g_k (b_k^\dagger + b_k) + \sum_k h \omega_k b_k^\dagger b_k,$$  \hspace{1cm} (1)

where $N$ is the total number of spins and $g_k$ is a coupling constant. When $g_k = 0$ and the Ising coupling $J_{ij}$ is treated in the mean field approximation (MFA), we obtain the phase diagram, depicted in Fig.1.

Several limiting cases of Eq(1) have received attention in recent literature. If $J_{ij} = 0$ and the term $s_i^x$ in the coupling with the heat bath is replaced by $s_i^z$, Eq(1) yields the celebrated spin-boson model of quantum dissipation. When the Ising interaction is replaced by a Zeeman coupling with an external field $h$ on a single ($N = 1$) spin, the model in Eq(1) describes low-temperature dissipative quantum tunneling in an asymmetric double well. Additionally, if $h = 0$, one has a symmetric double-well, tunneling in which can be impeded, leading to localization, when the coupling with the bosonic bath exceeds a certain critical value. This case is also relevant for a Kondo impurity of spin one-half (described by $s$), in interaction with a conduction electron bath, which can be modeled in terms of bosons as far as the electron-hole excitations near the Fermi surface are concerned. For $\Gamma = 0$, this model can be viewed as an generalization of the Dicke model of ‘superradiance’ where the Ising term includes an additional atom-atom interaction.

One other important application of Eq(1) ensues in the case wherein the range of the Ising interaction is infinite,
on this tunneling behavior switched on, enables us to treat the effect of dissipation.

we shall analyze diverse mean-field and low-temperature current topics in the condensed matter. In the sequel Hamiltonian then underscore the versatility and relevance of the model. A rowed from the literature on polaron-physics tion on the Hamiltonian in Eq(1), that has been bor-

the paper is organized as follows. With this background to the formalisms developed here, we notice that the total Hamiltonian can be diagonalized to a unitary transformation, whence the original coupling constant \( g_k \) (proportional to a variational parameter \( f_k \)) to be elevated to an exponential function, thus facilitating an analysis that works even in the regime of strong coupling to the bath. The transformed Hamiltonian, rewritten in terms of \( f_k \), is then treated in section III in the MFA to evaluate the associated density matrix from which the Helmholtz free-energy and an equation of state for the magnetization can be calculated. In section IV we focus our attention to spin wave like excitations near the absolute zero of temperature. The section V is devoted to a novel aspect of entanglement, important in the contemporary issue of quantum information process, in the context of two coupled Ising spins. Finally in section VI we present a brief summary.

II. EFFECTIVE SPIN-HAMILTONIAN

In this section we derive an effective Hamiltonian of the spin system, starting from Eq[1]. For this, we integrate out the bosonic degrees of freedom of the heat bath. Here we adopt the usual definition of the 'effective partition function' \( Z_{\text{eff}} \) of the spin system,

\[
Z_{\text{eff}} = Tr S_B e^{-\beta (H_s + H_B + H_I)}/ Tr e^{-\beta H_B}.
\]

where \( H_s \) is the Hamiltonian of the 'transverse field Ising model', \( H_B \) describes the non-interacting bosonic degrees of the heat bath and \( H_I \) is the interaction term with the heat bath. To decouple the spin-system from the bosonic heat bath, we subject the original Hamiltonian to a unitary transformation,

\[
\hat{H} = U H U^{-1},
\]

where,

\[
U = e^{\exp \left[ -\sum_i s_i^z \sum_k \frac{f_k}{\sqrt{N}} (b_k - b_k^\dagger) \right]}.
\]

At this stage we treat \( f_k \) as a variational parameter which can be determined from the minimization of the free-energy of the total system. As a special case, if we consider a non-interacting spin system by setting \( J_{ij} = 0 \), we notice that the total Hamiltonian can be diagonalized by the unitary transformation given in Eq.(7) with \( f_k = g_k/\hbar \omega_k \). Motivated by this observation, the above mentioned variational method has been applied to a single-spin \((N = 1)\) 'spin-boson' model, which successfully captures the 'Kondo' like localization transition.

Following the unitary transformation in Eq.(7) on
Eq. (4),

\[ \hat{H} = -\frac{J}{2N} \left[ \frac{N}{4} + S_x^2 + \frac{1}{2}(S_x^2 - S_y^2)\cosh(-2\hat{F}) - 1 \right] - \frac{1}{2}(S_xS_y + S_yS_z)\sinh(-2\hat{F}) - \Gamma S_x \\
+ \frac{S_x}{\sqrt{N}} \sum_k (b_k + b_k^\dagger)(g_k - \hbar\omega_k f_k) \\
- \frac{S_x^2}{N} \sum_k (2f_k g_k - \hbar\omega_k f_k^2) + \sum_k \hbar\omega_k b_k^\dagger b_k \quad (8) \]

where, \( \hat{F} = \sum_k \frac{J}{\sqrt{N}} (b_k - b_k^\dagger) \), and \( S_{x,z} \) are components of the total spin as given in Eq. (4). Now we approximate the total density-matrix as a direct product form \( \rho = \rho_s \otimes \rho_B \), where \( \rho_s \) and \( \rho_B \) denote the density matrices of the spin system and free bosons respectively. After integrating out the bosonic modes of the system, we obtain the effective Hamiltonian \( H_{eff} \) of the spin system:

\[ Tr_B \hat{H} = H_{eff} + \frac{J}{8} + F_B, \quad (9) \]

where \( F_B \) is the free energy of noninteracting bosons and,

\[ H_{eff} = -K_x S_x^2 - K_y S_y^2 - K_z S_z^2 - \Gamma S_x. \quad (10) \]

The Hamiltonian above describes a fully anisotropic Heisenberg-model in the presence of a magnetic field along the x-axis, when it is written using Eq. (3). Effective coupling strengths are given by,

\[ K_x = \frac{1}{N} \sum_k (2f_k g_k - \hbar\omega_k f_k^2), \]

\[ K_y = \frac{J}{4N} \left[ 1 - \exp\left\{ -\frac{2}{N} \sum_k f_k^2 \coth(\hbar\omega_k/2k_BT) \right\} \right], \]

\[ K_z = \frac{J}{4N} \left[ 1 + \exp\left\{ -\frac{2}{N} \sum_k f_k^2 \coth(\hbar\omega_k/2k_BT) \right\} \right]. \quad (11) \]

Going back to the notation of total spins (see Eq. (3)) we see that the terms proportional to \( K_x \) and \( K_y \) generate new physics in the context of molecular magnets. From minimization of the free-energy of the total system with respect to the variational parameter \( f_k \), we obtain,

\[ f_k = \frac{2g_k \langle S_z^2 \rangle/N}{[2\hbar\omega_k \langle S_z^2 \rangle/N + \frac{4\hbar^2}{\hbar^2} (\langle S_z^2 \rangle - \langle S_z^2 \rangle) \coth(\hbar\omega_k/2k_BT)]}, \quad (12) \]

where

\[ \tilde{K} = \exp\left[ -\frac{2}{N} \sum_k f_k^2 \coth(\hbar\omega_k/2k_BT) \right] \quad (13) \]

III. PHASE TRANSITION OF QUANTUM ISING MODEL: \( N \to \infty \)

In this section we consider the case wherein the number of spins \( N \to \infty \), which corresponds to an ‘infinite range’ quantum ising model in the thermodynamic limit. For the corresponding classical model it is known that the MFA is exact in this limit.

A. Mean-Field approximation

Within the MFA the phase diagram of the above model can be solved analytically. Here we assume that the density matrix of the spin system can be written as product of the density matrix of each spins \( \rho_s = \prod_i \rho_i \). The density matrix of each spin can further be expressed as,

\[ \rho_i = \frac{1}{2}(1 + m\sigma_z + \chi\sigma_z), \quad (14) \]

where the order parameters are \( \langle \sigma_i^z \rangle = m/2 \) and \( \langle \sigma_i^z \rangle = \chi/2 \). From the selfconsistency equation for \( f_k \) Eq. (12) we obtain,

\[ f_k = \chi g_k \left[ \frac{Jm^2\tilde{K}}{2N} \coth(\hbar\omega_k/2k_BT) + \chi^2\hbar\omega_k \right]^{-1}. \quad (15) \]

If we assume \( \frac{1}{N} \sum_k g_k^2/\hbar^2\omega_k^2 \to 0 \), in the limit of \( N \to \infty \), we have \( \tilde{K} \to 1 \).

The free energy of the system is given by,

\[ F = \frac{JNm^2}{8} - \Gamma N\chi - \frac{N\chi^2}{4} \sum_k (2f_k g_k - \hbar\omega_k f_k^2) \]

\[ + k_BT N \left[ \frac{1}{2}(1 + \xi) \ln(1 + \xi) + \frac{1}{2}(1 - \xi) \ln(1 - \xi) - \ln 2 \right] + F_B \quad (16) \]

where \( \xi = \sqrt{m^2 + \chi^2} \) and \( F_B \) is the free energy of the bosons. Minimizing the free energy per particle \( F/N \) with respect to the parameters \( m \) and \( \chi \) we obtain:

\[ -\frac{Jm}{4} + \frac{kTm}{2\xi} \ln \frac{1 + \xi}{1 - \xi} = 0 \quad (17) \]

\[ -\Gamma - \frac{\chi^2}{2} + \frac{kT\chi}{2\xi} \ln \frac{1 + \xi}{1 - \xi} = 0 \quad (18) \]

where \( \lambda = \sum_k g_k^2/\hbar\omega_k \). The ferromagnetic phase is defined by \( m \neq 0 \) and the phase boundary can be obtained from

\[ \Gamma/(J/2 - \lambda) = \tanh \left[ \frac{J\Gamma}{4kB_T(J/2 - \lambda)} \right]. \quad (19) \]

A three-dimensional plot of the phase diagram is shown in Fig. 2. As expected of course, for \( \lambda = 0 \), Eq. (19) yields the usual equation of state and the phase diagram depicted earlier in Fig. 1. The latter also bears out the expectation that as the strength of the coupling to the heat bath \( \lambda \) increases, the QCP on the \( \Gamma \)-axis is suppressed.
B. Semiclassical analysis

In this subsection we calculate the effective partition function of the spin-system semi-classically and establish the validity of the variational method and MFA for a large spin system \((N \to \infty)\). We can decompose a direct product of \(N\) spin \(1/2\) systems into a direct sum of total spin \(S\), where the magnitude of the spin varies from \(N/2\) to 0 or 1/2 (depending on even or odd \(N\)). In terms of the total spin operator \(S\), the partition function can be written as:

\[
Z = \sum_{S=0,1/2}^N \text{Tr}_S e^{-\beta H} = \sum_{S=0,1/2}^{N/2} P(S)\text{Tr}_S T r_B
\]

\[
e^{-\beta \left[ -\frac{S^2}{2} - J S_z + \lambda \sum_b (b_q b_q^\dagger + b_q^\dagger b_q) \right]}
\]

where \(P(S)\) is the number of ways the total spin \(S\) can be formed and is given by

\[
P(S) = \frac{(2S+1)N!}{(N/2-S)! (N/2+S+1)!}.
\]

For large \(S\) we can calculate the partition function of spin systems classically and replace the trace by the integral,

\[
\text{Tr}_S \to \frac{2S+1}{4\pi} \int \sin \theta d\theta d\phi
\]

where the components of the total spin \(\vec{S}\) are given by \((S \sin \theta \cos \phi, S \sin \theta \sin \phi, S \cos \theta)\). Using the coherent states for bosons we integrate out the bosonic degrees of freedom and obtain the effective partition function of the spin system as,

\[
Z_{\text{eff}} = \sum_{S=0,1/2}^{N/2} P(S) \left( \frac{2S+1}{4\pi} \right) \int \sin \theta d\theta d\phi e^{-\beta \left[ -\frac{S^2}{2} \cos^2 \theta - \Gamma S \sin \theta \cos \phi - \frac{\lambda}{2} S^2 \sin^2 \theta \cos^2 \phi \right]}
\]

The minimum value of the energy is obtained for \(\phi = 0\). For large \(S\) we introduce two variables \(r = \frac{S}{\sqrt{2}N}\) and \(x = \cos \theta\). In terms of these two variables we can write the effective partition function as:

\[
Z_{\text{eff}} = N_r \int_0^1 dr dx e^{-\beta N f(r,x)},
\]

where \(f(r, x)\) is the free energy per particle and \(N_r\) is the constant. Here we have used Stirling’s approximation,

\[
\ln P(S) \approx -\frac{N}{2} \left[ (1+r) \ln (1+r) + (1-r) \ln (1-r) - 2 \ln 2 \right],
\]

which is the entropy in MFA. The free energy per particle is given by:

\[
f(r, x) = -\frac{J}{8} r^2 x^2 - \frac{\Gamma r}{2} \sqrt{1-x^2} - \frac{\lambda}{4} r^2 (1-x^2) + \frac{k_B T}{2} \left[ (1+r) \ln (1+r) + (1-r) \ln (1-r) \right].
\]

Within a saddle point approximation we minimize \(f(r, x)\) and obtain the following set of equations:

\[
\frac{\partial f(r, x)}{\partial x} = 0,
\]

\[
r \sqrt{1-x^2} = \frac{\Gamma}{J/2-\lambda},
\]

\[
\frac{\partial f(r, x)}{\partial r} = 0,
\]

\[
-\frac{J}{2} r^2 - \Gamma \sqrt{1-x^2} - \lambda r (1-x^2) + \frac{k_B T}{2} \ln \left[ \frac{1+r}{1-r} \right] = 0.
\]

Defining the magnetisation along z-axis as \(m/2 = r x/2\), and using Eq(28) at the saddle point of the free energy function \(f(r, x)\) we obtain,

\[
\left[ m^2 + \frac{\Gamma^2}{(J/2-\lambda)^2} \right]^{1/2} = \tanh \left[ \frac{J}{4k_B T} \sqrt{m^2 + \frac{\Gamma^2}{(J/2-\lambda)^2}} \right]
\]

which checks with the earlier Eq(19).

IV. COLLECTIVE EXCITATIONS NEAR \(T = 0\)

In this section we analyze the low temperature spin wave like excitations by calculating the low-lying energies of the spin system coupled to a heat bath. In the
previous section we used the total spin representation $S^a = \sum_i s^a_i$, where $a = x, y, z$, to evaluate the partition function semiclassically. Quantum effects become relevant near zero temperature and consequently, the total spin takes a large value $S \sim N/2$, in order to minimize the free energy. In the limit of large $S$, we can derive an effective Hamiltonian describing the quantum fluctuations at low temperatures. In the Holstein-Primakoff representation, the components of total spin $S$ are given by,

\[
S_- = S - b^\dagger b,
\]

\[
S_z = b^\dagger \sqrt{2S-b^\dagger b},
\]

\[
S_+ = \sqrt{2S-b^\dagger b}b.
\]

where $b(b^\dagger)$ satisfy bosonic commutation rules. We assume that the total spin vector $S$ makes an angle $\theta$ with the $z$-axis and its projection on $x$-$y$ plane makes an angle $\phi$ with the $x$-axis. Now we can write:

\[
S^{xz} = (S - b^\dagger b) \cos \theta - \sqrt{S^2/2}(b^\dagger + b) \sin \theta
\]

\[
S^{xy} = \sqrt{S^2/2} \{[b^\dagger b] \cos \theta \cos \phi + i(b - b^\dagger) \sin \phi \}
\] + $(S - b^\dagger b) \sin \theta \cos \phi$ \tag{34}

Substituting Eq\(\ref{33}\) in the Hamiltonian Eq.\(\ref{41}\), we obtain,

\[
H = \frac{J}{8} - \frac{J}{2N}[(S - b^\dagger b)^2 \cos^2 \theta + \frac{S}{2}(b + b^\dagger)^2 \sin^2 \theta
\]

\[-2 \sqrt{\frac{S}{2}}(S - b^\dagger b)(b^\dagger + b) \cos \theta \sin \theta]

\[-\sqrt{\frac{S}{2}} \{[b^\dagger b] \cos \theta \cos \phi + i(b - b^\dagger) \sin \phi \}
\] + $(S - b^\dagger b) \sin \theta \cos \phi \times [\Gamma - \frac{1}{N} \sum_k g_k(b^\dagger_k + b_k)]
\]

\[+ \sum_k \hbar \omega_k b^\dagger_k b_k + O(1/S).
\]

After performing a transformation $c^\dagger = \frac{S}{\sqrt{\hbar \omega_k}}g_k \sin \theta \cos \phi$, we obtain from the above Hamiltonian the classical energy of large spin $S$ (part of the Hamiltonian which is proportional to $N$ for large $S$),

\[
H_0 = -\frac{JS^2}{2N} \cos^2 \theta - \Gamma S \sin \theta \cos \phi - \frac{\lambda}{N} S^2 \sin^2 \theta \cos^2 \phi
\]

\[
\text{Again, a minimum of } H_0 \text{ can be found for } \phi = 0 \text{ and } \cos \theta = 0 \text{ for the paramagnetic state. In the ferromagnetic state of spin } S, \text{ the minimum energy can be achieved for } \phi = 0 \text{ and,}
\]

\[
\frac{2N}{S}(\frac{J}{2} - \lambda) \sin \theta = \Gamma.
\]

The above equation determing the minimum classical energy is equivalent to the equation derived from the saddle point approximation to the free energy described in sec.IIB (see Eq\(\ref{28}\)). With this choice of $\theta$ and $\phi$, we notice that the terms linear in fluctuation operators ($b$ and $c_k$) in the Hamiltonian (Eq\(\ref{35}\)) vanish.

Finally the Hamiltonian describing the fluctuations can be exactly mapped on to the Caldeira-Leggett model\(\ref{22}\) describing an oscillator coupled to a heat bath,

\[
H_2 = \epsilon(b^\dagger b + \Delta(b^\dagger b)^2
\]

\[+ \epsilon(b + b^\dagger) \sum_k g_k(c^\dagger_k + c_k) + \hbar \omega_k c^\dagger_k c_k \] \tag{38}

where different parameters in the above Hamiltonian can be written in terms of the spin $S$ and its orientation $\theta$, \(\epsilon = \frac{JS}{N} \cos^2 \theta + \frac{2\lambda S}{N} \sin^2 \theta + \Gamma \sin \theta, \tag{39}\)

\[\Delta = -\frac{JS}{4N} \sin^2 \theta, \tag{40}\]

\[\tilde{g}_k = \frac{g_k}{\sqrt{S/2}} \cos \theta. \tag{41}\]

Following the work of Ambegaokar and Hakim\(\ref{22}\), we can diagonalize the above Hamiltonian quadratic in bosonic operators by means of the canonical transformations,

\[
c_\alpha = \sum_\beta A_{\alpha \beta} \tilde{c}_\beta + B_{\alpha \beta} \tilde{c}^\dagger_\beta,
\]

\[
c_\alpha^\dagger = \sum_\beta B_{\alpha \beta} \tilde{c}_\beta + A_{\alpha \beta} \tilde{c}^\dagger_\beta,
\]

\[
\text{where we denote the original bosonic operators as } \{c_\alpha\} = \{b, c_k\}. \text{ The Hamiltonian can be written in diagonal form } H_2 = \sum_\beta E_\beta \tilde{c}^\dagger_\beta \tilde{c}_\beta \text{ in terms of a new set of bosonic operators. The set of energies } E_\beta \text{ describes the low-lying excitation energies of the many-body system. Equation of motion of the operators can be obtained from the Hamiltonian, yielding}
\]

\[
\frac{db}{dt} = \epsilon b + 2\Delta(b^\dagger b) + \sum_k g_k(c^\dagger_k + c_k), \tag{43}\]

\[
\frac{d\tilde{c}_k}{dt} = \hbar \omega_k c_k + g_k(b^\dagger b), \tag{44}\]

\[
\frac{d\tilde{c}^\dagger_\beta}{dt} = -E_\beta \tilde{c}^\dagger_\beta. \tag{45}\]

Substituting Eq\(\ref{12}\) in Eq\(\ref{13}\), the matrix elements of the canonical transformation are obtained as,

\[
E_\beta A_{\alpha \beta} = \epsilon A_{\alpha \beta} + 2\Delta(A_{\alpha \beta} + B_{\alpha \beta}) + \sum_k g_k(A_{\alpha \beta} + B_{\alpha \beta}),
\]

\[-E_\beta B_{\alpha \beta} = \epsilon B_{\alpha \beta} + 2\Delta(A_{\alpha \beta} + B_{\alpha \beta}) + \sum_k g_k(A_{\alpha \beta} + B_{\alpha \beta}),
\]

\[
E_\beta A_{k \beta} = \hbar \omega_k A_{k \beta} + g_k(A_{k \beta} + B_{k \beta}),
\]

\[-E_\beta B_{k \beta} = \hbar \omega_k B_{k \beta} + g_k(A_{k \beta} + B_{k \beta}). \tag{46}\]
Finally, we obtain the equation determining the excitation energies \( E_\beta \),

\[
E_\beta^2 - c^2 - 4c\Delta = 4c \sum_k \frac{\tilde{g}_k^2 \hbar \omega_k}{E_\beta^2 - \hbar^2 \omega_k^2}.
\]

At \( T = 0 \) the total spin is \( S = N/2 \), and we can define two phases according to the orientation of the large spin. When \( \Gamma > J/2 - \lambda \), all spins are aligned along the \( x \)-axis, hence \( \cos \theta = 0 \) solution minimizes the classical energy of the system. The QCP is given by,

\[
\Gamma_c = J/2 - \lambda.
\]

In the paramagnetic phase the fluctuation operators of spin \((b^\dagger, b)\) decouple from the modes of the bath in the leading order, since

\[
\tilde{g}_k = g_k \cos \theta/2 = 0
\]

and they are coupled in the higher order terms of the Hamiltonian which are suppressed by a factor of \( 1/\sqrt{N} \). Quantum fluctuations of the paramagnet can then be described by an effective harmonic oscillator

\[
H_2 = \hbar \omega_p \tilde{c}^\dagger \tilde{c}_0,
\]

with the excitation energy:

\[
\hbar \omega_p = \sqrt{(\lambda + \Gamma)(\lambda + \Gamma - J/2)}.
\]

Here we note that although the fluctuations of spins are decoupled from the bath modes, the frequency \( \omega_p \) depends on dissipation. At the QCP, the excitation energy vanishes as \((\Gamma - \Gamma_c)^{1/2}\), which is in accordance with mean-field behavior.

V. GENERALIZED SPIN-BOSON MODEL FOR TWO INTERACTING SPINS

In this section we address the issue of quantum information and Schrödinger cat like state by considering our model-Hamiltonian Eq. (11) with \( N = 2 \), which represents two interacting spins attached to a heat bath. This generalizes the usual spin-boson model by including spin-spin interaction terms.

Following the procedure of unitary transformation and then tracing out the bosonic degrees of freedom (as mentioned in section II), we obtain the following effective Hamiltonian for the total spin,

\[
H = -K_z S_z^2 - K_y S_y^2 - K_x S_x^2 - \Gamma S_z,
\]

which follows directly from Eq. (10). Since at zero temperature, the triplet state plays an important role, we obtain the following eigenvalues and eigenfunctions for the triplet state:

\[
\epsilon_\pm = -(K_z + K_y)/2 - K_x \pm \sqrt{(K_z - K_y)^2 / 4 + \Gamma^2}
\]

FIG. 3: Variation of the parameter \( \cos \theta \) of the wavefunction \( |-\rangle \) (which is also the ‘concurrence’ of the wavefunction) with the change of the coupling \( \alpha \) with the heat bath, for \( \Gamma/J = 10^{-2} \), \( \Omega_c/J = 10^2 \) (solid line), \( \Gamma/J = 10^{-4} \), \( \Omega_c/J = 10^2 \) (dashed line) and \( \Gamma/J = 10^{-4} \), \( \Omega_c/J = 1 \) (dot-dashed line).

\[
|\pm \rangle = \frac{\cos \theta_\pm}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) + \frac{\sin \theta_\pm}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),
\]

with \( \tan \theta_\pm = -(\epsilon_\pm + K_z + K_x)/\Gamma \), and,

\[
\epsilon_2 = -K_z - K_y,
\]

\[
|2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle).
\]

The singlet state has zero energy. For a system without dissipation( when \( K_y = 0 \), \( K_x = 0 \) the ground state wave function is \( |-\rangle \), with

\[
\cos(2\theta) = \frac{K_z/2}{\sqrt{K_z^2/4 + \Gamma^2}}.
\]

Unlike the Ising system in the thermodynamic limit (for \( N \rightarrow \infty \)), this state does not have a net magnetization and \( \langle S_z \rangle = 0 \). The magnetization along the \( x \)-axis increases smoothly with increasing the magnetic field \( \Gamma \) and finally for very large \( \Gamma \), the state \( |-\rangle \) is factorized in two spin states directed along \( x \) axis.

Now switching on the coupling with the heat bath, we obtain,

\[
\cos(2\theta) = \frac{JK/8}{\sqrt{(JK/8)^2 + \Gamma^2}}.
\]

For the state \( |-\rangle, \langle S_z^2 \rangle = 1, \langle S_z^2 \rangle = \cos^2 \theta, \langle S_y^2 \rangle = \sin^2 \theta \), and hence, \( f_k = g_k/\hbar \omega_k + JK \cos(2\theta)/4 \). Further, for an Ohmic heat bath, the spectral density is given by,

\[
\sum_k g_k^2 \delta(\omega - \omega_k) = \alpha \omega \theta(\Omega - \omega),
\]

where \( \alpha \) is the coupling strength and \( \Omega \) is the cut-off frequency of the bath. The renormalization factor \( \tilde{K} \) can
be obtained from the self-consistent equation,

\[ \tilde{K} = \left( \frac{J\tilde{K}\cos(2\theta)/4}{\Omega + J\tilde{K}\cos(2\theta)/4} \right)^{\alpha} \exp \left[ \frac{\alpha\Omega}{\Omega + J\tilde{K}\cos(2\theta)/4} \right]. \]  

(60)

Like in the spin-boson model, the parameter \( \tilde{K} \) shows a crossover behavior below \( \alpha \approx 1 \). Two spins become parallel to x-direction and the magnetization along x-axis sharply becomes 1 when \( \alpha \) crosses a critical value \( \alpha_c \) for finite field \( \Gamma \).

Also we focus on another aspect of this wavefunction. For \( \theta = 0 \), the wavefunction \( |\rangle \) is maximally entangled as in a ‘cat-state’. In the other limit \( \theta = \pi/4 \) (in the limit of large field \( \Gamma \)), the ground state \( |\rangle \) can be factorized. The amount of entanglement between the two spins can be quantitatively calculated from the ‘concurrence’ of the wave function of two spins, which is \( \cos(2\theta) \).

It is interesting to notice that the concurrence shows a crossover at \( T = 0 \) by tuning the dissipative coupling strength \( \alpha \) as shown in Fig.3.

VI. SUMMARY AND OUTLOOK

In this paper we analyze the effect of dissipation on a model quantum system which undergoes a QPT. In various limiting cases this model represents different physical systems, starting from a TIM in the thermodynamic limit (when the number of spins \( N \to \infty \)) to a generalized spin-boson model of two interacting spins (for \( N = 2 \)). Also this model can be viewed as an interacting version of a multi-mode Dicke model describing the atom-photon interaction. In the context of a large spin our model has application to the area of nanomagnets. We have developed a self-consistent variational method to study the system for any number of spins, and the spin system can be described by an effective anisotropic Heisenberg model after integrating out the bosonic modes. The effective Heisenberg model shows two interesting effects, the renormalization of the original coupling between the z-component of spins similar to the spin-boson model and generation of extra terms which couple the x and y-components and of spins. In the thermodynamic limit \( (N \to \infty) \) the model represents a TIM in presence of dissipation and the phase diagram has been obtained within a variational-MFA which is in accordance with the saddle point approximation. It is interesting to note that for large \( N \), a new coupling between the x component of spins plays an important role and destroys the ferromagnetic phase for a smaller value of transverse field. The QPT of the model at zero temperature has been studied using the Holstein-Primakoff transformation. The excitation energy vanishes as \( (\Gamma - \Gamma_c)^{1/2} \) at the quantum critical point.

In the other limit, for \( N = 2 \), the model generalizes the usual spin-boson model by including a spin-spin interaction along the z axis. Unlike a TIM, this two-spin system shows zero magnetization along the z-axis but displays a sharp change in magnetization along the x-axis. For small \( N \), strong renormalization of spin-spin coupling of the original Hamiltonian becomes important for the crossover phenomenon similar to the original spin-boson model. From the point of view of quantum information theory the ground state of two-spins shows a transition from a maximally entangled ‘cat-state’ to a factorizable state where both spins are aligned along the x-axis.

In conclusion we have studied the effect of dissipation on a simple model of an interacting spin system which undergoes QPT. Further extension of the model for analyzing the effect of dissipation in dynamical quenching across a quantum critical point that is also relevant for the dynamics and thermodynamics of nanomagnets, as alluded to in sec. I, is left for future work.

1 S. Sachdev, Quantum Phase Transitions, Cambridge University Press, Cambridge (1999).
2 S. L. Sondhi, et al., Rev. Mod. Phys. 69, 315 (1997).
3 U. Weiss, Quantum Dissipative Systems, World Scientific.
4 S. Dattagupta and S. Puri, Dissipative phenomena in condensed matter physics, Springer-Verlag, Heidelberg (2004).
5 G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. 6, 504 (1965); G. W. Ford, J. T. Lewis, and R. F. O’Connell, Phys. Rev. A 37, 4419 (1988).
6 A. O. Caldeira, and A. J. Leggett, Physica A 121, 587 (1983).
7 S. Dattagupta, A paradigm called magnetism, World Scientific, Singapore (2008).
8 A. J. Leggett, et al, Rev. Mod. Phys. 59, 1 (1987).
9 H. Grabert, and U. Weiss, Phys. Rev. Lett. 54, 1605 (1985).
10 U. Weiss and M. Wollensak, Phys. Rev. Lett. 62, 1663 (1989).
11 S. Dattagupta, H. Grabert and R. Jung, J. Phys. Condens. Matter 1, 1405 (1989).
12 S. Chakravarty, Phys. Rev. Lett. 49, 681 (1982).
13 A. J. Bray, and M. A. Moore, Phys. Rev. Lett. 49, 1546 (1982).
14 L. D. Chang, and S. Chakravarty, Phys. Rev. B 31, 154 (1985).
15 R. Dicke, Phys. Rev. 93, 99 (1954).
16 L. Thomas et al, Nature 383, 145 (1996); E. M. Chudnovsky, and D. A. Garanin, Phys. Rev. Lett. 89, 157201 (2002); S. Urazhdin et al, Phys. Rev. Lett. 91, 146803 (2003).
17 T. Holstein, Ann. Phys. (N.Y.) 8, 325 (1959); 8, 343 (1959).
18 R. Silbey and R. Harris, J. Chem. Phys. 80, 2615 (1984).
19 J. L. Garcia-Palacios and S. Dattagupta, Phys. Rev. Lett. 95, 190401 (2005).
20 A. Chin, and M. Turlakov, Phys. Rev. B 73, 075311 (2006).
21 K. Hepp, and E. H. Lieb, Annals of Physics, 76, 360 (1973).
22 V. Hakim and V. Ambegaokar, Phys. Rev. A, 32, 423
Schrödinger, E. Die Gegenwärtige Situation in der Quantenmechanik. Naturwissenschaften 23, 807 (1935).

W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

S. Sinha, S. Dattagupta and A. Ghosh, in preparation.