NOTE ON TAUPOLOGICAL CLASSES
OF MODULI OF K3 SURFACES

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Abstract. In this note we prove some cycle class relations on moduli of K3 surfaces.

1. Introduction

This note deals with a few properties of tautological classes on moduli spaces of K3 surfaces. Let $M_{2d}$ denote a moduli stack of K3 surfaces over an algebraically closed field with a polarization of degree $2d$ prime to the characteristic of the field. The Chern classes of the relative cotangent bundle $\Omega^1_{X/M}$ of the universal K3 surface $X$ define classes $t_1$ and $t_2$ in the Chow groups $\text{CH}_i(X_{2d})$ of the universal K3 surface over $M_{2d}$. The class $t_1$ is the pull back from $M_{2d}$ of the first Chern class $c_1(V)$ of the Hodge line bundle $V = \pi_*(\Omega^2_{X/M})$. We use Grothendieck-Riemann-Roch to determine the push forwards of the powers of $t_2$. These are powers of $v$. We then prove that $v^{18} = 0$ in the Chow group with rational coefficients of $M_{2d}$. We show that this implies that a complete subvariety of $M_{2d}$ has dimension at most 17 and that this bound is sharp. These results are in line with those for moduli of abelian varieties. There the top Chern class $\lambda_g$ of the Hodge bundle vanishes in the Chow group with rational coefficients. The idea is that if the boundary of the Baily-Borel compactification has co-dimension $r$ then some tautological class of co-dimension $r$ vanishes. Our result means that $v^{18}$ is a torsion class. It would be very interesting to determine the order of this class as well as explicit representations of this class as a cycle on the boundary, cf., [2, 3].

2. The Moduli Space $M_{2d}$

Let $k$ be an algebraically closed field. We consider the moduli space $M_{2d}$ of polarized K3 surfaces over $k$ with a primitive polarization of degree $2d$. This is a 19-dimensional algebraic space. Over the complex numbers we can describe it as an orbifold quotient $\Gamma_{2d}\backslash \Omega_{2d}$, where $\Omega_{2d}$ is a bounded symmetric domain and $\Gamma_{2d}$ an arithmetic subgroup of $SO(3, 19)$ obtained as follows. Consider the lattice $U^3 \oplus E_8^2$, where $U$ denotes a hyperbolic plane and $E_8$ the usual rank 8 lattice. Let $h$ be an element of this lattice with $\langle h, h \rangle = 2d$. Then $L_{2d} = h^\perp \cong U \perp E_8^2 \perp Zu$ with $\langle u, u \rangle = -2d$ is a lattice of signature $(2, 19)$ and we put

$$\Omega_{2d} = \{[\omega] \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) : \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle > 0 \}.$$ 

The group $\Gamma_{2d}$ is the automorphism group of $L_{2d}$. It acts on $\Omega_{2d}$ and the quotient (an orbifold) is the analytic space of $M_{2d}$. It is well-known by Baily-Borel that
the sections of a sufficiently high power of $V$ give an embedding of $\Gamma_{2d}\setminus \Omega_{2d}$ as a quasi-projective variety.

3. GRR APPLIED TO THE SHEAF $\Omega^1_{X/M}$

In order to determine the push forward $\pi_*(t^n_2)$ under $\pi : X_{2d} \to M := M_{2d}$ we apply Grothendieck-Riemann-Roch to the structure sheaf of the universal (polarized) K3-surface $\pi : X \to M$. We work in the Chow ring with rational coefficients.

We have

$$\text{ch}(\pi_*\mathcal{O}_X) = \pi_*(\text{ch}(\mathcal{O}_X) \text{Td}^\vee(\Omega^1_{X/M})) = \pi_*(\text{Td}^\vee(\Omega^1_{X/M})).$$

As to the left-hand-side we have $\pi_\ast \mathcal{O}_X = 1 + V^\vee$, where $V = R^0\pi_\ast \mathcal{O}^2_{X/M}$ is the line bundle with fibre $H^0(X, \Omega^2_X)$ over $[X]$. We write $v$ for the first Chern class of this bundle on $M$. So the left hand side is $1 + e^{-v}$. For the right hand side, remark that $\Omega^1_{X/M}$ has as determinant a line bundle that is trivial on each K3-surface that is a fibre of $\pi$. Therefore, this line bundle is a pull back from $M$ and we can identify it with $\pi^\ast(V)$. If we denote by $t_i = c_i(\Omega^1_{X/M})$ the Chern classes of $\Omega^1_{X/M}$, the right hand side has the following form

$$\pi_*(1 - t_1/2 + (t_1^2 + t_2)/12 - t_1 t_2/24 + ...).$$

Comparing the terms of degree 0 gives $1+1 = 24/12$ since $c_1^2(X) = 0$ and $c_2(X) = 24$ for a K3 surface. For the terms of degree 1 we find: $-v = \pi_*(t_1 t_2)/24 = -v \cdot (\pi_*(t_2)/24)$ and this is in agreement. Degree 2 terms yield $\pi_*(t_2^2) = 88 t_1^2$. This checks with the next term:

$$v^3/6 = \frac{1}{1440} \pi_*(3t_2^2 t_1 - t_2 t_1^3).$$

Continuing this way we can determine $\pi_*(t^n_2)$ for all $j \geq 1$. More precisely, put $B(x) = x/(1 - e^{-x})$ and write $\gamma_1, \gamma_2$ for the Chern roots of $\Omega^1_{X/M}$. Then

$$\text{Td}^\vee(\Omega^1_{X/M}) = B(\gamma_1) B(\gamma_2) = \sum_{n,j : 0 \leq j \leq n} c(n,j)(\gamma_1 + \gamma_2)^{n-j}(\gamma_1 \gamma_2)^j$$

with $t_1 = \gamma_1 + \gamma_2$ and $t_2 = \gamma_1 \gamma_2$ and the Riemann-Roch identity says that if $\pi_*(t_2^{n+1}) = a_n v^{2n}$ the $a_n$ satisfy the relation:

$$\sum_{j \geq 0} a_j c_{n,j} = \begin{cases} 1/(n-2)! & \text{for } n \equiv 0(\mod 2), \, n \geq 2, \\ 2 & \text{for } n = 2. \end{cases}$$

Denoting $\pi_*(t_2^{n+1}) = a_n v^{2n}$ we find the following values for $a_n$.

| $n$   | $\pi_*(t_2^{n+1})/v^{2n}$   |
|-------|-----------------------------|
| 0     | 24                          |
| 1     | 88                          |
| 2     | 184                         |
| 3     | 352                         |
| 4     | 736                         |
| 5     | 1295488/691                 |
| 6     | 4292224/691                 |
| 7     | 68418650624/2499347         |
| 8     | 171412311922527744/109638854849 |
| 9     | 22654813560476770158592/19144150084038739 |
Proposition 3.1. Write $\pi_*(t^{n+1}) = a_n v^n$ for $n \geq 0$. The generating function

$$A(t) = \sum_{n=0}^{\infty} a_n t^n = 24 + 88 t + 184 t^2 + \ldots$$

is characterised uniquely by the property that for every $n > 0$ the coefficient of $t^{2n-1}$ in

$$\frac{2 - t}{1 - t} A \left( \frac{-t^2}{1 - t} \right)$$

is equal to $4n/B_{2n}$. Here $B_m$ denotes the $m$-th Bernoulli number.

Although the numbers $a_n$ are defined for all $n \geq 0$ they have a geometric interpretation for $n \leq 9$ only, apparently.

We now apply Grothendieck-Riemann-Roch to the sheaf $\Omega^1_{X/\mathcal{M}}$, or equivalently, to its dual $\Theta_{X/\mathcal{M}}$. It says

$$\text{ch}(\pi_*(\Theta_{X/\mathcal{M}})) = \pi_*\left(\text{ch}(\Theta_{X/\mathcal{M}}) \text{Td}^\vee(\Omega^1_{X/\mathcal{M}})\right).$$

Note that in the left hand side $\pi_*(\Theta_{X/\mathcal{M}}) = R^1\pi_*\Theta_{X/\mathcal{M}}$ since a K3 surface has no non-zero vector fields, $\emptyset$. Since the push forward of powers of $v$ and $t_1 = \pi^*(v)$ we see that $\text{ch}(\pi_*(\Theta_{X/\mathcal{M}}))$ is a polynomial in $v$. This can be determined by looking at cohomology once we show that the tautological ring of $\mathcal{M}_{2d}$ is $\mathbb{Q}[v]/v^{18}$.

Note that the fibre of $R^1\pi_*\Theta_{X/\mathcal{M}}$ is $H^1(X, \Theta_X)$, the space of infinitesimal deformations of $X$. The tangent space to $\mathcal{M}$ at $[X]$ can be identified with the orthogonal complement of $h$, the hyperplane class in $H^1(X, \Omega^1_X) = H^1(X, \Theta_X)$. On the other hand we know that by Hodge theory the following description for this tangent space. Let

$$0 \subset F^2 \subset F^1 = (F^2) \perp \subset H^2_{\text{DR}}$$

be the Hodge filtration on $H^2_{\text{DR}}(X)$ and let $h$ be the hyperplane class which gives a section of $H^2_{\text{DR}} \otimes O_{\mathcal{M}}$. Then the tangent space to $\mathcal{M}$ can be identified with $\text{Hom}(F^2, (F^1/h) \perp / F^2)$. Using the cup product we can identify $(F^2)^\vee$ with $H^2_{\text{DR}}/F^1$, i.e., in the Grothendieck group we have $[H^2_{\text{DR}}] = V^\vee + V^\perp$, where we identify $F^2$ with $V$. We now restrict to the orthogonal subbundle $h^\perp$ of the hyperplane class $h$ whose class in the Grothendieck group is $[H^2_{\text{DR}}] - 1$. Therefore we find $[\Theta_{\mathcal{M}}] = [H^2_{\text{DR}} - 1 - V - V^{-1}] \otimes V^{-1}$.

Proposition 3.2. In the Grothendieck group of $\mathcal{M}$ we have the relation $[\Theta_{\mathcal{M}}] = [H^2_{\text{DR}} - 1] \otimes V^{-1} - 1 - V^{-2}$.

In view of the Gauss-Manin connection on $H^2_{\text{DR}}$ we see that the Chern classes vanish in cohomology and that the total Chern class of the bundle $F^1$ on $\mathcal{M}$ is $1/(1 - v)$ and $\text{ch}(\Theta_{\mathcal{M}}) = -1 + 21e^{-v} - e^{-2v}$ and in particular we find $c_1^1(\Theta_{X/\mathcal{M}}) = -19v$. We already saw that the total Chern class of $R^1\pi_*\Omega^1_{X/\mathcal{M}}$ is $1/(1 - v^2)$ and so its first Chern class vanishes. This checks with global duality $(R^1\pi_*\Omega^1_{X/\mathcal{M}})^\vee \cong (R^1\pi_*\Omega^1_{X/\mathcal{M}}) \otimes V$. 
4. Vanishing of tautological classes in characteristic zero

Let \( II_{3,19} \) be the unique even unimodular lattice of signature \((3, 19)\) and let \( S \) be some Lorentzian sublattice of \( II_{3,19} \), say of signature \((1, m)\). Recall that an \( S \)-K3 surface \( X \) is a K3 surface with a fixed primitive embedding of \( S \) into the Picard group such that the image of \( S \) contains a semi-ample class, i.e., a class \( D \) such that \( D^2 > 0 \) and \( D \cdot C \geq 0 \) for all curves \( C \) on the K3 surface \( X \), cf., [1]. The period space \( Y \) of marked \( S \)-K3 surfaces is an orbifold which is a quotient of a hermitean symmetric domain of dimension \( 19 - m \) by an arithmetic subgroup of \( \text{Aut}(S^\perp) \).

**Theorem 4.1.** For \( m \leq 16 \) the cycle class \( v^{18-m} \) vanishes in the Chow group \( CH^\perp_{Q}(Y) \) with rational coefficients.

**Proof.** By imposing a level structure we can replace our period space by a finite cover and assume that we are working with a fine moduli space.

The proof is by descending induction on \( m \). For \( m = 16 \) the period domain can be identified with the Siegel upper half space \( \mathcal{H}_2 \) and the orbifold \( Y \) can be viewed as a moduli space of abelian surfaces. It thus carries a natural vector bundle, the Hodge bundle \( \pi^* (\Omega^1_X/Y) \) with Chern classes \( \lambda_1 \) and \( \lambda_2 \). One shows that \( \lambda_1 = v \) by comparing the factors of automorphy or by noticing that \( H^0(X, \Omega^2_X) \cong \wedge^2 H^1(\Omega^1_X) \) for an abelian surface. Furthermore, it is known that \( \lambda_2 \) vanishes by [4], Prop. 2.2. We conclude that \( v \) vanishes.

The induction step is now provided by Theorem 1.2 of [1]. There exists a modular form \( \Phi \) of weight \( k \geq 12 \) whose zero-divisor is of the form \( \sum m_i W_i \) with \( m_i \in \mathbb{Z}_{>0} \) with orbifolds \( W_i \) that are images in \( Y \) of quotients \( \Gamma_{L_i} \backslash \Omega_{L_i} \) under finite maps. Here \( \Omega_{L_i} \) is a hermitean symmetric domain of dimension one less than the original domain \( \Omega \) and the quotient parametrizes a family of \( S' \)-K3-surfaces with \( S' \supset S \) of signature \((1, m + 1)\). By induction we know that on each of the orbifolds \( \Gamma_{L_i} \backslash \Omega_{L_i} \) the class \( v^{17-m} \) vanishes. The zero-divisor of \( \Phi \) represents the class \( k v \). We thus find that a non-zero multiple of \( v^{18-m} = v \cdot v^{17-m} \) vanishes. \( \square \)

In characteristic 0 we can use the existence of the Satake compactification whose boundary is 1-dimensional to conclude that intersecting twice with a sufficiently general hyperplane yields a complete 17-dimensional subvariety of \( \mathcal{M} \). Since by Baily-Borel the class \( v \) is ample this shows that \( v^{17} \neq 0 \).

**Corollary 4.2.** The tautological ring of \( \mathcal{M}_{2d} \) is \( \mathbb{Q}[v]/(v^{18}) \).

**Corollary 4.3.** The maximal dimension of a complete subvariety of \( \mathcal{M}_{2d} \) is 17.

In positive characteristic the locus of K3-surfaces with height \( \geq 3 \) defines a complete subvariety of dimension 17, cf. [5].

If \( \mathcal{M}^* \) is the Baily-Borel compactification of \( \mathcal{M} \) then the ‘boundary’ is a 1-dimensional cycle. In the Chow group \( CH^\perp_{Q}(\mathcal{M}^*) \) the class \( v^{18} \) is represented by a 1-cycle with support on the boundary.

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