Clifford-Wolf homogeneous left invariant \((\alpha, \beta)\)-metrics on compact semi-simple Lie groups

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Abstract

Let \((M, F)\) be a connected Finsler space. An isometry of \((M, F)\) is called a Clifford-Wolf translation (or simply CW-translation) if it moves all points the same distance. The compact Finsler space \((M, F)\) is called restrictively Clifford-Wolf homogeneous (restrictively CW-homogeneous) if for any two sufficiently close points \(x_1, x_2 \in M\), there exists a CW-translation \(\sigma\) such that \(\sigma(x_1) = x_2\). In this paper, we define the good normalized datum for a homogeneous non-Riemannian \((\alpha, \beta)\)-space, and use it to study the restrictive CW-homogeneity of left invariant \((\alpha, \beta)\)-metrics on a compact connected semisimple Lie group. We prove that a left invariant restrictively CW-homogeneous \((\alpha, \beta)\)-metric on a compact semisimple Lie group must be of the Randers type. This gives a complete classification of left invariant \((\alpha, \beta)\)-metrics on compact semi-simple Lie groups which are restrictively Clifford-Wolf homogeneous.

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1 Introduction

The goal of this paper is to study left invariant restrictively Clifford-Wolf homogeneous (restrictively CW-homogeneous) \((\alpha, \beta)\)-metrics on compact connected semi-simple Lie groups. Recall that an isometry \(\sigma\) of a metric space \((X, d)\) is called a Clifford-Wolf translation (CW-translation) if the function \(d(x, \sigma(x)), x \in X\), is a constant. A metric space is called CW-homogeneous if given any two points \(x_1, x_2 \in M\), there is a CW-translation \(\sigma\) such that \(\sigma(x_1) = x_2\); see [BP99]. There is a slightly weaker version of CW-homogeneity, called restrictive CW-homogeneity, which only requires the existence

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of the CW-translation $\sigma$ with $\sigma(x_1) = x_2$ for sufficiently close $x_1$ and $x_2$; see Definition 2.2 below for the precise statement. Although a Finsler metric is not reversible in general, the above definitions can be adapted to Finsler spaces by a word by word restatement.

The study of CW-translations has important merits in the investigations of the space forms in Riemannian geometry; see Wolf’s book [WO10] for an excellent survey. The related results have motivated a lot of mathematical activities; see for example [WO62, FR63, OZ69, OZ74, DMW86] for the determination of CW-translations of explicit Riemannian manifolds; see also [HE74, AW76] for the applications of these results to the study of homogeneous Riemannian manifolds of negative (non-positive) curvatures.

Recently, Berestovskii and Nikonorov studied the local one-parameter groups of CW-translations of general Riemannian manifolds and established a correspondence between local one-parameter groups of CW-translations and Killing vector fields of constant length (KVFCLs); see [BN08-1, BN08-2, BN09]. The above research leads to a classification of connected simply connected CW-homogeneous Riemannian manifolds. The list consists of the Euclidean spaces, odd-dimensional spheres with constant curvature, compact connected simply-connected Lie groups with bi-invariant Riemannian metrics and Riemannian products of the above manifolds. Notice that for simply-connected Riemannian manifolds, CW-homogeneity is equivalent to restrictive CW-homogeneity.

More recently, we initiated the study of CW-translations of Finsler spaces; see [DX02, DX03-1]. The relation between local one-parameter group of CW-translations and KVFCLs was generalized to the Finslerian case. We classified CW-homogeneous left invariant Randers metrics on compact simple Lie groups [DX03-2] and all CW-homogeneous Randers metrics on simply-connected manifolds [XD03]. In this paper, we will discuss a more generalized class of Finsler metrics, $(\alpha, \beta)$-metrics. The main theorem is the following

**Theorem 1.1** Suppose $F = \alpha \phi(\beta/\alpha)$ is a left invariant restrictively CW-homogeneous $(\alpha, \beta)$-metrics on a compact connected simple Lie group $G$, then $F$ must be a Randers metric.

Combined with the classification theorem in [XD03] (which is still correct with CW-homogeneity changed to restrictive CW-homogeneity), this theorem provides a complete classification of restrictively CW-homogeneous left invariant $(\alpha, \beta)$-metrics on compact semi-simple Lie groups. Using some similar arguments as in [DX03-2] or [XD03], we can prove that a left invariant restrictively CW-homogeneous Finsler metric on a compact semisimple Lie group is actually CW-homogeneous. Therefore, Theorem 1.1 also gives a complete classification of CW-homogeneous left invariant $(\alpha, \beta)$-metrics on compact semisimple Lie groups.

Theorem 1.1 is not valid for a general compact Lie group. For example, let $G = G' \times S^1$, where $G'$ is a compact semi-simple Lie group, $\alpha$ is a bi-invariant metric on $G'$, and $\beta$ is a $\alpha$-parallel 1-form induced by the standard 1-form on the $S^1$-factor, then for any smooth function satisfies the condition below, the $(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$ is CW-homogeneous. Note that such a metric must be a Berwald metric.
A very interesting and difficult problem is to classify all the CW-homogeneous Finsler spaces. It seems much more difficult than the same problem for Randers spaces.

In Section 2, we present some known results on related topics. In Section 3, we define the good normalized datum for a homogeneous non-Riemannian \((\alpha,\beta)\)-space \((M,F)\), and gives some method to find such datum. In Section 4, we use the good normalized datum to study the space of KVFCLs of a left invariant restrictively CW-homogeneous \((\alpha,\beta)\)-metric on a compact connected simple Lie group and prove Theorem 1.1 for compact connected simple Lie groups. Finally, in Section 5, we prove Theorem 1.1 for all compact connected semi-simple Lie groups by mathematical induction.

2 Preliminaries

2.1 The definition and examples of Finsler metrics

A Minkowski norm on a \(n\)-dimensional real linear space \(V\) is a continuous function \(F: V \rightarrow [0, +\infty)\) satisfying the following conditions:

1. (Positivity) \(F(y)\) is a positive smooth function on \(V \setminus 0\).
2. (Positive homogeneity) \(F(\lambda y) = \lambda F(y)\) for any \(\lambda > 0\).
3. (Strong convexity) The Hessian matrix

\[
(g_{ij}(y)) = \left(\frac{1}{2}[F^2(y)]_{y^i y^j}\right)
\]

is positive definite on \(V \setminus 0\).

The Minkowski norm \(F\) is called Euclidean or a linear metric if its Hessian matrix is independent of \(y\), i.e., if \(F^2 = g_{ij} y^i y^j\) is defined by an inner product on \(V\).

Let \(M\) be a connected smooth manifold. A Finsler metric on \(M\) is a continuous function \(F: TM \rightarrow [0, +\infty)\) which is smooth on the slit tangent bundle \(TM \setminus 0\), such that its restriction to each tangent space is a Minkowski norm.

The pair \((M,F)\) is called a Finsler space or a Finsler manifold. It is a Riemannian manifold if its restriction in each tangent space is an Euclidean norm (a linear metric).

The most important examples of non-Riemannian Finsler metrics are Randers metrics. A Randers metric is a Finsler metric of the form \(F = \alpha + \beta\), where \(\alpha\) is a Riemannian metric and \(\beta\) is a 1-form whose \(\alpha\)-length is less than 1 everywhere. Randers metrics were introduced by G. Randers in 1941, in his study of general relativity [RA41].

There are a more generalized class of Finsler metrics which have been studied extensively in the literature. Let \(\alpha\) be a Riemannian metric and \(\beta\) a smooth 1-form on the manifold \(M\). An \((\alpha,\beta)\)-metric is a Finsler metric of the form \(F = \alpha \phi(\beta/\alpha)\), where \(\phi\) is a positive function on \(\mathbb{R}\). The condition for \(F\) to define a Finsler metric on \(M\) can be stated as follows (see [CS05]). Denote \(\epsilon_0 = \sup_{(x,y) \in TM \setminus 0} \beta(x,y)/\alpha(x,y)\). If \(\epsilon_0\) can be attained at certain point \((x_0,y_0)\) and it is positive, then \(\phi\) is required to be smooth on \(I = [-\epsilon_0, \epsilon_0]\) and satisfies

\[
\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,
\]

(2.2)
for all $b$ and $s$ such that $|s| \leq |b| \leq \epsilon_0$; If $\epsilon_0$ can not be attained at any point, then $\phi$ is required to be positive and smooth on $I = (-\epsilon_0, \epsilon_0)$ ($I = \mathbb{R}$ when $\epsilon_0 = \infty$), and (2.2) is satisfied for all $b$ and $s$ with $|s| \leq |b| < \epsilon_0$. Notice that the Riemannian metric $\alpha$ or the 1-form $\beta$ in the definition of a $(\alpha, \beta)$-metric may not be unique. When $\beta$ is identically $0$, the metric $F$ is Riemannian. If $\phi$ is a linear function, then $F$ is a Randers metric.

2.2 Homogeneous Finsler spaces

On a Finsler space $(M, F)$ one can define the arc length of a piecewise smooth path. Let $x, x' \in M$. Then the distance $d(x, x')$ is defined to be the supremum of the arc lengths of all piecewise smooth paths from $x$ to $x'$. Notice that in general we do not have the reversibility $d(x, x') \equiv d(x', x)$, unless $F$ is reversible, i.e., $F(x, y) = F(x, -y)$ for any $x \in M$ and $y \in T_x M$. An isometry $\varphi$ of $(M, F)$ is a diffeomorphism of $M$ such that $\varphi^* F = F$. Equivalently, an isometry is a homeomorphism of $M$ onto itself such that $d(x, x') = d(\varphi(x), \varphi(x'))$ for any $x, x' \in M$ (see [DH02]). It was proven by Deng and Hou that the group $I(M, F)$ of all isometries of $(M, F)$, endowed with the open-compact topology, is a Lie group [DH02]. The space $(M, F)$ is called a homogeneous Finsler space if $I(M, F)$ acts transitively on $M$. In this case the manifold $M$ can be written as a coset space $G/H$, where $G$ is a closed subgroup of $I(M, F)$ which acts transitively on $M$ and $H$ is the isotropy subgroup of $G$ at a point $x_0 \in M$. In general, there may be more than one way to write $M$ as $G/H$. Since in this paper we will only consider connected manifolds, the subgroup $G$ can also be chosen to be a closed connected subgroup of the connected isometry group $I_0(M, F)$.

Let us give some examples of homogeneous Finsler spaces.

Let $G$ be a a Lie group. A Finsler metric $F$ on $G$ is called left invariant if $L(G) \subset I(G, F)$. Then $(G, F)$ is obviously homogeneous.

The second example is a homogeneous Randers metric $F = \alpha + \beta$ on $M = G/H$. The uniqueness of the presentation of $F$ indicates that both $\alpha$ and $\beta$ are preserved under the action of $I(M, F)$. By a 1-to-1 correspondence, the metric $F$ is determined by the restrictions of $\alpha$ and $\beta$ in $T_{x_0} M = m = g/h$, i.e. an $\text{Ad}(H)$-invariant linear metric on $m$ and an $\text{Ad}(H)$-invariant vector in $m^*$.

Now we consider a homogeneous $(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$ on $M = G/H$. In general there exists more than one way to write $F$ as an $(\alpha, \beta)$-metric, hence $\alpha$ and $\beta$ may not be $G$-invariant, or equivalently, their restrictions in $m$ may not be $\text{Ad}(H)$-invariant. To tackle this problem, we introduce the notion of a good datum. A triple $(\phi, \alpha, \beta)$ is called a good datum of the homogeneous non-Riemannian $(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$ if both $\alpha$ and $\beta$ are invariant under the action of $I_0(M, F)$. The following properties of a good datum are easy to verify:

1. For any closed connected subgroup $G \subset I_0(M, F)$ which acts transitively on $M$, the restrictions of $\alpha$ and $\beta$ in $m$ are $\text{Ad}(H)$ invariant, where $H$ is isotropy subgroup of $G$ at $x_0 \in M$.

2. The isometry group of $(M, F)$ can be identified with the closed subgroup of $I(M, \alpha)$ which keeps $\beta$ invariant. A vector field $X$ is a Killing vector field of $(M, F)$ if and only if $X$ is a Killing vector field of $(M, \alpha)$ and $L_X \beta = 0$. 

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We will show how to find a good datum for a homogeneous \((\alpha, \beta)\)-metric in the next section.

2.3 CW-translations and CW-homogeneity of Finsler spaces

Although the distance function of a Finsler space is generally not symmetric, the notions of CW-translations and CW-homogeneity of Finsler spaces can be defined in the same way as for metric spaces. For the completeness of the article we briefly recall the definitions below.

**Definition 2.1** An isometry \(\rho\) of a Finsler space \((M, F)\) is called a Clifford-Wolf translation (CW-translation) if \(d(x, \rho(x))\) is a constant function for \(x \in M\).

**Definition 2.2** A Finsler space \((M, F)\) is called Clifford-Wolf homogeneous (CW-homogeneous) if for any pair of points \(x, x' \in M\), there is a CW-translation \(\rho\) which sends \(x\) to \(x'\). It is called restrictively CW-homogeneous, if for any \(x\), there is a neighborhood \(U\) of \(x\), such that for any pair of points \(x_1\) and \(x_2\) in \(U\), there is a CW-translation \(\rho\) of \((M, F)\), such that \(\rho(x_1) = x_2\).

The main tool to study CW-translations and CW-homogeneity in Finsler geometry is a natural interrelation between Killing vector fields of constant length (KVFCLs) and local one-parameter semigroups of CW-translations. We now recall the main results in [DX02].

**Theorem 2.3** Suppose \((M, F)\) is a complete Finsler manifold with positive injective radius. If \(X\) is a KVFCL of \((M, F)\) and \(\varphi_t\) is the flow generated by \(X\), then \(\varphi_t\) is a Clifford-Wolf translation for any sufficiently small \(t > 0\).

**Theorem 2.4** Let \((M, F)\) be a compact Finsler space. Then there is a \(\delta > 0\), such that any CW-translation \(\rho\) with \(d(x, \rho(x)) < \delta\) is contained in a local one-parameter semigroup of CW-translations generated by a KVFCL.

Notice that Theorem 2.4 is still correct if we replace the compactness of \(M\) by the homogeneity of \((M, F)\); see [XD03].

Based on these interrelation theorems, we have an equivalent description of the restrictive CW-homogeneity.

**Proposition 2.5** Let \((M, F)\) be a compact connected homogeneous Finsler space. Then it is restrictively CW-homogeneous if and only if any tangent vector can be extended to a KVFCL of \((M, F)\).

3 Good normalized data of a homogeneous non-Riemannian \((\alpha, \beta)\)-space

3.1 Non-Riemannian \((\alpha, \beta)\)-norms, linear isometry groups and normalized data

Before discussing homogeneous \((\alpha, \beta)\)-spaces, let us look at its local model.
Let $F$ be a Minkowski norm on a real linear space $V$. Denote by $L(V, F)$ the group of linear isometries of $(V, F)$ (which is a compact Lie group), and by $L_0(V, F)$ the unity component of $L(V, F)$. Then we have

**Lemma 3.1** (1) Suppose $\dim V = n > 1$. Then the Minkowski norm $F$ is an $(\alpha, \beta)$-norm if and only if there is a linear metric $\alpha$ and an $\alpha$-orthogonal decomposition $V = V_1 \oplus V_2$, with $\dim V_1 = n - 1$, such that $L_0(V, F)$ contains $SO(V_1, \alpha)$, the maximal connected subgroup of linear isomorphisms which preserve $\alpha$ and act trivially on $V_2$.

(2) An $(\alpha, \beta)$-norm $F$ is Riemannian if and only if $\dim L_0(V, F) > \dim SO(V_1, \alpha)$.

**Proof.** (1) Suppose $F = \alpha \phi(\beta/\alpha)$ is an $(\alpha, \beta)$-norm. If $\beta = 0$ we can choose any $\alpha$-orthogonal decomposition as indicated in the lemma. Then we have

$$SO(V_1, \alpha) \subset SO(V, \alpha) = L_0(V, F).$$

If $\beta \neq 0$, we can take $V_1 = \ker \beta$ and $V_2$ to be the $\alpha$-orthogonal complement of $V_1$. The functions $\alpha$ and $\beta$ take the same value on each $SO(V_1, \alpha)$-orbit, so does $F$. Thus $L_0(V, F)$ contains the connected subgroup $SO(V_1, \alpha)$.

Conversely, assume that we can find $\alpha$ and an $\alpha$-orthogonal decomposition $V = V_1 \oplus V_2$ with $\dim V_1 = n - 1$, such that $SO(V_1, \alpha) \subset L_0(V, F)$. Then we can choose a nonzero $\beta \in V^*$ such that $\ker \beta = V_1$. If $y_1, y_2 \in V$ and $\alpha(y_1) = \beta(y_2)$, then $y_1$ and $y_2$ must be contained in the same orbit of $SO(V_1, \alpha)$. Since $SO(V_1, \alpha) \subset L_0(V, F)$, we have $F(y_1) = F(y_2)$. Thus $F$ only depends on the values of $\alpha$ and $\beta$. Hence we can find a suitable function $\phi$ such that $F = \alpha \phi(\beta/\alpha)$.

(2) Up to conjugation, $SO(V, \alpha)$ is just the standard special orthogonal subgroup $SO(n)$ and $SO(V_1, \alpha)$ the standard subgroup $SO(n - 1) \subset SO(n)$. We have seen in the above argument that, if $F$ is Riemannian, then $L_0(V, F)$, which is isomorphic to $SO(n)$, has a larger dimension than $SO(V_1, \alpha)$.

Conversely, assume that $\dim L_0(V, F) > \dim SO(V_1, \alpha)$. Then we can find an infinitesimal generator $X$ of $L_0(V, F)$, nonzero vectors $V_1 \in V_1$ and $V_2 \in V_2$, such that $X(V_1) = V_2$ and $X(V_2) = -V_1$. Now $X$ generates an one-parameter of isometries, which is just the action of $S^1$ as rotations on the 2-dimensional subspace $W$ generated by $V_1$ and $V_2$. Then the restriction of $F$ to $W$ is invariant under the rotations generated by $X$, hence $F|_W$ it is a Euclidean norm. Therefore $F$ must of be the form $\sqrt{a \alpha^2 + b \beta^2}$ for some constants $a$ and $b$, and it is a linear metric on $V$.

According to Lemma 3.1, when $\dim V > 2$, a non-Riemannian $(\alpha, \beta)$-norm $F = \alpha \phi(\beta/\alpha)$ on $V$ determines a unique decomposition of $V$ into the direct sum of irreducible representations of $L_0(V, F)$, i.e., $V = V_1 + V_2$, such that $V_1$ is $(n - 1)$-dimensional with the natural action of $L_0(V, F)$, and $V_2$ is 1-dimensional with the trivial action of $L_0(V, F)$. Since $\ker \beta = V_1$, $\beta$ is uniquely determined by $F$ up to a scalar multiplication. Moreover, the linear metric $\alpha$ is also uniquely determined by $F$ in the sense that there are two positive scalars $c_1, c_2$ such that $\alpha|_{V_1} = c_1 F|_{V_1}$ and $\alpha|_{V_2} = c_2 F|_{V_2}$.

A triple $(\phi, \alpha, \beta)$ is called a normalized datum if we have

$$\alpha|_{V_1} = F|_{V_1},$$

and for any $y \in V_2$ with $\beta(y) > 0$, we have

$$\alpha(y) = \beta(y) = F(y).$$
For a normalized datum \((\phi, \alpha, \beta)\), we have \(\|\beta\|_\alpha = 1\). Thus \(\phi\) is a smooth function on \([-1, 1]\), and \(\phi(0) = \phi(1) = 1\).

The following corollary is obvious.

**Corollary 3.2** Let \(F\) be a non-Riemannian \((\alpha, \beta)\)-norm on a real linear space \(V\), with \(\dim V > 2\). Then there are at most two normalized data of \(F\). Moreover, in any normalized datum \((\phi, \alpha, \beta)\) of \(F\), \(\alpha\) and \(\beta\) are invariant under the action of \(L_0(V, F)\).

### 3.2 An existence theorem

Now we turn back to homogeneous non-Riemannian \((\alpha, \beta)\)-spaces. The following theorem tells us that in most cases a good datum exists. Moreover, the proof of the following problem shows how to find a good normalized datum.

**Theorem 3.3** Let \((M, F)\) be a homogeneous non-Riemannian Finsler space such that the restriction of \(F\) to any tangent space is an \((\alpha, \beta)\)-norm. Suppose there is a closed connected subgroup \(G\) of \(I_0(M, F)\) which acts transitively on \(M\) such that the isotropy subgroup \(H\) at a point \(x_0 \in M\) is connected. Then \(F\) is an \((\alpha, \beta)\)-metric. Moreover, there is a good global datum \((\phi, \alpha, \beta)\) of \(F\) such that the restriction of the datum to any tangent space is a normalized datum.

**Proof.** First we construct the global datum \((\phi, \alpha, \beta)\) for \(F\). Notice that since \(F\) is non-Riemannian, the restriction of \(F\) to a tangent space cannot be a linear metric. In particular, the restriction of \(F\) to \(T_{x_0}M\) is a non-euclidean \((\alpha, \beta)\)-norm, hence there exists a normalized datum \((\phi, \alpha, \beta)\) for \(F(x_0, \cdot)\). Then for any \(g \in G\), \((\phi, g^*\alpha, g^*\beta)\) is a normalized datum for \(F(g^{-1}x_0, \cdot)\). Now for any two elements \(g\) and \(g'\) in \(G\) such that \(g^{-1}x_0 = g'^{-1}x_0\), \(gg'^{-1} \in H\) defines an element in \(L_0(TM_{x_0}, F(x_0, \cdot))\) by the connectedness of \(H\). So we have \(gg'^{-1}\alpha = \alpha\) and \(gg'^{-1}\beta = \beta\). Thus the normalized data \((\phi, g^*\alpha, g^*\beta)\) and \((\phi, g'^*\alpha, g'^*\beta)\) coincide. By the smoothness of the action, these data form a smooth global datum \((\phi, \alpha, \beta)\) for \(F = \alpha\phi(\beta/\alpha)\). Therefore \(F\) is an \((\alpha, \beta)\)-metric.

Given \(\rho \in I_0(M, F)\), there is a continuous family \(\rho_t \in I_0(M, F)\) such that \(\rho_0 = \id\) and \(\rho_1 = \rho\). For each \(x \in M\), \((\phi, \rho_t^*(\alpha|_{\rho_t(x)}), \rho_t^*(\beta|_{\rho_t(x)})\)) is a continuous family of normalized data for the \((\alpha, \beta)\)-norm \(F(x, \cdot)\). It must be a constant family. Thus \(\rho^*\alpha = \alpha\) and \(\rho^*\beta = \beta\). Hence the normalized datum \((\phi, \alpha, \beta)\) is a good datum. □

As an example, let \(F\) be a left invariant non-Riemannian \((\alpha, \beta)\)-metric on a compact connected Lie group \(G\). Denote \(G' = I_0(G, F)\). Then \(L(G) \subset G'\), and the Lie group \(G\) can be written as \(G = G'/H\), where \(H\) is the isotropy subgroup of \(G'\) at \(e \in G\). Since \(G'\) is diffeomorphic to the product of \(G\) and \(H\), \(H\) is connected. By Theorem 3.3, we can find a good normalized datum for \(F\).

### 4 Restrictive CW-homogeneity and left invariant \((\alpha, \beta)\)-metrics on a compact connected simple \(G\)

#### 4.1 Some notations

We first introduce some notations which will be used throughout this section.
Let $G$ be a compact connected simple Lie group, and $F$ a left invariant non-Riemannian $(\alpha, \beta)$-metric on $G$. Theorem 3.3 indicates that we can find a good normalized datum for $F$. We denote the restriction of the datum to $T_x G = \mathfrak{g}$ as $(\phi, \alpha, \beta)$. Meanwhile, the $(\alpha, \beta)$-norm defined by $F$ in $\mathfrak{g}$ will also be denoted as $F = \alpha \phi(\beta/\alpha)$.

By Theorem 3.3 and [OT76], we have $I_0(G, F) \subset L(G)R(G)$. Let $G'$ be the maximal connected closed subgroup of $G$, such that $R(G')$ consists of isometric right translations. Then $I_0(G, F) = L(G)R(G')$. Denote $\text{Lie}(G) = \mathfrak{g}$ and $\text{Lie}(G') = \mathfrak{g}'$. The space of Killing vector fields of $(G, F)$ can be identified with the Lie algebra of $I_0(G, F)$, i.e., $\mathfrak{g} \oplus \mathfrak{g}'$. The isotropy subgroup at $e \in G$ is isomorphic to $G'$, whose Lie algebra is $\{(X', X')|X' \in \mathfrak{g}'\}$. The group $G'$ can also be identified with the maximal connected closed subgroup of $G$ whose Ad-action preserves the functions $\alpha$ and $\beta$ on $\mathfrak{g}$.

The inner product defined by $\alpha$ on $\mathfrak{g}$ will be denoted as $\langle \cdot, \cdot \rangle$, and the inner product corresponding to the bi-invariant linear metric $|| \cdot ||_{\mathfrak{b}i}$ will be denoted as $\langle \cdot, \cdot \rangle_{\mathfrak{b}i}$. Let $v$ and $v'$ be the nonzero vectors in $\mathfrak{g}$ such that $\beta(\cdot) = \langle v, \cdot \rangle = \langle v', \cdot \rangle_{\mathfrak{b}i}$. Then it is easy to see that $G' \subset C_G(v)$ and $\mathfrak{g} \subset c_\mathfrak{g}(v)$.

### 4.2 The decomposition of the set of KVFCLs

The interrelation between CW-translations and KVFCLs, and in particular Proposition 2.5 suggest that we should study the set of KVFCLs.

We first prove a similar criterion for a Killing vector field $(X, X') \in \mathfrak{g} \oplus \mathfrak{g}'$ to have constant length as in the Randers case [DX03-2].

**Theorem 4.1** If $(X, X') \in \mathfrak{g} \oplus \mathfrak{g}'$ defines a KVFCL of $F$, then either $X = 0$ or $X' \in c(\mathfrak{g}')$.

**Proof.** The Killing vector field defined by $(X, X')$ has the $F$-length $F(\text{Ad}(g)X - \text{Ad}(g')X')$ at $gg^{-1}$, for $g \in G$ and $g' \in G'$. If $(X, X')$ defines a KVFCL, then

$$\alpha(\text{Ad}(g)X - \text{Ad}(g')X') \phi\left(\frac{\beta(\text{Ad}(g)X - \text{Ad}(g')X')}{\alpha(\text{Ad}(g)X - \text{Ad}(g')X')}\right) = \text{const}, \forall g \in G, g' \in G'. \quad (4.4)$$

Thus for a fixed $g \in G$, $\beta(\text{Ad}(g)X - \text{Ad}(g')X') = \beta(\text{Ad}(g)X - X')$ is a constant function of $g'$. For $Y \in \mathfrak{g}'$ and $g_0' \in G'$, denote

$$X_{g_0'} = \text{Ad}(g)X - \text{Ad}(\exp(tY)g_0')X', \quad (4.5)$$

$$s_{g_0,Y} = \beta(X_{g_0',Y})/\alpha(X_{g_0',Y}), \text{ and}$$

$$s_0 = \frac{\beta(\text{Ad}(g)X - \text{Ad}(g')X')}{\alpha(\text{Ad}(g)X - \text{Ad}(g')X')} \quad (4.6)$$

Setting $g' = \exp(tY)g_0'$ in (4.4), taking the differential with respect to $t$ and considering the value at $t = 0$, we have

$$\left(\phi(s_0) - s_0\phi'(s_0)\right) \frac{d}{dt} \alpha(X_{g_0',Y})|_{t=0} = 0, \forall Y \in \mathfrak{g}', g_0 \in G'. \quad (4.7)$$

By (4.4) and (4.3), $\alpha(\text{Ad}(g)X - \text{Ad}(g')X')$ must also be a constant function of $g'$. Note that neither $\alpha(\text{Ad}(g)X)$ nor $\alpha(\text{Ad}(g')X') = \alpha(X')$ depends on $g'$, so $\langle \text{Ad}(g)X, \text{Ad}(g')X' \rangle$
is a constant function of $g'$. Thus for any $g \in G$, $\text{Ad}(g)X$ is $\alpha$-orthogonal to the ideal generated by $[X', g']$ in $g'$. Now change $g$ arbitrarily, we can prove that the ideal of $g$ generated by $[X, g]$ is $\alpha$-orthogonal to the ideal of $g'$ generated by $[X', g']$. If $X \neq 0$, then $X'$ generates the $0$ ideal in $g'$. Thus $X' \in \mathfrak{c}(g')$. This completes the proof. ■

For simplicity, we denote the set of KVFCLs of the metric $F$ as $\mathcal{K}_F$. Theorem 4.1 implies that $\mathcal{K}_F$ can be decomposed into the union of $\mathcal{K}_{F;1}$ and $\mathcal{K}_{F;2}$, where $\mathcal{K}_{F;1}$ is the closure of the set of KVFCLs $(X, X')$ with $X \neq 0$, and $\mathcal{K}_{F;2}$ is the linear subspace $0 \oplus \mathfrak{g}'$. The following lemma shows that $\mathcal{K}_{F;1} \cap \mathcal{K}_{F;2} = \{0\}$.

**Lemma 4.2** There is a constant $C > 0$, such that for any KVFCL $(X, X') \in \mathcal{K}_{F;1}$, we have $\|X\|_{\text{bi}} < C \|X\|_{\text{bi}}$.

**Proof.** The Lie algebra $\mathfrak{g}$ will be viewed as a flat manifold with the metric $\langle \cdot, \cdot \rangle_{\text{bi}}$, and any submanifold in it will be endowed with the induced metric.

Suppose conversely that the constant $C > 0$ indicated in the lemma does not exist. Then there is a sequence of $(X_n, X'_n) \in \mathcal{K}_{F;1}$ such that $\|X_n\|_{\text{bi}} = 1$, $X'_n \in \mathfrak{c}(g')$ with $\lim_{n \to \infty} \|X'_n\|_{\text{bi}} = \infty$. Denote $F(\text{Ad}(g)X_n - X'_n) = l_n$. Then the sequence $\{l_n\}$ also diverges to $\infty$. The $\text{Ad}(G)$-orbit $O_{X_n}$ is contained in the hypersurface

$$S_n = \{Y | F(Y - X'_n) = l_n\} \subset \mathfrak{g}, \tag{4.8}$$

on which the $C^0$-norm of all principal curvatures converges to $0$ when $n \to \infty$. Taking a suitable sequence if necessary, we can assume that $\lim_{n \to \infty} X_n = X$. Then in the closed round ball with center $0$ and radius $3$ (with respect to the bi-invariant metric), the hypersurfaces $S_n$ converges to a flat hyperplane $\mathcal{S}$ of codimension $1$ in $\mathfrak{g}$. Hence the hyperplane $\mathcal{S}$ contains the $\text{Ad}(G)$-orbit $O_X$ of the nonzero vector $X$. This can not happen for a compact connected simple $G$. ■

The KVFCLs in $\mathcal{K}_{F;2}$ or the CW-translations generated by them are relevant to $I_0(G, F)$ rather than $F$ itself. Therefore they are of little interest to our study. The following corollary shows that in most cases, we only need to consider the KVFCLs in $\mathcal{K}_{F;1}$.

**Corollary 4.3** Keep all the notations as above. The metric $F$ is restrictively CW-homogeneous if and only if any nonzero tangent vector can be extended to a KVFCL $(X, X')$ with $X \neq 0$.

**Proof.** We only need to prove the “only if” part. Suppose $F$ is restrictively CW-homogeneous. Notice that $\mathcal{K}_{F;2}$ can only cover tangent vectors in a subspace with positive codimension in each tangent space. For a nonzero tangent vector outside those subspaces, the existence of the extension follows directly from the restrictive CW-homogeneity of $F$. Now consider an arbitrary nonzero $u \in \mathfrak{g}' \subset T_eG$. One can find a sequence of tangent vectors $u_n \in T_eG$ with $u_n \notin \mathfrak{g}'$, $\forall n$, such that $u = \lim_{n \to \infty} u_n$.

Each tangent vector $u_n$ can be extended to a KVFCL $(X_n, X'_n) \in \mathcal{K}_{F;1}$. By taking a subsequence, we can have

$$\lim_{n \to \infty} (X_n, X'_n) = (X, X') \in \mathcal{K}_{F;1} \setminus 0, \tag{4.9}$$

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which takes the value $u$ at $e$. By Lemma 4.2, $X \neq 0$. For tangent vectors at other points, the argument is similar. This completes the proof of the corollary. □

The following theorem implies, for each $X \in \mathfrak{g}$, there are not too much $X' \in c(\mathfrak{g}')$ such that $(X, X') \in \mathcal{K}_{F;1}$.

**Theorem 4.4** Keep all notations as above. Suppose both $(X, X')$ and $(X, X'')$ define KVFCILs in $\mathcal{K}_{F;1}$. Then there exists $c \in \mathbb{R}$ such that $X' - X'' = cv$, where $v$ is the $\alpha$-dual of $\beta$.

**Proof.** Lemma 4.2 indicates when $X = 0$, we have $X' = X'' = 0$, so it is obvious. We can assume $X \neq 0$. The condition that $(X, X') \in \mathcal{K}_{F;1}$ implies

$$\alpha(\text{Ad}(\exp(tY)g)X - X')\phi(\frac{\beta(\text{Ad}(\exp(tY)g)X - X'')}{\alpha(\text{Ad}(\exp(tY)g)X - X')}) = \text{const.} \tag{4.10}$$

Differentiate (4.10) with respect to $t$ and take $t = 0$, then we get

$$\frac{\phi(s) - s\phi'(s)}{\alpha(\text{Ad}(g)X - X')}\langle [Y, \text{Ad}(g)X], \text{Ad}(g)X - X' \rangle + \phi'(s)\beta([Y, \text{Ad}(g)X]) = 0, \tag{4.11}$$

where $s = \beta(\text{Ad}(g)X - X')/\alpha(\text{Ad}(g)X - X')$. A similar equality holds for $(X, X'')$. Notice $\phi(s) - s\phi'(s) > 0$, So for any $g \in G$ and $Y \in \mathfrak{g}$, if $\beta([Y, \text{Ad}(g)X]) = 0$, we have

$$\langle [Y, \text{Ad}(g)X], \text{Ad}(g)X - X' \rangle = 0, \tag{4.12}$$

and then

$$\langle [Y, \text{Ad}(g)X], X' - X'' \rangle = 0. \tag{4.13}$$

To finish the proof for the theorem, we only need to prove the set

$$\mathcal{S} = \bigcup_{g \in G}([g, \text{Ad}(g)X] \cap \ker \beta) \tag{4.14}$$

span the subspace $V_1 = \ker \beta = V'\perp_{bi}$, in which $V'$ is the dual of $\ker \beta$ with respect to the bi-invariant metric. Assume on the contrary, there is a nonzero $v'' \in V_1$, such that

$$v'' \in ([g, \text{Ad}(g)X] \cap V_1)\perp_{bi} = [g, \text{Ad}(g)X]\perp_{bi} + \mathbb{R}v', \tag{4.15}$$

$\forall g \in G$. So $v''$ is contained in

$$\bigcap_{g \in G}([g, \text{Ad}(g)X]\perp_{bi} + \mathbb{R}v') = \bigcap_{g \in G}(\text{Ad}(g)c_{\mathfrak{g}}(X) + \mathbb{R}v'). \tag{4.16}$$

Notice the nonzero vectors $v''$ and $v'$ are linearly independent to each other. Thus for any $g \in G$, $\text{Ad}(g)c_{\mathfrak{g}}(X)$ has a nonzero intersection with the 2-dimensional real linear space spanned by $v'$ and $v''$. The next lemma states it is impossible. □

**Lemma 4.5** Let $G$ be a compact connected simple Lie group with Lie algebra $\mathfrak{g}$, $X \in \mathfrak{g}$ be a nonzero vector, and $\mathbf{L} \subset \mathfrak{g}$ be a real subspace with $\dim \mathbf{L} = 2$. Then there exists $g \in G$, such that $\mathbf{L} \cap \text{Ad}(g)c_{\mathfrak{g}}(X) = \{0\}$.
Proof. We will prove the lemma by contradiction. Assume on the contrary

\[ L \cap \text{Ad}(g)c_g(X) \neq \{0\}, \forall g \in G. \]  (4.17)

Then the minimum of \( \dim L \cap \text{Ad}(g)c_g(X) \), \( \forall g \in G \), is 1 or 2. If it is 2, then \( L \) is contained in the center of \( \mathfrak{g} \), which is a contradiction. So we can suitably change \( X \) by conjugations, such that \( \dim L \cap c_g(X) = 1 \). By the semi-continuity, for all \( g \in G \) sufficiently close to \( e \), we also have \( \dim L \cap \text{Ad}(g)c_g(X) = 1 \).

Assume \( U \in L \) generates \( L \cap c_g(X) \). Let \( \{U, U'\} \) be a basis of \( L \). Then there is a smooth real function \( f(g) \) for \( g \in G \) sufficiently close to \( e \), such that \( f(e) = 0 \) and \( L \cap \text{Ad}(g)c_g(X) \) is generated by \( U + f(g)U' \).

Take \( g = \exp(tY) \), and differentiate \([U + f(g)U', \text{Ad}(g)X] = 0 \) with respect to \( t \) at \( t = 0 \), we have

\[ [U, [Y, X]] + Df(Y)[U', X] = 0, \forall Y \in \mathfrak{g}. \]  (4.18)

in which \( Df : \mathfrak{g} \rightarrow \mathbb{R} \) is the differential of \( f \) at \( e \). Thus \( \dim[U, [X, \mathfrak{g}]] \leq 1 \). Because \( \dim[U, [X, \mathfrak{g}]] \) is even, so it must be 0. We have \( \{U, [X, \mathfrak{g}]\} = 0 \) and \([U', X] \neq 0 \), then \( Df \equiv 0 \). If we change \( X \) to \( \text{Ad}(g)X \), in which \( g \in G \) is sufficiently close to \( e \), we can get the same property for the corresponding differential. This implies \( f \equiv 0 \) around \( e \), i.e. \([U, \text{Ad}(g)X] = 0 \) for \( g \) sufficiently close to \( e \). This happens only when \( U \in c(\mathfrak{g}) \), which is a contradiction. ■

4.3 The properties of \( K_{F;1} \) when \( F \) is restrictive CW-homogeneous

We keep all notations as before, and further assume \( F \) is restrictively CW-homogeneous. Then the set \( K_{F;1} \) of KVFCFLs satisfies the following properties.

Lemma 4.6 Keep all notations as before and assume \( F \) is restrictively CW-homogeneous, then we have

1. The function \( \phi \) is real analytic on \([-1, 1]\).
2. The subset \( K_{F;1}\backslash\{0\} \subset (\mathfrak{g} \oplus \mathfrak{g}')\backslash\{0\} \) is a closed real analytic subvariety.
3. For any \( X \in \mathfrak{g} \), there are at most finite different \( X' \)'s, such that \( (X, X') \in K_{F;1} \).

Proof. (1) By Corollary 4.3 for any \( s_0 \in [-1, 1] \), we can find a tangent vector \( u \) with \( F(u) = 1 \), such that \( s_0 = \beta(u)/\alpha(u) \), and the tangent vector \( u \) can be extended to a Killing vector field \((X, X') \in K_{F;1} \) with \( X \neq 0 \), i.e.

\[ \alpha(\text{Ad}(g)X - X')\phi(\beta(\text{Ad}(g)X - X')/\alpha(\text{Ad}(g)X - X')) = 1, \forall g \in G. \]  (4.19)

The function \( s(g) = \beta(\text{Ad}(g)X - X')/\alpha(\text{Ad}(g)X - X') \) can not be a constant function for \( g \in G \). Otherwise by (4.19), \( \beta(\text{Ad}(g)X - X') \) is a constant function for \( g \in G \). Then the \( \text{Ad}(G) \)-orbit \( O_X \) is contained in a flat hyperplane, which is a contradiction with the assumption that \( X \neq 0 \) and \( G \) is simple. We will denote the range of \( s(g) \) as \( \mathcal{I}_{(X, X')} \), which is a closed interval.

For any side of \( s_0 \) which is contained in \( \mathcal{I}_{(X, X')} \), the positive side for example, we can choose \( X \) within the orbit \( O_X \) and find a vector \( Y \in \mathfrak{g} \) such that the real analytic function \( f(t) = s(\exp(tY)) \) satisfies for some \( k \in \mathbb{N} \),

\[ f(0) = s_0, f'(0) = f''(0) = \cdots = f^{(k-1)}(0) = 0, f^{(k)}(0) > 0. \]  (4.20)

We can find a suitable real analytic change of variable \( \tilde{t} = \tilde{t}(t) \), \( \tilde{t}(0) = 0 \), such that

\[
f(t) = f(0) + \tilde{t}^k \quad \text{around} \quad \tilde{t} = 0.
\]

The equality (4.19) can also be written as

\[
\phi(s_0 + \tilde{t}^k) = \frac{1}{\alpha(\text{Ad}(\exp(iY)) X - X')}.
\]  

(4.21)

Around \( \tilde{t} = 0 \), the left side of (4.21) is a smooth function of \( \tilde{t} \), which derivatives with respect to \( \tilde{t} \) at \( \tilde{t} = 0 \) vanishes except those with \( k \)-multiple degrees, and then the right side is a real analytic function of \( \tilde{t} \) with the same properties for its derivatives at \( \tilde{t} = 0 \). Thus the right side is a real analytic function of \( \tilde{t} = f(t) \) at the positive side of \( f(0) = s_0 \), and so does \( \phi(s) \) at the positive side of \( s_0 \). The proof for the negative side of \( s_0 \) is similar, we just need to require \( f^{(k)}(0) < 0 \) in (4.20) and take \( \tilde{t} = f(t) = f(0) - \tilde{t}^k \).

If \( s_0 \) is an endpoint of \( \mathcal{I}_{(X,X')} \), the argument above guarantees \( \phi(s) \) is real analytic at one side of \( s_0 \). We will see how to use Lemma 4.2 to prove the real analytic property of \( \phi(s) \) for the other side, the negative side for example. If there is another Killing vector field \( (X_0, X'_0) \) in \( \mathcal{K}_{F;1} \) such that an open neighborhood of \( s_0 \) is contained in \( \mathcal{I}_{(X_0,X'_0)} \), then it is done. Otherwise we can find a sequence \( s_n \) approaching \( s_0 \) from below. For each \( s_n \), we can find a KVFCCL \( (X_n, X'_n) \in \mathcal{K}_{F;1} \) with length 1, such that \( s_n \) is contained in \( \mathcal{I}_{(X_n,X'_n)} \) which lies below \( s_0 \). By taking a subsequence, this sequence of KVFCCLs converges to a KVFCCL \( (X_0, X'_0) \in \mathcal{K}_{F;1}\{0\} \), such that \( \mathcal{I}_{(X_0,X'_0)} \) contains the negative side of the endpoint \( s_0 \).

To summarize, the smooth function \( \phi(s) \) is real analytic for both sides of each point in \([-1,1] \), so it is a real analytic function on \([-1,1] \).

(2) Around any \((X,X') \in \mathcal{K}_{F;1}\{0\}\), the equations defining \( \mathcal{K}_{F;1} \) can be presented as

\[
\alpha(\text{Ad}(g) X - X') \phi\left( \frac{\beta(\text{Ad}(g) X - X')}{\alpha(\text{Ad}(g) X - X')} \right) = \alpha(X - X') \phi\left( \frac{\beta(X - X')}{\alpha(X - X')} \right), \quad \forall g \in G, \tag{4.22}
\]

which are real analytic equations for \( X \) and \( X' \), because \( \phi \) is real analytic on \([-1,1] \). So \( \mathcal{K}_{F;1}\{0\} \) is a closed real analytic subvariety of \( \{g \oplus g'\}\{0\} \).

(3) When \( X = 0 \), the assertion follows Lemma 4.2 directly. Now assume \( X \neq 0 \).

If on the contrary there are a sequence of different \( X'_n \)'s such that \((X, X'_n) \in \mathcal{K}_{F;1} \). By Lemma 4.2, taking a subsequence if necessary, we can assume \( \lim_{n \to \infty} X'_n = X' \). By Theorem 4.4 there is a sequence \( \{t_n\} \subset \mathbb{R}\{0\} \), such that \( \lim_{n \to \infty} t_n = 0 \), and \( X'_n = X' - t_nv \). So we have

\[
F(\text{Ad}(g) X - X' + t_nv) \equiv C_n, \quad \forall g \in G. \tag{4.23}
\]

Since \( \phi \) is real analytic, the continuous function \( F(\text{Ad}(g) X - X' + tv) \) of \( g \) and \( t \) is real analytic whenever \( \text{Ad}(g) X - X' + tv \neq 0 \). Because \((X, X') \in \mathcal{K}_{F;1}\{0\} \), for \( t \) sufficiently close to 0, \( F(\text{Ad}(g) X - X' + tv) \neq 0 \) for all \( g \in G \), and then \( F(\text{Ad}(g) X - X' + tv) \) is a constant function of \( g \). If there is a number \( t_0 = \inf\{t > 0\mid F(\text{Ad}(g) X - X' + tv) = 0, \text{ for some } g \in G\} \), then \( t_0 > 0 \) and there is \( g_0 \in G \) such that \( \text{Ad}(g_0) X - X' + t_0v = 0 \). For any \( t \in [0, t_0) \), \( F(\text{Ad}(g) X - X' + tv) \) is a constant function of \( g \). By the continuity,

\[
\text{Ad}(g) X - X' + t_0v = 0, \quad \forall g \in G, \tag{4.24}
\]
i.e. the Ad(G)-orbit $O_X$ is contained in a line, which is impossible when $X \neq 0$ and $G$ is compact simple. So Ad$(g)X - X' + tv \neq 0$ and $(X, X' - tv) \in K_{F,1}$ for all $t \geq 0$. 

This is a contradiction with Theorem \ref{thm:main_result}. 

There are two natural projections from $K_{F,1}\{0\}$ to $g\{0\}$, namely,

$$\pi_1(X, X') = X - X', \text{ and } \pi_2(X, X') = X. \tag{4.25}$$

The first projection maps each Killing vector field to its value at $e$. When $(G, F)$ is restrictive CW-homogeneous, by Corollary \ref{cor:global} the map $\pi_1$ is surjective. Thus the dimension of the real analytic variety $K_{F,1}\{0\}$ is no less than dim$g$. Lemma \ref{lem:analytic_dim} indicates the map $\pi_2$ has a finite pre-image for each $X$, which implies the dimension of $K_{F,1}\{0\}$ must be exactly dim$g$. Whitney’s theorem on the local stratification of analytic varieties \cite{WH65} indicates locally $K_{F,1}\{0\}$ can be decomposed as the disjoint union of finite smooth manifolds, among which there is one with the same dimension as $g$. Restricted to this subset, the finite map $\pi_2$ must be regular somewhere. So $\pi_2(K_{F,1}\{0\})$ contains a nonempty open subset $\mathcal{U} \subset g\{0\}$. The Ad($G$)-actions on the first factor preserve $K_{F,1}\{0\}$, so we can assume $\mathcal{U}$ is an Ad($G$)-invariant nonempty open subset of $g\{0\}$.

Let $t$ be any Cartan subalgebra of $g$. Then $\mathcal{U}' = \mathcal{U} \cap t$ is a nonempty open subset of $t$. For any nonzero $X$ in $\mathcal{U}'$, there is a $X' \in \mathcal{U}'$, such that $(X, X') \in K_{F,1}$. Let $V_1 = \ker \beta$, and $\text{pr}_1$ be the $\alpha$-orthogonal projection from $g$ to $V_1$. Though there maybe many choices for $X'$, they have the same $\text{pr}_1 X'$ by Theorem \ref{thm:main_result}.

From \ref{eq:l(X)}, we have seen $l(X) = \text{pr}_1 X'$ for $X \in \mathcal{U}'$ satisfies the following condition

$$\langle [g, \text{Ad}(g)X] \cap V_1, \text{Ad}(g)X - l(X) \rangle = 0. \tag{4.26}$$

Now we will see $l(X)$ can be extended to a linear map on $t$ with \ref{eq:l(X)} satisfied.

Choose a basis $\{X_1, \ldots, X_m\}$ of $t$ from the regular vectors in $\mathcal{U}'$. For each $X_i$, there is $X_i'' = l(X_i)$ such that $\langle [g, \text{Ad}(g)X_i] \cap V_1, \text{Ad}(g)X_i - X_i'' \rangle = 0$. For $X = \sum_{i=1}^m c_i X_i$, let $X'' = \sum_{i=1}^m c_i X_i''$. Because $[g, \text{Ad}(g)X] \subset [g, \text{Ad}(g)X_i], \forall i$, we have

$$\langle [g, \text{Ad}(g)X] \cap V_1, \text{Ad}(g)X_i - X_i'' \rangle = 0. \tag{4.27}$$

Take the linear combination of the above equalities for each $i$, we get

$$\langle [g, \text{Ad}(g)X] \cap V_1, \text{Ad}(g)X - X'' \rangle = 0. \tag{4.28}$$

This defines a linear map from $X$ to $X''$, satisfying \ref{eq:l(X)}. From the proof of Theorem \ref{thm:main_result}, this linear map coincides with $\text{pr}_1$ when $(X, X') \in K_{F,1}$.

For any $X_1, X_2 \in \pi_2(K_{F,1}\{0\})$ in the same orbit of Weyl group actions, they share the same $X'$ such that $(X_1, X'), (X_2, X') \in K_{F,1}$. So $l(X_1) = l(X_2)$, i.e. the linear map $l$ on $t$ is invariant for the Weyl group actions, which must be the 0 map.

Change $t$ arbitrarily, then we have

**Lemma 4.7** Keep all notations of this subsection. For any $(X, X') \in K_{F,1}$, we have $X'$ is a scalar multiple of $v$, the $\alpha$-dual of $\beta$. 

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4.4 Proof of Theorem 1.1 for compact connected simple $G$

To prove the main theorem for compact connected simple $G$, we only need to consider non-Riemannian metrics. So we will assume $F$ is a restrictive CW-homogeneous left invariant non-Riemannian ($\alpha, \beta$)-metric on a compact connected simple Lie group $G$, and keep all notations as before. We will see how the properties of $K_{F;1}$ can determine the metric $\alpha$ and help us prove the theorem.

For any nonzero $X \in \mathcal{U}$, we can find a pair $(X, X') \in K_{F;1}$. We have just proven $X'$ is a scalar multiple of $V$. The equality (4.11), with $g = e$, indicates the following
condition is satisfied,

$$\exists c \in \mathbb{R}, \text{ such that } \langle [Y, X], X - cv \rangle = 0, \forall Y \in \mathfrak{g}. \quad (4.29)$$

In fact $c$ can be determined by

$$cv = \frac{\phi(s) - s\phi'(s)}{\alpha(X - X')}X' - \phi'(s)v, \quad (4.30)$$

in which $s = \beta(X-X')/\alpha(X-X')$.

The next lemma indicates (4.29) can be satisfied for all $X \in \mathfrak{g}$, and it can define a linear function.

**Lemma 4.8** Keep all notations as before. Then there is a linear function $c(\cdot) : \mathfrak{g} \rightarrow \mathbb{R}$, such that

$$\langle [Y, X], X - c(X)v \rangle = 0, \forall Y \in \mathfrak{g} \quad (4.31)$$

**Proof.** We will first construct a function $c(X)$ satisfying (4.31). Then we will further refine it to be linear.

For any $X \in \mathfrak{g}$, let $t$ be a Cartan subalgebra containing $X$, and $\{X_1, \ldots, X_m\}$ a basis of $t$, in which each $X_i$ is a regular vector in $\mathcal{U} \cap t$. For each $X_i$, the corresponding $c_i = c(X_i)$ indicated by the lemma can be found. Assume $X = \sum_{i=1}^m a_iX_i$, then take $c = \sum_{i=1}^m a_ic_i$. Because $[\mathfrak{g}, X] \subset [\mathfrak{g}, X_i]$ for each $i$, for any $Y \in \mathfrak{g}$, we can find $Y_i \in \mathfrak{g}$ such that $[Y, X] = [Y_i, X_i]$, so we have

$$\langle [Y, X], X_i - c_i v \rangle = \langle [Y_i, X_i], X_i - c_i v \rangle = 0. \quad (4.32)$$

Take the linear combination of (4.32) for each $i$, we see the constant $c$ given above satisfies (4.31) for $X$, which can define the function $c(X)$ on $\mathfrak{g}$. If $X$ is contained by more than one Cartan subalgebra, and there are different $c_1$ and $c_2$ such that

$$\langle [Y, X], X - c_1 v \rangle = \langle [Y, X], X - c_2 v \rangle = 0, \forall Y \in \mathfrak{g}, \quad (4.33)$$

then it is easy to see $c(X) = 0$ satisfies (4.31).

Denote the dual of $\beta$ with respect to the bi-invariant metric as $\nu'$. Let $Y_0 \in \mathfrak{g}$ be any vector with $[Y_0, \nu'] \neq 0$, or equivalently $\langle [Y_0, X], V \rangle \neq 0$ for some $X$. From (4.31), the function $f_0(X) = \langle [Y_0, X], V \rangle$ vanishes on the codimension 1 linear subspace

$$\{X | \langle [Y_0, X], V \rangle = 0 \} \subset \mathfrak{g}. \quad (4.34)$$
This can only happen when \( f_0(X) \) splits as the product of two linear factors. Up to scalar multiplications, one is \( \langle [Y_0, X], V \rangle \), and the other is \( \tilde{c}(X) \) which coincides with \( c(X) \) on the nonempty open subset

\[
\{X\mid \langle [Y_0, X], V \rangle \neq 0\} \subset g. \tag{4.35}
\]

For \( X \) in this open set, we have

\[
\langle [Y, X], X \rangle = \langle [Y, X], \tilde{c}(X) V \rangle, \forall Y \in g, \tag{4.36}
\]

so it is still valid for all \( X, Y \in g \). With \( c(X) \) changed to \( \tilde{c}(X) \), we have finished the proof for the lemma. ■

Let \( l_0 : g \to g \) be the linear isomorphism defined by \( \langle X, Y \rangle = \langle X, l_0(Y) \rangle_{\text{bi}} \), and \( f : g \times g \to \mathbb{R} \) the bi-linear function defined by

\[
f(X, Y) = \langle X - c(X)V, Y \rangle = \langle l_0(X - c(X)V), Y \rangle_{\text{bi}}, \tag{4.37}
\]

in which \( c(\cdot) \) is the linear function indicated by Lemma 4.8. Let \( l_1(X) = l_0(X - c(X)V) \), then \( 4.31 \) indicates \( l_1 \) maps the regular vectors in any Cartan subalgebra \( t \) back to \( t \) itself. So it preserves each Cartan subalgebra. There is a nonzero vector \( X \in g \), such that \( \mathbb{R}X \) is the intersection of some Cartan subalgebras of \( g \). Then any vector on the \( \text{Ad}(G) \)-orbit \( O_X \) is an eigenvector of \( l_1 \). Because \( X \neq 0 \) and \( g \) is compact simple, this can only happen when \( l_1 \) is a scalar multiple of the identity map. So \( f(X, Y) \) is a bi-invariant inner product on \( g \).

Choose \( (X, Y) \in V_1 \times V_1 \), or \( (X, Y) \in V_2 \times V_1 \), in which \( V_1 = \ker \beta \) and \( V_2 = \mathbb{R}V \) for \( f(X, Y) \), we see immediately \( V_1 \) and \( V_2 \) are orthogonal with respect to both inner products from \( \alpha \) and the bi-invariant metric, and restricted to \( V_1 \), \( \alpha \) only differs from the bi-invariant metric by a scalar multiplication. To summarize we have

**Lemma 4.9** Keep all notations as above, then there are constants \( a \) and \( b \), such that \( \alpha^2(X) = a||X||^2_{\text{bi}} + b\beta^2(X) \).

By Lemma 4.9 the \((\alpha, \beta)\)-norm \( F \) on \( g \) can also be presented as \( F = \tilde{\alpha} \tilde{\phi}(\tilde{\beta}/\tilde{\alpha}) \), in which \( \tilde{\alpha} \) is the bi-invariant metric with \( \alpha^2(X) = a\tilde{\alpha}^2(X) + b\beta^2(X) \), \( \tilde{\beta} = \beta \), and \( \tilde{\phi}(s) = \sqrt{a + bs^2}\phi(s/\sqrt{a + bs^2}) \).

If there is a KVFCL of the form \((X, 0)\) with \( X \neq 0 \), then we have

\[
\tilde{\alpha}(\text{Ad}(g)X)\tilde{\phi}(\frac{\beta(\text{Ad}(g))}{\alpha(\text{Ad}(g)X)}) = \text{const}, \forall g \in G. \tag{4.38}
\]

Since \( \tilde{\alpha}(\text{Ad}(g)X) = \tilde{\alpha}(X) \) is a nonzero constant function of \( g \), and \( \beta(\text{Ad}(g)X) \) is not a constant function, the real analytic function \( \tilde{\phi} \) must be a constant function, i.e. the left invariant metric \( F \) must be a bi-invariant Riemannian metric. Though it is a contradiction with the assumption that \( F \) is non-Riemannian, it helps us with the discussion in the next case.

If there is a KVFCL of the form \((X, \lambda V)\) with \( X \neq 0 \) and \( \lambda \neq 0 \), and assume its \( F \)-length function is constantly 1, i.e.

\[
F(\text{Ad}(g)X - \lambda V) = 1, \forall g \in G. \tag{4.39}
\]
The strong convexity of $F$ implies $F(-\lambda V) < 1$. Applying a navigation transformation to $F$ which set the origin at $-\lambda V$, we get a new left invariant $(\alpha, \beta)$-metric $F'$. The $(\alpha, \beta)$-norm in $\mathfrak{g}$ defined by $F'$ is also denoted as $F'$. In $T_eG = \mathfrak{g}$, the indicatrix of $F'$ is a parallel shift of that of $F$, with $-\lambda V$ shifted to 0. While presenting $F'$, we can keep $\alpha$ and $\beta$ in the good normalized datum for $F$ and just change the function $\phi$. So any isometry of $(G, F)$, which preserves $\alpha$ and $\beta$, is also an isometry of $(G, F')$, and any Killing vector field of $(G, F)$ is still a Killing vector field of $(G, F')$. Because of (4.39),

$$F'(\text{Ad}(g)X) = 1, \forall g \in G,$$

(4.40)

so $(X, 0)$ defines Killing vector field of constant length 1 for $F'$.

There is another presentation of $F'$ using the bi-invariant $\tilde{\alpha}$ and $\tilde{\beta} = \beta$. By the discussion in the last case, $F'$ is a bi-invariant Riemannian metric and then $F$ is a Randers metric. This finishes the proof of Theorem 1.1 in the case that $G$ is a compact connected simple Lie group.

5 Proof of Theorem 1.1 for a compact connected semi-simple $G$

5.1 Notations and assumptions

Let $G$ be a compact connected semi-simple Lie group and $F$ be a left invariant restrictively CW-homogeneous $(\alpha, \beta)$-metric on $G$. There is no harm that we assume $F$ is non-Riemannian, otherwise the main theorem needs no proof.

When $G$ is not simply-connected, there is a connected simply-connected $\hat{G}$ covering $G$. Let $\hat{F}$ be the induced metric on $\hat{G}$, then $\hat{F}$ is also a non-Riemannian left invariant $(\alpha, \beta)$-metric. Any KVFCF for $(G, F)$ induces a KVFCF for $(\hat{G}, \hat{F})$ which exhausts all tangent vectors of $\hat{G}$ as well as $G$. So by Proposition 2.5 $\hat{F}$ is also restrictively CW-homogeneous. We only need to prove $\hat{F}$ is Randers, then so does $F$. So we will further assume $G$ is simply-connected.

We wish $I_0(G, F)$ be contained by $L(G)R(G)$, then we can have an explicit description of Killing vector fields, study the set of all KVFCFLs, and then the restrictive CW-homogeneity. Though it may not be correct when we consider semi-simple $G$ rather than the simple ones, we can change $F$ to $F'$ by a diffeomorphism, such that

$$L(G) \subset I_0(G, F') \subset L(G)R(G)$$

(5.41)

is satisfied.

Lemma 5.1 Let $F$ be a left invariant Finsler metric on the compact connected simply-connected group $G$, then there is a diffeomorphism $f$ on $G$, such that $F' = f^*F$ satisfies $L(G) \subset I_0(G, F') \subset L(G)R(G)$.

Proof. The proof is very similar to the one in the Riemannian case. Let $\mathfrak{k}$ be the Lie algebra of $I_0(G, F)$, then we have a linear space decomposition $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$, in which $\mathfrak{g}$ is in fact the Lie algebra of $L(G)$. Ozeki’s theorem [OZ77] states that we can find an ideal $\mathfrak{g}'$ of $\mathfrak{k}$ which is isometric to $\mathfrak{g}$ and $\mathfrak{g}' \cap \mathfrak{h} = 0$. Let $G'$ be the subgroup
of $I_0(G, F)$ corresponding to $g'$. It acts transitively on $G$. The map $\tilde{f} : G' \to G$ by $\tilde{f}(g') = g'(e), \forall g' \in G'$ is a covering map. By the simply-connectedness of $G$, it is a constant function of $g$. We keep all notations and assumptions as in the last subsection.

5.2 Finishing the Proof of Theorem 1.1

Obviously $F' = f^*F$ is a non-Riemannian metric, an $(\alpha, \beta)$-metric or a Randers metric if and only if $F$ is respectively. Any KVFCFL $X$ of $(G, F)$ one-to-one corresponds to the KVFCFL $f^{-1}X$ of $(G, f^*F)$, so $F$ is restrictively CW-homogeneous if and only if $F$ is. So we only need to prove the main theorem with the condition $I_0(G, F) \subset L(G)R(G)$.

To summarize, we only need to prove Theorem 1.1 with the following assumptions: $G$ is not simple, $G$ is simply-connected, $F$ is non-Riemannian, and $I_0(G, F) \subset L(G)R(G)$.

The space of Killing vector fields for $(G, F)$ can then be presented explicitly. Let $G'$ be the closed connected subgroup of $G$ such that $R(G')$ is the maximal connected subgroup of isometric right translations, and $g' = \text{Lie}(G')$, then the Lie algebra of $I_0(G, F)$, i.e. the space of Killing vector fields for $(G, F)$ is the direct sum $g \oplus g'$, in which $g$ corresponds to left translations and $g'$ corresponds to isometric right translations. Denote $\beta(u) = \langle u, v \rangle = \langle u, v' \rangle_{bi}, \forall u \in g$, then $g'$ is a subalgebra of $\mathfrak{c}_g(v)$ and $\mathfrak{c}_g(v')$.

Evaluation of the $F$-length function of a Killing vector field $(X, X') \in g \oplus g'$ at the point $g'' = gg'^{-1} \in G$, in which $g \in G$ and $g' \in G'$, is $F(\text{Ad}(g)X - \text{Ad}(g')X')$. Because $F$ is $\text{Ad}(G')$-invariant in $g = TG_e$, there is no contradiction when we use different $g$ and $g'$.

5.2 Finishing the Proof of Theorem 1.1

We keep all notations and assumptions as in the last subsection.

Let $g = g_1 \oplus g_2$ be any direct sum of nontrivial ideals and correspondingly $G = G_1 \times G_2$ the product of closed subgroups. On $G_1$ (or $G_2$), the metric $F$ induces a left invariant $(\alpha, \beta)$-metric $F|_{G_1}$. When $F$ is restrictively CW-homogeneous, then its restriction on $G_1$ is also restrictively CW-homogeneous. To see this, choose any nonzero tangent vector $X''_i \in g_1 = T_eG_1$, we can extended it to a KVFCFL of $(G, F)$, defined by

$$X = (X_1, X_2, X'_1, X'_2) \in g \oplus g' \subset g_1 \oplus g_2 \oplus g_1 \oplus g_2,$$

in which $X_1$ and $X_2$ are for left translations, and $X'_1$ and $X'_2$ are for right translations, on $G_1$ and $G_2$ respectively, $X_1 - X'_1 = X''_1$, and $X_2 = X'_2$. It is of constant length implies for any $(g'_1, g'_2) \in G'$, and any $(g_1, g_2) \in G$ with $g_2 = g'_2$, we have

$$F((\text{Ad}(g_1)X_1 - \text{Ad}(g'_1)X'_1, \text{Ad}(g_2)X_2 - \text{Ad}(g'_2)X'_2)) = F((\text{Ad}(g_1)X_1 - \text{Ad}(g'_1)X'_1, 0))$$

is a constant function of $g_1$ and $g'_1$. So $(X_1, X'_1)$ defines a KVFCFL of $(G_1, F|_{G_1})$. It can exhaust all tangent vectors $X''_1 = X_1 - X'_1$, so the restriction of $F$ to $G_1$ is restrictively CW-homogeneous. We have proven in the last section that $F|_{G_1}$ is Randers. Let
$(\phi, \alpha, \beta)$ be a good normalized datum for $F$, then only the values of $\phi$ for $s \in [-1,1]$ are used to define $F$.

Let $v = v_1 + v_2$ be the decomposition of the $\alpha$-dual of $\beta$ with respect to the decomposition of $g$. If $v_1 \neq 0$ and $v_2 = 0$, then $(\phi, \alpha|_{G_1}, \beta|_{G_1})$ is a good normalized datum of $F|_{G_1}$, i.e. the pointwise norms $||\beta|_{G_1}||_{a|G_1}(\cdot)$ are constantly $1$, and all values of $\phi$ for $s \in [-1,1]$ are used to define $F|_{G_1}$. When $F|_{G_1}$ is Randers, $\phi(s) = \sqrt{k_1 + k_2 s^2 + k_3}$ for some constants $k_1$, $k_2$ and $k_3$, $\forall s \in [-1,1]$, i.e. the same function $\phi$ defines a Randers metric $F$ on $G$.

By the observations above, we can prove Theorem 1 for semi-simples $G$ by mathematical induction. As we have mentioned above, we can assume $G$ is a compact connected simply connected Lie group, and the left invariant restrictively CW-homogeneous $F$ is non-Riemannian.

Let $G = G_1 \times G_2 \times \cdots \times G_n$, in which all $G_i$s are nontrivial simple Lie groups. Correspondingly we have the direct sum decomposition $g = g_1 \oplus \cdots \oplus g_n$ for the Lie algebra.

When $n = 1$, i.e. $G$ is simple, we have proven $F$ is a Randers metric in the last section. Assume we can prove $F$ is a Randers metric when $n = k$, then we need to prove $F$ is a Randers metric for $n = k + 1 > 1$.

As we have argued, we only need to prove the statement with the assumption $F$ is non-Riemannian and $L(G) \subset I_0(G, F) \subset L(G)R(G)$. The space of Killing vector fields of $(G, F)$ can be identified as a direct sum of Lie algebras $g \oplus g'$. Let $v = v_1 + \cdots + v_n$ with respect to the decomposition of $g$, then

$$g' \subset c_g(v) = \oplus_{i=1}^{n_i} c_{g_i}(v_i). \quad (5.43)$$

By the inductive assumption, $F|_{G_1 \times \cdots \times G_{n-1}}$ is Randers. If any $v_i = 0$, for example $v_n = 0$, by the above argument, we have seen the metric $F$ on $G$ is also Randers which finished mathematical induction. Now we will assume $v_i \neq 0$, $\forall i = 1,2,\ldots,n$. Then we have

**Lemma 5.2** Keep all notations and assumptions for $G$ and $F$ as above, let $(\phi, \alpha, \beta)$ be a good normalized datum of $F$, and assume $v = v_1 + \cdots + v_n$, $v_i \neq 0$, $\forall i = 1,2,\ldots,n$ for the $\alpha$-dual $v$ of $\beta$ then the function $\phi$ is real analytic in $(-1,1)$.

**Proof.** Let $g_0 \neq g$ be the largest ideal of $g$ contained by $\ker \beta$. For any KVFCFL given by $(X, X') \in g \oplus g'$, with $X \notin g_0$, the range $I_{(X,X')}$ of

$$s(g) = \beta(\text{Ad}(g)X - X')/\alpha(\text{Ad}(g)X - X'), \forall g \in G, \quad (5.44)$$

is a closed interval with positive length. Otherwise, $s(g)$ is a constant function of $g$, and then so is $\beta(\text{Ad}(g)X - X')$. It implies the ideal generated by $[X, g]$ is contained in $\ker \beta$, which is a contradiction with that $X \notin g_0$.

Consider the open subset

$$\mathcal{U} = g \setminus (g_0 + g') \quad (5.45)$$

in $g$. By (5.43), $g_0 + g'$ has a codimensions bigger than 1 in $g$, so $\mathcal{U}$ is a connected dense open subset of $g$. For any $s_0 \in (-1,1)$, we can find a tangent vector $u \in \mathcal{U} \subset T_eG$ such that $\beta(u)/\alpha(u) = s_0$. Let $(X, X')$ be a KVFCFL which value at $e$ is $u$, then $X \neq 0$ and
Because $X - X' = u \notin g_0 + g'$, we have $X \notin g_0$, i.e. $I_{(X,X')}$ is a closed interval with a positive length. Using only these KVFCIs, the proof can be carried out exactly as the one for (1) of Lemma 4.6.

The next lemma indicates the real analytic property of $\phi$ guarantees $\phi$ defines a Randers norm in $g = T_eG$, and by the homogeneity of $(G, F)$, finishes the mathematical induction.

**Lemma 5.3** Let $F$ be an $(\alpha, \beta)$-norm on $V$, $\dim V = n > 2$, which is non-Riemannian. Assume $(\phi, \alpha, \beta)$ is a normalized datum defining $F$, in which $\phi \in C^\infty[-1, 1]$ is real analytic on $(-1, 1)$, and there is a linear subspace $V' \in V$, $\dim V' = m > 1$, such that $V'$ is not contained by $\ker \beta$ and the restriction of $F$ in $V'$ is Randers, then the norm $F$ on $V$ is Randers.

**Proof.** Let the restrictions of $F, \alpha$ and $\beta$ in $V'$ be $\tilde{F}, \tilde{\alpha}$ and $\tilde{\beta}$ respectively. Because $\ker \beta$ does not contain $V'$, i.e. $\tilde{\beta}$ is not constantly 0 on $V'$. Direct calculation indicates if $\tilde{F} = \tilde{\alpha} \phi (\tilde{\beta} / \tilde{\alpha})$ is Randers, then $\phi (s) = \sqrt{k_1 + k_2 s^2 + k_3 s}$ around $s = 0$, for some constants $k_1, k_2$ and $k_3$. Because $\phi$ is real analytic on $(-1, 1)$ and smooth on $[-1, 1]$, it must satisfy the same formula for all $s \in [-1, 1]$, which implies $F$ is Randers.

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