FACE NUMBERS OF
BARYCENTRIC SUBDIVISIONS OF CUBICAL COMPLEXES

BY

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ABSTRACT

The $h$-polynomial of the barycentric subdivision of any $n$-dimensional cubical complex with nonnegative cubical $h$-vector is shown to have only real roots and to be interlaced by the Eulerian polynomial of type $B_n$. This result applies to barycentric subdivisions of shellable cubical complexes and, in particular, to barycentric subdivisions of cubical convex polytopes and answers affirmatively a question of Brenti, Mohammadi and Welker.

1. Introduction

A fundamental problem in algebraic and geometric combinatorics is to characterize, or at least obtain significant information about, the face enumerating vectors of triangulations of various topological spaces, such as balls and spheres [28]. Face enumerating vectors are often presented in the form of the $h$-polynomial (see Section 2 for definitions). Properties such as unimodality, log-concavity, $\gamma$-positivity and real-rootedness have been of primary interest [3, 8, 10, 26]. One expects that the ‘nicer’ the triangulation is combinatorially and the space being triangulated is topologically, the better the behavior of the $h$-polynomial is.
Following this line of thought, Brenti and Welker [11] considered an important and well studied triangulation in mathematics, namely the barycentric subdivision. They studied the transformation of the \( h \)-polynomial of a simplicial complex \( \Delta \) under barycentric subdivision and showed that the resulting \( h \)-polynomial has only real roots (a property with strong implications) for every simplicial complex \( \Delta \) with nonnegative \( h \)-polynomial. They asked [11, Question 3.10] whether the \( h \)-polynomial of the barycentric subdivision of any convex polytope has only real roots, suspecting an affirmative answer (see [21, p. 105]). This question was raised again by Mohammadi and Welker [21, Question 35] and, as is typically the case in face enumeration, it is far more interesting and more challenging for general polytopes and polyhedral complexes, than it is for simplicial polytopes and simplicial complexes. Somewhat surprisingly, no strong evidence has been provided in the literature that such a result may (or may not) hold beyond the simplicial setting. One should also note that barycentric subdivisions of boundary complexes of polytopes form a special class of flag triangulations of spheres and that the real-rootedness property fails for the \( h \)-polynomials of this more general class of triangulations in dimensions higher than four [14]. At present, it is unclear where the borderline between positive and negative results lies.

Mohammadi and Welker (based on earlier discussions with Brenti) suggested the class of cubical polytopes as another good test case; see [21, p. 105]. Cubical complexes and polytopes are important and mysterious objects with highly nontrivial combinatorial properties (see, for instance, [1, 5, 18, 19]). They have been studied both for their own independent interest, and for the role they play in other areas of mathematics. Given the intricacy of their combinatorics, it comes as no surprise that the question of Brenti and Welker turns out to be more difficult for them than for simplicial complexes. The following theorem provides the first general positive result on this question, since [11] appeared, and suggests that an affirmative answer should be expected at least for broad classes of nonsimplicial convex polytopes (or even more general cell complexes and posets).

**Theorem 1.1:** The \( h \)-polynomial of the barycentric subdivision of any shellable cubical complex has only real roots. In particular, barycentric subdivisions of cubical polytopes have this property.
The case of cubical polytopes was also studied recently by Hlavacex and Solus [17]. Using the concept of shellability and the theory of interlacing polynomials, they gave an affirmative answer for cubical complexes which admit a special type of shelling and applied their result to certain families of cubical polytopes, such as cuboids, capped cubical polytopes and neighborly cubical polytopes.

The proof of the aforementioned result of [11] applies a theorem of Brändén [7] on the subdivision operator [8, Section 3.3] to a formula for the $h$-polynomial of the barycentric subdivision of a simplicial complex (see Remark 3.7). The proof of Theorem 1.1 is motivated by the proof of the result of [11], given and extended to the setting of uniform triangulations of simplicial complexes in [4] (the latter was partially motivated by [8, Example 8.1]). To explain further, we let

$$h(\Delta, x) = \sum_{i=0}^{n+1} h_i(\Delta)x^i$$

denote the $h$-polynomial and $\text{sd}(\Delta)$ denote the barycentric subdivision of an $n$-dimensional simplicial complex $\Delta$. As already shown in [11], there exist polynomials with nonnegative coefficients $p_{n,k}(x)$ for $k \in \{0, 1, \ldots, n + 1\}$, which depend only on $n$ and $k$, such that

$$h(\text{sd}(\Delta), x) = \sum_{k=0}^{n+1} h_k(\Delta)p_{n,k}(x)$$

for every $n$-dimensional simplicial complex $\Delta$. For every $n \in \mathbb{N}$, the polynomials $p_{n,k}(x)$ can be shown [8, Example 8.1] to have only real roots and to form an interlacing sequence. This implies that their nonnegative linear combination $h(\text{sd}(\Delta), x)$ also has only real (negative) roots and that it is interlaced by $p_{n,0}(x)$, which equals the classical $(n + 1)$st Eulerian polynomial $A_{n+1}(x)$ [29, Section 1.4]. The interlacing condition implies that the roots of $h(\text{sd}(\Delta), x)$ are not arbitrary, but rather that they lie in certain intervals that depend only on the dimension $n$, formed by zero and the roots of $A_{n+1}(x)$. The polynomial $p_{n,k}(x)$ can be interpreted as the $h$-polynomial of the relative simplicial complex obtained from the barycentric subdivision of the $n$-dimensional simplex by removing all faces lying on $k$ facets of the simplex [4, Section 5], [17, Section 4.2].
This paper presents a similar picture for cubical complexes. We define (see Definition 3.1) polynomials $p_{n,k}^B(x)$ for $k \in \{0, 1, \ldots, n+1\}$ as the $h$-polynomials of relative simplicial complexes obtained from the barycentric subdivision of the $n$-dimensional cube by removing all faces lying on certain facets of the cube and prove (see Theorem 3.2) that Equation (1) continues to hold when $\Delta$ is replaced by an $n$-dimensional cubical complex $L$, $p_{n,k}(x)$ is replaced by $p_{n,k}^B(x)$ and the $h_k(\Delta)$ are replaced by the entries of the (normalized) cubical $h$-vector of $L$, introduced and studied by Adin [1]. We provide recurrences (see Proposition 3.3) for the polynomials $p_{n,k}^B(x)$ which guarantee that they form an interlacing sequence for every $n \in \mathbb{N}$ and conclude that $h(\text{sd}(L), x)$ has only real (negative) roots and that it is interlaced by the $n$th Eulerian polynomial $B_n(x)$ of type $B$ for every $n$-dimensional cubical complex $L$ with nonnegative cubical $h$-vector (see Corollary 3.5). This implies Theorem 1.1, since shellable cubical complexes were shown [1] to have nonnegative cubical $h$-vector and boundary complexes of convex polytopes are shellable [12].

The main results of this paper apply to cubical regular cell complexes (equivalently, to cubical posets) and will be stated at this level of generality. What comes perhaps unexpectedly is the fact that the transformation of a cubical $h$-polynomial into a simplicial one can be so well behaved. Corollary 3.5 has nontrivial applications to triangulations of simplicial complexes as well; see Remark 3.6.

2. Face enumeration of simplicial and cubical complexes

This section recalls some definitions and background on the face enumeration of simplicial and cubical complexes and their triangulations, and shellability. For more information and any undefined terminology, we recommend the books [16, 28]. All cell complexes considered here are assumed to be finite. Throughout this paper, we set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and denote by $|S|$ the cardinality of a finite set $S$.

2.1. SIMPLICIAL COMPLEXES. An $n$-dimensional relative simplicial complex [28, Section III.7] is a pair $(\Delta, \Gamma)$, denoted $\Delta/\Gamma$, where $\Delta$ is an (abstract) $n$-dimensional simplicial complex and $\Gamma$ is a subcomplex of $\Delta$. The $f$-polynomial of $\Delta/\Gamma$ is defined as

$$f(\Delta/\Gamma, x) = \sum_{i=0}^{n+1} f_{i-1}(\Delta/\Gamma)x^i,$$
where \( f_j(\Delta/\Gamma) \) is the number of \( j \)-dimensional faces of \( \Delta \) which do not belong to \( \Gamma \). The \textbf{\( h \)-polynomial} is defined as

\[
h(\Delta/\Gamma, x) = (1 - x)^{n+1} f(\Delta/\Gamma, \frac{x}{1-x}) = \sum_{i=0}^{n+1} f_{n-i}(\Delta/\Gamma) x^i (1 - x)^{n+1-i}
\]

\[
= \sum_{F \in \Delta/\Gamma} x^{|F|} (1 - x)^{n+1-|F|} := \sum_{k=0}^{n+1} h_k(\Delta/\Gamma) x^k
\]

and the sequence

\[
h(\Delta/\Gamma) := (h_0(\Delta/\Gamma), h_1(\Delta/\Gamma), \ldots, h_{n+1}(\Delta/\Gamma))
\]

is called the \textbf{\( h \)-vector} of \( \Delta/\Gamma \). Note that \( f(\Delta/\Gamma, x) \) has only real roots if and only if so does \( h(\Delta/\Gamma, x) \). When \( \Gamma \) is empty, we get the corresponding invariants of \( \Delta \) and drop \( \Gamma \) from the notation. Thus, for example, \( h(\Delta, x) = \sum_{k=0}^{n+1} h_k(\Delta) x^k \) is the (usual) \( h \)-polynomial of \( \Delta \).

Suppose now that \( \Delta \) triangulates an \( n \)-dimensional ball. Then, the boundary complex \( \partial \Delta \) is a triangulation of an \((n-1)\)-dimensional sphere and the \textbf{interior \( h \)-polynomial} of \( \Delta \) is defined as \( h^\circ(\Delta, x) = h(\Delta/\partial \Delta, x) \). The following statement is a special case of [25, Lemma 6.2].

\[ \text{Proposition 2.1 ([25])}: \text{ Let } \Delta \text{ be a triangulation of an } n \text{-dimensional ball. Let } \Gamma \text{ be a subcomplex of } \partial \Delta \text{ which is homeomorphic to an } (n-1) \text{-dimensional ball and } \bar{\Gamma} \text{ be the subcomplex of } \partial \Delta \text{ whose facets are those of } \partial \Delta \text{ which do not belong to } \Gamma. \text{ Then, } h(\Delta/\bar{\Gamma}, x) = x^{n+1} h(\Delta/\Gamma, 1/x). \]

Moreover, \( h^\circ(\Delta, x) = x^{n+1} h(\Delta, 1/x) \).

2.2. Cubical complexes. A \textbf{regular cell complex} [6, Section 4.7] is a (finite) collection \( \mathcal{L} \) of subspaces of a Hausdorff space \( X \), called \textbf{cells} or \textbf{faces}, each homeomorphic to a closed unit ball in some finite-dimensional Euclidean space, such that: (a) \( \emptyset \in \mathcal{L} \); (b) the relative interiors of the nonempty cells partition \( X \); and (c) the boundary of any cell in \( \mathcal{L} \) is a union of cells in \( \mathcal{L} \).

The \textbf{boundary complex} of \( \sigma \in \mathcal{L} \), denoted by \( \partial \sigma \), is defined as the regular cell complex consisting of all faces of \( \mathcal{L} \) properly contained in \( \sigma \). A regular cell complex \( \mathcal{L} \) is called \textbf{cubical} if every nonempty face of \( \mathcal{L} \) is combinatorially isomorphic to a cube. A convex polytope is called \textbf{cubical} if so is its boundary complex.
Given a cubical complex $L$ of dimension $n$, we denote by $f_k(L)$ the number of $k$-dimensional faces of $L$. The cubical $h$-polynomial was introduced and studied by Adin [1] as a (well behaved) analogue of the (simplicial) $h$-polynomial of a simplicial complex. Following [13, Section 4], we define the (normalized) cubical $h$-polynomial of $L$ as

$$(1 + x)h(L, x) = 1 + \sum_{k=0}^{n} f_k(L)x^{k+1}\left(\frac{1-x}{2}\right)^{n-k} + (-1)^n\tilde{\chi}(L)x^{n+2},$$

where $\tilde{\chi}(L) = -1 + \sum_{k=0}^{n}(-1)^kf_k(L)$ is the reduced Euler characteristic of $L$ (the only difference from Adin’s definition is that all coefficients of $h(L, x)$ have been divided by $2^n$ and, therefore, are not necessarily integers). We note that $h(L, x)$ is indeed a polynomial in $x$ of degree at most $n + 1$. The (normalized) cubical $h$-vector of $L$ is the sequence $h(L) = (h_0(L), h_1(L), \ldots, h_{n+1}(L))$, where $h(L, x) = \sum_{k=0}^{n+1} h_k(L)x^k$.

Adin showed that $h(L, x)$ has nonnegative coefficients for every shellable cubical complex $L$ [1, Theorem 5 (iii)] (his result is stated for abstract cubical complexes with the intersection property, but the proof is valid without assuming the latter). He asked whether the same holds whenever $L$ is Cohen–Macaulay [1, Question 1]. The coefficient $h_k(L)$ is known to be nonnegative for every Cohen–Macaulay $L$ for $k \in \{0, 1\}$, since $h_0(L) = 1$ and $h_1(L) = (f_0(L) - 2^n)/2^n$, for $k = n$ [2, Corollary 1.2] and for $k = n + 1$, since $h_{n+1}(L) = (-1)^n\tilde{\chi}(L)$, and for every $k$ in the special case that $L$ is the cubical barycentric subdivision of a Cohen–Macaulay simplicial complex [15] (see also Remark 3.6).

### 2.3. Barycentric Subdivision and Shellability

The barycentric subdivision of a regular cell complex $L$ is denoted by $\text{sd}(L)$ and defined as the abstract simplicial complex whose faces are the chains $\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_k$ of nonempty faces of $L$. The natural restriction of $\text{sd}(L)$ to a nonempty face $\sigma \in L$ is exactly the barycentric subdivision $\text{sd}(\sigma)$ of (the complex of faces of) $\sigma$.

Similarly, by the barycentric subdivision $\text{sd}(Q)$ of a convex polytope $Q$ we mean that of the complex of faces of $Q$. Since $\text{sd}(Q)$ is a cone over $\text{sd}(\partial Q)$, we have

$$h(\text{sd}(Q), x) = h(\text{sd}(\partial Q), x).$$

For the $n$-dimensional cube $Q$ we have

$$h(\text{sd}(Q), x) = B_n(x),$$
where $B_n(x)$ is the Eulerian polynomial which counts signed permutations of \{1, 2, \ldots, n\} by the number of descents of type $B$; see, for instance, [24, Chapter 11]. The following well known type $B$ analogue of Worpitzky’s identity [24, Equation (13.3)]

\[
\frac{B_n(x)}{(1 - x)^{n+1}} = \sum_{m \geq 0} (2m + 1)^n x^m
\]

will make computations in the following section easier.

A regular cell complex $\mathcal{L}$ is called pure if all its facets (faces which are maximal with respect to inclusion) have the same dimension. Such a complex $\mathcal{L}$ is called shellable if either it is zero-dimensional, or else there exists a linear ordering $\tau_1, \tau_2, \ldots, \tau_m$ of its facets, called a shelling, such that (a) $\partial\tau_1$ is shellable; and (b) for $2 \leq j \leq m$, the complex of faces of $\partial\tau_j$ which are contained in $\tau_1 \cup \tau_2 \cup \cdots \cup \tau_{j-1}$ is pure, of the same dimension as $\partial\tau_j$, and there exists a shelling of $\partial\tau_j$ for which the facets of $\partial\tau_j$ contained in $\tau_1 \cup \tau_2 \cup \cdots \cup \tau_{j-1}$ form an initial segment. A fundamental result of Bruggesser and Mani [12] states that $\partial Q$ is shellable for every convex polytope $Q$. For the shellability of cubical complexes in particular, see [13, Section 3], [17, Section 3].

### 3. The $h$-vector transformation

This section studies the transformation which maps the cubical $h$-vector of a cubical complex $\mathcal{L}$ to the (simplicial) $h$-vector of the barycentric subdivision $\text{sd}(\mathcal{L})$ and deduces Theorem 1.1 from its properties. We begin with an important definition.

**Definition 3.1:** For $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n + 1\}$ we denote by $\mathcal{C}_{n,k}$ the relative simplicial complex which is obtained from the barycentric subdivision of the $n$-dimensional cube by removing

- no face, if $k = 0$,
- all faces which lie in one facet and $k - 1$ pairs of antipodal facets of the cube (making a total of $2k - 1$ facets), if $k \in \{1, 2, \ldots, n\}$,
- all faces on the boundary of the cube, if $k = n + 1$. 


We define
\[ p_{n,k}^B(x) = h(C_{n,k}, x) \]
for \( k \in \{0, n+1\} \), and
\[ p_{n,k}^B(x) = 2h(C_{n,k}, x) \]
for \( k \in \{1, 2, \ldots, n\} \).

The polynomials \( p_{n,k}^B(x) \) are shown on Table 1 for \( n \leq 3 \). For \( n = 4 \),
\[
p_{4,k}^B(x) = \begin{cases} 
1 + 76x + 230x^2 + 76x^3 + x^4, & \text{if } k = 0, \\
108x + 460x^2 + 196x^3 + 4x^4, & \text{if } k = 1, \\
36x + 420x^2 + 300x^3 + 12x^4, & \text{if } k = 2, \\
12x + 300x^2 + 420x^3 + 36x^4, & \text{if } k = 3, \\
4x + 196x^2 + 460x^3 + 108x^4, & \text{if } k = 4, \\
x + 76x^2 + 230x^3 + 76x^4 + x^5, & \text{if } k = 5. 
\end{cases}
\]

Their significance stems from the following theorem.

**Theorem 3.2:** For every \( n \)-dimensional cubical complex \( L \),
\[
h(sd(L), x) = \sum_{k=0}^{n+1} h_k(L)p_{n,k}^B(x).
\]

| \( n \) | \( k = 0 \) | \( k = 1 \) | \( k = 2 \) |
|-------|----------|----------|----------|
| 0     | 1        |          |          |
| 1     | 1 + x    | 4x       | \( x + x^2 \) |
| 2     | 1 + 6x + x^2 | 12x + 4x^2 | 4x + 12x^2 |
| 3     | 1 + 23x + 23x^2 + x^3 | 36x + 56x^2 + 4x^3 | 12x + 72x^2 + 12x^3 |

| \( n \) | \( k = 3 \) | \( k = 4 \) |
|-------|----------|----------|
| 0     |          |          |
| 1     |          |          |
| 2     | \( x + 6x^2 + x^3 \) |          |
| 3     | \( 4x + 56x^2 + 36x^3 \) | \( x + 23x^2 + 23x^3 + x^4 \) |
The proof requires a few preliminary results. We first summarize some of the main properties of $p_{n,k}^B(x)$.

**Proposition 3.3:** For every $n \in \mathbb{N}$:

(a) The polynomial $p_{n,k}^B(x)$ has nonnegative coefficients for every $k \in \{0, 1, \ldots, n+1\}$; its degree is equal to $n + 1$, if $k = n + 1$, and to $n$ otherwise.

(b) $p_{n,n+1-k}^B(x) = x^{n+1}p_{n,k}^B(1/x)$ for every $k \in \{0, 1, \ldots, n+1\}$.

(c) $p_{n,0}^B(x) = B_n(x)$, $p_{n,n+1}^B(x) = xB_n(x)$ and $\sum_{k=0}^{n+1} p_{n,k}^B(x) = B_{n+1}(x)$.

(d) We have

\[
p_{n+1,k+1}^B(x) = \begin{cases} 
2p_{n+1,0}^B(x) + 2(x-1)p_{n,0}^B(x), & \text{if } k = 0, \\
p_{n+1,k}^B(x) + 2(x-1)p_{n,k}^B(x), & \text{if } 1 \leq k \leq n, \\
(1/2) \cdot p_{n+1,n+1}^B(x) + (x-1)p_{n,n+1}^B(x), & \text{if } k = n+1.
\end{cases}
\]

(e) The recurrence

\[
p_{n+1,k}^B(x) = \begin{cases} 
\sum_{i=0}^{n+1} p_{n,i}^B(x), & \text{if } k = 0, \\
2x\sum_{i=0}^{k-1} p_{n,i}^B(x) + 2\sum_{i=k}^{n+1} p_{n,i}^B(x), & \text{if } 1 \leq k \leq n+1, \\
x\sum_{i=0}^{n+1} p_{n,i}^B(x), & \text{if } k = n+2
\end{cases}
\]

holds for $k \in \{0, 1, \ldots, n+1\}$.

(f) We have

\[
p_{n,k}^B(x)/(1-x)^n = \begin{cases} 
\sum_{m \geq 0} (2m+1)^n x^m, & \text{if } k = 0, \\
\sum_{m \geq 0} (4m)(2m-1)^{k-1}(2m+1)^n x^m, & \text{if } 1 \leq k \leq n, \\
\sum_{m \geq 1} (2m-1)^n x^m, & \text{if } k = n+1.
\end{cases}
\]

**Proof.** We first note that, as discussed in Section 2.3,

\[p_{n,0}^B(x) = h(C_{n,0}, x) = B_n(x).
\]

Part (d) follows from Definition 3.1 and the definition of the $h$-polynomial of a relative simplicial complex. Indeed, for $1 \leq k \leq n$, we have

\[f(C_{n+1,k+1}, x) = f(C_{n+1,k}, x) - 2f(C_{n,k}, x).
\]
Hence,
\[
h(C_{n+1,k+1}, x) = (1-x)^{n+2} f\left(C_{n+1,k+1}, \frac{x}{1-x}\right)
= (1-x)^{n+2} f\left(C_{n+1,k}, \frac{x}{1-x}\right) - 2(1-x) \cdot (1-x)^{n+1} f\left(C_{n,k}, \frac{x}{1-x}\right)
= h(C_{n+1,k}, x) + 2(x-1)h(C_{n,k}, x)
\]
and
\[
p^B_{n+1,k+1}(x) = 2h(C_{n+1,k+1}, x)
= 2h(C_{n+1,k}, x) + 4(x-1)h(C_{n,k}, x)
= p^B_{n+1,k}(x) + 2(x-1)p^B_{n,k}(x).
\]

The same argument, similar to that in the proof of [4, Corollary 5.6], works for \(k \in \{0, n+1\}\). Part (f) follows from part (d) by straightforward induction on \(k\) (for fixed \(n\)), where the base \(k = 0\) of the induction holds because of Equation (3).

For part (c) we first note that
\[
p^B_{n,n+1}(x) = h(C_{n,n+1}, x) = h^\circ(C_{n,0}, x)
= x^{n+1} h(C_{n,0}, 1/x) = x^{n+1} B_n(1/x) = xB_n(x).
\]

The identity for the sum of the \(p^B_{n,k}(x)\) can be verified directly by summing that of part (f). For a more conceptual proof, one can use an obvious shelling of the boundary complex of the \((n+1)\)-dimensional cube to write, as explained in [17, Section 3], the \(h\)-polynomial \(B_{n+1}(x)\) of its barycentric subdivision as a sum of \(h\)-polynomials of relative simplicial complexes, each one combinatorially isomorphic to one of the \(C_{n,k}\). The details are left to the interested reader.

Given (c), the recursion of part (e) follows easily by induction on \(k\) from part (d) (this parallels the proof of [4, Lemma 6.3]).

Part (b) is a consequence of Definition 3.1 and Proposition 2.1. Alternatively, it follows from part (f) and standard facts about rational generating functions; see [29, Proposition 4.2.3]. The nonnegativity of the coefficients of \(p^B_{n,k}(x)\), claimed in part (a), follows from the recursion of part (e), as well as from general results [28, Corollary III.7.3] on the nonnegativity of \(h\)-vectors of Cohen–Macaulay relative simplicial complexes. The statement about the degree of \(p^B_{n,k}(x)\), claimed there, follows from either of parts (d), (e) or (f).
We leave the problem to find a combinatorial interpretation of \( p_{B_{n,k}}^B(x) \) open. Given part (c) of the proposition, one naturally expects that there is such an interpretation which refines one of the known combinatorial interpretations of \( B_{n+1}(x) \).

The following statement is a consequence of a more general result [27, Proposition 7.6] of Stanley on subdivisions of CW-posets. To keep this paper self-contained, we include a proof.

**Proposition 3.4**: For every \( n \)-dimensional cubical complex \( L \),

\[
h(sd(L), x) = (1 - x)^{n+1} + x \sum_{k=0}^{n} f_k(L)(1 - x)^{n-k}B_k(x).
\]

**Proof.** Since every face of \( sd(L) \) is an interior face of the restriction \( sd(\sigma) \) of \( sd(L) \) to a unique face \( \sigma \in L \), we have

\[
f(sd(L), x) = \sum_{\sigma \in L} f^\circ(\sigma, x) = 1 + \sum_{\sigma \in L \setminus \{\emptyset\}} f^\circ(\sigma, x).
\]

Transforming \( f \)-polynomials to \( h \)-polynomials in this equation and recalling from Section 2 that

\[
h^\circ(\sigma, x) = x^{k+1}h(\sigma, 1/x) = x^{k+1}B_k(1/x) = xB_k(x)
\]

for every nonempty \( k \)-dimensional face \( \sigma \in L \), we get

\[
h(sd(L), x) = (1 - x)^{n+1} f\left(sd(L), \frac{x}{1-x}\right)
\]

\[
= (1 - x)^{n+1} + \sum_{\sigma \in L \setminus \{\emptyset\}} (1 - x)^{n+1} f^\circ\left(sd(\sigma), \frac{x}{1-x}\right)
\]

\[
= (1 - x)^{n+1} + \sum_{\sigma \in L \setminus \{\emptyset\}} (1 - x)^{n-\text{dim}(\sigma)}h^\circ(\sigma, x)
\]

\[
= (1 - x)^{n+1} + \sum_{k=0}^{n} f_k(L)(1 - x)^{n-k}xB_k(x)
\]

and the proof follows.

**Proof of Theorem 3.2.** Let us denote by \( p(L, x) \) the right-hand side of the desired equality. Clearly, it suffices to show that

\[
h(sd(L), x)/(1 - x)^{n+1} = p(L, x)/(1 - x)^{n+1}.
\]
From Proposition 3.4 and Equation (3) we deduce that
\[
\frac{h(sd(L), x)}{(1 - x)^{n+1}} = 1 + x \sum_{k=0}^{n} f_k(L) \frac{B_k(x)}{(1 - x)^{k+1}}
\]
\[
= 1 + \sum_{m \geq 0} \left( \sum_{k=0}^{n} f_k(L)(2m + 1)^k \right) x^{m+1}
\]
\[
= 1 + \sum_{m \geq 1} \left( \sum_{k=0}^{n} f_k(L)(2m - 1)^k \right) x^{m}.
\]

Similarly, from part (f) of Proposition 3.3 we get
\[
\frac{p(L, x)}{(1 - x)^{n+1}} = \sum_{k=0}^{n+1} h_k(L) \frac{p^B_{n,k}(x)}{(1 - x)^{n+1}} = 1 + \sum_{m \geq 1} a_L(m)x^m,
\]
where
\[
a_L(y) := h_0(L)(2y+1)^n + \sum_{k=1}^{n} h_k(L)(4y)(2y-1)^{k-1}(2y+1)^{n-k} + h_{n+1}(L)(2y-1)^n.
\]

Thus, it remains to show that
\[
\sum_{k=0}^{n} f_k(L)(2y - 1)^k = a_L(y).
\]

We claim that this is, essentially, the defining Equation (2) of the cubical h-polynomial of L in disguised form. Indeed, cancelling first the summand
\[
1 + h_{n+1}(L)x^{n+2} = 1 + (-1)^{n} \widetilde{\chi}(L)x^{n+2},
\]
and then a factor of x, from both sides of (2) gives that
\[
\sum_{k=0}^{n} (h_k(L) + h_{k+1}(L))x^k = \left( \frac{1 - x}{2} \right)^n \sum_{k=0}^{n} f_k(L) \left( \frac{2x}{1 - x} \right)^k.
\]
Setting \(2x/(1-x) = 2y-1\), so that \(x = (2y-1)/(2y+1)\) and \((1-x)/2 = 1/(2y+1)\), the previous identity can be rewritten as
\[
\sum_{k=0}^{n} f_k(L)(2y - 1)^k = \sum_{k=0}^{n} (h_k(L) + h_{k+1}(L))(2y - 1)^k(2y + 1)^{n-k}.
\]

Since the right-hand side is readily equal to \(a_L(y)\), this proves Equation (4) and the theorem as well. \[\blacksquare\]
To deduce Theorem 1.1 from Theorem 3.2 and Proposition 3.3, we need to recall a few definitions and facts from the theory of interlacing polynomials; for more information, see [8, Section 7.8] and references therein. A polynomial \( p(x) \in \mathbb{R}[x] \) is called real-rooted if either it is the zero polynomial, or every complex root of \( p(x) \) is real. Given two real-rooted polynomials \( p(x), q(x) \in \mathbb{R}[x] \), we say that \( p(x) \) interlaces \( q(x) \) if the roots \( \{\alpha_i\} \) of \( p(x) \) interlace (or alternate to the left of) the roots \( \{\beta_j\} \) of \( q(x) \), in the sense that they can be listed as

\[
\cdots \leq \beta_3 \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1 \leq 0.
\]

A sequence \( (p_0(x), p_1(x), \ldots, p_n(x)) \) of real-rooted polynomials is called interlacing if \( p_i(x) \) interlaces \( p_j(x) \) for all \( 0 \leq i < j \leq n \). Assuming also that these polynomials have positive leading coefficients, every nonnegative linear combination of \( p_0(x), p_1(x), \ldots, p_n(x) \) is real-rooted and interlaced by \( p_0(x) \).

A standard way to produce interlacing sequences in combinatorics is the following. Suppose that \( p_0(x), p_1(x), \ldots, p_n(x) \) are real-rooted polynomials with nonnegative coefficients and set

\[
q_k(x) = x \sum_{i=0}^{k-1} p_i(x) + \sum_{i=k}^{n} p_i(x)
\]

for \( k \in \{0, 1, \ldots, n + 1\} \). Then, if the sequence \( (p_0(x), p_1(x), \ldots, p_n(x)) \) is interlacing, so is \( (q_0(x), q_1(x), \ldots, q_{n+1}(x)) \); see [8, Corollary 8.7] for a more general statement.

The following result is a stronger version of Theorem 1.1.

**Corollary 3.5.** The polynomial \( h(\text{sd}(L), x) \) is real-rooted and interlaced by the Eulerian polynomial \( B_n(x) \) for every \( n \)-dimensional cubical complex \( L \) with nonnegative cubical \( h \)-vector.

In particular, \( h(\text{sd}(L), x) \) and \( h(\text{sd}(Q), x) \) are real-rooted and interlaced by \( B_n(x) \) for every shellable, \( n \)-dimensional cubical complex \( L \) and every cubical polytope \( Q \) of dimension \( n + 1 \), respectively.

**Proof.** By an application of the lemma on interlacing sequences just discussed, the recurrence of part (e) of Proposition 3.3 implies that

\[
(p_{n,0}^B(x), p_{n,1}^B(x), \ldots, p_{n,n+1}^B(x))
\]
is interlacing for every \( n \in \mathbb{N} \) by induction on \( n \). Therefore, being a nonnegative linear combination of the elements of the sequence by Theorem 3.2, \( h(\text{sd}(\mathcal{L}), x) \) is real-rooted and interlaced by \( p^R_{n,0}(x) = B_n(x) \) for every \( n \)-dimensional cubical complex \( \mathcal{L} \) with nonnegative cubical \( h \)-vector. This proves the first statement.

The second statement follows from the first since shellable cubical complexes are known to have nonnegative cubical \( h \)-vector [1, Theorem 5 (iii)], \( h(\text{sd}(Q), x) = h(\text{sd}(\partial Q), x) \) for every convex polytope \( Q \) and because boundary complexes of polytopes are shellable.

**Remark 3.6:** Let \( \Delta \) be a simplicial complex with nonnegative \( h \)-vector and \( \mathcal{L} \) be a cubical complex which is obtained from \( \Delta \) by any operation which preserves nonnegativity of \( h \)-vectors. Corollary 3.5 implies that \( h(\text{sd}(\mathcal{L}), x) \) is real-rooted.

By a result of Hetyei [15], such an operation is the cubical barycentric subdivision \( \mathcal{L} = \text{sd}_c(\Delta) \) (see [3, p. 44]), also known as barycentric cover [5, Section 2.3], of \( \Delta \). Then, \( \text{sd}(\mathcal{L}) \) becomes the interval triangulation of \( \Delta \) [21, Section 3.3]. This argument shows that the interval triangulation of \( \Delta \) has a real-rooted \( h \)-polynomial for every simplicial complex \( \Delta \) with nonnegative \( h \)-vector and answers in the affirmative the question of [21, Problem 33]. Although there are other proofs of this fact in the literature (see [4] and references therein), the approach via Corollary 3.5 allows for more general results, e.g., by applying further cubical subdivisions of \( \text{sd}_c(\Delta) \) which preserve the nonnegativity of the cubical \( h \)-vector.

**Remark 3.7:** Applying the reasoning of the proof of Proposition 3.4 and of the first few lines of the proof of Theorem 3.2 to an \( n \)-dimensional simplicial complex \( \Delta \) gives that

\[
h(\text{sd}(\Delta), x) = (1 - x)^{n+1} + x \sum_{k=0}^{n} f_k(\Delta)(1 - x)^{n-k} A_{k+1}(x)
\]

and

\[
\frac{h(\text{sd}(\Delta), x)}{(1 - x)^{n+2}} = \sum_{m \geq 0} \left( \sum_{k=0}^{n+1} f_{k-1}(\Delta)m^k \right)x^m
\]

\[
= \sum_{m \geq 0} \left( \sum_{k=0}^{n+1} h_k(\Delta)m^k(m+1)^{n+1-k} \right)x^m.
\]

This is the expression at which Brenti and Welker arrived [11, Equation (5)] via a different route and which they used to show that \( h(\text{sd}(\Delta), x) \) has only real roots, provided that \( h_k(\Delta) \geq 0 \) for all \( k \).
Remark 3.8: Replacing $2y - 1$ by $x$ in (5) shows that the equation

$$\sum_{k=0}^{n} f_k(\mathcal{L}) x^k = \sum_{k=0}^{n} (h_k(\mathcal{L}) + h_{k+1}(\mathcal{L})) x^k (x + 2)^{n-k},$$

together with the condition $h_0(\mathcal{L}) = 1$, gives an equivalent way to define the normalized cubical $h$-vector of an $n$-dimensional cubical complex $\mathcal{L}$.

4. Closing remarks

There is a large body of literature on the barycentric subdivision of simplicial complexes which relates to the work [11]. Many of the questions addressed there make sense for cubical complexes. We only consider a couple of them here.

(a) Being real-rooted, $h(\text{sd}(\Delta), x)$ is unimodal for every $n$-dimensional simplicial complex $\Delta$ with nonnegative $h$-vector. Kubitzke and Nevo showed [20, Corollary 4.7] that the corresponding $h$-vector $(h_i(\text{sd}(\Delta)))_{0 \leq i \leq n+1}$ has a peak at $i = (n+1)/2$ if $n$ is odd, and at $i = n/2$ or $i = n/2 + 1$ if $n$ is even. The analogous statement for cubical complexes follows from Theorem 3.2 since, as in the simplicial setting, the unimodal polynomial $p_{n,k}^{B}(x)$ has a peak at $i = (n+1)/2$, if $n$ is odd, at $i = n/2$ if $n$ is even and $k \leq n/2$, and at $i = n/2 + 1$, if $n$ is even and $k \geq n/2 + 1$. The latter claim can be deduced from the recursion of part (e) of Proposition 3.3 by mimicking the argument given in the simplicial setting in [22, Section 2]. For general results on the unimodality of $h$-vectors of barycentric subdivisions of Cohen–Macaulay regular cell complexes, proven by algebraic methods, see Corollaries 1.2 and 5.12 in [23].

(b) The main result of [9] implies (see [4, Section 8]) that $h(\text{sd}(\Delta), x)$ has a nonnegative real-rooted symmetric decomposition with respect to $n$ for every triangulation $\Delta$ of an $n$-dimensional ball. Does this hold if $\Delta$ is replaced by any cubical subdivision of the $n$-dimensional ball? Are these symmetric decompositions interlacing? Do the polynomials $p_{n,k}^{B}(x)$ have such properties?

(c) The subdivision operator (see [8, Section 3.3]) has a natural generalization in the context of uniform triangulations of simplicial complexes [4, Section 5] which plays a role in that theory. It may be worth studying the cubical analogue of this operator further.
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