RELAXATION OF \( p \)-GROWTH INTEGRAL FUNCTIONALS UNDER SPACE-DEPENDENT DIFFERENTIAL CONSTRAINTS

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Abstract. A representation formula for the relaxation of integral energies
\[
(u, v) \mapsto \int_{\Omega} f(x, u(x), v(x)) \, dx,
\]
is obtained, where \( f \) satisfies \( p \)-growth assumptions, \( 1 < p < +\infty \), and the fields \( v \) are subject to space-dependent first order linear differential constraints in the framework of \( \mathcal{A} \)-quasiconvexity with variable coefficients.

1. Introduction

The analysis of constrained relaxation problems is a central question in materials science. Many applications in continuum mechanics and, in particular, in magnetoelasticity, rely on the characterization of minimizers of non-convex multiple integrals of the type
\[
u \mapsto \int_{\Omega} f(x, \nabla u(x), \ldots, \nabla^k u(x)) \, dx
\]
or
\[
(u, v) \mapsto \int_{\Omega} f(x, u(x), v(x)) \, dx,
\] (1.1)
where \( \Omega \) is an open, bounded subset of \( \mathbb{R}^N \), \( u : \Omega \to \mathbb{R}^m \), \( m \in \mathbb{N} \), and the fields \( v : \Omega \to \mathbb{R}^d \), \( d \in \mathbb{N} \), satisfy partial differential constraints of the type “\( \mathcal{A} v = 0 \)” other than \( \text{curl}\ v = 0 \) (see e.g. [5, 9]).

In this paper we provide a representation formula for the relaxation of non-convex integral energies of the form (1.1), in the case in which the energy density \( f \) satisfies \( p \)-growth assumptions, and the fields \( v \) are subject to linear first-order space-dependent differential constraints.

The natural framework to study this family of relaxation problems is within the theory of \( \mathcal{A} \)-quasiconvexity with variable coefficients. In order to present this notion, we need to introduce some notation.

For \( i = 1 \cdots N \), let \( A^i \in C^\infty(\mathbb{R}^N; \mathbb{M}^{l \times d}) \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{M}^{l \times d}) \), let \( 1 < p < +\infty \), and consider the differential operator
\[
\mathcal{A} : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{M}^l), \quad d, l \in \mathbb{N},
\] defined as
\[
\mathcal{A} v := \sum_{i=1}^{N} A^i(x) \frac{\partial v(x)}{\partial x_i}
\] (1.2)
for every \( v \in L^p(\Omega; \mathbb{R}^d) \), where (1.2) is to be interpreted in the sense of distributions. Assume that the symbol \( A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{M}^{l \times d} \),
\[
A(x, w) := \sum_{i=1}^{N} A^i(x) w_i \quad \text{for } (x, w) \in \mathbb{R}^N \times \mathbb{R}^N,
\]
satisfies the uniform constant rank condition (see [22])

$$\text{rank } A(x, w) = r \quad \text{for every } x \in \mathbb{R}^N \text{ and } w \in S^{n-1}.$$  

(1.3)

Let $Q$ be the unit cube in $\mathbb{R}^N$ with sides parallel to the coordinate axis, i.e.,

$$Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^N.$$

Denote by $C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^m)$ the set of $\mathbb{R}^m$-valued smooth maps that are $Q$-periodic in $\mathbb{R}^N$, and for every $x \in \Omega$ consider the set

$$C_x := \left\{ w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^m) : \int_Q w(y) \, dy = 0, \quad \text{and} \quad \sum_{i=1}^N A^i(x) \frac{\partial w(y)}{\partial y_i} = 0 \right\}.$$

Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)$ be a Carathéodory function. The $\mathcal{A}$-quasiconvex envelope of $f(x, u, \cdot)$ for $x \in \Omega$ and $u \in \mathbb{R}^m$ is defined for $\xi \in \mathbb{R}^d$ as

$$Q_{\mathcal{A}(x)} f(x, u, \xi) := \inf \left\{ \int_Q f(x, u, \xi + w(y)) \, dy : w \in C_x \right\}.$$

We say that $f$ is $\mathcal{A}$-quasiconvex if $f(x, u, \xi) = Q_{\mathcal{A}(x)} f(x, u, \xi)$ for a.e. $x \in \Omega$, and for all $u \in \mathbb{R}^m$ and $\xi \in \mathbb{R}^d$.

The notion of $\mathcal{A}$-quasiconvexity was first introduced by B. Dacorogna in [8], and extensively characterized in [17] by I. Fonseca and S. Müller for operators $\mathcal{A}$ defined as in (1.2), satisfying the constant rank condition (1.3), and having constant coefficients,

$$A^i(x) \equiv A^i \in \mathbb{M}^{l \times d} \quad \text{for every } x \in \mathbb{R}^N, \quad i = 1, \ldots, N.$$

In that paper the authors proved (see [17, Theorems 3.6 and 3.7]) that under $p$-growth assumptions on the energy density $f$, $\mathcal{A}$-quasiconvexity is necessary and sufficient for the lower-semicontinuity of integral functionals

$$I(u, v) := \int_\Omega f(x, u(x), v(x)) \, dx \quad \text{for every } (u, v) \in L^p(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^d)$$

along sequences $(u^n, v^n)$ satisfying $u^n \to u$ in measure, $v^n \to v$ in $L^p(\Omega; \mathbb{R}^d)$, and $\mathcal{A} v^n \to 0$ in $W^{-1,p}(\Omega)$. We remark that in the framework $\mathcal{A} = \text{curl}$, i.e., when $v^n = \nabla \phi^n$ for some $\phi^n \in W^{1,p}(\Omega; \mathbb{R}^m)$, $d = n \times m$, $\mathcal{A}$-quasiconvexity reduces to Morrey’s notion of quasiconvexity.

The analysis of properties of $\mathcal{A}$-quasiconvexity for operators with constant coefficients was extended in the subsequent paper [6], where A. Braides, I. Fonseca and G. Leoni provided an integral representation formula for relaxation problems under $p$-growth assumptions on the energy density, and presented (via $\Gamma$-convergence) homogenization results for periodic integrands evaluated along $\mathcal{A}$-free fields. These homogenization results were later generalized in [13], where I. Fonseca and S. Krömer worked under weaker assumptions on the energy density $f$. In [19, 20], simultaneous homogenization and dimension reduction was studied in the framework of $\mathcal{A}$-quasiconvexity with constant coefficients. Oscillations and concentrations generated by $\mathcal{A}$-free mappings are the subject of [14]. Very recently an analysis of the case in which the energy density is nonpositive has been carried out in [18], and applications to the theory of compressible Euler systems have been studied in [7]. A parallel analysis for operators with constant coefficients and under linear growth assumptions for the energy density has been developed in [1, 4, 15, 21]. A very general characterization in this setting has been obtained in [2], following the new insight in [12].

The theory of $\mathcal{A}$-quasiconvexity for operators with variable coefficients has been characterized by P. Santos in [23]. Homogenization results in this setting have been obtained in [10] and [11].
This paper is devoted to proving a representation result for the relaxation of integral energies in the framework of \(\mathcal{A}\)-quasiconvexity with variable coefficients. To be precise, let \(1 < p, q < +\infty\), \(d, m, l \in \mathbb{N}\), and consider a Carathéodory function \(f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)\) satisfying
\[
(H) \quad 0 \leq f(x, u, v) \leq C(1 + |u|^{p} + |v|^{q}), \quad 1 < p, q < +\infty,
\]
for a.e. \(x \in \Omega\), and all \((u, v) \in \mathbb{R}^m \times \mathbb{R}^d\), with \(C > 0\).

Denoting by \(\mathcal{O}(\Omega)\) the collection of open subsets of \(\Omega\), for every \(D \in \mathcal{O}(\Omega)\), \(u \in L^p(\Omega; \mathbb{R}^m)\) and \(v \in L^q(\Omega; \mathbb{R}^d)\) with \(\mathcal{A}v = 0\), we define
\[
\mathcal{I}((u, v), D) := \inf \left\{ \liminf_{n \to +\infty} \int_D f(x, u_n(x), v_n(x)) : u_n \to u \text{ strongly in } L^p(\Omega; \mathbb{R}^m), \right. \\
\left. v_n \rightharpoonup v \text{ weakly in } L^q(\Omega; \mathbb{R}^d) \text{ and } \mathcal{A}v_n \to 0 \text{ strongly in } W^{-1,q}(\Omega; \mathbb{R}^d) \right\}. \tag{1.4}
\]

Our main result is the following.

**Theorem 1.1.** Let \(\mathcal{A}\) be a first order differential operator with variable coefficients, satisfying (1.3). Let \(f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d \to [0, +\infty)\) be a Carathéodory function satisfying (H). Then,
\[
\int_D Q_{\mathcal{A}(x)} f(x, u(x), v(x)) \, dx = \mathcal{I}((u, v), D)
\]
for all \(D \in \mathcal{O}(\Omega)\), \(u \in L^p(\Omega; \mathbb{R}^m)\) and \(v \in L^q(\Omega; \mathbb{R}^d)\) with \(\mathcal{A}v = 0\).

Adopting the “blow-up” method introduced in [16], the proof of the theorem consists in showing that the functional \(\mathcal{I}((u, v), \cdot)\) is the trace of a Radon measure absolutely continuous with respect to the restriction of the Lebesgue measure \(\mathcal{L}^N\) to \(\Omega\), and proving that for a.e. \(x \in \Omega\) the Radon-Nicodym derivative \(d\mathcal{I}((u, v), \cdot)(x) / d\mathcal{L}^N\) coincides with the \(\mathcal{A}\)-quasiconvex envelope of \(f\).

The arguments used are a combination of the ideas from [6, Theorem 1.1] and from [23]. The main difference with [6, Theorem 1.1], which reduces to our setting in the case in which the operator \(\mathcal{A}\) has constant coefficients, is in the fact that while defining the operator \(\mathcal{I}\) in (1.4) we can not work with exact solutions of the PDE, but instead we need to study sequences of asymptotically \(\mathcal{A}\)-vanishing fields. As pointed out in [23], in the case of variable coefficients the natural framework is the context of pseudo-differential operators. In this setting, we don’t know how to project directly onto the kernel of the differential constraint, but we are able to construct an “approximate” projection operator \(P\) such that for every field \(v \in L^p\), the \(W^{-1,p}\) norm of \(\mathcal{A}Pv\) is controlled by the \(W^{-1,p}\) norm of \(v\) itself (we refer to [23, Subsection 2.1] for a detailed explanation of this issue and to the references therein for a treatment of the main properties of pseudo-differential operators). For the same reason, in the proof of the inequality
\[
\frac{d\mathcal{I}((u, v), \cdot)(x)}{d\mathcal{L}^N} \leq Q_{\mathcal{A}(x)} f(x, u(x), v(x)) \quad \text{for a.e. } x \in \Omega,
\]
an equi-integrability argument is needed (see Proposition 3.2). We also point out that the representation formula in Theorem 1.1 was obtained in a simplified setting in [11] as a corollary of the main homogenization result. Here we provide an alternative, direct proof, which does not rely on homogenization techniques.

The paper is organized as follows: in Section 2 we establish the main assumptions on the differential operator \(\mathcal{A}\) and we recall some preliminary results on \(\mathcal{A}\)-quasiconvexity with variable coefficients. Section 3 is devoted to the proof of Theorem 1.1.

**Notation**
Throughout the paper \(\Omega \subset \mathbb{R}^N\) is a bounded open set, \(1 < p, q < +\infty\), \(\mathcal{O}(\Omega)\) is the set of open subsets of \(\Omega\), \(Q\) denotes the unit cube in \(\mathbb{R}^N\), \(Q(x_0, r)\) and \(B(x_0, r)\) are, respectively, the open cube and the
open ball in $\mathbb{R}^N$, with center $x_0$ and radius $r$. Given an exponent $1 < q < +\infty$, we denote by $q'$ its conjugate exponent, i.e., $q' \in (1, +\infty)$ is such that

$$\frac{1}{q} + \frac{1}{q'} = 1.$$ 

Whenever a map $v \in L^q, C^\infty$, $\cdots$ is $Q$–periodic, that is

$$v(x + e_i) = v(x) \quad i = 1, \cdots, N,$$

for $a.e. \ x \in \mathbb{R}^N$, $\{e_1, \cdots, e_N\}$ being the standard basis of $\mathbb{R}^N$, we write $v \in L^q_{\text{per}}, C^\infty_{\text{per}}, \cdots$ We implicitly identify the spaces $L^q(Q)$ and $L^q_{\text{per}}(\mathbb{R}^N)$.

We adopt the convention that $C$ will denote a generic constant, whose value may change from line to line in the same formula.

2. Preliminary results

In this section we introduce the main assumptions on the differential operator $\mathcal{A}$ and we recall some preliminary results about $\mathcal{A}$–quasiconvexity.

For $i = 1, \cdots, N$, $x \in \mathbb{R}^N$, consider the linear operators $A^i(x) \in M^{l \times d}$, with $A^i \in C^\infty(\mathbb{R}^N; M^{l \times d}) \cap W^{1, \infty}(\mathbb{R}^N; M^{l \times d})$. For every $v \in L^q(\Omega; \mathbb{R}^d)$ we set

$$\mathcal{A} v := \sum_{i=1}^N A^i(x) \frac{\partial v(x)}{\partial x_i} \in W^{-1, q}(\Omega; \mathbb{R}^l).$$

The symbol $\mathcal{A} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\} \to M^{l \times d}$ associated to the differential operator $\mathcal{A}$ is

$$\mathcal{A}(x, \lambda) := \sum_{i=1}^N A^i(x) \lambda_i \in M^{l \times d}$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$. We assume that $\mathcal{A}$ satisfies the following uniform constant rank condition:

$$\text{rank} \left( \sum_{i=1}^N A^i(x) \lambda_i \right) = r \quad \text{for all } x \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}^N \setminus \{0\}. \quad (2.1)$$

For every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$, let $P(x, \lambda) : \mathbb{R}^d \to \mathbb{R}^d$ be the linear projection on Ker $\mathcal{A}(x, \lambda)$, and let $Q(x, \lambda) : \mathbb{R}^l \to \mathbb{R}^d$ be the linear operator given by

$$Q(x, \lambda) \mathcal{A}(x, \lambda) v := v - P(x, \lambda) v \quad \text{for all } v \in \mathbb{R}^d,$$

$$Q(x, \lambda) \xi = 0 \quad \text{if } \xi \notin \text{Range } \mathcal{A}(x, \lambda).$$

The main properties of $P(\cdot, \cdot)$ and $Q(\cdot, \cdot)$ are recalled in the following proposition (see e.g. [23, Subsection 2.1]).

**Proposition 2.1.** Under the constant rank condition (2.1), for every $x \in \mathbb{R}^N$ the operators $P(x, \cdot)$ and $Q(x, \cdot)$ are, respectively, $0$-homogeneous and $(-1)$-homogeneous. In addition, $P \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; M^{d \times d})$ and $Q \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}; M^{d \times l})$.

Let $\eta \in C^\infty_c(\Omega; [0, 1])$, $\eta = 1$ in $\Omega'$ for some $\Omega' \subset \subset \Omega$. We denote by $\mathcal{A}_\eta$ the symbol

$$\mathcal{A}_\eta(x, \lambda) := \sum_{i=1}^N \eta(x) A^i(x) \lambda_i, \quad (2.2)$$

for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$, and by $\mathcal{A}_\eta$ the corresponding pseudo-differential operator (see [23, Subsection 2.1] for an overview of the main properties of pseudo-differential operators). Let $\chi \in C^\infty(\mathbb{R}^+; \mathbb{R})$ be such that $\chi(|\lambda|) = 0$ for $|\lambda| < 1$ and $\chi(|\lambda|) = 1$ for $|\lambda| > 2$. Let also $P_\eta$ be the operator associated to the symbol

$$P_\eta(x, \lambda) := \eta^2(x) P(x, \lambda) \chi(|\lambda|) \quad (2.3)$$
for every $x \in \mathbb{R}^N$, $\lambda \in \mathbb{R}^N \setminus \{0\}$. The following proposition (see [23, Theorem 2.2 and Subsection 2.1]) collects the main properties of the operators $P_\eta$ and $A_\eta$.

**Proposition 2.2.** Let $1 < q < +\infty$, and let $A_\eta$ and $P_\eta$ be the pseudo-differential operators associated with the symbols (2.2) and (2.3), respectively. Then there exists a constant $C$ such that

$$
\|P_\eta v\|_{L^q(\Omega; \mathbb{R}^d)} \leq C \|v\|_{L^s(\Omega; \mathbb{R}^d)}
$$

for every $v \in L^q(\Omega; \mathbb{R}^d)$, and

$$
\|P_\eta v\|_{L^{q}(\Omega; \mathbb{R}^d)} \leq C \|v\|_{L^{q}(\Omega; \mathbb{R}^d)},
$$

$$
\|v - P_\eta v\|_{L^{q}(\Omega; \mathbb{R}^d)} \leq C (\|A_\eta v\|_{L^{q}(\Omega; \mathbb{R}^d)} + \|v\|_{L^{q}(\Omega; \mathbb{R}^d)}),
$$

$$
\|A_\eta P_\eta v\|_{L^{q}(\Omega; \mathbb{R}^d)} \leq C \|v\|_{L^{q}(\Omega; \mathbb{R}^d)}
$$

for every $v \in W^{-1,q}(\Omega; \mathbb{R}^d)$.

3. **Proof of Theorem 1.1**

Before proving Theorem 1.1 we state and prove a decomposition lemma, which generalizes [17, Lemma 2.15] to the case of operators with variable coefficients.

**Lemma 3.1.** Let $1 < q < +\infty$. Let $A$ be a first order differential operator with variable coefficients, satisfying (2.1). Let $v \in L^q(\Omega; \mathbb{R}^d)$, and let $\{v_n\}$ be a bounded sequence in $L^q(\Omega; \mathbb{R}^d)$ such that

$$
v_n \to v \quad \text{weakly in } L^q(\Omega; \mathbb{R}^d),
$$

$$
A v_n \to 0 \quad \text{strongly in } W^{-1,q}(\Omega; \mathbb{R}^d),
$$

$$
\{v_n\} \text{ generates the Young measure } \nu.
$$

Then, there exists a $q$-equiintegrable sequence $\{\tilde{v}_n\} \subset L^q(\Omega; \mathbb{R}^d)$ such that

$$
A \tilde{v}_n \to 0 \quad \text{strongly in } W^{-1,s}(\Omega; \mathbb{R}^d) \quad \text{for every } 1 < s < q,
$$

$$
\int_{\Omega} \tilde{v}_n(x) \, dx = \int_{\Omega} v(x) \, dx,
$$

$$
\tilde{v}_n - v_n \to 0 \quad \text{strongly in } L^s(\Omega; \mathbb{R}^d) \quad \text{for every } 1 < s < q,
$$

$$
\tilde{v}_n \to v \quad \text{weakly in } L^q(\Omega; \mathbb{R}^d).
$$

In addition, if $\Omega \subset Q$ then we can construct the sequence $\{\tilde{v}^n\}$ so that $\tilde{v}_n - v \in L^q_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)$ for every $n \in \mathbb{N}$.

**Proof.** Arguing as in the first part of [23, Proof of Theorem 1.1], we construct a $q$-equiintegrable sequence $\{\tilde{v}_n\}$ satisfying (3.1), (3.2) and (3.3). The conclusion follows by setting $\tilde{v}_n := \tilde{v}_n - \int_{\Omega} \tilde{v}_n(x) \, dx + \int_{\Omega} v(x) \, dx$.

In the case in which $\Omega \subset Q$, let $\{\varphi^i\}$ be a sequence of cut-off functions in $Q$ with $0 \leq \varphi^i \leq 1$ in $Q$, such that $\varphi^i = 0$ on $Q \setminus \Omega$ and $\varphi^i \to 1$ pointwise in $\Omega$. Define $w_n^i := \varphi^i(\tilde{v}_n - v)$. By (3.3) for every $\psi \in L^q(\Omega; \mathbb{R}^d)$ we have

$$
\lim_{i \to +\infty} \lim_{n \to +\infty} \int_{\Omega} w_n^i(x) \psi(x) \, dx = 0.
$$

By (3.1), (3.2), and the compact embedding of $L^q(\Omega; \mathbb{R}^d)$ into $W^{-1,q}(\Omega; \mathbb{R}^d)$, there holds

$$
A w_n^i = \varphi^i A \tilde{v}_n + \left( \sum_{j=1}^N A^i_j \frac{\partial \varphi^i}{\partial x_j} \right) \tilde{v}_n \to 0 \quad \text{strongly in } W^{-1,s}(\Omega; \mathbb{R}^d)
$$

as $n \to +\infty$, for every $1 < s < q$. Extending the maps $w_n^i$ outside $Q$ by periodicity, by the metrizability of the weak topology on bounded sets and by Attouch’s diagonalization lemma (see [3, Lemma 1.15 and Corollary 1.16]), we obtain a sequence

$$
w_n := w_n^{i(n)},
$$
with \( \{ w_n \} \subset L^{q}_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d) \), and such that \( w_n + v \) satisfies (3.1), (3.2) and (3.3). The thesis follows by setting

\[
\tilde{v}_n := w_n - \int_{\Omega} w_n(x) \, dx + v.
\]

\[\square\]

The following proposition will allow us to neglect vanishing perturbations of \( q \)-equiintegrable sequences.

**Proposition 3.2.** For every \( n \in \mathbb{N} \), let \( f_n : Q \times \mathbb{R}^d \to [0, +\infty) \) be a continuous function. Assume that there exists a constant \( C > 0 \) such that, for \( q > 1 \),

\[
\sup_{n \in \mathbb{N}} f_n(y, \xi) \leq C(1 + |\xi|^q) \quad \text{for every } y \in Q \text{ and } \xi \in \mathbb{R}^d,
\]

and that the sequence \( \{ f_n(y, \cdot) \} \) is equicontinuous in \( \mathbb{R}^d \), uniformly in \( y \). Let \( \{ w_n \} \) be a \( q \)-equiintegrable sequence in \( L^q(Q; \mathbb{R}^d) \), and let \( \{ v_n \} \subset L^q(Q; \mathbb{R}^d) \) be such that

\[
v_n \to 0 \quad \text{strongly in } L^q(Q; \mathbb{R}^d).
\]

Then

\[
\lim_{n \to +\infty} \left| \int_{Q} f_n(y, w_n(y)) \, dy - \int_{Q} f_n(y, v_n(y) + w_n(y)) \, dy \right| = 0.
\]

**Proof.** Fix \( \eta > 0 \). In view of (3.5), the sequence \( \{ C(1 + |v_n|^q + |w_n|^q) \} \) is equiintegrable in \( Q \), thus there exists \( 0 < \varepsilon < \frac{\eta}{3} \) such that

\[
\sup_{n \in \mathbb{N}} \int_A C(1 + |v_n(y)|^q + |w_n(y)|^q) \, dy < \frac{\eta}{3}
\]

for every \( A \subset Q \) with \( |A| < \varepsilon \). By the \( q \)-equiintegrability of \( \{ w_n \} \) and \( \{ v_n \} \), and by Chebyshev’s inequality there holds

\[
|Q \cap \left( \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \} \right) \leq \frac{1}{M^q} \int_Q (|w_n(y)|^q + |v_n(y)|^q) \, dy \leq \frac{C}{M^q}
\]

for every \( n \in \mathbb{N} \). Therefore, there exists \( M_0 \) satisfying

\[
\sup_{n \in \mathbb{N}} |Q \cap \left( \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \} \right) | \leq \frac{\varepsilon}{2}.
\]

By the uniform equicontinuity of the sequence \( \{ f_n(y, \cdot) \} \), there exists \( \delta > 0 \) such that, for every \( \xi_1, \xi_2 \in B(0, M_0) \), with \( |\xi_1 - \xi_2| < \delta \), we have

\[
\sup_{y \in Q} |f_n(y, \xi_1) - f_n(y, \xi_2)| < \varepsilon
\]

for every \( n \in \mathbb{N} \). By (3.5) and Egoroff’s theorem, there exists a set \( E_\varepsilon \subset Q \), \( |E_\varepsilon| < \frac{\varepsilon}{2} \), such that

\[
v_n \to 0 \quad \text{uniformly in } Q \setminus E_\varepsilon,
\]

and, in particular,

\[
|v_n(x)| < \delta \quad \text{for a.e. } x \in Q \setminus E_\varepsilon,
\]

for every \( n \geq n_0 \), for some \( n_0 \in \mathbb{N} \).

We observe that

\[
\int_{Q} f_n(y, v_n(y) + w_n(y)) \, dy = \int_{Q \cap \{|w_n| \leq M_0 \} \cup \{|v_n| \leq M_0 \}} f_n(y, v_n(y) + w_n(y)) \, dy
\]

\[
+ \int_{Q \cap \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \}} f_n(y, v_n(y) + w_n(y)) \, dy.
\]
The first term in the right-hand side of (3.10) can be further decomposed as
\[
\int_{Q \cap \{|w_n| \leq M_0 \} \cap \{|v_n| \leq M_0 \}} f_n(y, v_n(y) + w_n(y)) \, dy
\]
\[
= \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| \leq M_0 \} \cap \{|v_n| \leq M_0 \}} f_n(y, v_n(y) + w_n(y)) \, dy
\]
\[
+ \int_{E_\varepsilon \cap \{|w_n| \leq M_0 \} \cap \{|v_n| \leq M_0 \}} f_n(y, v_n(y) + w_n(y)) \, dy
\]
\[
= \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| \leq M_0 \} \cap \{|v_n| \leq M_0 \}} f_n(y, w_n(y)) \, dy
\]
\[
+ \int_{(Q \setminus E_\varepsilon) \cap \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \}} (f_n(y, v_n(y) + w_n(y)) - f_n(y, w_n(y))) \, dy
\]
\[
+ \int_{E_\varepsilon \cap \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \}} f_n(y, v_n(y) + w_n(y)) \, dy.
\]
We observe that by (3.7)
\[
|E_\varepsilon \cup \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \}| < \varepsilon.
\]
Hence, for \( n \geq n_0 \), by (3.4), (3.6), (3.8), and (3.9) we deduce the estimate
\[
\left| \int_Q f_n(y, w_n(y)) \, dy - \int_Q f_n(y, v_n(y)) \, dy \right| \leq \varepsilon + \int_{E_\varepsilon \cup \{|w_n| > M_0 \} \cup \{|v_n| > M_0 \}} 2C(1 + |w_n(y)|^p + |v_n(y)|^p) \, dy \leq \varepsilon + \frac{2\eta}{3}.
\] (3.11)
The thesis follows by the arbitrariness of \( \eta \).

We now prove our main result.

**Proof of Theorem 1.1.** The proof is subdivided into 4 steps. Steps 1 and 2 follow along the lines of [6, Proof of Theorem 1.1]. Step 3 is obtained by modifying [6, Lemma 3.5], whereas Step 4 follows by adapting an argument in [23, Proof of Theorem 1.2]. We only outline the main ideas of Steps 1 and 2 for convenience of the reader, whilst we provide more details for Steps 3 and 4.

**Step 1:**

The first step consists in showing that
\[
\mathcal{I}(u,v,D) = \inf \left\{ \liminf_{n \to +\infty} \int_D f(x,u(x),v_n(x)) \, dx : \{v_n\} \text{ is } q \text{-equiintegrable} \right\}
\]
\[
\Rightarrow \text{if } u_n \to 0 \text{ strongly in } W^{1,s}(D;\mathbb{R}^d) \text{ for every } 1 < s < q
\]
\[
\text{and } v_n \rightharpoonup v \text{ weakly in } L^q(D;\mathbb{R}^d).
\]
This identification is proved by adapting [6, Proof of Lemma 3.1]. The only difference is the application of Lemma 3.1 instead of [6, Proposition 2.3 (i)].

**Step 2:**

The second step is the proof that \( \mathcal{I}(u,v,\cdot) \) is the trace of a Radon measure absolutely continuous
with respect to $L^N[\Omega]$. This follows as a straightforward adaptation of [6, Lemma 3.4]. The only modifications are due to the fact that [6, Proposition 2.3 (i)] and [6, Lemma 3.1] are now replaced by Lemma 3.1 and Step 1.

**Step 3:**

We claim that

$$
\frac{dI((u,v),\cdot)}{d\mathcal{L}^N}(x_0) \geq Q_{\mathcal{A}(x_0)}f(x_0, u(x_0), v(x_0)) \quad \text{for a.e. } x_0 \in \Omega.
$$

(3.12)

Indeed, since $g(x, \xi) := f(x, u(x), \xi)$ is a Carathéodory function, by Scorza-Dragoni Theorem there exists a sequence of compact sets $K_j \subset \Omega$ such that

$$
|\Omega \setminus K_j| \leq \frac{1}{j}
$$

and the restriction of $g$ to $K_j \times \mathbb{R}^d$ is continuous. Hence, the set

$$
\omega := \bigcup_{j=1}^{+\infty} (K_j \cap K_j^*) \cap \mathcal{L}(u,v),
$$

(3.13)

where $K_j^*$ is the set of Lebesgue point for the characteristic function of $K_j$ and $\mathcal{L}(u,v)$ is the set of Lebesgue points of $u$ and $v$, is such that

$$
|\Omega \setminus \omega| \leq |\Omega \setminus K_j| \leq \frac{1}{j}
$$

for every $j$, and so $|\Omega \setminus \omega| = 0$. Let $x_0 \in \omega$ be such that

$$
\lim_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0,r)} |u(x) - u(x_0)|^p \, dx = \lim_{r \to 0^+} \frac{1}{r^N} \int_{Q(x_0,r)} |v(x) - v(x_0)|^q \, dx = 0,
$$

(3.14)

and

$$
\frac{dI((u,v),\cdot)}{d\mathcal{L}^N}(x_0) = \lim_{r \to 0^+} \frac{I((u,v), Q(x_0,r))}{r^N} < +\infty,
$$

(3.15)

where the sequence of radii $r$ is such that $I((u,v), \partial Q(x_0,r)) = 0$ for every $r$. (Such a choice of the sequence is possible due to Step 2).

By Step 1, for every $r$ there exists a $q$–equiintegrable sequence $\{v_{n,r}\}$ such that

$$
v_{n,r} \rightharpoonup v \quad \text{weakly in } L^q(Q(x_0,r); \mathbb{R}^d),
$$

$$
\mathcal{A} v_{n,r} \to 0 \quad \text{strongly in } W^{1,s}(Q(x_0,r); \mathbb{R}^d) \quad \text{for every } 1 < s < q
$$

(3.16)

as $n \to +\infty$, and

$$
\lim_{n \to +\infty} \int_{Q(x_0,r)} g(x, v_{n,r}(x)) \, dx \leq I((u,v), Q(x_0,r)) + r^{N+1}.
$$

A change of variables yields

$$
\frac{dI((u,v),\cdot)}{d\mathcal{L}^N}(x_0) \geq \lim_{r \to 0^+} \lim_{n \to +\infty} \int_{Q} g(x_0 + ry, v(x_0) + w_{n,r}(y)) \, dy,
$$

where

$$
w_{n,r}(y) := v_{n,r}(x_0 + ry) - v(x_0) \quad \text{for a.e. } y \in Q.
$$

Arguing as in [6, Proof of Lemma 3.5], Hölder’s inequality and a change of variables imply

$$
w_{n,r} \rightharpoonup 0 \quad \text{weakly in } L^q(Q; \mathbb{R}^d)
$$

(3.17)

as $n \to +\infty$ and $r \to 0^+$, in this order. We claim that

$$
\mathcal{A}(x_0 + r) w_{n,r} \to 0 \quad \text{strongly in } W^{1,s}(Q; \mathbb{R}^d),
$$

(3.18)

as $n \to +\infty$, for every $r$ and every $1 < s < q$. 

Indeed, let \( \varphi \in W^{1,s}_0(Q; \mathbb{R}^d) \). There holds
\[
\langle \mathcal{A}(x_0 + r \cdot) w_{n,r}, \varphi \rangle_{W^{-1,s}(Q; \mathbb{R}^l), W^{1,s}_0(\mathbb{R}^d)} = - \sum_{i=1}^{N} \left\{ \int_{Q} A^i(x_0 + ry) \cdot \frac{\partial \varphi(y)}{\partial x_i} dy \right\} + \int_{Q} A^i(x_0 + ry) \cdot \frac{\partial \varphi(y)}{\partial y_i} dx,
\]
where \( \psi_r(x) := \varphi(\frac{x-x_0}{r}) \) for a.e. \( x \in Q(x_0, r) \). Since \( \psi_r \in W^{1,s}_0(Q(x_0, r); \mathbb{R}^d) \) and
\[
\| \psi_r \|_{W^{1,s}_0(Q(x_0, r); \mathbb{R}^d)} \leq C(r) \| \varphi \|_{W^{1,s}_0(\mathbb{R}^d)},
\]
we obtain the estimate
\[
\| \mathcal{A}(x_0 + r \cdot) w_{n,r} \|_{W^{-1,s}(Q; \mathbb{R}^l)} \leq C(r) \| \mathcal{A} \varphi \|_{W^{1,s}_0(\mathbb{R}^d)}.
\]
Claim (3.18) follows by (3.16).
In view of (3.17) and (3.18), a diagonalization procedure yields a \( q \)--equiintegrable sequence \( \{ \hat{w}_k \} \subset L^q(Q; \mathbb{R}^d) \) satisfying
\[
\hat{w}_k \to 0 \quad \text{weakly in } L^q(Q; \mathbb{R}^d),
\]
\[
\langle \mathcal{A}(x_0 + r \cdot) \hat{w}_k, \varphi \rangle_{W^{-1,s}(Q; \mathbb{R}^l)} \to 0 \quad \text{strongly in } W^{-1,s}(Q; \mathbb{R}^l) \quad \text{for every } 1 < s < q,
\]
and
\[
\frac{dI((u,v), \cdot)}{d\mathcal{L}^N}(x_0) \geq \liminf_{k \to +\infty} \int_{Q} g(x_0 + r_k y, v(x_0) + \hat{w}_k(y)) dy.
\]
For every \( \varphi \in W^{1,s}_0(Q; \mathbb{R}^l) \), \( 1 < s < q \), there holds
\[
\langle (\mathcal{A}(x_0 + r \cdot) - \mathcal{A}(x_0)) \hat{w}_k, \varphi \rangle_{W^{-1,s}(Q; \mathbb{R}^l), W^{1,s}_0(\mathbb{R}^d)} = - \sum_{i=1}^{N} \left\{ \int_{Q} A^i(x_0 + ry) \cdot \hat{w}_k(y) \cdot \varphi(y) dy \right\} + \int_{Q} (A^i(x_0 + r_k y) - A^i(x_0)) \hat{w}_k(y) \cdot \frac{\partial \varphi(y)}{\partial y_i} dy.
\]
Thus,
\[
\| (\mathcal{A}(x_0 + r \cdot) - \mathcal{A}(x_0)) \hat{w}_k \|_{W^{-1,s}(Q; \mathbb{R}^l)} \leq r_k \sum_{i=1}^{N} \| A^i \|_{W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^{d \times d})} \| \hat{w}_k \|_{L^q(Q; \mathbb{R}^d)}
\]
for every \( 1 < s < q \). By (3.19) and (3.20) we conclude that
\[
\mathcal{A}(x_0) \hat{w}_k \to 0 \quad \text{strongly in } W^{-1,s}(Q; \mathbb{R}^l) \quad \text{for every } 1 < s < q.
\]
In view of (3.19) and (3.22), an adaptation of [6, Corollary 3.3] yields a \( q \)--equiintegrable sequence \( \{ w_k \} \) such that
\[
w_k \to 0 \quad \text{weakly in } L^q(Q; \mathbb{R}^d),
\]
\[
\int_{Q} w_k(y) dy = 0 \quad \text{for every } k,
\]
\[
\mathcal{A}(x_0) w_k = 0 \quad \text{for every } k,
\]
and
\[
\liminf_{k \to +\infty} \int_{Q} g(x_0, v(x_0) + w_k(y)) dy \leq \liminf_{k \to +\infty} \int_{Q} g(x_0 + r_k y, v(x_0) + \hat{w}_k(y)) dy.
\]
Finally, by combining (3.21), (3.23), and (3.24), and by the definition of \(\mathcal{A}\)-quasiconvex envelope for operators with constant coefficients, we obtain
\[
\frac{dI((u, v), \cdot)}{d\mathcal{L}^N}(x_0) \geq \liminf_{k \to +\infty} \int_Q g(x_0, v(x_0) + w_k(y)) \, dy
= \liminf_{k \to +\infty} \int_Q f(x_0, u(x_0), v(x_0) + w_k(y)) \, dy \geq Q_{\mathcal{A}(x_0)}(x_0, u(x_0), v(x_0))
\]
for a.e. \(x_0 \in \Omega\). This concludes the proof of Claim (3.12).

**Step 4:**
To complete the proof of the theorem we need to show that
\[
\frac{dI((u, v), \cdot)}{d\mathcal{L}^N}(x_0) \leq Q_{\mathcal{A}(x_0)}(x_0, u(x_0), v(x_0)) \quad \text{for a.e. } x_0 \in \Omega.
\] (3.25)

To this aim, let \(\mu > 0\), and \(x_0 \in \omega\) be such that (3.14) and (3.15) hold. Let \(w \in C^\infty_{\text{per}}(\mathbb{R}^N; \mathbb{R}^d)\) be such that
\[
\int_Q w(y) \, dy = 0, \quad \mathcal{A}(x_0)w = 0,
\] (3.26)
and
\[
\int_Q f(x_0, u(x_0), v(x_0) + w(y)) \, dy \leq Q_{\mathcal{A}(x_0)}(x_0, u(x_0), v(x_0)) + \mu.
\] (3.27)

Let \(\eta \in C^\infty_c(\Omega; [0, 1])\) be such that \(\eta \equiv 1\) in a neighborhood of \(x_0\) and let \(r\) be small enough so that
\[
Q(x_0, r) \subset \{ x : \eta(x) = 1 \} \quad \text{and} \quad Q(x_0, 2r) \subset \subset \Omega.
\] (3.28)

Consider a map \(\varphi \in C^\infty_c(Q(x_0, r); [0, 1])\) satisfying
\[
\mathcal{L}^N(Q(x_0, r) \cap \{ \varphi \neq 1 \}) < \mu r^N,
\] (3.29)
and define
\[
z^r_m(x) := \varphi(x) w \left( \frac{m(x - x_0)}{r} \right) \quad \text{for } x \in \mathbb{R}^N.
\] (3.30)

We observe that \(z^r_m \in L^q(\Omega; \mathbb{R}^d)\), and for \(\psi \in L^q(\Omega; \mathbb{R}^d)\) we have
\[
\int_\Omega z^r_m(x) \cdot \psi(x) \, dx = \int_\Omega \varphi(x) w \left( \frac{m(x - x_0)}{r} \right) \cdot \psi(x) \, dx
= r^N \int_Q \varphi(x_0 + ry) w(my) \cdot \psi(x_0 + ry) \, dy.
\]

By (3.26) and by the Riemann-Lebesgue lemma we have
\[
z^r_m \to 0 \quad \text{weakly in } L^q(\Omega; \mathbb{R}^d)
\] (3.31)
as \(m \to +\infty\). We claim that
\[
\limsup_{m \to +\infty} \| \mathcal{A}_{\eta} z^r_m \|_{W^{1,q}(\Omega; \mathbb{R}^d)} \leq C r^{\frac{N}{q} + 1},
\] (3.32)
where \(\mathcal{A}_{\eta}\) is the pseudo-differential operator defined in (2.2). Indeed, by (3.28) we obtain
\[
\mathcal{A}_{\eta} z^r_m = \mathcal{A} z^r_m - \mathcal{A}(x_0) z^r_m + \mathcal{A}(x_0) z^r_m
= \sum_{i=1}^N \frac{\partial (A^i(x) - A^i(x_0)) z^r_m(x)}{\partial x_i} + \sum_{i=1}^N A^i(x_0) \frac{\partial z^r_m(x)}{\partial x_i} - \sum_{i=1}^N \frac{\partial A^i(x)}{\partial x_i} z^r_m(x).
\] (3.33)
By the regularity of the operators $A^i$ and by a change of variables, the first term in the right-hand side of (3.33) is estimated as
\[
\left\| \sum_{i=1}^{N} \frac{\partial (A^i(x) - A^i(x_0))}{\partial x_i} z_m^r(x) \right\|_{W^{1,q}(\Omega;\mathbb{R}^d)} \leq \sum_{i=1}^{N} \left\| (A^i(x) - A^i(x_0)) \varphi(x) w \left( \frac{m(x - x_0)}{r} \right) \right\|_{L^q(Q(x_0, r); \mathbb{R}^d)} \leq \sum_{i=1}^{N} \| A^i \|_{W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^d)} \| \varphi \|_{L^q(Q(x_0, r))} \| w(m \cdot) \|_{L^q(Q; \mathbb{R}^d)} \frac{L}{r} + 1 \leq C_L \frac{L}{r} + 1.
\] (3.34)

In view of (3.26) the second term in the right-hand side of (3.33) becomes
\[
\sum_{i=1}^{N} A^i(x_0) \frac{\partial z_m^r(x)}{\partial x_i} = \sum_{i=1}^{N} A^i(x_0) \frac{\partial \varphi(x)}{\partial x_i} w \left( \frac{m(x - x_0)}{r} \right),
\]
and thus converges to zero weakly in $L^q(\Omega; \mathbb{R}^d)$, as $m \to +\infty$, due to (3.26) and by the Riemann-Lebesgue lemma. Hence,
\[
\left\| \sum_{i=1}^{N} A^i(x_0) \frac{\partial z_m^r(x)}{\partial x_i} \right\|_{W^{-1,q}(\Omega;\mathbb{R}^d)} \to 0 \quad \text{as} \quad m \to +\infty
\] (3.35)
by the compact embedding of $L^q(\Omega; \mathbb{R}^d)$ into $W^{-1,q}(\Omega; \mathbb{R}^d)$. Finally, the third term in the right-hand side of (3.33) satisfies
\[
\left\| \sum_{i=1}^{N} \frac{\partial A^i(x)}{\partial x_i} z_m^r(x) \right\|_{W^{-1,q}(\Omega;\mathbb{R}^d)} \to 0 \quad \text{as} \quad m \to +\infty.
\] (3.36)
Claim (3.32) follows by combining (3.34)–(3.36).

Consider the maps
\[
v_m^r := P_\eta z_m^r,
\]
where $P_\eta$ is the projection operator introduced in (2.3). By Proposition 2.2 we have
\[
\| v_m^r \|_{L^q(Q(x_0, r); \mathbb{R}^d)} \leq C \| z_m^r \|_{L^q(\Omega; \mathbb{R}^d)}, \quad \| v_m^r \|_{W^{-1,q}(Q(x_0, r); \mathbb{R}^d)} \leq C \| z_m^r \|_{W^{-1,q}(\Omega; \mathbb{R}^d)}, \quad \| \mathcal{A} v_m^r \|_{W^{-1,q}(Q(x_0, r); \mathbb{R}^d)} \leq C \| z_m^r \|_{W^{-1,q}(\Omega; \mathbb{R}^d)}, \quad \| v_m^r - z_m^r \|_{L^q(Q(x_0, r); \mathbb{R}^d)} \leq C(\| \mathcal{A} v_m^r \|_{W^{-1,q}(\Omega; \mathbb{R}^d)} + \| z_m^r \|_{W^{-1,q}(\Omega; \mathbb{R}^d)}).
\] (3.37)–(3.40)
By (3.31) and (3.37), the sequence $\{v_m^r\}$ is uniformly bounded in $L^q(Q(x_0, r); \mathbb{R}^d)$. Thus, there exists a map $v^r \in L^q(Q(x_0, r); \mathbb{R}^d)$ such that, up to the extraction of a (not relabelled) subsequence,
\[
v_m^r \rightharpoonup v^r \quad \text{in} \quad L^q(Q(x_0, r); \mathbb{R}^d)
\] (3.41)
as $m \to +\infty$. Again by (3.31), and by the compact embedding of $L^q$ into $W^{-1,q}$, we deduce that
\[
z_m^r \to 0 \quad \text{strongly in} \quad W^{-1,q}(\Omega; \mathbb{R}^d)
\] (3.42)
as $m \to +\infty$. Therefore, by combining (3.38) and (3.41), we conclude that
\[
v_m^r \to 0 \quad \text{weakly in} \quad L^q(Q(x_0, r); \mathbb{R}^d)
\]
as \( m \to +\infty \), and the convergence holds for the entire sequence. Additionally, by (3.28), (3.39), and (3.42), we obtain  
\[
\mathcal{A}^r_n v^r_m = \mathcal{A}^r_n v^r_m \to 0 \quad \text{strongly in } W^{-1,q}(Q(x_0,r) ; \mathbb{R}^d)
\]
as \( m \to +\infty \). Finally, by (3.32), (3.40), and (3.42), there holds  
\[
\lim_{r \to 0} \lim_{m \to +\infty} r^{-\frac{N}{q}} \| v^r_m - z^r_m \|_{L^q(Q(x_0,r) ; \mathbb{R}^d)} = 0. 
\] (3.43)

We recall that, since \( x_0 \) satisfies (3.15), Step 1 yields  
\[
dI(u,v)(x_0) = \lim_{r \to 0^+} \frac{I((u,v);Q(x_0,r))}{r^N} \leq \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0,r)} f(x,u(x),v(x) + v^r_m(x)) \, dx. 
\] (3.44)

We claim that  
\[
dI(u,v)(x_0) = \lim_{r \to 0^+} \frac{I((u,v);Q(x_0,r))}{r^N} \leq \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0,r)} g(x,v(x) + z^r_m(x)) \, dx, 
\] (3.45)

where \( g \) is the function introduced in Step 3. Indeed, for every \( r \in \mathbb{R} \), consider the function \( g^r : Q \times \mathbb{R}^d \to [0, +\infty) \) defined as  
\[
g^r(y,\xi) := g(x_0 + ry,\xi) \quad \text{for every } y \in Q, \xi \in \mathbb{R}^d.
\]

Since \( x_0 \in \omega \), by (3.13) there exists \( K_j \) such that \( x_0 \in K_j \). In particular, this yields the existence of \( r_0 > 0 \) such that for \( r \leq r_0 \), the maps \( g^r \) are continuous on \( Q \times \mathbb{R}^d \), and the family \( \{g^r(\cdot,\cdot)\} \) is equicontinuous in \( \mathbb{R}^d \), uniformly with respect to \( y \). A change of variables yields  
\[
\frac{1}{r^N} \int_{Q(x_0,r)} f(x,u(x),v(x) + v^r_m(x)) \, dx - \int_{Q(x_0,r)} f(x,u(x),v(x) + z^r_m(x)) \, dx 
\]
\[
= \left| \int_{Q} g^r(y,v(x_0 + ry) + v^r_m(x_0 + ry)) \, dy - \int_{Q} g^r(y,v(x_0 + ry) + z^r_m(x_0 + ry)) \, dy \right|.
\]

On the other hand, by (3.43) we have  
\[
\lim_{r \to 0} \lim_{m \to +\infty} \| z^r_m(x_0 + r \cdot) - v^r_m(x_0 + r \cdot) \|_{L^q(Q ; \mathbb{R}^d)} = \lim_{r \to 0} \lim_{m \to +\infty} r^{-\frac{N}{q}} \| z^r_m - v^r_m \|_{L^q(Q(x_0,r) ; \mathbb{R}^d)} = 0.
\]

Therefore, by a diagonal procedure we extract a subsequence \( \{m_r\} \) such that  
\[
\limsup_{r \to 0} \limsup_{m \to +\infty} \left| \int_{Q} g^r(y,v(x_0 + ry) + v^r_m(x_0 + ry)) \, dy - \int_{Q} g^r(y,v(x_0 + ry) + z^r_m(x_0 + ry)) \, dy \right| 
\]
\[
= \lim_{r \to 0} \left| \int_{Q} g^r(y,v(x_0 + ry) + v^r_m(x_0 + ry)) \, dy - \int_{Q} g^r(y,v(x_0 + ry) + z^r_m(x_0 + ry)) \, dy \right|, 
\] (3.46)

and  
\[
z^r_{m_r}(x_0 + r \cdot) - v^r_{m_r}(x_0 + r \cdot) \to 0 \quad \text{strongly in } L^q(Q ; \mathbb{R}^d).
\]

In view of (3.14), (3.30) and the Riemann-Lebesgue lemma, the sequence \( \{v(x_0 + r \cdot) + z^r_{m_r}(x_0 + r \cdot)\} \) is \( q \)-equiintegrable in \( Q \). Hence, by (H) we are under the assumptions of Proposition 3.2, and we conclude that  
\[
\lim_{r \to 0} \left| \int_{Q} g^r(y,v(x_0 + ry) + v^r_{m_r}(x_0 + ry)) \, dy - \int_{Q} g^r(y,v(x_0 + ry) + z^r_{m_r}(x_0 + ry)) \, dy \right| = 0. 
\] (3.47)

Claim (3.45) follows by combining (3.46) with (3.47).
Arguing as in [6, Proof of Lemma 3.5], for every $x_0 \in \omega$ (where $\omega$ is the set defined in (3.13)) we have

$$\liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x, u(x), v(x) + z_r^m(x)) \, dx$$

$$\leq \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + z_r^m(x)) \, dx,$$

hence by (3.45) we deduce that

$$\frac{dI(u, v)}{d\mathcal{L}^N}(x_0) \leq \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + z_r^m(x)) \, dx.$$

By (3.30) we obtain

$$\frac{dI(u, v)}{d\mathcal{L}^N}(x_0) \leq \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + z_r^m(x)) \, dx$$

$$\leq \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \left\{ \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + w\left(\frac{m(x - x_0)}{r}\right)) \, dx \right\}$$

$$+ \int_{Q(x_0, r) \cap \{ \varphi \neq 1 \}} f(x_0, u(x_0), v(x_0) + \varphi(x)w\left(\frac{m(x - x_0)}{r}\right)) \, dx \right\}.$$ 

The growth assumption (H) and estimate (3.29) yield

$$\int_{Q(x_0, r) \cap \{ \varphi \neq 1 \}} f(x_0, u(x_0), v(x_0) + \varphi(x)w\left(\frac{m(x - x_0)}{r}\right)) \, dx$$

$$\leq C \int_{Q(x_0, r) \cap \{ \varphi \neq 1 \}} \left(1 + \left| w\left(\frac{m(x - x_0)}{r}\right)\right|^q \right) \, dx$$

$$\leq C(1 + \| w \|_{\infty, (\mathbb{R}^N, \mathcal{L}^N)}^q L^N(Q(x_0, r) \cap \{ \varphi \neq 1 \}) \leq C \mu r^N.$$

Thus, by (3.48), the periodicity of $w$, and Riemann-Lebesgue lemma, we deduce

$$\frac{dI(u, v)}{d\mathcal{L}^N}(x_0) \leq C\mu + \liminf_{r \to 0^+} \liminf_{m \to +\infty} \frac{1}{r^N} \int_{Q(x_0, r)} f(x_0, u(x_0), v(x_0) + w\left(\frac{m(x - x_0)}{r}\right)) \, dx$$

$$= C\mu + \liminf_{m \to +\infty} \int_{Q} f(x_0, u(x_0), v(x_0) + w(y)) \, dy$$

$$= C\mu + \int_{Q} f(x_0, u(x_0), v(x_0) + w(y)) \, dy$$

$$\leq C\mu + Q_{\mathcal{A}}(x_0) f(x_0, u(x_0), v(x_0)),$$

where the last inequality is due to (3.27). Letting $\mu \to 0^+$ we conclude (3.25).

\[\square\]

**Acknowledgements**

The authors thank the Center for Nonlinear Analysis (NSF Grant No. DMS-0635983), where this research was carried out, and also acknowledge support of the National Science Foundation under the PIRE Grant No. OISE-0967140. The research of I. Fonseca and E. Davoli was funded by the National Science Foundation under Grant No. DMS-0905778. E. Davoli acknowledges the support of the Austrian Science Fund (FWF) projects P 27052 and I 2375. The research of I. Fonseca was further partially supported by the National Science Foundation under Grant No. DMS-1411646.
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