Jumping flatness and Aluthge transform of recursive weighted shifts

Hamza El Azhar\textsuperscript{a}, Kaissar Idrissi\textsuperscript{a}, El Hassan Zerouali\textsuperscript{a,∗}

\textsuperscript{a}Mohammed V University in Rabat, Rabat, Morocco.

Abstract

We devote this paper to Hamburger type weighted shifts. We give in particular an affirmative answer to a problem concerning subnormality of the Aluthge transform of Hamburger moment measures with finite support. We also extend the notion flatness, “jumping flatness property” introduced recently by Exner et al. for Hamburger-type weighted shift and provide obtain several results related to the representing measure of such weighted shifts.

Keywords: Jumping flatness, Aluthge transforms, Recursive sequences, Hamburger weighted type sequences, subnormal weighted shift.

2020 MSC: Primary 11B99; 44A60; 47B37 Secondary 30C15, 40A99.

1. Introduction

Let us denote \( \mathcal{H} \) an infinite dimensional Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) be the space of all bounded linear operators on \( \mathcal{H} \). An operator \( T \in \mathcal{L}(\mathcal{H}) \) is normal if \( TT^* = T^*T \), is subnormal if it is the restriction of some normal operators and is hyponormal if \( T^*T - TT^* \geq 0 \). Here \( T^* \) stands for the usual adjoint operator of \( T \). The polar decomposition of an operator is given by the unique representation \( T = U|T| \), where \( |T| = (T^*T)^{\frac{1}{2}} \) and \( U \) is a partial isometry satisfying \( kerU = kerT \) and \( kerU^* = kerT^* \). The Aluthge transform is then given by the expression

\[ \tilde{T} = |T|^\frac{1}{2}U|T|^\frac{1}{2}. \]

\textsuperscript{∗}The authors are supported by the CeReMaR and the Hassan II Academy of sciences. The last author is supported by African university of Sciences and technology-Abuja. Nigeria.

\textsuperscript{∗}Corresponding author

Email addresses: elazharhamza@gmail.com (Hamza El Azhar), kaissar.idrissi@gmail.com (Kaissar Idrissi), elhassan.zerouali@um5.ac.ma (El Hassan Zerouali)
The Aluthge transform was introduced in [1] by Aluthge, in order to extend several inequalities valid for hyponormal operators, and has received deep attention in the recent years.

We consider below $H = l^2(\mathbb{Z}_+)$ endowed by some orthonormal basis $\{e_n\}_{n \in \mathbb{Z}_+}$. The forward shift operator $W_\alpha$ is defined on the basis by $W_\alpha e_n = \alpha_n e_{n+1}$, where $\alpha = \{\alpha_n\}_{n \geq 0}$ is a given a sequence of positive real numbers (called weights). We associate with $W_\alpha$ the moments sequence obtained by

$$\gamma_0 = 1 \text{ and } \gamma_k \equiv \gamma_k(\alpha) := \alpha_0^2\alpha_1^2\cdots\alpha_{k-1}^2 \text{ for } k \geq 1.$$ 

We will say that a sequence $\gamma$ admits a representing signed measure (called also a charge) supported in $K \subset \mathbb{R}$, if

$$\gamma_n = \int_K t^n \mu(t) \text{ for every } n \geq 0 \text{ and } \text{supp}(\mu) \subset K. \quad (1.1)$$

It is known since 1938 that every sequence is a charge moment sequence supported in the real field. [11].

The weighted shift operator is bounded if and only if $\|W_\alpha\| = \sup_{n \geq 0} \alpha_n < +\infty$. It is clear that $W_\alpha$ is never normal and that $W_\alpha$ is hyponormal precisely when $\alpha_n$ is non decreasing. On the other hand, Berger’s Theorem says, $W_\alpha$ is a subnormal operator, if and only if, there exists a nonnegative Borelean measure $\mu$ (called Berger measure), which is a representing measure of $\{\gamma_n\}_{n \geq 0}$ and such that $\text{supp}(\mu) \subset [0, \|W_\alpha\|^2]$. The latter is equivalent to the positivity of the two Hankel matrices $(\gamma_{i+j})_{i,j \geq 0}$ and $(\gamma_{i+j+1})_{i,j \geq 0}$. It is also known that $\{\gamma_n\}_{n \geq 0}$ admits a nonnegative representing measure supported in $\mathbb{R}$ if and only if $(\gamma_{i+j})_{i,j \geq 0}$. Such a sequence will said to be Hamburger sequence.

We accord to the sequence $\gamma \equiv \{\gamma_n\}_{n \in \mathbb{Z}_+}$ the following matrices

$$M_n(k) = \begin{pmatrix}
\gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n} \\
\gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{k+n} & \gamma_{k+n+1} & \cdots & \gamma_{k+2n}
\end{pmatrix}, \text{ for } n, k \in \mathbb{Z}_+. \quad (1.2)$$

Recall the following definitions from [9].

2
Definition 1.1. A weighted shift $W_\alpha$ has property $H(n)$ [resp., property $\tilde{H}(n)$] if $M_n(k) \geq 0$ for all $k = 0, 2, 4, ...$ [resp., $\tilde{M}_n(k) \geq 0$ for all $k = 1, 3, 5, ...$]. And $W_\alpha$ has property $H(\infty)$ [resp., property $\tilde{H}(\infty)$] if it has property $H(n)$ [resp., property $\tilde{H}(n)$] for all $n \in \mathbb{Z}_+$. In particular, $W_\alpha$ is a Hamburger-type weighted shift if $W_\alpha$ has property $H(\infty)$ and is subnormal if $W_\alpha$ has both $H(\infty)$ and $\tilde{H}(\infty)$.

For a large family of shifts, a flatness phenomena occurs, when two successive weight are equal. More precisely

Proposition 1.2. Let $W_\alpha$ be a weighted shift such that $\alpha_{n_0} = \alpha_{n_0 + 1}$ for some $n_0 \geq 1$. We have

(i) (12, Theorem 6). If $W_\alpha$ subnormal, then $\alpha$ is flat, i.e, $\alpha_1 = \cdots = \alpha_n = \cdots$

(ii) (4, Corollary 6). If $W_\alpha$ has property $H(2)$ and $\tilde{H}(2)$ then $\alpha$ is flat.

(iii) (8, Theorem 4.2). If $W_\alpha$ has property $H(3)$ then $\alpha$ is flat.

Jumping flatness of weighted shifts was considered in some recent paper by Exner et al. It is defined as follows,

Definition 1.3. A weighted shift $W_\alpha$ with weight sequence $\alpha = (\alpha_n)_{n \geq 0}$ is said to have the jumping flatness property if $\alpha_n = \alpha_{n+2}$ for every $n \geq 1$. In addition, $W_\alpha$ has jumping flatness property of type I, if $\alpha_0 < \alpha_2$, and has jumping flatness property of type II, if $\alpha_0 = \alpha_2$.

We outline the definition above to give the next extension of jumping flatness,

Definition 1.4. A weighted shift $W_\alpha$, with weight sequence $(\alpha_n)_{n \geq 0}$, has the $k$-jumping flatness property if $\alpha_n = \alpha_{n+k}$ for every integer $n \geq 1$.

Clearly, flatness is 1-jumping flatness property and the Jumping flatness property, introduced in [9], coincides with 2-jumping flatness property. Moreover, subnormal weighted shifts with $k$-jumping property are flat, since their weights are non-decreasing. Hence, $k$-jumping flatness property is consistent only for non-subnormal weighted shifts.

The Aluthge transform $\tilde{W}_\alpha$ of a weighted shift $W_\alpha$ is also a weighted shift, denoted below $W_{\tilde{\alpha}}$. Indeed, it is easy to check that $|W_\alpha|e_n = \alpha_n e_n$ and that $U e_n = e_{n+1}$. It follows then that

$$\tilde{W}_\alpha e_n = \sqrt{\alpha_0 \alpha_{n+1}} e_{n+1} = \tilde{\alpha}_n e_{n+1} = W_{\tilde{\alpha}} e_n.$$
Notice also that $\gamma_n^2 = \frac{1}{\alpha n} \gamma_n \gamma_{n+1}$. In several recent papers, the problem of subnormality of the Aluthge transform of weighted shifts was considered. See [8], for example. In [8], the next question was considered

**HP** Under what conditions is the Hamburger-type property of a weighted shift preserved under the Aluthge transform?

A special attention was devoted to weighted shifts with 2-jumping flatness property. Since in this case, as it will be shown below, the representing measure has only two or 3 atoms, the next general problem is stated.

**Problem 5.8.** Let $W_\alpha$ be a weighted shift with the associated Hamburger moment measure $\mu := \phi \delta_p + \psi \delta_r + \rho \delta_q$ for some $p > 0$ and $-p < r < q$. Is it true that $\tilde{W}_\alpha$ is subnormal if and only if $r = 0$ and $p = q$?

This paper is organised as follows, We solve first problem 5.8 by giving an affirmative answer in the next section. We also give several results concerning (HP). Section 3 is devoted to $k-$ jumping flatness property for weighted shifts satisfying $H(n)$ for some adequate $n$. It is shown that a propagation phenomena occurs in this case.

## 2. Weighted shifts with subnormal Aluthge transform

Let $W_\alpha$ be a weighted shift with associated representing moment measure

$$\mu := a \delta_{-p} + b \delta_r + c \delta_q$$

for some $p > 0$ and $-p < r < q$. The problem is to find conditions on the parameters $p, q$ and $r$ equivalent to $\tilde{W}_\alpha$ is subnormal. We start with the next affirmative answer to Problem 5.8.

**Theorem 2.1.** Let $W_\alpha$ be a weighted shift with the associated representing moment measure

$$\mu := a \delta_{-p} + b \delta_r + c \delta_q$$

for some $p > 0$ and $-p < r < q$. Then $\tilde{W}_\alpha$ is subnormal if and only if $r = 0$ and $p = q$.

**Proof.** The following proposition that extends the subnormal case treated in [2], Proposition 1.2], is the key point of our proof:
Proposition 2.2. Let $W_\alpha$ be a Hamburger weighted shift with associated representing moment measure $\mu$. Then $\tilde{W}_\alpha$ admits a representing moment measure if and only if there exists a probability measure $\nu$ such that $\nu \ast \nu = \mu \ast t\mu$. Furthermore, $\tilde{W}_\alpha$ is subnormal if and only if $\nu$ is $\mathbb{R}^+$-supported.

Where $\ast$ denotes the multiplicative convolution

$$[\nu \ast \mu](A) = \int_{\mathbb{R}} \chi_A(xy)d\nu(x)d\mu(y)$$

Proof of Proposition 2.2. For the direct implication, suppose there exists a measure $\nu$ such that $\nu \ast \nu = \mu \ast t\mu$. Then

$$\alpha_0 \gamma_n^2 = \gamma_n \gamma_{n+1} = \left( \int_{\mathbb{R}} t^n d\mu(t) \right) \left( \int_{\mathbb{R}} t^{n+1} d\mu(t) \right)$$

$$= \left( \int_{\mathbb{R}} t^n d\mu(t) \right) \left( \int_{\mathbb{R}} s^n d\mu(s) \right) = \int_{\mathbb{R}} \int_{\mathbb{R}} (st)^n d\mu(t)s d\mu(s)$$

$$= \int_{\mathbb{R}} u^n d((\mu \ast t\mu)(u)) = \int_{\mathbb{R}} u^n d(\nu \ast \nu)(u) = \left( \int_{\mathbb{R}} u^n d\nu(u) \right)^2,$$

and hence $\nu$ is representing for $\tilde{W}_\alpha$.

Conversely, assume that $\tilde{W}_\alpha$ admits a representing measure. Since $W_\alpha$ admits a representing measure, then $W_\alpha^2 = W_\alpha S(\alpha)$ has as representing measure $\mu \ast t\mu$. Where $S(\alpha) = (\alpha_{n+1})_n$.

Thus we have

$$\int_{\mathbb{R}} u^n d(\nu \ast \nu)(u) = \left( \int_{\mathbb{R}} u^n d\nu(u) \right)^2 = \int_{\mathbb{R}} u^n d(\mu \ast t\mu)(u)$$

Since the weighted shift $W_\alpha$ is bounded, we get the the support of $\mu$ and $\nu$ are compact [8], so by Riesz representation theorem, we have $\nu \ast \nu = \mu \ast t\mu$.

For the last point, since the measure $\nu$ is Hamburger determinate we get that the Aluthge transform is subnormal if and only if $\nu$ is supported on $\mathbb{R}^+$. Sufficiency is obtained by [9, Theorem 5.7].

Assume now that $\tilde{W}_\alpha$ is subnormal. There exists a measure $\nu$ supported on $\mathbb{R}^+$, such that $\mu \ast t\mu = \nu \ast \nu$, in particular the support of $\mu \ast t\mu$ is in $\mathbb{R}^+$. Moreover,

$$\mu \ast t\mu = -pa^2 \delta_{\rho^2} + rb^2 \delta_{\rho^2} + qe^2 \delta_{\rho^2} + ab(r-p)\delta_{-pr} + ac(q-p)\delta_{-pq} + bc(q+r)\delta_{rq} \quad (2.1)$$
Since $q > 0$ (because weights are positive), $-pq < 0$, so $ac(q - p) = 0$ this is equivalent to say that $p = q$. Hence (2.1) transforms to

$$\mu * t\mu = p(c^2 - a^2)\delta_{p^2} + rb^2\delta_{r^2} + ab(r - p)\delta_{-pr} + bc(p + r)\delta_{rp}. \quad (2.2)$$

If we assume that $r \neq 0$ then because of the term $ab(r - p)\delta_{-pr}$, we get $r = p = q$ which is impossible ($r < q$). Finally $p = q$ and $r = 0$.

We deduce the next corollary,

**Corollary 2.3.** Let $W_\alpha$ be a Hamburger recursively generated weighted shift and let $\mu$ be its associated representing moment measure. If $\tilde{W}_\alpha$ is of Hamburger type with associated measure $\nu$, then $\nu$ is discrete, Furthermore

$$|\text{supp}(\nu)| = |\text{supp}(\mu)|.$$ 

Where

$$|\text{supp}(\mu)| = \{|\lambda| : \lambda \in \text{supp}(\mu)\}.$$

The previous proposition yields a more general result of independent interest.

**Theorem 2.4.** Let $W_\alpha$ be a non subnormal Hamburger recursively generated weighted shift and let $\mu = k \sum a_i \delta_{\lambda_i}$ be its associated representing moment measure. If for every $i \neq j, k$ satisfying $\lambda_i \leq 0$, we have $\lambda_i \neq \lambda_j \lambda_k$, then $\tilde{W}_\alpha$ is not a Hamburger type shift.

**Proof.** We have $\mu := \sum a_i \delta_{\lambda_i}$ for some $a_i > 0$, and $\lambda_i \in \mathbb{R}$. We conclude by observing that

$$\mu * t\mu = \sum_{i,j=0}^k a_i a_j \lambda_{i+j} \delta_{\lambda_i \lambda_j} \quad (2.3)$$

Indeed, suppose $\tilde{W}_\alpha$ is of Hamburger type, We obtain the measure of $\mu * t\mu$ is non negative, since $W_\alpha$ is non subnormal there is $\lambda_i < 0$. Then the quantity $a_i^2 \lambda_i \delta_{\lambda_i^2}$ in the expression of $\mu * t\mu$, contradicts $\mu$ is non negative.

Concerning the problem of preservation of Hamburger type property by the Aluthge transform in (HP), we have the following result.
Theorem 2.5. Let $W_\alpha$ be a Hamburger recursively generated weighted shift such that the associated measure has four atoms. Then $\tilde{W}_\alpha$ is not a Hamburger type shift.

Proof. We pose $\mu = a_1\delta_{\lambda_1} + a_2\delta_{\lambda_2} + a_3\delta_{\lambda_3} + a_4\delta_{\lambda_4}$, with $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and $a_i > 0$ for $1 \leq i \leq 4$. We will distinguish various cases.

1. $\text{supp}(\mu) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbb{R}_+$. We will get $W_\alpha$ is subnormal, and hence from Proposition 4.6] is not recursive. We conclude from Corollary 2.3 that $\tilde{W}_\alpha$ is not a Hamburger type.

2. $\lambda_1 < 0 \leq \lambda_2 < \lambda_3 < \lambda_4$. Assume that $\tilde{W}_\alpha$ is Hamburger type and let $\nu$ be the associated measure.

Writing

$$\nu * \nu = \mu * t\mu = a_1^2\lambda_1\delta_{\lambda_1^2} + a_2^2\lambda_2\delta_{\lambda_2^2} + a_3^2\lambda_3\delta_{\lambda_3^2} + a_4^2\lambda_4\delta_{\lambda_4^2} + \sum_{1 \leq i < j \leq 4} a_ia_j(\lambda_i + \lambda_j)\delta_{\lambda_i\lambda_j},$$

we obtain $\mu * t\mu$ is non negative, and then we derive, $-\lambda_1 \leq \lambda_2$. If $-\lambda_1 < \lambda_2$, we will contradict again $\mu * t\mu$ is non negative, because of the term $a_1^2\lambda_1\delta_{\lambda_1^2}$ in $\mu * t\mu$. Thus $\lambda_1 = -\lambda_2$. It will come

$$\mu * t\mu = (a_1^2 - a_2^2)\lambda_2\delta_{\lambda_2^2} + a_3^2\lambda_3\delta_{\lambda_3^2} + a_4^2\lambda_4\delta_{\lambda_4^2} + a_1a_3(\lambda_3 - \lambda_2)\delta_{-\lambda_2\lambda_3} + a_1a_4(\lambda_4 - \lambda_2)\delta_{-\lambda_2\lambda_4}$$

$$+ a_2a_3(\lambda_3 + \lambda_2)\delta_{\lambda_2\lambda_3} + a_2a_4(\lambda_4 + \lambda_2)\delta_{\lambda_2\lambda_4} + a_4a_3(\lambda_3 + \lambda_4)\delta_{\lambda_3\lambda_4}.$$

In particular,

$$\{\lambda_3^2, \lambda_1^2, \pm\lambda_2\lambda_3, \pm\lambda_2\lambda_4, \lambda_3\lambda_4\} \subset \text{supp}(\mu * t\mu) \subset \{\lambda_2^2, \lambda_3^2, \lambda_4^2, \pm\lambda_2\lambda_3, \pm\lambda_2\lambda_4, \lambda_3\lambda_4\}$$

Using Corollary 2.3 we get

$$\nu = b_2\delta_{\lambda_2} + b_3\delta_{\lambda_3} + b_4\delta_{\lambda_4} + c_2\delta_{-\lambda_2} + c_3\delta_{-\lambda_3} + c_4\delta_{-\lambda_4}$$

with $b_i \geq 0$ and $c_i \geq 0$ for $2 \leq i \leq 4$.

From $\nu * \nu = \mu * t\mu$ we deduce that :

$$\{\lambda_3^2, \lambda_1^2, \pm\lambda_2\lambda_3, \pm\lambda_2\lambda_4, \lambda_3\lambda_4\} \subset \text{supp}(\mu * \nu) \subset \{\lambda_2^2, \lambda_3^2, \lambda_4^2, \pm\lambda_2\lambda_3, \pm\lambda_2\lambda_4, \lambda_3\lambda_4\} \quad (2.4)$$

But,

$$\nu * \nu = (b_2^2 + c_2^2)\delta_{\lambda_2^2} + (b_3^2 + c_3^2)\delta_{\lambda_3^2} + (b_4^2 + c_4^2)\delta_{\lambda_4^2} + 2[(b_2b_3 + c_2c_3)\delta_{\lambda_2\lambda_3} + (b_3b_4 + c_3c_4)\delta_{\lambda_3\lambda_4}$$

$$+ (b_2b_4 + c_2c_4)\delta_{\lambda_2\lambda_4} + (b_3b_2 + c_3c_2)\delta_{-\lambda_2\lambda_3} + (b_3b_4 + c_3c_4)\delta_{-\lambda_3\lambda_4} + (b_2c_4 + c_2b_4)\delta_{-\lambda_2\lambda_4}$$

$$+ b_2c_2\delta_{-\lambda_2^2} + b_3c_3\delta_{-\lambda_3^2} + b_4c_4\delta_{-\lambda_4^2}]$$
We derive the next equations:

\[
\begin{aligned}
    b_4c_4 &= 0 \\
    b_3c_4 + c_3b_4 &= 0 \\
    b_2c_2 &= 0
\end{aligned}
\]

This is equivalent to

\[
b_2c_2 = 0 \text{ and } ((b_3, b_4) = (0, 0) \text{ or } (c_3, c_4) = (0, 0)).
\]

We derive four different cases

I)\(-\langle b_2, b_3, b_4 \rangle = (0, 0, 0)\) in this case supp(\(\nu\)) \(\subset \mathbb{R}_-\), this implies that supp(\(\nu \ast \nu\)) \(\subset \mathbb{R}_+\) which is impossible by \([2.4]\).

II)\(-\langle c_2, c_3, c_4 \rangle = (0, 0, 0)\) Similarly in this case supp(\(\nu\)) \(\subset \mathbb{R}_+\), this implies that supp(\(\nu \ast \nu\)) \(\subset \mathbb{R}_+\) which is impossible by \([2.4]\).

III)\(-\langle b_2, c_3, c_4 \rangle = (0, 0, 0)\) In this case \(\nu = c_2\delta_{-\lambda_2} + b_3\delta_{\lambda_3} + b_4\delta_{\lambda_4}\), this implies that \(\lambda_2\lambda_3 \not\in \text{supp}(\nu \ast \nu)\) which contradict \([2.4]\).

IV)\(-\langle c_2, b_3, b_4 \rangle = (0, 0, 0)\) Similarly in this case \(\nu = b_2\delta_{\lambda_2} + c_3\delta_{-\lambda_3} + c_4\delta_{-\lambda_4}\), this implies that \(\lambda_2\lambda_3 \not\in \text{supp}(\nu \ast \nu)\) which contradict \([2.4]\).

We conclude from this discussion that \(\hat{W}_\alpha\) is not of Hamburger type.

3. \(\lambda_1 < \lambda_2 < 0 \leq \lambda_3 < \lambda_4\). Assume that \(\hat{W}_\alpha\) is Hamburger type and let \(\nu\) be the associated measure.

As in the second case, we drive that \(\lambda_3 \geq -\lambda_2\) and \(\lambda_4 \geq -\lambda_1\), if \(\lambda_4 > -\lambda_1\), because of the term \(a_2^2\lambda_1\delta_{\lambda_1^2}\) in \(\mu \ast t\mu\) this one will be non positive measure. Thus \(\lambda_4 = -\lambda_1\). Similarly, we get that \(\lambda_3 = -\lambda_2\). Write \(\mu = a_1\delta_{-\lambda_4} + a_2\delta_{-\lambda_3} + a_3\delta_{\lambda_3} + a_4\delta_{\lambda_4}\) and then \(\nu = b_1\delta_{-\lambda_4} + b_2\delta_{-\lambda_3} + b_3\delta_{\lambda_3} + b_4\delta_{\lambda_4}\). It will follow,

\[
\begin{aligned}
    \mu \ast t\mu &= \lambda_4(a_1^2 - a_1^2)\delta_{\lambda_4^2} + \lambda_3(a_3^2 - a_3^2)\delta_{\lambda_3^2} \\
    &+ (\lambda_3 + \lambda_4)(a_3a_4 - a_1a_2)\delta_{\lambda_3\lambda_4} + (\lambda_4 - \lambda_3)(a_2a_4 - a_1a_3)\delta_{-\lambda_3\lambda_4},
\end{aligned}
\]

and

\[
\begin{aligned}
    \nu \ast \nu &= (b_1^2 + b_2^2)\delta_{\lambda_4^2} + (b_2^2 + b_4^2)\delta_{\lambda_3^2} + 2(b_1b_2 + b_3b_4)\delta_{\lambda_3\lambda_4} + 2(b_1b_3 + b_2b_4)\delta_{-\lambda_3\lambda_4} \\
    &+ 2b_1b_4\delta_{-\lambda_4^2} + 2b_2b_3\delta_{-\lambda_3^2}.
\end{aligned}
\]
Since \([-\lambda_2^2, -\lambda_3^2] \not\subset supp(\mu)\), we get that
\[
\begin{align*}
  b_1 b_4 &= 0 \\
  b_2 b_3 &= 0
\end{align*}
\]
As for the second case we have four cases, \((b_1, b_2) = (0, 0)\), \((b_1, b_3) = (0, 0)\), \((b_4, b_2) = (0, 0)\), and \((b_4, b_3) = (0, 0)\). The proof run similarly for the four cases.
For \((b_1, b_2) = (0, 0)\), from the equality \(\nu^* \nu = \mu^* t \mu\) we derive that
\[
\begin{align*}
  \lambda_4 (a_2^2 - a_1^2) &= b_3^2 \\
  \lambda_3 (a_3^2 - a_2^2) &= b_4^2 \\
  (\lambda_3 + \lambda_4)(a_3 a_4 - a_1 a_2) &= 2 b_3 b_4 \\
  (\lambda_4 - \lambda_3)(a_2 a_4 - a_1 a_3) &= 0
\end{align*}
\]
Since \(\lambda_3 > 0\) and \(\lambda_4 > 0\). This implies that
\[
\begin{align*}
  \lambda_3 + \lambda_4)^2 (a_3 a_4 - a_1 a_2)^2 &= 4 pq (a_2^2 - a_1^2)(a_3^2 - a_2^2) \\
  a_2 a_4 - a_1 a_3 &= 0
\end{align*}
\]
We get
\[
(\lambda_3 + \lambda_4)^2 (a_3 a_4 - a_1 a_2)^2 = 4 pq (a_3 a_4 - a_1 a_2)^2 \iff (\lambda_3 - \lambda_4)^2 (a_3 a_4 - a_1 a_2)^2 = 0
\]
Using the third equation we conclude that \(b_3 b_4 = 0\). Finally, \(\mu^* t \mu = 0\), which is impossible by Proposition 2.2.

4. \(\lambda_1 < \lambda_2 < \lambda_3 < 0 \leq \lambda_4\). Assume that \(\tilde{W}_\alpha\) is Hamburger type and let \(\nu\) be the associated measure.
As in the second case, we drive that \(\lambda_4 \geq -\lambda_1\), if \(\lambda_4 > -\lambda_1\), because of the term \(a_2^2 \lambda_1^4 \delta_{\lambda_1^2}\) in \(\mu^* t \mu\) this one will be non positive measure. Thus \(\lambda_4 = -\lambda_1\). Similarly, because of the term \(a_2 a_3 (\lambda_2 + \lambda_3) \delta_{\lambda_2 \lambda_3}\) in \(\mu^* t \mu\) this one will be non positive measure.

\[\square\]

**Remark 2.6.**
1. Theorem 2.4 together with [Prop. 4.11] sheds some light on questions (HP) and (SP) in [4] concerning the preservation of subnormality and Hamburger type property by the Aluthge transform of weighted shifts.
2. Theorem 2.4 allows to produce easily a whole class of examples of Hamburger type shift for which the Aluthge transform is not of Hamburger type.
3. **k-Jumping flatness property for weighted shifts**

In various research papers related to the subnormal completion problem in one variable, the recursiveness plays a central role in the explicit calculation of the subnormal completion of weighted shifts (see [2, 6]). A sequence is recursive when it satisfies following recursive relation,

$$
\gamma_{n+1} = a_0\gamma_n + a_1\gamma_{n-1} + \cdots + a_{r-1}\gamma_{n-r+1} \quad \text{for every} \quad n \geq r,
$$

(3.1)

where the coefficients $a_0, a_1, \cdots, a_{r-1}$ are some fixed numbers and $\gamma_0, \gamma_1, \cdots, \gamma_{r-1}$ are the initial conditions. When $\gamma$ is recursive, we say that the associated weighted shift $W_\alpha$ is recursively generated weighted shift. The polynomial $P(z) = z^r - a_0z^{r-1} - \cdots - a_{r-2}z - a_{r-1}$, is said to be generating for $\gamma$. The characteristic polynomial of $\gamma$ is the unique generating polynomial $P_\gamma(z)$ with minimal degree. It is usefully applied in the determination of the explicit expression of the general term $\gamma_n$, by considering the (characteristic) roots $\lambda_1, \lambda_2, \cdots, \lambda_s$ of $P_\gamma(z)$, with multiplicities $m_1, m_2, \ldots, m_s$ (respectively).

For weighted shifts, we have the following

**Proposition 3.1.** Let $W_\alpha$ be recursively generated weighted shift with characteristic polynomial $P_\gamma$. Then, the following are equivalent

1. $P_\gamma$ has only simple roots $\lambda_1, \cdots, \lambda_r$.
2. $W_\alpha$ admits a representing measure $\mu = c_1\delta_{\lambda_1} + \cdots + c_r\delta_{\lambda_r}$ for some suitable constants $c_1, \cdots, c_r$.

Notice in passing that

**Proposition 3.2.** Let $W_\alpha$ be a weighted shift with jumping flatness property and set $p = \alpha_1\alpha_2$. Then $\gamma$ satisfies the next recursive relation $\gamma_{n+3} = p^2\gamma_{n+1}$ for every $n \in \mathbb{Z}_+$. In particular, we have

1. If $W_\alpha$ has jumping flatness of type I, then $P_\gamma(X) = X(X - p)(X + p)$
2. If $W_\alpha$ has jumping flatness of type II, then $P_\gamma(X) = (X - p)(X + p)$

We deduce the next corollary

**Corollary 3.3.** Let $W_\alpha$ be a weighted shift with jumping flatness property. Then $W_\alpha$ admits a finite atomic representing measure $\mu$. Moreover,
1. If $W_\alpha$ has jumping flatness of type I, then $\mu = c_{-1}\delta_{-p} + c_0\delta_0 + c_1\delta_p$

2. If $W_\alpha$ has jumping flatness of type II, then then $\mu = c_{-1}\delta_{-p} + c_1\delta_p$.

for some suitable constants $c_{-1}, c_0$ and $c_1$ and with $p = \alpha_1\alpha_2$.

Without loss of generality, we assume in the sequel that $\gamma \equiv \{\gamma_n\}_{n \in \mathbb{Z}_+}$ is a recursive sequence of order $r$ satisfying (3.1) with $a_{r-1} \neq 0$, and $\gamma_0, \gamma_1, \ldots, \gamma_{r-1}$ are the initial conditions. Note that the Equality (3.1) yields that

$$\gamma_{n-r+1} = \frac{1}{a_{r-1}}\gamma_{n+1} - \frac{a_0}{a_{r-1}}\gamma_n - \cdots - \frac{a_{r-2}}{a_{r-1}}\gamma_{n+2}$$

$$= \sum_{i=1}^{r} \alpha_i' \gamma_{n-r+1+i}. \quad (3.2)$$

**Lemma 3.4.** Let $\gamma \equiv \{\gamma_n\}_{n \in \mathbb{Z}_+}$ be as in (3.1). If $M_{r-1}(2n_0) \geq 0$ for some integer $n_0 \geq 0$, then $M_\infty(\gamma) \geq 0$.

**Proof.** It suffice to show that if we have $M_{r-1}(2n_0) \geq 0$ then $M_r(2n_0 - 2) \geq 0$ and $M_r(2n_0 + 2) \geq 0$. To this aim, consider $x = (x_0, x_1, \ldots, x_r) \in \mathbb{R}^{r+1}$. We have

$$x^T M_r(2n_0 - 2)x = \sum_{i,j=0}^{r} x_j \gamma_{2n_0-2+i+j} x_i$$

$$= \sum_{i,j=1}^{r} x_j \gamma_{2n_0-2+i+j} x_i + \sum_{i=0}^{r} x_0^2 \gamma_{2n_0-2+i+j} x_i + \sum_{j=0}^{r} x_j^2 \gamma_{2n_0-2+i+j} x_i + \sum_{i=0}^{r} x_0 \gamma_{2n_0-2+i+j} x_i + \sum_{i=0}^{r} x_0 \gamma_{2n_0-2+i+j} x_i.$$

By virtue of (3.2), one have

$$\sum_{i=1}^{r} x_0 \gamma_{2n_0-2+i+j} x_i = \sum_{i=1}^{r} x_0 \left( \sum_{t=1}^{r} \alpha_i' \gamma_{2n_0-2+i+j} x_i \right) = \sum_{i,j=1}^{r} x_0 \alpha_i' \gamma_{2n_0-2+i+j} x_i;$$

$$x_0 \gamma_{2n_0-2} x_0 = x_0 \left( \sum_{t=1}^{r} \alpha_i' \gamma_{2n_0-2+i+j} \right) x_0 = x_0 \left( \sum_{i=1}^{r} \alpha_i' \sum_{j=1}^{r} \alpha_j' \gamma_{2n_0-2+i+j} \right) x_0$$

$$= \sum_{i,j=1}^{r} x_0 \alpha_i' \gamma_{2n_0-2+i+j} x_0 \alpha_i'.$$

Let $y = (y_1, y_2, \ldots, y_{r-1}) \in \mathbb{R}^{r+1}$ be given by $y_i = 2x_0 \alpha_i' + x_i$ for $j = 0, 1, \ldots, r - 1$. It follows from above that $x^T M_r(2n_0 - 2)x = y^T M_{r-1}(2n_0)x$. Since $M_r(2n_0 - 2)$ is positive semidefinite, then so is $M_{r-1}(2n_0)$. 

11
To show that $M_{r-1}(2n_0) \geq 0 \Rightarrow M_r(2n_0 + 2) \geq 0$ one repeats the above proof and using Equality (3.1) instead of Equality (3.2). This completes the proof.

\[ \square \]

**Theorem 3.5.** Let $W_\alpha$ be a weighted shift with property $H([\frac{3k}{2}] + 1)$. If there exists $n_0 \in \mathbb{Z}^+$ such that

$$\alpha_{n_0 + j} = \alpha_{n_0 + k + j}$$

for $j = 0, 1, \ldots, k - 1$,

then $W_\alpha$ has $k$-jumping flatness property.

**Proof.** (Outer propagation) Set $m_0 = 2[\frac{3k}{2}]$. We have $M_{[\frac{3k}{2}] + 1}(m_0) \geq 0$, then Smul’jan’s theorem yields

$$\gamma_{m_0 + [\frac{3k}{2}] + 1 + i} = \sum_{j=0}^{[\frac{3k}{2}]} a_j \gamma_{m_0 + [\frac{3k}{2}] + i - j} \quad (i = 0, 1, \ldots, \lfloor \frac{3k}{2} \rfloor) \quad (3.3)$$

for some real numbers $a_0, a_1, \ldots, a_{\lfloor \frac{3k}{2} \rfloor}$. Let us consider the recursive sequence $\tilde{\gamma} = \{\tilde{\gamma}_i\}_{i \in \mathbb{Z}^+}$ defined as follows

$$\begin{cases} 
\tilde{\gamma}_l = \gamma_l, & l = m_0, m_0 + 1, \ldots, m_0 + \lfloor \frac{3k}{2} \rfloor; \\
\tilde{\gamma}_{m_0 + [\frac{3k}{2}] + i + 1} = a_0 \tilde{\gamma}_{m_0 + [\frac{3k}{2}] + i} + a_1 \tilde{\gamma}_{m_0 + [\frac{3k}{2}] + i - 1} + \ldots + a_{\lfloor \frac{3k}{2} \rfloor} \tilde{\gamma}_i \text{ for all } i \in \mathbb{Z}^+. 
\end{cases} \quad (3.4)$$

Clearly, Formulas (3.3) and (3.4) yield

$$\tilde{\gamma}_i = \gamma_i \text{ for all } i = m_0, m_0 + 1, \ldots, m_0 + \lfloor \frac{3k}{2} \rfloor + 1. \quad (3.5)$$

Hence $M_{[\frac{3k}{2}]}(\gamma)(m_0) = M_{[\frac{3k}{2}]}(\gamma)(m_0)$. Since $M_{[\frac{3k}{2}]}(\gamma)(m_0) \geq 0$, then Lemma 3.4 implies that $M_\infty(\tilde{\gamma}) \geq 0$. Also, the sequence $\tilde{\gamma}$ is recursive, and then, according to \[ \text{Theorems 3.1 and 3.9}, \] $\tilde{\gamma}$ has a finitely atomic representing measure, say $\mu = \sum_{t=0}^{r-1} \rho_t \delta_{\lambda_t}$. In symbols

$$\tilde{\gamma}_i = \int_{\mathbb{R}} x^i d\mu = \sum_{t=0}^{r-1} \rho_t \lambda_t^i. \quad (3.6)$$

We have

$$\frac{\gamma_{n_0 + k + 1 + j}}{\gamma_{n_0 + j}} = \alpha_{n_0 + j} = \alpha_{n_0 + k + j} = \frac{\gamma_{n_0 + k + 1 + j}}{\gamma_{n_0 + k + j}} \text{ for all } j = 0, 1, \ldots, k - 1,$n_0 \gamma_{n_0 + 2k} = \gamma_{n_0 + k}^2. \quad (3.7)$$

By using (3.5), (3.6) and (3.7), one obtain

$$\left( \sum_{t=0}^{r-1} \rho_t (\lambda_t^{k})^{n_0} \right) \left( \sum_{t=0}^{r-1} \rho_t (\lambda_t^{k})^{n_0 + 2} \right) = \left( \sum_{t=0}^{r-1} \rho_t (\lambda_t^{k})^{n_0 + 1} \right)^2.$$
That implies
\[
\sum_{0 \leq i < j \leq r - 1} \rho_i \rho_j (\lambda^k_i \lambda^k_j)^{n_0 + 1} (\lambda^k_i - \lambda^k_j)^2 = 0,
\]
which gives \( \lambda^k_i = \lambda^k_j \) whenever \( \lambda_i \lambda_j \neq 0 \).

Hence
\[
\mu = \rho_0 \delta_0 + \rho_1 \delta_\lambda + \rho_2 \delta_{-\lambda} \quad \text{with} \quad \lambda > 0.
\]

Note in passing that if \( k \) is odd then \( \rho_2 = 0 \). Thereby,
\[
\tilde{\gamma}_0 = \rho_0 + \rho_1 + \rho_2 \quad \text{and} \quad \tilde{\gamma}_n = \rho_1 (\lambda)^n + \rho_2 (-\lambda)^n \quad \text{for all integer} \ n \geq 1.
\]

For all \( j = 0, 1, \ldots, k - 1 \), the Equality \((3.8)\) yields
\[
\alpha_{n_0 + j}^2 = \alpha_{n_0 + k + j}^2 = \alpha_{n_0 + 2k + j} = \alpha_{n_0 + 3k + j} = \ldots, \quad \text{for} \ j = 0, 1, \ldots, k - 1. \quad (3.10)
\]

(*Inner propagation*) Now, let \( n'_0 \) be the smallest integer such that \( n_0 \leq kn'_0 \). According to \((3.10)\), we have
\[
\alpha_{kn'_0 + j} = \alpha_{(k+1)n'_0 + j}, \quad j = 0, 1, \ldots, k - 1.
\]

Setting \( m'_0 = \left\lfloor \frac{2(k - 1)n'_0}{2} \right\rfloor \). The property \( H([\frac{3k}{2}] + 1) \) implies \( M([\frac{3k}{2}] + 1)(m'_0) \geq 0 \). By using Smul’Jan’s theorem, one obtain
\[
\gamma_{m'_0 + [\frac{3k}{2}] + 1 + i} = \sum_{j = 0}^{[\frac{3k}{2}]} b_j \gamma_{m'_0 + [\frac{3k}{2}] + i - j} \quad (i = 0, 1, \ldots, [\frac{3k}{2}]), \quad (3.11)
\]

where \( b_0, \ldots, b_{[\frac{3k}{2}]} \) are real numbers.

Consider the recursive sequence \( \hat{\gamma} = \{\hat{\gamma}_l\}_{l \in \mathbb{Z}_+} \), defined by
\[
\begin{cases}
\hat{\gamma}_l = \gamma_l, & l = m'_0, m'_0 + 1, \ldots, m'_0 + [\frac{3k}{2}]; \\
\hat{\gamma}_{[\frac{3k}{2}] + i + 1} = b_0 \hat{\gamma}_{[\frac{3k}{2}] + i} + b_1 \gamma_{[\frac{3k}{2}] + i - 1} + \ldots + b_{[\frac{3k}{2}]} \hat{\gamma}_i & \text{for all} \ i \in \mathbb{Z}_+.
\end{cases} \quad (3.12)
\]
Remark that $\hat{\gamma}_i = \gamma_i$ for all $i = m'_0, \ldots, m'_0 + 2\left\lceil \frac{3k}{2} \right\rceil + 1$.

In a similar manner as in the proof of the "outer propagation", we get

\[ \alpha_{(k-1)n'_0+j} = \alpha_{kn'_0+j}(= \alpha_{(k+1)n'_0+j}) \quad \text{for } j = 0, \ldots, k - 1. \]

Repeating the same process, we obtain the desired result. This completes the proof. \qed

Note that the above proof furnishes more interesting results. Indeed, if the hypothesis of Theorem 3.5 are verified and $k$ is an odd integer. Then, as observed above, the measure given in (3.9) is represented by $\mu = \rho_0\delta_0 + \rho_1\delta_\lambda$ where $\lambda$ is a positive number. Therefore, via (3.9), one has $\alpha_{n_0+1} = \alpha_{n_0+2}$. By applying Theorem 3.5, the operator $W_\alpha$ is subnormal and has 1-jumping flatness property.

Now, let us assume that the hypothesis of Theorem 3.5 are verified and $k$ is an even integer. In a similar way, one obtain $\alpha_{n_0} = \alpha_{n_0+2}$ and $\alpha_{n_0+1} = \alpha_{n_0+3}$. Using again Theorem 3.5 the operator $W_\alpha$ is of Hamburger-type and has 2-jumping flatness property.

**Theorem 3.6.** Let $W_\alpha$ be a weighed shift with property $H(\left\lceil \frac{3k}{2} \right\rceil + 1)$ such that there exists $n_0 \in \mathbb{Z}_+$ satisfying

\[ \alpha_{n_0+j} = \alpha_{n_0+k+j} \quad \text{for } j = 0, 1, \ldots, k - 1. \]

Then

- if $k$ is odd, then $W_\alpha$ is subnormal and is flat (has the 1-jumping flatness property);
- if $k$ is even, then $W_\alpha$ is a Hamburger-type operator and has a 2-jumping flatness property.

**Remark 3.7.** In the previous theorem, for weighted shifts with $k = 2$, we have $H(n) = H(\left\lceil \frac{3k}{2} \right\rceil + 1) = H(4)$ which coincides with the condition $H(4)$ as assumed in [9]. It is natural to ask if the property $H(\left\lceil \frac{3k}{2} \right\rceil + 1)$ above can be weakened to $H(\phi(n))$ for some $\phi(n) < \left\lceil \frac{3k}{2} \right\rceil + 1$.

**References**

[1] A. Aluthge, *On p-polynomial operators for $0 < p < 1$*, Integral Equations Operator Theory 13 (1990), 307-315.
[2] R. Ben Taher, M. Rachidi and H. Zerouali, *Recursive subnormal completion and truncated moment problem*, Bull. London Math. Soc. 33 (2001), 425-432.

[3] E. E. Chidume, M. Rachidi and E. H. Zerouali, *Solving the general truncated moment problem by $r$-generalized Fibonacci sequences method*, J. of Mathematical Analysis and Applications 256 (2001), 625-63.

[4] R. E. Curto, *Quadratically hyponormal weighted shifts*, Integral Equations Operator Theory 13 (1990), no. 1, 49-66.

[5] R. Curto and L. Fialkow, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math. 17(1991), 603-635.

[6] R.E Curto and L. A. Fialkow, *Recursively generated weighted shifts and the subnormal completion*, Integral Equations and Operator Theory 17 (1993), 202-246.

[7] R. Curto, J. Kim and J. Yoon, *The Aluthge transform of unilateral weighted shifts and the Square Root Problem for finitely atomic measures*, Math. Nachr. 292 (2019) 2352–2368.

[8] G.R. Exner, J.Y. Jin, I.B. Jung and M.R. Lee, *Weighted shifts induced by Hamburger moment sequences*, J. Math. Anal. Appl. 427 (2015) 581–599.

[9] G.R. Exner, J.Y. Jin, I.B. Jung and J. E. Lee, *Hamburger-type weighted shifts: Jumping flatness and Aluthge transforms*, J. Math. Anal. Appl. 494 (2021)

[10] S. H. Lee, W. Y. Lee and J. Yoon *Subnormality of Aluthge transforms of weighted shifts*, Integral Equations and Operator Theory 72 (2012), 241-251.

[11] G. Pólya. Sur l’indétermination d’un problème voisin du problème des moments. *CR Acad. Sci. Paris*, 207:708–711, 1938.

[12] J. Stampfli, *Which weighted shifts are subnormal?*, Pacific J. Math. 17 (1966), 367–379.