IMPROVED BOUND IN THE BENJAMIN AND LIGHTHILL CONJECTURE

EVGENIY LOKHARU

Abstract. The classical Benjamin and Lighthill conjecture about steady water waves states that the non-dimensional flow force constant of a solution is bounded by the corresponding constants of the supercritical and subcritical uniform streams respectively. These inequalities determine a parameter region that covers all steady motions. In fact not all points of the region determine a steady wave. In this note we prove a new and explicit lower bound for the flow force constant, which is asymptotically sharp in a certain sense. In particular, this recovers the well known inequality $F < 2$ for the Froude number, while significantly reducing the parameter region supporting steady waves.

1. Introduction

We consider the classical problem for steady waves on the surface of an ideal fluid above a flat bottom. In non-dimensional variables, when the mass flux and the gravitational constant are scaled to 1, every flow has two constants of motion, which are the total head $r$ and the flow-force $S$. In 1954 Benjamin and Lighthill [1] made a conjecture about the independent possible ranges of $r$ and $S$. Specifically, every steady flow, irrespective of amplitude or wavelength, is to realize a point $(r,S)$ within a certain region in $(r,S)$-plane. The latter is determined by all points $(r,S)$ for which

$$S_-(r) \leq S \leq S_+(r),$$

where $S_-(r)$ and $S_+(r)$ are the flow force constants corresponding to the subcritical and supercritical laminar flows respectively. The boundary of this region is a cusped curve representing all uniform streams (see Figure 1). This conjecture was verified by Benjamin for all irrotational Stokes waves in [5]. The left inequality in (1.1) was obtained earlier by Keady & Norbury [7] (also for periodic wavetrains). Recently Kozlov and Kuznetsov [9, 10] proved (1.1) for arbitrary solutions under weak regularity assumptions, provided the Bernoulli constant $r$ is close to it’s critical value $r_c$; it was extended to the rotational setting in [12], again for $r \approx r_c$; the latter condition guarantees that solutions are of small amplitude. The left inequality in (1.1) for periodic waves with a favorable vorticity was obtained by Keady & Norbury [8]. It was recently proved in [15] that (1.1) remains true for arbitrary waves with vorticity. In particular, all irrotational steady waves (not necessarily periodic) correspond to points in the region.

As it was recently proved in [13] that all solitary waves correspond to points on a part of lower boundary $S = S_-(r)$ (dashed curve in Figure 1). It is believed (but unproved) that the dashed curve ends with the point that determines the highest solitary wave. Anyway, the well known bound $F < 2$ (see [17]) for the Froude number implies that no solitary waves presented on $S = S_-(r)$ for large $r$. Besides, there are no small-amplitude periodic waves with $S \sim S_-(r)$. Together this suggests that no steady waves are determined by points in a neighborhood of the lower boundary $S = S_-(r)$ when $r$ is large.

In the present paper we prove the existence of a barrier, which is explicitly given by $S = \frac{1}{2} r^2$. This barrier intersects the lower boundary at a point with $F = 2$ and is asymptotically close to the upper boundary $S = S_+(r)$, which shows that all steady waves, including waves of greatest height, correspond to a smaller part of the region (grey region in Figure 1) from the Benjamin and Lighthill conjecture. Our proof is of special interest by itself. It is based on the flow force function formulation, introduced recently in [3]. This new reformulation is very similar to the problem for steady waves with positive constant vorticity. Our idea is essentially based on that similarity. It is known from [11] that the Bernoulli constant for waves with positive vorticity
Figure 1. Parameter region for steady water waves

is bounded from above; see also [14] for the case of negative vorticity. Thus, using a similar argument we can show that the flow force function formulation (after a rescaling) admit an explicit upper bound for the Bernoulli constant, which is equivalent to the desired inequality \( S > \frac{1}{2}r^2 \).

2. Statement of the problem

A non-dimensional version of the stream function formulation for two-dimensional water waves has the following form (see [5]):

\[
\begin{align*}
\Delta \psi &= 0 \quad \text{for } 0 < y < \eta, \quad (2.1a) \\
\frac{1}{2} |\nabla \psi|^2 + y &= r \quad \text{on } y = \eta, \quad (2.1b) \\
\psi &= 1 \quad \text{on } y = \eta, \quad (2.1c) \\
\psi &= 0 \quad \text{on } y = 0. \quad (2.1d)
\end{align*}
\]

Here \( r \) is referred to as the Bernoulli constant, problem’s parameter. The latter problem admits another constant (see [4] for more details), the flow force, defined as

\[
S = \int_0^\eta \left( \frac{1}{2}(\psi_y^2 - \psi_x^2) - y + r \right) dy. \quad (2.2)
\]

After taking \( x \)-derivative in (2.2) and using (2.1a) together with the boundary relation (2.1b), one verifies that \( S \) is a constant of motion independent of \( x \). Our main result can now be stated as follows.

**Theorem 2.1.** Let \((\psi, \eta) \in C^2(\mathbb{D}_\eta) \times C^2(\mathbb{R})\) be a solution to (2.1) which is not a laminar flow. Then \( S > \frac{1}{2}r^2 \), where \( r \) and \( S \) are the Bernoulli and the flow force constants respectively, defined by (2.1b) and (2.2).

Let us compare the quantity \( S_+(r) \) and the bound \( \frac{1}{2}r^2 \) from the theorem. Note that

\[
S_+(r) = \frac{1}{2d_+(r)} - \frac{1}{2}d_+^2(r) + rd_+(r),
\]

where \( d_+(r) > 1 \) is a root of \( \frac{1}{2}d^{-2} + d = r \). Thus, for large \( r \), we find \( d_+(r) \sim r \) and more precisely,

\[
d_+(r) = r - \frac{1}{2r^2} + O(r^{-3}) \quad \text{as } r \to +\infty.
\]

Therefore, we obtain

\[
\begin{align*}
S_+(r) &= \frac{1}{2r} \left( 1 + O(r^{-3}) \right) - \frac{1}{2} \left( r^2 - \frac{1}{r} + O(r^{-2}) \right) + r \left( r - \frac{1}{2r^2} + O(r^{-3}) \right) \\
&= r^2 \left( \frac{1}{2} + \frac{1}{2r} + O(r^{-2}) \right).
\end{align*}
\]
We see that the curve $S = \frac{1}{2} r^2$ from Theorem 2.1 is below the upper boundary $S = S_+(r)$ (see Figure 1) for large $r$ and is asymptotically accurate. On the other hand, curves $S = S_-(r)$ and $S = \frac{1}{2} r^2$ have one point of intersection, for which the Froude number $F := d^{-3/2} = 2$. This recovers the well known bound $F < 2$ for solitary waves; see [17, 2, 16].

It is quite interesting to find that for large $r$ an arbitrary solution, including highest waves, must have the flow force constant close to $S + \left(\frac{1}{2} r^2\right)$. At this point we can make a hypothesis that all steady waves for large $r$ are of small-amplitude.

3. Proof of Theorem 2.1

3.1. Flow force function formulation. Based on the definition for the flow force constant (2.2), we introduce the corresponding flow force function

$$ F(x, y) = \int_0^y \left(\frac{1}{2} \left(\psi_y^2(x, y') - \psi_x^2(x, y')\right) - y' + r\right) dy'. $$

(3.1)

Just as in [3] we can reformulate the water wave problem in terms of the function $F$. It is straightforward to obtain

$$ F_x = \psi_x \psi_y, \quad F_y = \frac{1}{2} \left(\psi_y^2 - \psi_x^2\right) - y + r. $$

(3.2)

Thus, we arrive to an equivalent formulation given by

$$ \Delta F + 1 = 0 \quad \text{for} \quad 0 < y < \eta, $$

(3.3a)

$$ \frac{1}{2} (-\eta' F_x + F_y) + y = r \quad \text{on} \quad y = \eta, $$

(3.3b)

$$ F = S \quad \text{on} \quad y = \eta, $$

(3.3c)

$$ F = 0 \quad \text{on} \quad y = 0. $$

(3.3d)

It follows immediately from the maximum principle that $F > 0$ for $y > 0$. In fact one can show that

$$ F_y \geq \psi_y^2 > 0 \quad \text{for} \quad 0 \leq y \leq \eta. $$

(3.4)

To see that it is enough to apply the maximum principle to the superharmonic function $\Phi = \frac{1}{2} (\psi_x^2 + \psi_y^2) + y$. It is straightforward that $\Phi_y = 0$ on $y$ and $\Phi = r$ on $y = \eta$, so that $\Phi \leq r$ everywhere. This yields (3.4).

In what follows we want to treat the system (3.3) as the water wave problem with constant vorticity. Thus, it is convenient to have the "mass flux" $F = S$ scaled to 1. This suggests new variables

$$ X = \sqrt{S}^{-1} x, \quad Y = \sqrt{S}^{-1} y, \quad \zeta(X) = \sqrt{S}^{-1} \eta(x), \quad F(X, Y) = S^{-1} F(x, y). $$

The scaled problems is

$$ \Delta \tilde{F} + 1 = 0 \quad \text{for} \quad 0 < Y < \zeta, $$

(3.5a)

$$ \frac{1}{2} (-\zeta X \tilde{F}_X + \tilde{F}_Y) + Y = R := r \sqrt{S}^{-1} \quad \text{on} \quad Y = \zeta, $$

(3.5b)

$$ \tilde{F} = 1 \quad \text{on} \quad Y = \zeta, $$

(3.5c)

$$ \tilde{F} = 0 \quad \text{on} \quad Y = 0. $$

(3.5d)

Furthermore, in view of (3.4), we additionally have

$$ F_Y > 0 \quad \text{for} \quad 0 \leq Y \leq \zeta. $$

(3.5e)

We are going to prove certain bounds for the "Bernoulli constant" $R$ in (3.5b). Note that the system (3.5) is very similar to the stream function formulation of the water problem with constant vorticity, for which the desired bounds were obtained in [11]. Thus, a similar argument can be applied here and we adapt it below.
3.2. Stream solutions. In order to obtain bounds for $R$ we need to study stream solutions to (3.5). These are pairs $\vec{F} = U(Y; d)$, $\zeta(X) = d$, parametrized by a constant “depth” $d > 0$. Using this ansatz in (3.5), one finds

$$U(Y; d) = -\frac{1}{2} Y^2 + (d^{-1} + \frac{1}{2} d) Y.$$ 

The corresponding Bernoulli constant is given by

$$R(d) = \frac{1}{2d} + \frac{3d}{4}.$$ 

We are interested in unidirectional solutions only, for which $U_Y > 0$ on $[0, d]$. As a result we obtain a restriction on $d$:

$$0 < d < d_0 = \sqrt{2}.$$ 

This critical value is characterized by the relation $U_Y(d_0; d_0) = 0$, while $U_Y(Y; d_0) > 0$ for $Y \in [0, d_0)$. Let us denote

$$R_0 = R(d_0) = \sqrt{2} \quad \text{and} \quad R_c = \sqrt{\frac{3}{2}}, \quad d_c = \sqrt{\frac{2}{3}}.$$ 

Note that $R_c = R(d_c)$ is the global minimum of $R(d)$. Thus, for any $R \in (R_c, R_0)$ there are two solutions $d = d_-(R)$ and $d = R+(d)$ with $d_-(R) < d+(R)$ to the equation

$$R = R(d).$$ 

These depths are similar to the subcritical and supercritical depths of conjugate laminar flows of the original water wave problem.

3.3. Bounds for the Bernoulli constant. Our aim is to prove the following theorem:

**Theorem 3.1.** Let $(\vec{F}, \zeta)$ be an arbitrary non-trivial (other than a stream solution) solution to (3.5). Then the corresponding Bernoulli constant $R$ is subject to the inequalities $R_c < R < R_0$.

Note that the statement of Theorem 2.1 follows directly from the upper bound $R < R_0$. Indeed, a non-trivial solution $(\psi, \eta)$ of the original system (2.1) with the Bernoulli constant $r$ generates a solution to (3.5) with $R = r\sqrt{S}^{-1}$. Now Theorem 3.1 gives $r\sqrt{S}^{-1} < R_0 = \sqrt{2}$, which is equivalent to $S > \frac{1}{2} r^2$ as stated in Theorem 2.1.

**Proof of Theorem 3.1.** Our argument is based on a comparison of a given solution $(\vec{F}, \zeta)$ to different stream solutions from Section 3.2. For this purpose we will apply the partial hodograph transformation introduced by Dubreil-Jacotin [6] but for the flow force function formulation (3.5). This is possible since $\vec{F}_Y > 0$ everywhere by (3.4). Thus, we introduce new independent variables

$$q = X, \quad p = \vec{F}(X, Y),$$ 

while new unknown function $h(q, p)$ (height function) is defined from the identity

$$h(q, p) = y.$$ 

Note that it is related to the flow force function $\vec{F}$ through the formulas

$$\vec{F}_X = -\frac{h_q}{h_p}, \quad \vec{F}_Y = \frac{1}{h_p},$$ 

where

$$h_p > 0 \quad \text{throughout new fixed domain} \ S = \mathbb{R} \times (0, 1).$$ 

An equivalent problem for $h(q, p)$ is given by

$$\left(1 + \frac{h_q^2}{2h_p^2} + p\right)_p - \left(\frac{h_q}{h_p}\right)_q = 0 \quad \text{in} \ S,$$ 

$$\frac{1 + h_q^2}{2h_p} + h = R \quad \text{on} \ p = 1,$$ 

$$h = 0 \quad \text{on} \ p = 0.$$ 


The surface profile \( \zeta \) becomes the boundary value of \( h \) on \( p = 1 \):

\[
h(q, 1) = \zeta(q), \quad q \in \mathbb{R}.
\]

For a detailed derivation of (3.7) we refer to [3]. Applying a similar transformation for stream functions \( U(Y; d) \), we obtain the corresponding height functions \( H(p; d) \), subject to

\[
\left( \frac{1}{2 H_p^2} + p \right)_p = 0, \quad H(0) = 0, \quad H(1) = d, \quad \frac{1}{2 H_p(1; d)} + d = R(d).
\]

Now because the domain for \( h \) and \( H \) is the same, we can compare these functions using the maximum principle. More precisely, we put

\[
w^{(d)} = h - H
\]

and using the corresponding equations for \( h \) and \( H \) one finds that \( w^{(d)} \) solves a homogeneous elliptic equation

\[
\frac{1}{h_p^2} w^{(d)}_{pp} - \frac{2}{h_p} w^{(d)}_{pq} + w^{(d)}_q - w^{(d)} + \frac{(w^{(d)}_q)^2 H_{pp}}{h_p^2} - \frac{w^{(d)}_p (h_p + H_p) H_{pp}}{h_p^2 H_{pp}^2} = 0. \tag{3.8}
\]

Thus, every \( w^{(d)} \) is subject to a maximum principle; see [18] for an elliptic maximum principle in unbounded domains.

Now we can prove Theorem 3.1. Note that when \( d \to d_0 = \sqrt{2} \) we have \( H_p(1; d) \to +\infty \). Thus, we can choose \( d < \sqrt{2} \) for which \( w^{(d)}(d) < 0 \) everywhere on \( p = 1 \). Therefore, \( w^{(d)} < 0 \) by the maximum principle, because otherwise we would obtain a contradiction with the Hopf lemma. Thus, we have \( \zeta(q) < d \) for all \( q \in \mathbb{R} \). By the continuity we can find \( d_* < d < d_0 \) such that \( \sup_{\mathbb{R}} \zeta = d_* \). In this case we can always find a sequence \( q_j \), possibly unbounded, such that

\[
\lim_{j \to +\infty} \zeta(q_j) = d_*, \quad \lim_{j \to +\infty} w^{(d)}_q(q_j, 1) = 0, \quad \lim_{j \to +\infty} w^{(d)}_p(q_j, 1) \geq 0.
\]

Taking the corresponding limit in (3.7) and using relations from above we obtain \( R \leq R(d_*) < R_0 \). The bottom inequality \( R > R_* \) can be proved just the same way by choosing \( d_1 = \inf_{\mathbb{R}} \zeta \) and repeating a similar argument for \( w^{(d_1)} \) (note that \( \zeta \) is separated from zero).

\[\square\]

References

[1] On cnoidal waves and bores, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 224 (1954), pp. 448–460.
[2] C. J. AMICK and J. F. TOLAND, On solitary water-waves of finite amplitude, Arch. Rational Mech. Anal., 76 (1981), pp. 9–95.
[3] B. BASU, A flow force reformulation for steady irrotational water waves, Journal of Differential Equations, 268 (2020), pp. 7417–7452.
[4] T. B. BENJAMIN, Impulse, flow force and variational principles, IMA Journal of Applied Mathematics, 32 (1984), pp. 3–68.
[5] T. B. BENJAMIN, Verification of the Benjamin-Lighthill conjecture about steady water waves., J. Fluid Mech., 295 (1995), pp. 357–356.
[6] M. L. DUBREIL-JACOTIN, Sur la détermination rigoureuse des ondes permanentes périodiques d’ampleur finite., J. Math. Pures Appl., 13 (1934), pp. 217–291.
[7] G. KEADY and J. NORBURY, Water waves and conjugate streams, Journal of Fluid Mechanics, 70 (1975), pp. 663–671.
[8] , Waves and conjugate streams with vorticity, Mathematika, 25 (1978), pp. 129–150.
[9] V. KOZLOV and N. KUZNETSOV, The benjamin–lighthill conjecture for near-critical values of bernoulli’s constant, Archive for Rational Mechanics and Analysis, 197 (2009), pp. 433–488.
[10] , The benjamin–lighthill conjecture for steady water waves (revisited), Archive for Rational Mechanics and Analysis, 201 (2011), pp. 631–645.
[11] V. KOZLOV, N. KUZNETSOV, and E. LOKHARU, On bounds and non-existence in the problem of steady waves with vorticity, Journal of Fluid Mechanics, 765 (2015).
[12] , On the benjamin–lighthill conjecture for water waves with vorticity, Journal of Fluid Mechanics, 825 (2017), pp. 961–1001.
[13] V. KOZLOV, E. LOKHARU, and M. H. WHEELER, Nonexistence of subcritical solitary waves, (2020).
[14] E. LOKHARU, Nonexistence of steady waves with negative vorticity, arXiv:2005.08666.
[15] E. Lokharu, *A sharp version of the benjamin and lighthill conjecture for steady waves with vorticity*, submitted to Acta Mathematica, (2020).

[16] J. B. McLeod, *The Froude number for solitary waves*, Proc. Roy. Soc. Edinburgh Sect. A, 97 (1984), pp. 193–197.

[17] V. P. Starr, *Momentum and energy integrals for gravity waves of finite height*, J. Mar. Res., 6 (1947), pp. 175–193.

[18] A. Vitolo, *A note on the maximum principle for second-order elliptic equations in general domains*, Acta Mathematica Sinica, English Series, 23 (2007), pp. 1955–1966.

**Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden**