Direct Calculation of the Critical Effective Potential

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The critical effective potential is the nonperturbative part of the effective action at a phase transition. It equals the scale invariant effective average potential and can be calculated from the renormalization group flow of the effective average action. In some cases this requires only the solution of an ordinary differential equation without actually simulating the renormalization group flow. Here the Ising model is examined beyond leading order and with full field dependent effective potential.

Introduction

The effective potential in statistical physics and quantum field theory \cite{1} includes fluctuations of all wavelengths as opposed to the microscopic potential. It is therefore relevant for the computation of macroscopic properties.

The usual way to access the effective potential is to start with the potential given by the microscopic theory and follow the renormalization group flow to the infrared \cite{3}. As the potential then depends on field variables and the scale, this requires solving a partial differential equation.

However, an especially interesting case is a second order phase transition where the correlation length diverges. The system becomes therefore scale invariant and the renormalization group flow ends up in a fix point, the effective critical potential. Its calculation requires only the solution of an ordinary differential equation \cite{4, 5, 6, 7, 8, 9, 10, 11}.

Effective average action

The effective average action (see e.g. \cite{12} and references therein) \( \Gamma_k \) interpolates between the microscopical or classical action \( S = \Gamma_\infty \) and the effective action \( \Gamma = \Gamma_0 \). By construction it includes only fluctuations with momenta larger than \( k \) and this transition is described by an exact renormalization group flow equation:

\[
\partial_k \Gamma_k = \text{Tr} \left( (U_k^{(2)} + R_k)^{-1} \partial_k R_k \right)
\]

Here \( \Gamma_k^{(2)} \) is the two point function and \( R_k \) is an arbitrary momentum cutoff with the properties \( R_k \rightarrow 0 \) for \( k \rightarrow 0 \), \( R_k \rightarrow \infty \) for \( k \rightarrow \infty \) and \( R_k(q^2) > 0 \) for \( q^2 \rightarrow 0 \).

We want to consider a particular interesting and common model, the three dimensional Ising model. The O(1) symmetry of this theory can be exploited for a lowest order derivative expansion of the effective average action:

\[
\Gamma_k = \int d^d x \left( U_k(\rho) + Z_k \frac{1}{2} \partial \mu \varphi(x) \partial \mu \varphi(x) \right)
\]

in terms of the most general form of the nonperturbative effective average potential \( U_k(\rho) \) and a field independent but scale dependent wave function renormalization \( Z_k \).

When inserted into the flow equation for the effective average action this ansatz gives the flow of the effective average potential:

\[
\partial_k U_k = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\partial \rho R_k(q^2)}{U_k^{(2)} + 2\rho U_k^{(2)} + q^2 Z_k + R_k(q^2)}
\]

As the potential is only determined up to a constant, it is convenient to consider its first derivative instead. The corresponding flow equation is given by straightforward differentiation:

\[
\partial_k U_k' = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left( -\frac{3}{2} U_k^{(2)} \partial_k R_k(q^2) \right)
\]

Switching to dimensionless quantities \( U' = U_k'/(Z_k k^2) \) and choosing a linear cutoff function \( R_k = Z_k (k^2 - q^2) \Theta(k^2 - q^2) \) \cite{13, 14} the flow equation becomes explicitly scale invariant:

\[
\partial_k u'(x) = \left( -2 + \eta \right) u' + (d - 2 + \eta) x u'' - 4 \nu (d + 2 - \eta) \frac{3 u''' + 2 x u'''}{(1 + u' + 2 x u'')^2}
\]

Here \( t = \ln(k) \), \( x = Z_k k^{d-\rho} \), \( v_{\nu}^{-1} = 2^{d+1} \pi^{d/2} \Gamma(d/2) \) and \( \eta \) denotes the anomalous dimension \( \eta = \partial_t \ln Z_k \).

The latter one can either be taken self consistently from

\[
\eta = \frac{8 \nu d \kappa (3 \lambda + 2 u(3))^2}{(1 + 2 \lambda \kappa)}
\]

( where \( \kappa(t) \) denotes the running minimum ( \( u'(\kappa) = 0 \) ) and \( \lambda = u''(\kappa) \), \( u^{(3)}(\kappa) \) ) or from the quite accurate approximation

\[
\eta = d^{-d}
\]

which will be used in this paper.

For comparison with literature we also use a sharp cutoff, where in leading order

\[
\partial_k u' = -2 u' + (d - 2) x u'' - 2 \nu \left( \frac{3 u'' + 2 x u'''}{1 + u' + 2 x u'} \right)
\]
Scaling solutions

The scale invariance of some system at a second order phase transition manifests itself in a fix point in the renormalization group flow of the dimensionless, scale invariant flow equations. We are thus interested in scaling solutions $u'$ such that

$$0 = \partial_t u' = acu' + bcxu'' + c\frac{3u'' + 2xu'''}{(1+u'+2xu'')^2}$$

where

$$a = \frac{d(d+2)}{4v^2} \frac{2-n}{d+2-\eta} \quad b = \frac{d(d+2)}{4v^2} \frac{d+2-n}{d+2-\eta}$$

This can be rewritten canonically in terms of first order ordinary differential equations for $u'(x)$ and $u''(x)$:

$$\partial_x u' = u''$$

$$\partial_x u'' = \frac{1}{2x} \left( (au' + bxu'')(1 + u' + 2xu'')^2 + 3u'' \right)$$

Given initial values $u'_0$, $u''_0$ at some point $x_0$ the differential equations yield a local solution $u'$, $u''$ in a vicinity around $x_0$. The space of such solutions has dimension two, its elements being uniquely defined by $u'_0$, $u''_0$.

However, we are interested only in physical solutions which are continuations of the local solutions to the whole positive real axis. In other words, we must find a point $x_0 \geq 0$ and initial values $u'_0$, $u''_0$ such that $u'$ is a smooth function for all $x \geq 0$.

Such a search is nontrivial, though, even numerically. As soon as the space of local solutions has a dimension larger than one, global solutions can be missed even if one is already close to the correct initial values. Fortunately, by using the constraint of requiring a global solution we are able to eliminate one of the two integration constants:

If $\partial_x u' = 0 \forall x$, this holds true especially for $x_0 = 0$:

$$0 = \partial_t u' \mid_{x_0} = acu'_0 + c\frac{3u''_0}{(1+u'_0)^2}$$

And hence

$$u''_0 = -\frac{3}{2}u'_0(1 + u'_0)^2$$

In the analogous calculation for the sharp cutoff we find:

$$0 = \partial_t u' = acu' + bcxu'' + c\frac{3u'' + 2xu'''}{1+u'+2xu''}$$

where $a = v_d^{-1}$, $b = v_d^{-1}(1 - d/2)$ such that

$$\partial_x u' = u''$$

$$\partial_x u'' = \frac{1}{2x} \left( (au' + bxu'')(1 + u' + 2xu'') + 3u'' \right)$$

$$u''_0 = -\frac{3}{2}u'_0(1 + u'_0)$$

Furthermore, we have additional constraints on $u'_0$: the physically interesting $u'$ has a single zero and is positive for large $x$. Thus $u'_0 < 0$. On the other hand, the denominator $1 + u' + 2xu''$ must be nonzero for all $x$. Therefore $u'_0 > -1$. Now the global solution can be found by varying only $u'_0$, in the range $-1 < u'_0 < 0$, until the above requirements are met. This can be done numerically efficiently.

For any initial $u'_0$ the solution $u'$ extends to some $x_{max}$. For too small $u'_0$ the flow diverges to positive infinity. On the other hand, for large initial values the flow crosses zero a second time. These two cases are separated by a critical initial value which leads to the physical solution we are interested in. The plot of $x_{max}$ as a function of $u'_0$ is then sharply peaked around this critical value:

In this way we find:

linear cutoff, $\eta = 1/27$ (a) \hspace{1cm} $u'_0 = -0.16902863438...$

linear cutoff, $\eta = 0$ (b) \hspace{1cm} $u'_0 = -0.18606424944...$

sharp cutoff, $\eta = 0$ (c) \hspace{1cm} $u'_0 = -0.46153372007...$

In perfect agreement with

reference [11] \hspace{0.5cm} ($\eta = 0$) \hspace{1cm} $\lambda_{1s} = -0.1860642...$

reference [5] \hspace{0.5cm} ($\eta = 0$) \hspace{1cm} $\sigma = -0.46153372...$
Effective scaling potentials

We can now plot the solutions $u'$ for the above initial values:

Due to the numerically limited accuracy of the initial value and the integration we are able to track these solutions only up to $x \approx 0.12$ in case of the linear cutoff and $x \approx 0.2$ in case of the sharp cutoff.

The second quantity to be read off from the critical effective potential is its zero $\kappa$ where $u'(\kappa) = 0$:

- linear cutoff, $\eta = 1/27$ (a) $\kappa=0.03060106819...$
- linear cutoff, $\eta = 0$ (b) $\kappa=0.03064764922...$
- sharp cutoff, $\eta = 0$ (c) $\kappa=0.0471134650...$

The value $\kappa=0.03060...$ is in good agreement with the fix point value $\kappa^*=0.03053...$ from the dynamical method.

As one can see from the different solutions for the linear and the sharp cutoff, the effective potential is not cutoff independent. Interestingly however, the location of the minimum hardly depends on the anomalous dimension in case of the linear cutoff. Moreover, it is also only weakly dependent on the initial value $u'_0$.

The critical exponent $\delta$

That the effective scaling potential in fact describes critical phenomena can be seen from the power law $u' \sim x^{(\delta-1)/2}$ for large $x$.

The critical exponents $\delta$ and $\eta$ are related by

$$\delta = \frac{d+2-\eta}{d-2+\eta}$$

Hence we expect

$$\lim_{x \to \infty} 1 + 2 \frac{\partial \ln(u')}{\partial \ln(x)} = 5$$

in case $\eta = 0$ and

$$\lim_{x \to \infty} 1 + 2 \frac{\partial \ln(u')}{\partial \ln(x)} = \frac{65}{11}$$

in case $\eta = 1/27$.

These asymptotic values are plotted as horizontal lines:

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Comparison to the dynamical method

In order to assess the advantage in computing complexity of the method presented, we also take a look at the usual dynamical method [3, 15]. In this case one starts with an initial potential $u'_A = \lambda_A (x - \kappa_A)$ at some ultraviolet scale $\Lambda$ with an initial expectation value of the fields $\kappa_A$ and some quartic coupling constant $\lambda_A$.

With this initial potential the partial differential equation for $\partial_t u'(x)$ is computed on a discretized grid. The renormalization group flow for $t \to -\infty$ then either leads to the symmetric phase ($\kappa \to 0$) or to the phase with spontaneous symmetry breaking ($\kappa \to \infty$).

These phases are separated by a second order phase transition. Correspondingly, there is a critical $\kappa_A = \kappa_{cr}$. Fine tuning $\kappa_A$ this value can be found. The renormalization group flow near the phase transition will then come close to the fix point and the scaling potential can be read out.

The fine tuning of $\kappa_A$ is now replaced by fine tuning $f_0$. In effect, the solution of a partial differential equation is replaced by the solution of an ordinary differential equation. This is much simpler and may reduce computing time by several orders of magnitude.

Conclusion and outlook

The method allows the calculation of the effective scaling potential with very little effort. The integrable range seems to depend only on numerical precision and could maybe be improved by the use of high accuracy floating point libraries.

The effective scaling potential could also be used as an initial value in a dynamical simulation, allowing the determination of critical exponents dependend on small deviations from the critical point, while still overcoming the time consuming necessity for a dynamical fix point search.

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