BISIMPLICIAL COMPLEXES AND ASPHERICITY

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Abstract. We present a discrete Morse-theoretic method for proving that a regular CW complex is homeomorphic to a sphere. We use this method to define bisimplices, the cells of a class of regular CW complexes we call bisimplicial complexes. The 1-skeleta of bisimplices are complete bipartite graphs making them suitable in constructing higher dimensional skeleta for bipartite graphs. We show that the flag bisimplicial completion of a finite bipartite bi-dismantlable graph is collapsible. We use this to show that the flag bisimplicial completion of a quadric complex is contractible and to construct a compact $K(G,1)$ for $G$ a torsion-free quadric group.

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1. Introduction

CW complexes are typically constructed by gluing together Euclidean polyhedra along faces. A Euclidean polyhedron is the convex hull of a finite point set in a Euclidean space, e.g., simplices and cubes. However, not all CW structures on cells of a CW complex arise as Euclidean polyhedra and for some applications it is natural to use nonpolyhedral cells. In this paper we construct an infinite family of nonpolyhedral CW balls called bisimplices. The 1-skeleta of bisimplices are connected complete bipartite graphs and so we consider them as bipartite analogs of simplices. Our motivation for this construction is to find a natural contractible higher dimensional skeleton for quadric complexes.

Quadric complexes are locally finite simply connected square complexes satisfying a certain combinatorial nonpositive curvature condition. A group is quadric if it acts properly and cocompactly on a quadric complex. Quadric complexes are examples of the generalized $(4,4)$-complexes of Wise and were first studied in

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Bisimplices are essentially constructed by starting with a $K_{m,n}$, $m,n \geq 2$, inductively spanning a biclique on each proper $K_{m',n'}$ subgraph, $m', n' \geq 2$, and then taking the cone of the result. The difficulty lies in showing that the base of this cone is homeomorphic to $S^{m+n-3}$ and hence that the cone has the structure of a regular CW complex with a single top dimensional cell of dimension $m+n-2$. This is trivial for $(m,n)$ equal to $(2,2)$ or $(2,3)$, as seen in the figure. To prove it for general $(m,n)$ is not quite so easy.

They generalize the folder complexes of Chepoi [5] and may be considered as square analogs of the 2-skeleta of systolic complexes [12]. They can be characterized by their 1-skeleta, which are precisely the hereditary modular graphs of metric graph theory [1].

1.1. Summary of Results. In contrast to simplices which are indexed by dimension, bisimplices are indexed by two natural numbers $m, n \geq 1$. For each dimension $d \geq 2$ there are $\lceil \frac{d-1}{2} \rceil$ bisimplices of dimension $d$. Recall that a CW complex is regular if the characteristic maps of its cells are injective. See Figure 1.

**Theorem A** (Theorem 4.1). There exists a family $\{\Sigma^{m,n}\}_{m,n \geq 1}$ of regular CW complexes called bisimplices satisfying the following conditions.

- $\Sigma^{m,n}$ has a unique maximal cell and so $\Sigma^{m,n}$ is homeomorphic to a ball.
- $\Sigma^{m,n}$ has dimension $m+n$.
- The 1-skeleton of $\Sigma^{m,n}$ is the complete bipartite graph $K_{m+1,n+1}$.

Moreover, the cells of a bisimplex $\Sigma$ are also bisimplices and these cells are precisely the full subcomplexes of $\Sigma$, aside from a few degenerate cases such as the $K_{0,\ell}$ and $K_{1,\ell}$ subgraphs of the 1-skeleton. We consider vertices and edges to be bisimplices also. These properties uniquely determine the cell posets of the bisimplices. However, proving that this family of posets is indeed a family of cell posets is not at all trivial and is an interesting application of the discrete Morse theory of Forman [3] and the Generalized Poincaré Conjecture. Specifically, we prove Theorem A by inductively applying the following theorem, which we expect to have applications elsewhere.

**Theorem B** (Theorem 3.6 and Remark 3.7). Let $P$ be a poset such that the order complexes of the under sets of $P$ are PL-triangulated spheres. If $P$ and all of its over sets admit spherical matchings then the order complex of $P$ is a PL-triangulated sphere.
A **spherical matching** is a combinatorial structure on the Hasse diagram of the cell poset of a regular CW complex $X$. This combinatorial structure is essentially a discrete Morse function on $X$ having exactly two critical cells and so, by the Sphere Theorem of Forman [8], implies that $X$ is homotopy equivalent to a sphere. See Section 3 for an introduction to discrete Morse theory and the definition of spherical matching. The other terminology of Theorem B is defined in Section 2.

Having constructed the family of bisimplices, we may construct regular CW complexes having bisimplices as cells. We call these **bisimplicial complexes** when the intersection of any two bisimplices is a full subcomplex. Given a bipartite graph $\Gamma$ there is a natural bisimplicial complex $\nabla(\Gamma)$ called the **flag bisimplicial completion** of $\Gamma$. The flag bisimplicial completion is defined analogously to the flag simplicial completion, also known as the clique complex, of a graph.

Our primary motivation for the definition of the flag bisimplicial completion is to apply it to the bipartite 1-skeleta of quadric complexes. **Quadric complexes** may be defined as simply connected 2-dimensional CW complexes whose minimal area disk diagrams are CAT(0) square complexes. We would like a natural way to glue higher dimensional cells to a quadric complex to obtain a contractible supercomplex. The 1-skeleton of a quadric complex $X$ is bipartite and may contain $K_{2,3}$ so it is not possible to extend $X$ simplicially or cubically. However, $X$ equals the 2-skeleton of $\Sigma(X^1)$, so a natural candidate for a contractible supercomplex is $\nabla(\Sigma(X^1))$.

**Theorem C** (Theorem 6.6). Let $X$ be a nonempty quadric complex. Then the flag bisimplicial completion $\Delta(X^1)$ is contractible.

Metric balls in $X^1$ induce finite quadric subcomplexes of $X$ and finite quadric complexes have bi-dismantlable 1-skeleta [1, 10]. A finite bipartite graph is bi-dismantlable if it can be reduced to a nonempty connected complete bipartite graph by successively deleting a vertex whose neighbourhood is contained in the neighborhood of another vertex. Theorem C then follows from Theorem D below whose proof is another application of the discrete Morse theory of Forman.

**Theorem D** (Theorem 5.2). Let $X$ be a flag, nonempty finite bisimplicial complex. If $X^1$ is bipartite and bi-dismantlable then $X$ is collapsible.

This method of proving contractibility mirrors that of Chepoi and Osajda for weakly systolic complexes [9] via *LC-contructibility* [7, 14].

As pointed out to the present author by Damian Osajda, a quadric complex $X$ may also naturally be made contractible by extending each connected complete bipartite subgraph of $X^1$ to a complete subgraph and then taking the flag simplicial completion of the resulting graph. However, this operation preserves neither the 1-skeleton nor the 2-skeleton of $X$. Moreover, the resulting complex has higher dimension than the flag bisimplicial completion $\Sigma(X^1)$.

If $X$ is a compact locally quadric complex, the construction of the bisimplicial completion of the universal cover $\tilde{X}$ has a corresponding construction in the base. We obtain from $X$ a compact complex $X^+$ whose 2-skeleton is $X$ and whose higher cells are obtained by successively gluing in higher dimensional bisimplices along immersions of their boundaries. Then applying Theorem C we obtain the following.

**Theorem E** (Theorem 6.10). Let $X$ be a compact locally quadric complex. If $\pi_1(X)$ is torsion-free then $X^+$ is a compact $K(\pi_1(X), 1)$. 
Note that $\pi_1(X)$ in Theorem 2 is torsion-free if and only if the automorphism group of every immersion of the 2-skeleton of a bisimplex into $X$ is trivial. This is a consequence of the invariant biclique theorem for quadric complexes [10]. Moreover, every torsion-free quadric group is the fundamental group of some locally quadric complex.

1.2. Structure of the Paper. In Section 2 we give some background on posets, regular CW complexes and PL-triangulated spheres. In Section 3 we present basic theorems of the discrete Morse theory of Forman. We apply these theorems and the topological Generalized Poincaré Conjecture to prove a discrete Morse-theoretic sphere recognition theorem. We use our sphere recognition theorem in Section 4 to construct the infinite family of bisimplices. We then prove some basic facts about bisimplices. In Section 5 we introduce bisimplicial complexes and prove that flag finite bisimplicial complexes with dismantlable 1-skeleta are collapsible, again making use of discrete Morse theory. Finally, in Section 6 we prove that the flag bisimplicial completion of a quadric complex is contractible and describe how to construct a $K(G,1)$ for a torsion-free quadric group $G$.

2. Posets and Regular CW Complexes

Let $P$ be a poset. The covering relation $C_P$ on $P$ is the following binary relation.

$$C_P(x, y) \iff x < y \text{ and there is no } z \text{ satisfying } x < z < y$$

A poset $P$ is graded if every element $x \in P$ is assigned a grade $|x| \in \mathbb{N}$ such that the following conditions hold.

$$C_P(x, y) \implies |x| + 1 = |y|$$

$$x < y \implies |x| < |y|$$

Let $P$ be a poset. For $x \in P$, the over set $O_x$ and under set $U_x$ of $P$ at $x$ are the following subsets of $P$.

$$O_x = \{ y \in P : y > x \} \quad U_x = \{ y \in P : y < x \}$$

We may write $O^P_x$ and $U^P_x$ if the poset is not clear from the context. Note that making the inequalities in the definitions of $O_x$ and $U_x$ nonstrict would give what are usually referred to as the principal ideal and principal filter having principal element $x$. For $x, y \in P$, the strict interval $(x, y)$ of $P$ between $x$ and $y$ is the subset of $P$ defined as follows.

$$(x, y) = O_x \cap U_y$$

The over sets, under sets and strict intervals of $P$ are themselves posets by restricting the order relation. If $P$ is graded then the over sets, under sets and strict intervals of $P$ are likewise themselves graded.

Let $P$ be a poset. The set of nonempty chains of $P$ form an abstract simplicial complex. Its associated simplicial complex is the order complex $\Delta_P$ of $P$.

A CW complex is regular if the characteristic maps of its cells are embeddings. Let $X$ be a regular CW complex. The cells of $X$ are regular CW subcomplexes. Viewing a cell $x$ as a ball, we denote its boundary by $\partial x$ and its interior by $x^\circ$. The $k$-skeleton $X^k$ of $X$ is the subcomplex of $X$ formed by the union of the cells of $X$ of dimension at most $k$. The cell poset $P_X$ of $X$ is the set of cells of $X$ ordered by inclusion. Cell posets are equipped with a natural grading, namely dimension: $|x| = \dim x$. A cell poset $P$ uniquely determines its regular CW complex $X_P$. The
order complex of the cell poset of a regular CW complex $X$ is isomorphic to the barycentric subdivision of $X$. A subset $Q$ of $P$ is the cell poset of a subcomplex of $X_P$ iff $Q$ is downward closed, meaning the following.

$$x \in Q \text{ and } y < x \implies y \in Q$$

The following theorem of Björner characterizes the cell posets.

**Theorem 2.1** (Björner [2, Proposition 3.1]). Let $P$ be a poset. Then $P$ is a cell poset iff the order complexes of its under sets are homeomorphic to spheres.

**Proof.** If $P$ is the cell poset of a regular CW complex $X$ then its under sets are the cell posets of the boundaries of its cells. The order complexes of these under sets are the barycentric subdivisions of these cells and so are homeomorphic to spheres.

To prove the converse, we construct $X_P$ inductively on dimension. Define the height function $h : P \to \mathbb{N}$ as follows.

$$h(x) = \max\{|C| - 1 : C \text{ is a chain in } P \text{ with maximum } x\}$$

Note that $h(x)$ is finite because, otherwise, the order complex of the under set at $x$ would be infinite dimensional. We have that $h(x) = 0$ for minimal elements of $P$ and that $h(x) < h(y)$ for $x < y$. We will show that $h$ is a grading on $P$. Let $x \in P$. Since the under set $U_x$ has order complex homeomorphic to a sphere $S^\ell$, the maximal chains of $U_x$ must all have size $\ell + 1$. Hence $h(x) = \ell + 1$ and $h(y) = \ell$ for any element of $P$ that is covered by $x$.

Let $P^k \subseteq P$ be defined by the following.

$$P^k = \{ x \in P : h(x) \leq k\}$$

We will construct $X_P$ such that its $k$-skeleton $X^k_P$ has cell poset $P^k$. We begin the induction by letting $X^0_P = P^0$. Suppose we have constructed $X^k_P$ having cell poset $P^k$. Let $x \in P$ with $h(x) = k + 1$. Since $h$ is a grading, we have $U_x \subseteq P^k$. So $U_x$ is the cell poset of a subcomplex $A$ of $X^k_P$. The barycentric subdivision of $A_x$ is isomorphic to the order complex of $U_x$ which, by hypothesis, is homeomorphic to a sphere. This sphere has dimension $k$ since that is the height of a maximal chain in $U_x$. We construct $X^{k+1}_P$ from $X^k_P$ by attaching a $(k + 1)$-ball along its boundary to each $A_x$ with $h(x) = k + 1$. Then $P^{k+1}$ is the cell poset of $X^{k+1}_P$. Having inductively defined the skeleta

$$X^0_P \subseteq X^1_P \subseteq X^2_P \subseteq \cdots$$

we obtain $X_P$ as the colimit. \qed

We consider the empty space to be the sphere of dimension $-1$.

Let $P$ be a cell poset. The under sets of $P$ are also cell posets. More precisely, the under set $U_x$ at a cell $x$ is the cell poset of the regular CW complex structure on the boundary of $x$.

The order complex of the over set $O_x$ is isomorphic to the link of the barycenter of $x$ in the barycentric subdivision of $X_P$. However, because of the existence of homology spheres, $O_x$ need not be a cell poset. A homology sphere is a manifold with the homology of a sphere but which is not homeomorphic to a sphere. The double suspension of a homology sphere is homeomorphic to a sphere, as first proved in full generality by Cannon [3]. The Poincaré homology sphere $X$, also known as the spherical dodecahedron space, is a homology 3-sphere that has a simplicial triangulation [19, Section 62]. Let $B$ be the regular CW complex with a single cell
of dimension 6 and whose boundary \( \partial B \) has the structure of the simplicial double suspension of \( X \). Let \( e \) be a 1-simplex of \( B \) joining two of the suspension points of \( \partial B \). Then the link \( \text{lk} \, e \) is isomorphic to \( X \) and so the over set \( O_e \) of \( e \) in \( P_B \) is the cell poset of \( X \) augmented with a new maximum element corresponding to the top-dimensional cell of \( B \). The under set of \( B \) in \( O_e \) then has order complex homeomorphic to \( X \) and not a sphere and hence \( O_e \) is not a cell poset.

If \( X \) is a simplicial complex, then the over set \( O_x \) at a cell \( x \) is also a cell poset. In fact, \( O_x \) is the cell poset of the link \( \text{lk} \, x \) of \( x \). Theorem 2.5 characterizes the cell posets in which this holds.

**Proposition 2.2.** Let \( P \) be a cell poset and suppose \( X \) is connected. If the order complexes of the over sets of \( P \) at its minimal elements have order complexes homeomorphic to spheres then \( X \) is a manifold.

**Proof.** The order complexes of the over sets of \( P \) at minimal elements are isomorphic to the links of vertices of the order complex of \( P \). Hence the links of vertices of the barycentric subdivision of \( X \) are homeomorphic to spheres. That \( X \) is connected ensures that these spheres all have the same dimension, say \( d-1 \), and that \( X \) is second countable. Then \( X \) is a \( d \)-manifold. \( \square \)

A **PL-triangulated manifold** is a simplicial complex \( X \) that is homeomorphic to a manifold such that the link of every simplex of \( X \) is homeomorphic to a sphere [11]. PL-triangulated manifolds are referred to as **combinatorial manifolds** in the PL-topology literature.

**Proposition 2.3.** Let \( P \) be a cell poset and suppose \( X \) is connected. Then the order complex of \( P \) is a PL-triangulated manifold iff the order complexes of the strict intervals and over sets of \( P \) are homeomorphic to spheres.

Proposition 2.3 is an immediate consequence of the following lemma.

**Lemma 2.4.** Let \( P \) be a cell poset and suppose \( X \) is connected. Let \( P' \) be the cell poset of the order complex \( \Delta P \). The order complexes of the over sets of \( P' \) are homeomorphic to spheres iff the order complexes of the strict intervals and over sets of \( P' \) are homeomorphic to spheres.

Indeed, the over sets of \( P' \) are the cell posets of the links of the simplices of \( \Delta P \). If \( \Delta P \) is a PL-triangulated manifold then these are all homeomorphic to spheres. Conversely, if the links are all homeomorphic to spheres then since \( X \) is connected, Proposition 2.2 ensures that \( \Delta P \) is homeomorphic to a manifold. Then, by definition, \( X \) is a PL-triangulated manifold.

Before proving Lemma 2.4 we study the over sets of cell posets of order complexes. Let \( P \) be a poset and let \( P' \) be the cell poset of the order complex of \( P \). Every \( c \in P' \) is a nonempty chain

\[ c = \{c_0, c_1, \ldots, c_k\} \subset P \]

with

\[ c_0 < c_1 < \cdots < c_k \]

in \( P \) and these chains are ordered by inclusion. Each element of the over set \( O_c \) is a chain in \( P \) containing \( c \) and so is determined by its intersections with \( U_{c_0} \), with \( O_{c_0} \) and with the strict intervals \((c_{i-1}, c_i)\). It follows that \( O_c \) embeds in the componentwise product order

\[ (U_{c_0})' \times (c_0, c_1)' \times (c_1, c_2)' \times (c_2, c_3)' \times \cdots \times (c_{k-1}, c_k)' \times (O_{c_k})' \]
where $Q'_1$ denotes the poset of all chains (including the empty chain) in a poset $Q$. Aside from the presence of a minimum element corresponding to the empty simplex, this product is isomorphic to the cell poset of the simplicial join of the order complexes of $U_{c_0}, O_{c_k}$ and the $(c_{i-1}, c_i)$. The complement of this minimum element is the image of $O_c$ under its embedding in the product. Hence, $X_{O_c}$ is isomorphic to the simplicial join

$$X_{O_c} \cong O_{U_{c_0}} \bowtie O_{(c_0, c_1)} \bowtie O_{(c_1, c_2)} \cdots \bowtie O_{(c_{k-1}, c_k)} \bowtie O_{c_k}$$

of the order complexes of $U_{c_0}, O_{c_k}$ and the $(c_{i-1}, c_i)$.

**Proof of Lemma 2.4.** The joins of spheres are spheres so, by the discussion above, the “if” part of Lemma 2.4 has been established. It remains to prove the “only if” part.

Assume that the order complexes of the over sets of $P'$ are homeomorphic to spheres. Let $x, y, z \in P$ with $x < y$. Let $c^x \in P'$ and $c^z \in P'$ be maximal chains of $P$ that have $x$ and $z$ as their maximums. Let $c_y \in P'$ be a maximal chain of $P$ that has $y$ as its minimum. Then we have

$$X_{O_{c_y \cup c^x}} \cong O_{(x, y)}$$

and

$$X_{O_{c^z}} \cong \Delta_{O_z}$$

and so $O_{(x, y)}$ and $O_{O_c}$ are homeomorphic to spheres.

A PL-triangulated sphere is a PL-triangulated manifold that is homeomorphic to a sphere.

**Theorem 2.5.** Let $P$ be a cell poset. The following conditions are equivalent.

1. The order complexes of under sets of $P$ are PL-triangulated spheres.
2. The order complexes of strict intervals of $P$ are PL-triangulated spheres.
3. The order complexes of strict intervals of $P$ are homeomorphic to spheres.
4. The over sets of $P$ are cell posets.

Theorem 2.5 characterizes the regular CW complexes $X$ for which each $d$-cell $x$ may be associated a link having the structure of a regular CW complex in which the $(k - d - 1)$-cells naturally correspond to the $k$-cells of $X$ that are incident to $x$. Theorem 2.5 says that this holds precisely when the boundaries of the cells of $X$ are PL-triangulated spheres.

**Proof of Theorem 2.5.**

1. $\implies$ 2. Let $(x, y)$ be a strict interval of $P$. Then $(x, y)$ is the over set of $U_y$ at $x$. So, by Proposition 2.3, the order complex of $(x, y)$ is homeomorphic to a sphere. Every over set and strict interval of $(x, y)$ has order complexes homeomorphic to a sphere.

2. $\implies$ 3. This is clear.

3. $\implies$ 4. Let $O_x$ be a over set of $P$. By Theorem 2.1 we need only show that the under set of $O_x$ at any $y \in O_x$ has order complex homeomorphic to a sphere. But the under set of $O_x$ at $y$ is the strict interval $(x, y)$ of $P$ and so this holds.

4. $\implies$ 1. Let $U_z$ be a under set of $P$. By Theorem 2.1 the order complex of $U_z$ is homeomorphic to a sphere. To prove that it is a PL-triangulated sphere it suffices, by Proposition 2.3, to show that, for $x < y < z$, the strict interval $(x, y)$ and the over set $O_{U_y}^x$ of $U_y$ at $x$ have order complexes homeomorphic to spheres.
But $(x, y)$ and $O_U^y$ are equal to the under sets $U^O_x$ and $U^O_y$ of $O_x$ and so, by Theorem 2.1, they have order complexes homeomorphic to spheres. □

3. Forman Morse Theory and the Recognition of Spheres

Let $P$ be a finite graded poset. The Hasse diagram $\Gamma_P$ of $P$ is the covering relation $C_P$ viewed as a directed graph. A matching $M$ on $P$ is a set of pairwise disjoint closed edges of $\Gamma_P$. An element $x \in P$ is matched by $M$ if it is contained in an edge of $M$. A matching $M$ on $P$ is acyclic if the directed graph $\Gamma_M^P$ obtained from $\Gamma_P$ by reversing the direction on the edges of $M$ has no directed cycles. An element $x \in P$ is a critical element of $M$ if $x$ is not matched by $M$. Acyclic matchings are also known as Morse matchings. If $P$ is a cell poset then an acyclic matching $M$ on $P$ determines a Forman discrete Morse function [8] on $X_P$ with the same set of critical cells [4]. The language of acyclic matchings for discrete Morse theory is due to Chari [4].

**Proposition 3.1.** Let $P$ be a finite graded poset, let $M$ be a matching on $P$ and let $(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k)$ be the sequence of edges of a directed cycle $\gamma$ of $\Gamma_M^P$. Then no consecutive pair of edges $(\gamma_i, \gamma_{i+1})$—indices modulo $k$—has both edges in $M$ or both edges in the complement of $M$.

**Proof.** Following an edge of $M$ causes a unit decrease in the grading. Following an edge not in $M$ causes a unit increase in the grading. Hence $\gamma$ must contain the same number of edges in $M$ as it does edges not in $M$. Since $M$ is a matching, there is no consecutive pair of edges of $\gamma$ both contained in $M$. Suppose we have a consecutive pair of edges $(\gamma_i, \gamma_{i+1})$ neither of which are contained in $M$. Then there are two more $M$-edges in $(\gamma_i+2, \gamma_{i+3}, \ldots, \gamma_{i+k})$ then there are non-$M$-edges. Hence there is a consecutive pair of edges of $\gamma$ both contained in $M$, contradicting the hypothesis that $M$ is a matching. □

We require the following basic theorems of Forman discrete Morse theory.

**Theorem 3.2** (Forman [8]). Let $P$ be a cell poset. Let $M$ be an acyclic matching on $P$ and let $Q$ be the set of critical cells of $M$. If $Q$ is downward closed, then $X_Q$ can be obtained from $X_P$ by a sequence of elementary collapses. In particular, $X_Q$ is homotopy equivalent to $X_P$.

**Theorem 3.3** (Forman [8, Corollary 3.5]). Let $P$ be a cell poset and let $M$ be an acyclic matching on $P$. Then $X_P$ is homotopy equivalent to a CW complex with as many cells of each dimension as $M$ has critical cells of that dimension.

Let $P$ be a finite graded poset. A spherical matching on $P$ is an acyclic matching $M$ on $P$ with two critical cells.

**Theorem 3.4** (Sphere Theorem of Forman [8, Theorem 5.1(1)]). Let $P$ be a cell poset. If $P$ has a spherical matching then $X_P$ is homotopy equivalent to a sphere.

We also require the Generalized Poincaré Conjecture for topological manifolds.

**Theorem 3.5** (Topological Generalized Poincaré Conjecture). A closed topological manifold $X$ is homotopy equivalent to the $d$-sphere iff it is homeomorphic to the $d$-sphere.
The first breakthrough in the proof of the Generalized Poincaré Conjecture was made by Smale, who proved that a PL-triangulated manifold \( X \) that is homotopy equivalent to the \( d \)-sphere is homeomorphic to the \( d \)-sphere, for \( d \geq 5 \) [20]. Stallings gave a different proof of this fact for \( d \geq 7 \) using an “engulfing” method [21]. This method was later extended by Zeeman to prove the cases \( d = 5 \) and \( d = 6 \) [23]. Newman generalized the engulfing method to topological manifolds and thus completing the proof of the Topological Generalized Poincaré Conjecture for \( d \geq 5 \) [15, Theorem 7]. In dimension 4 the conjecture was proved by Freedman [9]. In dimension 3 it was proved by Perelman [16, 18, 17]. In dimensions at most 2, it follows from the classification of manifolds.

**Theorem 3.6.** Let \( P \) be a poset such that the order complexes of the under sets of \( P \) are PL-triangulated spheres. The order complex of \( P \) is a PL-triangulated sphere iff the order complexes of \( P \) and all of its over sets are homotopy equivalent to spheres.

**Remark 3.7.** Let \( P \) be as in Theorem 3.6. By Theorem 2.1 and Theorem 2.5, \( P \) and its over sets are all cell posets. So if \( Q \) is equal to \( P \) or to one of its over sets then to show that the order complex of \( Q \) is homotopy equivalent to a sphere it suffices to show that \( Q \) has a spherical matching. This holds by Theorem 3.4 and the fact that the order complex of \( Q \) is the barycentric subdivision of \( X_Q \).

**Proof of Theorem 3.6.** The proof is by induction on the maximum size \( k \) of a chain in \( P \). If \( k = 0 \) then \( P \) is the empty poset and so has the PL-triangulated \(-1\)-sphere (i.e. the empty simplicial complex) as its order complex.

Suppose \( k > 0 \) and that the theorem holds for all lesser values of \( k \). Take \( x \in P \).

We show that \( O_x \) satisfies the conditions of the theorem. The under sets of \( O_x \) are strict intervals of \( P \) and so, by Theorem 2.5, the order complexes of the under sets of \( O_x \) are PL-triangulated spheres. By assumption \( O_x \) is homotopy equivalent to a sphere as are its over sets since they are also over sets of \( P \). Hence, by the inductive hypothesis, the order complex of \( O_x \) is a PL-triangulated sphere. By Theorem 2.5 the strict intervals of \( P \) are also PL-triangulated spheres and so, by Proposition 2.3 the order complex \( O_P \) of \( P \) is a PL-triangulated manifold. By assumption, \( O_P \) is homotopy equivalent to a sphere so, by Theorem 3.5 \( O_P \) is homeomorphic to a sphere and so is a PL-triangulated sphere. \( \Box \)

4. **Bisimplices**

A *bipartitioned set* is a set \( S \) along with a bipartition \( S = A \sqcup B \) into subsets that are possibly empty. A subset \( T \subseteq S \) is considered to be a bipartitioned set with its induced bipartition \( T = (T \cap A) \sqcup (T \cap B) \). Our goal is to span cells on certain subsets of a bipartitioned set, just as simplices are spanned on subsets of vertices of a simplicial complex. In our case, however, not all subsets are eligible to span a cell so we introduce the term spanworthy. A bipartitioned set \( S = A \sqcup B \) is *spanworthy* if \( S \neq \emptyset \) and the following holds.

\[
|A| \leq 1 \iff |B| \leq 1
\]

Spanworthiness excludes precisely the following cases.

- \( \emptyset \sqcup \emptyset \)
- \( A \sqcup \emptyset \) with \( |A| \geq 2 \)
- \( \emptyset \sqcup B \) with \( |B| \geq 2 \)
• $A \cup \{b\}$ with $|A| \geq 2$
• $\{a\} \sqcup B$ with $|B| \geq 2$

In particular, it excludes any $S$ of cardinality 3.

**Theorem 4.1.** Let $S = A \cup B$ be a spanworthy bipartitioned set and let $P$ be the collection of spanworthy proper subsets of $S$ ordered by inclusion. Then $P$ is a cell poset and $O_P$ is a PL-triangulated sphere.

**Proof.** Let $A = \{a_0, a_1, \ldots, a_m\}$ and $B = \{b_0, b_1, \ldots, b_n\}$.

If $m = -1$ or $n = -1$ then, by spanworthiness, $S$ is a singleton and so $P$ is empty. Then $P$ is the cell poset of the empty simplicial complex, i.e., the PL-triangulated $-1$-sphere. The PL-triangulated $-1$-sphere is equal to its own barycentric subdivision $O_P$ so the theorem holds in this case.

Assume that $m \geq 0$ and $n \geq 0$. If $m = 0$ or $n = 0$ then, by spanworthiness, $m = n = 0$ and $P$ is the poset with two incomparable elements. This is the cell poset of the two point simplicial complex, i.e., the PL-triangulated 0-sphere. The PL-triangulated 0-sphere is equal to its own barycentric subdivision $O_P$ so the theorem holds in this case.

Assume that $m \geq 1$ and $n \geq 1$. If $m = n = 1$ then the elements of $P$ are the singletons and the $\{a_i\} \sqcup \{b_j\}$ for $i$ and $j$ ranging over 0 and 1. So $P$ is isomorphic to the cell poset of the 4-cycle and so the theorem holds.

So, by symmetry, we may assume that $m \geq 1$ and $n > 1$. Assume that the theorem holds for all $S$ of lesser cardinality. Then the order complexes of the under sets of $P$ are PL-triangulated spheres. So, by Theorem 2.1, $P$ is a cell poset and, by Theorem 3.6, it suffices to show that the order complexes of $P$ and all of its over sets are homotopy equivalent to spheres.

Let $T \in P$. Spanworthiness implies that $|T| \neq 3$. If $|T| > 3$ then every proper subset of $S$ containing $T$ is spanworthy and so $O_T$ is isomorphic to the cell poset of the boundary of a simplex and so has order complex homeomorphic to a sphere. So it remains only to show that the order complexes of $P$ and $O_T$ for $|T| \in \{1, 2\}$ are homotopy equivalent to spheres. By Remark 3.7, it suffices to show that $P$ and such $O_T$ have spherical matchings. By symmetry we need only consider the cases $T = \{a_0\} \sqcup \emptyset$, $T = \emptyset \sqcup \{b_0\}$ and $T = \{a_0\} \sqcup \{b_0\}$.

Consider the following families of edges of the Hasse diagram $\Gamma_P$ of $P$.

$M_1 = \left\{ \{a_i\} \sqcup \emptyset \rightarrow \{a_i\} \sqcup \{b_n\} \right\}$

$M_2 = \left\{ \emptyset \sqcup \{b_j\} \rightarrow \{a_m\} \sqcup \{b_j\} : j \neq n \right\}$

$M_3 = \left\{ \{a_i\} \sqcup \{b_j\} \rightarrow \{a_i, a_m\} \sqcup \{b_j, b_n\} : i \neq m, j \neq n \right\}$

$M_4 = \left\{ A' \sqcup \{b_j, b_n\} \rightarrow (A' \cup \{a_m\}) \sqcup \{b_j, b_n\} : \begin{array}{l} a_m \notin A', j \neq n \\ |A'| \geq 2 \end{array} \right\}$

$M_5 = \left\{ A' \sqcup B' \rightarrow A' \sqcup (B' \cup \{b_n\}) : \begin{array}{l} A' \sqcup B' \neq A \sqcup (B \setminus \{b_n\}) \\ |A'| \geq 2, |B'| \geq 2 \end{array} \right\}$
Note that \( n > 1 \) ensures that the terminal endpoints of edges in \( M_3 \) and \( M_4 \) are proper subsets of \( A \cup B \) and hence elements of \( P \).

The endpoints of these edges from different families or from different ends of edges in the same family can be distinguished by the cardinality of their parts and by the presence of \( a_m \) and \( b_n \), as shown in Table 1. Moreover the initial endpoints of two edges from the same family are equal if and only if their terminal endpoints are equal. Hence we see that \( M = \bigcup_i M_i \) forms a matching on \( P \).

Let \( \Gamma_P^M \) be the directed graph obtained from \( \Gamma_P \) by reversing the direction of each edge in \( M \). By Proposition 3.1 to show that \( M \) is an acyclic matching we need only show that \( \Gamma_P^M \) does not contain any directed cycles whose edges alternate between being contained and not contained in \( M \). To do this it suffices to define a function \( \alpha : P \to \mathbb{N} \) such that \( \alpha(T_2) < \alpha(T_0) \) for any directed path

\[
T_0 \xrightarrow{e_0} T_1 \xrightarrow{e_1} T_2
\]

of \( \Gamma_P^M \) with \( e_0 \in M \) and \( e_1 \notin M \). Note that in \( \Gamma_P \), \( e_0 \) is directed from \( T_1 \) to \( T_0 \) so we have the following inclusions.

\[
T_0 \supseteq T_1 \subseteq T_2
\]

We define \( \alpha \) as follows.

\[
\alpha(T) = \begin{cases} 
0, & a_m \notin T \text{ and } b_n \notin T \\
1, & a_m \in T \text{ and } b_n \notin T \\
2, & a_m \notin T \text{ and } b_n \in T \\
3, & a_m \in T \text{ and } b_n \in T 
\end{cases}
\]

We may think of \( \alpha \) as a function summing the weights on the elements of \( T \), where \( a_m \) is assigned a weight of 1 and \( b_n \) is assigned a weight of 2 and all remaining elements have zero weight. Since \( T_1 = T_0 \setminus A \) for some nonempty \( A \subseteq \{a_m, b_n\} \), we have \( \alpha(T_1) < \alpha(T_0) \). Suppose \( \alpha(T_2) = \alpha(T_0) \). Then \( T_1 \cup A \subseteq T_2 \) and, since \( |T_0| - |T_1| = |T_2| - |T_1| \), we have \( T_0 = T_2 \). This is a contradiction since a pair of vertices of \( \Gamma_P^M \) may be joined by at most one edge and this edge is directed in a unique way. Suppose \( \alpha(T_2) > \alpha(T_0) \). Then \( T_1 = T_0 \setminus \{a_m\} \) and \( T_2 = T_1 \cup \{b_n\} \). The equality \( T_1 = T_0 \setminus \{a_m\} \) implies that \( e_0 \) is the reverse of an edge in \( M_2 \) or \( M_4 \). The equality \( T_2 = T_1 \cup \{b_n\} \) and the fact that \( T_2 \neq T_1 \) implies that \( b_n \notin T_1 \). This

| Family | Initial Endpoint | Terminal Endpoint |
|--------|------------------|-------------------|
|        | \( a_m \) \( b_n \) | \( a_m \) \( b_n \) |
| \( M_1 \) | 1 0 | ⊥ 1 1 | \( T \) |
| \( M_2 \) | 0 1 | ⊥ ⊥ 1 1 | \( T \) ⊥ |
| \( M_3 \) | 1 1 | ⊥ ⊥ 2 2 | \( T \) \( T \) |
| \( M_4 \) | \( n \geq 2 \) 2 | ⊥ \( T \) 2 2 | \( T \) \( T \) |
| \( M_5 \) | \( n \geq 2 \) 2 | ⊥ \( \geq 2 \) \( T \) 2 2 | \( T \) |

Table 1. Distinguishing characteristics of the endpoints of edges in the families of edges described in the proof of Theorem 4.1. Under \( a_m \) or \( b_n \), the symbol \( T \) indicates that this element is present in every member of the family and the symbol \( ⊥ \) indicates that this element is not present in any member of the family.
rules out the possibility that $e_0$ is the reverse of an edge in $M_4$. Thus we have

$$\{a_m\} \sqcup \{b_j\} \xrightarrow{c_0} \emptyset \sqcup \{b_j\}$$

and so $T_1 = \emptyset \sqcup \{b_j\}$ and $T_2 = \emptyset \sqcup \{b_j, b_n\}$ which is not spanworthy, a contradiction. We have established that $M$ is an acyclic matching.

Let $T \in P$. Since the Hasse diagram of $O_T$ is an induced subgraph of $\Gamma_P$, the subset $M_T \subset M$ consisting of all edges both of whose endpoints are contained in $O_T$ is an acyclic matching on $O_T$. It remains only to show that $M$ is spherical on $P$ and that $M_T$ is spherical on $O_T$ for $T = \{a_0\} \sqcup \emptyset$, $T = \emptyset \sqcup \{b_0\}$ and $T = \{a_0\} \sqcup \{b_0\}$. In fact it will suffice to prove that $M$ is spherical on $P$ with critical elements $\emptyset \sqcup \{b_n\}$ and $A \sqcup (B \setminus \{b_n\})$. Indeed, in this case the only critical element of $M$ contained in $O_T$ would be $A \sqcup (B \setminus \{b_n\})$. The only other possible critical elements of $O_T$ would arise from edges of $M$ having one endpoint in $O_T$ and the other endpoint in $P \setminus O_T$. But there is a unique such edge of $M$, namely the edge with initial endpoint $T$. Hence $M_T$ would have two critical elements.

We now prove that $M$ is spherical with critical elements $\emptyset \sqcup \{b_n\}$ and $A \sqcup (B \setminus \{b_n\})$. First we verify that these elements are indeed unmatched by $M$. Singletons in $B$ appear as endpoints only in $M_2$ where $\emptyset \sqcup \{b_n\}$ is not present so $\emptyset \sqcup \{b_n\}$ is critical. The element $A \sqcup (B \setminus \{b_n\})$ is maximal in $P$ and so may only appear as a terminal endpoint of an edge of $M$. These all contain $b_n$ except those in $M_2$ where they have the form $\{a_m\} \sqcup \{b_j\}$. Such an element cannot be equal to $A \sqcup (B \setminus \{b_n\})$ since then $A \sqcup B = \{a_m\} \sqcup \{b_j, b_n\}$ which is not spanworthy.

Now, suppose $T = A' \sqcup B'$ is an element of $P$ that is not equal to $\emptyset \sqcup \{b_n\}$ or $A \sqcup (B \setminus \{b_n\})$. We will show that $T$ is matched in $M$. We consider the following cases separately: (I) $|T| = 1$, (II) $|T| = 2$, (III) $|T| = 4$, (IV) $|T| > 4$ and $|B'| = 2$, (V) $|T| > 4$ and $|B'| > 2$.

Case I. $|T| = 1$. If $|A'| = 1$ then $T$ is an initial endpoint in $M_1$. Otherwise $|B'| = 1$ and $T$ is an initial endpoint in $M_2$.

Case II. $|T| = 2$. Then $|A'| = |B'| = 1$ by spanworthiness. If $b_n \in T$ then $T$ is a terminal endpoint of $M_1$. Otherwise $T$ is a terminal endpoint in $M_2$ if $a_m \in T$ and $T$ is a terminal endpoint in $M_3$ if $a_m \notin T$.

Case III. $|T| = 4$. Then $|A'| = |B'| = 2$ by spanworthiness. If $b_n \notin T$ then $T$ is an initial endpoint of $M_5$. Otherwise $T$ is a terminal endpoint in $M_3$ if $a_m \in T$ and $T$ is an initial endpoint in $M_4$ if $a_m \notin T$.

Case IV. $|T| > 4$ and $|B'| = 2$. Then $|A'| > 2$. If $b_n \notin T$ then $T$ is an initial endpoint of $M_5$. Otherwise $T$ is a terminal endpoint in $M_4$ if $a_m \in T$ and $T$ is an initial endpoint in $M_1$ if $a_m \notin T$.

Case V. $|T| > 4$ and $|B'| > 2$. Then $|A'| \geq 2$ by spanworthiness. If $b_n \in T$ then $T$ is a terminal endpoint in $M_5$. Otherwise $T$ is an initial endpoint in $M_5$.  

**Corollary 4.2.** Let $S = A \sqcup B$ be a spanworthy bipartitioned set and let $P'$ be the collection of spanworthy subsets of $S$ ordered by inclusion. Then $P'$ is the cell poset of a regular CW complex homeomorphic to a ball.

Note that the difference between $P'$ in Corollary 4.2 and $P$ in Theorem 4.1 is that $P'$ contains $S$.

**Proof of Corollary 4.2.** By Theorem 4.1 it suffices to show that the order complexes of the under sets of $P'$ are homeomorphic to spheres. Since $S$ is the maximum in $P'$, the under sets of $P'$ are $P = P' \setminus \{S\}$ and the under sets of $P$. By
Theorem 4.1, we know that $P$ is the cell poset of a regular CW complex $X_P$ that is homeomorphic to a sphere. Hence, by Theorem 2.2, the under sets of $P$ have order complexes homeomorphic to spheres and the order complex of $P$ is the barycentric subdivision of $X_P$ and so is also homeomorphic to a sphere. □

Let $S = A \cup B$ be a spanworthy bipartitioned set and let $P'$ be the collection of spanworthy subsets of $S$ ordered by inclusion. By Corollary 4.2, $P'$ is the cell poset of a regular CW complex $X_{P'}$ that is homeomorphic to a ball. A regular CW complex isomorphic to $X_{P'}$ is an $(m,n)$-bisimplex where $m = |A| - 1$ and $n = |B| - 1$. We let $\Sigma^{m,n}$ denote an $(m,n)$-bisimplex. See Figure 1.

Proposition 4.3. There is an isomorphism $\Sigma^{m,n} \cong \Sigma^{n,m}$.

Proof. This is clear from the symmetry of the definition. □

Proposition 4.4. The cells of a bisimplex are all bisimplices.

Proof. The cell poset $P$ of a bisimplex $\Sigma$ is isomorphic to the poset of spanworthy subsets of a spanworthy set $A \cup B$. The cell poset of a cell $x$ of $\Sigma$ corresponds to $P' = U_T \cup \{T\}$ for some spanworthy $T$. Hence, the cell poset of $x$ is isomorphic to the poset of spanworthy subsets of the spanworthy set $T$. □

Proposition 4.5. The 1-skeleton of $\Sigma^{m,n}$ is isomorphic to the complete bipartite graph $K_{m+1,n+1}$ on $m + 1$ and $n + 1$ vertices.

Proof. Let $P$ be the cell poset of $\Sigma^{m,n}$ viewed as the poset of spanworthy subsets of $A \cup B$ with $|A| = m + 1$ and $|B| = n + 1$. The 0-cells of $\Sigma^{m,n}$ correspond to the minimal elements of $P$. These are precisely the singletons in $A \cup B$ and so we may identify the 0-skeleton of $\Sigma^{m,n}$ with $A \cup B$. The 1-cells of $\Sigma^{m,n}$ correspond to those elements of $P$ that cover singletons. These are precisely the sets $\{a\} \sqcup \{b\}$ with $a \in A$ and $b \in B$. □

Proposition 4.6. Let $m \geq 1$ and let $n \geq 1$. The dimension of $\Sigma^{m,n}$ is $m + n$.

Proof. Let $P$ be the poset of spanworthy subsets of $A \cup B$ with $A = \{a_0, a_1, \ldots, a_m\}$ and $B = \{b_0, b_1, \ldots b_n\}$. Identify $P$ with the cell poset of $\Sigma^{m,n}$. Since $\Sigma^{m,n}$ is homeomorphic to a ball of some dimension $k$, the maximal chains of $P$ all have cardinality $k + 1$. One such maximal chain is the following.

\[
\begin{align*}
\{a_0\} \sqcup \emptyset &\subsetneq \{a_0\} \sqcup \{b_0\} \subsetneq \{a_0, a_1\} \sqcup \{b_0, b_1\} \\
&\subsetneq \{a_0, a_1, a_2\} \sqcup \{b_0, b_1\} \subsetneq \{a_0, a_1, a_2, a_3\} \sqcup \{b_0, b_1\} \subsetneq \cdots \subsetneq \{a_0, \ldots, a_m\} \sqcup \{b_0, b_1\} \\
&\subsetneq \{a_0, \ldots, a_m\} \sqcup \{b_0, b_1, b_2\} \subsetneq \cdots \subsetneq \{a_0, \ldots, a_m\} \sqcup \{b_0, \ldots, b_n\}
\end{align*}
\]

This chain has cardinality $|A| + |B| - 1$ where the $-1$ is due to the jump $\{a_0\} \sqcup \{b_0\} \subsetneq \{a_0, a_1\} \sqcup \{b_0, b_1\}$. Hence $\Sigma^{m,n}$ has dimension $k + 1 - 1 = |A| + |B| - 1 - 1 = m + n$. □

5. Bisimplicial Complexes

A full subcomplex $Y$ of a regular CW complex $X$ is full if $\partial x \subset Y$ implies $x \subset Y$ for any cell $x$ of $X$. A bisimplicial complex is a regular CW complex $X$ such that each cell $x$ of $X$ is isomorphic to a bisimplex and, for any two bisimplices $x$ and $y$ of $X$, the intersection $x \cap y$ is a full subcomplex of $X$. Note that this implies that the bisimplices themselves are full subcomplexes and, furthermore, that any finite intersection of bisimplices is full.
A complete bipartite graph $K$ is spanworthy if it is nonempty, connected and the bipartition on its vertex set is spanworthy. A spanworthy complete bipartite subgraph $K$ of the 1-skeleton $X^1$ of a bisimplicial complex $X$ spans a bisimplicial $\Sigma$ of $X$ if the 1-skeleton $(\Sigma)_1$ of $\Sigma$ is equal to $K$. Note that at most one bisimplicial may span $K$ since the intersection of two distinct bisimplices $\Sigma$ and $\Sigma'$ spanning $K$ would be full in neither $\Sigma$ nor $\Sigma'$. A bisimplicial complex $X$ is flag if every spanworthy complete bipartite subgraph $K$ of $X^1$ spans a bisimplicial $\Sigma$. We use the notation $\Sigma(A; B)$ to denote $\Sigma$, where $A \sqcup B$ is the bipartitioned vertex set of $K$.

**Definition 5.1.** Let $\Gamma$ be a graph. The flag bisimplicial completion $\Sigma(\Gamma)$ of $\Gamma$ is a flag bisimplicial complex defined inductively as follows. The 1-skeleton of $\Sigma(\Gamma)$ is $\Gamma$. Now, assume the $(k-1)$-skeleton of $\Sigma(\Gamma)$ has been defined. The $k$-skeleton is obtained by the following operation. To each subcomplex isomorphic to some $\Delta^m,n$ with $\dim(\Delta^m,n) = k$, glue in a copy of $\Sigma^m,n$ along the isomorphism.

Note that if $X$ is a flag bisimplicial complex then $X = \Sigma(X^1)$.

Let $\Gamma$ be a finite bipartite graph. The metric sphere $S_r(u) \subseteq \Gamma^0$ of radius $r$ about $u \in \Gamma^0$ is the set of vertices of $\Gamma$ at distance $r$ from $u$. If $u$ and $v$ are distinct vertices of $\Gamma$ then $u$ is dominated by $v$ if there is an inclusion $S_1(u) \subseteq S_1(v)$ of neighbourhoods.

A finite bipartite graph $\Gamma$ is bi-dismantlable if there exists a sequence $\Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_n$ of graphs ending on a nonempty connected complete bipartite graph such that, for each $i < n$, $\Gamma_{i+1}$ is a subgraph of $\Gamma_i$ induced on the complement of $\{v_i\}$ for some $v_i$ dominated in $\Gamma_i$.

**Theorem 5.2.** Let $X$ be a finite flag bisimplicial complex with $X^1$ bipartite. If $X^1$ is bi-dismantlable then $X$ is collapsible.

**Proof.** The proof is by induction on the length of the bi-dismantling sequence.

In the base case, $X^1$ is a nonempty connected complete bipartite graph on some bipartitioned vertex set $S = A \sqcup B$. Let $A = \{a_0, \ldots, a_m\}$ and $B = \{b_0, \ldots, b_n\}$. Without loss of generality $|A| \leq |B|$. If $X^1$ is not spanworthy then, as it is nonempty and connected, we have $|A| = 1$ and $|B| \geq 2$. Then the only spanworthy subgraphs of $X^1$ are its edges and vertices and so $X = X^1$. But $X^1$ is a tree and so $X$ is collapsible.

Suppose now that $X^1$ is spanworthy. By flagness, $X$ is a bisimplicial $\Sigma(A; B)$. If $|A| + |B| \leq 4$ then $\Sigma(A; B)$ is isomorphic to a vertex, an edge or a square. These are collapsible. So we assume that $|A| \geq 2$ and $|B| > 2$. Identify the cell poset of $\Sigma(A; B)$ with the poset $P'$ of nonempty spanworthy subsets of $A \sqcup B$. Then the poset $P = P' \setminus \{A \sqcup B\}$ is the cell poset of $\partial \Sigma(A; B)$. The proof of Theorem 4.1 gives a spherical matching $M$ on $P$. Let $M'$ be the matching obtained from $M$ by adding the following edge.

$$A \sqcup (B \setminus \{b_n\}) \to A \sqcup B$$

Then $M'$ is acyclic and leaves only $\emptyset \sqcup \{b_n\}$ unmatched. Hence, by Theorem 3.2 $X = \Sigma(A; B)$ is collapsible.

Now, suppose suppose the bi-dismantling sequence has nonzero length with $v$ the first dominated vertex in the sequence. Let $u$ be a dominator of $v$ in $X^1$. Let $P$ be the cell poset of $X$. Consider the downward closed subset $Q$ of $P$ defined as follows.

$$Q = \{x \in P : x \not\geq v\}$$
Then the subcomplex $X_Q = \bigcup Q$ is the full subcomplex of $X$ induced on $X^0 \setminus \{v\}$ and so $X_Q$ is flag. Moreover, $X^1_Q$ is the induced subgraph of $X^1$ obtained from $X^1$ by deleting $v$ and so $X^1_Q$ is dismantlable. Hence $X^1_Q$ is collapsible by induction. Therefore, by Theorem 3.2, it suffices to construct an acyclic matching $M$ on $P$ whose set of critical elements is $Q$.

Let $w$ be any neighbour of $v$ in $X^1$. Note that the vertices of any connected complete bipartite subgraph of $X^1$ containing $v$ are at distance at most 2 from $v$. Consider the following families of edges in the Hasse diagram $\Gamma_P$ of $P$.

\begin{align*}
M_1 &= \left\{ \Sigma(\{v\}; \emptyset) \to \Sigma(\{v\}; \{w\}) \right\} \\
M_2 &= \left\{ \Sigma(\{v\}; \{x\}) \to \Sigma(\{u, v\}; \{w, x\}) : x \in S_1(v) \setminus \{w\} \right\} \\
M_3 &= \left\{ \Sigma(\{u, v\}; N) \to \Sigma(\{u, v\}; \{w\} \cup N) : N \subseteq S_1(v) \setminus \{w\}, |N| \geq 2 \right\} \\
M_4 &= \left\{ \Sigma(T \cup \{v\}; N) \to \Sigma(T \cup \{u, v\}; N) : T \subseteq S_2(v) \setminus \{u\}, N \subseteq \bigcap_{y \in T \cup \{v\}} S_1(y), |T| \geq 1, |N| \geq 2 \right\}
\end{align*}

The union $M = \bigcup_i M_i$ is an acyclic matching on $P$ whose set of critical elements is $Q$. The argument is very similar to that in the proof of Theorem 4.1 with $w$ playing the role of $a_m$ and $u$ playing the role of $b_n$. $\square$
6. QUADRIC COMPLEXES AND ASPHERICITY

**Definition 6.1.** A locally quadric complex is a locally finite square complex $X$ with immersed cells such that no reduced disk diagram in $X$ has the form of Figure 2 and any immersed disk diagram of a form on the left-hand side of Figure 3 has a replacement on the right-hand side with the same boundary path. If, in addition, $X$ is simply connected then $X$ is quadric. A group $G$ is quadric if it acts properly and cocompactly on a quadric complex.

For a full introduction to quadric complexes and groups see prior work of the present author [10].

A square complex $X$ is flag if each square of $X$ is bounded by an embedded 4-cycle and each embedded 4-cycle of $X$ bounds a unique square of $X$.

**Proposition 6.2** ([10, Proposition 1.18]). Let $X$ be a connected square complex. Then $X$ is quadric if and only if $X$ is flag and every isometrically embedded cycle of $X$ has length 4.

It follows from Proposition 6.2 that a quadric complex $X$ is the 2-skeleton of the flag bisimplicial completion $\Delta(\square X)$. See Definition 5.1.

**Theorem 6.3** (Bandelt [1, Theorem 1]). A graph is hereditary modular if and only if it is connected and every isometrically embedded cycle has length 4.

A graph is modular if for every triple of vertices $u, v, w$ there exists a vertex $x$ which lies on some geodesic between each pair of vertices in the triple. A graph is hereditary modular if each of its isometrically embedded subgraphs is modular.

The metric ball of radius $r \in \mathbb{N}$ centered at a vertex $v$ of a graph (bisimplicial complex) is the induced (full) subgraph (subcomplex) on the set of vertices of distance at most $r$ to $v$ (in the 1-skeleton).

**Remark 6.4.** Let $\Gamma$ be a modular graph. Then the metric balls of $\Gamma$ are isometrically embedded. In particular, if $\Gamma$ is hereditary modular then its metric balls are hereditary modular.

**Theorem 6.5** (Bandelt [1, Theorem 2]). Let $\Gamma$ be a finite nonempty hereditary modular graph. Then $\Gamma$ is bi-dismantlable.

**Theorem 6.6.** Let $X$ be a nonempty quadric complex. Then the flag bisimplicial completion $\Delta(X)$ is contractible.

**Proof.** The metric balls of $X$ are quadric by Proposition 6.2, Theorem 6.3 and Remark 6.4. These balls are finite since quadric complexes are locally finite. Hence balls in $X$ are collapsible by Proposition 6.2, Theorem 6.3, Theorem 6.5 and Theorem 5.2. The balls of $X$ centered at a fixed vertex give an ascending exhaustion of $X$ by contractible subcomplex and so $X$ is contractible. \qed

6.1. A $K(G, 1)$ for Torsion-Free Quadric Groups. Let $X$ be a locally quadric complex. Then the universal cover $\tilde{X}$ is quadric and so $\pi_1 X$ is quadric. Let $\square_{m,n}$ denote the 2-skeleton of the bisimplex $\Sigma^{m,n}$. Let $\square_{m,n} \to \tilde{X}$ be an immersion with $m, n \geq 2$. Since $\square_{m,n}$ is simply connected it lifts to $\tilde{X}$. Since quadric complexes do not contain loops or bigons [10], every lift $\square_{m,n} \to \tilde{X}$ is an embedding. Every torsion-free quadric group is the fundamental group of a compact locally quadric complex. However, a compact locally quadric complex may have a fundamental
group with torsion. The following theorem allows to understand when this is the case.

**Theorem 6.7** (Invariant Biclique Theorem [10]). Let $F$ be a finite group acting on a quadric complex $\tilde{X}$. Then $F$ stabilizes a nonempty connected complete bipartite subgraph of $\tilde{X}$.

**Corollary 6.8.** Let $X$ be a compact locally quadric complex. Then $\pi_1(X)$ has torsion if and only if $\text{Aut}(\Box_{m,n} \to X)$ is nontrivial for some immersion $\Box_{m,n} \to X$ with $m, n \geq 1$.

**Proof.** Suppose $g \in \pi_1(X) \setminus \{1\}$ has finite order. Then $\langle g \rangle$ stabilizes a nonempty connected complete bipartite subgraph $K_{m+1,n+1}$ of $\tilde{X}^3$. The action is free so $g$ does not fix this $K_{m+1,n+1}$ pointwise. In particular, $m, n \geq 1$. The full subcomplex induced by this $K_{m+1,n+1}$ is a $\Box_{m,n}$. Restricting the covering map to $\Box_{m,n}$ we have an immersion $\Box_{m,n} \to X$. Restricting the action of $g$ to $\Box_{m,n}$ we obtain a nontrivial automorphism of $\Box_{m,n} \to X$.

Now, suppose there is a nontrivial automorphism $\varphi: \Box_{m,n} \to \Box_{m,n}$ of an immersion $\Box_{m,n} \to X$. Let $f: \Box_{m,n} \to \tilde{X}$ be a lift of this immersion and identify $\Box_{m,n}$ with its image under $f$. Then $\varphi$ extends to a nontrivial deck transformation which must have finite order. \qed

Let $X$ be a compact locally quadric complex. We will construct a compact complex $X^+$ having $X$ as its 2-skeleton such that $X^+$ is a $K(\pi_1(X), 1)$ if $\pi_1(X)$ is torsion-free. The construction is by induction on dimension. The 2-skeleton of $X^+$ is $X$. Now suppose we have already constructed the $(k - 1)$-skeleton $(X^+)^{k-1}$ of $X^+$. To obtain the $k$-skeleton of $X^+$ perform the following operation. Along each immersion $\partial \Sigma_{m,n} \to (X^+)^{k-1}$ with $\dim(\Sigma_{m,n}) = k$ glue in a copy of $\Sigma_{m,n}$. For the purposes of this operation, two immersions which are isomorphic over $(X^+)^{k-1}$ are considered identical and so result in only a single gluing. Since $X$ is compact there is a bound on the size of a connected complete bipartite graph which can immerse in $X$. Hence $X^+$ is compact.

**Lemma 6.9.** Let $X$ be a compact locally quadric complex with $\pi_1(X)$ torsion-free. Then the universal cover $\tilde{X}^+$ is isomorphic to the bisimplicial completion $\Sigma(\tilde{X}^1)$.

**Proof.** The proof is by induction on skeleta. The 2-skeleta of $X^+$ and $\Sigma(\tilde{X}^1)$ are $X$ and $\tilde{X}$ so the base case holds. Assume the statement holds for the $(k - 1)$-skeleta: $(\tilde{X}^1)^{k-1} \cong \Sigma(\tilde{X}^1)^{k-1}$. Each $\partial \Sigma_{m,n}$ subcomplex, $\dim(\Sigma_{m,n}) = k$, of $\Sigma(\tilde{X}^1)^{k-1}$ immerses into $(X^+)^{k-1}$ under the covering map and so spans a $\Sigma_{m,n}$ in $(\tilde{X}^1)^k$. On the other hand, each immersion $\partial \Sigma_{m,n} \to (X^+)^{k-1}$ with $\dim(\Sigma_{m,n}) = k$ lifts to an embedding in $\Sigma(\tilde{X}^1)^{k-1}$ whose image thus spans a unique $\Sigma_{m,n}$ in $\Sigma(\tilde{X}^1)^k$. So the set of boundaries of the $k$-dimensional bisimplices of $(\tilde{X}^1)^k$ and $\Sigma(\tilde{X}^1)^k$. No two $k$-dimensional bisimplices have the same boundary in $\Sigma(\tilde{X}^1)^k$. The same holds for $(\tilde{X}^1)^k$ by Corollary 6.8 and so $(\tilde{X}^1)^k \cong \Sigma(\tilde{X}^1)^k$. \qed

The main theorem of this section follows immediately from Lemma 6.9 and Theorem 6.6.

**Theorem 6.10.** Let $X$ be a compact locally quadric complex. If $\pi_1(X)$ is torsion-free then $X^+$ is a compact $K(\pi_1(X), 1)$. 
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