Harmonic Measure and Winding of Conformally Invariant Curves

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The exact joint multifractal distribution for the scaling and winding of the electrostatic potential lines near any conformally invariant scaling curve is derived in two dimensions. Its spectrum \( f(\alpha, \lambda) \) gives the Hausdorff dimension of the points where the potential scales with distance \( r \) as \( H \sim r^\alpha \) while the curve logarithmically spirals with a rotation angle \( \varphi = \lambda \ln r \). It obeys the scaling law \( f(\alpha, \lambda) = (1 + \lambda^2)f(\tilde{\alpha}) - b\lambda^2 \) with \( \tilde{\alpha} = \alpha/(1 + \lambda^2) \) and \( b = (25 - c)/12 \), and where \( f(\alpha) \equiv f(\alpha, 0) \) is the pure harmonic measure spectrum, and \( c \) the conformal central charge. The results apply to \( O(N) \) and Potts models, as well as to SLE\(_\infty\).

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The geometric description of the random fractals arising in Nature is a fascinating subject. Among these, the study of the particular class of random clusters or fractal curves arising in critical phenomena has led to fundamental advances in mathematical physics. In two dimensions (2D), conformal field theory (CFT) has in particular demonstrated that statistical systems at their critical point produce conformally invariant (CI) fractal structures, examples of which are the continuum scaling limits of random walks (RW), i.e., Brownian motion, self-avoiding walks (SAW), and critical Ising or Potts clusters. A wealth of exact methods has been devised for their study: Coulomb gas, conformal invariance, and quantum gravity and Coulomb gas methods), which allow the description of Brownian paths interacting and winding with CI curves, thereby providing a probabilistic description of the potential map.

Harmonic Measure and Rotations. Consider a single (CI) critical random cluster, generically called \( \mathcal{C} \). Let \( H(z) \) be the potential at the exterior point \( z \in \mathbb{C} \), with Dirichlet boundary conditions \( H(w \in \partial \mathcal{C}) = 0 \) on the outer (simply connected) boundary \( \partial \mathcal{C} \) of \( \mathcal{C} \), and \( H(w) = 1 \) on a circle “at \( \infty \), i.e., of a large radius scaling like the average size \( R \) of \( \mathcal{C} \). As is well-known, \( H(z) \) is identical to the probability that a Brownian path starting at \( z \) escapes to “\( \infty \)” without having hit \( \mathcal{C} \).

Let us now consider the degree with which the curves wind in the complex plane about point \( w \) and call \( \varphi(z) = \arg(z - w) \). The multifractal formalism (see also [6]), here generalized to take into account rotations, characterizes subsets \( \partial \mathcal{C}_{\alpha, \lambda} \) of boundary sites by a Hölder exponent \( \alpha \), and a rotation rate \( \lambda \), such that their potential lines respectively scale and

\[
H(z \to w \in \partial \mathcal{C}_{\alpha, \lambda}) \sim r^\alpha, \\
\varphi(z \to w \in \partial \mathcal{C}_{\alpha, \lambda}) \sim \lambda \ln r,
\]

in the scaling limit \( a_0 \ll r = |z - w| \ll R \), where \( a_0 \) is the lattice mesh, if any. The Hausdorff dimension \( \dim(\partial \mathcal{C}_{\alpha, \lambda}) = f(\alpha, \lambda) \) defines the mixed MF spectrum,
which is CI since under a conformal map both \( \alpha \) and \( \lambda \) are locally invariant.

Reversing the escaping Brownian path which represents the potential, one can also consider the harmonic measure \( H(w, r) \), which is the probability that such a path starting at distance \( R \) first hits the boundary in the disk \( B(w, r) \) of radius \( r \) centered at \( w \in \partial C \), and \( \varphi(w, r) \) the associated winding angle of the path down to distance \( r \) from \( w \). The mixed moments of \( H \) and \( e^\varphi \), averaged over all realizations of \( C \), are defined as

\[
Z_{n, p} = \left\langle \sum_{w \in \partial C} H^n(w, r) \exp(p \varphi(w, r)) \right\rangle \approx (r/R)^{\tau(n, p)},
\]

where the sum runs over the centers of a covering of the boundary by disks of radius \( r \), and where \( n \) and \( p \) are real numbers. The scaling limit involves multifractal scaling exponents \( \tau(n, p) \) which vary in a non-linear way with \( n \) and \( p \) \([13, 14, 15, 16] \). They obey the symmetric double Legendre transform

\[
\alpha = \frac{\partial \tau}{\partial n}(n, p), \quad \lambda = \frac{\partial \tau}{\partial p}(n, p), \quad f(\alpha, \lambda) = \alpha n + \lambda p - \tau(n, p),
\]

\[
n = \frac{\partial f}{\partial \alpha}(\alpha, \lambda), \quad p = \frac{\partial f}{\partial \lambda}(\alpha, \lambda).
\]

Because of the ensemble average \([17]\), values of \( f(\alpha, \lambda) \) can become negative for some values of \( \alpha \).

**Exact Mixed Multifractal Spectra.** Each 2D conformally invariant random statistical system can be labelled by its central charge \( c, c \leq 1 \). Our main result is the following exact scaling law:

\[
f(\alpha, \lambda) = (1 + \lambda^2) f \left( \frac{\alpha}{1 + \lambda^2} \right) - b \lambda^2, \quad (3)
\]

\[
b = \frac{25 - c}{12} \geq 2,
\]

where \( f(\alpha) \equiv f(\alpha, \lambda = 0) \) is the usual harmonic MF spectrum in the absence of prescribed winding, first obtained in \([18]\), which can be recast as:

\[
f(\alpha) = \alpha + b - \frac{b \alpha^2}{2a - 1} \quad (4)
\]

We thus arrive at the very simple formula:

\[
f(\alpha, \lambda) = \alpha + b - \frac{b \alpha^2}{2a - 1 - \lambda^2} \quad (5)
\]

Notice that by conformal symmetry \( \sup_\lambda f(\alpha, \lambda) = f(\alpha, \lambda = 0) \), i.e., the most likely situation in the absence of prescribed rotation is the same as \( \lambda = 0 \), i.e. winding-free. The domain of definition of the usual \( f(\alpha) \) \([19]\) is \( \alpha \geq 1/2 \) \([18]\), thus for \( \lambda \)-spiralling points Eq. \([18]\) gives

\[
\alpha \geq \frac{1}{2}(1 + \lambda^2), \quad (6)
\]

in agreement with a theorem by Beurling \([13, 15]\).

There is a geometrical meaning to the exponent \( \alpha \). For an angle with opening \( \theta \), \( \alpha = \pi/\theta \), thus the quantity \( \pi/\alpha \) can be regarded as a local generalized angle with respect to the harmonic measure. The geometrical MF spectrum of the boundary subset with such opening angle \( \theta \) and spiralling rate \( \lambda \) reads from \([16]\)

\[
f(\theta, \lambda) \equiv f(\alpha = \pi/\theta, \lambda) = \pi/\theta + b - \theta^2 \left( \frac{1}{\theta} + \frac{1}{1 + \lambda^2} - \theta \right).
\]

As in \([16]\), the domain of definition in the \( \theta \) variable is \( 0 \leq \theta \leq \theta(\lambda) \), with \( \theta(\lambda) = 2\pi/(1 + \lambda^2) \). The maximum is reached when the two frontier strands about point \( w \) locally collapse into a single \( \lambda \)-spiral, whose inner opening angle is \( \theta(\lambda) \) \([18]\).

In the absence of prescribed winding \( (\lambda = 0) \), the maximum \( D_{\text{EP}} \equiv D_{\text{EP}}(0) = \sup_\alpha f(\alpha, \lambda = 0) \) gives the dimension of the external perimeter of the fractal cluster, which is a simple curve without double points, and may differ from the full hull \([17, 19]\). Its dimension reads \([16]\)

\[
D_{\text{EP}} = \frac{1}{2} (1 + b) - \frac{1}{2} \sqrt{b(b - 2)}. \quad \text{(7)}
\]

This corresponds to typical values \( \hat{\alpha}(\lambda) = (1 + \lambda^2) \hat{\alpha}, \) and \( \theta(\lambda) = \hat{\theta}/(1 + \lambda^2) \). Since \( b \geq 2 \) and \( D_{\text{EP}} \leq 3/2, \) the EP dimension decreases with spiralling rate, in a simple parabolic way.

Fig. 1 displays typical multifractal functions \( f(\alpha, \lambda; c) \). The example chosen, \( c = 0 \), corresponds to the cases of a SAW, or of a percolation EP, the scaling limits of which both coincide with the Brownian frontier \([17, 19]\).
The original singularity at $\alpha = \frac{1}{2}$ in the rotation free MF functions $f(\alpha, 0)$, which describes boundary points with a needle local geometry, is shifted for $\lambda \neq 0$ towards the minimal value (6). The right branch of $f$-$\alpha$ with the maximal value $\lambda$ and scaling curve, with the maximum value $D_{\text{EP}} = 3/2$, the minimal value (6). Thus the $\lambda$-curves all become parallel for $\alpha \to +\infty$, i.e., $\theta \to 0^+$, corresponding to deep fjords where winding is easiest.

Limit multifractal spectra are obtained for $c = 1$, which exhibit exact examples of left-sided MF spectra, with a horizontal asymptote $f(\alpha \to +\infty, \lambda; c = 1) = \frac{3}{2} - \frac{1}{2} \lambda^2$ (Fig. 2). This corresponds to the frontier of a $Q = 4$ Potts cluster (i.e., the SLE$_{\kappa = 4}$), a universal random scaling curve, with the maximum value $D_{\text{EP}} = 3/2$, and a vanishing typical opening angle $\theta = 0$, i.e., the “ultimate Norway” where the EP is dominated by “fjords” everywhere $\Box$.

Fig. 3 displays the dimension $D_{\text{EP}}(\lambda)$ as a function of the rotation rate $\lambda$, for various values of $c \leq 1$, corresponding to different statistical systems. Again, the $c = 1$ case shows the least decay with $\lambda$, as expected from the predominance of fjords there.

Conformal Invariance and Quantum Gravity. We now give the main lines of the derivation of the exponent $\tau(n, p)$, hence $f(\alpha, \lambda)$, by generalized conformal invariance. By definition of the $H$-measure, $n$ independent Brownian paths $B$, starting a small distance $r$ away from a point $w$ of the frontier $\partial C$, and diffusing without hitting $\partial C$, give a geometric representation of the $n^{th}$ moment, $H^n$, in Eq.(6) for $n$ integer. Convexity yields analytic continuation for arbitrary $n$’s. Let us introduce an abstract (conformal) field operator $\Phi_{\partial C \wedge n}$, characterizing the presence of a vertex where $n$ such Brownian paths and the cluster’s frontier diffuse away from each other in a mutually-avoiding configuration noted $\partial C \wedge n$ $\Box$; to this operator is associated a scaling dimension $\kappa(n)$. To measure rotations as in moments $\Box$ we have to consider expectation values with insertion of the mixed operator

$$\Phi_{\partial C \wedge n} e^{p \arg(\partial C \wedge n)} \to \kappa(n, p) = \tau(n, p) + 2,$$

where $\arg(\partial C \wedge n)$ is the winding angle common to the frontier and to the Brownian paths, and where $\kappa(n, p)$ is the scaling dimension. One has $\kappa(n, p = 0) = \kappa(n)$, and $\tau(n, p = 0) \equiv \tau(n) = \kappa(n) - 2$.

Let us now use a fundamental mapping of the CFT in the plane $\mathbb{R}^2$ to the CFT on a fluctuating abstract random Riemann surface, i.e., in presence of 2D quantum gravity (QG) $\Box$. Two universal functions $U$ and $V$, acting on scaling dimensions, describe this map:

$$U(x) = \frac{x - \gamma}{1 - \gamma}, \quad V(x) = \frac{x^2 - \gamma^2}{4 - 1 - \gamma},$$

with $V(x) \equiv U\left(\frac{1}{2} (x + \gamma)\right)$ $\Box$. The parameter $\gamma$ is the solution of $c = 1 - 6\gamma^2 (1 - \gamma)^{-1}$, $\gamma \leq 0$.

For the purely harmonic exponents $\kappa(n)$, describing the mutually-avoiding set $\partial C \wedge n$, we have $\Box$

$$\kappa(n) = 2V[2U^{-1}(\kappa_1) + U^{-1}(\kappa_n)],$$

where $U^{-1}(x)$ is the positive inverse of $U$

$$2U^{-1}(x) = \sqrt{4(1 - \gamma)x + \gamma^2 + \gamma}.$$

In $\Box$, the arguments $\kappa_1$ and $n$ are respectively the boundary scaling dimensions (b.s.d.) of the simple path $S_1$ representing a semi-infinite random frontier (such that $\partial C \equiv S_1 \wedge S_1$), and of the packet of $n$ Brownian paths, both diffusing into the upper half-plane $\mathbb{R}^2$. The function $U^{-1}$ maps these half-plane b.s.d.’s to the corresponding b.s.d.’s in quantum gravity, the linear combination of which gives, still in QG, the b.s.d. of the mutually-avoiding set $\partial C \wedge n = (\wedge S_1)^2 \wedge n$. The function $V$ finally maps the latter b.s.d. into the scaling dimension in $\mathbb{R}^2$. The path b.s.d. $\kappa_1$ obeys $U^{-1}(\kappa_1) = (1 - \gamma)/2$ $\Box$.

![FIG. 2: Left-sided multifractal spectra $f(\alpha, \lambda)$ for the limit case $c = 1$ (frontier of a $Q = 4$ Potts cluster or SLE$_{\kappa = 4}$).](image1)

![FIG. 3: Dimensions $D_{\text{EP}}(\lambda)$ of the external frontiers as a function of rotation rate. The curves are indexed by the central charge $c$, and correspond respectively to: loop-erased RW ($c = -2$; SLE$_2$); Brownian or percolation external frontiers, and self-avoiding walk ($c = 0$; SLE$_{\kappa = 3}$); Ising clusters ($c = \frac{1}{2}$; SLE$_3$); $Q = 4$ Potts clusters ($c = 1$; SLE$_4$).](image2)
It is now useful to consider \( k \) semi-infinite random paths \( S_k \) joined at a single vertex in a \textit{mutually-avoiding} star configuration \( S_k = S_1 \cap S_1 \cap \cdots \cap S_1 = (\cup S_k)^k \). Its scaling dimension can be obtained from the same b.s.d. additivity rule in quantum gravity, as in \[ 10 \] 

\[
x(S_k) = 2V \left[ k U^{-1}(\tilde{x}_i) \right].
\]

The scaling dimensions \[ 10 \] and \[ 11 \] coincide when 

\[
x(n) = x(S_k(n)), \, k(n) = 2 + \frac{U^{-1}(n)}{U^{-1}(x_1)}. \quad (12)
\]

Thus we state the \textit{scaling star-equivalence} \( \partial C \land n \leftrightarrow S_k(n) \), of two simple paths \( S_1 \) avoiding \( n \) Brownian motions to \( k(n) \) simple paths in a mutually-avoiding star configuration, an equivalence which will also play an essential role in the complete rotation spectrum \[ 8 \].

\textit{Rotation scaling exponents.} The Gaussian distribution of the winding angle about the \textit{extremity} of a scaling path, like \( S_1 \), was derived in \[ 21 \], using exact Coulomb gas methods. The argument can be generalized to the winding angle of a star \( S_k \) about its center \[ 22 \], where one finds that the angular variance is reduced by a factor \( 1/k^2 \) (see also \[ 23 \]). The scaling dimension associated with the rotation scaling operator \( \Phi_{k,\tilde{c},p,\arg(S_k)} \) is found by analytic continuation of the Fourier transforms evaluated there \[ 22 \]:

\[
x(S_k; p) = x(S_k) - \frac{2}{1 - \gamma} \frac{p^2}{k^2},
\]

i.e., given by a quadratic shift in the star scaling exponent. To calculate the scaling dimension \[ 8 \], it is sufficient to use the star-equivalence \[ 3 \] above to conclude that 

\[
x(n, p) = x(S_k(n); p) = x(n) - \frac{2}{1 - \gamma} \frac{p^2}{k^2(n)},
\]

which is the key to our problem. Using Eqs \[ 12 \], \[ 10 \], and \[ 8 \] gives the useful identity:

\[
\frac{1}{8} (1 - \gamma)k^2(n) = x(n) - 2 + b,
\]

with \( b = \frac{1}{2} (2 - \gamma)^2 = \frac{25 - c}{12} \). Recalling \[ 8 \], we arrive at the multifractal result:

\[
\tau(n, p) = \tau(n) - \frac{1}{4} \frac{p^2}{\tau(n) + b}, \quad (13)
\]

where \( \tau(n) = x(n) - 2 \) corresponds to the purely harmonic spectrum with no prescribed rotation.

\textit{Legendre transform.} The structure of the full \( \tau \)-function \[ 13 \] leads by a formal Legendre transform \[ 4 \] directly to the identity

\[
f(\alpha, \lambda) = (1 + \lambda^2)f(\alpha) - b\lambda^2,
\]

where \( f(\alpha) \equiv \alpha n - \tau(n) \), with \( \alpha = d\tau(n)/dn \), is the purely harmonic MF function. It depends on the natural reduced variable \( \alpha \) \textit{à la} Beurling (\( \alpha \in \left[ \frac{1}{2}, +\infty \right) \))

\[
\alpha \equiv \frac{\alpha}{1 + \frac{\lambda^2}{2}} \frac{dx}{dn}(n) = \frac{1}{2} + \frac{1}{2} \sqrt{\frac{b}{2n + b - 2}},
\]

whose expression is found explicitly from \[ 10 \]. Whence Eq. \( 8 \), \textit{QED}.

\textit{O(N) and Potts models, SLE_\kappa}. Our results apply to the critical \( O(N) \) loop model, or to the EP’s of critical Fortuin-Kasteleyn (FK) clusters in the \( Q \)-Potts model, all described in terms of Coulomb gas with some coupling constant \( g \) \[ 9 \]. SLE_\kappa paths also describe cluster frontiers or hulls. One has the correspondence \( \kappa = 4g / \gamma \), with a central charge \( c = (3 - 2g)(3 - 2g') = \frac{1}{4}(6 - \kappa)(6 - \kappa') \), symmetric under the \textit{duality} \( gg' = 1 \) or \( \kappa \kappa' = 16 \). This duality gives FK-EP’s as some simple random \( O(N) \) loops, or, equivalently, the SLE_{\kappa' \leq 4} as the simple frontier of the SLE_{\kappa \geq 4} \[ 12 \].

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