Some Estimates Regarding Integrated density of States for Random Schrödinger Operator with decaying Random Potentials

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Abstract
We investigate some bounds for the density of states in the pure point regime for the random Schrödinger operators $H^\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} a_n q_n(\omega)$, acting on $\ell^2(\mathbb{Z}^d)$, where $\{q_n\}$ are iid random variables and $a_n \approx |n|^{-\alpha}$, $\alpha > 0$.

1. Introduction
The random Schrödinger operator with decaying randomness is a random Hamiltonian $H^\omega$ on $\ell^2(\mathbb{Z}^d)$ given by
\begin{equation}
(1.1) \quad H^\omega = -\Delta + V^\omega, \quad \omega \in \Omega.
\end{equation}
$\Delta$ is adjacency operator defined by
\begin{equation}
(\Delta u)(n) = \sum_{|m-n|=1} u(m) \forall u \in \ell^2(\mathbb{Z}^d).
\end{equation}
The random potential $V^\omega$ which is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ by the sequence $\{a_n q_n(\omega)\}_{n \in \mathbb{Z}^d}$ defined by
\begin{equation}
(1.2) \quad V^\omega = \sum_{n \in \mathbb{Z}^d} a_n q_n(\omega) |\delta_n\rangle \langle \delta_n|.
\end{equation}
where $\{\delta_n\}_{n \in \mathbb{Z}^d}$ be the standard basis for $\ell^2(\mathbb{Z}^d)$. Here $\{a_n\}_{n \in \mathbb{Z}^d}$ is a sequence of positive real numbers such that $a_n \to 0$ as $|n| \to \infty$ and $\{q_n\}_{n \in \mathbb{Z}^d}$ are real valued iid random variables with an absolutely continuous probability distribution $\mu$ which has bounded density. We realize $q_n$ as $\omega(n)$ on $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}_{\mathbb{R}^{\mathbb{Z}^d}}, \mathbb{P})$, $\mathbb{P} = \bigotimes \mu$ construct via Kolmogorov theorem we will refer to this probability space as $(\Omega, \mathcal{B}, \mathbb{P})$ henceforth.

For any $B \subset \mathbb{Z}^d$ we consider the orthogonal projection $\chi_B$ onto $\ell^2(B)$ and define the matrices
\begin{equation}
(1.3) \quad H^\omega_B = (\langle \delta_n, H^\omega_B \delta_m \rangle)_{n,m \in B}, \quad G^B(z;n,m) = \langle \delta_n, (H^\omega_B - z)^{-1} \delta_m \rangle, \quad G^B(z) = (H^\omega_B - z)^{-1}.
\end{equation}
$G(z) = (H^\omega - z)^{-1}$, $G(z;n,m) = \langle \delta_n, (H^\omega - z)^{-1} \delta_m \rangle, z \in \mathbb{C}^+.$

Note that $H^\omega_B$ is the matrix
\begin{equation}
H^\omega_B = \chi_B H^\omega \chi_B : \ell^2(\mathbb{Z}^d) \to \ell^2(B), \text{ a.e } \omega.
\end{equation}
We note that by an assumption on $V^\omega$, the operator $H^\omega$ are self adjoint a.e $\omega$ and have a common core domain consisting of vectors of finite support.

Let $\Lambda_L$ denote the d-dimension box of side length given by
\begin{equation}
\Lambda_L = \{(n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d : |n_i| \leq L \} \subset \mathbb{Z}^d.
\end{equation}
We then assume that.

**Hypothesis 1.1.** (1) The measure $\mu$ is absolute continuous with density satisfies
\begin{equation}
(1.4) \quad \rho(x) = \begin{cases} 
0 & \text{if } |x| < 1 \\
\frac{1}{|x|^\delta} & \text{if } |x| \geq 1, \text{ for some } \delta > 1.
\end{cases}
\end{equation}
(2) The sequence \( a_n \) satisfy \( a_n \simeq |n|^{-\alpha}, \alpha > 0 \).
(3) The pair \((\alpha, \delta)\) are s.t

\[
(1.5) \quad \beta_L = \sum_{n \in \Lambda_L} |n|^{-\alpha(\delta-1)} = O\left( (2L+1)^{d-\alpha(\delta-1)} \right).
\]

Remark 1.2. In the above we take explicit expression for \( \rho(x) = \frac{d\mu}{dx}(x) \) for simplification of calculation which we goint to do in the proofs of our results. Our result is still can be achieved for \( \frac{d\mu}{dx}(x) = O\left( \frac{1}{|x|^\delta} \right) \), \( \delta > 1 \) as \( |x| \to \infty \).

In [19] Kirsch-Krishna-Obermeit consider \( H^\omega = -\Delta + V^\omega \) on \( \ell^2(\mathbb{Z}^d) \) with same \( V^\omega \) defined in (1.2) shown that \( \sigma(H^\omega) = \mathbb{R} \) and \( \sigma_c(H^\omega) \subseteq [-2d, 2d] \) a.e \( \omega \), under some conditions on \( \{a_n\} \) and \( \mu \) (The density of \( \mu \) should not decay too fast at infinity and \( a_n \) should not decay too fast). For mathamatical formulation of above condition on \( a_n \)’s and \( \mu \) we refer Definition 2.1 in [19].

To show the existence of point spectrum outside \([-2d, 2d]\) they verifed Simon-Wolf criterion [23, Theorem 12.5] by showing exponential decay of the fractional moment of the Green function [19, Lemma 3.2]. This will give whenever \(|n - m| > 2R\) and the energy \( E \) is outside of \([-2d, 2d]\), the spectrum of \( \Delta \) we have

\[
(1.6) \quad \mathbb{E}^\omega(|G^{\Lambda^\epsilon}(E+i\epsilon : n, m)|^s) \leq D_{P(n,m)} e^{-c\left(\frac{|n-m|}{2}\right)}, \quad E \in \mathbb{R} \setminus [-2d, 2d]
\]

where \( \epsilon > 0 \), \( 0 < s < 1 \), \( c \) is positive constant and \( R \in \mathbb{Z}^+ \). Here \( D_{P(n,m)} \) is a constant independent of \( E, \epsilon \), but polynomially bounded in \( |n| \) and \( |m| \).

Jakšić-Last showed in [13, Theorem 1.2] that for \( d \geq 3 \) if \( a_n \simeq |n|^{-\alpha} \) \( \alpha > 1 \) then there is no singular spectrum inside \((-2d, 2d)\) of \( H^\omega \).

In this work we verifed the condition given in [19, Corollary 2.5] with \( a_n \) and \( \mu \) as in Hypothesis[1.1] for which the spectrum of \( H^\omega \) is \( \mathbb{R} \) and \( \sigma_c(H^\omega) \subseteq [-2d, 2d] \) a.e \( \omega \). Then we shown that the average spacing of eigenvalues of \( H^\omega_{\Lambda_L} \) near the energy \( \lambda \in \mathbb{R} \setminus [-2d, 2d] \) is of order \( \beta_L^{-1} \) and near \( \lambda \in [-2d, 2d] \) is of order \( \frac{1}{(2L+1)^d} \). Which shows that the eigenvalues of \( H^\omega_{\Lambda_L} \) are more densely distributed inside \([-2d, 2d] \), the continuous spectrum of \( H^\omega \) than the pure point regime i.e outside \([-2d, 2d] \).

Define

\[
(1.7) \quad N^\omega_{L}(\lambda) = \# \{ j : E_j \leq \lambda, \ E_j \in \sigma(H^\omega_{\Lambda_L}) \}.
\]

\[
(1.8) \quad \tilde{N}^\omega_{L}(\lambda) = \# \{ j : E_j \geq \lambda, \ E_j \in \sigma(H^\omega_{\Lambda_L}) \}.
\]

\[
(1.9) \quad \gamma_L(.) = \frac{1}{\beta_L} \sum_{n \in \Lambda_L} \mathbb{E}^\omega\left(\langle \delta_n, E_{H^\omega_{\Lambda_L}}(.)\delta_n \rangle\right)
\]

It is well known result that for the ergodic Anderson model (when \( a_n \) = constant, i.e \( \alpha = 0 \)) for almost all \( \omega \) the macroscopic limit

\[
N(E) := \lim_{L \to \infty} \frac{N^\omega_{L}(E)}{(2L+1)^d}
\]
exist and non random for $E \in \mathbb{R}$ (see [H]). This limit is called the integrated density of states. Since $N(E)$ is a non-decreasing function of $E$, it is differentiable almost everywhere, and if its derivative $n(E) = dN(E)/dE$ exists at $E \in \mathbb{R}$, we call it the density of states at the energy $E$.

Our results can be summarized as follows

**Theorem 1.3.** If $E = -2d - \epsilon < -2d$ for some $\epsilon > 0$ then we have

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{\delta - 1}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \frac{1}{(\delta - 1)^{\epsilon^{\delta - 1}}}.$$  

For $E = 2d + \epsilon > 2d$ we have

$$\frac{1}{(\delta - 1)(4d + \epsilon)^{\delta - 1}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_L^\omega(E)) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_L^\omega(E)) \leq \frac{1}{(\delta - 1)^{\epsilon^{\delta - 1}}}.$$  

Now we will investigate the average numbers of eigenvalues of $H^\omega_{N,L}$ inside $[-2d, 2d]$, the continuous spectrm of $H^\omega$, which can be given by following.

**Corollary 1.4.** For any interval $(M_1, M_2) \supset [-2d, 2d]$ we have

$$\lim_{L \to \infty} \frac{1}{(2L + 1)^d} \mathbb{E}^\omega(\# \{ \sigma(H^\omega_{N,L}) \cap (M_1, M_2) \}) = 1.$$  

**Corollary 1.5.** If $M_1 < -2d$ and $M_2 > 2d$ then we have

$$\lim_{L \to \infty} \gamma_L((-\infty, M_1] \cup [M_2, \infty)) \leq \frac{1}{\delta - 1} \left[ \frac{1}{(-2d - M_1)^{\delta - 1}} + \frac{1}{(M_2 - 2d)^{\delta - 1}} \right].$$  

For any interval $I \subseteq \mathbb{R} \setminus [-2d, 2d]$ with length $|I| > 4d$ there is a constant $C_I > 0$ such that

$$\lim_{L \to \infty} \gamma_L(I) \geq C_I > 0.$$  

**Corollary 1.6.** The sequence of measure $\{ \gamma_L \mid (M_1, M_2) \}$ admits a subsequence which will converge vaguely to a non-trivial measure say $\gamma$.

Here $M_1 < -2d$ and $M_2 > 2d$ and $\gamma_L \mid (M_1, M_2)$ denote the restriction of $\gamma_L$ to $\mathbb{R} \setminus (M_1, M_2)$.

The above theorem give us the estimate for the average of $N_L^\omega(\lambda)$ and $\tilde{N}_L^\omega(\lambda)$ but we can also do the pointwise estimate of above quantities which is given by following theorem.

**Theorem 1.7.** If $d \geq 2$, $0 < \alpha < \frac{1}{2}$ and $1 < \delta < \frac{1}{2\alpha}$ then for almost all $\omega$

$$\frac{1}{(\delta - 1)(2d - E)^{\delta - 1}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} N_L^\omega(E) \leq \frac{1}{(\delta - 1)(-2d - E)^{\delta - 1}}$$  

for $E < -2d$

$$\frac{1}{(\delta - 1)(2d + E)^{\delta - 1}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_L^\omega(E) \leq \frac{1}{(\delta - 1)(E - 2d)^{\delta - 1}}$$  

for $E > 2d$.  

In [9] Figotin-Germinet-Klein-Müller studied the Anderson Model on $L^2(\mathbb{R}^d)$ with decaying random potentials given by

$$H^\omega = -\Delta + \lambda \gamma_\alpha V^\omega \text{ on } L^2(\mathbb{R}^d),$$

where $\lambda > 0$ is the disorder parameter, $\gamma_\alpha$ is the envelope function

$$\gamma_\alpha(x) := (1 + |x|^2)^{-\frac{\alpha}{2}}, \quad \alpha \geq 0.$$ 

They assumed that density of single site distribution is in $L^\infty(\mathbb{R}^d)$ and has compact support. They showed that if $\alpha \in (0, 2)$ then $H^\omega$ has infinitely many eigenvalues in $(-\infty, 0)$ a.e $\omega$. In [9, Theorem 3] they gave the bound for $N^\omega(E)$, $E < 0$ (number of eigen values of $H^\omega$ below $E$) in terms of density of states for stationary (iid case) Model.

In [12] Gordon-Jakšić-Molcanov-Simon studied the Model given by

$$H^\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} (1 + |n|^\alpha)q_n(\omega), \quad \alpha > 0 \text{ on } l^2(\mathbb{Z}^d),$$

where $\{q_n\}$ are iid random variables uniformly distributed on $[0, 1]$.

They showed that if $\alpha > d$ then $H^\omega$ has discrete spectrum a.e $\omega$. For the case when $\alpha \leq d$ they construct a strictly decreasing sequence $\{a_k\}_{k \in \mathbb{N}}$ of positive numbers such that if $\frac{d}{k} \geq \alpha > \frac{d}{k+1}$ then for a.e $\omega$ we have the followings.

(i) $\sigma(H^\omega) = \sigma_{pp}(H^\omega)$ and eigen functions of $H^\omega$ decay exponentially,

(i) $\sigma_{ess}(H^\omega) = [a_k, \infty),$ 

(iii) $\#\sigma_{disc}(H^\omega) < \infty.$

They also showed that

(a) If $\frac{d}{k} > \alpha > \frac{d}{k+1}$ and $E \in (a_j, a_{j-1}), \ 1 \leq j \leq k,$ then

$$\lim_{L \to \infty} \frac{N_j^\omega(E)}{L^{d-j\alpha}} = N_j(E)$$

exists for a.e $\omega$ and is a non random function.

(b) If $\alpha = \frac{d}{k}$ and $E \in (a_j, a_{j-1}), \ 1 \leq j < k$ the above is valid. If $E \in (a_k, a_{k-1})$ then

$$\lim_{L \to \infty} \frac{N_k^\omega(E)}{lnL} = N_k(E)$$

exists for a.e $\omega$ and is a non random function.

In this work, essentially we show that for decaying potentials the confinement length is $(2L + 1)^d$ inside $[-2d, 2d]$ and $\beta_L$ outside $[-2d, 2d]$. Whereas for the growing potentials (as in [12]) the confinement length is changing as energy varies from interval to interval.

2. **On the pure point and continuous spectrum**

Now we want to identify the spectrum of $H^\omega$ (describe as in [14]) under the Hypothesis [14].
Let $x < 0$ and $\epsilon > 0$ such that $x + \epsilon < 0$ then for large enough $|n| \geq M$ we have $a_n^{-1}(x + \epsilon) \leq -1$ since $a_n^{-1} \to \infty$ as $|n| \to \infty$. Then we have for $|n| \geq M$

$$
\mu \left( \frac{1}{a_n}(x - \epsilon, x + \epsilon) \right) = \int_{a_n^{-1}(x-\epsilon)}^{a_n^{-1}(x+\epsilon)} \rho(t) dt
$$

$$
= a_n^{(\delta-1)} \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^{\delta}} dt.
$$

Therefore,

(2.1) $\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n}(x - \epsilon, x + \epsilon) \right) \geq \int_{x-\epsilon}^{x+\epsilon} \frac{1}{|t|^{\delta}} dt \sum_{|n| \geq M} a_n^{(\delta-1)} = \infty,$

since $\beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \to \infty$ as $L \to \infty$ (using 1.5).

For $x > 0$, similar calculation as above will give

(2.2) $\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n}(x - \epsilon, x + \epsilon) \right) = \infty.$

Now let $\epsilon > 0$, since $a_n^{-1} \to \infty$ as $|n| \to \infty$ then there exist $M$ such that $a_n^{-1}\epsilon > 1$ for $|n| \geq M$. So we have

$$
\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n}(-\epsilon, \epsilon) \right) \geq \sum_{|n| \geq M} \mu(-a_n^{-1}\epsilon, a_n^{-1}\epsilon)
$$

$$
= 2 \sum_{|n| \geq M} \int_{1}^{a_n^{-1}\epsilon} \frac{1}{t^{\delta}} dt
$$

$$
= \frac{2}{\delta - 1} \sum_{|n| \geq M} (1 - \epsilon^{1-\delta} a_n^{\delta-1})
$$

Since $\frac{2}{\delta - 1} \sum_{n \in \Lambda_L} (1 - \epsilon^{1-\delta} a_n^{\delta-1}) \approx \frac{2}{\delta - 1} [(2L + 1)^d - (2L + 1)^d - \alpha^{(\delta-1)}]$ then we have from above

(2.3) $\sum_{n \in \mathbb{Z}^d} \mu \left( \frac{1}{a_n}(-\epsilon, \epsilon) \right) = \infty.$

Now we have if $0 < \epsilon_1 < \epsilon_2$ then we have

$$
\mu \left( a_n^{-1}(x - \epsilon_1, x + \epsilon_1) \right) \leq \mu \left( a_n^{-1}(x - \epsilon_2, x + \epsilon_2) \right) \forall x \in \mathbb{R}.
$$

Now using above together with (2.1), (2.2) and (2.3) we have for all $\epsilon > 0$

(2.4) $\sum_{n \in \mathbb{Z}^d} \mu \left( a_n^{-1}(x - \epsilon, x + \epsilon) \right) = \infty, \forall x \in \mathbb{R}.$

Then using (2.4) from [19, Definition 2.1] we see that

$$
M = \cap_{k \in \mathbb{Z}^+} (a_kn - \text{supp}) = \mathbb{R}.
$$
Therefore [19, Corollary 2.5] and [19, Theorem 2.3] will give the following description about the spectrum of $H^\omega$.

$$\sigma_{\text{ess}}(H^\omega) = [-2d, 2d] + \mathbb{R} = \mathbb{R} \text{ and } \sigma_c(H^\omega) \subseteq [-2d, 2d] \text{ a.e } \omega.$$

### 2.1. Proof of Theorem 1.3

Define

$$A^\omega_{L, \pm} = \pm 2d + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{\delta_n}.$$ 

Define

$$N^\omega_{\pm, L}(E) = \# \{ j; E_j \leq E, E_j \in \sigma(A^\omega_{L, \pm}) \}, \quad N^\omega_L(E) = \# \{ j; E_j \leq E, E_j \in \sigma(H^\omega_{\Lambda_L}) \}.$$ 

Since $\sigma(\Delta) = [-2d, 2d]$ it is easy to see the following operator inequality

$$A^\omega_{L, -} \leq H^\omega_{\Lambda_L} \leq A^\omega_{L, +},$$

where $H^\omega_{\Lambda_L} = \chi_{\Lambda_L} H^\omega \chi_{\Lambda_L}$ is given by

$$H^\omega_{\Lambda_L} = \chi_{\Lambda_L} \Delta \chi_{\Lambda_L} + \sum_{n \in \Lambda_L} a_n q_n(\omega) P_{\delta_n}.$$ 

Simple application of the min-max principle [14, Theorem 6.44] will provide

$$N^\omega_{\Lambda_L}(E) \leq N^\omega_{L}(E) \leq N^\omega_{\Lambda_L}(E).$$

Now the spectrum $\sigma(A^\omega_{L, \pm})$ of $A^\omega_{L, \pm}$ consists of only eigen values and is given by

$$\sigma(A^\omega_{L, \pm}) = \{ n \in \Lambda_L : \pm 2d + a_n q_n(\omega) \}$$

Let $E < -2d$ with $E = -2d - \epsilon$ for some $\epsilon > 0$.

$$N^\omega_{\Lambda_L}(E) = \# \{ n \in \Lambda_L : -2d + a_n q_n(\omega) \leq -2d - \epsilon \}$$

$$= \# \{ n \in \Lambda_L : q_n(\omega) \in (-\infty, -a_n^{-1} \epsilon) \}$$

$$= \sum_{n \in \Lambda_L} \chi_{\{ \omega : q_n(\omega) \in (-\infty, -a_n^{-1} \epsilon) \}}$$

Since $q_n$ are i.i.d so if we take expectation of both side of above we get

$$\mathbb{E}^\omega(N^\omega_{\Lambda_L}(E)) = \sum_{n \in \Lambda_L} \mu(-\infty, -a_n^{-1} \epsilon)$$

$$= \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1} \epsilon} \rho(x) dx$$

since $a_n^{-1} \to \infty$ as $|n| \to \infty$ and $\epsilon > 0$ so there exist a $M \in \mathbb{N}$ such that

$$a_n^{-1} \epsilon > 1, \quad -a_n^{-1} \epsilon < -1 \forall |n| > M.$$
So from (2.7) for large enough \( L \), we get

\[
\mathbb{E}^\omega(N_{-L}^\omega(E)) = \sum_{n \in \Lambda_L} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx
\]

(2.9)

\[
= \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx + \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx.
\]

(2.10)

Since \( \# \{ n \in \mathbb{Z}^d : |n| \leq M \} \leq (2M + 1)^d \) then we have

\[
\sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx \leq (2M + 1)^d \int_{-\infty}^{-1} \rho(x) dx
\]

(2.11)

\[
= (2M + 1)^d \int_{-\infty}^{-1} \frac{1}{|x|^\delta} dx
\]

\[
= \frac{(2M + 1)^d}{(\delta - 1)}, \delta > 1 \text{ is given.}
\]

If we take \( \beta_L = \sum_{n \in \Lambda_L} a_n^{(\delta-1)} \) then \( \beta_L \to \infty \) as \( L \to \infty \) and we have from above

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| \leq M} \int_{-\infty}^{-1} \rho(x) dx = 0.
\]

(2.12)

Now

\[
\sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx = \sum_{n \in \Lambda_L, |n| > M} a_n^{-1} \int_{-\infty}^{-\epsilon} \rho(a_n^{-1}t) dt
\]

\[
= \sum_{n \in \Lambda_L, |n| > M} a_n^{-1} \int_{-\infty}^{-\epsilon} \frac{1}{|a_n^{-1}t|^\delta} dt
\]

\[
= \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)} \int_{-\infty}^{-\epsilon} \frac{1}{|t|^\delta} dt
\]

\[
= \frac{\epsilon^{\delta-1}}{1-\delta} \sum_{n \in \Lambda_L, |n| > M} a_n^{(\delta-1)}, \delta > 1
\]

This will give

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \sum_{n \in \Lambda_L, |n| > M} \int_{-\infty}^{-a_n^{-1}\epsilon} \rho(x) dx = \frac{\epsilon^{1-\delta}}{\delta - 1}.
\]

(2.13)

using (2.12) and (2.14) in (2.9) we get

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{-L}^\omega(E)) = \frac{\epsilon^{1-\delta}}{\delta - 1} = \frac{1}{(\delta - 1)\epsilon^{(\delta-1)}} > 0.
\]

(2.15)

Now a similar calculation with \( \mathbb{E}^\omega(N_{+L}^\omega(E)) \) will give

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{+L}^\omega(E)) = \frac{(4d + \epsilon)^{1-\delta}}{\delta - 1} = \frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta-1)}} > 0.
\]

(2.16)
Now using (2.15) and (2.16), from (2.6) we can conclude the following

\[
\frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(E)) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta - 1)}}.
\]

If we define

\[
\tilde{N}_{L,\pm}^\omega(E) = \#\{j : E_j \geq E, E_j \in \sigma(A_{L,\pm}^\omega)\},
\]

Now Min-max theorem and (2.5) will give

\[
\tilde{N}_{L,\pm}^\omega(E) \leq \tilde{N}_L^\omega(E) \leq \tilde{N}_{L,+}^\omega(E).
\]

Now if \(E = 2d + \epsilon > 2d\) for some \(\epsilon > 0\) then similar calculation as above will give

\[
\frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_L^\omega(E)) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_L^\omega(E)) \leq \frac{1}{(\delta - 1)\epsilon^{(\delta - 1)}}.
\]

Now (2.17) and (2.20) will give the Theorem 1.3.

**Corollary 1.4.**
Since \(H_{L,\pm}^\omega\) is a matrix of order \((2 + 1)^d\) then we have \(#\sigma(H_{L,\pm}^\omega) = (2 + 1)^d\).

If \(M_1 < -2d\) and \(M_2 > 2d\) then

\[
#\left\{\sigma(H_{L,\pm}^\omega) \cap (-\infty, M_1]\right\} + #\left\{\sigma(H_{L,\pm}^\omega) \cap (M_1, M_2]\right\} + #\left\{\sigma(H_{L,\pm}^\omega) \cap [M_2, \infty)\right\} = (2 + 1)^d.
\]

Now

\[
\frac{1}{(2 + 1)^d} \mathbb{E}^\omega\left\{\sigma(H_{L,\pm}^\omega) \cap (-\infty, M_1]\right\} = \frac{\beta_L}{(2 + 1)^d} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(M_1)).
\]

Now from (2.17) and Hypothesis 1.1 we have

\[
\lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_L^\omega(M_1)) < \infty, \text{ and } \lim_{L \to \infty} \frac{\beta_L}{(2 + 1)^d} = 0.
\]

Now it follows from above and (2.22) that

\[
\lim_{L \to \infty} \frac{1}{(2 + 1)^d} \mathbb{E}^\omega\left\{\sigma(H_{L,\pm}^\omega) \cap (-\infty, M_1]\right\} = 0.
\]

Similarly using (2.20) we can get

\[
\lim_{L \to \infty} \frac{1}{(2 + 1)^d} \mathbb{E}^\omega\left\{\sigma(H_{L,\pm}^\omega) \cap [M_2, \infty)\right\} = 0.
\]

Now using (2.21), (2.23) and (2.24) we get that for any interval \((M_1, M_2)\) which contain \([-2d, 2d]\)

\[
\lim_{L \to \infty} \frac{1}{(2 + 1)^d} \mathbb{E}^\omega\left\{\#\left\{\sigma(H_{L,\pm}^\omega) \cap (M_1, M_2]\right\}\right\} = 1.
\]
Corollary 1.5.
If $M_1 < -2d$ then from (1.9) we have
\[
\gamma_L(-\infty, M_1] = \frac{1}{\beta_L} \mathbb{E}[\text{Tr}(E_{H\mathbb{X}}(-\infty, M_1))]
\]
\[
= \frac{1}{\beta_L} \mathbb{E}[N^\omega_L(M_1)]. \text{ (using (1.7))}
\]
Now the above together with (2.17) will give
\[
\lim_{L \to \infty} \gamma_L(-\infty, M_1] \leq \frac{1}{(\delta - 1)(-2d - M_1)^{\delta - 1}} \text{ (using } \epsilon = -2d - M_1). \tag{2.26}
\]
Similarly for $M_2 > 2d$ using (2.20) we get
\[
\lim_{L \to \infty} \gamma_L(M_2, \infty) \leq \frac{1}{(\delta - 1)(M_2 - 2d)^{\delta - 1}} \text{ (using } \epsilon = M_2 - 2d). \tag{2.27}
\]
Now (2.26) and (2.27) will give (1.11).

Let $J = [E_1, E_2] \subset (-\infty, -2d)$ with $|J| > 4d$, set $E_1 = -2d - \epsilon_1$, $E_2 = -2d - \epsilon_2$ such that $\epsilon_1 - \epsilon_2 > 4d$.
\[
\gamma_L(J) = \frac{1}{\beta_L} \mathbb{E}[N^\omega_L(E_2)] - \frac{1}{\beta_L} \mathbb{E}[N^\omega_L(E_1)]
\]
\[
\geq \frac{1}{\beta_L} \mathbb{E}[N^\omega_L(E_2)] - \frac{1}{\beta_L} \mathbb{E}[N^\omega_L(E_1)]. \text{ (using (2.6))} \tag{2.28}
\]
Now we have using (2.16) and (2.15) we get (1.12), i.e.
\[
\lim_{L \to \infty} \gamma_L(J) \geq \frac{1}{(\delta - 1)} \left[ \frac{1}{(4d + \epsilon_2)^{\delta - 1}} - \frac{1}{(\epsilon_1)^{\delta - 1}} \right] > 0.
\]
Similar result holds even when $J \subset (2d, \infty)$ with $|J| > 4d$.

Corollary 1.6.
It follows from (1.11) that
\[
\sup_L \gamma_L((-\infty, M_1] \cup [M_2, \infty)) < \infty. \tag{2.29}
\]
We can write $\mathbb{R} \setminus (M_1, M_2) = \bigcup_n A_n$, countable union of compact sets. Now $\gamma_L|_{A_n}$ (restriction of $\gamma_L$ to $A_n$) admits a weakly convergence subsequence by Banach-Alaoglu Theorems. Then by diagonal argument we can select a subsequence of $\{\gamma_L\}$ which will converge vaguely to a non-trivial measure say $\gamma$ on $\mathbb{R} \setminus (M_1, M_2)$. Now non-triviality of $\gamma$ is given by the fact that if $J \subset \mathbb{R} \setminus (M_1, M_2)$ is an interval such that $4d < |J| < \infty$ then from (1.12) we get
\[
\inf_L \gamma_L(J) > 0.
\]
Before goes to the proof of the Theorem 1.7 we will prove the following lemma, which is required in the proof of the theorem.
Lemma 2.1. Let \( \{X_n\} \) be sequence of random variables on a probability space \((\Omega, \mathcal{B}, \mathbb{P})\) then if for each \( \epsilon > 0 \)
\[
\sum_{n=1}^{\infty} \mathbb{P}(\omega : |X_n(\omega) - X(\omega)| > \epsilon) < \infty.
\]
Then \( X_n \xrightarrow{n \to \infty} X \) a.e \( \omega \).

Proof:- Define \( A_n(\epsilon) = \{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \} \).

Now if
\[
\sum_{n=1}^{\infty} \mathbb{P}(A_n(\epsilon)) = \sum_{n=1}^{\infty} \mathbb{P}(\omega : |X_n(\omega) - X(\omega)| > \epsilon) < \infty.
\]
then Borel-Cantelli lemma will give
\[
\mathbb{P}(A(\epsilon)) = 0, \text{ where } A(\epsilon) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_n(\epsilon).
\]

Now we have
\[
\mathbb{P}(B(\epsilon)) = 1 \text{ where } B(\epsilon) = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_n(\epsilon)^c.
\]

Now for each \( N \in \mathbb{N} \) we define
\[
B_N = B(1/N) \text{ and } B = \bigcap_{N=1}^{\infty} B_N \text{ then } \mathbb{P}(B) = 1, \text{ since } \mathbb{P}(B_N) = 1.
\]

For any \( \delta > 0 \) we can choose \( M \in \mathbb{N} \) such that \( \frac{1}{M} < \delta \). Now if \( \omega \in B \) then \( \omega \in B_N \forall \ N \in \mathbb{N} \), from construction of \( B_M \) there exist a \( K \in \mathbb{N} \) such that
\[
|X_m(\omega) - X(\omega)| \leq \frac{1}{M} < \delta \forall \ m \geq K.
\]
So we have
\[
X_m \xrightarrow{m \to \infty} X \text{ on } B \text{ with } \mathbb{P}(B) = 1.
\]

Hence the lemma.

2.2. Proof of Theorem 1.7.

Let \( E = -2d - \epsilon \) for some \( \epsilon > 0 \) and define
\[
(2.30) \quad X_n(\omega) := \chi_{\{\omega: g_n(\omega) \leq -a_n^{-1}\epsilon\}}.
\]

Since \( \{g_n\}_n \) are iid so we have \( \{X_n\}_n \) is a sequence of independent random variables. Now from (2.7) we have
\[
(2.31) \quad N_{-L}(E) = \sum_{n \in \Lambda_L} X_n(\omega)
\]

We want to prove the following
\[
(2.32) \quad \lim_{L \to \infty} \frac{N_{-L}(E) - \mathbb{E}^\omega(N_{-L}(E))}{\beta_L} = 0 \text{ a.e } \omega.
\]
Using Lemma 2.1 to prove the above it is enough to show the followings

\[
\sum_{L=1}^{\infty} P\left(\omega : \left| \frac{N_{\omega,L}(E) - \mathbb{E}^{\omega}(N_{\omega,L}(E))}{\beta_L} \right| > \eta \right) < \infty \quad \forall \ \eta > 0.
\]

Now using Chebyshev’s inequality we get

\[
\sum_{L=1}^{\infty} P\left(\omega : \left| \frac{N_{\omega,L}(E) - \mathbb{E}^{\omega}(N_{\omega,L}(E))}{\beta_L} \right| > \eta \right) \leq \sum_{L=1}^{\infty} \frac{1}{\eta^2 \beta_L^2} \mathbb{E}^{\omega}\left( N_{\omega,L}(E) - \mathbb{E}^{\omega}(N_{\omega,L}(E)) \right)^2
\]

\[
= \sum_{n \in \Lambda_L} \mathbb{E}^{\omega}\left( X_n(\omega) - \mathbb{E}^{\omega}(X_n) \right)^2 \quad \text{(using (2.31))}
\]

Now using above in (2.34) we get

\[
\sum_{L=1}^{\infty} P\left(\omega : \left| \frac{N_{\omega,L}(E) - \mathbb{E}^{\omega}(N_{\omega,L}(E))}{\beta_L} \right| > \eta \right) \leq \frac{1}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L} \mathbb{E}^{\omega}\left( N_{\omega,L}(E) \right) \quad \text{(using (2.13))}
\]

\[
\leq \frac{C}{\eta^2} \sum_{L=1}^{\infty} \frac{1}{\beta_L} \quad \text{(using (1.3))}
\]

As we assume in the theorem that \(0 < \alpha < \frac{1}{2}, \ 1 < \delta < \frac{1}{\alpha} \) and \(d \geq 2\) then we have \(d - \alpha(\delta - 1) > 1\). So (2.33) is true from (2.35).
Now from (2.32) for almost all \( \omega \) we have
\[
\lim_{L \to \infty} \frac{1}{\beta_L} N_{\omega, -, L}^L(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{\omega, -, L}^L(E)) = \frac{1}{(\delta - 1)e^{(\delta - 1)}} \quad (\text{using (2.13)})
\]
\[
= \frac{1}{(\delta - 1)(-2d - E)^{(\delta - 1)}} (E = -2d - \epsilon).
\]
Now exactly simillar way we can get, for almost all \( \omega \)
\[
\lim_{L \to \infty} \frac{1}{\beta_L} N_{\omega, +, L}^L(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(N_{\omega, +, L}^L(E)) = \frac{1}{(\delta - 1)(4d + \epsilon)^{(\delta - 1)}} \quad (\text{using (2.16)})
\]
\[
= \frac{1}{(\delta - 1)(2d - E)^{(\delta - 1)}} (E = -2d - \epsilon).
\]
Now (2.36), (2.37) together with (2.6) will give,
when \( E < -2d \) for almost all \( \omega \)
\[
\frac{1}{(\delta - 1)(2d - E)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} N_{\omega, L}^L(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} N_{\omega}^L(E) \leq \frac{1}{(\delta - 1)(-2d - E)^{(\delta - 1)}}.
\]
When \( E = 2d + \epsilon > 2d \) then we can deal with \( \tilde{N}_{\omega, L}^\omega(E) \) (as in (2.18)) and similarly as above we can prove that,
For almost all \( \omega \)
\[
\lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_{\omega, +, L}^\omega(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_{\omega, +, L}^\omega(E)) = \frac{1}{(\delta - 1)(E - 2d)^{(\delta - 1)}}
\]
and
\[
\lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_{\omega, -, L}^\omega(E) = \lim_{L \to \infty} \frac{1}{\beta_L} \mathbb{E}^\omega(\tilde{N}_{\omega, -, L}^\omega(E)) = \frac{1}{(\delta - 1)(2d + E)^{(\delta - 1)}}
\]
Now the above and (2.19) will give the following.
For \( E > 2d \) almost all \( \omega \)
\[
\frac{1}{(\delta - 1)(2d + E)^{(\delta - 1)}} \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_{\omega}^L(E) \leq \lim_{L \to \infty} \frac{1}{\beta_L} \tilde{N}_{\omega}^L(E) \leq \frac{1}{(\delta - 1)(E - 2d)^{(\delta - 1)}.}
\]
From (2.38) and (2.39) our theorem is over.

References

[1] Anderson, P.W: Absence of diffusion in certain random lattices, Phys. Rev. 109, 1492-1505, 1958.

[2] Aizenman, Michael; Molchanov, Stanislav: Localization at large disorder and at extreme energies: an elementary derivation, Commun. Math. Phys. 157(2), 245-278, 1993.
[3] Aizenman, Michael; Warzel, Simone: *The Canopy Graph and Level Statistics for Random Operators on Trees*, Mathematical Physics, Analysis and Geometry, 9(4), 291-333, 2006.

[4] Carmona, René; Lacroix, Jean: *Spectral theory of random Schrödinger operators*, Boston, Birkhauser, 1990.

[5] Combes, Jean-Michel; Gérminet, François; Klein, Abel: *Generalized Eigenvalue-Counting Estimates for the Anderson Model*, J Stat Physics 135(2), 201-216, 2009.

[6] Demuth, Michael; Krishna, M: *Determining Spectra in Quantum Theory*, Progress in Mathematical Physics. 44, Birkhäuser, Boston, 2004.

[7] Dolai, Dhriti; Krishna, M: *Level Repulsion for a class of decaying random potentials*, Markov Processes and Related Fields (to appear), arXiv:1305.5619 [math.SP].

[8] Daley, D.J; Vere-Jones: *An Introduction to the Theory of Point Processes II*, General theory and structure, Springer, New York, 2008.

[9] Figotin, Alexander; Gérminet, François; Klein, Abel; Müller, Peter: *Persistence of Anderson localization in Schrödinger operators with decaying random potentials*, Ark. Mat. 45(1), 15-30, 2007.

[10] Gérminet, François; Klopp, Frédéric: *Spectral statistics for the discrete Anderson model in the localized regime*, Spectra of random operators and related topics, 11-24, RIMS Kôkyûroku Bessatsu, B27, Res. Inst. Math. Sci. (RIMS), Kyoto, 2011.

[11] Gordon, A. Ya; Molchanov, S. A; Tsagani, B: *Spectral theory of one-dimensional Schrödinger operators with strongly fluctuating potentials*, Funct. Anal. Appl. 25(3), 236-238, 1991.

[12] Gordon, Y. A; Jakšić, V; Molčanov, S; Simon, B: *Spectral properties of random Schrödinger operators with unbounded potentials*, Comm. Math. Phys. 157(1), 23-50, 1993.

[13] Jakšić, Vojkan; Last, Yoram: *Spectral structure of Anderson type Hamiltonians*, Invent. Math, 141(3), 561-577, 2000.

[14] Kato, Tosio: *Perturbation theory for Linear operators*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[15] Krishna M: *Anderson model with decaying randomness existence of extended states*, Proc. Indian Acad. Sci. Math. Sci. 100, 285-294 1990.

[16] Krishna, M: *Continuity of integrated density of states-independent randomness*, Proc. Ind. Acad. Sci. 117(3), 401-410, 2007.

[17] Killip, Rowan; Nakano, Fumihiko: *Eigenfunction Statistics in the Localized Anderson Model*, Ann. Henri Poincare 8(1), 27-36, 2007.

[18] Kotani, S; Nakano, Fumihiko: *Level statistics of one-dimensional Schrödinger operators with random decaying potential*, Preprint, (2012).

[19] Kirsch, W; Krishna, M; Obermeit, J: *Anderson model with decaying randomness: mobility edge*, Math.Z. 235(3), 421-433, 2000.

[20] Kotani, S; Ushirota, N: *One-dimensional Schrodinger operators with random decaying potentials*, Commun. Math. Phys. 115(2), 247-266, 1988.

[21] Minami, Nariyuki: *Local Fluctuation of the Spectrum of a Multidimensional Anderson Tight Binding Model*, Commun. Math. Phys. 177(3), 709-725, 1996.

[22] Reed, Michael; Simon, Barry: *Method of modern mathematical physics I*, Functional Analysis, Academic Press, 1978.

[23] Simon, Barry: *Trace ideals and their applications*, Mathematical Surveys and Monographs, 120, American Mathematical Society, Providence, RI, 2005. viii+150 pp.

[24] Simon, Barry; Wolff, Tom: *Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians*, Comm. Pure and Appl. Math. 39(1), 75-90, 1986.