TWO-STEP HOMOGENEOUS GEODESICS IN SOME HOMOGENEOUS FINSLER MANIFOLDS

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Abstract. A natural extension of a homogeneous geodesic in homogeneous Riemannian spaces $G/H$, known as a two-step homogeneous geodesic, can be expressed of the form $\gamma(t) = \pi(\exp(tx)\exp(ty))$, where $x$ and $y$ are elements of the Lie algebra of $G$. This paper aims to expand this concept to homogeneous Finsler spaces. We provide certain sufficient conditions for $(\alpha, \beta)$ spaces and decomposable cubic spaces to possess a one-parameter family of invariant Finsler metrics that can be classified as two-step Finsler geodesic orbit spaces. Additionally, we present some illustrative examples of these spaces.

1. Introduction

A homogeneous Riemannian manifold is defined as a connected Riemannian manifold $(M, h)$ where the largest connected group of isometries $G$ acts transitively on $M$. Such a manifold can be represented as the homogeneous space $G/H$, where $o$ is an arbitrary point of $M$ and $H$ is the isotropy group at $o$. There exists an $\text{Ad}(H)$-invariant decomposition $g = m \oplus h$, where $g$ and $h$ are the Lie algebras of $G$ and $H$, respectively, and $m$ is a linear subspace of $g$. The tangent space $T_oM$ can be identified with the subspace $m$ using the natural projection $\pi: G \longrightarrow G/H$.

A geodesic $\gamma(t)$ through the origin $o$ of $M = G/H$ is called a homogeneous geodesic if there exists a nonzero vector $x \in g$ such that $\gamma(t) = \pi(\exp(tx))$ for $t \in \mathbb{R}$. The nonzero vector $x$ is referred to as a geodesic vector. There is a one-to-one correspondence between the set of geodesic vectors and the set of homogeneous geodesics through the origin $o$ (see [5], [14] and [16]). Kowalski and Vanhecke established that a vector $0 \neq x \in g$ is a geodesic vector if and only if $\langle x_m, [x, z]_m \rangle = 0$ for any $z \in g$, where $\langle , \rangle$ is the inner product induced by the Riemannian metric $h$, and the subscript $m$ denotes the projection into $m$ with respect to the decomposition $g = m \oplus h$ (see proposition 2.1 of [15]). In [16], Kowalski and Szenthe showed that every homogeneous Riemannian manifold admits a homogeneous geodesic through any point $o \in M$. This result is generalized to the case of pseudo-Riemannian manifolds by Dusek in [8]. Yan and Deng generalized the result to Randers metrics [22]. Dusek proved the same result for odd-dimensionaential Finsler metrics [9, 10]. Yan and Huang proved the result in general regular Finsler spaces [23]. The result is generalized to a special type of non-regular Finsler metrics (Kropina metric) by the authors (see [11]).

In [3], Arvanitoyeorgos and Panagiotis Souris studied a generalization of the concept of homogeneous geodesics, of the form

$$(1.1) \quad \gamma(t) = \pi(\exp(tx)\exp(ty)), \quad x, y \in g,$$

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which is called two-step homogeneous geodesic. They gave sufficient conditions on homogeneous Riemannian manifolds to admit two-step homogeneous geodesics. Also they studied two-step g.o. spaces.

In this paper, we extend this study to the case of Finsler spaces and investigate the properties of two-step homogeneous geodesics. Section 2 provides some preliminaries on Finsler geometry. Section 3 offers a brief overview of naturally reductive Finsler spaces and geodesic vectors in the context of \((\alpha, \beta)\)-metrics. In section 4, we present our main results, including the sufficient conditions for \((\alpha, \beta)\) spaces and decomposable cubic spaces to be two-step Finsler g.o. spaces. Additionally, we provide examples of such spaces.

## 2. Preliminaries

Let \(M\) denote a smooth manifold with a continuous function \(F : TM \rightarrow [0, +\infty)\) that satisfies the following conditions:

1. \(F\) is differentiable on \(TM \setminus 0\),
2. For any \(x \in M, y \in T_x M, \) and \(\lambda \geq 0, F(x, \lambda y) = \lambda F(x, y),\)
3. For any \((x, y) \in TM \setminus 0\), the Hessian matrix \((g_{ij}(x, y)) = (1/2 \partial^2 F^2(x, y) \partial y^i \partial y^j)\), where \((x^1, \cdots, x^n)\) is a local coordinate system on an open subset \(U\) of \(M\), and \((y^1, \cdots, y^n)\) is the natural coordinate system on \(TU\), is positive definite.

Then the function \(F\) and the pair \((M, F)\) are called a Finsler metric and a Finsler manifold, respectively.

A notable class of Finsler metrics is the family of \((\alpha, \beta)\)-metrics, which are defined by combining a Riemannian metric and a one-form. Let \(h\) be a Riemannian metric on the manifold \(M\), and for any \((x, y) \in TM\), consider \(\alpha(x, y) = \sqrt{h(y, y)}\). Suppose \(\beta\) is a one-form on \(M\), and \(\phi : (-b_0, b_0) \rightarrow \mathbb{R}^+\) is a \(C^\infty\) function that satisfies the condition:

\[
\phi(s) - s \phi'(s) + (b^2 - s^2) \phi''(s) > 0, \quad |s| \leq b < b_0,
\]

where \(||\beta||_\alpha < b_0\). In this case, we can define the function \(F = \alpha \phi(\beta/\alpha)\), which represents a Finsler metric on \(M\). It is worth noting that the one-form \(\beta\) can be replaced by a vector field \(X\) such that \(\beta(x, y) = \langle X(x), y \rangle\).

For any Finsler manifold \((M, F)\), we can define the fundamental tensor \(g\) and the Cartan tensor \(C\) on the pull-back tangent bundle \(\pi^*TM\) over \(TM \setminus 0\) as follows:

\[
g_y(u, v) = g_{ij}(x, y) u^i v^j, \\
g_y(u, v, w) = C_{ijk}(x, y) u^i v^j w^k,
\]

where \(g_{ij}(x, y) = (1/2 F^2) y^i y^j\) and \(C_{ijk}(x, y) = (1/2 F^2) y^i y^j y^k\).

Let \((g^{ij})\) be the inverse matrix of \((g_{ij})\). For any \(y = y^i \frac{\partial}{\partial x^i} \in T_x M \setminus 0\), we define the following quantities:

\[
\gamma^i_{jk} = \frac{1}{2} g^{is} (\frac{\partial g_{si}}{\partial x^k} - \frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{sk}}{\partial x^j}), \\
N^i_j = \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_\ell y^\ell y^s,
\]

where \(C^i_{jk} = g^{is} C_{sjk}\).

The Chern connection is a unique linear connection on the pull-back tangent bundle \(\pi^*TM\).
such that its coefficients on the standard coordinate system are defined by:

\[
\Gamma^i_{jk} = \gamma^i_{jk} - g^{il} (C_{ljk} N^l_k - C_{jk} N^l_i + C_{kls} N^l_j).
\]

Suppose that \( V \) and \( W \) are two vector fields defined along a smooth curve \( \sigma : [0, r] \rightarrow M \). If \( T = T(t) = \dot{\sigma}(t) \) denotes the velocity field of the curve \( \sigma \), then we define \( DT \) with reference vector \( W \) by the following equation:

\[
DT = \left( \frac{dV^i}{dt} + V^j T^k (\Gamma^i_{jk})(\sigma, W) \right) \frac{\partial}{\partial x^i} |_{\sigma(t)}.
\]

A curve \( \sigma \) is called a geodesic (Finslerian geodesic) if \( DT \left( \frac{T}{F} \right) = 0 \). We recall that a Finsler metric \( F \) is called of Berwald type (or Berwaldian) if its Chern connection coefficients \( \Gamma^i_{jk} \), in the standard coordinate system, are functions of \( x \) only. It is also called of Douglas type if it is projectively equivalent to a Riemannian metric on \( M \).

3. Naturally reductive Finsler spaces and geodesic vectors

In this brief section, we study the concepts of geodesic vectors and naturally reductive spaces within the context of homogeneous \((\alpha, \beta)\) spaces. More precisely, we consider a homogeneous Finsler space \((M = G/H, F)\) where the Finsler metric \( F \) is defined by an invariant Riemannian metric \( h \) and a vector field \( X \). In a previous work by the second author and Parhizkar [19], they established that, under certain conditions, any geodesic vector of \((M, F)\) is also a geodesic vector of \((M, h)\) and vice versa. In the following proposition, we demonstrate that one of these conditions is not essential.

**Proposition 3.1.** Let \((M = G/H, F)\) be a homogeneous Finsler space with a reductive decomposition \( g = h \oplus m \), where \( F \) is an invariant \((\alpha, \beta)\)-metric characterized by an invariant Riemannian metric \( h \) and an invariant vector field \( X \). Assume that \( 0 \neq \gamma \in m \) and that \( X \) is orthogonal to \([y, m]_m\). Then, \( y \) is a geodesic vector of \((M, F)\) if and only if it is a geodesic vector of \((M, h)\).

**Proof.** In the proof of Theorem 2.3 presented in [19], it is demonstrated that

\[
g_{\gamma m}(y_m, [y, z]_m) = h(y_m, [y, z]_m) \left( \phi^2(r_m) - \phi(r_m) \phi'(r_m) r_m \right),
\]

where \( r_m = \frac{h(X, y_m)}{\sqrt{h(y_m, y_m)}} \). Let us introduce the function \( f(s) = \phi(s) - s \phi'(s) \), where \( \phi \) is the function used in the definition of the \((\alpha, \beta)\)-metric \( F \). Elementary calculus, leveraging the equation (2.1), confirms that \( f \) is a positive function. Now, the fact that the function \( \phi \) is positive implies that \( \phi^2(r_m) - \phi(r_m) \phi'(r_m) r_m > 0 \), which concludes the proof.

The proposition above demonstrates that Theorem 2.3 from [19] remains valid even without the condition \( \phi''(r_m) \leq 0 \). A similar result can be established for invariant \((\alpha, \beta)\)-metrics of Douglas type as follows.

**Proposition 3.2.** Let \((M = G/H, F)\) be a homogeneous Finsler space, where \( F \) is an invariant \((\alpha, \beta)\)-metric of Douglas type. Then, there exists an invariant Riemannian metric \( h \) on \( M \) such that \((M, F)\) and \((M, h)\) share the same geodesic vectors.
Proof. According to Theorem 1.1 in [18], any \((\alpha, \beta)\)-metric of Douglas type is either of Berwald type or Randers type. If \(F\) is of Berwald type, then the Finsler metric \(F\) and the Riemannian metric corresponding to \(\alpha\) have identical geodesics. On the other hand, if \(F\) is a Douglas metric of Randers type, then there exists an invariant Riemannian metric \(h\) and an invariant vector field \(X\) orthogonal to \([m, m]_m\) such that \(F(x, y) = \sqrt{h(y, y)} + h(X, y)\). Consequently, the previous proposition completes the proof. □

Remark 3.3. Hence, in both cases presented in the above propositions, all the results of geodesic vectors in the Riemannian case extend automatically to the Finsler case.

Now, let us proceed to the study of naturally reductive Finsler spaces. We begin by defining naturally reductive Riemannian spaces.

**Definition 3.4.** A homogeneous Riemannian manifold \((M = G/H, h)\) is called naturally reductive if there exists an \(\text{Ad}(H)\)-invariant decomposition \(\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\) such that

\[
\langle [x, y]_m, z \rangle + \langle y, [x, z]_m \rangle = 0 \quad \forall, x, y, z \in \mathfrak{m},
\]

where \(\langle , \rangle\) denotes the inner product on \(\mathfrak{m}\) induced by \(h\), and \([,]_m\) represents the projection to \(\mathfrak{m}\) with respect to the aforementioned decomposition.

If all geodesics of a homogeneous Riemannian manifold, under the influence of the largest connected group of isometries, are homogeneous geodesics, then the homogeneous Riemannian manifold is referred to as a geodesic orbit space (g.o. space). It has been shown that any naturally reductive homogeneous Riemannian space is a g.o. space (see [14]). However, there exist g.o. spaces that are not naturally reductive in any manner (see [13]). In [7], the aforementioned definition of naturally reductive homogeneous Riemannian spaces is extended to Finsler spaces as follows.

**Definition 3.5.** Let \(G/H\) be a homogeneous manifold equipped with an invariant Finsler metric \(F\). It is called a naturally reductive Finsler space if there exists an invariant Riemannian metric \(\tilde{h}\) on \(G/H\) such that \((G/H, \tilde{h})\) is naturally reductive, and the Levi-Civita connection of \(\tilde{h}\) coincides with the Chern connection of \(F\).

Based on this definition, a naturally reductive Finsler space must be Berwaldian.

**Remark 3.6.** We can see that if \((M = G/H, F)\) is a naturally reductive \((\alpha, \beta)\) space defined by a Riemannian metric \(h\) and a vector field \(X\), then the homogeneous Riemannian space \((M, h)\) is naturally reductive. If \((M = G/H, F)\) is naturally reductive then, there exists a Riemannian metric \(\tilde{h}\) such that \((M, \tilde{h})\) is a naturally reductive homogeneous Riemannian space, and the Levi-Civita connection of \(\tilde{h}\) and the Chern connection of \(F\) coincide. On the other hand, \((M, F)\) is of Berwald type so the Levi-Civita connection of \(h\) and the Chern connection of \(F\) also coincide. In fact, \((M, \tilde{h})\) and \((M, h)\) have the same Levi-Civita connection. It is well known that a Levi-Civita connection determines the Riemannian metric up to a constant conformal factor (see [20]). Therefore, there exists a positive real number \(\mu\) such that \(h = \mu \tilde{h}\) and so \((M, h)\) is also naturally reductive.
4. Two-step homogeneous g.o. Finsler spaces

As previously mentioned, the concept of a two-step homogeneous geodesic was introduced by Arvanitoyeorgos and Panagiotis Souris in [3]. In this section, we aim to explore this notion in the context of homogeneous Finsler spaces. Consider a homogeneous Finsler manifold \((G/H, F)\), where \(\pi : G \rightarrow G/H\) denotes the projection map, and let \(o = \pi(e)\) represent the origin of \(G/H\).

**Definition 4.1.** A geodesic \(\gamma\) on \(G/H\) is said to be two-step homogeneous if there exist non-zero vectors \(x, y \in \mathfrak{g}\) such that

\[
\gamma(t) = \pi(\exp tx \exp ty) \quad \forall t \in \mathbb{R}.
\]

**Definition 4.2.** A homogeneous Finsler space \((G/H, F)\), is called a two-step homogeneous g.o. space (two-step homogeneous geodesic orbit space) if all geodesics \(\gamma\) with \(\gamma(0) = o\) are two-step homogeneous.

In the following, we investigate the existence of two-step homogeneous g.o. Finsler spaces and provide some examples of such spaces.

**Theorem 4.3.** Assume that \((M = G/H, F)\) is a naturally reductive Finsler space, where \(F\) represents an invariant \((\alpha, \beta)\)-metric derived from an invariant Riemannian metric \(h\) and an invariant vector field \(X\). Let \(\langle , \rangle\) denote the corresponding inner product on \(m = T_0(G/H)\) defined by \(h\). If \(m = m_1 \oplus m_2\) represents an Ad\((H)\)-invariant orthogonal decomposition of \(m\) and \([m_1, m_2] \subseteq m_1\), with \(X\) belonging to \(m_2\), then \(M\) possesses a one-parameter family of invariant Finsler metrics \(F_\lambda\), where \(\lambda \in \mathbb{R}^+\), such that \((M, F_\lambda)\) is a two-step Finsler g.o. space. Each metric \(F_\lambda\) takes the following form:

- For \(0 < \lambda < 1\), \(F_\lambda = \alpha_\lambda \phi\left(\frac{\lambda}{\alpha_\lambda}\right)\).
- For \(\lambda > 1\), \(F_\lambda = \alpha_\lambda \phi\left(\frac{\lambda}{\alpha_\lambda}\right)\).

Here \(\alpha_\lambda\) is the norm of the Riemannian metric on \(M\) which corresponds to the inner product \(\langle , \rangle_\lambda = \langle , \rangle|_{m_1} \oplus \lambda \langle , \rangle|_{m_2}\) and \(\beta_\lambda\) is the one-form that corresponds to the vector field \(X_\lambda = \frac{1}{\sqrt{\lambda}}X\).

**Proof.** Since the homogeneous Finsler space \((M, F)\) is naturally reductive, we can apply Remark 3.6 to conclude that the homogeneous Riemannian space \((M, h)\) is also naturally reductive. By Corollary 2.4 in [3], it follows that \((M, h_\lambda)\) is a two-step geodesic orbit (g.o.) space, where \(h_\lambda\) is the Riemannian metric corresponding to the inner product \(\langle , \rangle_\lambda\) on \(T_0(G/H)\). Now, let us consider the Finsler metric \(F_\lambda\) as mentioned above. We will prove that \(F_\lambda\) is of Berwald type in each case. If we have \(y = y_1 + y_2\) and \(z = z_1 + z_2\), where \(y_1, z_1 \in m_1\) and \(y_2, z_2 \in m_2\), then we can express the inner product \(\langle y, z \rangle_\lambda\) as follows:

\[
\langle y, z \rangle_\lambda = \langle y_1, z_1 \rangle + \lambda \langle y_2, z_2 \rangle = (1 - \lambda)\langle y_1, z_1 \rangle + \lambda \langle y, z \rangle = \langle y, z \rangle + (\lambda - 1)\langle y_2, z_2 \rangle.
\]

On the other hand, we know that \(F\) is an \((\alpha, \beta)\)-metric of Berwald type. Referring to Proposition 3.1 in [4], we have the following relationships:

\[
\langle [y, X]_m, z \rangle + \langle [z, X]_m, y \rangle = 0, \quad \langle [y, z]_m, X \rangle = 0 \quad \forall y, z \in m.
\]

If \(X \in m_2\), then we can evaluate \(\langle [y, z]_m, X \rangle_\lambda\) as follows:

\[
\langle [y, z]_m, X \rangle_\lambda = (1 - \lambda)\langle [y, z]_m, 0 \rangle + \lambda \langle [y, z]_m, X \rangle = 0.
\]
In addition, we can compute \( \langle [y, X]_m, z \rangle + \langle [z, X]_m, y \rangle \) as follows:

\[
\langle [y, X]_m, z \rangle + \langle [z, X]_m, y \rangle = \langle [y, X]_m, z \rangle + (\lambda - 1)(\langle [y, X]_m, z \rangle)
+ \langle [z, X]_m, y \rangle + (\lambda - 1)(\langle [z, X]_m, y \rangle)
= (\lambda - 1)(\langle [y_1, X]_m, z_2 \rangle + \langle [y_2, X]_m, z_2 \rangle)
+ \langle [z_1, X]_m, y_2 \rangle + \langle [z_2, X]_m, y_2 \rangle)
= (\lambda - 1)(\langle [y_1, X]_m, z_2 \rangle + \langle [z_1, X]_m, y_2 \rangle).
\]

Since \([m_1, m_2] \subseteq m_1\), the aforementioned relation is equivalent to zero. Furthermore,

\[
\langle X, X \rangle = \lambda \langle X, X \rangle < \lambda b_0.
\]

Hence, in each scenario, \( F_\lambda \) is an \((\alpha, \beta)\)-metric of Berwald type, thereby implying that \((M, F_\lambda)\)
and \((M, h_\lambda)\) possess identical geodesics. \(\square\)

Next, we present a similar result for homogeneous decomposable cubic metric spaces. A
metric \( F \) is referred to as an \( m \)-th root Finsler metric if \( F = \sqrt[3]{T} \) and \( T = h_{i_1 \cdots i_m}(x)y^{i_1} \cdots y^{i_m} \),
where \( h_{i_1 \cdots i_m} \), with all its indices, is symmetric. The cubic metrics are specifically those
metrics that correspond to the third root. If \( T \) decomposes as \( T = h.b \), where \( h = h_{ij}y^i y^j \) is a
Riemannian metric and \( b = b_i(x)y^i \) is a one-form such that \( \|b\|^2 = h^{ij}b_ib_j = 1 \), then the cubic
metric \( F = \sqrt[3]{T} \) is called decomposable.

**Lemma 4.4.** Consider an invariant decomposable cubic space \((M = G/H, F = \sqrt[3]{T})\) where \( T = h.b \), with \( h = h_{ij}y^i y^j \) being an invariant Riemannian metric and \( b = b_i(x)y^i \) being an
invariant one-form. Let \( X \) denote the invariant vector field associated with \( b \). The metric \( F \)
is of Berwald type if and only if

\[
\langle [y, X]_m, z \rangle + \langle [z, X]_m, y \rangle = 0, \quad \langle [y, z]_m, X \rangle = 0 \quad \forall y, z \in m.
\]

**Proof.** The validity of \( F \) being of Berwald type is established by Theorem 9 in [6], which states
that \( F \) is of Berwald type if and only if \( b \) is parallel with respect to \( h \). To prove this, we can
employ the same argument as in the proof of Theorem 3.1 in [1]. \(\square\)

**Lemma 4.5.** Assume that \((M, h)\) and \((M, \bar{h})\) are two Riemannian manifolds sharing the same
godesics. Then, the Levi-Civita connections of \((M, h)\) and \((M, \bar{h})\) coincide.

**Proof.** Let \( \nabla \) and \( \bar{\nabla} \) denote the Levi-Civita connections of \((M, h)\) and \((M, \bar{h})\), respectively.
Both \( \nabla \) and \( \bar{\nabla} \) are symmetric. Consequently, Proposition 4.10 in [20] demonstrates that
\( \nabla = \bar{\nabla} \). \(\square\)

**Theorem 4.6.** Consider a naturally reductive decomposable cubic space \((M = G/H, F = \sqrt[3]{T})\)
where \( T = h.b \), with \( h = h_{ij}y^i y^j \) being an invariant Riemannian metric and \( b = b_i(x)y^i \) being an
invariant one-form satisfying \( \|b\|^2 = h^{ij}b_ib_j = 1 \). Let \( X \) denote the vector field corresponding
to \( b \), and let \( \langle , \rangle \) represent the corresponding inner product on \( m = T_o(G/H) \) concerning \( h \). Suppose that \( m = m_1 \oplus m_2 \) is an \( Ad(H) \)-invariant orthogonal decomposition of \( m \), such that
\([m_1, m_2] \subseteq m_1 \) and \( X \) belongs to \( m_2 \). Then, \( M \) admits a one-parameter family of invariant
Finsler metrics \( F_\lambda = \sqrt[3]{T_\lambda} \), where \( \lambda \in \mathbb{R}^+ \), such that \((M, F_\lambda)\) is a two-step Finsler g.o. space.
Here, \( T_\lambda = h_\lambda b_\lambda \), \( h_\lambda \) represents the Riemannian metric on \( M \) corresponding to the inner product

\[
\langle \cdot, \cdot \rangle_\lambda = \langle \cdot \rangle_{m_1} \oplus \lambda \langle \cdot \rangle_{m_2},
\]
and \( b_\lambda \) is a one-form associated with the vector field \( X_\lambda = \frac{1}{\sqrt{\langle X, X \rangle_\lambda}} X \).

**Proof.** Any naturally reductive Finsler space is of Berwald type. Therefore, \( F \) is a Berwaldian decomposable cubic metric. Remark 10 in [6] states that the geodesics of \((M, F)\) and \((M, h)\) coincide. Moreover, \( F \) is naturally reductive, meaning that there exists a naturally reductive Riemannian metric \( \tilde{h} \) where the Chern connection of \( F \) and the Levi-Civita connection of \( \tilde{h} \) coincide. Thus, Lemma 4.5 implies that \( h \) and \( \tilde{h} \) have the same connection. Remark 3.6 shows that \((M, h)\) is naturally reductive. By applying Corollary 2.4 in [3], \((M, h_\lambda)\) is a two-step g.o. space. Since \( F \) is of Berwald type, Lemma 4.4 and the proof of Theorem 4.3 imply that \( F_\lambda \) is a Berwald metric. Therefore, Theorem 9 in [6] concludes the proof. \[ \square \]

**Example 4.7.** Consider a Lie group \((G, h)\) equipped with a bi-invariant Riemannian metric. Let \( K \) be a connected subgroup of \( G \), and let \( g \) and \( \mathfrak{k} \) represent the Lie algebras of \( G \) and \( K \), respectively. The bi-invariant metric \( h \) induces an \( Ad \)-invariant positive definite inner product \( \langle , \rangle \) on \( g \). Furthermore, it allows for an orthogonal decomposition of \( g \) into \( \mathfrak{k} \) and \( m \). This decomposition is \( Ad(K) \)-invariant. For any \( x_t, y_t \in \mathfrak{k} \) and \( x_m \in m \), it follows that \( \langle [x_t, x_m], y_t \rangle = -\langle x_m, [x_t, y_t] \rangle = -\langle x_m, z_t \rangle = 0 \). Therefore, \([x_t, x_m] \in m\), and since \( K \) is connected, \( Ad(K)m \subseteq m \). Let \( F \) be an \((\alpha, \beta)\)-metric on \( G \) defined by \( h \) and a left invariant vector field \( X \in \mathfrak{k} \cap Z(g) \). The Riemannian metric \( h \) and the vector field \( X \) are both right invariant, making the Finsler metric \( F \) bi-invariant. Based on Theorem 5 in [17], we can conclude that \((G, F)\) is a naturally reductive Finsler space. Now, by applying Theorem 4.3, we consider \( \langle , \rangle_\lambda \):

\[
\langle , \rangle_\lambda = \langle , \rangle_{\mathfrak{m} \oplus \lambda(,)_{\mathfrak{k}}}.
\]

Hence, we can assert that \((G, F_\lambda)\) is a two-step homogeneous g.o. space.

**Example 4.8.** Consider a Lie group \( G \) with a bi-invariant Riemannian metric \( h \) and let \( g \) denote its Lie algebra. It is known that \( g = Z(g) \oplus g' \), which represents an orthogonal decomposition of \( g \) (see [2]). Let us introduce an \((\alpha, \beta)\)-metric \( F \), induced by the Riemannian metric \( h \), and a vector field \( X \in Z(g) \) on \( G \). Since both \( h \) and \( X \) are bi-invariant, it follows that the \((\alpha, \beta)\)-metric \( F \) is also bi-invariant. According to Theorem 5 in [17], \( F \) can be classified as a naturally reductive Finsler metric. Now, let \( h_\lambda \) be a Riemannian metric on \( G \) that corresponds to the inner product \( \langle , \rangle_\lambda = \langle , \rangle_{g' \oplus \lambda(,)_{Z(g)}} \) on \( g \). The Theorem 4.3 demonstrates that \( G \), equipped with the \((\alpha, \beta)\)-metric \( F_\lambda \) induced by the Riemannian metric \( h_\lambda \) and the vector \( X_\lambda \in Z(g) \), is a Berwald space. Consequently, \( F_\lambda \) and \( h_\lambda \) share the same geodesics. Furthermore, according to Theorem 2.3 of [3], \((G, F_\lambda)\) can be classified as a two-step g.o. space.

The next theorem can be obtained directly from Proposition 5.1 in [3] and Theorem 4.3.

**Theorem 4.9.** Consider a Lie group \( G \) equipped with a bi-invariant Riemannian metric \( h \). Let \( H \subset K \) be two closed connected subgroups of \( G \). Denote by \( \langle , \rangle \) the \( Ad \)-invariant inner product on the Lie algebra \( g \), which corresponds to the bi-invariant Riemannian metric \( h \). We define \( T_o(G/H) = \mathfrak{m}, T_o(G/K) = \mathfrak{m}_1, \) and \( T_o(K/H) = \mathfrak{m}_2 \), where \( \mathfrak{m}, \mathfrak{m}_1, \) and \( \mathfrak{m}_2 \) are subspaces of \( g \) and \( \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \). Let \( F_\lambda \) be the \( G \)-invariant \((\alpha, \beta)\)-metric on \( G/H \), corresponding to the \( Ad(H) \)-invariant positive definite inner product

\[
\langle , \rangle_\lambda = \langle , \rangle_{\mathfrak{m}_1} + \lambda \langle , \rangle_{\mathfrak{m}_2} \quad \lambda > 0
\]
on \( m \), and let \( X \in m_2 \) be a vector field orthogonal to \([m,m]_m\) with respect to \( \langle ., . \rangle \). Then, \((G/H,F)\) is a two-step g.o. space.

**Proof.** Let \( g, \mathfrak{t}, \) and \( h \) denote the Lie algebras of the Lie groups \( G, K, \) and \( H \), respectively. Given the \( \text{Ad} \)-invariant inner product \( \langle , \rangle \), we can identify orthogonal decompositions of \( g, \mathfrak{t}, \) and \( h \) as follows: \( g = \mathfrak{t} \oplus m_1 \) and \( h = \mathfrak{h} \oplus m_2 \). Notice that \( m = m_1 \oplus m_2 \) serves as an orthogonal decomposition with respect to \( \langle , \rangle |_m \), where \([m_1,m_2] \subseteq m_2\). Let \( h_\lambda \) be the Riemannian metric on \( G/H \) derived from \( \langle , \rangle \). By the proof of Proposition 5.1 in [3], we deduce that \((G/H,h_\lambda)\) qualifies as a two-step g.o. space. Given that \( \langle , \rangle \) is an \( \text{Ad} \)-invariant inner product, if \( X \in [m,m]_m \), it follows that

\[
\langle [y,X]_m,z \rangle + \langle [z,X]_m,y \rangle = 0, \quad \langle [y,z]_m,X \rangle = 0 \quad \forall y, z \in m.
\]

Furthermore, if we additionally consider \( X \in m_2 \), employing a similar calculation as in the proof of Theorem 4.3, we ascertain that

\[
\langle [y,X]_m,z \rangle_\lambda + \langle [z,X]_m,y \rangle_\lambda = 0, \quad \langle [y,z]_m,X \rangle_\lambda = 0 \quad \forall y, z \in m.
\]

Assume that \( F_\lambda \) denotes the \((\alpha, \beta)\)-metric associated with the inner product \( \langle , \rangle_\lambda \) and the vector \( X \in m_2 \cap [m,m]^{\perp} \). Consequently, \( F_\lambda \) qualifies as a Berwald type metric, and both the Finsler metric \( F_\lambda \) and the Riemannian metric \( h_\lambda \) share the same geodesics. \( \square \)

Finally, we present a family of two-step geodesic orbit spaces through an extension of the concept of navigation data in Randers metrics [12]. Let \((M,F)\) be a Finsler space and \( W \) be a vector field such that, for all \( x \in M \), \( F(x,W(x)) < 1 \). A Finsler metric \( \tilde{F} \) is said to possess the navigation data \((F,W)\) if and only if

\[
\tilde{F}(x,y) = 1 \iff F(x,y - W(x)) = 1.
\]

**Theorem 4.10.** Let \((M = G/H,F)\) be a two-step geodesic orbit space and \( W \) be a \( G \)-invariant Killing vector field on \( M \). Consider the Finsler space \((M, \tilde{F})\) equipped with the navigation data \((W,F)\). Then every geodesic of \((M, \tilde{F})\) is a two-step homogeneous geodesic.

**Proof.** Let \( \phi_t \) denote the flow of the Killing vector field \( W \). Suppose \( G' \) is the group generated by \( \phi_t \) and \( G \). If \( H' \) is the isotropy subgroup of \( G' \) at \( o = eH \), then we have \( H \subset H' \) and \( g' = h' \oplus m \) represents a reductive decomposition of \( G'/H' \). Let \( W_0 \) be the element corresponding to \( \phi_t \) in \( g' \). By virtue of Theorem 6.12 in [21], every geodesic \( \gamma \) of \((M, \tilde{F})\) passing through \( p \) has the following form:

\[
\gamma(t) = \exp(tW_0) \rho(t)
\]

where \( \rho(t) = \exp(tx) \exp(ty).p \) denotes a geodesic of \((M,F)\). Additionally, since \( W \) is \( G \)-invariant, it follows that \( W_0 \) belongs to the center of \( g' \). Consequently,

\[
\gamma(t) = \exp(tW_0) \exp(tx) \exp(ty).p = \exp(t(W_0 + x)) \exp(ty).p
\]

which concludes the proof. \( \square \)

**Corollary 4.11.** Consider \((M = G/H,h)\) as a Riemannian two-step geodesic orbit space, with \( W \) as a \( G \)-invariant Killing vector field on \( M \). If \( \|W\|_h < 1 \), then \((M,F)\), where \( F \) is a Randers metric with navigation data \((h,W)\), also qualifies as a two-step geodesic orbit space.
Corollary 4.12. Let \((M = G/H, h)\) be a Riemannian two-step geodesic orbit space, and \(W\) be a \(G\)-invariant Killing vector field on \(M\). If \(F\) represents a Kropina metric with navigation data \((h, \frac{W}{\|W\|_h})\), then \((M, F)\) is a two-step geodesic orbit space as well.

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