A boundary control problem associated to the Rayleigh-Bénard-Marangoni system

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Abstract

A boundary control problem associated to the stationary Rayleigh-Bénard-Marangoni (RBM) system is studied. Controls for the velocity and the temperature on parts of the boundary are considered, and the existence of optimal solutions is proved. By using the principle of Lagrange multipliers an optimality system is derived. By the way, the existence, uniqueness and regularity of weak solutions for the stationary RBM system in a polyhedral domain of $\mathbb{R}^3$ is analyzed. In particular, we deal with an elliptic regularity problem with mixed (Dirichlet, Neumann and Robin) boundary data on polyhedral domains.

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1 Introduction

Fluid movement by temperature gradients, also called thermal convection, is an important process in nature. Its main applications appear in industry, as for instance, in the growth of semiconductor crystals, as well as, thermal convection is the basis for the interpretation of various natural phenomena such as the movement of the earth’s plates, the solar activity, large scale circulations of the oceans, movement in the atmosphere, among others. A model of particular interest consists of a horizontal layer of a fluid in a container heated uniformly from below, with the bottom surface and the lateral walls rigid and the upper surface open to the atmosphere. Due to heating, the fluid in the bottom surface expands and it becomes lighter than the fluid in the upper surface, so that, by effect of the buoyancy, the liquid is potentially unstable. Due to the instability, the fluid tends to redistribute. However, this natural tendency will be controlled by its own viscosity. On the other hand, the upper surface which is free to the atmosphere experiences changes in its surface tension due to the temperature gradients in the surface. Then, it is expected that the temperature gradient exceeds a critical value, above which the instability can manifest.

The first experiments to demonstrate the beginning of Thermal instability in fluids were developed by Henri Bénard in 1900 (see [4]). In his experiments, Bénard considered a very thin layer of liquid, about $1 \text{ mm}$ of depth, in a metal plate maintained at a constant temperature. The upper surface was usually free and it was in contact with the air, which was at a lower temperature. Bénard experimented with a variety of liquids with different physical characteristics, primarily interested in the effect of viscosity on the convection, using liquids of high viscosity like wax whale melted and paraffin. In all these cases, Bénard found that when the temperature of the plate gradually increased, at a certain moment, the layer lost stability and formed patterns of hexagonal cells, all alike and correctly aligned.

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A first theoretical interpretation of thermal convection was provided by Lord Rayleigh in 1916 (see [31]), whose analysis was inspired by Bénard’s experiment. Rayleigh assumed that the fluid was confined between two horizontal thermally conductive plates and the fluid was being heated from below. Rayleigh considered that the effect of buoyancy is the only one responsible of the beginning of the instability, and theoretically the results coincided with the reported by Bénard, giving the impression that his model was correct. However, it is known now that Rayleigh’s theory is not adequate for explaining the convective mechanism observed by Bénard. In fact, in Bénard’s experiments, the free surface was in contact with the atmosphere which generates a surface tension, and Rayleigh, using a plate in the upper surface, eliminated the surface tension’s effects.

It should be noted that the surface tension is not constant and it may depend on the temperature or contaminants in the surface. This dependence is called capillarity or Marangoni effect [21, 26]. The importance of the Marangoni effect in Bénard’s experiments was established by Block in [5] from an experimental point of view, and by Pearson [30] from a theoretical point of view. Now is recognized that the Marangoni effect is the main cause of instability and convection in the Bénard original experiments.

Considering the above, we consider the physical situation of a horizontal layer of a fluid in a cubic container of height \(d\) (\(x_3\)-coordinate), of length \(L_1\) (\(x_2\)-coordinate) and width \(l_1\) (\(x_1\)-coordinate). The bottom surface of the container and the lateral walls are rigid and the upper surface is open to the atmosphere. In order to describe the system, we use the Oberbeck-Boussinesq approximation [7], which assumes that the thermodynamical coefficients are constant, except in the case of the density in the buoyancy term, which is considered as being \(\rho_0 [1 - \alpha (\theta - \theta_a)]\). Here \(\rho_0\) is the mean density, \(\theta_a\) is the temperature of the environment and \(\alpha\) is the thermal expansion coefficient, which is positive for most liquids. Moreover, we assume that the surface tension is a function of the temperature, and it is approximated by \(\sigma = \sigma_0 - \gamma (\theta - \theta_a)\). Here, \(\sigma_0\) is the surface tension at temperature \(\theta_a\), and \(\gamma\) is the ratio of change of the surface tension with the temperature (\(\gamma\) is positive for the more commonly used liquids). Also the free surface is presumed not to be distorted, that is, the vertical component of the velocity in the free surface always will be zero. Then we consider that the domain in which the fluid is confined, is given by \(\Omega = (0, l_1) \times (0, L_1) \times (0, d)\). However, the analysis developed in this work allows to consider a domain \(\hat{\Omega}\) with more general geometries, specifically, we can consider \(\Omega = \hat{\Omega} \times (0, d)\), being \(\hat{\Omega}\) a bounded domain of \(\mathbb{R}^3\) with Lipschitz boundary \(\partial \hat{\Omega}\).

In stationary regime, the RBM system is given by the following coupling between the Navier-Stokes equations and energy equation:

\[
\rho_0 (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} = \rho_0 [1 - \alpha (\theta - \theta_a)] \vec{g} \quad \text{in} \quad \Omega, \tag{1.1}
\]

\[
\text{div} \mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{1.2}
\]

\[
\rho_0 C_p (\mathbf{u} \cdot \nabla \theta) = K \Delta \theta \quad \text{in} \quad \Omega, \tag{1.3}
\]

where the unknowns are \(\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), u_3(\mathbf{x})) \in \mathbb{R}^3\), \(\theta(\mathbf{x}) \in \mathbb{R}\) and \(p(\mathbf{x}) \in \mathbb{R}\), which represent the velocity field, the temperature and the hydrostatic pressure of the fluid, respectively, at a point \(\mathbf{x} \in \Omega\). The constant \(C_p\) is the heat capacity per unit mass of the fluid, \(\mu\) its viscosity and \(K\) its thermal conductivity.

In order to express the system in adimensional form, we make the following changes of variables:

\[
x'_1 = \frac{x_1}{d}, \quad x'_2 = \frac{x_2}{d}, \quad x'_3 = \frac{x_3}{d}, \quad u'_1 = \frac{du_1}{\kappa}, \quad u'_2 = \frac{du_2}{\kappa}, \quad u'_3 = \frac{du_3}{\kappa},
\]

\[
\theta' = \frac{\theta - \theta_a}{\theta_a}, \quad p' = \frac{d^2 p}{\rho_0 \kappa}, \quad t' = \frac{\kappa t}{d^2},
\]

where \(\theta_a\) is the temperature at the bottom plate, \(\theta_u = \theta_c - \theta_a\), \(\kappa = \frac{K}{\rho_0 C_p}\) and \(\nu = \frac{\mu}{\rho_0}\).
Thus, removing the “primes”, from (1.1)-(1.3) we get

\[(u \cdot \nabla) u = Pr [(b + R \theta) \hat{e}_3 - \nabla p + \Delta u] \text{ in } \Omega,\]

\[(u \cdot \nabla) \theta = \Delta \theta \text{ in } \Omega,\]

\[\text{div } u = 0 \text{ in } \Omega,\]

with \(\Omega = (0, l) \times (0, L) \times (0, 1),\) where \(l = \frac{l_1}{d}\) and \(L = \frac{L_1}{d} \). Moreover, \(Pr = \frac{\nu}{\kappa}, \ R = \frac{|g| d^3}{\kappa \nu} \) and \(b = -\frac{|g| d^3}{\kappa \nu}.\) The number \(R\) is known as the Rayleigh number and it measures the effect of buoyancy; \(Pr\) is known as the Prandtl number and represents the relationship between the speed of diffusion of momentum and the rate of diffusion of heat in the fluid; the field \(\hat{g}\) is the acceleration due to gravity.

Let us denote by \(\partial \Omega\) the boundary of \(\Omega\) and let \(\Gamma_1 := \partial \Omega \cap \{x_3 = 1\}\) and \(\Gamma_0 := \partial \Omega \setminus \Gamma_1.\) Then the following boundary conditions are imposed:

\[u_i \big|_{\Gamma_0} = 0, \ i = 1, 2, \quad u_3 \big|_{\partial \Omega} = 0, \quad (1.7)\]

\[\left(\frac{\partial u_i}{\partial n} + M \frac{\partial \theta}{\partial x_i}\right) \big|_{\Gamma_1} = 0, \quad i = 1, 2, \quad (1.8)\]

\[\frac{\partial \theta}{\partial n} \big|_{\Gamma_0 \setminus \{x_3 = 0\}} = 0, \quad \left(\frac{\partial \theta}{\partial n} + B \theta\right) \big|_{\Gamma_1} = 0, \quad \theta \big|_{\{x_3 = 0\}} = \theta_c, \quad (1.9)\]

where \(n = (n_1, n_2, n_3)\) is the normal vector pointing outward, \(M = \frac{\kappa \theta_d d^2 \rho}{\nu \kappa}\) and \(h\) is the heat exchange coefficient of the surface with the atmosphere.

The boundary conditions for the velocity in (1.7) are no slip conditions on the rigid and free surface. The condition (1.8) takes into account the Marangoni effect, which represents the mass transfer at an interface between two fluids due to a surface tension gradient. Conditions (1.9) say that on the lateral surfaces there is not heat flow (adiabatics), on the free surface is allowed the heat flow, and the bottom surface is maintained at temperature \(\theta_c\) (isothermal).

From the point of view of the existence of solution of RBM problem, recently in [29] was discussed a bifurcation problem in which, considering either the Rayleigh number or the Prandtl number as bifurcation parameters, and using the local bifurcation theory Crandall-Rabinowitz [8], was showed the existence of stationary solutions to the problem (1.4) - (1.9), which bifurcate from a basic state of heat conduction. For basic state, we refer to the exact solution of the problem (1.4) -(1.9), which is given by

\[u_b = 0, \quad \theta_b = \theta_c - \frac{\theta_c B}{1 + B} x_3 \quad \text{and} \quad p_b = p_1 x_3 + p_2 x_3^2. \quad (1.10)\]

Previously to [29], in [9, 16, 17, 20] were obtained numerical results on the existence of solutions that bifurcate of the basic stationary states, instability and patter formation problems and validation of initial and boundary conditions. However, from a theoretical point of view, no more results are available in the literature. The main difficulty of RBM problem (1.4)-(1.9), beyond the coupling between Navier-Stokes equations and heat equations, is the mixed nature of the boundary conditions (1.9) and the geometry of the domain. Indeed, an open problem related to this model consists for finding strong stationary solutions (in Sobolev spaces \(H^2\), see [27, 28]), and also, the existence of strong local solutions (or global for small data) of the associated nonstationary model. The difficulty of these problems is in the necessity to attack problems of elliptic regularity in Sobolev spaces \(W^{k,p}\) for the Laplace and Stokes equations, in polyhedral domains, with mixed and nonhomogeneous boundary conditions. Elliptic regularly results are few and depend deeply on the geometry of the domain, see [10, 11, 13] for elliptic results associated to the Laplace
equation, and [10] [11] [13] [27] [28] for the Stokes equation; see also [15] [32] for related problems associated to the Hydrostatic Stokes equations and Boussinesq system. Moreover, contrary to what happens with the Navier-Stokes stationary equations, from the point of view of optimal control theory, as far as we known, results on boundary control problems in which the cost functional is subject to state equations governed by RBM system are not known. Some optimal control results associated with the Boussinesq equations, which also involve the velocity and the temperature, are known, see for example [11] [2] [8] [22] [23] [24] [25]; however, they differ substantially in the type of boundary conditions.

In this paper we analyze some extremal problems for which the linear velocity and the temperature of the fluid are controlled by boundary data along portions of the boundary; the objective functional is given by a sum of functionals which measure, in $L^p$ norm, the difference between the flow velocity (respectively, the temperature) and a given prescribed velocity (respectively, a prescribed temperature). The fluid motion is constrained to satisfy the stationary system of RBM. The exact mathematical formulation will be given in Section 2. We will prove the solvability of the optimal control problem and state the first-order optimality conditions by using the Lagrange multiplier method; we derive an optimality system in a weak and strong formulation. By the way, we need to prove the existence of weak solutions for RBM system with nonhomogeneous boundary data, as well as, we need to analyze the uniqueness and the regularity of weak solutions. It is worthwhile to remark that the proof of existence and regularity of weak solution for RBM system is not a simple generalization of the similar ones to deal with Navier-Stokes or related models in fluid mechanics [33]. The $H^2$-regularity for temperature is required in order to consider tangential derivatives of the temperature at the boundary in the trace sense (cf. [13], [19] and Lemma 3.2 below). In order to analyze the regularity problem carefully, we adapt regularity results for the Laplace equation with Dirichlet-Neumann boundary homogeneous conditions in corner domains of [10] [11] [13], and some ideas of [18] for to treat the Robin conditions and the Neumann nonhomogeneous conditions.

The outline of this paper is as follows: In Section 2 we give a precise definition of the optimal control problem to be studied and, in Section 3, we prove the existence of weak solutions, as well as, the uniqueness and a result of regularity. In Section 4 we prove the existence of optimal solutions. In section 5 we obtain the first-order optimality conditions, and by using the Lagrange multipliers method we derive an optimality system.

2 Statement of the boundary control problem

Throughout this paper we use the Sobolev space $H^m(\Omega)$, and $L^p(\Omega)$, $1 \leq p \leq \infty$, with the usual notations for norms $\| \cdot \|_{H^m}$ and $\| \cdot \|_{L^p}$ respectively. If $H$ is a Hilbert space we denote its inner product by $(\cdot, \cdot)_H$; in particular, the $L^2(\Omega)$-inner product will be represented by $(\cdot, \cdot)$. If $X$ is a general Banach space, its topological dual will be denoted by $X'$ and the duality product by $(\cdot, \cdot)_{X' \times X}$. Corresponding Sobolev spaces of vector valued functions will be denoted by $H^1(\Omega)$, $H^2(\Omega)$, $L^2(\Omega)$, and so on. We also will use the following Banach spaces

\[
X := \{ u \in H^1(\Omega) : \text{div } u = 0, u_3 = 0 \text{ on } \Gamma_1 \},
\]

\[
X_0 := \{ u \in H^1(\Omega) : \text{div } u = 0, u_3 = 0 \text{ on } \Gamma_0, u_3 = 0 \text{ on } \Gamma_1 \},
\]

\[
Y := \{ S \in H^1(\Omega) : S = 0 \text{ on } \{ x_3 = 0 \} \},
\]

\[
Z := \{ v = (v_1, v_2, v_3) \in L^4(\Omega) : \text{div } v = 0, v = 0 \text{ on } \Gamma_0, v_3 = 0 \text{ on } \Gamma_1 \},
\]

\[
\tilde{X} := \left\{ u = (u_1, u_2, u_3) \in X : \int_{\partial \Omega} u \cdot n \, dS = 0, \text{ and } u \cdot n = 0 \text{ on } \Gamma_0 \setminus \{ x_3 = 0 \} \right\},
\]

\[
\tilde{H}^1(\partial \Omega) := \{ \gamma_{|\partial \Omega} (u) : u \in \tilde{X} \},
\]
where $\gamma|_{\partial \Omega} : \tilde{X} \to \tilde{H}^\frac{1}{2}(\partial \Omega)$ denotes the usual trace operator. Moreover, if $\Gamma$ is an arbitrary subset of $\partial \Omega$, we use the notation $\langle f, g \rangle_\Gamma$ to represent the integral $\int_{\Gamma} f g \, dS$. In the paper, the letter $C$ will denote diverse positive constants which may change from line to line or even within a same line. In order to establish the boundary control problem, we consider the following stationary model related to (1.4)-(1.9) with nonhomogeneous boundary data:

$$\begin{align*}
\langle u \cdot \nabla \rangle u &= Pr \left[ (b + R \theta) \varepsilon \cdot \nabla p + \Delta u \right] \quad \text{in } \Omega, \\
(u \cdot \nabla) \theta &= \Delta \theta \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
\nabla \cdot u |_{\Gamma_0}^x &= g, \\
u_i |_{\Gamma_0}^x &= u^0, \\
u_3 |_{\Gamma_1} &= 0, \\
(\partial u_i / \partial n) + M (\partial \theta / \partial x_i) |_{\Gamma_1} &= 0, \quad i = 1, 2, \\
(\partial \theta / \partial n) |_{\Gamma_0 \{x_3 = 0\}} &= \phi_1, \\
(\partial \theta / \partial n) + B \theta |_{\Gamma_1} &= 0, \quad \theta |_{\{x_3 = 0\}} = \phi_2,
\end{align*}$$

where $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2$ and $\Gamma_1^1 \subseteq \partial \Omega$; the vector $u^0 = (u^0_1, u^0_2, u^0_3) \in H^\frac{1}{2}(\Gamma_0^2)$ is a Dirichlet condition for the velocity $u$ on $\Gamma_0^2 \subseteq \partial \Omega$; the field $g = (g_1, g_2, g_3) \in H^\frac{1}{2}(\Gamma_0^1)$ is given and denotes a control for $u$ on $\Gamma_0^1$; additionally, $\phi_1 \in H^\frac{1}{2}(\Gamma_0 \{x_3 = 0\})$ is a given function which denotes a Neumann control to temperature $\theta$ on $\Gamma_0 \{x_3 = 0\}$, and $\phi_2 \in H^\frac{1}{2}(\{x_3 = 0\})$ is a Dirichlet control to temperature $\theta$ on $\{x_3 = 0\} \subseteq \partial \Omega$.

Suppose that $U_1 \subset \tilde{H}^\frac{1}{2}(\Gamma_0^1)$, $U_2 \subset H^\frac{1}{2}(\Gamma_0 \{x_3 = 0\})$ and $U_3 \subset H^\frac{1}{2}(\{x_3 = 0\})$ are nonempty sets, and $\gamma_i$, $i = 1, \ldots, 6$, are constants. Assume one of the following conditions:

(i) $\gamma_i \geq 0$, for $i = 1, 2, \ldots, 6$, and $U_1$, $U_2$ and $U_3$ are bounded closed convex sets;

(ii) $\gamma_i \geq 0$ for $i = 1, 2, 3, \gamma_j > 0$ for $j = 4, 5, 6$ and $U_1$, $U_2$ and $U_3$ are closed convex sets.

We study the following constrained minimization problem on weak solutions to problem (2.1)-(2.8), for fixed data $u^0 \in \tilde{H}^\frac{1}{2}(\Gamma_0^2)$:

$$\begin{align*}
\text{Find } (u, \theta, g, \phi_1, \phi_2) \in \tilde{X} \times H^1(\Omega) \times U_1 \times U_2 \times U_3 \text{ such that, for } 2 \leq p, l \leq 6, \\
\text{the functional } J(u, \theta, g, \phi_1, \phi_2) &= \frac{2}{\nu} \| \text{rot } u \|_{L^2(\Omega)}^2 + \frac{2 \nu}{\nu} \| u - u_d \|_{L^p(\Omega)}^p + \frac{2}{\nu} \| \theta - \theta_d \|_{L^l(\Omega)}^l + \frac{2 \nu}{\nu} \| g \|_{H^\frac{1}{2}(\Gamma_0^2)}^2 \\
&\quad + \frac{2}{\nu} \| \phi_1 \|_{H^\frac{1}{2}(\Gamma_0 \{x_3 = 0\})}^2 + \frac{2 \nu}{\nu} \| \phi_2 \|_{H^\frac{1}{2}(\{x_3 = 0\})}^2,
\end{align*}$$

is minimized subject to the constraint that $(u, \theta)$ is a weak solution of (2.1) - (2.8). Here $u_d \in L^p(\Omega)$ and $\theta_d \in L^l(\Omega)$ are given.

## 3 Weak solutions, uniqueness and regularity

In this section we analyze the existence of weak solution for system (2.1) - (2.8), as well as, the uniqueness and the regularity of weak solutions. It is worthwhile to remark that the proof of existence and regularity of weak solution for RBM system (2.1) - (2.8) is not a simple generalization of the similar ones to deal with Navier-Stokes or related models in fluid mechanics. We start by giving the respective weak formulation. For that we introduce the bilinear and trilinear forms $a : \tilde{X} \times \tilde{X} \to \mathbb{R}$, $c : \tilde{X} \times \tilde{X} \times \tilde{X} \to \mathbb{R}$, $a_1 : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $b_1 : H^1(\Omega) \times \tilde{X} \to \mathbb{R}$, $y c_1 : \tilde{X} \times H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, for the velocity and temperature:
Lemma 3.1. The following relations hold for $c$ and $c_1$:

\[ c(u, v, z) = -c(u, z, v), \quad c(u, v, v) = 0; \quad \forall u \in Z, \; \forall v, z \in H^1(\Omega), \]  
(3.1)

\[ c_1(u, \theta, W) = -c_1(u, W, \theta), \quad c_1(u, \theta, \theta) = 0; \quad \forall u \in Z, \; \forall \theta, W \in H^1(\Omega). \]  
(3.2)

Proof: Considering that $u \in Z$, i.e., $u = 0$ on $\Gamma_0$, $u_\nu = 0$ on $\Gamma_1$ and $\text{div } u = 0$, and the normal vector $n$ on $\Gamma_1$ is $n = (0, 0, 1)$, we obtain that $u \cdot n = 0$ on $\Gamma^2$. Therefore, the proof follows as in Lemma 2.2 in [12], p. 285.

Lemma 3.2. [29] Assume that $\Omega$ is a bounded domain with boundary $\partial \Omega$ Lipschitz, and $\partial \Omega = \Gamma_0 \cup \Gamma_1$ with $\Gamma_1 \subseteq \{x_3 = 0\}$, $C$ a constant. If $\theta \in H^2(\Omega)$ then

\[ \int_{\Gamma_1} \frac{\partial \theta}{\partial x_1} v_1 + \frac{\partial \theta}{\partial x_2} v_2 dS = \int_{\Omega} \nabla \theta \cdot \frac{\partial \overline{v}}{\partial x_3} d\Omega, \quad \forall v \in X_0. \]

Motivated by the formula of integration by parts and using Lemma 3.2, we obtain the following weak formulation of (2.1)-(2.8).

Definition 3.3. A pair $(u, \theta) \in X \times H^1(\Omega)$ is said a weak solution of (2.1)-(2.8) if

\[ Pr a(u, v) + Pr M b_1(\theta, v) + c(u, u, v) = \langle f(\theta), v \rangle, \quad \forall v \in X_0, \]  
(3.3)

\[ c_1(u, \theta, W) + a_1(\theta, W) + \langle B \theta, W \rangle_{\Gamma_1} = \langle \phi_1, W \rangle_{\Gamma_0 \{x_3 = 0\}}, \quad \forall W \in Y, \]  
(3.4)

\[ u = u_0^g \text{ on } \Gamma_0 \quad \text{and} \quad \theta = \phi_2 \text{ on } \{x_3 = 0\}. \]  
(3.5)

Here $\langle f(\theta), v \rangle = \int_{\Omega} Pr (b + R \theta) c_1 \cdot \overline{v} d\Omega$, $\langle B \theta, W \rangle_{\Gamma_1} = \int_{\Gamma_1} B \theta \cdot W dS$, $\langle \phi_1, W \rangle_{\Gamma_0 \{x_3 = 0\}} = \int_{\Gamma_0 \{x_3 = 0\}} \phi_1 \cdot W dS$, and $u_0^g$ is a function such that $u_0^g = g$ on $\Gamma_0$ and $u_0^g = u^0$ on $\Gamma_2$.

3.1 Existence of weak solutions to the linear Problem

In this subsection we consider the case in which $u_0^g = 0$ on $\Gamma_0$ and $\phi_2 = 0$ on $\{x_3 = 0\}$. Let $\tilde{u} \in Z$ a given function. The weak statement of the linear Problem (3.3) - (3.5) consists in to find a pair $(u, \theta) \in X_0 \times Y$ such that

\[ Pr a(u, v) + Pr M b_1(\theta, v) + c(\tilde{u}, u, v) = \langle f(\theta), v \rangle, \quad \forall v \in X_0, \]  
(3.6)

\[ c_1(u, \theta, W) + a_1(\theta, W) + \langle B \theta, W \rangle_{\Gamma_1} = \langle \phi_1, W \rangle_{\Gamma_0 \{x_3 = 0\}}, \quad \forall W \in Y. \]  
(3.7)

We consider the bilinear continuous mappings $\tilde{a} : X_0 \times X_0 \rightarrow \mathbb{R}$ and $\tilde{a}_1 : Y \times Y \rightarrow \mathbb{R}$, given by

\[ \tilde{a}(u, v) = Pr a(u, v) + c(\tilde{u}, u, v), \quad \forall u, v \in X_0, \]

\[ \tilde{a}_1(\theta, W) = c_1(\tilde{u}, \theta, W) + a_1(\theta, W) + \langle B \theta, W \rangle_{\Gamma_1}, \quad \forall \theta, W \in Y. \]

Consequently we rewrite (3.6) and (3.7) as

\[ \tilde{a}(u, v) = \langle f(\theta), v \rangle - Pr M b_1(\theta, v), \quad \forall v \in X_0, \]  
(3.8)

\[ \tilde{a}_1(\theta, W) = \langle \phi_1, W \rangle_{\Gamma_0 \{x_3 = 0\}}, \quad \forall W \in Y. \]  
(3.9)
We can verify that the operator bilinear \( \hat{a}_1 \) is continuous and coercive on \( Y \), and that \( \phi_1 \in Y' \). Indeed, the continuity of \( \hat{a}_1 \) and \( \phi_1 \) it follows from the Hölder inequality and Sobolev embeddings. Moreover, the coercivity of \( \hat{a}_1 \) follows from (3.2) and the Poincaré inequality. Therefore, by the Lax-Milgram theorem, there is a unique \( \theta \in Y \) which satisfies the equation (3.9), and the following estimate holds:

\[
\|\theta\|_{H^1(\Omega)} \leq C \|\phi_1\|_{H^1_0(\Gamma_0 \setminus \{x_3 = 0\})}.
\]  

(3.10)

Here, \( C \) is a constant independent of \( \bar{u}, \phi_1, f, \) and \( \theta \). Knowing \( \theta \) and inserting it in the equation (3.8), we rewrite the corresponding problem for finding \( u \in X_0 \) such that

\[
\hat{a}(u, v) = \langle \theta, v \rangle, \quad \forall v \in X_0,
\]

(3.11)

where \( \langle \theta, v \rangle = \langle f(\theta), v \rangle - Pr M b_1(\theta, v), \) for all \( v \in X_0 \). By using the Hölder inequality and Sobolev embeddings we can verify that the operator bilinear \( \hat{a} \) is continuous and coercive on \( X_0 \) and \( l_\theta \in X_0' \). Moreover, coercivity of \( \hat{a} \) it follows from (3.1) and the Poincaré inequality. Therefore, by the Lax-Milgram theorem there is a unique \( u \in X_0 \) which satisfies the equation (3.11), and the following estimate holds:

\[
\|u\|_{H^1(\Omega)} \leq C \left( Pr b + Pr R \|\phi_1\|_{H^1_0(\Gamma_0 \setminus \{x_3 = 0\})} + Pr M \|\phi_1\|_{H^1_0(\Gamma_0 \setminus \{x_3 = 0\})} \right).
\]

(3.12)

Here \( C \) is a constant independent of \( \bar{u} \) and \( \phi_1 \). Thus we have proved the following lemma.

**Lemma 3.4.** Let \( \phi_1 \in H^1_0(\Gamma_0 \setminus \{x_3 = 0\}) \). Then there is a unique weak solution \( (u, \theta) \in X_0 \times Y \) of problem (3.6) - (3.7). Moreover, the solution satisfies (3.10) and (3.13).

### 3.2 Existence of weak solutions to the nonlinear Problem

We first consider the case in which \( u_0^0 = 0 \) on \( \Gamma_0 \) and \( \phi_2 = 0 \) on \( \{x_3 = 0\} \). In order to prove existence of a solution \( (u, \theta) \in X_0 \times Y \) of (3.3) - (3.4), we introduce the mapping \( F : Z \to Z \) defined by \( F(k) = z, \) \( k \in Z \), such that \( (z, \theta) \in X_0 \times Y \subset Z \times \bar{Y} \) is the solution of the following linear problem:

\[
Pr a(z, v) + Pr M b_1(\theta, v) + c(k, z, v) = \langle f(\theta), v \rangle, \quad \forall v \in X_0,
\]

(3.13)

\[
c_1(k, \theta, W) + a_1(\theta, W) + \langle B \theta, W \rangle_{\Gamma_1} = \langle \phi_1, W \rangle_{\Gamma_0 \setminus \{x_3 = 0\}}, \quad \forall W \in Y.
\]

(3.14)

It is not difficult to see that the mapping \( F \) is well defined. Indeed, it follows from Subsection 3.1 that, for each vector \( k \in Z \), the equation (3.14) has a unique solution \( \theta = \theta_k \in Y \) satisfying (3.10). Inserting the functions \( k \) and \( \theta_k \) in (3.13), we obtain that the equation (3.13) has a unique solution \( z = z_k \in X_0 \) satisfying (3.12) with \( u = z \). Moreover, the estimate (3.12) and the continuous embedding \( H^1(\Omega) \hookrightarrow L^4(\Omega) \) imply the bound

\[
\|z\|_Z = \|z\|_{L^4(\Omega)} \leq C \|z\|_{H^1(\Omega)} \leq C M,
\]

(3.15)

where \( M = Pr b + Pr R \|\phi_1\|_{H^1_0(\Gamma_0 \setminus \{x_3 = 0\})} + Pr M \|\phi_1\|_{H^1_0(\Gamma_0 \setminus \{x_3 = 0\})} \) and \( C \) is a constant independent of \( k \).

Then, we consider the ball \( B_r = \{k \in Z : \|k\|_Z \leq r \} \subset Z \), with \( r = CM \). It follows from (3.15) and the construction of \( B_r \) that \( F(B_r) \subseteq B_r \). Moreover \( F \) is continuous. Indeed, if \( k_1, k_2 \in Z \), from Subsection 3.1 exist \( \theta_1, \theta_2 \in Y \) and \( z_1, z_2 \in X_0 \) such that

\[
Pr a(z_1, v) + Pr M b_1(\theta_1, v) + c(k_1, z_1, v) = \langle f(\theta_1), v \rangle, \quad \forall v \in X_0,
\]

(3.16)

\[
Pr a(z_2, v) + Pr M b_1(\theta_2, v) + c(k_2, z_2, v) = \langle f(\theta_2), v \rangle, \quad \forall v \in X_0,
\]

(3.17)

\[
c_1(k_1, \theta_1, W) + a_1(\theta_1, W) + \langle B \theta_1, W \rangle_{\Gamma_1} = \langle \phi_1, W \rangle_{\Gamma_0 \setminus \{x_3 = 0\}}, \quad \forall W \in Y,
\]

(3.18)

\[
c_1(k_2, \theta_2, W) + a_1(\theta_2, W) + \langle B \theta_2, W \rangle_{\Gamma_1} = \langle \phi_1, W \rangle_{\Gamma_0 \setminus \{x_3 = 0\}}, \quad \forall W \in Y.
\]

(3.19)

Subtracting the equations (3.18) and (3.19), we get
Moreover, the solution satisfies (3.10) and (3.12).

Thus, taking \( \phi \) as above by using the Schauder Theorem. We state this result in the following theorem:

**Proposition 3.5.** Let \( \phi_1 \in H^2(\Omega \setminus \{x_3 = 0\}) \), \( \phi_2 \in H^2(\{x_3 = 0\}) \) and \( u_g^0 \in H^2(\Gamma_0) \). Then there is a weak solution \( (u, \theta) \in X_0 \times Y \) of problem (3.3) - (3.5) with \( u_g^0 = 0 \) on \( \Gamma_0 \) and \( \phi_2 = 0 \) on \( \{x_3 = 0\} \). Moreover, the solution satisfies (3.11) and (3.17).

In order to prove the existence of a solution to the problem (3.3) - (3.5) in the general case, that is \( u_g^0 \neq 0 \) and \( \phi_2 \neq 0 \), we reduce the problem to an auxiliary problem with homogeneous conditions for the velocity \( u \) on \( \Gamma_0 \) and the temperature \( \theta \) on \( \{x_3 = 0\} \). To this end, using the Hopf Lemma (see [14]), we introduce a function \( u_{\xi} = (u_{\xi 1}, u_{\xi 2}, u_{\xi 3}) \) which satisfies the conditions

\[
|c(v, u_{\xi} , v)| \leq \varepsilon \|v\|^2_{H^1(\Omega)}, \quad \text{for all } v \in X_0.
\]

Here, \( \varepsilon \) is an arbitrarily small number and \( u_{\xi}^g \in H^2(\Omega) \) is a function such that \( u_{\xi}^g = u_{\xi}^0 \) on \( \Gamma_0 \), \( u_{\xi}^g = 0 \) on \( \Gamma_1 \) and the compatibility condition \( \int_{\Gamma_0} u_{\xi}^g \cdot n \, dS = 0 \) is satisfied. Moreover, arguing as in [10], we construct a function \( \theta_{\xi} \in H^1(\Omega) \) such that

\[
\theta_{\xi} = \phi_2 \quad \text{on } \{x_3 = 0\}, \quad \frac{\partial \theta_{\xi}}{\partial n} + B \theta_{\xi} = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial \theta_{\xi}}{\partial n} = \phi_1 \quad \text{on } \Gamma_0 \setminus \{x_3 = 0\},
\]

\[
\|\theta_{\xi}\|_{L^2(\Omega)} \leq \delta, \quad \|\theta_{\xi}\|_{H^1(\Omega)} \leq C \left( \|\phi_1\|_{H^2(\Gamma_0 \setminus \{x_3 = 0\})} + \|\phi_2\|_{H^2(\{x_3 = 0\})} \right).
\]

Here \( \delta \) is an arbitrarily small number and the constant \( C \) depends on \( \delta \). Rewriting \( (u, \theta) \in X \times H^1(\Omega) \) in the form \( u = u_{\xi} + u \) and \( \theta = \theta_{\xi} + \theta \) with \( u_{\xi} \in X_0 \) and \( \theta \in Y \) new unknown functions, we obtain a nonlinear equation for their determination. We can prove existence of its solution as above by using the Schauder Theorem. We state this result in the following following:
Theorem 3.6. Let $\phi_1 \in H^r_0(\Gamma_0 \setminus \{x_3 = 0\})$, $\phi_2 \in H^r_0(\{x_3 = 0\})$ and $u_0^y \in H^r_0(\Gamma_0)$ such that $\int_{\Gamma_0} u_0^y \cdot n \, dS = 0$ and $u_0^y \cdot n = 0$ on $\Gamma_0 \setminus \{x_3 = 0\}$. Then there is at least one solution $(u, \theta) \in X \times H^1(\Omega)$ of problem (3.3) - (3.5), and the following estimate holds:

$$\|u\|_{H^r(\Gamma)} + \|\theta\|_{H^1(\Omega)} \leq C \left( \|u_0\|_{H^r(\Gamma)} + \|\phi_1\|_{H^r_0(\Gamma_0 \setminus \{x_3 = 0\})} + \|\phi_2\|_{H^r_0(\{x_3 = 0\})} \right).$$  

(3.21)

Here the constant $C$ depends linearly of the parameters $M$, $B$ and $R$ and the norm $\|u_0\|_{H^r(\Gamma)}$ depends on $\|u_0\|_{H^r_0(\Gamma_0)}$.

Taking into account the spaces $\tilde{X}$ and $\tilde{H}^r(\partial \Omega)$, we rewrite the Theorem as follows:

Theorem 3.7. Let $\phi_1 \in H^r_0(\Gamma_0 \setminus \{x_3 = 0\})$, $\phi_2 \in H^r_0(\{x_3 = 0\})$ and $u_0^y \in H^r_0(\Gamma_0)$. Then there is at least one solution $(u, \theta) \in \tilde{X} \times H^1(\Omega)$ of problem (3.3) - (3.5) and the estimate (3.21) holds.

3.3 Uniqueness of the weak solutions

The purpose of this section is to determine conditions on the boundary data and parameters which guarantee the uniqueness of the weak solution $(u, \theta) \in \tilde{X} \times H^1(\Omega)$ to the problem (3.3) - (3.5). For that suppose that there exist $(u_1, \theta_1), (u_2, \theta_2) \in \tilde{X} \times H^1(\Omega)$ weak solutions of system (3.3) - (3.5). Then, defining $u = u_1 - u_2$ and $\theta = \theta_1 - \theta_2$, we obtain that $(u, \theta) \in \tilde{X} \times H^1(\Omega)$ solves the system

$$Pr a(u, v) + Pr M b_1(\theta, v) + c(u, u, v) + c(u_2, u, v) = \int_\Omega Pr R \cdot \nabla v \, d\Omega, \quad \forall v \in X_0, \quad (3.22)$$

$$c_1(u, \theta_1, W) + c_1(u_2, \theta, W) + a_1(\theta, W) + (B \theta, W)_{\Gamma_1} = 0, \quad \forall W \in Y, \quad (3.23)$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad \theta = 0 \quad \text{on} \quad \{x_3 = 0\}. \quad (3.24)$$

We can easily see that if $u_2 \in \tilde{X}$, $u \in X_0$ and $\theta \in Y$, then $c(u_2, u, u) = 0$ and $c_1(u_2, \theta, \theta) = 0$. Thus, substituting $u \in X_0$ for $v$ in (3.22), $\theta \in Y$ for $W$ in (3.23), and using the Hölder inequality, Sobolev embeddings and the Poincaré inequality, we deduce

$$Pr \|\nabla u\|_{L^2(\Omega)} \leq Pr M \|\nabla\theta\|_{L^2(\Omega)} + C \|\nabla u\|_{L^2(\Omega)} \|\nabla u_1\|_{L^2(\Omega)} + CPR \|\nabla\theta\|_{L^2(\Omega)}, \quad (3.25)$$

$$\|\nabla\theta\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)} \|\nabla\theta_1\|_{L^2(\Omega)}. \quad (3.26)$$

Using (3.20) in (3.25), we find

$$Pr \|\nabla u\|_{L^2(\Omega)} \leq C \left( Pr M \|\nabla\theta_1\|_{L^2(\Omega)} + \|\nabla u_1\|_{L^2(\Omega)} + Pr R \|\nabla\theta_1\|_{L^2(\Omega)} \right) \|\nabla u\|_{L^2(\Omega)}.$$

Taking into account that $(u_1, \theta_1)$ and $(u_2, \theta_2)$ are weak solutions to the problem (3.3) - (3.5), then from Theorem 3.7 we have that $(u_1, \theta_1)$ and $(u_2, \theta_2)$ satisfy the estimate (3.21), which imply that

$$Pr \|\nabla u\|_{L^2(\Omega)} \leq (C Pr M + C_1 + CPR) \|\nabla u_1\|_{H^1(\Omega)} + \|\nabla u_2\|_{H^1(\Omega)} + \|\nabla\theta_1\|_{H^1(\Gamma_0 \setminus \{x_3 = 0\})} + \|\nabla\theta_2\|_{H^1(\{x_3 = 0\})} \|\nabla u\|_{L^2(\Omega)}, \quad (3.27)$$

where the constant $C_1$ is obtained from the estimate (3.21) and it depends almost linearly of the parameters $M, B$ and $R$. Therefore, if the condition

$$Pr - (C Pr M + C_1 + CPR) \|\nabla u_1\|_{H^1(\Omega)} + \|\nabla u_2\|_{H^1(\{x_3 = 0\})} + \|\nabla\theta_1\|_{H^1(\Gamma_0 \setminus \{x_3 = 0\})} + \|\nabla\theta_2\|_{H^1(\{x_3 = 0\})} > 0, \quad (3.27)$$

is satisfied, we conclude that $\|\nabla u\|_{L^2(\Omega)} = 0$, and consequently $u = 0$, which implies that $\theta = 0$. Moreover, using this fact in (3.20), we obtain that $\|\nabla\theta\|_{L^2(\Omega)} = 0$, and consequently $\theta = 0$, which implies that $\theta_1 = \theta_2$. Thus we have proved the following theorem:
Theorem 3.8. Let \( \phi_1 \in H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\}) \), \( \phi_2 \in H^{\frac{1}{2}}(\{x_3 = 0\}) \) and \( u_0^g \in \widetilde{H}^{\frac{1}{2}}(\Gamma_0) \). If the condition (3.27) is satisfied, then the problem (3.3) - (3.5) has a unique solution \((u, \theta) \in \tilde{X} \times H^1(\Omega)\). Moreover, the solution \((u, \theta)\) satisfies the estimate (3.21).

Remark 3.9. Observe that the condition

\[
Pr - (CPrM + C_1 + CPrR)[\|u\|_{H^1(\Omega)} + \|\phi_1\|_{H^{\frac{1}{2}}(\Gamma_0 \setminus \{x_3 = 0\})} + \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3 = 0\})}] > 0
\]

is verified if either the functions \(u_\varepsilon\), \(\phi_1\) and \(\phi_2\) are small, or if the coefficients \(M\), \(R\) and \(B\) are small. In particular, for small values of \(M\), \(R\) and \(B\) and boundary data \(u_0 = 0\), \(\phi_1 = 0\) and \(\phi_2 = \theta_c\), the basic solution \((u_b, \theta_b, p_b)\) given by (1.10) is unique.

3.4 Regularity

In Subsection 3.2 was demonstrated the existence of a weak solution \((u, \theta) \in \tilde{X} \times H^1(\Omega)\) to the problem (3.3) - (3.5); however, taking into account the tangential and normal derivatives of the temperature at the boundary, we need to prove that \(\theta \in H^2(\Omega)\) (see Lemma 3.2). In this subsection we analyze the following regularity problem for the weak solution \(\theta \in H^1(\Omega)\): Given \(u \in \tilde{X}\), find \(\theta \in H^2(\Omega)\) such that

\[
-\Delta \theta = -(u \cdot \nabla)\theta \quad \text{in } \Omega, \\
\frac{\partial \theta}{\partial n} + B\theta = 0 \quad \text{on } \Gamma_1, \\
\frac{\partial \theta}{\partial n} = \phi_1 \quad \text{on } \Gamma_2, \\
\theta = \phi_2 \quad \text{on } \Gamma_3,
\]

where \(\Gamma_1 := \{x_3 = 1\}\), \(\Gamma_3 := \{x_3 = 0\}\) and \(\Gamma_2 := \partial \Omega \setminus \{\Gamma_1 \cup \Gamma_3\}\) (see Figure 1).

![Figure 1: Representation of \(\partial \Omega\).](image)

Theorem 3.10. Let \(\phi_1 \in H^{\frac{1}{2}}(\Gamma_2)\), \(\phi_2 \in H^{\frac{1}{2}}(\Gamma_3)\) and \(f \in L^p(\Omega)\), with \(\frac{6}{5} < p \leq 2\). Then, the system

\[
-\Delta \theta = f \quad \text{in } \Omega, \\
\frac{\partial \theta}{\partial n} + B\theta = 0 \quad \text{on } \Gamma_1, \\
\frac{\partial \theta}{\partial n} = \phi_1 \quad \text{on } \Gamma_2, \\
\theta = \phi_2 \quad \text{on } \Gamma_3,
\]

has a solution \(\theta \in W^{2,p}(\Omega)\), with \(\frac{6}{5} < p \leq 2\).

Proof: We first convert the problem (3.29) with Robin, Neumann and Dirichlet conditions, in a boundary problem with only Dirichlet and Neumann conditions. For that, we will adapt the ideas of [18], Section 2. First we consider the functions \(\eta(x_3)\) and \(\tilde{\theta}\) defined by:

\[
\eta(x_3) = \exp[B(2x_3 - \frac{1}{2}x_3^2)] \quad \text{and} \quad \tilde{\theta} = \eta \theta.
\]
Since $B$ is constant, it is easy to check that problem (3.29) is equivalent to find $\tilde{\theta} \in W^{2,p}(\Omega)$ with $\frac{6}{5} < p \leq 2$, such that

$$
\begin{cases}
-\Delta \tilde{\theta} = \tilde{f} & \text{in } \Omega, \\
\frac{\partial \tilde{\theta}}{\partial n} = 0 & \text{on } \Gamma_1, \\
\frac{\partial \tilde{\theta}}{\partial n} = \tilde{\phi}_1 & \text{on } \Gamma_2, \\
\tilde{\theta} = \phi_2 & \text{on } \Gamma_3,
\end{cases}
$$

(3.31)

where $\tilde{f} = -\theta \eta'' - 2 \frac{\partial \theta}{\partial x_3} \eta' + \eta f$ and $\tilde{\phi}_1 = \eta \phi_1$. Taking into account that $\phi_2 \in H^\frac{3}{2}(\Gamma_3)$, we consider the function $\tilde{\phi}_2 \in H^\frac{3}{2}(\partial \Omega)$ given by $\tilde{\phi}_2 = \phi_2$ on $\Gamma_3$, and $\tilde{\phi}_2 = 0$ on $\partial \Omega \setminus \Gamma_3$. By the lifting Theorem, we have that exists $\Phi_2 \in H^2(\Omega)$ such that $\gamma(\Phi_2) = \tilde{\phi}_2$. Considering $\hat{\theta} = \tilde{\theta} - \Phi_2$, it is not difficult to verify that problem (3.31) is equivalent to find $\hat{\theta} \in W^{2,p}(\Omega)$ with $\frac{6}{5} < p \leq 2$, such that

$$
\begin{cases}
-\Delta \hat{\theta} = \hat{f} & \text{in } \Omega, \\
\frac{\partial \hat{\theta}}{\partial n} = \phi_3 & \text{on } \Gamma_1, \\
\frac{\partial \hat{\theta}}{\partial n} = \tilde{\phi}_1 & \text{on } \Gamma_2, \\
\hat{\theta} = 0 & \text{on } \Gamma_3,
\end{cases}
$$

(3.32)

where $\hat{f} = \tilde{f} + \Delta \Phi_2$, $\phi_3 = - \frac{\partial \Phi_2}{\partial n} \big|_{\Gamma_3}$ and $\tilde{\phi}_1 = \tilde{\phi}_1 - \frac{\partial \Phi_2}{\partial n} \big|_{\Gamma_2}$. In order to find the solution $\hat{\theta}$ of problem (3.32), we decompose $\hat{\theta}$ as the sum $\hat{\theta} = \theta_1 + \theta_2 + \theta_3$, where $\theta_1$, $\theta_2$ and $\theta_3$ solve respectively the following problems:

$$
\begin{cases}
-\Delta \theta_1 = \tilde{f} & \text{in } \Omega, \\
\frac{\partial \theta_1}{\partial n} = 0 & \text{on } \Gamma_1, \\
\frac{\partial \theta_1}{\partial n} = 0 & \text{on } \Gamma_2, \\
\theta_1 = 0 & \text{on } \Gamma_3,
\end{cases}
$$

(3.33)

$$
\begin{cases}
-\Delta \theta_2 = 0 & \text{in } \Omega, \\
\frac{\partial \theta_2}{\partial n} = 0 & \text{on } \Gamma_1, \\
\frac{\partial \theta_2}{\partial n} = \tilde{\phi}_1 & \text{on } \Gamma_2, \\
\theta_2 = 0 & \text{on } \Gamma_3,
\end{cases}
$$

(3.34)

$$
\begin{cases}
-\Delta \theta_3 = 0 & \text{in } \Omega, \\
\frac{\partial \theta_3}{\partial n} = \phi_3 & \text{on } \Gamma_1, \\
\frac{\partial \theta_3}{\partial n} = 0 & \text{on } \Gamma_2, \\
\theta_3 = 0 & \text{on } \Gamma_3.
\end{cases}
$$

(3.35)

In order to prove the existence of $\theta_1, \theta_2, \theta_3 \in W^{2,p}(\Omega)$, we require the following preliminary result:

**Theorem 3.11.** If $F \in L^p(\Omega)$ with $\frac{6}{5} < p < \infty$, then the weak solution to the problem

$$
\begin{cases}
-\Delta \omega = F & \text{in } \Omega, \\
\frac{\partial \omega}{\partial n} = 0 & \text{on } \Gamma_1 \cup \Gamma_2, \\
\omega = 0 & \text{on } \Gamma_3,
\end{cases}
$$

(3.36)

belongs to the space $W^{2,p}(\Omega)$. 
Proof: The proof follows from Theorem 1 and from Section 5 of [10]. See also [11]. \[\square\]

Thus, by Theorem 3.11 if \( \hat{f} \in L^p(\Omega) \) with \( \frac{2}{5} < p < \infty \), then the system \( \text{(3.33)} \) has solution \( \theta_1 \in W^{2,p}(\Omega) \). We remember that \( \hat{f} = -\theta_n'' - 2\frac{\partial \theta}{\partial x_3}n' + \eta f + \Delta \Phi_2 \). Observe that as \( \theta \in H^1(\Omega) \), \( \frac{\partial \theta}{\partial x_3} \in L^2(\Omega) \). Moreover since \( \eta(x) = \exp[B(2x_3 - \frac{2}{3}x_3^2)] \), then \( \eta' = B(2 - x_3)\eta \) and \( \eta'' = -B\eta + B^2(2 - x_3)^2\eta \). Recalling that \( 0 \leq x_3 \leq 1 \), we deduce that \( \eta, \eta' \) and \( \eta'' \) belong to \( L^2(\Omega) \). Finally, as \( \Phi_2 \in H^4(\Omega) \), \( \Delta \Phi_2 \in L^2(\Omega) \), and as by initial hypothesis \( f \in L^p(\Omega) \) with \( \frac{2}{5} < p < 2 \), we conclude that \( \hat{f} \in L^p(\Omega) \) with \( \frac{2}{5} < p < 2 \).

On the other hand, observe that for finding \( \theta_2, \theta_3 \in W^{2,p}(\Omega) \) solutions of \( \text{(3.34)} \) and \( \text{(3.35)} \) respectively, we can not use directly Theorem 3.11 because these systems have nonhomogeneous boundary conditions. Therefore, for solving the problem \( \text{(3.33)} \), we first divide \( \Gamma_2 \) in four parts \( \Gamma_i^2 \) with \( i = 1, 2, 3, 4 \), as showed in Figure 2 and then we decompose the solution \( \theta_i \) as the sum \( \theta_i = \theta_{2i} + \theta_{22} + \theta_{23} + \theta_{24} \), where \( \theta_{2i}, \theta_{22}, \theta_{23} \) and \( \theta_{24} \) solve respectively the following systems:

\[
\begin{align*}
-\Delta \theta_{21} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \theta_{21}}{\partial n} &= \hat{\phi}_1^i \quad \text{on } \Gamma_i^2, \\
\frac{\partial \theta_{21}}{\partial n} &= 0 \quad \text{on } \Gamma_1 \cup (\Gamma_2 \setminus \Gamma_i^2), \\
\theta_{21} &= 0 \quad \text{on } \Gamma_3, \\
\end{align*}
\]  
(3.37)

\[
\begin{align*}
-\Delta \theta_{22} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \theta_{22}}{\partial n} &= \hat{\phi}_2^i \quad \text{on } \Gamma_i^2, \\
\frac{\partial \theta_{22}}{\partial n} &= 0 \quad \text{on } \Gamma_1 \cup (\Gamma_2 \setminus \Gamma_i^2), \\
\theta_{22} &= 0 \quad \text{on } \Gamma_3, \\
\end{align*}
\]  
(3.38)

\[
\begin{align*}
-\Delta \theta_{23} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \theta_{23}}{\partial n} &= \hat{\phi}_3^i \quad \text{on } \Gamma_i^2, \\
\frac{\partial \theta_{23}}{\partial n} &= 0 \quad \text{on } \Gamma_1 \cup (\Gamma_2 \setminus \Gamma_i^2), \\
\theta_{23} &= 0 \quad \text{on } \Gamma_3, \\
\end{align*}
\]  
(3.39)

\[
\begin{align*}
-\Delta \theta_{24} &= 0 \quad \text{in } \Omega, \\
\frac{\partial \theta_{24}}{\partial n} &= \hat{\phi}_4^i \quad \text{on } \Gamma_i^2, \\
\frac{\partial \theta_{24}}{\partial n} &= 0 \quad \text{on } \Gamma_1 \cup (\Gamma_2 \setminus \Gamma_i^2), \\
\theta_{24} &= 0 \quad \text{on } \Gamma_3, \\
\end{align*}
\]  
(3.40)

where \( \hat{\phi}_j^i \) is defined by \( \hat{\phi}_j^i = \phi_1|_{\Gamma_j^i} \) on \( \Gamma_j^i \), and \( \hat{\phi}_j^i = 0 \) on \( \partial \Omega \setminus \Gamma_j^i, \ i = 1, 2, 3, 4 \).

\[\text{Figure 2: Division of } \Gamma_2.\]
For solving problem (3.37) we will adapt the ideas of [13], Section 2. For that, we divide the boundary of $\Gamma^1_2$, denoted by $\partial \Gamma^1_2$, as follows: $\partial \Gamma^1_2 = \Gamma^1_{12} \cup \Gamma^1_{13} \cup \Gamma^1_{14}$ (see Figure 3), and we construct a function $\psi_1$ as a solution of the heat equation:

\[
\begin{align*}
\frac{\partial \psi_1}{\partial t} &= \Delta \psi_1 & \text{in } \Gamma^1_1 \times (0, \infty), \\
\frac{\partial \psi_1}{\partial n} &= 0 & \text{on } \Gamma^1_{ij} \times (0, \infty), \quad i = 1, 2, 4, \\
\psi_1 &= 0 & \text{on } \Gamma^1_2 \times (0, \infty), \\
\psi_1(x_1, 0, x_3) &= \phi_1(x_1, x_3) & \text{on } \Gamma^1_2.
\end{align*}
\]

(3.41)

\[
\begin{figure}
\centering
\includegraphics[width=\textwidth]{circle.png}
\caption{Division de $\partial \Gamma^1_2$.}
\end{figure}

We easily see that $T_1$ satisfies the boundary conditions in (3.37). Moreover, taking into account that $\psi_1 \in H^2(\Omega)$, we deduce that $T_1 \in H^2(\Omega)$. Additionally, as $T_1 \in H^2(\Omega)$ then $-\Delta T_1 \in L^2(\Omega)$, and consequently $-\Delta T_1 \in L^p(\Omega)$ for $\frac{6}{5} < p < 2$. Thus, by Theorem 3.11 the solution $\tilde{T}_1$ of the system

\[
\begin{align*}
\Delta \tilde{T}_1 &= -\Delta T_1 & \text{in } \Omega, \\
\frac{\partial \tilde{T}_1}{\partial n} &= 0 & \text{on } \Gamma_1 \cup \Gamma_2, \\
\tilde{T}_1 &= 0 & \text{on } \Gamma_3.
\end{align*}
\]

(3.43)

belongs to $W^{2,p}(\Omega)$ with $\frac{6}{5} < p \leq 2$. In conclusion, considering $\theta_2 = T_1 + \tilde{T}_1$ we obtain that $\theta_2 \in W^{2,p}(\Omega)$, $\frac{6}{5} < p \leq 2$, and $\theta_2$ satisfies the system (3.37). Analogously we can find solutions $\theta_2$, $\theta_3$, $\theta_4$, and $\theta_5$ in $W^{2,p}(\Omega)$, $\frac{6}{5} < p \leq 2$, for problems (3.38), (3.39), (3.40) and (3.35) respectively. Thus, considering $\theta_2 = \theta_2 + \theta_3 + \theta_4 + \theta_5$, we deduce that $\theta_2 \in W^{2,p}(\Omega)$ is a solution to the system (3.34). Thus, it was verified the existence of $\theta_1, \theta_2, \theta_3 \in W^{2,p}(\Omega)$ solutions of (3.33), (3.34) and (3.35) respectively, and the theorem is proven.

Taking into account Theorem 3.10 then we prove the following theorem which guarantees the existence of solution of problem (3.28).

**Theorem 3.12.** Let $\phi_1 \in H^\frac{7}{2}(\Gamma_2)$, $\phi_2 \in H^2(\Gamma_3)$, $u \in X$ and $\theta \in H^1(\Omega)$ weak solution of system (3.28). Then, the solution $\theta$ belongs to the space $H^2(\Omega)$.

**Proof:** First, observe that as $u \in \tilde{X} \subset H^1(\Omega)$ then using Sobolev embeddings we obtain that $u \in L^6(\Omega)$. Moreover, as $\theta \in H^1(\Omega)$, $\nabla \theta \in L^2(\Omega)$ and consequently $-(u \cdot \nabla) \theta \in L^2(\Omega)$. Thus, by Theorem 3.10 we conclude that the problem (3.28) has solution $\theta \in W^{2,\frac{7}{2}}(\Omega)$. Analogously, since $\theta \in W^{2,\frac{7}{2}}(\Omega)$, $\nabla \theta \in W^{1,2}(\Omega)$, and consequently, using the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ we deduce that $\nabla \theta \in L^2(\Omega)$. Therefore, $-(u \cdot \nabla) \theta \in L^2(\Omega)$ and from Theorem 3.10 we conclude that the solution $\theta$ of problem (3.28) belongs to $W^{2,2}(\Omega) = H^2(\Omega)$.

$\square$
4 Existence of optimal solutions

In this section we will prove the existence of an optimal solution for Problem (2.9). We define the set of admissible solutions of Problem (2.9) as follows:

\[ Z_{ad} := \{ z \equiv (u, \theta, g, \phi_1, \phi_2) \in \tilde{X} \times H^1(\Omega) \times U_1 \times U_2 \times U_3 \text{ such that } J(z) < \infty \text{ and the equations (3.3) - (3.5) hold} \}. \]  

Then we have the following result

**Theorem 4.1.** Suppose that one of the conditions (i) and (ii) in (2.9) is satisfied. Then the problem (2.9) has at least one solution, that is, there exists at least a \( \hat{z} \equiv (\hat{u}, \hat{\theta}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \in Z_{ad} \) such that

\[ J(\hat{z}) = \min_{z \in Z_{ad}} J(z). \]  

**Proof:** From Theorem 3.7 we have that \( Z_{ad} \) is nonempty. Denote by \( (z_m) = (u_m, \theta_m, g_m, \phi_{1m}, \phi_{2m}) \subset Z_{ad}, m \in \mathbb{N}, \) a minimizing sequence for which \( \lim_{m \to \infty} J(z_m) = \min_{z \in Z_{ad}} J(z) \). If one of the conditions (i) and (ii) is satisfied, then there exist constants \( C_1, C_2 \) and \( C_3, \) independent of \( m, \) such that \( \|g_m\|_{H^2(\Omega)} \leq C_1, \|\phi_{1m}\|_{H^2(\Gamma_0 \setminus \{x_3 = 0\})} \leq C_2 \) and \( \|\phi_{2m}\|_{H^2(\Gamma_3 \setminus \{x_3 = 0\})} \leq C_3. \) Thus, from Theorem 3.7 we conclude that there exist constants \( C_4 \) and \( C_5, \) independent of \( m, \) such that \( \|u_m\|_{H^1(\Omega)} \leq C_4 \) and \( \|\theta_m\|_{H^1(\Omega)} \leq C_5. \)

Therefore, since \( U_1, U_2 \) and \( U_3 \) are closed convex subsets of \( H^2(\Gamma_0), H^2(\Gamma_0 \setminus \{x_3 = 0\}) \) and \( H^2(\{x_3 = 0\}), \) respectively, we obtain \( \hat{z} \equiv (\hat{u}, \hat{\theta}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \in Z_{ad} \subset H^1(\Omega) \times H^1(\Omega) \times U_1 \times U_2 \times U_3 \) such that, for some subsequence of \( (z_m)_{m \in \mathbb{N}} \subset Z_{ad} \) still denoted by \( (z_m)_{m \in \mathbb{N}}, \) we have

\begin{align*}
    u_m &\to \hat{u} \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^p(\Omega), \ 2 \leq p < 6, \\
    \theta_m &\to \hat{\theta} \text{ weakly in } H^1(\Omega) \text{ and strongly in } L^6(\Omega), \ 2 \leq l < 6, \\
    g_m &\to \hat{g} \text{ weakly in } H^2(\Gamma_0) \text{ and strongly in } L^2(\Gamma_0), \\
    \phi_{1m} &\to \hat{\phi}_1 \text{ weakly in } H^2(\Gamma_0 \setminus \{x_3 = 0\}) \text{ and strongly in } L^2(\Gamma_0 \setminus \{x_3 = 0\}), \\
    \phi_{2m} &\to \hat{\phi}_2 \text{ weakly in } H^2(\{x_3 = 0\}) \text{ and strongly in } L^2(\{x_3 = 0\}).
\end{align*}

Since \( \gamma \mid_{\Gamma_0} \ u_m = g_m, \gamma \mid_{\Gamma_2} \ u_m = u_0 \) and \( \gamma \mid_{\{x_3 = 0\}} \ \theta_m = \phi_{2m}, \) it follows from the properties of the trace operators that \( \gamma \mid_{\Gamma_0} \ \hat{u} = \hat{g}, \gamma \mid_{\Gamma_2} \ \hat{u} = u_0 \) and \( \gamma \mid_{\{x_3 = 0\}} \ \hat{\theta} = \hat{\phi}_2; \) so \( \hat{z} \) satisfies the boundary conditions (3.5). Moreover, the third component of \( u_m \) denoted by \( u_{m3} \) is equal to \( 0 \) on \( \Gamma_1, \) for all \( m \in \mathbb{N}, \) then from the continuity of the trace operators we obtain \( \hat{u}_3 = 0 \) on \( \Gamma_1. \) Also, using (3.3) we obtain that \( \text{div } u_m \to \text{div } \hat{u} \text{ weakly in } L^2(\Omega), \) and given that \( \text{div } u_m = 0, \) for all \( m \in \mathbb{N}, \) we conclude that \( \text{div } \hat{u} = 0. \) Moreover, as \( \hat{u} = \hat{g} \) on \( \Gamma_0, \ \hat{u} = u_0 \) on \( \Gamma_2 \) and \( \hat{u} \cdot n = 0 \) on \( \Gamma_0, \) we obtain that \( \int_{\Gamma_0} \hat{u} \cdot n \ d\Gamma = 0 \) and \( \hat{\theta} \mid_{\{x_3 = 0\}} = 0. \) Therefore, we conclude that \( \hat{u} \equiv \tilde{X}. \) A standard procedure permits to pass the limit, as \( m \) goes to \( \infty, \) in the variational formulation (3.3) - (3.5), and we obtain that \( \hat{z} \) satisfies the weak formulation (3.3) - (3.5). Consequently we have that \( \hat{z} \equiv (\hat{u}, \hat{\theta}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \in Z_{ad}, \) and then

\[ J(\hat{z}) \geq \inf_{z \in Z_{ad}} J(z). \]  

Finally, recalling that the functional \( J \) is weakly lower semicontinuous, we have that

\[ J(\hat{z}) = \min_{z \in Z_{ad}} J(z). \]  

\[ \square \]

**Remark 4.2.** Let \( (u_0, \theta_b) \) the basic solution to the problem (1.4) - (1.5) given by (1.10). From Theorem 4.1, we can obtain the existence of controls \( (g, \phi_1, \phi_2) \in U_1 \times U_2 \times U_3 \) and a weak solution \((u, \theta) \in \tilde{X} \times H^1(\Omega)\) of the problem (3.3) - (3.5), such that the functional (2.7) is minimized if we consider \( u_1 = u_0 \) and \( \theta_d = \theta_b, \) the basic state.
5 Necessary optimality conditions and an optimality system

5.1 Existence of Lagrange multipliers

This section is devoted to obtain an optimality system to problem (2.9). We wish to use the method of Lagrange multipliers to turn the constrained optimization problem (2.9) into an unconstrained one. We start denoting by \( G \) the Hilbert space \( G = \tilde{X} \times H^1(\Omega) \), with norm

\[
\|x\|_G = \left( \|u\|_{\tilde{X}}^2 + \|\theta\|_{H^1(\Omega)}^2 \right)^\frac{1}{2}, \quad x = (u, \theta) \in G.
\] (5.1)

We consider the following operators \( F_1 : G \times U_1 \times U_2 \times U_3 \to \tilde{X}_0, F_2 : G \times U_1 \times U_2 \times U_3 \to Y', \ F_3 : G \times U_1 \times U_2 \times U_3 \to \tilde{H}^{\frac{1}{2}}(\Gamma_0) \) and \( F_4 : G \times U_1 \times U_2 \times U_3 \to H^{\frac{1}{2}}(\{x_3 = 0\}) \), defined at each point \( z := (x, g, \phi_1, \phi_2) = (u, \theta, g, \phi_1, \phi_2) \) by:

\[
\begin{cases}
\langle F_1(z), v \rangle &= Pr \alpha(u, v) - b \langle \theta, v \rangle + c(u, u, v) - \langle f(\theta), v \rangle, \quad \forall v \in X_0, \\
\langle F_2(z), W \rangle &= c_1(u, \theta, W) + a_1(\theta, W) + (B\theta, W)_{\Gamma_1} - (\phi_1, W)_{\Gamma_0 \\{x_3 = 0\}}, \quad \forall W \in Y, \\
F_3(z) &= \gamma \|r_\theta u - u_g\|^2_{L^2(\Gamma_0)}, \\
F_4(z) &= \gamma \|z_{x_3 = 0}\|^2_{H^\frac{1}{2}(\{x_3 = 0\})},
\end{cases}
\] (5.2)

In order to simplify the notation, let us denote by \( M \) the space

\[
M \equiv \tilde{X}_0 \times Y' \times \tilde{H}^{\frac{1}{2}}(\Gamma_0) \times H^{\frac{1}{2}}(\{x_3 = 0\}),
\]

and define the operator

\[
F : G \times U_1 \times U_2 \times U_3 \to M, \text{ such that } F(z) := (F_1(z), F_2(z), F_3(z), F_4(z)).
\]

Then the optimal control problem (2.9) is equivalent to:

Find \( z := (x, g, \phi_1, \phi_2) \in G \times U_1 \times U_2 \times U_3 \) such that, for \( 2 \leq p, l \leq 6 \), the functional

\[
\mathcal{J}(u, \theta, g, \phi_1, \phi_2) = \int_{\Omega} \frac{1}{2} \|\text{rot } u\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u - u_g\|_{L^p(\Omega)}^2 + \frac{1}{2} \|\theta - \theta_d\|_{L^l(\Omega)}^2 + \frac{1}{2} \|g\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_0)}^2
\]

\[
+ \frac{1}{2} \|\phi_1\|_{H^\frac{1}{2}(\{x_3 = 0\})}^2 + \frac{1}{2} \|\phi_2\|_{H^{\frac{1}{2}}(\{x_3 = 0\})}^2
\] (5.3)

is minimized subject to \( \langle F(z), (v, W) \rangle = (\langle F_1(z), v \rangle, \langle F_2(z), W \rangle, F_3(z), F_4(z) \rangle = (0, 0, 0, 0) \).

We denote by \( \eta := (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R} \times \tilde{X}_0 \times Y \times (\tilde{H}^{\frac{1}{2}}(\Gamma_0))^\prime \times H^{-\frac{1}{2}}(\{x_3 = 0\}) \). Then, for the problem (5.3), the Lagrange functional \( \mathcal{L} \) is defined by:

\[
\mathcal{L} : G \times U_1 \times U_2 \times U_3 \times \mathbb{R} \times \tilde{X}_0 \times Y \times (\tilde{H}^{\frac{1}{2}}(\Gamma_0))^\prime \times H^{-\frac{1}{2}}(\{x_3 = 0\}) \to \mathbb{R}
\]

such that

\[
\mathcal{L}(z, \eta) = \lambda_0 \mathcal{J}(z) - \langle F_1(z), \lambda_1 \rangle_{\tilde{X}_0, \tilde{X}_0} - \langle F_2(z), \lambda_2 \rangle_{Y', Y} - \langle \lambda_3, F_3(z) \rangle_{\tilde{H}^{\frac{1}{2}}(\Gamma_0), \tilde{H}^{\frac{1}{2}}(\Gamma_0)}
\]

\[
- \langle \lambda_4, F_4(z) \rangle_{H^{-\frac{1}{2}}(\{x_3 = 0\}), H^{-\frac{1}{2}}(\{x_3 = 0\})}.
\] (5.4)

**Lemma 5.1.** The operator \( F \) is Fréchet differentiable with respect to \( x = (u, \theta) \in \tilde{X} \times H^1(\Omega) = G \). Moreover, at an arbitrary point \( \tilde{z} = (\tilde{x}, \tilde{g}, \tilde{\phi}_1, \tilde{\phi}_2) \in \tilde{X} \times H^1(\Omega) = G \), the Fréchet derivative operator of \( F \)
with respect to $x$ is the linear and bounded operator $F_\gamma(\hat{z}) : G \to \mathbb{M}$ which, to each element $h = (h_1, h_2) \in G$, is defined by:

\[
\begin{align*}
F_1(\hat{z})h &= Pr a(h_1, v) + Pr M b_1(h_2, v) + c(\hat{u}, h_1, v) + c(h_1, \hat{u}, v) - Pr R(h_2, v_3) \quad \forall v \in X_0, \\
F_2(\hat{z})h &= a_1(h_2, w) + a_1(h_2, w) + (Bh_2, W)_{\Gamma_1}, \forall W \in Y, \\
F_3(\hat{z})h &= \gamma_1(h_1, h_1), \\
F_4(\hat{z})h &= \gamma_1(h_2, h_2).
\end{align*}
\]

**Proof:** By using the definition of $F_1$ we obtain that,

\[
\langle F_1(\hat{x} + h, g, \hat{\phi}_1, \hat{\phi}_2) - F_1(\hat{z}), v \rangle = Pr a(h_1, v) + Pr M b_1(h_2, v) + c(\hat{u}, h_1, v) + c(h_1, \hat{u}, v) + c(h_1, h_1, v) - Pr R(h_2, v_3).
\]

Then we get

\[
[|F_1(\hat{x} + h, g, \hat{\phi}_1, \hat{\phi}_2) - F_1(\hat{z}), v| = |c(h_1, h_1, v)| \leq \|h_1\|_{L^2(\Omega)} \|\nabla h_1\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C \|h_1\|_X^2 \|v\|_{X_0} \leq C \|h_1\|_X^2 \leq C \|h\|_G^2,
\]

from which

\[
\lim_{\|h\|_G \to 0} \frac{|\langle F_1(\hat{x} + h, g, \hat{\phi}_1, \hat{\phi}_2) - F_1(\hat{z}), v \rangle - Pr a(h_1, v) - Pr M b_1(h_2, v) - c(\hat{u}, h_1, v) - c(h_1, \hat{u}, v) - c(h_1, h_1, v) - Pr R(h_2, v_3)\rangle}{\|h\|_G} = 0.
\]

Thus, by definition of the Fréchet derivative we conclude that

\[
\langle F_1(\hat{x}), h \rangle = Pr a(h_1, v) + Pr M b_1(h_2, v) + c(\hat{u}, h_1, v) + c(h_1, \hat{u}, v) - Pr R(h_2, v_3), \forall v \in X_0.
\]

Analogously can be determined $F_2(\hat{x})h$, $F_3(\hat{x})h$ and $F_4(\hat{x})h$. Thus, the proof is concluded. \qed

**Lemma 5.2.** The functional $J$ is Fréchet differentiable with respect to $x = (u, \theta) \in G$. Moreover, at an arbitrary point $\hat{z} = (\hat{x}, \hat{y}, \hat{\phi}_1, \hat{\phi}_2)$ it holds:

(i) The Fréchet derivative operator of $J$ with respect to $u$ is the linear and bounded operator $J_u(\hat{z}) : X \to \mathbb{R}$ defined by

\[
J_u(\hat{z})h_1 = \gamma_1(\text{rot} \hat{u}, \text{rot} h_1)_{L^2(\Omega)} + \gamma_2 \int_{\Omega} |\hat{u} - u_0|^{p-2} sgn(\hat{u} - u_0) \cdot h_1 \, d\Omega.
\]

(ii) The Fréchet derivative operator of $J$ with respect to $\theta$ is the linear and bounded operator $J_\theta(\hat{z}) : H^1(\Omega) \to \mathbb{R}$ defined by

\[
J_\theta(\hat{z})h_2 = \gamma_3 \int_{\Omega} |\hat{\theta} - \theta_0|^{p-2} sgn(\hat{\theta} - \theta_0) \cdot h_2 \, d\Omega.
\]

(iii) The Fréchet derivative operator of $J$ with respect to $x$ can be computed as

\[
J_x(\hat{z})h = J_u(\hat{z})h_1 + J_\theta(\hat{z})h_2,
\]

where $h = (h_1, h_2) \in G$.

**Proof:** It is not difficult to prove that if $J_1 : X \to \mathbb{R}$ is an operator defined by $J_1(u) = \frac{\gamma_2}{2} \|\text{rot} \, u\|_{L^2(\Omega)}^2$, then the Fréchet derivative is given by $J_{1u}(h_1) = \gamma_1(\text{rot} \hat{u}, \text{rot} h_1)$. On the other hand, if $J_3 : X \to \mathbb{R}$ is defined by $J_3(u) = \|u - u_0\|_{L^p(\Omega)}^p$, with $2 \leq p \leq 6$, in order to find the Fréchet derivative, first we
consider the operator \( \xi : L^p(\Omega) \to L^q(\Omega) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \), defined by \( \xi(u) = \frac{|u - u_0|^p - |u_0|^p}{u_0} \) if \( u \neq u_0 \) and \( \lim_{u \to u_0} \xi(u) = \frac{|u - u_0|^p - |u_0|^p}{u_0} \) for all \( u \) in \( \tilde{X} \). It is not difficult to see that \( \xi \) is continuous at the point \( u_0 \), and moreover \( \lim_{u \to u_0} \xi(u) = \text{sgn} (u - u_0) \) if \( 2 \leq p \leq 6 \). Also, taking into account that \( u, \tilde{u} \in \tilde{X} \), we can easily verify that \( \xi(u) \in L^q(\Omega) \), for all \( u \in L^p(\Omega) \). Moreover, observe that for all \( h \in L^p(\Omega) \), with \( h \neq 0 \), we have that

\[
\tilde{J}_2(\tilde{u} + h) - \tilde{J}_2(\tilde{u}) - \langle \xi(\tilde{u} + h), h \rangle = \int \frac{\tilde{u} + h - u_d|^p - |\tilde{u} - u_d|^p}{\tilde{u} + h - u} d\Omega - \int \frac{\tilde{u} + h - u_d|^p - |\tilde{u} - u_d|^p}{\tilde{u} + h - u} h d\Omega = 0.
\]

Thus, if we define the operator \( \varphi : L^p(\Omega) \to L^q(\Omega) \) by \( \varphi(h) = \xi(u + h) - \xi(\tilde{u}) \), for all \( h \in L^p(\Omega) \), we obtain that \( \tilde{J}(\tilde{u} + h) - \tilde{J}(\tilde{u}) - \langle \xi(\tilde{u}), h \rangle = \langle \varphi(h), h \rangle \), \( \forall h \in L^p(\Omega) \), \( h \neq 0 \), and consequently

\[
\frac{\| \tilde{J}(\tilde{u} + h) - \tilde{J}(\tilde{u}) - \langle \xi(\tilde{u}), h \rangle \|}{\| h \|_{\tilde{X}}} = \frac{\| \langle \varphi(h), h \rangle \|}{\| h \|_{L^p(\Omega)}} \leq \frac{\| \varphi(h) \|}{\| h \|_{L^p(\Omega)}} \quad \forall h \in \tilde{X} \hookrightarrow L^p(\Omega), \ h \neq 0.
\]

(5.7)

Taking into account that \( \xi \) is continuous in \( \tilde{u} \), we obtain \( \varphi(h) \to 0 \), when \( h \to 0 \). Finally, as \( \varphi(h) \in L^q(\Omega) \), for all \( h \in L^p(\Omega) \), we conclude that \( \langle \varphi(h), h \rangle \to 0 \), when \( h \to 0 \). Then, we deduce that

\[
\lim_{\| h \|_{\tilde{X}} \to 0} \frac{\| \tilde{J}(\tilde{u} + h) - \tilde{J}(\tilde{u}) - \langle \xi(\tilde{u}), h \rangle \|}{\| h \|_{\tilde{X}}} = 0,
\]

(5.8)

where \( \langle \xi(\tilde{u}), h \rangle = \langle p|\tilde{u} - u_d|^p - \text{sgn}(\tilde{u} - u_d), h \rangle \). Thus, we conclude (i). Analogously we can demonstrate (ii). The proof of (iii) follows without difficulties taking into account that the functional \( \tilde{J} \) depends linearly of \( u \) and \( \theta \).

\begin{remark}
(i) The functional \( \tilde{J} \) is convex with respect to \( g \in U_1 \), \( \phi_1 \in U_2 \) and \( \phi_2 \in U_3 \), since \( U_1 \), \( U_2 \), and \( U_3 \) are convex sets, and the norms \( \| \cdot \|_{H^2(\Gamma_0)} \), \( \| \cdot \|_{H^1(\Gamma_0 \setminus \{x_3=0\})} \), and \( \| \cdot \|_{H^1(\{x_3=0\})} \) are convex.

(ii) The operator \( F \) is convex, since \( F_1 \) does not depend on the controls, \( U_1 \times U_2 \times U_3 \) is convex set and the operators \( F_2, F_3 \) and \( F_4 \) depend linearly on the controls \( (g, \phi_1, \phi_2) \in U_1 \times U_2 \times U_3 \).
\end{remark}

\begin{lemma}
Let \( \bar{z} = (\bar{x}, \bar{y}, \bar{\phi}_1, \bar{\phi}_2) \in Z_{ad} \) an optimal solution of problem [DMM]. Then, the rang of the linear operator \( F_\bar{x}(\bar{z}) : G \to M \) has finite codimension.
\end{lemma}

\begin{proof}
Notice that for \( (v, W) \in X_0 \times Y \) we have

\[
\langle F_\bar{x}(\bar{z})h, (v, W) \rangle = \begin{pmatrix}
Pr a(h_1, v) + PrMh_1(h_2, v) - PrR(h_2, v_3)_{L^2(\Omega)} \\
\gamma|_{\Gamma_0} h_1 \\
\gamma|_{\{x_3=0\}} h_2
\end{pmatrix}
+ \begin{pmatrix}
c(\tilde{u}, h_1, v) + c(h_1, \tilde{u}, v) \\
c_1(h_2, W) + c_1(h_1, \theta, W)
\end{pmatrix}.
\]

It is not difficult to see that the operator \( T_1 : G \to M \) defined by

\[
T_1(h) = \begin{pmatrix}
Pr a(h_1, v) + PrMh_1(h_2, v) - PrR(h_2, v_3)_{L^2(\Omega)} \\
\gamma|_{\Gamma_0} h_1 \\
\gamma|_{\{x_3=0\}} h_2
\end{pmatrix}
\]
is an isomorphism and the operator \( I_2 : G \to M \)

\[
I_2(h) = \begin{pmatrix}
  c(\hat{u}, h_1, v) + c(h_1, \hat{u}, v) \\
c_1(\hat{u}, h_2, W) + c_1(h_1, \hat{\theta}, W) \\
0 \\
0
\end{pmatrix}
\]

is compact. Thus, \( F_x(\hat{z})h = I_1(h) + I_2(h) \) is a Fredholm operator because it is a sum of an isomorphism and a compact continuous operator. Therefore, \( F_x(\hat{z}) : G \to M \) has finite codimension in \( M \).

The following result guarantee the existence of Lagrange multipliers.

**Theorem 5.5.** Let \( \hat{z} = (\hat{x}, \hat{y}, \phi_1, \phi_2) \in Z_{ad} \) an optimal solution to the problem \( \text{(5.3)} \). Then, there exists a non zero Lagrange multiplier \( \eta := (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{R}^5 \times \{0\} \times X_0 \times Y \times (H^2(\Gamma_0))' \times H^{-\frac{1}{2}}(\{x_3 = 0\}) \) such that the Fréchet derivative of the Lagrange functional \( L \) with respect to \( x \) satisfies

\[
L_x(\hat{x}, \eta) h = \lambda_0 F_x(\hat{z})h - \langle F_{x1}(\hat{z})h, \lambda_1 \rangle x_0 - \langle F_{x2}(\hat{z})h, \lambda_2 \rangle y_0 - \langle \lambda_3, F_{x3}(\hat{z})h \rangle (\bar{H}^1(\Gamma_0), \bar{H}^2(\Gamma_0)) = 0, \quad (5.10)
\]

for all \( h \in G \). Moreover,

\[
\mathcal{L}(\hat{x}, \hat{y}, \phi_1, \phi_2, \eta) = \min_{(\hat{g}, \phi_1, \phi_2) \in U_1 \times U_2 \times U_3} \mathcal{L}(\hat{x}, \hat{y}, \phi_1, \phi_2, \eta). \quad (5.11)
\]

**Proof:** The proof follows from Lemma \([5.1]\) Lemma \([5.2]\) Remark \([5.3]\) Lemma \([5.4]\) and Theorem 3 in \([19]\).

5.2 Optimality system

From \((5.10)\) we can derive a system which is fulfilled by \( \lambda_0, \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \). Using \((5.5)\) and \((5.6)\), we rewrite \((5.10)\) as:

\[
\lambda_0 \gamma_1 (\hat{u}, \hat{h}_1)_{L^2(\Omega)} + \lambda_0 \gamma_2 \int_\Omega |\hat{u} - u_d|^{p-1} \text{sgn}(\hat{u} - u_d) \cdot \hat{h}_1 \, d\Omega + \gamma_3 \int_\Omega |\hat{\theta} - \theta_d|^{p-1} \text{sgn}(\hat{\theta} - \theta_d) \cdot h_2 \, d\Omega
\]

\[
-Pr a(h_1, 1) - Pr M b_1(h_2, 1) - c(h_1, \hat{u}, \hat{h}_1) - c(h_1, \hat{\theta}, \hat{h}_1) + Pr R(h_2, \lambda_1)_{L^2(\Omega)} - c_1(\hat{u}, h_1, 2)
\]

\[
-c_1(h_1, \hat{\theta}, \lambda_2) - a_1(h_2, 2) - \langle B h_2, \lambda_2 \rangle_{\Gamma_1} - \langle \lambda_4, \gamma |_{x_3 = 0} h_1 \rangle (\bar{H}^2(\Gamma_0), \bar{H}^2(\Gamma_0)) = 0, \quad (5.12)
\]

Putting in \((5.12)\) first \( h_2 = 0 \) and then \( h_1 = 0 \) we obtain

\[
\lambda_0 \gamma_1 (\hat{u}, \hat{h}_1)_{L^2(\Omega)} + \lambda_0 \gamma_2 \int_\Omega |\hat{u} - u_d|^{p-1} \text{sgn}(\hat{u} - u_d) \cdot \hat{h}_1 \, d\Omega - Pr a(h_1, 1) - c(h_1, \hat{u}, \hat{h}_1)
\]

\[
- c(h_1, \hat{u}, \lambda_1) - c_1(h_1, \hat{\theta}, \lambda_2) - \langle \lambda_4, \gamma |_{x_3 = 0} h_1 \rangle (\bar{H}^2(\Gamma_0), \bar{H}^2(\Gamma_0)) = 0, \quad (5.13)
\]

\[
\lambda_0 \gamma_3 \int_\Omega |\hat{\theta} - \theta_d|^{p-1} \text{sgn}(\hat{\theta} - \theta_d) \cdot h_2 \, d\Omega - Pr M b_1(h_2, 1) + Pr R(h_2, \lambda_1)_{L^2(\Omega)} - c_1(\hat{u}, h_2, 2)
\]

\[
-a_1(h_2, 2) - \langle B h_2, \lambda_2 \rangle_{\Gamma_1} - \langle \lambda_4, \gamma |_{x_3 = 0} h_2 \rangle_{H^{-\frac{1}{2}}(\{x_3 = 0\}), H^{\frac{1}{2}}(\{x_3 = 0\})} = 0, \quad (5.14)
\]
On the other hand, from the minimum principle (5.11), the definition of $L$ given in (5.2) and recalling that the controls $g$, $\phi_1$ and $\phi_2$ act explicitly in the expressions for $F_2$, $F_3$ and $F_4$, we get

\[
\langle F_2(\hat{x}, \hat{g}, \phi_1, \phi_2) - F_2(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2), \lambda_2 \rangle_{Y^*Y} + \langle \lambda_3, F_3(\hat{x}, \hat{g}, \phi_1, \phi_2) - F_3(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \rangle_{\hat{H}^\perp(\Gamma_0) : \hat{H}^\perp(\Gamma_0)} \\
+ \langle \lambda_4, F_4(\hat{x}, \hat{g}, \phi_1, \phi_2) - F_4(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \rangle_{H^{-1/2}(x_3 = 0), H^{1/2}(x_3 = 0)} \leq \lambda_0 \{ \mathcal{J}(\hat{x}, \hat{g}, \phi_1, \phi_2) - \mathcal{J}(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \}, \quad \forall (g, \phi_1, \phi_2) \in U_1 \times U_2 \times U_3.
\]  

(5.15)

Considering the definitions of the operators $F_2$, $F_3$ and $F_4$ given in (5.22) and the definition of $\mathcal{J}$ in (2.9), from (5.15) we obtain

\[
\frac{\lambda_0 \gamma_4}{2} (\|g - \hat{g}\|^2_{\hat{H}^\perp(\Gamma_0)}) + \frac{\lambda_0 \gamma_5}{2} (\|\phi_1 - \hat{\phi}_1\|^2_{\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})}) + \frac{\lambda_0 \gamma_6}{2} (\|\phi_2 - \hat{\phi}_2\|^2_{\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})}) \\
+ \lambda_3, g - \hat{g}, \hat{\phi}_1 - \hat{\phi}_1, \hat{\phi}_2 - \hat{\phi}_2, \lambda_2, \lambda_4 \in (\Gamma_0 \setminus \{x_3 = 0\}), H^{1/2}(\{x_3 = 0\}) \geq 0,
\]  

for all $(g, \phi_1, \phi_2) \in U_1 \times U_2 \times U_3$, and using the equality $2(\hat{v} - v, \hat{v})_{H^{1/2}(\partial\Omega)} = \|\hat{v}\|_{H^{1/2}(\partial\Omega)}^2 - \|v\|^2_{H^{1/2}(\partial\Omega)} + \|\hat{v} - v\|_{H^{1/2}(\partial\Omega)}^2$, we deduce that

\[
\frac{\lambda_0 \gamma_4}{2} (\|g - \hat{g}\|^2_{\hat{H}^\perp(\Gamma_0)}) + \frac{2(\|g - \hat{g}\|^2_{\hat{H}^\perp(\Gamma_0)})}{\lambda_0 \gamma_5} (\|\phi_1 - \hat{\phi}_1\|^2_{\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})}) + \frac{2(\|\phi_2 - \hat{\phi}_2\|^2_{\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})})}{\lambda_0 \gamma_6} (\|\phi_2 - \hat{\phi}_2\|^2_{\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})}) \\
+ \lambda_3, g - \hat{g}, \hat{\phi}_1 - \hat{\phi}_1, \hat{\phi}_2 - \hat{\phi}_2, \lambda_2, \lambda_4 \in (\Gamma_0 \setminus \{x_3 = 0\}), H^{1/2}(\{x_3 = 0\}) \geq 0,
\]  

where $(g - \hat{g}, \hat{\phi}_1 - \hat{\phi}_1, \hat{\phi}_2 - \hat{\phi}_2)_{H^{1/2}(\{x_3 = 0\})}$ and $(\phi_2 - \hat{\phi}_2)_{H^{1/2}(\{x_3 = 0\})}$ represent the inner product in the spaces $\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})$ and $H^{1/2}(\{x_3 = 0\})$ respectively. Thus, the optimality conditions are given by

\[
\langle \lambda_3 + \lambda_0 \gamma_4 g - \hat{g}, \hat{\phi}_1 - \hat{\phi}_1, \hat{\phi}_2 - \hat{\phi}_2 \rangle_{\hat{H}^\perp(\Gamma_0) : \hat{H}^\perp(\Gamma_0)} \geq 0, \quad \forall g \in U_1,
\]  

(5.17)

\[
\langle \phi_1 - \hat{\phi}_1, \lambda_2, \Gamma_0 \setminus \{x_3 = 0\} \rangle \geq 0, \quad \langle \lambda_0 \gamma_5 \phi_1 - \hat{\phi}_1, \hat{\phi}_2 - \hat{\phi}_2 \rangle_{\hat{H}^\perp(\Gamma_0 \setminus \{x_3 = 0\})} \geq 0, \quad \forall \phi_1 \in U_2,
\]  

(5.18)

\[
\langle \lambda_4, \lambda_0 \gamma_6 \phi_2, \phi_2 - \hat{\phi}_2 \rangle_{H^{-1/2}(\{x_3 = 0\})} \geq 0, \quad \forall \phi_2 \in U_3.
\]  

(5.19)

The optimality system is given by the state equations (5.3) - (5.5), together with the relations (5.13) - (5.14) and the optimality conditions (5.17) - (5.19).

**Remark 5.6.** Denoting by $\tilde{\eta} := \langle \lambda_1, \lambda_2, \lambda_3, \lambda_4 \rangle$, according to the Theorem 5.2 observe that in the Lagrange multiplier $(\lambda_0, \tilde{\eta})$, we can distinguish two cases: $\lambda_0 > 0$ and $\lambda_0 = 0$. In the first case, replacing $\tilde{\eta}$ por $\frac{1}{\lambda_0} \tilde{\eta}$ in (5.10) and (5.11), we can assume that $\lambda_0 = 1$. In the second case, if $\lambda_0 = 0$ the equation (5.10) takes the form

\[
L(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) = -\langle F_{1*}(\hat{z}) h, \lambda_1, \hat{x}_3, \hat{x}_3 \rangle - \langle F_{2*}(\hat{z}) h, \lambda_2 \rangle_{Y^*Y} \\
- \langle \lambda_3, F_{3*}(\hat{z}) h \rangle_{\hat{H}^\perp(\Gamma_0) : \hat{H}^\perp(\Gamma_0)} - \langle \lambda_4, F_{4*}(\hat{z}) h \rangle_{H^{-1/2}(\{x_3 = 0\})} = 0,
\]  

(5.20)

for all $h \in G$, which does not provide interesting information about a local minimum for the problem (5.9), because the term $\lambda_0 F_{4*}(\hat{z}) h$ in (5.10) disappears. Therefore, it is interesting to find out conditions on $(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2)$ as well as on $U_1 \times U_2 \times U_3$ under which we have $\lambda_0 > 0$ by necessity.
Similarly as in [1] we say the sets $U_1, U_2$ and $U_3$ possesses Property $C$ at a point $z^* = (u^*, \theta^*, g^*, \phi_1^*, \phi_2^*)$, if for every nonzero solution $\hat{z} \in X_0 \times Y \times (H^2(\Gamma_0))' \times H^{-\frac{1}{2}}(\{x_3=0\})$ of the equation (5.20) there are elements $g^d \in U_1, \phi^d \in U_2$ and $\phi_2^0 \in U_4$ such that

$$\langle \phi_1 - \phi_1^0, \lambda_2 \rangle_{\Gamma_0 \setminus \{x_3=0\}} > 0, \quad \langle \lambda_3, \hat{g} - g^d \rangle_{\Gamma_0} > 0 \quad \text{and} \quad \langle \lambda_4, \hat{\phi}_2 - \phi_2^0 \rangle_{\{x_3=0\}} > 0. \quad (5.21)$$

We easily see that if the sets $U_1, U_2$ and $U_3$ possesses Property $C$ in $(\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2)$ then $\lambda_0 > 0$ by necessity. Indeed, if we suppose to the contrary that there is a Lagrange multiplier $(0, \hat{\eta})$ then, by Theorem 5.26 the equation (5.20) and the inequality (5.16) hold for $\lambda_0 = 0$, which contradicts (5.21).

### 5.3 Derivation of differential equations and boundary conditions for Lagrange multipliers

We introduce the continuous linear operators $\hat{A} \equiv A(\hat{u}) : \hat{X} \rightarrow X'_0, \hat{A}_1 \equiv A_1(\hat{u}) : H^1(\Omega) \rightarrow Y', \hat{B} : H^1(\Omega) \rightarrow X'_0, \hat{B}_1 : H^1(\Omega) \rightarrow X'_0, \hat{C} \equiv C(\hat{u}) : \hat{X} \rightarrow Y$ defined by:

$$\begin{align*}
\langle \hat{A}_1 h_1, \lambda_1 \rangle &= Pr a(h_1, \lambda_1) + c(\hat{u}, h_1, \lambda_1), \quad \forall h_1 \in \hat{X}, \lambda_1 \in X_0, \\
\langle \hat{A}_2 h_2, \lambda_2 \rangle &= \langle \hat{u}, h_2 \lambda_2 \rangle + a_1(h_2, \lambda_2) + \langle B h_2, \lambda_2 \rangle_{\Gamma_1}, \quad \forall h_2 \in H^1(\Omega), \lambda_2 \in Y, \\
\langle \hat{B}_1 h_1, \lambda_1 \rangle &= \langle Pr R (h_1, \lambda_1), \lambda_1 \rangle_{\Gamma_2(\Omega)}, \quad \forall h_1 \in H^1(\Omega), \lambda_1 \in X_0, \\
\langle \hat{B}_2 h_2, \lambda_2 \rangle &= \langle Pr M b(h_2, \lambda_2), \lambda_2 \rangle_{\Gamma_2(\Omega)}, \quad \forall h_2 \in H^1(\Omega), \lambda_2 \in X_0, \\
\langle \hat{C}_1 h_1, \lambda_2 \rangle &= \langle \hat{u} h_1, \lambda_1 \rangle, \quad \forall h_1 \in \hat{X}, \lambda_1 \in X_0, \\
\langle \hat{C}_2 h_2, \lambda_2 \rangle &= \langle \hat{u} h_2, \lambda_2 \rangle, \quad \forall h_2 \in \hat{X}, \lambda_2 \in Y.
\end{align*}
$$

Using (5.22), we rewrite (5.13) and (5.14) as

$$\begin{align*}
\langle \lambda_0 \mathcal{J}_u(z) - \hat{A}_1^* \lambda_1 - \hat{C}_1^* \lambda_2, h_1 \rangle - \langle \lambda_3, \gamma \rangle_{\Gamma_0} h_1 &= 0, \quad \forall h_1 \in \hat{X}, \\
\langle \lambda_0 \mathcal{J}_\theta(z) - \hat{B}_1^* \lambda_1 + \hat{B}_1^* \lambda_2, h_2 \rangle - \langle \lambda_4, \gamma \rangle_{\{x_3=0\}} h_2 &= 0, \quad \forall h_2 \in H^1(\Omega).
\end{align*}
$$

Here $\hat{A}^*, \hat{A}_1^*, \hat{B}^*, \hat{B}_1^*, \hat{C}^*$ and $\hat{C}_1^*$ are the adjoints of the operators (5.22). Moreover, we introduce the linear operators $S_H : (\hat{X})' \rightarrow H^{-\frac{1}{2}}(\Omega)$ and $S_I : (H^1(\Omega))' \rightarrow H^{-\frac{1}{2}}(\Omega)$ defined by:

$$\begin{align*}
\langle S_H l, h_1 \rangle_{H^{-\frac{1}{2}}(\Omega)} &= \langle l, h_1 \rangle_{(\hat{X})'}, \\ &\quad \forall h_1 \in H^1(\Omega) \subset \hat{X}, \lambda \in (\hat{X})', \\
\langle S_I l, h_2 \rangle_{H^{-\frac{1}{2}}(\Omega)} &= \langle l, h_2 \rangle_{(H^1(\Omega))'}, \\ &\quad \forall h_2 \in H^1(\Omega) \subset H^1(\Omega), l \in (H^1(\Omega))'.
\end{align*}
$$

Using the Green formula and considering the restrictions of the functionals on the left-hand side of (5.23) and (5.24) to $H^1(\Omega)$ and $H^1(\Omega)$ respectively, we easily find that

$$\begin{align*}
Pr \Delta \lambda_1 + (\hat{u} \cdot \nabla) \lambda_1 - (\nabla \hat{u})^T \cdot \lambda_1 - \lambda_2 \nabla \hat{\theta} &= \lambda_0 \gamma_1 \text{rot}(\text{rot} \hat{u}) - \lambda_0 \gamma_2 (\hat{u} - u_d)^{p-1} \text{sgn}(\hat{u} - u_d) \text{ en } H^{-1}(\Omega), \\
\hat{u} \lambda_2 + \Delta \lambda_2 + Pr M \text{div} \left( \frac{\partial \lambda_1}{\partial x_3} \right) + Pr R \lambda_{13} &= -\lambda_0 \gamma_1 \hat{\theta} - \hat{\theta}_d |^{p-1} \text{sgn}(\hat{\theta} - \theta_d) \text{ en } H^{-1}(\Omega).
\end{align*}
$$

Thus, we have proven the following theorem:

**Theorem 5.7.** Let $\hat{z} = (\hat{x}, \hat{g}, \hat{\phi}_1, \hat{\phi}_2) \in Z_{ad}$ an optimal solution to the problem (5.20). Then there exist functions (Lagrange multipliers) $\lambda_0 \in \mathbb{R}^+ \cup \{0\}, \lambda_1 \in X_0, \lambda_2 \in Y, \lambda_3 \in (H^2(\Gamma_0))'$, and $\lambda_4 \in H^{-\frac{1}{2}}(\{x_3=0\})$ which, together with the element $\hat{z}$, satisfy the equations (5.27) and (5.28), the integral identities (5.13) and (5.14), and the optimality conditions (5.17) and (5.19).
Additionally, if we suppose that \( \hat{\mathbf{u}}, \lambda_1 \in H^2(\Omega) \) and \( \lambda_2 \in H^2(\Omega) \), then from (5.13), (5.14), (5.27) and (5.28) we can obtain the “pointwise” boundary relations for the Lagrange multipliers. Indeed, under the above conditions, we can multiply (5.27) and (5.28) by functions \( h_1 \in \tilde{X} \) and \( h_2 \in H^1(\Omega) \) respectively, and integrate the result over \( \Omega \). Applying the Green formula and subtracting the resultant relations from (5.13) and (5.14), we obtain

\[
\langle \lambda_3, \gamma \rangle_{\Gamma_0} + Pr \int_{\partial \Omega} \frac{\partial \lambda_1}{\partial \mathbf{n}} \cdot \mathbf{h}_1 \, dS = \lambda_0 \gamma_1 (\mathbf{rot} \hat{\mathbf{u}} \times \mathbf{n}) \cdot \mathbf{h}_1 \, dS, \quad \forall \mathbf{h}_1 \in \tilde{X},
\]

(5.29)

and

\[
(B \lambda_2, \gamma)_{\Gamma_1} + \langle \lambda_4, \gamma \rangle_{\{x_3=0\}} h_2 \mathbf{H}^{\mathbf{u}}(x_3=0) + Pr \int_{\partial \Omega} \frac{\partial \lambda_2}{\partial \mathbf{n}} : h_2 \, dS + Pr M \int_{\partial \Omega} \left( \frac{\partial \lambda_1}{\partial x_3} \right) \cdot h_2 \, dS = 0, \quad \forall \mathbf{h}_2 \in H^1(\Omega).
\]

(5.30)

Therefore, from (5.29) and (5.30) we arrive at the following boundary relations:

\[
\lambda_3 = \lambda_0 \gamma_1 (\mathbf{rot} \hat{\mathbf{u}} \times \mathbf{n}) - Pr (\nabla \lambda_1 \cdot \mathbf{n}) \quad \text{on } \Gamma_0,
\]

(5.31)

\[
\lambda_0 \gamma_1 (\mathbf{rot} \hat{\mathbf{u}} \times \mathbf{n}) - Pr (\nabla \lambda_1 \cdot \mathbf{n}) = 0 \quad \text{on } \Gamma_1,
\]

(5.32)

\[
\lambda_4 = - \left( \nabla \lambda_2 + Pr M \nabla \left( \frac{\partial \lambda_1}{\partial x_3} \right) \right) \cdot \mathbf{n} \quad \text{on } \{x_3=0\},
\]

(5.33)

\[
B \lambda_2 + \left( \nabla \lambda_2 + Pr M \nabla \left( \frac{\partial \lambda_1}{\partial x_3} \right) \right) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_1,
\]

(5.34)

\[
\left( \nabla \lambda_2 + Pr M \nabla \left( \frac{\partial \lambda_1}{\partial x_3} \right) \right) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_0 \setminus \{x_3=0\}.
\]

(5.35)

Thus, if \( \hat{\mathbf{u}}, \lambda_1 \in H^2(\Omega) \) and \( \lambda_2 \in H^2(\Omega) \), we conclude that the set of Lagrange multipliers \( \lambda_0, \lambda_1, \lambda_2, \lambda_3 \) and \( \lambda_4 \) satisfies the equations (5.27) and (5.28), the boundary relations (5.31) - (5.35) and the optimality conditions (5.17) - (5.19).

**Remark 5.8.** The conditions \( \hat{\mathbf{u}}, \lambda_1 \in H^2(\Omega) \) and \( \lambda_2 \in H^2(\Omega) \) can be relaxed. In fact, it is enough to observe that the forms in (5.29) and (5.30) have meaning.

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