Global passive system approximation

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Abstract

In this paper we present a new approach towards global passive approximation in order to find a passive transfer function \( G(s) \) that is nearest in some well-defined matrix norm sense to a non-passive transfer function \( H(s) \). It is based on existing solutions to pertinent matrix nearness problems. It is shown that the key point in constructing the nearest passive transfer function, is to find a good rational approximation of the well-known ramp function over an interval defined by the minimum and maximum dissipation of \( H(s) \). The proposed algorithms rely on the stable anti-stable projection of a given transfer function. Pertinent examples are given to show the scope and accuracy of the proposed algorithms.

Key words: Passivity, positive-real lemma, rational approximation

1. INTRODUCTION

For linear time-invariant systems, passivity guarantees stability and the possibility of synthesis of a transfer function by means of a lossy physical network of resistors, capacitors, inductors and transformers [1]. Therefore, passivity enforcement [2] and passification (passivation) [3] have become important issues in recent years [4–8], especially as more and more software tools render transfer functions which need passivity enforcement as a postprocessing step in order to generate reliable physical models. However, most of the techniques [2–7] are local perturbative and/or feedback approaches with fixed poles, while [8] is based on Fourier approximation, yielding passivated systems with a large number of poles. In this paper we present a new global approach in the sense that we find a passive transfer function \( G(s) \) that is nearest in a well-defined matrix norm sense to a non-passive transfer function \( H(s) \). It is based on existing solutions to some pertinent matrix nearness problems [9, 10]. We show that the key point in constructing the nearest passive transfer function \( G(s) \), is to find a good rational approximation for the ramp function \( \max(0, x) \) over an interval defined by the minimum and maximum dissipation of the non-passive transfer function \( H(s) \). It is also shown that in the Chebyshev or minimax sense this requires finding a rational Chebyshev approximation of the square root \( \sqrt{x} \) over the interval \([0, 1]\). The proposed algorithms rely heavily on the stable anti-stable projection [11, 12] of a given transfer function. Finally, five pertinent examples, both SISO and MIMO, are given to show the accuracy and relevance of the proposed algorithms.

2. PASSIVITY AND DISSIPATION

Notation: Throughout the paper \( X^T \) and \( X^H \) respectively denote the transpose and Hermitian transpose of a matrix \( X \), and \( I_n \) denotes the identity matrix of dimension \( n \). The Frobenius norm is defined as \( \|X\|_F = \sqrt{\text{tr} X^H X} \) and the spectral norm (or 2-norm or maximum singular value) is defined as \( \|X\|_2 = \|

\( \sqrt{\lambda_{\text{max}}(X^HX)} \). It is easy to show that \( \|X^H\|_F = \|X\|_F \) and \( \|X^H\|_2 = \|X\|_2 \). For two Hermitian matrices \( X \) and \( Y \), the matrix inequalities \( X > Y \) or \( X \geq Y \) mean that \( X - Y \) is respectively positive definite or positive semidefinite. The closed right halfplane \( \Re\{s\} \geq 0 \) is denoted \( \mathbb{C}_+ \).

For the real system with minimal realization

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

where \( B \neq 0, C \neq 0 \) are respectively \( n \times p \) and \( p \times n \) real matrices and \( A \neq 0 \) is a \( n \times n \) real matrix, to be passive, it is required that the \( p \times p \) transfer function

\[ H(s) = C(sI_n - A)^{-1}B + D \]

is analytic in \( \mathbb{C}_+ \), such that

\[ H(i\omega) + H(i\omega)^H \geq 0 \quad \forall \omega \in \mathbb{R} \]

It is well-known [13] that the positive-real lemma in linear matrix inequality (LMI) format: \( \exists P^T = P > 0 \) such that

\[
\begin{bmatrix}
A^TP + PA & PB - CT \\
B^TP - C & -D - DT
\end{bmatrix} \leq 0
\]

guarantees the passivity of the system [11]. A necessary, but not sufficient, condition for passivity is that \( A \) is stable, i.e., its eigenvalues are located in the closed left halfplane. In the sequel we will always suppose that \( A \) is Hurwitz stable, i.e., its eigenvalues are located in the open left halfplane. We will also assume, unless otherwise stated, that \( H(s) \) is non-passive, and devise ways of finding another as close as possible passive transfer function \( G(s) \).

In order to measure how far a given system is from passive we define the minimum dissipation \( \delta_-(H) \) [14] as

\[
\delta_-(H) = \min_{\omega \in \mathbb{R}} \lambda_{\text{min}}(R(\omega))
\]

where

\[ R(\omega) = H(i\omega) + H(i\omega)^H \]

Similarly, we also define the maximum dissipation \( \delta_+(H) \) as

\[
\delta_+(H) = \max_{\omega \in \mathbb{R}} \lambda_{\text{max}}(R(\omega))
\]

It is clear that the system is passive if and only if \( \delta_-(H) \geq 0 \). If \( \delta_-(H) < 0 \) the system is non-passive, and if \( \delta_+(H) \leq 0 \), the system is anti-passive, in the sense that then the system with transfer function \(-H(s)\) is passive.

In the sequel we will assume, unless otherwise stated, that the system is non-passive but passifiable, i.e., \(-\infty < \delta_-(H) < 0 < \delta_+(H) < \infty \). To obtain \( \delta_-(H) \) (or similarly \( \delta_+(H) \)), a simple bisection algorithm, based on the existence (or non-existence) of imaginary eigenvalues of the one-parameter Hamiltonian matrix

\[
N_\delta = \begin{bmatrix}
A & 0 \\
0 & -A^T
\end{bmatrix} + \begin{bmatrix}
B & -C^T
\end{bmatrix}(\delta I_p - D - DT)^{-1} \begin{bmatrix}
C & B^T
\end{bmatrix}
\]

was proposed in [14]. We have

**Proposition 2.1.** \( \delta > \delta_-(H) \) if and only if \( N_\delta \) admits purely imaginary eigenvalues.

*Proof.* See [14]. \( \square \)

It is clear that Proposition 2.1 always allows to decide, by checking the eigenvalues of \( N_\delta \), whether \( \delta > \delta_-(H) \) or not. This forms the basis of the bisection algorithm of [14]. The only problem is to start with a so-called bracket, i.e., provable lower and upper bounds for \( \delta_-(H) \). For that purpose we have
Proposition 2.2.

\[-2\|H\|_\infty \leq \delta_-(H) \leq \lambda_{\min}(D + D^T) \leq \lambda_{\max}(D + D^T) \leq \delta_+(H) \leq 2\|H\|_\infty\]  \hspace{1cm} (2)

Proof. Straightforward. Here the infinity norm \(\|H\|_\infty\) is defined as
\[\|H\|_\infty = \max_{\omega \in \mathbb{R}} \|H(i\omega)\|_2\]

Note that we can replace \(\|H\|_\infty\) in (2) by an upper bound such as the one given in [14].

3. MATRIX NEARNESS CONSIDERATIONS

Theorem 3.1. Let \(A = A^H\) be any Hermitian matrix with eigendecomposition \(A = U\Lambda U^H\), with \(U\) a unitary and \(\Lambda\) a real diagonal matrix. Then the positive semidefinite Hermitian matrix nearest to \(A\), both with respect to the Frobenius and spectral norms, is given by \(A_+ = U\max(0, \Lambda)U^H\).

Proof. First we give the proof for the Frobenius norm. We need to find
\[\min_{X \geq 0} \|X - A\|_F\]
Putting \(X = UY U^H\), and exploiting the unitary invariance of the Frobenius norm, we obtain
\[\|X - A\|_F^2 = \|Y - \Lambda\|_F^2 = \sum_{i \neq j} |Y_{ij}|^2 + \sum_i |Y_{ii} - \Lambda_{ii}|^2\]

It is clear that the minimum occurs when \(Y_{ij} = 0\) for \(i \neq j\), in other words when \(Y\) is diagonal. Hence we obtain
\[\|X - A\|_F^2 = \|Y - \Lambda\|_F^2 = \sum_i |Y_{ii} - \Lambda_{ii}|^2\]

It is easy to see that we must take \(Y_{ii} = \max(0, \Lambda_{ii})\) and this completes the proof for the Frobenius norm. Note that
\[\min_{X \geq 0} \|X - A\|_F = \sqrt{\sum_{\lambda_i(A) < 0} \lambda_i(A)^2}\]

For the spectral norm, it is known [9, 10] that
\[\min_{X \geq 0} \|X - A\|_2 = \inf\{r \geq 0 : A + rI \geq 0\}\]

In other words,
\[\min_{X \geq 0} \|X - A\|_2 = \max(0, -\lambda_{\min}(A))\]

Now
\[\|A_+ - A\|_2 = \max_{\lambda_i(A) < 0} |\lambda_i(A)|\]

which is zero when there are no negative eigenvalues, and \(-\lambda_{\min}(A)\) when there are negative eigenvalues. \(\square\)

Remark 3.1. From Theorem 3.1 it is possible to find the point-wise nearest positive semidefinite matrix for the Hermitian matrix \(R(\omega) = H(i\omega) + H(i\omega)^H\). Obviously, if we decompose \(R(\omega)\) as
\[R(\omega) = U(\omega)\Lambda(\omega)U(\omega)^H\]
then the point-wise nearest positive semidefinite matrix is
\[R_+(\omega) = U(\omega)\max(\Lambda(\omega), 0)U(\omega)^H\]
Unfortunately, in general, the entries of \(R_+(\omega)\) will not consist of rational functions and therefore cannot represent the transfer function of an LTI model on the imaginary axis. This problem, which in fact amounts to a rational approximation problem, will be addressed in the sequel.
4. RATIONAL APPROXIMATIONS

**Theorem 4.1.** Let \( H(s) \) be passifiable, i.e., \(-\infty < \delta_-(H) < 0 < \delta_+(H) < \infty\), and let \( R(\omega) = H(i\omega) + H(i\omega)^H \). Let further \( f(x) \) be a real-rational function\(^2\) satisfying
\[
\alpha \geq f(x) - \max(x, 0) \geq 0 \quad \forall x \in [\delta_-(H), \delta_+(H)]
\]
for some finite positive \( \alpha \). Then \( f(R(\omega)) \) is positive semidefinite for all \( \omega \in \mathbb{R} \). Furthermore we have
\[
\| f(R(\omega)) - R_+(\omega) \|_2 \leq \alpha \quad \forall \omega \in \mathbb{R}
\]

**Proof.** We have
\[
f(R(\omega)) - R_+(\omega) = U(\omega) \{ f(\Lambda(\omega)) - \max(\Lambda(\omega), 0) \} U(\omega)^H \geq 0
\]
Since \( R_+(\omega) \) is positive semidefinite, the same holds for \( f(R(\omega)) \). Now, since the spectral norm is unitarily invariant, we have
\[
\| f(R(\omega)) - R_+(\omega) \|_2 \leq \max_{\omega \in \mathbb{R}} | f(\lambda_i(\omega)) - \max(\lambda_i(\omega), 0) |
\]
\[
\leq \max_{\omega \in \mathbb{R}} \max_{i} | f(\lambda_i(\omega)) - \max(\lambda_i(\omega), 0) |
\]
\[
\leq \max_{x \in [\delta_-(H), \delta_+(H)]} \{ f(x) - \max(x, 0) \} \leq \alpha
\]
where the last inequality follows from the fact that all \( \lambda_i(\omega) \) are inside the interval \([\delta_-(H), \delta_+(H)]\). This completes the proof.

**Theorem 4.1** shows that the matrix \( R_+(\omega) \) can be approximated from above by the matrix \( f(R(\omega)) \). The problem is to find a suitable real-rational function \( f(x) \). We have the following:

**Theorem 4.2.** Let \( \zeta_n(x) = x(1 + x)^n / ((1 + x)^n - 1) \). Then
\[
\frac{1}{n} \geq \zeta_n(x) - \max(x, 0) \geq 0 \quad \forall x \geq -1, \quad n = 1, 2, 3, \ldots
\]

**Proof.** First we prove that \( \zeta_n(x) - x \geq 0 \) for \( x \geq 0 \). We have
\[
\zeta_n(x) - x = \left( \frac{(1 + x)^n - 1}{x} \right)^{-1}
\]
which is a positive and decreasing function for \( x \geq 0 \). Next we prove that \( \zeta_n(x) \) is increasing for all \( x \geq -1 \). This is equivalent to proving that \( \zeta_n(t - 1) = (t^{n+1} - t^n) / (t^n - 1) \) is increasing for all \( t \geq 0 \). This is clearly the case for \( n = 1 \). Taking derivatives, we have
\[
\frac{d}{dt} \zeta_n(t - 1) = \left[ n - (n + 1)t + t^{n+1} \right] \frac{t^{n-1}}{(t^n - 1)^2}
\]
Now \( n - (n + 1)t + t^{n+1} = 0 \) when \( t = 0 \) and \( \infty \) when \( t = \infty \). Since the derivative of \( n - (n + 1)t + t^{n+1} \) is \( (n + 1)(t^n - 1) \), the function \( n - (n + 1)t + t^{n+1} \) attains its unique minimum (with value zero) at \( t = 1 \). Hence \( \zeta_n(x) \) is increasing for all \( x \geq -1 \). We therefore conclude that \( \zeta_n(x) - \max(x, 0) \) increases from 0 to 1 in the interval \([-1, 1]\), and decreases from 1 to 0 in the interval \([0, \infty]\), which completes the proof.

**Corollary 4.1.** Let \( H(s) \) be passifiable. Then the real-rational function \( f(x) = \nu \zeta_n(x/\nu) \) with \( \nu = |\delta_-(H)| \) satisfies the premises of Theorem 4.1 with \( \alpha = \nu / n \).

\(^2\) A real-rational function \( f(x) \) is a rational function assuming only real values for all real \( x \).
Proof. Straightforward. □

Also, we need to find ways and means to define the matrix \( f(R(\omega)) = f(H(i\omega) + H(i\omega)^H) \) in the whole \( s \)-plane and then to extract a Hurwitz stable transfer function from it. By analytical continuation, we find the transfer function \( V(s) = f(H(s) + H(-s)^T) \) in the entire \( s \)-plane. Since \( f(x) \) is real-rational, the transfer function \( V(s) \) represents the realization of a per-symmetric LTI model, i.e., satisfying \( V(s) = V(-s)^T \). This implies that the poles of \( V(s) \) admit the imaginary axis as symmetry axis. The following proposition indicates how, starting from a per-symmetric LTI model \( V(s) \) we can find a Hurwitz stable transfer function by additive decomposition \[ 11 \, 12 \].

**Proposition 4.1.** Let \( V(s) \) be per-symmetric, i.e., \( V(s) = V(-s)^T \), such that \( V(s) \) has no poles on the imaginary axis. Then \( V(s) \) can be decomposed as \( V(s) = X(s) + X(-s)^T \), with \( X(s) \) is Hurwitz stable.

**Proof.** Putting \( V(s) = V_0(s) + D \), where \( V_0(s) \) is strictly proper and \( D = D^T = V(\infty) \), we can decompose \( V_0(s) \) uniquely into its stable and anti-stable parts, i.e.,

\[
V_0(s) = X_{\text{stab}}(s) + X_{\text{anti}}(s)
\]

Since \( V_0(s) \) is per-symmetric we have

\[
X_{\text{stab}}(s) + X_{\text{anti}}(s) = X_{\text{stab}}(-s)^T + X_{\text{anti}}(-s)^T
\]

and hence \( X_{\text{anti}}(s) = X_{\text{stab}}(-s)^T \). It follows that \( V(s) \) can be decomposed as \( V(s) = X(s) + X(-s)^T \), with \( X(s) = X_{\text{stab}}(s) + \frac{1}{2}D + E \), where \( E \) is an arbitrary skew-symmetric matrix. It should be noted that the procedure is unique when the skew-symmetric matrix \( E \) is known a priori.

**Remark 4.1.** Proposition \[ 4.2 \] assumes that \( V(s) \), in our case \( V(s) = f(H(s) + H(-s)^T) \), does not admit poles on the imaginary axis. By the inequality constraints \[ 3 \] we know that

\[
\alpha \geq f(\lambda_i(\omega)) - \max(\lambda_i(\omega), 0) \geq 0
\]

for all eigenvalues \( \lambda_i(\omega) \) of \( R(\omega) \). Since \( H(s) \) is assumed Hurwitz stable, \( R(\omega) = H(i\omega) + H(i\omega)^H \) cannot admit real poles, and hence, by the inequalities \[ 4 \], the functions \( f(\lambda_i(\omega)) \) are bounded. It follows that all entries of \( V(i\omega) = f(R(\omega)) \) are bounded, which implies that \( V(s) \) cannot have poles on the imaginary axis.

In the sequel we will use the Matlab® Robust Control Toolbox \[ 15 \] routine \texttt{stabproj} based on the stable, anti-stable decomposition algorithm \[ 12 \].

5. TWO ALGORITHMS

By Theorem \[ 4.1 \] and Corollary \[ 4.1 \] we need to find an LTI model with transfer function \( \phi_n \left( H(s) + H^T(-s) \right) \) where the real-rational function \( \phi_n(x) \) of denominator degree \( n \) and numerator degree \( n + 1 \) is

\[
\phi_n(x) = \nu \zeta_n(x/\nu) = \frac{x(1 + x/\nu)^n}{(1 + x/\nu)^n - 1}
\]

where \( \nu = |\delta_-(H)| \). Now it is easy to show that the following recurrence relationship holds:

\[
\phi_{2n}(x) = \frac{\phi_n(x)^2}{2\phi_n(x) - x} \quad n = 1, 2, 3, \ldots
\]

with \( \phi_1(x) = x + \nu \).

A first algorithm (Algorithm 1) that comes readily to the mind with \( Z(s) = H(s) + H^T(-s) \) is:

Initial value:

\[
Z_0(s) = Z(s) + \nu I_p
\]

\[
5
\]
The number of poles of the transfer function $H$ is well known \cite{16}, that when a transfer function $H(s)$ has the same number of poles as $\tilde{H}(s)$, $H(s)$ and $\tilde{H}(s)$ are expressed in the partial fractions as

$$H(s) = \frac{\alpha_0}{s - \beta_0} + \sum_{k=1}^{M} \frac{\alpha_k}{s - \beta_k}$$

Now if the original Hurwitz stable transfer function $H(s)$ has $N$ poles, then the transfer function $Z(s) = H(s) + H(-s)^T$ has $2N$ poles. Also, $f(Z(s))$ can be written as

$$f(Z(s)) = \alpha_0 I_p + \beta_0 Z(s) + \sum_{k=1}^{M} \alpha_k (Z(s) - \beta_k I_p)^{-1}$$

Hence, the set of poles of $f(Z(s))$ is at most the union of the sets of poles of $Z(s)$ and $(Z(s) - \beta_k I_p)^{-1}$. It is well known \cite{10}, that when a transfer function $H(s)$ is such that $H(\infty)$ is invertible, then $H(s)^{-1}$ exists and has the same number of poles as $H(s)$. Therefore, the number of poles of $f(Z(s))$, not considering potential cancellations, is $2N(M + 1)$. Finally, after the stable anti-stable decomposition, this number is to be divided by two, to yield $N(M + 1)$ poles for the final passivated transfer function $G(s)$. Of course the number $N(M + 1)$ is only an estimate, since pole-zero cancellations can occur. If for some reason, the number of poles of the explicitly proved passive transfer function $G(s)$ appears to be unacceptable high, a final judiciously chosen passivity preserving passive model order reduction step \cite{17, 20} can be applied.

Hence, in order to find a workable algorithm, we have to find the partial fractions decomposition of $f(x) = \phi_n(x) = \nu \zeta_n(x/\nu)$. If we restrict ourselves to even $n = 2m \geq 2$, we have the partial fraction expansion

$$\zeta_m(x) = x + \frac{1}{m} \left( \frac{1}{x + 2} + \Re \sum_{k=1}^{m-1} \frac{e^{2\pi ik/m} - e^{\pi ik/m}}{x + 1 - e^{\pi ik/m}} \right)$$
Hence, with \( f(x) = \nu \zeta_{2m}(x/\nu) \) we have

\[
f(Z(s)) = Z(s) + \nu^2 \frac{1}{m} \left( [Z(s) + 2\nu I_p]^{-1} + \Re \sum_{k=1}^{m-1} \left( e^{2\pi ik/m} - e^{-\pi ik/m} \right) [Z(s) + \nu(1 - e^{-\pi ik/m})I_p]^{-1} \right)
\]  

Algorithm 2 performs the state space addition as is, i.e., we add the realizations of \( Z(s) \), \( (\nu^2/m)[Z(s) + 2\nu I_p]^{-1} \), etc., to obtain \( f(Z(s)) \). The explicit state space form for the terms

\[
\Re \left[ \left( e^{2\pi ik/m} - e^{-\pi ik/m} \right) [Z(s) + \nu(1 - e^{-\pi ik/m})I_p]^{-1} \right]
\]

in formula (5) is obtained by the state space technique described in the Appendix. Finally, the stable anti-stable projection yields the passivated transfer function \( G(s) \).

5.1. Numerical Examples

We will consider only reciprocal non-passive systems, i.e., systems with \( H(s) = H(s)^T \), as these systems are representative of LTI systems satisfying the electromagnetic condition known as Lorentz reciprocity [22]. Of course the theory also remains valid for non-reciprocal LTI systems. Since for reciprocal systems \( R(\omega) \) is real and even, this explains why the plots in the sequel only show values for non-negative frequencies.

5.1.1. First example
As a first example we take the SISO Hurwitz stable non-passive transfer function

\[
H(s) = \frac{s^5 + 7.2s^4 + 47.01s^3 + 230.8s^2 + 536.6s + 587.1}{s^5 + 3.2s^4 + 32.61s^3 + 43.63s^2 + 117.5s + 104.3}
\]

We use the approach of Algorithm 1 with \( n_1 = 2 \). The passivated approximation \( G(s) \) has a non-minimal realization with 65 poles which are reduced to 20 by the routine \texttt{minreal} [16]. The real and imaginary parts of the original transfer function \( H(s) \) vs. the passivated transfer function \( G(s) \) are shown in Figs 1 and 2.

5.1.2. Second example
As a second example we take the SISO Hurwitz stable minimum phase non-passive transfer function

\[
H(s) = \frac{(s + 1)(s + 3)(s + 90)(s + 95)(s + 100)}{(s + 25)(s + 35)(s + 38)(s + 180)(s + 185)}
\]

Figure 1: Real part of \( G(s) \) vs. \( H(s) \)
We use the approach of Algorithm 2 with $m = 5$. The passivated approximation $G(s)$ has a realization with 50 poles. The real and imaginary parts of the original transfer function $H(s)$ vs. the passivated transfer function $G(s)$ are shown in Figs. 2 and 3.

As a third example we take the $2 \times 2$ MIMO Hurwitz stable non-passive transfer function

$$
H(s) = \begin{bmatrix}
2 + \frac{12}{s^2 + 3s + 2} & -\frac{2s + 10}{s + 6} \\
-\frac{2s + 10}{s + 6} & 2 - \frac{s + 3}{s^2 + 3s + 2}
\end{bmatrix}
$$

(8)

We use the approach of Algorithm 2 with $m = 4$. The passivated approximation $G(s)$ has a realization with 48 poles. Fig. 5 plots the values of $\lambda_{\min}(G(i\omega) + G(i\omega)^H)$ vs. $\lambda_{\min}(H(i\omega) + H(i\omega)^H)$. To show the nearness of the original and passivated transfer functions $H(s)$ and $G(s)$, we plot the relative error $\|G(i\omega) - H(i\omega)\|_2 / \|H(i\omega)\|_2$ in Fig. 6.

5.1.3. Third example

As a third example we take the $2 \times 2$ MIMO Hurwitz stable non-passive transfer function

$$
H(s) = \begin{bmatrix}
2 + \frac{12}{s^2 + 3s + 2} & -\frac{2s + 10}{s + 6} \\
-\frac{2s + 10}{s + 6} & 2 - \frac{s + 3}{s^2 + 3s + 2}
\end{bmatrix}
$$

(8)
6. MINIMAX ALGORITHM

The starting point for finding a passive approximant is to find a real-rational function \( f(x) \) that satisfies
\[
\alpha \geq f(x) - \max(x, 0) \geq 0 \quad \forall x \in [-a, b] \tag{9}
\]
where \( a = -\delta_-(H) = |\delta_-(H)| \) and \( b = \delta_+(H) \). Since \( \max(x, 0) = (|x| + x)/2 \), this can be written as
\[
\alpha \geq 2f(x) - x - |x| - \alpha \geq -\alpha \quad \forall x \in [-a, b] \tag{10}
\]
Putting \( r(x) = 2f(x) - x - \alpha \), and since our aim is to find the smallest positive \( \alpha \) such that (10) is satisfied, it is seen that we must find the rational minimax or Chebyshev approximant, i.e.,
\[
\min_{r} \max_{x \in [-a, b]} |r(x) - |x||
\]
Let us first treat the case \( a = b = 1 \), which is well-documented in the literature [23, 22]. Since \(|x|\) is even and the interval \([-1, 1]\) is symmetric with respect to 0, it is clear that \( r(x) \) must be an even rational function, i.e.,
r(x) = \rho(x^2). If we take \rho(t) irreducible with numerator and denominator of exact degree n, the minimax problem can be reformulated as:

\[
\min_{\rho} \max_{0 \leq t \leq 1} |\rho(t) - \sqrt{t}|
\]

Calling \(E_n\) the value obtained by the minimax problem (11), it is clear that at the minimum we must have

\[
E_n \geq \rho(t) - \sqrt{t} \geq -E_n \quad \text{for} \quad 0 \leq t \leq 1
\]  

Furthermore, the Remes condition \[25, 26\] requires that there are exactly \(2n + 2\) point \(t_k\) inside \([0, 1]\) where the equality

\[
\sqrt{t_k} - \rho(t_k) = (-1)^k E_n \quad k = 1, 2, \ldots, 2n + 2
\]

is satisfied. This allows an iterative approach \[25\] to find the optimal \(E_n\) and \(\rho(t)\). The poles and zeros of \(\rho(t)\) are all simple and intertwined on the negative real axis \[27\]. It follows that in general \(\rho(t)\) can be written as

\[
\rho(t) = a_0 - \sum_{k=1}^{n} \frac{a_k}{t + b_k}
\]

where all \(a_k, b_k\) are positive. For \(n = 4\) the coefficients \(a_k, b_k\) with \(b_0 = E_n\) are given in Table 1

| k  | 0     | 1     | 2     | 3     | 4     |
|----|-------|-------|-------|-------|-------|
| a_k| 2.6397296257 | 1.4034219887 \times 10^{-6} | 0.0003730797 | 0.0290141901 | 5.6266532592 |
| b_k| 0.0007365636 | 0.0000917473 | 0.0049831021 | 0.1014048457 | 2.4866930733 |

shows the approximation error \(\rho(t) - \sqrt{t}\) and the equioscillation property. Note that the asymptotic formula of \(E_n\) is known \[28\], i.e., we have \(E_n \approx 8e^{-\pi \sqrt{2n}}\) for \(n \to \infty\).

Formula (12) implies

\[
2E_n \geq \rho(x^2) + x + E_n - |x| \geq 0 \quad \text{for} \quad -1 \leq x \leq 1
\]

or

\[
E_n \geq \frac{\rho(x^2) + x + E_n}{2} - \max(0, x) \geq 0 \quad \text{for} \quad -1 \leq x \leq 1
\]  

Figure 6: Relative error between passivated and original transfer functions

![Figure 6: Relative error between passivated and original transfer functions](image)
For $a = b = 1$, the best rational function $f(x)$ satisfying \( \beta \) is therefore $f(x) = \frac{1}{2}(x^2 + x + E_n)$ with $\alpha = E_n$. It should be noted that $f(x)$ has numerator degree $2n + 1$ and denominator degree $2n$. The case of the general interval $[-a, b]$ instead of $[-1, 1]$ is treated by the following.

**Theorem 6.1.** Let $a, b > 0$ and $f(x)$ a real-rational function such that

$$\alpha \geq f(x) - \max(x, 0) \geq 0 \quad \text{for} \quad -1 \leq x \leq 1$$

Then the real-rational function

$$f_{a,b}(x) = \frac{x(b-a) + 2ab}{a+b} f\left(\frac{x(b+a)}{x(b-a) + 2ab}\right)$$

is such that

$$\alpha \max(a, b) \geq f_{a,b}(x) - \max(x, 0) \geq 0 \quad \text{for} \quad -a \leq x \leq b$$

**Proof.** The bilinear transformation $g(x) = x(b+a)/(x(b-a) + 2ab)$ maps the interval $[-a, b]$ onto the interval $[-1, 1]$. Moreover, the linear function $x(b-a) + 2ab$ is positive over $[-a, b]$ since it is positive at the endpoints. Hence

$$\alpha \geq f(g(x)) - \max(g(x), 0) \geq 0 \quad \text{for} \quad -a \leq x \leq b$$

implying

$$\alpha \frac{x(b-a) + 2ab}{a+b} \geq \frac{x(b-a) + 2ab}{b+a} f(g(x)) - \max(x, 0) \geq 0 \quad \text{for} \quad -a \leq x \leq b$$

which completes the proof. Note that, if the denominator degree of $f(x)$ is $m$ and the numerator degree is $m+1$, then the same holds for $f_{a,b}(x)$. \hfill \Box

In light of formula (13), we take $f(x) = \frac{1}{2}(x^2 + x + E_n)$ and $\alpha = E_n$. The function $f_{a,b}(x)$ can be conveniently written as

$$f_{a,b}(x) = (\tau x + \kappa) f\left(\frac{x}{\tau x + \kappa}\right)$$

where

$$\tau = \frac{\delta_+(H) - |\delta_-(H)|}{|\delta_+(H)| + |\delta_-(H)|}$$

and

$$\kappa = \frac{2 \delta_+(H) \delta_-(H)}{|\delta_+(H)| + |\delta_-(H)|}$$
For the function $f^\ell(x) = 1/(x^2 + \ell)$, the partial fraction expansion of $f^\ell_{a,b}(x)$ is given by

$$f^\ell_{a,b}(x) = \frac{(\tau x + \kappa)^3}{x^2 + \ell(\tau x + \kappa)^2} = x \frac{\tau^3}{1 + \tau^2\ell} + \kappa \frac{\tau^2(3 + \tau^2\ell)}{(1 + \tau^2\ell)^2} + \Re \left\{ \frac{\eta(\tau, \kappa, \ell)}{x - \xi(\tau, \kappa, \ell)} \right\}$$

where

$$\xi(\tau, \kappa, \ell) = \frac{\kappa \sqrt{\ell}}{i - \tau \sqrt{\ell}} \quad \eta(\tau, \kappa, \ell) = \frac{\xi(\tau, \kappa, \ell)^3}{\kappa \ell^2}$$

Hence for the function $f(x) = \frac{1}{2}(\rho(x^2) + x + E_n)$, the transformed function $f_{a,b}(x)$ can be written as

$$f_{a,b}(x) = \frac{1}{2} \left[ x + (E_n + a_0)(\tau x + \kappa) - \sum_{k=1}^{n} a_k f_{a,b}^k(x) \right]$$

(14)

The partial fraction expansion of (14) is the key of Algorithm 3, since we ultimately have to calculate $f_{a,b}(Z(s))$, where $Z(s) = H(s) + H(-s)^T$. The linear terms of (14) all add up to the compound linear term

$$f_{a,b}^{\text{linear}}(x) = \frac{1}{2} \left[ x + (E_n + a_0)(\tau x + \kappa) - \sum_{k=1}^{n} a_k \left\{ x \frac{\tau^3}{1 + \tau^2 b_k} + \kappa \frac{\tau^2(3 + \tau^2 b_k)}{(1 + \tau^2 b_k)^2} \right\} \right]$$

leading to a linear term $f_{a,b}^{\text{linear}}(Z(s)) = k_1 Z(s) + k_2 I_p$. The remaining terms, obtained by evaluating

$$\Re \left\{ \eta(\tau, \kappa, b_k)(Z(s) - \xi(\tau, \kappa, b_k)I_p)^{-1} \right\}$$

are obtained by the state space technique described in the Appendix. Finally, as in Algorithm 2, the stable anti-stable projection of $f_{a,b}(Z(s))$ is performed in order to obtain the passivated transfer function $G(s)$.

6.1. Numerical Examples

6.1.1. First example

As our first example we again take the SISO Hurwitz stable minimum phase non-passive transfer function (7), but here we use Algorithm 3 with $n = 4$ and the coefficients of Table 1. The passivated approximation $G(s)$ has a realization with 45 poles. The real and imaginary parts of the original transfer function $H(s)$ vs. the passivated transfer function $G(s)$ are shown in Figs 8 and 9. It is seen by comparing with Figs 3 and 4 that the approximation is better, while requiring 5 poles less.
6.1.2. Second example

For the second example we again take the MIMO Hurwitz stable non-passive transfer function (8), but here we use Algorithm 3 with \( n = 4 \) and the coefficients of Table 1. The passivated approximation \( G(s) \) has a realization with 46 poles. Fig. 10 plots the values of \( \lambda_{\min}(G(i\omega) + G(i\omega)^H) \) vs. \( \lambda_{\min}(H(i\omega) + H(i\omega)^H) \). To show the nearness of the original and passivated transfer functions \( H(s) \) and \( G(s) \), we plot the relative error \( \|G(i\omega) - H(i\omega)\|_2/\|H(i\omega)\|_2 \) in Fig. 11. It is seen by comparing with Figs 5 and 6 that the approximation is more or less similar, but requires 2 poles less.

![Figure 9: Imaginary part of \( G(s) \) vs. \( H(s) \)](image)

![Figure 10: Minimum eigenvalues of passivated vs. original transfer functions](image)

7. CONCLUSION

We have presented a new global passification approach towards finding a passive transfer function \( G(s) \) that is nearest in some well-defined matrix norm sense to a given non-passive transfer function \( H(s) \). It is shown that the key point in constructing the nearest passivated transfer function \( G(s) \), is to find a good rational approximation to the well-known ramp function over an interval defined by the minimum and
maximum dissipation of the given non-passive transfer function $H(s)$. It is also shown that in the Chebyshev or minimax sense this requires finding a rational Chebyshev approximation of the square root function over the unit interval. The proposed algorithms rely strongly on the stable anti-stable projection of a given transfer function. Five pertinent examples, both SISO and MIMO, are given to show the accuracy and relevance of the proposed algorithms.

8. APPENDIX

Suppose we have the real-rational transfer function $H(s) = C(sI_n - A)^{-1}B + D$, and we need to evaluate

\[ \Re \eta(H(s) - \xi I_p)^{-1} \]

where $\eta$ and $\xi$ are complex numbers. Suppose also that $D - \xi I_p$ is invertible. Putting $D_\xi = D - \xi I_p$, we have \[10\] that the complex state space transfer function $\eta(H(s) - \xi I_p)^{-1}$ is given by $\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}$ where

\[
\tilde{A} = A - BD_\xi^{-1}C, \quad \tilde{B} = \eta BD_\xi^{-1}, \quad \tilde{C} = -D_\xi^{-1}C, \quad \tilde{D} = \eta D_\xi^{-1}
\]

In state space form we have

\[
\begin{align*}
\dot{x} & = \tilde{A}x + \tilde{B}u \\
y & = \tilde{C}x + \tilde{D}u
\end{align*}
\]

The input $u$ is real, but the output $y$ is complex. Putting $y = y_1 + iy_2$, it is clear that we are only interested in $y_1$ as output. Decomposing all complex vectors and matrices in their real and imaginary components, we obtain

\[
\begin{align*}
\dot{x}_1 + i\dot{x}_2 & = (A_1 + iA_2)(x_1 + ix_2) + (B_1 + iB_2)u \\
y_1 + iy_2 & = (C_1 + iC_2)(x_1 + ix_2) + (D_1 + iD_2)u
\end{align*}
\]

Hence the state space equations with $u$ as input and $y_1$ as output are simply:

\[
\begin{align*}
\dot{x}_1 & = A_1 x_1 - A_2 x_2 + B_1 u \\
\dot{x}_2 & = A_2 x_1 + A_1 x_2 + B_2 u \\
y_1 & = C_1 x_1 - C_2 x_2 + D_1 u
\end{align*}
\]

\[^3\text{Considering } s = \frac{d}{dt} \text{ as a real operator.}\]
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