A CONDITION ON DELAY FOR DIFFERENTIAL EQUATIONS WITH DISCRETE STATE-DEPENDENT DELAY

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Abstract. Parabolic differential equations with discrete state-dependent delay are studied. The approach, based on an additional condition on the delay function introduced in [A.V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, Nonlinear Analysis: Theory, Methods and Applications, 70 (11) (2009), 3978-3986] is developed. We propose and study a state-dependent analogue of the condition which is sufficient for the well-posedness of the corresponding initial value problem on the whole space of continuous functions $C$. The dynamical system is constructed in $C$ and the existence of a compact global attractor is proved.

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1 Introduction

Delay differential equations is one of the oldest branches of the theory of infinite dimensional dynamical systems - theory which describes qualitative properties of systems, changing in time.

We refer to the classical monographs on the theory of ordinary (O.D.E.) delay equations [11, 12, 8, 2, 20]. The theory of partial (P.D.E.) delay equations is essentially less developed since such equations are infinite-dimensional in both time (as delay equations) and space (as P.D.E.s) variables, which makes the analysis more difficult. We refer to some works which are close to the present research [5, 6, 4, 24] and to the monograph [39].

A new class of equations with delays has recently attracted attention of many researchers. These equations have a delay term that may depend on the state of the system, i.e. the delay is state-dependent (SDD). Due to this type of delays such equations are inherently nonlinear and their study has begun in the case of ordinary differential equations [21, 23, 22, 17, 33, 34, 18] (for more details see also a recent survey [13], articles [35, 36] and references therein).

Investigations of these equations essentially differ from the ones of equations with constant or time-dependent delays. The underlying main mathematical difficulty of the theory lies in the fact that delay terms with discrete state-dependent delays are not Lipschitz continuous on the space of continuous functions - the main space, on which the classical theory of equations with delays is developed (see [38] for an explicit example of
the non-uniqueness and [13] for more details). It is a common point of view [13] that the corresponding initial value problem (IVP) is not generally well-posed in the sense of J. Hadamard [9, 10] in the space of continuous functions (C). This leads to the search of (particular) classes of equations which may be well-posed in the space of continuous functions (C).

Results for partial differential equations with SDD have been obtained only recently in [25] (case of distributed delays, weak solutions), [16] (mild solutions, unbounded discrete delay), and [26] (weak solutions, bounded discrete and distributed delays).

The main goal of the present paper is to develop an alternative approach, based on an additional condition (see (H) below) introduced in [27]. We propose and study a state-dependent analogue of the condition which is sufficient for the well-posedness of the corresponding initial value problem in the space C. The presented approach includes the possibility when the state-dependent delay function does not satisfy the condition on a subset of the phase space C, but the IVP still be well-posed in the whole space C. This is our second goal which is to connect the approach developed for ODEs (a restriction to a subset of Lipschitz continuous functions) and the approach [27] of a different nature.

Discussing the meaning of the main assumptions (H) and (Ĥ) (see below) for applied problems, we hope that these assumptions are the natural mathematical expression of the fact that many differential equations encountered in modeling real world phenomena have a parameter (time \( \eta_{ign} > 0 \) or \( \Theta^\ell > 0 \)) which is necessary to take into considerations the time changes in the system. The changes not always can be taken into considerations immediately. To this end, the existence of \( \eta_{ign} > 0 \) or \( \Theta^\ell > 0 \) (no matter how small the values of \( \eta_{ign} > 0 \) or \( \Theta^\ell > 0 \) are!) makes the corresponding initial value problem well-posed.

Having the well-posedness proved, we study the long-time asymptotic behavior of the correspond dynamical system and prove the existence of a compact global attractor.

2 Formulation of the model with state-dependent discrete delay

Let us consider the following parabolic partial differential equation with delay

\[
\frac{\partial}{\partial t} u(t,x) + Au(t,x) + du(t,x) = (F(u_t))(x), \quad x \in \Omega,
\]

where \( A \) is a densely-defined self-adjoint positive linear operator with domain \( D(A) \subset L^2(\Omega) \) and with compact resolvent, so \( A : D(A) \to L^2(\Omega) \) generates an analytic semigroup, \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \), \( d \) is a non-negative constant. As usually for delay equations, we denote by \( u_t \) the function of \( \theta \in [-r,0] \) by the formula \( u_t \equiv u_t(\theta) \equiv u(t+\theta) \).

We denote for short \( C \equiv C([-r,0];L^2(\Omega)) \). The norms in \( L^2(\Omega) \) and \( C \) are denoted by \( || \cdot || \) and \( || \cdot ||_C \) respectively.

The (nonlinear) delay term \( F : C([-r,0];L^2(\Omega)) \to L^2(\Omega) \) has the form

\[
F(\varphi) = B(\varphi(-\eta(\varphi))),
\]

where (nonlinear) mapping \( B : L^2(\Omega) \to L^2(\Omega) \) is Lipschitz continuous

\[
||B(v^1) - B(v^2)|| \leq L_B ||v^1 - v^2||, \quad \forall v^1, v^2 \in L^2(\Omega).
\]
The function \( \eta(\cdot) : C([-r, 0]; L^2(\Omega)) \to [0, r] \subset R_+ \) represents the state-dependent discrete delay. It is important to notice that \( F \) is nonlinear even in the case of linear \( B \).

We consider equation (1) with the following initial condition

\[
u|_{[-r,0]} = \varphi \in C \equiv C([-r, 0]; L^2(\Omega)). \tag{4}
\]

**Remark 1.** The results presented in this paper could be easily extended to the case of nonlinearity \( F \) of the form \( F(\varphi) = \sum_k B^k(\varphi(-\eta^k(\varphi))) \) as well as to O.D.E.s, for example, of the following form [29]

\[
\dot{u}(t) + Au(t) + d \cdot u(t) = b(u(t - \eta(u_t))), \quad u(\cdot) \in R^n, d \geq 0. \tag{5}
\]

In the last case one simply needs to substitute \( L^2(\Omega) \) by \( R^n \) and use \( C \equiv C([-r, 0]; R^n) \) instead of \( C([-r, 0]; L^2(\Omega)) \). The function \( b : R^n \to R^n \) is locally Lipschitz continuous and satisfies \( ||b(w)||_{R^n} \leq C_1||w||_{R^n} + C_b \) with \( C_1, C_b \geq 0 \); \( A \) is a matrix.

**Remark 2.** As an example we could consider nonlocal delay term \( F \) (see [2]) with the following mapping

\[
B(v)(x) \equiv \int_{\Omega} b(v(y))f(x - y)dy, \quad x \in \Omega,
\]

where \( f : \Omega - \Omega \to R \) is a bounded and measurable function \( ||f(z)|| \leq M_f, \forall z \in \Omega - \Omega \) and \( b : R \to R \) is a (locally) Lipschitz mapping, satisfying \( ||b(w)||_{R^n} \leq C_1||w||_{R^n} + C_b \) with \( C_b \geq 0 \). In this case equation (4) has the form

\[
\frac{\partial}{\partial t} u(t, x) + Au(t, x) + d u(t, x) = \int_{\Omega} b(u(t - \eta(u_t), y))f(x - y)dy, \quad x \in \Omega.
\]

One can easily check that \( B \) satisfies (3) with \( L_B \equiv L_b M_f |\Omega| \), where \( L_b \) is the Lipschitz constant of \( b \), and \( |\Omega| \equiv \int_{\Omega} 1 \, dx \).

Another example is a (local) delay term \( F \) (see [2]) with \( B(v)(x) \equiv b(v(x)), x \in \Omega \). Equation (4) has the form

\[
\frac{\partial}{\partial t} u(t, x) + Au(t, x) + d u(t, x) = b(u(t - \eta(u_t), x)), \quad x \in \Omega.
\]

An easy calculation show that (3) is satisfied with \( L_B \equiv L_b \).

The methods used in our work can be applied to other types of nonlinear and delay P.D.E.s (as well as O.D.E.s). We choose a particular form of nonlinear delay terms \( F \) for simplicity and to illustrate our approach on the diffusive Nicholson’s blowflies equation (see the end of the article for more details).

### 3 The existence of mild solutions

In our study we use the standard

**Definition 1.** A function \( u \in C([-r, T]; L^2(\Omega)) \) is called a mild solution on \([-r, T]\) of the initial value problem (1), (4) if it satisfies (4) and

\[
u(t) = e^{-At} \varphi(0) + \int_{0}^{t} e^{-A(t-s)} \{F(u_s) - d \cdot u(s)\} \, ds, \quad t \in [0, T]. \tag{6}
\]
Proposition 1 [27]. Assume the mapping $B$ is Lipschitz continuous (see (3)) and delay function $\eta(\cdot) : C([-r, 0]; L^2(\Omega)) \to [0, r] \subset \mathbb{R}_+$ is continuous.

Then for any initial function $\varphi \in C$, initial value problem (1), (4) has a global mild solution which satisfies $u \in C([-r, +\infty); L^2(\Omega))$.

The existence of a mild solution is a consequence of the continuity of $F : C \to L^2(\Omega)$ (see (1)) which gives the possibility to use the standard method based on Schauder fixed point theorem (see e.g. [39, theorem 2.1, p.46]). The solution is also global (is defined for all $t \geq -r$) since (3) implies $\|F(\varphi)\| \leq L_B \|\varphi\|_C + \|B(0)\|$ and one can apply, for example, [39, theorem 2.3, p.49].

Remark 3. It is important to notice that even in the case of ordinary differential equations (even scalar) the mapping of the form $\tilde{F}(\varphi) = \tilde{f}(\varphi(-r(\varphi))) : C([-r_0, 0]; \mathbb{R}) \to \mathbb{R}$ has a very unpleasant property. The authors in [19, p.3] write "Notice that the functional $\tilde{F}$ is defined on $C([-r_0, 0]; \mathbb{R})$, but it is clear that it is neither differentiable nor locally Lipschitz continuous, whatever the smoothness of $\tilde{f}$ and $r$." As a consequence, the Cauchy problem associated with equations with such a nonlinearity "...is not well-posed in the space of continuous functions, due to the non-uniqueness of solutions whatever the regularity of the functions $\tilde{f}$ and $r$" [19, p.2]. See also a detailed discussion in [13].

Remark 4. For a study of solutions to equations with a state-dependent delay in the space $C([-r, 0]; E)$ with $E$ not necessarily finite-dimensional Banach space see e.g. [1]

In this work we concentrate on conditions for the IVP (1), (4) to be well-posed.

4 Main results: uniqueness, well-posedness and asymptotic behavior

As in the previous section, we assume that $\eta : C \to [0, r]$ is continuous and $B$ is Lipschitz. Unlike to the existence of solutions, the uniqueness is essentially more delicate question in the presence of discrete state-dependent delay (see a classical example of the non-uniqueness in [38]).

Let us remind an important additional assumption on the delay function $\eta$, as it was introduced in [27]:

• $\exists \eta_{ign} > 0$ such that $\eta$ "ignores" values of $\varphi(\theta)$ for $\theta \in (-\eta_{ign}, 0]$ i.e.

$$\exists \eta_{ign} > 0 : \forall \varphi^1, \varphi^2 \in C : \forall \theta \in [-r, -\eta_{ign}], \varphi^1(\theta) = \varphi^2(\theta) \implies \eta(\varphi^1) = \eta(\varphi^2).$$

$(H)$

For examples of delay functions satisfying $(H)$ and the proof of the uniqueness of mild solutions (given by Proposition 1) as well as the well-posedness of the IVP (1), (4) see [27].

Remark 5. It is important to notice that, discussing the condition $(H)$ and its dependence on the value $\eta_{ign}$, we see that in the case $\eta_{ign} > r$, one has that the delay function $\eta$ ignores all values of $\varphi(\theta), \forall \theta \in [-r, 0]$, so $\eta(\varphi) \equiv \text{const,} \forall \varphi \in C$ i.e. equation (1) becomes an equation with constant (!) delay. On the other hand, the analogue of assumption
$(H)$ with $\eta_{ign} = 0$, is trivial since $\varphi^1(\theta) = \varphi^2(\theta)$ for all $\theta \in [-r, 0]$ means $\varphi^1 = \varphi^2$ in $C$, so $\eta(\varphi^1) = \eta(\varphi^2)$.

**Remark 6.** It is worth mentioning that the classical case of constant delay (see the previous remark) and the corresponding theory forms the basement for the discussed approach, but could be mixed with the approach of non-vanishing delays. In our case the delay $\eta$ may vanish (we do not assume the existence of $r_0 > 0$ such that $\eta(\varphi) \geq r_0, \forall \varphi$).

In the above condition (H) the semi-interval $(-\eta_{ign}, 0]$ is fixed (we remind that the value $\eta_{ign}$ could be arbitrary small).

Our goal is to extend the approach based on the condition (H) to a more wide class of state-dependent delay functions where the value $\eta_{ign}$ is not a constant any more, but a function of the state. Moreover, as an easy additional extension, we also allow the upper bound of the delayed segment to be state-dependent. More precisely, we consider two functions $\Theta^\ell, \Theta^u : C \rightarrow [0, r]$ (upper and low functions), satisfying

\[\forall \varphi \in C \quad 0 \leq \Theta^\ell(\varphi) \leq \Theta^u(\varphi) \leq r.\]

Now we are ready to introduce [29] the following state-dependent condition for the state-dependent delay function $\eta : C \rightarrow [0, r]$ (c.f. (H)):

- $\eta$ ”ignores” values of $\varphi(\theta)$ for $\theta \notin [-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$ i.e.

\[\forall \psi \in C \text{ such that } \forall \theta \in [-\Theta^u(\varphi), -\Theta^\ell(\varphi)] \Rightarrow \psi(\theta) = \varphi(\theta) \quad \implies \quad \eta(\psi) = \eta(\varphi). \quad (\hat{H})\]

The above condition means that state-dependent delay function $\eta$ ”ignores” all values of its argument $\varphi$ outside of $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)] \subset [-r, 0]$ and this delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$ is state-dependent. We could illustrate this property on the picture.

**Remark 7.** One could see that $(H)$ is a particular case of $(\hat{H})$ with $\Theta^\ell(\varphi) \equiv \eta_{ign}$ and $\Theta^u(\varphi) \equiv r, \forall \varphi \in C.$
Examples. It is easy to present many examples of (delay) functions $\eta$, which satisfy assumption $(\hat{H})$. The simplest one is

$$\eta(\varphi) = p_1(\varphi(-\chi(\varphi(-r)))) \text{ with } p_1 : L^2(\Omega) \to [0, r]$$

and given $\chi : L^2(\Omega) \to [0, r]$. Here $\Theta^f(\varphi) \equiv \chi(\varphi(-r))$ and $\Theta^u(\varphi) = r$. It is easy to see that the above delay function $\eta$ ignores values of $\varphi$ at points $\theta \in (-r, -\chi(\varphi(-r))) \cup (-\chi(\varphi(-r)), 0]$ and uses just two values of $\varphi$ at points $\theta = -r, \theta = -\chi(\varphi(-r))$. In our notations, the delayed segment $[-\Theta^u(\varphi), -\Theta^f(\varphi)] = [-r, -\chi(\varphi(-r))]$ is state-dependent.

In the same way, one has

$$\eta(\varphi) = \sum_{k=1}^{N} p_k(\varphi(-\chi^k(\varphi(-r)))) \text{ with } p_k, \chi^k : L^2(\Omega) \to [0, r].$$

In this case $[-\Theta^u(\varphi), -\Theta^f(\varphi)] = [-r, -\min k\{\chi^k(\varphi(-r))\}]$. A slightly more general example is

$$\eta(\varphi) = \sum_{k=1}^{N} p_k(\varphi(-\chi^k(\varphi(-r^k)))) \text{ with } p_k, \chi^k : L^2(\Omega) \to [0, r], \min r^k \in (0, r].$$

Here $\Theta^u(\varphi) = \max\{r^1, ..., r^N, \chi^1(\varphi(-r^1)), ..., \chi^N(\varphi(-r^N))\}$ and $\Theta^f(\varphi) = \min\{r^1, ..., r^N, \chi^1(\varphi(-r^1)), ..., \chi^N(\varphi(-r^N))\}$.

Examples of integral delay terms are as follows

$$\eta(\varphi) = \int_{-\chi^1(\varphi(-r^1))}^{-\chi^2(\varphi(-r^2))} p_1(\varphi(\theta))g(\theta) \, d\theta; \text{ and } \eta(\varphi) = p_1 \left( \int_{-\chi^1(\varphi(-r^1))}^{-\chi^2(\varphi(-r^2))} \varphi(\theta)g(\theta) \, d\theta \right).$$

Similar to the previous example, $\Theta^u(\varphi) = \max\{r^1, r^2, \chi^1(\varphi(-r^1)), \chi^2(\varphi(-r^2))\}$ and $\Theta^f(\varphi) = \min\{r^1, r^2, \chi^1(\varphi(-r^1)), \chi^2(\varphi(-r^2))\}$.

Remark 8. It is interesting to notice that an assumption similar to the existence of upper function $\Theta^u(\cdot)$ is used in [27] for ODEs with SDD (locally bounded delay). On the other hand, an assumption similar to $(H)$ is used in [14] for neutral ODEs (see $(A_4)(ii)$ in [14]), but together with another assumption on SDD to be bounded from below by a constant $r_0 > 0$ (c.f. remark 6).

Following [27, theorem 1] we have the first result

**Theorem 1.** Let both upper and low functions $\Theta^u, \Theta^f : C \to [0, r]$ be continuous and $\Theta^f(\varphi) > 0, \forall \varphi \in C$. Assume the delay function $\eta : C \to [0, r] \subset R_+$ is continuous and satisfies assumption $(\hat{H})$; the mapping $B$ is Lipschitz continuous (see [3]).

Then for any initial function $\varphi \in C$, initial value problem (I), (4) has an unique mild solution $u(t), t \geq 0$ (given by proposition 1).

If we define the evolution operator $S_t : C \to C$ by the formula $S_t \varphi \equiv u_t$, where $u(t)$ is the unique mild solution of (I), (4) with initial function $\varphi$, then the pair $(S_t, C)$ constitutes a dynamical system i.e. the following properties are satisfied:

1. $S_0 = Id$ (identity operator in $C$);
2. $\forall \, t, \tau \geq 0 \quad \Rightarrow \quad S_t S_\tau = S_{t+\tau}$;
3. $t \mapsto S_t$ is a strongly continuous in $C$ mapping;

4. for any $t \geq 0$ the evolution operator $S_t$ is continuous in $C$ i.e. for any $\{\varphi^n\}_{n=1}^\infty \subset C$ such that $||\varphi^n - \varphi||_C \to 0$ as $n \to \infty$, one has $||S_t\varphi^n - S_t\varphi||_C \to 0$ as $n \to \infty$.

The proof follows the line of [27, theorem 1] taking into account that condition $\Theta^\ell(\varphi) > 0, \forall \varphi \in C$ implies that for any fixed $\varphi \in C$, due to the continuity of $\Theta^\ell : C \to [0, r]$, there exists a neighbourhood $U(\varphi) \subset C$ such that for all $\psi \in U(\varphi)$ one has $\Theta^\ell(\psi) \geq \frac{1}{2}\Theta^\ell(\varphi) > 0$. That means that in $U(\varphi) \subset C$ we have the (state-independent) condition (H) with $\eta_{ign} = \frac{1}{2}\Theta^\ell(\varphi) > 0$ and all the arguments presented in [27, theorem 1] could be directly applied to this case. \[\blacksquare\]

**Remark 9.** We do not assume that the upper and low functions $\Theta^u, \Theta^\ell$ (which are used in ($\bar{H}$) to present the delayed segment $[-\Theta^u(\varphi), -\Theta^\ell(\varphi)]$) are the functions presenting the smallest possible delayed segment. More precisely, it is possible that there exist two other functions $\Theta^\lambda, \Theta^\rho$ such that for all $\varphi \in C$ one has $0 \leq \Theta^\ell(\varphi) \leq \Theta^\lambda(\varphi) \leq \Theta^u(\varphi) \leq \Theta^\rho(\varphi) \leq r$ and the same delay $\eta$ satisfies ($\bar{H}$) with $\Theta^\lambda, \Theta^\rho$ as well.

Our next step in studying the state-dependent condition ($\bar{H}$) is an attempt to avoid the condition $\Theta^\ell(\varphi) > 0, \forall \varphi \in C$. We are going to consider the general case $\Theta^\ell(\varphi) \geq 0, \forall \varphi \in C$ with a non-empty set $Z \equiv \{\varphi \in C : \Theta^\ell(\varphi) = 0\} \neq \emptyset$.

**Theorem 2.** Assume the mapping $B$ is Lipschitz continuous (see (3)). Moreover, let the following conditions be satisfied:

1) both upper and low functions $\Theta^u, \Theta^\ell : C \to [0, r]$ are continuous;

2) $Z \equiv \{\varphi \in C : \Theta^\ell(\varphi) = 0\} \subset C\mathcal{L}_L \equiv \{\varphi \in C : \sup_{t \neq s} \frac{||\varphi(t) - \varphi(s)||}{|t-s|} \leq L\}$;

3) delay function $\eta : C \to [0, r] \subset R_+$ is continuous and satisfies assumption ($\bar{H}$);

4) $\forall \varphi \in Z \Rightarrow \eta(\varphi) > 0$;

5) $\exists U_\omega(Z) \equiv \{\chi \in C : \exists \nu \in Z : ||\chi - \nu||_C \leq \omega\}; \exists L_\eta > 0 : \forall \varphi, \psi \in U_\omega(Z) \Rightarrow$

$$||\eta(\varphi) - \eta(\psi)|| \leq L_\eta \cdot ||\varphi - \psi||_C.$$

Then for any initial function $\varphi \in C$, initial value problem (1), (4) has an unique mild solution $u(t), t \geq 0$ (given by proposition 1). Moreover, the pair $(S_t, C)$ constitutes a dynamical system (see thm 1).

**Proof of theorem 2.** Let us consider $\varphi \in C$ which is an initial condition (see (1)). We start with the simple case $\varphi \notin Z$. By definition of $Z$, we have $\Theta^\ell(\varphi) > 0$. Hence we apply the same arguments as in the proof of theorem 1 (the state-independent condition ($\bar{H}$) is satisfied locally).

The rest of the proof is devoted to the case $\varphi \in Z$. We remind some estimates similar to estimates (6)-(13) in [27]. Denote by $u^k(t)$ any solution of (1), (4) with the initial function $\varphi^k$ and by $u(t)$ any solution of (1), (4) with the initial function $\varphi$.

We use the variation of constants formula for parabolic equation (with $\tilde{A} \equiv A + d \cdot E$)

$$u(t) = e^{-\tilde{A}t}u(0) + \int_0^t e^{-\tilde{A}(t-\tau)} B(u(\tau - \eta(u_\tau))) d\tau,$$  \hspace{1cm} (8)
\[ u^k(t) = e^{-\tilde{A} t} u^k(0) + \int_0^t e^{-\tilde{A}(t-\tau)} B(u^k(\tau - \eta(u^k_\tau))) \, d\tau. \] (9)

Using \( ||e^{-\tilde{A} t}|| \leq 1 \) and \( ||e^{-\tilde{A}(t-\tau)}|| \leq 1 \), we get

\[
||u^k(t) - u(t)|| \leq ||u^k(0) - u(0)|| + \int_0^t ||B(u^k(\tau - \eta(u^k_\tau))) - B(u(\tau - \eta(u_\tau)))|| \, d\tau
= ||\varphi^k(0) - \varphi(0)|| + J^k_1(t) + J^k_2(t),
\] (10)

where we denote (for \( s \geq 0, x \in \Omega \))

\[
J^k_1(s) \equiv J^k_1(s)(x) \equiv \int_0^s ||B(u^k(\tau - \eta(u^k_\tau))) - B(u(\tau - \eta(u_\tau)))|| \, d\tau,
\] (11)

\[
J^k_2(s) \equiv J^k_2(s)(x) \equiv \int_0^s ||B(u(\tau - \eta(u_\tau)))\, B(u(\tau - \eta(u_\tau)))|| \, d\tau.
\] (12)

Using the Lipschitz property \( \text{(3)} \) of \( B \), one easily gets

\[
J^k_1(t) \leq L_B \int_0^t ||u^k(\tau - \eta(u^k_\tau)) - u(\tau - \eta(u_\tau))|| \, d\tau
\leq L_B t \max_{s \in [-r,t]} ||u^k(s) - u(s)||.
\] (13)

Estimates \( \text{(13), (10)} \) and property \( J^k_2(s) \leq J^k_2(t) \) for \( s \leq t \leq t_0 \) give

\[
\max_{t \in [0,t_0]} ||u^k(t) - u(t)|| \leq ||\varphi^k(0) - \varphi(0)|| + L_B t_0 \max_{s \in [-r,t_0]} ||u^k(s) - u(s)|| + J^k_2(t_0).
\]

Hence

\[
\max_{s \in [-r,t_0]} ||u^k(s) - u(s)|| \leq ||\varphi^k - \varphi||_C + L_B t_0 \max_{s \in [-r,t_0]} ||u^k(s) - u(s)|| + J^k_2(t_0).
\] (14)

Now we study properties of \( J^k_2 \) which essentially differ from the ones in \( \text{[27]} \) since (H) is not satisfied. The Lipschitz property of \( B \) implies

\[
J^k_2(t_0) \leq L_B \int_0^{t_0} ||u(\tau - \eta(u_\tau))\, B(u(\tau - \eta(u_\tau)))|| \, d\tau.
\] (15)

Since \( \varphi \in Z \), property 4) gives \( \eta(\varphi) > 0 \). Due to the continuity of \( \eta \) (see 3)),

\[
\exists U_\alpha(\varphi) \equiv \{ \psi \in C : ||\varphi - \psi||_C \leq \alpha \} : \forall \psi \in U_\alpha(\varphi) \Rightarrow \eta(\psi) \geq \frac{3}{4} \eta(\varphi) > 0.
\] (16)

We choose \( \alpha < \omega \) (see property 5). By definition, a solution is strongly continuous function (with values in \( L^2(\Omega) \)), hence for any two solutions \( u(t) \) and \( u^k(t) \) there exist two time moments \( t_\varphi, t_{\varphi^k} > 0 \) such that for all \( t \in (0, t_\varphi] \) one has \( u_t \in U_\alpha(\varphi) \) and for all \( t \in (0, t_{\varphi^k}] \) one has \( u^k_t \in U_\alpha(\varphi) \).

**Remark 10.** More precisely, we assume that \( \exists N_\alpha \in N \) such that for all \( k \geq N_\alpha \) one has \( \varphi^k \in U_{\alpha/2}(\varphi) \) and hence there exists time moment \( t_{\varphi^k} \in (0, t_0] \) such that for all
\( t \in (0, t_\varphi) \) one has \( u_k^t \in U_\alpha(\varphi) \). The last assumption (\( \exists N_\alpha \in N : \forall k \geq N_\alpha \Rightarrow \varphi^k \in U_{\alpha/2}(\varphi) \)) is not restrictive since for the uniqueness of solutions we have \( \varphi^k = \varphi \) while for the continuity with respect to initial data (see below) we have \( \varphi^k \to \varphi \) in \( C \).

**Remark 11.** It is important to notice that we take any solution from the set of solutions of IVP (1), (4) with the initial function \( \varphi \) and have \( \varphi^k \to \varphi \) with respect to initial data (see below) we have \( \varphi^k \to \varphi \) in \( C \).

These and (16) imply that for all \( \tau \in [0, t_1] \), with \( t_1 \leq \min \{ t_\varphi ; t_\varphi^k ; 3_2 \eta(\varphi) \} \) one gets \( \tau - \eta(u_\tau) \leq 0, \tau - \eta(u_\tau^k) \leq 0 \) and \( u(\tau - \eta(u_\tau)) = \varphi(\tau - \eta(u_\tau)), u(\tau - \eta(u_\tau^k)) = \varphi(\tau - \eta(u_\tau^k)) \).

Hence, see (15) and properties 2), 5),

\[
J^k_2(t_1) \leq L_B \int_0^{t_1} ||\varphi(\tau - \eta(u_\tau^k)) - \varphi(\tau - \eta(u_\tau))|| d\tau \leq L_B L \int_0^{t_1} |\eta(u_\tau^k) - \eta(u_\tau)| d\tau
\]

\[
\leq L_B L \eta t_1 \max_{s \in [-r, t_1]} ||u_k^s(s) - u(s)||.
\]

Finally, we get (see the last estimate and (14))

\[
(1 - L_B t_1 [1 + LL_\eta]) \max_{s \in [-r, t_1]} ||u_k^s(s) - u(s)|| \leq ||\varphi - \varphi||_C.
\]

Choosing small enough \( t_1 > 0 \) (to have \( 1 - L_B t_1 [1 + LL_\eta] > 0 \)) i.e.

\[
t_1 \equiv \min \left\{ t_\varphi ; t_\varphi^k ; 3_4 \eta(\varphi) ; qL_B [1 + LL_\eta]^{-1} \right\}
\]

for any fixed \( q \in (0, 1) \), (17)

we get

\[
\max_{s \in [-r, t_1]} ||u_k^s(s) - u(s)|| \leq (1 - L_B t_1 [1 + LL_\eta]^{-1}) ||\varphi - \varphi||_C.
\]

(18)

It is easy to see that (18) particularly implies the uniqueness of mild solutions to I.V.P. (1), (4) in case when \( \varphi^k = \varphi \).

It gives us the possibility to define the evolution operator \( S_t : C \to C \) by the formula \( S_t \varphi \equiv u_t \), where \( u(t) \) is the unique mild solution of (1), (4) with initial function \( \varphi \).

Our next goal is to prove that pair \((S_t, C)\) constitutes a dynamical system (see the properties 1. – 4. as they are formulated in theorem 1). As in [27] p.3981, properties 1, 2 are consequences of the uniqueness of mild solutions. Property 3 is given by Proposition 1 since the solution is a continuous function \( u \in C([-r, T]; L^2(\Omega)) \).

Let us prove property 4. We consider any sequence \( \{ \varphi^k \}_{k=1}^\infty \subset C \), which converges (in space \( C \)) to \( \varphi \). Denote by \( u_k^t \) the (unique!) mild solution of (1), (4) with the initial function \( \varphi^k \) and by \( u(t) \) the (unique!) mild solution of (1), (4) with the initial function \( \varphi \).

One could think that (18) already provides the continuity with respect to initial data, but there is an important technical property used in developing (18) i.e. the choice of \( t_1 \) (see (17) and remark 11). In contrast to the previous study, now we have infinite set of functions \( \{ \varphi^k \}_{k=1}^\infty \subset C \), so it may happen that \( t_1 = t_1^k \to 0 \) when \( k \to \infty \).

We remind (see the text after (13)) that two time moments \( t_\varphi, t_\varphi^k > 0 \) have been chosen such that for all \( t \in (0, t_\varphi) \) one has \( u_t \in U_\alpha(\varphi) \) and for all \( t \in (0, t_\varphi^k) \) one has \( u_t^k \in U_\alpha(\varphi) \). Now our goal is to show that infinite number of moments \( t_\varphi, \{ t_\varphi^k \}_{k=1}^\infty \) could be chosen in such a way that \( t_2 \equiv \inf \{ t_{\varphi^k} \} > 0 \) and \( u_t^k, u_t^k \in U_\alpha(\varphi) \) for all
t ∈ (0, t_2]. To get this, we use the standard proof of the existence of a mild solution by a fixed point argument (see e.g. [39, p.46, thm 2.1]). More precisely, let U be an open subset of C and \( \tilde{F} : [0, b] \times U \to L^2(\Omega) \) be continuous. For \( \varphi \in C \) and any \( y \in Y_1 \equiv \{ y \in C([-r, t_3]; L^2(\Omega)) : y(0) = \varphi(0) \} \) we consider the extension function \( \hat{y} \) as follows

\[
\hat{y}(s) = \begin{cases} 
\varphi(s) & \text{for } s \in [-r,0]; \\
y(t) & \text{for } s \in (0,t_3]. 
\end{cases}
\]

Let \( Y_2 \equiv \{ y \in Y_1 : \hat{y}_t \in \overline{B_\delta(\varphi)} \text{ for } t \in [0, t_3] \} \). Consider a mapping \( G \) on \( Y_2 \) as follows

\[
G(y)(t) \equiv e^{-\hat{A}t}y(0) + \int_0^t e^{-\hat{A}(t-\tau)}\tilde{F}(\hat{y}_\tau) d\tau.
\]

One can check (see [39, p.46,47, thm 2.1]), that \( G \) maps \( Y_2 \) into \( Y_2 \) provided \( t_3 \equiv \min\{t'; b; \delta/(3N); \delta\} \). Here we use notations of [39, p.46] chosen as follows. Constants \( \delta > 0 \) and \( N > 0 \) are such that \( ||\tilde{F}(\psi)|| \leq N \) for all \( \psi \in B_\delta(\varphi) \equiv \{ \psi \in C : ||\psi - \varphi||_C \leq \delta \} \), \( ||e^{-\hat{A}t}|| \leq M = 1 \). The time moment \( t' < r \) is chosen so that if \( 0 \leq t \leq t' \) then \( ||\varphi(t + \theta) - \varphi(\theta)|| < \delta/3 \) and \( ||e^{-\hat{A}t}\varphi(0) - \varphi(0)|| < \delta/3 \). The solution is given by a fixed point \( y = G(y) \). For our goal it is sufficient to choose \( \delta \leq \alpha \) and \( t_2 \leq t_3 \) to get \( u_t, u_t^k \in U_\alpha(\varphi) \) for all \( t \in (0,t_2) \). Here we use \( \varphi^k \) instead of \( \varphi \) when necessary. The crucial point here is the possibility to choose \( t' \) (and hence \( t_3 \) and \( t_2 \)) independent of \( k \in N \). The choice of \( t' < r \) so that if \( 0 \leq t \leq t' \) then \( ||\varphi(t + \theta) - \varphi(\theta)|| < \delta/3 \) and \( ||\varphi^k(t + \theta) - \varphi^k(\theta)|| < \delta/3 \) for all \( k \in N \) is possible due to the convergence of \( \varphi^k \) (to \( \varphi \) in \( C \)). Since any convergent sequence is a pre-compact set in \( C \), the desired property is the equicontinuity given by the Arzela-Ascoli theorem. Now estimate (18) can be applied to our case and this completes the proof of property 4 and theorem 2.

Discussing assumptions of theorem 2, let us present a constructive example of low function \( \Theta^f \) which satisfies assumption 2). Consider any compact and convex set \( K_C \subset CL_C \subset C \). For example, for any compact and convex set \( K \in L^2(\Omega) \), the set \( \{ \varphi \in C : \varphi \in CL, \forall \theta \in [-r,0] \Rightarrow \varphi(\theta) \in K \} \) is compact (by Arzela-Ascoli theorem) and convex. First, constructing \( \Theta^f \), we set \( \Theta^f(\varphi) = 0 \) for all \( \varphi \in K_C \). Second, we take any \( p \in (0,r] \) and set \( \Theta^f(\varphi) = p \) for all \( \varphi \in C \) such that \( \text{dist}(\varphi, K_C) \geq 1 \). Third, for any \( \varphi \in C \) such that \( \text{dist}(\varphi, K_C) \in (0,1) \) we find an unique \( \hat{\varphi} \in K_C \) such that \( \text{dist}(\varphi, K_C) = ||\varphi - \hat{\varphi}||_C \). Such \( \hat{\varphi} \in K_C \) exists by the classical Weierstrass theorem since \( f(\psi) \equiv \text{dist}(\varphi, \psi) : K_C \to (0,1) \) is continuous (\( \varphi \) is fixed) and \( K_C \) is compact. The uniqueness of \( \hat{\varphi} \) follows from the convexity of \( K_C \). Finally, we set \( \Theta^f(\varphi) = p \cdot \text{dist}(\varphi, \hat{\varphi}) \in (0,p) \) for all \( \varphi \in C : \text{dist}(\varphi, K_C) \in (0,1) \). By construction, \( \Theta^f \) satisfies 2).

As for asymptotic behavior, we study of the long-time behavior of the dynamical system \((S_t, C)\), constructed in theorems 1 and 2. Similar to [27, theorem 2] we have the following result.

**Theorem 3.** Assume all the assumptions of theorems 1 or 2 are satisfied and additionally mapping \( B \) (see (23)) is bounded. Then the dynamical system \((S_t, C)\) has a compact global attractor \( \mathcal{A} \) which is a compact set in all spaces \( C_\delta \equiv C([-r, 0]; D(A^\delta)), \forall \delta \in (0, \frac{1}{2}) \).

**Lemma.** Let all the assumptions of theorem 2 be satisfied. Then the global attractor \( \mathcal{A} \) (see theorem 3) is a subset of \( CL_C \) (c.f. condition 2 in theorem 2).
Remark 12. Lemma gives a possibility to consider system (7), (4) with a state-dependent delay function $\eta$ which does not ignore values of its argument $\varphi$ for all points $\varphi \in A$ i.e. no information is lost on the global attractor $A$.

Proof of lemma. Consider any solution $u_t \in A$. Let us denote $f(t) \equiv F(u_t)$ and prove that $f$ is Hölder continuous.

We will need the following property, proved in [27] estimate (29) with $\delta = 0$

$$||u(t_1) - u(t_2)|| \leq L_0 |t_1 - t_2|^{1/2} \quad (19)$$

for any solution, belonging to the ball of dissipation (particularly, for any solution belonging to the attractor). Here $L_0$ is independent of solution $u$.

One can check that

$$||f(t_1) - f(t_2)|| \leq L_B \cdot ||u(t_1 - \eta(u_{t_1})) - u(t_2 - \eta(u_{t_2}))||. \quad (20)$$

Using (19), the Lipschitz property of $\eta$ (see 5 in theorem 2), we get from (20) that

$$||f(t_1) - f(t_2)|| \leq L_B L_0 \cdot |t_1 - \eta(u_{t_1}) - (t_2 - \eta(u_{t_2}))|^{1/2} \leq \left[ \text{ using 5 in theorem 2 } \right] \leq L_B L_0 \cdot (|t_1 - t_2| + |\eta(u_{t_1}) - \eta(u_{t_2})|)^{1/2} \leq \left[ \text{ using (19) } \right] \leq \frac{L_B L_0 \cdot (|t_1 - t_2| + L_\eta |u_{t_1} - u_{t_2}|)^{1/2}}{L_B L_0 \cdot (|t_1 - t_2|^{1/2} + (L_\eta L_0)^{1/2}|t_1 - t_2|^{1/4})}.$$

Finally, for $|t_1 - t_2| < 1$ one has

$$||f(t_1) - f(t_2)|| \leq L_B L_0 \cdot \left\{ 1 + (L_\eta L_0)^{1/2} \right\} |t_1 - t_2|^{1/4}. \quad (21)$$

Let us consider $\forall \psi \in A$. It is well-known that the attractor consists of whole trajectories i.e. $u_s \in A, \forall s \in R$. We take any $t_0 > r > 0$ and get $\varphi \in A$ such that $S_{r \varphi} = \psi$. Consider the variation of constants formula for parabolic equation (with $\tilde{A} \equiv A + d \cdot E$ see (8))

$$u(t) = e^{-\tilde{A} t} \varphi(0) + \int_0^t e^{-\tilde{A} (t-\tau)} F(u_{\tau}) \, d\tau. \quad (22)$$

The first term in the above formula (22) is Lipschitz for $t > t_0$ due to the standard estimate $||e^{-\tilde{A} t} v - e^{-\tilde{A} t_2 v}|| \leq (t_1 e^{-t_1 v}) \cdot |t_1 - t_2|, 0 < t_1 < t_2$. Moreover it is uniformly Lipschitz for any $v = \varphi(0), \varphi \in A$ since $||e^{-\tilde{A} t} \varphi(0), -e^{-\tilde{A} t_2} \varphi(0)|| \leq (r e^{-r_1 v}) \cdot |t_1 - t_2|, r < t_1 < t_2$. Here $||\varphi(0)|| \leq C(0)$ due to the dissipativeness of the dynamical system $(S_t; C)$ (for more details see [27] estimate (23)).

To prove that the second term in (22) is Lipschitz for $t > t_0$ we need the following

Proposition [15] lemma 3.2.1. Let $\tilde{A}$ be a sectorial operator in Banach space $X$. Assume function $f : (0, T) \to X$ is locally Hölder continuous and $\int_0^T ||f(s)||_X ds < \infty$ for some $\rho > 0$. Denote by $\Phi(t) \equiv \int_0^t e^{-\tilde{A} (t-s)} f(s) ds$. Then function $\Phi(\cdot)$ is continuous on $[0, T)$, continuously differentiable on $(0, T)$, $\Phi(t) \in D(\tilde{A})$ for $0 < t < T$ and $d\Phi(t)/dt + A\Phi(t) = f(t)$ for $0 < t < T$ and $\Phi(t) \to 0$ in $X$ as $t \to 0+$.
Remark 13. Our operator $\tilde{A}$ is sectorial since any self-adjoint densely defined bounded from below operator in a Hilbert space is sectorial (see e.g. [15, example 2, p.26]).

We apply the above proposition to $f(t) \equiv F(u_t)$ and use (21). The property $\int_{\rho}^0 \|f(s)\|_X ds < \infty$ for some $\rho > 0$ follows from the dissipativeness $\|u(t)\| \leq C(0)$, the continuity of $F : C \to L^2(\Omega)$ and the strong continuity of mild solution $u$. One uses the continuous differentiability of $\Phi$ on $[t_0 - r, t_0] \subset (0, T)$ which implies that $\max_{t \in [t_0 - r, t_0]} \|\Phi'(t)\| \equiv M_{\Phi;1} < \infty$. In our case $\Phi$ represents the second term in (22) which is proved to be Lipschitz continuous with Lipschitz constant $M_{\Phi;1}$ independent of $u$. The proof of lemma is complete.

Remark 14. One can also easily extend the method developed here to the case of non-autonomous nonlinear delay terms, for example, using another nonlinear function $\hat{b} : R \times R \to R$ (see remark 2) instead of $b$ to have $(\tilde{\Phi}(t, u_t))(x) = \hat{b}(t, u(t - \eta(u_t), x))$ or $(\tilde{\Phi}(t, u_t))(x) = \int_{\Omega} \hat{b}(t, u(t - \eta(u_t), y))f(x - y)dy$ in equation (1).

As an application we can consider the diffusive Nicholson’s blowflies equation (see e.g. [31] with state-dependent delays. More precisely, we consider equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset R^n$ is a bounded domain with a smooth boundary, the function $f$ (see remark 2) can be, for example, $f(s) = \frac{1}{\sqrt{4\pi\alpha}} e^{-s^2/4\alpha}$, as in [30] (for the non-local in space variable nonlinearity) or Dirac delta-function to get the local in space variable nonlinearity, the nonlinear function $b$ is given by $b(w) = p \cdot w e^{-w}$. Function $b$ is bounded, so for any continuous delay function $\eta$, satisfying $(\tilde{H})$, the conditions of theorems 1,2 are valid. As a result, we conclude that the initial value problem (1),(4) is well-posed in $C$ and the dynamical system $(S, C)$ has a global attractor (theorem 3).

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