Linear conductance of an interacting carbon nanotube ring

Matthis Eroms,
Institut für Physikalische Chemie, Universität Heidelberg, 69120 Heidelberg, Germany
Leonhard Mayrhofer and Milena Grifoni
Institut für Theoretische Physik, Universität Regensburg, 93035 Regensburg, Germany
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Linear transport through a single-walled carbon nanotube ring, pierced by a magnetic field and capacitively coupled to a gate voltage source, is investigated starting from a model of interacting $p_z$-electrons. The dc-conductance, calculated in the limit of weak tunneling between the ring and the leads, displays a periodic resonance pattern determined by the interplay between Coulomb interactions and quantum interference phenomena. Coulomb blockade effects are manifested in the absence of resonances for any applied flux in some gate voltage regions; the periodicity as a function of the applied flux can be smaller or larger than a flux quantum depending on the nanotube band mismatch.

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I. INTRODUCTION

Mesoscopic rings threaded by a magnetic field represent an important tool for the investigation of quantum interference phenomena. The archetype example is the well known Aharonov-Bohm effect, where the conductance of a clean ring exhibits a periodicity of one flux quantum $\Phi_0 = \hbar/e$. However, impurities and interactions can change this periodicity. In particular when transport through a one-dimensional ring is considered, interactions lead to spin-charge separation, such that the most important contribution to the conductance arises when both charge and spin excitations propagate from one contact to the other arriving at the drain at the same time. Additionally, the dc-conductance of an interacting one-dimensional ring shows Coulomb oscillations with peak positions depending on the applied magnetic field and interaction strength.

Among quasi-one dimensional systems, single-walled carbon nanotubes (SWNTs) have been proved to be extremely interesting to probe electron-electron correlation effects. Luttinger liquid behavior, leading to power-law dependence of various quantities, has been predicted theoretically and observed experimentally in long, straight nanotubes.

Moreover, as typical of low-dimensional systems, short, straight carbon nanotubes weakly attached to leads exhibit Coulomb blockade at low temperatures. In metallic SWNTs two bands cross at the Fermi energy. Together with the spin degree this leads to the formation of electron shells, each accommodating up to four electrons. As a result, a characteristic even-odd or fourfold periodicity of the Coulomb diamond size as a function of the gate voltage is found. Recently, spin-orbit effects in carbon nanotube quantum dots have been observed as well. While the Coulomb blockade can be explained merely by the ground state properties of a SWNT, the determination of the current at higher bias voltages requires the inclusion of transitions of the system to electronic excitations. In a mean-field treatment of the electron-electron interactions has been invoked to calculate the energy spectrum, while a bosonization approach, valid for nanotubes with moderate-to-large diameters ($\phi > 1.5$nm), has been used. Within the bosonization approach the fermionic ground state as well as the fermionic and bosonic excitations can be calculated. For small diameter nanotubes, short range interactions lead to an experimentally detectable singlet-triplet splitting.

Generally, nanotubes have linear or curved shape. Individual circular single-wall carbon nanotubes have been observed in. However, these systems have been poorly experimentally investigated so far. Also from the theoretical point of view, only few works address the properties of toroidal nanotubes. Moreover, except for the studies on the persistent current and on the conductance of toroidal SWNTs, respectively, all the remaining theoretical works neglect
electron-electron interactions effects. However, as for the case of straight SWNTs discussed above, electron correlation effects are expected to crucially influence the energy spectrum and transport properties of toroidal SWNTs. Indeed, in Ref. 26 it is found that for interacting SWNTs rings the persistent current pattern corresponds to the constant interaction model, with a fine structure stemming from exchange correlations. Results for the conductance of a SWNT ring weakly contacted to rings have been presented so far only in the short report 26, where a conductance resonance pattern depending on the interaction strength is reported. Moreover, a detailed derivation of the conductance formula for SWNT rings is missing in the short report 26. In this work we generalize the analysis of Ref. 26 on the conductance of interacting spinless electrons in a one-dimensional ring to the case of a three-dimensional toroidal, metallic SWNT at low energies. To this extent we start from a model of interacting $p_z$ electrons on a graphene lattice at low energies and impose periodic boundary conditions along the nanotube circumference and twisting boundary conditions along the tube axis. At low energies only the lowest transverse energy sub-bands of the ring contribute to transport, so that the problem becomes effectively one-dimensional in momentum space. The three-dimensional shape of the SWNT orbitals in real space, however, crucially determines the final conductance formula. Indeed, in contrast to Ref. 26, we find the absence of interference terms between anti-clockwise and clockwise circulating electrons in the conductance formula. This is due to the localized character of the $p_z$ orbitals and the fact that for a realistic nanotube the contacts are extended. Additionally, we predict an eight orbitals and the fact that for a realistic nanotube the spectrum and transport properties of toroidal SWNTs.

II. THE TOTAL HAMILTONIAN

We consider a ring made of a toroidal metallic single-wall carbon nanotube coupled to a source and drain electrode via tunneling contacts, cf. figure II. The torus is capacitively connected to a gate electrode with gate voltage $V_g$, which changes the chemical potential of the ring. Further, a magnetic flux $\Phi$ threads the center of the torus. We wish to study how the two parameters $V_g$ and $\Phi$ influence the linear conductance of the system. The model Hamiltonian reads

$$\hat{H} = \hat{H}_{\text{ring}} + \hat{H}_L + \hat{H}_R + \hat{H}_T + \hat{H}_{\text{ext}},$$

(1)

where $\hat{H}_{\text{ring}}$, which also includes the parameters $V_g$ and $\Phi$, describes the physics of the isolated SWNT ring and will be discussed in the next section. The second and third term refer to the metallic left and right contacts, described here as Fermi gases of non-interacting electrons. They read

$$\hat{H}_L = \sum_{\alpha} c_{\alpha \vec{q}}^\dagger \epsilon_{\alpha}(\vec{q}) c_{\alpha \vec{q}} - e\phi \sum_{\alpha} \epsilon_{\alpha}(\vec{q}) f_{\alpha \vec{q}} c_{\alpha \vec{q}}^\dagger c_{\alpha \vec{q}}^\dagger,$$

where $\epsilon_{\alpha}(\vec{q})$ is the energy dispersion relation of the lead $\alpha$ and $f_{\alpha \vec{q}}$ is an operator annihilating an electron with wave vector $\vec{q}$ and spin $\sigma$.

The term $\hat{H}_T$ is the Hamiltonian describing tunneling between the ring and the leads. Therefore it consists of two terms, $H_T = H_{TL} + H_{TR}$, and reads

$$\hat{H}_T = \sum_{\alpha=R,L} \int d^3r (T_{\alpha}(\vec{r}) \hat{\Psi}_\alpha^\dagger(\vec{r}) \hat{\Phi}_\alpha(\vec{r}) + \text{h.c.}),$$

(2)

where $\hat{\Psi}_\alpha^\dagger(\vec{r})$ and $\hat{\Phi}_\alpha(\vec{r}) = \sum_{\vec{r}} \phi_{\alpha\vec{q}}(\vec{r}) c_{\alpha \vec{q}}$ are electron operators in the dot and in the leads, respectively, and $T_{\alpha}(\vec{r})$ describes the generally position dependent transparency of the tunneling contact at lead $\alpha$. In the following we choose the representation where $\vec{r} = (x, r_\perp)$, with $x$ being directed along the tube axis, while $r_\perp$ is the coordinate on the nanotube cross-section. With $L$ being the circumference of the SWNT ring, the contacts are positioned in the region about $x = 0$ and $x = L/2$, see Fig. I. Finally, $\hat{H}_{\text{ext}}$ accounts for the energy dependence of the system on the external voltage sources controlling the chemical potential in the leads: $\hat{H}_{\text{ext}} = -e \sum_{\alpha} V_{\alpha} \sum_{\vec{q}} c_{\alpha \vec{q}}^\dagger c_{\alpha \vec{q}}^\dagger$.

III. LOW ENERGY DESCRIPTION OF METALLIC SWNT RINGS

In this section we derive the low energy Hamiltonian $\hat{H}_{\text{ring}}(\Phi) = \hat{H}_{\text{kin}} + \hat{V}_{e-e} + \hat{H}_{\text{gate}}$ of the SWNT ring and diagonalize it. The ring Hamiltonian includes a kinetic term, the electron-electron interactions as well as the effects of a capacitively applied gate voltage $V_g$ and of a magnetic flux $\Phi$. We assume a metallic SWNT. Hence, depending on the diameter of the SWNT, ”low energies” means an energy range of the order of 1 eV around the Fermi energy, where the dispersion relation for the non-interacting electrons near the Fermi points is linear and only the two lowest subbands touching at the
Fermi points can be considered. The effects of the magnetic field are included by introducing twisted boundary conditions (TBC)\(^{22}\). Curvature and Zeeman effects are neglected here. Indeed the Zeeman splitting yields a contribution inversely proportional to the square of the ring radius\(^{22}\) and is relevant only for very small rings. Finally, the linear dispersion relation around the Fermi points allows bosonization\(^{12}\) of the interacting Hamiltonian and its successive diagonalization when only the forward scattering part of the Coulomb interaction is included. The latter approximation is justified for SWNTs with medium-to-large cross-sections (\(0 > 1.5\text{nm}\))\(^{22}\).

A. Twisted boundary conditions

The band structure of non-interacting electrons in a SWNT ring is conveniently derived from the band structure of the \(p_z\) electrons in a graphene lattice. Since each unit cell of the graphene lattice contains two carbon atoms, there are a valence and a conduction band touching at the corner points of the first Brillouin zone. Only two of these Fermi points, \(\pm K_0 = \pm \frac{2\pi}{3\sqrt{3}a_0}c_1\), are independent. Since SWNTs are graphene sheets rolled up into a cylinder, we can obtain the electronic properties of a SWNT by imposing quantization of the wave vector \(\vec{k} = k_x \hat{e}_x + k_y \hat{e}_y + k_z \hat{e}_z\) around the tube waist, i.e., denoting the SWNT circumference with \(L\), one finds \(k_z = \frac{2\pi}{L}n\), \(m = 0, \pm 1, \pm 2, \ldots\), which leads to the formation of several subbands labelled by \(m\). We consider here only SWNTs of the armchair type, \(\hat{e}_x = \hat{e}_y\), which are all metallic. In this case, only the two sub-bands touching at the Fermi points \(\pm K_0 \hat{e}_z\) are relevant.

Figure 2 shows the linear valence and conduction bands of an armchair SWNT. Corresponding to the direction of motion of the electrons, there are two branches \(b = R, L\) with positive and negative slope, respectively. The linear dispersion relation reads

\[\varepsilon(b, \kappa) = \hbar v_F \text{sgn}(b) \kappa,\]

where \(\kappa\) measures the distance between \(k ||\) and the Fermi points \(F = \pm K_0\), i.e., \(k || = \kappa + F\), and \(v_F\) is the Fermi velocity, \(v_F = 8.1 \times 10^5 \text{m/s}\). Moreover, we use the convention \(b = r/l = \pm 1\). The corresponding Bloch waves are of the form

\[\varphi_{Fb\kappa}(\vec{r}) = e^{i\kappa x} \varphi_{Fb\kappa}(\vec{r}) = e^{i\kappa x} \frac{1}{\sqrt{N_L}} \sum_{\vec{R},p} f_{pFb} e^{i Fb \pi} \Pi(\vec{r} - \vec{R} - \vec{r}_p).\]

Here, \(N_L\) is the number of lattice points in the nanotube lattice. The index \(p = 1, 2\) refers to the two graphene sublattices and \(\vec{r}_p\) points from a lattice point \(\vec{R}\) to a carbon atom of sublattice \(p\). For an armchair SWNT, the parameters \(f_{pFb}\) are found to be \(f_{1Fb} = 1/\sqrt{2}\), \(f_{2Fb} = -1/\sqrt{2} \text{sgn}(Fb)\). Finally, we have the \(p_z\) electron orbital \(\Pi\).

As we consider a ring, we can impose in the absence of a magnetic field periodic boundary conditions along the direction parallel to the tube axis, yielding \(k || = \frac{2\pi}{L}n_k\), \(n_k \in \mathbb{Z}\). As the Fermi points \(\pm K_0\) are not necessarily an integer multiple of this step size, we introduce a mismatch parameter \(\delta\), \(\delta \in [-0.5, 0.5]\), so that we can write \(K_0 = \frac{2\pi}{L}n_k \pm \delta\). We still have to take account of the magnetic field. After travelling one time around the ring, an electron picks up a phase \(2\pi \frac{\delta}{\phi_0}\), where \(\phi_0 = \frac{\pi}{e}\) is the flux quantum. This fact can be taken into account by introducing twisted boundary conditions (TBC), \(\Psi(\vec{r} + L\hat{z}) = e^{i\frac{2\pi}{L} \delta} \Psi(\vec{r})\), instead of periodic ones\(^{22}\). Under TBC the allowed values of the momentum parallel to the tube axis \(\kappa\) are

\[\kappa = \frac{2\pi}{L} \left( n_{\kappa} + \frac{\phi}{\phi_0} + \delta \right), \ n_{\kappa} \in \mathbb{Z}\]

\[= \frac{2\pi}{L} \left( n_{\kappa} + \hat{\phi} \right), \ m_{\kappa} \in \mathbb{Z}.\]

Thus the electron operator \(\Psi\) including the electron spin \(\sigma = \uparrow, \downarrow\) reads

\[\Psi(\vec{r}) = \sum_{\sigma} \Psi_{\sigma}(\vec{r}) = \sum_{Fb\kappa} \varphi_{Fb\kappa}(\vec{r}) c_{Fb\kappa}\]

\[= \sum_{Fb\kappa} e^{i\kappa x} \varphi_{Fb\kappa}(\vec{r}) c_{Fb\kappa} = \sqrt{L} \sum_{Fb\kappa} \varphi_{Fb\kappa}(\vec{r}) \psi_{Fb\kappa}(x),\]

where \(c_{Fb\kappa}\) annihilates an electron in state \(|Fb\kappa\rangle\) and \(\psi_{Fb\kappa}(x)\) is a slowly varying 1D operator. Later on the
bosonisability of $\psi_{F\sigma}(x)$ will be of importance to evaluate the interference properties of the interacting SWNT ring.

B. Kinetic energy and gate Hamiltonian

The kinetic energy Hamiltonian in second quantization can in principle be read off from the dispersion relation for the low energy states \[^{[3]}\] :

$$\hat{H}_{\text{kin}} = \hbar v_F \sum_{F,\sigma} \text{sgn}(b) \kappa^{\dagger}_{F\sigma b\kappa} \kappa_{F\sigma b\kappa}. \quad (7)$$

Inserting the $\kappa$-quantization \[^{[4]}\], introducing the number counting operator $N_{F\sigma b} = \sum_{\kappa} \kappa^{\dagger}_{F\sigma b\kappa} \kappa_{F\sigma b\kappa}$ and using the abbreviation $\varepsilon_0 \equiv \hbar v_F \frac{2}{\epsilon}$ we get

$$\hat{H}_{\text{kin}} = \varepsilon_0 \sum_{F,\sigma} \text{sgn}(b) \left( \sum_{\kappa} m_{\kappa} \kappa^{\dagger}_{F\sigma b\kappa} \kappa_{F\sigma b\kappa} + \phi N_{F\sigma b} \right). \quad (8)$$

Gate voltage effects are included in the Hamiltonian $\hat{H}_{\text{gate}} = -\mu_\varepsilon N = -e\alpha V_\varepsilon$, where $N = \sum_{F\sigma b} N_{F\sigma b}$ is the total electron operator of the SWNT; $\mu_\varepsilon$ is the SWNT chemical potential and $\alpha$ a conversion factor.

C. Interaction Hamiltonian

A general form of the Hamiltonian $\hat{V}_{c-e}$ describing interactions between the electrons is

$$\hat{V}_{c-e} = \frac{1}{2} \sum_{\sigma \sigma'} \int d^3r \int d^3r' \psi^\dagger_{\sigma'}(\vec{r}') \Psi^{\dagger}_{\sigma}(\vec{r}) U(\vec{r}-\vec{r}') \Psi_{\sigma}(\vec{r}') \psi_{\sigma'}(\vec{r}),$$

where $U(\vec{r}-\vec{r}')$ is the possibly screened Coulomb potential. By expressing the 3D electron operator in terms of the 1D one, cf. Eq. \[^{[6]}\], and integrating over the components of $\vec{r}$ and $\vec{r}'$ perpendicular to the tube axis one obtains an effective one-dimensional interaction. As discussed \[^{[2]}\], this yields in general interlattice and intralattice interactions. In \[^{[20]}\] it is shown that when the tube diameter is large enough, the distinction between intra and inter-lattice interactions is no longer relevant, and only forward scattering processes, where the number of electrons in each branch remains constant, are relevant.

Introducing the 1D electron density operator $\rho_{F\sigma b}(x) = \psi^\dagger_{F\sigma b}(x) \psi_{F\sigma b}(x)$ and retaining only forward scattering processes we obtain for the interaction Hamiltonian

$$\hat{V}_{c-e} \approx \frac{1}{2} \sum_{F,\sigma} \sum_{b' \sigma'} \int_0^L dx \int_0^L dx' \rho_{F\sigma b}(x) \rho_{F\sigma' b'}(x') V_0(x, x'), \quad (10)$$

with the effective 1D potential $V_0(x, x')$ defined as

$$V_0(x, x') = \frac{L^2}{N_0^2} \int_{R, R'} d^2r_\bot \int d^2r'_\bot \times \left| \langle \Pi(\vec{r}-\vec{R})^2 U(\vec{r}-\vec{r}') \Pi(\vec{r}'-\vec{R}') \rangle^2 \right|. \quad (11)$$

D. Bosonization

The problem of a one-dimensional system of interacting bosons is often conveniently formulated in terms of bosonic operators. A didactic overview can be found \[^{[31]}\]. To this extent we assume a linear dispersion relation over all values of the momentum $\kappa$. This is justified by the long-range character of the interaction, leading to a momentum cut-off still lying in the low-energy range around the Fermi energy. The states with negative energy are assumed to be filled in the ground state, the Fermi sea, of the system. As in \[^{[31]}\], we now introduce new bosonic operators

$$b_{F\sigma q} := \frac{1}{\sqrt{|n_q|}} \left\{ \sum_{\kappa} \kappa^{\dagger}_{F\sigma q\kappa} \kappa_{F\sigma q\kappa} \quad q > 0 \right\} \left\{ \sum_{\kappa} \kappa^{\dagger}_{F\sigma q\kappa} \kappa_{F\sigma q\kappa} \quad q < 0 \right\}, \quad (12)$$

with $q = \frac{2\pi}{L} n_q$, $n_q \in \mathbb{Z}$, and obeying the canonical bosonic commutation relations

$$[b_{F,\sigma q}, b_{F,\sigma q'}^{\dagger}] = \delta_{F,F'} \delta_{\sigma,\sigma'} \delta_{q,q'}, \quad [b_{F,\sigma q}, b_{F,\sigma q'}] = 0. \quad (13)$$

In terms of the bosonic operators \[^{[12]}\] the kinetic energy Hamiltonian \[^{[8]}\] assumes the form

$$\hat{H}_{\text{kin}} := \varepsilon_0 \sum_{F,\sigma} \sum_{q \neq 0} \left[ n_q b_{F,\sigma}^{\dagger} b_{F,\sigma} + \phi \text{sgn}(b) N_{F\sigma b} \right]. \quad (14)$$

For the bosonization of the interaction part \[^{[10]}\], we decompose the density operator $\rho_{F\sigma b}$ into its Fourier components, which are proportional to the bosonic operators \[^{[12]}\] :

$$\rho_{F\sigma b}(x) = \psi^\dagger_{F\sigma b}(x) \psi_{F\sigma b}(x) = N_{F\sigma b}$$

$$+ \sum_{q > 0} \sqrt{|n_q|} \left( e^{i \text{sgn}(b) q x} b_{F\sigma q}(b) + e^{-i \text{sgn}(b) q x} b_{F\sigma q}^{\dagger}(b) \right). \quad (15)$$

By inserting \[^{[15]}\] into \[^{[10]}\] we get

$$\hat{V} = \frac{1}{2} W_0 N^2 + \sum_{F,\sigma} \sum_{F',\sigma'} \frac{1}{2} W_{-q |n_q|} \times$$

$$\times \left( b_{F,\sigma} + b_{F,\sigma-q} \right) \left( b_{F',\sigma'} - b_{F',\sigma'-q} \right),$$

with the total number of electrons $N = \sum_{F\sigma b} N_{F\sigma b}$ in the ring. The effective interaction $V_0(x, x')$ is absorbed
ubov transformation. More details can be found in the
approximate linear combinations of the number counting op-
ors yield
or shortly: \( \hat{F}_{jq} \), \( \tilde{F} = \oplus/\ominus \), \( j = c,s \). The advantage
of this transformation is that the term quadratic in the
bosonic operators \( \hat{b} \) associated to spin, charge and
orbital degrees of freedom:

\[
\begin{align*}
\hat{a}_{\ominus c q} & := \frac{1}{2} (b_{K_0|q} + b_{K_0|q} + b_{-K_0|q} + b_{-K_0|q}), \\
\hat{a}_{\ominus s q} & := \frac{1}{2} (b_{K_0|q} + b_{K_0|q} + b_{-K_0|q} - b_{-K_0|q}), \\
\hat{a}_{\ominus s q} & := \frac{1}{2} (b_{K_0|q} - b_{K_0|q} + b_{-K_0|q} - b_{-K_0|q}), \\
\hat{a}_{\ominus c q} & := \frac{1}{2} (b_{K_0|q} - b_{K_0|q} + b_{-K_0|q} + b_{-K_0|q}), \\
\hat{a}_{\ominus c q} & := \frac{1}{2} (b_{K_0|q} + b_{K_0|q} + b_{-K_0|q} - b_{-K_0|q}),
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\hat{a}_{\ominus s q} & := \frac{1}{2} (b_{K_0|q} - b_{K_0|q} + b_{-K_0|q} - b_{-K_0|q}), \\
\hat{a}_{\ominus c q} & := \frac{1}{2} (b_{K_0|q} - b_{K_0|q} + b_{-K_0|q} + b_{-K_0|q}), \\
\hat{a}_{\ominus c q} & := \frac{1}{2} (b_{K_0|q} + b_{K_0|q} + b_{-K_0|q} - b_{-K_0|q}),
\end{align*}
\]

(\( \ominus sq \)) denote the neutral modes. In total, there are
one charged and three neutral modes.
In order to investigate the combined effects of gate volt-
age and magnetic flux, it is convenient to introduce ap-
propriate linear combinations of the number counting op-
\( \hat{N}_{F r \sigma} : \)
\[
\begin{align*}
\hat{N} & := \sum_{F b \sigma} \sgn(b) \hat{N}_{F b \sigma}, \\
\hat{J} & := \sum_{F b \sigma} \sgn(b) \hat{N}_{F b \sigma},
\end{align*}
\]

being the total particle and total current operators and

\[
\begin{align*}
\hat{N}_{\ominus s} & := \sum_{F b \sigma} \sgn(b) \hat{N}_{F b \sigma}, \\
\hat{J}_{\ominus s} & := \sum_{F b \sigma} \sgn(b) \hat{N}_{F b \sigma},
\end{align*}
\]

Expressing \( \hat{H}_{\text{ring}} \) in terms of these newly defined operators yields

\[
\begin{align*}
\hat{H}_{\text{ring}} & := \hat{H}_{\text{ring}, b} + \hat{H}_{\text{ring}, f} = \sum_{F j} \sum_{q \neq 0} \varepsilon_{F j q} a_{F j q}^\dagger a_{F j q} + \varepsilon_0 \sum_{F b \sigma} \frac{N_{F b \sigma}^2}{2} + \frac{\varepsilon_0}{16} \left( \tilde{W} \hat{N} - k_{\mu_q} \right)^2 + (\hat{J} - \phi)^2 + \frac{\varepsilon_0}{16} \sum_{F j \neq c} (\hat{N}_{F j}^2 + \hat{J}_{F j}^2) + \text{const},
\end{align*}
\]

where \( \tilde{W} = 1 + \frac{8W_0}{\varepsilon_0}, k_{\mu_q} = \frac{\mu_q}{\varepsilon_0 + \varepsilon_0/8} \) and \( \phi = -8\phi \).
The first line corresponds to the bosonic part of the
Hamiltonian, \( \hat{H}_{\text{ring}, b} \), and the second and third lines to the
fermionic part \( \hat{H}_{\text{ring}, f} \). An eigenbasis of \( \hat{H}_{\text{ring}} \) is thus
formed by the states

\[
\left\{ \prod_{F j, q \neq 0} \left( \frac{a_{F j q}^\dagger}{\sqrt{m_{F j q}}} \right)^{m_{F j q}} \left| \tilde{N} \right> \right\} = \{ \left| \tilde{N}, \tilde{m} \right> \},
\]

where \( \left| \tilde{N}, \tilde{m} \right> \) has no bosonic excitations. In general, any
number of electrons can be distributed in many different
ways on the branches \( (F b \sigma) \), which is described by the
set of vectors

\[
\tilde{N} = (N_{K_0r \uparrow}, N_{K_0s \uparrow}, N_{K_0d \uparrow}, N_{K_0d \downarrow}, N_{-K_0r \uparrow}, N_{-K_0s \uparrow}, N_{-K_0d \uparrow}, N_{-K_0d \downarrow}).
\]

For each value of \( \tilde{N} \) one has to include the states which contain
the bosonic excitations of the interacting el-
teons, where the the parameter \( m_{F j q} \) counts the number of
excitations in the channel \( F j \) with momentum \( q \).
IV. LINEAR TRANSPORT

We have now all the ingredients to evaluate the linear transport characteristics of the interacting SWNT ring by use of the Kubo formula.

A. Conductance formula

The current operator at lead $L$ is defined as the rate of change of particles at the contact $L$, i.e., $\hat{I}_L = -e\hat{\dot{N}}_L = -\frac{e}{\hbar}[H, \hat{N}_L] = -\frac{e}{\hbar}[H_{TL}, \hat{N}_L]$ where $-e$ is the electron charge. It yields

$$\hat{I}_L = -\frac{i e}{\hbar} \sum_{\sigma} \int d^3 r (T_L(\vec{r})\Psi^\dagger_\sigma(\vec{r})\Phi_{\sigma L}(\vec{r}) - h.c.).$$

(24)

Analogously the current operator at the right lead is

$$\hat{I}_R = \frac{i e}{\hbar} \sum_{\sigma} \int d^3 r (T_R(\vec{r})\Psi^\dagger_\sigma(\vec{r})\Phi_{\sigma R}(\vec{r}) - h.c.).$$

(25)

To proceed, we use relation (6), relating the 3D operator $\Psi(\vec{r})$ to the slowly varying 1D one $\psi_{F\sigma}(x)$. We assume that the latter does not change significantly in the tunneling region. This yields for the current operators

$$\hat{I}_L = -\frac{i e}{\hbar} \sum_{F\sigma q} \sum_{q} \times$$

$$\left(\sqrt{T} \int d^3 r T_L(\vec{r})\varphi_{F}(\vec{r})\phi_{F\sigma}(\vec{r})\psi^\dagger_{F\sigma L}(x_L)c_{q\sigma L} - h.c.\right)$$

$$= -\frac{i e}{\hbar} \sum_{F\sigma q} \sum_{q} \left(T_{LF\sigma q}\psi^\dagger_{F\sigma L}(x_L)c_{q\sigma L} - h.c.\right),$$

(26)

$$\hat{I}_R = \frac{i e}{\hbar} \sum_{F\sigma q} \sum_{q} \left(T_{RF\sigma q}\psi^\dagger_{F\sigma R}(x_R)c_{q\sigma R} - h.c.\right),$$

(27)

where $x_L$ and $x_R$ are in the middle of the respective tunneling regions. The above expressions for the current operators can in turn be used to evaluate the DC-conductance of the interacting SWNT ring in terms of the Kubo formula

$$G = \lim_{\Omega \to 0} \text{Re} \frac{1}{\hbar \Omega} \int_{-\infty}^{t} dt' e^{i \Omega(t'-t)} \langle [\hat{I}_L(t), \hat{I}_R(t')] \rangle_{eq},$$

(28)

with $\hat{I}_L(t)$ and $\hat{I}_R(t)$ being current operators in the interaction representation, where the interaction is represented by the Hamiltonian $H_{ext}$. Hence, $\langle \cdot \rangle_{eq}$ describes the average with respect to the equilibrium density operator $\hat{\rho}_{eq} := Z^{-1} e^{-\beta(H_{ring} + H_T + H_{L} + H_{R})}$, with $\beta$ the inverse temperature and $Z$ the partition function. It is convenient to introduce the linear susceptibility

$$\chi(t) = -\frac{i}{\hbar} \theta(t) \langle [\hat{I}_L(t), \hat{I}_R(0)] \rangle_{eq}$$

in terms of which (28) assumes the compact form

$$G = \frac{e^2}{\hbar} \lim_{\Omega \to 0} \text{Im} \frac{\chi(\Omega)}{\Omega}.$$  

(29)

Here $\chi(\Omega)$ is the Fourier transform of the response function $\chi(t)$. In (24) the linear susceptibility at imaginary times

$$\chi(\tau_1 - \tau_2) = -\langle T_r [\hat{I}_L(\tau_1), \hat{I}_R(\tau_2)] \rangle_{eq},$$

(30)

where $T_r$ indicates time-ordering, has been evaluated for a spinless Luttinger liquid. Generalizing to the multi-channel situation represented by a 3D toroidal SWNT, an expression for the linear susceptibility can be obtained to lowest non vanishing order in the tunneling matrix elements $T_{\alpha F\beta q}$. A summary of the calculation, where we made use of the explicit form of the nanotube Bloch wave functions, can be found in the subsection below. It delivers for the conductance the remarkably simple result

$$G \approx \frac{e^2}{\hbar} \frac{\chi_{\sigma L}^2}{\hbar^2} \sum_{F\sigma q} \int_{-\infty}^{\infty} d\omega \left( \frac{\partial n_{F\omega}(\omega)}{\partial \omega} \right) |G_{F\sigma L}^{\text{ret}}(\omega, x_L - x_R)|^2,$$

(31)

where $n_{F\omega}(\omega)$ is the Fermi function and $G_{F\sigma L}^{\text{ret}}(\omega, x_L - x_R)$, the retarded Green’s function for the interacting electrons on the SWNT ring, is entirely determined by the slowly varying 1D part of the electron operator (6) as it is the Fourier transform of

$$G_{F\sigma L}^{\text{ret}}(t, x - x') = -\frac{i}{\hbar} \theta(t) \langle \{ \psi_{F\sigma L}(x, t), \psi_{F\sigma L}(x', 0) \} \rangle.$$  

(32)

As the number $N$ of electrons in the dot can vary, $\langle \cdot \rangle$ indicates the thermal average with respect to the grandcanonical equilibrium density matrix $\rho_{\text{ring}, N} = Z_{\text{ring}}^{-1} \text{exp}^{-\beta H_{\text{ring}}}$. In contrast, knowledge of the 3D character of the SWNT Bloch wave functions is encapsulated in the tunneling functions $\Phi_R$ and $\Phi_L$, see Eq. (12) below. We notice that Eq. (31) is not trivial, as it predicts the absence of interference between Green’s functions with different indices, in contrast to the case of a strictly one-dimensional ring considered in (24) and the formula given in (28). This result has its origin in the very strongly localized character of the $p_z$ orbitals, and on the fact that we assumed extended contacts coupling equally to both sublattices of the underlying graphene structure. For temperatures further than $k_B T$ from a resonance (31) further simplifies to

$$G \approx \frac{e^2}{\hbar} \frac{\chi_{\sigma L}^2}{\hbar^2} \sum_{F\sigma q} |G_{F\sigma L}^{\text{ret}}(0, x_L - x_R)|^2.$$  

(33)

B. Proof of the conductance formula (31)

In this subsection, which the hurried reader can skip, the linear susceptibility $\chi(\Omega)$ entering the conductance
In frequency space this yields

\[
\chi(t_2 - t_1) = -\frac{e^2}{\hbar^2} \int_0^{h\beta} d\tau \int_0^{h\beta} d\tau' \sum_{F_1 b_1} \sum_{F_2 b_2} \sum_{F_3 b_3} \sum_{\sigma' \sigma} T_{LF_1 b_1 \bar{q}} T_{LF_2 b_2 \bar{q} + i} T_{RF_3 b_3 \bar{q} - i} T_{RF_4 b_4 \bar{q}} \times
\]

\[
\left\{ \left( T_{\tau' \bar{q} \bar{q}} \psi_{F_1 b_1 \sigma}(\tau, x_L) \psi_{F_2 b_2 \sigma}(\tau, x_L) \psi_{F_3 b_3 \sigma}(\tau', x_R) \psi_{F_4 b_4 \sigma}(\tau', x_R) \right) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau_1) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau_2) + \right.
\]

\[
\left. - \left( T_{\tau' \bar{q} \bar{q}} \psi_{F_1 b_1 \sigma}(\tau, x_L) \psi_{F_2 b_2 \sigma}(\tau, x_L) \psi_{F_3 b_3 \sigma}(\tau', x_R) \psi_{F_4 b_4 \sigma}(\tau', x_R) \right) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau_1) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau') \right.
\]

\[
\left. - \left( T_{\tau' \bar{q} \bar{q}} \psi_{F_1 b_1 \sigma}(\tau, x_L) \psi_{F_2 b_2 \sigma}(\tau, x_L) \psi_{F_3 b_3 \sigma}(\tau', x_R) \psi_{F_4 b_4 \sigma}(\tau', x_R) \right) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau_1) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau_2) + \right.
\]

\[
\left. + \left( T_{\tau' \bar{q} \bar{q}} \psi_{F_1 b_1 \sigma}(\tau, x_L) \psi_{F_2 b_2 \sigma}(\tau, x_L) \psi_{F_3 b_3 \sigma}(\tau', x_R) \psi_{F_4 b_4 \sigma}(\tau', x_R) \right) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau_1) G_{\bar{q} \bar{q}}(\bar{q}, \tau - \tau') \right\},
\]

(34)

where \( G_{\alpha \alpha}(\bar{q}, \tau - \tau') = -\langle T_{\tau'}(c_{\bar{q} \alpha}(\tau) c_{\bar{q} \alpha}^\dagger(\tau')) \rangle_\alpha \) denotes the Green’s function for the free electrons of lead \( \alpha \). In \( \langle \cdot \rangle_\alpha \) describes the average with respect to the equilibrium density operator \( Z^{-1}_\alpha e^{-\beta H_\alpha} \) of the lead \( \alpha \).

In this approximation multiple interference processes are neglected. The conductance is thus expressed in terms of the single particle Green’s functions

\[
G_{Fb \sigma}(\tau - \tau', x_\alpha - x_\beta) := \langle T_{\tau'} \psi_{Fb \sigma}(\tau, x_\alpha) \psi_{Fb \sigma}^\dagger(\tau', x_\beta) \rangle.
\]

(36)

In frequency space this yields

\[
\tilde{\chi}(i\Omega_n) = -\frac{e^2}{\hbar^2} \frac{1}{\hbar\beta} \sum_{\omega, \omega'} \delta_{\Omega_n, \omega - \omega'}
\]

\[
\times \sum_{f b} \sum_{f' b'} \sum_{\sigma' \sigma} T_{LFb f} T_{LF' b' f'} T_{RF b' \sigma} T_{RF' \sigma f'}
\]

\[
\times G_{F' b' \sigma}(\omega, x_L - x_R) G_{F b \sigma}(\omega - i\Omega_n, x_R - x_L)
\]

\[
\times [G_{L \sigma}(\bar{q}, \omega - i\Omega_n) - G_{L \sigma}(\bar{q}, \omega)]
\]

\[
\times [G_{R \sigma}(\bar{q} + i\Omega_n) - G_{R \sigma}(\bar{q}, \omega + i\Omega_n)].
\]

(37)

As a further step we perform the summation over the fermionic frequencies and carry out the analytical continuation \( i\Omega_n \rightarrow \Omega \) to find \( \tilde{\chi}(\Omega) \). From (29) the conductance then follows as

\[
G = \frac{1}{\hbar^2} \sum_{Fb} \sum_{F' b'} \sum_{\sigma} T_{LFb f} T_{LF' b' f'} T_{RF b' \sigma} T_{RF' \sigma f'}
\]

\[
\times \int_{-\infty}^{\infty} d\omega G_{Fb \sigma}(\omega, x_L - x_R) G_{F' b' \sigma}(\omega, x_R - x_L)
\]

\[
\times \left(-\frac{\partial n_F(\omega)}{\partial \omega}\right) 4 \text{Im} G_{LF b}^{\text{ret}}(\bar{q}, \omega) \text{Im} G_{RF b}^{\text{ret}}(\bar{q}, \omega),
\]

where the superscript "ret/adv" refer to retarded/advanced Green’s functions, respectively. Such expression can be simplified further by replacing the summation over the \( \bar{q} \)-values with an integral over energies:

\[
\sum_{\bar{q}} = \int d\bar{q} \rho_L(\bar{q}) \sum_{\bar{q}},
\]

where \( \rho_L(\bar{q}) \) is the density of states of lead \( L \). Analogous procedure holds for the right lead. Recalling the explicit expression of
the tunneling function $T_{\alpha F b q}$ one then finds

$$G = \frac{L^2 e^2}{\hbar \pi^2} \sum_{Fb} \sum_{F' b'} \sum_{\sigma} \int d\varepsilon \rho_L(\varepsilon) \sum_{\tilde{q}\varepsilon} \int d\varepsilon' \rho_R(\varepsilon') \sum_{\tilde{q}\varepsilon'}$$

\begin{equation}
\times \int d^3 r_1 T_L(\tilde{r}_1) \varphi_{F b}^*(\tilde{r}_1) \frac{1}{\sqrt{|V|}} e^{i\tilde{q}\varepsilon \cdot \tilde{r}_1}
\end{equation}

\begin{equation}
\times \int d^3 r_2 T_L^*(\tilde{r}_2) \varphi_{F' b'}(\tilde{r}_2) \frac{1}{\sqrt{|V|}} e^{-i\tilde{q}\varepsilon' \cdot r_2}
\end{equation}

\begin{equation}
\times \int d^3 r_3 T_R(\tilde{r}_3) \varphi_{F b}^*(\tilde{r}_3) \frac{1}{\sqrt{|V|}} e^{i\tilde{q}\varepsilon \cdot \tilde{r}_3}
\end{equation}

\begin{equation}
\times \int d^3 r_4 T_R^*(\tilde{r}_4) \varphi_{F' b'}(\tilde{r}_4) \frac{1}{\sqrt{|V|}} e^{-i\tilde{q}\varepsilon' \cdot r_4}
\end{equation}

\begin{equation}
\times \int_{-\infty}^{\infty} d\omega G_{F b\sigma}^{\text{ret}}(\omega, x_L - x_R) G_{F' b' \sigma}^{\text{adv}}(\omega, x_R - x_L)
\end{equation}

\begin{equation}
\times \left(-\frac{\partial N_F(\omega)}{\partial \omega}\right) 4\pi^2 \delta(\omega - \varepsilon / \hbar) \delta(\omega - \varepsilon' / \hbar),
\end{equation}

where we made use of the expression for the spectral density of a free electron gas $A_{\alpha\sigma}(\tilde{q}, \omega) = -2\text{Im} (G_{\alpha\sigma}^{\text{ret}}(\tilde{q}, \omega)) = 2\pi \delta(\omega - \varepsilon_{\tilde{q}/\hbar})$, and we assumed that the leads wave functions are well approximated by plane waves. A crucial simplification follows now from the observation that the $p_z$ orbitals, entering the products $\varphi_{F b}^*(\tilde{r}_1) \varphi_{F b}(\tilde{r}_4)$, are strongly localized at the carbon atoms. The summation over the wave numbers $\tilde{q}$ associated with an energy $\varepsilon$ yields

\begin{equation}
\sum_{\tilde{q}\varepsilon} \frac{1}{\sqrt{L}} e^{i\tilde{q}\varepsilon \cdot (\tilde{R} + \tilde{r}_p - \tilde{R}' - \tilde{r}_p'^{\prime})}
\end{equation}

\begin{equation}
= \frac{4\pi \sin(q_\parallel |\tilde{R} + \tilde{r}_p - \tilde{R}' - \tilde{r}_p'^{\prime}|)}{q_\parallel |\tilde{R} + \tilde{r}_p - \tilde{R}' - \tilde{r}_p'^{\prime}|} \approx 4\pi \delta_{\tilde{R}, \tilde{R}'}, \delta_{p, p^{\prime}},
\end{equation}

where the latter equation means that we assume the lead wave vectors $\tilde{q}_i$ at the Fermi level to be larger than $1/a_0$, with $a_0$ the nearest neighbour distance on the graphene lattice. This is satisfied e.g. for conventional gold leads. We thus obtain

\begin{equation}
G = \left(\frac{2\pi}{\hbar}\right)^2 C^2 L^2 \sum_{F b} \sum_{F' b'} \sum_{\sigma} \sum_{pp'} f_{F b p} f_{F' b' p'} f_{F'' b'' p'} f_{F'' b'' p''} \times \int d\omega \rho_L(\hbar\omega) \rho_R(\hbar\omega) \left(-\frac{\partial N_F(\omega)}{\partial \omega}\right)
\end{equation}

\begin{equation}
\times \int_{-\infty}^{\infty} d\omega G_{F b\sigma}^{\text{ret}}(\omega, x_L - x_R) G_{F' b' \sigma}^{\text{adv}}(\omega, x_R - x_L),
\end{equation}

where the constant $C$ results from the integration over the $p_z$ orbitals. Since we assume an extended tunneling region, the fast oscillating terms with $F = -F'$ are supposed to cancel. Furthermore, we assume that both sublattices are equally coupled to the contacts, such that the sum over $\tilde{R}$ ($\tilde{R}'$) should give approximately the same results for $p = 1$ and $p = 2$. Hence, we can separate the sum over $p$ and $p'$ from the rest and exploit the relation

\begin{equation}
\sum_p f_{F b p} f_{F' b' p'} = \delta_{b, b'}.
\end{equation}

Introducing now the transmission function

\begin{equation}
\Phi_L = \frac{C_L}{N_L} \sum_{\tilde{R}, p} |T_L(\tilde{R} + \tau_p)|^2 \rho_L(\varepsilon_F),
\end{equation}

and the analogous one, $\Phi_R$, for the right lead we obtain Eq. (31) when the lead density of states is approximated with its value at the Fermi level (wide band approximation).

V. THE RETARDED GREEN’S FUNCTION

As the essential information on the conductance is contained in $G_{F b\sigma}^{\text{ret}}$, the main task lies in obtaining an expression for this retarded Green’s function in the case of a toroidal SWNT.

A. Time dependence

First we need to determine the time-dependence of the 1D electron operator introduced in (6). To this extent, it is convenient to make use of the so called bosonization identity, where the 1D electron operator is written as a product of a fermionic operator, the Klein factor, and a bosonic field operator. A derivation of this identity can be found e.g. in [31]. It yields

\begin{equation}
\psi_{F b\sigma}(x) = \eta_{F b\sigma} e^{i\frac{\pi}{4} N_F b_{x} \text{sgn}(b) + \frac{\pi}{4} x} e^{i\phi_{F b\sigma}(x)},
\end{equation}

where the Klein factor $\eta_{F b}$ lowers the number of electrons with given quantum numbers $F b \sigma$ by one, and

\begin{equation}
\phi_{F b\sigma}^\dagger(x) = \int_{q > 0} \frac{1}{\sqrt{|q|}} e^{-i\text{sgn}(b) q x} b_{F x \text{sgn}(b) q} e^{-\alpha q^2 / 2}.
\end{equation}

The infinitesimally small number $\alpha > 0$ in (44) serves to avoid divergences due to the extension of the linearised dispersion relation. Thus the time evolution is determined by the evolution of the Klein factors $\eta_{F b\sigma}(t)$ and by that of the bosonic fields $\phi_{F b\sigma}(t)$ (43). Let us then have a closer look at the Hamiltonian (21). It is made up of a part depending on the bosonic operators and one depending on the number counting operators. These two types of operators commute with each other. Therefore, the time evolution of the operators $\eta_{F b\sigma}(t)$ and $\phi_{F b\sigma}(x, t)$ can be treated separately. The bosonic part $H_{\text{ring}, b}$ yields the simple time evolution $a_{\tilde{F}jq}^\dagger(t) = e^{-i\tilde{F}jq t} a_{\tilde{F}jq}^\dagger$ and
analogously for the operator $a_{F,b}^\dagger(t)$. This can be related to the time evolution of the operators $b_{F,\sigma}$ appearing in the fields $\phi_{F,\sigma}$. The corresponding calculation for the Klein factors is a bit longer and yields

$$
\eta_{F,b\sigma}(t) = e^{-\text{tr} \left[ \frac{\tau_0}{2} (N_{F,b\sigma} + \frac{1}{2} + \tilde{\phi}_F(b)) + \tau_0 (N + \frac{1}{2}) - \mu_s \right] t} \eta_{F,b\sigma} =: Y_{F,b\sigma}(t) \eta_{F,b\sigma}(0), \tag{45}
$$

as well as

$$
\eta_{F,b\sigma}^+(t) = e^{\text{tr} \left[ \frac{\tau_0}{2} (N_{F,b\sigma} - \frac{1}{2} + \tilde{\phi}_F(b)) + \tau_0 (N - \frac{1}{2}) - \mu_s \right] t} \eta_{F,b\sigma}^+ =: Y_{F,b\sigma}^+(t) \eta_{F,b\sigma}^+(0). \tag{46}
$$

**B. Separation of the correlation function**

Let us now turn to the averaging process. As the number of electrons $N$ in the dot can vary we have to work in the grand canonical ensemble, summing over all values of $N$. The correlation function

$$
\langle \psi_{F,b\sigma}^\dagger(x',0) \psi_{F,b\sigma}(x,t) \rangle = \text{Tr} \left\{ \hat{\rho}_{\text{ring},N} \psi_{F,b\sigma}^\dagger(x',0) \psi_{F,b\sigma}(x,t) \right\}, \tag{47}
$$

is a thermal expectation value with respect to the grand-canonical density matrix operator \(\hat{\rho}_{\text{ring},N} = \frac{e^{-\beta \hat{H}_{\text{ring}}}}{Z_f}\), and the trace is over the many body states \(\langle \vec{N}, \vec{m} \rangle\), cf. \[22\], where \(\vec{N}\) and \(\vec{m}\) define the fermionic and bosonic configurations, respectively. Accordingly, the trace is

$$
\text{Tr} \{ \ldots \} \equiv \sum_N \sum_{\{N\}_N} \sum_{\vec{m}} \langle \vec{N}, \vec{m} \rangle \langle \vec{N}, \vec{m} \rangle. \tag{48}
$$

In order to make use of the unitarity of the Klein factors, we exploit the fact that the above correlator only depends on the time difference. Thus, with the help of the definitions \[13\], \[41\] and \[40\] we get

$$
\langle \psi_{F,b\sigma}^\dagger(x',0) \psi_{F,b\sigma}(x,t) \rangle = \langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle = \sum_N \sum_{\{N\}_N} \sum_{\vec{m}} \langle \vec{N}, \vec{m} \rangle \langle \vec{N}, \vec{m} \rangle \eta_{F,b\sigma}^+(0) \eta_{F,b\sigma}(0)

\times e^{-i 2 \pi (N_{F,b\sigma} \text{sgn}(b) + \tilde{\phi}) x} Y_{F,b\sigma}^+(t) \eta_{F,b\sigma}(0) \eta_{F,b\sigma}(0)

\times e^{i 2 \pi (N_{F,b\sigma} \text{sgn}(b) + \tilde{\phi}) x} e^{i \phi_{F,b\sigma}(x,0)} e^{i \phi_{F,b\sigma}(x,0)} \langle \vec{N}, \vec{m} \rangle. \tag{49}
$$

As the bosonic and the fermionic operators commute, the correlation function can now be separated into a fermionic and a bosonic part:

$$
\langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle = \langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle_f \langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle_b,
$$

where, with $E_{\text{ring},f}(\vec{N})$ being the eigenvalues of $\hat{H}_{\text{ring},f}$,

$$
\langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle_f = \sum_N \sum_{\{N\}_N} \frac{1}{Z_f} e^{-\beta E_{\text{ring},f}}

\times Y_{F,b\sigma}^+(t) e^{-i 2 \pi (N_{F,b\sigma} \text{sgn}(b) + \tilde{\phi}) (x' - x)}, \tag{50}
$$

and

$$
\langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle_b = \frac{1}{Z_b} \sum_{\vec{m}} \langle \vec{0}, \vec{m} \rangle e^{-\beta \hat{H}_{\text{ring},b}}

\times e^{-i \phi_{F,b\sigma}(x',-t)} e^{-i \phi_{F,b\sigma}(x',-t)} e^{i \phi_{F,b\sigma}(x,0)} e^{i \phi_{F,b\sigma}(x,0)} \langle \vec{0}, \vec{m} \rangle. \tag{51}
$$

In the bosonic part we only need to take care of the state $\langle \vec{0}, \vec{m} \rangle$, as the bosonic excitation spectrum is the same for all charge states. The partition functions read

$$
Z_f = \sum_{\{N\}_N} \langle \vec{N}, 0 \rangle e^{-\beta \hat{H}_{\text{ring},f}} \langle \vec{N}, 0 \rangle, \tag{52}
$$

$$
Z_b = \sum_{\vec{m}} \langle \vec{0}, \vec{m} \rangle e^{-\beta \hat{H}_{\text{ring},b}} \langle \vec{0}, \vec{m} \rangle. \tag{53}
$$

In order to evaluate the Green’s function we also need the correlator $\langle \psi_{F,b\sigma}(x,t) \psi_{F,b\sigma}(x') \rangle$. As above, it can be factorized in a fermionic and a bosonic part. It holds,

$$
\langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle_b = \langle \psi_{F,b\sigma}(x,t) \psi_{F,b\sigma}^\dagger(x',0) \rangle_b^* \tag{54}
$$

For the fermionic part we find

$$
\langle \psi_{F,b\sigma}(x,t) \psi_{F,b\sigma}^\dagger(x',0) \rangle_f = \sum_{\{N\}_N} \frac{1}{Z_f} e^{-\beta E_{\text{ring},f}}

\times Y_{F,b\sigma}(t) e^{-i 2 \pi (N_{F,b\sigma} \text{sgn}(b) + \tilde{\phi}) (x' - x)}, \tag{55}
$$

C. The bosonic part of the Green’s function

The evaluation of the bosonic part of the Green’s function is lengthy but standard. Following e.g. \[31\] we obtain

$$
\langle \psi_{F,b\sigma}^\dagger(x',-t) \psi_{F,b\sigma}(x,0) \rangle_b = \frac{1}{1 - e^{-\alpha \frac{\pi}{2}}} \tag{56}
$$

$$
\times \exp \left\{ -\frac{1}{4} \sum_{q > 0} \frac{e^{-\alpha q}}{n_q} \left[ \sum_{F_j} Q_{F_j q} (-t, \text{sgn}(b)(x' - x)) + 2 \mathcal{C}_q^2 \left\langle Q_{F_j q} (-t, (x' - x)) + Q_{F_j q} (-t, -(x' - x)) \right\rangle \right] \right\},
$$

where we introduced the correlation function

$$
Q_{\gamma q}(t, x) := \sin \left( \frac{\varepsilon_{\gamma q} t}{\hbar} - qx \right) \coth \left( \frac{\beta \varepsilon_{\gamma q}}{2} \right). \tag{57}
$$
The parameters $C_{\alpha \beta \gamma}^2$ entering in the Bogoliubov transformation which diagonalizes the bosonic part of the Hamiltonian, cf. (19), are defined in the Appendix A. To proceed in the analytic evaluation of the bosonic part of the Green's function we perform the following two approximations. The first one consists in linearising the Hamiltonian, cf. (19), are defined in the Appendix A.

\[ \varepsilon_{eq} = \hbar v_c |q| = \hbar \frac{v_F}{K_c} |q|, \]

(58)

where we introduced the charge velocity $v_c$. The coupling constant $K_c$ is given by $K_c = \left( 1 + \frac{8q_0^2}{\varepsilon_{eq}^2} \right)^{-1/2}$, $q_0 = 2\pi/L$, and is determined by the interaction for small values of the momentum $q$. In case of a repulsive interaction $K_c$ is always smaller than 1, yielding $v_c > v_F$, $\varepsilon_{eq} > \varepsilon_{q0}$ and $\varepsilon_c = \varepsilon_0/K_c > \varepsilon_0$.

The second approximation refers to the parameter $C_{\alpha \beta \gamma}^2$. We already assumed in section III C that, due to the long range character of the interaction, a momentum cut-off $q_c$ can be introduced. For high values of the momentum $q$, the parameter $C_{\alpha \beta \gamma}^2$ which depends on the interaction, has to vanish. We assume an exponential decay in $q$-space, $C_{\alpha \beta \gamma}^2 = C_{\alpha \beta \gamma}^2 e^{-q/q_c}$. The relation between $C_{\alpha \beta \gamma}^2$ and $K_c$ is found with the help of the transformation parameters $\alpha_q$ and $\beta_q$:

\[ \alpha_q = \frac{1}{2} \sqrt{\frac{\varepsilon_{0q}}{2 \varepsilon_{eq}}} \quad \beta_q = \frac{1}{2} \sqrt{\frac{\varepsilon_{0q}}{2 \varepsilon_{eq}}}. \]

For $q = q_0$ they read: $\alpha_{q0} = \frac{1}{2} \sqrt{K_c/2}$, $\beta_{q0} = 1/(2 \sqrt{2K_c})$, leading to the expression

\[ C_{\alpha \beta \gamma}^2 = \frac{1}{64} \left( \frac{1}{\alpha_{q0}} - \frac{1}{\beta_{q0}} \right)^2 = \frac{1}{8} \left( K_c^{-1/2} - K_c^{-1/2} \right)^2. \]

(59)

With these two approximations we can evaluate the bosonic correlation function at low temperatures, $k_B T \ll \varepsilon_0$ (in this regime $\coth(\beta \varepsilon_0/2) \approx 1$). We find, see Eq. (B7) in Appendix B

\[ \langle \psi_{Fb\sigma}^\dagger(x',-t)\psi_{Fb\sigma}(x,0) \rangle_b = \left[ 1 - z \left( \alpha + \frac{1}{q_c} \right) \right] C_{\alpha \beta \gamma}^2 \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} F_{k,l}(b, x' - x) e^{\frac{\pi}{2} \alpha k l} e^{\frac{\pi}{2} \beta k l}, \]

(60)

where $z(\alpha, x) = e^{\alpha x^2} e^{\frac{\pi}{2} x}$.

D. The Fourier transform of the Green’s function

Despite its intricate form, see (B7), the time dependence of the bosonic part of the Green’s function is trivial. Gathering then together bosonic and fermionic contributions we obtain for the Fourier transform of the SWNT Green’s function

\[ G_{Fb\sigma}^{\text{ret}}(\omega, x' - x) = \left[ 1 - z \left( \alpha + \frac{1}{q_c} \right) \right] C_{\alpha \beta \gamma}^2 \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{\omega - E_{Fb\sigma}} e^{-i \sqrt{2D} (\mathbf{q}_b \cdot \mathbf{g}(b) + \tilde{\phi})(x' - x)} \]

\[ \times \sum_{k,l=0}^{\infty} \left[ \frac{F_{k,l}(b, x' - x)}{\omega - \tilde{E}_{k,l}(N_{Fb\sigma}, N, \mu_g, \tilde{\phi}) + i\eta} + \frac{F_{k,l}(b, x' - x)}{\omega - \tilde{E}_{-k,-l}(N_{Fb\sigma} - 1, N + 1, \mu_g, \tilde{\phi}) + i\eta} \right], \]

(61)

where $\eta$ is a positive infinitesimal and the energy

\[ E_{-k,-l}(N_{Fb\sigma} + 1, N + 1, \mu_g, \tilde{\phi}) = -\mu_g + \varepsilon_0(N_{Fb\sigma} + k + \frac{1}{2} + \tilde{\phi} sgn(b)) + W_{00}(N + \frac{1}{2} + \varepsilon_c l) \]

is the energy needed to add a particle in the branch ($Fb\sigma$) to a system with $N$ electrons together with a difference of $k$ neutral and $l$ charged bosonic excitations. Likewise, $-\tilde{E}_{k,l}(N_{Fb\sigma}, N, \mu_g, \phi)$ is the energy gained by removing a particle on branch ($Fb\sigma$) from a system with $N$ electrons plus bosonic excitations.

VI. RESULTS FOR THE CONDUCTANCE

We are now finally at a stage in which we can discuss some features of the conductance of a toroidal SWNT, as they follow from the conductance formula (31) together with the expression for the Fourier transform of the Green’s function (61).

A. Summation over the relevant configurations

The expression (61) is quite intricate, as it requires a summation over all the fermionic and bosonic configurations. A noticeable simplification, however, occurs in the low temperature regime $\varepsilon_0 > k_B T$, which we shall address in the following. In this regime in fact we need to consider ground state configurations only of the SWNT ring, i.e., we can neglect bosonic and fermionic excitations. Neglecting bosonic excitations means to set $k = l = 0$ in the sum in (61), such that the only effect of the bosonic part of the Green’s function is in the interaction-dependent prefactor $\left[ 1 - z \left( \alpha + \frac{1}{q_c} \right) \right] C_{\alpha \beta \gamma}^2$. With $\tilde{\alpha} = \alpha + q_c^{-1}$ it yields a conductance suppression proportional to $\tilde{\alpha}/L)^{2C_{\alpha \beta \gamma}^2} = (\tilde{\alpha}/L)^2 (K_c + K_c^{-1} - 2)/4$ with respect to the non-interacting case. To find the relevant fermionic configurations we observe that at low temperatures the contribution to the trace comes from those values of the total electron number $N$ and of the total current $J$ which minimize the energy $E_{\text{ring}}(J,N)$. This energy is provided by the fermionic part of the Hamiltonian (21) and is determined by the applied gate voltage and magnetic field.
where now \( k \) fermonic species.

**FIG. 3:** Ground states resonances for a ring with two fermionic species \( b = R, L \). Along the lines the energies are degenerate and transport is there maximal. Continuous lines correspond to the interaction parameter \( \tilde{W} = 2 \), dotted lines to the non-interacting case, \( \tilde{W} = 1 \).

1. Spinless case

To clarify the ideas, let us first consider a ring with only two species of spinless electrons characterized by \( b = R, L \). Then, \( \hat{H}_{\text{ring},t} \) simplifies to

\[
\hat{H}_{\text{ring},t} = \frac{\tilde{\varepsilon}_0}{4} \left[ \tilde{W} (\mathcal{N} - k_{\mu_g})^2 + (\mathcal{J} - k_\phi)^2 \right],
\]

where now \( k_{\mu_g} \) and \( k_\phi = -2\tilde{\phi} + \tilde{W} = 1 + \frac{2W_{\text{eff}}}{\varepsilon_0} \) hold. Figure 3 shows which ground state \( (\mathcal{N}, \mathcal{J}) \) is occupied in the \( (k_\phi, k_{\mu_g}) \)-plane. The continuous lines are obtained by setting \( \tilde{W} = 2 \), whereas the dotted ones belong to the non-interacting case \( \tilde{W} = 1 \). Along such lines the energies of neighboring ground states are degenerate. At zero temperature these are the conductance resonances. Figure 4 shows the ground state energies \( E(k_{\mu_g}, \mathcal{N}, \mathcal{J}) \) for \( k_\phi = 0 \). For values of \( k_{\mu_g} \) around zero the ring is in the state \( (\mathcal{N} = 0, \mathcal{J} = 0) \). Raising the external gate voltage above \( k_{\mu_g} = 0.75 \) leads to a transition to a charged state with one electron. Above \( k_{\mu_g} = 1.25 \) the state \( (\mathcal{N} = 2, \mathcal{J} = 0) \) is the one with the lowest energy. We observe that along \( k_\phi = 0 \) for the states with an odd number of electrons there is also a degeneracy between the states with the same charge but different current directions.

Let us now turn back to the SWNT ring. Due to the presence of the spin and orbital degrees of freedom, the fermionic part of the Hamiltonian is quadratic in all of the variables \( \mathcal{N}_F, \mathcal{J}_F \), cf. [21]. However, it remains gapped only with respect to the total charge and current operators. Let us then introduce for given \( \mu_g \) and \( \tilde{\phi} \) the integer numbers \( N_c \) which minimize \( E_{\text{ring},t} \). Outside resonance, only one couple \( (N_c, J_c) \) contributes to the grandcanonical sum in (61). Nearby resonance we have to include states which differ by one electron and with corresponding current configurations differing by plus/minus one. Moreover, in the low temperature regime \( k_B T < \varepsilon_0 \) the quadratic forms in \( \tilde{H}_{\text{ring},t} \) impose that we only have to take into account those configurations \( \{ \mathcal{N} \} \), where the number of electrons in each configuration \( (F' b' \sigma') \) differs at most by one (i.e., we neglect "fermionic excitations"). From now on we measure the number of excess electrons or holes from the charge neutrality point where \( \tilde{\mathcal{N}} = 0 \). Thus, \( N_{F' b' \sigma'} \) can only assume the values \( m \) and \( m + 1 \), where \( m \geq 0 \) above and \( m < 0 \) below the charge neutrality point. This means that \( N_c \), obtained by summing over the occupation of all the branches \( (F' b' \sigma') \), can be written as \( N_c = 8m + l \), \( m \in \mathbb{Z}, \quad l = 0, 1, \ldots, 7 \). Notice that this implies an eight-electron periodicity as a function of the gate voltage. In contrast, as \( J_c \) counts the occupation difference between the orbital branches, \( b = R, L \), it can only take values between \(-4\) and \(+4\). We can thus simplify the trace operation considerably by summing only over the dominant configurations in (61). These are the configurations where \( N_c \rightarrow N_c + 1 \) with \( N_{F b \sigma} = m \), and
\( N_c + 1 \rightarrow N_c \) with \( N_{Fb\sigma} = m + 1 \). We obtain

\[
G^{\text{ret}}_{Fb\sigma}(\omega, x' - x) = \left[ 1 - z \left( \alpha + \frac{1}{g_c} \right) \right] C^2_{\omega = 0} \times e^{-\beta E_{\text{ring},f}(N_{N_c+1})} + e^{-\beta E_{\text{ring},f}(N_{N_c})} e^{-\frac{i}{2} \text{sgn}(b)(x' - x)}
\]

\[
e^{-\beta E_{\text{ring},f}(N_{N_c}) + e^{-\beta E_{\text{ring},f}(N_{N_c+1})}}
\]

\[
h\omega - E_{00}(m + 1, N_c + 1, \mu_g, \phi) + i\eta.
\]

Recalling that \( E_{\text{ring},f}(N_{N_c+1}) - E_{\text{ring},f}(N_{N_c}) = E_{00}(m + 1, N_c + 1, \mu_g, \phi) \equiv E_0 \), we obtain

\[
|G^{\text{ret}}_{Fb\sigma}(\omega, x' - x)|^2 = \left[ 1 - z \left( \alpha + \frac{1}{g_c} \right) \right] C^2_{\omega = 0} \times \cos^2(\Delta x) + \sin^2(\Delta x) \tanh^2 \left( \frac{\beta E_0}{2} \right)
\]

\[
\times \frac{\hbar^2}{|h\omega - E_{00}(m + 1, N_c + 1, \mu_g, \phi) + i\eta|^2}.
\]

where \( \Delta x := \frac{1}{2} (\text{sgn}(b)(x' - x)) \). When Eq. (65) is inserted in Eq. (61) the conductance diverges, as a consequence of the presence of the infinitesimal convergence factor \( i\eta \) rather than of a finite line-width \( \Delta x \). Nevertheless, Eq. (65) allows already to discuss the main qualitative features of the conductance, namely the resonance pattern as a function of the applied gate voltage and magnetic field.

To be definite, conductance resonances occur at a degeneracy for the removal of either a counter-clockwise or a clockwise propagating electron. Moreover, as in the spinless case, double resonances occur for special values of the applied magnetic flux and gate voltage, such that clockwise and anti-clockwise propagating modes are simultaneously at resonance. As the ring energy is independent of the spin and Fermi number degrees of freedom, a double resonance is uniquely fixed by \( N_c, J_c \). Let us consider e.g. \( N_c = 0, J_c = 0 \) and \( N_c + 1 = 1, J_c = \pm 1 \). Then it only exists a configuration with \( N_c = 0 \), while there are four equivalent configurations with \( N_c = +1, J_c = 1 \) and four with \( N_c = -1, J_c = -1 \) contributing to the double resonance.

The resonant pattern is easily obtained by approximating the derivative of the Fermi function in Eq. (61) with a delta function such that \( G \propto \sum_{Fb\sigma} |G^\text{ret}_{Fb\sigma}(0, x' - x)|^2 \), and then evaluating the poles of (65) as a function of \( k_{\mu}\) and \( k_{\phi} \). Upon recalling that \( N_c = 8m + l \), with \( m \in \mathbb{Z} \) and \( l = 0, 1, \ldots, 7 \), we find

\[
k_{\mu} = 8m + \frac{4}{\bar{W}} \left( 1 + \frac{W_0}{\varepsilon_0} \right) + \frac{8W_0}{\varepsilon_0^2 W} - k_{\phi} \text{sgn}(b).
\]

Notice that this implies a distance between adjacent resonant lines, with \( l \) and \( l + 1 \), of \( \Delta k_{\mu_l} = \frac{4W_0}{\varepsilon_0^2 W} \). The corresponding results are shown in figure 6. We choose a value of \( \bar{W} = 2 \) for the plot. The solutions for the non-interacting system are shown with dotted lines. In the non-interacting case the ring is occupied by a multiple of four electrons. From one square to the next other four electrons can be accommodated at one time, as there is no charging energy. In the presence of interactions, however, the picture is entirely different. The former squares are deformed to hexagons which only touch at their vertical boundary. There, the two ground states are degenerate. The conductance is zero here. In the parallel stripes in figure 6 we indicated one possible fermionic configuration only. There is no difference in the energy if we add the first electron to a spin-down state instead of a spin-up one as we did. The choice of \( F \) is arbitrary as well. The main difference to the non-interacting case is that, similar to the spinless case, for certain values of \( k_{\mu}\) no conductance resonances can be achieved.

Finally, Eq. (65) predicts a periodic pattern of the conductance as a function of the distance between left and right leads, with maxima when the distance between leads is \( x_R - x_L = L/2 \), cf. Fig. 1.

\section{Conclusions}

In this work the linear conductance of an interacting toroidal SWNT weakly coupled to leads and pierced by a magnetic field has been analyzed. We focussed on low temperatures \( k_B T < \varepsilon_0 \), where \( \varepsilon_0 \) is the mean level spacing. In this regime Coulomb blockade effects are known to crucially affect the conductance of a straight SWNT.
as a function of an applied gate voltage.\textsuperscript{13,16} Likewise, Coulomb interactions strongly influence the conductance characteristics of the ring. Due to charging effects, the electrons can enter the ring only one by one and not up to four at the time as in the non-interacting case. While in straight tubes the conductance pattern has a four-electron periodicity, for toroidal SWNTs we predict a periodicity of eight electrons. The conductance exhibits Coulomb resonances, and the resonance pattern is a function of the applied magnetic field $\phi$ and gate voltage $\mu_g$. In contrast to the non-interacting case, where the parameters $\phi$ and $\mu_g$ can always be tuned to match a resonance condition, in the interacting system this is no longer possible and we find that at certain values of $\mu_g$ a window opens where no conductance resonances can be obtained. Interference effects are manifested in the occurrence of double resonances where electron trajectories propagating clockwise and anti-clockwise are degenerate. Finally, the conductance as a function of the distance between left and right leads is maximal when the distance between leads is half of the circumference length, i.e., when clockwise and anti-clockwise propagating electrons have to cover the same source to drain distance.

The conductance of toroidal SWTNs has already been measured in the experiments\textsuperscript{22}, though not in the Coulomb blockade regime, thus we believe that the possible verification of our predictions is within the reach of present experiments.

**APPENDIX A: BOGOLIUBOV TRANSFORMATION**

The non-diagonal part of \textsuperscript{18} is given by

$$\hat{H}_{\text{nd}} = \varepsilon_0 \sum_{q \neq 0} n_q |n_q|^2 \left\{ b_{\uparrow \downarrow, q}^\dagger b_{\uparrow \downarrow, q} + \frac{1}{2} \sum_{q \neq 0} 4|n_q|W_{q,-q} \times \right\}$$

$$\times \left( b_{\uparrow \downarrow, q} b_{\uparrow \downarrow, -q} + b_{\uparrow \downarrow, q} b_{\downarrow \uparrow, -q} + b_{\downarrow \uparrow, -q} b_{\downarrow \uparrow, q} + b_{\downarrow \uparrow, -q} b_{\uparrow \downarrow, q} \right).$$

(A1)

The goal of the Bogoliubov transformation is the derivation of new operators, in terms of which $\hat{H}_{\text{nd}}$ is diagonal. In general, we can write them as a linear combination of the old bosonic operators $b_{\uparrow \downarrow, q}$ and $b_{\downarrow \uparrow, q}$:

$$a_{\uparrow \downarrow, q}^\dagger = \sum_{q'} \left( U_{qq'} b_{\uparrow \downarrow, q'}^\dagger - V_{qq'} b_{\downarrow \uparrow, q'} \right),$$

(A2)

$$a_{\downarrow \uparrow, q} = \sum_{q'} \left( -V_{qq'} b_{\uparrow \downarrow, q'}^\dagger + U_{qq'} b_{\downarrow \uparrow, q'} \right).$$

(A3)

Of course, these operators have to fulfill the bosonic commutation relations:

$$[a_{\uparrow \downarrow, q}, a_{\uparrow \downarrow, q'}^\dagger] = \delta_{qq'},$$

(A4)

$$[a_{\downarrow \uparrow, q}, a_{\downarrow \uparrow, q'}^\dagger] = [a_{\uparrow \downarrow, q}, a_{\uparrow \downarrow, q'}^\dagger] = 0.$$
The inverse transformation can also be written in a more concise way:

\[ b_{Fj} = \sum_{q} \left( S_{Fjqq} a_{Fj} + C_{Fjqq} a_{Fj}^\dagger \right), \]

where we have for the spin related coefficients \( S \) and \( C \):

\[ S_{\oplus qq'} = S_{\ominus qq'} = \delta_{qq'}; \quad C_{\oplus qq'} = C_{\ominus qq'} = 0, \]

while for the charge related ones, we have to take care of the sign of \( q \). Let us choose \( q > 0 \):

\[ S_{\ominus qq'} = \delta_{qq} \frac{\alpha + \beta_q}{8\alpha_3} - \delta_{q-q'} \frac{\alpha + \beta_q}{8\alpha_3}, \]
\[ C_{\ominus qq'} = \delta_{qq} \frac{\alpha - \beta_q}{8\alpha_3} + \delta_{q-q'} \frac{\alpha - \beta_q}{8\alpha_3}, \]
\[ S_{\ominus qq'} = \delta_{qq} \frac{\alpha + \beta_q}{8\alpha_3} + \delta_{q-q'} \frac{\alpha + \beta_q}{8\alpha_3}, \]
\[ C_{\ominus qq'} = \delta_{qq} \frac{\alpha - \beta_q}{8\alpha_3} - \delta_{q-q'} \frac{\alpha - \beta_q}{8\alpha_3}. \]

The final result for the diagonalised Hamiltonian is then

\[
\hat{H}_{\text{ring}} = \sum_{F_{jq}} \varepsilon F_{jq} a_{Fj}^\dagger a_{Fj} + \varepsilon_0 \sum_{F_{rs}} \frac{N_{F_{rs}}^2}{2} + \\
+ \varepsilon_0 \sum_{F_{rs}} \text{sgn}(r) N_{F_{rs}} + \frac{1}{2} W_{00} N^2, \tag{A10}
\]

where we introduced the energies

\[ \varepsilon_{\oplus q} = \varepsilon_{\ominus q} = \varepsilon_{\oplus q} = \varepsilon_{0q} \text{ and } \varepsilon_{\ominus q} = \varepsilon_{q}. \tag{A11} \]

**APPENDIX B: THE BOSONIC CORRELATOR**

We evaluate in this appendix explicitly the bosonic part of the Green’s function, Eq. (54), in the so-called Luttinger limit, where i) we assume a linear dispersion: \( \varepsilon_q = \varepsilon_0 q / K_c \), with coupling constant \( K_c = \sqrt{1 + \omega_{00}^2 / \varepsilon_0} \); ii) we assume an exponential decay of the parameter \( B_{\ominus qq} \) entering the Bogoliubov transformation: \( C_{\ominus qq}^2 = C_{\ominus qq}^2 e^{-q^2 / q_c} \), with \( q_c \) a momentum cut-off. Let us then start from Eq. (53):

\[
\left\langle \psi_{F_{bs}}^\dagger (x', -t) \psi_{F_{bs}} (x, 0) \right\rangle_b = \frac{1}{1 - e^{-\alpha_2 t / 2}}, \tag{B1}
\]

\[
\times \left\{ \frac{1}{4} \sum_{q > 0} e^{-\alpha q} \left[ \sum_{F_{j}} Q_{F_{j}} (t, \text{sgn}(b)(x' - x)) \right. \\
+ 2 C_{\ominus qq}^2 (Q_{\ominus q}(t, (x' - x)) + Q_{\ominus q}(t, -(x' - x))) \right\}, \tag{B2}
\]

where we introduced the correlation function

\[ Q_{\gamma q}(t, x) := i \sin \left( \frac{\varepsilon_{\gamma q} t}{\hbar} - q x \right) \]
\[ + \left[ 1 - \cos \left( \frac{\varepsilon_{\gamma q} t}{\hbar} - q x \right) \right] \coth \left( \frac{\beta \varepsilon_{\gamma q}}{2} \right). \]

In the low temperature regime we can approximate \( \coth(\beta \varepsilon_0q / 2) \) and \( \coth(\beta \varepsilon_{q} / 2) \) by 1. We perform now the Luttinger approximation and recall that \( q = 2\pi n / L \).

Rewriting the \( \sin \) and \( \cos \) functions as exponentials and remembering the formula

\[
\sum_{n > 0} e^{in} \frac{1}{n} = -\ln(1 - e^n), \quad |e^n| < 1, \tag{B3}
\]

we arrive at

\[
\left\langle \psi_{F_{bs}}^\dagger (x', -t) \psi_{F_{bs}} (x, 0) \right\rangle_b = \left[ 1 - z (\alpha + \frac{1}{q_c}, 0) \right] C_{\ominus q}^2 \tag{B4}
\]

where we have defined the function

\[ z(\alpha, x) := e^{-(\alpha + ix) \frac{\hbar}{2}}. \tag{B5} \]

Because the functions entering [B4] are periodic in time, we can write the bosonic correlation function as a product of Fourier series. To this extent we use the relation

\[ (1 - z)^\gamma = \sum_{k = 0}^{\infty} \frac{\Gamma(k - \gamma)}{k! \Gamma(-\gamma)} z^k, \quad |z| < 1, \tag{B6} \]

which yields the final result

\[
\left\langle \psi_{F_{bs}}^\dagger (x', -t) \psi_{F_{bs}} (x, 0) \right\rangle_b = \left[ 1 - z \left( \alpha + \frac{1}{q_c}, 0 \right) \right] C_{\ominus q}^2 \times \sum_{k = 0}^{\infty} r_k \left( \alpha, -\frac{3}{4}, \text{sgn}(b)(x' - x) \right) e^{\frac{\hbar}{2} \varepsilon_{0k} t} \times \sum_{l = 0}^{\infty} s_l (b, x' - x) e^{\frac{\hbar}{2} \varepsilon_{0l} t}. \tag{B7}
\]

Here we have defined

\[ r_k (\alpha, x, y) := \frac{\Gamma(k - \gamma)}{k! \Gamma(-\gamma)} e^{k \frac{\hbar}{2} \varepsilon_{0l} (\alpha + ix)}, \]
\[ s_l (b, x' - x) := \sum_{n_1 = 0}^{l} r_{n_1} \left( \alpha, -\frac{1}{4}, \text{sgn}(b)(x' - x) \right) \times \sum_{n_2 = 0}^{l} r_{n_2} \left( \alpha + \frac{1}{q_c} - \frac{1}{2}, C_{\ominus q}^2, (x' - x) \right) \times \sum_{n_3 = 0}^{l\ominus n_2 - n_1} r_{n_3} \left( \alpha + \frac{1}{q_c}, -\frac{1}{2}, C_{\ominus q}^2, -(x' - x) \right), \]

where \( \Gamma \) is the gamma function.
1. Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).
2. B. L. Al'tschuler, A. G. Aronov and B. Z. Spivak, Pis'ma Zh. Eksp. Teor. Fiz. 33, 101 (1981) [JETP Lett. 33, 94 (1981)].
3. E. A. Jagla and C. A. Balseiro, Phys. Rev. Lett. 68, 1220 (1992).
4. J. M. Kinaret, M. Jonson and R. I. Shekhter, Phys. Rev. B 57, 3777 (1998).
5. M. Pletyukhov, V. Gritsev and N. Pauget, Phys. Rev. B 74, 045301 (2006).
6. R. Saito, G. Dresselhaus and M. Dresselhaus, Physical Properties of Carbon Nanotubes (Imperial College Press, London, 1998).
7. R. Egger and A. O. Gogolin, Phys. Rev. Lett. 79, 5082 (1997); Eur. Phys. J. B 3, 281 (1998).
8. C. Kane, L. Balents and M. P. A. Fisher, Phys. Rev. Lett. 79, 5086 (1997).
9. S. J. Tans et al., Nature (London) 386, 474 (1997).
10. M. Bockrath et al., Nature (London) 397, 598 (1999).
11. H. W. Ch. Postma et al., Science 293, 76 (2001).
12. J. Lee et al., Phys. Rev. Lett. 93, 166403 (2004).
13. D. Cobden and J. Nygård, Phys. Rev. Lett. 89, 046803 (2002).
14. W. Liang, M. Bockrath and H. Park, Phys. Rev. Lett. 88, 126801 (2002).
15. S. Sapmaz, P. Jarillo-Herrero, J. Kong, C. Dekker, L. P. Kouwenhoven and H. S. J. van der Zant, Phys. Rev. B 71, 153402 (2005).
16. S. Moriyama, T. Fuse, M. Suzuki, Y. A. Yagi and K. Ishibashi, Phys. Rev. Lett. 94, 186806 (2005).
17. F. Kuemmeth, S. Ilani, D. C. Ralph and P. L. McEuen, Nature 452, 448 (2008).
18. Y. Oreg, K. Byczuk and B. I. Halperin, Phys. Rev. Lett. 85, 365 (2000).
19. L. Mayrhofer and M. Grifoni, Phys. Rev. B 74, 121403R (2006); Europ. Phys. J. B 56, 107 (2007).
20. L. Mayrhofer and M. Grifoni, Eur. Phys. J. B in press and arXiv:0804.3360.
21. J. Liu et al., Nature 385, 780 (1997).
22. R. Martel, H. R. Shea and Ph. Avouris, Nature 398, 299 (1999).
23. H. R. Shea, R. Martel and Ph. Avouris, Phys. Rev. Lett. 84, 4441 (2000).
24. M. F. Lin and D. S. Chuu, Phys. Rev. B 57, 6731 (1998).
25. A. A. Odintsov, W. Smit and H. Yoshioka, Europhysics Lett. 45, 598 (1999).
26. J. Rollbühler and A. A. Odintsov, Physica B 280, 386 (1999).
27. S. Latil, S. Roche and A. Rubio, Phys. Rev. B 67, 165420 (2003).
28. G. C. Rocha, M. Pacheco, Z. Particevic and A. Latgé, Phys. Rev. B 70, 233402 (2004).
29. H.-K. Zhao and J. Wang, Phys. Lett. A 325, 407 (2004).
30. M. Jack and M. Encinosa, arXiv:0709.0760v1.
31. J. van Delft and H. Schoeller, Ann. Phys. 7, 225 (1998).
32. D. Loss, Phys. Rev. Lett. 69, 343 (1992).
33. J. Avery, Creation and Annihilation Operators (McGraw-Hill, New York 1976).