THE ENERGY IDENTITY FOR A SEQUENCE OF YANG-MILLS
α-CONNECTIONS

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Abstract. We prove that the Yang-Mills α-functional satisfies the Palais-
Smale condition, implying the existence of critical points, which are called
Yang-Mills α-connections. It was shown in [10] that as α → 1, a sequence
of Yang-Mills α-connections converges to a Yang-Mills connection away from
finitely many points. We prove an energy identity for such a sequence of Yang-
Mills α-connections. As an application, we also prove an energy identity for
the Yang-Mills flow at the maximal existence time.

1. Introduction

Let M be a compact, connected, oriented four-dimensional Riemannian manifold
and E a vector bundle over M with structure group G, where G is a compact Lie
group compatible with the natural inner product on E. For each connection A, the
Yang-Mills functional is defined by

\[ YM(A) = \int_M |F_A|^2 \, dV, \]

where \( F_A \) is the curvature of A. A connection A is a Yang-Mills connection if it
satisfies the Yang-Mills equation

\[ D^*_AF_A = 0. \tag{1} \]

There have been many significant results on the existence of smooth Yang-Mills
connections on 4-manifolds. In a pioneering work, Atiyah, Hitchin, Drinfel’d and
Manin [2] constructed self-dual Yang-Mills connections on \( S^4 \). Taubes [29]
established the existence of self-dual (anti-self-dual) connections over compact 4-
manifolds with positive (negative) definite intersection form. In [1], Atiyah and
Bott established the fundamental result of Morse theory for the Yang-Mills equa-
tions on Riemann surfaces. Since the Yang-Mills functional does not satisfy the
Palais-Smale condition in dimension four, Taubes [30] suggested a new framework
for Morse theory for the Yang-Mills functional. However, it seems very challeng-
ing to find many applications of the new theory. Using \( m \)-equivariant connections,
Sibner, Sibner and Uhlenbeck [32] proved the existence of non-self-dual Yang-Mills
connections on the trivial \( SU(2) \) bundle over \( S^4 \). There are further results [21, 34] about Morse theory for the equivariant Yang-Mills functional on 4-manifolds.

On the other hand, the theory of Yang-Mills connections in dimension four has
many similarities with the theory of harmonic maps in dimension two. For maps
\( u : M \to N \) from a two-dimensional manifold M to another manifold N, the energy
functional \( E(u) = \int |\nabla u|^2 \) also fails to satisfy the Palais-Smale condition. For this
reason, Sacks and Uhlenbeck [23] introduced an α-functional \( E_\alpha = \int (1 + |\nabla u|^2)^\alpha \),
where \( \alpha > 1 \), to prove the existence of harmonic maps on surfaces. This new
approach via the Sacks-Uhlenbeck functional $E_\alpha$ is very powerful and has produced tremendous applications in harmonic maps and related topics. Specifically, the Sack-Uhlenbeck approach is described as follows. Let $u_\alpha: \tilde{M} \to N$ be a sequence of maps, where $u_\alpha$ is a critical point of the functional $E_\alpha$ and $\alpha > 1$. Sacks and Uhlenbeck proved that for a sequence with $\alpha \to 1$ and having uniformly bounded energies, the sequence $\{u_\alpha\}$ converges smoothly to a harmonic map $u_\infty$ away from at most finitely many points. Moreover, a sophisticated blow-up phenomenon occurs around singularities.

It is natural to ask whether energy is conserved in the blow-up process. Parker [20] established the energy identity for a sequence of smooth harmonic maps, and Ding and Tian [5] proved an energy identity for a sequence of smooth approximate harmonic maps. It was a long-standing open question to establish the energy identity for the above sequence of critical points $\{u_\alpha\}$ of the Sacks-Uhlenbeck functional $E_\alpha$. That is, given that the energies are uniformly bounded as $\alpha \to 1$, do there exist harmonic maps $\omega_i: S^2 \to N$ with $i = 1, \ldots, l$ such that

$$\lim_{\alpha \to 1} E(u_\alpha) = E(u_\infty) + \sum_{i=1}^{l} E(\omega_i)?$$

For this question, Li and Wang [15] established the weak identity for the limit $\lim_{\alpha \to 1} E_\alpha(u_\alpha)$. Under an additional assumption, Lamm [14] and Li and Wang [15] proved that the energy identity (2) holds. However, in a recent preprint [16], Li and Wang constructed a sequence of critical points $u_\alpha$ which provides a counterexample and shows that (2) is not true.

Following the above Sacks-Uhlenbeck approach, Hong, Tian and Yin [10] introduced the Yang-Mills $\alpha$-functional to establish the existence of Yang-Mills connections. For $\alpha > 1$, the Yang-Mills $\alpha$-functional $YM_\alpha$ is defined by

$$YM_\alpha(A) = \int_M (1 + |F_A|^2)^\alpha dV.$$ 

In several respects, the Yang-Mills $\alpha$-functional has nicer properties than the Yang-Mills functional. It was shown in [10] that the Yang-Mills $\alpha$-flow associated to (3) admits a global smooth solution, whose limit set contains a smooth critical point of the Yang-Mills $\alpha$-functional. By considering the limit $\alpha \to 1$, the authors were then able to obtain existence results for Yang-Mills connections and the Yang-Mills flow. Furthermore, Sedlacek [25] applied the weak compactness result of Uhlenbeck [34] to prove that a minimizing sequence $A_i$ of the Yang-Mills functional converges weakly in $W^{1,2}(M \setminus \{x_1, \ldots, x_l\})$ to a limiting connection $A_\infty$, which can be extended to a Yang-Mills connection in a (possibly) new bundle $E'$ over $M$. Hong, Tian and Yin [10] used the Yang-Mills $\alpha$-flow to find a modified minimizing sequence, which converges to the same limit in the smooth topology up to gauge transformations away from finitely many singular points, which improved the result in [25]. Moreover, for the modified minimizing sequence, an energy identity was proved.

In this paper, we continue the program of the Yang-Mills $\alpha$-functional and study the critical points of the Yang-Mills $\alpha$-functional directly. First, we show

**Theorem 1.** The functional $YM_\alpha$ satisfies the Palais-Smale condition for every $\alpha > 1$. 
The Palais-Smale condition (e.g. [19], [28], [14]) guarantees the existence of a critical point $A_\alpha$ of $YM_\alpha$; i.e. $A_\alpha$ satisfies
\begin{equation}
D^*_A \left((1 + |F_A|^2)^{\alpha - 1} F_A\right) = 0
\end{equation}
in the weak sense. In fact, the smoothness of the Yang-Mills $\alpha$-connection is essentially due to Isobe in [12], and we prove it in the Appendix. We call $A_\alpha$ a Yang-Mills $\alpha$-connection if it satisfies (4).

In analogy with the above Sacks-Uhlenbeck program, we consider attempting to construct a Yang-Mills connection as the limit as $\alpha \to 1$ of a sequence $A_\alpha$ of critical points of $YM_\alpha$. In fact, it was proved in [10] that for a sequence $\alpha_k \to 1$, the sequence $\{A_{\alpha_k}\}$ sub-converges smoothly, up to gauge transformations, to a Yang-Mills connection $A_\infty$ away from at most finitely many points. A natural question to ask is whether energy is lost during the blowup process. Fortunately, we are able to prove the following energy identity:

**Theorem 2.** Let $A_\alpha$ be a sequence of Yang-Mills $\alpha$-connections on $E$ with $\alpha \to 1$, and $YM(A_\alpha)$ uniformly bounded. Then there exists a finite set $S \subset M$ and a sequence of gauge transformations $\phi_\alpha$ defined on $M \setminus S$, such that for any compact $K \subset M \setminus S$, a subsequence of $\phi_\alpha^* A_\alpha$ converges to $A_\infty$ smoothly in $K$. Moreover, there are a finite number of bubble bundles $E_1, \ldots, E_l$ over $S$ and Yang-Mills connections $\tilde{A}_1, \ldots, \tilde{A}_l, A_\infty$ such that
\begin{equation}
\lim_{\alpha \to 1} YM(A_\alpha) = YM(A_\infty) + \sum_{i=1}^l YM(\tilde{A}_i, A_\infty).
\end{equation}

This result is surprising in comparison with those of Li-Wang [15], [16] and Lamm [14], since we are able to prove the full energy identity for Yang-Mills $\alpha$-connections. In fact, Li-Wang [15] and Lamm [14] used the idea of Sacks-Uhlenbeck [23] on removable singularities of harmonic maps in 2 dimensions to obtain that
\begin{equation}
\lim_{\delta \to 0} \lim_{R \to \infty} \lim_{\alpha \to 1} \int_{B_R \setminus B_{R\alpha} (x_\alpha)} |u_{\alpha, \theta}|^2 \, dx = 0,
\end{equation}
where $x_\alpha$ tends to a singularity $x_i$.

But they had to handle the term
\begin{equation}
\lim_{\delta \to 0} \lim_{R \to \infty} \lim_{\alpha \to 1} \int_{B_R \setminus B_{R\alpha} (x_\alpha)} \left|\frac{\partial u_\alpha}{\partial r}\right|^2 \, dx
\end{equation}
by using a type of Pohozaev’s identity which involves a very ‘bad’ term, so they could not prove the energy identity. In contrast, the idea of Uhlenbeck [35] on removable singularities of Yang-Mills connections in 4 dimensions is to construct a broken Hodge gauge, which is different from one in [23]. We can apply the idea of the broken Hodge gauge to show that
\begin{equation}
\lim_{R \to \infty} \lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{B_R \setminus B_{R\alpha} (x_\alpha)} |F_{A_\alpha}|^2 \, dV = 0.
\end{equation}
See Lemma 3.7.

Tian [31] first established the energy identity of a sequence of Yang-Mills $\Omega$-self-dual connections. Rivière [22] proved the energy identity for a sequence of smooth Yang-Mills connections in 4 dimensions. As a consequence of Theorem 2, we can give a different proof of this result (see Corollary 3.7). As another application, we
are also able to give a new proof of the energy identity results of [10] (see Proposition 3.8).

It is also possible to consider whether an energy identity holds at the blowup time for the corresponding heat flows. Struwe [26] established the global existence of the harmonic map flow in 2 dimensions. Ding-Tian [5] (see also [17]) established the energy identity for the harmonic map flow at the maximal existence time. Motivated these results, we can prove a similar result for the Yang-Mills flow in 4 dimensions. In fact, Struwe [27] established the global existence of weak solutions of the Yang-Mills flow

\[ \frac{\partial A}{\partial t} = -D^*_A F_A \quad \text{in } M \times [0, \infty) \]

with initial value \( A(0) = A_0 \). Struwe proved that the weak solution \( A(t) \) of the Yang-Mills flow is, up-to gauge transformations, smooth in \( M \times (0, T) \) for some maximal time \( T > 0 \). Moreover, as \( t \to T \) the solution \( A(t) \) converges, up-to gauge transformations, to a connection \( A(T) \), smoothly away from at most finitely many points. Schlatter [24] gave the details for the blow-up analysis of the Yang-Mills flow at the singular time \( T \), but there is no result concerning the energy identity of the Yang-Mills flow at the time \( T \). We can apply the proof of Theorem 2 to establish an energy identity for the Yang-Mills flow as follows:

**Theorem 3.** Let \( A(t) \) be a solution to the Yang-Mills flow (6) in \( M \times [0, T) \), where \( T \in (0, \infty) \) is the maximal existence time, and \( A(t) \) converges weakly as \( t \to T \) to a connection \( A(T) \). Then there are a finite number of bubble bundles \( E_1, \ldots, E_l \) over \( S^4 \) and Yang-Mills connections \( \tilde{A}_1, \ldots, \tilde{A}_l \) such that

\[
\lim_{t \to T} Y M(A(t)) = Y M(A(T)) + \sum_{i=1}^l Y M(\tilde{A}_i).
\]

In particular, if \( T = \infty \), Theorem 3 implies a new proof of an identity for a sequence of the second Chern numbers of holomorphic vector bundles over Kähler surfaces (Theorem 11 of [9], Theorem 4 of [11]).

The paper is organised as follows. In section 2, we recall some necessary background and prove Theorem 1. In section 3, we prove the energy identities Theorem 2 and Theorem 3. Finally, in the Appendix, we show the regularity of weak solutions to the Euler-Lagrange equations.

## 2. Preliminaries and the Palais-Smale condition

We begin by recalling background and introducing the notation we will require. Denote by \( \mathcal{A} \) the affine space of metric connections on \( E \). Let \( A \in \mathcal{A} \) be a metric connection. After fixing a reference connection \( D_0 = D_{A_0} \), we write \( D_A = D_0 + A \) where \( A \in \Gamma(\text{Ad} E \otimes T^* M) \). The curvature of \( A \) is \( F_A \in \Gamma(\text{Ad} E \otimes \Lambda^2 T^* M) \). Here we denote by \( \text{Ad} E \) the adjoint bundle which has fibre \( g \), the Lie algebra of \( G \).

The inner products on a pair of vector bundles induce an inner product on their tensor product. In the case of \( \text{Ad} E \subset \text{End} E = E \otimes E^* \), this is just the negative of the Killing form. This combines with the inner product on \( T^* M \) to define an inner product on \( \text{Ad} E \otimes^k T^* M \). Similarly, the connections on a pair of vector bundles induce a connection on their tensor product. Thus we obtain a connection on \( \text{Ad} E \subset \text{End} E \) and, using the Levi-Civita connection, a connection
on \( \text{Ad } E \otimes T^*M \). The Levi-Civita connection also allows us to define multiple iterations of the covariant derivative on any vector bundle.

We denote by \( D_A \) both the exterior covariant derivative on \( \Gamma(E \otimes \Omega^p(M)) \), and the exterior covariant derivative on \( \Gamma(\text{End } E \otimes \Omega^p(M)) \). The functional \( YM_\alpha \) has a Fréchet derivative

\[
(7) \quad dYM_{\alpha,A}(B) = \left. \frac{d}{dt} \right|_{t=0} YM_\alpha(A + tB) = \int_M 2\alpha \left( 1 + |F_A|^2 \right)^{\alpha-1} \langle F_A, D_A B \rangle \, dV
\]

so the Euler-Lagrange equation for \( YM_\alpha \) is (4). Note that solutions to (4) are smooth up to gauge by the results of the appendix. Then since \((1 + |F_A|^2)^{\alpha-1}\) is bounded below, this equation is equivalent to

\[
(8) \quad D_A^* F_A - (\alpha - 1) \frac{d|F_A|^2 \wedge \star F_A}{1 + |F_A|^2} = 0.
\]

The functional \( YM_\alpha \) is invariant under the action of the gauge group \( \mathcal{G} \), which consists of those automorphisms of \( E \) which preserve the inner product. A gauge transformation is thus a section of the bundle \( \text{Aut } E \), which has fibre \( G \). The gauge group acts on \( \mathcal{A} \) by

\[
s^* D_A = s^{-1} \circ D_A \circ s.
\]

Locally, we write \( D_A = d + A \). \( s^* D_A = d + s^* A \), we find

\[
(9) \quad s^* A = s^{-1} ds + s^{-1} A ds.
\]

The functional \( YM_\alpha \) is gauge invariant, so it is more appropriately viewed as a functional

\[
YM_\alpha : \mathcal{A}/\mathcal{G} \to \mathbb{R},
\]

where \( \mathcal{A}/\mathcal{G} \) is a space with gauge equivalence and imbued with the quotient topology, but not a smooth manifold.

We first must topologise \( \mathcal{A} \) by a suitable choice of norm. Given a section \( u \) of a vector bundle \( V \) with a smooth connection \( D_0 = D_{A_0} \), we define the \( W^{k,p} \) Sobolev norm by

\[
\|u\|_{W^{k,p}} = \left( \sum_{|\alpha| = 0}^k \int_M |\nabla^\alpha_{A_0} u|^p \right)^{1/p}.
\]

Different choices of reference connection lead to equivalent norms. We then define the Sobolev space \( W^{k,p}(V) \) to be the completion of the smooth sections of \( V \) by the \( W^{k,p} \) norm.

We may define Sobolev norms on the space of connections using the vector bundle \( \text{Ad } E \otimes T^*M \). A different choice of reference connection taken as the origin of the affine space \( \mathcal{A} \) also leads to an equivalent norm. We denote by \( \mathcal{A}^{1,p} = W^{1,p}(\mathcal{A}) \) the space of connections of class \( W^{1,p} \).

Since a gauge transformation \( s \) can be expressed locally over \( U \subset M \) as a map \( s : U \to G \), we will say that a gauge transformation is of class \( W^{k,p} \) if its local restrictions are of class \( W^{k,p} \), and \( s(x) \in G \) almost everywhere. Due to the derivative of the gauge transformation appearing in (9), we need to control two derivatives of the gauge transformation, so we consider \( \mathcal{G}^{2,p} \). We will choose \( p = 2\alpha \), and consider the space \( \mathcal{A}^{1,2\alpha} \) and its quotient \( \mathcal{A}^{1,2\alpha}/\mathcal{G}^{2,2\alpha} \).
Now, we prove that the Yang-Mills $\alpha$-functional satisfies the Palais-Smale condition on the quotient space.

**Proof of Theorem 1.** We say that a sequence $A_n \in \mathcal{A}^{1,2\alpha}/G^{2,2\alpha}$ is a Palais-Smale sequence if it satisfies

i) $\text{YM}_\alpha(A_n) \leq C,$

ii) $\|d\text{YM}_\alpha A_n\|_{W^{1,q}_{A_n}} \to 0,$

where $q$ is the dual number of $p = 2\alpha$; i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Similarly to one in [32], the norm $\| \cdot \|_{W^{1,q}_{A_n}}$ here is the dual norm of $W^{1,2\alpha}_{A_n}$ defined by

$$\|d\text{YM}_\alpha A\|_{W^{1,q}_{A}} = \frac{\max |\partial_x d\text{YM}_\alpha(A + \epsilon \delta A)|_{\epsilon=0}}{\|\delta A\|_{W^{1,2\alpha}_{A}}},$$

where

$$\|\delta A\|_{W^{1,2\alpha}_{A}} = \int_M |\nabla_A \delta A|^{2\alpha} + |\delta A|^{2\alpha} dV.$$ 

We say that the functional (5) satisfies the Palais-Smale condition if every Palais-Smale sequence contains a subsequence that converges in $W^{1,2\alpha}$ up to gauge transformations. That is, passing to a subsequence, $s_n^* A_n$ converges for some sequence of gauge transformations $s_n \in G^{2,p}$. Note that this condition implies Condition C of [13].

First we observe that the $L^2$ norm of the curvature cannot concentrate, since by Hölder’s inequality,

$$\int_{B_r} |F_{A_n}|^2 dV \leq \left( \int_{B_r} |F_{A_n}|^{2\alpha} dV \right)^{\frac{1}{\alpha}} |B_r|^{\frac{2\alpha-1}{\alpha}} \to 0$$

as $r \to 0$. For a sufficiently small ball $U = B_r$, the bundle $E$ can be trivialized so that we can assume $D_{A_n} = d + A_n$ on $U$. Then we apply the gauge-fixing theorem of Uhlenbeck (Theorem 1.3 of [34]) to obtain that there are local gauge transformations $s_n$ defined on $U$ such that

$$d^*(s_n^* A_n)|_U = 0, \quad s_n^* A_n \cdot \nu|_{\partial U} = 0$$

and

$$\|s_n^* A_n\|_{W^{1,2\alpha}(U)} \leq C \|F_{A_n}\|_{L^{2\alpha}(U)}.$$ 

Since this result is local, we must do this in each element $U_j$ of a cover of $M$, and check that in the limit the local connections can be sewn together to yield a global connection. In particular, one must check that the transition functions between elements of the cover are converging. This procedure is described by Sedlacek [25].

Since $M$ is compact, there is a finite open cover $\{U_j\}_{j=1}^L$ of $M$, where $U_j = B_{r_j}(x_j)$ for some sufficiently small $r > 0$, and at each $x \in M$, at most a finite number of the balls intersect. From the boundedness of $\text{YM}_\alpha(A_n)$, we have that $F_{A_n}$ is bounded in $W^{1,2\alpha}$. For simplicity, we define $A_{n,j} = s_n^* A_n$ on each ball $U_j$. Then $d + A_{n,j}$ can be regarded as a local representative of the global connection $D_{A_n}$ of $E$ over $U_j$. Moreover, in the overlap $U_i \cap U_j$ of two balls, $A_{n,i}$ and $A_{n,j}$ can be identified as the same by a gauge transformation $s_{ij} \in G^{2,2\alpha}$ between $E_{U_i}$ and $E_{U_j}$ (see Lemma 3.5 of [34], also [25]). Thus we can glue $A_j$ together to obtain a global connection $D_{\tilde{A}}$, which is $G^{2,2\alpha}$-gauge equivalent to $D_{A_n}$ in the local trivialization of $E_{U_j}$. Passing to a subsequence, it follows from the Rellich-Kondrachov Theorem...
that $\tilde{A}_{n,j}$ converges strongly in $L^{2\alpha}(U_j)$, implying that $\tilde{A}_{n,j}$ converges in $C^0$ and are uniformly bounded.

Since $YM_{\alpha}(\tilde{A}_n; M)$ and $\|\tilde{A}_{n,j} - \tilde{A}_{m,j}\|_{W^{1,2\alpha}(U_j)}$ are bounded, $\|\tilde{A}_{n,j} - \tilde{A}_{m,j}\|_{W^{1,2\alpha}(U_j)}$ is also bounded. Then we also have

$$ (10) \quad |dYM_{\alpha,\tilde{A}}(\tilde{A}_n - \tilde{A}_m)| \leq \|dYM_{\alpha,\tilde{A}}\|_{W^{-1,\alpha}} \sum_{j=1}^{L} \|\tilde{A}_{n,j} - \tilde{A}_{m,j}\|_{W^{1,2\alpha}(U_j)} \to 0. $$

Using Hölder’s inequality and the fact that $\|\tilde{A}_{n,j} - \tilde{A}_{m,j}\|_{L^{2\alpha}(U_j)} \to 0$, we find

$$ \int_{U_j} (1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} |F_{\tilde{A}_n}| |\tilde{A}_{n,j} - \tilde{A}_{m,j}| dV \leq \left( \int_{U_j} (1 + |F_{\tilde{A}_n}|^2)^{\alpha} dV \right)^{2\alpha-1} \left( \int_{U_j} |\tilde{A}_{n,j}|^{2\alpha} |\tilde{A}_{n,j} - \tilde{A}_{m,j}|^{2\alpha} dV \right)^{\frac{1}{2\alpha}} \to 0, \quad \text{as } m, n \to \infty. $$

Denote by $o(1)$ terms which are going to zero as $n, m \to \infty$. Then

$$ dYM_{\alpha,\tilde{A}}(\tilde{A}_n - \tilde{A}_m) = \int_M 2\alpha \left\langle D_{\tilde{A}_n}[(1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} F_{\tilde{A}_n}], (\tilde{A}_n - \tilde{A}_m) \right\rangle dV $$

$$ = \int_M 2\alpha (1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} \langle F_{\tilde{A}_n}, D_{\tilde{A}_n}(\tilde{A}_n - \tilde{A}_m) \rangle dV $$

$$ = \int_M 2\alpha (1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} \langle F_{\tilde{A}_n}, D_{\tilde{A}_0}(\tilde{A}_n - \tilde{A}_m) \rangle dV $$

$$ + \int_M 2\alpha (1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} \langle F_{\tilde{A}_n}, \tilde{A}_n#(\tilde{A}_n - \tilde{A}_m) \rangle dV $$

$$ = \int_M 2\alpha (1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} \langle F_{\tilde{A}_n}, F_{\tilde{A}_m} - F_{\tilde{A}_m} \rangle dV + o(1), $$

since $A \land A - B \land B = A \land (A - B) - (B - A) \land B$.

It is well-known that there is a constant $c > 0$ such that

$$ \langle (1 + |b|^2)^{\alpha-1} b - (1 + |a|^2)^{\alpha-1} a, b - a \rangle \geq c |b - a|^\alpha $$

for any two constants $a, b \in \mathbb{R}^k$. Using this inequality, we obtain

$$ (dYM_{\alpha,\tilde{A}_n} - dYM_{\alpha,\tilde{A}_m})(\tilde{A}_n - \tilde{A}_m) $$

$$ = \int_M 2\alpha \left\langle (1 + |F_{\tilde{A}_n}|^2)^{\alpha-1} F_{\tilde{A}_n} - (1 + |F_{\tilde{A}_m}|^2)^{\alpha-1} F_{\tilde{A}_m}, F_{\tilde{A}_n} - F_{\tilde{A}_m} \right\rangle dV + o(1) $$

$$ \geq 2\alpha c \int_M |F_{\tilde{A}_n} - F_{\tilde{A}_m}|^{2\alpha} dV + o(1) $$

$$ \geq 2\alpha c \int_{U_j} |F_{\tilde{A}_{n,j}} - F_{\tilde{A}_{m,j}}|^{2\alpha} dV + o(1). $$

Note that $F_{\tilde{A}_{n,j}} = d\tilde{A}_{n,j} + [\tilde{A}_{n,j}, \tilde{A}_{n,j}]$ and $F_{\tilde{A}_{m,j}} = d\tilde{A}_{m,j} + [\tilde{A}_{m,j}, \tilde{A}_{m,j}]$ in $U_j$. Then

$$ \int_{U_j} |d(\tilde{A}_{n,j} - \tilde{A}_{m,j})|^{2\alpha} dV = o(1). $$
Furthermore, we have the following remark.

\[ \alpha \text{-connections (Theorem 2), and the energy identity for the Yang-Mills flow at the maximal existence time (Theorem 3).} \]

Since \( d^*(\bar{A}_{n,j} - \bar{A}_{m,j}) = 0 \) in \( U_j \) and \( (\bar{A}_{n,j} - \bar{A}_{m,j}) \cdot v = 0 \) on \( \partial U_j \), it follows from Lemma 2.5 of [34] that

\[ \int_{U_j} |\nabla (\bar{A}_{n,j} - \bar{A}_{m,j})|^2 dV \leq C \int_{U_j} |d(\bar{A}_{n,j} - \bar{A}_{m,j})|^{2\alpha} dV \to 0. \]

Thus \( \bar{A}_{n,j} = s^*_n A_n \) is converging strongly in \( W^{1,2\alpha}(U_j) \) as required. After sewing together the patches \( U_j \) in the covering, we get a globally defined connection \( \bar{A}_\infty \). For any smooth \( B \) with support in \( U_i \), we have

\[ dYM_{\alpha,A_n}(B) = \int_M 2\alpha \left( 1 + |F_{A_n}|^2 \right)^{\alpha-1} \langle F_{A_n}, D A \rangle dV \]

Since \( \|dYM_{\alpha,A_n}\|_{W^{-1,\alpha}} \to 0 \), the strong convergence of \( A_n \) in \( W^{1,2\alpha} \) implies that

\[ \int_M \left( 1 + |F_{A_\infty}|^2 \right)^{\alpha-1} \langle F_{A_\infty}, D A \rangle dV = 0. \]

Therefore \( A_\infty \) is a critical point of \( YM_\alpha \). This completes a proof of Theorem 1. □

The Palais-Smale condition implies the existence of a critical point. In particular, given a min-max sequence of \( YM_\alpha \) (e.g. [28], [14], [32]), the Palais-Smale condition then ensures, for each \( \alpha > 1 \), the existence of a critical point \( A_\alpha \) of \( YM_\alpha \). Furthermore, we have the following remark.

Remark By using an idea of Struwe [28] (see also Lamm [13]), one expects that there exists a sequence of critical points \( A_\alpha \) of the Yang-Mills \( \alpha \)-functional with \( \alpha \to 1 \) such that

\[ \lim_{\alpha \to 1} \inf(\alpha - 1) \int_M \log(1 + |F_{A_n}|^2)(1 + |F_{A_\alpha}|^2)^\alpha dV = 0. \]

However, we do not require this result for this paper, so we will not give a proof here.

3. Energy Identity

In this section, we establish the energy identity for a sequence of Yang-Mills \( \alpha \)-connections (Theorem 2), and the energy identity for the Yang-Mills flow at the maximal existence time (Theorem 3).

Consider now a sequence \( A_\alpha \) of connections, where \( A_\alpha \) is a smooth critical point of \( YM_\alpha \), and \( \alpha \to 1 \). We suppose that they have uniformly bounded energy \( YM(A_\alpha) \leq K \) for some constant \( K \). We now study the limit of the sequence as \( \alpha \to 1 \). In fact, it was shown in Lemmas 3.5-3.6 of [10] that a subsequence of \( A_\alpha \), up to gauge transformations, converges smoothly to a connection \( \bar{A}_\infty \) away from finitely many points \( x_i \in M, 1 \leq i \leq l \). Moreover, there is a constant \( \varepsilon_0 > 0 \) such that the singular points \( \{x_i\} \) are defined by the condition

\[ \limsup_{\alpha \to 1} YM(A_\alpha; B_R(x_i)) \geq \varepsilon_0 \]

for any \( R \in (0, R_0] \), with some fixed \( R_0 > 0 \).

We now consider the energy identity [3] for the above sequence of Yang-Mills \( \alpha \)-connections \( A_\alpha \). We fix an energy concentration (singular) point \( x_i \), and choose \( R_0 \) so that \( B_{R_0}(x_i) \) contains no concentration points other than \( x_i \).

In order to establish the energy identity, we recall the removable singularity theorem of Uhlenbeck [33] and the gap theorem of Bourguignon and Lawson [3]:
There is a constant \( \varepsilon_0 > 0 \) such that if \( A \) is a Yang-Mills connection on \( S^4 \) satisfying 
\[
\int_{S^4} |F_A|^2 < \varepsilon_0,
\]
then \( A \) is flat; i.e. \( F_A = 0 \) on \( S^4 \).

We also have:

**Lemma 3.1.** \((\varepsilon\text{-regularity estimate})\) There exists \( \varepsilon_0 > 0 \) such that if \( A_\alpha \) is a smooth Yang-Mills \( \alpha \)-connection satisfying 
\[
\int_{B_R(x_0)} |F_{A_\alpha}|^2 dV < \varepsilon_0,
\]
then we have 
\[
\sup_{B_{R/2}(x_0)} |F_{A_\alpha}| \leq C \left( \int_{B_R(x_0)} |F_{A_\alpha}|^2 dV \right)^{1/2}.
\]

**Proof.** Without loss of generality, we can assume \( R = 1 \). Using Lemma 3.5 of [10], 
\( F_{A_\alpha} \) is bounded in \( B_{3/4}(x_0) \) by a constant \( C \). Recalled that each \( \alpha \)-connection \( A_\alpha \) has the Bochner type formula (see Lemma 3.2 of [10]): For \( \alpha - 1 \) sufficiently small, there is a constant \( C \) such that 
\[
- \nabla_{e_i} \left( \delta_{ij} + 2(\alpha - 1) \frac{\langle F_{ij}, F_{kl} \rangle}{1 + |F|} \nabla_{e_j} |F|^2 \right) \leq C |F|^2 (1 + |F|),
\]
where \( F = F_{A_\alpha} \). Then applying a variant of the Moser-Harnack estimate, we have 
\[
\sup_{B_{1/2}(x_0)} |F_{A_\alpha}| \leq C \left( \int_{B_1(x_0)} |F_{A_\alpha}|^2 dV \right)^{1/2}.
\]

\( \square \)

### 3.1. Bubble-neck decomposition

It is well-known that the bubble-neck decomposition holds for a sequence of smooth harmonic maps. There are two kind of methods on constructing the bubble tree and neck decomposition in harmonic maps, one present by Parker [20] and another one by Ding-Tian [5]. In fact, Parker [20] pointed out that a bubble procedure is also true for Yang-Mills connections (without details). Rivière [22] pointed out that using the idea of Ding-Tian [5], multiple bubbles can be simplified to a single bubble for a sequence of Yang-Mills connections.

In order to make proofs completely, we here give detailed proof on constructing the bubble-neck decomposition for a sequence of Yang-Mills \( \alpha \)-connections by following the idea of Ding-Tian [5]. Some details are similar to those in the Appendix of [16]. The bubble tree procedure is divided into three steps.

**Step 1.** To find a maximal (top) bubble at the level one (first re-scaling).

After passing to a subsequence we know that \( A_\alpha \rightarrow A_\infty \), up-to gauge transformations, smoothly in \( B_{R_0} \) away from \( x_i \), so we have
\[
Y M(A_\alpha; B_{R_0} \setminus B_\delta(x_i)) \rightarrow Y M(A_\infty; B_{R_0} \setminus B_\delta(x_i))
\]
for any \( \delta \in (0, R_0) \), where \( A_\infty \) is a Yang-Mills connection and the singularity of \( A_\infty \) can be removed (35).

Since \( x_i \) is a concentration point, we find sequences \( x_\alpha \rightarrow x_i \) such that 
\[
|F_{A_\alpha}(x_\alpha)| = \max_{B_{R_0}(x_i)} |F_{A_\alpha}|, \quad r_\alpha = \frac{1}{|F_{A_\alpha}(x_\alpha)|^{1/2}} \rightarrow 0.
\]
In the neighborhood of the singularity \( x_i \), \( D_{A_\alpha} = d + A_\alpha \) with \( A_\alpha = A_{\alpha,k}(x)dx^k \).

Then we define the rescaled connection

\[
D_{\hat{A}_\alpha}(x) = d + \hat{A}_\alpha(x) := d + r_\alpha^1 A_{\alpha,k}(x^1 + r_\alpha^1 x)dx^k.
\]

The connection \( \hat{A}_\alpha = A_\alpha(x^1 + r_\alpha^1 x) \) satisfies

\[
(15) \quad - D_{\hat{A}_\alpha}^* F_{\hat{A}_\alpha} + (\alpha - 1) \frac{(*d |F_{\hat{A}_\alpha}|^2) \wedge *F_{\hat{A}_\alpha}}{(r_\alpha^1)^4 + |F_{\hat{A}_\alpha}|^2} = 0.
\]

Since \( \|F_{\hat{A}_\alpha}\|_{L^\infty} = 1 \). Using Lemmas 3.6-3.7 of [10] again, \( \hat{A}_\alpha \) sub-converges smoothly, up-to gauge transformations, to \( \hat{A}_{1,\infty} \) locally in \( \mathbb{R}^4 \) as \( \alpha \to 1 \), and \( \hat{A}_{1,\infty} \) can be extended to a connection on \( S^4 \) (see [35]) and nontrivial. We call \( \hat{A}_{1,\infty} \) to be the first bubble, which satisfies

\[
(16) \quad YM(\hat{A}_{1,\infty}; \mathbb{R}^4) = \lim_{R \to \infty} \lim_{\alpha \to 1} YM(\hat{A}_\alpha; B\mathbb{R}(0)) = \lim_{R \to \infty} \lim_{\alpha \to 1} YM(A_\alpha; B_{R^1\alpha}(x)) + \ YM(\hat{A}_{1,\infty}; \mathbb{R}^4) + \lim_{R \to \infty} \lim_{\delta \to 0} \lim_{\alpha \to 1} YM(A_\alpha; B_\delta \setminus B_{R^1\alpha}(x^1)).
\]

**Step 2.** To find out new bubbles at the second level (second re-scaling).

Assume that for a fixed small constant \( \varepsilon > 0 \) (to be chosen later), there exist two positive constants \( \delta \) and \( R \) with \( Rr_\alpha^1 < 4\delta \) such that for all \( \alpha \) sufficiently close to 1, we have

\[
(17) \quad \int_{B_2 \setminus B_1(x^1)} |F_{A_\alpha}|^2 dV \leq \varepsilon
\]

for all \( r \in (\frac{Rr_\alpha^1}{2}, 2\delta) \).

If (17) is true, it follows from (13) and (16) that

\[
\lim_{\alpha \to 1} YM(A_\alpha; B_{R_\alpha}(x^1)) = YM(A_{\infty}; B_{R_\alpha}(x^1)) + YM(\hat{A}_{1,\infty}; \mathbb{R}^4)
\]

\[
+ \lim_{R \to \infty} \lim_{\delta \to 0} \lim_{\alpha \to 1} YM(A_\alpha; B_\delta \setminus B_{R^1\alpha}(x^1))
\]

We can show (see below Lemma 3.6) that there is no missing energy in the neck region, i.e.

\[
(18) \quad \lim_{R \to \infty} \lim_{\delta \to 0} \lim_{\alpha \to 1} YM(A_\alpha; B_\delta \setminus B_{R^1\alpha}(x^1)) = 0.
\]

Therefore, the energy identity follows. In this case, this means that there is only single bubble \( \hat{A}_{1,\infty} \) around \( x_i \).

If the assumption (17) is not true, then for any two constants \( R \) and \( \delta \) with \( Rr_\alpha^1 < 4\delta \), there is a constant \( r_\alpha^2 \in (\frac{Rr_\alpha^1}{2}, 2\delta) \) such that

\[
(19) \quad \lim_{\alpha \to 1} \int_{B_{r_\alpha^2 \alpha} \setminus B_{r_\alpha^1}} |F_{A_\alpha}|^2 > \varepsilon.
\]

If \( \lim_{\alpha \to 1} \frac{r_\alpha^2}{r_\alpha^1} \leq C \) for a finite constant \( C \), it is not a problem since \( \hat{A}_\alpha \) converges smoothly, up-to gauge transformations, to \( \hat{A}_{1,\infty} \) locally in \( \mathbb{R}^4 \). If \( \lim_{\alpha \to 1} \frac{r_\alpha^2}{r_\alpha^1} \neq 0 \), it can be ruled out by choosing \( \delta \) sufficiently small in (17). Therefore, we can assume that \( \lim_{\alpha \to 1} \frac{r_\alpha^2}{r_\alpha^1} = \infty \) and \( \lim_{\alpha \to 1} r_\alpha^2 = 0 \).

Since there might be many different constants \( r_\alpha^2 \in (\frac{Rr_\alpha^1}{2}, 2\delta) \) satisfying (19), we must classify these numbers. For any two constants \( r_\alpha^2 \) and \( \tilde{r}_\alpha^2 \) in \( (\frac{Rr_\alpha^1}{2}, 2\delta) \) satisfying
they can be classified in different groups by the following properties:

\begin{align*}
(20) \quad & \lim_{\alpha \to 1} \frac{r_\alpha^2}{r_{\lambda_\alpha}} = +\infty \quad \text{or} \quad \lim_{\alpha \to 1} \frac{r_{\lambda_\alpha}}{r_\alpha^2} = 0; \\
(21) \quad & 0 < a \leq \lim_{\alpha \to 1} \frac{r_\alpha^2}{r_{\lambda_\alpha}} < b \quad \text{or} \quad 0 < a \leq \lim_{\alpha \to 1} \frac{r_{\lambda_\alpha}}{r_\alpha^2} < b
\end{align*}

for finite constants \(a\) and \(b\). We say that \(r_\alpha^2\) and \(r_{\lambda_\alpha}\) are in the same group if they satisfy (21). Otherwise, they are in different groups if they satisfy (20).

Since there is a uniformly bounded energy \(Y M(A_\alpha) \leq K\) for some constant \(K\) and \(\varepsilon\) is a fixed constant, the above different groups of \(r_\alpha^2\) satisfying (19) must be finite, so that we can choose a smallest number \(r_\alpha^2\) of different groups as a re-scaling scale, but \(r_\alpha^2\) is not in the group of \(r_\alpha^2\); i.e. \(\lim_{\alpha \to 1} \frac{r_{\lambda_\alpha}}{r_\alpha^2} = \infty\) and \(\lim_{\alpha \to 1} \frac{r_\alpha^2}{r_{\lambda_\alpha}} = 0\).

There might be many other numbers \(r_{\lambda_\alpha}^2\) satisfying (19) in the same group of \(r_\alpha^2\). Because of (21), \(r_{\lambda_\alpha}^2/r_\alpha^2\) are bounded as \(\alpha \to 1\), but these numbers \(r_{\lambda_\alpha}^2\) can be ruled out by the following procedure. Set

\[ \tilde{A}_{2,\alpha}(x) = A_\alpha(x^1_\alpha + r_\alpha^2 x). \]

Passing to a subsequence, \(\tilde{A}_{2,\alpha}\) converges, up-to gauge transformations away from a finite concentration set of \(\{\tilde{A}_{2,\alpha}\}\), to a Yang-Mills connection \(\tilde{A}_{2,\infty}\) locally in \(\mathbb{R}^4\). Those numbers \(r_{\lambda_\alpha}^2\) in the same group \(r_\alpha^2\) have been handled out. If \(\{\tilde{A}_{2,\alpha}\}\) is non-trivial, then \(\tilde{A}_{2,\alpha}\) is a new bubble, which is different from the bubble \(\{\tilde{A}_{1,\alpha}\}\). We must point out that the above bubble connection \(\tilde{A}_{2,\infty}\) might be trivial, called a ‘ghost bubble’. In this case, there is at least a concentration point \(p \in B_2 \setminus B_1\) of \(\{\tilde{A}_{2,\alpha}\}\) due to (19).

At each concentration point \(p\) of \(\tilde{A}_{2,\alpha}\), we can repeat the procedure in Step 1; i.e. at each concentration point \(p\) of \(\tilde{A}_{2,\alpha}\), there are sequences \(x_\alpha^p \to p\) and \(\lambda_\alpha^p \to 0\) such that

\[ \tilde{A}_{2,\alpha}(x_\alpha^p + \lambda_\alpha^p x) \to \tilde{A}_{2,\infty} \]

up-to gauge transformation, where \(\tilde{A}_{2,\infty}\) is a Yang-Mills connection on \(\mathbb{R}^4\). Note that \(\tilde{A}_{2,\infty}\) is also a bubble for the sequence \(\{A_\alpha(x^1_\alpha + r_\alpha^2 x^p_\alpha + r_\alpha^2 \lambda_\alpha^p x)\}\).

Set \(x_\alpha^p = x^1_\alpha + r_\alpha^2 x^p_\alpha\). If \(p \neq 0\), then

\[ \frac{|x_\alpha^1 - x_\alpha^{2,p}|}{r_\alpha} = \frac{r_\alpha^2}{r_\alpha^2} |x^p_\alpha| \to \infty \text{ as } \alpha \to 1. \]

Therefore, the bubble \(\{\tilde{A}_{2,\infty}\}\) at \(p \neq 0\) is different from the bubble \(\tilde{A}_{1,\infty}\).

Since \(\lim_{\alpha \to 1} \frac{r_\alpha^2}{r_{\lambda_\alpha}} = 0\) and \(\tilde{A}_{1,\infty}\) is a bubble limiting connection for the sequence \(\{A_\alpha(x^1_\alpha + r_\alpha^2 x) = \tilde{A}_{2,\alpha}(\frac{r_\alpha^2}{r_{\lambda_\alpha}} x)\}\), \(p = 0\) is a concentration point of \(\tilde{A}_{2,\alpha}\) on \(\mathbb{R}^4\). Like Step 1, there is a small \(R^2 > 0\) such that the ball \(B_{R^2}(0)\) contains only the isolated concentration point \(0\) of \(\tilde{A}_{2,\alpha}\). Then, it can be seen from Step 1 that

\[ \lambda_\alpha^0 = \frac{1}{\max_{B_{R^2}(0)} |F_{\tilde{A}_{2,\alpha}}|^2} \geq \frac{1}{r_\alpha^2 \max_{B_{R^2}(0)} |F_{A_\alpha}|^2} = \frac{1}{r_\alpha^2} \]

Similarly to Step 1, the bubble \(\tilde{A}_{2,0,\infty}\) is chosen as the maximal (top) bubble for \(\tilde{A}_{2,\alpha}\) at \(p = 0\). Therefore the bubble \(\tilde{A}_{2,0,\infty}\) must be the same bubble \(\tilde{A}_{1,\infty}\). We can keep it there without a problem.
Next, we must continue the above procedure for possible new multiple bubbles at each blow-up point \( p \) again. Since there is a uniform bound \( K \) for \( YM(A_\alpha; M) \) and each non-trivial bubble on \( S^4 \) costs at least \( \varepsilon_0 \) of the energy by the gap theorem of Bourgiguon and Lawson \( [3] \), the above process must stop after finite steps.

**Step 3.** To find out all multiple bubbles.

Let \( r_\alpha^3 \) be in the second small group of numbers satisfying \( (19) \) with \( \lim_{\alpha \to 1} \frac{r_\alpha^3}{r_\alpha^3} = \infty \) and \( \lim_{\alpha \to 1} r_\alpha^3 = 0 \). Set

\[
\tilde{A}_{3,\alpha}(x) = A_\alpha(x_\alpha^1 + r_\alpha^3 x).
\]

Passing to a subsequence, \( \tilde{A}_{3,\alpha} \) converges, up-to gauge transformations away from a finite concentration set of \( \{\tilde{A}_{3,\alpha}\} \), locally to a Yang-Mills connection \( \tilde{A}_{3,\infty} \) in a bundle over \( \mathbb{R}^4 \). Then we can repeat the argument of Steps 1-2. All bubbles produced by \( \tilde{A}_{3,\alpha} \), except for those concentrated in 0, are different from Steps 1-2. By induction, we can find out all bubbles in all cases of the finite different groups. The above process must stop after finite steps by the gap theorem.

In summary, at each group level \( k \), the blow-up happens, there are finitely many blow-up points and Yang-Mills bubbles on \( \mathbb{R}^4 \). At each level \( k \) and each bubble point \( p_{k,l} \), there are sequences \( x_{\alpha}^{k,l} \to p_{k,l} \) and \( r_\alpha^k \to 0 \) such that passing to a subsequence, \( \tilde{A}_{\alpha,k,l}(x) = A_\alpha(x_\alpha^1 + r_\alpha^k x) \) converges, up-to gauge transformations, to \( \tilde{A}_{k,l,\infty} \), where \( \tilde{A}_{k,l,\infty} \) is a Yang-Mills connection in a bundle over \( \mathbb{R}^4 \).

(The above procedure of the bubble decomposition can be illustrated in the following figures. The above shadow parts in the figure stand for the neck regions. The above figures with number 1 can be viewed as the case of a single bubble and the figures with numbers 2-4 can be seen as a special example of three different bubbles, but real cases of multiple bubbles are much more complicated.)

In conclusion, there are finite numbers \( r_\alpha^k \), finite points \( x_{\alpha}^{k,l} \), positive constants \( R_{k,l}, \delta_{k,l} \) and a finite number of non-trivial Yang-Mills connections \( \tilde{A}_{k,l,\infty} \) on \( S^4 \) such that

\[
\lim_{\alpha \to 1} YM(A_\alpha; B_{R_0}(x_i)) = YM(A_\infty; B_{R_0}(x_i)) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} YM(\tilde{A}_{k,l,\infty}; S^4) + \sum_{k=1}^{L} \sum_{l=1}^{J_k} \lim_{R_{k,l} \to \infty} \lim_{\delta_{k,l} \to 0} \lim_{\alpha \to 1} YM(\tilde{A}_{k,l,\alpha}; B_{\delta_{k,l}} \backslash B_{R_{k,l} r_\alpha^k(x_{\alpha}^{k,l})}).
\]
Moreover, at each neck region \( B_{h,\delta} \setminus B_{R_\epsilon,\epsilon} \) in (22), for all \( \alpha \) sufficiently close to 1, we have

\[
\int_{B_{R,\epsilon}(x_{\alpha}^k)} |F_{\alpha,\epsilon,\alpha}|^2 dV \leq \varepsilon
\]

for all \( r \in (\frac{R_\epsilon,\epsilon}{2}, 2\delta_{h,\epsilon}) \), where \( \varepsilon \) is a fixed constant to be chosen sufficiently small.

Finally, we point out that the above bubble-neck decomposition could be formulated by the method of Parker [20].

3.2. No new bubble on each neck region. According to the previous result on the bubble-neck decomposition (22) with the property (23), we will prove no new bubble and energy loss on each neck region \( B_{h,\delta} \setminus B_{R_\epsilon,\epsilon} \) in (22). For simplicity of notations, we may take \( x_{\alpha}^k \) to be the origin of our coordinate system, and denote \( B_\delta = B_{h,\delta}(x_{\alpha}^k) \). Furthermore, if we choose coordinates which are orthonormal at \( x_{\alpha} \), then in the limit \( \delta \to 0 \) the metric on \( B_\delta \) will approach the Euclidean metric. Therefore, in this section we assume without loss of generality that the metric on \( B_\delta \) is given by \( g_{ij} = \delta_{ij} \).

Noting the assumption (23) on each annulus \( B_{h,\delta}(0) \) and applying Lemma 3.1, we have that

\[
\int_{B_{r,\epsilon}(x_{\alpha}^k)} |F_{\alpha,\epsilon,\alpha}|^2 dV \leq \gamma \sqrt{\varepsilon}
\]

for a sufficiently small constant \( \varepsilon > 0 \).

We choose polar coordinates \((r, \theta)\) on \( B_\delta \), where \( \theta = (\theta_1, \theta_2, \theta_3) \) are coordinates on \( S^3_r \). Our connection 1-form can be decomposed as

\[
F = F_r dr + A_\theta,
\]

where \( F_r = (A_r dr, dr) \) denotes the radial part and \( A_\theta = \sum_{j=1}^{3} A_{\theta j} d\theta^j \) denotes the \( S^3_r \)-part. For a vector field \( X = X_k \frac{1}{r^2} \), we define

\[
X \cdot F_A = F_A(X, \cdot) = \frac{1}{2} X_k F^i j dx^j,
\]

which belongs to \( \Gamma(\text{Ad } E \otimes T^* M) \). We write

\[
F = F_r dr \wedge d\theta + \frac{1}{2} F_{\theta i, \theta j} d\theta^i \wedge d\theta^j.
\]

Then \( \frac{\partial}{\partial r} \cdot F_A = F_{A, r \theta} d\theta \).

Following Uhlenbeck in [33], we will construct a broken Hodge gauge. We recall Theorem 2.8 and Corollary 2.9 of [33] in the following:

**Lemma 3.2.** Let \( D \) be a covariant derivative in a vector bundle \( E \) over \( U = \{ x : 1 \leq |x| \leq 2 \} \).

There exists a constant \( \gamma > 0 \) such that if \( \| F_A \|_{L^\infty(U)} \leq \gamma \), then there is a gauge transformation in which \( D = d + A \), \( d^* A = 0 \) on \( U \), \( d^*_A A_\theta = 0 \) on \( S_1 \) and \( S_2 \), and \( \int_{S_1} A_r = \int_{S_2} A_r = 0 \). Moreover, \( \| A \|_{L^\infty(U)} \leq C \| F_A \|_{L^\infty(U)} \) and

\[
(\lambda_4 - k') \| F_A \|_{L^\infty(U)}^2 \int_U |A|^2 \leq \int_U |F_A|^2,
\]

where \( \lambda_4 \) and \( k' \) are positive constants defined in Corollary 2.9 of [33].
For \( l = \{-1, 0, 1, 2, \ldots\} \), we define
\[
\mathcal{U}_l = \{ x : 2^{-l-1}\delta \leq |x| \leq 2^{-l}\delta \},
\]
\[
S_l = \{ x : |x| = 2^{-l}\delta \}.
\]
We choose \( n \) such that
\[
B_\delta \setminus B_{R\rho} \subseteq \sum_{l=0}^{n} \mathcal{U}_l \subseteq B_\delta \setminus B_{2R\rho}.
\]
A broken Hodge gauge is a continuous gauge transformation \( s \) which satisfies:

a) \( d^*A_l = 0 \) with \( A_l = s^*A_0|_{\mathcal{U}_l} \);

b) \( A_{l,\theta}|_{S_l} = A_{l-1,\theta}|_{S_l} \);

c) \( d^*_lA_{l,\theta} = 0 \) on \( S_l \) and \( S_{l+1} \);

d) \( \int_{S_l} A_{l,r} dS = \int_{S_{l+1}} A_{l,r} dS = 0 \), where \( dS \) is the measure on the boundary \( S^3 \).

Since \( \epsilon R^2 \|F_A(x)\|_{L^\infty} \) can be chosen sufficiently small, we can apply Theorem 4.6 of [35] to obtain a broken Hodge gauge on \( B_\delta \setminus B_{4R\rho} \). We henceforth assume that the connection \( A_\rho \) is expressed in this gauge.

We prove the following lemma which will be used in the proof of Lemma 3.3.

**Lemma 3.3.** Let \( B_R \) be the geodesic ball of radius \( R \) at a point \( x_0 \in M \), and \( \phi \) a smooth cut-off function on \( B_R \) with \( \phi = 1 \) on \( B_{kR} \), \( k < 1 \), and \( |d\phi| \leq cR^{-1} \). Let \( A \) be a connection on \( B_R \). There exists a constant \( \varepsilon_1 > 0 \) such that if
\[
\int_{B_R} |F_A|^2 dV < \varepsilon_1,
\]
then we have
\[
\int_{B_R} \|\nabla A F_A\|^2 \phi^2 dV \leq C \int_{B_R} |D_A F_A|^2 \phi^2 dV + C(1 + R^{-2}) \int_{B_R} |F_A|^2 dV.
\]

**Proof.** We compute
\[
\int_{B_R} \|\nabla A F_A\|^2 \phi^2 dV \leq \int_{B_R} \left( \langle \nabla A F_A, \nabla_A (\phi^2 F_A) \rangle + 2\phi |d\phi| |F| \|\nabla A F_A\| \right) dV
\]
\[
\leq \int_{B_R} \left( \langle \nabla_A \nabla_A F_A, \phi^2 F_A \rangle + C\phi R^{-1} |F| \|\nabla A F_A\| \right) dV.
\]
The second term is handled using Young’s inequality with \( \varepsilon \):
\[
\phi R^{-1} |F| \|\nabla A F_A\| \leq C\varepsilon^{-1} R^{-2} |F|^2 + C\varepsilon \|\nabla A F_A\|^2 \phi^2,
\]
for \( \varepsilon > 0 \) small. For the first term, it follows from using the Weitzenböck formula and the Bianchi identity that
\[
\int_{B_R} \langle \nabla_A \nabla_A F_A, \phi^2 F_A \rangle dV = \int_{B_R} \langle \Delta A F_A + F_A \# F_A + R_M \# F_A, \phi^2 F_A \rangle dV
\]
\[
\leq \int_{B_R} \left( \langle D_A F_A, D_A (\phi^2 F_A) \rangle + \langle D_A F_A, D_A (\phi^2 F_A) \rangle + C\phi^2 |F_A|^2 + c\phi^2 |F_A|^3 \right) dV
\]
\[
\leq \int_{B_R} \left( \langle D_A F_A \rangle^2 \phi^2 + C(1 + R^{-2}) |F_A|^2 \phi^2 + C |F_A|^3 \phi^2 + C\phi R^{-1} |F| \|D_A F_A\| \right) dV.
\]
The last term is handled using Young’s inequality. For the cubic term, using the Hölder and Sobolev inequalities, we have
\[
\int_{B_R} |F_A|^3 \phi^2 dV \leq \left( \int_{B_R} |F_A|^2 dV \right)^{1/2} \left( \int_{B_R} |F_A|^4 \phi^4 dV \right)^{1/2}
\]
\[
\leq C \left( \int_{B_R} |F_A|^2 dV \right)^{1/2} \left( \int_{B_R} |F_A|^2 \phi^2 dV + \int_{B_R} |\nabla_A(F_A\phi)|^2 dV \right)
\]
\[
\leq C\varepsilon_1^2 \left( \int_{B_R} |F_A|^2 \phi^2 dV + \int_{B_R} |\nabla_A F_A|^2 \phi^2 dV + CR^{-2} \int_{B_R} |F_A|^2 dV \right).
\]

Then choosing \( \varepsilon_1 \) small enough, the result follows. \( \Box \)

**Lemma 3.4.** There exists a positive constant \( \varepsilon_2 \) such that if
\[
\| F_{A_n} \|_{L^\infty(\Omega_t)} \leq \varepsilon_2 2^{4i} \delta^{-2},
\]
and \( \alpha \) is sufficiently close to 1, we have
\[
\int_{B_i \setminus B_{R}} |A_\alpha| |\nabla A_{\alpha} F_{A_n}| dV \leq C \int_{B_{2\delta} \setminus B_{6\delta}} |F_{A_n}|^2 dV.
\]

**Proof.** It follows from using Hölder’s inequality that
\[
\int_{B_i \setminus B_{R}} |A_\alpha| |\nabla A_{\alpha} F_{A_n}| dV \leq \sum_{l=0}^{n} \int_{\Omega_l} |A_\alpha| |\nabla A_{\alpha} F_{A_n}| dV
\]
\[
\leq \sum_{l=0}^{n} \left( \int_{\Omega_l} |A_\alpha|^2 dV \right)^{1/2} \left( \int_{\Omega_l} |\nabla A_{\alpha} F_{A_n}|^2 dV \right)^{1/2}.
\]

For the first factor, scaling in Lemma 3.2 (see also Corollary 2.9 of [55]), we have
\[
\int_{\Omega_l} |A_\alpha|^2 dV \leq C2^{-2l} \delta^2 (\lambda_4 - k'2^{-4l} \delta^2 \| F_{A_n} \|_{L^\infty(\Omega_l)})^{-1} \int_{\Omega_l} |F_{A_n}|^2 dV
\]
(25)
\[
\leq C2^{-2l} \delta^2 \int_{\Omega_l} |F_{A_n}|^2 dV,
\]

after choosing \( \varepsilon_2 \) sufficiently small. To deal with the second factor, we cover \( \Omega_l \) with finitely many open balls \( B_{2R/3}(i) \) such that \( \cup_i B_R(i) \subset W_l = \cup_{l-1} \cup \Omega_l \cup \Omega_{l+1} \), for some \( R \) with \( \frac{3\delta}{2R^{1/2}} \leq R < \frac{3\delta}{2\delta_0} \). We let \( \phi_i \) be a smooth cut-off function on \( B_R(i) \) with \( \phi = 1 \) on \( B_{2R/3}(i) \) and \( \|d\phi_i\| \leq CR^{-1} \). Note that from [55], we have
\[
|D^*_{A_n} F_{A_n}| \leq C(\alpha - 1) |\nabla A_{\alpha} F_{A_n}|.
\]

Applying Lemma 3.3 to the ball \( B_R(i) \) yields
\[
\int_{B_R(i)} |\nabla A_{\alpha} F_{A_n}|^2 \phi_i^2 dV
\]
\[
\leq C \int_{B_R(i)} |D^*_{A_n} F_{A_n}|^2 \phi_i^2 dV + C(1 + R^{-2}) \int_{B_R(i)} |F_{A_n}|^2 dV
\]
\[
\leq C(\alpha - 1) \int_{B_R(i)} |\nabla A_{\alpha} F_{A_n}|^2 \phi_i^2 dV + C(1 + 2^2 \delta^{-2}) \int_{B_R(i)} |F_{A_n}|^2 dV.
\]
Then for \( \alpha \) sufficiently close to 1, we find
\[
\int_{B_R(i)} |\nabla A |^2 dV \leq C(1 + 2^{2\delta} - 2) \int_{B_R(i)} |F_{A_\alpha}|^2 dV.
\]
Summing over \( i \), it follows that
\[
\int_{U_l} |\nabla A |^2 dV \leq C(1 + 2^{2\delta} - 2) \int_{W_l} |F_{A_\alpha}|^2 dV.
\]
Finally, combining the above we have
\[
\int_{B_\delta \setminus B_{Rr_\alpha}} |F_{A_\alpha}|^2 dV \leq C(1 + 2^{2\delta} - 2) \int_{W_l} |F_{A_\alpha}|^2 dV.
\]
which implies the desired result.

**Lemma 3.5.** There are a sequence \( \alpha \to 0 \) and a subsequence \( \alpha_k \to 1 \) such that if \( \int_{B_{2r_\alpha \setminus B_r}} |F_A|^2 \leq \varepsilon_3 \) for all \( r \in (\frac{1}{2} Rr_\alpha, 2\delta) \) and for a sufficiently small \( \varepsilon_3 > 0 \), the boundary integral satisfies
\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{\partial B_r} \langle A_{\alpha, \theta}, F_{A_{\alpha, \theta}} \rangle dS = 0.
\]

**Proof.** Since \( YM(A_\alpha) \leq K \) for each \( \alpha \), it follows from Fatou’s lemma that
\[
\int_0^{R_0} \liminf_{\alpha \to 1} \int_{\partial B_r} |F_{A_\alpha, \theta}|^2 dS d r \leq K.
\]
It follows that
\[
\lim_{\delta \to 0} \left( \delta \liminf_{\alpha \to 1} \int_{\partial B_r} |F_{A_\alpha, \theta}|^2 dS \right) = 0.
\]
From Theorem 4.6 of [35] (see also Lemma 3.2), in the Hodge gauge, we have
\[
\delta \| A_{\alpha, \theta} \|_{L^\infty(\partial B_r)} \leq C \delta^2 \| F_{A_\alpha} \|_{L^\infty(B_r \setminus B_\delta)} \leq C.
\]
This implies
\[
\int_{\partial B_r} |A_{\alpha, \theta}|^2 dS \leq C \delta.
\]
By Hölder’s inequality, we obtain
\[
\int_{\partial B_r} \langle A_{\alpha, \theta}, F_{A_{\alpha, \theta}} \rangle dS \leq \left( \int_{\partial B_r} |A_{\alpha, \theta}|^2 dS \right)^{\frac{1}{2}} \left( \int_{\partial B_r} |F_{A_{\alpha, \theta}}|^2 dS \right)^{\frac{1}{2}} \leq C \left( \delta \right)^{\frac{1}{2}} \left( \int_{\partial B_r} |F_{A_{\alpha, \theta}}|^2 dS \right)^{\frac{1}{2}}.
\]
Choosing a sequence \( \delta \to 0 \) and a suitable subsequence of \( \alpha_k \to 1 \), the claim follows from (26).

**Lemma 3.6.** There exists a sufficiently small \( \varepsilon > 0 \) such that if \( \int_{B_{2r \setminus B_r}} |F_A|^2 \leq \varepsilon \) for all \( r \in (\frac{1}{2} Rr_\alpha, 2\delta) \), then
\[
\lim_{\delta \to 0} \lim_{\alpha \to 1} \int_{B_\delta \setminus B_{Rr_\alpha}} |F_{A_\alpha}|^2 dV = 0.
\]
where the subsequence \( \alpha \to 1 \) is suitably chosen.

**Proof.** Under the assumption, \[(27)\] holds, so the above broken Hodge gauge exists by choosing \( \varepsilon \) sufficiently small. On each annulus \( \Omega_l \), we calculate
\[
\int_{\Omega_l} |F_{A_{\alpha}}|^2 \, dV = \int_{\Omega_l} \langle dA_{\alpha} + [A_{\alpha}, A_{\alpha}], F_{A_{\alpha}} \rangle \, dV \\
= \int_{\Omega_l} \langle D_{A_{\alpha}} A_{\alpha} + A_{\alpha} \# A_{\alpha}, F_{A_{\alpha}} \rangle \, dV \\
= \int_{\Omega_l} \langle A_{\alpha}, D^*_{A_{\alpha}} F_{A_{\alpha}} \rangle \, dV + \int_{\Omega_l} \langle A_{\alpha} \# A_{\alpha}, F_{A_{\alpha}} \rangle \, dV \\
+ \int_{S_l} \langle A_{\alpha, \theta}, F_{A_{\alpha, r\theta}} \rangle \, dS - \int_{S_{l+1}} \langle A_{\alpha, \theta}, F_{A_{\alpha, r\theta}} \rangle \, dS,
\]
(27)
where \( A_{\alpha} \# A_{\alpha} \) denotes a multi-linear combination of \( A_{\alpha} \) and \( A_{\alpha} \). The boundary terms here can be derived by constructing a radial cut-off function \( \phi(r) \) with \( \phi = 1 \) on \( \Omega_l \), and \( \phi = 0 \) outside of a slightly larger annulus
\[
\Omega'_l = \{ x : 2^{-l-1} \delta - \varepsilon \leq |x| \leq 2^{-l} \delta + \varepsilon \}.
\]
The derivative is \( d\phi = \frac{1}{r} dr \) on \( \Omega'_l \setminus \Omega_l \), and zero elsewhere. Then
\[
\int_{\Omega'_l} \langle D_{A_{\alpha}} A_{\alpha}, F_{A_{\alpha}} \rangle \phi dV = \int_{\Omega'_l} \langle A_{\alpha}, D^*_{A_{\alpha}} F_{A_{\alpha}} \rangle \phi dV - \int_{\Omega'_l} \langle d\phi \wedge A_{\alpha}, F_{A_{\alpha}} \rangle \, dV,
\]
where the last term will become the boundary terms of \[(27)\] in the limit \( \varepsilon \to 0 \). Summing \[(27)\] over \( l \) and using equation \[(8)\], we find
\[
\sum_{l=0}^{n} \int_{\Omega_l} |F_{A_{\alpha}}|^2 \, dV = \sum_{l=0}^{n} \int_{\Omega_l} \langle A_{\alpha}, D^*_{A_{\alpha}} F_{A_{\alpha}} \rangle \, dV + \int_{S_0} \langle A_{\alpha, \theta}, F_{A_{\alpha, r\theta}} \rangle \, dS \\
- \int_{S_{l+1}} \langle A_{\alpha, \theta}, F_{A_{\alpha, r\theta}} \rangle \, dS + (\alpha - 1) \sum_{l=0}^{n} \int_{\Omega_l} \left\langle A_{\alpha}, \frac{(d|F_{A_{\alpha}}|^2 \wedge *F_{A_{\alpha}})}{1 + |F_{A_{\alpha}}|^2} \right\rangle \, dV.
\]
Choosing a suitable subsequence of \( \alpha \to 1 \), two boundary terms are going to zero by Lemma \[3.3\]. Using Lemma \[3.4\] the final term is bounded by
\[
C(\alpha - 1) \sum_{l} \int_{\Omega_l} |A_{\alpha}| \| \nabla A_{\alpha} F_{A_{\alpha}} \| \, dV \leq C(\alpha - 1) \int_{B_1 \setminus B_{\varepsilon \delta}} |F_{A_{\alpha}}|^2 \, dV,
\]
which is going to zero as \( \alpha \to 1 \). For the first term, recalling \[(25)\],
\[
\int_{\Omega_l} \langle A_{\alpha} \# A_{\alpha}, F_{A_{\alpha}} \rangle \leq C \| F_{A_{\alpha}} \|_{L^\infty(\Omega_l)} \int_{\Omega_l} |A_{\alpha}|^2 \, dV \\
\leq C 2^{-2l} \| F_{A_{\alpha}} \|_{L^\infty(\Omega_l)} \int_{\Omega_l} |A_{\alpha}|^2 \, dV.
\]
Note that \( 2^{-2l} \| F_{A_{\alpha}} \|_{L^\infty(\Omega_l)} \) can be made sufficiently small, so that this term may be absorbed into the left hand side. The required result follows from Lemma \[3.5\] and the uniform bound of \( \int_{\Omega_l} |F_{A_{\alpha}}|^2 \, dV \).

We now complete the proof of Theorem \[2\].

**Proof of Theorem \[2\].** Theorem \[2\] follows from Lemma \[3.6\] and the bubble-neck decomposition \[(22)\] by choosing \( \varepsilon \) sufficiently small. \( \square \)
By letting $\alpha_i = 1$ for all $i$, Theorem 2 yields a new proof of a result of Rivière [22] on sequences of Yang-Mills connections:

**Corollary 3.7.** Let $A_i$ be a sequence of smooth Yang-Mills connections on $E$. Then there exists a finite set $S \subset M$, and a sequence of gauge transformations $s_i$ defined on $M \setminus S$, such that for any compact $K \subset M \setminus S$, $s_i^* A_i$ converges to $A_\infty$ smoothly in $K$. Moreover, there are a finite number of bubble bundles $E_1, \ldots, E_l$ over $S$ and Yang-Mills connections $\tilde{A}_1, \ldots, \tilde{A}_l$ such that

$$\lim_{i \to \infty} YM(A_i) = YM(A_\infty) + \sum_{i=1}^l YM(\tilde{A}_i).$$

As a second consequence of Theorem 2, we can also give a simple proof of the energy identity for a minimizing sequence (see [13], [10]):

**Proposition 3.8.** Let $E$ be a vector bundle over $M$. Assume that $A_i$ is a minimizing sequence of the Yang-Mills functional $YM$ among smooth connections on $E$, which converges weakly to some limit connection $A_\infty$ by Sedlacek’s result [25]. Then there are a finite number of bubble bundles $E_1, \ldots, E_l$ over $S$ and Yang-Mills connections $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_l$ such that

$$\lim_{i \to \infty} YM(A_i) = YM(A_\infty) + \sum_{i=1}^l YM(\tilde{A}_i).$$

**Proof.** We choose a sequence $\alpha_i \to 1$, and let $A_{\alpha_i}(t)$ be a solution to the Yang-Mills $\alpha_i$-flow of [10] with initial condition $A_{\alpha_i}(0) = A_i$ (see a similar argument for harmonic maps in [11]). Then $A_{\alpha_i}$ is a smooth solution to the flow equation

$$\frac{\partial A_{\alpha_i}}{\partial t} = -D_{A_{\alpha_i}}^* F_{A_{\alpha_i}} + (\alpha_i - 1) \frac{(d|F_{A_{\alpha_i}}|^2 \wedge \ast F_{A_{\alpha_i}})}{1 + |F_{A_{\alpha_i}}|^2}.$$  \hspace{1cm} (28)

Since $A_i$ is a minimizing sequence, by suitably choosing the sequence $\alpha_i \to 1$, there exists at least one $t_0 \in [1/2, 1]$ such that for all $i$,

$$\int_M |\partial_t A_{\alpha_i}(\cdot, t_0)|^2 dV \to 0.$$  \hspace{1cm} (29)

The result then follows from the same arguments as for Theorem 2. See [10] for more details. \hfill \Box

### 3.3. Applications to the Yang-Mills flow

In order to prove Theorem 3, we need the following local energy inequality:

**Lemma 3.9.** Let $A(t)$ be a solution to the Yang-Mills flow (20) in $M \times [0, T)$ for $A \in M \times [0, T)$ with initial value $A(0) = A_0$. For any $x_0$ with $B_{2R}(x_0) \subset X$ and for any two $s, \tau \in [0, T)$ with $s < \tau$, we have

$$\int_M |F_A|^2(\cdot, s) dV + \int_s^\tau \int_M |\partial_t A|^2 dV dt \leq \int_M |F_{A_0}|^2 dV,$$

$$\int_{B_R(x_0)} |F_A|^2(\cdot, \tau) dV \leq \int_{B_{2R}(x_0)} |F_A|^2(\cdot, s) dV + \frac{C(\tau - s)}{R^2} \int_M |F_{A_0}|^2 dV$$
and
\[
\int_{B_R(x_0)} |F_A|^2(r, s) \, dV \leq \int_{B_{2R}(x_0)} |F_A|^2(r, \tau) \, dV + C \int_s^\tau \int_{B_{2R}(x_0)} |\partial_t A|^2 \, dV \, dt
\]
\[
+ C \left( \frac{(r - s)}{R^2} \int_{B_{2R}(x_0)} |F_{A_0}|^2 \, dV \int_s^\tau \int_{B_{2R}(x_0)} |\partial_t A|^2 \, dV \, dt \right)^{1/2}.
\]

**Proof.** The proof can be found in [27] and Lemma 5 of [9]. \(\square\)

Now we present a proof of Theorem 3.

**Proof of Theorem 3.** Let \(T\) be the maximal existence time of the Yang-Mills flow. By the result of Struwe in [27], there are a finite number of points \(\{x_1, ..., x_l\}\) such that \(A(t)\) converges, up-to gauge transformations, to a connection \(A(T)\) in \(C_0^\infty(M, \{x_1, ..., x_l\})\) as \(t \to T\). In a local trivialization of \(E\) near each singularity \(x_j\), the connection \(D(t)\) can be expressed by \(d + A(t)\) with \(A(t) = A_i(x, t)dx^i\). At each singularity \(x_j\), there is a \(R_0 > 0\) such that there is no singularity inside \(B_{R_0}(x_j)\). Then there is a \(\Theta > 0\) such that as \(t \to T\)
\[
|F_{A(t)}|^2 dV \to \Theta \delta_{x_j} + |F_{A(T)}|^2 dV,
\]
where \(\delta_{x_j}\) denotes the Dirac mass at the singularity \(x_j\). This can be proved by using Lemma 3.9 (e.g. [17]). Then there exist sequences \(r_k \to 0, t_k \to T\) such that as \(k \to \infty\),
\[
\lim_{k \to \infty} \int_{B_{r_k}(x_j)} |F_{A(t_k)}|^2 \, dV = \Theta.
\]

We consider the rescaled connection
\[
\tilde{A}_k(t) = d + \tilde{A}_k(x, t) := d + r_k A_i(x_j + r_k x, t_k + r_k^2 t) dx^i.
\]
Then \(\tilde{A}_k(t)\) satisfies
\[
\frac{\partial \tilde{A}_k}{\partial t} = -D^*_A \tilde{F}_A \quad \text{in} \quad B_{r_k^{-1}(0)} \times [-2, 0]
\]
and
\[
\int_{-2}^0 \int_{B_{r_k^{-1}(0)}(-2)} |\partial_t \tilde{A}_k|^2 \, dV \leq \int_{t_k - 2r_k^2}^{t_k} \int_{B_{r_k^{-1}(0)}(-2)} |\partial_t A(t)|^2 \, dV \leq \int_{t_k - 2r_k^2}^{t_k} \int_M |\partial_t A(t)|^2 \, dV \to 0
\]
as \(k \to \infty\). Then there is a \(\tilde{t}_k \in (-1, 0)\) such that
\[
\int_{B_{r_k^{-1}(0)}(-2)} |\partial_t \tilde{A}_k(\tilde{t}_k)|^2 \to 0, \quad \lim_{k \to \infty} \int_{B_1(0)} |F_{A_k}(\cdot, 0)|^2 = \Theta.
\]
By applying Lemma 3.9 again, we have
\[
\int_{B_R(0)} |F_{A_k}(\cdot, \tilde{t}_k)|^2 \geq \int_{B_1(0)} |F_{A_k}(\cdot, 0)|^2 - \frac{C}{R^2} \int_M |F_{A_0}|^2 \, dV
\]
for \(R > 2\). This implies
\[
\lim_{k \to \infty} \int_{B_R(0)} |F_{A_k}(\cdot, \tilde{t}_k)|^2 \geq \Theta.
\]
By (30), we know
\[ \lim_{k \to \infty} \int_{B_R(0)} |F_{\tilde{A}_k}(t_k)|^2 = \Theta. \]
Since \( \tilde{A}_k(t) \) satisfies the Yang-Mills flow (32), a parabolic \( \varepsilon \)-regularity estimate holds; i.e. there is a constant \( \varepsilon > 0 \) such that if
\[ \sup_{t_k - 4r^2 \leq t \leq t_k} \int_{B_r(0)} |F_{\tilde{A}_k}(t_k)|^2 dV < \varepsilon \]
for some small \( r > 0 \), then we have
\[ |F_{\tilde{A}_k}(0, t_k)|^2 \leq \frac{C}{r^6} \int_{P_{t_k}(0)} |F_{\tilde{A}_k}(t)|^2 dV dt \leq \frac{C\varepsilon}{r^4}, \]
where \( P_{t_k}(0) = B_r(0) \times (t_k - r^2, t_k] \). For the above \( \varepsilon \) in (35), there is a constant \( R \) satisfying
\[ \int_{B_{2r}(0)} |F_{\tilde{A}_k}(t_k)|^2 dV < \varepsilon. \]
Using (34) we apply Lemma 3.9 to obtain that
\[ \int_{B_{2r}(0)} |F_{\tilde{A}_k}(t_k)|^2 dV \leq C \int_{B_{2r}(0)} |F_{\tilde{A}_k}(t)|^2 dV dt \]
for sufficiently large \( k \). With these estimates, the bubble tree procedure also works for the sequence of connections \( \tilde{A}_k(t_k) \) on each \( B_R(0) \). Therefore, using the proof of Theorem 2 on each \( B_R(0) \), there is a finite number \( l_R \) bubbling Yang-Mills connections \( \tilde{A}_{i,R} \) on \( S^4 \) such that
\[ \lim_{k \to \infty} \int_{B_R(0)} |F_{\tilde{A}_k}(t_k)|^2 = \sum_{i=1}^{l_R} YM(\tilde{A}_{i,R}; S^4). \]
By using (31), there is a constant \( \varepsilon_0 > 0 \) such that any non-trivial Yang-Mills connection \( \tilde{A}_{i,R} \) on \( S^4 \) has
\[ YM(\tilde{A}_{i,R}; S^4) \geq \varepsilon_0, \]
which implies \( 1 \leq l_R \leq \frac{C}{\varepsilon_0} \). As \( R \to \infty \), it follows from Corollary 3.7 that
\[ \Theta = \lim_{R \to \infty} \lim_{k \to \infty} \int_{B_R(0)} |F_{\tilde{A}_k}(t_k)|^2 \]
\[ = \lim_{R \to \infty} \sum_{i=1}^{l_R} YM(\tilde{A}_{i,R}; S^4) = \sum_{j=1}^{l} YM(\tilde{A}_{j,\infty}; S^4), \]
where each \( \tilde{A}_{j,\infty} \) is a Yang-Mills connections in a vector bundle \( E_j \) over \( S^4 \). This proves Theorem 3. \( \square \)
Finally, we make a remark about the case of $T = \infty$. Let $E$ be a complex vector bundle over a 4-manifold $M$. Let $A$ be a global smooth solution of the Yang-Mills flow equation \((3)\) in $M \times [0, \infty)$ with initial value $A_0$. For any a sequence \(\{t_k\}\) with $t_k \to \infty$, there is a subsequence, still denoted by \(\{t_k\}\), such that as $k \to \infty$, $A(x, t_k)$ converges in $C^\infty(M \setminus \Sigma)$ to a solution $A_\infty$ of the Yang-Mills equation \((1.2)\), where $\Sigma$ is a finite set of singularities in $M$. At each singularity $x_i$, \[
abla \liminf_{k \to \infty} \int_{B_r(x_i)} |F_A|^2 \left(\cdot, t_k\right) dV \geq \varepsilon_0 \]
for a constant $\varepsilon_0 > 0$. The second Chern classes (e.g. [31]) is defined by
\[
C_2(E) = \frac{1}{8\pi^2} \left[ \text{tr}(F_A \wedge F_A) - \text{tr}F_A \wedge \text{tr}F_A \right].
\]
Theorem 3 implies that $C_2(E; M) = \frac{1}{8\pi^2} \lim_{k \to \infty} \int_M [\text{tr}(F_{A(t_k)} \wedge F_{A(t_k)}) - \text{tr}F_{A(t_k)} \wedge \text{tr}F_{A(t_k)}]$
\[
= \frac{1}{8\pi^2} \int_M [\text{tr}(F_{A_\infty} \wedge F_{A_\infty}) - \text{tr}F_{A_\infty} \wedge \text{tr}F_{A_\infty}]
+ \frac{1}{8\pi^2} \sum_{i=1}^l \int_{S^4} [\text{tr}(F_{\tilde{A}_i} \wedge F_{\tilde{A}_i}) - \text{tr}F_{\tilde{A}_i} \wedge \text{tr}F_{\tilde{A}_i}]
= C_2(E_\infty; M) + \sum_{i=1}^l C_2(E_i; S^4),
\]
where $C_2(E_\infty; M)$ is the second Chern number of the limiting bundle induced by $A_\infty$, and $C_2(E_i; S^4)$ is the second Chern number of the bubbling bundle $E_i$ over $S^4$.

4. Appendix: Regularity

In this section we prove that a Yang-Mills $\alpha$-connection, a weak solution to the Euler-Lagrange equation \((3)\), is gauge-equivalent to a smooth solution. We say that $A \in A^{1,2\alpha}$ is a weak solution to \((3)\) if \[
\int_M \left( 1 + |F_A|^2 \right)^{\alpha-1} (F_A, D_A B) dV = 0
\]
for any $B \in C^\infty(A)$; i.e. $A$ satisfies
\[
\left(37\right) 
\frac{d}{d\alpha} \left( 1 + |F_A|^2 \right)^{\alpha-1} F_A = 0 \quad \text{in } U
\]
in the weak sense.

The proof is essentially similar to the one in [12] for $p$-Yang-Mills connections (see a similar approach in [13]). Since there are some differences, we would like to outline the main points. Let $x_0$ be a point in $M$. For any $\varepsilon_0$, there is a sufficiently small $R_0$ so that for each $R > 0$ with $R \leq R_0$,
\[
\int_{B_R(x_0)} |F_A|^2 \leq \int_{B_{R_0}(x_0)} |F_A|^2 \leq \varepsilon_0.
\]
For a sufficiently small $\varepsilon_0$, there is a gauge transformation $\sigma$ (see [34]) such that $\sigma^*(D_A) = d + A$ with $d^* A = 0$ in $B_R(x_0)$, $A \cdot \nu = 0$ on $\partial B_R(x_0)$, and
\[
\int_{B_R(x_0)} R^{-2\alpha} |A|^{2\alpha} + |\nabla A|^{2\alpha} \leq \int_{B_R(x_0)} |F_A|^{2\alpha}.
\]
Similarly to Lemma 2.1 in [12], there is a $B$ such that
\[
\begin{align*}
(38) & \quad d^* ((1 + |dB|^2)^{\alpha - 1} dB) = 0 \quad \text{in } B_R(x_0), \\
(39) & \quad d^* B = 0 \quad \text{in } B_R(x_0), \\
(40) & \quad \iota^* B = \iota^* A \quad \text{on } \partial B_R(x_0).
\end{align*}
\]
By the result of [33], we have
\[
\int_{B_\rho(x_0)} |dB|^{2\alpha} dV \leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |dB|^{2\alpha} dV \leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |dA|^{2\alpha} dV
\]
for any $\rho < R \leq R_0$. To compare the above equations of $A$ and $B$, we have
\[
\begin{align*}
& \int_{B_R(x_0)} |F_A - dB|^{2\alpha} dV \\
& \leq C \int_{B_R(x_0)} ((1 + |A|^2)^{\alpha - 1} F_A - (1 + |dB|^2)^{\alpha - 1} dB, F_A - dB) \ dV \\
& \leq C \int_{B_R(x_0)} |(1 + |A|^2)^{\alpha - 1} F_A - (1 + |dB|^2)^{\alpha - 1} dB| |A|^2 \ dV \\
& \leq \varepsilon \int_{B_R(x_0)} (1 + |A|^2)^\alpha \ dV + C \int_{B_R(x_0)} (1 + |A|^{4\alpha}) \ dV
\end{align*}
\]
for a sufficiently small $\varepsilon$, which will be determined later. By the Hölder and Sobolev inequalities, we have
\[
\begin{align*}
& \int_{B_R(x_0)} |A|^{4\alpha} \ dV \leq CR^{4(\alpha - 1)} \left( \int_{B_R(x_0)} |A|^{4\alpha - \frac{4}{\alpha}} \ dV \right)^{2 - \alpha} \\
& \leq CR^{4(\alpha - 1)} \left( \int_{B_R(x_0)} R^{-2\alpha} |A|^{2\alpha} + |\nabla A|^{2\alpha} \ dV \right)^2 \\
& \leq CR^{4(\alpha - 1)} \left( \int_{B_R(x_0)} |F_A|^{2\alpha} \ dV \right)^2 \leq C_1 R^{4(\alpha - 1)} \int_{B_R(x_0)} |F_A|^{2\alpha} \ dV.
\end{align*}
\]
Combining the above two estimates, we can get that for any $\rho < R \leq R_0$,
\[
\begin{align*}
(41) & \quad \int_{B_\rho(x_0)} (1 + |F_A|^2)^\alpha \ dV \leq C \left[ \left( \frac{\rho}{R} \right)^4 + \varepsilon + R^{4(\alpha - 1)} \right] \int_{B_R(x_0)} (1 + |F_A|^2)^\alpha \ dV + CR^4.
\end{align*}
\]
For a sufficiently small $R_0$ and $\varepsilon$, a well-known lemma (see Lemma 2.1 of Chapter III in [6]) implies that for any small constant $\delta > 0$, there is a constant $C$ such that
\[
\int_{B_\rho(x_0)} |F_A|^{2\alpha} \leq \int_{B_\rho(x_0)} (1 + |F_A|^2)^\alpha \leq C \left( \frac{\rho}{R} \right)^{4 - \delta} \int_{B_R(x_0)} (1 + |F_A|^2)^\alpha + C\rho^{4 - \delta}
\]
for any $\rho < R \leq R_0$. 
For a sufficiently small $R_0$, there is a gauge transformation $\sigma$ (see [34]) such that

$$\int_{B_R(x_0)} R_0^{-2\alpha} |A|^{2\alpha} + |\nabla A|^{2\alpha} \leq \int_{B_R(x_0)} |F_A|^{2\alpha}.$$  

For each $R \leq R_0$ we use Lemma 2.3 of [12] to show that there is a $B \in W^{1,p}(B_R(x_0))$ such that

$$dB = 0, \quad d^* B = 0 \quad \text{in } B_R(x_0)$$

with $B \cdot \nu = A \cdot \nu$ on $\partial B_R(x_0)$. Then we have

$$\int_{B_\rho(x_0)} |B|^{4\alpha} \leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |B|^{4\alpha}.$$  

Since $(A - B) \cdot \nu = 0$ on $\partial B_R(x_0)$ and $d^*(A - B) = 0$ in $B_R(x_0)$, we can apply Lemma 2.5 of [34] and the Sobolev embedding theorem to get

$$\int_{B_R(x_0)} |A - B|^{4\alpha} \leq CR^{4\alpha-4} \left( \int_{B_R(x_0)} |dA - dB|^{2\alpha} \right)^2 \leq C \left( \int_{B_R(x_0)} |dA|^{2\alpha} \right)^2.$$  

This implies that for any $\rho < R \leq R_0$,

$$\int_{B_\rho(x_0)} |A|^{4\alpha} \leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |A|^{4\alpha} + C \left( \int_{B_R(x_0)} |dA|^{2\alpha} \right)^2.$$  

From [11] and (42), we get

$$\int_{B_\rho(x_0)} |dA|^{2\alpha} + |A|^{4\alpha} \, dV$$

$$\leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |dA|^{2\alpha} + \varepsilon + R^{4(\alpha-1)} \right) \int_{B_R(x_0)} \left( |dA|^{2\alpha} + |A|^{4\alpha} \right) \, dV$$

$$+ CR^4.$$  

For each small $\varepsilon$, there is a sufficiently small $R_0$ such that

$$\int_{B_{R_0}(x_0)} |\nabla A|^2 \leq C \int_{B_{R_0}(x_0)} |F_A|^2 \leq \varepsilon.$$  

Then applying Lemma 2.1 of Chapter III in [6] again, we have

$$\int_{B_\rho(x_0)} |dA|^{2\alpha} + |A|^{4\alpha} \leq C \rho^{4-\delta}$$

for a small $\delta > 0$ and for any $\rho \leq R_0$. Then it follows from Lemma 2.3 of [34] that

$$\int_{B_\rho(x_0)} |\nabla A|^2 \, dV \leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |\nabla A|^2 + C \int_{B_R(x_0)} |\nabla (A - B)|^2 \, dV$$

$$\leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |\nabla A|^2 + C \int_{B_R(x_0)} |dA|^2 \, dV$$

$$\leq C \left( \frac{\rho}{R} \right)^4 \int_{B_R(x_0)} |\nabla A|^2 + C_2 R^{4-\frac{\delta}{2}}$$
for any $\rho < R \leq R_0$. Applying Lemma 2.1 of Chapter III in [6] again, we obtain
\[
\int_{B_{\rho}(x_0)} |\nabla A|^2 \, dV \leq C \rho^{4 - \frac{d}{n}} \quad \forall \rho \leq R_0.
\]
The Morrey growth lemma implies that $A$ is Hölder continuous at a neighborhood of the point $x_0 \in M$. A standard procedure (e.g. [3], [12], [7]) yields that $\nabla A$ is Hölder continuous and then smooth.

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