Tensor Perturbations in Starobinski’s Inflation

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Abstract. In an almost de Sitter space-time, the stochastic semiclassical Einstein-Langevin equations with cosmological constant non-zero \( \Lambda \) have been written in the TT-gauge in a perturbative way (e.g. as zero-order and first order equations). Applying order reduction to zero and first order equations we found approximate solutions to those equations. To understand how large are the physical perturbations on different time scales, a two point correlation function of the intrinsic and induced fluctuations have been computed explicitly and their spectrum as well.

1. Introduction
A mathematically consistent and fairly well understood theory of gravity is the semiclassical one, which is based on the semiclassical Einstein equations.

The semiclassical Einstein equations are a generalization to the Einstein equations for the classical metric when the expectation value of the stress-momentum tensor of the quantum fields are source of curvature. Nevertheless, it has pointed out that this semiclassical theory may not be valid when the matter fields have important quantum stress-energy fluctuations. Because, in that case, the stress-energy fluctuations may have back-reaction effects on the space-time in the form of induced gravitational perturbations. Several authors have discussed this topic in cosmology and flat space-time and the necessity to extend the semiclassical theory to the effects of such fluctuations became obvious.

To address this problem Hu [1] proposed a theory, in the context of semiclassical cosmology, which leads to the influence functional formalism of Feynman and Vernon [2]. The theory is based on the observation that the semiclassical equations can be derived directly from the effective action of Feynman and Vernon [3], [4], [5].

An alternative approach have been introduced by Verdaguer [4]. He proposed a stochastic semi-classical theory of gravity as perturbative generalization of the semiclassical theory of gravity to describe the back reaction of the stress-tensor fluctuations. The resulting equations are the semiclassical Einstein-Lagewan equations which incorporate in a consistent way the stochastic tensor field as the source of linear perturbations to the semiclassical metric.

The theory has interesting applications in black hole physics and in cosmology, particularly in view of the problem of structure formation. Some examples have already been done [6].

The paper is organized as follows: In Sec.2., we present some aspects of the stochastic semiclassical gravity and write the Einstein-Langevin equations with non-zero cosmological constant in a perturbative way. In Sec.3., applying the so called order reduction to our higher than two derivatives Einstein-Langevin equations, we found approximately the scale factor \( a(\tau) \) as
solution to the zero-order equation and, later on, an approximate solution to the first-order inhomogeneous Einstein-Langevin equations with the aid of a retarded Green function adapting the so called Bunch-Davis initial conditions. A two point correlation function of the intrinsic and induced fluctuations have been computed explicitly and their spectrum as well. The conclusions are presented in Sect.4.

2. Gravitational waves in an almost de Sitter Universe-The equations

We assume that our universe is described by a homogeneous FRW space-time which is spatially flat with small metric perturbations

\[ ds^2 \equiv \tilde{g}_{\mu\nu}dx^\mu dx^\nu \equiv a(\eta)^2[\eta_{\mu\nu} + h_{\mu\nu}(x)]dx^\mu dx^\nu \]  

where \( \eta_{\mu\nu} = diag\{-1, 1, 1, 1\} \), \( h_{\mu\nu} \) is a symmetric tensor representing small perturbations to the background flat metric, \( a(\eta) = e^{\omega(\eta)} \) is the conformal scale factor and \( \tilde{g}_{\mu\nu} = a(\eta)^2[\eta_{\mu\nu} + h_{\mu\nu}(x)] \) is the physical metric. Notice that in general it is assumed that \( \omega(x) \) is a scalar function independent of the metric and depends on the space time point only. In case of the spatially flat FRW it will be a function of the cosmological time \( t \) or conformal time \( \eta \) which is given by the expression \( d\eta = dt/a(t) \).

We are interested in deriving the semiclassical corrections to Einstein’s equations due to the quantum effect of the scalar field keeping the gravitational field classical. This may be achieved using the semiclassical Einstein’s equations with cosmological constant \( \Lambda \), since it is known [4] that quantum effects may be produced by the breaking of conformal flatness due to the coupling of the quantum field with the gravitational perturbations. These are:

\[ \bar{G}_{\mu\nu} \equiv \bar{R}_{\mu\nu} = \frac{1}{2}\tilde{g}_{\mu\nu}\bar{R} + \Lambda\tilde{g}_{\mu\nu} = 8\pi G_N \langle \bar{T}_{\mu\nu} \rangle \]

where tilde quantities refer to the physical metric \( \tilde{g}_{\mu\nu}, \bar{G}_{\mu\nu} \) is the Einstein’s tensor, \( \bar{R}_{\mu\nu} \) and \( \bar{R} \) are the Ricci tensor and the Ricci scalar for the physical metric, respectively and \( \langle \bar{T}_{\mu\nu} \rangle \) is the expectation value of the energy-momentum tensor. The \( \langle \bar{T}_{\mu\nu} \rangle \), is written as \( \langle \bar{T}_{\mu\nu} \rangle = < T_{\mu\nu}^{(0)} > + < T_{\mu\nu}^{(1)} > \) where \( < T_{\mu\nu}^{(0)} > \) represents the zero order vacuum correction to the classical stress-tensor due to the spatially flat background and \( < T_{\mu\nu}^{(1)} > \) is the first order vacuum expectation value of the stress-tensor. Subscripts (0) and (1) refer, respectively, to background and linear order metric perturbations.

Having the metric in the form (1), we modify the semiclassical Einstein equations (2) and write them in perturbative way. In this case the simplest equation to the first-order is derived by adding to the right of the Eq.(2) the stochastic tensor which characterizes the quantum fluctuations of the stress tensor.

The obtained stochastic equation up to the first order in \( h_{\mu\nu} \), known as Einstein-Langevin equation, is

\[ \{ \bar{G}_{\mu\nu}^{(1)}[g + h] + \Lambda e^{-2\omega}h_{\mu\nu} \} - 8\pi G_N < \bar{T}_{\mu\nu}^{(1)}[g + h] >_{\text{ren}} = 8\pi G_N \xi_{\mu\nu}[g] \]

where the subscript "ren" means renormalized, the field \( \xi_{\mu\nu}[g] \) is a Gaussian stochastic source defined by the following correlators (for details see Ref. [5],Sec.IV and references therein)

\[ < \xi_{\mu\nu}(x) >_\xi = 0, \quad < \xi_{\mu\nu}(x)\xi_{\alpha\beta}(y) >_\xi = N_{\mu\nu\alpha\beta}(x,y) \]

\( <>_\xi \) means statistical average \(^1\), \( N_{\mu\nu\alpha\beta}(x,y) \) is the noise kernel which describes the quantum fluctuations of the stress tensor operator

\[ 8N_{abcd}(x,y) \equiv < \{ \hat{f}_{ab}(x), \hat{f}_{cd}(y) \} > [g](\Omega) \]

\(^1\) In general, the two point correlation function of a stochastic tensor field \( \xi_{ab} \) must be a symmetric in the sense \( < \xi_{ab}(x)\xi_{cd} >_{\xi} = < \xi_{cd}(x)\xi_{ab} >_{\xi} \), and positive semi-definite real bi-tensor field
Their explicit expression are decomposed in the fashion $\{\}$ means anticommutator and $\tilde{E}_{ab}(x;\Omega) \equiv \tilde{T}_{ab}^R(x;\Omega)$ and $\tilde{T}_{ab}^R(x;\Omega)$ is renormalized and regularized operator depending on the regulator $\Omega$.2

The Einstein-Langevin equation Eq.(3), which are gauge invariant, must be thought as a linear equation for the metric perturbations $h_{\mu\nu}$ which will behave as a stochastic field tensor.

Thus the homogeneous Eq.(3) give semiclassical equations [8]-[15]

\[
e^{i\omega}\{[\tilde{G}^{\mu\nu}_{(0)} + \tilde{G}^{\mu\nu}_{(1)} + \Lambda e^{-2\omega}(\eta^{\mu\nu} - h^{\mu\nu})] - \frac{8\pi G_N k_1}{6} (\tilde{B}^{\mu\nu}_{(0)} + \tilde{B}^{\mu\nu}_{(1)}) \\
+ 8\pi G_N k_3 (\tilde{H}^{\mu\nu}_{(0)} + \tilde{H}^{\mu\nu}_{(1)})\} - 2(8\pi G_N k_3)\tilde{B}^{(0)}_{a\beta} C^{(1)}_{\mu\alpha\nu\beta} \\
+ (8\pi G_N k_2)[-4(\omega C^{(1)}_{\mu\alpha\nu\beta};a\beta) + \int d^4y A^{\mu\nu}_{(1)}(y) H(x-y;\bar{\mu})] = 0
\]

(6)

where $k_1$ is a numerical coefficient which has to be determined by experiment, $k_2$, $k_3$ are also numerical coefficients given by the relations [7]

\[
k_1 = \frac{1}{2880\pi^2}(N_0 + 11N_{1/2} + 62N_1), \quad k_2 = \frac{3}{2880\pi^2}(N_0 + 6N_{1/2} + 12N_1)\]

(7)

with $N_0,N_{1/2}$ and $N_1$ quantum numbers with spins 0, 1/2 and 1, respectively. Also, in Eq.(6) [8],[16],

\[
H(x-y;\bar{\mu}) = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)}[ln|\mu|^2 + i\pi \theta(-p^2)sgn(-p^0)]
\]

(8)

$G^{\mu\nu}(x)$ is the Einstein’s tensor, $C^{\mu\nu\alpha\beta}(x)$ Weyl’s tensor [17], $B^{\mu\nu}(x)$ and $A^{\mu\nu}(x)$ are the space time tensors given by the variation of $\int d^4x R^2(x)$ and $\int C^{\mu\nu\alpha\beta}(x)C_{\mu\nu\alpha\beta}(x)$ respectively with respect to the metric tensor $g_{\mu\nu}(x)$ and are expressed in terms of the metric tensor and its derivatives higher than two, while [4]

\[
H^{\mu\nu}(x) = -R^\mu\alpha R_\alpha^\nu + \frac{2}{3} RR^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{4} g^{\mu\nu} R^2
\]

(9)

Their explicit expression are decomposed in the fashion $B^{\mu\nu}(x) \equiv B^{\mu\nu}_{(0)} + B^{\mu\nu}_{(1)}$, $A^{\mu\nu}(x) \equiv A^{\mu\nu}_{(0)} + A^{\mu\nu}_{(1)}$ and $H^{\mu\nu}(x) \equiv H^{\mu\nu}_{(0)} + H^{\mu\nu}_{(1)}$, where the bracket sub-indices are referred to zero (0) and (1) order perturbations.

We will focus only in perturbations up to the first order and we will study the correlation of the small perturbation $h_{\mu\nu}$ in an almost de Sitter universe (1). For our convenience we introduce the notations

\[
H_0^2 = \frac{1}{8\pi G_N k_3}, \quad M^2 = \frac{1}{8\pi G_N k_1}, \quad H_1^2 = \frac{1}{8\pi G_N k_2}
\]

(10)

Thus, starting with Eqs.(2) which do not include the stochastic source and adapting the so called TT-gauge which means that $\tilde{h}^{(0)} = 0$, the trace $\bar{h} = h^{\alpha\alpha}$ and $h^{\mu\nu} = 0$, we obtain [18]-[19]:

I. Zero-order equations:

\[
\tilde{G}^{\mu\nu}_{(0)} + \Lambda e^{-2\omega}\eta^{\mu\nu} = 8\pi G_N <\tilde{T}^{\mu\nu}_{(0)}> = \frac{1}{H_0^2} \tilde{H}^{\mu\nu}_{(0)} - \frac{1}{6M^2} \tilde{B}^{\mu\nu}_{(0)}
\]

(11)

II. First-order equations:

\[
\tilde{G}^{\mu\nu}_{(1)} - \Lambda e^{-2\omega}h^{\mu\nu}
\]

2 For further details see Ref. [10],Sec.II.
\[ \frac{1}{2} \eta^{\sigma\tau} h_{\sigma\tau}^{\mu\nu} + h_{,0}^{\mu\nu} \omega,0 + h^{\mu\nu}(2 \omega_{,00} + \omega,0) - \Lambda e^{2\omega} h^{\mu\nu} \]

\[ = \frac{1}{H_0^2} e^{-2\omega} \left\{ -\eta^{\sigma\tau} h_{,\sigma\tau}^{\mu\nu}(2 \omega_{,00} - \omega,0) + 2 h_{,0}^{\mu\nu} \omega,0 \omega_{,00} \right\} \]

\[ + \frac{1}{M^2} e^{-2\omega} \left\{ \eta^{\sigma\tau} h_{,\sigma\tau}^{\mu\nu}(\omega_{,00} + \omega,0) - h_{,0}^{\mu\nu}(\omega_{,0000} + 2 \omega_{,0000} \omega,0) \right\} \]

\[ - h^{\mu\nu}[2\omega_{,0000} - 2 \omega_{,0000} \omega,0 - 12 \omega_{,0000} \omega,0 + \omega_{,0000}^2 + 3 \omega_{,0000}] \]

\[ + \frac{1}{H_1^2} e^{-2\omega} \left\{ [-2 \omega_{,0} \eta^{\sigma\tau} h_{,\sigma\tau}^{\mu\nu} + 2(h_{,00} + \frac{1}{2} \eta^{\sigma\tau} h_{,\sigma\tau}^{\mu\nu}) \omega,00] \right\} \]

\[ + \int d^4 y A_{(1)}^{\mu\nu}(y) H(x - y; \bar{\mu}) \] \hspace{1cm} (14)

Decomposing the perturbation \( h_{\mu\nu} \) into modes of the form \( h_{\mu\nu}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 \vec{k} e^{i \vec{k} \cdot \vec{x}} g_k(\eta) e^{i \vec{k} \cdot \vec{x}} \), Eq.(14) simplifies considerably and the functions \( g_k(\eta) \) satisfy the equation \([15], [20]\) \(^3\)

\[ \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 \vec{k} e^{i \vec{k} \cdot \vec{x}} e^{\mu\nu}(\vec{k}) \{ g''_k + \frac{1}{3} R_k + \frac{1}{3} R^0_R(R_0^R - R) \] 

\[ + g_0' \left[ - \frac{2a'}{a} (1 + \frac{1}{3} R - \frac{1}{3} R_0^0) + \frac{1}{3} R \right] \] 

\[ + g_k k^2 [1 + \frac{1}{3} R + \frac{1}{3} R_0^0 (R - 6 R_0^0)] \] 

\[ - 2 \frac{1}{H_1^2 a^2} (\frac{2a'}{a} (g''_k + k^2 g_k) + (g''_k - k^2 g_k) (\frac{a'}{a}) \} \] 

\[ = 2 \frac{1}{a^2 H_1^2} \int d^4 y A_{(1)}^{\mu\nu}(y) H(x - y; \bar{\mu}) \] \hspace{1cm} (15)

where prime stands for time derivatives in terms of the conformal time \( \eta \), \( R_0^0 = \frac{2}{a} (a a'' - a'^2) \) is the mixed 00–component of the Ricci tensor, \( R = \frac{6}{a^2} a'' \) is the scalar curvature, \( R' = a a'' - \frac{18a' a''}{a^2} \), \( g_k(\eta) \) is the amplitude of the gravitational wave (GW) and the last term, in the right hand side

\(^3\) Note that in Eq.(14) the term with the cosmological constant \( \Lambda \) does not exist because we have applied the Eq.(11) into Eq.(14).
of the Eq.(15) is the so called non-local term with

\[
H(x - y; \bar{\mu}) = \frac{1}{960\pi^2} \left( \mathcal{P}f\left(\frac{1}{\pi} \theta(x^0 - y^0)\right) \frac{d}{d(x-y)^2} \delta((x-y)^2) \right) \\
+ (1 - \gamma - \ln(\epsilon \bar{\mu})) \delta^4(x-y)
\]

\[
= \lim_{\epsilon \to 0^+} \frac{1}{960\pi^2} \left\{ \frac{1}{\pi} \theta(x^0 - y^0) \theta(|\vec{x} - \vec{y}| - \epsilon) \frac{d}{d((x-y)^2)} \delta((x-y)^2) \right\}
\]

\[
+ [1 - \gamma - \ln(\epsilon \bar{\mu})] \delta^4(x-y)
\]

(16)

\(\bar{\mu}\) is a renormalization parameter and \(\mathcal{P}f\) denotes the Hadamard finite part, which gives well defined expressions in the sense of distributions [21], [22].

At this stage we have to point out that in Eq.(15), if the right-hand side does not exist and the coefficient \(\frac{1}{3M^2}\) is replaced by \(-\frac{1}{3M^2}\), we obtain Vilenkin’s equation corrected by a higher-order term proportional to \(1/H_0^2\). In Eq.(15), if the terms proportional to \(1/H_0^2\) do not exist and we replace \(\frac{1}{3M^2}\) with \(-\frac{1}{3M^2}\), we recover the equation presented in Ref. [20].

3. The homogeneous Einstein-Langevin equations with \(\Lambda \neq 0\)

We start with the zero-order equation (13). Since \(a = e^{\omega(t)}\), \(d\eta = dt/a(t)\) and \(H = \frac{\dot{a}}{a}\) (dot stands for derivatives in terms of the coordinate time \(t\)) we find \(\omega,0 = \frac{\dot{a}}{a} = Ha\) and Eq.(13) becomes

\[
\omega_{,0}^2 - \Lambda e^{2\omega} = \frac{1}{H_0^2 e^{2\omega}} \omega_{,00}^4 + \frac{1}{M^2 e^{2\omega}} (\omega_{,00}^2 + 3\omega_{,0}^4 - 2\omega_{,00} \omega_{,000})
\]

(17)

If we consider that the background is driven by the cosmological constant \(\Lambda \neq 0\) and the terms proportional to \(1/H_0^2\) and \(1/M^2\) which include higher derivatives, perturbations to the background, we may apply order reduction as follows:

We consider the equation

\[
\omega_{,0}^2 - \Lambda e^{2\omega} = O\left(1/H_0^2, 1/M^2\right)
\]

(18)

From the above equation we find that

\[
\omega,0 = \pm \sqrt{\frac{\Lambda}{3}} e^{\omega}
\]

(19)

Taking the second and third derivative of Eq.(19) and substituting them into Eq.(17) we find

\[
\omega_{,0}^2 = \frac{\Lambda}{3} e^{2\omega} + \frac{1}{H_0^2} (\frac{\Lambda}{3})^2 e^{2\omega}
\]

(20)

In the last equation setting \(\omega,0 = \frac{\dot{a}}{a} = Ha\) we find

\[
H = \frac{\dot{a}}{a} = \pm \sqrt{\frac{\Lambda}{3} \left[ 1 + \frac{1}{H_0^2} (\frac{\Lambda}{3})^2 \right]}^{1/2}
\]

(21)

which leads to the solution \(a(t) = a_0 e^{Ht}\) if we assume that at \(t = 0\) the universe has size \(a(t = 0) = a_0\). Eventually, the scale factor in terms of the conformal time reads

\[
a(\eta) = -\frac{1}{H\eta}, \quad \forall \eta \in (-\infty, 0]
\]

(22)
Since we have the explicit expressions for the scale factor, we will compute from Eq.(15) the function \( g_k(\eta) \).

Thus we have

\[
\frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3k e^{i\vec{k} \cdot \vec{x}} e^{\mu\nu}(k) \{ g''_k[1 + \frac{1}{3M^2} R + \frac{1}{3H_0^2} (2R_0^0 - R)] + g'_k \left[ \frac{2a'}{a} \left( 1 + \frac{1}{3M^2} R - 2 \frac{1}{3H_0^2} R_0^0 + \frac{1}{3M^2} R' \right) \right] + g_k k^2 [1 + \frac{1}{3M^2} R + \frac{1}{3H_0^2} (R - 6R_0^0)] - \frac{2}{H_1^2 a^2} \left[ \frac{2a'}{a} (g''_k + k^2 g'_k) + (g''_k - k^2 g_k)(\frac{a'}{a})' \right] \} = \frac{2}{H_1^2 a^2} \left[ \frac{1}{2} \int_{-\infty}^{\infty} d^3k e^{\mu\nu}(k)e^{i\vec{k} \cdot \vec{x}} N_k^{(nl)} \right] \tag{23}
\]

where

\[
N_k^{(nl)} \equiv \{- \int_{-\infty}^{\infty} dy^0 \sin k(x^0 - y^0) [g''_{kk} (y^0) + 2k^2 g''_k (y^0)] + [1 - \gamma - \ln (\tilde{\mu})][g''_{kk} + 2k^2 g''_k + k^4 g_k] \} \tag{24}
\]

To make further progress with Eq.(23) we will proceed as follows:

We will apply order reduction to the right-hand side of Eq.(23), e.g. to the non-local term, and to the left-hand side of the Eq.(23). Then, we will discuss the obtained results.

Eq.(23) has derivatives higher than two. This is well known in the semiclassical content and it is common in back reaction problems. The higher derivative terms are responsible for spurious solutions. However, approximate solution to the Eq.(23) may be found, applying the method of "order reduction" [9], [22], [23] of the differential equations(see Appendix A), since the terms in Eq.(23) multiplied by \( \frac{1}{M^2}, \frac{1}{H_0^2}, \frac{1}{H_1^2} \) are corrections to the differential equation

\[
g''_k + \frac{2a'}{a} g'_k + k^2 g_k = O \left( \frac{1}{M^2}, \frac{1}{H_0^2}, \frac{1}{H_1^2} \right) \tag{25}
\]

Taking higher order derivatives of Eq.(25) and applying them only in the non-local term in Eq.(23) we find that the non-local term vanishes e.g. the non-local term in a de Sitter space-time with the scale factor being \( a = e^{\omega(\eta)} = -\frac{1}{H_0^2} \) vanishes. Furthermore, applying once more order reduction to the left hand side of the Eq.(23) we find that the \( g_k(\eta) \) satisfies the equation

\[
g''_k - \frac{2}{\eta} (1 - \nu_0) g'_k + k^2 (1 - 2\nu_0) g_k = 0 \tag{26}
\]

where \( \nu_0 \equiv \frac{2H_0^2}{H_1^2} \) and prime stand for derivatives in terms of the conformal time \( \eta \).

For further study of Eq.(26) we adapt the so called Bunch-Davies vacuum initial conditions, which are [24]-[27]

\[
z_k(\eta_i) = \frac{1}{\sqrt{\omega_k}}, \quad z_k(\eta_i)' = i\sqrt{\omega_k} \tag{27}
\]

where \( \eta = \eta_i \) is some initial time.

To compute the \( z_k \) and \( \omega_k \) appearing in (27) we work as follows:
In Eq. (26) we set
\[ g_k(\eta) = \eta^{(1-\nu_0)} z_k(\eta) \] (28)
and find the equation
\[ z_k''(\eta) + \omega_k^2 z_k(\eta) = 0 \] (29)
where
\[ \omega_k^2 = k^2 (1 - 2\nu_0) \left[ 1 - \frac{(\nu_0 - 1)(\nu_0 - 2)}{(k\eta)^2 (1 - 2\nu_0)} \right]^{-1} \] (30)

Eq. (29) admits the general solution of the form
\[ z_k(\eta) = c_{1k} z_k^{(1)}(\eta) + c_{2k} z_k^{(2)}(\eta) \] (31)
where
\[ z_k^{(1)}(\eta) = \sqrt{\eta} J_{-3/2+\nu_0} \sqrt{(1 - 2\nu_0) k\eta}, \]
\[ z_k^{(2)}(\eta) = \sqrt{\eta} Y_{-3/2+\nu_0} \sqrt{(1 - 2\nu_0) k\eta} \] (32)
and \( J_m(x) \) and \( Y_m(x) \) are the Bessel functions of first and second order, respectively. We compute arbitrary constants \( c_{1k}, c_{2k} \) using the Bunch-Davies initial conditions in vacuum Eq. (31) and find [24]:
\[ c_{1k} = \frac{1}{W_i \sqrt{\omega_k}} [z_k^{(2)}(\eta_i)' - i\omega_k z_k^{(2)}(\eta_i)] \]
\[ c_{2k} = -\frac{1}{W_i \sqrt{\omega_k}} [z_k^{(1)}(\eta_i)' - i\omega_k z_k^{(1)}(\eta_i)] \] (33)
with \( W_i \) being the Wronskian of the solutions \( z_k^{(1)}(\eta), z_k^{(2)}(\eta) \) evaluated at the initial time \( \eta = \eta_i \).

Here we have to point out that \( z_k(\eta, \eta_i) \) and its complex conjugate \( z_k(\eta, \eta_i) \) are two linearly independent solutions of the Eq. (29) which are the same for all Fourier modes with the given magnitude of the wave vector \( k = |\vec{k}| \). Their Wronskian \( W[z_k, \bar{z}_k] \) is different from zero but its derivative in terms of the conformal time \( \eta \) is zero which means that the Wronskian \( W[z_k, \bar{z}_k] \) is constant.

From Eqs. (31-33) we find the general solution of the Eq. (29) corresponding to the positive frequency \( k \)-mode to a graviton in the Bunch-Davies vacuum state in an almost de Sitter background with scale factor \( a(\eta) = -\frac{1}{k\eta} \) which reads:
\[ z_k(\eta, \eta_i) = \frac{1}{W_i \sqrt{\omega_k}} \left[ \frac{z_k^{(2)}(\eta_i)' z_k^{(1)}(\eta) - z_k^{(1)}(\eta_i)' z_k^{(2)}(\eta)}{\sqrt{(1 - 2\nu_0)}} \right] 
- i\omega_k z_k^{(2)}(\eta_i) - z_k^{(1)}(\eta_i) z_k^{(2)}(\eta) \]
\[ = -\frac{1}{\sqrt{k\eta_i} (1 - 2\nu_0)^{-3/4}} \left[ 1 - \frac{(1 - \nu_0)(2 - \nu_0)}{k^2 \eta_i^2 (1 - 2\nu_0)} \right]^{-1/4} \]
\[ \times \left[ \frac{1}{k\eta_i (1 - 2\nu_0)^{1/4}} + i \left[ 1 - \frac{(1 - \nu_0)(2 - \nu_0)}{k^2 \eta_i^2 (1 - 2\nu_0)} \right]^{1/2} \right] \]
\[ \times k\sqrt{1 - 2\nu_0} \sqrt{\eta_i} [Y_{-3/2+\nu_0}(x_i)] J_{-3/2+\nu_0}(x) - J_{-3/2+\nu_0}(x_i) Y_{-3/2+\nu_0}(x) \]
\[ + k\sqrt{1 - 2\nu_0} \sqrt{\eta_i} [Y_{-1/2+\nu_0}(x_i)] J_{-3/2+\nu_0}(x) - J_{-1/2+\nu_0}(x_i) Y_{-3/2+\nu_0}(x) \]
\[ \times [Y_{-1/2+\nu_0}(x_i)] J_{-1/2+\nu_0}(x_i) - J_{-1/2+\nu_0}(x_i) Y_{-1/2+\nu_0}(x_i)]^{-1} \] (34)
where \( x = k \eta \sqrt{1 - 2 \nu_0} \), \( x_i = k \eta_i \sqrt{1 - 2 \nu_0} \), \( \nu_0 = \frac{2 h^2}{H_i^2} \) and \( H = \sqrt{\Lambda/3} \left[ 1 + \frac{\Lambda}{H_0^2} \right]^{1/2} \).

From Eq.(28) and (34) we obtain the solution of the Eq.(26).

For gravitational waves (GWs) with \( k |\eta| \gg 1 \), Eq.(34) gives:

\[
G_{k} (\eta, \eta_i) = \frac{(1 - 2 \nu_0)^{-1/4}}{\sqrt{k}} \left[ 1 + \frac{i (1 - \nu_0) (2 - \nu_0)}{2 k \eta} e^{i k \sqrt{1 - 2 \nu_0} (\eta - \eta_i)} \right] \tag{35}
\]

For \( \nu_0 = 0 \), Eq.(35) reduces to the equation:

\[
G_{k} (\eta, \eta_i) = \frac{1}{\sqrt{k}} \left[ 1 + \frac{i}{k \eta} e^{i k (\eta - \eta_i)} \right] \tag{36}
\]

which corresponds to Mukhanov’s equation (8.122) [26].

### 3.1. A Green function to the inhomogeneous Einstein-Langevin equation with \( \Lambda \neq 0 \).

Our aim is to find the correlation function for the metric perturbations since they describe how large are the physical perturbations on different time scales. This can achieved using stochastic’s gravity approaches which is different from the semiclassical gravity. Thus, starting with the Einstein-Langevin Eq.(3) we lead to an inhomogeneous Eq.(23).

In Eq.(23) we apply order reduction and considering the scale factor (22) and non-zero \( \nu_0 \) propagator for the inhomogeneous Eq.(26) which includes a driving term due to the stochastic source which contributes to the metric.

Next, we will obtain the metric perturbations \( g_k \) in terms of the stochastic source.

To obtain the metric perturbations in terms of the stochastic source we need the retarded propagator for the \( g_k \) i.e. the required Green function to solve the inhomogeneous Eq.(26) with the appropriate boundary conditions. Thus, the problem now is to find a Green function to the inhomogeneous Eq.(26).

We consider the equation, [25]

\[
G_{(k) \text{Ret}}^{\nu}\nu (\eta - \eta') = \frac{2}{\eta} (1 - \nu_0) G_{(k) \text{Ret}}^\nu (\eta - \eta') + k^2 (1 - 2 \nu_0) G_{(k) \text{Ret}}^\nu (\eta - \eta') - \delta (\eta - \eta') \tag{37}
\]

Since the Green function \( G_{(k) \text{Ret}}(\eta - \eta') \) is proportional to the general solution of the Eq.(26), we find

\[
G_{(k) \text{Ret}} (\eta, \eta') = - \left[ \frac{a(\eta')}{a(\eta)} \right]^{(3/2 - \nu_0)} \Theta (\eta - \eta') \tag{38}
\]

\[
\times \left[ J_{-3/2 + \nu_0} (x) J_{-3/2 + \nu_0} (x') - J_{-3/2 + \nu_0} (x) J_{-3/2 + \nu_0} (x') \right]
\]

where \( W (x') = Y_{-1/2 + \nu_0} (x') J_{1/2 + \nu_0} (x') - Y_{1/2 + \nu_0} (x') J_{-1/2 + \nu_0} (x') \) and \( x' = k \eta' \sqrt{1 - 2 \nu_0} \).

Eq.(38) is the retarded propagator appropriate to study correlation of perturbations using the Einstein-Langevin Eq.(3) in an expanding universe with scale factor (22) and non-zero cosmological constant. From Eq.(38) we have \( G_{(k) \text{Ret}} (\eta, \eta') = 0 \) for \( \eta < \eta_i \) while for \( \eta > \eta_i \) we have a non-zero \( G_{(k) \text{Ret}} (\eta, \eta') \) due to the effect of the stochastic source after \( \eta_i \).

Now we go back to the inhomogeneous Einstein-Langevin equation Eq.(3). Its final form in a perturbative way includes the zero-order Eq.(11) and the inhomogeneous first-order Eq.(12) which now reads [5]:

\[
G_{(1) \mu\nu} = \Lambda e^{-2 \omega} \hat{h}_{\mu\nu} \tag{39}
\]

\[
= 8 \pi G_N < \hat{T}_{(1)}^{\mu\nu} > + \frac{1}{H_0^2} (\hat{H}_{(1)}^{(\mu\nu)} - 2 \hat{R}_{(0)}^{(0)} \hat{C}_{(1)}^{\mu\nu}) - \frac{1}{6 M^2} \hat{B}_{(1)}^{\mu\nu}
\]

\[
+ \frac{1}{H_1^2} e^{-6 \omega} \left[ -4 (\omega C_{(1)}^{\mu\nu, \beta})_{, \alpha \beta} + \int d^4 y A_{(1)}^{\mu\nu} \left( y \right) H (x - y, \mu) + F^{\mu\nu} \right] \]
In Eq.(39), the tensor \( F^{\mu\nu}(x) \) results from the functional variation of the stochastic term and its explicit expression is

\[
F^{\mu\nu}(x) = -16\pi G_N e^{-6\omega} \partial_\alpha \partial_\beta \xi^{\mu\alpha\nu\beta}(x)
\]  

(40)

In Eq.(40), the tensor \( F^{\mu\nu}(x) \) is symmetric and traceless, i.e. \( F^{\mu\nu}(x) = F^{\nu\mu}(x) \) and \( F^{\mu}_\mu(x) = 0 \) which means that there is no stochastic correction to the trace anomaly. Also, in the same equation, the tensor \( \xi^{\mu\alpha\nu\beta} \) plays the role of a stochastic source with the Gaussian probability given by Eq.(4.7) in Ref. [5]. This tensor has the symmetries of the Weyl tensor i.e. it has the symmetries of the Riemann tensor and vanishing trace in all its indices. This stochastic source is completely characterized by the relations

\[
< \xi_{\mu\nu\alpha\beta} > = 0,
\]

\[
< \xi_{\mu\nu\alpha\beta}(x) \xi_{\rho\sigma\lambda\delta}(y) > = T_{\mu\nu\rho\sigma\lambda\delta} N(x - y)
\]

(41)

where \( N(x - y) \) is the noise kernel corresponding to the two-point correlation function of the stochastic source since the probability distribution is Gaussian and the tensor \( T_{\mu\nu\rho\sigma\lambda\delta} \) is the product of four metric tensors in such a combination that the right-hand side of the equation satisfies the Weyl symmetries of the two stochastic fields on the left-hand side. Its explicit expression is given in the Appendix of Ref. [5].

In Eq.(39), the stochastic term \( F^{\mu\nu} \) will produce a stochastic contribution \( h^{\mu\nu}_{(inh)} \) to the space time inhomogeneity

\[
h^{\mu\nu}(\eta, \vec{x}) = h^{\mu\nu}_{(h)}(\eta, \vec{x}) + h^{\mu\nu}_{(inh)}(\eta, \vec{x})
\]

(42)

where \( h^{\mu\nu}_{(h)}(\eta, \vec{x}) \) is the solution to the homogeneous Einstein-Langevin Eq.(39) which is identical to the Eq.(6) and its explicit expression is

\[
h^{\mu\nu}_{(h)}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 k e^{i\vec{k} \cdot \vec{x}} \tilde{g}_{\mu\nu}(\vec{k}) e^{i\vec{k} \cdot \vec{x}} \eta^{(1 - \nu_0)} z_k(\eta)
\]

(43)

with \( z_k(\eta) \) given by Eq.(34).

Furthermore, substituting Eq.(42) into eq.(39), taking into account that \( h^{\mu\nu}_{(h)}(\eta, \vec{x}) \) is the solution to the homogeneous Einstein-Langevin and repeating the computations of Sec.3., Eq.(39) becomes an inhomogeneous differential equation of the form

\[
g^{\mu\nu}_{(inh)} k - \frac{2}{\eta} (1 - \nu_0) g^{\mu\nu}_{(inh)} k + k^2 (1 - 2\nu_0) g^{\mu\nu}_{(inh)} k = 16\pi G_N F_k(\eta)
\]

(44)

where we decomposed

\[
h^{\mu\nu}_{(inh)}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 k e^{i\vec{k} \cdot \vec{x}} g^{(inh)}_{\mu\nu}(\vec{k})
\]

\[
F^{\mu\nu}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 k e^{i\vec{k} \cdot \vec{x}} F_k(\eta)
\]

(45)

and took into account the source term (40).

The left-hand side of Eq.(44) is of the same form as in Eq.(26) and because of its non-zero right-hand side, admits a Green function of the form (38) and thus the solution to the Eq.(44)

\footnote{To find the explicit expression of the \( h^{\mu\nu}_{(inh)}(\eta, \vec{x}) \), we assume that the \( h^{\mu\nu}_{(inh)}(\eta, \vec{x}) \) satisfy the TT-gauge and impose the boundary conditions \( h^{\mu\nu}_{(inh)}(\eta = -\infty, \vec{x}) = 0 \) and \( \partial_\nu h^{\mu\nu}_{(inh)}(\eta = -\infty, \vec{x}) = 0 \).}
reads:  

\[ h_{(inh)}^{\mu \nu}(\eta, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 \vec{k} e^{i \vec{k} \cdot \vec{x}} g_{(inh)k}(\eta) \]

\[ = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3 \vec{k} e^{i \vec{k} \cdot \vec{x}} e^{ik \eta} \times \left\{ \frac{16\pi G_N}{4\pi} \int G_{(ret)k}(\eta, \eta') E_{k}(\eta')d\eta' \right\} \quad (46) \]

To understand how large are the physical perturbations on different time scales, we will calculate the two-point correlation function of the metric perturbations which is given by

\[ < h^{\mu \nu}(\eta_1; \vec{x}) h^{\lambda \theta}(\eta_2; \vec{y}) > = < h_{(h)}^{\mu \nu}(\eta_1; \vec{x}) h_{(h)}^{\lambda \theta}(\eta_2; \vec{y}) > + < h_{(inh)}^{\mu \nu}(\eta_1; \vec{x}) h_{(inh)}^{\lambda \theta}(\eta_2; \vec{y}) > \quad (47) \]

In Eq.(47) there are two different contributions to the two-point correlation functions.

The first one, corresponding to the homogeneous Einstein-Langevin Eq.(26), is connected to the fluctuations of the initial state of the metric perturbations, we refer to them as intrinsic fluctuations. The second contribution, corresponding to the inhomogeneous Einstein-Langevin Eq.(3), is proportional to the noise kernel and thus connected with the fluctuations of the quantum fields, we will refer to them as induced fluctuations [28].

Now we will concentrate on the intrinsic fluctuations. The intrinsic fluctuations are the expectation value of two-point correlations at equal times and are given by the relation [26]

\[ < h_{(h)}^{\mu}(\eta, \vec{x}) h_{(h)}^{\nu}(\eta, \vec{y}) > = \frac{8}{\pi^2 a^2(1-\nu_0)} \int_0^{\infty} \frac{k^3 dk}{k^2} \left| z_k(\eta) \right|^2 \sin kr \quad (48) \]

where \( r \equiv |\vec{x} - \vec{y}| \) and \( z_k(\eta) \) is given by the Eq.(34).

However, the power spectrum of those fluctuations with comoving wave number \( k \) may be computed from the equation [24]

\[ \delta^2(k, \eta) = \frac{8k^3}{\pi a^2(1-\nu_0)} \left| z_k(\eta) \right|^2 \]

where \( z_k(\eta) \) is given by Eq.(34). For the Gws with \( k|\eta| \gg 1 \) (e.g. in scales inside the horizon) we find:

\[ \delta_h = \frac{8H^2}{\pi}(1-2\nu_0)^{-1/2}k^2\eta^2 \left[ 1 + \frac{(1-\nu_0)^2(2-\nu_0)^2}{4k^2\eta^2} \right] \alpha^{2\nu_0} \quad (50) \]

where \( H \) is given by Eq.(21). Expressing Eq.(50) in terms of the physical wave number \( k_{ph} = \frac{k}{a} \) we have

\[ \delta_h^2 = \frac{8}{\pi} \frac{H^2}{\sqrt{(1-2\nu_0)}} \left[ \frac{k_{ph}^2}{H^2} + \frac{(1-\nu_0)^2(2-\nu_0)^2}{4} \right] \alpha^{2\nu_0} \]

\[ = \frac{8H^2}{\pi} \left[ 1 + \frac{k_{ph}^2}{H^2} + \nu_0[-2 + \frac{k_{ph}^2}{H^2} + 2 \ln a(1 + \frac{k_{ph}^2}{H^2})] + O(\nu_0^2) \right] \quad (51) \]

where \( \nu_0 = 2\frac{H^2}{H^2} \ll 1, \alpha = -\frac{1}{\sqrt{H^2}} \) and \( H^2 = \frac{\Lambda}{3} \left[ 1 + \frac{\Lambda}{3H^2} \right] \) (see Eq.(12)).

Setting \( \nu_0 = 0 \) in Eq.(51), we obtain Mukhanov’s formula (8.123) in Ref. [24].

\(^5\) Eq.(44) is linear in \( g_{(inh)k} \) and resembles to the inhomogeneous equations (26). Thus its solution may be computed with the initial conditions (27).
From Eq.(50) we compute the tensor spectra index ([24])

\[ n_T \equiv \frac{d \ln \delta_h^2}{d \ln k} = \left[ 1 + \frac{1}{k^2} \left( 3 \nu_0 + \frac{3}{4} \nu_0^2 - \frac{3}{2} \nu_0^3 + \frac{1}{4} \nu_0^4 \right) \right]^{-1} \]

\[ \sim \left[ 1 + \frac{H^2}{k^2} \left( 1 - 3 \nu_0 \right) \right]^{-1} \quad (52) \]

To compute the two-point correlation function of the induced fluctuations at equal times we use Eq.(46) and find:

\[ < h_{(inh)}^{\mu \nu}(\eta, \vec{x}) h_{(inh)}^{\rho \lambda}(\eta, \vec{y}) > = \frac{(8\pi G_N)^2}{(2\pi)^5} \int_{-\infty}^{\infty} d^3k \int_{-\infty}^{\infty} d^3k' \int_{-\infty}^{\infty} d^3k'' \int_{-\infty}^{\infty} d^3k''' \int_0^\eta d\eta'' \int_0^\eta d\eta''' \left\{ G_{(k)Rep}(\eta, \eta') G_{(k')Rep}(\eta, \eta'') \right\} \]

where we used the equation

\[ e^{\mu \nu}(\vec{k}) F_{k}(\eta') = \frac{1}{(2\pi)^{3/2}} \int d^3x' F^{\mu \nu}(\eta', \vec{x}') e^{-i\vec{k} \cdot \vec{x}'} \quad (54) \]

In Eq.(53), we perform some integrations and after some straightforward computations we find the two-point correlation function of the induced fluctuations at equal times

\[ < h_{(inh)}^{\mu \nu}(\eta, \vec{x}) h_{(inh)}^{\rho \lambda}(\eta, \vec{y}) > = 6\alpha (8\pi G_N)^2 \int_0^\infty k^3 dk \frac{\sin (kr)}{kr} \left\{ k_\alpha k_\beta k_\gamma k_\delta T^{\mu \nu \rho \gamma \delta} \right\} \]

\[ \times \int_0^\eta d\eta' e^{-12\omega(\eta')} G_{(k)Rep}(\eta, \eta') G_{(-k)Rep}(\eta, \eta') \] \quad (55)

4. Conclusions

In the context of stochastic semiclassical theory, we wrote the Einstein-Langevin equation with the cosmological constant \( \Lambda \) in an almost de Sitter universe which has been perturbed by the metric perturbations \( h_{\mu \nu} \).

The equations have been written in a perturbative way, in the TT-gauge. Ignoring higher than the first order perturbation and keeping only first order corrections, we derived a zero order equation which gives the evolution of the scale factor and a first order equations which describes the evolution of the metric perturbation when \( \Lambda \neq 0 \).

In the case where \( \Lambda \neq 0 \), we have applied order reduction to the Eq.(13) and found that the zero order equation admits the solution \( H = \sqrt{\Lambda/3} \left[ 1 + \frac{1}{H_0^2} \Lambda^{1/2} \right] \) which shows that the Hubble parameter is shifted by the quantum terms (as it was expected) and the scale factor is given by the Eq.(22). The result corresponds to an expanding for ever universe.

Subsequently, we found explicit expression for the kernel \( H(x - y; \bar{\mu}) \) appearing on the right-hand side of the Eq.(15) and applying order reduction to the first order Eq.(23) we verified
that, in a de Sitter space-time with a scale factor like Eq.(22), the non-local term vanishes and eventually the first order equation reduce to Eq.(26).

Eq.(26) is the homogeneous Einstein-Langevin equation with non-zero cosmological constant after order reduction. Its general solution has been computed uniquely in terms of Bessel functions of first and second kind adapting the Bunch-Davies initial conditions.

Since we are interested in finding the correlation function of the metric perturbations $h_{\mu\nu}$, we have considered the inhomogeneous Einstein-Langevin equation (3), which leads to the Eq.(44) e.g. the inhomogeneous Eq.(26), and computed its retarded Green function $G_{k(\text{Ret})}(\eta, \eta^\prime)$(Eq.(38)).

As we have pointed out in the section 2, Eq.(3) is a stochastic equation which predicts that the gravitational field has stochastic fluctuations over the background space-time due to the stochastic term $F_{\mu\nu}$. The stochastic term, produces the stochastic contribution $h_{\mu\nu}(\text{inh})$ to the space time inhomogeneity in the fashion described by Eq.(42). Because of the form of Eq.(42), we found two different contributions to the two-point correlation function: One is due to the intrinsic fluctuations and is given by Eq.(48). For scales inside the horizon, the power spectrum of those fluctuations have been computed (see Eq.(50) and an approximate formula in terms of the physical wave number has been computed. The second contribution is due to the induced fluctuations, a closed form of which is given by Eq.(56). The explicit expression of this contribution is now under consideration.

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5. Appendix A
Order reduction: The method has been applied by several authors [22], [23] in theoretical physics.

The method has as follows. Modify the original equation so that the new one is second order and as accurate as the original one. Thus, if we have an equation of the form

$$\frac{d^n x}{dt^n} = P\left[\frac{d^k x}{dt^k}\right] + \tau Q\left[\frac{d^m x}{dt^m}\right] + O(\tau^2)$$  \hfill (56)

where $k < m$, $m > n$, differentiating Eq.(56) $m - n$ times, substituting $\frac{d^m x}{dt^m}$ into $Q$ to order zero, and repeating the process up until $Q$ has lower derivative than $n$ we obtain

$$\frac{d^n x}{dt^n} = P\left[\frac{d^k x}{dt^k}\right] + \tau Q\left[\frac{d^{(n-1)} x}{dt^{(n-1)}}\right] + O(\tau^2)$$  \hfill (57)

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