Complementary observables can be measured simultaneously with less than perfect accuracy, but the uncertainty added in the observation to each observable can be removed by a proper data inversion. We show that complementarity manifests in that the inferred joint distribution after the inversion is pathological in the sense of not being able to represent a true joint distribution. This is closely related to the fact that the observed joint distribution is not separable. We apply this program to the paradigmatic example of complementarity: the path-interference duality in a Young interferometer. A key feature is that the conclusions hold equally well in the quantum and classical theories. Complementarity is also examined in terms of Wigner-like functions and duality relations.

I. INTRODUCTION

Complementarity is a profound idea expressing the impossibility of a joint simultaneous determination of two or more observables, say $A$ and $B$. This is usually understood to be a purely quantum phenomenon, and it is said that the quantum theory precludes the simultaneous exact measurement of conjugate observables. These two ideas are often exemplified by the standard uncertainty relation $\Delta A \Delta B \geq |\langle [A,B] \rangle|/2$.

Beyond this observation it is worth noting that:

(i) Complementary variables can be observed simultaneously with less than perfect accuracy [1–18]. Intuitively the uncertainties $\Delta A$ and $\Delta B$ represent such lack of precision unavoidably introduced by the observation.

(ii) Complementarity also holds in classical optics [19–22].

This work elaborate on these two points investigating the parallels between quantum and classical complementarity aiming a better understanding of this basic phenomenon. To this end two ingredients will be essential.

On the one hand, concerning point (i), in most scenarios the extra uncertainty added by the simultaneous observation can be removed to get the exact distributions of both observables separately. Then, this inversion can be legitimately applied to the measured joint distribution. When applied within classical statistical theories, this program works providing the exact joint probability distribution [23–26]. But in the quantum domain this attempts to obtain the quantum impossible, the joint distribution of incompatible observables. So, when this program provides a pathological inferred distribution we get a confirmation of nonclassical behaviour [24–26]. Actually, this is the only way nonclassical behaviour can consistently emerge.

On the other hand, following point (ii) the above program of data inversion can be equally well applied in classical optics, replacing probabilities by light intensity. We will find that the pathology in the classical sector is related to the radiance problem: the impossible determination of the amount of light emitted from a point in a given direction when the field is partially coherent [27–33].

We apply this inversion procedure to a seminal example of complementarity: the Young interferometer. The two conjugate observables are the light at the apertures and light at the interference plane. Their joint observation will be allowed by marking the light at each aperture by imprinting a different polarization state. Then the interference is observed keeping track of the polarization. We carry out this program both in quantum and classical optics. In the quantum sector we deal with photon probabilities, while in the classical sector we deal with light intensities. Interestingly, these magnitudes are closely related, as photon probabilities are usually considered as proportional to light intensities. In both cases we look for pathological results for the inferred joint distribution. Moreover, we show that pathologies can be ascribed to lack of separability of the observed joint distribution.

After presenting the basis of the inversion procedure in Sec. II we focus on the classical-optics realm in Sec. III showing the negativity in the inferred joint distribution. We also show there that the source of this pathology is that the observed joint distribution is not separable. Moreover, we analyze duality relations and the fact that the inferred distribution is actually a Wigner-like function, as a useful tool to combine complementary variables. Finally, we address the quantum sector in Sec. IV showing that it closely follows the classical complementarity.
II. INVERSION PROCEDURE

Let us recall the basis of the inversion procedure to be applied. This is equally valid both in the quantum and classical domains. We consider two complementary variables $A$ and $B$, taking values $a$ and $b$, which in the quantum sector are usually the eigenvalues of two Hermitian operators representing $A$ and $B$, respectively. We denote by $O_X(x)$ some observable depending on these variables, with $X = A, B$ and $x = a, b$. This observable will be probability in the quantum sector $O \rightarrow P$, and light intensity in the classical scenario $O \rightarrow I$. Our practical setting provides the joint observation of two variables $A$ and $B$, that can be considered as blurry counterparts of $A$ and $B$, respectively. Thus, after observation and measurement we get a well behaved operational joint distribution $\hat{O}(a, b)$ with marginals

$$\hat{O}_A(a) = \sum_b \hat{O}(a, b), \quad \hat{O}_B(b) = \sum_a \hat{O}(a, b). \quad (2.1)$$

We assume that these marginals provide complete information about two observables $A$ and $B$. This is to say that there are functions $M_X(x, x')$ such that the exact $O_A(a)$ and $O_B(b)$ can be retrieved from the observed marginals $(2.1)$:

$$O_X(x) = \sum_{x'} M_X(x, x') \hat{O}(x'), \quad (2.2)$$

where the functions $M_X(x, x')$ are completely known as far as we know all the details about the measurement being performed.

The key idea is to extend this inversion $(2.2)$ from the marginals to the complete joint distribution to obtain a joint distribution $O(a, b)$ for $A$ and $B$ as $[1, 14, 23, 27]$:

$$O(a, b) = \sum_{a', b'} M_A(a, a') M_B(b, b') \hat{O}(a', b'). \quad (2.3)$$

This distribution $O(a, b)$ is the one we expect to be pathological by taking negative values, since probabilities and intensities are expected to be nonnegative. This pathology will be related with the lack of separability of $O(a, b)$ in Sec. IIID below for the classical case, and in Sec. IV the quantum regime. Parallels can be drawn with the construction joint probability distributions via the inversion of moments $[54, 55]$.

III. CLASSICAL SECTOR

A. Settings. Exact, unobserved scenario

Our physical system is an standard Young interferometer with two small enough apertures to be labelled by the index $z = \pm 1$, which are illuminated by a monochromatic wave. Therefore, there are no temporal coherence issues and we work in space-frequency domain always within a purely classical-optics scenario.

The field at the apertures will be denoted as $E_z$ with light intensities

$$I_Z(z) = \langle |E_z|^2 \rangle, \quad (3.1)$$

where the angular brackets denote ensemble averages. Interference is observed in the far field leading in the usual way to the intensity distribution, in the appropriate units,

$$I_\phi(\phi) = \frac{1}{2\pi} \langle |E_1 + E_{-1} e^{-i\phi}|^2 \rangle, \quad (3.2)$$

this is to say

$$I_\phi(\phi) = \frac{1}{2\pi} \{ I_Z(1) + I_Z(-1) + 2 |\mu| \sqrt{I_Z(1)I_Z(-1)} \cos(\phi + \delta) \}, \quad (3.3)$$

where $\phi$ is the phase difference acquired from the slits to the observation point, $\mu$ is the complex degree of coherence and $\delta$ its phase

$$\mu = |\mu| e^{i\delta} = \frac{\langle E_1 E_1^* \rangle}{\sqrt{\langle |E_1|^2 \rangle \langle |E_{-1}|^2 \rangle}}. \quad (3.4)$$

As the result of the inversion procedure in Sec. II the goal is to obtain a joint distribution $I(z, \phi)$ meaning the amount of light leaving the aperture $z$ in the direction specified by $\phi$, so that the intensities $(3.1)$ and $(3.3)$ are its marginals

$$I_Z(z) = \int_{2\pi} d\phi I(z, \phi), \quad I_\phi(\phi) = \sum_{z=1,-1} I(z, \phi). \quad (3.5)$$

Before proceeding let us formalize the setting to gain insight and parallel the quantum scenario. The second-order statistics at the apertures can be represented by the $2 \times 2$ cross-spectral density matrix, with matrix elements $\Gamma_{i,j} = \langle E_i E_j^* \rangle$:

$$\Gamma = \begin{pmatrix} I_Z(1) & \mu \sqrt{I_Z(1)I_Z(-1)} \\ \mu^* \sqrt{I_Z(1)I_Z(-1)} & I_Z(-1) \end{pmatrix}, \quad (3.6)$$

or, equivalently,

$$\Gamma = \begin{pmatrix} I_Z(1) & \mu \sqrt{I_Z(1)I_Z(-1)} \\ \mu^* \sqrt{I_Z(1)I_Z(-1)} & I_Z(-1) \end{pmatrix}, \quad (3.7)$$

where $|E_1\rangle = |E_1| + |E_{-1} - 1\rangle$, with $|z = \pm 1\rangle$ a ket notation for the usual basis vectors

$$|z = 1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |z = -1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.8)$$

Since $\Gamma$ is Hermitian and positive semidefinite it has the properties to be a density matrix $\rho$ in quantum mechanics after a suitable normalization

$$\rho = \frac{1}{\text{tr} \Gamma} \Gamma. \quad (3.9)$$
To exploit this equivalence further let us consider units so that
\[ \text{tr}\Gamma = I_Z(1) + I_Z(-1) = \int_{2\pi} d\phi I_\phi(\phi) = 1. \] (3.10)

With all this, our basic quantities look very much like quantum probabilities since they can be expressed as
\[ I_Z(z) = \langle z|\Gamma|z \rangle, \quad I_\phi = \langle \phi|\Gamma|\phi \rangle, \] (3.11)
where the states representing interference \(|\phi \rangle\) are the phase states [33 12]
\[ |\phi \rangle = \frac{1}{\sqrt{2\pi}} \left( 1/e^{i\phi} \right). \] (3.12)

Notice that \(|z\rangle\) and \(|\phi \rangle\) are complementary \(|\langle \phi|z \rangle| = 1/\sqrt{2\pi}\) mutually unbiased bases in the two-dimensional Hilbert space \(\mathcal{H}_s\) describing this two-beam interference setting. So classical two-beam interference mimics a qubit.

### B. Joint observation and inversion

To perform a simultaneous observation of \(I_Z(z)\) and \(I_\phi(\phi)\) we must involve additional degrees of freedom. Let us transfer information about light at the apertures to the polarization state, so that \(I_Z(z)\) will be inferred from polarization measurements, while \(I_\phi(\phi)\) will be determined from the intensity distribution at the observation screen disregarding polarization.

Polarization can be suitably described by a two-dimensional Hilbert space \(\mathcal{H}_p\) with two orthogonal basis vectors \(|\rightarrow\rangle\) and \(|\uparrow\rangle\), representing horizontal and vertical linear polarization for example. The light illuminating the slits has horizontal polarization \(|\rightarrow\rangle\). Then, the transfer of information about \(I_Z(z)\) to the polarization state can be easily achieved in practice by placing a half-wave plate in one of the apertures and nothing on the other.

Now the scenario has been enlarged to be described by the Hilbert space \(\mathcal{H}_s \otimes \mathcal{H}_p\). So the field state on the apertures including polarization is now described by the 4 \(\times\) 4 cross-spectral density matrix
\[ \tilde{\Gamma} = \langle \tilde{E}|\langle \tilde{E} \rangle, \quad |\tilde{E} \rangle = E_1|1\rangle u_1 + E_{-1}|1\rangle u_{-1}, \] (3.13)
where
\[ |u_1 \rangle = \cos \theta |\rightarrow\rangle + \sin \theta |\uparrow\rangle, \quad |u_{-1} \rangle = |\rightarrow\rangle \] (3.14)
and \(\theta\) represents the polarization change induced by the half-wave plate. On the plane where we observe the interference now we place an ideal polarizer. We record the intensity \(\tilde{I}(p, \phi)\) for two orthogonal orientations of the polarization axis \(p = \pm 1\) represented by the unit vectors
\[ |p = 1 \rangle = \cos \vartheta |\rightarrow\rangle + \sin \vartheta |\uparrow\rangle, \] \[ |p = -1 \rangle = -\sin \vartheta |\rightarrow\rangle + \cos \vartheta |\uparrow\rangle, \] (3.15)
where \(\vartheta\) is an arbitrary angle. Thus, the recorded intensity is
\[ \tilde{I}(p, \phi) = \langle p|\langle \phi|\tilde{\Gamma}|\phi \rangle|p \rangle, \] (3.16)
leading to
\[ \tilde{I}(1, \phi) = \frac{1}{\sqrt{2\pi}} \left[ \cos^2 (\vartheta - \theta) I_Z(1) + \cos^2 \vartheta I_Z(-1) + 2|\mu| \cos \vartheta \cos (\vartheta - \theta) \cos (\phi + \delta) \right], \] (3.17)
\[ \tilde{I}(-1, \phi) = \frac{1}{\sqrt{2\pi}} \left[ \sin^2 (\vartheta - \theta) I_Z(1) + \sin^2 \vartheta I_Z(-1) - 2|\mu| \sin \vartheta \sin (\vartheta - \theta) \cos (\phi + \delta) \right]. \]

The corresponding marginals for polarization and interferometric intensity are
\[ \tilde{I}_P(p) = \int_{2\pi} d\phi \tilde{I}(p, \phi), \quad \tilde{I}_\Phi(\phi) = \sum_{p=\pm 1} \tilde{I}(p, \phi), \] (3.18)
with
\[ \tilde{I}_P(1) = \cos^2 (\vartheta - \theta) I_Z(1) + \cos^2 \vartheta I_Z(-1), \] \[ \tilde{I}_P(-1) = \sin^2 (\vartheta - \theta) I_Z(1) + \sin^2 \vartheta I_Z(-1), \] (3.19)
or in matrix form
\[ \left( \begin{array}{c} \tilde{I}_P(1) \\ \tilde{I}_P(-1) \end{array} \right) = \left( \begin{array}{cc} \cos^2 (\vartheta - \theta) & \cos^2 \vartheta \\ \sin^2 (\vartheta - \theta) & \sin^2 \vartheta \end{array} \right) \left( \begin{array}{c} I_Z(1) \\ I_Z(-1) \end{array} \right), \] (3.20)
while for the phase marginal distribution we have
\[ \tilde{I}_\Phi(\phi) = \frac{1}{\sqrt{2\pi}} \left[ I_Z(1) + I_Z(-1) + 2|\mu| \sqrt{I_Z(1)I_Z(-1)} \cos (\phi + \delta) \right]. \] (3.21)

The idea is that the polarization measurement \(\tilde{I}_P(p)\) represents the imperfect observation of \(I_Z(z)\), while \(\tilde{I}_\Phi(\phi)\) is an imperfect observation of \(I_\Phi(\phi)\), as a particular realization of the observables \(\tilde{A}\) and \(\tilde{B}\) mentioned in the introduction. Therefore, we have to look for the inverting functions \(M_{Z,\Phi}\) that carry out the inversions
\[ I_Z(z) = \sum_{p=\pm 1} M_{Z}(z, p) \tilde{I}_P(p), \] (3.22)
\[ I_\Phi(\phi) = \int_{2\pi} d\phi' M_{\Phi}(\phi, \phi') \tilde{I}_\Phi(\phi'). \]
These are, in matrix form for the slit-polarization pair,
\[ \left( \begin{array}{c} I_Z(1) \\ I_Z(-1) \end{array} \right) = \left( \begin{array}{cc} M_{Z}(1, 1) & M_{Z}(1, -1) \\ M_{Z}(-1, 1) & M_{Z}(-1, -1) \end{array} \right) \left( \begin{array}{c} \tilde{I}_P(1) \\ \tilde{I}_P(-1) \end{array} \right), \] (3.23)
with
\[
M_Z(z, p) = \begin{pmatrix}
\sin^2 \theta & -\cos^2 \theta \\
\cos^2 \theta & \sin^2 \theta
\end{pmatrix}
\begin{pmatrix}
\sin \theta \sin(2\theta - \phi) \\
\sin \theta \sin(2\theta - \phi)
\end{pmatrix}, \tag{3.24}
\]
while for the interferometric phase-variable
\[
M_\phi(\phi, \phi') = \frac{1}{2\pi} \left( 1 + \frac{2}{\cos \theta} \cos (\phi - \phi') \right). \tag{3.25}
\]

Then we extend the inversion procedure to the joint distributions as
\[
I(z, \phi) = \sum_{p=\pm 1} \int_{2\pi} d\phi' M_Z(z, p) M_\phi(\phi, \phi') \tilde{I}(p, \phi'), \tag{3.26}
\]
to obtain
\[
I(1, \phi) = \frac{1}{2\pi} \left[ I_Z(1) + \csc (2\theta - \phi) \sec \theta \sin(2\theta) \right.
\times |\mu| \sqrt{I_Z(1)I_Z(-1)} \cos (\phi + \delta), \tag{3.27}
\]
\[
I(-1, \phi) = \frac{1}{2\pi} \left[ I_Z(-1) + \csc (2\theta - \phi) \sec \theta \sin(2\theta - 2\theta) \right.
\times |\mu| \sqrt{I_Z(1)I_Z(-1)} \cos (\phi + \delta). \tag{3.28}
\]

Not every choice of \( \theta \) works equally well in order to infer \( I_z(z) \) in terms of \( I_P(p) \). In particular, the optimum choice for \( \theta \) may be given when the matrix in Eq. \( \text{3.20} \) is closer to the identity matrix. As shown in appendix A of Ref. \[43\], this holds for \( 2\theta - \theta = \pi/2 \). To gain insight, and without loss of generality, we can consider such a case to get from Eq. \( \text{3.27} \)
\[
I(z, \phi) = \frac{1}{2\pi} \left[ I_Z(z) + |\mu| \sqrt{I_Z(1)I_Z(-1)} \cos (\phi + \delta) \right]. \tag{3.28}
\]

C. Pathology

The pathological results we are looking for hold when \( I(z, \phi) < 0 \) for some particular values of \( z \) and \( \phi \). To this end, let us choose simply \( \phi = -\delta + \pi \) so that \( \cos (\phi + \delta) = -1 \) and then we have either \( I(1, \phi) < 0 \) or \( I(-1, \phi) < 0 \) provided that \( |\mu|^2 > \min \{I_Z(1)/I_Z(-1), I_Z(-1)/I_Z(1)\} \).

It is worth noting that the pathological behaviour requires some threshold value for the degree of coherence. In other words, every pathology disappears in the limit of vanishing coherence. This could be expected from a radiance-problem perspective. On the other hand, the pathology requires some intensity unbalance \( I_Z(1) \neq I_Z(-1) \), so that when \( I_Z(1) = I_Z(-1) \) we have a well-defined radiance function for every state of coherence.

D. Separability

In Refs. \[24, 26\] it has been shown that pathological behavior in the quantum sector holds when the observed statistics is not separable, as recalled in Sec. IVB below. Let us translate this idea to the classical sector here. We may say that the intensity-polarization distribution \( I(p, \phi) \) is separable in the \( p, \phi \) variables when we can find positive weights \( w_\lambda \) and positive semidefinite distributions \( I_P(p|\lambda) \) and \( I_\Phi(\phi|\lambda) \) such that
\[
\tilde{I}(p, \phi) = \sum_\lambda w_\lambda I_P(p|\lambda)I_\Phi(\phi|\lambda). \tag{3.29}
\]
This is equivalent to say that our problem is separable if
\[
\tilde{\Gamma} = \sum_\lambda w_\lambda \Gamma_\lambda \otimes \gamma_\lambda, \tag{3.30}
\]
where \( \Gamma_\lambda \) are legitimate cross-spectral density matrices in \( \mathcal{H}_\lambda \), while \( \gamma_\lambda \) are proper polarization matrices in \( \mathcal{H}_p \), with \( I_\Phi(\phi|\lambda) = \langle \phi | \Gamma_\lambda | \phi \rangle \) and \( I_P(p|\lambda) = \langle \rho | \gamma_\lambda | \rho \rangle \). In a quantum scenario the variables \( \lambda \) are usually referred to as hidden variables and are essentially the variables that make up the phase space of the problem.

In this classical-optics sector \( \lambda \) can be regarded as a suitable decomposition of \( \tilde{\Gamma} \) in polarized-field modes \[44\].

Deep down, the lack of separability must be ascribed to the entangled nature of the classical field state \( |\tilde{E} \rangle \) in Eq. \( \text{3.13} \).

This properly mimics the equivalent definition of separability in the quantum sector in terms of density matrices. So the results in the preceding section indicate that the classical electromagnetism is not always separable. This might offer a new perspective to better understand entanglement in classical and quantum optics and their differences \[20, 45, 51\].

E. Duality relations

It is natural to look for quantitative expressions of complementarity as exemplified by the standard Heisenberg uncertainty relation. This encounters the additional challenge that in finite-dimensional systems variances do not provide a meaningful uncertainty relation and alternative measures must be used. This has been successfully achieved in the quantum \[8, 11, 12, 13\] and classical realms \[16, 22\]. A suitable option to assess uncertainty is in terms of characteristics functions, as they have already been used in the quantum domain \[16, 17\]. Let us show that this actually works in the classical-optics scenario.

In this spirit we define the intrinsic certainties \( C_Z \) and \( C_\Phi \) for \( I_Z(z) \) and \( I_\Phi(\phi) \) as, always with the total intensity normalization \( \text{3.10} \),
\[
C_Z = |\sum_{z=\pm 1} e^{i\pi z/2} I_Z(z)| = |I_Z(1) - I_Z(-1)|, \tag{3.31}
\]
\[
C_\Phi = \left| \int_{2\pi} d\phi e^{i\phi} I_\Phi(\phi) \right| = |\mu| \sqrt{I_Z(1)I_Z(-1)},
\]
that are also known in the literature as distinguishability and visibility, respectively. We refer to them as certainties since they may express the degree of certainty one can have concerning the value of the corresponding observable. They take the maximum value unity, maximum certainty, when all the distribution is concentrated in a single value, while it takes the minimum value zero, minimum certainty, when the observable is uniformly distributed. The key point is that the satisfy the certainty or duality relation

\[ C_Z^2 + 4C_\phi^2 = \mathcal{P}^2, \] (3.32)

where \( \mathcal{P} \) is the analog of the degree of polarization if \( \Gamma \) were a polarization matrix \([19, 22]\)

\[ \mathcal{P}^2 = \frac{2\text{tr}(\Gamma^2)}{(\text{tr}\Gamma)^2} - 1, \] (3.33)

which is actually a measure of the purity of the state \( \rho \) in Eq. (3.22). Let us apply relations (3.31) to the marginals of the measured intensity-polarization distribution \( \hat{I}(p, \phi) \) to obtain observed certainties \( \hat{C}_Z \) and \( \hat{C}_\phi \). To this end, we consider the optimal measurement \( 2\theta - \theta = \pi/2 \) leading to

\[ \hat{C}_Z = |\hat{I}_p(1) - \hat{I}_p(-1)| = \mathcal{V}_Z C_Z, \] (3.34)

\[ \hat{C}_\phi = \left| \int_{2\pi} d\phi e^{i\phi} \hat{I}_\phi(\phi) \right| = \mathcal{V}_\phi C_\phi, \]

where

\[ \mathcal{V}_Z = |\sin \theta|, \quad \mathcal{V}_\phi = |\cos \theta|. \] (3.35)

We can appreciate several interesting features. On the one hand, the observed certainties are always lesser than or equal to the exact ones \( \hat{C}_Z \leq C_Z \) and \( \hat{C}_\phi \leq C_\phi \). This makes sense from the understanding that the joint measurement introduces additional uncertainty. On the other hand, it is quite nice that the noise added satisfies by itself a kind of certainty relation:

\[ \mathcal{V}_Z^2 + \mathcal{V}_\phi^2 = 1, \] (3.36)

a reminiscence that the noise can be traced back to complementarity in the apparatus variables.

### F. Wigner-like functions

The procedure we have followed defines a joint distribution \( I(z, \phi) \) for complementary variables with exact marginals. This recalls the idea of Wigner function both in the classical \([27, 53]\) and quantum \([52, 57]\) domains. A key feature is that Wigner functions combine elements from different theories. In the quantum case these are the quantum state vector and the classical phase space. In classical optics these are the wave and geometrical pictures of light propagation.

We can check that the result obtained for \( I(z, \phi) \) does not fit with the standard definition of Wigner function in classical optics as presented in Ref. [33]. For the Young interferometer the standard approach leads to a fictitious source of dark rays with positive and negative intensities at the midpoint between the apertures. Instead, the \( I(z, \phi) \) obtained here recalls Wigner functions introduced in quantum optics for number and phase variables \([57, 58]\), where the variable \( z \) takes the place of the number variable.

More specifically, let us translate the definition of Wigner function introduced in Eq. (9) of Ref. [58] to our scenario as

\[ Q_T(z, \phi) = \frac{1}{4\pi} \left( 2\Gamma_{z,z} + e^{-i\phi}\Gamma_{-1,1} + e^{i\phi}\Gamma_{1,-1} \right), \] (3.37)

which is exactly our Eq. (3.28) \( I(z, \phi) = Q_T(z, \phi) \). We can check whether \( Q_T(z, \phi) \) satisfies some of the desirable properties for a Wigner function \( W_T(z, \phi) \) depending on these variables, as listed for example in Ref. [57]. These are:

(i) Reality: If \( \Gamma^i = \Gamma \) then \( W_T^i(z, \phi) = W_T(z, \phi) \).

(ii) Correct marginals:

\[ \int_{2\pi} W_T^i(z, \phi) = \langle z|\Gamma|z \rangle, \] and

\[ \sum_{z \pm 1} W_T(z, \phi) = \langle \phi|\Gamma|\phi \rangle. \] (3.37)

(iii) Proper transformation laws under basic operations: a) phase shifts, if \( \Gamma^i \rightarrow \Gamma^i \rightarrow \Gamma_{\pm 1, \pm 1}e^{\pm i\phi} \) and \( \Gamma_{\pm 1, \pm 1} \rightarrow z \) then \( W_T(z, \phi) = W_T(z, \phi) \), shifts in \( z \), that in this case equals \( z \) exchange, if \( \Gamma_{i,j} \rightarrow \Gamma_{-i,-j} \), then \( W_T(z, \phi) = W_T(z, \phi) \).

(iv) Conjugation: if \( \Gamma^i \rightarrow \Gamma^i \rightarrow \Gamma_{i,j} \) then \( W_T(z, \phi) = W_T(z, \phi) \).

(v) Overlap: for all \( \Gamma \) and \( \Gamma^i \) we have \( \text{tr} (\Gamma\Gamma^i) = 2\pi \int_{2\pi} d\phi \sum_{z \pm 1} W_T(z, \phi)W_T(z, \phi) \).

We may add the following property. (vi) Completeness: \( W_T(z, \phi) \) determines \( \Gamma \) completely.

We can easily check that \( Q_T(z, \phi) \) satisfies (i), (ii), (iii), (iv), (v), but lacks (iii), where we have \( Q_T(z, \phi) = Q_T(z, -\phi) \), and (v).

As an alternative approach we may consider that Eq. (3.34) is an example of the Margenau-Hill-Terletsky distribution \( T(z, \phi) \), this is \( I(z, \phi) = T(z, \phi) = \text{Re}\{S(z, \phi)\} \), where \( S(z, \phi) \) is the complex Kirkwood distribution \([59–62]\)

\[ S(z, \phi) = \langle \phi|z \rangle\langle z|\phi \rangle = \frac{1}{2\pi} \left( \Gamma_{z,z} + e^{i\phi}\Gamma_{z,-z} \right). \] (3.38)

Regarding properties, the distribution \( T \) naturally satisfies the same properties as \( Q \). On the other hand \( S \) lacks reality (i), but satisfies (ii), (iii), (vi), and some slightly modified versions of the other properties. Instead of (iii) we have \( S_T(z, \phi) = S_T(-z, -\phi) \), while for (iv) \( S_T(z, \phi) = S_T^\dagger(z, -\phi) \), as well as a version of the overlap (v) in the form \( \text{tr}(\Gamma\Gamma^i) = 2\pi \int_{2\pi} d\phi \sum_{z \pm 1} S_T(z, \phi)S_T^\dagger(z, \phi) \).
IV. QUANTUM SECTOR

A. Settings. Exact, unobserved scenario

Focusing in the case of just one photon, the quantum sector emerges by the thorough replacement of field intensity by photon probability \( I \rightarrow P \) and cross-spectral density tensor by density matrix \( \Gamma \rightarrow \rho \), which in this context is customarily expressed in terms of the Pauli matrices \( \sigma \)

\[
\rho = \frac{1}{2} (\sigma_0 + s \cdot \sigma), \quad (4.1)
\]

where \( \sigma_0 \) is the 2 \( \times \) 2 identity, and \( s = (\sigma) \) is a three-dimensional real vector with \( |s| \leq 1 \). Within this context the observable \( Z \) is represented by the third Pauli matrix \( \sigma_z \), with eigenstates \( |z = \pm 1\rangle \), with \( z = 1 \) meaning that the photon is found in the upper aperture and \( z = -1 \) in the lower aperture. The corresponding probabilities are

\[
P_Z(z) = \langle z|\rho|z\rangle = \frac{1}{2} (1 + zs_z). \quad (4.2)
\]

The quantum description of phase-like variables is a rather tricky point since there is no simple operator for the phase or phase difference \( [35–42] \). Maybe the best quantum description of relative phase suited for our purposes is given by a positive-operator-valued measure. More specifically, the phase statistics \( P_\Phi(\phi) \) is given by projection of the photon state on the nonorthogonal phase states in Eq. \( 3.12 \) \( 35 \), this is

\[
|\phi\rangle = \frac{1}{\sqrt{2\pi}} \left( |z = 1\rangle + e^{i\phi}|z = -1\rangle \right), \quad (4.3)
\]

so that, the exact phase distribution is

\[
P_\Phi(\phi) = \langle \phi|\rho|\phi\rangle = \frac{1}{2\pi} (1 + \cos \phi s_x + \sin \phi s_y). \quad (4.4)
\]

B. Joint observation and inversion

The joint observation of \( \Phi \) and \( Z \) follows exactly the same steps and settings of the classical case so we can proceed directly to the statistics for the so constructed joint observation of interference and polarization as

\[
\tilde{P}(p, \phi) = \langle p|\langle \phi|\rho|\phi\rangle|p\rangle, \quad (4.5)
\]

where \( |p\rangle \) are defined as in Eq. \( 3.15 \), and \( |\phi\rangle \) are in Eq. \( 4.3 \), leading to

\[
\tilde{P}(p, \phi) = \frac{1}{2\pi} \left[ \gamma_0(z) + \gamma_X(z) \cos \phi s_x + \gamma_X(z) \sin \phi s_y + z\gamma_Z(z)s_z \right], \quad (4.6)
\]

where the functions \( \gamma \) are

\[
\begin{align*}
\gamma_0(1) &= \frac{1}{2} \left[ \cos^2(\theta - \theta) + \cos^2 \vartheta \right], \\
\gamma_0(-1) &= \frac{1}{2} \left[ \sin^2(\theta - \theta) + \sin^2 \vartheta \right], \\
\gamma_X(1) &= \cos(\theta - \theta) \cos \vartheta, \\
\gamma_X(-1) &= \sin(\theta - \theta) \sin \vartheta, \\
\gamma_Z(1) &= \frac{1}{2} \left[ \cos^2(\theta - \theta) - \cos^2 \vartheta \right], \\
\gamma_Z(-1) &= \frac{1}{2} \left[ -\sin^2(\theta - \theta) + \sin^2 \vartheta \right].
\end{align*} \quad (4.7)
\]

The observed marginal for \( Z \) is

\[
\tilde{P}_Z(z) = \gamma_0(z) + z\gamma_Z(z)s_z, \quad (4.8)
\]

while the observed marginal for the phase is

\[
\tilde{P}_\Phi(\phi) = \frac{1}{2\pi} \left[ 1 + \cos \theta \left( \cos \phi s_x + \sin \phi s_y \right) \right]. \quad (4.9)
\]

Then it is possible to obtain the exact statistics \( 4.2 \) and \( 4.3 \) from the operational ones in Eqs. \( 4.8 \) and \( 4.9 \) as

\[
P_Z(z) = \sum_p M_Z(z, p) \tilde{P}_Z(p), \quad (4.10)
\]

with the same \( M_Z(z, p) \) in Eq. \( 3.12 \), while the phase distribution can be inverted as

\[
P_\Phi(\phi) = \int d\phi' M_\Phi(\phi, \phi') \tilde{P}_\Phi(\phi'), \quad (4.11)
\]

with the same \( M_\Phi \) in Eq. \( 6.26 \). Finally we extend the inversion \( (4.10) \) and \( (4.11) \) from the marginals to the complete joint distribution to obtain a joint distribution \( P(z, \phi) \) as:

\[
P(z, \phi) = \sum_p \int d\phi' M_Z(z, p) M_\Phi(\phi, \phi') \tilde{P}(p, \phi'), \quad (4.12)
\]

leading to

\[
P(z, \phi) = \frac{1}{4\pi} \left[ 1 + \nu(z) \left( \cos \phi s_x + \sin \phi s_y + zs_z \right) \right], \quad (4.13)
\]

where

\[
\nu(1) = \frac{\sin(2\theta)}{\cos \theta \sin(2\theta - \theta)}, \quad \nu(-1) = \frac{\sin(2\theta - 2\theta)}{\cos \theta \sin(2\theta - \theta)}. \quad (4.14)
\]

Since \( \nu(1) + \nu(-1) = 2 \) we can appreciate that \( P(z, \phi) \) provides the correct exact marginals \( 4.2 \), \( 4.4 \) for both observables.

As in the classical case let us consider the optimum observation of \( P_Z(z) \) holds for \( 2\theta - \theta = \pi/2 \). In such a case, \( \nu(1) = \nu(-1) = 1 \) and an extremely simple expression for \( P(\phi, z) \) is obtained

\[
P(\phi, z) = \frac{1}{4\pi} \left[ 1 + \cos \phi s_x + \sin \phi s_y + zs_z \right]. \quad (4.15)
\]
C. Pathology

Let us examine whether $P(z, \phi)$ can take negative values. The minimum in Eq. (4.15) is

$$P_{\text{min}} = \frac{1}{4\pi} \left( 1 - |s_z| - \sqrt{s_x^2 + s_y^2} \right). \tag{4.16}$$

Therefore, $P_{\text{min}} < 0$ provided that

$$|\mu|^2 > \frac{1 - |s_z|}{1 + |s_z|} = \min \{ P_Z(1)/P_Z(-1), P_Z(-1)/P_Z(1) \}, \tag{4.17}$$

where $|\mu|$ has exactly the same meaning than in the classical case

$$|\mu|^2 = \frac{|\rho_{1,1}|^2}{\rho_{1,1} \rho_{-1,-1}} = \frac{s_x^2 + s_y^2}{1 - s_z^2}. \tag{4.18}$$

So condition (4.17) for pathological behaviour is exactly the same than in the classical sector.

The negativity does not mean that $P(z, \phi)$ is meaningless. This is useful for understanding basic concepts such as Bell tests and diverse basic nonclassical features [63 63]. We may say that this is actually the hallmark of quantumness and the way nonclassical states are defined in quantum optics [66 69].

D. Duality relations and Wigner-like function

It is clear that the same results regarding duality relations and Wigner-like functions hold in this quantum sector in exactly the same terms of the classical case. In particular, we have that for the usual case $P_Z(1) = P_Z(-1)$ the so defined $Q$ and $T$ function are nonnegative, so we may have a classical-like statistical model for complementarity.

E. Classical statistics and separability

Next we show that in classical physics the observed probabilities are always separable. To show this, we consider that classically the state of the system can be completely described by a legitimate probability distribution $w_\lambda$, where $\lambda$ runs over all admissible classical states for the system, this is the points of the corresponding phase space, assumed to form a discrete set for simplicity and without loss of generality. There is no limit to the number of points $\lambda$ so it may approach a continuum if necessary.

In a classical-physics scenario every phase-space point $\lambda$ has a perfectly defined nonrandom value for every observable, independent of the values taken by other variables, and any two observables $A$, $B$ can be observed without mutual disturbances, so the factorization of joint conditional probabilities holds $\tilde{P}(a, b | \lambda) = \tilde{P}_A(a | \lambda) \tilde{P}_B(b | \lambda)$, where $\tilde{P}_X(x | \lambda)$ is the conditional probability that the observable $X$ takes the value $x$ when the system state is $\lambda$. Strictly speaking, in the ideal case the conditional probabilities $\tilde{P}_X(x | \lambda)$ would be actually Dirac delta functions. Nevertheless, for the sake of completeness we include the possibility of some extra uncertainty introduced by the measurement process, which brings us closer to the quantum scenario. So, in classical physics the observed joint statistics for two generic observables $A$ and $B$ can be always expressed as

$$\tilde{P}(a, b) = \sum_\lambda w_\lambda \tilde{P}_A(a | \lambda) \tilde{P}_B(b | \lambda). \tag{4.19}$$

The point is that we can regard $\tilde{P}_X(x | \lambda)$ as an imperfect version of the exact distribution $P_X(x | \lambda)$, so we can apply the inversion procedure [22] leading to

$$P_X(x | \lambda) = \sum_{x'} M_X(x, x') \tilde{P}_X(x' | \lambda), \tag{4.20}$$

where $P_X(x | \lambda)$ is the exact statistics for $X$ when the system state is $\lambda$. Thus, because of the separable form (4.19) we readily get from Eq. (4.20) that the result of the inversion is the actual joint distribution for $A$ and $B$

$$P(a, b) = \sum_\lambda w_\lambda P_A(a | \lambda) P_B(b | \lambda), \tag{4.21}$$

and therefore a legitimate statistics. Consequently, lack of positivity or any other pathology of the retrieved joint distribution $P(a, b)$ is then a signature of nonclassical behavior. Note that this is a different situation from the one considered in Sec. III, where we considered intensities instead of probabilities.

V. CONCLUSIONS

Most practical and meaningful observations in quantum and classical physics are indirect in the sense that the desired information is retrieved after a suitable data analysis. This idea allows us to approach the joint distribution for conjugate observables by removing the instrumental effects of their imperfect simultaneous measurement. This can be done in classical optics as well as in quantum optics for a single photon, and we actually find exactly the same results mutatis mutandis. Thus, complementary is just a classical effect linked to wave behavior.

Let us emphasize a central point of the method: the pathology holds provided that the observed distribution is not separable. We hope that this analysis may enlighten the actual borderline between classical and quantum light.

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