Gauge and Cutoff Function Dependence of the Ultraviolet Fixed Point in Quantum Gravity

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Abstract

The exact renormalization group equation for pure quantum gravity is derived for an arbitrary gauge parameter in the space-time dimension $d = 4$. This equation is given by a non-linear functional differential equation for the effective average action. An action functional of the effective average action is approximated by the same functional space of the Einstein-Hilbert action. From this approximation, $\beta$-functions for the dimensionless Newton constant and cosmological constant are derived non-perturbatively. These are used for an analysis of the phase structure and the ultraviolet non-Gaussian fixed point of the dimensionless Newton constant. This fixed point strongly depends on the gauge parameter and the cutoff function. However, this fixed point exists without these ambiguities, except for some gauges. Hence, it is possible that pure quantum gravity in $d = 4$ is an asymptotically safe theory and non-perturbatively renormalizable.

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1 Introduction

As is well known, quantum gravity (QG) must be treated non-perturbatively. This is because, in the space-time dimension $d = 4$, the $L$-loop perturbative calculations in Einstein gravity cause divergences that are proportional to the $L + 1$ powers of the curvature tensor. Hence, the renormalization of these terms requires infinite number of couplings. Thus, QG is called (perturbatively) non-renormalizable. However, if QG is an asymptotically safe theory, it becomes (non-perturbatively) renormalizable [1]. An asymptotically safe theory is specified by the existence of a ultraviolet (UV) non-Gaussian fixed point (NGFP). The asymptotically safe nature of QG is suggested by $d = 2 + \epsilon$ gravity theory [1, 2]. It is also expected that this nature will be maintained in $d = 4$. However, an ordinary perturbative $\epsilon$-expansion is an asymptotic expansion. Thus, the large order behavior of $\epsilon$ is not reliable. Hence, to guarantee the existence of the UV NGFP in more higher dimensions, the appropriate method such as the Borel resummation must be applied.

The exact renormalization group equation (ERGE) [3] used in this article is one of the non-perturbative methods in field theories. In addition, the applicability of the ERGE exceeds the $\epsilon$-expansion and the $1/N$-expansion [4]. In the previous article [5], we used the formulation of the ERGE for pure QG [6], and clarified that QG has the UV NGFP in $2 < d \leq 4$. This result suggests that QG is an asymptotically safe theory and (non-perturbatively) renormalizable. However, this UV NGFP depends on the gauge parameter and the cutoff function. Hence, the purpose of this article is to study the effect of that dependence to the UV NGFP.

The cutoff function dependence is an artifact problem. The origin of this problem is stated as follows. The ERGE is formulated as a non-linear functional differential equation for the effective average action. The effective average action is an infrared (IR) cutoffed 1PI effective action, and corresponds to a coarse grained free energy. To formulate the effective average action, we must introduce the cutoff function. As we will discuss in Sec. 2, the profile of the cutoff function is arbitrary. If we can solve the ERGE without any approximations, this problem will not appear [7]. However, it is impossible to solve the ERGE without any approximations. Hence, to reduce this equation to the calculable form, we must truncate the functional space of the effective average action. For this truncation, the non-linear functional differential equation is reduced to a set of the non-linear differential equations. This truncation causes the cutoff function dependence.

The origin of the gauge dependence in the ERGE is same as that of the ordinary field theory.
If we only discuss the existence of the NGFP, this problem is not so serious. This is because, the non-physical quantity such as the $\beta$-function and the fixed point (FP) may depend on the gauge. However, if this dependence causes the disappearance of the NGFP, it becomes problem, because the gauge dependence changes the phase structure of the theory. We have been believed that Abelian and Yang-Mills gauge theories have only the GFP. Thus, the gauge dependence of the NGFP has not been discussed. However, the existence of the NGFP is an important point in QG. Though we expect that the NGFP should remain for any gauges, a strict proof is not known.

This paper is organized as follows. In the next section, the formulation of the ERGE for pure QG is reviewed (for details see [3]). In Sec. 3, the functional space is approximated by the same space of the Einstein-Hilbert action. In there, the $\beta$-functions for the dimensionless Newton constant and cosmological constant including the constant gauge parameter are derived by a slightly different method from other formulations [8, 9, 10]. In Sec. 4, these $\beta$-functions are used to study the effect of the gauge and cutoff function dependence to the UV NGFP. Section 5 is devoted to summary and discussion.

2 Formulation and approximation of the ERGE

To derive the ERGE for pure QG, we define the scale dependent generating functional $W_k$ of the connected Green functions. Here, $k$ is the IR cutoff scale in the Euclidean momentum space. Hence, $W_k$ is the IR cutoffed generating functional and defined by

$$e^{W_k[t,\bar{\sigma},\sigma;\beta,\tau;\bar{g}]} = \int \mathcal{D}h\mathcal{D}C\mathcal{D}\bar{C} \exp \left\{ -S_{\text{grav}}[\gamma] - S_{\text{g.f.}}[h;\bar{g}] - S_{\text{F.P.}}[h,C,\bar{C};\bar{g}] - S_{\text{e.s.}}[t,\bar{\sigma},\sigma;\beta,\tau;\bar{g}] - \Delta_k S_{\text{grav}}[h;\bar{g}] - \Delta_k S_{\text{gh}}[C,\bar{C};\bar{g}] \right\}. \tag{1}$$

Here, $S_{\text{grav}}[\gamma]$ is a general functional of a quantum metric $\gamma_{\mu\nu}$. In the background field method, the quantum metric is decomposed as the background metric $\bar{g}_{\mu\nu}$ and the fluctuation from the background $h_{\mu\nu}$. The functional $S_{\text{g.f.}}[h;\bar{g}]$ in Eq. (1) is the gauge fixing term given by

$$S_{\text{g.f.}}[h;\bar{g}] = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}} g^{\mu\nu} F_\mu F_\nu,$$

where $\alpha$ is a gauge parameter. In the harmonic gauge $F_\mu = 0$, the explicit form of $F_\mu$ is given by

$$F_\mu = \sqrt{2\kappa} F_\mu^{\alpha\beta}[\bar{g}] h_{\alpha\beta} = \sqrt{2\kappa} \left( \delta_\mu^\rho g^{\alpha\gamma} \bar{D}_\gamma - \frac{1}{2} g^{\alpha\beta} D_\mu \right) h_{\alpha\beta},$$
where \( \kappa = (32\pi G)^{-1/2} \) and \( G \) is the bare Newton constant. The covariant derivative \( \bar{D} \) is constructed from \( \bar{g}_{\mu
u} \). In Eq. (1), \( S_{\text{F.P.}}[h, C; \bar{g}] \) is the Faddeev-Popov ghost term given by

\[
S_{\text{F.P.}}[h, C; \bar{g}] = -\sqrt{2} \int d^d x \sqrt{\bar{g}} \bar{C}_\mu \mathcal{M}[\gamma; \bar{g}]^{\mu\nu} C^\nu,
\]

where \( C^\mu \) and \( \bar{C}_\mu \) are the ghost and anti-ghost fields respectively. Here,

\[
\mathcal{M}[\gamma; \bar{g}]^{\mu\nu} = \bar{g}^{\mu\rho} \bar{g}^{\sigma\lambda} \bar{D}_\lambda (\gamma_{\rho\sigma} \bar{D}_\nu + \gamma_{\sigma\nu} \bar{D}_\rho) - \bar{g}^{\mu\rho} \bar{g}^{\sigma\lambda} \bar{D}_\lambda \gamma_{\sigma\rho} \bar{D}_\nu.
\]

The functional \( S_{\text{e.s.}} \) in Eq. (1) is the external source term and given by

\[
S_{\text{e.s.}}[t, \bar{t}, \sigma, \bar{\sigma}; \tau, \bar{\tau}; \bar{g}] = -\int d^d x \sqrt{\bar{g}} \{ t^{\mu\nu} h_{\mu\nu} + \bar{\sigma}_\mu C^\mu + \sigma^{\mu} \bar{C}_\mu \\
+ \beta^{\mu\nu} (\delta_{\text{B}} h_{\mu\nu}) + \tau_\mu (\delta_{\text{B}} C^\mu) \}.
\]

Here the sources \( \beta^{\mu\nu} \) and \( \tau_\mu \) couple to the BRST variations of \( h_{\mu\nu} \) and \( C^\mu \) respectively.

In Eq. (1), \( \Delta_k S_{\text{grav}}[h; \bar{g}] \) and \( \Delta_k S_{\text{gh}}[C, \bar{C}; \bar{g}] \) are the cutoff actions. These terms control the propagation of fields to have the momentum \( k < p < k_0 \). Here \( k_0 \) is a UV cutoff scale of the theory. Hence these are given by a quadratic form of \( h_{\mu\nu} \) and the ghost fields. The explicit forms of these terms are given by

\[
\Delta_k S_{\text{grav}}[h; \bar{g}] = \frac{1}{2} \kappa^2 \int d^d x \sqrt{\bar{g}} h_{\mu\nu} (R_{\text{grav}}^k [\bar{g}])^{\mu\rho\sigma} h_{\rho\sigma},
\]

\[
\Delta_k S_{\text{gh}}[C, \bar{C}; \bar{g}] = \sqrt{2} \int d^d x \sqrt{\bar{g}} \bar{C}_\mu R_{\text{gh}}^k [\bar{g}] C^\mu,
\]

where the cutoff operator \( R_{\text{grav}}^k [\bar{g}] \) and \( R_{\text{gh}}^k [\bar{g}] \) are defined by

\[
R_{\text{grav}}^k [\bar{g}] = (Z_{\text{grav}}^k)^{\mu\rho\sigma} k^2 R^{(0)} (-\bar{D}^2/k^2),
\]

\[
R_{\text{gh}}^k [\bar{g}] = Z_{\text{gh}}^k k^2 R^{(0)} (-\bar{D}^2/k^2).
\]

Here, \( (Z_{\text{grav}}^k)^{\mu\rho\sigma} \) and \( Z_{\text{gh}}^k \) is the renormalization factor of \( h_{\mu\nu} \) and the ghost field respectively. A convenient choice of the cutoff function \( R^{(0)}(-\bar{D}^2/k^2) \) is

\[
R^{(0)}(-\bar{D}^2/k^2) = \frac{1 + e^{-\bar{D}^2/k^2} - e^{-\bar{D}^2/k_0^2}}{e^{-\bar{D}^2/k_0^2} - e^{-\bar{D}^2/k^2}}.
\]

If we take the limit \( k_0 \to \infty \), we have

\[
R^{(0)}(-\bar{D}^2/k^2) = \frac{-\bar{D}^2/k^2}{e^{-\bar{D}^2/k^2} - 1}.
\]

Hence, the constraints of the cutoff function \( R^{(0)}(u) \) are given by

\[
\lim_{u \to 0} R^{(0)}(u) = 1, \quad \lim_{u \to \infty} R^{(0)}(u) = 0.
\]
Any functions satisfying Eq. (3) are applicable. Hence, the profile of the cutoff function is arbitrary. In subsection 4.2, the effect of this ambiguity to the UV NGFP will be considered.

The effective average action $\Gamma_k$ is defined by

$$
\Gamma_k[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}] = \int d^d x \sqrt{\bar{g}} \left\{ \left( \mu^\nu \bar{h}^\mu_\nu + \bar{\sigma}_\mu \xi^\mu + \sigma^\mu \bar{\xi}_\mu \right) - W_k[t, \sigma, \bar{\sigma}; \beta, \tau; \bar{g}] \right\} - \Delta_k S^{grav}[\bar{h}; \bar{g}] - \Delta_k S^{gh}[\xi; \bar{g}],
$$

(7)

Here, scale dependent classical fields are given by

$$
\bar{h}^\mu_\nu = \frac{1}{\sqrt{\bar{g}}} \left( \frac{\delta W_k}{\delta \mu^\nu} \right), \quad \xi^\mu = \frac{1}{\sqrt{\bar{g}}} \left( \frac{\delta W_k}{\delta \sigma}_\mu \right), \quad \bar{\xi}_\mu = \frac{1}{\sqrt{\bar{g}}} \left( \frac{\delta W_k}{\delta \bar{\sigma}}^\mu \right).
$$

In addition, the classical field corresponding to $\gamma^\mu_\nu$ is introduced as

$$
g^\mu_\nu(x) = \bar{g}^\mu_\nu(x) + \bar{h}^\mu_\nu(x).
$$

The effective average action has two boundary conditions. One is given by

$$
\lim_{k \to 0} \Gamma_k[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}] = \Gamma_{1PI}[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}],
$$

in the limit $k \to 0$. This is because, all quantum corrections are included in $\Gamma_{k=0}$ in this limit. Thus this is equivalent to the ordinary 1PI effective action. The other is given by

$$
\lim_{k \to \infty} \Gamma_k[g, \xi, \bar{\xi}; \beta, \tau; \bar{g}] = S_{grav}[g] + S_{k.f.}[g - \bar{g}; \bar{g}] + S_{F.P.}[g - \bar{g}, \xi, \bar{\xi}; \bar{g}]
$$

$$
- \int \! d^d x \sqrt{\bar{g}} \left\{ \beta^\mu_\nu (\delta B)^\mu_\nu + \tau_\mu (\delta B \xi^\mu) \right\},
$$

(8)

in the limit $k \to \infty$. Here, we denote $\bar{h}^\mu_\nu$ as $g^\mu_\nu - \bar{g}^\mu_\nu$. Equation (8) means that $\Gamma_k$ is coincide with the bare action in this limit, since there are no quantum corrections.

If Eq. (3) is differentiated with respect to $t = \ln k$, and Legendre transformed by Eq. (7), we obtain

$$
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)} + \kappa^2 R_k^{grav} \right)^{-1} \bar{h}^\mu_\nu \left( \partial_t \kappa^2 R_k^{grav} \right)^\mu_\nu \bar{\sigma} \right]
$$

$$
- \frac{1}{2} \text{Tr} \left\{ \left( \Gamma_k^{(2)} + \sqrt{2} R_k^{gh} \right)^{-1} \left( \Gamma_k^{(2)} + \sqrt{2} R_k^{gh} \right)^{-1} \left( \partial_{t} \sqrt{2} R_k^{gh} \right) \left( \partial_{t} \sqrt{2} R_k^{gh} \right) \right\}.
$$

(9)

This is the ERGE for pure QG. Here $\Gamma_k^{(2)}$ is the Hessian of $\Gamma_k$ with respect to the subscript.

Though Eq. (9) is non-perturbatively exact, the manifest BRST invariance is broken by the explicit IR cutoff. To see this case more detail, we consider the Ward-Takahashi (WT) identity. Now, the WT identity is given by

$$
0 = \langle \delta_B S_{c.s.} + \delta_B \Delta_k S^{grav} + \delta_B \Delta_k S^{gh} \rangle.
$$
If this is written in terms of the effective average action, we have

\[
\int d^d x \frac{1}{\sqrt{g}} \left\{ \frac{\delta \Gamma_k'}{\delta h_{\mu\nu}} \frac{\delta \Gamma_k'}{\delta \beta^{\mu\nu}} + \frac{\delta \Gamma_k'}{\delta \xi^\mu} \frac{\delta \Gamma_k'}{\delta \tau^\mu} \right\} = Y_k \left( R_k^{\text{grav}}[\bar{g}], R_k^{\text{gh}}[\bar{g}] \right),
\]

(10)

where \( \Gamma_k' = \Gamma_k - S_{\text{g.f.}} \) is introduced. In the usual field theories, the RHS of Eq. (10) equals to zero. In the present case, the existence of the cutoff action makes it proportional to cutoff operators. The RHS of Eq. (10) goes to zero in the limit \( k \to 0 \), because cutoff operators goes to zero in this limit. Hence, an usual field theory is recovered. However, \( Y_k \) does not disappear in the intermediate scale \( k \). Thus, the BRST symmetry is broken in this scale.

Now, to get the BRST invariant RG flows, we approximate the functional space. As a first step approximation, we neglect the evolution of the ghost action and the external source fields. From this approximation, the effective averaged action is expected as

\[
\Gamma_k[g, \xi; \beta, \tau; \bar{g}] = \tilde{\Gamma}_k[g] + \hat{\Gamma}_k[g, \bar{g}] + S_{\text{g.f.}}[g - \bar{g}; \bar{g}] + S_{\text{F.P.}}[g - \bar{g}, \xi; \bar{g}] - \int d^d x \sqrt{g} \left\{ \beta^{\mu\nu} (\delta_B \bar{h}_{\mu\nu}) + \tau^\mu (\delta_B \xi^\mu) \right\}.
\]

(11)

Here \( S_{\text{g.f.}} \) and \( S_{\text{F.P.}} \) have the same form as in the bare action. The coupling to the BRST variations also has the same form as in the bare action. The remaining term is decomposed into \( \tilde{\Gamma}_k[g] \) and \( \hat{\Gamma}_k[g, \bar{g}] \). Here, \( \hat{\Gamma}_k[g, \bar{g}] \) contains the deviations for \( g_{\mu\nu} \neq \bar{g}_{\mu\nu} \), and satisfies \( \hat{\Gamma}_k[g, \bar{g}] = 0 \). The approximated effective average action given by Eq. (11) satisfies the boundary condition of Eq. (8), if these terms satisfy

\[
\lim_{k \to \infty} \tilde{\Gamma}_k[g] = S_{\text{grav}}[g], \quad \lim_{k \to \infty} \hat{\Gamma}_k[g, \bar{g}] = 0.
\]

These conditions suggest that setting \( \hat{\Gamma}_k[g, \bar{g}] = 0 \) for all \( k \) is the candidate to get the BRST invariant RG flows. Substituting Eq. (11) into Eq. (10), we obtain

\[
\int d^d x L_\xi g_{\mu\nu} \frac{\delta \tilde{\Gamma}_k[g, \bar{g}]}{\delta g_{\mu\nu}(x)} = -Y_k \left( R_k^{\text{grav}}[\bar{g}], R_k^{\text{gh}}[\bar{g}] \right),
\]

(12)

where \( L_\xi \) means the Lie derivative with respect to \( \xi^\mu \). The RHS of Eq. (12) is regarded as the higher loop corrections if it is evaluated perturbatively (for details see [6]). Hence to neglect \( Y_k \) is acceptable in the first approximation. This is consistent with setting \( \hat{\Gamma}_k = 0 \) for all scales. These approximation means that the RG flows are projected on \( g_{\mu\nu} = \bar{g}_{\mu\nu} \) in all scales. In the background spaces the BRST invariance is preserved. Hence, the projected RG flows moving in the background space are regarded as the BRST invariant.

If Eq. (11) is inserted into Eq. (9), the approximated ERGE becomes

\[
\partial_t \Gamma_k[g; \bar{g}] = S_k^{\text{grav}} - S_k^{\text{gh}},
\]

(13)
where,
\[
\Gamma_k[g; \bar{g}] = \bar{\Gamma}_k[g] + \dot{\Gamma}[g; \bar{g}] + S_{k.f.}[g - \bar{g}; \bar{g}].
\] (14)

This has the boundary condition
\[
\lim_{k \to \infty} \Gamma_k[g; \bar{g}] = S_{grav}[ar{g}] + S_{gh}^s[g - \bar{g}; \bar{g}].
\]

In Eq. (13), \(S_{grav}^k\) and \(S_{gh}^k\) correspond to the gravitational sector and the ghost sector respectively, and are given by
\[
S_{grav}^k = \frac{1}{2} \text{Tr} \left[ \left( \kappa^{-2} \Gamma_k^{(2)}[g; \bar{g}] + R_{k}^{grav}[\bar{g}] \right)^{-1} (\partial_t R_{k}^{grav}[\bar{g}]) \right],
\] (15)
\[
S_{gh}^k = -\text{Tr} \left[ \left( -\mathcal{M}[g; \bar{g}] + R_{k}^{gh}[\bar{g}] \right)^{-1} (\partial_t R_{k}^{gh}[\bar{g}]) \right].
\] (16)

In Eq. (15), \(\Gamma_k^{(2)}[g; \bar{g}]\) represents the Hessian of \(\Gamma_k[g; \bar{g}]\) with respect to \(g_{\mu\nu}\) at fixed \(\bar{g}_{\mu\nu}\).

### 3 Einstein-Hilbert truncation

In below we consider the case \(d = 4\). Now, to make problems easier, we truncate the functional space of \(\Gamma_k[g; \bar{g}]\). The most naive truncation is to take the functional space as the same space of the Einstein-Hilbert action. Thus the bare action is given by
\[
S_{grav}^k[g] = \frac{1}{16\pi\bar{G}} \int d^4x \sqrt{g} \left\{ -R(g) + 2\bar{\lambda} \right\},
\]
where \(\bar{\lambda}\) is the bare cosmological constant. Now, we define the scale dependent couplings as
\[
\bar{G} \to G_k = Z_{N_k}^{-1} \bar{G}, \quad \bar{\lambda} \to \bar{\lambda}_k, \quad \alpha \to \alpha_k = Z_{N_k}^{-1} \alpha.
\]

Hence, from Eq. (14), \(\Gamma_k[g; \bar{g}]\) is expected as
\[
\Gamma_k[g; \bar{g}] = 2\kappa^2 Z_{N_k} \int d^4x \sqrt{\bar{g}} \left\{ -R(g) + 2\bar{\lambda}_k \right\}
\]
\[
\quad + \frac{\kappa^2}{\alpha} Z_{N_k} \int d^4x \sqrt{\bar{g}} g^{\mu\nu}(\mathcal{F}_{\mu}^{\alpha\beta} g_{\alpha\beta})(\mathcal{F}_{\nu}^{\rho\sigma} g_{\rho\sigma}).
\] (17)

Here \(Z_{N_k}\) is a renormalization factor. If Eq. (17) is differentiated with respect to \(t\) and projected on \(g_{\mu\nu} = \bar{g}_{\mu\nu}\), we have
\[
\partial_t \Gamma_k[g; \bar{g}] = 2\kappa^2 \int d^4x \sqrt{\bar{g}} \left[ -R(g) \partial_t Z_{N_k} + 2\partial_t (Z_{N_k}\bar{\lambda}_k) \right].
\] (18)

This is the LHS of Eq. (13). Though the differentiation with respect to \(t\) brings the term that is proportional to the ghost action, this term disappears on \(g_{\mu\nu} = \bar{g}_{\mu\nu}\) because \(\mathcal{F}_{\mu}^{\alpha\beta} g_{\alpha\beta}|_{g=\bar{g}} = 0\). Thus the gauge parameter is treated as a constant. This is a problem of the present formulation.
Next step is to get the RHS of Eq. (13). Firstly, we calculate $S_k^{\text{grav}}$ given by Eq. (15). Now, we can naively write Eq. (15) as

$$S_k^{\text{grav}} = \frac{1}{2} \partial_t \text{Tr} \ln \left( \kappa^{-2} \Gamma_k^{(2)}[\bar{g}] + R_k^{\text{grav}}[\bar{g}] \right) - \frac{1}{2} \text{Tr} \left[ \left( \kappa^{-2} \Gamma_k^{(2)}[\bar{g}] + R_k^{\text{grav}}[\bar{g}] \right)^{-1} \left( \kappa^{-2} \partial_t \Gamma_k^{(2)}[\bar{g}] \right) \right].$$

(19)

The FP action $\Gamma^*[\bar{g}; \bar{g}]$ satisfies $\partial_t \Gamma^*[\bar{g}; \bar{g}] = 0$. Hence, if we are interested in only the FP solution, we can neglect the second term in the RHS of Eq. (19). Thus, we have

$$S_k^{\text{grav}} = -\partial_t \ln I_k^{\text{grav}}[\bar{g}],$$

(20)

where,

$$I_k^{\text{grav}}[\bar{g}] = \int \mathcal{D} \bar{h}_{\mu\nu} \exp \left\{ -\Gamma_k^{\text{quad}}[\bar{g}; \bar{g}] - \Delta_k S_k^{\text{grav}}[\bar{g}; \bar{g}] \right\}.$$  

(21)

Here, $\Delta_k S_k^{\text{grav}}$ is given similarly to Eq. (2) except for the change of the field: $h_{\mu\nu} \to \bar{h}_{\mu\nu}$, and $\Gamma_k^{\text{quad}}[\bar{g}; \bar{g}]$ is defined by

$$\kappa^{-2} \Gamma_k[\bar{g} = \bar{g} + \bar{h}; \bar{g}] = \Gamma_k[\bar{g}] + O(\bar{h}) + \Gamma_k^{\text{quad}}[\bar{h}; \bar{g}] + O(\bar{h}^3).$$

The explicit form is given by

$$\Gamma_k^{\text{quad}}[\bar{h}; \bar{g}] = Z_{N_k} \int \mathcal{D} \bar{g} \mathcal{D} \bar{h}_{\mu\nu} \mathcal{D} \bar{g} \mathcal{D} \bar{h}_{\rho\sigma} \left[ -K_{\mu\nu}^{\rho\sigma} \bar{D}^2 + U_{\rho\sigma} \bar{h}_{\mu\nu} \right] - \left( 1 - \frac{1}{\alpha} \right) Z_{N_k} \int \mathcal{D} \bar{g} \mathcal{D} \bar{h}_{\mu\nu} \mathcal{D} \bar{g} \mathcal{D} \bar{h}_{\rho\sigma} \left( \mathcal{F}_\alpha^{\beta\gamma} \bar{h}_{\alpha\beta} \right) \left( \mathcal{F}_{\rho\sigma}^{\alpha\beta} \bar{h}_{\rho\sigma} \right),$$

(22)

where,

$$K_{\mu\nu}^{\rho\sigma} = \frac{1}{4} \delta^{\mu\nu}_{\rho\sigma} + \frac{1}{2} \delta^{\mu\nu}_{\rho\sigma},$$

$$U_{\mu\nu}^{\rho\sigma} = K_{\mu\nu}^{\rho\sigma} (\bar{R} - 2\lambda_k) + \frac{1}{2} (\bar{g}^{\mu\nu} \bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma} \bar{R}^{\mu\nu}) - \frac{1}{2} \left( \bar{R}^{\mu\nu} \bar{R}_{\rho\sigma} + \bar{R}^{\rho\sigma} \bar{R}_{\mu\nu} \right).$$

Hence, if the one-loop effective action $I_k^{\text{grav}}$ is calculated, we get $S_k^{\text{grav}}$ from Eq. (20). Now, to calculate the one-loop effective action, we decompose $\bar{h}_{\mu\nu}$ as

$$\bar{h}_{\mu\nu} = \bar{h}_{\mu\nu}^{\perp} + \left( \bar{D}_{\mu} \xi_{\nu}^{\perp} + \bar{D}_{\nu} \xi_{\mu}^{\perp} \right) + \left( \bar{D}_{\mu} \bar{D}_{\nu} - \frac{1}{4} \bar{g}_{\mu\nu} \bar{D}^2 \right) \sigma + \frac{1}{4} \bar{g}_{\mu\nu} \phi,$$

(23)

where $\phi = \bar{g}^{\mu\nu} \bar{h}_{\mu\nu}$, and $\xi_{\mu}^{\perp}$ satisfies $\bar{D}^{\mu} \xi_{\mu}^{\perp} = 0$ [8, 11]. In addition, $\hat{h}_{\mu\nu}$ satisfies $\bar{D}^{\mu} \hat{h}_{\mu\nu} = 0$ and $\bar{g}^{\mu\nu} \hat{h}_{\mu\nu}^{\perp} = 0$. The resulting measure and Jacobian are given by

$$\mathcal{D} \bar{h}_{\mu\nu} \to \mathcal{D} \bar{h}_{\mu\nu}^{\perp} \mathcal{D} \xi_{\mu}^{\perp} \mathcal{D} \sigma \left[ \text{det} J \right]^{1/2},$$

(24)

$$J = \Delta_1 \left( -\frac{1}{4} \bar{R} \right) \otimes \Delta_0 \left( -\frac{1}{3} \bar{R} \right) \otimes \Delta_0 (0).$$

(25)
Furthermore, we take the background as the maximally symmetric. In this background, the Riemann tensor \( \bar{R}_{\mu\nu\rho\sigma} \) and the Ricchi tensor \( \bar{R}_{\mu\nu} \) are given by

\[
\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{12} (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma} \bar{g}_{\nu\rho}) \bar{R},
\]

\[
\bar{R}_{\mu\nu} = \frac{1}{4} \bar{g}_{\mu\nu} \bar{R}.
\]

Here, the scalar curvature \( \bar{R} \) characterizes the space. Now, we introduce the constrained operators as

\[
\Delta_0(X) \phi = (-\bar{D}^2 + X) \phi,
\]

\[
\Delta_{1\mu
u}(X) \xi_{\nu}^\perp = (-\bar{D}^2_{\mu\nu} + \bar{g}_{\mu\nu} X) \xi_{\nu}^\perp,
\]

\[
\Delta_{2\alpha\beta}(X) \hat{h}_{\mu\nu}^\perp = (-\bar{D}^2_{\alpha\beta} + \delta_{\alpha}^\mu \delta_{\beta}^\nu X) \hat{h}_{\mu\nu}^\perp.
\]

From these manipulations, Eq. (22) becomes

\[
\Gamma_k^{\text{quad}}[\bar{h}; \bar{g}] = \int d^4 x \sqrt{\bar{g}} \left\{ \frac{1}{2} \bar{h}_{\mu\nu} \Delta_2 \left( -2\bar{\lambda}_k + \frac{2}{3} \bar{R} \right) \hat{h}_{\mu\nu}^\perp + \frac{1}{\alpha} \xi_{\mu}^\perp \Delta_1 \left( -2\alpha\bar{\lambda}_k + \frac{2\alpha - 1}{4} \bar{R} \right) \xi_{\mu}^\perp - \frac{3(\alpha - 3)}{16\alpha} \left[ \hat{\sigma} \Delta_0 \left( + \frac{4\alpha}{\alpha - 3} \bar{\lambda}_k - \frac{\alpha - 1}{\alpha - 3} \bar{R} \right) \hat{\sigma} 
+ \frac{2(\alpha - 1)}{\alpha - 3} \hat{\sigma} \sqrt{\Delta_0(0)} \sqrt{\Delta_0(-\bar{R}/3)} \phi 
+ \frac{3\alpha - 1}{3(\alpha - 3)} \phi \Delta_0 \left( - \frac{2\alpha}{3\alpha - 1} \bar{\lambda}_k \right) \phi \right] \right\}. 
\]

Here, we introduced

\[
\hat{\xi}_{\mu}^\perp = \sqrt{\Delta_1 \left( -\frac{1}{4} \bar{R} \right)} \xi_{\mu}^\perp, \quad \hat{\sigma} = \sqrt{\Delta_0(0)} \sqrt{\Delta_0 \left( -\frac{1}{3} \bar{R} \right)} \sigma.
\]

The changes of variables in Eq. (27) cancel the jacobian in Eqs. (24) and (25).

To get the one-loop effective action given by Eq. (21), we decompose \( \bar{h}_{\mu\nu} \) in \( \Delta_k S_{\text{grav}}^\text{grav}[\bar{h}; \bar{g}] \). Up to now, the form of \( Z_k^{\text{grav}} \) in Eq. (3) is not specified. Now, we consider \( Z_k^{\text{grav}} \) as the non-trivial projection operator which makes \( \Delta_k S_{\text{grav}}^\text{grav}[\bar{h}; \bar{g}] \) to

\[
\Delta_k S_{\text{grav}}^\text{grav}[\bar{h}; \bar{g}] = \int d^4 x \sqrt{\bar{g}} \left\{ \frac{1}{2} \bar{h}_{\mu\nu} \left( k^2 R(0) \right) \hat{h}_{\mu\nu}^\perp + \frac{1}{\alpha} \hat{\xi}_{\mu}^\perp \left( k^2 R(0) \right) \hat{\xi}_{\mu}^\perp - \frac{3(\alpha - 3)}{16\alpha} \left\{ \hat{\sigma} \left( k^2 R(0) \right) \hat{\sigma} + \frac{3\alpha - 1}{3(\alpha - 1)} \phi \left( k^2 R(0) \right) \phi 
+ \frac{2(\alpha - 1)}{\alpha - 3} \hat{\sigma} \sqrt{\Delta_0(k^2 R(0))} \sqrt{\Delta_0 \left( k^2 R(0) - \bar{R}/3 \right)} \phi \right\} \right\}. 
\]
Here, we introduced the shorthand notation $R^{(0)}(-D^2/k^2) \equiv R^{(0)} \equiv R$. Hence, from Eqs. (26) and (28), we can calculate $I_k^{\text{grav}}$ by the Gaussian integral. However, that result includes the additional zero-modes introduced by the decomposition in Eq. (23). Then to remove these modes [8, 11], we introduce unconstrained operators as

\[
\det \Delta_S (X) = \det \Delta_0 (X), \\
\det \Delta_V (X) = \det \Delta_1 (X) \det \Delta_0 \left( X - \frac{1}{4} \bar{R} \right), \\
\det \Delta_T (X) = \det \Delta_2 (X) \det \Delta_1 \left( X - \frac{5}{12} \bar{R} \right) \det \Delta_0 \left( X - \frac{2}{3} \bar{R} \right).
\]

Hence, we have

\[
I_k^{\text{grav}}[\bar{g}] = \left[ \det Z_{Nk} \Delta_T \left( k^2 R^{(0)} - 2 \bar{\lambda}_k + \frac{2}{3} \bar{R} \right) \right]^{-\frac{1}{2}}
\cdot \left[ \det Z_{Nk} \Delta_V \left( k^2 R^{(0)} - 2 \alpha \bar{\lambda}_k + \frac{2 \alpha - 1}{4} \bar{R} \right) \right]^{-\frac{1}{2}}
\cdot \left[ \det Z_{Nk} \Delta_S \left( k^2 R^{(0)} - 2 \bar{\lambda}_k \right) \right]^{-\frac{1}{2}}.
\]

Therefore, from Eq. (21),

\[
S_k^{\text{grav}} = \frac{1}{2} \text{Tr}_T \left[ \mathcal{N} \left( A + \frac{2}{3} \bar{R} \right)^{-1} \right] + \frac{1}{2} \text{Tr}_V \left[ \mathcal{N} \left( A_{\alpha} + \frac{2 \alpha - 1}{4} \bar{R} \right)^{-1} \right]
\cdot \frac{1}{2} \text{Tr}_S \left[ \mathcal{N} A^{-1} \right],
\]

where, $A$, $A_{\alpha}$ and $\mathcal{N}$ are given by

\[
A = -D^2 + k^2 R^{(0)}(-D^2/k^2) - 2 \bar{\lambda}_k, \\
A_{\alpha} = -D^2 + k^2 R^{(0)}(-D^2/k^2) - 2 \alpha \bar{\lambda}_k, \\
\mathcal{N} = \left( 1 - \frac{\bar{T}}{2} \right) k^2 R^{(0)}(-D^2/k^2) + D^2 R^{(0)}(-D^2/k^2).
\]

Here, the prime means the differentiation with respect to the argument. The anomalous dimension $\eta$ is defined by $\eta = -\partial_t \ln Z_{Nk}$. In Eq. (22), $g_{\mu\nu} = \bar{g}_{\mu\nu}$ is taken. In below, we omit the bars from the metric and the scalar curvature.

The remaining term of the RHS of Eq. (13) is $S_k^{\text{gh}}$. For the Faddeev-Popov ghost term, we do not take into account the renormalization of these field. Hence, $R_k^{\text{gh}} = k^2 R^{(0)}(-D^2/k^2)$. In the present background, $M = -D^2 - R/4$. Therefore,

\[
S_k^{\text{gh}} = -\text{Tr}_V \left[ \mathcal{N}_0 \left( A_0 - \frac{1}{4} \bar{R} \right)^{-1} \right].
\]
Here, $\mathcal{A}_0$ and $\mathcal{N}_0$ in Eq. (31) are defined similarly to $\mathcal{A}$ and $\mathcal{N}$ except for $\lambda_k = 0$ and $\eta = 0$. In below, we denote the RHS of Eq. (13) as $\mathcal{S}_R = \mathcal{S}_k^{\text{grav}} + \mathcal{S}_k^{\text{sh}}$.

Now to get the coefficients of $\sqrt{g}$ and $\sqrt{g}R$, we expand $\mathcal{S}_R$ in terms of the scalar curvature $R$,\[\mathcal{S}_R = \frac{1}{2} \text{Tr}_T \left[ \mathcal{N} \mathcal{A}^{-1} \right] + \frac{1}{2} \text{Tr}_V \left[ \mathcal{N} \mathcal{A}_0^{-1} \right] - \frac{1}{2} \text{Tr}_V \left[ \mathcal{N} \mathcal{A}^{-1} \right] \]
\[\quad + \frac{1}{2} \text{Tr}_S \left[ \mathcal{N} \mathcal{A}^{-1} \right] - \text{Tr}_V \left[ \mathcal{N}_0 \mathcal{A}_0^{-1} \right] \]
\[\quad - R \left\{ \frac{1}{3} \text{Tr}_T \left[ \mathcal{N} \mathcal{A}^{-2} \right] + \frac{2\alpha - 1}{8} \text{Tr}_V \left[ \mathcal{N} \mathcal{A}_0^{-2} \right] \right\} + O(R^2). \tag{31} \]

Furthermore, to calculate the traces in Eq. (31), we use the heat kernel expansion:
\[\text{Tr}_j \left[ W(-D^2) \right] = (4\pi)^{-2} \text{tr}_j (I) \left\{ Q_2[W] \int d^4x \sqrt{g} \right. \]
\[\quad \left. + \frac{1}{6} Q_4[W] \int d^4x \sqrt{g}R + O(R^2) \right\}, \tag{32} \]
where, $I$ is a unit matrix and $j = T, V, S$ mean the tensor, vector and scalar respectively. In Eq. (32), $\text{tr}_j (I)$ simply counts the number of independent degrees of freedom of these quantity:
\[\text{tr}_T (I) = 9, \quad \text{tr}_V (I) = 4, \quad \text{tr}_S (I) = 1. \]

In Eq. (32), $Q_i[W], \ (i = 1, 2)$ is the Mellin transform of $W$, and given by\[Q_0[W] = W(0), \]
\[Q_n[W] = \frac{1}{\Gamma(n)} \int_0^\infty dzz^{n-1}W(z), \ (n > 0). \]

Therefore, if we insert Eq. (32) into Eq. (31), and compare it with Eq. (18),\[\partial_t (Z_{Nk}\Lambda_k) = \frac{1}{4\pi^2} \frac{k^4}{(4\pi^2)^2} \left[ 6\Phi_2^1(-2\Lambda_k/k^2) + 4\Phi_2^1(-2\alpha\Lambda_k/k^2) \right. \]
\[\quad \left. - \eta \left\{ 3\Phi_2^1(-2\Lambda_k/k^2) + 3\Phi_2^1(-2\alpha\Lambda_k/k^2) \right\} \right], \tag{33} \]
\[\partial_t Z_{Nk} = -\frac{1}{12\pi^2} \frac{k^2}{(4\pi^2)^2} \left[ -15\Phi_2^1(-2\Lambda_k/k^2) - 3(2\alpha - 1)\Phi_2^1(-2\alpha\Lambda_k/k^2) \right. \]
\[\quad + \Phi_1^1(-2\Lambda_k/k^2) + 2\Phi_1^1(-2\alpha\Lambda_k/k^2) - 12\Phi_2^1(0) - 8\Phi_1^1(0) \]
\[\quad - \eta \left\{ -15\Phi_2^1(-2\Lambda_k/k^2) - 3(2\alpha - 1)\Phi_2^1(-2\alpha\Lambda_k/k^2) \right. \]
\[\quad + 3\Phi_1^1(-2\Lambda_k/k^2) + 3\Phi_1^1(-2\alpha\Lambda_k/k^2) \left. \right\}. \tag{34} \]
Here \( \Phi_n^p(w) \) and \( \tilde{\Phi}_n^p(w) \) are concerning with the integrals of the cutoff function, and defined by

\[
\Phi_n^p(w) = \frac{1}{\Gamma(n)} \int_0^\infty dx x^{n-1} \frac{R^{(0)}(x) - xR^{(0)'}(x)}{[x + R^{(0)}(x) + w]^p},
\]

\[
\tilde{\Phi}_n^p(w) = \frac{1}{\Gamma(n)} \int_0^\infty dx x^{n-1} \frac{R^{(0)}(x)}{[x + R^{(0)}(x) + w]^p}.
\]

Here, the cutoff function \( R^{(0)}(x) \) satisfies constraints given by Eq. (6). In [6], this is given by

\[
R^{(0)}(x) = \frac{x}{\exp(x) - 1}.
\]  

Now, we introduce the dimensionless Newton constant \( g_k \) and cosmological constant \( \lambda_k \),

\[
g_k = k^2 G_k = k^2 Z_{N_k}^{-1} \tilde{G}, \quad \lambda_k = k^{-2} \tilde{\lambda}_k.
\]

If Eqs. (33) and (34) are expressed in terms of dimensionless couplings, we obtain

\[
\beta_g = \partial_t g_k = (2 + \eta) g_k, \quad \beta_\lambda = \partial_t \lambda_k = -(2 - \eta) \lambda_k + g_k B_3(\lambda_k, \alpha).
\]  

In this case, the anomalous dimension is given by

\[
\eta = \frac{g_k B_1(\lambda_k, \alpha)}{1 - g_k B_2(\lambda_k, \alpha)}. \tag{38}
\]

Now, \( B_i(\lambda_k, \alpha), \) \( i = 1, 2, 3 \) in Eqs. (37) and (38) is defined by

\[
B_1(\lambda_k, \alpha) = \frac{1}{3\pi} \left[ 3\Phi_1(-2\lambda_k) + 2\Phi_1(-2\alpha \lambda_k) - 4\Phi_1^2(0) 
- 15\Phi_2^2(-2\lambda_k) - 3(2\alpha - 1)\Phi_2^2(-2\alpha \lambda_k) - 6\Phi_2^2(0) \right],
\]

\[
B_2(\lambda_k, \alpha) = -\frac{1}{6\pi} \left[ 3\tilde{\Phi}_1(-2\lambda_k) + 2\tilde{\Phi}_1(-2\alpha \lambda_k) 
- 15\tilde{\Phi}_2^2(-2\lambda_k) - 3(2\alpha - 1)\tilde{\Phi}_2^2(-2\alpha \lambda_k) \right],
\]

\[
B_3(\lambda_k, \alpha) = \frac{1}{2\pi} \left[ 6\Phi_1^3(-2\lambda_k) + 4\Phi_1^3(-2\alpha \lambda_k) - 8\Phi_2^3(0) 
- \eta \left\{ 3\tilde{\Phi}_1(-2\lambda_k) + 2\tilde{\Phi}_2(-2\alpha \lambda_k) \right\} \right].
\]

These results are same as that of [8]. If \( \alpha = 1 \), these reproduce the results of [6].

4 The UV NGFP of QG
4.1 Gauge dependence

In below, we ignore the dimensionless cosmological constant. This approximation is reliable when the dimensionless cosmological constant is much smaller than the IR cutoff scale: $\lambda_k \ll k$. This approximation is applicable if we are interested in the local structure of the Universe.

On the FP, the scale invariance is preserved. Hence, if we denote $g^*$ as the FP of the dimensionless Newton constant, this satisfies

$$0 = (2 + \eta^*) g^*. \quad (39)$$

From Eq. (39), it is recognized that the candidates of the FP are $g^* = 0$ and $\eta^* = -2$. Here the former is the GFP and exists independently of $\alpha$. The latter is the candidate of the NGFP. The condition $\eta^* = -2$ reads to

$$g^* = \frac{-2}{B_1(\alpha) - 2B_2(\alpha)}. \quad (40)$$

Here, $B_i(\alpha), \ (i = 1, 2)$ is given by

$$B_1(\alpha) = \frac{1}{3\pi} \left[ \Phi_1^1(0) - 6(\alpha + 3)\Phi_2^2(0) \right],$$

$$B_2(\alpha) = -\frac{1}{6\pi} \left[ 5\Phi_1^1(0) - 6(\alpha + 2)\Phi_2^2(0) \right].$$

Now, to calculate $\Phi_i^1(0)$ and $\Phi_i^2(0) \ (i = 1, 2)$, we use the cutoff function given by Eq. (35). Hence we have

$$\Phi_1^1(0) = \frac{\pi^2}{6}, \quad \Phi_2^2(0) = 1, \quad \Phi_1^1(0) = 1, \quad \Phi_2^2(0) = \frac{1}{2}.$$ 

Therefore, Eq. (40) becomes

$$g^* = \frac{2\pi}{3} \left( \alpha - \frac{\pi^2 - 114}{54} \right)^{-1}. \quad (41)$$

As immediately recognize, Eq. (41) has the singularity at $\alpha_{\text{sing}} = (\pi^2 - 114)/54$. The existence of a singularity is not the problem, since this point is not the GFP. In the range $\alpha_{\text{sing}} < \alpha$, $g^*$ has a positive value except for the limit $\alpha \to \infty$. On the other hand, in the range $\alpha < \alpha_{\text{sing}}$, $g^*$ has a negative value except for the limit $\alpha \to -\infty$. In the limit $\alpha \to \pm \infty$, the NGFP merges to the GFP. This problem will be discussed in Sec. 5. The behavior of Eq. (41) is shown in Fig. 1 (a). In this figure, a vertical long dashed line corresponds to the singular point. A solid line is the positive NGFPs, and a dashed line is the negative NGFPs. In below, we consider only the positive NGFP. Hence, the shadowed region in this figure is ignored.
Fig. 1: The gauge dependence of the UV NGFP of the dimensionless Newton constant in $d = 4$ (a), and the RG flows of it in $\alpha = 1$ (b).

The typical RG flows are shown in Fig. 1 (b), if $\alpha$ is fixed to unity. Under the present approximation given by Eq. (20), only the FPs are reliable. However, when $\alpha = 1$, it is possible to study the behavior of the RG flows [5, 6]. In this figure, the horizontal bold solid line, $g^* = 0.7152$, corresponds to the UV NGFP (the circle in Fig. 1 (a)). This line separates the phase space into two regions; the strong coupling phase and the weak coupling phase.

### 4.2 Cutoff function dependence

As mentioned previously, constraints for the cutoff function are given by Eq. (33). Hence, any functions satisfying these conditions are applicable. Now, we slightly modify Eq. (35) as

$$R^{(0)}_1(x, s) = \frac{sx}{\exp(sx) - 1}, \quad s > 0.$$  

Here, $s$ parameterizes the profile of the cutoff function. As same as the previous subsection, the candidate of the UV NGFP is given by

$$g^* = \frac{-2}{B_1(\alpha, s) - 2B_2(\alpha, s)}.$$  

Here, $B_i(\alpha, s), \ (i = 1, 2)$ is defined similarly to $B_i(\alpha)$, except for $\Phi^i_1(0, s)$ and $\tilde{\Phi}^i_1(0, s), \ (i = 1, 2)$ depending on $s$.

If we numerically calculate the integral of $\Phi_1^i(0, s)$ and $\tilde{\Phi}_1^i(0, s)$ and insert these into Eq. (42), we get Fig. 2. In this figure, the surface represents the position of the UV NGFPs. The region
above the surface is the strong coupling phase, and that below the surface is the weak coupling phase. From this figure, it is confirmed that the UV NGFPs exist in a global range of $\alpha$ and $s$. The gauge dependence is same as the previous subsection. For the cutoff function dependence, it is recognized that the UV NGFPs merge to the GFPs in the limit $s \to 0$. The reason is expressed as follows. In the limit $s \to 0$, the integration of the cutoff function diverges, therefore, $B_1(\alpha, 0)$ function diverges. Hence, the denominator in Eq. (42) goes to infinity, and $g^*$ goes to zero. However, there is no IR cutoff in the limit $s \to 0$. Hence, this limit is out of the applicability of the ERGE. If we take another type of the cutoff functions such as $\exp(-sx)$, same structure is observed. Therefore, the cutoff function dependence does not cause the disappearance of the UV NGFP and the change of the phase structure.

5 Summary and discussion

In this article, we consider the gauge and cutoff function dependence of the UV NGFP. For the cutoff function dependence, the UV NGFP strongly depends on the profile of the cutoff function. However, this dependence does not change the phase structure of pure QG. Though there are many cutoff functions, the phase structure will not be changed by this dependence.

For the gauge dependence, the UV NGFP exists in a global range of $\alpha$ except for $\alpha = \pm \infty$. This gauge, $\alpha = \pm \infty$, is a bad gauge in this formulation. The disappearance of the UV NGFP
for this gauge is a serious problem. This is because, the phase structure of pure QG is changed in this gauge. This problem may be due to the treatment of the constant gauge parameter. Hence, if we will be able to treat the gauge parameter as the running gauge parameter, this problem may disappear.

In this article, only the operators that are invariant under general coordinate transformations are considered, because the gauge symmetry is preserved by the projection on \( g_{\mu\nu} = \bar{g}_{\mu\nu} \). However, if we improve this approximation to treat the running gauge parameter, the gauge symmetry is not manifestly maintained. Thus, the operators that can not preserve general coordinate transformations must be included in the functional space \([12]\).

The other improvement is to formulate the ERGE without the gauge fixing. Recently, for pure \( SU(N) \) gauge theory, Tim R. Morris proposed the formulation of the gauge invariant ERGE \([13]\). This formulation does not need the gauge fixing. Hence, if we can formulate the ERGE for QG in this formulation and show the existence of the UV NGFP, QG becomes an asymptotically safe theory and (non-perturbatively) renormalizable \([14]\).

In this article, the Einstein-Hilbert truncation is applied. However, for these two improvements, to accurately study the existence of the UV NGFP and the phase structure, we need the extension of the functional space. One possibility is to include the \( R^2 \)-terms. The other is to use the method so called frame transformations \([15]\). By this transformations, the higher derivative gravity is reduced to the Einstein gravity with the auxiliary tensor matter field. Hence, if we study this reduced theory, the phase structure of the higher derivative gravity will be clarified.

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