REES ALGEBRAS AND RESOLUTION OF SINGULARITIES

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Abstract. Embedded principalization of ideals in smooth schemes, also known as Log-resolutions of ideals, play a central role in algebraic geometry. If two sheaves of ideals, say \( I_1 \) and \( I_2 \), over a smooth scheme \( V \) have the same integral closure, it is well known that Log-resolution of one of them induces a Log-resolution of the other. On the other hand, in case \( V \) is smooth over a field of characteristic zero, an algorithm of desingularization provides, for each sheaf of ideals, a unique Log-resolution.

In this paper we show that algorithms of desingularization define the same Log-resolution for two ideals having the same integral closure. We prove this result here by using the form of induction introduced by Włodarczyk.

We extend the notion of Log-resolution of ideals over a smooth scheme \( V \), to that of Rees algebras over \( V \); and then we show that two Rees algebras with the same integral closure undergo the same constructive resolution. The key point is the interplay of integral closure with differential operators.

Contents

Introduction. 1
1. Monoidal transformations and Hironaka’s topology. 2
2. On Hironaka pairs and Rees algebras. 5
3. On differential Rees algebras and Kollár’s tuned ideals. 7
4. On differential Rees algebras and monoidal transformations. 9
5. Idealistic exponents versus basic objects. 11
6. Functions on pairs and simple Rees algebras 12
7. Reduction to the simple case 14
References 18

Introduction.

Differential operators on smooth schemes have played a central role in the study of embedded desingularization.

Date: August 2006.
Key words and phrases. Resolution of singularities. Desingularization.
2000 Mathematics subject classification. 14E15.
J. Giraud provides an alternative approach to the form of induction used by Hironaka in his Desingularization Theorem (over fields of characteristic zero). In doing so, Giraud introduced technics based on differential operators ([5], [6]). This result was important for the development of algorithms of desingularization in the late 80's (i.e. for constructive proofs of Hironaka's theorem).

Differential operators appeared in the work of J. Włodarczyk ([20]), and also on the notes of J. Kollár ([13]); where algorithms of resolution are developed.

The notions of Rees algebras over smooth schemes, and that of Rees algebras closed by higher order differentials, already appear in Hironaka's study on infinitely near points ([10]; [11]), and more recently in Kawanoue's work in [12].

A Log-resolution, or embedded principalization, of an ideal $I$ on a smooth scheme $V$, is a proper birational morphism of smooth schemes, say $V' \to V$, so that the total transform of $I$ is an invertible ideals in $V'$ supported on smooth hypersurfaces having only normal crossings. When $V$ is smooth over a field of characteristic zero there are algorithms that provide a Log-resolution of an ideal $I$. We shall make use of Rees algebras in proving that two ideals with the same integral closure undergo the same algorithmic Log-resolution (7.18).

The paper is organized so as to motivate the extension of Log-resolution theorems of ideals over fields of characteristic zero, to the case Rees algebras, this is done in Sections 1 and 2. In Sections 3 and 4, the reader is introduced to the fascinating relation of differential operators acting on Rees algebras, with the notion of integral closure of these algebras. These first 4 sections are included for self-containment. We refer to [18], or [19], for details.

In Section 5 we discuss some natural equivalence relation on Rees algebras when it comes to desingularization. Finally, in sections 6 and 7 we discuss the main results.

In this paper we always consider smooth schemes over fields of characteristic zero, however the extension of resolution theorems to Rees algebras, treated in this work, is also motivated by recent development of invariants over arbitrary fields.

We refer to [17] where a link of differential operators with elimination theory is presented. In that paper elimination of one variable is formulated in terms of Rees algebras. Over fields of characteristic zero this elimination recovers Hironaka's form of induction in desingularization theorems. However new invariants arise from this form of elimination, defined entirely in terms of Rees algebras, over fields of positive characteristic.

1. Monoidal transformations and Hironaka’s topology.

1.1. Fix a smooth scheme $V$ over a field $k$, an ideal $J \subset \mathcal{O}_V$, and a positive integer $b$. Hironaka attaches to these data, say $(J, b)$, a Zariski closed set in $V$, say

$$\text{Sing}(J, b) := \{x \in V / \nu_x(J_x) \geq b\}$$

where $\nu_x(J_x)$ denotes the order of $J$ at the local regular ring $\mathcal{O}_{V,x}$.

Given $(J_1, b_1)$ and $(J_2, b_2)$, then

$$\text{Sing}(J_1, b_1) \cap \text{Sing}(J_2, b_2) = \text{Sing}(K, c)$$

where $K = J_1^{b_2} + J_2^{b_1}$, and $c = b_1 \cdot b_2$. Set formally $(J_1, b_1) \odot (J_2, b_2) = (K, c)$. 

There is also a notion of permissible transformation on these data \((J,b)\). Let \(Y\) be a smooth subscheme in \(V\), included in the closed \(\text{Sing}(J,b)\), and let

\[
V \xleftarrow{\pi} V' \\
\cup \quad \cup \\
Y \quad H' = \pi^{-1}(Y),
\]

be the blow up of \(V\) at a smooth sub-scheme \(Y\). Note that

\[
J\mathcal{O}_{V'} = I(H')^b J',
\]

where \(I(H')\) is the sheaf of functions vanishing along the exceptional hypersurface \(H'\). We call \((J',b)\) the transform of \((J,b)\) by the permissible monoidal transformation.

If \(\pi\) is permissible for both \((J_1, b_1)\) and \((J_2, b_2)\), then it is permissible for \((K, c)\). Moreover, if \((J'_1, b_1), (J'_2, b_2)\), and \((K', c)\) denote the transforms, then \((J'_1, b_1) \otimes (J'_2, b_2) = (K', c)\).

1.2. We will consider \(\mathbb{N}\)-graded algebras. Fix a variable \(W\) and define a Rees algebra over \(V\) to be a graded noetherian subring of \(\mathcal{O}_V[W]\), say:

\[
\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n,
\]

where \(I_0 = \mathcal{O}_V\) and each \(I_n\) is a sheaf of ideals. By assumption, at every affine open set \(U \subset V\) there is a finite set

\[
\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\},
\]

\(n_i \geq 1\) and \(f_i \in \mathcal{O}_V(U)\), so that the restriction of \(\mathcal{G}\) to \(U\) is

\[
\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}] (\subset \mathcal{O}_V(U)[W]).
\]

To a Rees algebra \(\mathcal{G}\) we attach a closed set:

\[
\text{Sing}(\mathcal{G}) := \{x \in V/\nu_x(I_n) \geq n, \text{ for every } n \geq 1\},
\]

where \(\nu_x(I_n)\) denotes the order of the ideal \(I_n\) at the local regular ring \(\mathcal{O}_{V,x}\).

**Remark 1.3.** Rees algebras are related to Rees rings of ideals. A Rees algebra is a Rees ring if, given any affine open set \(U \subset V\), and \(\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}\) as above, all degrees \(n_i\) are one. In such case it is the Rees ring of the ideal \(I = \langle f_1, \ldots, f_s \rangle\). The integral closure of a Rees ring of an ideal is no longer a Rees ring of another ideal, however it is within the class of rings we consider here: the integral closure of a Rees ring is a Rees algebra.

In general Rees algebras are, in some sense, integral over Rees rings. In fact, if \(N\) is a positive integer divisible by all \(n_i\), it is easy to check that

\[
\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}] = \bigoplus_{n \geq 0} I_n W^n (\subset \mathcal{O}_V(U)[W]),
\]

is integral over the Rees sub-ring \(\mathcal{O}_V(U)[I_N W^N](\subset \mathcal{O}_V(U)[W^N])\). In fact, \(\mathcal{O}_V(U)[I_N W^N] \subset \mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}]\), and \((f_i W^{n_i})^{N/n_i} \in \mathcal{O}_V(U)[I_N W^N]\).
Proposition 1.4. Given an affine open $U \subset V$, and $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$ as above,

$$\text{Sing}(\mathcal{G}) \cap U = \bigcap_{1 \leq i \leq s} \{x \in U \mid \nu_x(f_i) \geq n_i\}.$$ 

Proof. It is clear that $\nu_x(f_i) \geq n_i$ for $x \in \text{Sing}(\mathcal{G})$, $0 \leq i \leq s$. So

$$\text{Sing}(\mathcal{G}) \cap U \subset \bigcap_{1 \leq i \leq s} \{x \in U \mid \nu_x(f_i) \geq n_i\}.$$ 

On the other hand, for every index $N \geq 1$, $I_N(U)W^N$ is generated by elements of the form $G_N(f_1 W^{n_1}, \ldots, f_s W^{n_s})$, where $G_N(Y_1, \ldots, Y_s) \in \mathcal{O}_U[Y_1, \ldots, Y_s]$ is weighted homogeneous of degree $N$, provided each $Y_j$ has weight $n_j$. The reverse inclusion is now clear. 

1.5. A monoidal transformation (1.1.1) is said to be permissible for $\mathcal{G}$ if $Y \subset \text{Sing}(\mathcal{G})$. In such case, for each index $n \geq 1$, there is a sheaf of ideals, say $I_n \subset \mathcal{O}_{V'}$, so that

$$I_n \mathcal{O}_{V'} = I(H')^n I_n.'$$ 

We define the total transform of $\mathcal{G}$ to be $\bigoplus_{n \geq 0} I_n \mathcal{O}_{V'} W^n$. On the other hand we define weighted transform of $\mathcal{G}$ as:

$$\mathcal{G}' = \bigoplus_{n \geq 0} I_n' W^n;$$

which is a Rees algebra over $V'$ (see 1.6).

Let $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ be a Rees algebra on $V$, and set $V \xleftarrow{\pi} V'$ a permissible transformation of $\mathcal{G}$. Let $U \subset V$ be affine open set, and $\mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$ be such that the restriction of $\mathcal{G}$ to $U$ is $\mathcal{O}_V(U)[f_1 W^{n_1}, \ldots, f_s W^{n_s}](\subset \mathcal{O}_V(U)[W])$. Note that the total transform of $\mathcal{G}$, restricted to the open set $\pi^{-1}(U)(\subset V')$, is also generated by $\{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}(\subset \mathcal{O}_{V'}(\pi^{-1}(U))[W])$.

Proposition 1.6. With the setting as above, there is an open covering of $\pi^{-1}(U)$ by affine sets $U^{(\ell)}$, so that:

1. $\langle f_i \rangle = I(H' \cap U^{(\ell)})^{n_i} \langle f_i^{\ell} \rangle$ for suitable $f_i^{\ell} \in \mathcal{O}_{V'}(U^{(\ell)}).$

2. The restriction of the weighted transform, say $\mathcal{G}'$, to each open set $U^{(\ell)}$ is

$$\mathcal{O}_{V'}(U^{(\ell)})[f_1^{\ell} W^{n_1}, \ldots, f_s^{\ell} W^{n_s}](\subset \mathcal{O}_{V'}(U^{(\ell)})[W]).$$

Proof. (1) Follows from proposition 1.4, since every $f_i$ has order at least $n_i$ along the center $Y$. For (2) argue as in the proof of Proposition 1.4, by using the fact that each ideal $I_N$ is generated by weighted homogeneous polynomials on the element of $\mathcal{F}$. 

Given two Rees algebras over $V$, say $\mathcal{G}_1 = \bigoplus_{n \geq 0} I_n W^n$ and $\mathcal{G}_2 = \bigoplus_{n \geq 0} J_n W^n$, set $K_n = I_n + J_n$ in $\mathcal{O}_V$, and define:

$$\mathcal{G}_1 \odot \mathcal{G}_2 = \bigoplus_{n \geq 0} K_n W^n,$$

as the subalgebra of $\mathcal{O}_V[W]$ generated by $\{K_n W^n, n \geq 0\}$. One can check that:
(1) \( \text{Sing}(G_1 \circ G_2) = \text{Sing}(G_1) \cap \text{Sing}(G_2) \). In particular, if \( \pi \) in (1.1.1) is permissible for \( G_1 \circ G_2 \), it is also permissible for \( G_1 \) and \( G_2 \).

(2) Set \( \pi \) as in (1), and let \( (G_1 \circ G_2)' \), \( G'_1 \), and \( G'_2 \) denote the transforms at \( V' \). Then:
\[
(G_1 \circ G_2)' = G'_1 \circ G'_2.
\]

2. On Hironaka pairs and Rees algebras.

Recall that two ideals, say \( I \) and \( J \), in a normal domain \( R \) have the same integral closure if they are equal for every extension to a valuation ring (i.e. if \( IS = JS \) for every ring homomorphism \( R \to S \) on a valuation ring \( S \)).

Hironaka considers the following equivalence on pairs \( (J, b) \) over a smooth scheme \( V \).

**Definition 2.1.** The pairs \( (J_1, b_1) \) and \( (J_2, b_2) \) are idealistic equivalent on \( V \) if \( J_{1b_2} \) and \( J_{2b_1} \) have the same integral closure.

Among Rees algebras the equivalence relation, also defined in terms of integral closure, is:

**Definition 2.2.** We say that two Rees algebras over \( V \), say \( G_1 = \bigoplus_{n \geq 0} I_n W^n \) and \( G_2 = \bigoplus_{n \geq 0} J_n W^n \), are integrally equivalent, if both have the same integral closure in \( O_V[W] \).

In general we want to identify two Rees algebras if they have the same integral closure. This notion will be revisited in section 5, where it will be linked with a weaker equivalence relation.

2.3. We assign to a pair \( (J, b) \) over a smooth scheme \( V \) the Rees algebra, say:
\[
G_{(J,b)} = O_V[JW^b].
\]

Note that \( G_{(J,b)} \) is a Rees ring of an ideal in \( O_V[W^b] \), but we can consider it as a graded subalgebra in \( O_V[W] \). Remark 1.3 shows that every Rees algebra is, in this sense, integrally equivalent to the Rees ring attached to a pair. In fact if \( G = \bigoplus_{n \geq 0} I_n W^n \), then it has the same integral closure as \( G_{(IN,N)} \) for a suitable \( N \).

2.4. A key point in our development is to attach invariants or geometric objects to Rees algebras. This will be always be done subject to the following two requirements:

(1) Every construction or invariant attached to a Rees algebra will be the same for two integrally equivalent Rees algebras.

(2) To all construction and invariants we present for Rees algebras, there will be a similar one on the class of idealistic pairs.

For example the operator \( \circ \) fulfills our requirement:

**Proposition 2.5.** Set \( G_1 = G_{(J_1,b_1)} \) and \( G_2 = G_{(J_2,b_2)} \) (i.e. the Rees algebras corresponding to Hironaka’s pairs \( (J_1, b_1) \) and \( (J_2, b_2) \)), then \( G_1 \circ G_2 \) is equivalent to the Rees algebra assigned to \( (J_1, b_1) \circ (J_2, b_2) \). Furthermore, this relation is preserved by transformations.

**Proof.** Fix an affine open set \( U \) in \( V \), \( \{f_1, \ldots, f_s\} \in O_V(U) \) generators of \( J_1(U) \), and \( \{g_1, \ldots, g_r\} \in O_V(U) \) generators of \( J_2(U) \). Then:
In 1.3 we have shown that for infinitely many suitable choices of $N$

Given a pair $(J, b)$ Rees equivalent (2.1). (1) and (2) show that $G$ associated Rees algebras $J$

In the particular case in which the Rees algebra is the Rees ring of an ideal, it turns out that

Proposition 2.9. The next result follows from the previous discussion.

As all algebras considered here are finitely generated over $O$

Proposition 2.6. Let $G_{ij}, i, j = 1, 2$, be Rees algebras such that $G_{ij}$ and $G_{ji}$ are integrally equivalent, $j = 1, 2$. Then the Rees algebras $G_{11} \circ G_{12}$ and $G_{21} \circ G_{22}$ are integrally equivalent.

Definition 2.7. Let $G = \bigoplus_n I_nW^n$ be a Rees algebra. For every $m \in \mathbb{Z}$, $m > 0$ we set

$$(2.7.1) \quad V^{(m)}(\bigoplus_n I_nW^n) = \bigoplus_{n \geq 0} I_{mn}W^{mn} (\subset O_V[W^m])$$

2.8. For every $f_nW^n \in G$, $(f_nW^n)m \in V^{(m)}(G)$; which shows that $V^{(m)}(G) \subset G$ is an integral extension for all $m$. So that $G$ and $V^{(m)}(G)$ are integrally equivalent (2.2).

The following properties hold for $V^{(m)}$:

1. If $G_1$ and $G_2$ are integrally equivalent Rees algebras then $V^{(m)}(G_1)$ and $V^{(m)}(G_2)$ are integrally equivalent (i.e. $V^{(m)}$ is compatible with integral closure (2.4)).

2. If $G = G_{(J, b)}$ then $V^{(m)}(G)$ and $G_{(J^m, bm)}$ are integrally equivalent.

In fact, if $\bigoplus_n I_nW^n \subset \bigoplus_n J_nW^n$ is the extension defined by taking integral closures, then

$$V^{(m)}(\bigoplus_n J_nW^n) = \text{the integral closure of } V^{(m)}(\bigoplus_n I_nW^n) \text{ in } O_V[W^m].$$

In the particular case in which the Rees algebra is the Rees ring of an ideal, it turns out that $J_n$ is the integral closure of $I_n$. In general, each $J_n$ contains the integral closure of $I_n$.

In 1.3 we have shown that for infinitely many suitable choices of $N$: $V^{(N)}(G)$ is integral over the Rees ring $O_V[I_NW^N]$; namely, that $G_{(I_N, N)} \subset G$ is a (finite) integral extension.

Given a pair $(J, b)$ and a positive integer $m$, the pairs $(J, b)$ and $(J^m, bm)$ are idealistic equivalent (2.1). (1) and (2) show that $G_{(J, b)}$ and $G_{(J^m, bm)}$ are integrally equivalent (2.2).

As all algebras considered here are finitely generated over $O_V$, for $N$ suitably big: $V^{(N)}(G) = O_V[I_NW^N]$.

The next result follows from the previous discussion.

Proposition 2.9. Two pairs $(J_1, b_1)$ and $(J_2, b_2)$ are idealistic equivalent if and only if the associated Rees algebras $G_1 = G_{(J_1, b_1)}$ and $G_2 = G_{(J_2, b_2)}$ are integrally equivalent.
Proof. Note that the following are equivalent:

- The pairs \((J_1, b_1)\) and \((J_2, b_2)\) are idealistic equivalent.
- The ideals \(J_{1}^{\mathfrak{n}}\) and \(J_{2}^{\mathfrak{n}}\) have the same integral closure.
- The Rees algebras \(V^{(b_2)}(G_1)\) and \(V^{(b_1)}(G_2)\) have the same integral closure.
- The Rees algebras \(G_1\) and \(G_2\) have the same integral closure.

\[\square\]

3. ON DIFFERENTIAL REES ALGEBRAS AND KOLÁR’S TUNED IDEALS.

Here \(V\) is smooth over a field \(k\), so for each non-negative integer \(r\) there is a locally free sheaf of differential operators of order \(r\), say \(\text{Diff}_k^{(r)}\).

**Definition 3.1.** We say that a Rees algebra \(G = \bigoplus I_n W^n\) is a differential Rees algebra, or simply a Diff-algebra, relative to the field \(k\), if:

- \((i)\): \(I_n \supset I_{n+1}\).
- \((ii)\): There is open covering of \(V\) by affine open sets \(\{U_i\}\), and for every \(D \in \text{Diff}^{(r)}(U_i)\), and \(h \in I_n(U_i)\), then \(D(h) \in I_{n-r}(U_i)\) provided \(n \geq r\).

Given a sheaf of ideals \(I \subset \mathcal{O}_V\) there is a natural definition of an extension, say \(\text{Diff}^{(r)}(I)\) (see Introduction). Note that \((ii)\) can be reformulated by

\((ii)^{'})\) \(\text{Diff}^{(r)}(I_n) \subset I_{n-r}\) for each \(n\), and \(0 \leq r \leq n\).

Diff-algebras are called differential structures in [19].

**Remark 3.2.** As we will view Rees algebras up to integral closure, it is not hard to check that condition \((i)\) can be imposed for Rees algebras. In fact, given \(G = \bigoplus I_n W^n\), we define \(G^\circ = \bigoplus I_n W^n\) by setting

\[
I_n^\circ = \sum_{r \geq n} I_r.
\]

If \(G = \bigoplus I_n W^n\) is generated by \(\mathcal{F} = \{g_{n_i} W^{m_i}, 1 \leq i \leq m, n_i > 0\}\). Namely, if:

\[
\bigoplus_{n \geq 0} I_n W^n = \mathcal{O}_V[\{g_{n_i} W^{m_i}\}_{i=1}^m],
\]

then \(G^\circ = \bigoplus I_n^\circ W^n\) is generated by the finite set \(\{g_{n_i} W^{m_i}, 1 \leq i \leq m, 1 \leq n_i' \leq n_i\}\). Note that \(I_n^\circ \supset I_{n+1}^\circ\), and that \(\bigoplus I_n W^n \subset \bigoplus I_n^\circ W^n\) is a finite extension. In fact, it suffices to check that given an element \(g \in I_n\), then \(g W^{n-1}\) is integral over \(\bigoplus I_n W^n\). One can check that

\[
g \in I_n \Rightarrow g^{n-1} \in I_{n(n-1)} \Rightarrow g^n \in I_{n(n-1)};
\]

so \(g \cdot W^{n-1}\) fulfills the equation \(Z^n - (g^n \cdot W^{n(n-1)}) = 0\).

**3.3.** Fix a closed point \(x \in V\), and a regular system of parameters \(\{x_1, \ldots, x_d\}\) at \(\mathcal{O}_{V,x}\). The residue field, say \(k'\) is a finite extension of \(k\), and the completion \(\hat{\mathcal{O}}_{V,x} = k'[\{x_1, \ldots, x_d\}]\).
The Taylor development is the continuous $k'$-linear ring homomorphism:

$$\text{Tay} : k'[[x_1, \ldots, x_d]] \rightarrow k'[[[x_1, \ldots, x_d, T_1, \ldots, T_d]]$$

that maps $x_i$ to $x_i + T_i$, $1 \leq i \leq d$. So for $f \in k'[[[x_1, \ldots, x_d]]$, $\text{Tay}(f(x)) = \sum_{\alpha \in \mathbb{N}^d} g_\alpha T^\alpha$, with $g_\alpha \in k'[[[x_1, \ldots, x_d]]$. Define, for each $\alpha \in \mathbb{N}^d$, $\Delta^\alpha$.

$$\Delta^\alpha(\mathcal{O}_{V,x}) \subset \mathcal{O}_{V,x},$$

and that $\{\Delta^\alpha, \alpha \in \mathbb{N}^d, 0 \leq |\alpha| \leq r\}$ generate the $\mathcal{O}_{V,x}$-module $\text{Diff}^{(r)}(\mathcal{O}_{V,x})$ (i.e. generate $\text{Diff}^{(r)}$ locally at $x$).

We finally introduce an operator on Rees algebras, which parallels Giraud’s extensions of ideals via differential operators.

**Theorem 3.4.** For every Rees algebra $G$ over a smooth scheme $V$, there is a Diff-algebra, say $G(G)$, such that:

(i): $G \subset G(G)$.

(ii): If $G \subset G'$ and $G'$ is a Diff-algebra, then $G(G) \subset G'$.

Furthermore, if $x \in V$ is a closed point, and $\{x_1, \ldots, x_d\}$ is a regular system of parameters at $\mathcal{O}_{V,x}$, and $G$ is locally generated by

$$\mathcal{F} = \{g_i w^{n_i}, n_i > 0, 1 \leq i \leq m\},$$

then

$$(3.4.1) \quad \mathcal{F}' = \{\Delta^\alpha(g_i) w^{n_i - \alpha} | g_i w^{n_i} \in \mathcal{F}, \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d, \ 0 \leq |\alpha| < n'_i \leq n_i\}$$

generates $G(G)$ locally at $x$.

**Remark 3.5.** The local description in the Theorem shows that $\text{Sing}(G) = \text{Sing}(G(G))$.

In fact, as $G \subset G(G)$, it is clear that $\text{Sing}(G) \supset \text{Sing}(G(G))$. For the converse note that if $\nu_x(g_{n_i}) \geq n_i$, then $\Delta^\alpha(g_{n_i})$ has order at least $n_i - |\alpha|$ at the local ring $\mathcal{O}_{V,x}$.

3.6. In general $G \subset G(G)$, and equality holds if $G$ is already a Diff-algebra.

Let $G = \bigoplus_{n \geq 0} I_n W^n$ be a Diff-algebra, in particular it is integral over a Rees subring, say $\mathcal{O}_V[I_N W^N]$ for suitable $N$ (see 1.3). These ideals $I_N$ are called *tuned ideals* in [13], page 45.

The previous Theorem defines an operator $G$ that extends Rees algebras into Diff-algebras. Another natural operator we have considered on Rees algebras is that defined by taking normalization. The next Theorem relates both notions of extensions.

**Theorem 3.7.** [18, Th 6.12] Let $G_1$ and $G_2$ be integrally equivalent Rees algebras on a smooth scheme $V$, then $G(G_1)$ and $G(G_2)$ are also integrally equivalent (2.2).
4. ON DIFFERENTIAL REES ALGEBRAS AND MONOIDAL TRANSFORMATIONS.

Let us briefly recall some previous results, where now \( J \subset \mathcal{O}_V \) be the sheaf of ideals defining a hypersurface \( X \) in the smooth scheme \( V \).

So \( \text{Diff}^0(J) = J \), and for each positive integer \( s \) there is an inclusion \( \text{Diff}^s(J) \subset \text{Diff}^{s+1}(J) \) as sheaves of ideals in \( \mathcal{O}_V \), and hence \( V(\text{Diff}^s(J)) \supset V(\text{Diff}^{s+1}(J)) \).

Recall that \( b \) is the highest multiplicity at points of \( X \), if and only if \( V(\text{Diff}^b(J)) = \emptyset \) and \( V(\text{Diff}^{b-1}(J)) \neq \emptyset \) (i.e. if and only if \( \text{Diff}^b(J) = \mathcal{O}_V \) and \( \text{Diff}^{b-1}(J) \) is a proper sheaf of ideals).

The closed set of interest is the set of \( b \)-fold points of \( X \) (i.e. \( V(\text{Diff}^{b-1}(J)) \)). Consider now a \( b \)-permissible transformation, say \( V' \to V \) as in 1.1.1. Recall that \( J\mathcal{O}_{W_i} = I(H')^bJ' \).

In this case \( J' \) is the sheaf of ideals defining a hypersurface \( X' \subset V' \), which is the strict transform of the hypersurface \( X \).

It is not hard to check that \( J' \) has at most order \( b \) at points of \( V' \) (i.e. that \( V(\text{Diff}^b(J')) = \emptyset \)).

If, in addition, \( J' \) has no point of order \( b \), then we say that \( \pi \) defines a \( b \)-simplification of \( J \).

At any rate, the closed set of interest is the set of \( b \)-fold points \( X' \).

If \( V(\text{Diff}^{b-1}(J')) \neq \emptyset \), let \( V' \xleftarrow{\pi'} V'' \) denote the monoidal transformation with some center \( Y' \subset V(\text{Diff}^{b-1}(J')) \). So \( \pi' \) is \( b \)-permissible, and set

\[
J'\mathcal{O}_{V'} = I(H'')^bJ''
\]

where \( I(H'') \) is the sheaf of functions vanishing along the exceptional hypersurface \( H'' \) of \( \pi'' \).

So again \( J'' \) has at most points of order \( b \), and if it does, define a \( b \)-permissible transformation at some smooth center \( Y'' \subset V(\text{Diff}^{b-1}(J'')) \).

So for \( J \) and \( b \) as before, we define, by iteration, a \( b \)-permissible sequence

\[
(4.0.1) \quad V \xleftarrow{\pi} V' \xleftarrow{\pi'} V'' \xleftarrow{\pi''} \ldots \xleftarrow{\pi^{(r-1)}} V^{(r)} \xleftarrow{\pi^{(r)}} V^{(r+1)}
\]

and factorizations \( J^{(i-1)}\mathcal{O}_{V^{(i)}} = I(H^{(i)})^bJ^{(i)} \). Where \( J^{(i)} \) is the sheaf of ideals defining a hypersurface \( X^{(i)} \subset V^{(i)} \), which is the strict transform of \( X \).

From the point of view of resolution it is clear that our interest is to define a \( b \)-permissible sequence so that \( X_{r+1} \) has no \( b \)-fold points.

We say that a \( b \)-permissible sequence \((4.0.1)\) defines a \( b \)-simplification of \( J \subset \mathcal{O}_W \) if the jacobian of \( V \leftarrow V^{(r+1)} \) has normal crossings, and \( V(\text{Diff}^{b-1}(J^{(r+1)})) = \emptyset \) (i.e. if \( X^{(r+1)} \) has at most points of multiplicity \( b - 1 \)).

Hironaka attaches to the original data \( J \) and \( b \) the pair \((J, b)\) (1.1). The closed set assigned to this pair in \( V \) is \( \text{Sing}(J, b) = V(\text{Diff}^{b-1}(J)) \). In our case, the \( b \)-fold points of the hypersurface \( X \).

Here we attach to the original data a Rees algebra (up to integral closure), namely \( \mathcal{G} = \mathcal{O}_V[\mathcal{JW}^b] \). And to this Rees algebra a closed set in \( V \), namely \( \text{Sing}(\mathcal{G}) \), which is again \( V(\text{Diff}^{b-1}(J)) \).

Moreover, we extended \( \mathcal{G} \) to a Diff-algebra \( G(\mathcal{G}) \), and \( \text{Sing}(\mathcal{G}) = \text{Sing}(G(\mathcal{G})) \) (3.4).
Let us focus on one \( b \)-permissible transformation \( \pi \) (1.1.1). The transform of Hironaka’s pair is the pair \((J', b)\). The transformation \( \pi \) is also permissible for both \( G \) and \( G(G) \), defining transforms of Rees algebras, say \( G' \) and \( G(G)' \) on \( V' \).

Note that, in our setting, \( J' \) is the ideal defining \( X' \), which is the strict transform of \( X \). The closed set assigned to \((J', b)\) is the set of \( b \)-fold points of \( X' \). On the other hand, \( G' = \mathcal{O}_{V'}[J'W^b] \), is such that \( \text{Sing}(G') \) is again the set of \( b \)-fold points \( X' \). A similar relation holds between pairs \((J^{(i)}, b)\) and the Rees algebras \( G^{(i)} \) (transform of \( G \)), for every \( b \)-permissible sequence (4.0.1).

The natural question is on how do the successive transforms of \( G(G) \) relate to the transforms of \( G \). The following theorem will address this question. It proves that the \( \Gamma \)-operator on Rees algebras is, in a natural way, compatible with transformation.

**Theorem 4.1.** (Giraud) Let \( G \) be a Rees algebra on a smooth scheme \( V \), and let \( V \leftarrow V' \) be a permissible (monoidal) transformation for \( G \). Let \( G' \) and \( G(G)' \) denote the transforms of \( G \) and \( G(G) \). Then:

\[ G' \subset G(G)' \subset G(G') \]

so that we have \( G(G)' = G(G(G)) \).

**Proof.** It is enough to prove that \( G(G)' \subset G(G') \).

Assume that \( G = \bigoplus_n J_n W^n \) and that the monoidal transformation has center \( Y \subset \text{Sing}(G) \). Let \( x' \in V' \) a point mapped to \( x \in V \). If \( f \in J_n \) at \( x \) then by hypothesis \( f \in I(Y)^n \). The total transform of \( f \in \mathcal{O}_{V', x'} \) is \( f = (h')^n f' \) where \( h' \) is a generator of \( I(H') \). It can be checked that

\[(4.1.1) \quad \text{Diff}_{\mathcal{O}_{V', x'}/k}^r(f) \subset I(H')^{n-r} \text{Diff}_{\mathcal{O}_{V', x'}/k}^r(f') \]

Then for every \( D \in \text{Diff}_{\mathcal{O}_{V', x'}/k}^r \), we have that for the total transform \( D(f) \) in \( \mathcal{O}_{V', x'} \): \( D(f) = (h')^{n-r} D'(f') \) where \( D' \in \text{Diff}_{\mathcal{O}_{V', x'}/k}^r \).

Assume that \( \mathcal{F} = \{f_1 W^{n_1}, \ldots, f_s W^{n_s}\} \) is a set of generators of \( G_x \). Note that the set \( \mathcal{F}' = \{f_1' W^{n_1}, \ldots, f_s' W^{n_s}\} \) generates \( G_x' \). Recall that a set of generators of \( G(G) \) is

\[ \{D(f_i)W^{m-r} | D \in \text{Diff}_{\mathcal{O}_{V', x}}^r, 0 \leq r < m \leq n_i, i = 1, \ldots, s\} \]

and a set of generators of the transform \( G(G) \) is

\[ \{(h')^{n_i-r} D'(f_i') W^{m-r} | 0 \leq r < m \leq n_i, i = 1, \ldots, s\} \]

where \( D' \) are some differential operators in \( \text{Diff}_{\mathcal{O}_{V', x'}}^r \).

\[ \square \]

**Definition 4.2.** A resolution of a Rees algebra \( G \) is a sequence of transformations, say

\[(4.2.1) \quad V \leftarrow V' \leftarrow \cdots \leftarrow V^{(r)} \leftarrow V^{(r+1)} \]

such that \( \text{Sing}(G^{(r+1)}) = \emptyset \), and the exceptional locus of \( V \leftarrow V^{(r+1)} \) is a union of smooth hypersurfaces with normal crossings.
5. IDEALISTIC EXPONENTS VERSUS BASIC OBJECTS.

Recall the definition of idealistic equivalence 2.1.

**Proposition 5.1.** Let \((J_1, b_1)\) and \((J_2, b_2)\) be idealistic equivalent. Then:

1. \(\text{Sing}(J_1, b_1) = \text{Sing}(J_2, b_2)\).
   
   Note, in particular, that every monoidal transform \(V \leftarrow V'\) on a center \(Y \subset \text{Sing}(J_1, b_1) = \text{Sing}(J_2, b_2)\) defines transforms, say \((J'_1, b_1)\) and \((J'_2, b_2)\) on \(V'\).

2. The pairs \((J'_1, b_1)\) and \((J'_2, b_2)\) are idealistic equivalent on \(V'\).

**5.2.** If two pairs \((J_1, b_1)\) and \((J_2, b_2)\) are idealistic equivalent over \(V\), the same holds for the restrictions to every open subset of \(V\), and also for restrictions in the sense of étale topology, and even for smooth topology (i.e. pull-backs by smooth morphisms \(V' \rightarrow V\)).

Note that if \((J_1, b_1)\) and \((J_2, b_2)\) are idealistic equivalent, they define the same closed set on \(V\) (i.e. \(\text{Sing}(J_1, b_1) = \text{Sing}(J_2, b_2)\)), and the same holds for monoidal transformations, pull-backs by smooth schemes, and hence by concatenation of both kinds of transformations. When this last condition holds on the singular locus of two pairs we say that they define the same closed sets.

**Definition 5.3.** Two pairs \((J_1, b_1)\) and \((J_2, b_2)\) are basically equivalent on \(V\), if they define the same closed sets.

Proposition 5.1 says that if two pairs are idealistic equivalent over \(V\), then they are basically equivalent.

An idealistic exponent, as defined by Hironaka in [9], is an equivalence class of pairs in the sense of idealistic equivalence. Whereas the notion of equivalence among basic objects (see [15] or [16]) is 5.3. In fact, the key point for constructive desingularization was to define an algorithm of resolutions of pairs \((J, b)\), so that two basically equivalent pairs undergo exactly the same resolution.

Recall now the definition of integrally equivalence on Rees algebras 2.2.

**Proposition 5.4.** Let \(G_1\) and \(G_2\) be two integrally equivalent Rees algebras over \(V\). Then:

1. \(\text{Sing}(G_1) = \text{Sing}(G_2)\).

   Note, in particular, that every monoidal transform \(V \leftarrow V'\) on a center \(Y \subset \text{Sing}(G_1) = \text{Sing}(G_2)\) defines transforms, say \(G'_1\) and \(G'_2\) on \(V'\).

2. \(G'_1\) and \(G'_2\) are integrally equivalent on \(V'\).

If \(G_1\) and \(G_2\) are integrally equivalent on \(V\), the same holds for every open restriction, and also for pull-backs by smooth morphisms \(V' \rightarrow V\).

On the other hand, as \(G'_1\) and \(G'_2\) are integrally equivalent, they define the same closed set on \(V'\) (the same singular locus), and the same holds for further monoidal transformations, pull-backs by smooth schemes, and concatenations of both kinds of transformations. When this condition holds on the singular locus of two Rees algebras over \(V\), we say that they define the same closed sets.
Definition 5.5. Two Rees algebras over \( V \), say \( G_1 = \bigoplus_{n \geq 0} I_n W^n \) and \( G_2 = \bigoplus_{n \geq 0} J_n W^n \), are \textit{basically} equivalent, if both define the same closed sets.

The previous Proposition asserts that if \( G_1 = \bigoplus_{n \geq 0} I_n W^n \) and \( G_2 = \bigoplus_{n \geq 0} J_n W^n \) are integrally equivalent, then they are basically equivalent.

Proposition 5.6. Let \((J_1, b_1)\) and \((J_2, b_2)\) be two pairs.

1. The pairs \((J_1, b_1)\) and \((J_2, b_2)\) are idealistically equivalent over a smooth scheme \( V \), if and only if the Rees algebras \( G_{(J_1, b_1)} \) and \( G_{(J_2, b_2)} \) are integrally equivalent (2.3).

2. The pairs \((J_1, b_1)\) and \((J_2, b_2)\) are basically equivalent over \( V \), if and only if the Rees algebras \( G_{(J_1, b_1)} \) and \( G_{(J_2, b_2)} \) are basically equivalent.

\[\text{Proof.} \text{ Note that (1) was already proved in 2.9. (2) follows from the fact that if } (J', b) \text{ is the transform of } (J, b) \text{ then } G_{(J', b)} \text{ is the transform of } G_{(J, b)}.\]

6. Functions on pairs and simple Rees algebras

Definition 6.1. Let \( V \) be a noetherian topological space [7, p. 5] and let \((\Lambda, \leq)\) be a totally ordered set. A function \( f : V \rightarrow (\Lambda, \leq) \) is said to be upper-semi-continuous (u.s.c.) if it takes finitely many values, and for every \( \alpha \in \Lambda \) the set \( \{ x \in V \mid f(x) \geq \alpha \} \) is closed.

We denote by \( \text{max } f \) the biggest value and \( \text{Max } f = \{ x \in V \mid f(x) = \text{max } f \} \), which is closed in \( V \).

Remark 6.2. The goal is to define an u.s.c function, say \( f_G \), for every Rees algebra \( G \), such that \( \text{Max } f_G \) is a permissible center. Set \( V' \rightarrow V \) the transformation with center \( \text{Max } f_G \). Then the function \( f_{G^r} \) is such that \( \text{max } f_G > \text{max } f_{G^r} \).

An algorithm of resolution of Rees algebras is the assignment of functions \( f_G \), for each \( G \), such that a sequence (4.2.1) is defined by setting each transformation in 4.2.1 with center \( \text{Max } f_{G^{(r)}} \). Furthermore, inequalities \( \text{max } f_G > \text{max } f_{G^r} > \cdots > \text{max } f_{G^{(r)}} \) will hold, and the sequence (4.2.1) defined in this way is a resolution for suitable \( r = r_G \).

6.3. We introduce a function, again a natural analog to that defined for idealistic exponents. Fix \( x \in V \). Given \( f W^n \in I_n W^n \), where \( G = \bigoplus_{n \geq 0} I_n W^n \), set

\[\text{ord}_x(f W^n) = \frac{\nu_x(f)}{n} \in \mathbb{Q};\]

called the order of \( f \) (weighted by \( n \)), where \( \nu_x \) denotes the order at the local regular ring \( O_{V,x} \). Note that \( x \in \text{Sing}(G) \) if and only if \( \text{ord}_x(f W^n) \geq 1 \) for all \( n \geq 1 \). We also define

\[\text{ord}_x(G) = \inf \{ \text{ord}_x(f W^n) \mid f W^n \in I_n W^n, \ n \geq 1 \}.\]

So, in general \( \text{ord}_x(G) \geq 1 \) iff \( x \in \text{Sing}(G) \). A Rees algebra \( G \) is said to be simple at \( x \) if \( \text{ord}_x(G) = 1 \).
Proposition 6.4. (1) If $\mathcal{G}$ is a Rees algebra generated over $O_V$ by $F = \{g_i W^{n_i} \mid 1 \leq i \leq m\}$, then $\text{ord}_x(\mathcal{G}) = \min\{\text{ord}_x(g_i W^{n_i}) \mid 1 \leq i \leq m\}$. If $N$ is a common multiple of all $n_i$, $1 \leq i \leq m$, then

$$\text{ord}_x(\mathcal{G}) = \frac{\nu_x(I_N)}{N}.$$  

(2) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are Rees algebras with the same integral closure (e.g. if $\mathcal{G}_1 \subset \mathcal{G}_2$ is a finite extension), then, for all $x \in \text{Sing}(\mathcal{G}_1)(= \text{Sing}(\mathcal{G}_2))$

$$\text{ord}_x(\mathcal{G}_1) = \text{ord}_x(\mathcal{G}_2).$$  

In particular, the function is compatible with integral equivalence (see 2.4).

(3) Let $\mathcal{G}(\mathcal{G})$ be the extension of $\mathcal{G}$ to a differential Rees algebra relative to $k$, then for all $x \in \text{Sing}(\mathcal{G}(\mathcal{G})) = \text{Sing}(\mathcal{G}(\mathcal{G}))$.

$$\text{ord}_x(\mathcal{G}) = \text{ord}_x(\mathcal{G}(\mathcal{G})).$$

Remark 6.5. The function $\text{ord}$ was introduced by Hironaka in the context of pairs. Given a pair $(J, b)$ as in 1.1; and assume that $J \subset O_V$ is a non-zero sheaf of ideals, a function $\text{ord}_x(J, b) : \text{Sing}(J, b) \to \mathbb{Q}$ is defined by setting

$$\text{ord}_x(J, b) = \frac{\nu_x(J)}{b}.$$  

Note that if $\mathcal{G}_{(J, b)}$ is the Rees algebra attached to $(J, b)$, then $\text{Sing}(J, b) = \text{Sing}(\mathcal{G}_{(J, b)})$, and for all $x \in \text{Sing}(J, b)$: $\text{ord}_x(J, b) = \text{ord}_x(\mathcal{G}_{(J, b)})(2.3)$.

Proposition 6.6. Let $\mathcal{G}$ be a Rees algebra and $V' \to V$ be a permissible transformation with center $Y$. Denote by $\mathcal{G}'$ the transform of $\mathcal{G}$.

(1) If the function $\text{ord}(\mathcal{G})$ is constant along $Y$ then, for all $x' \in V'$ mapping to $x \in V$:

$$\text{ord}_x(\mathcal{G}) \geq \text{ord}_{x'}(\mathcal{G}')$$

(2) If $\mathcal{G}$ is simple at $x$, then $\mathcal{G}'$ is simple at $x'$.

(3) If $\text{codim}_V(Y) = 1$ then $V' \to V$ is an isomorphism but $\mathcal{G}'$ and $\mathcal{G}$ are different, in fact

$$\text{ord}_{x'}(\mathcal{G}) = \text{ord}_x(\mathcal{G}) - 1, \quad \forall x = x' \in V = V'.$$

Proof. If $\mathcal{G} = \mathcal{G}_{(J, b)}$ then $\mathcal{G}' = \mathcal{G}_{(J', b)}$ and it is well known that $\nu_x(J) \geq \nu_{x'}(J')$.

In the general case $\mathcal{G}$ is integral over some $\mathcal{G}_{(J_N, N)}$ (1.3), and (1) follows from 6.4.

(2) is a consequence of (1), and (3) follows from the definition of transformation 1.5. ⊳

Proposition 6.7. Assume that the ground field $k$ is of characteristic zero. Fix $x \in \text{Sing}(\mathcal{G}) \subset V).$ If $\text{ord}(\mathcal{G})(x) = 1$ then there is, locally at $x$, a smooth hypersurface $Z$ such that

(1) $O_V[I(Z)W] \subset G(\mathcal{G}).$ And hence $\text{Sing}(\mathcal{G}) \subset Z$.

(2) If $V' \to V$ is a blow-up at a permissible center $C \subset \text{Sing}(\mathcal{G})$, and if $\mathcal{G}'$ is the transform of $\mathcal{G}$ and $Z' \subset V'$ is the strict transform of $Z$, then

$$O_{V'}[I(Z')W] \subset G(\mathcal{G}').$$
**Proof.** Fix a regular system of parameters \(\{x_1, \ldots, x_d\}\) at \(\mathcal{O}_{V,x}\), and an element \(f_nW^n \in \mathcal{G}\) so that \(f_n\) has order \(n\) at \(\mathcal{O}_{V,x}\). Note that \(\Delta^\alpha(f_n)W \in G(\mathcal{G})\) for \(|\alpha| = n - 1\), which can be chosen so that \(\Delta^\alpha(f_n)\) has order one, defining a smooth hypersurface \(Z\) locally at \(x\) (3.3).

**Theorem 6.8.** (Wlodarczyk, [20]) Let \(x \in V\) be a simple point of \(\mathcal{G}\), and assume that locally at \(x\), there are two hypersurfaces \(Z_1, Z_2 \subset V\) such that

\[
\mathcal{O}_V[I(Z_i)W] \subset G(\mathcal{G}) \quad i = 1, 2
\]

Then there exist étale neighborhoods \(\phi_1, \phi_2 : U \to V\) of \(x = \phi_1(y) = \phi_2(y)\), where \(y \in U\), such that \(\phi_i^*(Z_1) = \phi_2^*(Z_2)\) and \(\phi_i^*(G(\mathcal{G})) = \phi_2^*(G(\mathcal{G}))\).

**6.9.** If \(\mathcal{G}\) is simple at \(x \in \text{Sing}(\mathcal{G})\), then resolution of \(\mathcal{G}\) will reduce, locally, to the resolution of the restriction, say \(G(\mathcal{G})_Z\), of \(G(\mathcal{G})\) to a hypersurface \(Z\). Where \(Z\) is such that \(\mathcal{O}_V[I(Z)W] \subset G(\mathcal{G})\). By theorem 3.7 this procedure is well defined up to integral closure. If \(\mathcal{G}_1\) and \(\mathcal{G}_2\) have the same integral closure then \(G(\mathcal{G}_1)_Z\) and \(G(\mathcal{G}_2)_Z\) have the same integral closure.

By theorem 6.8 this procedure does not depend on the choice of the hypersurface \(Z\). In fact, if two possible hypersurfaces \(Z_1\) and \(Z_2\) fulfill the previous condition, then the restrictions \(G(\mathcal{G})_{Z_1}\) and \(G(\mathcal{G})_{Z_2}\) are the same (étale locally at \(y\)).

### 7. Reduction to the simple case

Given a Rees algebra \(\mathcal{G}\) we have defined \(\text{ord}_x(\mathcal{G}) \in \mathbb{Q} \geq 1\) for every point \(x \in \text{Sing}(\mathcal{G})\).

Suppose that \(\mathcal{G} = \mathcal{G}(J,b)\) and set \(\omega = \text{ord}_x(J,b) \geq 1\). Given an idealistic pair \((J,a)\) we may consider a new pair locally at \(x\), with order one, say \((J,a)\), where \(a = \omega b\). The next definition is the analogous formulation, now for the case of Rees algebras.

**Definition 7.1.** Let \(\mathcal{G} = \bigoplus_{n \geq 0} J_nW^n\) be a Rees algebra on \(V\) and fix \(\omega \in \mathbb{Q}, \omega > 0\). We define the twisted algebra \(\mathcal{G}(\omega)\) as follows

\[
\mathcal{G}(\omega) = \bigoplus_{n \geq 0} J_{\omega n}W^n
\]

where we set \(J_{\omega n} = 0\) whenever \(\frac{n}{\omega}\) is not an integer.

More precisely: if \(\omega = \frac{a}{b}\) where \(a\) and \(b\) are integers with no common factors, and if \(n = am\) for some \(m \in \mathbb{Z}\), then \(J_{\omega n} = J_{bn}\) and \(J_{\omega n} = 0\) if \(a\) not divide \(n\).

**Remark 7.2.** \(\mathcal{G}(\omega)\) can be constructed as follows: If \(\omega = \frac{a}{b}\) with \((a,b) = 1\), then consider the isomorphism \(\Phi : \mathcal{O}_V[W^b] \to \mathcal{O}_V[W^a]\) by sending \(W^b\) to \(W^a\). Then the twisted Rees algebra is

\[
\mathcal{G}(\omega) = \Phi(V^{(b)}(\mathcal{G}))
\]

**Proposition 7.3.** The definition 7.1 fulfills the requirement of (2.4).

1. If \(\mathcal{G} = \mathcal{G}(J,b)\) and \(\omega \in \mathbb{Q}\) is such that \(b\omega \in \mathbb{Z}\) then \(\mathcal{G}(\omega) = \mathcal{G}(J,b\omega)\).
2. If \(\mathcal{G}_1\) and \(\mathcal{G}_2\) have the same integral closure then \(\mathcal{G}_1(\omega)\) and \(\mathcal{G}_2(\omega)\) have the same integral closure.
Proof. (1) follows straightforward. (2) follows from 7.2 and 2.8, which asserts that the operator $V^n$ is compatible with integral closure.

**Proposition 7.4.** $G(\omega)$ is a Rees algebra, and $\omega \ord_x(G(\omega)) = \ord_x(G)$ at every point $x \in V$.

**Proof.** Set $G = \bigoplus I_n W^n$. As $G$ is finitely generated it is a finite extension of $G(I_{N}, N)$ for a suitable choice of $N$. 7.3 shows that $G(\omega)$ is a finite extension of $G(I_{N}, \omega N)$, therefore $G(\omega)$ is a Rees algebra.

For all $x \in V$ we have

\[
\alpha = \ord_x(G) \iff \nu_x(J_n) \geq \alpha n \ \forall n > 1, \ \text{and} \ \exists n_0 \ \text{with} \ \nu_x(J_{n_0}) = \alpha n_0 \\
\iff \nu_x(J_{bn}) \geq \alpha bn \ \forall n > 1, \ \text{and} \ \exists n_0 \ \text{with} \ \nu_x(J_{bn_0}) = \alpha bn_0 \\
\iff \nu_x(J_{\omega n}) \geq \frac{\alpha an}{\omega} \ \text{and} \ \exists n_0 \ \text{with} \ \nu_x(J_{\omega an_0}) = \frac{\alpha an_0}{\omega} \\
\iff \frac{\alpha}{\omega} = \ord(G(\omega))
\]

Note also that if $\omega \geq 1$ then $\text{Sing}(G(\omega)) \subset \text{Sing}(G)$, but if $\omega < 1$ then this inclusion may not be satisfied.

**Corollary 7.5.** If $\omega = \ord_x(G)$ then $\ord_x(G(\omega)) = 1$.

In order to achieve a resolution of a Rees algebra $G$, we will attach to $G$ some additional data. The first one will be a set of irreducible hypersurfaces having normal crossings, say $E = \{H_1, \ldots, H_r\}$.

**Definition 7.6.** Let $G = \bigoplus I_n W^n$ be a Rees algebra, $E = \{H_1, \ldots, H_r\}$ a set of irreducible hypersurfaces having normal crossings, and $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Q}^r$.

We say that the formal monomial $I(H_1)^{a_1} \cdots I(H_r)^{a_r}$ divides the algebra $G$ (or $E^\underline{a}$ divides $G$) if, for every $n$:

\[
J_n \subset I(H_1)^{[na_1]} \cdots I(H_r)^{[na_r]}
\]

where $[na_i] \leq na_i$ is the integer part of $na_i$, $i = 1, \ldots, r$.

**7.7.** We now show that Definition 7.6 fulfills the requirement of 2.4.

1. Let $G = G(J_{b \underline{a}})$ and $\underline{a} \in \mathbb{Q}^r$ be such that $b \underline{a} \in \mathbb{Z}^r$. The monomial $E^{\underline{a}}$ divides $G$ if and only if the monomial $E^{b \underline{a}}$ divides the ideal $J$.

2. Let $G_1$ and $G_2$ be two Rees algebras which are integrally equivalent. Then $E^{\underline{a}}$ divides $G_1$ if and only if $E^{\underline{a}}$ divides $G_2$.

**7.8.** Let $N$ be such that $G(J_{N,N}) \subset G = \bigoplus I_n W^n$ is finite. Set $\alpha_1, \ldots, \alpha_r$ be the biggest integers for which $J_N \subset I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r}$ and set $a_i = \frac{\alpha_i}{N} \in \mathbb{Q}$. Then $\underline{a} = (a_1, \ldots, a_r) \in \mathbb{Q}^r$ is the maximal vector such that the monomial $E^{\underline{a}}$ divides $G$. In the sense that if $\underline{a}' \neq \underline{a}$ and $a_i' \geq a_i$ then $E^{\underline{a}'}$ does not divide $G$. 


We claim that this assertion is independent of the choice of $N$ with the previous property. Note that $\alpha_i$ is the valuation of $J_N$ at the generic point of the hypersurface. On the other hand Proposition 2.9 says that if a Rees algebra $G$ is integrally equivalent to two different algebras attached to two pairs, say $G_{(j_1,b_1)}$ and to $G_{(j_2,b_2)}$, then the pairs $(j_1,b_1)$ and $(j_2,b_2)$ are idealistic equivalent (2.1). In particular $J_1^{b_2}$ and $J_2^{b_1}$ vanish along $H_i$ with the same order.

**Definition 7.9.** Let $G = \bigoplus_n J_n W^n$ be a Rees algebra and $E$ be a set of irreducible hypersurfaces having only normal crossings. Let $\omega$ be the maximal vector such that $E^\omega$ divides $G$. For all $n$ there is an ideal $I_n$ such that

$$J_n = I(H_1)^{[n\alpha_1]} \cdots I(H_r)^{[n\alpha_r]} I_n$$

Set $G^\gamma = \bigoplus_n I_n W^n$ and define the function $w\cdot \text{ord}(G) : V \to \mathbb{Q}$ to be $w\cdot \text{ord}(G) = \text{ord}(G^\gamma)$.

It follows from 7.8 that $G^\gamma$ is finitely generated, and hence a Rees algebra. Using notation as in 7.8, we have that $J_N = I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r} I_N$ and $G_{(I_N,N)} \subset G^\gamma$ is integral.

**7.10.** Definition 7.9 satisfies requirement in 2.4.

1. Let $(J, b)$ be a pair such that

$$J = I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r} I$$

and $I \not\subseteq I(H_i)$ for $i = 1, \ldots, r$. If $G = G_{(J,b)}$ then $G^\gamma = G_{(J,b)}$.

Moreover, this function $w\cdot \text{ord}(G)$ coincides with the function $w\cdot \text{ord}_{(J,b)}$, defined in terms of the pair $(J, b)$ in [3].

2. If $G_1$ and $G_2$ are integrally equivalent then $G_1^\gamma$ and $G_2^\gamma$ are integrally equivalent.

In particular $w\cdot \text{ord}(G_1) = w\cdot \text{ord}(G_2)$ (both functions are equal).

**7.11.** Let $V' \to V$ be a transformation with irreducible center $C$. We assume that $C \subset \text{Sing}(G)$, and that the function $w\cdot \text{ord}(G)$ is constant along $C$ and takes the value, say $\omega \in \mathbb{Q}$. Let $\omega$ be the maximal vector such that $E^\omega$ divides $G$. Let $G'$ be the transform of $G$, and set $E' = \{H'_1, \ldots, H'_r, H'_{r+1}\}$ where $H'_{r+1}$ is the exceptional divisor of $V' \to V$, and $H'_i$ is the strict transform of $H_i$, $i = 1, \ldots, r$. Then:

1. The monomial $(E')^{\omega'}$ is the maximal monomial which divides $G'$.

Where $\omega' = (\omega_1', \ldots, \omega_r', \omega_{r+1}')$, $\omega'_i = \omega_i$ for $i = 1, \ldots, r$ and

$$\omega'_{r+1} = \omega - 1 + \sum_{H_i \subset C} a_i$$

2. The twisted Rees algebra $G^\gamma(\omega)$ is simple and its transform is

$$(G^\gamma(\omega))' = (G')^\gamma(\omega)$$

**7.12.** We add some more information to a the Rees algebra in addition to $E$, say $(G, D, E)$ where $D \subset E$ is a suitable subset of hypersurfaces. Given a monoidal transformation we define the transform of $(G, D, E)$ to be $(G', D', E')$ where $G'$ is the transform of $G$, $E'$ is as in 7.11 and $D'$ consist of the strict transforms of $D$ if max $w\cdot \text{ord}(G) = \max w\cdot \text{ord}(G')$ and $D' = E'$ otherwise.
Definition 7.13. Given \((G, D, E)\) we define the function \(t(G) = (w-\text{ord}(G), n(G))\) where \(n(G)(x)\) is the number of hypersurfaces of \(D\) passing through \(x \in V\).

7.14. The function \(t(G)\) satisfies the requirements of 2.4.

1. If \(G = G(J, b)\) then \(t(G)\) is the function \(t(J, b)\) defined in terms of the pair in [3].
2. If \(G_1\) and \(G_2\) are integrally equivalent then \(t(G_1) = t(G_2)\).

7.15. Given \((G, D, E)\), if \(\max t(G) = (\omega, m)\) then set:

\[
T(G) = G \odot G^\omega \odot D_m
\]
where \(D_m\) is the Rees ring of the ideal

\[
\prod_{\{H_1, \ldots, H_m\} \subseteq D} \left( \sum_{j=1}^{m} I(H_j) \right)
\]

see [3, definition 15.14].

The Rees algebra \(T(G)\) is simple and \(\text{Sing}(T(G)) = \text{Max} t(G)\).

7.16. The algebra \(T(G)\) satisfies the requirements of 2.4.

1. If \(G = G(J, b)\), then \(T(G) = G_t(J, b)\). Where here \(t(J, b)\) denotes the pair attached to \((J, b)\) in [3, 15.14.2]; defined in terms of the function \(t\) of the pair \((J, b)\).
2. If \(G_1\) and \(G_2\) are integrally equivalent then \(T(G_1)\) and \(T(G_2)\) are integrally equivalent.

7.17. If \(V' \to V\) is a transformation with center \(C \subset \text{Sing}(T(G)) = \text{Max} t(G)\), \((G', D', E')\) is the transform of \((G, D, E)\), and \(T(G)'\) is the transform of \(T(G)\), then we have the same relation among the transforms, namely:

\[
T(G)' = G' \odot (G')^\omega \odot D'_m
\]

So that

1. If \(\max t(G) = \max t(G')\) then \(T(G)' = T(G')\).
2. If \(\max t(G) > \max t(G')\) then \(\text{Sing}(T(G')) = \emptyset\) and \(T(G')\) is a new Rees algebra such that \(\text{Sing}(T(G')) = \text{Max} t(G')\).

Theorem 7.18. The function \(t\) extends naturally, by induction on the dimension of the ambient space, to an algorithm of resolution of Rees algebras (6.2).

If \(G_1\) and \(G_2\) are integrally equivalent then the algorithm defines the same resolution for both Rees algebras.

Proof. The algorithm is defined by induction on \(\dim V\).

If \(\dim V = 1\), then \(\text{Sing}(G)\) consists of finitely many points, and every such point is the center of the transformation. Theorem follows in this case from 6.6(3).

Assume that an algorithm is defined for Rees algebras over smooth schemes of dimension \(n - 1\).

Let \(G\) be a Rees algebra on \(V\), \(\dim V = n\). The Rees algebra \(T(G)\) is simple, so that locally we may choose a smooth hypersurface \(Z \subset V\) and consider the resolution of the restriction
$G(\mathcal{T}(G))_Z$ of $G(\mathcal{T}(G))$ to $Z$, which exists by induction. Wlodarczyk's theorem 6.8 asserts that such local procedures globalize.

In particular there is a sequence of blow-ups, say $V' \to V$, such that the transform $\mathcal{T}(G)'$ has empty singular locus. This means that the transform $G'$ is such that $\max t(G) > \max t(G')$.

We continue, similarly, with the simple Rees algebra $\mathcal{T}(G')$, and so on.

This procedure defines a sequence of transformations for $G$, such that the function $t(G)$ drops after finitely many steps.

Now this procedure shall stop since the function $t(G)$ may not drop infinitely many times. This follows from the fact that $t(G)$ is the function associated to some suitable pair $(J_N, N)$; and for pairs the function $t$ may not drop infinitely many times, see [3].

Last assertion of the theorem follows from the fact that all constructions satisfy requirements in 2.4.

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