Ranks of Identity Difference Transformation Semigroup

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Abstract-This study focuses on the ranks of identity difference transformation semigroup. The ideals of all the (sub) semigroups: identity difference full transformation semigroup (\( IDT_n \)), identity difference order preserving transformation semigroup, \( IDO_n \), identity difference symmetric inverse transformation semigroup( \( IDI_n \)), identity difference partial order preserving symmetric inverse transformation semigroup \( IDPO_n \) and identity difference partial order preserving transformation semigroup \( IDPOn \) were investigated for rank and their combinatorial results obtained respectively.

Keywords: Transformation Semigroup, Identity Difference, Rank.

I. INTRODUCTION

The rank of transformation semigroup on a set \( X_n \) has been widely studied. Amongst the earliest studies, Gomes and Howie obtained the rank and idempotent rank of partial order preserving transformation semigroups \( POn \) to be \( (2n-1) \) and \( (3n-2) \) and that of \( On \) to be respectively \( (n) \) and \( (2n-1) \). Also they showed in the case of ‘strictly partial order preserving maps’ \( SPO_n = POn \backslash On \) while rank of \( SPO_n \) is \( 2(n-1) \) for all \( n \geq 2 \).

A generalization of this study initiated by Garba in 1994 showed that the semigroups \( L(n,r), M(n,r) \) and \( N(n,r) \) have equal rank with idempotent rank. That is, \( L(n,r) = \binom{2n}{n} \), \( M(n,r) = \sum_{k=r}^{n} \binom{n+k}{k} \binom{k-1}{r-1} \) and \( N(n,r) = \sum_{k=r}^{n} \binom{n+k}{k} \binom{k-1}{r-1} \) respectively.

The relative rank \( (S;A) \) of a subset \( A \) of a semigroup \( S \) is the minimum cardinality of a set \( B \) such that, \( \langle A \cup B \rangle = S \). The monoid of all contraction in \( T_n \) is of uncountable relative rank.

[9] The term compression map in partial order-preserving transformation semigroup was used by Zhao and Yang in 2012 to be \( CPOn = \{ \alpha \in POn; (\forall x,y \in dom \alpha), |x \alpha - y \alpha| \leq |x - y| \} \) where they characterized the Green’s relation on \( CPOn \) and the regularity of the elements of \( CPOn \).

[13] The definition of compression maps as in [13] is the same as contraction maps in [9],[6],[3], etc. The study in [13] pave way for further characterization of the Green’s relation on the subsemigroups \( CT_n \) and \( OCT_n \), which gave birth also to the characterization of both \( CT_n \) and \( OCT_n \) on the Green’s starred relations [6]. The cardinalities of these subsemigroups \( OCT_n \) and \( ORCT_n \) were investigation and \( |OCT_n| = (n+1)2^{n-2} \) and \( |ORCT_n| = (n+1)2^{n-1} - n \) were obtained [3].

Symons in [15] denoted the full transformation semigroup to be \( T(X) \) and studied the automorphism and isomorphism of \( T(X), Y = X \) with \( Y \) a fixed nonempty subset of \( X \). In [12] they characterized the regular elements of \( T(X,Y) \), amongst other results obtained from the set \( F(X,Y) = \{ \rho \in T(X,Y); \rho = Y \rho \} \) which consists of regular elements in \( T(X,Y) \). Sanwong and Sommane [14] characterized the Green’s relation on the subsemigroup \( T(X,Y) \) and proved that \( F(X,Y) \) is the largest regular subsemigroup of \( T(X,Y) \). Also, they obtained the rank and idempotent rank of \( F(X,Y) \) ideals with \( X \) a finite set. The subsemigroups \( P(X,Y) \) and \( I(X,Y) \) of the partial transformation semigroup \( P(X) \) was defined in [5] as follows; \( P(X,Y) = \{ \rho \in P(X); imp \subseteq Y \} \) \( I(X,Y) = \{ \rho \in I(X); imp \subseteq Y \} \) and further obtained that \( PF(X,Y) = \{ \rho \in P(X,Y); \rho = Y \rho \} \) is the largest regular subsemigroup of \( P(X,Y) \), and \( I(Y) \) is the largest inverse subsemigroup of \( I(X,Y) \) [5].

In [1] the identity difference transformation semigroups were introduced by using the formulae \( max(\{im(\alpha) - min(\{im(\alpha) \}) \leq 1 \) and the combinatorial results for \( |IDT_n|, |IDO_n|, |IDP_n|, |OIDT_n|, |OID_1|, \) and \( |OID_n| \) was obtained respectively. More combinatorial results for the nilpotents, \( |N_n| \), idempotents, \( |E(S)| \), and fix of the subsemigroups of the transformation semigroups was also obtained.

In [2] it was shown that \( IDT_n \) is a subsemigroup of the full transformation semigroup \( T_n \) and congruence property of the Green’s relations \( \mathcal{L} \) and \( \mathcal{R} \) examined on \( IDT_n \).

In [11] the study of the semigroup \( \mathcal{S} \) yield some combinatorial results for \( \mathcal{S}, W(\mathcal{S}) \) and \( \mathcal{W}(\mathcal{S}) \) for all \( n \geq 3 \). \( \mathcal{S} = 3 + \frac{1}{2}(n-1)(1+1) \) \( \mathcal{W}(\mathcal{S}) = n^2 - 3n + 2 \) \( \mathcal{W}(\mathcal{S}) = n^2 - 3n + 2 \) average work done.

This study investigates the identity difference transformation semigroup for ranks for \( n \geq 3 \).

where Section 2 of this work deals with the definition of terms, section 3 shows the properties of the ranks of the identity difference transformation semigroup, section 4 provides the main results and conclusion of the study respectively.
II. DEFINITION OF TERMS

A transformation is a map from a set X to itself (self-map). That is \( f: X \to X \).

The partial transformation semigroup \( PT_n \) is the semigroup of all partial transformations on the finite set \( N = \{1, 2, \ldots, n\} \) with respect to the composition of map. The Full transformation semigroup \( T_n \) on a set \( n \) is the semigroup of all transformations on \( X \) (that is, all mappings from \( X \) to itself) under the operation of composition of map. The Symmetric group \( S_n \) on a set \( X \) is the group consisting of all bijection from the set \( X \), where \( |X| = X \) to itself with function composition as the group operation. Note that \( S_n = T_n \cap I_n \). The set of all partial bijection(s) on a set \( X \), where \( |X| = X \) (that is, one-to-one partial transformation) forms an inverse semigroup called the symmetric inverse semigroup \( I_n \).

A binary relation \( \leq_X \) on a set \( X \) is said to be a (partial) order if

I. \( x \leq x \) \( \forall x \in X \). (\( \leq_X \) is reflexive).

II. \( x \leq y \) \( \text{and} \ y \leq x \) \( \Rightarrow \ x = y \). (\( \leq_X \) is antisymmetric).

III. \( x \leq y \) \( \text{and} \ y \leq z \) \( \Rightarrow \ x \leq z \). (\( \leq_X \) is transitive).

Let \( (A, \leq_A) \) and \( (B, \leq_B) \) be partially ordered sets. The map \( \alpha: A \to B \) is order preserving if \( a \leq_A a' \) implies \( \alpha(a) \leq_B \alpha(a') \) \( (\forall a, a' \in A) \).

A mapping \( \alpha \in T_n \) is said to be order preserving if for any \( x, y \in N \), then \( x \leq y \Rightarrow \alpha(x) \leq \alpha(y) \).

The set of all order preserving maps forms a semigroup and is denoted by \( O_n \).

Let \( \alpha: W \to X \) and \( \beta: X \to Y \) then the composition of these functions \( \alpha \circ \beta: W \to Y \), can be defined as \( x(\alpha \circ \beta) = x(\alpha)\beta \) \( (\forall x \in W) \).

Also, the composition of functions is associative if there exist another function \( \gamma: Y \to Z \), then \( (x(\alpha \circ \beta) \circ \gamma) = (x(\alpha)\beta)\gamma = x(\alpha \circ \circ \gamma) \) \( (\forall x \in W) \).

The term identity difference is defined by the rule \( W^+(\alpha) - W^-(\alpha) \leq 1 \) or \( \max \text{(ima)} - \min \text{(ima)} \leq 1 \).

The map \( \alpha \in PT_n \) is said to be contraction if \( |\alpha(a) - \alpha(b)| \leq |a - b| \ \forall a, b \in X_n \).

Let \( S = IDPT_n \), the \( \text{Rank}(S) = \min\{|A|: A \subseteq S \text{ and } |A| < A| = S\} \).

For more definitions see [10], [18], [40], [41], etc.

III. PROPERTIES OF THE RANK OF IDENTITY DIFFERENCE TRANSFORMATION SEMIGROUP

In this section we shall investigate the semigroups \( IDT_n, IDO_n, IDI_n, IDPOI_n \) and \( IDPO_n \) for rank and summarize their properties using \( \mathcal{R} \) classes of each of the semigroups under consideration.

For simplicity’s sake we shall arrange the element of the said semigroup in their respective \( \mathcal{R} \) classes.

A. IDENTITY DIFFERENCE FULL TRANSFORMATION SEMIGROUP \( IDT_n \)

Let the element \( \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix} \) be represented by \( (112) \) and \( IDT_n = S \).

Consider the \( \mathcal{R} \) classes of \( S \) to be \( R_1, R_2, R_3 \) and \( R_4 \) for \( n = 3 \) where

\[
\begin{align*}
R_1 &= \{(112), (221), (223), (332), \}, \\
R_2 &= \{(121), (212), (232), (323), \}, \\
R_3 &= \{(122), (211), (233), (322), \} \text{ and} \\
R_4 &= \{(111), (222), (333), \}.
\end{align*}
\]

In \( S, R_4 \) Contains the elements with the constant maps \( \zeta \).

The minimum generating set for \( S \) is \( A = \{(112), (233), (323), \} \).

By observation the constant map \( \zeta \notin A \). Hence \( \text{Rank}(S) = 3 \forall n = 3 \).

The above illustration is same for identity difference order-preserving transformation semigroup \( IDO_n \).

B. IDENTITY DIFFERENCE SYMMETRIC INVERSE TRANSFORMATION SEMIGROUP

Let \( S = IDI_n \).

Consider the \( \mathcal{R} \) classes \( (R_1, R_2, R_3) \) of the semigroup \( S \) for \( n = 3 \)

\[
\begin{align*}
R_1 &= \{(12), (21), (23), (32), \}, \\
R_2 &= \{(1-2), (2-1), (2-3), (3-2), \}, \\
R_3 &= \{(1-2), (2-3), (2-3), \} \text{ and} \\
R_4 &= \{(1-3), (2-3), (2-3), \}.
\end{align*}
\]

are the elements of \( IDI_n \) in their respective \( \mathcal{R} \) classes.

\( R_4, R_5, R_6 \) and \( R_7 \) Contains the elements with the identity, and empty maps. After some investigations we observe that the generating set can be obtained by picking an element randomly from each of the \( \mathcal{R} \) classes \( (R_1, R_2 \text{ and } R_3) \). Hence \( A \) contains 3 elements for \( n \geq 3 \).

In general, the minimum generating set \( A \) for \( S \) is \( 3, 6, 10, 15, \ldots \forall n \geq 3 \).

The illustration for \( IDI_n \) is same for \( IDPOI_n \) and they also have same sequence of minimum generating sets. Hence, \( IDI_n \) and \( IDPOI_n \) has equal ranks.

C. IDENTITY DIFFERENCE PARTIAL ORDER SEMIGROUP

SYMMETRIC INVERSE TRANSFORMATION SEMIGROUP \( IDPOI_n \)
Consider the $\mathcal{R}$ classes $(R_1, R_2, R_3$ and $R_7)$ of the semigroup $S$ for $n = 3$

$R_1 = \{(12 -), (23 -), \}$,
$R_2 = \{(1 -2), (2 -3), \}$,
$R_3 = \{(-12), (-23), \}$ and
$R_4 = \{(1 -), (2 -), (3 -), \}$.
$R_5 = \{(1 -1), (2 -2), (3 -3), \}$, $R_7 = \{(- -)\}$

D. IDENTITY DIFFERENCE PARTIAL-ORDER TRANSFORMATION SEMIGROUP IDPO$_n$

$R_1 = \{(12,),(223),\}$, $R_2 = \{(122), (233),\}$
$R_3 = \{(12 -), (23 -),\}$, $R_4 = \{(-12), (-23),\}$
$R_5 = \{(1 -2), (2 -3),\}$
$R_6 = \{(1 -1), (2 -2), (3 -3),\}$
$R_7 = \{(- -)\}$ are the elements of IDPO$_3$ in their respective $\mathcal{R}$ classes.

$R_8, R_9, \ldots R_{13}$ are selected for each $\mathcal{R}$ class.

$\mathcal{R}$ is the generating set for $S$ since one element is selected at a constant map elements). Suppose also that the choice of picking $\alpha, \beta, b, d, \ldots \in (\alpha, \beta, (a, b), (c, d), \ldots \in S$ is done randomly, then the products $(a \alpha, b \beta, c \alpha, a \alpha, b \beta, c \alpha, d \alpha, \ldots \in \mathcal{R}$ must generate the set of elements in $S$. Hence $(\alpha, b, d, \ldots \in \mathcal{R}$ is the generating set for $S$ since one element is selected at random from each $\mathcal{R}$ which excludes the $\mathcal{R}$ classes containing the constant or identity maps.

In contradiction, Since $(\alpha, \beta, (a, b), (c, d), \ldots \in \mathcal{R}$ if the choice of selecting the elements of $A$ from different sets of $\mathcal{R}$ are random and $(\alpha, \beta, (a, b), (c, d), \ldots \in \mathcal{R}$ for $A$. Hence, $A$ is not a generating set for $S$ since the choice of elements does not consider the above stated conditions.

Thus, $\alpha \in R(S)$ iff $\alpha \in A$ and $A \subseteq S$. □

THEOREM 1

Let $\alpha \in S$ and $R(S)$ denote the rank of $S$. $\alpha \in R(S)$ if $\alpha \in A$ is the generating set of $S$.

Proof

Let $S$ be the identity difference transformation semigroup. Suppose $(\alpha, \beta, (a, b), (c, d), \ldots \in \mathcal{R}$ of $S$. Then $(\alpha, b, d, \ldots \in A$ if $(\alpha, b, d, \ldots \in \mathcal{R}$, (where $\mathcal{R}$ is a constant map elements). Suppose also that the choice of picking $\alpha, b, d, \ldots \in (\alpha, \beta, (a, b), (c, d), \ldots \in S$ is done randomly, then the products $(a \alpha, b \beta, c \alpha, a \alpha, b \beta, c \alpha, d \alpha, \ldots \in \mathcal{R}$ must generate the set of elements in $S$. Hence $(\alpha, b, d, \ldots \in \mathcal{R}$ is the generating set for $S$ since one element is selected at random from each $\mathcal{R}$ which excludes the $\mathcal{R}$ classes containing the constant or identity maps.

IV. THE RANKS OF IDENTITY DIFFERENCE TRANSFORMATION SEMIGROUP

In this section we shall give the theoretical and combinatorial proves for the ranks of IDT$_n$, IDO$_n$, IDI$_n$, IDPO$_n$, and IDPO$_n$ respectively.

A. RANK OF IDENTITY DIFFERENCE FULL TRANSFORMATION SEMIGROUP IDT$_n$

THEOREM 2

Let $S = IDT_n$. If $|\mathcal{R}|$ be the order of $\mathcal{R}$ in $S$. Then $\text{Rank} (S) = (|\mathcal{R}| - 1) = 2^{(n-1)} - 1$.

Proof

Let $uS, \ldots, n + (n + 1)(2^n - 2)S$ be the set of right ideals in $S \forall u \in S$. If $aS = bS = cS = \cdots nS (\forall a, b, c, \ldots, n \in S)$; then the elements in the set $(a, b, c, \ldots, n)$ are in same $\mathcal{R}$.

Suppose $a, b, c, \ldots, n$ are set of idempotent elements, then $|\text{im}e| = 1$ where $e$ is a constant map. Hence $a, b, c, \ldots, n \in E(S)$. Let $uS = vS = \cdots 2(n - 1)S$ be set of equal right ideals in $S$ where $p, q, \ldots, 2(n - 1)$.

Let $uS = vS = \cdots 2(n - 1)S \in E(S)$.

Then $p, q, \ldots, 2(n - 1)$ are the elements in the same $\mathcal{R}$ where $\mathcal{R}$ is a set of $\mathcal{R}$ containing the constant map.

Suppose that $|\mathcal{R}|$ in $S$ contain the set $\mathcal{R}$ such that $\forall \alpha \in S a(X_1, X_2, \ldots X_n) = i, i \geq 1$ is a constant map. Then $\text{Rank} (S) = \min\{|A|: A \subseteq S, \leq A \subseteq \mathcal{R} \}$ is $|\mathcal{R}| - |\mathcal{R} | = |\mathcal{R} | - 1$.

Without loss of generality, for $n \geq 3$, $(aS = bS = \cdots = nS), (pS = qS = \cdots = 2(n - 1), (mS = nS = \cdots = 2(n - 1), 2^{(n-1)}$ implying that there are $2^{(n-1)}$ $\mathcal{R}$ in $S$.

Hence the $\text{Rank} (S) = |\mathcal{R}| - 1 = 2^{(n-1)} - 1 = 3, 7, 15, 31, 63, \ldots \forall n \geq 3$. □

See the table below;

| $n \geq 3$ | $\text{Rank} = (|\mathcal{R}| - 1) = 2^{(n-1)} - 1$ |
|---|---|
| 3 | 3 |
| 4 | 7 |
| 5 | 15 |
| 6 | 31 |
| 7 | 63 |
| 8 | 127 |
| 9 | 255 |
| 10 | 511 |

THEOREM 3

Let $S$ be identity difference full transformation semigroup. The $\text{Rank}(S) = \frac{1}{2} \sum_{\mathcal{R}=0}^{n} \binom{n}{\mathcal{R}} - \binom{n}{\mathcal{R}} = 2^{(n-1)} - 1$
Let $R(S) = \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} - \binom{n}{p}$. 

Implying that there are $n$ ways $P$ (respectively $n$) can be represented since $\binom{n}{p}$ and $\binom{n}{p}$. 

Recall the identity 

$$\sum_{p=0}^{n} \binom{n}{p} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$ 

So that, 

$$\frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} = \frac{1}{2} \left[ \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \right] = \frac{1}{2} [2^n] = 2^{-1} \cdot 2^n = 2^{n-1}.$$ 

Therefore $\frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} - \binom{n}{n} = 2^{n-1} - 2^n = 2^{n-1} - 1.$

\[ \square \]

**B. RANK OF IDENTITY DIFFERENCE ORDER**

**THEOREM 4**

Let $S = IDO_n$. If $|\mathcal{R}|$ be the order of $\mathcal{R}$ in $S$. Then $\text{Rank}(S)$ is defined by the rule $2^{n-1} - 1$.

**LEMMA 1**

Let $S$ be $IDO_n$ and $IDO_n$. If $nR_{\text{classes}}$ is the number of set of $R-\text{classes}$ in $S$. Then $\text{Rank}(S)$ is defined by the rule $(nR_{\text{classes}} - 1)$.

**THEOREM 5**

$\text{Rank}(S) = \binom{n-1}{n-2} = (n-1)$. 

Proof

The expression $\binom{n-1}{n-2}$ is true that $(n-1)$ can be represented in $(n-2)$ ways and as such we have,

$$\binom{n-1}{n-2} = \frac{(n-1)!}{(n-2)! \cdot 1!} = \frac{(n-1)!}{(n-2)! (n-1)!} = (n-1).$$

Hence $\binom{n-1}{n-2} = (n-1)$.

\[ \square \]

**C. RANK OF IDENTITY DIFFERENCE SYMMETRIC INVERSE TRANSFORMATION SEMIGROUP IDO_n**

**THEOREM 6**

Let $IDO_n = S$ be the identity difference symmetric inverse transformation semigroup. If $|\mathcal{R}|$ be the order of set of $R$-classes in $S$ then $\text{Rank}(S) = (|\mathcal{R}| - (n+1)) = (n-1) + \sum_{i=1}^{n} i$.

Proof

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}, \alpha_{n+1} = S$ be the right ideals of $S$ Satisfy $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in S$ Satisfy $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} = S$.

Where $\alpha_1 = (x_1, x_2, \ldots, x_n), \alpha_2 = (x_2, x_3, \ldots, x_n), \alpha_3 = (x_3, x_4, \ldots, x_n), \alpha_4 = (x_4, x_5, \ldots, x_n), \ldots, \alpha_{n-1} = (x_{n-1}, x_{n-2}, \ldots, x_n), \alpha_n = (x_{n}, x_{n-1}, \ldots, x_n).$

Similarly, let $\beta_1, \beta_2, \ldots, \beta_n \in S$ such that $\beta_1 = (x_1, x_2, \ldots, x_n), \beta_2 = (x_2, x_3, \ldots, x_n), \ldots, \beta_{n-1} = (x_{n-1}, x_{n-2}, \ldots, x_n), \beta_n = (x_n, x_{n-1}, \ldots, x_n)$.

Observe that, for all elements from $\alpha_1, \alpha_2 \in S$ there are precisely two points of maps from domain to codomain and every other point in each element are empty. As such, all elements from $\alpha_1, \alpha_2 \ldots \alpha_n \in S$ Satisfy $\alpha_1, \alpha_2 \ldots \alpha_n \in S$.

Recall that in $S$, there exist the empty map $\xi$ such that $\xi(x_1) = \xi, \xi(x_2) = \xi, \ldots, \xi(x_n) = \xi$. Hence $\xi$ is a set in $\mathcal{R}$.

Therefore, 

$|\mathcal{R}| = \{ 1 \}$

$\text{Rank}(S) = \text{Min}(|\mathcal{R}|: A, B) = \text{Min}(|\mathcal{R}|: A, B)$.

Then $\text{Rank}(S)$ is obtained by picking an element from each set of $\mathcal{R}$ (say $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n+1} = S$) whose element must contain precisely two points of mappings and every other point empty map. Also, the choice of picking the elements of $A$ is of high importance since $A, B = S$.

Without loss of generality, in $\mathcal{R}(S)$
Let $S = IDPO_n$. Then $\text{Rank}(S) = \left[ \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} \right] \binom{n-2}{n-3} + \binom{n}{n-2} = 2^{n-1}(n-2) + 1$

Proof

$\text{Rank}(S) = \left[ \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} \right] \binom{n-2}{n-3} + \binom{n}{n-2} = \frac{1}{2} \binom{n}{n} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p}$

$= \frac{1}{2} \cdot 2^n(n-2) + 1 = 2^{n-1}(n-2) + 1.$

Therefore, $\text{Rank}(S) = \left[ \frac{1}{2} \sum_{p=0}^{n} \binom{n}{p} \right] \binom{n-2}{n-3} + \binom{n}{n} = 2^{n-1}(n-2) + 1.$

□

THEOREM 10

Let $IDPO_n = S$ be the identity difference partial order preserving transformation semigroup.

$\text{Rank}(S) = |\mathcal{R}| - 2^n = (n-2)2^{n-1} + 1$.

Proof

Let $a_1, a_2, \ldots, a_{2^{n-1}+2(n-2)(n^2-n)}$ be arbitrary elements of $S$. Suppose there are $n.2^{n-1} + 1$ sets of $\mathcal{R}$ in $S \forall n \geq 3$, then the sets must contain the partial identity maps say $\alpha_1 = (x_1 x_2 \ldots x_n), \alpha_2 = (x_{1+i} x_2 \ldots x_n), \ldots, \alpha_n = (x_1 x_2 \ldots x_n)$, the identity maps say $\beta_1 = (x_1 \ldots x_n), \ldots, \beta_n = (x_{1+i} \ldots x_n)$ and the empty map $\xi = (x_1 \ldots x_n)$ where $\{\alpha_1, \alpha_2, \ldots, \alpha_n; \beta_1, \beta_2, \ldots, \beta_n; \ldots, \xi\} \in \mathcal{R}$.

Let $|\mathcal{R}| = n.2^{(n-1)} + 1$ and $|\emptyset| = 2^n$. Then
Rank(S) = |R| - |∅| = (n.2^{(n-1)} + 1) - 2^n = n.2^{n-1} + 1 - 2^n = n.2^{n-1} - 2^n + 1 = 2^n(n.2^{-1} - 1) + 1 = 2^n \left(\frac{n}{2} - 1\right) + 1 = 2^n \left(\frac{n-2}{2}\right) + 1 = 2^{n-1}(n-2) + 1.

See table below;

| n ≥ 3 | |R| = n.2^{n-1} + 1 | Rank(S) = 2^{(n-1)}(n-2) + 1 | Rank − |R| = 2^n |
|-------|------------------|-------------------|-----------------|
| 3     | 13               | 5                 | 8               |
| 4     | 33               | 17                | 16              |
| 5     | 81               | 49                | 32              |
| 6     | 193              | 129               | 64              |
| 7     | 449              | 321               | 128             |
| 8     | 1025             | 769               | 256             |
| 9     | 2305             | 1793              | 512             |
| 10    | 5121             | 4097              | 1024            |

V. CONCLUSION

We therefore conclude that the rank of identity difference transformation semigroup exist and can be easily obtained using the R - classes of the respective subsemigroups as shown in section 3 and 4.

Conflict of Interest Statement:
This is to affirm that there is no conflict of interest amongst the authors or whosoever.

REFERENCES
[1] A. O. Adeniji, (2012). “Identity Difference Transformation Semigroups”. Ph.D. thesis. Department of mathematics, faculty of science, university of Ilorin, Ilorin, Nigeria.
[2] A. O. Adeniji, and S.O. Makanjuola, (2013). “Congruence in Identity difference full transformation semigroup”. International Journal of Algebra, Vol. 7, 2013, no. 12, 563-572.
[3] A. D. Adeshola and A. Umar (2013). “Combinatorial results for certain semigroups of order-preserving full contraction mappings of a finite chain”. arXiv:1303.7428v2[math.Co].
[4] A. J. Cain. (2015), “Lecture notes for a tour through semigroups”, Porto & Lisbon.
[5] V.H. Fernandes, and J. Sanwong (2014). “On the rank of semigroups of transformations on a finite set with restricted range”. Algebra colloq., (21), 497-510.
[6] G. U. Garba, Muhammad Jamilu Ibrahim, Abdussamad Tanko Imam, (2017). “On certain semigroups of full contraction maps of a finite chain”. Turkish Journal of Mathematics. 41:500-507.
[7] G. U. Garba (1997): “Idempotents, nilpotents, rank and order in finite transformation semigroups”. PhD thesis at the department of mathematical and computational sciences, University of St Andrews.
[8] Gomes, Gracinda M.S. and John M. Howie (1987): “On the ranks of certain finite semigroups of transformations”. Math. Proc. Cambridge Phil. Soc. 101:395 - 403.
[9] P. M. Higgins, J. M. Howie, J. D. Mitchell and N. Ruskuc, (2003). “Countable versus uncountable ranks in infinite semigroups of transformations and relations”. Proceedings of the Edinburgh mathematical society.
[10] J. M. Howie, (1995). “Fundamentals of Semigroup Theory”. Vol. 12 of London Mathematical Society Monographs (New Series), Clarendon Press, Oxford University Press, New York.
[11] J. A. Omelebele and U. I. Asibong-Ibe, (2019) “A case of Non-identity difference order preserving transformation semigroup”. Journal of Semigroup Theory and Applications. http://doi.org/10.28919/jsta/4133, ISSN:2051-2937.
[12] S. Nenthein, P. Youngkhong, and Y. Kemprasit, (2005). “Regular elements of some transformation semigroups”. Pure math. Appl. 16(3), 307-314.
[13] P. Zhao and M. Yang (2012). “Regularity And Green’s Relations On Semigroups Of Transformation Preserving Order And Compression”. Bull. Korean Math. Soc. 49 No. 5, pp. 1015–1025 http://dx.doi.org/10.4134/BKMS.2012.49.5.1015.
[14] J. Sanwong and W. Sommanee, (2008). “Regularity and Green’s relations on a semigroup of transformations with restricted range”. Int. J. Math. Sci, doi: 10.1155/794013.
[15] J. S. V. Symons, (1975). “Some results concerning a transformation semigroup”. J. Aust. Math. Soc. (Series A, 19(4), 413-425.
[16] T. Harju, (1996); “Semigroups (Lecture note)”. Department of mathematics, University of Turku.FIN-20014 Turku, Finland.
[17] G. Vicky (2000) Semigroup theory. Lecture note.

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