IN Variant Hypersurfaces Of Endomorphisms Of Projective Varieties

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Abstract. We consider surjective endomorphisms $f$ of degree $> 1$ on projective manifolds $X$ of Picard number one and their $f^{-1}$-stable hypersurfaces $V$, and show that $V$ is rationally chain connected. Also given is an optimal upper bound for the number of $f^{-1}$-stable prime divisors on (not necessarily smooth) projective varieties.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. Theorems 1.1 ∼ 1.3 below are our main results. We refer to [19, Definition 2.34] for the definitions of Kawamata log terminal (klt) and log canonical singularities. See S.-W. Zhang [32, §1.2, §4.1] for the Dynamic Manin-Mumford conjecture solved for the pair $(X, f)$ as in the conclusion part of Theorem 1.1 below, and [5] for a related result on endomorphisms of (not necessarily projective) compact complex manifolds.

Theorem 1.1. Let $X$ be a normal projective variety of dimension $n \geq 2$, $V_i$ ($1 \leq i \leq s$) prime divisors, $H$ an ample Cartier divisor, and $f : X \to X$ an endomorphism with $\deg(f) = q^n > 1$ such that (for all $i$):

1. $X$ has only log canonical singularities around $\cup V_i$;
2. $V_i$ is Cartier and $V_i \equiv d_i H$ (numerically) for some $d_i > 0$; and
3. $f^{-1}(V_i) = V_i$.

Then $s \leq n + 1$. Further, the equality $s = n + 1$ holds if and only if:

$X = \mathbb{P}^n$, $V_i = \{X_i = 0\}$ ($1 \leq i \leq n + 1$)

(in suitable projective coordinates), and $f$ is given by

$f : [X_1, \ldots, X_{n+1}] \to [X'^n_1, \ldots, X'^q_{n+1}]$.

The conditions (1) and (2) in Theorem 1.1 are satisfied if $X$ is smooth with Picard number $\rho(X) = 1$. The ampleness of $V_i$ in Theorem 1.1 above and the related result
Proposition 2.12 below (with the Cartier-ness of $V_i$ replaced by the weaker $\mathbb{Q}$-Cartier-ness) is quite necessary because there are endomorphisms $f$ of degree $>1$ on toric surfaces whose boundary divisors have as many irreducible components as you like and are all stabilized by $f^{-1}$. The condition (1) is used to guarantee the inversion of adjunction (cf. [14]) and can be removed in dimension two (cf. [9, Theorem B], [25, Theorem 4.3.1]).

A projective variety $X$ is rationally chain connected if every two points $x_i \in X$ are contained in a connected chain of rational curves on $X$. When $X$ is smooth, $X$ is rationally chain connected if and only if $X$ is rationally connected, in the sense of Campana, and Kollár-Miyaoka-Mori ([4], [18]).

The condition (1) below is satisfied if $X$ is $\mathbb{Q}$-factorial with Picard number $\rho(X) = 1$, while the smoothness (or at least the mildness of singularities) of $X$ in (3) is necessary (cf. Remark 1.8).

Theorem 1.2. Let $X$ be a normal projective variety of dimension $n \geq 2$, $f : X \to X$ an endomorphism of degree $>1$, $(0 \neq) V = \sum_i V_i \subset X$ a reduced divisor with $f^{-1}(V) = V$, and $H \subset X$ an ample Cartier divisor. Assume the three conditions below (for all $i$):

1. $-K_X \sim_\mathbb{Q} rH$ (\$\mathbb{Q}\$-linear equivalence) and $V_i \sim_\mathbb{Q} d_i H$ for some $r, d_i \in \mathbb{Q}$;
2. $X$ has only log canonical singularities around $V$; and
3. $X$ is further assumed to be smooth if: $V = V_1$ (i.e., $V$ is irreducible), $K_X + V \sim_\mathbb{Q} 0$ and $f$ is étale outside $V \cup f^{-1}(\text{Sing} \ X)$.

Then $X$, each irreducible component $V_i$ and the normalization of $V_i$ are all rationally chain connected. Further, $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor, i.e., $r > 0$ in (1).

A morphism $f : X \to X$ is polarized (by $H$) if

$$f^* H \sim qH$$

for some ample Cartier divisor $H$ and some $q > 0$; then

$$\deg(f) = q^{\dim X}.$$ 

For instance, every non-constant endomorphism of a projective variety $X$ of Picard number $\rho(X) = 1$, is polarized; an $f$-stable subvariety $X \subset \mathbb{P}^n$ for a non-constant endomorphism $f : \mathbb{P}^n \to \mathbb{P}^n$, has the restriction $f|_X : X \to X$ polarized by the hyperplane; the multiplication map

$$m_A : A \to A \ (x \mapsto mx)$$

By the recent paper of A. Broustet and A. Hoering "Singularities of varieties admitting an endomorphism," arXiv:1304.4013, the condition (1) in Theorem 1.1, condition (2) in Theorem 1.2 and similar conditions in Propositions 2.1 and 2.12 are automatically satisfied if $X$ is $\mathbb{Q}$-Gorenstein and has a polarized endomorphism of degree $>1$. 

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(with $m \neq 0$) of an abelian variety $A$ is polarized by any $H = L + (-1)^*L$ with $L$ an ample divisor, so that $m_A^*H \sim m^2H$.

In Theorems 1.1 and 1.3, we give upper bounds for the number of $f^{-1}$-stable prime divisors on a (not necessarily smooth) projective variety; the bounds are optimal, and the second possibility in Theorem 1.3(2) does occur (cf. Examples 1.9 and 1.10). One may remove Hyp(A) in Theorem 1.3, when the Picard number $\rho(X) = 1$, or $X$ is a weak $\mathbb{Q}$-Fano variety, or the closed cone $\overline{NE}(X)$ of effective curves has only finitely many extremal rays (cf. Remark 1.8). Denote by

$$ N^1(X) := \text{NS}(X) \otimes \mathbb{Z} \mathbb{R} $$

the Néron-Severi group (over $\mathbb{R}$) with $\rho(X) := \text{rank}_\mathbb{R} N^1(X)$ the Picard number.

**Theorem 1.3.** Let $X$ be a projective variety of dimension $n$ with only $\mathbb{Q}$-factorial Kawa-
mata log terminal singularities, and $f : X \to X$ a polarized endomorphism with $\text{deg}(f) = q^n > 1$. Assume Hyp(A) : either $f^*|_{N^1(X)} = q \text{id}_{N^1(X)}$, or $n \leq 3$. Then we have (with $\rho := \rho(X)$):

1. Let $V_i \subset X$ ($1 \leq i \leq c$) be prime divisors with $f^{-1}(V_i) = V_i$. Then $c \leq n + \rho$. Further, if $c \geq 1$, then the pair $(X, \sum V_i)$ is log canonical and $X$ is uniruled.
2. Suppose $c \geq n + \rho - 2$. Then either $X$ is rationally connected, or there is a fibration $X \to E$ onto an elliptic curve $E$ so that every fibre is normal and rationally connected and some positive power $f^k$ descends to an $f_E : E \to E$ of degree $q$.
3. Suppose that $c \geq n + \rho - 1$. Then $X$ is rationally connected.
4. Suppose that $c \geq n + \rho$. Then $c = n + \rho$, (for some $t > 0$)

$$ K_X + \sum_{i=1}^{n+\rho} V_i \sim_{\mathbb{Q}} 0, \quad (f^t)^*|_{\text{Pic}(X)} = q^t \text{id}_{\text{Pic}(X)}; $$

$f$ is étale outside $(\bigcup V_i) \cup f^{-1}(\text{Sing} X)$ (and $X$ is a toric surface with $\sum V_i$ its boundary divisor, when $\dim X = 2$).

Corollary 1.4 below is a special case of Theorem 1.2 and is known for $X = \mathbb{P}^n$ with $n \leq 3$ (cf. [10], [26]); the smoothness and Picard number one assumption on $X$ are necessary (cf. Remark 1.8 and Example 1.10). In Corollary 1.4 $X$ is indeed a Fano manifold; but one would like to know more about the $V$ and even expects $X = \mathbb{P}^n$ and $V$ be a hyperplane; see [30] and the references therein. Such an expectation is very hard to prove even in dimension three and proving the smoothness of $V$ is the key, hence the relevance of Proposition 2.1 below.

**Corollary 1.4.** Let $X$ be a projective manifold of dimension $n \geq 2$ and Picard number one, $f : X \to X$ an endomorphism of degree $> 1$, and $V \subset X$ a prime divisor
with $f^{-1}(V) = V$. Then $X$, $V$ and the normalization $V'$ of $V$ are all rationally chain connected.

**Corollary 1.5.** With the notation and assumptions in Corollary 1.4, both $X$ and $V$ are simply connected, while $V'$ has a finite (topological) fundamental group.

1.6. **Main ingredients of the proofs.** The results of Favre [9], Nakayama [25] and Wahl [28] are very inspiring about the restriction of the singularity type of a normal surface imposed by the existence of an endomorphism of degree $> 1$ on the surface. For the proof of our results, the basic ingredients are: a log canonical singularity criterion (Proposition 2.1), the inversion of adjunction in Kawakita [14], a rational connectedness criterion of Qi Zhang [31] and its generalization in Hacon-McKernan [12], the characterization in Mori [23] on hypersurfaces in weighted projective spaces, and the equivariant Minimal Model Program in our early paper [29].

Theorems 1.1 and 1.3 motivate the question below (without assuming the Hyp(A) in Theorem 1.3). Question 1.7 (2) is Shokurov’s conjecture (cf. [27, Thm 6.4]).

**Question 1.7.** Suppose that a projective $n$-fold ($n \geq 3$) $X$ has only $\mathbb{Q}$-factorial Kawamata log terminal singularities, $f : X \to X$ a polarized endomorphism of degree $> 1$, and $V_i \subset X$ ($1 \leq i \leq s$) prime divisors with $f^{-1}(V_i) = V_i$.

(1) Is it true that $s \leq n + \rho(X)$?

(2) If $s = n + \rho(X)$, is it true that $X$ is a toric variety with $\sum V_i$ its boundary divisor?

**Remark 1.8.** (1) In Corollary 1.4, it is necessary to assume that $\rho(X) = 1$ (cf. Example 1.10), and $X$ is smooth or at least Kawamata log terminal (klt). Indeed, for every projective cone $Y$ over an elliptic curve and every section $V \subset Y$ (away from the vertex), there is an endomorphism $f : Y \to Y$ of $\deg(f) > 1$ and with $f^{-1}(V) = V$ (cf. [25, Theorem 7.1.1, or Corollary 5.2.3]). The cone $Y$ has Picard number one and a log canonical singularity at its vertex. Of course, $V$ is an elliptic curve, and is not rationally chain connected. By the way, $Y$ is rationally chain connected, but is not rationally connected. Observe that $K_Y + V \sim_{\mathbb{Q}} 0$, in connection with the condition (3) in Theorem 1.2 which is a stronger version of Corollary 1.4.

(2) Let $X$ be a projective variety with only klt singularities. If the closed cone $\overline{NE}(X)$ of effective curves has only finitely many extremal rays, then every polarized endomorphism $f : X \to X$ satisfies

$$f^*|_{N_1(X)} = q \, \text{id}_{N_1(X)}, \quad \deg(f) = q^{\dim X}$$

after replacing $f$ by its power, so that we can apply Theorem 1.3 (cf. [26, Lemma 2.1]). For instance, if $X$ or $(X, \Delta)$ is weak $\mathbb{Q}$-Fano, i.e., $X$ (resp. $(X, \Delta)$) has only klt singularities
and $-K_X$ (resp. $-(K_X + \Delta)$) is nef and big, then $\overline{\text{NE}}(X)$ has only finitely many extremal rays.

(3) By Example 1.9 it is necessary to assume the local factoriality of $X$ or the Cartier-ness of $V_i$ in Theorem 1.1, even when $X$ has only klt singularities. We remark that a Q-factorial Gorenstein terminal threefold is locally factorial.

A smooth hypersurface $X$ in $\mathbb{P}^{n+1}$ with $\deg(X) \geq 3$ and $n \geq 2$, has no endomorphism $f_X : X \to X$ of degree $> 1$ (cf. [1], [6]). However, singular $X$ may have plenty of endomorphisms $f_X$ of arbitrary degrees as shown in Example 1.9 below.

**Example 1.9.** We now construct many polarized endomorphisms for some degree $n+1$ singular hypersurface $X \subset \mathbb{P}^{n+1}$. Let

$$f = (F_0, \ldots, F_n) : \mathbb{P}^n \to \mathbb{P}^n \ (n \geq 2)$$

with $F_i = F_i(X_0, \ldots, X_n)$ homogeneous, be any endomorphism of degree $q^n > 1$, such that $f^{-1}(S) = S$ for a reduced degree $n+1$ hypersurface $S = \{S(X_0, \ldots, X_n) = 0\}$. So $S$ must be normal crossing and linear: $S = \sum_{i=0}^n S_i$ (cf. [26, Thm 1.5 in arXiv version 1]). Thus we may assume that $f = (X_0^q, \ldots, X_n^q)$ and $S_i = \{X_i = 0\}$. The relation $S \sim (n+1)H$ with $H \subset \mathbb{P}^n$ a hyperplane, defines

$$\pi : X = \text{Spec} \oplus_{i=0}^n O(-iH) \to \mathbb{P}^n$$

which is a Galois $\mathbb{Z}/(n+1)$-cover branched over $S$ so that $\pi^*S_i = (n+1)T_i$ with the restriction $\pi|_{T_i} : T_i \to S_i$ an isomorphism.

This $X$ is identifiable with the degree $n+1$ hypersurface

$$\{Z^{n+1} = S(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}$$

and has singularity of type $z^{n+1} = xy$ over the intersection points of $S$ locally defined as $xy = 0$. We may assume that $f^*S(X_0, \ldots, X_n) = S(X_0, \ldots, X_n)^q$ after replacing $S(X_0, \ldots, X_n)$ by a scalar multiple, so $f$ lifts to an endomorphism

$$g = (Z^q, F_0, \ldots, F_n)$$

of $\mathbb{P}^{n+1}$ (with homogeneous coordinates $[Z, X_0, \ldots, X_n]$), stabilizing $X$, so that $g_X := g|_X : X \to X$ is a polarized endomorphism of $\deg(g_X) = q^n$ (cf. [26, Lemma 2.1]). Note that $g^{-1}(X)$ is the union of $q$ distinct hypersurfaces

$$\{Z^{n+1} = \zeta^iS(X_0, \ldots, X_n)\} \subset \mathbb{P}^{n+1}$$

(all isomorphic to $X$), where $\zeta := \exp(2\pi i/q)$.

This $X$ has only Kawamata log terminal singularities and $\text{Pic} \ X = (\text{Pic} \mathbb{P}^{n+1})|_X$ $(n \geq 2)$ is of rank one (using Lefschetz type theorem [20, Example 3.1.25] when $n \geq 3$). We have
\( f^{-1}(S_i) = S_i \) and \( g_X^{-1}(T_i) = T_i \) \((0 \leq i \leq n)\). Note that \((n + 1)T_i = \pi^*S_i\) is Cartier, but \(T_i\) is not Cartier; of course \(X \not\cong \mathbb{P}^n\) (compare with Theorem 1.1).

If \(n = 2\), the relation \((n + 1)(T_i - T_0) \sim 0\) gives rise to an étale-in-codimension-one \(\mathbb{Z}/(n + 1)\)-cover
\[
\tau : \mathbb{P}^n \simeq \tilde{X} \rightarrow X
\]
so that \(\sum_{i=0}^{n} \tau^{-1}T_i\) is a union of \(n + 1\) normal crossing hyperplanes; indeed, \(\tau\) restricted over \(X \setminus \text{Sing} X\), is its universal cover (cf. [21] Lemma 6]), so that \(g_X\) lifts up to \(\tilde{X}\). A similar result seems to be true for \(n \geq 3\), by considering the ‘composite’ of the \(\mathbb{Z}/(n + 1)\)-covers given by \((n + 1)(T_i - T_0) \sim 0\) \((1 \leq i < n)\); see Question 1.7.

The simple Example 1.10 below shows that the conditions in Theorem 1.3 (2)(3), or the condition \(\rho(X) = 1\) in Corollary 1.4, is necessary.

**Example 1.10.** Let \(m_A : A \rightarrow A\) \((x \mapsto mx)\) with \(m \geq 2\), be the multiplication map of an abelian variety \(A\) of dimension \(\geq 1\) and Picard number one. Let \(v \geq 1, q := m^2\) and
\[
g : \mathbb{P}^v \rightarrow \mathbb{P}^v \ ([X_1, \ldots, X_{v+1}] \mapsto [X_1^q, \ldots, X_{v+1}^q]).
\]
Then
\[
f = (m_A \times g) : X = A \times \mathbb{P}^v \rightarrow X
\]
is a polarized endomorphism with \(f^*|_{N^1(X)} = \text{diag}[q, q]\), and \(f^{-1}\) stabilizes \(v + 1\) prime divisors \(V_i = A \times \{X_i = 0\} \subset X\) and no others; indeed, \(f\) is étale outside \(\cup V_i\). Note that \(X\) and \(V_i \simeq A \times \mathbb{P}^{v-1}\) are not rationally chain connected, and
\[
v + 1 = \dim X + \rho(X) - (1 + \dim A).
\]

**Acknowledgement.** I would like to thank N. Nakayama for the comments and informing me about Shokurov’s conjecture (cf. 1.7) and Wahl’s result [28] Corollary, page 626], and the referee for very careful reading and valuable suggestions to improve the paper.

### 2. Proofs of Theorems 1.1 ∼ 1.3

We use the standard notation in Hartshorne’s book and [19] or [16]. For a finite morphism \(f : X \rightarrow Y\) between normal varieties (especially for a surjective endomorphism \(f : X \rightarrow X\) of a normal projective variety \(X\)), we can define the pullback \(f^*L\) on \(X\) of a Weil divisor \(L\) on \(Y\), as the Zariski-closure of \((f|_U)^*(L|_U)\) where \(U \subset X\) (resp. \(V \subset Y\)) is a smooth Zariski-open subset of codimension \(\geq 2\) in \(X\) (resp. \(Y\)). When \(L\) is \(\mathbb{Q}\)-Cartier, our \(f^*L\) coincides with the usual pullback (or total transform) of \(L\).

In §2, we shall prove 1.1 ∼ 1.5 in the Introduction, and Propositions 2.1, 2.2 and 2.12 below.
The following log canonical singularity criterion is frequently used in proving the main results and should be of interest in its own right.

**Proposition 2.1.** Let $X$ be a normal (algebraic or analytic) variety, $f : X \to X$ a surjective endomorphism of $\deg(f) > 1$ and $(0 \neq) D$ a reduced divisor with $f^{-1}(D) = D$. Assume:

1. $X$ is log canonical around $D$ (cf. [19, Definition 2.34]);
2. $D$ is $\mathbb{Q}$-Cartier; and
3. $f$ is ramified around $D$.

Then the pair $(X, D)$ is log canonical around $D$. In particular, $D$ is normal crossing outside the union of $\text{Sing} X$ and a codimension three subset of $X$.

Proposition 2.2 below is used in the proof of Theorem 1.1. When $\dim X = 2$, Propositions 2.1 and 2.2 are shown by Nakayama [25, Theorem 4.3.1], and the proof of [25, Lemma 2.7.9] which seems to be effective in higher dimensions, as commented by Nakayama (cf. also Wahl [28, page 626] and Favre [9, Theorem B]); but the log canonical modification used in [25, Theorem 4.3.1] was not available then in higher dimensions. To avoid such problem, in our proof of Propositions 2.1 and 2.2, we compute log canonical threshold and discrepancy in the spirit of [19, Proposition 5.20].

**Proposition 2.2.** Let $X$ be a normal (algebraic or analytic) variety, $f : X \to X$ a surjective endomorphism of $\deg(f) > 1$ and $(0 \neq) D$ a reduced divisor with $f^{-1}(D) = D$. Assume:

1. There are effective $\mathbb{Q}$-divisors $G$ and $\Delta$ such that the pair $(X, G)$ has only log canonical singularities around $D$, and $K_X + G + D = f^*(K_X + G + D) + \Delta$, i.e., the ramification divisor $R_f = f^*(G + D) - (G + D) + \Delta$;
2. $D$ is $\mathbb{Q}$-Cartier; and
3. $f$ is ramified around $D$.

Then the pair $(X, G)$ has only purely log terminal singularities around $D$ (cf. [19, Def 2.34]). In particular, the structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_D$ are Cohen-Macaulay around $D$.

**2.3.** We now prove Propositions 2.1 and 2.2. We prove Proposition 2.1 first. Since the result is local in nature, we may assume that $X$ is log canonical. Consider the following log canonical threshold of $(X, D)$:

$$c := \max\{t \in \mathbb{R} \mid (X, tD) \text{ is log canonical}\}.$$ 

Then $0 \leq c \leq 1$. We may assume that $c < 1$ and shall reach a contradiction late. Let $D = \sum D_i$ be the irreducible decomposition with $D_i \neq D_j$ when $i \neq j$. Since
\( f^{-1}(D) = D \), we may assume that \( f^{-1}(D_i) = D_i \) after replacing \( f \) by its power. Since \( f \) is ramified around \( D \), we may write \( f^*D_i = q_iD_i \) for some \( q_i > 1 \). Thus
\[
K_X + D = f^*(K_X + D) + \Delta
\]
where \( \Delta \) is an effective integral Weil divisor not containing any \( D_i \), so that
\[
R_f := \sum (q_i - 1)D_i + \Delta
\]
is the ramification divisor of \( f \). We can write
\[
K_X + cD - \Delta - (1 - c) \sum (q_i - 1)D_i = f^*(K_X + cD).
\]
To distinguish the source and target of \( f \), we denote by \( f : X_1 = X \to X_2 = X \). For the pairs \((X_i, \Gamma_i)\) with
\[
X_i := X, \quad \Gamma_1 := cD - \Delta - (1 - c) \sum (q_i - 1)D_i
\]
(which is \( \mathbb{Q} \)-Cartier because so are \( K_X \) and \( f^*(K_X + cD) \)) and \( \Gamma_2 := cD \), we apply [19 Proposition 5.20]. By the definition of the log canonical threshold \( c \), there is an exceptional divisor \( E_2 \) (in a blowup of \( X_2 \)) with its image (the centre) contained in \( D \subset X_2 \) such that the discrepancy \( a(E_2, X_2, \Gamma_2) = -1 \). Let \( E_1 \) be an exceptional divisor (in a blowup of \( X_1 \)) which dominates \( E_2 \), via a lifting \( f' \) of \( f \), and hence has image (on \( X_1 \)) contained in \( D \). Here we use the assumption that \( f^{-1}(D) = D \). On the one hand, [19 Proposition 5.20] shows that
\[
Eq(2.1.1) \quad a(E_1, X_1, \Gamma_1) + 1 = r(a(E_2, X_2, \Gamma_2) + 1) = 0
\]
where \( r \geq 1 \) is the ramification index of \( f' \) along \( E_1 \). On the other hand, by [19 Lemma 2.27] and noting that \( E_1 \) has image in \( D \) and hence in the support of the effective divisor
\[
\Delta + (1 - c) \sum (q_i - 1)D_i = cD - \Gamma_1
\]
(which is \( \mathbb{Q} \)-Cartier because so is \( \Gamma_1 \) as mentioned early on), we have
\[
a(E_1, X_1, \Gamma_1) > a(E_1, X_1, cD) \geq -1
\]
since \((X, cD)\) is log canonical. This contradicts the display Eq(2.1.1) above. Therefore, \( c \geq 1 \) and \((X, D)\) is log canonical. This proves Proposition 2.1.

For Proposition 2.2 consider, in the notation above,
\[
K_X + G - \Delta - \sum (q_i - 1)D_i = f^*(K_X + G)
\]
and pairs \((X, G - \Delta - \sum (q_i - 1)D_i)\) and \((X, G)\). Then, using [19 Proposition 5.20, Lemma 2.27, Corollary 5.25], Proposition 2.2 can be proved as above.
2.4. We prove Theorem 1.2. By the assumption, \( f^{-1} \) stabilizes the reduced divisor \( V = \sum_i V_i, -K_X \sim_Q rH \) and \( V_i \sim_Q d_iH \) for some \( r, d_i, \in \mathbb{Q} \) and an ample Cartier divisor \( H \). Replacing \( f \) by its power, we may assume \( f(V_i) = V_i \) so that \( f^*V_i = q_iV_i \) for some \( q_i > 0 \). Since \( K_X \) and \( V_i \)’s are all proportional to \( H \), all \( q_i \)’s are the same, \( f^*K_X \sim_Q qK_X \) and \( f^*H \sim_Q qH \) with \( q := q_i = \sqrt{\deg(f)} > 1 \). Write

\[
K_X = f^*K_X + R_f
\]

with \( R_f \) the (effective) ramification divisor. Then \( R_f = (q - 1)V + \Delta \) with \( \Delta \) an effective Weil divisor which does not contain any \( V_i \). Thus

\[
K_X + V = f^*(K_X + V) + \Delta
\]

and

\[
0 \leq \Delta \sim_Q (1 - q)(K_X + V) \sim_Q (q - 1)(r - d)H
\]

where \( d := \sum d_i \). So \( r \geq d > 0 \). Let

\[
\sigma : V'_1 \to V_1
\]

be the normalization. By the subadjunction (cf. [17] Corollary 16.7]), we have

\[
K_{V'_1} + C' = \sigma^*(K_X + V)|_{V_1} \sim_Q -(r - d)\sigma^*(H|_{V_1})
\]

where \( C' \) is the sum of \( \sigma^*(V - V_1)|_{V_1} \), some non-negative contribution from the singularity of the pair \((X, V)\), and the conductor of \( V'_1 \) over \( V_1 \) (an integral effective Weil divisor).

We set \( \sigma = \text{id} \) when \( V_1 \) is normal.

We apply Proposition 2.1 to \((X, D, f) = (X, V, f)\). Since \( f^*V = qV \) with \( q > 1 \), our \( f \) is ramified along \( V \) with ramification index \( q \). Thus all conditions of Proposition 2.1 are satisfied; hence \((X, D)\) is log canonical around \( V \). By [14] Theorem, the pair \((V'_1, C')\) is also log canonical. If \( \Delta > 0 \), i.e., \( r > d \) \((> 0)\), then both \(-K_X \sim_Q rH \) and

\[
-(K_{V'_1} + C') \sim_Q (r - d)\sigma^*(H|_{V_1})
\]

are ample, so \( X, V'_1 \) (and hence \( V_1 \)) are all rationally chain connected by [12] Cor 1.3.

Suppose \( \Delta = 0 \), i.e., \( r = d \) \((> 0)\). Then \( K_X + V \sim_Q 0 \), and \( f \) is étale outside \( V \cup f^{-1}(\text{Sing} X) \) since the ramification divisor \( R_f = (q - 1)V \) now and by the purity of branch loci. If \( V = V_1 \) then \( X \) is smooth by the assumption, so \( X = \mathbb{P}^n \) and \( V \) is a union of \( n + 1 \) hyperplanes (cf. [13] Theorem 2.1]), contradicting the irreducibility of \( V \). If \( V \geq V_1 + V_2 \), then \( C'' := C' - \sigma^*(V_2|_{V_1}) \geq 0 \), the pair \((V'_1, C'')\) is log canonical (cf. [19] Corollary 2.35]) and \( -(K_{V'_1} + C'') \sim_Q \sigma^*(V_2|_{V_1}) \) is ample, so \( V'_1 \) is rationally chain connected by [12] Corollary 1.3]. This proves Theorem 1.2.
2.5. We prove Corollary 1.5 by a well known result of Campana [4], a rationally chain connected normal projective variety $Y$ has a finite (topological) fundamental group $\pi_1(Y)$; further, $\pi_1(Y) = \{1\}$ for smooth $Y$. This, Lefschetz hyperplane section theorem [20, Theorem 3.1.21] and Theorem 1.2 imply Corollary 1.5 except the triviality of $\pi_1(V)$ when $\dim X = 2$. Now assume $\dim X = 2$. Since $X$ is smooth and rationally chain connected, $X$ is rational. Thus $X \simeq \mathbb{P}^2$ since $X$ has Picard number one. Hence $V$, being $f^{-1}$-stabilized, is a line (cf. e.g. [26, Thm 1.5 in arXiv version 1]). So $V \simeq \mathbb{P}^1$ is simply connected. This proves Corollary 1.5.

2.6. The results below are used in the proof of Theorem 1.1 and Proposition 2.12.

Let us now define numerical equivalence on a normal projective surface $S$. First, one can define intersection form on $S$, using Mumford pullback. To be precise, let $\tau : S' \to S$ be a minimal resolution. For a Weil divisor $D$ on $S$, define the pullback $\tau^*D := \tau'^*D + \sum a_i E_i$ where $\tau' D$ is the proper transform of $D$ and $E_i$ are $\tau$-exceptional curves, and $a_i \in \mathbb{R}$ are uniquely determined (by the negativity of the matrix $(E_i E_j)$) and the condition $\tau^*D. E_j = 0$ for all $j$. We define the intersection $D_1, D_2 := \tau'^* D_1, \tau'^* D_2$. Weil divisors $D_1$ and $D_2$ on $S$ are called numerically equivalent if $D_1.C = D_2.C$ for every curve $C$ on $S$. This way, we have defined an equivalence relation among Weil divisors on $S$. The equivalence class containing $D$ is called the numerical Weil divisor class containing $D$.

Lemma 2.7 below is known to Iitaka, Sommese, Y. Fujimoto, and Nakayama [25, Lemma 3.7.1], . . . . We reprove it here for the convenience of the readers.

**Lemma 2.7.** Let $X$ be a normal projective variety of dimension $n$ and $f : X \to X$ an endomorphism with $\deg(f) \geq 2$. Supposer that the canonical (Weil) divisor $K_X$ is pseudo-effective (see [24, Ch II, Definition 5.5]). Then $f$ is étale in codimension one.

**Proof.** Write $K_X = f^* K_X + R_f$ with $R_f \geq 0$ the ramification (integral) divisor, noting that the pullback $f^*$ is defined at the beginning of §2. Substituting this expression of $K_X$ to the right hand side $(s-1)$-times, we get $K_X = (f^s)^* K_X + \sum_{i=0}^{s-1} (f^i)^* R_f$. Take an ample Cartier divisor $H$ on $X$. If $R_f = 0$, then we are done. Otherwise, the pseudo-effectivity of $K_X$ and [3] imply that $(R_f$ being an integral Weil divisor)

$$K_X.H^{n-1} = (f^s)^* K_X.H^{n-1} + \sum_{i=0}^{s-1} (f^i)^* R_f.H^{n-1} \geq s.$$ 

Let $s \to \infty$. We get a contradiction. \hfill \Box

**Lemma 2.8.** Let $S$ be a normal projective surface, $\tau : S' \to S$ the minimal resolution, $f : S \to S$ a polarized endomorphism with $\deg(f) = q^2 > 1$, and $D$, $\Delta$ effective Weil divisors such that $K_S + D = f^* (K_S + D) + \Delta$, i.e., the ramification divisor $R_f = f^* D -$
$D + \Delta$. Suppose that $D$ is an integral divisor, and $\text{Supp } D = \bigcup_{i=1}^{r} D_i$ has $r \geq 3$ irreducible components and (the dual graph of) it contains a loop. Replacing $f$ by its power, we have:

1. $S$ is klt. The pair $(S, D)$ has only log canonical singularities; so no three of $D_i$ share the same point.
2. $D$ is reduced, and $f^* D_i = q D_i$ for every $i$.
3. $\Delta = 0$, $K_S + D \sim 0$, and $f$ is étale outside $D \cup f^{-1}(\text{Sing } S)$.
4. $S$ is a rational surface. Every singularity of $S$ is either Du Val and away from $D$, or is a cyclic quotient singularity and lies in $\text{Sing } D$. $\tau^{-1} D$ is a simple loop of $\mathbb{P}^1$'s.
5. $f^* L \sim q L$ (resp. $f^* L \sim_{Q} q L$) for every Cartier (resp. Weil) divisor $L$ on $S$, so $f^* = q \text{id}$ on $\text{Pic } S$ (resp. on Weil divisor classes).

**Proof.** For (1), by [25, Theorem 4.3.1] or [9, Theorem B], both $S$ and the pair $(S, D)$ have only log canonical singularities. We will see that $S$ is klt in (4).

For (2), see [26, Lemmas 5.3 and 2.1 in arXiv version 1].

By (2), $f$ is not étale in codimension one, and hence $K_S$ is not pseudo-effective (cf. Lemma 2.7 or [25, Lemma 3.7.1]). So $S'$ is a ruled surface. Also, $f^* = q \text{id}$, on the numerical Weil divisor classes, after $f$ is replaced by its power (cf. [29, Theorem 2.7]). Thus

$$E(q(2.8.1)) \quad 0 \leq \Delta \equiv -(q - 1)(K_S + D), \quad -K_S \equiv D + \Delta/(q - 1).$$

Since $f^* = q \text{id}$ and $K_S$ is not pseudo-effective, the classification result of [23, Theorem 6.3.1] says that $S$ is either a rational surface, or an elliptic (smooth minimal) ruled surface, or a cone over an elliptic curve. If $S$ is elliptic ruled with $F$ a general fibre, then intersecting $F$ with Eq(2.8.1) above and noting that $D$ contains a loop, we may assume that $F(D_1 + D_2) = 2$ and $F.D_j = 0 (j = 3, \ldots, r)$; then $K_S + D_1 + D_2$ is pseudo-effective by using Hartshorne’s book, Chapter V, Propositions 2.20 and 2.21, which gives a contradiction:

$$K_S + D_1 + D_2 \equiv -\left(\sum_{i \geq 3} D_i + \Delta/(q - 1)\right) < 0.$$

If $S$ is a cone then $K_S + D_1$ is pseudo-effective, since $D$ contains a loop and hence we can find some $D_1 \leq D$ horizontal to generating lines, a contradiction as above.

Thus $S$ is a rational surface. Write

$$\tau^*(K_S + D) = K_S' + D' + \Sigma_1 + \Sigma_2 + \Sigma_3$$
where $D' = \tau' D$ is the proper transform of $D$, $\Sigma_i \geq 0$,

\[
\operatorname{Supp} \Sigma_1 = \tau^{-1}((\text{Sing } S) \cap (\text{Sing } D)),
\]

\[
\operatorname{Supp} \Sigma_2 = \tau^{-1}((\text{Sing } S) \cap (D \setminus \text{Sing } D)),
\]

\[
\operatorname{Supp} \Sigma_3 \subset \tau^{-1}((\text{Sing } S) \setminus D).
\]

By the results on (1) in the first paragraph and [15, Theorem 9.6], $(D$ and hence) $D' + \Sigma_1$ are reduced and contain a loop. Thus $K_{S'} + D' + \Sigma_1 \sim G \geq 0$ by the Riemann-Roch theorem (cf. [7, Lemma 2.3]). Pushing forward, we get $K_S + D \sim \tau_* G \geq 0$. This and the displayed Eq(2.8.1) above imply $\Delta = 0 = \tau_* G$, so (3) is true by the assumption on $R_f$ and the purity of branch loci.

Since $0 \sim \tau^* (K_S + D) = K_{S'} + D' + \Sigma_1 + \Sigma_2 + \Sigma_3 \sim G + \Sigma_2 + \Sigma_3$

we have $G = 0 = \Sigma_i$ ($i = 2, 3$). Now (4) follows from $\Sigma_i = 0$ ($i = 2, 3$), the results on (1) in the first paragraph, and the Riemann-Roch theorem (cf. the proofs of [7, Lemmas 2.2 and 2.3]).

(5) follows from [29, Theorem 2.7]. Indeed, $S$ is klt and hence $\mathbb{Q}$-factorial. The argument below is valid in any dimension for later use: since $S$ is klt (and rational, i.e. rational connected), $S'$ is rational (i.e., rational connected, cf. [12, Corollary 1.5]); thus $\pi_1(S')$ (and hence $\pi_1(S)$) are trivial (hence $q(S) = 0$), by a well-known result of Campana [4]; so $\operatorname{Pic} S$ is torsion free. $\square$

2.9. Proof of Theorem 1.1

By the assumption, $f^{-1}(V_i) = V_i$ and $V_i \equiv d_i H$ for some $d_i > 0$, so each $V_i$ is an ample Cartier divisor. Suppose there are $s \geq n + 1$ of such $V_i$. We have $f^* V_i = q V_i$ since $q^n = \deg(f) = (f^* V_i)^n / V_i^n$. So $f$ is polarized by $V_1$. We may assume that $H = V_1$, since all $V_i$ are (numerically) proportional to each other by the assumption. We shall inductively construct log canonical pairs $(X_i, D_i)$ ($1 \leq i \leq n - 2$) with $\dim X_i = n - i$.

Let

\[
X_0 := X, \quad D_0 := \sum_{i=1}^s V_i.
\]

By Proposition 2.1, the pair $(X_0, D_0)$ is log canonical around $D_0$. Let $\sigma_1 : X_1 \to X_0$ be the normalization of $V_1 \subset X_0$. Write

\[
K_{X_1} + D_1 = \sigma_1^* (K_{X_0} + D_0)
\]

so that the pair $(X_1, D_1)$ is again log canonical (cf. [14, Theorem]). Let $\Gamma_1 \subset X_1$ be the conductor divisor of $\sigma_1$. Set $\Gamma_1 = 0$ when $V_1$ is normal. By the calculation of the different
in [17, Corollary 16.7],

\[ D_1 \geq \Gamma_1 + \sum_{k=2}^{s} \sigma_1^* V_k; \]

hence the right hand side is reduced and each of its last \( s-1 \) term is nonzero and connected by the ampleness of \( V_k (\equiv d_kH) \). Our \( f \) lifts to an endomorphism \( f_1 : X_1 \to X_1 \) polarized by \( \sigma_1^* H \) so that \( f_1^{-1} \) stabilizes \( \Gamma_1 \) and \( \sigma_1^* V_k (k \geq 2) \) after replacing \( f \) by its power (cf. [26, Proposition 5.4 in arXiv version 1]). We repeat the process. Let \( \sigma_2 : X_2 \to X_1 \) be the normalization of an irreducible component of \( \sigma_1^* V_2 \subset X_1 \) which meets \( \Gamma_1 \) when it is nonzero; here we use the ampleness of \( V_2 (\equiv d_2H) \). Write

\[ K_{X_2} + D_2 = \sigma_2^*(K_{X_1} + D_1) \]

so that the pair \((X_2, D_2)\) is again log canonical. We have

\[ D_2 \geq \sigma_2^* \Gamma_1 + \sum_{k=3}^{s} \sigma_2^* \sigma_1^* V_k. \]

Our \( f_1 \) lifts to an endomorphism \( f_2 : X_2 \to X_2 \) polarized by \( \sigma_2^* \sigma_1^* H \) so that \( f_2^{-1} \) stabilizes each term of \( \sigma_2^* \Gamma_1 + \sum_{k=3}^{s} \sigma_2^* \sigma_1^* V_k \) after replacing \( f \) by its power. Thus we can construct normalizations (onto the images)

\[ \sigma_i : X_i \to X_{i-1} \quad (1 \leq i \leq n-2), \]

log canonical pairs \((X_i, D_i)\) with

\[ D_i \geq (\sigma_2 \cdots \sigma_i)^* \Gamma_1 + \sum_{k=i+1}^{s} (\sigma_1 \cdots \sigma_i)^* V_k \]

and endomorphisms \( f_i : X_i \to X_i \) polarized by the pullback of \( H \) and hence of \( \deg(f_i) = q^{\dim X_i} = q^{n-i} \) (cf. [26, Lemma 2.1]).

\( S := X_{n-2} \) is a normal surface with ample reduced (Cartier) divisors

\[ C_i := (\sigma_1 \cdots \sigma_{n-2})^* V_i \quad (n-1 \leq i \leq s) \]

so that \((S, D_{n-2}), (S, C)\) and \( S \) are all log canonical, where

\[ C := \sum_{i=n-1}^{s} C_i \leq D_{n-2} \]

(cf. [19, Notation 4.1], or [25, Remark 2.7.3, Theorem 2.7.4]). By the construction, \( f_{n-2}^{-1} \) stabilizes \( C_i \) and hence its irreducible components \( C_{ij} \) (after replacing \( f \) by its power), so \( f_{n-2}^* C_{ij} = qC_{ij} \) (cf. [26, Lemma 2.1]). Write

\[ K_S + C = f_{n-2}^*(K_S + C) + \Delta \]

with an effective Weil divisor \( \Delta \) containing no any \( C_{ij} \).
Claim 2.10. The following are true.

1. \( s = n + 1, K_S + C \sim \mathcal{O}, \Delta = 0, \) and \( f_{n-2} : S \to S \) is étale outside \( C \cup f_{n-2}^{-1}(\text{Sing } S) \).
2. \( S = X_{n-2} \cong \mathbb{P}^2, \) and \( C = \sum C_i = \sum_{i=n-1}^{n+1} (\sigma_1 \cdots \sigma_{n-2})^* V_i \) is the sum of three normal crossing lines.

Proof. Since each \( C_i \) is ample, \( \sum_{i=n-1}^{n+1} C_i \) contains a loop. Then (1) follows from Lemma 2.8 noting that \( f_{n-2}^* C_i = qC_i \) (for \( i = n - 1, \ldots, s \)) with \( s \geq n + 1 \). Since \( -K_S \sim C \), our \( S \) is Gorenstein and also klt by Lemma 2.8, so \( S \) is contracted to a point on \( X \). Thus \( \sigma (X) \) is normal (and also connected by the ampleness of \( S \)). The second assertion follows from the first with \( n = 2 \) so that \( f_{n-2}^* \Theta = q \Theta \) (cf. [26, Lemma 2.1]), contradicting Claim 2.10. Thus, by Serre’s criterion, this reduced divisor is normal (and also connected by the ampleness of \( V_i \)). The second assertion follows from the first with \( k = 0 \) and by relabeling \( V_k \).

Claim 2.11. Every \( \sigma_{k+1} \) \((0 \leq k \leq n-3)\) is an embedding onto its image, i.e., \((\sigma_1 \cdots \sigma_k)^* V_{k+1}\) is an (irreducible) normal Cohen-Macaulay variety. Hence every \( V_i \) \((1 \leq i \leq s = n + 1)\) is a normal variety.

Proof. Since \((X_k, D_k)\) is log canonical and \( D_k = (\sigma_1 \cdots \sigma_k)^* V_{k+1}\) (other effective divisor), \((X_k, D_k - (\sigma_1 \cdots \sigma_k)^* V_{k+1})\) is also log canonical (cf. [19, Corollary 2.35]). By Proposition 2.12 (for its condition (1) about \( R_{f_k} \), see the proof of Proposition 2.12 below), the reduced divisor \((\sigma_1 \cdots \sigma_k)^* V_{k+1}\) is Cohen-Macaulay. If this reduced divisor is not regular in codimension one, then the conductor divisor of \( X_{k+1} \) over it will give rise to an effective divisor \( \Theta \) in \( D_{n-2} - C \) (as we did for \( \Gamma_1 \)) which is preserved by \( f_{n-2}^{-1} \) so that \( f_{n-2}^* \Theta = q \Theta \) (cf. [26, Lemma 2.1]), contradicting Claim 2.10. Thus, by Serre’s \( R_1 + S_2 \) criterion, this reduced divisor is normal (and also connected by the ampleness of \( V_i \)). The second assertion follows from the first with \( k = 0 \) and by relabeling \( V_k \).

We continue the proof of Theorem 1.1. By Claim 2.11, \( X_k \) \((1 \leq k \leq n - 2)\) is equal to \( \cap_{i=1}^{k} V_i \) and Cohen-Macaulay. We now apply [26, Theorem 3.6] to show inductively the assertion that

\[
(X_i, \mathcal{O}(V_{i+1}|X_i)) \cong (\mathbb{P}^{n-i}, \mathcal{O}(1)) \quad (0 \leq i \leq n - 2).
\]
Relabel $V_i$ so that $d_{n+1} \geq d_n \geq \cdots \geq d_1$. By Claim 2.10

$$\left(X_{n-2}, \mathcal{O}(V_{n-2}|X_{n-2})\right) \simeq (\mathbb{P}^2, \mathcal{O}(1)).$$

Note that $X_{n-2} = V_{n-2}|X_{n-3}$, and $\mathcal{O}(X_{n-2}|X_{n-2}) \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ because

$$1 \leq (X_{n-2}|X_{n-2})^2 = (V_{n-2}|X_{n-2})^2 \leq (V_{n-1}|S)^2 = C_{n-1}^2 = 1$$

(here we used $d_{n-1} \geq d_{n-2}$). Thus, by [ibid.],

$$\left(X_{n-3}, \mathcal{O}(V_{n-2}|X_{n-3})\right) \simeq (\mathbb{P}^3, \mathcal{O}(1)).$$

Suppose the assertion is true for $i \geq k$. Then

$$\mathbb{P}^{n-k} \simeq X_k = V_k|X_{k-1}, \quad \mathcal{O}(X_k|X_k) \simeq \mathcal{O}_{\mathbb{P}^{n-k}}(1)$$

because

$$1 \leq (X_k|X_k)^{n-k} = (V_k|X_k)^{n-k} \leq (V_{k+1}|X_k)(V_k|X_k)^{n-k-1} = X_{k+1}(V_k|X_k)^{n-k-1}$$

$$(V_k|X_{k+1})^{n-k-1} \leq (V_{k+1}|X_{k+1})^{n-k-1} \leq \cdots \leq (V_{n-2}|X_{n-2})^2 = 1.$$ 

Thus the assertion is true for $i = k - 1$ by [ibid.]. This proves the assertion.

Now take $H \subset X = \mathbb{P}^n$ to be the hyperplane and $d_i = \deg(V_i)$. We have

$$K_X + \sum V_i = f^*(K_X + \sum V_i) + N$$

where $N$ is an effective Weil divisor. Thus

$$0 \leq N \sim (1 - q)(K_X + \sum V_i) \sim (q - 1)(n + 1 - \sum d_i)H$$

and hence $n + 1 \geq \sum_{i=1}^{n+1} d_i \geq n + 1$. So $d_i = 1$. By [25, Thm 1.5 in arXiv version 1], $\bigcup V_i$ is a normal crossing union of $n + 1$ hyperplanes, so that we may assume that $V_i = \{X_i = 0\}$, and also $f^*X_i = X_i^t$ (after replacing $X_i$ by a scalar multiple) since $f^*V_i = qV_i$. This proves Theorem 1.1 because the last ‘if part’ is clear.

If the Cartier-ness of $V_i$ in Theorem 1.1 is replaced by the weaker $\mathbb{Q}$-Cartier-ness, we have the following, where the condition (2) is true when $\rho(X) = 1$ and $X$ is $\mathbb{Q}$-factorial.

**Proposition 2.12.** Let $X$ be a normal projective variety of dimension $n \geq 2$, $V_i$ ($1 \leq i \leq s$) prime divisors, and $f : X \to X$ an endomorphism with $\deg(f) = q^n > 1$ such that:

1. $X$ has only log canonical singularities around $\bigcup V_i$;
2. every $V_i$ is $\mathbb{Q}$-Cartier and ample; and
3. $f^{-1}(V_i) = V_i$ for all $i$.

Then $s \leq n+1$ and $s = n+1$ only if: $f$ is étale outside $(\bigcup V_i) \cup f^{-1}(\text{Sing } X)$, and $\bigcap_{i=1}^t V_{b_i} \subset X$ is a normal (irreducible) subvariety for every subset $\{b_1, \ldots, b_t\} \subseteq \{1, \ldots, n + 1\}$ with $1 \leq t \leq n - 2$ (and further, $K_X + \sum_{i=1}^{n+1} V_i \equiv 0$ provided that $\rho(X) = 1$ or $f^*K_X \equiv qK_X$).
Proof. We assume that \( s \geq n + 1 \) and use the notation and steps in the proof of Theorem \[ 1 \] Then \( f^*V_i = qV_i \) and \( f \) is polarized by a multiple \( H \) of \( V_1 \). Write

\[
K_X + D_0 = f^*(K_X + D_0) + \Delta_f
\]

with \( \Delta_f \) an effective divisor containing no any \( V_i \). Pulling back by the normalization (onto \( V_i \)) \( \sigma_1 : X_1 \to X_0 = X \), we have

\[
K_{X_1} + D_1 = f_1^*(K_{X_1} + D_1) + \sigma_1^*\Delta_f
\]

where \( f_1 : X_1 \to X_1 \) is lifted from \( f|_{V_1} \) and polarized by \( \sigma_1^*H \). By \[ 26 \] Lemmas 5.3 and 2.1, proof of Proposition 5.4 in arXiv version \[ 1 \], \( \sigma_1^*\Delta_f \) contains no any component of \( D_1 \), our \( D_1 \) is reduced, and \( f_1^*D_{1j} = qD_{1j} \) for every irreducible component \( D_{1j} \) of \( D_1 \) (after \( f \) is replaced by its power). As in the proof of Theorem \[ 1 \] for \( 1 \leq i \leq n - 2 \), we have log canonical pairs \( (X_i, D_i) \) with \( D_i = \sum_j D_{ij} \) reduced, and normalizations (onto the images) \( \sigma_i : X_i \to X_{i-1} \).

We still have Claims \[ 2.10(1) \] (hence \( s = n+1 \)) and \[ 2.11 \] with the same proof, noting that \( (\sigma_1 \cdots \sigma_i)^*V_{i+v} \) (\( v \geq 1 \)) are reduced by the argument above for all pairs \( (X, \sum_{k=0}^{i} V_k) \). So \( \cap_{k=1}^{i} V_k = X_i \) (a normal variety). By the construction, inductively, we can write

\[
\text{Eq(2.12.1) } K_{X_i} + D_i = f_i^*(K_{X_i} + D_i) + \Delta_{f|X_i}
\]

so that \( f_i^*D_{ij} = qD_{ij} \) and \( R_{f_i} = (q - 1)D_i + \Delta_{f|X_i} \). These, together with \( D_{n-2} \geq C \) and Claim \[ 2.10(1) \], imply that \( D_{n-2} = C \) and \( \Delta_{f|X_{n-2}} = 0 \), so \( \Delta_f = 0 \) by the ampleness of \( V_i \) and hence \( R_f = (q - 1)D_0 \). For the second assertion, we use the purity of branch loci, Claim \[ 2.11 \] and the relabeling of \( V_k \).

If \( f^*K_X = qK_X \) or \( \rho(X) = 1 \) (and using \[ 26 \] Lemma 2.1), the Eq(2.12.1) above (with \( i = 0 \)) implies \( K_X + D_0 = q(K_X + D_0) \) and hence the last part of the proposition. \[ \square \]

2.13. Proof of Theorem \[ 1.3 \]

By the assumption, \( f : X \to X \) is a polarized endomorphism with \( \deg(f) = q^n > 1 \); and either \( n = \dim X \leq 3 \), or \( f^*|_{N^1(X)} = q \text{id}_{N^1(X)} \). We need to prove the four assertions in Theorem \[ 1.3 \] Our proof will be by the induction on \( \dim X \). The case \( \dim X = 1 \) follows from the Hurwitz formula. Suppose Theorem \[ 1.3 \] is true for those \( X' \) with \( \dim X' \leq n - 1 \). Consider the case \( \dim X = n \geq 2 \). We may assume that there are prime divisors \( V_j \) (\( 1 \leq j \leq s \)) with \( f^{-1}(V_j) = V_j \) for some \( s \geq \rho(X) + n - 2 \geq 1 \). We will mainly prove (1) and (4) of Theorem \[ 1.3 \] because (2) and (3) are similar and easier. So we may assume that \( s \geq n + \rho(X) \geq n + 1 \geq 3 \). By the assumption, \( f^*H \sim qH \), with an ample Cartier divisor \( H \) and \( q = \sqrt[\rho(f)]{\deg(f)} > 1 \); further, \( f^*V_j = qV_j \) (cf. \[ 26 \] Lemma 2.1). So one may
compute the ramification divisor of $f$ as:

$$R_f = (q - 1) \sum V_i + \Delta$$

with $\Delta$ an effective Weil divisor containing no any $V_j$, and hence

$$K_X + \sum V_j = f^*(K_X + \sum V_j) + \Delta.$$ 

Further, the second part of (1) follows from Proposition 2.1. Indeed, since $R_f > 0$, i.e., $f$ is not étale in codimension one, $K_X$ is not pseudo-effective (cf. Lemma 2.7 or [25 Lemma 3.7.1]). Hence $X$ is uniruled by the well known results of Mori-Mukai and Boucksom-Demailly-Paun-Peternell ([22, 3]).

Let $X \rightarrow X_1$ be a divisorial extremal contraction or a flip (and then $n \geq 3$). Let $V(1)_j \subset X_1$ be the image of $V_j$ when $V_j$ is not exceptional over $X_1$. Thus $s_1 \geq n + \rho(X_1)$ since $s_1 \geq s - 1$ and $\rho(X_1) = \rho(X) - 1$ (resp. $s_1 = s$ and $\rho(X_1) = \rho(X)$) when $X \rightarrow X_1$ is divisorial (resp. flip). The map $f$, replaced by its power, descends to a holomorphic endomorphism $f_1 : X_1 \rightarrow X_1$ of degree $q^n$, by using [29 Theorem 1.1, Lemmas 3.6 and 3.7] (under the condition $n = 3$ or $f^*|N^1(X) = q \text{id}_{N^1(X)}$) and [25 Proposition 3.6.8] (saying that negative curves are $f^{-1}$-periodic, under the condition $n = 2$). Clearly, $f_1^{-1}V(1)_j = V(1)_j$. Continuing the process, we have a composition

$$X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_r$$

of divisorial contractions and flips, holomorphic maps $f_i : X_i \rightarrow X_i$ induced from $f$ (replaced by its power), and prime divisors $V(i)_j \subset X_i$ (1 $\leq j \leq s_i$) which are the images of $V_j$, for some $s_i \geq n + \rho(X_i) \geq 3$. Note that $f_i^*V(i)_j = qV(i)_j$ and hence

$$R_{f_i} \geq (q - 1) \sum V(i)_j > 0.$$ 

Thus, as reasoned above for $X$, our $K_{X_i}$ is not pseudo-effective and hence $X_i$ is uniruled. Further, denoting by $\Delta(i)$ the image of $\Delta$, we have

$$K_{X_i} + \sum V(i)_j = f_i^*(K_{X_i} + \sum V(i)_j) + \Delta(i)$$

By the minimal model program and [24 the proof of Corollary 1.3.2] and since $K_X$ is not pseudo-effective as mentioned early on, we may assume that some $W := X_r$ has an extremal contraction

$$\pi : W = X_r \rightarrow Y$$

of fibration type so that $\dim Y \leq n - 1$, and $Y$ and $X_i$ all have only $\mathbb{Q}$-factorial Kawamata log terminal (klt) singularities (cf. [16 Lemma 5-1-5, Propositions 5-1-6 and 5-1-11], [24 Ch VII, Corollary 3.3]). Further, $f_r$ is polarized by some ample divisor $H_W$ of degree
and it descends to an endomorphism (polarized by some ample Cartier divisor \( H_Y \) of degree \( q^{\dim Y} \)):

\[
f_Y : Y \to Y;
\]

see [29] Theorem 2.13, Lemma 2.12; Theorems 1.1 and 2.7 for \( n \leq 3 \); [26] Lemma 2.3; and note that if \( f^*|_{\mathcal{N}(X)} = q \, \text{id}_{\mathcal{N}(X)} \) on \( X \) then the same is true on all of \( X_i \) and \( Y \). This is the second place we use the Hyp(A) in Theorem 1.3.

2.14. The case \( \dim Y = 0 \). Then \( \rho(W) = 1 \). Thus \(-K_W\) is ample (\( W \) being uniruled) and \( W \) and hence \( X \) are rationally connected (cf. [31] Theorem 1, [12] Corollary 1.5]), so \( q(X) = q(W) = 0 \) (cf. the proof of Lemma 2.8). Since \( s_r \geq n + \rho(X_r) = n + 1 \), by Proposition 2.12 \( s_r = n + 1 \) and \( K_W + \sum_{j \in I} V(r)_j \) is numerically (and hence \( \mathbb{Q} \)-linearly, \( q(W) \) being zero) equivalent to zero. Then, by the construction, \( s_i = n + \rho(X_i) \) for all \( i \) (so Theorem 1.3(1) is true) and the exceptional divisor of \( X_i \to X_{i+1} \) is contained in \( \sum_{j} V(i)_j \) when the map is divisorial. Hence \( \sum V_j \) is the sum of the proper transform of \( \sum_{j} V(r)_j \) and the exceptional divisors of the composite

\[
\delta : X = X_0 \dashrightarrow X_r = W.
\]

Write

\[
K_X + \sum_{j} V_j = \delta^*(K_W + \sum_{j} V(r)_j) + E_2 - E_1 \sim_{\mathbb{Q}} E_2 - E_1
\]

for some effective \( \delta \)-exceptional divisors \( E_i \) (with no common components) whose supports are hence contained in \( \cup V_j \), so \( f^*E_i = qE_i \); here the \( \delta \)-pullback is well defined since \( \delta \) involves only flips and holomorphic maps. By the display (*) above,

\[
E_2 - E_1 - \Delta \sim_{\mathbb{Q}} K_X + \sum_{j} V_j - \Delta = f^*(K_X + \sum_{j} V_j) \sim_{\mathbb{Q}} f^*(E_2 - E_1) = q(E_2 - E_1).
\]

Thus \( \Delta + (q-1)E_2 \sim_{\mathbb{Q}} (q-1)E_1 \). So \( E_1 = E_2 = \Delta = 0 \), because \( E_1 \) is \( \delta \)-exceptional and hence has Iitaka D-dimension \( \kappa(X, E_1) = 0 \), and \( \text{Supp} E_1 \subseteq \cup V_j \) and \( \text{Supp}(E_2 + \Delta) \) have no common components. Now \( K_X + \sum_{j} V_j \sim_{\mathbb{Q}} 0 \), so \( R_f = (q-1)\sum V_j \) and \( f \) is étale outside \( (\cup V_j) \cup f^{-1}(\text{Sing} X) \) by the purity of branch loci.

Since \( \rho(W) = 1 \) (and \( q(W) = 0 \)) and hence \( \text{Pic}(X) \) is spanned (over \( \mathbb{Q} \)) by \( H_W \) and \( V_j \) with \( f^*H_W \sim qH_W \) and \( f^*V_j = qV_j \), we have \( f^*|_{\text{Pic} X} = q \text{id}_{\text{Pic} X} \) (with \( f \) having been replaced by its power), by the proof of Lemma 2.8. This proves Theorem 1.3(4) in the present case; see [27] Theorem 6.4] for the assertion about \( (X, \sum V_i) \) being a toric pair when \( \dim X = 2 \).

2.15. The case \( 1 \leq \dim Y \leq n - 1 \). By [8] Theorem 5.1, the \( f_Y \)-periodic points are dense and we let \( y_0 \) be a general one of them so that \( f_Y(y_0) = y_0 \) (after replacing \( f \) by
its power). Then $W_0 := \pi^{-1}(y_0)$ is a klt Fano variety. The restriction
\[ f_{W_0} = f_{r|W_0} : W_0 \to W_0 \]
is an endomorphism of degree $q^{\dim W_0} > 1$ and polarized by $H_{W|W_0}$.

If the restriction $\pi : V(r)_i \to Y$ $(1 \leq i \leq s_r(1))$ is not surjective, then $V(r)_{i|W_0} = 0$ and $V(r)_i$ is perpendicular to any fibre of $\pi : W \to Y$ so that
\[ V(r)_i = \pi^* G_i \]
for some prime divisor $G_i \subset Y$ (since the relative Picard number $\rho(W/Y) = 1$) and also $f_Y^{-1}(G_i) = G_i$; by the inductive hypothesis, $s_r(1) \leq \dim Y + \rho(Y)$.

If the restriction
\[ \pi : V(r)_j \to Y \quad (s_r(1) + 1 \leq j \leq s_r(1) + s_r(2) = s_r) \]
is surjective, then these $V(r)_{j|W_0}$ are ample since $\rho(W/Y) = 1$, and they share no common irreducible component by the general choice of $W_0 = \pi^{-1}(y_0)$; by the proof of Proposition 2.12, $s_r(2) \leq \dim W_0 + 1$. Thus
\[ n + \rho(X_r) \leq s_r \leq \dim Y + \rho(Y) + \dim W_0 + 1 = n + \rho(X_r) \]
and all inequalities are actually equalities, so $s_i = n + \rho(X_i)$ for all $i$ as above, and Theorem 1.3(1) is true. Applying the inductive hypothesis on $Y$ we conclude that:

\[ (***) \quad s_r(1) = \dim Y + \rho(Y), \quad K_Y + \sum_{i=1}^{s_r(1)} G_i \sim_Q 0, \quad f_{Y|\text{Pic} Y}^{*} = q \text{ id}_{\text{Pic} Y} \]
(with $f$ replaced by its power) and that $Y$ is rationally connected, so $W$ (and hence $X$) are rationally connected by [11].

(For Theorem 1.3(2) or (3), we have $s_i - (\dim X_i + \rho(X_i) - 2) \geq 0$, or $\geq 1$, and hence $s_r(1) - (\dim Y + \rho(Y) - 2) \geq 0$, or $\geq 1$, respectively, by the upper bound of $s_r(2)$ above, so we can also apply the induction on $Y$).

Since Pic $W$ is spanned, over $\mathbb{Q}$, by $H_W$ and the pullback of Pic $Y$, we have $f_{r|\text{Pic} W}^{*} = q \text{ id}_{\text{Pic} W}$ and hence $f_{r|\text{Pic} X}^{*} = q \text{ id}_{\text{Pic} X}$ as reasoned in the case $\dim Y = 0$ (cf. the (***) above and the proof of Lemma 2.8). Since $s_r(2) = \dim W_0 + 1$, applying the proof of Proposition 2.12 to the pair $(W_0, f_{W_0})$ and noting that $f_{W_0|K_{W_0}}^{*} K_{W_0} \sim_Q qK_{W_0}$ for $K_{W_0} = K_{W|W_0}$, we have

\[ (***) \quad K_{W_0} + \sum_{\ell=1}^{s_r} V(r)_{\ell|W_0} \equiv 0. \]

Using this and restricting the display (*) above to $W_0$, we get $\Delta(r)_{W_0} = 0$ so that $\Delta(r) = \pi^* \Delta_Y$ for some effective divisor $\Delta_Y \subset Y$. 
We now show that \( \Delta(r) = 0 \) (i.e., \( \Delta_Y = 0 \)). Let

\[ \sigma(s_r) : W(s_r) \to W \]

be the normalization of \( V(r)_{s_r} \subset W \). Pulling back the display (*) above, we get (with \( k = s_r \)):

\[ \begin{align*}
(* ** *) & \quad K_W(k) + D(k) = f(k)^* (K_W(k) + D(k)) + \Delta(r)_k \\
\end{align*} \]

where \( f(k) \) is the lifting of \( f_k V(r)_k \) (polarized by the pullback of \( H_W \)), and \( \Delta(r)_k \) is the pullback of \( \Delta(r) \). As in the proof of Proposition 2.12, \((W(s_r), D(s_r))\) is log canonical with a reduced divisor

\[ D(s_r) \geq \sum_{t=1}^{s_r-1} \sigma(s_r)^* V(r)_t \]

where each term in the summand with \( t > s_r(1) \) is nonzero because \( V(r)_t' | W_0 \) (\( t' = t, r \)) are ample and \( \sigma(s_r) \) is finite over \( V(r)_{s_r} \cap W_0 \). Next, consider the normalization

\[ \sigma(k) : W(k) \to W(k + 1) \]

(with \( k = s_r - 1 \)) of an irreducible component of \( \sigma(k + 1)^* V(r)_k \subset W_{k+1} \). Thus for

\[ k = s_r, s_r - 1, \ldots, k_0 := s_r - \dim W_0 + 1 = s_r(1) + 2 \]

we have normalizations

\[ \sigma(k) : W(k) \to W(k + 1), \]

log canonical pairs \((W(k), D(k))\) with a reduced divisor

\[ D(k) \geq \text{(the pullback of } \sum_{i=1}^{k-1} V(r)_i), \]

and the display (****) above for all these \( k \). The natural composition \( \tau_{k_0} : W(k_0) \to Y \) is generically finite and surjective. Pushing forward the display (****) above with \( (\pi_{k_0}/(\deg(\tau_{k_0})) \) and noting that \( V(r)_i = \pi^* G_i \) (\( 1 \leq i \leq s_r(1) \)) and \( \Delta(r) = \pi^* \Delta_Y \), we get

\[ K_Y + \sum_{i=1}^{s_r(1)} G_i + C = f_\tau^* (K_Y + \sum_{i=1}^{s_r(1)} G_i + C) + \Delta_Y \]

where \( C \) is an effective divisor contributed from the branch locus of \( \tau_{k_0} \) and others. This and the display (**) above imply \( 0 + C \equiv f_\tau^* C + \Delta_Y \) and \( 0 \equiv (q - 1)C + \Delta_Y \). Thus \( C = 0 = \Delta_Y \). Hence \( \Delta(r) = 0 \).

Now \( K_W + \sum_{\ell=1}^{n+q(X)} V(r)_\ell \) restricts to zero on \( W_0 \) by the (***) above, so it is \( \mathbb{Q} \)-linearly equivalent to some pullback \( \pi^* L \). Thus the display (*) above becomes

\[ \pi^* L \sim_{\mathbb{Q}} f_\tau^* \pi^* L + 0 = \pi^* f_\tau^* L \sim_{\mathbb{Q}} q \pi^* L. \]

Hence \( 0 \sim_{\mathbb{Q}} \pi^* L \sim K_W + \sum_\ell V(r)_\ell \). This will deduce Theorem 1.3(4) as we did in the case \( \dim Y = 0 \).
For Theorem 1.3(2), by the above induction, either two general points of $X$ are connected by a chain of rational curves and hence $X$ is rationally connected by [12] Corollary 1.5, or there is a fibration $\eta : X \to E$ onto a smooth projective curve $E$ such that some power $f^k$ descends to some $f_E : E \to E$ of degree $q > 1$ (and with genus $g(E) \geq 1$, so $g(E) = 1$) and a general fibre $X_e$ is rationally connected; here we used again [11]. In the latter case, let $\Sigma := \{e \in E | X_e$ is not rationally connected}. Then $f_E^{-1}(\Sigma) \subseteq \Sigma$.

Applying $f_E^{-1}$ a few times and comparing the cardinalities of the sets involved, we see that $f_E^{-1}(\Sigma) = \Sigma$. So $\Sigma = \emptyset$ (and hence every fibre $X_e$ is rationally connected) since $f_E$ is étale. Similarly, every fibre $X_e$ is irreducible and normal (cf. [26] Lemma 4.7)). This proves Theorem 1.3.

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