We determine the exact global structure of the moduli space of $N=2$ supersymmetric $SO(n)$ and $USp(2n)$ gauge theories with matter hypermultiplets in the fundamental representations, using the non-renormalization theorem for the Higgs branches and the exact solutions for the Coulomb branches. By adding an ($N=2$)–breaking mass term for the adjoint chiral field and varying the mass, the $N=2$ theories can be made to flow to either an “electric” $N=1$ supersymmetric QCD or its $N=1$ dual “magnetic” version. We thus obtain a derivation of the $N=1$ dualities of [2].
1. Introduction and Discussion

Over the past two years much progress has been made in our understanding of the vacuum structure of supersymmetric gauge theories (for a review see [1]). One of the most interesting new phenomena uncovered is “$N=1$ duality” [2], where two different microscopic gauge theories have the same infrared behavior. The most striking examples of $N=1$ dual pairs involve one theory which is an asymptotically-free (AF) non-Abelian gauge theory and a second, dual, infrared-free theory with a different gauge group. In such cases the dual theory gives an explicit description of the low-energy physics at strong coupling. The identification of this free gauge group within the context of the microscopic AF theory is difficult. Indeed, the dual IR-free gauge group is magnetic compared to the electric AF group.

In the case of $SU(n_c)$ $N=1$ super-QCD with $n_f$ flavors, the dual gauge theory $SU(n_f - n_c)$ with $n_f$ flavors has been derived [4] by flowing down from the $N=2$ supersymmetric version of the theory. This paper extends that analysis to the $SO(n_c)$ and $USp(2n_c)$ gauge groups with matter in the defining representations. In these cases, the dual groups according to [2] and [3] are, respectively, $SO(n_f - n_c + 4)$ (for $n_f > n_c - 2$) and $USp(2n_f - 2n_c - 4)$ (for $n_f > n_c + 2$).

The bulk of the paper is concerned with mapping out a global description of the $N=2$ moduli space of these theories. We do this by first solving the classical vacuum equations, and then extending these solutions to the quantum theory using nonrenormalization arguments as well as the known exact solutions [5] for the Coulomb branches. Along the way we obtain a compact gauge-invariant description of the $N=2$ moduli space, which turns out to be quite a bit simpler for these gauge groups than for the $SU(n)$ case.

With the $N=2$ moduli space in hand, we then break to $N=1$ supersymmetry by turning on a bare mass $\mu$ for the $N=1$ adjoint chiral multiplet part of the $N=2$ vector multiplet. If the AF $N=2$ theory is characterized by a strong-coupling scale $\Lambda$, then for $\mu \gg \Lambda$ we flow to the corresponding microscopic (AF) $N=1$ theory. For $\mu \ll \Lambda$, on the other hand, we first integrate out the degrees of freedom with mass of order $\Lambda$, thus

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1 Here $USp(2n)$ denotes the unitary symplectic group of rank $n$. 

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effectively recovering the $N=2$ moduli space derived earlier. The small ($N=2$)-breaking mass $\mu$ generically lifts these vacua, except for special points, among which is the point of maximal, mutually local degeneration. This point corresponds to an $N=2$ vacuum with precisely the dual (IR-free) gauge group of $[2]$ and $[3]$.

Flowing to the IR limit in each of the cases $\mu \gg \Lambda$ and $\mu \ll \Lambda$, we are led to a derivation of the $N=1$ dualities of $[2]$ and $[3]$. Because $N=1$ supersymmetry prevents a phase transition between small and large $\mu$, the IR theories obtained in these two different limits must be equivalent. This equivalence goes beyond the the earlier arguments of $[2]$ and $[3]$, which essentially only found an equivalence between the chiral rings of the two theories. However, in at least one respect, our argument is incomplete: we obtain no information on the extra gauge singlet fields appearing in the dual theories $[2]$ and $[3]$.

One striking feature of the corresponding analysis in the $SU(n)$ case $[4]$ was that the IR-free $N=1$ dual gauge theory can be continuously deformed through the larger $N=2$ moduli space to a theory whose gauge group is a subgroup of the microscopic (AF) gauge group. The existence of a continuous interpolation between electric and magnetic degrees of freedom is allowed because there is no phase transition separating Higgs and confining phases (condensation of electric and magnetic charges, respectively) in $SU(n)$ with fundamental scalars. In $SO(n)$ with matter in the vector ($n$) representation, however, there is such a distinction, since a Wilson loop in a spinor representation cannot be screened by vector charges. Thus, in the $SO(n)$ case one expects that any such interpolation must pass through a phase transition.

This is indeed what we find: the branches of the $SO(n)$ $N=2$ moduli space that connect to the IR-free non-Abelian vacuum in question all have unbroken $U(1)$ gauge factors. So as we deform $N=1$ electric Higgs or confining (magnetic Higgs) vacua to the corresponding $N=2$ vacua, they gain $U(1)$ factors and so the asymptotic behavior of spinorial Wilson loops in these vacua changes abruptly to Coulomb-like behavior as $N=2$ symmetry is restored. This is because the charges in such a Wilson loop are charged under all the $U(1)$ factors in the Cartan subalgebra of the microscopic $SO(n)$ group—the spinor weights of $SO(n)$ are $(\pm \frac{1}{2}, \ldots, \pm \frac{1}{2})$.

However, the observation that the electric and magnetic Higgs phases are distinct
does not invalidate our duality argument. Because there is no electric/magnetic phase transition in the Coulomb phase, we can interpolate continuously between the dual IR theories, provided that during the interpolation we remain in the Coulomb phase at all times.

2. Classical $N=2$ $SO(n_c)$ Moduli Space

$N=2$ supersymmetric $SO(n_c)$ Yang-Mills theory is described in terms of $N=1$ superfields by a field strength chiral multiplet $W_{ab}^\alpha$ and a scalar chiral multiplet $\Phi_{ab}$, both in the adjoint representation of the gauge group, which together form an $N=2$ vector multiplet. Here $a, b = 1, \ldots, n_c$ are color indices. The Lagrangian is

$$\mathcal{L}_{YM} = \text{tr} \text{Im} \left[ \tau \int d^2\theta d^2\overline{\theta} \Phi e^V \Phi + \tau \int d^2\theta \frac{1}{2} \text{tr} W^2 \right],$$

(2.1)

where $\tau$ is the gauge coupling and theta angle $\tau = (\theta/\pi) + i(8\pi/g^2)$.

Matter in the $n_c$ representation of the gauge group is made up of the $N=1$ chiral "quark" multiplets $Q^i_a$, $i = 1, \ldots, 2n_f$, pairs of which ($Q^i_a, Q^{i+n_f}_a$) together make up $N=2$ hypermultiplets. Matter couples to the Yang-Mills fields via the usual kinetic terms and a cubic superpotential:

$$\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\overline{\theta} Q^i_a \overline{Q}^i_b (e^V)_{ab} + \int d^2\theta \sqrt{2} Q^i_a \Phi_{ab} Q^j_b \mathbb{J}^{ij} + \text{c.c.},$$

(2.2)

where $\mathbb{J}$ is the symplectic metric $\mathbb{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{I}_{n_f \times n_f}$, and $\mathbb{I}$ is the identity matrix.

The theory has a global $USp(2n_f)$ flavor symmetry (when there are no bare quark masses) as well as a $U(1)_R \times SU(2)_R$ chiral R-symmetry. $USp(2n_f)$ is the group of $2n_f \times 2n_f$ complex matrices $M$ satisfying $M \cdot \mathbb{I} \cdot \mathbb{M} = \mathbb{I}$ and $M \cdot \mathbb{J} \cdot \mathbb{M} = \mathbb{J}$, and thus preserving both the hermitian $Q_a \cdot \overline{Q}_b$ and symplectic $Q_a \cdot J Q_b$ inner products of complex $2n_f$-component vectors. Mass terms and instanton corrections break $U(1)_R$. When $n_f < n_c - 2$, the theory is asymptotically free and generates a strong-coupling scale $\Lambda$, the instanton factor is proportional to $\Lambda^{2n_c-2n_f-4}$, and the $U(1)_R$ symmetry is anomalous, being broken by instantons down to a discrete $\mathbb{Z}_{2n_c-2n_f-4}$ symmetry. For $n_f = n_c - 2$ the theory is scale-invariant and the $U(1)_R$ is not anomalous. In this case no strong coupling scale is generated, and the theory is described in terms of its bare coupling $\tau$. 

3
The classical vacua are the zeroes of the scalar potential, found by setting the $D$- and $F$-terms to zero. The $D$-term equations are

$$0 = [\Phi, \Phi],$$
$$0 = \text{Im}(Q_a \cdot \overline{Q}_b),$$

and the $F$-term equations are

$$0 = Q_a \cdot J_{Q_b},$$
$$0 = \Phi_{ab} Q_i^b.$$  \hspace{1cm} (2.3)

These vacuum equations imply that the fields $\Phi$ and $Q$ may get vevs, which we denote by the same symbols. The solutions to the vacuum equations form various “branches” corresponding to different phases of the theory. The Coulomb branch is defined as the set of solutions with $Q = 0$, Higgs branches are those with $\Phi = 0$, and mixed branches are those with both $\Phi$ and $Q$ nonvanishing.

**Coulomb Branch:** The Coulomb branch satisfies $[\Phi, \Phi] = 0$ with $Q = 0$, implying that $\Phi$ can be skew-diagonalized by a color rotation to a complex matrix

$$\Phi = \begin{pmatrix}
0 & \phi_1 & & \\
-\phi_1 & 0 & & \\
& & \ddots & \\
& & & 0 & \phi_{[n_c/2]} \\
& & & -\phi_{[n_c/2]} & 0
\end{pmatrix}. \hspace{1cm} (2.5)$$

This vev generically breaks $SO(n_c) \rightarrow U(1)^{[n_c/2]}$, motivating the name for this branch.

For $n_c$ even, gauge transformations in the Weyl group of $SO(n_c)$ are generated by permutations, and by simultaneous sign changes of pairs of the $\phi_a$. So the symmetric polynomials $S_\ell$ of the $\phi_a^2$, generated by $\sum_\ell S_\ell x^{[n_c/2]-\ell} = \prod_a (x-\phi_a^2)$, are “glue” gauge invariants. In addition to the $S_\ell$, there is one “extra” Weyl invariant $T = \phi_1 \phi_2 \cdots \phi_{n_c/2} = \pm \sqrt{S_{n_c/2}}$.

For $n_c$ odd, there is an extra row and column of zeroes in (2.5). The Weyl group is generated by permutations and individual sign flips of the $\phi_a$, so the symmetric polynomials $S_\ell$ in the $\phi_a^2$ form a complete basis of glue invariants on the Coulomb branch.

This classical moduli space has orbifold singularities along submanifolds where some of the $\phi_a$’s are equal or vanish. In this case some of the non-Abelian gauge symmetry is
restored. If \( k \phi_a \)'s are equal and non-zero, there is an enhanced \( SU(k) \) gauge symmetry. If they are also zero, then there is an enhanced \( SO(2k) \) or \( SO(2k+1) \), depending on whether \( n_c \) is even or odd, respectively. In this case, the glue invariants \( S_\ell = 0 \) for \( \ell > [n_c/2] - k \).

**Higgs Branches:** The Higgs branch is the space of solutions to the second equation in (2.3) and the first in (2.4) since \( \Phi = 0 \). Describe the squark fields as complex matrices with \( n_c \) rows and \( 2n_f \) columns. Any solution of the Higgs branch equations can be put in the following form by a combination of flavor and gauge rotations:

\[
Q = \begin{pmatrix} q_1 \\ \vdots \\ q_{n_f} \end{pmatrix}, \quad q_a \in \mathbb{R}^+. \tag{2.6}
\]

In (2.6) we assumed \( n_c > n_f \); if \( n_c < n_f \), then there will be \( n_c \) entries on the diagonal. Call such a solution with \( r \) of the \( q_i \) non-zero the \( r \)-Higgs branch. Thus, on the \( r \)-Higgs branch an \( SO(n_c-r) \) gauge symmetry is unbroken, with \( n_f-r \) massless hypermultiplets transforming in its vector representation. So, by the Higgs mechanism, of the \( n_f n_c - (n_f-r)(n_c-r) \) neutral hypermultiplets, \( \frac{1}{2} n_c(n_c-1) - \frac{1}{2}(n_c-r)(n_c-r-1) \) are given a mass, leaving \( \mathcal{H} = r n_f - \frac{1}{2} r(r-1) \) massless neutral hypermultiplets—the quaternionic dimension of the \( r \)-Higgs branch. (As we will see later, this counting is really only accurate for \( n_c-r \) even.)

In order to identify the unbroken global symmetries on the \( r \)-Higgs branches, it is useful to define a basis of gauge-invariant quantities made from the squark vevs, the meson and baryon fields

\[
M^{ij} = Q^i_a Q^j_a, \quad B[i_1...i_{n_c}] = Q^{i_1}_{a_1} \cdots Q^{i_{n_c}}_{a_{n_c}} \epsilon_{a_1...a_{n_c}}. \tag{2.7}
\]

The baryon field is defined for \( 2n_f \geq n_c \). From our solution for the \( r \)-Higgs branch squark vevs we see \( B \neq 0 \) only when \( r = n_c \leq n_f \). The meson field is diagonal with \( r q_i^2 \)'s along the diagonal. It therefore leaves a \( USp(2n_f-2r) \) global flavor symmetry unbroken. (A non-vanishing baryon field does not break this symmetry.) Thus the number of real goldstone modes is \( \mathcal{G} = n_f(2n_f+1) - (n_f-r)(2n_f-2r+1) \), and the number of real parameters describing the Higgs branch is \( \mathcal{P} = r \). It is a check on our counting that \( \mathcal{G} + \mathcal{P} = 4\mathcal{H} \).
**Mixed Branches:** Using the antisymmetry of $\Phi$, it follows from the second $F$-term equation in (2.4) that on an $r$-Higgs branch $\Phi$ must be zero except in a lower right-hand $(n_c-r) \times (n_c-r)$ block, which can be skew diagonalized by the unbroken $SO(n_c-r)$ rotations. Call the mixed branch which emanates from an $r$-Higgs branch simply the $r$-branch. When $n_c-r$ is odd, there will be a row and column of zeros in $\Phi$, with a corresponding row of zeros in $Q$. Thus, such an $r$-branch is really just a submanifold of the $r+1$-branch, and not a separate branch. From now on we denote by “$r$-branches” only those with $n_c-r$ even.

With non-zero vevs for $\Phi$ and $Q$ we can define, in addition to the glue invariants $S_\ell$ and meson $M$, a set of “baryonic” invariants

$$B[^{i_1...i_{n_c-2\ell}}_\ell] = \Phi_{a_1a_2} \cdots \Phi_{a_{2\ell-1}a_{2\ell}} Q_{a_{2\ell+1}}^{i_1} \cdots Q_{a_{n_c}}^{i_{n_c-2\ell}} \epsilon_{a_1...a_{n_c}}. \tag{2.8}$$

Note that $\ell = 0$ corresponds to the usual baryon, while $\ell = n_c/2$ (for $n_c$ even) gives the “extra” Coulomb-branch invariant $T$. From the block form of the $Q$ and $\Phi$ vevs found above on the $r$-branch, it follows that all the baryon invariants vanish except for $B^{(n_c-r)/2}$. Also, the glue invariants $S_\ell = 0$ for $\ell > (n_c-r)/2$.

**Gauge-Invariant Description of Moduli Space:** We have found above only representative solutions for the $Q$ and $\Phi$ vevs, since global symmetry transformations on these solutions will relate them to distinct points in the moduli space. To have a global description, it is useful to describe the various branches in terms of constraints on the gauge-invariant meson, glue, and baryon order parameters. In particular, setting the $D$-terms to zero and identifying orbits of the gauge group is equivalent to dividing out the space of $Q$ and $\Phi$ vevs by the action of the complexified gauge group. The latter operation may be achieved by expressing the vevs in terms of holomorphic gauge-invariant coordinates, which, however, are not independent as functions of the $Q$ and $\Phi$ vevs, but satisfy a set of polynomial relations. Below we find a set of generators for the constraints following from these relations and the $F$-term equations.

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2 There is an exception: if $n_c-n_f$ is odd, then for the maximal value of $r$, $r=n_f$, there is no $n_f+1$-branch for the $n_f$-branch to be a submanifold of.
Since the product of two color epsilon-tensors is the antisymmetrized sum of Kronecker deltas, it follows that

\[ B_k B_\ell = \delta_{k\ell} S_{\ell} \ast (M^{n_c-2\ell}), \quad 0 \leq \ell \leq [n_c/2], \quad (2.9) \]

where the "\ast" denotes the antisymmetrization of the product of M’s on half their flavor indices, and flavor indices are uncontracted. Note that the F-term equation \( \Phi \cdot Q = 0 \) has been used in deriving (2.9). Also, since any expression antisymmetrized on \( n_c+1 \) color indices must vanish, it follows that any product of M’s and B’s antisymmetrized on \( n_c+1 \) flavor indices must vanish. (2.9) can be used to eliminate all the B’s from such constraints, leading to one other independent constraint \( \ast (M^{n_c+1}) = 0 \). Another set of independent constraints follows from contracting the color identity \( 0 = \delta_{[b_1}^{a_1} \cdots \delta_{b_{n_c+1}]}^{a_{n_c+1}} \) with \( 2\ell \) \( \Phi \)'s and \((2n_c+2-4\ell)\) Q’s. Using \( \Phi \cdot Q = 0 \) one then finds

\[ 0 = S_{\ell} \ast (M^{n_c-2\ell+1}), \quad 0 \leq \ell \leq [n_c/2]. \quad (2.10) \]

Note that when \( \ell = 0 \) (\( S_0 \equiv 1 \)) this is equivalent to \( \ast (M^{n_c+1}) = 0 \).

The F-term equation \( Q \cdot J Q = 0 \) implies the further constraint

\[ M \cdot J M = 0. \quad (2.11) \]

The other constraints following from the F-term equation, \( B_\ell \cdot J M = B_\ell \cdot J B_k = 0 \), are not independent of (2.9) and (2.11). Thus (2.9), (2.10), and (2.11) form a complete set of constraints describing the classical moduli space.

We can solve these constraints and recover the properties of the \( r \)-branches found above. Eq. (2.11) implies \( M \) can be diagonalized to \( r \) positive real entries with \( r \leq n_f. \)

\[ \text{To see this, note that } (2.11) \text{ implies that the image of } M \text{ (viewed as a linear transformation) is symplectically orthogonal; in particular, } r \equiv \dim(\text{im } M) \leq n_f. \text{ So we can choose an orthonormal basis of } \text{im } M \text{ and extend it to an ortho-symplectic basis of the full space. Thus there is a } USp(2n_f) \text{ similarity transformation expressing } M \text{ in this basis where its last } 2n_f-r \text{ columns, and hence rows by its symmetry, vanish. It follows that } M \text{ is zero except for a complex symmetric upper left-hand } r \times r \text{ block, which can be diagonalized to non-negative real entries by a } U(r) \subset USp(2n_f) \text{ similarity transformation.} \]

\[ 7 \]
Furthermore, from (2.10) with $\ell=0$ we learn that actually $r \leq \min\{n_f, n_c\}$, reproducing the form of the meson fields on the $r$-branches. By (2.9), if any one baryon invariant does not vanish, say $B_\ell \neq 0$, then all the other $B_k = 0$ for $k \neq \ell$. Since on an $r$-branch $\text{rank}(M) = r$, we have that $\ast(M^{n_c-2k}) = 0$ for $k < (n_c-r)/2$ and are non-vanishing otherwise. Then (2.10) implies that $S_k = 0$ for $k > (n_c-r)/2$, and from (2.9) we learn that the one non-vanishing $B_\ell$ must have $\ell = (n_c-r)/2$ (for $n_c-r$ even).

**Summary:** We have found that the $SO(n_c)$ theory with $n_f$ vector flavors has a moduli space made up of $r$-branches with $0 \leq r \leq \min\{n_f, n_c\}$ with $n_c-r$ even. The $r$-branch has hypermultiplet dimension $\mathcal{H} = rn_f - \frac{1}{2}r(r-1)$ and vector multiplet dimension $\mathcal{V} = \frac{1}{2}(n_c-r)$. Thus, the $(r=0)$-branch is the Coulomb branch, the $(r=1)$-branch includes the Coulomb branch as a submanifold, while for $r=n_c$ we obtain a pure Higgs branch. The “root” of an $r$-branch is its submanifold of intersection with the Coulomb branch. Thus, the $r$-branch root has quaternionic dimension $(n_c-r)/2$ and has an $SO(r) \times U(1)^{(n_c-r)/2}$ unbroken gauge group classically.

### 3. Quantum $N=2$ $SO(n_c)$ Moduli Space

A non-renormalization theorem [4] implies that quantum mechanically the $r$-branches retain their Coulomb $\times$ Higgs product structure, the Higgs factors are not renormalized and do not depend on the quark masses, and the Coulomb factors are given by submanifolds of the quantum Coulomb branch.

We have seen that classically there exist $r$-branches for $0 \leq r \leq n_f$ with $n_c-r$ even which meet the Coulomb branch along submanifolds with gauge group $SO(r) \times U(1)^{(n_c-r)/2}$ with $n_f$ vector flavors. Since $SO(n_c)$ gauge theories are only asymptotically free when $n_f \leq n_c-2$, the $SO(r)$ factors at the roots of the $r$-branches are all IR-free, and will remain unbroken quantum-mechanically.

Submanifolds of the quantum Coulomb branch with unbroken $SO(r)$ gauge factors are easy to identify explicitly using the exact solution for the Coulomb branch found in [3]. The generic vacuum on the Coulomb branch is a $U(1)^{[n_c/2]}$ pure Abelian gauge theory characterized by an effective coupling $\tau_{ij}$ between the $i$th and $j$th $U(1)$ factors, which,
due to the ambiguity of electric-magnetic duality rotations in the $U(1)$ factors, forms a section of an $Sp(2[n_c/2],\mathbb{Z})$ bundle over the Coulomb branch. An explicit description of the Coulomb branch is given by associating to each point of the Coulomb branch a genus $[n_c/2]$ Riemann surface whose complex structure is the low energy coupling $\tau_{ij}$. Globally the quantum Coulomb branch can still be characterized by $[n_c/2]$ complex numbers $\phi_a$ (up to permutations and sign flips) just as in the classical analysis of the last section. The family of Riemann surfaces describing the effective action on the Coulomb branch with $n_f$ massless flavors is then

$$y^2 = x^{[n_c/2]} \prod_{a=1}^{[n_c/2]} (x - \phi_a^2)^2 - 4\Lambda^2(n_c-2-n_f)x^{n_f+2+\epsilon},$$

(3.1)

where $\epsilon = 1$ if $n_c$ is even, and $\epsilon = 0$ if $n_c$ is odd. This form of the solution is valid for all AF values of $n_f$. In the finite case, when $n_f = n_c-2$, $\Lambda^0$ should be replaced by a known function of the bare coupling. In the IR-free case, when $n_f > n_c-2$, (3.1) is valid in a sufficiently small neighborhood of $x = \phi_a = 0$. In particular, in the IR-free case, the form of the curve at the origin of moduli space (where the $SO(n_c)$ gauge symmetry is restored) is simply

$$y^2 = x^{n_c+\epsilon}(1 - 4\Lambda^2(n_c-2-n_f)x^{n_f-n_c+2}), \quad n_f > n_c-2.$$  

(3.2)

When two or more of the branch points of (3.1) collide as we vary the moduli, the Riemann surface degenerates, giving a singularity in the effective action corresponding to additional $N=2$ multiplets becoming massless. When $n_s$ independent pairs of branch points collide there will be generically $n_s$ hypermultiplet states becoming massless (with $U(1)$ charges proportional to the homology classes of the vanishing cycles on the Riemann surface). More complicated singularities will generally lead to different physics.

In particular, from (3.2) we see that when $r$ branch points coincide, one may expect an unbroken $SO(r)$ or $SO(r-1)$ gauge symmetry. Such singularities are easy to find in the AF curves (3.1) with $n_f \leq n_c-2$: just set some of the $\phi_a = 0$. Thus, on the submanifold with all but $(n_c-r)/2$ of the $\phi_a = 0$ (where $n_c-r$ is even), the curve becomes

$$y^2 = x^{r+\epsilon} \left[ \prod_{a=1}^{(n_c-r)/2} (x - \phi_a^2)^2 - 4\Lambda^2(n_c-2-n_f)x^{n_f-r+2} \right],$$

(3.3)
suggesting vacua with an unbroken $SO(r) \times U(1)^{(n_c-r)/2}$ gauge group. This interpretation is confirmed by the fact that these singular submanifolds reach far out on the Coulomb branch ($\phi_a \gg \Lambda$) where they have the semi-classical interpretation as the submanifolds of the Coulomb branch where an IR-free $SO(r) \times U(1)^{(n_c-r)/2}$ group is left unbroken.

We have thus located the roots of the $r$-branches in the full quantum theory, and found that the structure of the quantum moduli space is qualitatively much the same as its classical structure. It is easy to check that the IR–free vacua at the $r$-branch roots indeed have mixed Higgs-Coulomb branches emanating from them which are precisely the same as the $r$-branches determined classically in the last section.

Since the theories at the $r$-branch roots are IR–free, their gauge symmetry will survive quantum-mechanically. Quantum effects could, however, change this effective theory by bringing down additional light degrees of freedom. In particular, there may be points on the $r$-branch root submanifolds where (monopole) singlets charged under the $U(1)$ factors become massless. Such points are located where the factor in square brackets in (3.3) becomes singular due to pairs of its zeros coinciding. The maximal such singularity occurs when that factor is a perfect square, corresponding to $(n_c-r)/2$ hypermultiplets becoming simultaneously massless. We will see in the next section that these vacua are especially interesting since they remain vacua upon breaking to $N=1$ supersymmetry.

Expanding out the terms in square brackets in (3.3), the condition that they form a perfect square is

$$
\left[ x^{(n_c-r)/2} + s_1 x^{(n_c-r)/2-1} + \ldots + s_{(n_c-r)/2} \right]^2 - 4\Lambda^{2(n_c-2-n_f)} x^n r^{r+2} = \left[ x^{(n_c-r)/2} + \tilde{s}_1 x^{(n_c-r)/2-1} + \ldots + \tilde{s}_{(n_c-r)/2} \right]^2
$$

for some $s_\ell$ and $\tilde{s}_\ell$. Moving the first term on the left to the right and factorizing, it is then easy to show that if $r > 2n_f-n_c+4$ or $n_f \geq n_c-2$ there is no solution, and if $r \leq 2n_f-n_c+4$ the only solution is $s_{n_c-2-n_f} = \Lambda^{2(n_c-2-n_f)}$ with all the other $s_\ell = 0$.

Plugging this solution into (3.3) gives the curve

$$
y^2 = x^{2n_f-n_c+4+\epsilon} \left( x^{n_c-2-n_f} - \Lambda^{2(n_c-2-n_f)} \right)^2,
$$

in which $r$ has dropped out. Thus, we have located the unique point on the $SO(n_c)$ Coulomb branch with $SO(r) \times U(1)^{(n_c-r)/2}$ unbroken IR gauge group and the maximal
number \((n_c-r)/2\) of singlets charged under the \(U(1)\)’s. By comparison with (3.3) we see that (3.5) corresponds to \(r = 2n_f - n_c + 4\) and
\[
\phi_a^2 = \Lambda^2(0, \ldots, 0, \omega, \omega^2, \ldots, \omega^{n_c-2-n_f})
\]  
(3.6)

where \(\omega = \exp\{2\pi i/(n_c-2-n_f)\}\).

A simple contour argument shows that we can pick a basis in which the singlets have a diagonal charge matrix, with each singlet having charge 1 under only one of the \(U(1)\)’s. The squarks are neutral under the \(U(1)\)’s since they are in a real flavor representation. These charges can be summarized as follows:

| \(2n_f \times Q\) | \(SO(2n_f-n_c+4)\) | \(U(1)_1\) | \(\cdots\) | \(U(1)_{n_c-2-n_f}\) |
|-------------------|-------------------|-----------|-----------|-------------------|
| \(e_1\)           | 1                 | 0         | \(\cdots\) | 0                 |
| \(e_2\)           | \(\cdots\)       | \(\cdots\) | \(\cdots\) | \(\cdots\)       |
| \(e_{n_c-2-n_f}\) | 1                 | 0         | \(\cdots\) | 1                 |

\[ (3.7) \]

4. Breaking \(N=2\) \(SO(n_c)\) to \(N=1\)

In this section we break to \(N=1\) supersymmetry by turning on bare masses for the adjoint superfield \(\Phi\). Since \(\Phi\) is part of the \(N=2\) vector multiplet \((\Phi, W_\alpha)\), giving it a mass explicitly breaks \(N=2\) supersymmetry. In the microscopic theory, this corresponds to an \(N=1\) theory with a superpotential
\[
\mathcal{W} = \sqrt{2} Q_a \cdot J Q_b \Phi_{ab} - \frac{\mu}{2} \Phi_{ab} \Phi_{ab}.\]
\[ (4.1) \]

For \(\mu \gg \Lambda\) we can integrate \(\Phi\) out in a weak-coupling approximation, obtaining an effective superpotential that vanishes as \(\mu \to \infty\). We are thus left with \(N=1\) \(SO(n_c)\) super–QCD with \(2n_f\) flavors and no superpotential at scales above the strong-coupling scale \(\Lambda_1\) of the \(N=1\) theory. If the strong coupling scale of the \(N=2\) theory is \(\Lambda\), then by a one-loop matching, the \(N=1\) scale is \(\Lambda_1^{3(n_c-2)-2n_f} = \mu^{n_c-2} \Lambda^{2(n_c-2)-2n_f}\). The appropriate scaling limit sends \(\mu \to \infty\) and \(\Lambda \to 0\) keeping \(\Lambda_1\) fixed, so the model is described by the \(N=1\)

\[ \text{\footnote{In } N=1 \text{ theories we count flavors by the number of squark chiral multiplets. Thus, by this counting the } N=2 \text{ theory with } n_f \text{ hypermultiplets has } 2n_f \text{ } N=1 \text{ flavors.}} \]

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theory on scales between $\mu$ and $\Lambda_1$, below which the strongly-coupled dynamics of the $N=1$ theory takes over.

We can also study the breaking to $N=1$ by beginning with $\mu \ll \Lambda$. In this case we should study the effects of $\mu$ on the low-energy $N=2$ theory obtained in the previous sections. $N=1$ supersymmetry prevents a phase transition as we vary $\mu$, hence we should obtain the same result as that obtained for $\mu \gg \Lambda$.

We will now explain why generic vacua of the $N=2$ theory are lifted by nonzero $\mu$, and also why the special point we found on the $r = 2n_f - n_c + 4$ $r$-branch is not. We thus study the effects of the breaking to $N=1$ in the effective theories at the roots of the $r$-branches, which we saw have unbroken gauge groups of the form $SO(r) \times U(1)^{(n_c-r)/2}$. Let $\phi$ denote the adjoint scalar in the $SO(r)$ vector multiplets, and $\psi_k$ the adjoint scalars for each of the $U(1)$ vector multiplets. Then the microscopic mass term $(\mu/2)\text{tr}\phi^2$ becomes $\mu(\Lambda \sum x_i \psi_i + \frac{1}{2} \text{tr}\phi^2 + \ldots)$, where the dots denote higher-order terms, and $x_i$ are dimensionless numbers. (From the $\Phi$ vev (3.6) at the special point, we see that all $x_i \sim 1$.)

Note that at any point on an $r$-branch root for which there are fewer than $(n_c-r)/2$ massless singlets, $e_k$, charged under the $U(1)$’s, then the $N=2$ vacuum is lifted. This can be seen as follows. If there are $n_s$ singlets with $n_s < (n_c-r)/2$, a basis of the $U(1)$’s can be chosen to diagonalize the charges of the singlets, and the superpotential becomes

$$W = \sqrt{2} \text{tr}(Q \cdot J \phi) + \sqrt{2} \sum_{k=1}^{n_s} \psi_k e_k \tilde{e}_k + \mu \left( \Lambda \sum_{i=0}^{(n_c-r)/2} x_i \psi_i + \frac{1}{2} \text{tr}\phi^2 \right).$$

(4.2)

The $F$-term equations following from taking derivatives with respect to the $\psi_i$ then have no solution.

Therefore only the special vacuum (3.7) will lead to an $N=1$ vacuum. In this case the $e_k$ all get vevs, Higgsing all the $U(1)$ factors. Thus, when $\mu \neq 0$ the $e_k$ and $\psi_i$ fields are massive and can be integrated out, leaving the effective superpotential

$$W' = \sqrt{2} \text{tr}(Q \cdot J \phi) + \frac{\mu}{2} \text{tr}\phi^2,$$

(4.3)

It may be that certain other $N=2$ vacua corresponding to non-trivial fixed points can also remain $N=1$ vacua.
for an $N=1$ $SO(2n_f-n_c+4)$ super-QCD with $2n_f$ flavors. This is precisely the dual gauge group for an even number of flavors of $[2]$.

We should re-emphasize that the arguments given here show that the microscopic (AF) theory is IR-equivalent to another theory with the derived dual gauge group, whereas the earlier arguments of $[2]$ essentially only showed this to be true for the chiral rings of the two theories. However, we obtain no information on the extra gauge singlet fields appearing in the dual theory found in $[2]$. Also, much of the rich structure $[2,7]$ of the $N=1$ $SO(n)$ moduli spaces concerning the interplay of their Higgs and confining branches is missed in our analysis. Presumably a similar analysis including bare quark masses would enable us to recover much of this information.

5. $N=2$ Moduli Space and $N=1$ Duality for $USp(2n_c)$

In $N=2$ supersymmetric $USp(2n_c)$ QCD, matter in the $2n_c$ representation of the gauge group is made up of the $N=1$ chiral “quark” multiplets $Q^i_a$, $i = 1, \ldots, 2n_f$, pairs of which $(Q^2_i - 1, Q^2_i)$ together make up $N=2$ hypermultiplets, and which couple to the Yang-Mills fields as

$$
\mathcal{L}_{\text{matter}} = \int d^2 \theta d^2 \overline{\theta} \overline{Q}^i_a (e^V)^{ab} Q^i_b + \int d^2 \theta \sqrt{2} Q^i_a \Phi^{ab} Q^i_b + c.c.,
$$

(5.1)

where the symplectic metric $\mathbb{J}_{ab} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbb{I}_{n_c \times n_c}$ is used to raise and lower $USp(2n_c)$ color indices. Classically (and with no masses) the theory has a global $O(2n_f) \simeq SO(2n_f) \times \mathbb{Z}_2$ flavor symmetry as well as a $U(1)_R \times SU(2)_R$ chiral R-symmetry. Mass terms and instanton corrections break $U(1)_R$ and the $\mathbb{Z}_2$ of the flavor symmetry. When $n_f < 2n_c+2$, the theory is asymptotically free and generates a strong-coupling scale $\Lambda$, the instanton factor is proportional to $\Lambda^{2n_c+2-n_f}$, and the $U(1)_R$ symmetry is anomalous, being broken by instantons down to a discrete $\mathbb{Z}_{2n_c+2-n_f}$ symmetry.

The classical vacua are the solutions to the $D$-term equations,

$$
0 = \Phi_{ab} \Phi_c^b + \Phi_{cb} \Phi_a^b,
$$

$$
0 = Q_a \cdot \overline{Q}_b + Q_b \cdot \overline{Q}_a,
$$

(5.2)
and the $F$-term equations,
\begin{align}
0 &= Q_a \cdot Q_b, \\
0 &= Q^i_a \Phi^a_b.
\end{align}
(5.3)
The Coulomb branch satisfies the first $D$-term equation, implying that $\Phi$ can be diagonalized by a color rotation to
\[ \Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n_c} \end{pmatrix}, \quad \phi_a \in \mathbb{C}, \]
(5.4)
breaking $USp(2n_c) \rightarrow U(1)^{n_c}$, except when $k>1$ of the $\phi_a$'s are equal or vanish, in which case an $SU(k)$ or $USp(2k)$ gauge symmetry is restored, respectively. The Weyl group of $USp(2n_c)$ is generated by permutations and by sign flips of the $\phi_a$, so the symmetric polynomials $S_\ell$, $\ell = 1, \ldots, n_c$ in $\phi_a^2$ are “glue” gauge invariants. Along submanifolds of enhanced $USp(2k)$ symmetry, $S_\ell = 0$ for $\ell > n_c-k$.

The Higgs branches comprise the space of solutions to the second $D$-term equation and the first $F$-term equation since $\Phi = 0$. Describing the squark fields as complex matrices with $2n_c$ rows and $2n_f$ columns, any solution of the Higgs branch equations can be put in the following form by a combination of flavor and gauge rotations:
\[ Q = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes \begin{pmatrix} q_1 \\ \vdots \\ q_r \end{pmatrix}, \quad q_i \in \mathbb{R}^+, \]
(5.5)
where $r \leq \min\{n_c, n_f/2\}$. On this $r$-Higgs branch a $USp(2n_c-2r)$ gauge symmetry is unbroken, with $n_f-2r$ massless hypermultiplets transforming in its fundamental representation. By the Higgs mechanism, of the $2n_c n_f - 2(n_c-r)(n_f-2r)$ neutral hypermultiplets, $n_c (2n_c+1) - (n_c-r)(2n_c-2r+1)$ are given a mass, leaving $\mathcal{H} = 2rn_f - r(2r+1)$ massless neutral hypermultiplets.

Gauge-invariant quantities made from the squark vevs can all be made from the meson field $M^{ij} = Q_{a}^{i} J^{ab} Q_{b}^{j}$, since the antisymmetric tensor on $2n_c$ color indices is just the exterior
product of $n_c$ symplectic metrics. By (5.5), the meson field is

$$M = \begin{pmatrix}
-1 & -i \\
-i & 1 \\
i & -1
\end{pmatrix} \otimes \begin{pmatrix}
q_1^2 \\
\ddots \\
q_r^2
\end{pmatrix}, \quad (5.6)$$

therefore leaving unbroken an $SU(2)^r \times SO(2n_f-4r)$ global flavor symmetry. Thus the number of real goldstone modes is $G = n_f(2n_f-1) - (n_f-2r)(2n_f-4r-1) - 3r$, and the number of real parameters describing the Higgs branch is $P = r$. It is a check on our counting that $G + P = 4H$.

It follows from the second $F$-term equation that on an $r$-Higgs branch $\Phi$ must be zero except in a lower right-hand $(2n_c-2r) \times (2n_c-2r)$ block, which can be diagonalized by the unbroken $USp(2n_c-2r)$ rotations. A gauge-invariant description of these $r$-branches is given by a set of constraints on the glue and meson fields generated by

$$0 = S_\ell \ast (M^{n_c-\ell+1}), \quad \ell \leq n_c,$$

$$0 = M \cdot M, \quad (5.7)$$

analogous to (2.10) and (2.11) in the $SO(n_c)$ case.

The non-renormalization theorem implies that only the Coulomb factors of the $r$-branches can be modified quantum mechanically. We have seen that classically there exist $r$-branches for $0 \leq r \leq \min\{n_c, n_f/2\}$ which meet the Coulomb branch along submanifolds with gauge group $USp(2r) \times U(1)^{n_c-r}$ with $n_f$ fundamental flavors. The AF (or finite) microscopic theories have $n_f \leq 2n_c+2$. Thus, the $USp(2r)$ factors at the roots of the $r$-branches are all IR-free (or finite) and so will remain unbroken quantum-mechanically, with one exception. This is the branch with $r = [n_f/2]$, which is AF. As we will see, in this case the classical gauge group is broken quantum mechanically to a maximal subgroup leading to a non-AF theory.

Submanifolds of the quantum Coulomb branch with unbroken $USp(2r)$ gauge factors are easy to identify explicitly using the exact solution \[^3\] in terms of Riemann surfaces

\[^6\] The three $SO(4)$ generators commuting with the $4 \times 4$ block in (5.6) are $(-1,1) \otimes (1,1)$, $(-1,1) \otimes (1,1)$, and $(1,1) \otimes (1,1)$.
describing the effective action on the Coulomb branch with \( n_f > 0 \) massless flavors:

\[
y^2 = x \prod_{a=1}^{n_c} (x - \phi_a^2)^2 - 4\Lambda^2(2n_c+2-n_f)x^{n_f-1}.
\] (5.8)

In the finite case, \( n_f = 2n_c+2 \), \( \Lambda^0 \) should be replaced by a known function of the bare coupling; in the IR-free case, (5.8) is valid in a sufficiently small neighborhood of \( x = \phi_a = 0 \) where the curve has the simple form \( y^2 \propto x^{2n_c+1} \). Thus, when \( 2r+1 \) branch points of (5.8) coincide, one may expect an unbroken \( USp(2r) \) gauge symmetry. On the submanifold with \( r < [n_f/2] \) of the \( \phi_a = 0 \), the curve becomes

\[
y^2 = x^{2r+1} \prod_{a=1}^{n_c-r} (x - \phi_a^2)^2 - 4\Lambda^2(2n_c+2-n_f)x^{n_f-2r-2},
\] (5.9)

giving vacua with an unbroken \( USp(2r) \times U(1)^{n_c-r} \) gauge group, and so locating the roots of the \( r \)-branches in the full quantum theory for \( r < [n_f/2] \).

Though these theories at the \( r \)-branch roots are IR–free, quantum effects can change the IR theory at points where singlets, \( e_k \), charged under the \( U(1) \) factors become massless. Such points are located where the factor in square brackets in (5.9) becomes singular due to pairs of its zeros coinciding. The maximal such singularity occurs when that factor is a perfect square, corresponding to \( n_c-r \) hypermultiplets becoming simultaneously massless. As in the \( SO(n_c) \) case, there is a single solution to this condition, namely \( r = n_f-n_c-2 \) and \( \phi_a^2 = \Lambda^2(0, \ldots, 0, \omega, \omega^2, \ldots, \omega^{2n_c+2-n_f}) \) where \( \omega = \exp\{2\pi i/(2n_c+2-n_f)\} \). In an appropriate \( U(1) \) basis, the gauge charges of the light degrees of freedom at this special point are

\[
\begin{array}{ccccccc}
2n_f \times Q & USp(2n_f-2n_c-4) & \times & U(1)_1 & \times & \cdots & \times & U(1)_{2n_c+2-n_f} \\
e_1 & 2n_f-2n_c-4 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
e_{2n_c+2-n_f} & 1 & 0 & \cdots & 1
\end{array}
\] (5.10)

\footnote{In the special case \( r = [n_f/2] \), the form of the curve corresponds to the unbroken symmetry \( USp(n_f-2) \times U(1)^{n_c-n_f/2+1} \) (for \( n_f \) even). The classical symmetry \( USp(n_f) \) is broken quantum mechanically to \( USp(n_f-2) \), which is finite with \( n_f \) flavors. In fact, \( USp(n_f-2) \) is the maximal subgroup leading to a non-AF theory.}
We now break to $N=1$ supersymmetry by turning on a bare mass for the adjoint superfield $\Phi$, corresponding to adding a mass term $(\mu/2)\text{tr}\Phi^2$ to the superpotential of the microscopic theory. For $\mu \gg \Lambda$, we can integrate $\Phi$ out in a weak-coupling approximation, obtaining an effective superpotential which vanishes as $\mu \to \infty$. At scales above the strong-coupling scale $\Lambda_1$ of the $N=1$ theory, we obtain $N=1$ USp$(2n_c)$ super–QCD with $n_f$ flavors and no superpotential. If the strong coupling scale of the $N=2$ theory is $\Lambda$, then by a one-loop matching, the $N=1$ scale is $\Lambda_1^{2(3n_c+3-n_f)} = \mu^{2n_c+2} \Lambda^{2(2n_c+2-n_f)}$. The appropriate scaling limit sends $\mu \to \infty$ and $\Lambda \to 0$ keeping $\Lambda_1$ fixed, so the model is described by the $N=1$ theory at scales between $\mu$ and $\Lambda_1$, below which the strongly-coupled dynamics of the $N=1$ theory takes over.

We can also study the breaking to $N=1$ by beginning with $\mu \ll \Lambda$. In this case we should study the effects of $\mu$ on the $N=2$ vacua described above. It is easy to see that generic vacua of the $N=2$ theory are lifted by nonzero $\mu$; however, the special point we found on the $r = n_f - n_c - 2$ $r$-branch is not. Let $\phi$ denote the adjoint scalar in the USp$(2r)$ vector multiplets, and $\psi_k$ the adjoint scalars for each of the $U(1)$ vector multiplets for the unbroken USp$(2r) \times U(1)^{n_c-r}$ symmetry at the roots of the $r$-branches. The microscopic mass term $(\mu/2)\text{tr}\Phi^2$ becomes $\mu(\Lambda \sum_i x_i \psi_i + \frac{1}{2} \text{tr}\phi^2 + \ldots)$, where the dots denote higher-order terms, and $x_i$ are dimensionless numbers. At any point on an $r$-branch root for which there are $n_s$ massless singlets, $e_k$, charged under the $U(1)$’s, with $n_s \leq n_c - r$, a basis of the $U(1)$’s can be chosen to diagonalize the charges of the singlets. The $F$-terms following from the resulting superpotential have no solution unless $n_s = n_c - r$, showing that only the special vacuum (5.10) leads to an $N=1$ vacuum. In this case the $e_k$, $\psi_i$, and $\phi$ fields are massive and can be integrated out, leaving an effective $N=1$ USp$(2n_f - 2n_c - 4)$ super–QCD with $n_f$ flavors. This is precisely the dual gauge group of $[2][3]$.

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8 In $N=1$ theories we count flavors by pairs of squark chiral multiplets since an odd number of fundamental chiral multiplets is anomalous in USp$(2n_c)$ $[13]$; thus this counting is the same as the counting of hypermultiplet flavors in the $N=2$ theory.
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