On uniformization of Burnside’s curve \( y^2 = x^5 - x \)

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Main objects of uniformization of the curve \( y^2 = x^5 - x \) are studied: its Burnside’s parametrization, corresponding Schwarz’s equation, and accessory parameters. As a result we obtain the first examples of solvable Fuchsian equations on torus and exhibit number-theoretic integer \( q \)-series for uniformizing functions, relevant modular forms, and analytic series for holomorphic Abelian integrals. A conjecture of Whittaker for hyperelliptic curves and its hypergeometric reducibility are discussed. We also consider the conversion between Burnside’s and Whittaker’s uniformizations.

I. INTRODUCTION

Presently, one example of an algebraic curve of genus \( g > 1 \)

\[ y^2 = x^5 - x \]  \hspace{1cm} (1)

is known where one is able to present all the key objects of its uniformization in an explicit form. These include: 1) parametrization \( x(\tau), y(\tau) \) in terms of known special functions; 2) closed differential calculus associated with these functions; 3) pictures of conformal representations resulting from the uniformizing functions; 4) differential equations on functions \( x(\tau) \) and \( y(\tau) \), their solutions, and; 5) matrix representations for monodromy groups of related Fuchsian equations.

Somewhat surprising facet is the fact that, except for parametrizations by means of modular functions (the theory of modular equations \([9, 17]\)), the straightforward statement of these questions was not considered in the literature, including even the famous Klein curve \( y^2 = x^3 - x^2 \) \([20]\). The parametrization property for the curve (1) was found by William S. Burnside in 1893 \([3]\), but this result has, however, received almost no mention in the papers and has not appeared in the monographic literature devoted to uniformization, automorphic functions or other relevant material. To the best of our knowledge, since 1893 the work \([3]\) has been mentioned by R. Rankin in 1958 (see e. g. \([23]\)) and comparatively recently he drew attention to it in work \([24]\). On the other hand, explicit instances of all of these objects in the case of higher genera would enable one to construct nice applications wherever algebraic functions and Riemann surfaces appear. Such topics include algebraic-geometric integration, conformal field theories \([1]\), integrable quantum/classical dynamical systems and nonlinear PDE’s, equations ofPicard–Fuchs \([13, 14]\), second order linear ordinary differential equations (ODE’s) of Fuchsian type \([21]\), number theory, “Monstrous Moonshine” \([13]\), and many others. Yet another and deep application of uniformization originated in works of Takhtajan and collaborators in the late 1980’s and for further applications of the general uniformization theory see work \([1]\) and references therein. However, it is pertinent to note that the general theory has long experienced the lack of nontrivial illustrative instances.

History of the curve (1) goes back to the work of Bolza \([2]\) where he obtained period matrix for this curve and its automorphism group. Geometry of fundamental domains of Fuchsian groups (hyperbolic polygons) for curves of lower genera, including the curve (1), has been studied extensively in the literature and rather well developed. See for example works \([18, 21]\) and references therein. In 1958 Rankin found the correct Fuchsian equation uniformizing this curve in the framework of an approach of Whittaker \([20]\). More recently,

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Rankin simplified [24] (inserting radicals however) Burnside’s parametrizing functions and considered some of their group properties: that is, the structure of automorphism groups of the functions \( x(\tau) \) and \( y(\tau) \). The group of the function \( x(\tau) \) was described earlier by Klein & Fricke [15] and used by Rankin and Burnside himself [3]. At present, this is as far as we known about uniformization of the curve (1).

In this note, we would like to draw attention to this remarkable example in a classical framework and to exhibit the differential properties of Burnside’s Riemann surface. It is widely known that these properties are fully determined by the second order linear differential equations of Fuchsian class on plane [10] (Secs. II, III) in so far as the monodromical groups of these equations give, broadly speaking, a matrix 2×2-representation of fundamental group \( \pi_1 \) of the Riemann surface. If the surface (a curve) may be realized as a cover of a Riemann surface of genus \( g = 1 \) (torus) then these equations can be transformed to Fuchsian equations on this torus. We explain this in Sec. IV and consider also solutions of these equations. In addition to this, in Sec. III, we discuss relation with a conjecture of Whittaker [30, 31]. Automorphic forms and analytical integer \( q \)-series for functions we construct are considered in Sec. V. All the results are new.

As well as being the first completely describable example, Burnside’s curve also provides rich material for different kind nontrivial generalizations and observations, some of which we expound below. Throughout the paper we have adhered to basic and standard terminology in the theory of automorphic functions and uniformization [9, 10, 21, 28]. The classical bibliography is listed in the book [10] and the works [4, 14, 21] provide additional and modern references.

Let \( x = x(\tau) \) be meromorphic automorphic function on a Riemann surface of the algebraic curve \( F(x, y) = 0 \) with a local uniformizing parameter \( \tau \). Then \( x(\tau) \) satisfies the following nonlinear autonomous differential equation of the third order [10, 22, 23] (we call it Schwarz’s equation)

\[
\frac{\{x, \tau\}}{x_\tau^2} = Q(x, y), \quad \{x, \tau\} := \frac{x_{\tau\tau\tau}}{x_\tau} - \frac{3}{2} \left( \frac{x_{\tau\tau}}{x_\tau} \right)^2,
\]

wherein \( Q(x, y) \) is some rational function of \( x \) and \( y \) [10, 22]. To determine its coefficients, so that monodromy group of the associated Fuchsian equation

\[
\Psi_{xx} = \frac{1}{2} Q(x, y) \Psi, \quad \tau = \frac{\Psi_1(x)}{\Psi_2(x)}
\]

shall be Fuchsian, is the celebrated problem of accessory parameters considered in the context of algebraic curves [22, pp. 222–228]. With the correct parameters in hand the quantity \( \tau \) becomes the global uniformizing parameter.

Compared to the theory elliptic functions, currently available analytical description of uniformizing functions for the genera higher than unity is poorly developed. By this we mean: 1) determining oDE’s [2], their solutions, and correct accessory parameters; 2) effective series expansions and numerical computations; 3) inversion problems in fundamental polygons, i.e. search for solutions \( \tau \) to equations of the type \( x(\tau) = A \); 4) conformal representations; 5) Abelian integrals as functions of the global parameter \( \tau \); 6) addition theorems for these Abelian integrals and relation with the Jacobi inversion problem. In the following sections we shall fill up some of these gaps in the example of the curve (1). The availability of all of these attributes in the case of genus \( g = 1 \) provides the great efficiency of elliptic functions and their numerous applications, which cannot be said of the cases \( g > 1 \) for reasons of the insufficiently advanced analytic tools.

In this paper we follow classical Burnside’s \( \wp \)-formulae, although they have a simpler representation in terms of jacobian \( \vartheta \)-constants. Such and exhaustive \( \vartheta \)-function description

\[\text{1 See for example last sentence in the book [10] and Weyl’s emphasis on pp. 176–177 in the very first monograph on Riemann surfaces [28].}\]
of the example (1) (without intersections with the present work) have been detailed in the work [4]. In the same place further generalizations and extended bibliography are presented.

II. BURNSIDE’S UNIFORMIZATION AND THE SCHWARZ EQUATION

In this section we show that Burnside’s result provides the first example of nontrivial algebraic curve where one is known both corresponding Fuchsian equation with correct accessory parameters and uniformizing functions, and also complete differential calculus of these functions. We reproduce a parametrization of the curve (1) in an explicit form more suitable for our purpose. Let \( \tau \) be a complex variable with nonzero imaginary part \( \Im(\tau) \neq 0 \) and Weierstrass’s \( \sigma, \zeta, \wp, \wp’ \)-functions are taken with half-periods \( (\omega, \omega’) = (2, 2\tau) \). For example

\[
\wp(z|2, 2\tau) = \wp(z; g_2(2, 2\tau), g_3(2, 2\tau)) \quad (=: \wp(z)).
\]

In this notation the parametrization of Burnside [3] has the form

\[
\begin{align*}
\frac{x}{x^2} &= \frac{\wp(1) - \wp(2)}{\wp(\tau) - \wp(2)}, \\
y &= 4i \left[ \wp(\tau) - \wp(2\tau) \right] \left[ \wp \left( \frac{\tau}{2} \right) - \wp(\tau) \right] \left[ \wp \left( \frac{\tau}{2} + \frac{\tau}{2} \right) - \wp(2\tau + 1) \right] \left[ \wp \left( \frac{\tau}{2} \right) - \wp(1) \right].
\end{align*}
\]

Proposition 1. The Schwarz equation (2) and, therefore, the accessory parameters for Burnside’s parametrization (1), (8) have the form

\[
\{x, \tau\} = -\frac{1}{2} \left\{ \frac{1}{x} + \frac{1}{(x - 1)^2} + \frac{1}{(x + 1)^2} + \frac{1}{(x - i)^2} + \frac{1}{(x + i)^2} - \frac{4}{x^5 - x} \right\}.
\]

Of interest is to give a direct proof, although the verification of the calculations is not straightforward because the \( \zeta, \wp, \wp’ \)-functions are not closed under differentiation with respect to \( \tau \). For example, we shall require a closed system of differential equations satisfied by \( g_{2,3}(\omega, \omega’) \) and the periods \( \eta, \eta’(\omega, \omega’) \) of the elliptic integral \( \zeta(z|\omega, \omega’) \):

\[
\frac{dg_2}{d\tau} = \frac{i}{\pi} \left( 8g_2 \eta - 12g_3 \right), \quad \frac{dg_3}{d\tau} = \frac{i}{\pi} \left( 12g_3 \eta - \frac{2}{3}g_2 \right), \quad \frac{d\eta}{d\tau} = \frac{i}{\pi} \left( 2\eta^2 - \frac{1}{6}g_2 \right).
\]

where we use the notation \( \eta = \eta(\tau) \) for \( g_{2,3}(1, \tau) \) and the same for \( \eta’ \). Some of the intermediate results in the proof will be required later. Since the arguments of Weierstrass’s functions will constantly appear in the following calculations, we adopt the concise notation \( \wp_\alpha := \wp(\alpha|2, 2\tau) \) etc. Throughout the paper in the calculations that follows (being automated with a computer), we use intensively known and new properties of elliptic and modular functions collected in a form of reference source in work [5].

Proof. A straightforward substitution of (8) into (2) generates, apart from cumbrous formulae, many variables, including \( \tau \) in an explicit way. But they are not algebraically independent. Using the addition theorems for the functions \( \zeta(2\alpha), \wp(2\alpha), \wp’(2\alpha) \) at points \( \alpha = 1 \) and \( \tau \), we obtain six identities. We shall need also important formulae for the invariants \( g_{2,3} \)

\[
g_2 = \wp_\alpha^3 - \wp_\nu^3 - \wp_\alpha^2 - \wp_\nu^2 - \wp_\alpha - \wp_\nu, \quad g_3 = \wp_\nu \wp_\alpha^2 - \wp_\alpha \wp_\nu^2 - 4 \wp_\alpha \wp_\nu (\wp_\alpha + \wp_\nu),
\]

derivable from the relations \( \wp_\alpha^2 = 4\wp_\alpha^3 - g_2 \wp_\alpha - g_3 \), and which are valid under \( \alpha \neq \nu \). We use them with \( \alpha = 1 \) and \( \nu = 2 \). Now two of the six above identities are transformed to
The formula (4) is obtained after substitution of these derivatives into (2).

\[ \varphi' = \frac{2}{\eta' - 4\zeta' \tau} (3 \varphi_1^2 + \varphi_2^2 - 2 \varphi_1 \varphi_2 - 2 \varphi_2^2), \]

\[ \varphi_1^2 + 2 \varphi_1 \varphi_2^3 + 6(\varphi_1^2 - 2 \varphi_1 \varphi_2 - \varphi_2^2) \varphi_2^2 - 2 \varphi_2 (3 \varphi_1^2 - 6 \varphi_1 \varphi_2 - 4 \varphi_2^2) \varphi_2 + \]

\[ + \varphi_1^4 - 4 \varphi_1^3 \varphi_2 + 6 \varphi_1^2 \varphi_2^2 - 4 \varphi_1 \varphi_2^3 - 4 \varphi_2^4 = 0. \]

The equations (5)–(6), together with the rules of differentiation of Weierstrassian functions [2], contain all the information required for the proof. Using the first formula in (3) we obtain the following useful expressions:

\[ \varphi' = \frac{144}{\pi^2} \frac{x^4 - 1}{(x^4 - 6 x^3 + 6 x^2 - 6 x + 1)^2}, \quad \varphi_1 = \frac{x^4 - 6 x^3 + 6 x^2 - 6 x + 1}{x^4 + 6 x^2 + 1} \varphi_2, \quad \frac{\varphi_1}{\varphi_2} = \frac{x^4 - 5}{x^4 + 6 x^2 + 1}, \quad \frac{\varphi_2}{\varphi_1} = \frac{x^4 - 6 x^3 + 6 x^2 - 6 x + 1}{x^4 + 6 x^2 + 1}, \]

whereupon three derivatives of the \( x(\tau) \)-function acquire the form

\[ x_{\tau} = \frac{24 \pi}{x^5 - x} \frac{x^5 - x}{x^3 + 6 x^2 + 1} \varphi_2, \]

\[ x_{\tau\tau} = - \frac{96}{\pi^2} \frac{(x^4 - 6 x^2 + 1) \eta + 2 (5 x^4 - 1) \varphi_2 (x^5 - x) \varphi_2}{(x^4 + 6 x^2 + 1)^2}, \]

\[ x_{\tau\tau\tau} = - \frac{576}{\pi^2} \left\{ \frac{(x^4 + 6 x^2 + 1) \eta + 4 (5 x^4 - 1) \varphi_2}{(x^4 + 6 x^2 + 1)^2} \eta + 8 (11 x^8 - 26 x^4 - 1) \frac{\varphi_2}{x^4 + 6 x^2 + 1} \right\} (x^5 - x) \varphi_2. \]

The formula (4) is obtained after substitution of these derivatives into (2).

**Remark** 1. We can view the last identity in (10) as a plain curve in projective coordinates \( (\varphi_\tau : \varphi_1 : \varphi_2) \). It has genus \( g = 0 \). First formula in (10) and second one in (10) yield

\[ \frac{\varphi_\tau}{\varphi_2} = \frac{x^4 - 5}{x^4 + 6 x^2 + 1}, \quad \frac{\varphi_1}{\varphi_2} = \frac{x^4 - 6 x^3 + 6 x^2 - 6 x + 1}{x^4 + 6 x^2 + 1} \]

and therefore the quantity \( x \) may be considered as a global parameter in rational parametrization of this curve. Variation of \( x \in \mathbb{C} \) is equivalent, through (3), to variation of \( \tau \in \mathbb{H} \) in fundamental polygon for group \( \Gamma(4) \) and thereby correct variation of quantities \( (\varphi_\tau, \varphi_1, \varphi_2) \).

### III. FUCHSIAN EQUATIONS AND A CONJECTURE OF WHITTAKER

#### A. Fuchsian equations associated to Burnside’s parametrization

Integrability of the Fuchsian equation associated with the formula (4)

\[ \Psi_{xx} = - \frac{1}{2} Q(x) \Psi = \]

\[ = - \frac{1}{4} \frac{x^8 + 14 x^4 + 1}{(x^5 - x)^2} \Psi \]

(11)
immediately follows from known properties of the Schwarzian [2]. Denoting for brevity \( \sqrt[3]{z} \) as \( -2 \sqrt[3]{z} \) we have the well-known identity [10] \[
( -2 \sqrt[3]{\tau x} )_{xx} = \frac{1}{2} Q(x) \cdot -2 \sqrt[3]{\tau_x} .
\]

Thus, setting \( \Psi(x) = -2 \sqrt[3]{\tau x} \) we obtain an integral of equation (11). There are numerous forms for integrals of Fuchsian solvable equations and we, following (3), give them in Burnside’s \( \wp \)-manner.

**Proposition 2.** The general multi-valued integral of equation (11) is given by the formula

\[
\Psi(x) = \sqrt[4]{x^5 - x} \sqrt[4]{\wp(\tau(x)|2, 2\tau(x))} \left( A \tau(x) + B \right),
\]

where function \( \tau(x) \) is determined by the inversion of the expression

\[
x = \frac{\wp(1|2, 2\tau) - \wp(2|2, 2\tau)}{\wp(2|2, 2\tau) - \wp(2|2, 2\tau)}.
\]

Proof follows from the formulae (9) and (10). The second Fuchsian equation is an equation on the function \( \tilde{\Psi}(y) \) defined by the rule \( \tilde{\Psi}(y) = -2 \sqrt[3]{\tau_y} \). It has a form of equation with algebraic coefficients:

\[
\tilde{\Psi}_{yy} = \frac{1}{2} Q(x, y) \tilde{\Psi} = \frac{1}{4} \frac{5^4 x y^6 + 415 x^2 y^4 - 511 x^3 y^2 + 255 x^4 + 1}{(5^4 x y^6 + 1375 x^2 y^4 + 1025 x^3 y^2 + 255 x^4 + 1) y^2} \tilde{\Psi}.
\]

General representation for its integral is given by the formula

\[
\tilde{\Psi}(y) = \sqrt[3]{\tau_y} \left( A \tau(y) + B \right),
\]

where \( \tau(y) \) is an inversion of the second formula in (9). Schwarz’s equation corresponding to the second function \( y(\tau) \) has the form [2] with \( Q(x, y) \)-function defined by the right hand side of expressions (14).

The group of automorphisms \( \text{Aut}(x(\tau)) \) of Burnside’s function \( x(\tau) \) is of index 24 sub-group in the full modular group \( \text{PSL}_2(Z) \) having the Klein invariant \( J(\tau) \) as a Hauptmodulus [3, 17]. Indeed, with the proof of Proposition 1 we obtain the formula

\[
\frac{g_2^2}{g_2^2 - 27 g_3^2} = \frac{1}{108} \frac{(x^8 + 14 x^4 + 1)^3}{(x^5 - x)^4} = J(\tau).
\]

The factor group \( \text{PSL}_2(Z)/\text{Aut}(x(\tau)) \) is the octahedral group [17] of order 24 [2]. Taking into account a permutation of sheets \( y(\tau + 4) = -y(\tau) \) [24], it becomes the maximal automorphism group of order 48 for the curves of genus two.

**B. Whittaker’s conjecture**

In 1929 E. Whittaker [30] proposed a pure algebraic solutions of the transcendental problem of accessory parameters for certain Fuchsian groups of the first kind without parabolic edges [10]. Namely, from words of his son J. M. Whittaker [31], he suggested that for hyper-elliptic algebraic curves

\[
y^2 = (x - e_1) \cdots (x - e_{2g+1}) \quad (= f(x))
\]
the $Q$-function is given by the formula
\[ Q(x) = -\frac{3}{8} \left\{ \frac{f_x^2}{f^2} - \frac{2g + 2}{2g + 1} \frac{f_{xx}}{f} \right\}. \] (17)

The conjecture was checked for some curves in the 1930's by Whittaker's collaborators [8, 31]. What is more, by considering the hyperlemniscate algebraic curve
\[ y^2 = x^5 + 1, \] (18)
Whittaker [30] reduced the associated Fuchsian linear ordinary differential equation
\[ \Psi_{xx} = -\frac{3}{16} \left\{ \sum_{k=1}^{5} \frac{1}{(x - e_k)^2} - \frac{4x^3 + A_1 x^2 + A_2 x + A_3}{(x - e_1) \cdots (x - e_5)} \right\} \Psi \] (19)
to a hypergeometric equation and, in fact, explicitly demonstrated the first nontrivial example of integrability of equation (19). Note that even number of regular singular points is essential in the context of uniformization by Whittaker’s approach [29], as hyperelliptic curves have always even number of branch points.

C. Where do hypergeometric equations come from?

Whittaker does not elucidate the nature of his conjecture but the idea (not formulated) goes back to H. Weber (see formulae (10–16) in [27]), although Weber considered a conformal representation of multi-connected areas. The change of variables $x \mapsto y$ and the substitution
\[ \Psi(x) = \sqrt{y} \widetilde{\Psi}(y), \] (20)
lead to the appearance of the hypergeometric equation [27, 30]
\[ y (y - 1) \widetilde{\Psi}_{yy} + [(a + b + 1) y - c] \widetilde{\Psi}_y + a b \widetilde{\Psi} = 0. \] (21)

Such a reduction is not a common property of hyperelliptic curves and the relation between equations (19) and (21) (when it exists) depends on the kind of substitution. Nevertheless, the question arises concerning Burnside’s formulae [8]. An attempt to apply such an argument (including the substitution (20) to the curve (11) was undertaken in [26] but the conjecture certainly does not fit our example, because the group contains a parabolic element: $x(\tau + 4) = x(\tau)$. We observe however that the $Q$-functions (17) and (4) differ from each other by only the numeric multiplier $\frac{4}{3}$. Without taking into account this multiplier, all the accessory parameters $A_{1,2,3}$ in Fuchsian equations (19) are equal to zero (the same zero as in formula (4)). The examples of Burnside (1) and Whittaker (18) are thus the simplest ones with known monodromies.

An explicit connection of the example in question with the modular group and the hypergeometric equation is given by the formula (15). Indeed, function $J(\tau)$ satisfies the Fuchsian equation of hypergeometric type (21) with $(a = b = \frac{1}{12}, c = \frac{3}{2})$ and therefore $J$ is defined as inversion of a quotient of its two solutions. We do not display here numerous forms of such representations (see for example Klein's formulae [17, p. 61] or formulae (22)–(25) in [8, 14.6.2]). From (15) we deduce that $x^4 = z$ is a root of the polynomial
\[ (z^2 + 14z + 1)^3 - 108z(z - 1)^4 \cdot J(\tau) = 0. \]

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2 We were unsuccessful in finding any unpublished manuscript material of Edmund Whittaker on his conjecture in Edinburgh and London. Nevertheless the author thanks the staff of the London Mathematical Society Archives for their help in seeking such information.
Hence by this construction $\sqrt{z}(\tau)$ is a *globally single-valued* function no matter which branch of this root is chosen. This is a hidden form of Burnside’s function $x(\tau) = \sqrt{z}(\tau)$ in (3).

There is a simpler relation and a reduction of the Fuchsian equation (11) to the hypergeometric equation. Namely, the function $\psi(z)$ of this root is chosen. This is a hidden form of Burnside’s function $x(z)$.

Whittaker’s one? One way is to make use of the substitution (22), whereupon we get an equation

$$\psi(y) = \sqrt{y},$$

(and compare (24) with his conjecture (17)

$$y = \frac{1}{2} \left( \frac{y^2 + 3}{(y^2 - 1)^2} \sqrt{y} \right),$$

and its solution, say, in terms of hypergeometric series:

$$\psi_1(y) = (y^2 - 1)^{\frac{1}{2}} \cdot 2F_1 \left( \frac{1}{5}, \frac{4}{5} \mid \frac{2}{5} \right),$$

$$\psi_2(y) = (y - 1)^{\frac{1}{2}} (y^2 - 1)^{\frac{1}{2}} \cdot 2F_1 \left( \frac{3}{5}, \frac{6}{5} \mid \frac{1}{2} \right).$$

The monodromy group of the hypergeometric equation (23) is always Fuchsian if the difference of exponents is the reciprocal of an integer 1, 2, …, etc. In the case (23) the Fuchsian exponents for all the points $y = \{1, -1, \infty\}$ are

$$\left\{ \frac{2}{5}, \frac{3}{5} \right\},$$

so that the monodromy groups for Whittaker’s equations (14) and (21) have both genus zero. Their intersection is a genus 2 subgroup, i.e. group with four generators uniformizing equation (18). In fact, this was done by Whittaker [31].

**D. The conjecture and hypergeometric reducibility**

Whittaker’s conjecture admits an additional treatment. Let us write equation of the curve (10) in the form $y^2 = x^{2g+1} + E(x)$, where $E(x)$ is a polynomial of degree 2 $g$. Bring Whittaker’s $Q(x)$-function (19) into the form

$$Q(x) = -\frac{3}{8} \left\{ \frac{f^2}{f^2} - 4g(g + 1) \frac{x^{2g-1}}{f} + \frac{E'' + A(x)}{f} \right\},$$

and compare (24) with his conjecture (17)

$$Q(x) = -\frac{3}{8} \left\{ \frac{f^2}{f^2} - 4g(g + 1) \frac{x^{2g-1}}{f} + \frac{2g + 2E''}{2g + 1} \right\}. \tag{25}$$

It immediately follows that accessory polynomial is a symmetrical polynomial in branch points $e_j$, i.e. $(2g + 1) A(x) = E''$. (The polynomial $A(x)$, as symmetrical one in $e_j$, was considered in work [31]). When is it possible to reduce equation (19) to the hypergeometric one? One way is to make use of the substitution (22), whereupon we get

$$\Psi_{yy} = -\frac{1}{4} \frac{4g(g + 1)(2g + 1)^2 y^6 - (2g + 1)(8x^3 E''' + 3(2g + 1) x^2 A(x) + \cdots) y^4 + \cdots}{y^2 x^2 + xE' - (2g + 1) E^4} \Psi.$$
If we take into account that this rational function has not to depend on \(x\), we obtain from the denominator that \(E(x) = a\), where \(a\) is a constant. One can give a rigorous form to this reasoning but we speak here about only motivation. We get

\[
\tilde{\Psi}_{yy} = -\frac{1}{4} \frac{4 g (g + 1) (y^2 + 3a) - 3 x^2 A(x)}{(2 g + 1)^2 (y^2 - a)^2} \tilde{\Psi},
\]

so that the only possibility is to put the accessory polynomial \(A(x)\) equal to zero

\[
\tilde{\Psi}_{yy} = -\frac{g (g + 1) y^2 + 3a}{2 g + 1} \tilde{\Psi}
\]

and the monodromy automatically becomes Fuchsian. Moreover it becomes triangle group with angles \(\frac{2\pi}{2g+1}\). This is nothing else but all the examples of the papers [8, 30, 31] and elementary treatment of the conjecture from the standpoint of such a reduction.

We may reverse the reasoning. Let us make the hyperelliptic change of variables \(y \mapsto x\) by the rule \(y^2 = x^2 + E\) and the reverse substitution (22) into the hypergeometric equation (26) with the monodromy known to be Fuchsian. We shall obtain new Fuchsian equation in variable \(x\). Require that this equation has singularities at hyperelliptic branch points \(x = \epsilon_j\) and coincides with Whittaker’s form (24). What accessory polynomial \(A(x)\) does satisfy these requirements? The answer, as it follows from the previous constructions, is the Whittaker conjecture (25).

The conclusion is not changed if we consider the parabolic version

\[
Q(x) = \frac{1}{2} \left\{ \frac{f^2}{f^2} - \frac{2 g + 2 f_{xx}}{2 g + 1} \right\},
\]

but Burnside’s example does not lead to a simple reduction of the type \(x \mapsto y\) (see (14)), although it admits another one. See [4] for details on the transformations of the type (22) and additional discussion the conjecture in [7]. The conjecture is not true in general [7] but the question “when and why does it work?” remains still open. Yet another and perhaps simplest motivation to the hypergeometric reducibility is the fact that equations (18) and \(y^2 = x^2 + 1\) represent an algebraic function \(x(y) = \sqrt{y^2 - 1}\) with three branch points \(-1, 1, \infty\). But three points, independently of kind of uniformization (Whittaker’s or parabolic), always lead to a hypergeometry \(2F_1\). Correlation between the conjecture and equations with three branch-points was also remarked in work [7].

IV. UNIFORMIZATION AND COVERS OF TORI

If a curve covers an elliptic torus then an integrable Fuchsian equation on torus is naturally to be expected. This is the topic of the present section.

Riemann surfaces of higher genera have negative curvature. Therefore the simplest way to get nontrivial Fuchsian equations and uniformization over tori is to consider torus with at least one puncture. Such a problem is described by equation \(\Xi_{\alpha\alpha} = (-\frac{1}{4} \psi(\alpha) + A) \Xi\) which was already considered in the literature. First paper on this topic was the work [16] (see also subsequent works of L. Keen) however, up till now, no example with explicit analytic formulae has been obtained. In addition to the lemniscate and equi-unharmonic cases found in [16], brothers Chudnovsky revealed [6] two more exceptional cases when an accessory parameter is known and associated Fuchsian group has an arithmetic-algebraic nature. See their work [6] for more references and additional discussion to this problem. Solutions of Fuchsian equations, the problem of inversion and, what is important, transformations between the equations are not considered in these works. On the other hand, if we have correct accessory parameters and solvable Fuchsian uniformization for some algebraic curve and, in turn, this curve can be realized as a cover over torus, then we could automatically obtain non-trivial examples on torus. The general mode of getting such results is as follows.
Let we have a Fuchsian equation in variable \( x \in \mathbb{C} \), i.e. Fuchsian equation on plane, say, \((19)\). Let now \( R(x, \alpha) = 0 \) be a formula for the cover wherein \( \alpha \) is the global parameter on torus being covered. In most general case such formulae have the form

\[
\tilde{R}(x, \varphi(\alpha), \varphi'(\alpha)) = 0
\]  

with polynomial function \( \tilde{R} \). Considering equation \((27)\) as a transcendental (non algebraic) change of variables \( x \mapsto \alpha \) we can transform the initial Fuchsian “\( x \)-equation” into an equation in \( \alpha \) (“\( \alpha \)-equation). If “\( x \)-equation” was of Fuchsian class with correct accessory parameters then the “\( \alpha \)-equation” will be of the same class. Burnside’s curve is just the case. In other words, from the uniformization point of view, not merely punctured tori should be searched for, but situations when the tori are covered by nontrivial algebraic curves.

Since the examples that follows are the first exactly solvable ones along these lines we expound all this at greater length in Sec. IV.C.

A. Cover of torus

First (non-complete but hyper- and non-hyperelliptic) examples of cover of torus \( g = 1 \) by curves of higher genera were obtained in the first volume of Legendre’s *Traité* \(19\). We consider a hyperelliptic example of Legendre generalized by Jacobi \(15, \text{ Werke I}: pp. 375–381\):  

\[
y^2 = x(x-1)(x-a)(x-b)(x-ab).
\]  

Jacobi found a substitution (simpler version of Legendre’s one \(19, \text{ I}: p. 259\)) of second degree \( x \mapsto \lambda \):

\[
\lambda = \frac{(1-a)(1-b) x}{(x-a)(x-b)},
\]  

which leads to the fact that both holomorphic differentials for the curve \((28)\) reduce to elliptic differentials for the tori

\[
\left( \lambda, \mu \right)_{\pm} : \quad \mu^2 = \lambda(\lambda-1)(k \lambda-1), \quad k = k_{\pm} = -\frac{(\sqrt{a} \pm \sqrt{b})^2}{(a-1)(b-1)}.
\]  

We transform this equation into the canonical Weierstrassian form

\[
w^2 = 4 \left( z - \frac{2k-1}{3} \right) \left( z - \frac{2-k}{3} \right) \left( z + \frac{k+1}{3} \right)
\]

with the help of obvious scale transformation and subsequent parametrization:

\[
(z = k \lambda - \frac{k+1}{3}, \quad w = 2k \mu) \quad \Rightarrow \quad \left( \varphi(\alpha), \varphi'(\alpha) \right).
\]

The Burnside curve corresponds to the following parameters

\[
a = -1, \quad b = i, \quad k_{\pm} = \frac{1 \pm \sqrt{2}}{2}, \quad g_2 = \frac{5}{3}, \quad g_3 = \frac{7}{27} \sqrt{2}
\]

and the two tori are isomorphic to the one classical torus with a complex multiplication:

\[
J \left( \frac{\omega'}{\omega} \right) = \frac{5^3}{3^3}.
\]

\[3\] In the same place on p. 377 Whittaker’s curve \(18\) appeared. In this section we return to the standard conventions for Weierstrassian \( \zeta, \varphi \)-functions: \( \varphi(\alpha) = \varphi(\alpha|\omega, \omega'), \ e = \varphi(\omega|\omega, \omega') \), etc.
It has another standard form $\varphi'^2 = 4\varphi^3 - 30\varphi - 28$ due to formula

$$\varphi\left(\pm \frac{5}{3}, \frac{7\sqrt{2}}{27}\right) = \frac{\sqrt{2}}{6} \varphi\left(\frac{\sqrt{2}}{12} \varphi_{12}; 30, 28\right).$$

The values of corresponding periods are computed with use of modular inversion problem:

$$\omega = \sqrt{\frac{12}{5}} g_2(\sqrt{2}i), \quad \omega' = i\sqrt{2} \omega.$$

Using some properties of modular functions one can show [5] that this Burnside’s torus has an exact solution to the constant $\omega$ since

$$g_2(\sqrt{2}i) = \frac{5}{3} \pi^4 \hat{\eta}^8(\sqrt{2}i).$$

We get

$$\omega = \pi \sqrt{2} \hat{\eta}^8(\sqrt{2}i) = 2.118 156 723 947 863 188 505 038 347 005 72\ldots,$$

$$\omega' = -2 \sqrt{2} i \omega, \quad \omega'' = -\omega - \omega'.$$

where, to avoid confusion between Weierstrass’s and Dedekind’s standard notations for their $\eta$-functions, we denoted Dedekind’s one as $\hat{\eta}(\tau)$:

$$\hat{\eta}(\tau) = e^{\pi i \tau} \prod \left(1 - e^{2\pi i n\tau}\right).$$

Assuming all the introduced parameters to be fixed and taking upper sign in $k_{\pm}$, we view the Jacobi substitution (29) as an explicit 2-sheeted cover of the torus

$$\varphi'(\alpha)^2 = 4\left(\varphi(\alpha) - \frac{\sqrt{6}}{3}\right)\left(\varphi(\alpha) + \frac{3 + \sqrt{2}}{6}\right)\left(\varphi(\alpha) - \frac{3 - \sqrt{2}}{6}\right)$$

by the $x$-planes or a fundamental 10-gon for the function $x(\tau)$ in $\tau$-plane [4]. More precisely, the substitution (29), i.e. equation (27), has the form

$$\varphi(\alpha) = e' + \frac{1 + 3 e}{x(\tau) - i} - i \frac{1 + 3 e}{x(\tau) + 1},$$

where the quantities $(e, e', e'')$ are taken from (31) in that order. The formula (32) constitutes another representation of a Riemann surface defined by the Burnside curve or, which is the same, an equivalent transcendental representation $R(\alpha, \tau) = 0$ of the curve (1) itself in terms of meromorphic functions on covering and covered surfaces. Omitting argument $\tau$ in (32) we shall deal with algebraic-transcendental version $R(\alpha, x) = 0$ of this representation.

### B. Structure of the cover (32)

Let us consider ramification schemes of equation (32). The ramification points $\alpha_j$ of this representation, as a map $\alpha \mapsto x$, are determined from the equation

$$(6 \varphi(\alpha) - 3 + \sqrt{2})(6 \varphi(\alpha) + 21 + 13 \sqrt{2}) = 0$$

(this is discriminant of the formula (32)). Therefore its solutions have the form

$$\left(\alpha_1 = \omega'', \ x = -i \sqrt{i}, \quad \alpha_{2,3} = \pm \alpha, \ x = i \sqrt{i}\right),$$

where $\pm \alpha$ are solutions of the transcendental equation

$$\varphi(\alpha) = -\frac{7}{2} - \frac{13}{6} \sqrt{2} \quad \left(\Rightarrow \varphi'(\alpha) = \sqrt{32} \left(7 + 5 \sqrt{2}\right)i\right).$$

At the point $\alpha_1 = \omega''$ we have two independent holomorphic series:

$$x(\alpha) = \frac{1 - i}{2} \left(\sqrt{2} \pm \sqrt{2} (\alpha - \omega'') + \frac{1}{2} (\alpha - \omega'')^2 \pm \cdots\right)$$
and hence the ramification scheme is \(\{1, 1\}\) that means two non-permutable holomorphic branches. At each point \(±\alpha\) the ramification scheme is \(\{2\}\) (two permutable holomorphic branches). This follows obviously from the expansions
\[
x(\alpha) = i \sqrt{3} + \sqrt{32} \sqrt{\alpha - \alpha} + \cdots, \quad x(\alpha) = i \sqrt{3} - i \sqrt{32} \sqrt{\alpha + \alpha} + \cdots.
\]

Ramification scheme at infinity \(\phi(\alpha) = \infty\) is trivial since there is no ramifying here in fact:
\((x - i)(x + 1) \sim \alpha^2 + \cdots\). A check of the Riemann–Hurwitz formula
\[
\tilde{g} = \frac{1}{2} \sum q_j (q_j - 1) + N (g - 1) + 1 \quad (33)
\]
gives
\[
\tilde{g} = \frac{1}{2} \left( (1 - 1) + (2 - 1) 2 \right) + 2 (1 - 1) + 1 = 2
\]
as it must. Here \(\tilde{g}\) and \(g = 1\) are the genera of the cover and surface under covering respectively, \(N = 2\) is number of sheets of the cover \((32)\), and \(q_j\) are indices of the ramification at all the branch points \(\alpha_{1,2,3}\).

In the back direction \(x \mapsto \alpha\) the representation \((32)\) is also \((1 \mapsto 2)\)-map since two copies of torus cover the \(x\)-plane. Ramification schemes are as follows

\[
\begin{align*}
\phi' (\alpha) &= 0:\quad \begin{cases} (\alpha_1 = \omega, \ x = \{1, -i\}), \quad \{2\}, \{2\} \\ (\alpha_2 = \omega', \ x = \{0, \infty\}), \quad \{2\}, \{2\} \\ (\alpha_3 = \omega'', \ x = -i \sqrt{3}), \quad \{1, 1\} \end{cases} \\
\phi' (\alpha) &= \infty:\quad (\alpha_4 = 0, \ x = \{-1, i\}), \quad \{2\}, \{2\}.
\end{align*}
\]

For example in the neighborhood of the first branch point \(\alpha = \omega, \ x = 1\) we compute formally Puiseux series
\[
\alpha (x) = \omega - i \sqrt{2 + i} \sqrt{x - 1} + i \frac{\sqrt{26 \sqrt{2} + 34} - \sqrt{26 \sqrt{2} - 14}}{24} \sqrt{x - 1}^3 + \cdots. \quad (34)
\]

These series are important and discussed in Sec. V.D. Checking the formula \((33)\) under \(g = 0\) and \(N = 2\) (order of the function \(\phi(\alpha)\)) we obtain again
\[
\tilde{g} = \frac{1}{2} \left( (2 - 1) 4 + (1 - 1) 1 + (2 - 1) 2 \right) + 2 (0 - 1) + 1 = 2.
\]

C. Solvable Fuchsian equations on torus

A direct consequence of the preceding arguments is a linear ordinary differential equation of Fuchsian type on the torus \((31)\) with well defined accessory parameters. Indeed, making use of Jacobi’s substitution \((29)\) and changing \(\Psi\) in \((11)\) by the rule
\[
\Psi \mapsto \Lambda:\quad \Lambda (\lambda) = \sqrt{\lambda} x \Psi (x) = \frac{\sqrt{x^2 + i}}{(x + 1)(x - i)} \Psi (x),
\]
we arrive at equation with five regular singularities \(\lambda_j = \{0, 1, -2 \pm 2 \sqrt{2}, \infty\}\):
\[
\begin{align*}
\Lambda_{\lambda\lambda} &= \frac{1}{2} Q(\lambda) \Lambda = \\
&= -\frac{1}{4} \frac{\lambda^6 + 4 \lambda^5 + 16 \lambda^4 - 56 \lambda^3 + 68 \lambda^2 - 48 \lambda + 16}{\lambda^2 (\lambda - 1)^2 (\lambda^2 + 4 \lambda - 4)^2} \Lambda.
\end{align*}
\]
This equation is remarkable in itself because not every algebraic change of variables (formula in our case) in Fuchsian equation with rational coefficients and Fuchsian monodromy leads to equation, again, with rational coefficients and Fuchsian monodromy. Singularities \( \lambda_j = -2 \pm 2 \sqrt{2} \) correspond to elliptic edges of the second order and \( \lambda_j = \{0, 1, \infty\} \) are parabolic singularities. See [4] for further application of the Fuchsian equation (35) to uniformization.

Considering the parametrization of the torus (30)

\[
\lambda = 2 \left( \sqrt{2} - 1 \right) \varphi(\alpha) + \frac{2 \sqrt{2} - 1}{3}
\]
as a subsequent change \( \lambda \mapsto \alpha \) and supplementing with the formula

\[
\Lambda \mapsto \Xi : \Xi(\alpha) = \sqrt{\alpha} \Lambda(\lambda) = \text{const} \cdot \sqrt[3]{2} \varphi'(\alpha) \Lambda(\lambda),
\]
we get the equation

\[
\Xi_{\alpha\alpha} = \frac{1}{2} Q(\alpha) \Xi, \tag{36}
\]

where function \( Q(\alpha) \) (it must be elliptic one) is given by the following expression:

\[
Q(\alpha) = -\{\lambda, \alpha\} + \lambda^{\alpha} Q(\lambda) = 6 \varphi(2\alpha) + \frac{4}{3} (2 \sqrt{2} - 1)^2 \varphi'(\alpha)^2 Q(\lambda).
\]

Carrying out some simplifications we obtain the sought-for result.

**Proposition 3.** Fuchsian equation on torus corresponding to Burnside’s parametrization (3) is a linear ordinary differential equation on torus (31) with five regular singularities. The equation and accessory parameters have the form

\[
\Xi_{\alpha\alpha} = \left\{-\frac{1}{4} (\varphi(\alpha) + \varphi(\alpha - \omega) + \varphi(\alpha - \omega')) - \frac{3}{16} (\varphi(\alpha - \omega) + \varphi(\alpha + \omega)) + \frac{9}{64} \sqrt{8} i (\zeta(\alpha - \omega) - \zeta(\alpha + \omega)) + \frac{9}{32} (\sqrt{8} i \zeta(\omega) + 2 \sqrt{2} + 2) \right\} \Xi, \tag{37}
\]

**Remark 2.** Accessory parameters in Fuchsian ODE on torus (36) are by definition coefficients in front of \( \alpha^{-1} \) in function \( Q(\alpha) \). These are multipliers of \( \zeta \)-functions in (37) and a free term.

In so far as Burnside’s example is a curve with maximal symmetries, this equation provides, perhaps, the simplest example of non-triviality: solvable Fuchsian monodromy, solution, explicitly known inversion (see Sec. V.D), and algebraic curve. Solution of equation (37), in many forms, is given by back transformations written above. Invoking solutions of equation (11) described in [4] we readily obtain one of such forms:

\[
\Xi(\alpha) = \sqrt{\alpha} \Psi(x) = \sqrt{(x - i \sqrt{1}) y \cdot (A K(x^2) + B K'(x^2))},
\]

where \( K \) and \( K' \) are complete elliptic integrals [9] and the quantities \( x, x^2, y \), as functions of \( \alpha \), have to be expressed from the following pair of algebraic equations (consequence of the equations (1), (31), and (32)):

\[
x^2 = (i - 1) \frac{\varphi(\alpha) + \epsilon' - 2 \epsilon}{\varphi(\alpha) - \epsilon'} x + i, \quad \varphi'(\alpha) = -\frac{6 \epsilon + 2}{\sqrt{1 + i}} \frac{(x \pm i \sqrt{1}) y}{(x - i)^2 (x + 1)^2} \tag{38}
\]

The ultimate formula can be represented in various simplified forms but solutions of all the Fuchsian equations we consider are essentially multi-valued functions. This indispensable property reflects the required character of their monodromy groups.
Corollary. The quotient

$$\tau = \frac{\Xi_2(\alpha)}{\Xi_1(\alpha)} \iff (32) \text{ and } (11)$$

(39)

is the global parameter on both the Burnside Riemann surface and corresponding orbifold $\mathfrak{F}$, defined by monodromy group of the equation (37). Inversion of the ratio (39) is, by construction, a globally single-valued analytic function $\alpha = \alpha(\tau)$.

Another form of equation (37) can be useful:

$$\Xi_{\alpha\alpha} = \left\{ -\frac{1}{4} \left( \wp(\alpha) + \wp(\alpha - \omega) + \wp(\alpha - \omega^\prime) \right) - \frac{3}{16} \left( \wp(\alpha - \omega) + \wp(\alpha + \omega) + \frac{3(7 + 5\sqrt{2})}{\wp(\alpha) - \wp(\omega)} - 3\sqrt{2} - 3 \right) \right\} \Xi.$$  

By renormalizing the global parameter on torus $\alpha \mapsto \tilde{\alpha}$ by the rule $\alpha = \omega^\prime \tilde{\alpha}$ and bringing Weierstrassian functions $\wp(z|\omega, \omega^\prime)$ into the form $\wp(z|1, \mu) =: \tilde{\wp}(z|\mu)$ with one modulus:

$$\zeta(\alpha|\omega, \omega^\prime) = \frac{1}{\omega^\prime} \tilde{\zeta}(\tilde{\alpha}|\sqrt{2}i), \quad \wp(\alpha|\omega, \omega^\prime) = \frac{1}{\omega^\prime 2} \tilde{\wp}(\tilde{\alpha}|\sqrt{2}i)$$

we can present equation (37) in another canonical form:

$$\Xi_{\tilde{\alpha}\tilde{\alpha}} = \left\{ -\frac{1}{4} \left( \tilde{\wp}(\tilde{\alpha}) + \tilde{\wp}(\tilde{\alpha} - 1) + \tilde{\wp}(\tilde{\alpha} - \sqrt{2}i) \right) - \frac{3}{16} \left( \tilde{\wp}(\tilde{\alpha} - \tilde{\omega}) + \tilde{\wp}(\tilde{\alpha} + \tilde{\omega}) \right) - \frac{9}{64} M \left( \tilde{\zeta}(\tilde{\alpha} - \tilde{\omega}) - \tilde{\zeta}(\tilde{\alpha} + \tilde{\omega}) \right) - \frac{9}{32} M \left( \tilde{\zeta}(\tilde{\omega}) + (\tilde{\sqrt{2}} + 1) M \right) \right\} \Xi,$$

where $M := \sqrt[8]{2} \omega = \sqrt[8]{8 \pi \eta^2(\sqrt{2}i)}$. Note that all the accessory parameters are the real quantities as $\tilde{\zeta}(\tilde{\omega}) = 3.83102282421...$ and this equation, under real $\alpha$, is defined by a real even function of $\alpha$.

Clearly, the points $\alpha_j = \{0, \omega, \omega^\prime\}$ correspond to three punctures on the torus, because in the vicinity of these points we have expansions of the form

$$\frac{1}{2} Q(\alpha) = -\frac{1}{4} \left( \frac{1}{(\alpha - \alpha_j)^2} + \cdots \right).$$

The remaining points $\pm \tilde{\omega}$ correspond (locally) to elliptic edges of second order:

$$\frac{1}{2} Q(\alpha) = -\frac{3}{16} \left( \frac{1}{(\alpha + \tilde{\omega})^2} \pm \frac{9}{64} \sqrt{8} \frac{1}{(\alpha + \tilde{\omega})} \right) + \cdots.$$

The Fig. 1 illustrates these remarks. First example of Poincaré’s metric on the “toroidal” orbifold $\mathfrak{F}$ is presented in the work [4].

In order to obtain Fuchsian equation on torus, it is not necessary for the torus to be punctured. One may add elliptic singularities of finite order. Rankin [23] showed that

![Diagram of orbifold $\mathfrak{F}$ with fundamental group $\pi_1(\mathfrak{F})$ of rank 6 (Proposition 7) defined by monodromy group of equation (37). Crosses “×” stand for elliptic edges of 2nd order.](image-url)
Whittaker’s conjecture holds for Burnside’s curve (1) and accessory parameters $A_{1,2,3}$ in equation (10) have also zero values. Therefore we can derive a Fuchsian equation on torus corresponding to this kind of uniformization. Carrying out analogous calculations, we get yet another example of solvable Fuchsian equation on torus.

**Proposition 4.** The orbifold $X$ corresponding to Whittaker’s uniformization of Burnside’s Riemann surface (32) is defined by a Fuchsian equation on torus (31) with two singularities and three accessory parameters. The equation has the form

$$
\Xi_{\alpha\alpha} = -\frac{3}{16} \left\{ \wp(\alpha - \omega) + \wp(\alpha + \omega) - \sqrt{32} i \cdot \zeta(\alpha - \omega) + \sqrt{32} i \cdot \zeta(\alpha + \omega) - \left( \sqrt{2} i \zeta(\omega) + \sqrt{2} + 1 \right) \right\} \Xi
$$

or, equivalently,

$$
\Xi_{\alpha\alpha} = -\frac{3}{16} \left\{ \wp(\alpha - \omega) + \wp(\alpha + \omega) + \frac{7 + 5\sqrt{2}}{\wp(\alpha) - \wp(\omega)} - \sqrt{2} + 1 \right\} \Xi.
$$

Explicit description of monodromies (their genera, generators, arithmetical properties, if any, etc) of $X$ defined by equations (37) and (40) is open question. Erasing punctures in Eq. 1 we obtain illustration of this orbifold. Compared to equation (37), the equation (40) has only two singularities however the explicit expression for inversion of the ratio (39) for this equation is unknown. This important point manifests itself in the fact that none of the uniformizing functions for any non-modular Fuchsian equation is known hitherto.

**V. ANALYTICAL SERIES**

Power series are the widely exploited tools in the theory of elliptic functions and have enormous number of applications [13]. In this section we develop corresponding technique for the curve (1).

In order to obtain the series expansion of the function $x(\tau)$ we may not make use of well-known Laurent’s expansions of Weierstrassian functions in so far as such expansions turn formally $\wp$-functions in (3) into modular forms of fictitiously infinite weight. This is of course not the case. Explanation is that the pole $\tau = 2$ of the function $x(\tau)$ lies on a singularity line (real axis) of $\wp$-functions in (3). The same complications occur for other branch points of the curve (1); that is, roots of the polynomial $x^5 - x$. For the same reasons the function (12) has no seemingly singularities at points $x^4 = 5$. They are compensated by a singularity of the function $\wp(\tau(x)|2, 2\tau(x))$.

To overcome this obstacle we should use Schwarz’s equation (4) satisfied by the function $x(\tau)$. The series representation depend on the location of origin of the expansion. It can lie inside the fundamental circle or on its border.

We shall use the natural notation $\omega_j$ for pre-images of branch points $e_j$:

$$
x(\omega_0) = \infty, \quad x(\omega_j) = e_j, \quad e_j = \{0, 1, -1, i, -i\}.
$$

**A. Meromorphic derivative**

Let us define the **meromorphic derivative** $\mathcal{D}$ of nonconstant function $x = x(\tau)$ by the formula

$$
\mathcal{D} : \ [x, \tau] := \frac{\{x, \tau\}}{x_2} \quad \Leftrightarrow \quad \{x, \tau\} = -\{x, \tau\}.
$$

Motivation for introducing this object is the fact that $\mathcal{D}$ is a simplest and lowest order differential combination of a meromorphic function on a curve which is meromorphic function
as well. It also determines Schwarz’s equation \([2]\). Properties of the object \(\mathcal{D}\) follow from properties of the Schwarzian:

\[
\left[ X(q(\tau)), \tau \right] = \left\{ \left[ X(q), q \right] + \frac{1}{X^3} \left[ q(\tau), \tau \right] \right\}_{q=q(\tau)}.
\] (41)

Since we shall deal with meromorphic functions on Riemann surfaces (meromorphic automorphic functions, holomorphic and meromorphic Abelian integrals), without loss of generality, by virtue of (41), we may consider expansions only of the form

\[
X = q^{-n} \left( A + Bq + Cq^2 + \cdots \right), \quad n \in \mathbb{Z}, \ A \neq 0.
\] (42)

It follows that

\[
\left[ X, q \right] = \frac{1}{A^2} \left\{ \left( 1 - \frac{n^2}{2} \right) + \frac{n^2 - 1}{n^2} \frac{B}{A} q + \right. \\
\left. + \frac{2n(n^3 - n - 6)}{2n^4} \frac{AC}{A} - 3 \left( \frac{n^4 - n^2 - 2n + 2}{2} \right) \frac{B^2}{q^2} + \cdots \right\} q^{2n}.
\] (43)

This property entails the fact that \(\mathcal{D}\) has a pole/zero if, and only if the function \(X(q)\) has a fold zero/pole, i.e. \(n \neq \pm 1\), or a fold \(a\)-point (this is a point \(\tau\) where the function \(x(\tau)\) takes the value \(a\)). Poles and zeroes of \(\mathcal{D}\) are always fold. The fold \(a\)-points of analytic functions lead to violation of conformality of analytic maps and, in the uniformization theory, correspond to branch points of covers and singularities in Fuchsian equations. Conversely, let \(x(\tau)\) be meromorphic automorphic function on some algebraic curve. Being analytic function of \(\tau\), the function \(x(\tau)\) has only finitely many \(a\)-points. From (41)–(43) it follows that its \(\mathcal{D}\)-derivative is meromorphic function: that is, rational function of \((x,y)\). This is nothing but equation \([2]\) without recourse to auxiliary Fuchsian linear ODE.

**B. Local and global parameters**

To determine the behavior of a local parameter \(q = q(\tau)\) and, therefore, the type of point, we set \(x(\tau) = X(q)\). Let us apply the object \(\mathcal{D}\) to our main example. Let \(\omega_j\) be a zero or pole of the function \(x - e_j\). For example the pole \([12]\). Using the properties \([11], [13]\), an expansion of both sides of \([1]\) produces

\[
\frac{1}{A^2} \left\{ \left( \frac{1 - n^2}{2n^2} + \frac{B}{A} \frac{n^2 - 1}{n^2} q + \cdots \right) + \frac{1}{n^2} \left[ q, \tau \right] q^2 \left( 1 - 2 \frac{B}{A} \frac{n - 1}{n} q + \cdots \right) \right\} q^{2n} = \\
= \frac{1}{A^2} \left\{ -\frac{1}{2} + \frac{B}{A} q + \cdots \right\} q^{2n}.
\]

Balancing degrees with respect to \(q\) entails

\[
\left[ q, \tau \right] q^2 = -\frac{1}{2} + \cdots.
\] (44)

Integrating this equation by series \(q = a + b\tau + \cdots\) shows that \(a, b \neq 0\). Moreover, the inverse function \(\tau(q)\) is never meromorphic in \(q\) as the ansatz \(\tau = q^{-k} + \cdots\) involves incompatibility with \([11]\). Hence, locally, \(q\) has an exponential behavior

\[
q(\tau) = \exp \left( \frac{a\tau + b}{c\tau + d} \right) + \cdots \text{ holomorphic part}.
\] (45)

Choice of local parameters has no restrictions except for condition of being locally single-valued. Therefore we may omit all the dots in \([15]\) and obtain the following representation:

\[
q(\tau) = \exp \left( \frac{a\tau + b}{c\tau + d} \right),
\] (46)
where \((a, b, c, d)\) are arbitrary constants. Therefore \(\omega_0\) and also all the pre-images \(\omega_j\) of other branch points are edges of parabolic cycles. We shall call the formula (46) \textit{q-representation of the global coordinate} \(\tau\). Note that (46) is the usual form for local parameters in the vicinity of such edges [10, §41] and the order \(n\) is not determined.

In the case of Whittaker’s uniformization (19) we would have \(n = \pm 2\) and an equation for the local parameter in the form

\[
[q, \tau] = 0 + \cdots \Rightarrow q(\tau) = \frac{a \tau + b}{c \tau + d} + \cdots,
\]

so it is not necessary to change the global parameter \(\tau\) to the local \(q\). This is not surprising, because the pre-images \(\omega_j\) lie inside the fundamental polygon [29, 30] and, as is well-known, they are Weierstrass’s points, and the meromorphic function \(x - e_j\) on the curve has a 2-nd order \((n = \pm 2)\) pole/zeroes at the points \(\omega_j\).

Schwarz’s equation (2), after the substitution (46), acquires the form

\[
\frac{1}{2} q^2 X^2 - [X, q] = Q(X, Y) \quad (48)
\]

with the \(Q\)-function having the following structure

\[
Q(X, Y) = \frac{\mu}{(X - e_j)^{2n}} + \cdots.
\]

We must have a restriction on \(\mu\) because we look for meromorphic solutions (42) of equation (48), so that \(n\) has to be integer:

\[n^2 (2 \mu + 1) = 1.\]

The \(Q\)-function for any Fuchsian equation with Fuchsian monodromy must satisfy this relation. If \(\mu = -\frac{1}{2}\) we arrive at the parabolicity condition (44)–(46), i.e. \(n = \infty\). Otherwise we arrive at (47) and determine the order \(n\). For a Fuchsian group without parabolic edge at the point \(e_j\) we obtain: 1) \(\mu = -\frac{1}{3}\) \((n = \pm 2)\): hyperelliptic curves; 2) \(\mu = -\frac{1}{9}\) \((n = \pm 3)\): curves where \(x\)-function has a 3-nd order pole/zero (for example trigonal curves); etc.

C. Integer \(q\)-series for genus two (Burnside’s curve)

The general solution of equation (48) is determined up to the “exponential-linear” substitution

\[
q \mapsto \exp\left(\frac{a \ln q + b}{c \ln q + d}\right).
\]

In order to have a meromorphic function we have to obtain analytic Laurent series for the function \(X(q)\) in the neighborhood of \(q = 0\). The constants \((a, b, c, d)\) in (46) cannot be determined from the differential equation (48). However, well choice of these constants would allow one to construct analogues of the celebrated modular integer \(q\)-series but, in our case, they would correspond to a nontrivial group of genus two. Running ahead note that it is not evident at all that these series should be integer. If so, we shall call such series \textit{canonical representation} (see remark 4 further below).

The uniformness of the functions \(X = x(\tau)\) and \(Y = y(\tau)\) entails the following ansatz for the polar expansion (42)

\[
X = c_{-2} q^{-2} + c_{-1} q^{-1} + c_0 + c_1 q + \cdots.
\]

Appropriate normalization of the constants \((a, b, c, d)\) for this pole is as follows:

\[
q = e^{\frac{\pi i \tau}{\tau - \tau_0}}, \quad \tau \to 2 + i 0.
\]
Substituting this ansatz into (4) we get an integer series indeed:

\[
X = \frac{1}{2} q^{-2} \left( 1 + 2 q^8 - q^{16} - 2 q^{24} + 3 q^{32} + 2 q^{40} - 4 q^{48} - 4 q^{56} + \cdots \right). \tag{49}
\]

The general solution \(x(\tau)\) of equation (4) is obtained after the substitution (46). The \(q\)-series for the second uniformizing function \(y(\tau)\) is derived from equations (14), (48) or the identity \(Y^2 = X^5 - X\). We find that \(Y\) is also determined by an integer series:

\[
Y = 2^{-1/8} q^{-5} \left( 1 - 3 q^8 - 3 q^{16} + 14 q^{24} + 6 q^{32} - 33 q^{40} - 20 q^{48} + \cdots \right). \tag{50}
\]

Being the single-valued functions of \(\tau\), the series (49)–(50) are the sought for canonical representations of functions (3) in the vicinity of their pole \(\omega_0 = 2\) with correctly chosen constants \((a, b, c, d)\). Analyzing Burnside’s formulae and applying the same technique, one obtains canonical expansions for all other branch points \(e_j\):

\[
x(0) = 0, \quad x\left(\frac{1}{2}\right) = 1, \quad x(\infty) = -1, \quad x(1) = i, \quad x(-1) = -i. \tag{51}
\]

**Remark 3.** Note \(x(1) = i\) rather than 1 as it might seem from (18). The point \(\tau = 1 \not\in \mathbb{H}^+\) and we must use a limiting passage to get correct value \(x(1)\). Similarly, \(x(-1) \neq x(1)\) despite the formal fact that \(x(\tau) = x(\tau)\). Transformation \(\tau \mapsto -\tau\) preserves shape of function (18) but sends \(\mathbb{H}^+ \mapsto \mathbb{H}^-\) and therefore does not belong to group \(\text{PSL}_2(\mathbb{Z})\). We see that this sole point forbids such an automorphism as it must.

**Proposition 5.** The canonical representations for Laurent’s series for Burnside’s functions (3) do exist. The developments are determined by the formulae (49)–(51) in the vicinity of the pole \(\omega_0 = 2\) and coordinate

\[
q = e^{\frac{2\pi i}{\tau - \tau_i}}, \quad \tau \to 0 + i 0 \tag{52}
\]

corresponds to canonical representation in the vicinity of the zero \(\omega_1 = 0:\)

\[
\begin{cases}
X = e^{\frac{2\pi i}{\tau}} q^2 \left( 1 + 2 q^8 + 5 q^{16} + 10 q^{24} + 18 q^{32} + 32 q^{40} + \cdots \right) \\
Y = e^{-\frac{2\pi i}{\tau}} \sqrt{2} q \left( 1 + 9 q^8 + 42 q^{16} + 147 q^{24} + 444 q^{32} + 1206 q^{40} + \cdots \right).
\end{cases} \tag{53}
\]

The developments at branch points \(e_j\) \(= \{\pm 1, \pm i\}\) are determined, up to multipliers, by the one canonical series:

\[
\begin{cases}
X = \left\{ \pm 1, \pm i \right\} \left( 1 + 4 q^2 + 8 q^4 + 16 q^6 + 32 q^8 + 56 q^{10} + 96 q^{12} + \cdots \right) \\
Y = 4 \left\{ \sqrt{\pm 1}, \sqrt{\pm i} \right\} q \left( 1 + 6 q^2 + 24 q^4 + 80 q^6 + 231 q^8 + 606 q^{10} + \cdots \right),
\end{cases} \tag{54}
\]

where coordinates \(q\), according to (51), have the form

\[
\begin{align*}
\left\{ q = e^{\frac{2\pi i}{\tau - \tau_i}}, \quad \tau \to \frac{1}{2} + i 0 \right\}, & \quad \left\{ q = e^{\frac{2\pi i}{\tau - \tau_i}}, \quad \tau \to 1 + i 0 \right\}, \\
\left\{ q = e^{\frac{2\pi i}{\tau - \tau_i}}, \quad \tau \to -i \infty \right\}, & \quad \left\{ q = e^{\frac{2\pi i}{\tau - \tau_i}}, \quad \tau \to -1 + i 0 \right\}.
\end{align*}
\]

All the functions are holomorphic at \(\mathbb{H}^+\) and form the field \(\mathbb{C}(x, y)\) of meromorphic functions on the curve (1): \(R_1(x(\tau)) + y(\tau) R_2(x(\tau))\).

**Remark 4.** Complete proof of the fact that all the series are integer \(q\)-series goes beyond the scope of the present work. It exploits some manipulations with Weierstrassian functions, \(\vartheta\)-constants, and their transformations in \(\text{PSL}_2(\mathbb{Z})\) (3). As an example we exhibit exact representations only for the series (54).
Proposition 6. The integer series \( [31] \) have an exact representation in form of products:

\[
X = \left\{ \pm 1, \pm i \right\} \prod_{k=1}^{\infty} (1 + q^{4k})^2 (1 + q^{4k-2})^4
\]

\[
Y = 4 \{ \sqrt{\pm 1}, \sqrt{\pm i} \} q \prod_{k=1}^{\infty} (1 + q^{4k})^3 (1 + q^{2k})^6
\]

These formulae can be considered as an alternative and much simpler version of Burnside’s parametrization itself: \( Y^2 = X^3 - X \) is an identity between the products \( [32] \). Note that the series \( [19] \), \( [30] \) are alternating ones and exponential multipliers in the series \( [33] \) have been introduced in order that the series be positive definite.

Since equation \( [48] \) is a rational function of the quantities \( (\alpha, \tau) \) arbitrary Abelian differential \( d\alpha \) on Burnside’s Riemann surface and gives the inversion of the ratio \( [39] \) in

\[
\int_{x}^{x+\sqrt{1+i\sqrt{3}}} \frac{dx}{\sqrt{x^3 - x}} = \sqrt{1+i\sqrt{3}}^{-1} \left( \frac{1 \pm \sqrt{2}}{1 - i} \right) \left( 1 + q^4 \right)^2 \left( 1 + q^{4k-2} \right)^2 \left( 1 - q^{4k} \right)^3 \cdot dq
\]

This leads, after some simplification, to the following formula

\[
\int_{x}^{x+\sqrt{1+i\sqrt{3}}} \frac{dx}{\sqrt{x^3 - x}} = \sqrt{1+i\sqrt{3}}^{-1} \left( \frac{1 \pm \sqrt{2}}{1 - i} \right) \left( 1 + q^4 \right)^2 \left( 1 + q^{4k-2} \right)^2 \left( 1 - q^{4k} \right)^3 \cdot dq
\]

On the other hand explicitly solvable Fuchsian equations on torus described in Sec. IV.C allow one to construct \( q \)-series for these integrals and to get other information.

Proposition 7. The solution \( \alpha = \alpha(\tau) \) of Schwarz’s equation

\[
[\alpha, \tau] = -\frac{1}{2} (\varphi(\alpha) + \varphi(\alpha - \omega) + \varphi(\alpha - \omega')) - \frac{3}{8} (\varphi(\alpha - \omega) + \varphi(\alpha + \omega)) +
\]

\[
+ \frac{9}{32} \sqrt{3} i (\zeta(\alpha - \omega) - \zeta(\alpha + \omega)) + \frac{9}{16} \left( \sqrt{3} i \zeta(\omega) + 2\sqrt{2} + 2 \right),
\]

is a holomorphic additive function with respect to rank six group defined by monodromy of equation \( [37] \). Arbitrary Abelian differential \( d\alpha(\tau) \) is represented by the two integer \( q \)-series:

\[
\begin{align*}
\frac{dX}{Y} &= \left\{ \sqrt{\pm 1}, \sqrt{\pm i} \right\} 2 \prod_{k=1}^{\infty} (1 - q^{4k})^2 (1 - q^{4k-2})^2 (1 - q^{8k})^3 \cdot dq \\
X \frac{dX}{Y} &= \left\{ \pm \sqrt{\pm 1}, -\sqrt{\pm i} \right\} 2 \prod_{k=1}^{\infty} (1 - q^{4k})^2 (1 - q^{4k-2})^2 (1 - q^{8k})^3 \cdot dq
\end{align*}
\]

Proof. By virtue of the formula \( [59] \) the quantity \( \alpha(\tau) \) is proportional to the holomorphic Abelian integral on Burnside’s Riemann surface and gives the inversion of the ratio \( [39] \) in
the following explicit form
\[ \alpha^\pm(\tau) = q^{-1} \left( \frac{(1 \pm \sqrt{2})(1-i)x(\tau)}{x(\tau) - 1} - \frac{3 \pm \sqrt{2}}{6} \frac{i}{27} \frac{\mp 7\sqrt{2}}{2} \right). \] (58)

Hence, being analytic function, it is everywhere finite in a domain of its existence including the limiting points \( \omega_j \). The closed loops surrounding the points \( \alpha_j = \{0, \omega, \omega', \pm \omega\} \) determine five \( 2 \times 2 \)-matrix representations of generators of automorphisms of the function \( \alpha(\tau) \). These constitute a subgroup. The shifts \( \alpha \mapsto \alpha + \{2\omega, 2\omega'\} \) yield two remaining generators of the full monodromy group. Rank of the group (it is not free) is equal to six as there are a puncture and one obvious relation between these seven elements of the monodromy.

Again, as in Proposition 6, some routine manipulations with modular \( \vartheta \)-forms and (55) lead to formulae (57). Combining these formulae with (56)–(58) we get

\[ \alpha^\pm(\tau) = \frac{X \mp i \sqrt{1+i} \; dX}{Y}. \]

Series of a similar nature (without poles) can be obtained for points \( e_j = \{0, \infty\} \).

We see that both the Puiseux series for multi-valued functions considered in Sec. IV.B and Puiseux series for holomorphic integrals (56) are transformed into single-valued series when we use any of the global coordinates. In particular, we obtain \( q \)-representation for complicated and seemingly chaotic Puiseux series (54). The representation has a quite regular structure defined by the two integer \( q \)-series (57):

\[ d\alpha^+(\tau) = M \left\{ 1 - 2\gamma q^2 - q^8 + 6\gamma q^{10} - 6q^{16} - 2\gamma q^{18} + 5q^{24} - 4\gamma q^{26} + 12q^{32} - 6q^{40} - 10\gamma q^{42} - 7q^{48} + 12\gamma q^{50} - 4q^{56} + 6\gamma q^{58} + \cdots \right\} dq, \]

\[ \alpha^+(\tau) = \omega + M \left\{ q^2 - 2\gamma q^3 - q^9 + 6\gamma q^{11} - 6q^{17} - 2\gamma q^{19} + 5q^{25} - 4\gamma q^{27} + \cdots \right\}, \] (59)

where

\[ \gamma = \sqrt{2} - 1, \quad M = 2\sqrt{2} + 1, \quad q = e^{\frac{i\pi}{\tau}}. \]

Note that the term \( (\cdots + 12q^{32} - 6q^{40} - \cdots) \) is not a misprint. The quantities \( \gamma^1 \) and \( \gamma^0 \) do not alternate each other.

We remark that the complete group and Moonshine’s treatment of this nontrivial case of genus two is of independent interest. One complication occurring in a higher-genera Moonshine is that there is no canonical choice for Hauptmoduli [13]. However the genus two curves have a unique hyperelliptic shape and, as we have seen now, Burnside’s example can be thought of as the canonical in all respects. What is more, Moonshine’s treatment, if any, of the holomorphic “toroidal Hauptmodulus” \( \alpha(\tau) \) (59), its differential (57), and orbifold [37] would be of special interest.

Special attention must be given to the fact that once an explicit formula for uniformizing function, series for holomorphic integral, accessory parameters, or \( \Psi \)-function has been obtained, all the series are recomputed one through another. But for the reasons pointed out above, the formula for holomorphic integrals (in form of series (57), say) should be considered as the primary object of the theory. This remains valid even though we have no an explicit formula for the integrals in terms of elliptic ones like (56); that is, when the cover of torus does not exist.

E. Modular forms for Burnside’s function

Poincaré’s method of construction of automorphic functions as ratios of automorphic forms is widely known [14]. These forms are analytic functions \( \Theta(\tau) \) with the property:

\[ \theta^a(\tau + b) = (c\tau + d)^n \theta(\tau), \]
where \( n \) is the weight of the automorphic form \( \Theta \) and \((a, b, c, d)\) are substitutions of a group. Burnside’s example fits in this classical construction and generates integer \( q \)-series for modular forms.

**Proposition 8.** The function of Burnside \( x(\tau) \) is a ratio of two automorphic holomorphic modular forms of weight \( n = 2 \) with respect to group \( \Gamma(4) \):

\[
x(\tau) = \frac{\Theta_1(\tau)}{\Theta_2(\tau)},
\]

where \( \Theta_1(\tau) = \varphi(1) - \varphi(2), \quad \Theta_2(\tau) = \varphi(\tau) - \varphi(2) \).

**Proof.** Derivative of any automorphic function is an automorphic form of weight \( n = 2 \) (Abelian differential) with respect to automorphism group of the function. In our case, this is the monodromy group of the equation (11) — the group \( \Gamma(4) \). From (8), we have that \( \varphi(2) \) is a form of weight \( n = 2 \). From (7) we have the same for \( \varphi(1) \) and, from (10), for \( \varphi(\tau) \). Making use of the formulae (7)–(10) we have

\[
\Theta_1(\tau) = \frac{9 g_2}{4 g_2} \left( x^3 + x(x^8 + 14 x^4 + 1) \right) = \frac{1}{4} \frac{\pi i x_\tau}{x^2 - 1}. \tag{60}
\]

Correlation the series/products \((49)–(55)\) with \((57)\) (we omit details) shows that \( \Theta_1(\tau) \) is everywhere finite. The same is true for the form \( \Theta_2(\tau) \).

We exhibit some of such series only for the form \( \Theta_1 \). All the representations can be derived from the formula (60). For example the infinite point \( \tau \to +i \infty \) with the coordinate \( q = e^{\frac{2\pi i}{\tau - 3}} \) yields the following \( q \)-expansion:

\[
\Theta_1(\tau) = \frac{\pi^2}{16} \frac{q x_q}{x^2 - 1} = \frac{\pi^2}{16} \left( 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \right)^2 = \frac{\pi^2}{16} \left( 1 + 8 q^8 + 24 q^{16} + 32 q^{24} + 24 q^{32} + 48 q^{40} + 96 q^{48} + 64 q^{56} + \cdots \right).
\]

Zero of the form \( \Theta_1 \) corresponds to the cusp \( \infty \) and the following series

\[
\Theta_1(\tau) = \frac{q x_q}{x^2 - 1} \ln^2 q = -4 q^2 \ln^2 q \cdot \left( 1 + 4 q^4 + 6 q^8 + 8 q^{12} + 13 q^{16} + 12 q^{20} + 14 q^{24} + \cdots \right).
\]

One of the explicit analytic expressions for this series is given by the formula

\[
\Theta_1(\tau) = -4 q^2 \ln^2 q \cdot \sum_{k=0}^{\infty} \sigma_1(2 k + 1) q^{4k},
\]

where \( \sigma_1(n) \) is sum of positive divisors of \( n \). The form \( \Theta_2(\tau) \) and all other points \( \omega_j \) are treated in a similar manner but we do not write up here exact representations (which we have determined) to them in terms of number-theoretic functions or Jacobi’s \( \vartheta \)-constants. See works \([4, 21]\) for relevant information.

Alternatively, we could represent the function \( x(\tau) \) in somewhat unusual way. Namely in a form of ratio of two automorphic functions \( z = \frac{\varphi_1}{\varphi_2} - 1, \quad w = \frac{\varphi_1}{\varphi_2} - 1 \) rather than ratio of forms. Corresponding Schwarz’s equations and their monodromies for the functions \( z \) and \( w \) are derived from the formulae \([6]–[10]\). The following Schwarz’s equation elucidates completely this remark:

\[
z = 2 \frac{\varphi_1}{\varphi_2} + 2 \quad \Rightarrow \quad [z, \tau] = -\frac{27}{2} \frac{z^2 + 3}{z^2(z^2 - 9)^2}.
\]

Indeed, \( \text{Aut}(z(\tau)) = \Gamma(2) \supset \Gamma(4) = \text{Aut}(x(\tau)) \). \([3, 4, 17, 24]\).
F. Conversion between uniformizations of different kinds

The fact that both Whittaker’s and Burnside’s equations (11) and (19) describe uniformizations in their own rights means that there is a one-to-one conformal transformation between their global coordinates, i.e. bi-holomorphic equivalence.

Let $x(\tau)$ and $x(\mu)$ be Burnside’s and Whittaker’s uniformizing functions respectively. We are looking for a relation between $\tau$ and $\mu$: $\mu = \varphi(\tau)$. The function $\varphi$ and its inversion must be everywhere holomorphic functions because we deal with Fuchsian groups of first kind: that is, groups having invariant circles in the planes $\tau$ (holomorphically) and $\mu$ (anti-holomorphically).

Proposition 9. The coordinate $\mu$ (up to a linear-fractional transformation) is everywhere holomorphic function of $\tau$ satisfying the following differential equation

$$\varphi: \quad [\tau, \mu] = -\frac{3}{8} \frac{g_2(\tau)}{\pi^2}. \quad (61)$$

Proof. We have $x(\tau) = x(\mu)$ and

$$[x, \tau] = -\frac{1}{2} \frac{x^8 + 14x^4 + 1}{(x^5 - x)^2}, \quad [x, \mu] = -\frac{3}{8} \frac{x^8 + 14x^4 + 1}{(x^5 - x)^2}. \quad (62)$$

Invoking the transformation rule (11) of the object $\mathcal{D}$ we deduce that

$$[x, \tau] = [x(\mu), \tau] = [x(\mu), \mu] + \frac{1}{x_\mu^2} [\mu, \tau] =$$

$$= -\frac{3}{8} \frac{x^8 + 14x^4 + 1}{(x^5 - x)^2} + \frac{\mu_\tau^2}{x_\mu^2} [\mu, \tau] = -\frac{1}{2} \frac{x^8 + 14x^4 + 1}{(x^5 - x)^2}. \quad (63)$$

It follows that

$$\{\mu, \tau\} = -\frac{1}{8} \frac{x^8 + 14x^4 + 1}{(x^5 - x)^2} x_\tau^2.$$

Correlating this expression with formulae 5 and 11 we arrive at formula (61). The form $g_2(\tau)$ is everywhere finite at $\mathbb{H}^+$ and hence the Schwarzian $\{\mu, \tau\}$ and $\mu$ itself are finite as well. For the same reason function $\tau = \varphi^{-1}(\mu)$ has no fold $a$-points (see 13) and is reversible into the function of the same kind.

Explicitly solvable Schwarz’s equations of the form $[\tau, \mu] = Q(\tau)$ with holomorphic right-hand side $Q(\tau)$ do exist. One of such examples is the nice equation

$$[\tau, \mu] = -\frac{2}{3} \frac{g_2(\tau)}{\pi^2} \Rightarrow \frac{a}{c} \frac{\mu + b}{\mu + d} = \frac{\eta^2(\tau)}{\tau} \int_\tau^\eta \Psi d\tau \quad (64)$$

which comes from the following linear ODE with everywhere holomorphic coefficients:

$$\Psi_{\tau\tau} + \frac{n + 2}{\pi i} \eta(\tau) \Psi_{\tau} - \frac{n}{6\pi^2} g_2(\tau) \Psi = 0.$$

A direct check, with use of rules 5 and known relation $\pi \ln \eta, \tilde{\eta}(\tau) = i \eta(\tau)$, shows that $\Psi = \tilde{\eta}^n(\tau)$ solves this equation and general integral (62) corresponds to the case $n = -2$.

A remarkable fact is that equation (61) can also be exactly integrated but we shall present this material in a separate work. Here we restrict ourselves to series representations. Analyzing this equation one can show that in the neighborhood of infinity $\tau \to +i \infty$ the global coordinate $q$ must have the form $q = e^{\pi i \tau}$. Invoking the well-known expansion

$$g_2(\tau) = 20 \pi^4 \left( \frac{1}{240} + q^8 + 9q^{16} + 28q^{24} + 73q^{32} + 126q^{40} + 252q^{48} + \cdots \right)$$
we find that the series solutions have the following forms

\[
\tilde{\mu} = q - \frac{5}{21} q^9 - \frac{78}{833} q^{17} + \frac{4001}{39445} q^{25} + \frac{168948}{1711915} q^{33} + \frac{42752022}{491319031} q^{41} - \cdots,
\]

\[
q = \tilde{\mu} + \frac{5}{21} \tilde{\mu}^9 + \frac{503}{833} \tilde{\mu}^{17} + \frac{4138924}{2011695} \tilde{\mu}^{25} + \frac{6383638315}{788768067} \tilde{\mu}^{33} + \cdots \quad (\tilde{\mu} = \pi i \frac{\mu}{4}).
\]

Similar bi-holomorphic series exist for other cusps but their integer series realization, if any, is open question.

VI. REMARKS CONCERNING SOME OF THE LITERATURE

Some of the expressions in the Secs. II, III and ground forms appeared in (7)–(8) have already occurred in the literature [2, p. 59]. For example \( s^8 - 14 s^4 + 1 \), which differs from \( x^8 + 14 x^4 + 1 \) by a multiplier \( \sqrt{-1} \), appears throughout the Schwarz Abhandlungen [25] in different contexts. Formula (15) was obtained by him [25], Klein [17], and Brioschi (1877) in relation to the groups of Platonic solids, without mention of the explicit uniformization or function \( x(\tau) \). Slightly different (Legendre’s) form of (15) appeared in an earlier paper of Burnside on p. 176 in The Messenger of Math. XXI (1892). All this entails, in a hidden form, some results of Secs. II and V. We should also remark Schwarz’s comments on pp. 364–365 in [25 II], where other candidates for a complete uniformization can be found. We especially note the examples of Forsyth [11, Ex. 2–3, p. 188], [12, Ex. 13–14, pp. 242–243] which are not accompanied by any references or comments however.

Some of the integer sequences presented above can be found in The On-Line Encyclopedia of Integer Sequences by N. J. A. Sloane which currently available at http://www.research.att.com/~njas/sequences. Holomorphic and meromorphic Abelian integrals associated with Whittaker’s curve (18) were considered by Legendre (with numerous details but for the most part numerically) in Troisième Supplément to [19, see pp. 207–269].

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