Critical load and buckling of the single pile foundation subjected to the vertical load

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Abstract. In this study, the critical load and buckling of the single pile foundation subjected to the vertical load are investigated. Considering the second-order moment of the soil-structure interaction, the refined model of the single pile foundation is derived. Then, the critical load and buckling phenomenon of the single pile foundation is examined. Moreover, the effects of the vertical load and the foundation parameters on the critical load and buckling of the single pile foundation are systematically investigated.

1. Introduction
In the civil engineering, the soil-structure interaction problem is an important part of the dynamic design and analyze of the structure. In general, the buckling of the substructure plays an important role in the seismic behavior of the civil engineering. Based on the engineering practice, the foundation of the structure usually can define as the slender structure. Obviously, the pile foundation is a typical vertical slender structure.

The pile foundation is the usual deep foundation, which can bear a variety of the static and dynamic load. As we know, numerous studies have addressed the theory research of the pile foundation\textsuperscript{[1, 2]}. To meet the needs of the design of the pile foundation, some researches of the dynamic stability and buckling of the pile foundation have been published. Gabr and his co-workers proposed the computation model of the critical load of the slender friction pile foundation\textsuperscript{[3, 4]}. Based on the engineering facts, Kuo et al studied the minimum embedded length of the monopiles subjected to the cyclic horizontal load\textsuperscript{[5]}. However, there only few studies have focused on the buckling of the pile foundation with the second-order moment of the soil-structure interaction.

In this paper, considering the second-order moment of soil-structure interaction, the refined model of the pile foundation is proposed. Then, the critical load and buckling of the pile are investigated. Finally, numerical analysis of the effect of the main parameters on the buckling phenomena of the pile foundation is examined.
2. Theoretical procedure

2.1. Geometry and Loading of the pile

In this study, the attention is focused on the slender circular pile foundation with length H, diameter 2R, and subjected to the lateral load \( p(t) = P \cos \Omega t \), where \( P \) and \( \Omega \) denote the amplitude and frequency of the excitation) and the vertical load \( F(t) \). As show in Fig. 1, a cylindrical coordinate system \( O-r\varphi z \) is chosen, with the origin \( O \) placed at the upper end of the undeformed pile and the \( z \) direction along the axes of pile. In this paper, the subgrade reaction of the elastic foundation denoted as \( q(z, t) = k_0 v(z, t) - k_1 v''(z, t) \), where \( k_0 \) and \( k_1 \) are the Winkler parameter and the shear parameter of the soil medium, respectively [6]. Obviously, the displacements of the pile and the soil medium around the pile can be denoted as the separable functions of the cylindrical coordinate \( r, \varphi \) and \( z \). Accordingly, the displacements of the pile can be assumed as \( v(z, t) \) and \( w(z, t) \), which are the displacement along the pile axis and the vertical displacement of the pile, respectively.

![Figure 1. Pile-soil interaction system.](image)

2.2. Strain and stress of the pile

Referring to Fig. 1, the axial elongation of the pile element can be written as [7]

\[
e = \sqrt{(1 + w')^2 + v'^2} - 1
\]

where the prime indicates differentiation with respect to \( z \).

Moreover, based on the displacement vector of the pile in the local coordinate system, the nonzero axial strain of the pile can be obtained

\[
\varepsilon_z = e - r \cos(\varphi) \kappa
\]

where \( \kappa = \theta' \) is the curvature, \( \theta \) is the rotation angle.

2.3. Equation of motion

Considering the compressive deformation of the pile, the strain energy \( U \) of the pile can be written as

\[
U = \int_0^H \frac{1}{2} EI \kappa^2 \, dz + \int_0^H \frac{1}{2} EA e \, dz
\]

where \( E \) is the Young’s modulus of the pile; \( I \) is the moment of inertia of the cross-section; \( A \) is the area of the cross-section.
The kinetic energy $T$ is:

$$T = \int_0^H \frac{1}{2} [\rho A(\dot{w}^2 + \dot{v}^2) + j \dot{\theta}^2] \, dz$$

(4)

where $\rho$ is the density of the pile; $j$ is the rotary inertia.

Applying the Hamilton principle, the motion equation of the pile foundation can be obtained

$$\rho A\ddot{w} + c\dot{w} + EI\dddot{v} - k_0 v - k_1 \dddot{v} + EAF(t)\dddot{v} = \rho \delta(z) - R[v'(k_0 v - k_1 \dddot{v})]' + \frac{EA}{2H} \dddot{v} \int_0^H \psi^2 \, dz$$

(5)

where the damping term has been added with the viscous damping coefficient $c$; $\delta(z)$ is the Dirac $\delta$ function.

To obtain a more general conclusion of the dynamic response of the pile foundation, the following nondimensional variables are introduced

$$z^* = \frac{z}{H}; \quad v^* = \frac{v}{H}; \quad \lambda = \frac{R}{H}; \quad J = \frac{j}{\rho AH^2}; \quad t^* = t\sqrt{\frac{EI}{\rho AH^4}}; \quad F^* = \frac{FH^2}{r^2}$$

$$c^* = \frac{cH^2}{\sqrt{\rho AEI}}; \quad K_0 = \frac{k_0 H^4}{EI}; \quad K_1 = \frac{k_1 H^2}{EI}; \quad P^* = \frac{PH^4}{EI}; \quad \Omega^* = \Omega \sqrt{\frac{\rho AH^4}{EI}}$$

(6)

where $r = \sqrt{T/A}$ is the radius of gyration. Substituting Eq. (6) into Eq. (5), we can obtain

$$\ddot{v} + c\dot{v} + Lv = P\delta(z) \cos \Omega t + G_2(v, v) + \frac{2v''}{\lambda^2} \int_0^1 \psi'^2 \, dz,$$

(7)

where the terms of rotary inertia has been neglected; the asterisks are dropped for simplicity; $L$ is the linear operator, $Lv = \dddot{v} + Fv'' - K_1 \dddot{v} + K_0 v$; $G_2(v, v) = -\lambda(K_0 v' - K_1 v'')'$. Accordingly, the boundary conditions can be written as

$$z = 0, 1: \quad v(z, t)\psi''(z, t) = 0; \quad v(z, t)\psi''(z, t) = 0; \quad v'(z, t)\psi''(z, t) = 0.$$  

(8)

2.4. Critical load and buckling

Referring to Eq. (7), the critical load exists which can be obtained relevant to the buckling of the pile, when the pile subjected to the lateral and vertical loads. If all the damping and lateral load terms, the term related to the variable $t$, and the second-order moment term are neglected, the motion equation of the pile can be written as

$$\psi''' + (F - K_1)\psi'' - \frac{2\psi''}{\lambda^2} \int_0^1 \psi'^2 \, dz + K_0\psi = 0,$$

(9)

where $\psi(z)$ is the buckling shape of the pile. If the $\psi(z)$ is a known function, the following parameter can be included

$$\Gamma = \frac{2}{\lambda^2} \int_0^1 \psi'^2 \, dz,$$

(10)

Substituting Eq. (10) into Eq. (9), we can obtained

$$\psi''' + T^2\psi'' + K_0\psi = 0,$$

(11)

where $T^2 = F - K_1 - \Gamma$ is the constant value, which can be used to determine the critical load.
3. Results and discussions

Based on the characteristic equation, Eq. (11), we investigate the buckling phenomenon and frequency of the pile foundation. In this numerical analysis, the dimensional parameters and material properties of the pile and the soil medium are as follows: $E_s = 23.94\, \text{MPa}$, $\nu_s = 0.2$, $\nu = 0.25$, $H = 6.096\, \text{m}$, $R = 0.243\, \text{m}$, $\rho = 2.403 \times 10^3\, \text{kg/m}^3$, $E = 2.482 \times 10^3\, \text{MPa}$, where $E_s$ is the Young’s modulus of the soil medium; $\nu_s$ is the Poisson ratio of the soil medium; $\nu$ is the Poisson ratio of the pile. In the following numerical analysis, we consider two cases of the boundary conditions: free-hinged and free-free. Moreover, the effect of the boundary constraint on the buckling of the pile also be investigated.

Fig. 2 shows the bifurcation phenomena and the critical load of the pile. In general, the pile foundation not exhibits buckling and the static deflection (stability), until the vertical load greater than the critical load $F_{cr}$. Referring to Fig. 2, when the vertical load beyond the second-order critical load $F_2 = 135.787\, (145.424)$, there are three cases of the equilibrium state, which including the line mode (instability) and the buckling modes of the first- or second-order critical load. Moreover, compared to the free-hinged pile foundation, the static deflection of the free-free pile foundation quickly increases with the increase of vertical load.

![Figure 2. Bifurcation of the static deflection and the critical load of the pile: a) free-hinged; b) free-free.](image)

![Figure 3. Effect of the foundation parameters on the critical load: a) free-hinged; b) free-free.](image)
To reveal the effect of the foundation parameters on the critical load $F_{cr}$, Fig. 3 shows the variation of the first three order critical loads with the Winkler parameter of the foundation. Obviously, the critical load increases with the increase of the shear parameter $K_1$, and the increase value equal to $K_1 (F_{cr} = K_1 + T^2)$. In general, the relaxation of the boundary constraint causes the increase of the critical load. As shown in Fig. 3, the growth trend of the critical load is very weak when the Winkler parameter $K_0 < 100$. However, when the Winkler parameter increases beyond the specific value, the effect of the Winkler parameter on the critical load is very significant. On the other hand, it is interesting that the first and second-order critical load is very closer to each other, when the pile foundation with the free-free constraint and $K_0 = 794(6310)$.

4. Conclusions
An analytical model of the pile foundation subjected to the vertical load considering the second-order moment of the soil-structure interaction is proposed in this study. Then, the critical load and buckling of the pile are investigated. Numerical results shown that the effect of the vertical load on the buckling is significant, the boundary conditions and the foundation parameters are also the important factors in the analysis of the buckling of the pile foundation. Moreover, the quadratic nonlinear term included in the motion equation of the pile foundation, we would better to research the nonlinear response of the pile foundation in the following study.

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