ON FUNCTION THEORY IN QUANTUM DISC: q-DIFFERENTIAL EQUATIONS AND FOURIER TRANSFORM

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1 Green function and Poisson equation

It was shown in [7] that the Laplace-Beltrami operator \( \Box : L^2(d\nu)_q \to L^2(d\nu)_q \) has a bounded inverse. Hence, for any function \( f \in L^2(d\nu)_q \), there exists a unique solution \( u \in L^2(d\nu)_q \) of Poisson equation \( \Box u = f \).

**Proposition 1.1** \( \Box^{-1} f_0 = -(1 - q^2) \sum_{m=1}^{\infty} \frac{q^{-2} - 1}{q^{-2m} - 1} (1 - zz^*)^m. \)

**Proof.** It was shown in [7, section 5] that the ‘radial part’ \( \Box^{(0)} : L^2(d\nu)_q \to L^2(d\nu)_q \) of the Laplace-Beltrami operator \( \Box \) is given by \( \Box^{(0)} = Dx(q^{-1}x - 1)D \), with \( x = (1 - zz^*)^{-1} \). Hence, \( \Box^{-1} f_0 = \psi(x) \),

\[
\begin{cases}
  x(q^{-1}x - 1)D\psi(x) = q^{-1} - q \\
  \sum_{j=0}^{\infty} |\psi(q^{-2j})|^2 \cdot q^{-2j} < \infty
\end{cases}
\]

Thus, for all \( x \in q^{-2Z} \), one has

\[
(q^{-2}x - 1)(\psi(q^{-2}x) - \psi(x)) = (q^{-1} - q)^2,
\]

\[
\psi(x) = \psi(q^{-2}x) - (q^{-2} - 1)^2 \frac{q^4x^{-1}}{1 - q^2x^{-1}}. \quad (1.2)
\]

Now use (1.1) and (1.2) to get

\[
\psi(x) = -(q^{-2} - 1)^2 q^2 \sum_{j=1}^{\infty} \frac{q^{2j}x^{-1}}{1 - q^{2j}x^{-1}} = -(q^{-2} - 1)^2 q^2 \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} q^{2jm} x^{-m} = \\
= -(q^{-2} - 1)^2 q^2 \sum_{m=1}^{\infty} \frac{q^{2m}}{1 - q^{2m}(1 - zz^*)^m}. \quad \Box
\]

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Consider the integral operator $I_m : D(U)_q \to D(U)'_q$ with the kernel $G_m \in D(U \times U)'_q$ given by

$$G_m = \left\{ \left( 1 - \zeta^* \right)^m \cdot \left( 1 - z^* z \right)^m \right\}.$$

The following statement was announced in [6, Theorem 3.5]

**Theorem 1.2** For all $f \in D(U)_q$

$$\Box^{-1} f = - \sum_{m=1}^{\infty} \frac{q^{-2} - 1}{q^{-2m} - 1} I_m f.$$  \(1.4\)

To prove this theorem we need the following auxiliary result

**Lemma 1.3** $G_m$ is an invariant of the $U_q \mathfrak{sl}_2$-module $D(U \times U)'_q$.

**Proof of lemma.** The following invariants were introduced in [3]:

$$k_{22}^{-m} k_{11}^{-m} = q^{2m} \left\{ (1 - \zeta^*)^m \cdot \sum_{j=0}^{\infty} \frac{(q^{2m}; q^2)_j (q^{2(-m+1)} z^* \zeta)^j \cdot \sum_{n=0}^{\infty} \frac{(q^{2m}; q^2)_n (q^{-2m} z \zeta)^n (1 - z^* z)^m}{(q^2; q^2)_n} \right\}.$$

By a virtue of the q-binomial theorem (see [4]),

$$\sum_{i=0}^{\infty} \frac{(q^{2m}; q^2)_i t^i}{(q^2; q^2)_i} = t_0(q^{2m}; -; q^2, t) = (q^{2m} t; q^2)_\infty / (t; q^2)_\infty = (t; q^2)^{-1}.$$

Hence,

$$k_{22}^{-m} k_{11}^{-m} = q^{2m} \left\{ (1 - \zeta^*)^m (q^{-2(m+1)} z^* \zeta)^{-1} \cdot \sum_{n=0}^{\infty} \frac{(q^{2m}; q^2)_n (q^{-2m} z \zeta)^n (1 - z^* z)^m}{(q^2; q^2)_n} \right\}.$$

On the other hand, in $\text{Pol}(\mathbb{C})_q$ one has $(1 - \zeta^*) \zeta = q^2 \zeta (1 - \zeta^*)$, and in $\text{Pol}(\mathbb{C})_q^\text{op}$, respectively, $z (1 - z^* z) = q^2 (1 - z^* z) z$, whence

$$k_{22}^{-m} k_{11}^{-m} = q^{2m} \left\{ (1 - \zeta^*) (1 - z^* \zeta)^{-1} \right\}.$$

The invariance of $G_m$ follows from the invariance of $k_{22}^{-m} k_{11}^{-m}$. \qed

**Proof of theorem 1.2.** In the special case $f = f_0$ one has $I_m f_0 = (1 - q^2) (1 - z z^*)^m$ since $\zeta f_0 = f_0 \zeta = 0$, $\int f_0 d\nu = 1 - q^2$. Hence in that special case \(1.4\) follows from proposition 1.1.

By [6, proposition 3.9] $f_0$ generates the $U_q \mathfrak{sl}_2$-module $D(U)_q$. What remains is to show that the linear operators $\Box^{-1}$ and $- \sum_{m=1}^{\infty} \frac{q^{-2} - 1}{q^{-2m} - 1} I_m$ are morphisms of $U_q \mathfrak{sl}_2$-modules. For the first operator this follows from [6, proposition 4.3] and for the second one from lemma 1.3. \qed
2 Cauchy-Green formula

Let \( f \in D(U)_q \). This section presents a solution of the \( \overline{\partial} \)-problem in \( L^2(d\mu)_q \):

\[
\frac{\partial (r)}{\partial z^*}u = f, \quad u \perp \text{Ker} \left( \frac{\partial (r)}{\partial z^*} \right). \tag{2.1}
\]

Our aim is to prove the following statement (see [6, proposition 4.1])

**Theorem 2.1.** Let \( f \in D(U)_q \). Then

1. There exists a unique solution \( u \in L^2(d\mu)_q \) of the \( \overline{\partial} \)-problem \( \overline{\partial}u = f \), which is orthogonal to the kernel of \( \overline{\partial} \).

2. \( u = \frac{1}{2\pi i} \int_{U_q} d\zeta \frac{\partial (l)}{\partial z} G(z, \zeta) f d\zeta^* \), with \( G \in D(U \times U)'_q \) being the Green function of the Poisson equation.

3. \( f = -\frac{1}{2\pi i} \int_{U_q} (1 - z\zeta^*)^{-1}(1 - q^{-2}z\zeta^*)^{-1} d\zeta f(\zeta)d\zeta^* - \frac{1}{2\pi i} \int_{U_q} d\zeta \frac{\partial (l)}{\partial z} G(z, \zeta) \cdot \frac{\partial (r)}{\partial \zeta^*}d\zeta^* \).

To clarify the symmetry of this problem, pass from the partial derivative to the differential, and from functions to differential forms.

Consider the morphism of \( U_q\mathfrak{su}(1, 1) \)-modules \( \overline{\partial} : \Omega(U)^{(1, 0)}_q \rightarrow \Omega(U)^{(1, 1)}_q \). By a virtue of the canonical isomorphisms of covariant \( D(U)_q \)-bimodules \( \Omega(U)^{(0, j)}_q \simeq \Omega(U)^{(1, j)}_q \), \( j = 0, 1 \), \( f v_2 \mapsto f dz, f \in \Omega(U)^{(0, s)}_q \), the following scalar products are \( U_q\mathfrak{su}(1, 1) \)-invariant (see [7]):

\[
(f_1 dz, f_2 dz) = \int_{U_q} f_1^* f_2 (1 - z z^*)^2 d\nu, \quad (f_1 dz d\zeta, f_2 dz d\zeta^*) = \int_{U_q} f_2^* f_1 (1 - z z^*)^4 d\nu.
\]

The completions of pre-Hilbert spaces \( \Omega(U)^{(1, 0)}_q, \Omega(U)^{(1, 1)}_q \), are canonically isomorphic to the Hilbert spaces \( L^2(d\mu)_q, L^2((1 - z z^*)^2 d\mu)_q \), respectively \( (i_0 : f dz \mapsto f; \quad i_1 : f dz d\zeta \mapsto f \) are just those isomorphisms).

We may reduce solving the problem (2.1) to solving the following problem:

\[
\overline{\partial}u = f dz d\zeta^*, \quad u \perp \text{Ker} (\overline{\partial}), \tag{2.2}
\]

where the orthogonality means that the above invariant scalar product in the space of \( (1, 0) \)-forms vanishes.

To solve this problem, we need auxiliary linear operators \( \overline{\partial}^*, \overline{\square}^{(1, 1)} = -\overline{\partial} \cdot \overline{\partial}^* \). Turn to studying these operators.

**Lemma 2.2** For all \( f \in D(U)_q \), \( \frac{\partial (l)}{\partial z} f^* = \left( \frac{\partial (r)}{\partial z} \right)^* f^* \).

**Proof.** \( dz^* : \frac{\partial (l)}{\partial z} f^* = \overline{\partial} f^* = (\overline{\partial} f)^* = \left( \frac{\partial (r)}{\partial z} \cdot dz \right)^* = dz^* \cdot \left( \frac{\partial (r)}{\partial z} \right)^* \). \( \square \)
Lemma 2.3 For all $f_1, f_2 \in D(U)_q$, \( \overline{\partial}(f_1 dz), f_2 dz dz^* \) = \( f_1 dz, q^2 \frac{\partial(r)}{\partial z}(f_2 \cdot (1 - zz^*)^2)dz \).

**Proof.** An application of lemma 2.2 and the q-analogue of Green’s formula (see appendix in [R]) allows one to get for all $f_1, f_2 \in D(U)_q$:

\[
\left( \overline{\partial}(f_1 dz), f_2 dz dz^* \right) = -q^2 \int_U f_2^* \frac{\partial(r)}{\partial z^*}(1 - zz^*)^2 d\mu = -q^2 \int_U (1 - zz^*)^2 f_2^* \frac{\partial(r)}{\partial z} f_1 d\mu = \]

\[
= q^2 \frac{2i\pi}{\int_U} \int_U dz(1 - zz^*)^2 f_2^* \overline{\partial} f_1 = -q^2 \frac{2i\pi}{\int_U} \int_U dz \overline{\partial}((1 - zz^*)^2 f_2^*) f_1 = q^2 \int_U \frac{\partial(l)}{\partial z^*}((1 - zz^*)^2 f_2^*) f_1 d\mu = \]

\[
= q^2 \int_U \left( \frac{\partial(r)}{\partial z} (f_2(1 - zz^*)^2) \right)^* f_1 d\mu = q^2 \left( f_1 dz, \frac{\partial(r)}{\partial z}(f_2 \cdot (1 - zz^*)^2)dz \right). \quad \Box
\]

Corollary 2.4 The linear operator

\[
\overline{\partial}^* : \Omega(U)^{(1,1)}_q \to \Omega(U)^{(1,0)}_q; \quad \overline{\partial}^* : f dz dz^* \mapsto q^2 \frac{\partial(r)}{\partial z}(f \cdot (1 - zz^*)^2)dz,
\]

is a morphism of $U_q\mathfrak{sl}_2$-modules.

Corollary 2.5 The linear operator $\Box^{(1,1)} : \Omega(U)^{(1,1)}_q \to \Omega(U)^{(1,1)}_q$ given by $\Box^{(1,1)} : f dz dz^* \mapsto q^4 \frac{\partial(r)}{\partial z} \frac{\partial(r)}{\partial z}(f(1 - zz^*)^2)dz dz^*$, $f \in D(U)_q$, is an endomorphism of $U_q\mathfrak{sl}_2$-modules.

The relation $\Box^{(1,1)} = -\overline{\partial} \cdot \overline{\partial}^*$ allows one to get a solution of the $\overline{\partial}$-problem in the form $u = -\overline{\partial}^* \omega$, with $\omega$ being a solution of the Poisson equation $\Box^{(1,1)} \omega = f dz dz^*$.

Find a solution of the latter equation.

Lemma 2.6 The elements $\{z^m\}_{m>0}, z^* z, \{z^* m\}_{m>0}$, generate the $U_q\mathfrak{sl}_2$-module Pol($\mathbb{C}$)$_q$.

**Proof** reduces to reproducing the argument used while proving [R, theorem 3.9]. \( \Box \)

Lemma 2.7 The linear operator $\Box' : D(U)_q' \to D(U)_q'$ given by $\Box' : f \mapsto q^4 \left( \frac{\partial(r)}{\partial z} \frac{\partial(r)}{\partial z} f \right) (1 - zz^*)^2$, is an endomorphism of the $U_q\mathfrak{sl}_2$-module $D(U)_q'$.

**Proof.** Consider the isomorphism of $U_q\mathfrak{sl}_2$-modules $i : \Omega(U)^{(0,0)}_q \to \Omega(U)^{(1,1)}_q$ given by $i : f \mapsto f \cdot (1 - zz^*)^{-2} dz dz^*$. Obviously, $\Box' = i^{-1} \Box^{(1,1)} i$. What remains is to refer to corollary 2.3. \( \Box \)

Proposition 2.8 $q^2 \Box = \Box'$. 

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Before proving this proposition, we deduce its corollaries.

**Corollary 2.9** \( □^{(1,1)} = q^2 \Box i^{-1} \).

**Corollary 2.10** \( □ f = q^2 \left( \frac{\partial^{(r)}(z^{*})}{\partial z^{*}} \right) (1 - zz^{*})^2, f \in D(U)^{q}_{\eta} \).

Since \( i \) is an isometry, and \( 0 < c_1 \leq -\Box \leq c_2 \) (see [7]), one has

**Corollary 2.11** \( 0 < c_1 \leq q^{-2} \Box' \cdot \Box' \leq c_2 \).

Note that we have proved the boundedness of the linear map \( \Box' \) from the completion of \( \Omega(U)^{(1,1)}_{\eta} \) to the completion of \( \Omega(U)^{(1,0)}_{\eta} \).

**Proof of proposition 2.8**. Let \( f = z^{*}z \). By a virtue of [7, lemma 5.1] one has \( \Box(z^{*}z) = -q^2 \Box x^{-1} = q^2(1 - zz^{*})^2 = q^2 \Box f \). Thus, the relation \( \Box f = q^{-2} \Box' f \) is proved in the special case \( f = z^{*}z \). In the two another special cases \( f \in \{ z^m \}_{m \geq 0}, f \in \{ z^{*}m \}_{m \geq 0} \) the above relation follows from \( \Omega f = \Box f = \Box' f = 0 \), with \( \Omega \) being the Casimir element (see [7]). Hence, by virtue of lemmas 2.6, 2.7, the relation \( \Box f = q^{-2} \Box' f \) is valid for all the polynomials \( f \in \text{Pol}(\mathbb{C})_{q} \).

What remains is to apply the continuity of the linear maps \( \Box, \Box' \) in the topological vector space \( D(U)^{q}_{\eta} \) together with the density of \( \text{Pol}(\mathbb{C})_{q} \) in \( D(U)^{q}_{\eta} \).

The following result, together with its proof attached below, are due to S. Klimek and A. Lesniewski [4].

**Proposition 2.12** Consider the orthogonal projection \( P \) from \( L^2(d\mu_{q}) \) onto the subspace \( H^2(d\mu_{q}) \) generated by the monomials \( \{ z^m \}_{m \geq 0} \). For all \( f \in D(U)^{q}_{\eta} \) one has \( Pf = \int (1 - \zeta z^{*})^{-1}(1 - q^2 z^{*})^{-1} f(\zeta) d\mu(\zeta) \).

**Proof.** An application of [8, lemma 7.1] and the q-binomial theorem (see [3]) yield the following explicit expression for the kernel of the integral operator \( P \):

\[
\sum_{m=0}^{\infty} \frac{(q^4; q^2)^m (z^{*})^*}{(q^2; q^2)^m} = (q^4 z^{*}; q^2)^{1} \cdot (z^{*}; q^2)^{1}.
\]

**Remark 2.13** Another proof of proposition 2.12, which involves no properties of q-special functions, will be presented in appendix of [7].

**Proof of theorem 2.1**. By corollary 2.10, \( \Omega(U)^{(1,1)}_{\eta} \) contains a unique solution \( \omega \) of the Poisson equation \( \Box^{(1,1)} \omega = f dz dz^{*} \). It is given by

\[
\omega = q^{-2} \left( \int_{U_{q}} G(z, \zeta)f(\zeta)(1 - \zeta^{*})^2 d\nu \right) (1 - zz^{*})^{-2} dz dz^{*},
\]
with $G \in D(U \times U)'$, $G = -\sum_{m=1}^{\infty} \frac{q^{-2m} - 1}{q^{-2m} - 1} G_m$, being the Green function found in section 1.

By lemma 2.3 and corollary 2.11 the $(1,0)$-form $\left(-\int_{\tilde{U}_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} f(\zeta) d\mu \right) dz$ is a solution of the $\bar{\partial}$-problem (2.2). Hence, the function $u = -\int_{\tilde{U}_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} f(\zeta) d\mu$ is a solution of the $\bar{\partial}$-problem (2.1). Since the uniqueness of a solution of this $\bar{\partial}$-problem is obvious, we have proved the first two statements of theorem 2.1.

Let $f \in D(U)_q$, and $u = -\int_{\tilde{U}_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} \frac{\partial^{(r)} f}{\partial \zeta^{*}} d\mu$ be the above solution of the $\bar{\partial}$-problem $\frac{\partial^{(r)} u}{\partial z^{*}} = \frac{\partial^{(r)} f}{\partial z^{*}}$, $u \perp \text{Ker} \left( \frac{\partial^{(r)} f}{\partial z} \right)$. Then $u \perp H^2(d\mu)_q$, $f - u \in H^2(d\mu)_q$, and hence $Pf = P(f - u) = f - u$. Thus, $f = u + Pf$, and by a virtue of proposition 2.12

$$f = \int_{\tilde{U}_q} (1 - z\zeta^{*})^{-1} (1 - q^{-2}z\zeta^{*})^{-1} f(\zeta) d\mu(\zeta) - \int_{\tilde{U}_q} \frac{\partial^{(r)} G(z,\zeta)}{\partial z} \frac{\partial^{(r)} f}{\partial \zeta^{*}} d\mu.$$ 

This relation implies the third statement of theorem 2.1, the Green formula.

3 Eigenfunctions of the operator □

It follows from [9, section 5] that $q\square f = \Omega f$, $f \in D(U)'_q$, with $\Omega \in U_q\mathfrak{sl}_2$ being the Casimir element. Our purpose is to produce distributions $f \in D(U)'_q$ for which $\Omega f = \lambda f$ for some $\lambda \in \mathbb{C}$. More exactly, we shall prove the following result (it was announced in [9, proposition 5.1])

**Theorem 3.1.** For all $f \in \mathbb{C}[\partial U]_q$ the element

$$u = \int_{\partial U} P_{l+1}(z, e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}$$

(5.3)

of $D(U)'_q$ is an eigenvector of $\square$:

$$\square u = \lambda(l) u, \quad \lambda(l) = -\frac{(1 - q^{-2l})(1 - q^{2l+2})}{(1 - q^2)^2}.$$ 

We start with a similar problem for the quantum cone and, as in [8], consider the spaces $F(\Xi)_q^{(l)} \subset F(\tilde{\Xi})_q$ of degree $2l$ homogeneous functions on the quantum cone $\tilde{\Xi}$. Impose also the notation $F(\Xi)_q = F(\tilde{\Xi})_q \cap D(\Xi)'_q$, $F(\Xi)_q^{(l)} = F(\tilde{\Xi})_q^{(l)} \cap D(\Xi)'_q$, $l \in \mathbb{C}$.

By the construction, $F(\Xi)_q^{(0)}$ is a covariant $*$-algebra. We intend to give its description in terms of generators and relations.
Proposition 3.2 The bilateral ideal $J \subset \text{Pol}(\mathbb{C})_q$ generated by the single element $1 - zz^* = 0$, is a $U_q\mathfrak{sl}_2$-submodule of the $U_q\mathfrak{sl}_2$-module $\text{Pol}(\mathbb{C})_q$.

**Proof** is derivable from the explicit formulae

$$Hz = 2z, \quad X^- z = q^{1/2}, \quad X^+ z = -q^{-1/2}z^2,$$
$$Hz^* = -2z, \quad X^+ z^* = q^{-1/2}, \quad X^- z^* = -q^{1/2}z^2 \quad \square$$

Corollary 3.3 The $\ast$-algebra $\mathbb{C}\lbrack\partial U\rbrack_q \simeq \text{Pol}(\mathbb{C})_q/J$ considered in [6] is a covariant $\ast$-algebra.

Remind that in $\mathbb{C}\lbrack\partial U\rbrack_q$ one has $zz^* = z^*z = 1$.

An application of the relation (1.3) of [8] yields

Proposition 3.4 The covariant $\ast$-algebra $\mathbb{C}\lbrack\partial U\rbrack_q$ is isomorphic to the covariant $\ast$-algebra $F(\Xi)_q$ as follows:

$$i_0 : \mathbb{C}\lbrack\partial U\rbrack_q \to F(\Xi)_q^0, \quad i_0 : z \mapsto qt_{11}t_{12}^{-1}, \quad i_0 : z^* \mapsto t_{21}^{-1}t_{22}.$$ 

Note that the vector spaces $F(\Xi)_q^l$, $l \in \mathbb{C}$, are covariant $F(\Xi)_q^0$-bimodules, and the vector space $F(\Xi)_q$ is a covariant $\ast$-algebra. We identify the elements of $\mathbb{C}\lbrack\partial U\rbrack_q$ and their images under the embedding $i : \mathbb{C}\lbrack\partial U\rbrack_q \hookrightarrow F(\Xi)_q$.

Let $l \in \mathbb{C}$, $x = t_{12}t_{12}^* = -qt_{12}t_{21}$. Apply the relation (1.3) of [8] to get a description of the covariant bimodule $F(\Xi)_q^l$.

Proposition 3.5 For all $l \in \mathbb{C}$, $x^l \in F(\Xi)_q^l$ one has

$$zx^l = q^{2l}xz, \quad z^*x^l = q^{-2l}x^lz^*$$

$$X^+(x^l) = q^{-3/2}q^{-2l} \frac{q^{-1} - 1}{q^{-2} - 1} x^l$$

$$X^-(x^l) = q^{3/2} \frac{q^{2l} - 1}{1 - q^2} z^* x^l$$

$$H(x^l) = 0 \quad (3.4)$$

The covariant bimodules $F(\Xi)_q^l$ are, in particular, $U_q\mathfrak{sl}_2$-modules. The associated representations of $U_q\mathfrak{sl}_2$ are called the representations of the principal series. These are irreducible for some open dense set of $l \in \mathbb{C}$. By a virtue of relation (5.8) of [9], for those $l \in \mathbb{C}$, and hence for all $l \in \mathbb{C}$ and all $f \in F(\Xi)_q^l$, one has

$$\Omega f = \Lambda(l)f, \quad \Lambda(l) = \frac{(q^{-l} - q^l)(q^{-l} - q^{l+1})}{(q^{-1} - q)^2}.$$ 

Let $V^l$ be the $U_q\mathfrak{sl}_2$-modules considered in [9]. One can easily deduce from (3.3), (3.4), (3.5) the following.
Corollary 3.6 For all $l \in \mathbb{C}$, the linear map $i_l: V^{(l)} \to F(\Xi)^{(l)}_q$; $i_l: X^\pm m e_0 \mapsto X^\pm m(x^l)$, $m \in \mathbb{Z}_+$, are the isomorphisms of $U_q\mathfrak{sl}_2$-modules.

Proof of theorem 3.1. Let us turn to a construction of distributions $f \in D(X)^{(l)}_q$ on the quantum hyperboloid, which satisfy the equation $\Omega f = \Lambda(l)f$ for some $l \in \mathbb{C}$.

By the results of [3] section 6, the element
\[
q^{-l}k_{11}^l \defeq q^{-2l}q^{1} \sum_{j=0}^\infty \frac{\left(q^{-2l}; q^2\right)_j \left(q^{2(l+1)}z^*\zeta^*\right)}{\left(q^2; q^2\right)_j} \sum_{m=0}^\infty \frac{\left(q^{-2l}; q^2\right)_m \left(q^{2l}z\zeta^*\right)^m (1 - z^*z)^{-l}}{\left(q^2; q^2\right)_m}
\]
of the completion of $F(X)^{op} \otimes F(\Xi)^{(l)}_q$ is an invariant. (Here $z, z^* \in F(X)^{op}$, $\zeta, \zeta^* \in F(\Xi)_q$ are the elements given by explicit formulae in [3], section 6).

It follows from the results of [3] section 4 that the linear functional $\eta: F(\Xi)^{(l)}_q \to \mathbb{C}$,
\[
\int_{\Xi} \left( \sum_{m=\infty}^\infty a_m \zeta^m \right) \zeta^{-1} d\eta = a_0,
\]
is an invariant integral. Hence, the linear integral operator
\[
F(\Xi)^{(l)}_q \to F(X)^{(l)}_q; \quad f \mapsto \int_{\Xi} \{k_{11}^l\} f d\eta
\]
is a morphism of $U_q\mathfrak{sl}_2$-modules. By a virtue of (3.3), for any trigonometric polynomial $f(\zeta) \in \mathbb{C}[\partial U]_q$, the function
\[
\int_{\Xi} \{k_{11}^l\} f \zeta^{-l+1} d\eta = q^{-2l} \int_{\partial U} P_{-l}(z, e^{iq}) f(e^{iq}) \frac{d\theta}{2\pi}
\]
is an eigenfunction of the Laplace-Beltrami operator. Here $P_{-l}$ is a $q$-analogue of the Poisson kernel (see [3], section 5)). Now a passage from the quantum hyperboloid $X$ to the quantum disc $U$ via the isomorphism of $U_q\mathfrak{sl}_2$-modules $i: D(U)^{(l)}_q \to D(X)^{(l)}_q$ (see [3]) yields the statement of theorem 3.1.

Denote by $\mathbb{C}[\partial U]_{q,l}$ the vector space $\mathbb{C}[\partial U]_q$ equipped by the structure of $U_q\mathfrak{sl}_2$-module in such a way that the map $\mathbb{C}[\partial U]_{q,l} \to F(\Xi)^{(l)}_q; \quad f(z) \mapsto f(z)x^l$, is a morphism of $U_q\mathfrak{sl}_2$-modules.

An application of (3.1), (3.4) gives
\[
X^+ f(z) = -q^{-1/2}z^2(D f)(z) + q^{-3/2} \frac{q^{-2l} - 1}{q^2 - 1} f(qz),
\]
\[
X^- f(z) = q^{1/2} (D f)(z) + q^{3/2} \frac{1 - q^{2l}}{1 - q^2} f(qz),
\]
\[
H f(z) = 2z \frac{d}{dz} f(z),
\]
with $D: f(z) \mapsto (f(q^{-1} z) - f(z))/q^{-1} z - qz)$, $f \in \mathbb{C}[\partial U]_{q,l}$.

Let $Re l > -\frac{1}{2}$. With the notation of [3] being implicit, introduce a linear operator $I_l$ in $\mathbb{C}[\partial U]_{q,l}$ given by
\[
I_l f = \frac{\Gamma_{q^2}^2(l + 1)}{\Gamma_{q^2}(2l + 1)} \lim_{r \to 1} \frac{1}{1 - r^2} \sum_{m=\infty}^\infty (1 - r^2)^m b_{r,u},
\]
(3.6)
with \( u = \int_{\partial U} P_{l+1}(z,e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi}, f \in \mathbb{C}[\partial U]_q, l. \)

Our aim now is to prove the following result (see [3, proposition 5.3])

**Theorem 3.7.** \( I_l f = f \)

**Proof.** The theorem will be proved if we establish the existence of the limit in the right hand side of (3.6) and show that \( I_l \) is the identity operator.

Let \( L \subset \mathbb{C}[\partial U]_{q,l-1} \) be the linear subspace of all those elements \( f \in \mathbb{C}[\partial U]_q \) for which the both above statements are valid. By a virtue of [6, lemma 5.4],

\[
\lim_{z \to q^\infty} \varphi_l \left( \frac{1}{x} \right) = \frac{\Gamma_q^2(2l+1)}{\Gamma_q^2(l+1)} = 1, \tag{3.7}
\]

for \( \text{Re} l > -\frac{1}{2} \), with \( \varphi_l = \int_{\partial U} P_{l+1}(z,e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi} \). Thus, \( 1 \in L \). Moreover, an application of this lemma and the fact that the linear operator

\[
j_l : \mathbb{C}[\partial U]_{q,l} \to D(U)_q, \quad j_l : f \mapsto \int_{\partial U} P_{l+1}(z,e^{i\theta})f(e^{i\theta})\frac{d\theta}{2\pi}
\]

is an isomorphism of \( U_q \mathfrak{sl}_2 \)-modules, allows one to prove that \( L \) is a submodule of the \( U_q \mathfrak{sl}_2 \)-module \( \mathbb{C}[\partial U]_{q,l} \). On the other hand, with \( l \notin \mathbb{Z}_+ + \frac{\pi}{\ln(q^{-1})} \mathbb{Z} \), the \( U_q \mathfrak{sl}_2 \)-module \( \mathbb{C}[\partial U]_{q,l} \simeq V^{(l)} \) is simple. Hence, for \( l \) as above one has \( L = \mathbb{C}[U]_{q,l} \), and thus the theorem is proved. \( \Box \)

**Remark 3.8.** Let \( m \in \mathbb{Z}_+ \), and \( \psi(x) \) be a function on \( q^{-2\mathbb{Z}_+} \) such that \( z^m \cdot \psi(y^{-1}) = \int_{\partial U} P_{l+1}(z,e^{i\theta})e^{\beta m \theta}\frac{d\theta}{2\pi} \). Another way of proving the existence of the limit in the right hand side of (3.6) is based on producing a fundamental system of solutions of the difference equation for \( \psi(x) \). (This difference equation is a consequence of the relation \( \Omega(z^m \psi(y^{-1})) = \Lambda(l)(z^m \psi(y^{-1})) \). It is easy to prove the existence of such fundamental system of solutions \( \psi_1, \psi_2 \) that

\[
\lim_{x \to +\infty} \frac{\psi_1(x)}{x^l} = \lim_{x \to +\infty} \frac{\psi_2(x)}{x^{-l-1}} = 1.
\]

What remains is to use the relation \( \text{Re} l > -\frac{1}{2} \).

### 4 Decomposing in eigenfunctions of the operator \( \Box^{(0)} \)

One can find in [3, section 5] a description of the bounded linear operator \( \Box^{(0)} : f(x) \mapsto D x (q^{-1} x - 1) D f(x) \) in the Hilbert space \( L^2(d\nu)_q^{(0)} \) of such functions on \( q^{-2\mathbb{Z}_+} \) that \( \|f\| = \left( \int_1^\infty |f(x)|^2 d_{q^{-2}} x \right)^{1/2} < \infty \). That section also contains the relation (5.9) which determines the eigenfunctions \( \Phi_I(x) \) of \( \Box^{(0)} \). Besides, a unitary operator \( u : L^2(d\nu)_q^{(0)} \to L^2(dm) \) that
realizes a decomposition in those eigenfunctions was constructed. Remind that \( u \) could be defined by (5.10), and \( dm \) is a Borel measure on a compact \( \mathcal{L}_0 \) introduced by (5.7) in [3].

In this section, explicit formulae for eigenfunctions \( \Phi_l(x) \) and the spectral measure will be found; [6, proposition 3.2] will be proved.

**Proposition 4.1** \( \Phi_l(x) = 3 \Phi_2 \left[ \frac{x, q^{-2l}, q^{2(l+1)}; q^2; q^2}{q^2, 0} \right] \).

**Proof.** By a virtue of [6, corollary 5.2], the distribution
\[
3 \Phi_2 \left[ (1 - zz^*)^{-1}, q^{-2l}, q^{2(l+1)}; q^2; q^2 \right] \in D(U)_q
\]
is an eigenfunction of \( \square \). What remains is to apply the definition of \( \Phi_l(x) \) and the evident relation
\[
3 \Phi_2 \left[ 1, q^{-2l}, q^{2(l+1)}; q^2; q^2 \right] = 1. \]
\( \square \)

**Corollary 4.2** The spectrum of \( \square^{(0)} \) coincides with the segment \( \left[ -\frac{1}{(1-q)^2}, -\frac{1}{(1+q)^2} \right] \).

**Proof.** It follows from [3, section 5] that the continuous spectrum of \( \square^{(0)} \) fills this segment. So we are to prove that the discrete spectrum of \( \square^{(0)} \) is void, that is \( \Phi_l \notin L^2(d\nu)_q^{(0)} \) for \( \text{Re} l > -\frac{1}{2} \). This can be deduced from proposition 5.1 and lemma 5.4 of [3]. \( \square \)

By corollary 4.2, the carrier of \( dm \) coincides with the segment \( \{ l \in \mathbb{C} \mid \text{Re} l = -\frac{1}{2}, 0 \leq \text{Im} l \leq \frac{\pi}{h} \} \), with \( h = -2 \ln q \). Hence,
\[
\frac{1}{(1+q)^2} \leq -\square^{(0)} \leq \frac{1}{(1-q)^2}.
\]
This inequality implies [3, proposition 3.2].

We intend to obtain an explicit formula for the kernel \( G(x, \xi, l) \) of the integral operator \( (\square^{(0)} - \lambda(l)I)^{-1} \) in \( L^2(d\nu)_q^{(0)} \). By corollary 4.2, the ‘Green function’ \( G(x, \xi, l) \) is well defined and holomorphic in \( l \) for \( x, \xi \in q^{-2\mathbb{Z}+}, \text{Re} l \neq -\frac{1}{2} \).

Remind the notation \( [a]_q = (q^{-a} - q^{a})/(q^{-1} - q) \), and choose the branch of \( x^l \) in the half-plane \( \text{Re} x > 0 \): \( x^l = e^{\ln x \cdot l} \), with \( \ln x \) being the principal branch of the logarithm.

**Lemma 4.3** With \( |x| > q^2, \text{Re} x > 0 \), the function
\[
\psi_l(x) = x^l \cdot 2 \Phi_1 \left( \frac{q^{-2l}, q^{-2l}; q^2; q^2x^{-1}}{q^{-4l}} \right)
\]
satisfies the difference equation
\[
D x(q^{-1}x - 1)D\psi_l(x) = \lambda(l)\psi_l(x).
\]
Proof. The right hand side of (4.1) is of the form \( x^l \sum_{m=0}^{\infty} \frac{a_m}{x_m}, a_m \in \mathbb{C} \). Its substitution into (4.2) gives
\[
\frac{a_{m+1}}{a_m} = q \frac{[l-m]^2}{[l-1-m][l][l+1]} = q \frac{[l-m]}{[m+1][m-2l]}
\]
\[
= q^2 \frac{(1-\frac{2}{(1-q^{2(m+1)})(1-q^{4l+2m})})}{-\frac{1}{2}}.
\]
What remains is to use the definition of the basic hypergeometric series \(_2\Phi_1\) (see [3]).

This lemma and the definition of the Green function \( G(z, \xi, l) \) imply

Proposition 4.4
1) For \( \text{Re} \ l > -\frac{1}{2} \)
\[
G(x, \xi, l) = c_1(l) \begin{cases} 
\psi_l(\xi)f_l(x), & x \leq \xi \\
f_l(\xi)\psi_l(x), & x \geq \xi 
\end{cases} \quad (4.3)
\]
2) For \( \text{Re} \ l < -\frac{1}{2} \)
\[
G(x, \xi, l) = c_2(l) \begin{cases} 
\psi_{-l-1}(\xi)f_l(x), & x \leq \xi \\
f_l(\xi)\psi_{-l-1}(x), & x \geq \xi 
\end{cases} \quad (4.4)
\]
Here \( x, \xi \in q^{-2\mathbb{Z}} \), \( c_1(l), c_2(l) \in \mathbb{C} \).

Find the ‘constants’ \( c_1(l), c_2(l) \).

Lemma 4.5 For any two functions \( u, v \) on the semi-axis \( x > 0 \),
\[
Du(x) \cdot v(x) = D(u(x)v(qx)) - qu(qx)(Dv(qx)).
\]

Proof. The following q-analogue of Leibnitz formula is directly from the definition of \( D \):
\[
D(u(x)v(x)) = (Du(x) \cdot v(q^{-1}x) + u(qx)(Dv)(x)).
\]
Replace \( v(x) \) by \( v(qx) \) to get
\[
(Du(x)v(x)) = D(u(x)v(qx)) - u(qx)D(v(qx)).
\]
What remains is to apply the straightforward relation \( D(v(qx)) = q(Dv)(qx) \). \( \square \)

Let \( l \in \mathbb{C}, x \in q^{-2\mathbb{Z}} \), and \( \varphi_1(x), \varphi_2(x) \) be solutions of the difference equation \( Dx(q^{-1}x-1)D\varphi = \lambda(l)\varphi \).

Lemma 4.6
\[
W(\varphi_1, \varphi_2) = x(q^{-2}x-1) \left( \frac{\varphi_1(q^{-2}x) - \varphi_1(x)}{q^{-2}x-x} \cdot \varphi_2(x) - \varphi_1(x) \frac{\varphi_2(q^{-2}x) - \varphi_2(x)}{q^{-2}x-x} \right)
\]
does not depend on \( x \in q^{-2\mathbb{Z}} \).
**Proof.** Evidently,
\[0 = (Dx(q^{-1}x - 1)D\varphi_1)\varphi_2 - \varphi_1(Dx(q^{-1}x - 1)D\varphi_2).\]
Hence, by a virtue of lemma 4.5,
\[0 = D(x(q^{-1}x - 1)D\varphi_1(x) \cdot \varphi_2(qx)) - D(x(q^{-1}x - 1)D\varphi_2(x) \cdot \varphi_1(qx)).\]
That is,
\[D(x(q^{-1}x - 1)(D\varphi_1(x) \cdot \varphi_2(qx) - \varphi_1(qx)D\varphi_2(x)) = 0.\]
Hence, \(q^{-1}x(q^{-2}x - 1)((D\varphi_1(q^{-1}x) \cdot \varphi_2(x) - \varphi_1(x)(D\varphi_2(q^{-1}x)))\) is a constant.

Let \(\varphi_1(x), \varphi_2(x)\) be the eigenfunctions involved in the formulation of the previous lemma, and set up
\[\Phi(x, \xi) = \begin{cases} \varphi_1(x)\varphi_2(\xi), & x \geq \xi \\ \varphi_1(\xi)\varphi_2(x), & x \leq \xi \end{cases}.\]

**Lemma 4.7**
\[Dx(q^{-1}x - 1)D\Phi(x, \xi)\bigg|_{x=\xi} = \begin{cases} \frac{W(\varphi_1, \varphi_2)}{(1-q^2)x} + \lambda \Phi(\xi, \xi), & x = \xi \\ \lambda \Phi(x, \xi), & x \neq \xi \end{cases}.\]

**Proof.** Let \(x = \xi:\)
\[Dx(q^{-1}x - 1)D\Phi\bigg|_{x=\xi} = \frac{q^{-1}x(q^{-2}x - 1)\frac{\Phi(q^{-2}x, \xi) - \Phi(x, \xi)}{q^{-2}x - x} - x(x - 1)\frac{\Phi(x, \xi) - \Phi(q^{-2}x, \xi)}{x - q^{-2}x}}{q^{-1}x - qx}\bigg|_{x=\xi} = \]
\[= \frac{1}{(q^{-1} - q)\xi} \left( q^{-1}\xi(q^{-2}\xi - 1)\frac{\varphi_1(q^{-2}\xi) - \varphi_1(\xi)}{q^{-2}\xi - \xi} \cdot \varphi_2(\xi) - q\xi(\xi - 1)\varphi_1(\xi)\frac{\varphi_2(\xi) - \varphi_2(q^{-2}\xi)}{\xi - q^{-2}\xi} + \\
+ q^{-1}\xi(q^{-2}\xi - 1)\varphi_1(\xi)\frac{\varphi_2(q^{-2}\xi) - \varphi_2(\xi)}{q^{-2}\xi - \xi} - q^{-1}\xi(q^{-2}\xi - 1)\varphi_1(\xi)\frac{\varphi_2(q^{-2}\xi) - \varphi_2(\xi)}{q^{-2}\xi - \xi} \right).\]
We did not break the equality since we have added and then subtracted from its right hand side the same expression:
\[\frac{1}{(q^{-1} - q)\xi} q^{-1}\xi(q^{-2}\xi - 1)\varphi_1(\xi)\frac{\varphi_2(q^{-2}\xi) - \varphi_2(\xi)}{q^{-2}\xi - \xi}.\]
Thus we get
\[Dx(q^{-1}x - 1)D\Phi\bigg|_{x=\xi} = \frac{q^{-1}}{(q^{-1} - q)\xi} \cdot W(\varphi_1, \varphi_2) + \lambda \cdot \Phi(\xi, \xi).\]
In the case \(x \neq \xi\) the statement of the lemma is evident.

**Corollary 4.8**
1) For \(Re l > -\frac{1}{2}\), \(W(\psi_1, f_l) \neq 0, c_1(l) = \frac{1}{W(\psi_1, f_l)}\).
2) For \(Re l < -\frac{1}{2}\), \(W(\psi_{-1-l}, f_l) \neq 0, c_2(l) = \frac{1}{W(\psi_{-1-l}, f_l)}\).
Find $W(\psi_l, f_l)$, $W(\psi_{-1-l}, f_l)$ as functions of an indeterminate $l$. Remind the notation (see [4, section 6]):

$$c(l) = \frac{\Gamma q^2(2l + 1)}{(\Gamma q^2(l + 1))^2} = \frac{(q^{2(l+1)}; q^2)^2_\infty}{(q^{2(2l+1)}; q^2)^2_\infty(q^2; q^2)^2_\infty}. \quad (4.5)$$

**Lemma 4.9** For all $l \notin \frac{1}{2} + \mathbb{Z}$, $f_l(x) = c(l)\psi_l(x) + c(-1 - l)\psi_{-1-l}(x)$.

**Proof.** Consider the functions $f_l, \psi_l, \psi_{-1-l}$ holomorphic in the domain $l \notin \frac{1}{2} + \mathbb{Z}$. Evidently, $\{\psi_l, \psi_{-1-l}\}$ form the base in the vector space of solutions for the equation $Dx(q^{-1}x - 1)D\psi = \lambda(l)\psi$ in the space of functions on $q^{-2\mathbb{Z}}$. Hence $f_l(x) = a(l)\psi_l(x) + b(l)\psi_{-1-l}(x)$, with $a(l), b(l)$ being holomorphic in the domain $l \notin \frac{1}{2} + \mathbb{Z}$. Let $x \in q^{-2\mathbb{Z}}$ go to infinity. By a virtue of [6, lemma 5.4], $a(l) = c(l)$ for $\text{Re}l > -\frac{1}{2}$, and $b(l) = c(-1 - l)$ for $\text{Re}l < -\frac{1}{2}$. What remains is to apply the holomorphy of $a(l), b(l), c(l), c(-1 - l)$ in the domain $l \notin \frac{1}{2} + \mathbb{Z}$. \hfill \Box

**Lemma 4.10** $W(\psi_l, \psi_{-1-l}) = [2l + 1]_q$.

**Proof.** With $x \in q^{-2\mathbb{Z}}, x \to +\infty$, one has

$$\psi_l(x) \sim x^l, \quad \frac{\psi_l(q^{-2}x) - \psi_l(x)}{q^{-2}x - x} \sim \frac{q^{-2l} - 1}{q^2 - 1} x^{l-1}.$$ 

Hence by lemma 4.6,

$$W(\psi_l, \psi_{-1-l}) = \lim_{x \to +\infty, x \in q^{-2\mathbb{Z}}} x(1 - q^{-2}x) \left( \frac{q^{-2l} - 1}{q^{-2} - 1} - \frac{q^{-2(-1-l)} - 1}{q^{-2} - 1} \right) x^{-2}. \quad \Box$$

Lemmas 4.9, 4.10 and corollary 4.8 imply

**Proposition 4.11** The constants in (4.3) and (4.4) are given by

$$c_1(l) = \frac{1}{c(-1 - l)[2l + 1]_q}, \quad c_2(l) = -\frac{1}{c(l)[2l + 1]_q}, \quad (4.6)$$

with $c(l)$ being the $q$-analogue of Harish-Chandra’s $c$-function determined by (4.3).

The conclusion is as follows. For $\text{Re} l \neq -\frac{1}{2}$ the operator $\square^{(0)} - \lambda(l)I$ in the Hilbert space $L^2(d\nu)_q^{(0)}$ has a bounded inverse operator given by

$$((\square^{(0)} - \lambda(l)I)^{-1}\psi)(x) = \int G(x, \xi, l)\psi(\xi) d\nu_2 \xi, \quad \psi \in L^2(d\nu)_q^{(0)}.$$

The Green function is given by the explicit formulae (4.3), (4.4), (4.6).

Find the spectral projections of $\square^{(0)}$.

The following well known result follows from the Stieltjes inversion formula (see [3]).
Proposition 4.12 Let $A$ be a bounded selfadjoint operator with simple purely continuous spectrum. For any interval $(a_1, a_2)$ on the real axis, one has

$$E((a,b)) = \lim_{\epsilon \to +0} \frac{1}{2\pi i} \int_{a_1}^{a_2} (R_{\lambda-i\epsilon} - R_{\lambda+i\epsilon}) d\lambda,$$

with $R_{\lambda} = (A-\lambda I)^{-1}$.

Remark 4.13. There is an extension of proposition 4.12 to the case of an arbitrary selfadjoint operator (see [2, chapter 10, section 6]).

Proposition 4.14 Let $x, \xi \in q^{-2Z_+}$, $\text{Re} \ l = -\frac{1}{2}$. Then

$$\lim_{\epsilon \to +0} \left( G(x, \xi, l + \epsilon) - G(x, \xi, l - \epsilon) \right) = \frac{f_1(\xi)f_1(x)}{c(l)c(-1-l)[2l+1]_q}. \quad (4.7)$$

Proof. In the case $x \leq \xi$ one has due to (4.3), (4.4), (4.6):

$$\lim_{\epsilon \to +0} \left( G(x, \xi, l + \epsilon) - G(x, \xi, l - \epsilon) \right) = \frac{1}{c(l)c(-1-l)[2l+1]_q} \left( \frac{1}{c(l)}\psi_{1-l}(\xi) + \frac{1}{c(-1-l)}\psi(\xi) \right) f_1(x). \quad (4.8)$$

Now (4.7) follows from (4.8) and lemma 4.9. The case $x \geq \xi$ is completely similar to the case $x \leq \xi$. \qed

Remark 4.15. There is a natural generalization of proposition 4.14. Let $\text{Re} \ l = -\frac{1}{2}$ and let $\gamma(\epsilon)$ be such a parametric smooth curve on the complex plane that $\gamma(0) = l$, $\frac{d\gamma(0)}{d\epsilon} > 0$. Then

$$\lim_{\epsilon \to +0} \left( G(x, \xi, \gamma(-\epsilon)) - G(x, \xi, \gamma(\epsilon)) \right) = \frac{f_1(\xi)f_1(x)}{c(l)c(-1-l)[2l+1]_q}. \quad (4.9)$$

Remind that the spectrum of $\Box^{(0)}$ is simple, purely continuous and fills a segment. This segment was parametrized as follows:

$$\lambda(l) = -\frac{(1 - q^{-2l})(1 - q^{2l+2})}{(1 - q^2)^2}, \quad l = -\frac{1}{2} + i\rho, \ 0 \leq \rho \leq \frac{\pi}{h}.$$

Here, as before, $h = -2\ln q$. Note that

$$\frac{d\lambda}{dl} = \frac{1}{(1 - q^2)^2} d(q^{-2l} + q^{2l+2}) = \frac{h}{(1 - q^2)^2} (q^{-2l} - q^{2l+2}). \quad (4.10)$$

Apply proposition 4.12 to $\Box^{(0)}$. An application of (4.9), (4.10) yields the main result of this section, which was kindly communicated to the authors by L. I. Korogodsky.

Associate to each finitely supported function $f(x)$ on $q^{-2Z_+}$ the function

$$\hat{f}(\rho) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi_2 \left[ x, q^{-2l}, q^{2(l+1)}; q^2; q^2 \right] f(x) d^2 x.$$
on the segment $[0, \frac{\pi}{h}]$. Here $l = -\frac{1}{2} + i\rho$, $h = -2\ln q$.

**Example 4.16.** Let $f_0(x) = \begin{cases} 1, & x = 1 \\ 0, & x \neq 1 \end{cases}$. Then

\[ \hat{f}_0(\rho) = 1 - q^2. \]

(4.11)

Remind a well known result of operator theory ([1]):

**Proposition 4.17** Let $A$ be a bounded selfadjoint operator with simple spectrum in a Hilbert space $H$, $E_t$ the spectral measure of $A$, and $g$ such a vector that the linear span of $\{A^m g\}_{m \in \mathbb{Z}}$ is dense in $H$. With $\sigma(t) = (E_t g, g)$, the map

\[ f(t) \mapsto \int_{-\infty}^{\infty} f(t) dE_t g \]

is a unitary operator from $L^2_{\sigma}(\mathbb{R})$ onto $H$. This unitary map sets up the equivalence of $A$ and the multiplication operator $f(t) \mapsto tf(t)$ in $L^2_{\sigma}(\mathbb{R})$.

Now one can prove the following

**Proposition 4.18** Consider a Borel measure

\[ d\sigma(\rho) = \frac{1}{2\pi} \frac{h}{1 - q^2} \cdot \frac{d\rho}{c(-\frac{1}{2} + i\rho)c(-\frac{1}{2} - i\rho)} \]

(4.12)

on the segment $[0, \frac{\pi}{h}]$. The linear operator $f \mapsto \hat{f}$ is extendable by a continuity up to a unitary operator $u : L^2(d\nu)_{q}^{(0)} \to L^2(d\sigma)$. For all $f \in L^2(d\nu)_{q}^{(0)}$,

\[ u \cdot \Box^{(0)} f = \lambda(l)uf. \]

To conclude, note that the measure $dm(l)$ could be derived from the measure $d\sigma(\rho)$ via the substitution $l = -\frac{1}{2} + i\rho$.

### 5 Fourier transform

In [7, section 5] a unitary operator

\[ \mathbb{T} : L^2(d\nu)_q \to \bigoplus_{\mathbb{L}_0} \int_{\mathbb{L}_0} \mathbb{V}^{(l)} dm(l) \]

(5.1)

was constructed, with $\mathbb{V}^{(l)}$ being a completion of the $U_q \mathfrak{su}(1,1)$-module $V^{(l)}$, equipped with an invariant scalar product. By the results of the previous section,

\[ \bigoplus_{\mathbb{L}_0} \int_{\mathbb{L}_0} \mathbb{V}^{(l)} dm(l) \simeq \bigoplus_{0}^{\pi/h} \int_{0}^{\pi/h} \mathbb{V}^{(-\frac{1}{2} + i\rho)} d\sigma(\rho), \]
with \(d\sigma\) being the measure (4.12), and the modules \(V^{(-\frac{1}{2}+i\rho)}\) could be replaced by the isomorphic modules \(\mathbb{C}[z]_{q,-\frac{1}{2}+i\rho}\). The linear operator \(\mathbf{7}\) is replaced by a completion in \(L^2(d\nu)_q\) of a morphism of \(U_q\mathfrak{sl}_2\)-modules given by \(i\mathbf{f}_0 = 1 - q^2\). (This relation follows from (4.11); it determines unambiguously a morphism of \(U_q\mathfrak{sl}_2\)-modules by [4, proposition 3.9]).

Remind the notation (see [3]):

\[
P^t_l = (q^2 z^* \zeta^*; q^2)^{-1} \cdot (z\zeta^*; q^2)^{-1}(1 - \zeta\zeta^*)^l \in D(\Xi \times X)_q.
\]

**Proposition 5.1** For all \(f \in D(U)_q\),

\[
i f = \int_{U_q} P^t_{\frac{1}{2}+i\rho}(z, \zeta) f(\zeta) d\nu
\]

**Proof.** It is easy to show that for all \(\rho \in [0, \frac{\pi}{h}]\) the linear integral operator \(i_\rho : f \mapsto \int_{U_q} P^t_{\frac{1}{2}+i\rho}(z, \zeta) f(\zeta) d\nu\) maps the vector space \(D(U)_q\) into \(\mathbb{C}[[\partial U]]_{q,-\frac{1}{2}+i\rho}\). Now our statement follows from the following two lemmas.

**Lemma 5.2** \(i_\rho \mathbf{f}_0 = 1 - q^2\).

**Proof.** Apply the decomposition

\[
P^t_l = \sum_{j>0} \zeta^j \cdot \psi_j(\xi) + \psi_0(\xi) + \sum_{j>0} \psi_{-j}(\xi) \zeta^j,
\]

described in [3]. It is easy to show that only the term \(\psi_0(\xi)\) contributes to the integral \(i_\rho \mathbf{f}_0\). On the other hand, \(\psi_0(1) = 1\), \(\int_{U_q} 1 \cdot \mathbf{f}_0 d\nu = 1 - q^2\).

**Lemma 5.3** The linear operator \(i_\rho : D(U)_q \to \mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho}\) is a morphism of \(U_q\mathfrak{sl}_2\)-modules.

**Proof.** Consider the integral operator

\[
j_\rho : \mathbb{C}[\partial U]_{q,l} \to D(U)_q', \quad j_\rho : f \mapsto \int_0^{\frac{\pi}{h}} P^t_{\frac{1}{2}-i\rho}(z, e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}.
\]

It is a morphism of \(U_q\mathfrak{sl}_2\)-modules, as it was noted in section 3. Equip the \(U_q\mathfrak{su}(1,1)\)-modules \(D(U)_q, \mathbb{C}[\partial U]_{q,l}\) with invariant scalar products

\[
D(U)_q \times D(U)_q \to \mathbb{C}, \quad f_1 \times f_2 \mapsto \int_{U_q} f_2^* f_1 d\nu,
\]

\[
\mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho} \times \mathbb{C}[\partial U]_{q,-\frac{1}{2}+i\rho} \to \mathbb{C}, \quad f_1 \times f_2 \mapsto \int_{\partial \mathcal{U}} f_2^* f_1 \frac{d\theta}{2\pi}.
\]
It follows from the definitions that the integral operator with a kernel $K = \sum_i k''_i \otimes k'_i$ is conjugate to the integral operator with the kernel $K^t = \sum_i k'_i \otimes k''_i$. Hence $i_\rho = j^*_\rho$, and $i_\rho$ is a morphism of $U_q\mathfrak{sl}_2$-modules since this is the property of $j_\rho$ (see [7, section 5]). □

It follows from the proof of lemma 5.3 that $\overline{j} = \overline{i}^* \overline{*}$ is the integral operator

$$\overline{j} : f(e^{i\theta}, \zeta) \mapsto \int_0^{\pi/2} \int_0^{2\pi} P_{\frac{\pi}{2} - i\rho}(z, e^{i\theta}) f(e^{i\theta}, \rho) \frac{d\theta}{2\pi} d\sigma(\rho).$$

Since $\overline{i}$ is unitary (see [7]), $\overline{i} \cdot \overline{j} = \overline{j} \cdot \overline{i} = 1$. Hence $\overline{i}, \overline{j}$ coincide with the operators $F, F^{-1}$ introduced in [6], respectively. This implies the statement of [6, proposition 6.1].

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