The Structure of Configurations in One-Dimensional Majority Cellular Automata: From Cell Stability to Configuration Periodicity

Yonatan Nakar  
Tel-Aviv University  
yonatannakar@mail.tau.ac.il

Dana Ron  
Tel-Aviv University  
danaron@tau.ac.il

Abstract

We study the dynamics of (synchronous) one-dimensional cellular automata with cyclical boundary conditions that evolve according to the majority rule with radius $r$. We introduce a notion that we term cell stability with which we express the structure of the possible configurations that could emerge in this setting. Our main finding is that apart from the configurations of the form $(0^{r+1}0^* + 1^{r+1}1^*)^*$, which are always fixed-points, the other configurations that the automata could possibly converge to, which are known to be either fixed-points or 2-cycles, have a particular spatially periodic structure. Namely, each of these configurations is of the form $s^*$ where $s$ consists of $O(r^2)$ consecutive sequences of cells with the same state, each such sequence is of length at most $r$, and the total length of $s$ is $O(r^2)$ as well. We show that an analogous result also holds for the minority rule.
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1 Introduction

Dynamic processes that evolve according to the majority rule arise in various settings and as such have received wide attention in the past, primarily within the context of propagation of information or influence (e.g., [GM13, Pel02, Zeh19]). Here we consider perhaps the most basic case, that of one-dimensional cellular automata, where our focus is on analyzing the structure of the configuration space. Specifically, we analyze the configuration space of one-dimensional cellular automata with cyclical boundary conditions that evolve according to the majority rule with radius $r$.

It is well-known [GO81, PS83] that these processes always converge to configurations that correspond to cycles either of length 1 (fixed-points) or of length 2 (period-2 cycles). In particular, it is easy to verify (see, e.g., [TA04]) that configurations in which each cell belongs to a consecutive sequence of at least $r + 1$ cells with the same state are fixed-points. Not much is currently understood, however, about the structure of the other fixed-point configurations or of configurations that correspond to cycles of length 2.

The reason for this gap in understanding is largely due to the fact that most previous research has made assumptions about the mechanism producing the initial configuration. Namely, it is usually assumed that the state of each cell in the initial configuration is randomly chosen, independently from the other cells. See, for instance, the theoretical analysis in [TA04] and the experimental results in [TR11], both for one-dimensional majority cellular automata (and also the references within Section 1.6 for examples in other models). Under such assumptions, as shown in [TA04], these other configurations are indeed rarely encountered.

In this work, we tackle the problem of understanding the structure of the possible configurations without making assumptions about the mechanism behind the generation of the initial configuration. One of our main results (stated formally in Theorem 1) is that all period-2 configurations and all fixed-point configurations (other than those mentioned above) have a very special structure. Specifically, they have a “spatially” periodic structure with a period that is quadratic in the radius $r$. In the course of the proof of this result, we introduce several notions and prove several claims, which we believe are of interest in their own right as they shed light on the dynamics of the majority rule in cellular automata (and not only on the configurations they converge to).

1.1 The majority rule with radius $r$

In all that follows, when performing operations on cells $i \in \mathbb{Z}_n$, these operations are modulo $n$.

**Definition 1** (cell interval). For a pair of cells $i, j \in \mathbb{Z}_n$ we use $[i, j]$ to denote the sequence $i, i+1, \ldots, j$ (so that it is possible that $j < i$), which we refer to as a cell interval.

For an integer $n$, we refer to a function $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ as a configuration and view $\sigma$ as a (cyclic) binary string of length $n$.

**Definition 2** (neighborhood). For a cell $i \in \mathbb{Z}_n$ and an integer $r$, the $r$-neighborhood of $i$, denoted $\Gamma_r(i)$, is the cell interval $[i - r, i + r]$. For a set of cells $I \subseteq \mathbb{Z}_n$, we let $\Gamma_r(I)$ denote the set of cells in the union of cell intervals $[i - r, i + r]$ taken over all $i \in I$.

Given a state $\beta \in \{0, 1\}$, a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ and a cell interval $[i, j]$, we denote by $\#_\beta(\sigma[i, j])$ the number of cells $\ell \in [i, j]$ such that $\sigma(\ell) = \beta$.

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1In this work, a state is a value in $\{0, 1\}$.
Definition 3 (the majority rule). Denote by $\text{MAJ}_r$ the majority rule with radius $r$. That is, for a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$, $\text{MAJ}_r(\sigma)$ is the configuration $\sigma'$ in which for each cell $i \in \mathbb{Z}_n$,

$$
\sigma'(i) = \begin{cases} 
0 & \text{if } \#_0(\sigma[\Gamma_r(i)]) > \#_1(\sigma[\Gamma_r(i)]) \\
1 & \text{otherwise}
\end{cases}
$$

For each $t \geq 0$, denote by $\text{MAJ}_t^r(\sigma)$ the result of repeatedly applying the majority rule with radius $r$, starting from the configuration $\sigma$. In particular, $\text{MAJ}_0^r(\sigma) = \sigma$ and $\text{MAJ}_1^r(\sigma) = \text{MAJ}_r(\sigma)$.

1.2 Temporal and spatial periodicity

Eventually, for every initial configuration, the majority rule, and, in fact, any rule, reaches a cycle: a periodic sequence of configurations. As mentioned earlier, in the case of the majority rule, that cycle is always either a 2-cycle or a fixed-point.

Definition 4 (fixed-point). We say that a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ is a fixed-point if $\text{MAJ}_r(\sigma) = \sigma$.

Definition 5 (2-cycle). We say that a pair of distinct configurations $\sigma, \sigma' : \mathbb{Z}_n \rightarrow \{0, 1\}$ is a 2-cycle if $\text{MAJ}_r(\sigma) = \sigma'$ and $\text{MAJ}_r(\sigma') = \sigma$.

We refer to the configurations that constitute a cycle as temporally periodic configurations. That is,

Definition 6 (temporally periodic). We say that a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ is temporally periodic if $\text{MAJ}_2^r(\sigma) = \sigma$.

Note that if a configuration $\sigma$ is temporally periodic, then it is either the case that $\text{MAJ}_r(\sigma) = \sigma$ (i.e., $\sigma$ is a fixed-point), or $\text{MAJ}_r(\sigma) = \sigma'$ for $\sigma' \neq \sigma$, in which case $\sigma$ and $\sigma'$ constitute a 2-cycle.

Definition 7 (transient). If a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ is not temporally periodic, we say that $\sigma$ is transient.

Definitions 4-7 are all related to the notion of temporal periodicity, i.e., periodicity that occurs over time. In this paper, we relate temporal periodicity to spatial periodicity, i.e., periodic behavior exhibited within individual configurations. Formally,

Definition 8 (spatial period). We say that a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ has spatial period $p$ if $p$ is the minimum positive integer such that for every cell $i \in \mathbb{Z}_n$, $\sigma(i + p) = \sigma(i)$.

Definition 9 (spatially periodic). We say that a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ is spatially periodic if its spatial period $p$ satisfies $p < n$.

1.3 Our main result and the notion of cell stability

In this section we state our main result, Theorem 1. In order to state Theorem 1, we introduce the notion of a cell’s stability within a configuration via Definitions 10-12 (illustrated in Figure 1.1).

Definition 10 (unstable). We say that a cell $i \in \mathbb{Z}_n$ is unstable with respect to a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$ if $\sigma(i) \neq \sigma''(i)$ where $\sigma'' = \text{MAJ}_r^2(\sigma)$.
Recall that after a finite number of steps\(^2\), a one-dimensional cellular automaton that evolves according to the majority rule, reaches either a fixed-point or a 2-cycle. Thus, a configuration \(\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}\) is transient if and only if it contains unstable cells.

As for the “stable” cells, we define two variants: strongly stable and weakly stable.

**Definition 11** (strongly stable). We say that a cell \(i \in \mathbb{Z}_n\) is strongly stable with respect to a configuration \(\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}\) if there exists a cell interval \([a, b]\) of length at least \(r + 1\) such that \(i \in [a, b]\) and for each \(j \in [a, b]\), \(\sigma(i) = \sigma(j)\).

**Definition 12** (weakly stable). We say that a cell \(i \in \mathbb{Z}_n\) is weakly stable with respect to a configuration \(\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}\) if \(i\) is not strongly stable with respect to \(\sigma\), but \(\sigma(i) = \sigma''(i)\) where \(\sigma'' = \text{MAJ}_2^r(\sigma)\).

Figure 1.1: The evolution under the majority rule with \(r = 2\). Gray squares correspond to state-0 cells and dark squares correspond to state-1 cells. Each cell is labeled by a letter indicating the cell’s stability, where \(S\) stands for Strongly stable, \(W\) for Weakly stable and \(U\) for Unstable.

The crucial property of the strongly stable cells is that their states, unlike the states of the weakly stable cells, cannot change in later configurations. In that sense, their stability is “stronger” than that of the weakly stable cells. It is worth noting, though, that if a cell lies within a long cell interval of weakly stable cells, then that cell remains weakly stable, alternating between the same pair of states, for a number of steps that depends on the cell interval length.

Accordingly, given a configuration \(\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}\), we say that a cell interval \([i, j]\) is strongly stable, weakly stable or unstable if all the cells in that cell interval are, respectively, strongly stable, weakly stable or unstable.

Considering complete configurations, observe that all the configurations of the form \((0^{r+1}0^* + 1^{r+1}1^*)^*\) contain only strongly stable cells. As noted previously and explained in the characterization provided in [TA04], these configurations are always fixed-points, which means that they are, in particular, also temporally periodic (with a period of 1). However, there are more forms of temporally periodic configurations, both period-1 and period-2, that contain only weakly stable cells and are not addressed by [TA04]’s characterization, as the authors of [TA04] were only interested in “typical” configurations, which are not of that kind.\(^3\)

**Theorem 1** complements [TA04]’s characterization by additionally specifying the structure of the remaining temporally periodic configurations. In addition to temporally periodic configurations, **Theorem 1** also includes a property of the transient configurations that is related to the dynamics by which they eventually converge.

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\(^2\)Which is shown in Section 3 to be at most linear in \(n\).

\(^3\)Indeed, it is shown in [TA04] that the probability that a randomly selected configuration of length \(n\) being transient approaches 1 as \(n \to \infty\). As such, the additional temporally periodic configurations that we address in this work are, in a sense, not “typical”. We, in contrast to [TA04], make no assumption about the distribution of the configuration space, and are therefore interested in understanding the structure of all configurations, not only the “typical” ones.
Theorem 1. For any configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$, exactly one of the following must hold:

1. The configuration $\sigma$ is a temporally periodic configuration and it is either the case that:
   
   (a) all the cells in $\sigma$ are strongly stable, in which case $\sigma$ is of the form $(0^{r+1}0^*+1^{r+1}1^*)^*$, or
   
   (b) all the cells in $\sigma$ are weakly stable, in which case $\sigma$ is spatially periodic with spatial period at most $2r(r+1)$.

2. The configuration $\sigma$ is a transient configuration and the length of every unstable cell interval in $\sigma$ is at most $2r$.

Under the assumption that $r$ is a constant, Theorem 1 directly yields an output-sensitive algorithm that, given $n$, generates all the temporally periodic configurations of length $n$. The running-time of the algorithm is linear in the number of temporally periodic configurations.

Turning to transient configurations, recall that all transient configurations contain unstable cells, and the evolution of the transient configurations can be described using the notion of cell stability. Namely, the following is shown (in Section 3) regarding any transient configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$. First, the configuration $\text{MAJ}_r(\sigma)$ contains strictly fewer unstable cells than $\sigma$. Second, if $\sigma$ contains strongly stable cells, then $\text{MAJ}_r(\sigma)$ contains even more strongly stable cells than $\sigma$, and the automaton eventually converges to a fixed-point of the form defined in Case (1a). Third, if there are no strongly stable cells in $\sigma$, then there are cases in which the automaton eventually converges to a fixed-point of the form defined in Case (1a)\textsuperscript{4} and there are also cases in which it eventually converges to a fixed-point or to a 2-cycle of the form defined in Case (1b)\textsuperscript{5}.

1.4 Illustrating Theorem 1 for $r = 1, 2, 3$

To get a feel for the nature of the statement in Theorem 1, we demonstrate some of its aspects for $r = 1, 2, 3$.

1. For $r = 1$, the temporally periodic configurations are either

   (a) of the form $(000^*+111^*)^*$, or

   (b) of the form $(01)^*$.\textsuperscript{6}

2. For $r = 2$, the temporally periodic configurations are either

   (a) of the form $(000^*+1111^*)^*$, or

   (b) of one of the following forms: $(01)^*$, $(0011)^*$, $(001101)^*$, $(001011)^*$.

3. For $r = 3$, the temporally periodic configurations are either

   (a) of the form $(00000^*+11111^*)^*$, or

\textsuperscript{4}e.g., for $r = 3$, the transient configuration 001001001001001001 converges after one step to the fixed-point configuration $(0)^*$.

\textsuperscript{5}e.g., for $r = 4$, the transient configuration 0010110010110010110010110010111 converges after one step to the 2-cycle consisting of $(111000)^6$ and $(000111)^6$.

\textsuperscript{6}Also $(10)^*$, but since the configurations are cyclic, the patterns $(01)^*$ and $(10)^*$ correspond to equivalent sets of configurations.
(b) of the form \(s^*,\) where \(s\) belongs to the set:\(^7\)

\[
\{01, \\
0011, \\
010011, 010110, 001110, \\
01011001, 10100101, 10100110, 01011100, 10010011, 00011101, 10110001, \\
0011001110, 1000111001 \}
\]

1.5 Minority

An analog of Theorem 1 holds for the minority rule as well, with exactly the same variants of cell stability as those of the majority rule. In particular, Definitions 10-12 can be used verbatim to describe the evolution according to the minority rule, with the only difference being the temporal period of the weakly stable and the strongly stable cells.

Namely, in the minority rule, the strongly stable cells have a temporal period of 2 instead of 1, implying that the temporally periodic configurations of the form defined in Case (1a) of Theorem 1, rather than being fixed-points as in the majority rule, become the constituents of 2-cycles. Likewise, for every configuration, the weakly stable cells that would have had a temporal period of 1 under the majority rule, have a temporal period of 2 under the minority rule, and vice versa. This implies that the temporally periodic configurations of the form defined in Case (1b) of Theorem 1, while generated by exactly the same patterns, and hence having precisely the same form, have the opposite temporal period to that they would have had under the majority rule: the fixed-points become 2-cycles and the 2-cycles become fixed-points.

See illustrations in Figures 1.2 and 1.3.

1.6 Related work

The main focus of most of the research on majority/minority (and more generally, threshold) cellular automata so far has been on the convergence time (e.g., \([FGW83, FKW13, PW19]\)) and on the dominance problem\(^8\) (e.g., \([BBDCM12, FKR+03, MPGP17]\)).

As mentioned earlier, most of the work on the problem of understanding the structure of the configuration space is based on the assumption that the initial configuration is random. For the one-dimensional case, the case with which the current paper is concerned, this includes the paper of Tosic and Agha \([TA04]\). In their paper, they distinguish between synchronous/sequential and finite/infinite majority cellular automata with radius \(r\), and our work can be viewed as extending their result for the finite and synchronous case.

They show that whereas 2-cycles cannot emerge under the sequential model, in the synchronous model (the one we focus on in this paper), 2-cycles exist even for \(r = 1\). They also show that a randomly picked configuration is a transient configuration (and, in particular, not a 2-cycle) with probability approaching 1 (both for finite and infinite configurations), and it can additionally be shown that the probability that such a random transient configuration eventually converges to a 2-cycle approaches 0. Finally, they characterize the “common” forms of fixed-point configurations (those that in our paper are described in Case (1a) of Theorem 1).

\(^7\) The string \(s\) could also be the mirror or the complement of any of the specified patterns, which we omit for the sake of conciseness. For example, since we explicitly specified that \(s\) could be 010011, it means that \(s\) could also be 110010 (which is the mirror of 010011) or 101100 (which is the complement of 010011), even though these two are not explicitly specified.

\(^8\) In the dominance problem, one asks how many cells must initially be at a certain state so that eventually all cells have the same state.
Figures 1.2 and 1.3 depict the evolution under the majority rule (Figure 1.2) and under the minority rule (Figure 1.3) starting from the same initial configuration. Gray squares correspond to state-0 cells and dark squares correspond to state-1 cells. Each cell is labeled by a letter indicating the cell’s stability, where S stands for Strongly stable, W for Weakly stable and U for Unstable. While the two trajectories appear quite different, they are completely equivalent when viewed under the “stability mask”. In particular, cell $i$ at time $t$ has the same stability label in both figures for every $t, i$ pair. Yet, each pair of corresponding stable cells in the two figures have opposite periods. Thus, the fixed-point configuration to which the majority rule in Figure 1.2 converges to corresponds to a 2-cycle in Figure 1.3.

Their theoretical result is supplemented by a later experimental work [TR11], showing that in practice, convergence to these “common” fixed-point configurations occurs relatively quickly. Namely, the simulations in [TR11] demonstrate that convergence tends to occur in less than five steps for $n = 1000$ and $1 \leq r \leq 5$.

Additional work beyond the one-dimensional case includes [GZ21] for two-dimensional majority cellular automata, [GZ18] for majority in random regular graphs, [Zeh20] for majority in Erdős–Rényi graphs as well as expander graphs.

One notable work that does not rely on the assumption that the initial configuration is random is Turau’s work [Tur22] on characterizing all the temporally periodic configurations for majority and minority processes on trees. The characterization presented in [Tur22] also yields an output-sensitive algorithm for generating these configurations.

1.7 Some high-level ideas

As mentioned previously, in proving Theorem 1, we define a number of notions and establish several claims, some of which we believe are valuable in and of themselves. In this section we have chosen to highlight the high-level idea behind one of the key tools we utilize, which is a
mapping we introduce between maximal homogeneous blocks of consecutive configurations.

Given a configuration $\sigma: \mathbb{Z}_n \rightarrow \{0,1\}$, we say that a cell interval $[i,j]$ is a maximal homogeneous block in $\sigma$ with value $\beta \in \{0,1\}$ if for every cell $\ell \in [i,j]$, $\sigma(\ell) = \beta$, and also $\sigma(i-1) = \sigma(j+1) \neq \beta$ if the length of $[i,j]$ is less than $n$.

We refer to this mapping, defined below (and illustrated in Figure 1.5), as the alignment mapping. The alignment mapping, beyond being essential for the proof of Theorem 1, has several features that make it useful for reasoning about the dynamics of the majority rule, which is why we present its definition here.

**Definition 13** (alignment mapping). Let $\sigma$ and $\sigma'$ be a pair of configurations satisfying $\text{MAJ}_r(\sigma) = \sigma'$. Given a maximal homogeneous block $[i',j']$ in $\sigma'$, let $I$ be the maximal homogeneous block in $\sigma$ that contains the cell $i + r$ and let $J$ be the maximal homogeneous block in $\sigma$ that contains the cell $j - r$. The **alignment mapping** maps the maximal homogeneous block $[i',j']$ (in $\sigma'$) to the middle maximal homogeneous block $[i,j]$ between $I$ and $J$ in $\sigma$.

![Figure 1.5: The alignment mapping.](image)

We stress that the alignment mapping, as defined above (as well as in Section 6), is a backward mapping, in the sense that, given a configuration $\sigma'$, it maps all maximal homogeneous blocks in $\sigma'$ into those of the configuration $\sigma$ that precedes $\sigma'$. This naturally suggests defining the notion of the forward alignment mapping as the inverse function of the backward alignment mapping that would map the maximal homogeneous blocks of the configuration $\sigma$ to those of the configuration $\sigma'$ that follows $\sigma$ (for example, in Figure 1.5, the forward alignment mapping maps $[i,j]$ in $\sigma$ to $[i',j']$ in $\sigma'$).

However, while it can be shown that the backward alignment mapping is always one-to-one, it is not necessarily onto (unless we apply it within a pair of temporally periodic configurations). Hence, under our definition of the forward alignment mapping, not all blocks will be mapped forward.

Formally, let $\sigma_0,...,\sigma_m$ be a sequence of configurations where $\text{MAJ}_r(\sigma_{t-1}) = \sigma_t$ for each $1 \leq t \leq m$. We define the step-$t$ forward alignment mapping, denoted $\varphi_t$, as follows. Given a maximal homogeneous block $[i,j]$ in $\sigma_t$, if there is a maximal homogeneous block $[i',j']$ in $\sigma_{t+1}$ such that the backward alignment mapping between the configuration pair $\sigma_t, \sigma_{t+1}$ maps $[i',j']$
Since $\varphi_t([i,j]) = [i',j']$. Otherwise, $\varphi_t([i,j]) = \bot$. In the case in which $\varphi_t([i,j]) \neq \bot$, we also define $\varphi^2_t([i,j]) = \varphi_{t+1}(\varphi_t([i,j]))$.

One notable property of the forward alignment mapping is what we refer to as “identity preservation in stable intervals”. Roughly speaking, consider any maximal homogeneous block $[i,j]$ residing in a sufficiently long weakly stable or strongly stable cell interval of $\sigma_t$. Then $\varphi_t([i,j]) \neq \bot$, and hence $\varphi^2_t([i,j])$ is defined and is equal to the same maximal homogeneous block $[i,j]$ we started with. In particular, for a pair of configurations comprising a 2-cycle, applying the forward alignment mapping twice essentially maps each maximal homogeneous block to itself.

In the proof of Theorem 1, we essentially use the forward alignment mapping and its properties to show that for a configuration in which all maximal homogeneous blocks are of length at most $r$, if the configuration is temporally periodic, then it is also spatially periodic. We achieve this through three steps.

In the first step (Sections 7-8), we employ the alignment mapping to express the length of each of the configuration’s maximal homogeneous blocks in terms of the lengths of other $O(r)$ maximal homogeneous blocks in the preceding configuration. Specifically, given a pair of temporally periodic configurations $\sigma_t$ and $\sigma_{t+1}$, we obtain a relationship between the length of each maximal homogeneous block $[i,j]$ in $\sigma_t$ and the lengths of $O(r)$ consecutive maximal homogeneous blocks, belonging to a block sequence centered at the maximal homogeneous block $\varphi_t([i,j])$, in the configuration $\sigma_{t+1}$.

In the second step (Sections 9-10), we look at the difference between the length of each maximal homogeneous block $[i,j]$ and the lengths of the maximal homogeneous blocks at the two ends of the sequence mentioned above, and define aligned difference vectors, whose entries are these differences. We use the properties of the forward alignment mapping to establish that the aligned difference vectors (defined formally in Section 10) are spatially periodic with a spatial period that is linear in $r$.

In the third and final step (Section 11), by applying the relationship between aligned difference vectors iteratively, we use the spatial periodicity of the aligned difference vectors to establish that the configurations themselves are spatially periodic as well, and that each configuration’s spatial period must be quadratic in $r$.

2 Switch Points

For a value $\beta \in \{0, 1\}$, we denote $\bar{\beta} = 1 - \beta$.

**Definition 14** (switch point). Let $\sigma$ be a configuration. We say that a pair of consecutive cells $i, i+1$ constitute a switch point in $\sigma$ if $\sigma(i) \neq \sigma(i+1)$.

We now prove two properties of switch points, the first of which we refer to as the switch point property.

**Claim 1** (the switch point property). Let $\sigma$ and $\sigma'$ be a pair of configurations satisfying $\text{MAJ}_r(\sigma) = \sigma'$. If a pair of consecutive cells $i, i+1$ constitutes a switch point in $\sigma'$, then $\sigma(i-r) = \sigma'(i) \text{ and } \sigma(i+1+r) = \sigma'(i+1)$.

**Proof.** We only prove that $\sigma(i-r) = \sigma'(i)$, since the proof that $\sigma(i+1+r) = \sigma'(i+1)$ is symmetric.

Suppose $\sigma'(i+1) = \beta$ for some $\beta \in \{0, 1\}$. Since the pair $i, i+1$ constitutes a switch point in $\sigma'$, it must be the case that $\sigma'(i) = \bar{\beta}$. Assume, contrary to the claim, that $\sigma(i-r) = \beta$. Since $\Gamma_r(i+1) = \Gamma_r(i) \cup \{i+1+r\} \setminus \{i-r\}$ and $\sigma(i-r) = \beta$, it must hold that $\#_\beta(\sigma(\Gamma_r(i+1))) \leq \#_\beta(\sigma(\Gamma_r(i)))$.  

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This implies that, by the definition of $\text{MAJ}_r$, since $\sigma'(i+1) = \beta$, it must also hold that $\sigma'(i) = \beta$, and we reach a contradiction.

**Claim 2.** Let $\sigma$ and $\sigma'$ be a pair of configurations where $\text{MAJ}_r(\sigma) = \sigma'$. If a pair of consecutive cells $i, i+1$ constitutes a switch point in $\sigma'$, then

$$\#_0(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) = \#_1(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]).$$

Proof. Assume the contrary, and let $\beta \in \{0, 1\}$ be the majority value in $\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]$. That is,

$$\#_\beta(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) > \#_\beta(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]).$$

Hence, since $|\Gamma_r(i) \cap \Gamma_r(i+1)| = |[i-r+1, i+r]| = 2r,$

$$\#_\beta(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) \geq r+1.$$

Thus, since $\Gamma_r(i) \cap \Gamma_r(i+1) \subseteq \Gamma_r(i)$ and $\Gamma_r(i+1) \cap \Gamma_r(i) \subseteq \Gamma_r(i+1),$

$$\#_\beta(\Gamma_r(i)) \geq \#_\beta(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) \geq r+1$$

and

$$\#_\beta(\Gamma_r(i+1)) \geq \#_\beta(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) \geq r+1.$$

Therefore, by the definition of $\text{MAJ}_r$, $\sigma(i) = \sigma(i+1) = \beta$ and we reach a contradiction to the assumption that the pair $i, i+1$ is a switch point. \qed

The two properties of switch points captured by Claim 1 and Claim 2 together correspond, in fact, to a characterization of switch points, as formalized in Claim 3.

**Claim 3.** Let $\sigma$ and $\sigma'$ be a pair of configurations where $\text{MAJ}_r(\sigma) = \sigma'$. If for a cell $i \in \mathbb{Z}_n$, $\#_0(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) = \#_1(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)])$ and $\sigma(i-r) \neq \sigma(i+1+r)$, then the pair $i, i+1$ constitutes a switch point in $\sigma'$ in which $\sigma'(i) = \sigma(i-r)$ and $\sigma'(i+1) = \sigma(i+1+r)$.

Proof. Let $\beta \in \{0, 1\}$ be the value that $\sigma(i-r) = \beta$ and $\sigma(i+1+r) = \bar{\beta}$. Since $|\Gamma_r(i) \cap \Gamma_r(i+1)| = |[i-r+1, i+r]| = 2r,$

$$\#_\beta(\sigma[\Gamma_r(i) \cap \Gamma_r(i+1)]) = r.$$

Since $\Gamma_r(i) = (\Gamma_r(i) \cap \Gamma_r(i+1)) \cup \{i-r\}$, it must be the case that $\#_\beta(\Gamma_r(i)) = r+1$. Similarly, since $\Gamma_r(i+1) = (\Gamma_r(i) \cap \Gamma_r(i+1)) \cup \{i+1+r\}$, it must be the case that $\#_\beta(\Gamma_r(i+1)) = r+1$. Hence, by the definition of $\text{MAJ}_r$, $\sigma'(i) = \beta, \sigma'(i+1) = \bar{\beta}$ and the pair $i, i+1$ constitutes a switch point in $\sigma'$. \qed

### 3 Temporally periodic configurations

**Definition 15.** Given a configuration $\sigma$ and a value $\beta \in \{0, 1\}$, we denote by $B^\beta(\sigma)$ the set of maximal homogeneous blocks with value $\beta$ in $\sigma$. That is,

$$B^\beta(\sigma) = \{[i, j] : \forall k \in [i, j], \sigma(k) = \beta, \sigma(i-1) = \sigma(j+1) = \bar{\beta}\}.$$

Also, let $B(\sigma) = B^0(\sigma) \cup B^1(\sigma)$. 

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When we refer to an interval $[i, j] \subseteq \mathbb{Z}_n$, we denote its length by $|[i, j]|$. Note that it is not necessarily the case that $|[i, j]| = j - i + 1$ because of the cyclical boundary conditions.

In [GO81], it has been shown that a more general class of cellular automata that includes $\text{MAJ}_r$, always reach a cycle of temporal period 1 or 2. Nevertheless, we provide a proof tailored for our special case, $\text{MAJ}_r$, because it is simpler and shorter than the general proof in [GO81].

**Claim 4.** For every integer $r \geq 1$, the rule $\text{MAJ}_r$ has temporal period 2.

**Proof.** Let $\sigma_0 : \mathbb{Z}_n \to \{0, 1\}$ be any initial configuration, and for any integer $t \geq 0$, let $\sigma_t = \text{MAJ}_r^t(\sigma_0)$. We define a potential function $\phi : \{0, 1\}^n \times \{0, 1\}^n \to \mathbb{Z}$ over pairs of consecutive configurations in the sequence $\{\sigma_t\}_{t=0}^\infty$.

We shall use the following shorthand (where $i + x$ is computed mod $n$):

$$g_t(i) = \sum_{j=i-r}^{i+r} \sigma_t(j),$$  \hspace{1cm} (3.1)

where we observe that if $g_t(i) \geq r + 1$, then $\sigma_{t+1}(i) = 1$, while if $g_t(i) < r + 1$, then $\sigma_{t+1}(i) = 0$.

The potential function is defined as follows.

$$\phi(\sigma_t, \sigma_{t-1}) = \sum_{i=0}^{n-1} (\sigma_t(i) \cdot g_{t-1}(i)) - (r + 1/2) \sum_{i=0}^{n-1} (\sigma_t(i) + \sigma_{t-1}(i)) .$$  \hspace{1cm} (3.2)

Observe that

$$\sum_{i=0}^{n-1} (\sigma_t(i) \cdot g_{t-1}(i)) = \sum_{i=0}^{n-1} (\sigma_{t-1}(i) \cdot g_t(i)).$$  \hspace{1cm} (3.3)

(It is easiest to see this if we think of pairs $\sigma_t(i) = 1$ and $\sigma_{t-1}(j) = 1$ such that $|i - j| \leq r$ as edges.)

Now consider the change in the potential function:

$$\phi(\sigma_{t+1}, \sigma_t) - \phi(\sigma_t, \sigma_{t-1})$$

$$= \sum_{i=0}^{n-1} (\sigma_{t+1}(i) \cdot g_t(i)) - (r + 1/2) \sum_{i=0}^{n-1} (\sigma_{t+1}(i) + \sigma_t(i))$$

$$- \left( \sum_{i=0}^{n-1} (\sigma_t(i) \cdot g_{t-1}(i)) - (r + 1/2) \sum_{i=0}^{n-1} (\sigma_t(i) + \sigma_{t-1}(i)) \right) \hspace{1cm} (3.4)$$

$$= \sum_{i=0}^{n-1} ((\sigma_{t+1}(i) \cdot g_t(i)) - (\sigma_t(i) \cdot g_{t-1}(i))) - (r + 1/2) \sum_{i=0}^{n-1} (\sigma_{t+1}(i) - \sigma_{t-1}(i)) \hspace{1cm} (3.5)$$

$$= \sum_{i=0}^{n-1} (\sigma_{t+1}(i) - \sigma_{t-1}(i)) \cdot g_t(i) - (r + 1/2) \sum_{i=0}^{n-1} (\sigma_{t+1}(i) - \sigma_{t-1}(i)) \hspace{1cm} (3.6)$$

$$= \sum_{i=0}^{n-1} (\sigma_{t+1}(i) - \sigma_{t-1}(i)) \cdot (g_t(i) - (r + 1/2)) \hspace{1cm} (3.7)$$

$$= \sum_{i: \sigma_{t+1}(i) \neq \sigma_{t-1}(i)} |g_t(i) - (r + 1/2)| \hspace{1cm} (3.8)$$

$$\geq \frac{1}{2} |\{i : \sigma_{t+1}(i) \neq \sigma_{t-1}(i)\}| . \hspace{1cm} (3.9)$$

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That is, the value of $\phi$ is increasing at each step by $\frac{1}{2} \left| \left\{ i : \sigma_{t+1}(i) \neq \sigma_t(i) \right\} \right|$, and since $\phi$ is by definition a bounded function, it must be the case that there exists an integer $t^*$ s.t. for every $t > t^*$, $\left\{ i : \sigma_{t+1}(i) \neq \sigma_t(i) \right\} = \emptyset$. Hence, $\text{MAJ}_r$ has temporal period 2.

**Observation 5.** Let $\sigma$ be a configuration and let $[i, j]$ be an interval of cells such that for every cell $\ell \in [i, j]$, $\sigma(\ell) = \beta$ for some $\beta \in \{0, 1\}$. For an integer $t \geq 0$, let $\sigma_t = \text{MAJ}_{r}(\sigma)$. If the length of the interval $[i, j]$ is at least $r + 1$, then for every $t \geq 0$ and for every cell $\ell \in [i, j]$, $\sigma_t(\ell) = \beta$.

Recall from Section 1.3 that, given a configuration $\sigma : \mathbb{Z}_n \rightarrow \{0, 1\}$, we say that a cell interval $[i, j]$ is strongly stable, weakly stable or unstable if all the cells in that cell interval are, respectively, strongly stable, weakly stable or unstable. The definition applies to complete configurations as well. In particular, we make a distinction that partitions the temporally periodic configurations into two classes: strongly stable configurations (configurations of the form $(0^r1^r+1^r1^r)^*)$ and weakly stable configurations (temporally periodic configurations that are not strongly stable). We also define unstable configurations as transient configurations.

**Observation 6.** If a configuration is strongly stable, then it is also a fixed-point.

**Observation 7.** Let $\sigma$ be a configuration. If $B(\sigma)$ contains a block of length at least $r + 1$, then there exists an integer $t \geq 0$ such that $\text{MAJ}_{r}(\sigma)$ is strongly stable.

**Claim 8.** If $\sigma$ is a temporally periodic configuration, then it is either the case that for every block $[i, j] \in B(\sigma)$, $|[i, j]| \leq r$ or that for every block $[i, j] \in B(\sigma)$, $|[i, j]| > r$.

**Proof.** Let $\sigma' = \text{MAJ}_{r}(\sigma)$ and $\sigma'' = \text{MAJ}_{r}(\sigma')$. Suppose by way of contradiction that $B(\sigma)$ contains both a block of length at most $r$ and a block of length at least $r + 1$. Then there must be such a pair of consecutive blocks. Let $[a, b]$ and $[c, d]$ be the these two consecutive blocks where $[a, b]$ is the block whose length is at most $r$ and $[c, d]$ is the block whose length is at least $r + 1$. Since the two blocks are adjacent, it is either the case that $c = b + 1$ or that $a = d + 1$. Suppose without loss of generality that $c = b + 1$. Let $\beta$ be the value of the block $[a, b]$. Since the block $[c, d]$ is adjacent to the block $[a, b]$, the value of $[c, d]$ must be $\beta$.

Since the cell $b$ belongs to $[a, b]$, it must be the case that $\sigma(b) = \beta$. However, for each cell $i \in [b + 1, b + r]$, since $|[c, d]| > r$, it must be the case that $\sigma(i) = \beta$. Also, since $|[a, b]| \leq r$ and $a - 1 \in [b - r, b]$, it must be the case that $a - 1 \in \Gamma_r(b)$, so $\{a - 1\} \cup [b + 1, b + r] \in \Gamma_r(b)$. As $\sigma(a - 1) = \beta$, it is the case that $\#_{\beta}(\Gamma_r(b)) \geq r + 1$. Thus, $\sigma'(b) = \beta$.

Since $|[c, d]| \geq r + 1$ and for every cell $\ell \in [c, d]$, $\sigma(\ell) = \beta$, by Observation 5, it is also the case that for every cell $\ell \in [c, d]$, $\sigma'(\ell) = \beta$ as well. Now, since $|[b, d]| \geq r + 1$ as well, again, by Observation 5, it must hold that for every cell $\ell \in [b, d]$, $\sigma''(\ell) = \beta$ too. In particular, $\sigma''(b) = \beta \neq \sigma(b)$, and so $\sigma$ cannot be temporally periodic and we reach a contradiction.

**Corollary 2.** If a configuration $\sigma$ is weakly stable, then for each block $[i, j] \in B(\sigma)$, $|[i, j]| \leq r$.

**Definition 16** (temporally periodic configuration pair). We say that a pair of configurations $\sigma, \sigma'$ is a temporally periodic configuration pair if $\text{MAJ}_{r}(\sigma) = \sigma'$ and $\text{MAJ}_{r}(\sigma') = \text{MAJ}_{r}(\sigma)$.

**Claim 9.** Let $\sigma, \sigma'$ be a temporally periodic configuration pair. For every $[i, j] \in B(\sigma')$, each of the intervals $[i - r - 1, j - r]$ and $[i + r, j + r + 1]$ contains exactly one switch point in $\sigma$.

**Proof.** We prove the claim for the interval $[i - r - 1, j - r]$, as the proof for the interval $[i + r, j + r + 1]$ is analogous. Let $\beta \in \{0, 1\}$ be the value such that $[i, j] \in B^3(\sigma')$. Since $[i, j] \in B^3(\sigma')$, each of the pairs $(i - 1, i)$ and $(j, j + 1)$ is a switch point in $\sigma'$. Since the pair $\sigma, \sigma'$ is a temporally
periodic configuration pair, $\text{MAJ}_r(\sigma') = \sigma$. Hence, by the switch point property, it must be the case that \(\sigma(i - r - 1) = \bar{\beta}\) and \(\sigma(j - r) = \beta\).

Therefore, there must be a cell \(\ell \in [i - r - 1, j - r]\) such that \(\sigma(\ell) = \bar{\beta}\) and \(\sigma(\ell + 1) = \beta\). That is, the pair \(\ell, \ell + 1\) is a switch point in \(\sigma\) in the interval \([i - r - 1, j - r]\).

It is left to show that the pair \(\ell, \ell + 1\) is the only switch point in the interval \([i - r - 1, j - r]\) in \(\sigma\). Suppose by way of contradiction that this is not the case. Let \(\ell', \ell' + 1\) be the switch point in \(\sigma\) in \([i - r - 1, j - r]\) that is closest to \(\ell\). Since \(\ell, \ell + 1\) is a switch point of values \(\bar{\beta}, \beta\), it must be the case that \(\ell', \ell' + 1\) is a switch point of values \(\beta, \bar{\beta}\). That is, \(\sigma(\ell' + 1) = \bar{\beta}\). Since the pair \(\sigma, \sigma'\) is a temporally periodic configuration pair, \(\text{MAJ}_r(\sigma) = \sigma'\). Hence, by the switch point property, it must be the case that \(\sigma'(\ell' + 1 + r) = \bar{\beta}\). However, since \(\ell' \in [i - r - 1, j - r]\), it follows that \(\ell' + 1 \in [i - r, j - r]\), so \(\ell' + 1 + r \in [i, j]\). That is, the conclusion that \(\sigma'(\ell' + 1 + r) = \bar{\beta}\) is in contradiction to the assumption that \([i, j] \in B^3(\sigma')\).

\[\Box\]

4 On the lengths of maximal homogeneous blocks

**Definition 17** (balanced). We say that a configuration \(\sigma\) is balanced if \(\#_0(\sigma) = \#_1(\sigma)\). Similarly, we say that an interval \([i, j]\) is balanced in the configuration \(\sigma\) if \(\#_0(\sigma[i, j]) = \#_1(\sigma[i, j])\).

**Definition 18** (bias). Given a configuration \(\sigma\), we define its bias as \(\#_0(\sigma) - \#_1(\sigma)\). Similarly, given an interval \([i, j]\), we define its bias in \(\sigma\) as \(\#_0(\sigma[i, j]) - \#_1(\sigma[i, j])\).

That is, a balanced configuration is a configuration whose bias is 0. Similarly, an interval is balanced in a configuration if its bias is 0 in that configuration.

**Claim 10.** Let \(\sigma\) and \(\sigma'\) be a pair of configurations where \(\text{MAJ}_r(\sigma) = \sigma'\) and let \([i, j] \in B^3(\sigma')\) for some \(\beta \in \{0, 1\}\). If \(|[i, j]| \leq 2r + 1\), then:

\[|[i, j]| = \#_\beta(\sigma[i - r, j + r]) - \#_{\bar{\beta}}(\sigma[j - r, i + r]).\]

**Proof.** By the definition of \(B^3\), the interval \([i, j]\) is a maximal homogeneous block in \(\sigma'\). Also, \(|[i, j]| \leq 2r + 1 < n\). Hence, the pair \((i - 1, i)\) is a switch point in \(\sigma'\). Therefore, by Claim 2, the interval \([i - r, i + r - 1]\) is balanced in \(\sigma\). Similarly, since the pair \((j, j + 1)\) is also a switch point in \(\sigma'\), again, by Claim 2, the interval \([j - r + 1, j + r]\) is balanced in \(\sigma\).

We claim that the intervals \([i - r, j - r]\) and \([i + r, j + r]\) have the same bias in \(\sigma\). To see why, first observe that

\[\begin{align*}
[i - r, j - r] &= [i - r, i + r - 1] \setminus [j - r + 1, i + r - 1], \\
[i + r, j + r] &= [j - r + 1, j + r] \setminus [j - r + 1, i + r - 1].
\end{align*}\]

Since \(|[i - r, i + r - 1]| = |[j - r + 1, j + r]| = 2r\) and both \([i - r, i + r - 1]\) and \([j - r + 1, j + r]\) are balanced in \(\sigma\), by equations (4.1) and (4.2), the intervals \([i - r, j - r]\) and \([i + r, j + r]\) must have the same bias in \(\sigma\). The reason is that each of \([i - r, j - r]\) and \([i + r, j + r]\) equal the difference between a balanced interval of length \(2r\) and the common sub-interval \([j - r + 1, i + r - 1]\).

Hence, since \(|[i - r, j - r]| = |[i + r, j + r]|\),

\[\#_0(\sigma[i + r, j + r]) = \#_0(\sigma[i - r, j - r])\] and
\[\#_1(\sigma[i + r, j + r]) = \#_1(\sigma[i - r, j - r]).\]

Since \([j - r + 1, j + r] = [j - r + 1, i + r - 1] \cup [i + r, j + r]\) and \([j - r + 1, j + r]\) is balanced, it must be the case that
Hence,

\[
\#_\beta(\sigma[i + r, j + r]) = 
\#_\beta(\sigma[j - r + 1, i + r - 1]) + \#_\beta(\sigma[i + r, j + r]) - \#_\beta(\sigma[j - r + 1, i + r - 1]).
\]

We now express the length of \([i,j]\).

\[
|[i,j]| = |[i-r,j-r]| = \#_0(\sigma[i-r,j-r]) + \#_1(\sigma[i-r,j-r])
= \#_\beta(\sigma[i-r,j-r]) + \#_\beta(\sigma[i+r,j+r])
= \#_\beta(\sigma[i-r,j-r])
\]

\[
+ \#_\beta(\sigma[j-r+1,i+r-1])
\]

\[
+ \#_\beta(\sigma[i+r,j+r])
\]

\[
- \#_\bar{\beta}(\sigma[j-r+1,i+r-1])
\]

\[
= \#_\beta(\sigma[i-r,j+r]) - \#_\beta(\sigma[j-r+1,i+r-1])
\]

\[
= \#_\beta(\sigma[i-r,j+r]) - \#_\beta(\sigma[j-r,i+r]),
\]

as claimed, where:

- (4.8) follows from applying equations (4.3) and (4.4).
- (4.9) follows from applying Equation (4.5).
- (4.10) follows from the fact that

\[
[i-r,j-r] \cup [j-r+1,i+r-1] \cup [i+r,j+r] = [i-r,j+r].
\]

- (4.11) follows from the observation that, by the switch point property, since the pair

\((i,i-1)\) is a switch point in \(\sigma'\), \(\sigma(i+r) = \sigma'(i) = \beta\), and, similarly, since the pair \((j,j+1)\) is a switch point in \(\sigma'\), \(\sigma(j-r) = \sigma'(j) = \beta\) as well. Hence, \(\#_\beta(\sigma[j-r+1,i+r-1]) = \#_\bar{\beta}(\sigma[j-r,i+r]).\)

We have thus established Claim 10. \(\square\)

5 Block intervals defined by the left and right mappings

**Definition 19.** Given a configuration \(\sigma\), let \(B\) be a sequence of maximal homogeneous blocks in \(\sigma\). For two maximal homogeneous blocks \(X = [x, y]\) and \(X' = [x', y']\), not necessarily belonging to \(B\), the **block-interval defined by the pair** \(X, X'\), denoted \([X, X']_B\), is the following sequence of maximal homogeneous blocks:

\[
[X, X']_B = \{[i,j] \in B : [i,j] \subseteq [x,y]\}.
\]
Definition 20. Let \( \sigma \) and \( \sigma' \) be a pair of configurations where \( \text{MAJ}_r(\sigma) = \sigma' \). Given a maximal homogeneous block \( [i', j'] \in B(\sigma') \), let \( f_{\sigma,\sigma'}^+(i', j') \) be the maximal homogeneous block in \( B(\sigma) \) that contains the cell \( i = j' - r \). Similarly, let \( f_{\sigma,\sigma'}^+(i', j') \) be the maximal homogeneous block in \( B(\sigma) \) that contains the cell \( j = i' + r \).

We refer to the function \( f_{\sigma,\sigma'}^+ \) as the left mapping from \( \sigma' \) to \( \sigma \), and, similarly, we refer to the function \( f_{\sigma,\sigma'}^- \) as the right mapping from \( \sigma' \) to \( \sigma \).

Claim 11. For every pair of configurations \( \sigma \) and \( \sigma' \) satisfying \( \text{MAJ}_r(\sigma) = \sigma' \), if \( [i', j'] \in B(\sigma') \), then the value of \( \sigma' \) at \( [i', j'] \) equals the value of \( \sigma \) at \( f_{\sigma,\sigma'}^+(i', j') \) as well as at \( f_{\sigma,\sigma'}^-(i', j') \).

Proof. By the definition of \( B(\sigma') \), the interval \([i', j']\) is a maximal homogeneous block. Therefore, the pair \((i' - 1, i')\) is a switch point in \( \sigma' \), so by the switch point property, \( \sigma(i' + r) = \sigma'(i') \). A similar argument holds for \( \sigma(j' - r) \), and the claim follows.

Claim 12. For every pair of configurations \( \sigma \) and \( \sigma' \) satisfying \( \text{MAJ}_r(\sigma) = \sigma' \) and a maximal homogeneous block \( [i', j'] \in B(\sigma') \), the number of homogeneous blocks in the block interval \( [f^-_{\sigma,\sigma'}([i', j'])] \) is even.

Proof. By Claim 11, the value of \( f^+_{\sigma,\sigma'}([i, j]) \) equals the value of \( f^-_{\sigma,\sigma'}([i', j']) \). Hence, since the values of maximal homogeneous blocks in a configuration alternate, the number of maximal homogeneous blocks in the block interval \( [f^-_{\sigma,\sigma'}([i', j'])] \) must be even.

Claim 13. For every pair of configurations \( \sigma \) and \( \sigma' \) satisfying \( \text{MAJ}_r(\sigma) = \sigma' \), each of the functions \( f^+_{\sigma,\sigma'}([i', j']) \) and \( f^-_{\sigma,\sigma'}([i', j']) \) is one-to-one.

Proof. We prove the claim for \( f^+_{\sigma,\sigma'}([i', j']) \) and denote it by \( f \) for short. Let \( [i, j] \in B(\sigma) \) and suppose by way of contradiction that there are two different blocks \( [i', j'], [i'', j''] \in B(\sigma) \) s.t. \( f([i', j']) = f([i'', j'']) = [i, j] \). Let \( \beta \) be the value of \( \sigma \) at the block \([i, j]\). By Claim 11, the value of \( \sigma' \) at both \([i', j']\) and \([i'', j'']\) is also \( \beta \). By the definition of the mapping, the fact that \( f([i', j']) = f([i'', j'']) = [i, j] \) implies that both \( i' + r \) and \( i'' + r \) belong to \([i, j]\). Hence, either \( [i' + r, i'' + r] \subseteq [i, j] \) or \( [i'' + r, i' + r] \subseteq [i, j] \). Assume without loss of generality that \( [i' + r, i'' + r] \subseteq [i, j] \). Hence, there must exist a maximal homogeneous block \([i^*, j^*]\) in \( \sigma' \) where the value of \( \sigma' \) at \([i^*, j^*]\) is \( \beta \) and \( i^* \in [i', i''] \). Hence, it must hold that \( i^* + r \in [i' + r, i'' + r] \subseteq [i, j] \). That is, \( f([i^*, j^*]) = [i, j] \), in contradiction to Claim 11.

Claim 14. For every pair of configurations \( \sigma \) and \( \sigma' \) satisfying \( \text{MAJ}_r(\sigma) = \sigma' \), it must be the case that \( |B(\sigma')| \leq |B(\sigma)| \).

Proof. By Claim 13, there is a one-to-one mapping from \( B(\sigma') \) to \( B(\sigma) \). Hence, \( |B(\sigma')| \leq |B(\sigma)| \).

Claim 15. If \( \sigma \) and \( \sigma' \) constitute a temporally periodic configuration pair, then for every pair of blocks \([a, b], [c, d] \in B(\sigma) \) and \([a', b'] \in B(\sigma) \), the number of blocks in the block interval \( [f^-_{\sigma,\sigma'}([a, b]), f^-_{\sigma,\sigma'}([a', b'])] \) equals the number of blocks in \( [f^-_{\sigma,\sigma'}([c, d]), f^-_{\sigma,\sigma'}([c', d'])] \).

Moreover, for every pair of blocks \([a, b] \in B(\sigma) \) and \([a', b'] \in B(\sigma) \), the number of blocks in the block interval \( [f^-_{\sigma,\sigma'}([a', b']), f^+_{\sigma,\sigma'}([a', b'])] \) equals the number of blocks in \( [f^-_{\sigma,\sigma'}([a, b]), f^+_{\sigma,\sigma'}([a, b'])] \).

Proof. Let \( k \) be the number of maximal homogeneous blocks in \( \sigma \). By Claim 13, the functions \( f^-_{\sigma,\sigma'} \) and \( f^+_{\sigma,\sigma'} \) are one-to-one, and therefore the number of maximal homogeneous blocks in \( \sigma' \) is \( k \) as well. Denote by \([a_1, b_1], \ldots, [a_k, b_k] \) the sequence of maximal homogeneous blocks in \( \sigma \) starting from an arbitrary block \([a_1, b_1] \) such that for each \( 1 \leq i \leq k - 1 \), \( a_{i+1} = b_i + 1 \) (that is, the blocks \([a_i, b_i] \) and \([a_{i+1}, b_{i+1}] \) are consecutive).
We claim that for every $1 \leq i \leq k$, if $f_{\sigma,\sigma'}^+(\{a_i, b_i\}) = \{a'_i, b'_i\}$ and $f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\}) = \{a'_{i+1}, b'_{i+1}\}$, then $a'_{i+1} = b'_i + 1$ (that is, the blocks $f_{\sigma,\sigma'}^+(\{a_i, b_i\})$ and $f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\})$ are consecutive). We also claim that for every $1 \leq i \leq k$, if $f_{\sigma,\sigma'}^+(\{a_i, b_i\}) = \{a''_i, b''_i\}$ and $f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\}) = \{a''_{i+1}, b''_{i+1}\}$, then $a'_{i+1} = b''_i + 1$. We prove the former (since the proof of the latter is analogous).

Suppose towards a contradiction that for some $1 \leq i \leq k - 1$, $a'_{i+1} \neq b'_i + 1$. In that case, there must exist an integer $2 \leq j \leq k - i$ for which

$$f_{\sigma,\sigma'}^+(\{a_{i+j}, b_{i+j}\}) \subseteq \{a'_i, b'_i\}.$$

Since $b_{i+j} - r \in f_{\sigma,\sigma'}^+(\{a_{i+j}, b_{i+j}\})$, this means that $b_{i+j} - r \in \{a'_i, b'_i\}$, and hence $b_{i+j} - r \in [b_i - r, b_{i+1} - r]$. Thus,

$$b_{i+j} \in [b_i, b_{i+1}] \subseteq [a_i, b_i] \cup [a_{i+1}, b_{i+1}],$$

in contradiction to $j \geq 2$ (in other words, in contradiction to $[a_{i+j}, b_{i+j}]$ being distinct from $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$).

This establishes the claim that $a'_{i+1} = b'_i + 1$, and an analogous argument implies that $a''_{i+1} = b''_i + 1$ as well. Hence,

$$[f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\}), f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\})]_{B(\sigma)} = [f_{\sigma,\sigma'}^+(\{a_i, b_i\}), f_{\sigma,\sigma'}^+(\{a_i, b_i\})]_{B(\sigma)} \cup \{a'_i, b'_i\} \leq \{a''_i, b''_i\}.$$

Thus,

$$|[f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\}), f_{\sigma,\sigma'}^+(\{a_{i+1}, b_{i+1}\})]_{B(\sigma)}| = |[f_{\sigma,\sigma'}^+(\{a_i, b_i\}), f_{\sigma,\sigma'}^+(\{a_i, b_i\})]_{B(\sigma)}|. \tag{5.1}$$

Since Equation (5.1) holds for every $1 \leq i \leq k$, it must be the case that for every pair of blocks $[a, b], [c, d] \in B(\sigma)$, the number of blocks in the block interval $[f_{\sigma,\sigma'}^+(\{a, b\}), f_{\sigma,\sigma'}^+(\{a, b\})]_{B(\sigma)}$ equals the number of blocks in the block interval $[f_{\sigma,\sigma'}^+(\{c, d\}), f_{\sigma,\sigma'}^+(\{c, d\})]_{B(\sigma)}$, which establishes the first part of the claim.

\section{The alignment mapping}

\textbf{Definition 21} (alignment mapping). Let $\sigma$ and $\sigma'$ be a pair of configurations satisfying $\text{MAJ}_r(\sigma) = \sigma'$. Given a maximal homogeneous block $[i, j] \in B(\sigma')$, let $\varphi_{\sigma,\sigma'}([i, j])$ be the middle block in the block interval $[f_{\sigma,\sigma'}^+(\{i, j\}), f_{\sigma,\sigma'}^+(\{i, j\})]_{B(\sigma)}$ (the middle block is well-defined, since, by Claim 12, the number of blocks in that interval is odd).

We refer to the function $\varphi_{\sigma,\sigma'}$ as the alignment mapping from $\sigma'$ to $\sigma$.

\textbf{Claim 16.} For every pair of configurations $\sigma$ and $\sigma'$ satisfying $\text{MAJ}_r(\sigma) = \sigma'$, the alignment mapping $\varphi_{\sigma,\sigma'}$ is one-to-one.

\textbf{Proof.} Suppose that, contrary to the claim, $\varphi_{\sigma,\sigma'}$ is not one-to-one. That is, there are two distinct blocks $[a, b], [c, d] \in B(\sigma')$ such that

$$\varphi_{\sigma,\sigma'}([a, b]) = \varphi_{\sigma,\sigma'}([c, d]).$$
That being the case, denote by $I \in B(\sigma)$ the block satisfying $I = \varphi_{\sigma,\sigma'}([a, b]) = \varphi_{\sigma,\sigma'}([c, d])$.

Since $[a, b]$ and $[c, d]$ are maximal homogeneous blocks, it is either the case that the intervals $[b, c]$ and $[a, d]$ satisfy $[b, c] \subseteq [a, d]$ or that they satisfy $[a, d] \subseteq [b, c]$. We assume, then, without loss of generality, that

$$[b, c] \subseteq [a, d].$$  \hspace{1cm} (6.1)

By Claim 12, for every maximal homogeneous block $[i, j] \in B(\sigma')$, the number of blocks in $[f_{\sigma,\sigma'}^{-1}([i, j]), f_{\sigma,\sigma'}^{-1}([i, j])]_{B(\sigma)}$ is odd. Let $\delta_{[a,b]}$ be the integer satisfying

$$|[f_{\sigma,\sigma'}^{-1}([a, b]), f_{\sigma,\sigma'}([a, b])]_{B(\sigma)}| = 2\delta_{[a,b]} + 1,$$

and, similarly, let $\delta_{[c,d]}$ be the integer satisfying

$$|[f_{\sigma,\sigma'}^{-1}([c, d]), f_{\sigma,\sigma'}([c, d])]_{B(\sigma)}| = 2\delta_{[c,d]} + 1.$$

By Definition 21, the block $I$ defined above is the middle block of the block interval $[f_{\sigma,\sigma'}^{-1}([a, b]), f_{\sigma,\sigma'}^{-1}([a, b])]_{B(\sigma)}$ as well as of the block interval $[f_{\sigma,\sigma'}^{-1}([c, d]), f_{\sigma,\sigma'}^{-1}([c, d])]_{B(\sigma)}$. This means that the block $f_{\sigma,\sigma'}^{-1}([a, b])$ is located $\delta_{[a,b]}$ blocks away from $I$ to its left side and that the block $f_{\sigma,\sigma'}^{-1}([a, b])$ is located $\delta_{[a,b]}$ blocks away from $I$ to its right side. Similarly, the block $f_{\sigma,\sigma'}^{-1}([c, d])$ is located $\delta_{[c,d]}$ blocks away from $I$ to the left, and that the block $f_{\sigma,\sigma'}^{-1}([c, d])$ is located $\delta_{[c,d]}$ blocks away from $I$ to the right.

Thus, if $\delta_{[a,b]} \leq \delta_{[c,d]}$, then

$$[f_{\sigma,\sigma'}^{-1}([a, b]), f_{\sigma,\sigma'}([a, b])]_{B(\sigma)} \subseteq [f_{\sigma,\sigma'}^{-1}([c, d]), f_{\sigma,\sigma'}([c, d])]_{B(\sigma)},$$ \hspace{1cm} (6.2)

and if $\delta_{[a,b]} \geq \delta_{[c,d]}$, then

$$[f_{\sigma,\sigma'}^{-1}([c, d]), f_{\sigma,\sigma'}([c, d])]_{B(\sigma)} \subseteq [f_{\sigma,\sigma'}^{-1}([a, b]), f_{\sigma,\sigma'}([a, b])]_{B(\sigma)}.$$ \hspace{1cm} (6.3)

In the former case (Equation (6.2)),

$$[b - r, a + r] \subseteq [d - r, c + r].$$ \hspace{1cm} (6.4)

This is because, by Definition 20,

$$b - r \in f_{\sigma,\sigma'}^{-1}([a, b]),$$

$$a + r \in f_{\sigma,\sigma'}^{-1}([a, b]),$$

$$d - r \in f_{\sigma,\sigma'}^{-1}([c, d]),$$ and

$$c + r \in f_{\sigma,\sigma'}^{-1}([c, d]).$$

For the same reason, in the latter case (Equation (6.3)),

$$[d - r, c + r] \subseteq [b - r, a + r].$$ \hspace{1cm} (6.5)
However, as we show next, both \([b-r, a+r] \subseteq [d-r, c+r]\) (Equation (6.4)) and \([d-r, c+r] \subseteq [b-r, a+r]\) (Equation (6.5)) are impossible given the assumption that \([b, c] \subseteq [a, d]\) (Equation (6.1)).

To see why \([b-r, a+r] \subseteq [d-r, c+r]\) contradicts \([b, c] \subseteq [a, d]\), observe that \([b, c] \subseteq [a, d]\) implies \([d-r, c+r] \subseteq [d-r, d+r]\), and since \([d-r, d+r] = 2r + 1\), it must be the case that \([|d-r, c+r| < 2r]\).

Additionally, \([b, c] \subseteq [a, d]\) also implies \([b-r, b+r] \subseteq [b-r, c+r] \cup [b-r, b+r] = 2r + 1\), so \([|b-r, c+r| > 2r]\).

However, \([b-r, a+r] \subseteq [d-r, c+r]\) implies that \([b-r, c+r] \subseteq [d-r, c+r]\), so it cannot be the case that both \([|d-r, c+r| < 2r]\) and \([|b-r, c+r| > 2r]\).

The case in which \([d-r, c+r] \subseteq [b-r, a+r]\) can similarly be shown to contradict \([b, c] \subseteq [a, d]\), so we reach a contradiction in either case.

\[\square\]

**Claim 17.** Let \(\sigma, \sigma'\) be a temporally periodic configuration pair and let \([a, b], [c, d] \in B(\sigma')\) be two adjacent blocks in which \(c = b + 1\). If \(\varphi_{\sigma,\sigma'}([a, b]) = [a', b']\) and \(\varphi_{\sigma,\sigma'}([c, d]) = [c', d']\), then the blocks \([a', b']\) and \([c', d']\) are also adjacent and \(c' = b' + 1\).

**Proof.** Let \(\beta \in \{0, 1\}\) be the value of the block \([a, b]\) in \(\sigma'\). Since the block \([c, d]\) is adjacent to the block \([a, b]\), the value of the block \([c, d]\) in \(\sigma'\) must be \(\beta\). By the switch point property, \(\sigma(a+r) = \beta\) and \(\sigma(c+r) = \beta\), so the value of the block \(f_{\sigma,\sigma'}^+(a, b)\) in \(\sigma\) is \(\beta\) and the value of the block \(f_{\sigma,\sigma'}^-(c, d)\) in \(\sigma'\) is \(\beta\).

By Claim 9, the interval \([a+r, b+r+1]\) contains exactly one switch point in \(\sigma\). Since \(c = b + 1\), the interval \([a+r, c+r]\) contains exactly one switch point in \(\sigma\). As both \(f_{\sigma,\sigma'}^+(a, b)\) and \(f_{\sigma,\sigma'}^-(c, d)\) intersect with the interval \([a+r, c+r]\), that single switch point must be \(\beta\), the switch point between \(f_{\sigma,\sigma'}^+(a, b)\) and \(f_{\sigma,\sigma'}^-(c, d)\). Hence, the blocks \(f_{\sigma,\sigma'}^+(a, b)\) and \(f_{\sigma,\sigma'}^-(c, d)\) are adjacent with the block \(f_{\sigma,\sigma'}^+(a, b)\) preceding the block \(f_{\sigma,\sigma'}^-(c, d)\).

By a similar argument, the blocks \(f_{\sigma,\sigma'}^+(a, b)\) and \(f_{\sigma,\sigma'}^-(c, d)\) are adjacent with the block \(f_{\sigma,\sigma'}^+(a, b)\) preceding the block \(f_{\sigma,\sigma'}^-(c, d)\).

By the definition of the alignment mapping, the block \([a', b']\) is the middle block of the block interval \([f_{\sigma,\sigma'}^+(a, b), f_{\sigma,\sigma'}^-(a, b)\)] \(B(\sigma)\), and the block \([c', d']\) is the middle block of the block interval \([f_{\sigma,\sigma'}^+(c, d), f_{\sigma,\sigma'}^-(c, d)\)] \(B(\sigma)\).

Therefore, the blocks \([a', b']\) and \([c', d']\) must be adjacent to each other and it must also be the case that \(c' = b' + 1\), as claimed.

\[\square\]

**Definition 22.** Let \(\sigma, \sigma'\) be a temporally periodic configuration pair. Given an integer \(k\), we define \(\varphi_{\sigma,\sigma'}^k\) as follows. For every block \([i, j] \in B(\sigma)\),

1. \(\varphi_{\sigma,\sigma'}^0([i, j]) = [i, j]\).
2. \(\varphi_{\sigma,\sigma'}^1([i, j]) = \varphi_{\sigma,\sigma'}([i, j])\).
3. for \(k > 1\), if \(k\) is odd, then \(\varphi_{\sigma,\sigma'}^k([i, j]) = \varphi_{\sigma,\sigma'}^k(\varphi_{\sigma,\sigma'}^{k-1}([i, j]))\).
4. for \(k > 1\), if \(k\) is even, then \(\varphi_{\sigma,\sigma'}^k([i, j]) = \varphi_{\sigma,\sigma'}^k(\varphi_{\sigma,\sigma'}^{k-1}([i, j]))\).

**Observation 18.** Let \(\sigma, \sigma'\) be a temporally periodic configuration pair. For every block \([i, j] \in B(\sigma')\), \(\varphi_{\sigma,\sigma'}^k([i, j]) = [i, j]\).

**Definition 23.** Let \(\sigma_0 : \mathbb{Z}_n \rightarrow \{0, 1\}\) be any initial configuration, and for any integer \(t \geq 0\), let \(\sigma_t = MA^t(\sigma_0)\). Given a time step \(t \geq 1\), we define \(\varphi_t\) as \(\varphi_{\sigma_{t-1},\sigma_t}([i, j])\) for every block \([i, j] \in B(\sigma_t)\). Given an integer \(k\) and a time step \(t\) s.t. \(t \geq k\), we define \(\varphi_t^k\) as follows. For every block \([i, j] \in B(\sigma_t)\):
1. $\varphi^0_{k}(\sigma[i,j]) = [i,j]$.

2. for $k > 1$, $\varphi^k_{k+1}(\sigma[i,j]) = \varphi^{k-1}_{k+1}(\sigma)$.

7 Block lengths in temporally periodic configurations

Claim 19. Let $\sigma, \sigma'$ be a temporally periodic configuration pair of weakly stable configurations and let $\beta \in \{0,1\}$. For every block $[i,j] \in B^\beta(\sigma')$,

$$ |[i,j]| = \sum_{[i',j'] \in A^\beta} ||i',j'|| - \sum_{[i',j'] \in A^\beta} ||i',j'||, \tag{7.1} $$

where

$$ A^\beta = [f_{\sigma,\sigma'}^\beta([i,j]), f_{\sigma,\sigma'}^\beta([i,j])]_{B^\beta(\sigma)}, $$

and, similarly,

$$ A^\beta = [f_{\sigma,\sigma'}^\beta([i,j]), f_{\sigma,\sigma'}^\beta([i,j])]_{B^\beta(\sigma)}. $$

Proof. Since $\sigma'$ is a temporally periodic weakly stable configuration, by Claim 8, $|[i,j]| \leq r < 2r + 1$. Therefore, the conditions for applying Claim 10 hold. So, by Claim 10,

$$ |[i,j]| = \#_\beta(\sigma[i-r,j+r]) - \#_\beta(\sigma[j-r,i+r]). \tag{7.2} $$

To prove the claim, we relate Equation (7.2) to Equation (7.1) by defining four sets of cells:

$$ X^\beta = \{ \ell \in [i',j'] : [i',j'] \in A^\beta \}, $$

$$ X^\beta = \{ \ell \in [i',j'] : [i',j'] \in A^\beta \}, $$

$$ Y^\beta = \{ \ell \in [i-r,j+r] | \sigma(\ell) = \beta \}, $$

$$ Y^\beta = \{ \ell \in [j-r,i+r] | \sigma(\ell) = \beta \}. $$

In order to prove the claim, it is sufficient to show that

$$ X^\beta = Y^\beta, $$

$$ X^\beta = Y^\beta. $$

We first show that $X^\beta = Y^\beta$.

Let $i^*$ be the leftmost cell in the block $f_{\sigma,\sigma'}^{\beta}([i,j])$ and let $j^*$ be the rightmost cell in the block $f_{\sigma,\sigma'}^{\beta}([i,j])$. By the definition of $A^\beta$,

$$ X^\beta = \{ \ell \in [i^*,j^*] : \sigma(\ell) = \beta \}. $$

Since $[i,j] \in B^\beta(\sigma')$, each of the pairs $(i-1,i)$ and $(j,j+1)$ is a switch point in $\sigma'$, so by the switch point property, $\sigma(j-r) = \sigma(i+r) = \beta$. As $f_{\sigma,\sigma'}^{\beta}([i,j])$ and $f_{\sigma,\sigma'}^{\beta}([i,j])$ are by definition the two maximal homogeneous blocks in $\sigma$ that contain the cells $j-r$ and $i+r$ respectively, it must be the case that for each cell $\ell \in [i^*,j^*] \cup [i+r,j^*]$, $\sigma(\ell) = \beta$. Hence, if a cell $\ell \in [i^*,j^*]$ satisfies $\sigma(\ell) = \beta$, then $\ell \in [j-r+1,i+r-1] \subseteq [j-r,i+r]$. That is,
We now show that $X^\beta = Y^\beta$.

Recall that $i^*$ is the leftmost cell in the block $f^+_{\sigma,\sigma'}([i, j])$ and that $j^*$ is the rightmost cell in the block $f^-_{\sigma,\sigma'}([i, j])$, which means that $X^\beta = \{ \ell \in [i^*, j^*] : \sigma(\ell) = \beta \}$.

Since $(i-1, i)$ is a switch point in $\sigma'$, by the switch point property, $\sigma(i-1) = \beta$. This implies that $i^* \in [i-r, j-r]$. As $(i^*-1, i^*)$ is a switch point in $\sigma$ of type $(\beta, \beta)$, by Claim 9, there is exactly one switch point in $[i-r-1, j-r]$, and therefore it must be the case that

$$\{ \ell \in [i-r, j-r] : \sigma(\ell) = \beta \} = [i^*, j^*-1].$$

(7.3)

Similarly, since $(j, j+1)$ is a switch point in $\sigma'$, by the switch point property, $\sigma(j+r+1) = \beta$, which implies that $j^* \in [i+r, j+r]$. Because $(i^*-1, i^*)$ is a switch point in $\sigma$ of type $(\beta, \beta)$, by Claim 9, there is exactly one switch point in $[i+r, j+r+1]$, and therefore it must be the case that

$$\{ \ell \in [i+r, j+r] : \sigma(\ell) = \beta \} = [i+r, j^*].$$

(7.4)

Consequently,

$$Y^\beta = \{ \ell \in [i-r, j+r] : \sigma(\ell) = \beta \} = \{ \ell \in [i^*, j^*] : \sigma(\ell) = \beta \} = X^\beta,$$

(7.5)

(7.6)

(7.7)

where Equation (7.6) follows from equations (7.3) and (7.5) together with the observations that $i^* \in [i-r, j-r]$ and $j^* \in [i+r, j+r]$.

We’ve shown that $X^\beta = Y^\beta$ and $X^\beta = Y^\beta$, so the claim follows. 

8 The block-length vectors of temporally periodic configuration pairs

Definition 24 (block-length vector). Given a configuration $\sigma : \mathbb{Z}_n \to \{0, 1\}$, we define its block-length vector $\bar{v} : \mathbb{Z}_{|B(\sigma)|} \to \mathbb{N}$ as the cyclic sequence of the lengths of the configuration’s maximal homogeneous blocks.

That is, $\bar{v}(0) = |[i, j]|$ for an arbitrary block $[i, j] \in B(\sigma)$, and for each $k \in \mathbb{Z}_{|B(\sigma)|}$, if $[a, b] \in B(\sigma)$ is the block for which $\bar{v}(k-1) = |[a, b]|$ and $[c, d] \in B(\sigma)$ is the block satisfying $c = b + 1$, then $\bar{v}(k) = |[c, d]|$.

We note that every possible block-length vector, viewed as a ring of integers, corresponds to at most two configurations (up to cyclic shifts): one where the blocks at the odd positions have a value of 0 and one where the blocks at the even positions have a value of 0.\(^{10}\) So when we say that a block-length vector corresponds to a configuration $\sigma$, it means that the configuration $\sigma$ can be either of the at most two possibilities.

\(^{10}\)These two possible configurations collapse into one in the case in which the block-length vector $\bar{v}$ equals the concatenation of some block-length vector $\bar{u}$ to itself (i.e., $\bar{v} = \bar{u}\bar{u}$), where $\bar{u}$ is of odd length (for instance, the block-length vector 123123 corresponds to exactly one configuration, up to a cyclic shift, as the configuration resulting from assigning the value 0 to the first block is the same configuration resulting from assigning the value 1 to the first block).
Definition 25. We define the length of a block-length vector \( \vec{v} \), denoted by \(|\vec{v}|\), as the number of entries in the vector. That is, if \( \vec{v} \) is the block-length vector that corresponds to the configuration \( \sigma \), then \(|\vec{v}| = |B(\sigma)|\).

Claim 20. If \( \vec{v} \) and \( \vec{v}' \) are a pair of block-length vectors corresponding to a temporally periodic configuration pair, then \(|\vec{v}| = |\vec{v}'|\).

Proof. Let \( \sigma \) be a configuration that corresponds to the block-length vector \( \vec{v} \) and let \( \sigma' = \text{MAJ}_r(\sigma) \). Clearly, \( \vec{v}' \) is the block-length vector of the configuration \( \sigma' \), and the pair \( \sigma, \sigma' \) is a temporally periodic configuration pair.

Since \( \sigma' = \text{MAJ}_r(\sigma) \), by Claim 14, \(|B(\sigma')| \leq |B(\sigma)|\). Since \( \sigma, \sigma' \) are a temporally periodic configuration pair, it is also the case that \( \sigma = \text{MAJ}_r(\sigma') \), so again by Claim 14, \(|B(\sigma)| \leq |B(\sigma')|\).

That is, \(|B(\sigma)| = |B(\sigma')|\) and the claim follows.

Definition 26 (aligned). Let \( \vec{v} \) and \( \vec{v}' \) be a pair of block-length vectors of length \( k \) each corresponding to a temporally periodic configuration pair \( \sigma \) and \( \sigma' \). For each \( i \in \mathbb{Z}_k \), let \( I_i \) the maximal homogeneous block in \( \sigma \) that corresponds to \( \vec{v}_i \), and let \( I'_i \) be the maximal homogeneous block in \( \sigma' \) that corresponds to \( \vec{v}'_i \). We say that the pair \( \vec{v} \) and \( \vec{v}' \) are aligned if for each \( i \in \mathbb{Z}_k \), \( \varphi_{\sigma,\sigma'}(I'_i) = I_i \) (where \( \varphi_{\sigma,\sigma'} \) is the alignment mapping from \( \sigma' \) to \( \sigma \)).

9 The horizon of block-length vectors

Definition 27 (horizon). Let \( \vec{v} \) and \( \vec{v}' \) be a pair of block-length vectors of lengths \( k \) and \( k' \) each, corresponding to a pair of configurations \( \sigma \) and \( \sigma' \) satisfying \( \text{MAJ}_r(\sigma) = \sigma' \). For each entry \( i \in \mathbb{Z}_k \) of \( \vec{v}' \), we define the horizon of \( i \) in \( \vec{v}' \), denoted by \( \delta_{\vec{v}}(i) \), as follows: \( \delta_{\vec{v}}(i) \) is the value that satisfies

\[
|f_{\sigma,\sigma'}^r([a, b], f_{\sigma,\sigma'}^r([a, b]))_{B(\sigma')}| = 2\delta_{\vec{v}}(i) + 1,
\]

where \([a, b] \in B(\sigma')\) is the block in \( \sigma' \) that corresponds to entry \( i \) of \( \vec{v}' \).

Observation 21. If \( \vec{v} \) is the block-length vector corresponding to a temporally periodic configuration \( \sigma \), then for every \( i \in \mathbb{Z}_k \), where \( k \) is the length of \( \vec{v} \), the horizon \( \delta_{\vec{v}}(i) \) satisfies \( \delta_{\vec{v}}(i) \leq r \).

Claim 22. Let \( \vec{v}, \vec{v}' \) be a pair of block-length vectors corresponding to a temporally periodic configuration pair \( \sigma, \sigma' \). For every pair \( i, j \in \mathbb{Z}_{|\vec{v}|} \),

\[
\delta_{\vec{v}}(i) = \delta_{\vec{v}}(j).
\]

Moreover, for every \( i \in \mathbb{Z}_{|\vec{v}|} \),

\[
\delta_{\vec{v}}(i) = \delta'_{\vec{v}}(i).
\]

Proof. Let \([a, b] \in B(\sigma)\) be the block corresponding to entry \( i \) of the block-length vector \( \vec{v} \) and let \([c, d] \in B(\sigma)\) be the block corresponding to entry \( j \) of the block-length vector \( \vec{v} \). By Claim 15, the number of blocks in the block interval \([f_{\sigma,\sigma'}^r([a, b]), f_{\sigma,\sigma'}^r([a, b])]_{B(\sigma')}\) equals the number of blocks in the block interval \([f_{\sigma,\sigma'}^r([c, d]), f_{\sigma,\sigma'}^r([c, d])]_{B(\sigma')}\). That is, \(2\delta_{\vec{v}}(i) + 1 = 2\delta_{\vec{v}}(j) + 1\). Hence, \(\delta_{\vec{v}}(i) = \delta_{\vec{v}}(j)\), as claimed.

The second part of the claim similarly follows from the second part of Claim 15.

\[\text{The horizon } \delta'_{\vec{v}}(i) \text{ is well defined because, by Claim 12, the number of blocks in } [f_{\sigma,\sigma'}^r([a, b]), f_{\sigma,\sigma'}^r([a, b])]_{B(\sigma)} \text{ is odd.}\]

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Claim 23. If \( \vec{v} \) is the block-length vector corresponding to a weakly stable configuration \( \sigma \), then the total length of every sequence of \( 2\delta \) maximal homogeneous blocks in \( \sigma \) is at most \( 2r \), where \( \delta \) is the horizon of \( \vec{v} \) (which, by Claim 22, is the same for all maximal homogeneous blocks in the configuration \( \sigma \)).

Proof. We show that for every sequence of \( 2\delta + 1 \) consecutive maximal homogeneous blocks in the configuration \( \sigma \), if we remove the leftmost maximal homogeneous block from the sequence, then the total length of the remaining maximal homogeneous blocks is at most \( 2r \).

Let \( \sigma' = \text{MAJ}_{r} (\sigma) \). Since \( \sigma, \sigma' \) comprise a temporally periodic configuration pair of weakly stable configuration, every sequence of \( 2\delta + 1 \) of consecutive maximal homogeneous blocks in \( \sigma \) is of the form \([f_{\sigma,\sigma'}^{-\varepsilon}([a', b']), f_{\sigma,\sigma'}^{-\varepsilon}([a', b'])]_{B(\sigma)}\) for some maximal homogeneous block \([a', b'] \in B(\sigma')\). Let \( I \) be the cell interval composed of the maximal homogeneous blocks in the set \([f_{\sigma,\sigma'}^{-\varepsilon}([a', b']), f_{\sigma,\sigma'}^{-\varepsilon}([a', b'])]_{B(\sigma)} \setminus \{ f_{\sigma,\sigma'}^{-\varepsilon}([a', b']) \}\). We prove that \(|I| \leq 2r\).

Let \([a, b] = f_{\sigma,\sigma'}^{-\varepsilon}([a', b'])\). That is, \([a, b]\) is the maximal homogeneous block in \( B(\sigma) \) that contains the cell \( a' + r \). By Claim 17, if \( I_{B+r} \) is the maximal homogeneous block that starts at the cell \( b' + 1 \) in \( \sigma' \), then \( f_{\sigma,\sigma'}^{-\varepsilon}(I_{B+r}) \) is the maximal homogeneous block that follows \([a, b]\) in \( \sigma \), which means that \( b \in [a' + r, b' + 1 + r] \).

Thus,

\[
|a' + r, b| \leq b' + 1 + r - (a' + r) = |a', b'|. 
\] (9.1)

We conclude by bounding the length of the interval \( I \).

\[
|I| \leq |b' - r, b| 
\] (9.2)
\[
\leq |b' - r, a' + r| + |a' + r, b| 
\] (9.3)
\[
\leq (2r - |[a', b']|) + |[a', b']| 
\] (9.4)
\[
= 2r, 
\] (9.5)

where Equation (9.2) follows from the observation that \( I \subseteq [b' - r, b] \), and Equation (9.4) follows from noting that \(|b' - r, a' + r| = 2r - |[a', b']| \) and \(|a' + r, b| \leq |[a', b']|\).

Claim 24. If \( \vec{v} \) and \( \vec{v'} \) are a pair of aligned block-length vectors of length \( k \) and horizon \( \delta \) corresponding to a temporally periodic configuration pair, then for every \( i \in \mathbb{Z}_k \),

\[
\vec{v'}_i = \sum_{j=-\delta}^{\delta} (-1)^{j+i} \vec{v}_{i+j}. 
\]

Proof. Let \( \sigma, \sigma' \) be the temporally periodic configuration pair that corresponds to the block-length vector pair \( \vec{v}, \vec{v'} \) respectively. By Claim 19, for \( \beta \in \{0,1\} \) and for every block \([a', b'] \in B^\beta(\sigma')\),

\[
|[a', b']| = \sum_{x \in A^\beta} |x| - \sum_{x \in A^\beta} |x|,
\]

where

\[
A^\beta = [f_{\sigma,\sigma'}^{-\varepsilon}([a', b']), f_{\sigma,\sigma'}^{-\varepsilon}([a', b'])]_{B^\beta(\sigma)},
\]
\[
A^\beta = [f_{\sigma,\sigma'}^{-\varepsilon}([a', b']), f_{\sigma,\sigma'}^{-\varepsilon}([a', b'])]_{B^\beta(\sigma)}.
\]

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By the switch point property, the value of the block \( f_{\sigma,\sigma'}(\lfloor a',b' \rfloor) \) as well as the value of the block \( f_{\sigma,\sigma'}^{\rightarrow}(\lfloor a',b' \rfloor) \) is \( \beta \). In more detail, if \( j \) is the rightmost cell of the block \( \lfloor a',b' \rfloor \), then the pair \( j, j+1 \) is a switch point in the configuration \( \sigma' \), so by the switch point property, \( \sigma(j-r) = \sigma'(j) \), which implies that the value of the block \( f_{\sigma,\sigma'}^{\rightarrow}(\lfloor a',b' \rfloor) \) is \( \beta \) because \( j-r \in f_{\sigma,\sigma'}^{\rightarrow}(\lfloor a',b' \rfloor) \). A similar argument holds for the block \( f_{\sigma,\sigma'}^{\rightarrow}(\lfloor a',b' \rfloor) \).

The other block-values in the length \( 2\delta + 1 \) interval \( [f_{\sigma,\sigma'}^{\rightarrow}(\lfloor a',b' \rfloor), f_{\sigma,\sigma'}^{\rightarrow}(\lfloor a',b' \rfloor)]_{B(\sigma)} \) alternate between \( \bar{\beta} \) and \( \beta \).

Hence, for every \( i \in \mathbb{Z}_k \),

\[
\vec{v}_i = \sum_{j=-\delta}^{\delta} (-1)^{i+j} \vec{u}_{i+j}.
\]

Claim 25. If \( \vec{v} \) and \( \vec{v}' \) are a pair of aligned block-length vectors corresponding to a temporally periodic configuration pair, where the vectors are of length \( k \) and horizon \( \delta \), then for every \( i \in \mathbb{Z}_k \),

\[
\vec{v}'_i + \vec{v}'_{i+1} = \vec{v}_{i-\delta} + \vec{v}_{i+\delta+1}.
\]

Similarly,

\[
\vec{v}_i + \vec{v}_{i+1} = \vec{v}'_{i-\delta} + \vec{v}'_{i+\delta+1}.
\]

Proof. By Claim 24, for every \( i \in \mathbb{Z}_k \),

\[
\vec{v}'_i = \sum_{j=-\delta}^{\delta} (-1)^{i+j} \vec{u}_{i+j},
\]

\[
\vec{v}'_{i+1} = \sum_{j=-\delta}^{\delta} (-1)^{i+j} \vec{u}_{i+1+j}.
\]

Summing up the two equations we get:

\[
\vec{v}'_i + \vec{v}'_{i+1} = \vec{v}_{i-\delta} + \vec{v}_{i+\delta+1}.
\]

And by symmetry,

\[
\vec{v}_i + \vec{v}_{i+1} = \vec{v}'_{i-\delta} + \vec{v}'_{i+\delta+1}.
\]

The claim follows.

Claim 26. If \( \sigma \) is a temporally periodic configuration with block lengths at most \( r \) each, then \( \sigma \) is balanced.

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Proof. Let \( \sigma' = \text{MAJ}_r(\sigma) \). Let \( \vec{v} \) and \( \vec{v}' \) be the corresponding block-length vectors of the temporally periodic configuration pair \( \sigma, \sigma' \).

By Claim 20, \( |\vec{v}| = \vec{v}' \). Denote that length by \( 2k \) (note that since configurations are cyclical, it is either the case that the configuration is homogeneous or that the length is even, and in the former case the claim trivially holds). Let

\[
x = \sum_{i=0}^{k-1} \vec{v}_{2i}, \quad y = \sum_{i=0}^{k-1} \vec{v}_{2i+1}
\]

\[
x' = \sum_{i=0}^{k-1} \vec{v}'_{2i}, \quad y' = \sum_{i=0}^{k-1} \vec{v}'_{2i+1}.
\]

By Claim 24,

\[
x = (\delta + 1)x' - \delta y'
\]

\[
x = (\delta + 1)((\delta + 1)x - \delta y) - \delta((\delta + 1)y - \delta x)
\]

\[
x = y.
\]

The claim follows. \( \square \)

10 The difference vectors of weakly stable configurations are spatially periodic

Definition 28. Let \( \vec{v} \) and \( \vec{v}' \) be a pair of aligned block-length vectors of length \( k \) and horizon \( \delta \) corresponding to a temporally periodic configuration pair of weakly stable configurations. We define the pair of \((\delta + 1)\)-steps difference vectors \( \Delta \) and \( \Delta' \) between \( \vec{v} \) and \( \vec{v}' \). For each \( i \in \mathbb{Z}_k \),

\[
\Delta_i = \vec{v}'_{i+\delta+1} - \vec{v}_i,
\]

\[
\Delta'_i = \vec{v}_i + \delta + 1 - \vec{v'}_{i+\delta+1}.
\]

Claim 27. If \( \vec{v} \) and \( \vec{v}' \) are a pair of aligned block-length vectors of length \( k \) and horizon \( \delta \) corresponding to a temporally periodic configuration pair of weakly stable configurations, and \( \Delta \) and \( \Delta' \) are the pair’s two \((\delta + 1)\)-steps difference vectors, then for every \( i \in \mathbb{Z}_k \) and every integer \( m \geq 0 \),

\[
\sum_{j=0}^{m-1} \Delta_{i+2j(\delta+1)} + \sum_{j=0}^{m-1} \Delta'_{i+(2j+1)(\delta+1)} = \vec{v}_{i+2m(\delta+1)} - \vec{v}_i.
\]

Proof. We prove the claim by induction on \( m \). For \( m = 0 \), the sums in the left-hand side are empty and the right-hand side consist of a difference between two equal terms, so the equation clearly holds. We assume as our induction hypothesis that the equation holds for \( m \), and prove it for \( m + 1 \).
\[
\sum_{j=0}^{m} \Delta_i + 2j(\delta + 1) + \sum_{j=0}^{m} \Delta'_i + (2j+1)(\delta + 1) = \sum_{j=0}^{m-1} \Delta_i + 2j(\delta + 1) + \sum_{j=0}^{m-1} \Delta'_i + (2j+1)(\delta + 1)
\]

\[
+ \Delta_i + 2m(\delta + 1) + \Delta'_{i+(2m+1)(\delta + 1)}
\]

\[
= \vec{v}_{i+(2m+1)(\delta + 1)} - \vec{v}_i + \Delta_i + 2m(\delta + 1) + \Delta'_{i+(2m+1)(\delta + 1)}
\]

\[
= \left( \vec{v}_{i+(2m+1)(\delta + 1)} - \Delta_i + 2m(\delta + 1) \right) - \Delta'_{i+(2m+1)(\delta + 1)} - \vec{v}_i + \Delta_i + 2m(\delta + 1) + \Delta'_{i+(2m+1)(\delta + 1)}
\]

\[
= \vec{v}_{i+(2m+2)(\delta + 1)} - \vec{v}_i,
\]

where Equation (10.3) follows from applying the induction hypothesis, and Equations (10.4) and (10.6) follow directly from applying Definition 28 to the block-length vector \( \vec{v}_{i+2m(\delta + 1)} \) and then to the block-length vector \( \vec{v}_{j+(2m+1)(\delta + 1)} \). □

**Claim 28.** If \( \vec{v} \) and \( \vec{v}' \) are a pair of aligned block-length vectors of length \( k \) and horizon \( \delta \) corresponding to a temporarily periodic configuration pair of weakly stable configurations, then each of the pair’s two \( (\delta + 1) \)-steps difference vectors \( \Delta \) and \( \Delta' \) are spatially periodic with a spatial period that divides \( 2\delta \). In other words, for every \( i \in \mathbb{Z}_k \),

\[
\Delta_i = \Delta_i + 2\delta,
\]

\[
\Delta'_i = \Delta'_{i+2\delta}.
\]

**Proof.** By Claim 25, for every \( i \in \mathbb{Z}_k \),

\[
\vec{v}'_i + \vec{v}'_{i+1} = \vec{v}_{i-\delta} + \vec{v}_{i+\delta+1}.
\]

Hence,

\[
\vec{v}_{i+\delta+1} - \vec{v}'_i = \vec{v}'_{i+1} - \vec{v}_{i-\delta}.
\]

By definition, \( \Delta'_i = \vec{v}_{i+\delta+1} - \vec{v}'_i \) and \( \Delta_{i-\delta} = \vec{v}'_{i+1} - \vec{v}_{i-\delta} \). Thus,

\[
\Delta'_i = \Delta_{i-\delta}.
\]

Similarly, also by Claim 25, for every \( i \in \mathbb{Z}_k \),

\[
\vec{v}_i + \vec{v}_{i+1} = \vec{v}'_{i-\delta} + \vec{v}'_{i+\delta+1},
\]

which means that

\[
\vec{v}'_{i+\delta+1} - \vec{v}'_i = \vec{v}_{i+1} - \vec{v}'_{i-\delta}.
\]
as well, and since \( \Delta_i = \vec{v}_{i+\delta+1} - \vec{v}_i \) and \( \Delta'_{i-\delta} = \vec{v}_{i+1} - \vec{v}_{i-\delta} \) we also get
\[
\Delta_i = \Delta'_{i-\delta}. \tag{10.10}
\]
Combining equations (10.9) and (10.10) for \( i + 2\delta \) and for \( i + \delta \),
\[
\Delta_{i+2\delta} = \Delta'_{i+2\delta} = \Delta_i,
\]
\[
\Delta'_{i+2\delta} = \Delta_{i+\delta} = \Delta'_i.
\]
That is, the spatial period of each of the vectors \( \Delta \) and \( \Delta' \) is at most \( 2\delta \).

Claim 29. Let \( \vec{u} \) and \( \vec{u}' \) be a pair of length-\( k \) horizon-\( \delta \) aligned block-length vectors corresponding to a temporally periodic configuration pair of weakly stable configurations, and let \( \Delta \) and \( \Delta' \) be the pair’s two \((\delta + 1)\)-steps difference vectors. For every \( i \in \mathbb{Z}_k \), define
\[
\varsigma(i) = \sum_{j=0}^{2\delta-1} \Delta_{i+2j+1} + \sum_{j=0}^{2\delta-1} \Delta'_{i+(2j+1)(\delta+1)}.
\]
Then \( \varsigma(i) = \varsigma(i') \) for every pair \( i, i' \in \mathbb{Z}_k \).

Proof. We show that for every \( i \in \mathbb{Z}_k \), the value of \( \varsigma(i) \) does not depend on \( i \).
\[
\varsigma(i) = \sum_{j=0}^{2\delta-1} \Delta_{i+2j+1} + \sum_{j=0}^{2\delta-1} \Delta'_{i+(2j+1)(\delta+1)} \tag{10.11}
\]
\[
= \sum_{j=0}^{2\delta-1} \Delta_{i+2j} + \sum_{j=0}^{2\delta-1} \Delta'_{i+2\delta+2j+1} \tag{10.12}
\]
\[
= \sum_{j=0}^{2\delta-1} \Delta_{i+2j} + \sum_{j=0}^{2\delta-1} \Delta_{i+2j} \tag{10.13}
\]
\[
= \sum_{j=0}^{2\delta-1} \Delta_{i+j} \tag{10.14}
\]
\[
= \sum_{j=0}^{2\delta-1} \Delta_{j+1} \tag{10.15}
\]
where:

- Equation (10.12) follows from noting that, by Claim 28, the vector \( \Delta \) is spatially periodic with a spatial period that divides \( 2\delta \), so \( \Delta_{i+2j}(\delta+1) = \Delta_i+2j \) and \( \Delta'_i+(2j+1)(\delta+1) = \Delta'_{i+\delta+2j+1} \) for every \( 0 \leq j \leq 2\delta - 1 \).

- Equation (10.13) follows from the observation that, by Equation (10.9) in the proof of Claim 28, \( \Delta'_{i+\delta+2j+1} = \Delta_{i+2j+1} \) for every \( 0 \leq j \leq m - 1 \).

- Equation (10.14) follows from combining the two summations.

- Equation (10.15) follows again from the property that the vector \( \Delta \) is spatially periodic with a spatial period that divides \( 2\delta \) (Claim 28).

\[ \square \]
11 The weakly stable configurations are spatially periodic

**Claim 30.** The spatial period of every weakly stable configuration is at most \(2r(r+1)\).

*Proof.* Let \(\sigma, \sigma'\) be a temporally periodic configuration pair of weakly stable configurations, and let \(\vec{v}\) and \(\vec{v}'\) be the corresponding aligned block-length vectors. Let \(k\) be the length of \(\vec{v}\) and \(\vec{v}'\) and let \(\delta\) be their horizon.

Let \(\Delta\) and \(\Delta'\) be the two \((\delta + 1)\)-steps difference vectors of the pair \(\vec{v}, \vec{v}'\). By Claim 27, for every integer \(m \geq 0\),

\[
\vec{v}_i + \sum_{j=0}^{m-1} \Delta_{i+2j(\delta+1)} + \sum_{j=0}^{m-1} \Delta'_{i+(2j+1)(\delta+1)} = \vec{v}_{i+2m(\delta+1)}.
\]

If we assign \(m = 2\delta\), by Claim 29, the expression consisting of the two sums in Equation (11.1) does not depend on \(i\), so we denote that expression by \(\varsigma\). Thus, for every \(i \in \mathbb{Z}_k\),

\[
\vec{v}_i + \varsigma = \vec{v}_{i+2\delta(\delta+1)}.
\]

Since \(\vec{v}\) is a block-length vector, the values in its entries are bounded, and so it must be the case that \(\varsigma = 0\). Thus, for every \(i \in \mathbb{Z}_k\),

\[
\vec{v}_i = \vec{v}_{i+2\delta(\delta+1)}.
\]

That is, the block-length vector \(\vec{v}\) is spatially periodic with spatial period at most \(2\delta(\delta+1)\). By Claim 23, the total length of every sequence of \(2\delta\) maximal homogeneous blocks in \(\sigma\) is at most \(2r\), and, by Observation 21, \(\delta \leq r\), implying that the configuration \(\sigma\) is spatially periodic with spatial period at most \(2r(r+1)\). \(\Box\)

12 Putting it all together: Proving Theorem 1

*Proof of Theorem 1.* Let \(\sigma : \mathbb{Z}_n \to \{0, 1\}\) be any configuration, let \(\sigma' = \text{MAJ}_r(\sigma)\) and let \(\sigma'' = \text{MAJ}_r(\sigma')\).

Suppose first that the configuration \(\sigma\) is temporally periodic. If all the cells in \(\sigma\) are strongly stable, then \(\sigma\) is by definition of the form \((0^{r+1}+1^r)\), and by Observation 6, it is a fixed-point. Otherwise, still assuming that \(\sigma\) is temporally periodic, it must be the case that \(\sigma\) is weakly stable, which means that, by Corollary 2, all the cells in \(\sigma\) are weakly stable, and by Claim 30, the configuration \(\sigma\) is spatially periodic with spatial period at most \(2r(r+1)\).

Suppose now that the configuration \(\sigma\) is transient. By Claim 8, the length of every maximal homogeneous block in \(B(\sigma)\) is at most \(r\). We say that a cell interval \([i, j] \subseteq \mathbb{Z}_n\) is unstable with respect to \(\sigma\) if for every \(\ell \in [i, j]\), \(\sigma(\ell) \neq \sigma''(\ell)\). We claim that every unstable cell interval with respect to \(\sigma\) contains at least one switch point.

Suppose by way of contradiction that there exists an unstable cell interval with respect to \(\sigma\) that contains at least two switch points. Let \((i, i+1)\) and \((j-1, j)\) be the first two switch points in the unstable cell interval (without loss of generality, \((i, i+1)\) is to the left of \((j-1, j)\) and there is no other switch point between them). Hence, for some \(\beta \in \{0, 1\}\), it holds that \(\sigma(i) = \sigma(j) = \beta\), and for every \(\ell \in (i, j)\), \(\sigma(\ell) = \overline{\beta}\).

Since \([i, j]\) is also an unstable cell interval, it must be the case that \(\sigma''(i) = \sigma''(i+1) = \overline{\beta}\) and for every \(\ell \in (i, j)\), \(\sigma''(\ell) = \beta\). Hence, both \((i, i+1)\) and \((j-1, j)\) are switch points in \(\sigma''\). By the switch point property, \(\sigma'(i+1+r) = \beta\) and \(\sigma'(j+r) = \overline{\beta}\).
Let \( \ell' \) be the rightmost cell in the open interval \((i + r, j + r)\) where \( \sigma'(\ell') = \beta \). Since the pair \((\ell', \ell' + 1)\) constitutes a switch point in \(\sigma'\), by the switch point property, it must hold that \( \sigma(\ell' - r) = \beta \). However, since \( \ell' - r \in (i, j) \) (because \( \ell' \in (i + 1, j + r) \)), we reach a contradiction to the assumption that for every \( \ell \in (i, j) \), it must hold that \( \sigma(\ell) = \beta \).

Thus, every unstable cell interval with respect to \(\sigma\) contains at most one switch point, which implies that the maximum possible length of an unstable cell interval is \(2r\). \(\square\)
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