Rodrigues, Olinde: “Des lois géométriques qui régissent les déplacements d’un système solide.,” translation and commentary

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Abstract

I provide for the first time in English a line-by-line translation of the entire text of the monumental 1840 memoir of Olinde Rodrigues, “On the geometrical laws governing the motions of a solid system...” published in French in the Journal de Mathématiques Pures et Appliquées. I accompany the translation with copious notes in italics, in which I explicate some passages whose meaning is obscure in the direct translation, supply detailed proofs, where lacking, of assertions in the original, and clarify the overall organization of the memoir and the relation of its sections to one another. (In my notes, Rodrigues himself is consistently called “the author”.) I often supply a rendering in modern vector notation, for equations and formulas in the original text in which vectors are expressed laboriously in terms of their projections on fixed axes.

Benjamin Olinde Rodrigues (1795-1851) was a successful banker and ardent social critic descended from a Sephardic family long resident in France. He associated himself for many years with the utopian philosophy of Henri de Saint-Simon, supporting the latter’s movement both financially and administratively. Physics students beyond the introductory level are likely to know of his formula for the Legendre polynomial of any order, but unlikely to know anything about its author. This formula was contained in his 1815 thesis as Docteur ès-Sciences at the University of Paris, a thesis containing much material of far greater depth. Among his subsequent writings for over twenty years are articles favoring the equality of women, articles on Saint-Simonism, articles on the reform of the banking laws, and other social issues, but none on science or mathematics. Then in the last eight years of his life he published a number of mathematical papers, including the magnum opus which is the subject of this translation.

The paper is a double tour-de-force in which the same material is expounded first geometrically in Sections 1-14 and then algebraically in Sections 15-22. In reading the second part, one will necessarily experience some déjà-vu, but one should keep in mind that although the same assertions or formulas reappear, the logic binding them is not necessarily the same, as the starting assumptions may be different.
Then Sections 23-33 contain a kind of coda in which new issues are presented; these sections are well worth study for themselves alone, especially Section 33 which gives a meaning to Lagrange parameters surely at odds with that intended by Lagrange himself.

The paper now follows in English translation.

1 General idea of translation and rotation of a solid system.

I understand by a solid system any assemblage of points, either continuous or discontinuous, that are mutually bound in a fixed way, such that if any three of these points are located at positions not in a straight line, and all their distances from other points of the system are given, the placement of the system will be completely determined for any placement of the triangle formed by these three points.

Such a system can actually exist, since on a given triangular base, with given lengths of its sides, one can construct only a single pyramid identical to - that is, superposable on - another given pyramid. A pyramid obtained by reflecting the first pyramid through its base would not be considered superposable. Accordingly, three noncollinear points of a solid being fixed, no displacement of the solid is possible.

But if only two points of the system are required to stay unmoved, by fixing the distances of all other points from these two, one assures, to begin with, that all points in a line with these two are unmovable. This line becomes a fixed axis, and any other point of the system can only move on the circumference of a circle concentric with and normal to the axis. Since all the points of the system are bound unchangeably to any one of them and to the fixed axis, the rotation of one implies the rotation of all, and the amplitude of this rotation is the same for all the points of the system.

Any displacement of a solid about two fixed points reduces, therefore, to a rotation, of equal amplitude and in the same sense for all the points of the system, about the axis formed by the two fixed points.

Here it must be remarked that any given rotation can be replaced by a rotation of the opposite sense, of an amplitude complementary to that of the first rotation with respect to 360°.

Different rotations about the same axis result in a rotation equal to their sum; here one must take care to assign contrary signs to the amplitudes of rotation that are effected in opposite directions, but otherwise the order of succession of the rotations remains arbitrary.

If the amplitude of the rotations is infinitely small, the arcs described by the points of the solid located at a finite distance from the axis become indistinguishable from their chords, while the latter are variously inclined according to the angles of the rays drawn from the axis to the points of the system.

But suppose that the axis, while still firmly bound to the system under consideration, is infinitely far removed from it, and that the system undergoes an infinitely small rotation about the axis, of an infinitesimal
order reciprocal to that of the distance of the axis of the system. The effect will be to make all those points describe equal and parallel straight lines, so that the system will have simply undergone a translation, that is to say a displacement resulting in an equal transport of all its points in a certain direction.

Thus any translation of a system can be considered rigorously as a rotation of infinitesimal amplitude about an axis infinitely far off and normal to the direction of that translation.

It is no surprise, then, to find moreover that all the properties of translations are implied by those of rotations, just as those of a straight line are implied by those of a large circle to which the line is tangent. We need not linger over this.

We shall complete this general exposition of the displacement about a fixed axis by the following theorem, which is evident from the figure and whose consequences will be of use in what is to follow.

1.1 Movement of the axis of rotation without changing its direction.

The rotation of a solid system about a fixed axis can be replaced by an equal rotation about another parallel axis, followed by a translation of the system equal and parallel to the chord of the arc described by a point of the second axis about the first. Or, what comes to the same thing, to the chord of the arc that would be described by a point of the first axis about the second, except that the direction of the translation must be reversed.

The author apparently considers this theorem as having been proved by the preceding discussion, along with the figure referred to. In fact, the theorem is an immediate corollary to the theorem on couples to be discussed in 10.1.

2 Displacements by translation.

If two situations of the same solid are such that all the lines joining a point of the solid in one situation to the corresponding point in the other are equal and parallel, the solid may clearly arrive from the first to the second situation by sliding parallel to itself along one of these lines. The length and direction of the line will measure those of the translation of the system.

2.1 General law of composition of successive translations.

If the system undergoes several consecutive translations, differing both in direction and in extent, it is evident that all these translations sum up to a single unique translation, equal and parallel to the line that joins a point of the first situation to its corresponding point in the second. This line would close the polygon traced by the successive translations of the point in question, and its length and direction, as is well known, depend only
on those of the various other sides of the polygon and are independent of their order of succession.

By means of this law, one can reciprocally decompose any given translation into a succession of diverse translations, provided only that the sum of their projections on three perpendicular axes, or more generally on any arbitrary axis, is equal to the projection of the given translation on the same axis, or to the sum of its projections on the three axes. *The projections are here being “added” in what we would call a “vectorial” sense.*

One may call this law of composition the law of the polygon of translations.

3 On the displacement of a system about a fixed point.

What follows is a remarkably concise proof of a celebrated theorem due to Euler.[1]

Let two situations of the same system be given, sharing a point O which remains fixed in passing from one situation to the other. It should be understood that the displacement can be expressed in some way as a succession of rotations, with no reflection. Consider two arbitrary points A, B in the first situation, different from O and not collinear with it. In the second situation the corresponding points are A′, B′. Here and elsewhere I have used letters to designate points, lines, etc. as an aid to the reader’s comprehension. In the French original no letters are used, all geometrical entities being described purely by words.

The isosceles triangles AOA′, lying in plane Pa, and BOB′, lying in plane Pb, share a common vertex O. Through O pass a plane Na which is normal to Pa and bisects the vertex angle θa. See Fig. 1Ai and 1Aii. In Fig. 1Aii the points A, A′ are superimposed, as the line AA′ is normal to the plane of the diagram. The line marked xA is actually the common perpendicular to AA′ and L, a line defined below.

Likewise a plane Nb normal to Pb and bisecting θb. See Fig. 1Bi and 1Bii. Again, xB denotes the common perpendicular to BB′ and L.

The intersection of Na and Nb will be a line L (See Fig. 1C) passing through O and normal to both Pa and Pb. In Fig. 1C, it must be understood, if we regard the line L as ‘vertical’, that the triangle OAA′ is not “horizontal”, although its base AA′ is horizontal. Likewise OBB′ is not horizontal, although its base BB′ is. Also, although AA′ and BB′ are both horizontal, they lie in general in different horizontal planes and consequently do not intersect.

Any point S on L (being equidistant from A and A′, as well as from B and B′) can be considered as the common summit of two identical or superposable pyramids OSAB and OSA′B′ see the second paragraph of Section 1, having as bases the triangles OAB and O′A′B′; so that the line L, being invariably bound to the displaced system, remains unchanged by the displacement. Hence this displacement reduces to a rotation around the fixed axis L. (*L is prevented from sliding along its length by the*
immobility of $O$. This short passage is the whole proof, different from that
given by Euler.)

Two pyramids are congruent if they agree in the lengths of all six edges.
The edges $OA, OB, AB$ are equal respectively to $OA', OB', A'B'$ because
the displacement is rigid. The equality $SA = SA'$ follows from the con-
struction of the plane $N_a$, and $SB = SB'$ from that of $N_b$. The edge
$OS$ is the same in both pyramids because $S$ is defined as a point on $L$
a certain distance from $O$. The line $L$, however, has been located by a
construction drawing on both the initial and the final configurations. But
now the pyramid $OSAB$ makes it possible to determine the position of $L$
in terms of the initial configuration alone; and alternatively $OSA'B'$ by
the final configuration alone. The congruence of the two pyramids then
ensures that the location of $S$ is the same (both in the body and in space)
in both configurations, so that $L$ is unmoved by the displacement. (The
author’s primary emphasis on pyramids was already evident in the second
sentence of 1.)

In the singular case (it could be avoided if we wished) in which the
planes $N_a$ and $N_b$ coincide in a single plane $N$, we see that the axis is
simply the line of intersection of the planes $Q, Q'$ containing the original
triangles $OAB$ and $OA'B'$. Any other line, should it lie in the plane $N$,
forms with the sides of these triangles two trihedral angles symmetrically
related but not superposable.

To sum up, any displacement of a system (achievable by rotations)
about a fixed point reduces to a rotation about a fixed axis passing through
this point. Or, more generally, it reduces to an equal rotation about a
different fixed axis parallel to the first one, but otherwise located where
one will, provided that the rotation be followed by a translation of the
same extent and direction as the chord of the arc described by the original
fixed point $O$ under the rotation about the new axis; but the sense of the
translation must be opposite to that of the chord. (This generalization
follows from the theorem in Section 1 relating to the parallel transport of
an axis of rotation.)

4 On the most general displacement of a
solid system in space.

Now let us consider any two situations whatsoever of the same solid, and
seek the simplest mode of displacement that can bring the solid from
one situation to the other. Select any point $O_1$ of the solid in the first
situation, and imagine that for each point $A_1$, a straight line is drawn
from $O_1$ that is equal and parallel to the line $O_2A_2$ in the second situation.
Denote by $A_{12}$ the termination of the line thus drawn. Thus $O_1A_{12}$ and
$O_2A_2$ are opposite sides of a parallelogram, whence the same is true of
$A_{12}A_2$ and $O_1O_2$. We have thus constructed an intermediate assemblage
of points, $O_{12}, A_{12}, ..., $ forming a solid entirely identical to the one under
consideration, but lying in a situation intermediate between the two given
ones. Since $O_{12} = O$ (by considering the case $A_{12} = O_1$), the intermediate
situation can be derived from the first situation (in view of the theorem
of the preceding section) by means of a certain rotation through an angle \( \theta \) about a fixed axis \( L \) passing through \( O_1 \) (\textit{this is indeed the fixed point theorem of Euler, proved by the author in 3}) while the passage from the intermediate to the second situation requires only a translation whose extent and direction are the same as those of the line \( O_1O_2 \) (\textit{on account of the parallelogram} \( O_1O_2A_2A_1 \)).

Moreover, we observe that in view of the theorem on parallel transport in Section 1, nothing prevents us from supposing that this intermediate situation of the solid is reached from the first situation by a rotation through the same angle \( \theta \) about an arbitrary line \( L' \) parallel to \( L \), followed by a translation equal to the chord of the arc that would have been described by a point on \( L' \) in making the rotation about \( L \) that would take the first to the intermediate situation. \textit{The whole displacement taking each} \( A_1 \) to \( A_2 \) \textit{has already been decomposed as} \( R+T \) \textit{where} \( R \) \textit{is the rotation through} \( \theta \) \textit{about} \( L \) \textit{and} \( T \) \textit{is the translation taking} \( O_1 \) to \( O_2 \). By the theorem on parallel transport, \( R \) \textit{can be decomposed as} \( R'+T_{LL'} \) \textit{where} \( R' \) \textit{is the rotation through} \( \theta \) \textit{about} \( L' \) \textit{and} \( T_{LL'} \) \textit{is a translation normal to} \( L \) \textit{and} \( L' \).

But this translation and the following one taking the intermediate to the second situation combine to make a single translation equal and parallel to the line joining any point on \( L' \) to its corresponding point in the second situation. \textit{Thus the whole displacement is} \( R'+T_{LL'}+T = R'+T' \) \textit{where} \( T' = T_{LL'}+T \) \textit{is the translation that would take} \( O'_1 \) to \( O'_2 \). Now, for any origin chosen instead of \( O \), there is only one axis of rotation possible; therefore we have completely demonstrated the following theorem, indisputably one of the most beautiful in geometry, which deserves to be considered the fundamental basis of the geometric laws of the movement \textit{[of a rigid body in 3 dimensions]}.

\subsection{Fundamental theorem.}

However a solid has been transported from one place to another, the displacement can always be considered as resulting from two consecutive displacements, a rotation and a translation. The rotation takes place about a fixed axis passing through an arbitrarily chosen point in the initial situation and parallel to a certain direction. This direction, as well as the amplitude and sense of the rotation, is invariably determined by the initial and final situations. The translation takes place parallel to the line joining a point on the said axis to its corresponding point in the second situation, and its length is the length of that line.

The order of these displacements can be reversed: the translation can precede the rotation, but the latter then takes place about an axis passing through the point in the final situation that corresponds to the point that was taken to be the origin in the initial situation. In addition, the direction of the axis of rotation and its amplitude and sense are the same for all the points of the system, whether before or after the translation.

In this succession of displacements, let us observe that the line joining any point of the initial situation to its corresponding point in the final situation, that is the resultant line really traced by this point, forms the third side of a triangle, of which the first side \textit{representing the effect of the}
rotation) is variable for different points of the system but always normal to the axis of rotation, and the second side, constant for all the points of the system, measures the translation of the system.

The projection of this resultant line on the axis of rotation is therefore constant for all the points of the solid. This constancy can be achieved only relative to the direction of the axis of rotation, for the projection on an arbitrary direction is the sum of two projections, that of the chord of the arc of the rotation and that of the line traversed in translation. The first of these is variable, the second is constant. Their sum cannot, therefore by constant for any direction other than that which causes the first of these projections to vanish, that is, the direction of the axis of rotation.

All the points of a solid system displaced in an arbitrary manner are therefore equally transported relatively to the (invariably determined) direction of the axis of rotation.

If this constant projection vanishes, the displacement reduces to a rotation about a certain (preferred) axis without any translation. The transverse location of this axis is easily found by considering a plane normal to the common direction of all the possible axes of rotation. The straight line from any initial point in this plane to its corresponding final point will be the base of an isosceles triangle lying in the plane, of which the angle at the vertex will equal the amplitude of the rotation, and the vertex itself will lie on the (preferred) axis to be found.

In the more general case in which the displacement is not a pure rotation, this constant projection is the measure of the absolute translation of the system, by which is meant the minimum translation among all those associated with the possible axes of rotation, variously located but all having a common direction; it is none other than the translation of those points of the system that are displaced parallel to [this common direction]. If any of these points is chosen as the origin $O$, the axis of rotation, which we shall distinguish by calling it the central axis of the displacement, is also the axis of translation.

Thus the displacement reduces, with respect to its central axis, to turning about this axis while sliding parallel to its direction: a kind of movement that has been compared to that of a screw turning in its nut. This is the simplest expression of the fundamental theorem, in which the two displacements of rotation and translation take place simultaneously and orthogonally. Here the author attributes the theorem to M. Chasles.

This theorem may be viewed as the natural generalization of Euler's fixed point theorem - a generalization in which the displacement is assumed to result from an arbitrary series of rotations (excluding reflections) about axes that are not assumed to have a common point of intersection.

5 Locating the central axis.

But we have now to find this central axis, that is to find those points of the system carried by the displacement along a line parallel to the common direction of the possible axes of rotation. Now, one may arrive at this by

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the following construction.

Let $A$ be an arbitrary point in the first situation, and $A'$ its corresponding point in the second. Let $L$ be the line through $A$ in the common direction which is the (known) direction of the central axis. Let us think of this direction as “vertical”. Let $P$ be the (vertical) plane containing $L$ and $AA'$. (See Fig. 2A.) Within $P$, erect from $A$ a (horizontal) perpendicular to $L$ which terminates at the point $Z$ chosen so that $ZA'$ is a right angle. We may introduce Cartesian coordinates $x, y, z$, with origin at $A$, and let $L$ be the $z$-axis and $AZ$ the $x$-axis. Then $P$ is the $(x,z)$ plane.

Now let $R$ be the plane containing $AZ$ and normal to $L$. $R$ is the $(x,y)$ plane. See Fig. 2B. Note that $Z$ and $A'$ are both in the $(x,z)$ plane $P$ and that the line $ZA'$ is perpendicular to the $x$-axis $AZ$; therefore $ZA'$ is vertical, parallel to the $z$-axis $L$. Hence $ZA'$ is normal to the horizontal plane $R$. In other words, $Z$ is the foot of the perpendicular dropped from $A'$ to $R$.

Within $R$, construct an isosceles triangle having $AZ$ as base and vertex angle equal to the (known) amplitude $\theta$ of the rotation. The vertex $V$ of this triangle will lie on the central axis, provided only that the isosceles triangle is placed, relatively to $AZ$, in the sense of the rotation. On a given base in a given plane, there are two ways to erect an isosceles triangle with a given vertex angle, related by reflection in the base. Only one is right.

For it is evident that this vertex $V$, turning through $\theta$ about $L$, will reach a point $\bar{V}$ in the plane $R$, such that the chord from $V$ to $\bar{V}$ is equal and parallel to the line from $Z$ to $A$ because the triangle $VAV$ is congruent to the triangle $ZVA$, and moreover that the translation from $\bar{V}$ to the image $V'$ of $V$ in the second situation is equal and parallel to the line from $A$ to $A'$. The whole motion from $V$ to $V'$ is made by the rotation carrying $V$ to $\bar{V}$, followed by the translation from $\bar{V}$ to $V'$; whereas the motion of $A$ to $A'$ consists entirely of the translation. Since a translation affects all points equally, it follows that $V'V$ is equal and parallel to $AA'$. See Fig. 2C. Hence the resultant motion from $V$ to $V'$ is equal and parallel to the resultant motion from $Z$ to $A'$ that is, $VV' + V'V = ZA + AA'$ which in turn is parallel to $L$ as noted in previous paragraph but one. But this is just the condition satisfied by points on the central axis.

And reciprocally, taking $VV'$ as the axis of rotation, the point $A$ on rotation through $\theta$ about $V$ will travel to $Z$, and then by the translation parallel to $VV'$ will travel from $Z$ to $A'$, reaching its given destination.

And this construction shows that when the axis of relative translation $AA'$ is normal to the axis of rotation $L$, the whole displacement reduces to a simple rotation about the central axis, since then $Z$ coincides with $A'$ and so $V$ and $V'$ are the same.

Also, if a displacement of the solid is such that all the points of the solid remain in mutually parallel planes, the displacement reduces to a rotation about some fixed axis normal to these planes.

Although the foregoing construction correctly locates the central axis, the information required to carry it out as well as the reasoning to justify it are drawn from the whole of the preceding paragraphs of the essay. But if one studies the construction in this light, one is forced to keep in mind much redundant material. If, on the other hand, we disregard the preceding
paragraphs, we are in danger of drawing false inferences. For example, it looks as if this construction requires only one pair $A, A'$ of corresponding points. But that is not so: the fixed axis theorem of Euler, proved in 3, requires two such pairs, and the result is a necessary part of the proof of the author's construction to locate the central axis. Therefore he gives another construction, entirely self-contained and requiring only the location of two pairs $A, A'$ and $B, B'$ as well as the direction of the central axis. He attributes this construction to "mon ami M. Lévy".

Drop from $A$ a line perpendicular to the central axis at a point $C$, and likewise from $A'$ a perpendicular at $C'$, and consider the quadrilateral $ACC'A'$. Let $A^m, C^m$ be the midpoints of $AA', CC'$; then the line $A^mC^m$ will be perpendicular to both $AA'$ and $CC'$. This follows from the symmetry of the figure with respect to a $180^\circ$ rotation about $A^mC^m$. This property gives the following construction of the central axis, being given only the points $A, A'$, another corresponding pair $B, B'$, and the direction of the central axis:

Through each point $A_m$ (resp. $B_m$), pass a line $L_a$ (resp. $L_b$) parallel to the central axis, as well as a line $N_a$ (resp. $N_b$) normal to both $AA'$ and $L_a$ (resp. $BB'$ and $L_b$). Let $P_a$ (resp. $P_b$) be the plane formed by $L_a$ and $N_a$ (resp. $L_b$ and $N_b$). Then the planes $P_a$ and $P_b$ will intersect precisely on the central axis.

To justify the construction, we must refine our notation to take into account the use of two pairs: the quadrilateral based on $A$ must now be called $ACC'A'$, and that based on $B$ must be called $BCB'C'$. The points $C_a, C_b$ are not necessarily the same, but both lie on the central axis, as well as $C'_a, C'_b$.

Since $L_a$ is parallel to $C_aC'_a$, $N_a$ is normal to $C_aC'_a$ as well as to $AA'$; thus $N_a$ is the very line of which $A^mC^m$ is a segment. Therefore $C^m$ lies on $N_a$. But then the plane $P_a$ can equally well be described as formed by $AA'$ and $C_aC'_a$ instead of by $AA'$ and $L_a$, so that it contains the whole line $C_aC'_a$, that is the whole central axis. Likewise the plane $P_b$ is formed by $BB'$ and $C_bC'_b$ and also contains the central axis. Hence $P_a$ and $P_b$ intersect on the central axis, q.e.d.

There are degenerate cases, which will be left to the reader's study.

6 Consequences of the Fundamental Theorem.

It now behooves us to set forth the principal corollaries that follow from the fundamental theorem, relating to the particular displacements of the points, the lines, and the planes of a solid system.

(a) The distances separating each point of the solid in the first situation from its corresponding point in the second all have equal projections on the direction of the central axis; this common projection is the measure of the absolute translation of the system.

(b) As any line belonging to the displaced system does no more, with respect to its direction, than turn about the axis of rotation, there results a very simple relation between the angle formed by this line with
the axis of rotation and that formed between the initial and final directions of this line, to wit:

“The sine of the half-angle of displacement of any line belonging to a displaced system is equal to the sine of the half-rotation of the system, multiplied by the sine of the angle between this line and the axis of rotation.”

The author here is concerned only with the directions of the line in the initial and final configurations. Now, the set of all possible directions can be mapped onto the points on the surface of a sphere (say of radius $r$) by mapping each direction $D$ to the unique point $P$ for which the line from $O$, the center of the sphere, to $P$ has the direction $D$. The initial direction of the line in question thus maps to a point $A$ on the surface of the sphere, and its final direction to a point $B$. The direction of the central axis (the “axis of rotation”) is mapped to a point $C$, which we shall call the North Pole so as to make use of the ideas of latitude and longitude. The central axis is then the diameter from North to South Pole, and the rotation about this axis clearly does not change the angle made with it by the line in question. Thus $A$ and $B$ lie on the same circle of latitude, which we shall call $L$, and the arc from any point $P$ on $L$ to the North Pole is of a fixed length, making a constant angle $POC$ which we shall call $\chi$. The “angle of rotation of the system” is the difference in longitude between $A$ and $B$, which we may call $\phi$, and the “angle of displacement” of the line in question is the angle $\theta = AOB$, which measures the geodesic distance from $A$ to $B$.

With these definitions, the proposition rendered above in words may be expressed in symbols as

$$\sin(\theta/2) = \sin(\phi/2) \sin \chi. \tag{1}$$

We can understand the role of $\chi$ by noting that when $\chi$ becomes small, the geodesic distance from $A$ to $B$ becomes small because of the pinching of the base of the isosceles triangle $ABC$, even though the difference $\phi$ in longitude is kept constant.

The equation (1) can be derived as a restriction of the spherical Law of Cosines to isosceles triangles; but the author gives no indication of having such reasoning in mind. Instead, he says briefly that the proposition “is made evident by observing the figure”. Unfortunately, the figures originally appearing in the Journal de Mathématiques have been lost, but I believe I have closely reconstructed the one referred to in this passage, with the aid of my associate Dr. Hamilton.

The easy demonstration of (1) depends on the construction of planes and straight lines in the interior of the sphere (see Fig. 3A), particularly the straight line $\overline{AB}$. On the one hand, this line is a chord of the great circle $G$, of radius $r$ and center $O$ (see Fig. 3B) upon which both $A$ and $B$ are located; since this chord subtends an angle $\theta$ at $O$, we have

$$\overline{AB} = 2r \sin(\theta/2). \tag{2}$$

On the other hand, this same line is a chord of the small circle of latitude, $L$ (see Fig. 3C), whose center $K$ lies on the axis $OC$ and whose radius
we shall call $\rho$. Since the angle subtended at $K$ is $\phi$, we have

$$AB = 2\rho \sin(\phi/2).$$

(3)

Comparing (3) with (2), we find

$$\frac{\sin(\theta/2)}{\sin(\phi/2)} = \frac{\rho}{r}.$$  

(4)

Finally we determine $\rho/r$ by considering the right triangle $OKP$ (see Fig. 3D) for an arbitrary point $P$ on $L$. The hypotenuse $OP$ has length $r$, the angle $POK$ is $\chi$, and the side $KP$ opposite this angle has length $\rho$. Therefore

$$\rho/r = \sin \chi.$$  

(5)

Substituting (5) into (4), we obtain (1).

(c) Any line parallel to the axis of rotation is transported parallel to itself, while any line normal to that axis suffers an angular displacement equal to the amplitude of the rotation. (Special cases of (b)]

(d) Any plane invariably bound to the displaced system, and normal to the axis of rotation, is therefore transported into a plane parallel to the initial one, at a distance equal to the absolute translation of the system.

(e) The midpoint of the line that joins any point of the system to its correspondent i.e., to its final position after displacement is the point of that line that approaches most closely to the central axis of the displacement.

(f) the midpoints of all the lines that join the various points of a plane figure to their correspondants after an arbitrary displacement lie in a single plane, which also contains the midpoints of the lines joining any point outside the plane figure to its symmetric correspondent.

This plane makes equal angles with the planes of the two plane figures, as well as with the corresponding lines bound to the two figures, but if not within their respective planes, then symmetrically inclined.

The meaning of “symmetrically” in (f) is “by reflection in the special plane under consideration.” I leave the study of (f) to the most ambitious of readers.

7 The decomposition of any displacement into two pure rotations.

Having presented the fundamental geometric law of the passage of a solid from one given situation to another, also given in an arbitrary way, we have now to study the law of composition of successive displacements; by means of this law one can construct or calculate the elements of the composite displacement, that is the position of its central axis, the amplitude of its rotation, and the extent of its translation.

We have already presented the law of composition of translations; we shall next give that of rotations about different fixed axes, and finally that of arbitrary displacements, each resulting from a combined translation and rotation.
From this law of composition of rotations about different axes, we shall deduce an important transformation of the fundamental theorem (4.1), to wit:

“Any displacement of a solid system can be represented, in an infinite number of ways, as the composition of two successive rotations of the system about two nonintersecting fixed axes. The product of the sines of half these rotations, multiplied by the sine of the angle between the two axes and by the minimum distance between them, is equal - for each of these conjugate pairs of axes - to the product of the sine of half the angle of rotation of the system about the central axis of the (total) displacement with half the length of its absolute translation.”

Let the arbitrary (total) displacement be characterized by a central axis \( \mathcal{C} \), a rotation angle \( \theta_\mathcal{C} \), and a translation distance \( T \). It is asserted that there are infinitely many “conjugate pairs”, each of which consists of a rotation \( \theta_A \) about an axis \( A \) followed by a rotation \( \theta_B \) about an axis \( B \), such that the composition of these two rotations is equivalent to the total displacement under consideration. Each of these conjugate pairs is related to the total displacement by the equation

\[
D \sin \nu \sin(\theta_A/2) \sin(\theta_B/2) = (1/2)T \sin(\theta_\mathcal{C}/2),
\]

where \( D \) is the minimal distance between the two axes and \( \nu \) is the angle between their directions.

The author only states this remarkable theorem here, deferring its proof to 11.

To put it another way, the volume of the tetrahedron of which two opposite edges lie anywhere along the respective conjugate axes, provided that the length of each of these edges is proportional to the sine of the corresponding half-angle of rotation, is the same for all conjugate pairs of rotations whose composition is equivalent to a given displacement.

The equivalence of this second statement of the theorem to the first is based on a theorem of geometry, that the volume of any tetrahedron is one-sixth the product of the lengths of any two opposite edges, times the minimal distance between them, times the sine of the angle between them. This formula can be established by a variety of methods; further study is left to the reader.

Thus, any displacement of a solid system reduces to a rotation about one or two fixed axes.

In the case where one of these axes is parallel to the central axis (of the total displacement), it follows from the law of composition of rotations that its conjugate is situated at infinity and that the rotation corresponding to it becomes infinitely small and therefore amounts to a simple translation. This leads to the original version (4.1) of the fundamental theorem, so that that version is no more than a particular case of the theorem we have just given.

8 The composition of two given rotations.

We have now to present the law of composition of successive rotations of a solid about different axes. The author now begins the succession of
composition theorems that will lead ultimately to the decomposition theorem stated in the previous section.

Let us begin by considering only two intersecting axes, and let us seek to determine the resultant axis of these two rotations - the one about which the given solid will be finally found to have turned, in order to arrive from the initial to the final situation. (A problem closely related to this was studied in [3].)

This resultant axis must be placed in such a way that in being subjected to the two rotations indicated about the supposed intersecting axes, it comes back to its initial position. If, therefore, through each of the given axes one passes a plane that makes an angle with the plane of the two axes that is equal to half the rotation about that same axis, the intersection of these two planes will be the resultant we seek, as it arrives by virtue of the first rotation at the position symmetric by reflection in the plane of the two axes, and returns by the second rotation to its original position.

Call the two intersecting axes \( A \) and \( B \); they determine a plane that we shall call \( AB \). The resultant axis we call \( C \), and let \( AC, BC \) denote the planes formed respectively by \( A \) and \( C \), and by \( B \) and \( C \). The angles of rotation about \( A \) and \( B \) are \( \theta_A \) and \( \theta_B \).

It is assumed that the axes \( A \) and \( B \), intersecting at the origin \( 0 \), and the amplitudes of rotation \( \theta_A \) and \( \theta_B \), are given, and the problem posed is to determine the axis \( C \) and rotation amplitude \( \theta_C \) such that the rotation \( \theta_A \) about \( A \) followed by the rotation \( \theta_B \) about \( B \) will produce as resultant the rotation \( \theta_C \) about \( C \). By Euler’s theorem (3) such a \( C \) exists and passes through the origin \( 0 \).

In understanding the solution proposed by the author, it is necessary to distinguish the angle between two axes (e. g. between \( C \) and \( B \)) from the angle between two planes (e. g. between \( AC \) and \( AB \)).

The solution proposed is that \( AC \) should make an angle \( \theta_A/2 \) with \( AB \), and \( BC \) should make an angle \( \theta_B/2 \) with \( AB \). In this way the planes \( AC \) and \( BC \) are determined, and \( C \) is their intersection.

The argument is that if \( C^* \) is defined as the reflection of \( C \) in \( AB \), then the angle between \( AC \) and \( AC^* \) will be twice that between \( AC \) and \( AB \), that is twice \( \theta_A/2 \), so that the first rotation (of \( \theta_A \) about \( A \)) will bring \( AC \) to \( AC^* \) and hence \( C \) to \( C^* \) (since the angle between \( C \) and \( A \) is unchanged by the rotation). Then by a similar argument, the second rotation (of \( \theta_B \) about \( B \)) will take \( C^* \) back to \( C \). So the combined effect of the two partial rotations will be to leave \( C \) unaltered. But this is what is required in order that \( C \) be the resultant axis.

(It may well be objected that it has not been shown that the axis \( C \) is restored to its original position in the same sense in which it began. But if not, the situation may be remedied by taking the appropriate sign of \( \theta_C \).)

At the same time one sees that the angle between the two planes \( \langle AC \) and \( BC \rangle \) will be half the angle of the resultant rotation \( (\theta_C) \). For the first axis, which does not move under the first rotation, is displaced only by the second, and it describes about the resultant axis, determined as shown above, an angle twice that between the two planes.

The phrase “determined as shown above” refers to the determination of \( C \) as the intersection of two planes, of which in particular the plane
forms an angle $\theta_B/2$ with the plane $(AB)$. It follows that a rotation about $B$ through an angle $\theta_B$ brings the plane $(AB)$ to its reflection in $(BC)$. and in particular it brings $A$ to its reflection $A^*$ in $(BC)$.

But since $A$ was unchanged by the first rotation about itself, this second rotation about $B$ must have the same effect on $A$ as does the composite rotation through the angle $\theta_C$ (to be found) about $C$. That is, this composite rotation must take $A$ to $A^*$. But this requires that $\theta_C$ be twice the angle between $(AC)$ and $(BC)$, as asserted by the author.

Here let us note that the half-rotation (of a plane) about each axis can be measured equally well by the interior or the exterior angle of the two planes passing through that axis, only the sense of the rotation depending on which measure one adopts, since any rotation (of a point) about an axis in one sense is equivalent to a rotation in the opposite sense with an amplitude complementary to the first by $360^\circ$.

The conscientious reader may wish to ascertain that if a consistent sign convention be followed, whereby the angle of rotation about an axis is measured either always clockwise or always counterclockwise looking along the direction of the axis, this construction will yield the correct sign of $\theta_C$ in relation to those of $\theta_A$ and $\theta_B$.

Furthermore, as to the order of succession of these rotations, it comes about that if the two rotations are supposed to take place in a certain order, leading to a particular resultant axis $C$, then by reflecting this axis in the plane of the of the two given axes one obtains the resultant $C^*$ of the same two rotations in the inverse order. From this we see that the amplitude of the resultant rotation is independent of the order of the two given ones, but that the position of the resultant axis depends on this order, and that in the composition of more than two rotations about arbitrary intersecting axes, the order cannot be modified without altering both the position of the resultant axis and the amplitude of the resultant rotation.

The last statement is a bit too strong: the resultant amplitude will be unaltered if the order is completely inverted, as from $1234$ to $4321$, no matter how long the succession is. Likewise the amplitude is preserved under a cyclic permutation as from $12345$ to $34512$. A corollary of these two facts is that the amplitude cannot be altered by any permutation of the composing displacements unless they number at least 4.

Such is the characteristic difference between the composition of rotations and that of successive translations. In fact, these two kinds of composition are analogous in a way similar to the properties of a plane triangle and those of a spherical triangle. For if one compares the translations parallel to the three sides of a planar triangle to the sines of the half-angles of rotations effected around the three sides of a trihedral angle, the values of these translations and those of these sines are in equal manner proportional to the sines of the angles opposite to the respective sides of the planar triangle and of the trihedral angle.

The author is essentially comparing the Law of Sines for a planar triangle to that for a spherical triangle. But he adds the complication of associating the sides of the triangle to the corresponding translations or rotations of a solid system.
9 Composition of infinitesimal rotations.

But these two resultant axes $C$ and $C^*$, corresponding to the same two rotations in two different orders of succession, will coincide in the plane of the two axes if the rotations become infinitely small, and from this there follow two important consequences:

First, the order of succession of infinitesimal rotations about two intersecting axes (and, as it follows, about as many such axes as one wishes) is immaterial. And second, the axis and amplitude of the infinitesimal rotation resulting from the succession of two infinitesimal rotations $A$ and $B$ about two intersecting axes are determined in the same way as the axis and translation length that would result from two successive translations proportional to the given rotations and parallel to their axes.

The author is referring here to the theorem enunciated in Section 1: “Thus any translation of a system can be considered rigorously as a rotation of infinitesimal amplitude about an axis infinitely far off and normal to the direction of that translation.”

Note that this theorem concerns a single axis, and in the present context it applies separately to $A$ and to $B$. Note also that the angle between $A$ and $B$ is finite, unlike that between $C$ and $C^*$.

Since, by removing the axes of rotation far away, one may transform the infinitesimal rotations into finite translations perpendicular to these axes and inclined one to the other just as the axes are to each other, one achieves all the generality of the law of composition of finite rotations, which by mediation of the infinitesimal rotations includes also the law of composition of translations.

The author evidently does not mean that the two axes are removed far from each other, since they continue to intersect. Rather, he is observing the action of the infinitesimal rotations at points far from both of the two axes. But these points are regarded, for the present purpose, as “here”, while the two axes with their intersection are “there”, that is removed to infinity.

The author’s point is that the law of composition of finite rotations is so powerful that by suitable applications of it one can derive that of composition of finite translations as well.

10 Of the composition of rotations about two parallel axes.

All the points of the system displaced by two consecutive rotations about two parallel axes remain within parallel planes normal to these axes. Therefore the displacement reduces to a simple rotation about a certain axis parallel to the first two. This being admitted, the mode of determination and of construction of the resultant axis of two intersecting axes applies equally well to this case, and yields a resultant axis parallel to the first two and a composite rotation equal to the sum or the difference of the given rotations, according as they act in the same or opposite senses.
10.1 Couples of parallel rotations.

But here we encounter a remarkable case, that in which the two rotations are equal and of opposite sense. The composite rotation is then null and the composite axis is placed at infinity, which causes the displacement to amount to a simple translation. To be precise, each point of the displaced solid has traversed a line of length and direction constant for all points of the system; the direction of this line is normal to the two axes but makes an angle with the normal to the plane of the two axes, equal to half the given angle of rotation about each axis; and its length is the product of the distance between the axes by twice the sine of half the rotation angle.

Using rectilinear coordinates \( x, y, z \), let the two axes point in the \( z \)-direction and their separation \( d \) in the \( x \)-direction, so that they lie in the \( x-z \) plane; then the normal to their plane is in the \( y \)-direction. Since the coordinate \( z \) is unchanged in each rotation, we may regard the whole operation as confined to the \( x-y \) plane. Take any initial point \( P \) in this plane. Under the rotation about the first axis \( A \) it describes an arc of amplitude \( \theta \) ending at a point \( P^o \). Then the rotation about the second axis \( B \) carries the point along an arc also of amplitude \( \theta \) but in the opposite sense, from \( P^o \) to its final position \( P' \). The straight line from \( P \) to \( P' \) is asserted by the author to have the same length and direction for all points \( P \) - a fact by no means obvious. The following proof can best be followed by consulting Figs. 4a and 4b.

Let the axes \( A \) and \( B \) cut the \( x-y \) plane at the points \( A \) and \( B \), so that the distance \( d \) between the axes is the line segment \( AB \). And let the distances from \( P^o \) to the axes \( A \) and \( B \) be respectively \( r_A \) and \( r_B \). Then the isosceles triangles \( \triangle PAP \) and \( \triangle PBP \) are similar, having the same vertex angle \( \theta \); and their scale ratio is \( r_A/r_B \). Therefore \( PP^o/P^oP' = r_A/r_B \).

The base angles of the isosceles triangles pertaining to various initial points \( P \) are all equal to \( \phi = (180^\circ - \theta)/2 \). Therefore the angles \( \angle APB \) and \( \angle P'P \) are equal, being both equal to \( \phi + \chi \) where \( \chi \) is the (undetermined) angle \( AP^oP' \). (This assertion merits close examination of Fig. 4a.)

Hence the triangles \( \triangle APB \) and \( \triangle PBP' \) are similar, having two sides in the same ratio \( AP^o/BP^o = PP^o/P^oP' = r_A/r_B \) and agreeing in the included angle.

It follows that \( PP'/AB = PP^o/AP^o \) or \( PP' = (AB)(PP^o/AP^o) = AB(2\sin(\theta/2)) \), as asserted by the author. Note that this equation does not involve the angle \( \chi \).

Furthermore, the angle between the directions \( PP' \) and \( AB \) is the angle through which the triangle \( PP^oP' \) must be turned about \( P^o \) so as to make the angles \( \angle APB \) and \( \angle PP'P \) coincide. Clearly this angle is \( \phi \), so that \( PP' \) makes an angle \( \phi = (180^\circ - \theta)/2 = 90^\circ - \theta/2 \) with the \( x \)-axis. Therefore it makes an angle \( \theta/2 \) with the \( y \)-axis, as claimed.

Figs. 4a and 4b show two realizations of this construction. Using the line segment \( AB \) as the reference for length and direction, one sees that the point \( P \) is placed differently in the two diagrams, and that the ratio \( r_A/r_B \) is also quite different, as well as the size of the angle \( \chi \). Nevertheless, the dotted line from \( P \) to \( P' \) has the same length as well as the same direction in both diagrams, and in each diagram the triangle \( PP^oP' \) is similar to the triangle \( AP^oB \) as found in the above proof.
It may help the understanding to distinguish between **determinate** ratios, which for a given \( \theta \) are independent of the choice of \( P \), and **indeterminate** ratios, which are affected by that choice. The two isosceles triangles in the diagram have determinate shape, which yields the ratio \( 2 \sin(\theta/2) \) (base to sides), but their relative size is indeterminate, depending on the ratio \( r_A/r_B \). For the two symmetric triangles, however, the situation is reversed: the shape of these triangles involves \( r_A/r_B \) and is hence indeterminate, while the ratio of their size is determinate as the diagram exhibits it as tied to the shape of the isosceles triangles.

Once it is established that the direction and length of \( PP' \) is the same for all \( P \), the author’s statements about this direction and length can be derived very easily by setting \( P = A \). For then one has obviously \( P^0 = P \), and \( PP' = P^0P' \) which is the base of an isosceles triangle with vertex angle \( \theta \) at \( B \) and side \( AB \). The author’s statements follow.

The order of succession of the two rotations makes a difference; if the order is reversed, the composite line of translation is reflected about the normal to the two axes.

All this follows easily from a comparison of similar triangles (as shown above); and then if the rotations are infinitesimal, the order of succession becomes immaterial, and the translation acts along the normal to the plane of the two axes.

Thus any **couple of parallel rotations** (not necessarily infinitesimal) is equivalent to a simple translation, and reciprocally, any translation can be replaced in an infinite number of ways by a couple of this kind. The word “couple” is meant to imply that the two rotations are equal and opposite.

These couples of parallel rotations compose and decompose, in accordance with the law of translations, in an arbitrary order of succession, acting in all the positions that correspond in length and direction to a particular translation; compositions and decompositions which can be found by substituting for the couples the translations that they represent. The order of different couples is arbitrary; the order of the two rotations forming a single couple is not, if the rotations are finite.

Thus we have generalized to couples of finite rotations the law of composition which M. Poinsot, I believe, was the first to state for couples of infinitesimal rotations.

### 11 Proof of the general decomposition theorem

As any displacement of a solid system can be reduced to a rotation followed by a translation (see 4), and this translation can always be replaced by a **couple of rotations** (see 10.1), one of whose axes intersects the given rotation axis of the system, and the rotations about these two intersecting axes can be composed (see 8) into a single rotation, there results immediately the proof of the transformation stated above (see 7) of the fundamental theorem (see 4.1), to wit: that any displacement of a solid system can be accomplished in an infinite number of ways by the success-
sion of two rotations about two fixed nonintersecting axes.

The axes will be nonintersecting unless the translation is null, in which case one rotation suffices. This one-sentence proof can benefit, as usual, by some expansion of the reasoning and naming of the geometrical entities.

Let the axis of the given rotation be called $C$ and the accompanying translation be called $T$. As shown in 10.1, $T$ can be replaced by a couple of equal but opposite rotations about two parallel axes which we may call $B'$ and $B$, of which $B'$ may be located so as to intersect $C$. (The axis $B'$ is the one that was called $A$ in 10.1; here we shall define an entirely different axis as $A$.)

Then, by the method of 8, the rotations about $C$ and $B'$ can be composed to make one about an axis we shall call $A$; the two rotations about the nonintersecting axes $A$ and $B$, performed successively, generate the same displacement as $C$ and $T$.

This is quite clear as far as it goes, but it gives no clue as how to derive eq. (6) of 7. This will be done in 13; it is unnecessary to carry out the demonstration also in the present context, as it would involve the same steps sometimes done in reverse order.

12 Rotations about an arbitrary number of fixed nonintersecting axes.

Finally, there is the composition of rotations about an arbitrary number of fixed nonintersecting axes. Let us take a point in the space, upon which we shall study the effect of all these rotations in their order. We have seen that any rotation about a fixed axis can be replaced by another equal rotation, accomplished about another axis parallel to the first, followed by a translation equal to the chord of the arc described by a point of the new axis about the first in consequence of the rotation given at the outset. We have also seen that a translation followed by a rotation about an axis passing through the endpoint of the axis of translation can instead be preceded by it, if the axis of rotation passes through the origin of the axis of translation.

This last statement deserves careful examination. Let a translation $T$ be followed by a rotation $R$ whose axis is $L$. Decompose $T$ into $T_1$ parallel to $L$ and $T_2$ transverse to $L$. Since $T_1$ commutes with $R$, it suffices to consider $T_2$ and to project the whole situation onto a transverse plane.

We may represent points on this plane by complex numbers, and for simplicity let us represent the displacement $T_2$ by the number 1. Take $R$ to be a rotation of the plane through an angle $\theta$ about the point 1, and identify the author’s phrase “axis of translation” as the line from 0 to 1. Thus the endpoint of the axis of translation is at 1, and its origin is at 0. The claim is that the effect of $T_2$, followed by $R$ is the same as that of $R'$ followed by $T - t$, where $R'$ is the rotation through $\theta$ about 0.

But this is easily proven. Start with a point $z$ and first perform $R'$; this takes $z$ to $ze^{i\theta}$. Then perform $T_1$; the result is $ze^{i\theta} + 1$. On the other hand, $T_1$ acting first on $z$ produces $z + 1$, and the radius vector from 1 to $z + 1$ is $z$ so that the second transformation $R$ replaces the term $z$ by $ze^{i\theta}$.
yielding the final result $ze^{i\theta} + 1$ as before.

This being given, if, through the point of origin we have chosen for study, we pass axes parallel to each of the given nonintersecting axes, the displacement of the system operates successively about these axes, by means of the transport of the rotations to the intersecting axes respectively parallel to the original ones. By virtue of the successive replacement of rotations about axes passing through the endpoints of the translation lines by rotations about axes passing through their origins, the displacement of the system will be partitioned into a series of rotations respectively equal to those originally given, taking place successively about intersecting axes parallel to the first series, followed by a series of translations resulting from the chords successively traversed by the chosen point about the original nonintersecting axes in the order ascribed to the rotations.

One may ask whether the final composite translation would be the same if a different point of origin “p” had been chosen for study. The answer is yes, because a change in p can be simulated by keeping p unchanged and rigidly changing the positions of all the rotation axes.

(The composition of rotations about intersecting axes and that of translations will take place in the manner described above; in this case the composite displacement will reduce to a rotation and a translation whose axes both pass through the point of intersection.)

We see from (the general) construction that the elements of the final composite rotation depend only on the amplitude and direction of the individual rotations, and are not changed by any parallel movement of their axes; while the length and direction of the composite translation depend, as well, on the positions of the individual rotations, seeing that the chords successively described by the chosen point vary in length and direction according to the successive positions that the displaced point takes relatively to the various given axes.

If, in the system of these axes, there are found consecutive pairs that form couples of parallel rotations, it is evident that these couples do not contribute anything to the determination of the direction and amplitude of the resultant rotation, and that they influence only the length and direction of the resultant translation, as the point whose successive rotations determine this translation will be found, upon completion of each couple, to have described the translation equivalent to the couple.

13 The case of only two nonintersecting fixed axes.

The author intends in this section to give a second proof of the “two-axis” theorem of 7, which has already been proved in 8-11. But in that first proof, he started with a displacement described in screw form with the central axis C and rotation angle $\theta_C$ given, and constructed the two (usually) nonintersecting axes $A, B$ of rotations $\theta_A, \theta_B$ whose composition gives this displacement. In this second proof he assumes that $A, B$ are given, along with $\theta_A, \theta_B$, and constructs $C$ and $\theta_C$. Thus the decomposition theorem of 7 is replaced in the present section by a composition
theorem, a generalization of the theorem of 8 in that the axes A and B are no longer required to intersect. And this time the author carries through the derivation of eq. [6] in 7, which was omitted in 11.

Consider two nonintersecting axes and their shortest distance D, and take as the origin of the displacement the end A of that shortest distance lying on the first axis of rotation A - that is, the rotation to be first executed. Then, on passing through that origin an axis B′ parallel to the second one B given, the two intersecting axes will be composed into a third, which will be the axis of rotation of the displacement relative to that origin A.

Let the symbol D refer equally to the shortest distance between A and B and to the line segment of that length, pointing from its intersection A with A to its intersection B with B. The author has introduced a substitute axis B′, parallel to B but passing through A. Regarding A as Euler’s (3) fixed center, the fixed-point theorem says that the rotations θA,θB′ about the intersecting axes A, B′ can be composed to make a rotation θC′ about a third axis C′, also passing through A. The author calls C′ the “axis of rotation of the displacement relative to A.”

At the same time (see 1.1 and 4), the rotation about B can be accomplished by performing the rotation about B′ followed by a translation, which we may call T′. Thus the entire displacement (A,θA) followed by (B,θB) is equivalent to the three actions (A,θA), (B′,θB′), T′ taken in sequence, which in turn is the same as (C′,θC′) followed by T′. More compactly, we can write this result as

\[(A,θA)(B,θB) = (C′,θC′)T′.\] (7)

The accompanying axis of translation T′ will be given (1.1) as the chord of the arc described by this same origin A in consequence of the rotation about the second (B) of the two axes given.

It is desired, however, to express the whole displacement in terms of the central axis C and its associated quantities; thus we must have

\[(A,θA)(B,θB) = (C,θC)T,\] (8)

where T, called the absolute translation of the displacement, is directed along the axis C.

Comparing [8] to [7], we find that the rotation (C,θC) differs from the rotation (C′,θC′) only by a translation, which may be written T′ − T since translations compose by addition. It follows immediately that C and C′ are parallel and that θC = θC′. Furthermore, both these rotations move points only within planes perpendicular to C and to C′. Therefore the difference T′ − T is perpendicular to C, hence perpendicular to T which lies along C. It follows that T is the projection of T′ onto T, as stated in the author’s next remark:

The projection of this chord onto the composed axis of rotation, determined as above, measures the absolute translation of the (composite) displacement. It is equal to the sum of the projections of the two sides of the isosceles triangle of which it is the base. The two sides are equal.
to the shortest distance between the two given nonintersecting axes.

Let the whole displacement under study carry the point $A$ to its final position $A_f$. Since the first rotation about $A$ left $A$ unmoved, the movement from $A$ to $A_f$ is accomplished entirely by the second rotation, about $B$. But the point $B$, lying on $B$, is unmoved by this rotation; since the displacement is rigid, the distances $AB$ and $A_fB$ are equal. That is, the triangle $ABA_f$ is isosceles, with vertex at $B$ and sides $AB$, $A_fB$ both equal to $D$ as defined above.

The base of this triangle is the chord $AA_f$, which has previously been identified (author’s comment after eq. 7) as giving the translation $T'$. Its projection on $C$ is equal to the sum of the projections of $AB$ and $BA_f$. (This is readily understood by thinking of $AB, BA_f, AA_f$ as vectors.)

It is easily demonstrated in addition that the two sides are equally inclined to the composed axis.

This is an important claim. We see at once that the projection of $AB$ on $C$ is $D \cos S$ where $S$ is the angle between $C$ and $D$. The author wishes to establish that $BA_f$ makes the same angle with $C$ and therefore has the same projection. This will establish the important equation $T = 2D \cos S$.

In fact, this shortest distance ($D$) is normal to the plane $AB'$ of the two intersecting axes. Now, in considering the angle formed by this normal $D$ with the reflection $C'$ of the resultant axis $C$ in this same plane $AB'$, one sees that this angle does not change when one supposes it mobile and displaced rigidly by the second rotation (about $B$), which brings the reflected line $C'$ of which we speak into coincidence with the resultant axis $C$.

But, in this rotation, the normal $D$ is rotated through a plane perpendicular to the second axis $B$ (sweeping out a cone with vertex at $B$) so as to become parallel to the second side $A_fB$ of the isosceles triangle we are considering, and as it is evident that the angle of the normal $D$ with the resultant axis $C$ is supplementary to that which it forms with the reflection $C'$ of that axis, one sees that the resultant axis $C$ is, as we have just said, equally inclined with respect to the two sides $AB, BA_f$ of this isosceles triangle. The projection of $C'$ onto the initial position of $D = AB$ is the negative of the projection of $C$; that is, it equals the projection of $C$ onto $BA_f$. Therefore the $B$ rotation sweeps the angle under discussion to the angle between $C$ and $BA_f$ as asserted.

The preceding argument is best understood by comparing it with the construction in 8, where $A$ and $B$ intersect; $B$ and $B'$ coincide, $D$ vanishes but its direction is still defined as the normal to the plane $AB$; $A_f$ is identical to $A$ since both rotations leave $A$ fixed; and there is no translation $T'$ or $T$. The key specification is that $\theta_A$ is twice the angle between the planes $AC$ and $AB$, and $\theta_B$ is twice the angle between the planes $BC$ and $AB$, so that $C$ is reflected in $AB'$ by the $A$-rotation and reverse-reflected by the $B$-rotation. The author’s determination of $\theta_C$ depends on computing the final position of a line whose initial position was $A$, deduced on the one hand from the effect of the $B$-rotation and on the other hand from that of the $C$-rotation.

In the present case the plane of reflection is taken as $AB'$, but it could be any plane parallel to both $A$ and $B$, without changing the angles. Instead
of A, the initial position of the moving line is taken as D, and again a comparison is made between two ways of finding its final position. The required agreement between the two ways, as in 8, yields the result, which in this case is the equality of the angles made by AB and BA with the resultant axis C.

From which one finally concludes that the absolute translation of a solid system arising from the succession of two rotations about two fixed nonintersecting axes is equal to double the distance between the two axes, projected on the direction of the composite, or resultant, axis. That is, if the side AB of the isosceles triangle ABA is extended to twice its length, the projection of this doubled side onto the direction of C will fall on A.

The resulting equation is

$$T = 2D \cos S$$

as anticipated above.

But it is evident that the cosine of the angle S of this distance D with the composite axis C is equal to the sine of the angle made by C with the plane of the two composing axes A, B (since D itself is normal to this plane), which is found to be equal to the product of the sines of the given half-rotations by the sine of the angle between the two axes, divided by that of the composite half-rotation. That is,

$$\cos S = \sin h = \sin(\theta_A/2) \sin(\theta_B/2) \sin \nu / \sin(\theta_C/2),$$

where h = ₃(C, AB) and ν = ₃(A, B).

Equation (10) follows from the law of proportion of the sines of the half-rotations to those of the angles included between the opposing axes. Here again the author has compressed many steps into one. We can understand (10) more readily by noting that D and T no longer appear, so that the equation involves only directions. This enables us to associate the directions A, B, C with points A, B, C on a sphere, forming the vertices of a spherical triangle. Since we are dealing only with directions, we need not distinguish between C and C' or between B and B'. The angle ν = ₃(A, B) is just the arc c, the side of the triangle opposite to C. Thus (10) becomes

$$\sin h = \sin(\theta_A/2) \sin(\theta_B/2) \sin c / \sin(\theta_C/2).$$

where h is what we might call the altitude of the triangle, that is the arc running from C to c and making a right angle with the latter.

The author asserts that (11) is a consequence of the “law of proportion...” which can be stated as

$$\frac{\sin(\theta_A/2)}{\sin A} = \frac{\sin(\theta_B/2)}{\sin B} = \frac{\sin(\theta_C/2)}{\sin C}.$$
which is just the Law of Sines for a spherical triangle. (I am unable to see how the author could have arrived at (12) without relying on this law from spherical trigonometry.)

Now we must show how (12) leads to (11). If $h$ were the altitude of a planar triangle we would obviously have $h = a \sin B$. The spherical analogue is $\sin h = a \sin B$, also an application of the Law of Sines to the special case of a right triangle. But since $B = \theta B/2$ from (8), we have

\[
\sin h = \sin a \sin(\theta B/2) = \sin(\theta A/2) \sin(\theta B/2) \frac{\sin a}{\sin(\theta A/2)}. \tag{14}
\]

Now, applying (12), we may replace $\frac{\sin a}{\sin(\theta A/2)}$ by $\frac{\sin c}{\sin(\theta C/2)}$, obtaining (11) as asserted by the author.

From which we arrive at the modified Fundamental Theorem in the form (7) in which we have already pronounced it, namely:

“any displacement of a solid system can always arise, in an infinite number of ways, from the succession of two rotations about two non-intersecting fixed axes, provided that the product of the sines of the successive half-rotations by the distance between the two conjugate axes and by the sine of the angle of these axes is equal to the product of the absolute half-translation of the displaced system by the sine of the resultant half-rotation.”

That is, $\sin(\theta A/2) \sin(\theta B/2) D \sin \nu = (1/2) T \sin(\theta C/2)$, where $D$ is the distance and $\nu$ the angle between the two axes. (Having established (11), we can discard the spherical triangle representation, writing once more $\nu$ in place of $c$, and substitute (10) into (9). This gives indeed

\[
2D \sin(\theta A/2) \sin(\theta B/2) \sin \nu = T \sin(\theta C/2) \tag{15}
\]

in agreement with eq. (6) of (7.)

14 Composition of general displacements.

We are now in a position to resolve completely the following general problem, in which one considers the succession of (an arbitrary number of) displacements of the same solid.

Being given the axes of rotation and translation as well as the amplitude of the rotations and extent of the translations for each successive displacement of a system, it is required that we construct the axes [[and amplitudes]] of rotation and translation of this system relative to a given origin.

The solution of this problem is evidently the same as that of the previous one, where it was only a matter of rotations about fixed axes, since the translations can be replaced by couples of rotations about fixed axes. We have briefly indicated the solution in the last paragraph of 12. We therefore need not linger over it further.
14.1 The particular case of infinitely small displacements.

The solution is considerably simplified when one considers only infinitely small displacements. First of all, the order of the rotations is indifferent, and their composition by whatever number around intersecting axes operates like that of translations proportional to these rotations and parallel to these axes. Second, the order of the rotations and translations successively accomplished by the origin of the displacement is equally indifferent, and each of these rotations and translations can be established directly and separately, as though the point to be displaced were displaced only alternately and not successively, which follows from the fact that the space traversed by each of these displacements is infinitely small. The composition of these partial translations resulting from withdrawing from the given axes, or from the translations themselves that are joined to the rotations, acts in accordance with the same law as that of the rotations.

15

We have now to apply calculation to the geometric laws that we have just presented concerning the general displacements of a solid system. We shall start by deriving the formulas for change of coordinates of points in the solid system, which hold such a large place in analytical mechanics.

Let \( x, y, z \) and \( x + \Delta x, y + \Delta y, z + \Delta z \) be the coordinates of two points of which the first is moved to the second by the displacement of the system, and let \( \xi, \eta, \zeta \) be the coordinates of the midpoint of the line joining the two, so that

\[
\xi = x + (1/2)\Delta x, \quad \eta = y + (1/2)\Delta y, \quad \zeta = z + (1/2)\Delta z. \tag{16}
\]

Furthermore, let \( g, h, l \) be the angles formed by the direction of the axis of rotation with the three coordinate axes, \( \theta \) the amplitude of the rotation, \( t \) the absolute size of the translation, and \( X, Y, Z \) the coordinates of an arbitrary point on the central axis of the displacement. In much of what follows, the origin of coordinates may be assumed to lie on the central axis; that is, we can take \( X = Y = Z = 0 \), or in the vector notation to be introduced, \( \vec{W} = 0 \).

Consider the right triangle whose hypotenuse is formed by the line joining the initial and final point and whose sides are given, one by the arc of the chord described by the initial point under the rotation \( \theta \), and the other by the line traversed in a translation by this same point after undergoing the rotation. Clearly, the changes \( \Delta x, \Delta y, \Delta z \) are respectively equal to the projections of this hypotenuse, that is to the sum of the projections of the other two sides of this triangle on the respective coordinate axes.

Now, the side equal and parallel to the absolute translation \( t \) gives the three projections \( \cos g, \cos h, \cos l \); the other side is equal to \( 2u \tan(\theta/2) \), \( u \) denoting the distance from the central axis to this same side (the one formed by the chord). Let us call \( G, H, L \) the angles between this side and
the coordinate axes. Then we have immediately

\[ \Delta x = t \cos g + 2u \tan(\theta/2) \cos G \]
\[ \Delta y = t \cos h + 2u \tan(\theta/2) \cos H \]
\[ \Delta z = t \cos l + 2u \tan(\theta/2) \cos L. \]

(17)

This and the following equations can be better understood if translated into modern vector notation. Let us define a right-handed orthonormal system \( \hat{t}, \hat{u}, \hat{v} \) where \( \hat{t} \) points along the central axis; \( \hat{v} \) along the chord; and \( \hat{u} \), perpendicular to both, points to the midpoint of the chord from the base of the perpendicular dropped from that midpoint to the central axis. The whole displacement \((\Delta x, \Delta y, \Delta z)\) may be designated as \( \vec{\Delta} \); then the above equations say that

\[ \vec{\Delta} = \vec{t} + 2\hat{v}u \tan(\theta/2), \]

(18)

where \( \vec{t} = \hat{t} \) is the translation vector.

Since one has necessarily

\[ \cos g \cos G + \cos h \cos H + \cos l \cos L = 0, \]

(19)

(this says that \( \hat{t} \cdot \hat{v} = 0 \)) one deduces

\[ \Delta x \cos g + \Delta y \cos h + \Delta z \cos l = t \]

(20)

\[ (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = t^2 + 4u^2 \tan^2(\theta/2). \]

(21)

That is, \( \vec{\Delta} \cdot \vec{t} = t \) and \( \vec{\Delta} \cdot \vec{\Delta} = t^2 + (2u \tan(\theta/2))^2. \)

The first terms \((\text{of } (17))\) \( t \cos g, t \cos h, t \cos l \) represent the part of the changes that arise from the absolute translation displacement; the second \((\text{set of three})\) terms, the part due to the rotation performed by the displacement. In comparing these first terms to the second, one finds that the first, which measure the effect or moment of the translation of the system, have for value the projections of this translation on each axis of the coordinates, while the second, which represent for each point the effect or moment of the rotation of the system, have for value the projection upon each coordinate axis of \((\text{the area of an isosceles})\) triangle whose vertex is the midpoint of the line finally traversed by the point considered, and whose base is a line \([\text{segment}]\) directed along the central axis and of length \(4 \tan(\theta/2)\).

In the case of an infinitely small displacement, this midpoint \((\text{the vertex of the triangle})\) coincides with the initial point, and consequently the moment of the rotation, relative to any given direction, is equal to double the projection on that direction of \((\text{the area of})\) a triangle whose vertex is the point \((\text{under consideration})\) and whose base taken on the central axis is equal to the rotation of the system.

This explains how the theory of projections applies to the laws of translation through linear projections, and to those of rotation through the projection of areas. \( A \text{ possible influence of Grassmann here? or independent? Let us continue.} \)

(Wordy explanation here omitted: terms in \( t \) represent the effect of translation, those in \( u \) the effect of rotation.)
The chord $2\hat{u}\hat{v}\tan(\theta/2)$ being normal both to the central axis and to the perpendicular dropped from the point $\xi, \eta, \zeta$ to this axis, which has length $u$, we have
\[
\begin{align*}
    u\cos G &= (\eta - Y)\cos l - (\zeta - Z)\cos h \\
    u\cos H &= (\zeta - Z)\cos g - (\xi - X)\cos l \\
    u\cos L &= (\xi - X)\cos h - (\eta - Y)\cos g.
\end{align*}
\tag{22}
\]

Here we need to define more vectors. Let $\vec{W} = (X, Y, Z)$ represent the point on the central axis that has been identified as locating that axis with respect to the immovable space within which the solid exists. Let $\vec{w}$ stand for the base of the perpendicular dropped from the midpoint of the chord of rotation to the central axis. ($\vec{W}$, although chosen arbitrarily along the axis, is fixed for a particular displacement of the solid, whereas $\vec{w}$ slides along the axis as we consider the trajectories of different points of the solid.) And let $\vec{\omega} = (\xi, \eta, \zeta)$ represent the position of the midpoint of the chord. Then (22) tells us that
\[
\hat{v}\hat{u} = \hat{t} \times (\vec{\omega} - \vec{W}).
\tag{23}
\]

I note here that all the author’s equations starting with (22) are consistent with $(\hat{t}, \hat{u}, \hat{v})$ being a left-handed system, whereas the motion described in the Chasles theorem at the end of 4 (“mouvement ... de la vis dans son écrou”, movement of a screw in its nut) makes it definitely right-handed. I choose to write vector equations in the original right-handed notation. When equation arrays are written out in the original in terms of the components, I shall reproduce them without change; but in writing these arrays in vector notation I shall reverse the order of all cross-products. Thus, I shall interpret (22) as
\[
\hat{v}\hat{u} = \hat{t} \times (\vec{\omega} - \vec{W})
\tag{24}
\]
instead of as (23). I shall do this consistently without further comment.

Moreover, $\hat{t} \times (\vec{\omega} - \vec{W}) = 0$ since $\vec{w} - \vec{W} \perp \text{central axis}$. Therefore $\vec{\omega} - \vec{W}$ may be replaced by $\vec{\omega} - \vec{w}$ which is just $\vec{u} = \hat{u}u$. So (22) becomes
\[
\hat{v}\hat{u} = \hat{t} \times \vec{u}
\tag{25}
\]
which need not surprise us.

And in consequence,
\[
\begin{align*}
    \Delta x &= A + p\eta - n\zeta, \\
    \Delta y &= B + m\zeta - p\xi, \\
    \Delta z &= C + n\xi - m\eta.
\end{align*}
\tag{26}
\]

$A, B, C, m, n, p$ being six constants that depend on the position of the central axis, the length of the translation, and the amplitude of the rotation, as follows:
\[
\begin{align*}
    A &= t\cos g + 2\tan(\theta/2)(Z\cos h - Y\cos l), \\
    B &= t\cos h + 2\tan(\theta/2)(X\cos l - Z\cos g), \\
    C &= t\cos l + 2\tan(\theta/2)(Y\cos g - X\cos h).
\end{align*}
\tag{27}
\]
By including “the position of the central axis” as a variable, the author signals that the following calculations do not assume that this axis passes through the origin. Indeed, much of what follows in this Section becomes trivial if that assumption (the “null central axis” assumption or NCA) is made. For example, (27) becomes

\[ A = t \cos g, \]

etc.

Let us put \( \vec{\Gamma} = (A, B, C), \vec{q} = (m, n, p); \) then these definitions become

\[ \vec{\Gamma} = \vec{t} + 2\hat{W} \tan(\theta/2) \times \hat{t} \] (29)

and

\[ \vec{q} = 2\hat{t} \tan(\theta/2); \] (30)

under NCA, (29) becomes \( \vec{\Gamma} = \vec{t} \).

Going back to (26), in vector notation it reduces to

\[ \vec{\Delta} = \vec{\Gamma} + \vec{q} \times \vec{\omega} \] (31)

(\( \vec{\Delta} = \vec{\Gamma} \) under NCA).

Applying (29) and (30), this becomes

\[ \vec{\Delta} = \vec{t} - 2\hat{t} \times \hat{W} \tan(\theta/2) + 2\hat{t} \tan(\theta/2) \times \vec{\omega} \]

\[ = \vec{t} + 2\hat{t} \times (\vec{\omega} - \hat{W}) \tan(\theta/2), \] (32)

agreeing with (18) in view of (24).

If we denote by \( \alpha, \beta, \gamma \) “les variations des coordonnées de l’origine des axes coordonnés” (the coordinates of the point to which the origin of coordinates is carried by the displacement), we have the following relation:

\[ \alpha = A + (1/2)(p\beta - n\gamma) \]

\[ \beta = B + (1/2)(m\gamma - p\alpha) \]

\[ \gamma = C + (1/2)(n\alpha - m\beta). \] (33)

Let us introduce the vector \( \vec{\delta} = (\alpha, \beta, \gamma). \) Then (33) becomes

\[ \vec{\delta} = \vec{\Gamma} + (1/2)\vec{q} \times \vec{\delta}. \] (34)

To arrive at this equation, consider (31) and recall that \( \omega = \vec{t} + (1/2)\vec{\Delta}, \) where the displacement carries \( \vec{t} \) into \( \vec{t} + \vec{\Delta}. \) Now take the special case \( \vec{t} = 0 \) (the origin of coordinates). In this case \( \vec{\Delta} \) takes the value of \( \vec{\delta} \), by definition of the latter. Thus (31) reduces to \( \vec{\delta} = \vec{\Gamma} + (1/2)\vec{q} \times \vec{\delta} \) which is exactly (34). (Under NCA, one has simply \( \vec{\delta} = \vec{\Gamma}. \)

Without NCA, equation (34) looks indeterminate since \( \vec{\delta} \) is defined in terms of itself. But actually the system (33) provides three linear equations in the three unknowns \( (\alpha, \beta, \gamma) \) which are perfectly determinate.
The solution is obtained transparently by vector algebra. Let \( \vec{U} \) be the position vector of the point on the central axis closest to the origin, and \( \vec{V} = \hat{t} \times \vec{U} \), then \( \hat{t}, \vec{U}, \vec{V} \) are orthonormal. Moreover, (20) reduces to
\[
\vec{\Gamma} = \hat{t} + 2\vec{U} \times \hat{t} \tan(\theta/2)
\]
since \( \vec{W} \) and \( \vec{U} \) are both on the central axis. This can be written
\[
\vec{\Gamma} = \hat{t} - 2\vec{U} \tan(\theta/2) = \hat{t} - Uq\vec{V}
\]
in view of (30). From this we have
\[
\vec{\Gamma} : \vec{U} = 0
\]
and also
\[
\vec{\Gamma} : \hat{t} = t.
\]
Then from (34) we find
\[
\hat{\delta} : \hat{t} = t,
\]
and also, in view of (37),
\[
\hat{\delta} : \vec{U} = -(1/2)\hat{\delta} \times \vec{q} : \vec{U} = -(1/2)\hat{\delta} \times \vec{U} = -(1/2)\hat{\delta} : Uq\vec{V}.
\]
which can be rewritten as
\[
\hat{\delta} : \vec{U} = -\hat{\delta} : \vec{V} \tan(\theta/2).
\]
On the other hand, (36) also gives
\[
\vec{\Gamma} : \vec{V} = -Uq
\]
whence
\[
\hat{\delta} : \vec{V} = -Uq + \hat{\delta} : \vec{U} \tan(\theta/2).
\]
Substituting (41) into (43) gives
\[
\hat{\delta} : \vec{V} = -Uq - \hat{\delta} : \vec{V} \tan^2(\theta/2),
\]
or
\[
\hat{\delta} : \vec{V} = -Uq \cos^2(\theta/2) = -2U \cos^2(\theta/2) \tan(\theta/2) = -U \sin \theta
\]
whence by (41)
\[
\hat{\delta} : \vec{U} = 2U \sin(\theta/2) \cos(\theta/2) \tan(\theta/2) = U(1 - \cos \theta).
\]
Combining (39), (45), and (46), we obtain the formula
\[
\hat{\delta} = \hat{t} + \vec{U} - U(\vec{U} \cos \theta + \vec{V} \sin \theta),
\]
which tells us that the endpoint of \( \hat{\delta} \) can be located by passing through the origin \( 0 \) a circle in the \( \vec{U}, \vec{V} \) plane with center at \( \vec{U} \) on the central axis, moving counterclockwise on this circle from \( 0 \) through an arc subtending an angle \( \theta \) at the center, and erecting on the endpoint of this arc the translation vector \( \hat{t} \). Indeed, this is just the operation (rotation \( \theta \) about the
central axis followed by translation $\vec{t}$) that takes the origin of coordinates into its image under the displacement considered.

By means of (33) one may replace, if one wishes, the constants $A, B, C$ by their values in terms of $\alpha, \beta, \gamma$. This leads to

$$
\Delta x = \alpha + 2 \tan(\theta/2)(\eta - \beta/2) \cos l - (\zeta - \gamma/2) \cos h
$$

$$
\Delta y = \beta + 2 \tan(\theta/2)(\zeta - \gamma/2) \cos g - (\xi - \alpha/2) \cos l
$$

$$
\Delta z = \gamma + 2 \tan(\theta/2)(\xi - \alpha/2) \cos h - (\eta - \beta/2) \cos g
$$

(48)

where the first terms $\alpha, \beta, \gamma$ express the translation relative to the origin of coordinates (the length being $\sqrt{\alpha^2 + \beta^2 + \gamma^2}$) and those pertaining to the rotation express the rotation about an axis passing through the origin.

To understand the last remark, let us call the direction of the central axis "vertical" and recall the statement of the "Fundamental Theorem" in 4. There it is pointed out that any vertical axis can be chosen as the rotation axis, and that the accompanying translation, while constant for all points considered, is not vertical unless the central axis is the one chosen. In general, it is only the projection of this translation on the axis of rotation that is vertical. There is an additional horizontal translation that compensates for the change of rotation axis.

The equation (48) can be written in vector notation as

$$
\vec{\Delta} = \vec{\delta} + \vec{q} \times (\vec{\omega} - (1/2)\vec{\delta}),
$$

(49)

where $\vec{\delta}$ is given by (47). In (47) the term $\vec{t}$ can be taken as a vertical translation and the remaining term as the trajectory of the initial point $\vec{r} = 0$ under rotation about the central axis. But now the author desires to consider the vertical axis through the origin (call it the $0$-axis) as the axis of rotation. Then the initial point $\vec{r} = 0$ does not move under the rotation, and hence the entire expression (47) must be viewed as translation, containing both a vertical and a horizontal part.

For a general point $\vec{r}$, still taking the $0$-axis as the axis of rotation, the whole of $\vec{\delta}$ is still translation and therefore the second term of (49) gives the rotation about the $0$-axis.

Of course, under NCA there is no change of (47) and one still has $\vec{\Delta} = \vec{\delta} - \vec{t}$.

We could have established these formulas directly, as well as those which precede, and which we have constructed on the central axis.

### 16 Equations of the central axis.

The equations of the central axis follow in their turn from the above formulas, with the greatest simplicity; for the rotation has no effect on any point lying on that axis, and one then has $\Delta x = t \cos g$, $\Delta y = t \cos h$, $\Delta z = t \cos l$. (That is, $\vec{\Delta} = \vec{t}$.) The coordinates $x, y, z$ describe some point on that axis (I shall denote this point by $\vec{r}$) and taking this
into account one has the equations sought, expressed by means of $\alpha, \beta, \gamma$:

$$
p_{y} - n_{z} + A = t \cos g = \frac{m(Am + Bn + Cp)}{m^2 + n^2 + p^2} = \frac{m(\alpha m + \beta n + \gamma p)}{m^2 + n^2 + p^2},
$$

$$
m_{z} - p_{x} + B = t \cos h = \frac{n(Am + Bn + Cp)}{m^2 + n^2 + p^2} = \frac{n(\alpha m + \beta n + \gamma p)}{m^2 + n^2 + p^2},
$$

$$
n_{x} - m_{y} + C = t \cos l = \frac{p(Am + Bn + Cp)}{m^2 + n^2 + p^2} = \frac{p(\alpha m + \beta n + \gamma p)}{m^2 + n^2 + p^2}. \tag{50}
$$

In vector notation this array becomes

$$
\vec{\Gamma} + \vec{q} \times \vec{r} = \vec{t} = \vec{q} \cdot \vec{q}/q^2 = \vec{q} \cdot \vec{q}/q^2, \tag{51}
$$

where the first equality is simply a rearrangement of (29) taking account that both $\vec{r}$ and $\vec{W}$ are arbitrary points on the central axis, and also using (30); the second equality follows from (30) which gives $\vec{q} \cdot \vec{q} = q^2 \hat{t} \cdot \vec{\Gamma}$; and the third from (30) with (38).

This can be simplified by eliminating the constants $A, B, C$; we find (using (33))

$$
x - (1/2)\alpha - \frac{p_{z} - n_{y}}{m^2 + n^2 + p^2} = \frac{y - (1/2)\beta - \frac{n_{y} - p_{x}}{m^2 + n^2 + p^2}}{n} = \frac{z - (1/2)\gamma - \frac{m_{x} - n_{y}}{m^2 + n^2 + p^2}}{p}. \tag{52}
$$

To derive this, we combine (51) with (34), and eliminating $\vec{\Gamma}$, we have

$$
\vec{q} - (1/2)\vec{q} \times \vec{\delta} + \vec{q} \times \vec{r} = \vec{q} \cdot \vec{q}/q^2 \tag{53}
$$

which can be rearranged, using the identity $(\vec{\delta} \times \vec{q}) \times \vec{q} = \vec{q} \cdot \vec{\delta} - \vec{q} \cdot \vec{\delta}$, to yield

$$
(\vec{r} - \vec{r}_0) \times \vec{q} = 0 \tag{54}
$$

where

$$
\vec{r}_0 = (1/2)\vec{\delta} - \vec{\delta} \times \vec{q}/q^2; \tag{55}
$$

these two equations are exactly the content of (52), which says that $\vec{r} - \vec{r}_0$ is parallel to $\vec{q}$ when $\vec{r}_0$ is given by (55).

In these equations the coordinates subtracted from $x, y, z$ (that is, the components of $\vec{r}_0$) are precisely those of the vertex of an isosceles triangle normal to the plane of the two relative axes of translation and of rotation passing through the origin of the coordinate axes, raised on a base perpendicular to the relative axis of rotation, cutting at its midpoint the relative axis of translation and equal in length to the translation itself, the vertex angle being equal to the amplitude of rotation $\theta$; all this in agreement with the construction given previously.

I find this description difficult to visualize. Again I resort to the equations to find the geometric meaning of $\vec{r}_0$.

From (39), (45), and (46), let us extract the half-angle formula for $\vec{\delta}$:

$$
\vec{\delta} = \vec{t} + 2\hat{U} \sin(\theta/2) - 2U \hat{V} \sin(\theta/2) \cos(\theta/2). \tag{56}
$$
Then, since $\vec{q}/q^2 = \vec{t}/(2 \tan(\theta/2))$,

$$
\vec{\delta} \times \vec{q}/q^2 = \vec{\delta} \times \vec{t}/(2 \tan(\theta/2)) = [2\vec{U} \sin^2(\theta/2) - 2UV \sin(\theta/2) \cos(\theta/2)] \times \vec{i}/(2 \tan(\theta/2)) = [\vec{U} \sin(\theta/2) \cos(\theta/2) - UV \cos^2(\theta/2)] \times \vec{i} = -UV \sin(\theta/2) \cos(\theta/2) - \vec{U} \cos^2(\theta/2) 
$$

(57)

Now (57) and (55) combine harmoniously to yield

$$
\vec{r}_0 = \frac{1}{2} \vec{\delta} - [-UV \sin(\theta/2) \cos(\theta/2) - \vec{U} \cos^2(\theta/2)] = \frac{1}{2}[\vec{i} + 2\vec{U} \sin^2(\theta/2) - 2UV \sin(\theta/2) \cos(\theta/2)] + [UV \sin(\theta/2) \cos(\theta/2) + \vec{U} \cos^2(\theta/2)] = (\vec{i} + \vec{U})(\sin^2(\theta/2) + \cos^2(\theta/2)) + UV(\sin(\theta/2) \cos(\theta/2) - \sin(\theta/2) \cos(\theta/2)) = \frac{1}{2}\vec{i} + \vec{U}
$$

(58)

saying that $\vec{r}_0$ is the midpoint of the line joining the point $\vec{U}$, the point on the central axis closest to the origin, to the point $\vec{r} + \vec{\delta} = \vec{U} + \vec{i}$ to which the point $\vec{U}$ is carried by the displacement. Under NCA, $\vec{U} = 0$ and $r_0 = \vec{i}/2$.

These equations (referring to (52)) can also be written as follows, on introducing the rotation angle and the direction of the axis of rotation:

$$
x - \frac{1}{2}a - (1/2) \cot(\theta/2)(\beta \cos l - \gamma \cos h) = 0
$$

$$
y - (1/2)\beta - (1/2) \cot(\theta/2)(\gamma \cos g - \alpha \cos l) = 0
$$

$$
z - (1/2)\gamma - (1/2) \cot(\theta/2)(\alpha \cos h - \beta \cos g) = 0
$$

(59)

These equations follow immediately from (52) by using (28). In modern notation they become just another way of saying that $\vec{r}_0$ is given by (55).

16.1 General equation of the central axis

But one may represent these three equations for the projections of the central axis (it is not clear which three equations are meant, since (59) consists of only two equations) by a single equation with undetermined coefficients, namely

$$
\cos a \Delta x + \cos b \Delta y + \cos c \Delta z = t \cos(t, a, b, c)
$$

(60)

where

$$
\cos(t, a, b, c) = \cos a \cos g + \cos b \cos h + \cos a \cos l,
$$

(61)
a, b, c being the three angles formed by any direction whatever with the coordinate axes. That is, if $\vec{d} = \cos a, \cos b, \cos c$ is a unit vector in any direction whatsoever, then $\Delta \cdot \vec{d} = t \vec{i}$, which is just the property of initial points $\vec{r}$ that lie on the central axis.
17 The case of an infinitesimal displacement.

If one considers only an infinitesimal displacements, that is where $\Delta x, \Delta y, \Delta z$ are infinitely small and can be replaced by $dx, dy, dz$ and correspondingly the constants $\alpha, \beta, \gamma, \theta$ are small of the same order, then, neglecting terms of the second or higher order, the equations (48) and (28) reduce to

$$dx = \alpha + py - nz, \quad dy = \beta + mz - px, \quad dz = \gamma + nx - my; \quad (62)$$

and for the equations of the central axis,

$$\frac{\theta x + \gamma \cos h - \beta \cos l}{\cos g} = \frac{\theta y + \alpha \cos l - \gamma \cos g}{\cos h} = \frac{\theta z + \beta \cos g - \alpha \cos h}{\cos l}. \quad (63)$$

18 Let us return to the general formulas (48). From them one derives

$$\begin{align*}
(\vec{\Delta} - \vec{\delta})^2 &= 4 \tan^2(\theta/2) \left[ (\vec{\omega} - (1/2)\vec{\delta})^2 - ((\vec{\omega} - (1/2)\vec{\delta}) \cdot \hat{t})^2 \right]. \quad (65)
\end{align*}$$

I shall henceforth omit some of the more complex equations given in the original text, supplying only my transcription into modern notation. The derivation of (65) may proceed as follows: (48) can be transcribed to

$$\vec{\Delta} = \vec{\delta} + 2 \tan(\theta/2)\hat{t} \times (\vec{\omega} - (1/2)\vec{\delta}) \quad (66)$$

where $\vec{\delta}$ is given by (47). To see how (66) leads to (65), let us first define $\vec{\omega}' = \vec{\omega} - (1/2)\vec{\delta}$. Then (66) can be written

$$\vec{\Delta} = \vec{\delta} + 2 \tan(\theta/2)\hat{t} \times \vec{\omega}' \quad (67)$$

which leads immediately to

$$\begin{align*}
(\vec{\Delta} - \vec{\delta})^2 &= 4 \tan^2(\theta/2) (\hat{t} \times \vec{\omega}')^2, \quad (68)
\end{align*}$$

whereas (65) becomes

$$\begin{align*}
(\vec{\Delta} - \vec{\delta})^2 &= 4 \tan^2(\theta/2) [(\vec{\omega}'^2 - (\vec{\omega}' \cdot \hat{t})^2]. \quad (69)
\end{align*}$$

But (68) and (69) are identical, from the familiar identity $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$. Thus (65) is a consequence of previously derived formulas.

To the equation (69) (which gives the magnitude of $\vec{\Delta} - \vec{\delta}$) one may adjoin the two equations

$$\begin{align*}
(\vec{\Delta} - \vec{\delta}) \cdot \hat{t} &= 0 \quad (70) \\
(\vec{\Delta} - \vec{\delta}) \cdot \vec{\omega}' &= 0 \quad (71)
\end{align*}$$
(which give its direction, normal to both \( \dot{t} \) and \( \vec{\omega}' \); these equations follow directly from (67).)

But now let us replace \( \vec{\omega} \) by its value \( \vec{r}'+(1/2)\vec{\Delta} \). We thus obtain from (67)

\[
\vec{\Delta} - \vec{\delta} = 2 \tan(\theta/2) \hat{t} \times \vec{r} + \tan(\theta/2) \hat{t} \times (\vec{\Delta} - \vec{\delta})
\]

(I have dissected my own composite \( \vec{\omega}' \) as well as making the author’s substitution) and from this in turn

\[
\vec{\Delta} = \vec{\delta} + (\sin \theta) \hat{t} \times \vec{r} + 2 \sin^2(\theta/2)[\hat{t} \cdot \vec{r} - \vec{r}],
\]

of which only the first two terms will survive on passing from finite to infinitesimal displacements.

In order to derive (73), one must substitute (72) into itself as follows:

Evaluate

\[
\hat{t} \times (\vec{\Delta} - \vec{\delta}) = \hat{t} \times [2 \tan(\theta/2) \hat{t} \times \vec{r}] + \hat{t} \times [\tan(\theta/2) \hat{t} \times (\vec{\Delta} - \vec{\delta})]
\]

\[
= 2 \tan(\theta/2)[\hat{t} \cdot \vec{r} - \vec{r}] + \tan(\theta/2)[\hat{t} \cdot (\vec{\Delta} - \vec{\delta}) - (\vec{\Delta} - \vec{\delta})]
\]

\[
= 2 \tan(\theta/2)[\hat{t} \cdot \vec{r} - \vec{r}] + \tan(\theta/2)[0 - (\vec{\Delta} - \vec{\delta})].
\]

Now substitute this expression into the last term of (72):

\[
\vec{\Delta} - \vec{\delta} = 2 \tan(\theta/2) \hat{t} \times \vec{r} + 2 \tan^2(\theta/2)(\hat{t} \cdot \vec{r} - \vec{r}) - \tan^2(\theta/2)(\vec{\Delta} - \vec{\delta}).
\]

Transposing the last term to the left side of the equation, and noting that

\[
1 + \tan^2(\theta/2) = 1/\cos^2(\theta/2),
\]

we find

\[
(\vec{\Delta} - \vec{\delta})/\cos^2(\theta/2) = 2 \tan(\theta/2) \hat{t} \times \vec{r} + 2 \tan^2(\theta/2)(\hat{t} \cdot \vec{r} - \vec{r})
\]

Finally multiplying the equation by \( \cos^2(\theta/2) \), we obtain (73).

18.1 Expressions for finite displacements as rational functions of \( \vec{\Delta} \) and \( \vec{q} \)

If one retains in these formulas the primitive constants \( m, n, p \), and directly extracts the values of \( \Delta_x, \Delta_y, \Delta_z, \ldots \), (the author now gives formulas which I shall abbreviate by recalling that these three \( \Delta \)'s form a single vector \( \vec{\Delta} \), that \( \alpha, \beta, \gamma \) form the vector \( \vec{\delta} \), and that \( m, n, p \) are the components of \( \vec{q} = q \hat{t} \), and which I shall derive as follows: Let us write all the trigonometric functions in (76) in terms of \( \tan(\theta/2) \), thus:

\[
\vec{\Delta} - \vec{\delta} = [2 \tan(\theta/2) \hat{t} \times \vec{r} + 2 \tan^2(\theta/2)(\hat{t} \cdot \vec{r} - \vec{r})]/(1 + \tan^2(\theta/2)).
\]

Then recalling the definition (30), we have

\[
\vec{\Delta} = \vec{\delta} + [\vec{q} \times \vec{r} + (1/2)(\vec{q} \cdot \vec{r} - q^2 \vec{r})](1 + (1/4)q^2)
\]

in which the displacement \( \vec{\Delta} \) is given as a rational function of \( \vec{\delta} \) and \( \vec{q} \). That is, in the words of the author),

one has the following expressions (for \( \Delta_x, \Delta_y, \Delta_z \)) in rational functions of the six constants \( \alpha, \beta, \gamma, m, n, p \): (the expressions he gives are essentially those of (78) written out in components.)
18.2 Important consequence relating to the formulas for transforming rectangular coordinates.

Comparing these expressions with those that one would obtain by considering the transformation of rectangular coordinates, as this will be indicated in 26, one obtains a way of reducing the nine coefficients that enter into the formulas for this transformation to three independent variables \( m, n, p \), entirely free of irrationals, which I believe has not been given before.

Until now, the author has been occupied with calculating \( \overrightarrow{\Delta} \), the straight-line displacement from an initial point \( \overrightarrow{r} \) to its destination \( \overrightarrow{r}' \). He now turns his attention to the transformation of \( \overrightarrow{r} \) to \( \overrightarrow{r}' \), in the case of a pure rotation. Nowadays we think of this transformation as given by a \( 3 \times 3 \) matrix transforming \((x, y, z)\) to \((x', y', z')\). The author frames this in a complementary way as making a transformation of the coordinate system \( \hat{x}, \hat{y}, \hat{z} \), such that if \( \overrightarrow{r} = x \hat{x} + y \hat{y} + z \hat{z} \) then \( \overrightarrow{r}' = x' \hat{x} + y' \hat{y} + z' \hat{z} \) - the coordinates, not the components, being altered. The “nine coefficients” he speaks of are the nine elements of the matrix

\[
(M) = \begin{pmatrix}
x \cdot x' & y \cdot x' & z \cdot x' \\
x \cdot y' & y \cdot y' & z \cdot y' \\
x \cdot z' & y \cdot z' & z \cdot z'
\end{pmatrix}
\]

The author gives formulas for these nine elements which have three notable features: (1) they involve no input other than the three parameters \( m, n, p \); (2) they contain no irrational expressions; and (3) the three parameters are completely independent. He simply lists these nine formulas; I shall display them in matrix form, but I shall write each matrix element exactly as it appears in the original text except that in order to save space I shall write \( \frac{mn}{2} \) rather than \( \frac{mn}{2} \), etc., and I shall place the common denominator \( 1 + (m^2 + n^2 + p^2)/4 \) outside the matrix as a prefix. Here is the result:

\[
(M) = \begin{pmatrix}
1 + (m^2 - n^2 - p^2)/4 & mn/2 - p & pm/2 + n \\
mn/2 + p & 1 + (n^2 - p^2 - m^2)/4 & np/2 - m \\
np/2 - m & np/2 + m & 1 + (p^2 - m^2 - n^2)/4
\end{pmatrix}.
\]

On eliminating \( m, n, p \) from the (off-diagonal) coefficients, one obtains the formulas of Monge, in irrational functions, for the three (diagonal) coefficients.

It is of some interest to decompose \( (M) = (M)_1 + (M)_2 + (M)_3 \) where

\[
(M)_1 = \frac{[1 - (m^2 + n^2 + p^2)/4]}{[1 + (m^2 + n^2 + p^2)/4]} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
(M)_2 = \frac{[1 + (m^2 + n^2 + p^2)/4]}{[1 + (m^2 + n^2 + p^2)/4]}^{-1} \begin{pmatrix} m^2/2 & mn/2 & pm/2 \\ mn/2 & n^2/2 & np/2 \\ pm/2 & np/2 & p^2/2 \end{pmatrix},
\]

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\[
(M)_3 = [1 + (m^2 + n^2 + p^2)/4]^{-1} \begin{pmatrix}
  0 & -p & n \\
  p & 0 & -m \\
  -n & m & 0 
\end{pmatrix}.
\]

Passing over to modern vector notation, this gives us
\[
\vec{r}' = (M) \cdot \vec{r} = (1 + q^2/4)^{-1}[(1 - q^2/4)\vec{r} + (1/2)\vec{q} \cdot \vec{r} + \vec{q} \times \vec{r}]. \quad (79)
\]

Now, if we readmit the trigonometric functions via \((30)\), we have
\[
[1 - q^2/4]/(1 + q^2/4) = (1 - \tan^2(\theta/2))/(1 + \tan^2(\theta/2)) = \cos \theta, \quad (80)
\]
\[
(1/2)\vec{q} \times (1 + q^2/4) = 2\hat{t} \tan^2(\theta/2)/\sec^2(\theta/2) = 2\hat{t} \sin^2(\theta/2) = \hat{t}(1 - \cos \theta), \quad (81)
\]
\[
\vec{q}/(1 + q^2/4) = 2\hat{t} \tan(\theta/2) \cos^2(\theta/2) = \hat{t} \sin \theta; \quad (82)
\]
and so we arrive at the celebrated Rodrigues rotation formula
\[
\vec{r}' = \vec{r} \cos \theta + \hat{t} \cdot \vec{r}(1 - \cos \theta) + \hat{t} \times \vec{r} \sin \theta. \quad (83)
\]

19 On the composition of two displacements.

The following three paragraphs summarize the entire essay, looking backward to the beginning as well as forward to the end.

From a geometric consideration of the displacement of a solid system, we began (this essay) by deducing the characteristic properties or general laws of the displacement, which always reduces to a rotation followed by a translation, or equivalently to a single couple of rotations about two fixed axes. (The two axes may be either parallel or not parallel; in the first case the displacement reduces to a simple translation, provided that the two rotations are equal and in opposite directions.)

From these properties we have now derived the analytic expression for the transformation, either finite or infinitely small, of the coordinates of a solid system undergoing an arbitrary displacement.

The author seems to be referring here to the formula for the matrix \((M)\), which turned out in modern notation to be the rotation formula \((83)\). But in the reasoning to follow, he works not from \((83)\) but from the earlier formula \((19)\), which gives the simple displacement \(\Delta\) in terms of \(\vec{r}\) and \(\vec{q}\). One readily sees that if one sets \(\vec{q} = 0\) one obtains \(\Delta = \delta\), which is a pure translation since the components \(\alpha, \beta, \gamma\) of \(\delta\) do not depend on the location of \(\vec{r}\); and that if \(\delta = 0\) then \(\Delta = \vec{ω} \times \vec{r}\), which describes a pure rotation since it vanishes at the origin \(\vec{r} = 0\).

We shall now deduce, from this expression, the laws of composition of rotations and of translations that we previously exhibited synthetically. And finally we shall establish these same formulas directly, by three distinct analytic procedures, making use exclusively of the invariance of the distances between points of this system.
Let us designate by $\vec{\Delta}', \vec{\Delta}''$ the two successive changes of positions of a point in the displaced system, and by $\vec{\Delta}$ the resultant change. Likewise by $\vec{\omega}', \vec{\omega}''$ the positions of the midpoints of the two straight-line displacements of the point, and by $\vec{\omega}$ the midpoint of the resultant displacement. Thus one has

$$\vec{\omega}' = \vec{r} + (1/2)\vec{\Delta}', \quad (84)$$

$$\vec{\omega}'' = \vec{r} + \vec{\Delta}' + (1/2)\vec{\Delta}'', \quad (85)$$

and

$$\vec{\omega} = \vec{r} + (1/2)\Delta \vec{r}. \quad (86)$$

Moreover,

$$\vec{\Delta} = \vec{\Delta}' + \vec{\Delta}'', \quad (87)$$

I am routinely putting all equations in modern form.

Furthermore, let us designate by $\hat{\vec{t}}', \hat{\vec{t}}''$ the absolute translations, and by $\hat{\theta}', \hat{\theta}''$ and $\hat{q}', \hat{q}''$ the rotations and the directions of the rotation axes of each of the two consecutive displacements under consideration; and by $\hat{T}, \hat{\Theta}, \hat{Q}$ the analogous elements of the composite displacement.

First we shall examine separately the composition of simple translations and that of rotations without translation. If we suppose that the displacements consist purely of translations, we have

$$\vec{\Delta}' = \hat{\vec{t}}', \vec{\Delta}'' = \hat{\vec{t}}'', \vec{\Delta} = \hat{T} = \hat{\vec{t}}' + \hat{\vec{t}}'', \quad (88)$$

from which we see that the composite translation is nothing other than the third side of the triangle formed by the successive passages of a point of the system by reason of the two given translations. To compose more than two given translations, one may easily generalize from the triangle to the polygon of translations.

Now consider the composition of two rotations, without translation, about two axes that intersect at the origin of coordinates. We shall have

$$\vec{\Delta}' = \hat{\vec{q}}' \times \vec{\omega}', \quad \vec{\Delta}'' = \hat{\vec{q}}'' \times \vec{\omega}'', \quad \vec{\Delta} = \hat{\vec{Q}} \times \vec{\omega}, \quad (89)$$

where

$$\hat{\vec{q}}' = \hat{q}'2\tan(\theta'/2), \quad \hat{\vec{q}}'' = \hat{q}''2\tan(\theta''/2), \quad \hat{\vec{Q}} = \hat{Q}'2\tan(\Theta/2), \quad (90)$$

To understand these equations, consider the third line of (89). The part of $\vec{\omega}$ parallel to $\hat{\vec{Q}}$ does not contribute to $\hat{\vec{Q}} \times \vec{\omega}$. Therefore the expression $\hat{\vec{Q}} \times \vec{\omega}$ simply rotates $\vec{\omega}$ through a right angle in the plane perpendicular to $\hat{\vec{Q}}$. This gives a vector with the right direction (along $\vec{\Delta}$) but the wrong magnitude. The magnitude of $\vec{\Delta}$ is obtained by inserting a
factor $2 \tan(\Theta/2)$, that is by replacing $\hat{Q}$ with $\vec{Q}$ as indicated in the third line of (90). Likewise for the first and second lines of these equations.

In all these discussions $\hat{q}$ is the direction of the rotation axis, and $\hat{t}$ of the translation. But sometimes, as in 18, it is assumed that one is dealing with the central axis, so that $\hat{q} = \hat{t}$. Now, however, we have two partial rotation axes intersecting at $0$, which does not necessarily lie on the central axis of either partial displacement; therefore $\hat{q}$ must be distinguished from $\hat{t}$, for each partial displacement and for the resultant. Indeed, for the remainder of this Section the translations $\vec{t}', \vec{t}'', \vec{T}$ are taken as zero, so that $\hat{t}', \hat{t}'', \hat{T}$ are indeterminate.

We need to determine the composite parameters $\Theta, \vec{Q}$ as functions of the partial ones $\theta', \vec{q}'$ and $\theta'', \vec{q}''$. It will be useful to begin by eliminating the variables $\vec{\omega}', \vec{\omega}''$ from (89), using the relations (84) through (87). This elimination is the key to the author’s solution of the problem posed. He writes down the elimination formulas without showing a derivation; here I shall present a geometrical derivation.

Let $A, B, C$ denote respectively the points whose position vectors are $\vec{r}, \vec{r} + \vec{\Delta}', \vec{r} + \vec{\Delta}$. Then $\vec{\omega}', \vec{\omega}'', \vec{\omega}$ are the position vectors of the midpoints of $AB, BC, AC$. To eliminate $\vec{\omega}'$, consider the triangle formed by $A, \vec{\omega}', \vec{\omega}$. It is similar to $ABC$ but half as large in linear dimension. Therefore the vector $\vec{\omega}' - \vec{\omega}$ is just half the vector from $C$ to $B$ which is $-\vec{\Delta}''$. So we have

$$\vec{\omega}' = \vec{\omega} - (1/2)\vec{\Delta}''$$

(91)

To eliminate $\vec{\omega}''$, consider the triangle formed by $\vec{\omega}, \vec{\omega}''$, $C$. It is also similar to $ABC$ but half as large in linear dimension. Therefore the vector $\vec{\omega}'' - \vec{\omega}$ is just half the vector from $A$ to $B$ which is $+\vec{\Delta}'$. So we have

$$\vec{\omega}'' = \vec{\omega} + (1/2)\vec{\Delta}'$$

(92)

Note the opposite placement of $'$ and $''$, as well as the change of sign of the displacement term between (91) and (92). These features will have important consequences later on.

Substituting (91) and (92) into (89), we obtain

$$\vec{\Delta}' = \vec{q}' \times \vec{\omega} - (1/2) \vec{q}' \times [\vec{q}'' \times \vec{\omega}]$$

(93)

$$\vec{\Delta}'' = \vec{q}'' \times \vec{\omega} + (1/2) \vec{q}'' \times \vec{\Delta}'$$

(94)

from which we deduce the following values for the partial displacements:

Substituting (91) into (93), we have

$$\vec{\Delta}' = [\vec{\Delta}' - \vec{\Delta}'] = \vec{q}' \times \vec{\omega} - (1/2) \vec{q}' \times [\vec{q}'' \times \vec{\omega}]$$

$$= \vec{a}' + \vec{b}' - (1/4) \vec{q}' \times [\vec{q}'' \times \vec{\Delta}']$$

(95)

where

$$\vec{a}' = \vec{q}' \times \vec{\omega}, \quad \vec{b}' = -(1/2) \vec{q}' \times [\vec{q}'' \times \vec{\omega}]$$

(96)
and substituting in the reverse direction,

\[ \Delta'' = \vec{q}'' \times \vec{\omega} + (1/2) \vec{q}'' \times [\vec{q} \times \vec{\omega}] - (1/2) \vec{q}' \times \Delta'' \]

\[ = \vec{a}'' + \vec{b}'' - (1/4) \vec{q}'' \times [\vec{q} \times \Delta''] \]  

(97)

where

\[ \vec{a}'' = \vec{q}'' \times \vec{\omega}, \quad \vec{b}'' = +(1/2) \vec{q}'' \times [\vec{q} \times \vec{\omega}] \].  

(98)

The exchange of order of the two partial displacements carries \( a' \) into \( a'' \) but \( b' \) into \(-b''\); this change of sign is inherited from the one between (91) and (92). We can express the change by rewriting

\[ b' = +(1/2) \vec{q}' \times [\vec{q} \times \vec{\omega}] \].

In view of (87), we now have

\[ \Delta = \vec{a} + \vec{b} - (1/4) [\vec{q} \times (\vec{a} \times \Delta') + \vec{q}'' \times (\vec{q}' \times \Delta')] = \vec{a} + \vec{b} + (1/4) \vec{q}' \cdot \vec{q}'' \Delta' \]  

(99)

where we have used the fact that \( \vec{q}' \cdot \Delta' \vec{r} = \vec{q}'' \cdot \Delta'' \vec{r} = 0 \), and where

\[ \vec{a} = \vec{a}' + \vec{a}'' = (\vec{q}' + \vec{q}'') \times \vec{\omega} \]  

(100)

and

\[ \vec{b} = \vec{b}' + \vec{b}'' = (1/2) [\vec{q} \times (\vec{q}' \times \vec{r}) + \vec{q}'' \times (\vec{q}' \times \vec{r})] = -(1/2) \vec{\omega} \times (\vec{q}'' \times \vec{q}') \].

(101)

In the last step I have used the Jacobi identity for the cyclic sum of a double cross product.

The last term in (99) can be transposed to the left side, giving

\[ \Delta(1 - (1/4) \vec{q}' \cdot \vec{q}'') = \vec{a} + \vec{b} \]  

(102)

The appearance of the common factor \( \vec{\omega} \) in (102) is the major fruit of (91) and (92).

Now by comparison of this expression for \( \Delta \vec{r} \) with the postulated expression for the same quantity in terms of \( \Theta, T \), we arrive at the following relation: The author wishes to compare (102) with the third equation in (89). This comparison yields the equation

\[ \vec{Q} \times \vec{\omega} = \vec{s} \times \vec{\omega} \]  

(103)

where

\[ \vec{s} = \frac{\vec{q}' + \vec{q}'' + (1/2) \vec{q}'' \times \vec{q}'}{1 - (1/4) \vec{q}' \cdot \vec{q}''}. \]  

(104)

One must now remember that \( \vec{Q} \), as well as the variables entering into \( \vec{s} \), are fixed for a particular displacement of the whole solid, whereas \( \vec{r} \), which enters into \( \vec{\omega} \), can be any point in the solid. Therefore (103) holds for every possible \( \vec{\omega} \). This implies the author’s “following relation”, namely

\[ \vec{Q} = \vec{s}, \]  

(105)
from which one deduces, for the value of the resultant rotation,
\[
\cos(\Theta/2) = \cos(\theta'/2) \cos(\theta''/2) - \sin(\theta'/2) \sin(\theta''/2) \cos \nu, \tag{106}
\]
where \( \nu \) is the angle between the two axes of rotation, thus \( \cos \nu = \hat{q}' \cdot \hat{q}'' \).

The deduction may proceed as follows. First we replace the three \( q \)-vectors by their definitions: generically, \( \hat{q} = 2 \hat{q} \tan(\theta/2) \). Thus \( \tan^2(\Theta/2) = Q^2/4 \), and
\[
\cos^2(\Theta/2) = (1 + (Q^2/4))^{-1}. \tag{107}
\]
Meanwhile we may write \( \hat{s} = 2 \hat{N}/D \), so that substituting \( \tag{105} \) into \( \tag{107} \) we have
\[
\cos^2(\Theta/2) = (1 + (s^2/4))^{-1} = \frac{D^2}{N^2 + D^2} \tag{108}
\]
where
\[
\hat{N} = \hat{q}'/2 + \hat{q}''/2 + \hat{q}' \times \hat{q}''/4, \tag{109}
\]
\[
D = 1 - (1/4) \cos \nu. \tag{110}
\]

From \( \tag{108} \) we see that if \( N^2 + D^2 \) can be exhibited as a perfect square we can obtain \( \cos(\Theta/2) \) without radicals. The third term of \( \tag{109} \) is orthogonal to both of the first two; therefore
\[
N^2 = \left[ (\hat{q}'/2 + \hat{q}''/2)^2 + (\hat{q}' \times \hat{q}'')^2 \right]/16
= [\tan^2(\theta'/2) + \tan^2(\theta''/2) + 2 \tan(\theta'/2) \tan(\theta''/2) \cos \nu]
+ \tan^2(\theta'/2) \tan^2(\theta''/2) (1 - \cos^2 \nu); \tag{111}
\]
of course \( (\hat{q}' \times \hat{q}'')^2 = \sin^2 \nu = 1 - \cos^2 \nu \). At the same time \( \tag{110} \) gives us
\[
D^2 = 1 - 2 \tan(\theta'/2) \tan(\theta''/2) \cos \nu + \tan^2(\theta'/2) \tan^2(\theta''/2) \cos^2 \nu. \tag{112}
\]
The terms in \( \cos \nu \) and \( \cos^2 \nu \) obligingly cancel between \( \tag{111} \) and \( \tag{112} \), leaving
\[
N^2 + D^2 = 1 + \tan^2(\theta'/2) + \tan^2(\theta''/2) + \tan^2(\theta'/2) \tan^2(\theta''/2)
= (1 + \tan^2(\theta'/2))(1 + \tan^2(\theta''/2)) = 1/\cos^2(\theta'/2) \cos^2(\theta''/2). \tag{113}
\]
This enables us to take the square root of \( \tag{108} \) :
\[
\cos(\Theta/2) = \cos(\theta'/2) \cos(\theta''/2)(1 - \tan(\theta'/2) \tan(\theta''/2) \cos \nu)
= \cos(\theta'/2) \cos(\theta''/2) - \sin(\theta'/2) \sin(\theta''/2) \cos \nu, \tag{114}
\]
in agreement with \( \tag{106} \). This startling relation was deduced from \( \tag{105} \) by the author, who shows no intermediate steps in the text. One can only wonder at his ability to navigate the maze of substitutions without the help of our vector relations.

- and for the inclination of the resultant axis,
\[
\dot{Q} \sin(\Theta/2) = \hat{q}' \sin(\theta'/2) \cos(\theta''/2) + \hat{q}'' \sin(\theta'/2) \cos(\theta''/2) + \hat{q}' \times \hat{q}'' \sin(\theta'/2) \sin(\theta''/2). \tag{115}
\]
Recalling the definitions given after \( \tag{89} \),
\[
\dot{Q} = 2Q \tan(\theta/2), \quad \hat{q}' = 2\hat{q} \tan(\theta/2) \quad \hat{q}'' = 2\hat{q} \tan(\theta'/2). \tag{116}
\]
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we can write \((105)\) as

\[
\hat{Q} \tan(\Theta/2) = \hat{q} \tan(\theta'/2) + \hat{q}'' \tan(\theta''/2) + \hat{q}' \times \hat{q} \tan(\theta'/2) \tan(\theta''/2) /
1 - \hat{q}' \cdot \hat{q} \tan(\theta'/2) \tan(\theta''/2)
\]

\(117\)

and \((106)\) as

\[
\cos(\Theta/2) = \cos(\theta'/2) \cos(\theta''/2) (1 - \hat{q}' \cdot \hat{q} \tan(\theta'/2) \tan(\theta''/2)).
\]

\(118\)

Multiplying these two equations, we obtain \((115)\).

In these formulas one notices immediately that the order of succession of the rotations \(\theta', \theta''\) has no effect on the amplitude \(\Theta\) of the resultant rotation (see \((114)\)), but that it does affect the direction of the axis of that rotation (see the last term of \((115)\)), unless the rotations \(\theta', \theta''\) are infinitely small.

Now, the expression for \(\cos(\Theta/2)\) holds if \(\Theta/2\) is an angle of a spherical triangle, of which the opposing side is \(\nu\) and the two other angles are \(\theta'/2, \theta''/2\). This situation is dual to the one described by the usual spherical law of cosines, which gives a side in terms of the opposite angle and the two other sides. That is why the formula \((114)\) has a minus sign before the second term instead of a plus sign. To specify more precisely the position of the resultant axis in relation to the two given axes, let us suppose, which is always possible, that

\[
\cos l = \cos l' = 0, \cos h' = 0, \cos g' = 1, \cos g'' = \cos \nu, \cos h'' = \sin \nu.
\]

\(119\)

Generally, the author has regarded the \(x, y, z\) coordinate system as fixed and independent of what displacement is being applied to the solid system. Occasionally he has made slight departures: in the previous section he discussed linear orthogonal transformations of the coordinates, and in the present inquiry he has already specified that the origin of coordinates is at the intersection of the two given (and hence also of the resultant) axes of rotation. Now he chooses an orientation of the coordinate system completely tailored to the problem at hand. He has the \(x\)-axis coinciding with the axis of the rotation \(\theta'\), and the \(z\)-axis perpendicular to both rotation axes \(\theta, \theta''\). Thus the axis of rotation \(\theta''\) lies in the \(x\)-\(y\) plane making an angle \(\nu\) with the \(x\)-axis. This will simplify his calculations, but at the cost of making it impossible to apply modern vector notation to his formulas; they will have to be written out in components as in the original text.

One now has

\[
\sin(\Theta/2) \cos G = \sin(\theta'/2) \cos(\theta''/2) + \sin(\theta''/2) \cos(\theta'/2) \cos \nu,
\sin(\Theta/2) \cos H = \sin(\theta''/2) \cos(\theta'/2) \sin \nu,
\sin(\Theta/2) \cos L = \sin(\theta'/2) \sin(\theta''/2) \sin \nu.
\]

\(120\)

These equations can be obtained by resolving \((115)\) into its \(x\)-, \(y\)-, and \(z\)-components, and applying \((119)\).

If the order of rotations is reversed, the only change in \((120)\) is that the sign of \(\sin L\) becomes negative, from which it follows that the new resultant axis is placed in a symmetric position to the old one relative to the plane of the two given axes.
This result can be obtained directly from \((115)\) more easily than by struggling through \((120)\). The sum of the first two terms on the right side of \((115)\) is symmetric in \(\hat{q}'\) and \(\hat{q}''\), and both lie in the \(\hat{q}'-\hat{q}''\) plane; the last term is antisymmetric and perpendicular to this plane. Therefore the interchange of \(\hat{q}'\) with \(\hat{q}''\) causes \(Q\) to be reflected in the \(\hat{q}'-\hat{q}''\) plane.

If we denote by \(H'\) the angle formed by the resultant axis with the axis of the rotation \(\theta''\), we shall have
\[
\cos H' = \cos G \cos \theta'' + \cos H \cos \theta'' = \frac{\sin(\theta''/2) \cos(\theta''/2) + \sin(\theta''/2) \cos(\theta''/2) \cos \nu}{\sin(\theta/2)}
\]
\((121)\)

(the first equality comes from the decomposition \(\hat{q}'' = \hat{x} \cos \theta'' + \hat{y} \cos \theta''\), the second by applying \((119)\) to \((120)\)); besides this we have
\[
\sin^2(\theta'/2) = \sin^2(\theta'/2) + \sin^2(\theta'/2) + \sin^2(\theta'/2) \cos(\theta'/2) \cos \nu^2
\]
\(= \sin^2(\theta'/2) \sin^2 \nu + [\sin(\theta'/2) \cos(\theta'/2) + \sin(\theta'/2) \cos(\theta'/2) \cos \nu]^2
\)
\((122)\)

and consequently
\[
\sin^2 G = \frac{\sin^2(\theta'/2) \sin^2 \nu}{\sin^2(\theta/2)}, \quad \sin^2 H' = \frac{\sin^2(\theta'/2) \sin^2 \nu}{\sin^2(\theta/2)}
\]
\((123)\)
equations that establish the proportionality of the sines of the half-rotations to those of the angles formed by the resultant axis with the given axes inversely corresponding \(\text{(that is, } G \text{ to } \theta'' \text{ and } H' \text{ to } \theta')\), and which lead to the construction that we have indicated from the outset for the composition of rotations.

The first line of \((122)\) is obtained by summing the squares of the three equations in \((120)\). The second line follows from the first in view of \((114)\) which gives \(\Theta\) as a symmetric function of \(\theta'\) and \(\theta''\). In regard to \((123)\), note that \(G\) bears the same relation to \(\theta'\) as \(H'\) to \(\theta''\), in view of the condition \(\cos \theta'' = 1\) from \((119)\). Of course the author means to consider the square root of \((123)\). The resulting formula is the one already described in Section 6, corollary \# 2.

If we were to follow an analogous procedure for the composition of rotations about an arbitrary number of intersecting axes, the resulting formulas would be rendered exceedingly complex by the terms of second order; hence we omit that subject.

20 On the analytic composition of rotations about nonintersecting axes.

As for the composition of rotations about nonintersecting axes, and generally of an arbitrary succession of displacements of a solid system, given by individual displacements \(\Delta', \Delta'', \text{etc.}\) whose analytic form is known, we shall have
\[
\Delta x = \Delta' x + \Delta'' x + \Delta''' x + \ldots = A + 2 \tan(\Theta/2)(Y \cos L - Z \cos H),
\]
\[
\Delta y = \Delta' y + \Delta'' y + \Delta''' y + \ldots = B + 2 \tan(\Theta/2)(Z \cos G - X \cos L),
\]
\[
\Delta z = \Delta' z + \Delta'' z + \Delta''' z + \ldots = C + 2 \tan(\Theta/2)(X \cos H - Y \cos G).
\]
\((124)\)
The constants $A, B, C$ etc. (pertaining to the resultant displacement) are to be found in terms of analogous constants $A', B', C'$ etc.; $A'', B'', C''$ etc.; ... belonging to each of the consecutive displacements to be combined, and of the other elements of these displacements. The expressions beginning $A, B, C$ are closely related to those seen in (26) and (28). But it is evident that the elements $\Theta, G, H, L$ of the resultant rotation depend only on the rotational elements of these displacements, just as we have seen earlier from geometric considerations.

Having posed a problem of sweeping generality, the author proceeds to solve only the simplest case, that of composing two rotations about non-intersecting axes. Moreover, he will select a made-to-order orientation of the coordinate system similar to the one he introduced to treat two intersecting axes. Whereas in the previous section both axes passed through the point $(0, 0, 0)$, and gave their direction cosines as $(1, 0, 0)$ and $(\cos \nu, \sin \nu, 0)$, in the nonintersecting case (119) will still give the direction cosines but the second axis will pass not through $(0, 0, 0)$ but through $(0, 0, u)$ where $u$ is geometrically the closest distance of approach between the two axes.

Let us take, for example, the composition of rotations about two fixed nonintersecting axes, one identified with the $x$-axis and the other normal to the $z$-axis and cutting that axis at a distance $u$ from the origin. We shall, as before, denote by $\nu$ the angle between these two rotation axes and by $\theta', \theta''$ the amplitudes of the respective rotations. To begin with, we shall have for the amplitude of the resultant rotation and the direction of its axis, the same formulas as obtained in the previous section. To fix the position of the central (i.e. composite) axis, as well as to find the displacements of the coordinates, one needs only to calculate the displacements $\alpha, \beta, \gamma$ from the origin of coordinates. Now, those displacements arising from the first rotation about the $x$ axis are null; hence it suffices to calculate those arising from the rotation $\theta''$, for which one has in general (cf. (48) near the end of Section 15)

\[
\Delta''x = \alpha + 2\tan(\theta''/2)[(Y'' - (1/2)\beta) \cos \rho'' - (Z'' - (1/2)\gamma) \cos \varphi'']
\]
\[
\Delta''y = \beta + 2\tan(\theta''/2)[(Z'' - (1/2)\gamma) \cos \varphi'' - (X'' - (1/2)\alpha) \cos \rho'']
\]
\[
\Delta''z = \gamma + 2\tan(\theta''/2)[(X'' - (1/2)\alpha) \cos \rho'' - (Y'' - (1/2)\beta) \cos \varphi'']
\]

these displacements must vanish for all points on the axis of rotation $\theta''$, for which one has

\[
\cos \rho'' = 0, \ \cos \varphi'' = \sin \nu, \ \cos \varphi'' = \cos \nu, \ Y'' = X'' \tan \nu, \ Z'' = u.
\]

From this there result the following values for $\alpha, \beta, \gamma$, which one could easily derive from the construction itself,

\[
\alpha = u \sin \nu \sin \theta'', \ \beta = -u \cos \nu \sin \theta'', \ \gamma = 2u \sin^2(\theta''/2),
\]

in agreement with the theorems of Section 13.

Thus the resultant axis and the two given divergent axes become more nearly parallel to a single plane, the greater the distance between the two latter axes in comparison with the absolute translation of the resultant displacement. That is, $\cos L$ approaches zero as $u >> T$. 

42
Since the two rotation axes do not intersect, Euler’s fixed point theorem (Section 3) does not hold, and the resultant displacement is not a pure rotation but contains an absolute translation \( \mathbf{T} = \mathbf{T} \).

In (125) the coordinates of the point \( \xi, \eta, \zeta \) defined in (16) have been replaced in (48) by \( X, Y, Z \) as given in (126). Note that in the case \( \mu = 0 \) (two intersecting axes) \( X, Y, Z \) would be strictly proportional to \( \cos g'', \cos h'', \cos l'' \) and consequently the terms in these quantities would vanish; this would leave only \((\alpha, \beta, \gamma)\) on the right side of (125) so that the left side is made to vanish by setting \((\alpha, \beta, \gamma) = 0\). This is why \((\alpha, \beta, \gamma)\) do not appear in Section 19.

For \( \mu \neq 0 \), only the terms in \( X, Y \) cancel out, and to solve for \( \alpha, \beta, \gamma \) it suffices to make \( \Delta''(x, y, z) \) vanish when \((X, Y, Z) = (0, 0, \mu)\). This yields (using (126) as well)

\[
0 = \alpha + 2 \tan(\theta''/2)[-(\mu - (1/2)\gamma) \sin \nu] \\
0 = \beta + 2 \tan(\theta''/2)[\mu - (1/2)\gamma \cos \nu] \\
0 = \gamma + 2 \tan(\theta''/2)[-\left( (-1/2)\alpha \sin \nu - (-1/2)\beta \cos \nu \right)]
\]

(128)

The first two lines give

\[
\alpha = (2u - \gamma) \tan(\theta''/2) \sin \nu, \quad \beta = -(2u - \gamma) \tan(\theta''/2) \cos \nu
\]

and the third gives

\[
\gamma = (2u - \gamma) \tan^2(\theta''/2)(\sin^2 \nu + \cos^2 \nu) = 2u \tan^2(\theta/2) - \gamma \tan^2(\theta/2). 
\]

(130)

Transposing the last term to the left, and multiplying the equation by \( \cos^2(\theta''/2) \), we have

\[
\gamma = 2u \tan^2(\theta''/2) \cos^2(\theta''/2) = 2u \sin^2(\theta''/2),
\]

(131)

and substituting (131) into (129) gives

\[
\alpha = u \sin \nu \sin \theta'', \quad \beta = -u \cos \nu \sin \theta''
\]

(132)

as in (127). (The text has a mistake: \( \sin \nu \) instead of \( \cos \nu \) in \( \beta \).)

The analogy between this calculation and the one discussed in the last paragraphs of Section 15 may be obscured by the fact that then only a single rotation was under consideration, whereas now there are two. But there is no difference if one makes the appropriate correspondences. The role played by the central axis in Section 15 is here played by the second of the two axes, having direction \( \hat{q}'' \) and rotation angle \( \theta'' \). Call this axis the active axis. In both cases the active axis does not pass through the origin of coordinates. In both cases a substitute axis is introduced, parallel to the active axis but passing through the origin of coordinates. In both cases \((\alpha, \beta, \gamma) = \delta \) is the displacement of a point starting at the origin. In the present case there is an additional axis, the first, which plays no part in determining \( \delta \) because it lies on the substitute axis and its action on the point of origin is null.

In the present case, the values of \( \alpha, \beta, \gamma \) can be easily obtained because the direction \( \hat{q}'' = (\cos \nu, \sin \nu, 0) \) is given. Starting at the origin \((0, 0, 0)\), the trajectory of rotation about the active axis describes an arc
θ" of the circle whose center is at (0, 0, u) and whose plane is normal to q". This gives immediately γ = u(1 − cos θ") = 2u sin²(θ''/2) as well as \(\sqrt{α² + β²} = u \sin \theta"\). From the direction of q" we infer that α : β : γ = sin ν : (−cos ν) and so \(α = u \sin \theta" \sin \nu", β = −u \sin \theta" \cos \nu\). In this way [131] and [132] are derived from geometry alone.

From (127) one derives, for the value \(T\) of the absolute translation resulting from two rotations about two nonintersecting fixed axes:

\[
T = \alpha \cos G + \beta \cos H + γ \cos L
= \frac{2u \sin \nu \sin(\theta'/2) \sin(\theta''/2)}{\sin(\Theta/2)}
= 2u \cos \nu ,
\]

This array contains three equalities. To prove the first equality, one notes that \(T = \hat{T} \cdot \hat{T}\) where \(\hat{T}\) is the displacement of the composite axis along its length due to the successive actions of the two partial rotations. We now treat the composite axis just as we treated the second partial axis in deriving [131] and [132]; we introduce a substitute composite axis parallel to the true composite axis but passing through the origin. Referring rotations to this substitute axis, we deduce as before that \((α, β, γ) = \hat{δ}\) is the displacement undergone by the point originally at (0, 0, 0). That is, \(\hat{δ}\) is the translation, common to all points, that must be added to rotation about the substitute axis in order to simulate the original composite displacement \(\hat{T}\). But rotation about the substitute axis, applied to an arbitrary point, produces a trajectory perpendicular to its direction \(\hat{T}\), so that \(\hat{T} \cdot \hat{T} = \hat{δ} \cdot \hat{T}\). Hence \(T = \hat{δ} \cdot \hat{T}\), which is the first equality.

The second equality is obtained by taking \((α, β, γ)\) from (127) and \(\hat{T}\) from (120). This gives

\[
\begin{align*}
α \cos G \sin(\Theta/2) &= u \sin \nu \sin \theta"[\sin(\theta'/2) \cos(\theta''/2) + \cos(\theta'/2) \sin(\theta''/2) \cos \nu] \\
β \cos H \sin(\Theta/2) &= −u \cos \nu \sin \theta"[\cos(\theta'/2) \sin(\theta''/2) \sin \nu] \\
γ \cos L \sin(\Theta/2) &= 2u \sin^2(\theta'/2) \sin(\theta''/2) \sin \nu
\end{align*}
\]

(134)

The second term in the first line cancels the second line so that

\[
α \cos G \sin(\Theta/2) + β \cos H \sin(\Theta/2) = u \sin \nu \sin \theta"[\sin(\theta'/2) \cos(\theta''/2) \cos \nu] = 2u \sin \nu \sin(\theta'/2) \sin(\theta''/2) \cos^2(\theta''/2)
\]

(135)

and adding the third line we obtain the second equality of (133). The third equality is obtained by simply comparing the numerator of the second line of (133) with the third line of (135).

The equations of the resultant axis are obtained by substituting for \(α, β, γ\) their values from (127) in the general equations

\[
\begin{align*}
x &= \frac{(1/2) \alpha − (1/2) \cot(\Theta/2)(β \cos L + γ \cos H)}{\cos G} \\
y &= \frac{(1/2) \beta − (1/2) \cot(\Theta/2)(γ \cos G + α \cos L)}{\cos H} \\
z &= \frac{(1/2) \gamma − (1/2) \cot(\Theta/2)(α \cos H + β \cos G)}{\cos L}
\end{align*}
\]

(136)

These equations are the same as (52) with g, h, l, θ capitalized.
In the case of infinitely small rotations these formulas are considerably simplified, and it is found that the \((\text{resultant})\) central axis is parallel to the plane of the two contributing axes and intersects their shortest distance. Effectively, in this case, with neglect of infinitesimals of second order, one has

\[
\cos G = \frac{\theta' + \theta'' \cos \nu}{\Theta}, \quad \cos H = \frac{\theta'' \sin \nu}{\Theta}, \quad \cos L = \frac{\theta' \theta'' \sin \nu}{2\Theta},
\]

\[
\Theta^2 = \theta'^2 + \theta''^2 + 2\theta' \theta'' \cos \nu,
\]

\[
\alpha = u \theta'' \sin \nu, \quad \beta = -u \sin \theta'' \cos \nu, \quad \gamma = 0, \quad T = \frac{u \theta' \theta'' \sin \nu}{\Theta} \tag{137}
\]

\(I\ have\ corrected\ three\ errors\ in\ the\ French\ text\ of\ this\ array:\ (i)\ top\ line,\ \cos H\ \text{numerator,} \ \sin \nu\ \text{incorrectly\ given\ as}\ \sin^2 \nu;\ (ii)\ second\ line,\ \text{left}\ \text{side\ of\ equation,} \ \Theta^2\ \text{incorrectly\ given\ as}\ \Theta;\ (iii)\ bottom\ line,\ \text{T\ numerator}\ \text{incorrectly\ given\ as}\ \theta \theta \theta \sin \nu\)

and for the equations of the \((\text{resultant})\) central axis,

\[
y = \frac{x \theta'' \sin \nu}{\theta' + \theta'' \cos \nu}, \quad z = \frac{u \theta' \theta'' \sin \nu}{\Theta^2}. \tag{138}
\]

21 Composition of successive rotations about three perpendicular axes.

We shall end this subject by giving the following formulas for the composition of three successive rotations \(\theta, \theta', \theta''\) about the three coordinate axes \(x, y, z\). The elements \(\Theta, G, H, L\) of the composite rotation are expressed as follows:

\[
\cos \left(\frac{\Theta}{2}\right) = \cos \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta'}{2}\right) \cos \left(\frac{\theta''}{2}\right) - \sin \left(\frac{\theta}{2}\right) \sin \left(\frac{\theta'}{2}\right) \sin \left(\frac{\theta''}{2}\right),
\]

\[
\sin^2 G = 1 - \frac{\cos \theta' \cos \theta''}{2 \sin^2 \left(\frac{\Theta}{2}\right)},
\]

\[
\sin^2 H = 1 - \frac{\cos \theta \cos \theta'' + \sin \left(\frac{\Theta}{2}\right) \sin \left(\frac{\theta'}{2}\right) \sin \left(\frac{\theta''}{2}\right)}{2 \sin^2 \left(\frac{\Theta}{2}\right)},
\]

\[
\sin^2 L = 1 - \frac{\cos \theta \cos \theta'}{2 \sin^2 \left(\frac{\Theta}{2}\right)} \tag{140}
\]

These formulas become symmetric with respect to each of the three successive rotations only when, the rotations being infinitely small, the term \(\sin \left(\frac{\Theta}{2}\right) \sin \left(\frac{\theta'}{2}\right) \sin \left(\frac{\theta''}{2}\right)\) vanishes in \(\sin^2 H\). This vanishing also makes the order of these rotations indifferent; one then finds, in accordance with the law of composition of infinitesimally small rotations,

\[
\Theta^2 = \theta^2 + \theta'^2 + \theta''^2, \quad \cos G = \theta / \Theta, \quad \cos H = \theta' / \Theta, \quad \cos L = \theta'' / \Theta. \tag{141}
\]

The problem inverse to the one we have just solved would have for its object the \textit{decomposition} of a finite rotation about a given axis into three rotations about the three coordinate axes. This amounts to solving the
above equations for $\theta, \theta', \theta''$ in terms of $\Theta, G, H, L$, which cannot be done for finite rotations but is utterly simple when the rotations are infinitely small.

22 On the composition of successive infinitesimal displacements of a solid system.

We shall now consider, directly and with a special extension, the laws of composition and decomposition of successive infinitesimal displacements, drawn from the analytic expression for the infinitely small coordinate changes of a solid system.

These changes are generally expressed in the following way,

$$
\delta x = \alpha + py - nz, \quad \delta y = \beta + mz - px, \quad \delta z = \gamma + nx - my,
$$

(142)

$$
(\bar{\Delta} = \bar{\delta} + \bar{r} \times \bar{q}, \text{where } \bar{\Delta} \text{ and } \bar{q} \text{ are infinitesimal})
$$

as linear functions of the infinitesimal elements of the displacement, $\alpha, \beta, \gamma, m, n, p$ (that is, $\delta$ and $q$). It results that the changes arising from several successive infinitesimal displacements combine by adding together the changes due separately to each of these successive displacements, referred to the original situation of the system. The elements of the composite displacement are the sums of the analogous elements of the partial displacements.

This is the (usual) way in which the complete differential of a function of several variables is formed by adding the partial differentials relative to each of those variables. And since one neglects infinitesimals of the second order, it makes no difference whether the differentials are expressed in terms of the starting values of the finite variables, or in terms of their values successively augmented by the infinitesimal increments they undergo.

Hence if one denotes by $\vec{\delta}', \vec{q}'; \vec{\delta}'', \vec{q}''; \vec{\delta}''', \vec{q}'''$ etc. the elements of the successive displacements to be composed, for each of which one has

$$
\bar{\Delta}' = \bar{\delta}' + \bar{r} \times \bar{q}', \quad \bar{\Delta}'' = \bar{\delta}'' + \bar{q}'' \times \bar{r}, \text{etc.}
$$

(143)

the elements of the composite displacement, represented by $A, B, C, M, N, P$ (that is, by $\bar{\Gamma}, \bar{Q}$), will be respectively the sums of the given partial elements; one will have

$$
A = \alpha' + \alpha'' + \ldots = \Sigma \alpha, \quad B = \Sigma \beta, \quad C = \Sigma \gamma, \quad M = \Sigma m, \quad N = \Sigma n, \quad P = \Sigma p,
$$

(144)

and for the expressions of the composite (infinitesimal) variations of the coordinates $x, y, z$,

$$
\bar{\Delta} = \bar{\Gamma} + \bar{r} \times \bar{Q} = \bar{\Gamma} + \bar{\Theta} \bar{r} \times \bar{Q}
$$

(145)

where we have introduced the (infinitesimal) rotation $\Theta$ and the direction $\bar{Q}$.

The author now points out that since the displacements are infinitesimal, one might equally decompose the same total displacement (resulting
from pure rotations about arbitrarily many axes arbitrarily placed and directed) into exactly three partial rotations taken successively about the \( x, y, z \) axes, with infinitesimal angles \( m, n, p \), each accompanied by the appropriate "screw" translation \( \vec{t} \cdot \hat{i}, \vec{t} \cdot \hat{j}, \vec{t} \cdot \hat{k} \).

## 23 Geometry and analytical mechanics.

These successive rotations \( m, n, p \) are known in mechanics by the name of **elementary rotations**, and considered as **simultaneous** in the passage from the geometric to the mechanical laws of the displacement of bodies, notwithstanding that geometry cannot take account of them except by supposing them to be **successive**. For it is evident that the system, in turning about the axis whose rotation is \( \theta \) and whose angles with the \( x, y, z \) axes are \( g, h, l \), does not achieve **at the same time** the three rotations \( m, n, p \) about those coordinate axes: this would require four axes of rotation instead of a single one (Mécanique Analytique, vol. 1, p. 52).

This reference is to the epoch-making two-volume treatise (1788–9) by Joseph-Louis Lagrange, drawing on discoveries and insights from Euler, D’Alembert, the Bernoulli brothers, and others, which for the first time showed that all mechanical properties of a system could be derived from a single algebraic formula and embodied in a single differential equation, without (according to Lagrange himself) requiring “either geometrical or mechanical constructions or reasoning”.

We encounter here a fundamental point in the philosophy of mathematics, that which separates geometry from mechanics, and the importance of which it is the object of this Memoir to establish in its totality.

The rotation \( \theta \) results from the successive composition of the rotations \( \theta \cos g, \theta \cos h, \theta \cos l \), because the displacement due to this rotation \( \theta \) is for each coordinate axis the sum of the displacements which would be due separately to each of the elementary rotations. In fact, if the system were to turn only about the \( x \)-axis with a rotation \( \theta \cos g \), one would have for this displacement

\[
\delta' x = 0, \quad \delta' y = \theta z \cos g, \quad \delta' z = -\theta y \cos g; \tag{146}
\]

if on the contrary the system were to turn only about the \( y \)-axis with a rotation \( \theta \cos h \), one would have for this displacement

\[
\delta'' x = -\theta z \cos h, \quad \delta'' y = 0, \quad \delta'' z = \theta x \cos h; \tag{147}
\]

(\textit{the expressions for } \delta'' x \textit{ and } \delta'' z \textit{ in the French text contain slight errors}) and finally the rotation \( \theta \cos l \) about the \( z \)-axis. considered alone, would give

\[
\delta''' x = \theta y \cos l, \quad \delta''' y = -\theta x \cos l, \quad \delta''' z = 0. \tag{148}
\]

The sum of these composite displacements relative to each axis will therefore give for the whole displacement, effected definitively by the rotation \( \theta \) about the axis \((g, h, l)\), the following expression:

\[
\delta \vec{r} = \theta \vec{r} \times \hat{\imath}. \tag{149}
\]
This is the sum, in modern notation, of $\delta'\vec{r} + \delta''\vec{r} + \delta'''\vec{r}$ as given by the three previous equations, where $\hat{t} = (\cos g, \cos h, \cos l)$. It appears that the author is continuing to make all displacements infinitesimal as in the previous section.

24 Successive infinitesimal displacements

Let us return to the composition of arbitrary displacements, or changes, given successively for the same system. The formula for a single such displacement can be written, in conformity with paragraph 15, as

\[
\begin{align*}
\delta x &= \alpha + \theta(y \cos l - z \cos h) = t \cos g + \theta u \cos G \\
\delta y &= \beta + \theta(z \cos g - x \cos l) = t \cos h + \theta u \cos H \\
\delta z &= \gamma + \theta(x \cos h - y \cos g) = t \cos l + \theta u \cos L.
\end{align*}
\] (150)

These equations, in which $u$ denotes the distance from the point $(x, y, z)$ to the central axis of the displacement, and $(G, H, L)$ are the angles formed with the coordinate axes by the direction of the infinitely small arc $u\theta$, describe a rotation about that central axis. In modern form the above array becomes

\[
\delta\vec{r} = \vec{r} + \theta\vec{r} \times \hat{t} = t\hat{t} - \theta\vec{u} \times \hat{t}. \quad (151)
\]

We are dealing again with a substitute axis having the same direction $\hat{t}$ as the central axis, but displaced so as to pass through the origin $\vec{r} = 0$. The rotation about the central axis is equivalent to the same rotation about the substitute axis, augmented by a translation $\vec{r}$ that is common to all points $\vec{r}$. In the case $\vec{r} = 0$ the substitute rotation has no effect and $\vec{r}$ is the whole displacement. For general $\vec{r}$ one has also the substitute rotation $\theta\vec{r} \times \hat{t}$.

The expression after the second $=$ sign is obtained by decomposing $\vec{r}$ into a part parallel and a part perpendicular to the central axis. The parallel part $\vec{r} \cdot \hat{t} = tt$ is the translation of the central axis along itself, which in general accompanies the rotation about the central axis in accordance with the “screw” principle. The perpendicular part describes the rotation of the origin of coordinates about the central axis; since $\vec{u}$ is the perpendicular from the point $\vec{r}$ to the central axis, $\vec{r} + \vec{u}$ is the perpendicular from the origin to the central axis. Therefore the perpendicular part of $\vec{r}$ is $-\theta(\vec{r} + \vec{u}) \times \hat{t}$. Combining the two parts, we obtain $\vec{r} = tt - \theta r \times \hat{t} - \theta \vec{u} \times \hat{t}$ or

\[
\vec{r} + \theta\vec{r} \times \hat{t} = tt - \theta \vec{u} \times \hat{t}
\] (152)

in keeping with [151].

In general, the displacement relative to an arbitrary direction $s$, making angles $a, b, c$ with the $x, y, z$ coordinates, is given by

\[
\delta s = t \cos(t, s) + \theta u \cos(tu, s)
\] (153)

where $\cos(t, s), \cos(tu, s)$ are the cosines of the angles made by this direction with the central axis and with the infinitely small arc of rotation $\theta$. ][If we put this equation in modern form,

\[
\hat{s} \cdot \delta \vec{r} = t \hat{s} \cdot \hat{t} + \theta u \hat{s} \cdot (-\vec{u} \times \hat{t}),
\] (154)
we see that it results from taking the dot product of \([151]\) with \(\hat{s}\). We can also recover the three equations \([150]\) by replacing \(s\) with \(x, y,\) or \(z\).

But if two lines are given in the space, one knows that the (shortest) distance from a (particular) point of one line to the other line is reciprocal to the sine of the angle formed by the first line with the plane containing the second line and the point (of the first line) under consideration. Or equivalently, the product of this sine and this distance is constantly equal to the product of the shortest distance between the two lines and the sine of their inclination (that is, of the angle between their directions). If, therefore, we define \(D\) as the distance between the central axis and the line passing through the point of the system under consideration (that is, the point \(\vec{r}\)) in the direction \(s\), and \(\nu\) as the angle between this line and the central axis, we shall have

\[
\begin{align*}
  u \cos(tu, s) &= D \sin \nu, \\
  \delta s &= t \cos \nu + D \theta \sin \nu. 
\end{align*}
\]

Here the author liberates himself from the given point \(\vec{r}\) and refers in his description to whatever point of the \(s\)-line through \(\vec{r}\) is closest to the central axis. This completes his analysis of a single displacement.

If we now consider the successive (infinitesimal) displacements of the system about central axes whose elements are \(t, \theta, g, h, l;\) \(t', \theta', g', h', l';\) \(t'', \theta'', g'', h'', l'';\) etc., and denote by \(\delta S\) the resultant displacement of a point of the system relative to that same fixed direction \(s\) and by \(T, \Theta, G, H, L\) the elements of the resultant central axis, we shall have

\[
\delta S = \Sigma t \cos \nu + \Sigma D \theta \sin \nu = T \cos \nu + \Theta D \sin \nu, 
\]

\(D\) being the distance from the resultant axis to the line \(s\) drawn through the point \((x, y, z)\).

This equation, owing to the indeterminate parameters \(a, b, c\) implicit in it and to the fact that it must hold for all points of the system, is equivalent to the following six equations which give the position of the resultant central axis and the resultant translation and rotation:

\[
\begin{align*}
  \Sigma \alpha - T \cos G + \Theta Y \cos L - \Theta Z \cos H &= 0, \\
  \Sigma \beta - T \cos H + \Theta Z \cos L - \Theta X \cos L &= 0, \\
  \Sigma \gamma - T \cos L + \Theta X \cos L - \Theta Y \cos G &= 0, \\
  \Theta \cos G &= \Sigma \theta \cos g, \Theta \cos H &= \Sigma \theta \cos h, \Theta \cos L = \Sigma \theta \cos l, 
\end{align*}
\]

where \(X, Y, Z\) are the coordinates of an arbitrary point on the resultant central axis.

Without loss of rigor, one could consider only pure rotations in this analysis, since the translations \(t, t',...\) can always be represented by couples of rotations; and simply put

\[
\delta S = T \cos V + \Theta D \sin V = \Sigma \theta \Delta \sin \nu. 
\]

Here I omit some commentary that seems to me both tedious and repetitious.
25 Conditions for equilibrium from many infinitely small successive displacements.

We are now led to seek out what conditions need to be satisfied by the elements of the successive displacements proposed for a solid system, in order that the system, passing successively through various infinitesimally neighboring situations, should return to its initial position; this would amount to a condition of equilibrium, or neutralization, on the totality of the successive displacements. Now, it is evident that all the relevant conditions are contained in a single equation, which can be decomposed into six others on account of the indeterminate quantities implicit in it, to wit:

\[ \delta S = 0, \]  

(159)

since this equation expresses that each point of the system has returned to its initial position.

The six equations hidden in \( \delta S = 0 \) are

\[
\begin{align*}
\delta x_0 & = \Sigma \alpha = 0, \\
\delta y_0 & = \Sigma \beta = 0, \\
\delta z_0 & = \Sigma \gamma = 0, \\
\Sigma \theta \cos g & = 0, \\
\Sigma \theta \cos h & = 0, \\
\Sigma \theta \cos l & = 0,
\end{align*}
\]

(160)

where \( \delta x_0, \delta y_0, \delta z_0 \) stand for the resultant variations of the coordinates of the origin. The first three equations express the immobility of the origin of the coordinates; and the the three others, that no resultant rotation has occurred in the displaced system (i.e. no rotation about a resultant fixed axis through the origin, such as Euler’s theorem permits). This double condition excludes the possibility of any resultant displacement whatever.

The double condition is an immediate consequence of the relations (158) and (159), which cannot be satisfied for every point in the system unless

\[ T = 0, \Theta = 0. \]  

(161)

For \( \Theta = 0 \) implies the three last equations of (160), while the first three follow from (21) of Section 15.

These six equations of equilibrium are analytically contained in a single equation which expresses the general law of this equilibrium in the simplest way (in terms of the elements of the successive displacements), namely

\[ \Sigma \theta D \sin \nu = 0. \]  

(162)

But the equilibrium of these infinitesimal successive displacements, all else remaining the same, will continue to hold no matter how rapidly they succeed one another. Passing to the limit, one arrives at the identity of the laws of equilibrium due to successive infinitesimal displacements with those due to simultaneous infinitesimal displacements. This equivalence is needed in order to justify the analogy between geometry and mechanics, to be presented in the following section.
26 Analogy of these (geometric) laws of composition and equilibrium with those of composition and equilibrium of forces applied to an immovable system.

The analogy between this general (geometric) law and that of the equilibrium of forces applied to an immovable system is striking. Let the applied forces follow the axes of rotation and suppose them proportional to those rotations; then the moment of a force on the system is exactly proportional to that of the corresponding rotation, and each translation is replaced by a couple (in the sense of earlier sections) of applied forces following the axes of the couple of rotations equivalent to the translation. The analogy, however, extends to the laws of composition and may be stated thus:

A system of successive displacements being given to be composed into a resultant displacement, and at the same time a system of forces proportional to the successive rotations given for each displacement and applied following the same axes as the rotations, the translations of the successive displacements, supposing that they are not implicitly included in the rotations by being represented by couples of rotations, being then represented in the system of forces under consideration by couples of forces whose moments would be equal to those of the translations, relatively to the three coordinate axes, the system of displacements will amount to a resultant displacement composed of a rotation and an absolute translation relative to the central axis of rotation; just as the system of forces will resolve itself by the successive composition of its elements into a single force and a single couple of forces situated in a plane normal to the resultant force. This resultant force will be applied at the central axis of the resultant displacement, which will be at the same time the central axis of the static system; it will be proportional to the resultant rotation, and the moment of the couple normal to this force will be proportional to the absolute translation of the system that operates in a way parallel to the central axis. Should the axes of the rotations to be combined be all parallel and pass through determined points, the resultant central axis is also parallel to them and passes through a certain point that corresponds to the center of the parallel forces, the same point no matter what be the direction of the axes of rotation, and which is nothing other than the center of gravity of the points on the composing axes of rotation that are determined when all the rotations are equal.

This passage, as translated faithfully above, is somewhat in need of a retranslation or decipherment. The basic key to the passage is that the phrase “moment d’une force suivant un axe” means, at least in modern French, the torque \( \vec{r} \times \vec{F} \) about that axis, supposing that the force \( \vec{F} \) is applied to or through a point whose position vector relative to some point on the axis is \( \vec{r} \). Thus, in general, the force is not applied along the axis, nor even toward some point on the axis; if it were, the “moment” would be zero. Of course the words “torque” and “moment” were not as cleanly defined in the author’s time as now, but this identification does clarify the
With this key in hand, we can appreciate what otherwise would be perplexing: that a single force is repeatedly associated with a geometric rotation, while a geometric translation corresponds to the moment of a pair or couple of forces acting in a plane normal to the geometric translation. As to the geometric meaning of the word “moment”, it may be helpful to refer to the discussion following (21) in 15. Of course the modern form of a cross-product using the right-hand rule was not available; instead the author measures the “moment” of a rotation by an area, an idea introduced independently by H. Grassmann almost at the same time.

In fact, the equations of the resultant central axis of the composition of the fixed axes of rotation reduce in this case to

$$\vec{W} \times \hat{T} + \frac{\Sigma \delta}{\Sigma \theta} = 0,$$

and we have

\[
\begin{align*}
T &= 0, \quad \Theta = \Sigma \theta \\
= \Sigma \alpha &= \Sigma \theta (Z \cos h - Y \cos h), \\
= \Sigma \beta &= \Sigma \theta (X \cos h - Z \cos h), \\
= \Sigma \gamma &= \Sigma \theta (Y \cos h - X \cos h)
\end{align*}
\]

(164)

where \(X,Y,Z\) denote the coordinates of the axis of rotation \(\theta\). Hence the resultant axis passes through the point whose coordinates are

\[
\begin{align*}
x &= \frac{\Sigma \theta X}{\Sigma \theta}, \quad y = \frac{\Sigma \theta Y}{\Sigma \theta}, \quad z = \frac{\Sigma \theta Z}{\Sigma \theta}.
\end{align*}
\]

(165)

27 Determination of the changes in coordinates of a solid due to an arbitrary displacement, deduced analytically from the conditions of invariability of the system.

We consider the displacement of the coordinate axes, given that they are rigidly attached to the system as it is displaced. This leads immediately to the algebraic expression for the changes in the coordinates of any point, and there remains only to reduce to a minimum the number of arbitrary constants that enter the calculation, as we shall now see.

The author now proposes to derive the preceding results without using the formulas derived from (49), but by a different method.

Let us denote by \(a,b,c, a',b',c'\); \(a'',b'',c''\) the cosines of the angles formed by the displaced coordinate axes with their original directions. Then the new coordinates \(x + \Delta x, y + \Delta y, z + \Delta z\), of a point of the system after displacement relative to the “old” axes, will be expressed as a function of the same displaced coordinates relative to the new axes. But this will be the same function, already well known, that gives the
new axes in terms of the old, that is
\[ \begin{align*}
  x + \Delta x &= \alpha + ax + by + cz, \\
y + \Delta y &= \beta + a'x + b'y + c'z, \\
z + \Delta z &= \gamma + a''x + b''y + c''z.
\end{align*} \quad (166)
\]

The idea is that since the axes are rigidly attached to the system, they suffer the same displacement from old to new as does the point under consideration. So if one compares the final position of the point to its old position, both relative to the old axes, one finds the same transformation formulas as in comparing the coordinates of the same new point, relative to the old axes, with its coordinates relative to the new ones.

These formulas express \( x + \Delta x, y + \Delta y, z + \Delta z \) as linear functions of the initial coordinates \( x, y, z \) for any displacement of the solid whatever. They contain 12 arbitrary coefficients, but really only 6 of these are independent. The first three, \( \alpha, \beta, \gamma \), determine the movement of the origin, and the three “diagonal” coefficients \( a, b', c' \) serve to define the directions of the three displaced axes relative to the old ones. In terms of the latter, one may eliminate the six off-diagonal elements by the formulas of Monge.

But the reduction of twelve constants to six can be achieved even more simply, without using the formulas of Monge (complicated by radicals) and by a route that leads to the simplest possible expressions for \( \Delta x, \Delta y, \Delta z \) in terms of \( x, y, z \). We introduce into (166) the coordinates \( \xi, \eta, \zeta \) of the midpoint of the line joining the initial to the final position of the point in question. This gives us
\[ \begin{align*}
x &= \xi - (1/2)\Delta x, & y &= \eta - (1/2)\Delta y, & z &= \zeta - (1/2)\Delta z. \quad (167)
\end{align*} \]

Then we unite these three equations into one by multiplying them respectively by three indeterminate factors \( \mu, \nu, \pi \) and adding together the results. We now have the single equation
\[ \begin{align*}
(1/2)\Delta x(\bar{a} + \mu) + (1/2)\Delta y(\bar{b} + \nu) + (1/2)\Delta z(\bar{c} + \pi) &= (\alpha \mu + \beta \nu + \gamma \pi) \\
+ \xi(\bar{a} - \mu) + \eta(\bar{b} - \nu) + \zeta(\bar{c} - \pi), \quad (168)
\end{align*} \]
where we define
\[ \begin{align*}
\bar{a} &= a\mu + a'\nu + a''\pi, & \bar{b} &= b\mu + b'\nu + b''\pi, & \bar{c} &= c\mu + c'\nu + c''\pi. \quad (169)
\end{align*} \]

To determine (for example) \( \Delta x \), we must assign to \( \mu, \nu, \pi \) those values that cause the coefficients of \( \Delta y \) and \( \Delta z \) to vanish; that is, we must set
\[ \bar{b} + \nu = \bar{c} + \pi = 0, \quad (170) \]
from which we find
\[ \mu = (1 + b')(1 + c'') - c'b'', \quad \nu = b''c - b(1 + c''), \quad \pi = bc' - c(1 + b'). \quad (171) \]

(French text has incorrectly \( a'' \) instead of \( b''c \) in \( \nu \)).

Now, the nine cosines \( \bar{a} = (a, a', a''), \bar{b} = (b, b', b''), \bar{c} = (c, c', c'') \) satisfy, as is well known, the relations
\[ \begin{align*}
\bar{a} &= \bar{c} \times \bar{b}, \\
\bar{b} &= \bar{a} \times \bar{c}, \\
\bar{c} &= \bar{b} \times \bar{a}. \quad (172)
\end{align*} \]
These can be deduced from six others,

$$|\vec{a}|^2 = |\vec{b}|^2 = |\vec{c}|^2 = 1,$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0,$$  \hspace{1cm} (173)

which say that the coordinate axes, both the old and the new, form orthonormal systems. The ambiguity of signs that enter into the deduction is resolved by another condition altogether necessary in considering the displacement of a solid system, namely that the system of new axes arising from the displacement must always remain in the condition of superposition with the old axes that is possible for each respective axis and its correspondent.

I have adhered to my usual convention of reversing cross-products so as to maintain right-handedness (see 15). I hope that by doing so I satisfy the author’s condition above.

This being understood, it is clear that the above relations lead to

$$\mu = 1 + a + b' + c'', \nu = a' - b, \pi = a'' - c,$$  \hspace{1cm} (174)

from which

$$\tilde{a} = 1 + a + b' + c'' = \mu$$

and hence

$$\Delta x - \alpha = \frac{2[(\eta - (1/2)\beta)(b - a') - (\zeta - (1/2)\gamma)(a'' - c)]}{1 + a + b' + c''}. \hspace{1cm} (175)$$

In eqs (170), (171), (174), (175), and (176), the author has broken the threefold symmetry of (168) by singling out $\Delta x$ as the component to be determined. He now retreats from that choice by considering in turn each of the other components $\Delta y, \Delta z$.

Similarly one can obtain

$$\Delta y - \beta = \frac{2[(\xi - (1/2)\alpha)(c' - b'') - (\eta - (1/2)\gamma)(b' - a')]}{1 + a + b' + c''}. \hspace{1cm} (176)$$

$$\Delta z - \gamma = \frac{2[(\xi - (1/2)\alpha)(a'' - c) - (\eta - (1/2)\beta)(a' - b')]}{1 + a + b' + c''}. \hspace{1cm} (177)$$

These formulas are identical to those of 15], provided that we set

$$m = \frac{2(c' - b'')}{1 + a + b' + c''}, \hspace{0.5cm} n = \frac{2(a'' - c)}{1 + a + b' + c''}, \hspace{0.5cm} p = \frac{2(b - a')}{1 + a + b' + c''}. \hspace{1cm} (178)$$

28 Infinitesimal version of 27.

In the case of infinitesimal displacements, we neglect angular displacements of second order and directly obtain $a = b' = c'' = 1$, from which

$$\delta x = \alpha + by + cz, \hspace{0.5cm} \delta y = \beta + c' z + a' x, \hspace{0.5cm} \delta z = \gamma + a'' x + b'' y. \hspace{1cm} (180)$$

Also, the distance from any point to the origin, when the latter is drawn along with the displacement of the system, is invariable, so that (vectorially) $\vec{r} \cdot (\delta \vec{r} - \vec{\delta}) = 0$ for any arbitrary point $\vec{r}$. This necessitates
the following relations among the “off-diagonal” first-order infinitesimal cosines:

\[ b + a' = c + a'' = c' + b'' = 0. \]  \(\text{(181)}\)

Hence by setting \(m = c' = -b'', n = a'' = -c, p = b = -a'\), we have finally

\[ \delta x = \alpha + py - nz, \quad \delta y = \beta + mz - px, \quad \delta z = \gamma + nx - my. \]  \(\text{(182)}\)

29 Algebraic deduction of coordinate changes.

And now, yet another method.

But it is interesting to arrive at these same formulas for finite or infinitesimal coordinate changes by a purely algebraic route, independent of any geometric consideration (other than the Pythagorean formula for the distance between two points) starting from the invariability of the distances between points of the solid.

So let \(\vec{r}_0, \vec{r}_1, \vec{r}_2, \vec{r}\) be the positions of four points invariably linked together and belonging to the solid system. The first three points and their displacements are to be considered as known, and the displacement of the fourth point is to be calculated as a function of its starting position and the known quantities.

The distances between these four points will remain constant under an arbitrary displacement of the system; this condition, when expressed algebraically, will give the following six equations:

\[ |\vec{r}_1 + \Delta \vec{r}_1 - \vec{r}_0 - \Delta \vec{r}_0|^2 = |\vec{r}_1 - \vec{r}_0|^2 \]
\[ |\vec{r}_2 + \Delta \vec{r}_2 - \vec{r}_0 - \Delta \vec{r}_0|^2 = |\vec{r}_2 - \vec{r}_0|^2 \]
\[ |\vec{r}_2 + \Delta \vec{r}_2 - \vec{r}_1 - \Delta \vec{r}_1|^2 = |\vec{r}_2 - \vec{r}_1|^2 \]
\[ |\vec{r} + \Delta \vec{r} - \vec{r}_0 - \Delta \vec{r}_0|^2 = |\vec{r} - \vec{r}_0|^2 \]
\[ |\vec{r} + \Delta \vec{r} - \vec{r}_1 - \Delta \vec{r}_1|^2 = |\vec{r} - \vec{r}_1|^2 \]
\[ |\vec{r} + \Delta \vec{r} - \vec{r}_2 - \Delta \vec{r}_2|^2 = |\vec{r} - \vec{r}_2|^2 \].  \(\text{(184)}\)

I am rendering equations when possible in vector form for brevity and ease of reading. The Pythagorean formula for distance is implicit in the vectors; thus \(|\vec{r} - \vec{r}_0|^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\), etc.

We may think of a tetrahedron with vertex at \(\vec{r}\) and base having the three points \(\vec{r}_0, \vec{r}_1, \vec{r}_2\). Then the first three equations of \(\text{(184)}\) pertain to the edges of the base triangle, and the last three to the edges that meet at the vertex.

The six equations are quadratic in the displacements \(\Delta \vec{r}_0, \Delta \vec{r}_1, \Delta \vec{r}_2, \Delta \vec{r}\) but can be made linear by introducing the midpoint position \(\vec{\omega} = \vec{r} + \)
(1/2)\Delta r^*, and likewise for the three base points. One then obtains

\[ 2(\vec{\omega} - \vec{\omega}_0) \cdot (\Delta \vec{r} - \Delta \vec{r}_0) = 0, \]

\[ 2(\vec{\omega}_1 - \vec{\omega}_0) \cdot (\Delta \vec{r}_1 - \Delta \vec{r}_0) = 0, \]

\[ 2(\vec{\omega}_2 - \vec{\omega}_0) \cdot (\Delta \vec{r}_2 - \Delta \vec{r}_0) = 0, \]

\[ (\vec{\omega} - \vec{\omega}_0) \cdot (\Delta \vec{r}_1 - \Delta \vec{r}_0) + (\Delta \vec{r} - \Delta \vec{r}_0) \cdot (\vec{\omega}_1 - \vec{\omega}_0) = 0, \]

\[ (\vec{\omega} - \vec{\omega}_0) \cdot (\Delta \vec{r}_2 - \Delta \vec{r}_0) + (\Delta \vec{r} - \Delta \vec{r}_0) \cdot (\vec{\omega}_2 - \vec{\omega}_0) = 0, \]

\[ (\vec{\omega}_1 - \vec{\omega}_0) \cdot (\Delta \vec{r}_2 - \Delta \vec{r}_0) + (\Delta \vec{r}_1 - \Delta \vec{r}_0) \cdot (\vec{\omega}_2 - \vec{\omega}_0) = 0. \quad (185) \]

These equations may be derived as follows. We note that each linear factor in (185) contains either \( \vec{\omega}_0 \) or \( \Delta \vec{r}_0 \). Let \( (\vec{R}, \vec{\Omega}) \) and \( (\vec{R}', \vec{\Omega}') \) be any of the pairs \((\vec{r}, \vec{\omega})\), \((\vec{r}_1, \vec{\omega}_1)\), \((\vec{r}_2, \vec{\omega}_2)\). From the definition \( \vec{\Omega} = \vec{R} + (1/2)\Delta \vec{R} \) we have

\[ \vec{R} = \vec{\Omega} - (1/2)\Delta \vec{R}, \quad \vec{R} + \Delta \vec{R} = \vec{\Omega} + (1/2)\Delta \vec{R}, \quad \text{and likewise with} \quad \vec{R}', \vec{\Omega}' \quad \text{in place of} \quad \vec{R}, \vec{\Omega}. \]

Therefore

\[ [(\vec{R} + \Delta \vec{R}) - (\vec{r}_0 + \Delta \vec{r}_0)] \cdot [(\vec{R} + \Delta \vec{R}) - (\vec{r}_1 + \Delta \vec{r}_0)] - (\vec{R} - \vec{r}_0) \cdot (\vec{R} - \vec{r}_0) = 0, \]

\[ [(\vec{\Omega} + (1/2)\Delta \vec{R}) - (\vec{\omega}_0 + (1/2)\Delta \vec{r}_0)] \cdot [(\vec{\Omega} + (1/2)\Delta \vec{R}) - (\vec{\omega}_0 + (1/2)\Delta \vec{r}_0)] - (\vec{\Omega} - \vec{\omega}_0) \cdot (\vec{\Omega} - \vec{\omega}_0) = 0, \]

\[ [(\vec{\Omega}' - (1/2)\Delta \vec{R}) - (\vec{\omega}_0 - (1/2)\Delta \vec{r}_0)] \cdot [(\vec{\Omega}' - (1/2)\Delta \vec{R}) - (\vec{\omega}_0 - (1/2)\Delta \vec{r}_0)] - (\vec{\Omega}' - \vec{\omega}_0) \cdot (\vec{\Omega}' - \vec{\omega}_0) = 0. \quad (186) \]

and since the top line of (187) vanishes by (184), the bottom line vanishes, giving us the first three lines of (185) if \( (R, \Omega) \) and \( (R', \Omega') \) are the same, and the last three if they are different.

It will be observed that the author has selected the pair \((\vec{r}_0, \vec{\omega}_0)\) to play the starring rôle in (185), inasmuch as it is the only pair that is represented in every linear factor in every equation. Other equations could be written down, but they can be deduced from the six displayed. For example, the equation \((\vec{\omega}_0 - \vec{\omega}) \cdot (\Delta \vec{r}_1 - \Delta \vec{r}) + (\Delta \vec{r}_0 - \Delta \vec{r}) \cdot (\vec{\omega}_1 - \vec{\omega}) = 0, \) obtained by interchanging \( \vec{r}, \vec{\omega} \) with \( \vec{r}_0, \vec{\omega}_0 \) in the fourth line of (185), can be deduced by subtracting that line from the top line.

We now multiply these six equations by the respective six factors \( \mu^2, \nu^2, \pi^2, \mu \nu, \mu \pi, \nu \pi \) and add them together. This results in the following single equation which (when \( \mu, \nu, \pi \) vary independently) contains the foregoing six:

\[ [\mu(\vec{\omega} - \vec{\omega}_0) + \nu(\vec{\omega}_1 - \vec{\omega}_0) + \pi(\vec{\omega}_2 - \vec{\omega}_0)] [\mu(\Delta \vec{r} - \Delta \vec{r}_0) + \nu(\Delta \vec{r}_1 - \Delta \vec{r}_0) + \pi(\Delta \vec{r}_2 - \Delta \vec{r}_0)] = 0. \quad (188) \]

When a similar operation was performed in 27, the coefficients \( \mu, \nu, \pi \) were attached to the \( x, y, z \)-components and the resulting manipulations could not be written vectorially. Here, the three coefficients are attached to three different vectors.

By a suitable choice of \( \mu, \nu, \pi \), this equation can be made to contain only \( \Delta \vec{r} \) [and not \( \Delta \vec{r}_1 \) or \( \Delta \vec{r}_2 \)] and the value of this displacement can be found in the simplest way.

Having thus far allowed us the luxury of our vector notation, the author is now going to forbid it by singling out one component \( x \) for study. What makes this necessary is that he wishes to use the principle that the vanishing of a product of two scalar quantities implies the vanishing of
These equations evidently imply the following ones: at least one of the factors, and this is not true of the dot product of two vectors.

To wit, suppose we set

\[ \mu(\eta - \eta_0) + \nu(\eta_1 - \eta_0) + \pi(\eta_2 - \eta_0) = 0, \]
\[ \mu(\zeta - \zeta_0) + \nu(\zeta_1 - \zeta_0) + \pi(\zeta_2 - \zeta_0) = 0 : \tag{189} \]

then \(188\) reduces to

\[ [\mu(\xi - \xi_0) + \nu(\xi_1 - \xi_0) + \pi(\xi_2 - \xi_0)] \] (188) \[ [\mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0)] = 0. \tag{190} \]

He has arranged \(\mu, \nu, \pi\) to make the first factor of the \(y\) and \(z\) parts of \(188\) vanish, and so the corresponding products vanish; therefore the product in the \(x\) part must also vanish. The principle of factorization then dictates that at least one of the \(x\) factors vanish, but he claims (deferring proof to the following section) that the first factor cannot vanish if the second does not, and so infers that the second factor vanishes:

Now, the first factor of \(190\) cannot be zero unless the second is also, as we shall demonstrate hereafter; we therefore have the simultaneous equations

\[ \mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0) = 0, \]
\[ \mu(\eta - \eta_0) + \nu(\eta_1 - \eta_0) + \pi(\eta_2 - \eta_0) = 0, \]
\[ \mu(\zeta - \zeta_0) + \nu(\zeta_1 - \zeta_0) + \pi(\zeta_2 - \zeta_0) = 0. \tag{191} \]

These equations evidently imply the following ones:

\[ \Delta x_1 - \Delta x_0 = \eta(\eta_1 - \eta_0) - n(\zeta_1 - \zeta_0), \]
\[ \Delta x_2 - \Delta x_0 = \eta(\eta_2 - \eta_0) - n(\zeta_2 - \zeta_0), \]
\[ \Delta x - \Delta x_0 = \eta(\eta_0) - n(\zeta - \zeta_0). \tag{192} \]

(French text has mistakenly \(p(\xi_1 - \xi_0)\) in third equation) where \(n\) and \(p\) are two constants bound to the displacements of the three first points \((\vec{r}_1, \vec{r}_2, \vec{r}_0, \text{the base of the tetrahedron})\) by the first two of these three equations.

The three simultaneous equations \((191)\) are homogeneous in \(\mu, \nu, \pi\); therefore the determinant must vanish. Interchanging rows and columns, we deduce that the three homogeneous equations

\[ (\Delta x - \Delta x_0)\vec{m} + (\eta - \eta_0)\vec{n} + (\zeta - \zeta_0)\vec{p} = 0, \]
\[ (\Delta x_1 - \Delta x_0)\vec{m} + (\eta_1 - \eta_0)\vec{n} + (\zeta - \zeta_0)\vec{p} = 0, \]
\[ (\Delta x_2 - \Delta x_0)\vec{m} + (\eta_2 - \eta_0)\vec{n} + (\zeta_2 - \zeta_0)\vec{p} = 0 \tag{193} \]

have a solution \(\vec{m}, \vec{n}, \vec{p}\). Defining \(n = \vec{p}/\vec{m}, p = -\vec{n}/\vec{m}, \) we obtain \(192\).

The same analysis (if \(y\) or \(z\) instead of \(x\) had been singled out) would give

\[ \Delta y - \Delta y_0 = m'(\zeta - \zeta_0) - p'(\xi - \xi_0), \]
\[ \Delta z - \Delta z_0 = n''(\xi - \xi_0) - m''(\eta - \eta_0). \tag{194} \]
But we have also (from the top line of (185))

\[(\xi - \xi_0)(\Delta x - \Delta x_0) + (\eta - \eta_0)(\Delta y - \Delta y_0) + (\zeta - \zeta_0)(\Delta z - \Delta z_0) = 0; \quad (195)\]

it follows that \(m = m' = m''\), \(n = n' = n''\), \(p = p' = p''\), and so we have at last

\[
\begin{align*}
\Delta x - \Delta x_0 &= A + p\eta - n\zeta, \\
\Delta y - \Delta y_0 &= B + m\zeta - p\xi, \\
\Delta z - \Delta z_0 &= C + n\xi - m\eta,
\end{align*}
\]  

(196)

where the six constants \(A, B, C, m, n, p\) are functions of the displacements \(\Delta \vec{r}_0, \Delta \vec{r}_1, \Delta \vec{r}_2\).

Substituting (194) and the third line of (192) into (195), we find for example that the product \((\xi - \xi_0)(\eta - \eta_0)\) appears with coefficient \(p' - p\) so that \(p' = p\). This is one of six equations that together justify dropping all the primes in (196).

### 30 Proof of prior claim.

But it remains to prove what we have claimed, that in (190) the first factor \([\mu(\xi - \xi_0) + \nu(\xi_1 - \xi_0) + \pi(\xi_2 - \xi_0)]\) cannot vanish unless the second factor \([\mu(\Delta x - \Delta x_0) + \nu(\Delta x_1 - \Delta x_0) + \pi(\Delta x_2 - \Delta x_0)]\) does so as well.

The three simultaneous equations (they are easily expressed as one, in vector notation)

\[
\mu(\vec{w} - \vec{w}_0) + \nu(\vec{w}_1 - \vec{w}_0) + \pi(\vec{w}_2 - \vec{w}_0) = 0
\]  

(197)

express that the midpoints of the lines traversed by the four points we are considering lie in the same plane. But this unusual condition can be fulfilled only if either, on the one hand, the four points themselves also lie in one plane, or on the other hand, the pyramid formed by these points after displacement is not genuinely superposable on the one formed initially, but is only symmetric to it. The second hypothesis is not admissible in our problem, but the first must be examined.

Consider the phrase “midpoint plane” as referring to the plane containing the four midpoints \(\vec{w}\), etc. If we examine the four points of the tetrahedron in relation to the midpoint plane, we see that the only way to make the actual midpoints lie in this plane is to make the displacement take each vertex of the tetrahedron to the opposite side of the plane. But the resulting pyramid is not “superposable” on the original one (see 1, second paragraph) unless all four points already lie in the midpoint plane, so that the tetrahedron is identical to its mirror image. This exceptional case is precisely the “first hypothesis” advanced by the author.

We shall now prove that if the four points of the tetrahedron, as well as those to which they are displaced, all lie in one plane, then both factors of (190) vanish.
Let \((x, y, z), (x_1, y_1, z_1), (x_2, y_2, z_2), (x_0, y_0, z_0)\) denote the initial positions of the four points, and \((x', y', z')\), etc. their positions after displacement. If the four points are initially in the same plane, there must be coefficients \(g, h, l\) such that

\[
\begin{align*}
g(x - x_0) + h(x_1 - x_0) + l(x_2 - x_0) &= 0, \\
g(y - y_0) + h(y_1 - y_0) + l(y_2 - y_0) &= 0, \\
g(z - z_0) + h(z_1 - z_0) + l(z_2 - z_0) &= 0. \quad (198)
\end{align*}
\]

Likewise, if the four points after displacement are in the same plane, there must be \(g', h', l'\) such that

\[
\begin{align*}
g'(x' - x_0') + h'(x_1' - x_0') + l'(x_2' - x_0') &= 0, \\
g'(y' - y_0') + h'(y_1' - y_0') + l'(y_2' - y_0') &= 0, \\
g'(z' - z_0') + h'(z_1' - z_0') + l'(z_2' - z_0') &= 0. \quad (199)
\end{align*}
\]

Now, from the invariability of distances we must have

\[
\begin{align*}
|\vec{r} - \vec{r}_0|^2 &= |\vec{r}' - \vec{r}_0'|^2, \\
|\vec{r}_1 - \vec{r}_0|^2 &= |\vec{r}_1' - \vec{r}_0'|^2, \\
|\vec{r}_2 - \vec{r}_0|^2 &= |\vec{r}_2' - \vec{r}_0'|^2 \quad (200)
\end{align*}
\]

as well as

\[
\begin{align*}
(\vec{r} - \vec{r}_0) \cdot (\vec{r}_1 - \vec{r}_0) &= (\vec{r}' - \vec{r}_0') \cdot (\vec{r}_1' - \vec{r}_0'), \\
(\vec{r} - \vec{r}_0) \cdot (\vec{r}_2 - \vec{r}_0) &= (\vec{r}' - \vec{r}_0') \cdot (\vec{r}_2' - \vec{r}_0'), \\
(\vec{r}_1 - \vec{r}_0) \cdot (\vec{r}_2 - \vec{r}_0) &= (\vec{r}_1' - \vec{r}_0') \cdot (\vec{r}_2' - \vec{r}_0'). \quad (201)
\end{align*}
\]

But the coefficients \(g', h', l'\) satisfy the same relations \((199)\) to \(\vec{r}' - \vec{r}_0', \vec{r}_1' - \vec{r}_0', \vec{r}_2' - \vec{r}_0'\) as do the coefficients \(g, h, l\) (see \((198)\)) to \(\vec{r} - \vec{r}_0, \vec{r}_1 - \vec{r}_0, \vec{r}_2 - \vec{r}_0\). Consequently \(g', h', l'\) are proportional to \(g, h, l\) and we may as well set the former equal to the latter.

In vector notation, \((198)\) and \((199)\) become

\[
\begin{align*}
g(\vec{r} - \vec{r}_0) + h(\vec{r}_1 - \vec{r}_0) + l(\vec{r}_2 - \vec{r}_0) &= 0 \quad (202) \\
g(\vec{r}' - \vec{r}_0') + h(\vec{r}_1' - \vec{r}_0') + l(\vec{r}_2' - \vec{r}_0') &= 0
\end{align*}
\]

Putting \((\vec{r}, \vec{r}')\), etc. in terms of \((\vec{w}, \Delta \vec{r})\), etc. we have finally

\[
\begin{align*}
g(\vec{w} - \vec{w}_0) + h(\vec{w}_1 - \vec{w}_0) + l(\vec{w}_2 - \vec{w}_0) &= 0, \\
g(\Delta \vec{r} - \Delta \vec{r}_0) + h(\Delta \vec{r}_1 - \Delta \vec{r}_0) + l(\Delta \vec{r}_2 - \Delta \vec{r}_0) &= 0. \quad (203)
\end{align*}
\]

Here the upper line, compared with \((197)\), shows that \(g, h, l\) are proportional to \(\mu, \nu, \pi\). Therefore the former can be replaced by the latter in the lower line, yielding

\[
\mu(\Delta \vec{r} - \Delta \vec{r}_0) + \nu(\Delta \vec{r}_1 - \Delta \vec{r}_0) + \pi(\Delta \vec{r}_2 - \Delta \vec{r}_0), \quad (204)
\]

Q. E. D.
31 Coordinate changes from infinitesimal analysis.

And a third way!

The two analytic methods that we have just presented for determining the formulas for the coordinate changes of solid system are based on purely algebraic procedures. From infinitesimal analysis we can derive an even simpler proof of these formulas, resembling that given by Lagrange in his Analytic Mechanics, and describing in the same analysis both the expression for finite changes and that for infinitesimal ones. Here is that demonstration.

Given a point $\vec{r}$ in the undisplaced system, consider a neighboring point $\vec{r} + d\vec{r}$ also before displacement, $d\vec{r}$ being infinitesimal. Let the changes due to displacement be denoted by a prefix $\Delta$ or $\delta$ according as the displacement is finite or infinitesimal. We wish to determine the general expression for $\Delta \vec{r}$ by means of the equation $\Delta |d\vec{r}|^2 = 0$.

Now, letting $\vec{\omega} = \vec{r} + (1/2) \Delta \vec{r}$, and using the commutation of the signs $d$ and $\Delta$, the foregoing equation becomes

$$d\vec{\omega} \cdot d\Delta \vec{r} = 0,$$

which is to be satisfied in the most general way (that is, for arbitrary choices of $d\vec{r}$).

The proof of (205) is made easier by defining $\vec{r}' = \vec{r} + \Delta \vec{r}$. Then

$$\Delta |d\vec{r}'|^2 = d\vec{r}' \cdot d\vec{r}' - d\vec{r} \cdot d\vec{r} = (d\vec{r}' + d\vec{r}) \cdot (d\vec{r}' - d\vec{r}) = 2d\vec{\omega} \cdot \Delta \vec{r},$$

which establishes (205).

To this end, let us consider $\Delta \vec{r}$ as a function of $\vec{\omega}$; then (205) can be written, taking $d\vec{\omega}$ as constant (that is, as unchanged by the displacement), as $\Delta(d\vec{\omega} \cdot d\vec{r}) = 0$, from which follows

$$\frac{d\vec{\omega} \cdot \Delta \vec{r}}{|d\vec{\omega}|} = \text{constant}$$

(in the same sense of “constant”).

This equation (207) represents algebraically the property of a quadrilateral two of whose sides are equal, that these two sides, and the two other sides as well, project equally (that is, each opposed pair makes a pair of equal projections) on the line joining the midpoints of the second pair of sides.

This geometrical theorem needs some interpretation as well as a proof. The two sides that are equal (in length) are the initial and final vectors $d\vec{r}$ and $d(\vec{r} + \Delta \vec{r})$, since the two neighboring points are rigidly connected during the displacement. The other two sides are the lines traversed by the two neighboring points, which may be finite; these two lines need not be of equal lengths (although the difference $d|\Delta \vec{r}|$ is infinitesimal) since different points can move differently under a given displacement. There is no requirement that the quadrilateral be all in one plane - this makes visualization even more challenging. I have been unable to find a geometric proof simpler than the algebraic one already given leading to (207).
However, it may help to restate the theorem without infinitesimals, but with vector notation. Let the four vertices of the quadrilateral be called \( \vec{r}_1, (\vec{r}_1)', r_2, (\vec{r}_2)'), and let the opposite lengths \(|\vec{r}_1 - \vec{r}_2|\) and \(|(\vec{r}_1)' - (\vec{r}_2)'|\) be equal. Let the second pair of opposite sides be called \( \Delta_1 = (\vec{r}_1)' - \vec{r}_1 \) and \( \Delta_2 = (\vec{r}_2)' - \vec{r}_2 \). Let \( \bar{\omega}_1, \bar{\omega}_2 \) be the midpoints of these two sides. Then the four vertices can be renamed \( \bar{\omega}_1 - (1/2) \Delta_1, \bar{\omega}_1 + (1/2) \Delta_1, \bar{\omega}_2 - (1/2) \Delta_2, \bar{\omega}_2 + (1/2) \Delta_2 \). The two equal lengths are now \(|(\bar{\omega}_1 - \bar{\omega}_2) - (1/2)(\Delta_1 - \Delta_2)|\) and \(|(\bar{\omega}_1 - \bar{\omega}_2) + (1/2)(\Delta_1 - \Delta_2)|\), from which it follows trivially that \( (\bar{\omega}_1 - \bar{\omega}_2) \) is orthogonal to \( (\Delta_1 - \Delta_2) \) as well as to \( \vec{r}_1 - \vec{r}_2 \), as stated by the theorem.

Consider now the equation

\[
d\xi d\Delta x + d\eta d\Delta y + d\zeta d\Delta z = 0. \tag{208}
\]

(This is eq. \(206\) written out in components.) We shall expand the complete differentials \(d\Delta x\), etc., in terms of partials \(\partial\). \(\text{(That is, } d\Delta x = d\xi \frac{\partial \Delta x}{\partial \xi} + d\eta \frac{\partial \Delta x}{\partial \eta} + d\zeta \frac{\partial \Delta x}{\partial \zeta}, \text{ etc.)}\) \(208\) thus becomes

\[
(d\xi)^2 \frac{\partial \Delta x}{\partial \xi} + (d\eta)^2 \frac{\partial \Delta y}{\partial \eta} + (d\zeta)^2 \frac{\partial \Delta z}{\partial \zeta} + d\xi d\eta \left( \frac{\partial \Delta x}{\partial \eta} + \frac{\partial \Delta y}{\partial \xi} \right) + d\xi d\zeta \left( \frac{\partial \Delta x}{\partial \zeta} + \frac{\partial \Delta y}{\partial \xi} \right) + d\eta d\zeta \left( \frac{\partial \Delta y}{\partial \zeta} + \frac{\partial \Delta z}{\partial \eta} \right) + d\eta d\zeta \left( \frac{\partial \Delta z}{\partial \xi} + \frac{\partial \Delta x}{\partial \eta} \right) = 0. \tag{209}
\]

Inasmuch as the differentials \(d\xi, d\eta, d\zeta\) are independent, \(209\) implies

\[
\frac{\partial \Delta x}{\partial \xi} = \frac{\partial \Delta y}{\partial \eta} = \frac{\partial \Delta z}{\partial \zeta} = 0,
\]

\[
\frac{\partial \Delta x}{\partial \eta} + \frac{\partial \Delta y}{\partial \zeta} = \frac{\partial \Delta z}{\partial \zeta} = \frac{\partial \Delta y}{\partial \xi} + \frac{\partial \Delta z}{\partial \eta} = 0. \tag{210}
\]

By setting \(d\eta = d\zeta = 0\), one establishes the vanishing of \(\frac{\partial \Delta x}{\partial \xi}\), similarly \(\frac{\partial \Delta y}{\partial \eta}\) and \(\frac{\partial \Delta z}{\partial \zeta}\). Then by constraining only \(d\xi\) to vanish, one isolates \(\frac{\partial \Delta x}{\partial \xi}\) and \(\frac{\partial \Delta y}{\partial \eta}\), and so forth.

This system of six equations can be easily integrated. The first shows that \(\Delta x\) is independent of \(\xi\), but then its derivatives \(\frac{\partial \Delta x}{\partial \eta}\), \(\frac{\partial \Delta x}{\partial \zeta}\), \(\frac{\partial \Delta y}{\partial \eta}\), \(\frac{\partial \Delta y}{\partial \zeta}\), \(\frac{\partial \Delta z}{\partial \eta}\), \(\frac{\partial \Delta z}{\partial \zeta}\) are also independent of \(\xi\), and likewise starting with the second or third term, so that each of the nine partial derivatives is independent of each of the independent variables \(\xi, \eta, \zeta\); \(i.e., each is constant.\) Moreover, the matrix of derivatives \(\begin{vmatrix} \frac{\partial \Delta x}{\partial \xi} & \frac{\partial \Delta y}{\partial \eta} & \frac{\partial \Delta z}{\partial \zeta} \\ \frac{\partial \Delta x}{\partial \eta} & \frac{\partial \Delta y}{\partial \zeta} & \frac{\partial \Delta z}{\partial \xi} \\ \frac{\partial \Delta x}{\partial \zeta} & \frac{\partial \Delta y}{\partial \xi} & \frac{\partial \Delta z}{\partial \eta} \end{vmatrix}\) is an antisymmetric matrix of constants, and this leads finally to the expressions at which we have previously arrived,

\[
\Delta x = A + p\eta - n\zeta, \quad \Delta y = B + m\zeta - p\xi, \quad \Delta z = C + n\xi - m\eta. \tag{211}
\]

Here the sense of “constant” is that the quantity so called vanishes under the operator \(d\), unlike its sense in \(207\). The point is that the equations \(205\) and \(208\) remain true with no change in form, if the “neighboring point” \(\vec{r} + d\vec{r}\) is replaced by a different neighboring point. Therefore all the consequences can be differentiated again and again if one wishes. In particular, \(210\) can be differentiated so as to show that all \(27\) second partial derivatives are zero, and consequently all nine first partial derivatives \(\begin{vmatrix} \frac{\partial \Delta x}{\partial \xi} & \frac{\partial \Delta y}{\partial \eta} & \frac{\partial \Delta z}{\partial \zeta} \\ \frac{\partial \Delta x}{\partial \eta} & \frac{\partial \Delta y}{\partial \zeta} & \frac{\partial \Delta z}{\partial \xi} \\ \frac{\partial \Delta x}{\partial \zeta} & \frac{\partial \Delta y}{\partial \xi} & \frac{\partial \Delta z}{\partial \eta} \end{vmatrix}\) are constant. Then appealing again to \(210\), one sees that this constant matrix is antisymmetric.
We shall not return to the transformations undergone by these formulas in reestablishing the variables \(x, y, z\); it suffices to have shown how the method of variations applies to the study of these formulas and gives in a single algebraic form both finite and infinitesimal coordinate changes in displaced points, these points being replaced in the case of finite displacements by the midpoints of the interval traversed in a straight line by them.

32 Recapitulation

To wind up this work it remains to deduce quickly, from the expression for these changes, the geometric laws for the displacement of solid bodies that we developed synthetically in the first place, and took as point of departure for our first analysis.

The formula

\[ \vec{\Delta} = \vec{\Gamma} + \vec{q} \times \vec{\omega} \]  

(212)

(this is the vectorial form of (211)) immediately gives the following fundamental relation:

\[ \vec{q} \cdot \vec{\Delta} = \vec{q} \cdot \vec{\Gamma}, \]  

(213)

from which one sees that the lines actually traversed by all the points of the system, in passing from one situation to the other, all have equal projections onto a particular direction \(\hat{q}\). Recall that the components of \(\vec{\Gamma}\) and those of \(\vec{q}\) are those six constants \(A, B, \Gamma, m, n, p\) which characterize the displacement as a whole. Therefore, (213) tells us that although different points \(\vec{r}\) traverse different lines \(\vec{\Delta}\), these lines all project equally on the fixed direction \(\hat{q}\). Denoting this projection by \(t\), one will have for all the points of the displaced system

\[ \hat{q} \cdot \vec{\Delta} = t \]  

(214)

and for two different points,

\[ \hat{q} \cdot (\vec{\Delta}_1 - \vec{\Delta}_2) = 0. \]  

(215)

Here \(\vec{\Delta}_1 - \vec{\Delta}_2\) is the chord of the arc that would be described by the first point about an axis of rotation drawn through the second point and parallel to the direction \(\hat{q}\); the two points can thereafter be brought to their final positions by translating them both by \(\Delta_2\).

If \(\theta\) represents the angle of that rotation, and \(u\) the distance from that chord to that axis of rotation, one has evidently

\[ 4u^2 \tan^2(\theta/2) = |\vec{\Delta}_1 - \vec{\Delta}_2|^2 = q^2[|\vec{\omega}_1 - \vec{\omega}_2|^2 - ((\vec{\omega}_1 - \vec{\omega}_2) \cdot \hat{q})^2], \]  

(216)

\[ u^2 = |\vec{\omega}_1 - \vec{\omega}_2|^2 - [(\vec{\omega}_1 - \vec{\omega}_2) \cdot \hat{q}]^2 \]  

(217)

and therefore, regardless of what two points are being considered,

\[ 4 \tan^2(\theta/2) = q^2. \]  

(218)
The author has worked his way back to what followed immediately from the definition \( \hat{q} = (m, n, p) \) given in Section 15. It should be noted that \( \hat{q} \) is the same unit vector as \( t \).

Thus, the displacement given to a solid, from one situation to another, can always be resolved into two consecutive displacements, one of rotation and one of translation, just as has been explained at the start of this treatise.

Furthermore, let \( \nu \) represent the amplitude of the angular displacement of a line within the solid, and \( \phi \) the angle formed by the axis of rotation with this line, then if \( \vec{r}_1, \vec{r}_2 \) are two points on the line we have

\[
\cos \phi = \frac{(\vec{r}_1 - \vec{r}_2) \cdot \hat{q}}{|\vec{r}_1 - \vec{r}_2|},
\]

and, in view of the invariability of the distances between points of the solid,

\[
\Delta \cos \phi = \frac{(|\vec{r}_1 - \vec{r}_2|)}{|\vec{r}_1 - \vec{r}_2|} = 0.
\]

The angle \( \phi \) is the same before and after the displacement. As for the angle \( \nu \), we have

\[
\cos \nu = \frac{(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2 + \vec{\Delta}_1 - \vec{\Delta}_2)}{|\vec{r}_1 - \vec{r}_2|^2},
\]

whence finally the remarkable relation

\[
\sin(\nu/2) = \sin \phi \sin(\theta/2)
\]

expressing the theorem stated in 5. (Actually the relation is given in 6.)

It is not trivial to derive (222) from the preceding equations. An essential preliminary step is to derive the identity

\[
(\vec{\omega}_1 - \vec{\omega}_2) \cdot (\vec{\Delta}_1 - \vec{\Delta}_2) = 0.
\]

This is easily done by noting that for any point \( \vec{r} \) we have \( \vec{r} = \vec{\omega} - (1/2)\vec{\Delta} \) and \( \vec{r} + \vec{\Delta} = \vec{\omega} + (1/2)\vec{\Delta} \), so that the equivalence of the two lengths \( |\vec{r}_1 - \vec{r}_2| \) and \( (|\vec{r}_1 + \vec{\Delta}_1| - (|\vec{r}_2 + \vec{\Delta}_2|) \) can be written as

\[
|(|\vec{\omega}_1 - (1/2)\vec{\Delta}_1) - (\vec{\omega}_2 - (1/2)\vec{\Delta}_2)|^2 = |(|\vec{\omega}_1 + (1/2)\vec{\Delta}_1) - (\vec{\omega}_2 + (1/2)\vec{\Delta}_2)|^2.
\]

which is equivalent to (222).

From (223) there follow the two useful relations

\[
(\vec{\omega}_1 - \vec{\omega}_2) \cdot \vec{\Delta}_1 = (\vec{\omega}_1 - \vec{\omega}_2) \cdot \vec{\Delta}_2
\]

and

\[
|\vec{\omega}_1 - \vec{\omega}_2| = |\vec{\omega}_1 - \vec{\omega}_2|^2 + (1/4)|\vec{\Delta}_1 - \vec{\Delta}_2|^2.
\]

Now consider (221). The numerator can be written as

\[
[(\vec{\omega}_1 - (1/2)\vec{\Delta}_1) - (\vec{\omega}_2 - (1/2)\vec{\Delta}_2)] \cdot [(\vec{\omega}_1 + (1/2)\vec{\Delta}_1) - (\vec{\omega}_2 + (1/2)\vec{\Delta}_2)] = |\vec{\omega}_1 - \vec{\omega}_2|^2 - (1/4)|\vec{\Delta}_1 - \vec{\Delta}_2|^2
\]

in view of (223), and the denominator as

\[
|\vec{\omega}_1 - \vec{\omega}_2|^2 + (1/4)|\vec{\Delta}_1 - \vec{\Delta}_2|^2
\]

(228)
by (226). Then

$$\sin^2(\nu/2) = (1/2)(1 - \cos \nu) = (1/4) \frac{||\Delta_1 - \Delta_2||^2}{||\omega_1 - \omega_2||^2 + (1/4)||\Delta_1 - \Delta_2||^2} = (1/4) \frac{||\Delta_1 - \Delta_2||^2}{||\vec{r}_1 - \vec{r}_2||^2}.$$  

(229)

In dealing with \(\phi\), we recall that it is the angle between the line containing \(\vec{r}_1\) and \(\vec{r}_2\) and the fixed direction \(\hat{q}\), also called the axis of rotation. If we think of \(\hat{q}\) as vertical, \(\vec{r}_1 - \vec{r}_2\) may be resolved into a vertical part \((\vec{r}_1 - \vec{r}_2) \cdot \hat{q}\) and a horizontal part which we may call \((\vec{r}_1 - \vec{r}_2)_{\perp}\). Then \(\cos \phi\) is given by (219), and for \(\sin \phi\) we have

$$\sin \phi = \frac{|\vec{r}_1 - \vec{r}_2|_{\perp}}{|\vec{r}_1 - \vec{r}_2|}. \quad \text{(230)}$$

Combining (230) with (229), we find

$$\sin^2(\nu/2) = (1/4) \frac{||\Delta_1 - \Delta_2||^2}{||\vec{r}_1 - \vec{r}_2||^2_{\perp}}. \quad \text{(231)}$$

To obtain \(\sin^2(\theta/2)\), it is not sufficient to proceed directly from (218), as this will involve a factor \(\vec{q}^2 = m^2 + n^2 + p^2\) which does not enter into (231). Instead, consider the arc swept out by rotating \(\vec{r}_1\) an angle \(\theta\) about the "vertical" axis passing through \(\vec{r}_2\). The radius of this arc is \(|\vec{r}_1 - \vec{r}_2|_{\perp}\), and the chord subtended by \(\theta\) has length \(|\Delta_1 - \Delta_2|_{\perp}\). Therefore

$$\sin^2(\theta/2) = (1/4) \frac{||\Delta_1 - \Delta_2||^2}{||\vec{r}_1 - \vec{r}_2||^2_{\perp}}. \quad \text{(232)}$$

Combining (232) with (231), we obtain (222).

The lines parallel to the direction of the axis of rotation are therefore transported parallel to themselves. Among all these lines there is one that simply glides upon itself; for this line the change \(\Delta \vec{r}\) is evidently in the direction \(\hat{q}\). The equation of this line is therefore

$$\Delta \vec{r} = \hat{q} \tau = \vec{t}; \quad \text{(233)}$$

and since

$$\vec{\omega} = \vec{r} + (1/2) \Delta \vec{r}, \quad \text{(234)}$$

we obtain the same equation already given in 16 for the central axis of the displacement:

$$\Gamma + \vec{q} \times \vec{r} = t \hat{q} = \vec{t}. \quad \text{(235)}$$

33 Conclusion - General law of Statics.

Geometry considers the displacements, finite or infinitesimal, of solid bodies, brought about by the successive action of causes or of forces capable of producing them.
Mechanics considers consecutive displacements of solid bodies, and more generally of arbitrary systems of points, brought about by the simultaneous and prolonged action of causes or of forces capable of producing them.

Statics is that most elementary part of Mechanics in which one considers only the possibility of infinitesimal or virtual displacements of these systems, resulting from the simultaneous and discontinuous action of those same causes.

The idea seems to be that the distinction between Geometry and Mechanics disappears when one considers only infinitesimal displacements in Geometry, or instantaneous applications of force in Mechanics.

Geometry teaches that the displacement of a solid body reduces to a turning about one or two fixed axes. (Two, if one wishes to eliminate translations.)

It follows that if the forces that act simultaneously on a solid system cannot impress on it any rotation about any fixed axis whatever, these forces equilibrate or neutralize one another, and the body remains at rest.

These forces, considered separately, can act only in two ways, either by tending to turn this solid body about a fixed axis, or by tending to displace a certain point of the system, or more expressively to change the coordinates of that point. That is the most general way to consider and to examine, in Mechanics, the action of forces.

At any rate the law of equilibrium is identical in these two modes of thought, as we shall see. (Probably the two modes are that of Geometry and that of Mechanics.)

If any forces or causes of displacement tend successively, or else simultaneously in passing to the limit (that is to say in passing from Geometry to Mechanics), to impress on a solid elementary or virtual rotations \( \theta, \theta', \theta'', \ldots \), about given fixed axes, the law of equilibrium of these forces is that the sum of the moments of these rotations should vanish relatively to any axis whatever. This law is rendered algebraically by the equation

\[ \Sigma \theta D \sin \nu = 0, \]

(236)
implying, on account of the indeterminacy of that arbitrary axis, six special equations, which reduce to three when the solid system reduces to a point.

Let us now examine what happens when the forces acting simultaneously on the solid are applied individually to various points of the system. As any displacement refers virtually to a fixed axis of rotation, it will suffice to consider for each point the change that can result, from the action of the forces affecting that point, in the coordinate of that point orthogonal to that fixed axis (and also orthogonal to the perpendicular dropped from the point to the axis), whose resistance is opposed to any change in its (the point's) other rectangular coordinates. The fixed axis is understood to be mechanically fixed: it cannot move either along its own length or perpendicularly to it. The only way the point can move without the axis moving is on the tangent to the circle it would describe if the system rotates about the axis.

Now, it is evident that (wordy redundant passage omitted).
From another point of view, on account of the rigidity of the system, it is evident that two forces equal (in magnitude) will, in their action on any given point, be in equilibrium (1) if they are applied through the point in opposite directions; (2) if they are applied in opposite sense through the extremities of a fixed line segment; (3) if they tend to turn in opposite sense a circumference whose center is fixed, in the plane of which they are applied tangentially; (4) if, more generally, they tend to turn oppositely a right cylinder whose axis is fixed, at whose surface they are applied tangentially and orthogonal to its axis.

It results from these propositions that if one considers all the forces that tend to displace the individual points of a solid system which contains a fixed axis, and acting in given directions, and to impress on them individually given virtual translations, there will be equilibrium among all these forces if, supposing that they are all applied to points equidistant from the fixed axis - which is always possible - the sum of the moments of the virtual translations that measure the effect of these forces is zero relative to that axis. For “moment”, read “torque: the total torque about the fixed axis should vanish, in order to produce equilibrium. I don’t understand the language about points equidistant from the fixed axis.

Passing from a fixed axis to an arbitrary axis, it will follow necessarily that the general equation (236) is the algebraic expression of the equilibrium of a set of forces capable of producing virtual or infinitesimal translations proportional to the rotations \( \theta, \theta', \theta'', \ldots \), those forces being applied about the axes of these rotations positively or negatively according to the signs of the rotations.

This explains the remarkable analogy between the laws of equilibrium (and consequently of the composition) of infinitesimal rotations and the laws of equilibrium and composition of forces, considered in Statics. This is the analogy discussed in 26.

If one denotes these forces by their finite magnitudes \( P, P', \ldots \), the equation of their equilibrium will be

\[
\Sigma PD \sin \nu = 0. \tag{237}
\]

Here each term \( PD \sin \nu \) expresses the static moment [[torque]] of the force \( P \) about the fixed axis; it is equal to the product of the distance from the point of application of that force to the fixed axis with the component of that force normal to that axis. The author here combines \( P \) and \( \nu \) into a single quantity, a component of the force. He should say also, normal to the distance \( D \).

Repetitive paragraph omitted.

The conditions of equilibrium of forces applied to a solid system, which the secondary but admirably ingenious consideration of couples reduces in finite terms to two conditions, are therefore comprised in a single law, similarly expressed in finite terms, that the sum of the moments of the forces be null about an arbitrary axis. This law is general and applies, as does that of the principle of virtual speeds - equivalent to an infinitesimal transformation - to the equilibrium of any system rigid or not, provided that the conditions of the connections among points of the system be replaced by the introduction of forces that make it possible to regard
these points as entirely free. Here the author definitely is speaking of
Mechanics; his idea is that even if the system is not solid but consists of
independently mobile points, it can be made to act like a solid if the forces
acting on it are such as to leave invariant all the distances $|\vec{r}_i - \vec{r}_j|$. These
forces are determined by analysis and eliminated in accordance with the
equations that give the equilibrium of each point.

By this means the law of equilibrium of a point immediately implies
that of a solid system. We shall not linger over this. We merely remark
that in the particular case of the forces introduced being equal and oppo-
site in pairs, the sum of the moments of all the forces applied to all the
points, which must be null for equilibrium, contains only the sum of the
given forces, and thus expresses, in identical form, the law of equilibrium
of one or many points entirely free and that of a rigid system.

33.1 On the equation of virtual speeds.

Consider an infinitesimal displacement of the system, producing changes
characterized by the symbol $\delta$ in the coordinates. If the force $P$ tends to
change the coordinate $p$, (we claim that) the product $P\delta p$ will be equal
to $PD\theta \sin \nu$, where $D\theta \sin \nu$ is the moment of the virtual rotation $\theta$.

In fact, since this infinitesimal displacement must reduce to either a
single or to two successive rotations, it need only be considered in its
unique rotation or in the first of the two. And so the infinitesimal arc
described by the point at which $P$ is applied, projected along the direction
of that force, will be equal to the infinitesimal change in the coordinate $p$
on which the force acts. This (indeed) gives $\delta p = \theta D \sin \nu$, or
\[
P\delta p = \theta PD \sin \nu.
\]

Hence the general equation (237) of equilibrium of forces
transforms into
\[
\sum P\delta p = 0,
\]
which says that if a solid system is in equilibrium under a set of
forces, and by any cause this system is infinitesimally dislodged
from its present position, the sum of the products of each force
with the infinitesimal distance traversed by the point (of appli-
cation of the force) of the system along the direction of that force
must be zero, and conversely; this is the principle of virtual
speeds. As we would say nowadays, the solid is in equilbrium under a set
of forces applied at certain points in a certain manner if and only if the
virtual work that would be done by these forces in any hypothetical in-
finitesimal displacement obeying the geometrical constraints of the
solid is zero.

Equation (239), although it is certainly superior, algebraically speak-
ing, to (237), is no more general at bottom; but it expresses in the simplest
possible way the law of equilibrium of any system in which the conditions
binding the parts together can be transformed into a set of linear equations
among the changes of coordinates of the various points of the system.

The remainder of this section, and hence of the whole treatise, is de-
voted to elucidating the above remark.
In fact, (suppose that) these conditions are expressed by equations such as
\[ \delta L = 0, \delta L' = 0, \delta L'' = 0, \text{ etc.} \]
or more generally by a single equation such as
\[ \lambda \delta L + \lambda' \delta L' + \lambda'' \delta L'' + \ldots = 0, \tag{240} \]
in which \( \lambda, \lambda', \lambda'' \) are arbitrary multipliers (what we nowadays call Lagrange multipliers). Let us denote by \( \bar{x}, \bar{y}, \bar{z} \) the coefficients of the changes \( \delta x, \delta y, \delta z \) (hidden within) this equation; by \( r \) a linear distance in the direction of \( (\bar{x}, \bar{y}, \bar{z}) \); and by \( R \) a force equal to \( \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2} \), applied at the point \((x, y, z)\) in the direction of the line that it tends to change. Then we have
\[ \delta r = \frac{\bar{x} \delta x + \bar{y} \delta y + \bar{z} \delta z}{\sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}} \tag{241} \]
and similarly for \( \delta r', \delta r'' \), etc. Equation (240) will now take the form
\[ R \delta r + R' \delta r' + R'' \delta r'' + \ldots = 0, \tag{242} \]
will have equal generality, the changes \( \delta p \) being limited in (239) by the equation of conditions (240), and in (242) being completely independent. Now, in the second case, (242) expresses the condition of equilibrium of all the points of the system, independent of all binding, but implied by the external forces \( P, P', P'' \), and by other forces \( R, R', R'' \) which statically have replaced the assumed conditions of binding.

The relation between (240) and (242) is the familiar one that arises in the Lagrange multiplier method: one relaxes the rigidity conditions requiring \( \delta L \), etc. to vanish, at the cost of introducing the extra terms \( R \delta r \), etc. into the equation to be solved. It is interesting that the whole procedure, nowadays presented as a purely mathematical manipulation in the spirit of [4], is here explicated in a totally physical way.

By eliminating these forces \( R, R', R'' \) from (242) one can obtain the definitive equations of equilibrium of the external forces subject to the binding of the system. And conversely, if these equations hold, there will be equilibrium, since the external forces are then determined, and the equations (242) on independent values of \( \delta r \), etc. establish the immobility of all the points of the system resulting from the action of the external forces plus that of the others that are statically equivalent to the given binding among the various points of the system.

When the system under consideration is continuous, the equations of binding contain definite integrals which represent in some way an infinite number of linear conditions among the changes of coordinates of the system. The arbitrary multipliers may be moved inside the integral sign, and it then remains to solve for the changes by a method entirely analytic and independent of any static consideration. One easily arrives at the following general formula:
\[ \delta S^a U dx_1 dx_2 dx_3 \ldots dx_n = S^a dx_1 dx_2 dx_3 \ldots dx_n \left[ \delta U + U \left( \frac{\partial \delta x_1}{\partial x_1} + \frac{\partial \delta x_2}{\partial x_2} + \ldots + \frac{\partial \delta x_n}{\partial x_n} \right) \right], \tag{243} \]
where \( U \) is an arbitrary function of the independent variables \( x_1, x_2, \ldots, x_n \), and the symbol \( S^a \) denotes a multiple definite integral of order \( n \).
In today’s notation \( \int dx_1 \ldots dx_n U(x_1, \ldots x_n) = \int dx_1 \ldots \int dx_n [\delta U(x_1, \ldots x_n) + U(x_1, \ldots x_n) \left( \frac{\partial \delta x_1}{\partial x_1} + \ldots + \frac{\partial \delta x_n}{\partial x_n} \right)] \). \( \text{(244)} \)

The point of interest is that \( U \) is an arbitrary function, not necessarily linear in the \( x_i \). This is why the term \( \delta U \) enters the right side of the equation.

December 5, 1840.

34 Acknowledgment

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[4] J. L. Lagrange, Mécanique Analytique, Tome 1. Courcier, Paris, 1776.

Figures
1) Section 3 (Euler)
2) Section 5 (Central axis)
3) Section 6 (spherical geodesic)
4) Section 10 (parallel axes)
Fig 2A: The x-z plane P

Fig 2B: The x-y plane R

Fig 2C: 3-dimensional view,
VV' (parallel to L) is the central axis
Fig 4A: PP₀P' Theorem

Fig 4B: PP₀P' Theorem