Optimal and Robust Quantum-Enhanced Metrology Without Echoes

Samuel P. Nolan,1,∗ Stuart S. Szigeti,1,2 and Simon A. Haine3

1School of Mathematics and Physics, The University of Queensland, Brisbane, Queensland, Australia
2ARC Centre for Engineered Quantum Systems, University of Queensland, Brisbane, Queensland, Australia
3Department of Physics and Astronomy, University of Sussex, Brighton, United Kingdom

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Unlocking the metrological potential of many-body entangled states is a significant challenge, primarily due to the fragility of these states to typical noise sources. Echo protocols, whereby perfect time reversal of the initial entangling operation is attempted, have been shown to be optimal (i.e. saturate the quantum Cramér-Rao bound) and to significantly increase the robustness of quantum-enhanced interferometers to detection noise. However, achieving the requisite time reversal is often difficult or impossible in practice. We relax this time-reversal assumption and show that optimal and robust quantum metrology can be achieved without echoes. Specifically, we show that, in principle, an echo - or indeed any interaction-based readout after the phase encoding - is entirely unnecessary for optimal quantum-enhanced interferometry. In the presence of detection noise, we consider a class of ‘pseudo-echoes’, which do not require any reversal of the initial entangling dynamics, as well as time-asymmetric protocols, and show that these are similarly robust or better than echo protocols over a wide range of realistic parameter regimes. This is especially the case if the overall interaction time is a fixed resource and the entangling operation is limited to relatively short durations. We therefore conclude that although echoes can be useful, they are often outperformed by other interaction-based readout protocols, and are sometimes entirely unnecessary.

Quantum-correlated states enable precision measurements below the standard quantum limit (SQL) [1–2]. However, despite many proof-of-principle experiments [3–7], a useful (i.e., high-precision) quantum-enhanced measurement has yet to be performed. This is partially due to the inherent fragility of quantum-correlated states to typical noise sources [8] and the difficulty in marrying quantum-state-generation protocols with the practical requirements of high-precision metrology [9,10]: addressing these issues continues to be a challenging and active area of research [11–18]. Another reason is that many estimation protocols are inherently suboptimal, in that they do not fully exploit the state’s quantum correlations. That is, the parameter estimation performs worse than the quantum Cramér-Rao bound (QCRB), which states that the most precise estimate of a classical parameter \( \phi \) is \( \Delta \phi = 1/\sqrt{F_Q} \), where \( F_Q \) is the quantum Fisher information (QFI) [8,19–21]. For example, consider the possible nonclassical states of N qubits generated via the one-axis twisting (OAT) Hamiltonian [22–25]. The usual procedure, known as spin squeezing, uses the expectation value of pseudo-spin as the estimator, which yields a maximum sensitivity of \( \Delta \phi \sim N^{-5/6} \). However, OAT is capable of producing entangled non-Gaussian states (ENGs), which can achieve the Heisenberg limit (HL) \( F_Q = N^2 \) and therefore have enormous metrological potential. Nevertheless, for ENGs an average pseudo-spin estimator yields a precision worse than the SQL [see also Fig. [1](a)].

However, the full metrological potential of ENGs can be harnessed via the inclusion of an appropriate unitary operation prior to the measurement. These so-called “interaction-based readout” protocols [26–29] are of the form

\[
|\psi_0\rangle = \hat{U}_2 \hat{U}_\phi \hat{U}_1 |\psi_0\rangle,
\]

(1)

where \( |\psi_0\rangle \) is the initial (unentangled) state, \( \hat{U}_1 \) the entangling operation (e.g. OAT), \( \hat{U}_\phi \) the phase-encoding unitary (e.g., Mach-Zehnder interferometry), and \( \hat{U}_2 \) the interaction-based readout made prior to the measurement. In particular, echo protocols [26–28,30,31] which perfectly time reverse the first entangling unitary (i.e. \( \hat{U}_2 = \hat{U}_1^\dagger \)) not only give improved sensitivities, but further robustify the quantum enhancement to detection noise that makes \( n \) particles indistinguishable from \( n \pm \sigma \) particles [27,28,32] (see also green triangles in Fig. 1). Detection noise is a key limitation in current experiments [7,33–38] and can significantly diminish the metrological gain for even modest values of \( \sigma \). Additionally, in the absence of detection noise, an echo is guaranteed to saturate the QCRB provided the measurement projects onto the initial state \( |\psi_0\rangle \) [26] (see red squares in Fig. 1). Echoes therefore deliver both optimal and robust quantum-enhanced interferometry.

Unfortunately, echoes are not always straightforward to implement. For example, the nonlinear atom-atom interactions that generate many-body entanglement in a Bose-Einstein condensate (BEC) can only be reversed by changing the inter- and intra-component couplings [59]. This typically requires a Feshbach resonance [40], which is not available to many commonly used atomic species, including \(^{87}\)Rb, and even if possible is limited to small condensates and squeezing durations due to inherent instabilities in attractive condensates [41,42] or instabilities and poor mode-matching in two-component mixtures [43,44]. Implementing echoes in soliton-based atom interferometers [44,46] and optical fibers [47,49] would be similarly impractical. Furthermore, the existence of extremely sensitive quantum-enhanced protocols that do not obey time-reversal symmetry [28,50,51] suggests that echoes may not be the only (or even best) route to optimal and robust quantum metrology.

In this Letter, we formally verify this intuition by demonstrating optimal and robust quantum metrology without echoes. By appealing to the classical Fisher information (CFI)
we show that it is possible to saturate the QCRB with the trivial protocol $\hat{U}_2 = 1$ - i.e., without an echo. This requires access to the full probability distribution of measurement outcomes in a particular basis, which is typically easily-accessible (e.g., a full spin-resolving measurement for an $N$ pseudo-spin system). In the presence of detection noise, we confirm that echo protocols provide significant robustness, although perhaps surprisingly interaction-based readout protocols that lack time-reversal symmetry can be similarly or more robust. Finally, we consider situations where the overall interferometer time is a fixed resource and show that echo protocols are suboptimal for short OAT times - which is the regime current experiments operate in. Our investigation shows that although echoes can be useful, there are many situations where other interaction-based readout protocols are superior.

Optimal interferometry without an echo.— Suppose some classical parameter $\phi$ is encoded onto the state $\hat{\rho}$ via the unitary $\hat{U}_\phi = \exp(-i\phi \hat{G})$. By subsequently performing a set of measurements in some complete, orthonormal basis \{m\}, the parameter $\phi$ can be estimated from the probabilities $P_m(\phi) = \langle m | \hat{U}_\phi \hat{\rho} \hat{U}_\phi^\dagger | m \rangle$. The precision of this estimate is bounded by the Cramér-Rao bound (CRB) $\Delta\phi \geq 1 / \sqrt{F_C(\phi)}$ where $F_C$ is the CFI. The CFI can be related to the probabilities via the Hellinger distance

$$d_H^2(\phi_1, \phi_2) \equiv 1 - \sum_m P_m(\phi_1)P_m(\phi_2),$$

since $d_H(0, \phi) = F_C(0)\phi^2 / 8 + O(\phi^3)$ \cite{hellinger2022interferometry} \cite{hellinger2020interferometry} \cite{hellinger2021interferometry}.

In general, there is no guarantee that the CFI associated with this measurement procedure is optimal (i.e. saturates the QCRB). However, we can show that the CFI always saturates the QCRB under the following modest assumptions:

1. the input state is a parity eigenstate \cite{hamburger2021parity}: $\hat{\Pi} \hat{\rho} = (-1)^p \hat{\rho}$ with $p = 0$ or 1 for parity operator $\hat{\Pi} = \sum_m (-1)^m |m\rangle\langle m|;$

2. the generator $\hat{G}$ flips parity (i.e. $\hat{\Pi}\hat{G}\hat{\Pi} = -\hat{G}$).

In principle, these conditions hold for most spin-squeezing interferometry experiments and for SU(1,1) interferometers \cite{hamburger2021parity}.

We proceed by expanding $P_m(\phi) = P_m(0) + \phi P'_m(0) + \phi^2 P''_m(0)/2 + O(\phi^3)$, where $P'_m(0) = i\langle m | \hat{G} \hat{\rho} | m \rangle - c.c.$ and $P''_m(0) = \langle m | \hat{G} \hat{G} \hat{\rho} | m \rangle - \langle m | G^2 \hat{\rho} | m \rangle + c.c.$, respectively, and c.c denotes the complex conjugate. Conditions (1) and (2) imply that $P'_m(0) = 0$, since $\langle m | \hat{G} \hat{G} | m \rangle = 0$. Furthermore, $P_m(0) = (-1)^m \langle m | \hat{G} \hat{\rho} | m \rangle$ and $\langle m | \hat{G} \hat{G} | m \rangle = (-1)^m P_m(0) \hat{G} \hat{\rho} \hat{G}$, which implies that the product $P_m(0) \hat{G} \hat{\rho} \hat{G}$ is always 0. Consequently, after keeping the leading term from a binomial expansion of the square root in Eq. \[2\], the Hellinger distance simplifies to

$$d_H^2(0, \phi) = 0 + O(\phi^3) \equiv \phi^2 \langle \hat{G}^2 \rangle \hat{\rho} \equiv \phi^2 F_C(0) / 8,$$

where $\langle \hat{G}^2 \rangle \hat{\rho} \equiv \text{Tr}(\hat{G}^2 \hat{\rho})$. Finally, our two assumptions ensure that $\langle \hat{G} \rangle \hat{\rho} = 0$, in which case the variance in $\hat{G}$ is simply $\text{Var}(\hat{G}) = \langle \hat{G}^2 \rangle \hat{\rho}$. Equating powers of $\phi$ in Eq. \[3\] gives

$$F_C(0) = 4 \text{Var}(\hat{G}) \hat{\rho}.$$

Since $F_C \leq F_Q$ and $F_Q \leq 4 \text{Var}(\hat{G}) \hat{\rho}$ \cite{hamburger2021parity}, we have shown that $F_C(0) = F_Q$, proving that our measurement procedure is optimal provided conditions (1) and (2) are true.

The implications of this result are profound. Firstly, it is not simply a proof that the QCRB is saturable. Rather, it concretely determines the optimal measurement basis \cite{hamburger2021parity}, which is typically easily accessible (e.g. a spin-resolving measurement for a pseudo-spin ensemble, as shown below). Secondly, it shows that inclusion of a second unitary $\hat{U}_2$ after the phase encoding, such that $P_m(\phi) = \langle m | \hat{U}_2 \hat{\Pi} \hat{G} \hat{\Pi} \hat{U}_2^\dagger | m \rangle$, leaves

FIG. 1. The metrological gain $\text{Gain}[\text{dB}] = -10 \log_{10}(N\Delta\phi^2)$ achievable by OAT evolution of a maximal $\hat{J}_z$ eigenstate of $N = 100$ particles, followed by a rotation that minimizes the variance along the $\hat{J}_z$ pseudo-spin axis - i.e. $\hat{U}_{\theta,\lambda} = \exp(-i\hat{J}_z \theta (N, \chi t)) \exp(-i\hat{J}_z^2 \chi t)$. Following the phase-encoding and interaction-based readout $\hat{U}_2$, measuring the signal $\hat{S}$ yields an estimation precision $\Delta\phi^2 = \text{Var}(\hat{S}(\phi))/\langle \partial_\phi (\hat{S}(\phi)) \rangle^2$. (a) Metrological gain as a function of squeezing strength $\chi t$ assuming perfect particle-detection resolution ($\sigma = 0$). Both conventional OAT (blue circles) and an echo with $\hat{S} = \hat{J}_y$ (green triangles) quickly reach the ENGS (‘over-squeezed’) regime. The QCRB, identified for the displayed schemes, can be saturated with an echo followed by a measurement that projects onto the initial state (i.e. that counts instances of maximal $\hat{J}_z$) (red squares) or, as we show below, without an echo by full spin-resolving measurement. (b) Dependence of gain on detection noise $\sigma$ for fixed $\chi t = 0.1$, indicated by the vertical black line in (a). An echo followed by a measurement of the average pseudo-spin (green triangles) is significantly more robust than conventional OAT and also an echo followed by a measurement projecting onto the initial state $|\psi_0\rangle$, which both require detection resolution at the single particle level.

$\hat{S}$

Legend

| $\hat{U}_2$ | $\hat{S}$ | Ref. |
| --- | --- | --- |
| $\hat{U}_1^\dagger$ | $|\psi_0\rangle\langle\psi_0|$ | [27] |
| $\hat{U}_1^\dagger$ | $\hat{J}_y$ | [26] |
| 1 | $\hat{J}_z$ | [22] |

$\chi_0$ 0 $\chi_1$ 0 $\chi_2$ 0 $\chi_3$ 0 $\chi_4$

$\chi_t$

$\chi_0$ 0 $\chi_1$ 0 $\chi_2$ 0 $\chi_3$ 0 $\chi_4$
the CFI unchanged provided $U_2$ conserves parity with respect to the measurement basis. This means that, fundamentally, an echo protocol is unnecessary; all interaction-based readout protocols have identical CFI, and indeed are equivalent to simply doing nothing after the phase encoding ($U_2 = 1$). More concretely, the three schemes listed in Fig. 1(a), which have wildly different phase sensitivities and experimental complexities, are all capable of saturating the QCRB if one has access to a full spin-resolving measurement.

**Application:** spin-squeezing interferometry.— A broad class of interferometry can be performed within two-bosonic-mode systems of $N$ particles. Provided $N$ is fixed, these systems can be described by the SU(2) Lie algebra $[\hat{J}_x, \hat{J}_y] = i\epsilon_{i,j,k} \hat{J}_k$, where $\epsilon_{i,j,k}$ is the Levi-Civita symbol, for $i = x, y, z$. Consequently, our above result implies that if the resultante is a spin-coherent state, then its parity with respect to the $\hat{J}_x$ eigenbasis remains unchanged under any of these protocols. Thereby, our above result implies that if the resultante spin-squeezed state is passed through a Mach-Zehnder interferometer $(\hat{G} = \hat{J}_y)$ and measurements are subsequently made in the $\hat{J}_x$ eigenbasis, then conditions (1) and (2) are met, the CFI saturates the QCRB and the best phase sensitivity attained. In fact, this is true for a spin-squeezing protocol described by the fanciful (although certainly not the most general) Hamiltonian

$$\hat{H} = \hat{c}^+_{n_x} (\hat{J}_x)^{n_x} + \hat{c}^+_{n_y} (\hat{J}_y)^{2n_y} + \hat{c}^+_{n_z} (\hat{J}_y)^{2n_z} + \hat{c}^+_{n_z} (\hat{J}_z)^{2n_z} + (\hat{c}^+_{n_x})^* (\hat{J}_x)^{2n_x} + \hat{c}^+_{n_y} \hat{c}^+_{n_z} \hat{J}_y \hat{J}_z + \hat{c}^+_{m_x} (\hat{J}_x)^{2n_x} + \hat{c}^+_{m_y} (\hat{J}_y)^{2n_y} + \hat{c}^+_{m_z} (\hat{J}_z)^{2n_z} + (\hat{c}^+_{m_x})^* (\hat{J}_x)^{2n_x} + \hat{c}^+_{m_y} \hat{c}^+_{m_z} \hat{J}_y \hat{J}_z + \hat{c}^+_{m_z} (\hat{J}_z)^{2n_z},$$

where $\hat{O}_t$ are coefficients chosen such that $\hat{H}$ is Hermitian and there are implicit summations over the non-negative integers $n_j$.

Spin-squeezing has been demonstrated in trapped ions [58-60], BECs [61-63], cold atoms in cavities [57 64-66], and optical systems [47, 48, 67], and has been used to enhance proof-of-principle interferometer measurements [23, 50, 68], including atomic clocks [69, 70] and magnetometers [71, 72]. Note the ‘proof-of-principle’ aspect to these experiments; spin squeezing has not yet resulted in a useful measurement that surpasses current shot-noise-limited high-precision devices. This is due to the fragility of spin-squeezed states, which to date has limited the degree of squeezing and/or particle numbers to relatively modest values. It is therefore of paramount importance to maximize the metrological benefits of squeezing, preferably with minimal increases in experimental complexity. Our above result suggests that implementing a full spin-resolving measurement, as reported in [52] for example, is important for achieving this goal.

Although we focus on spin-squeezing protocols for the remainder of this Letter, it is worth emphasizing the wider applicability of our result. For instance, SU(1,1) interferometers [23, 55, 73-75] (and related schemes [76, 78]) achieve sub-shot-noise sensitivities by generating quantum correlations within the interferometer via nonlinear active beamsplitters. These schemes are time-symmetric about the phase-encoding - a challenging feat to achieve in practice, and one that places practical limits on the achievable quantum enhancement [79-81]. However, our result proves that the CFI saturates the QCRB with or without the second nonlinear beamsplitter. This greatly reduces the technical complexity, and indeed is the route pursued by truncated SU(1,1) interferometry [51].

**Robustifying against detection noise.**— Although we have shown that a full spin-resolving measurement typically renders echoes unnecessary, in practice such a measurement would be difficult in the presence of detection noise. Echoes could therefore still play an important role in optimal parameter-estimation strategies. Here we investigate the CFI in the presence of detection noise, and although we do confirm that echoes can provide significant robustification, perhaps surprisingly we show that better sensitivities can be obtained with non-echo protocols.

For concreteness, we consider the quantum-correlated state generated by evolving a maximal $\hat{J}_x$ eigenstate under OAT for time $t$. After passing through a Mach-Zehnder interferometer, an interaction-based readout $U_2$ is applied (which leaves the QFI unchanged) and a spin-resolving measurement made in the optimal basis $\{|m\}$.

$$P_m(\phi) = \frac{\sqrt{2\pi}}{2}(\hat{c}_m^+ \hat{c}_m + \hat{c}_m \hat{c}_m^+)$$

for time $t$. After passing through a Mach-Zehnder interferometer, an interaction-based readout $U_2$ is applied (which leaves the QFI unchanged) and a spin-resolving measurement made in the optimal basis $\{|m\}$.

We model imperfect particle resolution in this measurement as a Gaussian noise of variance $\sigma^2$, corresponding to an uncertainty $\sigma$ in the measured particle number. This noise distorts the measured probabilities (and consequently the CFI), which we account for by replacing $P_m(\phi)$ with the conditional probabilities $P_m(\phi|\sigma) = \sum_{m'} C_{m'} \exp[-(m - m')^2/2\sigma^2] P_m(\phi)$ [78, 79] where $C_{m'} = \sum_{m} \exp[-(m - m')^2/2\sigma^2]$ normalizes the Gaussian probability distribution corresponding to the detection noise.

In Fig. 2 on the left we plot the CFI as a function of detection resolution for a range of interaction-based readout operations $U_2$. As expected, an echo ($U_2 = U_1^t$) provides significant robustification over no echo ($U_2 = 1$). We note that using the full probability distribution $P_m$ is vastly more robust than using only the $m = 0$ component, as originally proposed in [26]. However, we find a class of time-asymmetric protocols that outperform echoes. Specifically, if $U_1 = U_{OAT}(t_1)$ corresponds to an OAT operation of duration $t_1$, then $U_2 = U_{OAT}(t_2)$ with $t_2 > t_1$ generally outperforms an echo (i.e. $t_2 = t_1$) provided the squeezing strength $\chi t_1$ is modest. Furthermore, robustification to detection noise can be achieved via “pseudo-echoes”, $U_2 = U_{TNT}$, which do not reverse the time evolution of $U_1$ [28, 50]. Although these are not as effective as a true echo or asymmetric time-reversal protocol, these protocols nevertheless provide good robustification. Furthermore, pseudo-echoes are an excellent option for sys-
tems where time-reversal protocols are difficult or impossible to implement, such as OAT due to atom-atom interactions in a BEC [23, 62], bright-soliton atom interferometry [44–46], and polarization squeezing in optical fibers [47, 49].

For OAT, the creation of a GHZ state [83] provides an upper limit on the QFI (specifically, the Heisenberg limit $F_Q = N^2$), since at $\chi t > \pi/2$ the state revives towards the initial condition (assumed to be a maximal $J_y$ eigenstate). In principle, the most robust interaction-based readout for OAT is $\hat{U}_2 = \hat{U}_{\text{OAT}}(\chi t = \pi/2) \equiv \hat{U}_{\text{GHZ}}$, with a measurement in the $J_y$ basis, since this projects onto the initial state. Although implementing such a protocol is infeasible in current experiments, since sources of loss typically limit the squeezing operation to $\chi t \ll \pi/2$, this extreme case provides some conceptual insight into why these protocols successfully robustify OAT to detection inefficiency. On the right of Fig. 2 (green histograms) we can see that for a GHZ state and an echo the Hellinger distance between the probability distributions $P_m(0)$ and $P_m(\delta \phi)$ is large, for a small perturbation to the classical parameter $\delta \phi$, and remains large in the presence of detection noise. That is, these two probability distributions remain distinguishable, thereby maintaining the large CFI $F_C = N^2$, even in the presence of detection noise exceeding $\sqrt{N}$. In contrast, for $\hat{U}_2 = 1$ in the optimal measurement basis (here the $J_x$ eigenbasis) the states at $\phi = 0$ and $\phi = \delta \phi$ are distinguished by a decay of odd $P_m$. Even a small amount of detection noise amounts to a Gaussian convolution which obscures these fringes, making the two distributions virtually indistinguishable and thus resulting in a small CFI [see blue histogram and insets on the right of Fig. 2]. Perhaps counterintuitively, this means the GHZ state with an echo is more robust than a state prepared with a smaller $\chi t$.

Achieving good particle-resolving detection is experimentally challenging. Currently, the best particle-resolving detection schemes are limited to relatively small ensembles of particles [35, 36, 84]. Scaling up this detection resolution to the large-particle ensembles required for useful (i.e., high-precision) quantum-enhanced metrology is likely extremely difficult, if not impossible, in practice. Since both echo and non-echo interaction-based readouts enable sub-SQL sensitivities for $\sigma \sim \sqrt{N}$ detection noise, we anticipate their application in practical quantum sensors.

**Optimal protocols with total-time constraint.**— Typically, the overall duration of OAT experiments is limited due to, for example, the presence of particle losses and/or dephasing [24, 85, 86] and a desire to maintain a high repetition rate. Therefore, if an experiment is restricted to some fixed total squeezing time $T = t_1 + t_2$ there is potentially a trade-off between increasing $t_1$ in order to increase the CFI via $\hat{U}_1(t_1)$ and increasing $t_2$ in order to optimally tune the interaction-based readout $\hat{U}_2(t_2)$. We scanned over a range of $t_1$ values, and in Figure 3 we plot the best $F_C$ (top), and the corresponding $t_1$ value ($t_{\text{opt}}^1$, bottom). Intuitively, Fig. 3 shows that for sufficiently small detection
noise any interaction-based readout (i.e., \( t_2 > 0 \)) confers no benefit, and the best strategy is simply to maximize the CFI by choosing \( t_1 = T \). This is consistent with our proof above. In contrast, for large \( T \) (e.g., \( \chi T = \pi/2 \)) and non-negligible levels of detection noise an echo remains the best strategy up until the detection noise approaches \( N \). The reason is simple: when evolving an initial maximal \( J_x \) eigenstate under OAT, the QFI quickly reaches a plateau at \( N^{1/2} \) [see Fig. 1(a)]. If \( t_1 \) is sufficiently large such that the CFI lies on this plateau, there is little QFI to be gained from increasing \( t_1 \). In this situation there is no real trade-off between increasing the QFI via \( t_1 \) and increasing the robustness via \( t_2 \), and so an echo remains optimal. Interestingly, in this large-\( T \) regime a “pseudo-echo” performs as well as an echo, but is far simpler to implement.

For moderate protocol durations (e.g., \( \chi T = 0.1 \)) the story is less clear-cut. As shown in the middle panel in Fig. 3 there are regimes where it is beneficial to choose a time-asymmetric readout protocol such as \( \hat{U}_2 = \hat{U}_{OAT}^1(t_2) \) over an echo. For smaller values of \( T \), interaction-based readout protocols of the form \( \hat{U}_2 = \hat{U}_{OAT}(t_2) \), which include pseudo-echoes, perform poorly in comparison.

Current experiments are limited to relatively small squeezing durations. For instance, the landmark experiment of Ref. [23] was performed with \( \chi T \approx 0.01 \) (and \( N = 170 \)). Interestingly, for fixed, small squeezing times at this order of magnitude, Fig. 3 indicates the optimal strategy is \( \hat{U}_2 = 1 \) (i.e., no interaction-based readout), even in the presence of modest detection noise.

Conclusions and outlook — By considering the CFI we have shown that in the absence of detection noise, for many experiments echoes are not necessary for optimal quantum-enhanced metrology. Echoes are only useful in the presence of detection noise. However, we consider a class of asymmetric time-reversal protocols that are superior to echoes, and also show that pseudo-echoes, which do not require any time reversal, can provide comparable robustness. The latter class of protocols are advantageous in systems such as BECs, bright-soliton interferometers, and optical fibers, where it is difficult or impossible to subject the state to time reversal. Determining that echoes are not uniquely suited to optimal and robust quantum-enhanced metrology gives experimentalists additional flexibility in the design of protocols, and could find near-term applications in current spin-squeezing experiments that are restricted to short spin-squeezing durations.

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