Hofer geometry via toric degeneration

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Abstract

The main theme of this paper is to use toric degeneration to study Hofer geometry by producing distinct homogeneous quasimorphisms on the group of Hamiltonian diffeomorphisms. We focus on the (complex $n$-dimensional) quadric hypersurface and the del Pezzo surfaces, and study two classes of distinguished Lagrangian submanifolds that appear naturally in a toric degeneration, namely the Lagrangian torus which is the monotone fiber of a Lagrangian torus fibration, and the Lagrangian spheres that appear as vanishing cycles. For the quadrics, we prove that the group of Hamiltonian diffeomorphisms admits two distinct homogeneous quasimorphisms and derive some superheaviness results. Along the way, we show that the toric degeneration is compatible with the Biran decomposition. This implies that for $n = 2$, the Lagrangian fiber torus (Gelfand–Zeitlin torus) is Hamiltonian isotopic to the Chekanov torus, which answers a question of Y. Kim. We prove analogous results for the del Pezzo surfaces. We also discuss applications to $C^0$ symplectic topology.

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1 Introduction and overview of the results

1.1 Hofer geometry

The main theme of this paper is to use toric degeneration to study Hofer geometry by producing distinct homogeneous quasimorphisms on the group of Hamiltonian diffeomorphisms. The set of Hamiltonian diffeomorphisms of a closed symplectic manifold \( X = (X,\omega) \), denoted by \( \text{Ham}(X) \) (as well as its universal cover \( \tilde{\text{Ham}}(X) \)) forms a group and moreover, it has a remarkable bi-invariant metric called the Hofer metric \([Hof93]\). The geometry of \( \text{Ham}(X) \) (and \( \tilde{\text{Ham}}(X) \)) with respect to the Hofer metric has been an active and an important research topic, e.g. \([Pol01]\), which is now called the Hofer geometry. One of the first important questions in this topic was the so-called Hofer diameter conjecture which conjectures that the diameter of \( \text{Ham}(X) \) (and \( \tilde{\text{Ham}}(X) \)) with respect to the Hofer metric is infinity. Many fragmented cases were solved by various symplectic geometers before a major step forward was made by Entov–Polterovich in \([EP03]\), where they introduced the method of quasimorphisms to the study of Hofer geometry. This provided a systematic way to solve the Hofer diameter conjecture for a sufficiently large class of symplectic manifolds by means of Floer theory. However, beyond the Hofer diameter conjecture, not much is known about the Hofer geometry. For example, consider the following question, known as the Kapovich–Polterovich question:

**Question 1.1.1** (Kapovich–Polterovich question\(^1\)). For a closed symplectic manifold \( X \), is the group \( \text{Ham}(X) \) quasi-isometric to the real line \( \mathbb{R} \) with respect to the Hofer metric?

A negative answer to the Kapovich–Polterovich question will immediately solve the Hofer diameter conjecture, but extremely little is known about it. At the time of writing, the only cases for which Question \(^1\) has been answered (in the negative) are the cases of symplectic manifolds that satisfy some dynamical condition \([Ush13]\), \( X = S^2 \times S^2 \) Fukaya–Oh–Ohta–Ono \([FOOO19]\) (see also \([EliPol]\)), and \( X = S^2 \), which was done in 2021, independently by Cristofaro-Gardiner–Humilière–Seyfaddini \([CGHS]\) and Polterovich–Shelukhin \([PS]\). Other than that, even for the weaker version of Question \(^1\) where \( \text{Ham}(X) \) is replaced by \( \tilde{\text{Ham}}(X) \), the question remains widely open.

\(^1\)The original question was for the case \( X = S^2 \).
This paper attempts to use toric degeneration to obtain new insights about the Hofer geometry, especially in (real) dimension higher than 2. We focus on the quadric hypersurface
\[ Q^n := \{ [z_0 : z_1 : \cdots : z_{n+1}] \in \mathbb{C}P^{n+1} : z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0 \} \]
and the symplectic del Pezzo surfaces, and study two distinguished classes of Lagrangian submanifolds that appear naturally in a toric degeneration, namely the Lagrangian torus and the Lagrangian spheres that appear as the monotone fiber of a Lagrangian torus fibration and vanishing cycles, respectively.

1.2 Main results for quadrics
We first state the results for the quadric hypersurfaces.

**Theorem A.** The two Entov–Polterovich type homogeneous quasimorphisms
\[ \zeta_\pm : \widehat{\text{Ham}}(Q^n) \to \mathbb{R} \]
are distinct, i.e.
\[ \zeta_+ \neq \zeta_- \]
This immediately answers the \( \widehat{\text{Ham}}(X) \) version of Question 1.1.1 in the negative for \( X = Q^n \). Theorem A generalizes results of Eliashberg–Polterovich from [EliPol] and the author from [Kaw22A], where the \( n = 2 \)-case and the \( n = 4 \)-case were proven, respectively.

1.3 Lagrangians in quadric hypersurfaces
In the proof of Theorem A producing disjoint Lagrangian submanifolds is a key step and this is where the toric degeneration plays a crucial role. Recall that roughly speaking, a toric degeneration for a symplectic manifold \( X \) is a way to deform \( X \) into a toric variety \( X_0 \) which allows one to study the symplectic geometry of \( X \) via the toric geometry of \( X_0 \) (Section 3.1). There are two distinguished subsets for toric degenerations. The first is the monotone Lagrangian fiber torus; given a toric degeneration, one obtains a Lagrangian torus fibration for \( X \) and there is a unique fiber for which the Lagrangian fiber torus becomes monotone. The second is the vanishing locus, which is the set obtained by tracing back the points that get crushed into the singular locus of the toric variety \( X_0 \) in the process of the toric degeneration. Note that the vanishing locus is not necessarily a geometrically nice set, e.g. a submanifold, even though in practice it often turns out to be a Lagrangian cell-complex.

As for the quadric hypersurface \( Q^n \), the toric degeneration
\[ \mathcal{X} := \{(z,t) \in \mathbb{C}P^{n+1} \times \mathbb{C} : z_0^2 + z_1^2 + z_2^2 + t(z_3^2 + \cdots + z_{n+1}^2) = 0 \} \] \[ (1.3.1) \]
was introduced by Nishinou–Nohara–Ueda in [NNU] which is compatible with the celebrated Gelfand–Zeitlin (GZ) system on \( Q^n \). The GZ system defines a Lagrangian torus fibration on \( Q^n \) and there is one monotone torus fiber which we call the GZ torus and denote it by \( T^n_{\text{GZ}} \).
The \( n = 2 \) case deserves special attention, as it is well-known that \( Q^2 \) is symplectomorphic to the monotone \( S^2 \times S^2 \). Through this symplectomorphism, we can see the monotone GZ fiber torus \( T^2_{GZ} \) as a monotone Lagrangian torus in \( S^2 \times S^2 \). In [KimA], Yoosik Kim computed the superpotential for the monotone GZ fiber torus \( T^2_{GZ} \) and found that for \( n = 2 \), it agrees with the superpotential for the celebrated Chekanov torus in \( S^2 \times S^2 \). This lead him to ask the following question:

**Question 1.3.1 ([KimA, Section 4.2]).** In the monotone \( S^2 \times S^2 \), is the monotone GZ fiber torus \( T^2_{GZ} \) Hamiltonian isotopic to the Chekanov torus \( T^2_{Ch} \)?

We study some geometric properties of the distinguished monotone Lagrangian submanifolds \( T^n_{GZ} \) and \( S^n \) in \( Q^n \). Our second main result is the following.

**Theorem B.** The monotone Lagrangian torus fiber \( T^n_{GZ} \) in \( Q^n \) satisfies the following properties:

1. Consider the polarization \((X, \Sigma) = (Q^n, Q^{n-1})\). The monotone GZ fiber \( T^n_{GZ} \) in \( Q^n \) coincides with the distinguished monotone Lagrangian torus obtained from the Biran circle bundle construction from \( T^{n-1}_{GZ} \) in \( Q^{n-1} \), i.e.
   \[
   T^n_{GZ} = \tilde{T}^{n-1}_{GZ}.
   \]

2. The real Lagrangian sphere
   \[
   S^n_{\text{van}} := \{ z \in \mathbb{C}P^{n+1} : z_0^2 + \cdots + z_n^2 + z_{n+1}^2 = 0, \ z_0, \cdots, z_n \in \mathbb{R}, \ z_{n+1} \in i \mathbb{R} \}
   \]
   is a vanishing locus of the toric degeneration (1.3.1), and it is disjoint from \( T^n_{GZ} \), i.e.
   \[
   S^n_{\text{van}} \cap T^n_{GZ} = \emptyset.
   \]

3. The Gelfand–Zeitlin torus \( T^n_{GZ} \) is \( \zeta^+ \)-superheavy and the vanishing cycle \( S^n_{\text{van}} \) is \( \zeta^- \)-superheavy.

The third assertion of Theorem [B] generalizes a result of Eliashberg–Polterovich from [EliPol] where the \( n = 2 \) case was proven.

**Remark 1.3.2.**

1. For details about polarizations and the Biran circle bundle construction, see Section 2.2.

2. The singular toric variety which is the central fiber of the toric degeneration (1.3.1) has non-isolated singular locus. This makes it more difficult to understand the geometry of the vanishing locus, c.f. the second statement in Theorem [B] compared to the cases for del Pezzo surfaces that we consider later, where the singular loci are rational double points (\( A_m \)-singularities).

3. In [Kaw22A], it was proven that \( \zeta^+ \neq \zeta^- \) for \( n = 2, 4 \) but nothing about the superheaviness was proved. The new input is the use of some results from the mirror symmetry literature, e.g. the Auroux–Kontsevich–Seidel theorem, combined with the comparison technique of spectral invariants with different coefficients developed in [Kaw22A]; see Sections 4.1 and 4.2 for more on this.
4. In [Kaw22A], finding disjoint Lagrangians was also a key step, but in this paper, we focus on a different geometric structure and use different Lagrangians; e.g., for \( n = 4 \), [Kaw22A] focused on a Lagrangian diffeomorphic to \( S^1 \times S^3 \), which is different from the sphere we consider in Theorem B.

5. Theorem B has other applications; it plays an important role in [KawA].

The \( n = 2 \) case of the first assertion in Theorem B has the following corollary, which answers the aforementioned Question 1.3.1 of Yoosik Kim.

**Corollary 1.3.3** (Kim’s question 1.3.1). In the monotone \( S^2 \times S^2 \), the monotone GZ fiber \( T^2_{GZ} \) is Hamiltonian isotopic to the Chekanov torus \( T^2_{Ch} \).

**Remark 1.3.4.** The answer to Kim’s question 1.3.1, namely Corollary 1.3.3, was very recently obtained by Kim himself in [KimB], by a different approach.

### 1.4 Main results for the del Pezzo surfaces

In this section, we state the main results for the del Pezzo surfaces, which are analogous to results for quadrics. Recall that the del Pezzo surfaces are smooth (complex) two-dimensional Fano varieties, equipped with a monotone symplectic form. They are classified and consist of \( \mathbb{C}P^2 \), \( S^2 \times S^2 (= Q^2) \), and \( \mathbb{D}_k := \mathbb{C}P^2 \# k \mathbb{C}P^2 \), \( 1 \leq k \leq 8 \).

**Theorem C.** 1. For \( \mathbb{D}_2 \), there exist two distinct Entov–Polterovich type homogeneous quasimorphisms

\[ \zeta_j : \widehat{\text{Ham}}(\mathbb{D}_2) \to \mathbb{R}, \ j = 1, 2, \]

\[ \zeta_1 \neq \zeta_2. \]

2. For \( \mathbb{D}_k \), \( k = 3, 4 \), there exist three distinct Entov–Polterovich type homogeneous quasimorphisms

\[ \zeta_j : \widehat{\text{Ham}}(\mathbb{D}_k) \to \mathbb{R}, \ j = 1, 2, 3, \]

\[ \zeta_i \neq \zeta_j, \text{ if } i \neq j. \]

Theorem C leads to the following, which answers the Kapovich–Polterovich question in the negative.

**Theorem D** (Kapovich–Polterovich question). The group \( \text{Ham}(\mathbb{D}_k) \) is not quasi-isometric to the real line \( \mathbb{R} \) with respect to the Hofer metric when \( k = 3, 4 \).

**Remark 1.4.1.** As mentioned earlier in Section 1.1, to the author’s knowledge, the Kapovich–Polterovich question has been only answered (in the negative) for few cases.
1.5 Applications

We discuss some applications of Theorems A, B, C. The following question of Entov–Polterovich–Py from 2012 has been an important open question, which is referred to as the “Quasimorphism question” in the monograph of McDuff–Salamon [MS98, Chapter 14, Problem 23]:

**Question 1.5.1 (Entov–Polterovich–Py question [EPP12]).** For a closed symplectic manifold \((X, \omega)\), is there a non-trivial homogeneous quasimorphism on \(\text{Ham}(X, \omega)\) which is \(C^0\)-continuous? If yes, can it be Hofer Lipschitz continuous?

**Remark 1.5.2.**

1. The \(C^0\)-topology on \(\text{Ham}(X)\) is induced by the \(C^0\)-metric of Hamiltonian diffeomorphisms \(\phi, \psi \in \text{Ham}(M, \omega)\), which is defined by

\[
d_{C^0}(\phi, \psi) := \max_{x \in M} d(\phi(x), \psi(x)),
\]

where \(d\) denotes the distance on \(X\) induced by a fixed Riemannian metric on \(X\). Different choices of Riemannian metrics on \(X\) will induce equivalent \(C^0\)-topology.

2. One point of Question 1.5.1 is that currently the relation between the Hofer metric and the \(C^0\)-metric is not well understood. See, for example, [JS] for the latest progress on this topic.

3. The original question was posed for \(X = S^2\). This extended version was considered in [Kaw22A] before the final resolution of the original question by Cristofaro-Gardiner–Humilière–Mak–Seyfaddini–Smith in [CGHMSS].

In [Kaw22A], a positive answer to Question 1.5.1 was provided for \(Q^n\), \(n = 2, 4\). By using Theorems B and C we manage to generalize this to \(Q^n\) for all \(n\), and to some del Pezzo surfaces.

**Theorem E.** For \(X = Q^n\) and \(X = \mathbb{D}_k\), \(k = 2, 3, 4\), there exist non-trivial homogeneous quasimorphisms

\[
\mu : \text{Ham}(X) \to \mathbb{R}
\]

that are both Hofer Lipschitz and \(C^0\) continuous.

**Remark 1.5.3.**

1. We can actually construct two distinct quasimorphisms with the relevant continuity properties in Theorem E for \(\mathbb{D}_k\), \(k = 3, 4\).

2. It is also known that one can construct quasimorphisms for \(\mathbb{C}P^n\) that have continuity properties as in Question 1.5.1 by [KS].
The argument of the proof of Theorem E actually shows that the asymptotic spectral norm \( \overline{\gamma} \)
\[
\overline{\gamma} : \Ham(X) \to \mathbb{R}
\]
\[
\overline{\gamma}(\phi) := \lim_{k \to +\infty} \frac{\gamma(\phi^k)}{k}
\]
is \( C^0 \)-continuous for \( X = Q^n \) and \( X = \mathbb{D}_k, \ k = 2, 3, 4 \), even though the \( C^0 \)-continuity of the spectral norm \( \gamma \) for none of these symplectic manifolds is confirmed at the time of writing. Note that the \( C^0 \)-continuity of the spectral norm \( \gamma \) is currently proven for the closed surfaces [Sey13], symplectically aspherical manifolds [BHS21], \( CP^n \) [Sh22], and negative monotone symplectic manifolds [Kaw22B].

**Theorem F.** Let \((X, \omega)\) be a monotone symplectic manifold whose quantum cohomology ring \( QH(X; \Lambda) \) is semi-simple. Then, the asymptotic spectral norm
\[
\overline{\gamma} : \Ham(X) \to \mathbb{R}
\]
is \( C^0 \)-continuous. Moreover, for \( Q^n \) and \( \mathbb{D}_k, \ k = 2, 3, 4, \overline{\gamma} \) is non-trivial, i.e.
\[
\overline{\gamma} \neq 0.
\]

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### 2 Preliminaries

We will review results that we will use in the proofs. A special emphasis is put on comparing the conventions in [OU16] and other works, as we will use the convention from [OU16] in the proofs. We will assume that symplectic manifolds and Lagrangian submanifolds are all closed and monotone.

#### 2.1 Entov–Polterovich quasimorphisms and (super)heaviness

First, we review some quantitative Floer theory including spectral invariants by following [MS04]. It is well-known that on a symplectic manifold \((X, \omega)\), for a non-degenerate,
Hamiltonian $H := \{H_t : X \to \mathbb{R}\}_{t \in [0,1]}$ and a choice of a nice coefficient field $\Lambda^\downarrow$, which will be either the downward Laurent coefficients $\Lambda^\downarrow_{\text{Lau}}$

$$\Lambda^\downarrow_{\text{Lau}} := \left\{ \sum_{k \leq k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C} \right\},$$

or the downward Novikov coefficients $\Lambda^\downarrow_{\text{Nov}}$

$$\Lambda^\downarrow_{\text{Nov}} := \left\{ \sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to -\infty} \lambda_j = +\infty \right\},$$

one can construct a filtered Floer homology group $\{HF^\tau(H) := HF^\tau(H;\Lambda^\downarrow)\}_{\tau \in \mathbb{R}}$ where for two numbers $\tau < \tau'$, the groups $HF^\tau(H;\Lambda^\downarrow)$ and $HF^{\tau'}(H;\Lambda^\downarrow)$ are related by a map induced by the inclusion map on the chain level:

$$i_{\tau,\tau'} : HF^\tau(H;\Lambda^\downarrow) \longrightarrow HF^{\tau'}(H;\Lambda^\downarrow),$$

and especially we have

$$i_{\tau} : HF^\tau(H;\Lambda^\downarrow) \longrightarrow HF(H;\Lambda^\downarrow),$$

where $HF(H;\Lambda^\downarrow)$ is the Floer homology. There is a canonical ring isomorphism called the Piunikhin–Salamon–Schwarz (PSS)-map \cite{PSS96, MS04}

$$\text{PSS}_{H;\Lambda} : QH(X;\Lambda) \to HF(H;\Lambda^\downarrow),$$

where $QH(X;\Lambda)$ denotes the quantum cohomology ring of $X$ with $\Lambda$-coefficients, i.e.

$$QH(X;\Lambda) := H^\ast(X;\mathbb{C}) \otimes \Lambda.$$  

Here, $\Lambda$ is either the Laurent coefficients $\Lambda_{\text{Lau}}$

$$\Lambda_{\text{Lau}} := \left\{ \sum_{k \geq k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C} \right\},$$

or the Novikov coefficients $\Lambda_{\text{Nov}}$

$$\Lambda_{\text{Nov}} := \left\{ \sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to +\infty} \lambda_j = +\infty \right\}$$

chosen accordingly to the set-up of the Floer homology.

**Remark 2.1.1.**

1. The choice of the coefficient fields will eventually become very important, see Section 4.2. If we do not specify the choice of it and denote it by $\Lambda$, it means that the argument/result holds for both $\Lambda_{\text{Lau}}$ and $\Lambda_{\text{Nov}}$.

\[\text{\footnotesize Usually, the PSS-isomorphism is a ring isomorphism between the quantum homology ring and Floer homology ring } QH_\ast(X;\Lambda^\downarrow) \to HF(H;\Lambda^\downarrow), \text{ but here we compose this with the Poincaré duality isomorphism } PD : QH(X;\Lambda) \to QH_\ast(X;\Lambda^\downarrow) \text{ and call the composition the PSS-isomorphism, c.f. } [\text{MS04, Section 12}]\]
2. Nevertheless, it might be helpful to keep in mind that $\Lambda_{Lau}$ can be embedded to $\Lambda_{Nov}$ by the inclusion given by $t \mapsto T^{\lambda_0}$, where $\lambda_0$ is a positive generator of $\langle \omega, \pi_2(X) \rangle$.

3. The formal variable $t$ in $\Lambda_{Lau}$ should be seen as representing the sphere with the minimal possible positive area $\lambda_0 > 0$ (and thus, the minimal possible Maslov index as well, by monotonicity).

Spectral invariants, which were introduced by Schwarz [Sch00] and developed by Oh [Oh05] following the idea of Viterbo [Vit92], are real numbers $\{c(H, a) \in \mathbb{R}\}$ associated to a pair of a Hamiltonian $H$ and a class $a \in QH(X; \Lambda)$ in the following way:

$$c(H, a) := \inf\{\tau \in \mathbb{R} : PSS_{H; \Lambda}(a) \in \text{Im}(i_{\tau})\}.$$ 

**Remark 2.1.2.** Although the Floer homology is only defined for a non-degenerate Hamiltonian $H$, the spectral invariants can be defined for any Hamiltonian by using the following so-called Hofer continuity property:

$$c(H, a) - c(G, a) \leq \int_0^1 (H_t(x) - G_t(x)) \, dt$$

for any $a \in QH(X)$, $H$ and $G$.

**Triangle inequality.** For Hamiltonians $H$ and $G$, define their catenation $H \# G$ (which is a Hamiltonian that generates the path $t \mapsto \phi_H^t \circ \phi_G^t$) as

$$(H \# G)(t, x) := H(t, x) + G(t, (\phi_H^t)^{-1}(x)). \tag{2.1.2}$$

Now, given two classes $a, b \in QH(X; \Lambda)$, we have

$$c(H, a) + c(G, b) \geq c(H \# G, a \ast b). \tag{2.1.3}$$

**Valuation.** For any $a \in QH(X; \Lambda) \setminus \{0\}$,

$$c(0, a) = \nu(a) \tag{2.1.4}$$

where $0$ is the zero-function and $\nu : QH(X; \Lambda) \to \mathbb{R}$ is the natural valuation function

$$\nu : QH(X; \Lambda) \to \mathbb{R}$$

$$\nu(a) := \nu(\sum_{j=1}^{\infty} a_j T^{\lambda_j}) := \min\{\lambda_j : a_j \neq 0\}. \tag{2.1.5}$$

**Spectral norm.** The spectral pseudo-norm for a Hamiltonian $H$ is the sum of the spectral invariants for $H$ and the unit $1_X \in QH(X; \Lambda)$ of the quantum cohomology ring and for the inverse Hamiltonian $\overline{H}(t, x) := -H(t, (\phi_H^{-1})^t(x))$, which generates the Hamiltonian path $t \mapsto (\phi_H^{-1})^{-1}$, and the unit $1_X \in QH(X; \Lambda)$:

$$\gamma(H) := c(H, 1_X) + c(\overline{H}, 1_X). \tag{2.1.6}$$

Now, the spectral norm $\gamma(\phi)$ for a Hamiltonian diffeomorphism $\phi$ is defined as follows.
\[
\gamma : \text{Ham}(X) \longrightarrow \mathbb{R}
\]

\[
\phi \mapsto \gamma(\phi) := \inf_{\phi_H = \phi} \gamma(H).
\]

(2.1.7)

It is well-known that the spectral norm defines a non-degenerate metric
\[
d_{\gamma}(\phi, \phi') := \gamma(\phi - 1 \circ \phi')
\]
on \text{Ham}(X).

Entov–Polterovich quasimorphisms. Entov–Polterovich constructed a special map called the quasimorphism on \( \widetilde{\text{Ham}}(X) \) for under some assumptions via spectral invariants, which we will briefly review. Recall that a quasimorphism \( \mu \) on a group \( G \) is a map to the real line \( \mathbb{R} \) that satisfies the following two properties:

1. There exists a constant \( C > 0 \) such that
\[
|\mu(f \cdot g) - \mu(f) - \mu(g)| < C
\]
for any \( f, g \in G \).

2. For any \( k \in \mathbb{Z} \) and \( f \in G \), we have
\[
\mu(f^k) = k \cdot \mu(f).
\]

(2.1.9)

Theorem 2.1.3 ([EP03]). Suppose \( QH(X; \Lambda) \) has a field factor, i.e.
\[
QH(X; \Lambda) = Q \oplus A
\]
where \( Q \) is a field and \( A \) is some algebra. Decompose the unit \( 1_X \) of \( QH(X; \Lambda) \) with respect to this split, i.e.
\[
1_X = e + a.
\]
Then, the asymptotic spectral invariant of \( \widetilde{\phi} \) with respect to \( e \) defines a quasimorphism, i.e.
\[
\zeta_e : \widetilde{\text{Ham}}(X) \longrightarrow \mathbb{R}
\]

\[
\zeta_e(\widetilde{\phi}) := \lim_{k \to +\infty} \frac{c(H^{\#_k}, e)}{k}
\]

(2.1.10)

where \( H \) is any mean-normalized Hamiltonian such that the path \( t \mapsto \phi^1_H \) represents the class \( \widetilde{\phi} \) in \( \text{Ham}(X) \) and \( H^{\#_k} \) is the \( k \)-th catenation of \( H \), \( H^k := H\#H\#\cdots\#H \).

Remark 2.1.4. When we consider Entov–Polterovich quasimorphisms, we take the Laurent coefficients \( \Lambda_{\text{Lau}} \) (see Section 4.2) but the condition in Theorem 2.1.3 is equivalent for \( \Lambda_{\text{Lau}} \) and \( \Lambda_{\text{Nov}} \), see [EP08].

Example 2.1.5. As for the quadric hypersurface \( Q^n \), the quantum cohomology is semi-simple [Abr00] and with the Laurent coefficients, it splits into a direct sum of two fields (which could be easily seen by the fact that the minimal Chern number \( N_{Q^n} = n \))
\[
QH(Q^n; \Lambda_{\text{Lau}}) = Q_+ \oplus Q_-,
\]

(2.1.11)
where the unit $1_{Q^n}$ splits as

$$1_{Q^n} = e_+ + e_-,$$

$$e_{\pm} := \frac{1_{Q^n} \pm PD([pt])t}{2}.$$  (2.1.12)

Thus, we get two Entov–Polterovich quasimorphisms

$$\zeta_{\pm} := \zeta_{e_{\pm}} : \widetilde{\text{Ham}}(Q^n) \to \mathbb{R}$$  (2.1.13)

which we prove that they are distinct in Theorem \[A\].

**Remark 2.1.6.** By slight abuse of notation, we will also see $\zeta_e$ as a function on the set of time-independent Hamiltonians:

$$\zeta_e : C^\infty(X) \to \mathbb{R}$$

$$\zeta_e(H) := \lim_{k \to +\infty} \frac{c(H^k, e)}{k}.$$  (2.1.14)

**Spectral rigidity.** Not only that Entov–Polterovich defined quasimorphisms on $\widetilde{\text{Ham}}(X)$, they introduced a level of rigidity for subsets in $X$ called (super)heaviness.

**Definition 2.1.7** ([EP09],[EP06]). Take an idempotent $e \in QH(X; \Lambda)$ and denote the asymptotic spectral invariant with respect to $e$ by $\zeta_e$. A subset $S$ of $X$ is called

1. $e$-heavy (or heavy with respect to $e$) if for any time-independent Hamiltonian $H : X \to \mathbb{R}$, we have

$$\inf_{x \in S} H(x) \leq \zeta_e(H),$$

2. $e$-superheavy (or superheavy with respect to $e$) if for any time-independent Hamiltonian $H : X \to \mathbb{R}$, we have

$$\zeta_e(H) \leq \sup_{x \in S} H(x).$$

The following is an easy corollary of the definition of superheaviness which is useful.

**Proposition 2.1.8** ([EP09]). Assume the same condition on $QH(X; \Lambda)$ as in Theorem 2.1.3. Let $S$ be a subset of $X$ that is $\zeta$-superheavy. For a time-independent Hamiltonian $H : X \to \mathbb{R}$ whose restriction to $S$ is constant, i.e. $H|_S \equiv r$, $r \in \mathbb{R}$, we have

$$\zeta(H) = r.$$

We end this section by giving a criterion for heaviness for Lagrangians, proved by Fukaya–Oh–Ohta–Ono (there are earlier results with less generality, c.f. [Alb05]). Let $L$ be a monotone Lagrangian and suppose its Floer cohomology $HF^*(L, \rho)$ equipped with a $C^*$-local system $\rho : H_1(L; \mathbb{Z}) \to C^*$, c.f. [BC12], is non-zero. There is a ring homomorphism called the (length 0) closed-open map

$$\mathcal{CO}^0 : QH(X; \Lambda) \to HF^*(L, \rho)$$

which is a quantum analogue of the restriction map.

---

4The original notation used in [FOOO09] for $\mathcal{CO}^0$ is $i^{*}_{qm}$. 

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Theorem 2.1.9 ([FOOO19 Theorem 1.6]). If \( CO^0(e) \neq 0 \) for an idempotent \( e \in \text{QH}(X; \Lambda) \), then \( L \) is \( \zeta_e \)-heavy.

Remark 2.1.10. When \( \zeta_e \) is homogeneous, e.g. when \( e \) is a unit of a field factor of \( \text{QH}(X; \Lambda) \) and \( \zeta_e \) is an Entov–Polterovich quasimorphism, then heaviness and superheaviness are equivalent so Theorem 2.1.9 will be good enough to obtain the superheaviness of \( L \).

2.2 Biran decomposition

In this section, we briefly review the notion of Biran decomposition, which was established by Biran in [Bir01], while comparing the conventions in [Bir01] and [OU16]. We emphasis that in this paper, the convention of Oakley–Usher [OU16] is used.

We start with the Biran’s setting [Bir01 page 412]. Let \( X \) be a symplectic Kähler manifold \( X = (X, \omega_{\text{Bir}}) \) with an integral symplectic form, i.e. \( [\omega_{\text{Bir}}] \in H^2(X; \mathbb{Z}) \). Consider a (Donaldson) divisor \( \Sigma \) which gives a polarization of degree \( k \), i.e. the pair \( (X, \Sigma) \) satisfies

\[
P \text{D}([\Sigma]) = k[\omega_{\text{Bir}}] \in H^2(X; \mathbb{Z}).
\]

The complement \( X \setminus \Sigma \) has the structure of a Stein manifold and thus, one can define the unstable set of a plurisubharmonic function which is called the skeleton \( \Delta \). Biran proved that the complement of the skeleton \( \Delta \), namely \( X \setminus \Delta \), is symplectomorphic to a certain disk bundle (defined with appropriate connection/curvature: we require that the connection 1-form \( \alpha_{\text{Bir}} \) satisfies \( \int_{\partial D^2} = \{ |z| = 1 \} \alpha_{\text{Bir}} = 1 \)):

\[
D^2 = \{ w = r \cdot e^{2\pi \sqrt{-1} \theta} \in \mathbb{C} : r = |w| \leq 1 \} \hookrightarrow D \Sigma \xrightarrow{\pi} \Sigma
\]

\[w \mapsto (w, \zeta) \mapsto \zeta,
\]

equipped with the symplectic form

\[
\omega_{\text{can}, \text{Bir}} := \pi^* (k \omega_{\text{Bir}}|_{\Sigma}) + d(r^2 \alpha_{\text{Bir}}),
\]

where \( \omega_{\text{Bir}}|_{\Sigma} := i^* \omega_{\text{Bir}}, i : \Sigma \hookrightarrow X \), that is,

\[
(X \setminus \Delta, \omega_{\text{Bir}}) \xrightarrow{\sim} (D \Sigma, \frac{1}{k} \omega_{\text{can}, \text{Bir}}).
\]

Note that the area of a fiber disk of the disk bundle \( (D \Sigma, \omega_{\text{can}, \text{Bir}}) \) is

\[
\int_{D^2 = \{ |z| < 1 \}} d(r^2 \alpha_{\text{Bir}}) = \int_{\partial D^2} r^2 \alpha_{\text{Bir}} = 1.
\]

Thus, for the Biran decomposition with Biran’s convention \( (D \Sigma, \frac{1}{k} \omega_{\text{can}, \text{Bir}}) \), the area of the fiber disk is \( \frac{1}{k} \), i.e. the radius of the disk is \( 1/\sqrt{\pi k} \).

Next, we explain Oakley–Usher’s convention in [OU16]. The symplectic form of \( X \) \( \omega_{\text{OU}} \) is scaled so that

\[
\omega_{\text{OU}} = 2\pi \omega_{\text{Bir}}.
\]
Note that unlike $\omega_{\text{Bir}}$, $\omega_{\text{OU}}$ does not represent an integral cohomology class, i.e. $[\omega_{\text{OU}}] \notin H^2(X; \mathbb{Z})$. Oakley–Usher describes the aforementioned Biran decomposition from the relation

$$\tau \cdot PD([\Sigma]) = [\omega_{\text{OU}}]$$

for $\tau > 0$. Note that from the equations (2.2.1), (2.2.6), and (2.2.5), we obtain

$$k = \frac{2\pi}{\tau}.$$  

(2.2.7)

The Biran decomposition with respect to Oakley–Usher’s convention is as follows:

$$(X \setminus \Delta, \omega_{\text{OU}}) \simeq (D\Sigma, \frac{1}{k} \omega_{\text{can};\text{OU}})$$

(2.2.8)

where

$$\omega_{\text{can};\text{OU}} := \pi^*(k \omega_{\text{OU};\Sigma}) + d(r^2 \alpha_{\text{OU}})$$

$$\omega_{\text{OU};\Sigma} := i^* \omega_{\text{OU}}$$

where $i: \Sigma \hookrightarrow X$ and $\alpha_{\text{OU}}$ is the connection 1-form satisfying

$$\alpha_{\text{OU}} = 2\pi \alpha_{\text{Bir}}.$$  

(2.2.9)

From the equation (2.2.5), we have

$$\omega_{\text{can};\text{OU}} = 2\pi \omega_{\text{can};\text{Bir}}$$

and thus, by using (2.2.7), we have

$$\frac{1}{k} \omega_{\text{can};\text{OU}} = \frac{2\pi}{k} \omega_{\text{can};\text{Bir}} = \tau \omega_{\text{can};\text{Bir}}$$

(2.2.10)

which means that for the Biran decomposition with Oakley–Usher’s convention $(D\Sigma, \frac{1}{k} \omega_{\text{can};\text{OU}})$, the area of the disk is $\tau$, i.e. the radius of the fiber disk is $\sqrt{\tau/\pi}$.

We look at the examples we will be focusing on.

**Example 2.2.1.** For $X = \mathbb{C}P^n$, $\omega_{\text{Bir}}$ is the Fubini–Study form scaled so that

$$\int_{\mathbb{C}P^1} \omega_{\text{Bir}} = 1.$$  

whereas $\omega_{\text{OU}}$ satisfies

$$\int_{\mathbb{C}P^1} \omega_{\text{OU}} = 2\pi.$$  

For $X = \mathbb{Q}^n$, which is the main case of our interest, $\omega_{\text{Bir}}$ and $\omega_{\text{OU}}$ are

$$\omega_{\text{Bir}} = i^* \omega_{\mathbb{C}P^{n+1};\text{Bir}};$$

$$\omega_{\text{OU}} = i^* \omega_{\mathbb{C}P^{n+1};\text{OU}}$$

(2.2.11)

where $i: \mathbb{Q}^n \hookrightarrow \mathbb{C}P^{n+1}$.

Now, we look at some polarizations.
1. \((X, \Sigma) = (\mathbb{C}P^n, \mathbb{C}P^{n-1} = \{z_{n+1} = 0\})\), \((Q^n, Q^{n-1} = \{z_{n+1} = 0\} \cap Q^n)\): As the degree of this polarization is \(k = 1\), we have

\[
\tau = 2\pi,
\]

which implies that the fiber disk has radius \(\sqrt{2}\).

2. \((X, \Sigma) = (\mathbb{C}P^n, Q^{n-1})\): As the degree of this polarization is \(k = 2\), we have

\[
\tau = 2\pi/2 = \pi,
\]

which implies that the fiber disk has radius 1.

Biran decomposition has been also known as a nice way to construct interesting Lagrangians, which we call the Biran circle bundle construction and explain what it is. Let \(L\) be a Lagrangian submanifold in \(\Sigma\). Consider the radius \(r > 0\) circle bundle associated to the disk bundle \(D\Sigma\)

\[
D\Sigma_{|u|=r} := \{u \in D\Sigma : |u| = r\}.
\]

The set

\[
\tilde{L}_r := \pi_{|u|=r}^{-1}(L), \quad \pi_{|u|=r} : D\Sigma_{|u|=r} \to \Sigma
\]

defines a Lagrangian submanifold in \(D\Sigma\), which is a circle bundle over \(L\). Note that \(\pi_{|u|=r}\) denotes the restricted projection \(D\Sigma_{|u|=r} \to \Sigma\). Via the symplectic identification \((2.2.4)\), we can see \(\tilde{L}_r\) as a Lagrangian submanifold in \(X \setminus \Sigma\) or \(X\).

When \(L\) is a monotone in \(\Sigma\), then there is a distinguished radius \(r_0 > 0\) for which the lifted Lagrangian submanifold \(\tilde{L}\) becomes also monotone in \(X\) and according to [BC09, Proposition 6.4.1], it satisfies

\[
\left(2.2.12\right)
\]

\[
\tau^2_0 = \frac{2\kappa_L}{2\kappa_L + 1}
\]

where \(\kappa_L\) is the monotonicity constant for \(L\) in \(\Sigma\), i.e. \(\omega_{\Sigma}|_{\pi_2(\Sigma, L)} = \kappa_L \cdot \mu_L|_{\pi_2(\Sigma, L)}\). We sometimes call the radius \(r_0\) the monotone radius as well.

In the following, the lifted Lagrangian submanifold \(\tilde{L} := \tilde{L}_{r_0}\) will always be this distinguished monotone Lagrangian submanifold in \(X\).

### 2.3 Various ways to see quadrics

The aim of this section is to see that the quadric hypersurface \(Q^n\) can be identified to some coadjoint orbit \(O_\lambda\) of the Lie algebra \(\mathfrak{so}(n + 1; \mathbb{R})\).

First of all, as Oakley–Usher it is convenient to see \(Q^n\) as the following quotient: first, we have

\[
\mathbb{C}P^n = \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^*
\]

\[
= \{z \in \mathbb{C}^{n+1} : |z| = 1\}/S^1.
\]

Then,
\[ Q^n = \{ z \in \mathbb{C}P^{n+1} : z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0 \} \]
\[ = \{ z \in \mathbb{C}^{n+2} : |z| = 2, z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0 \}/S^1. \]  
(2.3.2)

This is the Oakley–Usher way to see \( Q^n \). Now, by writing \( z = x + iy \), we have

\[ Q^n = \{ z \in \mathbb{C}^{n+2} : |z| = 2, z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0 \}/S^1. \]  
(2.3.3)

Thus, a point in \( Q^n \) is a orthogonal frame \( x, y \) in \( \mathbb{R}^{n+2} \) where the rotations are identified, thus this defines a plane in \( \mathbb{R}^{n+2} \). Thus,

\[ Q^n = Gr_{\mathbb{R}}^+(2, n+2), \]

where \( Gr_{\mathbb{R}}^+(2, n+2) \) denotes the real oriented Grassmanian. Now, the group \( SO(n+2; \mathbb{R}) \) acts transitively on \( Gr_{\mathbb{R}}^+(2, n+2) \) and the isotropy subgroup of the plane spanned by \( (1, 0, \cdots, 0) \) and \( (0, 1, 0, \cdots, 0) \) is \( SO(2; \mathbb{R}) \times SO(n; \mathbb{R}) \), thus,

\[ Q^n = Gr_{\mathbb{R}}(2, n+2) = SO(n+2; \mathbb{R})/(SO(2; \mathbb{R}) \times SO(n; \mathbb{R})). \]  
(2.3.4)

Now, \( G := SO(n+2; \mathbb{R}) \) also acts transitively on \( so(n+2; \mathbb{R}) = \{ A : A^t = -A \} \) by the adjoint action. We try to find an element \( \lambda \in so(n+2; \mathbb{R}) \) so that the stabilizer of this action equals the stabilizer of the previous action, namely \( SO(2; \mathbb{R}) \times SO(n; \mathbb{R}) \). In order to do this, we can consider

\[
\lambda := \begin{pmatrix}
0 & \lambda_1 & 0 & \cdots & 0 \\
-\lambda_1 & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix} \in so(n+2; \mathbb{R}).
\]  
(2.3.5)

Thus, we now have

\[ SO(n+2; \mathbb{R})/SO(2; \mathbb{R}) \times SO(n; \mathbb{R}) \cong O_\lambda. \]  
(2.3.6)

From (2.3.4) and (2.3.6), we conclude

\[ Q^n \cong O_\lambda. \]  
(2.3.7)

To avoid confusion, we will denote this identification by \( A \):

\[ A : Q^n \cong O_\lambda, \]
\[ z \mapsto A(z). \]  
(2.3.8)

The coadjoint orbits are known to possess a canonical symplectic structure given by the so-called Kirillov–Kostant–Souriau (KKS) form \( \omega_{KKS} \) which satisfies

\[ [\omega_{KKS}] = c_1(TO_\lambda) \]

and thus via (2.3.8), one obtains a KKS form on \( Q^n \). The symplectic form we work with in this paper needs to be appropriately scaled, see Section 2.5 for this matter.
2.4 Gelfand–Zeitlin system for quadrics

In this section, we will review the completely integrable system called the Gelfand–Zeitlin (GZ) system for the quadric $Q^n$ via (2.3.8), as explained in [KimA, Section 2.2].

For any $z \in Q^n$, for each $2 \leq k \leq n + 2$, we take the left-upper $k \times k$-submatrix of $A(z)$ which we will denote by $A(z)^{(k)}$. As $A(z)$ is a skew-symmetric $(n + 2) \times (n + 2)$ matrix, each $A(z)^{(k)}$ is also skew-symmetric, and thus its eigenvalues are either all 0 or $\pm \sqrt{-\nu(z)^{(k)}}$, $0 \cdots, 0$ where $\nu(z)^{(k)} > 0$. Now, define the following map:

$$\Phi : Q^n \longrightarrow \mathbb{R}^n$$

where

$$\lambda_1^{(k)}(z) := \begin{cases} 
\nu^{(k+1)}(z) & \text{if either } k \geq 2 \text{ or } (k = 1 \text{ and } Pf(A(z)^{(2)}) \geq 0) \\
-\nu^{(2)}(z) & k = 1 \text{ and } Pf(A(z)^{(2)}) < 0,
\end{cases}$$

where $Pf$ is the Phaffian.

Guillemin–Sternberg proved that $\Phi$ forms a completely integrable system.

**Theorem 2.4.1 ([GS83]).** The map $\Phi = (\Phi_1, \cdots, \Phi_n)$ in (2.4.1) is a completely integrable system on $Q^n$. In fact, the $k$-th component $\Phi_k$ generates a Hamiltonian $S^1$-action on $\Phi^{-1}(\{u = (u_1, \cdots, u_n) \in \mathbb{R}^n : u_k \neq 0\})$.

In [NNU, Proposition 3.1], Nishinou–Nohara–Ueda computed the GZ system $\{\lambda_1^{(k)}\}_{2 \leq k \leq n + 1}$ for $Q^n$

$$\lambda_1^{(k)} : Q^n \longrightarrow \mathbb{R}$$

which is as follows:

$$\lambda_1^{(2)}(z) = \frac{\lambda}{|z|^2}(z_1\bar{z}_2 - \bar{z}_1z_2),$$

$$(\lambda_1^{(k)})^2(z) = -\sum_{1 \leq i < j \leq k} \left(\frac{z_i\bar{z}_j - \bar{z}_i z_j}{|z|^2}\right)^2$$
$$= \left(\frac{\lambda}{|z|^2}\right)^2 \left(k\sum_{j=1}^k |z_j|^2 - \sum_{j=1}^k |z_j|^2\right), \quad k \geq 3,$$

$$\lambda := 2.$$

**Remark 2.4.2.** In Kim’s paper [KimA], he uses the notation $\{u_k\}_{1 \leq k \leq n}$, where the correspondence is given by

$$\lambda_1^{(k)} = u_{k-1}.$$

The monotone GZ fiber is expressed as follows [KimA, Proposition 3.7]:

$$T^n_{GZ} := \Phi_Q^{-1} \left(0, \frac{1}{n}, 2/2, \cdots, 2/n, \frac{n-1}{n}\right)$$

where we used the notation

$$\Phi_Q : Q^n \longrightarrow \mathbb{R}^n$$

$$\Phi_Q(z) := (\lambda_1^{(2)}, \lambda_1^{(3)}, \cdots, \lambda_1^{(n+1)})(z).$$
2.5 Convention

In this section, we will remind some conventions that is used in Section 3 and 4, which are taken from [OU16].

For the quadric hypersurface $Q^n \subset \mathbb{CP}^{n+1}$, we equip a symplectic form $\omega$ such that

$$\omega = i^* \omega_{\mathbb{CP}^{n+1}}$$

where

$$i : Q^n \hookrightarrow \mathbb{CP}^{n+1}$$

is the inclusion and $\omega_{\mathbb{CP}^{n+1}}$ is the Fubini–Study form scaled so that

$$\int_{\mathbb{CP}^{n+1}} \omega_{\mathbb{CP}^{n+1}} = 2\pi.$$

This makes the monotonicity constant to be equal to $\frac{2\pi}{n}$, i.e.

$$\omega|_{\pi_2(Q^n)} = \frac{2\pi}{n} \cdot c_1(TQ^n)|_{\pi_2(Q^n)}.$$

In [KimA] and [NNU], they scale the symplectic form on $Q^n$ so that the monotonicity constant becomes 1. In fact, they use the Kirillov–Kostant–Souriau (KKS) form $\omega_{\text{KKS}}$ that satisfies

$$[\omega_{\text{KKS}}] = c_1(TQ^n).$$

These different choices of the normalization cause some rescaling in the results in [KimA] and [NNU] and in this paper, we use their results in the scaled form which amounts to putting $\lambda = 2\pi$ instead of $\lambda = n$ in their results.

3 Proofs of Theorem B (1), (2)

3.1 Toric degeneration

In this section, as we briefly review toric degenerations of symplectic manifolds and of integrable systems. The latter is a special class of the former which was introduced by Nishinou–Nohara–Ueda in [NNU10] that makes the integrable structure on a symplectic manifold and the toric integrable structure on its degenerated toric variety compatible. We study the particular case of the quadric hypersurface $Q^n$ and we specify the toric degeneration for the quadric hypersurface $Q^n$ which we will use in this paper. We define two distinguished Lagrangians in $Q^n$, namely the monotone GZ torus $T_{\text{GZ}}^n$ and the Lagrangian sphere $S_{\text{van}}^n := \{ x \in \mathbb{CP}^{n+1} : x_0^2 + x_1^2 + \cdots + x_{n-1}^2 = x_n^2, \ x_j \in \mathbb{R} \}$. We also discuss its superpotential, which was computed by Y. Kim in [KimA].

Definition 3.1.1 ([HK15 Definition 2], [Eva, Section 1]). A toric degeneration of a symplectic manifold $X$ is a flat family $\pi : \tilde{X} \to \mathbb{C}$ whose fibers $\{ X_t := \pi^{-1}(t) \}_{t \in \mathbb{C}}$ satisfy the following properties:

1. For $t \neq 0$, the fiber $X_t$ is smooth and $X_1$ is isomorphic to $X$.
2. For $t = 0$, the fiber $X_0$ is a toric variety that is not smooth.
3. The fibres $X_t$ are projective subvarieties of the same projective space, i.e. there is a morphism $f : X \to \mathbb{C}P^N$ such that for every $t \in \mathbb{C}$, $f|_{X_t} : X_t \to \mathbb{C}P^N$ is an embedding.

Remark 3.1.2. In fact, it turns out that all the smooth fibers are symplectomorphic to each other; see [Eva, Lemma 1.1].

We provide an example of a toric degeneration for the quadric hypersurface $Q^n$ considered by Nishinou–Nohara–Ueda in [NNU], which we will be using throughout the paper.

Example 3.1.3. Consider the following:

$$X := \{(z, t) \in \mathbb{C}P^{n+1} \times \mathbb{C} : z_0^2 + z_1^2 + z_2^2 + t(z_3^2 + \cdots + z_{n+1}^2) = 0\}. \quad (3.1.1)$$

We define

$$\pi : X \to \mathbb{C}
\begin{align*}
(z, t) \mapsto t.
\end{align*} \quad (3.1.2)$$

Then for $t = 1$, the fiber $X_1$ is isomorphic to $Q^n$ and for $t = 0$, the fiber $X_0$ is a singular toric variety (that is isomorphic to the weighted projective space $\mathbb{C}P(1, 1, 1, 2, \cdots, 2)$), which is an orbifold whose singular locus $X_0^{\text{sing}}$ is

$$X_0^{\text{sing}} = \{[0 : 0 : 0 : z_3 : \cdots : z_{n+1}] \in \mathbb{C}P^{n+1}\}. \quad (3.1.3)$$

Remark 3.1.4. In the toric degeneration considered in Example 3.1.3, the singular locus (3.1.3) is not an isolated set (each singular point is a transverse $A_1$-singularity). This is very different from the toric degenerations for del Pezzo surfaces that we will consider in Section 5 and makes the study of the geometry of vanishing cycles more difficult.

We now define a toric degeneration of a completely integrable system.

Definition 3.1.5 ([NNU10, Definition 1.1]). Let $X^{2n} = (X, \omega)$ be a symplectic manifold and $\Phi : X \to \mathbb{R}^n$ a completely integrable system with respect to $\omega$. A toric degeneration of the completely integrable system is a toric degeneration of $X$

$$\pi : \mathcal{X} \longrightarrow \mathbb{C}$$

with the following data:

- a (piecewise smooth) path $\gamma : [0, 1] \to \mathbb{C}$, $\gamma(0) = 1$, $\gamma(1) = 0$,
- a continuous map $\tilde{\Phi} : \mathcal{X}|_{\gamma([0, 1])} \to \mathbb{R}^n$,
- a flow $\phi_t$ on $\mathcal{X}|_{\gamma([0, 1])}$ defined away from the singular locus of $\mathcal{X}|_{\gamma([0, 1])}$,

satisfy the following properties:
1. For each $t$, $\Phi_t := \Phi|_{X_t} : X_t \to \mathbb{R}^n$ defines a completely integrable system where for $t = 0$, it coincides with the toric system $\Phi_0 : X_0 \to \mathbb{R}^n$ and for $t = 1$, it coincides with $\Phi : X \to \mathbb{R}^n$.

2. Away from the singular locus, the flow $\phi_t$ restricts to a symplectomorphism between $X_s$ and $X_{s-t}$ that preserves the integrable system:

$$
\begin{array}{c}
(X_s, \omega_s) \\
\Phi_t \\
\Phi_{s-t}
\end{array} 
\xrightarrow{\phi_t} 
\begin{array}{c}
(X_{s-t}, \omega_{s-t}) \\
\mathbb{R}^n
\end{array}
$$

We now would like to construct a toric degeneration for the GZ system on $Q^n$. The methods of Nishinou–Nohara–Ueda from [NNU10] and Harada–Kaveh from [HK15] which are based on the gradient-Hamiltonian vector field (due to Ruan [Rua01]) allow us to do that starting from the toric degeneration of $Q^n$ in Example 3.1.3 which is of our main interest.

We briefly review Ruan’s idea. Let $\mathcal{X}$ be an algebraic variety equipped with a Kähler form $\tilde{\omega}$. Let $\pi : \mathcal{X} \to \mathbb{C}$ be a morphism (i.e. an algebraic map) and $\nabla \text{Re}(\pi)$ be the gradient vector field on (the smooth locus of) $\mathcal{X}$ with respect to the Kähler form $\tilde{\omega}$ where $\text{Re}(\pi)$ is the real part of the holomorphic function $\pi$. Now, the gradient-Hamiltonian vector field $V$ on the smooth locus of $\mathcal{X}$ is defined by

$$
V := -\frac{\nabla \text{Re}(\pi)}{||\nabla \text{Re}(\pi)||^2}.
$$

From the normalization, it follows that

$$
V(\text{Re}(\pi)) = -1.
$$

Although the flow of the gradient-Hamiltonian vector field $V$ is not complete due to it not being defined on the singular loci, Harada–Kaveh [HK15] proved that one can extend the flow continuously on the whole $\mathcal{X}$ and obtain

$$
\phi^t : \mathcal{X} \to \mathcal{X}
$$

which satisfies $\phi^t(X_t) = X_0$.

By using this gradient-Hamiltonian flow applied to the toric degeneration of $Q^n$ in Example 3.1.3 Nishinou–Nohara–Ueda obtained a toric degeneration for the GZ system on $Q^n$.

**Proposition 3.1.6** ([NNU Proposition 3.1], [KimA Proposition 2.6]). The toric degeneration $\pi : \mathcal{X} \to \mathbb{C}$ in Example 3.1.3 together with the gradient-Hamiltonian flow $\phi_t$ of $\pi$ defines a toric degeneration of the GZ system on $Q^n$ (by taking the path $\gamma(t) := 1 - t$ and $\Phi_t := \Phi_0 \circ \phi_t$.)

The compatibility between the GZ system on $X = Q^n$ and the toric system on $X_0(\simeq \mathbb{C}P(1,1,1,2,\cdots, 2))$ implies that the monotone GZ torus is sent to a toric torus
through the toric degeneration with the gradient-Hamiltonian flow, i.e. the monotone GZ torus fiber $T_{GZ}^n = \Phi^{-1}(x_0)$ satisfies

$$\phi_1(T_{GZ}^n) = \Phi^{-1}(x_0),$$

where $x_0$ is the barycenter of the moment polytope of $\Phi_0$.

The map $\phi_1$ allows us to define the vanishing loci.

**Definition 3.1.7.** Let $S$ be a subset of the singular locus of $X_0$, i.e. $S \subset X_0^{\text{sing}}$. The vanishing locus corresponding to $S$ is the set $\phi_1^{-1}(S)$.

Now, we get back to the toric degeneration (3.1.3), (3.1.6) for $Q^n$ which is of our specific interest.

**Proposition 3.1.8.** The Lagrangian sphere

$$S^n = \{x \in \mathbb{C}P^{n+1} : x_0^2 + x_1^2 + \cdots + x_{n-1}^2 = x_n, \ x_j \in \mathbb{R}\} \subset Q^n$$

is the vanishing locus of the set

$$\{[0 : 0 : x_3 : \cdots : x_n : ix_{n+1}] \in \mathbb{C}P^{n+1} : x_j \in \mathbb{R}, \ \forall 3 \leq j \leq n + 1\} \subset X_0^{\text{sing}}.$$

**Remark 3.1.9.** Because of Proposition 3.1.8 we will denote the sphere $S^n$ by $S^n_{\text{van}}$.

**Proof of Proposition 3.1.8.** Consider the following anti-symplectic involution:

$$\tau : \mathcal{X} \to \mathcal{X},$$

$$([z_0 : z_1 : \cdots : z_n : z_{n+1}], t) \mapsto ([\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n} : -\overline{z_{n+1}}], \overline{t}).$$

(3.1.8)

For each $t \in [0, 1]$, this restricts to an anti-symplectic involution on $X_t :$

$$\tau : X_t \to X_t,$$

$$([z_0 : z_1 : \cdots : z_n : z_{n+1}], t) \mapsto ([\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n} : -\overline{z_{n+1}}], \overline{t}).$$

(3.1.9)

By using that the Kähler metric $g$ (corresponding to the Kähler form $\bar{\omega} = \omega_{FS} \oplus \omega_{\text{stand}}$) on $\mathcal{X}$ is preserved by the anti-symplectic involution $\tau$, i.e. $\tau^* g = g$, we can show that the gradient-Hamiltonian vector field $V$ along $\mathcal{X}|_{[0,1]}$ is also preserved by the anti-symplectic involution $\tau$, i.e. $\tau_* V(z, t) = V(z, t)$ for $(z, t) \in \mathcal{X}|_{[0,1]}$ by

$$g(\tau_* \nabla \text{Re}(\pi), W) = (\tau^* g)(\nabla \text{Re}(\pi), \tau_*^{-1} W)$$

$$= g(\nabla \text{Re}(\pi), \tau W)$$

$$= d\text{Re}(\pi)(\tau_* W)$$

$$= d(\text{Re}(\pi) \circ \tau)(W)$$

$$= d\text{Re}(\pi)(W)$$

$$= g(\nabla \text{Re}(\pi), W)$$

for any $W \in T_{(z,t)}\mathcal{X}$ with $(z, t) \in \mathcal{X}|_{[0,1]}$. Note that the fifth line uses that $(z, t) \in \mathcal{X}|_{[0,1]}$ (more precisely, that $t \in \mathbb{R}$).
Thus, \[ \phi^t(\text{Fix}(\tau) \cap X_1) = \text{Fix}(\tau) \cap X_{1-t}, \]
where
\[ \text{Fix}(\tau) \cap X_{1-t} = \{ x \in \mathbb{C}P^{n+1} : x_0^2 + x_1^2 + x_2^2 + (1 - t)(x_3^2 + \cdots + x_n^2 + (ix_{n+1})^2) = 0, \; x_j \in \mathbb{R} \}. \] (3.1.10)

The \( t = 1 \) case will give us that
\[ \phi^1(\text{Fix}(\tau) \cap X_1) = \text{Fix}(\tau) \cap X_0, \]
where
\[ \text{Fix}(\tau) \cap X_1 = \{ (x, 1) \in \mathbb{C}P^{n+1} \times \mathbb{C} : x_0^2 + x_1^2 + \cdots + x_n^2 = x_{n+1}^2 \}, \]
which is equivalent to the Lagrangian sphere in \( Q^n \) that we are interested in, namely
\[ S^n = \{ x \in \mathbb{C}P^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = x_{n+1}^2, \; x_j \in \mathbb{R} \} \]
under the identification between \( Q^n \) and \( X_1 \), and
\[ \text{Fix}(\tau) \cap X_0 = \{ [0 : 0 : x_3 : \cdots : x_{n} : ix_{n+1}] \in \mathbb{C}P^{n+1} : x_j \in \mathbb{R}, \; \forall 3 \leq j \leq n+1 \} \subset X_0^{\text{sing}}. \]

Thus, \( S^n \) is the vanishing locus of the singular locus
\[ \{ [0 : 0 : x_3 : \cdots : x_{n} : ix_{n+1}] \in \mathbb{C}P^{n+1} : x_j \in \mathbb{R}, \; \forall 3 \leq j \leq n+1 \}. \]

\[ \square \]

**Remark 3.1.10.** The argument in the above proof is essentially the same as [Eva, Lemma 1.20], which is a very useful method to identify vanishing cycles based on [Eva, Lemma 1.17].

**Remark 3.1.11.** Note that \( S^n \) is just a part of the vanishing locus of the singular locus of \( X_0 \) and we can detect other parts of the vanishing locus by taking different anti-symplectic involutions: for example, by looking at the fixed locus of
\[ \tau' : X_t \to X_t \]
\[ ([z_0 : z_1 : \cdots : z_n : z_{n+1}], t) \mapsto ([\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n} : -\overline{z_{n+1}}], t), \] (3.1.11)
we can conclude that
\[ \{ x \in \mathbb{C}P^{n+1} : x_0^2 + x_1^2 + \cdots + x_{n-1}^2 = x_n^2 + x_{n+1}^2, \; x_j \in \mathbb{R} \} \simeq S^1 \times S^{n-1} \]
is also a part of the vanishing locus.

Nishinou–Nohara–Ueda applied toric degenerations of GZ systems to compute the superpotential of the monotone Lagrangian torus fiber for some toric degenerations in [NNU10, NNU], which enlarged the cases where one can compute the superpotential. This idea inspired Fukaya–Oh–Ohta–Ono and Y. Kim to compute the superpotential
of the Chekanov torus in \( S^2 \times S^2 \cong Q^2 \) \cite{FOOO12} and the GZ torus \( T^n_{GZ} \) in \( Q^n \) \cite{KimA}, respectively. We now summarize Y. Kim's work from \cite{KimA}.

According to Y. Kim, the superpotential \( W_{T^n_{GZ}} : (\mathbb{C}^*)^n \to \mathbb{C} \) of \( T^n_{GZ} \) takes the following form:

\[
W_{T^n_{GZ}}(z) = \frac{1}{z_n} + \frac{z_n}{z_{n-1}} + \cdots + \frac{z_2}{z_1} + 2z_2 + z_1z_2.
\] (3.1.12)

This has \( n \) different (non-degenerate) critical points \( \rho_j, \ j = 0, 1, \cdots, n-1 \):

\[
\rho_j := (1, \xi_j^{-(n-1)}, \xi_j^{-(n-2)}, \cdots, \xi_j^{-2}, \xi_j^{-1})
\] (3.1.13)

where \( \xi_j = 4^{1/n} \exp(\frac{2\pi \sqrt{-1}}{n} \cdot j), \ j = 0, 1, \cdots, n-1 \) (each \( \xi_j \) satisfies \( \xi_j^n = 4 \)), and the critical values are as follows:

\[
W_{T^n_{GZ}}(\rho_j) = n \cdot \xi_j.
\] (3.1.14)

This implies that there are \( n \) different local systems \( \rho_j : H_1(T^n_{GZ}; \mathbb{Z}) \to \mathbb{C}^*, \ j = 1, \cdots, n \) so that the Floer homology of \( T^n_{GZ} \) with respect to these local systems are non-zero:

\[
HF(T^n_{GZ}, \rho_j) \neq 0.
\] (3.1.15)

Remark 3.1.12. As Y. Kim points out in \cite{KimA}, Fukaya–Oh–Ohta–Ono’s superpotential for the Chekanov torus in \( S^2 \times S^2 \cong Q^2 \) \cite{FOOO12} coincides with the \( n = 2 \) case of his superpotential (3.1.12) for \( T^n_{GZ} \). This made his ask whether the two dimensional GZ torus \( T^2_{GZ} \) is Hamiltonian isotopic to the Chekanov torus (Question 1.3.1). We will answer this in the positive in Corollary 1.3.3.

3.2 Proof

In this section, we prove the main results, namely Theorems A and B and Corollary 1.3.3. We first look at how Corollary 1.3.3 follows from Theorem B.

Proof of Corollary 1.3.3 First of all, note that \( Q^2 \) is symplectomorphic to the monotone \( S^2 \times S^2 \) (with appropriate normalization), i.e.

\[
\Phi : Q^2 \cong S^2 \times S^2,
\]

and through this identification, the polarization \( (Q^2, Q^1) \) gets translated to the polarization \( (S^2 \times S^2, \Delta) \) where \( \Delta \) denotes the diagonal sphere

\[
\Delta := \{(x, x) \in S^2 \times S^2\},
\]

i.e.

\[
\Phi : (Q^2, Q^1) \cong (S^2 \times S^2, \Delta).
\] (3.2.1)

Now, our aim is to prove that \( \Phi(T^2_{GZ}) \) is Hamiltonian isotopic to the Chekanov torus. From Theorem B we know that \( T^2_{GZ} \) in \( Q^2 \) is equal to the Biran circle fibration of \( T^1_{GZ} \) in \( Q^1 \), i.e.

\[
T^2_{GZ} = \widetilde{T^1_{GZ}}.
\]

5More precisely, the superpotential for a Lagrangian torus \( L \cong T^n \) is a map \( W_L : \text{Hom}(H_1(L; \mathbb{Z}), \mathbb{C}^*) \to \mathbb{C} \), but it is common to identify the set of local systems \( \text{Hom}(H_1(L; \mathbb{Z}), \mathbb{C}^*) \) with \( (\mathbb{C}^*)^n \) by taking a \( \mathbb{Z} \)-basis of \( H_1(L; \mathbb{Z}) \).
The circle $T^1_{GZ}$ in $Q^1$ is the equatorial circle $T^1_{eq}$ via the identification between $Q^1$ and $S^2$, i.e.
$$\Phi(T^1_{GZ}) = T^1_{eq}.$$ 

In [OU16], it was proven that for the polarization $(S^2 \times S^2, \Delta)$, the Lagrangian torus obtained as the Biran circle bundle of the equatorial circle in $\Delta \simeq S^2$, i.e. $T^1_{eq}$ which is denoted by $T_{BC}$ in the paper, is Hamiltonian isotopic to the Chekanov torus (and to the Fukaya–Oh–Ohta–Ono torus, Entov–Polterovich torus, Albers–Frauenfelder torus).

From the equivalence of polarizations (3.2.1), we have
$$\Phi(T^2_{GZ}) = \Phi(T^1_{GZ}) = \Phi(T^1_{eq}) = \tilde{T}^1_{eq} = T_{BC}. \tag{3.2.2}$$

Thus, we conclude that $\Phi(T^2_{GZ})$ is Hamiltonian isotopic to the Chekanov torus in $S^2 \times S^2$.

We now prove the first assertion of Theorem B.

**Proof of Theorem B (1).** Recall from Section 2.4 that the monotone GZ torus $T^n_{GZ}$ is defined as
$$T^n_{GZ} := \Phi^{-1}_{Q^n} \left( (0, 2 \cdot \frac{1}{n}, 2 \cdot \frac{2}{n}, \ldots, 2 \cdot \frac{n-1}{n}) \right) \tag{3.2.3}$$

where $\Phi_{Q^n}$ is
$$\Phi_{Q^n} : Q^n \longrightarrow \mathbb{R}^n$$
$$\Phi_{Q^n}(z) := (\lambda^{(2)}, \lambda^{(3)}, \ldots, \lambda^{(n+1)}(z), \ldots) \tag{3.2.4}$$

with $\lambda^{(k)} : Q^n \longrightarrow \mathbb{R}$ such that
$$\lambda^{(2)}(z) = \frac{\lambda}{|z|^2}i(z_1 \overline{z}_2 - \overline{z}_1 z_2),$$
$$(\lambda^{(k)})^2(z) = -\sum_{1 \leq i < j \leq k} \left( \frac{z_i \overline{z}_j - \overline{z}_i z_j}{|z|^2} \right)^2 \tag{3.2.5}$$
$$= \left( \frac{\lambda}{|z|^2} \right)^2 \left( |z_1|^2 - |z_2|^2 \right)^2, k \geq 3,$$
$$\lambda : = 2.$$

From now on, we will be looking at the GZ systems on $Q^n$ and its submanifold $Q^{n-1} = \{ z_{n+1} = 0 \}$ so we will introduce the notation
$$\lambda^{(k)}_{Q^n} : Q^n \longrightarrow \mathbb{R}$$
to denote the GZ system for $Q^n$ and
$$\lambda^{(k)}_{Q^{n-1}} : Q^{n-1} \longrightarrow \mathbb{R}$$
to denote the GZ system for $Q^{n-1}$.
Now, we prove that the monotone GZ fiber in $Q^n$ is equal to the monotone Biran circle fibration of the monotone GZ fiber in $Q^{n-1}$ for the polarization, i.e.

$$T^n_{GZ} = \tilde{T}^{n-1}_{GZ}.$$

Note that apriori, the two geometric constructions, namely the toric degeneration and the Biran decomposition has nothing to do with each other.

The Biran decomposition associated to the polarization $(Q^n, Q^{n-1})$ is expressed in terms of coordinate as follows [OU16, Section 4.2]:

$$DQ^{n-1} \cong Q^n \setminus \Delta$$

$$(w, \zeta) \mapsto [z_0 : \cdots : z_{n+1}] = \left[ \left(1 - \frac{|\zeta|^2}{4}\right) w - \frac{\zeta^2 w}{4} : \left(1 - \frac{|\zeta|^2}{4}\right)^{1/2} \zeta \right], \quad (3.2.6)$$

First, we focus on the Hamiltonian $\lambda^{(n+1)}_{Q^n}$. By using

$$\sum_{j=0}^{n+1} z_j^2 = 0,$$

the expression of $\lambda^{(n+1)}_{Q^n}$ can we rewritten as follows:

$$(\lambda^{(n+1)}_{Q^n})^2 = \left(\frac{\lambda}{|z|^2}\right)^2 \left((\sum_{j=0}^{n} |z_j|^2)^2 - (\sum_{j=0}^{n} |z_j|^2)^2\right)$$

$$= \left(\frac{\lambda}{|z|^2}\right)^2 \left((|z|^2 - |z_{n+1}|^2)^2 - | - z_{n+1}|^2\right)$$

$$= \left(\frac{\lambda}{|z|^2}\right)^2 \left(|z|^4 - 2|z|^2|z_{n+1}|^2\right)$$

$$= \lambda^2 \left(1 - 2\frac{|z_{n+1}|^2}{|z|^2}\right),$$

thus, we have

$$\lambda^{(n+1)}_{Q^n}(z) = \lambda \left(1 - 2\frac{|z_{n+1}|^2}{|z|^2}\right)^{1/2}.$$

By using that $|z|^2 = 2$ and that the symplectomorphism (3.2.6), we further have

$$\lambda^{(n+1)}_{Q^n}(z) = \lambda \left(1 - 2\frac{|z_{n+1}|^2}{|z|^2}\right)^{1/2}$$

$$= \lambda \left(1 - \frac{1}{2} \left(1 - \frac{|\zeta|^2}{4}\right) |\zeta|^2\right)^{1/2}$$

$$= \lambda \left(1 - \frac{|\zeta|^2 + \frac{|\zeta|^4}{4}}{2}\right)^{1/2}$$

$$= \lambda \left(1 - \frac{|\zeta|^2}{2}\right).$$

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From (2.4.4), the monotone GZ fiber satisfies
\[ \lambda^{(n+1)}_{Q^n}(z) = 2 \cdot \frac{n-1}{n} \]
which is equivalent to
\[ \lambda(1 - \frac{|\zeta|^2}{2}) = 2 \cdot \frac{n-1}{n} \]
which is (as \( \lambda = 2 \))
\[ \frac{|\zeta|^2}{2} = \frac{1}{n}. \quad (3.2.9) \]

We will see that this is precisely the monotone radius of the Biran circle bundle construction for \((Q^n, Q^{n-1})\). Indeed, by [OU16, Section 4.4] (see also [BC09, Proposition 6.4.1]), the monotone radius \(r_0\) satisfies
\[ \frac{r_0^2}{2} = \frac{2}{n} = \frac{1}{n} \]
where \(\kappa_{Q^n}\) is the monotonicity constant for \(Q^n\) (see our convention in Section 2.5). Thus,
\[ \frac{r_0^2}{2} = \frac{2\kappa_L}{2\kappa_L + 1} = \frac{2 \cdot 1/2(n-1)}{2 \cdot 1/2(n-1) + 1} = \frac{1}{n-1} \cdot \frac{n-1}{n} = \frac{1}{n} \quad (3.2.10) \]

Now, we shift our focus to \(\lambda^{(k)}_{Q^n}\) where \(2 \leq k \leq n\). By using the symplectomorphism (3.2.6), we rewrite the Hamiltonians \(\lambda^{(k)}_{Q^n}\) as follows:
\[ \lambda^{(k)}_{Q^n}(z) = -\frac{\lambda}{|z|^2} \left( \sum_{1 \leq i < j \leq k} (z_i \overline{z_j} - \overline{z_i} z_j)^2 \right)^{1/2} \]
\[ = -\frac{\lambda}{|z|^2} \left( \sum_{1 \leq i < j \leq k} (2i \text{Im}(z_i \overline{z_j}))^2 \right)^{1/2} \]
\[ = -\lambda \left( \sum_{1 \leq i < j \leq k} (i \text{Im}(z_i \overline{z_j}))^2 \right)^{1/2} \]
\[ = -\lambda \left( \sum_{1 \leq i < j \leq k} \left(1 - \frac{|\zeta|^2}{2}\right) i \text{Im}(w_i \overline{w_j})^2 \right)^{1/2} \]
\[ = -\lambda \left( 1 - \frac{|\zeta|^2}{2} \right) \left( \sum_{1 \leq i < j \leq k} (i \text{Im}(w_i \overline{w_j}))^2 \right)^{1/2} \]
\[ = \left(1 - \frac{|\zeta|^2}{2}\right) \lambda^{(k)}_{Q^{n-1}}(w). \quad (3.2.11) \]
By using (3.2.9), we have
\[
\lambda_{Q^n}(z) = \left(1 - \frac{|\zeta|^2}{2}\right) \lambda_{Q^{n-1}}(w) = \left(1 - \frac{1}{n}\right) \lambda_{Q^{n-1}}(w) = \frac{n-1}{n} \cdot \lambda_{Q^{n-1}}(w).
\] (3.2.12)

Finally, we compare the monotone GZ tori $T^n_{GZ}$ and $T^{n-1}_{GZ}$ by using (3.2.9) and (3.2.12). For $2 \leq k \leq n$, by (2.4.4) the monotone GZ torus $T^n_{GZ}$ satisfies
\[
\lambda_{Q^n}(z) = 2 \cdot \frac{k-2}{n}
\] (3.2.13)
which, according to (3.2.12), is equivalent to
\[
\lambda_{Q^{n-1}}(w) = 2 \cdot \frac{k-2}{n-1}
\] (3.2.14)
which is nothing but the description of $T^{n-1}_{GZ}$. This implies that $T^n_{GZ}$ is the monotone Biran circle fibration (i.e. the Biran circle fibration with radius as in (3.2.9)) over $T^{n-1}_{GZ}$, namely
\[
T^n_{GZ} = \tilde{T}^{n-1}_{GZ}.
\]
This completes the proof of the first assertion of Theorem B.

We now prove the second assertion of Theorem B.

Proof of Theorem B (2). From Proposition 3.1.8 we know that the vanishing sphere $S^n_{\text{van}}$ in $X = X_1$ gets mapped to the singular locus of $X_0$ by $\phi_1$, i.e.
\[
\phi_1(S^n_{\text{van}}) \subset X_0^{\text{sing}}.
\] (3.2.15)

On the other hand, we have seen in (3.1.7) that from Proposition 3.1.6 we get
\[
\phi_1(T^n_{GZ}) = \Phi^{-1}(x_0),
\] (3.2.16)
which implies
\[
\phi_1(T^n_{GZ}) \subset X_0 \setminus X_0^{\text{sing}}.
\] (3.2.17)
The properties (3.2.15) and (3.2.17) imply
\[
S^n_{\text{van}} \cap T^n_{GZ} = \emptyset.
\]
4 Proofs of Theorem \[\textbf{A}, \textbf{B}\] (3)

4.1 Proof–Part 1

Proof of Theorem \[\textbf{B}\] (3). We use the following theorem due to Auroux–Kontsevich–Seidel.

**Theorem 4.1.1** ([Aur07, Section 6], [Sh16, Lemma 2.7, Proposition 2.9]). Let \(X\) be a closed monotone symplectic manifold and let \(L\) be a monotone Lagrangian. Assume that for some local system \(\rho\), we have \(HF(L, \rho) \neq 0\). The (length 0) closed-open map \(\text{CO}^0 : QH(X; \Lambda_{Nov}) \to HF(L, \rho)\) has the following properties:

1. For \(c_1 := c_1(TX)\),
   \[
   \text{CO}^0(c_1) = WL(\rho) \cdot 1_{(L, \rho)}
   \]
   where \(WL(\rho)\) is the value of the superpotential \(W_L : \text{Hom}(H_1(L; \mathbb{Z}), \mathbb{C}^*) \to \mathbb{C}\) of the Lagrangian \(L\) equipped with a local system \(\rho\).

2. Consider the map
   \[
   c_1 * - : QH(X; \Lambda_{Nov}) \to QH(X; \Lambda_{Nov})
   \]
   and split \(QH(X; \Lambda_{Nov})\) into generalized eigenspaces with respect to \(c_1 * -\):
   \[
   QH(X; \Lambda_{Nov}) = \bigoplus_w QH(X; \Lambda_{Nov})_w
   \]
   where \(w\) is an eigenvalue of \(c_1 * -\). The map
   \[
   \text{CO}^0 : QH(X; \Lambda_{Nov})_w \to HF(L, \rho)
   \]
   is zero if \(w \neq WL(\rho)\) and is a unital homomorphism if \(w = WL(\rho)\).

**Remark 4.1.2.** Note that in Theorem 4.1.1 it is important that we take the universal Novikov field

\[
\Lambda_{Nov} := \{ \sum_{j=1}^{\infty} a_j T^{\lambda_j} : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to +\infty} \lambda_j = +\infty \}
\]

for the quantum cohomology instead of the field of the Laurent series

\[
\Lambda_{Lau} := \{ \sum_{k \geq k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C} \},
\]

as \(\Lambda_{Nov}\) is algebraically closed and \(\Lambda_{Lau}\) is not.

Now, we apply Theorem 4.1.1 to \(Q^n\). The eigenvalues of the map

\[
c_1 * - : QH(Q^n; \Lambda_{Nov}) \to QH(Q^n; \Lambda_{Nov})
\]

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are 0 and \( n \cdot \xi_j, \ j = 1, \cdots, n \) where \( \{ \xi_j \}_{j=1,\cdots,n} \) are solutions to \( \xi^n = 4 \) (see [Sh16, Corollary 1.14]). Thus, \( QH(X; \Lambda_{\text{Nov}}) \) splits into a direct sum of generalized eigenspaces with respect to \( c_1 \ast - \) as follows:

\[
QH(Q^n; \Lambda_{\text{Nov}}) = \bigoplus_{1 \leq j \leq n} QH(X; \Lambda_{\text{Nov}})_{n, \xi_j} \oplus QH(X; \Lambda_{\text{Nov}})_0.
\] (4.1.3)

We decompose the unit \( 1_{Q^n} \) with respect to this split:

\[
1_{Q^n} = \sum_{1 \leq j \leq n} e_{n, \xi_j} + e_0.
\] (4.1.4)

In view of the superpotential computation for \( T^n_{GZ} \) of Y. Kim (3.1.13), (3.1.14), Theorem 4.1.1 implies that

\[
\mathcal{CO}^0 : QH(X; \Lambda_{\text{Nov}})_0 \longrightarrow HF(S^n_{\text{van}})
\]

(4.1.7)

where

\[
\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}
\] (4.1.6)

Similarly, given that \( W_{S_{\text{van}}} = 0 \) (\( S_{\text{van}} \) does not bound any Maslov 2-disks), Theorem 4.1.1 implies

\[
\mathcal{CO}^0 : QH(X; \Lambda_{\text{Nov}})_0 \longrightarrow HF(S^n_{\text{van}})
\]

(4.1.7)

where

\[
\delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}
\] (4.1.6)

The heaviness criterion Theorem 2.1.9 applied to (4.1.5) and (4.1.7) imply that \( T^n_{GZ} \) is \( \zeta_{e_{n, \xi_j}} \)-heavy for all \( 1 \leq j \leq n \) and \( S^n_{\text{van}} \) is \( \zeta_{e_0} \)-heavy. Moreover, by using that \( QH(X; \Lambda_{\text{Nov}})_{n, \xi_j} \) is a field, we can conclude that \( \zeta_{e_{n, \xi_j}} \)-superheavy for all \( 1 \leq j \leq n \), see Remark 2.1.10. On the other hand, as \( QH(X; \Lambda_{\text{Nov}})_0 \) is not a field if \( n \) is even (see Section 4.2 for this in more detail), we will need some extra arguments to show that \( S^n_{\text{van}} \) is \( \zeta_{e_0} \)-superheavy, which is the purpose of Sections 4.2 and 4.3.

### 4.2 The Laurent and Novikov fields

The aim of this section is to clarify the relation between spectral invariants defined with respect to different coefficient fields. Note that \( \zeta_\pm : \overline{\text{Ham}}(Q^n) \to \mathbb{R} \) are defined with the Laurent coefficients

\[
\Lambda_{\text{Lau}} = \left\{ \sum_{k \geq k_0} b_k t^k : k_0 \in \mathbb{Z}, b_k \in \mathbb{C} \right\},
\]

while we have worked entirely with the Novikov coefficients

\[
\Lambda_{\text{Nov}} = \left\{ \sum_{j=1}^{\infty} a_j T^\lambda_j : a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, \lim_{j \to +\infty} \lambda_j = +\infty \right\}
\]

in Section 4.1 (see Remark 4.1.2). Recall that \( \Lambda_{\text{Lau}} \) can be embedded to \( \Lambda_{\text{Nov}} \) by the inclusion given by \( t \mapsto T^\lambda_0 \), and this inclusion extends to

\[
i : QH(X; \Lambda_{\text{Lau}}) \hookrightarrow QH(X; \Lambda_{\text{Nov}}).
\]

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This subtlety of the choice of the coefficient field, which was analyzed in [Kaw22A, Section 4.2, 4.5], might seem technical but turns out to be very useful and important. To summarize the points from [Kaw22A, Section 4.2, 4.5], to work with spectral invariants, e.g. Entov–Polterovich quasimorphisms, it is more convenient to work with the Laurent coefficient while Lagrangian Floer theory is more suited to work with the universal Novikov field, e.g. Theorem 4.1.

We will focus on the case of $Q^n$. With the Laurent coefficients, the quantum cohomology splits into a direct sum of two fields

$$QH(Q^n; \Lambda_{\text{Laun}}) = Q_+ \oplus Q_-,$$

where the unit $1_{Q^n}$ splits as

$$1_{Q^n} = e_+ + e_-,$$

$$e_\pm := \frac{1_{Q^n} \pm PD([pt])t}{2}. \quad (4.2.2)$$

However, when we consider $QH(Q^n; \Lambda_{\text{Nov}})$, $Q_+$ and $Q_-$ further splits into finer fields; $Q_+$ splits as a direct sum of $n$ fields

$$Q_+ = \bigoplus_{1 \leq i \leq n} Q_{+,i},$$

and $Q_-$ splits as

$$Q_- = \begin{cases} Q_-, \text{ if } n \text{ is odd} \\ Q_{-,1} \oplus Q_{-,2}, \text{ if } n \text{ is even.} \end{cases} \quad (4.2.3)$$

Thus, we have

$$QH(Q^n; \Lambda_{\text{Nov}}) = \begin{cases} (\bigoplus_{1 \leq i \leq n} Q_{+,i}) \oplus Q_-, \text{ if } n \text{ is odd} \\ (\bigoplus_{1 \leq i \leq n} Q_{+,i}) \oplus \left( \bigoplus_{j=1,2} Q_{-,j} \right), \text{ if } n \text{ is even} \end{cases} \quad (4.2.4)$$

where the unit $1_{Q^n}$ splits as

$$1_{Q^n} = \begin{cases} \sum_{1 \leq i \leq n} e_{+,i} + e_-, \text{ if } n \text{ is odd} \\ \sum_{1 \leq i \leq n} e_{+,i} + \sum_{j=1,2} e_{-,j}, \text{ if } n \text{ is even.} \end{cases} \quad (4.2.5)$$

$$1_{Q^n} = \sum_{1 \leq i \leq n} e_{+,i} + \sum_{j=1,2} e_{-,j},$$

$$i(e_+) = \sum_{1 \leq i \leq n} e_{+,i},$$

$$i(e_-) = \sum_{j=1,2} e_{-,j}. \quad (4.2.6)$$

**Remark 4.2.1.** When $n$ is even, the idempotents $e_{-,j}$, $j = 1, 2$ are given by

$$e_{-,j} := \frac{1}{2} \left( 1_{Q^n} \pm PD([S^n]) \cdot T^{\lambda_0/2} - PD([pt]) \cdot T^{\lambda_0} \right) \quad (4.2.7)$$
where \([S^n]\) is the homology class represented by the vanishing sphere \(S^n_{\text{van}}\). The precise expression for \(\{e_{+,i}\}\) will be omitted but can be obtained similarly.

In Section 4.1 we have considered yet another split of \(QH(Q^n; \Lambda_{\text{Nov}})\), namely the eigenvalue decomposition (4.1.3):

\[
QH(Q^n; \Lambda_{\text{Nov}}) = \bigoplus_{1 \leq j \leq n} QH(X; \Lambda_{\text{Nov}})_{n \cdot \xi_j} \oplus QH(X; \Lambda_{\text{Nov}})_0,
\]

and the unit \(1_{Q^n}\) is decomposed as

\[
1_{Q^n} = \sum_{1 \leq j \leq n} e_n \cdot \xi_j + e_0.
\]

One can check that \(c_1 \ast e_{+,i} = n \cdot \xi_i \cdot e_{+,i}\) and \(c_1 \ast e_{-,j} = 0\) and thus, the relation between the splits (4.2.1), (4.2.4) and (4.2.8) is

\[
e_{n \cdot \xi_i} = e_{+,i},
\]

\[
e_+ = \sum_{1 \leq i \leq n} e_{+,i},
\]

and

\[
e_- = e_0,
\]

\[
e_0 = \sum_{j=1,2} e_{-,j}, \text{ if } n \text{ is even}
\]

which means that \(QH(X; \Lambda_{\text{Nov}})_{n \cdot \xi_j}\) is a field for any \(1 \leq j \leq n\) and \(QH(X; \Lambda_{\text{Nov}})_0\) is not (it is a direct sum of two fields). All these are not trivial but easy to see from [Sh16, Section 7.4].

In [Kaw22A, Lemma 31, 32] (see also [Kaw22A, Proof of Theorem 6, Remark 44]), the author studied the relation between spectral invariants of a class seen as elements of quantum cohomology with different coefficient fields and the lemma implies the following.

**Theorem 4.2.2.** We have the following relation between asymptotic spectral invariants:

\[
\zeta_{e_{+,i}} = \zeta_{n \cdot \xi_i} = \zeta_{e_{+,i}}, \quad i = 1, \ldots, n,
\]

\[
\zeta_{e_-} = \zeta_0 = \zeta_{e_{-,j}}, \quad j = 1, 2.
\]

### 4.3 Proof–Part 2

In this section, we will combine results from Sections 4.1 and 4.2 to complete the proof of Theorem B.

In Section 4.1, we have shown that \(T^n_{\text{GZ}}\) is \(\zeta_{e_{n \cdot \xi_j}}\)-superheavy for any \(1 \leq j \leq n\) and \(S^n_{\text{van}}\) is \(\zeta_{e_0}\)-heavy and thus, Theorem 4.2.2 implies that \(T^n_{\text{GZ}}\) is \(\zeta_{e_+}\)-superheavy and \(S^n_{\text{van}}\) is \(\zeta_{e_-}\)-heavy. As \(e_-\) is a unit of a field factor of \(QH(X, \Lambda_{\text{Lau}})\), \(\zeta_{e_-}\) is homogeneous and thus \(S^n_{\text{van}}\) is actually \(\zeta_{e_-}\)-superheavy. We have completed the proof of Theorem B. □
4.4 Proof of Theorem [A]

We prove Theorem [A] by using Theorem [B].

**Proof of Theorem [A].** From Theorem [B], the Lagrangian sphere $S^n_{\text{van}}$ and the monotone GZ torus $T^n_{\text{GZ}}$ are $e_-$-superheavy and $e_+$-superheavy, respectively. Also by Theorem [B], $S^n_{\text{van}}$ and $T^n_{\text{GZ}}$ are disjoint so one can take a Hamiltonian $H$ on $Q^n$ such that it is time-independent and its restriction to $S^n_{\text{van}}$ and $T^n_{\text{GZ}}$ are 0 and 1, respectively, i.e.

$$H|_{S^n_{\text{van}}} \equiv 0, \quad H|_{T^n_{\text{GZ}}} \equiv 1.$$  

By Proposition 2.1.8, we have

$$\zeta_-(H) = 0, \quad \zeta_+(H) = 1,$$

which implies

$$\zeta_- \neq \zeta_+.$$

We have proven Theorem [A].

5 Proof of Theorem [C]

In this section, we prove Theorem [C]. In [Sun20], Y. Sun studied toric degenerations of del Pezzo surfaces and computed the superpotentials for the Lagrangian tori that are obtained by symplectically parallel transporting the Lagrangian fiber tori of the barycenters of the moment polytopes. Unlike the case of quadrics that we considered earlier, the singular loci for the toric degenerations of del Pezzo surfaces are isolated sets, and moreover, consist of $A_m$-singularities. Thus, we do not need to study the geometry of the vanishing cycles, as they are merely $A_m$-configurations of Lagrangian spheres.

We now explain how to use Y. Sun’s results to obtain Theorem [C].

**Proof of Theorem [C].** We start by summarizing Y. Sun’s result. In [Sun20], Y. Sun considered toric degenerations for the del Pezzo surfaces to compute the superpotentials of the Lagrangian tori therein. Y. Sun’s superpotentials (for the monotone torus fiber without bulk-deformation) have non-degenerate critical points and thus, the monotone Lagrangian torus fiber has non-zero self-Floer homology with respect to the local systems corresponding to the non-degenerate critical points of the superpotential. Y. Sun also classifies the singularities that appear in each toric degeneration, and the result is as follows:

1. For $D_2$, there is one $A_1$-singularity ([Sun20, Section 3.4]).
2. For $D_3$, there is one $A_1$-singularity and one $A_2$-singularity ([Sun20, Appendix B, case of $X_3$]).
3. For $D_4$, there is one $A_1$-singularity and one $A_2$-singularity ([Sun20, Appendix B, case of $X_6$]).

**Remark 5.0.1.**
1. Y. Sun considers $D_5$ as well, but we will not deal with this, as $QH(D_5)$ is not semi-simple.

2. For $D_3$ and $D_4$, there are several toric degenerations; see 
[Sun20], Appendix B, cases of $X_3, X_4, X_5$, 
[Sun20] Appendix B, case of $X_6, X_7$, respectively.

3. One difference between the case of the quadrics and the case of del Pezzo surfaces is that in the former, the singular locus was not a discrete set, while in the latter the singular locus is discrete.

By looking at the vanishing cycles of the $A_m$-singularities, one obtains a chain of Lagrangian spheres (i.e. an $A_m$-configuration), and in particular we have the following:

1. In $D_2$, there is one Lagrangian sphere $S_1$ that is disjoint from the monotone Lagrangian torus.

2. In $D_3$, there is one Lagrangian sphere $S_1$ (from the $A_1$-singularity) and one Lagrangian sphere $S_2$ (from the $A_2$-singularity) that are both disjoint to each other and also from the monotone Lagrangian torus.

3. In $D_4$, there is one Lagrangian sphere $S_1$ (from the $A_1$-singularity) and one Lagrangian sphere $S_2$ (from the $A_2$-singularity) that are both disjoint to each other and also from the monotone Lagrangian torus.

**Remark 5.0.2.** The properties of the spectral invariants of the Lagrangian spheres forming an $A_m$ configuration is studied in [KawB]. The relation between the spheres and the corresponding idempotents is also explained in more detail in [KawB].

Note that all the Lagrangian spheres that appear above have non-zero Floer homology, c.f. [BM15] Proposition A.2. Now, using that $QH(D_k)$ for $k = 2, 3, 4$ are semi-simple, c.f. [BM04], and by employing an analogous argument to Section 4, we have the following:

1. For $k = 2$, there exist two units of field factors of $QH(D_2)$ $e_1, e_2$ for which the Lagrangian sphere $S_1$ is $e_1$-superheavy and the monotone Lagrangian torus is $e_2$-superheavy.

2. For each $k = 3, 4$, there exist three units of field factors of $QH(D_k)$ $e_1, e_2, e_3$ for which the Lagrangian sphere $S_1$ is $e_1$-superheavy, the Lagrangian sphere $S_2$ is $e_2$-superheavy, and the monotone Lagrangian torus is $e_3$-superheavy.

As all the involved Lagrangians are disjoint, we have

1. For $k = 2$, $\zeta_{e_1} \neq \zeta_{e_2}$.

2. For $k = 3, 4$, $\zeta_{e_i} \neq \zeta_{e_j}$, $1 \leq i < j \leq 3$.

This finishes the proof of Theorem C.

**Remark 5.0.3.** For the del Pezzo surfaces, unlike the quadrics, we did not have to deal with the change of coefficient as in Section 4.2. This is because the minimal Chern number of the del Pezzo surfaces $D_k$, $k \geq 1$ is 1, and thus Laurent coefficients and Novikov coefficients will give the same split in quantum cohomology.
6 Proofs of Theorem D and Applications

In this section, we prove Theorems Theorem D, E and F.

We first prove Theorem E.

Proof of Theorem E. First, recall the following result from [Kaw22A]:

**Theorem 6.0.1 ([Kaw22A, Theorem 22]).** Let \((X, \omega)\) be a monotone symplectic manifold. Assume its quantum cohomology ring \(QH(X; \Lambda)\) is semi-simple i.e.

\[
QH(X; \Lambda) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_l
\]

for some \(l \in \mathbb{N}\) where each \(Q_j\) is a field. We decompose the unit \(1_X \in QH(X; \Lambda)\) into a sum of idempotents with respect to this split:

\[
1_X = e_1 + e_2 + \cdots + e_l, \quad e_j \in Q_j.
\]

Then for any \(i, j \in \{1, 2, \ldots, l\},\)

\[
\mu := \zeta e_i - \zeta e_j
\]

descends from \(\tilde{\text{Ham}}(X, \omega)\) to \(\text{Ham}(X, \omega)\) and defines a homogeneous quasimorphism on \(\text{Ham}(X, \omega)\) which is \(C^0\)-continuous i.e.

\[
\mu : (\text{Ham}(X, \omega), d_{C^0}) \to \mathbb{R}
\]

is continuous. Moreover, it is Hofer Lipschitz continuous.

Now, when \(X = Q^n\), we take

\[
\mu := \zeta - \zeta .
\]

Theorem 6.0.1 and Theorem A imply Theorem E.

When \(X = D_k, \ k = 2, 3, 4\), we take

\[
\mu_{i,j} := \zeta e_i - \zeta e_j
\]

with \(i \neq j\), where the Entov–Polterovich quasimorphisms \(\{\zeta e_i\}\) are taken as in Section 5. Theorem 6.0.1 and Theorem C imply Theorem E.

\[\square\]

Remark 6.0.2. Note that the above quasimorphism extends to the \(C^0\)-closure of \(\text{Ham}(X)\), so \(\mu = \zeta - \zeta \) and \(\mu_{i,j} = \zeta e_i - \zeta e_j\) give quasimorphisms on \(\text{Ham}^{C^0}(Q^n)\) and \(\text{Ham}^{C^0}(D_k), \ k = 2, 3, 4\), respectively.

We prove Theorem D.

Proof of Theorem D. When \(k = 3, 4\), in the final step of the proof of Theorem E we had

\[
\mu_{1,2}, \mu_{2,3} : \text{Ham}(D_k) \to \mathbb{R}
\]

\[
\mu_{1,2} \neq \mu_{2,3}, \quad (6.0.1)
\]
As $\mu_{1,2}$ and $\mu_{2,3}$ are both Hofer Lipschitz continuous and linearly independent, i.e. there is no constant $\alpha \in \mathbb{R}$ such that $\mu_{2,3}(\phi) = \alpha \cdot \mu_{1,2}(\phi)$ for every $\phi \in \text{Ham}(\mathbb{D}_k)$, we can conclude that $\text{Ham}(\mathbb{D}_k)$ is not quasi-isometric to the real line $\mathbb{R}$ with respect to the Hofer metric in the following way: first, we see that $\mathbb{Z}$ (and thus $\mathbb{R}$) quasi-isometrically embeds to $\text{Ham}(\mathbb{D}_k)$. Indeed, by taking $\phi \in \text{Ham}(\mathbb{D}_k)$ such that $\mu_{1,2}(\phi) \neq 0$, we can see that

$$Z \to \text{Ham}(\mathbb{D}_k)$$

$$k \mapsto \phi^k \quad (6.0.2)$$

defines a quasi-isometric embedding, as

$$|\mu_{1,2}(\phi)| \cdot |k - l| = |\mu_{1,2}(\phi^{k-l})|$$

$$\leq d_{\text{Hof}}(\text{id}, \phi^{k-l})$$

$$= d_{\text{Hof}}(\phi^k, \phi^l) \quad (6.0.3)$$

$$\leq d_{\text{Hof}}(\text{id}, \phi) \cdot |k - l|.$$

Now, we want to show that $\mathbb{Z}$ (thus $\mathbb{R}$) is not quasi-isometric to $\text{Ham}(\mathbb{D}_k)$. Without loss of generality, we can assume that $\mu_{2,3}(\phi) = 0$ (otherwise, we can consider $\mu_{2,3} - \frac{\mu_{2,3}(\phi)}{\mu_{1,2}(\phi)} \cdot \mu_{1,2}$ instead of $\mu_{2,3}$). As $\mu_{1,2}$ and $\mu_{2,3}$ are linearly independent, there is $\psi \in \text{Ham}(\mathbb{D}_k)$ such that $\mu_{2,3}(\psi) \neq 0$. Consider the following embedding of $\mathbb{Z}$ to $\text{Ham}(\mathbb{D}_k)$:

$$Z \to \text{Ham}(\mathbb{D}_k)$$

$$k \mapsto \psi^k \quad (6.0.4)$$

In order to show that $\mathbb{Z}$ (thus $\mathbb{R}$) is not quasi-isometric to $\text{Ham}(\mathbb{D}_k)$, it suffices to show that the orbit $\{\psi^k : k \in \mathbb{Z}\}$ gets arbitrary far from $\{\phi^m : m \in \mathbb{Z}\}$. To see this, for every $m \in \mathbb{Z}$, we have

$$d_{\text{Hof}}(\psi^k, \phi^m) = d_{\text{Hof}}(\text{id}, \phi^{-m} \psi^k)$$

$$\geq |\mu_{2,3}(\phi^{-m} \psi^k)|$$

$$\geq |\mu_{2,3}(\psi^k) - \mu_{2,3}(\phi^m)| - \text{Const.} \quad (6.0.5)$$

$$= |k \cdot \mu_{2,3}(\psi) - m \cdot 0| - \text{Const.}$$

$$= |k| \cdot |\mu_{2,3}(\psi)| - \text{Const.}$$

Note that the third line uses the property (2.1.8) of quasimorphisms and the constant only depends on $\mu_{2,3}$ and the fourth line uses the property (2.1.9) of quasimorphisms. Now, (6.0.5) shows that

$$\lim_{k \to \pm \infty} d_{\text{Hof}}(\psi^k, \phi^m) = +\infty$$

and thus $\mathbb{Z}$ (thus $\mathbb{R}$) is not quasi-isometric to $\text{Ham}(\mathbb{D}_k)$. This completes the proof of Theorem D.

We prove Theorem F.

Proof of Theorem F. We start with the following claim.
Claim 6.0.3. Suppose $QH(X)$ is semi-simple and denote the split of it by

$$QH(X; \Lambda) = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_l.$$  

For any Hamiltonian $H$, we have

$$\zeta_{1X}(H) = \max_{1 \leq j \leq l} \zeta_{e_j}(H).$$

Claim 6.0.3 implies that for any $\phi \in \text{Ham}(X)$,

$$\gamma(\phi) = \max_{1 \leq i, j \leq l} \mu_{i,j}(\phi)$$  \hspace{1cm} (6.0.6)

where

$$\mu_{i,j}(\phi_H) = \mu_{i,j}(H) := \zeta_{e_i}(H) - \zeta_{e_j}(H).$$

In fact,

$$\gamma(\phi_H) = \zeta_{1X}(H) + \zeta_{1X}(\overline{H})
= \max_{1 \leq i \leq l} \zeta_{e_i}(H) + \max_{1 \leq j \leq l} \zeta_{e_j}(\overline{H})
= \max_{1 \leq i, j \leq l} \{\zeta_{e_i}(H) - \zeta_{e_j}(H)\}
= \max_{1 \leq i, j \leq l} \mu_{i,j}(\phi).$$  \hspace{1cm} (6.0.7)

By Theorem 6.0.1, we know that for each $i, j$, $\mu_{i,j}$ is $C^0$-continuous and therefore, $\gamma$ is $C^0$-continuous. Theorem E implies $\gamma \neq 0$ for $X = Q^n$ and $X = \mathbb{D}_k$, $k = 2, 3, 4$. □

We prove Claim 6.0.3

**Proof of Claim 6.0.3**  We first prove $\zeta_{1X}(H) \geq \max_{1 \leq j \leq l} \zeta_{e_j}(H)$ for any $H$. By the triangle inequality, we get

$$c(H, 1_X) + \nu(e_j) \geq c(H, e_j)$$

for any $j$ and Hamiltonian $H$ and thus

$$\zeta_{1X}(H) \geq \zeta_{e_j}(H)$$

for any $j$ and Hamiltonian $H$. Therefore,

$$\zeta_{1X}(H) \geq \max_{1 \leq j \leq l} \zeta_{e_j}(H)$$

for any Hamiltonian $H$.

Next, we prove $\zeta_{1X}(H) \leq \max_{1 \leq j \leq l} \zeta_{e_j}(H)$. A standard property of spectral invariants implies

$$c(H, 1_X) \leq \max_{1 \leq j \leq l} c(H, e_j)$$

as $1_X = e_1 + e_2 + \cdots + e_l$ and thus

$$\zeta_{1X}(H) \leq \max_{1 \leq j \leq l} \zeta_{e_j}(H)$$

for any Hamiltonian $H$. This completes the proof of Claim 6.0.3. □
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