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Explicit matrix inverses for lower triangular matrices with entries involving continuous $q$-ultraspherical polynomials

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Abstract

For a one-parameter family of lower triangular matrices with entries involving continuous $q$-ultraspherical polynomials we give an explicit lower triangular inverse matrix, with entries involving again continuous $q$-ultraspherical functions. The matrices are $q$-analogues of results given by Cagliero and Koornwinder recently. The proofs are not $q$-analogues of the Cagliero-Koornwinder case, but are of a different nature involving $q$-Racah polynomials. Some applications of these new formulas are given. Also the limit $\beta \to 0$ is studied and gives rise to continuous $q$-Hermite polynomials for $0 < q < 1$ and $q > 1$.

1 Introduction

In [11] Koelink, van Pruijssen and Román needed to invert a lower triangular matrix with entries involving Gegenbauer (or ultraspherical) polynomials. The solution was given by Cagliero and Koornwinder [5] in the wider context of a two-parameter family of lower triangular matrices involving Jacobi polynomials. The inverse of this matrix is given in terms of Jacobi polynomials as well. Cagliero and Koornwinder [5] solved this problem using the Rodrigues formula for the Jacobi polynomials and some variations on the product rule. Thereafter Koelink, de los Ríos and Román [12] used the results of Cagliero and Koornwinder [5] with an extra free parameter.

In this paper we give a partial $q$-analogue of the result of Cagliero and Koornwinder [5]. In a forthcoming paper [1], which is a quantum analogue of [10, 11], the main Theorem 1.1 is used to obtain an inverse of a lower triangular matrix with entries involving continuous $q$-ultraspherical polynomials. Theorem 1.1 gives the inverse of this matrix in a more general situation. Theorem 1.1 is the main result of this paper.

Theorem 1.1. Let $\beta \in \mathbb{C} \setminus \{0\}$, $\beta \neq q^{\frac{1}{k}}$ for $k \in \mathbb{Z}$. Define doubly infinite lower triangular matrices $L^{\beta}(x)$ and $M^{\beta}(x)$ by

$$L^{\beta}(x)_{m,n} = \frac{1}{(\beta^2 q^{2n}; q)_{m-n}}C_{m-n}(x; \beta q^n | q), \quad n \leq m,$$

$$M^{\beta}(x)_{m,n} = \frac{\beta^{m-n} q^{(m-1)(m-n)}}{(\beta^2 q^{m-n-1}; q)_{m-n}}C_{m-n}(x; \beta^{-1} q^{1-m} | q), \quad n \leq m,$$

where $m, n \in \mathbb{Z}$ and $C_m(x; \beta | q)$ are the continuous $q$-ultraspherical polynomials defined in Section 2 for all $\beta$. Then $M^{\beta}(x)$ and $L^{\beta}(x)$ are each other’s inverse, i.e. $L^{\beta}(x)M^{\beta}(x) = I = M^{\beta}(x)L^{\beta}(x)$, where $I_{m,n} = \delta_{m,n}$ is the identity.

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The proof of Theorem 1.1 is given in Section 3.

Theorem 1.1 has a finite dimensional analogue, because the entries of $L^\beta M^\beta$ only involve finite sums of continuous $q$-ultraspherical polynomials. From Theorem 1.1 we have to following corollary.

**Corollary 1.2.** For a non-negative integer $N$ and $\beta \in \mathbb{C}\setminus\{0\}$ such that $\beta \neq q^{-\frac{N}{2}}$ for $k = 0, 1, \ldots, 2N - 2$. Define lower triangular matrices $L^\beta(x)$ and $M^\beta(x)$

\[
L^\beta(x)_{m,n} = \frac{1}{(\beta^2 q^{2n}; q)_{m-n}} C_{m-n}(x; \beta q^n | q), \quad 0 \leq n \leq m \leq N
\]

\[
M^\beta(x)_{m,n} = \frac{\beta^{m-n} q^{-\frac{1}{2}}(m-n)}{(\beta^2 q^{m-n+1}; q)_{m-n}} C_{m-n}(x; \beta^{-1}q^{1-m} | q), \quad 0 \leq n \leq m \leq N.
\]

Then $M^\beta(x)$ and $L^\beta(x)$ are each others inverse, i.e. $L^\beta(x)M^\beta(x) = I = M^\beta(x)L^\beta(x)$, where $I$ is the identity matrix.

The proof of Theorem 1.1 is not a straightforward $q$-analogue of the proof given by Cagliero and Koornwinder [5]. The proof uses $q$-Racah polynomials and does not use Rodrigues formulas or product rules of differentials which are the essential ingredients for the proof in [5]. In particular, the $q \to 1$ limit of the proof presented here gives an alternative proof of the special case $\alpha = \beta$ of Cagliero and Koornwinder [5].

We compute the coefficients of $e^{ik\theta}$ of products of two continuous $q$-ultraspherical polynomials and express the coefficients in terms of terminating balanced basic hypergeometric series $\phi_3$. For certain parameters this series transforms to a $q$-Racah polynomial. The orthogonality relations of the $q$-Racah polynomials then lead to Theorem 1.1. The proof of Theorem 1.1 for $q \to 1$, gives an interesting new proof of [5, Theorem 4.1] in the special case $\alpha = \beta$, showing that the coefficients of $e^{ik\theta}$ of products of certain Gegenbauer polynomials are actually Racah polynomials. The entries of the matrix identity $L(x)M(x) = I$ in [5, Theorem 4.1] correspond to orthogonality relations of Racah polynomials, see Example 4.1.

In Section 3 we study matrices $L^\beta$ and $M^\beta$ for Theorem 1.1 for a suitable limit $\beta \to 0$. The entries of $L^\beta$ become continuous $q$-Hermite polynomials and the entries of $M^\beta$ converge to continuous $q^{-1}$-Hermite polynomials as $\beta \to 0$.

We emphasise again that our proof is different than the proof of Cagliero and Koornwinder [5]. It is possible to extend the proof of Cagliero and Koornwinder [5] to a $q$-analogue for certain polynomials in the $q$-Askey scheme [9]. For example [5, Lemma 5.1] has a $q$-analogue for the $q$-derivative operator [6, Exercise 1.12]. Then with the use of Rodrigues’ formula and suitable parameters for the orthogonal polynomials it is possible to find $q$-analouges for [5, (4.1), (4.2)]. The author was able to extend [5, (4.1), (4.2)] to the little $q$-Jacobi polynomials. However these results involve different $q$-shifts in the $x$ of the polynomials and don’t seem to lead to a result similar to Theorem 1.1 or [5, Theorem 4.1]. Also Cagliero and Koornwinder [5] were motivated by [3, 11] to extend their formulas to a two parameter family of Jacobi polynomials. We lack this motivation and therefore decided not to include these results for the little $q$-Jacobi polynomials in this paper. We didn’t extend the results to other families of polynomials.

## 2 Preliminaries

We recall some facts on basic hypergeometric series and related polynomials, see Gasper and Rahman [6] and Koekoek, Lesky and Swarttouw [9]. We fix $0 < q < 1$ and we follow notation of [6].

For $\beta \in \mathbb{C}$, the continuous $q$-ultraspherical polynomials are given by

\[
C_n(x; \beta | q) = \sum_{k=0}^{n} \frac{(\beta; q)_k (\beta; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} e^{i(n-k)\theta}, \quad x = \cos(\theta),
\]

see [6, Exercise 1.28] and [9, §14.10]. Notice that the continuous $q$-ultraspherical polynomials are defined for
all $\beta$. A generating function for the continuous $q$-ultraspherical polynomials is

$$\sum_{n=0}^{\infty} C_n(x; \beta|q)^n = \frac{(\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} \quad |t| < 1, \quad x = \cos(\theta) \in [-1, 1],$$

(2.2)

see [6, Exercise 1.29] and [9, (14.10.27)].

For $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $q\alpha = q^{-N}$, $\beta \delta q = q^{-N}$ or $\gamma q = q^{-N}$, for $N \in \mathbb{N}$, define the $q$-Racah polynomials

$$R_n(\mu(x); \alpha, \beta, \gamma, \delta; q) = 4\phi_3\left(q^{N-n}\alpha^{-1}, q^{-n-1}\gamma^{-1}, q^{-n-1}\delta^{-1}; q^2, q^{-1}\gamma q^{2N+n}; q, q\right),$$

(2.3)

where $n = 0, 1, \ldots, N$. If $q\alpha = q^{-N}$ and $\beta = 1$ the $q$-Racah polynomials are not orthogonal with respect to a positive measure. Still the $q$-Racah polynomials are orthogonal

$$\sum_{x=0}^{N} \frac{(q^{-N}, q^{-N}; q)_{x}}{(q, q^{-N}; q)_{x}} (1 - \gamma \delta q^{2x+1}) q^{N\mu} R_n(\mu(x); q^{-N-1}, 1, \gamma, \delta; q) R_n(\mu(x); q^{-N-1}, 1, \gamma, \delta; q)$$

$$= \delta_{m,n} h_m(\gamma, \delta; N),$$

(2.4)

where $h_m(\gamma, \delta; N)$ is given in [6, §7.2] and [9, §14.2]. It follows that if $n = 0$ we have

$$\sum_{x=0}^{N} \frac{(q^{-N}, q^{-N}; q)_{x}}{(q, q^{-N}; q)_{x}} (1 - \gamma \delta q^{2x+1}) q^{N\mu} R_n(\mu(x); q^{-N-1}, 1, \gamma, \delta; q)$$

$$= \delta_{m,0} h_0(\gamma, \delta; N),$$

(2.5)

where $h_0(\gamma, \delta; N) = \delta_{N,0}$ if $\gamma \neq q^{-\ell}$ and $\delta \neq q^{-m}$ with $\ell, m = 1, 2, \ldots, N$.

Note that [24] can also be proved directly, also see [3]. To show this substitute [24] in [21] so that

$$\sum_{x=0}^{N} \frac{(q^{-N}, q^{-N}; q)_{x}}{(q, q^{-N}; q)_{x}} (1 - \gamma \delta q^{2x+1}) q^{N\mu} R_n(\mu(x); q^{-N-1}, 1, \gamma, \delta; q)$$

$$= \delta_{m,k} q^k.$$  

Then expand the left hand side of [24] in $q^\ell$ observing that it is a polynomial in $q^\ell$ of degree $N - k$. Finally applying the summation formula [6, (II.21)] on the $x$-sum gives the right hand side of [24].

**Remark 2.1.** One of the referees pointed out that if $q\alpha = q^{-N}$ and $\beta = 1$ then from [24] it follows that $R_n = R_{N-n}$. Therefore for $n > \frac{1}{2}N$ the polynomial $R_n$ will have degree $N - n < n$. So there can be no non-degenerate orthogonality. However, the system of polynomials $R_n$ for $n \leq \frac{1}{2}N$ can still be orthogonal with respect to positive weights.

Sears’ transformation formula, [6, (III.15) & (III.16)], for terminating balanced $4\phi_3$ series is

$$4\phi_3\left(q^{-n}, a, b, c ; q, q\right) = a_n (e - 1, f - 1, g - 1; g)_n 4\phi_3\left(q^{-n}, a, db^{-1}, dc^{-1} ; q, q\right)$$

$$= a_n (e - 1, f - 1, g - 1; g)_n 4\phi_3\left(q^{-n}, e - 1, f - 1, ef(a)^{-1} ; q, q\right),$$

(2.6)

where $abc = defq^n$.

### 3 Proof of Theorem 1.1

The idea of the proof of Theorem 1.1 is to first expand a sum of products of continuous $q$-ultraspherical polynomials in terms of $e^{ik\theta}$, where $x = \cos(\theta)$. We show that the coefficients of $e^{ik\theta}$ are balanced basic hypergeometric series $4\phi_3$. For the continuous $q$-ultraspherical polynomials with parameters as in Theorem 1.1 we show that the coefficients of $e^{ik\theta}$ correspond to the orthogonality relations for $q$-Racah polynomials. This proves the key Lemma 3.3 from which Theorem 1.1 follows.
Lemma 3.1. Take \( n \in \mathbb{N} \). Let \( \alpha_k, \beta_k \) and \( c(k) \) be constants for \( k = 0, 1, \ldots, n \). Then

\[
\sum_{k=0}^{n} c(k) C_{n-k}(x; \alpha_k|q) C_k(x; \beta_k|q) = \sum_{p=0}^{n} d(p) e^{i(n-2p)\theta}, \quad x = \cos(\theta), \tag{3.1}
\]

where \( d(p) \) is given by

\[
d(p) = \sum_{k=0}^{n-p} c(k) \frac{(\alpha_k; q)_p (\alpha_k; q)_{n-p-k} (\beta_k; q)_k}{(q; q)_p (q; q)_{n-p-k}} 4\phi_3 \left( \begin{array}{c} q^{-p}, \alpha_k q^{n-p-k}, q^{-k}, \beta_k \\ q^{-1-p} \alpha_k^{-1}, q^{-1-k} \beta_k^{-1}, q^{n-p-k+1} \end{array} q, q^2 \alpha_k^{-1} \beta_k^{-1} \right) + \sum_{k=n-p+1}^{n} c(k) \frac{(\alpha_k; q)_{n-k} (\beta_k; q)_{p-n-k} (\beta_k; q)_n}{(q; q)_{n-k} (q; q)_{n-p-k}} 4\phi_3 \left( \begin{array}{c} q^{k-n}, q^{p-n}, \alpha_k, \beta_k q^{k+p-n} \\ q^{-1-n-k} \alpha_k^{-1}, q^{-1-k} \beta_k^{-1}, q^{-1-n} \beta_k^{-1} \end{array} q, q^2 \alpha_k^{-1} \beta_k^{-1} \right). \tag{3.2}
\]

Proof. First expand the left hand side of (3.1) using (2.1), so that the left hand side of (3.1) equals

\[
\sum_{k=0}^{n} c(k) \sum_{s=0}^{n-k} \frac{(\alpha_k; q)_s (\alpha_k; q)_{n-k-s} (\beta_k; q)_{k-t} (\beta_k; q)_t}{(q; q)_s (q; q)_{n-k-s} (q; q)_t} e^{i(n-2(s+t))\theta}. \tag{3.3}
\]

Now fix \( p = s + t \) and substitute \( s = p - t \) in (3.3) so that the coefficient of \( e^{i(n-2p)\theta} \) becomes

\[
\sum_{k=0}^{n} c(k) \sum_{t=0}^{k\wedge p} \frac{(\alpha_k; q)_{p-t} (\alpha_k; q)_{n-k-p+t} (\beta_k; q)_t}{(q; q)_{p-t} (q; q)_{n-k-p+t}}. \tag{3.4}
\]

For \( 0 \leq k \leq n-p \) so that \( k+p-n \leq 0 \), the \( t \)-sum of (3.4) is, after simplifying the \( q \)-Pochhammer symbols, the balanced terminating \( 4\phi_3 \)

\[
\frac{(\alpha_k; q)_p (\alpha_k; q)_{n-p-k} (\beta_k; q)_k}{(q; q)_p (q; q)_{n-p-k}} 4\phi_3 \left( \begin{array}{c} q^{-p}, \alpha_k q^{n-p-k}, q^{-k}, \beta_k \\ q^{-1-p} \alpha_k^{-1}, q^{-1-k} \beta_k^{-1}, q^{n-p-k+1} \end{array} q, q^2 \alpha_k^{-1} \beta_k^{-1} \right). \tag{3.5}
\]

For \( n-p \leq k \leq n \) so that \( k+p-n \geq 0 \) substitute \( t \rightarrow t+k+p-n \) so that the \( t \)-sum of (3.4) is, after simplifying the \( q \)-Pochhammer symbols, the balanced terminating \( 4\phi_3 \)

\[
\frac{(\alpha_k; q)_{n-k} (\beta_k; q)_{p-n-k} (\beta_k; q)_n}{(q; q)_{n-k} (q; q)_{p-n-k}} 4\phi_3 \left( \begin{array}{c} q^{k-n}, q^{p-n}, \alpha_k, \beta_k q^{k+p-n} \\ q^{-1+n+k} \alpha_k^{-1}, q^{-1+k} \beta_k^{-1}, q^{-1+n} \beta_k^{-1} \end{array} q, q^2 \alpha_k^{-1} \beta_k^{-1} \right). \tag{3.6}
\]

Combining (3.5) and (3.6) gives (3.2).

Remark 3.2. Since the continuous \( q \)-ultraspherical polynomials are polynomials in \( x \) the coefficients of \( e^{i(n-p)\theta} \) and \( e^{ip\theta} \) of the left hand side of (3.1) must be equal. Therefore \( d(p) = d(n-p) \) and (3.1) can be rewritten in terms of Chebyshev polynomials \( T_p \) of the first kind, see [1] §9.8.2, as follows

\[
\sum_{k=0}^{n} c(k) C_{n-k}(x; \alpha_k|q) C_k(x; \beta_k|q) = \sum_{p=0}^{[\frac{n}{2}]} (2 - \delta_{n,2p})d(p)T_{n-2p}(x). \tag{3.7}
\]

Remark 3.3. It is possible to write (3.2) uniformly

\[
d(p) = \sum_{k=0}^{n} c(k) \frac{(\alpha_k; q)_p (\beta_k; q)_k}{(q; q)_p (q; q)_k} \frac{(\alpha_k; q)_\infty (\beta_k; q)_\infty}{(q; q)_\infty} \frac{(q^{n-k-p+1}; q)_\infty}{(q; q)_\infty} \times 4\phi_3 \left( \begin{array}{c} q^{-p}, \alpha_k q^{n-p-k}, q^{-k}, \beta_k \\ q^{-1-p} \alpha_k^{-1}, q^{-1-k} \beta_k^{-1}, q^{n-p-k+1} \end{array} q, q^2 \alpha_k^{-1} \beta_k^{-1} \right). \tag{3.8}
\]

4
If $0 \leq k \leq n - p$ we have that (3.3) is equal to

$$
\frac{(\alpha_k; q)_p (\beta_k; q)_k}{(q; q)_p (q; q)_k} \frac{(\alpha_k; q)_\infty (\alpha_k q^{n-k-p}; q)_\infty}{(q^n; q)_\infty} \frac{(q^{n-k-p+1}; q)_\infty}{(q; q)_\infty} 4\phi_3 \left( \frac{q^{-p}, \alpha_k q^{n-p-k}, \beta_k}{q^{-p} \alpha_k^{-1}, q^{-k} \beta_k^{-1}, q^{n-p-k+1}; q, q^2 \alpha_k^{-1} \beta_k^{-1}} \right). \tag{3.7}
$$

Use the convention

$$
\frac{(q^{1-N}; q)_\infty}{(q^{1-N}; q)_t} = (q^{1-N+t}; q)_\infty,
$$

so that for $n - p < k \leq n$

$$
\frac{(q^{1-N}; q)_\infty}{(q; q)_\infty} \sum_{t=0}^{\infty} \frac{C_t}{(q; q)_t} = \frac{(q^{N+1}; q)_\infty}{(q; q)_\infty} \sum_{t=0}^{\infty} \frac{C_{N+t}}{(q; q)^t},
$$

where $N \in \mathbb{N}$ and $C_t$ are arbitrary constants. Then for $N = k + p - n$ we have that (3.7) becomes

$$
\frac{(\alpha_k; q)_p (\beta_k; q)_k}{(q; q)_p (q; q)_k} \frac{(\alpha_k; q)_\infty (\alpha_k q^{n-k-p}; q)_\infty}{(q^n; q)_\infty} \times \sum_{t=0}^{\infty} \frac{(q^{-p}, \alpha_k q^{n-k-p}, \beta_k, q^{-k}; q)_t (q^{-p} \alpha_k^{-1}, q^{-k} \beta_k^{-1}; q)_t}{(q; q)_t} \left( q^2 \alpha_k^{-1} \beta_k^{-1} \right)^{t+k+p-n}
$$

$$
= \frac{(\alpha_k; q)_p (\beta_k; q)_k}{(q; q)_p (q; q)_k} \frac{(\alpha_k; q)_\infty (\alpha_k q^{n-k-p}; q)_\infty}{(q^n; q)_\infty} \times \sum_{t=0}^{\infty} \frac{(q^{-p} \alpha_k q^{n-k-p}, \beta_k, q^{-k}; q)_t (q^{-p} \alpha_k^{-1}, q^{-k} \beta_k^{-1}; q)_t}{(q; q)_t} \left( q^2 \alpha_k^{-1} \beta_k^{-1} \right)^{t+k+p-n}
$$

$$
\times 4\phi_3 \left( \frac{q^{-n}, \alpha_k, \beta_k q^{n-p}, q^{p-n}}{q^{k+p-n+1}, q^{k-n+1} \alpha_k^{-1}, q^{1+p-n} \beta_k^{-1}} \frac{q^2 \alpha_k^{-1} \beta_k^{-1}}{q}. \tag{3.8}
$$

Simplifying the $q$-Pochhammer symbols of (3.3) shows that (3.8) is equal to (3.6).

**Lemma 3.4.** For $m, n \in \mathbb{Z}$ such that $n \leq m$. Let $\beta \in \mathbb{C}$ such that $\beta^2 \neq q^{-2m+1}, q^{-2m+2}, \ldots, q^{-2n}$. Then

$$
\sum_{k=0}^{m-n} \frac{(1 - \beta^2 q^{2n+2k-1})}{(\beta^2 q^{2n+2k-1}; q)_{m-n+1}} q^{k(k+n-1)} C_{m-n-k}(x; \beta q^k|q) C_k(x; \beta^{-1} q^{1-k-n}|q) = \delta_{m,n}. \tag{3.9}
$$

**Proof.** Apply Lemma 3.1 with $n, \alpha_k, \beta_k$ specialised to $m-n, q^{k+n} \beta, 1-k-n \beta^{-1}$ so that in particular $\alpha_k \beta_k = q$ for all $k$. Then the left hand side of (3.9) is $\sum_{p=0}^{m-n} d(p) e^{(m-n-2p)\theta}$, where $x = \cos(\theta)$ and

$$
d(p) = \sum_{k=0}^{m-n-p} \frac{(1 - \beta^2 q^{2n+2k-1})}{(\beta^2 q^{2n+k-1}; q)_{m-n+1}} q^{k(k+n-1)}
$$

$$
\times \frac{(\beta q^{k+n}; q)_p (\beta q^{k+n}; q)_{m-n-p-k} (\beta^{-1} q^{1-k-n}; q)_k}{(q; q)_p (q; q)_{m-n-p-k} (q; q)_k}
$$

$$
\times 4\phi_3 \left( \frac{q^{-k}, q^{-p}, \beta q^{m-p}, q^{n-k+1} \beta^{-1}}{q^{p-n-k+1} \beta^{-1}, q^{m-n-k-p+1}; q} \right)
$$

$$
+ \sum_{k=m-n-p+1}^{m-n} \frac{(1 - \beta^2 q^{2n+2k-1})}{(\beta^2 q^{2n+k-1}; q)_{m-n+1}} q^{k(k+n-1)}
$$

$$
\times \frac{(q^{k+n} \beta; q)_{m-n-k} (q^{1-k-n} \beta^{-1}; q)_p (q^{k+n} \beta^{-1}; q)_m-n-p}{(q; q)_{m-n-k} (q; q)_p (q; q)_m-n-p}
$$

$$
\times 4\phi_3 \left( \frac{q^{-k+m+n}, q^{p-m+n}, q^{k+n} \beta, q^{p-m} \beta^{-1}}{q^{1-m} \beta^{-1}, q^{k+p-m+n+1} \beta q^{2n-m+p+k}; q} \right). \tag{3.10}
$$
We transform the basic hypergeometric series \( \psi_3 \) of (3.10). Apply Sears' transformation formula (2.5) to the first \( \psi_3 \) in (3.10) to see that the \( \psi_3 \) is equal to
\[
\frac{(q^{-n+k+1} \beta^{-1}, q^{-n-k}; q)_k}{(q^{-n-k+1} \beta^{-1}, q^{-n-k+1}; q)_k} q^{-pk} \psi_3 \left( q^{-k}, q^{-p}, q^{-n+m+p}, \beta^2 q^{2n+k-1} \beta q^n, \beta q^n, q^{-m} : q, q \right). \tag{3.11}
\]

Apply Sears' transformation formula (2.6) to the second \( \psi_3 \) in (3.10) in order to see that the \( \psi_3 \) is equal to
\[
\frac{(q^{1+p-m} \beta^{-1}, q^{1-m} \beta; q)_m}{(q^2, \beta q^n; q)_m} q^{n-m} \psi_3 \left( q^{n-m}, \beta^2 q^{2n+k-1}, q^{-p}, q^{-k} : q, q \right). \tag{3.12}
\]

The \( \psi_3 \) of (3.11) and (3.12) can be written as the \( q \)-Racah polynomial \( R_p(\mu(k); q^{m-1}, 1, \beta q^{n-1}, \beta q^{n-1}; q) \), see [23]. Therefore (3.10) becomes, after simplifying the \( q \)-Pochhammer symbols using \( (q^r \beta^{-1}; q) \ell = (-1)^{\ell} q^{\frac{1}{2} \ell(\ell-1)} \beta^{-\ell} (\beta q^{1-r}; q) \ell \) repeatedly,
\[
\frac{(\beta q^n; q)_p (\beta q^n; q)_{m-n-p}}{(\beta^2 q^{2n}; q)_m (q; q)_p (q; q)_{m-n-p}} \sum_{k=0}^{m-n} \frac{(\beta^2 q^{2n-1}, q^{n-m}; \beta q^n)_k (1 - \beta^2 q^{2n+2k-1})}{(q, \beta^2 q^{2n}; q)_k} q^{k(m-n)} \times R_p(\mu(k); q^{m-1}, 1, \beta q^{n-1}, \beta q^{n-1}; q). \tag{3.13}
\]

The \( k \)-sum of (3.13) corresponds to the orthogonality relations (2.4) for the \( q \)-Racah polynomial. Hence (3.13) becomes
\[
d(p) = \frac{(\beta q^n; q)_{m-n}}{(\beta^2 q^n; q)_{m-n}} \delta_{p,0} h_0(\beta q^n, \beta q^n; m-n).
\]

Since \( h_0(\beta q^n, \beta q^n; m-n) = 0 \) if \( n < m \) and \( h_0(\beta q^n, \beta q^n; m-n) = 1 \) if \( m = n \), the result follows.

Proof of Theorem 1.1. Multiplying the matrices \( L^\beta \) and \( M^\beta \) it is sufficient to evaluate the entries of \( L^\beta M^\beta \) for \( m \geq n \). Hence
\[
(L^\beta(x) M^\beta(x))_{m,n} = \sum_{k=n}^{m} \frac{\beta^{k-n} q^{(k-1)(k-n)}}{(\beta^2 q^{2k}; q)_{m-k}(\beta^2 q^{n+k-1}; q)_{k-n}} C_{m-k}(x; \beta q^n) C_{k-n}(x; q^{1-k} \beta^{-1} q).
\]

Applying Lemma 3.4 then yields the result.

4 Applications

Example 4.1. The limit \( q \to 1 \) in the proof of Theorem 1.1 gives a new proof for [5, Theorem 4.1] for
\[
\alpha = \beta. \quad \text{Lemma 3.1 gives}
\]
\[
\sum_{k=0}^{n} c(k) C^{(\alpha_k)}(x) C^{(\beta_k)}(x) = \sum_{p=0}^{n} d(p) e^{(n-2p)\theta}, \quad x = \cos(\theta),
\]
where \( C^{(\alpha)}(x) \) are the Gegenbauer polynomials, see [9] 9.8.1], and
\[
d(p) = \sum_{k=0}^{n} \frac{c(k) (\alpha_k)_{p-k} (\beta_k)_{k} \alpha_k}{p! (n-k)! k!} F_3 \left[ \begin{array}{c} -p, \alpha_k + n - p - k, -k, \beta_k \\ 1 - p - \alpha_k, 1 - k - \beta_k, n - p + k + 1 \end{array} ; 2 - \alpha_k - \beta_k \right] + \sum_{k=n-p+1}^{n} \frac{c(k) (\alpha_k)_{n-k} (\beta_k)_{p-n+k} (\beta_k)_{n-p}}{(n-k)! (p - n + k)! (p - k)!} \times F_3 \left[ \begin{array}{c} k - n, p - n, \alpha_k, \beta_k + k + p - n \\ 1 - n + k - \alpha_k, k + p - n + 1, 1 - n + p - \beta_k \end{array} ; 2 - \alpha_k - \beta_k \right].
\]

6
Then Lemma \[3.3\] yields, for \(0 \leq n \leq m\) and \(\alpha \in \mathbb{C}\), \(2\alpha \neq -2m+1, -2m+2, \ldots, -2n\),

\[
\sum_{k=0}^{m-n} \frac{(2n+2k+2\alpha-1)}{m-n+1} C_{m-n-k}^{(\alpha+k+n)}(x) C_{k}^{(1-k-n-\alpha)}(x) = \delta_{m,n},
\]

which is the key equation to show \[4.4\] Theorem 4.1] for the case \(\alpha = \beta\).

**Example 4.2.** The problem of finding an inverse of the matrix \(L^\beta\) in Theorem \[4.4] originally arose in \[1\] where the finite dimensional lower triangular matrix

\[
L(x)_{m,n} = q^{m-n} \frac{(q^2; q^2)_m (q^2; q^2)_{2n+1}}{(q^2; q^2)_m (q^2; q^2)_{2n+1} C_{m-n}(x; q^{2n+2}|q^2), \quad 0 \leq n \leq m \leq N,
\]

for arbitrary \(N \in \mathbb{N}\) appears. Using Corollary \[5.2\] in base \(q^2\) with \(\beta = q^2\) after conjugation with a diagonal matrix we find that the inverse matrix is given by

\[
(L(x))^{-1}_{m,n} = q^{(2m+1)(m-n)} \frac{(q^2; q^2)_m (q^2; q^2)_{m+n}}{(q^2; q^2)_2m (q^2; q^2)_n C_{m-n}(x; q^{-2m}|q^2), \quad 0 \leq n \leq m \leq N.
\]

Note that the entries of \(L(x)\) and its inverse \((L(x))^{-1}\) are independent of the size of \(N\).

**Example 4.3.** From the generating function \[6.2\] for the continuous \(q\)-ultraspherical polynomials it follows that

\[
\sum_{n=0}^{\infty} C_n(x; \alpha \beta | q) t^n = \frac{(\alpha \beta t^\theta; \alpha \beta e^{-i \theta}; q)_{\infty}}{(t e^{i \theta}; t e^{-i \theta}; q)_{\infty}} \left(\frac{q^2, \alpha \beta e^{-i \theta}; q}{\alpha \beta e^{i \theta}, \alpha \beta e^{-i \theta}; q}\right) = \sum_{m,n=0}^{\infty} C_m(x; \alpha | q) C_n(x; \beta | q) t^m (\alpha t)^n.
\]

Comparing the powers of \(t\) shows

\[
C_n(x; \alpha \beta | q) = \sum_{k=0}^{n} \alpha^k C_{n-k}(x; \alpha | q) C_k(x; \beta | q).
\]

Now take \(\beta = \alpha^{-1}\), then \[4.1\] for \(\beta = 1\) gives

\[
\delta_{n,0} = \sum_{k=0}^{n} \alpha^k C_{n-k}(x; \alpha | q) C_k(x; \alpha^{-1} | q).
\]

On the other hand from Lemma \[5.1\] it follows that

\[
\sum_{k=0}^{n} \alpha^k C_{n-k}(x; \alpha | q) C_k(x; \alpha^{-1} | q) = \sum_{p=0}^{n} d(p) e^{i(n-2p)\theta}, \quad x = \cos(\theta).
\]

Combining \[4.1\] and \[5.2\] it follows that \(d(p) = \delta_{n,0}\). Writing out the explicit expression of \(d(p)\) gives for \(n > 0\) the identity

\[
0 = \sum_{k=0}^{n-p} \frac{\alpha^k (\alpha; q)_p (\alpha; q)_{n-p-k} (\alpha^{-1}; q)_k}{(q; q)_p (q; q)_{n-p-k} (q; q)_k} 4\phi_3 \left( \frac{q^{-p}, \alpha q^{n-p-k}, q^{-k}, \alpha^{-1}}{q^{1-p}, \alpha q^{-k}, q^{n-p-k+1}; q, q^2} \right)
\]

\[
+ \sum_{k=n-p+1}^{n} \alpha^k (\alpha; q)_{n-k} (\alpha^{-1}; q)_{n-p+k} (\alpha^{-1}; q)_{n-p} 4\phi_3 \left( \frac{q^{k-n}, q^{p-n}, \alpha, q^{k+p-n} \alpha^{-1}}{q^{1-n+k}, \alpha^{-1}, q^{k+p-n+1}, \alpha q^{1-n+p}; q, q^2} \right).
\]

In particular if \(p = 0\)

\[
\sum_{k=0}^{n} \alpha^k (\alpha; q)_{n-k} (\alpha^{-1}; q)_k = 0.
\]
Remark that (4.3) also follows from applying (4.3) twice.

5 Limit case $\beta \to 0$

Define $L^0(x)$ and $M^0(x)$ by $L^0(x)_{m,n} = \lim_{\beta \to 0} L^\beta(x)_{m,n}$ and $M^0(x)_{m,n} = \lim_{\beta \to 0} M^\beta(x)_{m,n}$ where the limit is taken over $\beta \neq q^k$, where $k \in \mathbb{Z}$. We show that the limits exist, that the entries of $L^0(x)$ are given in terms of continuous $q$-Hermite polynomials and that the entries of $M^0(x)$ are a given in terms of continuous $q^{-1}$-Hermite polynomials.

The continuous $q$-Hermite polynomials are given by

$$H(x|q) = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} e^{i(2k-1)}, \quad x = \cos(\theta),$$

(5.1)

see [9] §14.26. The continuous $q$-Hermite polynomials are, apart from a different normalisation, the special case $\beta = 0$ of the continuous $q$-ultraspherical polynomials

$$C_n(x; 0|q) = \frac{H_n(x|q)}{(q; q)_n}.$$ 

(5.2)

The corresponding generating function for the continuous $q$-Hermite polynomials is

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q; q)_n} t^n = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}} \quad |t| < 1, \quad x = \cos(\theta),$$

(5.3)

see [9] (14.26.11).

The polynomials $H_n(x|q^{-1})$ are called the continuous $q^{-1}$-Hermite polynomials and are defined by taking $q \mapsto q^{-1}$ in (5.1), see [2]. The continuous $q$-Hermite polynomials are orthogonal with respect to a positive measure on $(-1, 1)$. However the continuous $q^{-1}$-Hermite polynomials are orthogonal on the imaginary axis and correspond to an indeterminate moment problem, see [2] and [8].

Theorem 5.1. The doubly infinite lower triangular matrices $L^0(x)$ and $M^0(x)$ are given by

$$L^0(x)_{m,n} = \frac{H_{m-n}(x|q)}{(q; q)_{m-n}}, \quad M^0(x)_{m,n} = (-1)^{m-n} q^{(m-n)} \frac{H_{m-n}(x|q^{-1})}{(q; q)_{m-n}}, \quad n \leq m,$$

where $m,n \in \mathbb{Z}$. $M^0(x)$ and $L^0(x)$ are each others inverse, i.e. $L^0(x)M^0(x) = I = M^0(x)L^0(x)$, where $I_{m,n} = \delta_{m,n}$ is the identity.

Proof. With (5.2) we have for $n \leq m$

$$L^0(x)_{m,n} = \lim_{\beta \to 0} L^\beta(x)_{m,n} = \lim_{\beta \to 0} \frac{1}{(\beta^2 q^{2n}; q)_{m-n}} C_{m-n}(x; \beta q^n|q) = \frac{H_{m-n}(x|q)}{(q; q)_{m-n}}.$$ 

From (2.1) it follows that $C_n(x; \beta|q) = (\beta q^{-1})^n C_n(x; \beta^{-1}|q^{-1})$. Therefore write $M^\beta(x)_{m,n}$ as

$$\frac{\beta^{m-n} q^{(m-1)(m-n)}}{(\beta^2 q^{m+n-1}; q)_{m-n}} (\beta^{-1} q^{-m})^{m-n} C_{m-n}(x; \beta q^{-m-1}|q^{-1}) = \frac{q^{-(m-n)}}{(\beta^2 q^{m+n-1}; q)_{m-n}} C_{m-n}(x; \beta q^{-m-1}|q^{-1}).$$
Upon taking the limit $\beta \to 0$ and using (5.2) we find

$$M^0(x)_{m,n} = \lim_{\beta \to 0} M^\beta(x)_{m,n} = q^{-(m-n)} \frac{H_{m-n}(x|q^{-1})}{(q^{-1};q^{-1})_{m-n}} = (-1)^{m-n} q^{\binom{m-n}{2}} \frac{H_{m-n}(x|q^{-1})}{(q;q)_{m-n}}.$$  

From Theorem 1.1 it follows that $L^0(x)M^0(x) = I = M^0(x)L^0(x)$. □

**Corollary 5.2.** For $N \in \mathbb{N}$ define lower triangular matrices $L^0(x)$ and $M^0(x)$

$$L^0(x)_{m,n} = \frac{H_{m-n}(x|q)}{(q;q)_{m-n}}, \quad M^0(x)_{m,n} = (-1)^{m-n} q^{\binom{m-n}{2}} \frac{H_{m-n}(x|q^{-1})}{(q;q)_{m-n}}, \quad 0 \leq n \leq m \leq N,$$

Then $M^0(x)$ and $L^0(x)$ are each others inverse, i.e. $L^0(x)M^0(x) = I = M^0(x)L^0(x)$, where $I$ is the identity matrix.

**Remark 5.3.** Theorem 5.1 also follows from a generating function for the continuous $q^{-1}$-Hermite polynomials. From [7] Theorem 21.2.1

$$\sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} H_n(x|q^{-1}) t^n = (te^{i\theta}, te^{-i\theta}; q)_\infty, \quad |t| < 1, \quad x = \cos(\theta). \quad (5.4)$$

Combining (5.3) and (5.4) it follows that for $|t| < 1$

$$1 = \frac{(te^{i\theta}, te^{-i\theta}; q)_\infty}{(te^{i\theta}, te^{-i\theta}; q)_\infty} = \left( \sum_{m=0}^{\infty} \frac{H_m(x|q)}{(q;q)_m} \right) \left( \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{H_n(x|q^{-1})}{(q;q)_n} \right) = \sum_{p=0}^{\infty} \left( \sum_{k=0}^{p} \frac{H_{p-k}(x|q)}{(q;q)_{p-k}} (-1)^k q^{\binom{k}{2}} \frac{H_k(x|q^{-1})}{(q;q)_k} \right) t^p.$$

Take $p = m - n$ so that we have

$$\sum_{k=0}^{m-n} \frac{H_{m-n-k}(x|q)}{(q;q)_{m-n-k}} (-1)^k q^{\binom{k}{2}} H_k(x|q^{-1}) = \delta_{m,n}. \quad (5.5)$$

From (5.5) Theorem 5.1 also follows.

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