REMARKS ON RAMANUJAM-KAWAMATA-VIEHWEG VANISHING THEOREM

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Abstract. In this article we prove a general result on a nef vector bundle $E$ on a projective manifold $X$ of dimension $n$ depending on the vector space $H^{n,n}(X, E)$. It is also shown that $H^{n,n}(X, E) = 0$ for an indecomposable nef rank 2 vector bundles $E$ on some specific type of $n$ dimensional projective manifold $X$. The same vanishing shown to hold for indecomposable nef and big rank 2 vector bundles on any variety with trivial canonical bundle.

1. Introduction

Let $X$ be a smooth projective complex manifold of dimension $n$. For any coherent sheaf $E$ on $X$, we denote $H^{p,q}(X, E)$ the cohomology group $H^q(X, E \otimes \Omega^p_X)$, where $\Omega^p_X$ is the sheaf of holomorphic differential forms of degree $p$ on $X$.

Akizuki-Kodaira-Nakano famous vanishing theorem says:

If $L$ is an ample line bundle on a projective manifold $X$ of dimension $n$, then

\[ H^{p,q}(X, L) = 0 \text{ for } p + q - n > 0. \]

The particular case $p = n$ is the Kodaira vanishing theorem. The Kodaira vanishing theorem was extended to nef and big line bundle on a smooth surface by Ramanujam [7] and for higher dimension by Kawamata [3] and Viehweg [9].

Ramanujam has given in [7] an example showing that in general, one does not expect Akizuki-Kodaira-Nakano type vanishing result for nef and big line bundle.

Le Potier [6] generalized the Akizuki-Kodaira-Nakano type vanishing theorem to the case of ample vector bundle as follows:

If $E$ is an ample vector bundle of rank $r$ on a projective manifold $X$ of dimension $n$, then

\[ H^{p,q}(X, E) = 0 \text{ for } p + q - n > r - 1. \]

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The vanishing results of Ramanujam-Kawamata-Viehweg and Le Potier naturally led to ask the following question:

Let $E$ be a nef and big vector bundle of rank $r$ on a projective manifold $X$ of dimension $n$. Is

\[(2) \quad H^{n,q}(X, E) = 0 \quad \text{for} \quad q > r - 1?\]

The example given by Ramanujam in [7] shows that one can not expect in general ”Akizuki-Kodaira-Nakano” type of vanishing for nef and big line bundle. The same example can also be used to show that the question (2) has a negative answer (see Example (4.2)).

Regarding the question (2) for a nef and big rank two vector bundle $E$ on a smooth surface $X$ the only group which one hope to vanish is the group $H^{2,2}(X, E)$. In trying to investigate this problem we obtained the following:

**Theorem 1.1.** Let $E$ be a nef vector bundle of rank $r$ on a projective manifold $X$ of dimension $n$. Set $k(E) := \dim H^{n,n}(X, E)$. Then $k(E) \leq r$ and $E$ admits a trivial bundle of rank $k(E)$ as quotient. In particular, $k(E) = r$ if and only if $E$ is isomorphic to trivial vector bundle of rank $r$.

**Corollary 1.2.** Let $E$ be an indecomposable nef vector bundle of rank $r$ on a projective manifold $X$ of dimension $n$. Assume that $c_r(E) \neq 0$. Then $H^{n,n}(X, E) = 0$.

For the case of rank 2 vector bundles we have the following:

**Theorem 1.3.** Let $E$ be an indecomposable nef vector bundle of rank 2 on a projective manifold $X$ of dimension $n$. If $H^{1}(X, \det(E)) = 0$, then $H^{n,n}(X, E) = 0$.

As a consequence we obtain:

**Corollary 1.4.** Let $X$ be a Grassmannian of dimension $n \geq 2$ or a complete intersection of dimension $n \geq 3$ in a Grassmannian. If $E$ is an indecomposable nef vector bundle of rank 2 on $X$, then $H^{n,n}(X, E) = 0$.

**Corollary 1.5.** Let $X$ be a projective manifold of dimension $n \geq 2$ with $K_X = \mathcal{O}_X$. If $E$ is an indecomposable nef and big vector bundle of rank 2 on $X$, then $H^{n,n}(X, E) = 0$.

We recall a vanishing theorem of Schneider [8] related to nef and big vector bundle, in a slightly different version from the original one, but follows from the proof given there.
**Theorem 1.6.** Let $E$ (resp. $L$) be a vector bundle (resp. line bundle) on a projective manifold $X$ of dimension $n$. If $E \otimes L$ is nef and big then
\[ H^{n,q}(X, S^k(E) \otimes \det(E) \otimes L) = 0, \quad \text{for} \quad q > 0. \]

2. **Notations and Definitions**

Throughout we work over the field of complex numbers.

For a vector bundle $E$ on a projective manifold $X$, we will denote by $E^\vee$ the dual of $E$, $c_i(E) \in H^{2i}(X, \mathbb{Z})$ is the $i$-th chern class of $E$, $\mathbb{P}(E)$ is the projective bundle whose fiber over a point $x \in X$ is the projective space of $1$-dimensional quotients of the vector space $E_x$, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient line bundle on $\mathbb{P}(E)$.

**Definition 2.1.** Let $X$ be a projective manifold of dimension $n$. A line bundle $L$ on $X$ is called nef, if for every irreducible curve $C$ in $X$, degree of $L|_C$ is non negative. A nef line bundle $L$ is called big if $c_1(L)n > 0$.

A vector bundle $E$ on $X$ is said to be nef if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is nef.

A nef vector bundle $E$ is said to be big if $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is big or equivalently
\[ s_n(E) = p^* (c_1(\mathcal{O}_{\mathbb{P}(E)}(1)))^{n+r-1} > 0, \]
where $s_n(E)$ is the $n$-th Segre class of $E$ and $p : \mathbb{P}(E) \to X$ be the natural projection.

3. **Proof of the results**

First we recall some results which we need.

**Proposition 3.1.** [5, Proposition 6.1.18 (i)] A vector bundle $E$ on $X$ is nef if and only if the following condition is satisfied: Given any morphism $f : C \to X$ finite onto its image from an irreducible smooth curve $C$ to $X$, and given any quotient line bundle $L$ of $f^*(E)$, then one has $\deg L \geq 0$.

**Lemma 3.2.** [1, Proposition 1.16] Let $E$ be a nef vector bundle on a projective manifold $X$ of dimension $n$. If $\sigma$ is a non-zero section of $E^\vee$ then $\sigma$ is nowhere vanishing on $X$.

**Proof:** The proof given in [1] uses analytic methods. Here we give an algebraic proof. First we prove the lemma when $X$ is a curve. In this case if $\sigma$ vanishes at some points, we get a positive degree line sub bundle of $E^\vee$. By dualizing we see that $E$ has a line bundle quotient of negative degree. This is a contradiction to the Proposition(3.1). Thus $\sigma$ is nowhere vanishing on $X$.  

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For the general case, assume $\sigma$ vanishes at some points and dimension of $X$ is greater than one. Let $Z$ be the subscheme of $X$ defined by the vanishing of $\sigma$ and $I_Z$ denotes its sheaf of ideals. The section $\sigma$ induces surjection
\begin{equation}
\sigma : E \to I_Z \to 0.
\end{equation}

Let $C$ be a smooth curve in $X$ with the property $D = C \cap Z$ is a non-empty proper closed subscheme of $C$. Then by restricting the surjective scheme $\sigma$ to $C$ and going modulo torsion we get a surjection:
\begin{equation}
\tau : E|_C \to \mathcal{O}_C(-D) \to 0.
\end{equation}

Since $C$ is a smooth curve $\mathcal{O}_C(-D)$ is a line bundle of negative degree, which is a contradiction to the fact that $E$ is nef. Hence we must have $Z = \emptyset$. □

**Lemma 3.3.** [10, see, Proposition 4.8.] If $E$ is a nef and big vector bundle on a Kähler manifold $X$, then the line bundle $\det(E)$ on $X$ is big.

The Dominance theorem [theorem 3.3] in [4] ensures that $\det(E)$ is nef.

We also need to recall the proposition [Prop. 1.15 (iii)] in [1]. We will state it in a different version, which follows immediately from the proof given there.

**Lemma 3.4.** Let
\begin{equation*}
0 \to F \to E \to Q \to 0
\end{equation*}
be an exact sequence of holomorphic vector bundles and $\text{rank}(E) = r, \text{rank}(F) = f$.

If $\wedge^{r-f+1} E \otimes \det Q^{-1}$ is nef (resp. ample), then $F$ is nef (resp. ample).

**Proof of Theorem (1.1):**

The proof is by induction on the $\text{rank}(E) = r$. If $r = 1$ and $k(E) = 0$ then there is nothing to prove.

If $k(E) > 0$, then by Lemma(3.2) there is a non zero homomorphism
\[ \sigma : \mathcal{O}_X \to E^\vee \]
which is nowhere vanishing. This implies that $E$ is a trivial bundle of rank one. Since $k(\mathcal{O}_X) = 1$, the Theorem follows in this case.

Let $r > 1$. We assume our Theorem holds for all nef vector bundles of rank less than or equal $r - 1$. Again, if $k(E) = 0$ there is nothing to
prove. So we assume \( k(E) > 0 \). Then applying Lemma \((3.2)\) we get an exact sequence

\[
0 \to \mathcal{O}_X \to E^\vee \to F^\vee \to 0,
\]

where \( F^\vee \) is a dual of vector bundle \( F \) of rank \( r - 1 \). Dualizing \((5)\) we get an exact sequence

\[
0 \to F \to E \to \mathcal{O}_X \to 0.
\]

By Lemma \((3.4)\) \( F \) is a nef vector bundle. Now since \( \text{rank}(F) = r - 1 \), we have by induction assumption \( k(F) \leq r - 1 \) and \( F \) admits a trivial quotient of rank \( k(F) \). This implies by duality \( F^\vee \) admits trivial subbundle of rank \( k(F) \).

Taking the inverse image of this \( V \) we see that \( E^\vee \) admits a subbundle \( S^\vee \) of rank \( k(E) \). Note that \( S^\vee \) is an extension of \( \mathcal{O}_X^{k(E)-1} \) by \( \mathcal{O}_X \). The dual \( S \) of \( S^\vee \) is nef, since it is an extension of trivial bundle of rank \( k(E) - 1 \) by a trivial bundle of rank 1. If \( k(E) < r \) then it follows by induction \( S \) is trivial. This proves the result.

If \( k(E) = r \) then again by induction \( F = \mathcal{O}_X^{r-1} \) and all the sections of \( F^\vee \) lifts to sections of \( E^\vee \), hence \( E^\vee \) and \( E \) are isomorphic to \( \mathcal{O}_X^r \).

**Proof of Theorem \((1.3)\):**

Assume \( H^{n,n}(X, E) \neq 0 \), then we get by Serre duality \( H^{0,0}(X, E^\vee) \neq 0 \). Let \( \sigma \) be a non-zero section of \( E^\vee \). Since \( E \) is nef by Lemma \((3.2)\) the section \( \sigma \) is nowhere vanishing, and gives an exact sequence

\[
0 \to \mathcal{O}_X \to E^\vee \to \det(E)^\vee \to 0.
\]

This extension gives a class in the cohomology group \( H^1(X, \det(E)) \). But by our assumption this group is zero and hence the extension splits. Thus \( E^\vee \) splits and hence \( E \) splits too, this is a contradiction.

**Proof of Corollary \((1.2)\):**

If \( H^{n,n}(X, E) \neq 0 \), then by Theorem \((1.1)\) we get an exact sequence

\[
0 \to F \to E \to \mathcal{O}_X \to 0.
\]

This implies \( c_r(E) = c_r(F) = 0 \), this is a contradiction.
Proof of Corollary (1.4): 

If $X$ is a Grassmannian of dimension $\geq 2$ or a complete intersection of dimension $\geq 3$ in a Grassmannian, then for any line bundle $L$, $H^1(X, L) = 0$. Hence if $E$ is an indecomposable vector bundle of rank two on $X$, then the hypothesis of Theorem(1.1) holds for $E$. \hfill \Box \\

Proof of Corollary (1.5): 

Assume $H^{n,n}(X, E) \neq 0$. Since $E$ is nef and big, $\det(E)$ is nef and big by the Lemma(3.3). Hence we have an exact sequence:

$$0 \to \det(E) \to E \to \mathcal{O}_X \to 0.$$ 

But $K_X$ is trivial implies $H^1(X, \det(E)) = 0$ by Kawamata-Ramanujam-Viehweg vanishing theorem. Hence that the exact sequence (10) splits and hence $E$ is decomposable, which is a contradiction. \hfill \Box \\

Remark 3.5. Corollary (1.5) applies for example to complex algebraic torus, $K3$ surfaces and Calabi-Yau manifolds. \hfill \Box \\

4. Counter examples of Ramanujam 

Example 4.1. The following example is due to Ramanujam [7]. Denote $\mathbb{P}^3$ blown up at a point by $X$ and $\pi : X \to \mathbb{P}^3$ be the natural morphism and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Clearly the line bundle $L$ is nef and big and hence $H^1(X, \Omega_X \otimes L^{-1}) \neq 0$. 

Example 4.2. Note that the variety $X$ in the Example(4.1) can be identified with $\mathbb{P}(E)$ in such a way that $L \simeq \mathcal{O}_{\mathbb{P}(E)}(1)$, where $E = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. Clearly the bundle $E$ on $\mathbb{P}^2$ is nef and big and $H^{2,2}(\mathbb{P}^2, E) \neq 0$. This shows that one can not expect Le Potier type vanishing result for nef and big vector bundle even for $p = n$. 

More general example: if $Y$ is a projective manifold of dimention $n$ and $H$ is an ample line bundle on $Y$, then the vector bundle $E = \mathcal{O}_Y \oplus H$ is nef and big vector bundle but $H^{n,n}(Y, E) \neq 0$. 

Remark 4.3. The non vanishing of $H^{1,1}(X, L^{-1})$ of Example (4.1) can be deduced from the non vanishing of the group $H^{2,2}(\mathbb{P}^2, E)$ in Example (4.2). Indeed:

$$H^2(X, \Omega_X^2 \otimes L) \simeq H^{2,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$$

by Le Potier isomorphism [6, Lemma 8].
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