Basic properties of Lie-orthogonal operators on a finite-dimensional Lie algebra are studied. In particular, the center, the radical and the components of the ascending central series prove to be invariant with respect to any Lie-orthogonal operator. Over an algebraically closed field of characteristic 0, only a solvable Lie algebra with solvability degree not greater than two possesses Lie-orthogonal operators whose all eigenvalues differ from \( \pm 1 \). The main result of the paper is that Lie-orthogonal operators on a simple Lie algebra are exhausted by trivial ones. This allows us to give the complete description of Lie-orthogonal operators for semi-simple and reductive algebras, as well as a preliminary description of Lie-orthogonal operators on Lie algebras with nontrivial Levi–Mal’tsev decomposition. The sets of Lie-orthogonal operators of some classes of Lie algebras (the Heisenberg algebras, the almost Abelian algebras, etc.) are directly computed.

1 Introduction

The condition of Lie orthogonality of operators defined of Lie algebras naturally arises in the course of the study of certain structures connected to the Lie algebras such as Kähler manifolds and Clifford structures [7, 8, 10]. The simplest among these structures are Abelian complex structures on real Lie algebras [7, 8], i.e., linear operators satisfying the following two conditions:

1) \( J^2 = -\text{Id} \),
2) \( [Jx, Jy] = [x, y] \).

The object studied in the present paper is a class of operators satisfying the second condition alone, which is called the condition of Lie orthogonality. The renunciation of the first condition helps us to understand what properties of Abelian complex structures are implied by the second condition. For example, Proposition 3.1 from [7] directly follows from Theorem 1 of the present paper.

Moreover, Lie-orthogonal operators are also related to a particular case of so-called generalized differentiations in the same way as automorphisms are related to usual differentiations. Recall that a linear operator \( D \in \text{End}(L) \) is called a generalized \((\alpha, \beta, \gamma)\)-differentiation on a Lie algebra \( L \) if for fixed constants \( \alpha, \beta \) and \( \gamma \) the equation \( \alpha D[x, y] = \beta [Dx, y] + \gamma [x, Dy] \) holds for any elements \( x \) and \( y \) from the algebra \( L \) [11]. Lie-orthogonal operators are associated with \((0, 1, 1)\)-differentiation.

In this paper we enhance and generalize results from [11, 13] and completely describe the sets of Lie-orthogonal operators for certain classes of Lie algebras.

The paper has a following structure: The simplest notions and properties related to Lie-orthogonality are considered in Section 2. Elementary algebraic structures on the sets of such operators are also discussed. Actions of Lie-orthogonal operators on the center and the radical of a Lie algebras as well as on ideals generated by root subspaces and the center are studied in Section 3. In the same section we introduce the notion of equivalence of Lie-orthogonal operators, which is especially important in the course of consideration of non-centerless Lie algebras. Lie-orthogonal automorphism are also studied. It is proved in Section 4 that any simple Lie possesses only the trivial Lie-orthogonal operators. Using this fact, we completely describe Lie-orthogonal operators on semi-simple and reductive Lie algebras and derive some results for general Lie
algebras. The basis approach is applied in Section \[5\] to the direct calculation of the sets of Lie-orthogonal operators for important classes of Lie algebras including the special linear algebras, Heisenberg algebras, the almost Abelian algebras, etc.

2 Elementary properties of Lie-orthogonal operators

In this section we consider an arbitrary Lie algebra \(L\) as the imposed restrictions on the dimension and the underlying field are not essential.

2.1 Definitions and examples

**Definition 1.** A linear operator \(J\) on \(L\) is called \textit{Lie-orthogonal} if \([Jx,Jy]=[x,y]\) for any \(x,y \in L\).

**Example 1.** Each linear operator on an Abelian Lie algebra is Lie-orthogonal. The zero operator is Lie-orthogonal if and only if the corresponding Lie algebra is Abelian.

**Example 2.** For any Lie algebra \(L\), the identical operator \(\text{Id}_L\) as well as \(-\text{Id}_L\) are Lie-orthogonal over \(L\). We will call these operators \textit{trivial} Lie-orthogonal operators on \(L\). This is why only the Lie-orthogonal operators which are different from the trivial ones are of interest.

For certain Lie algebras the associated sets of Lie-orthogonal operators are exhausted by trivial operators.

**Example 3.** Consider the Lie algebra \(L = \text{sl}_2\). We choose a basis \(\{e_1,e_2,e_3\}\) such that the associated commutation relations are \([e_1,e_2] = e_3, [e_2,e_3] = e_1, [e_3,e_1] = e_2\), i.e. \([e_i,e_j] = e_k\) for an even transposition of the indices \(i, j, k\). In other words, we use the representation of the algebra \(\text{sl}_2\) as the algebra of three-dimensional vectors with the vector product as a Lie bracket.

Let \(J\) be a Lie-orthogonal operator on \(\text{sl}_2\). By definition, \(e_k = [e_i,e_j] = [Je_i,Je_j]\). Hereby the vector \(e_k\) is orthogonal to the vectors \(Je_i\) and \(Je_j\). Recomposing these relations, we obtain that for any \(i\) the vector \(Je_i\) is orthogonal to \(e_j\) and \(e_k\), and hence it is proportional to \(e_i\): \(Je_i = \lambda_i e_i\) for some constant \(\lambda_i\). The commutation relations imply that \(\lambda_i \lambda_j = 1, i \neq j\), from which either \(\lambda_1 = \lambda_2 = \lambda_3 = 1\) or \(\lambda_1 = \lambda_2 = \lambda_3 = -1\). This means that either \(J = \text{Id}_L\) or \(J = -\text{Id}_L\).

Therefore, the algebra \(\text{sl}_2\) admits only trivial Lie-orthogonal operators.

**Proposition 1.** Suppose that \(L = L_1 \oplus \cdots \oplus L_k\), i.e. \(L\) is a direct sum of its ideals \(L_1, \ldots, L_k\), and \(J_i\) is a Lie-orthogonal operator on \(L_i\), \(i = 1, \ldots, k\). Then the operator \(J = J_1 \oplus \cdots \oplus J_k\) is Lie-orthogonal on \(L\).

**Example 4.** If the Lie algebra \(L\) is a direct sum of its ideals \(L_1, \ldots, L_k\) and for each \(i = 1, \ldots, k\) either \(J_i = \text{Id}_{L_i}\) or \(J_i = -\text{Id}_{L_i}\) then the operator \(J = J_1 \oplus \cdots \oplus J_k\) is Lie-orthogonal on \(L\).

2.2 Algebraic structures related to Lie-orthogonal operators

**Proposition 2.** If \(L\) is a Lie algebra, \(J\) is a Lie-orthogonal operator on \(L\) and \(S\) is a subalgebra of \(L\) which is invariant with respect to \(J\) then the restriction of the operator \(J\) on \(S\) is a Lie-orthogonal operator on the subalgebra \(S\).

**Proposition 3.** If \(J\) is a Lie-orthogonal operator on an Lie algebra \(L\), then the operator \(-J\) is also Lie-orthogonal on this algebra.

**Proposition 4.** Let \(J\) be a non-degenerate Lie-orthogonal operator on a Lie algebra \(L\). Then the inverse \(J^{-1}\) of \(J\) is also a Lie-orthogonal operator on the algebra \(L\). The condition of Lie orthogonality of the operator \(J\) is equivalent to the condition \([Jx,y]=[x,J^{-1}y]\) for any elements \(x,y \in L\).
Proposition 5. The composition of Lie-orthogonal operators on a Lie algebra $L$ is a Lie-orthogonal operator on this algebra. Therefore, the set of Lie-orthogonal operators on $L$ is a monoid with involution with respect to the composition of operators. Non-degenerate Lie-orthogonal operators on $L$ form a group with respect to the same operation.

Using arguments from the proof of Theorem 3 from [1], we derive the following assertion:

Proposition 6. Let $J$ be a Lie-orthogonal operator on a Lie algebra $L$ and let $S$ be its automorphism. Then the operator $S^{-1}JS$ is also Lie-orthogonal on $L$.

Proof. For any $x$ and $y$ from the algebra $L$ the operator $\tilde{J} = S^{-1}JS$ satisfies the condition of Lie orthogonality: $[\tilde{J}x, \tilde{J}y] = [S^{-1}JSx, S^{-1}JSy] = S^{-1}[JSx, JSy] = S^{-1}[x, y]$. □

Corollary 1. The set of Lie-orthogonal operators on a Lie algebra $L$ is a $G$-set with $G = \text{Aut}(L)$.

2.3 Basis approach

Consider an $n$-dimensional Lie algebra $L$ and a Lie-orthogonal operator $J$ on $L$. We fix a basis $e_1, \ldots, e_n$ in $L$. The commutation relations of $L$ in the fixed basis take the form $[e_i, e_j] = \epsilon_{ij}^k e_k$, where $\epsilon_{ij}^k$ are components of the structure constant tensor of $L$, and the matrix of the operator $J$ is $(J_{ij})$. Here and in what follows the indices $i$, $j$ and $k$ run from 1 to $n$, and we assume summation over repeated indices.

We write down the definitions of Lie orthogonality for each pair of basis elements and expand all involved elements of the algebra with respect to the basis:

$[Je_i, Je_j] = [J_{i'i'}, J_{j'j'}e_{i'}e_{j'}] = J_{i'i'}J_{j'j'}[e_{i'}, e_{j'}] = J_{i'i'}J_{j'j'}\epsilon_{i'j'}^k e_k = [e_i, e_j] = \epsilon_{ij}^k e_k$,

from which $J_{i'i'}J_{j'j'}\epsilon_{i'j'}^k = \epsilon_{ij}^k$. In the matrix notation the last condition can be represented in the form

$J^T C^k J = C^k$,

where for each fixed $k$ the matrix $C^k = (\epsilon_{ij}^k)$ is skew-symmetric. This condition appears to be the condition of the invariance of the bilinear skew-symmetric form associated with the matrix $C_k$ with respect to the operator $J$ for each $k$. This implies the system of at most $n^2(n-1)/2$ quadratic inhomogeneous equations for the coefficients of the matrix of the operator $J$. The solution of this system completely describes the set of Lie-orthogonal operators on the Lie algebra $L$. (See Section 5 for examples of calculations involving the basis approach.)

3 Invariant ideals of Lie-orthogonal operators

In the theory of Lie-orthogonal operators an important role is played by invariant subspaces of these operators, which are also ideals of the corresponding Lie algebras.

3.1 Center

A special place among ideals that are invariant with respect to all Lie-orthogonal operators over a given Lie algebra $L$ belongs to the center $Z$ of $L$. The properties of Lie-orthogonal operators essentially depend on whether or no the center of Lie algebra is zero.

Lemma 1. Let $L$ be a finite-dimensional Lie algebra and $J$ be a Lie-orthogonal operator on $L$. Then $J(Z) \subseteq Z$, where $Z = Z(L)$ is the center of $L$. 

3
Proof. Suppose that \( x \in Z \). Then \([x, y] = 0\) for any \( y \in L \). It is necessary to prove that any \( y \in L \) satisfies the condition \([Jx, y] = 0\). We write down the Fitting decomposition (see e.g. (3)) of the space \( L \) with respect to the operator \( J: L = L_0 + \hat{L} \), where \( L_0 \) and \( \hat{L} \) are invariant subspaces of \( J \), and the restriction of \( J \) on \( L_0 \) (resp. \( \hat{L} \)) is a nilpotent (resp. invertible) operator. Here and in what follows the symbol “\( \oplus \)” denotes the direct sum of vector spaces. Hence for each \( y \) from \( L \) we have \( y = y_0 + y_1 \), where \( y_0 \in L_0 \) and \( y_1 \in \hat{L} \). Properties of \( J \), the definition of \( y_1 \) and linearity of Lie bracket imply that \( 0 = [x, y] = [Jx, Jy] = [Jx, Jy_1] \). As the operator \( J \) is invertible on \( \hat{L} \), the element \( Jy_1 \) runs through \( \hat{L} \) when the element \( y_1 \) runs through \( \hat{L} \). Given an arbitrary element \( \tilde{y}_0 \in L_0 \), the element \( \tilde{y} = \tilde{y}_1 + \tilde{y}_0 \) runs through the entire space \( L \) and the following equality is true: \( 0 = [Jx, Jy_1] = [Jx, \tilde{y}_1] = [Jx, \tilde{y}_1] + [Jx, \tilde{y}_0] = [Jx, \tilde{y}] \). \( \square \)

Lemma 2. Suppose that \( L \) is a finite-dimensional Lie algebra with nonzero center, \( Z \neq \{0\} \). \( \tilde{L} \) is an arbitrary subspace of the space \( L \), such that \( L = Z + \tilde{L} \). If \( J \) is a Lie-orthogonal operator on \( L \) and an operator \( \tilde{J} \) on \( L \) satisfies the conditions \( \tilde{J}|_{\tilde{L}} = J|_{\tilde{L}} \) and \( \tilde{J}(Z) \subseteq Z \) then \( \tilde{J} \) is a Lie-orthogonal operator on \( L \).

In other words, a Lie-orthogonal operator can be redefined on the center of the corresponding Lie algebra as one likes.

Proof. We represent arbitrary elements \( x \) and \( y \) from \( L \) in the form \( x = x_0 + x_1 \) and \( y = y_0 + y_1 \), where \( x_0, y_0 \in Z \) and \( x_1, y_1 \in \tilde{L} \). Then
\[
[Jx, Jy] = [J x_0 + \tilde{J} x_1, J y_0 + \tilde{J} y_1] = [\tilde{J} x_1, \tilde{J} y_1] = [J x_1, J y_1] =
\]
\[
= [x_1, y_1] = [x_0 + x_1, y_0 + y_1] = [x, y].
\]

This means that \( \tilde{J} \) is a Lie orthogonal operator. \( \square \)

The following assertion generalizes Lemma 2.

Lemma 3. Suppose that \( L \) is a finite-dimensional Lie algebra with nonzero center, \( Z \neq \{0\} \). \( J \) is a Lie-orthogonal operator on \( L \) and \( J_0 \) is an operator on \( L \) with the image contained in \( Z \): \( J_0 L \subseteq Z \). Then \( J + J_0 \) is a Lie-orthogonal operator on \( L \).

Proof. We take arbitrary elements \( x \) and \( y \) from \( L \). As \( J_0 x \) and \( J_0 y \) belong to the center by lemma’s conditions, we have \([J + J_0)x, (J + J_0)y] = [Jx, Jy] + [J_0 x, Jy] + [Jx, J_0 y] + [J_0 x, J_0 y] = [Jx, Jy] = [x, y]. \) This completes the proof. \( \square \)

In Lemma 2 the operator \( J_0 \) equals 0 on the complement of the center. Lemma 3 allows us to introduce an equivalence relation on the set of Lie-orthogonal operators on a fixed algebra \( L \).

Definition 2. We call Lie-orthogonal operators \( J \) and \( \tilde{J} \) on a Lie algebra equivalent if the image of their difference is contained in the center of this algebra.

If the algebra is centerless, only coinciding operators are equivalent.

Lemma 4 (see (3)). Let \( L \) be a finite-dimensional Lie algebra and \( J \) be a Lie-orthogonal operator on \( L \). Then \( L_0 \subseteq Z \), where \( L_0 \) is a root subspace of the algebra \( L \) which is associated with the zero eigenvalue of the operator \( J \) (i.e., the zero Fitting component of this algebra with respect to \( J \)).

Proof. We proof this lemma in another way than that in (3), without using the Jordan normal form of \( J \). Let \( k \) be the nilpotency degree of the restriction of \( J \) on \( L_0 \). Given arbitrary elements \( x \in L_0 \) and \( y \in L \), we have \( J^k x = 0 \). Since \( J^k \) is a Lie-orthogonal operator on \( L \), as \( J \) is, we obtain \([x, y] = [J^k x, J^k y] = [0, J^k y] = 0\), and hence \( x \) belongs to the center \( Z \). \( \square \)
**Corollary 2.** Any Lie-orthogonal operator on a centerless Lie algebra is invertible. Lie-orthogonal operators on such an algebra form a group.

**Corollary 3.** Any Lie-orthogonal operator on a finite-dimensional Lie algebra is equivalent to some invertible Lie-orthogonal operator on the same algebra.

Given a decomposition of an $n$-dimensional algebra $L$ into the direct sum of the center $Z$ and its complement $\tilde{L}$ as vector spaces ($L = Z \oplus \tilde{L}$), the “essential” part of any Lie-orthogonal operator $J$ on $L$ is the operator $PJP$, where $P$ is the operator of projection onto $\tilde{L}$ in the above decomposition of $L$. The operator $PJP$ is associated with the factorized operator $J/Z$ on the factor-algebra $L/Z$ (cf. Section 3.2). After fixing a basis in such a way that the first $k$ elements of it form a basis of $Z$, where $k = \dim Z$, and the others form a basis of $\tilde{L}$, we obtain that the matrix of any Lie-orthogonal operator $J$ on $L$ in the chosen basis can be represented in the form

$$
\begin{pmatrix}
B_0 & B_1 \\
0 & \tilde{J}
\end{pmatrix},
$$

where $B_0$ and $B_1$ are arbitrary $k \times k$ and $k \times (n-k)$ matrices, respectively, $0$ is the zero $(n-k) \times k$ matrix, and $\tilde{J}$ is the matrix of the restriction of $PJP$ on $\tilde{L}$. The matrix $\tilde{J}$ can also be interpreted as the matrix of the factorized operator $J/Z$ on the factor-algebra $L/Z$.

### 3.2 Radical and other special ideals

**Lemma 5.** Let $L$ be a finite-dimensional Lie algebra and $J$ be a Lie-orthogonal operator on $L$. Then $J(R) \subseteq R$, where $R = R(L)$ is the radical of the algebra $L$, i.e., its maximal solvable ideal.

**Proof.** By $K(x, y)$ we denote the Killing form of the algebra $L$. The radical $R$ is the orthogonal complement to the derivative $L'$ of the algebra $L$ with respect to the Killing form $K$ [2], i.e.,

$$R = \{ x \in L \mid \forall y, z \in L: K(x, [y, z]) = 0 \}.$$

Given an arbitrary $x$ from $R$ and arbitrary $y$ and $z$ from $L$, the definition of Lie-orthogonal operator and the associativity of the Killing form with respect to the Lie bracket imply

$$K(Jx, [y, z]) = K(Jx, [Jy, Jz]) = K([Jx, Jy], Jz) = K([x, y], Jz) = K(x, [y, Jz]) = 0.$$

This means that $Jx \in R$. \qed

It is clear from the proof of Lemma 3 from [1] that this lemma is true not only for invertible Lie-orthogonal operators. Therefore the lemma can be reformulated in the following way:

**Lemma 6.** Suppose that $I$ is an ideal of a Lie algebra $L$ such that the center of the factor-algebra $L/I$ is zero. Then the ideal $I$ is invariant with respect to the action of each Lie-orthogonal operator on the algebra $L$.

**Proof.** In the course of the factorization with respect to the ideal $I$, elements of the center of the algebra $L$ are mapped into the center of the factor-algebra $L/I$, which is zero by the lemma conditions. This implies that the center of the algebra is contained in the ideal $I$. Hence this ideal is invariant with respect to a Lie-orthogonal operator $J$ if and only if it is invariant with respect to any Lie-orthogonal operator which is equivalent to $J$. This is why we can assume, without loss of generality, that the operator $J$ is invertible.

Using the reformulated definition of Lie-orthogonal operator, we obtain that for any $x \in I$ and $y \in L$ the commutator $[Jx, y] = [x, J^{-1}y]$ belongs to the ideal $I$. Therefore, the equivalence class $Jx + I$ belongs to the center of the factor-algebra $L/I$. This implies that $Jx \in I$. \qed
If an ideal $I$ of a Lie algebra $L$ is invariant with respect to the action of a Lie-orthogonal operator $J$ on $L$ then the operator $J$ can be factorized consistently with factorizing the algebra $L$. The factor-operator $J/I$ on the factor-algebra $L/I$ is also Lie-orthogonal. At the same time, factorized Lie-orthogonal operators do not exhaust all possible Lie-orthogonal operators on the factor-algebra.

Iteratively combining Lemma 1 with factorizing with respect to the corresponding centers, we obtain a generalization of Corollary 2 from [1].

**Proposition 7.** Each element of the ascending central series of a Lie algebra $L$ is invariant with respect to any Lie-orthogonal operator on $L$.

**Lemma 7.** Let $L$ be a finite-dimensional centerless Lie algebra, which is decomposed into the direct sum of its ideals $I_1, \ldots, I_k$: $L = I_1 \oplus \cdots \oplus I_k$. An operator $J$ is Lie-orthogonal on $L$ if and only if it can be represented as

$$J = J_1 \oplus \cdots \oplus J_k,$$

where for any $i = 1, \ldots, k$ the operator $J_i$ is Lie-orthogonal on $I_i$. If the center of the algebra $L$ is not zero then the same representation is true up to the equivalence relation of Lie-orthogonal operators.

**Proof.** The statement of the lemma is equivalent to the fact that each of the ideals is invariant (up to the equivalence relation) with respect to the action of any Lie-orthogonal operator. It is sufficient to proof the invariance of the ideals $I_1, \ldots, I_k$ for the particular case $k = 2$. The center $Z$ of the algebra $L = I_1 \oplus I_2$ can be represented in the form $Z = Z_1 \oplus Z_2$, where $Z_1 = Z \cap I_1$ and $Z_2 = Z \cap I_2$. Moreover, $L/I_1 \simeq I_2$ and $L/I_2 \simeq I_1$. Then, if the center of the algebra $L$ is zero, the invariance of the ideals $I_1$ and $I_2$ directly follows from the lemma [6], as the centers of the ideals $I_1$ and $I_2$ are zero.

Suppose that the center of the algebra $L$ is not zero. Up to the equivalence relation we can consider only invertible Lie-orthogonal operators. We fix such an operator $J$. By $P_1$ and $P_2$ we denote the operators of the projection on the ideals $I_1$ and $I_2$ respectively, which are associated with the decomposition $L = I_1 \oplus I_2$. We have $P_1 + P_2 = \text{Id}_L$. For arbitrary elements $x \in I_1$ and $y \in I_2$ we derive that the commutator $[Jx, y] = [x, J^{-1}y]$ belongs to the intersection of the ideals $I_1$ and $I_2$, therefore, it is equal to 0. This means that $P_2Jx \in Z_2$ and $P_1Jy \in Z_1$. Hence the images of the operators $P_1JP_2$ and $P_2JP_1$ are contained $Z$. Consider the operator $\tilde{J} = J - P_1JP_2 - P_2JP_1$. It is equivalent to the operator $J$ in view of the definition. Moreover, if $x \in I_1$ then $Jx = Jx - P_2Jx = P_1Jx \in I_1$. Analogously, if $y \in I_2$ then $Jy = Jy - P_1Jy = P_2Jy \in I_2$. \qed

### 3.3 Root subspaces of Lie-orthogonal operators

Suppose that the underlying field is of characteristic 0 and algebraically closed.

**Lemma 8 ([13]).** Let $L$ be a finite-dimensional Lie algebra and $J$ be a Lie-orthogonal operator on $L$. Suppose additionally that $L_\lambda$ and $L_\mu$ are root subspaces of the algebra $L$ with respect to the operator $J$ which correspond to such eigenvalues $\lambda$ and $\mu$ that $\lambda \mu \neq 1$. Then $[L_\lambda, L_\mu] = 0$.

**Proof.** We prove this lemma without choosing a basis of the algebra $L$. Suppose that $\lambda$ and $\mu$ are such eigenvalues of the operator $J$ that $\lambda \mu \neq 1$. The corresponding root subspaces $L_\lambda$ and $L_\mu$ can be represented in the form:

$$L_\lambda = \bigcup_{i=0}^{k_\lambda} \ker(J - \lambda E)^i, \quad L_\mu = \bigcup_{j=0}^{k_\mu} \ker(J - \mu E)^j,$$

where $k_\lambda$ and $k_\mu$ denote the multiplicity of $\lambda$ and $\mu$ as the roots of the characteristic polynomial of the operator $J$ and $E$ denotes $\text{Id}_L$. Therefore, it suffices to proof that for any $i$ and $j$ arbitrary
elements \( x \in \ker(J - \lambda E)^i \) and \( y \in \ker(J - \mu E)^j \) commutate. The last assertion is proved by induction with respect to \( m = i + j \). If \( m = 0 \) then \( i = j = 0 \), both the elements \( x \) and \( y \) are zero as elements of the kernel of the identity operator and hence \( [x, y] = 0 \). Supposing that this assertion is true for all \((i, j)\) with \( i + j < m \), we prove it for \( m \). The definition of Lie orthogonality implies

\[
[x, y] = [Jx, Jy] = [(J - \lambda E)x, (J - \mu E)y] + [(J - \lambda E)x, \mu y] + [\lambda x, (J - \mu E)y]
\]

\[
+ [\lambda x, \mu y] = \lambda \mu [x, y].
\]

The first three summands are zero by the induction hypothesis as \((J - \lambda E)x \in \ker(J - \lambda E)^{i-1}\) and \((J - \mu E)y \in \ker(J - \mu E)^{j-1}\). Taking into account that \( \lambda \mu \neq 1 \), we obtain \( [x, y] = 0 \). \( \square \)

We generalize Lemma 3 from [13] and the main theorem of the same paper via getting rid of the condition that the corresponding algebra is centerless.

**Lemma 9.** Let \( L \) be a finite-dimensional Lie algebra and \( J \) be a Lie-orthogonal operator on \( L \). For any eigenvalue \( \lambda \) of the operator \( J \), we consider the subspace \( I_\lambda \), where \( I_\lambda = L_\lambda \oplus L_{\lambda^{-1}} \oplus Z \) if \( \lambda \notin \{\pm 1, 0\} \), \( I_\lambda = L_\lambda \oplus Z \) if \( \lambda = \pm 1 \), \( I_\lambda = Z \) if \( \lambda = 0 \), and \( Z \) is the center of \( L \). Then the subspace \( I_\lambda \) is an ideal of the algebra \( L \).

**Proof.** For the case \( \lambda = 0 \) the lemma is obvious. Suppose that \( \lambda \neq 0 \). Consider the subspace \( S_\lambda = \sum_{\mu \neq \lambda} L_\mu \). In view of Lemma 3, the centralizer \( C_\lambda \) of \( S_\lambda \) in the algebra \( L \) contains \( I_\lambda \). If an element of the centralizer does not belong to \( I_\lambda \) then \( U = S_\lambda \cap C_\lambda \cap (L \setminus I_\lambda) \neq \emptyset \). The elements of the intersection \( U \) commute with all elements of the subspace \( S_\lambda \) by the definition of centralizer as well as with all elements of the subspace \( I_\lambda \) according to Lemma 3 and the definition of center. We also have \( I_\lambda + S_\lambda = L \). Therefore, the intersection \( U \) is contained in the center \( Z \) which is subset of \( I_\lambda \). At the same time, \( U \) is contained in the compliment to \( I_\lambda \) by its definition. As a result, \( C_\lambda = I_\lambda \) and hence \( I_\lambda \) is a subalgebra of the algebra \( L \). Then Lemma 3 implies that \( I_\lambda \) is an ideal of this algebra. \( \square \)

**Corollary 4.** If the condition \( \lambda \neq \pm 1 \) is true, then \( I_\lambda \) is a solvable ideal of the algebra \( L \) with the solvability degree not greater than two.

**Proof.** As \( \lambda \neq \pm 1 \), Lemma 3 implies that \( L_\lambda \) and \( L_{\lambda^{-1}} \) are Abelian subalgebras of the algebra \( L \) and, therefore, of the ideal \( I_\lambda \). We introduce the notation \( Z_\lambda = L_\lambda \cap Z \) and choose the subspace \( \tilde{Z}_\lambda \) of the center \( Z \) such that \( Z = Z_\lambda \oplus \tilde{Z}_\lambda = Z_\lambda \oplus \tilde{Z}_\lambda \). Then we have \( I_\lambda = L_\lambda \oplus \tilde{L}_\lambda \), where \( \tilde{L}_\lambda = L_{\lambda^{-1}} \oplus \tilde{Z}_\lambda \). As the subalgebras \( L_\lambda \) and \( \tilde{L}_\lambda \) are Abelian, the ideal \( I_\lambda \) is a sum of two Abelian subalgebras. Therefore, it is solvable and its solvability degree is not greater than two (see e.g. [4] or Lemma 1 in [12]). \( \square \)

**Corollary 5.** If \( L \) is a finite-dimensional simple Lie algebra and \( J \) is a Lie-orthogonal operator on \( L \) then all eigenvalues of the operator \( J \) are equal to either 1 or -1.

**Proof.** A simple algebra is centerless, not solvable and contains no ideals. Therefore, the operator \( J \) has a single root subspace. The corresponding eigenvalue equals either 1 or -1. \( \square \)

**Corollary 6.** If \( L \) is a finite-dimensional semi-simple Lie algebra and \( J \) is a Lie-orthogonal operator on \( L \) then each eigenvalue of the operator \( J \) is equal to either 1 or -1. Each simple component of the algebra \( L \) is contained in one of the root subspaces.

**Proof.** A semi-simple algebra is centerless and contains no nonzero solvable ideals. Therefore, the algebra \( L \) is a direct sum of two root subspaces, \( L_1 \) and \( L_{-1} \), which are also ideals and correspond to the eigenvalue 1 and -1, respectively. As any simple component of the algebra \( L \) is an ideal in this algebra and does not contain proper ideals, it either do not intersect with one of the above root subspaces or is contained in it. \( \square \)
Theorem 1. Let $L$ be a finite-dimensional Lie algebra and $J$ be a Lie-orthogonal operator on $L$ and all its eigenvalues be different from ±1. Then $L$ is a solvable Lie algebra of solvability degree not greater than two.

Proof. Theorem conditions imply that $L$ is a sum of the ideals $I_\lambda$, where $\lambda$ runs through nonzero eigenvalues of the operator $J$. (See the definition of $I_\lambda$ in Lemma 9.) Note that this sum is not direct as the intersection of every pair of these ideals coincides with the center $Z$ of the algebra $L$. For each $\lambda$ the ideal $I_\lambda$ is a solvable ideal of solvability degree not greater than two. These ideals commutate to each other. Therefore, the algebra $L$ is also solvable of solvability degree not greater than two.

3.4 Lie-orthogonal automorphisms

In this section we study the intersection of the automorphisms group of a Lie algebra $L$ and the set of its Lie-orthogonal operators.

Lemma 10. Let $J$ be a Lie-orthogonal automorphism on $L$ and $L' = [L, L]$ denote the derivative of $L$. Then $\ker(J - \text{Id}_L) \supset L'$ and $[\text{Im}(J - \text{Id}_L), L'] = 0$.

Proof. For any elements $x$ and $y$ from $L$ we have $J[x, y] = [Jx, Jy] = [x, y]$. This implies that $(J - \text{Id}_L)[x, y] = 0$. Therefore, $\ker(J - \text{Id}_L) \supset L'$. Analogously, for any elements $x, y$ and $z$ from $L$ we get $[[x, y], Jz] = [[Jx, Jy], Jz] = [J[x, y], Jz] = [[x, y], z]$, i.e., $[[x, y], Jz - z] = 0$. This means that $[\text{Im}(J - \text{Id}_L), L'] = 0$.

Corollary 7. Any Lie-orthogonal automorphism on a perfect algebra is identical.

Recall that a Lie algebra is called perfect if it coincides with its derivative.

Lemma 11. Let a Lie algebra $L$ be finite-dimensional and $J$ be a Lie-orthogonal automorphism on $L$. Then $\text{Im}(J - \text{Id}_L) \subset R$, where $R = R(L)$ is the radical of $L$.

Proof. We fix arbitrary elements $x, y$ and $z$ from $L$. Using the invariance of Killing form $K(x, y)$ of the algebra $L$ with respect to each automorphism of $L$, we derive that $K([Jx, Jy], Jz) = K([x, y], z)$. At the same time, Lie orthogonality of $J$ implies that $K([Jx, Jy], Jz) = K([x, y], Jz)$. Therefore, $K([x, y], Jz - z) = 0$, i.e., the image $\text{Im}(J - \text{Id}_L)$ is contained in the orthogonal complement to the derivative $L'$ with respect to the Killing form $K$. As well known, this complement coincides with the radical $R$.

Let $I_\mu$ denote the ideal of the algebra $L$, that is introduced in Lemma 9. Suppose that the underlying field of $L$ is algebraically closed.

Corollary 8. Let $J$ be a Lie-orthogonal automorphism of a finite-dimensional Lie algebra $L$. Then for any eigenvalue $\mu$ of $J$, which is not equal to one, the corresponding ideal $I_\mu$ is a nilpotent ideal of nilpotence degree not greater than two.

Proof. As $I_\mu$ is an ideal of $L$, the commutator $[I_\mu, I_\mu]$ is contained in $I_\mu$. Moreover, $[I_\mu, I_\mu] \subset L'$ by the definition of $L'$ and $L' \in I_1$ in view of Lemma 10. Hence $[I_\mu, I_\mu] \subset I_\mu \cap I_1 = Z$, where $Z$ is the center of $L$, i.e., the nilpotence degree of $I_\mu$ is not greater than two.

Corollary 9. All eigenvalues of any Lie-orthogonal automorphism of a centerless finite-dimensional Lie algebra are equal to one.

Proof. Let $J$ be a Lie-orthogonal automorphism on a centerless finite-dimensional Lie algebra $L$. As the center $Z$ of $L$ is zero, the operator $J$ has no zero eigenvalues. We fix a number $\nu$ that is not zero or one. Consider the corresponding ideal $I_\nu$. Analogously to the previous corollary, $[I_\nu, I_\nu] \subset Z$. As $[I_\nu, I_\nu] = 0$ for any eigenvalue $\mu$ that is not equal to $\nu$ or $\nu^{-1}$, and the algebra $L$
is a sum of the ideals $I_\mu$, where the subscript $\mu$ runs through the set of eigenvalues of the operator $J$, the ideal $I_\nu$ is contained in $Z$. Therefore, $I_\nu = \{0\}$, i.e., $\nu$ is not an eigenvalue of $J$. 

Lemma 12. Let $J$ be a Lie-orthogonal automorphism on a finite-dimensional Lie algebra $L$. Then $L$ can be represented as a sum of three ideals $(I_1 + I) \oplus Z_0$, where the restriction of $J$ on $I_1$ has only unit eigenvalues, $I$ is a nilpotent ideal of nilpotence degree not greater than two and the ideal $Z_0$ is Abelian. Moreover, $I_1 \cap I = L' \cap Z$.

Proof. We consider the ideal $Z_1 = Z \cap L'$ and take a subspace $Z_0$ of $Z$ such that $Z = Z_1 \oplus Z_0$. Then for each eigenvalue $\lambda$ of the operator $J$ we define subspaces $I_\lambda$ in a similar way as the ideals $I_\mu$ in Lemma 9 using $Z_0$ instead of $Z$, i.e., $I_\lambda = L_\lambda \oplus L_{\lambda-1} \oplus Z_0$ if $\lambda \notin \{\pm 1, 0\}$, $I_\lambda = L_\lambda \oplus Z_0$ if $\lambda = \pm 1$, $I_\lambda = Z_0$ if $\lambda = 0$. By $I$ we denote the sum of ideals $I_\lambda$ associated with eigenvalues $\lambda$ of $J$ which are not equal to one. In view of Lemma 8 and Corollary 8, $\hat{\lambda}$ is a nilpotent ideal of nilpotence degree 2. It is obvious that $I_1 \cap I = Z_1 = L' \cap Z$, $I_1 \cap Z_0 = I \cap Z_0 = \{0\}$. As the algebra $L$ can be represented as a sum of all the ideals $I_\mu$ and $Z_0$, we have $L = (I_1 + I) \oplus Z_0$. 

4 Lie-orthogonal operators on Lie algebras with nonzero Levi factor

4.1 Semi-simple algebras

The rigid structure of semi-simple Lie algebras imposes strong restrictions on the corresponding sets of Lie-orthogonal operators.

Lemma 13. The polynomial $x^2 - 1$ annuls any Lie-orthogonal operator on a semi-simple finite-dimensional Lie algebra.

Proof. Let $K(x, y)$ be a Killing form of the semi-simple Lie algebra $L$ and $J$ be a Lie-orthogonal operator on $L$. By the definition of Lie orthogonality and in view of the associativity of the Killing form with respect to Lie bracket [6], for arbitrary $x$, $y$ and $z$ from $L$ we have the following equalities:

\[ K([Jx, Jy], Jz) = K(Jx, [Jy, Jz]) = K(Jx, [y, z]) = K([Jx, y], z), \]

\[ K([Jx, Jy], Jz) = -K([Jy, Jx], Jz) = -K(Jy, [Jx, Jz]) = \]

\[ = -K(Jy, [x, z]) = -K([Jy, x], z) = K([x, Jy], z). \]

Therefore, $K([Jx, y], z) = K([x, Jy], z)$, i.e., $K([Jx, y] - [x, Jy], z) = 0$. As the element $z$ is arbitrary and the form $K$ is nondegenerate as a Killing form of a semi-simple algebra, we have $[Jx, y] - [x, Jy] = 0$, i.e., $[Jx, y] = [x, Jy]$ for any $x$ and $y$ from $L$. Combining this property with the definition of Lie-orthogonal operators implies $[x, y] = [Jx, Jy] = [x, J^2 y]$, whence $[x, J^2 y - y] = 0$ for all $y$ from $L$. This means that $J^2 - \text{Id}_L = 0$. In other words, the polynomial $x^2 - 1$ annuls $J$. 

It follows from Lemma 13 that each eigenvalues of the operator $J$ is equal to either 1 or $-1$. This agrees with Corollaries 5 and 6. At the same time, the statement of this lemma is much stronger: It guarantees the existence of a basis of $L$, which is formed by eigenvectors of $J$. Taking into account Corollary 6, we obtain the following assertions:

Theorem 2. Lie-orthogonal operators on any simple finite-dimensional Lie algebra are exhausted by the trivial operators $\text{Id}_L$ and $-\text{Id}_L$. 

9
Theorem 3. Let $L$ be a semi-simple finite-dimensional Lie algebra and $L = L_1 \oplus \cdots \oplus L_k$ be its decomposition into simple components. Then any Lie-orthogonal operator $J$ on $L$ can be represented in the form

$$J = J_1 \oplus \cdots \oplus J_k,$$

where for any $i = 1, \ldots, k$ we have either $J_i = \text{Id}_{L_i}$ or $J_i = -\text{Id}_{L_i}$.

In other words, any Lie-orthogonal operator $J$ leads to the partition of the semi-simple algebra $L$ into two ideals. The operator $J$ acts on one of the ideals as $\text{Id}$ and on other as $-\text{Id}$.

4.2 Reductive algebras

Combining Lemma 7 with Theorem 8 we can directly obtain the complete description of Lie-orthogonal operators on reductive Lie algebras.

Corollary 10. Let $L$ be a finite-dimensional reductive Lie algebra and $L = Z \oplus L_1 \oplus \cdots \oplus L_k$ be its decomposition into the center and simple components. An operator $J$ is Lie-orthogonal on $L$ if and only if it can be represented in the form

$$J = J_0 \oplus J_1 \oplus \cdots \oplus J_k + J_Z,$$

where for each $i = 1, \ldots, k$ we have either $J_i = \text{Id}_{L_i}$ or $J_i = -\text{Id}_{L_i}$, $J_0$ is the identically zero operator on $Z$ and $J_Z$ is an arbitrary operator on $L$ whose image is contained in the center $Z$.

4.3 Levi factor and root subspaces of operators

Suppose that the underlying field is of characteristic 0 and algebraically closed. In view of Lemma 9 a Lie algebra possessing a Lie-orthogonal operator $J$, can be represented as the sum of the ideals $I_\lambda$, each of which has either the form $I_\lambda = L_\lambda \oplus L_{\lambda^{-1}} \oplus Z$ if $\lambda \not\in \{\pm 1, 0\}$ or the form $I_\lambda = L_\lambda \oplus Z$ if $\lambda = \pm 1$. Here each $\lambda$ is a nonzero eigenvalue of the operator $J$. If $\lambda^{-1}$ is also an eigenvalue of this operator, we omit the corresponding ideal from the sum in order to avoid a repetition. This sum is not direct in the general case since each of these ideals contains the center $Z$ of $L$. Nevertheless, each of these ideals can be separately studied, up to the equivalence, with respect to the restrictions imposed on their structure by the operator $J$. This is possible because these ideals are invariant subspaces of the operator $J$ and additionally $[I_\lambda, I_\mu] = 0$ when $\lambda \not= \mu$ and $\mu \lambda \not= 1$.

Each ideal $I_\lambda$, where $\lambda \not= \pm 1$, is solvable and hence is contained in the radical $R$ of the algebra $L$. Therefore, only the ideals $I_1$ and $I_{-1}$ may have nonzero intersections with a Levi factor of $L$. Due to the involution $J \to -J$ on the set of Lie-orthogonal operators, it suffices to study the ideal $I_1$.

The intersections of the radical $R$ and a Levi factor $S$ of the algebra $L$ with $I_1$ are the radical $R_1$ and a Levi factor $S_1$ of $I_1$, respectively, see, e.g., Corollary 4 from [2]. Note that $S = S_1 \oplus S_{-1}$, where $S_{-1} = S \cap I_{-1}$. Lemma 5 implies that the radical $R_1$ is invariant with respect to the operator $J$ and hence with respect to the restriction of $J$ on $I_1$, too. The factor-algebra of the ideal $I_1$ with respect to its radical $R_1$ is isomorphic to the subalgebra $S_1$ and is obviously semi-simple. As the radical $R_1$ is invariant with respect to $J$, the factorization of the restriction $J_1$ of the operator $J$ on $I_1$ with respect to $R_1$ gives the well-defined operator $J_1/R_1$ on the factor-algebra $I_1/R_1$ with the single eigenvalue that is equal to 1. In view of Theorem 3 the factorized operator $J_1/R_1$ is identical. Therefore, the image of the operator $J_1 = \text{Id}_{I_1}$ is contained in $R_1$, and this operator is nilpotent by the definition of $I_1$ up to the equivalence relation. We sum up the obtained result as the following assertion:
Proposition 8. Let $J$ be a Lie-orthogonal operator on a finite-dimensional Lie algebra $L$ and $I_1$ be an ideal that corresponds to the eigenvalue 1. Then the restriction $J_1$ of the operator $J$ on the ideal $I_1$ can be represented, up to equivalence of Lie-orthogonal operators on $I_1$, in the form $J_1 = \text{Id}_{I_1} + N$, where $N$ is a nilpotent operator on $I_1$ and the image of $N$ is contained in the radical $R_1$ of the ideal $I_1$.

5 Direct calculation of Lie-orthogonal operators

For some classes of Lie algebras of simple structure, we can completely describe their Lie-orthogonal operators using only the definition of Lie-orthogonality and commutation relations in the canonical bases of these algebras. Such computation is often a necessary step in the study of Lie-orthogonal operators, giving a base for making conjectures about general properties of these operators.

5.1 Special linear algebras

For convenience we calculate Lie-orthogonal operators on $\text{gl}_n$ instead of $\text{sl}_n$. This is possible because the algebra $\text{gl}_n$ is a central extension of the algebra $\text{sl}_n$: $\text{gl}_n = \text{sl}_n \oplus \langle E_n \rangle$, where $E_n$ is the unit matrix of size $n$, which generates the center $\langle E_n \rangle$ of the algebra $\text{gl}_n$. Hence in view of Lemma 3 the operator $J$ is Lie-orthogonal on the algebra $\text{gl}_n$ if and only if the operator $PJ|_{\text{sl}_n}$ is Lie-orthogonal on the algebra $\text{sl}_n$. Here $P$ denotes the projection operator from $\text{gl}_n$ onto $\text{sl}_n$, which is associated with the above decomposition of $\text{gl}_n$.

Note that the Lie algebra $\text{sl}_n$ is simple for any $n$. Hence any Lie-orthogonal operator on it is nondegenerate. This is why we can always find a corresponding nondegenerate Lie-orthogonal operator on its central extension, which is $\text{gl}_n$. As a result, it is sufficient to consider only nondegenerate Lie-orthogonal operators on $\text{gl}_n$.

Let $J$ be such an operator. Then the condition of Lie orthogonality can be represented in the form $[Jx, y] = [x, J^{-1}y]$. We write down the last equality for the matrix units $x = E^{ij}$ and $y = E^{kl}$ of the algebra $\text{gl}_n$ for fixed values of the indices $i$, $j$, $k$ and $l$ from $\{1, \ldots, n\}$. For this we introduce the notations $JE^{ij} = A^{ij}$ and $J^{-1}E^{kl} = B^{kl}$. The expansion by the standard matrix basis for $A^{ij}$ and $B^{kl}$ takes the form $A^{ij} = A_{pq}^{ij}E^{pq}$ and $B^{kl} = B_{pq}^{kl}E^{pq}$. Here and in what follows we assume summation from 1 to $n$ by the repeated indices $p$ and $q$. Therefore, the Lie-orthogonality condition for the matrix units is $[A^{ij}, E^{kl}] - [E^{ij}, B^{kl}] = [A^{ij}, E^{kl}] + [B^{kl}, E^{ij}] = 0$.

After expanding of $A^{ij}$ and $B^{kl}$ by the basis, we obtain

$$A_{jk}^{ij}E^{pl} - A_{lj}^{ij}E^{kq} + B_{ps}^{kl}E^{pq} - B_{jq}^{kl}E^{iq} = 0.$$ 

Considering different possibilities for the values of the indices $i$, $j$, $k$ and $l$ from $\{1, \ldots, n\}$, we collect the coefficients of basis elements in the last equality and equate these coefficients to zero. As a result, we derive the following system:

$$k \neq i, \ l \neq j$$

$$E^{pl}, \ p \neq i, \ k: \ A_{jk}^{ij} = 0, \quad E^{iq}, \ q \neq j, \ l: \ B_{lj}^{kl} = 0,$$

$$E^{qk}, \ q \neq j, \ l: \ A_{lj}^{ij} = 0, \quad E^{iq}, \ q \neq j, \ l: \ B_{jq}^{kl} = 0,$$

$$E^{il}: \ A_{ik}^{ij} = B_{jl}^{kl}, \quad E^{kl}: \ A_{kk}^{ij} = A_{il}^{ij},$$

$$E^{kj}: \ A_{lj}^{ij} = B_{ki}^{kl}, \quad E^{ij}: \ B_{ii}^{kl} = B_{jj}^{kl},$$

$$k = i, \ l \neq j$$

$$E^{pl}, \ p \neq i: \ A_{pi}^{ij} = 0, \quad E^{p_{i}}, \ p \neq i: \ B_{pk}^{kl} = 0,$$

$$E^{qk}, \ q \neq j, \ l: \ A_{ij}^{ij} + B_{jq}^{kl} = 0,$$
\[ E^{il} : A^{ij}_{il} = A^{ij}_{ll} + B^{kl}_{jl}, \quad E^{ij} : B^{kl}_{kk} = A^{ij}_{ij} + B^{kl}_{jj}, \]

\( k \neq i, l = j \)

\[ E^{kq}, q \neq j : A^{ij}_{kj} = 0, \quad E^{iq}, q \neq j : B^{kl}_{iq} = 0, \]

\[ E^{pj} , p \neq k, i : A^{ij}_{pk} + B^{kl}_{pi} = 0, \]

\[ E^{ij} : A^{ij}_{kk} = A^{ij}_{ii} + B^{kl}_{jj}; \]

\( k = i, l = j \)

\[ E^{pj} , p \neq i : A^{ij}_{pi} + B^{kl}_{pk} = 0, \quad E^{iq}, q \neq j : A^{ij}_{jq} + B^{kl}_{iq} = 0, \]

As the system is symmetric with respect to \( A^{ij} \) and \( B^{kl} \), it suffices to study only the constraints, imposed on \( A^{ij} \). The system implies that \( A^{ij}_{pq} = 0 \) if \( p \neq q \) and \( (p, q) \neq (i, j) \); \( A^{ij}_{kk} = A^{ij}_{ii} \) if \( i \neq j \); and \( A^{ii}_{kk} = A^{ii}_{ii} \) if \( k, l \neq i \). This means that \( A^{ij} = \lambda_{ij} E^{ij} + \kappa_{ij} E_{ij} \), where \( \lambda_{ij} \) and \( \kappa_{ij} \) are constants. Since the values of the operator \( J \) on the basis elements are defined up to adding elements of the center, the constant \( \kappa_{ij} \) can be assumed zero, e.g., \( A^{ij} = JE^{ij} = \lambda_{ij} E^{ij} \). This means that each basis element \( E^{ij} \) is an eigenvector of the operator \( J \). Then for an arbitrary index triple \( (i, j, k) \) such that \( (i, k) \neq (k, j) \) we obtain \([E^{ik}, E^{kj}]= [JE^{ik}, JE^{kj}] = \lambda_{ik} \lambda_{kj} [E^{ik}, E^{kj}] \). Hence \( \lambda_{ik} \lambda_{kj} = 1 \) since \([E^{ik}, E^{kj}] \neq 0 \). Therefore, either all \( \lambda_{ij} \) equal 1 or they all equal -1.

As a result, we obtain Theorem 2 for the particular case \( L = sl_n \).

### 5.2 Heisenberg algebras

Consider the Heisenberg algebra \( h_n \) for a fixed values of \( n \). This is a nilpotent Lie algebra of dimension \( 2n + 1 \) and nilpotency degree 2 with one-dimensional center \( Z \). We fix a basis \( \{e, p_1, \ldots, p_n, q_1, \ldots, q_n\} \) of the algebra \( h_n \), in which the nonzero commutation relations take the canonical form

\[ [p_j, q_j] = e, \]

and hence the center of \( h_n \) is \( Z = \langle e \rangle \). Here and in what follows the indices \( i, j \) and \( k \) run from 1 to \( n \).

Let \( J \) be a Lie-orthogonal operator on \( h_n \) and \( \tilde{J} \) be its essential part, i.e., \( \tilde{J} = (P J)|_{\hat{h}} \), where \( \hat{h} = \langle p_i, q_j \rangle \) and \( P \) is the projection operator onto \( \hat{h} \) in the representation \( h_n \). We denote the matrix of the operator \( \tilde{J} \) in the canonical basis \( \{p_1, \ldots, p_n, q_1, \ldots, q_n\} \) by the same symbol as the operator. In accordance with the basis partition into \( \{p_1, \ldots, p_n\} \) and \( \{q_1, \ldots, q_n\} \), we split the matrix \( \tilde{J} \) into blocks:

\[ \tilde{J} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{where} \quad A, B, C, D \in M_n. \]

Then \( \tilde{J} p_j = p_i a_{ij} + q_i c_{ij} \) and \( \tilde{J} q_k = p_i b_{ik} + q_i d_{ik} \). Here and in what follows we assume with respect to repeated index \( i \). In view of the definition of Lie-orthogonal operators, we have

\[ [J p_j, J q_k] = (a_{ij} d_{ik} - c_{ij} b_{ik}) e = \delta_{ik} e, \]

\[ [J p_j, J p_k] = (a_{ij} c_{ik} - c_{ij} a_{ik}) e = 0, \]

\[ [J q_j, J q_k] = (b_{ij} d_{ik} - d_{ij} b_{ik}) e = 0, \]

where \( \delta_{jk} \) is the Kronecker delta. Using the matrix notation these equalities are written in the form

\[ A^T D - C^T B = E, \quad A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \]
where $E \in M_n$ is the unit matrix. This means that $\tilde{J} \in \text{Sp}_{2n}$. Indeed, let

$$S = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

be the matrix of the canonical symplectic form. Then

$$\tilde{J}^T S \tilde{J} = \begin{pmatrix} C^T A - A^T C & C^T B - A^T D \\ D^T A - B^T C & D^T B - B^T D \end{pmatrix} = S.$$  

As a result, the following assertion is true:

**Theorem 4.** An operator is Lie-orthogonal on the Heisenberg algebra $h_n$ if and only if in the canonical basis $\{e, p_1, \ldots, p_n, q_1, \ldots, q_n\}$ its matrix takes a form

$$\begin{pmatrix} r & R \\ 0 & \tilde{J} \end{pmatrix},$$

where $r$ is an arbitrary scalar, $R$ is an arbitrary matrix from $M_{1,2n}$, $0$ denotes the zero matrix from $M_{2n,1}$ and $\tilde{J}$ is an arbitrary matrix from $\text{Sp}_{2n}$.

### 5.3 Almost Abelian algebras

A Lie algebra is called almost Abelian if it contains an Abelian ideal of codimension 1. Any almost Abelian algebra is solvable of solvability degree not greater than two.

Let $L$ be an almost Abelian but not Abelian algebra of dimension $n$. We choose such a basis in $L$ that $e_1, \ldots, e_{n-1}$ form a basis of an $(n-1)$-dimensional Abelian ideal $I$ containing in $L$. Then the equations $[e_n, e_j] = \sum_{i=1}^{n-1} a_{ij} e_i$ exhaust all nonzero commutation relations of the algebra $L$. Here and in what follows the indices $i, j, i', j'$ run from 1 to $n-1$ and indices $k$ and $l$ run from 1 to $n$. We assume summation with respect to repeated indices. Thus, the nonzero matrix $A = (a_{ij}) \in M_{n-1}$ completely defines the almost Abelian algebra $L$. Note that the center of the algebra $L$ coincides with the kernel of the operator $A$, which acts on $I$ and is defined by the matrix $A$. Almost Abelian Lie algebras $L$ and $\tilde{L}$ are isomorphic if the corresponding matrices $A$ and $\tilde{A}$ are similar up to a constant multiplier, i.e., there exist a nonzero constant $\mu$ and a nondegenerate matrix $S \in M_{n-1}$ such that $\tilde{A} = \mu S^{-1} AS$.

**Proposition 9.** Let $L$ be an $n$-dimensional almost Abelian but not Abelian Lie algebra with a fixed basis such that $\langle e_1, \ldots, e_{n-1} \rangle$ is an Abelian ideal of $L$ and $\langle e_1, \ldots, e_k \rangle = Z$, where $Z$ is the center of $L$ and $k = \dim Z < n-1$. In the chosen basis the nonzero commutation relations take the form $[e_n, e_j] = a_{ij} e_i$ and define the matrix $A = (a_{ij})$, where the first $k$ columns of $A$ are zero and the rank of $A$ equals $n-k$. An operator $J$ on $L$ is Lie-orthogonal if and only if its matrix in the basis $\{e_1, \ldots, e_n\}$ has one of the following forms depending on $\dim Z$:

1. $k = \dim Z < n-2$:

$$\begin{pmatrix} B_0 & B_1 & B_2 \\ 0 & \mu E & B_3 \\ 0 & 0 & \mu^{-1} \end{pmatrix},$$

where $B_0$, $B_1$, $B_2$ and $B_3$ are arbitrary $k \times k$, $k \times (n-k-1)$, $k \times 1$ and $n-k-1 \times 1$ matrices, respectively, $\mu$ is an arbitrary nonzero constant, $E$ is the unit $n-k-1$ matrix and zeros denote zero matrices of appropriate dimensions.

2. $k = \dim Z = n-2$, i.e., the algebra $L$ is isomorphic to either $(n-2)g_1 \oplus g_2$ or $(n-3)g_1 \oplus h_3$:

$$\begin{pmatrix} B_0 & B_1 \\ 0 & C \end{pmatrix},$$

where $B_0$ and $B_1$ are arbitrary $(n-2) \times (n-2)$ and $(n-2) \times 2$ matrices, respectively, $0$ is the zero $2 \times 2n$ matrix, and $C$ is an arbitrary $2 \times 2$ matrix with the determinant equal to one, i.e., $C \in \text{SL}_2$. 

13
Proof. We split the matrix of the operator $J$ into the blocks with respect to the representation of the algebra $L$ as the direct sum of the spaces $I = \langle e_1, \ldots, e_{n-1} \rangle$ and $\langle e_n \rangle$:

$$J = \begin{pmatrix} J_{II} & J_{In} \\ J_{In} & J_{nn} \end{pmatrix}.$$ 

In view of the definition of Lie-orthogonal operators, we have

$$0 = [e_i, e_j] = [J e_i, J e_j] = [J_{i,i} e_i, J_{j,j} e_j] = (J_{i,j} J_{j,i} - J_{i,j} J_{i,j} \gamma_j) a_{j,j}' e_j',$$

$$a_{j,j}' e_j' = [e_n, e_j] = [J e_n, J e_j] = [J_{k,n} e_k, J_{j,j} e_j] = (J_{n,j} J_{j,j} - J_{n,j} J_{n,j} \gamma_j) a_{j,j}' e_j',$$

or, in the matrix notation,

$$J_{mi}(AJ_{II})_{j,j} = J_{nj}(AJ_{II})_{j,j},$$

$$J_{mn}(AJ_{II})_{j,j} - J_{nj}(AJ_{jn})_{j,j} = a_{j,j}'.$$ (1)

The system (1) implies that either $J_{nj} = 0$ for each $j$ or there is such $j_0$ that $J_{nj_0} \neq 0$. We consider these alternatives separately.

1. Suppose that $J_{nj} = 0$ for each $j$. Then it follows from (2) that $J_{mn} A J_{II} = A$. As the matrix $A$ is nonzero, we have $J_{n,n} \neq 0$. We denote $1/J_{nn}$ by $\mu$. The system (2) takes the form $A (J_{II} - \mu E) = 0$. Therefore, ker $A$, which coincides with the center $Z$ of $L$, contains the image of the operator $J_{II} - \mu E$. It proves the first case of the theorem.

2. Suppose that there is such $j_0$ that $J_{nj_0} \neq 0$. Then all tuples $(J_{II})_{1,j}, \ldots, (J_{II})_{n-1,j}, J_{nj})$ are proportional, i.e., $J_{nj} = \lambda_j J_{nj_0}$ and $(J_{II})_{ij} = \lambda_j (J_{II})_{ij_0}$ for some constants $\lambda_j$. Substituting the obtained expressions into the system (2), we obtain that $a_{ij} = \kappa_i \lambda_j$, where $\kappa_i = J_{nn} (J_{II})_{ij_0} - J_{nj_0} (J_{II})_{ij}$. As $A \neq 0$, this means that the rank of the matrix $A$ equals one. Up to the choice of basis, it can be assumed that either $A = E^{n-1,n-1}$ or $A = E^{n-2,n-1}$. In the term of $\kappa$ and $\lambda$ we have that $\lambda_1 = \cdots = \lambda_{n-2} = 0$ and hence $j_0 = n - 1$ and $\lambda_{n-1} = 1$. Moreover, either $\kappa_{n-1}$ or $\kappa_{n-2}$ is equal to one, respectively, and all the other $\kappa$’s are zero. The equation $J_{nj} = \lambda_j J_{n,n-1}$ implies that $J_{nj} = 0$ for each $j$ from 1 to $n - 2$. Analogously, the equation $(AJ_{II})_{ij} = \lambda_j (AJ_{II})_{i,n-1}$ for either $i = n - 1$ or $i = n - 2$ respectively gives that $J_{n,n-1} = 0$. One more equation $J_{n-1,n-1} J_{nn} - J_{n-1,n} J_{n,n-1} = 1$ follows from the definition of $\kappa$. The above equations form complete system on the entries of the matrix of $J$. This leads to the second case of the theorem. \qed

5.4 Solvable algebras with nilradicals of minimal dimension

It was proven by Mubarakzyanov [5] Theorem 5) that the dimension of the nilradical $N$ of an $n$-dimensional solvable Lie algebra $L$ over a field of characteristic zero is not greater than $n/2$. If $\dim N = n/2$ in the case of even $n$ then the algebra $L$ is the direct sum of $n/2$ copies of the two-dimensional non-Abelian Lie algebra $g_2$: $L = (n/2)g_2$ [5] Theorem 6]. Theorem 7 from the same paper [5] implies that in the case of odd $n$ the minimal dimension of the nilradical equals $[n/2] + 1 = (n+1)/2$ and the algebra $L$ with the nilradical of this dimension can be decomposed into the direct sum of a single one-dimensional (Abelian) Lie algebra $g_1$ and $n/2$ copies of the algebra $g_2$, i.e., $L = g_1 \oplus [n/2]g_2$. In view of Lemma [7] any Lie-orthogonal operator $J$ on the algebra $L$ are decomposed into the direct sum of Lie-orthogonal operators on the above components of the algebra $L$. In the case of odd dimension the algebra $L$ has the nonzero center $Z = g_1$ and so that decomposition is up to the Lie-orthogonal operators equivalence. As the algebra $g_2$ is centerless, Lie-orthogonal operators on this algebra form a group. It can be directly calculated (see also [1] Example 1) that this group coincides with $\text{SL}_2$. After taking into account Lemma [8] we obtain the following assertion.
Proposition 10. Suppose that the nilradical of an \( n \)-dimensional solvable Lie algebra \( L \) is of minimal dimension which is equal to \( [n+1]/2 \). If the dimension \( n \) is even, the algebra \( L \) is isomorphic to the algebra \( [n/2]g_2 \) and Lie-orthogonal operators on \( L \) form a group isomorphic to the direct sum of \( n/2 \) copies of the group \( SL_2 \). If \( n \) is odd, the algebra \( L \) is isomorphic to \( g_1 \oplus [n/2]g_2 \) and each Lie-orthogonal operator \( J \) on \( L \) can be represented as \( J = 0 \oplus J_1 \oplus \cdots \oplus J_{[n/2]} + J_0 \). Here 0 denotes the zero operator on the center \( Z = g_1 \) of \( L \), \( J_i \) is a Lie-orthogonal operator on the \( i \)th copy of \( g_2 \) (i.e., its matrix belongs to the group \( SL_2 \)) and \( J_0 \) is an operator on \( L \) whose image is contained in \( Z = g_1 \).

5.5 Low-dimensional non-solvable complex Lie algebras

Non-solvable complex Lie algebras of the dimension \( n \leq 5 \) are exhausted by algebras \( sl_2 \ (n = 3) \), \( sl_2 \oplus g_1 \ (n = 4) \), \( sl_2 \oplus 2g_1 \), \( sl_2 \oplus g_2 \) and \( sl_2 \notin 2g_1 \ (n = 5) \).

The algebra \( L = sl_2 \) is simple, hence in view of Theorem 2 all Lie-orthogonal operators on this algebra are the trivial ones, i.e. \( Id_L \) and \(-Id_L \). See also Subsection 5.1. The algebras \( sl_2 \oplus g_1 \) and \( sl_2 \oplus 2g_1 \) are reductive and Lie-orthogonal operators on reductive algebras are completely described in Corollary 7.

The algebra \( sl_2 \oplus g_2 \) has zero center and can be represented as a direct sum of its ideals \( sl_2 \) and \( g_2 \). We can now use Lemma 7 which implies that any Lie-orthogonal operator on this algebra is a direct sum of Lie-orthogonal operators on ideals \( sl_2 \) and \( g_2 \). The sets of Lie-orthogonal operators on these ideals are already described (see before and the previous subsection). Therefore, Lie-orthogonal operators on the algebra \( sl_2 \oplus g_2 \) form a group, which is isomorphic to \( Z_2 \times SL_2 \).

The only algebras left are \( L = sl_2 \oplus 2g_1 \). The nonzero commutation relations of this algebra in the canonical basis are exhausted by the following:

\[
\begin{align*}
[e_1, e_2] &= 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \\
[e_1, e_4] &= e_4, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5, \quad [e_1, e_5] = -e_5,
\end{align*}
\]

i.e., \( e_1, e_2 \) and \( e_3 \) form a basis of Levi factor, which is isomorphic to \( sl_2 \), \( e_4 \) and \( e_5 \) form a basis of the Abelian radical \( R \) of this algebra.

Let \( J \) be a Lie-orthogonal operator on \( L \). The only ideal of the algebra \( L \) containing Levi factor is the very algebra \( L \). Also, note that the algebra \( L \) has the nonzero center. Then, it follows from the content of Subsection 4.3 that the operator \( J \) has the only eigenvalue, which equals either 1 or \(-1 \). Up to involution, we can assume that this eigenvalue is 1. This means that the entire algebra \( L \) is contained in the root subspace of the operator \( J \) that corresponds to the eigenvalue 1. Moreover, Proposition 8 implies, that \( J = Id_L + N \), where \( N \) is a nilpotent operator with image contained in the radical \( R \). Therefore, the action of the operator \( J \) on elements of the basis can be represented in the form:

\[
\begin{align*}
Je_i &= e_i + a_1 e_4 + b_1 e_5, \quad i = 1, 2, 3, \\
Je_i &= a_4 e_4 + b_5 e_5, \quad i = 4, 5.
\end{align*}
\]

We apply the definition of Lie orthogonality to the different pairs of basis elements:

\[
\begin{align*}
[J e_1, J e_2] &= 2e_2 + a_2 e_4 - b_2 e_5 - b_1 e_4 = [e_1, e_2] = 2e_2, \\
[J e_1, J e_3] &= -2e_3 + a_3 e_4 - b_3 e_5 - a_1 e = [e_1, e_3] = -2e_3, \\
[J e_2, J e_3] &= e_1 + b_3 e_4 - a_2 e_5 = [e_2, e_3] = e_1, \\
[J e_1, J e_4] &= a_4 e_4 - b_4 e_5 = [e_1, e_4] = e_4, \\
[J e_1, J e_5] &= a_5 e_4 - b_5 e_5 = [e_1, e_5] = -e_5.
\end{align*}
\]

Thus we have the equations on the coefficients \( b_2 = 0, a_2 = b_1, a_3 = 0, a_1 = -b_3, b_3 = 0, a_2 = 0 \) (where from \( a_i = b_i = 0, i = 1, 2, 3 \), \( a_4 = 1, b_4 = 0, a_5 = 0 \) and \( b_5 = 1 \).

Summing up the calculation, we obtain the next proposition.
Proposition 11. Lie-orthogonal operators on the Lie algebra \( L = \mathfrak{sl}_2 \oplus \mathfrak{g}_1 \) are exhausted by the trivial \( \text{Id}_L \) and \( -\text{Id}_L \).

6 Conclusion

In this paper we have studied Lie-orthogonal operators on finite-dimensional Lie algebras over a field of characteristic 0. One of main directions of the study is the generalization of results obtained in [13, 1] via removing the restrictions on centers of algebras and/or of operator non-degeneracy. In particular, it has been proved that the center and all elements of the ascending central series of a Lie algebra are invariant with respect to any Lie-orthogonal operator on this algebra. If a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0 possesses a Lie-orthogonal operator with eigenvalues not equal to \( \pm 1 \) then this algebra is solvable of solvability degree 2. Proofs of a number of existing assertions have been simplified, including the lemma on that the zero root subspace is contained in the center of the algebra, the lemma on the commutation of pairs of root subspaces such that the product of the respective eigenvalues is not equal to 1, the lemma on the invariance of an ideal the factor-algebra of which is centerless.

The natural equivalence relation of Lie-orthogonal operators has been introduced. It plays an important role in the entire consideration as many assertions have been formulated up to this equivalence. In particular, it has been proved that the decomposition of a Lie algebra into the direct sum of the ideals implies the decomposition of any Lie-orthogonal operator on the algebra into the direct sum of Lie-orthogonal operators on these ideals up to the above equivalence.

Perhaps, the most interesting result of the paper is the complete description of Lie-orthogonal operators on semi-simple Lie algebras. It appears that Lie-orthogonal operators on a simple Lie algebra are exhausted by the trivial ones, i.e., identical and minus identical. Then any Lie-orthogonal operator on a semi-simple algebra can be represented as a direct sum of trivial operators on the simple components of the algebra. This result allowed us to completely describe Lie-orthogonal operators on reductive Lie algebras and to derive certain properties of Lie-orthogonal operators on algebras with nonzero Levi factors.

For some classes of Lie algebras, the respective sets of Lie-orthogonal operators have been found via direct calculations. The list of such classes includes the special linear algebras, the Heisenberg algebras, the almost Abelian algebras and all non-solvable algebras of dimension not higher than five. The calculation for the special linear algebras plays the test role for applying the basis approach. The introduced notion of equivalence allowed us to briefly formulate the assertion on Lie-orthogonal operators on a Heisenberg algebra: the essential parts of these operators form the symplectic group of the appropriate dimension (the algebra dimension minus one). The description of Lie-orthogonal operators on almost Abelian algebras in an important step to the complete description of Lie-orthogonal operators on low-dimensional algebras since almost Abelian algebras constitute a considerable portion of low-dimensional algebras. The description of Lie-orthogonal operators on low-dimensional algebras gives necessary material for suggesting conjectures on Lie-orthogonal operators with non-trivial Levi–Maltsev decomposition.

Using the procedure of the algebraic closure it is possible to extend results obtained for algebraically closed fields to fields which are not algebraically closed, although this extension may be nontrivial and hence requires a further investigation.

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