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1 Introduction

Denote by $M_g$ and $A_g$ the coarse moduli spaces of genus $g$ curves and principally polarized abelian $g$-folds, respectively, and let a superscript $S$ denote their Satake compactifications. The main result of [CSB] is that the intersection $(M^S_{g+m} \cap A^S_g)$, taken inside $A^S_{g+m}$, is far from transverse; it contains the $m$th order infinitesimal neighbourhood of $M^S_g$ in $A^S_g$. So $\cup_m (M^S_{g+m} \cap A^S_g)$ is the formal completion of $A^S_g$ along $M^S_g$ and there are no stable Siegel modular forms that vanish along every moduli space $M_g$. The proof depends upon the construction by Fay of certain, very special, degenerating families of curves for which he could establish a formula for (a part of) the derivative of the period matrix as a certain explicit tensor of rank one [F, p. 53]. For an arbitrary degeneration the derivative is a tensor of higher rank, usually maximal, and it is more difficult to make use of this; cf. the assertion on p. 1 of the erratum to [G-SM]. Interpreting Fay’s formula in terms of the projective geometry of the canonical model of the singular fibre then gives the result.

Here we prove similar results for the loci $V_{g,n,tot}$ in $M_g$ of $n$-gonal curves of genus $g$ with a point of total ramification, for any fixed $n \geq 3$, as follows.

**Theorem 1.1** (= 3.7) There is no stable Siegel modular form that vanishes on every locus $V_{g,n,tot}$. In particular, there is no stable Siegel modular form that vanishes on every trigonal locus.

This sharpens [CSB], but depends upon it. For hyperelliptic curves, however, Codogni has shown [C] that the story becomes very different. He has found many millions of stable modular forms that vanish on the hyperelliptic locus in every genus, for example, the difference $\Theta_P - \Theta_Q$ of two theta series where $P, Q$ are positive, even and unimodular quadratic forms of rank 32 with no roots.

Recall that a curve $C$ is $n$-gonal if there is a map $C \to \mathbb{P}^1$ of degree $n$. If $g = 2n - 2$ is even, then a general curve of genus $g$ is $n$-gonal in finitely many ways; if $g > 2n - 2$ then the $n$-gonal curves form a proper subvariety (the Hurwitz scheme) $V_{g,n}$ of $M_g$. The $n$-gonal curves for which the given map to $\mathbb{P}^1$ has a point of total ramification form the subvariety $V_{g,n,tot}$ mentioned above. Its closure in $M^S_g$ will be denoted by $V^S_{g,n,tot}$.

Compared to the arguments in [CSB], the proof here depends upon combining Fay’s construction with those by Schiffer to get certain variations of a curve where what is essentially the derivative of the period matrix can be calculated explicitly. Controlling the construction of these Fay–Schiffer variations (see below) is crucial in controlling the derivative.
2 Variations

Suppose that $C$ is a curve (= compact Riemann surface) of genus $g$, that $a, b, c$ are
distinct points of $C$ and that $z_a, z_b, z_c$ are local co-ordinates on $C$ at $a, b, c$ respec-
tively. There are various well known kinds of variation that can be constructed
from these data, and we recall some of them now.

The first is a Fay variation of $C$ centred at $(a, z_a; b, z_b)$. This is a particular
proper morphism $C \to \Delta$ from a smooth complex surface to a disc such that the
fibre over 0 is the nodal curve $C/\sim$ and for every $t \neq 0$ the fibre $C_t = C_t$ is
of genus $g + 1$. It is constructed as follows [F, p. 50].

Fix $\delta > 0$ with $\delta << 1$. Let $D_{\delta^2}$ be a disc of radius $\delta^2$ and complex co-
ordinate $t$. In $C \times D_{\delta^2}$ consider two closed subsets, one defined by the inequality
$|z_a| \leq |t|/\delta$ and the other by the inequality $|z_b| \leq |t|/\delta$. Delete these closed
subsets from $C \times D_{\delta^2}$ to get the complex manifold $C^0$. There are open subsets $U_a$
and $U_b$ of $C^0$ defined by the further inequalities $|z_a| < \delta$ and $|z_b| < \delta$, respectively.

Let $S$ be the open part of the complex surface with co-ordinates $X, Y$
declared by the inequalities $|X|, |Y| < \delta$. There is a morphism $S \to D_{\delta^2}$ given by
$t = XY$. Now map $U_a$ and $U_b$ to $S$ by the formulae

$$X = z_a, \ Y = t/z_a,$$

$$X = t/z_b, \ Y = z_b$$

and then glue $C^0$ to $S$ via these maps; by definition, the result is $C$, and $C$ is
provided with a proper morphism to $\Delta = D_{\delta^2}$.

Another kind is a Schiffer variation of $C$ centred at $(c, z_c)$. This is a particular
proper morphism $C \to \Delta$ where now all fibres are smooth of genus $g$.
It is also constructed via a glueing procedure.

Start with $C \times D_{\delta^2/4}$ and delete the closed subset defined by the inequality
$|z_c| \leq \sqrt{|t|}$ to obtain the complex manifold $C^0$. In $C^0$ there is the open subset $V_c$
declared by

$$\sqrt{|t|} < |z_c| < \delta - \sqrt{|t|}.$$

The principle of the argument says that, as $z_c$ goes once around the circle $R$ of
radius $\delta - \sqrt{|t|}$ and centre 0, so $w = z_c + t/z_c$ has exactly one zero inside $R$, so
that the image of $R$ in the $w$-plane is a simple closed curve $\Gamma(t)$ around 0, and
varies smoothly with $t$ for $0 \leq |t| < \delta^2/4$.

Say that $D(t)$ is the open neighbourhood of 0 with boundary $\Gamma(t)$. Then
$\cup_{0 \leq |t| < \delta^2/4} D(t)$ is an open submanifold $V$ of $C \times D_{\delta^2/4}$. Map $V_c$ to $V$ via $w = z_c + t/z_c$; this is unramified, since the branch locus is given by $z_c^2 + t = 0$, and
and glueing $C^0$ to $V$ via the map $V_c \to V$ that has just been constructed gives the
Schiffer variation of which we speak.

If now $(a_1, \ldots, a_n)$ are distinct points of $C$ and $z_{j} = z_{a_j}$ is a local co-
ordinate at each, then we can simultaneously construct a Fay variation centred at
(a_{n-1}, z_{n-1}; a_n, z_n) and a Schiffer variation centred at (a_1, z_1; \ldots; a_{n-2}, z_{n-2}). This is a proper map f : C^+ \to \Delta^{n-1}, where now \Delta^{n-1} is an (n-1)-dimensional complex polydisc with co-ordinates t_1, t_2, \ldots, t_{n-1}, the map f is smooth over the locus \ t_{n-1} \neq 0 and the fibres over \ t_{n-1} = 0 are copies of the nodal curve C/(a_{n-1} \sim a_n).

We call it the Fay–Schiffer variation of C centred at (a_1, z_1; \ldots; a_n, z_n).

**Theorem 2.1** With respect to a suitable fixed homology basis and a correspondingly normalized basis \( \omega = (\omega_1, \ldots, \omega_2) \) of the abelian differentials on C, the period matrix \( T(t) \) of \( C_t \) can be written in \( 2 \times 2 \) block form as

\[
T(t) = \begin{bmatrix}
\tau + \sum_{j=1}^{n-1} t_j \sigma_j & AJ(t) + t_{n-1}s \\
\frac{1}{2\pi i} (\log t_{n-1} + c_1 + c_2 t_{n-1}) & 1
\end{bmatrix} + O(t^2)
\]

where for 1 \leq j \leq n-2 the matrix \( \sigma_j \) is of rank 1 and is given by

\[
(\sigma_j)_{pq} = 2\pi i \left( \frac{\omega_p}{dz_j}(a_j) \frac{\omega_q}{dz_j}(a_j) \right),
\]

the matrix \( \sigma_{n-1} \) is of rank 1 and is given by

\[
(\sigma_{n-1})_{pq} = 2\pi i \left( \frac{\omega_p}{d\bar{z}_n}(a_{n-1}) - \frac{\omega_p}{d\bar{z}_n}(a_n) \right) \left( \frac{\omega_q}{d\bar{z}_{n-1}}(a_{n-1}) - \frac{\omega_q}{d\bar{z}_{n-1}}(a_n) \right),
\]

\( ^tM \) is the transpose of the matrix \( M, AJ(t) = AJ_0(a_n - a_{n-1}) + \sum_{j=1}^{n-2} t_j AJ_j, AJ_0 \) is the Abel–Jacobi map \( AJ_0(y - x) = \int_x^y \omega \) on C, each \( AJ_j \) is a holomorphic function of the parameters \( a_i, z_i \) for \( i = 1, \ldots, n - 2 \), \( s, c_1, c_2 \) are holomorphic functions of the parameters \( a_j, z_j \) in the construction and \( c_1 \) also depends on \( t_1, \ldots, t_{n-2} \).

**PROOF:** Consider the Schiffer variation of C centred at (a_1, z_1; \ldots; a_{n-2}, z_{n-2}). This gives a genus \( g \) family \( \Gamma \to \Delta^{n-2} \) where \( \Delta^{n-2} \) is an \((n-2)\)-dimensional polydisc with co-ordinates \( t_1, \ldots, t_{n-2} \) and the period matrix of \( \Gamma_t \) is

\[
\begin{bmatrix}
\tau + \sum_{j=1}^{n-2} t_j \sigma_j \\
\frac{1}{2\pi i} (\log t_{n-1} + c_1 + c_2 t_{n-1})
\end{bmatrix} + O(t^2).
\]

(This is due to Patt [P].) By construction, this Schiffer variation is trivial outside neighbourhoods of the points \( a_1, \ldots, a_{n-2}, \) and so the points \( a_{n-1}, a_n \) and the local co-ordinates \( z_{n-1}, z_n \) come along for the ride. So now we make a Fay variation of \( \Gamma \to \Delta^{n-2} \) centred at \( (a_{n-1}, z_{n-1}; a_n, z_n) \) to get \( C \to \Delta^{n-1} \). The period matrix \( T(t) \) of the curve \( C_t \) of genus \( g+1 \) is then

\[
\begin{bmatrix}
\tau + \sum_{j=1}^{n-2} t_j \sigma_j + t_{n-1} \sigma_{n-1} & AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t)) + t_{n-1}s \\
\frac{1}{2\pi i} (\log t_{n-1} + c_1 t_{n-1} + c_2) & 1
\end{bmatrix} + O(t^2)
\]

where \( AJ_{\Gamma_t} \) is the Abel–Jacobi map for the curve \( \Gamma_t \) and each of the terms \( AJ_{\Gamma_t}(a_n(t) - a_{n-1}(t)), s, c_1 \) and \( c_2 \) is a holomorphic function of \( t_1, \ldots, t_{n-2} \) and the
parameters \(a_1, \ldots, a_{n-2}\) and \(z_1, \ldots, z_{n-2}\). However, for \(t_1 = \cdots = t_{n-2} = 0\) the family \(C \to \Delta^{n-1}\) is just the usual Fay variation of \(C\) centred at \((a_{n-1}, z_{n-1}; a_n, z_n)\), and so the Abel–Jacobi term \(AJ_C((a_n(t) - a_{n-1}(t)))\) is independent of the \(a_j\) and the \(z_j\); \(AJ_C((a_n(0) - a_{n-1}(0))) = AJ_C(a_n - a_{n-1})\) and so

\[
AJ_C((a_n(t) - a_{n-1}(t))) = AJ_C(a_n - a_{n-1}) + \sum_{i=1}^{n-2} t_i AJ_C + O(t^2).
\]

Now suppose that \(h : C \to B\) is a morphism of Riemann surfaces of degree \(n\), that \(e \in B\) is a point over which \(h\) is unramified and that \(h^{-1}(e) = \{a_1, \ldots, a_n\}\). For any local co-ordinate \(z_e\) on \(B\) at \(e\), define the local co-ordinate \(z_j\) on \(C\) at \(a_j\) to be the pull-back of \(z_e\) restricted to a neighbourhood of \(a_j\).

Take the corresponding Fay–Schiffer variation \(C^+ \to \Delta^{n-1}\) of \(C\) centred at \((a_1, z_1; \ldots; a_n, z_n)\), and let \(C \to \Delta\) be the one-parameter family obtained by restricting \(C^+ \to \Delta^{n-1}\) to the diagonal disc \(\Delta\) in \(\Delta^{n-1}\) defined by \(t_1 = \cdots = t_{n-1} = t\). Let \(B \to \Delta\) be the Schiffer variation of \(B\) centred at \((e, z_e)\).

**Proposition 2.2** There is a degree \(n\) morphism \(H : C \to B\) relative to \(\Delta\) that at \(t = 0\) is the morphism \(C/(a_{n-1} \sim a_n) \to B\) induced by \(h\).

**Proof:** The Schiffer variation \(B \to \Delta\) is constructed by deleting a disc and then gluing in a new disc with co-ordinate \(w = z_e + t/z_e\); the variation \(C \to \Delta\) is constructed by the same formula except where the points \(a_{n-1}, a_n\) are identified over \(t = 0\). Here we have a complex surface \(S\) with co-ordinates \(X, Y\) with \(XY = t\), and the gluing was given by \(X = z_{n-1}, Y = t/z_{n-1}\) and \(X = t/z_n, Y = z_n\). So to construct \(H : C \to B\) it is enough to give the map from \(S\) to the \(w\)-disc. This is achieved by writing \(w = X + Y\).

Note that for all \(t\), including \(t = 0\), the morphism \(H_t : C_t \to B_t\) coincides with \(h\) outside a union of small open sets. In particular, the ramification data of \(H_t\) coincides with those of \(h\) away from this union.

**Proposition 2.3** \(V_{g,n,\text{tot}} \times M_1\) lies in the closure of \(V_{g+1,n,\text{tot}}\).

**Proof:** Suppose that the curve \(C\) is a point in \(V_{g,n,\text{tot}}\), that \(f : C \to \mathbb{P}^1\) is of degree \(n\) and that \(f\) is totally ramified at \(P \in C\). Say \(f(P) = e\), so that \(f^{-1}(e) = n[P]\). Suppose also that the curve \(E\) is a point in \(M_1\). Fix \(Q \in E\), and regard \(E\) as an elliptic curve with origin \(Q\). Then choose a primitive \(n\)-torsion point \(R\) on \(E\), so that \(n[Q] \sim n[R]\) and there is a rational function \(h : E \to \mathbb{P}^1\) such that \(h^{-1}(0) = n[Q]\) and \(h^{-1}(\infty) = n[R]\). We assume, as we may, that \(e \neq 0, \infty\).

We shall construct a variation similar (but not identical) to that described on pp. 37–41 of [F], omitting the topological details. Choose local co-ordinates \(z_e\) and \(z_0\) on \(\mathbb{P}^1\) at \(e\) and \(0\), respectively. Then there is a local co-ordinate \(w_P\) on \(C\) at \(P\) with \(z_e = w_P^0\) and a local co-ordinate \(w_Q\) on \(E\) at \(Q\) with \(z_0 = w_Q^0\). Use
these to construct variations $\mathcal{C} \to \Delta$ and $\mathcal{B} \to \Delta$, where $\mathcal{B}$ is obtained by gluing $\mathbb{P}^1 \times \Delta$ and $\mathbb{P}^1 \times \Delta$ to the surface $S_n = (X_nY_n = t^n)$ by

$$X_n = z_\epsilon, \quad Y_n = t^n/z_\epsilon,$$

and $\mathcal{C}$ is obtained by gluing $C \times \Delta$ and $E \times \Delta$ to the surface $S_1 = (X_1Y_1 = t)$ by

$$X_1 = \frac{w}{P}, \quad Y_1 = \frac{t}{wP},$$

$$X_1 = \frac{t}{wQ}, \quad Y_1 = wQ.$$

Via the morphism $S_1 \to S_n$ given by $X_n = X_1^n$, $Y_n = Y_1^n$ there is a morphism $\pi : \mathcal{C} \to \mathcal{B}$ obtained by gluing the morphisms $f \times 1_\Delta : C \times \Delta \to \mathbb{P}^1 \times \Delta$ and $h \times 1_\Delta : E \times \Delta \to \mathbb{P}^1 \times \Delta$. Moreover, since $h \times 1_\Delta$ is totally ramified along $\{R\} \times \Delta$ and the variation $\mathcal{C} \to \Delta$ is trivial outside neighbourhoods of $P$ and of $Q$, the morphism $\mathcal{C}_t \to \mathcal{B}_t$ is totally ramified somewhere. Since $\mathcal{B}_t \cong \mathbb{P}^1$, the result is proved.

3 Modular forms vanishing on $V_{g,n,\text{tot}}$

We fix an integer $n$ with $3 \leq n \leq g - 1$. We are especially interested in those values of $n$ for which a general curve of genus $g$ possesses at most finitely many $g^1_n$’s, so that $n \leq g/2 + 1$. Then if $C$ is a non-hyperelliptic curve possessing a pencil $\Pi$ that is a complete $g^1_n$, the linear span $\langle D \rangle$ of each element $D$ of $\Pi$ is a copy of $\mathbb{P}^{n-2}$, and as $D$ varies over $\Pi$ these copies sweep out a rational scroll $\Sigma(\Pi)$ of dimension $n - 1$ in $\mathbb{P}^{g-1}$. For example, if $n = 3$ then $\Sigma(\Pi)$ is a surface (and is the intersection of the quadrics that contain $C$).

Suppose that $G = G_{g+1}$ is a Siegel modular form on $A_{g+1}$ such that the restriction $G|_{M_{g+1}}$ of $G$ to $M_{g+1}$ has multiplicity at least $m$ along $V_{g+1,n,\text{tot}}$. That is, $G$ and all its partial derivatives $F$ of order $\leq m - 1$ with respect to the entries $T_{pq}$ of a period matrix $T$ in $\mathfrak{S}_{g+1}$ in directions tangent to $M_{g+1}$ vanish along $V_{g+1,n,\text{tot}}$. We can define the Siegel $\Phi$-operator on the derivatives by

$$\Phi(F)(\tau) = \lim_{t \to \infty} F \begin{pmatrix} \tau & 0 \\ 0 & t \end{pmatrix}.$$

Lemma 3.1 $\Phi(F)$ is a derivative of $\Phi(G)$ of order $\leq m - 1$ in directions tangent to $M_g$ and vanishes along $V_{g,n,\text{tot}}$.

PROOF: By construction, $\Phi(F)$ can be computed by restricting to $A_g \times A_1$, then restricting to $A_g \times \{j\}$ for some $j \in A_1$, and finally letting $j \to \infty$. Since the intersection of $M_{g+1}$ and $A_g \times A_1$ certainly contains $M_g \times M_1$, the first part of the lemma is proved. The second part then follows from Proposition 2.3.

□
Theorem 3.2  Under these assumptions, the restriction $\Phi(G) = G|_{M_g}$ has multiplicity at least $m + 1$ along $V_{g,n,tot}$.

PROOF: We need to show that $\Phi(F)$ is singular along $V_{g,n,tot}$. Now $F$ has a Fourier expansion

$$F(T) = \sum_{X \in S_g} a(X) \exp \pi i \text{tr}(XT),$$

where $T$ is a point in Siegel space $\mathfrak{h}_{g+1}$ and $S_n$ is the lattice of positive semi-definite $n \times n$ symmetric matrices over $\mathbb{Z}$ whose diagonal is even.

Take a curve $C$ in $V_{g,n,tot}$, and choose any reduced divisor $D = \sum a_j$ in the specified $g^1_n$ on $C$. Let $h : C \to B = \mathbb{P}^1$ be the morphism defined by this $g^1_n$ and say that $D = h^{-1}(e)$ and that $h$ is totally ramified at $P$. We have, according to Proposition 2.2, a 1-parameter Fay–Schiffer variation $C \to \Delta$ of $C$ centred at $(a_1, z_1; \ldots, a_n, z_n)$ with a degree $n$ morphism to the Fay–Schiffer variation $B \to \Delta$ of $B$ centred at $(e, z_e)$. Since $B = \mathbb{P}^1$, the variation $B \to \Delta$ is trivial, so that for $t \neq 0$ the curve $C_t$ lies in $V_{g+1,n}$. Moreover, because the variation is constructed to be trivial outside a neighbourhood of $D$, the curve $C_t$ lies in $V_{g+1,n,tot}$.

Now the argument follows [CSB] closely.

Take $T = T(t)$ to be the period matrix of $C_t$ as above. Note that since $t_1 = \cdots = t_{n-1} = t$, we can re-arrange $c_1$ and $c_2$ so that both of them are independent of $t$, and are holomorphic functions only of the parameters $(e, z_e)$. Then

$$F_{g+1}(T) = \sum_{X \in S_g} a(X) \exp \pi i \sum_{p,q=1}^{g+1} x_{pq}T_{pq}$$

where $X = (x_{pq})$. Our aim is to examine the coefficient of $t$ in the expansion of this expression in powers of $t$, so calculate modulo $t^2$. Since $\exp 2\pi iT_{g+1,1+1} \equiv t. \exp c_1. \exp(c_2t)$ modulo $t^2$, it follows that

$$(F_{g+1})(T) \equiv \sum_{x_{g+1,1+1}=0}^{x_{g+1,1+1}=2} + \sum_{x_{g+1,1+1}=2}$$

modulo $t^2$, since all terms with $x_{g+1,1+1} \geq 4$ vanish modulo $t^2$. Here $\sum_{x_{g+1,1+1}=r}$ denotes the sum over $X \in S_{g+1}$ with $x_{g+1,1+1} = r$, for $r = 0$ or 2. Therefore, modulo $t^2$,

$$\sum_{x_{g+1,1+1}=0} \equiv \sum_{X \in S_g} a(X) \exp \pi i \sum_{p,q=1}^{g} x_{pq}(\tau_{pq} + t\sigma_{pq})$$

and

$$\sum_{x_{g+1,1+1}=2} \equiv t. \exp c_1. \sum_{X \in S_{g+1}, x_{g+1,1+1}=2} a(X)$$

$$\cdot \exp \left(2\pi i \sum_{p=1}^{g} x_{p,1+1} \int_{a_{n-1}}^{a_n} \omega_p \right) \cdot \exp \left(\pi i \sum_{p,q=1}^{g} x_{pq}T_{pq}\right).$$
So the coefficient of $t$ is $A + B \exp c_1$, where

$$A = \sum_{x_{g+1, g+1}=0} a(X) \left( \pi i \sum_{p,q=1}^{g} x_{pq} \sigma_{pq} \right) \left( \exp \pi i \sum_{p,q=1}^{g} x_{pq} \tau_{pq} \right),$$

$$B = \sum_{x_{g+1, g+1}=2} a(X) \left( \exp 2\pi i \sum_{p=1}^{g} x_{p,g+1} \int_{\omega_p}^{\omega_{p+1}} \omega_p \right) \left( \exp \pi i \sum_{p,q=1}^{g} x_{pq} \tau_{pq} \right).$$

By assumption, $A + B \exp c_1$ vanishes identically.

Now rescale the local co-ordinate $z_e$; that is, given any non-zero scalar $\lambda$, replace $z_e$ by $\lambda^{x+1} - 1 z_e$. Such a rescaling will produce a different family $C \to \Delta$ with $C_t$ in $V_{g+1, \text{tot}}$ for all $t \neq 0$, but the quantity $A + (\exp c_1)B$ will still vanish for the rescaled family. Moreover, $B$ is invariant under this rescaling, as is revealed by a cursory inspection. Also $c_1$ is a holomorphic function of $\lambda$ because the entries of a period matrix are holomorphic functions of the parameters.

**Lemma 3.3** $A = B = 0$.

**Proof:** From the description above of $\sigma_{pq}$, this rescaling multiplies $\sigma_{pq}$ by $\lambda^2$, so that $A$ can be written as

$$A = C\lambda^2$$

with $C$ independent of $\lambda$. So we have an identity

$$C\lambda^2 = -B \exp(c_1(\lambda))$$

of holomorphic functions on the 1-dimensional algebraic torus $\mathbb{G}_m = \text{Spec} \mathbb{C}[\lambda^{\pm}]$, where $B, C$ are constant functions on $\mathbb{G}_m$. The result follows at once. \qed

Now $A$ can also be written as

$$A = \left. \frac{\partial}{\partial t} \right|_{t=0} \left( \sum_{X \in S_g} a(X) \exp \pi i \sum_{pq=1}^{g} x_{pq} (\tau_{pq} + t \sigma_{pq}) \right) = \left. \frac{\partial}{\partial t} \right|_{t=0} F_g(\tau + t \sigma).$$

That is, $\sigma$ lies in the Zariski tangent space $H$ at the point $\tau$ to the divisor in $\mathcal{S}_g$ defined by the function $F_g$. It is important to note that, from this description, $H$ depends upon $C$ but is independent of any of the other parameters (points, local co-ordinates) used to construct the variation. Thus $H$ contains every $\sigma$ that arises from different choices of these other parameters.

Assume that $C$ has no non-trivial automorphisms. Then there are the standard classical natural identifications of tangent spaces to moduli given by

$$T_{[C]} M_g = H^0(\Omega^1_C \otimes \Omega^2)^\vee,$$

$$T_{[C]} A_g = \text{Sym}^2 H^0(\Omega^1_C)^\vee.$$
The inclusion $T_{[C]}M_g \hookrightarrow T_{[C]}A_g$ is dual to the natural multiplication (which is surjective, by Max Noether’s theorem) $\text{Sym}^2 H^0(\Omega_C^1) \to H^0(\Omega_C^{1,2})$. So the vector space of quadrics in $\mathbb{P}^{n-1}$ can be regarded as the space of linear forms on $T_{[C]}A_g$, and then $T_{[C]}M_g$ is the subspace of $T_{[C]}A_g$ defined by the vanishing of those quadrics in $\mathbb{P}^{n-1}$ that contain $C$.

We know that the tangent space $H$ to the divisor $(F_g = 0)$ at the point $\tau$ in $\mathfrak{H}_g$ contains every matrix $\sigma$ that arises as above. Projectivize, and use the classical descriptions above of the tangent spaces to moduli. Then (the projectivization of) $H$ is a hyperplane in $\mathbb{P}(\text{Sym}^2 H^0(C, K_C))$ that contains every point $\sigma(n-1, n) = \sigma = (\sigma_{pq})$ of the form

$$
\sigma_{pq} = (\omega_p(a_n) - \omega_p(a_{n-1})) (\omega_q(a_n) - \omega_q(a_{n-1})) + \sum_{j=1}^{n-2} \omega_p(a_j) \omega_q(a_j),
$$

where we have omitted a factor of $2\pi i$ and the factors of $dz_c$ that should appear as denominators. We can also regard $H$ as a quadric in the $\mathbb{P}^{n-1}$ in which $C$ is canonically embedded, and then what we have to prove is that $H$ contains $C$.

We shall in fact prove a stronger statement, namely that $H$ contains the scroll $\Sigma(\Pi)$ (which certainly contains $C$) that is mentioned in the first paragraph of this section.

In $\mathbb{P}^{n-1}$, any element $D = \sum_{j=1}^n a_j$ of the given pencil $\Pi$ spans a copy $L = L_D$ of $\mathbb{P}^{n-2}$, the points $a_1, \ldots, a_n$ are, therefore, in general position in $L$. Regard $L$ as the projectivization of an $(n-1)$-dimensional vector space $W$ and the points $a_j$ as projectivizations of vectors $w_j = (\omega_1(a_j), \ldots, \omega_2(a_j))$ in $W$. Consider the second Veronese embedding $\text{Ver}_2(L_D)$ in a copy $\mathbb{P}^N_D$ of $\mathbb{P}^N$, where $N + 1 = n(n-1)/2$ and $\mathbb{P}^N_D$ is a linear subspace of the projectivized tangent space $\mathbb{P}(T_{[C]}A_g)$. Then $H$ contains the point (in the projectivization of $\text{Sym}^2 W$)

$$
\sigma(n-1, n) = (w_{n-1} - w_n)^2 + \sum_{i=1}^{n-2} w_i^2;
$$

the same argument shows that $H$ also contains every other point $\sigma(k, l)$, for $k < l$, that is obtained from $\sigma(n-1, n)$ by permutation of the vectors $w_1, \ldots, w_n$. The $\sigma(k, l)$ form a set of $N + 1$ points in $\mathbb{P}^N_D$.

**Lemma 3.4** These $N + 1$ points span $\mathbb{P}^N_D$.

**Proof:** We can assume that $\sum w_j = 0$. So $W$ is the irreducible $(n-1)$-dimensional representation of the symmetric group $\mathfrak{S}_n$ as a Coxeter group of type $A_{n-1}$. Let $\mathbf{1}$ denote the trivial 1-dimensional representation, so that $W \oplus \mathbf{1}$ is the standard permutation representation $V$ with standard basis $(v_1, \ldots, v_n)$ and $\mathbf{1}$ is the line generated by $e_1 = \sum v_i$. Let $\pi_1 : V \to W$ be the projection, so that $\pi_1(v_i) = w_i$. Let $V_2 \subset \text{Sym}^2 V$ be the module of cross-terms.
Lemma 3.5  Sym²V = e₁V ⊕ V₂.

PROOF:  It is enough to show that e₁V ∩ V₂ = 0, since both sides have the same dimension. So take Q ∈ e₁V ∩ V₂; evaluating the polynomial Q at each of the points (1, 0, ..., 0), ..., (0, ..., 0, 1) in turn shows that Q = 0, because Q ∈ V₂, and Lemma 3.5 is proved.

So the projection π₁ : V → W induces an inclusion π₂ : V₂ → Sym²W; since both terms have the same dimension, π₂ is an isomorphism. Hence the cross-terms w_kw_l form a basis of Sym²W.

Now σ(k, l) = -2π₂(e₂) - 2w_kw_l. Suppose that there is a linear relation \( \sum k,l \lambda_{kl} \sigma(k, l) = 0; \) then \( \sum k,l \lambda_{kl}w_kw_l. \) Since π₂(e₂) = \( \sum w_kw_l \) and the w_kw_l are linearly independent, we get

\[ \lambda_{ij} = -\sum_{k,l} \lambda_{kl} \]

for all i, j. Then \( \lambda_{ij} = 0, \) and the lemma is proved.

It follows that H contains \( \mathbb{P}^N_D, \) and therefore contains V_{r2}(L_D) for every reduced divisor D in Π, the g₁^n under consideration. So indeed H, when regarded as a quadric in \( \mathbb{P}^{g−1}, \) contains the rational scroll Σ(Π).

Corollary 3.6  Assume that n ≥ 3 and that m ≥ 1. Then the intersection \( V_{g+m,n,tot}^S \cap M_g \) contains the mth order infinitesimal neighbourhood of \( V_{g,n,tot} \) in \( M_g. \)

PROOF:  Suppose that Φ is some modular form on \( A_{g+1} \) such that (Φ)₀ ∩ M_{g+1} is singular, with multiplicity m, along \( V_{g+1,n,tot}. \) That is, Φ and all its derivatives of order at most m − 1, taken in directions along \( M_{g+1}, \) vanish along \( V_{g+1,n,tot}. \)

Suppose that F is such a derivative. Then it follows from what we have shown that the restriction \( F\big|_{M_g} \) is singular along \( V_{g,n,tot}. \) That is, the restriction \( Φ\big|_{A_g} \) of Φ to \( A_g \) and all derivatives of \( Φ\big|_{A_g} \) of order at most m, taken in directions along \( M_g, \) vanish along \( V_{g,n,tot}. \) This follows from Lemma 3.1.

Theorem 3.7  Fix n ≥ 3. Then there is no stable Siegel modular form that vanishes on the totally ramified n-gonal locus \( V_{g,n,tot} \) for every g.

PROOF:  Suppose that F is such a modular form. Then, by Corollary 3.6, F vanishes on \( M_g \) for every g. But the main result of [C-SB] is that then \( F = 0. \)

The main result of [G-SM] is that the the Schottky form \( F = Θ_{E_2} − Θ_{D_{16}^+} \) (the difference of two theta series associated to the positive even unimodular lattices \( E_2^\oplus \) and \( D_{16}^+ \) of rank 16) that, by results of Schottky [S] and Igusa [I1], [I2], defines \( M_4 \) inside \( A_4, \) does not vanish along \( M_5. \) They prove further that it cuts out the exactly trigonal locus \( V_{5,3} \) in \( M_5, \) and does so with multiplicity 1.
Corollary 3.8 In genus 6 the Schottky form $F$ does not vanish along the totally ramified trigonal locus.

Proof: Suppose that $F_6$ vanishes along $V_{6,3,\text{tot}}$. Then, by Theorem 3.2, the restriction $F_5|_{M_5}$ of $F_5$ to $M_5$ is singular along $V_{5,3,\text{tot}}$. Then the trigonal locus $V_{5,3}$ is singular along the subvariety $V_{5,3,\text{tot}}$. But the trigonal locus is smooth outside the hyperelliptic locus, and we are done.

For $g = 6$ there is another subvariety of $M_g$ that is distinguished by the fact that the canonical model is not an intersection of quadrics, namely the locus of plane quintics. Our techniques, however, cannot let us decide whether $F$ vanishes along this locus; more generally, they cannot handle $g_r^n$'s with $r \geq 2$.

4 The even genus case

Suppose that $g = 2(n - 1)$ is even. Then a general curve of genus $g$ has a finite, but non-zero, number of $g_1^n$'s, while the locus $V_{g+1,n}$ is an irreducible divisor in $M_{g+1}$ (and a general curve in $V_{g+1,n}$ has a unique $g_1^n$).

Fix a general curve $C$ of genus $g = 2(n - 1)$, and let $\Pi_1, \ldots, \Pi_r$ be the $g_1^n$'s on it. (The number $r$ is a known function of $g$, but all we need is that $r \geq 4$ when $g \geq 6$.) As above, the members of each $\Pi_i$ sweep out a scroll $\Sigma_i = \Sigma(\Pi_i)$ in $\mathbb{P}^{g-1}$ that contains $C$.

Lemma 4.1 If $g \geq 6$, then there is no quadric in $\mathbb{P}^{g-1}$ that contains every $\Sigma_i$.

Proof: Choose any $a \in C$. For every $i$ there is a unique $D_i \in \Pi_i$ passing through $a$. Say $D_i = a + \sum_{j=2}^{n} b_{ij}$ and $L_i = \langle D_i \rangle$. Suppose that there is a hyperplane $H$ in $\mathbb{P}^{g-1}$ that contains each $L_i$; then

$$H.C \geq a + \sum_{ij} b_{ij},$$

so that $2g - 2 \geq 1 + r(n - 1)$. Since $r \geq 4$ this is impossible, and there is no such hyperplane. Since the $L_i$ are linear, this means that $\cup L_i$ has embedding dimension $g - 1$ at $a$.

Now suppose that $Q$ is a quadric that contains every $\Sigma_i$. Then $Q$ contains $\cup L_i$, and so has embedding dimension $g - 1$ at every point of $C$. However, the singular locus of a quadric is linear, and we are done.

This is false for $g = 4$; there are two $g_1^3$'s, but the scrolls $\Sigma_1$ and $\Sigma_2$ coincide, and are the unique quadric containing $C$.

Theorem 4.2 The $n$-gonal divisor $V_{g+1,n}$ in $M_{g+1}$ has contact with $A_g$ along $M_g$.

Proof: We need to show that for any modular form $F = F_{g+1}$ on $A_{g+1}$ that vanishes along $V_{g+1,n}$, the restriction $F_g$ of $F$ to $A_g$ is singular along $M_g$. But this follows from the proof of Theorem 3.2 (the entire proof, except for the final
paragraph): if $F_g$ is smooth on $A_g$ at the point $[C]$ of $M_g$, then the tangent hyperplane $H$ to $A_g$ corresponds, if it is non-zero, to a quadric in $\mathbb{P}^{g-1}$ that contains every scroll $\Sigma(\Pi_i)$. But we have just seen that there is no such quadric.

\[\square\]

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