INTEGRABLE HAMILTONIAN SYSTEMS ON SYMMETRIC SPACES: JACOBI, KEPLER AND MOSER

VELIMIR JURDJEVIC

Dedicated to the memory of J. Moser.

Abstract. This paper defines a class of left invariant variational problems on a Lie group $G$ whose Lie algebra $\mathfrak{g}$ admits Cartan decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{t}$ with the usual Lie algebraic conditions

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{t}, \quad [\mathfrak{p}, \mathfrak{t}] \subseteq \mathfrak{p}, \quad [\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{t}.$$ 

The Maximum Principle of optimal control leads to the Hamiltonians $H$ on $\mathfrak{g}$ that admit spectral parameter representations with important contributions to the theory of integrable Hamiltonian systems. Particular cases provide natural explanations for the classical results of Fock and Moser linking Kepler’s problem to the geodesics on spaces of constant curvature and J.Moser’s work on integrability based on isospectral methods in which C. Newman’s mechanical problem on the sphere and C. L. Jacobi’s geodesic problem on an ellipsoid play the central role. The paper also shows the relevance of this class of Hamiltonians to the elastic curves on spaces of constant curvature.

1. Introduction

A Lie group $G$ with an involutive automorphism $\sigma$ admits several natural variational problems whose solutions provide new insights into the theory of integrable Hamiltonian systems and to the geometry of the associated homogeneous spaces. An involutive automorphism $\sigma$ on a Lie group $G$ induces a splitting $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ of $G$ with $\mathfrak{t}$ equal to the Lie algebra of the group $K$ of fixed points under $\sigma$. When $G$ is semisimple then $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{t}$ relative to the Killing form and $\mathfrak{p}$ and $\mathfrak{t}$ satisfy the following Lie algebraic relations:

$$[\mathfrak{p}, \mathfrak{p}] = \mathfrak{t}, \quad [\mathfrak{p}, \mathfrak{t}] = \mathfrak{p}.$$ 

The first relation implies that any two points of $G$ can be connected by a curve whose tangent takes values in the left invariant distribution $D(g) = \{gU : U \in \mathfrak{p}\}$.

In the case that $(G, K)$ is a Riemannian symmetric pair there is an $Ad_K$ invariant, positive definite quadratic form $\langle \ , \ \rangle$ on $\mathfrak{p}$ that induces a natural optimal control problem on $G$: minimize the integral $\int_0^T \langle U(t), U(t) \rangle dt$ among all curves $g(t) \in G$ that are the solutions of

$$\frac{dg}{dt} = g(t)U(t), \quad U(t) \in \mathfrak{p}, \quad t \in [0, T]$$ 

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with fixed boundary conditions \( g(0) = g_0, g(T) = g_1 \). Here \( g_0 \) and \( g_1 \) are arbitrary but fixed points in \( G \) and the terminal time \( T > 0 \) is also fixed. This problem, called the canonical sub-Riemannian problem on \( G \), is well defined in the sense that for any pair of boundary points in \( G \) there is an optimal solution. It is well known that optimal solutions are of the form

\[
g(t) = g_0 e^{t(P+Q)} e^{-tQ}
\]

for some elements \( P \in \mathfrak{p} \), and \( Q \in \mathfrak{k} \) \((1.7)\). The above implies that any element \( g \) in \( G \) can be represented as \( g = e^{t(P+Q)} e^{-tQ} \) for some elements \( P \in \mathfrak{p} \), and \( Q \in \mathfrak{k} \). This sub-Riemannian problem is naturally related to the canonical Riemannian problem on the quotient space \( M = G/K \) in the sense that the Riemannian geodesics are the projections of the above curves with \( Q = 0 \).

In this paper we will be interested in another optimal control problem defined by an affine distribution \( \mathcal{D}(g) = \{g(A+U) : U \in \mathfrak{k} \} \) with \( A \) a regular element in \( \mathfrak{p} \) under the assumption that the Killing form is definite on \( \mathfrak{k} \) (which is true when \( K \) is a compact subgroup of \( G \)). This optimal control problem consists of finding the minimum of \( \frac{1}{2} \int_0^T (U(t), U(t)) dt \) among all solution curves \( g(t) \in G \) of the affine control problem

\[
\frac{dg}{dt}(t) = g(t)(A + U(t)), U(t) \in \mathfrak{k}, \ t \in [0, T]
\]

subject to the given boundary conditions \( g(0) = g_0, g(T) = g_1 \) where the quadratic form \( \langle , \rangle \) denotes a scalar multiple of the Killing form that is positive definite on \( \mathfrak{k} \).

This problem might be regarded as the canonical affine problem on symmetric pairs \((G, K)\) for the following reasons. The reciprocal affine system with \( A \in \mathfrak{k} \) and \( U \in \mathfrak{p} \) is isomorphic to \((1.2)\) and bears little resemblance to the solutions of \((1.4)\). Moreover, two affine systems defined by regular elements \( A_1 \) and \( A_2 \) are conjugate.

The affine problem \((1.4)\) will be referred to as the Affine-Killing problem or \((\text{Aff})\) for brevity. It is first shown that \((\text{Aff})\) is well defined in the sense that for any pair of boundary conditions \((g_0, g_1)\) there exist \( T > 0 \) and a solution \( g(t) \) of \((1.4)\) that satisfies \( g(0) = g_0, g(T) = g_1 \) such that the control \( U(t) \) that generates \( g(t) \) minimizes the integral \( \frac{1}{2} \int_0^T (U(t), U(t)) dt \) among all other controls whose solution curves satisfy the same boundary data. Then it is shown that optimal solutions are the projections of the integral curves of a certain Hamiltonian system on the cotangent bundle \( T^*G \) of \( G \) obtained through the use of the Maximum Principle of Optimal Control.

To preserve the left invariant symmetries, \( T^*G \) is realized as the product \( G \times \mathfrak{g}^* \) with \( \mathfrak{g}^* \) equal to the dual of the Lie algebra \( \mathfrak{g} \) of \( G \) and then \( \mathfrak{g}^* \) is identified with \( \mathfrak{g} \) via the Killing form so that ultimately \( T^*G \) is realized as \( G \times \mathfrak{g} \). In this setting the Hamiltonian associated with \((\text{Aff})\) is of the form

\[
H = \frac{1}{2} \langle L_{\mathfrak{p}}, L_{\mathfrak{p}} \rangle + \langle A, L_{\mathfrak{k}} \rangle
\]

where \( L_{\mathfrak{p}} \) and \( L_{\mathfrak{k}} \) denote the projections of \( L \in \mathfrak{g} \) onto the factors \( \mathfrak{p} \) and \( \mathfrak{k} \).

Because \( K \) acts on \( \mathfrak{p} \) via the adjoint action, \( \mathfrak{g} \) as a vector space carries two Lie algebras: a Lie algebra of \( G \) and the Lie algebra \( \mathfrak{g}_s \) of the semidirect product
\( G_s = K \times p \). The affine problem then admits an analogous formulation on \( G_s \) and the Maximum Principle leads to the Hamiltonian \( H \) that formally looks the same as the one obtained in the semisimple case. We refer to the semidirect version of \((\text{Aff})\) as the semidirect shadow problem.

The essence of the paper lies in the integrability properties of the associated Hamiltonian flows \( \vec{H} \) and \( \vec{H}_s \). It is shown that each flow admits a spectral representation

\[
\frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda]
\]

where \( M_\lambda = \frac{1}{\lambda} (L_p - \epsilon A) \) and \( L_\lambda = L_p - \lambda L_k + (\lambda^2 - s)A \), with \( s = 1 \) in the semisimple case and \( s = 0 \) in the semidirect case. Hence, the spectral invariants are constants of motion for the associated Hamiltonian flows.

On spaces of constant curvature these results recover the integrability results associated with elastic curves and their mechanical counterparts (\cite{17}). Remarkably, the spectral invariants above also recover the classical integrability results C. Newmann for mechanical systems with quadratic potential (\cite{22}) and the related results of C.G.J. Jacobi concerning the geodesics on an ellipsoid. The present formalism also clarifies the contributions of J. Moser (\cite{20}) on integrability of Hamiltonian systems based on isospectral methods. More significantly, this study reveals a large class of integrable Hamiltonian systems in which these classical examples appear only as very particular cases.

2. Notations and the Background material

The basic setting is most naturally defined through the language of symmetric spaces. The essential ingredients are assembled below.

An involutive automorphism \( \sigma \) on \( G \) is an analytic mapping \( G \rightarrow G, \sigma \neq I \) that satisfies

\[
\sigma(g_2 g_1) = \sigma(g_2) \sigma(g_1), \quad \text{for all } g_1, g_2 \text{ in } G.
\]

Then the tangent map \( \sigma_* \) of \( \sigma \) induces a splitting \( g = p \oplus k \) of the Lie algebra \( g \) of \( G \) with

\[
p = \{ A \in g : \sigma_*(A) = -A \} \text{ and } k = \{ A \in g : \sigma_*(A) = A \}
\]

The fact that \( \sigma_* \) is a Lie algebra automorphism easily implies the following Lie algebraic relations

\[
[p, p] \subseteq \mathfrak{k}, \quad [p, \mathfrak{k}] \subseteq p, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}
\]

It follows that \( \mathfrak{k} \) is a Lie subalgebra of \( \mathfrak{g} \), equal to the Lie algebra of the group \( K = \{ g \in G : \sigma(g) = g \} \) and that \( p \) is an \( \text{Ad}_K \) invariant vector subspace of \( g \) in the sense that \( \text{Ad}_h(p) \subseteq p \) for any \( h \in K \). An \( \text{Ad}_K \) invariant non-degenerate quadratic form on \( p \) will be called pseudo Riemannian. It is easy to show by differentiating that an \( \text{Ad}_K \) invariant quadratic form \( <,> \) on \( p \) is invariant, in the sense that

\[
< A, [B, C] > = < [A, B], C >
\]

for any \( A, C \) in \( p \) and any \( B \) in \( \mathfrak{k} \).

A pseudo Riemannian form that is positive definite will be called Riemannian. In the literature of symmetric spaces (\cite{6}) the pair \((G, K)\), with \( K \) a closed subgroup of \( G \) obtained by an involutive automorphism on \( G \) described above, is called a symmetric pair. If in addition this pair admits an \( \text{Ad}_K \) invariant positive
definite quadratic form \( \langle \ , \ \rangle \) on \( p \) then it is called a Riemannian symmetric pair. Riemannian symmetric pairs can be characterized as follows.

**Proposition 1.** Let \( Ad_{h,p} \) denote the restriction of \( Ad_h \) to \( p \). Then a symmetric pair \((G, K)\) admits a Riemannian quadratic form \( \langle \ , \ \rangle \) on \( p \) if and only if \( \{ Ad_{h,p} : h \in K \} \) is a compact subgroup of \( Gl(p) \).

In the text below we will make use of the Killing form \( \langle A, B \rangle_k = Tr(adA \circ adB) \) for \( A \) and \( B \) in \( g \), where \( Tr(X) \) denotes the trace of a linear endomorphism \( X \). The Killing form is invariant under any automorphism \( \phi \) on \( g \) and in particular it is \( Ad_K \) and \( Ad_G \) invariant \([6]\). The invariance relative to \( Ad_G \) implies

\[
\langle A, [B, C] \rangle_k = \langle [A, B], C \rangle_k
\]

for any matrices \( A, B, C \) in \( g \). Spaces \( p \) and \( \mathfrak{k} \) are orthogonal relative to \( \langle \ , \ \rangle_k \) because \( \langle A, B \rangle_k = \langle \sigma_\ast(A), \sigma_\ast(B) \rangle_k = \langle -A, B \rangle_k = -\langle A, B \rangle_k \) for any \( A \) in \( p \) and any \( B \) in \( \mathfrak{k} \).

In this paper \((G, K)\) will be assumed a symmetric Riemannian pair with \( G \) semisimple and connected and \( K \) compact. Semisimplicity implies that the Killing form is non-degenerate, which then implies that its restriction to \( p \) is pseudo Riemannian. Semisimplicity also implies that the Cartan relations \((2.3)\) take on a stronger form

\[
[p, p] = \mathfrak{k}, \ [p, \mathfrak{k}] = p, \ \{ \mathfrak{k}, \mathfrak{k} \} \subseteq \mathfrak{k}.
\]

The fact that \( K \) is a compact subgroup of \( G \) implies that the Killing form is negative definite on \( \mathfrak{k} \ ([5], p. 56) \). In the sequel \( \langle \ , \ \rangle \) will denote any scalar multiple of the Killing form which is positive definite on \( \mathfrak{k} \). Under these conditions then \( ||U|| \) will denote the induced norm \( ||U|| = \sqrt{\langle U, U \rangle} \).

An element \( A \) in \( p \) is said to be regular if \( \{ B \in p : [A, B] = 0 \} \) is an abelian algebra. It follows that \( A \) is regular if and only if the algebra \( \mathfrak{a} \) spanned by \( \{ B \in p : [A, B] = 0 \} \) is a maximal abelian algebra in \( p \) that contains \( A \) \([5]\).

With these notions at our disposal we return now to the affine problem defined above. It will be convenient to adopt the language of control theory and regard \((1.2)\) as a control system with \( U(t) \) playing the role of control. In order to meet the conditions of the Maximum Principle control functions are assumed bounded and measurable on compact intervals \([0, T]\). Solutions of \((1.2)\) are called trajectories. A control \( U(t) \) is said to steer \( g_0 \) to \( g_1 \) in \( T \) units of time if the corresponding trajectory \( g(t), t \in [0, T] \) satisfies \( g(0) = g_0, \ g(T) = g_1 \). A trajectory \( g(t) \) generated by a control \( U(t) \) on an interval \([0, T]\) is optimal relative to the boundary conditions \((g_0, g_1)\) if the integral \( \frac{1}{T} \int_0^T ||U(t)||^2 dt \) is minimal among all other controls that steer \( g_0 \) to \( g_1 \) in \( T \) units of time. Controls that result in optimal trajectories are called optimal. Thus every optimal control \( U(t) \) gives rise to a unique optimal trajectory because the initial point \( g_0 \) is fixed.

3. The existence of optimal solutions

**Proposition 2.** If \( A \) is regular then \((\text{Aff})\) problem is well posed in the sense that for any pair of boundary conditions \((g_0, g_1)\) there exist a time \( T > 0 \) and a solution \( g(t) \) on the interval \([0, T]\) that is optimal relative to \( g_0 \) and \( g_1 \).

The proof of this proposition requires several auxiliary facts from the optimal control theory and from the theory of symmetric spaces. We begin first with the facts from the theory of symmetric spaces \([5, 6]\).
A Lie algebra \( g \) is said to be simple if it contains no ideals other than \( \{0\} \) and \( g \). A Lie group \( G \) is said to be simple if its Lie algebra is simple. The first fact is given by

**Lemma 1.** If \((G, K)\) is a symmetric pair with \( G \) simple then \( Ad_K \) acts irreducibly on \( p \).

**Proof.** Let \( V \) denote an \( Ad_K \) invariant vector subspace of \( p \). Denote by \( V^\perp \) the orthogonal complement of \( V \) in \( p \) relative to the Killing form \( \langle \cdot, \cdot \rangle_k \). Since
\[
\langle V, Ad_h(V^\perp) \rangle_k = \langle Ad_{h^{-1}}(V), V^\perp \rangle_k,
\]

it follows that \( V^\perp \) is also \( Ad_K \) invariant. Therefore, \([\mathfrak{t}, V] \subseteq V \) and \([\mathfrak{t}, V^\perp] \subseteq V^\perp \), which in turn implies that \( \langle \mathfrak{t}, [V, V^\perp] \rangle_k = 0 \). It follows that \([V, V^\perp] = 0 \) by semisimplicity of \( g \).

The above implies that \( V + [V, V] \) is an ideal in \( g \) that is orthogonal to \( V^\perp \). Since \( g \) is simple \( V + [V, V] = g \) and therefore \( V^\perp = 0 \). But then \( V = p \). \( \square \)

The other facts which are needed for the proof are assembled below in the forms of propositions.

**Proposition 3.** Suppose that \((G, K)\) is a symmetric Riemannain pair. There exist linear subspaces \( p_1, p_2, \ldots, p_m \) of \( p \) such that

1. \( p = p_1 \oplus p_2 \oplus \cdots \oplus p_m \) and
2. \( p_1, p_2, \ldots, p_m \) are pairwise orthogonal relative to the Killing form.
3. Each \( p_i \) is \( \text{ad}(\mathfrak{t}) \) invariant and contains no proper \( \text{ad}(\mathfrak{t}) \) invariant linear subspace.

**Proposition 4.** Let \( g_i = p_i + [p_i, p_i] \) \( i = 1, \ldots, m \) in a semisimple Lie algebra \( g \). Then, (1) Each \( g_i \) is an ideal of \( g \) and also a simple Lie algebra.

Moreover,

2. \( \langle g_i, g_j \rangle_k = 0 \) and \( \langle g_i, g_j \rangle_k = 0, i \neq j \), and \( \{X \in g : [X, Y] = 0, Y \in g\} \).

**Corollary 1.** Let \( A_i \) denote the projection of a regular element \( A \) in \( p \) on \( p_i \). Then each \( A_i \neq 0, i = 1, \ldots, m \).

**Proof.** If \( A_i \) were equal to 0 then \([A_i, p_i] = 0\) by (2) in Proposition 4. This would imply that \( p_i \) is abelian by regularity of \( A \), which in turn would imply that \( g_i = p_i \). But that would contradict (3) in Proposition 4. \( \square \)

We now turn attention to the pertinent ingredients from the accessibility theory of control systems. The Lie Saturate of a left invariant family of vector fields \( \mathcal{F} \) is the largest family of left invariant vector fields (in the sense of set inclusion) that leaves the closure of the reachable sets of \( \mathcal{F} \) invariant (\cite{15}). It is denoted by \( LS(\mathcal{F}) \).

Since left invariant vector fields are defined by their values at the identity, the Lie saturate admits a paraphrase in terms of the defining set \( \Gamma \) in \( g \). For the affine system (\cite{1,3}), \( \Gamma = \{A + B : B \in \mathfrak{t}\} \).

**Definition 1.** Let \( \Gamma \subseteq g \). The reachable set of \( \Gamma \) denoted by \( \Lambda(\Gamma) \) is the set of terminal points \( g(T) \in G \) corresponding to the absolutely continuous curves \( g(t) \) on intervals \([0, T]\) such that \( g(0) = I \) and \( \frac{dg}{dt}(t)g^{-1}(t) \in \Gamma \) for almost all \( t \in [0, T] \).

Then \( LS(\Gamma) \), the Lie Saturate of \( \Gamma \) can be described as the largest family in \( g \) such that

\[
(3.1) \quad \text{cl}(\Lambda(\mathcal{A}(LS(\Gamma)))) = \text{cl}(\Lambda(\Gamma)),
\]
where \( \text{cl}(X) \) denotes the topological closure of a set \( X \).

The following lemma is well known in control theory ([13]).

**Lemma 2.** a. \( LS(\Gamma) = \mathfrak{g} \) is a necessary and sufficient condition that \( A(\Gamma) = G \).

If \( C \) denotes the convex cone spanned by \( \sum_{i=1}^m \alpha_i \text{Ad}_{h_i}(A), \ h_i \in \mathfrak{k}, \ \alpha_i \geq 0, \ i = 1, 2, \ldots, m, m \in \mathbb{Z}^+ \) then

\[
(3.2) \quad C \cup \mathfrak{k} \subseteq LS(\Gamma)
\]

where \( \Gamma = \{ A + U : U \in \mathfrak{k} \} \).

With these results at our disposal let us turn to the proof of the proposition.

**Proof.** Let \( \text{Traj}(g_0, g_1) \) denote the set of solutions \( g(t) \) of \((1.3)\) that satisfy \( g(0) = g_0, \ g(T) = g_1 \) for some \( T > 0 \). If \( \text{Traj}(g_0, g_1) \) is not empty for any \( g_0 \) and \( g_1 \) in \( G \) then \((1.4)\) is said to be **controllable**. An argument based on weak compactness of closed balls in Hilbert spaces shows that there is an optimal trajectory \( \hat{\text{Traj}} \) of closed balls in \( G \). The following lemma is well known in control theory ([15]).

Now address the question of controllability. According to Lemma 2 it would follow that \( \text{Traj}(g_0, g_1) \) is absolutely continuous and hence belongs to \( L^\infty([0, T]) \). The above argument shows that controllability implies the existence of optimal trajectories.

3.1. **Semidirect products and the shadow problem.** Recall that if \( K_0 \) is a Lie group which acts linearly on a finite dimensional vector space \( V \) then the semidirect product \( G_\ast = V \rtimes K_0 \) consists of points \( (v, h) \) in \( V \times K_0 \) with the group operation \((v_1, h_1)(v_2, h_2) = (v_1 + h_1(v_2), h_1h_2)\). The Lie algebra \( \mathfrak{g}_\ast \) of \( G_\ast \) consists of pairs \((A, B)\) in \( V \times \mathfrak{k}_0 \) where \( \mathfrak{k}_0 \) denotes the Lie algebra of \( K_0 \), with the Lie bracket

\[
[(A_1, B_1), (A_2, B_2)]_s = (B_1(A_2) - B_2(A_1), [B_1, B_2]).
\]

Every semidirect product \( V \rtimes K_0 \) admits an involutive automorphism \( \sigma(x, h) = (-x, h) \) for every \( (x, h) \in V \times K_0 \). It follows that \( K = \{ 0 \} \times K_0 \) is the group of fixed points of \( \sigma \) and that

\[
(3.3) \quad \mathfrak{p} = V \times \{ 0 \} \quad \text{and} \quad \mathfrak{k} = \{ 0 \} \times \mathfrak{k}_0.
\]

It is easy to check that \( \text{Ad}_{h}(x, 0) = (h(x), 0) \) for every \( h \in K \) and every \( x \in V \). Therefore, \((G, K)\) is a symmetric Riemannian pair if and only if \( K_0 \) is a compact subgroup of \( Gl(V) \).

Every Lie group \( G \) that admits an involutive automorphism carries the semidirect product \( G_\ast = \mathfrak{p} \rtimes K \) because \( K \) acts linearly on the Cartan space \( \mathfrak{p} \) via the
transformation
\[(3.4)\quad h(A) = Ad_h(A), A \in \mathfrak{p}, \text{ for each } h \in K.\]

Therefore, the Lie bracket on \(\mathfrak{g}_s\) is given by
\[(3.5)\quad [(A_1, B_1), (A_2, B_2)]_s = ([\text{ad}B_1 A_2] - \text{ad}B_2 A_1), [B_1, B_2]).\]

If \((A, B)\) in \(\mathfrak{p} \times \mathfrak{k}\) is identified with \(A + B\) in \(\mathfrak{p} + \mathfrak{k}\) then the semidirect Lie bracket \([\ ,\ ]\) can be redefined as
\[(3.6)\quad [(A_1 + B_1), (A_2 + B_2)]_s = [B_1, A_2] - [B_2, A_1] + [B_1, B_2],\]

from which it follows that
\[(3.7)\quad [\mathfrak{p}, \mathfrak{p}]_s = 0, [\mathfrak{p}, \mathfrak{k}]_s = [\mathfrak{p}, \mathfrak{k}], [\mathfrak{k}, \mathfrak{k}]_s = [\mathfrak{k}, \mathfrak{k}].\]

Thus \(\mathfrak{g}\) is the underlying vector space for both Lie algebras \(\mathfrak{g}\) and \(\mathfrak{g}_s\), a fact which is important for the subsequent development. The passage from \(\mathfrak{g}_s\) to \(\mathfrak{g}\) can be described by a continuous parameter \(s\) by deforming the Lie algebra \(\mathfrak{g}_s\) to \(\mathfrak{g}\) via the Lie bracket \([\ ,\ ]\) : \begin{equation}
(3.8) \quad [(A_1 + B_1), (A_2 + B_2)]_s = [B_1, A_2] - [B_2, A_1] + [B_1, B_2] + s[A_1, A_2]
\end{equation}

We now return briefly to the affine problem \(\text{Aff}\) to note that the data which is required for its formulation on a semisimple Lie group \(G\) also permits a formulation on the semidirect product \(G_s\). The semidirect version consists of minimizing the integral \(\frac{1}{2} \int_0^T \langle U(t), U(t) \rangle \, dt\) over all solutions \(g(t)\) in \(G_s\) of
\[
\frac{d}{dt} g(t) = g(t)(A + U(t)), U(t) \in \mathfrak{k}, \ t \in [0, T]
\]

that meet the boundary conditions \(g(0) = g_0, g(T) = g_1\). This "shadow" problem will be referred to as \((\text{Aff}_s)\). The same arguments used in the semisimple case show that \((\text{Aff}_s)\) is also well defined in the sense of Proposition 2.

4. LEFT INVARIANT HAMILTONIAN SYSTEMS AND THE MAXIMUM PRINCIPLE

Consider now the necessary conditions of optimality provided by the Maximum Principle. The Maximum Principle states that each minimizer is the projection of an extremal curve in the cotangent bundle \(T^* G\) and each extremal curve is an integral curve of a certain Hamiltonian vector field on \(T^* G\). To state all this in more detail requires additional notation and terminology.

As already stated earlier, \(\mathfrak{g}^*\) denotes the dual of \(\mathfrak{g}\). The dual of a Lie algebra carries a Poisson structure inherited from the symplectic structure of \(T^* G\) realized as the product \(G \times \mathfrak{g}^*\) via the left translations. Functions on \(\mathfrak{g}^*\) are called left-invariant Hamiltonians. If \(f\) and \(h\) are left-invariant Hamiltonians then their Poisson bracket \(\{f, h\}\) is defined by \(\{f, h\}(l) = l([df, dh]), \text{ for } l \in \mathfrak{g}\).

On semisimple Lie algebras \(\mathfrak{g}^*\) can be identified with \(\mathfrak{g}\) via the quadratic form \(\langle \ , \ \rangle\) with \(\langle L, X \rangle = l(X)\) for all \(X \in \mathfrak{g}\). In this identification \(\mathfrak{p}^*\) and \(\mathfrak{k}^*\) are identified with \(\mathfrak{p}\) and \(\mathfrak{k}\) whenever \(\mathfrak{g}\) admits a Cartan decomposition \(\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}\). The above then implies that \(l = l_p + l_t\) with \(l_p \in \mathfrak{p}^*\) and \(l_t \in \mathfrak{k}^*\) is identified with \(L = L_p + L_t\) where \(L_p\) and \(L_t\) correspond to \(L_p \in \mathfrak{p}\) and \(L_t \in \mathfrak{k}\).

To preserve the left invariant symmetries \(T^* G\) will be trivialized by the left translations and considered as the product \(G \times \mathfrak{g}^*\). The advantage of the above choice of trivialization is that the Hamiltonian lift of a left invariant vector field becomes a linear function on \(\mathfrak{g}^*\). Recall that a Hamiltonian lift of a vector field
X on a manifold M is a function $H_X$ on $T^*M$ defined as $H_X(\xi) = \xi(X(x))$ for each $\xi \in T^*_x M$. If $X(g) = gA$ is a left invariant vector field on a Lie group $G$, then $H_X(g,l) = l(A), l \in g^*$. 

Any left invariant function $h$ generates a Hamiltonian vector field $\vec{h}$ on $G \times g^*$ whose integral curves $(g(t),l(t))$ are the solutions of the following differential equations

\begin{equation}
(4.1) \quad \frac{dg}{dt} = g(t)dh(l(t)), \quad \frac{dl}{dt} = -ad^*(dh(l))(l(t)),
\end{equation}

where $dh$ denotes the differential of $h$ considered as an element of $g$ under the isomorphism $(g^*)^* \leftrightarrow g$, and where $ad^*(L) : g^* \rightarrow g^*$ is defined by $ad^*(L)(l) = l \circ ad(L)$.

With these notations at our disposal we now apply the Maximum Principle to the affine problem. The affine problem defines ”cost-extended system” in $\mathbb{R} \times G$:

\begin{equation}
(4.2) \quad \frac{dx}{dt} = \frac{1}{2}\|U(t)\|^2, \quad \frac{dg}{dt}(t) = g(t)(A + U(t)), U(t) \in \mathfrak{t}.
\end{equation}

The Hamiltonian lift of the cost-extended system is given by:

\begin{equation}
(4.3) \quad H_U(\lambda, l) = \lambda \frac{1}{2}\|U\|^2 + l(A + U), \lambda \in \mathbb{R}, l \in g^*.
\end{equation}

The above is a function on $T^*(\mathbb{R} \times G)$ trivialized as $(\mathbb{R} \times \mathbb{R}) \times (G \times g^*)$ with coordinates $(x, \lambda, g, l)$. Each control function $U(t)$ generates a time varying Hamiltonian $H_{U(t)}(\lambda, l)$; the integral curves $\xi(t) = (x(t), \lambda(t), g(t), l(t))$ of the associated Hamiltonian vector field $\vec{H}_{U(t)}$ are the solutions of

\begin{equation}
(4.4) \quad \frac{dx}{dt} = \frac{\partial H_{U(t)}}{\partial \lambda}, \quad \frac{d\lambda}{dt} = -\frac{\partial H_{U(t)}}{\partial x}, \quad \frac{dg}{dt} = g(A + U(t)), \quad \frac{dl}{dt} = -ad^*(A + U(t))(l(t))
\end{equation}

It follows that $\lambda$ is constant for any solution $\xi(t)$ since $\frac{\partial H_{U(t)}}{\partial \lambda} = 0$.

**Proposition 5. The Maximum Principle.** Assume that $\vec{U}(t)$ is an optimal control that generates the trajectory $\vec{g}(t)$. Let $\bar{x}(t)$ denote its running cost $\int_0^t \frac{1}{2}\|U\|^2 dt$. Then $\bar{x}(U(t), \bar{g}(t))$ is the projection of an integral curve of $\vec{\xi}(t) = (\bar{x}(t), \bar{\lambda}(t), \bar{g}(t), \bar{l}(t))$ of $\vec{H}_{U(t)}$ that satisfies the following conditions:

\begin{equation}
(4.5) \quad \bar{\lambda} \leq 0. \text{ When } \lambda = 0 \text{ then, } \bar{l}(t) \neq 0.
\end{equation}

\begin{equation}
(4.6) \quad \vec{H}_{U(t)}(\bar{\lambda}, \bar{l}(t)) \geq H_U(\bar{\lambda}, \bar{l}(t)), U \in \mathfrak{t}, \text{ a.e. in } [0,T].
\end{equation}

In the literature on optimal control it is customary to consider only the projections $(g(t),l(t))$ of integral curves $\xi(t)$ of $\vec{H}_U(\lambda, l)$ which are parametrized by a non-positive parameter $\lambda$. Control functions $U(t)$ on $T^*G$ are called *extremal* if they generate solutions of (4.4) that satisfy conditions (4.5) and (4.6) of the Maximum Principle. Extremal curves that correspond to $\lambda = 0$ are called *abnormal* and those that correspond to $\lambda < 0$ are called *normal*. In the normal case $\lambda$ is reduced to $-1$ because of the homogeneity properties of $H_U(\lambda, l)$ with respect to $\lambda$.

The Maximum Principle can be restated in terms of the extremals by saying that each optimal trajectory is the projection of an extremal curve (normal or abnormal). Thus the Maximum Principle identifies two distinct Hamiltonians associated with each optimal control problem depending on the value of $\lambda$. 

Return now to the Hamiltonians of the affine problem. After the identifications of \( l = l_p + l_t \) with \( L = L_p + L_t \) the Hamiltonian \( H_U(\lambda, l) \) is identified with \( H(\lambda, L) = \lambda \frac{1}{2} ||U||^2 + \langle A, L_p \rangle + \langle U, L_t \rangle \).

In the normal case \( \lambda = -1 \), and the maximality condition (4.10) easily implies that each normal extremal curve \((g(t), L(t))\) is an integral curve of the Hamiltonian
\[
H = \frac{1}{2} ||L_t||^2 + \langle A, L_p \rangle
\]
generated by the extremal control \( U(t) = L_t(t) \). This Hamiltonian will be referred to as the affine Hamiltonian.

In the abnormal case the maximization relative to \( U \) results in a constraint \( L_t = 0 \) and does not directly yield the value for \( U \). Further investigations of these extremals will be deferred to the next section.

4.1. Extremal equations.

Hamiltonian equations (4.1) reveal their symmetries more readily when recast on the Lie algebras rather than on their duals. In order to treat the affine Hamiltonian both as a Hamiltonian on \( G \) and as a Hamiltonian on the semidirect product \( G_s \), equations (4.1) need to be recast on \( g \) and \( g_s \). Since the Lie bracket is different in two cases the differential equations take on different forms.

Recall that \([A, B]_s\) denotes the Lie bracket that deforms the semisimple Lie bracket when \( s = 1 \) to the semidirect Lie bracket when \( s = 0 \). Let \( dh = dh_p + dh_t \), \( L = L_p + L_t \), \( X = X_p + X_t \) denote the appropriate decompositions relative to \( p \) and \( t \).

Then the bracket \( \frac{dt}{dt}(X) = -ad^*(dh(l))(l(t))(X) = -l([dh, X]_s) \) corresponds to \( \langle \frac{dl}{dt}, X \rangle = -\langle L, [dh, X]_s \rangle \) ; the latter implies that \( \langle \frac{dl_p}{dt}, X_p \rangle + \langle \frac{dl_t}{dt}, X_t \rangle = -\langle L_p, [dh_p, X_t] + [dh_t, X_p] \rangle - \langle L_t, [dh_t, X_t] + s[dh_p, L_p] \rangle \), or
\[
\frac{dl_p}{dt}, X_p \rangle + \langle \frac{dl_t}{dt}, X_t \rangle = \langle [dh_t, L_t] + [dh_p, L_p], X_t \rangle + \langle [dh_t, L_p] + s[dh_p, L_t], X_p \rangle.
\]

Therefore,
\[
\frac{dL_t}{dt} = [dh_t, L_t] + [dh_p, L_p], \quad \frac{dL_p}{dt} = [dh_t, L_p] + s[dh_p, L_t].
\]

In the case of affine Hamiltonian \( H \) given by (4.11), \( dH = L_t + A \) and the preceding equations become
\[
\frac{dL_p}{dt}(t) = [L_t, L_p] + s[A, L_t] = [sA - L_p, L_t], \quad \frac{dL_t}{dt}(t) = [A, L_p].
\]

Abnormal extremals are the integral curves of \( H_0 = \langle L, A + U \rangle \) subject to the constraint \( L_t = 0 \), that is, abnormal extremal curves \((g(t), L(t))\) are the solutions of
\[
\frac{dg}{dt} = g(t)(A + U(t)),
\]
\[
\frac{dL_p}{dt}(t) = [U(t), L_p] + s[A, L_t], \quad \frac{dL_t}{dt}(t) = [A, L_p] + [U(t), L_t]
\]
subject to \( L_t(t) = 0 \) for all \( t \in [0, T] \). They are described by the following proposition.
Proposition 6. Abnormal extremal curves are the solutions of \(\frac{dg}{dt} = g(t)(A + U(t))\) generated by bounded and measurable controls \(U(t) \in \mathfrak{k}\) that satisfy the constraints

\[
[A, L_p] = 0, \quad [L_p, U(t)] = 0
\]

for some element \(L_p\) in \(\mathfrak{p}\).

If \(L_p\) is regular, then the corresponding abnormal extremal curve whose projection on \(G\) is optimal is also normal.

Proof. Suppose that \(g(t), L_p(t), L_e(t) = 0\) is an abnormal extremal curve generated by the control \(U(t) \in \mathfrak{k}\). Then equations \(1.10\) imply that \(\frac{dL_p}{dt} = [U(t), L_p]\) and \([A, L_p] = 0\). This means that \(L_p(t)\) belongs to the maximal abelian subalgebra \(\mathfrak{a}\) in \(\mathfrak{p}\) that contains \(A\). Therefore, \(H_p\) also belongs to \(\mathfrak{a}\).

If \(X\) is an arbitrary element of \(\mathfrak{a}\) then \([\langle U(t), L_p \rangle, X \rangle = \langle U(t), [L_p, X]\rangle\) = 0. Therefore, \(\langle X, \frac{dL_p}{dt}\rangle = 0\). Since the Killing form is nondegenerate on \(\mathfrak{a}\), \(\frac{dL_p}{dt} = 0\) and therefore, \(L_p(t)\) is constant. This proves the first part of the proposition.

To prove the second part assume that \(L_p\) is regular. Then \([A, L_p] = 0\) implies that \([L_p, [A, U(t)]]\) = 0. Since \(L_p\) is regular and belongs to \(\mathfrak{a}\), \([A, U(t)\]) also belongs to \(\mathfrak{a}\). It then follows that \([A, U(t)] = 0\) by the argument identical to the one used in the preceding paragraph.

It now follows that \(g(t) = g(0)e^{At}\) where \(h(t)\) denotes the solution of \(\frac{dh}{dt} = h(t)U(t),\ h(0) = I\). Let \(g_0\) and \(g_1\) denote the boundary points relative to which \(g(t)\) is optimal. Then \(h(t)\) is optimal relative to \(h(0) = I\) and \(h(T) = e^{-AT}g_1\). This means that \(h(t)\) is a geodesic in \(K\) relative to the bi-invariant metric induced by \(\langle \,, \rangle\). Hence the control that generates \(h(t)\) must be constant, i.e., \(h(t) = e^{Ut}\) for some element \(U\) in \(\mathfrak{k}\).

The reader can readily verify that each trajectory \((g(t), U(t))\) of the affine system in which the control \(U(t)\) is constant and commutes with \(A\) is the projection of a solution of \(1.10\).

\(\square\)

Corollary 2. If \(\mathfrak{p}\) is such that each non-zero element is regular, then each abnormal extremal that projects onto an optimal trajectory is also a projection of a normal extremal curve. In particular, on isometry groups of space forms (simply connected symmetric spaces of constant curvature) each optimal trajectory of the affine system is the projection of a normal extremal curve.

Proof. See the discussion on space forms in Section 6. \(\square\)

Remark 1. The above proposition raises an interesting question. Is every optimal trajectory on an arbitrary symmetric space the projection of a normal extremal curve?

It seems that \(G = SL_n(R)\) with \(\mathfrak{p}\) the space of symmetric matrices with trace zero and \(\mathfrak{k} = so_n(R)\) is a good testing ground for this question. In this situation there are plenty abnormal extremal curves but it is not clear exactly how they relate to optimality.
5. Spectral representation and its consequences

We will now recall an observation made in (18) that a system of differential equations of the form

\[ \frac{dX_0}{dt} = [X_0, X_1], \quad \frac{dX_1}{dt} = [X_0, X_2], \ldots, \quad \frac{dX_n}{dt} = [X_0, X_{n+1}], \quad \frac{dX_{n+1}}{dt} = 0 \]

admits a spectral representation

\[ \frac{dL_\lambda}{dt} = [M_\lambda, L_\lambda] \]

with \( M_\lambda = \frac{1}{\lambda} X_0 \) and \( L_\lambda = \frac{1}{\lambda} X_0 + X_1 + \lambda X_2 + \cdots + \lambda^n X_{n+1} \). This representation is a consequence of a dilational symmetry

\[ \tilde{X}_0 = \frac{1}{\lambda} X_0, \quad \tilde{X}_1 = X_1, \quad \tilde{X}_2 = \lambda X_2, \ldots, \quad \tilde{X}_{n+1} = \lambda^n X_{n+1}. \]

For then \( \tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n+1} \) also satisfy (5.1) and therefore, \( \frac{dL_\lambda}{dt} = [\tilde{X}_0, \tilde{L}] \) where \( \tilde{L} = \tilde{X}_0 + \tilde{X}_1 + \cdots + \tilde{X}_{n+1} \).

Extremal equations (4.9) are of the form (5.1) with \( X_0 = L_p - \epsilon A, \quad X_1 = -L_t \) and \( X_2 = A \). Since \( (5.2) \) is invariant under a multiplication by \( \lambda \) it will be convenient to redefine \( L_\lambda \) as \( L_\lambda = X_0 + \lambda X_1 + \lambda^2 X_2 + \cdots + \lambda^{n+1} X_{n+1} \) in which case

\[ L_\lambda = L_p - \lambda L_t + (\lambda^2 - \lambda) A \]

is a spectral matrix for equations (4.9), in the sense that the spectral invariants of \( L_\lambda \) are constants of motion for the corresponding Hamiltonian system. Moreover, these functions are in involution according to the following proposition.

**Proposition 7.** The spectral invariants of \( L_\lambda = L_p - \lambda L_t + (\lambda^2 - 1) A \) Poisson commute with each other relative to the semisimple Lie algebra structure, while the spectral invariants of \( L_\lambda = L_p - \lambda L_t + \lambda^2 A \) Poisson commute relative to the semidirect product structure.

The proof below is a minor adaptation of the one presented in (24) and (26).

**Proof.** Let \( T : g^* \to g^* \) be defined by \( T(p + k) = \frac{1}{\lambda} p - k + \mu a \) for \( p \in p^*, k \in t^* \). Here, \( a \) is a fixed element of \( p^* \) and \( \lambda \) and \( \mu \) are parameters. Then \( T^{-1} = \lambda p - k - \lambda \mu a \). This diffeomorphism extends to a diffeomorphism on forms according to the following formula:

\[ \{f, g\}_{\lambda, \mu}(\xi) = (T \circ \{f, g\})(\xi) = \{f \circ T^{-1}, g \circ T^{-1}\}(T(\xi)), \]

where \( \{ , \} \) denote the canonical Poisson form on \( g^* \) (relative to the semisimple structure). A simple calculation shows that

\[ \{f, g\}_{\lambda, \mu} = -\lambda^2 \{f, g\} - \lambda \mu \{f, g\}_a - (1 - \lambda^2) \{f, g\}_s \]

where \( \{f, g\}_a = \{f, g\}(a) \) and \( \{f, g\}_s \) is the Poisson bracket relative to the semidirect product structure. Relative to the semidirect structure \( \{f, g\}_s \) the shifted Poisson bracket \( \{f, g\}_{\lambda, \mu}(\xi) = (T \circ \{f, g\}_s)(\xi) \) takes on a slightly different form:\( \{f, g\}_{\lambda, \mu} = -\{f, g\}_s - \lambda \mu \{f, g\}_a \)

Functions on \( g^* \) which Poisson commute with any other function on \( g^* \) are called **Casimirs**, i.e., Casimirs are the elements of the center of the Poisson algebra \( C^\infty(g^*) \).
It $f$ is any Casimir then $f_{\lambda, \mu} = T \circ f$ satisfies \( \{ f_{\lambda, \mu}, g \}_{\lambda, \mu} = 0 \) for any function $g$ on $\mathfrak{g}$ and any parameters $\lambda$ and $\mu$. In the case that $g$ is another Casimir then $f_{\lambda, \mu}$ and $g_{\lambda, \mu} = T \circ g$ satisfy
\[
\{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_1, \mu_1} = \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_2, \mu_2} = 0.
\]

for any values $\lambda_1, \mu_1, \lambda_2, \mu_2$. The same applies to the semidirect Poisson bracket.

Suppose now that $\mu = 2^{-1} \lambda^2 - 1$. It follows from \((5.6)\) that $\{ f, g \}_{\lambda, \mu} = -\lambda^2 \{ f, g \} - (1 - \lambda^2) (\{ f, g \}_{\lambda} + \{ f, g \}_{\mu})$. Therefore,
\[
0 = \frac{1}{\lambda^2} \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_1, \mu_1} - \frac{1}{\lambda^2 - 1} \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_2, \mu_2} = \frac{\lambda^2 - \lambda^2}{(1 - \lambda^2)(1 - \lambda^2)} \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}.
\]

Since $\lambda_1$ and $\lambda_1$ are arbitrary $\{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \} = 0$. This argument proves the first part of the proposition because $\lambda T \neq L_p - \lambda L_t + (\lambda^2 - 1) A$ when $\mu = 2^{-1} \lambda$. after the identifications $p \to L_p$, $k \to L_k$ and $a \to A$.

In the semidirect product $\lambda T = L_p - \lambda L_t + \lambda^2 A$ when $\lambda = \mu$. Then,
\[
0 = \frac{1}{\lambda^2} \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_1, \mu_1} - \frac{1}{\lambda^2 - 1} \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_2, \mu_2} = \frac{\lambda^2 - \lambda^2}{(1 - \lambda^2)(1 - \lambda^2)} \{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_1, \mu_1},
\]

Therefore, $\{ f_{\lambda_1, \mu_1}, g_{\lambda_2, \mu_2} \}_{\lambda_1, \mu_1} = 0$.

\section{Specific cases}

It will be convenient to single out some symmetric pairs $(G, K)$ on which more detailed integrability investigations can be carried out.

\subsection{Non compact Riemannian symmetric spaces}

Every semidirect product $G = V \rtimes H$ admits an involutive automorphism $\sigma(v, h) = (-v, h), (v, h) \in V \rtimes H$ with $K = \{0\} \times H$ the group of fixed points under $\sigma$. The corresponding splitting is given by $p = V \times \{0\}$ and $t = \{0\} \times H$. The pair $(G, K)$ is a Euclidean symmetric pair. Below are some examples of non-Euclidean symmetric spaces.

**Selfadjoint groups.** A matrix group $G$ is called self adjoint if the transpose $g^T$ belongs to $G$ for every $g \in G$. Let $G \subseteq SL_n(R)$ be any self adjoint group. Define $\sigma : G \to G$ by $\sigma(g) = (g^T)^{-1}$. Then $\sigma(g) = g$ if and only if $g \in SO_n(R) \cap G$. Assuming that $G \neq SO_n(R)$, then $\sigma$ is an involutive automorphism and $(G, K)$ is a symmetric pair with $K = G \cap SO_n(R)$. The splitting of $\mathfrak{g}$ induced by $\sigma$ is given by
\[
\mathfrak{p} = \{ A \in \mathfrak{g} : A^T = A \}, \text{ and } \mathfrak{k} = \{ A \in \mathfrak{g} : A^T = -A \}.
\]

The quadratic form defined by
\[
(A, B) = \frac{1}{2} \text{Trace}(AB)
\]

is $Ad_K$ invariant and positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$. Hence, $(G, K)$ is a Riemannian symmetric pair. Below are some noteworthy special cases of self adjoint groups.

**Positive definite matrices.** $G = SL_n(R)$, $K = SO_n(R)$. Then $SL_n(R) / SO_n(R)$ can be identified with the space of positive definite $n \times n$ matrices with real entries.

**The generalized upper half plane.** $G = Sp_n, K = SO_{2n} \cap Sp_n = SU_n$. Recall that $Sp_n$ denotes the group that leaves the symplectic form $< x, y > = \sum_{i=1}^n x_i y_{i+n} - y_i x_{i+n}$ in $R^{2n}$ invariant. In this situation
\[
\mathfrak{p} = \{ \begin{pmatrix} A & B \\ B & -A^T \end{pmatrix} \}, \mathfrak{k} = \{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A^T = -A, B^T = B \}.
\]
The quotient $Sp_n/SU_n$ can be considered as the generalized upper half plane since it can be realized also as the space of complex $n \times n$ matrices $Z$ with
\begin{equation}
Z = X + iY
\end{equation}
with $X$ and $Y$ real $n \times n$ symmetric matrices and $Y$ positive definite.
For $n = 1$, $Sp_1 = SL_2$, $SU_1 = SO_2(R)$ and $Sp_1/SU_1$ coincides with $SL_2(R)/SO_2(R)$. The latter, with its Riemannian metric induced by $\langle \cdot, \cdot \rangle$, is identified with Poincaré’s upper half plane.

**Open sets of Grassmannians.** $G = SO(p, q)$, $K = (O_p(R) \times O_q(R)) \cap SO_{p+q}(R)$.

The quotient space $M^+_{p,q} = SO(p, q)/(O_p(R) \times O_q(R)) \cap SO_{p+q}(R)$ is identified with the open subset of Grassmannians $Gr_p(\mathbb{R}^{p+q})$ consisting of all $q$ dimensional subspaces in $\mathbb{R}^{p+q}$ on which the quadratic form $\langle x, y \rangle_{p,q} = -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{p+q} x_i y_i$ is positive definite. The corresponding Lie algebra splitting is given by
\[
\begin{align*}
p &= \{ M = \begin{pmatrix} 0 & X^T \\ X & 0 \end{pmatrix} : X \text{ any } p \times q \text{ matrix}, \} \\
t &= \{ M = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} : B^T = -B, C^T = -C \}.
\end{align*}
\]

The space $M^+_{1,n}$ can be identified with the hyperboloid $\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : x_0^2 - (x_1^2 + \cdots + x_n^2) = 1, x_0 > 0 \}$ via the following identification. Let $P$ denote the orthogonal complement relative to $\langle \cdot, \cdot \rangle_{1,n}$ of an $n$ dimensional subspace $Q$ in $\mathbb{R}^{n+1}$ on which $\langle x, y \rangle_{1,n} = -x_0 y_0 + \sum_{i=2}^{n} x_i y_i$ is positive definite. Since $\langle \cdot, \cdot \rangle_{1,n}$ is positive on $Q$ and non-degenerate on $\mathbb{R}^{n+1}$, $P$ is transversal to $Q$ and hence is one dimensional. Let $p = (p_0, \ldots, p_n)$ be any non-zero point of $P$. Since the form $\langle \cdot, \cdot \rangle_{1,n}$ is indefinite on $\mathbb{R}^{n+1}$, $\langle p, p \rangle_{1,n} < 0$. If $p$ is normalized so that $p_0 > 0$ and $\langle p, p \rangle = -1$ then $p \in \mathbb{H}^n$ and $Q$ is identified with the tangent space at $p$.

6.0.2. **Compact Riemannian symmetric spaces.**

**The Grassmannians.** Let $G = SO_{p+q}(R)$ with the automorphism $\sigma(g) = JgJ^{-1}$ where $J$ denotes a diagonal matrix with its first $p$ diagonal entries equal to 1 and the remaining diagonal entries equal to $-1$. It follows that $\sigma(g) = g$ if and only if $J = gJg^T$. An easy calculation shows that $J = gJg^T$ if and only if
\[
g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}
\]
where $g_1$ is a $p \times p$ matrix, $g_2$ is a $q \times q$ matrix and $g_1 = g_1^T$ and $g_2 = g_2^T$. Hence the isotropy group $K$ is equal to $(O_p(R) \times O_q(R)) \cap SO_{p+q}(R)$, which will be denoted by $S(O_p(R) \times O_q(R))$.

The tangent map of $\sigma$ splits $\mathfrak{g} = so_{p+q}(R)$ into $\mathfrak{p}$, the vector space of matrices $P = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$ where $B$ is a $p \times q$ matrix, and $\mathfrak{t}$, the Lie algebra of $K$ consisting of matrices $Q = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ with $A$ and $D$ antisymmetric.

The pair $(G, K)$ is a symmetric Riemannian pair with the metric on $\mathfrak{p}$ defined by the quadratic form $(P_1, P_2) = -\frac{1}{2} Tr(P_1 P_2)$. The homogeneous space $Gr_p = SO_{p+q}(R)/S(O_p(R) \times O_q(R))$ is the space of $p$ dimensional linear subspaces in $\mathbb{R}^{p+q}$ and it is a double cover of $SO_{p+q}(R)/SO_p(R) \times SO_q(R)$ . The latter is the space of oriented $p$ dimensional linear subspaces in $\mathbb{R}^{p+q}$. 




When \( p = 1 \) and \( q = n \), then the set of oriented lines in \( \mathbb{R}^{n+1} \) is identified with the sphere \( S^n \) and the above gives
\[
S^n = SO_{n+1}(R)/\{1\} \times SO_n(R).
\]

**Complex symmetric matrices.** Let \( G = SU_n \) with \( \sigma(g) = (g^T)^{-1} \). It follows that \( \sigma(g) = g \) if and only if \( g \in SO_n(R) \). The corresponding splitting of \( su_n \) identifies \( \mathfrak{f} \) with the real part of matrices in \( su_n \) and \( \mathfrak{p} \) with the imaginary matrices in \( su_n \), i.e., \( \mathfrak{p} = \{iY : Y = Y^T\} \) and \( \mathfrak{f} = \{X : XT = -X\} \).

The pair \( (SU_n, SO_n(R)) \) is a symmetric Riemannian pair with the metric induced by the trace form \( \langle A, B \rangle = -\frac{1}{2}Tr(AB) \). The quotient space \( M = SU_n/SO_n(R) \) can be identified with complex matrices of the form \( e^{iA} \) for some symmetric real matrix \( A \), because every matrix \( g \in SU_n \) can be written in its polar form as \( g = e^{iA}R \) for some \( R \in SO_n(R) \).

For \( n = 2 \), \( M \) is a two dimensional sphere as can be verified by the following argument. If \( A = i \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \) denote a matrix in \( \mathfrak{p} \) then \( A^2 = -(a^2 + b^2)I \) and therefore,
\[
e^{iA} = I \cos \sqrt{a^2 + b^2} + \frac{i}{\sqrt{a^2 + b^2}} A \sin \sqrt{a^2 + b^2},
\]
or
\[
e^{iA} = \cos \sqrt{a^2 + b^2} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \sqrt{a^2 + b^2} \left( \begin{array}{cc} a & b \\ b & -a \end{array} \right) \sin \sqrt{a^2 + b^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Then \( x = \cos \sqrt{a^2 + b^2} \), \( y = \frac{a}{\sqrt{a^2 + b^2}} \sin \sqrt{a^2 + b^2} \) and \( z = \frac{b}{\sqrt{a^2 + b^2}} \sin \sqrt{a^2 + b^2} \) identifies the above matrix with the sphere \( x^2 + y^2 + z^2 = 1 \). The decomposition \( g = e^{iA}R \) corresponds to the Hopf fibration \( S^3 \to S^2 \to S^1 \).

### 6.1. Space forms.
Simply connected Riemannian spaces of constant curvature, known as space forms, consist of hyperboloids (spaces of negative curvature), spheres (spaces of negative curvature) and Euclidean spaces (spaces of zero curvature). The normalized prototypes are the unit hyperboloid \( \mathbb{H}^n \), the unit sphere \( S^n \) and the Euclidean space \( \mathbb{E}^n \). It follows from above that
\[
S^n = SO_{n+1}/K, \quad \mathbb{H}^n = SO(n, 1)/K, \quad \mathbb{E}^n = \mathbb{R}^n \times SO_n(R)/K,
\]
where \( K = \{1\} \times SO_n(R) \).

The splitting of the corresponding algebras can be described in terms of the curvature parameter \( \epsilon = \pm 1, 0 \) with
\[
p_\epsilon = \{\begin{pmatrix} 0 & -p \epsilon^T \\ p & 0 \end{pmatrix}, \ p \in \mathbb{R}^n\}, \quad t_\epsilon = \{\begin{pmatrix} 0 & 0 \\ 0 & X \end{pmatrix}, \ X \in so_n(R)\}.
\]

It will be convenient to introduce a shorthand notation and write
\[
M_\epsilon = G_\epsilon/K, \quad K = \{1\} \times SO_n(R),
\]
with \( G_\epsilon \) equal to \( SO_{n+1}(R) \) when \( \epsilon = 1 \), \( SO(n, 1) \) when \( \epsilon = -1 \), and \( SE_n \) when \( \epsilon = 0 \).
7. Affine problem on space forms

7.1. Elastic curves and the pendulum. On space forms equations \((4.9)\) admit additional integrals of motion in involution with the spectral ones described by Proposition [7].

They are described as follows: let \(\mathfrak{k} = \{ M \in \mathfrak{g} : [M, A] = 0 \}\), and let \(\mathfrak{k}_{\perp}^A\) denote the orthogonal complement of \(\mathfrak{k}\) in \(\mathfrak{g}\) relative to \(\langle , \rangle\). It is easy to see that \(\mathfrak{k}\) is a Lie subalgebra of \(\mathfrak{g}\) and that \([A, L_p] \in \mathfrak{k}_{\perp}^A\). Therefore, the projection of \(L_t\) on \(\mathfrak{k}_{\perp}\) is constant along the solutions of \((4.9)\).

Recall now that an extremal control \(U(t)\) is equal to \(L_t(t)\) and that the corresponding extremal energy is equal to \(\frac{1}{2} \int_0^T ||U(t)||^2 dt\). Let \(L_t(t) = L_A + L_{\perp}^A(t)\) denote the decomposition of \(L_t(t)\) onto the factors \(\mathfrak{k}^A\) and \(\mathfrak{k}_{\perp}^A\). Then,\[
\frac{1}{2} \int_0^T ||U(t)||^2 dt = \frac{1}{2} \int_0^T (||L_A||^2 + ||L_{\perp}^A(t)||) dt = \frac{1}{2} \int_0^T ||L_{\perp}^A(t)||^2 + \text{constant}
\]
along each trajectory of \((4.9)\).

Remarkably, \(||L_{\perp}^A(t)||^2 = \kappa(t)^2\), where \(\kappa(t)\) denotes the geodesic curvature of the projected curve on \(G_t/K\), whenever \(L_A = 0\) and \(||A|| = 1\) (the norm of a matrix \(A = \begin{pmatrix} 0 & -eA^T \\ a & 0 \end{pmatrix}\) is given by \(\sqrt{\sum_{i=1}^n a_i^2}\)).

To demonstrate this fact, consider \(M_e\) as a principal \(G_e\) bundle with connection \(\mathcal{D}\) consisting of left invariant vector fields on \(G_e\) that take values in \(\mathfrak{p}\), at the group identity. In this setting curves \(g(t)\) in \(G_e/K\) can be represented by horizontal curves via the formula\[
x(t) = g(t)e_0, e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}.
\]

In this representation, \(||\frac{dx}{dt}||_e\), the Riemannian length in \(M_e\) of the tangent vector \(\frac{dx}{dt}\), is given by \(||A_e(t)|||\), where \(A_e(t) = \frac{dx}{dt}(g^{-1}(t))\).

Solution curves \(g(t)\) of the affine system \(\frac{dx}{dt} = g(t)(A + U(t))\) project onto the same curve \(x(t)\) as the associated horizontal curves \(\tilde{g}(t) = g(t)h(t)\), where \(h(t)\) is a solution in \(K\) of \(\frac{dh}{dt}(t) = h(t)U(t)\). It follows that \(\frac{dh}{dt}(t) = \tilde{g}(t)(h(t)A^{-1}(t))\).

Hence,\[
||\frac{dx}{dt}||_e = ||h(t)A^{-1}(t)|| = ||A|| = 1.
\]
Then, \(\frac{D}{dx} \frac{dx}{dt}\), the covariant derivative of \(\frac{dx}{dt}\) along \(x(t)\), is given by \(\frac{D}{dx} \frac{dx}{dt} = (\tilde{g}(t)(h(t)[U(t), A]h^{-1}(t))e_0)\).

Since \(||\frac{dx}{dt}||_e = ||A|| = 1\), the geodesic curvature \(\kappa(t)\) of \(x(t)\) is given by\[
\kappa^2(t) = ||\frac{D}{dx} \frac{dx}{dt}||^2 = ||[U(t), A]||^2
\]
In particular along the extremal curves \(U(t) = L_t\) hence, \(\kappa^2(t) = ||[L_{\perp}^A(t), A]||^2\) when \(L_A = 0\).
It remains to show that \([|\langle [L^1_A(t), A] \rangle|^2 = |L^1_A(t)|^2\). There exists \(h \in K\) such that \(h^{-1}Ah = E_1 = \begin{pmatrix} 0 & -ke_1^T \\ e_1 & 0 \end{pmatrix}\), a consequence of the fact that \(K\) acts transitively by adjoint action on the unit sphere in \(\mathfrak{p}_e\). Then,
\[0 = h^{-1}[A, \mathfrak{e}_g]h = [E_1, h^{-1}\mathfrak{e}_g h],\]
and therefore,
\[
h^{-1}\mathfrak{e}_g h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & so_{n-1}(R) \\ 0 & 0 & l \end{pmatrix}, h^{-1}\mathfrak{e}_g h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -l^T \\ 0 & l & 0 \end{pmatrix}, l \in \mathbb{R}^{n-1}.
\]

Hence,
\[
\kappa^2(t) = |\langle L^1_A(t), A \rangle|^2 = |h^{-1}L^1_A(t)h, E_1|^2 = ||e_1 \wedge l||^2 = ||l||^2.
\]

Therefore, \(|l|^2 = |L^1_A(t)|\), and the extremal energy \(\frac{1}{2} \int_0^T |L^1_A|^2 dt\) is given by
\[
\frac{1}{2} \int_0^T \kappa^2(t) dt.
\]

Curves \(x(t)\) in the base space \(G_{\mathbb{R}}/K\) which are the projections of these extremal curves are called elastic and \(\frac{1}{2} \int_0^T \kappa^2(t) dt\) is called their elastic energy (16). Equations (1.9) with \(L_A = 0\) can be rephrased as
\[
dL_p(t) = [L^1_A, L_p] + s[A, L^1_A], \quad \frac{dL^1_A}{dt}(t) = [A, L_p].
\]

We will return to these equations after a brief digression to mechanics and the connections between the elastic problem and the motions of a mathematical pendulum.

### 7.1.1. The pendulum.

There is a remarkable (and somewhat mysterious) connection between elastic curves and heavy tops that will be recalled below (see also (17) for a more general discussion). Consider first an \(n\) dimensional pendulum of unit length suspended at the origin of \(\mathbb{R}^n\) and acted upon by the ”gravitational force” \(\vec{F} = -e_1\), where \(e_1, \ldots, e_n\) denote the standard basis in \(\mathbb{R}^n\) (here, all physical constants are normalized to one).

The motions of the pendulum are confined to the unit sphere \(S^{n-1}\). For each curve \(q(t)\) on \(S^{n-1}\) let \(f_1(t), \ldots, f_n(t)\) denote an orthonormal frame, called the moving frame, adapted to \(q(t)\) by the constraint \(q(t) = f_1(t)\) and positively oriented relative to the absolute frame \(e_1, \ldots, e_n\), in the sense that the matrix \(R(t)\) defined by \(f_i(t) = R(t)e_i, i = 1, \ldots, n\), belongs to \(SO_n(R)\).

This choice of polarization identifies the sphere as the quotient \(G/K\) with \(G = SO_n(R)\) and \(K\) the isotropy subgroup of \(SO_n(R)\) defined by \(Ke_1 = e_1\). Evidently, \(K = \{1\} \times SO_{n-1}(R)\). Let \(\mathfrak{e}_0\) denote the Lie algebra of \(K\) and let \(\mathfrak{e}_1\) denote the orthogonal complement in \(\mathfrak{g} = so_n(R)\) relative to the trace form. Then
\[
\mathfrak{e}_1 = \left\{ \begin{pmatrix} 0 & -u^T \\ u & 0 \end{pmatrix}, u \in \mathbb{R}^{n-1} \right\}, \mathfrak{e}_0 = \begin{pmatrix} 0 & 0 \\ 0 & so_{n-1}(R) \end{pmatrix}.
\]

We will regard \(SO_{n}(R)\) as the principal \(SO_{n}(R)\) bundle (under the right action) over \(S^{n-1}\) with a connection \(\mathcal{D}\) consisting of the left invariant vector fields with values in \(\mathfrak{e}_1\). As usual, vector fields in \(\mathcal{D}\) and their integral curves will be called horizontal.
It follows that every curve \( q(t) \) on \( S^{n-1} \) can be lifted to a horizontal curve \( R(t) \), in the sense that \( q(t) = R(t)e_1 \), and \( \frac{dR}{dt} = R(t) \begin{pmatrix} 0 & -u^T(t) \\ u(t) & 0 \end{pmatrix} \) for some curve \( u(t) \) in \( \mathbb{R}^{n-1} \). Furthermore, it follows that any two such liftings are related by a left multiple by an element in \( K \).

The kinetic energy \( T \) associated with a path in \( S^{n-1} \) is given by

\[
T = \frac{1}{2} \left\| \frac{dq}{dt} \right\|^2 = \frac{1}{2} \left\| \frac{dR}{dt} e_1 \right\|^2 = \frac{1}{2} \left\| R(t) \begin{pmatrix} 0 & -u^T(t) \\ u(t) & 0 \end{pmatrix} \right\|^2 = \frac{1}{2} \left\| u(t) \right\|^2.
\]

The potential energy \( V(q) \) relative to a fixed point \( q_0 \) is given by \( V(q) = -\int_{t_0}^{q_0} \tilde{F} \cdot \frac{dq}{dt} dt \), where \( \sigma(t) \) is a path from \( q_0 \) to \( q \). It follows that \( V(q) = e_1 \cdot (q - q_0) \). It is convenient to take \( q_0 = -e_1 \) in which case \( V = e_1 \cdot q + 1 \).

The Principle of Least Action states that each extremal curve \((7.2)\) that the Lagrangian \( L \) is equal to \( \{ q(t), \frac{dR}{dt} \} dt \) over the paths from \( q(t_0) \) to \( q(t_1) \) for any \( t_0 \) and \( t_1 \) (sufficiently near each other to avoid conjugate points), where \( L \) denotes the Lagrangian \( L = T - V \). Thus motions of the pendulum can be viewed as the solutions of the following optimal control problem on \( SO_n(R) \):

Minimize the integral \( \int_{t_0}^{t_1} \frac{1}{2} \left\| u(t) \right\|^2 - (e_1 \cdot R e_1 + 1) dt \) over the solutions \( R(t) \in SO_n(R) \) of \( \frac{dR}{dt} = R(t)U_1(t), \ u_1(t) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix} \) subject to the boundary conditions \( R(t_0) \in \{ R : R e_0 = q_0 \}, R(t_1) \in \{ R : R e_0 = q_1 \} \).

The Maximum Principle then leads to the energy Hamiltonian \( \hat{H} \) on the cotangent bundle of \( SO_n(R) \). In the realization of the cotangent bundle as the product \( SO_n(R) \times so_n(R) \), further identified with the tangent bundle \( SO_n(R) \times so_n(R) \) via the trace form \( (A, B) = -\frac{1}{2} Tr(AB) \), the energy Hamiltonian is given by \( \hat{H} = \frac{1}{2}(Q_1, Q_1) + (e_1 \cdot R e_1 + 1) \) where \( Q_1 \) denotes the projection on \( \mathfrak{t}_1 \) of a matrix \( Q \) in \( so_n(R) \).

The Hamiltonian equations associated with \( \hat{H} \) (see [17], Ch. IV) for details) are given by:

\[
(7.2) \quad \frac{dR}{dt} = R(t)(Q_1(t)), \quad \frac{dQ}{dt} = [Q_1(t), Q(t)] - R^T(t)e_1 \wedge e_1
\]

It is evident from \((7.2)\) that the projection \( Q_0 \) of \( Q(t) \) on \( \mathfrak{t}_0 \) is constant. But then this constant must be zero because of the transversality condition imposed by the Maximum principle. To be more explicit, recall that the transversality condition states that each extremal curve \((R(t), Q(t))\) annihilates the tangent vectors of \( S_0 \) at \( R(t_0) \). Therefore, \( \langle Q(t_0), X \rangle = 0 \) for all \( X \in \mathfrak{t}_0 \) (since the tangent space at \( R(t_0) \) is equal to \( \{ R(t_0)X : X \in \mathfrak{t}_0 \} \)). The transversality condition at the terminal time \( t_1 \) reaffirms that \( Q_0 = 0 \); hence, it and is redundant in this case.

Equations \((7.2)\) can be lifted to the semidirect product \( \mathbb{R}^n \ltimes so_n(R) \) by identifying vector \( p(t) = -R^T(t)e_1 \) with matrix \( P(t) = \begin{pmatrix} 0 & 0 \\ p(t) & 0 \end{pmatrix} \) and vector \( e_1 \) with matrix \( E_1 = \begin{pmatrix} 0 & 0 \\ e_1 & 0 \end{pmatrix} \). Both matrices belong to the Cartan space \( \mathfrak{p} \) in the semidirect Lie algebra \( \{ \begin{pmatrix} 0 & 0 \\ x & X \end{pmatrix} : x \in \mathbb{R}^n, X \in so_n(R) \} \).
Let $\bar{Q} = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$ denote the embedding of $so_n(R)$ into the semidirect product algebra, and let $\bar{Q}_0$ and $\bar{Q}_1$ denote the embeddings of $Q_0$ and $Q_1$. Then

\begin{equation} (7.3) \begin{pmatrix} 0 & 0 \\ Q_1 p(t) & 0 \end{pmatrix} = [P(t), \bar{Q}_1], \text{ and } [E_1, P(t)] = \begin{pmatrix} 0 & 0 \\ 0 & p(t) \wedge e_1 \end{pmatrix}. \end{equation}

It follows that the extremal equations (7.2) can be written also as

\begin{equation} (7.4) \frac{dg}{dt} = g(t)(E_1 + \tilde{K}_1(t)), \frac{dP}{dt} = [P(t), \bar{Q}_1], \frac{d\bar{Q}_1}{dt} = [E_1, P(t)], \end{equation}

where $g(t) = \begin{pmatrix} 1 & 0 \\ q(t) & R(t) \end{pmatrix}$.

The Lie algebra part of equations (7.4) agree with equations (7.1) when $\epsilon = 1$ and $A = E_1$. So on the level of Lie algebras, the equations of the mathematical pendulum coincide with the equations for the Euclidean elastic curves.

Consider now the isospectral matrix $L_\lambda = L_p - \lambda L_{\tilde{A}} - (\lambda^2 - \epsilon)A$ associated with (7.1). Since the spectral invariants of $L_\lambda$ are invariant under conjugations by elements in $K$ there is no loss in generality if $A$ is taken to be $E_1$. Matrices in $p_c$, $\epsilon = \pm 1$ are of the form $\begin{pmatrix} 0 & -ep^T \\ p & 0 \end{pmatrix}$ and can be written as $\bar{p} \wedge \epsilon e_0$ where $\bar{p} = \begin{pmatrix} 0 \\ p \end{pmatrix}, p \in \mathbb{R}^n$ and where

\begin{equation} (7.5) (a \wedge b)x = (b, x)a - (a, x)b, \text{ with } (x, y)_\epsilon = x_0 y_0 + e p \sum_{i=1}^{n} x_i y_i, \end{equation}

while matrices in $L_{\tilde{A}}$ can be written as $\bar{x} \wedge e_1$ with $\bar{x} = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}, x \in \mathbb{R}^{n-1}$. It follows that the range of $L_\lambda$ is contained in the linear span of $\bar{p}, \bar{x}, e_0, e_1$. An easy calculation shows that the characteristic polynomial of the restriction of $L_\lambda$ to this vector space is given by

\begin{equation} (7.6) \xi^4 + c_1 \xi^2 + c_2 = 0 \end{equation}

with

$c_1 = \epsilon (\lambda^2 - s) + (\lambda^2 - s)H + s||L_{\tilde{A}}||^2 + ||L_p||^2, c_2 = \lambda^2(||L_{\tilde{A}}||^2 ||L_p||^2 - ||L_{\tilde{A}}||^2 ||L_p||^2)$.

The coefficients $c_1$ and $c_2$ reveal $I_2 = ||L_{\tilde{A}}||^2 ||L_p||^2 - ||L_{\tilde{A}}||^2 ||L_p||^2$ as a new integral of motion in addition to the Hamiltonian $\mathcal{H}$ and the Casimir $I_1 = s||L_{\tilde{A}}||^2 + ||L_p||^2$. This integral of motion admits a nice geometric interpretation relative to the underlying elastic curve $x(t)$:

$I_2 = \kappa^2(t) \tau(t)$

where $\tau(t)$ denotes the torsion of $x(t)$. Then $\xi(t) = \kappa^2(t)$ is a solution of

\begin{equation} (7.7) \frac{d\xi}{dt} = -\xi^3 + 4(\mathcal{H} - k)\xi^2 + 4(I_1 - \mathcal{H}^2)\xi + 4I_2 \end{equation}

and hence is solvable in terms of elliptic functions ([16], [17]).
These geometric identifications suggest an integrating procedure in terms of the Serret-Frenet frames along an elastic curve $x(t)$. Let $T, N, B$ denote the Serret-Frenet triad given by the standard formulas

$$
\frac{dT}{dt} = \kappa(t)N(t), \quad \frac{dN}{dt} = -\kappa(t)T(t) + \tau(t)B(t), \quad \frac{dB}{dt} = -\kappa(t)N(t).
$$

The tangent vector $T(t)$ can be identified with $T(t) = (h(t))E_1 h^{-1}(t))$ in the horizontal distribution $\mathcal{D}$ where $h(t)$ is a solution of $\frac{dh}{dt}(t) = h(t)(U(t))$ with $U(t) = L_A^1(t)^\ast$. Then the normal and the binormal vectors $N(t)$ and $B(t)$ can be easily obtained from (7.8). It was shown first in [9] and then in [16] that $\frac{dB}{dt}$ is contained in the linear span of $T(t), N(t), B(t)$. Hence, equations (7.8) carry complete information about elastic curves. By analogy, the equations of the mathematical pendulum are also integrable by an identical procedure.

It can be shown that the general case with $L_A$ an arbitrary constant is related to an $n$ dimensional heavy top with equal principal moments of inertia, but these details will not be addressed here.

**Remark 2.** The affine distribution $\mathcal{D}(g) = \{A + U : U \in \mathfrak{t}\}$ does not extend to the elastic problems on more general symmetric spaces because the isometry group $K$ does not act transitively on the spheres in the Cartan space $\mathfrak{p}$. Hence not every curve $x(t)$ in $G/K$ can be lifted to a horizontal curve $g(t)$ that is a solution of $\frac{dg}{dt} = g(t)(h(t)Ah^{-1}(t))$, for some curve $h(t)$ in $K$. This observation raises a question about the geometric significance of the affine problem for general symmetric spaces.

### 8. Affine problem on coadjoint orbits

Certain coadjoint orbits coincide with the cotangent bundles of quadric surfaces and the restriction of the affine Hamiltonian to these orbits coincides with the Hamiltonians associated with mechanical systems with quadratic potential. On these orbits the spectral invariants of [5,4] form Lagrangian submanifolds of the orbits, or, stated differently, the restrictions of the Hamiltonian to such manifolds become completely integrable. These findings provide a natural theoretical framework for several classically known integrability results and at the same time point to a larger class of systems that conform to the same integration procedures. The text below supports these claims in complete detail.

Recall that the coadjoint orbit $\mathcal{O}_G(l_0)$ of $G$ through $l_0 \in \mathfrak{g}^*$ is defined

$$
\mathcal{O}_G(l_0) = \{l : l = Ad_{g^{-1}}^\ast(l_0), g \in G\},
$$

where $Ad_{g^{-1}}^\ast(l_0)(X) = l_0(Ad_g^{-1}(X))$, $X \in \mathfrak{g}$. Also recall that $\mathfrak{g}^*$ is a Poisson manifold under the Poisson bracket $\{f, h\}(l) = l([df, dh])$, $l \in \mathfrak{g}^*$, and that $\mathfrak{g}^*$ is foliated by coadjoint orbits of $G$ each of which is symplectic. More precisely, the tangent space of $\mathcal{O}_G(l_0)$ at $l$ consists of vectors $v = ad^* M(l)$, $M \in \mathfrak{g}$ and the symplectic form $\omega$ at $l$ is given by

$$
\omega_1(v_1, v_2) = l([M_1, M_2]), \quad \text{with} \quad v_1 = ad^* M_1(l) \quad \text{and} \quad v_2 = ad^* M_2(l). \tag{8.1}
$$

[2, Appendix 2].

In the semisimple case $\mathcal{O}_G(l_0)$ is identified with the adjoint orbit $Ad_G(L_0)$ via the correspondence $(L, X) = l(X)$ for all $X \in \mathfrak{g}$. Consequently, each adjoint orbit in a semisimple Lie algebra is even dimensional. In this correspondence, tangent
further suppose that \( l \) describes their structure. These manifolds need not coincide with the adjoint orbits. The proposition below or-\( \text{bits through rank one matrices in } \frak{g} \) to the coadjoint orbit. The latter fact will be of central impor-\( \text{tance for the rest of products in some detail.} \)

\[ \text{Semidirect products of their own. The paragraph below describes these semidirect} \]

\[ \text{vectors at } l \text{ are identified with matrices } v = [L,M] \text{ and the symplectic form takes on its dual form } \omega_{L}(v_{1},v_{2}) = \langle L,[M_{1},M_{2}] \rangle. \]

When \((G,K)\) is a symmetric pair then \( \frak{g}^\ast \) carries another Poisson structure \{ , \} \( s \) induced by the semidirect product \( \frak{p} \ltimes \frak{k} \). As in the semisimple case the quadratic form \( \langle , \rangle \) can be used to identify the coadjoint orbits with certain submanifolds of \( \frak{g} \). Since the Killing form is not invariant relative to the semidirect Lie bracket, these manifolds need not coincide with the adjoint orbits. The proposition below describes their structure.

**Proposition 8.** Suppose that \( l_{0} \in \frak{g}^\ast \), \( g = (X,h) \in G_{s} \) and \( l = Ad_{g^{-1}}(l_{0}) \), and further suppose that \( l_{0} \rightarrow L_{0} = P_{0} + Q_{0} \), and \( l \rightarrow L = P + Q \) are the correspondences defined by the Killing form with \( P_{0} \) and \( P \) in \( \frak{p} \) and \( Q_{0} \) and \( Q \) in \( \frak{k} \).

Then
\[
(8.2) \quad P = Ad_{h}(P_{0}), \text{ and } Q = [Ad_{h}(P_{0}),X] + Ad_{h}(Q_{0}).
\]

**Proof.** Let \( Z = U + V \) be an arbitrary point of \( \frak{g} \) with \( U \in \frak{p} \) and \( V \in \frak{k} \).

Then
\[
Ad_{g^{-1}}(Z) = \frac{d}{dt}(g^{-1}(eU,e^{V})g)|_{t=0} = \frac{d}{dt}(-Ad_{h^{-1}}(X) + Ad_{h^{-1}}(eU + e^{V}X)e^{-V}h^{-1}e^{V}h) \\
= Ad_{h^{-1}}(U + [X,V]) + Ad_{h^{-1}}(V).
\]

Hence,
\[
l(Z) = l_{0}(Ad_{g^{-1}}(Z)) = \langle P_{0}, Ad_{h^{-1}}(U + [X,V]) \rangle + \langle Q_{0}, Ad_{h^{-1}}(V) \rangle \\
= \langle Ad_{h}(P_{0}),U \rangle + (Ad_{h}(P_{0}),[X,V]) + (Ad_{h}(Q_{0}),V) \\
= \langle Ad_{h}(P_{0}),U \rangle + ([Ad_{h}(P_{0}),X],V) + \langle Ad_{h}(Q_{0}),V \rangle = \langle P,U \rangle + \langle Q,V \rangle
\]

Therefore,
\[
P = Ad_{h}(P_{0}), \text{ and } Q = [Ad_{h}(P_{0}),X] + Ad_{h}(Q_{0}). \quad \Box
\]

For left invariant Hamiltonians \( H \) each coadjoint orbit is an integral manifold for the Hamiltonian vector field \( \vec{H} \). Moreover, the Hamiltonian vector field on a coadjoint orbit induced by the restriction of \( H \) coincides with the restriction of \( \vec{H} \) to the coadjoint orbit. The latter fact will be of central importance for the rest of the paper as it will be shown that the Hamiltonian \( \Box \) restricted to the coadjoint orbits through rank one matrices in \( SL_{n+1}(R) \) relate directly to Kepler’s problem, geodesic problem of Jacobi and the mechanical problem of Newman. The identification with the affine problem provides natural explanation for their integrability.

### 8.1. Coadjoint orbits on the vector space of matrices of trace zero.

The vector space \( V_{n} \) of \( n \times n \) matrices of trace zero admits several kinds of Lie algebras and each of these Lie algebras induces its own Poisson structure on the dual space \( V_{n}^\ast \). The most common Poisson structure is induced by the canonical Lie bracket, i.e., in which \( V_{n} \) as a Lie algebra is equal to \( sl_{n}(R) \). The decomposition of \( sl_{n}(R) \) as the sum of symmetric and skew-symmetric matrices (associated with the automorphism \( \sigma(g) = (g^{T})^{-1} \)) allows \( V_{n} \) to be considered also as the semidirect Lie algebra \( Sym_{n} \ltimes so_{n}(R) \), where \( Sym_{n} \) denotes the space of symmetric \( n \times n \) matrices of trace zero. There are also automorphisms of non-Riemannian type which induce semidirect products of their own. The paragraph below describes these semidirect products in some detail.

Let \( n = p + q \) and let \( \sigma : SL_{p+q}(R) \rightarrow SL_{p+q}(R) \) be defined by \( \sigma(g) = J((g^{T})^{-1})J^{-1} \) where \( J \) is diagonal matrix with its first \( p \) diagonal entries equal to 1 and the remaining diagonal entries equal to \(-1 \). It follows that \( \sigma(g) = g \) if and
only if $J = gJg^T$ or equivalently, $\sigma(g) = g$ if and only if $g \in SO(p, q)$. The tangent map $\sigma_*$ given by $\sigma_*(A) = -JA^TJ$ induces a decomposition $p \oplus \mathfrak{k}$ where $p$ consists of matrices $P$ such that $P = JP^TJ$ which implies that $P = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}$ with $A$ and $D$ symmetric and $B$ an arbitrary $p \times q$ matrix. The Lie algebra $\mathfrak{t}$ is the Lie algebra of $K = SO(p, q)$ and consists of matrices $Q = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ with $A$ and $D$ antisymmetric.

The symmetric pair $(SL_{p+q}(R), SO(p, q))$ is strictly pseudo- Riemannian because the subgroup of the restrictions of $Ad_K$ to $p$ is not a compact subgroup of $Gl(p)$ (the trace form is indefinite on $p$). Nevertheless, this automorphism endows $V_n$ with the semidirect Lie algebra $p \ltimes \mathfrak{k}$ which will be of some relevance for the material below.

8.1.1. **Coadjoint orbits through rank one matrices.** Consider first the symmetric pair $(SL_{n+1}(R), SO_{n+1}(R))$ with the quadratic form $(A, B) = -\frac{1}{2}Tr(AB)$ on $sl_{n+1}(R)$. This form is positive definite on the space of symmetric matrices $p$ and negative definite on $\mathfrak{k} = so_{n+1}(R)$. Suppose that $P = x \otimes x$ is a rank one symmetric matrix generated by a vector $x \in \mathbb{R}^{n+1}$. Then $P_0 = x \otimes x - \frac{||x||^2}{n+1}I$ is in $p$ since $Tr(x \otimes x) = \sum_{i=1}^{n+1} (e_i, x \otimes x e_i) = ||x||^2$. There are two coadjoint orbits through $P_0$, one relative to the action of $SL_{n+1}(R)$ and the other relative to the action of the semidirect product $p \ltimes SO_{n+1}(R)$.

**Proposition 9.** The coadjoint orbit $S$ through $P_0 = x_0 \otimes x_0 - \frac{||x_0||^2}{n+1}I$ is symplectomorphic to the cotangent bundle of the projective space $\mathbb{P}^{n+1}$ in the semisimple case, and is symplectomorphic to the cotangent bundle of the sphere $S^n$ in the semidirect case.

**Proof.** Let $S$ denote the coadjoint orbit of $SL_{n+1}(R)$ through $P_0 = x_0 \otimes x_0 - \frac{||x_0||^2}{n+1}I$. If $R \in SO_{n+1}(R)$ then $RP_0R^{-1} = Rx_0 \otimes Rx_0 - \frac{||x_0||^2}{n+1}I$. It follows that $x_0$ can be replaced by $||x_0||e_0$, because $SO_{n+1}(R)$ acts transitively on spheres. Therefore, $S$ is diffeomorphic to $||x_0||^2S(e_0 \otimes e_0)S^{-1}$, $S \in SL_{n+1}(R)$. Consider now the orbit through $e_0 \otimes e_0$. It follows that $S(e_0 \otimes e_0)S^{-1} = S(e_0 \otimes e_0)(ST)^{-1}e_0$ for any $S \in SL_{n+1}(R)$. It is easy to verify that for any $x \neq 0$ and any $y$ such that $x \cdot y = 1$ there exists a matrix $S \in SL_{n+1}(R)$ such that $Sc_0 = x$ and $STy = e_0$. Hence, $S(e_0 \otimes e_0)S^{-1} = x \otimes y$.

The set of matrices $\{x \otimes y - \frac{xy}{n+1}I : x \cdot y = 1\}$ can be identified with the set of lines $\{(ax, \frac{1}{a}y) : x \cdot y = 1\}$ which is symplectomorphic to the tangent bundle of $\mathbb{P}^{n+1}$. The latter is identified with the cotangent bundle via the ambient Euclidean inner product.

Consider now the coadjoint orbit relative to the semidirect case. It follows from [S, 2] that $S$ consists of matrices $P = Ad(h)P_0 = h(x_0) \otimes h(x_0) - \frac{||x_0||^2}{n+1}I$ and $Q = [Ad_h(P_0), X] = [x \otimes x, X] = Xx \wedge x$ since $Q_0 = 0$. Therefore, $P$ is a rank one matrix generated by $x = h(x_0)$, with $h \in SO_{n+1}(R)$ and $Q$ is a rank two antisymmetric matrix $x \wedge y$ with $y = -Xx$. Since $X$ is an arbitrary symmetric matrix of trace zero $y$ can be any point in $\mathbb{R}^{n+1}$. The correspondence between $(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow x \otimes x - \frac{||x||^2}{n+1}I + x \wedge y$ is one to one provided that $x \cdot y = 0$. Moreover, $x$ can be any point of the sphere $||x|| = ||x_0||$ since $SO_{n+1}(R)$ acts
transitively on spheres by conjugations. Therefore $S$ is identified with points $(x, y)$ such that $||x|| = ||x_0||$ and $x \cdot y = 0$ which is the tangent (cotangent) bundle of the sphere $S^n$ since the two bundles are identified via the Euclidean inner product in $\mathbb{R}^{n+1}$.

It may be instructive to show directly that the canonical symplectic form on the cotangent bundle of the sphere coincides with the symplectic form of the coadjoint orbit.

The tangent bundle of the cotangent bundle of the sphere is given by the vectors $(x, y)$ in $\mathbb{R}^{4n+1}$ subject to the constraints $||x|| = ||x_0||$, $x \cdot x = 0$, $x \cdot y = 0$ and $\dot{x} \cdot y + x \cdot \dot{y} = 0$ and these vectors are identified with matrices $\dot{x} \otimes x + x \otimes \dot{x} + x \wedge y + x \wedge y$ on the tangent bundle of the coadjoint orbit (we have omitted the trace factor since it is irrelevant for the these calculations). The canonical symplectic form on the cotangent bundle of the sphere is given by

$$\omega_{(x,y)}((\dot{x}_1, y_1), (\dot{x}_2, y_2)) = \dot{x}_1 y_2 - \dot{x}_2 y_1$$

It follows from above that $l \in O_G(l_0)$ is identified with $L = x \otimes x + x \wedge y$. Then tangent vectors $v = ad^*M(l)$ at $l$ are identified with matrices $V$ via the formula $\langle V, U \rangle = \langle L, [M, U] \rangle$ for all $U \in g_s$.

An easy calculation shows that $V = [x \otimes x, B] + [x \wedge y, A] + [x \otimes x, A]$ is the tangent vector at $L$ defined by $M = A + B$ with $A^T = -A, B^T = B$. Then, $V = \dot{x} \otimes x + x \otimes \dot{x} + \dot{x} \wedge y + x \wedge y$ implies that

$$\dot{x} = Ax, \dot{y} = Ay - Bx$$

It follows from (8.1) that the symplectic form on the coadjoint orbit is given by

$$\omega_L(V_1, V_2) = \langle L, [M_2, M_1] \rangle.$$ Then, $\omega_x \otimes x + x \wedge y(v_1, v_2) = \langle L, [B_2, A_1] + [A_2, B_1] + [A_2, A_1] \rangle = \langle x \otimes x, [B_2, A_1] + [A_2, B_1] + (x \wedge y, [A_2, A_1]) = A_2 \cdot B_2 \cdot x - A_1 \cdot x \cdot B_2 \cdot x - (A_2 x \cdot A_1 y - A_1 x \cdot A_2 y) = A_2 x \cdot (A_1 y - y_1) - A_1 x \cdot (A_2 y - y_2) - (A_2 x \cdot A_1 y - A_1 x \cdot A_2 y) = -A_2 x \cdot y_1 + A_1 x \cdot y_2 = (\dot{x} \cdot y_2 - \dot{x} \cdot y_1).$$

Next consider analogous orbits defined by the pseudo Riemannian symmetric pair $(SL_{n+1}(R), SO(1, n))$ with the Cartan space $p$ consisting of matrices $P = \begin{pmatrix} 0 & -p^T \\ p & 0 \end{pmatrix}$ with $p$ an $n \times 1$ matrix and $P_0$ an $n \times n$ symmetric matrix. These orbits will be defined through the hyperbolic inner product $(x, y)_{-1} = x \cdot J y = x_1 y_1 - \sum_{i=2}^{n+1} x_i y_i$, where $J = diag(1, -1, \ldots, -1)$. It is easy to verify that $P \in p$ if and only if $Tr(P) = 0$ and $(Px, y)_{-1} = (x, Py)_{-1}$; similarly, $A \in so(1, n)$ if and only if $(Ax, y)_{-1} = -(x, Ay)_{-1}$ for all $x, y$ in $\mathbb{R}^{n+1}$. Thus $p$ is the space of "hyperbolic symmetric" matrices.

**Definition 2.** The hyperbolic rank one matrices are matrices of the form $x \otimes x$ for some vector $x \in \mathbb{R}^{n+1}$. They will be denoted by $(x \otimes x)_{-1}$.

It follows that $(x \otimes x)_{-1} u = (x, u)_{-1} x$, and therefore,

$$(x \otimes x)_{-1} u = (x, u)_{-1} (x, v)_{-1} = (u, (x \otimes x)_{-1} v)_{-1}.$$

Since the trace of $(x \otimes x)_{-1}$ is equal to $||x||^2_{-1} = (x, x)_{-1}$, $P = (x \otimes x)_{-1} - \frac{||x||^2_{-1}}{(n+1)} I$ is in $p$. 

Now define rank two skew symmetric hyperbolic matrices \( Q = (x \wedge y)_{-1} = x \otimes Jy - y \otimes Jx \). An easy calculation shows that \( Q \in \mathfrak{so}(1,n) \). Then

**Proposition 10.** The coadjoint orbit \( S \) of the semidirect product \( G_\epsilon = \text{Sym}_{-1} \ltimes SO(1,n) \) through \( P_0 = (x_0 \otimes x_0)_{-1} - \frac{||x_0||^2}{n+1} I \) is equal to \( \{ (x \otimes x)_{-1} - \frac{||x||^2}{n+1} I + (x \wedge \dot{x})_{-1} : ||x||_{-1} = ||x_0||_{-1} \} \). The latter is symplectomorphic to the cotangent bundle of the hyperboloid \( \mathbb{H}^n = \{ (x,y) : ||x||_{-1} = ||x_0||_{-1}, (x,y)_{-1} = 0 \} \). The canonical symplectic form \( \omega \) on \( S \) is given by

\[
\omega_{x,y}((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)) = (\dot{x}_1, \dot{y}_2)_{-1} - (\dot{x}_2, \dot{y}_1)_{-1}
\]

where \((\dot{x}_1, \dot{y}_1)\) and \((\dot{x}_2, \dot{y}_2)\) denote tangent vectors at \((x,y)\).

**Proof.** The proof is analogous to the proof in the previous proposition and will be omitted. \( \square \)

The coadjoint orbits of the above semidirect products through matrices of rank one can be expressed in terms of a single parameter \( \epsilon = \pm 1 \) with \( SO_\epsilon = SO_{n+1}(R) \) for \( \epsilon = 1 \) and \( SO_\epsilon = SO(1,n) \) for \( \epsilon = -1 \). Then \( p_\epsilon \) will denote the vector space \( \{ \begin{pmatrix} 0 & \epsilon p^T \\ p & 0 \end{pmatrix} : p \in \mathbb{R}^n, p^T P_0 = P_0, Tr(P_0) = 0 \} \). A matrix \( X \) belongs to \( p_\epsilon \) if and only if it is symmetric relative to the quadratic form \( (x,y)_\epsilon = x_1 y_1 + \epsilon \sum_{i=2}^{n+1} x_i y_i \).

We will let \( S_\epsilon \) denote the coadjoint orbit of the semidirect product \( G_\epsilon = p_\epsilon \ltimes SO_\epsilon \) through \( P_0 = (x_0 \otimes x_0)_{-1} - \frac{||x_0||^2}{n+1} I \) where \( ||x_0||^2 = (x,x)_\epsilon \). The symplectic form \( \omega_{x,y}((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)) = (\dot{x}_2, \dot{y}_1)_{-1} - (\dot{x}_1, \dot{y}_2)_{-1} \) is dual to the Poisson form

\[
\{ f_1, f_2 \}_\epsilon = (\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y})_\epsilon - (\frac{\partial f_2}{\partial x}, \frac{\partial f_1}{\partial y})_\epsilon
\]

9. The affine Hamiltonian system on coadjoint orbits of rank one

Consider now the restriction of \( H = \frac{1}{2} (L_t, L_t) + (A, L_p) \) to the semidirect orbits \( S_\epsilon \) where \( (A,B) = -\frac{1}{4} Tr(AB) \). This trace form is negative definite on the space of symmetric matrices and positive definite on \( \mathfrak{so}_{n+1}(R) \), but in the pseudo-Riemannian case it is indefinite on both \( \mathfrak{p}_{-1} \) and \( \mathfrak{t}_{-1} \).

In what follows it will be convenient to relax the condition that \( Tr(A) = 0 \). It is clear that both \( A \) and \( A - \frac{Tr(A)}{n+1} I \) define the same affine Hamiltonian since \( \langle I, L_p \rangle = 0 \). Then the restrictions of \( L_t \) and \( L_p \) to \( S_\epsilon \) are given by \( L_t = (x \wedge y)_\epsilon \) and \( L_p = (x \otimes x)_\epsilon - \frac{||x||^2}{n+1} I \). An easy calculation shows that the restriction of \( H \) is given by

\[
H = \frac{1}{2} ||x||_\epsilon \cdot ||y||_\epsilon - \frac{1}{2} (Ax, x)_\epsilon.
\]

9.1. Mechanical system of Newmann. If we replace \( A \) by \(-A\) then the restriction of the affine Hamiltonian to \( S_\epsilon \) with \( \epsilon = 1 \) is given by \( H = \frac{1}{2} (||x||^2 ||y||^2 + \frac{1}{2} (x, Ax)) \) which coincides with the Hamiltonian of the mechanical problem on the sphere with a quadratic potential of C. Newmann \((21)\ and (22)\). Then \( H = \frac{1}{2} (||x||^2 + ||y||^2 + \frac{1}{2} (x, Ax))_{-1} \) can be considered as the hyperbolic analogue of the problem of Newmann.

The equations of motion (4.9) reduce to

\[
\frac{d}{dt} (x(t) \wedge y(t))_\epsilon = [-A_\epsilon (x \otimes y)_\epsilon] = (Ax(t) \wedge x(t))_\epsilon.
\]
and
\[
\frac{d}{dt}(x(t) \otimes x(t))_\epsilon = [(x(t) \wedge y(t))_\epsilon, (x(t) \otimes x(t))_\epsilon] = ||x||^2_\epsilon (x(t) \otimes y(t))_\epsilon + (y(t) \otimes x(t))_\epsilon.
\]

An easy calculation shows that the preceding equations are equivalent to
\[
\frac{dx(t)}{dt} = ||x||^2_\epsilon y(t), \quad \frac{dy(t)}{dt} = -Ax(t) + \frac{1}{||x||^2_\epsilon} (Ax(t), x(t))_\epsilon - ||y(t)||^2_\epsilon x(t).
\]

Equations (9.2) with \( \epsilon = 1 \) and \( ||x|| = 1 \) form a point of departure for J. Moser’s book on integrable Hamiltonian systems (24). We will presently show that equations (9.2) can also be derived in a self contained way from a ”mechanical point of view” through the Maximum Principle of control.

This mechanical problem will be phrased as an optimal control problem of minimizing the Lagrangian
\[
\frac{1}{2} \int_0^1 \left( ||u(t)||^2_\epsilon - (Ax, x)_\epsilon \right) dt
\]
over the absolutely continuous curves \( x(t) \) on an interval \( [0, T] \) that satisfy \( ||x||_\epsilon = ||x_0||_\epsilon, \frac{dx(t)}{dt} = u(t), \) and also satisfy fixed boundary conditions \( x(0) = x_0 \) and \( x(T) = x_1. \)

The constraint \( ||x||_\epsilon = ||x_0||_\epsilon \) implies that \( (u(t), x(t))_\epsilon = 0. \) The Maximum Principle of optimal control leads to the appropriate Hamiltonian on the cotangent bundle of the sphere \( ||x||_\epsilon = ||x_0||_\epsilon = 1 \) (the hyperbolic sphere is the unit hyperboloid \( \mathbb{H}^n \)).

We will use \( T^*S_\epsilon \) to denote this cotangent bundle. It will be identified with the subset of \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \) subject to the constraints \( G_1 = ||x||_\epsilon - 1 = 0 \) and \( G_2 = (y, x)_\epsilon = 0. \) The Maximum Principle states that the appropriate Hamiltonian for this problem is obtained by maximizing
\[
H_0 = -\frac{1}{2} (||u(t)||^2_\epsilon - (Ax, x)_\epsilon) + (y, u)_\epsilon \text{ relative to the controls } u \text{ that are subject to the constraint } G_0 = (x, u)_\epsilon = 0.
\]

According to the method of Lagrange the maximal Hamiltonian is obtained by maximizing \( H = H_0 + \lambda_0 G_0. \) It follows that the optimal control \( u \) occurs at \( u = y + \lambda_0 x. \) But then \( (u, x)_\epsilon = (y, x)_\epsilon + \lambda_0 ||x||_\epsilon \) implies that \( \lambda_0 = 0. \) Hence, the maximal value of \( H_0 \) is given by
\[
H_0 = \frac{1}{2} (||y||^2_\epsilon + (Ax, x)_\epsilon).
\]

The above Hamiltonian is to be taken as a Hamiltonian on \( T^*S_\epsilon; \) hence, \( T^*S_\epsilon \) must an invariant manifold for \( H_0. \) Therefore, integral curves of \( H_0 \) are the restrictions to \( T^*S_\epsilon \) of the integral curves of a modified Hamiltonian
\[
H = H_0 + \lambda_1 G_1 + \lambda_2 G_2
\]
in which the multipliers \( \lambda_1 \) and \( \lambda_2 \) are determined by requiring that \( T^*S_\epsilon \) be invariant for \( H. \) This requirement will be satisfied whenever the Poisson brackets \( \{H, G_1\} \) and \( \{H, G_2\} \) vanish on \( T^*S_\epsilon \). The vanishing of these Poisson brackets implies that \( \lambda_1 = -\frac{\{H_0, G_1\}}{\{G_2, G_1\}} \) and \( \lambda_2 = \frac{\{H_0, G_2\}}{\{G_2, G_1\}}. \)

An easy calculation based on (8.3) yields \( \{H_0, G_1\} = -2(y, x)_\epsilon, \{H_0, G_2\} = (Ax, x)_\epsilon - (y, y)_\epsilon \) and \( \{G_1, G_2\} = 2||x||^2_\epsilon = 2 \) from which it follows that
\[
\lambda_1 = (Ax, x)_\epsilon - (y, y)_\epsilon, \lambda_2 = (y, x)_\epsilon.
\]

Hence,
\[
H = \frac{1}{2} ||y||^2_\epsilon + \frac{1}{2} (Ax, x)_\epsilon - ((Ax, x)_\epsilon - (y, y)_\epsilon) G_1 - (y, x)_\epsilon G_2.
\]
The flow of $H$ restricted to $G_1 = 0, G_2 = 0$ is given by

\[
\frac{dx}{dt} = \frac{\partial H}{\partial y} = y, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x} = -Ax + \frac{1}{\|x|\epsilon^2}((Ax, x)\epsilon - (y, y)\epsilon)x(t)
\]

The preceding equations coincide with (9.2).

9.2. Integrability. The spectral invariants of $L_\lambda = L_p - \lambda L_t - \lambda^2 A$ naturally lead to the appropriate coordinates in terms of which the above equations can be integrated. It will be more convenient to divide $L_\lambda$ by $-\lambda^2$ and redefine $L_\lambda = -\frac{1}{\lambda}L_p + \frac{1}{\lambda}L_t + A$. Since the trace of $(x \otimes x)_\epsilon$ is a scalar multiple of the identity it is inessential for the calculations below and will be omitted. Then the spectrum of $L_\lambda$ is given by

\[
[0, \lambda_1), \lambda_1, \lambda_2), \lambda_2, \lambda_3), \ldots
\]

with $L_p = (x \otimes x)_\epsilon$ and $L_t = (x \wedge y)_\epsilon$. It follows that spectral calculations can be reduced to

\[
0 = \det(I - (zI - A)^{-1}(-\frac{1}{\lambda^2}L_p + \frac{1}{\lambda}L_t)), \quad \det(zI - A) \neq 0.
\]

Matrix $I - (zI - A)^{-1}(-\frac{1}{\lambda^2}L_p + \frac{1}{\lambda}L_t)$ is of the form

\[
I - R_z(x_1 \otimes \xi_1 + x_2 \otimes \xi_2)
\]

where $x_1 = x, x_2 = y, \xi_1 = -\frac{1}{\lambda}x_1 + \frac{1}{\lambda}x_2, \xi_2 = -\frac{1}{\lambda}x_1$ and $R_z = (zI - A)^{-1}$.

A simple argument involving a change of basis shows that the solution of equation (9.6) can be reduced to $\det(I - W_z) = 0$, where $W_z = (w_{ij})$ is a $2 \times 2$ matrix with entries $w_{ij}$ equal to the $(Rx_i, \xi_j)\epsilon$ (20).

It follows that

\[
W_z = \begin{pmatrix}
(R_z x, x)\epsilon & (R_z x, y)\epsilon \\
(R_z y, x)\epsilon & (R_z y, y)\epsilon
\end{pmatrix}
\]

or,

\[
W_z = \begin{pmatrix}
-\frac{1}{\lambda^2}(R_z x, x)\epsilon + \frac{1}{\lambda}(R_z x, y)\epsilon & -\frac{1}{\lambda}(R_z x, x)\epsilon + \frac{1}{\lambda^2}(R_z x, y)\epsilon \\
-\frac{1}{\lambda^2}(R_z y, x)\epsilon + \frac{1}{\lambda}(R_z y, y)\epsilon & -\frac{1}{\lambda}(R_z y, x)\epsilon + \frac{1}{\lambda^2}(R_z y, y)\epsilon
\end{pmatrix}.
\]

Then $\det(I - W_z) = 0$ if and only if $1 - Tr(W_z) + Det(W_z) = 0$ which in turn implies that $1 + \frac{1}{\lambda^2}(R_z x, x)\epsilon + \frac{1}{\lambda^2}(R_z x, x)\epsilon + (R_z x, y)\epsilon - (R_z x, y)^2 = 0$.

Let

\[
F = (R_z x, x)\epsilon + (R_z x, x)\epsilon + (R_z y, y)\epsilon - (R_z x, y)^2.
\]

It follows from above that $0 = \det(zI - L_\lambda)$ outside of the spectrum of $A$ if and only if $F(z) = -\lambda^2$. It is easy to verify that $\lim_{z \to \pm\infty} zF(z) = (x, x)\epsilon = 1$ which implies that $F(z)$ takes both positive and negative values for any $x \neq 0$. Therefore, $F$ is constant along any solution of (9.2).

Function $F$ is rational with poles at the eigenvalues of the matrix $A$. Hence, $F(z)$ will be constant along the solutions of (9.2) if and only if the residues of $F$ are constant along the solutions of (9.2).
In the Euclidean case the eigenvalues of $A$ are real and distinct since $A$ is symmetric and regular. Hence, there is no loss in generality in assuming that $A$ is diagonal. Let $\alpha_1, \ldots, \alpha_{n+1}$ denote its diagonal entries. Then,

$$F(z) = \sum_{k=0}^{n} \frac{F_k}{z - \alpha_k},$$

where $F_0, \ldots, F_n$ denote the residues of $F$. It follows that $F_k = \lim_{z \to \alpha_k} F(z)$.

Therefore, the spectral function

$$F(z) = \sum_{k=0}^{n} \frac{F_k}{z - \alpha_k},$$

are in involution relative to the canonical Poisson bracket in $\mathbb{R}^{n+1}$. Hence, there is no loss in generality in assuming that $F_0 = 0 = \ldots = F_n$.

Therefore, the spectral function

$$F(z) = \sum_{k=0}^{n} \frac{F_k}{z - \alpha_k},$$

are in involution relative to the canonical Poisson bracket in $\mathbb{R}^{n+1}$. Hence, there is no loss in generality in assuming that $F_0 = 0 = \ldots = F_n$.

Therefore, the spectral function

$$F(z) = \sum_{k=0}^{n} \frac{F_k}{z - \alpha_k},$$

are in involution relative to the canonical Poisson bracket in $\mathbb{R}^{n+1}$. Hence, there is no loss in generality in assuming that $F_0 = 0 = \ldots = F_n$.
The residues $F_0, \ldots, F_n$ defined by $F(z) = \sum_{k=0}^{n} \frac{F_k}{z - \alpha_k}$ are given by

$$F_k = v_k^2 + \sum_{j=0, j \neq k}^{n} \frac{(v_j v_k - v_k v_j)^2}{(\alpha_k - \alpha_j)}, \quad k = 0, \ldots, n.$$  \hfill (9.14)

Since $F(z)$ is real valued for real $z$, $F_1 = \bar{F}_0$ and each $F_k, k = 2, \ldots, n$, is real. It follows that

$$Re(F_0), Im(F_0), F_k, k = 2, \ldots, n$$

are integrals of motion for the hyperbolic Neumann problem.

We leave it to the reader to show that

$$F_0 = \phi_0 + \frac{1}{2} \sum_{k=2}^{n} \frac{1}{\alpha_k^2 + \alpha_k^2}((-x_0 y_k - x_k y_0) + i(x_1 y_k - y_1 x_k)^2(\alpha_k + i\alpha),$$

where $\phi_0 = \frac{1}{2}(x_0^2 - x_1^2) + i(x_0 x_1 + \frac{1}{2}(x_0 y_1 - x_1 y_0)^2$, and

$$F_k = \phi_k - \sum_{j=2, j \neq k}^{n} \frac{(x_j y_k - x_k y_j)^2}{(\alpha_k - \alpha_j)},$$

with $\phi_k$ equal to

$$-x_k^2 - \frac{1}{\alpha_k^2 + \alpha_k^2} \alpha_k((x_k y_0 - x_0 y_k)^2 - (x_k y_1 - y_1 x_k)^2) + 2\alpha((x_k y_0 - x_0 y_k)(x_k y_1 - y_1 x_k)$$

for $k \geq 2$.

9.3. Integration procedure. In the Euclidean case the integration procedure goes back to C.L. Jacobi in connection to the geodesic problem on an ellipsoid. Its modern version is presented in Moser’s papers ([21], [20]). Rather than just to refer to the classical literature for details, it seems worthwhile to proceed with the main ingredients of this procedure. For simplicity of exposition we will confine our attention to the Euclidean sphere; the passage to the hyperbolic case requires only minor modifications.

The integration is done on the manifold $S$ defined by

$$|x| = 1, (x, y) = 0, F_k = x_k^2 + \sum_{j=1, j \neq k}^{n+1} \frac{(x_j y_k - x_k y_j)^2}{(\alpha_k - \alpha_j)} = c_k, k = 1, \ldots, n + 1$$

defined by the numbers $c_1, \ldots, c_{n+1}$ that satisfy $\sum_{k=1}^{n+1} c_k = 1$.

The following auxiliary lemma will be useful for some calculations below

**Lemma 3.** Let $g(z) = \Pi_{k=1}^{n} (z - x_k)$ where $x_1, \ldots, x_n$ are any distinct $n$ numbers. Then

$$f(x) = \sum_{k=1}^{n} \frac{f(x_k)}{g'(x_k)(x - x_k)}.$$  \hfill (9.15)

and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \sum_{k=1}^{n} \frac{f(x_k)}{g'(x_k)}$$

for any polynomial function $f(z)$.

Furthermore, $\sum_{k=1}^{n} \frac{f(x_k)}{g'(x_k)} = 0$ if $\deg(f) < n - 1$ and $\sum_{k=1}^{n} \frac{f(x_k)}{g'(x_k)} = 1$ if $\deg(f) = n - 1$ and its leading coefficient is equal to one.

**Proof.** Follows easily from the partial fraction expansion of $\frac{f(x)}{g(x)}$. \hfill \Box
Following Jacobi, the integration will be carried out in terms of elliptic coordinates \( u_1, \ldots, u_n \) defined as the zeros of \( (R_x x, x) = \sum_{k=1}^{n+1} z_k^2 \) for each point on the sphere \(||x|| = 1\). In addition, use will be made of the zeros of the rational function \( \sum_{k=1}^{n+1} \frac{z_k^2}{z - \alpha_k} \). These zeros will be denoted by \( v_1, \ldots, v_n \) and will be assumed all distinct. Let

\[
(9.17) \quad m(z) = \Pi_{k=1}^n (z - u_k), \quad a(z) = \Pi_{k=1}^{n+1} (z - \alpha_k), \quad b(z) = \Pi_{k=1}^n (z - v_k).
\]

**Lemma 4.**

\[
\sum_{k=1}^{n+1} \frac{x_k^2}{z - \alpha_k} = \frac{m(z)}{a(z)}, \quad \text{and} \quad \sum_{k=1}^{n+1} \frac{c_k}{z - \alpha_k} = \frac{b(z)}{a(z)}.
\]

**Proof.** The fact that any rational function is determined up to a constant factor by its zeros and poles implies that \( \sum_{k=1}^{n+1} \frac{x_k^2}{z - \alpha_k} = \frac{m(z)}{a(z)} \), where \( c \) is a constant. It follows from Lemma 3 that \( c = 1 \), because

\[
1 = \sum_{k=0}^n x_k^2 = c \sum_{k=0}^n \frac{m(\alpha_k)}{a(\alpha_k)} = c.
\]

The same argument carries over to \( \sum_{k=1}^{n+1} \frac{c_k}{z - \alpha_k} \). \( \square \)

Since \( x_k^2 = \frac{m(\alpha_k)}{a(\alpha_k)} \), \( x_k \) can be recovered up to a sign from \( u_1, \ldots, u_n \).

Recall now the function \( F(z) = (1 + (R_y y, y)) (R_x x, x) - (R_x x, y)^2 \). Then

\[
F(u_k) = - (R_{u_k} x, y)^2 = \sum_{k=0}^n \frac{c_k}{z - \alpha_k} \quad \text{and therefore,}
\]

\[
(R_{u_k} x, y) = \pm \sqrt{\frac{b(u_k)}{a(u_k)}}, \quad k = 1, \ldots, n.
\]

Each choice of the sign defines a set of \( n \) linear equations for the variables \( y_1, \ldots, y_{n+1} \), which together with \( (x, y) = 0 \) determine \( y \) uniquely in terms of \( u = (u_1, \ldots, u_n) \), and each choice identifies \( (u_1, \ldots, u_n) \) as a system of coordinates for the Lagrangian manifold \( S \).

**Proposition 12.** Vectors \( \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_{n+1}} \) form an orthogonal frame on the sphere \(||x|| = 1\), with \( ||\frac{\partial x}{\partial u_k}|| = \frac{1}{2} m'(u_k) \). Suppose that \( (R_{u_k} x, y) = - \sqrt{\frac{b(u_k)}{a(u_k)}}, \quad k = 1, \ldots, n \). Then, the differential equation \( \frac{dx}{dt} = y \) is given on \( S \) by

\[
\frac{1}{2} m'(u_k) \frac{du_k}{dt} = -a(u_k) b(u_k), \quad k = 1, \ldots, n.
\]

The preceding equations can also be written as

\[
(9.18) \quad \sum_{k=1}^n \frac{u_k^{n-j}}{2 \sqrt{-a(u_k) b(u_k)}} \frac{du_k}{dt} = \delta_{1j}, \quad j = 1, \ldots, n - 1.
\]

**Proof.** An easy logarithmic differentiation of \( x_k^2 = \frac{m(\alpha_k)}{a(\alpha_k)} \) yields \( \frac{\partial x_k}{\partial u_j} = -\frac{x_k}{2(\alpha_k - u_j)} \).

This implies that

\[
\frac{\partial x}{\partial u_j} = \frac{1}{2} (u_j I - A)^{-1} x, \quad j = 1, \ldots, n.
\]

Therefore, \( \left( \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_{n+1}} \right) = \frac{1}{2} ((u_1 I - A)^{-1} x, (u_2 I - A)^{-1} x) = \frac{1}{2} ((u_3 I - A)^{-1} (u_2 I - A)^{-1} x, x) = \frac{1}{2} ((u_3 I - A)^{-1} x - (u_2 I - A)^{-1} x, x) =
\]
In fact he shows that the Jacobi map given by
\[ (R_{u_1} x, x) - (R_{u_k} x, x) \] is 0 for \( k \neq j \).
For \( j = k \),
\[ \frac{\partial}{\partial u_k} \frac{\partial}{\partial u_k} = \frac{1}{2} \left( (u_k I - A)^{-2} x, x \right) = -\frac{1}{4} \frac{d}{dx} (R_z x, x)|_{z=u_k} = -\frac{1}{4} \frac{d}{dz} \frac{m(z)}{a(z)}|_{z=u_k} = -\frac{1}{4} n(u_k) \] since \( m'(u_k) \phi(u_k) \neq 0 \), \( \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n} \) form an orthogonal frame on \( S \).
Let \( P_1, \ldots, P_n \) denote the coordinates of \( y \) relative to the frame \( \frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots, \frac{\partial x}{\partial u_n} \) for \( (x, y) \in S \). It follows that \( \frac{\partial x}{\partial u_k}, y \) = \( P_k \| \frac{\partial x}{\partial u_k} \|^2 \). But \( \frac{\partial x}{\partial u_k}, y \) = \( \frac{1}{2} ((u_k I - A)^{-1} x, y) = \frac{1}{2} (R_{u_k} x, x) = -\frac{1}{2} \sqrt{-\frac{b(u_k)}{\alpha(u_k)}} \).

Therefore,
\[ P_k = \frac{2}{m'(u_k)} \sqrt{-a(u_k)b(u_k)}. \]

Suppose now that \((x(t), y(t))\) is a curve in \( S \) with \( \frac{dx}{dt} = y \). Then, \( \frac{dx}{dt} = \sum_{k=1}^{n} \frac{\partial x}{\partial u_k} \frac{du_k}{dt} \).
Therefore, \( \frac{m'(u_k)}{du_k} \frac{du_k}{dt} = \sqrt{-a(u_k)b(u_k)} \).

Second expression follows from Lemma \( \Box \) which implies that \( \sum_{k=1}^{n} \frac{u_k^n}{m(u_k)} = \delta_{n-1, m}, m \leq n - 1 \) by taking \( f(z) = z^m \) and \( g(z) = m(z) \). Then, \( m'(u_k) = 2 \sqrt{-a(u_k)b(u_k)} \frac{du_k}{dt} \) which, after the substitution, leads to
\[ \sum_{k=1}^{n} \frac{u_k^{n-j}}{2 \sqrt{-a(u_k)b(u_k)}} \frac{du_k}{dt} = \delta_{1j}, j = 1, \ldots, n-1 \].

J. Moser points out that equation \( (9.18) \) is related to the Jacobi map of the Riemann surface
\[ (9.19) \]
\[ w^2 = -4a(z)b(z). \]

In fact he shows that the Jacobi map given by
\[ \sum_{k=1}^{n} \int_{(0,0)}^{(u_k, u_k)} \frac{z^{n-j} dz}{2 \sqrt{-a(z)b(z)}} \]
takes the divisor class defined by \( (u_k, 2 \sqrt{a(u_k)b(u_k)}) \), \( k = 1, \ldots, n \) into a point \( s \in C^n / \Gamma \) where \( \Gamma \) denotes the period lattice of the differentials of the first kind \( (21) \).

10. CONNECTION TO GEODESIC PROBLEMS ON QUADRIC SURFACES: KNORRER’S TRANSFORMATION

Jacobi’s geodesic problem on an ellipsoid \( S = \{ x \in \mathbb{R}^{n+1} : (x, A^{-1} x) = 1 \} \) consists of finding curves in \( S \) of minimal length, relative to the metric inherited from the ambient Euclidean metric in \( \mathbb{R}^{n+1} \), that connect a given pair of points in \( S \). Jacobi was able to show that the curves of minimal length can be obtained from the solutions of a first order partial differential equation, known today as the Hamilton-Jacobi equation; in the process, he discovered an ingenious choice of coordinates on the ellipsoid, known today as elliptic coordinates, in terms of which the associated partial differential equation becomes separable with its solutions given by hyperelliptic functions.

Alternatively, the geodesic equations can be represented by a Hamiltonian system
\[ \frac{dx}{dt} = p, \quad \frac{dp}{dt} = -\frac{(p, A^{-1} p)}{\|A^{-1} x\|^2} A^{-1} x. \]
on the cotangent bundle of $S$ realized as the subset of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ subject to $G_1 = (x, A^{-1}x) - 1 = 0, G_2 = (p, A^{-1}x) = 0$. Moser shows that the above equations are generated by the Hamiltonian

$$ H = \frac{1}{2} ||p||^2 + \frac{(p, A^{-1}p)}{2||A^{-1}x||^2} G_1 - \frac{(p, A^{-1}x)}{||A^{-1}x||^2} G_2 $$

in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ in the sense that,

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

but constrained to $G_1 = G_2 = 0$ and $H = \frac{1}{2}$, i.e., to $||p|| = 1$ (21).

Let us modify above equations by replacing the Euclidean inner product $(x, y)$ by the inner product $(x, y)_\epsilon$ that encompasses both the Euclidean and the hyperbolic inner product. Then, equations (10.1) take on the following form:

$$(10.2) \quad \frac{dx}{dt} = (p, A^{-1} p)_\epsilon A^{-1} x, \quad \frac{dp}{dt} = -\frac{(p, A^{-1} p)_\epsilon}{||A^{-1}x||^2} A^{-1} x,$$

assuming that $||A^{-1}x|| \neq 0$. It follows that $G_1 = (x, A^{-1}x)_\epsilon - 1 = 0, G_2 = (p, A^{-1}x)_\epsilon = 0$, $||p|| = 1$ is an invariant set for (10.2).

Remarkably, equations (10.2) can be transformed into the equations of (9.4) by a transformation discovered by H. Knorrer in (12), and the integrals of motion of the geodesic problem can be deduced from the integrals of motion associated with the mechanical problem on the sphere. In what follows we will consider the inverse of the Knorrer’s transformation and show that the integrals of motion for the geodesic problem can be deduced from the mechanical problem of Newmann. For that reason we will begin with equations (9.4) written as

$$(10.3) \quad \frac{du}{ds} = v, \quad \frac{dv}{ds} = -A^{-1} u + ((A^{-1} u, u)_\epsilon - ||v||^2) u, \quad ||v||_\epsilon = 1, \epsilon = \pm 1.$$

It is important to keep in mind that the matrix $A$ also depends on $\epsilon$ since it belongs to $p_\epsilon$ (modulo the trace). In what follows it will be necessary to assume that $(A u, u)_\epsilon > 0, u \neq 0$.

Recall that $F_0 = ((v, Av)_\epsilon - 1)(u, Au)_\epsilon - (u, Av)_\epsilon^2$ is an integral of motion for (10.3) as can be easily seen from (9.13) with $z = 0$. Let $\Phi(\lambda, u, v) = (x, p)$ denote the mapping from the manifold $N_0 = \{(\lambda, u, v) : ||u||_\epsilon = 1, (u, v)_\epsilon = 0, F_0 = 0, \lambda \in \mathbb{R}\}$ given by

$$(10.4) \quad x = \frac{Au}{\sqrt{(Au, u)_\epsilon}}, \quad p = \frac{\lambda}{\sqrt{(Au, u)_\epsilon}} (Av - \frac{(Au, v)_\epsilon}{(Au, u)_\epsilon} Au).$$

It follows that $(x, A^{-1}x)_\epsilon = 1$ and that $(p, A^{-1}x)_\epsilon = 0$, hence $\Phi$ maps into the tangent bundle of the quadric $(x, A^{-1}x)_\epsilon = 1$. We will show now that $\Phi$ is invertible and that its inverse is given by:

$$(10.5) \quad \lambda = \sqrt{\frac{(A^{-1} p, p)_\epsilon}{||A^{-1} x||^2}}_\epsilon, \quad u = \frac{A^{-1} x}{||A^{-1} x||^2}_\epsilon, \quad v = \frac{1}{\sqrt{(A^{-1} p, p)_\epsilon}} (A^{-1} p - (u, A^{-1} p)_\epsilon u).$$

It follows from (10.4) that $\frac{(Au, v)_\epsilon}{\lambda^2} (A^{-1} p, p)_\epsilon = (Av - \frac{(Au, v)_\epsilon}{(Au, u)_\epsilon} Au) (v - \frac{(Au, v)_\epsilon}{(Au, u)_\epsilon} u) = (v, Av)_\epsilon - \frac{(Au, v_u)_\epsilon^2}{(Au, u)_\epsilon}. $
Let \( v \) under the substitutions given by formulas (10.5). Then, (10.7)\[ dx = \frac{d\lambda}{(Au(s), v(s))} \lambda(s). \]Hence, (10.6)\[ \frac{d\lambda}{ds} = 2\frac{(Au(s), v(s))}{(Au(s), u(s))^2}\lambda(s). \]Then,\[ \frac{d\lambda}{ds} = \frac{Av}{\sqrt{(Au, u)_e} - \frac{(Av, v)_e}{(Au, u)_e}Au} = \frac{1}{\lambda}v, \text{ and} \]
\[ \frac{dp}{ds} = -\frac{\lambda}{\sqrt{(Au, u)_e}}(Av - \frac{(Av, v)_e}{(Au, u)_e}Au) + \frac{\lambda}{\sqrt{(Au, u)_e}}\frac{d\lambda}{ds}(Av - \frac{(Av, v)_e}{(Au, u)_e}Au) = -\frac{\lambda}{\sqrt{(Au, u)_e}}u, \text{ and} \]
\[ \frac{d\lambda}{ds} = \frac{Av}{\sqrt{(Au, u)_e}}(Av - \frac{(Av, v)_e}{(Au, u)_e}Au) + \frac{\lambda}{\sqrt{(Au, u)_e}}\frac{d\lambda}{ds}(Av - \frac{(Av, v)_e}{(Au, u)_e}Au) = -\frac{\lambda}{\sqrt{(Au, u)_e}}u. \]

Hence, (10.7)\[ \frac{dx}{ds} = \frac{1}{\lambda}v, \quad \frac{dp}{ds} = -\lambda A^{-1}x. \]

Now identify \( \frac{1}{\lambda(s)} \) with a function \( \frac{dt}{ds} \). It follows from (10.5) that \( \frac{dt^2}{ds^2} = \frac{1}{\lambda^2} = \frac{||A^{-1}x||^2}{||A^{-1}x||^2}. \) Hence, (10.6)\[ \frac{d\lambda}{ds} = \frac{Av}{||A^{-1}x||^2} = \lambda(s) \]and hence equations (10.7) coincide with equations (10.2) after the reparametrization \( t = \phi(s) \) with \( \frac{dt}{ds} = \frac{1}{\lambda(s)} \).

The integrals of motion obtained for the mechanical problem of Newmann have their analogues for the problem of Jacobi via the following proposition

**Proposition 13.** Let \( F(w) = (1 + (R_wv, v)_e)(R_wu, u)_e - (R_wu, v)_e^2 \), with \( R_w = (wI - A)^{-1} \). Then,

\[ F\left( \frac{1}{z} \right) = \frac{1}{||A^{-1}x||^2} \left( 1 + (S_x, x)_e \right) \left( S_z, p \right)_e - (S_z, p)_e^2, S_z = (z - A)^{-1}, \]

under the substitutions given by formulas (10.5).

**Proof.** It is easy to verify that \( F(w) \) is invariant under the change of variable \( v \rightarrow v + \alpha u \) with \( \alpha \) an arbitrary scalar. Hence, \( F(w) = (1 + (R_wV, V))(R_wu, u) - (R_wu, V)^2 \), where \( V = \frac{A^{-1}p}{\lambda ||A^{-1}x||^2} \). In the proof below we will use the identity

\[ \frac{1}{z} - A^{-1} = A = -(z - A)^{-1}A^2. \]

Then,

\[ 1 + (R_zV, V)_e = 1 + \left( \frac{1}{z} - A^{-1} \right)^{-1}V, V)_e = \]
\[ 1 - ((z - A)^{-1}A^2 \frac{A^{-1}p}{\lambda ||A^{-1}x||^2}, \frac{A^{-1}p}{\lambda ||A^{-1}x||^2})_e - \left( \frac{A^{-1}p}{\lambda ||A^{-1}x||^2}, \frac{A^{-1}p}{\lambda ||A^{-1}x||^2} \right)_e = \]
\[ -((z - A)^{-1}A^2 \frac{A^{-1}p}{\lambda ||A^{-1}x||^2}, \frac{A^{-1}p}{\lambda ||A^{-1}x||^2})_e = -\frac{1}{\lambda^2 ||A^{-1}x||^2} (S_z, p)_e. \]
Further,

\[ (R_{\frac{1}{\epsilon}} u, u) = -((z - A)^{-1}A^2\frac{A^{-1}x}{||(A^{-1}x)||_{\epsilon}}, \frac{A^{-1}x}{||(A^{-1}x)||_{\epsilon}})_{\epsilon} - \frac{1}{||A^{-1}x||_{\epsilon}^2}(1 + (S_{x, x})_{\epsilon}), \]

and

\[ (R_{\frac{1}{\epsilon}} u, V)_{\epsilon} = -((z - A)^{-1}A^2\frac{A^{-1}x}{||(A^{-1}x)||_{\epsilon}}, \frac{A^{-1}p}{||(A^{-1}x)||_{\epsilon}})_{\epsilon} - \frac{1}{||A^{-1}x||_{\epsilon}^2}(S_{x, p})_{\epsilon}. \]

Hence,

\[ F(\frac{1}{2}) = \frac{1}{\lambda^{||A^{-1}x||_{\epsilon}^2}}((1 + (S_{x, x}))_{\epsilon}(S_{x, p})_{\epsilon} + (S_{x, x})_{\epsilon}) = \frac{1}{||A^{-1}x||_{\epsilon}^2}((1 + (S_{x, x})_{\epsilon})(S_{x, p})_{\epsilon} - (S_{x, x})_{\epsilon}^2). \]

**Corollary 3.** Function \( G(z) = (1 + (S_{x, x})_{\epsilon})(S_{x, p})_{\epsilon} - (S_{x, x})_{\epsilon}^2 \) is constant along the geodesic flow \((10.2)\).

**Proof.** \( ||A^{-1}x||_{\epsilon}^2(A^{-1}p, p)_{\epsilon} \) is an integral of motion for \((10.2)\) because

\[ \frac{d}{dt}||A^{-1}x||_{\epsilon}^2(A^{-1}p, p)_{\epsilon} = 2(A^{-1}p, A^{-1}x)_{\epsilon}(A^{-1}p, p)_{\epsilon} - ||A^{-1}x||_{\epsilon}^2(A^{-1}p, (A^{-1}p, p), \frac{A^{-1}x}{||A^{-1}x||_{\epsilon}^2})_{\epsilon} = 0. \]

It follows that \( G(z) \) is constant along the solutions of \((10.2)\). Since \( F(\frac{1}{2}) \) is constant along the solutions of \((10.2)\). \( \square \)

**Remark 4.** Function \( ||A^{-1}x||_{\epsilon}^2(A^{-1}p, p)_{\epsilon} \) is known as Joachimsthal’s integral of motion \((14), (24)\).

In the Euclidean case the matrix \( A \) can be assumed diagonal with \( \alpha_1, \ldots, \alpha_{n+1} \) its eigenvalues. An argument identical to the one used above shows that

\[ G(z) = \sum_{k=1}^{n+1} \frac{G_k}{z - \alpha_k} \]

and that the residues \( G_k \) are given by

\[ G_k = p_k^2 + \sum_{j=1, j \neq k}^{n+1} \frac{(x_j p_k - x_k p_j)^2}{(\alpha_k - \alpha_j)}, \]

as reported in \((21)\). The hyperbolic case differs only in minor details due to different canonical structures of \( A \).

11. **The case \( A = 0 \) and the problem of Kepler**

Consider now the Hamiltonian \( H = \frac{1}{2}(L_p, L_t) \) on coadjoint orbits of \( p_\epsilon \times t_\epsilon \), \( \epsilon = \pm 1 \), through rank one matrices in \( p_\epsilon \). We will continue with the notations of the last two sections and consider the coadjoint orbits through matrices \( P_0 = (x_0 \otimes x_0)_{\epsilon} - \frac{||x_0||^2_{n+1}}{n+1} I, \epsilon = \pm 1 \). We have seen that the these coadjoint orbits consist of matrices

\[ L_p = (x \otimes x)_{\epsilon}, L_t = (x \wedge y)_{\epsilon}, ||x||_{\epsilon} = ||x_0||_{\epsilon}, (x, y)_{\epsilon} = 0, \]

that are symplectomorphic to the cotangent bundle of the “sphere” \( \{ (x, y) : ||x||_{\epsilon} = ||x_0||_{\epsilon}, (x, y)_{\epsilon} = 0 \} \). The Hamiltonian equations \((11)\) and \((12)\) reduce to

\[ \frac{dL_t}{dt} = [L_t, L_p], \frac{dL_p}{dt} = 0, \]
and

\begin{equation}
\frac{dx}{dt} = ||x||_2^2 y, \quad \frac{dy}{dt} = -||y||_2^2 x.
\end{equation}

It follows that the solutions satisfy

\[ \frac{d^2x}{dt^2} + ||x||_2^2 ||y||_2^2 x = 0 \]

The restriction of the Hamiltonian $H$ to these orbits is given by

\[ H = \frac{1}{2} ||x||_2^2 ||y||_2^2. \]

Hence, on energy level $H = \frac{\varepsilon}{2}$, the preceding equations reduce to

\begin{equation}
\frac{d^2x}{dt^2} + \varepsilon x = 0.
\end{equation}

It follows that the solutions of (11.3) are given by great circles $x(t) = a \cos(t) + b \sin(t)$ for $\varepsilon = 1$, and great hyperbolas $x(t) = a \cosh(t) + b \sinh(t)$ for $\varepsilon = -1$, with

\[ ||a||_2^2 + ||b||_2^2 = ||x_0||_2^2, \quad (a, b) \neq 0. \]

Recall now the Hamiltonian $E = \frac{1}{2}||p|| - \frac{1}{||q||}$ associated with the problem of Kepler in the phase space $(q, p) \in \mathbb{R}^n \times \mathbb{R}^n : q \neq 0$ corresponding to the normalized constants $m = kM = 1$ and the associated equations of motion

\begin{equation}
\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -\frac{1}{||q||^3} q.
\end{equation}

Below is a summary of the classical theory connected with Kepler’s problem.

1. $L = q \wedge p$ and $F = Lp - \frac{q}{||q||}$ are constants of motion for (11.4). $L$ is an $n$ dimensional generalization of the angular momentum $p \times q$. Its constancy implies that each solution remains in the plane spanned by $q(0)$ and $p(0)$.

The second vector is call the Runge-Lenz vector or, sometimes, the eccentricity vector. It lies in the plane spanned by $p$ and $q$.

2. Let $||F||^2 = 2||L||^2 E + 1$, where $||L||^2 = -\frac{1}{2}Tr(L^2) = ||q||^2||p||^2 - (q \cdot p)^2$. Then, $||F|| < 1$ whenever $E < 0$, $||F|| = 1$ whenever $E = 0$, and $||F|| > 1$ whenever $E > 0$.

3. A solution $(q(t), p(t))$ evolves on a line through the origin if and only if $q(0)$ and $p(0)$ are colinear, that is, whenever $L = 0$. In the case that $L \neq 0$

\[ ||q(t)|| = \frac{||L||}{1 + ||F||^2 \cos \phi(t)}, \]

where $\phi(t)$ denotes the angle between $F$ and $q(t)$. Therefore,

$q(t)$ traces an ellipse when $||F|| < 1$, a parabola when $||F|| = 1$ and a hyperbola when $||F|| > 1$.

There is a remarkable connection between the solutions of Kepler’s problem and the geodesic flows on space forms that was first reported by V.A. Fock in 1935 in connection with the theory of hydrogen atom ([8]) which then was rediscovered independently by J. Moser in ([12]) for the geodesics on a sphere. Moser’s study was later completed to all space forms by Y. Osipov in ([23]).

As brilliant as these contributions were, they, nevertheless, did not attempt any explanations in regard to this enigmatic connection between planetary motions and geodesics on space forms. This issue later inspired V. Guillemin and S. Sternberg to take up the problem of Kepler in ([10]) in a larger geometric context with Moser’s observation at the heart of the matter.
It seems altogether natural to include Kepler’s problem in this study. In this setting Kepler’s system is recognized within a large class of integrable systems and secondly, the focus on coadjoint representations provides natural explanations for its connections to the geodesic problems. Following Moser we will consider the stereographic projection from the sphere \(|x|_2^2 = h^2\) into \(\mathbb{R}^n\) given by

\[
\lambda(x - he_0) + he_0 = (0, p), \quad \text{where } \lambda = \frac{h}{h - x_0}.
\]

Here, \((x_0, x_1, \ldots, x_n)\) denote the coordinates of a point \(x\) in \(\mathbb{R}^{n+1}\) corresponding to the standard basis \(e_0, \ldots, e_n\). It follows that

\[
(11.5) \quad x_0 = \frac{h(||p||^2 - \varepsilon h^2)}{||p||^2 + \varepsilon h^2}, \quad \text{and} \quad \bar{x} = x - x_0e_0 = \frac{2\varepsilon h^2}{||p||^2 + \varepsilon h^2}p.
\]

Consider now the extension of this mapping to the cotangent bundle of \(|x|_2^2 = h^2\) that pulls back the canonical symplectic form in \(\mathbb{R}^n \times \mathbb{R}^n\) onto the symplectic form of the cotangent bundle of \(|x|_2^2 = h^2\). It suffices to find a mapping \(q = \Phi(x, y)\) such that

\[
(11.6) \quad \sum_{i=1}^n q_idp_i = (y, dx)_e = y_0dx_0 + \varepsilon \sum_{i=1}^n y_i dx_i
\]

because the symplectic forms are the exterior derivatives of the preceding forms. It turns out that such an extension is unique by the following arguments.

Let \(x = \Phi(p)\) denote the mapping given by \((11.5)\). Then,

\[
(11.7) \quad dx = \frac{\partial \Phi}{\partial p}_e dp = \left(\frac{4\varepsilon h^3}{(||p||^2 + \varepsilon h^2)^2}p \cdot dp, \frac{2\varepsilon h^2}{||p||^2 + \varepsilon h^2}dp - \frac{4\varepsilon h^2p \cdot dp}{(||p||^2 + \varepsilon h^2)^2}\right).
\]

It follows by an easy calculation that

\[
(11.8) \quad ||dx||_e^2 = dx_0^2 + \varepsilon \sum_{i=1}^n dx_i^2 = \frac{4h^4\varepsilon}{(||p||^2 + \varepsilon h^2)^2}||dp||^2.
\]

Since \((dx, dx)_e = ((\frac{\partial \Phi}{\partial p})_e dp, (\frac{\partial \Phi}{\partial p})_e dp)_e = ((\frac{\partial \Phi}{\partial p})_e^T dp, dp)_e = \frac{4h^4\varepsilon}{(||p||^2 + \varepsilon h^2)^2}||dp||^2\) it follows that \((\frac{\partial \Phi}{\partial p})_e^T I_n = \frac{4h^4\varepsilon}{(||p||^2 + \varepsilon h^2)^2} I_n\), where \(I_n\) denotes the \(n\) dimensional identity and \((\frac{\partial \Phi}{\partial p})_e^T\) the adjoint operator of \(\frac{\partial \Phi}{\partial p}\) relative to the inner product \((\ , \ )_e\), i.e., \((\frac{\partial \Phi}{\partial p})_e^T J^T_{\varepsilon} (\frac{\partial \Phi}{\partial p})_e = (\frac{\partial \Phi}{\partial p})_e^T J^T_{\varepsilon}\) with \((\frac{\partial \Phi}{\partial p})_e^T\) the transpose of \(\frac{\partial \Phi}{\partial p}\) and \(J_{\varepsilon}\) diagonal matrix with its diagonal entries \((1, \varepsilon, \ldots, \varepsilon)\).

Then,

\[
q \cdot dp = (y \cdot dx)_e = (y, \frac{\partial \Phi}{\partial p} dp)_e = (\frac{\partial \Phi}{\partial p})_e^T y, dp)_e \implies q = (\frac{\partial \Phi}{\partial p})_e^T y \quad \text{or,}
\]

\[
y = \varepsilon \frac{||(p||^2 + \varepsilon h^2)^2}{4h^4}(\frac{\partial \Phi}{\partial p})_e^T q,
\]

since \((\frac{\partial \Phi}{\partial p})_e^T y = \varepsilon \frac{||(p||^2 + \varepsilon h^2)^2}{4h^4}(\frac{\partial \Phi}{\partial p})_e^T q = q\).

Equations \((11.7)\) reveal that

\[
(11.9) \quad y = \frac{1}{h} q \cdot p, \quad \frac{||p||^2 + \varepsilon h^2}{2h^2}q - \frac{q \cdot p}{h^2}p)
\]

from which it follows that

\[
(11.10) \quad ||y||_e^2 = \varepsilon \frac{(||p||^2 + \varepsilon h^2)^2}{4h^4}||q||^2.
\]
To pass to the problem of Kepler, write the Hamiltonian \( H = \frac{1}{2}||x||^2 ||y||^2 \) in the variables \( (p, q) \). It follows that \( H = \frac{1}{2}h^2 \epsilon (||p||^2 + \epsilon h^2 q^2) \), \( ||q||^2 = \frac{1}{2}h^2 (||p||^2 + \epsilon h^2 q^2) ||q||^2 \).

The corresponding flow is given by

\[
\frac{dp}{ds} = \frac{\partial H}{\partial q} = \frac{\epsilon (||p||^2 + \epsilon h^2 q^2)^2}{4h^2} q, \quad \frac{dq}{ds} = -\frac{\partial H}{\partial p} = -\frac{\epsilon ||p||^2 + \epsilon h^2}{2h^2} ||q||^2 p.
\]

On energy level \( H = \frac{1}{2} ||x||^2 ||y||^2 = 1 \) and the preceding equations reduce to

\[
\frac{dp}{ds} = \epsilon q \frac{q}{h^2 ||q||^2}, \quad \frac{dq}{ds} = -\epsilon ||q|| \frac{q}{h^2 p}.
\]

The preceding equations coincide with the equations of Kepler’s problem (11.3) after the reparametrization by a parameter \( t = -\frac{\epsilon}{h^2} \int_0^s ||q||^2 d\tau \). For then, \( \frac{dt}{ds} = -\frac{\epsilon \cdot \bar{q}^2}{h^2 ||q||^2} \) and equations (11.11) become

\[
\frac{dp}{dt} = \frac{dp}{ds} \frac{ds}{dt}, \quad \frac{dq}{dt} = \frac{dq}{ds} \frac{ds}{dt} = p.
\]

Since \( \frac{(||p||^2 + \epsilon h^2 q^2)^2}{4} ||q||^2 = 1 \),

\[
E = \frac{1}{2} ||p||^2 \left( \frac{1}{2} ||q||^2 \right) (2 - \epsilon h^2 ||q|| - 2) = \frac{1}{2} (2 - \epsilon h^2 ||q|| - 2) = -\frac{1}{2} \epsilon h^2.
\]

So \( E < 0 \) in the spherical case and \( E > 0 \) in the hyperbolic case.

The Euclidean case \( E = 0 \) can be obtained by a limiting argument in which \( \epsilon \) is regarded as a continuous parameter which tends to zero. To explain in more detail, let \( \bar{x}(t) = x(t) - x_0(t) \epsilon e_0 \) where \( x(t) \) is a solution of (11.3) if \( w(t) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \bar{x}(t) \), then \( w(t) \) is a solution of

\[
\frac{d^2 w}{dt^2} = 0,
\]

that is, \( w(t) \) is a geodesic corresponding to the standard Euclidean metric. It then follows from (11.5) that \( \lim_{\epsilon \to 0} x_0 = h \) and

\[
w = \lim_{\epsilon \to 0} \frac{1}{h^2 \epsilon} \bar{x} = \lim_{\epsilon \to 0} \frac{2}{||p||^2 + \epsilon h^2 p} = \frac{2}{||p||^2}.
\]

Moreover, \( \lim_{\epsilon \to 0} dx_0 = 0 \) and \( \lim_{\epsilon \to 0} \frac{dx}{\epsilon} = dw = \frac{2}{||p||^2} ||dp|| dp - \frac{4p dp}{||p||^2} p \) as can be seen from (11.7). Therefore, \( ||dw||^2 = \frac{4}{||p||^2} ||dp||^2 \).

The transformation \( p \to w \) with \( w = \frac{2}{||p||^2} p \) is the inversion about the circle \( ||p||^2 = 2 \), and \( ||dw||^2 = \frac{4}{||p||^2} ||dp||^2 \) is the corresponding transformation of the Euclidean metric \( ||dp||^2 \). The Hamiltonian \( H_0 \) associated with this metric is equal to \( \frac{1}{2} ||\bar{x}||^2 ||q||^2 \).

This Hamiltonian can be also obtained as the limit of \( \frac{h^2}{\epsilon} \frac{(||p||^2 + \epsilon h^2 q^2)^2}{4h^2} ||q||^2 \) when \( \epsilon \to 0 \). On energy level \( H = \frac{1}{2} ||p||^2 ||q|| = 2 \) and therefore, \( E = 0 \).

The integrals of motion for the problem of Kepler are synonymous with the constancy of the matrix \( (x \wedge y)_\epsilon \) along the flow of (11.1). To be more specific, let \( x = x_0 e_0 + \bar{x} \) and \( y = y_0 e_0 + \bar{y} \). Then,

\[
(x \wedge y)_\epsilon = (x_0 e_0 + \bar{x} \wedge y_0 e_0 + \bar{y})_\epsilon = x_0 (e_0 \wedge \bar{y})_\epsilon + y_0 (e_0 \wedge \bar{x})_\epsilon + (\bar{x} \wedge \bar{y})_\epsilon.
\]

Since \( (x(t) \wedge y(t))_\epsilon \) is constant along the flows of (11.1), both \( x_0 (e_0 \wedge \bar{y})_\epsilon - y_0 (e_0 \wedge \bar{x})_\epsilon \)
and \((\bar{x} \wedge \bar{y})_x\) are also constant. But then the angular momentum \(L = q \wedge p\) and the Runge-Lenz vector \(F = Lp - \frac{e}{||q||}\) are given by

\[(11.13)\]

\[L = (\bar{y} \wedge \bar{x})_x \quad \text{and} \quad F = h(y_0(\epsilon_0 \wedge \bar{x}) - x_0(\epsilon_0 \wedge \bar{y})_x)_0.\]

The first equality is evident from equations (11.9) and (11.2) and the fact that \((\bar{y} \wedge \bar{x})_x = \epsilon(\bar{y} \wedge \bar{x})\). Second equality follows by the calculation below:

\[(y_0(\epsilon_0 \wedge \bar{x})_x - x_0(\epsilon_0 \wedge \bar{y})_x)_0 = -y_0\bar{x} + x_0\bar{y} = -\frac{1}{k}q \cdot p \frac{2h^2}{||q||^2 + \epsilon h^2} p + \frac{1}{2h^2} \left( ||q||^2 + \epsilon h^2 \right) q - \frac{2}{k} h \bar{p} = \frac{1}{k}(-q \cdot p + ||p||^2 - \epsilon q^2).\]

Since \(\frac{1}{2}||q||^2 + \epsilon h^2\ ||q|| = 1\), \(\epsilon h^2 = \frac{2}{||q||} - ||p||^2\), and therefore,

\[h(y_0(\epsilon_0 \wedge \bar{x})_x - x_0(\epsilon_0 \wedge \bar{y})_x)_0 = (-q \cdot p + ||p||^2 q - \frac{q}{||q||} = (q \wedge p) p - \frac{q}{||q||} = F.\]

11.0.1. **Conic sections and the geodesics.** The geodesics of the spaces of constant curvature are transformed into the conic sections of the problem of Kepler, a fact well known in conformal geometry. For the convenience of the reader not familiar with these facts and also for the completeness of the presentation we include the basic details.

In the spherical case, the great circle \(x = a \cos \omega t + b \sin \omega t\) with \(||a|| = ||b|| = \) \(h\), \(a \cdot b = 0\) can be rotated around \(\epsilon_0\) so that \(a\) and \(b\) are in the subspace spanned by \(\epsilon_0, \epsilon_1, \epsilon_2\). Moreover, such a rotation \(R\) can be chosen so that \(Ra = h\epsilon_1\).

Let \(\alpha\) denote the angle that the great circle makes with the plane \(x_0 = 0\). Then,

\[x_0 = h \sin \alpha \sin(\omega t), \quad x_1 = h \cos(\omega t), \quad x_2 = -h \cos \alpha \sin(\omega t), \quad x_i = 0, \quad i = 3, \ldots, n + 1,\]

because \(||x||^2||y||^2 = h^2\). Furthermore, \(- \frac{h}{h - x_0}(x - he_0) + he_0 = (0, p)\) implies that

\[p_1 = \frac{h}{1 - \sin \alpha \sin(\omega t)} \cos(\omega t), \quad p_2 = \frac{-h}{1 - \sin \alpha \sin(\omega t)} \cos \alpha \sin(\omega t), \quad p_i = 0, \quad i = 3, \ldots, n,\]

and \(y(t) = \frac{1}{h} \frac{dx}{dt}\) implies that

\[y = (\sin \alpha \cos(\omega t), -\sin(\omega t), -\cos \alpha \cos(\omega t), 0, \ldots, 0).\]

Then,

\[||p||^2 + h^2 = \frac{2h^2}{1 - \sin \alpha \sin(\omega t)} \quad \text{and} \quad \frac{p \cdot q}{h} = \sin \alpha \cos(\omega t),\]

(implied by equations (11.10)). Hence,

\[y_1 = \frac{q_1}{\sin \alpha \sin(\omega t)} - \frac{\sin \alpha \cos(\omega t)}{1 - \sin \alpha \sin(\omega t)} \cos(\omega t), \quad y_2 = \frac{q_2}{\sin \alpha \sin(\omega t)} + \frac{\sin \alpha \cos(\omega t)}{1 - \sin \alpha \sin(\omega t)} \cos \alpha \sin(\omega t).\]

It follows that \(- \sin(\omega t) (1 - \sin \alpha \sin(\omega t)) = q_1 - \sin \alpha \cos(\omega t) \cos(\omega t)\) and \(- \cos \alpha \cos(\omega t) (1 - \sin \alpha \sin(\omega t)) = q_2 + \sin \alpha \cos(\omega t) \cos \alpha \sin(\omega t)\) and therefore,

\[q_1 = - \sin(\omega t) \sin \alpha, \quad q_2 = - \cos \alpha \cos(\omega t).\]

Hence, the great circle is transformed into the ellipse

\[(q_1 - \sin \alpha)^2 + \frac{1}{\cos^2 \alpha} q_2^2 = 1\]

This ellipse degenerates into a line through the origin when \(\alpha = \frac{\pi}{2}\), or when the great circle passes through \(he_0\).

A similar argument shows that the hyperboloid \(x(t) = a \sin(\omega t) + b \cosh(\omega t)\) is transformed into the hyperbola \(- (q_1 - \sin \alpha)^2 + \frac{1}{\cos^2 \alpha} q_2^2 = 1\).
In the Euclidean case, the line $w = a + bt$ is transformed into the curve $\frac{2}{||p(t)||^2} p(t)$ via the mapping $w = \frac{2}{||p||^2} p$. Hence,

$$b = \frac{d}{dt} = \frac{2}{||p||^2} \frac{dp}{dt} - \frac{1}{||p||^2} (p, \frac{dp}{dt}) p.$$ 

and $\frac{dp}{dt} = \frac{4}{||p||^4} q$, $||p||^2 = \frac{4}{||w||^2}$. The above equation becomes

$$b = 2 \frac{q}{||w||^2} + 2 \frac{1}{||w||^2} (w, \frac{dw}{dt}) w.$$ 

Hence,

$$q = \frac{1}{2} (b(||a||^2 - ||b||^2 t^2) - 2a((a, b) + ||b||^2 t)).$$

This equation is a parabola in the $a, b$ plane.

12. Concluding remarks

The above exposition could be viewed as a first step in unifying various fragmented results in the theory of integrable systems. The fact that much of this theory is related to Lie groups and the associated Lie algebras has been recognized in one form or another for some time now (3 24 26 27). However, in contrast to the cited publications, the present study uses control theory and its Maximum Principle as a point of departure for geometric problems with nonholonomic constraints which greatly facilitates passage to the appropriate Hamiltonians and which, at the same time, clarifies the role of the Hamiltonians for the original problems.

Additionally, the ubiquitous presence of the affine problem on any symmetric space paves a way for new classes of integrable systems, for it seems very likely that the affine Hamiltonian is integrable on any coadjoint orbit (3). Further clarifications of this situation would be welcome additions to the theory of integrable systems. Along more specific lines, the study of Fedorov and Jovanovic (7) strongly suggests that the problem of Newmann on Steifel manifolds can be seen also as the restriction of the affine Hamiltonian to the coadjoint orbit of the semidirect product through an arbitrary symmetric matrix. It would be instructive to investigate this situation in some detail.

It might be also worthwhile to mention that the solutions of the affine problem on the unitary group would find direct applications in the emerging field of quantum control (4). This topic, however, because of its own intricacies is deferred to a separate study.

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Department of Mathematics, University of Toronto
E-mail address: jurdj@math.toronto.edu
Current address: Department of Mathematics, University of Toronto, 40 St. George St., Toronto
URL: http://www.jurdj.math.toronto.edu