A PATH INTEGRATION APPROACH TO THE CORRELATORS OF XY HEISENBERG MAGNET AND RANDOM WALKS

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Abstract

The path integral approach is used for the calculation of the correlation functions of the XY Heisenberg chain. The obtained answers for the two-point correlators of the XX magnet are of the determinantal form and are interpreted in terms of the generating functions for the random turns vicious walkers.

Keywords: Path Integration; XY Heisenberg Magnet; Random Walks.
1 Introduction

The problem of enumeration of paths of vicious walkers on one-dimensional lattice was formulated by M. Fisher [1] and since then continues to attract much attention (see refs. in [2, 3]). The walkers are called ‘vicious’ because they annihilate each other at the same lattice site, and their trajectories are thus non-intersecting. Similar problems appear in the theory of domain walls [4], directed percolation [5], self-organized criticality [6], and polymer theory [7]. It has been proposed Ref. [2] to use the XX Heisenberg chain to enumerate the paths of the random turns vicious walkers.

Approach based on path integration was developed in Refs. [8, 9] to calculate thermal correlation functions of the XY Heisenberg magnet. Dependence of the integration variables on the imaginary time is defined by special quasi-periodicity conditions. In the present paper, this method is used for the calculation of the two-point correlation functions of the XX model and the interpretation of the obtained answer in terms of the generating functions of the random turns vicious walkers is given.

2 The problem

The Hamiltonian of the periodic XY Heisenberg chain of “length” $M$ ($M$ is chosen to be even) in transverse magnetic field $h > 0$ is:

$$H = H_0 + \gamma H_1 - h S^z, \quad H_0 \equiv - \sum_{n,m=1}^{M} \Delta_{nm}^{(\pm)} \sigma_n^+ \sigma_m^-,$$

$$H_1 \equiv - \frac{1}{2} \sum_{n,m=1}^{M} \Delta_{nm}^{(\pm)} (\sigma_n^+ \sigma_m^+ + \sigma_n^- \sigma_m^-), \quad S^z \equiv \frac{1}{2} \sum_{n=1}^{M} \sigma_n^z.$$

Here $S^z$ is $z$-component of the total spin operator, and the entries of the so-called hopping matrix $\Delta_{nm}^{(s)}$ ($s = \pm$) are:

$$2 \Delta_{nm}^{(s)} \equiv \delta_{|n-m|,1} + s \delta_{|n-m|,M-1},$$

where $\delta_{n,l}$ is the Kronecker symbol. The Pauli matrices $\sigma_n^\pm = (1/2)(\sigma_n^x \pm i \sigma_n^y)$ and $\sigma_n^z$, where $n \in M \equiv \{1, \ldots, M\}$, satisfy the commutation relations: $[\sigma_k^+, \sigma_l^-] = \delta_{kl} \sigma_k^z$ and $[\sigma_k^z, \sigma_l^\pm] = \pm 2 \delta_{kl} \sigma_k^\pm$. The periodic boundary condition reads: $\sigma_{n+M}^a = \sigma_n^a, \forall n$. The Hamiltonian $H$ (1), taken at $\gamma = 0$ (the case of XX magnet), commutes with $S^z$.

Time-dependent thermal correlation functions are defined as follows:

$$G_{ab}(m,t) \equiv Z^{-1} \text{Tr} \left( \sigma_i^a e^{-\beta H} \sigma_k^b(t) e^{-\beta H} \right), \quad Z \equiv \text{Tr} \left( e^{-\beta H} \right),$$

where $\sigma_k^b(t) \equiv e^{itH} \sigma_k^b e^{-itH}$, $\beta = 1/T$ is inverse temperature, and $t$ is time. This correlator may be rewritten in terms of the canonical lattice Fermi fields $c_i, c_i^\dagger$, where $i, j \in M$, by means of the Jordan-Wigner map:

$$\sigma_n^+ = \left( \prod_{j=1}^{n-1} \sigma_j^z \right) c_n, \quad \sigma_n^- = c_n^\dagger \left( \prod_{j=1}^{n-1} \sigma_j^z \right), \quad n \in M,$$

where $\sigma_j^z = 1 - 2c_j c_j^\dagger$. The periodic conditions for the spin operators result in the boundary conditions for the fermions:

$$c_{M+1} = (-1)^N c_1, \quad c_{M+1}^\dagger = c_1^\dagger (-1)^N,$$
where $N = \sum_{n=1}^{M} c_n^\dagger c_n$ is the operator of the total number of particles. In the fermionic representation, $H$ (1) will take a form $H = H^+ P^+ + H^- P^-$, where $P^\pm = (1/2)(I \pm (-1)^N)$ are projectors [8]. The operators $H^s$ are of identical form with $s = \pm$ pointing out a correspondence between these operators and appropriate specification of the conditions (4): $c_{M+1} = -s c_1, c_{M+1}^\dagger = -s c_1^\dagger$.

Equation (3) for the $z$-components of spins, for instance, becomes:

$$G_{zz}(m,t) = 1 - 2 Z^{-1} \text{Tr} \left( c_{l+m}^\dagger c_{l+m} e^{-\beta H} \right) - 2 Z^{-1} \text{Tr} \left( c_{l}^\dagger c_{l} e^{-\beta H} \right)$$

$$+ 4 Z^{-1} \text{Tr} \left( c_{l+m}^\dagger c_{l+m} e^{iH} c_{l}^\dagger c_{l} e^{-(\beta+i)t)H} \right).$$

To evaluate (5), it is convenient to consider the generating functional:

$$G \equiv G(S,T|\mu,\nu) = Z^{-1} \text{Tr} \left( e^{\nu H} e^{T e^{-\mu H}} \right),$$

where $\mu, \nu$ are the complex parameters, $\mu + \nu = \beta$. Two operators, $S \equiv e^{\dagger S_c}$ and $T \equiv e^{\dagger T_c}$, are defined through the matrices $\hat S = \text{diag} \left\{ S_1, S_2, \ldots, S_M \right\}$, $\hat T = \text{diag} \left\{ T_1, T_2, \ldots, T_M \right\}$. For instance, the last term in R.H.S. of (5) is obtained from (6) in the following way:

$$\frac{\partial}{\partial S_k} \frac{\partial}{\partial T_l} G(S,T|\mu,\nu) \bigg|_{S_n, T_n, \forall n} \rightarrow 0 \quad \mu, \nu \rightarrow -it, \beta + it$$

As a result, we express the trace in R.H.S. of (6) in the form [8]:

$$\text{Tr} \left( e^{S} e^{-\mu H} e^{T} e^{-\nu H} \right) = \frac{1}{2} \left( G_F^+ Z_F^+ + G_F^- Z_F^- + G_B^+ Z_B^+ - G_B^- Z_B^- \right),$$

where

$$G_F^+ Z_F^+ = \text{Tr} \left( e^{S} e^{-\mu H^\pm} e^{T} e^{-\nu H^\pm} \right),$$

$$G_B^+ Z_B^+ = \text{Tr} \left( e^{S} e^{-\mu H^\pm} e^{T} (-1)^N e^{-\nu H^\pm} \right),$$

and $Z_F^\pm = \text{Tr} \left( e^{\pm \beta H^\pm} \right)$, $Z_B^\pm = \text{Tr} \left( (-1)^N e^{\pm \beta H^\pm} \right)$.

### 3 The path integral

We use the coherent states $|z\rangle \equiv \exp(c^\dagger z)|0\rangle$ and $\langle z^*| \equiv \langle 0| \exp(z^* c)$ generated from the Fock vacuum $|0\rangle$, $c_k|0\rangle = 0$, $\forall k$. We use the short-hand notations for the $M$-component objects, say, $z^* \equiv (z_1^*, \ldots, z_M^*)$ and $z \equiv (z_1, \ldots, z_M)$ formed by the independent Grassmann parameters $z_k, z^*_k$ ($k \in \mathcal{M}$). Besides, $\sum_{k=1}^{M} c_k^\dagger z_k \equiv c^\dagger z$, $\prod_{k=1}^{M} dz_k \equiv dz$, etc. Then, we shall represent [9] the trace of the operator in $G_F^\pm Z_F^\pm$ (8) by means of the Grassmann integration over $dz, dz^*$:

$$G_F^\pm Z_F^\pm = \int dz \, dz^* \, e^{\ast z} \langle z^*| e^{S} e^{-\mu H^\pm} e^{T} e^{-\nu H^\pm}|z\rangle.$$

For the sake of simplicity we shall consider the $XX$ model only and take those $H^\pm$ that correspond to $H$ (1) at $\gamma = 0$.

To represent R.H.S. of (9) as the path integral, we first introduce new coherent states $|x(I)\rangle$, $\langle x^*(I)|$, where $2L \times M$ independent Grassmann parameters are arranged in the form
of $2L$ “vectors” $x^*(I), x(I)$ ($I \in \{1, \ldots, L\}$). It allows to insert $L$ times the decompositions of unity
\[ \int dx^*(I) dx(I) \exp(-x^*(I)x(I)) |x(I)\rangle \langle x^*(I)| \]
into R.H.S. of (9). We define then the additional variables satisfying the quasi-periodicity conditions:
\[ -\hat{E} x(0) = x(L + 1) \equiv z, \quad x^*(L + 1) = x^*(0) \hat{E}^{-1} \equiv z^*. \tag{10} \]
Here, $\hat{E} \equiv e^S e^{-\mu \hat{H}^\pm} e^{\hat{T}}$ with the matrices $\hat{H}^\pm$ expressed [9] through the hopping matrices (2): $\hat{H}^\pm = -\hat{\Delta}^{(\mp)} + h\hat{T}$, where $\hat{T}$ is a unit $M \times M$ matrix. The described procedure allows to pass in the limit $L \to \infty$ from $(L + 1)$-fold integration to the continuous one over “infinite” product of the measures $d\lambda^\pm$ $x^\tau \tau 
\begin{align*}
G^\pm_F Z^\pm_F &= \int e^S d\lambda^* d\lambda \prod_\tau dx^\tau(\tau) dx(\tau).
\end{align*}

The integration over the auxiliary Grassmann variables $\lambda^*$, $\lambda$ guarantees the fulfilment of the continuous version of the constraints (10). The action functional is $S \equiv \int L(\tau) d\tau$, where $L(\tau)$ is the Lagrangian:
\[ L(\tau) \equiv x^\tau(\tau) \left( \frac{d}{d\tau} - \hat{H}^\pm \right) x(\tau) + J^\tau(\tau)x(\tau) + x^\tau(\tau)J(\tau), \]
\[ J^\tau(\tau) \equiv \lambda^*(\delta(\tau) \hat{T} + \delta(\tau - \nu) \hat{E}^{-1}), \quad \delta(\tau) \equiv \delta(\tau) \hat{T} + \delta(\tau - \nu) \hat{E} \lambda. \]

The $\delta$-functions reduce $\tau$ to the segment $[0, \beta]$. The stationary phase requirements $\delta S/\delta x^* = 0, \delta S/\delta x = 0$ yield the regularized answer [9]:
\[ G^\pm_F = \det \left( \hat{T} + \frac{e^{(\beta - \nu)\hat{H}^\pm} e^S e^{-\mu \hat{H}^\pm} e^{\hat{T}} - \hat{T}}{\hat{T} + e^{\beta \hat{H}^\pm}} \right). \]

The remainder correlators $G_{ab}(m, t)$ (3) (with $a, b \in \{+, -\}$) are obtained analogously.

## 4 Random walks

The evolution of the states obtained by selective flipping of the spins governed by the $XX$ Hamiltonian $H_0$ (1) is related to a model of a random turns vicious walkers [2,3]. Indeed, let us consider the following average over the ferromagnetic state vectors $\langle \uparrow |, | \uparrow \rangle$:
\[ F_{j;1}(\lambda) \equiv \langle \uparrow | \sigma^+_j e^{-\lambda H_0} \sigma^-_i | \uparrow \rangle, \tag{11} \]
where $| \uparrow \rangle \equiv \otimes^M_{n=1} | \uparrow \rangle_n$, i.e., all spins are up, and $\lambda$ is an “evolution” parameter. Spin up (or down) corresponds to empty (or occupied) site. Differentiating $F_{j;1}(\lambda)$ (11) and applying the commutator $[H_0, \sigma^+_j]$, we obtain the differential-difference equation (master equation):
\[ \frac{d}{d\lambda} F_{j;1}(\lambda) = \frac{1}{2} \left( F_{j+1;1}(\lambda) + F_{j-1;1}(\lambda) \right). \tag{12} \]

The average $F_{j;1}$ may be considered as the generating function of paths made by a random walker travelling from $t$th to $j$th site. Really, its $K$-th derivative has the form
\[ \frac{d^K}{d\lambda^K} F_{j;1}(\lambda) \bigg|_{\lambda=0} = \langle \uparrow | \sigma^+_j (-H_0)^K \sigma^-_i | \uparrow \rangle = \sum_{n_1, \ldots, n_K} \Delta^{(+)}_{j_{n_{K-1}}} \cdots \Delta^{(+)}_{h_{n_1}} \Delta^{(+)}_{n_1}. \]
A single step to one of the nearest sites is prescribed by the hopping matrix (2) with \( s = + \). After \( K \) steps, each path connecting \( l^{th} \) and \( j^{th} \) sites contributes into the sum. The \( N \)-point correlation function \((N \leq M)\),

\[
F_{j_1,j_2,\ldots,j_N,l_1,l_2,\ldots,l_N}(\lambda) = \langle \hat{\sigma}_{j_1}^+ \sigma_{j_2}^+ \cdots \sigma_{j_N}^+ e^{-\lambda H_0} \sigma_{l_1}^- \sigma_{l_2}^- \cdots \sigma_{l_N}^- | \hat{\psi} \rangle, \tag{13}
\]

enumerates the nests of the lattice paths of \( N \) random turns vicious walkers being initially located at the positions \( l_1 > l_2 > \cdots > l_N \) and, eventually, at \( j_1 > j_2 > \cdots > j_N \). It is expressed in the form [2]:

\[
F_{j_1,\ldots,j_N;l_1,\ldots,l_N}(\lambda) = \det (F_{j_r; l_s}(\lambda))_{1 \leq r, s \leq N}. \tag{14}
\]

The ground and the excited states of the XX chain at \( h = 0 \) with the total spin equal to \((M/2) - N\) are decomposed over a basis of states \( \sigma_{l_1}^- \sigma_{l_2}^- \cdots \sigma_{l_N}^- | \hat{\psi} \rangle \) with \( N \) spins flipped [10]. Therefore, the trace \( \tilde{F}_{m+1,1}(\lambda) \equiv \text{Tr} \left( \sigma_{m+1}^+ e^{-\lambda H_0} \sigma_1^- \right) \) is a linear combination of the generating functions (13) describing the evolution of \( N + 1 \) random turns walkers. The initial and the final positions of one of them are fixed at \( l_1 = 1 \) and \( j_1 = m + 1 \), respectively, while for the rest ones these positions are random. In the thermodynamic limit, the number of the virtual walkers tends to infinity. We apply the procedure described in 3 to calculation of \( \tilde{F}_{m+1,1}(\lambda) \) in the limit when \( M \) and \( N \) are large enough. In this limit, the contribution of the terms with the subindex ‘B’ become, with regard at (8), negligible in (7). We thus obtain:

\[
\begin{align*}
\tilde{F}_{m+1,1}(\lambda) &= \left[ \text{Tr} \left( e^{-\lambda \hat{H}_0} \tilde{e}_{1,m+1} \right) - \frac{d}{d\alpha} \right] \det \left( \hat{I} + \hat{U}_m + \frac{\alpha}{M} \hat{V}_m \right) \bigg|_{\alpha = 0} \\
&= \det \left( \hat{I} + \hat{U}_m \right) \left[ \text{Tr} \left( e^{-\lambda \hat{H}_0} \tilde{e}_{1,m+1} \right) - \frac{1}{M} \text{Tr} \left( \frac{\hat{V}_m}{\hat{I} + \hat{U}_m} \right) \right],
\end{align*}
\]

where \( \tilde{e}_{1,m+1} \equiv (\delta_{1,n} \delta_{m+1,l})_{1 \leq n, l \leq M} \), and the matrix \( \hat{H}_0 \) is used instead of \( \hat{H}^\pm \) since \( s \) can be taken zero at large enough \( M \). The traces of \( \lambda \)-dependent \( M \times M \) matrices \( \hat{U}_m \) and \( \hat{V}_m \) are given below. Differential equation analogous to (12) is fulfilled by \( \tilde{F}_{m+1,1}(\lambda) \). At large separation \( m \) it takes the form:

\[
\frac{d}{d\lambda} \tilde{F}_{m+1,1}(\lambda) = \frac{1}{2} \left( \tilde{F}_{m;1}(\lambda) + \tilde{F}_{m+2;1}(\lambda) \right) - \text{Tr} \left( H_0 \sigma_{m+1}^+ e^{-\lambda H_0} \sigma_1^- \right). \tag{15}
\]

We expand formally \( \tilde{F}_{m+1,1}(\lambda) \) with respect to \( \hat{U}_m \) and obtain the answer in two lowest orders as follows:

\[
\begin{align*}
\tilde{F}_{m+1,1}(\lambda) &\approx F_{m+1,1}(\lambda) + F_{m+1,1}(\lambda) \times \text{tr} \hat{U}_m - \frac{1}{M} \text{tr} \hat{V}_m, \\
\text{tr} \hat{U}_m &= (M - 2m) F_{1;1}(\lambda), \\
\frac{1}{M} \text{tr} \hat{V}_m &= F_{m+1,1}(2\lambda) - 2 \sum_{l=1}^{m} F_{m+1,l}(\lambda) F_{l;1}(\lambda). \tag{16}
\end{align*}
\]

Although \( M \) and \( m \) are chosen to be large in this expansion, the ratio \( m/M \) is kept bounded. In each order the master equation (15) is fulfilled by (16). The contribution of the second order can be re-expressed through the two-point functions \( F_{m+1;l;1,1}(\lambda) \) (see (13), (14)). Thus, summation over intermediate positions (of a virtual walker located at \( l^{th} \) site) arises in the second order. A similar picture is expected in next orders.
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