Higher-spin initial data in twistor space with complex star-\text{gen}values

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Abstract

This paper is a supplement to and extension of arXiv:1903.01399. In the internal twistor space of the 4D Vasiliev’s higher-spin gravity, we study the star-product eigenfunctions of number operators with generic complex eigenvalues. In particular, we focus on a set of eigenfunctions represented by generalized Laguerre functions, which is closed under star-multiplications with creation and annihilation operators to arbitrary complex powers. This set of eigenfunctions can be written as linear combinations of two subsets of eigenfunctions, one of which is closed under the star-multiplication with the creation operator to any arbitrary power – and the other with the annihilation operator. The two subsets intersect when the left and the right eigenvalues differ by an integer. We further briefly discuss some problems that we are facing in order to use these eigenfunctions as the initial data to construct solutions to Vasiliev’s equations.

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1 Introduction

Vasiliev’s equations of higher-spin gravity \[1\,2\] are known as an interacting theory with infinitely many higher-spin fields, which can be seen as a non-linear extension of (Fang)-Fronsdal equations of free higher-spin fields \[3\,4\]. General relativity, which in some sense can be seen as a non-linear extension of the spin-2 Fronsdal equation, has much richer physical content than merely spin-2 particles. Likewise, Vasiliev’s equations are not just about higher-spin particles, and in order to comprehend various aspects of their physical implications, efforts have been made to find and interpret their solutions \[5\,16\].

A frequently used method to solve Vasiliev’s equations, which was proposed in \[5\,20\], is that we can first construct solutions in the absence of ordinary spacetime (or more often referred to as “initial data” of solutions), and then turn on spacetime by doing gauge transformations (see \[9\,21\] for recent review and development).

To illustrate this method, let us look at the Vasiliev’s equations in 4D spacetime for all integers spins, written in their component equations:

\[
\begin{align*}
\partial_\mu U^\mu + U_\mu \ast U^\mu &= 0 , \\
\partial_\mu \Phi + U_\mu \ast \Phi - \Phi \ast \pi (U_\mu) &= 0 , \\
\partial_\mu V_\alpha - \partial_\alpha U_\mu + [U_\mu, V_\alpha] &= 0 \text{ and h.c. } , \\
\partial_\alpha V_{[\beta]} + V_{[\alpha} \ast V_{\beta]} + \frac{i}{4} \varepsilon_{\alpha\beta} \Phi \ast \kappa &= 0 \text{ and h.c. } , \\
\partial_\alpha V_{[\beta]} + V_{[\alpha} \ast \Phi - \Phi \ast \pi (V_\alpha) &= 0 \text{ and h.c. } , \\
\partial_\alpha \bar{V}_\alpha - \partial_\alpha V_\alpha + [V_\alpha, \bar{V}_\alpha] &= 0 ,
\end{align*}
\]

where all the fields can depend on three sets of coordinates: \(x^\mu\) for the ordinary 4D spacetime, \(Y^\alpha\) and \(Z^\alpha\) for the two 4-dimensional internal and external symplectic manifolds (often referred to as “twistor spaces”), which can be decomposed as \(Y^\alpha = (\tilde{y}^\alpha, \bar{y}^\alpha)\) and \(Z^\alpha = (\tilde{z}^\alpha, \bar{z}^\alpha)\) with \((\tilde{y}^\alpha)^\dagger = \bar{y}^\alpha\) and \((\tilde{z}^\alpha)^\dagger = -\bar{z}^\alpha\). The underlined indices are \(\text{Sp}(4,\mathbb{R})\) or \(\text{USp}(2,2)\) indices for the AdS or dS background, and \{\(\alpha, \beta, \ldots\)\} are \(\text{SL}(2,\mathbb{C})\) indices raised or lowered by the Levi-Civita symbols \(\varepsilon_{\alpha\beta}\), \(\varepsilon_{\tilde{\alpha}\tilde{\beta}}\) or \(\varepsilon_{\bar{\alpha}\bar{\beta}}\). The \(\text{SL}(2,\mathbb{C})\) indices that appear in (1.1) refer to those of \(Z\)-coordinates, while the \(Y\)-coordinates are building blocks of symmetry algebra generators, which are implicit in (1.1) and whose indices are contracted with the ones of field components. In the equations, \(\Phi\) is a zero-form, \(U_\mu\) and \(V_\alpha\) (with its hermitian conjugate \(\bar{V}_\alpha\)) are the spacetime and twistor-space components of a one-form field, functioning as gauge fields. The star-product between the fields can be formally defined as

\[
\begin{align*} 
 f_1 (y, \tilde{y}, z, \bar{\tilde{z}}) \ast f_2 (y, \tilde{y}, z, \bar{\tilde{z}}) &= \int d^2 \bar{u} d^2 \bar{v} d^2 \bar{\tilde{u}} d^2 \bar{\tilde{v}} (2\pi)^{-4} e^{i(\nu^a u_a + \bar{\nu}^a \bar{u}_a)} \\
&\cdot f_1 (y + u, \tilde{y} + \bar{u}; z + u, \bar{\tilde{z}} + \bar{\tilde{u}}) f_2 (y + v, \tilde{y} + \bar{v}; z - v, \bar{\tilde{z}} - \bar{\tilde{v}}) .
\end{align*}
\]

Furthermore, \(\kappa\) is the inner Klein operator satisfying

\[
\kappa = \kappa_y \ast \kappa_z , \quad \kappa_y = 2\pi \delta^2(y) , \quad \kappa_z = 2\pi \delta^2(z) ,
\]

(idem. \(\bar{\kappa}\)), where \(\delta^2\) is the two dimensional Dirac delta function, and \(\pi\) and \(\bar{\pi}\) are operations that

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\[\text{In this paper, we only focus on the 4D spacetime, though efforts have also been made to solve the 3D Prokushkin-Vasiliev theory} \[17\,19\].\]
For the bosonic model, the fields should satisfy
\[ \pi \left( x^\mu; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}} \right) = \left( x^\mu; -y^\alpha, \bar{y}^{\dot{\alpha}}; -z^\alpha, \bar{z}^{\dot{\alpha}} \right), \]
\[ \bar{\pi} \left( x^\mu; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}} \right) = \left( x^\mu; y^\alpha, -\bar{y}^{\dot{\alpha}}; z^\alpha, -\bar{z}^{\dot{\alpha}} \right). \]  
(1.4)

For the bosonic model, the fields should satisfy \( \pi (\Phi, U_\mu, V_\alpha) = (\Phi, U_\mu, -V_\alpha) \), so that non-integer-spin degrees of freedom are projected out, and depending on the background being AdS or dS, different reality conditions should be imposed on these fields (see [6,15] for details, which we will skip here).

The simplest solutions to (1.1) are vacuum solutions. As can be easily seen, if we set all fields to zero except the spacetime gauge field \( U_\mu \), then \( U_\mu \) becomes a pure-gauge independent of \( Z \)-coordinates, and thus
\[ U_\mu = L^{-1} \star \partial_\mu L, \]  
(1.5)
with an arbitrary gauge function \( L(x, Y) \) is a solution. Whether such a solution is AdS or dS or something else depends on the expression of \( L \).

To obtain solutions other than the vacua, we can do things in the opposite way: if we trivialize spacetime i.e. set \( U_\mu \) to zero and let \( \Phi \) and \( V \) be independent of \( x^\mu \), then the first three equations in (1.1) are directly solved and thus we can focus on solving the last three equations without involving spacetime (similar to the method of separating variables for solving differential equations). In other words, (1.1) can be solved by using the following ansatz:
\[ U_\mu = G^{-1} \star \partial_\mu G \]
\[ \Phi = G^{-1} \star \Phi' \star \pi(G) \]
\[ V_\alpha = G^{-1} \star V'_\alpha \star G + G^{-1} \star \partial_\alpha G, \]  
(1.6)
where the primed fields are independent of spacetime. After we obtain the solution for the primed fields, we can turn on spacetime by properly doing a gauge transformation\(^2\) with gauge function \( G \) that enables the extraction of Fronsdal fields. We often write \( G = L \star H \), where \( L \) is the vacuum gauge function in (1.5), so that we can do the gauge transformation in two steps, because doing the \( L \)-gauge transformation is much easier than the full gauge transformation \( G \), and with the \( L \)-gauge we can already study some spacetime properties of the solution at the linearized level (see e.g. [8,15,21]). Now by substituting (1.6) into (1.1), the first three equations of (1.1) are directly solved and the last three are converted to the same equations with all the fields primed:
\[ \partial_\alpha V'_\beta + V'_\alpha \star V'_\beta + \frac{i}{4} \epsilon_{\alpha\beta\gamma} \Phi' \star \kappa_\gamma = 0 \quad \text{and h.c.}, \]  
(1.7a)
\[ \partial_\alpha \Phi' + V'_\alpha \star \Phi' - \Phi' \star \bar{\pi}(V'_\alpha) = 0 \quad \text{and h.c.}, \]  
(1.7b)
\[ \partial_\alpha V'_\alpha - \partial_\alpha V'_\alpha + [V'_\alpha, V'_\alpha]_\star = 0. \]  
(1.7c)

The last two equations of (1.7) can be directly solved by the ansatz\(^3\)
\[ \Phi' = \Psi(y, \bar{y}) \star \kappa_y, \]
\[ V'_\alpha = \sum_{n=1}^{\infty} a^{(n)}_{\alpha}(z) \star \Psi(y, \bar{y})^{*n} \quad \text{with} \quad \kappa_z \star a^{(n)}_{\alpha}(z) \star \kappa_z = -a^{(n)}_{\alpha}(z), \]  
(1.8)
\(^2\)Due to the \( \pi, \bar{\pi} \)-automorphisms of the symmetry algebra, two different adjoint representations exist in Vasiliev’s equations, namely the adjoint and the twisted adjoint representations. \( \Phi \) lives in the twisted adjoint representation whose gauge transformation is modified with a \( \pi \) as shown in (1.6).
\(^3\)\( \Psi \) lives in the adjoint representation and \( \Phi' \) lives in the twisted one. The \( \star \kappa_y \) can be used to convert between them. \(^7\) This ansatz was first explicitly used in [8]. See also [9,15] for details.

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3\( \Psi \) lives in the adjoint representation and \( \Phi' \) lives in the twisted one. The \( \star \kappa_y \) can be used to convert between them. \(^7\) This ansatz was first explicitly used in [8]. See also [9,15] for details.
We can make two copies of the above set of creation, annihilation and number operators and let them be equal to the product between the two copies of stargenfunctions i.e.

\[ \Phi = \Phi' = \Phi'' \]

so that \([a^-, a^+] = 1\). The number operator in principle should be defined as \(a^+ \star a^-\), which is equal to \(a^+ a^- - \frac{1}{2}\), but for convenience we use as in [10] the shifted number operator instead (we drop the word “shifted” from now):

\[ w = a^+ a^- . \]

We denote the stargenfunction of the number operator \(f_{\lambda L | \lambda R}(a^+, a^-)\) with \(\lambda_L\) and \(\lambda_R\) being the left and the right stargenvalues and assume that the function depends on the \(Y\)-coordinates only via the creation and annihilation operators, and thus

\[
\begin{align*}
    w \star f_{\lambda L | \lambda R}(a^+, a^-) &= \lambda_L f_{\lambda L | \lambda R}(a^+, a^-), \\
    f_{\lambda L | \lambda R}(a^+, a^-) \star w &= \lambda_R f_{\lambda L | \lambda R}(a^+, a^-).
\end{align*}
\]

We can make two copies of the above set of creation, annihilation and number operators and stargenfunctions (label them with “1” and “2”), such that \(w_2 \pm w_1\) are a pair of Cartan generators\(^6\) of the \((A)dS\) isometry algebra, then a convenient way of constructing the initial data \(\Psi\) or \(\Phi'\), is to let them be equal to the product between the two copies of stargenfunctions i.e.\(^6\)

\[
f_{\lambda_{1L}, \lambda_{2L} | \lambda_{1R}, \lambda_{2R}}(a^1_+, a^1_-, a^2_+, a^2_-) = f_{\lambda_{1L} | \lambda_{1R}}(a^1_+, a^1_-) \star f_{\lambda_{2L} | \lambda_{2R}}(a^2_+, a^2_-). \]

For example, in the case that the energy and angular momentum operators are \(E = \frac{1}{2}(w_2 + w_1)\) and \(J = \frac{1}{2}(w_2 - w_1)\) and the stargenfunctions live in the adjoint representation, if we use diagonal stargenfunctions i.e. set \(\lambda_{1L} = \lambda_{1R} = \lambda_1\) and \(\lambda_{2L} = \lambda_{2R} = \lambda_2\), which leads to vanishing commutators and \(\text{tr}(\lambda_1) = \text{tr}(\lambda_2) = \text{tr}((\lambda_1 \lambda_2)^2) = 0\).

\(^4\)The solution is certain under the assumption that the expression of the gauge function \(G\) can be fixed by demanding a good physical interpretation of the solution. See [21] for a recent discussion.

\(^5\)The generators are selected among \(E, J, iB\) and \(iP\), which are time-translation, rotation, boost and spatial transvection respectively, and for non-compact generators \(B\) and \(P\) the imaginary factor \(i\) has to be multiplied for constructing the Fock spaces.

\(^6\)One can prove that the star-product on the r.h.s. is equal to the ordinary product, due to commutativity between the creation, annihilation and number operators from different copies.
\[ E, f_{\lambda_1,\lambda_2|\lambda_1,\lambda_2}, \quad [J, f_{\lambda_1,\lambda_2|\lambda_1,\lambda_2}], \]
then we expect the solution after switching on spacetime by a
gauge transformation should exhibit the symmetries of the time-translation and the rotation along
a certain spatial axis.

The paper \[10\] only allowed the star-gevvalues \(\lambda\)'s to be half-integers and mainly focused on
diagonal cases, which seemed to be too much constraint later when we studied fields on BTZ-like
backgrounds. It is well-known that the 3D BTZ black hole \[22\] can be obtained from AdS\(_3\)
by compactifying the non-compact spatial direction corresponding to a spatially transvectional
isometry \[23\]. Field fluctuations in the context of Vasiliev's equations on such a 3D background
were studied in \[18\], and in \[16\] we tried to study them in 4D instead with a BTZ-like background
obtained from AdS\(_4\) by the same kind of compactification \[24–26\]. Such compactification leads
to a periodicity condition on the fields, which leads to the requirement in the initial data that
the star-gevfunctions of the spatial transvection generator \(i\mathcal{P}\) should in general be non-diagonal
with complex star-gevvalues, and furthermore, the excitation of the (angular) momentum along the
compactified direction corresponds to adding quantized imaginary numbers to the star-gevvalues in
the initial data.

This motivates us to systematically study the solutions to \[1.11\] in a more general sense with
arbitrary complex left and right star-gevvalues and to investigate how far we can go to organize these
star-gevfunctions into Fock spaces. In Section 2, we present solutions to \[1.11\] in terms of special
functions, which are converted into integral representations in Section 3 for convenience of doing
star-products. Then in Section 4 we focus on a set of star-gevfunctions that can be transformed
from one to another by using creation and annihilation operators to generic complex powers. In
Section 5 we summarize the results of this paper and briefly discuss some problems of using these
star-gevfunctions to further construct valid solutions to Vasiliev's equations.

2 Solve for the star-gevfunctions

In order to solve \[1.11\], we first use the lemmas
\[
\begin{align*}
Y_{\alpha} \ast f(Y) &= Y_{\alpha}f(Y) + i\frac{\partial}{\partial Y_{\alpha}}f(Y), \quad (2.1a) \\
\ast Y_{\alpha} &= Y_{\alpha}f(Y) - i\frac{\partial}{\partial Y_{\alpha}}f(Y) \quad (2.1b)
\end{align*}
\]
and \[1.9\] to convert the star-products with number operators into derivatives w.r.t. the creation
and annihilation operators, which converts \[1.11\] into
\[
\begin{align*}
\left(a^+a^- + \frac{1}{2}a^+\frac{\partial}{\partial a^+} - \frac{1}{2}a^-\frac{\partial}{\partial a^-} - \frac{1}{4}\frac{\partial^2}{\partial a^+\partial a^-}\right)f_{\lambda_L|\lambda_R}(a^+, a^-) &= \lambda_L f_{\lambda_L|\lambda_R}(a^+, a^-), \quad (2.2a) \\
\left(a^+a^- - \frac{1}{2}a^+\frac{\partial}{\partial a^+} + \frac{1}{2}a^-\frac{\partial}{\partial a^-} - \frac{1}{4}\frac{\partial^2}{\partial a^+\partial a^-}\right)f_{\lambda_L|\lambda_R}(a^+, a^-) &= \lambda_R f_{\lambda_L|\lambda_R}(a^+, a^-), \quad (2.2b)
\end{align*}
\]
or equivalently by taking the sum and the difference:
\[
\begin{align*}
\left(a^+\frac{\partial}{\partial a^+} - a^-\frac{\partial}{\partial a^-}\right)f_{\lambda_L|\lambda_R}(a^+, a^-) &= (\lambda_L - \lambda_R) f_{\lambda_L|\lambda_R}(a^+, a^-), \quad (2.3) \\
\left(2a^+a^- - \frac{1}{2}\frac{\partial^2}{\partial a^+\partial a^-}\right)f_{\lambda_L|\lambda_R}(a^+, a^-) &= (\lambda_L + \lambda_R) f_{\lambda_L|\lambda_R}(a^+, a^-). \quad (2.4)
\end{align*}
\]
The solution to (2.3) can be written as

\[ f^+_{\lambda L|\lambda R} (a^+, a^-) = (a^+)^{\lambda_L - \lambda_R} g^+_{\lambda L|\lambda R} (a^+ a^-) \quad \text{or} \quad f^-_{\lambda L|\lambda R} (a^+, a^-) = (a^-)^{\lambda_R - \lambda_L} g^-_{\lambda L|\lambda R} (a^+ a^-). \] (2.5)

Note that the two equations stay the same by exchanging \( a^+ \leftrightarrow a^- \), \( \lambda_L \leftrightarrow \lambda_R \), corresponding to the exchange of the solutions \( f^+ \) and \( f^- \) above. By substituting (2.5) into (2.4) we get

\[ 2w g^{\pm}_{\lambda L|\lambda R} (w) - \frac{1}{2} (\pm \lambda_L + \lambda_R + 1) g^{\pm''}_{\lambda L|\lambda R} (w) - \frac{1}{2} w g^{\pm''}_{\lambda L|\lambda R} (w) = (\lambda_L + \lambda_R) g^{\pm}_{\lambda L|\lambda R} (w). \] (2.6)

For generic choices of the stargenvalues, solutions to (2.6) can be written as

\[ g^{\pm}_{\lambda L|\lambda R} (w) = C_1 e^{-2w} L^{\lambda_L - \lambda_R}_{\lambda_R - \frac{1}{2}} (4w) + C_2 w^{\lambda_R - \lambda_L} e^{-2w} L^{\lambda_R - \lambda_L}_{\lambda_L - \frac{1}{2}} (4w), \]
\[ g^{-}_{\lambda L|\lambda R} (w) = C_1 e^{-2w} L^{\lambda_R - \lambda_L}_{\lambda_L - \frac{1}{2}} (4w) + C_2 w^{\lambda_L - \lambda_R} e^{-2w} L^{\lambda_L - \lambda_R}_{\lambda_R - \frac{1}{2}} (4w), \] (2.7)

where the \( L \) is the generalized Laguerre function, and the \( C \)'s are integration constants. Then we can substitute either one of (2.7) into (2.5), which gives the solution to the original equations

\[ f_{\lambda L|\lambda R} (a^+, a^-) = C_1 (a^+)^{\lambda_L - \lambda_R} e^{-2w} L^{\lambda_L - \lambda_R}_{\lambda_R - \frac{1}{2}} (4w) + C_2 (a^-)^{\lambda_R - \lambda_L} e^{-2w} L^{\lambda_R - \lambda_L}_{\lambda_L - \frac{1}{2}} (4w). \] (2.8)

Because the Tricomi confluent hypergeometric function

\[ U \left( \frac{1}{2} - \lambda_R, 1 + \lambda_L - \lambda_R, 4w \right) = \csc \left[ \pi (\lambda_L - \lambda_R) \right] \left[ -\cos \left( \pi \lambda_L \right) \Gamma \left( \frac{1}{2} + \lambda_R \right) L^{\lambda_L - \lambda_R}_{\lambda_R - \frac{1}{2}} (4w) + (4w)^{\lambda_L - \lambda_R} \cos \left( \pi \lambda_R \right) \Gamma \left( \frac{1}{2} + \lambda_L \right) L^{\lambda_R - \lambda_L}_{\lambda_L - \frac{1}{2}} (4w) \right] \] (2.9)

we can alternatively write (2.8) as

\[ f_{\lambda L|\lambda R} (a^+, a^-) = (a^+)^{\lambda_L - \lambda_R} e^{-2w} \left[ C_3 L^{\lambda_L - \lambda_R}_{\lambda_R - \frac{1}{2}} (4w) + C_4 U \left( \frac{1}{2} - \lambda_R, 1 + \lambda_L - \lambda_R, 4w \right) \right]. \] (2.10)

There exists some subtlety that, for special stargenvalues, the two branches of solutions presented above may degenerate. For example, the two terms in (2.10) degenerate when \( \lambda_R + \frac{1}{2} \in \mathbb{Z} \) because of the factor \( \cos \left( \pi \lambda_R \right) \) in (2.9). For another example, the two terms in (2.8) degenerate when \( \lambda_R - \lambda_L \in \mathbb{Z} \), because

\[ L^{\lambda_L - \lambda_R}_{\lambda_R - \frac{1}{2}} (4w) = \frac{(4w)^{\lambda_R - \lambda_L}}{\left( \frac{1}{2} - \lambda_R \right) L^{\lambda_R - \lambda_L}_{\lambda_L - \frac{1}{2}} (4w)} \] \quad for \( \lambda_R - \lambda_L \in \mathbb{Z} \), \quad (2.11)

where \( (\cdot) \) is the Pochhammer symbol. In this paper, we will be interested only in the stargenfunctions that can be systematically brought from one to another by the creation and annihilation operators, so we will not involve further discussions on the missing branches in the case of degeneracy.

Note that when both stargenvalues \( \lambda_L \) and \( \lambda_R \) are half-integers, (2.8) and (2.10) reduce to the situation discussed in [10], and in [16] we only focused on the situation that either stargenvalue is complex with the other still being a half-integer. This was because we wanted to limit ourselves to the small closed contour integral representation of the stargenfunctions. However, in this paper, we let both stargenvalues be complex numbers in general, and use different integral representations as shown in the next section.
3 Integral representation

To enable star-product computation of special functions, we very often need to represent them by integrals with integrands where \( Y \)-coordinates only appear in exponents, which we shall do in this section. Note that usually the integrals cannot cover the whole parameter space, and the prescription of analytic continuation is needed for the rest of the parameter space after the integrals are done.

We first rewrite the generalized Laguerre function and the Tricomi confluent hypergeometric function in terms of integrals on the real axis

\[
e^{-2w} L_{\lambda_R - \frac{1}{2}}^{\lambda_L - \frac{1}{2}} (4w) = \frac{2^{\lambda_R - \lambda_L} \cos (\pi \lambda_R)}{\pi} \int_{-1}^{1} e^{2ws} \left| s - 1 \right|^{\lambda_L - \frac{1}{2}} \left| s + 1 \right|^{\lambda_R + \frac{1}{2}} ds, \tag{3.1}
\]

\[
e^{-2w} U \left( \frac{1}{2} - \lambda_R, 1 + \lambda_L - \lambda_R, 4w \right) = \frac{2^{\lambda_R - \lambda_L} \Gamma \left( \frac{1}{2} - \lambda_R \right)}{\Gamma \left( \frac{1}{2} - \lambda_L \right)} \int_{-\infty}^{-1} e^{2ws} \left| s - 1 \right|^{\lambda_L - \frac{1}{2}} \left| s + 1 \right|^{\lambda_R + \frac{1}{2}} ds. \tag{3.2}
\]

To be precise, (3.1) holds when \( \text{Re} \left( \lambda_R \right) < \frac{1}{2} \) and \( \text{Re} \left( \lambda_L \right) > -\frac{1}{2} \), and (3.2) holds when \( \text{Re} \left( \lambda_L \right) > -\frac{1}{2} \) and \( \text{Re} \left( w \right) > 0 \).

We can further convert them into contour integrals on the complex \( s \)-plane.

Let us define the contour integral (Figure 1)

\[
\oint_{C_{\infty, -1}} = \int_{-\infty - i\epsilon}^{-1 - i\epsilon} + \int_{-1 + i\epsilon}^{-1 + i\epsilon} + \int_{-1 - i\epsilon}^{\infty + i\epsilon}, \tag{3.3}
\]

where \( \epsilon \) is an infinitesimally small positive number, and \( \int_{-1 - i\epsilon}^{\infty + i\epsilon} \) is a half-circle surrounding the right side of \(-1\) on the complex plane. We adopt the branch cuts that are consistent with

\[
\ln (1 - s) \quad \text{and} \quad \ln (1 + s), \tag{3.4}
\]

where \( \ln \) uses the principal branch of the phase angle \((-\pi, \pi]\) of its parameter. With this convention we can prove that

\[
\int_{-\infty - i\epsilon}^{-1 - i\epsilon} e^{2ws} \frac{(1-s)^{\lambda_L - \frac{1}{2}}}{(1+s)^{\lambda_R + \frac{1}{2}}} ds \xrightarrow{\epsilon \to 0} \int_{-\infty}^{-1} e^{2ws} \frac{|s - 1|^{\lambda_L - \frac{1}{2}}}{|s + 1|^{\lambda_R + \frac{1}{2}}} e^{-i\pi (\lambda_R + \frac{1}{2})} ds, \tag{3.5}
\]

\[
\int_{-1 + i\epsilon}^{\infty + i\epsilon} e^{2ws} \frac{(1-s)^{\lambda_L - \frac{1}{2}}}{(1+s)^{\lambda_R + \frac{1}{2}}} ds \xrightarrow{\epsilon \to 0} \int_{-\infty}^{-1} e^{2ws} \frac{|s - 1|^{\lambda_L - \frac{1}{2}}}{|s + 1|^{\lambda_R + \frac{1}{2}}} e^{-i\pi (\lambda_R + \frac{1}{2})} ds.
\]
Furthermore, for the half-circle integral we can do a change of the variable \( s = \epsilon e^{i\theta} - 1 \), then

\[
\int_{-1-\epsilon i}^{1+\epsilon i} e^{2ws} \frac{(1-s)^{\lambda L - \frac{1}{2}}}{(1+s)^{\lambda R + \frac{1}{2}}} ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2w(\epsilon e^{i\theta} - 1)} \frac{(2-\epsilon e^{i\theta})^{\lambda L - \frac{1}{2}}}{(\epsilon e^{i\theta})^{\lambda R + \frac{1}{2}}} i\epsilon e^{i\theta} d\theta ,
\]

(3.6)

which, by counting the power of \( \epsilon \), obviously vanishes in the limit \( \epsilon \to 0 \) when \( \Re(\lambda R) < \frac{1}{2} \). Therefore, using (3.5) and (3.2) we can derive

\[
\oint_{C_{-\infty,-1}} e^{2ws} \frac{(1-s)^{\lambda L - \frac{1}{2}}}{(1+s)^{\lambda R + \frac{1}{2}}} ds = 2^{1+\lambda_L - \lambda_R} i \cos (\pi \lambda R) \Gamma \left( \frac{1}{2} - \lambda_R \right) e^{-2wU} \left( \frac{1}{2} - \lambda_R, 1 + \lambda_L - \lambda_R, 4w \right) .
\]

(3.7)

Figure 2:

Let us also define the contour integral (Figure 2)

\[
\oint_{C_{-\infty,1}} = \int_{-\infty - \epsilon i}^{1 - \epsilon i} + \int_{1 - \epsilon i}^{1 + \epsilon i} + \int_{-\infty + \epsilon i}^{1 + \epsilon i} ,
\]

(3.8)

where \( \int_{1 - \epsilon i}^{1 + \epsilon i} \) is a half-circle surrounding the right side of 1 on the complex plane. Here we adopt the branch cuts that are consistent with

\[
\ln(s-1) \quad \text{and} \quad \ln(s+1) .
\]

(3.9)

Then we can derive

\[
\int_{-\infty - \epsilon i}^{1 - \epsilon i} e^{2ws} \frac{(s-1)^{\lambda L - \frac{1}{2}}}{(s+1)^{\lambda R + \frac{1}{2}}} ds \to 0 = \int_{-\infty}^{1-\epsilon} e^{2ws} \frac{|s-1|^{\lambda L - \frac{1}{2}} e^{-i\pi(\lambda L - \frac{1}{2})}}{|s+1|^{\lambda R + \frac{1}{2}} e^{-i\pi(\lambda R + \frac{1}{2})}} ds ,
\]

\[
\int_{1 - \epsilon i}^{1 + \epsilon i} e^{2ws} \frac{(s-1)^{\lambda L - \frac{1}{2}}}{(s+1)^{\lambda R + \frac{1}{2}}} ds \to 0 = \int_{1}^{1+\epsilon} e^{2ws} \frac{|s-1|^{\lambda L - \frac{1}{2}} e^{-i\pi(\lambda L - \frac{1}{2})}}{|s+1|^{\lambda R + \frac{1}{2}} e^{-i\pi(\lambda R + \frac{1}{2})}} ds ,
\]

\[
\int_{-\infty + \epsilon i}^{1 + \epsilon i} e^{2ws} \frac{(s-1)^{\lambda L - \frac{1}{2}}}{(s+1)^{\lambda R + \frac{1}{2}}} ds \to 0 = \int_{0}^{1+\epsilon} e^{2ws} \frac{|s-1|^{\lambda L - \frac{1}{2}} e^{i\pi(\lambda L - \frac{1}{2})}}{|s+1|^{\lambda R + \frac{1}{2}} e^{i\pi(\lambda R + \frac{1}{2})}} ds ,
\]

\[
\int_{-\infty + \epsilon i}^{1 + \epsilon i} e^{2ws} \frac{(s-1)^{\lambda L - \frac{1}{2}}}{(s+1)^{\lambda R + \frac{1}{2}}} ds \to 0 = \int_{0}^{1+\epsilon} e^{2ws} \frac{|s-1|^{\lambda L - \frac{1}{2}} e^{i\pi(\lambda L - \frac{1}{2})}}{|s+1|^{\lambda R + \frac{1}{2}} e^{i\pi(\lambda R + \frac{1}{2})}} ds .
\]

(3.10)
Again, we can see with a change of the variable $s = \epsilon e^{i\theta} + 1$ that the half-circle integral

$$
\int_{1-i\epsilon}^{1+i\epsilon} e^{2ws} (s-1)^{\lambda_L - \frac{1}{2}} (s+1)^{\lambda_R + \frac{1}{2}} ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2w(\epsilon e^{i\theta} + 1)} \frac{(\epsilon e^{i\theta})^{\lambda_L - \frac{1}{2}}}{(\epsilon e^{i\theta} + 2)^{\lambda_R + \frac{1}{2}}} i\epsilon e^{i\theta} d\theta
$$

(3.11)

vanishes in the limit $\epsilon \to 0$ when $\text{Re}(\lambda_L) > -\frac{1}{2}$. Therefore, combining (3.10), (3.1) and (3.2) we can derive

$$
\int_{C_{\infty,1}} e^{2ws} (s-1)^{\lambda_L - \frac{1}{2}} (s+1)^{\lambda_R + \frac{1}{2}} ds = 2^{1+\lambda_L-\lambda_R} \pi i \cos (\pi \lambda_L) \sec (\pi \lambda_R) e^{-2wL^{\lambda_L-\lambda_R}} (4w) + 2^{1+\lambda_L-\lambda_R} i \sin [\pi (\lambda_L - \lambda_R)] \Gamma \left(\frac{1}{2} - \lambda_R\right) e^{-2wU} \left(\frac{1}{2} - \lambda_R, 1 + \lambda_L - \lambda_R, 4w\right).
$$

(3.12)

The second term on the r.h.s. is the result of $\int_{-1-i\epsilon}^{-1+i\epsilon} + \int_{1-i\epsilon}^{1+i\epsilon}$, and it vanishes when $\lambda_L - \lambda_R \in \mathbb{Z}$, which can be explained by the cancellation of the branch cuts where $s < -1$ on the real-axis.

To summarize, we can rewrite the solution (2.10) as

$$f_{\lambda_L\lambda_R}(a^+, a^-) = (a^+)^{\lambda_L-\lambda_R} \left[ C_5 \int_{C_{\infty,1}} e^{2ws} (s-1)^{\lambda_L - \frac{1}{2}} (s+1)^{\lambda_R + \frac{1}{2}} ds + C_6 \int_{C_{\infty,-1}} e^{2ws} (s-1)^{\lambda_L - \frac{1}{2}} (s+1)^{\lambda_R + \frac{1}{2}} ds \right],
$$

(3.13)

and it is important to notice that the convergence of the above contour integrals requires only $\text{Re}(w) > 0$ and has no restrictions on $\lambda_{L,R}$. In the diagonal case $\lambda_L = \lambda_R$, due to the cancellation between two branch cuts, $\int_{C_{\infty,1}}$ can be replaced by $\int_{C_{-1,1}}$, which is a contour integral surrounding the section of the real axis between $-1$ and $1$ (Figure 3).

Figure 3:

At the end of this section, let us briefly compare the contours in this paper and the previous ones in [10,16]. The stargenfunctions in [16] (or [10]) are just special cases in Section 2 with one (or two) of the left and right stargenvalues set to half-integers. Therefore, we should be able to reproduce these special cases also from the contour integrals. Take the first term on the r.h.s. of (3.13) for example, if we set $\lambda_L = \lambda_R \in \mathbb{Z} + \frac{1}{2}$, all brunch cuts can be dropped, then the contour reduces to two circles around the points $s = \pm 1$, one of which has zero residue, and thus the integral reproduces stargenfunctions with Laguerre polynomials as in [10]. Take the second term for another example, if we set $\lambda_R + \frac{1}{2} \in \mathbb{Z}^+$, the left-side brunch cut in Figure 1 goes away, then the contour reduces to a circle around $s = -1$, and thus by keeping $\lambda_L \in \mathbb{C}$ with the other brunch cut, the integral reproduces stargenfunctions with generalized Laguerre functions as in [16].

---

7 See [27] for relevant discussions in details.
4 Creation and annihilation operators

We now focus on the two branches of stargenfunctions given by the two terms in (2.8) (call them the “plus” and the “minus” branches). For convenience of star-product computation, we express them in the integral representation (3.1), i.e.

\[ f_{\lambda L|\lambda_R}^+ (a^+, a^-) = (a^+)^{\lambda_L - \lambda_R} \int_{-1}^1 e^{2ws}(1-s)^{\lambda_L - \frac{1}{2}} (1+s)^{\lambda_R + \frac{1}{2}} ds , \tag{4.1a} \]

\[ f_{\lambda L|\lambda_R}^- (a^+, a^-) = (a^-)^{\lambda_R - \lambda_L} \int_{-1}^1 e^{2ws}(1-s)^{\lambda_R - \frac{1}{2}} (1+s)^{\lambda_L + \frac{1}{2}} ds , \tag{4.1b} \]

where the constant overall factors have been dropped for simplicity. Note that the two branches are in general different, although they have the same stargenvalues, and that the two degenerate into one when \( \lambda_R - \lambda_L \in \mathbb{Z} \) as already discussed around (2.11). In this section, we will investigate how these functions (as if they were eigenstates of the number operator in quantum mechanics) can be related by the creation and annihilation operators.

We start from the diagonal case \( \lambda_L = \lambda_R = \lambda \)

\[ f_{\lambda|\lambda} (a^+, a^-) = \int_{-1}^1 e^{2ws}(1-s)^{\lambda - \frac{1}{2}} (1+s)^{\lambda + \frac{1}{2}} ds . \tag{4.2} \]

We would like to show that by doing a left or a right star-product with \( a^\pm \) to a complex power can shift the stargenvalue accordingly, and to this end we first express \( a^\pm \) with the power \( p \) using the Mellin transform (as done in \([16]\))

\[ (a^\pm)^p = \int_0^{+\infty} \frac{e^{-\tau a^\pm}}{\Gamma (-p)} d\tau , \tag{4.3} \]

so that we can pick out the factor \( e^{-\tau a^\pm} \) for the convenience of doing star-products. \(^8\)

Then by using (4.1), (4.3) and the following set of star-product results

\[ e^{-\tau a^+} * e^{2sw} = e^{2sw - \tau (1-s)a^+} , \tag{4.4a} \]

\[ e^{-\tau a^-} * e^{2sw} = e^{2sw - \tau (1+s)a^-} , \tag{4.4b} \]

\[ e^{2sw} * e^{-\tau a^+} = e^{2sw - \tau (1+s)a^+} , \tag{4.4c} \]

\[ e^{2sw} * e^{-\tau a^-} = e^{2sw - \tau (1-s)a^-} , \tag{4.4d} \]

we can easily prove that

\[ f_{\lambda L|\lambda_R}^+ = (a^+)^{\lambda_L - \lambda_R} * f_{\lambda R|\lambda_R}^- = f_{\lambda L|\lambda_R}^- * (a^+)^{\lambda_L - \lambda_R} , \tag{4.5a} \]

\[ f_{\lambda L|\lambda_R}^- = (a^-)^{\lambda_R - \lambda_L} * f_{\lambda R|\lambda_R}^+ = f_{\lambda L|\lambda_R}^+ * (a^-)^{\lambda_R - \lambda_L} . \tag{4.5b} \]

Thus the shift of the left (right) stargenvalue is equal to the power of the creation (annihilation) operator acting from the left (right); or equal to the opposite power if it is an annihilation (creation) operator.

\(^8\)We need the restrictions \( \left| \text{Re} \left( \lambda_{(L,R)} \right) \right| < \frac{1}{2} \), \( \text{Re} (a^\pm) > 0 \) and \( \text{Re} (p) < 0 \) for convergence of the above integrals, but after we have finished the star-product computation and completed the integrals, we obtain back formulas like \( (a^\pm)^p \) and the generalized Laguerre functions, then by analytic continuation we can lift these restrictions.
Another important conclusion from the above computation is that, starting from a diagonal eigenstate, the creation/annihilation operator to a generic complex power always brings it to a state of the plus/minus branch, no matter whether the star-product is done from the left or from the right. Then naturally we have to ask the question: what will happen if both the creation and the annihilation operators act on a diagonal state? The answer is: in general the result is a mixture between the plus and the minus branches.

To illustrate this, let us investigate how
\[(a^+)_{\lambda^+} \star (a^+)_{\lambda^\pm} \tag{4.6}\]
acts on \(f_{\lambda|\lambda}(a^+, a^-)\) by the star-product. By using (4.3) and the lemma
\[e^{-\tau^+ a^+} \star e^{-\tau^\pm a^\pm} = e^{-\tau^+ a^- - \tau^+ a^+ + \frac{1}{2}\tau^+ - \tau^-}, \tag{4.7}\]
we can derive the following result:

\[
\begin{align*}
(a^+)_{\lambda^+} \star (a^+)_{\lambda^\pm} & = \left(-\frac{1}{2}\right)^{\lambda^-} \Gamma(\lambda^- - \lambda^+) \left((a^+)_{\lambda^+ - \lambda^-} - \lambda^+\right)_1 F_1 \left(-\lambda^-; 1 - \lambda^- + \lambda^+, \mp 2w\right) \\
& + \left(-\frac{1}{2}\right)^{\lambda^+} \Gamma(\lambda^+ - \lambda^-) \left((a^-)_{\lambda^- - \lambda^+} - \lambda^-\right)_1 F_1 \left(-\lambda^-; 1 - \lambda^- + \lambda^-, \mp 2w\right). \tag{4.8}
\end{align*}
\]

As can be seen, \((a^+)_{\lambda^+} \star (a^+)_{\lambda^\pm}\) splits into two terms, one of which has a factor \((a^+)_{\lambda^+ - \lambda^-}\) and the other has \((a^-)_{\lambda^- - \lambda^+}.\) In the next paragraph, we will show that these two terms acting on \(f_{\lambda|\lambda}\) lead to the plus and the minus branches respectively. Thus, generally speaking, the result is a linear combination of the two branches. However, note that special cases exist. For example, if either \(\lambda^+\) or \(\lambda^-\) is a non-negative integer, one of the \(\Gamma\)-functions on the denominators blows up, then only the minus or the plus branch survives; and if both \(\lambda^+\) and \(\lambda^-\) are non-negative integers, numerators and denominators may both blow up, then (4.8) should be computed by the limit as \(\lambda^\pm\) approaching their values, which results in an ordinary polynomial of \(a^\pm\), and the resulting state remains degenerate between the plus and the minus branches.

Now we further massage the two terms in (4.8) to confirm their branches. We first express the hypergeometric function \(_1 F_1\) in its integral representation \([9]\) and then use (4.3) and (4.4) to convert the ordinary products between the \(a^\pm\)-factor and the \(w\)-factor into star-products, which gives:

\[
\begin{align*}
(a^-)_{\lambda^-} \star (a^+)_{\lambda^+} & = \left(-\frac{1}{2}\right)^{\lambda^-} \frac{\sin (\pi \lambda^+)}{\Gamma(-\lambda^-) \sin \left(\pi (\lambda^+ - \lambda^-)\right)} \int_{-1}^{1} ds \frac{(1 - s)_{\lambda^-}}{(1 + s)_{\lambda^+ + 1}} e^{-(1 + s)w} \star (a^+)_{\lambda^+ - \lambda^-} \\
& + \left(-\frac{1}{2}\right)^{\lambda^+} \frac{\sin (\pi \lambda^-)}{\Gamma(-\lambda^+) \sin \left(\pi (\lambda^- - \lambda^+)\right)} \int_{-1}^{1} ds \frac{(1 - s)_{\lambda^+}}{(1 + s)_{\lambda^- + 1}} (a^-)_{\lambda^- - \lambda^+} \star e^{-(1 + s)w}, \tag{4.9}
\end{align*}
\]

\([9]\) The result is obtained by integrating \(\int_0^{+\infty} \int_0^{+\infty} \frac{e^{(\nu)}_{\lambda^+ - 1} (\nu - \lambda^- - 1)}{\Gamma(-\lambda^-)} e^{-\nu a^- - \nu a^+ + e^{\nu} \tau^+ d\tau^+ d\tau^-}\) with \(\text{Re}(\lambda^\pm) < 0, \text{Re}(a^\pm) > 0, \text{Re}(c) < 0,\) then doing analytic continuation and setting \(c = \pm \frac{1}{2}\).

\([10]\) \(_1 F_1\) \((-\nu, \lambda + 1, z) = \frac{\Gamma(\lambda + 1) \Gamma(\nu + 1)}{\Gamma(\lambda + \nu + 1)} L^\nu_{\lambda}(z)\), and thus (4.1) can be used here.
and
\[
(a^+)^{\lambda^+} \star (a^-)^{\lambda^-} = \left(\frac{1}{2}\right)^{\lambda^-} \sin(\pi \lambda^-) \int_{-1}^1 ds \frac{(1 - s)^{\lambda^-}}{(1 + s)^{\lambda^+ + 1}} (a^+)^{\lambda^+ - \lambda^-} \star e^{(1+s)\text{w}}
\]
\[
+ \left(\frac{1}{2}\right)^{\lambda^+} \sin(\pi \lambda^+) \int_{-1}^1 ds \frac{(1 - s)^{\lambda^+}}{(1 + s)^{\lambda^- + 1}} e^{(1+s)\text{w}} \star (a^-)^{\lambda^- - \lambda^+},
\]
where the two terms in (4.9) and (4.10) are derived from the two terms in (4.8), respectively. Then by using the lemma
\[
e_c^{\text{w}} = \text{sech} \left(\frac{c}{2}\right) e^{2\tanh\left(\frac{c}{2}\right)\text{w}},
\]
or equivalently
\[
e^{\mp (1+s)\text{w}} = \frac{2}{\sqrt{(1 - s)(3 + s)}} e^{\mp 2\arctanh\left(\frac{1+s}{2}\right)\text{w}},
\]
we can further replace all the ordinary exponential functions of \(w\) with the star-exponential ones. Obviously, star-exponential functions of \(w\) acting on its eigenstate do not change the state, and \(a^+\) and \(a^-\) with generic complex powers acting on \(f_{\lambda|\lambda}\) lead to the plus and minus branches, respectively. In this way, it is clear that, with generic \(\lambda^\pm\), the first and the second term in (4.9), (4.10) and hence in (4.8) acting on \(f_{\lambda|\lambda}\) produce the plus and minus branches, respectively.

Finally, let us summarize that: (1) All the eigenstates of the types \(f^+\) and \(f^-\) (including the degenerate ones) are related by the creation and annihilation operators, respectively, as shown in (4.5); (2) Any number of factors of \(a^+\) and \(a^-\) to arbitrary powers star-multiplied with \(f_{\lambda|\lambda}\) leads to a linear combination of the types \(f^+\) and \(f^-\) thus in other words, the set of stargenfunctions of the types \(f^+, f^-\) and their linear combinations is closed under the star-multiplications with \(a^\pm\) to arbitrary powers.

## 5 Conclusion and discussion

In this paper, motivated by the construction of initial data in the holomorphic gauge to solve Vasiliev’s equations, we have studied the star-product stargenfunctions of the number operator, as defined in (1.11) with generic complex stargenvalues. We have re-written these stargenfunctions using integral representations and also expressed the creation and annihilation operators to complex powers by integrals via the Mellin transform, so that their star-products can be computed conveniently. We have in particular picked out a set of stargenfunctions (2.8) represented by generalized Laguerre functions, which has been shown to be closed under the star-multiplication with creation and annihilation operators to arbitrary complex powers. This set of stargenfunctions can be written as linear combinations of two subsets of stargenfunctions (the “plus” and “minus” branches), except that the two subsets intersect when left and the right stargenvalues differ by an integer. The whole plus (minus) branch can be related by – and is closed under – the star-multiplication with creation (annihilation) operators to generic complex powers.

\[\text{If factors of the type } (a^\pm)^{\lambda^\pm} \text{ appear on two different sides of } f_{\lambda|\lambda}, \text{ we can move them onto the same side by using (4.5), and if many such factors appear on the same side, we can always exploit the above computation for } (a^\pm)^{\lambda^\pm} \star (a^\pm)^{\lambda^\pm} \text{ to reduce them one by one, until only one factor of the type } (a^\pm)^{\lambda^\pm} \text{ remains in each branch.}\]
However, we are not yet able to use the above results to construct valid solutions to Vasiliev’s equations. One of the difficulties happens when we go beyond linear combinations of the stargenfunctions. As in [1.8] the twistor space gauge field $V'$ is expressed in terms of star-product series of the initial-data field $\Psi$. Thus, for example if we use the stargenfunctions $f_{\lambda R}^{\pm}$ above to construct $\Psi$, we will encounter star-multiplications to all orders between these stargenfunctions, and we have to make sure that such star-multiplications make sense. Let us take a look at the diagonal stargenfunctions to illustrate the problem. When $\lambda$ is a half-integer, according to [10], $f_{\lambda R}$ should act like a projector: $f_{\lambda R} \neq f_{\lambda R} \neq f_{\lambda R}$. Therefore, naively we expect the same thing should happen for generic complex $\lambda$, but it does not seem to be the case. If we multiply the expression (4.2) by itself, by using the lemma

$$e^{2sw} \ast e^{2'sw} = \frac{1}{1 + ss'} e^{2 + s'sw},$$

we get

$$f_{\lambda R} \ast f_{\lambda R} = \int_{-1}^{1} ds \int_{-1}^{1} ds'$$

$$e^{2sw} \ast e^{2'sw}(1 - s - \frac{1}{2}) (1 - s') \frac{1}{1 + ss'} e^{2 + s'sw}$$

$$= \int_{-1}^{1} ds \frac{1}{(1 + s)(1 - s)}$$

$$\int_{-1}^{1} du \frac{1}{(1 + u)} e^{2uw}$$

$$= f_{\lambda R} \int_{-1}^{1} ds \frac{1}{(1 + s)(1 - s)}.$$

where in the second last step the integration variable $s'$ has been changed by $s' = \frac{s - u}{ss'}$ or equivalently $u = \frac{s - s'}{ss}$. In the last line of (5.2), we obviously have a problem as the integral diverges. In the paper [10], when $\lambda$ is a half-integer, a prescription has been given based on contour integrals. In that case, we can replace $f_{\lambda R}$ with the contour integral around either $s = 1$ or $s = -1$ (the contour and the real-axis integrals can produce the same Laguerre polynomial from the beginning), thus in (5.2) we no longer need to worry about the divergence. However, when $\lambda$ is a generic complex number, we have to use the contour $C_{-1,1}$ as shown in Figure 3 instead, and consequently we end up with $f_{C_{-1,1}} ds \frac{1}{(s+1)(s-1)}$, which has two residues cancelled with each other. Thus with this kind of prescription we have $f_{\lambda R} \ast f_{\lambda R} = 0$. It is not yet certain to us how this should be interpreted – perhaps we should for consistency exclude half-integer stargenvalues when we use such stargenfunctions to construct the initial data, or perhaps we should find a better prescription that we are not yet aware of.

Another complication is the star-product with the inner Klein operators. As introduced around (1.12), to construct the initial data, we need to make two copies of the stargenfunctions. For each copy, there are two branches $(+, -)$, then there are in total four branches $(+, +, -, -)$ in the initial data. An stargenfunction of any particular branch star-multiplied by $\kappa_y$ or $\bar{\kappa}_y$ in general leads to a linear combination of all four branches. Thus the computation is very much involved, and we are not yet sure whether such $\ast \kappa_y$ and $\ast \bar{\kappa}_y$ are always well-defined, or whether additional conditions on the stargenvalues have to be imposed for consistency. We will continue these investigations in our future work.

---

Note that here the stargenvalues have continuous spectra. A naive guess is that, unlike the discrete case for half-integers, perhaps here with some prescriptions the star-products between stargenfunctions should lead to distributions like Dirac delta functions instead of Kronecker deltas, and thus infinities and zeros could be somewhat expected. Furthermore, note that in this paper we have not yet discussed stargenfunctions that are missing in the case of degeneracy, or those corresponding to distributions in $u$. There might be a chance that an stargenfunction that has additionally summed these missing components could be normalized more easily.
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