Control of robotic yoyo with energy compensation based on an integrated model of a robot and a yoyo

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Abstract. This paper presents an approach to control a periodic up-and-down yoyo motion with a robot arm which has a parallel link mechanism. For controlling it, an integrated model of the robot arm and the yoyo is developed which describes their mutual interference, and a controller which can compensate energy loss with the yoyo motion is designed based on the integrated model. The effectiveness of the proposed controller is verified through a numerical simulation. Its results show that the robot arm attains the continuous up-and-down yoyo motion and the trajectories of the rotation and the up-and-down motion are periodic. The proposed approach is effective in controlling the periodic up-and-down yoyo motion in conclusion.

Keywords: Robotic yoyo, Rhythmic task, Energy-based control, Modeling, Numerical simulation

1. Introduction

Robots have participated actively in a workplace such as a factory. They will coexist in a human living environment and work on behalf of and with humans in the future, and hence will be required to perform human skilled tasks. Throwing motion control for a manipulator [1, 2, 3] and a robotic flute performance [4] are examples of studies for robots with human skilled tasks.

A rhythmic task with a tool is one of the human skilled tasks. An example of the task is juggling [5, 6, 7] or playing with different toys [8, 9], like a yoyo [10, 11]. Since it requires that a robot motion is synchronized with complex behavior of a tool, it is easy for a human, but a complex task for a robot due to difference of motility and sensors.

Playing a yoyo is discussed as the rhythmic task. The yoyo is a toy made of two discs connected with a thin short axle and a string is tied to the axle. Although there are various yoyo operations like throwing it horizontally, an elementary operation is to attain a periodic up-and-down motion of the yoyo, which is attained by moving the string up and down. The
operation of the periodic up-and-down yoyo motion is called “gravity-pull” and requires the synchronization with the yoyo of an operator.

In the gravity-pull, the yoyo free-falls with converting its kinetic energy to potential energy and bounds at the bottom position. Some of its energy is lost due to friction between the yoyo discs and the string and bottom impact. Compensation for its energy loss, which should be completed effectively by dexterously synchronizing the operator with the yoyo, is hence required for attaining the periodic up-and-down motion.

This paper proposes a controller for the robotic yoyo operation by energy compensation based on a model which describes dynamics of both a robot and the yoyo and their mutual interference for dexterous synchronization. Our goal is to achieve the continuous gravity-pull with a robot.

Robotic yoyo has been investigated in literatures, e.g., [10, 12, 13, 14, 15, 16]. Jin has modeled complex yoyo dynamics by separating the yoyo motion into four phases and realized the robotic gravity-pull by designing a controller for a robot arm based on the simplified yoyo model [12, 13]. Hashimoto [10] and Žlajpah [14, 15] have also achieved it with a simplified model and visual feedback control. Mombaur has reported the gravity-pull with the biped robot HRP-2 [16]. In these studies, the robots control the up-and-down yoyo motion by detecting a yoyo height with a vision sensor and moving its hand depending on the yoyo height. Although these cases have adopted a hand height or acceleration of the robots as a target value for control, this paper adopts compensation for the energy loss with the yoyo motion as a control target. It enables control input to be determined without depending on positions of a hand and a yoyo. It is hence capable of controlling challenging yoyo motions which cannot be operated depending on yoyo positions and are performed without visual feedback. This paper, moreover, proposes an integrated model of a robot and the yoyo to consider robot dynamics, which is not regarded in the previous studies. The integrated model describes mutual interference between a robot and the yoyo and enables the dexterous synchronization. Besides, an integrated model of a robot and a tool is effective in model-based development of a robot through numerical simulation.

To achieve the gravity-pull, consideration of the mutual interference between a robot and the yoyo and controller design based on their dynamics are required. For such issues, an integrated model of a robot arm and a yoyo is developed in our previous studies [17, 18, 19]. The gravity-pull, moreover, has been achieved on simulations by designing a controller which can compensate the energy loss with its motion, based only on a yoyo model [20, 21]. In this paper, a robot arm which has a parallel link mechanism (called parallel link arm) is applied as a robot operating the yoyo, and an integrated model of the parallel link arm and the yoyo is developed with the above methods. A controller which allows the arm to compensate the energy loss by raising the yoyo is designed by extending the previous controller based on the integrated model.

In the paper, first, the integrated model of the parallel link arm and the yoyo is developed by integrating models of the arm and the yoyo as subsystems after each development of their model in order to describe their mutual interference. Next, the controller which can compensate the energy loss with the yoyo motion is designed based on the integrated model. The controller generates joint torque of the parallel link arm as control input based on their dynamics. The effectiveness of the proposed controller for the continuous gravity-pull is
verified through a numerical simulation.

2. Modeling the integrated system of the arm and the yoyo

The integrated model of the parallel link arm and the yoyo is developed to consider their mutual interference. Motion equations of models are derived by applying the projection method [22, 23, 24]. Assuming that a system is constituted of independent parts and they are connected by constraint conditions, the projection method yields a whole motion equation from their independent motion equations and their constraint conditions. The integrated motion equation is derived by integrating the parallel link arm and the yoyo as subsystems after each derivation of their motion equations [17, 18, 19]. This method has the advantage that such local systems as the arm and the yoyo can be modified easily.

2.1. The parallel link arm model

The motion equation of the parallel link arm shown in Fig. 1 is derived [19]. It has two counter weights to balance by adjusting their positions.

![Figure 1: Parallel link arm. The parallel link arm has two counter weights to balance by adjusting their positions.](image)

The model diagram of the parallel link arm is shown in Fig. 2, the physical parameters are shown in Table 1, and the variables are shown in Table 2. The rotational angles of the link 2-4, the weights and the hand are a relative angle for the link 1.

An unconstrained motion equation constituted of motion equations of independent parts is written. Generalized coordinates $\mathbf{x}_a$ are defined as

$$\mathbf{x}_a = \begin{bmatrix} \theta_{l1}, \theta_{l4}, \theta_{w1}, \theta_{w2}, \theta_p, x_{l1}, y_{l1}, x_{l4}, y_{l4}, x_{w1}, y_{w1}, x_{w2}, y_{w2}, x_p, y_p \end{bmatrix}^T. \quad (1)$$

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Figure 2: Model diagram of the parallel link arm. The rotational angles of the link 2-4, the weights and the hand are a relative angle for the link 1.

Table 1: Physical parameters of the parallel link arm (i = 1, · · · , 4, j = 1, 2)

| Parameter                                           | Symbol |
|-----------------------------------------------------|--------|
| Mass of links (kg)                                  | mi     |
| Mass of weights (kg)                                | mw,j   |
| Mass of hand (kg)                                   | mp     |
| Inertia moment of links (kgm²)                      | li     |
| Inertia moment of weights (kgm²)                    | lw,j   |
| Inertia moment of hand (kgm²)                       | lp     |
| Viscosity of links (Nms/rad)                        | ci     |
| Length of links (m)                                 | li     |
| Length from joint to center of gravity of link (m)  | lgil   |
| Length from joint to weight (m)                     | lgwj   |
| Length from joint to center of gravity of hand (m)  | lgp    |
| Gravity acceleration (m/s²)                         | g      |

Table 2: Variables of the parallel link arm (i = 1, · · · , 4, j = 1, 2)

| Parameter                                      | Symbol |
|------------------------------------------------|--------|
| Center of gravity coordinates of links (m)     | (xli, yli) |
| Center of gravity coordinates of weights (m)    | (xwj,j, ywj,j) |
| Center of gravity coordinates of hand (m)       | (xp, yp) |
| Rotational angle of links (rad)                 | θli    |
| Rotational angle of weights (rad)               | θwj,j  |
| Rotational angle of hand (rad)                  | θp     |
| Joint torque of link 1                          | τl1    |
A generalized mass matrix $M_a$ and a generalized force vector $h_a$ are given by

$$M_a = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix},$$

(2)

$$M_{11} := \begin{bmatrix} \sum_{i=1}^4 l_i^2 + \sum_{j=1}^2 l_{wj} + I_p \\ \sum_{i=1}^4 l_i^2 + l_{w2} + I_p \\ \sum_{i=1}^4 l_i^2 + l_{w2} + I_p \\ \sum_{i=1}^4 l_i^2 + l_{w2} + I_p \\ \sum_{i=1}^4 l_i^2 + l_{w2} + I_p \\ \sum_{i=1}^4 l_i^2 + l_{w2} + I_p \end{bmatrix} \begin{bmatrix} I_{4} \\ I_{4} \\ I_{4} \\ I_{4} \\ I_{4} \\ I_{4} \end{bmatrix} \begin{bmatrix} l_4 \\ I_{w1} \\ I_{w2} \\ I_{w2} \\ I_{w2} \\ I_p \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$M_{22} := \text{diag}\left(m_{11}, m_{12}, \cdots, m_{14}, m_{21}, m_{22}, m_{23}, m_{24}, m_{31}, m_{32}, m_{33}, m_{34}, m_{41}, m_{42}, m_{43}, m_{44}\right),$$

$$h_a = \begin{bmatrix} \tau_{11} - c_{11}\dot{\theta}_{11}, -c_{12}\dot{\theta}_{12}, -c_{13}\dot{\theta}_{13}, -c_{14}\dot{\theta}_{14}, 0, 0, 0, \\ 0, 0, 0, -m_{14}\dot{g}, 0, -m_{21}\dot{g}, 0, -m_{22}\dot{g}, 0 - m_p\dot{g} \end{bmatrix}^T.$$ (3)

The unconstrained motion equation is represented by $M_a\ddot{x}_a = h_a$.

The projection method leads a constrained motion equation by considering conditions which constrain system behavior including definitions of positional relationships between each part. The constraint conditions of the parallel link arm, which are definitions of positional and angular relationships between the links, the weights and the hand, are given by

$$\begin{align*}
\theta_{12} &= \pi/2 - \theta_{11}, \\
\theta_{13} &= \pi/2 + \theta_{11}, \\
\theta_{w1} &= \pi, \\
\theta_p &= -\pi/2, \\
x_{11} &= l_{g1}\cos \theta_{11}, \\
y_{11} &= l_{g1}\sin \theta_{11}, \\
x_{12} &= l_{11}\cos \theta_{11}, \\
y_{12} &= l_{11}\sin \theta_{11} + l_{g2}, \\
x_{13} &= (l_{11} - l_{g3})\cos \theta_{11}, \\
y_{13} &= (l_{11} - l_{g3})\sin \theta_{11} + l_{12}, \\
x_{14} &= (l_{11} - l_{13})\cos \theta_{11}, \\
y_{14} &= (l_{11} - l_{13})\sin \theta_{11} + l_{12} - l_{g4}, \\
x_{w1} &= -l_{gw1}\cos \theta_{11}, \\
y_{w1} &= -l_{gw1}\sin \theta_{11}, \\
x_{w2} &= -(l_{gw2} + l_{12} - l_{11})\cos \theta_{11}, \\
y_{w2} &= -(l_{gw2} + l_{12} - l_{11})\sin \theta_{11} + l_{12}, \\
x_{p} &= l_{11}\cos \theta_{11} + l_{g3}, \\
y_{p} &= l_{11}\sin \theta_{11} + l_{g3}. 
\end{align*}$$

(4)

A constraint matrix $C_a$ which should satisfy $C_a\ddot{x}_a = 0$ is obtained from the constraint conditions. They are combined as a constraint equation $\Phi_a = 0$ by moving the right member of each equation to the other side. The constraint matrix $C_a$ is thus represented by

$$C_a = \frac{\partial \Phi_a}{\partial \dot{x}_a}.$$ (5)

Applying the constraint matrix $C_a$ and Lagrange’s undetermined multipliers $\lambda_a$ yields a constrained system

$$M_a\ddot{x}_a = h_a + C_a^T\lambda_a.$$ (6)
Since (6) has redundant degrees of freedom, they are reduced.

An independent velocity under the constrained state $\dot{q}_a$ which is selected from $\dot{x}_a$ is defined as

$$\dot{q}_a = \dot{\theta}_1.$$  \hfill (7)

Given that $\dot{x}_a = [\dot{q}_a, \dot{v}_a]^T$, the constraint matrix $C_a$ can be represented by $C_a = [C_{a1}, C_{a2}]$ to satisfy $C_a \dot{x}_a = C_{a1} \dot{q}_a + C_{a2} \dot{v}_a$. From this relationship, an orthogonal matrix $D_a$ can be obtained so as to be $C_a D_a = 0$ and $\dot{x}_a = D_a \dot{q}_a$. Since $C_a \dot{x}_a = C_{a1} \dot{q}_a + C_{a2} \dot{v}_a = 0$ yields $v_a = -C_{a2}^{-1} C_{a1} \dot{q}_a$, the orthogonal matrix $D_a$ is obtained from $\dot{x}_a = [\dot{q}_a, v_a^T]^T = D_a \dot{q}_a$ as

$$D_a = \begin{bmatrix} 1 \\ -C_{a2}^{-1} C_{a1} \end{bmatrix}.$$  \hfill (8)

Besides, (8) satisfies

$$C_a D_a = C_{a1} - C_{a2} C_{a1}^{-1} C_{a1} = 0.$$

The constrained motion equation is derived by projecting the constrained system (6) on the space constrained by $D_a^T$ and transforming the coordinates of the component vectors. The motion equation of the parallel link arm is thereby derived as

$$D_a^T M_a D_a \ddot{q}_a + D_a^T M_a D_a \dot{q}_a = D_a^T h_a.$$  \hfill (9)

### 2.2. The yoyo model

The motion equation of the yoyo shown in Fig. 3 is derived [25]. The yoyo is painted white to estimate its center of gravity by image processing.

![Yoyo](image)

Figure 3: Yoyo. The yoyo is painted white to estimate its center of gravity by image processing.

The motion equation of the yoyo is derived based on following assumptions.

Assumption 1. The yoyo moves in a vertical two dimensional surface. The rotational axis is always perpendicular to the vertical two dimensional surface.

Assumption 2. The string is always perpendicular to the radius of the yoyo axle while the yoyo is moving up and down. The yoyo axle rotates by $\pi$ rad into another side of the string only when the whole string is unwound.

Assumption 3. The string is flexible and extensible.
Assumption 4. Diameter of the string is considered. Viscous friction for the winding string is proportional to the radius of the axle and the rotational velocity of the yoyo.

According to the assumptions 1 - 4, the model diagram of the yoyo is shown in Fig. 4, the physical parameters are shown in Table 3, and the variables are shown in Table 4. The connection point is a virtual mass point defined for considering flexibility and extensibility of the string. The translational distance from the hand to the point \( r_c \) is a relative distance for the hand. The rotational angle of the yoyo \( \theta_y \) is a relative angle for the string and should be 0 when it is at the bottom position.

![Model diagram of the yoyo](image)

**Figure 4: Model diagram of the yoyo.** Its motion is separated into the up-and-down motion phase (a) and (c) and the transition phase (b) depending on its rotational angle. The connection point is a virtual mass point defined for considering flexibility and extensibility of the string. The translational distance from the hand to the point \( r_c \) is a relative distance for the hand. The rotational angle of the yoyo \( \theta_y \) is a relative angle for the string and should be 0 when it is at the bottom position.

| Physical parameters of the yoyo | Description |
|--------------------------------|-------------|
| Mass of yoyo (kg)              | \( m_y \)   |
| Mass of hand (kg)              | \( m_h \)   |
| Mass of connection point (kg)  | \( m_c \)   |
| Inertia moment of yoyo (kgm\(^2\)) | \( I_y \) |
| Inertia moment of string (kgm\(^2\)) | \( I_s \) |
| Viscosity coefficient of yoyo (Ns/rad) | \( c_y \) |
| Viscosity of string (Nms/rad)  | \( c_s \)   |
| Damper coefficient of string (Ns/m) | \( d_c \) |
| Spring coefficient of string (N/m) | \( k_c \) |
| Maximum length of string (m)   | \( l_0 \)   |
| Radius of yoyo axle (m)        | \( r_0 \)   |
| Effective diameter of string (m/rad) | \( k_{ds} \) |

The first assumption enable the yoyo to rotate and move vertically and horizontally. The swinging of the yoyo is considered by defining the angle of the string for the vertical line \( \theta_s \).
Table 4: Variables of the yoyo

| Variable | Symbol |
|----------|--------|
| Center of gravity coordinates of yoyo (m) | \((x_y, y_y)\) |
| Center of gravity coordinates of hand (m) | \((x_h, y_h)\) |
| Rotational angle of yoyo (rad) | \(\theta_y\) |
| Rotational angle of string (rad) | \(\theta_s\) |
| Translational distance from hand to connection point (m) | \(r_c\) |

However, the yawing and pitching are neglected because these motions don’t occur when it rotates at high speed.

The assumption 2 allows us to separate the yoyo motion into the up-and-down motion phase, when the yoyo moves up and down, and the transition phase, when the yoyo axle rotates by \(\pi\) rad into another side of the string. The range of the up-and-down motion phase is \(|\theta_y| \geq \pi/2\), and the range of the transition phase is \(|\theta_y| < \pi/2\) as shown in Fig. 4. The positional relationship between the string and the yoyo axle is switched via the transition phase in \(\theta_y \leq -\pi/2\) and \(\theta_y \geq \pi/2\). The up motion and the down motion are switched when the yoyo reaches the top position and the bottom position. The yoyo state transition is shown in Fig. 5.

![Figure 5: Yoyo state transition. The yoyo repeats up-and-down motion until an operator catches it. The positional relationship between the string and the yoyo axle is switched via the transition phase. The up motion and the down motion are switched when the yoyo reaches the top position and the bottom position.](image)

According to the assumptions 3, a string tension is represented by a tension which is generated by a spring-damper model between the connection point and the hand while the translational distance \(r_c\) exceeds any value. It, however, is represented by a tension which is generated by a spring model while the string goes slack (\(\dot{r}_c < 0\)) because the string doesn’t have resistance to velocity. The tensionless string is represented by zeroing the tension between the connection point and the hand while the hand is closer to the yoyo than the connection point.

The assumption 4 allows the radius of the axle with the wound string to increase in proportion to the rotational angle of the yoyo. The changes of the axle radius and the string length are represented by setting the effective diameter of the string \(k_{ds}\) since the axle radius doesn’t change before increasing the string layer. Here friction for rotation of the yoyo

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The unconstrained motion equation is represented by assumption 4 by
\[
\dot{x}_y = \begin{bmatrix} \dot{\theta}_y, \dot{\theta}_s, \ddot{x}_c, \ddot{y}_c, x_h, y_h, x_y, y_y \end{bmatrix}^T. \tag{10}
\]
A generalized mass matrix \(\bar{M}_y\) and a generalized force vector \(\bar{h}_y\) are given by
\[
\bar{M}_y = \text{diag}(I_y, I_s, m_c, m_c, m_h, m_h, m_y, m_y), \tag{11}
\]
\[
\bar{h}_y = \begin{bmatrix}
-R_y(\dot{\theta}_y - \dot{\theta}_s) c_s(\dot{\theta}_y - \dot{\theta}_s), -c_s \dot{\theta}_s + R_y(\dot{\theta}_y - \dot{\theta}_s) c_s(\dot{\theta}_y - \dot{\theta}_s), -F_{kd}(r_c, \dot{r}_c) \sin \theta_s, \\
F_{kd}(r_c, \dot{r}_c) \cos \theta_s - m_y g, F_{kd}(r_c, \dot{r}_c) \sin \theta_s, -F_{kd}(r_c, \dot{r}_c) \cos \theta_s - m_h g, \\
0, -m_y g 
\end{bmatrix}^T, \tag{12}
\]
where the force for the string tension \(F_{kd}(r_c, \dot{r}_c)\) is given from the assumption 3 by
\[
F_{kd}(r_c, \dot{r}_c) = \begin{cases} 
 kc r_c + d_c \dot{r}_c & (r_c \geq 0, \dot{r}_c \geq 0) \\
 kc r_c & (r_c \geq 0, \dot{r}_c < 0) \\
 0 & (r_c < 0)
\end{cases}, \tag{13}
\]
and the radius of the axle \(R_y(\theta_y)\) and the length of the string \(L_s(\theta_y)\) are also given from the assumption 4 by
\[
R_y(\theta_y) = \begin{cases} 
 r_0 + k_d s \left( |\theta_y| - \frac{\pi}{2} \right) & \left( |\theta_y| \geq \frac{\pi}{2} \right) \\
 r_0 & \left( |\theta_y| < \frac{\pi}{2} \right)
\end{cases}, \tag{14}
\]
\[
L_s(\theta_y) = \begin{cases} 
 l_0 - \left( r_0 \left( |\theta_y| - \frac{\pi}{2} \right) + \frac{d_c}{k_d} \left( |\theta_y| - \frac{\pi}{2} \right)^2 \right) & \left( |\theta_y| \geq \frac{\pi}{2} \right) \\
 l_0 & \left( |\theta_y| < \frac{\pi}{2} \right)
\end{cases}. \tag{15}
\]
The unconstrained motion equation is represented by \(\bar{M}_y \ddot{x}_y = \bar{h}_y\). Here \((\ddot{x}_c, \ddot{y}_c)\) and \(\dot{\theta}_y\) are transformed into \(r_c\) and \(\theta_y\), respectively. Given that transformed generalized coordinates \(x_y\) are
\[
\ddot{x}_y = \begin{bmatrix} \dot{\theta}_y, \dot{\theta}_s, r_c, x_h, y_h, x_y, y_y \end{bmatrix}^T, \tag{16}
\]
\((\ddot{x}_c, \ddot{y}_c)\) and \(\dot{\theta}_y\) are represented by
\[
\begin{align*}
\ddot{x}_c &= x_h + r_c \sin \theta_s, \\
\ddot{y}_c &= y_h - r_c \cos \theta_s, \\
\dot{\theta}_y &= \theta_y + \dot{\theta}_s. \tag{17}
\end{align*}
\]
A velocity transformation matrix $A_y$ which should satisfy $\dot{x}_y = A_y \dot{x}_y$ can thus be given by substituting (17) into (10) as

$$A_y = \frac{\partial \dot{x}_y}{\partial x_y}. \quad (18)$$

Since $\ddot{x}_y = A_y \dot{x}_y + A_y \ddot{x}_y$, a transformed unconstrained motion equation is obtained as

$$A_y^T \ddot{M}_y A_y \dot{x}_y = A_y^T (\ddot{h}_y - \ddot{M}_y A_y \dot{x}_y). \quad (19)$$

Given that $M_y = A_y^T \ddot{M}_y A_y$ and $h_y = A_y^T (\ddot{h}_y - \ddot{M}_y A_y \dot{x}_y)$, (19) can be represented by

$$M_y \ddot{q}_y = h_y. \quad (20)$$

Constraint conditions of the yoyo are definitions of positional relationships between the yoyo and the hand. They are defined in the up-and-down motion phase and the transition phase, respectively.

### 2.2.1. The up-and-down motion phase ($|\theta_y| \geq \pi/2$)

The string is always perpendicular to the radius of the yoyo axle while $|\theta_y| \geq \pi/2$ as shown in Fig. 4(a) and Fig. 4(c). The constraint conditions while $|\theta_y| \geq \pi/2$ are given by

$$\begin{align*}
x_y &= x_h + (L_s(\theta_y) + r_c) \sin \theta_s + R_y(\theta_y) \cos(\theta_s) \text{sgn}(\theta_y) , \\
y_y &= y_h - (L_s(\theta_y) + r_c) \cos \theta_s + R_y(\theta_y) \sin(\theta_s) \text{sgn}(\theta_y).
\end{align*} \quad (20)$$

The constraint conditions (20) are combined as a constraint equation $\Phi_y = 0$, and the constraint matrix $C_y$ can be calculated by

$$C_y = \frac{\partial \Phi_y}{\partial x_y}. \quad (21)$$

A constrained system is given by

$$M_y \ddot{q}_y = h_y + C_y^T \lambda_y, \quad (22)$$

and an independent velocity vector under the constrained state $\dot{q}_y$ is defined as

$$\dot{q}_y = \begin{bmatrix} \dot{\theta}_y, \dot{\theta}_s, \dot{r}_c, \dot{x}_h, \dot{y}_h \end{bmatrix}^T. \quad (23)$$

An orthogonal matrix $D_y$ to reduce degrees of freedom in (22) is obtained as

$$D_y = \begin{bmatrix} I_5 \\ -C_y^{-1}C_y^1 \end{bmatrix}. \quad (24)$$

where $I$ denotes an identity matrix and an index of $I$ denotes a dimension of an identity matrix, consequently the motion equation of the yoyo is derived as

$$D_y^T M_y D_y \ddot{q}_y + D_y^T M_y D_y \dot{q}_y = D_y^T h_y. \quad (25)$$

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2.2.2. The transition phase (|\(\theta_y\)| < \(\pi/2\))

The yoyo axle rotates within \(-\pi/2 < \theta_y < \pi/2\) as shown in Fig. 4(b). The constraint conditions while |\(\theta_y\)| < \(\pi/2\) are given by

\[
\begin{align*}
    x_y &= x_h + (l_0 + r_c)\sin(\theta_s) + r_0\sin(\theta_s + \theta_y), \\
    y_y &= y_h - (l_0 + r_c)\cos(\theta_s) - r_0\cos(\theta_s + \theta_y).
\end{align*}
\]

(26)

The independent velocity vector \(\dot{\mathbf{q}}_y\) is identical with (23). The motion equation of the yoyo is consequently represented by (25) by deriving the constraint matrix \(\mathbf{C}_y\) and the orthogonal matrix \(\mathbf{D}_y\) from the constraint conditions (26) in the previous described method.

2.3. The integrated model of the arm and the yoyo

The integrated motion equation of the parallel link arm and the yoyo is derived by integrating the motion equations (9) and (25) as subsystems. Generalized coordinates \(\bar{\mathbf{x}}\), a generalized mass matrix \(\bar{\mathbf{M}}\) and a generalized force vector \(\bar{\mathbf{h}}\) are given by

\[
\bar{\mathbf{x}} = \begin{bmatrix} q_a, q_y^T \end{bmatrix}^T,
\]

(27)

\[
\bar{\mathbf{M}} = \begin{bmatrix}
    D_a^T M_a D_a & 0 \\
    0 & D_y^T M_y D_y
\end{bmatrix},
\]

(28)

\[
\bar{\mathbf{h}} = \left( (D_a^T (h_a - M_a \dot{D}_a \dot{q}_a)) \right)^T, \left( D_y^T (h_y - M_y \dot{D}_y \dot{q}_y) \right)^T
\]

(29)

thus the unconstrained motion equation is represented by \(\bar{\mathbf{M}} \ddot{\bar{\mathbf{x}}} = \bar{\mathbf{h}}\).

Constraint conditions that the hand coordinates defined in each motion equation are in agreement enlinks the yoyo to the arm and are given by

\[
\begin{align*}
    x_h = x_p &= l_{11} \cos \theta_{l1} + l_{g1}, \\
    y_h = y_p &= l_{11} \sin \theta_{l1} + l_{g2}.
\end{align*}
\]

(30)

A constraint matrix \(\bar{\mathbf{C}}\) can be obtained from the constraint conditions (30) in the previous described method.

Since

\[
\dot{\bar{\mathbf{x}}} = \begin{bmatrix} \dot{q}_a, \dot{q}_y^T \end{bmatrix}^T = \begin{bmatrix} \dot{\theta}_{l1}, \dot{\theta}_y, \dot{\theta}_s, \dot{r}_c, \dot{x}_h, \dot{y}_h \end{bmatrix}^T,
\]

(31)

an independent velocity \(\dot{\mathbf{q}}\) is defined as

\[
\dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_{l1}, \dot{\theta}_y, \dot{\theta}_s, \dot{r}_c \end{bmatrix}^T.
\]

(32)

An orthogonal matrix \(\bar{\mathbf{D}}\) is thus obtained as

\[
\bar{\mathbf{D}} = \begin{bmatrix}
    I_4 \\
    -\bar{\mathbf{C}}^{-1}_2 \bar{\mathbf{C}}_1
\end{bmatrix},
\]

(33)

and the integrated motion equation of the parallel link arm and the yoyo is represented by

\[
\bar{\mathbf{D}}^T \dot{\bar{\mathbf{M}}} \ddot{\bar{\mathbf{q}}} + \bar{\mathbf{D}}^T \dot{\bar{\mathbf{M}}} \dot{\bar{\mathbf{q}}} = \bar{\mathbf{D}}^T \bar{\mathbf{h}}.
\]

(34)
3. Controller design

Some of the energy of the yoyo is lost due to friction between the yoyo discs and the string and the bottom impact while an operator is performing the gravity-pull. The continuous gravity-pull requires keeping the yoyo motion in any limit cycle by compensating the energy loss. This chapter hence presents the controller which can compensate the energy loss [20, 21]. The controller specifically generates the joint torque of the parallel link arm which allows the kinetic energy of the yoyo to reach target energy when it reaches the bottom position.

This paper supposes that all rotational angles and velocities can be observed and the arm pulls up the yoyo immediately before it reaches the bottom position, which means that the arm pulls up it during its down motion, based on Žlajpah’s previous studies [14, 15].

3.1. Linearization of the parallel link arm

The motion equation of the parallel link arm is linearized, and the integrated model of the linearized arm model and the yoyo model is developed to facilitate the obtaining of the joint torque. The motion equation of the parallel link arm (9) can be represented by

\[ M(q_a) \ddot{q}_a + h(q_a, \dot{q}_a) = \tau_{l1}, \]  
(35)

\[ M(q_a) := D_T^a M_a D_a, \]
\[ h(q_a, \dot{q}_a) := D_T^a M_a D_a \dot{q}_a - (D_T^a h_a - \tau_{l1}). \]

Setting a new input \( u_q \) yields non-linear feedback for (35)

\[ M(q_a) u_q + h(q_a, \dot{q}_a) = \tau_{l1}, \]  
(36)

and substituting (36) into (35) yields its linearized motion equation

\[ \ddot{q}_a = u_q. \]  
(37)

Integrating the linearized parallel link arm motion equation (37) and the yoyo motion equation (25) modifies the generalized mass matrix \( \bar{M} \) and the generalized force vector \( \bar{h} \) as

\[ \bar{M} = \begin{bmatrix} 1 & 0 \\ 0 & D_T^y M_y D_y \end{bmatrix}, \]  
(38)

\[ \bar{h} = \begin{bmatrix} u_q, (D_T^y (h_y - M_y D_y \dot{q}_y)) \end{bmatrix}^T. \]  
(39)

The integrated motion equation of the linearized arm and the yoyo can be derived by (34) with (38) and (39).

3.2. Energy compensation control

The kinetic energy of the yoyo \( E \) is defined as

\[ E = \frac{1}{2} I_y \dot{\theta}_y^2. \]  
(40)
Eq. (40) should be maximum at the bottom position and its amplitude is maintained constant if the energy is conserved. Some of the energy, however, is lost due to the friction between the yo-yo discs and the string and the bottom impact. Its kinetic energy at the bottom position for the case of completing the gravity-pull is hence set as the target energy \( E_d \), and it is considered that \( E = E_d \) is satisfied at the bottom position.

An energy state function of the yo-yo \( V \) is defined as

\[
V = \frac{1}{2} (E - E_d)^2, \quad V = (E - E_d) E.
\]  

Eq. (41) is \( V > 0 \) obviously and satisfy \( V = 0 \) for \( E = E_d \). When \( V < 0 \), \( V \to 0 \) and \( E \to E_d \) are satisfied. The gravity-pull is hence achieved when the joint torque satisfying \( \dot{V} < 0 \) is given to the parallel link arm during the down motion of the yo-yo. Since the input is supplied during the down motion only, the integrated motion equation in the up-and-down motion phase is utilized. The motion equation in terms of \( \dot{\theta}_g \) is given from (34) by

\[
\ddot{\theta}_g = - \frac{R_y(\theta_y) \left( \frac{1}{l_1 \cos \theta_1} u_q + m_h l_1 \dot{\theta}_1^2 \sin \theta_1 - m_h g \right) \text{sgn} \theta_y + c_p \dot{\theta}_y}{m_y k_d^2 + I_y}.
\]  

where the acceleration of the joint, horizontal motion of the hand and rotation of the string are ignored, and the minute mass of the connection point is zero. The time derivative of the energy state function \( \dot{V} \) is written by applying (40), (41) and (42) as

\[
\dot{V} = - \frac{(E - E_d) L_p R_y(\theta_y) \left( \frac{1}{l_1 \cos \theta_1} u_q + m_h l_1 \dot{\theta}_1^2 \sin \theta_1 - m_h g \right) \dot{\theta}_y \text{sgn} \theta_y + c_p \dot{\theta}_y^2}{m_y k_d^2 + I_y} < 0. \tag{43}
\]

Since the input can only increase the energy, if \( E - E_d \geq 0 \), i.e. \( E \geq E_d \), then \( u_q = 0 \). If \( E - E_d < 0 \), i.e. \( E < E_d \), then (43) yields

\[
\left( \frac{1}{l_1 \cos \theta_1} u_q + m_h l_1 \dot{\theta}_1^2 \sin \theta_1 - m_h g \right) \dot{\theta}_y \text{sgn} \theta_y + c_p \dot{\theta}_y^2 < 0. \tag{44}
\]

since \(-((E - E_d) L_p R_y(\theta_y))/(m_y k_d^2 + I_y) > 0\). Here \( \dot{\theta}_y \text{sgn} \theta_y = -|\dot{\theta}_y| \) holds during the down motion of the yo-yo, and \( \cos \theta_1 > 0 \) holds by assuming that \(-\pi/2 < \theta_1 < \pi/2 \) as the range of joint motion. (44) is hence calculated as

\[
u_q > \left( c_p |\dot{\theta}_y| - m_h l_1 \dot{\theta}_1^2 \sin \theta_1 + m_h g \right) l_1 \cos \theta_1. \tag{45}
\]

The input is determined so as to satisfy (45). The input \( u_q \) is defined as

\[
u_q = \alpha_p \left( c_p |\dot{\theta}_y| - m_h l_1 \dot{\theta}_1^2 \sin \theta_1 + m_h g \right) l_1 \cos \theta_1. \tag{46}
\]

where \( \alpha_p \) is a gain to adjust the input, and \( 1 < \alpha_p < \infty \). To determine \( \alpha_p \), values of the energy state function \( V \) when the yo-yo reaches the bottom position are defined as the Poincare mapping \( \xi = \{ \xi_1, \xi_2, \cdots, \xi_k, \cdots \} \). Here \( \xi_k \) denotes a value of the energy state function \( V \) for a time \( t_k \) when the yo-yo reaches the bottom position. The control target is
achieved when the Poincare mapping is 0. The gain $\alpha_p$ settling the Poincare mapping to 0 is defined as

$$\alpha_p = \alpha_0 + \sum_{k=1}^{n} \left( K_p \Delta \xi_k + K_i \sum_{j=1}^{k} \Delta \xi_j + K_d \frac{\Delta \xi_k - \Delta \xi_{k-1}}{\Delta t_k} \right),$$

(47)

where $\Delta \xi_k = \xi_k - \xi_*$ is a difference between the value $\xi_k$ and the target value $\xi_* = 0$ of the energy state function $V$, $\Delta t_k = t_k - t_{k-1}$ is a difference between the time $t_k$ and the previous time $t_{k-1}$ when the yoyo reaches the bottom position, and $\alpha_0$, $K_p$, $K_d$ and $K_i$ are adjustable parameters and positive constant. The joint torque of the parallel link arm $\tau_f$ is thus given by substituting (46) into (36).

4. Simulation of the gravity-pull

The gravity-pull is simulated to verify the effectiveness of the proposed controller. In this simulation, the integrated model of the non-linear arm and the yoyo (34) is applied as a plant, and a position controller based on an error system is applied with the proposed controller to return the hand to an initial position and keep the position.

The simulation parameters of the arm and the yoyo are shown in Table 5 and Table 6. The parameters of the mass, the length and the radius in Table 5 and Table 6 are determined by measurement of the arm and the yoyo. The effective diameter of the string is calculated from its maximum length, its maximum winding numbers and the radius of the yoyo axle. The parameters of the inertia moment, the viscosity, the damper coefficient and the spring coefficient in Table 6 are determined by parameters identification based on motion measurement in the yoyo. The parameters of the inertia moment, the viscosity in Table 5 are any values because they have not been evaluated strictly in this paper.

The controller parameters are shown in Table 7. The target energy of the yoyo $E_d$ is determined based on its potential energy so as to lower its top position than the hand. The adjustable parameters $\alpha_0$, $K_p$, $K_d$ and $K_i$ are determined by trial and error.

| Parameters                                      | Number of $i$ or $j$ |
|------------------------------------------------|----------------------|
| Mass of link $i$ ($\times 10^{-2}$ kg)          | 4.05 5.00 4.05 100   |
| Mass of weight $j$ ($\times 10^{-3}$ kg)        | 14.1 14.1            |
| Inertia moment of link $i$ ($\times 10^{-2}$ kgm$^2$) | 2.00 5.00 2.00 0.00 |
| Inertia moment of weight $j$ ($\times 10^{-3}$ kgm$^2$) | 3.20 3.20            |
| Viscosity of link $i$ ($\times 10^{-6}$ Nms/rad) | 1.00 1.00 1.00       |
| Length of link $i$ ($\times 10^{-1}$ m)         | 2.50 1.00 2.50 1.00  |
| Length from joint to center of gravity of link $i$ ($\times 10^{-2}$ m) | 12.5 5.00 12.5 5.00  |
| Length from joint to weight $j$ ($\times 10^{-2}$ m) | 5.00 5.00            |
| Mass of hand ($\times 10^{-3}$ kg)              | 1.00                 |
| Inertia moment of hand ($\times 10^{-6}$ kgm$^2$) | 1.00                 |
| Length from joint to center of gravity of hand ($\times 10^{-2}$ m) | 1.00                 |
| Gravity acceleration (m/s$^2$)                  | 9.81                 |

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Table 6: Simulation parameters of the yoyo

| Parameter                                      | Value          |
|------------------------------------------------|----------------|
| Mass of yoyo (kg)                              | 5.46 × 10⁻²    |
| Mass of hand (kg)                              | 1.00           |
| Mass of connection point (kg)                  | 1.00 × 10⁻⁶    |
| Inertia moment of yoyo (kgm²)                  | 1.56 × 10⁻⁵    |
| Inertia moment of string (kgm²)                | 3.44 × 10⁻⁴    |
| Viscosity coefficient of yoyo (Ns/rad)         | 1.91 × 10⁻⁴    |
| Viscosity of string (Nms/rad)                  | 7.59 × 10⁻³    |
| Damper coefficient of string (Ns/m)            | 6.08           |
| Spring coefficient of string (N/m)             | 1.00 × 10³     |
| Maximum length of string (m)                   | 1.05           |
| Radius of yoyo axle (m)                        | 4.00 × 10⁻³    |
| Effective diameter of string (m/rad)           | 5.85 × 10⁻⁵    |

Table 7: Controller parameters

| Parameter | Value          |
|-----------|----------------|
| E_d       | 4.51 × 10⁻¹    |
| a₀        | 1.00 × 10⁴     |
| K_p       | 2.00 × 10²     |
| K_i       | 1.00           |
| K_d       | 1.00 × 10³     |

The initial rotational angles of the parallel link arm, the yoyo and the string are set at \( \theta_0 = 0.00 \) rad, \( \theta_y = -1.35 \times 10² \) rad and \( \theta_s = 0.00 \) rad, respectively. The initial translational distance from the hand to the connection point is set at \( r_c = 0.00 \) m.

The arm lets the yoyo free-fall and pulls it up while it moves down and the string length is greater than or equal to \( 15l_0/16 \).

Simulation results are shown in Figs. 6 - 11. Figure 6 shows the rotational angles of the yoyo and the string, Fig. 7 and Fig. 8 show the positions of the yoyo and the hand, Fig. 9 shows the angle of the arm joint, Fig. 10 shows the joint torque with the energy compensation control, Fig. 11 shows the error between the kinetic energy and the target energy of the yoyo and Fig. 12 shows the transition of the energy state function.

Figure 6 and Figure 7 show that the yoyo moves up and down periodically with rotating continuously. The yoyo swings at a constant frequency corresponding with its periodic up-and-down motion as shown in Fig. 8. From the above, the continuous gravity-pull is achieved.

Figure 9 shows that the joint of the parallel link arm moves within the range of the motion \(-\pi/2 < \theta_0 < \pi/2\). Figure 10 shows that the positive joint torque is supplied to its joint immediately before the yoyo reaches the bottom position. The input allows the error between the kinetic energy and the target energy of the yoyo to be more than 0 as shown in Fig. 11, which means that the kinetic energy exceeds the target energy. The kinetic energy is thus compensated by the proposed controller.

Although the energy state function \( V \) settles to the constant value shown in Fig. 12, it doesn’t reach the target value 0. The error between its value and the target value arises from the excessive recovery of the kinetic energy of the yoyo. The error, however, has low
Figure 6: Rotational angles of the yoyo and the string (thy: yoyo, ths: string). The rotational angle of the yoyo increases and decreases periodically with the central focus on 0.

Figure 7: Vertical positions of the yoyo and the hand (yy: yoyo, yh: hand). The yoyo moves with the constant amplitude within the constant distance from the hand.

Figure 8: Horizontal positions of the yoyo and the hand (xy: yoyo, xh: hand). The yoyo swings at a constant frequency corresponding with its periodic up-and-down motion.
Figure 9: Angle of the arm joint. The joint moves within the range of the motion $-\pi/2 < \theta_{l1} < \pi/2$.

Figure 10: Joint torque with energy compensation control. The positive joint torque is supplied to the parallel link arm immediately before the yoyo reaches the bottom position.

Figure 11: Error between the kinetic energy and the target energy of the yoyo. The error exceeds 0 by energy compensation before the yoyo reaches the bottom position.
Figure 12: Transition of the energy state function. The energy state function $V$ settles to the less value than $3.00 \times 10^{-4}$, however, it doesn’t reach the target value 0.

influence on the continuous gravity-pull, since the value less than the maximum value of the potential energy based on the bottom position is set as the target value and the error is less than $3.00 \times 10^{-4}$.

From the above results, the proposed controller is effective in controlling the periodic up-and-down yoyo motion.

5. Conclusion

This paper has presented an approach to achieve an operation of the periodic up-and-down yoyo motion called “gravity-pull” with a parallel link arm. First an integrated model of the parallel link arm and the yoyo has been developed by integrating each model of the arm and the yoyo as subsystems in order to describe their mutual interference. Next a controller which can compensate energy loss with the yoyo motion has been designed based on the integrated model in which the model of the parallel link arm has only been linearized. The effectiveness of the proposed controller for the continuous gravity-pull has been verified through a numerical simulation. The simulation results have shown that the parallel link arm has achieved the continuous gravity-pull and the trajectories of the rotation and the up-and-down motion of the yoyo are periodic. The proposed approach is effective in the controlling of the periodic up-and-down yoyo motion, like the continuous gravity-pull, in conclusion. The proposed controller will be implemented in the experimental platform and tested to prove the efficacy and validity of the proposed approach in the future work.

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