SUBDIVISIONS OF SHELLABLE COMPLEXES

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ABSTRACT. In geometric, algebraic, and topological combinatorics, the unimodality and log-concavity of polynomials is frequently studied. A popular approach is to show that the polynomial admits a stronger property; namely, real-rootedness. Currently, many of the open questions on real-rootedness and unimodality of combinatorial polynomials pertain to $f$- and $h$-polynomials, which enumerate the faces of cell complexes. When proving that a polynomial is real-rooted, we often rely on the theory of interlacing polynomials and their recursive nature. In this paper, we relate the recursive nature of interlacing polynomials to the recursive structure, typically termed ‘shellability,’ of cell complexes. We derive a sufficient condition for real-rootedness of the $h$-polynomial of a subdivision of a shellable complex. As applications, we solve two open problems: We positively answer a question posed first by Brenti and Welker, and then again by Mohammadi and Welker, on the barycentric subdivision of a cubical polytope. We also give a positive solution to second problem of Mohammadi and Welker on the edgewise subdivision of a cell complex.

1. Introduction

Many endeavors in modern combinatorics aim to derive inequalities that hold amongst a sequence of nonnegative numbers $p_0, \ldots, p_d$ that encode some algebraic, geometric, and/or topological data [10, 11, 13, 19, 37]. These inequalities are typically assigned to the generating polynomial $p = p_0 + p_1 x + \cdots + p_d x^d$ associated to the sequence $p_0, \ldots, p_d$. The generating polynomial $p$ is called unimodal if there exists $t \in [d]$ such that $p_0 \leq \cdots \leq p_t \geq \cdots \geq p_d$. It is called log-concave if $p_k^2 \geq p_{k-1} p_{k+1}$ for all $k \in [d]$, and it is called real-rooted if $p \equiv 0$ or $p$ has only real zeros. A classic result states that $p$ is both log-concave and unimodal whenever it is real-rooted [11, Theorem 1.2.1]. Since real-rootedness is the strongest of these three conditions, many conjectures in the literature ask when certain generating polynomials are not only unimodal or log-concave, but also real-rooted. Most proofs of such conjectures in turn rely on the theory of interlacing polynomials [19], which are inherently tied to recursions associated to the generating polynomials of interest.

In the field of algebraic, geometric, and topological combinatorics, the generating polynomials of interest are typically the $f$- or $h$-polynomial associated to a cell complex. A foundational result in the field, known as the $g$-theorem, implies that the $h$-polynomial associated to the boundary complex of a simplicial polytope is unimodal [37]. In the subsequent years following the proof of the $g$-theorem, extensions of the $g$-theorem via the relationship between $h$-polynomials of simplicial complexes and their subdivisions became of interest [3, 38]. In [14], the unimodality result implied by the $g$-theorem was strengthened significantly for one family of simplicial complexes by Brenti and Welker, who showed that the $h$-polynomial of

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the barycentric subdivision of a simplicial polytope is real-rooted. They then asked if their result generalizes to the boundary complex of any polytope [14 Question 1]. In [30], Mohammadi and Welker raised this same question again and they conjectured that cubical polytopes are likely a good starting point.

In the same way that proofs of real-rootedness via interlacing polynomials often rely on underlying recursions, proofs pertaining to the geometry of polytopal complexes often make use of the recursive structure of the complex, when it exists. This recursive property of polytopal complexes is termed shellability, and a classic result of Bruggesser and Mani [16] states that the boundary complex of a polytope always admits this property. In this paper, we relate the recursive structure of interlacing polynomials to the notion of shellability so as to derive a sufficient condition for the $h$-polynomial of a subdivision of a shellable complex to be real-rooted. As a result, we recover a positive answer to the conjecture on cubical polytopes raised by the questions of Brenti, Mohammadi and Welker [14, 30]. To do so, we will make use of some techniques drawn from discrete geometry and the theory of lattice point enumeration.

The remainder of the paper is structured as follows: In Section 2 we develop the necessary preliminaries pertaining to polytopal complexes, interlacing polynomials, and lattice point enumeration. In Section 3 we show that the $h$-polynomial of a subdivision of a shellable complex whose shelling order induces a collection of relative complexes with $h$-polynomials forming an interlacing sequence is real-rooted. In Section 4 we apply the results of Section 3 to deduce that the $h$-polynomial of the barycentric subdivision of a cubical polytope is real-rooted. We also apply these techniques to give an alternative proof of the original result of Brenti and Welker [14], so as to unify the solution to their proposed problem with its motivation. We also apply these methods to solve a second problem propose by Mohammadi and Welker [30] pertaining to edgewise subdivisions of cell complexes. In Section 5 we investigate some alternative proofs of the problem Brenti, Welker, and Mohammadi for well-studied families of cubical polytopes. Namely, we give an alternative proof for all capped cubical polytopes, and for families of cuboids. We end with a conjecture pertaining to the latter examples and their recursive definition.

2. Preliminaries

The results in this paper are concerned with the $f$- and $h$-polynomials of polytopal complexes. A collection $\mathcal{C}$ of polytopes is called a polytopal complex if

1. the empty polytope $\emptyset$ is in $\mathcal{C}$,
2. if $P \in \mathcal{C}$ then all faces of $P$ are also in $\mathcal{C}$, and if
3. if $P, Q \in \mathcal{C}$, then their intersection $P \cap Q$ is a face of both $P$ and $Q$.

The elements of $\mathcal{C}$ are called its faces, and its maximal faces (with respect to inclusion) are called its facets. If all the facets of $\mathcal{C}$ have the same dimension then $\mathcal{C}$ is called pure. When all facets of $\mathcal{C}$ are simplices, we call $\mathcal{C}$ a simplicial complex, and when all facets of $\mathcal{C}$ are cubes, we call $\mathcal{C}$ a cubical complex. Note that, in this definition, we do not require our polytopal complex to be embedded in some Euclidean space, but instead treat it as an abstract cell complex. Given a convex polytope $P \subset \mathbb{R}^n$, we can naturally produce two associated (abstract) polytopal complexes: the complex $\mathcal{C}(P)$ consisting of all faces in $P$ and the complex $\mathcal{C}(\partial P)$ consisting of all faces in $\partial P$, the boundary of $P$. We call $\mathcal{C}(\partial P)$ the boundary complex of $P$. Given a polytopal complex $\mathcal{C}$, a polytopal complex $\mathcal{D}$ is called a
subcomplex of \( C \) if every face of \( D \) is also a face of \( C \). We refer to the difference \( C \setminus D = \{ P \in C : P \notin D \} \) as a relative (polytopal) complex, and we define the dimension of \( C \setminus D \) to be the largest dimension of a polytope in \( C \setminus D \). When \( D = \emptyset \), note that \( C \setminus D = C \).

The \( f \)-polynomial of a \((d-1)\)-dimensional polytopal complex \( C \) is the polynomial

\[
f(C; x) := f_{-1}(C) + f_0(C)x + f_1(C)x^2 + \cdots + f_{d-1}(C)x^d,
\]

where \( f_{-1}(C) := 1 \) and \( f_k(C) \) denotes the number of \( k \)-dimensional faces of \( C \) for \( 0 \leq k \leq d-1 \). Given a subcomplex \( D \) of \( C \), the \( f \)-polynomial of the relative complex \( C \setminus D \) is then

\[
f(C \setminus D; x) := f(C; x) - f(D; x).
\]

The \( h \)-polynomial of the \((m-1)\)-dimensional relative complex \( C \setminus D \) is the polynomial

\[
h(C \setminus D; x) := (1 - x)^m f \left( \frac{C \setminus D}{x}; \frac{x}{1 - x} \right).
\]

We write \( h(C \setminus D; x) = h_0(C \setminus D) + h_1(C \setminus D)x + \cdots + h_m(C \setminus D)x^m \) when expressing \( h(C \setminus D; x) \) in the standard basis, and we similarly write \( f(C \setminus D; x) = f_0(C \setminus D) + f_1(C \setminus D)x + \cdots + f_m(C \setminus D)x^m \). The following lemma, whose proof is an exercise, relates the \( h \)-polynomial of a polytopal complex to those of its relative complexes.

**Lemma 2.1.** Let \( C \) be a \((d-1)\)-dimensional polytopal complex and suppose that \( C \) can be written as the disjoint union

\[
C = \bigsqcup_{i=1}^s R_i
\]

where \( R_i \) are relative \((d-1)\)-dimensional polytopal complexes. Then

\[
h(C; x) = \sum_{i=1}^s h(R_i; x).
\]

### 2.1. Subdivisions and local \( h \)-polynomials.

Given a polytopal complex \( C \), a (topological) subdivision of \( C \) is a polytopal complex \( C' \) such that each face of \( F \in C \) is subdivided into a ball by faces of \( C' \) such that the boundary of this ball is a subdivision of the boundary of \( F \). The subdivision is further called geometric if both \( C \) and \( C' \) admit geometric realizations, \( G \) and \( G' \), respectively; that is to say, each face of \( C \) and \( C' \) is realized by a convex polytope in some real-Euclidean space such that \( G \) and \( G' \) both have the same underlying set of vertices and each face of \( G' \) is contained in a face of \( G \). Given a subdivision \( C' \) of \( C \), we may refer to its associated inclusion map \( \varphi : C' \rightarrow C \). While the main result of this paper applies to general topological subdivisions, the applications of these results will pertain to some special families of subdivisions that are well-studied in the literature. These include the barycentric subdivision and the edgewise subdivision of a complex.

Aside from \( h \)-polynomials for relative complexes, we will also make use of another polynomial invariant associated to a subdivided simplex. Let \( 2^{[d]} \) be an (abstract) \((d-1)\)-dimensional simplex, and let \( \varphi : C \rightarrow 2^{[d]} \) be a subdivision of \( 2^{[d]} \). For a face \( F \in 2^{[d]} \) (i.e., a subset of \([d]\)), we let \( C_F := \varphi^{-1}(2^F) \) denote the subdivision of \( 2^{[d]} \) by \( C \) restricted to \( F \). The local \( h \)-polynomial of \( 2^{[d]} \) with respect to \( C \) is then defined to be

\[
\ell_{[d]}(C_[d]; x) := \sum_{F \subseteq [d]} (-1)^{d-|F|} h(C_F; x).
\]
Stanley gave the following expression of the $h$-polynomial of a simplicial subdivision of a simplicial complex in terms of local $h$-polynomials:

**Theorem 2.2.** [38, Theorem 3.2] Let $C$ be a pure $d$-dimensional simplicial complex and let $C'$ be a simplicial subdivision of $C$.

$$h(C'; x) = \sum_{\Delta \in C} h(\text{link}_C(\Delta); x) \ell(\Delta; C')$$

In Theorem 2.2, the complex $\text{link}_C(\Delta)$, which is called the link of $\Delta$ in $C$, is the simplicial complex formed from the simplices in $C$ that are disjoint from $\Delta$ but are contained in a face of $C$ that also contains $\Delta$.

### 2.2. Interlacing polynomials.

Two real-rooted polynomials $p, q \in \mathbb{R}[x]$ are said to interlace if there is a zero of $p$ between each pair of zeros of $q$ (counted with multiplicity) and vice versa. If $p$ and $q$ are interlacing, it follows that the Wronskian $W[p, q] = p'q - pq'$ is either nonpositive or nonnegative on all of $\mathbb{R}$. We will write $p \prec q$ if $p$ and $q$ are real-rooted, interlacing, and the Wronskian $W[p, q]$ is nonpositive on all of $\mathbb{R}$. We also assume that the zero polynomial $0$ is real-rooted and that $0 \prec p$ and $p \prec 0$ for any real-rooted polynomial $p$.

**Remark 2.1.** Notice that if the signs of the leading coefficients of two real-rooted polynomials $p$ and $q$ are both positive, then $p \prec q$ if and only if

$$\beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1,$$

where $\beta_2, \beta_1$ and $\alpha_2, \alpha_1$ are the zeros of $p$ and $q$, respectively. In particular, when we work with combinatorial generating polynomials, $p \prec q$ is equivalent to $p$ and $q$ being real-rooted and interlacing.

A polynomial $p \in \mathbb{C}[x]$ is called stable if $p$ is identically zero or if all of its zeros have nonpositive imaginary parts. The Hermite-Biehler Theorem relates the relation $p \prec q$ to stability in such a way that we can derive some useful tools for proving results about interlacing polynomials:

**Theorem 2.3.** [31, Theorem 6.3.4] If $p, q \in \mathbb{R}[x]$ then $p \prec q$ if and only if $q + ip$ is stable.

Remark 2.1 and Theorem 2.3 allow us to quickly derive some useful results.

**Lemma 2.4.** If $p$ and $q$ are real-rooted polynomials in $\mathbb{R}[x]$ then

1. $p \prec \alpha p$ for all $\alpha \in \mathbb{R}$,
2. $p \prec q$ if and only if $\alpha p \prec \alpha q$ for any $\alpha \in \mathbb{R} \setminus \{0\}$,
3. $p \prec q$ if and only if $-q \prec p$, and
4. if $p$ and $q$ have positive leading coefficients then $p \prec q$ if and only if $q \prec xp$.

We will also require the following proposition.

**Proposition 2.5.** [9, Lemma 2.6] Let $p$ be a real-rooted polynomial that is not identically zero. Then the following two sets are convex cones:

$$\{ q \in \mathbb{R}[x] : p \prec q \} \quad \text{and} \quad \{ q \in \mathbb{R}[x] : q \prec p \}.$$

It follows from Proposition 2.5 that we can sum a pair of interlacing polynomials to produce a new polynomial with only real-roots. More generally, we will work with recursions for which we need to sum several polynomials to produce a new
real-rooted polynomial. Let \((p_i)_{i=0}^s = (p_0, \ldots, p_s)\) be a sequence of real-rooted polynomials. We say that the sequence of polynomials \((p_i)_{i=0}^s\) is an interlacing sequence if \(p_i < p_j\) for all \(1 \leq i \leq j \leq s\). Note that, by Proposition 2.5, any convex combination of polynomials in an interlacing sequence is real-rooted.

For a polynomial \(p \in \mathbb{R}[x]\) of degree at most \(d\), we let \(I_d(p) := x^d p(1/x)\). When \(d\) is the degree of \(p\), then \(I_d(p)\) is the reciprocal of \(p\). A polynomial \(p = p_0 + p_1 x + \cdots + p_d x^d \in \mathbb{R}[x]\) is called symmetric with respect to degree \(d\) if \(p_k = p_{d-k}\) for all \(k = 0, \ldots, d\). If \(p\) is a degree \(d\) generating polynomial that is both real-rooted and symmetric with respect to \(d\) then \(I_d(p) < p\). However, non-symmetric polynomials also satisfy the latter condition, making it a natural generalization of symmetry for real-rooted polynomials. In [20], the authors characterized the condition \(I_d(p) < p\) in terms of the symmetric decomposition of \(p\), which has been of recent interest [2 4 5 6 7 20 35]. In this paper, the polynomials that we aim to show have only real zeros are known to be symmetric with respect to their degree. However, we will make use of the more general phenomena \(I_d(p) < p\) in some of the proofs.

2.3. Ehrhart Theory. In Section 4 we will use some techniques originating from discrete geometry and the theory of lattice point enumeration in convex polytopes. A subset \(P \subset \mathbb{R}^n\) is called a \(d\)-dimensional lattice polytope if it is the convex hull of finitely many points in \(\mathbb{Z}^n\) whose affine span is a \(d\)-dimensional affine subspace of \(\mathbb{R}^n\). For \(t \in \mathbb{Z}_{>0}\) we call \(tP := \{tp \in \mathbb{R}^n : p \in P\}\) the \(t^{th}\) dilate of \(P\), and we call the function \(i(P; t) := |tP \cap \mathbb{Z}^n|\) the Ehrhart function of \(P\). In an analogous fashion, we can let \(H = \{(a, y) = b\}\) be a subset of the facet-defining hyperplanes of \(P\) and set

\[ S_H := \{z \in \mathbb{R}^n : \langle a, z \rangle = b \text{ for some } \langle a, y \rangle = b \in H\}. \]

We then define the half-open polytope \(P \setminus S_H\), which we may denote by \(P \setminus H\) when we need to highlight the facet-defining hyperplanes that capture the points in \(S_H\). As before, the \(t^{th}\) dilate of \(P \setminus S_H\) is \(t(P \setminus S_H) := \{tp \in \mathbb{R}^n : p \in P \setminus S_H\}\) and the Ehrhart function of \(P \setminus S_H\) is defined to be \(i(P \setminus S_H; t) := |(P \setminus S_H) \cap \mathbb{Z}^n|\) for \(t > 0\) (see [3] Section 5.3). Notice that if \(H = \emptyset\) then \(P = P \setminus S_H\), and if \(H\) is the complete collection of facet-defining hyperplanes of \(P\) then \(P \setminus S_H =: P^o\), the relative interior of \(P\). The Ehrhart series of the relative interior of \(P\) is defined to be

\[ \text{Ehr}_{P^o}(x) := \sum_{t \geq 0} i(P^o; t) x^t. \]

In the case that \(H\) is not the complete set of facet-defining hyperplanes of \(P\), the Ehrhart series of \(P \setminus S_H\) is defined as

\[ \text{Ehr}_{P \setminus S_H}(x) := \sum_{t \geq 0} i(P \setminus S_H; t) x^t, \]

and the constant term is computed to be the Euler characteristic of \(P \setminus S\) (see [8] Theorem 5.1.8)). When written in a closed rational form, the Ehrhart series of \(P \setminus S_H\), for any choice of \(H\), is

\[ \text{Ehr}_{P \setminus S_H}(x) = \frac{h_0^* + h_1^* x + \cdots + h_d^* x^d}{(1 - x)^{d+1}}, \]

and the polynomial \(h^*(P \setminus S_H; x) := h_0^* + h_1^* x + \cdots + h_d^* x^d\) is called the (Ehrhart) \(h^\ast\)-polynomial of \(P \setminus S_H\). It is well-known that \(h^*(P \setminus S_H; x)\) has only nonnegative integral coefficients (see for instance [28]). Since a lattice polytope \(P\) is a subset
of $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, it is natural to consider its subdivisions into polyhedral complexes whose 0-dimensional faces correspond to the lattice points in $P \cap \mathbb{Z}^n$. When such a (geometric) subdivision consists of only simplices, we call it a triangulation of $P$. When each simplex $\Delta$ in a triangulation of $P$ has $h^*(\Delta; x) = 1$, we call it a unimodular triangulation of $P$. The following lemma is also well-known, and a proof appears in [7, Chapter 10].

**Lemma 2.6.** Let $P \subset \mathbb{R}^n$ be a $d$-dimensional lattice polytope and let $T$ be a unimodular triangulation of $P$. Then

$$h^*(P; x) = h(T; x).$$

We will need a slight generalization of Lemma 2.6 whose proof is analogous to that of Lemma 2.6. However, we provide it below for the sake of completeness. In the following, given a facet-defining hyperplane $H$ of $P$, we will let $F_H$ denote the facet of $P$ defined by $H$.

**Lemma 2.7.** Let $P \subset \mathbb{R}^n$ be a $d$-dimensional lattice polytope with a unimodular triangulation $T$, and let $H$ be a subset of its facet-defining hyperplanes. If the Euler characteristic of $P \setminus S_H$ is 0 then

$$h^*(P \setminus S_H; x) = h(T \setminus (\bigcup_{H \in S_H} T|_{F_H}); x).$$

**Proof.** To prove the claim, we first write $P \setminus S_H$ as a disjoint union of the (nonempty) open simplices in the relative complex $T \setminus (\bigcup_{H \in S_H} T|_{F_H})$:

$$P \setminus S_H = \bigcup_{\Delta \in T \setminus (\bigcup_{H \in S_H} T|_{F_H})} \Delta^\circ,$$

and we note that

$$i(P \setminus S_H; t) = \sum_{\Delta \in T \setminus (\bigcup_{H \in S_H} T|_{F_H})} i(\Delta^\circ; t).$$

It then follows that

$$\text{Ehr}_{P \setminus S_H}(x) = \sum_{n \geq 0} i(P \setminus S_H)x^n,$$

$$= i(P \setminus S_H; 0) + \sum_{n \geq 0} \left(\sum_{\Delta \in T \setminus (\bigcup_{H \in S_H} T|_{F_H})} i(\Delta^\circ; t)\right)x^n,$$

$$= i(P \setminus S_H; 0) + \left(\sum_{\Delta \in T \setminus (\bigcup_{H \in S_H} T|_{F_H})} \left(\sum_{n \geq 0} i(\Delta^\circ; t)x^n\right)\right),$$

$$= \sum_{\Delta \in T \setminus (\bigcup_{H \in S_H} T|_{F_H})} \text{Ehr}_{\Delta^\circ}(x),$$

where the last equality follows from the definition of the Ehrhart series of the relative interior of a lattice polytope and the fact that $i(P \setminus S_H; 0)$ is the Euler characteristic of $P \setminus S_H$ (which we have assumed to be zero). Since each simplex $\Delta^\circ$ is the interior of a unimodular simplex, it follows by Ehrhart-MacDonald reciprocity [7, Theorem 4.1] that

$$\text{Ehr}_{\Delta^\circ}(x) = \frac{x^{\dim(\Delta) + 1}}{(1 - x)^{\dim(\Delta) + 1}}.$$
Therefore, in analogous fashion to the proof of \cite{Ehr} Theorem 10.3], we have that
\[
\text{Ehr}_{P \setminus S_H}(x) = \sum_{\Delta \in T \setminus (\cup_{H \in S_H} T|_{F_H})} \text{Ehr}_{\Delta^*}(x),
\]
\[
= \sum_{\Delta \in T \setminus (\cup_{H \in S_H} T|_{F_H})} \frac{x^{\dim(\Delta)+1}}{(1-x)^{\dim(\Delta)+1}},
\]
\[
= \sum_{k=-1}^{d} f_k(T \setminus (\cup_{H \in S_H} T|_{F_H})) \left( \frac{x}{1-x} \right)^{k+1},
\]
\[
= \sum_{k=0}^{d+1} f_{k-1}(T \setminus (\cup_{H \in S_H} T|_{F_H})) x^{k+1} \left( 1 - x \right)^{d-k+1},
\]
\[
= h(T \setminus (\cup_{H \in S_H} T|_{F_H}); x)
\]
\[
= \frac{h(T \setminus (\cup_{H \in S_H} T|_{F_H}); x)}{(1-x)^{d+1}},
\]
which completes the proof. \[\square\]

To prove the desired results in Section 3 we will use well-chosen sets \( \mathcal{H} \) and \( S_H \). Let \( q \in \mathbb{R}^n \), and let \( P \subset \mathbb{R}^n \) a \( d \)-dimensional convex polytope. A point \( p \in P \) is called \textit{visible} from \( q \) if the open line segment \((q, p)\) in \( \mathbb{R}^n \) does not meet the interior of \( P \). Let \( B \subset \partial P \) denote the collection of all points visible from \( q \), and set \( D := \partial P \setminus B \), the closure of \( P \setminus B \). Given a facet \( F \) of \( P \), the point \( q \) is said to be \textit{beyond} \( F \) if \( q \not\in T_F(P) \), the tangent cone of \( F \) in \( P \). It follows that \( q \) is beyond \( F \) if and only if the closed line segment \([q, p]\) satisfies \([q, p] \cap P = \{p\}\) for all \( p \in F \) \cite{MacDonald} Section 3.7]. Otherwise, the point \( q \) is said to be \textit{beneath} \( F \). Hence, \( B \) consists of all points in \( \partial P \) that lie in a facet which \( q \) is beyond; that is,
\[
P \setminus B = P \setminus \mathcal{H}_B,
\]
where \( \mathcal{H}_B \) denotes the collection of facets which \( q \) is beyond. Similarly, \( D \) consists of all points in \( \partial P \) that lie in a facet which \( q \) is beneath; that is,
\[
P \setminus D = P \setminus \mathcal{H}_D,
\]
where \( \mathcal{H}_D \) denotes the collection of facets which \( q \) is beneath. Stanley observed in \cite{Stanley} Proposition 8.2], that \( i(P \setminus B; t) \) is a polynomial and that classical Ehrhart-MacDonald reciprocity \cite{Ehr} Theorem 4.1] can be extended to
\[
(-1)^d i(P \setminus D; t) = i(P \setminus B; -t).
\]
The following lemma translates this result into a statement about \( h^* \)-polynomials.

**Lemma 2.8.** Let \( P \subset \mathbb{R}^n \) be a \( d \)-dimensional lattice polytope, and let \( q \in \mathbb{R}^n \). Let \( B \) denote the points in \( \partial P \) that are visible from \( q \), and set \( D := \partial P \setminus B \). If \( B \) and \( D \) are both nonempty then
\[
h^*(P \setminus D; x) = \mathcal{I}_{d+1} h^*(P \setminus B; x).
\]

**Proof.** Since \( i(P \setminus B; t) \) is a polynomial in \( t \) then
\[
\sum_{t \geq 0} i(P \setminus B; t)x^t \quad \text{and} \quad \sum_{t < 0} i(P \setminus B; t)x^t
\]
both evaluate to rational functions, and
\[ \sum_{t \geq 0} i(P \setminus B; t)x^t + \sum_{t < 0} i(P \setminus B; t)x^t = 0. \]
(See, for instance, [7, Exercise 4.7]). Since \((-1)^d i(P \setminus D; t) = i(P \setminus B; -t)\), it follows that
\[
\text{Ehr}_{P \setminus B} \left( \frac{1}{x} \right) = \sum_{t \leq 0} i(P \setminus B; -t)x^t, \\
= - \sum_{t > 0} i(P \setminus B; -t)x^t, \\
= \sum_{t > 0} (-1)^{d+1} i(P \setminus D; t)x^t, \\
= (-1)^{d+1} \text{Ehr}_{P \setminus D}(x).
\]
Here, we used the fact that
\[ i(P \setminus B; 0) = \chi(P \setminus B) = 0 = \chi(P \setminus D) = i(P \setminus D; 0). \]
This follows from the fact that both \(B\) and \(D\) are assumed to be nonempty and that \(B, D\) and \(P\) are all contractible topological spaces. It then follows that
\[
\frac{h^*(P \setminus D; x)}{(1 - x)^{d+1}} = \text{Ehr}_{P \setminus D}(x) = (-1)^{d+1} \text{Ehr}_{P \setminus B} \left( \frac{1}{x} \right) = \frac{I_{d+1} h^*(P \setminus B; z)}{(1 - x)^{d+1}}.
\]
Thus, we conclude that
\[ h^*(P \setminus D; x) = I_{d+1} h^*(P \setminus B; x). \]

3. SUBDIVISIONS OF SHELLABLE COMPLEXES

In this section, we provide a sufficient condition for the \(h\)-polynomial of a subdivision of a polytopal complex to have only real zeros. As part of this condition, we will require the complex to be shellable.

**Definition 3.1.** Let \(C\) be a pure \(d\)-dimensional polytopal complex. A **shelling** of \(C\) is a linear ordering \((F_1, F_2, \ldots, F_s)\) of the facets of \(C\) such that either \(C\) is zero-dimensional (and thus the facets are points), or it satisfies the following two conditions:

1. The boundary complex \(C(\partial F_1)\) of the first facet in the linear ordering has a shelling, and
2. For \(j \in [s]\), the intersection of the facet \(F_j\) with the union of the previous facets is nonempty and it is the beginning segment of a shelling of the \((d - 1)\)-dimensional boundary complex of \(F_j\); that is,
\[ F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \cdots G_r \]
for some shelling \((G_1, \ldots, G_r)\) of the complex \(C(\partial F_j)\) and \(r \in [t]\).

A polytopal complex is **shellable** if it is pure and admits a shelling.

A shelling of a polytopal complex presents a natural way to decompose the complex into disjoint, relative polytopal complexes. Given a shelling order \((F_1, \ldots, F_s)\)
of a polytopal complex $C$, and a subdivision $\varphi : C' \to C$, we can let

$$R_i := C'|_{F_i} \setminus \left( \bigcup_{k=1}^{i-1} C'|_{F_k} \right),$$

to produce the decomposition of $C'$ into disjoint relative complexes

$$C' = \bigcup_{i=1}^{s} R_i,$$

with respect to the shelling $(F_1, \ldots, F_s)$ of $C$.

The recursive structure of shelling orders of polytopal complexes, and the addi-
tive nature of the $h$-polynomials of their associated relative simplicial complexes (see Lemma 2.1) pairs nicely with the properties of interlacing polynomials dis-
cussed in Subsection 2.2. In particular, Lemma 2.1 allows us to combine the fact that a shellable complex $C$ always admits a decomposition as in equation (1) with
the facts about interlacing sequences collected in Subsection 2.2. We can then
prove a theorem that directly relates the recursive nature of shelling orders to the
recursive nature of interlacing sequences of polynomials.

**Theorem 3.1.** Let $C$ be a shellable polytopal complex with shelling $(F_1, \ldots, F_s)$ and
subdivision $\varphi : C' \to C$, and let

$$R_i := C'|_{F_i} \setminus \left( \bigcup_{k=1}^{i-1} C'|_{F_k} \right),$$

for all $i \in [s]$. If $(h(R_{\sigma(i)}; x))_{i=1}^{s}$ is an interlacing sequence for some $\sigma \in S_s$, then

$h(C'; x)$ is real-rooted.

**Proof.** Notice first that since $C$ is a shellable polytopal complex of dimension $d$, and
since $C'$ is a (topological) subdivision of $C$, then each subcomplex $C'|_{F_i} := \varphi^{-1}(F_i)$ is also $d$-dimensional. Moreover, since $R_i$ only removes faces of dimension strictly
less than $d$, then each $R_i$ is a $d$-dimensional relative simplicial complex. So, by
Lemma 2.1 we have that

$$h(\Omega; x) = \sum_{i=1}^{s} h(R_i; x).$$

Supposing now that there exists $\sigma \in S_s$ such that $(h(R_{\sigma(i)}; x))_{i=1}^{s}$ is an interlacing sequence, it then follows from Proposition 2.5 that $h(\Omega; x)$ is real-rooted.

While the proof of Theorem 3.1 is straightforward to derive (once we have care-
fully defined and identified all of the necessary ingredients), its applications prove
to be fairly extensive. In the coming section, we apply it to answer two open
problems in the literature, and we additionally use it to reprove the results that
motivated these problems. From this perspective, it seems that Theorem 3.1 may
offer a unifying approach to questions on the real-rootedness of $h$-polynomials of
subdivisions of shellable complexes.

### 4. Applications to Boundary Complexes of Polytopes

In this section we apply Theorem 3.1 to some classical subdivisions of the bound-
dary complexes of polytopes that are of interest in algebraic, geometric, and topo-
logical combinatorics. In Subsection 4.1 we show that the boundary complex of

any cubical polytope (i.e., a polytope whose facets are all cubes) has only real zeros. This positively answers a question for cubical polytopes first raised by Brenti and Welker in [14, Question 1] that was later posed again in [30].

Problem 4.1. [14, Question 1] [30, Problem 35] Let $C$ be the boundary complex of an arbitrary polytope. Is the $h$-polynomial of the barycentric subdivision of $C$ real-rooted? In [30], cubical polytopes are proposed as the first case of interest.

Problem 4.1 was motivated by the results of Brenti and Welker [14], which showed that the $h$-polynomial of the boundary complex of a simplicial polytope has only real zeros. In Subsection 4.2, we use Theorem 3.1 to give an alternative proof of this result. This proof utilizes the geometric approach of Theorem 3.1 suggesting that Theorem 3.1 could be the correct approach for resolving Problem 4.1 in its fullest generality. At the same time, Theorem 3.1 is not only applicable to barycentric subdivisions. In Subsection 4.3, we apply Theorem 3.1 to the edgewise subdivision of a complex so as to solve a second problem of Mohammadi and Welker [30, Problem 27] for shellable complexes. Here, we are additionally able to give a solution to the problem in its fullest generality.

In the following, we will make use of some well-studied real-rooted polynomials, which can be defined as follows: For $d, r \geq 1$ and $0 \leq \ell \leq d$, let $A^{(r)}_{d,\ell}$ be the polynomial defined by the relation

$$
\sum_{t \geq 0} (rt)^\ell(r+1)^{d-\ell}x^t = \frac{A^{(r)}_{d,\ell}}{(1-x)^{d+1}}.
$$

We call $A^{(r)}_{d,\ell}$ the $d$th $r$-colored $\ell$-Eulerian polynomial. When $r = 1$ and $\ell = 0$, $A^{(r)}_{d,0}$ is the classical Eulerian polynomial, which enumerates the elements of $S_d$ by the excedance statistic. When $r = 2$ and $\ell = 0$, $A^{(r)}_{d,0}$ is the Type B Eulerian polynomial, which enumerates signed permutations. For $d \geq 1$, the polynomials $A^{(1)}_{d,0}$ and $A^{(2)}_{d,0}$ are symmetric with respect to degree $d-1$ and $d$, respectively. When $\ell = 0$ and $r \geq 1$, $A^{(r)}_{d,0}$ is the $d$th colored Eulerian polynomial, which enumerates the the elements of the wreath product $\mathbb{Z}_r \wr S_d$ with respect to their excedance statistic (see [20, Section 3], for example). It is an immediate consequence of [18, Theorem 4.6] that $A^{(r)}_{d,0}$ has only real, simple zeros for all $d, r \geq 1$ and $0 \leq \ell \leq d$.

Lemma 4.2. For $d, r \geq 1$ and $0 \leq \ell \leq d$, the polynomial $A^{(r)}_{d,\ell}$ has only simple, real zeros. Moreover, for a fixed $d, r \geq 1$, $(A^{(r)}_{d,\ell})_{\ell=0}^d$ is an interlacing sequence.

Lemma 4.2 will play a key role in the coming subsections.

4.1. Barycentric subdivisions of cubical polytopes. Given a polytopal complex $C$, let $\mathcal{F}(C)$ denote its face lattice with partial order $\prec_C$ given by inclusion. The barycentric subdivision of $C$ is the simplicial complex $\text{sd}(C)$ whose $k$-dimensional faces are the subsets $\{F_0, F_1, \ldots, F_k\}$ of faces of $C$ for which

$$
\emptyset \prec_C F_0 \prec_C F_1 \prec_C \cdots \prec_C F_k
$$

is a strictly increasing chain in $\mathcal{F}(C)$. Our goal in this subsection is to show that $h(\text{sd}(C); x)$ has only real, simple zeros when $C$ is that boundary complex of a cubical polytope. To do so, we need to first consider the $h$-polynomials of relative complexes.
of barycentrically subdivided cubes. To determine these polynomials, we will make use of the lemmata developed in Subsection 2.3. Since the results in this section pertain to lattice polytopes, which are embedded in real-Euclidean space, we will need to use a geometric realization of the barycentric subdivision of the $d$-cube. In the following, we will let $\square_d$ denote the (abstract) $d$-dimensional cube. The following result is well-known.

**Lemma 4.3.** For $d \geq 1$ we have that

$$h(sd(\partial \square_d); x) = A_{d,0}^{(2)} = h(sd(\square_d); x).$$

**Proof.** The first equality is noted to be well-known in [30]. The second equality follows from the general fact that if $C$ is a simplicial complex and $D$ is the simplicial complex produced by coning over $C$ then $h(C; x) = h(D; x)$. $\square$

We will also let $[-1,1]^d \subset \mathbb{R}^d$ denote the geometric realization of $\square_d$ in $d$-dimensional real-Euclidean space as the convex hull of all $(-1,1)$-vectors in $\mathbb{R}^d$. The following lemma is likely well-known to experts in the field, but we include a proof for the sake of completeness.

**Lemma 4.4.** Let $T_d$ denote the triangulation of the $d$-cube $[-1,1]^d$ that is induced by the hyperplanes $x_i = \pm x_j$ for $0 \leq i < j \leq d$ and $x_i = 0$ for $i \in [d]$. Then $T_d$ is abstractly isomorphic to the barycentric subdivision of the $d$-cube.

**Proof.** We will induct on $d$. Observe that the result holds for the base case of $d = 1$, and suppose it holds up to some $d - 1$. We first consider the triangulation $T_d$ of $[-1,1]^d$ restricted to each facet $F$ of the cube. Note that $F$ is a $(d-1)$-cube and can be described by intersecting $[-1,1]^d$ with either the hyperplane $x_i = 1$ or $x_i = -1$ for some fixed $i \in [d]$. We now focus on how the hyperplanes defining our triangulation intersect $F$. First observe that the induced triangulation of $F$ by $T_d$ is defined by hyperplanes of the form $x_i = \pm x_j$, for $j \neq i$. Notice that the hyperplanes that intersect the interior of $F$ are exactly those of the form $x_j = \pm x_k$ or $x_j = 0$, for $j, k \neq i$. Up to a possible change of coordinates, we see that this subdivision when restricted to $F$ is exactly the subdivision $T_{d-1}$ of the $(d-1)$-cube $[-1,1]^{d-1}$. By our inductive hypothesis, this is abstractly isomorphic to the barycentric subdivision of $F$. Thus, our given triangulation induces a barycentric subdivision of the boundary complex of $[-1,1]^d$. We now investigate how the given hyperplanes subdivide $[-1,1]^d$ as a whole. Since these hyperplanes meet at a unique point in the interior of $[-1,1]^d$ (the origin), the subdivision of $[-1,1]^d$ can be described by taking the induced subdivision of the boundary complex and coning over an interior point. Since the barycentric subdivision of a polytope is given by coning over the barycentric subdivision of its boundary, the result follows. $\square$

The triangulation $T_d$ from Lemma 4.3 has an $h$-polynomial with a well-known combinatorial interpretation: A signed permutation on $[d]$ is a pair $(\pi, \varepsilon) \in \tilde{S}_d \times \{-1,1\}^d$, which we sometimes denote as $\pi_1^{\varepsilon_1} \cdots \pi_d^{\varepsilon_d}$, where $\pi = \pi_1 \cdots \pi_d$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$. Set $\pi_0 := 0$ and $\varepsilon_0 := 1$ for all $(\pi, \varepsilon) \in \tilde{S}_d \times \{-1,1\}^d$ and all $d \geq 1$. Then $e \in [d - 1]_0 := \{0,1, \ldots, d-1\}$ is a descent of $(\pi, \varepsilon)$ if $\varepsilon_i \pi_i > \varepsilon_{i+1} \pi_{i+1}$. We also let

$$\text{Des}(\pi, \varepsilon) := \{i \in [d - 1]_0 : \varepsilon_i \pi_i > \varepsilon_{i+1} \pi_{i+1}\},$$

$$\text{des}(\pi, \varepsilon) := |\text{Des}(\pi, \varepsilon)|.$$
Let $0 \leq \ell \leq d$. Going one step further, we define the $\ell$-descent set of $(\pi, \varepsilon)$ to be
\[
\text{Des}_\ell(\pi, \varepsilon) := \begin{cases} 
\text{Des}(\pi, \varepsilon) \cup \{d\} & \text{if } d + 1 - \ell \leq \varepsilon_d \pi_d \leq d, \\
\text{Des}(\pi, \varepsilon) & \text{otherwise}.
\end{cases}
\]
We then set $\text{des}_\ell(\pi, \varepsilon) := |\text{Des}_\ell(\pi, \varepsilon)|$. The Type $B$ $\ell$-Eulerian polynomial is defined as
\[
B_{d, \ell} := \sum_{(\pi, \varepsilon) \in \Theta_d \times \{-1, 1\}^d, \varepsilon_d \pi_d = d + 1 - \ell} x^{\text{des}_\ell(\pi, \varepsilon)}.
\]
For $0 \leq \ell \leq d$ we define the half-open polytope
\[
[-1, 1]_\ell^d := [-1, 1]_\ell^{d-1} \setminus \{x_d = 1, \ldots, x_{d+1-\ell} = 1\},
\]
and for $d + 1 \leq \ell \leq 2d$ we define it to be
\[
[-1, 1]_\ell^d := [-1, 1]_\ell^{d-1} \setminus \{x_d = 1, \ldots, x_1 = 1\} \cup \{x_d = -1, \ldots, x_{2d+1-\ell} = 1\}.
\]
For $0 \leq \ell \leq d$, we can then make use of the following theorem from [6].

**Theorem 4.5.** [6] Theorem 5.1 | For $d \geq 1$ and $0 \leq \ell \leq d$,
\[
\text{Ehr}_{[-1, 1]_\ell^d}(x) = \frac{B_{d+1, \ell+1}}{(1 - x)^{d+1}}.
\]

In [6], it is further noted that for $0 \leq \ell \leq d$,
\[
i([-1, 1]_\ell^d; t) = \sum_{|S| \leq d} (2t)^{|S|} = (2t)\ell(2t + 1)^{d-\ell}.
\]
(3)
From this, it follows that $B_{d+1, \ell+1} = A_{d, \ell}^{(2)}$, for $0 \leq \ell \leq d$. The polynomials $B_{d+1, \ell+1}$ for $d + 1 \leq \ell \leq 2d$ can also be computed using the polynomials $A_{d, \ell}^{(2)}$. However, it requires a small geometric trick.

**Lemma 4.6.** For $d \geq 1$ and $0 \leq \ell < d$,
\[
h^*([-1, 1]_{2d-\ell}^d; x) = I_{d+1}h^*([-1, 1]_\ell^d; x).
\]
In particular, $h^*([-1, 1]_{2d-\ell}^d; x) = xI_{d}A_{d, \ell}^{(2)}$.

**Proof.** Notice first that for $0 \leq \ell \leq d$, the half-open polytope $[-1, 1]_\ell^d$ corresponds to $[-1, 1]_\ell^d \setminus H$ where we have removed all facets visible from a point $q \in \mathbb{R}^n$ for a fixed choice of $q$. Let $B_\ell$ denote the set of all such points visible from $q$ on $\partial[-1, 1]_\ell^d$, and let $D_\ell := \partial([-1, 1]_\ell^d \setminus B_\ell$. Notice also, if $[-1, 1]_\ell^d = [-1, 1]_\ell^d \setminus B_\ell$ then $[-1, 1]_{2d-\ell}^d$ is unimodularly equivalent to $[-1, 1]_\ell^d \setminus D_\ell$.

Next, consider the case when $\ell = 0$. Then the desired statement follows directly from classic Ehrhart-MacDonald reciprocity [7] Theorem 4.1. Thus, we need only prove the statement when $0 < \ell < d$. Assuming this is the case, we then know that $B_\ell$ and $D_\ell$ are both nonempty, and $i(P \setminus B; t)$ is a polynomial in $t$ (see equation (3)). Thus, by Lemma 2.8 we know that
\[
h^*([-1, 1]_{2d-\ell}^d; x) = h^*(P \setminus D_\ell; x) = I_{d+1}h^*(P \setminus B_\ell; x) = I_{d+1}h^*([-1, 1]_\ell^d; x).
\]
The fact that $h^*([-1, 1]_{2d-\ell}^d; x) = xI_{d}A_{d, \ell}^{(2)}$ then follows from equations (2) and (3), and the fact that $A_{d, \ell}^{(2)}$ has $d$ simple, real zeros. The fact that the zeros of $A_{d, \ell}^{(2)}$ are simple and real is noted in Lemma 1.2. The fact that there are $d$ such zeros can be derived from the combinatorial interpretation of $A_{d, \ell}^{(2)}$ as $B_{d+1, \ell+1}$.
Given our interpretation of the polynomials \(h^*[\ldbrack -\infty, \infty; x]^{d}_\ell; x\) for \(0 \leq \ell \leq 2d\) in terms of the polynomials \(A^{(2)}_{d,\ell}\), we can prove that all cubical polytopes have a barycentric subdivision with a real-rooted \(h\)-polynomial.

**Theorem 4.7.** Let \(\mathcal{C}\) be a the boundary complex of a \((d + 1)\)-dimensional cubical polytope and let \(\text{sd}(\mathcal{C})\) denote its barycentric subdivision. Then \(h(\text{sd}(\mathcal{C}); x)\) is real-rooted.

**Proof.** This proof relies on the classic result of Bruggesser and Mani [10], which states that all polytopes have a shellable boundary complex. It further requires the particular type of shelling they present, which is typically called a line shelling. The basic construction is as follows: Given a point \(p \in \mathbb{R}^d\) (and outside of \(P\)), we choose a line \(L\) through \(p\) and a point in general position in the polytope \(P\). The generality here indicates that the line \(L\) passes through the interior of \(P\), and that the intersection points of \(L\) with the facet-defining hyperplanes of \(P\) are all distinct. We also assume that \(L\) is not parallel to any of the facet-defining hyperplanes of \(P\). We then orient the line \(L\) such that it points from \(P\) and towards the point \(p\). We then start at the point \(p_1 := F_1 \cap L\) lying in the facet \(F_1\) of \(P\) such that if we follow the orientation of \(L\) from \(p_1\) we will not enter the interior of \(P\) (note that this point is unique). We continue following this line, and eventually we hit the second point of intersection \(p_2\) of a facet-defining hyperplane with \(L\). Suppose the facet of \(P\) defined by this hyperplane is \(F_2\). Then \(F_2\) is the second facet in our shelling of \(\mathcal{C}\).

We then continue in this fashion to shell the first “half” of \(P\); at each step we let \(p_j\) denote the point of intersection of \(L\) with the \(j^{\text{th}}\) facet-defining hyperplane, and we let \(F_j\) denote the facet of \(P\) defined by this hyperplane. The resulting order of the facets is \((F_1, F_2, \ldots, F_t)\). Notice now that for each facet \(F_j\) for \(j \in [t]\), we have that

\[
F_j \cap (F_1 \cup \cdots \cup F_{j-1}) = G_1 \cup \cdots \cup G_r,
\]

where \(G_1, \ldots, G_r\) are the facets of \(F_j\) that are visible from the point \(p_j\) in the affine subspace of \(\mathbb{R}^d\) given by the affine hull \(\text{aff}(F_j)\) of \(F_j\). For \(j \in [t]\), let \(B_j := G_1 \cup \cdots \cup G_r\) denote the collection of points on \(\partial F_j\) visible from \(p_j\) and define the relative simplicial complex \(R_j := \text{sd}(F_j) \setminus \text{sd}(B_j)\).

Continuing with the shelling, we can imagine that we have taken the one-point compactification of \(\mathbb{R}^d\), which we denote by \(\mathbb{R}^d \cup \{\infty\}\). The line \(L\) then passes through the point \(\infty\), and then begins to approach \(P\) from the other side. Let \(p_{t+1}\) denote the first point of intersection of \(L\) (after passing through \(\infty\)) with a facet-defining hyperplane of \(P\). Similar to before, we let \(F_{t+1}\) denote the facet defined by this hyperplane. The shelling of Bruggesser and Mani then takes \(F_{t+1}\) as the next facet in the shelling order of \(\mathcal{C}\). Continuing in this fashion, the resulting shelling order is \((F_1, \ldots, F_t, F_{t+1}, \ldots, F_s)\). In this case, for \(t + 1 \leq j \leq s\), we have that

\[
F_j \cap (F_1 \cup \cdots \cup F_{j-1}) = G_1 \cup \cdots \cup G_r,
\]

where \(G_1 \cup \cdots \cup G_r\) are the facets of \(F_j\) that are not visible from the point \(p_j\) in the affine subspace \(\text{aff}(F_j)\) of \(\mathbb{R}^d\). So for \(t + 1 \leq j \leq s\), we set \(B_j := G_1 \cup \cdots \cup G_r\), we let \(D_j := \partial F_j \setminus B_j\) denote the polytopal subcomplex of the boundary of \(F_j\) consisting of all facets other than \(G_1, \ldots, G_r\). We then define the relative simplicial complex \(R_j := \text{sd}(F_j) \setminus \text{sd}(D_j)\). It follows that

\[
\text{sd}(\mathcal{C}) = \bigcup_{j=1}^{s} R_j,
\]
and by Lemma \ref{lem:interlacing}
\[ h(\text{sd}(C); x) = \sum_{j=1}^{s} h(\mathcal{R}_j; x). \]

Therefore, by Theorem \ref{thm:interlacing} it suffices to show that the polynomials \( h(\mathcal{R}_j; x) \) for \( j \in [s] \) form an interlacing sequence. By construction however, each \( \mathcal{R}_j \) has a geometric realization as the subdivision of some \([-1,1]^d_\ell\) induced by the triangulation \( T_d \) of \([-1,1]^d\) from Lemma \ref{lem:triangulation} for some \( 0 \leq \ell \leq 2d \). Recall then that \( T_d \) is a unimodular triangulation of \([-1,1]^d\). So if \( \mathcal{R}_j \) has geometric realization \([-1,1]^d_\ell\) for \( 0 \leq \ell < 2d \), then \( \mathcal{R}_j \) has Euler characteristic 0. So by Lemma \ref{lem:combinatorial} it follows that
\[ h(\mathcal{R}_j; x) = h^*([-1,1]_\ell^d; x) \]
for some \( 0 < \ell < 2d \). In the case that \( \mathcal{R}_j \) has geometric realization \([-1,1]^d_0\) = \([-1,1]^d\), then by Lemma \ref{lem:combinatorial} we have that
\[ h(\mathcal{R}_j; x) = h^*([-1,1]_0^d; x) \]

Finally, in the case that \( \mathcal{R}_j \) has geometric realization \([-1,1]_{2d}^d\), then \( \mathcal{R}_j \) is the relative complex produced by taking the barycentric subdivision of the \( d \)-cube and then removing its subdivided boundary. Hence, the \( f \)-polynomial of \( \mathcal{R}_j \) satisfies
\[ f(\mathcal{R}_j; x) = xf(\text{sd}(\partial \square_d); x). \]

So by Lemma \ref{lem:polynomial} we know that \( h(\mathcal{R}_j; x) = xA_{d,0}^{(2)} \). Moreover, since \( A_{d,0}^{(2)} \) is known to by symmetric with respect to degree \( d \), it follows that \( h(\mathcal{R}_j; x) = xI_dA_{d,0}^{(2)} \). Hence, by Lemma \ref{lem:polynomial} it follows that \( h(\mathcal{R}_j; x) \) is of the form \( A_{d,\ell}^{(2)} \) or \( xI_dA_{d,\ell}^{(2)} \) for some \( 0 \leq \ell \leq 2d \) for all \( j \in [s] \). Thus, to complete the proof, it suffices to show that the sequence
\[ A_{d,0}^{(2)} < A_{d,1}^{(2)} < \cdots < A_{d,d-1}^{(2)} < A_{d,d}^{(2)} < xI_dA_{d,0}^{(2)} < xI_dA_{d,1}^{(2)} < \cdots < xI_dA_{d,d}^{(2)} < xI_dA_{d,0}^{(2)} \]
is an interlacing sequence. By \cite{15} Lemma 2.3, it suffices to check that each of the following interlacing relations are satisfied:

1. \( A_{d,0}^{(2)} < xI_dA_{d,0}^{(2)} \).
2. \( A_{d,\ell}^{(2)} < A_{d,k}^{(2)} \) for all \( 0 \leq \ell < k \leq d \).
3. \( A_{d,\ell}^{(2)} < xI_dA_{d,\ell}^{(2)} \) and
4. \( xI_dA_{d,\ell}^{(2)} < xI_dA_{d,k}^{(2)} \) for all \( 0 \leq \ell < k \leq d \).

Case (1) is immediate from the fact that \( A_{d,0}^{(2)} = I_dA_{d,0}^{(2)} \) and Lemma \ref{lem:interlacing} \cite{1}. Case (2) follows from Lemma \ref{lem:triangle}. Case (3) follows from \cite{20} Theorem 3.1, which shows that \( I_dA_{d,\ell}^{(2)} < A_{d,d}^{(2)} \) and Lemma \ref{lem:triangle} \cite{1}, and case (4) follows directly from case (2). Thus, since this sequence is interlacing, it follows that \( h(\text{sd}(C); x) \) is real-rooted, which completes the proof.

\[ \square \]

4.2. Barycentric subdivisions of simplicial polytopes. In this section, we apply Theorem \ref{thm:interlacing} to give an alternative proof of the result of \cite{14} that motivated Problem \ref{prob:interlacing}. Namely, we apply Theorem \ref{thm:interlacing} to show that the \( h \)-polynomial of the boundary complex of a simplicial polytope has only real zeros. We will prove this using a similar narrative as in Subsection \ref{subsec:interlacing} in that we will use a shelling argument to decompose the complex into relative simplicial complexes, and show that the \( h \)-polynomials of each complex form an interlacing family. In this case, the relative
complexes used in the decomposition will be subdivided half-open simplices. In the following, let $\Delta_d$ denote the $d$-dimensional simplex, and let $\Delta_{d,\ell}$ denote the relative simplicial complex given by removing $\ell$ of the facets of $\Delta_d$ for $0 \leq \ell \leq d + 1$. Note that $\Delta_{d,0} = \Delta_d$. We will require the following well-known result.

**Lemma 4.8.** Let $d \geq 1$ and $0 \leq \ell \leq d + 1$. Then $h(\Delta_{d,\ell}; x) = x^\ell$.

**Proof.** By using the Principle of Inclusion-Exclusion, we deduce that

$$f(\Delta_{d,\ell}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(\Delta_{d-j}; x).$$

Since $h(\Delta_d; x) = 1$ for all $d \geq 1$, it then follows that

$$h(\Delta_{d,\ell}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (1 - x)^j = x^\ell.$$  

□

To make use of this observation, we need to generalize the main result of [14] (i.e. [14, Theorem 1]) to relative complexes.

**Lemma 4.9.** Let $C$ be a $(d - 1)$-dimensional Boolean cell complex and $D$ a subcomplex of $C$. If the relative complex $C \setminus D$ is also $(d - 1)$-dimensional then

$$h(\text{sd}(C) \setminus \text{sd}(D); x) = \sum_{\ell=0}^{d} h_\ell(C \setminus D) A_{d,\ell}^{(1)}.$$ 

**Proof.** The definition of the $h$-polynomial of a relative complex $C \setminus D$ is known to be equivalent to the condition that

$$h_r(C \setminus D) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}$$

for all $r = 0, \ldots, d$. From this formula it follows that for all $r = 0, \ldots, d$

$$h_r(\text{sd}(C) \setminus \text{sd}(D)) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\text{sd}(C) \setminus \text{sd}(D)),$$

$$= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} (f_{i-1}(\text{sd}(C)) - f_{i-1}(\text{sd}(D))),$$

$$= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\text{sd}(C)) - \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\text{sd}(D)).$$

Since $\text{sd}(C)$ and $\text{sd}(D)$ are both Boolean cell complexes then we can apply [14 Lemma 1]. This yields

$$h_r(\text{sd}(C) \setminus \text{sd}(D)) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} f_{m-1}(\text{sd}(C)) S(m, k) m!,$$

$$- \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} f_{m-1}(\text{sd}(D)) S(m, k) m!.$$ 

Notice here that the formula for $f_{m-1}(\text{sd}(D))$ given in [14 Lemma 1] still holds with respect to degree $d$ even if $D$ has dimension less than $d$. This is immediate...
from the proof of [14, Lemma 1] when we remember that $f_{m-1}(D) = 0$ for all $m$ greater than the dimension of $D$. Hence, it follows that

$$h_r(sd(C) \setminus sd(D)) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} S(m, k) m! (f_{m-1}(C) - f_{m-1}(D)), $$

$$= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} S(m, k) m! f_{m-1}(C \setminus D). $$

Since we have assumed that $C \setminus D$ is $(d-1)$-dimensional, it follows that

$$f_{m-1}(C \setminus D) = \sum_{\ell=0}^{m} \binom{d-\ell}{d-m} h_\ell(C \setminus D),$$

and so

$$h_r(sd(C) \setminus sd(D)) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} S(m, k) m! \sum_{\ell=0}^{m} \binom{d-\ell}{d-m} h_\ell(C \setminus D), $$

$$= \sum_{\ell=0}^{d} \left( \sum_{m=0}^{d} \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \binom{d-\ell}{d-m} S(m, k) m! \right) h_\ell(C \setminus D),$$

In the proof of [14, Theorem 1], it is shown that the coefficient of $h_\ell(C \setminus D)$ in the above expression is equal to the $k^{th}$ coefficient of $A^{(1)}_{d, \ell}$. Hence,

$$h_r(sd(C) \setminus sd(D); x) = \sum_{\ell=0}^{d} h_\ell(C \setminus D) A^{(1)}_{d, \ell}. $$

By combining Lemma [4.8] and Lemma [4.9] we recover the following proposition:

**Proposition 4.10.** Let $\Delta_{d-1}$ be a $(d-1)$-dimensional simplex, and let $0 \leq \ell \leq d$ be the number of facets of $\Delta_{d-1}$ missing in $\Delta_{d-1, \ell}$. Then,

$$h_r(sd(\Delta_{d-1}), x) = A^{(1)}_{d, c}. $$

Applying Lemma [4.2] to Proposition [4.10] we see that the $h$-polynomials of the barycentric subdivision of a simplex restricted to half-open simplices form an inter-lacing family. Since a shelling of the boundary of a simplicial polytope will decompose this complex into such half-open simplices, the $h$-polynomial of the barycentric subdivision of the complex will be real-rooted. We summarize this observation in the following theorem, which is originally due to Brenti and Welker [14].

**Theorem 4.11.** Let $C$ be a $(d-1)$-dimensional shellable simplicial complex. Then $h(sd(C); x)$ is real-rooted. In particular, the $h$-polynomial of the barycentric subdivision of the boundary complex of a $d$-dimensional simplicial polytope has only real zeros.

**Proof.** The result is an immediate consequence of Theorem [3.1], Lemma [4.2], and Proposition [4.10]. The special case of boundary complexes of simplicial polytopes follows from the fact that the boundary complex of any polytope admits a shelling [16].
The proofs of Theorems 4.7 and 4.11 given here suggest that the shellability of all boundary complexes of polytopes could be key to answering Problem 4.11 in its fullest generality.

4.3. **Edgewise subdivisions of simplicial polytopes.** The edgewise subdivision of a simplicial complex is another well-studied subdivision that arises frequently in algebraic and topological contexts (see for instance [15, 17, 21, 23]). Within algebra, it is intimately tied to the Veronese construction, and it is considered to be the algebraic analogue of barycentric subdivision [15, Acknowledgements]. For $r \geq 1$, the $r^{th}$ edgewise subdivision of a simplex is defined as follows: Suppose that $\Delta := \text{conv}(e^{(1)}, \ldots, e^{(d)}) \subset \mathbb{R}^d$ is a $(d - 1)$-dimensional simplex with 0-dimensional faces $e^{(1)}, \ldots, e^{(d)}$, the standard basis vectors in $\mathbb{R}^d$. For $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we let

$$\text{supp}(x) := \{i \in [d] : x_i \neq 0\},$$

and we define the linear transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\varphi : x \mapsto (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_d).$$

The $r^{th}$ **edgewise subdivision** of $\Delta$ is the simplicial complex $\Delta^{(r)}$ whose set of $0$-dimensional faces are the lattice points in $r\Delta \cap \mathbb{Z}^d$ and for which $F \subset r\Delta \cap \mathbb{Z}^d$ is a face of $\Delta^{(r)}$ if and only if

$$\bigcup_{x \in F} \{\text{supp}(x)\} \in \Delta,$$

and for all $x, y \in F$

$$\varphi(x) - \varphi(y) \in \{0, 1\}^d \quad \text{or} \quad \varphi(y) - \varphi(x) \in \{0, 1\}^d.$$

Given a simplicial complex $\mathcal{C}$, the $r^{th}$ **edgewise subdivision** of $\mathcal{C}$, denoted $\mathcal{C}^{(r)}$, is given by gluing together the $r^{th}$ edgewise subdivisions of each of its facets. In [30], the authors proposed the following problem to which we will now give a positive answer. The first solution will apply to shellable simplicial complexes, and it will use the geometric approach given by Theorem [34]. The second solution will combine some recent enumerative results from the literature, which yield an answer to the problem in its fullest generality (i.e., for both shellable and nonshellable simplicial complexes).

**Problem 4.12.** [30, Problem 27] If $\mathcal{C}$ is a $d$-dimensional simplicial complex with $h_k(\mathcal{C}) \geq 0$ for all $0 \leq k \leq d + 1$, is $h(\mathcal{C}^{(r)}; x)$ real-rooted whenever $r > d$?

To answer this question, we first note that for every $r \geq 1$ and polynomial $p \in \mathbb{R}[x]$ there are uniquely determined polynomials $p^{(0)}, p^{(1)}, \ldots, p^{(r-1)} \in \mathbb{R}[x]$ satisfying

$$p = p^{(0)}(x^r) + xp^{(1)}(x^r) + x^2p^{(2)}(x^r) + \cdots + x^{r-1}p^{(r-1)}(x).$$

We then define the linear operator

$$\langle r, \ell \rangle : \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \quad \text{where} \quad \langle r, \ell \rangle : p \longrightarrow p^{(\ell)},$$

and the polynomial

$$p^{(r,d)} := (1 + x + \cdots + x^{r-1})^d.$$

It is well-known that the sequence

$$\left(p^{(r,d)}\right)^r_{\ell=1} = \left(p^{(r-r-\ell)}_{(r,d)}, p^{(r-r-\ell+1)}_{(r,d)}, \ldots, p^{(r,0)}_{(r,d)}\right),$$

(4)
is an interlacing sequence (see \cite{35} Remark 4.2 or \cite{27}, for instance). On the other hand, \cite{2} Equation 21 shows that for any $d$-dimensional simplicial complex $C$

\[
h(C^{(r)}; x) = ((1 + x + x^2 + \cdots + x^{r-1})^{d+1} h(\Delta; x))^{(r,0)}
\]

for all $r \geq 1$. As in Subsection 4.2, let $\Delta_d$ denote the $d$-dimensional simplex, and let $\Delta_{d,\ell}$ denote the relative simplicial complex given by removing $\ell$ of the facets of $\Delta_d$ for $0 \leq \ell \leq d + 1$.

**Lemma 4.13.** Let $d \geq 1$, $r > d$, and $0 < \ell \leq d + 1$. Then

\[
h(\Delta_{d,\ell}^{(r)}; x) = x^p_{(r, d+\ell)}.
\]

**Proof.** Notice first that the relative complex $\Delta_{d,\ell}^{(r)}$ can be constructed in two equivalent ways: Either we first remove the $\ell$ facets of $\Delta_d$ and then apply the subdivision procedure outlined in the definition of the edgewise subdivision to the corresponding geometric realization of the half-open simplex $\Delta_{d,\ell}$, or we first compute $\Delta_{d}^{(r)}$ and then remove the faces of $\Delta_{d}^{(r)}$ lying in the $\ell$ facets of $\Delta_d$ scheduled for removal. For the purposes of this proof, we work with the latter construction. Our first goal, then, is to prove the following fact in analogy to equation (5):

\[
h(\Delta_{d,\ell}^{(r)}; x) = ((1 + x + \cdots + x^{r-1})^{d+1} h(\Delta_{d,\ell}; x))^{(r,0)}.
\]

Given the chosen construction of $\Delta_{d,\ell}^{(r)}$, we know that

\[
f(\Delta_{d,\ell}^{(r)}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(\Delta_{d-j}^{(r)}; x),
\]

and so

\[
h(\Delta_{d,\ell}^{(r)}; x) = (1 - x)^{d+1} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(\Delta_{d-j}^{(r)}; \frac{x}{1 - x}),
\]

\[
= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (1 - x)^j h(\Delta_{d-j}^{(r)}; x).
\]

Since $\Delta_{d-j}$ is a simplicial complex, it follows from equation (5) that

\[
h(\Delta_{d-j}^{(r)}; x) = ((1 + x + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{(r,0)}.
\]

Since

\[
(1 - x)^j((1 + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{(r,0)}
\]

\[
= ((1 - x^r)^j(1 + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{(r,0)},
\]

it follows that

\[
h(\Delta_{d,\ell}^{(r)}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} ((1 - x^r)^j(1 + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{(r,0)},
\]

\[
= \left((1 + \cdots + x^{r-1})^{d+1} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (1 - x)^j h(\Delta_{d-j}; x)\right)^{(r,0)},
\]

\[
= ((1 + \cdots + x^{r-1})^{d+1} h(\Delta_{d,\ell}; x))^{(r,0)},
\]
as desired. We then note that \( h(\Delta_{d,\ell}; x) = x^\ell \), by Lemma 4.8. Since \( r > d \) and \( 0 < \ell \leq d + 1 \), it follows that
\[
h(\Delta_{d,\ell}; x) = x^p_{(r,d+1)},
\]
which completes the proof. \( \square \)

Lemma 4.13 gives the necessary tools to positively answer Problem 4.12 for shellable simplicial complexes.

**Theorem 4.14.** Let \( C \) be a \( d \)-dimensional shellable simplicial complex, and let \( r > d \). Then \( h(C^{(r)}; x) \) is real-rooted.

**Proof.** Since \( C \) is a shellable polytopal complex, then we can write \( C^{(r)} \) as a disjoint union of relative simplicial complexes \( R_i \), one for each facet in a shelling order \((F_1, \ldots, F_s)\) of \( C \), such that
\[
h(C^{(r)}; x) = \sum_{i=1}^{s} h(R_i; x),
\]
as in the hypothesis of Theorem 3.1. Since each \( R_i \) is equal to \( \Delta_{d,\ell}^{(r)} \) for some \( 0 \leq \ell \leq d + 1 \), then, by Lemma 4.13 and equation (5), \( h(C^{(r)}; x) \) is a convex combination of the polynomials in the sequence
\[
\left( p_{(r,0)}, x_{(r,d+1)}, x_{(r,d+1)}, \cdots, x_{(r,d+1)}, x_{(r,d+1)} \right).
\]
Since the sequence (4) is interlacing, it follows from Lemma 2.4 that this sequence is also interlacing. By Theorem 3.1, we conclude that \( h(C^{(r)}; x) \) is real-rooted. \( \square \)

As an immediate corollary to Theorem 4.14 we get that, for \( r \geq d \), the \( r \)-th edge-wise subdivision of the boundary complex of any \( d \)-dimensional simplicial polytope has a real-rooted \( h \)-polynomial.

**Corollary 4.15.** Let \( C \) be the boundary complex of a \( d \)-dimensional polytope. Then for \( r \geq d \), the edgewise subdivision \( C^{(r)} \) of \( C \) has a real-rooted \( h \)-polynomial.

On the other hand, Theorem 4.14 holds more generally. The following gives a complete answer to Problem 4.12 (i.e., for both shellable and nonshellable complexes).

**Theorem 4.16.** If \( C \) is a \( d \)-dimensional simplicial complex with \( h_{k}(C) \geq 0 \) for all \( 0 \leq k \leq d + 1 \) then \( h(C^{(r)}; x) \) is real-rooted whenever \( r > d \).

**Proof.** The proof is given by combining an observation of Athanasiadis in [2] with some recent results of Jochemko [27]. Combining [27, Theorem 1.1] with [27, Lemma 3.1] for \( i = 0 \), we see that the polynomial
\[
((1 + x + x^2 + \cdots + x^{r-1})^{d+1} p)^{(r,0)}
\]
has only real zeros whenever \( p \) is degree \( d + 1 \) with only nonnegative coefficients and \( r > d \). On the other hand, it follows from equation (5) that
\[
h(C^{(r)}; x) = ((1 + x + x^2 + \cdots + x^{r-1})^{d+1} h(C; x))^{(r,0)}
\]
for all \( r \geq 1 \). The result follows. \( \square \)
Theorem 4.16 shows that the geometric approach used in Theorem 4.14 was not necessary, as it was for the solution to Problem 4.1 for cubical complexes given in Theorem 4.7. On the other hand, the geometric proof of Theorem 4.14 highlights that the applications of Theorem 3.1 are not limited to barycentric subdivisions.

5. On Families of Cubical Polytopes

Despite the growing literature on cubical polytopes, there exist relatively few families of explicitly constructed cubical polytopes. This phenomenon is mentioned, and the short list of constructions for cubical polytopes is catalogued, in the thesis [34]. These constructions tend to be recursive in nature, and hence lend themselves to proofs of real-rootedness via interlacers. While Theorem 4.7 demonstrates that the $h$-polynomial of the barycentric subdivision of the boundary complex of any cubical polytope is real-rooted, it is natural to ask whether the known families of recursively constructible cubical polytopes admit an alternative proof of this result that is more intimately tied to their specific recursive nature. In this section, we study this question for two well-studied, recursively constructed, families of cubical polytopes: the capped cubical polytopes and the cuboids. In Subsection 5.1, we give an alternative proof of Theorem 4.7 for capped cubical polytopes based on their recursive definition. In Subsection 5.2, we give some partial results in this direction for the cuboids and make an associated conjecture.

5.1. Capped cubical polytopes. Capped cubical polytopes, or stacked cubical polytopes, are a classically studied family of cubical polytopes that are the cubical analogue to stacked simplicial polytopes [29]. A polytope $P$ is called capped over a given cubical polytope $Q$ if there is a combinatorial cube $C$ such that $P = Q \cup C$ and $F := Q \cap C$ is a facet of $Q$. In this case, we then think of $P$ as produced by capping $Q$ over $F$, and we write $P = \text{capped}(Q, F)$.

We say that a polytope is $\ell$-fold capped cubical for some $\ell \in \mathbb{Z}_{\geq 0}$ if it can be obtained from a combinatorial cube by $\ell$ capping operations.

In the following, let $C$ denote a $(d - 1)$-dimensional cubical complex that is the boundary complex of an $\ell$-fold capped polytope. Let $\Box_d$ denote the (combinatorial) $d$-cube and let $\partial \Box_d$ denote the boundary complex of $\Box_d$ with the interior of one facet removed. Our goal in this subsection is to show that $h(\text{sd}(\partial \Box_d); x)$ is real-rooted whenever $C$ is the boundary complex of an $\ell$-capped cubical polytope $P$ for some $\ell \in \mathbb{Z}_{\geq 0}$ by way of the recursive construction of $P$. To do so, we require some lemmata.

Let $A^{(2)}_d := A^{(2)}_{d,0}$ denote the $d^{th}$ Type B Eulerian polynomial, and recall from the introduction to Section 4 that $A^{(2)}_d$ is symmetric with respect to $d$, with nonnegative coefficients and only real-zeros [39]. For the next two lemmas we will need to use the theory of interlacing polynomials. In particular, we will make use of the properties of interlacing polynomials introduced in Subsection 2.2.

Lemma 5.1. For $d \geq 1$,

$$A^{(2)}_{d-1} \prec h(\text{sd}(\partial \Box_d); x).$$

Proof. First consider the $f$-polynomial $f(\text{sd}(\partial \Box_d); x)$. Since $\partial \Box_d$ is the boundary complex of $\Box_d$ with the interior of a facet removed, it follows that $f_0(\text{sd}(\partial \Box_d))$
is equal to $f_k(sd(\partial \varDelta_d))$ minus the number of $k$-dimensional faces intersecting the interior of the removed facet. Since the facet removed is a $(d-1)$-cube, we know that the number of such $k$-dimensional faces is equal to the number of $(k-1)$-dimensional faces in $sd(\partial \varDelta_{d-1})$, the subdivision of the boundary of the facet. Hence,

$$f(sd(\partial \varDelta_d); x) = f(sd(\partial \varDelta_d); x) - xf(sd(\partial \varDelta_{d-1}); x).$$

Since $sd(\partial \varDelta_d)$ and $sd(\partial \varDelta_d)$ are $(d-1)$-dimensional and $sd(\partial \varDelta_{d-1})$ is $(d-2)$-dimensional, applying the $f$-to-$h$ transformation yields

$$h(sd(\partial \varDelta_d); x) = h(sd(\partial \varDelta_d); x) - xh(sd(\partial \varDelta_{d-1}); x).$$

So by Lemma 4.3 we see that

$$h(sd(\partial \varDelta_d); x) = A^{(2)}_d - xA^{(2)}_{d-1}.$$

Thus, it suffices to show that $A^{(2)}_{d-1} < A^{(2)}_d - xA^{(2)}_{d-1}$. To see this, we first note that $A^{(2)}_{d-1} < A^{(2)}_d - A^{(2)}_{d-1}$. This follows since $A^{(2)}_{d-1} < A^{(2)}_d$ implies $A^{(2)}_{d-1} < -A^{(2)}_{d-1}$. Since, $A^{(2)}_{d-1} < A^{(2)}_d$ (by, for instance, [7] Theorem 4.6), combining these two observations with Proposition 2.5 yields

$$A^{(2)}_{d-1} < A^{(2)}_d - A^{(2)}_{d-1},$$

as desired. We now use this observation to show that $A^{(2)}_{d-1} - xA^{(2)}_{d-1}$. To do so, we first make a few observations about the polynomials $A^{(2)}_{d-1}$ and $A^{(2)}_d - A^{(2)}_{d-1}$. First, for all $d \geq 1$, $A^{(2)}_d$ is the $h$*-polynomial of the $d$-cube $[-1,1]^d$. Hence the constant term of $A^{(2)}_d$ is 1 (see, for instance, [7]). Since $A^{(2)}_d$ is symmetric with respect to $d$, we have that

$$\mathcal{I}_d A^{(2)}_d = A^{(2)}_d.$$

It follows from these observations that $A^{(2)}_d - A^{(2)}_{d-1}$ has only nonnegative integer coefficients and constant term 0. Thus, by Remark 2.1 the observation $A^{(2)}_{d-1} < A^{(2)}_d - A^{(2)}_{d-1}$ implies that $A^{(2)}_d - A^{(2)}_{d-1}$ has only nonpositive real zeros, which we denote as

$$\alpha_d \leq \alpha_{d-1} \leq \cdots \leq \alpha_2 \leq \alpha_1 = 0.$$

Since $A^{(2)}_{d-1}$ is real-rooted of degree $d-1$ with nonnegative coefficients and constant term 1, we know that it has only negative zeros, which we denote as

$$\beta_{d-1} \leq \cdots \leq \beta_2 \leq \beta_1 < 0.$$

Since $A^{(2)}_d - A^{(2)}_{d-1}$ we then have that

$$\alpha_d \leq \beta_{d-1} \leq \alpha_{d-1} \leq \cdots \leq \beta_2 \leq \alpha_2 \leq \beta_1 < \alpha_1 = 0.$$

(6)

Since for all $d \geq 1$, $A^{(2)}_d$ is symmetric with respect to $d$, it follows that

$$\mathcal{I}_d \left( A^{(2)}_d - A^{(2)}_{d-1} \right) = \mathcal{I}_d A^{(2)}_d - x \mathcal{I}_d A^{(2)}_{d-1} = A^{(2)}_d - x A^{(2)}_{d-1}.$$

It follows that the zeros of $A^{(2)}_d - x A^{(2)}_{d-1}$ are

$$\frac{1}{\alpha_2} \leq \cdots \leq \frac{1}{\alpha_{d-1}} \leq \frac{1}{\alpha_d} < 0,$$
and the zeros of $I_dA_{d-1}^{(2)} = x I_{d-1}A_{d-1}^{(2)}$ are
\[
\frac{1}{\beta_1} \leq \frac{1}{\beta_2} \leq \cdots \leq \frac{1}{\beta_{d-1}} < 0,
\]
plus the addition zero 0. It follows from (3) that
\[
\frac{1}{\beta_1} \leq \frac{1}{\alpha_2} \leq \frac{1}{\beta_2} \leq \cdots \leq \frac{1}{\beta_{d-2}} \leq \frac{1}{\alpha_{d-1}} \leq \frac{1}{\beta_{d-1}} \leq \frac{1}{\alpha_d} < 0.
\]
Since $I_dA_{d-1}^{(2)} = x I_{d-1}A_{d-1}^{(2)} = xA_{d-1}^{(2)}$, and $A_{d}^{(2)} - xA_{d-1}^{(2)}$ have positive leading coefficients, it follows from Remark 2.1 that
\[
A_{d}^{(2)} - xA_{d-1}^{(2)} \prec xA_{d-1}^{(2)}.
\]
Since both polynomials have only nonpositive zeros, we conclude, via Lemma 2.4, that
\[
A_{d-1}^{(2)} \prec A_{d}^{(2)} - xA_{d-1}^{(2)},
\]
which completes the proof. □

Since $\ell$-capped cubical polytopes can be constructed recursively, our goal is to use the result of the previous lemmas to verify that the $h$-polynomial of the barycentric subdivision of the boundary complex of such a polytope is real-rooted. To this end, we will let $\mathcal{C}$ denote the boundary complex of a $d$-dimensional $\ell$-capped cubical polytope $P$. Suppose that $P = \text{capped}(Q, F)$ and let $D$ denote the boundary complex of the $(\ell - 1)$-capped cubical polytope $Q$.

Since $\text{sd}(D)$ and $\text{sd}(F)$ are both $(d - 1)$-dimensional, it follows that
\[
h(\text{sd}(D) \setminus \text{sd}(F); x) = h(\text{sd}(D); x) - h(\text{sd}(F); x). \tag{7}
\]
Additionally, by linearity of the $f$-to-$h$ transformation and the fact that $\text{sd}(C)$, $\text{sd}(D) \setminus \text{sd}(F)$, and $\text{sd}(\partial \widetilde{D})$ are all $(d - 1)$-dimensional, it follows that
\[
h(\text{sd}(C); x) = h(\text{sd}(D) \setminus \text{sd}(F); x) + h(\text{sd}(\partial \widetilde{D}); x). \tag{8}
\]
Combining these observations, we can then prove the following theorem.

**Theorem 5.2.** Let $\mathcal{C}$ denote the boundary complex of a $d$-dimensional $\ell$-capped cubical polytope. Then $h(\text{sd}(\mathcal{C}); x)$ is real-rooted. Moreover,
\[
A_{\ell - 1}^{(2)} \prec h(\text{sd}(\mathcal{C}); x).
\]

**Proof.** Note that $\mathcal{C}$ is the boundary complex of a $d$-dimensional $\ell$-capped cubical polytope $P$, where $P = \text{capped}(Q, F)$ for some fact $F$ of a $d$-dimensional $(\ell - 1)$-capped cubical polytope $Q$. So it suffices to prove the result by induction on $\ell$.

First suppose that $\ell = 1$. Then, by definition of an $\ell$-capped cubical polytope, it follows that $Q$ is the $d$-cube. Let $\mathcal{C}$ denote the boundary complex of $P$ and let $D$ denote the boundary complex of $Q$. By Lemma 4.4 and (7), we have that
\[
h(\text{sd}(D) \setminus \text{sd}(F); x) = A_{d}^{(2)} - A_{d-1}^{(2)}.
\]
Since $A_{d-1}^{(2)} \prec -A_{d-1}^{(2)}$ and $A_{d}^{(2)} \prec A_{d-1}^{(2)}$, it follows from Proposition 2.3 that $A_{d-1}^{(2)} \prec h(\text{sd}(D) \setminus \text{sd}(F); x)$. By Lemma 5.1, we have that $A_{d-1}^{(2)} \prec h(\text{sd}(\partial \widetilde{D}); x)$. Thus, by (8) and Proposition 2.3
\[
A_{d-1}^{(2)} \prec h(\text{sd}(D) \setminus \text{sd}(F); x) + h(\text{sd}(\partial \widetilde{D}); x) = h(\text{sd}(\mathcal{C}); x),
\]
which completes the base case.
Now take $\ell > 1$ and suppose that the result holds for all $(\ell - 1)$-capped cubical polytopes of dimension $d$; that is, for the boundary complex $\mathcal{D}$ of any such polytope assume that

$$A_{d-1}^{(2)} < h(\text{sd}(\mathcal{D}); x).$$

Suppose that $P$ is $\ell$-capped and satisfies $P = \text{capped}(Q, F)$ for some facet $F$ of $Q$ where $Q$ is $(\ell - 1)$-capped. Assuming that $P$ has boundary complex $\mathcal{C}$ and $Q$ has boundary complex $\mathcal{D}$, it follows from the inductive hypothesis that $A_{d-1}^{(2)} < h(\text{sd}(\mathcal{D}); x)$.

Since $A_{d-1}^{(2)} < A_{d-1}^{(2)}$, Proposition 2.5 and (7) imply that

$$A_{d-1}^{(2)} \times h(\text{sd}(\mathcal{D}); x) - A_{d-1}^{(2)} = h(\text{sd}(\mathcal{D}); x) - h(\text{sd}(F); x) = h(\text{sd}(\mathcal{D}) \setminus \text{sd}(F); x).$$

Hence, by Lemma 5.1, Proposition 2.5, and (8), it follows that

$$A_{d-1}^{(2)} \times h(\text{sd}(\mathcal{D}) \setminus \text{sd}(F); x) + h(\text{sd}(\partial \mathcal{C}); x) = h(\text{sd}(\mathcal{C}); x).$$

In particular, the $h$-polynomial of the barycentric subdivision of an $\ell$-capped cubical polytope is real-rooted.

5.2. Cuboids. Cuboids are a family of cubical polytopes described by Grunbaum in [24]. For each dimension $d$, there are $d + 1$ cuboids, denoted $Q_d^\ell$ for $0 \leq \ell \leq d$. The first cuboid in dimension $d$, denoted $Q_0^d$, is the $d$-cube, and the rest are defined recursively as follows: To construct $Q_\ell^d$ for $\ell > 0$, glue two (combinatorial) copies of $Q_{\ell-1}^d$ at a common (combinatorial) $Q_{\ell-1}^{d-1}$. The boundary of the resulting complex is the boundary complex of the cuboid $Q_\ell^d$. Equivalently, to construct the $\ell$th $d$-dimensional cuboid for $0 \leq \ell \leq d$, start by taking the geometric realization $[-1, 1]^d$ of $Q_\ell^d$. Then consider the geometric subdivision of $[-1, 1]^d$ given by intersecting $[-1, 1]^d$ with the $\ell$ hyperplanes $x_1 = 0, x_2 = 0, \ldots, x_\ell = 0$. The boundary of the resulting cubical complex is the boundary complex of the cuboid $Q_\ell^d$.

In [34], Schwartz shows that $Q_d^\ell$ is the cubical barycentric subdivision (as defined in [34] Subsection 3.3) of the $d$-dimensional cross polytope; i.e., the convex hull of $\pm e_1, \ldots, \pm e_d$ where $e_1, \ldots, e_d \in \mathbb{R}^d$ are the standard basis vectors in $\mathbb{R}^d$. In the following, we use this fact to show that the barycentric subdivision of $Q_d^\ell$ has real-rooted $h$-polynomial. To do so, we will use local $h$-polynomials, which were introduced in Subsection 2.2 in [3]. Athanasiadis makes the following remark relating the local $h$-polynomial of the second edgewise subdivision (see Subsection 4.3) and the local $h$-polynomial of the barycentric subdivision of the cubical barycentric subdivision of a simplex.

**Lemma 5.3.** [3 Remark 4.5] Let $\Delta$ denote a $d$-dimensional simplex, and let $\Delta^{(2)}$ denote its second edgewise subdivision. Let $\text{sd}(\text{csd}(\Delta))$ denote the barycentric subdivision of the cubical barycentric subdivision of $\Delta$. Then,

$$\ell_\Delta(\text{sd}(\text{csd}(\Delta)); x) = \ell_\Delta(\Delta^{(2)}; x).$$

The last ingredient we need is a theorem of Hurwitz.

**Theorem 5.4.** [22] Theorem 7.64 Let $p \in \mathbb{R}[x]$ be a polynomial with only real, negative zeros. Then, $p^{(2.0)}$ and $p^{(2.1)}$ also have only real, negative zeros.

We can then use Lemma 5.3 to prove the following:

**Theorem 5.5.** Let $\mathcal{C}$ be the boundary complex of $Q_d^\ell$, where $d > 2$. Then $h(\text{sd}(\mathcal{C}); x)$ is real-rooted.
Proof. First, we let $D$ be the boundary complex of the $d$-dimensional cross polytope. Let $sd(csd(D))$ be the barycentric subdivision of the cubical barycentric subdivision of $D$, and let $D^{(2)}$ be the second edgewise subdivision of $D$. By Theorem 2.2, we have the following expressions for $h(sd(csd(D)); x)$ and $h(D^{(2)}; x)$ in terms of local $h$-polynomials:

$$h(sd(csd(D)); x) = \sum_{\Delta \in D} h(link_{D}(\Delta); x) \ell_{\Delta}(sd(csd(D))_{\Delta}; x),$$

and

$$h(D^{(2)}; x) = \sum_{\Delta \in D} h(link_{D}(\Delta); x) \ell_{\Delta}(D^{(2)}_{\Delta}; x).$$

By Lemma 5.3 for all $\Delta \in D$ we have that $\ell_{\Delta}(sd(csd(D))_{\Delta}; x) = \ell_{\Delta}(D^{(2)}_{\Delta}; x)$. Thus, we conclude that $h(sd(csd(D)); x) = h(D^{(2)}; x)$. By equation 5, we then have that

$$h(sd(csd(D)); x) = h(D^{(2)}; x) = ((1 + x)^d h(D; x))^{(2,0)}.$$ 

Since the $h$-polynomial of the boundary complex of the $d$-dimensional cross polytope is $(x + 1)^d$, it follows that

$$h(sd(csd(D)); x) = ((1 + x)^{2d})^{(2,0)}.$$ 

By Theorem 5.4 since $(1+x)^{2d}$ has only real, negative zeros, then $h(sd(csd(D)); x)$ must be real-rooted. As mentioned earlier, the boundary complex of $Q^d_k$ is isomorphic to the cubical barycentric subdivision of the cross polytope, so their barycentric subdivisions must have the same $h$-polynomial. This completes the proof. \qed

Notice that, the recursions defining the $d$-dimensional $\ell$-capped polytopes and the $d$-dimensional cuboids coincide for $\ell = 0, 1$. In particular, $\ell$-capped polytopes use the same recursion as cuboids but always glue along a common $Q^d_0$ as opposed to a common $Q^{d-1}_{k-1}$. Hence, one would hope that a similar argument to that used in Subsection 5.1 could be applied to cuboids. However, the technicalities of such a proof appear more difficult to sort out. Hence, we make the following conjecture.

Conjecture 5.6. Let $d \geq 1$ and $0 < k \leq d$. Let $C$ denote the boundary complex of $sd(Q^d_k)$ with a subcomplex isomorphic $sd(Q^{d-1}_{k-1})$ removed, and let $D$ denote the boundary complex of $sd(Q^d_{k-1})$ with only the interior of this subcomplex removed. Then $h(C; x)$ and $h(D; x)$ have a common interlacer; that is, there exists $p \in \mathbb{R}[x]$ such that $p \prec h(C; x)$ and $p \prec h(D; x)$.

We further suspect that the common interlacer in Conjecture 5.6 should be $A^{(2)}_{d-1}$, just as in Theorem 5.2.

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