Weak convergence of Euler scheme for SDEs with singular drift

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Abstract

In this paper, we investigate the weak convergence rate of Euler-Maruyama’s approximation for stochastic differential equations with irregular drifts. Explicit weak convergence rates are presented if drifts satisfy an integrability condition including discontinuous functions which can be non-piecewise continuous or in fractional Sobolev space.

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1 Introduction

Stochastic differential equations (SDEs for short) with singular coefficients have been extensively studied recently, see \cite{8,16,17,18,19} and references therein. Meanwhile, in order for one to understand the numerical approximation of SDEs with irregular coefficients, numerical schemes are established. The strong and weak convergence rate for singular SDEs are obtained, see \cite{2,3,6,7,9,10,11,12,13,14,15} for instance. Particularly, the strong convergence of Euler-Maruyama’s (abbreviated as EM’s) scheme for discontinuous monotone drifts was investigated in \cite{5}. \cite{13} obtained the strong convergence rates of EM’s scheme for SDEs with drifts satisfying the one-side Lipschitz condition, and \cite{14} investigated the one-dimensional setup without the one-side Lipschitz assumption. Recently, \cite{3} obtains strong convergence rates for multidimensional SDEs under an integrability condition by using the Krylov’s estimate and the result of Gaussian type heat kernel estimates established by the parametrix.
method in [11]. The weak convergence is concerned with the convergence of the distribution of the solutions of SDEs. In this paper, we shall investigate the weak error of EM’s scheme for the following SDE on \( \mathbb{R}^d \)

\[
dX_t = b(X_t)dt + \sigma dW_t, \quad X_0 = x \in \mathbb{R}^d,
\]

where \((W_t)_{t \geq 0}\) is a Brownian motion on \( \mathbb{R}^d \) with respect to a complete filtration probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})\). The associated EM’s scheme reads as follows: for any \( \delta \in (0, 1) \),

\[
dX^{(\delta)}_t = b(X^{(\delta)}_t)dt + \sigma dW_t, \quad X^{(\delta)}_0 = 0,
\]

where \( t^{\delta} = \lfloor t/\delta \rfloor \delta \) and \( \lfloor t/\delta \rfloor \) denotes the integral part of \( t/\delta \). The weak convergence rate is concerned with the approximation of \( \mathbb{E} f(X_t) \) by \( \mathbb{E} f(X^{(\delta)}_t) \) for a given function \( f \). The weak error has been obtained for some SDEs with discontinuous drifts in [6, 7]. However, they required that the given function \( f \) is Hölder continuous. The weak convergence rate with a measurable and bounded function \( f \) can be dated back to [1], where the coefficients of SDEs need to be smooth. Recently, [4, 15] established the weak convergence rate of EM’s scheme for SDEs with irregular coefficients by using Girsanov’s transformation. Inspired by [3] and [4, 15], we shall give a note on the weak error for (1.2) with a possibly discontinuous drift \( b \). Moreover, the given function \( f \) is only assumed to be bounded and measurable on \( \mathbb{R}^d \).

The remainder of this paper is organized as follows: The main result is presented in Section 2. All the proofs are given in Section 3.

## 2 Main Result and Examples

Let \( | \cdot | \) be the Euclidean norm, \( \langle \cdot, \cdot \rangle \) be the Euclidean product. \( \| \cdot \| \) denotes the operator norm. Throughout this paper, we assume the coefficients of (2.2) satisfy the following assumptions:

(H1) \( b : \mathbb{R}^d \to \mathbb{R}^d \) is measurable and \( \sigma \) is an invertible \( d \times d \)-matrix. There exist nonnegative constants \( L_1, L_2 \) such that

\[ |b(x)| \leq L_1 + L_2|x|. \]

(H2) There exist \( p_0 \geq 2, \alpha > 0 \) and \( \phi \in C([0, T]; (0, +\infty)) \) with \( \int_0^T \phi^2(s)ds < \infty \) such that

\[
\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} |b(y) - b(x)|^{p_0} e^{-\frac{|z-x|^2}{s^2}} \frac{|y-x|^2}{r^2} dx dy \leq \left( \phi(s)r^\alpha \right)^{p_0}, \quad s > 0, r \in [0, 1].
\]

By [19, Theorem 1.1], (1.1) has a unique strong solution under (H1). It is clear that (1.2) also has a unique strong solution. We denote \( \| f \|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)| \). We now formulate the main result.
Theorem 2.1. Assume (H1)-(H2). If
\[
TL_2\|\sigma^{-1}\|\|\sigma\|\frac{\sqrt{2(p_0+1)(p_0+3)}}{p_0 - 1} < 1, \tag{2.1}
\]
then for any bounded measurable function \(f\) on \(\mathbb{R}^d\), there exists a constant \(C_{T,p_0,\sigma,x}>0\) such that
\[
|E f(X_t) - E f(X_t^{(d)})| \leq C_{T,p_0,\sigma,x} \|f\|_\infty \delta^\alpha, \quad t \in [0, T]. \tag{2.2}
\]
Additionally, if \(b\) has sublinear growth, i.e. for any \(\epsilon > 0\), there exists \(L(\epsilon) > 0\) such that \(|b(x)| \leq L(\epsilon) + \epsilon|x|\), the convergence holds for any \(T > 0\).

Remark 2.1. When the drift \(b\) is non-regular, the boundedness on \(b\) is needed see e.g. [3, 6, 7]. Here, we allow that \(b\) has linear growth by (H1). If \(b\) is bounded or \(\beta\)-Hölder continuous with \(\beta < 1\), then \(b\) has sublinear growth.

In the condition (H2), if \(\alpha\) is a decreasing function of \(p_0\), then we can choose \(p_0 = 2\) without considering that \(T\) depends on \(p_0\) increasingly, see Example 2.3.

We give several examples to illustrate the condition (H2) and the convergence rate \(\alpha\).

Example 2.2. If \(b\) is the Hölder continuous with exponent \(\beta\), i.e.
\[
|b(y) - b(x)| \leq L|x - y|^\beta,
\]
then (H2) holds with \(\alpha = \frac{\beta}{2}\) and a constant function \(\phi(s)\). It is clear that \(b\) has sublinear growth if \(\beta < 1\). Then for any \(T > 0\), (2.2) holds with \(\alpha = \frac{\beta}{2}\).

Proof. By the Hölder continuity and the fact
\[
\sup_{x \geq 0} x^{\gamma'} e^{-\gamma x^2} = \left(\frac{\gamma'}{2 e \gamma}\right)^{\gamma'/2}, \quad \gamma', \gamma > 0, \tag{2.3}
\]
the assertion follows from the following inequality
\[
\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} |b(y) - b(x)|^p e^{-\frac{|x-z|^2}{s^2} - \frac{|y-z|^2}{r^2}} dx dy
\leq L^p \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^{\beta p} e^{-\frac{|x-z|^2}{s^2} - \frac{|y-z|^2}{r^2}} dx dy
\leq L^p \frac{1}{s^{\frac{\beta p}{2}}} \left(\frac{\beta p r}{e}\right)^{\frac{\beta p}{2}} \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-\frac{|x-z|^2}{s^2}} e^{-\frac{|y-z|^2}{r^2}} dx dy
\leq C L^p \left(\frac{\beta p r}{e}\right)^{\frac{\beta p}{2}}.
\]
\[\square\]
The following example shows that (H2) can hold even if \( b(x) \) is not piecewise continuous.

**Example 2.3.** Let \( A \) be the Smith-Volterra-Cantor set on \([0, 1]\), which is constructed in the following way. The first step, we let \( I_{1,1} = (\frac{3}{32}, \frac{5}{32}) \), \( J_{1,1} = [0, \frac{3}{32}] \), \( J_{1,2} = [\frac{5}{32}, 1] \) and remove the open interval \( I_{1,1} \). The second step, we remove the middle \( \frac{1}{32} \) open intervals, denoting by \( I_{2,1} \) and \( I_{2,2} \), from \( J_{1,1} \) and \( J_{1,2} \) respectively, i.e. \( I_{2,1} = (\frac{5}{32}, \frac{7}{32}) \), \( I_{2,2} = (\frac{25}{32}, \frac{27}{32}) \). The intervals left are denoted by \( J_{2,1}, J_{2,2}, J_{2,3}, J_{2,4}, \) i.e.

\[
J_{2,1} = \left[0, \frac{5}{32}\right], \quad J_{2,2} = \left[\frac{7}{32}, \frac{3}{8}\right], \quad J_{2,3} = \left[\frac{5}{8}, \frac{25}{27}\right], \quad J_{2,4} = \left[\frac{27}{32}, 1\right].
\]

For the \( n \)-th step, we remove the middle \( \frac{1}{2^n} \) open intervals \( I_{n,1}, \ldots, I_{n,2^{n-1}} \) from \( J_{n-1,1}, \ldots, J_{n-1,2^{n-1}} \) respectively, and the intervals left are denoted by \( J_{n,1}, \ldots, J_{n,2^n} \). Let

\[
A = \bigcap_{n=1}^{\infty} \left( \bigcup_{k=1}^{2^n} J_{n,k} \right).
\]

Then \( A \) is a nowhere dense set and the Lebesgue measure of \( A \) is \( 1/2 \). Define

\[
b(x) = \mathbb{1}_{[0,1]}(x) - \lim_{n \to \infty} \sum_{j=1}^{2^{n-1}} 2^{-(n+j)} \mathbb{1}_{I_{n,j}}(x)
\]

\[
= \mathbb{1}_A(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n-1}} (1 - 2^{-(n+j)}) \mathbb{1}_{I_{n,j}}(x).
\]

All of the endpoints of the intervals \( I_{n,j} \) are the discontinuous points of \( b \), which is dense in \( A \). For any interval \( I \subset [0,1] \) such that \( I \cap A \neq \emptyset \), it always contains the discontinuous points of \( b \). However, any interval \( I \subset [0,1] \) such that \( I \cap A = \emptyset \), it is a subset of some \( I_{n,j} \). Hence, \( b \) is not a piecewise continuous function. In the following, we shall show that \( b \) satisfies condition (H2) with \( p_0 = 2 \) and \( \alpha = \frac{1}{2} \) and \( \phi(s) = C s^\frac{1}{2} \).

**Proof.** For \( z > 0 \) and any interval \((a_1, a_2)\) (it is similar for \([a_1, a_2]\))

\[
\int_{-\infty}^{+\infty} \left| \mathbb{1}_{(a_1,a_2)}(x+u) - \mathbb{1}_{(a_1,a_2)}(x) \right|^2 dx
\]

\[
= \int_{-a_2}^{a_2} \mathbb{1}_{(a_1,a_2)}(x)dx + \int_{a_2}^{a_2} \mathbb{1}_{(a_1-a_2,u)}(x)dx
\]

\[
= \int_{-a_2}^{a_2} dx + \int_{(a_2-u)\cap a_1}^{a_2} dx
\]

\[
\leq 2(|u| \wedge (a_2-a_1)).
\]
For $z < 0$,
\[
\int_{-\infty}^{+\infty} \left| \mathbb{1}_{(a_1,a_2)}(x + u) - \mathbb{1}_{(a_1,a_2)}(x) \right|^2 \, dx
\]
\[
= \int_{-\infty}^{+\infty} \left| \mathbb{1}_{(a_1,a_2)}(v) - \mathbb{1}_{(a_1,a_2)}(v - u) \right|^2 \, dv \leq 2 (|u| \wedge (a_2 - a_1)).
\]
Hence, by Jessen’s inequality
\[
\int_{-\infty}^{+\infty} |b(x + u) - b(x)|^2 \, dx
\]
\[
\leq \int_{-\infty}^{+\infty} \left( \left| \mathbb{1}_{[0,1]}(x + u) - \mathbb{1}_{[0,1]}(x) \right| 
+ \sum_{n=1}^{+\infty} \sum_{j=1}^{2^{n-1}} 2^{-(n+j)} \left| \mathbb{1}_{I_n,j}(x + z) - \mathbb{1}_{I_n,j}(x) \right| \right)^2 \, dx
\]
\[
\leq \left( 1 + \sum_{n=1}^{+\infty} \sum_{j=1}^{2^{n-1}} 2^{-(n+j)} \int_{-\infty}^{+\infty} \left| \mathbb{1}_{[0,1]}(x + u) - \mathbb{1}_{[0,1]}(x) \right|^2 \, dx 
+ \sum_{n=1}^{+\infty} \sum_{j=1}^{2^{n-1}} 2^{-(n+j)} \int_{-\infty}^{+\infty} \left| \mathbb{1}_{I_n,j}(x + u) - \mathbb{1}_{I_n,j}(x) \right|^2 \, dx \right)
\]
\[
\leq 2 \left( 1 + \sum_{n=1}^{+\infty} \sum_{j=1}^{2^{n-1}} 2^{-(n+j)} \right)^2 |u| = 4|u|
\]
Therefore
\[
\sup_{z \in \mathbb{R}} \int_{\mathbb{R} \times \mathbb{R}} |b(y) - b(x)|^2 \frac{e^{-|x-z|^2}}{s^{\frac{d}{4}}} e^{-|y-x|^2} \, dx \, dy
\]
\[
\leq \frac{1}{s^{\frac{d}{4}}} \int_{\mathbb{R}} e^{-\frac{|u|^2}{s}} \int_{\mathbb{R}} |b(x + u) - b(x)|^2 \, dx \, du
\]
\[
\leq \frac{4}{s^{\frac{d}{2}}} \int_{\mathbb{R}} e^{-\frac{|u|^2}{s}} |u|^2 \, du = \left( Cs^{\frac{1}{4}} \left( \frac{1}{4} \right) \right)^2.
\]

A general class of functions that satisfies (H2) is the (fractional) Sobolev space $W^{\beta,p}(\mathbb{R}^d)$:

**Example 2.4.** If there exist $\beta > 0$ and $p \in (2, \infty) \cap (\frac{d}{2}, +\infty)$ such that the Gagliardo seminorm of $b$ is finite, i.e.
\[
[b]_{W^{\beta,p}} := \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|b(x) - b(y)|^p}{|x - y|^{d + \beta p}} \, dx \, dy \right)^{\frac{1}{p}} < \infty,
\]
then (H2) holds for any $2 \leq p' < p$ with $\alpha = \frac{\beta}{2}$ and $\phi(s) = Cs^{\frac{d}{4}}$. Hence, if $b$ satisfies (H1) and $[f]_{W^{\alpha,p}} < \infty$ with $p \in (2, \infty) \cap (\frac{d}{2}, +\infty)$, then (2.2) holds.
The proof of this example is similar to that of example in [3], we omit it here.

**Remark 2.2.** In [3], the strong convergence and the convergence rate are investigated under the drift satisfying an integrability condition and boundeness. Here we obtain the weak convergence rate, which the drift needs not be bounded and the function \( f \) in equation (2.2) is only bounded and measurable. From the examples above, one can see that the drift could be very irregular, this means that we have extended the results in [1] where the coefficients must be smooth. However, the method developed here seems difficult to deal with SDEs driven by multiplicative noise. Moreover, our method is not optimal in Lipschitz case since the classical weak rate is \( \alpha = 1 \) for SDEs with smooth coefficients in [1].

### 3 Proof of Theorem 2.1

The key point to prove the main result is to construct a reference SDE, which provides a new representation of (1.1) and its EM’s approximation SDE (1.2) under another probability measures which will be defined by the Girsanov theorem.

Let \( Y_t = \sigma W_t \), which is the reference SDE of (1.1). The first lemma is on the exponential estimate of \( |b(Y_t)| \). We use a weaker condition than that of (H1) on \( b \): there exist nonnegative constants \( L_1, L_2 \) and \( F \geq 0 \) satisfying \( \|F\|_{L^p[\mathbb{R}^d]} \) for some \( p_1 > d \) such that

\[
|b(x)| \leq L_1 + L_2|x| + F(x). \tag{3.1}
\]

By Krylov’s estimate, for any \( q \) such that \( d, p_1, + \frac{1}{q} < 1 \), there exists (see e.g. [8])

\[
\mathbb{E} \left[ \int_S^T F^2(Y_s) ds \right] \leq (T - S)^{\frac{1}{2}} \|F\|_{L^p},
\]

which yields the following Khasminskii’s estimate (see e.g. [8, Lemma 3.5]): for any \( C > 0 \)

\[
\mathbb{E} \exp \left\{ C \int_0^T F^2(Y_s) ds \right\} < \infty. \tag{3.2}
\]

**Lemma 3.1.** Assume (3.1) holds. If \( \lambda \) and \( T \) satisfy

\[
2T^2 \lambda L_2^2 \|\sigma^{-1}\|^2 \|\sigma\|^2 < 1, \tag{3.3}
\]

then

\[
\mathbb{E} \exp \left\{ \lambda \int_0^T |\sigma^{-1} b(Y_s)|^2 ds \right\} < \infty. \tag{3.4}
\]
Proof. By the H"older inequality, we derive that

\[
\mathbb{E} \exp \left\{ \lambda \int_0^T |\sigma^{-1}b(Y_s)|^2 ds \right\} \\
\leq \mathbb{E} \exp \left\{ \lambda \int_0^T \|\sigma^{-1}\|^2 \left( (L_1 + L_2|x|) + L_2|Y_s - x| + F(Y_s) \right)^2 ds \right\} \\
\leq \exp \{\lambda T\|\sigma^{-1}\|^2 (L_1 + L_2|x|)^2 (2 + \varepsilon_1^{-1}) \} \\
\times \left( \mathbb{E} \exp \left\{ \lambda (1 + \varepsilon_1 + \varepsilon_2)^2 L_2^2 \|\sigma^{-1}\|^2 \int_0^T |Y_s - x|^2 ds \right\} \right)^{\frac{1}{1+\varepsilon_1+\varepsilon_2}} \\
\times \left( \mathbb{E} \exp \left\{ \frac{\lambda (2 + \varepsilon_2^{-1})(1 + \varepsilon_1 + \varepsilon_2)}{\varepsilon_1 + \varepsilon_2} \|\sigma^{-1}\|^2 \int_0^T F^2(Y_s) ds \right\} \right)^{\frac{\varepsilon_1 + \varepsilon_2}{1+\varepsilon_1+\varepsilon_2}}. 
\tag{3.5}
\]

It follows from (3.2) that for any \( \varepsilon_2 > 0 \)

\[
\mathbb{E} \exp \left\{ \frac{2\lambda (1 + \varepsilon_2^{-1})(1 + \varepsilon_1 + \varepsilon_2)}{\varepsilon_1 + \varepsilon_2} \|\sigma^{-1}\|^2 \int_0^T F^2(Y_s) ds \right\} < \infty.
\]

Since (3.3), we can choose \( \varepsilon_1 \) and \( \varepsilon_2 \) such that

\[2T^2(1 + \varepsilon_1 + \varepsilon_2)^2 \lambda L_2^2 \|\sigma^{-1}\|^2 ||\sigma||^2 < 1.\]

Then, by the Jensen inequality,

\[
\mathbb{E} \exp \left\{ \lambda (1 + \varepsilon_1 + \varepsilon_2)^2 L_2^2 \|\sigma^{-1}\|^2 \int_0^T |Y_s - x|^2 ds \right\} \\
\leq \frac{1}{T} \int_0^T \mathbb{E} \exp \left\{ T\lambda (1 + \varepsilon_1 + \varepsilon_2)^2 L_2^2 \|\sigma^{-1}\|^2 |Y_s - x|^2 \right\} ds \\
= \int_0^T \int_{\mathbb{R}^d} \exp \left\{ T\lambda (1 + \varepsilon_1 + \varepsilon_2)^2 L_2^2 \|\sigma^{-1}\|^2 |y - x|^2 - \frac{|\sigma^{-1}(y-x)|^2}{2s} \right\} dy ds \\
< \infty. 
\tag{3.6}
\]

The proof is therefore complete. \( \square \)

For the process \( \{Y_t\}_{t \in [0,T]} \), we have the following estimate.

Lemma 3.2. Assume (H1) holds. If \( \lambda \) and \( T \) satisfy (3.3), then

\[
\sup_{\delta > 0} \mathbb{E} \exp \left\{ \lambda \int_0^T |\sigma^{-1}b(Y_{s+\delta})|^2 ds \right\} < \infty. 
\tag{3.7}
\]

Proof. In the same way as in (3.5), we have for any \( \varepsilon_1 > 0 \),

\[
\mathbb{E} \exp \left\{ \lambda \int_0^T |\sigma^{-1}b(Y_s)|^2 ds \right\} \leq \exp \{\lambda T\|\sigma^{-1}\|^2 (L_1 + L_2|x|)^2 (1 + \varepsilon_1^{-1}) \}.
\]
\[
\times \mathbb{E} \exp \left\{ \lambda (1 + \varepsilon_1) L_2^2 \| \sigma^{-1} \|^2 \int_0^T |Y_{s_2} - x|^2 ds \right\}.
\]

By the Jensen inequality, as (3.6), we can choose \( \varepsilon_1 \) such that
\[
\sup_{\delta > 0} \mathbb{E} \exp \left\{ \lambda (1 + \varepsilon_1) L_2^2 \| \sigma^{-1} \|^2 \int_0^T |Y_{s_2} - x|^2 ds \right\} < \infty.
\]

We can now give the Proof of Theorem 2.1. Let
\[
\hat{W}_t = W_t - \int_0^t \sigma^{-1} b(Y_s) ds, \quad \tilde{W}_t = W_t - \int_0^t \sigma^{-1} b(Y_s) ds,
\]
\[
R_{1,T} = \exp \left\{ \int_0^T \langle \sigma^{-1} b(Y_s), dW_s \rangle - \frac{1}{2} \int_0^T |\sigma^{-1} b(Y_s)|^2 ds \right\},
\]
\[
R_{2,T} = \exp \left\{ \int_0^T \langle \sigma^{-1} b(Y_s), dW_s \rangle - \frac{1}{2} \int_0^T |\sigma^{-1} b(Y_s)|^2 ds \right\}.
\]

**Proof of Theorem 2.1.** The proof is divided into two steps:

**Step (i),** we shall prove that the assertion holds under (H1) and (H2).

We first show that \( \{\hat{W}_t\}_{t \in [0,T]} \) is a Brownian motion under \( Q_1 := R_{1,T} \mathbb{P} \), and \( \{\tilde{W}_t\}_{t \in [0,T]} \) is a Brownian motion under \( R_{2,T} \mathbb{P} \). Since (2.1), we have by Lemma 3.1 that
\[
\mathbb{E} \exp \left\{ \frac{(p_0 + 3)(p_0 + 1)}{(p_0 - 1)^2} \int_0^T |\sigma^{-1} b(Y_s)|^2 ds \right\} < \infty.
\]

(3.8)

It is clear that \( \frac{(p_0 + 3)(p_0 + 1)}{(p_0 - 1)^2} > \frac{1}{2} \), so by Novikov’s condition \( \{R_{1,t}\}_{t \in [0,T]} \) is a martingale and \( \{\hat{W}_t\}_{t \in [0,T]} \) is a Brownian motion under \( Q_1 \). Similarly, it follows from (2.1), Lemma 3.1, and Novikov’s condition that \( \{\tilde{W}_t\}_{t \in [0,T]} \) is a Brownian motion under \( Q_2 := R_{2,T} \mathbb{P} \).

We can reformulate \( Y_t \) as follows:
\[
Y_t = x + \int_0^t b(Y_s) ds + \sigma \hat{W}_t,
\]
which means that \( Y_t \) is a weak solution of (1.1). Hence, \( Y_t \) under \( Q_1 \) has the same law of \( X_t \) under \( \mathbb{P} \), since the pathwise uniqueness of the solutions to (1.1).

Similarly, reformulating \( Y_t \) as follows:
\[
Y_t = x + \int_0^t b(Y_s) ds + \sigma \tilde{W}_t,
\]
(3.9)

\( (Y_t, \tilde{W}_t) \) under \( Q_2 \) is also a solution of (1.2), which has a pathwise unique solution. Hence \( Y_t \) under \( Q_2 \) has the same law of \( X_t^{(\delta)} \) under \( \mathbb{P} \).
For every bounded measurable function $f$ on $\mathbb{R}^d$
\[
|\mathbb{E}f(X_t) - \mathbb{E}f(X_t^{(d)})| = |\mathbb{E}Q_1 f(Y_t) - \mathbb{E}Q_2 f(Y_t)|
\leq \|f\|_\infty \mathbb{E}|R_{1,T} - R_{2,T}|
\leq \|f\|_\infty \mathbb{E}\left\{ (R_{1,T} \vee R_{2,T}) \left| \int_0^T \langle \sigma^{-1}b(Y_s) - b(Y_s) \rangle, dW_s \right| \right\}
\leq \|f\|_\infty \mathbb{E}\left\{ \left( \mathbb{E}|R_{1,T}|^{\frac{p_0}{p_0-1}} \right)^{\frac{p_0-1}{p_0}} + \left( \mathbb{E}|R_{2,T}|^{\frac{p_0}{p_0-1}} \right)^{\frac{p_0-1}{p_0}} \right\}
\times \left\{ \int_0^T \mathbb{E}\left| \sigma^{-1}b(Y_s) \right|^2 - |\sigma^{-1}b(Y_s)|^2 \mathbb{E}^{\frac{1}{2}} \right\}
\times \int_0^T \mathbb{E}\left| \sigma^{-1}b(Y_s) \right|^2 - |\sigma^{-1}b(Y_s)|^2 \mathbb{E}^{\frac{1}{2}} \right\}.
\]

Define the stopping time $\tau_{1,n} = \inf\{ t > 0 : \int_0^t |\sigma^{-1}b(Y_s)|^2 ds \geq n \}$. Then $\hat{W}_{n,t} = W_t - \int_0^{\tau_{1,n}} |\sigma^{-1}b(Y_s)|^2 ds$ is a Brownian motion under $Q_1$. By Hölder’s inequality, we arrive at
\[
\mathbb{E}R_{1,T}^{\frac{p_0}{p_0-1}} = \mathbb{E}\exp\left\{ \frac{p_0}{p_0-1} \int_0^{\tau_{1,n}} \langle \sigma^{-1}b(Y_s), dW_s \rangle \right\}
\leq \left( \mathbb{E}M_{1,T}^{\tau_{1,n}} \right)^{1/2} \left( \mathbb{E}\exp\left\{ \frac{p_0(p_0+1)}{(p_0-1)^2} \int_0^{\tau_{1,n}} |\sigma^{-1}b(Y_s)|^2 ds \right\} \right)^{1/2}
= \left( \mathbb{E}\exp\left\{ \frac{p_0(p_0+1)}{(p_0-1)^2} \int_0^{\tau_{1,n}} |\sigma^{-1}b(Y_s)|^2 ds \right\} \right)^{1/2},
\]
where
\[
M_{1,T}^{\tau_{1,n}} = \exp\left\{ \frac{2p_0}{p_0-1} \int_0^{\tau_{1,n}} \langle \sigma^{-1}b(Y_s), dW_s \rangle \right\}
- 2 \left( \frac{p_0}{p_0-1} \right)^2 \int_0^{\tau_{1,n}} |\sigma^{-1}b(Y_s)|^2 ds \right\},
\]
which is an exponential martingale. Similarly,
\[
\mathbb{E}R_{1,T}^{\frac{p_0+1}{p_0}} \leq \left( \mathbb{E}\exp\left\{ \frac{(p_0+3)(p_0+1)}{(p_0-1)^2} \int_0^{\tau_{1,n}} |\sigma^{-1}b(Y_s)|^2 ds \right\} \right)^{1/2}.
\]
It is clear that \( \frac{(p_0+3)(p_0+1)}{(p_0-1)^2} > \frac{p_0(p_0+1)}{(p_0-1)^2} \). Then, due to (2.1), Lemma 3.1 and (3.8), we have by letting \( n \to +\infty \) that
\[
\mathbb{E} \left( R_{1,T}^{p_0-1} + R_{1,T}^{p_0} \right) < \infty.
\]
Similarly, we can prove by Lemma 3.2 that
\[
\mathbb{E} \left( R_{2,T}^{p_0-1} + R_{2,T}^{p_0} \right) < \infty.
\]
By (H1), for \( s \geq \delta \), we have
\[
\mathbb{E} |b(Y_s) - b(Y_{s_0})|^{p_0} = \mathbb{E} |b(x + \sigma W_s) - b(x + \sigma W_{s_0})|^{p_0}
\]
\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(x+y) - b(x+z)|^{p_0} e^{\frac{|y-z|}{2\delta}} e^{\frac{|y-z|^2}{2(s-s_\delta)}} \, dx \, dy
\]
\[
\leq \frac{||\sigma||^{2d}}{\pi^d \det(\sigma \sigma^*)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(v) - b(u)|^{p_0} e^{\frac{|v-u|}{2\delta}} e^{\frac{|v-u|^2}{2(s-s_\delta)}} \, du \, dv
\]
\[
\leq \frac{||\sigma||^{2d}}{\pi^d \det(\sigma \sigma^*)} (\phi(2s_\delta||\sigma||^2)(2(s-s_\delta)||\sigma||^2)\alpha)^{p_0}.
\]
Noting that
\[
\lim_{\delta \to 0+} \sum_{k=1}^{[T/\delta]} \phi^2(2k\delta||\sigma||^2)\delta = \int_0^T \phi^2(2||\sigma||^2r) \, dr
\]
\[
= \frac{1}{2||\sigma||^2} \int_0^{2||\sigma||^2T} \phi^2(s) \, ds < \infty,
\]
the BDG inequality, (3.11), (3.12) and \( p_0 \geq 2 \) imply that
\[
G_{1,T} = \left( \mathbb{E} \left| \int_0^T \langle \sigma^{-1}(b(Y_s) - b(Y_{s_0})), dW_s \rangle \right|^{p_0} \right)^{1/p_0}
\]
\[
\leq \left( \frac{p_0}{p_0-1} \right)^{\frac{p_0}{2}} \left( \frac{p_0(p_0-1)}{2} \right)^{\frac{1}{2}} ||\sigma^{-1}|| \left( \int_0^T \mathbb{E} |b(Y_s) - b(Y_{s_0})|^{p_0} \, ds \right)^{\frac{1}{2}}
\]
\[
\leq \delta^\alpha 2^{\alpha-1} ||\sigma||^{\frac{2\alpha-1}{p_0}} ||\sigma^{-1}||^{\frac{1}{p_0}} \left( \frac{p_0}{p_0-1} \right)^{\frac{p_0}{2}} \left( \frac{p_0(p_0-1)}{2} \right)^{\frac{1}{2}} \int_0^{2||\sigma||^2T} \phi^2(s) \, ds
\]
\[
= C_{T,p_0,\alpha,\delta} \delta^\alpha.
\]
Noting that for any \( p \geq 1 \)
\[
\mathbb{E} |Y_1|^p \leq 2^{p-1} \left( |x|^p + (\sqrt{t}||\sigma||)^p \mathbb{E} |W_1|^p \right),
\]
\[
10
\]
we have
\[
\left( \mathbb{E}|b(Y_s) + b(Y_{s+})|^{\frac{p_0(p_0+1)}{p_0-1}} \right)^{\frac{p_0-1}{p_0(p_0+1)}} \\
\leq \left( \mathbb{E} (2L_1 + L_2(|Y_s| + |Y_{s+}|))^{\frac{p_0(p_0+1)}{p_0-1}} \right)^{\frac{p_0-1}{p_0(p_0+1)}} \\
\leq 6 \left\{ L_1 + 2L_2 \left( |x| + \sqrt{T} \| \sigma \| \left( \mathbb{E}|W_1|^{\frac{p_0(p_0+1)}{p_0-1}} \right)^{\frac{p_0-1}{p_0(p_0+1)}} \right) \right\} \\
=: C_{T,p_0,\sigma,L_1,L_2,x}.
\]

Combining this with (H1), (H2) and taking the similar argument as in (3.13), we obtain
\[
G_{2,T} = \frac{1}{2} \int_0^T \left( \mathbb{E}|\sigma^{-1}b(Y_s)|^2 - |\sigma^{-1}b(Y_s)|^2 \right)^{\frac{p_0+1}{2}} \frac{p_0+1}{p_0-1} ds \\
\leq \frac{1}{2} \int_0^T \left( \mathbb{E}|b(Y_s) - b(Y_{s+})|^{\frac{p_0+1}{2}} |b(Y_s) + b(Y_{s+})|^{\frac{p_0+1}{2}} \right)^{\frac{p_0}{p_0-1}} ds \\
\leq \frac{1}{2} \int_0^T \left( \mathbb{E}|b(Y_s) - b(Y_{s+})|^{\frac{p_0}{2}} \left( \mathbb{E}|b(Y_s) + b(Y_{s+})|^{\frac{p_0(p_0+1)}{p_0-1}} \right)^{\frac{p_0-1}{p_0(p_0+1)}} \right)^{\frac{p_0}{p_0-1}} ds \\
\leq \frac{1}{2} \|\sigma^{-1}\|^2 C_{T,p_0,\sigma,L_1,L_2,x} \int_0^T \left( \mathbb{E}|b(Y_s) - b(Y_{s+})|^{\frac{p_0}{2}} \right)^{\frac{1}{p_0}} ds \\
\leq C_{T,p_0,\sigma,L_1,L_2,\phi,x} \delta^\alpha, \tag{3.15}
\]

where
\[
C_{T,p_0,\sigma,L_1,L_2,\phi,x} = \frac{2^{\alpha-2} \|\sigma\|^{\frac{2\alpha}{p_0}+2\alpha-2} \|\sigma^{-1}\|^2 C_{T,p_0,\sigma,L_1,L_2,x} (\pi^d \det(\sigma\sigma^*))^{\frac{1}{p_0}}}{(\pi^d \det(\sigma\sigma^*))^{\frac{1}{p_0}}} \int_0^2 \|\sigma\|^2 \phi(s) ds.
\]

Therefore, the conclusion holds under (H1) and (H2).

Step (ii), we claim that if \( b \) satisfies the sublinear condition, then convergence holds for any \( T > 0 \). In fact, for any given \( T > 0 \), we can always choose \( L_2 = \epsilon > 0 \) small enough such that (2.1) holds. Then (2.2) holds.

The proof is therefore complete. \( \square \)

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