Improving quantum interferometry by using entanglement

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Abstract

We address the use of entanglement to improve the precision of generalized quantum interferometry, \textit{i.e.} of binary measurements aimed to determine whether or not a perturbation has been applied by a given device. For the most relevant operations in quantum optics, we evaluate the optimal detection strategy and the ultimate bounds to the minimum detectable perturbation. Our results indicate that entanglement-assisted strategies improve the discrimination in comparison with conventional schemes. A concrete setup to approach performances of the optimal strategies is also suggested.

I. INTRODUCTION

Any interferometric setup is devised to reveal minute perturbations to a given configuration. Such perturbations may be induced by the environment or by the action of a given device. In an interferometer, the internal quantum operation is monitored by probing the output state, which, in turn, results from the evolution of a given input. By suitably choosing the input signal and the detection stage one optimizes the interferometric measurement. Optimization has two main goals: i) to maximize the probability of revealing a perturbation, when it occurs, and ii) to minimize the value of the smallest perturbation that can be effectively detected.

In essence, any interferometric scheme may be viewed as a binary communication system \cite{1,2}, with the perturbation playing the role of the encoded information. In order to see better this analogy let us consider the scheme shown in Fig. 1a. A source $S$ of quantum state prepares the input signal, say $\rho_0$, which travels along the interferometer, and it is eventually measured by some detector, denoted by $D$. The detector is described by an operator-valued probability measure (POVM) $\Pi(x)$, with $x \in X$, $X$ being the manifold describing the possible detection outcomes. Inside the interferometer we have generic quantum device, which may or may not perturb the signal, \textit{i.e.} it performs the quantum operation described by the unitary $U_\lambda$. If a perturbation occurs the signal is modified and, at the output, we
have the state \( \varrho_\lambda = U_\lambda \varrho_0 U_\lambda^\dagger \). The aim of the detection stage is to discriminate between \( \varrho_0 \) and its perturbed version \( \varrho_\lambda \). An optimized interferometer is a device that is able to tell which \( \varrho \), for \( \lambda \) as small as possible. Posed in this way, interferometry is naturally viewed as a binary decision problem, and the detection stage can be described by a two-value POVM \( \{ \Pi_0, \Pi_\lambda \equiv \mathbb{I} - \Pi_0 \} \), which corresponds to the two possible inferences.

The main goal of the present paper is to demonstrate the benefits of entanglement in binary interferometry. We will show that distinguishability of the two hypothesis (\( H_0 \): nothing happened and \( H_\lambda \): a perturbation has occurred) can be improved by: i) using an input signal which is entangled with another subsystem, and ii) measuring the two systems jointly at the output of the interferometer (see Fig. 1b).

In order to optimize the detection strategies, and to show the benefits of entanglement, we will make use of results and methods from quantum detection theory applied to binary decision [3,4]. This approach is particularly useful for our purposes, since it does not refer to any specific detection scheme for the final stage of the interferometer, but rather, owing to its generality, it allows to find the ultimate quantum limits to interferometry for specific classes of quantum signals.

In the next Section, in order to establish notation, we briefly review the Neyman-Pearson approach to quantum binary decision, and state a lemma about minimum input-output overlap. Then, in Section III we apply these results to the interferometric detection of perturbations induced by the most relevant operations in quantum optics such as displacement, squeezing, mixing and phase-shifting. As we will see, entanglement-assisted interferometers provide better discrimination than conventional schemes. In Section IV we analyze an interferometric configuration that achieves, for the quantum operations discussed in Section III, the ultimate bounds to precision. Finally, in Section V we close the paper with some concluding remarks.

II. QUANTUM BINARY DECISIONS IN THE NEYMAN-PEARSON APPROACH

The problem that we are facing is to decide among two hypotheses \( H_0 \) and \( H_1 \) about the state of a system, which is described by a density operator \( \varrho \) on the Hilbert space of the system. To each hypotheses it will correspond a different density operator as follows

\[
\begin{align*}
H_0 : \text{the system is in the state } \varrho_0, \\
H_\lambda : \text{the system is in the state } \varrho_\lambda.
\end{align*}
\]

(1)

Of course, there are many different measurements which can provide information about the state of the system: each of them, however, can be recast mathematically as a two-value POVM, corresponding to the two possible inferences \( H_0 \) and \( H_1 \), namely

\[
\Pi_0, \Pi_\lambda \geq 0 \quad \Pi_0 + \Pi_\lambda = \mathbb{I}.
\]

(2)

One then needs an optimization strategy in order to determine the most reliable measurement discriminating between the two states. If \( \varrho_0 \) and \( \varrho_\lambda \) are orthogonal i.e. \( \varrho_0 \varrho_\lambda = \varrho_\lambda \varrho_0 = 0 \) the solution is trivial, since \( \Pi_0 \) is the projection into any subspace that contains the support of \( \varrho_0 \) and is orthogonal to the support of \( \varrho_\lambda \), and \( \Pi_\lambda \) is simply the complement \( \Pi_\lambda = \mathbb{I} - \Pi_0 \).
In most cases of interest, however, the states are not orthogonal and one has to apply an optimization strategy. Since interferometric schemes are frequently used for detecting low-rate events, we may want to look for a strategy that keeps a low-rate of false alarm, namely of wrong inference of perturbation occurrence. For this purpose, it is suitable to adopt a so-called Neyman-Pearson (NP) detection strategy, which consists in fixing a tolerable value of the false-alarm probability $Q_0$—the probability of inferring that the state of the system is $\varrho_\lambda$ while it is actually $\varrho_0$—and then maximizing the detection probability $Q_\lambda$, i.e. the probability of a correct inference of hypothesis $\mathcal{H}_\lambda$ [4]. It has been proved by Helstrom [3] and Holevo [4] that this problem can be solved by diagonalizing the operator

$$\varrho_\lambda - \mu \varrho_0,$$

$\mu$ playing the role of a Lagrange multiplier accounting for the bound of fixed false alarm probability. According to [3] the optimal POVM is the one in which $\Pi_\lambda$ is the projection onto the eigenspaces of (3) relative to positive eigenvalues and $\Pi_0 = I - \Pi_\lambda$. Unfortunately, the diagonalization of (3) is generally not easy. However, when $\varrho_0 = |\psi_0\rangle\langle\psi_0|$ and $\varrho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ are pure states it can be easily solved analytically, by expanding $|\psi_0\rangle$ and $|\psi_\lambda\rangle$ on the eigenvectors of the difference operator (3). In this way one can evaluate both $Q_0$ and $Q_\lambda$ versus $\mu$, and after eliminating $\mu$ from their expressions one obtains

$$Q_\lambda = \begin{cases} \sqrt{Q_0|\kappa|^2 + (1 - Q_0)(1 - |\kappa|^2)}^2 & \text{for } 0 \leq Q_0 \leq |\kappa|^2, \\ 1 & \text{for } |\kappa|^2 < Q_0 \leq 1. \end{cases}$$

(4)

where $|\kappa|^2 = |\langle\psi_0|\psi_\lambda\rangle|^2 = |\langle\psi_0|U_\lambda|\psi_0\rangle|^2$ is the overlap between the two states. The detection probability is a decreasing function of the overlap—the smaller the overlap, the easier the discrimination—since one can reach detection probability 1 while keeping a low false alarm probability. On the contrary, when the overlap approaches 1 one is forced to decrease the detection probability in order to keep the false alarm probability small.

The optimal choice of the probe that minimizes the overlap, depends on the eigenvalues of the unitary operation $U_\lambda$. In order to illustrate this, let us expand $U_\lambda$ in terms of its eigenvectors $U_\lambda = \sum_j e^{i\phi_j} |\varphi_j\rangle\langle\varphi_j|$ (with integrals replacing sums in case of continuous spectrum) and let’s denote by $\mathcal{O}(U_\lambda) = \min_\psi |\langle\psi|U_\lambda|\psi\rangle|^2$ the minimum overlap between the two possible outputs, as obtained by varying the probe state. Then we have the following overlap Lemma [5]: the minimum overlap $\mathcal{O}(U_\lambda)$ is given by the distance from the origin in the complex plane of the polygon whose vertices are the eigenvalues of $U_\lambda$. Therefore, the overlap is either zero (if the polygon includes the origin) or it is given by

$$\mathcal{O}(U_\lambda) = \cos^2 \frac{\Delta \varphi}{2},$$

(5)

where $\Delta \varphi$ is the angular spread of the eigenvalues. Zero overlap can be achieved with a probe state that is given by a superposition of at least three eigenvectors of $U_\lambda$, corresponding to eigenvalues that make a polygon that encloses the origin (or, if they exist, by a superposition of two of them corresponding to diametrically opposed eigenvalues). Instead, if the minimum overlap is not zero, it is achieved by the optimal probe state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\varphi_i\rangle + |\varphi_j\rangle),$$

(6)

with $\Delta \varphi = \varphi_i - \varphi_j.$
III. ENTANGLEMENT IN QUANTUM INTERFEROMETRY

In this section we compare the performances of single-mode (Fig. 1a) and entanglement-assisted interferometric schemes (Fig. 1b) in the detection of small perturbations induced by relevant quantum optical operations such as displacement, squeezing, mixing and phase-shifting. The comparison is made in terms of the detection sensitivity, namely, upon parametrizing the “size” of the perturbation—whence the corresponding output state—by a coupling parameter $\lambda$. In other words, the comparison is made in terms of the minimum detectable value $\lambda_{\text{min}}$ of $\lambda$ corresponding to output states that can be effectively discriminated while keeping the acceptance ratio $\gamma^*$ of the NP strategy large, namely $\gamma^* \equiv Q_\lambda/Q_0 \gg 1$. We will call the quantity $\lambda_{\text{min}}$ the “sensitivity” of the interferometric scheme. Using Eq. (4) the above condition can be written in term of the overlap as follows

$$|\kappa|^2 = 1 - \Lambda(Q_0, \gamma^*)$$

$$\Lambda(Q_0, \gamma^*) = Q_0 \left[ 1 + \gamma^*(1 - 2Q_0) - 2\sqrt{\gamma^*(1 - Q_0)(1 - \gamma^*Q_0)} \right].$$

For each class of transformations, we will make some general considerations and then focus our attention on sensitivity bounds that can be achieved using realistic (i.e. feasible with current technology) probe signals.

A. Perturbation made of a single-mode complex displacement

Let us first consider the case when the perturbation is imposed by the displacement operator $U_\alpha \equiv D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a)$. In principle, in this case, the discrimination can be done exactly with single-mode probe. This can be seen by writing the displacement as $U_\alpha = \exp(i2|\alpha|x_\theta)$, $x_\theta = 1/2(a^\dagger e^{i\theta} + ae^{-i\theta})$ being the quadrature operator, and $\theta = \arg(\alpha)$. Since the spectrum of the quadrature coincides with the real axis, the spectrum of $U_\alpha$ covers the whole unit circle, and, therefore, the states $|\psi_0\rangle$ and $|\psi_\alpha\rangle = U_\alpha|\psi_0\rangle$ can be discriminated with certainty either by choosing $|\psi_0\rangle$ as the eigenstate of the conjugated quadrature $x_{\theta + \pi/2}$, or, according to the overlap lemma, as a superposition of at least two eigenstates of the quadrature $x_\theta$. Unfortunately, such optimal states are unphysical, since they are not normalizable and have infinite energy. Moreover, even though we approximate them with physical states with finite energy, the identification of the optimal states would require the knowledge of the phase of the perturbation. In order to see that, let us rewrite the eigenvector $|0\rangle_{\theta + \pi/2}$ as the limiting case of a squeezed vacuum, $|0\rangle_{\theta + \pi/2} = \lim_{|\zeta| \rightarrow \infty} |\zeta\rangle = \lim_{|\zeta| \rightarrow \infty} S(\zeta)|0\rangle$, where $\theta = \arg(\zeta) + \pi/2$ is the argument of the squeezing parameter $\zeta$ of the squeezing operator given by

$$S(\zeta) = \exp[1/2(\zeta^2a^a + \bar{\zeta}a^\dagger)],$$

and $|0\rangle$ is the electromagnetic vacuum. Our squeezed vacuum has mean photon number $N = \sinh^2|\zeta|$. The overlap is readily evaluated as

$$|\kappa|^2 = |\langle\zeta|D(\alpha)|\zeta\rangle|^2 = \exp \left\{ -|\alpha|^2 \left[ 2N + 1 + \sqrt{N(N + 1)} \cos 2\delta \right] \right\}.$$
where $\delta = \arg(\zeta) - \arg(\alpha)$. By inserting the overlap in Eq. (7) we obtain the minimum detectable $|\alpha|^2$. However, Eq. (3) shows a very strong dependence of $|\alpha|^2_{\text{min}}$ on the phase parameter $\delta$, which makes the whole optimized scheme very unstable, namely one should know the phase of perturbation very precisely in order to get a truly optimized detection. Indeed, we have

$$|\alpha|^2_{\text{min}} \simeq \Lambda(Q_0, \gamma^*)/4N \quad \text{for} \quad \delta = \pi/2$$  \hspace{1cm} (10)

$$|\alpha|^2_{\text{min}} \simeq 4N\Lambda(Q_0, \gamma^*) \quad \text{for} \quad \delta = 0 ,$$  \hspace{1cm} (11)

with the second expression that shows an asymptotically divergent behavior in $N$.

Let us now consider an entanglement-assisted scheme. We suppose you have available a two-mode probe state $|\psi\rangle\rangle$ and consider the configuration $U_\alpha = D(\alpha) \otimes I$ in which the displacement perturbs one mode, say $a$, and the other mode is left unperturbed. As the probe state we consider the entangled state from parametric downconversion of vacuum for finite gain—the so-called “twin-beam” state

$$|x\rangle\rangle = \sqrt{1 - x^2} \sum_n x^n |nn\rangle\rangle \quad 0 \leq x < 1 ,$$  \hspace{1cm} (12)

where here $|nn\rangle = |n\rangle_a \otimes |n\rangle_b$. The twin-beam in Eq. (12) has mean photon number $N = 2x^2/(1 - x^2)$ and it is achieved starting from the vacuum via the unitary evolution $|x\rangle\rangle = \exp[x(a^b + ab)]|0\rangle\rangle$. In order to evaluate the sensitivity, the main task is now to calculate the overlap $|\kappa|^2 = |\langle x|U_\alpha|x\rangle|^2$. We have

$$\kappa = (1 - x^2) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+n} \langle mm|D(\alpha) \otimes I|nn\rangle =$$

$$= (1 - x^2) \sum_{n=0}^{\infty} x^{2n} \langle n|D(\alpha)|n\rangle = (1 - x^2) e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} x^{2n} L_n(|\alpha|^2) =$$

$$= \exp \left[-\frac{|\alpha|^2}{2} \frac{1 + x^2}{1 - x^2} \right] = \exp \left[-\frac{|\alpha|^2}{2} (N + 1) \right] .$$  \hspace{1cm} (13)

Equation (13) implies for $|\alpha|^2_{\text{min}}$ the scaling

$$|\alpha|^2_{\text{min}} \simeq \frac{\Lambda(Q_0, \gamma^*)}{N + 1} ,$$  \hspace{1cm} (14)

which is remarkably independent on the phase of perturbation, and thus represents a robust bound to the sensitivity of a single-mode displacement.

**B. Perturbation made of a single-mode squeezing (phase-sensitive amplifier)**

The second kind of perturbation that we analyze is the squeezing of a single radiation mode, which is described by the squeezing operator $S(\zeta)$ in Eq. (8). Without loss of generality we can consider $\zeta = \bar{\zeta} = r$ as real and use the notation $U_r$ to indicate the transformation, namely $U_r = \exp[-irA]$, with $A = i/2(a^+ - a^2)$. The spectrum of $A$ is
continuous and extends from $-\infty$ to $\infty$: this means that the eigenvalues of $U_r$ cover the whole unit circle. Therefore, it is possible in principle to discriminate the perturbation exactly, using as a probe either an eigenstate of the operator conjugated to $A$, or using a superposition of two or more eigenstates of $A$. However, analogously to the case of the displacement, such probe states would be non normalizable and have infinite energy, whence one must resort to physical approximations of such states. For a coherent probe the overlap can be calculated through the overlap of the corresponding Wigner functions, giving as a result

$$|\langle \alpha | U_r | \alpha \rangle|^2 = \exp \left[ -\frac{2N \cos^2 \phi (1 - \cosh r - \sinh r)^2}{1 + \exp(2r)} - \frac{2N \sin^2 \phi (1 - \cosh r + \sinh r)^2}{1 + \exp(-2r)} \right] ,$$

(15)

where $N = |\alpha|^2 = \langle \alpha | a^\dagger a | \alpha \rangle$ is the mean number of photons of the probe state. By expanding for small $r$ we have

$$|\langle \alpha | U_r | \alpha \rangle|^2 \simeq 1 - Nr^2 ,$$

(16)

and therefore the minimum detectable perturbation would be

$$r_{\text{min}} \simeq \sqrt{\frac{\Lambda(Q_0, \gamma^*)}{N}} .$$

(17)

For a squeezed vacuum probe $S(\zeta)|0\rangle$ one has

$$\kappa = \langle 0 | S^\dagger(\zeta) U_r S(\zeta) | 0 \rangle = [\cosh r + 2i \sinh |\zeta| \cosh |\zeta| \sinh r \sin \psi]^{-\frac{1}{2}}$$

(18)

where $\psi = \arg(\zeta)$ and correspondingly the minimum detectable $r$ is given by:

$$r_{\text{min}} = \begin{cases} \log\left[1 - \frac{1}{2\Lambda(Q_0, \gamma^*)} \right] & \text{for } \sin \psi = 0, \\ \sqrt{\frac{\Lambda(Q_0, \gamma^*)}{2}} N \sin \psi & \text{otherwise}, \end{cases}$$

(19)

with $N = \sinh^2 |\zeta|$. The bound in Eq. (19) strongly depends on the phase between the squeezing perturbation and the squeezing of the probe, and therefore cannot be achieved in practice without prior knowledge of the phase of the perturbation.

Let us now consider an entangled probe state in a twin beam state of the form (12). The input-output overlap is calculated as follows

$$\kappa = \langle x | U_r \otimes \mathbb{I} | x \rangle = (1 - x^2) \sum_{n=0}^{\infty} x^{2n} \langle n | U_r | n \rangle .$$

(20)

In order to calculate the matrix element $\langle n | U_r | n \rangle$ we use the identities $S(r) = e^{\frac{i}{2} \tanh(r) a^\dagger [\cosh(r)]^{-\frac{1}{2}} a - \frac{1}{2} e^{-\frac{1}{2} \tanh(r)} a^\dagger}$ and

$$e^{-\frac{1}{2} \tanh(r) a^\dagger} |n\rangle = \sum_{l=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-\tanh(r))^l}{2^l l!} \sqrt{\frac{n!}{(n-2l)!}} |n-2l\rangle ,$$

(21)
where $[m]$ indicates the integer part of $m$, and finally we get

$$\langle n|U_r|n\rangle = \frac{n!}{[\text{Ch}(r)]^{n+\frac{3}{2}}} \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l[\sinh(r)]^{2l}}{4^l(l!)^2(n-2l)!}. \tag{22}$$

Using Eq. (22) we calculate $\kappa$ by means of Eq. (20)

$$\kappa = (1 - x^2) \sum_{n=0}^{\infty} x^{2n} \frac{n!}{[\cosh(r)]^{n+\frac{1}{2}}} \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l[\sinh(r)]^{2l}}{4^l(l!)^2(n-2l)!} =$$

$$= \frac{(1 - x^2)}{[\cosh(r)]^{\frac{3}{2}}} \sum_{l=0}^{\infty} \left(-\frac{x^4 \sinh^2(r)}{4 \cosh^2(r)}\right)^l \frac{2l!}{l!^2} \sum_{n=0}^{\infty} \frac{(n + 2l)!}{n!2l!} \left(\frac{x^2}{\cosh(r)}\right)^n =$$

$$= \frac{(1 - x^2)}{[(x^4 + 1) \cosh(r) - 2x^2]^{\frac{1}{2}}}.$$

Inserting this expression in Eq. (7) we have for $r_{\text{min}}$ the scaling law

$$r_{\text{min}} \simeq \sqrt{\frac{\Lambda(Q_0, \gamma^*)}{1 - \Lambda(Q_0, \gamma^*)}} \sqrt{\frac{1}{N^2 + 2N + 2}} \simeq \sqrt{\frac{\Lambda(Q_0, \gamma^*)}{1 - \Lambda(Q_0, \gamma^*)}} \frac{2}{N}. \tag{23}$$

The same result is obtained by varying the phase of the squeezing amplitude $\zeta$, thus confirming the robustness of the bound (23) that is obtained using an entangled probe.

C. Perturbation made of a two-mode phase-shift

The third problem we address is that of a perturbation induced by the two-modes phase shift operator $a^\dagger b + ab^\dagger$, characterizing a mixer (beam splitter) or a Mach-Zehnder interferometer. This case differs from the previous ones in that the perturbation is represented by the two-modes unitary operator $V_\phi = \exp\{i\phi(a^\dagger b + ab^\dagger)\}$. In this case the spectrum is given by $\exp\{m\phi\}$, with $m \in \mathbb{Z}$ (see e.g. [11]). Therefore, if $\phi = (q/p)\pi$ with $q \in 2\mathbb{Z} + 1$ and $p \in \mathbb{Z}$ (but this is a null-measure set of values of $\phi$) then the optimal state is given by a superposition of two eigenstates of $V_\phi$ with eigenvalues differing by $\pi$. In the general case, the optimal state is any superposition of three or more eigenstates of $V_\phi$, such that the polygon of its eigenvalues on the unit circle encloses the origin [7]. Such optimal states are entangled, since they are obtained from the eigenstates $|n, d\rangle$ of $a^\dagger a - b^\dagger b$

$$(a^\dagger a - b^\dagger b) |n, d\rangle = d |n, d\rangle, \quad |n, d\rangle = \begin{cases} |n + d\rangle |n\rangle & \text{for } d \geq 0, \\ |n\rangle |n + d\rangle & \text{for } d < 0, \end{cases} \tag{24}$$

by the unitary transformation $\exp\{-(\pi/4)(a^\dagger b - ab^\dagger)\}$. Actually, the optimal states are far from being practically realizable. However, we have proved that they are entangled, and this suggests to explore the possibility of performing a reliable discrimination by physically realizable entangled states. For a twin-beam we have

$$\kappa = \langle x|V_\phi|x\rangle =$$

$$= (1 - x^2) \langle 00|e^{\gamma a b}e^{i\gamma_0 a^\dagger b}e^{\frac{1}{2}\gamma_1(a^\dagger b^\dagger)\gamma_1}e^{i\gamma_0 a^\dagger b^\dagger}}e^{x a b^\dagger}|00\rangle, \tag{25}$$
where $\gamma_0 = \tan \phi$ and $\gamma_1 = -\log(\cos^2 \phi)$. After some algebra we get

$$|\kappa|^2 = \frac{1}{1 + \frac{4x^2 \sin^2 \phi}{(1-x^2)^2}} = \frac{1}{1 + N(N+2) \sin^2 \phi}.$$  \hfill (26)

The minimum detectable $\phi$, according to (26), is thus given by

$$\phi_{\min} = \arcsin \left( \frac{\Lambda(Q_0, \gamma^*)}{\sqrt{N(N+2)}} \right) \approx \frac{\Lambda(Q_0, \gamma^*)}{N}.$$  \hfill (27)

The scaling in Eq. (27) does not depend on any parameter but the energy of the input state. This should be compared with the sensitivity of the customary single-mode interferometry \cite{11} based on squeezed states, where the same scaling is achieved only for a very precise tuning of the phase of the squeezing. This means that the entanglement-assisted interferometry provides a much more reliable and easily tunable scheme.

**IV. IMPLEMENTATION BY DIFFERENCE-PHOTOCURRENT INTERFEROMETRY**

In this section we present a concrete scheme for binary decision based on an entangled probe. The scheme should be feasible with current technology, and would allow to approach the ultimate precision bounds that have been obtained in the previous sections. In Fig. 2 we show a schematic diagram of the interferometric setup. The input state is the entangled twin-beam $\ket{x}$ produced by a nondegenerate optical parametric amplifier (NOPA). Such entangled probe is possibly subjected to the action of the unitary $U_\lambda$ (Figs. 2a and 2b describe the cases of a single mode and a two-mode perturbation respectively). At the output the two beams are detected and the difference photocurrent $D = a^\dagger a - b^\dagger b$ is measured. If no perturbation occurs, then the output state is still a twin-beam, and since $\ket{x}$ is an eigenstate of $D$ with zero eigenvalue we have a constant zero outcome for the difference photocurrent. On the other hand, when a perturbation occurs the output state is no longer an eigenstate of $D$, and we detect fluctuations which signal the presence of the perturbation itself. The false-alarm and the detection probabilities are given by

$$Q_0 = P(d \neq 0 | \text{not } U_\lambda) \equiv 0$$
$$Q_\lambda = P(d \neq 0 | U_\lambda) = 1 - P(d \equiv 0 | U_\lambda),$$  \hfill (28)

where the probability of observing zero counts at the output, after the action of $U_\lambda$, is given by

$$P(d \equiv 0 | U_\lambda) = \sum_n |\langle n, n | U_\lambda | x \rangle|^2,$$  \hfill (29)

since the eigenvalue $d = 0$ is degenerate. In this case the false-alarm probability is zero and therefore it is not necessary to introduce an acceptance ratio. The scaling of the minimum detectable perturbation can be obtained directly in term of the detection probability $Q_\lambda$ by Eqs. (29) and (30). For the three transformations considered in the previous Section we
have

**Displacement**

\[ P(d = 0|\alpha \neq 0) = \exp \left( -|\alpha|^2(1 + N) \right) I_0(|\alpha|^2\sqrt{N(N + 2)}) \quad \rightarrow \quad |\alpha|^2_{\text{min}} \simeq \frac{\sqrt{Q\lambda}}{N} \] (31)

**Squeezing**

\[ P(d = 0|r \neq 0) = 1 - r^2 N + O(r^2) \quad \rightarrow \quad r_{\text{min}} \simeq \frac{\sqrt{Q\lambda}}{N} \] (32)

**Two-mode phase-shift**

\[ P(d = 0|\phi \neq 0) = 1 - \frac{1}{2} \phi^2 N^2 + O(\phi^2) \quad \rightarrow \quad \phi_{\text{min}} \simeq \frac{\sqrt{2Q\lambda}}{N} . \] (33)

On can see that in all examples considered above, a realistic interferometer based on a difference-photocurrent measurement provides a precision that re-scales with the energy in the same way as the ultimate bounds obtained in the previous sections.

It is worth noticing that the experimental measurement of a modulated absorption based on entanglement-assisted difference-photocurrent detection has been already performed using the entangled beam exiting an amplifier above threshold (parametric oscillator, OPO) [12].

**V. SUMMARY AND CONCLUSIONS**

In this paper we have analyzed the effect of entanglement on the interferometric estimation of relevant quantum optical parameters such as displacing and squeezing amplitudes or interferometric phase-shift. We have evaluated the minimum detectable perturbation according to the Neyman-Pearson detection strategy, and have shown that entanglement always improves the detection in comparison with conventional schemes. In particular, for the case of estimation of the displacement and the squeezing amplitudes we have shown that the precision of the apparatus that use an entangled probe is independent of the phase of the perturbation, and is therefore more robust and reliable of a simple scheme based on single-mode probe states. Similarly, for the estimation of a two-mode phase-shift, entanglement the interferometer is much more stable when we use a twin beam that when we use squeezed states.

Since the Neyman-Pearson detection strategy does not correspond to a realistic detector, we proposed a feasible interferometric setup that is based on the measurement of the difference photocurrent on an entangled twin-beam. Remarkably, this scheme has the same energy-scaling of the ultimate precision bound, and at the same time is very stable. We conclude that the technology of entanglement can be of great help in improving precision and stability of quantum interferometers.

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REFERENCES

[1] J. N. Hollenhorst, *Phys. Rev. D* **19**, 1669 (1979)
[2] M. G. A. Paris, Phys. Lett. A **225**, 23 (1997).
[3] C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[4] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, (North-Holland, Amsterdam, 1982).
[5] J. Neyman, E. Pearson, *Phil. Trans. Roy. Soc. London* A**321**, 289 (1933)
[6] K. R. Parthasarathy, in *Stochastics in finite and infinite dimensions*, Rajput et al Eds., Trends in Mathematics, Birkhauser Boston, pp. 361-377 (2000).
[7] G. M. D’Ariano, P. Lo Presti, M. G. A. Paris, preprint quant-ph/0109040.
[8] C. G. Bollini and L. E. Oxman, Phys. Rev. A**47**, 2339 (2001).
[9] G. M. D’Ariano, *Int. J. Mod. Phys. B* **6**, 1291 (1992)
[10] G. M. D’Ariano, M. G. A. Paris, P. Perinotti, J. Opt. B **3**, 337 (2001).
[11] C. M. Caves, Phys. Rev. D **23**, 1693 (1981); R. S. Bondurant and J. H. Shapiro, Phys. Rev. A **30**, 2548 (1984).
[12] P. Souto-Ribeiro et al, Opt. Lett. **22**, 1893 (1997); J. R. Gao et al, Opt. Lett. **23**, 870 (1998).
FIG. 1. A generalized interferometer is a binary detection scheme aimed to check whether or not a given quantum device (the hexagon in the figure) has performed the quantum operations described by the unitary operator $U_{\lambda}$. The signal employed as a probe is prepared by the source $S$ and then enters the device, which may or may not apply $U_{\lambda}$. The two hypotheses: $H_0$ (the signal is unperturbed) and $H_\lambda$ ($U_{\lambda}$ has been applied) should be discriminated on the basis of the outcome of the detector $D$. (a): simple scheme involving a single-mode probe. (b): scheme involving entanglement-assisted binary detection.

FIG. 2. Interferometric scheme to achieve ultimate bounds on precision by means of an entangled probe. The NOPA generates a twin-beam which may be subjected to the action of the unitary $U_{\lambda}$. At the output the beams are detected and the difference photocurrent is measured. For an unperturbed interferometer the output is again a twin-beam state, and the scheme is designed in order to obtain a constant zero difference photocurrent, whereas a perturbation $U_{\lambda}$ would produce fluctuations in the difference photocurrent. (a) general scheme for single-mode perturbation; (b) scheme for two-mode perturbation.