Solution of Dirac Equation for Charged and Neutral Fermions with Anomalous Magnetic Moments in Uniform Magnetic Field

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(Dated: May 1, 2014)

The Dirac equation for charged and neutral fermions with anomalous magnetic moments is solved in a uniform magnetic field. We find the relativistic wave functions and energy spectra. In the non–relativistic limit the wave functions and energy spectra of charged fermions agree with the known solutions of the Schrödinger equation.

PACS: 12.15.Ff, 13.15.+g, 23.40.Bw, 26.65.+t

1. Introduction

The quantum field theoretic problem of the motion of charged fermions with spin \( \frac{1}{2} \) in a uniform magnetic field has been solved in the pioneering papers by Johnson and Lippmann [1]. The wave functions and energy spectra of the charged fermions have been obtained as solutions of the Dirac equation at the neglect of the Pauli term, describing the interaction of the anomalous magnetic moment of the fermion with a uniform magnetic field [2]. The results, obtained in [1], have been applied to the analysis of the neutron decay rates in very strong magnetic fields with field strengths of order of \( B \sim 10^{12} \text{T} \) [3]. Such strong magnetic fields may exist in neutron stars and white dwarfs [3]. The solution of the Dirac equation with the Pauli term for charged and neutral fermions with spin \( \frac{1}{2} \) has been proposed in [7]. In [8] only the relativistic energy spectra of charged fermions with the Pauli energy splitting were found, which are caused by the interaction of the anomalous magnetic moments of fermions with a uniform magnetic field. In [8] and [9] the relativistic energy spectra and the relativistic wave functions of charged fermions (see [8]) and charged and neutral fermions (see [9]) with anomalous magnetic moments were found in the “number of state” representation, reducing the problem of the motion of fermions with anomalous magnetic moments in a uniform magnetic field to the harmonic oscillator problem [10, 11].

In this paper we propose a detailed solution of the Dirac equation with the Pauli term for fermions with spin \( \frac{1}{2} \) and anomalous magnetic moment \( \kappa \). The fermions are either charged \( Ze \) with \( Z > 0 \) and \( Z < 0 \) for positively and negatively charged fermions, where \( e \) is the proton charge, or neutral with \( Z = 0 \), respectively. The solution of the Dirac equation runs as follows. We reduce the system of first order differential equations for coupled up and down components of the Dirac wave functions to the fourth order differential equations for the decoupled components, which in turn can be transformed into a system of second order differential equations for the decoupled spin up and spin down components of the Dirac wave functions. The solutions of these second order differential equations give correct relativistic energy spectra, splitted due to the interaction of the anomalous magnetic moments of fermions with a uniform magnetic field. The system of first order differential equations is used for the calculation of the normalisation constants of the relativistic wave functions. The obtained relativistic wave functions and energy spectra possess well–known non–relativistic limits [11]. We would like to note that our technique is similar to that, which has been used in [7]. Nevertheless, our intermediate calculations are much simpler as well as the derivation of the energy spectra. In addition we calculate the relativistic wave functions of the charged fermions that has not been done in [7]. As regards the results obtained in [8, 9], we would like to accentuate that we solve the Dirac equation in cylindrical coordinates and define the energy spectra in dependence on two quantum numbers, namely, the radial quantum number \( n_p = 0, 1, 2, \ldots \) and the magnetic quantum number \( m = 0, \pm 1, \pm 2, \ldots \). According to Mathews [10], the “one–dimensional harmonic oscillator” description of relativistic fermions, moving in a uniform magnetic field, cannot adequately cover the two–dimensional motion of fermions in the plane orthogonal to the magnetic field. Thus, following Mathews [10], only a two–dimensional description in cylindrical coordinates with two quantum numbers \((n_p, m)\) can be used for the correct analysis of relativistic quantum states of fermions in a uniform magnetic field.

Of course, one may also use the principal quantum number \( n \), which is a function of \( n_p \) and \( m \), and the magnetic quantum number \( m \). In comparison with the results obtained in [8, 9] we discuss in more detail the dependence of the relativistic wave functions on the magnetic quantum number, which testifies an infinite degeneration of the energy levels of fermions in the “one–dimensional harmonic oscillator” representation [10] in a uniform magnetic field. It also plays an important role for applications of the obtained results, for example to the neutron decay that requires the
calculation matrix elements by integrating over the spatial coordinates and summation over quantum numbers.

The paper is organised as follows. In section 2 we give a detailed solution of the Dirac equation for charged fermions with spin \( \frac{1}{2} \) and an anomalous magnetic moment \( \kappa \). In section 3 we propose the solution of the Dirac equation for neutral fermions with spin \( \frac{1}{2} \) and an anomalous magnetic moment \( \kappa \). In section 4 we analyse the non–relativistic limit of the relativistic wave functions and energy spectra, obtained in sections 2 and 3. In the Conclusion we discuss the obtained results.

2. Solution of Dirac equation for charged fermions with anomalous magnetic moments in uniform magnetic field

Let a fermion with mass \( M \), spin \( \frac{1}{2} \), electric charge \( Z e \) and an anomalous magnetic moment \( \kappa \) move in a uniform magnetic field \( \vec{B} = B \hat{e}_z \), directed along the \( z \)-axis, which coincides with the quantisation axis of the fermion spin. The Dirac equation takes the form [2]

\[
\left( \gamma^\mu (i \partial_\mu - ZeA_\mu (x)) + \lambda \sigma_{\mu\nu} F^{\mu\nu} (x) - M \right) \psi(x) = 0, \tag{1}
\]

where \( x = (t, \vec{r}) \), \( \gamma^\mu = (\gamma^0, \gamma^i) = (\beta, \beta \vec{\alpha}) \) and \( \sigma_{\mu\nu} = \frac{i}{2} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \) are Dirac matrices in the Dirac representation [2]. Then, \( \lambda \sigma_{\mu\nu} F^{\mu\nu} (x) \) is the Pauli term, \( \gamma^0 \) is the time–dependent part of the Dirac matrix, \( \beta = \frac{1}{2} \) and \( \lambda = \kappa Ze / 4M \), \( F^{\mu\nu} (x) = \partial^\mu A^\nu (x) - \partial^\nu A^\mu (x) \) is the electromagnetic field strength and \( A^\mu (x) = (A^0 (x), \vec{A} (x)) \) is the electromagnetic potential. For a uniform magnetic field, where \( A^0 (x) = 0 \) and \( \vec{A} (0, \vec{r}) = \frac{1}{\mu} (\vec{B} \times \vec{r}) \), Eq. (1) can be transcribed into the form

\[
\frac{\partial}{\partial t} \psi(x) = \left( \vec{\alpha} \cdot \vec{\pi} - 2\beta \lambda \vec{\sum} \cdot \vec{B} + \beta M \right) \psi(x). \tag{2}
\]

Here \( \vec{\pi} = -i \vec{\nabla} - Ze \vec{A} \) is the operator of the canonical momentum of the fermion with charge \( Ze \) in a magnetic field \( \vec{B} \) and \( 2\beta \lambda \vec{\sum} \cdot \vec{B} \) is the Pauli term, where \( \vec{\sum} = \gamma^0 \gamma^5 \) is a diagonal matrix \( \text{diag}(\sigma_x, \sigma_y, \sigma_z) \), the elements of which \( \sigma_x, \sigma_y, \sigma_z \) are Pauli \( 2 \times 2 \) matrices [2]. Since we search for stationary solutions, we take the wave function of the fermion in the following form

\[
\psi(x) = e^{-iEt} \begin{pmatrix} \varphi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix}, \tag{3}
\]

where \( \vec{r} = (\vec{r}_\perp, z) = (x, y, z) \) and \( \varphi(\vec{r}) \) and \( \chi(\vec{r}) \) are two–component wave functions, obeying the equations

\[
\begin{align*}
(E - M + 2\lambda \sigma_z B) \varphi(\vec{r}) &= \sigma_z \varphi(\vec{r}), \\
(E + M - 2\lambda \sigma_z B) \chi(\vec{r}) &= \sigma_z \chi(\vec{r}).
\end{align*} \tag{4}
\]

Taking into account that the magnetic field is directed along the \( z \)-axis, we can rewrite Eq. (4) as follows

\[
\begin{align*}
(E - M + 2\lambda \sigma_z B) \varphi(\vec{r}) &= \left( \sigma_+ \pi_- + \sigma_- \pi_+ + \sigma_z \pi_z \right) \varphi(\vec{r}), \\
(E + M - 2\lambda \sigma_z B) \chi(\vec{r}) &= \left( \sigma_+ \pi_- + \sigma_- \pi_+ + \sigma_z \pi_z \right) \chi(\vec{r}),
\end{align*} \tag{5}
\]

where \( \pi_\perp = \pi_x \pm i \pi_y \) and \( \pi_z = -i \partial_z \). Specifying the direction of the fermion spin “up” and “down” as \( \uparrow \) and \( \downarrow \), the two–component spinor wave functions take the form

\[
\begin{pmatrix} \varphi(\vec{r}) \\ \chi(\vec{r}) \end{pmatrix} = \begin{pmatrix} \varphi_{\uparrow}(\vec{r}) \\ \varphi_{\downarrow}(\vec{r}) \end{pmatrix}, \quad \begin{pmatrix} \chi_{\uparrow}(\vec{r}) \\ \chi_{\downarrow}(\vec{r}) \end{pmatrix}, \tag{6}
\]

where \( \varphi_{\uparrow, \downarrow} \) and \( \chi_{\uparrow, \downarrow} \) are eigenfunctions of the \( \sigma_z \)-operator, i.e. \( \sigma_z \varphi_{\uparrow} = + \varphi_{\uparrow}, \sigma_z \varphi_{\downarrow} = - \varphi_{\downarrow}, \sigma_z \chi_{\uparrow} = + \chi_{\uparrow} \) and \( \sigma_z \chi_{\downarrow} = - \chi_{\downarrow} \). Since the \( p_z \)-component of the fermion is conserved, the wave functions \( \varphi_{\uparrow, \downarrow}(\vec{r}) \) and \( \chi_{\uparrow, \downarrow}(\vec{r}) \) take the form

\[
\varphi_{\uparrow, \downarrow}(\vec{r}) = \varphi_{\uparrow, \downarrow}(\vec{r}_\perp) e^{ip_z z} \quad \text{and} \quad \chi_{\uparrow, \downarrow}(\vec{r}) = \chi_{\uparrow, \downarrow}(\vec{r}_\perp) e^{ip_z z},
\]

respectively. Substituting the wave functions Eq. (6) in
where $\Phi$ are the generalised Laguerre polynomials, defined by
\begin{equation}
\end{equation}

Acting on these equations with the operators $\pi_\pm$ and using the commutation relation $[\pi_+, \pi_-] = 2ZeB$ we arrive at the system of second order differential equations
\begin{equation}
\end{equation}

where the components $\uparrow$ and $\downarrow$ of the wave functions are decoupled.

The disentangled components $\varphi_\uparrow(\vec{r}_\perp)$, $\chi_\uparrow(\vec{r}_\perp)$ and $\varphi_\downarrow(\vec{r}_\perp), \chi_\downarrow(\vec{r}_\perp)$ of the fermion wave function are described by the fourth order differential equations
\begin{equation}
\end{equation}

which can be rewritten in the following form
\begin{equation}
\end{equation}

We obtain the solutions of these equations in cylindrical coordinates $\vec{r} = (\rho, \phi, z)$, where they read (see also [10])
\begin{equation}
\end{equation}

We obtain the unnormalised solutions $f_{n_\rho, m}(\vec{r}_\perp)$ of these equations in the form (see also [11, 12])
\begin{equation}
\end{equation}

where $n_\rho = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$ are the radial and magnetic quantum numbers, respectively, $L_{n_\rho}^{\pm m}(\sqrt{Z}eB\rho^2/2)$ are the generalised Laguerre polynomials, defined by
\begin{equation}
\end{equation}
where $\xi = |Z|eB\rho^2/2$ for a fermion in a uniform magnetic field and $\Gamma(n_\rho + 1)$ is the Euler $\Gamma$–function $\Gamma(n_\rho + 1) = n_\rho!$. According to the definition of the generalised Laguerre polynomials Eq.\[14\], the polynomials vanish for $n_\rho \leq -1$.

The energy spectra for the $\Psi^\pm_{\perp}(\vec{r}_\perp)$ and $\Phi^\pm_{\perp}(\vec{r}_\perp)$ components of the fermion wave functions are

$$
E_{\pm \uparrow, n_\rho, m}^{(\pm, Z>0)} = \sqrt{M^2 + ZeB (2n_\rho + |m| - m) \mp 2\lambda B},
$$

$$
E_{\pm \downarrow, n_\rho, m}^{(\pm, Z>0)} = \sqrt{M^2 + ZeB (2(n_\rho + 1) + |m| - m) \mp 2\lambda B},
$$

and

$$
E_{\pm \downarrow, n_\rho, m}^{(\pm, Z<0)} = \sqrt{M^2 + |Z|eB (2n_\rho + |m| + m) \mp 2\lambda B},
$$

$$
E_{\pm \downarrow, n_\rho, m}^{(\pm, Z<0)} = \sqrt{M^2 + |Z|eB (2(n_\rho + 1) + |m| + m) \mp 2\lambda B},
$$

for positively and negatively charged fermions, respectively.

In order to define the Dirac wave functions we have to take into account that $\varphi^\pm_{\perp\uparrow}(\vec{r}_\perp)$ and $\chi^\pm_{\perp\downarrow}(\vec{r}_\perp)$ satisfy the system of the first order differential equations Eq.\[7\]. For this aim we use the following relations

$$
\pi^+ f_{n_\rho, m}(\vec{r}_\perp) = -i \rho^{m-1} e^{-\frac{|Z|eB}{4} \rho^2 \epsilon^{(m+1)\phi}} \left[ \frac{(m)_{n_\rho} - (|m| - m)L^{(m)}_{n_\rho} - \frac{(|Z| - Z)eB}{2} \rho^2 L^{(m)}_{n_\rho} + \rho \frac{dL^{(m)}_{n_\rho}}{d\rho} \right],
$$

$$
\pi^- f_{n_\rho, m}(\vec{r}_\perp) = -i \rho^{m-1} e^{-\frac{|Z|eB}{4} \rho^2 \epsilon^{(m+1)\phi}} \left[ \frac{(m)_{n_\rho} + (|m| + m)L^{(m)}_{n_\rho} - \frac{(|Z| + Z)eB}{2} \rho^2 L^{(m)}_{n_\rho} + \rho \frac{dL^{(m)}_{n_\rho}}{d\rho} \right].
$$

For positively charged fermions $Z > 0$ and the states with a magnetic number $m \geq 0$ we get the relations

$$
\pi^+ f_{n_\rho, m}(\vec{r}_\perp) = +iZeB f_{n_\rho - 1, m+1}(\vec{r}_\perp),
$$

$$
\pi^- f_{n_\rho - 1, m+1}(\vec{r}_\perp) = -2i n_\rho f_{n_\rho, m}(\vec{r}_\perp).
$$

In turn for fermions with $Z > 0$ and the states with a magnetic number $m < 0$ the corresponding relations are

$$
\pi^+ f_{n_\rho, m}(\vec{r}_\perp) = -2i (n_\rho + |m|) f_{n_\rho, m+1}(\vec{r}_\perp),
$$

$$
\pi^- f_{n_\rho, m+1}(\vec{r}_\perp) = +i ZeB f_{n_\rho, m}(\vec{r}_\perp).
$$

For negatively charged fermions $Z < 0$ and the states with magnetic number $m \geq 0$ we obtain the relations

$$
\pi^+ f_{n_\rho, m}(\vec{r}_\perp) = +i |Z|eB f_{n_\rho, m+1}(\vec{r}_\perp),
$$

$$
\pi^- f_{n_\rho, m+1}(\vec{r}_\perp) = -2i (n_\rho + m + 1) f_{n_\rho, m}(\vec{r}_\perp),
$$

and for fermions with $Z < 0$ and magnetic number $m < 0$ we finally obtain

$$
\pi^+ f_{n_\rho, m}(\vec{r}_\perp) = -2i (n_\rho + 1) f_{n_\rho+1, m+1}(\vec{r}_\perp),
$$

$$
\pi^- f_{n_\rho+1, m+1}(\vec{r}_\perp) = +i |Z|eB f_{n_\rho, m}(\vec{r}_\perp).
$$

As a result the Dirac wave functions of charged fermions with anomalous magnetic moments are defined by

$$
\psi_{n_\rho, m \geq 0, p_z}^{(\pm, Z>0)}(x) = \begin{pmatrix} A_1 f_{n_\rho, m}(\vec{r}_\perp) \\ A_2 f_{n_\rho - 1, m+1}(\vec{r}_\perp) \\ A_3 f_{n_\rho, m}(\vec{r}_\perp) \\ A_4 f_{n_\rho - 1, m+1}(\vec{r}_\perp) \end{pmatrix} e^{-iE_{n_\rho, m \geq 0}^{(\pm, Z>0)}(x) + ip_z x}, \psi_{n_\rho, m < 0, p_z}^{(\pm, Z>0)}(x) = \begin{pmatrix} B_1 f_{n_\rho, m}(\vec{r}_\perp) \\ B_2 f_{n_\rho, m+1}(\vec{r}_\perp) \\ B_3 f_{n_\rho, m}(\vec{r}_\perp) \\ B_4 f_{n_\rho, m+1}(\vec{r}_\perp) \end{pmatrix} e^{-iE_{n_\rho, m < 0}^{(\pm, Z>0)}(x) + ip_z x},
$$

and

$$
\psi_{n_\rho, m \geq 0}^{(\pm, Z<0)}(x) = \begin{pmatrix} C_1 f_{n_\rho, m}(\vec{r}_\perp) \\ C_2 f_{n_\rho - 1, m+1}(\vec{r}_\perp) \\ C_3 f_{n_\rho, m}(\vec{r}_\perp) \\ C_4 f_{n_\rho - 1, m+1}(\vec{r}_\perp) \end{pmatrix} e^{-iE_{n_\rho, m \geq 0}^{(\pm, Z<0)}(x) + ip_z x}, \psi_{n_\rho, m < 0}^{(\pm, Z<0)}(x) = \begin{pmatrix} D_1 f_{n_\rho, m}(\vec{r}_\perp) \\ D_2 f_{n_\rho, m+1}(\vec{r}_\perp) \\ D_3 f_{n_\rho, m}(\vec{r}_\perp) \\ D_4 f_{n_\rho, m+1}(\vec{r}_\perp) \end{pmatrix} e^{-iE_{n_\rho, m < 0}^{(\pm, Z<0)}(x) + ip_z x},
$$

and
where \( n_\rho = 0, 1, \ldots \) and \( m = 0, \pm 1, \pm 2, \ldots \). The relativistic energy spectra are given for \( Z > 0 \) by

\[
E_{n_\rho,m}^{(\pm, Z > 0)} = \sqrt{(\sqrt{M^2 + |Z|eB (2n_\rho + |m| - m)} \mp 2\lambda B)^2 + p_z^2},
\]

and for \( Z < 0 \) by

\[
E_{n_\rho,m}^{(\pm, Z < 0)} = \sqrt{(\sqrt{M^2 + |Z|eB (2n_\rho + |m| + m)} \mp 2\lambda B)^2 + p_z^2}.
\]

The Dirac wave functions are normalised by

\[
\int d^3 x \psi_{n_\rho,m,p_z}^{(\sigma, Z)}(x)\psi_{n_\rho,m,p_z}^{(\sigma, Z)}(x) = 2E_{n_\rho,m}^{(\sigma, Z)}(2\pi)^3 \delta_{n_\rho,\rho'} \delta_{m,m'} \delta(p_z - p_{z'}),
\]

where \( \sigma = \pm \). For the calculation of the normalisation constants we use the following relation

\[
\int d^3 x f_{n_\rho,m}(\vec{r}_\perp)^2 = \pi(n_\rho + |m|)2^{m+1}n_\rho!(|Z|eB)^{|m|+1} \delta_{n_\rho,\rho'} \delta_{m,m'}.
\]

Substituting Eq. (22) and Eq. (23) into Eq. (11) we arrive at a system of algebraical equations for the normalisation constants \( A_i, B_i, C_i \) and \( D_i \) for \( i = 1, 2, 3, 4 \). Solving these equations together with the normalisation condition Eq. (26) and using Eq. (27) we obtain the following normalised Dirac wave functions of positively charged \((Z > 0)\) fermions

\[
\psi_{n_\rho,m>0,p_z}^{(\pm, Z > 0)}(x) = \sqrt{\pi(n_\rho - 1)!|Z|eB)^{m+2}} \frac{1}{2^{m+1}(n_\rho + m)!} \sqrt{(E_{n_\rho,m>0}^{(\pm, Z > 0)} \mp E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)})(E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)} \mp 2\lambda B)(E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)} \mp 2\lambda B \mp M)}
\]

\[
\times \left( \frac{2i(n_\rho E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)} \pm E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)}) f_{n_\rho,m}(\vec{r}_\perp)}{2i(n_\rho E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)} \pm E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)}) f_{n_\rho,m}(\vec{r}_\perp)} \pm p_z E_{\perp,n_\rho,m>0}^{(\pm, Z > 0)} \mp 2\lambda B \mp M) f_{n_\rho-1,m+1}(\vec{r}_\perp)} \right) e^{-iE_{\perp,n_\rho,m>0}^{(\pm, Z > 0)} t + ip_z z},
\]

and negatively charged \((Z < 0)\) fermions

\[
\psi_{n_\rho,m<0,p_z}^{(\pm, Z < 0)}(x) = \sqrt{\pi(n_\rho - 1)!|Z|eB)^{m+2}} \frac{1}{2^{m+1}(n_\rho + m)!} \sqrt{(E_{n_\rho,m<0}^{(\pm, Z < 0)} \mp E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)})(E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \mp 2\lambda B)(E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \mp 2\lambda B \mp M)}
\]

\[
\times \left( \frac{iZeB E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \pm E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)}) f_{n_\rho,m}(\vec{r}_\perp)}{iZeB E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \pm E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)}) f_{n_\rho,m}(\vec{r}_\perp)} \mp p_z E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \mp 2\lambda B \mp M) f_{n_\rho-1,m+1}(\vec{r}_\perp)} \right) e^{-iE_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} t + ip_z z},
\]

and positively charged \((Z > 0)\) fermions

\[
\psi_{n_\rho,m>0,p_z}^{(\pm, Z < 0)}(x) = \sqrt{\pi(n_\rho - 1)!|Z|eB)^{m+2}} \frac{1}{2^{m+1}(n_\rho + m)!} \sqrt{(E_{n_\rho,m<0}^{(\pm, Z < 0)} \mp E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)})(E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \mp 2\lambda B)(E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \mp 2\lambda B \mp M)}
\]

\[
\times \left( \frac{2i(n_\rho + m)(E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \pm E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)}) f_{n_\rho-1,m}(\vec{r}_\perp)}{2i(n_\rho + m)(E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \pm E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)}) f_{n_\rho-1,m}(\vec{r}_\perp)} \mp p_z E_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} \pm 2\lambda B \pm M) f_{n_\rho-1,m+1}(\vec{r}_\perp)} \right) e^{-iE_{\perp,n_\rho,m<0}^{(\pm, Z < 0)} t + ip_z z},
\]
and
\[
\psi_{n_\rho, m < 0, p_\perp}(x) = \sqrt{\frac{\pi n_\rho! |Z| eB|^{m|}}{2|m|-1(n_\rho + |m| - 1)!}} \left( \frac{1}{E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm E_{n_\rho, m < 0}^{(\pm, Z > 0)}(E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm 2\lambda B)(E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm 2\lambda B + M)} \right) \times \\
\times \left( i|Z| eB (E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm E_{n_\rho, m < 0}^{(\pm, Z > 0)} f_{n_\rho - 1, m}(\vec{r}_\perp) \\
+ p_\perp (E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm 2\lambda B + M) f_{n_\rho, m + 1}(\vec{r}_\perp) \\
i|Z| eB f_{n_\rho - 1, m}(\vec{r}_\perp) \\
\mp (E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm 2\lambda B + M)(E_{n_\rho, m < 0}^{(\pm, Z < 0)} \pm E_{n_\rho, m < 0}^{(\pm, Z > 0)} f_{n_\rho, m + 1}(\vec{r}_\perp)) \right) \right)
\]
where \( n_\rho = 0, 1, 2, \ldots \) and \( m = 0, \pm 1, \pm 2, \ldots \).

3. Solution of Dirac equation for neutral fermions with anomalous magnetic moments in uniform magnetic field

For a neutral fermion with mass \( M \) and an anomalous magnetic moment \( \kappa \), moving in a uniform magnetic field, the Dirac equation takes the form
\[
i\partial_t \psi(x) = \left( -i\vec{\alpha} \cdot \vec{\nabla} - 2\beta\lambda \vec{\Sigma} \cdot \vec{B} + \beta M \right) \psi(x),
\]
where \( \lambda = \kappa e/4M \). Directing a magnetic field along the \( z \)-axis \( \vec{B} = B\hat{e}_z \) and separating longitudinal and transverse degrees of freedom relative to the magnetic field, the Dirac equation Eq. (32) can be transformed into the system of first order differential equations, which can be obtained from Eq. (11) with \( Z = 0 \) with coupled large and small components of the Dirac bispinor wave function, and then to the second order differential equation with decoupled large and small components of the Dirac wave function
\[
\left( \frac{1}{2} \{\pi_+, \pi_-\} - (E_\perp \pm 2\lambda B)^2 + M^2 \right) \Phi^{(\pm)}(\vec{r}_\perp) = 0,
\]
which can be obtained from Eq. (11) with \( Z = 0 \). For neutral fermions the ”up” and ”down” solutions coincide, i.e. \( \Phi^{(\pm)}(\vec{r}_\perp) = \Phi^{(\pm)}(\vec{r}_\perp) = \Phi^{(\pm)}(\vec{r}_\perp) \). Since for neutral fermions the canonical momentum operator is \( \vec{p} = -i\vec{\nabla} \), in Cartesian coordinates Eq. (32) takes the form
\[
\left( \Delta_\perp + (E_\perp \pm 2\lambda B)^2 - M^2 \right) \Phi^{(\pm)}(\vec{r}_\perp) = 0,
\]
with \( \Delta_\perp = \partial_x^2 + \partial_y^2 \). The solution of this equation can be taken in the form of a plane wave \( \Phi^{(\pm)}(\vec{r}_\perp) \sim e^{i\vec{p}_\perp \cdot \vec{r}_\perp} \). As a result the energy spectrum of the transverse motion of neutral fermions with anomalous magnetic moments in a uniform magnetic field is given by
\[
E_\perp^{(\pm)} = \sqrt{M^2 + p_\perp^2 \mp 2\lambda B},
\]
with \( p_\perp^2 = p_x^2 + p_y^2 \). The Dirac bispinor wave functions of a neutral fermion with an anomalous magnetic moment \( \kappa \), moving in a uniform magnetic field, we search in the form
\[
\psi^{(\pm)}(x) = \left( \begin{array}{l} F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} \right) e^{-iE_\perp^{(\pm)} t + i\vec{p}_\perp \cdot \vec{r}}.
\]
The wave functions Eq. (36) are normalised as
\[
\int d^3x \psi^{(\pm)}(x) \psi^{(\sigma)}(x) = 2 E_\perp^{(\pm)} (2\pi)^3 \delta(3)(\vec{p}_\perp - \vec{p}_\perp) \delta_{\sigma, \sigma},
\]
where \( E_\perp^{(\pm)} = \sqrt{E_\perp^{(\pm)^2} + p_\perp^2} \) for \( \sigma = \pm \).
the normalised Dirac wave functions of neutral fermions

\[ \psi_{\mu}^{(\pm)}(x) = \frac{1}{\sqrt{2(E_{\mu}^{(\pm)} \pm E_{\perp}^{(\pm)})(E_{\perp}^{(\pm)} \pm 2\lambda B)(E_{\perp}^{(\pm)} \pm 2\lambda B \mp M)}} \times \left( \begin{array}{c} (p_x - ip_y)(E_{\mu}^{(\pm)} \pm E_{\perp}^{(\pm)}) \\
\mp p_z (E_{\mu}^{(\pm)} \pm 2\lambda B \mp M) \\
\pm (E_{\mu}^{(\pm)} \pm E_{\perp}^{(\pm)})(E_{\perp}^{(\pm)} \pm 2\lambda B \mp M) \end{array} \right) e^{-i(E_{\mu}^{(\pm)} t + ip_z \cdot \vec{r})}. \] (38)

Our solutions for the relativistic wave functions of the neutral fermions with anomalous magnetic moments, moving in a uniform magnetic field, agree well with those, obtained in [3].

4. Non–relativistic limit of solutions of Dirac equation for charged and neutral fermions with anomalous magnetic moments in uniform magnetic field

In this section we investigate the non–relativistic limits of the relativistic wave functions of the charged and neutral fermions and their energy spectra. For the positively and negatively charged fermions in the states \((\pm, Z > 0)\) and \((\pm, Z < 0)\) the wave functions take the form

\[
\psi_{n_\rho, m, p_z}^{(\pm, Z>0)}(x) = 2\pi \sqrt{2M} \sqrt{\frac{n_\rho!(ZeB)^{|m|+1}}{\pi(n_\rho + |m|)!2^{|m|+1}}} \frac{1}{0} \frac{0}{0} e^{i(m+1)\phi} e^{-\frac{1}{2}|ZeB\rho^2} e^{i\phi} \\
\times e^{-\frac{1}{2}|ZeB\rho^2} e^{i\phi} \\
\psi_{n_\rho, m>0, p_z}^{(-, Z>0)}(x) = 2\pi \sqrt{2M} \sqrt{\frac{(n_\rho - 1)!(ZeB)^{m+2}}{\pi(n_\rho + m)!2^{m+2}}} \frac{0}{1} \frac{1}{0} e^{i(m+1)\phi} e^{-\frac{1}{2}|ZeB\rho^2} e^{i\phi} \\
\times e^{-\frac{1}{2}|ZeB\rho^2} e^{i\phi} \\
\psi_{n_\rho, m<0, p_z}^{(-, Z>0)}(x) = 2\pi \sqrt{2M} \sqrt{\frac{n_\rho!(ZeB)^{|m|}}{\pi(n_\rho + |m| - 1)!2^{|m|}}} \frac{0}{1} \frac{1}{0} e^{i(m+1)\phi} e^{-\frac{1}{2}|ZeB\rho^2} e^{i\phi} \\
\times e^{-\frac{1}{2}|ZeB\rho^2} e^{i\phi} \]
(39)
and

\[ \psi_{n_{p}, m > 0, p_{z}}^{(+, \rho > 0)}(x) = 2\pi \sqrt{2M} \sqrt{\frac{(n_{p} - 1)!(|Z|eB)^{|m|} + 1}{\pi(n_{p} + |m| - 1)!2^{|m|} + 1}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rho^{m}_{n_{p}} L_{n_{p}-1}^{m} \left( \frac{|Z|eB^{2}}{2} \right) e^{-\frac{1}{2}|Z|eB^{2} e^{im\phi}} \times e^{-i\varepsilon_{n_{p, m > 0}}^{(+, \rho > 0)}t + ip_{z}z}, \]

\[ \psi_{n_{p}, m > 0, p_{z}}^{(-, \rho < 0)}(x) = 2\pi \sqrt{2M} \sqrt{\frac{(n_{p} - 1)!(|Z|eB)^{|m| + 2}}{\pi(n_{p} + |m| - 1)!2^{|m|} + 2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rho^{m + 1}_{n_{p}} L_{n_{p}-1}^{m + 1} \left( \frac{|Z|eB^{2}}{2} \right) e^{-\frac{1}{2}|Z|eB^{2} e^{i(m + 1)\phi}} \times e^{-i\varepsilon_{n_{p, m > 0}}^{(-, \rho < 0)}t + ip_{z}z}, \]

where \( n_{p} = 0, 1, 2, \ldots \) and \( m = 0, \pm 1, \pm 2, \ldots \). The wave functions Eq.\(39\) and Eq.\(40\) are defined up to unessential phase factors.

In the non–relativistic approximation the energy spectra \( \varepsilon_{n_{p, m}^{(+ Z > 0)}} = \varepsilon_{n_{p, m}^{(\pm Z \geq 0)}} - M \) of charged fermions in a uniform magnetic field can be written in the form

\[ \varepsilon_{n_{p, m}^{(+, Z > 0)}} = \frac{ZeB}{M} \left( n_{p} + \frac{|m| - m + 1}{2} \right) - \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

\[ \varepsilon_{n_{p, m}^{(+, Z > 0)}} = \frac{ZeB}{M} \left( n_{p} - 1 + \frac{(|m| + 1) - (m + 1) + 1}{2} \right) + \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

\[ \varepsilon_{n_{p, m}^{(-, Z > 0)}} = \frac{ZeB}{M} \left( n_{p} + \frac{(|m| - 1) - (m + 1) + 1}{2} \right) + \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

\[ \varepsilon_{n_{p, m}^{(+, Z < 0)}} = \frac{|Z|eB}{M} \left( n_{p} - 1 + \frac{|m| + m + 1}{2} \right) + \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

\[ \varepsilon_{n_{p, m}^{(-, Z < 0)}} = \frac{|Z|eB}{M} \left( n_{p} - 1 + \frac{(|m| + 1) + (m + 1) + 1}{2} \right) - \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

(41)

The terms, proportional to a total magnetic moment \( \mu = 1 + \kappa \), correspond to the Pauli energy splitting \[11\]. The non–relativistic energy spectra Eq.(41) correspond to the quantum states, described by the non–relativistic wave functions Eq.\(39\) and Eq.\(40\) of positively and negatively charged fermions.

Using the properties of the Laguerre polynomials \[12\] we may make a change of the quantum numbers \( n_{p} - 1 \rightarrow n_{p} \) for the second wave function of Eq.\(39\) as well as the first and second one of Eq.\(40\). Additionally, all wave functions with polarisation down are shifted \( m + 1 \rightarrow m \). This brings the non–relativistic wave functions of positively and negatively charged fermions in Eq.\(39\) and Eq.\(40\) to a unified form

\[ \psi_{n_{p, m}^{(\pm Z \geq 0)}}(x) = 2\pi \sqrt{2M} \sqrt{\frac{(n_{p} - 1)!(|Z|eB)^{|m| + 1}}{\pi(n_{p} + |m|)!2^{|m| + 1}}} \varphi_{\pm} \rho^{m}_{n_{p}} L_{n_{p}-1}^{m} \left( \frac{|Z|eB^{2}}{2} \right) e^{-\frac{1}{2}|Z|eB^{2} e^{im\phi}} e^{-i\varepsilon_{n_{p, m}^{(\pm Z \geq 0)}}t + ip_{z}z}, \]

agreeing fully with the non–relativistic wave functions obtained in \[11\], \( \varphi_{\pm} \) are the wave functions with elements \( \varphi_{+} = (1, 0, 0, 0) \) and \( \varphi_{-} = (0, 1, 0, 0) \), respectively. These wave functions describe the non–relativistic quantum states with the energy spectra

\[ \varepsilon_{n_{p, m}^{(+, Z > 0)}} = \frac{ZeB}{M} \left( n_{p} + \frac{|m| - m + 1}{2} \right) \pm \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

\[ \varepsilon_{n_{p, m}^{(-, Z < 0)}} = \frac{|Z|eB}{M} \left( n_{p} + \frac{(|m| + m + 1)}{2} \right) \pm \frac{1}{2} \mu M + \frac{p_{z}^{2}}{2M}, \]

(43)
for positively and negatively charged fermions, respectively, with quantum numbers $n_{\rho} = 0, 1, \ldots$ and $m = 0, \pm 1, \ldots$.

In the non–relativistic limit the wave functions of a neutral fermion with an anomalous magnetic moment $\kappa$ are equal to

$$
\psi_{\rho}^{(+)}(x) = \sqrt{2M} \begin{pmatrix} 1 \\
0 \\
0 
\end{pmatrix} e^{-i\kappa^{(+)} z t + i\vec{p} \cdot \vec{r}}, \quad \psi_{\rho}^{(-)}(x) = \sqrt{2M} \begin{pmatrix} 0 \\
1 \\
0 
\end{pmatrix} e^{-i\kappa^{(-)} z t + i\vec{p} \cdot \vec{r}}. \tag{44}
$$

The energy spectra $\varepsilon_{\rho}^{(\pm)}$ are given by

$$
\varepsilon_{\rho}^{(\pm)} = \frac{\vec{p}^2}{2M} \pm \frac{1}{2} \kappa eB/M. \tag{45}
$$

The Pauli energy splitting is defined by the last term in Eq.\,(44) \[11\].

5. Quantum states of charged fermions with anomalous magnetic moments in uniform magnetic field with radial quantum number $n_{\rho} = 0$

The analysis of quantum states with zero quantum numbers is one of the most important mathematical problems of Quantum mechanics \[11\] as a part of Mathematical physics \[12\]. In section 2 we have found the relativistic wave functions and energy spectra of charged fermions with anomalous magnetic moments coupled to a uniform magnetic field. These wave functions have a rather complicated dependence on the radial quantum number $n_{\rho}$, which does not make obvious the existence of the correct quantum states for $n_{\rho} = 0$. In this section, skipping rather tedious and cumbersome intermediate calculation we adduce the wave functions of quantum states of charged fermions with anomalous magnetic moments, moving in a uniform magnetic field with radial quantum number $n_{\rho} = 0$.

First, we consider the quantum states described by the wave functions Eq.\,(28) and Eq.\,(31), the energy spectra of which do not depend on the magnetic quantum number. Taking the wave functions Eq.\,(28) and Eq.\,(31) in the limit $n_{\rho} \to 0$ with arbitrary magnetic quantum numbers $m$ and skipping intermediate calculations we obtain the following expressions

$$
\psi_{0,m>0,p_z}(x) = \sqrt{\frac{\pi |ZeB|^{m+1}}{2^m m!}} \sqrt{E(+,Z>0) + E(+,0,m>0)} \begin{pmatrix} f_{0,m}(\vec{r}_\perp) \\
p_z/E_{0,m>0} + E(+,0,m>0) f_{0,m}(\vec{r}_\perp) \\
0 
\end{pmatrix} e^{-iE_{0,m>0} t + i p_z z},
$$

$$
\psi_{0,m>0,p_z}(x) = 0,
$$

$$
\psi_{0,m<0,p_z}(x) = 0,
$$

$$
\psi_{0,m<0,p_z}(x) = \sqrt{\frac{\pi |ZeB|^m}{2^m |m|!}} \sqrt{E(−,Z<0) + E(−,0,m<0)} \begin{pmatrix} 0 \\
f_{0,m+1}(\vec{r}_\perp) \\
-E_{0,m<0} + E(−,0,m<0) f_{0,m+1}(\vec{r}_\perp) 
\end{pmatrix} e^{-iE_{0,m<0} t + i p_z z}. \tag{46}
$$

The quantum states of positively charged fermions, described by the wave functions Eq.\,(29), depend on the radial quantum number $n_{\rho}$ and the magnetic quantum number $m < 0$, i.e. $m = -1, -2, \ldots$. Setting $n_{\rho} = 0$ we obtain

$$
\psi_{0,m<0,p_z}(x) = \sqrt{\frac{\pi |ZeB|^m}{2^m |m|!}} \sqrt{E(±,Z>0) + E(±,0,m<0)} \begin{pmatrix} \frac{1}{2} \kappa eB(M + 2AB) + 2AB \sqrt{E(±,Z>0) + 2AB} \sqrt{E(±,0,m<0) + 2AB} \\
E(±,0,m<0) + 2AB \sqrt{E(±,Z>0) + 2AB} \sqrt{E(±,0,m<0) + 2AB} \\
E(±,Z>0) + 2AB \sqrt{E(±,Z>0) + 2AB} \sqrt{E(±,0,m<0) + 2AB} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \kappa eB \sqrt{E(±,Z>0) + 2AB} \sqrt{E(±,0,m<0) + 2AB} \\
E(±,0,m<0) + 2AB \sqrt{E(±,Z>0) + 2AB} \sqrt{E(±,0,m<0) + 2AB} \\
E(±,Z>0) + 2AB \sqrt{E(±,Z>0) + 2AB} \sqrt{E(±,0,m<0) + 2AB} \end{pmatrix} e^{-iE_{0,m<0} t + i p_z z}. \tag{47}
$$
The wave functions Eq. (30) describe the quantum states of negatively charged fermions with anomalous magnetic moments, characterised by the radial quantum number \( n_\rho \) and the magnetic quantum number \( m \geq 0 \), i.e. \( m = 0, 1, 1, 2, \ldots \). The quantum states, characterised by the quantum numbers \( n_\rho = 0, m \geq 0 \) are defined by the wave functions
\[
\psi_{0,m \geq 0,\rho}^{(\pm Z < 0)}(x) = 0.
\]
(48)
The properties of the relativistic wave functions obtained at \( n_\rho = 0 \) \( q \), taken in the non–relativistic limit, agree well with the properties of the non–relativistic wave functions, given in Eq. (39) and Eq. (40).

6. Conclusive discussion

We have proposed a solution of the Dirac equation for charged and neutral fermions with spin \( \frac{1}{2} \) and anomalous magnetic moments, moving in a uniform magnetic field. It is well–known that the Dirac equation, describing the motion of a relativistic fermion with spin \( \frac{1}{2} \) in any external field, can be transformed into a system of first order differential equations for the wave functions of the coupled up and down spin states of the large and small components of the Dirac wave function [2]. The procedure of a disentanglement of the up and down spin states of the large and small components of the Dirac wave function depends on the structure of the external field and, of course, the properties of the fermions. For charged fermions without anomalous magnetic moment the differential equations, describing the wave functions of the disentangled up and down spin states, are of second order [1, 4]–[7]. As we have shown above, a non-vanishing anomalous magnetic moment leads to fourth order differential equations for the disentangled up and down spin states. This agrees well with the analysis of the energy spectra of charged fermions with spin \( \frac{1}{2} \) and anomalous magnetic moment carried out in [7]. These fourth order differential equations can be reduced to second order ones with eigenvalues yielding the Pauli energy splitting. The wave functions of the disentangled up and down spin states are calculated in cylindrical coordinates in terms of the generalised Laguerre polynomials \( L_{\rho}^{(m)} \) and circular functions \( e^{im\phi} \), depending on the radial quantum number \( n_\rho = 0, 1, 2, \ldots \) and the magnetic quantum number \( m = 0, \pm 1, \pm 2, \ldots \). According to [10], only the solutions of the Dirac equation in cylindrical coordinates for charged fermions, moving in a uniform magnetic field, can adequately describe a two–dimensional motion of fermions in the plane orthogonal to a uniform magnetic field.

For the calculation of the normalised Dirac wave functions we use the system of first order differential equations for the wave functions Eq. (7) of the coupled up and down spin states and the normalisation conditions Eq. (26) and Eq. (37) for charged and neutral fermions, respectively. In the limit of vanishing anomalous magnetic moment, the obtained Dirac wave functions coincide with those, given in [1, 3]–[7]. In comparison with the wave functions, obtained in [2], the solutions, proposed above, are more detailed and convenient for applications.

In the non–relativistic limit the wave functions of the charged fermions coincide with the well–known solutions of the Schrödinger equation [11]. The relativistic wave functions of a neutral fermion with an anomalous magnetic moment, moving in a uniform magnetic field, have a shape of plane waves with an energy spectrum, splitted by the interaction of its anomalous magnetic moment with a uniform magnetic field.

The relativistic energy spectra of charged fermions in a uniform magnetic field are calculated in dependence of the radial \( n_\rho = 0, 1, 2, \ldots \) and magnetic \( m = 0, \pm 1, \pm 2, \ldots \) quantum numbers. All of these energy levels have an additional splitting, which is caused by the interaction of the anomalous magnetic moments of fermions with a uniform magnetic field. The energy spectra of positively charged fermions with positive magnetic quantum number and negatively charged fermions with negative magnetic quantum numbers are infinitely degenerated [10]. In the non–relativistic limit the energy spectra reproduce the well–known Landau energy spectra with the Pauli energy splitting [11].

Of course, the wave functions and the energy spectra of charged fermions, given in terms of the radial \( n_\rho \) and magnetic \( m \) quantum numbers, can be presented in terms of the principal quantum number \( n \), expressed in terms of the radial and magnetic quantum numbers, and the magnetic quantum number \( m \). However, such a definition of the fermion states in a uniform magnetic field is not convenient for practical calculations.

Comparing our results with those, obtained in [7], we would like to accentuate that our technique of the solution of the Dirac equation is similar to that used in [7] but simpler. In addition to the energy spectra, which were obtained in [7], we have calculated the relativistic wave functions. As regards the results, obtained in [8], where the solutions of the Dirac equation were found in the “number of states” representation, we have calculated not only the wave functions and energy spectra, but also investigated a detailed dependence of the relativistic wave functions on the magnetic quantum number and analysed their non–relativistic limits. A detailed behaviour of the relativistic wave functions and their exact dependence on the quantum numbers play an important role for applications of the obtained results, for example, to the neutron \( \beta^- \)–decay.
This work was supported by the Austrian “Fonds zur Förderung der Wissenschaftlichen Forschung” (FWF) under contract I689-N16 and contract I862-N20 AXION and in part by the U.S. Department of Energy contract no. DE-FG02-08ER41531, no. DE-AC02-06CH11357 and by the Wisconsin Alumni Research Foundation.

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