PERMUTATION ORBIFOLDS OF THE HEISENBERG VERTEX ALGEBRA $\mathcal{H}(3)$

ANTUN MILAS, MICHAEL PENN, HANBO SHAO

Abstract. We study the $S_3$-orbifold of a rank three Heisenberg vertex algebras in terms of generators and relations. By using invariant theory we prove that the orbifold algebra has a minimal strongly generating set of vectors whose conformal weights are $1, 2, 3, 4, 5, 6^2$ (two generators of degree 6). The structure of the cyclic $Z_3$-orbifold is determined by similar methods. We also study modular properties of characters of modules for these vertex algebras.

1. Introduction

For every vertex operator algebra $V$, the $n$-fold tensor product $V^\otimes n$ has a natural vertex operator algebra structure. The symmetric group $S_n$ acts on $V(n) := V^\otimes n$ by permuting tensor factors and thus $S_n \subset \text{Aut}(V(n))$. Denote by $V(n)^{S_n}$ the fixed point vertex operator subalgebra called the $S_n$-orbifold of $V(n)$. It is an open problem to classify irreducible modules of $V(n)^{S_n}$ although it is widely believed that every such module should come from a $g$-twisted $V^\otimes n$-module for some $g \in S_n$. When it comes to inner structure of $V(n)^{S_n}$ (e.g. strong system of generators) this is wide open even for Heisenberg orbifolds $\mathcal{H}(n)^{S_n}$, where $\mathcal{H}$ is the rank one Heisenberg vertex algebra.

Finite and permutation orbifolds have been extensively studied both in physics [4, 7] and mathematics literature. In the literature on vertex algebras, the main focus has been on classification and construction of twisted modules starting with [14, 17]. Further developments include "Quantum Galois" theory developed in [10], work on twisted sectors of permutation orbifolds [2] (see also [9]), tensor category structure for general $G$-orbifolds of rational vertex algebras [12, 16], etc. There are also numerous papers on orbifold vertex algebras for abelian groups and groups of small order. Structure and representations of 2-permutation orbifolds were subjects of [1, 2, 11]; see also more recent work on 3-permutation orbifolds of lattice vertex algebras [13]. ADE orbifolds of rank one lattice vertex algebras (e.g. [9]) are important for classification of $c = 1$ rational vertex algebras. Orbifolds of irrational $C_2$-cofinite vertex algebras have been investigated in [3, 8] and [2]. A. Linshaw (see [8, 18, 19, 20], etc.) extensively studied orbifolds of "free field" vertex operator (super)algebras using classical invariant theory. As a consequence of the main result in [20], every finite orbifold of an affine vertex algebra is finitely strongly generated. In particular, this implies that $\mathcal{H}(n)^{S_n}$ has this property. Characters of orbifolds of affine vertex algebras were investigated earlier in [15].

In this paper we are concerned with the structure of one of the simplest non-abelian orbifolds, $\mathcal{H}(3)^{S_3}$ where $\mathcal{H}$ is the rank one Heisenberg vertex algebra (case $\mathcal{H}(2)^{S_2}$ is well-understood [11, 11]). We prove three main results. First result pertains to generators of $\mathcal{H}(3)^{S_3}$. We show in Theorem 4.1 that this vertex algebra is isomorphic to a $W$-algebra of type $(2, 3, 4, 5, 6^2)$ tensored with a rank one Heisenberg vertex algebra. Here labels $2, 3, 4, 5, 6^2$ indicate that this $W$-algebra is strongly generated by the Virasoro vector (of
degree two) and five primary vectors of degrees 3, 4, 5 and two of degree 6. These five generators are given explicitly in Section 4 where they are denoted by $J_1, J_2, C_1, C_2$ and $C_3$. Our second result pertains to the cyclic $\mathbb{Z}_2$-orbifold of $\mathcal{H}(3)$ (see Theorem 5.1 for details). In the last part we discuss characters of modules for vertex operator algebras $V(n)_{S_n}$ and especially for $n = 3$. It is expected that many irrational vertex algebras will enjoy modular invariance in a generalized sense involving iterated integrals instead of sums. For rank $n$ permutation orbifolds the character of a module $M$ should transform as

$$\text{ch}[M] \left( -\frac{1}{\tau} \right) = \sum_{i \geq 1} \chi_{S_{M, M_i}} \text{ch}[M_{\lambda_i}] d\lambda_i,$$

where $\lambda_i \in \mathbb{R}^i$, $1 \leq i \leq n$, parametrize certain $\mathcal{H}(n)_{S_n}$-modules. After outlining how characters of $\mathcal{H}(3)_{S_3}$-modules can be computed (see also [4]) we show that they indeed obey generalized modular invariance as above (see Theorem 7.1).

2. THE $S_n$-ORBIFOLD OF THE HEISENBERG VERTEX OPERATOR ALGEBRA

Let $\mathcal{H}$ denotes the rank one Heisenberg vertex operator algebra generated by $\alpha(-1)\mathbb{1}$, with the usual conformal vector (and grading) given by $\omega = \frac{1}{2}\alpha^2(-1)\mathbb{1}$. Let $\mathcal{H}(n) = \mathcal{H}^n$. For convenience, we suppress the tensor product symbol and let $\alpha_1(-1)\mathbb{1} := \alpha(-1) \mathbb{1} \otimes \cdots \otimes \mathbb{1} \in \mathcal{H}(n)$, and similarly we define $\alpha_i(-1)\mathbb{1}$, $i \geq 2$, such that $\mathcal{H}(n) = \langle \alpha_1(-1)\mathbb{1}, \cdots, \alpha_n(-1)\mathbb{1} \rangle$. We consider the natural action of $S_n$ on $\mathcal{H}(n)$ given by

$$\sigma \cdot \alpha_{i_1}(m_1) \cdots \alpha_{i_k}(m_k) \mathbb{1} = \alpha_{\sigma(i_1)}(m_1) \cdots \alpha_{\sigma(i_k)}(m_k) \mathbb{1},$$

for $1 \leq i_j \leq n$, $m_j < 0$, and $\sigma \in S_n$.

We clearly have a natural linear isomorphism

$$\mathcal{H}(n) \cong \mathbb{C}[x_i(m)| 1 \leq i \leq n, m \geq 0]$$

induced by $\alpha_i(-m - 1) \mapsto x_i(m)$ for $m \geq 0$. Using terminology in [19], we say that $\mathbb{C}[x_i(m)| 1 \leq i \leq n, m \geq 0]$ is the associated graded algebra of the vertex algebra $\mathcal{H}(n)$. Further, we may endow the polynomial algebra $\mathbb{C}[x_i(m)| 1 \leq i \leq n, m \geq 0]$ with the structure of a $\partial$-ring by defining the map

$$\partial : \mathbb{C}[x_i(m)| 1 \leq i \leq n, m \geq 0] \rightarrow \mathbb{C}[x_i(m)| 1 \leq i \leq n, m \geq 0]$$

$$x_i(m) \mapsto (m + 1)x_i(m + 1),$$

where the action of $\partial$ is extended to the whole space via the Leibniz rule. This definition of $\partial$ is compatible with the translation operator in $\mathcal{H}(n)$ given by $T(v) = v_{-2}\mathbb{1}$.

The following Lemma is from [13].

Lemma 2.1. Let $\mathcal{A}$ be a vertex algebra with a $\mathbb{Z}_{\geq 0}$ filtration, where $\tilde{\mathcal{A}}$ is the associated $\partial$-ring. If $\tilde{\mathcal{A}}$ generates $\tilde{\mathcal{A}}$ then $\{a_i| i \in I\}$ strongly generates $\mathcal{A}$, where $a_i$ and $\tilde{a}_i$ are related via the natural linear isomorphism described by the $\mathbb{Z}_{\geq 0}$ filtration.

Further, we recall that the invariant ring $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ has a variety of generating sets, including the power sum polynomials

$$p_i = x_1^i + \cdots + x_n^i, \; i \geq 1.$$

In addition to this, it is common to study the invariant theory of the ring of infinitely many commuting copies of this polynomial algebra, where we denote by $x_i(m)$ the copy of $x_i$ from
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the $m^{th}$ copy. A Theorem of Weyl [23] shows that $\mathbb{C}[x_{i}(m)|1 \leq i \leq n, m \geq 0]^{S_{n}}$ is generated by the polarizations of these polynomials

$$q_{k}(m_{1}, \ldots, m_{k}) = \sum_{i=1}^{n} x_{i}(m_{1}) \cdots x_{i}(m_{k}),$$

for $1 \leq k \leq n$. Now, applying Lemma 2.1, we have an initial strong generating set for the orbifold $\mathcal{H}(n)^{S_{n}}$ given by the vectors

$$\omega_{k}(m_{1}, \ldots, m_{k}) = \sum_{i=1}^{n} \alpha_{i}(-1 - m_{1}) \cdots \alpha_{i}(-1 - m_{k}) \mathbb{1},$$

for $1 \leq k \leq n$ and $m_{j} \geq 0$. It should be noted that the conformal vector of $\mathcal{H}(n)$ is $\frac{1}{2}\omega_{2}(0,0)$, making the orbifold a vertex operator algebra.

3. Warmup: $\mathcal{H}(2)^{S_{2}}$

Here we describe the structure of $\mathcal{H}(2)^{S_{2}}$. This case is well-known so we only provide a few details. Denote by $W(2,4) := \mathcal{H}(1)^{S_{2}}$ the fixed point subalgebra under the action $\alpha(-1)\mathbb{1} \rightarrow -\alpha(-1)\mathbb{1}$, where $\alpha$ is the Heisenberg generator. This vertex algebra was thoroughly studied in [11]. Let

$$h = \alpha_{1}(-1)\mathbb{1} + \alpha_{2}(-1)\mathbb{1}, \quad h^{\perp} = \alpha_{1}(-1)\mathbb{1} - \alpha_{2}(-1)\mathbb{1}, \quad \omega = \frac{1}{4}\alpha_{1}^{2}(-1)\mathbb{1} + \frac{1}{4}\alpha_{2}^{2}(-1)\mathbb{1} - \frac{1}{2}\alpha_{1}(-1)\alpha_{2}(-1)\mathbb{1}.$$

Observe that as vertex algebras

$$\mathcal{H}(2) = \mathcal{H}(1)_{h} \otimes \mathcal{H}(1)_{h^{\perp}}$$

where $\mathcal{H}(1)_{h} = \langle h \rangle$ and $\mathcal{H}(1)_{h^{\perp}} = \langle h^{\perp} \rangle$ with conformal vector $\omega$. The nontrivial element of the group $S_{2}$ fixes the first tensor factor and sends

$$h^{\perp} \rightarrow -h^{\perp}.$$

Thus we immediately get

$$\mathcal{H}(2)^{S_{2}} \cong \mathcal{H}(1) \otimes W(2,4).$$

It is easy to see that

$$H = \frac{1}{2} \left( \alpha_{1}(-1)^{4} + \alpha_{2}(-1)^{4} - 4\alpha_{1}^{3}(-1)\alpha_{2}(-1) - 4\alpha_{1}(-1)\alpha_{2}^{3}(-1) + 6\alpha_{1}^{2}(-1)\alpha_{2}^{2}(-1) + 3\alpha_{1}^{2}(-2) + 3\alpha_{2}^{2}(-2) \right.$$

$$\left. - 6\alpha_{1}(-2)\alpha_{2}(-2) + 2\alpha_{1}(-1)\alpha_{2}(-3) + 2\alpha_{1}(-3)\alpha_{2}(-1) - 2\alpha_{1}(-3)\alpha_{1}(-1) - 2\alpha_{2}(-3)\alpha_{2}(-1) \right) 1$$

is contained in $\langle h^{\perp} \rangle$ and is primary of conformal weight 4, thus a generator of the $W(2,4)$ algebra.
4. The Rank Three Case

4.1. The orbifold $\mathcal{H}(3)^{S_3}$. As described above we know that the orbifold $\mathcal{H}(3)^{S_3}$ will be strongly generated by the vectors

$$
\omega_1(a) = \sum_{i=1}^3 \alpha_i(-1-a) \mathbb{1}, \\
\omega_2(a, b) = \sum_{i=1}^3 \alpha_i(-1-a)\alpha_i(-1-b) \mathbb{1}, \\
\omega_3(a, b, c) = \sum_{i=1}^3 \alpha_i(-1-a)\alpha_i(-1-b)\alpha_i(-1-c) \mathbb{1},
$$

(4.1)

The following change of basis of generating set will allow for an efficient reduction in the generating set of the corresponding orbifold

$$
\beta_1(-1) \mathbb{1} = \frac{1}{\sqrt{3}}(\alpha_1(-1) \mathbb{1} + \alpha_2(-1) \mathbb{1} + \alpha_3(-1) \mathbb{1}) \\
\beta_2(-1) \mathbb{1} = \frac{1}{\sqrt{3}}(\alpha_1(-1) \mathbb{1} + \eta^2\alpha_2(-1) \mathbb{1} + \eta\alpha_3(-1) \mathbb{1}) \\
\beta_3(-1) \mathbb{1} = \frac{1}{\sqrt{3}}(\alpha_1(-1) \mathbb{1} + \eta\alpha_2(-1) \mathbb{1} + \eta^2\alpha_3(-1) \mathbb{1}),
$$

(4.2)

where $\eta$ is a primitive third root of unity. Using this generating set we have $\sigma \cdot \beta_1(-1) \mathbb{1} = \beta_1(-1) \mathbb{1}$ for all $\sigma \in S_3$. Further, examining the action of the generators of $S_3$ on the generators of $\mathcal{H}(3)$ we have

$$
(1\ 2) \cdot \beta_2(-1) \mathbb{1} = \beta_3(-1) \mathbb{1} \quad (1\ 2) \cdot \beta_3(-1) \mathbb{1} = \beta_2(-1) \mathbb{1}
$$

(4.3) and

$$
(1\ 2\ 3) \cdot \beta_2(-1) \mathbb{1} = \eta\beta_2(-1) \mathbb{1} \quad (1\ 2\ 3) \cdot \beta_3(-1) \mathbb{1} = \eta^2\beta_3(-1) \mathbb{1}
$$

(4.4)

From this action, we see that an initial generating set for the orbifold $\mathcal{H}(3)^{S_3}$ may be taken to be

$$
\omega^0_1(a) = \beta_1(-1-a) \mathbb{1} \\
\omega^0_2(a, b) = \beta_2(-1-a)\beta_3(-1-b) \mathbb{1} + \beta_3(-1-a)\beta_2(-1-b) \mathbb{1} \\
\omega^0_3(a, b, c) = \beta_2(-1-a)\beta_3(-1-b)\beta_2(-1-c) \mathbb{1} + \beta_3(-1-a)\beta_3(-1-b)\beta_3(-1-c) \mathbb{1},
$$

(4.5)

for $0 \leq a \leq b \leq c$. In fact, we can see explicitly write the relation between our original generators (4.1) and our new generators (4.3) as follows

$$
\omega_1(a) = \sqrt{3}\omega^0_1(a) \\
\omega_2(a, b) = \omega^0_2(a, b) + \omega^0_1(a) - \omega^0_1(b) \\
\omega_3(a, b, c) = \frac{1}{3}\omega^0_3(a, b, c) + \frac{1}{\sqrt{3}}(\omega^0_1(a) - \omega^0_1(b)) + \omega^0_1(b) - \omega^0_1(c) + \omega^0_1(c) - \omega^0_2(a, b) + \omega^0_1(a) - \omega^0_1(b) - \omega^0_1(c)
$$

(4.6)
We may make a similar rewriting of the generating set of \( \mathbb{C}[x_i(m)|1 \leq i \leq 3, m \geq 0] \) by setting
\[
y_1(m_1) = \frac{1}{\sqrt{3}}(x_1(m_1) + x_2(m_1) + x_3(m_1))
\]
and
\[
y_2(m_2) = \frac{1}{\sqrt{3}}(x_1(m_2) + \eta^2 x_2(m_2) + \eta x_3(m_2))
\]
(4.7)
\[
y_3(m_3) = \frac{1}{\sqrt{3}}(x_1(m_3) + \eta x_2(m_3) + \eta x_3(m_3)).
\]
for \( m_i \geq 0 \), and
\[
q_i^0(a) = y_1(a)
\]
(4.8)
\[
q_i^0(a, b) = y_2(a)y_3(b) + y_3(a)y_2(b)
\]
\[
q_i^0(a, b, c) = y_2(a)y_2(b)y_2(c) + y_3(a)y_3(b)y_3(c).
\]

Using the translation operator, \( T \), restricted to the orbifold \( \mathcal{H}(3)^{S_3} \), the initial strong generating \( (4.5) \) can be reduced per the following Lemma.

**Lemma 4.1.** The orbifold \( \mathcal{H}(3)^{S_3} \) is strongly generated by the vectors
\[
\omega_1^0(0)
\]
(4.9)
\[
\omega_2^0(0, 2a) \text{ for } a \geq 0
\]
\[
\omega_3^0(0, a, 2b) \text{ for } b \geq 0 \text{ and } 0 \leq a \leq b.
\]

**Proof.** It is clear that since
\[
\omega_1^0(a) - 1 = \frac{1}{a!}\omega_1^0(0) - a 1
\]
(4.10)
the linear portion of the generating set \( (4.5) \) can be immediately minimized to contain the single weight one element \( \omega_1(0) \).

Moving on toward the quadratic terms in \( (4.5) \), we consider the vector spaces
\[
A_2(m) = \text{span} \{\omega_2^0(a, b)|a + b = m\}
\]
and a natural family of subspaces
\[
\partial A_2(m) = \{v - 2 \mathbb{1}|v \in A_2(m)\}.
\]
(4.11)

Observe that
\[
\omega_2^0(a, b) - 2 \mathbb{1} = \text{Res} z^{-1} \frac{\partial}{\partial z} (Y(\beta_\zeta(-1-a)\beta_\zeta(-1-b)\mathbb{1}, z) + Y(\beta_\zeta(-1-a)\beta_\zeta(-1-b)\mathbb{1}, z))
\]
(4.13)
\[
= \frac{1}{a!b!} \text{Res} z^{-1} \left( z \frac{\partial}{\partial z} \beta_\zeta(z) \frac{\partial}{\partial z} \beta_\zeta(z) + z \frac{\partial}{\partial z} \beta_\zeta(z) \frac{\partial}{\partial z} \beta_\zeta(z) \right)
\]
\[
= (a + 1) \omega_2^0(a + 1, b) + (b + 1) \omega_2^0(a, b + 1),
\]
where \( \beta_\zeta(z) = Y(\beta_\zeta(-1)\mathbb{1}, z) \). Also notice that for all \( a, b \geq 0 \) we have \( \omega_2^0(a, b) = \omega_2^0(b, a) \).

We may take an initial basis for \( A_2(m) \) to be the set
\[
\mathcal{B}_2(m) = \{\omega_2^0(i, m - i)|0 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor \}
\]
(4.14)
which implies that
\[
\dim A(2n + 1) = \dim A(2n) = n + 1.
\]
(4.15)
Further the set
\[(4.16) \{v_{-2i} \mid v \in B_2(m)\}\]
is clearly a basis for $\partial A_2(m)$. It follows that
\[(4.17) A_2(2n) = \partial A_2(2n - 1) \oplus C\omega_2^0(0, 2n)\]
and thus by induction we may take a more convenient basis of $A_2(m)$ to be
\[(4.18) \{\omega_2^0(0, 2i - 1 - m + 2i) \mid 0 \leq i \leq \frac{m}{2}\}\].

Thus, a more efficient set of quadratic generators for the $\mathcal{H}(3)^S$ may be taken to be $\omega_2^0(0, 2a)$ for $a \geq 0$.

Finally, we consider the cubic generators. We set
\[(4.19) A_3(m) = \text{span} \{\omega_3^0(a, b, c) \mid a + b + c = m\}\]
and work towards finding a basis of $A(m)$ that has fewer parameters. An initial basis can be taken to be
\[(4.20) \{\omega_3^0(a, b, c) \mid 0 \leq a \leq b \leq c\}\].

Observe that
\[(4.21) \dim A_3(m) = p(m + 3)\]
where $p(n)$ is the number of partitions of $n$ in three parts. It is well known that $p(n)$ satisfies the recursion
\[(4.22) p(n) = p(n - 1) + \left\lfloor \frac{n + 2}{2} \right\rfloor - \left\lfloor \frac{n + 2}{3} \right\rfloor\].

Observe that $\partial A_3(m - 1) = \text{span} \{v_{-2i} \mid v \in A_3(m - 1)\}$ is a subspace of $A_3(m)$, as such we now consider the quotient vector space
\[(4.23) B_3(m) = A_3(m) / \partial A_3(m - 1)\].

If we show that the vector spaces $B(m)$ have bases given by
\[
\begin{align*}
B_{B_3(6n)} &= \{\omega_3^0(0, 2i, 6n - 2i) \mid 0 \leq i \leq n\}, \\
B_{B_3(6n+1)} &= \{\omega_3^0(0, 2i + 1, 6n - 2i) \mid 0 \leq i \leq n - 1\}, \\
B_{B_3(6n+2)} &= \{\omega_3^0(0, 2i, 6n + 2 - 2i) \mid 0 \leq i \leq n\}, \\
B_{B_3(6n+3)} &= \{\omega_3^0(0, 2i + 1, 6n + 2 - 2i) \mid 0 \leq i \leq n\}, \\
B_{B_3(6n+4)} &= \{\omega_3^0(0, 2i, 6n + 4 - 2i) \mid 0 \leq i \leq n\}, \\
B_{B_3(6n+5)} &= \{\omega_3^0(0, 2i + 1, 6n + 4 - 2i) \mid 0 \leq i \leq n\}
\end{align*}
\]
then we will have
\[(4.24) \dim B_3(m) = \left\lfloor \frac{m + 2}{2} \right\rfloor - \left\lfloor \frac{m + 2}{3} \right\rfloor\]
from which our result will follow.
Remark 4.1. Using (1.6) we may take

\[ \omega_1(0) \]
(4.25)
\[ \omega_2(0, 2a) \text{ for } a \geq 0 \]
\[ \omega_3(0, a, 2b) \text{ for } b \geq 0 \text{ and } 0 \leq a \leq b, \]
as our strong generating set. We take advantage of this translation tool as generators of this form are somewhat more natural to the parent algebra, \( \mathcal{H}(3) \).

Remark 4.2. This simplification of the generating set also holds if we consider the \( \partial \)-ring associated to \( \mathcal{H}(3)^{S_3} \), which has un-reduced generators given by (2.4), we may reduce these to a minimal generating set for \( \mathbb{C}[x_i(m)] \mid 1 \leq i \leq 3, m \geq 0 \)^{S_3} given by

\[ q_1(0) \]
(4.26)
\[ q_2(0, 2a) \text{ for } a \geq 0 \]
\[ q_3(0, a, 2b) \text{ for } b \geq 0 \text{ and } 0 \leq a \leq b. \]

We now present the following relations among the generators of \( \mathbb{C}[x_i(m)] \mid 1 \leq i \leq 3, m \geq 0 \)^{S_3}.

Lemma 4.2. For \( \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6) \) with \( a_1, a_2, a_3, a_4, a_5, a_6 \geq 0 \) we have

\[ D_6^{C,1}(\mathbf{a}) = q_2^0(a_1, a_2)q_2^0(a_3, a_4)q_2^0(a_5, a_6) - q_2^0(a_1, a_2)q_2^0(a_3, a_6)q_2^0(a_4, a_5) \]
(4.27)
\[ + q_2^0(a_1, a_4)q_2^0(a_2, a_6)q_2^0(a_3, a_5) - q_2^0(a_1, a_4)q_2^0(a_2, a_3)q_2^0(a_5, a_6) \]
\[ + q_2^0(a_1, a_5)q_2^0(a_2, a_4)q_2^0(a_3, a_6) - q_2^0(a_1, a_5)q_2^0(a_2, a_6)q_2^0(a_3, a_4) \]
\[ + q_2^0(a_1, a_6)q_2^0(a_2, a_3)q_2^0(a_4, a_5) - q_2^0(a_1, a_6)q_2^0(a_2, a_4)q_2^0(a_3, a_5) = 0, \]

\[ D_6^{C,2}(\mathbf{a}) = q_3^0(a_1, a_2, a_3)q_3^0(a_4, a_5, a_6) - q_3^0(a_1, a_2, a_4)q_3^0(a_3, a_5, a_6) \]
(4.28)
\[ + \frac{1}{2} q_2^0(a_1, a_3)q_2^0(a_2, a_6)q_2^0(a_4, a_5) - \frac{1}{2} q_2^0(a_1, a_4)q_2^0(a_2, a_5)q_2^0(a_3, a_6) \]
\[ + \frac{1}{2} q_2^0(a_1, a_5)q_2^0(a_2, a_3)q_2^0(a_4, a_6) - \frac{1}{2} q_2^0(a_1, a_5)q_2^0(a_2, a_4)q_2^0(a_3, a_6) = 0, \]
and

\[ D_5^C(\mathbf{a}) = q_2^0(a_1, a_2)q_3^0(a_3, a_4, a_5) - q_2^0(a_1, a_5)q_3^0(a_2, a_3, a_4) \]
(4.29)
\[ - q_2^0(a_2, a_5)q_3^0(a_1, a_3, a_4) - q_2^0(a_3, a_4)q_3^0(a_1, a_2, a_5) \]
\[ + q_2^0(a_3, a_5)q_3^0(a_1, a_2, a_4) + q_2^0(a_4, a_5)q_3^0(a_1, a_2, a_3) = 0, \]
for \( \mathbf{a} = (a_1, a_2, a_3, a_4, a_5) \) with \( a_1, a_2, a_3, a_4, a_5 = 0 \)

Remark 4.3. These relations play the role of determinant (and similar) relations in the invariant theory of the classical Lie groups. We do not claim that these two families of expressions generate all relations in this case. In fact, there are such relations at every degree and since the Frobenius number of 5 and 6 is 19, there are relations at least up to this degree.
We now pull the relations from Lemma 4.2 to expressions involving the generators of the orbifold \( \mathcal{H}(3)^{S_3} \). For \( a = (a_1, a_2, a_3, a_4, a_5, a_6) \) with \( a_1, a_2, a_3, a_4, a_5, a_6 \geq 0 \) set

\[
D_6^1(a) = \sum_{i=1}^6 \omega^0_i(a_1, a_2, a_3, a_4, a_5, a_6) - \omega^0_i(a_1, a_2) - 1 \omega^0_i(a_3, a_4) - 1 \omega^0_i(a_5, a_6) - \omega^0_i(a_1, a_2, a_3, a_4, a_5, a_6) + \omega^0_i(a_1, a_4) - 1 \omega^0_i(a_2, a_6) - 1 \omega^0_i(a_3, a_5) - \omega^0_i(a_1, a_4) - 1 \omega^0_i(a_2, a_3) - 1 \omega^0_i(a_5, a_6) + \omega^0_i(a_1, a_4) - 1 \omega^0_i(a_2, a_4) - 1 \omega^0_i(a_3, a_5) - \omega^0_i(a_1, a_6) - 1 \omega^0_i(a_2, a_4) - 1 \omega^0_i(a_3, a_5),
\]

(4.30)

\[
D_6^2(a) = \omega^0_i(a_1, a_2, a_3, a_4, a_5, a_6) - \omega^0_i(a_1, a_2, a_3, a_5, a_6) + \frac{1}{2} \omega^0_i(a_1, a_3) - 1 \omega^0_i(a_2, a_6) - 1 \omega^0_i(a_4, a_5) - \frac{1}{2} \omega^0_i(a_1, a_4) - 1 \omega^0_i(a_2, a_5) - 1 \omega^0_i(a_3, a_6) + \frac{1}{2} \omega^0_i(a_1, a_5) - 1 \omega^0_i(a_2, a_3) - 1 \omega^0_i(a_4, a_6) - \frac{1}{2} \omega^0_i(a_1, a_5) - 1 \omega^0_i(a_2, a_4) - 1 \omega^0_i(a_3, a_5),
\]

(4.31)

and

\[
D_5(a) = \omega^0_i(a_1, a_2) - 1 \omega^0_i(a_3, a_4) - \omega^0_i(a_1, a_2, a_3, a_4) - 1 \omega^0_i(a_2, a_5),
\]

(4.32)

for \( a = (a_1, a_2, a_3, a_4, a_5) \) with \( a_1, a_2, a_3, a_4, a_5 \).

Observe that for \( i \in \{1, 2\} \), the weight of the expression \( D_6^i(a) \) is \(|a| + 6\) while the weight of \( D_5(a) \) is \(|a| + 5\), where we take \(|u_1, \ldots, u_n| = u_1 + \cdots + u_n\) for any multi-index. By Lemma 4.2 and repeated applications of the weak associativity properties of vertex algebras, along with our linear isomorphism (2.22) we see that for \( i \in \{1, 2\} \), we may rewrite

\[
D_6^i(a) = D_6^{(4)}(a) + D_6^{(2)}(a),
\]

(4.33)

where

\[
D_2^{(4)}(a) = \sum_{a, b, c, d \geq 0} \mu_{a,b,c,d}^{(4)} \omega^0_0(a, b) - 1 \omega^0_0(c, d)
\]

(4.34)

and

\[
D_2^{(2)}(a) = \sum_{a, b \geq 0} \mu_{a,b}^{(2)} \omega^0_2(a, b),
\]

(4.35)

where \( \mu_{a,b,c,d}^{(4)} \) and \( \mu_{a,b}^{(2)} \) are appropriate constants. Importantly, this expansion has no terms that are “cubic” in the generators \( \omega^0_2(a, b) \) or “quadratic” in the generators \( \omega^0_0(a, b, c) \), each of which would contain a combination of six of the original \( \beta_i \) vectors with \( i \in \{2, 3\} \). We have a similar (and simper) decomposition

\[
D_5(a) = D_5^{(3)}(a),
\]

(4.36)

where

\[
D_3^{(3)}(a) = \sum_{a, b, c \geq 0} \mu_{a,b,c}^{(3)} \omega^0_0(a, b, c),
\]

(4.37)
where \( \mu_{a,b,c} \) are constants. Again, Lemma 4.2 weak associativity, and \( (2.2) \) implies that the expansion of \( D_3(a) \) will not contain terms that are “products” of the generators \( \omega^0_2(a, b) \) and \( \omega^0_3(a, b, c) \) or otherwise a combination of five of the original vectors \( \beta_2 \) and \( \beta_3 \).

Now we are poised to use the expressions \( D_6^b(a) \) and \( D_5(a) \) to further reduce the generating set of \( \mathcal{H}(3)^S \) as described in Lemma 4.1.

**Lemma 4.3.** The full list of quadratic generators described in Lemma 4.1 can be replaced with the set
\[ \{ \omega^0_2(0,0), \omega^0_2(0,2), \omega^0_2(0,4), \} \].

**Proof.** Using the expression \( D^1_6(0,0,0,0,1,1) \), the decomposition described in \( (4.33) \) may be used to construct the following equation
\[ \omega^0_2(0,6) = \frac{53880}{371} \omega^0_2(0,4)-3\mathbb{I} - \frac{57210}{371} \omega^0_2(0,2)-5\mathbb{I} + \frac{174150}{371} \omega^0_2(0,0)-7\mathbb{I} 
- \frac{300}{371} \omega^0_2(0,4)-1\mathbb{I}\omega^0_2(0,0) - \frac{900}{371} \omega^0_2(0,3)-1\mathbb{I}\omega^0_2(0,1) + \frac{1290}{371} \omega^0_2(0,1)-1\mathbb{I}\omega^0_2(0,3) - \frac{1290}{371} \omega^0_2(0,0)-1\mathbb{I}\omega^0_2(1,3) 
- \frac{60}{371} \omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,2) + \frac{120}{371} \omega^0_2(0,2)-1\mathbb{I}\omega^0_2(0,2) - \frac{705}{371} \omega^0_2(0,2)-1\mathbb{I}\omega^0_2(1,1) - \frac{165}{371} \omega^0_2(1,2)-1\mathbb{I}\omega^0_2(0,1) 
+ \frac{165}{371} \omega^0_2(1,1)-1\mathbb{I}\omega^0_2(1,2) - \frac{90}{371} \omega^0_2(1,1)-1\mathbb{I}\omega^0_2(1,1) 
+ \frac{45}{371} \omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,1)-1\mathbb{I}\omega^0_2(1,1) - \frac{15}{371} \omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,0) - \frac{15}{371} \omega^0_2(0,1)-1\mathbb{I}\omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,1) 
- \frac{15}{371} \omega^0_2(0,1)-1\mathbb{I}\omega^0_2(0,1)-1\mathbb{I}\omega^0_2(0,0) + \frac{60}{371} \omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,1)-1\mathbb{I}\omega^0_2(0,1) - \frac{60}{371} \omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,0)-1\mathbb{I}\omega^0_2(0,1), \]
where this calculation was performed using \( [22] \). In particular, we can write
\[ \omega^0_2(0,6) = P(\omega^0_2(0,0), \omega^0_2(0,2), \omega^0_2(0,4), \omega^0_3(a, b, c)), \]
where \( P(\cdot) \) is a vertex algebraic polynomial. Now we observe that the family of operators
\[ \Psi_a = \frac{1}{c_a} \left( -\frac{3}{2} \omega^0_2(0,2)_1 + \omega^0_2(0,4)_3 \right) \]
with
\[ c_a = \frac{3}{4} (1 + 2a) (11 + 28a + 24a^2 + 8a^3), \]
have the property that
\[ \Psi_a \omega^0_2(0,2a) = \omega^0_2(0,2a + 2). \]
Further, iterative applying this operator to \( (4.39) \) proves our claim.

**Lemma 4.4.** The full list of cubic generators described in Lemma 4.1 can be replaced with the set \( \{ \omega^0_2(0,0,0), \omega^0_2(0,0,2), \omega^0_3(0,1,2), \} \).

**Proof.** Using \( D_5(0,0,0,1,1) \) and the decomposition described in \( (4.30),(4.37) \) we have
\[ \omega^0_3(0,0,4) = -\frac{16}{15} \omega^0_2(0,1,2)-2\mathbb{I} + \frac{4}{15} \omega^0_2(0,0,2)-3\mathbb{I} + \frac{24}{45} \omega^0_2(0,0,0)-5\mathbb{I} - \frac{2}{5} \omega^0_2(0,0)-1\omega^0_3(0,1,1) 
+ \frac{4}{5} \omega^0_2(0,1,1)-1\omega^0_3(0,0,1) - \frac{2}{5} \omega^0_2(0,1,1)-1\omega^0_3(0,0,0), \]
Proof. Clearly, by a result from [10], this vertex algebra is simple. By Lemma 4.3, Lemma 4.4 and earlier discussion we see that \( H \) generating set is minimal.

From the character formula (see Proposition 6.1) we see that

\[
\text{Theorem 4.1.} \quad (i) \text{ The vertex operator algebra } \mathcal{H}(3)^{S_3} \text{ is simple of type } (1, 2, 3, 4, 5, 6^2), \text{ i.e. it is strongly generated by seven vectors whose conformal weights are: } 1, 2, 3, 4, 5, 6, 6. \text{ This generating set is minimal.}
\]

(ii) \( \mathcal{H}(3)^{S_3} \) is isomorphic to \( \mathcal{H}(1) \otimes W \), where \( W \) is of type \( (2, 3, 4, 5, 6^2) \).

(iii) \( \mathcal{H}(3)^{S_3} \) is not freely generated (by any set of generators).

Proof. Clearly, by a result from [10], this vertex algebra is simple. By Lemma 4.3, Lemma 4.4 and earlier discussion we see that \( \mathcal{H}(3)^{S_3} \) is strongly generated by vectors of conformal weights: 1 (linear generator), 2, 4, 6 (quadratic generators) and 3, 5, 6 (cubic generators).

From the character formula (see Proposition 6.1) we see that

\[
q^{1/8} \text{ch}[\mathcal{H}(3)^{S_3}](\tau) = \frac{1}{(q; q)_\infty(q^2; q)_\infty(q^3; q)_\infty(q^4; q)_\infty(q^5; q)_\infty(q^6; q)_\infty} = O(q^9),
\]

where \((a; q)_\infty := \prod_{n \geq 0}(1 - aq^n)\). Easy analysis shows that dropping one (or more) generators from this generating set would imply that certain graded dimensions of \( \mathcal{H}(3)^{S_3} \) are strictly bigger than the corresponding graded dimension for the smaller subalgebra. Thus, the
proposed set of generators must be a minimal generating set. Clearly, this vertex algebra is not freely generated by these generators due to

\[ q^{1/8} \text{ch} \left[ H(3)^{S_3} \right](\tau) - \frac{1 - q^9}{(q; q)_\infty (q^2; q)_\infty (q^3; q)_\infty (q^4; q)_\infty (q^6; q)_\infty (q^9; q)_\infty} = O(q^{10}), \]

which implies that there must be a nontrivial relation at degree 9. In fact, one can quickly argue this is impossible simply from the fact that ch[H(3)^{S_3}] is modular, which is impossible to achieve with a freely generating set.

For (ii), we first observe that \langle \omega_1(0) \rangle is isomorphic to H(1). By taking the commutant \( W := \text{Comm}(H(1), H(3)^{S_3}) \), we get \( H(1) \otimes W \cong H(3)^{S_3} \). Each generator of \( H(3)^{S_3} \), except \( \omega_1(0) \), can be written as a linear combination of elements in \( H(1) \otimes W \) with a nonzero component in \( W \) (otherwise it would imply that some of the generator are contained \( \langle \omega_1(0) \rangle \), contradicting to minimality in (i)). These nonzero components form a generating set of \( W \).

\[ \square \]

Using (4.10), we may take the original invariants \( \omega_1(0), \omega_2(0, 0), \omega_2(0, 2), \omega_2(0, 4), \omega_3(0, 0, 0), \omega_3(0, 0, 2), \) and \( \omega_3(0, 1, 2) \) as our minimal strong generating set. Furthermore, the following change of variables allows us to express the orbifold in terms of primary generators.

\[ h = \omega_1(0), \quad \omega = \frac{1}{2} \omega_2(0, 0), \quad J_1 = 2\omega_2(0, 2) - \frac{24}{37} \omega_{-1} - \frac{30}{37} \omega_{-3} \mathbb{1}, \]
\[ J_2 = 24\omega_2(0, 4) - \frac{256}{37} \omega_{-1} - \frac{256}{27} \omega_1 + \frac{364}{2403} \omega_{-2} - \frac{4074}{2403} \omega_{-3} + \frac{3472}{2403} \omega_{-3} \mathbb{1} \]
\[ C_1 = \omega_3(0, 0, 0) - \frac{1}{3} h_{-1} h_{-1} \mathbb{1} \]
\[ C_2 = 2\omega_3(0, 0, 2) - \frac{2}{15} \omega_3(0, 0, 0) - \frac{8}{45} h_{-1} h_{-1} + \frac{2}{15} h_{-2} h_{-2} \]
\[ - \frac{16}{45} \omega_{-1} \omega_3(0, 0, 0) + \frac{16}{135} h_{-1} h_{-1} h_{-1} \]
\[ C_3 = 2\omega_3(0, 1, 2) - \frac{1}{10} \omega_3(0, 0, 2) - \frac{6}{25} \omega_3(0, 0, 0) - \frac{2}{25} \omega_{-1} \omega_3(0, 0, 0) - \frac{3}{25} \omega_{-2} \omega_3(0, 0, 0) - h_{-1} h_{-1} + \frac{2}{25} h_{-2} h_{-2} h_{-1} \]
\[ - \frac{2}{25} h_{-2} h_{-1} h_{-1} + \frac{2}{25} h_{-3} h_{-2} + \frac{6}{75} h_{-4} h_{-1} + \frac{7}{75} h_{-5} \mathbb{1} \]

5. The orbifold \( H(3)^{Z_3} \)

We now consider the orbifold of \( H(3) \) under the action of the cyclic subgroup \( Z_3 \cong \langle (1 \ 2 \ 3) \rangle \subset S_3 \). Using a similar strategy to the previous subsection we may take

\[ \omega_0^1 = \beta_1(1) \mathbb{1}, \quad \omega_0^2(a, b, c) = \beta_2(-1 - a) \beta_3(-1 - b) \mathbb{1}, \quad \omega_0^{0, 2}(a, b, c) = \beta_2(-1 - a) \beta_3(-1 - b) \beta_4(-1 - c) \mathbb{1} \]

as our initial generating set. Parallel to Lemma 4.1, we have the following initial reduction of the generating set.
Lemma 5.1. The orbifold \( \mathcal{H}(3)^{\mathbb{Z}_3} \) is strongly generated by vectors\( \varepsilon_0(0), \varepsilon_1(0, a) \) for \( a \geq 0 \)\( \varepsilon_2(0, a, 2b) \) for \( b \geq 0 \) and \( 0 \leq a \leq b \), \( \varepsilon_{3,3,3}(0, a, 2b) \) for \( b \geq 0 \) and \( 0 \leq a \leq b \).

The overall reduction of this initial generating set is similar to the strategy used in the previous sub-section and so we provide fewer details.

Lemma 5.2. The full list of quadratic generators described in (5.2) can be replaced with\( \varepsilon_{2,3}(0, 0), \varepsilon_{2,3}(0, 1), \varepsilon_{2,3}(0, 2), \) and \( \varepsilon_{2,3}(0, 3) \).

Proof. Using\( \varepsilon_{2,3}(0, 4) = \frac{1}{2} \varepsilon_{2,3}(0, 0) - \varepsilon_{2,3}(1, 1) - \frac{1}{2} \varepsilon_{2,3}(0, 1) - \varepsilon_{2,3}(1, 0) + \frac{1}{2} \varepsilon_{2,3}(0, 3) - 2 \)

we can eliminate the vector \( \varepsilon_{2,3}(0, 4) \) from the generating set.

Now we consider the operator

\[
\Psi_a = \frac{1}{a^2 + a} (2 \varepsilon_{2,3}(0, 0) + \varepsilon_{2,3}(0, 1))
\]

and observe that\( \Psi_a \varepsilon_{2,3}(0, a) = \varepsilon_{2,3}(0, a + 1) \)

and so, when iteratively applied to \( \varepsilon_{2,3}(0, 4) \) we finish the proof. \( \square \)

Lemma 5.3. The full list of cubic generators described in (5.2) can be replaced with\( \varepsilon_{2,2,2}(0, 0, 0), \varepsilon_{2,2,2}(0, 0, 2), \varepsilon_{3,3,3}(0, 0, 0), \varepsilon_{3,3,3}(0, 0, 2) \).

Proof. We focus on the \( \varepsilon_{2,2,2}(0, b, 2a) \) terms as the \( \varepsilon_{3,3,3}(0, b, 2a) \) are similar. We begin with

\[
\varepsilon_{2,2,2}(0, 1, 2) = -\frac{3}{20} \varepsilon_{2,2,2}(0, 0, 0) - \varepsilon_{2,3}(1, 0) + \frac{3}{20} \varepsilon_{2,2,2}(0, 0, 1) - \varepsilon_{2,3}(0, 0)
\]

\[
- \frac{11}{20} \varepsilon_{2,2,2}(0, 0, 2) - 2 \varepsilon_{2,3}(0, 0, 0) - 4 \varepsilon_{2,3}(0, 0)
\]

and

\[
\varepsilon_{2,2,2}(0, 0, 4) = -\varepsilon_{2,2,2}(0, 0, 2) - \varepsilon_{2,3}(0, 0, 0) + \varepsilon_{2,3}(2, 0) - \varepsilon_{2,3}(0, 2)
\]

\[
+ \frac{8}{27} \varepsilon_{2,2,2}(0, 1, 2) - 2 \varepsilon_{2,3}(0, 0, 0) - 3 \varepsilon_{2,3}(0, 0, 0) - 5 \varepsilon_{2,3}(0, 0, 0) - 5 \varepsilon_{2,3}(0, 0, 0)
\]

Now using operators similar to those used in the proof of Lemma 5.2 will finish the argument. \( \square \)

Theorem 5.1. (i) The vertex operator algebra \( \mathcal{H}(3)^{\mathbb{Z}_3} \) is simple of type \((1, 2, 3^3, 4, 5^3)\), i.e. it is strongly generated by seven vectors whose conformal weights are: \(1, 2, 3, 3, 4, 5, 5, 5\). This generating set is minimal.

(ii) \( \mathcal{H}(3)^{\mathbb{Z}_3} \) is isomorphic to \( \mathcal{H}(1) \otimes W \), where \( W \) is of type \((2, 3^3, 4, 5^3)\).

(iii) \( \mathcal{H}(3)^{\mathbb{Z}_3} \) is not freely generated (by any set of generators).

Proof. This proof is similar to the proof of Theorem [1]. In particular, we use the fact that the free \( W \)-algebra on a generating set containing the same weight elements with any one removed will have certain graded dimensions strictly less than those described by the character of \( \mathcal{H}(3)^{\mathbb{Z}_3} \), [6,1]. \( \square \)
Finally, the orbifold $\mathcal{H}(3)^{\mathbb{Z}_3}$ may be described in terms of primary generators with the following vectors.

\begin{align}
(5.10) \\
h &= \omega^0(0), \quad \omega = \omega^0(0, 0) + \frac{1}{2} \omega^0(0) - 1 \omega^0(0), \quad J_1 = \omega^0(0, 1) - \frac{1}{2} \omega^0 - 1 h - 2 h \\
J_2 &= 2 \omega^0(0, 2) - \omega^0(0, 1) - 1 + 2 \omega^0 - 1 h - 2 h - 2 h - 1 h - 1 h + 77 \omega - 7 h - 3 h \\
&- \frac{12}{3} h - 2 h, \\
J_3 &= 6 \omega^0(0, 3) - \frac{8}{3} \omega^0(0, 2) - 1 + 4 h - 2 h - 2 h + 8 h - 1 h - 1 h - 2 h - 1 h \\
&- 16 h - 3 h + 8 h - 1 h - 3 h + 4 h - 1 h - 3 h - 1 h + 4 h - 1 h \\
&+ \frac{88}{5} h - 5 h, \\
C_1^{(2)} &= \omega^0_{2,2,2}(0, 0, 0), \quad C_2^{(2)} = \omega^0_{2,2,2}(0, 0, 2) - \frac{1}{15} \omega^0_{2,2,2}(0, 0, 0) - 3 h - 3 h - 1 h + 4 h - 1 h \\
&- \frac{15}{8} \omega^0_{2,2,2}(0, 0, 0) \\
C_1^{(3)} &= \omega^0_{3,3,3}(0, 0, 0), \quad C_2^{(3)} = \omega^0_{3,3,3}(0, 0, 2) - \frac{1}{15} \omega^0_{3,3,3}(0, 0, 0) - 3 h - 3 h - 1 h + 4 h - 1 h \\
&- \frac{15}{8} \omega^0_{3,3,3}(0, 0, 0)
\end{align}

6. The Character of $\mathcal{H}(n)^{S_n}$

We first derive a general formula for the character of the $S_n$-orbifold of a vertex algebra. Although formulas of this form already appeared in the literature (e.g. [4] or [15]) we include the proof for completeness.

**Theorem 6.1.** Let $V$ be a VOA and $f^{S_n}(q)$ the character of the $S_n$-fixed point subalgebra $V(n)^{S_n} \subset V^{\otimes n}$ (under the usual action), then

$$f^{S_n}(q) = \frac{1}{n!} \sum_{g \in S_n} \text{tr}_{V^{\otimes n}} g q^{L(0) - c/24}$$

where $g$ acts by permutation of tensor factors.

**Proof.** Consider the group algebra $\mathbb{C}[S_n]$ and the idempotent

$$e = \frac{1}{|S_n|} \sum_{g \in S_n} g \in \mathbb{C}[S_n], \quad e^2 = e$$

where $|S_n| = n!$. The space $\text{Im}(e)$ is precisely the $S_n$-fixed subVOA of $V^{\otimes n}$. As all eigenvalues of $e$ are 1 or 0 (and 1 for the $S_n$-fixed subalgebra), the character can be computed simply by taking the trace of $e$:

$$\text{tr}_{V^{\otimes n}} e q^{L(0) - c/24} = \text{tr}_{V(n)^{S_n}} q^{L(0) - c/24}. \quad \square$$

Now we specialize to $n = 3$ and $V = \mathcal{H}$.

**Proposition 6.1.**

$$\text{ch}([\mathcal{H}(3)^{S_3}](\tau)) = \frac{q^{-1/8}}{6} \left( \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^2} + 3 \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n})} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)} + 2 \prod_{n=1}^{\infty} \frac{1}{(1 - q^{3n})} \right).$$
One similarly argues for 3-cycles that
\[ \text{tr}(12) \text{ on the following basis of } \mathcal{H}(3) \]
and standard representation
\[ \text{tr}(123) \text{ in this basis has a non-zero entry on the diagonal } (=1) \text{ if and only if the corresponding basis element is } \]
\[ p_\lambda \otimes p_\lambda \otimes p_\nu. \]

Since all \( p_\lambda \) are generated by \( \alpha(-i) \), vectors contributing to the trace are generated by \( \alpha(-i) \otimes \alpha(-i) \otimes \alpha(-j) \) Thus
\[ \text{tr}(12) q^{L(0)-3/24} = q^{-1/8} \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-q^{2i}}. \]

One similarly argues for 3-cycles that
\[ \text{tr}(123) q^{L(0)-3/24} = q^{-1/8} \prod_{i=1}^{\infty} \frac{1}{1-q^{3i}}. \]

\[ \square \]

**Corollary 6.1.**

\[ \text{ch}[\mathcal{H}(3)^{S_3}] = \frac{q^{-1/8}}{3} \left( \prod_{n=1}^{\infty} \frac{1}{1-q^n} + 2 \prod_{n=1}^{\infty} \frac{1}{1-q^{3n}} \right). \]

7. **Modular invariance property of characters of \( \mathcal{H}(3)^{S_3} \)-modules**

By a general result we have a decomposition of \( (S_3, \mathcal{H}(3)^{S_3}) \)-module:
\[ \mathcal{H}(3) = \mathcal{H}(3)^{S_3} \oplus \mathcal{H}(3)^{S_3, \text{sgn}} \oplus V(2)^{\otimes 2} \otimes \mathcal{H}(3)^{S_3, \text{st}} \]

where \( V(2) \) is the 2-dimensional standard representation of \( S_3 \), and \( \mathcal{H}(3)^{S_3, \text{sgn}} \) and \( \mathcal{H}(3)^{S_3, \text{st}} \) are irreducible \( \mathcal{H}(3)^{S_3} \)-modules \[ 16 \]. By using Young symmetrizers for the sign representation \text{sgn} and standard representation \text{st} we quickly get
\[ \text{ch}[\mathcal{H}(3)^{S_3, \text{sgn}}] = \frac{q^{-1/8}}{6} \left( \prod_{n=1}^{\infty} \frac{1}{1-q^n} - 3 \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} + 2 \prod_{n=1}^{\infty} \frac{1}{1-q^{3n}} \right) \]
\[ \text{ch}[\mathcal{H}(3)^{S_3, \text{st}}] = \frac{q^{-1/8}}{6} \left( 2 \prod_{n=1}^{\infty} \frac{1}{1-q^n} - 2 \prod_{n=1}^{\infty} \frac{1}{1-q^{3n}} \right). \]

Next we compute modular transformation properties of characters of these modules and show that the space closes under the \( SL(2, \mathbb{Z}) \) action. We need relations
\[ \eta(-1/\tau)^n = (\sqrt{-i\eta})^n \eta(\tau)^n, \quad n \geq 1, \]
where as usual \( \eta(\tau) = q^{1/24}(q;q)_\infty \), and higher dimensional Gauss' integral formula

\[
\int_{\mathbb{R}^n} q^{w_1^2/2 + \cdots + w_n^2/2} dw = \frac{1}{(\sqrt{-i\tau})^n}.
\]

Notice also asymptotic behavior (as \( t \to 0^+ \))

\[ (7.1) \quad \eta(it)^n \sim \left( \frac{1}{t^2} \right) e^{-\frac{\pi i n}{t^2}}. \]

From these formulas we easily get

\[
\frac{1}{\eta(-1/\tau)^3} = \int_{\mathbb{R}^3} q^{w_1^2/2+w_2^2/2+w_3^2/2} \eta(\tau)^3 dw,
\]

\[
\frac{1}{\eta(-1/\tau)^2 \eta(-2/\tau)} = \sqrt{2} \int_{\mathbb{R}^2} q^{w_2^2/2+w_3^2/2} \eta(\tau)\eta(\tau/2) dw.
\]

\[
\frac{1}{\eta(-3/\tau)} = \sqrt{3} \int_{\mathbb{R}} q^{w_1^2/2} \eta(\tau/3) dw.
\]

Now we interpret these formula in terms of characters of irreducible \( g \)-twisted \( \mathcal{H}(3) \)-modules (thus ordinary \( \mathcal{H}(3)^{S^3} \)-modules) - this will allow us to prove a modular-invariance type theorem.

Consider \( (w_1, w_2, w_3) \) parametrizing \( \mathbb{C}^3 \). For each such triple we have a Fock space \( \mathcal{H}(3) \)-module \( F_{w_1,w_2,w_3} \). Away from hyperplanes \( w_1 - w_2 = 0, w_1 - w_3 = 0, w_2 - w_3 = 0 \), that is, when the parameters are distinct \( F_{w_1,w_2,w_3} \) is irreducible also as an \( \mathcal{H}(3)^{S^3} \) module (this can be proven by using [10]). We call these representations typical. We clearly have

\[
\text{ch}[F_{\lambda,\mu,\nu}](\tau) = \frac{q^{w_1^2/2+w_2^2/2+w_3^2/2}}{\eta(\tau)^3}.
\]

The hyperplanes carry information about atypical representations. We first consider \( (w_1, w_2, w_3) \) when two \( w_i \) and \( w_j \) are equal but they do not equal \( w_k \) (geometrically these triples lie on a hyperplane without the line \( w_1 = w_2 = w_3 \)). Without loss of generality we take \( w_1 = w_2 \neq w_3 \). For each such triple we have a \( \sigma \)-twisted \( \mathcal{H}(3) \)-modules \( \theta F_{w_1,w_2,w_3} \), which is constructed through isomorphism in Section 2. Thus we have, in the notation of [11] a twisted module \( F_{w_1,w_3}(\theta) = F_{w_1} \otimes M(1)(\theta) \otimes F_{w_2} \), where \( M(1)(\theta) \) is a unique irreducible twisted \( \mathcal{H} \)-module of conformal weight \( \frac{16}{(123)} \). Then we get

\[
\text{ch}[F_{w_1,w_3}(\theta)] = \frac{q^{w_1^2+w_3^2}}{\eta(\tau/2)\eta(\tau)}.
\]

Lastly we consider \( w = w_1 = w_2 = w_3 \) and \( \sigma \)-twisted \( \mathcal{H}(3) \)-modules \( F_w(\sigma) \). These modules constructed below are in one-to-one correspondence with \( \mathcal{H}(1) \)-modules [5] so they depend on a single parameter \( w \). We claim

\[ (7.2) \quad \text{ch}[F_w(\sigma)](\tau) = \frac{q^{w^2/2}}{\eta(\tau/3)}.
\]

First we construct the \( \sigma := (123) \)-twisted Heisenberg algebra \( \mathfrak{h}^\sigma \) [17]. It is isomorphic to \( \mathbb{C}[t, t^{-1}] \otimes t^1/3 \mathbb{C}[t, t^{-1}] \otimes t^{2/3} \mathbb{C}[t, t^{-1}] \otimes \mathbb{C} k \) as a graded Lie algebra. For the twisted Fock space \( F_w(\sigma) \) we have

\[
\text{ch}[F_w(\sigma)](\tau) = \frac{q^{w^2/2+h-1/8}}{(q;q)_{\infty}(q^{1/3};q)_{\infty}(q^{2/3};q)_{\infty}(q^{4/3};q)^{3/4}} = \frac{q^{w^2/2+h-3/24}}{(q^{1/3};q^{1/3})_{\infty}}.
\]
It remains to compute the conformal anomaly $h$ for an automorphism of order $p$, which is given by $h = \frac{1}{p^2} \sum_{i=1}^{p-1} (p-i)r_i$, where $r_i$ are dimensions of eigenspaces. Plugging in $p = 3$ and $r_i = 1$ we obtain $h = \frac{1}{9}$ and thus (7.2) follows. Above calculations allow us to conclude

**Theorem 7.1.** The character of $\mathcal{H}(3)^{S_3}$ is modular invariant, in the sense of (7.1).

In fact this result holds for any irreducible $\mathcal{H}(3)^{S_3}$-module discussed above (and thus conjecturally for all irreducible modules).

For a $V$-module $M$ we define its quantum dimension:

$$\text{qdim}[M] = \lim_{t \to 0^+} \frac{\text{ch}[M](it)}{\text{ch}[V](it)}.$$ 

For general $M$ this limit might not exist or it might be infinite. We now compute quantum dimensions of $\mathcal{H}(3)^{S_3}$-modules discussed earlier.

**Proposition 7.1.** For typical Fock representations we have

$$\text{qdim}[F_{w_1,w_2,w_3}] = 6.$$ 

For atypical modules appearing inside $\mathcal{H}(3)$:

$$\text{qdim}[\mathcal{H}(3)^{S_3,sgn}] = 1 \quad \text{and} \quad \text{qdim}[\mathcal{H}(3)^{S_3,st}] = 2$$

All atypical representation coming from $g$-twisted $\mathcal{H}(3)$-modules with $g \neq 1$ (whose parameters lie on the hyperplanes) have quantum dimension $+\infty$.

**Proof.** This essentially follows from the asymptotic formula (7.1). For $g$-twisted $\mathcal{H}(3)$-modules, $g \neq 1$, we get

$$\frac{1}{\eta(\tau) \eta(\tau/2)} \sim \frac{1}{t} e^{\frac{\pi}{3}} \quad \text{and} \quad \frac{1}{\eta(\tau/3)} \sim \frac{1}{\sqrt{3}} e^{\frac{\pi}{3}},$$

so both have faster growth than $\frac{1}{\eta(\tau)} \sim t^{3/2} e^{\frac{\pi}{3}}$.

□

8. Future work

**Remark 8.1.** (1.) In order to prove that every irreducible $\mathcal{H}(3)^{S_3}$-module appears in the decomposition of a $g$-twisted $\mathcal{H}(3)$-module, $g \in S_3$, we need to describe Zhu’s algebra. This will be studied in our future work.

(2.) In [21] we determine strong minimal generating sets for the permutation orbifold $\mathcal{F}(3)^{S_3}$, where $\mathcal{F}$ is the free fermion vertex algebra, and for $\mathcal{SF}(3)^{S_3}$, where $\mathcal{SF}$ is the rank one symplectic fermion [8]. We prove that these vertex algebras are of type $\frac{1}{2}, 2, 4, \frac{9}{2}$ and $1^2, 2, 3^3, 4^3, 5^5, 6^4$, respectively.

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Department of Mathematics and Statistics, SUNY-Albany
E-mail address: amilas@albany.edu

Mathematics Department, Randolph College
E-mail address: mpenn@randolphcollege.edu

Department of Mathematics and Computer Science, Colorado College
E-mail address: b_shao@coloradocollege.edu