Quantitative Simplification of Filtered Simplicial Complexes

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Abstract
We introduce a new invariant defined on the vertices of a given filtered simplicial complex, called codensity, which controls the impact of removing vertices on the persistent homology of this filtered complex. We achieve this control through the use of an interleaving type of distance between filtered simplicial complexes. We study the special case of Vietoris–Rips filtrations and show that our bounds offer a significant improvement over the immediate bounds coming from considerations related to the Gromov–Hausdorff distance. Based on these ideas we give an iterative method for the practical simplification of filtered simplicial complexes. As a byproduct of our analysis we identify a notion of core of a filtered simplicial complex which admits the interpretation as a minimalistic simplicial filtration which retains all the persistent homology information.

Keywords Filtration · Persistence · Interleaving

Mathematics Subject Classification 55U10

1 Introduction

Topological Data Analysis tries to combine and take advantage of both the quantitative (but albeit often noisy) nature of Data and the qualitative nature of Topology [12]. This is done through a machinery that first assigns a scale dependent family of topological...
spaces to a given dataset and then studies how topological properties behave as we change the scale.

For a subset $I$ of $\mathbb{R}$, a filtered simplicial complex indexed over $I$ is a family $(X^t)_{t \in I}$ of simplicial complexes such that for each $t \leq t'$ in $I$, $X^t$ is contained in $X^{t'}$. Filtered simplicial complexes arise in topological data analysis for example as Vietoris–Rips or Čech complexes of metric spaces [21]. Simplicial complexes have the advantage of admitting a discrete description, hence they are naturally better suited for computations when compared to arbitrary topological spaces.

A useful and computationally feasible way of analyzing the scale dependent features of a filtered simplicial complex is through persistent homology and persistence diagrams/barcodes [12,21]. Given a filtered simplicial complex $X^*$, for a given $k \in \mathbb{N}$, efficient computation of its $k$-th dimensional persistent homology $\text{PH}_k(X^*)$ is studied in many papers, for example [18,22,23,36]: Persistent homology can be computed in time cubic in the number of simplices.

Given this computational complexity, in the interest of being able to process large datasets, an important task is that of simplifying filtered simplicial complexes (that is, reducing the total number of total simplices) in a way such that it is possible to precisely quantify the trade-off between degree of simplification and loss/distortion of homological features [9,13,18,20,22,26,34].

**Related Work.** Given an $n$-point metric space $(X, d)$, Sheehy [34] constructs a filtered simplicial complex of size $O(n)$, whose persistence diagram is an approximation of the Vietoris–Rips filtration. Here the approximation is between the respective persistence diagrams, but in a (multiplicative) sense different from the one provided by the usual bottleneck distance. He calls this type of approximation a $c$-approximation, where $c$ is the multiplicative constant.

Using a net-tree, Sheehy assigns a non-negative deletion time to each point of the metric space $(X, d)$. For a scale $\alpha \geq 0$, he defines $N_\alpha$ as the subset of points with deletion time greater than $\alpha$. Furthermore, for each scale $\alpha$, he defines a relaxed metric $d_\alpha$ on $N_\alpha$ such that $d_0 = d$ and the distance between two points is determined by the scale $\alpha$, and the deletion times of points. Through this relaxed family of metric spaces he obtains a zigzag filtration, whose persistence module is non-zigzag, i.e. the arrows in the backwards direction are isomorphisms. Then he constructs a non-zigzag filtration $S$ with the same persistent homology. He shows that $S$ is a $c = 1/(1 - 2\varepsilon)$ approximation of the Vietoris–Rips filtration of $X$, where $0 < \varepsilon \leq 1/3$ is a user defined precision constant. He also shows that $S$ is of size $O(n)$ where the constant depends on the precision $\varepsilon$ and the doubling dimension of the metric space.

In [26], the authors extend the work in [34] to Čech complexes of metric spaces. They construct an approximate family of simplicial complexes using a net-tree and a structure which they called Well-Separated Simplicial Decomposition (WSSD). This approximate family is of size $O(n)$ where $n$ is the number of points in the original metric space. As in [34], they use a multiplicative approximation for persistence diagrams. In the last part of the paper, they give a generalization of the multiplicative approximation relation between Čech and Vietoris–Rips complexes. First, they observe that the Vietoris–Rips complex can be thought as the 1-completion of the Čech complex and a similar completion can be applied in higher dimensions. Here $k$-completion of
a simplicial complex means the largest simplicial complex whose $k$-skeleton is same with the original one. Then they show that the $\left(\lfloor 1/(2\varepsilon + \varepsilon^2) + 1 \rfloor - 1 \right)$-completion of the Čech complex is a $(1 + \varepsilon)$-approximation of the Čech filtration.

In [18] the authors study the general problem of computing the persistent homology of a given family of simplicial complexes connected by simplicial maps. It is well known that given a simplicial map between simplicial complexes, the induced map between homology groups can be calculated more efficiently if the map is an inclusion. Here, the authors describe a way of calculating the induced homology map of a general simplicial map by using only simplicial inclusions. They introduce two elementary types of simplicial maps: elementary inclusions and elementary collapses. They first show that every simplicial map can be expressed as the composition of such maps. Then they study how to compute the induced homology maps of elementary simplicial maps effectively. In particular, they show how one can calculate the induced homology map of a simple collapse from a simplicial inclusion. The authors then note that the simplicial maps introduced in the approximations given by [34] are not necessarily inclusions, hence their methods can be useful here. Furthermore, they introduce two more approximation schemes for Vietoris–Rips filtration similar to [34].

In [20] the authors modify their sparsification method for approximating the Vietoris–Rips complex described in [18] with a more strict one which they call Batch-Collapse. They obtain a similar approximation bound while obtaining a smaller family of simplicial complexes.

The authors of [9] consider approximating Čech complex of a finite set of points in the Euclidean space. They first prove a sandwich theorem for sequences of covers of space giving a multiplicative approximation relation between the sequence of nerves of given sequences of covers. Using this theorem, first they obtain an approximation for the Čech filtration by using a net-tree construction as in [34]. Then they develop a method of coarsening a sequence of covers and apply this to get an approximation to Čech filtration. Although the latter construction does not have the theoretical guarantees that the first have in terms of size, the authors report that in practice it works much faster.

In [9,18,26,34], approximation results are proved by constructing somewhat intricate simplicial maps. Furthermore, the cases of Vietoris–Rips and Čech complexes required different treatment. In [13], although not proving new approximation results, the authors unify previous results through the notion of convex metrics and also give much simpler approximation proofs by using the neighborhoods in the ambient space instead of simplicial complexes.

Our method of simplifying a filtered simplicial complex is by removing nice vertices from it so that the removal has a small effect on the persistent homology. This can be considered as an extension/quantification of the idea of removing vertices from simplicial complexes while preserving its homotopy type, which is considered in the papers [3,4,30,35]. The way we quantify the effect of the removal of a vertex can be seen as a generalization of the idea of $\varepsilon$-crushings introduced by Latschev [27] which is applied to the Vietoris–Rips filtration of metric spaces.

**Contributions and Structure of the Paper.** In this paper we consider the effect on persistent homology of removing a vertex and all cells containing it. In this respect,
These two finite spaces have the same Vietoris–Rips \( \text{PH}_{\geq 1} \), see Example 4.3.
We start Sect. 2 by reviewing the generalization of the Gromov–Hausdorff distance to filtered simplicial complexes given in [31]. We show that it generalizes the Gromov–Hausdorff distance between metric spaces in the sense that the Gromov–Hausdorff distance between metric spaces is equal to the Gromov–Hausdorff distance between their Vietoris–Rips complexes, using the ideas in [31, Prop. 5.1]. We then introduce an interleaving type pseudo-distance $d^F$ for filtered simplicial complexes. By its categorical nature, interleaving type distances appear in many different settings [6,7,10,15,29,32].

It is known that the interleaving distance between the persistent homology of Vietoris–Rips complexes of metric spaces is less than or equal to twice the Gromov–Hausdorff distance between the spaces, see [14] and [16, Lem. 4.3]. However, we have the following general theorem which establishes that $d^F$ mediates between the interleaving distance and (twice) the Gromov–Hausdorff distance but in general differs from both:

**Theorem 1.1** (Stability) Let $X^*, Y^*$ be constructible filtered simplicial complexes. Then, for every $k \in \mathbb{N}$ we have

$$d^I_1(\text{PH}_k(X^*), \text{PH}_k(Y^*)) \leq d^F(X^*, Y^*) \leq 2d_{GH}(X^*, Y^*).$$

The proof of this theorem is given in Sect. 2.4. In the construction of the metric $d^F$ we had to pay special attention to the notion of contiguity, a coarse way in which homotopy arises between simplicial maps (see [33, Thm. 12.5]). In the context of computational topology, related studies appear in [7,8,34].

In Sect. 3, given a filtered simplicial complex $X^*$, we introduce an invariant $\delta_X(v, w) \geq 0$, called the vertex quasi-distance of $X^*$, defined for each pair of vertices $v$ and $w$. We then define $\delta_X(v)$, the codensity of the vertex $v$, as the minimal of $\delta_X(v, w)$ as $w$ ranges over all vertices distinct from $v$. We show that this invariant controls the contribution of a vertex to the persistent homology, in a way described in the following proposition:

**Proposition 1.2** (Removal of a vertex) Let $v$ be a vertex of $X^*$ and $(X - \{v\})^*$ be the full filtered subcomplex of $X^*$ obtained by removing the vertex $v$. Then,

$$d^F_1(X^*, (X - \{v\})^*) \leq \delta_X(v).$$

This proposition shows that by computing $\delta_X(v, w)$ for all $v, w$ we have a method for simplifying a filtered simplicial complex while keeping definite guarantees in terms of the approximation error of the persistent homology. We then discuss how we can make the calculation of $\delta_X(v, w)$ simpler and how to make $\delta_X(v, w)$ smaller if we are only interested in persistent homology of certain degrees only (e.g. $\text{PH}_1$).

In Sect. 4 we then show what our constructions correspond to for Vietoris–Rips complexes of finite metric spaces and give an example showing the advantages of our simplification guarantees to those given by the Gromov–Hausdorff based bounds of [14]. We then introduce a class of metric graphs which we call simple graphs and as an application of our results we characterize their persistent homology.
In Sect. 5, we introduce simple filtered simplicial complexes: We call a filtered simplicial complex $X^*$ simple if for each vertex $v$ its condensity $\delta_X(v) > 0$. Proposition 1.2 implies that any non-simple filtered simplicial complex can be reduced in size without changing its persistent homology. Then we show that this observation can be strengthened in the following way:

**Theorem 1.3** (Classification via cores) For each filtered finite simplicial complex $X^*$, there exists a unique (up to isomorphism) simple filtered complex $C^*$ such that $d^F_I(X^*, C^*) = 0$. Furthermore, $C^*$ is a full subcomplex of $X^*$.

Hence simple filtered complexes classify filtered complexes with respect to $d^F_I$. We denote $C^*$ described in Theorem 1.3 by $C(X^*) \subseteq X^*$ and call it the core of $X^*$.

Theorem 1.3 above can then be interpreted as follows. Equivalence (i.e. $d^F_I = 0$) between filtered simplicial complexes coincides with isomorphism between their respective cores: Namely $d^F_I(X^*, Y^*) = 0$ if and only if $C(X^*)$ and $C(Y^*)$ are isomorphic. In particular this implies that the number of elements in the core is a well defined invariant. More precisely, the core of a filtered simplicial complex $X^*$ coincides with the minimal cardinality filtered simplicial complex at zero $d^F_I$ distance from $X^*$ (Corollary 5.5).

We obtain Theorem 1.3 as a corollary of a more general statement (Proposition 5.3) which says that between simple filtered simplicial complexes, for small enough distances, $2d_{GH}$ and $d^F_I$ coincide and furthermore this coincidence is realized through specific bijective maps.

In Sect. 6, we give a construction depending on a parameter $r \geq 0$ which extends a filtered simplicial complex so that its $d^F_I$ distance to the original space is 0, while the $d_{GH}$ distance is at least $r/2$. This shows that $d^F_I$ can be much smaller than $d_{GH}$.

### 2 Gromov–Hausdorff and Interleaving Type Distances Between Filtered Simplicial Complexes

As we mentioned in the introduction, in this paper we only consider constructible filtered simplicial complexes. Let us give the definition here for reference.

**Definition 2.1** (Constructible filtered simplicial complex) A filtered simplicial complex is called constructible if it is pointwise finite dimensional, indexed over $\mathbb{R}_{\geq 0}$, and new simplices are added at finitely many indices.

Given a finite set $V$ we denote the power set of $V$ minus the empty set by $P(V)$. Given a metric space $(X, d_X)$ the diameter function is defined by $\text{diam}_X : P(X) \to \mathbb{R}_+$, where $\sigma \mapsto \max_{x, x' \in \sigma} d_X(x, x')$. By $\overline{\mathbb{R}}$ we will mean the extended reals $\mathbb{R} \cup \{-\infty, +\infty\}$.

**2.1 Gromov–Hausdorff Distance Between Filtered Simplicial Complexes**

We define the vertex set of a filtered simplicial complex as the union of the vertex sets of its components (i.e. individual $X^I$’s).
Definition 2.2  *(Size function)* Given a finite filtered simplicial complex \(X^*\) with vertex set \(V\), define the size function \(D_X : P(V) \rightarrow \mathbb{R}\) as follows: \(D_X(\alpha) := \inf\{r : \alpha \in X^r\}\).

Note that if \(\alpha\) is not contained in \(X^r\) for any \(r\) then \(D_X(\alpha) = \infty\) and if it is contained in all \(X^r\) then \(D_X(\alpha) = -\infty\). Also, by the constructibility the condition \(D_X(\alpha)\) is realized as the minimum if it is finite. Note that if \(\alpha \subseteq \alpha'\), then \(D_X(\alpha) \leq D_X(\alpha')\).

Remark 2.3  If \(X^*\) is the Vietoris–Rips complex of a metric space, then \(D_X \equiv \text{diam}_X\).

Conversely, if we have \(D : P(V) \rightarrow \mathbb{R}\) monotonic with respect to inclusion, then we can define a filtered simplicial complex \(X^*_D\) with the vertex set \(V\) by \(X^*_D := \{\alpha : D(\alpha) \leq r\}\).

Remark 2.4  These constructions are inverses of each other, more precisely \(D_D \equiv D\) and \(X^*_D = X^*_D\). Hence a filtered simplicial complex is uniquely determined by its size function.

We now review a notion of distance between filtered simplicial complexes \([31]\).

Definition 2.5  *(Tripods and distortion)* A tripod between \(X^*, Y^*\) with vertex sets \(V, W\) respectively is a finite set \(Z\) with surjective maps \(p_X : Z \rightarrow V\) and \(p_Y : Z \rightarrow W\). The distortion \(\text{dis}(Z)\) of a tripod \((Z, p_X, p_Y)\) is defined by \(\max_{\alpha \in P(Z)} |D_X(p_X(\alpha)) - D_Y(p_Y(\alpha))|\), where the convention \(\infty - \infty = 0\) is assumed.

Definition 2.6  *(Gromov–Hausdorff distance between filtered simplicial complexes)*

The Gromov–Hausdorff distance between the filtered simplicial complexes \(X^*\) and \(Y^*\) is

\[
d_{GH}(X^*, Y^*) := \frac{1}{2} \inf \left\{ \text{dis}(Z) : Z \text{ a tripod between } X^* \text{ and } Y^* \right\}.
\]

Note that the product of vertex sets with the projection maps gives a tripod and if the size functions are finite then the distortion of this tripod is finite. Hence, the Gromov–Hausdorff distance between filtered simplicial complexes with finite size functions is finite.

Remark 2.7  Given a tripod \((Z, p_X, p_Y)\), let \(R = \{(p_X(z), p_Y(z)) : z \in Z\} \subseteq V \times W\). If we denote the projection maps \(V \times W \rightarrow V, W\) by \(\pi_1, \pi_2\), then \((R, \pi_1, \pi_2)\) is a tripod between \(X^*, Y^*\). Furthermore, \(\text{dis}(Z) = \text{dis}(R)\). Since the vertex sets \(V, W\) are assumed to be finite, there are finitely many such \(R\)’s. Therefore, the infimum in the definition of \(d_{GH}\) is realized.

The definition of the Gromov–Hausdorff distance between filtered spaces generalizes the Gromov–Hausdorff distance between metric spaces (see \([11, \text{Sect. 7.3}]\)):

Proposition 2.8  *(Extension)* Let \(M, N\) be finite metric spaces and \(X^*, Y^*\) be their Vietoris–Rips complexes respectively. Then \(d_{GH}(M, N) = d_{GH}(X^*, Y^*)\).

We also have:
Proposition 2.9 \( d_{GH} \) is a (pseudo-)metric between filtered simplicial complexes.

Proofs of Proposition 2.8 and Proposition 2.9 are in the Appendix.

Definition 2.10 (Isomorphism and weak isomorphism) We call two filtered simplicial complexes isomorphic if there exists a size preserving bijection between their vertex sets. We call two filtered simplicial complexes weakly isomorphic if their Gromov–Hausdorff distance is 0.

Note that an isomorphism between filtered simplicial complexes means they only differ by a relabeling of their vertices. Hence we have the following remark.

Remark 2.11 Isomorphism implies weak isomorphism. This can be seen by taking the tripod \( (R, p, q) \) where \( R \) is the graph of the size preserving bijection between the vertex sets of the initial filtered simplicial complexes and \( p, q \) are the natural projection maps.

Now let us see that the converse is not true, i.e. there are weakly isomorphic filtered simplicial complexes which are not isomorphic.

Example 2.12 (A pair of non-isomorphic but weakly isomorphic filtered simplicial complexes) Given a positive integer \( n \) and a real number \( c \), let \( X^*(n, c) \) be the filtered simplicial complex with vertex set \( \{1, \ldots, n\} \) and the constant size function equal to \( c \). Note that for any \( n, m \in \mathbb{N} \) (possibly different), the tripod between \( X^*(n, c) \) and \( X^*(m, c) \) given by the product of their vertex sets has zero distortion. Hence, \( d_{GH}(X^*(n, c), X^*(m, c)) = 0 \) which means that \( X^*(n, c) \) and \( X^*(m, c) \) are weakly isomorphic.

2.2 The Interleaving Type Distance \( d^f_I \) Between Filtered Simplicial Complexes

Here we introduce an interleaving type distance between filtered simplicial complexes. For the interleaving distance between persistence modules see [5, Sect. 3.1].

Here, we introduce an interleaving type of distance between filtered simplicial complexes which interacts nicely with their persistent homology. We use the following notation/terminology: A persistence module (over \( \mathbb{R} \)) is a family of vector spaces \( (V_t)_{t \in \mathbb{R}} \) with linear maps \( f_t: V_t \to V_{t'} \) for \( t \leq t' \) such that \( f_{t':t} = id_{V_t} \) and for each \( t \leq t' \leq t'' \), \( f_{t'',t} = f_{t',t} \circ f_{t'',t'} \). By the functoriality of homology, for \( k \in \mathbb{N} \), the homology groups \( H_k(X^t) \) of a filtered simplicial complex \( X^* \) form a persistence module, where the linear maps are induced by the inclusion \( X^t \hookrightarrow X^{t'} \). This persistence module is called the \( k \)-th persistent homology of \( X^* \) and is denoted by \( PH_k(X^*) \).

A morphism between filtered simplicial complexes is a function between their vertex sets.

Definition 2.13 (Degree) Let \( f \) be a morphism from \( X^* \) to \( Y^* \). Given \( r \geq 0 \), we say that \( f \) is \( r \)-simplicial if \( D_Y(f(\alpha)) \leq D_X(\alpha) + r \) for each \( \alpha \). We define the degree \( \deg(f) \) of \( f \) by

\[
\deg(f) := \inf \{ r \geq 0 : f \text{ is } r\text{-simplicial} \}.
\]
By the constructibility assumption, \( f \) is \( \text{deg}(f) \)-simplicial.

Hence the degree of a morphism can be thought as a measure of the failure of the morphism from being simplicial.

**Remark 2.14** If \( f: X^* \to Y^* \) is \( r \)-simplicial, then it induces a morphism \( f_\alpha \) from the persistence module \( \text{PH}_k(X^*) \to \text{PH}_k(Y^{*+r}) \), induced by the simplicial maps \( X' \to Y'^{r+r}, \alpha \mapsto f(\alpha) \).

Recall that two simplicial maps \( f \) and \( g \) from a simplicial complex \( S \) to a simplicial complex \( T \) are called contiguous if for each simplex \( \sigma \) in \( X \), \( f(\sigma) \cup g(\sigma) \) is a simplex in \( T \). The following definition is a way of quantifying this classical concept for filtered simplicial complexes.

**Definition 2.15** (Codegree) Let \( f, g \) be morphisms from \( X^* \) to \( Y^* \). Given \( r \geq 0 \), we say that \( f, g \) are \( r \)-contiguous if \( D_Y(f(\alpha) \cup g(\alpha)) \leq D_X(\alpha) + r \) for each \( \alpha \). We define the codegree \( \text{codeg}(f,g) \) of \( f, g \) by

\[
\text{codeg}(f,g) := \inf\{r \geq 0 : f, g \text{ are } r\text{-contiguous}\}.
\]

By the constructibility assumption, \( f, g \) are \( \text{codeg}(f,g) \)-contiguous.

**Remark 2.16** Let \( f, g: X^* \to Y^* \) be morphisms of filtered simplicial complexes. Then,

1. \( \text{deg}(f) = \text{deg}(f, f) \leq \text{codeg}(f,g) \).
2. If \( f, g \) are \( r \)-contiguous, then they induce the same maps \( \text{PH}_k(X^*) \to \text{PH}_k(Y^{*+r}) \), as the maps \( X' \to Y'^{r+r} \) given by \( \alpha \mapsto f(\alpha), g(\alpha) \) are contiguous as simplicial maps.
3. For each morphism \( h: Z^* \to X^* \), we have \( \text{codeg}(f \circ h, g \circ h) \leq \text{codeg}(f, g) + \text{deg}(h) \).
4. For each morphism \( h: Y^* \to Z^* \), we have \( \text{codeg}(h \circ f, h \circ g) \leq \text{codeg}(f, g) + \text{deg}(h) \).

Assume we are given three morphisms \( f, g, h \) such that \( \text{codeg}(f, g) \leq r \) and \( \text{codeg}(g, h) \leq r \). Although it is possible that \( \text{codeg}(f, h) > r \), by part 2 of the remark above, \( f, g, h \) induce the same maps \( \text{PH}_k(X^*) \to \text{PH}_k(Y^{*+r}) \). The following definition is given to capture this type of situations, see Sect. 2.3 below.

**Definition 2.17** (\( \infty \)-codegree) Define the \( \infty \)-codegree of \( f \) and \( g \) as

\[
\text{codeg}_\infty(f, g) := \min_{f = f_0, \ldots, f_n = g} \max_{i=1, \ldots, n} \text{codeg}(f_{i-1}, f_i).
\]

**Proposition 2.18** Let \( f, g, h: X^* \to Y^* \) and \( f', g': Z^* \to X^* \) be morphisms of filtered simplicial complexes. Then,

1. \( \text{deg}(f) = \text{codeg}_\infty(f, f) \leq \text{codeg}_\infty(f, g) \leq \text{codeg}(f, g) \).
2. If \( \text{codeg}_\infty(f, g) \leq r \), then \( f, g \) induce the same maps from \( \text{PH}_k(X^*) \to \text{PH}_k(Y^{*+r}) \).
(3) (Ultrametricity) \(\text{codeg}^\infty(f, h) \leq \max(\text{codeg}^\infty(f, g), \text{codeg}^\infty(g, h))\).

(4) \(\text{codeg}^\infty(f \circ f', g \circ g') \leq \text{codeg}^\infty(f, g) + \text{codeg}^\infty(f', g')\).

**Proof** (1) \(\text{codeg}^\infty(f, g) \leq \text{codeg}(f, g)\) can be seen by taking \(f = f_0, f_1 = g\). \(\deg(f) \leq \text{codeg}(f, g)\) since for any \(f = f_0, \ldots, f_n = g\), \(\deg(f) \leq \text{codeg}(f_0, f_1)\). This also shows that \(\deg(f) = \text{codeg}^\infty(f, f) \leq \text{codeg}^\infty(f, g)\).

(2) There exists \(f = f_0, \ldots, f_n = g\) such that \(\text{codeg}(f_{i-1}, f_i) \leq r\). By Remark 2.16, \(f_{i-1}, f_i\) induce the same maps from \(\text{PH}(X^*)\) to \(\text{PH}(Y^{*+r})\). Hence \(f_0 = f, g = f_n\) also induce the same maps.

(3) Follows by concatenating sequences of functions.

(4) Let \(f = f_1, \ldots, f_n = g\) be the sequence realizing \(\text{codeg}^\infty(f, g)\). Then, for any morphism \(h\) whose range is same with the domain of \(f, g\), by Remark 2.16 we have

\[
\text{codeg}^\infty(f \circ h, g \circ h) \leq \max_i \text{codeg}(f_{i-1} \circ h, f_i \circ h)
\leq \max_i \text{codeg}(f_{i-1}, f_i) + \deg(h)
= \text{codeg}^\infty(f, g) + \deg(h)
\]

Similarly we have

\[
\text{codeg}^\infty(h \circ f, h \circ g) \leq \text{codeg}^\infty(f, g) + \deg(h).
\]

Now by using these and part (i), (iii) above, we get

\[
\text{codeg}^\infty(f \circ f', g \circ g') \leq \max(\text{codeg}^\infty(f \circ f', g \circ f'), \text{codeg}^\infty(g \circ f', g \circ g'))
\leq \max(\text{codeg}^\infty(f, g) + \deg(g'), \text{codeg}^\infty(f', g') + \deg(g))
\leq \text{codeg}^\infty(f, g) + \text{codeg}^\infty(f', g').
\]

**Definition 2.19** (Interleaving distance between filtered simplicial complexes) For \(\varepsilon \geq 0\), an \(\varepsilon\)-interleaving between \(X^*\) and \(Y^*\) consists of morphisms \(f: X^* \to Y^*, g: Y^* \to X^*\) such that

\[
\deg(f), \deg(g) \leq \varepsilon, \quad \text{and} \quad \text{codeg}^\infty(g \circ f, \text{id}_{X^*}), \text{codeg}^\infty(f \circ g, \text{id}_{Y^*}) \leq 2\varepsilon.
\]

In this case we say that \(X^*, Y^*\) are \(\varepsilon\)-interleaved. We define

\[
d_f^\varepsilon(X^*, Y^*) := \inf\{\varepsilon \geq 0 : X^*, Y^* \text{ are } \varepsilon\text{-interleaved}\}.
\]

We then have:

**Proposition 2.20** \(d_f^\varepsilon\) is a (pseudo-)distance between filtered simplicial complexes.
Non-negativity and symmetry follow from the definition. $d_I^F(X^*, X^*) = 0$ since $\text{id}_{X^*}$ gives a $0$-interleaving. Let us show the triangle inequality. Let $f : X^* \rightarrow Y^*$, $g : Y^* \rightarrow X^*$ be an $\epsilon$-interleaving between $X^*, Y^*$ and $f' : Y^* \rightarrow Z^*$, $g' : Z^* \rightarrow Y^*$ be an $\epsilon'$-interleaving between $Y^*, Z^*$. Let us show that $f' \circ f, g \circ g'$ is an $(\epsilon + \epsilon')$-interleaving between $X^*, Z^*$.

$$\deg(f' \circ f), \deg(g \circ g') \leq \epsilon + \epsilon',$$

which follows from the definition of degree. By Remark 2.18 we have

$$\text{codeg}^\infty(g \circ g' \circ f' \circ f, \text{id}_{X^*}) \leq \max(\text{deg}(g) + \text{codeg}^\infty(g' \circ f', \text{id}_{Y^*}) + \deg(f), 2\epsilon) \leq \max(\epsilon + 2\epsilon' + \epsilon, 2\epsilon) = 2(\epsilon + \epsilon').$$

Similarly

$$\text{codeg}^\infty(f' \circ f \circ g \circ g', \text{id}_{Z^*}) \leq 2(\epsilon + \epsilon').$$

This completes the proof.

**Definition 2.21** We call $X^*, Y^*$ equivalent if $d_I^F(X^*, Y^*) = 0$.

Note that the interleaving distance $d_I^F$ is different from the interleaving distance between the corresponding persistence homology modules, as there are simplicial complexes which are not contiguous but have same homology.

Because of Theorem 1.1, equivalent filtered simplicial complexes have the same persistent homologies. In the next section we see that weakly isomorphic filtered simplicial complexes (see Definition 2.10) are equivalent. For now, let us give an example to show that the converse is not true.

**Example 2.22** (Equivalence is weaker than weak isomorphism) Define $\Delta_n^*$ as the filtered simplicial complex with vertex set $\{0, \ldots, n\}$ and size function $D_n(\omega) := \max\{i : i \in \omega\}$ (see Fig. 2). Note that for any tripod $(R, p, q)$ between $\Delta_m^*, \Delta_n^*$, we have $\text{dis}(R) \geq |D_n(p(R)) - D_m(q(R))| = |m - n|$, hence $d_{GH}(\Delta_m^*, \Delta_n^*) \geq |m - n|/2$. We now show that $d_I^F(\Delta_m^*, \Delta_n^*) = 0$. The topological basis of this is the fact that any two maps onto a simplex are contiguous. Without loss of generality assume that $m \leq n$. Let $\iota : \Delta_m^* \rightarrow \Delta_n^*$ be the morphism given by the inclusion of the vertex set $\{0, \ldots, m\}$. This completes the proof.
and let \( \pi : \Delta^*_n \to \Delta^*_m \) be the map given by \( k \mapsto \min(m, k) \). Since both maps are size non-increasing and size functions are defined by the maximum, both maps have degree 0. Also note that (1) \( \pi \circ \iota = \id \), and (2) if \( \alpha \subseteq \{0, \ldots, n\} \) has maximal element \( i \), then so does \( \alpha \cup \iota \circ \pi(\alpha) \). Hence, \( D_n(\alpha) = D_n(\alpha \cup \iota \circ \pi(\alpha)) \). This shows that \( \text{codeg}(\iota \circ \pi, \id) = 0 \). Therefore \( d^F_1(\Delta^*_m, \Delta^*_n) = 0 \). In Sect. 6 we give a construction generalizing this one (see Remark 6.1).

### 2.3 Remarks About the Definition of \( d^F_1 \)

It is possible [17,19] to define a related but strictly stronger notion of \( \varepsilon \)-interleaving between filtered simplicial complexes than the one given in Definition 2.19. Given filtered simplicial complexes \( X^* \) and \( Y^* \), an \( \varepsilon \)-strong interleaving between \( X^*, Y^* \) is a pair of morphisms \( f : X^* \to Y^* \), \( g : Y^* \to X^* \) such that

\[
\text{deg}(f), \; \text{deg}(g) \leq \varepsilon, \; \text{and codeg}(g \circ f, \id_{X^*}), \; \text{codeg}(f \circ g, \id_{Y^*}) \leq 2\varepsilon.
\]

The difference with Definition 2.19 is that \( \text{codeg}^\infty \) has been replaced by (the generally larger number) \( \text{codeg} \). The problem with this definition is that it does not give a metric as we show next. Define \( \hat{d}^F_1(\Delta^*, \Delta^*) \) as the infimal \( \varepsilon \geq 0 \) such that \( X^* \) and \( Y^* \) are \( \varepsilon \)-strongly interleaved.

Note that the definition of \( \text{codeg}^\infty \) uses chains of morphisms. Such a sequence of morphisms used in the proof of Proposition 2.20 to show the triangle inequality for \( d^F_1 \). The topological basis of the necessity of considering chains is the following: If simplicial maps \( f, f' : S \to T \) are contiguous and \( g, g' : T \to U \) are contiguous, it does not necessarily follow that \( g \circ f, g' \circ f' : S \to U \) are contiguous. Instead, what we have is \( g \circ f \) is contiguous to \( g \circ f' \) which is in turn contiguous to \( g' \circ f' \). Note that contiguity is not an equivalence relation between simplicial morphisms.

\( \hat{d}^F_1 \) is not a Metric. Let us give a concrete example to show that \( \hat{d}^F_1 \) does not satisfy the triangle inequality. For a non-negative integer \( n \), let \( X^*_n \) be the filtered simplicial complex with vertex set \( \{v_0, \ldots, v_n\} \) and such that

1. the cells of \( X^*_0 \) coincide with the set of all edges of the form \( \{v_i, v_{i+1}\} \),
2. \( X^*_n \) is the full simplex for \( t \geq 1 \),
3. \( X^*_n = \emptyset \) for \( t < 0 \), and
4. \( X^*_n = X^*_0 \) for \( 0 \leq t < 1 \).

Note that \( X^*_n \) is included in \( X^*_n+1 \) via the morphism \( v_i \mapsto v_i \) for all \( i = 0, \ldots, n \). Also, \( X^*_{n+1} \) surjects onto \( X^*_n \) via the morphism \( v_i \mapsto v_i \) for \( i = 0, \ldots, n \) and \( v_{n+1} \mapsto v_n \). By using these maps, we see that \( \hat{d}^F_1(X^*_n, X^*_n) = 0 \). However, for \( n \geq 3 \), \( \hat{d}^F_1(X^*_n, X^*_{n+1}) \) is not 0, as no constant map from \( X^*_n \) to itself is contiguous to the identity. Therefore, \( \hat{d}^F_1 \) fails to satisfy the triangle inequality, for otherwise one would have

\[
0 < \hat{d}^F_1(X^*_0, X^*_3) \leq \hat{d}^F_1(X^*_0, X^*_1) + \hat{d}^F_1(X^*_1, X^*_2) + \hat{d}^F_1(X^*_2, X^*_3) = 0,
\]

a contradiction.
2.4 Stability and the Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first need the following

**Lemma 2.23** Let \( f : X^* \rightarrow Y^*, g : Y^* \rightarrow X^* \) be morphisms. Let \( R \) be the correspondence between the vertex sets of \( X^*, Y^* \) containing the graphs of \( f \) and \( g \). Then \( (f, g) \) is an \( \text{dis}(R) \)-interleaving.

A proof of Lemma 2.23 is in the Appendix.

**Proof of Theorem 1.1** By the definition of interleavings for filtered simplicial complexes and Remark 2.18, an \( \epsilon \)-interleaving between filtered simplicial complexes induces an \( \epsilon \)-interleaving between their persistence modules. Hence

\[ d_I(\text{PH}_k(X^*), \text{PH}_k(Y^*)) \leq d_F(X^*, Y^*). \]

Now let \( R \) be a correspondence between the vertex sets of \( X^*, Y^* \). Then there are morphism \( f : X^* \rightarrow Y^*, g : Y^* \rightarrow X^* \) such that \( R \) contains graphs of \( f \) and \( g \). By Lemma 2.23, \( X^*, Y^* \) are \( \text{dis}(R) \)-interleaved. Since \( R \) was an arbitrary correspondence, by Remark 2.7

\[ d_I(X^*, Y^*) \leq 2d_{GH}(X^*, Y^*). \]

\( \square \)

**Remark 2.24** Note that the main point of Theorem 1.1 is not that the interleaving distance between persistence homology modules are less than twice the Gromov Hausdorff distance between filtered simplicial complexes (which is already proven in the case of Vietoris–Rips complexes, see [14]), but the newly introduced interleaving distance between filtered simplicial complexes sits in between.

3 The Vertex Quasi-Distance and Simplification

Removing vertices from simplicial complexes while preserving the homotopy type is considered in the papers [3,4,30,35]. Latschev [27] applied a similar idea to the Vietoris–Rips filtration. Here we generalize it to arbitrary filtered simplicial complexes, while controlling the effect of a removal in terms of \( d_F \). Let us start with the following definition.

**Definition 3.1** (The vertex quasi-distance) Let \( X^* \) be a filtered simplicial complex. Given vertices \( v, w \) of \( X^* \), define the vertex quasi-distance \( \delta_X(v, w) \) to be the minimal \( \delta \geq 0 \) such that

\[ D_X(\alpha \cup \{v\}) + \delta \geq D_X(\alpha \cup \{w\}), \]

for each non-empty set of vertices \( \alpha \).
Note that taking $\alpha$ as the full vertex set already requires $\delta \geq 0$, hence we can equivalently define $\delta_X(v, w)$ by $\delta_X(v, w) := \max_{\alpha} (D_X(\alpha \cup \{w\}) - D_X(\alpha \cup \{v\}))$.

Although the vertex quasi-distance is not necessarily symmetric (i.e. $\delta_X(v, w)$ may be different from $\delta_X(w, v)$), the following remark shows that it satisfies other properties of a metric. Such structures are called \textit{quasimetric spaces}.

\begin{remark}[Quasimetric] For all vertices $v, v', v''$,
\begin{enumerate}
\item $\delta_X(v, v) = 0$.
\item $\delta_X(v, v') + \delta_X(v', v'') \geq \delta_X(v, v'')$.
\end{enumerate}
\end{remark}

\begin{definition}[Codensity function] For each vertex $v$ let
\begin{itemize}
\item $- \delta_X(v) := \min_{w \neq v} \delta_X(v, w)$. This is called the codensity of vertex $v$.
\item $\delta(X^*) := \min_v \delta_X(v)$ (minimal codensity of $X^*$).
\end{itemize}

We introduce this invariant to control the effect of removing a vertex from a filtered simplicial complex on its persistent homology. Before proving Proposition 1.2 let us precisely define what we mean by removing a vertex.

\begin{definition}[Filtered subcomplex] A filtered subcomplex of a filtered simplicial complex $X^*$ is a filtered simplicial complex $Y^*$ such that for each $t$, $Y^t$ is a subcomplex of $X^t$. We call $Y^*$ a full filtered subcomplex if each $Y^t$ is a full subcomplex of $X^t$, precisely a simplex of $X^t$ is a simplex of $Y^t$ if and only if its vertices are in $Y^t$. Note that a full subcomplex is determined by its vertex set. Therefore, if we take a vertex $v$ from a filtered simplicial complex $X^*$ with vertex set $V$, there exists a unique full filtered subcomplex of it such that the vertex set at index $t$ is the vertex set of $X^t$ minus $v$. We denote this subcomplex by $(X - \{v\})^*$.
\end{definition}

\begin{remark}[Restriction] The size function of $(X - \{v\})^*$ is the restriction of the size function of $X^*$.
\end{remark}

\begin{proof}[Proof of Proposition 1.2] Let $w \neq v$ be a vertex such that $\delta_X(v, w) = \delta_X(v)$. Let $f : X^* \to (X - \{v\})^*$ be the map which is identity on all vertices except $v$ and maps $v$ to $w$. Let $\iota : (X - \{v\})^* \to X^*$ be the inclusion map. Let us show that $(f, \iota)$ is a $\delta_X(v)$-interleaving. We have $\text{deg}(\iota) = 0$. Let $\alpha$ be a non-empty subset of the vertex set of $X^*$. If $v \notin \alpha$ then $f(\alpha) = \alpha$. If $v \in \alpha$, then $f(\alpha) \subseteq \alpha \cup \{w\}$, hence

$$D_X(f(\alpha)) \leq D_X(\alpha \cup \{w\}) \leq D_X(\alpha \cup \{v\}) + \delta_X(v) = D_X(\alpha) + \delta_X(v).$$

Hence $\text{deg}(f) \leq \delta_X(v)$. Since $f \circ \iota = \text{id}_{(X - \{v\})^*}$, $\text{codeg}^\infty(f \circ \iota, \text{id}_{(X - v)^*}) = 0$. If $v \notin \alpha$, then $\alpha \cup \iota \circ f(\alpha) = \alpha$. If $v \in \alpha$ then $\alpha \cup \iota \circ f(\alpha) = \alpha \cup \{w\}$, hence

$$D_X(\alpha \cup \iota \circ f(\alpha)) = D_X(\alpha \cup \{w\}) \leq D_X(\alpha \cup \{v\}) + \delta_X(v) = D_X(\alpha) + \delta_X(v).$$

Hence $\text{codeg}^\infty(\iota \circ f, \text{id}_{X^*}) \leq \delta_X(v)$. Therefore $X^*$, $(X - \{v\})^*$ are $\delta_X(v)$-interleaved.
\end{proof}
Listing 1  Simplification Method. Note: at the end of the execution $Y^*$ is full subcomplex of $X^*$ with vertex set $W$. Note: the procedure ComputeCodensityMatrix() is discussed in the next section.

**INPUT:** $X^*$, $N$: number of vertices to be removed

**OUTPUT:** $Y^*$, a full subcomplex and errorBound

**SET** $Q = $ ComputeCondensityMatrix($X^*$), errorBound = 0, $W$ = Vertex set of $X^*$.

for $k$ from 1 to $N$
  $(i, j)$ = index of the minimal nondiagonal element of $Q$
  errorBound = errorBound + $Q(i, j)$
  remove $W(i)$ from $W$
  remove $i$-th row and column from $Q$
endfor

Computational Consequences. Note that $\delta_X(v, w)$ can only become smaller after we remove a vertex since the maximum in the definition of vertex quasi-distance is now taken on a smaller set. However, this does not imply that $\delta_X(v)$ also become smaller after removing a vertex, since it is possible that the removed vertex $w$ is the vertex realizing $\delta_X(v) = \delta_X(v, w)$. Still, the observation of the monotonicity of $\delta_X(v, w)$ gives us a method to simplify a filtered simplicial complex while bounding the approximation error in the persistent homology. Let us enumerate the vertex set of $X^*$ as $(v_1, \ldots, v_n)$ and let $Q(X^*)$ be the $n \times n$ matrix given by $\delta_X(v_i, v_j)$. This method is streamlined in Listing 1. Note that when the procedure terminates, the interleaving distance $d^I_F(X^*, Y^*) \leq$ errorBound. In the following subsections we discuss how to decrease the time complexity and/or error bound obtained from this method, if we are only interested in certain degrees of homology.

3.1 Computing $\delta_X(v, w)$: The Procedure ComputeCodensityMatrix() 

The simplification method given in Listing 1 calls the procedure ComputeCodensityMatrix() which calculates the matrix $[\delta_X(v_i, v_j)]$. In this section we explain the mathematical ideas behind it. We will not provide pseudo-code as the procedure will be made evident.

Normally, the definition of $\delta_X(v, w)$ (Definition 3.1) requires checking all non-empty subsets of the vertex set $V$ of $X^*$, which in total gives us a complexity of $O(2^n)$. However, we can achieve a better complexity if our filtered simplicial complex has some structure.

Proposition 3.6 below shows that if the filtered simplicial complex is clique, then we only need to check singletons in order to calculate $\delta_X(v, w)$. Recall that a simplicial complex is called clique if a simplex is included in it whenever its 1-skeleton is included. A filtered simplicial complex $X^*$ is called clique if each $X^t$ is clique.

**Proposition 3.6** (The case of clique filtered simplicial complexes) Let $X^*$ be a clique filtered simplicial complex with vertex set $V$ and size function $D_X$. Then

$$\delta_X(v, v') = \max_{w \in V} (D_X([v', w]) - D_X([v, w])).$$

We have the following lemma whose proof we omit:
Lemma 3.7 Let $X^*$ be a clique filtered simplicial complex with the size function $D_X$. Then,

$$D_X(\alpha) = \max_{v, w \in \alpha} D_X([v, w]).$$

Proof of Proposition 3.6 Let $\varepsilon := \max_{w \in V} \left( D_X([v', w]) - D_X([v, w]) \right)$. Recall that

$$\delta_X(v, v') = \max_{\alpha \subseteq V, \alpha \neq \emptyset} D_X(\alpha \cup \{v'\}) - D_X(\alpha \cup \{v\}),$$

hence $\delta_X(v, v') \geq \varepsilon$. Let us show that $\delta_X(v, v') \leq \varepsilon$. Let $\alpha$ be a non-empty subset of $V$. Then by Lemma 3.7 we have

$$D_X(\alpha \cup \{v'\}) = \max \left( \max_{w, w' \in \alpha} D_X([w, w']), \max_{w \in \alpha} D_X([v', w]) \right) \leq \max \left( \max_{w, w' \in \alpha} D_X([w, w']), \max_{w \in \alpha} D_X([v, w]) \right) + \varepsilon \leq D_X(\alpha \cup \{v\}) + \varepsilon.$$

Since $\alpha$ was arbitrary, $\delta_X(v, v') \leq \varepsilon$. □

Definition 3.8 As a generalization of the concept of a clique complex, let us call a simplicial complex $k$-clique if a simplex is contained in it if and only if its $k$-skeleton is contained in it. A clique complex is 1-clique with respect to this definition.

By a proof similar to that of Proposition 3.6, we can obtain the following generalization:

Proposition 3.9 (The case of $k$-clique filtered simplicial complexes) Let $k$ be a positive integer and let $X^*$ be a filtered simplicial complex such that for each $t$, $X^t$ is $k$-clique. Then for all vertices $v$ and $v'$,

$$\delta_X(v, v') = \max_{\alpha, 0 < |\alpha| \leq k} \left( D_X(\alpha \cup \{v'\}) - D_X(\alpha \cup \{v\}) \right).$$

Remark 3.10 By Proposition 3.9, Lemma 7 of the paper [34] is equivalent in our terminology to saying $\delta(p, \pi_{\alpha}(p)) = 0$, hence the codensity $\delta(p) = 0$. The homology isomorphism in Lemma 8 of [34] then follows from the fact that removing a codensity zero point does not affect homology.

Now we use Proposition 3.9 to show we can turn a given filtered simplicial complex into one satisfying the assumptions in Proposition 3.9 without losing persistent homology information in degrees less than $k$.

Proposition 3.11 Let $X^*$ be a filtered simplicial complex. Let $Y^*$ be the filtered simplicial complex with the same vertex set as $X^*$ such that for each $t$, a simplex is in $Y^t$ if and only if its $k$-skeleton is in $X^t$. Note that $Y^*$ is well defined and $X^t \subseteq Y^t$ for each $t$. We have:
1) \(Y^t\) is \(k\)-clique for all \(t\).
2) \(\text{PH}_{\leq k}(X^*) \cong \text{PH}_{\leq k}(Y^*)\).
3) \(D_Y(\alpha) = \max_{\beta \leq \alpha, 0 < |\beta| \leq k + 1} D_X(\beta)\).

**Proof** (1) Assume the \(k\)-skeleton \(\alpha^k\) of \(\alpha\) is contained in \(Y^i\). Then the \(k\)-skeleton of \(\alpha^k\), which is \(\alpha^k\) itself, is contained in \(X^i\). Therefore \(\alpha\) is contained in \(Y^i\).

(2) Note that if \(\alpha\) is a simplex of dimension less than or equal to \(k\), then its \(k\)-skeleton is itself, hence it is contained in \(X^i\) if and only if it is contained in \(Y^i\). Therefore the inclusion \(X^* \rightarrow Y^*\) is identity in the level of \(k\)-skeleton. Therefore, it induces an isomorphism between homology groups of degree less than \(k\).

(3) Let \(r := \max_{\beta \leq \alpha, 0 < |\beta| \leq k + 1} D_X(\beta)\). Since the \(k\)-skeletons of \(X^*, Y^*\) are the same, for \(|\beta| \leq k + 1\), we have \(D_X(\beta) = D_Y(\beta)\). Therefore

\[
D_Y(\alpha) \geq \max_{\beta \leq \alpha, 0 < |\beta| \leq k + 1} D_Y(\beta) = \max_{\beta \leq \alpha, 0 < |\beta| \leq k + 1} D_X(\beta) = r.
\]

Now let us show that \(D_Y(\alpha) \leq r\). We need to show that \(\alpha \in Y^r\). By the definition of \(r\), the \(k\)-th skeleton of \(\alpha\) is contained in \(X^r\). This implies that \(\alpha\) is in \(Y^r\). \(\Box\)

### 3.2 Specializing \(\delta_X(v, w)\) According to Homology Degree

In this section we refine our ideas so that given a filtered simplicial complex \(X^*\) and \(k \in \mathbb{N}\), the bound given in Proposition 1.2 is better adapted to scenarios when one only wishes to compute persistent homologies \(\text{PH}_j(X^*)\) for \(j \geq k\).

Given a filtered simplicial complex and \(k \in \mathbb{N}\), let \(Y^* = \mathcal{T}_k(X^*)\) be the filtered simplicial complex with the same vertex such that a simplex is in \(Y^i\) if it is in \(X^i\) and each simplex in its \(k\)-skeleton is contained in a \(k\)-simplex of \(X^i\). In other words, we remove simplices from \(X^i\) which have dimension less than \(k\) and are not contained in any \(k\)-simplex of \(X^i\). Note that \(Y^*\) is well defined and \(Y^t \subseteq X^t\) for each \(t\).

**Proposition 3.12** Denote the vertex quasi-distance for \(X^*\) by \(\delta_X\), and by \(\delta_Y\) denote the vertex quasidistance of \(Y^* = \mathcal{T}_k(X^*)\). Then, we have:

1) \(\text{PH}_{\geq k}(Y^*) = \text{PH}_{\geq k}(X^*)\).
2) \(D_Y(\alpha) = \min_{\beta \geq \alpha, |\beta| \geq k + 1} D_X(\beta)\).
3) \(\delta_Y(v, w) \leq \delta_X(v, w)\).
4) Let \(m \geq k\) be a non-negative integer. If \(X^*\) satisfies the property that for each \(t\), \(\alpha \in X^i\) if and only if the \(m\)-skeleton of \(\alpha\) is in \(X^i\), then \(Y^*\) satisfies this property too.

**Proof** (1) Note that for \(k' \geq k\), the \(k'\)-simplices of \(X^i\) is same with the \(k'\)-simplices of \(Y^i\). Since the homology of degree \(k'\) is determined by such cells, the inclusion \(X^* \subseteq X^*\) induces the isomorphism \(\text{PH}_{\geq k}(Y^*) \rightarrow \text{PH}_{\geq k}(X^*)\).

(2) By the identity mentioned above, for \(\beta\) with \(|\beta| \geq k + 1\) we have \(D_Y(\beta) = D_X(\beta)\). Let us denote \(r := \min_{\beta \geq \alpha, |\beta| \geq k + 1} D_X(\beta)\). Then we have

\[
D_Y(\alpha) \leq \min_{\beta \geq \alpha, |\beta| \geq k + 1} D_Y(\beta) = \min_{\beta \geq \alpha, |\beta| \geq k + 1} D_X(\beta) = r.
\]
Let us show that $D_Y(\alpha) \geq r$. Let $s < r$. Let us show that $D_Y(\alpha) > s$. If the dimension of $\alpha$ is greater than or equal to $k$, then $D_Y(\alpha) = D_X(\alpha) = r > s$. Now assume that $\alpha$ has dimension less than $k$. By definition of $r$, every $k$-cell containing $\alpha$ has size strictly greater than $s$, therefore $\alpha$ is not contained in $X^s$. Hence, we have $D_Y(\alpha) > s$. Since $s < r$ was arbitrary $D_Y(\alpha) \geq r$.

(3) Let $\alpha$ be a simplex and $\beta$ be the simplex with dimension greater than or equal to $k$ containing $\alpha \cup \{v\}$ such that $D_Y(\alpha \cup \{v\}) = D_X(\beta)$. Then we have

$$D_Y(\alpha \cup \{w\}) - D_Y(\alpha \cup \{v\}) \leq D_Y(\beta \cup \{w\}) - D_Y(\alpha \cup \{v\})$$
$$= D_X(\beta \cup \{w\}) - D_X(\beta)$$
$$= D_X(\beta \cup \{w\}) - D_X(\beta \cup \{v\})$$
$$\leq \delta_X(v, w).$$

Since $\alpha$ was arbitrary, $\delta_Y(v, w) \leq \delta_X(v, w)$.

(4) Let $\alpha$ be a simplex whose $m$-skeleton $\alpha^m$ is in $Y^t$. Let us show that $\alpha \in Y^t$. Note that $\alpha$ is in $X^t$. If the dimension of $\alpha$ is greater or equal than $k$, then $\alpha$ is in $Y^t$. Now assume that the dimension of $\alpha$ is less than $k$. Then we have that $\alpha^m = \alpha$ is in $Y^t$. □

**Summary.** If we are only interested in degree $k$ persistent homology, then we can first apply the *clique-fication* process described in Proposition 3.11 for $k + 1$ so that the calculation of each entry $\delta_X(v, w)$ of the matrix $Q(X^*)$ becomes a $O(n^k)$ task instead of $O(2^n)$, where $n$ is the number of vertices. Then we can apply the process described in Proposition 3.12 so that we lower the values of $\delta_X(v, w)$ and get a better error bound for the simplification process. Then we can start our simplification process. After removing a vertex we have two options, we can either keep working with the original codensity matrix to get the upper bound on the change in persistent homology, or we may want to compute the codensity matrix again, since its elements may decrease after removal. Note that if we remove a vertex from a $k$-clique filtered simplicial complex, it will still be $k$-clique. Hence calculating the codensity matrix does not become more costly after removing a vertex.

### 4 An Application to the Vietoris–Rips Filtration of Finite Metric Spaces and Graphs

In this section, we apply the ideas we developed in previous sections to the Vietoris–Rips filtration of finite metric spaces and graphs.

#### 4.1 Finite Metric Spaces

We show can we get a much smaller upper bound (compared to the Gromov–Hausdorff bound) on the effect of the removal of a vertex on the persistent homology through codensity considerations, if we restrict our attention to persistent homology of degree $\geq 1$. 
Remark 4.1 Let \( X^* \) be the Vietoris–Rips complex of a finite metric space \((M, d_M)\). Then for each \( x, y \in M \) we have
\[
\delta_X(x, y) = d_M(x, y).
\]
In particular, this implies that the codensity of a point is the distance to the nearest neighbor.

Proof For the Vietoris–Rips filtration, the size function is the diameter. Note that for \( \alpha = \{x\}, \text{diam}_M(\alpha \cup \{y\}) - \text{diam}_M(\alpha \cup \{x\}) = d_M(x, y) - 0 = d_M(x, y), \) hence \( \delta_X(x, y) \geq d_M(x, y) \). Note that, by triangle inequality for any \( z \) we have \( d_M(y, z) \leq d_M(x, z) + d_M(x, y) \) and this implies that for any subset \( \alpha \) we have \( \text{diam}_M(\alpha \cup \{y\}) \leq \text{diam}_M(\alpha \cup \{x\}) + d_M(x, y) \) and this implies that \( \delta_X(x, y) \leq d_M(x, y) \). Hence \( \delta_X(x, y) = d_M(x, y) \).

Remark 4.1 shows that the codensity bound on the effect of removing a vertex directly from the Vietoris–Rips filtration is not better than the Gromov–Hausdorff bound. However, let us see how we can get much better bounds by applying methods mentioned Sect. 3.2 if we are only interested in persistence homology of \( X^* \).

Let \((M, d_M)\) be a metric space and denote the modified size function described in Proposition 3.12 by \( \text{diam}_{M,k} \). More precisely, for \( \alpha \subseteq M \), we have
\[
\text{diam}_{M,k}(\alpha) := \min_{\beta \supseteq \alpha, |\beta| \geq k+1} \text{diam}_M(\beta).
\]

Note that \( \text{diam}_{M,0} = \text{diam}_M \). Let us denote the corresponding filtered simplicial complex by \( X^*_k \). By Proposition 3.12, degree \( \geq k \) persistent homology of \( X^* = \text{VR}(M) \) is the same as that of \( X^*_k \). Therefore, if we are interested in persistence homology of degree at least 1, then instead of working with the Vietoris–Rips complex, we can work with \( X^*_1 \) which has the advantage of having a smaller codensity function.

Proposition 4.2 Let \( M \) be a metric space, \( X^*_1 \) be the modified Vietoris–Rips complex described as above and \( \delta_1 \) be the codensity function of \( X^*_1 \). Let \( x \) be a point in \( M \) and \( y \) be the closest point to \( x \). Then
\[
\delta_1(x, y) = \max\left( 0, \max_{p \neq x, y} (d_M(y, p) - d_M(x, p)) \right).
\]

Proof Since the Vietoris–Rips filtration is clique, by Proposition 3.12, \( X^*_1 \) is also clique. By Proposition 3.6, we have \( \delta_1(x, y) = \max_{p \in M} (\text{diam}_1(\{y, p\}) - \text{diam}_1(\{x, p\})) \). Hence,
\[
\delta_1(x, y) = \max_{p \in M} \left( \text{diam}_{M,1}(\{p, y\}) - \text{diam}_{M,1}(\{p, x\}) \right)
= \max \left( d_M(x, y) - \text{diam}_{M,1}(\{x\}), \text{diam}_{M,1}(\{y\}) \right)
- d_M(x, y), \max_{p \neq x, y} d_M(p, y) - d_M(x, y) \right)
= \max \left( 0, \max_{p \neq x, y} (d_M(y, p) - d_M(x, p)) \right).
\]

\( \square \)
The following is an example where the modified codensity described in Proposition 4.2 is significantly smaller than the original codensity, which is equal to the distance to the nearest neighbor for the Vietoris–Rips filtration by Remark 4.1.

Example 4.3 (Circle with flares (see Fig. 1)) Let $M$ be a finite metric space described as follows: It is a finite set of points selected from a circle and some flares attached to it, see Fig. 1. Let us show that for an endpoint $x$ of a flare in $M$, $\delta_1(x) = 0$. Note that this implies that our method (see Listing 1) will inductively remove all points in flares without any cost on $\text{PH}_{\geq 1}(\text{VR}^*(M))$ until only the points on the circle are left. Note that this is significantly less than both the Gromov–Hausdorff distance between the original space $M$ and the final space $M'$, and the sum of Gromov–Hausdorff costs of successively removing single points.

Let $y$ be the closest to point to $x$ in $M$. Since $x$ is a endpoint in a flare, for each $p \neq x$ we have $d_M(x, p) = d_M(x, y) + d_M(y, p)$, in particular $d_M(y, p) \leq d_M(x, p)$. Therefore, by Proposition 4.2 we have $\delta_1(x) \leq \delta_1(x, y) = 0$.

4.2 Application to Metric Graphs

In this section we use our results to provide some characterization results related to the Vietoris–Rips persistent homology of metric graphs along the lines of [2].

Recall that a metric graph is a topological graph with a length structure. Up to isometry, a metric graph is determined by the lengths of its edges. Here we only consider finite graphs, i.e. ones having finitely many edges.

Let us call a metric graph simple if it can be constructed by inductively wedging circles and closed intervals (see Fig. 3). Recall that the wedge sum of two pointed topological spaces $(X, p)$, $(Y, q)$ is defined by

$$X \vee Y := ((X, p) \coprod (Y, q))/\{p, q\}.$$  

It is a pointed topological space where the chosen point is the point representing the identified points $p, q$. Furthermore, if $(X, d_X)$, $(Y, d_Y)$ are metric spaces, then the wedge sum carries the metric $d$ which reduces to $d_X$ and $d_Y$ on the copies of $X, Y$, and for each $x \in X, y \in Y$, $d(x, y) := d_X(x, p) + d_Y(y, q)$.

The main proposition we prove in this section is the following. A similar result can be found in [2, Proposition 13]. Here we do not put any restriction on edge lengths.

Proposition 4.4 Let $G$ be a simple metric graph obtained by using circles $(C_i)_{i=1}^n$ and some intervals. Then we have

$$\text{PH}_{\geq 1}(\text{VR}^*(G)) = \bigoplus_{i=1}^n \text{PH}_{\geq 1}(\text{VR}^*(C_i)).$$

In particular, since a metric tree is a simple graph without circles, a metric tree has trivial $\text{PH}_{\geq 1}$. 

We use the following lemma in the proof of Proposition 4.4. This lemma is in itself an application of Theorem 1.1 and Proposition 1.2. A similar result is stated in [2]
Fig. 3 A simple metric graph

which requires elements from the theory of folds, elementary simplicial collapses, and LC reductions. It also follows from [25, Proposition 2.2]. Our proof is self-contained.

**Lemma 4.5** Let $I$ be a closed interval in $\mathbb{R}$. Then $\text{PH}_{\geq 1}(\text{VR}^*(I)) = 0$.

In the proof of Lemma 4.5, we use the following statement.

**Proposition 4.6** Let $X$ be a metric space, $k$ be a non-negative integer and $r$ be a non-negative real number. Let $R$ be a ring, $J$ be the poset of finite subsets of $X$ and $F : J \to R - \text{Mod}$ be the functor $A \mapsto H_k(\text{VR}^r(A))$ where the coefficients of homology is taken in the ring $R$. Then $\text{colim} F = H_k(\text{VR}^r(X))$.

The fact that the Vietoris–Rips complex of a metric space is the colimit of the Vietoris–Rips complexes of its finite subspaces is mentioned in [1, Sect. 7]. One may be able to prove Proposition 4.6 starting from this fact but we contribute a different more direct proof which we give in the Appendix.

**Proof of Lemma 4.5** By Proposition 4.6, it is enough to show that for any finite subset $A$ of $I$, $k \geq 1$ and $r \geq 0$, we have $H_k(\text{VR}^r(A)) = 0$. Fix $A, r$ as above. We proceed by induction. Order $A$ as $x_1 < \cdots < x_n$. The case $n = 1$ is obvious. Let $n > 1$. By Proposition 4.2, $\delta_1(x_n, x_{n-1}) = 0$, where $\delta_1$ is defined as in Sect. 4.1. Hence, if we let $A' = A - \{x_n\}$, then by Theorem 1.1, Propositions 1.2 and 3.12 there is a 0-interleaving between $\text{PH}_{\geq 1}(\text{VR}^*(A))$ and $\text{PH}_{\geq 1}(\text{VR}^*(A'))$. This completes the proof. 

The other main ingredient of the proof of Proposition 4.4 is the following:

**Proposition 4.7** Let $X, Y$ be metric spaces. Then $\text{VR}^*(X \vee Y)$ has the same persistent homology with the wedge sum $\text{VR}^*(X) \vee \text{VR}^*(Y)$.

Proposition 4.7 can be found in [2, Prop. 4]. In the Appendix, we give an alternative proof via barycentric subdivision which is more concise. Now, we can start the proof of Proposition 4.4.

**Proof of Proposition 4.4** Recall that given two simplicial complexes $S, T$, $H_{\geq 1}(S \vee T)$ is naturally isomorphic to $H_{\geq 1}(S) \bigoplus H_{\geq 1}(T)$ (see [24, Corr. 2.25]). The result then follows from Lemma 4.5 and Proposition 4.7. 

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5 Classification of Filtered Simplicial Complexes via $d_F^I$

The idea of removing vertices from a simplicial complex one by one without changing its homotopy type to get a minimal core determined up to isomorphism is used in [3,4,30,35]. Here we show that a similar result holds for filtered complexes and in this way we prove Theorem 1.3.

**Definition 5.1** (Simple filtered simplicial complex) A filtered simplicial complex $X^*$ is called simple if $\delta(X^*) > 0$.

**Lemma 5.2** (Non-identity morphisms) Every non-identity morphism $f : X^* \to X^*$ has

$$\text{codeg}^\infty(f, \text{id}_{X^*}) \geq \delta(X^*).$$

**Proof** Let $\text{id}_{X^*} = f_0, f_1, \ldots, f_n = f$ be a family of morphisms realizing $\delta := \text{codeg}^\infty(f, \text{id}_{X^*})$. Without loss of generality, we can assume that $f_1$ is non-identity. Note that $\text{codeg}(f_1, \text{id}_{X^*}) \leq \delta$. Let $v$ be a vertex such that $w := f_1(v)$ different from $v$. Now, we have

$$D_X(\alpha \cup \{w\}) \leq D_X((\alpha \cup \{v\}) \cup (f_1(\alpha) \cup \{w\}))$$

$$\leq D_X(\alpha \cup \{v\}) + \delta.$$

Since $\alpha$ was arbitrary,

$$\text{codeg}^\infty(f, \text{id}_{X^*}) = \delta \geq \delta_X(v, w) \geq \delta(X^*).$$

**Proposition 5.3** Let $X^*, Y^*$ be simple filtered simplicial complexes such that for some $r \geq 0$,

$$\min(\delta(X^*), \delta(Y^*)) > r.$$

If $d_F^I(X^*, Y^*) \leq r/2$, then $2d_{GH}(X^*, Y^*) = d_F^I(X^*, Y^*)$. Furthermore, in this case there exists an invertible morphism $f : X^* \to Y^*$ with inverse $g : Y^* \to X^*$ such that the value above is equal to max($\text{deg}(f)$, $\text{deg}(g)$).

**Proof** By Theorem 1.1, we already know that $2d_{GH}(X^*, Y^*) \geq d_F^I(X^*, Y^*)$. Let us show that $2d_{GH}(X^*, Y^*) \leq d_F^I(X^*, Y^*)$.

Let $f : X^* \to Y^*$, $g : Y^* \to X^*$ be morphisms realizing the interleaving distance $\varepsilon := d_F^I(X^*, Y^*)$. Then,

$$\text{codeg}^\infty(g \circ f, \text{id}_{X^*}) \leq 2\varepsilon \leq r < \delta(X^*),$$

hence by Lemma 5.2, $g \circ f = \text{id}_{X^*}$. Similarly $f \circ g = \text{id}_{Y^*}$. Note that this implies that $\varepsilon = \max(\text{deg}(f), \text{deg}(g))$. If we define $R$ as the graph of $f$, then $R$ is a correspondence between the vertex sets of $X^*, Y^*$. It is enough to show that $\text{dis}(R) \leq \varepsilon$. 

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Let \( \beta \) be a non-empty subset of \( R \). Let us denote the projection maps from \( R \) to the vertex sets of \( X^*, Y^* \) by \( p_X, p_Y \) respectively. Let \( \alpha := p_X(\beta) \). Since \( R \) is the graph of \( f \), 
\[
P_Y(\beta) = f(\alpha).
\]
Now we have,
\[
D_Y^*(p_Y(\beta)) - D_X^*(p_X(\beta)) = D_Y^*(f(\alpha)) - D_X^*(\alpha) 
\leq \deg(f) \leq \varepsilon,
\]
and
\[
D_X^*(p_X(\beta)) - D_Y^*(p_Y(\beta)) = D_X^*(\alpha) - D_Y^*(f(\alpha)) 
= D_X^*(g(f(\alpha))) - D_Y^*(f(\alpha)) 
\leq \deg(g) \leq \varepsilon,
\]
hence
\[
|D_X^*(p_X(\beta)) - D_Y^*(p_Y(\beta))| \leq \varepsilon.
\]
Since \( \beta \) was arbitrary, we have
\[
2d_{GH}(X^*, Y^*) \leq \text{dis}(R) \leq \varepsilon = d^F_1(X^*, Y^*). \quad \square
\]

**Proof of Theorem 1.3**

Existence: By Proposition 1.2, by removing \( v \) such that \( \delta_X(v) = 0 \) one by one, we get a simple filtered simplicial complex \( C^* \) such that \( X^* \) is equivalent to \( C^* \), i.e. \( d^F_1(X^*, C^*) = 0 \). Note that for a filtered simplicial complex \( P^* \) with a single vertex, \( \delta_X(P^*) = \infty \) hence it is simple. Since \( C^* \) is obtained from \( X^* \) by removing vertices, it is a full subcomplex.

Uniqueness: Assume \( C^*, T^* \) are simple filtered simplicial complexes equivalent to \( X^* \). Then, by the triangle inequality for \( d^F_1 \) they are equivalent to each other. Hence by Proposition 5.3, taking \( r = 0 \), we see that \( C^*, T^* \) are isomorphic, since the map \( f \) becomes a size preserving bijection as both \( f \) and its inverse has degree 0. \( \square \)

**Remark 5.4** (Cores and isomorphism) As it is explained in the proof above, we obtain the core of \( X^* \) by removing vertices \( v \) with \( \delta_X(v) = 0 \) one by one. Since the core is determined up to isomorphism, the order in which we remove the points does not matter, in any case we remove the same number of points and although we may reach different subcomplexes, they will be necessarily isomorphic.

Theorem 1.3 implies the following. Let \( C(X) = \{ C^* | d^F_1(X^*, C^*) = 0 \} \), that is, \( C(X^*) \) consists of all the filtered simplicial complexes equivalent to \( X^* \). Let \( m(X^*) \) be the minimal possible cardinality over all vertex sets of elements in \( C(X^*) \).

**Corollary 5.5** (The core is minimal) The vertex set of the core \( C(X^*) \) has minimal cardinality \( m(X^*) \).
These two filtered simplicial complexes are at 0 $d^F_I$-distance while they are at Gromov–Hausdorff distance at least $\frac{1}{2}$.

**Proof** Let $C^* \in C(X^*)$ be such that its vertex sets has minimal cardinality $m(X^*)$. It follows that $C^*$ is simple for otherwise, according to Proposition 1.2, we would be able to reduce its size. Then, by the triangle inequality for $d^F_I$ (Proposition 2.20) and Theorem 1.3, the distance between $C(X^*)$ and $C^*$ is also zero. But since both $C^*$ and $C(X^*)$ are simple, Theorem 1.3 implies that they have to be isomorphic. In particular, their vertex sets ought to have the same cardinality. \qed

### 6 An Example Where $d^F_I \ll d_{GH}$

We now give a generalization of Example 2.22 which shows that $d^F_I$ can be much smaller than $d_{GH}$.

Let $X^*$ be a filtered simplicial complex with the size function $D_X$. Given a vertex $w$ and a real number $r \geq 0$, we define a *single vertex extension* $X^*_{w,r}$ as follows. The underlying vertex set is the vertex set of $X^*$ plus a new vertex $v_0$. We define a size function $\tilde{D}$ on this new vertex set as follows. We set $\tilde{D}(\{v_0\}) := D_X(\{w\}) + r$ and for a nonempty subset $\alpha$ of the vertex set of $X^*$, we set $\tilde{D}(\alpha) := D_X(\alpha)$, $\tilde{D}(\alpha \cup \{v_0\}) := D_X(\alpha \cup \{w\}) + r$ (see Fig. 4). Let us show that $\tilde{D}$ is monotonic with respect to inclusion, Let $\alpha$ be nonempty subset of the vertex set of $X^*_{w,r}$. If $v_0 \in \alpha$, then $\tilde{D}(\alpha \cup \{v_0\}) = \tilde{D}(\alpha)$. If $v_0 \notin \alpha$, then

$$\tilde{D}(\alpha \cup \{v_0\}) = D_X(\alpha \cup \{w\}) + r \geq D_X(\alpha) = \tilde{D}(\alpha).$$

Hence, in any case $\tilde{D}(\alpha \cup \{v_0\}) \geq \tilde{D}(\alpha)$. Now let $\alpha \subseteq \beta$. If $v_0 \notin \beta$, then

$$\tilde{D}(\beta) = D_X(\beta) \geq D_X(\alpha) = \tilde{D}(\alpha).$$

If $v_0 \in \beta$, then

$$\tilde{D}(\beta) = D_X(\beta \cup \{w\} - \{v_0\}) + r \geq D_X(\alpha \cup \{w\} - \{v_0\}) + r = \tilde{D}(\alpha \cup \{v_0\}) \geq \tilde{D}(\alpha).$$
Hence $\tilde{D}$ is a size function and $X^\ast_{w,r}$ is a filtered simplicial complex. Note that $X^\ast$ is a full subcomplex of $X^\ast_{w,r}$ obtained by removing the vertex $v_0$. Let us show that $\delta_{X^\ast_{w,r}}(v_0) = 0$. For any non-empty subset $\alpha$ of the vertex set of $X^\ast_{w,r}$, we have

$$\tilde{D}(\alpha \cup \{w\}) \leq \tilde{D}(\alpha \cup \{v_0, w\}) = D_X(\alpha \cup \{v_0\} \cup \{w\}) + r = \tilde{D}(\alpha \cup \{v_0\}).$$

Hence $\delta_{X^\ast_{w,r}}(v_0, w) = 0$, so $\delta_{w,r}(v_0) = 0$. By Proposition 1.2, $d^F(X^\ast_{w,r}, X^\ast) = 0$.

Now let us show that $d^H(X^\ast_{w,r}, X^\ast) \geq r/2$. Let $(Z, p, \tilde{p})$ be any tripod between the vertex sets $V, \tilde{V}$ of $X^\ast, X^\ast_{w,r}$. Then

$$\text{dis}(Z) \geq \tilde{D}(\tilde{p}(Z)) - D_X(p(Z)) = \tilde{D}(\tilde{V}) - D_X(V) = r.$$

Since $Z$ was arbitrary, $d^H(X^\ast_{w,r}, X^\ast) \geq r/2$. Therefore, if we take $r \gg 0$, then

$$d^F(X^\ast_{w,r}, X^\ast) = 0 \leq r/2 \leq d^H(X^\ast_{w,r}, X^\ast).$$

**Remark 6.1** Recall $\Delta^\ast_n$ from Example 2.22, with the vertex set $\{0, \ldots, n\}$ and the size function given by maximum. Note that $\Delta^\ast_{n+1} = (\Delta^\ast_n)_{w=n, r=1}$. This also shows that $d^F(\Delta^\ast_m, \Delta^\ast_n) = 0$.

**Appendix**

**Proof of Proposition 2.8** Let $R$ be a correspondence between $M, N$ (i.e. $R \subseteq M \times N$ and $\pi_M(R) = M, \pi_N(R) = N$). Note that $R$ can be considered as a tripod between $X^\ast, Y^\ast$. By Remark 2.7, it is enough to show that the distortion of $R$ as a metric correspondence between $M, N$ is same with the distortion of $R$ as a tripod between $X^\ast, Y^\ast$. Let us denote the first one by $\text{dis}^\text{met}(R)$ and the second one by $\text{dis}^\text{tri}(R)$.

**Claim** $\text{dis}^\text{tri}(R) \geq \text{dis}^\text{met}(R)$.

**Proof** By Remark 2.3, the size functions of $X^\ast, Y^\ast$ are given by the diameter. Hence we have:

$$\text{dis}^\text{tri}(R) \geq \max_{(x,y),(x',y') \in R} |\text{diam}_{M}(x,x') - \text{diam}_{N}(y,y')|$$

$$= \max_{(x,y),(x',y') \in R} |d_{M}(x,x') - d_{N}(y,y')|$$

$$= \text{dis}^\text{met}(R).$$

**Claim** $\text{dis}^\text{tri}(R) \leq \text{dis}^\text{met}(R)$.

**Proof** Let $\alpha \in P(R)$. Let $\alpha \in P(R)$ such that

$$\text{dis}^\text{tri}(R) = |\text{diam}_{M}(\pi_{M}(\alpha)) - \text{diam}_{N}(\pi_{N}(\alpha))|. \quad \Box$$
Without loss of generality, we can assume that
\[ \text{diam}_M(\pi_M(\alpha)) \geq \text{diam}_N(\pi_N(\alpha)). \]

Let \( x, x' \) be points in \( \pi_M(\alpha) \) so that
\[ \text{diam}_M(\pi_M(\alpha)) = d_M(x, x'). \]

There exists points \( y, y' \) in \( N \) such that \((x, y), (x', y') \in \alpha\). Then we have
\[ \text{dis}(R) = \text{diam}_M(\pi_M(\alpha)) - \text{diam}_N(\pi_N(\alpha)) = d_M(x, x') - d_N(y, y') \leq \text{dis}^\text{met}(R). \]

**Proof of Proposition 2.9** Non-negativity and symmetry properties follows from the definition. \( d_{GH}(X^*, X^*) = 0 \) since the distortion of the identity tripod on the vertex set of \( X \) is 0. Let us show the triangle inequality. Let \((Z, p, p')\) be a tripod between \( X^*, X'^* \) and \((Z', q', q'')\) be a tripod between \( X'^*, X''^* \). Let \( Z'' \) be the fiber product
\[ Z'' = Z_{p'} \times_{q'} Z'. \]

Then \((Z'', p \circ \pi_Z, q'' \circ \pi_{Z'})\) is a tripod between \( X^*, X''^* \). Given \( \alpha \in P(Z'') \), we have
\[
|D_X(p \circ \pi_Z(\alpha)) - D_{X''}(q'' \circ \pi_{Z'}(\alpha))| 
\leq |D_X(p \circ \pi_Z(\alpha)) - D_{X'}(p' \circ \pi_Z(\alpha))| 
+ |D_{X'}(p' \circ \pi_Z(\alpha)) - D_{X''}(q'' \circ \pi_{Z'}(\alpha))| 
\leq |D_X(p \circ \pi_Z(\alpha)) - D_{X'}(p' \circ \pi_Z(\alpha))| 
+ |D_{X'}(q' \circ \pi_{Z'}(\alpha)) - D_{X''}(q'' \circ \pi_{Z'}(\alpha))| 
\leq \text{dis}(Z) + \text{dis}(Z').
\]

Since \( \alpha \in P(Z'') \) was arbitrary, \( \text{dis}(Z'') \leq \text{dis}(Z') + \text{dis}(Z''). \) Since the tripods \( Z, Z' \) were arbitrary, \( d_{GH}(X^*, X''^*) \leq d_{GH}(X^*, X'^*) + d_{GH}(X'^*, X''^*). \)

**Proof of Lemma 2.23** Let \( p_X \) (resp. \( p_Y \)) be the projection map from \( R \) to the vertex set of \( X^* \) (resp. \( Y^* \)). Let \( \alpha \) be a non-empty subset of the vertex set of \( X^* \). Note that
\[ f(\alpha) \subseteq p_Y(p_X^{-1}(\alpha)). \]

Let \( \varepsilon := \text{dis}(R) \). We have
\[
D_{Y^*}(f(\alpha)) \leq D_{Y^*}(p_Y(p_X^{-1}(\alpha))) 
\leq D_{X^*}(p_X(p_X^{-1}(\alpha)) + \varepsilon 
= D_{X^*}(\alpha) + \varepsilon.
\]
Hence \( \deg(f) \leq \varepsilon \). Similarly \( \deg(g) \leq \varepsilon \).

Note that \( p_X(p_Y^{-1}(f(\alpha))) \) contains both \( \alpha \) and \( g \circ f(\alpha) \). Hence

\[
D_{X^*}(g \circ f(\alpha) \cup \alpha) \leq D_{X^*}(p_X(p_Y^{-1}(f(\alpha)))) \\
\leq D_{Y^*}(p_Y(p_Y^{-1}(f(\alpha)))) + \varepsilon \\
= D_{Y^*}(f(\alpha)) + \varepsilon \\
\leq D_{X^*}(\alpha) + 2\varepsilon.
\]

This shows that

\[
\text{codeg}^\infty(g \circ f, \text{id}_{X^*}) \leq \text{codeg}(g \circ f, \text{id}_{X^*}) \leq 2\varepsilon.
\]

Similarly,

\[
\text{codeg}^\infty(f \circ g, \text{id}_{Y^*}) \leq 2\varepsilon.
\]

This completes the proof. \( \Box \)

**Proof of Proposition 4.6** Let us start by fixing some notation. Let \( A \) be a finite subset of \( X \). We denote the homology maps induced by the inclusion \( A \hookrightarrow X \) by \( \iota_A : H_k(VR' (A)) \to H_k(VR' (X)) \). If \( B \) is another finite subset of \( X \) such that \( A \subseteq B \), we denote the homology map induced by this inclusion by \( \iota_{A,B} : H_k(VR' (A)) \to H_k(VR' (B)) \). If \( z \) is a cycle in \( VR' (X) \) such that the vertices of \( z \) is contained in \( A \), we denote the homology class it represents in \( H_k(VR' (A)) \) by \([z]_A \).

The homology maps \( \iota_A : H_k(VR' (A)) \to H_k(VR' (X)) \) shows that \( H_k(VR' (X)) \) is a cocone for \( J \). Let us show that it is universal. Let \( M \) be another cocone for \( J \) with morphisms \( \phi_A : H_k(VR' (A)) \to M \). Define a map \( u : H_k(VR' (X)) \to M \) as follows. Given a homology class \( c \) in \( H_k(VR' (X)) \), let \( z \) be a cycle representing it and let \( A \) be a finite subset containing the vertices of \( z \). Define \( u(c) := \phi_A([z]_A) \). Let us show that this map is well defined. Let \( z' \) be another cycle representing \( c \) and \( A' \) be a finite subset containing the vertices of \( z' \). Note that \( z - z' \) is a boundary in \( VR' (X) \), let \( w \) be a chain in \( VR' (X) \) so that \( \partial w = z - z' \). Let \( B \) a finite subspace of \( X \) containing \( A, A' \) and vertices of \( w \). Note that \([z]_B = [z']_B\) as \( w \) is contained in \( B \). We have

\[
\phi_A([z]_A) = \phi_B \circ \iota_{A,B}([z]_A) = \phi_B([z]_B) = \phi_B([z']_B) \\
= \phi_B \circ \iota_{A',B}([z']_B) = \phi_{A'}([z']_{A'}). 
\]

This shows that \( u : H_k(VR' (X)) \to M \) is well defined. It is an \( R \)-module homomorphism since if \( c, c' \) are homology classes in \( H_k(VR' (X)) \) represented by cycles \( z, z' \) whose vertices are contained in a finite subspace \( A \), then for any \( \lambda \in R \) we have

\[
u(c + \lambda \ c') = \phi_A([z]_A + \lambda [z']_A) \\
= \phi_A([z]_A) + \lambda \phi_A([z']_A) = u(c) + \lambda u(c'). \]
Given a homology class \( c \) in \( H_k(\text{VR}^r(A)) \) which is represented by a cycle \( z \), we have
\[
u \circ i_A(c) = \phi_A([z]_A) = \phi_A(c),\]
hence \( u \) commutes with structure maps. The uniqueness of such \( u \) follows from the fact that for every homology class \( c \) in \( H_k(\text{VR}^r(X)) \) there exists a finite subset \( A \) such that \( c \) is contained in the image of \( i_A: H_k(\text{VR}^r(A)) \to H_k(\text{VR}^r(X)) \).

\[ \qed \]

**Proof of Proposition 4.7** Let \( p \in X, q \in Y \) be the chosen points for the wedge sum. Note that \( \text{VR}^r(X) \lor \text{VR}^r(Y) \) is contained in \( \text{VR}^r(X \lor Y) \) for each \( r \geq 0 \). Let us show that this inclusion induces an isomorphism between homology groups. Note that this is enough for our proof as these inclusions commutes with the structure maps of both filtered simplicial complexes.

Order the disjoint union of \( X \lor Y \) and denote the smallest element of a finite subset \( \sigma \) by \( \min(\sigma) \). It is known that for a simplicial complex \( S \) with ordered vertices, the map \( B(S) \to S \) from the barycentric subdivision \( B(S) \) of \( S \) to \( S \) defined by \( \sigma \mapsto \min(\sigma) \) is simplicial and induces an isomorphism between homology groups [28, pp. 166, 167].

Consider the map \( f: B(\text{VR}^r(X \lor Y)) \to \text{VR}^r(X) \lor \text{VR}^r(Y) \) defined as follows: \( f(\sigma) = \min(\sigma) \) if \( \sigma \) is contained in \( X \) or \( Y \), \( f(\sigma) = p = q \) else. Let us see that this map is simplicial. Take a simplex \( S = (\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_n) \) in \( B(\text{VR}^r(X \lor Y)) \). Without loss of generality, assume that \( k \) is the maximal integer such that for \( i \leq k \), \( \sigma_i \subseteq X \) and furthermore assume that for \( i > k \), \( \sigma_i \) is neither contained in \( X \) nor in \( Y \). If \( k = n \), then \( f(S) \subseteq \sigma_n \), hence it is a simplex in \( \text{VR}^r(X) \lor \text{VR}^r(Y) \). If \( k \neq n \), then \( \sigma_n \) contains elements from both \( X \) and \( Y \), hence \( \sigma_n \cup \{p\} \) is a simplex in \( \text{VR}^r(X \lor Y) \), which in turn implies that \( \sigma_k \cup \{p\} \) is a simplex in \( \text{VR}^r(X) \). Hence, \( f(S) \subseteq \sigma_k \cup \{p\} \) is a simplex in \( \text{VR}^r(X \lor Y) \). Therefore, \( f \) is simplicial.

Consider the following (non-commutative) diagram:

\[
\begin{array}{ccc}
\text{VR}^r(X \lor \text{VR}^r(Y)) & \xrightarrow{i} & \text{VR}^r(X \lor Y) \\
\psi \uparrow & & \psi \uparrow \\
B(\text{VR}^r(X)) \lor B(\text{VR}^r(Y)) & \xleftarrow{j} & B(\text{VR}^r(X \lor Y)).
\end{array}
\]

Here, \( i, j \) are inclusions and \( \varphi, \psi \) are maps defined by \( \sigma \mapsto \min(\sigma) \). This diagram is non-commutative only because \( i \circ f \) is not equal to \( \psi \). All other commutativity relations hold. Let us show that \( i \circ f \) is contiguous to \( \psi \), hence the non-commutativity disappears when we pass to homology.

As above, take a simplex \( S = (\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_n) \) in \( B(\text{VR}^r(X \lor Y)) \). Without loss of generality, assume that \( k \) is the maximal integer such that for \( i \leq k \), \( \sigma_i \subseteq X \) and furthermore assume that for \( i > k \), \( \sigma_i \) is neither contained in \( X \) nor in \( Y \). If \( k = n \), then \( S \) is in the image of \( j \), therefore \( i \circ f(S) = \psi(S) \). If \( k < n \), then we have
\[
i \circ f(S) \cup \psi(S) \subseteq \sigma_n \cup \{p\} \in \text{VR}^r(X \lor Y).
\]
This shows the contiguity. Now we have the following commutative diagram:

\[
\begin{align*}
H_*(VR^r(X) \lor VR^r(Y)) & \xrightarrow{i_*} H_*(VR^r(X \lor Y)) \\
& \xleftarrow{\psi_*} H_*(B(VR^r(X)) \lor B(VR^r(Y))) \\
& \xrightarrow{\phi_*} H_*(B(VR^r(X \lor Y)))
\end{align*}
\]

Now, the induced map \( f_\ast \) is surjective since \( \phi_* \) is surjective and \( f_* \) is injective since \( \psi_* \) is injective. Hence \( f_* \) is an isomorphism, which implies that \( i_* = \psi_* \circ f_*^{-1} \) is an isomorphism. \( \square \)

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