BEYOND RAID 6 — AN EFFICIENT SYSTEMATIC CODE PROTECTING AGAINST MULTIPLE ERRORS, ERASURES, AND SILENT DATA CORRUPTION

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Abstract. We describe a replacement for RAID 6, based on a new linear, systematic code, which detects and corrects any combination of $E$ errors (unknown location) and $Z$ erasures (known location) provided that $Z + 2E \leq 4$. We investigate some scenarios for error correction beyond the code’s minimum distance, using list decoding. We describe a decoding algorithm with quasi-logarithmic time complexity, when parallel processing is used: $\approx O(\log N)$ where $N$ is the number of disks in the array (similar to RAID 6).

By comparison, the error correcting code implemented by RAID 6 allows error detection and correction only when $(E, Z) = (1, 0)$, $(0, 1)$, or $(0, 2)$. Hence, when in degraded mode (i.e., when $Z \geq 1$), RAID 6 loses its ability for detecting and correcting random errors (i.e., $E = 0$), leading to data loss known as silent data corruption. In contrast, the proposed code does not experience silent data corruption unless $Z \geq 3$.

The aforementioned properties of our code, the relative simplicity of implementation, vastly improved data protection, and low computational complexity of the decoding algorithm, make our code a natural successor to RAID 6. As this code is based on the use of quintuple parity, this justifies the name PentaRAID™ for the RAID technology implementing the ideas of the current paper.

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2010 Mathematics Subject Classification. 94B05, 94B35.

Key words and phrases. RAID 6 replacement, RAID, error-correcting codes, erasure codes, silent data corruption, fault tolerance, storage array, disk array, Reed-Solomon coding.
1. Introduction

RAID (Redundant Arrays of Inexpensive Drives) was introduced as a method of increasing reliability of data storage systems [6]. The original data is stored on \( k \) devices, for simplicity called “drives”. Redundancy is added by using extra \( p \) drives storing summary information (“parity”) computed from the original data. Parity information is used to reconstruct data lost due to failure of some of the drives. The practicality of this approach depends on the existence of efficient algorithms for computing parity and reconstructing lost data. There is a trade-off involving efficiency, protection from data loss, and cost. As RAID evolved, several specific schemes were proposed, known as RAID levels. RAID 6 is one popular scheme, in which \( p = 2 \) and \( k \) is arbitrary [11, 1]. RAID 6 is limited in its ability to protect data by \( p = 2 \), which makes it possible to recover from erasures \( Z \) and errors \( E \) provided that \( Z + 2E \leq 2 \). The need for RAID 6 replacement has been understood for quite a while [7, 8]; it has even been predicted that RAID 6 will cease to work in year approximately 2019, due to evolving storage capacity and application needs [3]. As \( k \) increases, the probability of having 2 failed drives simultaneously increases. It should be noted that recovery takes a significant amount of time with current large drives (say, > 1 day for modest values of \( k \)).

There exist codes which allow arbitrary \( p \), such as Reed-Solomon codes, but they come with significant computational overhead and added complexity of implementation. In the current paper we introduce a code which uses \( p = 5 \), allows recovery from up to 2 drive failures at unknown locations and up to 4 drives at known locations, and which is nearly as easy to implement as the RAID 6 code. Moreover, the knowledge required to implement it is similar, and it amounts to familiarity with Galois field arithmetic in the scope of the popular Anvin’s paper [1]. Thus, we hope that PentaRAID™ will successfully fill the gap between RAID 6 and Reed-Solomon codes [1]. It should be noted that our algorithm admits an implementation which uses a constant, very small number of Galois field operations per error, independent of the size of the underlying Galois field, if the arithmetical operations in the field have constant time (e.g. use lookup tables). Thus the algorithm has better computational complexity than the alternatives, and easily supports a large number of drives, e.g. 254 = 2^8 − 2 if the underlying Galois field is \( GF(2^{256}) \), and 65,533 = 2^{16} − 2 if the field is \( GF(2^{16}) \).

The remainder of the paper is organized as follows. Section 2 introduces notations and some preliminary considerations. In Section 3 we prove the main theoretical result of the current paper, Theorem 1. A special case in which \( E = 0 \) (i.e erasure code) is discussed in Section 4. An efficient way of solving a quadratic equations over a field of characteristic 2 is presented in Section 5. Section 6 describes in detail the decoding algorithm for the proposed code. In sections 7, 8 we present the decoding technique for the case of dual disk corruption (i.e \( E = 2 \) and \( Z = 0 \)), and its computational complexity, respectively. In the last three sections 9, 10, and 11 we investigate the error correction capabilities beyond the minimum distance.

2. Preliminary Considerations and Notations

The bulk of the algebraic operations in the current paper are over the finite field \( \mathbb{F} \) of characteristic two, and thus addition is the same as subtraction. In addition to that, we will

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1PentaRAID™ is the trademark used by Xoralgo Inc., a company formed by the authors in collaboration with the University of Arizona, to pursue commercial implementations of the technology based on the research described in the current paper.
use frequently the Frobenius identity \((a + b)^2 = a^2 + b^2\). In particular, we can apply our results to the most commonly used finite field: the Galois field \(GF(2^m)\), where \(m \geq 1\) is otherwise an arbitrary integer, and the results of the paper are applicable to all these fields.

The main object studied in this paper is a linear, systematic code over the Galois field \(F\). As it is customary, we define the code by its generator and parity matrices. Consider the generator matrix

\[
G = \left( \frac{I_{k \times k}}{P_{5 \times k}} \right),
\]

and the parity check matrix

\[
H = \left( \begin{array}{c|c} P_{5 \times k} & I_{5 \times 5} \end{array} \right),
\]

where

\[
P_{5 \times k} = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_k \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_k^2 \\
\alpha_1^3 & \alpha_2^3 & \cdots & \alpha_k^3 \\
\alpha_1^2 + \alpha_1 & \alpha_2^2 + \alpha_2 & \cdots & \alpha_k^2 + \alpha_k
\end{pmatrix}.
\]

As we will often refer to the structure of the column of this matrix, it will be convenient to introduce a vector depending on \(\rho \in F\):

\[
P^{(\rho)} = \begin{pmatrix}
1 & \rho & \rho^2 & \rho^3 & \rho(\rho + 1)
\end{pmatrix}^T.
\]

Using this notation, we can define the columns of \(P = P_{5 \times k}\) to be \(P_i = P^{(\alpha_i)}\).

The entries \(\alpha_i, i = 1, \ldots, k\), are distinct elements of \(F\), excluding the zero element, and any one of the 3\(^{rd}\) roots of unity distinct from 1 (if such a root exists in \(F\)). So, we have the following bound:

\[
k \leq k_{\text{max}}(F) = \begin{cases}
|F| - 1 & \text{if there is no } 3^{rd} \text{ roots of unity in } F, \\
|F| - 2 & \text{if there is a } 3^{rd} \text{ root of unity in } F.
\end{cases}
\]

For example, if \(F = GF(256) = GF(2^8)\), the equation \(\zeta^3 - 1 = (\zeta - 1)(\zeta^2 + \zeta + 1)\) has two solutions in \(F\) distinct from 1, and thus \(k \leq k_{\text{max}} = 254\). The reason behind the exclusion of a 3\(^{rd}\) root of unity will be clarified later.

It should be noted that, in contrast with most linear codes considered in literature, the rows of the matrix \(P_{5 \times k}\) are not linearly independent: the sum of rows 2 and 3 is equal to row 5. Also, by dropping the 5\(^{th}\) row, we obtain a 4 \(\times\) 4 Vandermonde matrix.

For \(k\) data disks \(D_1, \ldots, D_k\) where \(k \leq k_{\text{max}}\), let

\[
m = \begin{pmatrix}
D_1 & D_2 & \cdots & D_k
\end{pmatrix}^T
\]

be the original message (original values of the data drives), and

\[
t = G \cdot m = \begin{pmatrix}
D_1 & D_2 & \cdots & D_k & P_1 & P_2 & P_3 & P_4 & P_5
\end{pmatrix}^T
\]

be the transmitted message. Note that \(H \cdot G\) is a zero matrix, since the field is of characteristic 2, and thus \(H \cdot t = H \cdot G \cdot m = 0\).
Consider the received message $r$ to be the transmitted message, plus an error message $r = t + e$, where $e$ is an element of the vector space $\mathbb{F}^{k+5}$. As it is customary, we think of $e$ as random, thus to be modeled with probability theory.

The *syndrome* of a received message is

$$s = H \cdot r = H (t + e) = H \cdot t + H \cdot e = 0 + H \cdot e = H \cdot e.$$  

In short, the syndrome of a message is $s = H \cdot e$. It is a column vector of size 5 by 1.

In this paper, we prove that the linear code defined by the above pair of matrices $G$ and $H$ can be uniquely decoded using an approach widely known as *syndrome decoding*. We will provide an algorithm which will perform the decoding task. As the first step, we prove the following theorem:

**Theorem 1** (Injectivity of error-to-syndrome mapping). For one-error and two-error patterns, the function (the parity check matrix $H$) that maps the error $e$ into the the syndrome $s = H \cdot e$, is an injective transformation. Thus, having the syndrome $s$, can tell us the locations of the errors and the error values.

The proof of Theorem 1 will occupy the entire next section, as the proof is divided into a significant number of distinct cases which require detailed analysis.

The following theorem describes our code in classical terms:

**Theorem 2** (On dimension, length and distance of the code). The linear code given by generator matrix $G$ and the parity check matrix $H$ is a systematic linear code with length $k + 5$, rank (or dimension) $k$ and distance 5.

*Proof.* We defer the proof of the statement concerning the distance to Proposition 1. The length of the code is clearly $k + 5$. The rank of the code is $k$ as the columns of the generator matrix are linearly independent. It is clear that rank $G \leq k$, the number of columns. Erasing parity matrix $P_{5 \times k}$ from the generator matrix leaves us with the identity matrix $I_{k \times k}$, so rank $G \geq k$. Combining these estimates, rank $G = k$.

In some applications the locations of the failed (or erased) drives are known. The following theorem is an easy consequence of Theorem 2. We address this case in Section 3 in particular, in Theorem 6, proving that up to 4 erasures at known locations can be corrected.

In general, it is advantageous for a linear, forward error correcting code (FEC) to have distance as large as possible. An $[N, K, D]$ linear FEC, where $N$ is the length, $K$ is the rank and $D$ is the distance, can correct $E$ errors and $Z$ (known) erasures if the condition $Z + 2 \times E \leq D - 1$ is satisfied ([5], pp. 104–105). As an example, a liner FEC with distance $D = 5$ can correct a combination of $Z = 2$ (known) erasures and $E = 1$ error (at unknown location).

### 3. Proofs of the Main Results

In this section we prove the main theoretical result of the current paper, Theorem 1. Let us begin with stating useful results from coding theory.

**Definition 1** (Weight of a vector). The weight of a vector $e \in \mathbb{F}^n$ is the number of non-zero entries:

$$\text{weight } e = \# \{ i \in \{1, 2, \ldots, n \} : e_i \neq 0 \}.$$
Let $\mathcal{S}_{\leq \nu} \subseteq \mathbb{F}^n$ be the set of all vectors $e$ with at most $\nu$ non-zero entries, i.e. whose weight is at most $\nu$. In order to prove Theorem 1, we will prove that the function
\begin{equation}
\mathbb{F}^{k+5} \ni \mathcal{S}_{\leq \nu}^{k+5} \ni e \mapsto H \cdot e \in \mathbb{F}^5
\end{equation}
is injective.

Let us recall a known theorem from the theory of linear codes [5], p. 88, Theorem 3.3:

**Theorem 3.** For a linear code $C \subseteq \mathbb{F}^n$ with a parity check matrix $H$, the minimum distance is $d$ if and only if both of the following conditions hold:

1. every set of $d-1$ columns of $H$ is linearly independent;
2. some set of $d$ columns of $H$ is linearly dependent.

We also have the theorem stating the connection between the distance and capability to detect and correct errors at unknown locations [5], p. 101.

**Theorem 4.** Let $C \subseteq \mathbb{F}^{k+r}$ be a linear code with distance $d$. Then the mapping
\begin{equation}
\mathbb{F}^{k+r} \ni \mathcal{S}_{\leq \nu}^{k+r} \ni e \mapsto H \cdot e \in \mathbb{F}^r
\end{equation}
is injective if
\[
\nu \leq \left\lfloor \frac{d-1}{2} \right\rfloor.
\]

**Proposition 1.** Let $C \subseteq \mathbb{F}^{k+5}$ be the code defined by the generator matrix $G$ and parity check matrix $H$, given by equations (1), (2) and (3). Then the minimum distance of $C$ is 5.

**Proof.** First, let us show that any combination of 4 distinct columns of $H$ is linearly independent. Any 4 columns of $H$ are obtained by choosing $r \leq 4$ columns of the identity matrix $I_{5 \times 5}$, and then choosing $t = 4 - r$ columns of $P_{5 \times k}$, i.e. vectors of the form
\[
P_i^\top = \left(1 \ \alpha_i \ \alpha_i^2 \ \alpha_i^3 \ \alpha_i(\alpha_i + 1)\right)^\top.
\]
To explain the method of proof, let us first consider a special case, when 2 columns of $I_5$ are selected,
\[
I_2^\top = \left(0 \ 1 \ 0 \ 0 \ 0\right)^\top,
\]
\[
I_4^\top = \left(0 \ 0 \ 0 \ 1 \ 0\right)^\top.
\]
Let us consider the matrix $A$ which contains the vectors whose independence we study. It is our prerogative to order the vectors in any order, and we choose to write the ones coming from the identity matrix before the ones coming from the parity matrix. Also, we choose to perform column rather than row reduction to find the echelon form.
\[
A = \left( I_2 \ I_4 \ P_i \ P_j \right) = \begin{pmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & \alpha_i & \alpha_j \\
0 & 0 & \alpha_i^2 & \alpha_j^2 \\
0 & 1 & \alpha_i^3 & \alpha_j^3 \\
0 & 0 & \alpha_i(\alpha_i + 1) & \alpha_j(\alpha_j + 1)
\end{pmatrix}.
\]
We need to prove that this matrix has rank 4. As rank is invariant under elementary column operations, we can subtract $\alpha_i \times column_1$ from column operations.
4. Similarly, we can subtract multiples of \( \text{column}_2 \) from \( \text{column}_3 \) and \( \text{column}_4 \) to eliminate entries in row 4. The resulting matrix is:

\[
A^{(1)} = \begin{pmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & \alpha_i^2 & \alpha_j^2 \\
0 & 1 & 0 & 0 \\
0 & 0 & \alpha_i(\alpha_i + 1) & \alpha_j(\alpha_j + 1)
\end{pmatrix}.
\]

We can rearrange the rows of \( A^{(1)} \) to bring it to the column echelon form:

\[
A^{(2)} = \begin{pmatrix}
I_{2 \times 2} & 0 \\
0 & B
\end{pmatrix}, \quad B = \begin{pmatrix}
1 \\
\alpha_i^2 \\
\alpha_i(\alpha_i + 1) \\
\alpha_j(\alpha_j + 1)
\end{pmatrix}.
\]

It is easy to see that \( \text{rank} \ A^{(2)} = \text{rank} \ I_{2 \times 2} + \text{rank} \ B = 2 + \text{rank} \ B \). Therefore we need to show that \( \text{rank} \ B = 2 \).

We observe that matrix \( B \) is obtained by erasing 2 rows of the matrix \( C \) given by:

\[
C = \begin{pmatrix}
1 & 1 \\
\alpha_i & \alpha_j \\
\alpha_i^2 & \alpha_j^2 \\
\alpha_i(\alpha_i + 1) & \alpha_j(\alpha_j + 1)
\end{pmatrix}.
\]

The general case is also reduced to calculating rank of a specific matrix \( B \). We consider the matrix

\[
C = \begin{pmatrix}
P_{i_1} & P_{i_2} & \ldots & P_{i_t}
\end{pmatrix}
\]

obtained from \( P_{5 \times k} \) by taking any subset of \( t \leq 4 \) columns, and proving that each of its submatrices obtained by erasing \( r = 4 - t \) rows has rank \( t = 4 - r \). As \( r \) corresponds to the number of columns taken from the identity matrix \( I_{5 \times 5} \), the total rank is \( r + t = (4 - t) + t = 4 \) as claimed.

Let us consider various cases as \( t = 0, 1, 2, 3, 4 \).

If \( t = 0 \) then the matrix has 0 columns, and the rank is 0.

If \( t = 1 \) then \( r = 3 \) entries are erased from a single-column matrix \( C = P_i \). At least one of the two remaining entries is non-zero so \( \text{rank} \ C = 1 = t \), as required.

\[\]
If \( t = 2 \) then \( r = 2 \). The matrix \( C^{lm} \), obtained by erasing two rows \( l < m \), from \( C \), is thus \( 3 \times 2 \), and is in one of the \( \binom{5}{2} = 10 \) forms:

\[
C^{45} = \begin{pmatrix} 1 & 1 \\ \alpha_i & \alpha_j \\ \alpha_i^2 & \alpha_j^2 \end{pmatrix}, \quad C^{35} = \begin{pmatrix} 1 & 1 \\ \alpha_i & \alpha_j \\ \alpha_i^2 & \alpha_j^2 \end{pmatrix}, \quad C^{34} = \begin{pmatrix} 1 & 1 \\ \alpha_i & \alpha_j \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix},
\]

\[
C^{25} = \begin{pmatrix} 1 & 1 \\ \alpha_i^2 & \alpha_j^2 \\ \alpha_i & \alpha_j \end{pmatrix}, \quad C^{24} = \begin{pmatrix} 1 & 1 \\ \alpha_i^2 & \alpha_j^2 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix}, \quad C^{23} = \begin{pmatrix} 1 & 1 \\ \alpha_i^3 & \alpha_j^3 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix},
\]

\[
C^{15} = \begin{pmatrix} \alpha_i & \alpha_j \\ \alpha_i^2 & \alpha_j^2 \\ \alpha_i^3 & \alpha_j^3 \end{pmatrix}, \quad C^{14} = \begin{pmatrix} \alpha_i & \alpha_j \\ \alpha_i^2 & \alpha_j^2 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix}, \quad C^{13} = \begin{pmatrix} \alpha_i & \alpha_j \\ \alpha_i^3 & \alpha_j^3 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix},
\]

\[
C^{12} = \begin{pmatrix} \alpha_i^2 & \alpha_j^2 \\ \alpha_i^3 & \alpha_j^3 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix}.
\]

In order for each of the matrices to have the required rank 2, every one of the above matrices should have a \( 2 \times 2 \) non-singular submatrix.

First 3 matrices, \( C^{45}, C^{35} \) and \( C^{34} \), contain the matrix

\[
\begin{pmatrix} 1 & 1 \\ \alpha_i & \alpha_j \end{pmatrix}
\]

which has determinant \( \alpha_j - \alpha_i \neq 0 \), in view of our assumption that all \( \alpha_i \) are distinct.

Matrices \( C^{25} \) and \( C^{24} \) contain the matrix

\[
\begin{pmatrix} 1 & 1 \\ \alpha_i^2 & \alpha_j^2 \end{pmatrix}
\]

which has determinant \( \alpha_j^2 - \alpha_i^2 = (\alpha_j - \alpha_i)^2 \) by the Frobenius identity. Hence, it is also \( \neq 0 \).

Matrices \( C^{15} \) and \( C^{14} \) contain the matrix

\[
\begin{pmatrix} \alpha_i & \alpha_j \\ \alpha_i^2 & \alpha_j^2 \end{pmatrix}
\]

which has determinant \( \alpha_i\alpha_j^2 - \alpha_j\alpha_i^2 = \alpha_i\alpha_j(\alpha_j - \alpha_i) \neq 0 \), as \( \alpha_i \) are all non-zero and distinct.

Matrix \( C^{13} \) contains the matrix

\[
\begin{pmatrix} \alpha_i & \alpha_j \\ \alpha_i^3 & \alpha_j^3 \end{pmatrix}
\]

which has determinant \( \alpha_i\alpha_j^3 - \alpha_j\alpha_i^3 = \alpha_i\alpha_j(\alpha_j^2 - \alpha_i^2) = \alpha_i\alpha_j(\alpha_j - \alpha_i)^2 \neq 0 \) in view of \( \alpha_i, \alpha_j \neq 0 \) and \( \alpha_i \neq \alpha_j \).

Matrix \( C^{12} \) contains the matrix

\[
\begin{pmatrix} \alpha_i^2 & \alpha_j^2 \\ \alpha_i^3 & \alpha_j^3 \end{pmatrix}
\]

which has determinant \( \alpha_i^2\alpha_j^3 - \alpha_j^2\alpha_i^3 = \alpha_i^2\alpha_j^2(\alpha_j - \alpha_i) \neq 0 \) in view of \( \alpha_i, \alpha_j \neq 0 \) and \( \alpha_i \neq \alpha_j \).

The only matrix left in this group of 10 is \( C^{23} \). It has three \( 2 \times 2 \) submatrices, some of which can have the determinant 0. We claim that the following two submatrices of \( C^{23} \),

\[
\begin{pmatrix} 1 & 1 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix}, \quad \begin{pmatrix} \alpha_i^3 & \alpha_j^3 \\ \alpha_i (\alpha_i + 1) & \alpha_j (\alpha_j + 1) \end{pmatrix},
\]

are non singular.
cannot be both singular. Indeed, their determinants are:
\[ \alpha_j (\alpha_j + 1) + \alpha_i (\alpha_i + 1), \quad \alpha_i^3 \alpha_j (\alpha_j + 1) + \alpha_j^3 \alpha_i (\alpha_i + 1). \]

Let us suppose that both determinants are 0. We then have (after dividing the second one by \( \alpha_i \alpha_j \neq 0 \)):
\[
\begin{cases}
\alpha_j (\alpha_j + 1) = \alpha_i (\alpha_i + 1) \\
\alpha_i^3 (\alpha_j + 1) = \alpha_j^3 (\alpha_i + 1).
\end{cases}
\]

The first equation and Frobenius identity imply:
\[ 0 = \alpha_i^2 + \alpha_j^2 + \alpha_i + \alpha_j = (\alpha_i + \alpha_j)^2 + (\alpha_i + \alpha_j). \]

Thus, \( \alpha_i + \alpha_j = 0 \) or \( \alpha_i + \alpha_j = 1 \). Since \( \alpha_i \neq \alpha_j \), \( \alpha_i + \alpha_j \neq 0 \). Thus \( \alpha_i + \alpha_j = 1 \).

If \( \alpha_i = 1 \) then \( 0 = \alpha_j (\alpha_j + 1) \). Hence \( \alpha_j = 1 \) (\( \alpha_j \neq 0 \) by assumption). But then \( \alpha_i = \alpha_j = 1 \) which contradicts the assumption that \( \alpha \)'s are distinct. Hence, we can divide equations side by side and obtain
\[ \frac{\alpha_j}{\alpha_i^2} = \frac{\alpha_i}{\alpha_j^2}, \]

or \( \alpha_i^3 = \alpha_j^3 \). Therefore,
\[ 1 = (\alpha_i + \alpha_j)^3 = \alpha_i^3 + 3 \alpha_i^2 \alpha_j + 3 \alpha_i \alpha_j^2 + \alpha_j^3 = \alpha_i^3 + \alpha_i^2 \alpha_j + \alpha_i \alpha_j^2 + \alpha_j^3 = \alpha_i \alpha_j (\alpha_i + \alpha_j) = \alpha_i \alpha_j. \]

Hence \( \alpha_i \alpha_j = 1 \) and \( \alpha_i + \alpha_j = 1 \), which implies that \( \alpha_i \) and \( \alpha_j \) are the distinct roots of the equation \( \alpha^2 + \alpha + 1 = 0 \), i.e. are the distinct 3rd roots of unity in \( \mathbb{F} \). But we excluded one of the 3rd roots of unity from amongst \( \alpha_j, j = 1, 2, \ldots, k \), so we obtained a contradiction. Thus at least one of the matrices is invertible.

If \( t = 3 \) then \( r = 4 - t = 1 \). In this case, matrix \( C \) is given by:
\[
C = \begin{pmatrix}
1 & 1 & 1 \\
\alpha_i & \alpha_j & \alpha_i \\
\alpha_i^2 & \alpha_i^2 & \alpha_i^2 \\
\alpha_i^3 & \alpha_i^3 & \alpha_i^3
\end{pmatrix}.
\]

The submatrices obtained by erasing a single row from \( C \) are:
\[
\begin{align*}
C^5 &= \begin{pmatrix}
1 & 1 & 1 \\
\alpha_i & \alpha_j & \alpha_i \\
\alpha_i^2 & \alpha_i^2 & \alpha_i^2 \\
\alpha_i^3 & \alpha_i^3 & \alpha_i^3
\end{pmatrix}, \\
C^4 &= \begin{pmatrix}
1 & 1 & 1 \\
\alpha_i & \alpha_j & \alpha_i \\
\alpha_i^2 & \alpha_i^2 & \alpha_i^2 \\
\alpha_i^3 & \alpha_i^3 & \alpha_i^3
\end{pmatrix}^{(\alpha_i (\alpha_i + 1) \alpha_j (\alpha_j + 1) \alpha_i (\alpha_i + 1))}, \\
C^3 &= \begin{pmatrix}
1 & 1 & 1 \\
\alpha_i & \alpha_j & \alpha_i \\
\alpha_i^2 & \alpha_i^2 & \alpha_i^2 \\
\alpha_i^3 & \alpha_i^3 & \alpha_i^3
\end{pmatrix}^{(\alpha_i (\alpha_i + 1) \alpha_j (\alpha_j + 1) \alpha_i (\alpha_i + 1))}, \\
C^2 &= \begin{pmatrix}
1 & 1 & 1 \\
\alpha_i^2 & \alpha_i^2 & \alpha_i^2 \\
\alpha_i^3 & \alpha_i^3 & \alpha_i^3 \\
\alpha_i^4 & \alpha_i^4 & \alpha_i^4
\end{pmatrix}^{(\alpha_i (\alpha_i + 1) \alpha_j (\alpha_j + 1) \alpha_i (\alpha_i + 1))}, \\
C^1 &= \begin{pmatrix}
\alpha_i^2 & \alpha_i^2 & \alpha_i^2 \\
\alpha_i^3 & \alpha_i^3 & \alpha_i^3 \\
\alpha_i^4 & \alpha_i^4 & \alpha_i^4
\end{pmatrix}^{(\alpha_i (\alpha_i + 1) \alpha_j (\alpha_j + 1) \alpha_i (\alpha_i + 1))}. 
\end{align*}
\]
Of these 5 matrices, matrix $C^5$ and $C^4$ contain the $3 \times 3$ Vandermonde matrix:

$$
\begin{pmatrix}
\alpha_i & \alpha_j & \alpha_l \\
\alpha_i^2 & \alpha_j^2 & \alpha_l^2 \\
\alpha_i^3 & \alpha_j^3 & \alpha_l^3
\end{pmatrix}.
$$

which has the determinant $(\alpha_j - \alpha_i)(\alpha_l - \alpha_i)(\alpha_l - \alpha_j) \neq 0$, in view of the fact that $\alpha_i$ are all distinct. Matrix $C^1$ of the 5 contains a matrix related to Vandermonde,

$$
\begin{pmatrix}
\alpha_i & \alpha_j & \alpha_l \\
\alpha_i^2 & \alpha_j^2 & \alpha_l^2 \\
\alpha_i^3 & \alpha_j^3 & \alpha_l^3
\end{pmatrix}
$$

which has the determinant $\alpha_i\alpha_j\alpha_l(\alpha_j - \alpha_i)(\alpha_l - \alpha_i)(\alpha_l - \alpha_j) \neq 0$.

This leaves matrix $C^3$ and $C^2$. We find, using CAS, that

$$
\begin{vmatrix}
\alpha_i & \alpha_j & \alpha_l \\
\alpha_i^3 & \alpha_j^3 & \alpha_l^3 \\
\alpha_i(\alpha_i + 1) & \alpha_j(\alpha_j + 1) & \alpha_l(\alpha_l + 1)
\end{vmatrix} = \alpha_i\alpha_j\alpha_l(\alpha_i - \alpha_j)(\alpha_i - \alpha_l)(\alpha_l - \alpha_j) \neq 0,
$$

(this can also be seen by subtracting row 1 from row 3, and then swapping rows 2 and 3; the matrix becomes the same as $C^1$) and

$$
\begin{vmatrix}
\alpha_i^2 & \alpha_j^2 & \alpha_l^2 \\
\alpha_i^3 & \alpha_j^3 & \alpha_l^3 \\
\alpha_i(\alpha_i + 1) & \alpha_j(\alpha_j + 1) & \alpha_l(\alpha_l + 1)
\end{vmatrix} = \alpha_i\alpha_j\alpha_l(\alpha_i - \alpha_j)(\alpha_i - \alpha_l)(\alpha_l - \alpha_j) \neq 0.
$$

(this factorization is valid over a field of any characteristic; we can also use the following argument: subtract row 1 from row 3, then swap rows 1 and 3; the resulting matrix is now the same as matrix 5). Since the matrix under the above determinant is a submatrix of both matrices $C^3$ and $C^2$, they both have rank 3.

Finally, when $t = 4$, we need to prove that $C$ itself has rank 4. However, we observe that the first four rows of this matrix form a $4 \times 4$ Vandermonde matrix, which is non-singular in view of $\alpha_i$ being distinct.

Clearly, any set of 5 columns of the $P_{5 \times k}$ matrix is linearly dependent because row 5 is a sum of row 2 and 3. Hence, the proof is complete. \qed

As a corollary of the proof we obtain the following useful criterion:

**Theorem 5** (A criterion of a systematic code to have distance $d + 1$). A systematic code with parity matrix $P$ has distance $d + 1$ iff every matrix $P'$ obtained from $P$ by taking $t \leq d$ columns of $P$ and deleting $d - t$ rows of $P$ has rank $t$, i.e. $P'$ has a non-singular submatrix of size $t \times t$.

4. **RECOVERY FROM UP TO 4 ERASURES (KNOWN LOCATIONS ERRORS)**

The following theorem addresses the situation when up to 4 drives at known locations have been erased (or corrupted).

**Theorem 6** (Correctability of up to 4 erasures). The code with generator matrix $G$ defined by $[1]$ allows recovery from up to 4 drive failures at known locations.
Proof. The code is a systematic code with distance \( d = 5 \). The proof given below works for an arbitrary code with distance \( d \) and parity check matrix \( H \). Hence, any \( d - 1 \) columns of \( H \) are linearly independent. If the locations of the erased drives are \( i_1, i_2, \ldots, i_q \) then \( e_i = 0 \) if \( i \notin \{i_1, i_2, \ldots, i_q\} \). Therefore

\[
s = H e = \sum_{m=1}^{q} H_{im} e_{im}.
\]

Let \( H' = \left( \begin{array}{c|c|c} H_{i_1} & H_{i_2} & \cdots & H_{i_q} \end{array} \right) \) be the submatrix of \( H \). If \( q \leq d - 1 \) then there is a subset of \( q \leq d - 1 \) rows of \( H' \) such that the resulting matrix, which we will call \( H'' \), is non-singular. In view of

\[
H'' \cdot e' = s'
\]

where \( s' \) is obtained from \( e \) and \( s \) by selecting the same rows, and \( e' = ( e_{i_1}, e_{i_2}, \ldots, e_{i_q} ) \), has a unique solution \( e' \). Clearly, we can complete \( e' \) to \( e \) in a unique way, by setting

\[
e_i = \begin{cases} e'_m & \text{if } i = i_m \text{ for some } m, 1 \leq m \leq q, \\ 0 & \text{otherwise.} \end{cases}
\]

The transmitted vector \( t \) is found from \( r = t + e \), yielding \( t = r - e = r \). Note that \( t = r + e \) if the field \( \mathbb{F} \) has characteristic 2. \( \square \)

Remark 1 (On the choice of a non-singular submatrix). The choice of a specific \( 4 \times 4 \) non-singular submatrix in the proof of Theorem 6 was further discussed, and made explicit, in the proof of Proposition 1.

4.1. Recovery when positions of errors are known: Let us state a general principle which works when positions of all errors are known, and so \( E = 0 \). Suppose that we have a systematic code of distance \( d + 1 \) and we have a failure of \( t \) data disks at known locations (known erasures). Let \( I \subseteq \{1, 2, \ldots, k\} \) be the set of data error locations. In addition, let \( J \) be a subset of no more than \( d - t \) known parity error locations. We know that a submatrix \( P' \) of \( P \) (the parity matrix) obtained by taking only \( t \) columns with indices in \( I \) and deleting \( d - t \) rows of \( P \) contains a \( t \times t \) submatrix matrix \( P'' \) which is non-singular (Theorem 5). Thus we may choose \( P'' \) whose row indices are not in \( J \), i.e. do not correspond to known parity errors. Let the set of row indices be \( K \), where \( |K| = t \) and \( K \cap J = \emptyset \). We can use \( P'' \) to find the error values of data errors. Then we may find the parity error values for all parities \( j \notin K \). We simply use the equation:

\[
e_{k+j} = s_j - \sum_{i \in I} p_{ji} e_i
\]

These values, \( e_{k+j}, j \notin K \) may or may not be 0, which determines the actual number of parity errors. In order to satisfy the assumptions about parity errors, \( e_{k+j} = 0 \) for \( j \notin J \) is required. Hence, when all errors are erasures, the decoding problem is solved by linear algebra methods. As we will see, in other cases non-linear methods are required, requiring sometimes delicate analysis of systems of polynomial equations.

5. Solving Quadratic Equations over \( GF(2) \)

The equation \( a x^2 + b x + c = 0 \) is solved by the quadratic formula

\[
x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
over any field of characteristic \( \neq 2 \). If the characteristic is 2, the quadratic formula obviously cannot work as the denominator is 0. In the current section we develop a replacement for the quadratic formula, which will allow us to solve quadratic equations.

**Lemma 1** (On the difference of the roots of a quadratic equation). *Let \( a, b, c \) be constants in a Galois field \( \mathbb{F} \) of characteristic 2 and let \( x \) be a variable. If \( x_1 \in \mathbb{F} \) is a root of the quadratic equation*

\[
ax^2 + bx + c = 0, \quad a \neq 0
\]

*then the second root is given by the equation:*

\[
x_2 = x_1 + \frac{b}{a}.
\]

**Proof.** By direct calculation:

\[
a x_2^2 + b x_2 + c = a \left( x_1 + \frac{b}{a} \right)^2 + b \left( x_1 + \frac{b}{a} \right) + c
\]

\[
= a \left( x_1^2 + \frac{b^2}{a^2} \right) + b \left( x_1 + \frac{b}{a} \right) + c
\]

\[
= ax_1^2 + \frac{b^2}{a} + bx_1 + \frac{b^2}{a} + c
\]

\[
= \left( ax_1^2 + bx_1 + c \right) + 2\frac{b^2}{a} = 0.
\]

We shall focus on the algorithmic aspects of solving the quadratic equation. An obvious algorithm over a finite field is obtained by trying all the elements until we find a root, i.e. perform a search. The cost of this grows as the size of the field grows. In this section we will develop an algorithm which uses constant time and linear memory.

Our first observation is that if \( b = 0 \) then the quadratic equation becomes \( ax^2 + c = 0 \), or

\[
x^2 = \frac{c}{a}.
\]

If \( c = 0 \) then \( x = 0 \). if \( c \neq 0 \) then we need to compute the square root. The Frobenius map \( J : \mathbb{F} \to \mathbb{F} \) given by

\[
J(x) = x^2
\]

is an automorphism of \( \mathbb{F} \) and it is a linear map when \( \mathbb{F} \) is treated as a vector space over the field \( GF(2) \). Therefore, solving the equation \( J(x) = y \) is tantamount to inverting a matrix over \( GF(2) \). If \( \mathbb{F} = GF(2^m) \) then \( \mathbb{F} \) has dimension \( m \), and \( J \) is a \( m \times m \) matrix of elements of \( GF(2) \). The only solution to equation \((10)\) is thus:

\[
x = J^{-1} \left( \frac{c}{a} \right).
\]

Of course, matrix \( J \) may be precomputed, and matrix multiplication can be used to find \( x \). Alternatively, we can tabulate all square roots, and use a lookup table.

In the remainder of the paper we will write the unique solution to \( x^2 = y \) as \( \sqrt{y} \).
We may assume that \( a = 1 \) (i.e. make quadratic equation monic), by dividing \((9)\) by \( a \). As the next step, we scale \( x \), so that \( b = 1 \). Let \( x = b y \). We obtain:

\[
x^2 + b x + c = b^2 y^2 + b^2 y + c = b^2 y(y + 1) + c,
\]

\[
y(y + 1) = \frac{c}{b^2}.
\]

Hence, we reduced the quadratic equation \((9)\) to:

\[
y(y + 1) = d
\]

where \( d \in \mathbb{F} \) is given and seek \( y \). It turns out that this equation is also easy to solve. If \( d = 0 \) then \( y = 0 \) or \( y = 1 \). If \( d \neq 0 \) then we write the equation in terms of the Frobenius automorphism \( J \):

\[
y(y + 1) = y^2 + y = J(y) + y = (J + I) y
\]

where \( I : \mathbb{F} \to \mathbb{F} \) is the identity map. We note that \( J + I \) is a linear transformation over \( GF(2) \) given by a square matrix. We can solve the equation \((J + I) y = d\) as a linear system of \( m \) equations in \( m \) unknowns over \( GF(2) \). We note that

\[
\ker(J + I) = \{ y : (J + I) y = 0 \} = \{0, 1\} \subseteq \mathbb{F},
\]

as \((J + I) y = 0\) is equivalent to \( y(y + 1) = 0 \). Hence, the kernel is 1-dimensional as a linear subspace of the vector space \( \mathbb{F} \) over the field \( GF(2) \). This implies that the solution set of \((J + I) y = d\) is a 1-dimensional coset \( y_0 + \ker(J + I) \), where \( y_0 \) is a particular solution, or no solution exists. As 1-dimensional subspaces in characteristic 2 are 2-element sets, there are thus exactly 2 solutions to the equation \( y(y + 1) = d \) for every \( y \). Moreover, if the two roots are \( y_1 \) and \( y_2 \) then \( y_1 - y_2 \in \{0, 1\} \) (i.e. \( y_1 - y_2 \in \ker(J + I) \)). Hence, if \( y_1(y_1 + 1) = d \) then \( y_2 = y_1 + 1 \) is the second solution, and thus \( y_1 y_2 = d \).

From the above discussion it follows that \( J + I \) has nullity 1 and thus by the Rank-Nullity Theorem its rank is \( m - 1 \), as the dimension of \( \mathbb{F} \) as a vector space over \( GF(2) \) is \( m \). Thus \( \text{im}(J + I) \) has codimension 1. Hence, the equation \( y(y + 1) = d \) has a solution for exactly a half of the elements \( d \in \mathbb{F} \). Linear algebra tells us that there exists a linear functional \( \varphi : \mathbb{F} \to GF(2) \) such that

\[
y(y + 1) = d \quad \text{has a solution, iff} \quad \varphi(d) = 0.
\]

Moreover, \( \varphi \) is the unique non-zero solution to \((J^* + I)\varphi = 0\), where \( J^* : \mathbb{F}^* \to \mathbb{F}^* \) is the dual operator of \( J \), acting on the dual space \( \mathbb{F}^* \). Equivalently, \( \varphi \) is a linear functional such that \( \varphi \circ (J + I) = 0 \).

The above somewhat abstract discussion can be made concrete, if \( \mathbb{F} = GF(2^m) \), and \( D \) is a primitive element satisfying the equation \( g(D) = 0 \), where \( g \) is the chosen primitive polynomial \( g(x) = \sum_{j=0}^{m} g_j x^j \) \((g_j \in GF(2) \text{ for } j = 0, 1, \ldots, m \text{ and } g_m = 1)\). The set \( \{1, D, D^2, \ldots, D^{m-1}\} \) is a basis of \( \mathbb{F} \) as a vector space over \( GF(2) \). This basis is used to identify \( d = \sum_{j=0}^{m-1} d_j D^j \) with a vector \((d_0, d_1, \ldots, d_{m-1})\) in the vector space \( GF(2)^m \). The linear map \( J \) is represented with respect to this basis by the matrix \((J_{ij})\), \( i, j = 0, 1, \ldots, m-1 \), where the entries \( J_{ij} \) are found from the formula:

\[
D^{2j} = \sum_{i=0}^{m-1} J_{ij} D^i, \quad i, j = 0, 1, \ldots, m - 1.
\]

Let \( z \) be the left eigenvector of \( J \) for eigenvalue 1, i.e. \((J + I)^\top z = 0\). In characteristic 2 this vector is unique. The functional \( \varphi \) is then given by \( \varphi(x) = z^\top x \). The condition of solvability
of \((J + I)y = d\) is \(\varphi(z) = z^\top d = 0\). If \(d = \sum_{j=0}^{m-1} d_j D^j\), where \(D\) is the primitive element generating \(\mathbb{F}\), and \(z = (z_0, z_1, \ldots, z_{m-1})\) then the condition of solvability is

\[
(15) \quad \sum_{j=0}^{m-1} z_j d_j = 0.
\]

Let \(Z = \{j \in \{0, 1, \ldots, m\} : z_j = 1\}\) be the set of locations of all non-zero coordinates of \(z\) (counting from 0). Then the condition \((15)\) can also be written as:

\[
(16) \quad \sum_{j \in Z} d_j = 0.
\]

Thus, we may view the condition of solvability as a kind of parity check on a subset of the coefficients \(d_j \in \{0, 1\}\) representing \(d \in \mathbb{F}\).

**Example 1** (Operator \(J + I\) for \(m = 4\)). If \(\mathbb{F} = GF(2^4)\), \(g(x) = x^4 + x + 1\), we find that the matrix of \(J + I\) is:

\[
J = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad J + I = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

If \(D\) is the primitive element then the columns of \(J\) are the coefficients of polynomials obtained by formally dividing \(1, D^2, D^4, D^6\) by \(g(D)\) and writing the coefficients of the remainder in ascending order of powers. Since \(1, D^2\) are powers of \(D\) below the degree of the primitive polynomial, they yield first two columns \((1, 0, 0, 0)\) and \((0, 0, 1, 0)\). Furthermore, long division over \(GF(2)\) yields:

\[
D^4 = D^2 + 1 \mod g(D)
\]

\[
D^6 = D^2 \cdot D^4 = D^2 \cdot (D + 1) = D^3 + D^2 \mod g(D).
\]

Hence, the \(3^{rd}\) column of \(J\) is \((1, 1, 0, 0)\) and the \(4^{th}\) is \((0, 0, 1, 1)\). We find \(z\) by solving \((J + I)^\top z = 0\). We observe that \(J + I\) has a row of zeros, so \((J + I)^\top\) has a column of zeros, the \(4^{th}\) column. Hence \(z = (0, 0, 0, 1)\) is a solution, and, on general grounds, this solution is unique. Hence, the condition of solvability of \(y(y+1) = d\), where \(d = d_0 + d_1 D + d_2 D^2 + d_3 D^3\), is: \(d_3 = 0\).

**Example 2** (Solvability for \(m = 8\) and other values of \(m\)). The most commonly used field is \(\mathbb{F} = GF(2^8)\) with \(g(x) = x^8 + x^4 + x^3 + x^2 + 1\). The condition of solvability of \(y(y+1) = d\), where \(d = \sum_{j=0}^{7} d_j D^j\), is \(d_5 = 0\) and is found by the same approach. For some other values of \(m\), we obtain: \(m = 5\) with \(g(x) = x^5 + x^2 + 1\) yields \(d_1 + d_3 = 0\), \(m = 6\) with \(g(x) = x^6 + x + 1\) yields \(d_5 = 0\). The case of \(m = 5\) demonstrates that the parity check \(z^\top d = 0\) may involve more than 1 coefficient \(d_j\).

### 6. The Decoding Algorithm

In this section we will describe in detail the decoding algorithm for the code identified by the generator matrix \(G\) and parity check matrix \(H\), given by equations \((1\), \((2)\) and \((3)\).

The decoder algorithm depends on the number of zeros and the patterns we have noticed on the elements of the syndromes. We keep it in mind that \(\alpha_i\) are non-zero, distinct elements of the Galois field \(\mathbb{F}\).
Given the parity check matrix $H$ and the transmitted vector $t$, error vector $e$ and received message vector $r = t + e$, we consider the syndrome vector:

$$ s = H \cdot r = H \cdot (t + e) = H \cdot e = (s_1 \ s_2 \ s_3 \ s_4 \ s_5)^\top $$

6.1. **The case of the zero syndrome vector.** Let us dispose of the easiest case of decoding first, that of $s = 0$.

**Lemma 2** (On the zero syndrome vector). *If all the entries of the syndrome are zeros, then either we have no errors or we have silent data corruption that is not detected. Silent data corruption is possible when the number of errors is at least 5.*

**Proof.** Since we have $H \cdot r = H \cdot e = 0$, $r$ is a valid codeword (belongs to the columnspace of $G$). If $r \neq t$ then we must have $H \cdot r = H \cdot t$, i.e. $H \cdot (r - t) = 0$. Therefore, $r - t$ is a vector of weight at least 5. i.e. at least 5 errors occurred. □

6.2. **A brief survey of syndrome decoding.** Syndrome decoding depends on the following observation: if $i_1, i_2, \ldots, i_q$ are locations of the failed drives then we may be able to determine the non-zero error values $e_{i_j}$, and study a subsystem of $H \cdot e = s$

$$ (H_{i_1} \ | \ H_{i_2} \ | \ \cdots \ | \ H_{i_q}) \cdot \begin{pmatrix} e_{i_1} \\ e_{i_2} \\ \vdots \\ e_{i_q} \end{pmatrix} = s $$

where $H_i$ denotes the $i$-th column of $H$. We also use the consequence of the fact that the code is systematic, that

$$ H_i = \begin{cases} P_i & 1 \leq i \leq k, \\ I_j & i = k + j, \ j = 1, 2, \ldots, 5. \end{cases} $$

where $P_i$ denotes the $i$-th column of the parity matrix $P_{5 \times k}$, and $I_j$ denotes the $j$-th column of the identity matrix $I_{5 \times 5}$. We recall that

$$ P_i = \begin{pmatrix} 1 \\ \alpha_i \\ \alpha_i^2 \\ \alpha_i^3 \\ \alpha_i(\alpha_i + 1) \end{pmatrix}, \quad I_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \ldots, \quad I_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. $$

If the number of failed data drives is $r \leq q$ then we can split the set of locations into $\{i_1, i_2, \ldots, i_r\}$ and $\{k + j_1, k + j_2, \ldots, k + j_{q-r}\}$, where $j_l = i_{k+l} - k$ for $l = 1, 2, \ldots, q-r$. Therefore, the linear system (17) can be further specialized for systematic codes as:

$$ (P_{i_1} \ | \ P_{i_2} \ | \ \cdots \ | \ P_{i_r}) \cdot \begin{pmatrix} e_{i_1} \\ e_{i_2} \\ \vdots \\ e_{i_r} \end{pmatrix} + \begin{pmatrix} e_{j_1} \\ e_{j_2} \\ \vdots \\ e_{j_{q-r}} \end{pmatrix} = s. $$

We will abbreviate this system to

$$ \tilde{P} \cdot e_{\text{data}} + e_{\text{parity}} = s $$
where $\tilde{P}$ is the submatrix obtained from $P_{5 \times k}$ by keeping only columns at failed drive locations, $i_1, i_2, \ldots, i_r$.

6.3. The case of two failed parity drives. The next observation is that, since we are interested only in reconstructing of only up to 2 drives, if there is a failed parity drive $j$ then the vector $\tilde{s} = s - e_{k+j}I_j$ is either 0 or is proportional to one of the vectors $P_i$ or $I_l$ ($l \neq j$). In the latter case,

$$0 = \tilde{s} - e_{k+l}I_l = s - e_{k+j}I_j - e_{k+l}I_l.$$  (20)

i.e. $s = e_{k+j}I_j + e_{k+l}I_l$. This means that the failed parity drives are $j$ and $l$, and their error values $e_{k+j}, e_{k+l}$ are found from these simple equations:

$$e_{k+j} = s_{k+j},$$

$$e_{k+l} = s_{k+l}.$$  

Hence, we disposed of the case when both failed drives are parity drives.

6.4. Failure of one parity and one data drives. Let us suppose that the failed drives are the $i$-th data drive $i \leq k$ and $(k+j)$-th (parity) drive. We will simply say that the $j$-th parity drive failed. We have the following equation relating errors and syndromes

$$e_i P_i + e_{k+j}I_j = s.$$  (21)

where $I_j$ is the $j$-th column of the identity matrix $I_{5 \times 5}$, and $e_i \neq 0$. Thus, $e_i\alpha_i^{l-1} = s_l$ for $l \neq j$, $l = 1, 2, 3, 4$, and $e_5\alpha_i(\alpha_i + 1) = s_5$ if $5 \neq j$. In particular $s_l \neq 0$ for $l = 1, 2, 3, 4$, $l \neq j$. Thus, if $s_l = 0$ for some $l \in \{1, 2, 3, 4\}$ then automatically $j = l$.

It is also true that one of the following holds:

(1) $\alpha_i \neq 1$ and $s_5 = e_i\alpha_i(\alpha_i + 1)$;

(2) $\alpha_i = 1$ and $s_5 = 0$.

Using this information, we may find $\alpha_i$ as follows:

(1) If $j = 1$ then $\alpha_i = s_3/s_2$, $s_4/s_3 = \alpha_i = s_3/s_2$ and $s_5/\alpha_i = s_5/(s_3/s_2) = s_5 s_2/s_3 = \alpha_i + 1 = s_3/s_2 + 1$.

(2) If $j = 2$ then $\alpha_i = s_4/s_3$, $s_3/s_1 = \alpha_i^2$, $s_5/\alpha_i = s_5/(s_4/s_3) = s_5 s_3/s_4 = \alpha_i + 1 = s_4/s_1 + 1$.

(3) If $j = 3, 4, 5$ then $\alpha_i = s_2/s_1$; $s_l/s_1 = \alpha_i^{l-1} = (s_2/s_1)^{l-1}$ if $l \neq j$, $l < 5$; also $s_5/s_2 = \alpha_i + 1 = (s_2/s_1) + 1$ if $j \neq 5$.

Once we have found $\alpha_i$, we set $e_l = s_l/\alpha_i^{l-1}$ using one of the $l$ values found. Then we set

$$e_{k+j} = s_j - \alpha_i^{j-1} e_i$$ if $j \leq 4$, or $e_{k+5} = s_5 - \alpha_i(\alpha_i + 1) e_i$ if $j = 5$.

Algorithm B implements the above method for finding the location of the failed parity and data drives, based on the syndrome vector $s$. It solves a slightly more general equation:

$$x P^{(p)} + y I_j = s.$$  (22)

It not only finds $x$ and $y$, but also $j$ and $\rho$, which is crucial to finding the locations of the failed drives. The algorithm returns the quadruple $(j, \rho, x, y)$. Algorithm B rejects solutions in which $x = 0$ or $y = 0$. Thus, it does not handle the (easy) case when $s$ has weight 1, which should be treated separately. Hence, the input vector $s \in \mathbb{F}^5$ should have weight at least 2. It should be noted that for some syndrome vectors $s$ two solutions of equation (22) exist and have $\rho \neq 0$. This can only occur when $\rho$ is a cubic root of unity $\neq 1$. Since we excluded
one of the cubic roots from the set $\alpha$, the $\rho$ which belongs to $\alpha$ is unique. Algorithm \textup{[1]} needs the explicit knowledge of the excluded root, so that it can omit it in its search for solutions. We pass the excluded cubic root of unity in the second argument to the function \texttt{LOCATEFAILEDPARTYANDDATA}(s, $X$), where $X$ should be either empty set or a 1-element set containing the excluded root.

Algorithm \textup{[2]} uses $j$, $\rho$, $x$ and $y$ found by Algorithm \textup{[1]} to compute the error vector, as indicated above.

6.5. \textbf{Failure of two data drives}. The only cases left to consider are those in which the only failed drives are data drives. Not surprisingly, this is the most delicate case to analyze.

We will somewhat relax the above assumption, by assuming only that up to 2 data drives have failed. If the locations of the failed drives (if any) are at locations $i$ and $j$, the linear system \textup{(19)} is:

$$
\begin{pmatrix}
1 & 1 \\
\alpha_i & \alpha_j \\
\alpha_i^2 & \alpha_j^2 \\
\alpha_i^3 & \alpha_j^3 \\
\alpha_i(\alpha_i + 1) & \alpha_j(\alpha_j + 1)
\end{pmatrix} \cdot \begin{pmatrix}
e_i \\
e_j
\end{pmatrix} = \begin{pmatrix}s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5
\end{pmatrix}
$$

The problem reduces to the following: given $s$, find the locations $i$ and $j$, and the error values $e_i$ and $e_j$. The most helpful result comes from linear algebra:

\textbf{Theorem 7} (A criterion of solvability of a linear system). A linear system $A \cdot x = b$ has a solution iff $N^Tb = 0$, where $N$ is a matrix whose columns form a basis of the nullspace of $A^\top$.

This theorem is typically stated as $\text{im}(A) = \ker(A^\top)$.

We proceed to calculate the basis of the nullspace of $A^\top$. Using elementary matrices, we find the reduced row echelon form without row exchanges:

$$
\begin{pmatrix}
1 & \alpha_i \\
0 & 1 \\
0 & \alpha_j + \alpha_i
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & \alpha_j + \alpha_i + \alpha_j
\end{pmatrix}
\begin{pmatrix}
1 & \alpha_i & \alpha_i^2 & \alpha_i^3 & \alpha_i(\alpha_i + 1) \\
1 & \alpha_j & \alpha_j^2 & \alpha_j^3 & \alpha_j(\alpha_j + 1)
\end{pmatrix}
$$

A simple rearrangement of the entries of this matrix yields:

$$
N = \begin{pmatrix}
\alpha_i \alpha_j & \alpha_i \alpha_j(\alpha_i + \alpha_j) & \alpha_i \alpha_j \\
\alpha_i + \alpha_j & \alpha_i^2 + \alpha_i \alpha_j + \alpha_j^2 & \alpha_i + \alpha_j + 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

Thus $\text{im}(N) = \ker(A^\top)$. Now the sufficient and necessary condition of solvability of $A \cdot x = b$ is $N^Tb = 0$, or, after some simple transformations:

$$
\begin{align*}
s_3 &= s_1 \alpha_i \alpha_j + s_2(\alpha_i + \alpha_j) \\
s_4 &= s_1 \alpha_i \alpha_j(\alpha_i + \alpha_j) + s_2(\alpha_i^2 + \alpha_i \alpha_j + \alpha_j^2) \\
s_5 - s_2 &= s_1 \alpha_i \alpha_j + s_2(\alpha_i + \alpha_j)
\end{align*}
$$
Turning to the first and last equation in this system, we observe that they have the same right-hand side. Thus, a necessary condition of solvability is \( s_5 - s_2 = s_3 \) or \( s_2 + s_5 + s_3 = 0 \). We note that this is the equation which involves the parity check involving only the parity drives: the 5\(^{th} \) row of the parity matrix is the sum of the 2\(^{nd} \) and 3\(^{rd} \) rows.

If \( s_2 + s_5 + s_3 = 0 \) then the last equation can be discarded, and we obtain the following system:

\[
\begin{align*}
    s_3 &= s_1 \alpha_i \alpha_j + s_2 (\alpha_i + \alpha_j) \\
    s_4 &= s_1 \alpha_i \alpha_j (\alpha_i + \alpha_j) + s_2 (\alpha_i^2 + \alpha_i \alpha_j + \alpha_j^2)
\end{align*}
\]

from which we are able to find \( \alpha_i \) and \( \alpha_j \). We can express this system in terms of the symmetric polynomials:

\[
\begin{align*}
    \sigma_1 &= \alpha_i + \alpha_j, \\
    \sigma_2 &= \alpha_i \cdot \alpha_j.
\end{align*}
\]

It should be noted that we are only interested in solution where \( \alpha_i \neq \alpha_j \), so \( \sigma_1 \neq 0 \). Also, \( \sigma_2 = \alpha_i \alpha_j \neq 0 \) as all \( \alpha_i \) are non-zero. In short, we are also interested in vectors \( (\sigma_1, \sigma_2) \neq 0 \).

We utilize the Frobenius identity to rewrite \( \sigma_1^2 + \sigma_2^2 \) in the second equation as \( \sigma_1^2 \). We obtain

\[
\begin{align*}
    s_3 &= s_1 \sigma_2 + s_2 \sigma_1 \\
    s_4 &= s_1 \sigma_1 \sigma_2 + s_2 \left( \sigma_1^2 + \sigma_2 \right)
\end{align*}
\]

The second equation can also be written as \( s_4 = (s_1 \sigma_2 + s_2 \sigma_1) \sigma_1 + s_2 \sigma_2 = s_3 \sigma_1 + s_2 \sigma_2 \).

Using the first equation, we obtain the system:

\[
\begin{align*}
    s_3 &= s_1 \sigma_2 + s_2 \sigma_1 \\
    s_4 &= s_3 \sigma_1 + s_2 \sigma_2
\end{align*}
\]

This is a linear system for \( (\sigma_1, \sigma_2) \) which in matrix form is:

\[
\begin{pmatrix}
    s_2 & s_1 \\
    s_3 & s_2
\end{pmatrix}
\begin{pmatrix}
    \sigma_1 \\
    \sigma_2
\end{pmatrix}
=
\begin{pmatrix}
    s_3 \\
    s_4
\end{pmatrix}
\]

The determinant of the matrix of the system is \( D = s_2^2 - s_1 s_3 \). Therefore, we have two distinct cases: \( D = 0 \) and \( D \neq 0 \). Let us analyze the singular case \( D = 0 \) first.

**Lemma 3** (On the singular case \( D = 0 \)). Let us assume that the system \( (23) \) is consistent (has a solution). Let \( D = s_2^2 - s_1 s_3 \) be the determinant of the matrix of the system \( (25) \). If \( D = 0 \), where then either \( s = 0 \) or \( s \) comes from a single failed data drive \( i \) and \( s = s_1 P_i \).

**Proof.** The necessary condition for a solution of \( (25) \) to exist is that the two determinants \( D_1 \) and \( D_2 \) are also zero, where \( D_1 \) and \( D_2 \) come from Cramer’s rule vanish:

\[
D_1 = \begin{vmatrix}
    s_3 & s_1 \\
    s_4 & s_2
\end{vmatrix}
= s_2 s_3 - s_1 s_4 = 0,
\quad
D_2 = \begin{vmatrix}
    s_2 & s_3 \\
    s_3 & s_4
\end{vmatrix}
= s_2 s_4 - s_3^2 = 0.
\]

Thus, we have a system of equations:

\[
\begin{align*}
    s_1 s_3 &= s_2^2 \\
    s_2 s_3 &= s_1 s_4 \\
    s_2 s_4 &= s_3^2
\end{align*}
\]

Let us analyze possible solutions of \( (26) \).
If \( s_1 = 0 \) then \( s_2 = s_3 = 0 \). Also, \( s_5 = s_2 + s_3 = 0 \). Hence, \( s_4 \) can be the only non-zero syndrome. The first two equations of the system (23)

\[
\begin{pmatrix}
1 & 1 \\
\alpha_i & \alpha_j
\end{pmatrix}
\begin{pmatrix}
e_i \\
e_j
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

with determinant \( \alpha_j - \alpha_i \neq 0 \) imply that \( e_i = e_j = 0 \) (note: \( \alpha_i \neq \alpha_j \)). Hence, \( s_4 = 0 \) and \( s = 0 \).

Let us thus assume \( s_1 \neq 0 \). If \( s_2 = 0 \) then \( s_3 = s_4 = s_5 = 0 \). Hence \( s_1 \) would be the only non-zero syndrome. The second and third equation of the system (23)

\[
\begin{pmatrix}
\alpha_i & \alpha_j \\
\alpha_i^2 & \alpha_j^2
\end{pmatrix}
\begin{pmatrix}
e_i \\
e_j
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

with determinant \( \alpha_i \alpha_j^2 - \alpha_j \alpha_i^2 = \alpha_i \alpha_j (\alpha_i - \alpha_j) \neq 0 \) imply that \( e_i = e_j = 0 \) (note: \( \alpha_i \neq \alpha_j \)).

If \( s_1 \neq 0 \) and \( s_2 \neq 0 \) then \( s_3 \neq 0 \) and \( s_4 \neq 0 \). Hence, \( s_1, s_2, s_3 \) and \( s_4 \) are all non-zero. We find from the first equation of (20) that \( s_3 = s_2^2/s_1 \). Plugging into the second and third equations, we get \( s_2 (s_2^2/s_1) = s_1 s_4 \) and \( s_2 s_4 = (s_2^2/s_1)^2 \). Simplifying, \( s_2^3 = s_2^2 s_4 \) and \( s_2^2 s_4 = s_2^3 \). Both equations are identical.

In summary, if the linear system (25) is singular then either \( s = 0 \) and \( e_i = e_j = 0 \), or \( s \neq 0 \) and we have both \( s_3 s_1 = s_2^2 \) and \( s_1^2 s_4 = s_2^3 \). In the latter case, \( s_1, s_2, s_3 \) and \( s_4 \) are all non-zero. We rewrite these equations as:

\[
\frac{s_2}{s_1} = \frac{s_3}{s_2} = \frac{s_4}{s_3}
\]

(Note: \( s_4/s_3 = (s_2^2/s_1^2)/(s_2^2/s_1) = s_2/s_1 \)). Let \( \rho \) be the common value of these ratios. Then

\[
s_5 = s_2 + s_3 = s_1 ((s_2/s_1) + (s_3/s_2) (s_2/s_1)) = s_1 \rho (\rho + 1).
\]

Hence, there is a \( \rho \in \mathbb{F}^* \) such that \( s \) is proportional to the vector \( P^{(\rho)} \) defined by equation \((\text{I})\).

If \( \rho = \alpha_i \) for some \( i \) then \( s = s_1 P_i \). This implies that \( e \) has weight 1 and \( e_i = s_1 \), and in fact only one data drive failed. If \( \rho \neq \alpha_i \) for all \( \alpha_i \) then \( s = s_1 P^{(\rho)} \) cannot be a linear combination of \( P_i \) and \( P_j \) for any combination of \( i \) and \( j \). Indeed, the matrix

\[
\begin{pmatrix}
P_i & P_j & P^{(\rho)}
\end{pmatrix}
\]

contains a non-singular \( 3 \times 3 \) Vandermonde matrix, and its columns are thus linearly independent. \( \square \)

Thus, we may assume \( D \neq 0 \) and obtain the solution of the non-singular system (25) by Cramer’s rule:

\[
\begin{align*}
\sigma_1 &= \frac{D_1}{D} = \frac{s_2 s_3 - s_1 s_4}{s_2^2 - s_1 s_3}, \\
\sigma_2 &= \frac{D_2}{D} = \frac{s_2 s_4 - s_1 s_3}{s_2^2 - s_1 s_3}.
\end{align*}
\]

Once we have found \( \sigma_1 \) and \( \sigma_2 \), we find the roots \( \alpha_i \) and \( \alpha_j \), in view of the Vieta identity

\[
(\zeta - \alpha_1) (\zeta - \alpha_2) = \zeta^2 - \sigma_1 \zeta + \sigma_2,
\]

from the quadratic equation

\[
\zeta^2 - \sigma_1 \zeta + \sigma_2 = 0.
\]
This yields \(i\) and \(j\), the locations of the failed drives. The error values can be obtained from the first two equations of (23):

\[
\begin{pmatrix}
1 & 1 \\
\alpha_i & \alpha_j
\end{pmatrix} \cdot \begin{pmatrix}
e_i \\
e_j
\end{pmatrix} = \begin{pmatrix}s_1 \\
s_2
\end{pmatrix}
\]

Explicitly given, they are:

\[
\begin{align*}
e_i &= \frac{s_1\alpha_j - s_2}{\alpha_j - \alpha_i}, \\
e_j &= \frac{s_2 - \alpha_i s_1}{\alpha_j - \alpha_i}.
\end{align*}
\]

7. The Main Algorithm: Recovery from up to 2 Errors (unknown locations errors)

Algorithm 5 defines the overall flow control structure, but does little work on its own. It uses several other algorithms for which we do not define any pseudo-code, as they are straightforward once a particular implementation strategy is chosen. Here we list them with their signature and requirements:

1. A function \(\text{Lookup}(\alpha, \zeta)\), which returns an index \(i\), \(1 \leq i \leq k\), such that \(\alpha_i = \zeta\), or \(\emptyset\) if \(\zeta \notin \alpha\). Here \(\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}\) is the set of elements of the underlying Galois field of characteristic 2. For example, \(\alpha\) could be implemented as a map \(\alpha : \{1, 2, \ldots, k\} \rightarrow \mathbb{F}\).

2. A function \(\text{FindNonZeros}(s)\), which returns a list of indices \(i\) of a Galois vector \(s \in \mathbb{F}^5\) such that \(s_i \neq 0\).

3. A function \(\text{ParityCheckMatrix}(\alpha)\), which returns the \(5 \times (k + 5)\) parity check matrix \(H\) of our code, defined by equation (2).

4. A function \(\text{NumEROFELEMENTS}(collection)\), which returns the number of elements of generic collections of objects, such as sets, lists and vectors.

5. A function \(\text{SOLVEQUADRATICEQUATION}(a, b, c)\), which returns the two roots of the equation \(f(x) = ax^2 + bx + c = 0\), where \(a, b, c \in \mathbb{F}\) and \(x\) is a variable ranging over \(\mathbb{F}\). We outlined two algorithms in Section 5 for doing this.

6. A function \(\text{EXCLUDEDCUBICROOTSOFTHEUNITY}(\alpha)\), which returns the list of roots of the equation \(\zeta^2 + \zeta + 1 = 0\) which are not in the set \(\alpha\). This list is possibly empty, and has not more than 1 element, if our exclusion rules are observed.

Remark 2. Generalization to any field \(GF(2^m)\):

The algebra rules used in the current paper are applicable to any field \(GF(2^m)\). For \(\mathbb{F} = GF(2^8)\) the maximum number of drives supported is 254. If we need to build a RAID array with more than 254 drives, we can choose \(\mathbb{F} = GF(2^m)\) with \(m > 8\) and also excludes one of the \(3^{rd}\) roots of unity if exist, yielding limit of up to \(2^m - 1\) or \(2^m - 2\) possible drives.

For example, If \(\mathbb{F} = GF(2^{16})\), \(2^{16} = 65,536\). We have to exclude the zero element and one of the \(3^{rd}\) roots of unity and have a limit of up to 65,534 drives.

Remark 3. What if we do not want to exclude one of the two \(3^{rd}\) roots of unity?

Let us assume that we are using all of the 255 drives for \(\mathbb{F} = GF(2^8)\). Then, the algorithm still functions and has a very low probability of not working correctly!!
This algorithm will fail only in the case of having two failed data drives, whose locations correspond to both of the 3rd root of unity, with equal error values.

Therefore, the probability that our algorithm fails due to non-exclusion of a 3rd root of unity is

\[
\frac{1}{\binom{2^m-1}{2} \cdot (2^m - 1)}.
\]

(This is a conditional probability, under the assumption that failure indeed occurs.)

For example, if \( F = GF(256) \), the risk is \( \approx 10^{-7} \), and for the \( F = GF(2^{16}) \), the risk would be \( \approx 7.1 \times 10^{-15} \).

8. Computational Complexity

It is clear that algorithm 5 involves a constant (very small) number of Galois field operations (additions, multiplications and divisions) over the field \( F \). If we choose to solve quadratic equations using Gaussian elimination, the number of operations in \( GF(2) \) is \( O((\log |F|)^3) \), which is the computational complexity of Gaussian elimination, while the lookup table approach is constant time. Thus, the algorithm corrects a single stripe containing an error in constant time, independent of the size of the field, assuming lookup table implementation.

A more-in-depth analysis of computational complexity requires taking into account the fact that with a growing number of disks we must also allow the field to grow. If \( F = GF(2^m) \), and \( N = k + 5 \) is the total number of disks in the array, we must use a field for which the number \( Q = 2^m \) is equal to \( N \) up to 1 or 2 disks.

The complexity of a RAID method implementing striping typically is computed as the time or space required to encode/decode a single codeword, which is a stripe. As an example, RAID 6 requires a fixed number of Galois field operations (addition, multiplication, division, logarithm lookup) when decoding a received vector, not counting the computation of parities or syndromes. This correlates with the number of CPU cycles and the time required to decode a received vector. The number of operations does not depend on the number of disks in the array \( N \). Also, constant time access is assumed to array elements. However, as \( N \) increases, it is necessary to use a larger Galois field \( GF(Q) \) with \( Q > N \). The number of bits \( B = \lceil \log_2(Q) \rceil \) per Galois field element grows, thus requiring more time per Galois field operation. Addition, which is identical to XOR, is \( O(B) \). Multiplication has complexity \( O(B \cdot \log(B) \cdot \log(\log(B))) \), according to the state of the art \cite{9}. Being close to \( O(B) \), this kind of complexity is referred to as quasilinear time complexity. Thus, the time complexity of RAID 6 decoding is \( O(\log(N) \cdot \log \log(N) \cdot \log \log \log(N)) \) rather than constant, and will be called quasi-logarithmic in the paper. Some operations, such as solving a quadratic equation in \( GF(Q) \), where \( Q = 2^m \), require inverting a matrix with coefficients in \( GF(2) \) of size \( O(m) \). The complexity of matrix inversion is \( O(m^p) \) where the best \( p \leq 3 \), and the best known value of \( p \) known today is \( p \approx 2.373 \). Thus, the complexity in terms of \( Q \) of solving quadratic equation is \( O(\log(Q)^p) \). It is possible to choose \( Q \) arbitrarily large, independently of the number of disks \( N \), incurring large computational cost. With an optimal choice of \( Q \), \( Q \leq 2 \times N \) and the computational cost is \( O(\log(N)^p) \). However, the matrix inversion for solving quadratic equation can be performed only once, with its result stored in a lookup table of size \( O(m^2) \), thus not affecting run time, assuming lookup time \( O(1) \) or even \( O(\log m) \),
if binary search needs to be used. Thus, having an algorithm which performs a constant number of Galois field operations, and solves a fixed number of quadratic equations, remains quasi-logarithmic in \( N \).

It should be noted that calculating parities for FEC codes requires \( O(N) \) operations (\( N \) multiplications, \( N \) additions to add the results). However, the summation step on a parallel computer can be reduced to \( O(\log(N)) \) by requiring \( N \) parallel processors and shared memory (PRAM). The \( N \) multiplications can be performed in parallel, in quasi-logarithmic time \( O(\log(\log N) \cdot \log \log \log N)) \). Therefore, if error correction can be performed in a fixed number of Galois operations (not depending on \( N \)), the overall algorithm remains quasi-logarithmic on a parallel computer with \( N \) processors.

### 9. Error Correcting Capabilities for 3 Failed Drives

In this section we obtain results on detecting and correcting of 3 errors. Clearly, for a code of distance \( d = 5 \) code, one can expect to be able to correct only \( \lfloor (d-1)/2 \rfloor = 2 \) errors by a decoder which searches for the nearest valid codeword (minimum distance decoder), such as ours. However, since we have 5 parities, it turns out that our code has an advantage over a hypothetical code with 4 parities, when it comes to detecting and correcting 3 failing drives.

The main idea is that of list decoding. When the syndrome vector \( s \in \mathbb{F}^5 \) is determined not to be consistent with 2 errors (any combination of data and parity errors), we are able to find all possible vectors \( e \) of weight 3 such that \( s = H e \). This strategy may be successful if the set of possible solutions is not too large, and that there exists an efficient algorithm to compute this set.

Let us consider the case when 3 data drives failed first. We note that \( s_2 + s_3 + s_5 = 0 \) is a necessary condition for a syndrome vector \( s \) to be due to pure data drive failures. Therefore, we can drop \( s_5 \) after checking this condition.

**Theorem 8 (On three failed data drives).** Let us consider the code introduced by equations 1, 2 and 3. Let us suppose that a syndrome vector \( s = H \cdot e \) comes from failure of exactly 3 data drives at locations \( i, j \) and \( l \), and there is no failure of 2 drives which results in \( s \). Then \( s_2 + s_3 + s_5 = 0 \) and

\[
(29) \quad s_4 - \sigma_1 s_3 + \sigma_2 s_2 - \sigma_3 s_1 = 0
\]

where \( \sigma_m, m = 1, 2, 3 \), are the symmetric polynomials of \( \alpha_i, \alpha_j \) and \( \alpha_l \):

\[
\sigma_1 = \alpha_i + \alpha_j + \alpha_l,
\]

\[
\sigma_2 = \alpha_i \alpha_j + \alpha_i \alpha_l + \alpha_j \alpha_l,
\]

\[
\sigma_3 = \alpha_i \alpha_j \alpha_l.
\]

**Proof.** Let us set \( a = \alpha_i, b = \alpha_j \) and \( c = \alpha_l \). We need to study the solutions of the system

\[
x \cdot P^{(a)} + y \cdot P^{(b)} + z \cdot P^{(c)} = s
\]
which in full form is:

\[ x + y + z = s_1, \]
\[ a x + b y + c z = s_2, \]
\[ a^2 x + b^2 y + c^2 z = s_3, \]
\[ a^3 x + b^3 y + c^3 z = s_4, \]
\[ a(a+1) x + b(b+1) y + c(c+1) z = s_5. \]

The last equation, in view of \( s_5 = s_2 + s_3 \), can be dropped, as it is the sum of the second and third equation. The first four equations form a linear system for \( x, y, \) and \( z \). which is overdetermined. Moreover, the first 3 equations have a coefficient matrix which is a \( 3 \times 3 \) Vandezmonde matrix with determinant \( D_0 = (b-a)(c-a)(c-b) \). We may assume that \( D_0 \neq 0 \) as \( a, b, \) and \( c \) range over distinct elements of the set \( \alpha \). Thus, the system is consistent iff the \( 4 \times 4 \) augmented coefficient matrix has rank 3. This determinant can be calculated using CAS, and is:

\[ D = D_0 \cdot (s_4 - s_3 \sigma_1 + s_2 \sigma_2 - s_1 \sigma_3) \]

Hence, the consistency condition is equivalent to equation (29). \( \square \)

Theorem 8 limits the number of triples \((a, b, c)\) to \( k(k - 1)/2 \), because by trying all combinations \( \{a, b\} \subseteq \alpha \) we find \( c \) from (29).

In underlying applications to RAID, it may be possible to obtain several syndromes which allow us to further limit the failed data drive locations. In fact, most commonly we will have a sample of several syndrome vectors due to failed drives. We can also obtain a sample by repeatedly reading and writing suspect stripes of data on the drives. This approach may quickly succeed. The criterion of success is given in the next theorem, and it roughly consists in checking linear independence of syndromes, which is a straightforward task.

**Theorem 9.** Let \( s^{(m)}_5 = (s^{(m)}_j)_{j=1}^5, m = 1, 2, \ldots, M, \) be syndrome vectors, i.e. any vectors in the range of \( H \), and

\[ s^{(m)}_5 = s^{(m)}_2 + s^{(m)}_3 \quad \text{for } m = 1, 2, \ldots, M. \]

Then the following is true:

1. If \( M = 3 \) and the matrix \((s^{(m)}_j)_{j,m=1}^3\) is non-singular, then the locations \((i, j, l)\) of the failed drives which may result in these syndromes can be found by first finding \((\sigma_1, \sigma_2, \sigma_3)\) from the linear system:

\[ \sum_{j=1}^3 (-1)^{j-1} s^{(m)}_{4-j} \sigma_j = s^{(m)}_4, \quad m = 1, 2, 3 \]

and then solving the cubic equation:

\[ (\zeta - a)(\zeta - b)(\zeta - c) = \zeta^3 - \sigma_1 \zeta^2 + \sigma_2 \zeta - \sigma_3 = 0. \]

The solution is the unique triple \((a, b, c)\). We may determine \((i, j, l)\) by lookup, equating \( \alpha_i = a, \alpha_j = b \) and \( \alpha_l = c \).

2. If \( M = 4 \) and the syndrome vectors are linearly independent then cannot come from a failure of \( \leq 3 \) data drives.
(3) If \( M = 2 \) and such syndrome vectors are found such that
\[
(s_1^{(1)}, s_2^{(1)}, s_3^{(1)}) \neq (s_1^{(2)}, s_2^{(2)}, s_3^{(2)})
\]
where both vectors are non-zero, then the number of triples \((i, j, l)\) of failed data drives which can result in those syndromes does not exceed \( k \).

Proof. The case of 3 syndromes is obvious, in view of our preceding analysis. If there are 4 linearly independent syndromes then there is no solution for \((\sigma_1, \sigma_2, \sigma_3)\). If there are 2 syndromes are found as described in the theorem then there are 2 linearly independent syndromes then the set of triples \((\sigma_1, \sigma_2, \sigma_3)\) form a 1-dimensional affine subspace of \( \mathbb{F}^3 \). Thus, varying one of the variables \( \sigma_1, \sigma_2 \) or \( \sigma_3 \) (the free variable) over the set \( \alpha \) and finding the other two from the system of linear equations, yields not more than \( k \) solutions. \( \square \)

Remark 4 (On locating and recovery of 3 failed data drives). Based on Theorem 9 we have a clear strategy to locate and correct 3 failed data drives. We simply collect syndromes and observe their projections onto the first 3 coordinates. Once we find 3 independent vectors in our collection, we can locate the failed drives. We can clearly correct the resulting errors, as we can correct up to 4 errors at known locations.

The relevant bound for all other cases is the subject of our next theorem.

Theorem 10 (On number of solutions for 3-drive failure). Let \( s \in \mathbb{F}^5 \) be a vector such that there is no vector \( e \in \mathbb{F}^{k+5} \) of weight 2 for which \( He = s \). Then the number of triples \((i, j, l)\), \(1 \leq i < j < l \leq k\), such that there is a vector \( e \) of weight 3, such that

- (1) \( s \) is a syndrome vector for \( e \), i.e. \( He = s \);
- (2) \( e_m = 0 \) unless \( m \in \{i, j, l\} \) \((1 \leq m \leq k + 5)\);
- (3) \( l > k \), i.e. not all three failed drives are data drives;

is not more than \( 2k + 4 \). For given \( s \) and triple \((i, j, l)\), the vector \( e \) with the above properties is unique.

Proof. Cases of failure of 3 drives can be divided according to the number of failed parity drives.

If 3 parity drives fail, at positions \( k + i, k + j, k + m \), where \( 1 \leq i < j < m \leq 5 \) the equation is
\[
x \cdot I_i + y \cdot I_j + z \cdot I_m = s.
\]
This equation implies that \( s \) has weight 3 and syndromes \( i, j \) and \( m \) are the non-zero syndromes. Moreover, \( x = s_i \), \( y = s_j \) and \( z = s_m \). Thus, for every \( s \) of weight 3 there exists a unique solution of this type. The error vector satisfies \( e_{k+i} = x, e_{k+j} = y \) and \( e_{k+m} = z \), and \( e_l = 0 \) for \( l \notin \{i, j, m\} \).

If 2 parity drives fail, at positions \( k + j, k + l \), where \( 1 \leq j < l \leq 5 \), along with data drive \( i \) then we have
\[
x \cdot P^{(\rho)} + y \cdot I_j + z \cdot I_m = s
\]
where \( \rho = \alpha_i \). Let \( P^{(\rho)}_{j,m} \) be the vector \( P^{(\rho)} \) with entries \( j \) and \( l \) erased. Thus
\[
x \cdot P^{(\rho)}_{j,l} = s^{(j,l)}
\]
where \( s^{(j,l)} \) is the syndrome vector \( s \) with entries \( j \) and \( m \) erased. This gives us 3 equations for \( x \) and \( \rho \), which should allow us to solve the problem. Table I contains the results of careful analysis of all cases for distinct pairs \((j, l)\). As we can see, in each case we have multiple
Table 1. Systems of equations for recovery from a failure of 3 drives, one data and 2 parity at location \( k + j, k + l, j, l = 1, 2, 3, 4, 5, j < l \). The equations in the third column contain \( \rho \). The fourth column contains a constraint obtained by elimination of \( \rho \) from equations in column 2.

| \( j \) | \( l \) | System of equations for \( \rho \) | Constraints on \( s \) |
|---|---|---|---|
| 1 | 2 | \( s_5 \rho + s_4 + s_3 \) \( s_3 \rho + s_4 \) | \( s_4 s_5 + s_3 s_4 + s_3^2 \) |
| 1 | 3 | \( s_2 \rho + s_5 + s_2 \) \( s_5 \rho + s_5 + s_4 + s_2 \) | \( s_5^2 + s_2 s_4 + s_2^2 \) |
| 1 | 4 | \((s_5 + s_3) \rho + s_3\) | \( s_5 + s_3 + s_2 \) |
| 1 | 5 | \( s_2 \rho + s_3 \) \( s_3 \rho + s_4 \) | \( s_2 s_4 + s_3^2 \) |
| 2 | 3 | \( (s_5 + s_1) \rho + s_5 + s_4 \) \( s_5 \rho^2 + s_4 \rho + s_4 \) \( (s_5^2 + s_4 s_5) \rho + s_4^2 + s_1 s_4 \) | \( s_5^3 + s_1 s_4 s_5 + s_1 s_4^2 + s_1^2 s_4 \) |
| 2 | 4 | \( s_1 \rho + s_5 + s_3 \) \( (s_5 + s_3) \rho + s_3 \) | \( s_5^2 + s_3^2 + s_1 s_3 \) |
| 2 | 5 | \( s_1 \rho^2 + s_3 \) \( s_3 \rho + s_4 \) \( s_1 s_4 \rho + s_4^2 \) | \( s_1 s_4^2 + s_3^2 \) |
| 3 | 4 | \( s_1 \rho + s_2 \) \( s_2 \rho + s_5 + s_2 \) | \( s_1 s_5 + s_2^2 + s_1 s_2 \) |
| 3 | 5 | \( s_1 \rho + s_2 \) \( s_2 \rho^2 + s_4 \) \( s_2 \rho + s_1 s_4 \) | \( s_1^2 s_4 + s_2^2 \) |
| 4 | 5 | \( s_1 \rho + s_2 \) \( s_2 \rho + s_3 \) | \( s_1 s_3 + s_2^2 \) |

(two or three) equations which \( \rho \) satisfies, with coefficients dependent on the syndromes \( s_t, t = 1, 2, \ldots, 5 \) (column 3 of the table). We also can obtain constraints on the syndromes by eliminating \( \rho \) from the equations in column 2. These are listed in column 3. As we can see, in each case there is exactly one constraint. A lengthy analysis shows that \( \rho \) is unique, except for the degenerate situation, when \( s \) is a syndrome vector for an error vector of weight \( \leq 2 \). The arguments are straightforward but lengthy, and are omitted. We only mention the case \( j = 2 \) and \( l = 3 \) as it is different from other cases in one respect, that it relies upon the exclusion rule for cubic roots of unity. If \( \rho \) is non-unique then \( s_5 + s_1 = 0, s_5 + s_4 = 0, s_5 + s_4 s_5 = 0, s_4^2 + s_1 s_4 = 0 \) and \( s_3^2 + s_1 s_4 s_5 + s_1 s_4^2 + s_1^2 s_4 = 0 \). These equations imply \( s_1 = s_4 \). Also \( s_5 (s_5 + s_4) = 0 \). If \( s_5 = 0 \) then also \( s_1 = 0 \) and \( s_4 = 0 \). Thus \( s \) has weight 2, which is a syndrome vector for \( \leq 2 \) failed parity drives with numbers in the set \{2, 3\}. Thus, we may assume \( s_5 \neq 0 \). Then \( s_5 = s_4 \) and thus \( s_1 = s_4 = s_5 \). The second equation in column 2 reduces to \( s_5 (\rho^2 + \rho + 1) = 0 \), which implies \( \rho^2 + \rho + 1 = 0 \). Therefore \( \rho \) is a cubic root of unity \( \neq 1 \). It must therefore be the root of unity different from the excluded one. This makes \( \rho \) unique.

Hence, there is at most one solution with 2 failed data and 1 failed parity drive.
If 1 parity drive fails at position $j$, along with 2 data drives, we have

$$x \cdot P(a) + y \cdot P(b) + z \cdot I_j = s.$$  

**Table 2.** Systems of equations for recovery from a failure of 3 drives, two data and 1 parity at location $k+j$, $j = 1, 2, 3, 4, 5$. The equations are expressed in terms of symmetric polynomials $\sigma_1 = a + b$ and $\sigma_2 = a \cdot b$.

| $j$ | System of equations for $\sigma_1$ and $\sigma_2$ | Constraint on $s$ |
|-----|-----------------------------------------------|------------------|
| 5   | $s_3 + \sigma_1 s_2 + s_1 \sigma_2$          |                  |
|     | $s_4 + \sigma_1 s_3 + \sigma_2 s_2$         |                  |
|     | $s_2 s_4 + s_2^3 + \sigma_2 (s_1 s_3 + s_2^2)$ |                  |
| 4   | $\sigma_1 (s_5 + s_3) + s_3 + s_1 \sigma_2$ | $s_5 + s_3 + s_2$|
| 3   | $s_5 + \sigma_1 s_2 + s_2 + s_1 \sigma_2$   |                  |
|     | $\sigma_1 s_5 + s_5 + s_4 + \sigma_2 (s_2 + s_1) + s_2$ |                  |
|     | $s_2^3 + \sigma_2 (s_1 s_5 + s_2^2 + s_1 s_2) + s_2 s_4 + s_2^2$ |                  |
| 2   | $\sigma_2 (s_5 + s_3 + s_1) + \sigma_1 s_5 + s_4 + s_3$ |                  |
|     | $\sigma_2 (s_2^3 + s_3 (s_3 + s_1)) + s_4 s_5 + s_3 s_4 + s_3^2$ |                  |
| 1   | $\sigma_2 (s_5 + s_3) + s_4 + \sigma_1 s_3$   | $s_5 + s_3 + s_2$|

For fixed $j$, this is a system of equations for $x$, $y$, $a$, $b$ and $z$, i.e. 5 equations in 5 unknowns, i.e. the problem is well-posed. The case is thus subdivided into subcases according to the value of $j$. We preprocessed the equations with CAS, by first erasing equation in row $j$ (which eliminates $z$), and then eliminating variables $x$ and $y$. Also, since the system is symmetric with respect to $a$ and $b$, we expressed the equations in terms of symmetric polynomials $\sigma_1 = a + b$ and $\sigma_2 = a \cdot b$. The result is in Table 2. It should be noted that in each case we have a linear system of equations for $(\sigma_1, \sigma_2)$. Each solution of the linear system yields a single solution $(a, b)$ up to swapping $a$ and $b$. We proceed to more precisely determine the number of solutions. The analysis of subcases for $j = 5, 4, 3, 2, 1$ is as follows:

**Case** $j = 5$. There are 3 linear equations for $(\sigma_1, \sigma_2)$. The solution of the system is non-unique iff $s_1 s_3 + s_2^2 = 0$ and $s_2 s_4 + s_3^2 = 0$. This system of equations can also be written as

$$s_1 s_3 = s_2^2,$$
$$s_2 s_4 = s_3^2.$$

If $s_2 = 0$ then $s_3 = 0$, and $s_4 = 0$. Thus $s$ has weight at most 2, and it matches the case of 2 failed parity drives, which is a contradiction. Therefore, $s_2 \neq 0$ and $s_1 \neq 0$, $s_3 \neq 0$, $s_4 = s_3^2/s_2 \neq 0$. Let us define $\rho = s_2/s_1 \neq 0$. We have $s_2 = s_1 \rho$, $s_3 = s_2^2/s_1 = s_1 \rho^2$, $s_4 = s_3^2/s_2 = s_1 \rho^3$. Hence, $s = x P(\rho) + z I_5$, where $x = s_1$, for some $z$. This is also a contradiction. Hence, we may assume that the linear system for $j = 5$ is non-singular. Hence, there is a unique solution $(\sigma_1, \sigma_2)$.

We conclude that there exists a unique solution with $j = 5$.  

Case $j = 4$. The first equation, $s_2 + s_3 + s_5 = 0$, is a necessary condition on the syndromes for this case to be possible. The second equation yields a relation between $a$ and $b$, more precisely, a linear relationship between the symmetric polynomials $\sigma_1$ and $\sigma_2$, which can be rewritten as $s_1\sigma_2 + s_2\sigma_1 + s_3 = 0$. Unless $s_1 = s_2 = 0$ this constraint is non-degenerate. If $s_1 = s_2 = 0$, also $s_3 = 0$. Hence $s_5 = s_2 + s_3 = 0$. Hence, $s_4$ can be the only non-zero syndrome. In this case $a$ and $b$ are arbitrary. However, $x$ and $y$ are determined to be 0, so no data drives have failed. Thus, in contradiction with our assumption, there is only one failed drive: parity drive at position $k + 4$. Hence, we may assume that either $s_1 \neq 0$ or $s_2 \neq 0$. By letting $b$ assume all $k$ values $\alpha_1$, $\alpha_2$, ..., $\alpha_k$, we determine $a$ from the linear equation $(s_1 b + s_2) a = s_3 + s_2 b$. If $s_1 = 0$, $a = s_3/s_2 + b$. If $s_1 \neq 0$ then $a = (s_3 + s_2 b)/(s_1 b + s_2)$ is unique for $b \neq s_3/s_2$. If $b = s_3/s_2$ then $a$ is $b$ and otherwise arbitrary. This leads to $2(k - 1)$ pairs $(a, b)$. Also, there is a symmetry: if $(a, b)$ is a solution, so is $(b, a)$. This symmetry shows that every solution is repeated twice in the above procedure. Thus the number of solutions for $j = 4$ is bounded by $k - 1$ in total.

Case $j = 3$. Condition of non-unique solution is that $s_1 s_5 + s_1 s_2 + s_2^2 = 0$, or $s_1 (s_2 + s_5) = s_2^2$, and $s_2 s_4 + s_2^2 + s_5^2 = 0$, or $s_2 (s_2 + s_4) = s_5^2$. Thus,

$$s_1 (s_2 + s_5) = s_2^2,$$

$$s_2 (s_2 + s_4) = s_5^2.$$

Let us suppose that $s_2 \neq 0$. Then $s_1 \neq 0$ and $s_2 + s_5 \neq 0$. Let us define $\rho = s_2/s_1 \neq 0$. Then $s_1 (s_1 \rho + s_5) = s_2^2 \rho^2$. Hence $s_1 s_5 = s_2^2 (\rho^2 + \rho)$, or $s_5 = s_1 \rho (\rho + 1)$. Furthermore,

$$s_4 = s_2^2/s_2 - s_2 = s_1^2 \rho^2 (\rho^2 + 1)/(s_1 \rho) - s_1 \rho = s_1 (\rho^3 + \rho) - s_1 \rho = s_1 \rho^3.$$

We thus have proven $s = s_1 (1, \rho, s_3/s_1, \rho^3, s_1 \rho (\rho + 1))$. This implies $s = xP^{(\rho)} + z I_3$ where $x = s_1$ and $z = s_3 - x \rho^3$. This means that $s$ matches a solution with just two drives failed, which is a contradiction.

Let us suppose that $s_2 = 0$. Then $s_1 s_5 = 0$ and $s_1 s_4 = 0$. If $s_1 \neq 0$ then $s_2 = s_5 = s_4 = 0$ and $s$ has weight $\leq 2$ which is consistent with 2 parity drive failure. So $s_1 = s_2 = 0$. Also $s_5 = 0$ and $s_4 = 0$. Thus $s = 0$, which is consistent with no drive failing, which is again a contradiction.

The solution for $j = 3$ is therefore unique.

Case $j = 2$. The linear system for $\sigma_1$ and $\sigma_2$ is singular iff $s_1 s_2 - (s_3 + s_5)^2 = 0$. Moreover, the coefficients of the last equation simultaneously vanish iff $s_2^2 + s_3 (s_3 + s_1) = 0$ and $s_4 s_5 + s_3 s_4 + s_3^2 = 0$. This system is equivalent to

$$s_3 (s_1 + s_3) = s_5^2,$$

$$s_3 (s_3 + s_4) = s_4 s_5.$$

Let us suppose $s_5 \neq 0$. Then $s_3 \neq 0$ and $s_1 + s_3 \neq 0$. If $s_4 = 0$ then the second equation yields $s_3^2 = 0$, which implies $s_3 = s_4 = 0$. Hence, $\sigma_2 s_5 = 0$ by the third equation in Table 2 But $\sigma_2 \neq 0$, as only $a, b \neq 0$ are solutions. Therefore $s_5 = 0$, which contradicts our assumption that $s_5 \neq 0$. Hence $s_4 \neq 0$. This implies $s_3 \neq 0$ and $s_3 + s_4 \neq 0$. If $s_1 = 0$ then the first equation implies $s_3^2 = s_5^2$, and thus $s_3 = s_5$ (we used Frobenius identity). Last second equation yields $s_3 + s_4 = s_4$, i.e. $s_3 = 0$, which would be a contradiction. Hence, $s_1 \neq 0$. This implies that $s_1, s_3, s_4, s_5 \neq 0$. Let $\rho = s_4/s_3 \neq 0$. Then $s_3 (s_3 + s_3 \rho) = s_3 \rho s_5$. Thus $s_5 = s_3 (1 + \rho^{-1})$. Also $s_3 (s_1 + s_3) = s_5^2 = s_3^2 (1 + \rho^{-2})$. Hence, $s_1 + s_3 = s_3 (1 + \rho^{-2})$ and
s_1 = s_3 \rho^{-2}$. Thus $s_3 = s_1 \rho^2$. Hence $s = s_1(1, s_2/s_1, \rho^2, \rho^3, \rho(\rho+1))$. Again, there is a solution to $xP(\rho)+zI_2 = 0$, which is a contradiction.

Let us suppose $s_5 = 0$. But then the system becomes

$$
\begin{align*}
    s_3^2 &= s_1 s_3, \\
    s_3 s_4 &= s_2^2.
\end{align*}
$$

If $s_3 \neq 0$ (in addition to $s_5 \neq 0$) then $s_1, s_4 \neq 0$. Again, we define $\rho = s_4/s_3 \neq 0$ and obtain $s_1 = s_3^2/s_4 = s_3/\rho^2$. So $s = s_1(1, s_2/s_1, \rho^2, \rho^3, 0)$. But also $s_3 = s_1$ which implies $\rho^2 = 1$. Hence, $\rho = 1$. This implies that $s = xP(1)+zI_2$ has a solution, which is a contradiction.

If $s_3 = 0$ (and $s_5 = 0$ by assumption) then $s_4 = 0$ by the third equation in Table 2. But then $s$ has weight at most 2, and it matches 2 failed parity drives, which contradicts our assumptions.

We proved that all degenerate cases come from syndrome vectors which match failure of fewer than 3 drives.

The solution for $j = 2$ is therefore unique.

**Case** $j = 1$. In this case, equation $s_5 = s_2 + s_3$ is required for the solution to exist. The second equation yields a linear constraint on $(\sigma_1, \sigma_2)$:

$$
    s_2 \sigma_2 + s_3 \sigma_1 + s_4 = 0.
$$

This constraint is consistent and non-trivial unless $s_2 = s_3 = s_4 = 0$. In this case, also $s_5 = s_2 + s_3 = 0$. Hence $s = s_1(1, 0, 0, 0, 0)$. But this means $s = s_1 I_1$, i.e. $s$ is consistent with one parity drive failure, which is a contradiction.

Therefore, we may assume that the constraint is non-trivial and the set of admissible pairs $(\sigma_1, \sigma_2)$ forms a 1-dimensional linear subspace of $\mathbb{F}^2$. Moreover, since $(\zeta - a)(\zeta - b) = \zeta^2 - \sigma_1 \zeta + \sigma_2 = 0$, for every admissible pair $(\sigma_1, \sigma_2)$ there are at most 1 solutions $(a, b)$, up to swapping $a$ and $b$. Hence, the total number of solutions is $\leq 1$. A sharper estimate is obtained by considering setting $b = \alpha_i$, $i = 1, 2, \ldots, k$. For fixed $b$, the constraint is $s_2 (a b) + s_3 (a + b) + s_4 = 0$ or $(s_2 b + s_3) a + s_4 = 0$. If $b = s_3/s_2$ and $s_4 = 0$ then $a$ is not equal to $b$ and otherwise arbitrary, yielding $k - 1$ possible solutions. If $b \neq s_3/s_2$, $a = s_4/(s_2 b + s_3)$ is unique. Therefore, the number of pairs $(a, b)$ of this type is again $k - 1$. Again, due to symmetry, we can eliminate half the pairs $(a, b)$ The total number of solutions of $j = 1$ is thus not greater than $k - 1$.

Finally, the total number of solutions with fixed $s$, which may match all of the cases, is bounded by:

$$
2(k - 1) + 1 + 5 + 3 \cdot 1 = 2k + 4.
$$

The techniques of this proof are easily implemented as a collection of algorithms. The top-level algorithm is Algorithm 14. This algorithm can be invoked after trying all 2-disk failures against the syndrome vector $s$, to create a list of all matching error vectors of weight $\leq 3$, except for a simultaneous failure of 3 data drives.

**Remark** 5. How can Theorem 10 be applied?

Recovery of data based on Theorem 10 will vary depending on the drive characteristics (note that “drive” is used in a broad sense, to mean any kind of storage device or even a communications channel).
Example 3 (Recovery from a 3-drive failure). Since Theorem 10 yields a list of triples suspected of failure, we can save the relevant data and write new data to the suspected locations. Assuming that failed drives are the triple \((i, j, l)\), this triple will be repeated for each syndrome obtained in the course of this experiment. If one assume that can repeatedly generate errors from the failing drives, we will be able to find the failed drives by an elimination process.

10. Degraded Modes

When some drives are removed from a RAID array, we say that the array is in “degraded mode”. For example RAID 6 can operate with two drives removed, but it loses all error detecting and correcting capabilities. It is implied that the locations of the removed drives are known. This modifies the recovery problem. We are interested in a maximum likelihood algorithm, which seeks a solution with the highest likelihood, which is equivalent to giving priority to solutions with a smaller number of failed drives. The \textit{a priori} knowledge that some drives are missing changes the order in which the solutions are presented. It is assumed that parities are calculating without the missing drives, or equivalently, the data from the missing drives is replaced with zeros. Loosely speaking, we treat the missing drives as having been erased.

The algorithm based on the parity check matrix (2) creates a number of special cases, depending on the number and role (parity or data) of the missing drives.

10.1. A method to handle \textbf{combinations of erasures and errors}. We recall that equations (18) and (19) can be used to perform syndrome decoding. We will further develop notations helpful in describing decoding of arbitrary combinations of erasures at known locations and errors at unknown locations.

Let \(I \subseteq 1, 2, \ldots, k\) and \(J \subseteq 1, 2, \ldots, 5\) by any subsets satisfying

\[
|I| + |J| \leq 5
\]

where \(I \cup J\) is the set of locations of known erasures. It is easy to see that all solutions to the equation \(H \cdot e = s\) are obtained by first solving

\[
P_{J,I} e_I = s_J
\]

where \(e_I\) is the vector obtained from the full error vector by keeping entries in the set \(I\), \(P_{J,I}\) is obtained from the parity matrix \(P\) by keeping only columns with indices in the set \(I\) and deleting rows with indices in the set \(J\), and \(s_J\) is the vector obtained from the syndrome vector \(s\) by deleting entries in the set \(J\). In short, equations with indices in the set \(J\) are deleted. After solving equation (30) we construct \(e\) by setting \(e_i = 0\) for \(i \notin I\), \(1 \leq i \leq k\). Entries \(e_{k+j}\) are then uniquely determined by the \(j\)-th equation of the system \(H \cdot e = s\), which reduces to:

\[
e_{k+j} = s_j - \sum_{i \in I} p_{j,i} e_i, \quad j \in J.
\]

The matrix equation (30) is a system of \textit{non-linear} equations for the error locators \(\rho_i\), and the error values \(x_i, i \in I\). It will be analyzed by methods of algebra. Some of the systems are hard enough to require advanced methods, such as Gröbner basis calculations and elimination theory. Sometimes the calculations are lengthy enough to be performed with the aid of a Computer Algebra System (CAS). In our calculations we used the free, open source CAS.
Maxima [4], containing a Gröbner basis package written by one of the authors of this paper (Rychlik). Let us write down the explicit form of system (30):

\[
\begin{pmatrix}
  p_{j_1,i_1} & p_{j_1,i_2} & \cdots & p_{j_1,i_r} \\
  p_{j_2,i_1} & p_{j_2,i_2} & \cdots & p_{j_2,i_r} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{j_t,i_1} & p_{j_t,i_2} & \cdots & p_{j_t,i_r}
\end{pmatrix}
\begin{pmatrix}
  e_{i_1} \\
  e_{i_2} \\
  \vdots \\
  e_{i_r}
\end{pmatrix}
=
\begin{pmatrix}
  s_{j_1} \\
  s_{j_2} \\
  \vdots \\
  s_{j_t}
\end{pmatrix}
\]

(31)

where \( P = [p_{j,i}] \) is the parity matrix, \( r = |I|, \ t = 5 - |J|, \) and

\[
\{i_1, i_2, \ldots, i_r\} = I, \\
\{j_1, j_2, \ldots, j_t\} = \{1, 2, \ldots, 5\} \setminus J.
\]

Clearly, this notation generalizes any systematic code with 5 parities, and can be further generalized to any number of parities. Matrix equation (31) can also be written using summation notation:

\[
\sum_{\mu=1}^{r} p_{j_\mu,i_\mu} e_{i_\mu} = s_{j_\mu}, \quad \nu = 1, 2, \ldots, t.
\]

(32)

When \( P \) is an algebraic function of locators \( \rho_i, \ i = 1, 2, \ldots, k \) of data, this is an algebraic system. Our matrix is explicitly an algebraic function of locators, and thus system (32) is thus explicitly a polynomial system of equations. It should be noted that any function on a vector space \( \mathbb{F}^k \) with values in \( \mathbb{F} \) has a polynomial representation, and thus the method of reducing the decoding problem to a system of polynomial equations is universally applicable to all linear, systematic codes, and even more general classes of codes. However, the computational complexity of the decoding algorithm depends on the algebraic complexity of the system (this term used loosely, as there is no rigorous, universal notion of algebraic complexity). We recall that for the code given by (2) we have

\[
p_{j,i} = \begin{cases} 
\alpha_i^{j-1}, & j = 1, 2, 3, 4, \\
\alpha_i(\alpha_i + 1), & j = 5.
\end{cases}
\]

(33)

resulting in a system (32) of total degree at most 4 in variables \( \rho_\mu = \alpha_{i_\mu} \) (error locators) and \( x_\mu = e_{i_\mu} \) (error values), for \( i = 1, 2, \ldots, k \). By methods of elimination theory, solving these systems reduces to solving polynomial equations in 1 variable. The practical implementation of elimination theory in computational algebraic geometry is provided by Gröbner basis [2].

It should be noted that unknown location parity errors add discrete variables \( j \) in the range \( 1 \leq j \leq 5 \). They cannot be handled by algebraic methods, or at least are inconvenient to handle. However, we may simulate such errors by branching (as in branch-and-bound) on all possible values.

It will be generally advantageous for given \( |J| = t \) to consider the maximum possible \( r = |I| \) for which a unique solution of (32) exists, because smaller \( r \) are special cases obtained by setting some \( x_\mu \) to 0 and thus are a part of the analysis for the maximum \( r \).

Also, it should be noted that there is one case when (32) is linear, namely when all data errors are known erasures. In this case all error locators are known and we solve a linear system for the error values. As this is done by the standard methods of linear algebra, it should be considered relatively easy.
Generally, error locators for fixed $r$ are treated identically, and thus it is beneficial to re-write (32) in terms of the elementary symmetric polynomials of the error locators in order to lower the degree of the system.

10.2. One parity drive missing. If $j$, $1 \leq j \leq 5$, is the index of the missing parity drive then the equation $s = H \cdot e$ is analyzed by eliminating the variable $e_{k+j}$, which reduces to the equation

$$P^j \cdot e^j = s^j$$

in which the superscript means that row $j$ of the corresponding matrix has been erased. The above equation involves only data drives. Two variables are associated with every drive (the locator $\rho_i$ and the error value $x_i$, $i = 1, 2, \ldots, k$). Since we have 4 equations, we can in principle accommodate two failed data drives. Thus we consider the equation

$$x \, P^{(u)} + y \, P^{(v)} = s$$

using the notation introduced by (4). The details of recovery depend on which parity drive is missing.

$j = 1$. The system of equations in this case is:

$$
\begin{pmatrix}
  u & v \\
  u^2 & v^2 \\
  u^3 & v^3 \\
  u(u+1) & v(v+1)
\end{pmatrix}
\cdot
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
s_2 \\
s_3 \\
s_4 \\
s_5
\end{pmatrix},
$$

In order for this system to be consistent, we have to have $s_2 + s_3 = s_5$. Then the last equation is dependent and can be discarded, resulting in:

$$
\begin{pmatrix}
u & v \\
u^2 & v^2 \\
u^3 & v^3
\end{pmatrix}
\cdot
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
s_2 \\
s_3 \\
s_4
\end{pmatrix}.
$$

With the aid of a CAS, we obtain the system for $u$ and $v$ alone:

$$s_2 u v + s_3 (u + v) + s_4 = 0.$$

We use the Vieta substitution $\sigma_1 = u + v$ and $\sigma_2 = u \cdot v$. We can write the above equation as a linear relationship between $\sigma_1$ and $\sigma_2$:

$$s_3 \sigma_1 + s_2 \sigma_2 = s_4.$$

Note that if this relationship is trivial and consistent then $s_2 = s_3 = s_4 = 0$ and $x = y = 0$ is a solution. As $s_2 + s_3 = s_5$, also $s_5 = 0$. Hence, $s$ has weight $\leq 1$ and it has a solution with no missing data drives, and this is the solution with maximum likelihood.

If $u$ is known then

$$(s_2 u + s_3)v = s_3 u + s_4,$$

Given that $(s_2 \cdot u + s_3) \neq 0$, we have a unique solution

$$v = \frac{s_3 u + s_4}{s_2 \cdot u + s_3}.$$
We would like to emphasize that uniqueness does not mean existence. The equation is inconsistent if \( s_3 u + s_4 \neq 0 \) and \( s_2 \cdot u + s_3 = 0 \). Therefore, non-uniqueness is only possible when

\[
\begin{align*}
\text{either} & \quad s_2 u + s_3 = 0, \\
\text{or} & \quad s_3 u + s_4 = 0.
\end{align*}
\]

In particular \( u = \frac{s_3}{s_2} = \frac{s_4}{s_3} \), or \( s_2 s_4 = s_2^2 \). This last equation is another necessary condition to have a solution with parity \( j = 1 \) missing, and a data drive missing whose locator is \( u = \frac{s_3}{s_2} = \frac{s_4}{s_3} \). In particular \( s_2, s_3, s_4 \neq 0 \). If this condition is satisfied then \( x \) and \( y \) are found by linear algebra:

\[
\begin{align*}
x &= \frac{s_2 u + s_3}{u(u + v)}, \\
y &= \frac{s_2 u + s_3}{v(u + v)}.
\end{align*}
\]

This works when \( u, v \neq 0 \) and \( u \neq v \), all of which can be assumed.

Hence, two-data recovery (in addition to missing parity, for a total of three failed drives) does not work (i.e. result in a unique solution), unless we have another missing data drive. Hence, with only one drive missing, we can only recover one data drive. The condition

\[
\begin{align*}
s_2 + s_3 &= s_5,
\end{align*}
\]

serves as a parity check. We can solve the equation with one failed data drive:

\[
x P^{(u)} = s
\]

without using component \( j = 1 \), which leads to

\[
x \cdot \begin{pmatrix} u \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} s_2 \\ s_3 \\ s_4 \end{pmatrix}.
\]

In particular, if \( x \neq 0 \) then \( s_l \neq 0 \) for \( l = 2, 3, 4 \) and:

\[
\begin{align*}
u &= \frac{s_3}{s_2}, \\
x &= \frac{s_2}{u}.
\end{align*}
\]

\( j = 2 \). The system is

\[
\begin{pmatrix}
1 & 1 \\
\frac{1}{u^2} & \frac{1}{v^2} \\
\frac{1}{u^3} & \frac{1}{v^3} \\
\frac{1}{u(u + 1)} & \frac{1}{v(v + 1)}
\end{pmatrix}
\begin{pmatrix}
x \\ y
\end{pmatrix}
= \begin{pmatrix}
s_1 \\ s_3 \\ s_4 \\ s_5
\end{pmatrix}.
\]

Elimination using CAS produces these equations, with \( \sigma_1 = u + v \) and \( \sigma_2 = u \cdot v \) being the symmetric polynomials:

\[
\begin{align*}
\sigma_1 s_5 + \sigma_2 (s_1 + s_3 + s_5) &= s_4 + s_3, \\
\sigma_2 (s_5^2 + s_3^2 + s_1 s_3) &= s_4 s_5 + s_3 s_4 + s_3^2, \\
\sigma_1 s_3 + \sigma_2 (s_5 + s_3) &= s_4.
\end{align*}
\]

(34)
This system has a unique solution, up to exchanging $u$ and $v$, unless

\[
\begin{align*}
& s_5^2 + s_3^2 + s_1 s_3 = 0, \\
& s_4 s_5 + s_3 s_4 + s_3^2 = 0
\end{align*}
\]

Simplifying:

\[
\begin{align*}
& (s_5 + s_3)^2 = s_1 s_3, \\
& s_4 (s_5 + s_3) = s_3^2
\end{align*}
\]

We note that if $s_1 = 0$ then $s_5 + s_3 = 0$ and also $s_3 = 0$, and $s_5 = 0$. This leaves $s_4$ arbitrary. The system (34) reduces to

\[
\begin{align*}
& 0 = s_4, \\
& 0 = 0, \\
& 0 = s_4.
\end{align*}
\]

This implies that all syndromes are 0, which has a solution with no errors, which is always most likely. Hence, we may assume that $s_1 \neq 0$. If $s_3 = 0$ then $s_3 + s_5 = 0$ (assuming $s_1 \neq 0$), and thus $s_5 = 0$. This leaves $s_1$ and $s_4$ arbitrary. The system (34) reduces to

\[
\begin{align*}
& \sigma_2 s_1 = s_4, \\
& 0 = 0, \\
& 0 = s_4.
\end{align*}
\]

Therefore $s_4 = 0$. But then $\sigma_2 = 0$, so either $u = 0$ or $v = 0$. But this is an invalid locator, so it is rejected. Hence, we assume $s_1 \neq 0$ and $s_3 \neq 0$. Also, $s_4 \neq 0$ and $s_3 + s_5 \neq 0$, i.e. $s_3 \neq s_5$. Eliminating $s_5 + s_3$ we obtain $(s_3^2/s_4)^2 = s_1 s_3$ or $s_3^2 = s_1 s_4^2$. We notice that under the degeneracy condition the first and third equation of system (34) form a singular linear system (by checking the determinant is 0). Hence, the condition indeed yields non-unique solution.

Finally, we obtain the unique solution

\[
\begin{align*}
& \sigma_1 = s_3 (s_3 + s_5) + s_1 s_4, \\
& \sigma_2 = s_4 (s_3 + s_5) + s_3^2, \\
& (s_3 + s_5)^2 + s_1 s_3
\end{align*}
\]

given that $s_3 \neq 0$ and $s_1 s_3 \neq (s_3 + s_5)^2$. As usual, $u$ and $v$ are the roots of the quadratic equation

\[
\zeta^2 - \sigma_1 \zeta + \sigma_2 = 0.
\]

( Again, if another missing drive is known then one of the roots is known and the solution is unique. We will use this fact later on.)

\[
\begin{align*}
& j = 3. \text{ The system in this case is:} \\
& \begin{pmatrix}
& 1 & 1 \\
& u & v \\
& u^3 & v^3 \\
& u(u + 1) & v(v + 1)
\end{pmatrix}
& \begin{pmatrix}
& x \\
& y
\end{pmatrix}
& =
\begin{pmatrix}
& s_1 \\
& s_2 \\
& s_4 \\
& s_5
\end{pmatrix}.
\end{align*}
\]
Using similar methods as in other cases, we obtain the unique solution
\[
\sigma_1 = s_2(s_2 + s_5) + s_1 s_4 \overline{s_2} s_2 + s_1 s_5 + s_2^2, \\
\sigma_2 = (s_2 + s_5)^2 + s_2 s_4 \overline{s_1 s_2 + s_5} + s_2^2 \overline{s_2}.
\]
subject to the condition \( s_1(s_2 + s_5) + s_2^2 \neq 0 \).

\( j = 4 \). The system in this case is:
\[
\begin{pmatrix}
1 & 1 \\
\overline{u} & \overline{v} \\
\overline{u^2} & \overline{v^2} \\
\overline{u(u+1)} & \overline{v(v+1)}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
\overline{s_1} \\
\overline{s_2} \\
\overline{s_3} \\
\overline{s_5}
\end{pmatrix}.
\]
In this case, we have a solvability condition \( s_1 + s_2 = s_5 \). The last linear equation drops out, yielding:
\[
\begin{pmatrix}
1 & 1 \\
\overline{u} & \overline{v} \\
\overline{u^2} & \overline{v^2}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
\overline{s_1} \\
\overline{s_2} \\
\overline{s_3}
\end{pmatrix}.
\]
and the condition on \( \sigma_1, \sigma_2 \):
\[
s_2 + \sigma_1 s_2 + s_1 \sigma_2 = 0
\]
This equation is only useful assuming that we know one of the two locators \( u \) and \( v \), i.e. that there is another missing disk.

\( j = 5 \). The system in this case is:
\[
\begin{pmatrix}
1 & 1 \\
\overline{u} & \overline{v} \\
\overline{u^2} & \overline{v^2} \\
\overline{u^3} & \overline{v^3}
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
\overline{s_1} \\
\overline{s_2} \\
\overline{s_3} \\
\overline{s_4}
\end{pmatrix}.
\]
The analysis yields the unique solution:
\[
\sigma_1 = s_1 s_4 + s_1 s_3 \overline{s_1 s_3 + s_2^2}, \\
\sigma_2 = s_2 s_4 + s_3^2 \overline{s_1 s_3 + s_2^2}.
\]
This is subject to the condition: \( s_1 s_3 + s_2^2 = 0 \).

**11. Error Correcting Capabilities for 4 Failed Drives**

The method is essentially the same as for 3 disks, so we quickly get to the point, by establishing notation and analyzing the systems of algebraic equations covering all cases. We note that the inequality \( Z + 2 E \leq 4 \) when \( Z = 4 \), does not allow any errors at unknown locations. Therefore, the positions of all failed drives are assumed to be known. The problem of finding error values is then a linear problem, and all ingredients to solving it are now available in the proof of Proposition [1]. It should be noted that our code uses quintuple parity, which means that with 4 known erasures the code has still an error detecting capability, roughly equivalent to 1 parity check.
11.1. **One parity drive missing.** Five systems of equations are obtained from the general system involving data error locators \((u_1, u_2, u_3)\) and data error values \((x_1, x_2, x_3)\), by starting with the basic system

\[
\begin{pmatrix}
1 & 1 & 1 \\
u_1 & u_2 & u_3 \\
u_1^2 & u_2^2 & u_3^2 \\
u_1^3 & u_2^3 & u_3^3 \\
u_1(u_1 + 1) & u_2(u_2 + 1) & u_3(u_3 + 1)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5
\end{pmatrix}.
\]

We consider subsystems obtained by deleting one equation, which is an overdetermined system with 4 equations. We know that the coefficient matrix after deletion of a row has a 3 × 3 submatrix which is non-singular, thus has rank 3 (see proof of Proposition 1 and Theorem 5). Hence, the consistency condition is that the 4 × 4 augmented matrix has rank 3, i.e. the determinant is 0. Hence, for \(j = 1, 2, 3, 4, 5\) we have a single polynomial which is the sufficient condition of consistency. We thus require for a parity disk \(j\) to be the failed parity that the 4 × 4 minors of the matrix below be singular:

\[
\begin{pmatrix}
1 & 1 & 1 & s_1 \\
u_1 & u_2 & u_3 & s_2 \\
u_1^2 & u_2^2 & u_3^2 & s_3 \\
u_1^3 & u_2^3 & u_3^3 & s_4 \\
u_1(u_1 + 1) & u_2(u_2 + 1) & u_3(u_3 + 1) & s_5
\end{pmatrix}
\]

One way to find the 4 polynomials is to form a matrix by adding a column of indeterminates \(w_j\), \(j = 1, 2, 3, 4, 5\), and considering the determinant:

\[
\begin{vmatrix}
1 & 1 & 1 & s_1 & w_1 \\
u_1 & u_2 & u_3 & s_2 & w_2 \\
u_1^2 & u_2^2 & u_3^2 & s_3 & w_3 \\
u_1^3 & u_2^3 & u_3^3 & s_4 & w_4 \\
u_1(u_1 + 1) & u_2(u_2 + 1) & u_3(u_3 + 1) & s_5 & w_5
\end{vmatrix}
\]

Then the polynomial equivalent to consistency with parity \(j\) error is the coefficient at \(w_j\) in the above determinant. Moreover, the coefficients are symmetric functions of \(u_1, u_2, u_3\) and as such can be expressed in terms of elementary symmetric polynomials.

**Missing parity \(j = 1\).** With the aid of CAS, we obtain the coefficient at \(w_1\):

\[
u_1 u_2 (u_2 - u_1) u_3 (u_3 - u_1) (u_3 - u_2) (s_5 - s_3 - s_2)
\]

Apparently, it is 0 only if

\[
s_2 + s_3 + s_5 = 0.
\]

**Missing parity \(j = 2\).** In this case, coefficient at \(w_2\) is:

\[-(u_2 - u_1) (u_3 - u_1) (u_3 - u_2)
\]

\[(u_2 u_3 s_5 + u_1 u_3 s_5 + u_1 u_2 s_5 + s_4 - u_2 s_3 u_3 - u_1 s_3 u_3 - s_3 u_3 - u_1 u_2 s_3 - u_2 s_3 - u_1 s_3)
\]

Only the last factor contributes a non-trivial condition (after rewriting in terms of the elementary symmetric polynomials \(\sigma_1 = u_1 + u_2 + u_3\) and \(\sigma_2 = u_1 u_2 + u_1 u_3 + u_2 u_3\)):

\[
\sigma_2 (s_5 + s_3) + s_4 + \sigma_1 s_3 = 0
\]
Missing parity \( j = 3, 4, 5 \). In these cases, the coefficient at \( w_j \) is 0, i.e. the existence and uniqueness is automatic.

Table 3. A table of equations related to decoding 4 errors, where 3 are data errors and 1 is a parity error. The listing of conditions on the syndromes \( s_j \) and data error locators \( u_1, u_2 \) and \( u_3 \) to have a solution with one parity error at position \( j \), written in terms of the elementary symmetric polynomials \( \sigma_1 = u_1 + u_2 + u_3 \) and \( \sigma_2 = u_1 u_2 + u_1 u_3 + u_2 u_3 \).

| \( j \) | Consistency condition |
|-------|----------------------|
| 1     | \( s_2 + s_3 + s_5 = 0 \) |
| 2     | \( \sigma_2 (s_5 + s_3) + s_4 + \sigma_1 s_3 = 0 \) |
| 3, 4, 5 | Empty |

11.2. Two parity drives missing. Following the method for a single missing parity, we consider a determinant:

\[
\begin{vmatrix}
1 & 1 & s_1 & w_1 & z_1 \\
u_1 & u_2 & s_2 & w_2 & z_2 \\
u_2 & u_2 & s_3 & w_3 & z_3 \\
u_3 & u_3 & s_4 & w_4 & z_4 \\
u_1(u_1 + 1) & u_2(u_2 + 1) & s_5 & w_5 & z_5
\end{vmatrix}
\]

A solution to the equation \( He = s \) exists with data error locations given by data error locators \( u_1, u_2 \), with parity errors at positions \( j \) and \( l \), iff the coefficient at \( w_j z_l \) of the above polynomial is 0. These coefficients are listed in Table 4. It should be noted that all consistency conditions are equations which are either linear or quadratic in \((u_1, u_2)\) (the exception is pair \((1, 4)\) which is never satisfied; the equation is \(1 = 0\)). Therefore, if only one of the data disks is a known erasure, these equations limit the second data disk to at most 2 positions, which provides a viable method to repair RAID with 3 erasures and 1 failure at unknown location.

What is important about the degeneracy condition in the fourth column of the table is that only when the syndrome vector satisfies this condition the equation in the second column degenerates enough to allow the possibility of more than 2 solutions. It is clear that the consistency equation when treated as function of \( u_1 \) is a quadratic equation, and only when all coefficients of it are 0 the degeneracy occurs. By comparing with Table 2 we can see that the degeneracy conditions are the consistency conditions for that case (except for parity pair \((4, 1)\) which is never consistent with 4 errors, 2-parity). Hence, Table 4 does not allow more than 2 combinations of data errors, where only 1 data error is at a known location. If the degeneracy condition is satisfied, there is a 2 data, 1 parity error consistent with the syndromes, which is more likely.

Appendix A. Additional properties

We formulate several results without a proof, which address several specific situations which may occur when more than 2 drives fail. There are cases where recovery is possible. In other cases, we cannot recover the content of lost drives. Our results are summarized in Table 5.
The degeneracy condition in the last column is a condition for the equation expressing consistency to have more than 2 solutions.

| $j$ | $l$ | Consistency condition | Degeneracy condition |
|-----|-----|------------------------|----------------------|
| 1   | 2   | $\sigma_2 (s_5 + s_3) + s_4 + \sigma_1 s_3$ | $s_4 s_5 + s_3 s_4 + s_2^2$ |
| 1   | 3   | $\sigma_1 (s_5 + s_2) + s_4 + s_2 \sigma_2$ | $s_5^2 + s_2 s_4 + s_2^2$ |
| 1   | 4   | 1                       | 1                    |
| 1   | 5   | $s_4 + \sigma_1 s_3 + s_2 \sigma_2$ | $s_2 s_4 + s_3^2$ |
| 2   | 3   | $\sigma_2 s_5 + \sigma_1^2 s_5 + \sigma_1 s_4 + s_4 + s_1 \sigma_1^2 + s_1 \sigma_1 \sigma_2$ | $s_4^3 + s_1 s_4 s_5 + s_1 s_3^2 + s_1^2 s_4$ |
| 2   | 4   | $\sigma_1 (s_5 + s_3) + s_3 + s_1 \sigma_2$ | $s_5^2 + s_3^2 + s_1 s_3$ |
| 2   | 5   | $\sigma_1 s_4 + \sigma_2 s_3 + \sigma_1^2 s_3 + s_1 \sigma_1 \sigma_2$ | $s_1 s_4^2 + s_3^2$ |
| 3   | 4   | $s_5 + \sigma_1 \sigma_2 + \sigma_1 s_2 + s_2$ | $s_1 s_5 + s_3^2 + s_1 s_2$ |
| 3   | 5   | $s_4 + s_2 \sigma_2 + \sigma_1 \sigma_2 + \sigma_1^2 s_2$ | $s_1^2 s_4 + s_3^2$ |
| 4   | 5   | $s_3 + \sigma_1 s_2 + \sigma_1 s_2$ | $s_1 s_3 + s_2^2$ |

It should be noted that our primary algorithm, Algorithm 5, searches for the error vector $e$ of minimum weight, matching given syndrome vector $s$. Given that the probability of an individual disk failure is sufficiently low, this leads to maximum likelihood decoding, where most likely errors are given priority over less likely errors. This results in a unique solution if the number of failed drives is not more than 2. If the number reaches 3, it may happen that there is an error vector $e$ of weight 3, but this vector may not be unique. Table 5 identifies situations in which it is possible to identify most likely error vectors $e$ by an algebraic procedure based on solving a linear system of type (10), but there may be many choices of columns of $H$ which result in equally likely solutions. The idea of a decoder producing many solutions in descending order of likelihood is that of list decoding [10]. Thus, Table 5 is helpful in constructing a list decoder. Of course, a list decoder can be based on brute force search, which always works, but it has expensive exponential run time.

**Remark 6 (On non-linear nature of list decoding).** It is worth noting that a list decoder must find solutions to systems of algebraic equations. In fact, we had to solve a non-linear system of algebraic equations in order to find the locations of the failed drives. It is an important observation in this paper that this can be done by solving a quadratic equation for a particular code given by parity check matrix [2].

**Appendix B. Intellectual property status disclosure**

An earlier version of this paper was submitted on January 24, 2017 to USPTO with a provisional patent application (application number: 62/449,920); and on January 19, 2018 to USPTO with a full patent application (International application number: PCT/US18/14420).

PentaRAID™ is the trademark used by Xoralgo Inc., a company formed by the authors in collaboration with the University of Arizona, to pursue commercial implementations of the technology based on the research described in the current paper.
Table 5. We describe various situations in which recovery of data may be possible.

| Failed Drive # | Analysis of recovery options | Recoverable? |
|----------------|------------------------------|--------------|
| Data | Parity | | |
| 5 | 0 | There is no way to recover; the fifth row of $P$ is a sum of second and third row, and thus the relevant matrix is a singular matrix. | No |
| 0 | 5 | We can recover, by recomputing all parity. | Yes |
| 1 | 4 | We can recover the data drive first using the non-failed parity drive, and then recomputing the other 4 failed drives. | Yes |
| 4 | 1 | We can recover if the failed parity is the second, third, or fifth. | Yes |
| | | No way to recover if the failed parity is the first, or fourth (determinant always zero). | No |
| 2 | 3 | If both of the second and third parity are among the three failed parities, then we might be able to recover (based on the determinant). | Maybe |
| | | If at most one of the second and third parity are among the three failed parities, we can recover. | Yes |

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Algorithm 1 This algorithm yields the solution \((j, \rho, x, y)\) of the equation \(x P(\rho) + y I_j = s\), where \(P(\rho)\) is a vector given by equation (1), \(I_j\) is the \(j\)-th column of the \(5 \times 5\) and \(x, y \in \mathbb{F}\). The first input is a vector \(s \in \mathbb{F}^5\). The second input is a set (possibly empty) \(X\), consisting of cubic roots of unity which are not acceptable values of \(\rho\). If there is a (unique) solution such that \(x \neq 0\) and \(\rho \neq 0\), the quadruple \((j, \rho, x, y)\) is returned. If there is no solution with these properties, \(j = 0\) and \((0, 0, 0, 0, 0)\) is returned. Any \(\rho\) which may be otherwise be a solution, but belongs to \(X\), will not be considered a solution, and another \(\rho\) will be tried.

1: function LocateFailedParityAndData(s,X)
2:     \(j \leftarrow 0\)
3:     \(\rho \leftarrow 0\) \hspace{1cm} \triangleright Zero in \(\mathbb{F}\).
4:     \(x \leftarrow 0\) \hspace{1cm} \triangleright Zero in \(\mathbb{F}\).
5:     \(y \leftarrow 0\) \hspace{1cm} \triangleright Zero in \(\mathbb{F}\).
6:     for \(m = 1, 3\) do \hspace{1cm} \triangleright If a solution exists, \(s_1 \cdot s_2 \neq 0\) or \(s_3 \cdot s_4 \neq 0\).
7:         if \(s_m = 0\) or \(s_{m+1} = 0\) then \hspace{1cm} \triangleright Hence, if \(m = 1\), try \(m = 3\).
8:             continue
9:         end if
10:        \(\rho \leftarrow s_{m+1}/s_m\) \hspace{1cm} \triangleright A candidate for \(\rho\), \(\rho \neq 0\).
11:        \(P \leftarrow (1, \rho, \rho^2, \rho^3, \rho(\rho + 1))\) \hspace{1cm} \triangleright Note: \(P = P(\rho)\).
12:        if \(P_4 = 1\) \& \(\rho \in X\) then \hspace{1cm} \triangleright Potential \(\rho\) is an excluded root of unity.
13:            continue;
14:        end if
15:        \(x \leftarrow s_m/P_m\) \hspace{1cm} \triangleright A candidate for \(x\), \(x \neq 0\).
16:        failcount \leftarrow 0 \hspace{1cm} \triangleright Initialize failed parity drive count.
17:        for \(t = 1, 2, 3, 4, 5\), such that \(t \neq m, m+1\) do \hspace{1cm} \triangleright Consistency violation with \(s = x P\) at position \(t\).
18:            if \(s_t \neq x P_t\) then \hspace{1cm} \triangleright Increment failed parity drive count.
19:                failcount \leftarrow failcount + 1
20:                \(j \leftarrow t\) \hspace{1cm} \triangleright Make parity drive \(t\) a candidate for a failed parity drive.
21:            end if
22:        end for
23:        if \(failcount = 0\) then \hspace{1cm} \triangleright No failed parity drive.
24:            \(j \leftarrow 6\)
25:            go to 33
26:        else if \(failcount > 1\) then \hspace{1cm} \triangleright More than 1 failure, no solution.
27:            \(j \leftarrow 0\) \hspace{1cm} \triangleright Signal failure, or try another \(\rho\).
28:        else \hspace{1cm} \triangleright failcount = 1.
29:            \(y \leftarrow s_j - x \cdot P_j\) \hspace{1cm} \triangleright Parity drive \(j\) is the only inconsistent parity drive.
30:            go to 33
31:        end if
32:    end for
33:    return \((j, \rho, x, y)\)
34: end function
Algorithm 2 Recovering from a failure of one parity and one data drive at unknown locations, or a single data drive failure at unknown location. The algorithm accepts as input the set $\alpha$ of elements of $\mathbb{F}$ and a syndrome vector $s \in \mathbb{F}^5$. It finds the locations $i$ and $j$ of the failed drives, and determines the error vector $e \in \mathbb{F}^{k+5}$ such that $e_i P_i + e_{k+j} I_j = s$. Upon success, it returns $(e, \text{true})$. If no solution exists, $(0, \text{false})$ is returned. The algorithm handles correctly the case of a single data drive failure, by finding a solution to $e_i P_i = s$ and setting $e_{k+j} = 0$ for $j = 1, 2, 3, 4, 5$.

1: function RecoverFailedParityAndData($\alpha, s$)  
2: $k \leftarrow \text{NumberOfElements}(\alpha)$  
3: $e \leftarrow 0$  \hspace{1cm} $\triangleright$ 0 in $\mathbb{F}^{k+5}$.  
4: $X \leftarrow \text{ExcludedCubicRootsOfUnity}(\alpha)$  
5: $(j, \rho, x, y) \leftarrow \text{LocateFailedParityAndData}(s, X)$  
6: if $j = 0$ then  \hspace{1cm} $\triangleright s$ does not come from a single parity, single data drive failure.  
7: | return $(e, \text{false})$  \hspace{1cm} $\triangleright$ Still, $e = 0$.  
8: end if  
9: $i \leftarrow \text{Lookup}(\alpha, \rho)$  \hspace{1cm} $\triangleright$ Find $i$ such that $\alpha_i = \rho$  
10: if $i = \emptyset$ then  
11: | return $(e, \text{false})$  \hspace{1cm} $\triangleright$ Still, $e = 0$.  
12: end if  
13: $e_i \leftarrow x$  
14: if $j \leq 5$ then  
15: | $e_{k+j} \leftarrow y$  
16: end if  
17: return $(e, \text{true})$  
18: end function
Algorithm 3 This algorithm solves the equation \( x P(u) + y P(v) = s \), where \( P(u), P(v) \) are given by equation (4). The input is the syndrome vector \( s \in \mathbb{F}_5 \). If a solution exists, we return the quadruple \((u, v, x, y)\). The algorithm handles only cases requiring \( x \) and \( y \) to be non-zero, i.e. two failed data drives. It is assumed that \( s_2 + s_3 + s_5 = 0 \), as otherwise the syndrome vector would imply at least one parity drive failure. We also assume \( D, D_1, D_2 \neq 0 \), as otherwise the system (23) has been shown to be either inconsistent, or \( s = 0 \), or \( s \) comes from a single data drive failure. This algorithm utilizes formulas (27) and (28).

1: function \textsc{LocateTwoFailedDataDrives}(s)
2:   \( u \leftarrow 0 \)
3:   \( v \leftarrow 0 \)
4:   if \( s_2 + s_3 + s_5 \neq 0 \) then \( \triangleright \) This syndrome implies a failure of parity drive.
5:      go to 18 \( \triangleright \) Signal failure by returning \( u = v = 0 \).
6:   end if
7:   \( D \leftarrow s_2^2 - s_1 s_3 \)
8:   \( D_1 \leftarrow s_2 s_3 - s_1 s_4 \)
9:   \( D_2 \leftarrow s_2 s_4 - s_3^2 \)
10:  if \( D = 0 \) or \( D_1 = 0 \) or \( D_2 = 0 \) then \( \triangleright \) Not handled by this algorithm.
11:     go to 18 \( \triangleright \) Signal failure by returning \( u = v = 0 \).
12:  end if
13:  \( \sigma_1 \leftarrow D_1 / D \) \( \triangleright \sigma_1 \neq 0 \)
14:  \( \sigma_2 \leftarrow D_2 / D \) \( \triangleright \sigma_2 \neq 0 \)
15:  \( \{u, v\} \leftarrow \textsc{SolveQuadraticEquation}(1, -\sigma_1, \sigma_2) \) \( \triangleright \) Solve \( \zeta^2 - \sigma_1 \zeta + \sigma_2 = 0 \).
16:  \( x \leftarrow (v \cdot s_1 - s_2) / (v - u) \)
17:  \( y \leftarrow (s_2 - u \cdot s_1) / (v - u) \)
18:  return \( (u, v, x, y) \)
19: end function
**Algorithm 4** Recovery from a failure of two data drives at unknown locations. The input consists of the set $\alpha \subseteq \mathbb{F}$ and the syndrome vector $s \in \mathbb{F}^5$. Upon success, the algorithm returns the error vector $e \in \mathbb{F}^{k+5}$, where $k$ is the number of elements of $\alpha$. The algorithm solves the equation $e_i p_i + e_j p_j = s$, when there is a solution with $e_i, e_j \neq 0$, i.e. the syndrome comes from two failing data drives, but does not come from either $s = 0$ or a single drive failure (parity or data). If there is no solution satisfying these properties $(0, \text{false})$ is returned.

```
1: function RECOVERTWOFAILEDDATADRIVES(\alpha, s) 
2:   \begin{align*} 
3:     (u, v, x, y) &\leftarrow \text{LOCATETWOFAILEDDATADRIVES}(s) \\
4:     k &\leftarrow \text{NUMBEROFELEMENTS}(\alpha) \\
5:     i &\leftarrow \text{LOOKUP}(\alpha, u) \\
6:     j &\leftarrow \text{LOOKUP}(\alpha, v) \\
7:     e &\leftarrow 0 \quad \triangleright \text{This is } 0 \in \mathbb{F}^{k+5}. \\
8:     \text{if } i \neq \emptyset \& j \neq \emptyset \text{ then} \\
9:       e_i &\leftarrow x \\
10:      e_j &\leftarrow y \\
11:     \quad \text{return } (e, \text{true}) \\
12:   \text{else} \\
13:     \quad \text{return } (e, \text{false}) \\
14: \end{align*} 
15: end function
```
Algorithm 5 The decoding algorithm for code defined by the parity matrix \( \mathbf{H} \). The inputs are: a subset \( \alpha \) of non-zero elements of the Galois field \( \mathbb{F} \) and the received vector \( r \). The output is either \((t, \text{true})\), there \( t \) is the transmitted vector, or \((r, \text{false})\) if corruption of more than 2 drives is detected. If no more than 2 drives had an error, \( t \) is guaranteed to be correct.

1: function \( \text{RAIDDecode}(\alpha, r) \)
2: \( H \leftarrow \text{ParityCheckMatrix}(\alpha) \quad \triangleright \) Obtain parity check matrix.
3: \( s \leftarrow H \cdot r \quad \triangleright \) Compute the syndrome.
4: \( \text{lst} \leftarrow \text{FindNonZeros}(s) \quad \triangleright \) Get indices of non-zero elements of \( s \).
5: \( nz \leftarrow \text{NumberOfElements}(\text{lst}) \quad \triangleright \) Find number of non-zeros (weight of \( s \)).
6: if \( nz = 0 \) then
7: \( \quad \text{else if } nz = 1 \) then
8: \( \quad \quad i \leftarrow \text{lst}_1 \)
9: \( \quad \quad e_{k+i} \leftarrow s_i \)
10: \( \quad \) \( \text{else if } nz = 2 \) then
11: \( \quad \quad i \leftarrow \text{lst}_1 \)
12: \( \quad \quad j \leftarrow \text{lst}_2 \)
13: \( \quad \quad e_{k+i} \leftarrow s_i \)
14: \( \quad \quad e_{k+j} \leftarrow s_j \)
15: \( \) \( \text{else} \)
16: \( \quad (e, \text{status}) \leftarrow \text{RecoverFailedParityAndData}(\alpha, s) \)
17: \( \quad \text{if } \text{status} = \text{true} \) then \( \triangleright \) One parity and one data drive failed, and recovered.
18: \( \quad \quad \text{go to } 26 \)
19: \( \) \( \text{end if} \)
20: \( \quad (e, \text{status}) \leftarrow \text{RecoverTwoFailedDataDrives}(\alpha, s) \)
21: \( \quad \text{if } \text{status} = \text{true} \) then \( \triangleright \) Two data drives failed, and recovered.
22: \( \quad \quad \text{go to } 26 \)
23: \( \) \( \text{end if} \)
24: \( \) \( \text{return } (r, \text{false}) \quad \triangleright \) Return received message and signal failure.
25: \( \) \( \text{end if} \)
26: \( t \leftarrow r + e \quad \triangleright \) Compute the transmitted vector \( t \) by correcting errors in \( r \).
27: \( \text{return } (t, \text{true}) \quad \triangleright \) Return transmitted vector and signal success.
28: end function
Algorithm 6 An algorithm which produces the augmented matrix $C = [A | b]$ of the linear system $A \cdot \sigma = b$, where $\sigma = (\sigma_1, \sigma_2)$ and $\sigma_1 = u + v, \sigma_2 = u v$ are symmetric polynomials of $u$ and $v$. Galois field element $\text{cond}$ represents the extra constraint value that must be 0 for the system to be consistent. This algorithm is based on Table 2. Matrix $C$ has either 1 or 2 rows.

1: function AugmatrixForParity($s, j$)
2:     if $j = 5$ then
3:         $C \leftarrow \begin{pmatrix} s_2 & s_1 & s_3 \\ s_3 & s_2 & s_4 \end{pmatrix}$
4:         $\text{cond} \leftarrow 0$
5:     else if $j = 4$ then
6:         $C \leftarrow \begin{pmatrix} s_2 & s_1 & s_3 \\ s_5 & s_1 + s_2 & s_2 + s_4 + s_5 \end{pmatrix}$
7:         $\text{cond} \leftarrow s_2 + s_3 + s_5$
8:     else if $j = 3$ then
9:         $C \leftarrow \begin{pmatrix} s_2 & s_1 & s_2 + s_5 \\ s_5 & s_1 + s_2 & s_2 + s_4 + s_5 \end{pmatrix}$
10:        $\text{cond} \leftarrow 0$
11:     else if $j = 2$ then
12:        $C \leftarrow \begin{pmatrix} s_5 & s_1 + s_3 + s_5 & s_3 + s_4 \\ s_3 & s_3 + s_5 & s_4 \end{pmatrix}$
13:        $\text{cond} \leftarrow 0$
14:     else if $j = 1$ then
15:        $C \leftarrow \begin{pmatrix} s_3 & s_2 & s_4 \end{pmatrix}$
16:        $\text{cond} \leftarrow s_2 + s_3 + s_5$
17:     end if
18:     return $(C, \text{cont})$
19: end function
Algorithm 7 Solve the equation \( x P(u) + y P(v) + z I_j = s \). Return all solutions in vectors \( u, v, x, y, \) and \( z \). Also, return the solution count \( \text{cnt} \). If \( \text{cnt} = 0 \) then \( u = v = x = y = z = [] \) (empty vector).

1: function `LocateTwoDataForParity(\( \alpha, s, j \))`
2: \( u \leftarrow [] \); \( v \leftarrow [] \)  \hspace{1cm} \triangleright \text{Initialize to empty vectors.}
3: \( x \leftarrow [] \); \( y \leftarrow [] \); \( z \leftarrow [] \)  \hspace{1cm} \triangleright \text{Initialize to empty vectors.}
4: \( \text{cnt} \leftarrow 0 \)
5: \((C, \text{cond}) \leftarrow \text{AugmatrixForParity}(s, j)\)
6: \( \text{cnt} \leftarrow 0 \);
7: \text{if} \ \text{cond} \neq 0 \ \text{then}
8: \hspace{1cm} \text{go to 20}
9: \text{end if}
10: \text{if} \ \text{NumberOfRows}(C) = 2 \ \text{then}
11: \hspace{1cm} \{u, v\} \leftarrow \text{LocateTwoDataWhenDetermined}(C)
12: \hspace{1cm} \text{if} \ u = 0 \ & \ v = 0 \ \text{then}
13: \hspace{2cm} \text{go to 20}
14: \hspace{1cm} \text{end if}
15: \hspace{1cm} (x, y, z) \leftarrow \text{CalculateCoefficients}(u, v, s, j)\)
16: \hspace{1cm} \text{cnt} \leftarrow 1
17: \text{else if} \ \text{NumberOfRows}(C) = 1 \ \text{then}
18: \hspace{1cm} (u, v, x, y, z, \text{cnt}) \leftarrow \text{LocateTwoDataWhenUnderdetermined}(\alpha, s, j, C)\)
19: \text{end if}
20: \text{return} (u, v, x, y, z, \text{cnt})  \hspace{1cm} \triangleright \text{No solution exists}
21: \text{end function}
Algorithm 8  An algorithm implementing a helper function for Algorithm 7. This simple algorithm first solves a $2 \times 2$ linear system by Cramer’s Rule and then solves a quadratic equation to find $u$ and $v$.

1: function LocateTwoDataWhenDetermined($C$) 
2: \hspace{1em} $u \leftarrow 0; v \leftarrow 0$ \hfill $\triangleright$ Zero in Galois field. 
3: \hspace{1em} $D \leftarrow C_{11} C_{22} - C_{21} C_{12}$ 
4: \hspace{1em} $D_1 \leftarrow C_{13} C_{22} - C_{23} C_{12}$ 
5: \hspace{1em} $D_2 \leftarrow C_{11} C_{23} - C_{21} C_{13}$ 
6: \hspace{1em} if $D = 0 \text{ or } D_1 = 0 \text{ or } D_2 = 0$ then 
7: \hspace{2em} go to 18 \hfill 18: end function 
8: end if 
9: \hspace{1em} $\sigma_1 \leftarrow D_1 / D$ 
10: \hspace{1em} $\sigma_2 \leftarrow D_2 / D$ 
11: \hspace{1em} $\text{roots} \leftarrow \text{SOLVEQUADRATIC}\text{EQUATION}(1, -\sigma_1, \sigma_2)$ 
12: \hspace{1em} if $\text{roots} = \emptyset$ then 
13: \hspace{2em} $\triangleright$ In view of $\sigma_1 \neq 0$, there are two distinct roots. 
14: \hspace{2em} go to 18 
15: \hspace{2em} $u \leftarrow \text{roots}_1$ 
16: \hspace{2em} $v \leftarrow \text{roots}_2$ 
17: end if 
18: return $(u, v)$
Algorithm 9 If $C = [A | b]$ and $A = [c_1, c_2]$, $b = c_3$ then the system $A \sigma = b$ has only one equation $c_1 \sigma_1 + c_2 \sigma_2 = c_3$. We find $c_1 (u + v) + c_2 u v = c_3$, $(c_1 + c_2 v) u = c_3 - c_1 v$, $u = (c_3 - c_1 v)/(c_1 + c_2 v)$. The above formula yields unique $u$ when $v \neq -c_1/c_2$. Otherwise, $u$ is arbitrary if $c_3 - c_1 (-c_1/c_2) = c_3 - c_2^2/c_2 = 0$. Thus, $u$ is arbitrary if $c_3 c_2 = c_1^2$.

1: function LocateTwoDataWhenUnderdetermined($\alpha, s, j, c$)  
2: \hspace{1em} cnt ← 0  
3: \hspace{1em} $k \leftarrow \text{NumberOfElements}(\alpha)$  
4: \hspace{1em} for $l = 1, 2, \ldots, k$ do  
5: \hspace{2em} $v \leftarrow \alpha_l$  
6: \hspace{2em} $N \leftarrow c_3 - c_1 v$  
7: \hspace{2em} $D \leftarrow c_1 + c_2 v$  
8: \hspace{2em} if $D \neq 0 \& N \neq 0$ then  
9: \hspace{3em} $u \leftarrow N/D$  
10: \hspace{3em} if $u < v$ then  
11: \hspace{4em} $(x, y, z) = \text{CalculateCoefficients}(u, v, s, j)$;  
12: \hspace{4em} cnt $\leftarrow$ cnt + 1  
13: \hspace{4em} $uLst_{\text{cnt}} = u; vLst_{\text{cnt}} = v; xLst_{\text{cnt}} = x; yLst_{\text{cnt}} = y; zLst_{\text{cnt}} = z$  
14: \hspace{3em} end if  
15: \hspace{2em} else if $N = 0 \& D = 0$ then \ (> Now $u$ is arbitrary, the only constraint is $u \neq v$.  
16: \hspace{3em} for $u \in \alpha$ do \ (> Sort pairs to avoid duplicates.  
17: \hspace{4em} \hspace{1em} if $u < v$ then \ (> Sort pairs to avoid duplicates.  
18: \hspace{5em} \hspace{1em} $(x, y, z) = \text{CalculateCoefficients}(u, v, s, j)$  
19: \hspace{5em} \hspace{1em} cnt $\leftarrow$ cnt + 1  
20: \hspace{5em} \hspace{1em} $uLst_{\text{cnt}} = u; vLst_{\text{cnt}} = v; xLst_{\text{cnt}} = x; yLst_{\text{cnt}} = y; zLst_{\text{cnt}} = z$  
21: \hspace{4em} \hspace{1em} end if  
22: \hspace{3em} \hspace{1em} end if  
23: \hspace{2em} \hspace{1em} end for  
24: \hspace{1em} \hspace{1em} end for  
25: \hspace{1em} end function
Algorithm 10 This helper algorithm implements function \textsc{CalculateCoefficients} which finds the coefficients $x$, $y$ and $z$ for two data and one parity error. A call $(x, y, z) \leftarrow \textsc{CalculateCoefficients}(u, v, s, j)$ solves the vector equation in Galois field: $x P^u + y P^v + z I_j = s$ The arguments $u$ and $v$ must be non-zero and distinct, and $J$ must be in the range $1 \leq j \leq 5$.

1: function \textsc{CalculateCoefficients}(u,v,s,j)  
2:  \hspace{1em} \textbf{if} $j > 2$ \textbf{then} \hspace{2em} \triangleright \text{Use rows 1 and 2.}  
3:  \hspace{2em} $D \leftarrow v - u$  
4:  \hspace{2em} $x \leftarrow (s_1 v - s_2)/D$  
5:  \hspace{2em} $y \leftarrow (s_1 u - s_2)/D$  
6:  \hspace{1em} \textbf{else if} $j = 2$ \textbf{then} \hspace{2em} \triangleright \text{Use rows 1 and 3.}  
7:  \hspace{2em} $D \leftarrow (v - u)^2$  
8:  \hspace{2em} $x \leftarrow (s_1 v^2 - s_3)/D$  
9:  \hspace{2em} $y \leftarrow (s_1 u^2 - s_3)/D$  
10:  \hspace{1em} \textbf{else if} $j = 1$ \textbf{then} \hspace{2em} \triangleright \text{Use rows 2 and 3.}  
11:  \hspace{2em} $D \leftarrow u v (v - u)$  
12:  \hspace{2em} $x \leftarrow (s_2 v^2 - s_3 v)/D$  
13:  \hspace{2em} $y \leftarrow (s_2 u^2 - s_3 u)/D$  
14:  \hspace{1em} \textbf{end if}  
15:  \hspace{1em} \textbf{if} $j < 5$ \textbf{then} \hspace{2em} \triangleright \text{$j = 5$}  
16:  \hspace{2em} $z \leftarrow s_j - x u^{j-1} - y v^{j-1}$  
17:  \hspace{1em} \textbf{else}  
18:  \hspace{2em} $z \leftarrow s_j - x u (u + 1) - y v (v + 1)$  
19:  \hspace{1em} \textbf{end if}  
20: \textbf{end function}
Algorithm 11 implements recovery of three failed drives, case of 2 parity and 1 data error.
It accepts as arguments the set $\alpha \subset \mathbb{F}$, the syndrome vector $s \in \mathbb{F}^5$, the index of the broken parities $j$ and $l$ ($1 \leq j < l \leq 5$), and the set of excluded cubic roots of unity $X$. It returns the error vector $e \in \mathbb{F}^{k+5}$.

1: function RecoverDataForTwoParity($\alpha, s, j, l, X$)  
2: \hspace{1em} $k \leftarrow \text{NumberOfElements}(\alpha)$; $e \leftarrow \[]$; $i \leftarrow \[]$; $\rho \leftarrow 0$ \hspace{1em} $\triangleright 0 \in \mathbb{F}$
3: \hspace{1em} if $j = 1$ then
4: \hspace{2em} if $l = 2$ \& $s_4 s_5 + s_3 s_4 + s_3^2 = 0$ then
5: \hspace{3em} \hspace{1em} if $s_3 \neq 0$ then
6: \hspace{4em} \hspace{1em} \hspace{1em} $\rho \leftarrow s_4/s_3$
7: \hspace{3em} \hspace{1em} else if $s_5 \neq 0$ then
8: \hspace{4em} \hspace{1em} \hspace{1em} $\rho \leftarrow (s_3 + s_4)/s_5$
9: \hspace{3em} \hspace{1em} end if
10: \hspace{1em} else if $l = 3$ \& $s_5^2 + s_2 s_4 + s_2^2 = 0$ then
11: \hspace{2em} \hspace{1em} if $s_2 \neq 0$ then
12: \hspace{3em} \hspace{1em} \hspace{1em} $\rho \leftarrow (s_2 + s_5)/s_2$
13: \hspace{2em} \hspace{1em} else if $s_5 \neq 0$ then
14: \hspace{3em} \hspace{1em} \hspace{1em} $\rho \leftarrow (s_2 + s_5 + s_5)/s_5$
15: \hspace{2em} \hspace{1em} end if
16: \hspace{1em} else if $l = 4$ \& $s_2 + s_3 + s_5 = 0$ then
17: \hspace{2em} \hspace{1em} if $s_2 \neq 0$ then
18: \hspace{3em} \hspace{1em} \hspace{1em} $\rho \leftarrow s_3/s_2$
19: \hspace{2em} \hspace{1em} end if
20: \hspace{1em} else if $l = 5$ \& $s_2 \neq 0$ \& $s_2 s_4 + s_3^2 = 0$ then
21: \hspace{2em} \hspace{1em} if $s_2 \neq 0$ then
22: \hspace{3em} \hspace{1em} \hspace{1em} $\rho \leftarrow s_3/s_2$
23: \hspace{1em} else if $j = 2$ then
24: \hspace{2em} if $l = 3$ \& $(s_5^2 + s_1 s_4) s_5 + s_1 s_4 (s_1 + s_4) = 0$ then
25: \hspace{3em} \hspace{1em} if $s_1 + s_5 \neq 0$ then
26: \hspace{4em} \hspace{1em} \hspace{1em} $\rho \leftarrow (s_4 + s_5)/(s_1 + s_5)$
27: \hspace{3em} \hspace{1em} else if $s_5 \neq 0$ then \hspace{1em} $\triangleright s_1 = s_5 = s_4$ and $\rho^2 + \rho + 1 = 0$
28: \hspace{4em} \hspace{1em} \hspace{1em} if $X \neq \emptyset$ then
29: \hspace{5em} \hspace{1em} \hspace{1em} \hspace{1em} $\rho \leftarrow 1 - X_1$
30: \hspace{4em} \hspace{1em} \hspace{1em} end if
31: \hspace{3em} \hspace{1em} end if
32: \hspace{2em} else if $l = 4$ \& $(s_3 + s_5)^2 + s_1 s_3 = 0$ then
33: \hspace{3em} \hspace{1em} if $s_3 + s_5 \neq 0$ then
34: \hspace{4em} \hspace{1em} \hspace{1em} $\rho \leftarrow s_3/(s_3 + s_5)$
35: \hspace{3em} \hspace{1em} end if
36: \hspace{2em} else if $l = 5$ \& $s_1 s_4^2 + s_3^2 = 0$ then
37: \hspace{3em} \hspace{1em} if $s_3 \neq 0$ then
38: \hspace{4em} \hspace{1em} \hspace{1em} $\rho \leftarrow s_4/s_3$
39: \hspace{3em} \hspace{1em} end if
40: \hspace{1em} end if
Algorithm 12 Algorithm[11] part 2

41: else if \( j = 3 \) then
42:   if \( l = 4 \) \& \( s_1 (s_5 + s_2) + s_2^2 = 0 \) then
43:     if \( s_2 \neq 0 \) then
44:       \( \rho \leftarrow (s_2 + s_5)/s_2 \)
45:     end if
46:   else if \( l = 5 \) \& \( s_1^2 s_4 + s_2^3 = 0 \) then
47:     if \( s_2 \neq 0 \) then
48:       \( \rho \leftarrow s_1 s_4/s_2^2 \)
49:     end if
50:   end if
51: else if \( j = 4 \) then
52:   if \( l = 5 \) \& \( s_1 s_3 + s_2^2 = 0 \) then
53:     if \( s_2 \neq 0 \) then
54:       \( \rho \leftarrow s_3/s_2 \)
55:     end if
56:   end if
57: end if
58: if \( \rho = 0 \) then
59:   return
60: end if
61: \( i \leftarrow \text{LOOKUP}(\alpha, \rho) \)
62: if \( i \neq \emptyset \) then
63:   \( p \leftarrow P(\rho) \)
64:   \( \{q, t, w\} \leftarrow \{1, 2, 3, 4, 5\} \setminus \{j, l\} \) \quad \text{▷ The sorted complement of} \ \{j, l\}, \ q < t < w.
65:   \( s' \leftarrow (s_q, s_t, s_w) \)
66:   \( p' \leftarrow (p_q, p_t, p_w) \)
67:   \( x \leftarrow \frac{s_q}{p_q} \) \quad \text{▷ Prospective} \ x. \ \text{Note that} \ q < 5 \implies p_q \neq 0, \ \text{as} \ \rho \neq 0.
68:   if \( s_t = x p_t \) \& \( s_w = x p_w \) then \quad \text{▷ We indeed have a solution.}
69:     \( e \leftarrow 0 \)
70:     \( e_i \leftarrow x \)
71:     \( e_{k+j} \leftarrow s_j - x p_j \)
72:     \( e_{k+l} \leftarrow s_l - x p_l \)
73:   end if
74: end if
75: end function
Algorithm 13 Implements recovery of three failed drives, case of 1 parity and 2 data errors. Accepts as arguments the set $\alpha \subset \mathbb{F}$, the syndrome vector $s \in \mathbb{F}^5$, and the broken parity drive index $j$ ($1 \leq j \leq 5$). It returns the error matrix $e$ which has size $(k+5) \times cnt$, where $cnt$ is the count of possible error vectors which result in $s$. Thus, $H \cdot e = [s|s|\cdots|s]$ where $s$ is repeated $cnt$ times. We note that $cnt = 0$ is possible, when there is no solution meeting our specification.

1: function RecoverTwoDataForParity($\alpha, s, j$)
2:   $(u, v, x, y, z, cnt) \leftarrow$ LocateTwoDataForParity($\alpha, s, j$)
3:   $k \leftarrow$ NumberOfElements($\alpha$)
4:   $e = 0$ \hspace{1cm} $\triangleright$ A $(k+5) \times cnt$ matrix of 0 $\in \mathbb{F}$.
5:   $t \leftarrow 0$
6:   for $l = 1, 2, \ldots, cnt$ do
7:     $i_1 \leftarrow$ Lookup($\alpha, u_l$)
8:     $i_2 \leftarrow$ Lookup($\alpha, v_l$)
9:       if $i_1 = \emptyset$ or $i_2 = \emptyset$ then
10:          continue;
11:       end if
12:     $t \leftarrow t + 1$
13:     $e_{i_1,t} \leftarrow x_l$
14:     $e_{i_2,t} \leftarrow y_l$
15:     $e_{k+j,t} \leftarrow z_l$
16:   end for
17:   $e \leftarrow$ SelectColumns($e, 1, t$) \hspace{1cm} $\triangleright$ Keep only columns 1–$t$.
18:   return $e$
19: end function
Algorithm 14 Implements recovery for 3-drive failure where at least one of the drives is a parity drive. The arguments are: the set $\alpha \subset \mathbb{F}$, the syndrome vector $s \in \mathbb{F}^5$ and the set of excluded cubic roots of unity, $X$ (with 0 or 1 element). The result is a matrix $e$ whose columns are all possible error vectors satisfying our specification. Hence, this algorithm is a partial list decoder for all 3-drive errors, with a notable exception when the 3 drives are data drives.

```
1: function RECOVERTHREEFAILEDDISKS($\alpha, s, X$)
2:     $lst \leftarrow$ FINDNONZEROS($s$)
3:     $nz \leftarrow$ NUMBEROFELEMENTS($lst$)
4:     $k \leftarrow$ NUMBEROFELEMENTS($\alpha$);
5:     $e \leftarrow []$
6:     if $nz = 3$ then ▷ Find a solution with 3 parity errors.
7:         $f \leftarrow 0$ ▷ Vector of $k + 5$ zeros in $\mathbb{F}$.
8:         for $t \in lst$ do
9:             $f_{k+t} \leftarrow s_t$
10:         end for
11:         $e \leftarrow$ APPENDCOLUMNS($e, f$)
12:     end if
13:     for $j = 1, 2, 3, 4$ do ▷ Find all solutions with 2 parity errors.
14:         for $l = j + 1, j + 2, \ldots, 5$ do
15:             $f \leftarrow$ RECOVERDATAFORTWOPARITY($\alpha, s, j, l, X$)
16:             $e \leftarrow$ APPENDCOLUMNS($e, f$)
17:         end for
18:     end for
19:     return $e$
20:     for $j = 1, 2, 3, 4, 5$ do ▷ Find all solutions with 1 parity error.
21:         $f \leftarrow$ RECOVERTWODATAFORPARITY($\alpha, s, j$)
22:         $e \leftarrow$ APPENDCOLUMNS($e, f$)
23:     end for
24: end function
```