Cauchy–Dirichlet problems for a class of hypoelliptic equation in \( \mathbb{R}^d \): a new probabilistic representation formula for the gradient of the solutions

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Abstract

We are concerned with an Ornstein–Uhlenbeck process \( X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}\sqrt{C}dW(s) \) in \( \mathbb{R}^d \), \( d \geq 1 \), where \( A \) and \( C \) are \( d \times d \) matrices, \( C \) being semidefinite positive. Our basic assumption is that the matrix \( Q_t = \int_0^t e^{sA}Ce^{sA^*}ds \) is non singular for all \( t > 0 \); this implies that the corresponding Kolmogorov operator is hypoelliptic. Then we consider the stopped semigroup \( R^{\theta_r}_T \phi(x) = \mathbb{E}[\phi(X(T, x)) \mathbb{1}_{T \leq \tau_r}] \), \( T \geq 0 \) where \( \theta_r = \{ g < r \} \) is bounded, \( g \) is convex, and \( \tau_r^T = \inf\{ t > 0 : X(t, x) \in \theta_r \} \). We prove the existence and a new representation formula for the gradient of \( R^{\theta_r}_T \phi \), where \( T > 0 \) and \( \phi \) is bounded and Borel.

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1 Introduction and setting of the problem

We are here concerned with an Ornstein–Uhlenbeck process in $\mathbb{R}^d = H$, $d \in \mathbb{N}$,

$$X(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} \sqrt{C} \, dW(s), \quad x \in H, \; t \geq 0,$$

where $A$ and $C$ are $d \times d$ matrices, $C$ being symmetric and semi-definite positive. Moreover $W(t)$, $t \geq 0$, represents an $H$–valued standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $W_A$ the stochastic convolution

$$W_A(t) := \int_0^t e^{(t-s)A} \sqrt{C} \, dW(s), \quad t \geq 0.$$

Our basic assumption is the following.

Hypothesis 1. The matrix $Q_t := \int_0^t e^{sA}Ce^{sA^*} \, ds$ is non singular for all $t > 0$.

Remark 1. Hypothesis 1 arises in controllability problems for the deterministic system $D_t\xi = A\xi + \sqrt{C}u$, where $u$ is a control. See e.g. [Za92].

As well known, the transition semigroup $R_t$, $t \geq 0$, corresponding to the process $X(t, x)$ is given by

$$R_t \varphi(x) = \int_H \varphi(y)N_{e^{tA}x,Q_t}(dy), \quad t \geq 0, \; x \in H, \; \varphi \in B_b(H),$$

where $N_{e^{tA}x,Q_t}$ is the gaussian probability measure on $H$ of mean $e^{tA}x$ and covariance $Q_t$ and $B_b(H)$ denotes the space of all mappings $H \to H$ which are bounded and Borel. By Hypothesis 1 the matrix $\Lambda_t := Q_t^{-1/2}e^{tA}$ is non singular for all $t > 0$; consequently, by the Cameron–Martin Theorem it follows that $N_{e^{tA}x,Q_t} \ll N_{Q_t}$ and

$$\frac{dN_{e^{tA}x,Q_t}}{dN_{Q_t}}(y) = e^{-\frac{1}{2}||\Lambda_t x||^2_H + \langle \Lambda_t x, Q_t^{-1/2}y \rangle_H}, \quad t > 0, \; y \in H.$$

Therefore a well known representation formula for $R_t$ follows by (3),

$$R_t \varphi(x) = \int_H \varphi(y)e^{-\frac{1}{2}||\Lambda_t x||^2_H + \langle \Lambda_t x, Q_t^{-1/2}y \rangle_H}N_{Q_t}(dy), \quad t \geq 0, \; x \in H, \; \varphi \in B_b(H).$$

By (5) we can deduce that $R_t \varphi$ is differentiable infinitely many times, in particular it is strongly Feller.
The goal of this paper is to generalise the above regularity results to the stopped semigroup $R_T^{O_r}$, $T \geq 0$, defined by

$$R_T^{O_r} \varphi(x) = \mathbb{E}[\varphi(X(T, x)) \mathbb{1}_{T \leq \tau_x}], \quad T \geq 0, \quad \varphi \in B_b(\partial O_r),$$

where $\partial O_r$ is an open convex bounded subset of $H$ and $\tau_x$ is the exit time from $\partial O_r$.

More precisely, we shall assume

**Hypothesis 2.**

(i) $g : H \to \mathbb{R}$ is a convex function of class $C^1$ such that $g(0) = 0$, $g(x) > 0$ and $g'(x) \neq 0$ for all $x \neq 0$. For any $r > 0$ we set $O_r = \{g < r\}$, $\overline{O_r} = \{g \leq r\}$ and $\partial O_r = \{g^{-1}(r)\}$. Moreover, $O_r$ is bounded.

(ii) There exist $a, b > 0$ such that $|g(x)| + |g'(x)|_H \leq a + e^{b|x|_H}$ for all $x \in H$.

The semigroup $R_T^{O_r}$, $T \geq 0$, is related, as well known, to the Dirichlet problem in $\overline{O_r}$ for the Kolmogorov operator,

$$\mathcal{K} \varphi := \frac{1}{2} \text{Tr}[CD^2 \varphi] + \langle Ax, D\varphi \rangle.$$  

This problem is elliptic when the matrix $C$ is non singular, otherwise is hypoelliptic. In the last case the existence of the gradient of $R_T^{O_r} \varphi$, $T \geq 0$, when $\varphi$ is namely bounded and Borel, is more challenging.

Here is a simple example where Hypothesis 1 is fulfilled.

**Example 2.** Let $d = 2$ and $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then we have

$$Q_t = \int_0^t e^{sA}Ce^{sA^*}ds = \int_0^t \begin{pmatrix} 1 & s \\ s & s^2 \end{pmatrix} ds = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix}. $$

Therefore $\det Q_t > 0$ for any $t > 0$. If $f \in B_b(H)$ and $t > 0$, we conclude that $R_tf$ is of class $C^\infty$. Note that $u(t, \xi) = R_tf(\xi)$, is the solution of the well known Kolmogorov equation

$$\begin{cases}
D_t u(t, \xi_1, \xi_2) = \frac{1}{2} D_{\xi_1}^2 u(t, \xi_1, \xi_2) + \xi_1 D_{\xi_2} u(t, \xi_1, \xi_2) \\
u(0, \xi) = f(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,
\end{cases}$$

which is hypoelliptic.

Let us explain our result. We start from an obvious consequence of (6),

$$R_T^{O_r} \varphi(x) = \int_{\{g(e^{sA}x + h(s)) \leq r, \forall s \in [0, T]\}} \varphi(h(T) + e^{TA}x)N_{Q_T}(dh), \quad \varphi \in B_b(\partial O_r),$$

where $\varphi \in B_b(\partial O_r)$. 

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where $N_{Q_T}$ is the law of $W_A(\cdot)$ in $X := L^2(0,T;H)$ or in $E := C([0,T];H)$, see Lemma 3 below.

Let $x \in H$; then we cannot eliminate $x$ in identity (9) making the translation $h \to h - e^T x$ and using the Cameron–Martin formula because the measures $N_{e^T A x, Q}$ and $N_Q$ are singular. For this reason we look for another translation $h \to h - a(x, \cdot)$ such that $a(x, \cdot)$ belongs to $Q_T(X)$ for all $x \in H$ (and a–fortiori to $Q_T^{1/2}(X)$, the Cameron–Martin space of $N_{Q_T}$) and such that:

$$a(x, T) = e^T x, \quad \forall x \in H$$  \hfill (10)

(see Proposition 3 below). Then the measures $N_{a(x, \cdot), Q_T}$ and $N_{Q_T}$ are equivalent, so that by the Cameron–Martin Theorem we have

$$\frac{dN_{a(x, \cdot), Q_T}}{dN_{Q_T}}(h) = \exp \left\{ -\frac{1}{2} |Q_T^{-1/2} a(x, \cdot)|^2_X + W_{Q_T^{-1/2} a(x, \cdot)}(h) \right\}, \quad x \in H, \ h \in X;$$  \hfill (11)

where $Q_T^{-1/2}$ is the pseudo inverse of $Q_T^{1/2}$, see e.g. [DaZa14, Theorem 2.23]. Now we take advantage of the special form of $a(x, \cdot)$ for simplifying identity (9). We write

$$|Q_T^{-1/2} a(x, \cdot)|_X^2 = \langle Q_T^{-1} a(x, \cdot) , a(x, \cdot) \rangle =: F(x),$$  \hfill (12)

and,

$$W_{Q_T^{-1/2} a(x, \cdot)}(h) = \langle Q_T^{-1/2} a(x, \cdot), Q_T^{-1/2} h \rangle_X = \langle Q_T^{-1} a(x, \cdot), h \rangle_X =: G(x, h).$$  \hfill (13)

Note that both $F$ and $G$ are regular. Now (11) becomes

$$\frac{dN_{a(x, \cdot), Q_T}}{dN_{Q_T}}(h) = \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\}, \quad x \in H, \ h \in X;$$  \hfill (14)

and

$$R_T^ \partial \varphi(x) = \int_{\{\Gamma(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} N_{Q_T}(dh), \quad \varphi \in B(\overline{\cal O}_r),$$  \hfill (15)

where

$$\Gamma(h + d(x, \cdot)) = \sup_{t \in [0,T]} g(h(t) + d(x,t)), \quad k \in E, \quad d(x,t) = e^T x - a(x, t), \quad t \in [0,T].$$  \hfill (16)

In the integral (15) the variable $x$ does not appear under the argument of $\varphi$. Since the mapping $x \to \Gamma(h + d(x, \cdot))$ is continuous this fact implies that the semigroup $R_T^ \partial$, $T > 0$, is strong Feller that is $\varphi \in B_b(H) \Rightarrow R_T^ \partial \varphi \in C_b(H)$ for all $T > 0$, see Proposition 7 below.

More difficult is to show that $R_T^ \partial \varphi$ is differentiable for all $T > 0$. As it is expected, this will produce a surface integral which, unfortunately, is not covered by the classical assumptions from Airault–Malliavin,
To overcome this difficulty we introduce in Section 3 an approximating semigroup \( R_{T,n}^{Or}, T > 0 \), for all decomposition \( \{t_j = \frac{jT}{2^n}, j = 0, 1, \ldots, 2^n\} \) of \([0, T]\), namely by approximating any function \( h \) from \( E \) by step functions. Then we arrive to an identity for \( R_{T,n}^{Or}\varphi(x) \) (see (44)) that can be easily differentiated with respect to \( x \), see identity (47). It remains to let \( n \to \infty \); this is not easy due to the factor

\[
\langle Q^{-1/2}_{T,n}(d_x(x, \cdot)y), Q^{-1/2}_{T,n}h \rangle_{H^{2n}}
\]

which appears in the identity (47) because \( d_x(x, \cdot)y \) does not belong to the Cameron–Martin space of \( N_{Q_T} \). Some additional work is required, based on the Ehrhard inequality for the gaussian measure \( N_{Q_T} \) and the selection principle of Helly. After some manipulations, we arrive at the representation formula (71) which is the main result of the paper. Our procedure was partially inspired by a paper by Linde \([Li86]\), which was dealing, however, with a completely different situation.

We believe that our method could be extended to more general Kolmogorov operators of the form

\[
\mathcal{K}_1\varphi = \frac{1}{2} \text{Tr} [CD^2\varphi] + \langle Ax + b(x), D\varphi \rangle, \quad \varphi \in C^2(E).
\] (17)

where \( b : H \to H \) is suitable nonlinear mapping. This will be the object of a future work.

We end this section with some notation. For any \( T > 0 \) we consider the law of \( X(\cdot, x) \) both in the Banach space \( E = C([0, T]; H) \) and in the Hilbert space \( X = L^2(0, T; H) \) (in the second case it is concentrated on \( E \) which is a Borel subset of \( X \)). We shall denote by \( | \cdot |_X \) (resp. \( | \cdot |_E \)) the norm of \( X \) (resp. of \( E \)). The scalar product from two elements \( x, y \in H \) (resp. \( X \)) will be denoted either by \( \langle x, y \rangle_H \) (resp. \( \langle x, y \rangle_X \)) or by \( x \cdot y \). If \( \varphi \in C^1_b(E) \) and \( \eta \in E \) we denote by \( D\varphi(h) \cdot \eta \) the derivative of \( \varphi \) at \( h \) in the direction \( \eta \).

In what follows several integrals with respect to \( dN_Q \) will be considered, according to the convenience, both in \( X \) and in \( E \).

## 2 Strong Feller property of \( R_{T}^{Or}, T > 0 \)

We first recall some properties of the gaussian measure \( Q_T \). The following lemma is well known, see e.g. \([DaZa14\text{, Theorem 5.2}].\]

**Lemma 3.** The law of \( W_A(\cdot) \) is gaussian \( N_{Q_T} \), both in \( E \) and in \( X \), where \( Q_T \) is given by

\[
(Q_T h)(t) = \int_0^T K(t, s)h(s) \, ds, \quad t \in [0, T], \; h \in X
\] (18)

where

\[
K(t, s) = \begin{cases} 
\int_0^se^{(t-r)A}Ce^{(s-r)A^*}dr & \text{if } 0 \leq s \leq t \leq T \\
\int_0^t e^{(t-r)A}Ce^{(s-r)A^*}dr & \text{if } 0 \leq t \leq s \leq T.
\end{cases}
\] (19)
We note that there exists an orthonormal basis \((e_j)\) on \(X\) and a sequence \((\lambda_j)\) of nonnegative numbers such that
\[ Q^T e_j = \lambda_j e_j, \quad j \in \mathbb{N}, \]
and an integer \(k_0 \geq 0\) such that
\[
\lambda_1 = \lambda_2 = \cdots = \lambda_{k_0} = 0, \quad \lambda_j > 0, \quad \forall j > k_0.
\]
If \(k_0 = 0\) then \(Q^T\) is non degenerate.

We shall denote by \(L_T\) the linear operator from \(X\) into itself defined by
\[
L_T h(t) = \int_0^t e^{(t-s)A} \sqrt{C} h(s) ds, \quad h \in X, \quad t \in [0,T].
\]
(20)
Its adjoint \(L_T^*\) is given by
\[
L_T^* g(t) = \int_t^T \sqrt{C} e^{(s-t)A^*} g(s) ds, \quad g \in X, \quad t \in [0,T].
\]
(21)
It is easily checked that \(Q^T = L_T L_T^*\). Moreover, by [DaZa14, Corollary B5], the Cameron–Martin space of the Gaussian measure \(N_{Q^T}\) is given by
\[
Q^{1/2}_T(X) = L_T(X),
\]
(22)
both in \(E\) and in \(X\).

**Remark 4.** If \(\det C = 0\) one checks easily that the gaussian measure \(Q^T\) is degenerate and
\[
\ker Q = \{h \in X : L_T^* h = 0\}.
\]

For instance, coming back to Example 2 we see that in that case
\[
\ker Q = \{h = (h_1, h_2) \in E : h_1'(t) = h_2(t)\}.
\]

We shall denote in what follows by \(Q_T^{-1}\) (resp. \(Q_T^{-1/2}\)) the pseudo–inverse of \(Q_T\) (resp. the pseudo–inverse of \(Q_T^{1/2}\)) \footnote{Let \(S : X \to Y\) be a linear, bounded and compact operator; the pseudo–inverse \(S^{-1}\) of \(S\) is defined as follows. For any \(y \in S(X)\) we denote by \(S^{-1} y\) the element of minimal norm from the convex set \(\{x \in X : S(x) = y\}\).} Clearly, the domain of \(Q_T^{-1}\) is equal to \(Q_T(X)\) and \(h \in Q_T^{-1}(X)\) if and only if the following series is convergent in \(X\)
\[
Q_T^{-1} h = \sum_{j=k_0+1}^{\infty} \lambda_j^{-1} \langle h, e_j \rangle_X e_j.
\]

Similar assertion holds for \(Q_T^{-1/2}\).

Now to introduce the required translation; first we need a lemma.
Lemma 5. Define $U := \int_0^T r e^{rA}Ce^{rA^*} dr$. Then $\det U > 0$.

Proof. We have in fact

$$U \geq \frac{T}{2} \int_{T/2}^T e^{rA}Ce^{rA^*} dr = \frac{T}{2} \int_0^{T/2} e^{(T/2+z)A}Ce^{(T/2+z)A^*} dz = \frac{T}{2} e^{AT/2} Q_{T/2} e^{A^*T/2}.$$ 

It follows that $\det U \geq \frac{T}{2} e^{TTrA} \det Q_{T/2} > 0$, as claimed. \hfill \Box

Proposition 6. For all $x \in H$ set

$$u(x,t) := e^{(T-t)A^*}U^{-1}e^{TA}x, \quad t \in [0,T]$$

and define $a(x,\cdot) := Q_T u(x,\cdot)$. Then it results $a(x,T) = e^{TA}x$. Moreover, there is $c_1, c_1, c_2 > 0$ such that

$$|u(t,x)|_H \leq c_1 T |x|_H, \quad \forall t \in [0,T], \; x \in H,$$ (23)

and

$$|a(x,t)|_H \leq c_1 T |x|_H, \quad \forall t \in [0,T], \; x \in H.$$ (24)

Proof. Write

$$a(x,T) = \int_0^T K(T,s) u(x,s) \, ds = \int_0^T \left( \int_0^s e^{(T-r)A}Ce^{(s-r)A^*} \, dr \right) e^{(T-s)A^*}U^{-1}e^{TA}x \, ds$$

$$= \int_0^T \left( \int_0^s e^{(T-r)A}Ce^{(T-r)A^*} \, dr \right) U^{-1}e^{TA}x \, ds = \int_0^T (T-r) e^{(T-r)A}Ce^{(T-r)A^*} dr U^{-1}e^{TA}x = e^{TA}x,$$

as required. Finally,

$$|u(x,t)|_H \leq \sup_{s \in [0,T]} \|e^{sA}\|_{\mathcal{L}(H)} \|U^{-1}\|_{\mathcal{L}(H)} |x|_H, \quad t \in [0,T],$$

so that (24) and (25) follow easily. \hfill \Box

Now we prove the first new result of the paper.

Proposition 7. Under Hypotheses 1 and 2 the semigroup $R_T^{0r}$, $T > 0$, is strong Feller.

Proof. Let $\varphi \in B(\overline{\mathcal{D}}_r)$ and $x_0, x \in \overline{\mathcal{D}}_r$. Then by (15) we find

$$|R_T^0 \varphi(x) - R_T^0 \varphi(x_0)| \leq \|\varphi\| \int_X \left| \exp \left\{ -\frac{1}{2} F(x) + G(x,h) \right\} - \exp \left\{ -\frac{1}{2} F(x_0) + G(x_0,h) \right\} \right| N_{Q_T}(dh)$$

$$+ \|\varphi\| \int_X \mathbb{1}_{\{\Gamma(h+d(x_0)) \leq r\} \cup \{\Gamma(h+d(x)) \leq r\}} \exp \left\{ -\frac{1}{2} F(x_0) + G(x_0,h) \right\} N_{Q_T}(dh) =: A_1 + A_2.$$
Taking into account (24) we have
\[ F(x) = \langle Q^{-1}_T a(x, \cdot), a(x, \cdot) \rangle_X = \langle u(x, \cdot), Q_T u(x, \cdot) \rangle_H \leq \|Q_T\|_{\mathcal{L}(H)} c_T^2 |x|^2_H \]
and
\[ |G(x, h)| = |\langle Q^{-1}_T a(x, \cdot), h \rangle_X| \leq c_T |x| |h|_X \]
Therefore
\[ \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} - \exp \left\{ -\frac{1}{2} F(x_0) + G(x_0, h) \right\} = \int_0^1 \exp \left\{ -\frac{1}{2} F((1 - \alpha) x_0 + \alpha x) + G((1 - \alpha) x_0 + \alpha x, h) \right\} (x - x_0) \, d\alpha \leq \int_0^1 \exp \{ G((1 - \xi) x_0 + \xi x, h) \} |x - x_0| \, d\alpha \leq \exp \{ c_T T |x| |h|_X \} |x - x_0|. \]
It follows that
\[ A_1 \leq \|\varphi\|_{\infty} \int_X \exp \{ c_T T |x| |h|_X \} dN_Q_T |x - x_0|. \]
Since the integral above is finite we have \( \lim_{x \to x_0} A_1 = 0. \) Concerning \( A_2 \) we have \( \lim_{x \to x_0} A_2 = 0 \) by the continuity of \( d(x, \cdot) \) and the dominated convergence theorem. The proof is complete. \( \square \)

3 Approximating semigroup

We define an approximating semigroup \( R_{T,n}^a \varphi, T > 0, \) on \( B_b(\overline{\Omega}_r) \) setting for all \( n \in \mathbb{N}, \)
\[ R_{T,n}^a \varphi(x) = \int_{\{\Gamma_n(x+d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \{ -\frac{1}{2} F(x) + G^n(x, h) \} N_Q_T(dh), \quad (26) \]
where \( F(x), \ x \in \overline{\Omega}_r \) is defined by (12), \( d(x, t) \) by (10) and \( \Gamma_n \) by
\[ \Gamma_n(x+d(x, \cdot)) = \sup \{ g(h(t_j) + d(x, t_j)), \ t_j = \frac{r}{2^n}, \ j = 0, 1, \ldots, 2^n \}, \quad \forall h \in E, \ n \in \mathbb{N} \quad (27) \]
and
\[ G^n(x, h) = \sum_{j=1}^{2^n} (u(x, t_j) \cdot h(t_j)) (t_j - t_{j-1}), \quad x \in \overline{\Omega}_r, \ h \in E. \quad (28) \]

Lemma 8. (i) It results
\[ |\Gamma_n(h + d(x, \cdot)) - \Gamma_n(h_1 + d(x, \cdot))| \leq a + be^{|h_1|_E + |h|_E}, \quad h_1, h_2 \in E. \quad (29) \]
(ii) Moreover \( h \to \Gamma_n(h + d(x, \cdot)) \) belongs to \( W^{1,2}(E, N_Q) \)
\[ |\Gamma'_n(h + d(x, \cdot)) \cdot d(x, \cdot)| \leq (a + be^{|h|_E} |d(x, \cdot)|. \]
Proof. (i) follows from Hypothesis 2(ii) and (ii) is a well known consequence of the local lipschitzianity of \( \Gamma_n \).

Proposition 9. Under Hypotheses 1, 2 for all \( \varphi \in B(\partial_r) \) it results
\[
\lim_{n \to \infty} R^{\partial_r}_{T,n} \varphi(x) = R^{\partial_r}_T \varphi(x), \quad \forall \ x \in \partial_r.
\]

Proof. Let \( \varphi \in B_b(\partial_r) \). Then
\[
|R^{\partial_r}_{T,n} \varphi(x) - R^{\partial_r}_T \varphi(x)| \leq \|\varphi\|_\infty \int_{\{\Gamma_n(h+d(x,\cdot)) \leq r\}} \left| \exp\left(-\frac{1}{2} F(x)\right) \right| \exp\{G^n(x,h)\} - \exp\{G(x,h)\} N_{Q_T}(dh) \]  
\[ + \|\varphi\|_\infty \int_{\{\Gamma_n(h+d(x,\cdot)) \leq r\} \setminus \{\Gamma(h+d(x,\cdot)) \leq r\}} \exp\left(-\frac{1}{2} F(x) + G(x,h)\right) N_{Q_T}(dh). \tag{30}
\]

Taking into account (24), yields
\[
|G^n(x,h)| \leq \sum_{j=1}^{2^n} |u(x,t_j)|_H |h(t_j)|_H (t_j - t_{j-1}) \leq c_T |x|_{C(\partial_r)} |h|_E, \quad x \in \partial_r, \ h \in E \tag{31}
\]

Now, set
\[
B_n := \{h : g(h(t_i)) + d(x,t_i) \leq r, \ t_j = \frac{jT}{2^n}, j = 0, 1, ... , 2^n\}, \quad B = \{h : g(h(t)) + d(x,t) \leq r, \ t \in [0,T]\}.
\]

Then \( B \subset B_n \) and \( \bigcap_{n \in \mathbb{N}} B_n = B \), so that \( N_{Q}(B_n) \downarrow N_{Q}(B) \) as \( n \to \infty \).

Moreover,
\[
\lim_{n \to \infty} G^n(x,h) = G(x,h), \quad \forall \ h \in E, \ x \in \partial_r \tag{32}
\]
and by (23) there is \( c_T > 0 \) such that
\[
\exp\{G^n(x,h)\} \leq e^{c_T|h|_E}, \quad \forall \ h \in E, \forall \ x \in \partial_r. \tag{33}
\]

The conclusion follows from the dominated convergence theorem.

It useful to write an expression of \( R^{\partial_r}_{T,n} \varphi \) as a finite dimensional integral. To this purpose we consider the linear mapping
\[
E = C([0,T]; H) \to H^{2^n}, \quad h \to (h(t_1), h(t_2), \ldots, h(t_{2^n})), \quad t_j = \frac{jT}{2^n}, \ j = 0, 1, ... , 2^n, \tag{34}
\]
whose law is obviously gaussian, say \( N_{Q_{T,n}} \). Then for any \( n \in \mathbb{N} \) and any \( \varphi : H^{2n} \to \mathbb{R} \) bounded and Borel we have,

\[
\int_E \varphi(h(t_1), h(t_2), \ldots, h(t_{2^n})) N_{Q_{T}}(dh) = \mathbb{E}[\varphi(W_A(t_1), W_A(t_2), \ldots, W_A(t_{2^n}))]
\]

\[= \int_{H^{2n}} \varphi(\xi_1, \ldots, \xi_{2^n}) N_{Q_{T,n}}(d\xi_1 \cdots d\xi_{2^n}). \tag{35}\]

Now we can write the approximating semigroup as an integral over \( H^{2n} \), namely

\[
R_{T,n}^\varphi(x) = \int_{\{\Gamma_n(\xi + d(x, \cdot)) \leq r\}} \varphi(\xi_{2^n}) \exp\{-\frac{1}{2} F(x) + G^n(x, \xi)\} N_{Q_{T,n}}(d\xi), \quad \xi \in H^{2n}, \tag{36}\]

where

\[
\Gamma_n(\xi + d(x, \cdot)) = \sup\{g(\xi + d(x, t_j)), \quad t_j = \frac{jT}{2^n}, \quad j = 0, 1, \ldots, 2^n\}
\]

and

\[
G^n(x, \xi) = \sum_{j=1}^{2^n} (u(x, t_j) \cdot \xi_j) (t_j - t_{j-1}), \quad x \in \overline{\mathcal{O}}, \quad \xi \in H^{2n}.
\]

**Proposition 10.** \( Q_{T,n} \) has a bounded inverse for all \( n \in \mathbb{N} \), so the Cameron–Martin space of \( N_{Q_{T,n}} \) is the whole \( H^{2n} \).

**Proof.** Let \( n \in \mathbb{N} \), then by (35) we have

\[
\mathbb{E}[e^{i\lambda \sum_{h=1}^{2^n} \langle \xi_h, W_A(t_h) \rangle_H}] = e^{-\frac{1}{2} \lambda^2 \langle Q_{T,n} \xi, \xi \rangle_{H^{2n}}}, \quad \xi = (\xi_1, \ldots, \xi_{2^n})
\]

We claim that if \( Q_{T,n} \xi = 0 \) then \( \xi = 0 \). In fact, if \( Q_{T,n} \xi = 0 \), we have \( \mathbb{E}[e^{i\lambda \sum_{h=1}^{2^n} \langle \xi_h, W_A(t_h) \rangle_H}] = 1 \) and so,

\[
\sum_{h=1}^{2^n} \langle \xi_h, W_A(t_h) \rangle_H = 0, \quad N_{Q_{T,n}}-\text{a.s.}. \tag{37}\]

Now, setting \( L(t) = \int_0^t e^{-sA} dW(s) \), and \( \rho_i = e^{-t_i A^*} \xi_i \), we have

\[
\langle \rho_1 + \rho_2 + \cdots + \rho_{2^n}, L(t_1) \rangle + \langle \rho_2 + \cdots + \rho_{2^n}, L(t_2) - L(t_1) \rangle + \cdots + \langle \rho_{2^n}, L(t_{2^n}) - L(t_{2^n-1}) \rangle = 0.
\]

Multiplying both sides by \( \langle \rho_1 + \rho_2 + \cdots + \rho_{2^n}, L(t_1) \rangle \) and taking expectation, yields

\[
\rho_1 + \rho_2 + \cdots + \rho_{2^n} = 0,
\]

since \( \mathbb{E}(\langle v, L(t_1) \rangle^2) = \langle Q_{t_1} v, v \rangle \) and \( Q_{t_1} \) is non singular by Hypothesis \( \Box \)

Similarly we obtain \( \rho_k + \rho_{k+1} + \cdots + \rho_{2^n} = 0 \) for \( k = 2, 3, \ldots, 2^n \), which implies finally \( \xi = 0 \). \( \Box \)
4 Differentiating the approximating semigroup

First note that by (28) we have
\[(G^n_h(x, h) \cdot (d_x(x, \cdot)y) = \sum_{j=1}^{2^n} (u(x, t_j) \cdot d_x(x, t_j)y)(t_j - t_{j-1}), \quad x \in \Omega, \ h \in E. \quad (38)\]

**Lemma 11.** For all $x \in \Omega, \ y \in H, \ n \in \mathbb{N}, \ \varphi \in B_b(\Omega)$ we have
\[
D_x R_{T,n}^\varphi \cdot y =: M_1(n, x, y) + M_2(n, x, y),
\]
where
\[
M_1(n, x, y) = \int_{\{\Gamma_n(h+d(x,\cdot))\leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \times \left( -\frac{1}{2} F_x(x)y + G^n_x(x, h)y - G^n_h(x, h) \cdot (d_x(x, \cdot)y) \right) N_{Q_T}(dh).
\]
and
\[
M_2(n, x, y) = \int_{\{\Gamma_n(h+d(x,\cdot))\leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \left( \langle Q^{-1/2}_{T,n}(d_x(x, \cdot)y), Q^{-1/2}_{T,n}h \rangle_{H^{2^n}} \right) N_{Q_T}(dh),
\]
where
\[
\langle Q^{-1/2}_{T,n}(d_x(x, \cdot)y), Q^{-1/2}_{T,n}h \rangle_{H^{2^n}} = \sum_{i,j=1}^{2^n} (Q^{-1}_{T,n})_{i,j}(d_x(x, t_i)y) \cdot h(t_j).
\]
and $(\psi_j)$ is the standard orthogonal basis of $H^{2^n}$.

**Proof.** We first write identity (26) as
\[
R_{T,n}^\varphi \cdot x = \int_{\{\Gamma_n(h+d(x,\cdot))\leq r\}} \varphi(\xi^{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{Q_T}(d\xi),
\]
Then we drop the dependence on $x$ under the domain of integration by making the translation $\xi \to \xi - d(x, \cdot)$ and recalling that $d(x, T) = 0$; we write
\[
R_{T,n}^\varphi(x) = \int_{\{\Gamma_n(\xi)\leq r\}} \varphi(\xi^{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi - d(x, \cdot)) \right\} N_{d(x,\cdot),Q_T}(d\xi).
\]
so, using again the Cameron–Martin Theorem (this is possible thanks to Proposition [10]), we have

$$R_{T,n}^{\xi} \varphi(x) = \int_{\{\Gamma_n(\xi) \leq r\}} \varphi(\xi_2^n) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi - d(x, \cdot)) \right\} \chi_n(x, \xi) N_{Q_T, n}(d\xi),$$

(44)

where

$$\chi_n(x, \xi) = \exp \left\{ -\frac{1}{2} \|Q_{T,n}^{-1/2} d(x, \cdot)\|_{H^{2^n}} + \langle Q_{T,n}^{-1/2} d(x, \cdot), Q_{T,n}^{-1/2} \xi \rangle_{H^{2^n}} \right\}.$$  

(45)

We now can differentiate $R_{T,n}^{\xi} \varphi(x)$ at any given direction $y \in H$. Taking into account that for any $x, y \in H$ we have,

$$D_x \chi_n(x, \xi) \cdot y = \langle Q_{T,n}^{-1/2}(d_x(x, \cdot)y), Q_{T,n}^{-1/2}(\xi - d(x, \cdot)) \rangle_{H^{2^n}} \chi_n(x, \xi).$$

(46)

we find

$$D_x R_{T,n}^{\xi} \varphi(x) \cdot y = \int_{\{\Gamma_n(\xi) \leq r\}} \varphi(\xi_2^n) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi - d(x, \cdot)) \right\}$$

$$\times \left[ -\frac{1}{2} F_x(y) + G^n_x(x, \xi - d(x, \cdot))y - \langle G^n_\xi(x, \xi - d(x, \cdot)), d_x(x, \cdot)y \rangle_{H^{2^n}} \right]$$

$$+ \langle Q_{T,n}^{-1/2}(d_x(x, \cdot)y), Q_{T,n}^{-1/2}(\xi - d(x, \cdot)) \rangle_{H^{2^n}} \chi_n(x, \xi) N_{Q_T, n}(d\xi).$$

(47)

Here $F_x$ and $G_x$ denote the derivatives with respect to $x$ of $F$ and $G$ respectively, whereas $G_\xi$ is the derivative with respect to $\xi$. Now making the opposite translation $\xi_j \to \xi_j + d(x, t_j)$, $j = 0, 1, \ldots, 2^n$, we obtain

$$D_x R_{T,n}^{\xi} \varphi(x) \cdot y = \int_{\{\Gamma_n(\xi + d(x, \cdot)) \leq r\}} \varphi(\xi_2^n) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\}$$

$$\times \left( -\frac{1}{2} F_x(y) + G^n_x(x, \xi) y - \langle G^n_\xi(x, \xi), d_x(x, \cdot)y \rangle_{H^{2^n}} \right) N_{Q_T, n}(d\xi)$$

$$+ \int_{\{\Gamma_n(\xi + d(x, \cdot)) \leq r\}} \varphi(\xi_2^n) \exp \left\{ -\frac{1}{2} F(x) + G^n(h, \xi) \right\} \langle Q_{T,n}^{-1/2}(d_x(x, \cdot)y), Q_{T,n}^{-1/2}(\xi) \rangle_{H^{2^n}} N_{Q_T, n}(d\xi).$$

(48)

We arrive at the conclusion making the change of variables [34].

In Section 4 we shall easily prove the existence of the limit of $M_1(n, x, y)$ as $n \to \infty$. Instead a problem arises, as said in the introduction, for the term $M_2(n, x, y)$ due to the factor

$$\langle Q_{T,n}^{-1/2}(d_x(x, \cdot)y), Q_{T,n}^{-1/2}(h) \rangle_{H^{2^n}},$$

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because \( d_x(x, \cdot) y \) does not belong to \( Q_{1/2}^T(X) \). So, we look in the next Lemma 16 for a different expression of \( M_2(n, x, y) \) that does not contain this term. Before we need to recall the definition and some properties of the Sobolev space \( W^{1,p}(E, N_{Q_T}) \). We shall need a result which is a straightforward generalisation of [Ce01, Proposition 6.1.5]

**Lemma 12.** For any \( \varphi \in C^1_b(E) \) there exists a sequence \( (\varphi_n) \in C^1_b(X) \) such that

(i) \( \lim_{n \to \infty} \varphi_n(h) \to \varphi(h), \quad \forall \ h \in E \).

(ii) \( \lim_{n \to \infty} \langle D\varphi_n(h), \eta \rangle_X = D\varphi(h) \cdot \eta, \quad \forall \ h, \eta \in E \).

**Proof.** For any \( \varphi \in C^1_b(E) \) set

\[
\varphi_n : H \to E, \quad x \to \varphi_n(x)(t) = \frac{n}{2} \int_{t - \frac{1}{n}}^{t + \frac{1}{n}} \hat{\varphi}(s) \, ds, \quad t \in [0, T],
\]

and \( \hat{\varphi}(s) \) is the extension by oddness of \( \varphi(s) \), for \( s \in (-T, 0) \) and \( s \in (T, 2T) \). Then it is easy to check that \( (\varphi_n) \) fulfills (i) and (ii). \( \square \)

The following result is similar to [BoDaTu18, Proposition 4.2].

**Proposition 13.** For all \( \varphi \in C^1_b(E) \) and any \( \eta \in Q_{1/2}^1(X) \subset E \) the following integration by parts formula holds

\[
\int_E D\varphi(h) \cdot \eta N_{Q_T}(dh) = \int_E \varphi(h) \langle Q^{-1/2} h, Q^{-1/2} \eta \rangle_H N_{Q_T}(dh).
\]

**Proof.** Let \( \varphi_n \in C^1_b(X) \) be a sequence as in Lemma 12 then we have

\[
\int_H \langle D\varphi_n(h), \eta \rangle_X N_{Q_T}(dh) = \int_H \varphi_n(x) \langle Q^{-1/2} h, Q^{-1/2} \eta \rangle_X dN_{Q_T}(dh).
\]

The conclusion follows letting \( n \to \infty \). \( \square \)

**Corollary 14.** For all \( \varphi, \psi \in C^1_b(E) \) and any \( \eta \in Q_{1/2}^1(X) \subset E \) the following integration by parts formula holds

\[
\int_E D\varphi \cdot \eta \psi dN_{Q_T} = - \int_E D\psi \cdot \eta \varphi dN_{Q_T} + \int_E \varphi \psi \langle Q^{-1/2} x, Q^{-1/2} z \rangle_X dN_{Q_T}.
\]

**Remark 15.** By (51) it follows, by standard arguments, that the gradient operator \( D \) is closable in \( L^p(E, N_Q) \) for any \( p \geq 1 \); we shall still denote by \( D \) its closure and by \( W^{1,p}(E, N_Q) \) its domain. Finally, it is well known that all Lipschitz continuous function \( \varphi : E \to \mathbb{R} \) belongs to \( W^{1,p}(E, N_Q) \). See also Lemma 8

Now we are ready to prove the announced lemma.
Lemma 16. \textit{Assume Hypotheses 1 and 2.} Let \( \theta \). Now, by a classical integration by parts formula we have, see e.g. [Bo 98].

\begin{align*}
M_2(n, x, y) &= \int_{\Gamma_n(h+d(x,\cdot)) \leq r} \varphi(h(T))D_h \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \cdot (d_x(x,\cdot)y) N_{\mathcal{Q}_T}(dh) \\
&\quad + \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon \leq \Gamma_n(h+d(x,\cdot)) \leq r+\epsilon\}} \varphi(h(T)) (\Gamma'_n(h+d(x,\cdot))) \cdot (d_x(x,\cdot)y) N_{\mathcal{Q}_T}(dh) \\
&= : M_{2,1}(n, x, y) + M_{2,2}(n, x, y).
\end{align*}

\textit{Proof.} By [Bo 98] we have

\begin{align*}
M_2(n, x, y) &= \int_{\{\Gamma_n(\xi+d(x,\cdot)) \leq r\}} \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} \langle \mathbb{Q}_{T,n}^{-1/2} (d_x(x,\cdot)y), \mathbb{Q}_{T,n}^{-1/2} (\xi) \rangle_{H^{2^n}} N_{\mathcal{Q}_{T,n}}(d\xi).
\end{align*}

Let us first assume in addition that \( \varphi \in C^1(\mathcal{O}_r) \). Then we argue similarly to [BoDaTu18 Proposition 4.5], defining a mapping \( \theta_\epsilon : \mathbb{R} \to \mathbb{R} \),

\begin{align*}
\theta_\epsilon(s) &= \begin{cases} 
0, & \text{if } s \leq r - \epsilon \\
\frac{1}{2\epsilon} (s - r + \epsilon), & \text{if } r - \epsilon \leq s \leq r + \epsilon \\
1, & \text{if } s \geq r + \epsilon.
\end{cases}
\end{align*}

Then we approximate \( M_2(n, x, y) \) by setting

\begin{align*}
M_2'(n, x, y) &= \int_{H^{2^n}} \theta_\epsilon(\Gamma_n(\xi+d(x,\cdot))) \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} \langle \mathbb{Q}_{T,n}^{-1/2} (d_x(x,\cdot)y), \mathbb{Q}_{T,n}^{-1/2} (\xi) \rangle_{H^{2^n}} N_{\mathcal{Q}_{T,n}}(d\xi),
\end{align*}

so that the \( \lim_{\epsilon \to 0} M_2'(n, x, y) \) exists and is given by

\begin{align*}
\lim_{\epsilon \to 0} M_2'(n, x, y) &= \int_{\{\Gamma_n(\xi+d(x,\cdot)) \leq r\}} \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} \langle \mathbb{Q}_{T,n}^{-1/2} (d_x(x,\cdot)y), \mathbb{Q}_{T,n}^{-1/2} (\xi) \rangle_{H^{2^n}} N_{\mathcal{Q}_{T,n}}(d\xi),
\end{align*}

Now, by a classical integration by parts formula we have, see e.g. [Bo 98].

\begin{align*}
M_2'(n, x, y) &= \int_{H^{2^n}} \theta_\epsilon(\Gamma_n(\xi+d(x,\cdot))) (D_h \varphi(\xi_{2^n}) \cdot d_x(x, T) y) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{\mathcal{Q}_{T,n}}(d\xi) \\
&\quad + \int_{H^{2^n}} \theta_\epsilon(\Gamma_n(\xi+d(x,\cdot))) \varphi(\xi_{2^n}) \langle D_h \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} \cdot d_x(x,\cdot)y) \rangle N_{\mathcal{Q}_{T,n}}(d\xi) \\
&\quad + \int_{H^{2^n}} (D_h \theta_\epsilon(\Gamma_n(\xi+d(x,\cdot))) \cdot (d_x(x,\cdot)y)) \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{\mathcal{Q}_{T,n}}(d\xi).
\end{align*}
Taking into account that the first integral vanishes, because $d_x(x, T) y = 0$, and that
\[
\langle D\theta_t^x(\Gamma_n a(\xi + d(x, \cdot))), (d(x, \cdot)) \rangle_{H^2^n} = \theta_t^x(\Gamma_n(\xi + d(x, \cdot))) \langle \Gamma_n^\prime(\xi + d(x, \cdot)), d_x(x, \cdot) y \rangle_{H^2^n}
\]
we deduce by (55), letting $\epsilon \to 0$, that
\[
M_2(n, x, y) = \int_{\{\Gamma_n(\xi + d(x, \cdot)) \leq r\}} \varphi(\xi^2^n) (D_h \exp \{-\frac{1}{2} F(x) + G^n(x, \xi)\} \cdot d_x(x, \cdot) y)) N_{Q_T, n}(d\xi)
\]
\[
+ \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\{\epsilon \leq \Gamma_n(\xi + d(x, \cdot)) \leq r + \epsilon\}} \langle \Gamma_n^\prime(\xi + d(x, \cdot)), d_x(x, \cdot) y \rangle_{H^2^n} \varphi(\xi^2^n) \exp \{-\frac{1}{2} F(x) + G^n(x, \xi)\} N_{Q_T, n}(d\xi).
\]
(56)

So the conclusion of the lemma follows, by the change of variables (54), when $\varphi \in C^1(\overline{O})$. The case when $\varphi \in C(\overline{O})$ can be handled by a uniform approximation of $\varphi$ by $C^1(\overline{O})$ functions. Finally, if $\varphi \in B_b(\mathcal{O})$ we conclude using the strong Feller property of the semigroup, see Proposition 7.

It remains to compute the limit in (52). This we will do using the Ehrhard inequality.

### 4.1 Applying the Ehrhard inequality

Define
\[
\Lambda_x(s) := N_{Q_T}(\Gamma(h + d(x, \cdot))) \leq s), \quad \Lambda_{n,x}(s) := N_{Q_T}(\Gamma_n(\xi + d(x, \cdot)) \leq s), \quad \forall s > 0, \ x \in \overline{O}, \ n \in \mathbb{N}. \quad (57)
\]

Since $g$ is convex by Hypothesis 2(i), the mapping $\Gamma(h + d(x, \cdot))$ (resp. $\Gamma_n(\xi + d(x, \cdot))$) is convex as well. By applying the Ehrhard inequality (see e.g. [Bo98, Th. 4.4.1]) we see that for any $x \in \overline{O}$ the real function
\[
[0, +\infty) \to \mathbb{R}, \ s \to S_x(s) := \Phi^{-1}(\Lambda_x(s)), (resp. [0, +\infty) \to \mathbb{R}, \ s \to S_{n,x}(s) := \Phi^{-1}(\Lambda_{n,x}(s))),
\]
where
\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}v^2} dv, \quad z \in \mathbb{R},
\]
is concave. Note that $\Phi^{-1} : (0, 1) \to (-\infty, +\infty)$. As a consequence, $\Lambda_x(\cdot)$ (resp. $\Lambda_{n,x}(\cdot)$) is differentiable at any $s > 0$ up to a discrete set $N_s$ where there exist the left and the right derivative; we shall denote by $D^+_r \Lambda_x(s)$ (resp $D^+_r \Lambda_{n,x}(s)$) the right derivatives at any discontinuity point, and also (with the same symbol) the derivative at the other points.

It follows that $N_{Q_T} \circ (\Gamma(h + d(x, \cdot)))^{-1}$ (resp. $N_{Q_T} \circ (\Gamma_n(h + d(x, \cdot)))^{-1}$) is absolutely continuous with respect to the Lebesgue measure $\ell$ and it results
\[
\frac{dN_{Q_T} \circ (\Gamma(h + d(x, \cdot)))^{-1}}{d\ell}(s) = D^+_r \Lambda_x(s), \quad (resp. \frac{dN_{Q_T} \circ (\Gamma_n(h + d(x, \cdot)))^{-1}}{d\ell}(s) = D^+_r \Lambda_{n,x}(s), \ s > 0.
\]
Note that for any \(x\), \(\Lambda_{n,x}(s)\) is increasing on \(s\) and decreasing on \(n\). Moreover, \(\Lambda_{n,x}(0) = 0\) and \(\Lambda_{n,x}(s) \uparrow 1\) as \(s \to \infty\). Also

\[
D^+ S_x(s) = \sqrt{2\pi} e^{\frac{s^2}{2}} D^+ \Lambda_x(s)
\]

(59)

Now we are going to estimate \(D^+_r \Lambda_{n,x}(s)\) independently of \(n, x\) and \(s \in \left[\frac{r}{2}, 3r/2\right]\). Then we shall show that \(D^+_r \Lambda_{n,x} \to D^+_r \Lambda_x\).

**Lemma 17.** There exists \(K_r > 0\) independent of \(x, n, s\) such that

\[
D^+_r \Lambda_{n,x}(s) \leq K_r, \quad \forall \ x \in \overline{O_r}, \forall \ n \in \mathbb{N}, \forall \ s \in \left[\frac{r}{2}, 3r/2\right].
\]

(60)

Moreover, it results

\[
\lim_{n \to \infty} D^+_r \Lambda_{n,x} = D^+_r \Lambda_x, \quad \forall \ x \in \overline{O_r}.
\]

(61)

**Proof.** We proceed in three steps.

**Step 1.** There is \(l_1 > 0\) such that

\[
0 < l_1 \leq \Lambda_{n,x}(s), \quad \forall \ x \in \overline{O_r}, \forall \ n \geq 2, \forall \ s \in \left[\frac{r}{2}, 3r/2\right].
\]

(62)

It is enough to show (62) for \(\Lambda_{2,x}(s)\), because \(\Lambda_{2,x}(s) \geq \Lambda_{n,x}(s)\) for \(n \geq 2\).

Since the convex set \(\{\xi \in H^{2n} : \Gamma_2(\xi + d(x, \cdot)) < s\}\) is open and non empty and the measure \(N_{Qr,2}\) is non degenerate by Proposition 10, it follows that there is \(l_1(x)\) such that

\[
0 < l_1(x) \leq \Lambda_{n,x}(s), \quad \forall \ x \in \overline{O_r}, \forall \ n \geq 2, \forall \ s \in \left[\frac{r}{2}, 3r/2\right].
\]

Since \(\overline{O_r}\) is compact, the conclusion follows.

**Step 2.** There is \(l_2 < 1\) such that

\[
\Lambda_{n,x}(s) < l_2 < 1, \quad \forall \ x \in \overline{O_r}, \forall \ n \in \mathbb{N}, \forall \ s \in \left[\frac{r}{2}, 3r/2\right].
\]

(63)

In fact, thanks to Hypothesis 2(ii), there exists \(M > 0\) such that

\[
\Lambda_{n,x}(s) \leq M, \quad \forall \ x \in \overline{O_r}, \forall \ n \in \mathbb{N}, \forall \ s \in \left[\frac{r}{2}, 3r/2\right].
\]

(64)

**Step 3.** Conclusion.

Note first that

\[
S_{n,x}(s) \downarrow S_x(s) \quad \text{as} \ n \to \infty, \ x \in \overline{O_r}.
\]
The sequence \((S_{n,x}(\cdot))\) is obviously increasing and also concave by the Ehrhard inequality. Therefore, all elements of \((S'_{n,x}(\cdot))\) are positive and decreasing; so, they are BV in the interval \([r/2, 3r/2]\).

We claim that the sequence \((S'_{n,x}(r))\) is equi-bounded in \([r/2, 3r/2]\) in BV norm. To show this fact it is enough to see that \((S'_{n,x}(r))\) is namely equi-bounded at \(r_1\) (because it is decreasing). In fact, since \(S_{n,x}\) is concave we have if \(0 < \epsilon \leq \frac{r}{2}\),
\[
S'_{n,x}(r_1) \leq \frac{1}{r}(S_{n,x}(r_1 + \epsilon) - S_{n,x}(r_1)) \leq \frac{2}{r} S_{n,x}(r_2) = \frac{2}{\epsilon} \Phi^{-1}(r_2).
\]
(65)

Therefore we can apply the selection principle of Helly, see e.g. [KoFo70, Theorem 5 page 372] to the sequence \((S'_{n,x}(\cdot))\) and conclude that there exists a subsequence of \((S'_{n,x}(\cdot))\) still denoted by \((S'_{n,x}(\cdot))\) that converges in all points of \([r/2, 3r/2]\) to a function \(f(x, \cdot)\).

We claim that \(f(x, s)\) is the derivative of \(S_{x}(s)\) in \([r/2, 3r/2]\). This follows by an elementary argument writing,
\[
S_{n,x}(s) = \int_{r_1}^{s} S'_{n,x}(v)dv, \quad s \in [r/2, 3r/2],
\]
(recall that \(S_{n,x}(\cdot)\) is absolutely continuous by [Bo98, Corollary 4.4.2]). By the Dominated Convergence theorem it follows that for \(k \to \infty\) we have
\[
S_{x}(s) = \int_{r_1}^{s} S'_{x}(v)dv, \quad s \in [r/2, 3r/2],
\]
which implies
\[
S'_{x}(s) = f(x, s), \quad s \in [r/2, 3r/2],
\]
as required.

Therefore there is a subsequence of \((S'_{n,x}(\cdot))\) which converges to \(S'_x\) and consequently all the sequence \((S'_{n,x}(\cdot))\) will converges to \(S'_x\) thus \(\Lambda_x(r)\) has the right derivative for \(r \in [r/2, 3r/2]\), \(x \in \overline{Q}_r\), and (61) follows.

Finally, taking into account (65) and (59), it results
\[
D^+_r \Lambda_x(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}S_2(r)} S'_x(r) = 1 + \frac{1}{\epsilon} \Phi^{-1}(r_2).
\]
So, (61) follows. \(\square\)

The next lemma is devoted to the computation of \(\lim_{\epsilon \to 0} M_{2,2}(n, x, y)\), defined by (52).

**Lemma 18.** Let \(n \in \mathbb{N}, r > 0, x \in \overline{Q}_r, y \in H\). Then it results
\[
M_{2,2}(n, x, y) = \mathbb{E}_N Q_r \left[ \varphi(h(T)) \left( \Gamma'_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y) | \Gamma_n(h + d(x, \cdot)) = r \right) \right] D^+_n \Lambda_x(r).
\]
(66)

**Proof.** Let us recall that by (52) we have,
\[
M_{2,2}(n, x, y) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon \leq \Gamma_n(h + d(x, \cdot)) \leq r + \epsilon\}} \varphi(h(T))(\Gamma'_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y) N_{Q_r}(dh),
\]

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Taking into account (58) it follows that

$$M_{2,2}(n, x, y) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{r-\epsilon}^{r+\epsilon} \mathbb{E}_{\mathcal{N}_T} [\varphi(h(T)) (\Gamma_n'(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y)|\Gamma_n(h + d(x, \cdot)) = s] \, D_{n,r} \Lambda_x(s) \, ds.$$ 

Note that the existence of a regular distribution of

$$\mathbb{E}_{\mathcal{N}_T} [\varphi(h(T)) (\Gamma_n'(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y)|\Gamma_n(h + d(x, \cdot)) = s]$$

is granted because $E$ is separable, see [Du02, 10.2.2].

It follows that

$$M_{2,2}(n, x, y) = \mathbb{E}_{\mathcal{N}_T} [\varphi(h(T)) (\Gamma_n'(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y)|\Gamma_n(h + d(x, \cdot)) = r] \, D_{n,r}^+ \Lambda_x(r),$$

by virtue of the dominated convergence theorem. \hfill \square

We prove now the following result.

**Proposition 19.** Assume Hypotheses 1 and 2 and let $n \in \mathbb{N}$. Then we have

$$D_x R_{T,n}^\theta \varphi(x) \cdot y = \int_{\{\Gamma_n(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \left( -\frac{1}{2} F_x(x)y + G^n_x(x, h)y \right) N_{\mathcal{Q}_T}(dh)$$

$$+ \mathbb{E}_{\mathcal{N}_T} [\varphi(h(T)) (\Gamma_n'(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y)|\Gamma_n(h + d(x, \cdot)) = r] \, D_{n,r}^+ \Lambda_x(r).$$

(67)

Moreover, there is $c_{2,T}(r) > 0$ such that the following estimate holds

$$|D_x R_{T,n}^\theta \varphi(x) \cdot y| \leq \|\varphi\|_\infty c_{2,T}(r) + \|\varphi\|_\infty \exp \left\{ -\frac{1}{2} F(x) \right\}$$

$$\times \int_{\{\Gamma_n(h + d(x, \cdot)) \leq r\}} \exp \left\{ c_T |h|_E \left( \frac{1}{2} \|F_x(x)\|_{\mathcal{L}(H)} + Tc_{2,T}(r) \|U^{-1}\|_{\mathcal{L}(H)} \|h\|_E \right) \right\} N_{\mathcal{Q}_T}(dh).$$

(68)

**Proof.** From Lemmas 11 and 18 we obtain

$$D_x R_{T,n}^\theta \varphi(x) \cdot y = M_{1,1}(n, x, y) + M_{2,1}(n, x, y) + M_{2,2}(n, x, y)$$

and so,

$$D_x R_{T,n}^\theta \varphi(x) \cdot y = \int_{\{\Gamma_n(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\}$$

$$\times \left( -\frac{1}{2} F_x(x)y + G^n_x(x, h)y - (G^n_h(x, h) \cdot (d_x(x, \cdot)y)) \right) N_{\mathcal{Q}_T}(dh)$$

$$+ \int_{\{\Gamma_n(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) (D_h \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \cdot (d_x(x, \cdot)y)) N_{\mathcal{Q}_T}(dh)$$

$$+ \mathbb{E}[\varphi(h(T)) (\Gamma_n'(h + d(x, \cdot)) \cdot (d_x(x, \cdot)y)|\Gamma_n(h + d(x, \cdot)) = r] \, D_{r}^+ \Lambda_{n,x}(r).$$
Since
\[ D_h \exp \{ G^n(x, h) \} \cdot (d_x(x, \cdot) y) = \exp \{ G^n(x, h) \} G_h^n(d_x(x, \cdot) y) \cdot (d_x(x, \cdot) y), \]
we obtain letting \( n \to \infty \), after some simplifications, identity (67). Finally, we prove (68). First by (33) we have
\[ \exp \{ G^n(x, h) \} \leq c_T |h|_E. \]
Moreover by (28) it follows that
\[ G^n(x, h) = \sum_{j=1}^{2^n} u_x(x, t_j) \cdot h(t_j) (t_j - t_{j-1}), \quad x \in \mathcal{O}_r, \ h \in E \]
and therefore we have
\[ |G^n(x, h)| \leq T \|u_x(x, \cdot)\|_{\mathscr{L}(H)} \|h\|_E \leq T c_T^2 \|U^{-1}\|_{\mathscr{L}(H)} \|h\|_E, \quad x \in \mathcal{O}_r, \ h \in E \quad (69) \]
Finally, by Hypothesis 2(ii) there exists \( c_{1,T} > 0 \) such that
\[ |\Gamma'_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y)| \leq c_{1,T} |h|_E. \]
Finally, taking into account (60) and Lemma 8 yields
\[ \left| \mathbb{E}_{N_{Q_T}} [\varphi(h(T)) (\Gamma'_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) |\Gamma_n(h + d(x, \cdot)) = r)] D^+_r \Lambda_{n,x}(r) \right| \]
\[ \leq \|\varphi\|_\infty c_{1,T} |D^+_r \Lambda_{x}(r)| \leq \|\varphi\|_\infty c_{1,T} K_r. \]
The conclusion follows. \( \square \)

5 Main results

Now we take \( \varphi \in B_b(H) \) and prove a representation formula for \( D_x R^\varphi_{T^r} \varphi(x) \).

**Theorem 20.** Assume Hypotheses 1 and 2. Then there exists the gradient of \( R^\varphi_{T^r} \varphi \) for all \( \varphi \in B_b(0) \) and it results
\[ D_x R^\varphi_{T^r} \varphi(x) \cdot y = \int_{\{\Gamma(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} \left( -\frac{1}{2} F_x(x) y + G_x(x, h) y \right) N_{Q_T}(dh) \]
\[ + \mathbb{E}_{N_{Q_T}} [\varphi(h(T)) (\Gamma'(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) |\Gamma(h + d(x, \cdot)) = r)] D^+_r \Lambda_{x}(r). \]
\[ (71) \]
Proof. We recall that by Proposition 19 we have
\[ D_x R^{E_r}_{T,n}\varphi(x) \cdot y = I(n, x, y) + J(n, x, y), \]
where
\[ I(n, x, y) = \int_{\{\Gamma_n(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{-\frac{1}{2} F(x) + G^n(x, h) \right\} \left(-\frac{1}{2} F_x(x) y + G^n_x(x, h) y \right) N_{Q_T}(dh) \]
and
\[ J(n, x, y) = \mathbb{E}_{N_{Q_T}} [\varphi(h(T)) \Gamma_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y)] \Gamma_n(h + d(x, \cdot)) = r] D_r \Lambda_{n,x}(r). \] (73)

Step 1. Convergence of \( I(n, x, y) \) as \( n \to \infty \).
For all \( x \in \overline{Q_T} \) and all \( y \in H \) we have
\[ \lim_{n \to \infty} I(n, x, y) = \int_{\{\Gamma(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{-\frac{1}{2} F(x) + G(x, h) \right\} \left(-\frac{1}{2} F_x(x) y + G_x(x, h) y \right) N_{Q_T}(dh). \] (74)

This follows by the dominated convergence theorem arguing as in the proof of Proposition 9.

Step 2. Convergence of \( J(n, x, y) \) as \( n \to \infty \). Let \( r > 0, x \in \overline{Q_T}, y \in H \). Then, we have,
\[ \lim_{n \to \infty} J(n, x, y) = -\mathbb{E} [\varphi(h(T)) (\Gamma'(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y)] \Gamma(h + d(x, \cdot)) = r] D_r \Lambda_{x}(r). \] (75)

First we notice that \( \Gamma_n(h + d(x, \cdot)) \) converges uniformly to \( \Gamma(h + d(x, \cdot)) \) for any \( x \). Moreover, since the function \( h \to \sup_{t \in [0,\infty)} h(t) \) is Lipschitz continuous in \( E \) and \( g \) fulfills Hypothesis 2(ii), it follows that \( \Gamma_n(h + d(x, \cdot)) \) belongs to a bounded subset of \( W^{1,2}(E, N_{Q_T}) \), by Lemma 8. So, a subsequence of \( (\Gamma_n(h + d(x, \cdot))) \) (which we still denote by \( (\Gamma_n'(h + d(x, \cdot))) \)) converges to \( \Gamma'(h + d(x, \cdot)) \) in \( L^1(E, N_{Q_T}) \).

Now we start from (73) which we write as
\[ J(n, x, y) = \mathbb{E} [\Psi_n(h)] \Gamma_n(h + d(x, \cdot)) = r] D_r \Lambda_n(x)(r), \]
where
\[ \Psi_n(h) = -\varphi(h(T)) (\Gamma'_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y)). \] (76)

Note that \( D_r \Lambda_n(x)(r) \to D_r \Lambda_x(r) \) as \( n \to \infty \) by Lemma 17.

By Hypothesis 2(ii) we have
\[ |\Psi_n(h)| \leq ||\varphi||_{\infty}(a + e^{b|x|E}), \quad \forall h \in E, \]
so that, there exists \( M > 0 \) such that \( |\Psi_n(h)|_{L^1(E, N_0)} \leq M, \forall n \in \mathbb{N} \). Also \( \Psi_n(h) \to \Psi(h) \) for all \( h \in E \) by Lemma 10(iii). Therefore \( \Psi_n \to \Psi \) in \( L^1(E, N_Q) \) by the dominated convergence theorem.
Now we can show that
\[\lim_{n \to \infty} \mathbb{E}[\Psi_n | \Gamma_n(h + d(x, \cdot)) = r] = \mathbb{E}[\Psi | \Gamma(h + d(x, \cdot)) = r]. \tag{77}\]
To this aim write
\[
|\mathbb{E}[\Psi_n | \Gamma_n(h + d(x, \cdot)) = r] - \mathbb{E}[\Psi | \Gamma(h + d(x, \cdot)) = r]| \\
\leq |\mathbb{E}[\Psi_n - \Psi | \Gamma_n(h + d(x, \cdot)) = r]| + |\mathbb{E}[\Psi | \Gamma_n(h + d(x, \cdot)) = r] - \mathbb{E}[\Psi | \Gamma(h + d(x, \cdot)) = r]| \\
:= J_1(n) + J_2(n).
\]
Since \(\Psi_n \to \Psi\) in \(L^1(E, N_Q)\) we have
\[|J_1(n)| \to 0 \quad \text{in} \quad L^1(E, N_Q) \quad \text{as} \quad n \to \infty. \tag{78}\]
For dealing with \(J_2(n)\), note that
\[\lim_{n \to \infty} \mathbb{E}[\Psi_n | \Gamma_n(h + d(x, \cdot)) = r] = \mathbb{E}[\Psi | \Gamma(h + d(x, \cdot)) = r] \tag{79}\]
because \(\Gamma_n\) is decreasing to \(\Gamma\), see e.g. [Du02, 10.1.7]. Now step 2 follows from (78) and (79).

**Step 3.** Existence of \(D_x R_T^\rho \varphi\), for all \(\varphi \in C_b(O_r)\).
Let us recall that by Proposition 9 and Steps 1,2 we know that
1. there exists the limit
   \[\lim_{n \to \infty} R_{T,n}^\rho \varphi(x) = R_T^\rho \varphi(x), \quad \forall \quad x \in \overline{O_r}.\]
2. there exists the limit
   \[\lim_{n \to \infty} D_x R_{T,n}^\rho \varphi(x) \cdot y =: \Xi(x) \cdot y \quad \forall \quad x \in \overline{O_r}, \quad y \in H.\]
3. There exists \(M_{\|\varphi\|} > 0\) such that
   \[|R_{T,n}^\rho \varphi(x)| + |D_x R_{T,n}^\rho \varphi(x)| \leq M_{\|\varphi\|}, \quad \forall \quad x \in \overline{O_r}. \tag{80}\]
Let now \(x, x_0 \in \overline{O_r}\). Since
\[
R_{T,n}^\rho \varphi(x) - R_{T,n}^\rho \varphi(x_0) = \int_0^1 (D_x R_{T,n}^\rho \varphi)(\alpha x + (1 - \alpha)x_0) \cdot (x - x_0) \, d\alpha.
\]
Letting \(n \to \infty\) we obtain, by the dominated convergence theorem
\[
R_T^\rho \varphi(x) - R_T^\rho \varphi(x_0) = \int_0^1 \Xi(\alpha x + (1 - \alpha)x_0) \cdot (x - x_0) \, d\alpha.
\]
This implies that $R_T^\theta \varphi(x)$ is differentiable at $x$ in the direction $y$ and
\[ DR_T^\theta \varphi(x) \cdot y = \Psi(x) \cdot y. \]

**Step 4.** $\varphi \in B_b(\overline{O}_r)$

Since $R_T^\theta$ is strong Feller (Proposition 7), we have $R_T^\theta/2 \in C_b(H)$, so, the conclusion follows starting from $T/2$.

The proof is complete. \qed

**Example 21.** We continue here Example 2. Let
\[ A = \begin{pmatrix} 0 & 0 \\ t & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]
Then we have $e^tA = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $e^{tA^*} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ We have seen that Hypothesis 1 is fulfilled. Let $U$ as in Lemma 5

\[ U = \int_0^T re^{rA}Ce^{-rA^*}dr = \int_0^T \begin{pmatrix} r & r^2 \\ r^2 & r^3 \end{pmatrix} ds = \frac{1}{12} \begin{pmatrix} 6T^2 & 4T^3 \\ 4T^3 & 3T^4 \end{pmatrix} \]
so that $\det U > 0$ and
\[ U^{-1} = \frac{6}{T^2} \begin{pmatrix} 3 & -\frac{4}{T} \\ -\frac{4}{T} & \frac{6}{T^2} \end{pmatrix} \]
Moreover, by Proposition 6 we have
\[ u(x, s) = \frac{6}{T^4} \begin{pmatrix} T^2 - 2Ts & 2(T - 3s) \\ 2T & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x \in H, \ s \in [0, T]. \]
So,
\[ |u(x, s)|_H \leq T^{-4}C_1(T)|x|_H \tag{82} \]
and
\[ |K(t, s)|_H \leq C_2(T), \quad \forall \ t, s \in [0, T], \tag{83} \]
where $C_1(T), C_2(T)$ are continuous in $(0, +\infty)$. Moreover
\[ |a(t, x)|_H = |Q_T u(t, x)|_H \leq T^{-3}C_1(T)|x|_H \tag{84} \]
Finally, $G(x, h) = \langle u(t, x), h \rangle_H \leq |h|_X |u(t, x)|$.

Concerning Hypothesis 2, assume that $g(x) = |x|^2$. Then
\[ \Lambda(x, r) = \int_{\|h + d(x, \cdot)\|_E \leq r} N_{Q_T}(dh) \]
Note that by (60) we know that $D_r\Lambda(x, r)$ is uniformly bounded in $x$. 22
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