Quasi-modular forms and trace functions associated to free boson and lattice vertex operator algebras

Chongying Dong\textsuperscript{1}, Geoffrey Mason\textsuperscript{2}
Department of Mathematics, University of California Santa Cruz, CA 95064, U.S.A
Kiyokazu Nagatomo\textsuperscript{3}
Department of Mathematics, Graduate School of Science, Osaka University
Osaka, Toyonaka 560-0043, Japan

Abstract: We study graded traces of vectors in free bosonic vertex operator algebras and lattice vertex operator algebras. We show in particular that trace functions in these two theories always have the shape $f(q)/\eta(q)^d$ where $f(q)$ is quasi-modular in the case of $d$ free bosons, and modular (i.e., a sum of holomorphic modular forms of various weights) in the case of theories based on a lattice $L$ of rank $d$. We also show how spherical harmonic polynomials with respect to $L$ are related to primary fields in lattice theories.

1 Introduction

The purpose of this paper is to study the space of 1-point correlation functions, or trace functions, which arise from certain vertex operator algebras.

Let us suppose that $V$ is a vertex operator algebra \cite{FLM} (also see \cite{B}, \cite{MN}) with standard $L(0)$-grading

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$ 

The most basic trace function is the formal graded character

$$\text{ch}_q V = \text{tr} V q^{L(0)-c/24} = q^{-c/24} \sum_{n \in \mathbb{Z}} (\dim V_n) q^n.$$ 

1Supported by NSF grant DMS-9700923 and a research grant from the Committee on Research, UC Santa Cruz.
2Supported by NSF grant DMS-9700909 and a research grant from the Committee on Research, UC Santa Cruz.
3Supported in part by Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.
where \( c \) is the central charge of \( V \).

For many well-known VOAs, the graded character has certain modular-invariance properties. For example, if \( V(\mathfrak{g}, l) \) is the generalized Verma module of level \( l \) associated to a (complex) Lie algebra \( \mathfrak{g} \) of dimension \( d \) (cf. [FZ], [L]) and trivial \( \mathfrak{g} \)-module, then \( V(\mathfrak{g}, l) \) is a vertex (operator) algebra satisfying

\[
\text{ch}_q V(\mathfrak{g}, l) = \eta(q)^{-d}
\]

where \( \eta(q) \) is the Dedekind eta-function. Similarly, if \( L \) is a positive-definite, even lattice of rank \( d \) and \( \theta_L(q) \) is the corresponding theta-function, and if \( V_L \) is the associated VOA (cf. [B], [FLM]), then

\[
\text{ch}_q V_L = \theta_L(q)/\eta(q)^d.
\]

So \( \text{ch}_q V_L \) is a modular function (i.e., of weight zero) on some congruence subgroup of the modular group \( SL(2, \mathbb{Z}) \), while \( \text{ch}_q V(\mathfrak{g}, l) \) is a modular form of weight \(-d/2\) on the full modular group \( \text{SL}(2, \mathbb{Z}) \). On the other hand, the VOA \( M(c, 0) \) associated to the Virasoro algebra of central charge \( c \) (cf. [FZ], [L]) satisfies

\[
\text{ch}_q M(c, 0) = (1 - q)q^{-c/24 + 1}/\eta(q)
\]

and hence is not modular. Here, of course, we are interpreting \( q \) in the usual way to be equal to \( e^{2\pi i \tau} \) with \( \tau \) in the complex upper-half plane \( \mathcal{H} \).

In the fundamental paper of Zhu [Zh] general correlation functions, which generalize the graded character, were studied. Recall (loc cit) that if \( v \) lies in \( V_k \) with vertex operator \( Y(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1} \), then the so-called zero mode \( o(v) = v(k - 1) \) is a linear operator on \( V \) which leaves invariant each homogeneous space \( V_n \), so that one can form the expression

\[
Z(v, q) = \text{tr}_V o(v)q^{L(0) - c/24} = q^{-c/24} \sum_{n \in \mathbb{Z}} (\text{tr}_V o(v))q^n.
\]

One calls the linear extension of \( Z(v, q) \) to all of \( V \) the (1-point) correlation function determined by \( V \). It is the purpose of this paper to understand the nature of this function in the case of the lattice VOAs \( V_L \) and the Heisenberg VOA \( M(1) \), which is the space \( V(\mathfrak{g}, l) \) in the case that \( \mathfrak{g} \) is an abelian Lie algebra equipped with a non-degenerate symmetric bilinear form \( (,\) and \( l = 1 \). \( M(1) \) is often referred to as the VOA of \( d \) free bosons if \( \text{dim} \mathfrak{g} = d \). In particular, we ask when \( Z(v, q) \) is modular in a suitable sense and, if not, how does it deviate from being modular?

In order to explain our results and methods, we need to introduce so-called quasi-modular forms [KZ], which play an interesting role in the present paper. For a positive

---

4We will have no need to concern ourselves with the fact that \( d \) may be odd, so that the weight may be a half integer.
integer $N$ and for a Dirichlet character $\epsilon$ modulo $N$, let $M(N, \epsilon)$ denote the ring of modular forms on the congruence subgroup $\Gamma_0(N)$ which transform according to the character $\epsilon$ and which are holomorphic in $H$. Let

\begin{equation}
    Q(N, \epsilon) = M(N, \epsilon)[E_2]
\end{equation}

be the space obtained by adjoining to $M(N, \epsilon)$ the Eisenstein series $E_2(q)$. For example, if $N = 1$, so that also $\epsilon = 1$, then we have

\begin{equation}
    Q = Q(1, 1) = C[E_2, E_4, E_6]
\end{equation}

the full ring of quasi-modular forms on $SL(2, \mathbb{Z})$. We can now state our first result:

**Theorem 1.** Let $V$ be the VOA of $d$ free bosons, with $Q$ as in (1.3).

(a) If $v$ is in $V$, then $Z(v, q)$ converges to a holomorphic function $f(v, q)/\eta(q)^d$ in the upper half plane for some $f(v, q) \in Q$.

(b) Every $f(q)$ in $Q$ may be realized as $f(v, q)$ for some $v$ in $V$.

We can restate Theorem 1 in the following form: there is a linear surjection

\begin{equation}
    t : V \to Q, \quad v \mapsto (\text{ch}_q V)^{-1}Z(v, q).
\end{equation}

In fact we will construct a section of the map $t$, more precisely we explicitly describe a subspace $W$ of $V$ such that the restriction of $t$ to $W$ is an isomorphism onto $Q$.

**Theorem 2.** Let $L$ be a positive-definite even lattice of rank $d$ and level $N$, so that the theta function $\theta_L(q)$ lies in $M(N, \epsilon)$ for suitable $\epsilon$. Then for every element $v$ in $V_L$, we have

\begin{equation}
    Z(v, q) = f(v, q)/\eta(q)^d
\end{equation}

for some $f(v, q)$ in $M(N, \epsilon)$. In particular, each $Z(v, q)$ is a sum of modular forms (of varying weights).

The role of quasi-modular forms in the proof of Theorem 2 is more hidden. To explain, we recall here (and discuss in greater detail in Section 4.1) that in [Zh], Zhu shows how to identify vectors in $V$ such that the corresponding trace function $Z(v, q)$ essentially behaves like a modular form of weight $k$ for some $k$. More precisely, if we think of $Z(v, q)$ as a function of $\tau$, then under the action of the modular group

$$
    \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : Z(v, \tau) \mapsto (c\tau + d)^{-k}Z(v, \gamma \tau), \quad \gamma \in SL(2, \mathbb{Z}),
$$

the $\gamma$-transforms of $Z(v, \tau)$ spans a finite-dimensional $SL(2, \mathbb{Z})$-module of holomorphic functions on $H$. Moreover each such $\gamma$ transform has a suitable q-expansion. Extending
the language of [KM], we call such functions \textit{generalized modular forms} of weight \( k \). A modular form has this property, however it is shown in [KM] that there are generalized modular forms which are not modular. Thus one cannot deduce from Zhu’s results alone that the trace functions \( Z(v,q) \) are modular, or even sums of modular forms. However, we will show below that the trace functions occurring in Theorem 2 also have the property that \( \eta(q)^d Z(v,q) \) lies in \( Q(N, \epsilon) \) so that in this regard they behave similarly to the trace functions in the free boson case. It is easy to see that a generalized modular form that is also a quasi-modular form is necessarily an ordinary modular form, and Theorem 2 then follows.

Our third main theorem has a slightly different flavor. In the paper [DM1] the space of trace functions \( Z(v,q) \) was determined in the case of the Moonshine module \( V^{\natural} \) [FLM], a prominent role being played by the primary fields i.e., the (homogeneous) vectors of \( V^{\natural} \) which are highest weight vectors for the Virasoro algebra. Based on this work and the calculations in [HL], we conjecture that if \( V \) is a holomorphic VOA then each cusp form on \( SL(2,\mathbb{Z}) \) (possibly with character) can be realized by a trace function \( Z(v,q) \) in which \( v \) is a primary field. The conjecture seems to be non-trivial for any holomorphic VOA. We will prove it for the lattice VOA \( V_{Es} \) based on the \( E_8 \) root lattice. Our approach uses the theory of spherical harmonics, and through this mechanism one sees that the conjecture may be viewed as a conformal field theoretic analog of the following problem in number theory: given a positive-definite self-dual even lattice \( L \), describe the space of modular forms \( \theta_L(P, \tau) \) obtained by modifying \( \theta_L(\tau) \) by \( P \), where \( P \) ranges over the homogeneous spherical harmonic functions with respect to \( L \). (Replacing “holomorphic” by “rational” in our conjecture corresponds to eliminating the self-duality of \( L \).) Waldspurger showed [W] that all cusp forms of level one can be obtained as \( \theta_{Es}(P, \tau) \) for suitable \( P \), and this leads to our result about \( V_{Es} \) because of the following.

\textbf{Theorem 3.} Let \( P \) be a homogeneous spherical harmonic of degree \( k \) with respect to the lattice \( L \) of rank \( d \). Then there is a primary field \( v_P \) in the lattice VOA \( V_L \) with the property that

\[ Z(v_P, q) = \theta_L(P, q)/\eta(q)^d. \]

There is a circle of ideas relating our results: in [EZ], Eichler and Zagier gave an approach to the result of Waldspurger using ideas from the theory of Jacobi forms. Jacobi forms and Jacobi-like forms [Za] are closely related to quasi-modular forms, and they also play a role in the results of [DM2] which identify the quasi-modular forms underlying the proof of theorem 2 and relate them to theta functions modified by a spherical harmonic. And the occurrence of Jacobi forms in string theory has been observed frequently in the past few years.
2 Vertex operator algebras and graded traces

In this section we briefly review from [Zh] the “bracket” vertex operator algebra $(V, Y[], 1, \omega - c/24)$ constructed from $(V, Y, 1, \omega)$. We also list several formulas involving the trace functions from [Zh] and present an easy corollary which is used frequently in the later sections.

2.1 Vertex operator algebras of genus one

Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra. Then $V$ may be regarded as a vertex operator algebra on the sphere. In order to study modular invariance in the theory of vertex operator algebras, a new vertex operator algebra on the torus was introduced in [Zh]. The new vertex operator algebra is $(V, Y[], 1, \omega - c/24)$ where $c$ is the central charge of $V$. The new vertex operator associated to a homogeneous element $a$ is given by

$$Y[a, z] = \sum_{n \in \mathbb{Z}} a[n] z^{-n-1} = Y(a, e^z - 1) e^z \omega^{\text{wt}(a)}$$

while a Virasoro element is $	ilde{\omega} = \omega - c/24$. Thus

$$a[m] = \text{Res}_z (Y(a, z)(\ln (1 + z))^m (1 + z)^{\omega^{\text{wt}(a)-1}})$$

and

$$a[m] = \sum_{i=m}^{\infty} c(\text{wt}(a), i, m)a(i)$$

for some scalars $c(\text{wt}(a), i, m)$ such that $c(\text{wt}(a), m, m) = 1$. In particular,

$$a[0] = \sum_{i \geq 0} \binom{\omega^{\text{wt}(a)} - 1}{i} a(i).$$

We also write

$$L[z] = Y[\omega, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.$$  

Then the $L[n]$ again generate a copy of the Virasoro algebra with the same central charge $c$. Now $V$ is graded by the $L[0]$-eigenvalues, that is

$$V = \bigoplus_{n \in \mathbb{Z}} V[n]$$
where $V_{[n]} = \{ v \in V | L[0]v = nv \}$. We also write $\text{wt}[a] = n$ if $a \in V_{[n]}$. It should be pointed out that for any $n \in \mathbb{Z}$ we have

$$\sum_{m \leq n} V_m = \sum_{m \leq n} V_{[n]}.$$  

We also recall the notion of $V$-module briefly. A $V$-module $M = (M, Y^M)$ is a $\mathbb{C}$-graded vector space

$$M = \oplus_{\lambda \in \mathbb{C}} M_{\lambda}$$

such that each $M_{\lambda}$ is finite-dimensional and $M_{\lambda+n}$ is zero if $n$ is a small enough integer. Furthermore $M_{\lambda}$ is the eigenspace of $L(0)$ with eigenvalue $\lambda$ where $L(0)$ is the component operator of $Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$. As in the case of vertex operator algebras, the component operator $o(a) = a^M(\text{wt}(a) - 1)$ preserves each homogeneous space $M_{\lambda}$ if $a \in V$ is homogeneous and $Y^M(a, z) = \sum_{n \in \mathbb{Z}} a^M(n) z^{-n-1}$. Again using the $\mathbb{C}$-linearity, we extend the operator $o(a)$ to all $a \in V$.

### 2.2 Graded traces

Next we state some results from [Zh]. We use the Eisenstein series $E_{2k}(\tau)$ normalized as in [DLM] equation (4.28). Thus:

$$E_{2k}(\tau) = \frac{-B_{2k}}{2k!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

where $\sigma_k(n)$ is the sum of the $k$-powers of the divisors of $n$ and $B_{2k}$ a Bernoulli number.

Let $M = (M, Y^M)$ be a $V$-module. For any $a \in V$ we define a formal power series in $q$:

$$Z_M(a, q) = \text{tr}_M o(a) q^{L(0)-c/24} = q^{-c/24} \sum_{\lambda \in \mathbb{C}} (\text{tr}_{M_{\lambda}} o(a)) q^\lambda.$$  

Then $Z_V(a, q)$ is exactly $Z(a, q)$ defined before.

**Proposition 2.2.1 ([Zh], Proposition 4.3.5, 4.3.6).** Let $M$ be a $V$-module. Then the following identities hold as formal power series for any $a, b \in V$.

1. $Z_M(a[0]b, q) = 0$,
2. $\text{tr}_M o(a)o(b) q^{L(0)-c/24} = Z_M(a[-1]b, q) - \sum_{k=1}^{\infty} E_{2k}(q) Z_M(a[2k-1]b, q)$,
3. $Z_M(a[-2]b, q) = -\sum_{k=2}^{\infty} (2k-1) E_{2k}(q) Z_M(a[2k-2]b, q)$.
The following corollary plays an important role in our arguments.

**Corollary 2.2.2.** Let notation be as before. Then for each positive integer \( r \) we have

\[
Z_M(a[−r]b, q) = \delta_{1,r} \text{tr}_M o(a)o(b)q^{L(0)−c/24} \\
+ (-1)^{r+1} \sum_{k>r/2} h(k, r) E_{2k}(q) Z_M(a[2k−r]b, q)
\]

where \( h(k, r) = \binom{2k-1}{r-1} \).

**Proof.** We prove the result by induction on \( r \). The cases \( r = 1, 2 \) are nothing but (2.2.2) and (2.2.3) respectively. Suppose the statement is true for \( r \geq 2 \). Then replacing \( a \) by \( L[-1]a \) and noting \( (L[-1]a)[n] = -na[n-1] \), we see that

\[
rZ_M(a[−r−1]b, q) = (-1)^{r+1} \sum_{k>r/2} (r − 2k)h(k, r) E_{2k}(q) Z_M(a[2k−r−1]b, q).
\]

Then by (2.2.1), we conclude that

\[
Z_M(a[−r−1]b, q) = (-1)^{r+2} \sum_{k>(r+1)/2} h(k, r+1) E_{2k}(q) Z_M(a[2k−r−1]b, q).
\]

\( \square \)

### 3 Quasi-modularity of trace functions for free boson VOAs

In this section, after reviewing from [FLM] the free boson vertex operator algebra \( M(1) \), we establish Theorem [1].

#### 3.1 Free boson VOAs

Let \( \mathfrak{h} \) be a \( d \)-dimensional vector space with a non-degenerate symmetric bilinear form \( ( , ) \) and \( \hat{\mathfrak{h}} \) be the corresponding affinization viewing \( \mathfrak{h} \) as an abelian Lie algebra: \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \) with commutator relations

\[
[h \otimes t^m, h' \otimes t^n] = (h, h')\delta_{m+n,0}K, \quad (h, h' \in \mathfrak{h}, m, n \in \mathbb{Z}), \\
[K, \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]] = 0.
\]

Consider the induced module

\[
M(1) = U(\hat{\mathfrak{h}}) \otimes_{\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]} \mathbb{C}
\]
where \( h \otimes \mathbb{C}[t] \) acts trivially on \( \mathbb{C} \) and \( K \) acts as 1. We denote by \( h(n) \) the action of \( h \otimes t^n \) on \( M(1) \). The space \( M(1) \) is linearly isomorphic to the symmetric algebra \( S(h \otimes t^{-1}\mathbb{C}[t^{-1}]) \). Thus setting \( 1 = 1 \otimes 1 \), any element in \( M(1) \) is a linear combination of elements of type

\[
v = a_1(-n_1) \cdots a_k(-n_k)1, \ (a_1, \ldots, a_k \in h, n_1, \ldots, n_k \in \mathbb{Z}_+).
\]

For such \( v \), we define

\[
Y(v, z) = \partial^{n_1-1}a_1(z) \cdots \partial^{n_k-1}a_k(z) \circ \partial^{(n)} = \frac{1}{n!} \left( \frac{d}{dz} \right)^n
\]

where

\[
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}
\]

and \( \circ \circ \) indicates the normal ordering procedure.

Now let \( \{h_i\}_{i=1}^d \) be an orthonormal basis of \( h \) and set \( \omega = \frac{1}{2} \sum_i h_i(-1)^21 \). Then \((M(1), Y, 1, \omega)\) is a vertex operator algebra with a vacuum \( 1 \) and Virasoro element \( \omega \) (see [FLM]). In particular,

\[
M(1) = \bigoplus_{n \geq 0} M(1)_n
\]

where \( M(1)_n = \langle a_1(-n_1) \cdots a_k(-n_k)1 | a_1, \ldots, a_k \in h, n_1, \ldots, n_k \in \mathbb{Z}_+, \sum n_i = n \rangle \). We will identify \( M(1)_1 \) with \( h \) in an obvious way.

Note that \( o(a) = 0 \) for \( a \in h \), so that (2.2.4) simplifies in the free boson case. Moreover, we see from Subsection 2.1 that \( a[0] = a(0) = 0 \). Thus we have the following commutator relation by noting that \( a[1]b = a(1)b = (a, b) \) for \( a, b \in h \):

\[
[a[m], b[n]] = m\delta_{m+n,0}(a, b).
\]

It is easy to see from this that \( M(1)_n \) is spanned by vectors \( a_1[-n_1] \cdots a_k[-n_k]1 \) for \( a_1, \ldots, a_k \in h, n_1, \ldots, n_k \in \mathbb{Z}_+ \) such that \( \sum n_i = n \).

### 3.2 Proof of Theorem 1

We first prove part (a) of Theorem \[\blacksquare\]. It is enough to prove it for an element of the spanning set defined in Section 3.1.

Let \( v = a_1[-r_1] \cdots a_k[-r_k]1 \) and define the length of \( v \) to be \( k \), denoting this by \( l(v) = k \). We prove the statement by induction on \( k \). If \( k = 0 \) then \( v = 1 \), \( Z(v, q) = 1/\eta(q)^d \) and there is nothing to prove. Now we assume that the statement holds for
With $l(v) \leq k$, and consider $v = a[-r]a_1[-r_1] \cdots a_k[-r_k]1 = a[-r]b$ where $b = a_1[-r_1] \cdots a_k[-r_k]1$. Since $o(a) = 0$, we see from equation (3.1.3) that

$$Z(a[-r]b, q) = \sum_{m>r/2} h(m, r)E_{2m}(q)Z(a[2m-r]b, q).$$

Since $2m - r > 0$, $a[2m-r]b$ is a linear combination of homogeneous vectors whose lengths are no greater than $k$. By induction, each $Z(a[2m-r]b, q)$ converges to a holomorphic function which is a quotient of a quasi-modular form by $\eta(q)^d$. Note that $E_{2m}(q)$ is in $Q$. It is immediate that $Z(v, q)$ converges to a holomorphic function of the same kind.

In order to prove part (b) of Theorem 1 we need several lemmas. From now on, we fix $a \in \mathfrak{h}$ such that $(a, a) = 1$.

**Lemma 3.2.1.** If $n > 0$, $r \geq 0$ are integers, we have

$$Z(a[-n]^{2r}1, q) = (-1)^{(n+1)r}n^r(2r-1)!!h(n, n)^rE_{2n}(q)^r/\eta(q)^d$$

where $(2r-1)!! = 1 \cdot 3 \cdot 5 \cdots (2r-3) \cdot (2r-1)$.

**Proof.** Recall Corollary 2.2.2 and the fact that $a[s]a[-t] = s\delta_{s,t}$ for positive integers $s, t$ (cf. (3.1.3)). We have

$$Z(a[-n]^{2r}1, q) = (-1)^{n+1}h(n, n)E_{2n}(q)Z(a[2n-n]a[-n]^{2r-1}1, q) = (-1)^{n+1}n(2r-1)h(n, n)E_{2n}(q)Z(a[-n]^{2r-2}1, q).$$

Now the result follows by induction, noting that $Z(1, q) = 1/\eta(q)^d$. $\square$

**Lemma 3.2.2.** Let $r, s \geq 0$ be integers. Then

$$Z(a[-1]^{2r}a[-2]^{2s}1, q) = (-6)^s(2r-1)!!(2s-1)!!E_2(q)^rE_4(q)^s/\eta(q)^d$$

and

$$Z(a[-2]^{2s}a[-3]^{2r}1, q) = (-6)^s(2s-1)!!(2t-1)!!(30)^tE_4(q)^sE_6(q)^t/\eta(q)^d.$$ 

**Proof.** Since the proofs of (3.2.2) and (3.2.3) are almost identical we only prove (3.2.2). The special case $r = 0$ follows from Lemma 3.2.1. So we can proceed by induction on $r$ and assume that $r \neq 0$. Then by Corollary 2.2.2 and Lemma 3.2.1 we have

$$Z(a[-1]^{2r}a[-2]^{2s}1, q) = (2r-1)h(1, 1)E_2(q)Z(a[-1]^{2r-2}a[-2]^{2s}1, q)$$

$$= (-6)^s(2r-1)!!(2s-1)!!E_2(q)^rE_4(q)^s/\eta(q)^d,$$

as required. $\square$
Lemma 3.2.3. Let \( r, s, t \geq 0 \) be integers. Then

\[
Z(a[-1]^{2r}a[-2]^{2s}a[-3]^{2t}1, q) = c_{r,s,t}E_2(q)^r E_4(q)^s E_6(q)^t / \eta(q)^d + \text{lower terms}
\]

where \( c_{r,s,t} \) is a nonzero constant and “lower terms” means a linear combination of terms of the form \( E_2(q)^{r'} E_4(q)^{s'} E_6(q)^{t'} / \eta(q)^d \) with \( 0 \leq r' < r \) if \( r \geq 1 \), and \( 0 \) if \( r = 0 \).

Proof. If \( r = 0 \) this is (3.2.3), so we may assume that \( r \geq 1 \). First we deal with the case \( s = 0 \). Following Corollary 2.2.2 and Lemma 3.2.1 we have

\[
Z(a[-1]^{2r}a[-2]^{2s}a[-3]^{2t}1, q) = (2r - 1)E_2(q)Z(a[-1]^{2r-2}a[-2]^{2s}a[-3]^{2t}1, q) + 6tE_4(q)Z(a[-1]^{2r-1}a[-2]^{2s}a[-3]^{2t-1}1, q).
\]

Continuing in this fashion, the result follows.

Now we can prove part (b) of Theorem 1. Since the space \( Q \) of quasi-modular forms on \( SL(2, \mathbb{Z}) \) is an algebra generated by \( E_2(q), E_4(q) \) and \( E_6(q) \), it is enough to prove (b) for \( f(q) = E_2(q)^r E_4(q)^s E_6(q)^t \) with non-negative integers \( r, s, t \). But this follows from Lemmas 3.2.2 and 3.2.3. Indeed, this shows that if \( W \) is the subspace of \( M(1) \) spanned by \( a[-1]^{2r}a[-2]^{2s}a[-3]^{2t}1 \) for \( r, s, t \geq 0 \) then the map \( t \) of (1.3) induces an isomorphism from \( W \) to \( Q \).

4 Modularity of trace functions for lattice vertex operator algebras

We study trace functions for lattice vertex operator algebras \( V_L \) ([B], [FLM]). We prove that \( Z(v, q) \) is quasi-modular for all \( v \in V_L \). This result together with a result in [DM2] yields a proof of Theorem 3. It should be mentioned that in fact, we compute the trace function \( Z_M(v, q) \) corresponding to an irreducible module \( M \). It is clear from our proof that each \( Z_M(v, q) \) is also modular.

4.1 Vertex operator algebras associated to lattices

We now work in the setting of Chapter 8 of [FLM]. In particular, \( L \) is a positive definite lattice, and \( h = L \otimes \mathbb{Z} \mathbb{C} \). Let \( \mathbb{C}[L] \) be the group algebra with a basis \( \{ e^\alpha | \alpha \in L \} \). Then the vertex operator algebra \( V_L \) associated to \( L \) is

\[
V_L = M(1) \otimes \mathbb{C}[L]
\]
as vector spaces. The vertex operator \( Y(v, z) \) for \( v \in M(1) \) is defined as in the case of \( M(1) \) (see formulas (3.1.1) and (3.1.2)). The operator \( a(n) \) for \( a \in h \) and \( n \neq 0 \) acts
on $V_L$ via its action on $M(1)$. The operator $a(0)$ acts on $V_L$ by acting on $\mathbb{C}[L]$ in the following way:

\begin{equation}
(4.1.1) \quad a(0)e^\alpha = (a, \alpha)e^\alpha
\end{equation}

for $\alpha \in L$. We identify $M(1)$ with $M(1) \otimes \varepsilon^0$. Then the vacuum of $V_L$ is the vacuum $1$ of $M(1)$ and the Virasoro element is the same as before. For the purposes of this paper we do not make explicit the expression for vertex operators $Y(v, z)$ for general $v$ except for those $v \in M(1)$. We remark that $V_L$ is a rational vertex operator algebra in the sense of [DLM].

Let $L^\circ = \{ \alpha \in L \otimes \mathbb{Z} \mid (\alpha, L) \subset \mathbb{Z} \}$ be the dual lattice of $L$ with coset decomposition $L^\circ = \cup_{i \in L^\circ/L}(L + \lambda_i)$. Then each space $V_{L+\lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$ is an irreducible $V_L$-module [FLM] where $\mathbb{C}[L + \lambda_i]$ is the subspace of the group algebra $\mathbb{C}[L^\circ]$ corresponding the coset $L + \lambda_i$. Moreover $\{V_{L+\lambda_i} \mid i \in L^\circ/L\}$ constitute the complete set of non-isomorphic irreducible $V_L$-modules [D]. Since $V_L$ also satisfies the $C_2$-condition (cf. [Zh], [DLM]) we have the following modular property for $V_L$ by [Zh]:

**Proposition 4.1.1.** Let $v \in (V_L)_{[n]}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$. Then $Z_i(v, q)$ converges to a holomorphic function in the upper half plane and there exist scalars $c_{ij}^\gamma$ independent of $v$ and $\tau$ such that

\[ Z_i(v, \tau + \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^n \sum_{j \in L^\circ/L} c_{i,j}^\gamma Z_j(v, \tau) \]

where $Z_i(v, \tau) = Z_i(v, q) = Z_{V_{L+\lambda_i}}(v, q)$.

### 4.2 Graded traces in $V_L$

In this subsection we determine $Z_i(v, q)$ for $i \in L^\circ/L$ and $v \in V_L$, and prove Theorem 4.

**Lemma 4.2.1.** Let $v \in M(1) \otimes e^\alpha$ for nonzero $\alpha \in L$. Then $Z_i(v, q) = 0$.

**Proof.** Recall from [FLM] the vertex operator $Y(v, z)$. It is easy to see that $v(n)M(1) \otimes e^\beta \subset M(1) \otimes e^{\alpha + \beta}$ for $\beta \in L + \lambda_i$ for all $n \in \mathbb{Z}$. So it is immediate that $Z_i(v, q) = 0$. \qed

It remains to determine $Z_i(v, q)$ for $v \in M(1)$. Note that as $M(1)$-module, $V_{L+\lambda_i} = \oplus_{\alpha \in L + \lambda_i} M(1) \otimes e^\alpha$ and each $M(1) \otimes e^\alpha$ is an irreducible $M(1)$-module. For any $v \in M(1)$ and $\alpha \in L^\circ$ we set

\[ Z_{\alpha}(v, q) = Z_{M(1) \otimes e^\alpha}(v, q). \]

Then

\[ Z_i(v, q) = \sum_{\alpha \in L + \lambda_i} Z_{\alpha}(v, q). \]
Lemma 4.2.2. Let $a \in \mathfrak{h}$ such that $(a,a) = 1$ and $\alpha \in L^\circ$. Then for any non-negative integer $r$, $Z_\alpha(a[-1]^r1, q)$ converges to holomorphic function in the upper half plane. Moreover there exist scalars $c_{r,r-2i}$ with $0 \leq i \leq r/2$ and $c_r = 1$ independent of $\alpha$ and $a$ such that

$$Z_\alpha(a[-1]^r1, q) = \left( \sum_{0 \leq i \leq r/2} c_{r,r-2i}(a, \alpha)^{r-2i} E_2(q)^i \right) q^{(\alpha,\alpha)/2} / \eta(q)^d.$$ 

Proof. We prove this by induction on $r$. If $r = 0$ the assertion is clear. If $r = 1$ the assertion follows from the fact that $o(a[-1]1) = (a, \alpha)$ on $M(1) \otimes e^\alpha$. Now we assume that $r \geq 2$. Using Corollary 2.2.2 gives

$$Z_\alpha(a[-1]^r1, q) = (a, \alpha) Z_\alpha(a[-1]^{r-1}1, q) + (r-1) E_2(q) Z_\alpha(a[-1]^{r-2}1, q).$$

By induction, both $Z_\alpha(a[-1]^{r-1}1, q)$ and $Z_\alpha(a[-1]^{r-2}1, q)$ converge to holomorphic functions in $H$, whence so does $Z_\alpha(a[-1]^r1, q)$. Indeed,

$$Z_\alpha(a[-1]^r1, q) = (a, \alpha) \left( \sum_{0 \leq i \leq (r-1)/2} c_{r-1,r-1-2i}(a, \alpha)^{r-1-2i} E_2(q)^i \right) q^{(\alpha,\alpha)/2} / \eta(q)^d$$

$$+ (r-1) E_2(q) \left( \sum_{0 \leq i \leq (r-2)/2} c_{r-2,r-2-2i}(a, \alpha)^{r-2-2i} E_2(q)^i \right) q^{(\alpha,\alpha)/2} / \eta(q)^d$$

$$= \left( \sum_{0 \leq i \leq r/2} c_{r,r-2i}(a, \alpha)^{r-2i} E_2(q)^i \right) q^{(\alpha,\alpha)/2} / \eta(q)^d.$$ 

It is easy to express $c_{r,r-2i}$ in terms of the $c_{r-1,r-2j}$ and $c_{r-2,r-2k}$, and so obtain a recursive formula. We leave this to the reader. Since $c_{r-1,r-2j}$ and $c_{r-2,r-2k}$ are independent of $a, \alpha, c_{r,r-2i}$ is also independent of $a, \alpha$. \hfill $\square$

For convenience, we set

$$f_{a,\alpha,r}(q) = \sum_{0 \leq i \leq r/2} c_{r,r-2i}(a, \alpha)^{r-2i} E_2(q)^i.$$ 

Recall that $\{h_1, ..., h_4\}$ is an orthonormal basis of $\mathfrak{h}$. Let $r_1, ..., r_d$ be non-negative integers. Then by Corollary 2.2.2 and Lemma 1.2, we obtain

Lemma 4.2.3. Let $v = h_1[-1]^{r_1} \cdots h_d[-1]^{r_d}1$. Then $Z_\alpha(v, q)$ converges to a holomorphic function in the upper half plane and

$$Z_\alpha(v, q) = f_{h_1,\alpha,r_1}(q) \cdots f_{h_d,\alpha,r_d}(q) q^{(\alpha,\alpha)/2} / \eta(q)^d.$$ 

12
Recall that $V_{L+\lambda_i} = \oplus_{\alpha \in L+\lambda_i} M(1) \otimes e^\alpha$. The following corollary is immediate:

**Corollary 4.2.4.** Let $v$ be as before. The function $Z_i(v, q)$ is equal to
\[
(\sum_{\alpha \in L+\lambda_i} f_{h_1,\alpha, r_1}(q) \cdots f_{h_d,\alpha, r_d}(q) q^{(\alpha,\alpha)/2})/\eta(q)^{d}.
\]

**Theorem 4.2.5.** Let $v \in M(1)$. Then $Z_i(v, q)$ can be expressed as a finite sum
\[
Z_i(v, q) = \sum_j f_j(q) \Theta_{L+\lambda_i, k_j}(a_j, q)/\eta(q)^{d}
\]
where $f_j(q) \in Q$, $k_j \geq 0$, $a_j \in h$ and
\[
\Theta_{L+\lambda_i, k_j}(a_j, q) = \sum_{\alpha \in L+\lambda_i} (a_j, \alpha)^{k_j} q^{(\alpha,\alpha)/2}.
\]

**Proof.** First we take $v = a[-n]^r \cdots a[-1]^r \cdot 1$ for $a \in h$ with $(a, a) = 1$. Then from Corollary 2.2.2 we have
\[
Z_i(v, q) = \sum_{s \geq 0} c_s g_s(q) Z_i(a[-1]^r \cdot 1, q)
\]
for some scalars $c_s$ and quasi-modular forms $g_s(q) \in Q$. Now the result follows from Corollary 4.2.4. For arbitrary $v$ we can assume that $v$ is a monomial in $h_j(-n)$ for $j = 1, \cdots, d$ and $n > 0$. Use the result for $a[-n]^r \cdots a[-1]^r \cdot 1$ and Corollary 2.2.2 again to see that $Z_i(v, q)$ is a linear combination of functions of the form
\[
f(q) \Theta_{L+\lambda_i, t_1, \cdots, t_d}(h_1, \cdots, h_d, q)/\eta(q)^{d}
\]
where $f(q)$ belong to $Q$, $t_j$ are non-negative integers and
\[
\Theta_{L+\lambda_i, t_1, \cdots, t_d}(h_1, \cdots, h_d, q) = \sum_{\alpha \in L+\lambda_i} (h_1, \alpha)^{t_1} \cdots (h_d, \alpha)^{t_d} q^{(\alpha,\alpha)/2}.
\]
It is easy to see that $\Theta_{L+\lambda_i, t_1, \cdots, t_d}(h_1, \cdots, h_d, q)$ is a linear combination of $\Theta_{L+\lambda_i, t_1+\cdots+t_d}(b, q)$ for $b \in h$. This completes the proof of the theorem.

We now concentrate on the case $i = 0$. Note that $Z_0(v, q) = Z(v, q)$. We prove

**Theorem 4.2.6.** Let $Q(N, \epsilon)$ be as in (1.4) in the introduction. For each $v$ in $V_L$, the function $Z(v, q)\eta(q)^d$ lies in $Q(N, \epsilon)$.
Proof. After theorem [4.2.5] it suffices to show that if $a$ in $\mathfrak{h}$ satisfies $(a,a) = 1$ then the function $\Theta_{L,k}(a,a)$ lies in $Q(N,\epsilon)$. This is essentially established in the paper [DM2]. In the notation of [DM2], the present function $\Theta_{L,k}(a,q)$ is denoted $\theta(Q,a,k,\tau)$ where $Q$ is the integral quadratic form which corresponds to $L$. If $k$ is odd then the function is identically zero. If $k = 2l$ is even then Theorem 2 of (loc cit) says that the function

$$\phi(Q,a,2l,\tau) = \sum_{t=0}^{l} \gamma(t,2l)E_2(\tau)^t\theta(Q,a,2l-2t,\tau)$$

is a holomorphic modular form in $M(N,\epsilon)$. In (4.2.1), $\gamma(t,2l)$ is a certain non-zero constant equal to 1 when $t = 0$. Moreover $\psi(Q,a,0,\tau)$ is just the theta function of $L$. Using this information, it follows from (4.2.1) and induction on $l$ that $\phi(Q,a,2l,\tau)$ indeed lies in $M(N,\epsilon)$, as required.

We are now in a position to complete the proof of Theorem 2. In Theorem 4.2.6 we have shown that each trace function $Z(v,q)$ satisfies (1.4) with $f(v,q)$ some element of $Q(N,\epsilon)$. We have to show that in fact $f(v,q)$ lies in $M(N,\epsilon)$. First note that we may assume without loss that $v$ lies in $(V_L)_k$ for some integer $k$ (cf. section 2.1). This puts us in a position to apply Proposition 4.1.1, which tell us that $Z(v,q)$ is a generalized modular form in the sense of the introduction. Thus the following result completes the proof of Theorem 2.

**Proposition 4.2.7.** An element of $Q(N,\epsilon)$ is a modular form of weight $k$ if, and only if, it is a generalized modular form of weight $k$.

**Proof.** It is enough to prove sufficiency. Let $f(\tau) \in Q(N,\epsilon)$, of weight $k$, be expressed in the form

$$f(\tau) = \sum_{i=0}^{m} f_i(\tau)E_2(\tau)^i$$

with each $f_i(\tau)$ a form in $M(N,\epsilon)$ of weight $k - 2i$. As $f(\tau)$ is also a generalized modular form of weight $k$, each of its $\gamma$-transforms $(\gamma \in SL(2,\mathbb{Z}))$ has a $q$-expansion. In particular, the $S$-transform of $f(\tau)$ yields an equality of the shape

$$\tau^{-k}f(S\tau) = \sum_{n} b(n)q^{n/N}.$$  

The $\gamma$-transform of $E_2(\tau)$ is well-known. In particular, we have

$$E_2(S\tau) = \tau^2E_2(\tau) - \tau/2\pi i.$$  

Comparing (4.2.2) - (4.2.4) we obtain an equality

$$\sum_{n} b(n)q^{n/N} = \sum_{i=0}^{m} \tau^{-k+2i}f_i(\tau)(E_2(\tau) - 1/\log q)^i.$$  

14
Since each $\tau^{-k+2i} f_i(S\tau)$ has a $q$-expansion, an equality like (4.2.5) can only hold if all $f_i(\tau)$ are zero for $i > 0$. Thus $f(\tau) = f_0(\tau)$ is indeed modular.

This completes the proof of Proposition 4.2.7, and hence that of Theorem 2, with the following caveat: in Theorem 4.2.6 and Proposition 4.2.7 we were implicitly dealing with modular forms of integral weight. But it is easy to see that these results and those of [DM2] extend to the half-integral case i.e., the case in which the lattice $L$ has odd rank. We leave the details to the interested reader.

5 Spherical harmonics and trace functions

In this section we first discuss the relation between spherical harmonics and highest weight vectors for the Virasoro algebra. Each spherical harmonic $P$ of degree $k$ gives rise to a highest weight vector $v_P$ of weight $k$ in a canonical way. We then prove Theorem 3.

5.1 Spherical harmonics and primary vectors

We continue our discussion of the vertex operator algebra $V_L$. In particular, $M(1)$ is a vertex operator subalgebra of $V_L$.

An element in $V_L$ is called a primary state (or singular vector or highest weight vector) if it satisfies $L(n)v = 0$ for all $n \in \mathbb{Z}_+$. Thanks to the Virasoro commutator relations, this is equivalent to $L(1)v = L(2)v = 0$.

Lemma 5.1.1. Let $v \in M(1)$ be a polynomial in the variables $h_i(-1)$, $(1 \leq i \leq d)$. Then $v$ is quasiprimary, i.e., $L(1)v = 0$.

Proof. Recall that $\{h_1, ..., h_d\}$ is an orthonormal basis of $\mathfrak{h}$. Note that

$$L(1) = \frac{1}{2} \sum_{i=1}^{d} \sum_{k \in \mathbb{Z}} \left( h_i(1-k)h_i(k) \right) = \sum_{i=1}^{d} \sum_{k \geq 1} h_i(1-k)h_i(k)$$

and both $h_i(k)$ for $k \geq 2$ and $h_i(0)$ annihilate $v$. Since all summands of $L(1)$ contain one of these as a factor, we immediately see that $L(1)v = 0$.

Lemma 5.1.2. Let $v \in M(1)$ be a polynomial in the variables $h_i(-1)$, $(1 \leq i \leq d)$. Then $v$ is primary if, and only if, $v$ is a spherical harmonic, i.e., $\Delta v = 0$ where

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial h_i(-1)^2}.$$
Proof. It suffices to consider the condition $L(2)v = 0$. Since

$$L(2) = \frac{1}{2} \sum_{i=1}^{d} h_i(1)^2 + \sum_{i=1}^{d} \sum_{k \geq 2} h_i(2-k)h_i(k)$$

and $h_i(k)$ annihilates $v$ if $k \geq 2$, we see that $L(2)v = 0$ is equivalent to $\sum_{i=1}^{d} h_i(1)^2 v = 0$. Now since $h_i(1)h_i(-1)^n 1 = nh_i(-1)^{n-1} 1$, $h_i(1)$ acts as $\partial/\partial h_i(-1)$ on $M(1)$. \qed

5.2 Proof of Theorem 3

Let $P$ be a spherical harmonic of degree $k$:

$$P = P(x_1, \ldots, x_d), \Delta P = 0$$

where $\Delta = \sum_{i=1}^{d} \partial^2/\partial x_i^2$. Let $v_P$ be the corresponding primary state in $M(1)$, i.e.,

$$v_P = P(h_1(-1), \ldots, h_d(-1)) 1.$$ 

We note that $o(v_P) = v_P(k - 1)$. Now let $P = \sum_{I} c_I x^I$ where $c_I \in \mathbb{C}$ and $x^I = x_1^{a_1} \cdots x_d^{a_d}$ and where $I = (a_1, \ldots, a_d)$ is some ordered tuple of non-negative integers such that $\sum a_i = k$. Then from Section 3.1,

$$Y(v_P, z) = \sum_{I} c_I Y(v_I, z) = \sum_{I} c_I \sum_{i} h_i(z)^{a_i}$$

where $h_i(z) = \prod_{i=1}^{d} h_i(z)^{a_i}$ and $h_i(z) = \sum_{n \in \mathbb{Z}} h_i(n) z^{-n-1}$. Therefore

$$o(v_P) = \sum_{I} c_I \sum_{i} h_i(n_{i} 1 \cdots h_i(n_{a_i}) h_2(n_{2a_1}) \cdots h_2(n_{2a_2}) \cdots h_d(n_{1a_1}) \cdots h_1(n_{da_d})$$

where the indices in the second sum range over all integers such that $\sum_{i=1}^{d} \sum_{j=1}^{a_i} n_{ij} = 0$. In particular, this includes the case with all $n_{ij} = 0$:

$$\sigma = \sum_{J} c_J h_1(0)^{a_1} \cdots h_d(0)^{a_d} = P(h_1(0), \ldots, h_d(0)).$$

As $h_i(0)$ operates on $u \otimes e^\alpha$ as in (4.1.1), we see that

$$\sigma : u \otimes e^\alpha \mapsto \left( \sum_{J} c_J (h_1, \alpha)^{a_1} \cdots (h_d, \alpha)^{a_d} \right) u \otimes e^\alpha.$$ 

But this is exactly $P(\alpha) u \otimes e^\alpha$. It follows that

$$\text{tr}_{V_L} q^{k_0 - \frac{d}{24}} = \sum_{\alpha \in L} P(\alpha) q^{(\alpha, \alpha)}/\eta(\tau)^d = \frac{\theta_L(P, \tau)}{\eta(\tau)^d}.$$
To prove the theorem, it remains to show that all other terms in (5.2.1) have trace 0 in their action on $V_L$. To see this, we adopt a different approach and note that to prove the theorem it is enough to establish it for a spanning set of spherical harmonics of degree $k$. These are given, for example [H], by

$$P = (t_1x_1 + \cdots + t_dx_d)^k$$

for $(t_1, \ldots, t_d) \in \mathbb{C}^d$ such that $\sum_{i=1}^d t_i^2 = 0$. Set $h_0 = \sum_{i=1}^d t_i h_i \in \mathfrak{h}$, and note that $(h_0, h_0) = \sum_{i=1}^d t_i^2 = 0$. Then

$$Y(v_P, z) = \circ (\sum_{i=1}^d t_i h_i(z))^k \circ = \circ h_0(z)^k \circ.$$

Clearly

$$o(v_p) = h_0(0)^k + \sum_{n \neq 0} \binom{k}{2} h_0(0)^{k-2} h_0(n) h_0(-n) + \cdots.$$

Then it is enough to show that each operator of the form

(5.2.2) $U = h_0(n_1)h_0(n_2) \cdots h_0(n_r), \quad (r \geq 2, \text{ not all } n_i \text{ are zero, } \sum n_i = 0)$

has trace 0 on $M(1)$. Let us take a basis $h_0, k_0, k_1, \ldots, k_{d-2}$ of $\mathfrak{h}$ with

$$(h_0, k_0) = 1, \quad (h_0, k_i) = 0, \quad (1 \leq i \leq d - 2).$$

Then $h_0(m)$ commutes with all $h_0(m), k_i(m), (1 \leq i \leq d, m \in \mathbb{Z})$. Now consider a vector in $M(1)$ of the form $p \cdot v$ where $p$ is a polynomial in $k_0(-m), m \in \mathbb{Z}_+$ and $v$ does not involve any mode $k_0(m), m \in \mathbb{Z}$. Since some $n_i$ occurring in (5.2.2) is positive, we see that the operator $U$ maps $p \cdot v$ to 0 or $p'v'$ where $p'$ has lower degree than $p$. Therefore the trace of $U$ on $M(1)$ is 0 as the operator $U$ never has non-zero eigenvalues. This completes the proof of Theorem 3.

As explained in the introduction, the result of Waldspurger [W] together with Theorem 3 implies the

**Corollary 5.2.1.** Let $f(q)$ be any cusp-form of level 1. There there is a primary field $v$ in the lattice VOA $V_{E_8}$ such that $Z(v, q) = f(q)/\eta(q)^8$.

**References**

[B] Borcherds, R.E., Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* 83 (1986), 3068-3071.
[D] Dong, C., Vertex algebras associated with even lattices, *J. Algebra* **160** (1993), 245–265.

[DLM] Dong, C., Li, H. and Mason, G., Modular invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* to appear, q-alg/9703010.

[DM1] Dong, C., Mason, G., Monstrous Moonshine of higher weight, math.QA/9803116.

[DM2] Dong, C., Mason, G., Transformation laws for theta-functions, math.QA/9903107.

[EZ] Eichler, M., Zagier, D., The Theory of Jacobi Forms, Progr. Math., 55, Birkhäuser, Boston, 1985.

[FLM] Frenkel, I.B., Lepowsky, J. and Meurman, A., Vertex operator algebras and the Monster, Academic Press, 1988.

[FZ] Frenkel, I.B., Zhu, Y., Vertex operator algebras associated to representations of affine and Virasoro algebras, *Duke Math. J.* **66** (1992), 123–168.

[HL] Harada, K., Lang, M., Modular forms associated with the Monster module. The Monster and Lie algebras (Columbus, OH, 1996), 59–83, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.

[H] Hecke, E., Analytische Arithmetik der positiven quadratischen Formen, in Mathematische Werke, Vandenhoeck and Ruprecht, Göttingen, 1983.

[KZ] Kaneko, M, Zagier, D., A generalized Jacobi theta function and quasimodular forms. The moduli space of curves (Texel Island, 1994), 165–172, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[KM] Knopp, M., Mason, G., Generalized modular forms, preprint.

[L] Li, H., Local systems of vertex operators, vertex superalgebras and modules, *J. Pure Appl. Algebra* **109** (1996), 143–195.

[MN] Matsuo, A., Nagatomo, K., Axioms for a Vertex Algebra and the Locality of Quantum Fields, MSJ Memoirs **4**, 1999.

[W] Waldspurger, J.L., Engendrement par des series theta de certains espaces de formes modulaires, *Invent. Math.* **50** (1979), 135–168.

[Za] Zagier, D., Modular forms and differential operators, *Proc. Indian Acad. Sci. Math. Sci.* **104** (1994), 57–75.
[Zh] Zhu, Y., Modular invariance of characters of vertex operator algebras, *J. AMS* 9 (1996), 237-301.