DEGREE OF IRRATIONALITY OF VERY GENERAL ABELIAN SURFACES

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1. Introduction

Given a projective variety $X$ of dimension $n$ which is not rational, one can try to quantify how far it is from being rational. When $n = 1$, the natural invariant is the gonality of a curve $C$, defined to be the smallest degree of a branched covering $C' \to \mathbb{P}^1$ (where $C'$ is the normalization of $C$). One generalization of gonality to higher dimensions is the degree of irrationality, defined as:

$$\text{irr}(X) = \min\{\delta > 0 \mid \exists \text{ a degree } \delta \text{ rational dominant map } X \to \mathbb{P}^n\}.$$

Recently, there has been significant progress in understanding the case of hypersurfaces of large degree (cf. [2], [3], [4]). The history behind the development of these ideas is described in [4]. The results of [2], [3], [4] depend on the positivity of the canonical bundles of the varieties in question, so it is interesting to consider what happens in the $K_X$-trivial case. Our purpose here is to prove the somewhat surprising fact that the degree of irrationality of a very general polarized abelian surface is uniformly bounded above, independently of the degree of the polarization.

To be precise, let $A = A_d$ be an abelian surface carrying a polarization $L = L_d$ of type $(1, d)$ and assume that $\text{NS}(A) = \mathbb{Z}[L]$. An argument of Stapleton [9] showed that there is a constant $C$ such that

$$\text{irr}(A) \leq C \cdot \sqrt{d}$$

for $d \gg 0$, and it was conjectured in [4] that equality holds asymptotically. Our main result shows that this is maximally false:

**Theorem 1.1.** For an abelian surface $A = A_d$ with Picard number $\rho = 1$, one has

$$\text{irr}(A) \leq 4.$$

We conjecture that in general equality holds. However, as far as we can see, the conjecture of [4] for polarized K3 surfaces $(S_d, B_d)$ of genus $d$ - namely, that there exist constants $C_1, C_2$ such that $C_1 \cdot \sqrt{d} \leq \text{irr}(S_d) \leq C_2 \cdot \sqrt{d}$ for $d \gg 0$ - remains plausible.\(^1\)

For an abelian variety $A$ of dimension $n$, it has been shown in [1] that $\text{irr}(A) \geq n + 1$ (for $n = 2$, one can also see this via Lemma 3.5). When $n = 2$, Yoshihara proved that $\text{irr}(A) = 3$ for abelian surfaces $A$ containing a smooth curve of genus 3 (cf. [11]). On a related note, Voisin [10] showed that the covering gonality of a very general abelian variety $A$ of dimension $n$ is bounded from below

\(^1\)In other words, $B_d$ is an ample line bundle on $S_d$ with $B_d^2 = 2d - 2$. 
by \( f(n) \), where \( f(n) \) grows like \( \log(n) \), and this lower bound was subsequently improved to \( \lceil \frac{1}{2}n + 1 \rceil \) by Martin [8].

In the proof of our theorem, assuming as we may that \( L \) is symmetric, we consider the space \( H^0(A, \mathcal{O}_A(2L))^+ \) of even sections of \( \mathcal{O}_A(2L) \). By imposing suitable multiplicities at the two-torsion points of \( A \), we construct a subspace \( V \subset H^0(A, \mathcal{O}_A(2L))^+ \) which numerically should define a rational map from \( A \) to a surface \( S \subset \mathbb{P}^N \). Using bounds on the degree of the map and the degree of \( S \), as well as projection from linear subspaces, we construct a degree 4 rational covering \( A \rightarrow \mathbb{P}^2 \).

The main difficulty is to deal with the possibility that \( \mathbb{P}_{sub}(V) \) has a fixed component; this approach was inspired in part by the work of Bauer in [5], [6].

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2. Set-up

Let \( A = A_d \) be an abelian surface with \( \rho(A) = 1 \). Assume \( \text{NS}(A) \cong \mathbb{Z}[L] \) where \( L \) is a polarization of type \((1, d)\) for some fixed \( d \geq 1 \), so that \( L^2 = 2d \) and \( h^0(L) = d \). Let

\[
\iota : A \rightarrow A, \quad x \mapsto -x
\]

be the inverse morphism, and let \( Z = \{p_1, \ldots, p_{16}\} \) be the set of two-torsion points of \( A \) (fixed points of \( \iota \)). We may assume that \( L \) is symmetric – that is, \( \iota^* \mathcal{O}_A(L) \cong \mathcal{O}_A(L) \) – by replacing \( L \) with a suitable translate. In particular, the cyclic group of order two acts on \( H^0(A, \mathcal{O}_A(2L))^+ \). The space of even sections \( H^0(A, \mathcal{O}_A(2L))^+ \) of the line bundle \( \mathcal{O}_A(2L) \) (sections \( s \) with the property that \( \iota^* s = s \)) has dimension

\[
h^0(A, 2L)^+ = 2d + 2
\]

(see [7, Corollary 4.6.6]). An even section of \( \mathcal{O}_A(2L) \) vanishes to even order at any two-torsion point, so we need to impose at most

\[
1 + 3 + \cdots + (2m - 1) = m^2
\]

conditions for every even section to vanish to order \( 2m \) at any fixed point \( p \in Z \) (see [5] for more details).

Fix any integer solutions \( a_1, \ldots, a_{16} \geq 0 \) to the equation

\[
\sum_{i=1}^{16} a_i^2 = 2d - 2,
\]

with \( a_{15} = 0 = a_{16} \). This is possible by Lagrange’s four-squares theorem. Let \( V \subset H^0(A, \mathcal{O}_A(2L))^+ \) be the space of even sections vanishing to order at least \( 2a_i \) at each point \( p_i \), such that

\[
\dim V \geq 2d + 2 - \sum_{i=1}^{16} a_i^2 \geq 4.
\]

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2. Covering gonality is defined as the minimum integer \( c > 0 \) such that given a general point \( x \in A \), there exists a curve \( C \) passing through \( x \) with gonality \( c \).

3. This assumption will be useful in Corollary 3.4. For larger values of \( d \), note that there are many solutions.
Let \( d = \mathbb{P}_{\text{sub}}(V) \subseteq |2L|^+ \) be the corresponding linear system of divisors, whose dimension is \( N := \dim d \geq 3 \). Write
\[
d_i := \text{mult}_{p_i} D
\]
for a general divisor \( D \in d \), so that \( d_i \geq 2a_i \).

**Remark 2.1.** From [7, Section 4.8], it follows that sections of \( V \) are pulled back from the singular Kummer surface \( A/\iota \), so any divisor \( D \in d \) is symmetric, i.e. \( \iota(D) = D \).

Let \( \varphi : A \rightarrow \mathbb{P}^N \) be the rational map given by the linear system \( d \) above, and write \( S := \overline{\text{Im}(\varphi)} \) for the image of \( \varphi \). Regardless of whether or not \( d \) has a fixed component, we find that:

**Proposition 2.2.** \( S \subset \mathbb{P}^N \) is an irreducible and nondegenerate surface.

**Proof.** Suppose for the sake of contradiction that \( \overline{\text{Im}(\varphi)} \) is a nondegenerate curve \( C \). Then \( \deg C \geq 3 \) since \( N \geq 3 \), and a hyperplane section of \( C \subset \mathbb{P}^N \) pulls back to a divisor with at least three irreducible components. This contradicts the fact that any divisor \( D(\sim \text{lin } 2L) \in d \) has at most two irreducible components since \( \text{NS}(A) = \mathbb{Z}[L] \). So the image of \( \varphi \) is a surface. \( \square \)

**Lemma 2.3.** Let \( \varphi : X \rightarrow \mathbb{P}^n \) be a rational map from a surface \( X \) to a projective space of dimension \( n \geq 2 \), and suppose that its image \( S := \overline{\text{Im}(\varphi)} \subset \mathbb{P}^n \) has dimension 2. Let \( d \) be the linear system corresponding to \( \varphi \) (assuming \( d \) has no base components). Then for any \( D \in d \),
\[
\deg \varphi \cdot \deg S \leq D^2.
\]

**Proof.** The indeterminacy locus of \( \varphi \) is a finite set. \( \square \)

### 3. Degree bounds

We now study the numerical properties of the linear series \( d \) constructed above. Keeping the notation as in §2:

**Lemma 3.1.** If \( d \) has no fixed components, then
\[
\deg \varphi \cdot \deg S \leq 8.
\]

**Proof.** By applying Proposition 2.2 and blowing-up \( A \) along the collection of two-torsion points \( Z \) to resolve some of the base points of \( d \), we arrive at the diagram
\[
\begin{array}{c}
\hat{A} := \text{Bl}_Z A \\
\Downarrow \pi \\
A \xrightarrow{\varphi} S \subset \mathbb{P}^N
\end{array}
\]

The linear system corresponding to \( \psi \) has no fixed components, so its divisors are of the form
\[
\hat{D} \sim \text{lin } \pi^* D - \sum_{i=1}^{16} d_i E_i,
\]
where \( \hat{D} \) denotes the strict transform of \( D \). By Lemma 2.3 applied to \( \psi \),
\[
\deg \varphi \cdot \deg S = \deg \psi \cdot \deg S \leq \hat{D}^2 = 4L^2 - \sum_{i=1}^{16} d_i^2 \leq 4 \left( 2d - \sum_{i=1}^{16} a_i^2 \right) = 8. \tag{1}
\]
The main work is to treat the case when \( \mathfrak{d} \) has a fixed divisor \( F \neq 0 \). In this situation, we may write:
\[
D = F + M \in \mathfrak{d} \quad \text{and} \quad \mathfrak{d} = F + b,
\]
where \( F \) and \( M \) are the fixed and movable components of \( \mathfrak{d} \), respectively. By definition, \( \dim \mathfrak{d} = \dim b \). Note that \( D \sim_{\text{lin}} 2L \) implies \( F, M \sim_{\text{alg}} L \) for all \( M \in b \). Choose a general divisor \( M \in b \) and write
\[
m_i := \text{mult}_{p_i} M \quad \text{and} \quad f_i := \text{mult}_{p_i} F,
\]
so that \( d_i = m_i + f_i \geq 2a_i \) for all \( i \). We claim that \( F \) must be symmetric as a divisor. If not, then
\[
\iota(M) + \iota(F) = \iota(D) = D = M + F \quad \text{for all} \quad D \in \mathfrak{d}.
\]
This implies that \( M = \iota(F) \) and \( F = \iota(M) \) for all \( M \in b \), which would mean that \( M \) must also be fixed, leading to a contradiction. Hence, \( F \) must be symmetric, and likewise for all \( M \in b \).

We first need an intermediate estimate:

**Proposition 3.2.** Assume \( \mathfrak{d} \) has a fixed component \( F \neq 0 \). Keeping the notation as above,
\[
\sum_{i=1}^{16} m_i^2 \geq 2d - 8.
\]

**Proof.** The idea here is to use the Kummer construction to push our fixed curve \( F \) onto a K3 surface and apply Riemann-Roch. This is analogous to a proof of Bauer’s in [6, Theorem 6.1]. Consider the smooth Kummer K3 surface \( K \) associated to \( A \):
\[
\begin{align*}
E \subset \hat{A} & \xrightarrow{\gamma} \hat{A}/\{1, \sigma\} =: K \\
\pi & \downarrow \\
Z \subset A
\end{align*}
\]
where \( \pi \) is the blow-up of \( A \) along the collection of two-torsion points \( Z \). Since the points in \( Z \) are \( \iota \)-invariant, \( \iota \) lifts to an involution \( \sigma \) on \( \hat{A} \) and the quotient \( K \) is a smooth K3 surface. Let \( E_i \) denote the exceptional curve over \( p_i \in Z \), so that \( E = \sum_{i=1}^{16} E_i \) is the exceptional divisor of \( \pi \). Since \( F \) is symmetric, its strict transform
\[
\hat{F} = \pi^* F - \sum_{i=1}^{16} f_i E_i,
\]
descends to an irreducible curve \( \hat{F} \subset K \). We claim that
\[
h^0(K, O_K(\hat{F})) = 1.
\]
In fact, if the linear system \( |O_K(\hat{F})| \) were to contain a pencil, then this would give us a pencil of symmetric curves in \( |O_A(F)| \) with the same multiplicities at the two-torsion points, which contradicts \( F \) being a fixed component of \( \mathfrak{d} \).
From the exact sequence $0 \to \mathcal{O}_K(-\bar{F}) \to \mathcal{O}_K \to \mathcal{O}_F \to 0$, it follows that $H^i(K, \mathcal{O}_K(\bar{F})) = 0$ for $i > 0$, so by Riemann-Roch
\[ 1 = h^0(K, \mathcal{O}_K(\bar{F})) = \chi(\mathcal{O}_K, \mathcal{O}_K(\bar{F})) = \frac{1}{2}(\bar{F})^2 + 2 \]
and therefore $(\bar{F})^2 = -2$. On the other hand, the equality
\[ -4 = 2(\bar{F})^2 = (\pi^*\bar{F})^2 = (\bar{F})^2 = F^2 - \sum_{i=1}^{16} f_i^2 = 2d - \sum_{i=1}^{16} f_i^2 \]
combined with $\sum_{i=1}^{16} f_i m_i \leq \sum_{i=1}^{16} (\frac{d}{2})^2$ yields
\[ \sum_{i=1}^{16} d_i^2 = \sum_{i=1}^{16} (f_i^2 + m_i^2 + 2f_im_i) \leq 2d + 4 + \sum_{i=1}^{16} m_i^2 \leq \frac{1}{2} \sum_{i=1}^{16} d_i^2. \]
After rearranging the terms, we find that
\[ \sum_{i=1}^{16} m_i^2 \geq -2d - 4 + \frac{1}{2} \sum_{i=1}^{16} d_i^2 \geq -2d - 4 + 2 \sum_{i=1}^{16} a_i^2 \geq 2d - 8 \]
for a general divisor $D = F + M \in \mathfrak{d}$, which is the desired inequality.

As an immediate consequence:

**Theorem 3.3.** Assume $\mathfrak{d}$ has a fixed component $F \neq 0$, and let $\mathfrak{b} = \mathfrak{d} - F$ be the linear system defining $\varphi : A \dashrightarrow S \subseteq \mathbb{P}^N$. Then
\[ \deg \varphi \cdot \deg S \leq 8. \]

**Proof.** As we saw in the proof of Lemma 3.1,
\[ \deg \varphi \cdot \deg S \leq M^2 - \sum_{i=1}^{16} m_i^2 \leq 2d - (2d - 8) = 8. \]  

**Corollary 3.4.** There exists a 4-to-1 rational map $\varphi : A \dashrightarrow \mathbb{P}^2$.

**Proof.** Recall that we chose the $a_i$ so that $a_{15} = 0 = a_{16}$. From Remark 2.1, it follows that $\varphi : A \dashrightarrow S \subset \mathbb{P}^N$ factors through the quotient $A \to A/\iota$, so $\deg \varphi$ must be even. The surface $S$ is nondegenerate, so $\deg S \geq 2$. By Lemma 3.5 below, it is impossible for $S$ to be rational together with $\deg \varphi = 2$, so $\{\deg \varphi = 2, \deg S = 2, 3\}$ is ruled out by the classification of quadric and cubic surfaces (using the fact that $\rho(A) = 1$).

Together with the upper bound $\deg \varphi \cdot \deg S \leq 8$ given by Lemma 3.1 and Theorem 3.3, there are two possibilities:
\[ \{\deg \varphi = 2, \deg S = 4\} \quad \text{and} \quad \{\deg \varphi = 4, \deg S = 2\}. \]

Either of these imply equality throughout (1) or (2), so that there is a morphism $\text{Bl}_Z A \to S$ which fits into the diagram:
where $K$ is the smooth Kummer K3 surface, $\gamma$ is a branched cover of degree 2, and $G_i := \gamma(E_i)$.

In the first case where $\deg \varphi = 2$ and $\deg S = 4$, from (1) and (2) it follows that $d_{15} = 0 = d_{16}$ or $n_{15} = 0 = n_{16}$. This implies that the curves $G_{15}, G_{16}$ are contracted and their images $q_{15}, q_{16}$ under $\alpha$ are double points on $S$ since $\alpha$ is a birational morphism. Projection from a general $(N - 3)$-plane containing one but not both of the $q_i$ defines a rational map $A \rightarrow \mathbb{P}^2$ of degree 2 (if $q_{15}$ is a cone point of $S$, pick a general plane passing through $q_{16}$, and vice versa). In the second case where $\deg \varphi = 4$ and $\deg S = 2$, note that $S$ is rational.

This immediately leads to Theorem 1.1. It is natural to ask what $\text{irr}(A_d)$ is equal to for a very general polarized abelian surface. At least one can see geometrically:

**Lemma 3.5.** There are no rational dominant maps $A \rightarrow \mathbb{P}^2$ of degree 2.

**Proof.** Suppose there exists such a map. We have the following diagram

$$
\begin{array}{ccc}
A[2] & \xrightarrow{\Sigma} & A \\
\downarrow g & & \downarrow h \\
A[2] \rightarrow \mathbb{P}^2 & \xrightarrow{\Sigma^{-1}(0)} & K[2](A)
\end{array}
$$


where $g$ is the pullback map on 0-cycles and $A[2]$ is the Hilbert scheme of 2 points on $A$. Since the rational map $\Sigma \circ g$ can be extended to a morphism, it must be constant. So $\text{Im}(g)$ is contained in a fiber $\Sigma^{-1}(0)$, which is a smooth Kummer K3 surface $K[2](A)$. Since $g$ is injective, it descends to an injective (and hence birational) map $h : \mathbb{P}^2 \rightarrow K[2](A)$, yielding a contradiction.

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