GEOMETRY OF WARPED PRODUCT AND CR-WARPED PRODUCT SUBMANIFOLDS IN KAHLER MANIFOLDS: MODIFIED VERSION

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Abstract. The warped product \( N_1 \times f N_2 \) of two Riemannian manifolds \((N_1, g_1)\) and \((N_2, g_2)\) is the product manifold \( N_1 \times N_2 \) equipped with the warped product metric \( g = g_1 + f^2 g_2 \), where \( f \) is a positive function on \( N_1 \). Warped products play very important roles in differential geometry as well as in physics. A submanifold \( M \) of a Kaehler manifold \( \tilde{M} \) is called a CR-warped product if it is a warped product \( M_T \times f N_\perp \) of a complex submanifold \( M_T \) and a totally real submanifold \( M_\perp \) of \( \tilde{M} \).

In this article we survey recent results on warped product and CR-warped product submanifolds in Kaehler manifolds. Several closely related results will also be presented.

1. Introduction

Let \( B \) and \( F \) be two Riemannian manifolds with Riemannian metrics \( g_B \) and \( g_F \), respectively, and let \( f \) be a positive function on \( B \). Consider the product manifold \( B \times F \) with its projection \( \pi : B \times F \to B \) and \( \eta : B \times F \to F \). The warped product \( M = B \times f F \) is the manifold \( B \times F \) equipped with the warped product Riemannian metric given by

\[
g = g_B + f^2 g_F
\]

We call the function \( f \) the warping function of the warped product \[4\]. The notion of warped products plays important roles in differential geometry as well as in physics, especially in the theory of general relativity (cf. \[30, 48\]).

A submanifold \( M \) of a Kaehler manifold \( (\tilde{M}, \tilde{g}, J) \) is called a CR-submanifold if there exist a holomorphic distribution \( \mathcal{D} \) and a totally real distribution \( \mathcal{D}^\perp \) on \( M \) such that \( TM = \mathcal{D} \oplus \mathcal{D}^\perp \), where \( TM \) denotes the tangent bundle of \( M \). The notion of CR-submanifolds was introduced by A. Bejancu (cf. \[8\]).

On the other hand, the author proved in \[17\] that there do not exist warped product submanifolds of the form \( M_L \times f N_T \) in any Kaehler manifold \( \tilde{M} \) such that \( N_T \) is a holomorphic submanifold and \( N_\perp \) is a totally real submanifold of \( \tilde{M} \). Moreover, the author introduced the notion of CR-warped products in \[17\] as follows. A submanifold \( M \) of a Kaehler manifold \( \tilde{M} \) is called a CR-warped product

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if it is a warped product $M_T \times_f N_\perp$ of a complex submanifold $M_T$ and a totally real submanifold $M_\perp$ of $\tilde{M}$.

A famous embedding theorem of J. F. Nash [47] states that every Riemannian manifold can be isometrically imbedded in a Euclidean space with sufficiently high codimension. In particular, the Nash theorem implies that every warped product manifold $N_1 \times_f N_2$ can be isometrically embedded as a Riemannian submanifold in a Euclidean space.

In view of Nash’s theorem, the author asked at the beginning of this century the following two fundamental questions (see [20, 21, 30]).

**Fundamental Question A.** What can we conclude from an arbitrary isometric immersion of a warped product manifold into a Euclidean space or more generally, into an arbitrary Riemannian manifold?

**Fundamental Question B.** What can we conclude from an arbitrary CR-warped product manifold into an arbitrary complex-space-form or more generally, into an arbitrary Kaehler manifold?

The study of these two questions was initiated by the author in a series of his articles [11, 13–15, 17–25, 29, 34]. Since then the study of warped product submanifolds has become an active research subject in differential geometry of submanifolds.

The purpose of this article is to survey recent results on warped product and CR-warped product submanifolds in Kaehler manifolds. Several closely related results will also be presented.

2. Preliminaries

In this section we provide some basic notations, formulas, definitions, and results.

For the submanifold $M$ we denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections of $M$ and $\tilde{M}^m$, respectively. The Gauss and Weingarten formulas are given respectively by (see, for instance, [6, 30, 31])

\[
\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\
\tilde{\nabla}_X \xi &= A_\xi X + D_X \xi
\end{align*}
\]

(2.1) \hspace{1cm} (2.2)

for any vector fields $X, Y$ tangent to $M$ and vector field $\xi$ normal to $M$, where $\sigma$ denotes the second fundamental form, $D$ the normal connection, and $A$ the shape operator of the submanifold.

Let $M$ be an $n$-dimensional submanifold of a Riemannian $m$-manifold $\tilde{M}^m$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ in $\tilde{M}^m$ such that, restricted to $M$, the vectors $e_1, \ldots, e_n$ are tangent to $M$ and hence $e_{n+1}, \ldots, e_m$ are normal to $M$. Let $\{\sigma^r_{ij}\}$, $i, j = 1, \ldots, n; r = n + 1, \ldots, m$, denote the coefficients of the second fundamental form $h$ with respect to $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$. Then we have

\[
\sigma^r_{ij} = \langle \sigma(e_i, e_j), e_r \rangle = \langle A_{e_r} e_i, e_j \rangle,
\]

where $\langle \ , \ \rangle$ denotes the inner product.
The mean curvature vector $\vec{H}$ is defined by
\begin{equation}
\vec{H} = \frac{1}{n} \text{trace } \sigma = \frac{1}{n} \sum_{i=1}^{n} \sigma(e_i, e_i),
\end{equation}
where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame of the tangent bundle $TM$ of $M$. The squared mean curvature is then given by
\begin{equation}
H^2 = \langle \vec{H}, \vec{H} \rangle.
\end{equation}
A submanifold $M$ is called minimal in $\tilde{M}^m$ if its mean curvature vector vanishes identically.

Let $R$ and $\tilde{R}$ denote the Riemann curvature tensors of $M$ and $\tilde{M}^m$, respectively. The equation of Gauss is given by
\begin{equation}
R(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \langle \sigma(X, W;), \sigma(Y, Z) \rangle - \langle \sigma(X, Z;), \sigma(Y, W) \rangle.
\end{equation}

Let $M$ be a Riemannian $p$-manifold and $e_1, \ldots, e_p$ be an orthonormal frame fields on $M$. For differentiable function $\varphi$ on $M$, the Laplacian $\Delta \varphi$ of $\varphi$ is defined by
\begin{equation}
\Delta \varphi = \sum_{j=1}^{p} \{ (\nabla e_j) \varphi - e_j \varphi \}.
\end{equation}

For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space $T_p M$ at a point $p \in M$, the scalar curvature $\tau$ of $M$ at $p$ is defined to be (cf. [9, 10, 26])
\begin{equation}
\tau(p) = \sum_{i<j} K(e_i \wedge e_j),
\end{equation}
where $K(e_i \wedge e_j)$ denotes the sectional curvature of the plane section spanned by $e_i$ and $e_j$.

3. Warped products in space forms

Let $M_1, \ldots, M_k$ be $k$ Riemannian manifolds and let
\begin{equation}
f: M_1 \times \cdots \times M_k \rightarrow \mathbb{E}^N
\end{equation}
be an isometric immersion of the Riemannian product $M_1 \times \cdots \times M_k$ into the Euclidean $N$-space $\mathbb{E}^N$. J. D. Moore [16] proved that if the second fundamental form $\sigma$ of $f$ has the property that $\sigma(X, Y) = 0$ for $X$ tangent to $M_i$ and $Y$ tangent to $M_j$, $i \neq j$, then $f$ is a product immersion, that is, there exist isometric immersions $f_i: M_i \rightarrow \mathbb{E}^{m_i}, 1 \leq i \leq k$, such that
\begin{equation}
f(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))
\end{equation}
when $x_i \in M_i$ for $1 \leq i \leq k$. 
Let $\phi : N_1 \times f N_2 \to R^m(c)$ be an isometric immersion of a warped product $N_1 \times f N_2$ into a Riemannian manifold with constant sectional curvature $c$. Denote by $\sigma$ the second fundamental form of $\phi$. The immersion $\phi : N_1 \times f N_2 \to R^m(c)$ is called mixed totally geodesic if $\sigma(X, Z) = 0$ for any $X$ in $D_1$ and $Z$ in $D_2$.

The next theorem provides a solution to Fundamental Question A.

**Theorem 3.1.** [20] For any isometric immersion $\phi : N_1 \times f N_2 \to R^m(c)$ of a warped product $N_1 \times f N_2$ into a Riemannian manifold of constant curvature $c$, we have

\[
\frac{\Delta f}{f} \leq \frac{n_2^2}{4n_2} H^2 + n_1 c,
\]

where $n_i = \dim N_i$, $n = n_1 + n_2$, $H^2$ is the squared mean curvature of $\phi$, and $\Delta f$ is the Laplacian of $f$ on $N_1$.

The equality sign of (3.2) holds identically if and only if $\iota : N_1 \times f N_2 \to R^m(c)$ is a mixed totally geodesic immersion satisfying $\text{trace} h_1 = \text{trace} h_2$, where $\text{trace} h_1$ and $\text{trace} h_2$ denote the trace of $\sigma$ restricted to $N_1$ and $N_2$, respectively.

By making a minor modification of the proof of Theorem 3.1 in [20], using the method of [28], we also have the following general solution from [44] to the Fundamental Question A.

**Theorem 3.2.** If $\tilde{M}_m$ is a Riemannian manifold with sectional curvatures bounded from above by a constant $c$, then for any isometric immersion $\phi : N_1 \times f N_2 \to \tilde{M}_m$ from a warped product $N_1 \times f N_2$ into $\tilde{M}_m$ the warping function $f$ satisfies

\[
\frac{\Delta f}{f} \leq \frac{n_2^2}{4n_2} H^2 + n_1 c,
\]

where $n_1 = \dim N_1$ and $n_2 = \dim N_2$.

An immediate consequence of Theorem 3.2 is the following.

**Corollary 3.1.** There do not exist minimal immersions of a Riemannian product $N_1 \times N_2$ of two Riemannian manifolds into a negatively curved Riemannian manifold $\tilde{M}$.

For arbitrary warped products submanifolds in complex hyperbolic spaces, we have the following general results from [23].

**Theorem 3.3.** Let $\phi : N_1 \times f N_2 \to CH^m(4c)$ be an arbitrary isometric immersion of a warped product $N_1 \times f N_2$ into the complex hyperbolic $m$-space $CH^m(4c)$ of constant holomorphic sectional curvature $4c$. Then we have

\[
\frac{\Delta f}{f} \leq \frac{(n_1 + n_2)^2}{4n_2} H^2 + n_1 c.
\]

The equality sign of (3.4) holds if and only if the following three conditions hold.

1. $\phi$ is mixed totally geodesic,
2. $\text{trace} h_1 = \text{trace} h_2$, and
3. $J D_1 \perp D_2$, where $J$ is the almost complex structure of $CH^m$. 

Some interesting immediate consequences of Theorem 3.3 are the following non-existence results [23].

**Corollary 3.2.** Let $N_1 \times_f N_2$ be a warped product whose warping function $f$ is harmonic. Then $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.

**Corollary 3.3.** If $f$ is an eigenfunction of Laplacian on $N_1$ with eigenvalue $\lambda > 0$, then $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.

**Corollary 3.4.** If $N_1$ is compact, then every warped product $N_1 \times_f N_2$ does not admit an isometric minimal immersion into any complex hyperbolic space.

For arbitrary warped products submanifolds in the complex projective $m$-space $CP^m(4c)$ with constant holomorphic sectional curvature $4c$, we have the following results from [25].

**Theorem 3.4.** Let $\phi : N_1 \times_f N_2 \rightarrow CP^m(4c)$ be an arbitrary isometric immersion of a warped product into the complex projective $m$-space $CP^m(4c)$ of constant holomorphic sectional curvature $4c$. Then we have

$$\Delta f \leq \frac{(n_1 + n_2)^2}{4n_2}H^2 + (3 + n_1)c. \tag{3.5}$$

The equality sign of (3.5) holds identically if and only if we have

1. $n_1 = n_2 = 1$,
2. $f$ is an eigenfunction of the Laplacian of $N_1$ with eigenvalue $4c$, and
3. $\phi$ is totally geodesic and holomorphic.

An immediate application of Theorem 3.4 is the following non-immersion result.

**Corollary 3.5.** If $f$ is a positive function on a Riemannian $n_1$-manifold $N_1$ such that $(\Delta f)/f > 3 + n_1$ at some point $p \in N_1$, then, for any Riemannian manifold $N_2$, the warped product $N_1 \times_f N_2$ does not admit any isometric minimal immersion into $CP^m(4)$ for any $m$.

Theorem 3.4 can be sharpened as the following theorem for totally real minimal immersions.

**Theorem 3.5.** If $f$ is a positive function on a Riemannian $n_1$-manifold $N_1$ such that $(\Delta f)/f > n_1$ at some point $p \in N_1$, then, for any Riemannian manifold $N_2$, the warped product $N_1 \times_f N_2$ does not admit any isometric totally real minimal immersion into $CP^m(4)$ for any $m$.

The following examples illustrate that Theorems 3.3, 3.4 and 3.5 are sharp.

**Example 3.1.** Let $I = (-\pi/4, \pi/4)$, $N_2 = S^1(1)$ and $f = \frac{1}{2}\cos 2s$. Then the warped product

$$N_1 \times_f N_2 =: I \times_{(\cos 2s)/2} S^1(1)$$

has constant sectional curvature $4$. Clearly, we have $(\Delta f)/f = 4$. If we define the complex structure $J$ on the warped product by $J \left(\frac{\partial}{\partial s}\right) = 2(\sec 2s)\frac{\partial}{\partial t}$, then
(I × (cos 2s)/2 S^1(1), g, J) is holomorphically isometric to a dense open subset of \( CP^1(4) \).

Let \( \phi : CP^1(4) \to CP^m(4) \) be a standard totally geodesic embedding of \( CP^1(4) \) into \( CP^m(4) \). Then the restriction of \( \phi \) to \( I \times (cos 2s)/2 S^1(1) \) gives rise to a minimal isometric immersion of \( I \times (cos 2s)/2 S^1 \) into \( CP^m(4) \) which satisfies the equality case of inequality (3.5) on \( I \times (cos 2s)/2 S^1(1) \) identically.

**Example 3.2.** Consider the same warped product \( N_1 \times_f N_2 = I \times (cos 2s)/2 S^1(1) \) as given in Example 5.1. Let \( \phi \) be a standard totally geodesic holomorphic embedding of \( CP^1(4) \) into \( CP^m(4) \). Then the restriction of \( \phi \) to \( N_1 \times_f N_2 \) is an isometric minimal immersion of \( N_1 \times_f N_2 \) into \( CP^m(4) \) which satisfies \( (\Delta f)/f = 3 + n_1 \) identically. This example shows that the assumption “\((\Delta f)/f > 3 + n_1\) at some point in \( N_1 \)” given in Theorem 3.4 is best possible.

**Example 3.3.** Let \( g_1 \) be the standard metric on \( S^n(1) \). Denote by \( N_1 \times_f N_2 \) the warped product given by \( N_1 = (-\pi/2, \pi/2), N_2 = S^n(1) \) and \( f = \cos s \). Then the warping function of this warped product satisfies \( \Delta f/f = n_1 \) identically. Moreover, it is easy to verify that this warped product is isometric to a dense open subset of \( S^n \). Let

\[ \phi : S^n(1) \xrightarrow{\text{projection to 2:1}} RP^n(4) \xrightarrow{\text{totally geodesic}} CP^m(4) \]

be a standard totally geodesic Lagrangian immersion of \( S^n(1) \) into \( CP^m(4) \). Then the restriction of \( \phi \) to \( N_1 \times_f N_2 \) is a totally real minimal immersion. This example illustrates that the assumption “\((\Delta f)/f > n_1\) at some point in \( N_1 \)” given in Theorem 3.5 is also sharp.

### 4. Segre Embedding and Its Converse

For simplicity, we denote \( S^n(1) \), \( RP^n(1) \), \( CP^n(4) \) and \( CH^n(-4) \) by \( S^n \), \( RP^n \), \( CP^n \) and \( CH^n \), respectively.

Let \( (z_0^i, \ldots, z_s^i) \), \( 1 \leq i \leq s \), be homogeneous coordinates of \( CP^{n_i} \). Define a map:

\[ S_{\alpha_1 \cdots \alpha_s} : CP^{n_1} \times \cdots \times CP^{n_s} \to CP^N, \quad N = \prod_{i=1}^s (n_i + 1) - 1, \]

which maps a point \( ((z_0^1, \ldots, z_s^1), \ldots, (z_0^s, \ldots, z_s^s)) \) in \( CP^{n_1} \times \cdots \times CP^{n_s} \) to the point \( (z_0^1 \cdots z_s^i)_{1 \leq i_1 \leq n_1, \ldots, 1 \leq i_s \leq n_s} \) in \( CP^N \). The map \( S_{\alpha_1 \cdots \alpha_s} \) is a Kaehler embedding which is known as the *Segre embedding*. The Segre embedding was constructed by C. Segre in 1891.

The following results from \([7, 39]\) established in 1981 can be regarded as the “converse” to Segre embedding constructed in 1891.

**Theorem 4.1.** Let \( M_1^{n_1}, \ldots, M_s^{n_s} \) be Kaehler manifolds of dimensions \( \alpha_1, \ldots, \alpha_s \), respectively. Then every holomorphically isometric immersion

\[ f : M_1^{n_1} \times \cdots \times M_s^{n_s} \to CP^N, \quad N = \prod_{i=1}^s (n_i + 1) - 1, \]
of $M_1^{\alpha_1} \times \cdots \times M_s^{\alpha_s}$ into $CP^N$ is locally the Segre embedding, i.e., $M_1^{\alpha_1},\ldots,M_s^{\alpha_s}$ are open portions of $CP^{\alpha_1},\ldots,CP^{\alpha_s}$, respectively. Moreover, $f$ is congruent to the Segre embedding.

Let $\overline{\nabla}^k\sigma$, $k = 0, 1, 2, \ldots$, denote the $k$-th covariant derivative of the second fundamental form. Denoted by $||\overline{\nabla}^k\sigma||^2$ the squared norm of $\overline{\nabla}^k\sigma$.

The following result was proved in [39].

**Theorem 4.2.** Let $M_1^{\alpha_1} \times \cdots \times M_s^{\alpha_s}$ be a product Kaehler submanifold of $CP^N$. Then

$$||\overline{\nabla}^{k-2}\sigma||^2 \geq k! 2^k \sum_{i_1<\cdots<i_k} \alpha_{i_1} \cdots \alpha_{i_k},$$

for $k = 2, 3, \ldots$.

The equality sign of (4.2) holds for some $k$ if and only if $M_1^{\alpha_1},\ldots,M_s^{\alpha_s}$ are open parts of $CP^{\alpha_1},\ldots,CP^{\alpha_s}$, respectively, and the immersion is congruent to the Segre embedding.

If $k = 2$, Theorem 4.2 reduces to the following result of [7].

**Theorem 4.3.** Let $M_1^h \times M_2^p$ be a product Kaehler submanifold of $CP^N$. Then we have

$$||\sigma||^2 \geq 8hp.$$  

The equality sign of inequality (4.3) holds if and only if $M_1^h$ and $M_2^p$ are open portions of $CP^h$ and $CP^p$, respectively, and moreover the immersion is congruent to the Segre embedding $S_{h,p}$.

We may extend Theorem 4.3 to the following for warped products.

**Theorem 4.4.** Let $(M_1^h, g_1)$ and $(M_2^p, g_2)$ be two Kaehler manifolds of complex dimension $h$ and $p$ respectively and let $f$ be a positive function on $M_1^h$. If $\phi : M_1^h \times_f M_2^p \rightarrow CP^N$ is a holomorphically isometric immersion of the warped product manifold $M_1^h \times_f M_2^p$ into $CP^N$. Then $f$ is a constant, say $c$. Moreover, we have

$$||\sigma||^2 \geq 8hp.$$  

The equality sign of (4.4) holds if and only if $(M_1^h, g_1)$ and $(M_2^p, cg_2)$ are open portions of $CP^h$ and $CP^p$, respectively, and moreover the immersion $\phi$ is congruent to the Segre embedding.

**Proof.** Under the hypothesis, the warped product manifold $M_1^h \times_f M_2^p$ must be a Kaehler manifold. Therefore, the warping function $f$ must be a positive constant. Consequently, the theorem follows from Theorem 4.3. \qed

5. **CR-products in Kaehler manifolds**

A submanifold $N$ in a Kaehler manifold $\tilde{M}$ is called a *totally real submanifold* if the almost complex structure $J$ of $\tilde{M}$ carries each tangent space $T_x N$ of $N$ into its corresponding normal space $T^\perp_x N$ [15, 19, 41]. The submanifold $N$ is called a
holomorphic submanifold (or Kaehler submanifold) if \( J \) carries each \( T_x N \) into itself. The submanifold \( N \) is called slant \([11]\) if for any nonzero vector \( X \) tangent to \( N \) the angle \( \theta(X) \) between \( JX \) and \( T_p N \) does not depend on the choice of the point \( p \in N \) and of the choice of the vector \( X \in T_p N \). On the other hand, it depends only on the point \( p \), then the submanifold \( N \) is called pointwise slant \([37]\).

Let \( M \) be a submanifold of a Kaehler manifold \( \tilde{M} \). For each point \( p \in M \), put
\[
\mathcal{H}_p = T_p M \cap J(T_p M),
\]
i.e., \( \mathcal{H}_p \) is the maximal holomorphic subspace of the tangent space \( T_p M \). If the dimension of \( \mathcal{H}_p \) remains the same for each \( p \in M \), then \( M \) is called a generic submanifold \([10]\).

A CR-submanifold of a Kaehler manifold \( \tilde{M} \) is called a CR-product \([7, 8]\) if it is a Riemannian product \( N_T \times N_\perp \) of a Kaehler submanifold \( N_T \) and a totally real submanifold \( N_\perp \). A CR-submanifold is called mixed totally geodesic if the second fundamental form of the CR-submanifold satisfying
\[
\sigma(X, Z) = 0
\]
for any \( X \in \mathcal{D} \) and \( Z \in \mathcal{D}_\perp \).

For CR-products in complex space forms, the following result are known.

**Theorem 5.1.** \([7]\) We have

(i) A CR-submanifold in the complex Euclidean \( m \)-space \( \mathbb{C}^m \) is a CR-product if and only if it is a direct sum of a Kaehler submanifold and a totally real submanifold of linear complex subspaces.

(ii) There do not exist CR-products in complex hyperbolic spaces other than Kaehler submanifolds and totally real submanifolds.

CR-products \( N_T \times N_\perp \) in \( \mathbb{CP}^{h+p+hp} \) are obtained from the Segre embedding \( S_{h,p} \); namely, we have the following results.

**Theorem 5.2.** \([7]\) Let \( N^h_T \times N^p_\perp \) be the CR-product in \( \mathbb{CP}^m \) with constant holomorphic sectional curvature \( A \). Then

\[
m \geq h + p + hp.
\]

The equality sign of (5.1) holds if and only if

(a) \( N^h_T \) is a totally geodesic Kaehler submanifold,

(b) \( N^p_\perp \) is a totally real submanifold, and

(c) the immersion is given by

\[
N^h_T \times N^p_\perp \to \mathbb{CP}^h \times \mathbb{CP}^p \xrightarrow{S_{h,p}} \mathbb{CP}^{h+p+hp}.
\]

**Theorem 5.3.** \([7]\) Let \( N^h_T \times N^p_\perp \) be a CR-product in \( \mathbb{CP}^m \). Then the squared norm of the second fundamental form satisfies

\[
||\sigma||^2 \geq 4hp.
\]

The equality sign of (5.2) holds if and only if

(a) \( N^h_T \) is a totally geodesic Kaehler submanifold,
(b) $N^T_\perp$ is a totally geodesic totally real submanifold, and
(c) the immersion is given by
$$N^h_T \times N^\perp_\perp \overset{\text{totally geodesic}}{\longrightarrow} CP^h \times CP^p \overset{\text{Segre imbedding}}{\longrightarrow} CP^{h+p+hp} \subset CP^m.$$  

6. Warped Product CR-submanifolds

In this section we present known results on warped product CR-submanifold in Kaehler manifolds. First, we mention the following result.

Theorem 6.1. [17] If $N^\perp_\perp \times f N^T_\perp$ is a warped product CR-submanifold of a Kaehler manifold $\tilde{M}$ such that $N^\perp_\perp$ is a totally real and $N^T_\perp$ a Kaehler submanifold of $\tilde{M}$, then it is a CR-product.

Theorem 6.1 shows that there does not exist warped product CR-submanifolds of the form $N^\perp_\perp \times f N^T_\perp$ other than CR-products. So, we only need to consider warped product CR-submanifolds of the form: $N^T_\perp \times f N^\perp_\perp$, by reversing the two factors $N^T_\perp$ and $N^\perp_\perp$ of the warped product. The author simply calls such CR-submanifolds CR-warped products in [17].

CR-warped products are simply characterized as follows.

Proposition 6.1. [17] A proper CR-submanifold $M$ of a Kaehler manifold $\tilde{M}$ is locally a CR-warped product if and only if the shape operator $A$ satisfies

$$A_{JZ}X = ((JX)\mu)Z, \quad X \in \mathcal{D}, \quad Z \in \mathcal{D}^\perp,$$

for some function $\mu$ on $M$ satisfying $W\mu = 0$, $\forall W \in \mathcal{D}^\perp$.

A fundamental result on CR-warped products in arbitrary Kaehler manifolds is the following theorem.

Theorem 6.2. [17, 34] Let $N^T_\perp \times f N^\perp_\perp$ be a CR-warped product submanifold in an arbitrary Kaehler manifold $\tilde{M}$. Then the second fundamental form $\sigma$ satisfies

$$||\sigma||^2 \geq 2p||\nabla(\ln f)||^2,$$

where $\nabla(\ln f)$ is the gradient of $\ln f$ on $N^T_\perp$ and $p = \dim N^\perp_\perp$.

If the equality sign of (6.2) holds identically, then $N^T_\perp$ is a totally geodesic Kaehler submanifold and $N^\perp_\perp$ is a totally umbilical totally real submanifold of $\tilde{M}$. Moreover, $N^T_\perp \times f N^\perp_\perp$ is minimal in $\tilde{M}$.

When $M$ is anti-holomorphic, i.e., when $J\mathcal{D}^\perp = T^\perp \perp N$, and $p > 1$. The equality sign of (6.2) holds identically if and only if $N^\perp_\perp$ is a totally umbilical submanifold of $\tilde{M}$.

Let $M$ be anti-holomorphic with $p = 1$. The equality sign of (6.2) holds identically if the characteristic vector field $J\xi$ of $M$ is a principal vector field with zero as its principal curvature. Conversely, if the equality sign of (6.2) holds, then the characteristic vector field $J\xi$ of $M$ is a principal vector field with zero as its principal curvature only if $M = N^T_\perp \times f N^\perp_\perp$ is a trivial CR-warped product immersed in $M$ as a totally geodesic hypersurface.

Also, when $M$ is anti-holomorphic with $p = 1$, the equality sign of (6.2) holds identically if and only if $M$ is a minimal hypersurface in $\tilde{M}$. 
CR-warped products in complex space forms satisfying the equality case of (6.2) have been completely classified in [17, 18].

**Theorem 6.3.** A CR-warped product \( N_T \times_f N_\perp \) in \( \mathbb{C}^m \) satisfies

\[
||\sigma||^2 = 2p||\nabla(\ln f)||^2
\]

identically if and only if the following four statements hold:

(i) \( N_T \) is an open portion of a complex Euclidean h-space \( \mathbb{C}^h \),

(ii) \( N_\perp \) is an open portion of the unit p-sphere \( S^p \),

(iii) there exists \( a = (a_1, \ldots, a_h) \in S^{h-1} \subset \mathbb{E}^h \) such that \( f = \sqrt{(a, z)^2 + (ia, z)^2} \)

for \( z = (z_1, \ldots, z_h) \in \mathbb{C}^h \), \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1} \), and

(iv) up to rigid motions, the immersion is given by

\[
\mathbf{x}(z, w) = \left( z_1 + (w_0 - 1)a_1 \sum_{j=1}^h a_j z_j, \ldots, z_h + (w_0 - 1)a_h \sum_{j=1}^h a_j z_j, w_1 \sum_{j=1}^h a_j z_j, \ldots, w_p \sum_{j=1}^h a_j z_j, 0, \ldots, 0 \right).
\]

A CR-warped product \( N_T \times_f N_\perp \) is said to be trivial if its warping function \( f \) is constant. A trivial CR-warped product \( N_T \times_f N_\perp \) is nothing but a CR-product \( N_T \times N_\perp^f \), where \( N_\perp^f \) is the manifold with metric \( f^2 g_{N_\perp} \) which is homothetic to the original metric \( g_{N_\perp} \) on \( N_\perp \).

The following result completely classifies CR-warped products in complex projective spaces satisfying the equality case of (6.2) identically.

**Theorem 6.4.** [18] A non-trivial CR-warped product \( N_T \times_f N_\perp \) in the complex projective m-space \( CP^m(4) \) satisfies the basic equality \( ||\sigma||^2 = 2p||\nabla(\ln f)||^2 \) if and only if we have

1. \( N_T \) is an open portion of complex Euclidean h-space \( \mathbb{C}^h \),
2. \( N_\perp \) is an open portion of a unit p-sphere \( S^p \), and
3. up to rigid motions, the immersion \( \mathbf{x} \) of \( N_T \times_f N_\perp \) into \( CP^m \) is the composition \( \pi \circ \mathbf{x} \), where

\[
\mathbf{x}(z, w) = \left( z_0 + (w_0 - 1)a_0 \sum_{j=0}^h a_j z_j, \ldots, z_h + (w_0 - 1)a_h \sum_{j=0}^h a_j z_j, w_1 \sum_{j=0}^h a_j z_j, \ldots, w_p \sum_{j=0}^h a_j z_j, 0, \ldots, 0 \right)
\]

\( \pi \) is the projection \( \pi : \mathbb{C}^{m+1} \rightarrow CP^m \), \( a_0, \ldots, a_h \) are real numbers satisfying \( a_0^2 + a_1^2 + \cdots + a_h^2 = 1 \), \( z = (z_0, z_1, \ldots, z_h) \in \mathbb{C}^{h+1} \) and \( w = (w_0, \ldots, w_p) \in S^p \subset \mathbb{E}^{p+1} \).

The following result completely classifies CR-warped products in complex hyperbolic spaces satisfying the equality case of (6.2) identically.
Theorem 6.5. \cite{18} A CR-warped product $N_T \times_f N_\perp$ in the complex hyperbolic m-space $CH^m(-4)$ satisfies the basic equality

$$||\sigma||^2 = 2p||\nabla(\ln f)||^2$$

if and only if one of the following two cases occurs:

1. $N_T$ is an open portion of complex Euclidean h-space $\mathbb{C}^h$, $N_\perp$ is an open portion of a unit p-sphere $S^p$ and, up to rigid motions, the immersion is the composition $\pi \circ \tilde{x}$, where $\pi$ is the projection $\pi : \mathbb{C}^{m+1} \to CH^m$ and

$$\tilde{x}(z,w) = \left( z_0 + a_0(1 - w_0) \sum_{j=0}^h a_j z_j, z_1 + a_1(1 - w_0) \sum_{j=0}^h a_j z_j, \ldots, 
\right)$$

for some real numbers $a_0, \ldots, a_h$ satisfying $a_0^2 - a_1^2 - \cdots - a_h^2 = -1$, where $z = (z_0, \ldots, z_h) \in \mathbb{C}_1^{h+1}$ and $w = (w_0, \ldots, w_p) \in S^p \subset E^{p+1}$.

2. $p = 1$, $N_T$ is an open portion of $\mathbb{C}^h$ and, up to rigid motions, the immersion is the composition $\pi \circ \tilde{x}$, where

$$\tilde{x}(z,t) = \left( z_0 + a_0(\cosh t - 1) \sum_{j=0}^h a_j z_j, z_1 + a_1(1 - \cosh t) \sum_{j=0}^h a_j z_j, 
\right)$$

for some real numbers $a_0, a_1, \ldots, a_{h+1}$ satisfying $a_0^2 - a_1^2 - \cdots - a_{h+1}^2 = 1$.

A multiply warped product $N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ in a Kaehler manifold $\tilde{M}$ is called a multiply CR-warped product if $N_T$ is a holomorphic submanifold and $N_\perp = f_2 N_2 \times \cdots \times f_k N_k$ is a totally real submanifold of $\tilde{M}$.

The next theorem extends \cite{6,2} for multiply CR-warped products.

Theorem 6.6. \cite{35} Let $N = N_T \times_{f_2} N_2 \times \cdots \times_{f_k} N_k$ be a multiply CR-warped product in an arbitrary Kaehler manifold $\tilde{M}$. Then the second fundamental form $\sigma$ and the warping functions $f_2, \ldots, f_k$ satisfy

$$||\sigma||^2 \geq 2 \sum_{i=2}^k n_i ||\nabla(\ln f_i)||^2.$$  

(6.4)

The equality sign of inequality \cite{6,4} holds identically if and only if the following four statements hold:

(a) $N_T$ is a totally geodesic holomorphic submanifold of $\tilde{M}$;

(b) For each $i \in \{2, \ldots, k\}$, $N_i$ is a totally umbilical submanifold of $\tilde{M}$ with $-\nabla(\ln f_i)$ as its mean curvature vector;

(c) $f_2 N_2 \times \cdots \times_{f_k} N_k$ is immersed as mixed totally geodesic submanifold in $\tilde{M}$; and

(d) For each point $p \in N$, the first normal space $\text{Im} \ h_p$ is a subspace of $J(T_p N_\perp)$. 
Remark 6.1. B. Sahin [50] extends Theorem 6.1 to the following.

Theorem 6.7. There exist no warped product submanifolds of the type \( M_\theta \times_f M_T \) and \( M_T \times f M_\theta \) in a Kaehler manifold, where \( M_\theta \) is a proper slant submanifold and \( M_T \) is a holomorphic submanifold of \( \tilde{M} \).

Remark 6.2. As an extension of Theorem 6.7 the following non-existence result was proved by K. A. Khan, S. Ali and N. Jamal.

Theorem 6.8. [45] There do not exist proper warped product submanifolds of the type \( N \times_f N_T \) and \( N_T \times f N \) in a Kaehler manifold, where \( N_T \) is a complex submanifold and \( N \) is any non-totally real generic submanifold of a Kaehler manifold \( \tilde{M} \).

Remark 6.3. B. Sahin proved the following.

Theorem 6.9. [51] There do not exist doubly warped product CR-submanifolds which are not (singly) warped product CR-submanifolds in the form \( f_1 M_T \times f_2 M_\perp \), where \( M_T \) is a holomorphic submanifold and \( M_\perp \) is a totally real submanifold of a Kaehler manifold \( \tilde{M} \).

7. CR-warpED PRODUCTS WITH COMPACT HOLOMORPHIC FACTOR

When the holomorphic factor \( N_T \) of a CR-warped product \( N_T \times_f \) \( N_\perp \) is compact, we have the following sharp results.

Theorem 7.1. [27] Let \( N_T \times_f N_\perp \) be a CR-warped product in the complex projective \( m \)-space \( CP^m(4) \) of constant holomorphic sectional curvature 4. If \( N_T \) is compact, then we have

\[
m \geq h + p + hp.
\]

Remark 7.1. The example mentioned in Statement (c) of Theorem 5.2 shows that Theorem 7.1 is sharp.

Theorem 7.2. [27] If \( N_T \times_f N_\perp \) is a CR-warped product in \( CP^{h+p+hp}(4) \) with compact \( N_T \), then \( N_T \) is holomorphically isometric to \( CP^h \).

Theorem 7.3. [27] For any CR-warped product \( N_T \times_f N_\perp \) in \( CP^m(4) \) with compact \( N_T \) and any \( q \in N_\perp \), we have

\[
\int_{N_T \times \{q\}} ||\sigma||^2 dV_T \geq 4hp \text{vol}(N_T),
\]

where \( ||\sigma|| \) is the norm of the second fundamental form, \( dV_T \) is the volume element of \( N_T \), and \( \text{vol}(N_T) \) is the volume of \( N_T \).

The equality sign of (7.1) holds identically if and only if we have:

1. The warping function \( f \) is constant.
2. \( (N_T, g_{N_T}) \) is holomorphically isometric to \( CP^h(4) \) and it is isometrically immersed in \( CP^m \) as a totally geodesic complex submanifold.
3. \( (N_\perp, f^2 g_{N_\perp}) \) is isometric to an open portion of the real projective \( p \)-space \( RP^p(1) \) of constant sectional curvature one and it is isometrically immersed in \( CP^m \) as a totally geodesic totally real submanifold.
(4) \( N_T \times_f N_\perp \) is immersed linearly fully in a complex subspace \( CP^{h+p+h_p}(4) \) of \( CP^m(4) \); and moreover, the immersion is rigid.

**Theorem 7.4.** [27] Let \( N_T \times_f N_\perp \) be a CR-warped product with compact \( N_T \) in \( CP^m(4) \). If the warping function \( f \) is a non-constant function, then for each \( q \in N_\perp \) we have

\[
\int_{N_T \times \{q\}} ||\sigma||^2 dV_T \geq 2p\lambda_1 \int_{N_T} (\ln f)^2 dV_T + 4hp \text{ vol}(N_T),
\]

where \( \lambda_1 \) is the first positive eigenvalue of the Laplacian \( \Delta \) of \( N_T \).

Moreover, the equality sign of (7.2) holds identically if and only if we have

1. \( \Delta \ln f = \lambda_1 \ln f \).
2. The CR-warped product is both \( N_T \)-totally geodesic and \( N_\perp \)-totally geodesic.

The following example shows that Theorems 7.3 and 7.4 are sharp.

**Example 7.1.** Let \( \iota_1 \) be the identity map of \( CP^h(4) \) and let

\[
\iota_2 : RPP(1) \rightarrow CP^p(4)
\]

be a totally geodesic Lagrangian embedding of \( RPP(1) \) into \( CP^p(4) \). Denote by

\[
\iota = (\iota_1, \iota_2) : CP^h(4) \times RPP(1) \rightarrow CP^h(4) \times CP^p(4)
\]

the product embedding of \( \iota_1 \) and \( \iota_2 \). Moreover, let \( S_{h,p} \) be the Segre embedding of \( CP^h(4) \times CP^p(4) \) into \( CP^{h+p+h_p}(4) \). Then the composition \( \phi = S_{h,p} \circ \iota \):

\[
CP^h(4) \times RPP(1) \xrightarrow{(\iota_1, \iota_2)} CP^h(4) \times CP^p(4) \xrightarrow{S_{h,p}} CP^{h+p+h_p}(4)
\]

is a CR-warped product in \( CP^{h+p+h_p}(4) \) whose holomorphic factor \( N_T = CP^h(4) \) is a compact manifold. Since the second fundamental form of \( \phi \) satisfies the equation: \( ||\sigma||^2 = 4hp \), we have the equality case of inequality (7.1) identically.

The next example shows that the assumption of compactness in Theorems 7.3 and 7.4 cannot be removed.

**Example 7.2.** Let \( \mathbb{C}^* = \mathbb{C} - \{0\} \) and \( \mathbb{C}_m^{m+1} = \mathbb{C}^{m+1} - \{0\} \). Denote by \( \{z_0, \ldots, z_h\} \) a natural complex coordinate system on \( \mathbb{C}_m^{m+1} \).

Consider the action of \( \mathbb{C}^* \) on \( \mathbb{C}_m^{m+1} \) given by

\[
\lambda \cdot (z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m)
\]

for \( \lambda \in \mathbb{C}^* \). Let \( \pi(z) \) denote the equivalent class containing \( z \) under this action. Then the set of equivalent classes is the complex projective \( m \)-space \( CP^m(4) \) with the complex structure induced from the complex structure on \( \mathbb{C}_m^{m+1} \).

For any two natural numbers \( h \) and \( p \), we define a map:

\[
\bar{\phi} : \mathbb{C}_m^{m+1} \times S^p(1) \rightarrow \mathbb{C}_m^{h+p+1}
\]

by

\[
\bar{\phi}(z_0, \ldots, z_h; w_0, \ldots, w_p) = (w_0 z_0, w_1 z_0, \ldots, w_p z_0, z_1, \ldots, z_h)
\]

for \( (z_0, \ldots, z_h) \) in \( \mathbb{C}_m^{h+1} \) and \( (w_0, \ldots, w_p) \) in \( S^p \) with \( \sum_{j=0}^p w_j^2 = 1 \).
Since the image of $\tilde{\phi}$ is invariant under the action of $C_*$, the composition:

$$\pi \circ \tilde{\phi} : \mathbb{C}^{h+1}_* \times S^p \xrightarrow{\tilde{\phi}} \mathbb{C}^{h+p+1}_* \xrightarrow{\pi} C P^{h+p}(4)$$

induces a CR-immersion of the product manifold $N_T \times S^p$ into $C P^{h+p}(4)$, where

$$N_T = \{(z_0, \ldots, z_h) \in C P^h : z_0 \neq 0\}$$

is a proper open subset of $C P^h(4)$. Clearly, the induced metric on $N_T \times S^p$ is a warped product metric and the holomorphic factor $N_T$ is non-compact.

Notice that the complex dimension of the ambient space is $h + p$; far less than $h + p + hp$.

8. ANOTHER OPTIMAL INEQUALITY FOR CR-WARPED PRODUCTS

All CR-warped products in complex space forms also satisfy another general optimal inequality obtained in [24].

**Theorem 8.1.** Let $N = N_T^h \times f N_p^h$ be a CR-warped product in a complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $c$. Then we have

$$||\sigma||^2 \geq 2p\{||\nabla (\ln f)||^2 + \Delta (\ln f) + 2hc\}. \tag{8.1}$$

If the equality sign of (8.1) holds identically, then $N_T$ is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold. Moreover, $N$ is a minimal submanifold in $\tilde{M}(4c)$.

The following three theorems completely classify all CR-warped products which satisfy the equality case of (8.1) identically.

**Theorem 8.2.** [24] Let $\phi : N_T^h \times f N_p^h \rightarrow \mathbb{C}^m$ be a CR-warped product in $\mathbb{C}^m$. Then we have

$$||\sigma||^2 \geq 2p\{||\nabla (\ln f)||^2 + \Delta (\ln f)\}. \tag{8.2}$$

The equality case of inequality (8.2) holds identically if and only if the following four statements hold.

1. $N_T$ is an open portion of $\mathbb{C}^h_* := \mathbb{C}^h - \{0\}$;
2. $N_\perp$ is an open portion of $S^p$;
3. There is $\alpha$, $1 \leq \alpha \leq h$, and complex Euclidean coordinates $\{z_1, \ldots, z_h\}$ on $\mathbb{C}^h$ such that $f = \sqrt{\sum_{j=1}^h z_j \bar{z}_j}$;
4. Up to rigid motions, the immersion $\phi$ is given by

$$\phi = (w_0 z_1, \ldots, w_p z_1, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0, \ldots, 0)$$

for $z = (z_1, \ldots, z_h) \in \mathbb{C}^h$ and $w = (w_0, \ldots, w_p) \in S^p(1) \subset E^{p+1}$.

**Theorem 8.3.** [24] Let $\phi : N_T^h \times f N_\perp \rightarrow C P^m(4)$ be a CR-warped product with $\dim_\mathbb{C} N_T = h$ and $\dim_\mathbb{R} N_\perp = p$. Then we have

$$||\sigma||^2 \geq 2p\{||\nabla (\ln f)||^2 + \Delta (\ln f) + 2h\}. \tag{8.3}$$

The CR-warped product satisfies the equality case of inequality (8.3) identically if and only if the following three statements hold.
(a) \( N_T \) is an open portion of complex projective \( h \)-space \( CP^h(4) \);
(b) \( N_\perp \) is an open portion of unit \( p \)-sphere \( S^p(1) \); and
(c) There exists a natural number \( \alpha \leq h \) such that, up to rigid motions, \( \phi \) is the composition \( \pi \circ \bar{\phi} \), where
\[
\bar{\phi}(z, w) = (w_0 z_0, \ldots, w_p z_0, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0 \ldots, 0)
\]
for \( z = (z_0, \ldots, z_h) \in \mathbb{C}_1^{h+1} \) and \( w = (w_0, \ldots, w_p) \in S^p(1) \subset \mathbb{E}^{p+1} \), where \( \pi \) is the projection \( \pi: \mathbb{C}_1^{m+1} \to CP^m \).

**Theorem 8.4.** \[23\] Let \( \phi: N_T \times f N_\perp \to CH^m(-4) \) be a CR-warped product with \( \dim_{\mathbb{C}} N_T = h \) and \( \dim_{\mathbb{R}} N_\perp = p \). Then we have
\[
||\sigma||^2 \geq 2p\{||\nabla (\ln f)||^2 + \Delta (\ln f) - 2h\}.
\]

The CR-warped product satisfies the equality case of (8.3) identically if and only if the following three statements hold.

(a) \( N_T \) is an open portion of complex hyperbolic \( h \)-space \( CH^h(-4) \);
(b) \( N_\perp \) is an open portion of unit \( p \)-sphere \( S^p(1) \) (or \( \mathbb{R} \), when \( p = 1 \)); and
(c) up to rigid motions, \( \phi \) is the composition \( \pi \circ \bar{\phi} \), where either \( \bar{\phi} \) is given by
\[
\bar{\phi}(z, w) = (z_0, \ldots, z_{\beta}, w_0 z_{\beta+1}, \ldots, w_p z_{\beta+1}, \ldots, w_0 z_h, \ldots, w_p z_h, 0 \ldots, 0)
\]
for \( 0 < \beta \leq h \), \( z = (z_0, \ldots, z_h) \in \mathbb{C}_1^{h+1} \) and \( w = (w_0, \ldots, w_p) \in S^p(1) \), or \( \bar{\phi} \) is given by
\[
\bar{\phi}(z, w) = (z_0 \cosh u, z_0 \sinh u, z_1 \cos u, z_1 \sin u, \ldots, z_\alpha \cos u, z_\alpha \sin u, z_{\alpha+1}, \ldots, z_h, 0 \ldots, 0)
\]
for \( z = (z_0, \ldots, z_h) \in \mathbb{C}_1^{h+1} \), where \( \pi \) is the projection \( \pi: \mathbb{C}_1^{m+1} \to CH^m(-4) \).

9. **Warped product real hypersurfaces in complex space forms**

For real hypersurfaces, we have the following non-existence theorem.

**Theorem 9.1.** \[40\] There do not exist real hypersurfaces in complex projective and complex hyperbolic spaces which are Riemannian products of two or more Riemannian manifolds of positive dimension.

In other words, every real hypersurface in a non-flat complex space form is irreducible.

A contact manifold is an odd-dimensional manifold \( M^{2n+1} \) with a 1-form \( \eta \) such that \( \eta \wedge (d\eta)^n \neq 0 \). A curve \( \gamma = \gamma(t) \) in a contact manifold is called a Legendre curve if \( \eta(\beta'(t)) = 0 \) along \( \beta \). Let \( S^{2n+1}(c) \) denote the hypersphere in \( \mathbb{C}^{n+1} \) with curvature \( c \) centered at the origin. Then \( S^{2n+1}(c) \) is a contact manifold endowed with a canonical contact structure which is the dual 1-form of the characteristic vector field \( J\xi \), where \( J \) is the complex structure and \( \xi \) the unit normal vector on \( S^{2n+1}(c) \).
Legendre curves are known to play an important role in the study of contact manifolds, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves.

Contrast to Theorem 9.1, there exist many warped product real hypersurfaces in complex space forms as given in the following three theorems from [22].

**Theorem 9.2.** Let \( \gamma : I \to S^3(a^2) \subset \mathbb{C}^2 \) defined on an open interval \( I \). Then
\[
(9.1) \quad x(z_1, \ldots, z_n, t) = (a_1(t)z_1, a_2(t)z_2, \ldots, z_n), \quad z_1 \neq 0
\]
defines a real hypersurface which is the warped product \( \mathbb{C}^n_\ast \times_{a|z_1|} I \) of a complex \( n \)-plane and \( I \), where \( \mathbb{C}^n_\ast = \{(z_1, \ldots, z_n) : z_1 \neq 0\} \).

Conversely, up to rigid motions of \( \mathbb{C}^{n+1} \), every real hypersurface in \( \mathbb{C}^{n+1} \) which is the warped product \( N \times_{f} I \) of a complex hypersurface \( N \) and an open interval \( I \) is either obtained in the way described above or given by the product submanifold \( \mathbb{C}^n \times C \subset \mathbb{C}^n \times \mathbb{C}^1 \) of \( \mathbb{C}^n \) and a real curve \( C \) in \( \mathbb{C}^1 \).

Let \( S^{2n+3}(1) \) denote the unit hypersphere in \( \mathbb{C}^{n+2} \) centered at the origin and put
\[
U(1) = \{ \lambda \in \mathbb{C} : \lambda \bar{\lambda} = 1 \}.
\]
Then there is a \( U(1) \)-action on \( S^{2n+3}(1) \) defined by \( z \mapsto \lambda z \). At \( z \in S^{2n+3}(1) \) the vector \( V = iz \) is tangent to the flow of the action. The quotient space \( S^{2n+3}(1)/\sim \), under the identification induced from the action, is a complex projective space \( CP^{n+1} \) which endows with the canonical Fubini-Study metric of constant holomorphic sectional curvature 4.

The almost complex structure \( J \) on \( CP^{n+1}(4) \) is induced from the complex structure \( J \) on \( \mathbb{C}^{n+2} \) via the Hopf fibration: \( \pi : S^{2n+3}(1) \to CP^{n+1}(4) \). It is well-known that the Hopf fibration \( \pi \) is a Riemannian submersion such that \( V = iz \) spans the vertical subspaces. Let \( \phi : M \to CP^{n+1}(4) \) be an isometric immersion. Then \( \hat{M} = \pi^{-1}(M) \) is a principal circle bundle over \( M \) with totally geodesic fibers. The lift \( \hat{\phi} : \hat{M} \to S^{2n+3}(1) \) of \( \phi \) is an isometric immersion so that the diagram:
\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\phi}} & S^{2n+3}(1) \\
\downarrow{\pi} & & \downarrow{\pi} \\
M & \xrightarrow{\phi} & CP^{n+1}(4)
\end{array}
\]
commutes.

Conversely, if \( \psi : \hat{M} \to S^{2n+3}(1) \) is an isometric immersion which is invariant under the \( U(1) \)-action, then there is a unique isometric immersion \( \psi_\pi : \pi(\hat{M}) \to CP^{n+1}(4) \) such that the associated diagram commutes. We simply call the immersion \( \psi_\pi : \pi(\hat{M}) \to CP^{n+1}(4) \) the projection of \( \psi : \hat{M} \to S^{2n+3}(1) \).

For a given vector \( X \in T_z(CP^{n+1}(4)) \) and a point \( u \in S^{2n+2}(1) \) with \( \pi(u) = z \), we denote by \( X_u^z \) the horizontal lift of \( X \) at \( u \) via \( \pi \). There exists a canonical orthogonal decomposition:
\[
(9.2) \quad T_uS^{2n+3}(1) = (T_{\pi(u)}CP^{n+1}(4))^u \oplus \text{Span} \{V_u\}.
\]
Since \( \pi \) is a Riemannian submersion, \( X \) and \( X_u^* \) have the same length.

We put

\[
S_{*}^{2n+1}(1) = \left\{ (z_0, \ldots, z_n) : \sum_{k=0}^{n} z_k \bar{z}_k = 1, \ z_0 \neq 0 \right\},
\]

\( CP_n^0 = \pi(S_{*}^{2n+1}(1)) \).

The next theorem classified all warped products hypersurfaces of the form \( N \times_f I \) in complex projective spaces.

**Theorem 9.3.** 22 Suppose that \( a \) is a positive number and \( \gamma(t) = (\Gamma_1(t), \Gamma_2(t)) \) is a unit speed Legendre curve \( \gamma : I \to S^3(a^2) \subset \mathbb{C}^2 \) defined on an open interval \( I \).

Let \( x : S_{*}^{2n+1} \times I \to \mathbb{C}^{n+2} \) be the map defined by

\[
x(z_0, \ldots, z_n, t) = (a\Gamma_1(t)z_0, a\Gamma_2(t)z_0, z_1, \ldots, z_n), \quad \sum_{k=0}^{n} z_k \bar{z}_k = 1.
\]

Then

1. \( x \) induces an isometric immersion \( \psi : S_{*}^{2n+1}(1) \times_{a|z_0|} I \to S^{2n+3}(1) \).
2. The image \( \psi(S_{*}^{2n+1}(1) \times_{a|z_0|} I) \) in \( S^{2n+3}(1) \) is invariant under the action of \( U(1) \).
3. the projection

\[
\psi_{\pi} : \pi(S_{*}^{2n+1}(1) \times_{a|z_0|} I) \to CP^{n+1}(4)
\]

of \( \psi \) via \( \pi \) is a warped product hypersurface \( CP_{n}^0 \times_{a|z_0|} I \) in \( CP^{n+1}(4) \).

Conversely, if a real hypersurface in \( CP^{n+1}(4) \) is a warped product \( N \times_f I \) of a complex hypersurface \( N \) of \( CP^{n+1}(4) \) and an open interval \( I \), then, up to rigid motions, it is locally obtained in the way described above.

In the complex pseudo-Euclidean space \( \mathbb{C}^{n+2}_{1} \) endowed with pseudo-Euclidean metric

\[
g_0 = -dz_0d \bar{z}_0 + \sum_{j=1}^{n+1} dz_j d \bar{z}_j,
\]

we define the anti-de Sitter space-time by

\[
H^{2n+3}_{1}(-1) = \left\{ (z_0, z_1, \ldots, z_{n+1}) : \langle z, z \rangle = -1 \right\}.
\]

It is known that \( H^{2n+3}_{1}(-1) \) has constant sectional curvature \(-1\). There is a \( U(1) \)-action on \( H^{2n+3}_{1}(-1) \) defined by \( z \mapsto \lambda z \). At a point \( z \in H^{2n+3}_{1}(-1), iz \) is tangent to the flow of the action. The orbit is given by \( z_t = e^{it}z \) with \( \frac{dz}{dt} = iz_t \) which lies in the negative-definite plane spanned by \( z \) and \( iz \).

The quotient space \( H^{2n+3}_{1}(-1)/\sim \) is the complex hyperbolic space \( CH^{n+1}(-4) \) which endows a canonical Kaehler metric of constant holomorphic sectional curvature \(-4\). The complex structure \( J \) on \( CH^{n+1}(-4) \) is induced from the canonical complex structure \( J \) on \( \mathbb{C}^{n+2} \) via the totally geodesic fibration: \( \pi : H^{2n+3}_{1} \to CH^{n+1}(-4) \).
Let $\phi: M \to CH^{n+1}(-4)$ be an isometric immersion. Then $\hat{M} = \pi^{-1}(M)$ is a principal circle bundle over $M$ with totally geodesic fibers. The lift $\hat{\phi}: \hat{M} \to H^{2n+3}_1(-1)$ of $\phi$ is an isometric immersion such that the diagram:

$$
\begin{array}{c}
\hat{M} \\ \downarrow \phi \\
H^{2n+3}_1(-1) \\
\downarrow \pi \\
M \\ \downarrow \phi \\
CH^{n+1}(-4)
\end{array}
$$

commutes.

Conversely, if $\psi: \hat{M} \to H^{2n+3}_1(-1)$ is an isometric immersion which is invariant under the $U(1)$-action, there is a unique isometric immersion $\psi_\pi: \pi(\hat{M}) \to CH^{n+1}(-4)$, called the projection of $\psi$ so that the associated diagram commutes.

We put

$$
(9.7) \quad H^{2n+1}_{1s}(-1) = \{(z_0, \ldots, z_n) \in H^{2n+1}_1(-1): z_n \neq 0\},
$$

$$
(9.8) \quad CH^n(-4) = \pi(H^{2n+1}_1(-1)).
$$

The following theorem classifies all warped products hypersurfaces of the form $N \times_f I$ in complex hyperbolic spaces.

**Theorem 9.4.** Suppose that $a$ is a positive number and $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ is a unit speed Legendre curve $\gamma: I \to S^3(a^2) \subset \mathbb{C}^2$. Let

$$
y: H^{2n+1}_{1s}(-1) \times I \to \mathbb{C}^{n+2}
$$

be the map defined by

$$
y(z_0, \ldots, z_n, t) = (z_0, \ldots, z_{n-1}, a\Gamma_1(t)z_n, a\Gamma_2(t)z_n),
$$

$$
(9.10) \quad z_0\bar{z}_0 - \sum_{k=1}^{n} z_k\bar{z}_k = 1.
$$

Then we have

1. $y$ induces an isometric immersion $\psi: H^{2n+1}_{1s}(-1) \times_{a|z_n|} I \to H^{2n+3}_1(-1)$.
2. The image $\psi(H^{2n+1}_{1s}(-1) \times_{a|z_n|} I)$ in $H^{2n+3}_1(-1)$ is invariant under the $U(1)$-action.
3. the projection $\psi_\pi: \pi(H^{2n+1}_{1s}(-1) \times_{a|z_n|} I) \to CH^{n+1}(-4)$ of $\psi$ via $\pi$ is a warped product hypersurface $CH^n(-4) \times_{a|z_n|} I$ in $CH^{n+1}(-4)$.

Conversely, if a real hypersurface in $CH^{n+1}(-4)$ is a warped product $N \times_f I$ of a complex hypersurface $N$ and an open interval $I$, then, up to rigid motions, it is locally obtained in the way described above.

10. **Twisted product $CR$-submanifolds of Kaehler manifolds**

Twisted products $B \times_\lambda F$ are natural extensions of warped products, namely the function may depend on both factors (cf. [9] page 66). When $\lambda$ depends only on $B$, the twisted product becomes a warped product. If $B$ is a point, the twisted product is nothing but a conformal change of metric on $F$. 
The study of twisted product CR-submanifolds was initiated by the author in 2000 (see [14]). In particular, the following results are obtained.

**Theorem 10.1.** [14] If \( M = N_\perp \times_\lambda N_T \) is a twisted product CR-submanifold of a Kähler manifold \( \tilde{M} \) such that \( N_\perp \) is a totally real submanifold and \( N_T \) is a holomorphic submanifold of \( \tilde{M} \), then \( M \) is a CR-product.

**Theorem 10.2.** [14] Let \( M = N_T \times_\lambda N_\perp \) be a twisted product CR-submanifold of a Kähler manifold \( \tilde{M} \) such that \( N_\perp \) is a totally real submanifold and \( N_T \) is a holomorphic submanifold of \( \tilde{M} \). Then we have

1. The squared norm of the second fundamental form of \( M \) in \( \tilde{M} \) satisfies
   \[
   ||\sigma||^2 \geq 2p ||\nabla^T (\ln \lambda)||^2,
   \]
   where \( \nabla^T (\ln \lambda) \) is the \( N^T \)-component of the gradient \( \nabla (\ln \lambda) \) of \( \ln \lambda \) and \( p \) is the dimension of \( N_\perp \).
2. If \( ||\sigma||^2 = 2p ||\nabla^T (\ln \lambda)||^2 \) holds identically, then \( N_T \) is a totally geodesic submanifold and \( N_\perp \) is a totally umbilical submanifold of \( \tilde{M} \).
3. If \( M \) is anti-holomorphic in \( \tilde{M} \) and \( \dim N_\perp > 1 \), then \( ||\sigma||^2 = 2p ||\nabla^T (\ln \lambda)||^2 \) holds identically if and only if \( N_T \) is a totally geodesic submanifold and \( N_\perp \) is a totally umbilical submanifold of \( \tilde{M} \).

For mixed foliate twisted product CR-submanifolds of Kähler manifolds, we have the following result.

**Theorem 10.3.** [14] Let \( M = N_T \times_\lambda N_\perp \) be a twisted product CR-submanifold of a Kähler manifold \( \tilde{M} \) such that \( N_\perp \) is a totally real submanifold and \( N_T \) is a holomorphic submanifold of \( \tilde{M} \). If \( M \) is mixed totally geodesic, then we have

1. The twisted function \( \lambda \) is a function depending only on \( N_\perp \).
2. \( N_T \times N^\lambda_\perp \) is a CR-product, where \( N^\lambda_\perp \) denotes the manifold \( N_\perp \) equipped with the metric \( g^\lambda_{N_\perp} = \lambda^2 g_{N_\perp} \).

Next, we provide ample examples of twisted product CR-submanifolds in complex Euclidean spaces which are not CR-warped product submanifolds.

Let \( z : N_T \to \mathbb{C}^m \) be a holomorphic submanifold of a complex Euclidean manifold \( \mathbb{C}^m \) and \( w : N^1_\perp \to \mathbb{C}^f \) be a totally real submanifold such that the image of \( N_T \times N^1_\perp \) under the product immersion \( \psi = (z, w) \) does not contain the origin \((0, 0)\) of \( \mathbb{C}^m \oplus \mathbb{C}^f \).

Let \( j : N^2_\perp \to S^{q-1} \subset \mathbb{E}^q \) be an isometric immersion of a Riemannian manifold \( N^2_\perp \) into the unit hypersphere \( S^{q-1} \) of \( \mathbb{E}^q \) centered at the origin.

Consider the map

\[
\phi = (z, w) \otimes j : N_T \times N^1_\perp \times N^2_\perp \to (\mathbb{C}^m \oplus \mathbb{C}^f) \otimes \mathbb{E}^q
\]

defined by

\[
\phi(p_1, p_2, p_3) = (z(p_1), z(p_2)) \otimes j(p_3),
\]

for \( p_1 \in N_T, p_2 \in N^1_\perp, p_3 \in N^2_\perp \).
On \((\mathbb{C}^m \oplus \mathbb{C}^f) \otimes \mathbb{E}^q\) we define a complex structure \(J\) by
\[
J((B, E) \otimes F) = (iB, iE) \otimes F, \quad i = \sqrt{-1},
\]
for any \(B \in \mathbb{C}^m,\ E \in \mathbb{C}^f\) and \(F \in \mathbb{E}^q\). Then \((\mathbb{C}^m \oplus \mathbb{C}^f) \otimes \mathbb{E}^q\) becomes a complex Euclidean \((m + f)q\)-space \(\mathbb{C}^{(m+f)q}\).

Let us put \(N_\perp = N^1_\perp \times N^2_\perp\). We denote by \(|z|\) the distance function from the origin of \(\mathbb{C}^m\) to the position of \(N_T\) in \(\mathbb{C}^m\) via \(z\); and denote by \(|w|\) the distance function from the origin of \(\mathbb{C}^f\) to the position of \(N^1_\perp\) in \(\mathbb{C}^f\) via \(w\). We define a function \(\lambda\) by \(\lambda = \sqrt{|z|^2 + |w|^2}\). Then \(\lambda > 0\) is a differentiable function on \(N_T \times N_\perp\), which depends on both \(N_T\) and \(N_\perp = N^1_\perp \times N^2_\perp\).

Let \(M\) denote the twisted product \(N_T \times_\lambda N_\perp\) with twisted function \(\lambda\). Clearly, \(M\) is not a warped product.

For such a twisted product \(N_T \times_\lambda N_\perp\) in \(\mathbb{C}^{(m+f)q}\) defined above we have the following.

**Proposition 10.1.** \([14]\) The map \(\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \to \mathbb{C}^{(m+f)q}\) defined by \((\ref{eq:product})\) satisfies the following properties:

1. \(\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \to \mathbb{C}^{(m+f)q}\) is an isometric immersion.
2. \(\phi = (z, w) \otimes j : N_T \times_\lambda N_\perp \to \mathbb{C}^{(m+f)q}\) is a twisted product \(CR\)-submanifold such that \(N_T\) is a holomorphic submanifold and \(N_\perp\) is a totally real submanifold of \(\mathbb{C}^{(m+f)q}\).

**Remark 10.1.** B. Sahin proved the following.

**Theorem 10.4.** \([51]\) There do not exist doubly twisted product \(CR\)-submanifolds in a \(K\)ahler manifold which are not (singly) twisted product \(CR\)-submanifolds in the form \(f_1 M_T \times f_2 M_\perp\), where \(M_T\) is a holomorphic submanifold and \(M_\perp\) is a totally real submanifold of the \(K\)ahler manifold \(\bar{M}\).

An almost Hermitian manifold \((M, g, J)\) with almost complex structure \(J\) is called a nearly \(K\)ahler manifold provided that \((\ref{eq:nearlykahler})\)
\[
(\nabla_X J)X = 0, \quad \forall X \in TM.
\]

**Remark 10.2.** Theorem \([10,4]\) was extended by S. Uddin in \([55]\) to doubly twisted product \(CR\)-submanifolds in a nearly \(K\)ahler manifold.
11. Warped products with a purely real factor in Kaehler manifolds

Recall from [10] that a submanifold \( N \) of an almost Hermitian manifold \( \tilde{M} \) is called purely real if the complex structure \( J \) on \( \tilde{M} \) carries the tangent bundle of \( N \) into a transversal bundle, i.e., \( J(TN) \cap TN = \{0\} \).

The next non-existence result is an extension of Theorem 10.4 of [50].

**Theorem 11.1.** [2, 45] There do not exist non-trivial warped product submanifolds \( N^0 \times_f N^T \) in a Kaehler manifold \( \tilde{M} \) such that \( N^T \) is a complex submanifold and \( N^0 \) is a proper purely real submanifold of \( \tilde{M} \).

A hemi-slant submanifold of an almost Hermitian manifold is a submanifold \( M \) with two orthogonal distributions \( H^\perp \) and \( H^\theta \) such that \( TM = H^\perp \oplus H^\theta \), \( JH^\perp \subset T^\perp M \) and \( H^\theta \) is \( \theta \)-slant.

**Remark 11.1.** Hemi-slant submanifolds were first defined in [5] under the name of anti-slant submanifolds.

The next non-existence result was proved in [52].

**Theorem 11.2.** There exist no warped product hemi-slant submanifolds \( N^\perp \times_f N^\theta \) in a Kaehler manifold \( \tilde{M} \) such that \( N^\perp \) is a totally real submanifold and \( N^\theta \) is a proper slant submanifold of \( \tilde{M} \).

For warped product submanifolds \( N^\theta \times_f N^\perp \), we also have the following result from [52].

**Theorem 11.3.** Let \( N \) be an \((m+n)\)-dimensional mixed geodesic warped product submanifold \( N^\theta \times_f N^\perp \) in a Kaehler manifold \( \tilde{M}^{m+n} \) such that \( N^\theta \) is a proper slant submanifold and \( N^\perp \) be a totally real submanifold of \( \tilde{M}^{m+n} \). Then we have:

1. The second fundamental form \( \sigma \) of \( N \) satisfies
   \[
   ||\sigma||^2 \geq m (\cot^2 \theta)|\nabla (\ln f)|^2, \quad m = \dim N^\perp.
   \]
2. If the equality of (11.1) holds identically, then \( N^\theta \) is a totally geodesic submanifold and \( N^\perp \) is a totally umbilical submanifold of \( \tilde{M} \). Moreover, \( M \) is never a minimal submanifold of \( \tilde{M} \).

A submanifold \( N \) of an almost Hermitian manifold \( \tilde{M} \) is called semi-slant [49], if the tangent bundle \( TN \) is the direct sum \( H \oplus H^\theta \) of two orthogonal distributions \( H \) and \( H^\theta \), where \( H \) is holomorphic, i.e. \( H \) is invariant with respect to the complex structure \( J \) of \( \tilde{M} \) and \( H^\theta \) is a \( \theta \)-slant distribution, i.e., the angle \( \theta \) between \( JX \) and \( H^\theta_x \) is constant for any unit vector \( X \in H^\theta_x \) and for any point \( x \in N \).

Contrast to CR-warped products \( N^T \times_f N^\perp \), we have the following non-existence result for semi-slant warped product submanifolds.

**Theorem 11.4.** [50] There do not exist warped product submanifolds of the form: \( N^T \times_f N^\theta \) in a Kaehler manifold \( \tilde{M} \) such that \( N^T \) is a complex submanifold and \( N^\theta \) is a proper slant submanifold of \( \tilde{M} \).
A natural extension of semi-slant submanifolds are pointwise semi-slant submanifolds defined and studied in [53].

A submanifold $N$ of an almost Hermitian manifold $\tilde{M}$ is called **pointwise semi-slant** if the tangent bundle $TN$ is the direct sum $\mathcal{H} \oplus \mathcal{H}^\theta$ of two orthogonal distributions $\mathcal{H}$ and $\mathcal{H}^\theta$, where $\mathcal{H}$ is a holomorphic distribution and $\mathcal{H}^\theta$ is a pointwise slant distribution, i.e., for any given point $x \in N$ the angle $\theta(x)$ between $JX$ and $\mathcal{H}^\theta_x$ is independent of the choice of the unit vector $X \in \mathcal{H}^\theta_x$.

For warped product pointwise semi-slant submanifolds, we have the following.

**Theorem 11.5.** [53] Let $N = N^T \times_f N^\theta$ be a non-trivial warped product pointwise semi-slant submanifold of a Kaehler manifold $\tilde{M}^{h+p}$, where $N^T$ is a complex submanifold with $\dim \mathbb{C} N^T = h$ and $N^\theta$ is a proper pointwise slant submanifold with $\dim N^\theta = p$. Then we have

1. The second fundamental form $\sigma$ of $N$ satisfies
   \begin{equation}
   ||\sigma||^2 \geq 2p(\csc^2 \theta + \cot^2 \theta)|\nabla^T (\ln f)|^2.
   \end{equation}
2. If the equality of (11.2) holds identically, then $N^T$ is a totally geodesic complex submanifold and $N^\theta$ is a totally umbilical submanifold of $\tilde{M}^{h+p}$. Moreover, $N$ is a minimal submanifold of $\tilde{M}^{h+p}$.

Pointwise bi-slant immersions are defined as follows (cf. [42]).

A submanifold $N$ of an almost Hermitian manifold $\tilde{M}$ is called **pointwise bi-slant** if there exists a pair of orthogonal distributions $\mathcal{H}_1$ and $\mathcal{H}_2$ of $M$ such that the following three conditions hold.

(a) $TN = \mathcal{H}_1 \oplus \mathcal{H}_2$;
(b) $J\mathcal{H}_1 \perp \mathcal{H}_2$ and $J\mathcal{H}_2 \perp \mathcal{H}_1$;
(c) Each distribution $\mathcal{H}_i$ is pointwise slant with slant function $\theta_i$ ($i = 1, 2$).

A pointwise bi-slant submanifold is a **bi-slant submanifold** if both slant functions $\theta_1$ and $\theta_2$ are constant.

Analogous to CR-warped products, Chen and Uddin defined warped product pointwise bi-slant submanifolds in [42] as follows.

A warped product $N_1 \times_f N_2$ in an almost Hermitian manifold $\tilde{M}$ is called a **warped product pointwise bi-slant submanifold** if both factors $N_1$ and $N_2$ are pointwise slant submanifolds of $\tilde{M}$. A warped product pointwise bi-slant submanifold $N_1 \times_f N_2$ is called **warped product bi-slant** if both $N_1$ and $N_2$ are slant submanifolds.

For warped product bi-slant submanifolds we have the following classification result.

**Theorem 11.6.** [42] Let $M = M_\theta_1 \times_f M_\theta_2$ be a warped product bi-slant submanifold in a Kaehler manifold $\tilde{M}$. Then one of the following two cases must occurs:

(i) The warping function $f$ is constant, i.e., $M$ is a Riemannian product of two slant submanifolds;
(ii) The slant angle of $M_\theta_2$ satisfies $\theta_2 = \pi/2$, i.e., $M$ is a warped product hemi-slant submanifold with $M_\theta_2$ being a totally real submanifold $M_\perp$ of $\tilde{M}$. 

For pointwise pseudo-slant warped product submanifolds in a Kähler Manifolds, we have the following two results from [54].

**Theorem 11.7.** Let \( M = F \times_f M_\theta \) be a mixed geodesic warped product submanifold of a \( 2m \)-dimensional Kähler manifold \( \tilde{M} \) such that \( F \) is a \( \gamma \)-dimensional totally real submanifold and \( M_\theta \) is a \( \beta \)-dimensional proper pointwise slant submanifold of \( M \). Then the squared norm of the second fundamental form \( ||\sigma||^2 \) of \( M \) satisfies

\[
||\sigma||^2 \geq \beta \cos^2 \theta ||\nabla (\ln f)||^2.
\]

**Theorem 11.8.** Let \( M = M_\theta \times_r F \) be a mixed geodesic warped product submanifold of an even-dimensional Kähler manifold \( \tilde{M} \) such that \( M_\theta \) is a \( \beta \)-dimensional proper pointwise slant submanifold of \( M \) and \( F \) is a \( \gamma \)-dimensional totally real submanifold. Then the squared norm of the second fundamental form \( ||\sigma||^2 \) of \( M \) satisfies

\[
||\sigma||^2 \geq \beta \cot^2 \theta ||\nabla (\ln f)||^2.
\]

**Remark 11.2.** Theorem 8.1 of [24] was extended to warped product pointwise semi-slant submanifolds by A. Ali, S. Uddin and A. M. Othman in [1].

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