An exotic Deligne-Langlands correspondence for symplectic groups

Syu Kato

October 8, 2018

Abstract

Let $G = Sp(2n, \mathbb{C})$ be a complex symplectic group. We introduce a $G \times (\mathbb{C}^\times)^{\ell+1}$-variety $\mathcal{N}_\ell$, which we call the $\ell$-exotic nilpotent cone. Then, we realize the Hecke algebra $\mathbb{H}$ of type $C_n^{(1)}$ with three parameters via equivariant algebraic $K$-theory in terms of the geometry of $\mathcal{N}_2$. This enables us to establish a Deligne-Langlands type classification of simple $\mathbb{H}$-modules under a mild assumption on parameters. As applications, we present a character formula and multiplicity formulas of $\mathbb{H}$-modules.

Table of Contents

§1 Preparatory materials
§2 Hecke algebras and exotic nilpotent cones
§3 Clan decomposition
§4 On stabilizers of exotic nilpotent orbits
§5 Semisimple elements attached to $G\backslash \mathcal{N}_1$
§6 A vanishing theorem
§7 Standard modules and an induction theorem
§8 Exotic Springer correspondence
§9 A deformation argument on parameters
§10 Main Theorems
§11 Consequences

Introduction

In their celebrated paper [KL87], Kazhdan and Lusztig gave a classification of simple modules of an affine Hecke algebra $\mathbb{H}$ with one-parameter in terms of the geometry of nilpotent cones. (It is also done by Ginzburg, c.f. [CG97].) Since some of the affine Hecke algebras admit two or three parameters, it is natural to extend their result to multi-parameter cases. (It is called the unequal parameter case.) Lusztig realized the “graded version” of $\mathbb{H}$ (with unequal parameters) via several geometric means [Lu88, Lu89, Lu95b] (c.f. [Lu03]) and classified their representations in certain cases. Unfortunately, his geometries admit essentially

$^*$Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Meguro Komaba 153-8914, Japan.
$^†$Current address: Research Institute for Mathematical Sciences, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502, Japan. E-mail:kato@kurims.kyoto-u.ac.jp
$^‡$The author was partially supported by JSPS Research Fellowship for Young Scientists (PD) 15-10371 and JSPS Grant-in-Aid for Young Scientists (B) 20-740011 during this research.
only one parameter. As a result, his classification is restricted to the case where all of the parameters are certain integral power of a single parameter. It is enough for his main interest, the study of representations of $p$-adic groups (c.f. [Lu95a]). However, there are many areas of mathematics which wait for the full-representation theory of Hecke algebras with unequal parameters (see e.g. Macdonald’s book [Mc03] and its featured review in MathSciNet).

In this paper, we give a realization of all simple modules of the Hecke algebra of type $C_n^{(1)}$ with three parameters by introducing a variety which we call the $\ell$-exotic nilpotent cone (c.f. §1.1). Our framework works for all parameters and realizes the whole Hecke algebra (Theorem A) and its specialization to each central character. Unfortunately, the study of our geometry becomes harder for some parameters and the result becomes less explicit in such cases. Even so, our result gives a definitive classification of simple modules of affine Hecke algebras of type $B_n^{(1)}$ and $C_n^{(1)}$ for almost all parameters including so-called real central character case. (See the argument after Theorem D.)

Let $G$ be the complex symplectic group $Sp(2n, \mathbb{C})$. We fix its Borel subgroup $B$ and a maximal torus $T \subset B$. Let $R$ be the root system of $(G, T)$. We embed $R$ into a $n$-dimensional Euclid space $\oplus_i \mathbb{C}e_i$ as $R = \{ \pm e_i \pm e_j \} \cup \{ \pm 2e_i \}$. We define $V_1 := \mathbb{C}^{2n}$ and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. For each non-negative integer $\ell$, we put $V_\ell := V_1^{\otimes \ell} \oplus V_2$ and call it the $\ell$-exotic representation. Let $V_\ell^+$ be the positive part of $V_\ell$ (for precise definition, see §1). We define

$$F_\ell := G \times B V_\ell^+ \subset G \times B V_\ell \cong G/B \times V_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow V_\ell.$$

We denote the image of $\mu_\ell$ by $\mathcal{R}_\ell$. This is the $G$-variety which we refer as the $\ell$-exotic nilpotent cone. We put $Z_\ell := F_\ell \times_{\mathcal{R}_\ell} F_\ell$. Let $G_\ell := G \times (\mathbb{C}^*)^{\ell+1}$. We have a natural $G_\ell$-action on $F_\ell$ (and $Z_\ell$). (In fact, the variety $F_\ell$ admits an action of $G \times GL(\ell, \mathbb{C}) \times \mathbb{C}^\times$. We use only a restricted action in this paper.)

Assume that $\mathcal{H}$ is the Hecke algebra with unequal parameters of type $C_n^{(1)}$ (c.f. Definition 2.1). This algebra has three parameters $q_0, q_1, q_2$. All affine Hecke algebras of classical type with two parameters are obtained from $\mathcal{H}$ by suitable specialization of parameters (c.f. Remark 2.2).

**Theorem A (= Theorem 2.8).** We have an isomorphism

$$\mathcal{H} \cong \mathbb{C} \otimes_{\mathbb{Z}} K^{G_2}(Z_2)$$

as algebras.

The Ginzburg theory suggests a classification of simple $\mathcal{H}$-modules by the $G$-conjugacy classes of the following Langlands parameters:

**Definition B** (Langlands parameters).

1) A triple $\vec{q} := (q_0, q_1, q_2) \in (\mathbb{C}^*)^3$ is said to be admissible if $q_0 \neq q_1, q_2$ is not a root of unity of order $\leq 2n$, $q_0 q_1^e \neq q_2^{n/2}$ for $|m| < n$;

2) A pair $(a, X) = (s, \vec{q}, X_0 \oplus X_1 \oplus X_2) \in G_2 \times \mathcal{R}_2$ is called an admissible parameter iff $s$ is semisimple, $\vec{q}$ is admissible, and $sX_i = q_i X_i$ for $i = 0, 1, 2$. 

2
For \( a = (s, \bar{q}) \in G_2 \), we put \( G(s) := Z_G(s) \) and \( G_2(a) := Z_{G_2}(a) \).

Notice that our Langlands parameters do not have additional data as in the usual Deligne-Langlands-Lusztig correspondence. This is because the (equivariant) fundamental groups of orbits are always trivial (c.f. Theorem 4.10).

Instead, we have the following kind of difficulty:

**Example C** (Non-regular parameters). Let \( G = Sp(4, \mathbb{C}) \) and let \( a = (\exp(r\epsilon_1 + (r + \pi\sqrt{-1})\epsilon_2), e^r, -e^r, -e^{2r}) \in T \times (\mathbb{C}^\times)^3 \) \((r \in \mathbb{C}\pi\sqrt{-1}\mathbb{I})\). Then, the number of \( G_2(a) \)-orbits in \( \mathfrak{H}_2^s \) is eight, while the number of corresponding representations of \( \mathbb{H} \) is six. (c.f. Enomoto [En06]) These orbits contain weight vectors of \( \epsilon_1 + \epsilon_2 \) or “\( \epsilon_1 & \epsilon_2 \)”. 

Now we state the main theorem of this paper:

**Theorem D** (= Theorem 10.2). The set of \( G \)-conjugacy classes of admissible parameters is in one-to-one correspondence with the set of isomorphism classes of simple \( \mathbb{H} \)-modules if \( q_2 \) is not a root of unity of order \( \leq 2n \), and \( \bar{q}_0q_1^{\pm 1} \neq q_2^4 \).

We treat a slightly more general case in Theorem 10.1 including Example C. Since the general condition is rather technical, we state only a part of it here.

By imposing an additional relation \( \bar{q}_0 + q_1 = 0 \), the algebra \( \mathbb{H} \) specializes to an extended Hecke algebra \( \mathbb{H}_B \) of type \( B_n^{(1)} \) with two-parameters. (c.f. Remark 2.2.) Therefore, Theorem D also gives a definitive classification of simple \( \mathbb{H}_B \)-modules except for \( -q_0^m = q_2^m \) \((|m| < n) \) or \( q_2 \) is a root of unity of order \( \leq 2n \).

Let us illustrate an example which (partly) explains the title “exotic”:

**Example E** (Equal parameter case). Let \( G = Sp(4, \mathbb{C}) \). Let \( s = \exp(r\epsilon_1 + r\epsilon_2) \in T \) \((r \in \mathbb{C}\pi\sqrt{-1}\mathbb{I})\). Fix \( a_0 = (s, e^{2r}) \in G \times \mathbb{C}^\times \) and \( a = (s, e^r, -e^r, e^{2r}) \in G_2 \). Let \( \mathcal{N} \) be the nilpotent cone of \( G \). Then, the sets of \( G(s) \)-orbits of \( \mathcal{N}^{a_0} \) and \( \mathfrak{H}_2^s \) are responsible for the usual and our exotic Deligne-Langlands correspondences. The number of \( G(s) \)-orbits in \( \mathcal{N}^{a_0} \) is three. (Corresponding to root vectors of \( \emptyset, 2\epsilon_1 \), and “\( 2\epsilon_1 & 2\epsilon_2 \)”) The number of \( G(s) \)-orbits in \( \mathfrak{H}_2^s \) is four. (Corresponding to weight vectors of \( \emptyset, \epsilon_1, \epsilon_1 + \epsilon_2 \), and “\( \epsilon_1 & \epsilon_1 + \epsilon_2 \)”). On the other hand, the actual number of simple modules arising in this way is four (c.f. Ram [Ra01] and [En06]).

The organization of this paper is as follows:

In §1, we fix notation and introduce exotic nilcones and related varieties. In particular, we present geometric structures involved in our varieties as much as we need in the later sections. In §2, we prove Theorem A, which connects our varieties with an affine Hecke algebra \( \mathbb{H} \) of type \( C_n^{(1)} \). In order to simplify the study of representation theory of \( \mathbb{H} \), we divide our varieties into a product of primitive ones in §3. In §4, we prove that the stabilizers of exotic nilpotent orbits are connected, which implies that “the Lusztig part” of the Deligne-Langlands-Lusztig parameter should be always trivial in our situation. Unfortunately, we have no nice parabolic subgroup as Kazhdan-Lusztig employed in [KL87]. We construct some explicit semisimple element out of each orbit in §5 for the sake of compensation. We introduce the notion of exotic Springer fibers and prove its odd-term vanishing result in §6, under the assumption that the parameters are sufficiently nice (including admissible case). Its proof essentially relies on the argument of §5. We define our standard modules as the total homology group of exotic Springer fibers in §7. At the same time, we present an induction
theorem, which claims that they behave well under inductions. In §8, we present 
an analogue of the Springer correspondence for exotic nilcones. In order to prove 
Theorem D, we still need two additional structural results. One is that our 
geometric structure is preserved by replacing the central character by a suitable 
real positive one. The other is that we can embed the corresponding finite Weyl 
group into the graded version of $\mathbb{H}$. Our proofs of both results essentially use 
admissibility of parameters. These results occupy §9. With the knowledge of all 
of the previous sections except for §7, we prove Theorem D in §10. The last 
section §11 concerns with applications, which are straightforward consequences 
of Ginzburg theory assuming the results presented in earlier sections.

Acknowledgment: The present from of this paper\textsuperscript{1} is heavily benefited from 
the comments from Pramod Achar, Susumu Ariki, Michel Brion, Masaki Kashiwara, 
Michael Finkelberg, Anthony Henderson, George Lusztig, Hiraku Nakajima, Eric Op-
dam, Midori Shiota, Toshiyuki Tanisaki, Masahiko Yoshinaga, and discussion with 
Naoya Enomoto, Hisayosi Matumoto, Eric Vasserot, Tonny A. Springer. The author 
wants to express his deep gratitude to all of them for their kindness, warmth, and 
tolerance. In particular, Professor Ariki kindly arranged him an opportunity to talk 
at a seminar at RIMS. The author wishes to express his gratitude to him and all the 
participants of the seminar. Last, but not least, the author would like to thank the 
referee for his kindness and patience.

1 Preparatory materials

Let $G := Sp(2n, \mathbb{C})$. Let $B$ be a Borel subgroup of $G$. Let $T$ be a maximal 
torus of $B$. Let $X^+(T)$ be the character group of $T$. Let $R$ be the root system 
of $(G,T)$ and let $R^+$ be its positive part defined by $B$. We embed $R$ and $R^+$ 
into a $n$-dimensional Euclid space $E = \oplus_i \mathbb{C} \epsilon_i$ with standard inner product as:

$$R^+ = \{ \epsilon_i \pm \epsilon_j \}_{i<j} \cup \{ 2 \epsilon_i \} \subset \{ \pm \epsilon_i \pm \epsilon_j \} \cup \{ \pm 2 \epsilon_i \} = R \subset E.$$  

By the inner product, we identify $\epsilon_i$ with its dual basis. We put $\epsilon_i := -\epsilon_{-i}$ when 
$-n \leq i < 0$. We put $\alpha_i := \epsilon_i - \epsilon_{i+1}$ ($i = 1, \ldots, n-1$) or $2\epsilon_n$ ($i = n$). Let $W$ be the 
Weyl group of $(G,T)$. For each $\alpha_i$, we denote the reflection of $E$ corresponding 
to $\alpha_i$ by $s_i$. Let $l : W \to \mathbb{Z}_{\geq 0}$ be the length function with respect to $(B,T)$. We 
denote by $w \in N_G(T)$ a lift of $w \in W$. For a subgroup $H \subset G$ containing $T$, we 
put $wH := \{ wHw^{-1} \}$. For a group $H$ and its element $s$, we put $H(s) := Z_H(s)$. 
For a subset $S \subset H$, we put $H(S) := \cap_{s \in S} H(s)$. We denote the identity 
component of $H$ by $H^0$. We denote by $R(H)$ and $R(H)_s$ the representation 
ring of $H$ and its localization along the evaluation at $s \in H$, respectively. For 
each $\alpha \in R$, we denote the corresponding one-parameter unipotent subgroup of 
$G$ (with respect to $T$) by $U_\alpha$. We define $g, t, g(s)$, etc... to be the Lie algebras 
of $G, T, G(s)$, etc..., respectively.

\textsuperscript{1}Note: After the original version of this paper is circulated (in 2006, with different 
argument and weaker conclusion in Theorem D, and consequently give a classification of $H$-
modules only with a help of Lusztig’s results [Lu88, Lu89, Lu95b]), there appeared two kinds 
of related works. One is the study of geometry which is connected to our nilcone by Achar-
Henderson [AH08], Enomoto [En08], Finkelberg-Ginzburg-Travkin [FGT08], Springer [Sp07], 
Travkin [Tr08], and the other is the classification of tempered dual by Opdam and Solleveld 
[OS07, OS08, So07]. For the former, I have included explanations about the situation as much 
as I could in order to avoid potential problems. For the latter, we are preparing another paper 
[CK] in this direction.
For a $T$-module $V$, we define its weight $\lambda$-part (with respect to $T$) as $V[\lambda]$. We define the positive part $V^+$ and negative part $V^-$ of $V$ as

$$V^+ := \bigoplus_{\lambda \in Q \cap R^+-\{0\}} V[\lambda], \text{ and } V^- := \bigoplus_{\lambda \in Q \cap R^+ - \{0\}} V[\lambda],$$

respectively. We denote the set of $T$-weights of $V$ by $\Psi(V)$.

In this paper, a segment is a set of integers $I$ written as $I = [i_1, i_2] \cap \mathbb{Z}$ for some integers $i_1 \leq i_2$. By abuse of notation, we may denote $I$ by $[i_1, i_2]$. For a segment $I$, we set $I^* := I$ (if $0 \not\in I$) or $I - \{0\}$ (if $0 \in I$). We denote the absolute value function by $| \cdot | : \mathbb{C} \to \mathbb{R}_{\geq 0}$. We set $\Gamma_0 := 2\pi \sqrt{-1} \mathbb{Z} \subset \mathbb{C}$ and set $\exp : \mathbb{E} \to T$ to be the exponential map. We normalize the map $\exp$ so that $\ker \exp \cong \sum_{i=1}^n \Gamma_0 \epsilon_i$.

A variety in this paper is a quasi-projective reduced scheme of finite type over $\mathbb{C}$. Its points are closed points. If an algebraic group $H$ acts on a variety $X$, then we denote the stabilizer of the $H$-action at $x \in X$ by $\text{Stab}_H x$. For each $h \in H$, we denote by $X^h$ the $h$-fixed point set of $X$. For a variety $X$, we denote by $H_*(X)$ the Borel-Moore homology groups with coefficients $\mathbb{C}$.

### 1.1 Exotic nilpotent cones

Let $\ell = 0, 1,$ or $2$. We define $V_1 := \mathbb{C}^{2n}$ (vector representation) and $V_2 := (\wedge^2 V_1)/\mathbb{C}$. These representations have $B$-highest weights $\epsilon_1$ and $\epsilon_1 + \epsilon_2$, respectively. We put $V_\ell := V_1^\ell_1 \oplus V_2$ and call it the $\ell$-exotic representation of $Sp(2n)$.

For $\ell \geq 1$, the set of non-zero weights of $V_\ell$ is in one-to-one correspondence with $R$ as

$$R \ni \begin{cases} \pm 2\epsilon_i & \leftrightarrow \pm \epsilon_i \\ \pm \epsilon_i \pm \epsilon_j & \leftrightarrow \pm \epsilon_i \pm \epsilon_j \end{cases} \in \Psi(V_1) \quad \in \Psi(V_2).$$

(1.1)

We define

$$F_\ell := G \times B \Psi_\ell^+ \subset G \times B \Psi_\ell \cong G/B \times \Psi_\ell.$$

Composing with the second projection, we have a map

$$\mu_\ell : F_\ell \longrightarrow \Psi_\ell.$$

We denote the image of $\mu_\ell$ by $\mathcal{N}_\ell$. We call this variety the $\ell$-exotic nilpotent cone. By abuse of notation, we may denote the map $F_\ell \to \mathcal{N}_\ell$ also by $\mu_\ell$.

**Convention 1.1.** For the sake of simplicity, we define objects $F$, $\mathcal{N}$, $\Psi$, $\mu$, etc... to be the objects $F_\ell$, $\mathcal{N}_\ell$, $\Psi_\ell$, $\mu_\ell$ etc... with $\ell = 1$.

We summarize some basic geometric properties of $\mathcal{N}_\ell$:

**Theorem 1.2** (Geometric properties of $\mathcal{N}_\ell$). We have the following:

1. The defining ideal of $\mathcal{N}_\ell$ is generated by $G$-invariant polynomials of $\mathbb{C}[\Psi_\ell]$ without constant terms;
2. The variety $\mathcal{N}_\ell$ is normal;
3. For $\ell = 1, 2$, the map $\mu_\ell$ is a birational projective morphism onto $\mathcal{N}_\ell$;
4. Every fiber of the map \( \mu_\ell \) is connected;

   In the below, we present properties which are valid only for the \( \ell = 1 \) case.

5. The set of \( G \)-orbits in \( N_1 \) is finite;

6. The map \( \mu_1 \) is stratified semi-small with respect to the stratification of \( N_1 \) given by \( G \)-orbits.

Proof. The proof is given after Lemma 1.5 since we need extra notation.

**Lemma 1.3.** We have a natural identification

\[
F_\ell \cong \{(gB, X) \in G/B \times \mathbb{V}_\ell; X \in g\mathbb{V}_\ell^+\}.
\]

Proof. Straightforward.

Let \( G_\ell := G \times (\mathbb{C}^*)^{\ell+1} \). We define a \( G_\ell \)-action on \( \mathfrak{g}_\ell \) as

\[
G_\ell \times \mathfrak{g}_\ell \ni (g, q_2-\ell, \ldots, q_2) \times (X_2-\ell \oplus \cdots \oplus X_2) \mapsto (q_2^{-1}gX_2-\ell \oplus \cdots \oplus q_2^{-1}gX_2) \in \mathfrak{g}_\ell.
\]

(Here we always regard \( X_2-\ell, \ldots, X_1 \in V_1 \) and \( X_2 \in V_2 \).) Similarly, we have a natural \( G_\ell \)-action on \( F_\ell \) which makes \( \mu_\ell \) a \( G_\ell \)-equivariant map. We define \( Z_\ell := F_\ell \times_{G_\ell} F_\ell \). By Lemma 1.3, we have

\[
Z_\ell := \{(g_1B, g_2B, X) \in (G/B)^3 \times \mathbb{V}_\ell; X \in g_1\mathbb{V}_\ell^+ \cap g_2\mathbb{V}_\ell^+\}.
\]

We put

\[
Z_{\ell}^{123} := \{(g_1B, g_2B, g_3B, X) \in (G/B)^3 \times \mathbb{V}_\ell; X \in g_1\mathbb{V}_\ell^+ \cap g_2\mathbb{V}_\ell^+ \cap g_3\mathbb{V}_\ell^+\}.
\]

We define \( p_i : Z_\ell \ni (g_1B, g_2B, X) \mapsto (g_iB, X) \in F_\ell \) and \( p_{ij} : Z_{\ell}^{123} \ni (g_1B, g_2B, g_3B, X) \mapsto (g_iB, g_jB, X) \in Z_\ell \) \((i, j \in \{1, 2, 3\})\). We also put \( \tilde{p}_i : F_\ell \times F_\ell \to F_\ell \) as the first and second projections \((i = 1, 2)\). (Notice that the meaning of \( p_i, \tilde{p}_i, p_{ij} \) depends on \( \ell \). The author hopes that there occurs no confusion on it.)

**Lemma 1.4.** The maps \( p_i \) and \( p_{ij} \) \((1 \leq i < j \leq 3)\) are projective.

Proof. The fibers of the above maps are given as the subsets of \( G/B \) defined by incidence relations. It is automatically closed, and we obtain the result.

We have a projection

\[
\pi_\ell : Z_\ell \ni (g_1B, g_2B, X) \mapsto (g_1B, g_2B) \in G/B \times G/B.
\]

For each \( w \in W \), we define a point \( p_w := B \times \hat{w}B \in G/B \times G/B \). This point is independent of the choice of \( \hat{w} \). We put \( O_w := Gp_w \subset G/B \times G/B \). By the Bruhat decomposition, we have

\[
G/B \times G/B = \bigsqcup_{w \in W} O_w.
\]

**Lemma 1.5.** The variety \( Z_\ell \) \((\ell = 1, 2)\) consists of \(|W|\)-irreducible components. Moreover, the dimensions of all of the irreducible components of \( Z \) are equal to \( \dim F \).
Proof. We first prove the assertion for \( Z = Z_1 \). By (1.2), the structure of \( Z \) is determined by the fibers over \( p_w \). We have

\[
\pi^{-1}(p_w) = \mathbb{V}^+ \cap \hat{w}\mathbb{V}^+.
\]

By the dimension counting using (1.1), we deduce

\[
\dim \mathbb{V}^+ \cap \hat{w}\mathbb{V}^+ = \dim \mathbb{V}_1^+ \cap \hat{w}\mathbb{V}_1^+ + \dim \mathbb{V}_2^+ \cap \hat{w}\mathbb{V}_2^+ = \#(R_1^+ \cap wR_1^+) + \#(R_s^+ \cap wR_s^+) = N - \ell(w),
\]

where \( N := \dim \mathbb{V}^+ = \dim G/B \) and \( R_1^+, R_s^+ \) are the sets of long and short positive roots, respectively. As a consequence, we deduce

\[
\dim \pi^{-1}(p_w) = N + \ell(w) + N - \ell(w) = 2N.
\]

Thus, each \( \pi^{-1}(O_w) \) is an irreducible component of \( Z \). Moreover, we have \( \pi^{-1}(O_1) \cong F \), which implies that the dimensions of irreducible components of \( Z \) are equal to \( \dim F \).

Next, we prove the assertion for \( Z_2 \). By forgetting the first \( V_1 \)-factor, we have a surjective map \( \eta : Z_2 \to Z \). We have a surjective map \( \eta' : Z \to Z_0 \) given by forgetting the \( V_1 \)-factor. The fiber of \( (\eta' \circ \eta) \) at \( x \in Z \) is isomorphic to the two-fold product of the fiber of \( \eta' \) at \( \eta'(x) \). The latter fiber is isomorphic to the vector space \( \mathbb{V}_1^+ \cap g\mathbb{V}_1^+ \) when \( \pi(x) = (1, g)p_1 \). Therefore, the preimage of each irreducible component of \( Z \) gives an irreducible component of \( Z_2 \). These irreducible components are distinct since their images under \( \eta \) must be distinct.

Hence, the number of irreducible components of \( Z_2 \) is equal to the number of irreducible components of \( Z \) as desired. \( \square \)

Proof of Theorem 1.2. The weight distribution of \( \mathbb{V}^+ \) and the Hesselink theory (c.f. [Po04] Theorem 1) claims that \( \mu \) gives a birational projective morphism onto an irreducible component of the Hilbert nilcone of \( \mathbb{V}_\ell \). Here the Hilbert nilcone of \( \mathbb{V}_\ell \) is an irreducible normal variety by Dadok-Kac [DK85] or Schwarz [Sc78]. In particular, our variety \( \mathcal{N}_\ell \subset \mathbb{V}_\ell \) is the Hilbert nilcone itself. Therefore, we obtain 1–3). 4) is an immediate consequence of 2), 3), and the Zariski main theorem (c.f. [CG97] 3.3.26). 5) is proved as a part of Proposition 1.1. We show 6). Let \( \hat{O} \) be the inverse image of a \( G \)-orbit \( G.X = \emptyset \subset \mathcal{N} \) under the map \( \mu \circ p_2 \). Then, we have

\[
\dim \emptyset + 2 \dim \mu^{-1}(X) \leq \dim \hat{O}.
\]

The dimension of the RHS is less than or equal to \( \dim F \), which is the (constant) dimension of irreducible components of \( Z \). In particular, we have

\[
\dim \emptyset + 2 \dim \mu^{-1}(X) \leq \dim \mathcal{N} = \dim F,
\]

which implies that \( \mu \) is semi-small. \( \square \)

By a general result of [Gi97] p135 (c.f. [CG97] 2.7), the \( G_\ell \)-equivariant \( K \)-group of \( Z_\ell \) becomes an associative algebra via the map

\[
*: K^{G_\ell}(Z_\ell) \times K^{G_\ell}(Z_\ell) \ni ([\mathcal{E}], [\mathcal{F}]) \mapsto \sum_{i \geq 0} (-1)^i [R^i(p_{13}), (p_{12}^*E \otimes p_{23}^*F)] \in K^{G_\ell}(Z_\ell).
\]
Moreover, the $G_\ell$-equivariant $K$-group of $F_\ell$ becomes a representation of $K^{G_\ell}(Z_\ell)$ as
\[ \circ : K^{G_\ell}(Z_\ell) \times K^{G_\ell}(F_\ell) \ni ([\mathcal{E}], [K]) \mapsto \sum_{i \geq 0} (-1)^i [\mathcal{R}^i(p_1)_*(\mathcal{E} \otimes L\tilde{p}_2^*K)] \in K^{G_\ell}(F_\ell). \]

Here we regard $\mathcal{E}$ as a sheaf over $F_\ell \times F_\ell$ via the natural embedding $Z_\ell \subset F_\ell \times F_\ell$.

### 1.2 Definition of parameters

In this subsection, we present the definitions of parameters which we need in the sequel. First, we put $a_0 := (1, 1, -1, 1) \in G_2$. (The value $a_0$ is special in the sense it naturally gives the Weyl group of type $C$ in our framework. C.f. §8)

**Definition 1.6 (Configuration of semisimple elements).**

1) An element $a = (s, q_0, q_1, q_2) \in G_2$ is called pre-admissible iff $s$ is semisimple, $q_0 \neq q_1, q_2$ is not a root of unity of order $\leq 2n$.

2) An element $a \in G_2$ is called finite if $\mathfrak{G}_a$ has only finitely many $G_2(a)$-orbit.

3) A pre-admissible element $a = (s, q_0, q_1, q_2)$ is called admissible if $q_0q_1^{\pm 1} \neq q_2^{\pm m}$ holds for every $0 \leq m < n$.

For a pre-admissible element $a = (s, q_0, q_1, q_2)$, we put
\[ V_a \equiv V_1^{(s,q_0)} \oplus V_1^{(s,q_1)} \oplus V_2^{(s,q_2)} \subset V_1 \oplus V_1 \oplus V_2 = V_2. \]

In the below, we may denote $(q_0, q_1, q_2) \in (\mathbb{C}^\times)^3$ by $\vec{q}$ for the sake of simplicity.

Let $a = (s, \vec{q}) \in G_2$ be a pre-admissible element such that $s \in T$. We sometimes denote it as
\[ s = \exp \left( \sum_{i=1}^n \log_i(s)e_i \right) \in \exp(\mathfrak{g}) \cong T, \]
where $\log_i(s) \in \mathbb{C}$.

**Remark 1.7.** The values of $\log_i(s)$ are determined modulo $\Gamma_0$. Here we understand that $\log_i(s)$ is a fixed choice of a representative in $\log_i(s) + \Gamma_0$.

**Definition 1.8 (Admissible parameters).**

1) A pre-admissible parameter is a pair $\nu = (a, X) = (s, \vec{q}, X_1 \oplus X_2) \in G_2 \times \mathfrak{G}_1$ such that $a$ is pre-admissible, $(s - q_0)(s - q_1)X_1 = 0$, and $sX_2 = q_2X_2$;

For a pre-admissible $a \in G_2$, we denote by $\Lambda_a$ the set of $G(s)$-conjugacy classes of pre-admissible parameters of the form $(a, Y)$, where $Y \in \mathbb{V}$.

2) A pre-admissible parameter $\nu = (a, X)$ is called admissible if $a$ is admissible.
1.3 Orbit structures arising from $\mathfrak{N}_{\ell}$

In the below, we fix vectors in $V_1$ and $V_2$ as follows:

- For each $i \in [-n, n]^*$, we define $0 \neq x_i \in V_1$ as a non-zero vector of weight $\epsilon_i$;
- For each distinct $i, j \in [-n, n]^*$, we define $y_{ij} \in V_2$ to be a non-zero vector of weight $\epsilon_i - \epsilon_j$.

The following is a slight enhancement of the good basis of Ohta [Oh86] (1.3).

**Definition 1.9 (Signed partitions).** Let $J := \{J_1, J_2, \ldots\}$ be a collection of sequence of elements of $[-n, n]^*$. (I.e. each $J_k \in J$ is a sequence $(J_1^k, J_2^k, \ldots)$ in $[-n, n]^*$.) We put $J_k^+ := (|J_1^k|, |J_2^k|, \ldots)$ for each $k = 1, 2, \ldots$. We call $J$ a signed partition of $n$ if and only if $\{J_1^+, J_2^+, \ldots\}$ gives a subdivision of $[1, n]$. I.e. we have

$$[1, n] = \bigcup_{k \geq 1} J_k^+ = \bigcup_{k \geq 1} \{|j|; j \in J_k\} \quad \text{and} \quad J_k^+ \cap J_{k'}^+ = \emptyset \quad \text{for} \quad k \neq k'.$$

For each member $J$ of a signed partition $J$, we define a subtorus

$$T_J := \exp \sum_{i \in J} C\epsilon_i \subset T.$$ 

Let $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of $n$. Then, we regard it as a signed partition by setting

$$J_j^+ := i + \sum_{k=1}^{j-1} \lambda_k \quad \text{if} \quad \lambda_j \neq 0 \quad \text{and} \quad 1 \leq i \leq \lambda_j.$$ 

**Definition 1.10 (Foot functions).** Let $\ell = 0, 1, \text{or} 2$. A collection of $\ell$-tuple of functions $\delta_k : [-n, n]^* \to \{0, 1\}$ for $1 \leq k \leq \ell$ is called a $\ell$-foot function of $n$. We denote a $\ell$-foot function $\{\delta_k\}_{k=1}^\ell$ by $\delta$.

Notice that Definition 1.10 claims that $\delta = \emptyset$ when $\ell = 0$.

**Definition 1.11 (Marked partitions, blocks, and normal forms).** Let $\ell$ be as in Definition 1.10. We refer a pair $\sigma = (J, \delta)$ consisting of a signed partition and a $\ell$-foot function of $n$ as a $\ell$-marked partition if the following condition holds:

- For each $J \in J$ and $m = 1, \ldots, \ell$, we have
  $$\#\{j \in J; \delta_m(j) = 1\} + \#\{j \in J; \delta_m(-j) = 1\} \leq 1.$$ 

For each $J \in J$, we define the $\ell$-block $v^\ell_{J} = v^1_{\sigma, 1} + v^1_{\sigma, 2} \in V$ associated to $(\sigma, J) = (\sigma, \{J^1, J^2, \ldots\})$ as:

$$v^1_{\sigma, 1} := \sum_{j \in J} \sum_{k=1}^\ell (\delta_k(j)x_j + \delta_k(-j)x_{-j}) \in V_1$$

$$v^1_{\sigma, 2} := \sum_{j \geq 1} y_{j, j+1} \in V_2,$$
where we regard \( y_{J_k} J_{k+1} \equiv 0 \) whenever \( J_k \) nor \( J_{k+1} \) is non-existent.

A \( \ell \)-normal form \( v_{\sigma} = v_{\sigma,1} + v_{\sigma,2} \in V \) associated to \( \sigma \) is defined as:

\[
\begin{align*}
    v_{\sigma,1} := \sum_{J \in J} v_{\sigma,1}^J \in V_1, \quad \text{and} \quad v_{\sigma,2} := \sum_{J \in J} v_{\sigma,2}^J \in V_2.
\end{align*}
\]

**Remark 1.12.** We regard that \( \ell \)-normal forms are elements of \( V = V_1 \), regardless the value of \( \ell \).

**Definition 1.13** (Strict normal forms). A \( \ell \)-marked partition \( \sigma = (J, \bar{\delta}) \) is called strict if and only if the following four conditions hold:

1. \( J \) is obtained from a partition \( \lambda \) of \( n \);
2. We have \( \delta_2 \equiv 0 \) and \( \delta_1(j) = 0 \) for every \( j \in [-n, -1] \);

Before stating the rest of the conditions, we introduce extra notation. Assume the above two conditions. If we have \( \delta_1(j) = 1 \) for \( j \in J \), then we set \( \# J := \# \{j' \in J; j' \leq j\} \) and \( \# J := \# \{j' \in J; j' > j\} \).

3. Let \( k < m \) be two integers and let \( J = \{J_1, J_2, \ldots\} \). Then, we have \( \delta_1 J_m \equiv 0 \) if \( \# J_k = \# J_m \);
4. Let \( J, J' \in J \) be a pair such that \( \delta_1(j) = 1 = \delta_1(j') \) for some \( j \in J \) and \( j' \in J' \). If \( \# J > \# J' \), then we have \( \# J > \# J' \) and \( \# J > \# J' \).

Conditions are not applicable when \( \delta_2 \) or \( \delta_1 \) are non-existent. Notice that only the first condition survives when \( \ell = 0 \). A normal form attached to a strict \( \ell \)-marked partition is called a \( \ell \)-strict normal form.

In the below, we refer foot functions, blocks, normal forms..., to be the 1-foot functions, 1-blocks, 1-normal forms..., respectively. Moreover, we naturally identify strict 1-normal forms and strict 2-normal forms since \( \delta_2 \equiv 0 \) for 2-strict marked partitions.

Let \( \text{Irrep}W \) be the set of isomorphism classes of irreducible \( W \)-modules.

**Theorem 1.14** (Orbit description of \( \mathcal{R}_1 \)). We have:

1. The set of strict 1-normal forms is in one-to-one correspondence with the set of \( G \)-orbits of \( \mathcal{R}_1 \);
2. We have \( \# (G \backslash \mathcal{R}_1) = \# \text{Irrep}W \);
3. For each \( X \in \mathcal{R}_1 \), the group \( \text{Stab}_{G} X \) is connected.

**Remark 1.15.** The original form of the proof of Theorem 1.14 (in [Ka06b]) employs explicit calculation using basis. In the meantime, Springer [Sp07] gives a base-free proof (with stronger consequences). The proof given here is somewhat the mixture of the both, which the author gives it for the sake of completeness. Note that the closure relation of the orbits of \( \mathcal{R}_1 \) is calculated by Achar-Henderson [AH08].

The proof of Theorem 1.14 is obtained as a combination of Proposition 4.5 and Theorem 8.3 by using the knowledge of the following:
Proposition 1.16 (Weak version of Theorem 1.14). We have:

1. Each $G$-orbit of $\mathcal{R}_1$ contains a strict normal form;

2. The number of elements of the set of strict marked partitions is less than or equal to $\#\text{Irrep} V$.

Proof. By a result of Ohta-Sekiguchi [Se84, Oh86], the set of strict 0-marked partitions are in one-to-one correspondence with the set of $G$-orbits of $\mathcal{R}_0$ via the assignment $\sigma \mapsto Gv_\sigma$. We have

$$\mathbb{C}[V]^G \cap \mathbb{C}[V_0] = \mathbb{C}[V_0]^G,$$

which gives the natural projection map

$$\mathcal{R}_1 \rightarrow \mathcal{R}_0$$

obtained from the natural projection $V_1 \rightarrow V_0$. (In fact we have $\mathbb{C}[V]^G = \mathbb{C}[V_0]^G$. But this fact is not used here.) It follows that each orbit of $\mathcal{R}_1$ contains a vector of type $v = v_1 \oplus V_2$, where $\lambda$ is a partition of $n$ regarded as a strict 0-marked partition $(J, \emptyset)$ in a natural way.

Consider the action of

$$G' := \text{Sp}(2\lambda_1) \times \text{Sp}(2\lambda_2) \times \cdots \subset \text{Sp}(2n),$$

which are embedded so that $T \subset G'$ and $V_1$ restricted to $G'$ has the form

$$\text{Res}^G_{G'} V_1 = \bigoplus_{k \geq 1} V_1^k$$

such that $V_1^k$ is a vector representation of $\text{Sp}(2\lambda_k)$ with $T$-weights $\pm \epsilon_i$ for $i = 1 + \sum_{j=1}^{k-1} \lambda_j, \ldots, \lambda_k + \sum_{j=1}^{k-1} \lambda_j$.

Let $\omega$ be the symplectic form on $V_1$ which is preserved by $G$. For each $k$, we put $\omega_k := \omega|_{V_1^k}$. We have $v = \sum_{k \geq 1} v_k$, where $v_k = v_{1,k} \oplus V_{2,\lambda_k} \in V_1^k \oplus \Lambda^2 V_1^k$. We consider an identification of $y_{ij}$ $(i, j \in J_k)$ with a matrix such that $y_{ij} x_k = c_{ij} x_i (k = j)$, $-c_{ij} x_j (k = -i)$, 0 (otherwise), for some $c_{ij} \in \mathbb{C}$. We arrange $\{c_{ij}\}_{i,j}$ so that $\Lambda^2 V_1^k$ is $\text{Sp}(2\lambda_k)$-equivariantly identified with the subset of $\text{End} V_1^k$ such that

$$\omega_k(y_{ij} x, x') = \omega_k(x, y_{ij} x')$$

for each $x, x' \in V_1^k$ and $i, j \in (J_k \cup -J_k)$.

The Ohta-Sekiguchi result asserts that this gives an identification of $\text{Sp}(2\lambda_k) V_{2,\lambda}$ and the set of linear nilpotent endomorphisms on $V_1^k$ of maximal rank ($= \dim V_1^k - 2$) which preserve $\omega_k$. Since $v_{1,k}$ can be complemented to a suitable choice of a standard basis of $V_1^k$ (as a symplectic vector space), we deduce that a suitable change of symplectic basis makes $v_{1,k}$ into one of $x_i (i > 0)$.

This implies that $v$ can be transformed into a 1-normal form associated to $(J, \delta_1) = (\lambda, \delta_1)$ which satisfies 1.13 1) and 2).

Now for each $k < k'$, we examine the $\text{Sp}(2\lambda_k + 2\lambda_{k'})$-orbit which contains $v_k + v_{k'} \in V_1^k \oplus V_1^{k'}$. We have $\lambda_k \geq \lambda_{k'}$ by 1.13 1). We put $\xi := V_{2,\sigma}^k + V_{2,\sigma'}^{k'}$.

The $\text{Sp}(2\lambda_k + 2\lambda_{k'})$-conjugacy class of $\xi$ is the set of nilpotent endomorphisms of $V_1^k \oplus V_1^{k'}$ which preserve $\omega$ and have $(\lambda_k, \lambda_{k'}, \lambda_{k'})$ as its (nilpotent) Jordan
form. If \( v_{1,k} = 0 \) or \( v_{1,k'} = 0 \) hold, then 1.13 3) and 4) are satisfied for the pair \((J_k, J_{k'})\). Hence, we assume \( v_{1,k} \neq 0 \neq v_{1,k'} \) in the below. We have

\[
\xi^{#J_k} v_{1,k} = 0, \xi^{#J_{k'}} v_{1,k'} \neq 0, \text{ and } \xi^{#J_{k'}} v_{1,k'} = 0, \xi^{#J_k - 1} v_{1,k} \neq 0
\]

\[v_{1,k} \in \text{Im} \xi^{#J_k}, v_{1,k'} \not\in \text{Im} \xi^{#J_{k'} + 1}, \text{ and } v_{1,k'} \in \text{Im} \xi^{#J_{k'}}, v_{1,k'} \not\in \text{Im} \xi^{#J_k + 1} \]

If \( \#J_k \leq \#J_{k'} \) or \( \#J_k \leq \#J_{k'} \) holds, then we can regard \( v_{1,k} + v_{1,k'} \) as a part of a standard basis of a \((V_{2,a})\)-stable symplectic subspace of \((V_{1}^{k'} \oplus V_{1}^{k})\) isomorphic to \(V_{1}^{k'}\) or \(V_{1}^{k}\), respectively. When \( \lambda_k = \lambda_{k'} \), we use this to change our normal form so that the corresponding marked partition satisfies 1.13 3). When \( \lambda_k > \lambda_{k'} \), we use this to transform our marked partition into another marked partition which satisfies 1.13 4) for the pair \((J_k, J_{k'})\). If we make changes to our marked partition in one of the above two procedures, then \( \mathbf{J} \) is unchanged, \( \delta_1 \) on one of \( \{J_k, J_{k'}\} \) is unchanged, but \( \delta_1 \) on the other becomes 0. By repeating these procedures for every possible pair \( k < k' \), we complete the proof of the first assertion.

For the second assertion, recall that \( \text{Irrep} W \) is parametrized by the set of ordered pair of partitions \((\lambda^1, \lambda^2)\) which sum up to \( n \). We define two-partitions out of a strict marked partition \( \sigma \) as

\[
\lambda_k^1 + \lambda_k^2 = \lambda_k, \text{ and } \lambda_k^2 := \begin{cases} 
\#J_k & \text{ (such that } \# \text{ and } \# \text{ are defined for both } J_k \text{ and } J_{k''}) \n\max\{0, \#J_{k'}, \lambda_k - \#J_{k''}; k' > k > k'' \} & \text{(otherwise)}
\end{cases}
\]

for each \( k \), where the set we choose its maximal is formed only from these \( J_{k'} \) and \( J_{k''} \) for which \( \# \) and \( \# \) are defined. It is clear that two sequences \( \lambda^1, \lambda^2 \) sum up to \( n \). By 1.13 4), we deduce that

\[
\lambda_k - \#J_{k''} < \#J_k \text{ (this is equivalent to } \#J_{k''} > \#J_k) \]

holds for \( k'' < k \) (such that \( \# \) and \( \# \) are defined for both \( J_k \) and \( J_{k''} \)). It follows that \( \lambda^2 \) is a partition. (I.e. \( \{\lambda_i^2\}_i \) is a decreasing sequence.) By the symmetry of \( \# \) and \( \# \) in 1.13, we conclude that \( \lambda^1 \) is also a partition.

Therefore, it suffices to prove that the pairs of partitions formed by strict marked partitions are equal only if the marked partitions are equal. (Since this gives the injectivity of the above assignment.) For this, we assume that two strict marked partitions \( \sigma = (\mathbf{J}, \delta_1) \) and \( \sigma' = (\mathbf{J}', \delta_1') \) gives the same pair \((\lambda^1, \lambda^2)\) to deduce contradiction. We can assume that \( \mathbf{J} = \mathbf{J}' \) since \( \lambda = \lambda^1 + \lambda^2 \). Hence, their difference is concentrated in their foot function. By 1.13 3) and 4), we deduce that the foot functions are non-trivial on \( J_k = J_k' \) if and only if

\[
\lambda_k^2 \neq \max\{\lambda_j^2, \lambda_k - \lambda_i^1; \lambda_j \neq \lambda_i, i < k < j\}
\]

and \( \lambda_k \neq \lambda_{k-1} \). Moreover, the value of the foot functions on \( J_k \) are determined by the value of \( \lambda_k^2 \) if they are non-trivial. Since this system has a unique solution, we deduce \( \sigma = \sigma' \), which is contradiction. Thus, the pair of partitions recovers a strict marked partition uniquely, which completes the proof of the second assertion.

**Theorem 1.17** (Orbit structure of \( \mathfrak{g}_2 \)). Let \( \nu = (a, X) = (s, \bar{q}, X) \) be an admissible parameter. Then, there exists \( g \in G \) such that:

\[
gsg^{-1} \in T \text{ and } gX \text{ is a normal form.}
\]
Lemma 1.18. Let \((w, \sigma)\) be a marked partition which is a \(W\)-translation of a strict marked partition. Then, we have
\[
\text{where we set}
\]
\[
\sum_{i=1}^{n} \sigma_i \delta_{-j} (j = \pm n)
\]
Using this, we define the \(W\)-action \(\cdot\) on the set of \(\ell\)-marked partitions as:
For \(w \in W\) and \(\sigma = (J, \delta) = (\{J_1, J_2, \ldots\}, \{\delta_1, \ldots, \delta_\ell\})\), we set
\[
w \cdot \sigma := (\{w \cdot J_1, w \cdot J_2, \ldots\}, \{w \cdot \delta_1, \ldots, w \cdot \delta_\ell\}),
\]
where we set
\[
w \cdot (J_1^1, J_1^2, \ldots) = ((w \cdot J_1)^1, (w \cdot J_1)^2, \ldots) := (w \cdot J_1^1, w \cdot J_1^2, \ldots)
\]
and \(w \cdot \delta_k(j) := \delta_k(w^{-1} \cdot j)\). Notice that we have \(w \cdot J_k = (w \cdot J)_k\) and \(w \cdot J_k^1 = (w \cdot J_k)^1\) for every \(k, j\) in this action.

**Lemma 1.18.** Let \(\sigma = (J, \delta)\) be a marked partition which is a \(W\)-translation of a strict marked partition. Then, we have
\[
C^X v_{1, \sigma} \oplus C^X v_{2, \sigma} \subset T v_{\sigma}, \quad \text{and } C^X v_{1, \sigma}^J \oplus C^X v_{2, \sigma}^J \subset T_J v_{\sigma}^J \text{ for each } J \in J.
\]

**Proof.** Since \(T_J \cap T_J' = \{1\}\), it suffices to prove the second assertion. Let \(v_{\sigma}^J = \sum_{\xi \in \Xi} v_{\xi}\) be the \(T\)-eigen-decomposition of \(v_{\sigma}^J\). Then, we have \#\(\Xi\) = \#\(J\) and \(\dim T_J = \dim T_J = \#J\). Moreover, the weights appearing in \(\Xi\) are linearly independent. Hence, we have the scalar multiplications of each \(v_{\xi}\), which implies the result.

**Corollary 1.19** (of the proof of Lemma 1.18). Let \(\sigma\) be a strict marked partition. Let \(w \in W\). Then, we have \(v_{w, \sigma} \in G v_{\sigma}\). □

1.4 Structure of simple modules

We put \(T_\ell := T \times (\mathbb{C}^\times)^{\ell+1}\). Let \(a \in T_\ell\). Let \(\mu^a : F^a_\ell \to \mathfrak{N}^a_{\ell}\) denote the restriction of \(\mu_\ell\) to \(a\)-fixed points.
We review the convolution realization of simple modules in our situation. The detailed constructions are found in [CG97] 5.11, 8.4 or [Gi97] §5. For its variant, see [Jo98].
The properties we used to apply the Ginzburg theory are: 1) \(Z_\ell = F_\ell \times_{\mathfrak{N}^a_{\ell}} F_\ell\); 2) \(F_\ell\) is smooth; 3) \(\mu_\ell\) is projective; 4) \(R(G_\ell) \subset K^{G_\ell}(Z_\ell)\) is central; and 5) \(H_\bullet(Z_\ell)\) is spanned by algebraic cycles.
Let \(C_a\) be the unique residual field of \(\mathbb{C} \otimes \mathbb{Z} R(G_\ell)_a\) or \(\mathbb{C} \otimes \mathbb{Z} R(T_\ell)_a\). The Thomason localization theorem yields ring isomorphisms
\[
C_a \otimes_{R(G_\ell)} K^{G_\ell}(Z_\ell) \cong C_a \otimes_{R(G_\ell(a))} K^{G_\ell(a)}(Z^a_\ell) \cong C_a \otimes_{R(T_\ell)} K^{T_\ell}(Z^a_\ell).
\]
Moreover, we have the Riemann-Roch isomorphism
\[
C_a \otimes_{R(T_\ell)} K^{T_\ell}(Z^a_\ell) \cong K(Z^a_\ell) \xrightarrow{RR} H_\bullet(Z^a_\ell) \cong \text{Ext}^\bullet(\mu^a_\bullet C_{F^a_\ell}, \mu^a_\bullet C_{F^a_\ell}).
\]
By the equivariant Beilinson-Bernstein-Deligne (-Gabber) decomposition theorem (c.f. Saito [Sa88] 5.4.8.2), we have

$$
\mu^\sharp C^\omega F^\ell \cong \bigoplus_{\mathfrak{m} \subseteq \mathfrak{M}_X} L_{\mathfrak{m},X} \otimes IC(\mathfrak{m},\chi)[d],
$$

where $\mathfrak{m} \subseteq \mathfrak{M}_X$ is a $G$-stable subset such that $\mu^\alpha$ is locally trivial along $\mathfrak{m}$, $\chi$ is an irreducible local system on $\mathfrak{m}$, $d$ is an integer, $L_{\mathfrak{m},X}d$ is a finite dimensional vector space, and $IC(\mathfrak{m},\chi)$ is the minimal extension of $\chi$. Moreover, the set of $\mathfrak{m}$'s such that $L_{\mathfrak{m},X} \neq 0$ (for some $\chi$ and $d$) forms a subset of an algebraic stratification in the sense of [CG97] 3.2.23. It follows that:

**Theorem 1.20** (Ginzburg [Gi97] Theorem 5.2). The set of simple modules of $K^{G_\ell}(Z_\ell)$ for which $R(G_\ell)$ acts as the evaluation at $a$ is in one-to-one correspondence with the set of isomorphism classes of irreducible $G_\ell(a)$-equivariant perverse sheaves appearing in $\mu^\sharp C^\omega F^\ell$ (up to degree shift).

## 2 Hecke algebras and exotic nilpotent cones

We retain the setting of the previous section. We put $G = G_2$, $T := T_2$, $G := F_2$, $\mu := \mu_2$, $Z := Z_2$, and $\pi := \pi_2$. Most of the arguments in this section are exactly the same as [CG97] 7.6 if we replace $G$ by $G \times \mathbb{C}^\times$, $\mathfrak{M}_2$ by the usual nilpotent cone, $\mu$ by the moment map, $F$ by the cotangent bundle of the flag variety, and $Z$ by the Steinberg variety. Therefore, we frequently omit the detail and make pointers to [CG97] 7.6 in which the reader can obtain a correct proof merely replacing the meaning of symbols as mentioned above.

We put $\mathcal{A}_Z := \mathbb{Z}[q^{\frac{1}{2}}_0, q^{\frac{1}{2}}_1, q^{\frac{1}{2}}_2]$ and $\mathcal{A} := \mathbb{C} \otimes \mathcal{A}_Z = \mathbb{C}[q^{\frac{1}{2}}_0, q^{\frac{1}{2}}_1, q^{\frac{1}{2}}_2]$.

**Definition 2.1** (Hecke algebra of type $C_n^{(1)}$). A Hecke algebra of type $C_n^{(1)}$ with three parameters is an associative algebra $\mathbb{H}$ over $\mathcal{A}$ generated by $(T_i)_i^{n\times}$ and $\{e^\lambda\}_{\lambda \in X^*(T)}$ subject to the following relations:

**Toric relations** For each $\lambda, \mu \in X^*(T)$, we have $e^\lambda \cdot e^\mu = e^\lambda + \mu$ (and $e^0 = 1$);

**The Hecke relations** We have

$$(T_i + 1)(T_i - q_2) = 0 \quad (1 \leq i < n) \quad \text{and} \quad (T_n + 1)(T_n + q_0q_1) = 0;$$

**The braid relations** We have

$$T_iT_j = T_jT_i \quad \text{if} \quad |i - j| > 1, \quad (T_nT_{n-1})^2 = (T_{n-1}T_n)^2, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad \text{if} \quad 1 \leq i < n - 1;$$

**The Bernstein-Lusztig relations** For each $\lambda \in X^*(T)$, we have

$$T_i e^\lambda - e^{s_i^\lambda} T_i = \begin{cases} 
(1 - q_2) \frac{e^{\lambda} - e^{s_i^\lambda}}{e^2 - 1} & (i \neq n) \\
(1 - q_0) \frac{e^{\lambda} - e^{s_n^\lambda}}{e^2 - 1} & (i = n)
\end{cases}$$

$$(T_n e^\lambda - e^{s_n^\lambda} T_n = \frac{1 - t_n^2 - t_n(t_n - t_0^{-1})e^{s_n^\lambda}}{e^{2s_n} - 1}(e^\lambda - e^{s_n^\lambda});$$
2) If $n = 1$, then we have $T_1 = T_n$ in Definition 2.1. In this case, we have $\mathbb{H} \cong \mathbb{C}[\mathfrak{d}_2^{\mathbb{A}_1}] \otimes_{\mathbb{C}} \mathbb{H}_0$, where $\mathbb{H}_0$ is the Hecke algebra of type $A_1^{(1)}$ with two-parameters $(q_0, q_1)$.

3) An extended Hecke algebra of type $B_n^{(1)}$ with two-parameters considered in [En06] is obtained by requiring $q_0 + q_1 = 0$. An equal parameter extended Hecke algebra of type $B_n^{(1)}$ is obtained by requiring $q_0 + q_1 = 0$ and $q_1^2 = q_2$. An equal parameter Hecke algebra of type $C_n^{(1)}$ is obtained by requiring $q_2 = -q_0q_1$ and $(1 + q_0)(1 + q_1) = 0$.

For each $w \in W$, we define two closed subvarieties of $Z$ as

$$Z_{\leq w} := \pi^{-1}(O_w) \text{ and } Z_{< w} := Z_{\leq w} \setminus \pi^{-1}(O_w).$$

Let $\lambda \in X^*(T)$. Let $\mathcal{L}_\lambda$ be the pullback of the line bundle $G \times B \lambda^{-1}$ over $G/B$ to $F$. Clearly $\mathcal{L}_\lambda$ admits a $G$-action by letting $(\mathbb{C}^*)^3$ act on $\mathcal{L}_\lambda$ trivially. We denote the operator $[\mathcal{L}_\lambda^* \mathcal{L}_\lambda \otimes \bullet]$ by $e^\lambda$. By abuse of notation, we may denote $e^\lambda(1)$ by $e^\lambda$ (in $K^G(Z)$). Let $q_0 \in R((1) \times \mathbb{C}^\times \times (1) \times \{1\}) \subset R(G)$, $q_1 \in R((1) \times (1) \times \mathbb{C}^\times \times (1)) \subset R(G)$, and $q_2 \in R((1) \times (1) \times (1) \times \mathbb{C}^\times) \subset R(G)$ be the inverse of degree-one characters. (i.e. $q_2$ corresponds to the inverse of the scalar multiplication on $V_2$.) By the operation $e^\lambda$ and the multiplication by $q_i$, each of $K^G(Z_{\leq w})$ admits a structure of $R(T)$-modules.

Each $Z_{\leq w} \setminus Z_{< w}$ is a $G$-equivariant vector bundle over an affine fibration over $G/B$ via the composition of $\pi$ and the second projection. Therefore, the cellular fibration Lemma (or the successive application of localization sequence) yields:

**Theorem 2.3** (c.f. [CG97] 7.6.11). We have

$$K^G(Z_{\leq w}) = \bigoplus_{v \in W; O_v \subset \pi^{-1}(O_w)} R(T) [O_{Z_{\leq w}}].$$

For each $i = 1, 2, \ldots, n$, we put $O_i := \pi^{-1}(O_{\lambda,i})$. We define $\tilde{T}_i := [O_{\lambda,i}]$ for each $i = 1, \ldots, n$.

**Theorem 2.4** (c.f. Proof of [CG97] 7.6.12). The set $\{[O_{Z_{\leq 1}}], \tilde{T}_i, e^\lambda; 1 \leq i \leq n, \lambda \in X^*(T)\}$ is a generator set of $K^G(Z)$ as $\mathcal{A}_G$-algebras.

**Proof.** The tensor product of structure sheaves corresponding to vector subspaces of a vector space is the structure sheaf of their intersection. Taking account into that, the proof of the assertion is exactly the same as [CG97] 7.6.12.

By the Thom isomorphism, we have an identification

$$K^G(F) \cong K^G(G/B) \cong R(T) = \mathcal{A}_G[T].$$

We normalize the images of $[\mathcal{L}_\lambda]$ and $q_i (i = 0, 1, 2)$ under (2.1) as $e^\lambda$ and $q_i$, respectively.

**Theorem 2.5** (c.f. [CG97] Claim 7.6.7). The homomorphism

$$o : K^G(Z) \rightarrow \text{End}_{R(G)}K^G(F)$$

is injective.
Proposition 2.6. We have
1. \([O_{Z_{\leq 1}}] = 1 \in \text{End}_{R(G)} K^G(F)\);
2. \( \hat{T}_i \circ e^\lambda = (1 - q_0 e^{s_0 \alpha_i}) e^{\frac{\lambda - e^{s_i \alpha_i}}{1 - e^{-\alpha_i}}} \) for every \( \lambda \in X^*(T) \) and every \( 1 \leq i < n \);
3. \( \hat{T}_n \circ e^\lambda = (1 - q_0 e^{s_0 \alpha_n}) (1 - q_1 e^{s_1 \alpha_n}) e^{\frac{\lambda - e^{s_n \alpha_n}}{1 - e^{-\alpha_n}}} \) for every \( \lambda \in X^*(T) \).

Proof. The component \( Z_{\leq 1} \) is equal to the diagonal embedding of \( F \). In particular, both of the first and the second projections give isomorphisms between \( Z_{\leq 1} \) and \( F \). It follows that
\[
(\hat{T}_i \circ [L_\lambda]) = \sum_{i \geq 0} (-1)^i [R^i(p_1)_* (O_{Z_{\leq 1}} \otimes \tilde{\rho}_i^s L_\lambda)]
\]
which proves 1). For each \( i = 1, \ldots, n \), we define \( \mathbb{V}^+(i) := V^+_2 \cap s_i V^+_2 \). Let \( P_i := \bar{B}s_i \bar{B} \cap B \) be a parabolic subgroup of \( G \) corresponding to \( s_i \). Each \( \mathbb{V}^+(i) \) is \( B \)-stable. Hence, it is \( P_i \)-stable. We have
\[
\pi(\mathcal{O}_i) = \mathcal{O}_{s_i} = G(1 \times P_i)p_1 \subset G/B \times G/B.
\]
The product \((1 \times P_i)p_1 \times \mathbb{V}^+(i)\) is a \( B \)-equivariant vector bundle. Here we have \( G \cap (B \times P_i) = B \). Hence, we can induce it up to a \( G \)-equivariant vector bundle \( \mathbb{V}(i) \) on \( \pi(\mathcal{O}_i) \). By means of the natural embedding of \( G \)-equivariant vector bundles
\[
F = G \times B \mathbb{V}_2^+ \hookrightarrow G \times B \mathbb{V}_2 \cong G \times \mathbb{V}_2,
\]
we can naturally identify \( \pi^{-1}(\mathbb{V}_s) \) with \( \mathbb{V}^+(i) \). Since \( \mathbb{V}^+(i) \) is \( P_i \)-stable, we conclude \( \pi^{-1}(\mathbb{V}_s) \cong \mathbb{V}^+(i) \) as \( P_i \)-modules. As a consequence, we conclude \( \mathbb{V}(i) \cong \mathcal{O}_i \). Let \( \bar{F}(i) := G \times B (\mathbb{V}_2^+ \cap \mathbb{V}^+(i)) \). It is a \( G \)-equivariant quotient bundle of \( F \). The rank of \( \bar{F}(i) \) is one \((1 \leq i < n)\) or two \((i = n)\). Let \( \bar{Z}_{\leq s_i} \) be the image of \( Z_{\leq s_i} \) under the quotient map \( F \times F \to \bar{F}(i) \times \bar{F}(i) \). We obtain the following commutative diagram:

Here the above objects are smooth \( \mathbb{V}^+(i) \)-fibrations over the bottom objects. Therefore, if suffices to compute the convolution operation of the bottom line. We have \( \bar{Z}_{\leq s_i} = \mathcal{O}_{s_i} \cup \Delta(\bar{F}(i)) \), where \( \Delta : \bar{F}(i) \hookrightarrow \bar{F}(i)^2 \) is the diagonal embedding. Let \( \tilde{p}_j : \bar{Z}_{s_i} \to \mathbb{G}/B \) \((j = 1, 2)\) be projections induced by the natural projections of \( \mathbb{G}/B \times \mathbb{G}/B \). By construction, each \( \tilde{p}_j \) is a \( \mathbb{G} \)-equivariant \( \mathbb{P}^1 \)-fibration. Let \( \bar{L}_\lambda \) be the pullback of \( \mathbb{G} \times B \lambda^{-1} \) to \( \bar{F}(i) \). We deduce
\[
(\hat{T}_i \circ [L_\lambda]) = \sum_{i \geq 0} (-1)^i [R^i(\tilde{p}_1)_* (O_{\bar{Z}_{s_i}} \otimes (O_{\bar{F}(i)} \boxtimes \bar{L}_\lambda))]
\]
\[
= \sum_{i \geq 0} (-1)^i [R^i(\tilde{p}_1)_* \tilde{p}_2^s \tilde{p}_s^s (G \times B \lambda^{-1})] = \left[ G \times B \left( \frac{e^\lambda - e^{s_i \alpha_i}}{1 - e^{-\alpha_i}} \right) \right],
\]
where \( i : G/B \leftrightarrow \tilde{F}(i) \) is the zero section, and \( \frac{\lambda}{1-e^{-\lambda}} \in R(T) \cong R(B) \) is a virtual \( B \)-module. Here the ideal sheaf associated to \( G/B \subset \tilde{F}(i) \) represents \( \mathfrak{q}_2[\mathcal{L}_i] \) in \( K^G(\tilde{F}(i)) \) \((1 \leq i < n)\) or corresponds to \( \mathfrak{q}_0\mathcal{L}_n + \mathfrak{q}_1\mathcal{L}_n \subset \mathcal{O}_{\tilde{F}(i)}\) \((i = n)\). In the latter case, divisors corresponding to \( \mathfrak{q}_0\mathcal{L}_n \) and \( \mathfrak{q}_1\mathcal{L}_n \) are normal crossing. Thus, we have \([\mathfrak{q}_0\mathcal{L}_n \cap \mathfrak{q}_1\mathcal{L}_n] = \mathfrak{q}_0\mathfrak{q}_1[\mathcal{L}_2n] \). In particular, we deduce

\[
[q_0\mathcal{L}_n + q_1\mathcal{L}_n] = q_0[\mathcal{L}_n] + q_1[\mathcal{L}_n] - q_0q_1[\mathcal{L}_2n] \in K^G(\tilde{F}(n)).
\]

Therefore, we conclude

\[
\tilde{T}_i \circ e^\lambda = \begin{cases} 
(1 - q_2e^{\alpha_i}) \frac{\lambda - e^{\alpha_i} - \alpha_i}{1 - e^{-\alpha_i}} & (1 \leq i < n) \\
(1 - q_0e^{\alpha_n}) (1 - q_1e^{\alpha_n}) \frac{\lambda - e^{\alpha_n} - \alpha_n}{1 - e^{-\alpha_n}} & (i = n)
\end{cases}
\]
as desired. \( \square \)

The following representation of \( \mathbb{H} \) is usually called the basic representation or the anti-spherical representation:

**Theorem 2.7** (Basic representation c.f. [Mc03] 4.3.10). There is an injective \( \mathbb{A} \)-algebra homomorphism

\[
\varepsilon : \mathbb{H} \to \text{End}_{\mathbb{A}[T]},
\]
defined as \( \varepsilon(e^\lambda) := e^\lambda \cdot (\lambda \in X^*(T)) \) and

\[
\varepsilon(T_i)e^\lambda := \begin{cases} 
\frac{\lambda - e^{\alpha_i}}{1 - e^{-\alpha_i}} - q_2e^{\alpha_i} \frac{\lambda - e^{\alpha_i} + \alpha_i}{1 - e^{-\alpha_i}} & (1 \leq i < n) \\
q_0q_1 \frac{\lambda - e^{\alpha_n} + \alpha_n}{1 - e^{-\alpha_n}} - (q_0 + q_1)e^{\alpha_n} \frac{\lambda - e^{\alpha_n}}{1 - e^{-\alpha_n}} & (i = n)
\end{cases}
\]

**Theorem 2.8** (Exotic geometric realization of Hecke algebras). We have an isomorphism

\[
\mathbb{H} \cong \mathbb{C} \otimes \mathbb{Z} \ K^G(\mathbb{Z}),
\]
as algebras.

**Proof.** Consider an assignment \( \vartheta \)

\[
\varepsilon, \tilde{T}_i : \mathbb{H} \to e^\lambda, T_i \mapsto \begin{cases} 
\tilde{T}_i - (1 - q_2(e^{\alpha_i} + 1)) & (1 \leq i < n) \\
\tilde{T}_i + (q_0 + q_1)e^{\alpha_n} - (1 + q_0q_1(e^{\alpha_n} + 1)) & (i = n)
\end{cases}
\]

By means of the Thom isomorphism, the above assignment gives an action of
an element of the set \( \{ e^\lambda \} \cup \{ T_n \}_{n=1}^\infty \) on \( \mathcal{A}[T] \). We have
\[
\vartheta (e^\lambda) e^\mu = e^{\lambda+\mu}
\]
\[
\vartheta (T_n) e^\lambda = \left( \tilde{T}_i - (1 - q_2(e^{\alpha_i} + 1)) \right) e^\lambda = (1 - q_2 e^{\alpha_i}) \left( \frac{e^\lambda - e^{\alpha_i} \lambda - \alpha_i}{1 - e^{-\alpha_i}} - \frac{e^\lambda + q_2(e^{\alpha_i} + 1)e^\lambda}{1 - e^{-\alpha_i}} \right) = (\frac{e^\lambda - e^{\alpha_i} \lambda - \alpha_i}{1 - e^{-\alpha_i}} - \frac{e^\lambda - e^{\alpha_i} \lambda - 2\alpha_i}{1 - e^{-\alpha_i}}) = \varepsilon (T_i) e^\lambda.
\]
This identifies \( \mathbb{C} \otimes_{\mathbb{Z}} K^G(F) \) with the basic representation of \( \mathbb{H} \) via the correspondence \( e^\lambda \mapsto e^\lambda \) and \( T_i \mapsto \tilde{T}_i \). In particular, it gives an inclusion \( \mathbb{H} \subset \mathbb{C} \otimes_{\mathbb{Z}} K^G(Z) \). Here we have \( T_i \in \tilde{T}_i + R(T) \) for \( 1 \leq i \leq n \). It follows that \( \mathbb{C} \otimes_{\mathbb{Z}} K^G(Z) \subset \mathbb{H} \), which yields the result.

\[\textbf{Theorem 2.9} \quad \text{(Bernstein c.f. [CG97] 7.1.14 and [Mc03] 4.2.10). The center } Z(\mathbb{H}) \text{ of } \mathbb{H} \text{ is naturally isomorphic to } \mathbb{C} \otimes_{\mathbb{Z}} R(G).\]

\[\textbf{Corollary 2.10.} \quad \text{The center of } K^G(Z) \text{ is } R(G).\]

For a semisimple element \( a \in G \), we define
\[
\mathbb{H}_a := \mathbb{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H} \quad \text{(c.f. §1.4)}
\]
and call it the specialized Hecke algebra.

\[\textbf{Theorem 2.11.} \quad \text{Let } a \in G \text{ be a semisimple element. We have an isomorphism } \mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(Z^a) \text{ as algebras.}\]

\[\textbf{Proof.} \quad \text{This is a combination of [CG97] 6.2.3 and 5.10.11. (See also [CG97] 8.1.6.)}\]

\[\textbf{Convention 2.12.} \quad \text{Let } a = (s, \tilde{q}) \in G \text{ be a pre-admissible element. We define } Z^a_+ \text{ to be the image of } Z^a \text{ under the natural projection defined by } Z \ni (g_1B, g_2B, X_0, X_1, X_2) \mapsto (g_1B, g_2B, X_0 + X_1, X_2) \in Z.\]

Let \( F^a_+ \) be the image of \( Z^a_+ \) via the first (or the second) projection. Let \( \mu^a_+ \) be the restriction of \( \mu \) to \( F^a_+ \). We denote its image by \( \mathfrak{N}^a_+ \). By the assumption \( q_0 \neq q_1 \), we have \( F^a_+ \cong F^a, \ Z^a_+ \cong Z^a \), and \( \mathfrak{N}^a_+ \cong \mathfrak{N}^a_0 \).

\[\textbf{Corollary 2.13.} \quad \text{Keep the setting of Convention 2.12. We have an isomorphism } \mathbb{H}_a \cong \mathbb{C} \otimes_{\mathbb{Z}} K(Z^a_+) \text{ as algebras.}\]
For the later use, let us introduce our last class of parameter here.

**Definition 2.14** (Regular parameters). A pre-admissible parameter \((a, X)\) is called regular iff there exists a direct factor \(A[d] \subset \mu^d_a \mathcal{C}_{\mathbb{F}_a^n}\), where \(A\) is a simple \(G(a)\)-equivariant perverse sheaf on \(\mathcal{R}_a^n\) such that \(\text{supp} A = \overline{G(a)X}\) and \(d\) is an integer.

We denote by \(\mathcal{R}_a\) the set of \(G(a)\)-conjugacy classes of regular pre-admissible parameters of the form \((a, X)\) \((X \in \mathcal{R}_a^n)\).

### 3 Clan decomposition

We work under the same setting as in §2.

**Definition 3.1** (Clans). Let \(a = (s, \vec{q})\) be a pre-admissible element such that \(s \in T\). Let \(q^2 = e^{r_2}\). We put \(\Gamma := r_2 \mathbb{Z} + \Gamma_0\). A clan associated to \(a\) is a maximal subset \(c \subset [1, n]\) with the following property: For each two elements \(i, j \in c\), there exists a sequence \(i = i_0, i_1, \ldots, i_m = j\) (in \(c\)) such that

\[
\{\log_{i_k}(s) + \log_{i_{k+1}}(s)\} \cap \{\pm r_2 + \Gamma_0\} \neq \emptyset \quad \text{for each } 0 \leq k < m.
\]

We have a disjoint decomposition

\[ [1, n] = \bigsqcup_{c \in \mathcal{C}_a} c, \]

where each \(c\) is a clan associated to \(a\) and \(\mathcal{C}_a\) is the set of clans associated to \(a\). For a clan \(c\), we put \(n^c := \#c\).

We assume the setting of Definition 3.1 in the rest of this section unless stated otherwise. At the level of Lie algebras, we have a decomposition

\[
g(s) = t \oplus \bigoplus_{i < j, \sigma_1, \sigma_2 \in \{\pm 1\}, \sigma_1 \log_i(s) + \sigma_2 \log_j(s) \equiv 0} g(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{i \in [1, n], \sigma \in \{\pm 1\}, 2 \log_i(s) \equiv 0} g(s)[\sigma 2 \epsilon_i],
\]

where \(\equiv\) means modulo \(\Gamma_0\). For each \(c \in \mathcal{C}_a\), we define a Lie algebra \(g(s)_c\) as the Lie subalgebra of \(g(s)\) defined as

\[
\bigoplus_{i \in c} C_{\epsilon_i} \oplus \bigoplus_{i < j \in c, \sigma_1, \sigma_2 \in \{\pm 1\}, \sigma_1 \log_i(s) + \sigma_2 \log_j(s) \equiv 0} g(s)[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \bigoplus_{i \in c, \sigma \in \{\pm 1\}, 2 \log_i(s) \equiv 0} g(s)[\sigma 2 \epsilon_i],
\]

where \(\equiv\) means modulo \(\Gamma_0\). Moreover, we have

\[
g(s) = \bigoplus_{c \in \mathcal{C}_a} g(s)_c. \quad (3.1)
\]

In particular, we have \([g(s)_c, g(s)_{c'}] = 0\) unless \(c = c'\). Let \(G(s)_c\) be the connected subgroup of \(G(s)\) which has \(g(s)_c\) as its Lie algebra.

The following theorem is a consequence of Steinberg’s centralizer theorem and the Borel-de Siebenthal theorem, for which we present a proof for the reference purpose.
Theorem 3.2 (Centralizer theorem for symplectic groups). Let \( A \subset T \) be an algebraic subgroup. Then, the group \( G(A) \) is connected.

Proof. By a Lie algebra calculation, the group \( G(A)^\circ \) is generated by \( T \) and unipotent one-parameter subgroups \( U_\alpha (\alpha \in R) \) such that \( \alpha(A) = \{1\} \). By a repeated use of the Borel-de Siebenthal theorem [BS49], the root system of \( G(A)^\circ \) is the product of the standard presentations of the root systems of

\[
GL(m, \mathbb{C}), SL(2, \mathbb{C}), \text{ or } Sp(2m, \mathbb{C}).
\]

(3.2)

In particular, the derived group of \( G(A)^\circ \) must be simply connected. Now we prove the theorem by induction on the cardinality \( k \) of a generating set of \( A \). (Notice that the word "generating" means the Zariski closure of the group generated by a given subset of \( T \) is \( A \). Hence, we can assume the finiteness of the cardinality of such a set.) The case \( k = 1 \) is an immediate consequence of Steinberg’s centralizer theorem (c.f. [Ca85] 3.5.6). If the assertion is true for smaller \( k \), then it suffices to consider the centralizer of a semi-simple element in a group listed at (3.2). This is again connected by Steinberg’s centralizer theorem. Therefore, the induction proceeds and we obtain the result. \( \square \)

Lemma 3.3. We have \( G(s) = \prod_{c \in C_a} G(s)_c \).

Proof. By (3.1), it is clear that \( \prod_{c \in C_a} G(s)_c \) is equal to the identity component of \( G(s) \). Since \( G \) is a simply connected semi-simple group, it follows that \( G(s) \) is connected by Theorem 3.2. In particular, we have \( G(s) \subset \prod_{c \in C_a} G(s)_c \) as desired.

We denote \( B \cap G(s)_c \) and \( wB \cap G(s)_c \) by \( B(s)_c \) and \( wB(s)_c \), respectively.

Convention 3.4. We denote by \( \mathbb{V}^a \) the image of \( \mathbb{V}^a_2 \) to \( \mathbb{V} \) via the map

\[
\mathbb{V}^a_2 \ni (X_0 \oplus X_1 \oplus X_2) \mapsto ((X_0 + X_1) \oplus X_2) \in \mathbb{V}.
\]

Since \( q_0 \neq q_1 \), we have \( \mathbb{V}^a \cong \mathbb{V}^a_2 \).

For each \( c \in C_a \), we define

\[
\mathbb{V}^a_c := \sum_{i,j \in \mathbb{C}, \sigma_1, \sigma_2, \sigma_3 \in \{\pm 1\}} \mathbb{V}^a[\sigma_1 \epsilon_i + \sigma_2 \epsilon_j] \oplus \mathbb{V}^a[\sigma_3 \epsilon_i].
\]

It is clear that \( \mathbb{V}^a = \bigoplus_{c \in C_a} \mathbb{V}^a_c \). By the comparison of weights, the \( g(s)_e \)-action on \( \mathbb{V}^a_c \) is trivial unless \( c = c^c \).

Remark 3.5. Since \( c \) is not an integer and we do not use \( \mathbb{V}_t \) in the rest of this paper (except for §7), we use the notation \( \mathbb{V}^a_c \). The author hopes the reader not to confuse \( \mathbb{V}^a_c \) with \( (\mathbb{V}_t)^a \).

Lemma 3.6. Let \( \mathcal{O} \subset \mathbb{V}^a_c \) be a \( G(a) \)-orbit. Let \( \mathcal{O}_c \) denote the image of \( \mathcal{O} \) under the natural projection \( \mathbb{V}^a \rightarrow \mathbb{V}^a_c \). Then, we have a product decomposition \( \mathcal{O} = \bigoplus_{c \in C_a} \mathcal{O}_c \).

Proof. Let \( X \in \mathbb{V}^a_c \). There exists a family \( \{X_c\}_{c \in C_a} (X_c \in \mathbb{V}^a_c) \) such that \( X = \sum_{c \in C_a} X_c \). We have \( G(s)_c X = \bigoplus_{c \in C_a} G(s)_c X_c \). For each of \( i = 0, 1 \), the clan \( c \in C_a \) such that \( (\mathbb{V}^a_c \cap \mathbb{V}^a) \neq \{0\} \) is at most one since clans are
determined by the $s$-eigenvalues of $V_1$. Let $c^i (i = 1, 2)$ be the unique clan such that $(V_1^{(s,q_i)} \cap \mathcal{V}_e) \neq \{0\}$. Let $G_e$ be the product of scalar multiplications of $V_1^{(s,q_i)}$ such that $V_1^{(s,q_i)} \cap \mathcal{V}_e \neq \{0\}$. Since the set of $a$-fixed points of a conic variety in $\mathcal{V}$ is conic, we have $(G(s)_e \times (\mathbb{C}^\times)^3)X_e = (G(s)_e \times G_e)X_e$. We have $\prod_{c \in C_a} (G(s)_c \times G_e) \subset G(a)$. It follows that

$$G(a)X = \bigoplus_{c \in C_a} G(a)X_e = \bigoplus_{c \in C_a} (G(s)_c \times G_e)X_e = \bigoplus_{c \in C_a} \mathcal{O}_c$$

as desired. \hfill \Box

For each $w \in W$, we define

$$F_{a+}^a(w) := G(s) \times^w B(s) (\hat{w}\mathcal{V}^+ \cap \mathcal{V}_a).$$

Similarly, we define

$$F_{a+}^a(w, c) := G(s)_c \times^w B(s)_c (\hat{w}\mathcal{V}^+ \cap \mathcal{V}_e)$$

for each $c \in C_a$.

**Lemma 3.7.** We have $F_{a+}^a = \cup_{w \in W} F_{a+}^a(w)$.

**Proof.** The set of $a$-fixed points of $G/B$ is a disjoint union of flag varieties of $G(s)$. It follows that each point of $F_{a+}^a$ is $G(s)$-conjugate to a point in the fiber over a $T$-fixed point of $G/B$. \hfill \Box

The local structures of these connected components are as follows.

**Lemma 3.8.** For each $w \in W$, we have

$$F_{a+}^a(w) \cong \prod_{c \in C_a} F_{a+}^a(w, c).$$

**Proof.** The set $\mathcal{V}_e^a$ is $T$-stable for each $c \in C_a$. Hence, we have

$$F_{a+}^a(w) = G(s) \times^w B(s) (\hat{w}\mathcal{V}^+ \cap \mathcal{V}_e^a) \cong G(s) \times^w B(s) (\bigoplus_{c \in C_a} (\hat{w}\mathcal{V}^+ \cap \mathcal{V}_e^a)).$$

Since we have $G(s)/B(s) \cong \prod_{c \in C_a} G(s)_c/B(s)_c$, we deduce

$$G(s) \times^w B(s) (\hat{w}\mathcal{V}^+ \cap \mathcal{V}_e^a) \cong \prod_{c \in C_a} G(s)_c \times^w B(s)_c (\hat{w}\mathcal{V}^+ \cap \mathcal{V}_e^a \cap \mathcal{V}_e^a).$$

Here the RHS is isomorphic to

$$F_{a+}^a(w, c) \times \bigoplus_{c \neq c'} G(s)_c \times^w B(s)_c.$$
We define a map \( w \mu_c^a \) by

\[
  w \mu_c^a : F_+^a(w, c) = G(s)_c \times {}^{wB(s)_c} (\hat{\mathcal{V}}^+ \cap \mathcal{V}_c^a) \longrightarrow \mathcal{V}_c^a.
\]

We put \( G_c := Sp(2n^c) \) and \( s_c := \exp(\sum_{i \in c} \log_i(s_i)) \in T \). We have embeddings

\[
s = \prod_{c \in C_a} s_c = \prod_{c \in C_a} Sp(2n^c) \subset Sp(2n),
\]

induced by the following identifications:

\[
  g(s)_c = g_c(s_c) \subset \left( \bigoplus_{c \in C_a} C_{c_i} \right) \oplus \bigoplus_{\alpha = \sigma_1 c_1 + \sigma_2 c_2 \neq 0 \atop \sigma_1, \sigma_2 \in \{ \pm 1 \}, i, j \in c} g[\alpha] = g_c.
\]

Note that we have \( G(s)_c = G_c(s_c) \subset G_c \) in general.

Let \( V(c) \) be the 1-exotic representation of \( G_c \). We have a natural embedding \( V_c^a \subset \mathcal{V}(c) \) of \( G(s)_c \)-modules. (The \( G(s)_c \)-module structure on the RHS is given by the restriction of the \( G_c \)-action.)

Let \( \nu = (a, X) \) be a pre-admissible parameter. We have a family of pre-admissible parameters \( \nu_c := (s_c, \bar{q}, X_c) \) of \( G_c \)'s such that \( s = \prod_c s_c, X = \oplus_c X_c \).

We denote \( \nu = \prod_{c \in C_a} \nu_c \)

and call it the clan decomposition of \( \nu \). Let \( W_a := \prod_{c \in C_a} N_{G_c}(T)/T \). By Lemma 3.8, we conclude that

\[
  \bigcup_{w \in W_a} F_+^a(w) \subset F_+^a
\]

is the product of the \( F_+^a \)'s obtained by replacing the pair \((G, \nu)\) by \((G_c, \nu_c)\) for all \( c \in C_a \).

**Proposition 3.9** (Clan decomposition of \( \mu^a \)). For each \( w \in W \), we have

\[
  \mu_c^a \mid_{F_+^a(w)} \cong \prod_{c \in C_a} w \mu_c^a.
\]

In particular, every irreducible direct summand \( A \) of \((\mu_c^a)_* C_{F_+^a}^c \) is written as an external product of \( G(s)_c \)-equivariant sheaves appearing in \((w \mu_c^a)_* C_{F_+^a}^c(w, c)\) (up to degree shift).

**Proof.** The first assertion follows from the combination of Lemma 3.6, Lemma 3.8, and the definition of \( w \mu_c^a \). We have \( C_{F_+^a}^c = \bigoplus_{F_+^a(w) \subset F_+^a} C_{F_+^a}^c(w) \). A direct summand of \((\mu_c^a)_* C_{F_+^a}^c \) is a direct summand of \((\mu_c^a)_* C_{F_+^a}^c(w)\) for some \( w \in W \). Since

\[
  (\mu_c^a)_* C_{F_+^a}^c(w) \cong \bigoplus_c (w \mu_c^a)_* C_{F_+^a}^c(w, c),
\]

the second assertion follows.

**Corollary 3.10.** Let \( \nu = (a, X) \) be a pre-admissible parameter. Then, it is regular if and only if \( \nu_c \) is a regular pre-admissible parameter of \( G_c \) for every \( c \in C_a \).
Proof. Let $W_0 := N_{G(a)}(T)/T \subset W$. We have a natural inclusion $W_0 \subset W$. Here we have
\[
\mu^a_+ = \bigsqcup_{w \in W/W_0} \mu^a_+|_{F^a_+(w)},
\]
where we regard $W/W_0 \subset W$ by taking some representative. For each $w \in W$, there exists $v \in W_0$ such that $wV^+ \cap V^a = vV^+ \cap V^a \subset V^a$. Moreover, we can choose $v$ so that $wB(s)_c = vB(s)_c$ holds for each $c \in C_a$. As a consequence, all $F^a_+(w)$ are isomorphic to one of $F^a_+(w)$ ($w \in W_0$) as $G(a)$-varieties, together with maps $\mu^a_+|_{F^a_+(w)}$ to $V^a$. Therefore, $\nu$ is regular if and only if an intersection cohomology complex with its support $\overline{G(a)}\nu$ (with degree shift) appears in $(\mu^a_+), C_{F^a_+(w)}$ for some $w \in W_a$. Hence, Proposition 3.9 implies the result. □

Corollary 3.10 reduces the analysis of the decomposition pattern of $(\mu^a_+), C_{F^a_+}$ into the case that $\nu$ has a unique clan.

4 On stabilizers of exotic nilpotent orbits

We retain the setting of §2.

Lemma 4.1. Let $H$ be a connected linear algebraic group and let $X$ be a variety with $H$-action. If $H = H_rH_u$ be a Levi decomposition of $H$ with $H_r$ its reductive part. If $\text{Stab}_Hx$ is connected for $x \in X$, then so is $\text{Stab}_Hx$.

Proof. Assume to the contrary to deduce contradiction. Let $h \in \text{Stab}_Hx$ be an element which is not in the identity component. Let $h = h_rh_u \in H_rH_u$ be its Levi decomposition. For some $k > 1$, we have $h^k \in (\text{Stab}_Hx)^o$. This implies the existence of $g \in (\text{Stab}_Hx)^o$ which satisfies $h^k = g^k$. Let $g = g_rg_u$ be the Levi decomposition. We have $H_u \triangleleft H$, which claims $h^k = g^k$. Replacing $h$ by $g^{-1}h$, we further assume $h^k = 1$. Here we have $h^k \in (\text{Stab}_Hx)^o = \text{Stab}_Hx$. Put $u := h^k \in \text{Stab}_Hx$. Let $U$ denote the group given as the Zariski closure of the group generated by $h$. We have its connected component decomposition $U = U_0 \cup U_1 \cup U_2 \cup \cdots$, where $1, u \in U_0$ and $h \in U_1$. Since $H_r$ is of finite order, $U_0$ is unipotent and each of $U_i$ is a homogeneous $U_0$-space. Let $U^{(m)}$ be the $m$-th lower central subgroup of $U_0$. For each $m$, the adjoint $h$-action preserves $U^{(m)}_0$. It follows that if $u \in U^{(m)}_0$ for some $m$, then we have $(hu_m)^k \in U^{(m)}_0$ for each $u_m \in U^{(m)}_0$. Moreover, we have $huh^{-1} \equiv u \mod U^{(m+1)}_0$ by $u = h^k$. Since $U^{(m)}_0/U^{(m+1)}_0$ is abelian, we deduce
\[
\{(hu_m)^k; u_m \in U^{(m)}_0\}/U^{(m+1)}_0 = \{\bar{u}_m \in U^{(m)}_0/U^{(m+1)}_0; h\bar{u}_mh^{-1} = \bar{u}_m\} \subset U^{(m)}_0/U^{(m+1)}_0
\]
for each $m$. The second term contains $1 \mod U^{(m+1)}_0$. We have $U^{(m)}_0 = \{1\}$ for $m \gg 0$. Hence, we can change $h$ if necessary to assume $h^k = 1$, which implies that $h$ is semisimple. Therefore, $h$ belongs to $\text{Stab}_Hx$. An element of finite order is always semisimple, hence its unipotent part is trivial. Thus, we have $h_u = 1$ if $h^r = 1$. Therefore, we have contradiction and the result follows. □

Theorem 4.2 (Igusa [Ig73] Lemma 8, Springer [Sp07]). Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ be a partition of $n$. We regard it as a 0-marked partition. Then, the reductive part of $\text{Stab}_Gv_\lambda$ is
\[
L_\lambda := \text{Sp}(2n_1, \mathbb{C}) \times \text{Sp}(2n_2, \mathbb{C}) \times \cdots
\]
where the sequence \((n_1, n_2, \ldots)\) are the number of \(\lambda_i\)'s which share the same value. Moreover, we have

\[
\text{Res}^G_{G_{\lambda}} V_1 = \bigoplus_{i \geq 1} V(i)^{\oplus \lambda_i},
\]

(4.1)

where \(V(i)\) is the vector representation of \(Sp(2n_i)\) with trivial actions of \(Sp(2n_j)\) \((j \neq i)\).

**Remark 4.3.** Igusa's result is not as precise as Theorem 4.2. But we can deduce from its proof without difficulty. Springer [Sp07] contains more precise statement.

**Corollary 4.4.** Keep the setting of Theorem 4.2. Then, we can choose a maximal torus of \(L_{\lambda}\) inside \(T\).

**Proof.** Let \(\sigma = (J, \emptyset)\) be the 0-marked partition corresponding to \(\lambda\). By Lemma 1.18, we have \(C^x \subset \text{Stab}_{T_J} v_{J,\sigma}^J\) for each \(J \in J\). It follows that \(L_{\lambda} \cap T\) contains a torus of dimension \((\sum_{i \geq 0} n_i)\), which implies the result. \(\square\)

**Proposition 4.5.** Let \(X \in \mathfrak{N}_2\). Then, \(\text{Stab}_G X\) is connected.

**Proof.** Let \(X = (X_0 \oplus X_1) \oplus v_{2,\lambda}\), where \(\lambda\) is a partition of \(n\) regarded as a 0-marked partition. It suffices to show that the action of \(\text{Stab}_G v_{2,\lambda}\) on \((X_0 \oplus X_1)\) has connected stabilizer. Let \(L_{\lambda} U_{\lambda}\) be the Levi decomposition of \(\text{Stab}_G v_{2,\lambda}\). By Lemma 4.1, it is sufficient to show that the stabilizer of \(L_{\lambda}\) on \((X_0 \oplus X_1)\) is connected. By Theorem 4.2, it suffices to prove that the \(G\)-stabilizer of finite set of elements in \(V_1\) is connected. By a repeated use of Lemma 4.1, it suffices to prove that the \(G\)-stabilizer of one element in \(V_1\) has \(Sp(2n-2)\) as its (reductive) Levi factor. We denote the element \(v \in V_1\) and fix a symplectic form on \(V_1\) which is preserved by \(G\). Then, it is easy to see that \(\text{Stab}_G v\) preserves \(Cv\) and the compliment space \(v^+\) of \(V_1\) with respect to the symplectic form. Thus, its Levi component is given as a subgroup of

\[
C^x \times Sp(2n-2) = (C^x \times GL(2n-2, \mathbb{C}) \times C^x) \cap Sp(2n) \subset GL(V_1),
\]

which fixes \(v\). (Here the middle group is the Levi component of \(GL(V_1)\) which preserves a partial flag \(\{0\} \subset Cv \subset v^+ \subset V_1\).) Therefore, it is \(Sp(2n-2)\) as desired. \(\square\)

**Remark 4.6.** Springer [Sp07] contains an explicit description of the \(G\)-stabilizer of each strict normal form. As is seen easily from the proof of Proposition 4.5, it is not hard to write down the \(G\)-stabilizer of a point of \(\mathfrak{N}_2\) assuming [Sp07].

**Corollary 4.7** (of the proof of Proposition 4.5). For each \(X \in \mathfrak{N}_2\), the reductive part of \(\text{Stab}_G X\) is a product of symplectic groups. \(\square\)

**Theorem 4.8** (Refined form of Theorem 1.17). Let \(\nu = (a, X) = (s, \vec{q}, X_1 \oplus X_2) \in T \times \mathfrak{N}\) be a pre-admissible parameter which is admissible or \(a = a_0\). Then, we have a clan decomposition

\[
\nu = \prod_{e \in \mathcal{C}_a} \nu^e = \prod_{e \in \mathcal{C}_a} (s_e, \vec{q}, X_e)
\]

with the following properties:
• Each \( \nu_e \) is an admissible parameter;

• There exists \( g \in G \) such that:

\[
gsg^{-1} \in T \text{ and each } gX_e \text{ is a strict normal form.}
\]

Proof. If \( a = a_0 \), then we have \( \mathfrak{g}_2 \cong \mathfrak{n} \). Hence, the result reduces to Proposition 1.16 1).

Thus, we assume that \( a \) is admissible. Since the admissibility condition depends only on the configuration of \( \tilde{q} \), the clan decomposition preserves admissibility. Hence, it suffices to prove the case that \( e = [1, n] \) is the unique clan of \( C_a \). Then, two distinct eigenvalues \( t_1, t_2 \) of \( s \) on \( V_1 \) satisfies

\[
t_1t_2 \text{ or } t_1/t_2 = q^{m^n}, \text{ where } |m| < n.
\]

It follows that at least one of \( q_0 \) or \( q_1 \) does not appear as a \( s \)-eigenvalue of \( V_1 \) by the admissibility condition. Therefore, we can assume \( (s - q_1)X_1 = 0 \) by swapping the roles of \( q_0 \) and \( q_1 \) if necessary.

Let us take \( G \)-conjugate to assume that \( X = v_\sigma \) for a strict marked partition \( \sigma = (J, \delta) \). By the description of the \( G \)-stabilizer of \( v_\sigma \), we deduce that we can choose a maximal torus of \( \text{Stab}_Gv_\sigma \) inside of \( T \). By Lemma 1.18 and the fact \( v_\sigma \) is a strict normal form, we deduce that a (possibly disconnected) maximal torus of \( \text{Stab}_Gv_\sigma \) is taken inside \( T \). Therefore, we conclude that \( (a, v_\sigma) \) is a strict normal form after taking conjugate of \( a \) by the \( \text{Stab}_Gv_\sigma \)-action (or the \( \text{Stab}_Gv_\sigma \)-action).

Corollary 4.9. Let \( a = (s, q_0, q_1, q_2) \in T \) be an admissible element. If \( C_a \) consists of a unique clan \([1, n]\), then we have either \( V_1^{(s, q_0)} = \{0\} \) or \( V_1^{(s, q_1)} = \{0\} \).

Proof. See the second paragraph of the proof of Proposition 4.8.

Theorem 4.10. Let \( (a, X) = (s, \tilde{q}, X) \) be a pre-admissible parameter. Then, \( \text{Stab}_{G(s)}X \) is connected.

Proof. The group \( \text{Stab}_{G(s)}X \) consists of elements of \( \text{Stab}_GX \) which commute with \( s \) inside \( G \). Moreover, this is equal to \( (G \cap (\text{Stab}_GX)(a)) \). Let \( L \) be the Levi part of \( \text{Stab}_GX \). By Lemma 4.1, the desired component group is the same as that of \( (G \cap L(a)) \). By Corollary 4.7, we deduce that \( L \) is a product of symplectic groups and a torus \( T^1 \) which injects into \((\mathbb{C}^\times)^3 \) via the second projection \( G = G \times (\mathbb{C}^\times)^3 \rightarrow (\mathbb{C}^\times)^3 \).

We take \( G \)-conjugation if necessary to assume that \( X_2 \) is a strict \( 0 \)-normal form corresponding to a partition \( \lambda \) of \( n \) and \( s \in T \) by Theorem 4.8. Then, the semi-simple groups contributing \( L \) are direct factors of the subgroup of the group \( L = L_\lambda \) borrowed from Theorem 4.2 which fix the both of \( X_0 \) and \( X_1 \). The \( T \)-action on \( V_1 \) is compatible with the restriction of \( (4.1) \). It follows that we have a sequence of semi-simple elements \( s_1, s_2, \ldots, \) in \( L \) such that

\[
Z_L(s) = \{ g \in L : gs = sg \} = \bigcap_{j \geq 1} L(s_j).
\]

Let \( A \) be the Zariski closure of the group generated by \( s_1, \ldots \) in \( L \). The condition that an element of \( L \) fixes \( X_0 \) or \( X_1 \) can be translated into a condition that a
collection of vectors \( \{X_0^1, X_0^2, \ldots\} \) or \( \{X_1^1, X_1^2, \ldots\} \) of \( \oplus_i V(i) \) obtained from 
\( X_0 \) or \( X_1 \) by (4.1) is fixed, respectively. We put \( S \subset \oplus_i V(i) \) to be the \( A \)-span of 
all the vectors in \( \{X_0^1, \ldots\} \cup \{X_1^1, \ldots\} \). The condition that an element of \( L(A) \) 
fixes \( X_0 \) and \( X_1 \) is the same as fixing each element of \( S \). Here the subgroup \( L' \) of \( L \) which 
fixes \( S \) is a product of (probably smaller) symplectic groups as in the proof of Proposition 4.5. Moreover, the subgroup of \( L(A) \) which fixes \( S \) is isomorphic \( L'(A') \) for some torus \( A' \subset L' \) obtained as the Zariski closure of 
elements of \( L' \) which acts as the same as \( s_1, \ldots, s_k \). (Here \( S^\perp \) is the 
orthogonal complement of \( S \) with respect to the \( G \)-invariant symplectic form on \( V_1 \)).

Therefore, we deduce that \( \text{Stab}_{L(A)}(X_0 \oplus X_1) \) is written as a product of the 
centralizer of some subgroups of maximal torus in symplectic groups. Since each 
of such groups are connected by Theorem 3.2, we conclude the result. \( \square \)

5 Semisimple elements attached to \( G \setminus \mathcal{H}_1 \)

We keep the setting of the previous section.

Let \( \sigma := (J, \delta) \) be a strict marked partition. Let \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \) be the 
partition of \( n \) corresponding to \( J = \{J_1, J_2, \ldots\} \).

We fix a sequence of positive real numbers \( \gamma_0, \gamma_1, \ldots, \gamma_n > (n + 1)\gamma \) such that
\[
\{2\gamma_i, 2\gamma_j, \gamma_i + \gamma_j, \gamma_i - \gamma_j\} \cap (\Gamma + \mathbb{Z}\gamma) = \emptyset
\]
holds for every pair of distinct numbers \( (i, j) \) in \([0, n] \).

Remark 5.1. Our choice of \( \{\gamma_k\}_k \) and \( \gamma \) are possible since \( \mathbb{C} \) is an extension of 
the field \( \mathbb{Q}(q_2, \sqrt{-1}, \pi) \) with infinite transcendental degree.

We define a semi-simple element \( s_\sigma \in T \) as follows:

- If \( \delta|_{J_k} \equiv 0 \), then we set \( \log_j s_\sigma = \gamma_k - j\gamma \) for each \( j \in J_k \);
- If \( \delta(J_0) = 1 \) for \( J_0 \in J_k \), then we set \( \log_j s_\sigma = \gamma_0 - (j - J_0)\gamma \) for each \( j \in J_k \).

By the definition of strict marked partitions, the choice of \( J_0 \) is unique for each \( J \in J \). Hence, \( s_\sigma \) is uniquely determined. We put \( a_\sigma := (s_\sigma, e^{\gamma_0}, 1, e^\gamma) \in T \).

Lemma 5.2. In the above setting, we have \( a_\sigma v_\sigma = v_\sigma \).

Proof. It suffices to prove \( (s_\sigma, e^{\gamma_0}, 1, e^\gamma)v^J_\sigma = v^J_\sigma \) for each \( J \in J \). Let \( J = J_k \).
Then, \( v^J_{2,\sigma} \) is a sum of \( y_{i,i+1} \) for \( i, i+1 \in J_k \), which has \( s_\sigma \)-eigenvalue
\[
e^{(\gamma_k - i\gamma - (\gamma_k - (i+1)\gamma))} = e^{\gamma} \text{ or } e^{(\gamma_0 - (i-j_0)\gamma - (\gamma_0 -(i+1-j_0)\gamma))} = e^{\gamma},
\]
where the latter case occurs only if \( \delta(j_0) = 1 \) for some \( j_0 \in J_k \). Hence, we have \( s_\sigma v^J_{2,\sigma} = e^{\gamma}v^J_{2,\sigma} \). Moreover, we have \( s_\sigma x_i = e^{\gamma_0}x_i \) if \( \delta_1(i) = 1 \). In particular, we 
have \( s_\sigma v^J_{1,\sigma} = e^{\gamma_0}v^J_{1,\sigma} \). These calculations imply the desired result. \( \square \)

Fix a real number \( r > 0 \). We define \( D_\sigma \in T \) to be
\[
\log_i D_\sigma = \begin{cases} 
0 & (\log_i s_\sigma \not\in \gamma_0 + \Gamma) \\
-r(j(J_k)) & (i \in J_k \not\supset \exists j_0, \delta_1(j_0) = 1)
\end{cases}
\]
Consider a parabolic subgroup $P_\sigma$ of $G(s_\sigma)$:

$$P_\sigma := \{g \in G(s_\sigma); \lim_{N \to \infty} \text{Ad}(D^N_\sigma)g \in G(s_\sigma)\}.$$

It is well-known that $P_\sigma$ is a parabolic subgroup of $G(s_\sigma)$. Let $w_\sigma$ be the shortest element of $W$ such that

$$\langle w_\sigma R^+, D_\sigma \rangle \leq 1.$$

It is straight-forward to see

$$w_\sigma B \cap G(s_\sigma) \subset P_\sigma.$$

**Lemma 5.3.** For a strict marked partition $\sigma$, we have $v_\sigma \in \mathcal{V}_a \sigma \cap w_\sigma \mathcal{V}^+.$

**Proof.** By Lemma 5.2, it suffices to prove $v_\sigma \in w_\sigma \mathcal{V}^+$. The definition of $w_\sigma$ implies that

1. $x_i \in w_\sigma \mathcal{V}^+$ if and only if a) $D_\sigma(\epsilon_i) < 1$ or b) $D_\sigma(\epsilon_i) = 1$ and $i > 0$;
2. $y_{ij} \in w_\sigma \mathcal{V}^+$ if and only if a) $D_\sigma(\epsilon_i - \epsilon_j) < 1$ or b) $D_\sigma(\epsilon_i - \epsilon_j) = 1$ and $\epsilon_i - \epsilon_j \in R^+.$

Since $v_{1,\sigma}$ is sum of $x_i$ with $D_\sigma$-eigenvalue $< 1$, we have $v_{1,\sigma} \in w_\sigma \mathcal{V}^+$. The vector $v_{2,\sigma}$ have $D_\sigma$-eigenvalue 1. By construction, a strict normal form is contained in $\mathcal{V}^+$. Therefore, we conclude $v_{2,\sigma} \in w_\sigma \mathcal{V}^+$, which completes the proof.  \[\square\]

**Proposition 5.4.** Let $\sigma = (J, \vec{\delta})$ be a strict marked partition. Then, we have an inclusion

$$P_\sigma v_\sigma \subset \mathcal{V}_a \sigma \cap w_\sigma \mathcal{V}^+,$$

which is dense open.

Before giving the proof of Proposition 5.4, we count the set of weights we concern in its proof:

**Lemma 5.5.** Keep the setting of Proposition 5.4. Then, the set $\Psi(\mathcal{V}_a \sigma \cap w_\sigma \mathcal{V}^+)$ is given by the following list:

1. $\epsilon_i - \epsilon_{i+1}$ for each $i, i+1 \in J \in J$;
2. $\epsilon_{i+1} - \epsilon_{i+1+1}$ if the following conditions hold:
   - $i + j_0, j_0 \in J_k$, and $i + j_1 + 1, j_1 \in J_{k'}$ for some $k, k'$;
   - $\delta_1(j_0) = 1 = \delta_1(j_1)$, and $\#J_k > \#J_{k'}$;
3. $\epsilon_{j_0}$ for each $j_0 \in J_k$ such that $\delta(j_0) = 1$.

**Proof.** In this proof, we assume all integer index (which are a priori not necessarily positive) to be positive. By the choice of the sequence $\{\gamma_k\}_k$, we have

$$|\langle s_\sigma, \epsilon_j + \epsilon_{j'} \rangle| \geq e^{-2n\gamma} \min\{e^{\gamma k + \gamma k'}, k, k' \in [1, n]\} > e^\gamma$$

for each $j, j'$. It follows that weights of the form $\pm(\epsilon_j + \epsilon_{j'})$ does not belong to $\Psi(\mathcal{V}_a \sigma)$. We examine the assertion $\pm\epsilon_j, \epsilon_j - \epsilon_{j'} \in \Psi(\mathcal{V}_a \sigma \cap w_\sigma \mathcal{V}^+)$ by the case-by-case analysis. We have three cases:
(Case \( j,j' \in J_k \)) We have \( \epsilon_j - \epsilon_j' \in \Psi(V_{\alpha}) \) if and only if
\[
\langle \epsilon_j - \epsilon_j', s_{\sigma} \rangle = e'(j'-j)\gamma = e^\gamma.
\]
This forces \( j' - j = 1 \). Hence, we put \( i := j, j' = i + 1 \). We have \( \langle \epsilon_i - \epsilon_{i+1}, D_{\sigma} \rangle = 1 \le 1 \), which verifies the first part of the assertion.

(Case \( i \in J_k \neq J_m \ni j \)) By (5.1), we deduce that
\[
\langle \epsilon_i - \epsilon_j, s_{\sigma} \rangle \in e^\gamma = q_2^Z
\]
if and only if \( j \in J_k \) or \( \delta_1(J_k) = \{0,1\} = \delta_1(J_m) \) holds. We choose \( j_0 \in J_k \) and \( j_1 \in J_m \) such that \( \delta_1(j_0) = 1 = \delta_1(j_1) \). We write \( j = i + j_0 \) and \( j' := i' + j_1 \).
Then, we need
\[
\langle \epsilon_{i+j_0} - \epsilon_{i'+j_1}, s_{\sigma} \rangle = e'((i'-i)\gamma = e^\gamma.
\]
This happens if and only if \( i' = i + 1 \). By the definition of \( D_{\sigma} \), we have
\[
\langle \epsilon_{i+j_0} - \epsilon_{i+j_1+1}, D_{\sigma} \rangle \le 1
\]
if and only if \( \# J_k \ge \# J_{k'} \). Since we assume \( (J, \tilde{\delta}) \) to be a strict marked partition, it follows that \( \# J_k \neq \# J_{k'} \) by 1.13 4). This verifies the second part of the assertion.

(Case \( j \in J_k \)) If \( \epsilon_j \in \Psi(V_{\alpha}) \) or \( \epsilon_j \in -\Psi(V_{\alpha}) \), then we have \( (s, \epsilon_j) = e^{70} \) or \( e^{-70} \), respectively. By (5.1), this forces \( \epsilon_j \in \Psi(V_{\alpha}) \) and \( \delta_1(J_k) = \{0,1\} \). Let \( j_0 \in J_k \) be such that \( \delta_1(j_0) = 1 \). Put \( j = i + j_0 \) for some \( i \in \mathbb{Z} \). Then, we have \( \langle \epsilon_{i+j_0}, s_{\sigma} \rangle = e^i\gamma + 70 = e^{70} \) if and only if \( i = 0 \). Moreover, we have \( \langle \epsilon_{j_0}, D_{\sigma} \rangle = 1 \le 1 \) and \( j_0 > 0 \), which verifies the final part of the assertion.

\[\text{Lemma 5.6.} \] The group \( P_{\sigma} \) satisfies the following conditions:

1. \( P_{\sigma} = TU_{\sigma} \subset B \), where \( U_{\sigma} \) is an unipotent subgroup of \( G \);

2. The Lie algebra \( u_{\sigma} \) of \( U_{\sigma} \) contains \( g[\alpha] \subset g \) if and only if \( \alpha = \epsilon_j - \epsilon_{j'} \), where \( j,j' \) are as follows:
   - \( j \in J_k \neq J_{k'} \ni j' \) for some \( k,k' \);
   - There exists \( j_0 \in J_k \) and \( j_1 \in J_{k'} \) such that \( \delta_1(j_0) = 1 = \delta_1(j_1) \);
   - \( j - j_0 = j' - j_1 \) and \( j < j' \).

\[\text{Proof.} \] Let \( L_{\sigma} \) be the reductive Levi component of \( P_{\sigma} \) which contains \( T \). Let \( A \) be the Zariski closure of the group generated by \( D_{\sigma} \) and \( s_{\sigma} \). Then, \( G(A) \) is connected by Theorem 3.2. Thus, we have \( L_{\sigma} = T \) if we have \( \alpha(A) \neq \{1\} \) for each \( \alpha \in R \). This is equivalent to \( \alpha(D_{\sigma}) \neq 1 \) or \( \alpha(s_{\sigma}) \neq 1 \) holds for each \( \alpha \in R \) since \( R \) is a finite set. By (5.1), we have \( \langle \epsilon_i - \epsilon_j, s_{\sigma} \rangle = 1 \) only if
\[
\exists \kappa \in \{\pm 1\} \text{ s.t. } \kappa i \in J_k \neq J_{k'} \ni \kappa j \text{ and } \delta_1(J_k) = \{0,1\} = \delta_1(J_{k'}), \tag{5.2}
\]
where \( J_k, J_{k'} \in J \). By 1.13 4), we deduce that
\[
\langle \epsilon_i, D_{\sigma} \rangle \neq \langle \epsilon_j, D_{\sigma} \rangle \text{ for each } i \in J_k, j \in J_{k'}.
\]
Therefore, we conclude \( L_{\sigma} = T \).
Since \( T \) normalizes the unipotent part of \( P_\sigma \), we describe all one-parameter unipotent subgroup of \( G \) belonging to \( P_\sigma \) in order to prove the assertion. This is equivalent to count the set of weight spaces \( g[\alpha] \subset g \) which is fixed by \( s_\sigma \) and has eigenvalue \( \leq 1 \) with respect to \( D_\sigma \). We examine the case \( \alpha = \epsilon_i - \epsilon_j \) with the assumption (5.2) for \( \kappa = +1 \). (This last part of the assumption is achieved by swapping the roles of \( i \) and \( j \) if necessary.) Fix \( j_0 \in J_k \) and \( j_1 \in J_{k'} \) such that \( \delta_i(j_0) = 1 = \delta_i(j_1). \) Then, the definition of \( s_\sigma \) further asserts \( i - j_0 = j - j_1. \) In order that \( D_\sigma \) has eigenvalue \( \leq 1 \), we need to have

\[
\langle \epsilon_i, D_\sigma \rangle \leq \langle \epsilon_j, D_\sigma \rangle,
\]

which is equivalent to \( \#J_k \geq \#J_{k'} \). This implies \( \#J_k > \#J_{k'} \) by 1.13 4). It follows that \( i < j \), which verifies the second condition. Since \( \alpha = \epsilon_i - \epsilon_j \in \mathbb{R}^+ \) in this case, we also deduce the first condition.

**Proof of Proposition 5.4.** Since \( P_\sigma \subset G(\mathbf{s}_\sigma) \), we have \( P_\sigma \mathbf{v}_\sigma \subset \mathbb{V}^{\mathbf{s}_\sigma} \). Since the reductive part of \( P_\sigma \) is equal to \( T \), we deduce \( P_\sigma \mathbf{v}_\sigma \subset \mathbb{V}^{\mathbf{s}} \mathbf{v}_\sigma \). Therefore, it suffices to prove the following equality at the level of tangent space

\[
T_{\mathbf{v}_\sigma}(P_\sigma \mathbf{v}_\sigma) \cong \mathfrak{p}_\sigma \mathbf{v}_\sigma = \mathbb{V}^{\mathbf{s}_\sigma} \cap \mathbb{V}^{\mathbf{s}} \mathbf{v}_\sigma \tag{5.3}
\]

in order to deduce the assertion. Consider a \( T \)-weight decomposition \( \mathbf{v}^J = \sum_{\beta \in \Xi_J, v_\beta} \), where \( J \in \mathbf{J} \) and \( 0 \neq v_\beta \in \mathbb{V}[\beta] \). Each \( \Xi_J \) consists of linearly independent weights of \( X^*(T_J) \). Moreover, we have \( X^*(T_J) \cap X^*(T_{J'}) = \{0\} \) in \( X^*(T) \) (by using the natural embeddings). Hence, we deduce

\[
t^J \mathbf{v}_\sigma = \sum_{k \geq 1} t^J_k \mathbf{v}_\sigma = \sum_{k \geq 1} \sum_{\beta \in \Xi_{J_k}} c_{\beta} v_\beta.
\]

It is easy to see that \( \bigcup_{k \geq 1} \Xi_{J_k} \) is precisely the set of \( T \)-weights described in Lemma 5.5 1) and 3).

In the below, we apply the action of \( u_\sigma \) (c.f. Lemma 5.6) to fill out each \( \mathbb{V}[\beta] \) for each \( T \)-weight \( \beta \) described in Lemma 5.5 2). Such a \( \beta \) is written as \( \epsilon_i - \epsilon_j \), where \( i \in J_k, j \in J_{k'} \) are as in Lemma 5.5 2). By explicit calculation, we have a non-zero element of \( g \) of weight \( \epsilon_{m+j_0} - \epsilon_{m+j_1} \) which satisfies

\[
\xi_m \mathbf{v}_\sigma = \begin{cases} 
 y_{m-1+j_0, m+j_1} - y_{m+j_0, m+j_1+1} & (m+j_0+1 \in J_{k'}) \\
 y_{m-1+j_0, m+j_1} & (m+j_0+1 \notin J_{k'})
\end{cases}.
\]

for each \( m+j_0 \in J_{k'} \). (Here we implicitly used \( \#J_k > \#J_{k'} \), which is deduced from \( \#J_k > \#J_{k'} \) by 1.13 4).) We know \( \xi_m \in \mathfrak{p}_\sigma \) by Lemma 5.6 2). We have

\[
\left( \sum_{m \in \mathbb{Z}, m+j_1 \in J_{k'}} c_{\xi_m} \right) \mathbf{v}_\sigma = \sum_{m \in \mathbb{Z}, m+j_1 \in J_{k'}} \mathbb{V}[\epsilon_{m-1+j_0} - \epsilon_{m+j_1}].
\]

By summing up for all possible pairs \( (J_k, J_{k'}) \in \mathbf{J} \), the set of \( T \)-weights appearing in the RHS exhausts the \( T \)-weights described in Lemma 5.5 2).

**Corollary 5.7.** Keep the setting of Proposition 5.4. Let \( \nu = (\mathbf{a}, \mathbf{v}_\sigma) = (s, q, \mathbf{v}_\sigma) \) be an admissible parameter. Then, the natural embedding

\[
P_\sigma(s) \mathbf{v}_\sigma \subset \mathbb{V}^{\mathbf{a}} \cap \mathbb{V}^{\mathbf{s}_\sigma} \cap \mathbb{V}^{\mathbf{s}} \mathbf{v}_\sigma
\]

is dense open.
Proof. The assertion follows merely by taking $a$-fixed part of (5.3) in the proof of Proposition 5.4.

6 A vanishing theorem

We retain the setting of the previous section.

**Definition 6.1** (Exotic Springer fibers). For each $X \in \mathfrak{H}_2$, we define $E_{X}$ as the image of the projection of

$$\mu^{-1}(X) \subset F = G \times B \mathfrak{g}^+$$

to $G/B$. For a pre-admissible parameter $\nu = (a, X)$, we have a subvariety $\mu^{-1}(X)^a \subset \mu^{-1}(X)$. We denote the image of $\mu^{-1}(X)^a$ under the projection to $G/B$ by $E_{X}^a$. By construction, we have $E_{X} \cong \mu^{-1}(X)$ and $E_{X}^a \cong \mu^{-1}(X)^a \subset F^a$. We call $E_{X}$ and $E_{X}^a$ exotic Springer fibers.

**Theorem 6.2** (Homology vanishing theorem). Let $\nu = (a, X)$ be a pre-admissible parameter which is admissible or $a = a_0$. Then, we have

$$H_{2i+1}(E_{X}^a) = 0 \text{ for every } i = 0, 1, \ldots.$$ 

Moreover, we have an isomorphism

$$\text{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K(E_{X}^a) \xrightarrow{\cong} H_{\ast}(E_{X}^a).$$

**Remark 6.3.** 1) The map $\text{ch}$ in Theorem 6.2 is the homology Chern character map. (See e.g. [CG97] §5.8.) It sends the class of the (embedded) structure sheaf $O_C$ for a closed subvariety $C \subset E_X^a$ to

$$\text{ch}[O_C] = [C] + \text{lower degree terms} \in H_{2\dim C}(E_{X}^a) \oplus \cdots \oplus H_0(E_{X}^a).$$

2) The first part of Theorem 6.2 is valid even for integral coefficient case when $G(s) \subset GL(n, \mathbb{C})$ (c.f. [AH08])\footnote{Previous versions of this paper also contain a similar result (since math.RT/0601155v3, April/2006). The author decided to drop it since it is unnecessary to prove our main theorems and [AH08] contains a better proof.}. Here we present a proof along the line of earlier versions of this paper, with an enhancement (the proof of Theorem 6.2 modulo Proposition 6.7 given in §6.2) informed to the author by Eric Vasserot.

6.1 Review of general theory on homology vanishing

In this subsection, we recall several definitions and results of [BH85] and [DLP88] which we need in the course of our proof of Theorem 6.2.

**Definition 6.4** ($\alpha$-partitions). A partition of a variety $X$ over $\mathbb{C}$ is said to be an $\alpha$-partition if it is indexed as $X_1, X_2, \ldots, X_k$ in such a way that $X_i \cup \cdots \cup X_i$ is closed for every $i = 1, \ldots, k$.

**Theorem 6.5** ([DLP88] 1.7–1.10). Let $X$ be a variety with $\alpha$-partition $X_1, X_2, \ldots, X_k$. If we have

$$H_{2i+1}(X_m) = 0 \text{ for every } i = 0, 1, \ldots$$
for each \( m = 1, \ldots, k \), then we have

\[
H_{2i+1}(X) = 0 \text{ for every } i = 0, 1, \ldots
\]

Moreover, we have

\[
\sum_{i \geq 0} \dim H_{2i}(X) = \sum_{m \geq 1} \sum_{i \geq 0} \dim H_{2i}(X_m).
\]

**Theorem 6.6** ([BH85] 9.1). Let \( Z \) be a smooth variety with \( \mathbb{G}_m \)-action. Assume that for some \( t \in \mathbb{G}_m \), we have

- \( Z^G_m = Z^t \) and \( \lim_{N \to \infty} t^Nz = z \) \( \forall z \in Z \);
- For each \( z_0 \in Z^t \), the set \( \{ z \in Z; \lim_{N \to \infty} t^Nz = z_0 \} \) defines an affine closed subscheme of \( Z \).

Then, \( Z \) is a vector bundle over \( Z^t \). In particular, the two conditions

\[
H_{2i+1}(Z) = 0 \text{ for every } i = 0, 1, \ldots, \text{ and } H_{2i+1}(Z^t) = 0 \text{ for every } i = 0, 1, \ldots
\]

are equivalent. Moreover, we have

\[
\sum_{i \geq 0} \dim H_{2i}(Z) = \sum_{i \geq 0} \dim H_{2i}(Z^t)
\]

if one of the above equivalent conditions hold.

### 6.2 Proof of vanishing theorem

This subsection is devoted to the proof of Theorem 6.2.

By taking \( G \)-conjugation if necessary, we assume \( a \in T \). We have

\[
(a, X) = (s, \tilde{q}, X) = \prod_{c \in C_a} (s_c, \tilde{q}, X_c).
\]

By the same argument as in the proof of Corollary 3.10, each connected component of \( \mu^{-1}(X)^a \) is a product of connected components of

\[
\mathcal{E}_{X_e}^{(s_c, \tilde{q})} \subset Sp(2n^c)/(B \cap Sp(2n^c)) \text{ for all } c \in C_a.
\]

Therefore, by the Künneth formula, it suffices to prove the assertion when \( C_a \) consists of a unique clan \([1, n]\). By Proposition 4.8, we further assume that \( s \in T \), and \( X = v_\sigma \) for a strict marked partition \( \sigma = (J, \delta) \) by taking \( G \)-conjugate if necessary.

**Proposition 6.7** (Weak version of Theorem 6.2). Let \( \nu = (a, X) = (s, \tilde{q}, v_\sigma) \) be a pre-admissible parameter which is admissible or \( a = a_0 \). Assume that \( s \in T \) and \( \sigma \) is a strict marked partition. For \( s_\sigma \in T \) defined in the above of Lemma 5.2:

- We have

\[
H_{2i+1}((\mathcal{E}_X^{(s_\sigma)})^{s_\sigma}) = 0 \text{ for every } i = 0, 1, \ldots;
\]
Each connected component of $((\mathcal{E}_k^X)^{\sigma})^\tau$ is smooth projective;

We have an isomorphism

$$\text{ch} : \mathbb{C} \otimes_{\mathbb{Z}} K((\mathcal{E}_k^X)^{\sigma}) \xrightarrow{\cong} H_\bullet((\mathcal{E}_k^X)^{\sigma}).$$

Before giving the proof of Proposition 6.7, we complete our proof of Theorem 6.2 for $(a, v_\sigma)$ assuming Proposition 6.7 for $(a, v_\sigma)$.

Proof of Theorem 6.2 for $(a, X) = (a, v_\sigma)$. Let $\mathcal{E}_1, \mathcal{E}_2, \ldots$ be a sequence of all connected components of $(\mathcal{E}_k^X)^{\sigma}$. For each $\mathcal{E}_k$, we set

$$\mathcal{B}_k := \{gB \in \mathcal{E}_k^X : \lim_{N \to \infty} s^{-N}_\sigma gB \in \mathcal{E}_k\}.$$

Let $\pi_k : \mathcal{B}_k \to \mathcal{E}_k$ be the $s^{-1}_\sigma$-attracting map. Let

$$P := \{g \in G : \lim_{N \to \infty} \text{Ad}(s^{-N}_\sigma)g \in G\}$$

be a parabolic subgroup of $G$. It is straight-forward to see

$$\lim_{N \to \infty} \text{Ad}(s^{-N}_\sigma)g \in G(s_\sigma)$$

for $g \in P$. It follows that each $\mathcal{B}_k$ intersects with a unique $P$-orbit in $G/B$. In particular, we can assume that the sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots$ forms an $\alpha$-partition of $\bigcup_{k \geq 1} \mathcal{B}_k \subset \mathcal{E}_k^X$ by rearranging the sequence if necessary.

We choose $\{\gamma_i\}^i$ (in the definition of $s_\sigma$) so that we have

$$\min\{(s_\sigma, \epsilon_i) : i \in J_k\} < \min\{(s_\sigma, \epsilon_i) : i \in J_{k'}\}, \quad \text{and}$$

$$\gamma \max\{(s_\sigma, \epsilon_i) : i \in J_k\} > \max\{(s_\sigma, \epsilon_i) : i \in J_{k'}\}$$

for each $k < k'$. (This choice is possible by $\#J_k \geq \#J_{k'}$ and Definition 1.13.)

Claim A. Each fiber of $\pi_k$ is an irreducible affine scheme.

Proof. Let $P = G(s_\sigma)U$ be the Levi decomposition. Let $gB \in \mathcal{E}_k$. We set $F(gB) := \{u \in U(s) ; X \in u gV^+\}$. This is a closed subset of $U(s)$. Set $U^\circ : = (U(s) \cap gUg^{-1})$. We have a free right $U^\circ$-action on $F(gB)$. We have $\pi^{-1}_k(gB) \cong F(gB)/U^\circ$. Since $\pi^{-1}_k(gB)$ is a closed subspace of an affine space $U(s)/U^\circ$, it suffices to prove that $F(gB)$ is an affine space. We have a product decomposition $U(s) = U_1 \cup U_2$, where $U_1$ is the product of $U_{\epsilon_i, \epsilon_j} \subset U(s)$ ($i, j > 0$) and $U_2$ is the product of $U_{\epsilon_i, -\epsilon_j}$ $U(s)$ ($i, j > 0$). By (6.1), the space $(U_1 X - X)$ is a linear subspace of $V^+$. Hence, $(U_1 X - X) \cap gV^+$ is an affine space. Here $\text{Stab}_{U} X$ is a unipotent group, which is automatically an affine space.

Since $U_2$ acts $V^+_1$ trivially, we have $u \in F(gB)$ only if $u \in U_2 u_1$ with $u_1 \in F(gB)$. The closed subset $(U_2 u_1 \cap F(gB)) \subset U_2 u_1$ define linear conditions on $U_2$ since $U_2$ is commutative. Let $A \subset T$ denote the Zariski closure of the group generated by $s$ and $s_\sigma$. The group $G(A)$ normalizes $U_1$ and $U_2$, and $U_1$ normalizes $U_2$. Hence, the condition along different points of $(U_1 \mathcal{E}_k \cap \mathcal{B}_k)$ are isomorphic via conjugation of $U_1 G(A)$. It follows that $F(gB)$ is a successive fibration of affine spaces by affine spaces, which is itself an affine space.

Claim B. The variety $\mathcal{B}_k$ is a smooth affine bundle over $\mathcal{E}_k$.  

32
Proof. We keep the setting of the proof of Claim A. Let \( f(gB) := \{ \xi \in \text{Lie}\mathcal{U}(a); \xi X \in g\mathcal{N}^+ \} \). Since \( gB \) is the unique \( s_\sigma \)-fixed point of \( F(gB) \), we deduce

\[
\dim F(gB) \leq \dim f(gB).
\]

(6.2)

Notice that \( U \) is invariant under the \( G(s_\sigma) \)-action. Thus, \( \dim f(gB) \) is invariant along \( \mathcal{E}_k \). In view of Claim A, the assertion follows if the equality of (6.2) holds for each \( gB \in \mathcal{E}_k \). If \( \xi \in f(gB) \) is a \( s_\sigma \)-eigenvector, then we have \( \exp(-\xi) = 1 - \xi \in F(gB) \). In particular, we have \( \dim F(gB) \geq \dim f(gB) \) since we have enough number of linearly independent tangent lines.

We return to the proof of Theorem 6.2

By Claim B and Theorem 6.6, the first assertion reduces to

\[
H_{2i+1}(\mathcal{E}_k) = 0 \quad \text{for every} \quad i = 0, 1, \ldots
\]

for each \( k \). Hence, Proposition 6.7 for \((a, X)\) yields the first assertion.

Similarly, Proposition 6.7 for \((a, X)\) and the Thom isomorphism give

\[
\text{ch} : \mathbb{C} \otimes \mathbb{Z} K(B_k) \xrightarrow{\mu} H_n(B_k)
\]

for each \( k \). The Chern character map commutes with localization sequences. (c.f. [CG97] §5.8.) Therefore, a successive application of localization sequences implies the second assertion.

The rest of this section is devoted to the proof of Proposition 6.7.

We set \((s_\sigma, \bar{q}_\sigma) := a_\sigma \in \mathbf{T} \) defined in §5. We have \( a_\sigma X = X \). Hence, \( a_\sigma \) acts on \( \mu^{-1}(X)^\sigma \). Its projection gives the \( s_\sigma \)-action on \( \mathcal{E}^\sigma \). Let \( \mathcal{A} \) be the Zariski closure of the subgroup of \( \mathbf{T} \) generated by \( a \) and \( a_\sigma \). We put \( \mathcal{W}_A := \{ w \in \mathcal{W}; B(A) \subset wB \} \). We put \( F^A(w) := G(A) \times B(A) (\mathcal{V}^A \cap w\mathcal{V}^+) \) for each \( w \in \mathcal{W}_A \). We have \( \bigcup_{w \in \mathcal{W}_A} F^A(w) = (G \times B \mathcal{V}^+)^A \). Consider the map

\[
\begin{array}{c}
w \mu^A : F^A(w) = G(A) \times B(A) (\mathcal{V}^A \cap w\mathcal{V}^+) \longrightarrow \mathcal{V}^A
\end{array}
\]

for each \( w \in \mathcal{W}_A \).

**Lemma 6.8 (Part of Proposition 6.7).** Each connected component of \((\mathcal{E}^\sigma)^{a_\sigma}\) is smooth projective.

**Proof.** Projectivity follows from that of \( \mathcal{E}_X \), which itself follows by Theorem 1.2 3). By Lemma 5.6 1) and Corollary 5.7, we deduce that

\[
\overline{B(A)}v_\sigma \subset \mathcal{V}^A
\]

is a linear subspace. It follows that \((w \mu^A)^{-1}(\overline{B(A)}v_\sigma)\) is a smooth subvariety of \( G(A) \times B(A) (\mathcal{V}^A \cap w\mathcal{V}^+) \). Hence, \((w \mu^A)^{-1}(B(A)v_\sigma)\) is a smooth subvariety of \( F^A(w) \). Since changing \( v_\sigma \) by \( B(A) \)-action gives isomorphic fibers, we deduce that \((w \mu^A)^{-1}(v_\sigma)\) is a smooth subvariety of \( F^A(w) \) as required.

**Corollary 6.9 (of the Proof of Lemma 6.8).** The variety \((w \mu^A)^{-1}(\overline{B(A)}v_\sigma)\) is smooth.
We return to the proof of Proposition 6.7.
We prove the rest of assertions by the induction on the cardinality $n(\sigma)$ of the set $$\mathcal{N}(\sigma) := \{ J \in \mathbf{J}; \delta_1(J) = \{0, 1\} \}.$$ In other words, we assume Theorem 6.2 for every admissible parameter of the form $(a, \nu_{\sigma'})$ such that $n(\sigma') < n(\sigma)$. If $n(\sigma) = 0$, then Lemma 5.6 2) asserts that $G(s_\sigma) = T$. This implies that $(\mathcal{E}_{\mathbf{X}}^n)^{\nu_{\sigma}}$ is a union of points. Thus, we obtain the assertion for $n(\sigma) = 0$.

We prove the assertion for $n(\sigma) = k$ by assuming that the assertion holds for all $n(\sigma) < k$. Let $J \in \mathcal{N}(\sigma)$ be the member such that $\# J \geq \# J'$ for every $J' \in \mathcal{N}(\sigma)$. Let $\sigma'$ be a strict marked partition obtained from $\sigma$ by replacing $\delta_1$ by $\delta_1'$ defined as:$$\delta_1'(J) = \{0\}, \text{ and } \delta_1'(j) = \delta_1(j) \text{ for all } j \in [1, n] \setminus J.$$ Let $j_0 \in J$ be the unique element such that $\delta_1(j_0) = 1$. By Lemma 1.18, there exists $t \in T_J$ such that$$\lim_{N \to \infty} t^N \nu_{\sigma} = \nu_{\sigma'} = \nu_{\sigma'}.$$ By Lemma 5.6 2), every $T$-weight of $P_\sigma$ containing $i \in J$ is of the form $\epsilon_i - \epsilon_j$ for some $j \in J'$. Moreover, we have $P_{\sigma'} \subset P_\sigma$. It follows that the action of $t \in T$ contracts $P_\sigma$ to $P_{\sigma'}$. By Corollary 5.7, the $t$-action also contracts $V^A \cap w_{\sigma'} V^+$ to$$V^A \cap V^0 \cap w_{\sigma'} V^+ = V^A' \cap w_{\sigma'} V^+,$$where $A'$ is the Zariski closure of $(a, a_{\sigma'}) \subset T$.

Therefore, the $t$-action contracts $(w_{\mu^A})^{-1}(B(A)\nu_{\sigma})$ to $(w_{\mu^A'})^{-1}(B(A')\nu_{\sigma'})$. By taking the quotient of $S := B(A)\nu_{\sigma} \cup B(A)\nu_{\sigma'}$ by $\text{Stab}_{B(A)} \nu_{\sigma}'$, we obtain an affine plane $\mathbb{A}^1$ with contracting $t$-action to the origin. Therefore, we obtain a smooth family of smooth projective varieties over $\mathbb{A}^1$ whose fiber over $0 \in \mathbb{A}^1$ is $\mathcal{E}_{\mathbf{X}_{\sigma}}^A$, and whose general fiber $\mathcal{E}_{\mathbf{X}_{\sigma}}^A$ contracting to $\mathcal{E}_{\mathbf{X}_{\sigma'}}^A$. Moving smooth projective varieties is the same as moving all cycles by rational equivalence. Therefore, it suffices to prove

$$\mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{E}_{\mathbf{X}_{\sigma}}^A) \xrightarrow{\cong} H_*(\mathcal{E}_{\mathbf{X}_{\sigma}}^A).$$

(6.3)

Since $\mathcal{E}_{\mathbf{X}_{\sigma}}^A$ is smooth, the Bialynicki-Birula theorem asserts that $\mathcal{E}_{\mathbf{X}_{\sigma'}}^A$ is a union of vector bundles over connected components of $\mathcal{E}_{\mathbf{X}_{\sigma'}}^A$. Since the Chern character map commutes with pullbacks and localization sequences, we deduce (6.3) from Theorem 6.6 and Proposition 6.7 for $(a, \nu_{\sigma'})$. Therefore, we have Proposition 6.7 for every admissible parameter of the form $(a, \nu_{\sigma})$ with $n(\sigma) = k$. Hence, the induction proceeds and we have proved Proposition 6.7 (and hence Theorem 6.2).
7 Standard modules and an induction theorem

We retain the setting of the previous section.

**Definition 7.1** (Standard modules). Let $\nu = (a, X)$ be a pre-admissible parameter. We define

$$M_\nu := \mathcal{H}_\bullet(\mathcal{E}_X^a)$$

and

$$M'_\nu := \mathcal{H}_\bullet(\mathcal{E}_X^{a \times X}).$$

By the Ginzburg theory [CG97] 8.6, each of $M_\nu$ or $M'_\nu$ is a $H$-module.

By the symmetry of the construction of varieties involved in $M(a,X)$ and $H_a$, we deduce $M(a,X) \cong M(Ad(g) a, gX)$ as $H_a = H_{Ad(g) a}$-modules for each $g \in G$.

Let $s_Q \in T(\mathbb{R})$ be an element such that

$$0 < \langle \alpha, s_Q \rangle \leq 1$$

for all $\alpha \in R^+$. (7.1)

Let $Q := G(s_Q)$ and $Q := Q \times (\mathbb{C}^\times)^3$. These are subgroups of $G$ and $G$, respectively. We put $V_Q := V_2^{s_Q}$ and $\mathfrak{N}_Q := V_2^{s_Q} \subset V_Q$. We put $F_Q := Q \times (Q \cap B) (V_Q \cap V_2^+)$. We have a map

$$\mu_Q : F_Q = Q \times (Q \cap B) (V_Q \cap V_2^+) \rightarrow \mathfrak{N}_Q.$$

We define $Z_Q := F_Q \times_{\mathfrak{N}_Q} F_Q$.

The natural inclusion map

$$F_Q = Q \times (Q \cap B) (V_Q \cap V_2^+) \rightarrow \bigcup_{w \in W} Q \times (w B) (V_Q \cap V_2^+) = F^{s_Q}$$

gives an identification of $F_Q$ with a connected component of $F^{s_Q}$. This equips an action of $Q$ on $\mathfrak{N}_Q$, $F_Q$, and $Z_Q$ by restricting the $G$-actions on their ambient spaces.

We put

$$\mathbb{H}_Q := \mathbb{C} \otimes_Z K^Q(Z_Q),$$

where the convolution algebra structure on $K^Q(Z_Q)$ are equipped by the restrictions of the maps $p_1$ and $p_2$ from $Z \rightarrow F$ to $Z_Q \rightarrow F_Q$.

**Lemma 7.2.** Keep the above setting. Form an increasing sequence of integers

$$1 \leq n_1 \leq n_2 \leq \cdots$$

by requiring that

$$\alpha_i(s_Q) < 1$$

if and only if $i = n_k$ for some $k$.

Then, we have

1. $\mathbb{H}_Q$ is a subalgebra of $\mathbb{H}$ generated by $\mathcal{A}[T]$ and the set

   $$\{T_i ; i \neq n_k \text{ for some } k\} ;$$

2. For a pre-admissible parameter $\nu = (s, \vec{q}, X)$ such that $s \in T$ and $X \in \mathfrak{N}_Q$, the vector space

   $$M^Q_\nu := H_\bullet(\mu_Q^{-1}(X))^{(s, \vec{q})}$$

   is a $\mathbb{H}_Q$-module.
Theorem 7.4 (Induction theorem). Let $\nu$ as an element, then we have an isomorphism $H^\nu = (\alpha, \beta, sQ) = 1$ for $\alpha, \beta \in \mathbb{R}^+$ if and only if $\langle \alpha, sQ \rangle = 1$ and $\langle \beta, sQ \rangle = 1$. This implies that $Q$ is generated by $T$ and the one-parameter unipotent subgroups corresponding to simple roots $\alpha_i$ (and $-\alpha_i$) such that $\alpha_i(sQ) = 1$.

The variety $F_Q$ decomposes into a product of vector bundles over the flag varieties of simple components of $Q$. By explicit computation, we deduce that the vector bundles we concern are either a) the cotangent bundle of the flag variety when the simple component is type $A$, or b) the variety $F$ for a (possibly smaller) symplectic group which arose as a simple component of $Q$. Moreover, the map $\mu_Q$ is the product of the moment maps of the cotangent bundles of flag varieties of type $A$ and our map $\mu$ (for some symplectic group).

Hence, taking account into the argument in §2, both statements are straightforward modifications of [CG97] §7.6 and §8.6. Thus, we leave the details to the reader.

Corollary 7.3. Under the assumption of Lemma 7.2, $QB$ is a parabolic subgroup of $G$.

Proof. See the first paragraph of the proof of Lemma 7.2.

Let $V_U$ be the unique $T$-equivariant splitting of the map $V_2^+ \to V_2^+/V_2^+$. Let $U$ be the unipotent radical of $QB$. If $X \in V$ satisfies $sQX = X$, then $sQ$ has eigenvalue $< 1$ on $uX$. Hence, we have necessarily $uX \subset V_U$.

For an admissible parameter $(s, \vec{q}, X)$, we can regard $X = (X_0 + X_1 + X_2) \in V$ as an element $X_0 \oplus X_1 \oplus X_2 \in V_2$ so that $sX_i = q_iX_i \in V_1$ for $i = 0, 1$.

Theorem 7.4 (Induction theorem). We put $P := QB$. Let $P = QU$ be its Levi decomposition. Let $\nu = (a, X) = (s, \vec{q}, X)$ be an admissible parameter regarded as an element of $G \times V_2$. Assume $s \in Q$ and $X \in \mathfrak{N}_Q$. If we have

$$V_U^\nu \subset uX,$$

then we have an isomorphism

$$\text{Ind}^G_{\nu} M^Q \cong M_{\nu},$$

as $\mathbb{H}$-modules, where $M^Q_{\nu}$ is as in Lemma 7.2.

The rest of this section is devoted to the proof of Theorem 7.4.

By taking $Q$-conjugation if necessary, we assume $X \in V_2^+$. Let $W_Q := N_Q(T)/T \subset W$. We define

$$W^Q := \{w \in W; \ell(w) \leq \ell(vw) \text{ for all } v \in W_Q\}.$$

Let $w \in W^Q$. Let $O_w$ be the $P$-orbit of $G/B$ which contains $wB$. By counting the weights, we have $(V_2^+ \cap \mathcal{V}_Q) \subset (V_2^+ \cap wV_2^+)$. It follows that $X \in (V_2^+ \cap wV_2^+)$. Hence, the map

$$(\mathcal{E}_X \cap O_1) = \mu_Q^{-1}(X) \ni gB \mapsto gwB \in \mathcal{E}_X \cap O_w$$

gives rise to an isomorphism $(\mathcal{E}_X \cap O_1) \cong \mathcal{E}_X \cap O_{w^Q}$. Let $B^-$ be the opposite Borel subgroup of $B$ with respect to $T$. We put $U_w := U \cap wB^-$. Since $sQ$ attracts points of $O_w$, we obtain a map

$$\psi_w : \mathcal{E}_X \cap O_w \to (\mathcal{E}_X \cap O_w)^{sQ} \cong (\mathcal{E}_X \cap O_1)$$
which sends a point $p$ to $\lim_{N \to \infty} s_Q^N p$. We have an expression of a point $guwB \in \mathcal{E}_X \cap \mathcal{O}_w$ as $g \in Q$, $gB \in \mu^{-1}_Q(X)$, and $u \in U_w$. Let $w_Q$ be the longest element of $W^Q$.

**Lemma 7.5.** The fiber of the map $\psi_w$ at $gB \in QB$ is given as

$$\psi_w^{-1}(gB) = \{ u \in gU_w g^{-1}; uX - X \in g^wV_2^+ \cap V_U \}.$$  

In particular, $\psi_w^{-1}(gB)$ is isomorphic to $\text{Stab}_U(X)$.

**Proof.** The variety $\mathcal{O}_w$ is a $U_w$-fibration over $\mathcal{O}_1$. The condition $X \in guwV_2^+$ is equivalent to $(gu^{-1}g^{-1})X - X \in guV_2^+$. Moreover, $U$ is $Q$-stable and $(gu^{-1}g^{-1})X - X \in V_U$, which implies the first result. Since $U_{w_Q} = U$ and $g^{w_Q}V_2^+ \cap V_U = \{0\}$, we conclude the second assertion. \hfill \Box

**Lemma 7.6.** We have

$$\dim H_\ast(\mathcal{E}^\alpha_X \cap \mathcal{O}_w) = \dim H_\ast(\mathcal{E}^\alpha_X \cap \mathcal{O}_1).$$

**Proof.** By the proof of Theorem 6.2 and [DLP88] 3.9, we have an $\alpha$-partition $X_1, X_2, \ldots$ of $\mathcal{E}^\alpha_X \cap \mathcal{O}_1$ such that each $X_m$ $(m = 1, 2, \ldots)$ is a smooth variety without odd-term homology. By condition (7.2), we see that

$$\dim (guwV^+ \cap (X + V_U^0)) = \dim (wV^+ \cap (X + V_U^0))$$

for all $g = hu \in QU(s)$ such that $hB \in \mathcal{E}^\alpha_X$. We denote its (common) dimension by $d$. Here $(guwV^+ \cap (X + V_U^0))$ is an affine space contained in $GX$. The fiber of the map

$$\varphi_w : (\mathcal{E}^\alpha_X \cap \mathcal{O}_w) \to (\mathcal{E}^\alpha_X \cap \mathcal{O}_w^Q) \cong (\mathcal{E}^\alpha_X \cap \mathcal{O}_1)$$

is isomorphic to a fiber of the following map at $X$:

$$U(s) \times (U(s) \cap \text{Ad}(guwB)) (guwV^+ \cap (X + V_U^0)) \to X + V_U^0 = U(s)X.$$  

In particular, $\varphi_w$ is a smooth affine fibration of relative dimension $\dim U_w(s) + d - \dim V_U^0$. Therefore, Theorem 6.6 implies that $\varphi_w^{-1}(X_m)$ is a vector bundle over $X_m$ for each $m$. Hence, we deduce that

$$\sum_{i \geq 0} \dim H_{2i}(X_m) = \sum_{i \geq 0} \dim H_{2i}(\varphi_w^{-1}(X_m))$$

and

$$H_{2i+1}(X_m) = H_{2i+1}(\varphi_w^{-1}(X_m)) = 0$$

for each $m$. Since $\varphi_w^{-1}(X_1), \varphi_w^{-1}(X_2), \ldots$ forms an $\alpha$-partition of $\mathcal{E}^\alpha_X \cap \mathcal{O}_w$, we obtain the result. \hfill \Box

We return to the proof of Theorem 7.4.

It is easy to see that

$$\mathcal{E}^\alpha_X = \bigsqcup_{w \in W^Q} (\mathcal{E}^\alpha_X \cap \mathcal{O}_w)$$

forms an $\alpha$-partition. Together with Theorem 6.2 and Lemma 7.6, this implies

$$\dim M_\nu = (\#W^Q) \dim M_\nu^Q = (\#W/\#W_Q) \dim M_\nu^Q. \hspace{1cm} (7.3)$$
Moreover, the natural map
\[ \iota : M^Q_\nu = H_\bullet (E_X^a \cap O_1) \hookrightarrow H_\bullet (E_X^a) = M_\nu \]
is injective. Since we have
\[ p_1(Z_{\leq s_i} \cap p_2^{-1}(O_1)) \subset O_1 \text{ if } i \neq n_k \text{ for some } k = 1, 2, \ldots , \]
the map \( \iota \) is an embedding of \( H_Q \)-modules. (The sequence \( \{ n_k \} \) is borrowed from Lemma 7.2.) Hence, we have an induced map \( \phi : \text{Ind}^H_{H_0} M^Q_\nu \rightarrow M_\nu \).

Thanks to (7.3), we have:

**Lemma 7.7.** Theorem 7.4 follows if \( \phi \) is surjective.

We return to the proof of Theorem 7.4.

For each \( w \in W^Q \), we define \( R_w := [O_{Z_{\leq w-1}}] \in K^G(Z) \).

By the construction of §2, we have
\[ R_w H_\bullet (E_X^a \cap O_1) \subset H_\bullet (E_X^a \cap \overline{O_w}) \subset H_\bullet (E_X^a). \]

Since \( W^Q \) has a partial order \( \leq_Q \) induced by the Bruhat order, we put
\[ H_\bullet (E_X^a)_{\leq w} := \sum_{v \leq_Q w : v \in W^Q} R_v H_\bullet (E_X^a \cap O_1). \]

Consider the composition map
\[ \tau_w : H_\bullet (E_X^a \cap O_1) \xrightarrow{R_w} H_\bullet (E_X^a \cap \overline{O_w}) \xrightarrow{\text{res}} H_\bullet (E_X^a \cap O_w). \]

**Lemma 7.8.** Theorem 7.4 follows if each \( \tau_w \) is surjective.

**Proof.** We have
\[ \dim \text{Im} \phi \geq \sum_{w \in W^Q} \dim \text{gr} H_\bullet (E_X^a)_{\leq w} = \sum_{w \in W^Q} \dim M^Q_\nu = (\# W^Q) \dim M^Q_\nu, \]
where we used the assumption at the first inequality. Here \( \text{gr} \) stands for the graded quotient with respect to some completed order on \( W^Q \) which extends \( \leq_Q \). By (7.3), we conclude that \( \phi \) must be surjective.

We return to the proof of Theorem 7.4.

We have only to prove that each \( \tau_w \) is surjective provided if (7.2) holds. We have an open embedding
\[ (p_2^{-1}(E_X \cap O_1) \cap \pi^{-1}(O_{w-1})) \subset (p_2^{-1}(E_X \cap O_1) \cap Z_{\leq w-1}). \]

**Lemma 7.9.** For each subset \( E \subset (E_X \cap O_1) \), we have \( (p_2^{-1}(E) \cap \pi^{-1}(O_{w-1})) \cong \psi_{w-1}^{-1} (E) \).
Proof. By definition, the LHS is written as:

\[ \{(g_1B, g_2B) \in \mu^{-1}(X) \times (\mathcal{E}_X \cap \mathcal{O}_1); g_1^{-1}g_2 \in B_{\psi^{-1}}B\} \]

Since \( B \cap Q = wB \cap Q \), we have \( B_{\psi^{-1}}B = B_{\psi^{-1}}U \). By taking the right \( B \)-translation if necessary, we can assume \( g_1 \in g_2U_{\psi} \). This forces \( g_1B \) to live in the fiber of the map \( \psi_w \). This implies that \( g_2B \) is completely determined by the data of \( \psi^{-1}_w(\mathcal{E}) \) and vice versa.

Let \( A \) be the Zariski closure of \( \langle a, s \rangle \subset T \). The set \( uX \subset V \) is an \( A \)-stable linear subspace. It follows that

\[ S := \mathbb{V}_U / uX \]

has a \( A \)-stable splitting in \( V \). Using this splitting, we define

\[ \mathcal{E}^-_X := \{(gB, X + y) \in F; gB \in (\mathcal{E}_X \cap \mathcal{O}_1), y \in S\} \]

Each element of \( V \) is contracted to 0 by the \( s \)-action. Hence, \( S \) has a contraction to 0 \( \in S \). This gives a contraction

\[ \theta : \mathcal{E}^-_X \longrightarrow (\mathcal{E}_X \cap \mathcal{O}_1) \]

given by collecting \( s \)-attracting points.

Theorem 7.10. For each \( w \in W^Q \), the intersection of \( \pi^{-1}(O^{-1}_w) \) and \((F \times \mathcal{E}^-_X)\) is transversal inside \( F^2 \).

Proof. We prove the assertion by induction. The case \( w = 1 \) is clear. Assume that

- \( w = w's \) by \( w' \in W^Q \) and \( s = s_i \) such that \( \ell(w) = \ell(w') + 1 \);
- The assertion holds for \( w' \);

and prove the assertion for \( w \). Let \( \delta_{in} \) be Kronecker’s delta which takes 1 if \( s = s_n \) and 0 otherwise. For \( v = w, w' \), we set

\[ \mathcal{E}^+(v) := \{(gB, X + y) \in p_1((\mathcal{F} \times \mathcal{E}^-_X) \cap \pi^{-1}(O_{v^{-1}}))\} \]

We denote the fibers of the maps \( \mathcal{E}^+(v) \rightarrow (\mathcal{E}_X \cap \mathcal{O}_v^Q) \) over \( gB \) as:

\[ F_{v}(gB) := \{(u, X + y) \in gU_vg^{-1} \times S; u^{-1}X - X \in g^v\mathcal{V}_2^+, y \in S \cap gu^v\mathcal{V}_2^+\} \]

We have

\[ \dim \pi^{-1}(O_{w^{-1}}) = \dim \pi^{-1}(O_{(w')^{-1}}) - \delta_{in} \]

Taking account into Lemma 7.9, the failure of the dimension condition of transversal intersection implies

\[ \dim F_{w}(gB) > \dim F_{w'}(gB) \text{ if } s \neq s_n \text{ or} \]
\[ \dim F_{w}(gB) \geq \dim F_{w'}(gB) \text{ if } s = s_n \]  \hspace{1cm} (7.4)

for some \( gB \in (\mathcal{E}_X \cap \mathcal{O}_1) \). We assume (7.4) (for some \( gB \)) to deduce contradiction. Being transversal intersection is an open condition. Since the intersection
Lemma 7.11. The map $\tau_w$ is an isomorphism.

Proof. By [CG97] 2.7.26 and Theorem 7.10, we deduce that the map $\tau_w$ induces an isomorphism

$$H_*(\mathcal{E}^\sim_X) \cong H_*(p_1((F \times \mathcal{E}^\sim_X) \cap \pi^{-1}(O_{w-1}))).$$

(7.5)
The spaces appearing in the homologies are given as fibrations over \((E_X \cap O_1)\) and \((E_X \cap O_w)\) with its fiber linear subspaces of \(S\). Here \(E_X\) has larger fiber. We switch to algebraic \(K\)-theory by Theorem 6.2 and Lemma 7.9. We have

\[ R(A)_a \otimes_{R(A)} K^A(E_X^a) \cong R(A)_a \otimes_{R(A)} K^A(E_X^a \cap O_1) \]

by the Thomason localization theorem and the fact that \(A\) fixes \((E_X^a \cap O_1)\). Hence, the map \(\tau_w\) itself is surjective if \([Y], [\theta^{-1}(Y)] \in R(A)_a \otimes_{R(A)} K^A(E_X^a)\) define the same cycle up to an invertible factor for each \(A\)-stable closed subvariety \(Y \subset (E_X \cap O_1)\). This is true if the alternating sum of the Koszul complex of \(S\) is invertible in \(R(A)_a\). This is equivalent to \(S^a = 0\), which is further re-phrased as

\[ V^a \subseteq uX. \]

This is (7.2).

We return to the proof of Theorem 7.4. Thanks to Lemma 7.11, we have finished the proof of Theorem 7.4 by Lemma 7.8.

8 Exotic Springer correspondence

We keep the setting of the previous section.

As we see in §1.4, we know that the action of \(H\) on \(M_\nu\) factors through the isomorphism

\[ C \otimes Z K(Z^a) \xrightarrow{RR} H_\bullet(Z^a) \cong \mathbb{H}_a. \]

Let \(\langle C[t]^W \rangle\) be the ideal of \(C[t]\) generated by the set of \(W\)-invariant polynomials without constant terms. Let \(C[W]#(C[t]/\langle C[t]^W \rangle)\) be the smash-product, which means that its product is given as

\[ (w_1, f_1)(w_2, f_2) := (w_1 w_2, f_1 w_1(f_2)) \quad \text{for } w_1, w_2 \in W, f_1, f_2 \in C[t]/\langle C[t]^W \rangle. \]

It is clear that \(F^a_0 \cong F\), \(Z^a_0 \cong Z\), and the restriction of the natural projections \(Z \to F\) restrict to natural projections \(Z \to F\).

Proposition 8.1. We have an isomorphism

\[ C[W]#(C[t]/\langle C[t]^W \rangle) \cong H_\bullet(Z^a_0) \]

as algebras.

Proof. We have

\[ H_\bullet(Z^a_0) \cong C_{a_0} \otimes_{R(G)} K^G(Z). \]

Here the RHS is written as

\[ C \otimes_{R(G)} \mathbb{H}/(q_0 = -q_1 = q_2 = 1). \]

Thus, we have

\[ H_\bullet(Z^a_0) \cong C \otimes_{R(G)} C[\widetilde{W}], \]
where \( \widetilde{W} := W \ltimes X(T) \) is the affine Weyl group of type \( C_n^{(1)} \). (Here \( C \) is the \( R(G) \)-module given by the evaluation at \( 1 \in G \). The algebra \( R(G) \) acts on \( C[\widetilde{W}] \) by \( R(G) \cong \mathbb{Z}[X(T)]^W \).) Thus, it suffices to show

\[
C[X^*(T)]_{m_1^-} \cong \mathbb{C}[t]/\langle C[t]^W \rangle,
\]

where \( m_1^- \subset C[X^*(T)]^W = C[T]^W \) is the defining ideal of the image of \( 1 \in T \) in \( \text{Spec} C[T]^W \). This follows from the fact that the neighborhoods of \( 1 \in T \) and \( 0 \in t \) are \( W \)-equivariantly diffeomorphic through the exponential map.

**Corollary 8.2.** Keep the setting of 8.1. We have a surjection

\[
H_u(\mathbb{Z}^a) \twoheadrightarrow \mathbb{C}[W].
\]

**Proof.** Keep the notation of the proof of Theorem 8.1. We have

\[
\langle C[t]^W \rangle \subset m_1 \subset C[t],
\]

where \( m_1 \) is the defining ideal of \( 0 \in t \). Since \( 0 \) is a \( W \)-fixed point of \( t \), we deduce that \( m_1 \) is a \( W \)-invariant maximal ideal. It follows that

\[
H_u(\mathbb{Z}^a) \cong \mathbb{C}[W] \# \langle C[t]/\langle C[t]^W \rangle \rangle \twoheadrightarrow \mathbb{C}[W] \# \langle C[t]/m_1 \rangle \cong \mathbb{C}[W]
\]

as desired. \( \square \)

**Theorem 8.3** (Exotic Springer correspondence). There exist one-to-one correspondences between the sets of the following three kinds of objects:

- a strict marked partition \( \sigma \);
- the \( G \)-orbit of \( \mathfrak{M} \) given as \( Gv_\sigma \);
- an irreducible \( W \)-module.

**Remark 8.4.** Our proof of Theorem 8.3 does not tell which representation is obtained from a given orbit. Such information can be found in [Ka08], which employs totally different argument.

**Proof of Theorem 8.3.** Let \( \mathcal{P} \) be the set of isomorphism classes of \( G \)-equivariant irreducible perverse sheaves on \( \mathfrak{M} \). Each \( I \in \mathcal{P} \) is isomorphic to the minimal extension from a \( G \)-orbit of \( \mathfrak{M} \). By Proposition 4.5, the (perverse) sheaf \( I \) must be the extension of a constant sheaf on a \( G \)-orbit. This implies \#\( \mathcal{P} \leq \#(G\setminus \mathfrak{M}) \). Let \( S \) be the set of strict normal forms. By Proposition 1.16 1), we have \#\( (G\setminus \mathfrak{M}) \leq \#S \). Hence, we have

\[
\#\text{Irrep}W \leq \#\mathcal{P} \leq \#(G\setminus \mathfrak{M}) \leq \#S \leq \#\text{Irrep}W, \tag{8.1}
\]

where the first inequality comes from Theorem 1.20 and the last inequality is Proposition 1.16 2). This forces all the inequalities in (8.1) to be equalities as required. \( \square \)

The following is a summary of the consequences of §1.4 Theorem 1.20:

**Theorem 8.5** (Ginzburg, [CG97] §8.5). Let \( a \) be a finite pre-admissible element. Let \( L \) be an irreducible \( \mathbb{H}^a \)-module. Then, there exists a unique \( G(a) \)-orbit \( \mathcal{O} \subset \mathfrak{M}^a \) with the following properties:
1. There exists a surjective \( \mathbb{H}_a \)-module homomorphism \( M(a,X) \to L \) for every \( X \in \mathcal{O} \);

2. If \( L \) appears in the composition factor of \( M(a,Y) \) as \( \mathbb{H}_a \)-modules for some \( Y \in \mathfrak{M}^a \), then we have \( Y \in \mathcal{U} \). □

Theorems 8.3 and 8.5 claim that each strict marked partition \( \sigma \) gives a unique simple quotient of \( M(a_0,v_\sigma) \). We denote this \( W \)-module by \( L_\sigma \) or \( L_X \) for \( X \in Gv_\sigma \), depending on the situation.

**Corollary 8.6.** Keep the setting of Theorem 8.3. A \( C[W] \)-module \( M(a_0,X) \) contains \( L_\sigma \) only if \( X \in \mathcal{O}Gv_\sigma \) holds. □

## 9 A deformation argument on parameters

We retain the setting of the previous section.

**Theorem 9.1.** Let \( a = (s,\bar{q}) \in T \) be an admissible element such that \( C_a = \{[1,n]\} \). Then, there exists an admissible element \( a' := (s',\bar{q}') \) such that

- The \( s' \)-action on \( V_1 \) has only positive real eigenvalues;
- We have \( q_0',q_1',q_2' \in \mathbb{R}_{>0} \);
- We have equalities \( \mathfrak{M}_a = \mathfrak{M}_a' \) and \( G(s) = G(s') \).

Moreover, we have an isomorphism \( \mathbb{H}_a \cong \mathbb{H}_a' \) as algebras.

**Proof.** Let \( N \) be the largest positive integer such that \( 1,q_2,\ldots,q_2^N \) are distinct. If \( q_2 \) is not a root of unity, then we regard \( N = \infty \). For each \( i = 1,\ldots,n \), we set \( \chi_i := \epsilon_i(s) \). By rearranging \( s \) by the \( W \)-action if necessary, we assume \( |\chi_i| \geq 1 \) (if \( N = \infty \) or \( \chi_i = q_2^j \) for some \( j \in \frac{1}{2}[0,N] \). We set \( E := \{\chi_i; 1 \leq i \leq n\} \). We choose a representative \( j_0 \in [1,n] \) which satisfies the following condition:

- If \( \pm 1 \in E \), then we require \( \chi_{j_0} = \pm 1 \);
- If \( \pm q_2^{1/2} \in E \), then we require \( \chi_{j_0} = \pm q_2^{1/2} \);
- If \( \pm q_2^{-1/2} \in E \) and \( \pm q_2^{1/2} \in E \), then we require \( \chi_{j_0} = \pm q_2^{-1/2} \);

For each pair \( i,j \in [1,n] \), we have

\[
\chi_i^{\kappa_{i,j}} = \chi_j^{\kappa_{i,j}} q_2^{m_{i,j}} \quad \text{for some } \kappa_{i,j},\kappa_{i,j}' \in \{\pm 1\}, m_{i,j} \in [0,n]. 
\] (9.1)

Since \( q_2 \) is not a root of unity of order \( \leq 2n \), it follows that the choice of \( m_{i,j} \) is at most one if \( (\kappa_{i,j},\kappa_{i,j}') \) is fixed. For each pair \( (i,j) \in [1,n] \), we set \( I(i,j) \) to be the set of triples \( (\kappa_{i,j},\kappa_{i,j}',m_{i,j}) \) which satisfies (9.1). Choose two real numbers \( q \gg q_2 \gg 1 \) such that \( q \) and \( q_2 \) have no algebraic relation. Then, we set

\[
(\chi_i')^{\kappa_{i,j}} := \begin{cases} 
(q_2)^{m_{i,j}} & (\chi_{j_0} = \pm 1) \\
q q_2^{m_{i,j} - \kappa_{i,j}/2} & (\chi_{j_0} = \pm q_2^{1/2}) \\
q(q_2)^{m_{i,j} + \kappa_{i,j}/2} & (\chi_{j_0} = \pm q_2^{-1/2}) \\
q q_2^{m_{i,j}} & (\chi_{j_0} \neq \pm 1, \pm q_2^{\pm 1/2})
\end{cases}
\]
Since the relation (9.1) for \((i, j)\) is determined by that of \((i_j, j_0)\) and \((j, j_0)\) for each pair \(i, j\) in \([1, n]\), it follows that

\[
(\chi_i')^{\kappa_{i,j}} = (\chi_j')^{\kappa_{i,j}} (q_2')^{m_{i,j}} \quad \text{for all } (\kappa_{i,j}, \kappa'_{i,j}, m_{i,j}) \in \mathcal{I}_{(i,j)}. \tag{9.2}
\]

Conversely, we have \((\kappa_{i,j}, \kappa'_{i,j}, m_{i,j}) \in \mathcal{I}_{(i,j)}\) if the relation (9.2) holds. It is clear that \(\chi_i^2 = 1\) if and only if \((\chi_i')^2 = 1\). We put \(s' \in T\) so that \(\kappa_i(s') = \chi_i'\) for each \(i = 1, 2, \ldots, n\). By the above consideration, it follows that \(g(s') = g(s)\). Since both \(G(s')\) and \(G(s)\) are connected by Theorem 3.2, we deduce \(G(s) = G(s')\).

Since the relation of (9.1) is preserved, we have \(V_2^{(s,q_2)} = V_2^{(s',q_2')}\). If we have \(\chi_{i,x} = q_k\) for some \(i \in [1, n]\), \(\kappa_i \in \pm 1\), and \(k = 0, 1\), then we set \(q_k' := (\chi_i')^{\kappa_i}\). Otherwise, we set \(q_k' \Rightarrow 0, 1\) to be an arbitrary real number which is not an eigenvalue of \(s'\) on \(V_1\). (I.e. not equal to any of \((\chi_i')^{\pm 1}\).) Since we have infinitely many possibilities, we can assume \(q_0 \neq q_1^2\) and \(q_0' \Rightarrow 1\) in this case. This gives \(\nu^\alpha = \nu^\alpha\) by setting \(a' := (s', q')\). We have \(q_0' \neq q_1^2\) in all cases since \(q_0 \neq q_1\).

Hence, the isomorphism \(V_2^\alpha \cong \nu^\alpha\) implies \(V_2^\alpha \cong \nu^\alpha\).

Therefore, as subvarieties of \(F\) and \(\mathfrak{M}_\alpha\), we have equalities

\[
F^\alpha = \bigcup_{w \in W} G(s) \times wB(s) (w\nu^+ \cap \nu^\alpha) = \bigcup_{w \in W} G(s') \times wB(s') (w\nu^+ \cap \nu^\alpha') = F'^{\alpha'}
\]

and \(\mathfrak{M}_\alpha^\alpha = \mathfrak{M}_\alpha'^{\alpha'}\).

The projection map \(F^\alpha \to \mathfrak{M}_\alpha^\alpha\) is induced by the projection \(\mu\). Hence, so is \(F'^{\alpha'} \to \mathfrak{M}_\alpha'^{\alpha'}\). Therefore, we have an equality of convolution algebras

\[
\mathbb{H}_\alpha \cong \mathbb{C} \otimes_k K(Z^\alpha) = \mathbb{C} \otimes_k K(Z'^{\alpha'}) \cong \mathbb{H}_{\alpha'},
\]

which proves the last assertion.

Since \(q, q_2' \gg 1\), each of \(q_i'\) \((i = 0, 1)\) is positive real. This verifies the requirement about \(q'\) as desired.

\begin{proof}
Proposition 9.2. Let \(a = (s, q_0, q_1, q_2) \in T\) be an admissible element such that:

- We have \(C_a = [1, n]\);
- The \(s\)-action on \(V_1\) has only positive real eigenvalues;
- We have \(V_1^{(s, q_1)} = \{0\}\);
- Each \(q_i\) \((i = 0, 1, 2)\) is a positive real number;

Let \(\varphi := (s, q_0, q_2)\) and let

\[
\log \varphi := (\log s, r_0, r_2), \quad \text{where } q_0 = e^{r_0}, q_2 = e^{r_2}.
\]

Let \(A\) be the Zariski closure of \(\varphi\) \(\subset T\). Then \(H^A_\varphi(Z)\) is a \(\mathbb{C}[a]\)-algebra such that

1. The quotient of \(H^A_\varphi(Z)\) by the ideal generated by functions of \(\mathbb{C}[a]\) which is zero along \(\log \varphi\) is isomorphic to \(H_\varphi(Z^\varphi)\).

\end{proof}
2. The images of the natural inclusions $C[W] \subset H_\bullet(Z) \subset H_\bullet^A(Z)$ induces an injection

$$C[W] \hookrightarrow H_\bullet(Z^a) = H_\bullet(Z^a).$$

Moreover, we have

$$C[a] \otimes H_\bullet(E_X) \cong H_\bullet(E_X^a) \text{ for } X \in \mathcal{X},$$

as a compatible $(C[W], C[a])$-module, where $W$ acts on $a$ trivially.

**Corollary 9.3.** Keep the setting of Proposition 9.2. We have

$$M(a_0, X) = H_\bullet(E_X) \cong H_\bullet(E_X^a) = M(a, X)$$

as $C[W]$-modules. \qed

The rest of this section is devoted to the proof of Proposition 9.2.

**Lemma 9.4.** Keep the setting of Proposition 9.2. Then, $A$ is connected.

**Proof.** The group $A$ is defined to be the spectrum of the quotient of $C[T_1]$ by the ideal generated by monomials $m$ such that $m(s, q_0, q_2) = 1$. Since all the values of $\epsilon_i(s)$, $q_0$, $q_1$ are positive real number, the conditions $m(s, q_0, q_2) = 1$ and $m(s', q'_0, q'_2) = 1$ are the same for all $r \in \mathbb{R}_{>0}$, where the branch of powers are taken so that all of $\epsilon_i(s'), q'_0, q'_2$ ($i = 1, \ldots, n$) are positive real numbers. It follows that a monomial $m \in C[T_1]$ satisfies $m(s, q_0, q_2)^k = 1$ for some positive integer $k$ if and only if $m(s, q_0, q_2) = 1$. Therefore, such monomials form a saturated $\mathbb{Z}$-sublattice of $X^*(T_1)$. In particular, its quotient lattice is a free $\mathbb{Z}$-lattice, which implies that $A$ is connected. \qed

We return to the proof of Proposition 9.2.

For each $m \geq 0$, let $ET_m := (C^m \setminus \{0\})^{\dim T}$ be a variety such that $i$-th $C^\times$-factor of $T = (C^\times)^{\dim T}$ acts as dilation of the $i$-th factor for each $1 \leq i \leq n + 3$. By the standard embedding $C^m \hookrightarrow C^{m+1}$ sending $(x)$ to $(x, 0)$, we form a sequence of $T$-varieties

$$\emptyset = ET_0 \hookrightarrow ET_1 \hookrightarrow ET_2 \hookrightarrow \cdots.$$ \hfill (9.3)

We define $ET := \lim_{\to m} ET_m$, which is an ind-quasiaffine scheme with free $T$-action. When we consider the homology of $ET$, we refer to the homology of its underlying classical topological space $\bigcup_{m \geq 0} ET_m$. Since $ET$ is contractible manifold with respect to the classical topology, we regard $ET$ as the universal vector bundle of each subgroup of $T$. (Hence we regard $BA := A \setminus ET$ in the below.)

**Corollary 9.5** (of Lemma 9.4). Keep the above setting. We have $H^{odd}(BA) = 0$. \qed

We return to the proof of Proposition 9.2.

For an $A$-variety $\mathcal{X}$, we set

$$\mathcal{X}_A := \Delta A \setminus (ET \times \mathcal{X}),$$

where $\Delta A$ represents the diagonal action of $A$. We have a forgetful map

$$f^A_\mathcal{X} : \mathcal{X}_A \to BA = A \setminus ET.$$
Let $\mathbb{D}_A^\perp$ be the relative dualizing sheaf with respect to $f_A^4$ (c.f. Bernstein-Lunts [BL94] §1.6). We define

$$H^A(X) \cong H^{-i}(X_A, \mathbb{D}_A^\perp).$$

In the below, we understand that $H^A(X) := \bigoplus_m H^A_m(X)$. Notice that this homology group is the same as the one obtained by replacing $ET$ with an ind-object of the direct system $\{ET_m\}$ and take the limit of the associated inverse system since $X$ is homotopic to a finite dimensional CW-complex. The projection maps $p_i : Z_A \to F_A$ ($i = 1, 2$) equip $H^A_m(Z)$ a structure of convolution algebra. It is straightforward to see that the diagonal subsets $\Delta F \subset Z$ and $((\Delta F)\_A) \subset Z_A$ represent $1 \in H^A_m(Z)$ and $1 \in H^A_m(Z)$, respectively.

**Lemma 9.6.** The algebra $H^A_m(Z)$ contains $H^A_m(Z)$ as its subalgebra. In particular, we have $\mathbb{C}[W] \subset H^A_m(Z)$ as subalgebras. Moreover, the center of $H^A_m(Z)$ contains $H^\bullet(BA)[[\Delta F]_A] \subset H^A_m(Z)$.

**Proof.** In the Leray spectral sequence

$$H^i(BA) \otimes H_j(Z) \Rightarrow H^{i+j}(Z),$$

we have $H^{odd}(BA) = 0$ and $H^{odd}(Z) = 0$ (since $Z$ is paved by affine spaces). It follows that this spectral sequence degenerates at the level of $E_2$-terms. Moreover, the image of the natural map $i : H_j(Z) \hookrightarrow H^A_m(Z)$ represents cycles which are locally constant fibration over the base $BA$. It follows that the map $i$ is an embedding of convolution algebras.

Multiplying $H^\bullet(BA)$ is an operation along the base $BA$, which commutes with the convolution operation (along the fibers of $f^4_2$). It follows that $H^\bullet(BA) \rightarrow H^\bullet(BA)[[\Delta F]_A] \subset H^A_m(Z)$ is central subalgebra as desired. $\square$

We return to the proof of Proposition 9.2.

By the Thomason localization theorem (see e.g. [CG97] §8.2), we have an isomorphism

$$R(A)_{\perp} \otimes_{R(A)} K^A_m(Z) \cong R(A)_{\perp} \otimes_{R(A)} K^A(Z)$$

as algebras. For each of $X = Z$, or $Z^\perp$, we have an embedding

$$i : K^A_m(X) \hookrightarrow \varprojlim_{m} K^A_m(ET_m \times X) \cong \varprojlim_{m} K(A \setminus (ET_m \times X))$$

obtained by pulling back an $A$-equivariant vector bundle on an irreducible component of $X$ to each $ET_m \times X$ by the second projection. The latter inverse limits are formed by the pullbacks via the closed embeddings coming from (9.3). Here the last inverse limit has a natural topology whose open sets are formed by the formal sum of vector bundles which are trivial on $K(A \setminus (ET_m \times X))$ for some fixed choice of $m$. By construction, the image of $i$ must be dense open with respect to the topology on the RHS.

We regard the RHS as a substitute of $K(X_A)$. Let $\mathbb{C}[a]_a$ and $\mathbb{C}[a]_a$ be the formal power series ring and the localized ring of $\mathbb{C}[a]$ along log $a$, respectively. The Chern character map relative to $BA$ gives an isomorphism

$$\mathbb{C}[a]_a \otimes_{\mathbb{C}[a]} H^A_m(Z) \cong \mathbb{C}[a]_a \otimes_{\mathbb{C}[a]} H^A_m(Z).$$
By restricting this to the sum of vectors of finitely many degrees, we obtain
\[
\mathbb{C}[a]_a \otimes_{\mathbb{C}[a]} H^A_\bullet(\mathbb{Z}^2) \cong \mathbb{C}[a]_a \otimes_{\mathbb{C}[a]} H^A_\bullet(\mathbb{Z}). \tag{9.4}
\]
Since localization along \(\mathbb{C}[a]_a\) commutes with the quotient by its unique maximal ideal, we deduce the first assertion.

The isomorphism (9.4) is an algebra isomorphism. This implies that \(1 \in \mathbb{C}[W]\) goes to \(1 \in H^A_\bullet(\mathbb{Z})\). It follows that each of \(s_i\) goes to a non-zero element of \(H^A_\bullet(\mathbb{Z})\) with its square equal to 1. By construction, there exists \(f_i \in \mathbb{C}[a]\) (since a polynomial ring is integrally closed), which implies that the images of \(1, s_1, \ldots, s_n \in H^A_\bullet(\mathbb{Z})\) are linearly independent. This verifies the second assertion.

10 Main Theorems

We retain the setting of §2.

**Theorem 10.1** (Deligne-Langlands type classification). Let \(a \in G\) be a finite pre-admissible element. Then, \(\mathcal{R}_a\) is in one-to-one correspondence with the set of isomorphism classes of simple \(\mathbb{H}_a\)-modules.

**Proof.** By definition, each element of \(\mathcal{R}_a\) corresponds to at least one isomorphism class of \(\mathbb{H}_a\)-modules. Since \(a\) is finite, each irreducible direct summand of \((\mu^+)_a)_{C_{F^2}}\) is the minimal extension of a local system (up to degree shift) from a \(G(a)\)-orbit \(\mathcal{O}\). By Theorem 4.10, a \(G(a)\)-equivariant local system on \(\mathcal{O}\) is a constant sheaf. As a result, every element of \(\mathcal{R}_a\) corresponds to at most one irreducible module as desired.

**Theorem 10.2** (Effective Deligne-Langlands type classification). Let \(a \in G\) be an admissible element. Then, the set \(\Lambda_a\) is in one-to-one correspondence with the set of isomorphism classes of simple \(\mathbb{H}_a\)-modules.

**Proof.** The proof is given at the end of this section.

As in Remark 2.2, the quotient \(\mathbb{H}/(q_0 + q_1)\) is isomorphic to an extended Hecke algebra \(\mathbb{H}_B\) of type \(B^{(1)}\) with two parameters. Hence, we have

**Corollary 10.3** (Effective Deligne-Langlands type classification for type \(B\)). Let \(a = (s, q_0, -q_0, q_2) \in G\) be a pre-admissible element such that \(-q_0^2 \neq q_2^{n+1}\) holds for every \(0 \leq m < n\). Then, the set \(\Lambda_a\) is in one-to-one correspondence with the set of isomorphism classes of simple \(\mathbb{H}_a\)-modules.
Proposition 10.5. In the course of the proof, we use:

\[ t_1^2 = q_2, t_n^2 = -q_0 q_1, t_n(t_0 - t_0^{-1}) = q_0 + q_1 \] (c.f. Remark 2.2 1)).

Let \( T_0, \ldots, T_n \) be the Iwahori-Matsumoto generators of \( \mathbb{H} \) (c.f. [Mc03, Lu03]). Their Hecke relations read

\[ (T_0 + 1)(T_0 - t_i^2) = (T_i + 1)(T_i - t_i^2) = (T_n + 1)(T_n - t_n^2) = 0, \]

where \( 1 \leq i < n \). The natural map \( \varphi(T_i) = T_{n-i} \) (\( 0 \leq i \leq n \)) extends to an algebra map \( \varphi : \mathbb{H} \rightarrow \mathbb{H}' \), where \( \mathbb{H}' \) is the Hecke algebra of type \( C_n^{(1)} \) with parameters \( t_n, t_1, t_0 \). We have \( t_n = \pm \sqrt{-q_0 q_1} \) and \( t_0 = \pm \sqrt{-q_0/q_1} \) or \( \pm \sqrt{-q_1/q_0} \). In particular, \( \varphi \) changes the parameters as \( (q_0, q_1, q_2) \rightarrow (q_0, q_1^{-1}, q_2) \) or \( (q_0^{-1}, q_1, q_2) \). Therefore, the representation theory of \( \mathbb{H}_n(a = (s, \bar{q})) \) is unchanged if we replace \( q_0 \) with \( q_0^{-1} \), or \( q_1 \) with \( q_1^{-1} \).

The rest of this section is devoted to the proof of Theorem 10.3. In the course of the proof, we use:

Proposition 10.5. Let \( a = (s, \bar{q}) \in T \) be an admissible element. Let \( O \subset \mathfrak{N} \) be a \( G \)-orbit. For any two distinct \( G(s) \)-orbits \( O_1, O_2 \subset O \cap \mathfrak{N}_+^a \), we have

\[ O_1 \cap O_2 = \emptyset. \]

Proof. By Proposition 4.8 and Lemma 1.18, we deduce that the scalar multiplication of a normal form of \( \mathfrak{N} \) is achieved by the action of \( T \). It follows that each \( G(s) \)-orbit of \( \mathfrak{N}^a \) is a \( Z_G(a) \)-orbit. Let \( X \in O_1 \). Let \( G_X \) be the stabilizer of \( X \) in \( G \). Assume that \( O_2 \cap \overline{O_1} \neq \emptyset \) to deduce contradiction. Since \( O_2 \) is a \( Z_G(a) \)-orbit, we have \( O_2 \subset \overline{O_1} \). Fix \( X' \in O_2 \). Consider an open neighborhood \( U \) of \( T \) in \( G \) (as complex analytic manifolds). Then, \( UX' \in O \) is an open neighborhood of \( X' \). It follows that \( UX' \cap O_1 \neq \emptyset \). We put \( g_{a, X'} := \text{Lie}G_{X'} + \text{Lie}Z_G(a) \). We have

\[ N_{O_2/O_1, X'} = g_{a, X'}. \]

Every non-zero vectors of \( N_{O_2/O_1, X'} \) is expressed as a linear combination of eigenvectors with respect to the \( a \)-action. These \( a \)-eigenvectors can be taken to have non-zero weights and does not contained in \( G_{X'} \). It follows that

\[ UX' \cap O_1 \not\subset \mathfrak{N}^a, \]

which is contradiction (for an arbitrary sufficiently small \( U \)). Hence, we have necessarily \( O_2 \cap \overline{O_1} = \emptyset \) as desired.

Proof of Theorem 10.2. By taking \( G \)-conjugate if necessary, we assume \( a \in T \). By Corollary 3.10, it suffices to prove Theorem 10.2 when \( C_n \) consists of a unique clan \([1, n] \). By Corollary 4.9, we can further assume \( V_t^{(s, n)} = \{0\} \) by swapping the roles of \( q_0 \) and \( q_1 \) if necessary. By Theorem 8.5 (c.f. Theorem 1.20), an admissible parameter \((a, X)\) is regular if there exists a simple \( \mathbb{H}_a \)-constituent of \( M(a, X) \) which does not appear in any \( M(a, X') \) such that \( G(s)X \nsubseteq G(s)X' \).
We apply Proposition 9.1 (if necessary) to modify $a$ so that the assumption of Proposition 9.2 is fulfilled. By Proposition 9.2, each $M_{(a,X)}$ has a $W$-module structure given by the restriction of the $H_a$-module structure. Moreover, the simple $W$-module $L_X$ corresponding to the $G$-orbit $G\cdot X$ (by the exotic Springer correspondence) appears in $M_{(a,X)}$. By Corollary 9.3, $M_{(a,X)}$ contains $L_X$ as $W$-modules. Hence, the simple $H_a$-constituent of $M_{(a,X)}$ which contains $L_X$ as $W$-type does not occur in any $M_{(a,X')}$ such that $G\cdot X \not= G\cdot X'$ as required.

11 Consequences

In this section, we present some of the consequences of our results. We retain the setting of the previous section.

**Definition 11.1.** Let $\nu = (a,X)$ be an admissible parameter. Let $L_\nu$ be the simple module of $H$ corresponding to $\nu$. Let $IC(\nu)$ be the corresponding $G(a)$-equivariant simple perverse sheaf on $\mathbb{H}_a$. (c.f. §1.4) We denote by $P_\nu$ the projective cover of $L_\nu$ as $H_a$-modules. (It exists since $H_a$ is finite dimensional.)

Let $K$ be a $H$-module and let $L$ be a simple $H$-module. We denote by $[K : L]$ the multiplicity of $L$ in $K$.

Applying [CG97] 8.6.23 to our situation, we obtain:

**Theorem 11.2 (The multiplicity formula of standard modules).** Let $\nu = (a,X)$, $\nu' = (a,X')$ be admissible parameters. We have:

$$[M_\nu : L_{\nu'}] = \sum_k \dim H^k(i_X^! IC(\nu'))$$

and

$$[M_{\nu'} : L_{\nu}] = \sum_k \dim H^k(i_X^* IC(\nu'))$$

where $i_X : \{X\} \hookrightarrow \mathfrak{R}_a$ is an inclusion.

The following result is a variant of the Lusztig-Ginzburg character formula of standard modules in our setting.

**Theorem 11.3 (The character formula of standard modules).** Let $\nu = (a,X) = (s,\vec{q},X)$ be an admissible parameter. Let $B_\nu$ be the set of connected components of $E_a^X$. For each $B \in B_\nu$, we define a linear form $\langle \bullet, s \rangle_B$ as a composition map

$$\begin{array}{c}
\langle \bullet, s \rangle_B : R(T) \xrightarrow{\cong} R(gBg^{-1}) \xrightarrow{ev_s} \mathbb{C} \\
\xrightarrow{R^+} \{ \text{weights of } gBg^{-1} \}
\end{array}$$

by some $g \in G$ such that $gB \in B$. Then, $\langle \bullet, s \rangle_B$ is independent of the choice of $g$ and the restriction of $M_\nu$ to $R(T)$ is given as

$$\text{Tr}(e^\lambda ; M_\nu) := \sum_{B \in B_\nu} \langle \lambda, s \rangle_B \sum_{j \geq 0} \dim H_{2j}(B, \mathbb{C}).$$

**Proof.** Taking account into Theorem 4.10, the proof is exactly the same as in [CG97] §8.2.
**Definition 11.4.** Let \( a = (s, q) \in T \) be an admissible element. We form three \( |\Lambda_a| \times |\Lambda_a| \) matrices

\[
P: L_{a,a} := [P_{a}, L_{a}'],
\]

\[
D_{a,a} := \delta_{a, a'} \chi_{c}(\nu),
\]

\[
IC_{a,a} := [M_{\nu, \nu}'],
\]

where \( \chi_{c}(\nu) := \sum_{i \geq 0} (-1)^{i} \dim H^{i}(G(a)X, \mathbb{C}) \) \( (\nu = (a, X)) \).

The following result is a special case of the Ginzburg theory [CG97] Theorem 8.7.5 applied to our particular setting:

**Theorem 11.5** (The multiplicity formula of projective modules). Keep the setting of Definition 11.4. We have

\[
P: L_{a} = IC_{a} \cdot D_{a} \cdot IC_{a}^t,
\]

where \( t \) denotes the transposition of matrices.

\[
\square
\]

**Index of notation**

(Sorted by the order of appearance)

- \( G, B, T, G(s), U_{a}, \ldots \) §1 \( p_{w} \in O_{w} \) §1.1 \( H_{\nu}, F_{a}^{\nu}, \mu_{a}^{\nu}, \mathfrak{m}_{a}^{\nu}, \ldots \) §2
- \( R, R^{+}, \varepsilon, \varepsilon, \alpha_{i} \) §1 ∗, o §1.1 \( \mathfrak{R}_{a} \) §2
- \( W, w \in NC(T), s_{i}, \ell \) §1 \( a_{0} := (1, 1, -1, 1) \) §1.2 \( c, n^{e}, \Gamma \) §3
- \( wH := \hat{w}Hw^{-1} \) §1 \( \tilde{q}, \log_{k}(s) (s \in T) \) §1.2 \( g(s)e, G(s)e \) §3
- \( \text{Stab}_{H} x \ (x \in X) \) §1 \( \Lambda_{a} \) §1.2 \( V_{a}, V_{e}, F_{a}^{+}, F_{+}^{a}(w) \) §3
- \( g, t, g(s), \ldots \) §1 \( x_{i}, y_{i}, j \in V \) §1.3 \( w_{a}^{\nu} \) §3
- \( V[\lambda], V^{+}, V^{-}, \Psi(V) \) §1 \( J, T_{j}, \delta \) §1.3 \( G_{a}, V(e), X_{a}, \ldots \) §3
- \( H_{a}(X), H_{a}(X, Z) \) §1 \( \sigma = (J, \delta) \) §1.3 \( \nu_{c} \) §3
- \( I, I^{*}, G_{0}, \exp \) §1 \( v_{\sigma}, v_{1, \sigma}, v_{2, \sigma}, \ldots \) §1.3 \( s_{\sigma}, D_{\sigma}, P_{0} \) §5
- \( V_{1} = \mathbb{C}^{n}, V_{2} = \wedge^{2}V_{1} \) §1.1 \( \overrightarrow{\Phi}_{J} = \Phi_{J} \ (J \in J) \) §1.3 \( E_{X}, E_{X}^{0}, \mathfrak{h} \) §6
- \( V_{e} : \ell \)-exotic rep. §1.1 \( T_{J}, F_{a}^{\nu}, \nu_{a}^{\nu}, \mathfrak{m}_{a}^{\nu}, \ldots \) §1.4 \( M_{\nu}, M_{\nu}^{*} \) §7
- \( F_{\ell}, \mu_{\ell}, \mathfrak{m}_{\ell} \) §1.1 \( G = G_{2}, T = T_{2}, \ldots \) §2 \( s_{Q}, V_{Q}, \mathbb{E}_{Q}, \ldots \) §7
- \( F_{\mu}, \mathfrak{m}_{\ldots} \) §1.1 \( \mathcal{A}, \mathfrak{H} \) §2 \( L_{\sigma} = L_{X} (X \in GV_{\sigma}) \) §8
- \( G_{\ell}, Z_{\ell}, \mathfrak{p}_{\ell}, \pi_{\ell} \) §1.1 \( T_{\ell}, q_{\ell}, e^{\lambda} \in \mathfrak{h} \) §2 \( ET, BA, H^{*}_{a}(X) \) §9
- \( C_{a} \) §1.1 \( Z_{\leq w}, O_{a}, \mathcal{T}, \ldots \) §2 \( L_{\sigma}, IC(\nu) \) §11

**References**

[AH08] Pramod N. Achar, and Anthony Henderson, Orbit closures in the enhanced nilpotent cone, Adv. Math. 219 (2008), 27–62

[BH85] Hyman Bass and William J. Haboush, Linearizing certain reductive group actions, Trans. Amer. Math. Soc. 292 no.2 463–482 (1985).

[BL94] Joseph Bernstein and Valery Lunts, Equivariant Sheaves and Functors, LNM 1578 Springer (1994).

[BS49] Armand Borel, and Jean de Siebenthal, Les sous-groupes fermes de rang maximum des groupes de Lie clos. Comment. Math. Helv. 23, (1949). 200–221.

[Ca85] Roger W. Cartier, Finite Groups of Lie type: Conjugacy classes and complex characters, Wiley, (1985). ISBN 0-471-50683-4
[CG97] Neil Chriss, and Victor Ginzburg, Representation theory and complex geometry. Birkhäuser Boston, Inc., Boston, MA, 1997. x+495 pp. ISBN 0-8176-3792-3

[CK] Dan Ciubotaru, and Syu Kato, in preparation.

[DLP88] Corrado De Concini, George Lusztig, and Claudio Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, J. Amer. Math. Soc. 1, (1988), no.1, 15–34.

[DK85] Jiri Dadok, and Victor Kac, Polar representations. J. Algebra 92 (1985), no. 2, 504–524.

[En06] Naoya Enomoto, Classification of the Irreducible Representations of Affine Hecke Algebras of Type $B_2$ with unequal parameters, J. Math. Kyoto Univ. 46 no.2. (2006) 259–273.

[En08] Naoya Enomoto, A quiver construction of symmetric crystals, arXiv:0806.3615, preprint.

[FGT08] Michael Finkelberg, Victor Ginzburg, and Roman Travkin, Mirabolic affine Grassmannian and character sheaves, arXiv:0802.1652.

[Gi97] Victor Ginzburg, Geometric methods in representation theory of Hecke algebras and quantum groups. Notes by Vladimir Baranovsky. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 127–183, Kluwer Acad. Publ., Dordrecht, 1998.

[Ha77] Robin Hartshorne, Algebraic Geometry, GTM 52. Springer-Verlag, 1977. xvi+496 pp. ISBN 0-387-90244-9.

[Ig73] Jun-ichi Igusa, Geometry of absolutely admissible representations, in: Number theory, algebraic geometry and commutative algebra, in honor of Yasuo Akizuki, 373–452, Kinokuniya, Tokyo, (1973).

[Jo98] Roy Joshua, Modules over convolution algebras from derived categories, I, J. Algebra, 203, 385–446, (1998).

[Ka06b] Syu Kato, An exotic Springer correspondence for symplectic groups, math.RT/0607478, preprint (never to appear).

[Ka08] Syu Kato, Deformations of nilpotent cones and Springer correspondences, arXiv:0801.3707, preprint.

[KL87] David Kazhdan, and George Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987), no. 1, 153–215.

[Lu88] George Lusztig, Cuspidal local systems and graded Hecke algebras. I, Inst. Hautes Études Sci. Publ. Math., 67, (1988), 145–202.

[Lu89] George Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. 2, (1989), no.3, 599–635.

[Lu95a] George Lusztig, Classification of unipotent representations of simple p-adic groups. Internat. Math. Res. Notices 1995, no. 11, 517–589.

[Lu95b] George Lusztig, Cuspidal local systems and graded Hecke algebras. II, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., 16, 217–275. (1995)

[Lu02] George Lusztig, Cuspidal local systems and graded Hecke algebras. III, Representations Theory 6, 202–242. (2002)

[Lu03] George Lusztig, Hecke algebras with unequal parameters, CRM Monograph Series, 18, American Mathematical Society, Providence, RI, (2003), vi+136, ISBN: 0-8218-3356-1

[Mc03] Ian G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Cambridge Tracts in Mathematics 157 (2003)

[Oh86] Takuya Ohta, The singularities of the closures of nilpotent orbits in certain symmetric pairs, Tôhoku Math. J., 38 (1986), 441-468.

[OS07] Eric Opdam, and Maarten Solleveld, Homological algebra for affine Hecke algebras, arXiv:0708.1722, to appear in Adv. in Math.

[OS08] Eric Opdam, and Maarten Solleveld, Discrete series characters for affine Hecke algebras and their formal degrees, arXiv:0804.0026.
[Po04] Vladimir L. Popov, The cone of Hilbert nullforms, Proc. Steklov Math. Inst. 241 (2003), 177-194.

[Ra01] Arun Ram, Representations of rank two affine Hecke algebras, in "Advances in Algebra and Geometry, University of Hyderabad conference 2001", C. Musili ed., Hindustan Book Agency, 2003, 57-91, available online at http://www.math.wisc.edu/~ram/bib.html

[Sa88] Morihiko Saito, Modules de Hodge polarisables, Publ. Res. Inst. Math. Sci. 24 (1988), no. 6, 849-995 (1989).

[Sc78] Gerald W. Schwarz, Representations of simple groups with a free module of covariants, Invent. Math. 49, 1–12 (1978).

[Se84] Jiro Sekiguchi, The nilpotent subvariety of the vector space associated to a symmetric pair, Publ. Res. Inst. Math. Sci. 20 (1984), 155–212.

[So07] Maarten Solleveld, Periodic cyclic homology of reductive p-adic groups, arXiv:0710.5815.

[Sp07] Tonny A. Springer, The exotic nilcone of a symplectic group, preprint.

[Sp82] Nicolas Spaltenstein, Classes unipotents et sous groupes de Borel, Lect. Notes. Math. 946 Springer-Verlag, 1982.

[Th86] Robert W. Thomason, Lefschetz-Riemann-Roch theorem and coherent trace formula, Invent. Math. 85 515–543 (1986).

[Tr08] Roman Travkin, Mirabolic Robinson-Schensted-Knuth correspondence, preprint.