A necessary flexibility condition for a nondegenerate suspension in Lobachevsky 3-space

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Abstract. We show that some combination of the lengths of all edges of the equator of a flexible suspension in Lobachevsky 3-space is equal to zero (each length is taken with a ‘plus’ or ‘minus’ sign in this combination).

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§ 1. Introduction

A polyhedron (more precisely, a polyhedral surface) is said to be flexible if its spatial shape can be changed continuously by changing just its dihedral angles, that is, if every face remains congruent to itself during the flex.

In 1897 Bricard [1] described all flexible octahedra in Euclidean 3-space. These were the first flexible polyhedra to be constructed and are special cases of Euclidean flexible suspensions. The modern method of constructing Bricard’s octahedra was first described by Lebesgue [2].

In 1974 Connelly [3] proved that for any flexible suspension in Euclidean 3-space such that the lengths between the poles of the suspension change during the flex, some combination of the lengths of all edges of its equator is equal to zero (each length is taken with a ‘plus’ or ‘minus’ sign in this combination). Connelly’s method was to reduce the problem to the study of an analytic function of a complex variable in neighbourhoods of its singular points.

In 2001 Mikhalev [4] gave a new proof of Connelly’s result using algebraic methods. He proved, in addition, that for every spatial quadrilateral formed by edges of a flexible suspension of the above type and containing its both poles there is a combination of the lengths (taken with ‘plus’ and ‘minus’ signs) of the edges of the quadrilateral which is equal to zero.

Our aim here is to prove a similar result for the equator of a flexible suspension in Lobachevsky 3-space, applying Connelly’s method [1].
§ 2. Formulating the main result

Let $\mathcal{K}$ be a simplicial complex. A polyhedron (a polyhedral surface) in Lobachevsky 3-space is a continuous map from $\mathcal{K}$ to $\mathbb{H}^3$ which sends every $k$-dimensional simplex of $\mathcal{K}$ to a subset of a $k$-dimensional plane in Lobachevsky space ($k \leq 2$). The images of topological 2-simplices are called faces, of topological 1-simplices are called edges, and of topological 0-simplices are called vertices of the polyhedron. Note that in our definition the image of a simplex can be degenerate (for instance, a face can lie on a hyperbolic straight line, and an edge can reduce to a point), and faces can intersect in their interior points. If $v_1, \ldots, v_W$ are the vertices of $\mathcal{K}$, and if $\mathcal{P}: \mathcal{K} \to \mathbb{H}^3$ is a polyhedron, then $\mathcal{P}$ is determined by the $W$ points $P_1, \ldots, P_W \in \mathbb{H}^3$, where $P_j \equiv \mathcal{P}(v_j)$, $j = 1, \ldots, W$.

If $\mathcal{P}: \mathcal{K} \to \mathbb{H}^3$ and $\mathcal{Q}: \mathcal{K} \to \mathbb{H}^3$ are two polyhedra, then we say that $\mathcal{P}$ and $\mathcal{Q}$ are congruent if there exists a motion $\mathcal{A}: \mathbb{H}^3 \to \mathbb{H}^3$ such that $\mathcal{Q} = \mathcal{A} \circ \mathcal{P}$ (that is, the isometric mapping $\mathcal{A}$ sends every vertex of $\mathcal{P}$ to a corresponding vertex of $\mathcal{Q}$: $Q_j = \mathcal{A}(P_j)$, or in other words $\mathcal{Q}(v_j) = \mathcal{A}(\mathcal{P}(v_j))$, $j = 1, \ldots, W$). We say that $\mathcal{P}$ and $\mathcal{Q}$ are isometric (in the intrinsic metric) if each edge of $\mathcal{P}$ has the same length as the corresponding edge of $\mathcal{Q}$, that is, if $\langle v_j, v_k \rangle$ is a 1-simplex of $\mathcal{K}$ then $d_{\mathbb{H}^3}(Q_j, Q_k) = d_{\mathbb{H}^3}(P_j, P_k)$, where $d_{\mathbb{H}^3}(\cdot, \cdot)$ stands for the distance in Lobachevsky space $\mathbb{H}^3$.

A polyhedron $\mathcal{P}$ is flexible if, for some continuous one-parameter family of polyhedra $\mathcal{P}_t: \mathcal{K} \to \mathbb{H}^3$, $0 \leq t \leq 1$, the following three conditions hold true: (1) $\mathcal{P}_0 = \mathcal{P}$; (2) each $\mathcal{P}_t$ is isometric to $\mathcal{P}_0$; (3) some $\mathcal{P}_t$ is not congruent to $\mathcal{P}_0$.

Let $\mathcal{K}$ be defined as follows: $\mathcal{K}$ has vertices $v_0, v_1, \ldots, v_V, v_{V+1}$, where $v_1, \ldots, v_V$ form a cycle ($v_j$ is adjacent to $v_{j+1}$, $j = 1, \ldots, V-1$, and $v_V$ is adjacent to $v_1$), and $v_0$ and $v_{V+1}$ are each adjacent to all of $v_1, \ldots, v_V$. Each polyhedron $\mathcal{P}$ based on $\mathcal{K}$ is called a suspension. Call $N \equiv \mathcal{P}(v_0)$ the north pole, and $S \equiv \mathcal{P}(v_{V+1})$ the south pole, and call $P_j \equiv \mathcal{P}(v_j)$, $j = 1, \ldots, V$, vertices of the equator of $\mathcal{P}$.

Assume that a suspension $\mathcal{P}$ is flexible. If we suppose the segment $NS$ to be an extra edge, then $\mathcal{P}$ becomes a set of $V$ tetrahedra glued cyclically along their common edge $NS$. We call a suspension nondegenerate if none of these tetrahedra lies on a hyperbolic 2-plane. Note that a nondegenerate suspension $\mathcal{P}$ does not flex if the distance between $N$ and $S$ remains constant. Therefore, as in the Euclidean case [3], we assume that the length of $NS$ varies as we flex $\mathcal{P}$. Examples of degenerate suspensions are a double covered cap: a suspension with coinciding poles (see Fig. 1), and a suspension with a wing: a suspension whose vertices $N$, $S$, $P_{i-1}$ and $P_{i+1}$ lie on a straight line for some $i$ (see Fig. 2). We will not study degenerate flexible suspensions in this paper.

The main result of this paper is

**Theorem 1.** Let $\mathcal{P}$ be a nondegenerate flexible suspension in Lobachevsky 3-space with poles $S$ and $N$, and with vertices of the equator $P_j$, $j = 1, \ldots, V$. Then for some set of signs $\sigma_{j,j+1} \in \{+1, -1\}$, $j = 1, \ldots, V$, the sum of the lengths $e_{j,j+1}$ of all edges $P_jP_{j+1}$ of the equator of $\mathcal{P}$, taken with the corresponding signs $\sigma_{j,j+1}$, is equal to zero. (Here and below, by definition, it is assumed that $P_VP_{V+1} \equiv P_VP_1$ and $\sigma_{V,V+1} \equiv \sigma_{V,1}$.)
§ 3. Connelly’s flexibility equation for a suspension

Connelly in [3] obtained a flexibility equation for a nondegenerate suspension in Euclidean 3-space. Following him, in this section we will obtain an flexibility equation for a nondegenerate suspension in Lobachevsky 3-space.

We place a nondegenerate suspension \( \mathcal{P} \) into the Poincaré upper half-space model [5] of Lobachevsky 3-space \( \mathbb{H}^3 \) in such a way that the poles \( N \) and \( S \) of \( \mathcal{P} \) lie on the \( z \)-axis of the Cartesian coordinate system of the Poincaré model (see Fig. 3). Let \( S \) have the coordinates \((0, 0, z_S)\), \( N \) the coordinates \((0, 0, z_N)\), and \( P_j \) the coordinates \((x_j, y_j, z_j)\), \( j = 1, \ldots, V \). Also we denote the length of the edge \( NP_j \) by \( e_j \), and of \( SP_j \) by \( e'_j \), \( j = 1, \ldots, V \) (here and below \( e_{V,V+1}^{\text{def}} = e_{V,1} \)).

Consider a Euclidean orthogonal projection \( \mathcal{P} \) of \( \mathcal{P} \) onto the \( xy \)-plane (see Fig. 4). Note that \( \mathcal{P} \) is also the hyperbolic orthogonal projection of \( \mathcal{P} \) onto the \( xy \)-plane from the only point at infinity of \( \mathbb{H}^3 \) which does not lie on this plane. This
projection sends the poles $N$ and $S$ of $\mathcal{P}$ to the origin $O(0, 0)$ on the $xy$-plane, $P_j$ to the point $\tilde{P}_j (x_j, y_j)$, the edges $NP_j$ and $SP_j$ to the Euclidean segment $O\tilde{P}_j$, and the edge $P_j P_{j+1}$ of the equator of $\mathcal{P}$ to the Euclidean segment $\tilde{P}_j \tilde{P}_{j+1}$, $j = 1, \ldots, V$ (here and below $P_{V+1} \overset{\text{def}}{=} P_1$, $\tilde{P}_{V+1} \overset{\text{def}}{=} \tilde{P}_1$, $x_{V+1} \overset{\text{def}}{=} x_1$, $y_{V+1} \overset{\text{def}}{=} y_1$, $z_{V+1} \overset{\text{def}}{=} z_1$).

The polar coordinates $(\rho_j, \theta_j)$ of $\tilde{P}_j$, $j = 1, \ldots, V$, are related to its Cartesian coordinates by the formulae

$$\rho_j = \sqrt{x_j^2 + y_j^2}, \quad \sin \theta_j = \frac{y_j}{\rho_j} = \frac{y_j}{\sqrt{x_j^2 + y_j^2}}, \quad \cos \theta_j = \frac{x_j}{\rho_j} = \frac{x_j}{\sqrt{x_j^2 + y_j^2}}.$$  

(3.1)

Note that by construction, the dihedral angle $\theta_{j,j+1}$ of the tetrahedron $NSP_jP_{j+1}$ at the edge $NS$ is equal to the flat angle $\angle \tilde{P}_jO\tilde{P}_{j+1}$, $j = 1, \ldots, V$, and

$$\theta_{j,j+1} = \theta_{j+1} - \theta_j.$$  

(3.2)

Note also that the value of $\theta_{j,j+1}$ can be negative. Applying the trigonometric relations for the difference of two angles (3.2) we get

$$\cos \theta_{j,j+1} = \cos \theta_{j+1} \cos \theta_{j} + \sin \theta_{j+1} \sin \theta_{j},$$

$$\sin \theta_{j,j+1} = \sin \theta_{j+1} \cos \theta_{j} - \cos \theta_{j+1} \sin \theta_{j}.$$  

(3.3)

Taking (3.1) into account, we can reduce (3.3) to

$$\cos \theta_{j,j+1} = \frac{x_j x_{j+1} + y_j y_{j+1}}{\sqrt{x_j^2 + y_j^2} \sqrt{x_{j+1}^2 + y_{j+1}^2}},$$

$$\sin \theta_{j,j+1} = \frac{x_j y_{j+1} - y_j x_{j+1}}{\sqrt{x_j^2 + y_j^2} \sqrt{x_{j+1}^2 + y_{j+1}^2}}.$$  

Then, according to Euler’s formula

$$e^{i\theta_{j,j+1}} = \cos \theta_{j,j+1} + i \sin \theta_{j,j+1} = \frac{(x_j x_{j+1} + y_j y_{j+1}) + i(x_j y_{j+1} - y_j x_{j+1})}{\sqrt{x_j^2 + y_j^2} \sqrt{x_{j+1}^2 + y_{j+1}^2}}.$$  

(3.4)

Following Connelly [3] we observe that the sum of the dihedral angles $\theta_{j,j+1}$ of all tetrahedra $NSP_jP_{j+1}$, $j = 1, \ldots, V$, at the edge $NS$ is constant and a multiple of $2\pi$ (here and below $\theta_{V,V+1} \overset{\text{def}}{=} \theta_{V,1}$, $\theta_{V+1} \overset{\text{def}}{=} \theta_1$, $\rho_{V+1} \overset{\text{def}}{=} \rho_1$), that is,

$$\sum_{j=1}^{V} \theta_{j,j+1} = 2\pi m \quad \text{for some integer } m,$$  

(3.5)

and remains so during the deformation of the suspension, when the values of the angles $\theta_{j,j+1}$, $j = 1, \ldots, V$, vary continuously.

We rewrite the flexibility equation (3.5) in a convenient form:

$$\prod_{j=1}^{V} e^{i\theta_{j,j+1}} = 1.$$  

(3.6)
Thus, taking (3.4) into account we see that the coordinates of the vertices of $P$ are related as follows:

$$\prod_{j=1}^{V} \frac{(x_jx_{j+1} + y_jy_{j+1}) + i(x_jy_{j+1} - y_jx_{j+1})}{x_j^2 + y_j^2} = 1,$$

or in different notation

$$\prod_{j=1}^{V} F_{j,j+1} = \prod_{j=1}^{V} \frac{G_{j,j+1}}{\rho_j \rho_{j+1}} = \prod_{j=1}^{V} \frac{G_{j,j+1}}{\rho_j^2} = 1,$$

where $G_{j,m} = (x_jx_m + y_jy_m) + i(x_jy_m - y_jx_m)$, $F_{j,m} = G_{j,m}/\rho_j \rho_m$, $j, m = 1, \ldots, V$, and $G_{V,V+1} \overset{\text{def}}{=} G_{V,1}$, $F_{V,V+1} \overset{\text{def}}{=} F_{V,1}$.

When we study the deformation $P_t$ of the suspension $P$, all objects and values related to $P_t$ are naturally inherited from the notation for the corresponding entities related to $P$. For example, the coordinate $x_j(t)$ of the point $P_j(t)$ of the deformation $P_t$ corresponds to the coordinate $x_j$ of the point $P_j$ of the suspension $P$, the dihedral angle $\theta_{j,j+1}(t)$ of the tetrahedron $N(t)S(t)P_j(t)P_{j+1}(t)$ at the edge $N(t)S(t)$ corresponds to the dihedral angle $\theta_{j,j+1}$ of the tetrahedron $NSP_jP_{j+1}$ at the edge $NS$, and so on.

### §4. The flexibility equation for a suspension in terms of the lengths of its edges

In this section we are going to express the flexibility equation for a suspension (3.7) in terms of the lengths of the edges of $P$. Recall that the lengths of the edges of $P$ remain constant during the flex. With this in mind we need to demonstrate that the following two statements hold. The first can be verified by direct calculation (see also Fig. 5).
Lemma 1. Given a Poincaré upper half-plane $\mathbb{H}^2$ with coordinates $(\rho, z)$ (that is, with the metric given by the formula $ds^2 = (d\rho^2 + dz^2)/z^2$), the distance between points $A(\rho_0, z_A)$ and $B(\rho_0, z_B)$ with the same first coordinate $\rho_0$ can be calculated using the formula

$$d_{\mathbb{H}^2}(A, B) = \left| \log \frac{z_B}{z_A} \right|. \quad (4.1)$$

Lemma 2. Given a Poincaré upper half-plane $\mathbb{H}^2$ with coordinates $(\rho, z)$ (that is, with the metric given by the formula $ds^2 = (d\rho^2 + dz^2)/z^2$), the distance between points $A(\rho_A, z_A)$ and $B(\rho_B, z_B)$, $l \overset{\text{def}}{=} d_{\mathbb{H}^2}(A, B)$, is related to their coordinates by the formula

$$(\rho_B - \rho_A)^2 + z_A^2 + z_B^2 = 2z_Az_B \cosh l. \quad (4.2)$$

Proof. According to part (2) of Corollary A.5.8 in [6], the distance between the points with coordinates $(x, t)$ and $(y, s)$ in the Poincaré upper half-space model $\mathbb{R}^n \times \mathbb{R}^+$ of Lobachevsky $(n + 1)$-space $\mathbb{H}^{n+1}$ is calculated using the formula

$$d_{\mathbb{H}^{n+1}}((x, t), (y, s)) = 2 \tanh^{-1} \left( \frac{\|x - y\|^2 + (t - s)^2}{\|x - y\|^2 + (t + s)^2} \right)^{1/2}, \quad (4.3)$$

where the symbol $\| \cdot \|$ stands for the standard Euclidean norm in $\mathbb{R}^n$.

By (4.3) the distance between the points $A$ and $B$ (see Fig. 6) is calculated using the formula

$$l = 2 \tanh^{-1} \left( \frac{(\rho_A - \rho_B)^2 + (z_A - z_B)^2}{(\rho_A - \rho_B)^2 + (z_A + z_B)^2} \right)^{1/2} \quad (4.4)$$

(where $n = 1$, $(x, t) = (\rho_A, z_A)$ and $(y, s) = (\rho_B, z_B)$).

After a series of transformations of (4.4) we get

$$(\rho_A - \rho_B)^2 \left( \cosh^2 \frac{l}{2} - \sinh^2 \frac{l}{2} \right) + (z_A^2 + z_B^2) \left( \cosh^2 \frac{l}{2} - \sinh^2 \frac{l}{2} \right) = 2z_Az_B \left( \cosh^2 \frac{l}{2} + \sinh^2 \frac{l}{2} \right).$$

By the identities $\cosh^2 l/2 - \sinh^2 l/2 = 1$ and $\cosh l = \cosh^2 l/2 + \sinh^2 l/2$, this reduces to (4.2).

We will now express $G_{j,j+1}$ and $\rho_j^2$ in terms of the lengths of edges of $\mathcal{P}$.

We assume that the coordinates of the south pole $S$ are $(0,0,1)$. Let $t \overset{\text{def}}{=} e^{d_{\mathbb{H}^3}(N,S)}$, where $d_{\mathbb{H}^3}(N,S)$ is the distance between the poles $N$ and $S$ of $\mathcal{P}$. Without loss of generality, we may assume that $z_N \geq z_S$. Then, by Lemma 1, the coordinates of $N$ are $(0,0,t)$.

Applying Lemma 2 to the points $S$ and $P_j$ lying on the hyperbolic plane SNP$_j$, by (4.2) we get

$$\rho_j^2 + z_j^2 + 1 = 2z_j \cosh e_j. \quad (4.5)$$

Now we apply Lemma 2 to the vertices $N$ and $P_j$:

$$\rho_j^2 + z_j^2 + t^2 = 2tz_j \cosh e_j. \quad (4.6)$$
Subtracting (4.5) from (4.6), under the assumption that \( t \cosh e_j \neq \cosh e_j' \), we get

\[
    z_j = \frac{t^2 - 1}{2(t \cosh e_j - \cosh e_j')}. \tag{4.7}
\]

Also, taking (4.5) and (4.7) into account, we obtain

\[
    \rho_j^2 = 2z_j \cosh e_j' - z_j^2 - 1 = \frac{(t^2 - 1) \cosh e_j'}{(t \cosh e_j - \cosh e_j')} - \frac{(t^2 - 1)^2}{4(t \cosh e_j - \cosh e_j')^2} - 1. \tag{4.8}
\]

Let \( \rho_{j,j+1} \) denote the Euclidean distance between the points \( \tilde{P}_j \) and \( \tilde{P}_{j+1} \), \( j = 1, \ldots, V \) (here and below \( \rho_{V,V+1} \) def = \( \rho_{V,1} \)). Applying Lemma 2 to the vertices \( P_j \) and \( P_{j+1} \), we get

\[
    \rho_{j,j+1}^2 = 2z_j z_{j+1} \cosh e_{j,j+1} - z_j^2 - z_{j+1}^2. \tag{4.9}
\]

By Pythagoras’s theorem \( \rho_{j,j+1} \) is related to the Cartesian coordinates of \( \tilde{P}_j \) and \( \tilde{P}_{j+1} \) by the formula

\[
    \rho_{j,j+1} = \sqrt{(x_j + x_{j+1})^2 + (y_j + y_{j+1})^2}. \tag{4.10}
\]

By (3.1) equation (4.10) reduces to

\[
    \rho_{j,j+1}^2 = (x_j^2 + y_j^2) + (x_{j+1}^2 + y_{j+1}^2) - 2(x_j x_{j+1} + y_j y_{j+1})
    = \rho_j^2 + \rho_{j+1}^2 - 2(x_j x_{j+1} + y_j y_{j+1}).
\]

Thus, taking (4.8) and (4.9) into account, the expression \( x_j x_{j+1} + y_j y_{j+1} \), which is a part of \( G_{j,j+1} \) in (3.8), is related to the lengths of the edges of \( \mathcal{P} \) by the formula

\[
    x_j x_{j+1} + y_j y_{j+1} = \frac{\rho_j^2 + \rho_{j+1}^2 - \rho_{j,j+1}^2}{2}
    = z_j \cosh e_j' + z_{j+1} \cosh e_{j+1}' - z_j z_{j+1} \cosh e_{j,j+1} - 1. \tag{4.11}
\]

Substituting (4.7) in (4.11) we get

\[
    x_j x_{j+1} + y_j y_{j+1} = \frac{1}{2} \left( \frac{(t^2 - 1) \cosh e_j'}{(t \cosh e_j - \cosh e_j')} + \frac{(t^2 - 1) \cosh e_{j+1}'}{(t \cosh e_{j+1} - \cosh e_{j+1}')}
    - \frac{(t^2 - 1)^2 \cosh e_{j,j+1}}{(t \cosh e_j - \cosh e_j')(t \cosh e_{j+1} - \cosh e_{j+1}')} - 2 \right). \tag{4.12}
\]

Now we express \( x_j y_{j+1} - y_j x_{j+1} \), which is also a part of \( G_{j,j+1} \), in terms of the length of the edges of \( \mathcal{P} \).

According to (3.4),

\[
    \cos \theta_{j,j+1} = \frac{x_j x_{j+1} + y_j y_{j+1}}{\rho_j \rho_{j+1}}, \quad \sin \theta_{j,j+1} = \frac{x_j y_{j+1} - y_j x_{j+1}}{\rho_j \rho_{j+1}}. \tag{4.13}
\]

Note that by definition (3.1), \( \rho_j > 0, j = 1, \ldots, V \).
By the basic trigonometric identity, the formula
\[ \sin \theta_{j,j+1} = \sigma_{j,j+1} \sqrt{1 - \cos^2 \theta_{j,j+1}} \]
holds true, where \( \sigma_{j,j+1} = 1 \) if \( \sin \theta_{j,j+1} \geq 0 \), and \( \sigma_{j,j+1} = -1 \) if \( \sin \theta_{j,j+1} < 0 \) (recall that \( \theta_{j,j+1} \) is determined in (3.2)). Then (4.13) and (4.14) imply that
\[
x_j y_{j+1} - y_j x_{j+1} = \rho_j \rho_{j+1} \sin \theta_{j,j+1} = \sigma_{j,j+1} \rho_j \rho_{j+1} \sqrt{1 - \cos^2 \theta_{j,j+1}}
\]
\[
= \sigma_{j,j+1} \rho_j \rho_{j+1} \sqrt{1 - \frac{(x_j x_{j+1} + y_j y_{j+1})^2}{\rho_j^2 \rho_{j+1}^2}}
\]
\[
= \sigma_{j,j+1} \sqrt{\rho_j^2 \rho_{j+1}^2 - (x_j x_{j+1} + y_j y_{j+1})^2}.
\] (4.15)
Substituting (4.8) and (4.12) in (4.15) we get
\[
x_j y_{j+1} - y_j x_{j+1} = \sigma_{j,j+1} \left[ \left( \frac{(t^2 - 1) \cosh e_j'}{(t \cosh e_j - \cosh e_j')^2} - \frac{(t^2 - 1)^2}{4(t \cosh e_j - \cosh e_j')^2} - 1 \right)
\]
\[
\times \left( \frac{(t^2 - 1) \cosh e_{j+1}'}{(t \cosh e_{j+1} - \cosh e_{j+1}')^2} - \frac{(t^2 - 1)^2}{4(t \cosh e_{j+1} - \cosh e_{j+1}')^2} - 1 \right)
\]
\[
- \frac{1}{4} \left( \frac{(t^2 - 1) \cosh e_j'}{(t \cosh e_j - \cosh e_j')^2} + \frac{(t^2 - 1) \cosh e_{j+1}'}{(t \cosh e_{j+1} - \cosh e_{j+1}')^2} \right)
\]
\[
- \frac{(t^2 - 1)^2 \cosh e_{j,j+1}}{2(t \cosh e_j - \cosh e_j')(t \cosh e_{j+1} - \cosh e_{j+1}')} - 2 \right)^{1/2}.
\] (4.16)
Substituting (4.8), (4.12) and (4.16) into (3.7) we obtain the flexibility equation for a suspension in terms of the lengths of the edges of \( \mathcal{P} \).

§ 5. Proof of the theorem

In order to prove Theorem 1 we shall study the singular points of the flexibility equation for a suspension.

Assume that a nondegenerate suspension \( \mathcal{P} \) flexes. Then, as we said in § 2, the distance \( l_{NS} \) between the poles of \( \mathcal{P} \) changes during the flex. Let \( t \stackrel{\text{def}}{=} e^{l_{NS}} \) be the parameter of the flex of \( \mathcal{P} \). Identity (3.8) holds true at every moment \( t \) of the flex, while the values of the expressions \( E_{j,j+1}, G_{j,j+1}, \rho_j^2, j = 1, \ldots, V \), which form part of (3.8) vary as \( t \) changes. Here the functions \( G_{j,j+1}(t) = [x_j x_{j+1} + y_j y_{j+1} + 1][t] + i[x_j y_{j+1} - y_j x_{j+1}][t] \) and \( \rho_j^2(t), j = 1, \ldots, V, \) are defined in (4.8), (4.12) and (4.16).

Assume now that for some \( j \in \{1, \ldots, V\} \) the dihedral angle \( \theta_{j,j+1}(t) \) remains constant (the value of \( \theta_{j,j+1}(t) \) can also be zero) as \( t \) changes. In this case the length of the edge \( N(t) S(t) \) of the tetrahedron \( N(t) S(t) P_j(t) P_{j+1}(t) \) must also be constant (all other edges of the tetrahedron are also the edges of \( \mathcal{P}_t \), therefore their lengths are fixed), that is, the value of \( t \) does not change. As we mentioned in § 2, in this case \( \mathcal{P} \) cannot be flexible. Thus we have a contradiction. Therefore, the values of the angles \( \theta_{j,j+1}(t), j = 1, \ldots, V, \) change continuously during the flex.
Hence, there exists an interval \((t_1, t_2)\) such that for all \(t \in (t_1, t_2)\) it is true that \(\theta_{j,j+1}(t) \neq 0\) for every \(j \in \{1, \ldots, V\}\).

We extend both sides of the flexibility equation (3.8) as functions of \(t\) to the whole complex plane \(\mathbb{C}\). By the theorem on the uniqueness of the analytic function [7], the expression (3.8) remains valid.

The analytic functions \(F_{j,j+1}(t), \ j = 1, \ldots, V\), have a finite number of algebraic singular points. Without loss of generality we can assume that none of these points lies in the interval \((t_1, t_2)\). For every \(F_{j,j+1}(t), \ j = 1, \ldots, V\), we choose a single-valued branch \((F_{j,j+1}(t), D)\), where \(D \subset \mathbb{C}\) is an unbounded domain containing \((t_1, t_2)\). Let \(\mathcal{W} \subset D\) be a path connecting \(t_0 \in (t_1, t_2)\) and \(\infty\) such that \(t_0\) is a unique real point of \(\mathcal{W}\). We will calculate the limit of \(F_{j,j+1}(t)\) as \(t \to \infty\) along \(\mathcal{W}\).

Taking (4.8) into account we get

\[
\lim_{t \to \infty} \frac{\rho_{j}^2(t)}{t^2} = \lim_{t \to \infty} \left[ \frac{1}{t^2} \left( \frac{(t^2 - 1) \cosh \sigma}{t \cosh e_j - \cosh \sigma} - \frac{(t^2 - 1)^2}{4(t \cosh e_j - \cosh \sigma)^2} - 1 \right) \right] = -\frac{1}{4 \cosh^2 e_j}. \tag{5.1}
\]

Similarly, from (4.12) we derive

\[
\lim_{t \to \infty} \frac{(x_j x_{j+1} + y_j y_{j+1})}{t^2} = -\frac{\cosh \psi_{j,j+1}}{4 \cosh e_j \cosh \psi_{j,j+1}}. \tag{5.2}
\]

Also from (4.15) and taking (5.1) and (5.2) into account we have

\[
\lim_{t \to \infty} \frac{(x_j y_{j+1} - y_j x_{j+1})^2}{t^4} = \lim_{t \to \infty} \left[ \frac{\rho_{j}^2(t) \rho_{j+1}^2(t) - (x_j x_{j+1} + y_j y_{j+1})^2(t)}{t^4} \right] = \frac{1}{16 \cosh^2 e_j \cosh^2 \psi_{j,j+1}} - \frac{\cosh^2 \psi_{j,j+1}}{16 \cosh^2 e_j \cosh^2 \psi_{j,j+1}} = \frac{1 - \cosh^2 \psi_{j,j+1}}{16 \cosh^2 e_j \cosh^2 \psi_{j,j+1}}.
\]

Hence,

\[
\lim_{t \to \infty} \frac{(x_j y_{j+1} - y_j x_{j+1})}{t^2} = i \sigma_{j,j+1} \sqrt{\frac{\cosh^2 \psi_{j,j+1} - 1}{4 \cosh e_j \cosh \psi_{j,j+1}}}, \tag{5.3}
\]

where \(\sigma_{j,j+1} \in \{+1, -1\}\) is determined by the single-valued branch \((F_{j,j+1}(t), D)\) and the path \(\mathcal{W}\).

By the definition of \(G_{j,j+1}(t)\), and (5.2) and (5.3) we get

\[
\lim_{t \to \infty} \frac{G_{j,j+1}(t)}{t^2} = -\frac{\cosh \psi_{j,j+1} + \sigma_{j,j+1} \sqrt{\cosh^2 \psi_{j,j+1} - 1}}{4 \cosh e_j \cosh \psi_{j,j+1}}. \tag{5.4}
\]

By (5.4) and (5.1), the limit of the left-hand side of (3.8) as \(t \to \infty\) is

\[
\lim_{t \to \infty} \prod_{j=1}^{V} F_{j,j+1}(t) = \lim_{t \to \infty} \prod_{j=1}^{V} \frac{F_{j,j+1}(t)/t^2}{\rho_{j}^2(t)/t^2} = \prod_{j=1}^{V} \left( \cosh \psi_{j,j+1} + \sigma_{j,j+1} \sqrt{\cosh^2 \psi_{j,j+1} - 1} \right),
\]

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and (3.8) as $t \to \infty$ transforms into

$$\prod_{j=1}^V \left( \cosh e_{j,j+1} + \sigma_{j,j+1} \sqrt{\cosh^2 e_{j,j+1} - 1} \right) = 1. \quad (5.5)$$

Using the hyperbolic trigonometric identity $\cosh^2 x - \sinh^2 x = 1$ and the fact that $e_{j,j+1} > 0$, we have

$$\sqrt{\cosh^2 e_{j,j+1} - 1} = \sqrt{\sinh^2 e_{j,j+1}} = \sinh e_{j,j+1}. \quad (5.6)$$

By (5.6) equation (5.5) transforms into

$$\prod_{j=1}^V \left( \cosh e_{j,j+1} + \sigma_{j,j+1} \sinh e_{j,j+1} \right) = 1. \quad (5.7)$$

As $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$, we have

$$\cosh e_{j,j+1} + \sigma_{j,j+1} \sinh e_{j,j+1} = \begin{cases} e^{e_{j,j+1}} & \text{for } \sigma_{j,j+1} = 1, \\ e^{-e_{j,j+1}} & \text{for } \sigma_{j,j+1} = -1 \end{cases} = e^{\sigma_{j,j+1} e_{j,j+1}}. \quad (5.8)$$

Substituting (5.8) in (5.5) and taking the logarithm of the resulting equation we obtain

$$\sum_{j=1}^V \sigma_{j,j+1} e_{j,j+1} = 0. \quad (5.9)$$

**Remark 1.** Studying the behaviour of the flexibility equation (3.8) in neighbourhoods of other singular points of the left-hand side of (3.8) gave us no interesting results: either we obtained trivial identities such as $1 = 1$ (for instance, as $t \to \pm 1$), or the limit of the left-hand side of the flexibility equation was too complicated to distinguish interesting patterns in it.

§6. The verification of the necessary flexibility condition for a nondegenerate suspension for the Bricard-Stachel octahedra in Lobachevsky 3-space

In 2002 Stachel [8] proved the flexibility of the analogues of Bricard’s octahedra in Lobachevsky 3-space. We shall verify that the necessary flexibility condition holds for a nondegenerate suspension for the Bricard-Stachel octahedra in Lobachevsky 3-space.

We define an octahedron $\vartheta$ as the suspension $NABCDS$ with poles $N$ and $S$, and with the vertices of the equator $A$, $B$, $C$, and $D$. Note that we can consider the vertices $A$ and $C$ as the poles of $\vartheta$ (in which case the quadrilateral $NDSB$ serves as the equator of $\vartheta$), or the vertices $B$ and $D$ (in which case the quadrilateral $NCSA$ serves as the equator of $\vartheta$).
6.1. Bricard-Stachel octahedra of types 1 and 2. The procedure for constructing Bricard-Stachel octahedra of types 1 and 2 in Lobachevsky 3-space is the same as for Bricard octahedra of types 1 and 2 in Euclidean 3-space [8], [9].

Any Bricard-Stachel octahedron of type 1 in $\mathbb{H}^3$ can be constructed in the following way. Consider a disk-homeomorphic piecewise linear surface $\mathcal{S}$ in $\mathbb{H}^3$ composed of four triangles $ABN$, $BCN$, $CDN$, and $DAN$ such that $d_{\mathbb{H}^3}(A,B) = d_{\mathbb{H}^3}(C,D)$ and $d_{\mathbb{H}^3}(B,C) = d_{\mathbb{H}^3}(D,A)$. It is known that a spatial quadrilateral $ABCD$ whose opposite sides have the same lengths is symmetric with respect to a line $L$ passing through the mid-points of its diagonals $AC$ and $BD$ (see Fig. 7; for a more precise analogy with the Euclidean case, in this figure as well as in the following figures we draw polyhedra in the Kleinian model of Lobachevsky space where lines and planes are intersections of Euclidean lines and planes with a fixed unit ball). Glue $\mathcal{S}$ to its symmetric image with respect to $L$ along $ABCD$. Denote the symmetric image of $N$ under the symmetry with respect to $L$ by $S$ (see Fig. 8). The resulting polyhedral surface $NABCDS$ with self-intersections is flexible (because $\mathcal{S}$ is flexible) and combinatorially it is an octahedron. We will call it a Bricard-Stachel octahedron of type 1. By construction it follows that $d_{\mathbb{H}^3}(A,N) = d_{\mathbb{H}^3}(C,S)$, $d_{\mathbb{H}^3}(B,N) = d_{\mathbb{H}^3}(D,S)$, $d_{\mathbb{H}^3}(C,N) = d_{\mathbb{H}^3}(A,S)$ and $d_{\mathbb{H}^3}(D,N) = d_{\mathbb{H}^3}(B,S)$.

Any Bricard-Stachel octahedron of type 2 in $\mathbb{H}^3$ can be constructed as follows. Consider a disk-homeomorphic piecewise linear surface $\mathcal{S}$ in $\mathbb{H}^3$ composed of four triangles $ABN$, $BCN$, $CDN$ and $DAN$ such that $d_{\mathbb{H}^3}(A,B) = d_{\mathbb{H}^3}(B,C)$ and $d_{\mathbb{H}^3}(C,D) = d_{\mathbb{H}^3}(D,A)$. It is known that a spatial quadrilateral $ABCD$ whose neighbouring sides at the vertices $B$ and $D$ have the same lengths is symmetric with respect to a plane $H$ which dissects the dihedral angle between the half-planes $ABD$ and $CBD$ (see Fig. 9). Glue $\mathcal{S}$ to its symmetric image with respect to $H$ along $ABCD$. Denote the symmetric image of $N$ under the symmetry with respect to $H$ by $S$ (see Fig. 10). The resulting polyhedral surface $NABCDS$ with self-intersections is flexible (because $\mathcal{S}$ is flexible) and combinatorially it is an octahedron. We will call it a Bricard-Stachel octahedron of type 2. By construction it follows that $d_{\mathbb{H}^3}(A,N) = d_{\mathbb{H}^3}(C,S)$, $d_{\mathbb{H}^3}(C,N) = d_{\mathbb{H}^3}(A,S)$, $d_{\mathbb{H}^3}(B,N) = d_{\mathbb{H}^3}(B,S)$ and $d_{\mathbb{H}^3}(D,N) = d_{\mathbb{H}^3}(D,S)$.
6.2. Bricard-Stachel octahedra of type 3. There are three subtypes of Bricard-Stachel octahedra of type 3 in Lobachevsky space [8]; their constructions are based on circles, horocycles or hypercircles, respectively. The process of construction is common to all the subtypes of the Bricard-Stachel octahedra of type 3 and is the same as for the Bricard’s octahedra of type 3 in Euclidean space.

Any Bricard-Stachel octahedron of type 3 in $\mathbb{H}^3$ can be constructed in the following way. Let $K_{AC}$ and $K_{AB}$ be two distinct circles (horocycles, hypercircles) in $\mathbb{H}^2$ with a common centre $M$ and let $A_1$ and $A_2$ be two distinct finite points outside $K_{AC}$ and $K_{AB}$. Suppose in addition that $K_{AC}$, $K_{AB}$, $A_1$ and $A_2$ are taken in such a way that the straight lines tangent to $K_{AB}$ and passing through $A_1$ and $A_2$ intersect pairwise at finite points of $\mathbb{H}^2$ and form a quadrilateral $A_1B_1A_2B_2$ tangent to $K_{AB}$, and also that the straight lines tangent to $K_{AC}$ and passing through $A_1$ and $A_2$ intersect pairwise at finite points of $\mathbb{H}^2$ and form a quadrilateral $A_1C_1A_2C_2$ tangent to $K_{AC}$ (see Fig. 11; for clarity, we have placed the circles $K_{AB}$ and $K_{AC}$ so that their common centre coincides with the centre of the Kleinian model of Lobachevsky space; in this case $K_{AB}$ and $K_{AC}$ are Euclidean circles as well). The polyhedron $\vartheta$ with vertices $A_i$, $B_j$, $C_k$, edges $A_iB_j$, $A_iC_k$, $B_jC_k$, and faces $\triangle A_iB_jC_k$, $i, j, k \in \{1, 2\}$, is an octahedron in the sense of the definition given above (see Fig. 12). The following pairs of vertices can serve as the poles of $\vartheta$: $(A_1, A_2)$ with corresponding equator $B_1C_1B_2C_2$; $(B_1, B_2)$ with equator $A_1C_1A_2C_2$; and $(C_1, C_2)$ with equator $A_1B_1A_2B_2$. Suppose in addition that $\vartheta$ does not have symmetries. We will call such an octahedron $\vartheta$ a Bricard-Stachel octahedron of type 3.
Figure 11. The construction of a Bricard-Stachel octahedron of type 3 based on circles. Step 1.

Figure 12. The construction of a Bricard-Stachel octahedron of type 3 based on circles. Step 2.
According to Stachel [8], $\mathcal{O}$ flexes continuously in $\mathbb{H}^3$. Moreover, $\mathcal{O}$ admits two flat positions during the flex (we have constructed $\mathcal{O}$ in one of these). Hence, for every equator of $\mathcal{O}$, $A_1B_1A_2B_2$, $B_1C_1B_2C_2$ and $A_1C_1A_2C_2$, all straight lines containing a side of the equator are tangent to some circle (horocycle, hypercircle) at least in one flat position of $\mathcal{O}$. Using this fact, we will prove that Theorem 1 is valid for the Bricard-Stachel octahedra of type 3. We have to consider three possible cases: when an equator of $\mathcal{O}$ is tangent to a circle, a horocycle, or a hypercircle in $\mathbb{H}^2$. Here we look at the most common situation, when no three vertices of an equator of a flexible octahedron in its flat position lie on a straight line.

6.2.1. An equator of a Bricard-Stachel octahedron of type 3 is tangent to a circle in $\mathbb{H}^2$. Let $M$ be the centre of the circle $K_{AB}$ with radius $R$ in $\mathbb{H}^2$ and let all straight lines containing a side of the quadrilateral $A_1B_1A_2B_2$ be tangent to $K_{AB}$. We draw segments $M_P_1$, $M_P_2$, $M_P_3$, $M_P_4$ connecting $M$ with the straight lines $A_1B_2$, $A_2B_2$, $A_2B_1$, $A_1B_1$ and which are perpendicular to the corresponding lines. By construction, $d_{\mathbb{H}^2}(M, P_1) = d_{\mathbb{H}^2}(M, P_2) = d_{\mathbb{H}^2}(M, P_3) = d_{\mathbb{H}^2}(M, P_4) = R$.

By Pythagoras’s theorem for a Lobachevsky space [10] applied to $\triangle A_1MP_1$ and $\triangle A_1MP_4$, we obtain:

$$\cosh d_{\mathbb{H}^2}(A_1, P_1) = \cosh d_{\mathbb{H}^2}(A_1, P_4) = \cosh d_{\mathbb{H}^2}(A_1, M)/\cosh R.$$

Then $a \overset{\text{def}}{=} d_{\mathbb{H}^2}(A_1, P_1) = d_{\mathbb{H}^2}(A_1, P_4)$. Similarly we get: $b \overset{\text{def}}{=} d_{\mathbb{H}^2}(B_2, P_1) = d_{\mathbb{H}^2}(B_2, P_2)$, $c \overset{\text{def}}{=} d_{\mathbb{H}^2}(A_2, P_2) = d_{\mathbb{H}^2}(A_2, P_3)$ and $d \overset{\text{def}}{=} d_{\mathbb{H}^2}(B_1, P_3) = d_{\mathbb{H}^2}(B_1, P_4)$.

If the circle $K_{AB}$ is inscribed in the quadrilateral $A_1B_1A_2B_2$ (see Fig. 11), then $d_{\mathbb{H}^2}(A_1, B_2) = a + b$, $d_{\mathbb{H}^2}(A_2, B_2) = b + c$, $d_{\mathbb{H}^2}(A_2, B_1) = c + d$, $d_{\mathbb{H}^2}(A_1, B_1) = a + d$ and the identity

$$d_{\mathbb{H}^2}(A_1, B_2) - d_{\mathbb{H}^2}(A_2, B_2) + d_{\mathbb{H}^2}(A_1, B_1) - d_{\mathbb{H}^2}(A_1, B_1) = 0 \quad (6.1)$$

holds true.

If the circle $K_{AB}$ is tangent to the quadrilateral $A_1B_1A_2B_2$ externally (this case corresponds to the quadrilateral $A_1C_1A_2C_2$ and to the circle $K_{AC}$ in Fig. 12), then $d_{\mathbb{H}^2}(A_1, B_2) = a - b$, $d_{\mathbb{H}^2}(A_2, B_2) = b + c$, $d_{\mathbb{H}^2}(A_2, B_1) = c - d$, $d_{\mathbb{H}^2}(A_1, B_1) = a + d$ and the identity

$$d_{\mathbb{H}^2}(A_1, B_2) + d_{\mathbb{H}^2}(A_2, B_2) - d_{\mathbb{H}^2}(A_1, B_1) - d_{\mathbb{H}^2}(A_1, B_1) = 0 \quad (6.2)$$

holds true.

By (6.1) and (6.2), Theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a circle in at least one of its flat positions.

6.2.2. An equator of a Bricard-Stachel octahedron of type 3 is tangent to a horocycle in $\mathbb{H}^2$. Consider the Poincaré upper half-plane model of the Lobachevsky plane $\mathbb{H}^2$ with coordinates $(\rho, z)$ (that is, with the metric given by the formula $ds^2 = (d\rho^2 + dz^2)/z^2$). Without loss of generality we can assume that the centre of the horocycle tangent to the equator of a Bricard-Stachel octahedron $\mathcal{O}$ of type 3 coincides with $\infty$, the (unique) point at infinity of $\mathbb{H}^2$ which does not lie on the Euclidean line $z = 0$. We denote the family of such horocycles by $K = \{\rho = R \mid R > 0\}$. Let $K_R \in K$ and let $A_1 = (\rho_{A_1}, z_{A_1})$ and $A_2 = (\rho_{A_2}, z_{A_2})$
be two opposite vertices of $\mathcal{O}$ such that the straight line (in $\mathbb{H}^2$) passing through $A_1$ and $A_2$ is not tangent to $K_R$. All the vertices of $\mathcal{O}$ are located outside $K_R$, hence $z_{A_1} < R$ and $z_{A_2} < R$. We will construct all possible quadrangles tangent to $K_R$ with the opposite vertices $A_1$ and $A_2$, that is, all quadrangles that can serve as equators of $\mathcal{O}$. Then we will verify that Theorem 1 holds for such quadrangles.

Let $T = (\rho_T, z_T)$ be a point in $\mathbb{H}^2$ and let $\Lambda$ be a straight line in $\mathbb{H}^2$ passing through $T$ which is realized in the Poincaré upper half-plane as the Euclidean semi-circle with radius $\sqrt{(\rho_T - \rho_T, \Lambda)^2 + z_T^2}$ and centre $O^{T}_{\Lambda} = (\rho_T, \Lambda, 0)$. Then the angle $\varphi_{T}^{\Lambda} \overset{\mathrm{def}}{=} \angle TO_{\Lambda}^{T} \rho \in (0, \pi)$ determines the position of $T$ on $\Lambda$ uniquely.

**Remark 2.** For every finite point $T = (\rho_T, z_T)$, $z_T < R$, there exist precisely two straight lines $\Lambda_{l}^{T}$ and $\Lambda_{r}^{T}$ tangent to the horocycle $K_R$ and containing $T$. They are realized in the Poincaré upper half-plane as the Euclidean semi-circles with radius $R$ and centres $O^{T}_{l} = (\rho_{T,l}, 0)$ and $O^{T}_{r} = (\rho_{T,r}, 0)$, $\rho_{T,l} \leq \rho_T \leq \rho_{T,r}$. The angles $\varphi_{T}^{l} \overset{\mathrm{def}}{=} \angle TO_{l}^{T} \rho$ and $\varphi_{T}^{r} \overset{\mathrm{def}}{=} \angle TO_{r}^{T} \rho$ serve as the coordinates of $T$ on $\Lambda_{l}^{T}$ and $\Lambda_{r}^{T}$, respectively. Then by construction and Pythagoras’s theorem for the Euclidean plane we see that $\varphi_{T}^{r} = \pi - \varphi_{T}^{l}$. Hence,

$$\cos \varphi_{T}^{r} = -\cos \varphi_{T}^{l}. \quad (6.3)$$

According to Remark 2, there are precisely two straight lines $\Lambda_{l}^{A_1}$ and $\Lambda_{r}^{A_1}$ passing through $A_1$ and tangent to $K_R$; these are realized in $\mathbb{H}^2$ as Euclidean semi-circles with radius $R$ and centres $O_{l}^{A_1} = (\rho_{A_1,l}, 0)$ and $O_{r}^{A_1} = (\rho_{A_1,r}, 0)$, $\rho_{A_1,l} \leq \rho_{A_1} \leq \rho_{A_1,r}$. The angles $\varphi_{A_1}^{l} \overset{\mathrm{def}}{=} \angle A_1 O_{l}^{A_1} \rho$ and $\varphi_{A_1}^{r} \overset{\mathrm{def}}{=} \angle A_1 O_{r}^{A_1} \rho$ serve as the coordinates of $A_1$ on $\Lambda_{l}^{A_1}$ and $\Lambda_{r}^{A_1}$, respectively. Moreover,

$$\cos \varphi_{A_1}^{l} = -\cos \varphi_{A_1}^{r}. \quad (6.4)$$

Similarly, there are exactly two straight lines $\Lambda_{l}^{A_2}$ and $\Lambda_{r}^{A_2}$ passing through $A_2$ and tangent to $K_R$; these are realized in $\mathbb{H}^2$ as Euclidean semi-circles with radius $R$ and centres $O_{l}^{A_2} = (\rho_{A_2,l}, 0)$ and $O_{r}^{A_2} = (\rho_{A_2,r}, 0)$, $\rho_{A_2,l} \leq \rho_{A_2} \leq \rho_{A_2,r}$. The angles $\varphi_{A_2}^{l} \overset{\mathrm{def}}{=} \angle A_2 O_{l}^{A_2} \rho$ and $\varphi_{A_2}^{r} \overset{\mathrm{def}}{=} \angle A_2 O_{r}^{A_2} \rho$ serve as the coordinates of $A_2$ on $\Lambda_{l}^{A_2}$ and $\Lambda_{r}^{A_2}$, respectively. Moreover,

$$\cos \varphi_{A_2}^{l} = -\cos \varphi_{A_2}^{r}. \quad (6.5)$$

Suppose that $\Lambda_{l}^{A_1}$ and $\Lambda_{l}^{A_2}$ intersect at a point $B_1$. Then the angles $\varphi_{B_1}^{A_1} \overset{\mathrm{def}}{=} \angle B_1 O_{l}^{A_1} \rho$ and $\varphi_{B_1}^{A_2} \overset{\mathrm{def}}{=} \angle B_1 O_{l}^{A_2} \rho$ serve as the coordinates of $B_1$ on $\Lambda_{l}^{A_1}$ and $\Lambda_{l}^{A_2}$, respectively. Moreover,

$$\cos \varphi_{B_1}^{A_2} = -\cos \varphi_{B_1}^{A_1}. \quad (6.6)$$

Suppose, also, that $\Lambda_{r}^{A_1}$ and $\Lambda_{r}^{A_2}$ intersect at a point $B_2$. Then the angles $\varphi_{B_2}^{A_1} \overset{\mathrm{def}}{=} \angle B_2 O_{r}^{A_1} \rho$ and $\varphi_{B_2}^{A_2} \overset{\mathrm{def}}{=} \angle B_2 O_{r}^{A_2} \rho$ serve as the coordinates of $B_2$ on $\Lambda_{r}^{A_1}$ and $\Lambda_{r}^{A_2}$, respectively. Moreover,

$$\cos \varphi_{B_2}^{A_2} = -\cos \varphi_{B_2}^{A_1}. \quad (6.7)$$
Let the straight lines $\Lambda_{r}^{A_1}$ and $\Lambda_{r}^{A_2}$ intersect at a point $C_1$. Then the angles $\varphi_{C_1}^{A_1} = \angle C_1 O_{r}^{A_1} \rho$ and $\varphi_{C_1}^{A_2} = \angle C_1 O_{r}^{A_2} \rho$ serve as the coordinates of $C_1$ on $\Lambda_{r}^{A_1}$ and $\Lambda_{r}^{A_2}$, respectively. Moreover,

$$\cos \varphi_{C_1}^{A_1} = - \cos \varphi_{C_1}^{A_2}. \quad (6.8)$$

Also, let the straight lines $\Lambda_{l}^{A_1}$ and $\Lambda_{l}^{A_2}$ intersect at a point $C_2$. Then the angles $\varphi_{C_2}^{A_1} = \angle C_2 O_{l}^{A_1} \rho$ and $\varphi_{C_2}^{A_2} = \angle C_2 O_{l}^{A_2} \rho$ serve as the coordinates of $C_2$ on $\Lambda_{l}^{A_1}$ and $\Lambda_{l}^{A_2}$, respectively. Moreover,

$$\cos \varphi_{C_2}^{A_2} = - \cos \varphi_{C_2}^{A_1}. \quad (6.9)$$

By construction, the quadrangles $A_1 B_1 A_2 B_2$ and $A_1 C_1 A_2 C_2$ are tangent to $K_R$, and the points $A_1, A_2$ are opposite vertices of each of these quadrangles. In order to verify that Theorem 1 holds for the flexible octahedra with the equator $A_1 B_1 A_2 B_2$ or $A_1 C_1 A_2 C_2$ we need to prove the following easy statement.

**Lemma 3.** Given a Poincaré upper half-plane $\mathbb{H}^2$ with coordinates $(\rho, z)$ (that is, with the metric given by the formula $ds^2 = (d\rho^2 + dz^2)/z^2$). Let $A$ and $B$ be points on the straight line $\Lambda$ realized in $\mathbb{H}^2$ as the Euclidean semi-circle with radius $R$ and centre $O_{\Lambda} = (\rho_{O_{\Lambda}}, 0)$, and let the angles $\varphi_A = \angle O_{\Lambda} \rho$ and $\varphi_B = \angle B \rho$ serve as the coordinates of $A$ and $B$, respectively, on $\Lambda$. Assume further that $0 < \varphi_A \leq \varphi_B < \pi$. Then the distance between $A$ and $B$ is calculated as follows:

$$d_{\mathbb{H}^2}(A, B) = \frac{1}{2} \log \left( \frac{1 + \cos \varphi_A}{1 + \cos \varphi_B} \right) \left( \frac{1 - \cos \varphi_B}{1 - \cos \varphi_A} \right). \quad (6.10)$$

**Proof.** The hyperbolic segment $\Lambda_{AB}$ connecting the points $A$ and $B$ is specified parametrically by the formulae $\Lambda_{AB}(t) : (\rho(\varphi), z(\varphi)), \varphi \in [\varphi_A, \varphi_B]$, where $\rho(\varphi) = \rho_{O_{\Lambda}} + R \cos \varphi, z(\varphi) = R \sin \varphi, A = \Lambda_{AB}(\varphi_A), B = \Lambda_{AB}(\varphi_B)$. The direct calculation shows that the length of $\Lambda_{AB}$ is equal to the right-hand side of (6.10)).

By Lemma 3, the lengths of the edges of the quadrilateral $A_1 B_1 A_2 B_2$ are calculated as follows:

$$d_{\mathbb{H}^2}(A_1, B_1) = \frac{1}{2} \log \left( \frac{1 + \cos \varphi_{A_1}^{A_1}}{1 + \cos \varphi_{B_1}^{A_1}} \right) \left( \frac{1 - \cos \varphi_{B_1}^{A_1}}{1 - \cos \varphi_{A_1}^{A_1}} \right), \quad (6.11)$$

$$d_{\mathbb{H}^2}(A_2, B_1) = \frac{1}{2} \log \left( \frac{1 + \cos \varphi_{A_2}^{A_2}}{1 + \cos \varphi_{B_1}^{A_2}} \right) \left( \frac{1 - \cos \varphi_{B_1}^{A_2}}{1 - \cos \varphi_{A_2}^{A_2}} \right), \quad (6.12)$$

$$d_{\mathbb{H}^2}(B_2, A_1) = \frac{1}{2} \log \left( \frac{1 + \cos \varphi_{B_2}^{A_1}}{1 + \cos \varphi_{A_1}^{A_1}} \right) \left( \frac{1 - \cos \varphi_{A_1}^{A_1}}{1 - \cos \varphi_{B_2}^{A_1}} \right), \quad (6.13)$$

$$d_{\mathbb{H}^2}(B_2, A_2) = \frac{1}{2} \log \left( \frac{1 + \cos \varphi_{B_2}^{A_2}}{1 + \cos \varphi_{A_2}^{A_2}} \right) \left( \frac{1 - \cos \varphi_{A_2}^{A_2}}{1 - \cos \varphi_{B_2}^{A_2}} \right). \quad (6.14)$$
Then, by (6.4)–(6.7), we see that
\[ d_{\mathbb{H}^2}(A_1, B_1) + d_{\mathbb{H}^2}(A_2, B_1) - d_{\mathbb{H}^2}(B_2, A_1) - d_{\mathbb{H}^2}(B_2, A_2) = 0. \tag{6.15} \]

By Lemma 3, the lengths of the edges of the quadrilateral \( A_1C_1A_2C_2 \) are calculated as follows:

\[ d_{\mathbb{H}^2}(C_1, A_1) = \frac{1}{2} \log \left[ \frac{1 + \cos \varphi_{C_1}^{A_1}}{1 + \cos \varphi_{A_1}^{A_1}} \cdot \frac{1 - \cos \varphi_{A_1}^{A_1}}{1 - \cos \varphi_{C_1}^{A_1}} \right], \tag{6.16} \]

\[ d_{\mathbb{H}^2}(C_2, A_1) = \frac{1}{2} \log \left[ \frac{1 + \cos \varphi_{C_2}^{A_1}}{1 + \cos \varphi_{A_1}^{A_1}} \cdot \frac{1 - \cos \varphi_{A_1}^{A_1}}{1 - \cos \varphi_{C_2}^{A_1}} \right], \tag{6.17} \]

\[ d_{\mathbb{H}^2}(A_2, C_1) = \frac{1}{2} \log \left[ \frac{1 + \cos \varphi_{A_2}^{A_2}}{1 + \cos \varphi_{C_1}^{A_2}} \cdot \frac{1 - \cos \varphi_{C_1}^{A_2}}{1 - \cos \varphi_{A_2}^{A_2}} \right], \tag{6.18} \]

\[ d_{\mathbb{H}^2}(A_2, C_2) = \frac{1}{2} \log \left[ \frac{1 + \cos \varphi_{A_2}^{A_2}}{1 + \cos \varphi_{C_2}^{A_2}} \cdot \frac{1 - \cos \varphi_{C_2}^{A_2}}{1 - \cos \varphi_{A_2}^{A_2}} \right]. \tag{6.19} \]

Using (6.4), (6.5), (6.8) and (6.9) it is easy to verify that
\[ d_{\mathbb{H}^2}(C_2, A_1) + d_{\mathbb{H}^2}(C_1, A_1) - d_{\mathbb{H}^2}(A_2, C_1) - d_{\mathbb{H}^2}(A_2, C_2) = 0. \tag{6.20} \]

Now, by (6.15) and (6.20), Theorem 1 holds for any equator of a Bricard-Stachel octahedron of type 3 tangent to a horocycle in at least one of its flat positions.

6.2.3. An equator of a Bricard-Stachel octahedron of type 3 is tangent to a horocircle in \( \mathbb{H}^2 \). Consider the Poincaré upper half-plane model of Lobachevsky plane \( \mathbb{H}^2 \) with coordinates \((\rho, z)\) (that is, with the metric given by the formula \( ds^2 = (d\rho^2 + dz^2)/z^2 \)). Without loss of generality we can assume that the hypercircle tangent to the equator of a Bricard-Stachel octahedron \( O \) of type 3 passes through \( \infty \), the (unique) point at infinity of \( \mathbb{H}^2 \) which does not lie on the Euclidean line \( z = 0 \), and through the point \( O = (0, 0) \) at infinity of \( \mathbb{H}^2 \). Every such hypercircle is specified by the equation \( z = \rho \tan \alpha \) for some \( \alpha \in (0, \pi/2) \) \( \cup \) \( (\pi/2, \pi) \). Because \( \mathbb{H}^2 \) is symmetric with respect to the straight line \( \rho = 0 \), it is sufficient to consider the family of hypercircles \( K = \{ z = \rho \tan \alpha \mid \alpha \in (0, \pi/2) \} \). Let \( K_\alpha \in K \). We will construct all possible quadrangles tangent to \( K_\alpha \) such that none of their vertices belong to \( K_\alpha \), that is, all quadrangles which can serve as equators of \( O \). Then we will verify that Theorem 1 holds for such quadrangles.

We will study the quadrangles based on the straight lines \( \Lambda_1^{A_1}, \Lambda_2^{A_1}, \Lambda_1^{A_2} \) and \( \Lambda_2^{A_2} \), tangent to \( K_\alpha \), which are realized in \( \mathbb{H}^2 \) as the Euclidean semi-circles with the centres \( O_1^{A_1} = (\rho_{A_1}, l, 0), O_2^{A_1} = (\rho_{A_1}, r, 0), O_1^{A_2} = (\rho_{A_2}, l, 0) \) and \( O_2^{A_2} = (\rho_{A_2}, r, 0) \). Also let \( \Lambda_1^{A_1} \) and \( \Lambda_2^{A_1} \) intersect at a point \( A_1 \), and let \( \Lambda_1^{A_2} \) and \( \Lambda_2^{A_2} \) intersect at a point \( A_2 \). Assume that \( A_1 \) and \( A_2 \) are two opposite vertices of \( O \) and, for definiteness, that \( 0 < \rho_{A_1}, l < \rho_{A_1}, r \) and \( 0 < \rho_{A_2}, l < \rho_{A_2}, r \).

**Remark 3.** Let \( T = (\rho_T, z_T) \) be a point in \( \mathbb{H}^2 \) at the intersection of the straight lines \( \Lambda_1^T \) and \( \Lambda_2^T \) tangent to a horocircle \( K_\alpha \), and let \( \Lambda_1^T \) and \( \Lambda_2^T \) be realized
in $\mathbb{H}^2$ as Euclidean semi-circles with centres $O_l^T = (\rho_T, 0)$ and $O_r^T = (\rho_T, 0)$ ($\rho_T < \rho_T$). Then, by Remark 2, the angles $\varphi_T^l \overset{\text{def}}{=} \angle TO_l^T \rho$ and $\varphi_T^r \overset{\text{def}}{=} \angle TO_r^T \rho$ uniquely determine the position of $T$ on $\Lambda_l^T$ and $\Lambda_r^T$, respectively. Moreover,

$$\cos \varphi_T^l = \frac{\rho_T}{\rho_T - \rho_T} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2},$$

$$\cos \varphi_T^r = \frac{\rho_T}{\rho_T - \rho_T} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \tag{6.21}$$

**Proof.** Since $\Lambda_l^T$ and $\Lambda_r^T$ are tangent to $K_\alpha$, the radii $R_l$ and $R_r$ of the semi-circles realizing $\Lambda_l^T$ and $\Lambda_r^T$ in $\mathbb{H}^2$ are determined by the formulae

$$R_l = \rho_T \sin \alpha, \quad R_r = \rho_T \sin \alpha. \tag{6.22}$$

Let $T_\infty$ be the point with coordinates $(\rho_T, 0)$. Applying the Euclidean Pythagoras’s theorem to $\triangle TT_\infty O_l^T$ and simplifying the expression obtained, we get

$$\rho_T^2 + z_T^2 = 2\rho_T \rho_T - \rho^2_T \cos^2 \alpha. \tag{6.23}$$

Similarly, from $\triangle TT_\infty O_r^T$ we find that

$$\rho_T^2 + z_T^2 = 2\rho_T \rho_T \rho_T - \rho^2_T \cos^2 \alpha. \tag{6.24}$$

Subtracting (6.23) from (6.24) we can easily deduce:

$$\rho_T = \frac{\rho_T + \rho_T}{2} \cos^2 \alpha. \tag{6.25}$$

From the definitions of the cosines $\cos \varphi_T^l = (\rho_T - \rho_T)/R_l$ and $\cos \varphi_T^r = (\rho_T - \rho_T)/R_r$ and taking (6.22) and (6.25) into account we obtain (6.21).

By Remark 3, the angles $\varphi_{A_1}^{A_1} \overset{\text{def}}{=} \angle A_1 O_l^{A_1} \rho$ and $\varphi_{A_1}^{A_1} \overset{\text{def}}{=} \angle A_1 O_r^{A_1} \rho$ uniquely determine the position of $A_1$ on $\Lambda_l^{A_1}$ and $\Lambda_r^{A_1}$, respectively. Moreover,

$$\cos \varphi_{A_1}^{A_1} = \frac{\rho_{A_1}}{\rho_{A_1} - \rho_{A_1}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2},$$

$$\cos \varphi_{A_1}^{A_1} = \frac{\rho_{A_1}}{\rho_{A_1} - \rho_{A_1}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \tag{6.26}$$

Similarly, the angles $\varphi_{A_2}^{A_2} \overset{\text{def}}{=} \angle A_2 O_l^{A_2} \rho$ and $\varphi_{A_2}^{A_2} \overset{\text{def}}{=} \angle A_2 O_r^{A_2} \rho$ serve as the coordinates of $A_2$ on $\Lambda_l^{A_2}$ and $\Lambda_r^{A_2}$, respectively. Moreover,

$$\cos \varphi_{A_2}^{A_2} = \frac{\rho_{A_2}}{\rho_{A_2} - \rho_{A_2}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2},$$

$$\cos \varphi_{A_2}^{A_2} = \frac{\rho_{A_2}}{\rho_{A_2} - \rho_{A_2}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2 \sin \alpha} - \frac{\sin \alpha}{2}. \tag{6.27}$$
Suppose that the straight lines $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point $B_1$. Then the angles $\varphi_{B_1}^{A_1} = \angle B_1 O_l^{A_1} \rho$ and $\varphi_{B_1}^{A_2} = \angle B_1 O_l^{A_2} \rho$ serve as the coordinates of $B_1$ on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$, respectively. Moreover,

$$\cos \varphi_{B_1}^{A_1} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2},$$

$$\cos \varphi_{B_1}^{A_2} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2}. \quad (6.28)$$

Suppose also that $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$ intersect at a point $B_2$. Then the angles $\varphi_{B_2}^{A_1} = \angle B_2 O_r^{A_1} \rho$ and $\varphi_{B_2}^{A_2} = \angle B_2 O_r^{A_2} \rho$ serve as the coordinates of $B_2$ on $\Lambda_r^{A_1}$ and $\Lambda_r^{A_2}$, respectively. Moreover,

$$\cos \varphi_{B_2}^{A_1} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2},$$

$$\cos \varphi_{B_2}^{A_2} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2}. \quad (6.29)$$

Suppose that $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point $C_1$. Then the angles $\varphi_{C_1}^{A_1} = \angle C_1 O_l^{A_1} \rho$ and $\varphi_{C_1}^{A_2} = \angle C_1 O_l^{A_2} \rho$ serve as the coordinates of $C_1$ on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$, respectively. Moreover,

$$\cos \varphi_{C_1}^{A_2} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2},$$

$$\cos \varphi_{C_1}^{A_1} = \frac{\rho_{A_2,l}}{\rho_{A_1,l}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2}. \quad (6.30)$$

Suppose also that $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$ intersect at a point $C_2$. Then the angles $\varphi_{C_2}^{A_1} = \angle C_2 O_l^{A_1} \rho$ and $\varphi_{C_2}^{A_2} = \angle C_2 O_l^{A_2} \rho$ serve as the coordinates of $C_2$ on $\Lambda_l^{A_1}$ and $\Lambda_l^{A_2}$, respectively. Moreover,

$$\cos \varphi_{C_2}^{A_2} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2},$$

$$\cos \varphi_{C_2}^{A_1} = \frac{\rho_{A_2,r}}{\rho_{A_1,r}} \frac{\cos^2 \alpha}{2 \sin \alpha} - \frac{1}{2} \frac{\sin \alpha}{2}. \quad (6.31)$$

As in the case of quadrangles tangent to a horocycle in $\mathbb{H}^2$, the lengths of the edges of $A_1 B_1 A_2 B_2$ are expressed in (6.11)–(6.14) and the lengths of the edges of $A_1 C_1 A_2 C_2$ are calculated in (6.16)–(6.19). Taking (6.26)–(6.31) into account, it is easy to state the validity of (6.15) and (6.20).

According to (6.15) and (6.20), Theorem 1 is valid for any equator of a Bricard-Stachel octahedron of type 3 tangent to a hypercircle in at least one of its flat positions.
The case when three vertices of an equator of a flexible octahedron in its flat position lie on a straight line is similar. The case when all four vertices of an equator lie on a straight line is trivial.

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