Frenkel electron on an arbitrary electromagnetic background and magnetic Zitterbewegung

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We present Lagrangian which implies both necessary constraints and dynamical equations for position and spin of relativistic spin one-half particle. The model is consistent for any value of magnetic moment \( \mu \) and for arbitrary electromagnetic background. Our equations coincide with those of Frenkel in the approximation in which the latter have been obtained by Frenkel. Transition from approximate to exact equations yields two structural modifications of the theory. First, Frenkel condition on spin-tensor turns into the Pirani condition. Second, canonical momentum is no more proportional to velocity. Due to this, even when \( \mu = 1 \) (Frenkel case), the complete and approximate equations predict different behavior of spinning particle. The difference between momentum and velocity means extra contribution into spin-orbit interaction. To estimate the contribution, we found exact solution to complete equations for the case of uniform magnetic field. While BMT electron moves around the circle, our particle experiences magnetic Zitterbewegung, that is oscillates in the direction of magnetic field with amplitude of order of Compton wavelength for the fast particle.

I. INTRODUCTION

Consistent and complete description of spin effects of the relativistic electron is achieved in QED on the base of Dirac equation. However, starting from the pioneer works [1–3] and up to present date, interpretation of final results in some cases is under permanent and controversial debates in various theoretical and experimental set-ups [4–13]. Understanding of the spin precession in the case of arbitrary magnetic moment in an external electromagnetic field is important in the development of experimental technics for measurements of anomalous magnetic moment [14, 15]. In accelerator physics [16] it is important to control resonances leading to depolarization of a beam. In the case of vertex electrons carrying arbitrary angular momentum, semiclassical description can also be useful [17]. So the relationship among classical and quantum descriptions remains an important step of analysis, providing the interpretation of results of QFT computations in usual terms: particles and their interactions. Hence an actual task is to develop, in a systematic form, the classical model of an electron [18–28, 30] which would be as close as possible to the Dirac equation. May be the best candidates for classical equations of relativistic electron are those of Frenkel [18, 19] and Bargmann, Michel and Telegdi (BMT) [20]. In this work we continue detailed analysis of these equations started in [31, 32], and construct Lagrangian which gives generalization of these equations on the case of arbitrary electromagnetic background.

Even for non interacting theory, search for Lagrangian which yields the right number of degrees of freedom and the right non relativistic limit represents rather non trivial problem [19, 22, 24, 30]. In [32] we have solved this problem, considering spin as a composed quantity (inner angular momentum) constructed from non-Grassmann vector-like variable and its conjugated momentum [27, 28, 30]. The desired properties are guaranteed by six constraints. So, in Hamiltonian formulation the model represents rather nontrivial example of a constrained system. Phase space of the model turns out to be curved manifold equipped, in a natural way, with the structure of fiber bundle. Detailed analysis of the underlying geometry has been presented in [31]. This allowed us to develop the proper quantization scheme. In [32] we have performed both canonical (in physical-time parametrization) and manifestly covariant (in arbitrary parametrization) quantization of free model, and established the relation with one-particle sector of Dirac equation as well as with quantum theory of two-component Klein-Gordon equation developed by Feynmann and Gel’-Mann [33]. It has been demonstrated that various known in the literature non-covariant, covariant and manifestly-covariant operators of position and spin acquire clear meaning and interpretation in the Lagrangian model of Frenkel electron. In particular, we have found the manifestly covariant form of position and spin operators in the space of positive-energy Dirac spinors.

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In the Hamiltonian formulation appeared two second-class constraints which, at the end, supply the Frenkel condi-
tion on spin-tensor. They depend on both position and spin-sector variables. This leads to new properties as
compared with non relativistic spin [31]. The constraints must be taken into account by transition from Poisson
to Dirac bracket, this leads to nonvanishing classical brackets for the position variables. In the result, the position
space is endowed, in a natural way, with noncommutative structure which originates from accounting of spin degrees
of freedom. Our model represents an example of situation when physically interesting noncommutative relativistic
particle emerges in a natural way. For the case, the "parameter of non-commutativity" is proportional to spin-tensor.
As a consequence, operators corresponding to position of the electron are non-commutative (they can be identified
with Fryce (d) operators). This implies that effects of non-commutativity could be presented at the Compton
wave length, in contrast to conventional expectations of non-commutativity at Planck length.

There are a lot of candidates for spin and position operators of the relativistic electron [1] [3] [21] [54]. Different
position observables coincide when we consider standard quasi-classical limit. So, in absence of a systematically
constructed classical model of an electron it is difficult to understand the difference between these operators. Our
approach allows us to do this, after realizing all them at the classical level. Besides, all the candidates obey the same
equations in free theory, so the question of which of them are the true position and spin is a matter of convention.
The situation changes in interacting theory, considered below, where we can distinguish the variables according
their classical dynamics in an external field.

In the present work we return to the classical mechanics and study an interacting theory. In section II we show
that our Lagrangian admits interaction with an arbitrary electromagnetic background1. The model contains two
coupling constants - charge $e$ and the interaction constant $\mu$ of basic spin variables with $F_{\mu\nu}$. Provisionally, we call
this magnetic moment. The theory is consistent for arbitrary values of $\mu$. For the position variable we have the
minimal interaction term $\frac{1}{2}A_{\mu}A^{\mu}$. As for spin, when the particle has non-vanishing magnetic moment, this interacts
with electromagnetic field in a highly nonlinear way. This turns out to be necessary for preservation of the number
and algebraic structure of constraints in the passage from free to interacting theory. In section III we present and
analyze equations of motion in Hamiltonian and Lagrangian formulations. We show that they follow from simple and
expected Hamiltonian (33), when we deal with the Dirac bracket. We compare our equations with those of Frenkel
[18] [19]. Frenkel considered the case $\mu = 1$, and found his equations in the quadratic approximation on spin-tensor. We
show that our exact equations coincide with those of Frenkel in these limits. Hence our Lagrangian gives complete
Frenkel equations for arbitrary field and magnetic moment.

Frenkel tensor can be used to construct BMT-type four-vector. We write the corresponding equations of motion.
While Hamiltonian equations can be rewritten in closed form in terms of BMT vector, see Eq. (60)-(63), we do not
achieved this for Lagrangian equations in our theory. In the Lagrangian form, the equation for BMT vector contains
Frenkel tensor, see Eq. (63). It seems that Frenkel spin in our theory represents more fundamental object as compared
with BMT spin.

In section IV we find exact solution to our equations in the uniform magnetic field. With respect to BMT equations
our model takes into account two physical effects. First, magnetic moment interacted with a magnetic field results in
additional mass of electron. Second, in the case of anomalous magnetic moment the velocity and the momentum are
not collinear, this modifies the Lorentz force. Our model naturally incorporates both these effects and leads to small
corrections of the trajectory and spin precession. While equations of motion have a rather complicated structure, in
the case of uniform magnetic field there are a lot of symmetries and hence integral of motions providing complete
analytical solution.

II. LAGRANGIAN AND HAMILTONIAN OF INTERACTING THEORY

To start with, we shortly describe the structure of free theory [32] [33]. Configuration space of the model consist
of the position $x^{\mu}(\tau) = (ct, x)$ and the vector-like variable of spin $\omega^{\mu}(\tau) = (\omega^{0}, \omega)$, $\omega = (\omega_{1}, \omega_{2}, \omega_{3})$ in an arbitrary
parametrization $\tau$. $p^{\mu}$ and $\pi^{\mu}$ are conjugate momenta for $x^{\mu}$ and $\omega^{\mu}$. The variables in the free theory are subject to the
constraints

$$T_{1} = p^{2} + (mc)^{2} = 0,$$

$$T_{3} = \pi^{2} - a_{3} = 0, \quad T_{4} = \omega^{2} - a_{4} = 0, \quad T_{5} = \omega\pi = 0, \quad \text{where} \quad a_{3} = \frac{3\hbar^{2}}{4a_{4}},$$

$$T_{6} = p\omega = 0, \quad T_{7} = p\pi = 0,$$

1 Interaction with an arbitrary curved background is presented in [35].
where $\omega = -\omega^0 \pi^0 + \omega \pi$ and so on. As the Hamiltonian action functional, we simply take $L_H = p \dot{x} + \pi \dot{\omega} - H$, with Hamiltonian $H = g_i T_i$ in the form of linear combination the constraints $T_i$ multiplied by auxiliary variables $g_i$, $i = 1, 3, 4, 5, 6, 7$. The constraint $T_3$ belong to first-class and is related with local spin-plane symmetry presented in the theory [31]. The basic spin-sector variables change under the symmetry, so they do not represent an observable quantities. As the observable quantity we take the Frenkel spin-tensor $J$:

$$J^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu).$$

The constraints (2) and (3) imply the following restrictions (in the free theory the conjugated momentum is proportional to four-velocity, $p^\mu \sim u^\mu$)

$$J^{\mu\nu} p_\nu = 0, \quad J^2 = 6\hbar^2.$$

Spacial components of the Frenkel tensor can be used to construct the quantity

$$S^i = \frac{1}{4} \epsilon^{ijk} J_{jk},$$

which we identify with non relativistic spin of Pauli theory. In the interacting theory $p^\mu$ turns into canonical momentum $\mathcal{P}^\mu$, which for non uniform fields or/and $\mu \neq 1$ does not proportional to four-velocity. Hence in this case the Frenkel condition $J^{\mu\nu} u_\nu = 0$ turns into the Pirani condition $\mathcal{P}^{\mu} = 0$.

Frenkel tensor can be used to construct four-vector $s^\mu(\tau) \equiv 1 \frac{1}{4 \sqrt{-p^2}} \epsilon^{\mu\alpha\beta\epsilon} p_\alpha J_{\beta\epsilon}$, then $s^\mu p_\mu = 0$, $s^2 = \frac{3\hbar^2}{4}$.

In our theory, even in the case of interaction, the condition $s^\mu \mathcal{P}_\mu = 0$ implies $s^\mu u_\mu = 0$, see Eq. (58) below. So we can identify $s^\mu$ with BMT vector [20]. In the rest frame, spacial components $s^i$ of BMT vector coincide with $S^i$. In arbitrary frame, they are related as follows:

$$S^i = \frac{p^0}{\sqrt{-p^2}} \left( \delta_{ij} - \frac{p_i p_j}{(p^0)^2} \right) s^j.$$

In the free theory the equation (7) can be inverted, $J^{\mu\nu} = -\frac{2}{\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_\alpha s_\beta$, so the two quantities are mathematically equivalent. In the interacting theory, we have $p \rightarrow \mathcal{P} = \mathcal{P}(u, J, F)$, see Eqs. (28) and (29), so (7) became non linear equation.

In [32] we developed Lagrangian formulation of the theory. Excluding conjugate momenta from $L_H$, we obtained the Lagrangian action. Further, excluding the auxiliary variables, one after another, we obtained various equivalent formulations of the model. In the end, we got the “minimal” formulation without auxiliary variables. This reads

$$S = \int d\tau - mc\sqrt{-x N x} + \sqrt{a_3} \sqrt{\omega N \omega} - \frac{1}{2} g_4 (\omega^2 - a_4),$$

where $N^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{\omega^{\mu} \omega^{\nu}}{\omega^2}$ is projector on the plane transverse to the direction of $\omega^\mu$. The action is written in a parametrization $\tau$ which obeys $\frac{d\tau}{dt} > 0$, this implies $g_4(\tau) > 0$ and $p^0 > 0$.

Interaction with external background should not spoil the number and algebraic properties of constraints (1)-(3). We do not know how to achieve this for the minimal action. For instance, the natural reparametrization-invariant interaction $\frac{2}{\hbar^2} A_{\mu} \ddot{x}^\mu + \mu F_{\mu\nu} \omega^\nu \omega^\nu$, even for vanishing magnetic moment, leads to the theory with the number and algebraic structure of constraints different from those of free theory. So we start with the equivalent Lagrangian with four auxiliary variables, $g_1, g_3, g_4$ and $g_7$, this turns out to be appropriate to our aims. Of course, the auxiliary variables will be excluded from final equations of motion, see Eqs. (45)-(51).

To introduce coupling of the position variable with an electro-magnetic field, we add the minimal interaction term $A_{\mu} \dot{x}^\mu$. As for spin, we propose to modify derivative of $\omega$ as follows

$$\dot{\omega}^\mu \rightarrow D\omega^\mu = \dot{\omega}^\mu - g_1 \frac{e H}{c} (F \omega)^\mu.$$

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2 The last term in (7) represents kinematic (velocity-independent) constraint which is well known from classical mechanics. So, we might follow the classical-mechanics prescription to exclude $g_4$ as well. But this would lead to lose of manifest covariance of the formalism.

3 Hanson an Regge [22] have found highly non linear interaction for the case of their relativistic top. It would be interesting to apply their formalism to our minimal action.
This is the only term which we have found to be consistent with the constraints $T_i$. Lagrangian reads
\[
L = \frac{1}{2 \det G} \left[ g_3 (\dot{x} N \dot{x}) - 2g_7 (\dot{x} N D \omega) + g_1 (D \omega N D \omega) \right] + \frac{e}{c} A_\mu \dot{x}^\mu - \frac{g_4}{2} (\omega^2 - a_4) - \frac{g_1}{2} m^2 c^2 + \frac{g_3}{2} a_3 ,
\]
where $\det \dot{G} = g_1 g_3 - g_2^2$. Since $L$ contains the auxiliary variables, even for $\mu = 0$ we have higher nonlinear interaction. As a consequence, motion of spin influences motion of the particle and vice versa.

We first establish whether our Lagrangian gives the desired constraints. The momenta read
\[
p^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{\det G} \left( g_3 N \dot{x}^\mu - g_7 N D \omega^\mu \right) + \frac{e}{c} A^\mu ,
p_i = \frac{\partial L}{\partial \dot{\omega}^\mu} = \frac{1}{\det G} \left( -g_7 N \dot{x}^\mu + g_1 N D \omega^\mu \right) ,
\]
\[
p_{\omega i} = \frac{\partial L}{\partial \ddot{g}_i} = 0 .
\]
According to (13), the momenta $p_{\omega i}$ represent primary constraints, $p_{\omega i} = 0$. Using the property $N \omega = 0$ of the projector $N$, from Eqs. (12) follow more primary constraints, $T_5 = \pi \omega = 0$ and $T_6 = \mathcal{P} \omega = 0$. It has been denoted
\[
\mathcal{P}^\mu = p^\mu - \frac{e}{c} A^\mu .
\]
To write Hamiltonian, we solve the system (12) with respect to projected velocities
\[
N \dot{x}^\mu = g_1 \mathcal{P}^\mu + g_7 p_i^\mu ,
\]
\[
ND \omega^\mu = g_3 p_{\omega i} + g_7 \mathcal{P}^\mu .
\]
Using these expressions as well as the identities $\mathcal{P} \dot{x} = \mathcal{P} N \dot{x}$, $\pi \omega = \pi N \dot{\omega}$ we obtain Hamiltonian $H = p \dot{x} + \pi \dot{\omega} - L + \lambda_\alpha \Phi_\alpha$ in the form
\[
H = \frac{g_3}{2} \left( \mathcal{P}^2 - \frac{\mu^c}{2e}(JF) + m^2 c^2 \right) + \frac{g_3}{2} (\pi^2 - a_3) + \frac{g_4}{2} (\omega^2 - a_4) + \lambda_5 (\omega \pi) + \lambda_6 (\mathcal{P} \omega) + g_7 (\mathcal{P} \pi) + \lambda_4 \pi \omega_i ,
\]
where $\lambda_5$ and $\lambda_6$ appear as Lagrangian multipliers for primary constraints $T_5$ and $T_6$. We denote $(JF) = J^{\mu \nu} F_{\mu \nu}$ and so on. From (16) we conclude that $T_1, T_3, T_4$ and $T_7$ appear as secondary constraints when we impose the compatibility conditions $\dot{\pi}_{\omega i} = \{ \pi_{\omega i}, H \} = 0$. The second, third and fourth stages of the Dirac-Bergmann algorithm can be resumed as follows
\[
T_1 = 0 \quad \Rightarrow \quad \lambda_6 C + g_7 D = 0 ,
\]
\[
T_3 = 0 \quad \Rightarrow \quad \lambda_5 = 0 ,
\]
\[
T_4 = 0 \quad \Rightarrow \quad \lambda_5 = 0 ,
\]
\[
T_5 = 0 \quad \Rightarrow \quad g_4 = \frac{a_3}{a_4} g_3 , \quad \Rightarrow \quad \lambda_{g_4} = \frac{a_3}{a_4} \lambda_{g_3} ,
\]
\[
T_6 = 0 \quad \Rightarrow \quad g_1 C - g_7 M^2 c^2 = 0 , \quad \Rightarrow \quad \lambda_{g_7} = f(\lambda_{g_1}) ,
\]
\[
T_7 = 0 \quad \Rightarrow \quad g_1 D + \lambda_6 M^2 c^2 = 0 .
\]
We have denoted
\[
M^2 = m^2 - \frac{e(2\mu + 1)}{4c^3} F_{\mu \nu} J^{\mu \nu} ,
\]
\[
C = -\frac{e}{c} (\mu - 1)(\omega F \mathcal{P}) + \frac{e}{4c}(\omega \partial)(JF) , \quad D = -\frac{e}{c} (\mu - 1)(\pi F \mathcal{P}) + \frac{e}{4c}(\pi \partial)(JF) .
\]
Eq. (17) turns out to be a consequence of (21) and (22), $\lambda_6 (21) + g_7 (22) = g_1 (17)$, and can be omitted. Eq. (21) determines $g_7 = \frac{C}{\lambda_{g_7} g_1}$ while (22) gives the lagrangian multiplier $\lambda_6 = -\frac{D}{\lambda_{g_7} g_1}$. The Dirac-Bergmann algorithm stops at the fourth stage. This yields all the desired constraints $T_a, a = 1, 3, 4, 5, 6, 7$. Two auxiliary variables, $g_1$ and $g_3$, and the corresponding Lagrange multipliers $\lambda_{g_1}$, $\lambda_{g_3}$, have not been determined.

It is useful to summarize the algebra of Poisson brackets between constraints in a compact form, see Table I. We note that Poisson brackets of $T_1$ and $T_3 = T_3 + \frac{a_4}{a_4} T_4$ vanish on the constraint surface, so they form the first-class subset. The presence of two first-class constraints is in correspondence with the fact that two lagrangian multipliers remain undetermined within the Dirac procedure. Matrix of Poisson brackets of the remaining constraints, $T_4, T_5, T_6$ and $T_7$, is non degenerated, so this is a set of second-class constraints. All this is in correspondence with free theory.

In resume, interaction introduced does not spoil the structure and algebraic properties of Hamiltonian constraints of the free theory.
TABLE I: Algebra of constraints

| $T_1 = \mathcal{P}^2 - \frac{\mu e}{2c} F^{\mu \nu} J_{\mu \nu} + m^2 c^2$ | $T_3$ | $T_4$ | $T_5$ | $T_6$ | $T_7$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | -2C | -2D |
| $T_3 = \pi^2 - a_3$ | 0 | 0 | -4T_5 | -2(a_3 + T_3) | -2T_7 | 0 |
| $T_4 = \omega^2 - a_4$ | 0 | 4T_5 | 0 | 2(T_4 + a_4) | 0 | 2T_6 |
| $T_5 = \omega \pi$ | 0 | 2(T_3 + a_3) | -2(a_4 + T_4) | 0 | -T_6 | T_7 |
| $T_6 = \mathcal{P} \omega$ | 2C | 2T_7 | 0 | T_5 | 0 | $T_1 - M^2 c^2$ |
| $T_7 = \mathcal{P} \pi$ | 2D | 0 | -2T_6 | -T_7 | -T_1 + M^2 c^2 | 0 |

### III. EXACT FRENKEL EQUATIONS ON ARBITRARY BACKGROUND

#### A. Hamiltonian equations of motion

The Hamiltonian \[ H_1 = \frac{g_1}{2} \left( \mathcal{P}^2 - \frac{\mu e}{2c} (JF) + m^2 c^2 \right) + \frac{g_3}{2} (J^2 - 8a_3 a_4). \] (25)

Equations of motion now can be obtained with help of $H_1$ and the Dirac bracket, $\dot{q} = \{q, H_1\}_D$. They read

\[
\dot{x}^\mu = g_1 u^\mu, \quad \dot{p}^\mu = g_1 \frac{e}{c} (F u)^\mu + g_1 \frac{\mu e}{4c} \partial^\mu (JF),
\] (26)

\[
\dot{\omega}^\mu = g_1 \frac{e}{c} (F \omega)^\mu + g_3 \pi^\mu + g_7 \mathcal{P}^\mu,
\]

\[
\dot{\pi}^\mu = g_1 \frac{e}{c} (F \pi)^\mu - \frac{a_3}{a_4} g_3 \omega^\mu - g_6 \mathcal{P}^\mu,
\] (27)

where $\partial^\mu (JF) = J^{\alpha \beta} \partial^\mu F_{\alpha \beta}$. According to (21) and (22), the four-velocity $u^\mu$ is not proportional to canonical momentum $\mathcal{P}^\mu$

\[
u^\mu = \mathcal{P}^\mu + \frac{g_7}{g_1} \pi^\mu + \frac{\lambda_6}{g_1} \omega^\mu = T^\mu_{\nu} \mathcal{P}^\nu + Y^\mu.
\] (28)

We have denoted

\[
T^\mu_{\nu} = \eta^\mu_{\nu} + \frac{e(\mu - 1)}{2c^2 M^2} (JF)^{\mu \nu}, \quad Y^\mu = -\frac{\mu e}{8M^2 c^2} J^{\mu \alpha} \partial_\alpha (JF).
\] (29)

All the basic variables have ambiguous evolution. $x^\mu$ and $\mathcal{P}^\mu$ have one-parametric ambiguity due to $g_1$ (they change under reparametrizations) while $\omega$ and $\pi$ have two-parametric ambiguity due to $g_1$ and $g_3$ (they change under reparametrizations and spin-plane symmetry). The quantities $x^\mu$, $\mathcal{P}^\mu$ and the spin-tensor $J^{\mu \nu}$ are spin-plane invariants. Their equations of motion form a closed system

\[
\dot{x}^\mu = g_1 \left[ \mathcal{P}^\mu + \frac{e(\mu - 1)}{2c^2 M^2} (JF)^{\mu \nu} - \frac{\mu e}{8M^2 c^2} J^{\mu \alpha} \partial_\alpha (JF) \right],
\] (30)

\[
\dot{p}^\mu = e \frac{c}{\mathcal{P} F_{\nu}^\mu} + g_1 \frac{\mu e}{4c} \partial^\mu (JF),
\] (31)

\[
J^{\mu \nu} = g_1 e \left[ \mu F_{\mu \nu} J^{\rho \alpha} - \frac{\mu - 1}{c^2 M^2} (JF)^{\mu \nu} \mathcal{P}^\rho + \frac{\mu}{4M^2 c^2} \mathcal{P}^{[\mu} J^{\nu]} \partial_\alpha (JF) \right].
\] (32)
The last term in (25) does not contribute to the equations of motion for \(x, P\) and \(J\), and can be omitted. Then the Hamiltonian for these variables acquires a simple and expected form

\[
H = \frac{g_1}{2} \left( p^2 - \frac{\mu e}{2c} (JF) + m^2 c^2 \right).
\] (33)

The interaction yields two essential structural modifications of the theory. Free theory implies the Frenkel condition, \(J^\mu \dot{x}_\nu = 0\), and \(p^\mu \sim \dot{x}^\mu\). Interaction modifies not only dynamical equations but also the Frenkel condition, the latter necessarily turns into the Pirani condition

\[
J^\mu P_\nu = 0,
\] (34)

where, due to (28), \(P\) is not proportional to \(\dot{x}^\mu\). Then Eqs. (30) and (31) imply that the interaction leads to modification of the Lorentz-force equation even for uniform field. Only for the non anomalous value of magnetic moment, \(\mu = 1\), and uniform electromagnetic field the equations (21) and (22) would be the same as in free theory, \(\lambda_6 = g_7 = 0\). Then \(T^\mu_\nu = \eta^\mu_\nu\), \(Y^\mu = 0\), and four-velocity becomes proportional to \(\frac{1}{c^2} \sim \frac{\lambda}{c}\), while the term with a gradient of field is proportional to \(\frac{\lambda^2}{c^2} \sim \frac{h^2}{c^2}\).

The remaining ambiguity due to \(g_1\) presented in the equations (30)-(32) reflects the reparametrization symmetry of the theory. Assuming that the functions \(x^\mu(\tau), p^\mu(\tau)\) and \(J^\mu_\nu(\tau)\) represent the physical variables \(x^i(t), p^\mu(t)\) and \(J^\mu_\nu(t)\) in the parametric form, their equations read

\[
\frac{dx^i}{dt} = c u^i u^0, \quad \frac{dx^0}{dt} = c,
\] (35)

\[
\frac{dP^\mu}{dt} = \frac{c}{a^2} F^{\mu\nu} u_\nu + \frac{\mu e}{4a^0} \delta^\mu (JF),
\] (36)

\[
\frac{dJ^\mu_\nu}{dt} = \frac{c}{g_1 u^0} j^{\mu_\nu}.
\] (37)

As it should be, they have unambiguous dynamics. Equations (30)- (32) are written in arbitrary parametrization of the world-line. In the next subsection we exclude \(P\) and \(g_1\), and then analyze the resulting equations in the proper-time parameterizations. This allow us to compare them with original Frenkel equations.

**B. Lagrangian form of equations**

Hamiltonian equations from the previous section can be rewritten in the Lagrangian form for the set \(x, J\). Let us analyze relation between velocity and momentum given by the Hamiltonian equation

\[
\dot{x}^\mu = g_1 (T^\mu_\nu P^\nu + Y^\mu),
\] (38)

where

\[
T^\mu_\nu = \eta^\mu_\nu - aJ^{\alpha_\mu} F^\alpha_\nu, \quad a = \frac{e(\mu - 1)}{2c^2 M^2}.
\] (39)

Matrix \(T\) is invertible, the inverse matrix \(\tilde{T}\) has the same structure (we used the identity \((JF)^{\mu_\nu} = -\frac{1}{2} (JF) J^{\mu_\nu}\) which implied by (4))

\[
\tilde{T}^{\mu_\nu} = \eta^{\mu_\nu} + b J^{\alpha_\mu} F^\alpha_\nu, \quad b = \frac{2a}{2 + a(JF)}.
\] (40)

Let us first exclude \(P\) and \(g_1\) from equation (31). From Eq. (38), we can express \(P\) through \(\dot{x}\)

\[
P^\mu = \frac{1}{g_1} \tilde{T}^\mu_\nu \dot{x}^\nu - \tilde{T}^\mu_\nu Y^\nu.
\] (41)

We can express \(g_1\) calculating square of the following expression

\[
P^\mu + \tilde{T}^\mu_\nu Y^\nu = \frac{1}{g_1} \tilde{T}^\mu_\nu \dot{x}^\nu,
\]
which yields

\[ \mathcal{P}^2 + (\dot{\mathcal{T}}Y)^\mu (\ddot{\mathcal{T}}Y)_\mu = \frac{1}{g_1^2} (\ddot{\mathcal{T}}\dot{x})^\mu (\dddot{\mathcal{T}}\dot{x})_\mu. \]

We used that \( \mathcal{P}_\mu \dddot{\mathcal{T}}^\nu = \mathcal{P}_\mu \) and \( \mathcal{P}_\mu Y^\mu = 0 \). From the last equation we find \( g_1 \)

\[ g_1 = \sqrt{\frac{\ddot{\mathcal{T}}\dot{x}^2}{(\dot{\mathcal{T}}Y)^2 - m^2 c^2 + \mu e \mathcal{F}^\mu}} = \sqrt{-\frac{\dot{g} \dot{\mathcal{T}}}{m_r c}}, \quad (42) \]

where we have introduced the symmetric matrix \( g_{\mu\nu} = (\dddot{T}T)_{\mu\nu} \),

and the radiation mass

\[ m_r^2 = m^2 - \frac{\mu e}{2 c^3} (\mathcal{J} F) - \frac{g_{YY}}{c^2}. \quad (44) \]

In the natural parametrization \( \sqrt{-\dot{g} \dot{\mathcal{T}}} = c \), we have \( g_1 = m_r^{-1} \), that is the auxiliary variable, which appeared in front of mass-shell constraint \( T_1 = 0 \), is the inverse radiation mass. Due to the identity \((\dot{\mathcal{T}}Y)^\mu = b Y^\mu \) we also can write \( g_{YY} = b^2 Y^2 \). Using \( 41 \) and \( 42 \) in \( 31 \) and \( 32 \) we write closed system of equations for \( x^\mu \) and \( J^{\mu \nu} \) in the form

\[ \frac{d}{d\tau} \left[ m_r c (\ddot{\mathcal{T}})_{\mu} \sqrt{-\dot{g} \dot{\mathcal{T}}} - (\dot{\mathcal{T}}Y)^\mu \right] = \frac{e}{c} (F \dot{x})^\mu + \frac{\mu e}{4 m_r c^2} \partial^\mu (\mathcal{F} \mathcal{T}), \quad (45) \]

\[ j^{\mu \nu} = e \left[ \frac{\sqrt{-\dot{g} \dot{\mathcal{T}}}}{m_r c} \right] \mathcal{F}^{[\mu}_{\alpha} J^{\nu \alpha]}, \quad (46) \]

\[ J^{\mu \nu} \dot{T}^\alpha (m_r c \sqrt{-\dot{g} \dot{\mathcal{T}}}) \dot{x}^\alpha - Y^\alpha = 0. \quad (47) \]

Let us compare them with Frenkel equations. Frenkel found equations of motion consistent with the condition \( J^{\mu \nu} \mathcal{U}_{\nu} = 0 \) up to order \( \mathcal{O}^4(J, F, \partial F) \). Besides, he considered the case \( \mu = 1 \). Taking these approximations in our equations in the proper-time parametrization \( \sqrt{-(\dot{x})^2} = c \), we arrive at those of Frenkel (our \( J \) is \( \frac{2m c}{e} \) of Frenkel \( J \))

\[ \frac{d}{d\tau} \left[ (m - \frac{e}{4 m c^2} (J F) x^\mu + \frac{e}{8 m^2 c^3} J^{\mu \nu} \partial^\mu (J F) \right] = \frac{e}{c} (F \dot{x})^\mu + \frac{e}{4 m c} \partial^\mu (J F), \quad (48) \]

\[ j^{\mu \nu} = \frac{e}{m c} \left[ F^{[\mu}_{\alpha} J^{\nu \alpha]} - \frac{1}{4 m^2 c^2} \dddot{x}^{[\mu} J^{\nu \alpha]} \partial^\alpha (J F) \right], \quad (46) \]

\[ J^{\mu \nu} \dddot{x}_\nu = 0. \quad (49) \]

In general case, our equations \( 45 \)-\( 47 \) involve two types of corrections as compared with those of Frenkel. First, energy of magnetic moment in non uniform field leads to the contribution \( -\frac{g_{YY}}{c^2} \) into the Frenkel radiation mass, see \( 43 \). Second, when \( \mu \neq 0 \), a contribution arises because the Frenkel condition which has been satisfied for the free particle, turns into Pirani condition in the interacting theory. Its Lagrangian form is written in \( 47 \). In the result, the components \( J^{0i} \) vanish in the frame \( \mathcal{F}^\mu = (\mathcal{P}^0, \mathcal{0}) \) instead of the rest frame. Hence our model predicts small dipole electric moment of the particle.

The structure of our equations simplified significantly for the stationary homogeneous field \( \partial^\alpha \mathcal{F}^{\mu \nu} = 0 \). In this case \( 45 \) and \( 46 \) read

\[ \left[ \frac{m_r (\ddot{\mathcal{T}} \dot{x})^\mu}{\sqrt{-(\dot{g} \dot{\mathcal{T}})}} \right] = \frac{e}{c^2} (F \dot{x})^\mu, \quad (50) \]
Due to Eqs. (28) and (29) together with (4), the Bargmann-Michel and Telegdi vector in our model. Using the identities

\( sB \equiv \frac{\mu}{m_c c^2} \left[ \mu \sqrt{-g} \tilde{F}^\mu \alpha J^\alpha - \frac{m_c^2 (\mu - 1)}{M^2 \sqrt{-g}} \left( J F \tilde{T} \right)^\mu \left( \tilde{T} \right)^\nu \right] \). (51)

\( (J \tilde{T}^\mu) = 0 \). (52)

Contract (50) with \( (\tilde{T} \mu) \). Then r.h.s. vanishes. Using the identity \( v_{\mu} \left[ \frac{\tilde{e}^\mu}{\sqrt{-g}} \right] = 0 \) at the l.h.s., we obtain

\[ \dot{m}_r \sqrt{-g} (\tilde{T} \dot{x}^2) = 0, \quad \dot{m}_r = 0, \] (53)
on-shell. Hence \( F^{\mu\nu} J_{\mu\nu} \) is a conserved quantity. Then \( T_1 = 0 \) implies that \( \mathcal{P}^2 \) is a conserved quantity as well.

Contracting (50) with \( T \) we can further simplify this equation

\[ \frac{d}{dT} \left[ \frac{m_r \dot{x}^\mu}{\sqrt{-g} \tilde{F}^\mu} \right] = \frac{e}{c^2} (F' \dot{x})^\mu, \quad F' = TF - \frac{m_r c^2}{e \sqrt{-g} \tilde{F}^\mu} x \tilde{T}. \] (54)

Let us choose a parametrization which implies

\[ g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -c^2. \] (55)

Since \( g \tilde{x}^2 = x^2 + O(J^2) \), in the linear approximation on \( J \) this is just the proper-time parametrization. Then the equations (51) and (54) read

\[ \frac{d(\tilde{T} \dot{x})^\mu}{dT} = \frac{e}{m_r c} (F' \dot{x})^\mu, \quad \tilde{x} = \frac{e}{m_r c} (F' \dot{x})^\mu, \quad F' = TF - \frac{m_r c^2}{e \sqrt{-g} \tilde{F}^\mu} T \tilde{T}, \] (56)

\[ j^{\mu\nu} = \frac{e}{m_r c} \left[ \mu F^{[\mu \alpha} J^{\alpha \nu]} - \frac{m_c^2 (\mu - 1)}{c^2 M^2} \left( J F \tilde{T} \right)^{\mu} \left( \tilde{T} \right)^{\nu} \right]. \] (57)

So, when \( \mu \neq 0 \), the exact equations differ from the approximate equations (48) and (49) even for uniform field.

C. BMT vector in Frenkel theory

Since \( J^{\mu\nu} \mathcal{P}_\nu = 0 \), the spin-tensor is equivalent to the four-vector where we replace \( p^\mu \rightarrow \mathcal{P}^\mu \). Then \( s^\mu \mathcal{P}_\mu = 0 \). Due to Eqs. (28) and (29) together with (4), \( s^\mu \) also obeys the condition

\[ s^\mu u_\mu = s^\mu \dot{x}_\mu = 0. \] (58)

The physical dynamics can be described using \( s^\mu \) instead of \( J^{\mu\nu} \). Eq. (58) suggests that \( s^\mu \) could be candidate for BMT-vector in our model. Using the identities

\[ J^{\mu\nu} = \frac{2}{\sqrt{-\mathcal{P}}} \epsilon^{\mu\nu\alpha\beta} s_\alpha \mathcal{P}_\beta, \quad \epsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} = \frac{4}{\sqrt{-\mathcal{P}}} \mathcal{P}^{[\mu \mathcal{P}^\nu]}, \] (59)

to represent \( J^{\mu\nu} \) through \( s^\mu \) in Eqs. (30)-(32), we obtain the closed system of equations for spin-plane invariant quantities

\[ \dot{x}^\mu = g_1 \left[ \mathcal{P}^\mu + \frac{c(\mu - 1)}{c^3 M^2 \sqrt{-\mathcal{P}}} \epsilon^{\mu\nu\lambda\gamma} \left( F \mathcal{P} \right)_\nu s_\lambda \mathcal{P}_\gamma + \frac{\mu c}{2 M^2} \epsilon^{\mu\nu\lambda\gamma} \epsilon^{\alpha\beta\gamma \delta} s_\gamma \mathcal{P}_\delta \mathcal{P}_\beta \partial_\alpha F_{\rho \beta} \right], \] (60)

\[ \dot{\mathcal{P}}^\mu = \frac{c}{\mathcal{P}} \left( F \mathcal{P} \right)^\mu + g_1 \frac{\mu c}{2 c \sqrt{-\mathcal{P}}} \epsilon_{\alpha\beta\gamma\rho \mathcal{P}^\mu \mathcal{P}^\nu} s_\gamma \mathcal{P}_\delta \mathcal{P}_\beta \partial_\alpha F_{\rho \beta}, \] (61)

\[ \dot{s}^\mu = g_1 \frac{c}{\mathcal{P}} \left[ \left( F \mathcal{P} \right)^\mu + \frac{1}{\mathcal{P}^2} \left( s F \mathcal{P} \right)^\mu \right] - \frac{1}{\mathcal{P}^2} \left( \dot{\mathcal{P}} \mathcal{P} \right)^\mu. \] (62)

These equations valid for arbitrary electro-magnetic fields. Let us consider the case of uniform field discussed by Bargmann Michel and Telegdi. Then we can compare these equations with BMT equations. First, we should exclude
\( \mathcal{P} \) and \( g_1 \) from equations \((61)\) and \((62)\) using \((60)\). In contrast to \((38)\), where \( \dot{z}^\mu \) is a linear function of \( \mathcal{P}^\mu \) and \( J^{\mu\nu} \), in \((60)\) \( \dot{z}^\mu \) is a non-linear function of \( \mathcal{P}^\mu \) and \( s^\mu \). Inverse function which express \( \mathcal{P}^\mu \) as a function of \( \dot{z}^\mu \), \( s^\mu \) exists, though we can't find its explicit form even in the case of uniform fields. Formally using \((41)\) and \((42)\) in the case of uniform fields, \( \partial_\alpha F^{\mu\nu} = 0 \), we get
\[
\dot{s}^\mu = \sqrt{-g} \dot{x}^\mu \frac{e^\mu}{m_e c^2} (Fs)^\mu - \frac{e}{m_e c^2 \sqrt{-g} \dot{x}^\mu} \left[ (\mu - 1)(sF^\mu) + \mu b(sF^\mu \dot{x}) \right] \sqrt{\mathcal{P}}^\mu, \quad s \dot{x} = 0. \tag{63}
\]
Eq. \((63)\) contains \( J \) but for weak fields the corresponding contribution can be neglected. In the uniform field and in the parametrization \((55)\) we have Eq. \((56)\) for \( x \) and
\[
\dot{s}^\mu = \frac{e^\mu}{m_e c} (Fs)^\mu - \frac{e}{m_e c^2} (\mu - 1)(sF^\mu) \dot{x}^\mu. \tag{64}
\]
This can be compared with BMT-equations
\[
\ddot{x}^\mu = \frac{e}{mc} (F^\mu), \tag{65}
\]
\[
\dot{s}^\mu = \frac{e^\mu}{m_e c} (Fs)^\mu - \frac{e}{mc^2} (\mu - 1)(sF^\mu) \dot{x}^\mu. \tag{66}
\]
We can also introduce BMT-tensor dual to \( s^\mu \)
\[
J_{BMT}^{\mu\nu} = \frac{2}{c} \epsilon^{\mu\nu\alpha\beta} s_{\alpha} \dot{x}_{\beta}. \tag{67}
\]
Due to \((60)\) this obeys the equation
\[
\ddot{J}_{BMT}^{\mu\nu} = \frac{e}{mc} \left[ \mu F^{[\mu}_{\alpha\beta} \dot{F}^{\alpha\beta]}_{BMT} - \frac{(\mu - 1)}{c^2} (J_{BMT} F^\mu)_{\mu} \dot{x}^\nu \right]. \tag{67}
\]
This can be compared with \((57)\).

Obtaining their equation \((66)\) in uniform field, Bargmann, Michel and Telegdi supposed that the motion of particle \((65)\) is independent from the motion of spin. Besides they looked for the equation for \( s^\mu \) linear on \( s \) and \( F \). Obtaining Eqs. \((56)\) and \((64)\) we have not made any supposition of such a kind. Our approach is based on the variational formulation which satisfies all the necessary symmetries. The exact equations \((66)\) and \((64)\) involve two types of essential corrections as compared with BMT equations. First, an energy of magnetic moment in electromagnetic formulation which satisfies all the necessary symmetries. The exact equations \((56)\) and \((64)\) involve two types of essential corrections as compared with BMT equations.

IV. EXACT SOLUTION IN UNIFORM MAGNETIC FIELD

BMT equations give important information about spin kinematics for relativistic particles. Exact solutions to the Dirac equation in a constant magnetic field can be used to obtain solutions to BMT equations \([39]\). Integrability of BMT equations in the case of rather general electromagnetic backgrounds were studied in \([40]\). In this section we would like to study the behavior of our model in more details. We take BMT vector of our model as the basic quantity, compare dynamics of our model with those of BMT. So we take for the analysis the closed system of equations \((60)\).

Consider a particle with initial momentum \( \mathcal{P}^\mu(0) \) and BMT spin \( s^\mu(0) \) moving in the uniform magnetic field directed along \( z \)-axis, \( \mathbf{B} = B \mathbf{e}_z \), of a laboratory Cartesian coordinate system defined by an orthonormal basis \( (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \). We already established that \( \mathcal{P}^2 \) and \( (FJ) \) are integrals of motion for uniform fields. In the case of uniform magnetic field we have
\[
(FJ) = 4 \gamma \left[ (BS) - (B\beta)(\beta S) \right].
\]
Initial values \( \mathcal{P}^\mu(0), s^\mu(0) \) should satisfy to the constraints of our model \( (T_1 = 0, s^2 = 3\hbar^2/4, \mathcal{P} s = 0) \).
Here and through the rest of this section we use the following notations

\[ \gamma = \frac{P^0}{\sqrt{P^2}}, \quad \beta = \frac{\vec{P}}{P^0}, \]

in accordance with our construction of Lorentz invariant SO(3) spin fiber bundle \[31\]. Here, \( \gamma \) plays a role of relativistic factor which in the limit of free electron reads \( \gamma = (1 - v^2/c^2)^{-1/2} \). Denote by \( \beta \) module of vector \( \beta \). The quantity \( \beta \) is also a constant, which practically (for the magnetic fields smaller than Schwinger field) can be taken as \( a = \frac{2e(\mu - 1)}{4m^2c^3 - e(2\mu + 1)(FJ)} \),

is also a constant, which practically (for the magnetic fields smaller than Schwinger field) can be taken as \( a \approx \frac{e(\mu - 1)}{2m^2c^3} \).

The Hamiltonian equations of motions \([60]-[62]\) written in the parametrization of physical time read

\[
\frac{dx}{dt} = \frac{c}{u^0} \left[ \begin{array}{c} u \\ v \\ w \end{array} \right], \quad \frac{dx^0}{dt} = c, \quad (68)
\]

\[
\frac{d\mathcal{P}}{dt} = \frac{e}{u^0} \left[ u, B \right], \quad \frac{dP^0}{dt} = 0, \quad (69)
\]

\[
\frac{dS}{dt} = \frac{e\mu}{u^0} \left[ \frac{1}{p^2} (S, \mathcal{P}, B) \mathcal{P} - \frac{e}{p^2} u^0 (S, u, B) \mathcal{P} \right], \quad (70)
\]

\[
\frac{dS^0}{dt} = \frac{eP^0}{u^0p^2} \left[ \frac{\mu(S, \mathcal{P}, B) - (S, u, B)}{\mu} \right], \quad (71)
\]

\[
u^0 = \mathcal{P}^0 \left[ 1 + 2a_0^2 (\beta^2 (BS) - (B\beta)(\mathcal{P}B)) \right], \quad (72)
\]

\[
u = \mathcal{P} \left[ 1 + 2a_0^2 ((BS) - (B\beta)(\mathcal{P}B)) \right] - \frac{2a_0}{\gamma} B (\mathcal{P}S), \quad (73)
\]

where \([S, B]\) and \((S, \mathcal{P}, B)\) mean the vector and mixed product of 3-dimensional vectors.

The velocity is not collinear to the momentum \(\mathcal{P}^\mu\), its projection to the 3-hyperplane orthogonal to \(\mathcal{P}^\mu\) is proportional to anomalously magnetic moment. As a result, in general there is no common rest frame for velocity and momentum.

From \([69]\) follows that \(\mathcal{P}^0 = \text{const}\), hence \(\mathcal{P}^2 = \text{const}\), \(\gamma = \text{const}\) and \(\beta^2 = \text{const}\). Another integral of motion is the projection of momentum to the magnetic field, \((\mathcal{P}B) = \text{const}\). To simplify our calculations we assume without losing generality that initial vector of momentum is orthogonal to magnetic field, \((\mathcal{P}B) = 0\). Indeed, other values of \((\mathcal{P}B)\) can be obtained by boosts along \(B\) which do not modify electromagnetic tensor \((B' = B, E' = E = 0)\).

For the motion with momentum orthogonal to magnetic field we obtain the following system of equations

\[
\frac{d\mathcal{P}}{dt} = \Omega_p [\mathcal{P}, e_z], \quad (74)
\]

\[
\frac{dx}{dt} = c \left[ \frac{\Omega_p}{eB} \mathcal{P} - \frac{2a_0\Omega_s}{\gamma} e_z (\mathcal{P}S) \right], \quad (75)
\]

\[
\frac{dS^0}{dt} = \Omega'_s (S, \beta, e_z), \quad (76)
\]

\[
\frac{dS}{dt} = \Omega_s [S, e_z] + \Omega'_s (S, \beta, e_z), \quad (77)
\]

where we use the following constants (frequencies)

\[
\Omega_p = \frac{eB(1 + 2a_0^2 (BS))}{\mathcal{P}^0 (1 + 2a_0^2 (BS))}, \quad (78)
\]

\[
\Omega_s = \frac{\mu e B}{\mathcal{P}^0 (1 + 2a_0^2 (BS))}, \quad (79)
\]

\[
\Omega'_s = \frac{eB\mathcal{P}^0 (\mu - 1 - 2a_0^2 (BS))}{\mathcal{P}^2 (1 + 2a_0^2 (BS))} = \gamma^2 (\Omega_s - \Omega_p). \quad (80)
\]

Multiplying \([77]\) by \(B\) we find that \((BS) = \text{const}\) as it should be. From \([74]\) we find that the vector \(\mathcal{P}\) rotates with a constant circular frequency \(\Omega_p\) in the plane, orthogonal to magnetic field. In the orthonormal basis \((e_x, e_y, e_z)\) solution for \(\mathcal{P}\) yields

\[
\mathcal{P} = |\mathcal{P}^{(0)}| \left[ e_x \cos (\Omega_p t + \phi_p) + e_y \sin (\Omega_p t + \phi_p) \right]. \quad (81)
\]
For simplicity we choose \( \mathbf{e}_x \) to provide \( \phi_p = 0 \).

To solve \([77]\) we first note that \( S_z = \text{const} \). From constraints \( \mathcal{P} = 0, S^2 = \frac{3}{4} \hbar^2 \) one can see that the spin \( S \) for a particle with momentum \( \mathcal{P} = \mathcal{P}_0 \beta \) belong to the following ellipsoid

\[
S_i (\delta^{ij} - \beta^i \beta^j) S_j = \frac{3}{4} \hbar^2,
\]

which is obtained from a sphere with radius \( \hbar \sqrt{3}/2 \) by stretching in \( \gamma \) times in the direction of \( \mathcal{P} \). The main principal axis of this ellipsoid is always directed along \( \mathcal{P} \). Therefore we write \( S_x \mathbf{e}_x + S_y \mathbf{e}_y = S_1 \tau_1 + S_2 \tau_2 \) where the new basis is

\[
\tau_1 = e_x \cos(\Omega_p t) + e_y \sin(\Omega_p t), \quad \text{then} \quad \frac{d\tau_1}{dt} = \Omega_p \tau_2,
\]

\[
\tau_2 = -e_x \sin(\Omega_p t) + e_y \cos(\Omega_p t), \quad \text{then} \quad \frac{d\tau_2}{dt} = -\Omega_p \tau_1,
\]

\[
\tau_3 = e_z,
\]

which is determined by two principal axes of the ellipsoid. Note, that we just choose convenient variables to solve the differential equation without transition to another reference frame. As a result we obtain simple equations

\[
\frac{dS_1}{dt} = -\Omega'_s S_2, \quad \frac{dS_2}{dt} = \frac{\Omega'_s}{\gamma^2} S_1,
\]

which describe evolution of spin on the ellipsoid \([82]\). Auxiliary radius vector \( S_1 \tau_1 + S_2 \tau_2 \) rotates with circular frequency \( \Omega'_s \gamma^{-1} \). Solutions to Eqs. \([83]\) read

\[
S_1(t) = S^{(0)} \cos \left( \frac{\Omega'_s}{\gamma} t + \phi \right), \quad S_2(t) = S^{(0)} \sin \left( \frac{\Omega'_s}{\gamma} t + \phi \right).
\]

The radius vector \( \vec{S} \) moves on the ellipse which is obtained as intersection of ellipsoid \([82]\) and plane \( S_z = \text{const} \). \( S^{(0)} \) is nothing but the semi-major axis of this ellipse, therefore it is restricted by the following interval \( 0 \leq S^{(0)} \leq \gamma \sqrt{3} \hbar/2 \).

In terms of initial variables spacial components of 4-vector \( S^\mu \) evolves as follows

\[
S(t) = e_x S^{(0)} \left[ \cos \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \cos (\Omega_p t) - \frac{1}{\gamma} \sin \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \sin (\Omega_p t) \right] +
\]

\[
e_y S^{(0)} \left[ \cos \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \sin (\Omega_p t) + \frac{1}{\gamma} \sin \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \cos (\Omega_p t) \right] + e_z S^{(0)} ,
\]

where constants \( S^{(0)}, S_z^{(0)}, \phi \) are restricted by \([82]\). Note that the angular velocity of precession of vector \( S \) around \( B \) is time-dependent. Nevertheless, the helicity \( \langle \mathcal{P} S \rangle \) changes with the constant rate, \( \Omega'_s/\gamma \). Indeed, \( \langle \mathcal{P} S \rangle = S^{(0)} |\mathcal{P}^{(0)}| \cos \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \).

Now we can substitute solutions \( S(t) \) and \( \mathcal{P}(t) \) into equation for \( \vec{x} \)

\[
\frac{dx}{dt} = c |\mathcal{P}^{(0)}| \left( \frac{\Omega_p}{eB} (e_x \cos(\Omega_p t) + e_y \sin(\Omega_p t)) - \frac{2a \Omega_s}{\gamma \mu e} S^{(0)} e_z \cos \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \right).
\]

Integrating the last equation we find the trajectory

\[
x(t) = \vec{x}_c + c |\mathcal{P}^{(0)}| \left( \frac{1}{eB} (e_x \sin(\Omega_p t) - e_y \cos(\Omega_p t)) - \frac{2a \Omega_s}{\gamma \mu e} S^{(0)} e_z \sin \left( \frac{\Omega'_s}{\gamma} t + \phi \right) \right).
\]

(85)

Trajectory represents sum of two motions: circular motion in the plane orthogonal to \( B \) and oscillations along \( B \). These oscillations accompany variations of the helicity. Vector \( \vec{x}_c \) defines the center of circle. The amplitude of oscillations along \( B \)

\[
\Delta z = \frac{2ca \Omega_s}{\Omega'_s \mu e} |\mathcal{P}^{(0)}| S^{(0)} \lesssim \beta \lambda c ,
\]

(86)
FIG. 1: Momentum, velocity, spin and trajectory of a charged spinning particle in the uniform magnetic field less than the Compton wavelength. The trajectory of the particle is shown in Figure 1.

Exact solution for this particular case demonstrates some important features of our model. There are two essentially different situations with $\mu = 1$ and $\mu \neq 1$. In the case of usual magnetic moment $\mu = 1$, helicity is an integral of motion, additional oscillations vanish and the particle moves along circular trajectory in the plane orthogonal to the magnetic field (dotted line in Figure 1).

In the case of anomalous magnetic moment $\mu - 1 \neq 0$, helicity oscillates and affects the trajectory of particle. For a small anomalous magnetic moment helicity and $z$-coordinate oscillate with a very slow rate. These slow oscillations of trajectory along $B$ with the amplitude of Compton wavelength can be called magnetic Zitterbewegung. Magnetic Zitterbewegung in our model appears due to modification of Lorentz force for the spinning particle. Correction to the angular velocity of orbital motion $\Omega_p$ is given by

$$\Omega_p \approx \frac{eB}{\gamma mc} \left( 1 + \frac{e\gamma(\mu - 1)}{m^2c^3}(SB) + o(h, \mu - 1, B) \right),$$

(87)

where $eB/(\gamma mc)$ is the angular velocity for spinless or BMT particle.

Frequency of helicity variations also corrected by high-order terms

$$\frac{\Omega'_s}{\gamma} = (\mu - 1)\frac{eB}{mc} \left( 1 - \frac{e\gamma}{m^2c^3}(1 + (\mu - 1)\beta^2)(SB) + o(h, \mu - 1, B) \right),$$

(88)

from the value $(\mu - 1)\frac{eB}{mc} = (\frac{2}{3} - 1)\frac{eB}{mc}$ computed by Bargmann, Michel and Telegdi [20]. The corrections are small, and for the experiments discussed by them, our equations gives practically the same results. Therefore our model is compatible with these experiments. We believe that other physical situations may be realized, where the corrections could became notable. For instance, this may be the case of quasiparticles with large magnetic moment [17].

V. CONCLUSIONS

In this work we have presented solution to the problem which has been posed by Frenkel in 1926. He noticed that search for variational formulation which takes into account the spin-tensor constraint $J^{\mu \nu} \dot{x}_\nu = 0$ represents rather non trivial problem. He found equations of motion consistent with this condition in the approximation $O^3(J, F, \partial F)$, and when anomalous magnetic moment is $\mu = 1$. We have found Lagrangian action [11] for charged spinning particle which implies all the desired constraints and equations of motion without approximation. They remain consistent for any value of magnetic moment and for an arbitrary electromagnetic background. Besides, due to the constraints
our action guarantees the right number of both spacial and spin degrees of freedom. In the above mentioned approximations, our equations coincide with those of Frenkel. In the recent work [33], we also demonstrated that the classical spinning particle has an expected behavior in arbitrary curved background.

With the Lagrangian and Hamiltonian formulations at hands, we can unambiguously construct quantum mechanics of the spinning particle and establish its relation with the Dirac equation. For the free theory, this has been done in the work [32]. We showed that this gives one-particle sector of the Dirac equation. Due to the second-class constraints \( \partial_{\alpha} F_{\mu \nu} = 0 \), the positions \( x_{i} \) obey to classical brackets with nonvanishing right hand side, see (90). So, in the Dirac theory they realized by non commutative operators which we identified with Pryce (d) center-of-mass [1]. Since namely \( x_{i} \) has an expected behavior (56) as the position of spinning particle in classical interacting theory, our model argue in favor of covariant Pryce (d) operator as the position operator of Dirac theory.

In resume, we have constructed variational formulation for relativistic spin one-half particle which is self consistent and has reasonable behavior on both classical and quantum level.

As we have seen, interaction necessarily modifies some basic relations of the model. In the free theory the conjugated momentum is proportional to velocity, \( p^{\mu} \sim \dot{x}^{\mu} \) and the Frenkel condition holds. This is no more true in interacting theory. The Frenkel condition turns into the Pirani condition, \( J^{\mu \nu} P_{\nu} = 0 \), where the canonical momentum is not collinear to velocity. The advantage of Lagrangian formulation is that this gives exact relation between them (see also Eqs. (38), (39) and (42))

\[
\dot{x}^{\mu} = g_{1}(T^{\mu \nu} P_{\nu} + Y^{\mu}), \quad T^{\mu \nu} = \eta^{\mu \nu} + O(\mu - 1, J), \quad Y^{\mu} = O(\partial F, J) .
\]  

(89)

Only when \( \mu = 1 \) and \( \partial_{\alpha} F_{\mu \nu} = 0 \), the interacting and free theory have the same structure. To see the meaning of the deformation, we can compare our Hamiltonian equations for \( Y^{\mu} = 0 \): \( \dot{x}^{\mu} = g_{1}(T^{\mu \nu} P_{\nu} + Y^{\mu}), \quad T^{\mu \nu} = \frac{2L}{c}(FP)^{\mu \nu} \) with the standard expressions \( \dot{x}^{\mu} = g_{1} P^{\mu}, \quad \dot{P}^{\mu} = \frac{2L}{c}(FP)^{\mu} \). From this it is clear that excluding \( P \) from our equations, we obtain extra contributions to the standard expression for the Lorentz force, \( \ddot{x}^{\mu} = \frac{2L}{c}(F\dot{x})^{\mu} + O(J) \). So the modification [89] mean that complete theory yields an extra spin-orbit interaction as compared with the approximate Frenkel and BMT equations.

We studied possible effects of this spin-orbit interaction in the case of uniform magnetic field. The exact analytical solution was obtained. Besides oscillations of the helicity first calculated by Bargmann, Michel and Telegdi, the particle with anomalous magnetic moment experiences effect of magnetic Zitterbewegung of the trajectory. Usual circular motion in the plane orthogonal to \( B \) perturbed by slow oscillations along \( B \) with the amplitude of order of Compton wavelength. The Larmor frequency [87] and the frequency of helicity oscillations [88] are also shifted by small corrections. It would be interesting to construct an experiment which could detect these possible corrections, for instance due to resonance effects. Another possibility is an artificial simulation of a point-like system with spin and a large anomalous magnetic moment. This can be inspired by simulations of Zitterbewegung itself with a trapped ions [12].

\section{VI. ACKNOWLEDGMENTS}

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Appendix. Dirac brackets

We construct Dirac brackets that take into account the second-class pairs $T_3$, $T_4$, $T_5$ and $T_6$. We will calculate them iteratively in the case of arbitrary electromagnetic background. Then Dirac brackets of the free theory can be obtained by substitution $F^{\mu \nu} = 0$. We start from the pair of second class constraints is $T_6$ and $T_7$,

$$\Delta_{67} = \{T_6, T_7\} = \mathcal{P}^2 + \frac{e}{4c} F_{\mu \nu} J^{\mu \nu}.$$

At the constraint surface $\Delta_{67} = -M^2 c^2$. The Poisson brackets of initial variables with constraints $T_6$ and $T_7$ are given in table [H]. Using table [H] we calculate the Dirac brackets of basic variables with respect to constraints $T_6$ and $T_7$

$$\{Q_1, Q_2\}_{67} = \{Q_1, Q_2\} + \frac{1}{\Delta_{67}} (\{Q_1, T_6\}\{T_7, Q_2\} - \{Q_1, T_7\}\{T_6, Q_2\}).$$
The brackets read

\[
\begin{align*}
\{x^\mu, x^\nu\}_{67} &= -\frac{J^{\mu\nu}}{2\Delta_{67}}, \\
\{x^\mu, p^\nu\}_{67} &= \eta^{\mu\nu} + \frac{e}{2c\Delta_{67}} F^{\mu\nu} \equiv T^{\mu\nu}_{(0)}, \\
\{x^\mu, \omega^\nu\}_{67} &= -\frac{\omega^{\mu\nu} p^\nu}{\Delta_{67}}, \\
\{x^\mu, \pi^\nu\}_{67} &= -\frac{\pi^{\mu\nu} p^\nu}{\Delta_{67}}, \\
\{x^\mu, J^{\alpha\beta}\}_{67} &= \frac{1}{\Delta_{67}} \left( J^{\mu\alpha} p^\beta - J^{\mu\beta} p^\alpha \right), \\
\{\omega^\mu, \omega^\nu\}_{67} &= 0, \\
\{\pi^\mu, \pi^\nu\}_{67} &= 0, \\
\{\omega^\mu, \pi^\nu\}_{67} &= \eta^{\mu\nu} - \frac{p^{\mu\nu}}{\Delta_{67}} \equiv G^{\mu\nu}, \\
\{\omega^\mu, J^{\alpha\beta}\}_{67} &= 2(\omega^{\alpha} G^{\mu\beta} - \omega^\beta G^{\alpha\mu}), \\
\{\pi^\mu, J^{\alpha\beta}\}_{67} &= 2(\pi^{\alpha} G^{\mu\beta} - \pi^\beta G^{\alpha\mu}), \\
\{J^{\mu\nu}, J^{\alpha\beta}\}_{67} &= 2(G^{\mu\alpha} J^{\nu\beta} - G^{\mu\beta} J^{\nu\alpha} - G^{\nu\alpha} J^{\mu\beta} + G^{\nu\beta} J^{\mu\alpha}), \\
\{p^\mu, p^\nu\}_{67} &= \frac{e}{c} F^{\mu\nu} + \frac{e^2}{2\Delta_{67}c^2} (FJF)^{\mu\nu} = \frac{e}{c} F^{\mu\nu} T^{\mu\nu}_{(0)}, \\
\{p^\mu, \omega^\nu\}_{67} &= -\frac{e}{\Delta_{67}c} F^{\mu\alpha} \omega^\alpha p^\nu, \\
\{p^\mu, \pi^\nu\}_{67} &= -\frac{e}{\Delta_{67}c} F^{\mu\alpha} \pi^\alpha p^\nu, \\
\{p^\mu, J^{\alpha\beta}\}_{67} &= -\frac{e}{\Delta_{67}c} F^{\mu\nu}(p^\alpha J^{\nu\beta} - p^\beta J^{\nu\alpha}).
\end{align*}
\]

We have defined tensor \( G^{\mu\nu} \) as the Dirac bracket of spin variables \( \omega^\mu \) and \( \pi^\nu \). Besides, \( T^{\mu\nu}_{(0)} = T^{\mu\nu}(\mu = 0) \), where \( T^{\mu\nu} = \eta^{\mu\nu} - \frac{e^2}{2\Delta_{67}} (JF)^{\mu\nu} \).

On the next step we calculate Dirac brackets for the pair \( \{T_4, T_5\}_{67} = 2(T_4 + a_4) \),

\[
\{Q_1, Q_2\}_{4567} = \{Q_1, Q_2\}_{67} + \frac{1}{2\omega^2} \left( \{Q_1, T_4\}_{67} \{T_5, Q_2\}_{67} - \{Q_1, T_5\}_{67} \{T_4, Q_2\}_{67} \right).
\]

The Dirac brackets \( \{., .\}_{67} \) of initial variables with \( T_4 \) and \( T_5 \) are given in table [III].

From table [III] it is seen that variables \( x^\mu, p^\mu, J^{\mu\nu} \) have vanishing Dirac brackets \( \{., .\}_{67} \) with constraints \( T_4 \) and \( T_5 \). Therefore, new Dirac brackets \( \{., .\}_{4567} \) coincide with old Dirac brackets \( \{., .\}_{67} \) when at least one of arguments is a function \( Z(x^\mu, p^\mu, J^{\mu\nu}) \) of variables \( x^\mu, p^\mu \) and \( J^{\mu\nu} \) only,

\[
\{Z, Q\}_{4567} = \{Z, Q\}_{67}, \quad Z = Z(x^\mu, p^\mu, J^{\mu\nu}).
\]
TABLE III: Constraints vs variables

| Variables | $\{\cdot,\cdot\}$ | $x^\mu$ | $P^\mu$ | $\pi^\mu$ | $\omega^\mu$ | $J^{\mu\nu}$ |
|-----------|------------------|---------|---------|-----------|------------|-------------|
| $T_1 = \omega^2 - \alpha_4$ | 0 | 0 | 2$\omega^\mu$ | 0 | 0 |
| $T_2 = \omega\pi$ | 0 | 0 | $\pi^\mu$ | $-\omega^\mu$ | 0 |

We omit subscripts of brackets, so that $\{\cdot,\cdot\}$ means $\{\cdot,\cdot\}^{4567}$. The only modification in the Dirac brackets comes from the basic variables in the spin sector

\[
\{\omega^\mu, \omega^\nu\} = 0, \quad \{\omega^\mu, \pi^\nu\} = \eta^{\mu\nu} - \frac{P^\mu P^\nu}{\Delta_{67}} - \frac{\omega^\mu \omega^\nu}{\omega^2}, \quad \{\pi^\mu, \pi^\nu\} = -\frac{J^{\mu\nu}}{2\omega^2}.
\]

Thus the complete list of Dirac brackets $\{\cdot,\cdot\}$ consist of expressions (90)-(94), (107)-(109) and (98)-(104).

Now the second class constraints can be put equal to zero, therefore we can rewrite the Hamiltonian as follows

\[
H = g_1 \frac{1}{2} (P^2 - \frac{e\mu}{2c} (FJ) + m^2 c^2).
\]

The constraint $T_3$ can also be excluded since it has zero Dirac brackets with $T_1$ and with all spin-plane invariant variables of the theory. This Hamiltonian generates evolution

\[
\dot{x}^\mu = \{x^\mu, H\}, \quad \dot{P}^\mu = \{P^\mu, H\}, \quad \dot{J}^{\mu\nu} = \{J^{\mu\nu}, H\}.
\]

To check consistency of our calculations, let us obtain equations of motion using the Dirac brackets $\{\cdot,\cdot\}$ (and taking into account that at the constrained surface $\Delta_{67} = -M^2 c^2 = -m^2 c^2 = \frac{\epsilon(2\mu + 1)}{4c^2} (FJ)$).

Equation for coordinate reads

\[
\dot{x}^\mu = \frac{g_1}{2} (x^\mu, P^2 - \frac{e\mu}{2c} (FJ)) =
\]

\[
g_1 \{x^\mu, P^\nu\} P_\nu - g_1 \frac{e\mu}{4c} \{x^\mu, J^{\alpha\beta}\} F_{\alpha\beta} - g_1 \frac{e\mu}{4c} \{x^\mu, F^{\alpha\beta}\} J_{\alpha\beta} =
\]

\[
g_1 \left( \eta^{\mu\nu} - \frac{e}{2M^2 c^3} J^{\mu\alpha} F_\alpha^\nu \right) P_\nu - g_1 \frac{e\mu}{4M^2 c^3} \left( F^\alpha J^{\mu\beta} - F^{\beta\mu} J^{\alpha\nu} \right) F_{\alpha\beta} - g_1 \frac{e\mu}{8M^2 c^3} J^{\mu\nu} \partial_\nu (FJ) =
\]

\[
g_1 \left( \eta^{\mu\nu} - \frac{e}{2M^2 c^3} J^{\mu\alpha} F_\alpha^\nu \right) P_\nu + g_1 \frac{e\mu}{2M^2 c^3} J^{\mu\beta} F_{\beta\alpha} P_\alpha - g_1 \frac{e\mu}{8M^2 c^3} J^{\mu\nu} \partial_\nu (FJ) =
\]

\[
g_1 \left( \eta^{\mu\nu} + \frac{e(\mu - 1)}{2M^2 c^3} J^{\mu\alpha} F_\alpha^\nu \right) P_\nu - g_1 \frac{e\mu}{8M^2 c^3} J^{\mu\nu} \partial_\nu (FJ) = g_1 u^\mu,
\]

Equation for momentum reads

\[
\dot{P}^\mu = \frac{g_1}{2} \{P^\mu, P^2 - \frac{e\mu}{2c} (FJ)\} =
\]

\[
g_1 \{P^\mu, P^\nu\} P_\nu - g_1 \frac{e\mu}{4c} \{P^\mu, J^{\alpha\beta}\} F_{\alpha\beta} - g_1 \frac{e\mu}{4c} \{P^\mu, x^\rho\} \partial_\rho F^{\alpha\beta} J_{\alpha\beta} =
\]

\[
g_1 \left( \frac{e}{c} F^{\mu\nu} + \frac{e^2}{2M^2 c^4} F^{\mu\alpha} F_\alpha^\nu J_{\alpha\beta} \right) P_\nu + g_1 \frac{e\mu}{2M^2 c^3} F^{\mu\nu} P_\nu J^{\beta\gamma} F_{\beta\gamma} +
\]
evolution we can impose the gauge condition form a pair of second class constraints. We have

\[ \text{Equation for spin-tensor reads} \]

\[ j^{\alpha\beta} = \frac{g_1}{2} \left( J^{\alpha\beta}, \mathcal{P}^2 - \frac{e\mu}{2c} (FJ) \right) = 0 \]

\[ = g_1 \{ J^{\alpha\beta}, \mathcal{P}^\mu \} \mathcal{P}_\mu - \frac{e\mu}{4c} \{ J^{\alpha\beta}, J^{\mu\nu} \} F_{\mu\nu} - g_1 \frac{e\mu}{4c} \{ J^{\alpha\beta}, x^\mu \} \partial_\mu (FJ) = 0 \]

The Hamiltonian is proportional to the first class constraint \( T_1 \). Therefore equations of motion contain arbitrary function \( g_1(\tau) \) which is related with reparameterization invariance of the model. To obtain unambiguous equations of evolution we can impose the gauge \( x^\mu = ct \). The gauge is often called canonical gauge. Constraint \( T_1 \) together with this condition form a pair of second class constraints. We have

\[ \{ x^0 - ct, T_1 \} = 2u^0, \]

and Dirac brackets in the canonical gauge read

\[ \{ Q_1, Q_2 \}_\tau = \{ Q_1, Q_2 \} + \frac{1}{2u^0} \left( \{ Q_1, G_1 \} \{ T_1, Q_2 \} - \{ Q_1, T_1 \} \{ G_1, Q_2 \} \right). \]

The Dirac brackets of constraint \( T_1 \) and canonical gauge \( G_1 \) with physical variables are given in table IV. There compact notations

\[ \hat{\mathcal{P}}^\mu \equiv \frac{g_1}{2} \{ \mathcal{P}^\mu, T_1 \}, \]

\[ j^{\mu\nu} \equiv \frac{g_1}{2} \{ J^{\mu\nu}, T_1 \}, \]

\[ u^\mu \equiv \left( \eta^{\mu\nu} + \frac{e(\mu - 1)}{2M^2c^3} J^{\mu\alpha} F_{\alpha\nu} \right) \mathcal{P}_\nu - \frac{e\mu}{8M^2c^3} J^{\mu\nu} \partial_\nu (FJ). \]
TABLE IV: Constraints vs variables (canonical gauge)

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\{,\} & \{\} & x^\mu & \mathcal{P}_\mu & \omega^{\mu} & \pi^\mu & J^{\mu\nu} \\
\hline
T_1 = \mathcal{P}^2 = \frac{e}{2c} (F \mathcal{P}) & -2u^\mu & -2 \frac{\mathcal{P}_\mu}{\Delta_{67}} & \frac{2 \omega^{\mu} (F \omega)^\mu}{\Delta_{67}} & \frac{2 \pi^{\mu}}{\Delta_{67}} & \frac{-2}{\Delta_{67}} & \frac{2}{\Delta_{67}} \\
G_1 = x^\beta - c \tau & \frac{1}{2 \Delta_{67}} J^{0\mu} & \frac{\tau}{\Delta_{67}} & \frac{\omega^{0} \mu}{\Delta_{67}} & \frac{\pi^{0} \mu}{\Delta_{67}} & \frac{-1}{\Delta_{67}} & \frac{1}{\Delta_{67}} \\
\hline
\end{array}
\]

\[
C = -\frac{e}{c} (\mu - 1) (\omega \mathcal{P}) + \frac{e \mu}{4c} (\omega \partial)(F \mathcal{P}),
\]
\[
D = -\frac{e}{c} (\mu - 1) (\pi \mathcal{P}) + \frac{e \mu}{4c} (\pi \partial)(F \mathcal{P}),
\]

were used.

The Dirac brackets which take into account the canonical gauge are as follow.

Spacial sector:

\[
\{x^\mu, x^\nu\}_\tau = \frac{-1}{2u^0 \Delta_{67}} \left( u^0 J^{\mu\nu} - u^\mu J^{0\nu} + u^\nu J^{0\mu} \right),
\]
(112)

\[
\{x^\mu, \mathcal{P}^\nu\}_\tau = \eta^{\mu\nu} - \frac{u^\mu}{u^0} \eta^{0\nu} + \frac{e}{2u^0 \Delta_{67}} \left( u^0 J^{\alpha\nu} - u^\mu J^{0\nu} + u^\nu J^{0\mu} \right) F^{\alpha\nu}_\tau - \frac{e \mu}{8u^0 e \Delta_{67}} J^{0\mu} \mathcal{P}^\nu (F \mathcal{P}),
\]
(113)

\[
\{\mathcal{P}^\mu, \mathcal{P}^\nu\}_\tau = \frac{e}{u^0 c} \left( u^0 F^{\mu}_\tau T^{\nu}_0 - \frac{c}{e g_1} \mathcal{P}^{\mu}_\tau T^{\nu}_0 + \frac{c}{e g_1} \mathcal{P}^{\nu}_\tau T^{\mu}_0 \right).
\]
(114)

Frenkel sector:

\[
\{J^{\mu\nu}, J^{\alpha\beta}\}_\tau = \{J^{\mu\nu}, J^{\alpha\beta}\} - \frac{1}{g_1 u^0 \Delta_{67}} \left( J^{\mu\nu} J^{0[\beta} \mathcal{P}^{\alpha]} - J^{\alpha\beta} J^{0[\nu} \mathcal{P}^{\mu]} \right),
\]
(115)

\[
\{x^\mu, J^{\alpha\beta}\}_\tau = \frac{-1}{u^0 \Delta_{67}} \left( u^\mu J^{0[\alpha} \mathcal{P}^{\beta]} - u^\beta J^{0[\alpha} \mathcal{P}^{\beta]} \right) - \frac{1}{2u^0 \Delta_{67} g_1} J^{0\mu} J^{\alpha\beta},
\]
(116)

\[
\{\mathcal{P}^\mu, J^{\alpha\beta}\}_\tau = \frac{e}{u^0 c \Delta_{67}} F^{\mu\nu}_\tau \left( u^0 \mathcal{P}^{[\alpha} J^{\beta\nu]} - u^\nu \mathcal{P}^{[\beta} J^{\alpha\nu]} \right) - \frac{1}{u^0 g_1} T^{0\mu}_0 J^{\alpha\beta} - \frac{e \mu}{4u^0 c \Delta_{67}} \left( F^{\mu} J^{\beta\alpha} \right) \mathcal{P}^{[\beta} J^{\alpha\nu]}.
\]
(117)

Basic spin variables:

\[
\{\omega^\mu, \omega^\nu\}_\tau = -\frac{e \mu \omega^0}{2u^0 \Delta_{67}} (F^{\mu\alpha} \omega^\nu - F^{\nu\alpha} \omega^\mu) \omega^\alpha,
\]
(118)

\[
\{\omega^\mu, \pi^\nu\}_\tau = \eta^{\mu\nu} - \frac{\mathcal{P}^{\mu\pi\nu}}{\Delta_{67}} \left( 1 + \frac{\omega^0 \mathcal{C} + \omega^0 \mathcal{D}}{u^0 \Delta_{67}} \right) - \frac{\omega^{\mu} \omega^{\nu}}{\omega^2} - \frac{e \mu \omega^0}{u^0 c \Delta_{67}} (F^{\mu} \omega^\alpha \omega^\nu - \omega^{0} \mathcal{P}^{\nu} \mathcal{P}^{\alpha})
\]
(119)

\[
\{\pi^\mu, \pi^\nu\}_\tau = \frac{-J^{\mu\nu}}{2\omega^2} - \frac{e \mu \omega^0}{2u^0 \Delta_{67}} (F^{\mu\nu} \mathcal{P}^{\nu} - F^{\nu\mu} \mathcal{P}^{\mu}) \mathcal{P}^{\alpha}.
\]
(120)

Other mixed brackets:

\[
\{x^\mu, \omega^\nu\}_\tau = \frac{-\omega^\mu \omega^\nu}{\Delta_{67}} + \frac{1}{u^0} \left( \frac{J^{0\mu}}{2\Delta_{67}} \left( C \mathcal{P}^{\nu} - \frac{e \mu}{c} (F \omega)^\nu \right) + \frac{u^\mu \omega^0 \mathcal{P}^{\nu}}{\Delta_{67}} \right),
\]
(121)

\[
\{x^\mu, \pi^\nu\}_\tau = \frac{-\pi^\mu \pi^\nu}{\Delta_{67}} + \frac{1}{u^0} \left( \frac{J^{0\mu}}{2\Delta_{67}} \left( D \mathcal{P}^{\nu} - \frac{e \mu}{c} (F \mu)^\nu \right) + \frac{u^\mu \pi^0 \mathcal{P}^{\nu}}{\Delta_{67}} \right),
\]
(122)

\[
\{\omega^\mu, J^{\alpha\beta}\}_\tau = 2(\omega^\alpha C^{\mu\beta} - \omega^\beta C^{\mu\alpha}) - \frac{1}{u^0 \Delta_{67}} \left( \omega^0 \mathcal{P}^{\mu} J^{\alpha\beta} \frac{g_1}{1} + \left( C \mathcal{P}^{\mu} \frac{\Delta_{67}}{\Delta_{67}} - \frac{e \mu}{c} (F \omega)^\mu \right) J^{0[\beta} \mathcal{P}^{\alpha]} \right),
\]
(123)

\[
\{\pi^\mu, J^{\alpha\beta}\}_\tau = 2(\pi^\alpha C^{\mu\beta} - \pi^\beta C^{\mu\alpha}) - \frac{1}{u^0 \Delta_{67}} \left( \pi^0 \mathcal{P}^{\mu} J^{\alpha\beta} \frac{g_1}{1} + \left( D \mathcal{P}^{\mu} \frac{\Delta_{67}}{\Delta_{67}} - \frac{e \mu}{c} (F \pi)^\mu \right) J^{0[\beta} \mathcal{P}^{\alpha]} \right),
\]
(124)

\[
\{\mathcal{P}^\mu, \omega^\nu\}_\tau = -\left( \frac{\omega^\mu \omega^\nu}{\Delta_{67}} - \frac{1}{u^0} \left( \frac{\mathcal{P}^{0\mu}}{\Delta_{67}} - \frac{e \mu}{c} (F \omega)^\nu \right) - \frac{\mathcal{P}^{\mu} \omega^0 \mathcal{P}^{\nu}}{g_1 \Delta_{67}} \right),
\]
(125)

\[
\{\mathcal{P}^\mu, \pi^\nu\}_\tau = -\left( \frac{\mathcal{P}^{0\mu}}{\Delta_{67}} - \frac{1}{u^0} \left( \frac{\mathcal{P}^{0\mu}}{\Delta_{67}} - \frac{e \mu}{c} (F \pi)^\nu \right) - \frac{\mathcal{P}^{\mu} \pi^0 \mathcal{P}^{\nu}}{g_1 \Delta_{67}} \right).
\]
(126)
In the free theory the algebra of Dirac brackets simplifies significantly. In this case $P_\mu = p_\mu$, $u_\mu = p_\mu$, $J_{\mu\nu} = \dot{P}_\mu = 0$, $\triangle_{\alpha\beta} = p^2$, and in an arbitrary parametrization $\tau$, we have the following brackets

Basic variables of spin:

$$\{\omega^\mu, \omega^\nu\} = 0, \quad \{\omega^\mu, \pi^\nu\} = g^{\mu\nu} - \frac{\omega^\mu\omega^\nu}{\omega^2}, \quad \{\pi^\mu, \pi^\nu\} = -\frac{1}{2\omega^2} J^{\mu\nu};$$  \hfill (127)

$$\{x^\mu, \omega^\nu\} = -\frac{\omega^\mu p^\nu}{p^2}, \quad \{x^\mu, \pi^\nu\} = -\frac{\pi^\mu p^\nu}{p^2};$$  \hfill (128)

Spacial sector:

$$\{x^\mu, x^\nu\} = -\frac{1}{2p^2} J^{\mu\nu}, \quad \{x^\mu, p^\nu\} = \eta^{\mu\nu}, \quad \{p^\mu, p^\nu\} = 0.$$  \hfill (129)

Frenkel sector:

$$\{J^{\mu\nu}, J^{\alpha\beta}\} = 2(g^{\mu\alpha}J^{\nu\beta} - g^{\mu\beta}J^{\nu\alpha} - g^{\nu\alpha}J^{\mu\beta} + g^{\nu\beta}J^{\mu\alpha}),$$  \hfill (130)

$$\{x^\mu, J^{\alpha\beta}\} = \frac{1}{p^2} (J^{\mu\alpha}p^\beta - J^{\mu\beta}p^\alpha),$$  \hfill (131)

BMT-sector: take $s_\mu = \frac{1}{4\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta}$, then

$$\{s^\mu, s^\nu\} = -\frac{1}{\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_\alpha s^\beta = \frac{1}{2} J^{\mu\nu},$$  \hfill (132)

$$\{x^\mu, s^\nu\} = -\frac{s^\mu p^\nu}{p^2} = -\frac{1}{4\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} - \frac{p^\mu s^\nu}{p^2}.$$  \hfill (133)

Other Dirac brackets vanish. In the equations (127) and (141) it has been denoted

$$g^{\mu\nu} = \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}.$$  \hfill (134)

Together with $\tilde{g}^{\mu\nu} \equiv \frac{p^\mu p^\nu}{p^2}$, this forms a pair of projectors

$$g + \tilde{g} = 1, \quad g^2 = g, \quad \tilde{g}^2 = \tilde{g}, \quad g\tilde{g} = 0.$$  \hfill (135)

The free Dirac brackets which take into account the canonical gauge are as follows.

Basic variables of spin:

$$\{\omega^\mu, \omega^\nu\}_\tau = 0, \quad \{\omega^\mu, \pi^\nu\}_\tau = g^{\mu\nu} - \frac{\omega^\mu\omega^\nu}{\omega^2}, \quad \{\pi^\mu, \pi^\nu\}_\tau = -\frac{1}{2\omega^2} J^{\mu\nu},$$  \hfill (136)

$$\{x^\mu, \omega^\nu\}_\tau = -\frac{\omega^\mu p^\nu}{p^2} + \frac{\omega^0 p^\mu p^\nu}{p^2 p^2},$$  \hfill (137)

$$\{x^\mu, \pi^\nu\}_\tau = -\frac{\pi^\mu p^\nu}{p^2} + \frac{\pi^0 p^\mu p^\nu}{p^2 p^2}.$$  \hfill (138)

Spacial sector:

$$\{x^\mu, x^\nu\}_\tau = -\frac{1}{2p^2 p^2} (p^0 J^{\mu\nu} - p^\mu J^{0\nu} - p^\nu J^{0\mu}) +$$  \hfill (139)

$$\{x^\mu, p^\nu\}_\tau = \eta^{\mu\nu} - \frac{p^\mu}{p^2} \eta^{0\nu}, \quad \{p^\mu, p^\nu\}_\tau = 0.$$  \hfill (140)
Frenkel sector:

\[
\{J^{\mu\nu}, J^{\alpha\beta}\}_r = 2(g^{\mu\alpha} J^{\nu\beta} - g^{\mu\beta} J^{\nu\alpha} - g^{\nu\alpha} J^{\mu\beta} + g^{\nu\beta} J^{\mu\alpha}),
\]

(141)

\[
\{x^{\mu}, J^{\alpha\beta}\}_r = \frac{-1}{p^0 p^2} \left( p^\mu J^{[\alpha \beta]} - p^{[\alpha} p^{\beta]} \right),
\]

(142)

BMT-sector:

\[
\{s^{\mu}, s^{\nu}\}_r = -\frac{1}{\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} p_\alpha s_\beta = \frac{1}{2} J^{\mu\nu},
\]

(143)

\[
\{x^{\mu}, s^{\nu}\}_r = -\frac{1}{4\sqrt{-p^2}} \epsilon^{\mu\nu\alpha\beta} J_{\alpha\beta} + \frac{p^\mu}{4p^0} \frac{\epsilon^{0\nu\alpha\beta} J_{\alpha\beta}}{\sqrt{-p^2}} = \frac{(s^0 p^\mu p^\nu - s^\mu p^0 p^\nu)}{p^0 p^2}.
\]

(144)

Other Dirac brackets vanish.

Here we define Dirac brackets for all phase space variables. After transition to the Dirac brackets the second-class constraints can be used as strong equalities, therefore it is enough to consider Dirac brackets at the constraint surface only. Then, explicit form of the Dirac brackets depends on the choice of independent variables. For instance, in the free theory considered in the gauge of physical time we can present \(s^0_{BMT}\) and \(p^0\) in terms of independent variables \(s_{BMT}, p, x\)

\[
s^0 = \frac{(s \cdot p)}{\sqrt{p^2 + (mc)^2}}, \quad p^0 = \sqrt{p^2 + (mc)^2}.
\]

(145)

The non vanishing Dirac brackets are

\[
\{x^i, x^j\}_D = \frac{\epsilon^{ijk} s_k}{mc p^0}, \quad \{x^i, p^j\}_r = \delta^{ij}, \quad \{p^i, p^j\}_r = 0,
\]

(146)

\[
\{s^i, s^j\}_r = \frac{p^0}{mc} \epsilon^{ijk} \left( s_k - \frac{(s \cdot p) p_k}{p_0^2} \right),
\]

(147)

\[
\{x^i, s^j\}_D = \left( s^i - \frac{(s \cdot p) p^i}{p_0^2} \right) \frac{p^j}{(mc)^2}.
\]

(148)