On an Extension of a Theorem of Eilenberg and a Characterization of Topological Connectedness

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December 24, 2019

Abstract: On taking a non-trivial and semi-transitive bi-relation constituted by two (hard and soft) binary relations, we report a (i) p-continuity assumption that guarantees the completeness and transitivity of its soft part, and a (ii) characterization of a connected topological space in terms of its attendant properties on the space. Our work generalizes antecedent results in applied mathematics, all following Eilenberg (1941), and now framed in the context of a parametrized-topological space. This re-framing is directly inspired by the continuity assumption in Wold (1943–44) and the mixture-space structure proposed in Herstein and Milnor (1953), and the unifying synthesis of these pioneering but neglected papers that it affords may have independent interest.

Mathematics Subject Classification. 91B55, 37E05.

Key Words: Bi-relation, non-trivial, semi-transitive, complete, transitive, connected, p-continuity, parametrically-topologized space

†The two theorems reported here were announced without proof at the 2019 NSF/NBER/CEME Conference on Mathematical Economics: In Honor of Robert M. Anderson in Berkeley, October 25, 2019; and more comprehensively, at a departmental seminar at the Australian National University on December 17, 2019. The authors acknowledge with gratitude extended correspondence and conversation with Professors Max Amarante, Yorgos Gerasimou, Alfio Giarlotta, Farhad Husseinov, Michael Mandler, Rich McLean and Debraj Ray. Needless to say, all errors of reading and interpretation are solely the authors’. We use the certified random order in order to list the authors; see Ray-Robson (2018, American Economic Review).

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1 Introduction

The question considered in this paper concerns a binary relation $R$ on a set $X$ conceived as a subset of $X \times X$, with its transpose, and upper and lower section at $x \in X$ respectively defined as $R^{-1} = \{(x, y) \mid (y, x) \in R\}$, $R(x) = \{y \mid (x, y) \in R\}$, and $R^{-1}(x) = \{y \mid (y, x) \in R\}$. With $\Delta = \{(x, x) \mid x \in X\}$, $R^c$ the complement of $R$, its symmetric and asymmetric parts respectively denoted as $I = R \cap R^{-1}$, and $P = R \setminus R^{-1}$, $I \cap P = \emptyset$, and $R = I \cup P$, we let the composition $R \circ R'$ be given by $(y, x) \in R \circ R'$ if $R^{-1}(x) \cap R'(y) \neq \emptyset$, for any two relations $R, R'$ on a $X$, and call a relation $R$ on $X$ non-trivial if $P \neq \emptyset$, semi-transitive if $I \circ P \subseteq P$ and $P \circ I \subseteq P$, transitive if $R \circ R \subseteq R$, (negatively transitive if $R^c$ is transitive), and complete if $R \cup R^{-1} = X \times X$. We can now ask for a sufficient condition in any register that ensures that a non-trivial, semi-transitive relation with a transitive symmetric part is complete and transitive. Thus, rather than denoting $R$ by $\preceq$, as is standard especially in the social sciences, this is to ask for any register that ensures, in the vernacular of set-theory,

\[(P \neq \emptyset) \land ((I \circ P \subseteq P) \land (P \circ I \subseteq P)) \land (I \circ I \subseteq I) \implies (R \cup R^{-1} = X \times X) \land (R \circ R \subseteq R).\]

In his remarkable paper, Eilenberg [1941] provided a partial answer to this question by invoking a topological register and assuming a continuous binary (preference) relation on a connected (choice) set. His answer inaugurated the study of a partially ordered topological space, and his work received important substantive extension at the hands of Debreu [1954, 1964], Ward [1954a and 1954b], Sonnenschein [1963, 1967], McCartan [1966] and Schmeidler [1971]. In the twin-registers of economic and decision theory, the extension involved a move to a setting of total pre-orders, which is to say from a setting of singleton indifference sets to a more general situation where the preference map of a consumer is delineated by indifference surfaces.

Recent work in both mathematical economics and mathematical psychology has rediscovered these original papers, and under the label of the Eilenberg-Sonnenschein (ES) research program, has been particularly stimulated by what it sees as the derivation of behavioral consequences of merely technical topological assumptions; see, for example, Khan and Uyanık [2019], Giarlotta and Watson [2019] and their references. Eilenberg, and following him Ward, phrased their results only in the language of topological structures, but Herstein and Milnor [1953], with Wold [1943–44] as their important precursor, focused on the functional representation of the relation, and shifted all topological assumptions on the choice set to that on the unit interval. This is to say, they focused their attention on the mixing operation, rather than on the objects of choice itself. Wold [1943–44] used a similar scalar-continuity property in his work, but with an additional monotonicity assumption; also see Fishburn [1982], Wakker [1989], Bridges and Mehta [1995], Herden and Pallack [2001] and Candeal, Induráin and Molina [2012], Galaabaatar, Khan and Uyanık [2019] on numerical representation of preferences. In addition to this parametrized topological setting, we also mention for the record, a third setting, that involves no topology at all, but constrains itself to a purely algebraic structure. This goes back at least

\[\vdash\]
to seminal paper of Holder (1901); also see Fishburn (1972) and Luce (2000).

In this paper we work with a richer order-theoretic structure defined by two binary relations instead of one, a structure that is only now being appreciated and given prominence; see Giarlotta and Greco (2013), Giarlotta (2014), Giarlotta and Watson (2019) and Yumur (2019) and their references; also see Chipman (1971) and Maceroni, Marinacci and Rustichini (2006) for an implicitly assumed bi-preference structure. In this richer structure, we introduce a parametric continuity concept for (bi-)relations that does not assume any structure on the choice set itself, and is weaker than the usual continuity properties that require it to have one of the topological or algebraic structures. We report two theorems, one of which provides a new characterization of topological connectedness, and show by examples that our results are non-vacuous. Even in the special case of a single binary relation, our first result provide a synthetic treatment of the antecedent ES literature by generalizing and unifying it, and in particular, our analytical treatment provides an alternative proof of the theorems of Eilenberg (1941), Sonnenschein (1965, 1967) and Rader (1963).

2 Notational and Conceptual Preliminaries

We elaborate the terminology presented in Section 1 to the setting of a bi-relation.

**Definition 1.** A bi-relation on a set $X$ is a pair $(R_H, R_S)$ of relations on $X$ such that $R_H \subseteq R_S$ and $P_S \subseteq P_H$. A bi-relation $(R_H, R_S)$ is non-trivial if $P_S \neq \emptyset$, and semi-transitive if

(i) $R_H$ is semi-transitive,

(ii) $I_S \circ R_H \subseteq R_S$ and $R_H \circ I_S \subseteq R_S$,

(iii) $P_H \cap (P_S \circ I_S) \subseteq P_S$ and $P_H \cap (I_S \circ P_S) \subseteq P_S$.

For any bi-relation $(R_H, R_S)$, we informally refer to $R_H$ as its hard part, and $R_S$ as its soft part. We now present the basic continuity assumption that motivates this paper and is the subject of its investigation.

**Definition 2.** A bi-relation $(R_H, R_S)$ on a set $X$ is $p$-continuous (parametrically continuous) if for all $x, y \in X$, there exist a topological space $\Lambda_{xy}$ and a function $f_{xy} : \Lambda_{xy} \to X$ with $x, y \in f_{xy}(\Lambda_{xy})$ such that for all $z \in X$,

(i) $f_{xy}^{-1}(R_H(z))$ and $f_{xy}^{-1}(R_H^{-1}(z))$ are closed, and

(ii) $f_{xy}^{-1}(P_S(z))$ and $f_{xy}^{-1}(P_S^{-1}(z))$ are open.

We call a $p$-continuous bi-relation connected if each space $\Lambda_{xy}$ in Definition 2 is connected. The concept is admittedly abstract, but one can get a feel for it by dissociating the two aspects of Herstein and Milnor (1953) that it generalizes: first, keeping the unit interval, and focusing on the fact that rather than a linear function, any function is being considered; and second, replacing the unit interval by any topological space, both operations localized by the dependence on the pair $(\Lambda_{xy}, f_{xy})$ on the two chosen points, $x$ and $y$.

We now recall for the reader the standard
Definition 3. A topological space $X$ is connected if it is not the union of two non-empty, disjoint open sets. A subset of $X$ is connected if it is connected as a subspace.

A relation $R$ on a set $X$ is $p$-continuous if the bi-relation $(R, R)$ is $p$-continuous, and it is connected if the bi-relation $(R, R)$ is connected. The reader may wish to contrast this definition to Fishburn (1972, p. 27). Moreover,

Definition 4. A relation on a topological space is continuous if it has closed sections and its asymmetric part has open sections.

3 The Results and their Proofs

We can now present our first result,

Theorem 1. The soft part of every non-trivial, semi-transitive and connected bi-relation is complete and transitive.

Next, we present a special case of Theorem 1 for a binary relation. Note that a relation $R$ on a set $X$ is non-trivial, semi-transitive, connected and has a transitive symmetric part if and only if the bi-relation $(R, R)$ on $X$ is non-trivial, semi-transitivity and connected. On setting $R_H = R_S$ in a bi-relation $(R_H, R_S)$, we obtain the following as a corollary of Theorem 1.

Corollary 1. Every non-trivial, semi-transitive and connected binary relation whose symmetric part is transitive, is complete and transitive.

In the light of the above discussion, we can now present

Theorem (Eilenberg 1941). Every anti-symmetric, complete and continuous relation on a connected set is transitive.

Theorem 1 provides a generalization and alternative proof not only of Eilenberg (1941, 2.1), but also of the extensions pursued in Rader (1963) and Sonnenschein (1965). Without any attempt to downplay the importance of the exercise, we invite the reader to show that the principal results in the literature follow as a consequence of Theorem 1 and Corollary 1 above, and Lemma 1 and Propositions 1 and 2 below: Rader (1963, Lemma), Sonnenschein (1965, Theorem 3), Schmeidler (1971, Theorem), Dubra (2011, Theorem 1), Karni and Safra (2015, Theorem 1), McCarthy and Mikkola (2018, Theorem 1), Khan and Uyanık (2019, Proposition 1); Giarlotta and Watson (2019, Theorem 5.2) and finally Uyanık (2019b, Theorems 1 and 2). In our judgment, this exercise testifies to the importance of the results presented here. (Note, however, that the authors of the above papers present their results in an extended form that also involve consideration not germane to those pursued here; our generalization concerns only the relevant part of their results.) In the sequel, we also illustrate the novelty of our proof-techniques by contrasting them with earlier proofs.

We now turn to the proofs, and begin by recalling the following.

Lemma 1. For any binary relation $R$ on a set $X$ the following is true:
(a) if $R$ is complete and semi-transitive, then $I$ is transitive;
(b) if $P$ is negatively transitive, then it is transitive and $R$ is semi-transitive;
(c) $R$ is transitive if and only if it is semi-transitive and $P, I$ are transitive.

For a proof, see Sen (1969, Theorem I) and Khan and Uyanık (2019, Proposition 2) on Sen’s deconstruction of the transitivity postulate.

We shall also need the following two claims concerning a non-trivial, semi-transitive and connected bi-relation $(R_H, R_S)$ on an arbitrary set $X$.

Claim 1. If $(y, x) \in P_S$, then $P_S(y) \cup \overline{P_S}(x) = X$.

Claim 2. $R_S \cup \overline{R_S} = X \times X$.

The proof of Theorem follows as a direct consequence of these preliminary results.

**Proof of Theorem** Claim 2 already establishes the completeness of the soft part of the bi-relation. Now negative transitivity of $P_S$ is equivalent to Claim Then (b) of Lemma implies $R_S$ is semi-transitive and $P_S$ is transitive. It follows from Claim semi-transitivity of $R_S$ and (a) of Lemma that $I_S$ is transitive. Then (c) of Lemma implies $R_S$ is transitive.

All that remains now are the proofs of the two claims.

**Proof of Claim** Pick $(y, x) \in P_S$. Then $(y, x) \in P_H$. It follows from semi-transitivity of the bi-relation that

$$R_H(y) \cup \overline{R_H}(x) = P_S(y) \cup \overline{P_S}(x).$$

(1)

One direction of the inclusion relationship immediately follows from the definition of bi-relation. In order to prove the other direction pick $z \in R_H^{-1}(x)$. Then either $z \in I_H(x)$ or $z \in \overline{R_H}(x)$.

Assume $z \in I_H(x)$. It follows from the definition of bi-relation that $z \in I_S(x)$. Moreover, $x \in P_H(y)$, $z \in I_H(x)$ and semi-transitivity of $R_H$ imply $(y, z) \in P_H$. Note that $z \in I_S(x)$ and $x \in P_S(y)$ imply $(y, z) \in I_S \circ P_S$. Then semi-transitivity of the bi-relation implies $z \in P_S(y)$. Now assume $z \in P_H^{-1}(x)$. Then the definition of bi-relation implies that either $x \in P_S(z)$ or $x \in I_S(z)$.

If $x \in P_S(z)$, then there is nothing to prove. Now assume $x \in I_S(z)$. Then semi-transitivity of the bi-relation, $z \in I_S(x)$ and $x \in P_H(y)$ imply that $z \in R_S(y)$. Hence either $z \in P_S(y)$ or $z \in I_S(y)$. If $z \in I_S(y)$, then semi-transitivity of the bi-relation, $x \in P_S(y), y \in I_S(z)$ and $x \in P_H(z)$ imply that $x \in P_S(z)$. This contradicts $x \in I_S(z)$. Hence, $z \in P_S(y)$ must hold. The proof for $z \in R_H(y)$ is analogous.

We next prove that $P_S(y) \cup \overline{P_S}(x) = X$. To this end pick $z \in X$. It follows from the connectedness of the bi-relation that there exist a connected topological space $\Lambda_{yz}$ and a function $f_{yz} : \Lambda_{yz} \to X$ satisfying the conditions in Definition above that

$$f_{yz}^{-1}(R_H(y)) \cup f_{yz}^{-1}(R_H^{-1}(x)) = P_S(y) \cup f_{yz}^{-1}(P_S^{-1}(x)).$$

(2)

It follows from $x \in P_S(y)$ and $y = f_{yz}(\lambda)$ for some $\lambda \in \Lambda_{yz}$ that $\lambda \in f_{yz}^{-1}(P_S^{-1}(x))$. Hence the set in Equation is non-empty. It follows from p-continuity that it is both closed and open in $\Lambda_{yz}$. Therefore, as a nonempty, closed and open subset of a connected set $\Lambda_{yz}$, the set $f_{yz}^{-1}(P_S(y)) \cup f_{yz}^{-1}(P_S^{-1}(x))$ is equal to $\Lambda_{yz}$. Then it follows from $z = f_{yz}(\delta)$ for some $\delta \in \Lambda_{yz}$ that $\delta \in f_{yz}^{-1}(P_S(y)) \cup f_{yz}^{-1}(P_S^{-1}(x))$, hence $z \in P_S(y)$ or $z \in P_S^{-1}(x)$.


Proof of Claim 2. Assume there exists \( u, v \in X \) such that \((u, v) \not\in R_S \cup R_S^{-1}\). Note that non-triviality of \( R_S \) implies that \((y, x) \in P_S \) for some \( x, y \in X \). Since \( P_S \) is negatively transitive, then \( u \in P_S^{-1}(\bar{x}) \) or \( u \in P_S(\bar{y}) \). Assume \( u \in P_S^{-1}(\bar{x}) \). Then negative transitivity of \( P_S \) implies that \( v \in P_S^{-1}(\bar{x}) \) or \( v \in P_S(u) \). Since \( u, v \not\in R_S \cup R_S^{-1} \), therefore \( v \in P_S^{-1}(\bar{x}) \). Hence \( \bar{x} \in P_S(u) \cap P_S(v) \).

The semi-transitivity of the bi-relation implies that

\[
I_H(u) \cap R_H(v) = \emptyset = R_H(u) \cap I_H(v).
\]

(3)

In order to prove the first equality, assume there exists \( z \in I_H(u) \cap R_H(v) \). If \( z \in H(v) \), then semi-transitivity of \( R_H \) implies that \( u \in P_H(v) \), hence \( u \in R_S(v) \). This yields a contradiction. Then assume \( z \in I_H(v) \). The definition of bi-relation implies that \( z \in I_S(v) \). Then it follows from semi-transitivity of bi-relation that \((u, v) \in R_S \). This yields a contradiction. The proof of the second equality is analogous.

Therefore, Equation 3 and the definition of the bi-relation imply that

\[
R_H(u) \cap R_H(v) = P_S(u) \cap P_S(v).
\]

(4)

It follows from the connectedness of the bi-relation that there exist a connected topological space \( \Lambda_{xu} \) and a function \( f_{xu} : \Lambda_{xu} \to X \) satisfying the conditions in Definition 2. It follows from Equation 1 above that

\[
f_{xu}^{-1}(R_H(u)) \cap f_{xu}^{-1}(R_H(v)) = f_{xu}^{-1}(P_S(u)) \cap f_{xu}^{-1}(P_S(v)).
\]

(5)

It follows from \( \bar{x} \in P_S(u) \cap P_S(v) \) and \( \bar{x} = f_{xu}(\lambda) \) for some \( \lambda \in \Lambda_{xu} \) that \( \lambda \in f_{xu}^{-1}(P_S(u)) \cap f_{xu}^{-1}(P_S(v)) \). Analogously it follows from \( u \not\in P_S(u) \) and \( u = f_{xu}(\delta) \) for some \( \delta \in \Lambda_{xu} \) that \( \delta \not\in f_{xu}^{-1}(P_S(u)) \cap f_{xu}^{-1}(P_S(v)) \). Hence the set if Equation 2 is non-empty and proper subset of \( \Lambda_{yz} \). It follows from \( \rho \)-continuity that it is both closed and open in \( \Lambda_{yz} \). This contradicts with the connectedness of \( \Lambda_{xu} \).

The proof is analogous for \( u \in P_S(\bar{y}) \). Therefore \( R_S \) is complete.

We now turn to the second main result of this paper. Note that the essential point of Theorem 4 is that its conclusion of completeness and transitivity of the soft relation do not call for any mathematical structure on the (choice) set on which the (preference) bi-relation is defined: a local topological structure on the local parameter space suffices enough to obtain. We now work towards a converse question first posed in the context of a single binary relation in Khan and Uyanık (2019). Our second theorem shows that topological connectedness is both necessary and sufficient for completeness and transitivity of a bi-relation. Towards this end, for any topology \( \tau \) on \( X \), let \( \mathcal{R}(X, \tau) \), denote the set of all non-trivial and semi-transitive bi-relations \((R_H, R_S)\) on \( X \) such that \( \tau(R_H, R_S) \subseteq \tau \), where \( \tau(R_H, R_S) \) denotes the coarsest topology on \( X \) containing all sections of \( P_S \) and the complements of the sections of \( R_H \). We can then present

**Theorem 2.** For any set \( X \) and topology \( \tau \) on it the following is equivalent:

(a) the space \((X, \tau)\) is connected;

(b) if \((R, R) \in \mathcal{R}(X, \tau)\), then \( R \) is complete and transitive;

(c) if \((R_H, R_S) \in \mathcal{R}(X, \tau)\), then \( R_S \) is complete and transitive.

Proof. The implication (a) \( \Rightarrow \) (c) follows from Theorem 4 above and Proposition 1 below, (c) \( \Rightarrow \) (b) from setting \( R_H = R_S \) and (b) \( \Rightarrow \) (a) from Khan and Uyanık (2019, Theorem 2).
We note that in a parallel but different path, McCartan (1966) provides a characterization of compactness and Hausdorff separation axiom by using the preferences defined on a choice set.

We have already observed above that the richer bi-preference structure that we work with enables different techniques of proof. We elaborate this observation here, beginning with Eilenberg’s method. As a preliminary remark, note that although Eilenberg assumes the relation to be complete, what follows shows that his method-of-proof extends to a situation where this is not so. Given his assumption on $R$ being an anti-symmetric and continuous relation on a connected set $X$, $R \cap R^{-1} \subseteq \Delta$. The method requires one to pick $x, y, z \in X$ such that $(z, y), (y, x) \in P$. The connectedness of $X$ and the continuity of $R$ imply that $R(y)$ is connected and contains $y$. Note first that if $y \notin R(y)$, then $x \in R(y), z \notin R(y)$ and continuity of $R$ imply that $R(y)$ is a closed, open, non-empty and a proper subset of $X$, which contradicts the connectedness of the space. Now assume $R(y)$ is disconnected. Then there exists non-empty, disjoint and closed subsets $A, B$ of the subspace $R(y)$ such that $A \cup B = R(y)$. Let $y \in B$. Note that $X = A \cup (B \cup R^{-1}(y) \cup (R(y) \cup R^{-1}(y))^c \cup \{y\}) = A \cup (B \cup R^{-1}(y) \cup (P(y) \cup P^{-1}(y))^c)$, where $A$ and the set in parenthesis are non-empty, disjoint and closed. This contradicts the connectedness of $X$. Finally, note that $R(y) = P(z) \cup P^{-1}(z) \cup (R \cup R^{-1})^c(z)$, and recall that $x, y \in R(y)$ and $y \in P(z)$. If $x \notin P(z)$, then this furnishes us a contradiction to the connectedness of $R(y)$. Therefore, $(z, x) \in P$. Since $R$ is anti-symmetric, it is transitive. The proof is complete.

Now consider the special case of the method we pursue in Theorem which begins with an anti-symmetric and continuous relation $R$ on a connected set $X$, and picks $x, y \in X$ such that $(y, x) \in P$. Then the anti-symmetry of $R$ implies that $R(y) \cup R^{-1}(x) = P(y) \cup P^{-1}(x)$. Therefore it follows from the continuity of $R$ and connectedness of $X$ that $P(y) \cup P^{-1}(x) = X$. Hence $P$ is negatively transitive, which implies $P$ is transitive. In order to see this, pick $(y, z), (z, x) \in P$. Negative transitivity of $P$ implies that $(y, x) \in P$ or $(z, y) \in P$. Therefore $(y, x) \in P$, hence $P$ is transitive. Since $R$ is anti-symmetric, it is transitive. Therefore the proof is complete.

Finally, we leave it to the reader to check that our method of proof has some similarities with the argument pursued by Schmeidler (1971), but there are also some subtle differences. In any case, is not the intricacy of the proofs but the surprise of the conclusion that is the point of all this work.

We conclude this section by listing possible future directions of research. Note that p-continuity does not require $f_{xy}$ to be mixture-linear, and more importantly, for all $x, y$ the function is allowed to be chosen differently. This has relevance to Diewert, Avriel and Zang (1981) which studies different forms of convexity assumptions on functions, and merits investigation. It will also be interesting to extend the results in this paper to $k$-connected spaces and component-wise non-trivial relations on an arbitrary topological spaces; see for example Khan and Uyanık (2019). Furthermore, given our tilt to the subject, generalizing Lorimer’s (1967) set-theoretic version of Eilenberg and Sonnenschein’s result is of interest. Finally, it will be interesting to explore how our results generalize to $n$-ary relations, given their importance in the literature of analytical philosophy; see Anand (1993); Temkin (1996, 2015) and their references.
4 Some Additional Considerations

In this section, we turn to the usual scalar and section continuity properties of a relation, and show that they are stronger than $p$-continuity, once relevant mathematical structures on the choice set are in place; see Uyanık (2019a) for a detailed discussion of the relationship between different continuity assumptions on preferences, and Ciesielski and Miller (2016) for those on functions.

**Proposition 1.** Every continuous relation on a topological space is $p$-continuous.

**Proof.** Let $R$ be a continuous relation on a topological space $X$. For all $x, y \in X$, setting $\Lambda_{xy} = X$ and $f_{xy}(a) = a$ for all $a \in \Lambda_{xy}$ finishes the proof. Finally, note that if $X$ is connected, then the bi-relation is connected.

For scalar continuity concepts, we say that a set $S$ is a mixture set if for any $x, y \in S$ and for any $\mu \in [0,1]$ we can associate another element, $x\mu y$, that is in $S$, and where for all $\lambda, \mu \in [0,1]$,

(S1) $x_1 y = x$, (S1) $x\mu y = y(1 - \mu)x$, and (S3) $(x\mu y)\lambda y = x(\lambda\mu)y$.

**Definition 5.** We refer a relation $R$ on a mixture set $S$ to be

(i) mixture-continuous if for all $x, y, z \in S$, the sets \{ $\lambda \in [0,1]$ | $x\lambda y \in R(z)$ \} and \{ $\lambda \in [0,1]$ | $x\lambda y \in R^{-1}(z)$ \} are closed,

(ii) Archimedean if for all $x, y, z \in S$ with $(y, x) \in P$, there exists $\lambda, \delta \in (0,1)$ such that $x\lambda z \in P(y)$ and $y\delta z \in P^{-1}(x)$.

Moreover, $R$ is scalarly continuous if it is mixture-continuous and Archimedean.

**Proposition 2.** Every scalarly continuous relation on a mixture set is connected, hence $p$-continuous.

**Proof.** Let $R$ be a scalarly continuous relation on a mixture set $X$ and $\Lambda_{xy} = [0,1]$ for all $x, y \in X$. Then, for all $x, y \in X$, $f_{xy}(\lambda) = x\lambda y$ for all $\lambda \in \Lambda_{xy}$. It is easy to observe that the closedness of the sets $f_{xy}^{-1}(R(z))$ and $f_{xy}^{-1}(R^{-1}(z))$ is equivalent to mixture-continuity property. Under the mixture continuity assumption, it follows from Galaabaatar, Khan and Uyanık (2019, Proposition 1) that Archimedean property is equivalent to the property that $f_{xy}^{-1}(P(z))$ and $f_{xy}^{-1}(P^{-1}(z))$ are open.

The following two definitions extends the usual continuity assumptions on uni-relations to bi-relations.

**Definition 6.** A bi-relation $(R_H, R_S)$ on a topological space is continuous if $R_H$ has closed sections and $P_S$ has open sections.

**Definition 7.** A bi-relation $(R_H, R_S)$ on a mixture set $S$ is scalarly continuous if $R_H$ is mixture continuous and for all $x, y, z \in S$, the sets \{ $\lambda \in [0,1]$ | $x\lambda y \in P_S(z)$ \} and \{ $\lambda \in [0,1]$ | $x\lambda y \in P_S^{-1}(z)$ \} are open.

The second part of scalar continuity is a property slightly stronger than $R_S$ being Archimedean. The Archimedean property and part (ii) of Definition 7 are equivalent if $R_S$ is mixture-continuous; see Galaabaatar, Khan and Uyanık (2019, Proposition 1) for details. We leave it to the reader to obtain the versions of the two propositions above for bi-relations.
5 Some Concluding Examples

We conclude the paper with three examples. The first is a modified version of a famous example due to [Genocchi and Peano (1884)] and illustrates a preference relation that is \( p \)-continuous but not continuous, thus ensuring that the two theorems reported in this paper are non-vacuous. The second example concerns the canonical order in \( \mathbb{R}^n \); it satisfies all of the assumptions of Theorem [I] except the openness requirements of the sections. The third example is a simple modification of the second that is phrased in terms of a relation that is both incomplete and non-transitive.

**Example 1.** Let \( X = [0, 1]^2 \) and \( f : X \to \mathbb{R} \) defined as
\[
f(x) = \frac{2x_1x_2}{x_1^2 + x_2^2} \quad \text{if} \ x \neq (0, 0) \quad \text{and} \ f(0, 0) = (1, 1).
\]
Note that \( f \) is not continuous in each variable, and hence not (jointly) continuous. Induce a relation \( R \) on \( X \) such that \((x, y) \in R \) if \( f(x) \leq f(y) \). It is clear that \( R \) is complete and transitive. However, \( R \) is not mixture-continuous, hence not continuous. In order to see this, let \( x = (0, 0) \) and \( y = (1, 0) \). Then the set \( \{ \lambda \in [0, 1] | x\lambda + (1 - \lambda)y \in R^{-1}(y) \} = (0, 1] \) is not open in \( X \). It is not difficult to show that \( R \) is \( p \)-continuous; simply for \( x, y \) above, set \( \Lambda_{xy} = [0, 1] \) and \( f_{xy}(\lambda) = (\lambda, \lambda) \) for \( \lambda < 0.5 \) and \( f_{xy}(\lambda) = (\lambda, 1 - \lambda) \) for \( \lambda \geq 0.5 \).

**Example 2.** Let \( X = \mathbb{R}^n_+ \) and \( R \) be the usual relation \( \geq \) on \( X \) defined as “\( x \geq y \) if \( x_i \geq y_i \) for all \( i \).” The asymmetric part \( P \) of \( R \) is the relation \( > \) defined as “\( x > y \) if \( x_i \geq y_i \) for all \( i \) and \( x_j > y_j \) for some \( j \).” It is clear that \( X \) is connected, \( R \) is anti-symmetric, transitive, non-trivial and has closed sections. However, the section of \( P \) are not open since \( P(x) = R(x) \setminus \{x\} \) and \( P^{-1}(x) = R^{-1}(x) \setminus \{x\} \). Clearly, \( R \) is incomplete.

**Example 3.** Let \( X = [0, 1] \) and \( R = \{(x, y) | x \leq 0.5, x \leq y \leq 0.5\} \cup \{(x, y) | x \geq 0.5, y \geq x\} \). It is clear that \( R \) is reflexive and anti-symmetric, hence \( R \) is semi-transitive and has a transitive indifference. Note that \( R^{-1}(0.5) = [0, 0.5] \) and \( R(0.5) = [0.5, 1] \). Since \( [0, 0.5] \times [0.5, 1] \not\subset R \) implies \( R \) is not transitive. It is clear that \( R \) is not complete. Since \( R \) has closed graph, it has closed sections. However, \( P \) does not have open sections. For example \( P(0.25) = (0.25, 0.5) \) and \( P^{-1}(0.75) = [0.5, 0.75] \) which are not open in \([0, 1]\).

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