FROM SYMPLECTIC COHOMOLOGY TO LAGRANGIAN ENUMERATIVE GEOMETRY

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Abstract. We build a bridge between Floer theory on open symplectic manifolds and the enumerative geometry of holomorphic disks inside their Fano compactifications, by detecting elements in symplectic cohomology which are mirror to Landau-Ginzburg potentials. We also treat the higher Maslov index versions of the potentials.

We discover a relation between higher disk potentials and symplectic cohomology rings of smooth anticanonical divisor complements (themselves conjecturally related to closed-string Gromov-Witten invariants), and explore several other applications to the geometry of Liouville domains.

1. Introduction

1.1. Overview. A recurring topic in symplectic geometry, closely related to mirror symmetry, is the interplay between the symplectic topology of open symplectic manifolds and that of their compactifications. The idea is that Floer theory on open manifolds is usually easier to understand, and Floer theory on the compactification can sometimes be seen as a deformation of the former one. This interplay has been the driving force behind many important developments, notably the proofs of homological mirror symmetry for the genus 2 curve and the quartic surface by Seidel [50, 52, 55] and for projective hypersurfaces by Sheridan [59, 58]; the works of Cieliebak and Latschev [21], Seidel [54, 56], Ganatra and Pomerleano [33]; and the ongoing work of Borman and Sheridan [12].

Our aim is to investigate this circle of ideas in a new setting that reveals a different aspect of mirror symmetry. Inside closed manifolds, we are interested in the enumerative geometry of holomorphic disks with boundary on a given (monotone) Lagrangian submanifold $L$; more precisely, in a specific and most basic invariant called the Landau-Ginzburg potential and its important generalisation explained later. Our main result links it to the wrapped symplectic geometry of open Liouville subdomains $M$ containing $L$, translating the Landau-Ginzburg potential into the language of symplectic cohomology and closed-open string maps on $M$. The result (Theorem 1.1) can be written as follows:

$$W_L = CO_L(\mathcal{B}S)$$

where $W_L$ is the potential, $\mathcal{B}S \in SH^0(M)$ is a deformation element called the Borman-Sheridan class (explored by Seidel [56] in the different context of Calabi-Yau manifolds), and $CO_L$ is the closed-open map. The precise statement appears later in the introduction. This theorem has a very clear mirror-symmetric interpretation which is the next thing we discuss, at a slightly informal level.

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1.2. **Mirror symmetry context.** Let $X$ be a (smooth compact) Fano variety of complex dimension $n$ equipped with a monotone Kähler symplectic form, and $\Sigma \subset X$ be a smooth anticanonical divisor. Roughly speaking, it is expected that the mirror $\hat{X}$ of $X \setminus \Sigma$ is a variety which carries a proper map

$$W: \hat{X} \to \mathbb{C},$$

therefore its ring of regular functions is isomorphic to the polynomial ring in one variable $r$:

$$\mathbb{C}[\hat{X}] \cong \mathbb{C}[r],$$

generated by the function $W$. The function $W$ is essentially canonical; the pair $(\hat{X}, W)$ is called a *Landau-Ginzburg model* and is mirror to the compact Fano variety $X$.

Consider a monotone Lagrangian torus $L \subset X$ which is exact in $X \setminus \Sigma$; for simplicity assume that $L$ is a fibre of a suitable SYZ fibration. The torus can be equipped with $\mathbb{C}^*$-local systems to produce an associated $(\mathbb{C}^*)^n$-chart in the mirror which is denoted by

$$(\mathbb{C}^*)^n_L \subset \hat{X}.$$

Our results, Theorem 1.1 and Proposition 2.3 can be interpreted as the following identity:

$$W|_{(\mathbb{C}^*)^n_L} = W_L.$$

In other words, start with the canonical regular function $W$ on $\hat{X}$ (the LG model) and restrict it to a $(\mathbb{C}^*)^n_L$-chart; the result can be written as a Laurent polynomial. The claim is that this Laurent polynomial is the LG potential of the torus $L$. It means that the LG potentials of monotone Lagrangian tori in $X \setminus \Sigma$ are in fact different ‘avatars’ of the same function $W$ defined on the whole mirror.

Furthermore, let $M \subset X \setminus \Sigma$ be a Liouville subdomain. Let $\hat{M}$ be its mirror, and consider the inclusions below:

$$M \hookrightarrow \hat{M} \hookrightarrow X \setminus \Sigma \hookrightarrow \hat{X}$$

As a general expectation, rings of regular functions are mirror to degree zero symplectic cohomology. (This was proven by Ganatra and Pomerleano [34] for the complement to the anticanonical divisor itself, see Section 2.) So one expects:

$$SH^0(M) \cong \mathbb{C}[\hat{M}].$$

These rings can be complicated (much bigger than $\mathbb{C}[r]$), but they carry a distinguished element, the restriction of $W$:

$$W|_{\hat{M}} \in \mathbb{C}[\hat{M}].$$

It is natural to ask what is the symplectic counterpart, or the mirror, of this element. This is answered by Theorem 1.1, which can be summarised in the language of mirror symmetry as follows:

*The Borman-Sheridan class $BS \in SH^0(M)$ is mirror to $W|_{\hat{M}}$.***

The actual scope of Theorem 1.1 is broader: $\Sigma$ can be of higher degree, and $L$ is not required to be a torus or provide an actual chart in the mirror.
1.3. **Overview, continued.** Following an idea that we learned from James Pascaleff we introduce, in a restricted setting, the *higher disk potentials* of a monotone Lagrangian submanifold which is disjoint from a smooth anticanonical divisor, and establish a similar theorem about them, roughly reading:

\[ W_{L,k} = \mathcal{O}_L(D_k) \]

where \( W_{L,k} \) is the Maslov index \( 2k \) potential and \( D_k \in \mathcal{SH}^0(M) \) is a deformation class. The case \( k = 1 \) corresponds to the previous theorem.

Our results provide a convenient tool in the study of symplectic cohomology, holomorphic disk counts, and the topology of Liouville domains. To demonstrate this, we explore several applications:

— A theorem that higher disk potentials determine the product structure on the symplectic cohomology ring of the complement \( X \setminus \Sigma \) of a smooth anticanonical divisor \( \Sigma \).

This is of special interest in view of a conjecture that we learned from Mark Gross and Bernd Siebert. According to it, the symplectic cohomology product can also be expressed in terms of closed-string log Gromov-Witten invariants of the pair \((X, \Sigma)\). Combined with this conjecture, our result would provide interesting identities between open- and closed-string GW theories of \( X \) in line with the general intuition that specific combinations of holomorphic disks can be glued to holomorphic spheres.

— An alternative proof of a general form of the wall-crossing formula for Lagrangian mutations due to Pascaleff and the author [47].

— A connection between seemingly distant properties of a Liouville domain \( M \): the existence of a Fano compactification, the existence of an exact Lagrangian torus inside, the finite-dimensionality of \( \mathcal{SH}^0(M) \), and split-generation of \( \mathcal{Fuk}(M) \) by simply-connected Lagrangians. Two sample outputs appear below.

— We show that the plumbing of two copies of \( T^* (S^2 \times S^2) \) does not contain an exact Lagrangian 4-torus.

— We prove that Vianna’s exact tori in del Pezzo surfaces minus an anticanonical divisor are not split-generated by spheres, which extends a result of Keating [38].

The proof strategies developed here are re-usable in different settings. Roughly speaking, they recast a stretching procedure for holomorphic curves, which would typically be performed in the framework of Symplectic Field Theory [14], within the world of symplectic cohomology. A major benefit is that it makes the moduli spaces unproblematically regular; and an equally important feature is that it gives access to the algebraic structures like the closed-open maps. While we employ the Hamiltonian stretching procedure which by itself is entirely standard (it is used in classical Floer theory), the main content of the proofs lies in analysing the broken curves to show that, in some sense, they behave analogously to what one would expect under SFT stretching. This idea is implemented in the specific setup of Landau-Ginzburg potentials but its outlook is far wider; for instance, we plan to develop it further in the work in progress [26].

1.4. **Main results.** Suppose \( X \) is a monotone symplectic manifold, and \( L \subset X \) is a monotone Lagrangian submanifold. Unless otherwise stated, we shall assume that \( X \) is closed; and all Lagrangian submanifolds are closed and carry a fixed
orientation and spin structure. We begin with a brief reminder of the Landau-Ginzburg potential of $L$ and its relation to local systems.

The simplest enumerative geometry problem relative to $L$ is to count holomorphic disks with boundary on $L$ which are of Maslov index 2, and whose boundary passes through a specified point on $L$. The answer to this problem can be packaged into a generating function called the Landau-Ginzburg superpotential, or simply the potential. It is a Laurent polynomial

$$W_L \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

where $m = \dim H_1(L; \mathbb{C})$. When $L$ is monotone, Maslov index 2 disks have minimal positive symplectic area, so the potential is invariant under the choice of a tame almost complex structure and of Hamiltonian isotopies of $L$. See Section 4 for a more extended reminder.

Let $\rho$ be a (rank one) $\mathbb{C}^*$-local system on $L$, and denote the pair $L = (L, \rho)$. One can view the superpotential as a function on the space of $\mathbb{C}^*$-local systems, meaning that one can evaluate $W_L(\rho) \in \mathbb{C}$ to a complex number: it counts the same holomorphic disks as described above, but their count is weighted using the monodromies of $\rho$ along the boundaries of the disks, and the result is a number.

We denote

$$m_0^L = W_L(\rho) \cdot 1_L \in HF^0_0(L, L)$$

where $1_L$ is the Floer cohomology unit. (This is the curvature of $L$ in the monotone Fukaya category of $X$.)

We are ready state our main theorem: we shall use the notion of a Donaldson hypersurface which is reminded in Section 4 and a technical notion of a grading-compatible embedding which is defined after the statement. The theorem roughly asserts that the symplectic cohomology of any (nice) Liouville domain $M \subset X$ lying away from a Donaldson hypersurface $\Sigma \subset X$ has a canonical deformation class, the Borman-Sheridan class, with the following property. For any monotone Lagrangian submanifold $L \subset X$ which happens to be contained in $M$ and becomes exact therein, the potential of $L$ can be computed by applying the closed-open map to this deformation class.

**Theorem 1.1.** Let $X$ be a simply-connected monotone symplectic manifold, and $\Sigma \subset X$ be a Donaldson hypersurface dual to $d\gamma_1(X)$ for some $d \in \mathbb{N}$. Fix a Liouville subdomain $M \subset X \setminus \Sigma$ such that $c_1(M) = 0 \in H^2(M; \mathbb{Z})$ and the embedding is grading-compatible. There exists an element $BS \in SH^0(M)$ which is called the Borman-Sheridan class, with the following property. For any monotone Lagrangian submanifold $L \subset X$ such that $L$ is contained in $M$ and is exact in $X \setminus \Sigma$ (automatically, $L$ is also exact in $M$). Let $W_L$ be the superpotential of $L$ computed inside $X$. Take a local system $\rho$ on $L$ and denote $L = (L, \rho)$. Then the image of $BS$ under the closed-open map to $L$ equals

$$\frac{1}{d} m_0^L = \frac{1}{d} W_L(\rho) \cdot 1_L:$$

(1.1)

We point out that the Floer cohomology $HF^0_M(L, L) \cong H^0(L) = \mathbb{C} \cdot 1_L$ is computed inside $M$. The Borman-Sheridan class depends on $M$ and its Liouville embedding into $X \setminus \Sigma$, but not on $L \subset M$. 
The name for the Borman-Sheridan originates from the ongoing work \[12\]; it has also recently appeared in the work of Seidel \[56\] in the Calabi-Yau setup. We present our version of the definition during the proof: specifically, in Proposition \[5.12\]. In Section 3 we collect several applications of the result to the symplectic topology of Liouville domains; they were mentioned above.

**Remark 1.1.** The theorem holds for all fields \(\mathbb{K}\) such that \(d \neq 0 \in \mathbb{K}\), but we stick to \(\mathbb{C}\) for concreteness.

**Remark 1.2.** If \(\Sigma\) is anticanonical \((d = 1)\), it is allowable to take \(M = X \setminus \Sigma\) in Theorem 1.1. In this important case, one can compute the Borman-Sheridan class explicitly: see Proposition 2.3 below. However, Theorem 1.1 does not apply to \(M = X \setminus \Sigma\) if \(d > 1\), because \(c_1(X \setminus \Sigma; \mathbb{Z})\) is torsion but non-zero in this case.

**Remark 1.3.** By a version of the Donaldson or the Auroux-Gayet-Mohsen theorem \[8, 18, \text{Theorem 3.6}\], given a monotone Lagrangian submanifold \(L \subset X\), one can find a Donaldson hypersurface \(\Sigma\) away from \(L\) and such that \(L \subset X \setminus \Sigma\) is exact, bringing us closer to setup of Theorem 1.1. However, the degree of the hypersurface may in general be large. See also the discussion in \[47\].

**Remark 1.4.** Suppose \(M_1 \subset M_2 \subset X \setminus \Sigma\) are nested embeddings of Liouville domains, and consider the Viterbo map
\[
\text{Vit} : \text{SH}^\ast(M_2) \to \text{SH}^\ast(M_1).
\]
Using the tools developed in the proof of Theorem 1.1 it easy to argue that the Viterbo map respects the Borman-Sheridan classes. In particular, if \(\Sigma\) is anticanonical, the Borman-Sheridan class of a Liouville subdomain \(M \subset X \setminus \Sigma\) is the Viterbo map image of the Borman-Sheridan class of \(X \setminus \Sigma\) itself, which is determined in Proposition 2.3.

For example, suppose that \(M_1 \cong T^*L\) is a Weinstein neighbourhood of a Lagrangian torus satisfying the conditions of Theorem 1.1 and \(M_2 = M\) is chosen as in Theorem 1.1 Then \(\text{SH}^\ast(M_1) \cong \mathbb{Z}[H_1(L; \mathbb{Z})] \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]\), and applying the composition
\[
\text{SH}^0(M) \xrightarrow{\text{Vit}} \text{SH}^0(M_1) \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]
\]to \(\text{BS}\) computes \(\frac{1}{d} W_L\) as a Laurent polynomial. An analogous statement holds for a Lagrangian \(L\) of general topology, with an extra detail which we will encounter in the beginning of Section 3.1. This way, Theorem 1.1 admits an equivalent reformulation using Viterbo maps instead of the closed-open maps.

Let us explain the grading conventions that are being used. The Floer cohomology \(HF^\ast(L, L)\) is graded in the way singular cohomology is, with the unit in degree 0. Our definition of symplectic cohomology and its grading follows e.g. the conventions of Ritter \[49\]: the unit has degree 0, the closed-open maps have degree 0, and the Viterbo isomorphism \[65, 1, 2, 3\] reads
\[
\text{SH}^\ast(T^*L) \cong H_{n-\ast}(LL)
\]where \(LL\) is the free loop space of a spin manifold \(L\). Since Theorem 1.1 assumes that \(c_1(M) = 0\), \(\text{SH}^\ast(M)\) is \(\mathbb{Z}\)-graded.

We come to the definition of what it means for \(M \subset X \setminus \Sigma\) to be grading-compatible. Let \(d\) be the degree of \(\Sigma\) so that \(|\Sigma|\) is Poincaré dual to \(dc_1(X)\). Then the
$d$th power of the canonical bundle $(K_X\setminus \Sigma)^d$ has a natural trivialisation $\eta$. Following [17], one says that $M \subset X \setminus \Sigma$ is grading-compatible if $\eta$ admits a $d$th root over $M$; that root provides a trivialisation of $K_M$ which is used to grade $SH^*(M)$. Grading-compatibility is a mild topological condition which is equivalent to the fact that $[\Sigma]$ is divisible by $d$ in $H_{2n-2}(X \setminus M; \mathbb{Z})$. It is satisfied in all interesting examples, in particular when $\Sigma$ is anticanonical; see [17, Section 2.2, Proposition 2.5] for further discussion.

In Section 6, we provide a version of Theorem 1.1 for partial compactifications, or equivalently for normal crossings divisors instead of smooth ones. A rough sketch is appears below.

**Theorem 1.2 (=Theorem 6.5).** When $L$ is disjoint from a normal crossings anticanonical divisor $\Sigma = \cup_{i \in I} \Sigma_i$ in a compact Fano variety $Y$, there holds a version of Theorem 1.1 concerning the potential of $L$ in the non-compact manifold $X = Y \setminus \cup_{i \in J} \Sigma_i$, for any subset $J \subset I$.

In Section 2, we introduce **higher disk potentials**

$$W_{L,k} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$$

which, roughly speaking, count Maslov index $2k$ disks with boundary on $L$, passing through a specified point of $L$, intersecting the given anticanonical divisor $\Sigma$ at a single point, and this intersection happens with order of tangency $k$, i.e. intersection multiplicity $k$. We need to impose restrictions making sure that these potentials are well-defined: $\Sigma$ must be (smooth) anticanonical, and either possess an almost complex structure without rational curves (which is satisfied when $\dim X = 4$ or 6), or we only allow $k$ to take values up to the minimal Chern number of $X$. It would be very interesting to lift these restrictions, but it falls outside the scope of this paper. Once they are lifted, we expect that the proof of Theorem 2.1 generalises accordingly.

We show in Section 2 that higher disk potentials can be used to compute the structure constants of the symplectic cohomology ring $SH^0(X \setminus \Sigma)$ with respect to its canonical additive basis formed by (perturbations of) the periodic orbits of the standard $S^1$-periodic Reeb flow around $\Sigma$. We postpone a detailed discussion to Section 2 and now mention a sketch of the main statement:

**Theorem 1.3 (=Corollary 2.5).** It holds that

$$(W_L)^k = W_{L,k} + \sum_{0 \leq i \leq k-1} c_{i,k} W_{L,i},$$

where $c_{i,k}$ are the structure coefficients computing the product on $SH^0(X \setminus \Sigma)$ with respect to the canonical basis.

According to the mentioned conjecture of Gross and Siebert, the same structure constants can be expressed in terms of certain closed-string log Gromov-Witten invariants of $(X, \Sigma)$, cf. [35, 4].

**Structure of the paper.** In Section 2 we discuss the higher disk potentials, a version of Theorem 1.1 for them, and show that higher disk potentials can be used to compute structure constants of symplectic cohomology rings. In Section 3 we re-prove the wall-crossing formula for Lagrangian mutations due to Pascaleff and the author [47], and explore several applications to the existence of exact Lagrangian tori in Liouville domains, and split-generation by simply-connected Lagrangians.
In Section 4, we set up the preliminary material for the proof of Theorem 1.1. In Section 5, we prove Theorems 1.1 and 2.1. In Section 6, we compute the Borman-Sheridan class in the case when $M = X \setminus \Sigma$ and discuss what happens when $\Sigma$ is a normal crossings divisor, rather than a smooth one.

**Related work.** In order to get control on the broken curves arising in the proof of Theorem 1.1, we use a collection of existing tools surrounding the theory of symplectic cohomology. Apart from the foundational material, these tools include confinement lemmas of Cieliebak and Oancea [24], and a lemma about intersection multiplicity with a divisor at constant a periodic orbit by Seidel [56]. Certain ingredients of our proof are also found in several recent papers on related subjects, for example: Diogo [25], Ganatra and Pomerleano [33], Gutt [36], Lazarev [39], Sylvan [61]. We expect a degree of likelihood with the methods developed in the ongoing work of Borman and Sheridan [12] as well.

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### 2. Higher disk potentials and anticanonical divisor complements

#### 2.1. Higher disk potentials.** A quick recollection of the usual Landau-Ginzburg potential is found in Section 4. Assuming familiarity with it, we will now introduce higher disk potentials.

Let $L \subset X$ be a monotone Lagrangian submanifold, $\Sigma \subset X$ a smooth anticanonical divisor disjoint from $L$ and such that $L \subset \Sigma \setminus X$ is exact. Fix a point $p \in L$. Fix a tame almost complex structure $J$ which is integrable in a neighbourhood of $\Sigma$ and makes $\Sigma$ complex. Consider a class $A \in H_2(X,L;\mathbb{Z})$, a positive integer $k$ and define the moduli space

$$\mathcal{M}(k, A) = \left\{ u: (D^2, \partial D^2) \to (X,L), \quad \bar{\partial} u = 0, \quad [u] = A, \begin{array}{l} u(1) = p, \quad u(0) \in \Sigma, \\ u \text{ has a } k\text{-fold tangency to } \Sigma \text{ at } u(0). \end{array} \right\}$$

The latter condition means that the local intersection number $u \cdot \Sigma$ at the point $u(0)$ equals $k$. For example, $k = 1$ means transverse intersection.

**Remark 2.1.** The homological intersection number of a Maslov index $2k$ disk with $\Sigma$ equals $k$ by [1.9] below. Due to positivity of intersections, Maslov index $2k$ holomorphic disks having an order $k$ tangency to $\Sigma$ are precisely the ones that have a unique geometric intersection point with $\Sigma$. 

Having \( k \)-fold tangency is a Fredholm problem \([22, 33]\), and for \( J \) generic outside a neighbourhood of \( \Sigma \), \( \mathcal{M}(k, A) \) is a manifold of dimension
\[
\dim \mathcal{M}(k, A) = \mu(A) - 2k.
\]
For a \( \mathbb{C}^* \)-local system \( \rho \) on \( L \), one would like to define the higher disk potential \( W_{L,k} \) by
\[
W_{L,k}(\rho) = \sum_{A: \mu(A) = 2k} \rho([\partial A]) \cdot \# \mathcal{M}(k, A) \in \mathbb{C},
\]
where \([\partial A] \in H_1(L; \mathbb{Z})\) is the boundary of \( A \), \( \rho([\partial A]) \in \mathbb{C}^* \) is the value of the monodromy of \( \rho \), and \( \# \mathcal{M}(k, A) \) is the signed count of the points in the (oriented zero-dimensional) moduli space. Alternatively, fixing a basis for \( H_1(L; \mathbb{Z})/\text{Torsion} \), one can package the same information into a Laurent polynomial
\[
W_{L,k} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]
\]
given by
\[
W_{L,k}(x) = \sum_{A: \mu(A) = 2k} x^{[\partial A]} \cdot \# \mathcal{M}(k, A),
\]
where \( x^l = x_1^l \ldots x_m^l \) and \([\partial A] \in \mathbb{Z}^m = H_1(L; \mathbb{Z})/\text{Torsion} \).

It is not immediately obvious that \( W_{L,k} \) is invariant under the choices of \( J \) (integrable in a neighbourhood of \( \Sigma \)) because of sphere bubbling at the point of tangency. When a sphere bubble falls entirely within \( \Sigma \), one loses control on the order of tangency of the remaining part of the curve, making it non-obvious why this bubbling does not happen for 1-dimensional families of \( J \)s. While it would be very interesting to develop the definition in general, for simplicity we shall restrict to either of the following cases when the unwanted sphere bubbling is clearly ruled out:

- Denoting the minimal Chern number of \( X \) by \( N \), we restrict to \( k \leq N \);
- We assume that \( \dim_X X = 4 \), in which case \( \Sigma \) is an elliptic curve therefore it is aspherical;
- We assume that \( \dim_X X = 6 \), in which case \( \Sigma \) is a Calabi-Yau surface therefore it admits integrable complex structures without holomorphic curves.

We also point out that \( \Sigma \) is anticanonical; higher degree divisors are disallowed.

**Theorem 2.1.** In the above setup, let \( M \subset X \setminus \Sigma \) be any Liouville subdomain. There exists a class \( D_k \in \text{SH}^0(M) \) such that the following holds.

Take any monotone Lagrangian submanifold \( L \subset X \) such that \( L \) is contained in \( M \) and is exact in \( X \setminus \Sigma \). Let \( W_{L,k} \) be the higher disk potential of \( L \) computed inside \( X \). Take a local system \( \rho \) on \( L \) and denote \( \mathbf{L} = (L, \rho) \). Then the image of \( D_k \) under the closed-open map to \( \mathbf{L} \) equals \( W_{L,k}(\rho) \cdot 1_L \):
\[
D_k \overset{\cong}{\longrightarrow} \text{SH}^0(M) \overset{\text{CO}}{\longrightarrow} \text{HF}^0_M(\mathbf{L}, \mathbf{L})
\]
Clearly, \( W_{L,1} = W_L \) is the usual Landau-Ginzburg potential, and one easily sees from the proof that \( D_1 = BS \).
2.2. Symplectic cohomology of anticanonical divisor complements. Let $X$ be a monotone symplectic manifold and $\Sigma \subset X$ a smooth Donaldson divisor of degree $d$. We remind that this means $[\Sigma] = dc_1(X)$ for $d \in \mathbb{Z}_{>0}$. The Reeb flow on the boundary-at-infinity of $X \setminus \Sigma$ is $S^1$-periodic, and the space of its Reeb orbits is homeomorphic to $S\Sigma$, the unit normal bundle to $\Sigma$. One can choose a Hamiltonian perturbation in such a way that on chain level, the complex computing $SH^*(X \setminus \Sigma)$ has a fractional $\frac{1}{d}\mathbb{Z}$-grading and is isomorphic, as a graded vector space, to:

$$CF^*(X \setminus \Sigma) \cong C^*(X \setminus \Sigma) \oplus (C^*(S\Sigma) \otimes \mathbb{C}[r]), \quad |r| = 1 - 1/d.$$  

(2.3)

See e.g. [51, 25, 33, 34]. Now suppose that $d = 1$, that is, $\Sigma$ is anticanonical. Then on chain level,

$$CF^0(X \setminus \Sigma) \cong \mathbb{C}[r], \quad CF^i(X \setminus \Sigma) = 0 \text{ for } i < 0.$$  

(2.4)

The theorem below is due to Ganatra and Pomerleano [34], compare [25].

**Theorem 2.2.** If $\Sigma$ is smooth anticanonical, then the symplectic cohomology differential on $CF^0(X \setminus \Sigma)$ from (2.3) vanishes. Consequently, as a vector space,

$$SH^0(X \setminus \Sigma) = CF^0(X \setminus \Sigma)$$

has the natural geometric basis $\{1, r, r^2, \ldots\}$ coming from (2.3), where we denote the element $1 \otimes r^i$ appearing in the right summand of (2.3) simply by $r^i$.

Moreover, as an algebra, $SH^0(X \setminus \Sigma)$ is generated by $r$ and is abstractly isomorphic to the polynomial algebra in one variable. $\square$

We wish to add several comments about the theorem. First, if the Floer differential vanishes on $CF^0(X \setminus \Sigma)$, it follows by (2.4) that $SH^0(X \setminus \Sigma) = CF^0(X \setminus \Sigma)$.

**Remark 2.2.** This has to be compared to the case when $\Sigma$ is a smooth symplectic divisor such that $[\Sigma] = \frac{1}{d}c_1(X)$. An example of this situation is when $\Sigma$ is a hyperplane in $\mathbb{C}P^n$; then $[\Sigma] = \frac{1}{n+1}c_1(\mathbb{C}P^n)$. The symplectic cohomology of $\mathbb{C}P^n \setminus \Sigma \cong \mathbb{R}^{2n}$ vanishes. In this case a version of (2.3) holds with $|r| = 1 - d$, and the complex $CF^*(X \setminus \Sigma)$ now has elements of negative degree. Therefore the Floer differential can hit the unit.

We now wish to explain the ‘moreover’ part of Theorem 2.2. By a suitable choice of a Hamiltonian perturbation realising the complex (2.3), one can arrange the actions of the generators $u^i$ appearing in Theorem 2.2 to be additive up an error:

$$A(r^{i+j}) = A(r^i) + A(r^j) + (\epsilon\text{-error}),$$

where the error can be made arbitrarily small, in particular separating the orbit actions. The chain-level differential and product on symplectic cohomology are action-non-decreasing. It follows that on chain level, the symplectic cohomology product $r^i \star r^j$ is a combination of the elements $r^k$ for $k \leq i + j$. (The actions of the $r^i$ are negative and tend to $-\infty$.) Moreover, the coefficient by $r^{i+j}$ is computed by low-energy curves, and choosing a suitable Hamiltonian (a convenient choice is to make it autonomous and work in the $S^1$-Morse-Bott setup, see e.g. [13, 15, 17]) one can show that there is a unique such low-energy curve.
It follows that the product structure on $SH^0(X \setminus \Sigma)$, written in the canonical basis $\{1, r, r^2, \ldots\}$, has the following form:

\begin{equation}
\label{product}
    r^k = r^k + \sum_{0 \leq i \leq k-1} c_{ik} r^i.
\end{equation}

Here we denote by $r^k = r \ast \ldots \ast r$ the $k$th power of $r$ with respect to the symplectic cohomology product. The structure constants $c_{ik}$ need not be trivial, and are of big interest as they are expected to encode certain closed log Gromov-Witten invariants of $X$.

### 2.3. Structure constants and higher potentials.

We take a different viewpoint and reveal that the structure constants above can be computed from the higher disk potentials of a monotone Lagrangian submanifold. The next proposition will be proved in Section 5.

**Proposition 2.3.** Let $\Sigma$ be a smooth anticanonical divisor and take $M = X \setminus \Sigma$. In the situation of Theorem 1.1, it holds that $BS = r$. In the situation of Theorem 2.1, it holds that $D_k = r^k$.

**Lemma 2.4.** In the setting of Theorem 2.1 and Proposition 2.3, let $\rho$ be any local system on $L$. Then

\[ CO(r^k) = (W_L(\rho))^k \cdot 1_L, \]

where $W_L$ is the usual LG potential of $L$ computed in $X$.

**Proof.** One has $CO(r) = W_L(\rho) \cdot 1_L$ by Theorem 1.1 and Proposition 2.3. The claim follows from the fact that $CO$ is a ring map. \hfill $\Box$

**Corollary 2.5.** Suppose $\Sigma \subset X$ is an anticanonical divisor, and $L \subset X \setminus \Sigma$ is a Lagrangian submanifold which is monotone in $X$ and exact in $X \setminus \Sigma$. Assume that we are in a setting when the higher disk potentials $W_{L,i}$ are defined for all $i \leq k$. Writing the disk potentials as Laurent polynomials, the following identity between Laurent polynomials holds:

\[ (W_L)^k = W_{L,k} + \sum_{0 \leq i \leq k-1} c_{ik} W_{L,i}, \]

where $W_L = W_{L,1}$ is the usual Landau-Ginzburg potential and $c_{ik}$ are the structure constants from (2.5). \hfill $\Box$

**Example 2.3.** Take the monotone Clifford torus $L \subset \mathbb{C}P^2$. There exists a smooth elliptic curve $\Sigma \subset \mathbb{C}P^2$ such that $L$ is exact in its complement. One has, in some basis for $H_1(L; \mathbb{Z})$:  

\[ W_L = x + y + 1/xy. \]

One computes:

\[ (W_L)^3 = 6 + (x^3 + y^3 + 1/x^3 y^3 + 3x^2 y + 3xy^2 + 3x/y + 3/xy^2 + 3y/x + 3/x^2 y) \]

It is expected by several methods that

\[ c_{0,3} = 6, \]

while $c_{1,3}, c_{2,3}$ vanish for minimal Chern number reasons. It follows that $W_{L,3} = (W_L)^3 - 6$.

Conversely, we hope that it is possible to compute the higher disk potentials of toric fibres relative to a smooth anticanonical divisor by other means, and deduce
the structure constants for symplectic cohomology using that. The main point is that it is not hard to determine the holomorphic disks of any Maslov index intersecting the singular toric boundary divisor at a single geometric point. However, the enumerative geometry changes when we pass to the smooth divisor, and one needs to understand this change. It is left as a subject for future research.

Remark 2.4. Recall that, unless dim\( R X \) = 4 or 6, Corollary 2.5 applies to \( k \leq N \) where \( N \) is the minimal Chern number of \( X \). For grading reasons, the first non-trivial structure constant is \( c_{0,N} \). So in higher dimensions, the applicability domain of Corollary 2.5 covers exactly the first non-trivial structure constant (while in dimensions 4 and 6 it covers all of them). As mentioned, it would be interesting to have an unrestricted definition of the higher disk potential, and to lift the corresponding restrictions from Corollary 2.5.

3. Wall-crossing and symplectic topology of Liouville domains

3.1. The wall-crossing formula. Theorem 1.1 provides an alternative proof of a general form of the wall-crossing formula due to Pascaleff and the author [47]. Let \( L_0, L_1 \subset M \subset X \) be two Lagrangian submanifolds satisfying the conditions of Theorem 1.1. Let \( \rho_i \) be a \( C^* \)-local system on \( L_i \) and denote \( L_i = (L_i, \rho_i) \).

Theorem 3.1 ([47]). Assume that the Floer cohomology of the pair below computed in \( M \) does not vanish:

\[ HF_M^*(L_0, L_1) \neq 0. \]

Then it holds that

\[ W_{L_0}(\rho_0) = W_{L_1}(\rho_1) \in \mathbb{C}, \]

where the potentials are computed inside \( X \).

We refer to [47] for the context surrounding this theorem, and its applications. In particular, we remind that in any given geometric setting, one still has to compute the pairs \( (\rho_1, \rho_2) \) for which the Floer cohomology appearing in the statement does not vanish; this computation determines the precise shape of the wall-crossing formula relating the potentials.

Let us prove Theorem 3.1 using Theorem 1.1. Consider the following diagram:

\[
\begin{array}{cccc}
\frac{1}{\theta}W_{L_0}(\rho_0) \cdot 1_{L_0} & \leftrightarrow & BS & \leftrightarrow & \frac{1}{\theta}W_{L_1}(\rho_1) \cdot 1_{L_1} \\
\cap & \cap & \cap & \cap & \\
HF_M^*(L_0, L_0) & \leftrightarrow & SH^0(M) & \leftrightarrow & HF_M^*(L_1, L_1)
\end{array}
\]

(3.1)

Above, the subscripts for the \( CO \)-maps are simply used to specify the target. Now pick any non-zero element

\[ 0 \neq x \in HF_M^*(L_0, L_1), \]

assuming that this Floer cohomology is non-vanishing. The next equality follows from the general fact that the closed-open maps turn Floer cohomologies into \( SH^*(M) \)-modules:

\[ \mu^2(CO_{L_0}(BS), x) = \mu^2(x, CO_{L_1}(BS)). \]

Here \( \mu^2 \) denotes the product in Floer cohomology. By (3.1), the equality above rewrites as

\[ \frac{1}{\theta}W_{L_0}(\rho_0) \cdot x = \frac{1}{\theta}W_{L_1}(\rho_1) \cdot x. \]

Since \( x \neq 0 \), it follows that \( W_{L_0}(\rho_0) = W_{L_1}(\rho_1) \).
While the original proof of Theorem 3.1 given in [47] is easier and less technical than that of Theorem 1.1, we hope that the argument presented here serves as a useful illustration of Theorem 1.1.

Figure 1. By “domain-stretching” the perturbation of the holomorphic equation, one makes holomorphic disks break this way in order to prove Theorem 1.1.

Let us summarize this illustration at an informal level, also explaining the main aspects of the proof of Theorem 1.1. Consider all holomorphic Maslov index 2 disks in $X$ with boundary on either $L_0$ and $L_1$, satisfying a boundary point constraint. See Figure 1, where we have depicted the Donaldson divisor $\Sigma$ which the disks will have to intersect.

In the proof of Theorem 1.1, we perform a Hamiltonian domain-stretching procedure upon the disks which makes them break into curves shown in Figure 1. The curves are broken along a periodic orbit of an $s$-shaped Hamiltonian which we explicitly choose. The counts of the left parts of the broken curves in Figure 1 are precisely the closed-open maps onto the $L_i$. It is crucial that the right parts of the broken curves are independent on the Lagrangian we started with. Their count is what is taken to be the definition of the Borman-Sheridan class $BS$.

3.2. Lagrangian embeddings with constant potential. In this subsection, we explain and elaborate on the following hypothesis about a Fano variety $X$: the potential of any monotone Lagrangian torus in $X$ must be non-constant. This hypothesis is needed for later referral, and while we expect it to hold for all Fanos, we prove it only for ones with semisimple quantum cohomology $\mathbb{Q}H^\ast(X; \mathbb{C})$.

We begin with a relatively distant background discussion. By a theorem of Viterbo [67] and Eliashberg [30], a manifold $L$ admitting a metric of negative sectional curvature does not admit Lagrangian embeddings into uniruled symplectic manifolds (this includes Fano manifolds). Observe that the Lagrangian in question need not be monotone.

Let us attempt to reproduce the Viterbo-Eliashberg theorem for monotone Lagrangian submanifolds using an elementary approach (without appealing to neck-stretching). Recall that the Morse indices of the geodesics of a metric of negative sectional curvature does not admit Lagrangian embeddings into uniruled symplectic manifolds (this includes Fano manifolds). Observe that the Lagrangian in question need not be monotone.

Let us attempt to reproduce the Viterbo-Eliashberg theorem for monotone Lagrangian submanifolds using an elementary approach (without appealing to neck-stretching). Recall that the Morse indices of the geodesics of a metric of negative sectional curvature (with respect to the length functional) are all equal to 0. Denoting by $\mathcal{LL}$ the free loop space of $L$, it follows that $H_s(\mathcal{LL})$ is concentrated in degree zero, except for the constant loop cycles $H_s(L) \subset H_s(\mathcal{LL})$. In particular, $H_0(\mathcal{LL})$ is 1-dimensional and generated by the unit.

Now let $L$ be an arbitrary connected oriented spin manifold. Let $\Omega L$ be the based loop space of $L$, then there is a fibration

$$L \to \mathcal{LL} \to \Omega L.$$
Consider the intersection with the fibre map:

$$i!: H_n(\mathcal{L}L) \to H_0(\Omega L) \cong \mathbb{Z}[\pi_1(L)].$$

**Lemma 3.2.** Suppose the image of $i!$ is trivial, i.e. additively generated by the trivial loop: $\text{img } i! = \mathbb{Z}[1]$. Then for any monotone Lagrangian embedding $L \subset X$, the potential of $L$ is constant.

**Remark 3.1.** We remind that the potential being constant means the following: the algebraic count of holomorphic Maslov index 2 disks with any fixed non-zero boundary homology class in $H_1(L; \mathbb{Z})/\text{Torsion}$ and passing through a prescribed point on $L$, vanishes.

**Example 3.2.** One can check that $\text{img } i!$ lies in the centre of $\mathbb{Z}[\pi_1(L)]$. In particular, if $L = L_1 \sharp L_2$ is a connected sum with $\pi_1(L_1), \pi_1(L_2) \neq 0$, $\text{img } i!$ is trivial. Another example when the condition of Lemma 3.2 holds is when $L$ has non-negative sectional curvature, since $H_n(\mathcal{L}L)$ is one-dimensional in this case.

**Proof of Lemma 3.2.** Let $\Sigma \subset X$ be a Donaldson hypersurface provided by Remark 1.2. Consider the moduli space of parameterised Maslov index 2 disks $u: (D, \partial D) \to (X, L)$ satisfying $u(0) \in \Sigma$, compare with the beginning of proof of Theorem 1.1. This moduli space is $n$-dimensional and carries a map to $\mathcal{L}L$ by restricting to $\partial D$, therefore defines a class in $H_n(\mathcal{L}L)$. The potential is the image of this class under the composition of $i!$ and the abelianisation map, up to a constant which is the degree of $\Sigma$. \qed

We state the following hypothesis for future reference.

**Hypothesis 3.3.** Let $X$ be a monotone symplectic $2n$-manifold. Every monotone Lagranian torus in $X$ has non-constant LG potential.

Our next step is to show that the hypothesis holds whenever the quantum cohomology $\mathcal{QH}^*(X; \mathbb{C})$ is semisimple. However, we expect the hypothesis to hold for all Fano varieties. We mention several reasons to expect this, emphasizing that these are only general ideas (we hope to develop them elsewhere):

- Since Fano manifolds are uniruled, Lagrangian tori in $X$ bound some holomorphic disks essentially by the original Viterbo-Eliashberg neck-stretching argument. An argument of Cieliebak and Mohnke [23] sometimes shows that monotone Lagrangian tori bound Maslov index 2 disks with non-trivial boundary. One hopes to show that these disks do not cancel algebraically.

- Suppose $L$ belongs to the complement of a smooth anticanical divisor and all higher disk potentials are well-defined. Suppose $W_L \equiv c \neq 0$, then from Corollary 2.5 one deduces that all structure coefficients for $SH^0(X \setminus \Sigma)$ are trivial. If $X$ is a toric Fano, this is provably a contradiction: looking at the potential of the standard toric fibre, one can argue that the constant terms appearing in its powers witness the non-triviality of the structure constants. If $W_L \equiv 0$, one can only argue that the structure constants $c_{0,k} = 0$ for all $k$, but this should be enough to provide a contradiction.

We will now prove Hypothesis 3.3 in the case when $\mathcal{QH}^*(X; \mathbb{C})$ is semisimple.

**Lemma 3.4.** Let $L \subset X$ be a monotone Lagrangian submanifold and $\rho_1 \neq \rho_2$ two different local systems on $L$. Denote $L_i = (L, \rho_i)$. Then $\dim HF^*(L_1, L_2) < \dim H^*(L)$. 

Proof. Recall that Oh’s spectral sequence \([44]\) converging to \(HF^*(L_1, L_2)\) begins with the first page \(C^*(L)\) carrying the singular differential of degree 1 twisted by the local system \(\rho_1 \rho_2^{-1}\). So it is enough to check that the twisted singular cohomology is smaller than \(\hat{H}^*(L)\). For example, pick a CW decomposition of \(L\) one of whose 1-handles corresponds to a loop supporting non-trivial monodromy of \(\rho_1 \rho_2^{-1}\). The twisted singular CW differential \(C^0(L) \to C^1(L)\) has non-zero image on that 1-cell, therefore the generator of \(C^0(L)\) is not closed, and the conclusion follows. \(\square\)

**Corollary 3.5.** If \(X\) is a Fano variety with semisimple \(QH^*(X; \mathbb{C})\), Hypothesis \([3.3]\) holds for it.

**Proof.** Let \(L \subset X\) be a monotone Lagrangian torus and \(\rho\) be any \(\mathbb{C}^*\)-local system on \(L\). Supposing \(W_L\) is constant, we claim that \(\dim HF^*(L, L) = \dim H^*(L)\). Because \(W_L\) is constant, the Maslov index 2 part of the Floer differential in Oh’s spectral sequence computing \(HF^*(L, L)\) is trivial. Since \(H^*(L)\) is generated by \(H^1(L)\), one has that \(HF^*(L, L) \neq 0\) by an argument of Biran and Cornea \([10]\).

Since \(QH^*(X; \mathbb{C})\) is semisimple, the monotone Fukaya category of \(X\) splits as a finite disjoint union of non-trivial categories, each isomorphic to the category of \(Cl(E)\)-modules, see e.g. \([60, 59]\). In particular, the Fukaya category has finitely many quasi-isomorphism types of objects whose endomorphism space is bounded in dimension by a given number.

On the other hand, if \(\rho_1, \rho_2\) are two different local systems on \(L\), the corresponding objects \(L_1, L_2\) of the Fukaya category are never quasi-isomorphic, by Lemma \([3.4]\) and what is shown in the beginning of this proof. This gives a contradiction. \(\square\)

### 3.3. Exact tori in Liouville domains

Below is the main result of this section, which we will use to draw several corollaries about the symplectic topology of Liouville domains.

**Proposition 3.6.** Suppose \(X\) is a simply-connected monotone symplectic manifold, and \(\Sigma \subset X\) is a Donaldson hypersurface. Suppose \(M \subset X \setminus \Sigma\) is a grading-compatible Liouville subdomain with \(c_1(M) = 0\), \(L \subset M\) is an exact torus with vanishing Maslov class, and:

— either \(SH^0(M)\) is finite-dimensional as a vector space,

— or there exist finitely many exact Lagrangian submanifolds \(K_i \subset M\) with \(H_1(K_i; \mathbb{C}) = 0\) satisfying the following property: for any \(\mathbb{C}^*\)-local system \(\rho\) on \(L\), the object \((L, \rho)\) is split-generated, in the compact exact Fukaya category of \(M\), by the \(K_i\).

Then the potential of the monotone torus \(L \subset X\) is constant. In particular, both cases provide a contradiction if Hypothesis \([3.3]\) holds for \(X\).

**Proof.** Suppose \(L \subset M\) is such a torus. Then it becomes monotone under the inclusion \(M \subset X\), compare \([18]\) Example 3.2, Lemma 3.4, \([47]\) Section 2]. We claim that either of the two given conditions imply that \(W_L\) is constant, where the potential \(W_L\) is computed in \(X\).

Under the first condition, assume \(W_L\) is a non-constant Laurent polynomial. Then the powers \((W_L)^k\) are linearly independent for all \(k\). It follows from Theorem \([1.1]\) that the powers \(BS^k \in SH^0(M)\) with respect to the symplectic cohomology product are linearly independent for all \(k\), hence \(SH^0(M)\) is infinite-dimensional.

Under the second condition, for any local system \(\rho\) there exists a Lagrangian \(K_i\) such that

\[
HF^*_M((L, \rho), K_i) \neq 0.
\]

(3.2)
For a fixed $K_i$, the set of all $\rho \in (\mathbb{C}^*)^n$ such that (3.2) holds is Zariski closed in $(\mathbb{C}^*)^n$. It follows that there exists a $K_i$ such that (3.2) holds for all local systems, because the union of finitely many Zariski closed sets cannot cover $(\mathbb{C}^*)^n$ unless one of the sets coincides with it. Since $H_1(K_i;\mathbb{C}) = 0$, the LG potential of $K \subset X$ is automatically constant: $W_K \equiv c$. By the wall-crossing formula (Theorem 3.1),

$$W_L(\rho) = c$$

for all $\rho$. It means that $W_L \equiv c$. \hfill $\square$

In the rest of the section we present several applications of Proposition 3.6 to the symplectic topology of Liouville domains, offering a fresh look on some of the known results in the field and providing new extensions thereof.

3.4. Generation by simply-connected Lagrangians.

**Corollary 3.7.** Suppose $X$ is simply-connected monotone symplectic manifold containing a monotone torus with non-constant potential, and $\Sigma \subset X$ is a Donaldson divisor such that the torus is exact in its complement. Then the compact exact Fukaya category of any grading-compatible Liouville subdomain of $X \setminus \Sigma$ with vanishing Chern class is not split-generated by simply-connected Lagrangians. \hfill $\square$

**Example 3.3.** Keating [37] showed that the 4-dimensional Milnor fibres of isolated complex singularities of modality one contain an exact Lagrangian torus $L$ not split-generated by the vanishing Lagrangian spheres. She explicitly showed that the locus in $(\mathbb{C}^*)^2$ consisting of local systems $\rho$ such that the Floer cohomology between $(L,\rho)$ and the vanishing Lagrangian spheres is 1-dimensional. On the other hand, split-generation would force the locus to be 2-dimensional.

The simplest modality one Milnor fibres are

$$\mathcal{T}_{3,3,3} = dP_6 \setminus \Sigma, \quad \mathcal{T}_{2,4,4} = dP_7 \setminus \Sigma, \quad \mathcal{T}_{2,3,6} = dP_8 \setminus \Sigma,$$

where $dP_i$ is the del Pezzo surface which is the blowup of $\mathbb{CP}^2$ at $i$ points, and $\Sigma$ is an anticanonical divisor in each of them. Vianna showed [63, 64] that each of these Milnor fibres contains infinitely many exact Lagrangian tori, whose potentials in the compactification $dP_i$ are non-constant. In view of Proposition 3.6 we arrive at the following generalisation of [38].

**Corollary 3.8.** Fix any finite collection of Lagrangian spheres in the Milnor fibre $\mathcal{T}_{3,3,3}$, $\mathcal{T}_{2,4,4}$ or $\mathcal{T}_{2,3,6}$. For almost all $\mathbb{C}^*$-local systems (precisely, for a non-empty Zariski open set of them) on each exact Vianna torus in the Milnor fibre, that torus is not split-generated by the fixed collection of spheres. \hfill $\square$

**Example 3.4.** Let $M$ be the plumbing of two copies of $T^*S^2$ at two points. It is Liouville isomorphic to

$$\{(x, y, z) \in \mathbb{C}^3 : xy = z^2 - 1\} \setminus \{xy = -1\}.$$ 

This can be seen using the Lefschetz fibration onto $\mathbb{C} \setminus \{0\}$ by projecting to the $z$-plane. Therefore $M$ embeds into the quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$ away from an anticanonical divisor (which can be smoothed). The domain $M$ contains an exact torus (the matching torus for a loop in the base of the Lefschetz fibration encircling both singular points, and enclosing the correct amount of area). Proposition 3.7 implies that this torus is not generated by any collection of Lagrangian spheres in $M$. This is totally expected because the torus is displaceable from both core spheres.
Remark 3.5. A subtlety must be pointed out here. Suppose \( X \) is a Fano projective hypersurface and \( \Sigma \subset X \) is an anticanonical divisor. Sheridan proved \([57, 59]\) that the compact exact Fukaya category of \( X \setminus \Sigma \) is split-generated by the union of certain Lagrangian spheres considered as a single immersed Lagrangian submanifold. This is different from being split-generated by the collection of the same Lagrangian spheres considered individually: already in the previous example, \( \mathcal{F}uk(M) \) is generated by the single object \( K = K_1 \cup K_2 \), the union of the two core spheres. In particular, the matching exact torus \( L \subset M \) is Hamiltonian displaceable from \( K_1 \) and \( K_2 \), but not from the union \( K_1 \cup K_2 \).

3.5. Non-existence of exact tori. We move on to the discussion along a different line: if it is known that \( \mathcal{F}uk(M) \) is split-generated by spheres, Proposition 3.6 may be invoked to show the non-existence of exact Lagrangian tori in \( M \). Current knowledge includes the following:

— by Ritter \([48]\), four-dimensional \( A_k \)-Milnor fibres do not contain exact Lagrangian tori;
— by Abouzaid and Smith \([5]\), \( A_2 \)-Milnor fibres in any dimension do not contain exact Lagrangian tori.

Recall that the \( A_k \)-Milnor fibre is a plumbing of \( k \) copies of \( T^*S^n \) according to the chain graph. The following was also proved in \([3]\).

**Theorem 3.9.** Let \( M \) be a plumbing of cotangent bundles of simply-connected manifolds according to a tree, and \( \dim_{\mathbb{R}} M \geq 6 \). Then the compact exact Fukaya category of \( M \) is generated by the cores of the plumbing. \( \square \)

There is an alternative way of deriving this theorem. By \([29\), Theorem 54\], the plumbing \( M \) satisfies Koszul duality in the sense of \([31]\); by adopting an argument from \([40\), Section 4.1\], this fact implies split-generation. The advantage of this argument is that it works in some 4-dimensional cases, too. Recall that by \([31]\), the plumbing of \( T^*S^2s \) according to a Dynkin tree satisfies Koszul duality, and an argument adopted from \([40\), Section 4.1\] implies the split-generation.

**Theorem 3.10.** Let \( M \) be a plumbing of \( T^*S^2s \) according to a Dynkin tree. Then the compact exact Fukaya category of \( M \) is split-generated by the cores of the plumbing. \( \square \)

**Example 3.6.** The del Pezzo surface \( X \) which is the blowup of \( \mathbb{C}P^2 \) at 3 points contains two Lagrangian spheres intersecting transversely once, hence it contains the \( A_2 \) Milnor fibre \( M \). Moreover, \( M \) lies away from an anticanonical divisor. Note that \( \mathbb{Q}H^*(X) \) is semisimple, so by Theorem 3.10, Corollary 3.7 and Corollary 3.5 \( M \) does not contain an exact Lagrangian torus with vanishing Maslov class. Although this was known, we can apply the same argument to the product (now using Theorem 3.9) and get the result below which seems to be new.

**Corollary 3.11.** The plumbing of two copies of \( T^*(S^2 \times S^2) \) along one point does not contain an exact Lagrangian 4-torus with vanishing Maslov class. \( \square \)

**Example 3.7.** Let \( X \) be an \( n \)-dimensional projective hypersurface of degree \( n \). The \( n \)-dimensional \( A_k \) Milnor fibre \( M \) embeds into \( X \) (away from an anticanonical divisor) when \( k \leq n \), see e.g. \([62]\). (The actual bound on \( k \) in terms of \( n \) is in fact much higher.) If Hypothesis 3.3 holds for \( X \), then Theorem 3.9, Corollary 3.7 and Corollary 3.5 imply that \( M \) does not contain an exact Lagrangian torus with vanishing Maslov class.
Remark 3.8. Instead of tori, one could speak of other Lagrangians $L$ such that $H^1(L; \mathbb{C}) \neq 0$, provided that suitable versions of Hypothesis 3.3 hold. Our proof of Corollary 3.5 used the fact that $H^* (L; \mathbb{C})$ is generated by $H^1 (L; \mathbb{C})$, and the torus is the most natural example having this property.

3.6. Concluding notes. Consider an affine variety $M$ which is the complement of an ample normal crossings divisor in a projective variety $X$, see e.g. [43]. Then $M$ is a Liouville domain, and after smoothing the given divisor, one obtains a Liouville embedding $M \subset X \setminus \Sigma$ where $\Sigma$ is smooth. The affine variety $M$ may admit many other compactifications; this is a broadly studied subject in algebraic geometry. Proposition 3.6 can be understood as a set of conditions that prevent an affine variety $M$ from admitting a Fano compactification.

Although constructing exact Lagragian tori in $M$ is complicated, see e.g. [38], one general approach to the construction is well known (compare [42]): one considers a Lefschetz fibration on $M$ and seeks to construct a Lagrangian torus as a matching path, reducing the problem to one inside the fibre.

We note that the applications of Proposition 3.6 provided above were only using the split-generation condition; we have not discussed the applications which would rely on the dimension of $SH^0 (M)$ instead. It may be interesting to look for such examples too, bearing in mind that $SH^0 (M)$ is also related to Lefschetz fibrations on $M$ by an exact sequence involving the monodromy map, due to McLean [42].

4. Preparations

In this section we set up the preparatory material for the proof of Theorem 1.1. We quickly remind the notions of superpotential, symplectic cohomology and the closed-open maps. Then we discuss positivity of intersections for Floer solutions and set down a class of $s$-shaped Hamiltonians which will be used in the proof.

4.1. The superpotential. Let $L \subset X$ be a monotone Lagrangian submanifold. Recall that it is assumed to be oriented and spin. Denote $m = \text{rk} \ H_1 (L; \mathbb{Z}) / \text{Torsion}$ and choose a basis of this group:

\begin{equation}
\mathbb{Z}^m \cong \text{ker} \ H_1 (L; \mathbb{Z}) / \text{Torsion}.
\end{equation}

The potential of $L$ with respect to the basis (4.1) is a Laurent polynomial

$$W_L : (\mathbb{C}^*)^m \rightarrow \mathbb{C}$$

defined by:

\begin{equation}
W_L (\mathbf{x}) = \sum_{l \in \mathbb{Z}^m} \mathbf{x}^l \cdot \# M^l_0
\end{equation}

where $\mathbf{x}^l = x_1^{l_1} \cdots x_m^{l_m}$ and $M^l_0$ is the moduli space of unparametrised $J$-holomorphic Maslov index 2 disks $(\partial D_1, \partial D_2) \subset (X, L)$ passing through a specified point $p \in L$, and whose boundary homology class $[\partial D]$ equals $l \in \mathbb{Z}^m$ in the chosen basis (4.1). The holomorphic disks are computed with respect to a regular tame almost complex structure $J$. Then $M^l_0$ is 0-dimensional and oriented, so the signed count $\# M^l_0$ is an integer.

For an equivalent way of defining the superpotential, let $\rho$ be a local system on $L$, by which we mean (in a slightly non-standard way) a map of Abelian groups

$$\rho : H_1 (L; \mathbb{Z}) / \text{Torsion} \rightarrow \mathbb{C}^*$$

where $C^*$ is the group of invertibles. Using the basis (4.1), one can view $\rho$ as a point:

$$\rho \in (C^*)^m,$$

by computing its values on the basis elements. We will use the two ways of looking at $\rho$ interchangeably. Let

$$\mathcal{M}_0 = \bigsqcup_{l \in \mathbb{Z}^m} \mathcal{M}_0^l$$

be the moduli space of all holomorphic Maslov index 2 disks as above, with any boundary homology class. Then one puts

$$W_L(\rho) = \#_\rho \mathcal{M}_0 \in \mathbb{C},$$

where the right hand side denotes the count of holomorphic disks in $\mathcal{M}_0$ weighted using the local system:

$$\#_\rho \mathcal{M}_0 = \sum_{l \in \mathbb{Z}^m} \# \mathcal{M}_0^l \cdot \rho(l)$$

Formula (4.3) defines the value of the potential at any point of $(C^*)^m$, and the resulting function is precisely the Laurent polynomial (4.2), so the two definitions of the potential are consistent.

If one changes the basis (4.1) by a matrix $(a_{ij}) \in SL(m; \mathbb{Z})$, the corresponding superpotentials differ by a change of co-ordinates given by the multiplicative action of $GL(m; \mathbb{Z})$ on $(C^*)^m$:

$$x_i \mapsto x_i' = \prod_{j=1}^m x_j^{a_{ij}}, \text{ so that } W_L'(x_1, \ldots, x_m) = W_L(x_1', \ldots, x_m').$$

Recall that $GL(m; \mathbb{Z})$ consists of integral matrices with determinant $\pm 1$. The proposition below is classical.

**Proposition 4.1.** For a monotone Lagrangian submanifold $L \subset X$, its superpotential $W_L$ up to the change of co-ordinates is invariant under Hamiltonian isotopies of $L$, and more generally under symplectomorphisms of $X$ applied to $L$. □

4.2. Symplectic cohomology. We assume that the reader is familiar with basic Floer theory, the definitions of symplectic cohomology, closed-open maps and related terminology, like Liouville domains. The reader can consult e.g. [32, 19, 66, 51, 20, 49] for the necessary background. Assuming familiarity with these notions, we give a quick overview in the amount required to set up the necessary notation.

Let $M$ be a Liouville domain with boundary $\partial M$. Its Liouville vector field gives a canonical parameterisation of the collar of $\partial M$ by

$$[1 - \delta, 1] \times \partial M \subset M.$$

Here $\{1\} \times \partial M$ is the actual boundary of $\partial M$. In our definition of symplectic cohomology, we work directly with $M$ and not its Liouville completion; both ways are of course equivalent.

To define symplectic cohomology, we begin with class of Hamiltonians $M \to \mathbb{R}$ which are zero away from the collar, are monotone functions of the collar co-ordinate on the collar, and which become linear of fixed slope in the collar coordinate starting from a certain distance to $\partial M$, say on $[1 - \delta/2, 1] \times \partial M$. In the setup of symplectic cohomology, the important quantity to keep track of is the
slope of the Hamiltonian near $\partial M$. For us it is more convenient to keep track of the maximum value of our Hamiltonians which is achieved on $\partial M$. We denote it by $l$ and call the height, see Figure 2. As $l$ tends to infinity so does the slope, which is good enough for the purpose of defining symplectic cohomology.

We perturb the Hamiltonians of the specified class by:

— a perturbation away from the collar which turns the Hamiltonians into Morse functions away from the collar;
— optionally, a non-autonomous ($S^1$-dependent) perturbation in the collar which makes the 1-periodic orbits of the Hamiltonian flow non-negenerate.

We denote the resulting function after any such perturbation by

$$\hat{H}_l: S^1 \times M \to \mathbb{R},$$

where $l$ is the maximum value (up to an $\epsilon$-error), or the height. All other choices, in particular the precise perturbations, are immaterial. The shape of such function is sketched in Figure 2 (left), and more crudely in Figure 2 (right). We will adopt the crude variants of the pictures in the future. The Floer complex $CF^*(\hat{H}_l)$ is generated, as a vector space over $\mathbb{C}$, by time-1 periodic orbits of the Hamiltonian vector field $X_{\hat{H}_l}$. These orbits come in two types:

1: Constant orbits that correspond to critical points of $\hat{H}_l$ away from the collar;
2: Orbits in the collar that correspond to Reeb orbits of $\partial M$ of various periods.

By the maximum principle, solutions to Floer’s equation never escape to $\partial M$, so there are well-defined Floer cohomology groups $HF^*(\hat{H}_l)$. We use the cohomological convention where positive punctures serve as inputs. This means that Floer’s differential is given by:

$$d\gamma = \sum \#\mathcal{M}(\gamma_+, \gamma_-) \cdot \gamma_-$$

where Floer solutions $u \in \mathcal{M}(\gamma_+, \gamma_-)$ have orbits $\gamma_{\pm}$ as their $s \to \pm\infty$ asymptotics, see Figure 3.

Figure 3. A curve for Floer’s differential.
When $l \leq l'$, it is easy to arrange that $\hat{H}_l \leq \hat{H}_{l'}$ everywhere. If one sets up Floer’s continuation equations using a homotopy $H(s)$ between $\hat{H}_l$ and $\hat{H}_l (l \leq l')$ such that $\partial_s H \leq 0$ everywhere, the solutions also obey the maximum principle. This means that there are well-defined continuation maps $HF^*(\hat{H}_l) \to HF^*(\hat{H}_{l'}), l \leq l'$. The symplectic cohomology is the direct limit with respect to these continuation maps:

$$SH^*(M) = \lim_{l \to +\infty} HF^*(\hat{H}_l).$$

Symplectic cohomology acquires a $\mathbb{Z}$-grading once a trivialisation of the canonical bundle $K_M$ is fixed.

4.3. Closed-open maps. Let $M$ be a Liouville manifold and $L \subset M$ be an exact Lagrangian submanifold equipped with a local system $\rho$. Assume that $c_1(M) = 0$, and fix a trivialisation of the canonical bundle $K_M$ in which $L$ has vanishing Maslov class. Denote the pair $(L, \rho) = L$. The closed-open map is a map of graded algebras

$$CO: SH^*(M) \to HF^*(L, L).$$

Here $HF^*(L, L)$ is the self-Floer cohomology. Because $L$ is exact, there is an algebra isomorphism

$$HF^*(L, L) \cong H^*(L; \mathbb{C})$$

for any $\rho$. We denote by $1_L \in HF^*(L, L)$ the unit. We shall write $HF^*_L(L, L)$ when we wish to emphasise that the Floer cohomology is computed inside $M$.

The definition of the closed-open map goes by counting maps from the half-cylinder

$$u: [0, +\infty) \times S^1 \to M$$

solving the usual Floer’s equation with Hamiltonian $\hat{H}_l$ (precisely the same one as used above), and with Lagrangian boundary condition $L$. The closed-open map to $HF^*(L, L) = H^*(L)$ is then obtained by evaluating the solutions $u$ at the fixed boundary point $\{s = t = 0\}$, and weighting the counts using the local system in the way it is done in (4.4). Restricting to degree zero, one can write:

$$CO_l(\gamma) = \# M(pt, \gamma) \cdot 1_L \in HF^0(L, L) \quad \text{for} \quad \gamma \in CF^0(\hat{H}_l),$$

where $M(pt, \gamma)$ counts Floer solutions $u$ on the half-cylinder as above, asymptotic to $\gamma$ as $s \to +\infty$ and sending the fixed boundary marked point $\{s = t = 0\}$ to a specified point $pt \in L$.

Because the maps $CO_l$ commute with the continuation maps, they define the closed-open map $CO: SH^*(M) \to HF^*(L, L)$ under the limit (4.7). The definition of the closed-open map just given can be found in e.g. [51].

4.4. Donaldson divisors. A Donaldson divisor $\Sigma$ in a closed monotone symplectic manifold $X$ is a smooth real codimension 2 symplectic submanifold whose homology class is dual to $d\Sigma(X)$, where $d$ is a positive integer called the degree of $\Sigma$. Donaldson proved that such divisors always exist [27], and the complement $X \setminus \Sigma$ has a canonical Liouville structure, see e.g. [15 Section 4.1].

By [8 [18], for a given monotone Lagrangian submanifold $L \subset X$, there exists a Donaldson divisor $\Sigma$ disjoint from it, and such that $L \subset X \setminus \Sigma$ is exact. Moreover, given any such Lagrangian submanifold, $\frac{1}{2d}[\Sigma]$ is dual to twice the Maslov class of $L$.

Namely, for a homology class $A \in H_2(X, L; \mathbb{Z})$ one has:

$$\mu(A) = A \cdot [\Sigma]/2d$$
where $\mu$ is the Maslov index and “” is the homology intersection, see e.g. [18 Lemma 3.4].

4.5. Positivity of intersections with a Hamiltonian term. Let $\Sigma \subset X$ be a symplectic hypersurface, and fix a tame almost complex structure $J$ such that $T\Sigma$ is $J$-invariant; one says that $J$ preserves $\Sigma$. It has been proved by Cieliebak and Mohnke [22] (using the Carleman similarity principle from McDuff and Salamon [41]) that all intersections of $J$-holomorphic curves with $\Sigma$ are positive. We will now discuss similar statements for solutions of Floer’s equation, both at interior points and at punctures. We begin with intersections at interior points.

Denote by $U(\Sigma)$ a tubular neighbourhood of $\Sigma$. Let $S \subset \mathbb{C}$ be a domain, and suppose $u : S \to U(\Sigma)$ solves Floer’s equation:

\[(4.10) \quad \partial_s u + J\partial_t u = JX_H(s,t).\]

Here $s = \text{Re} \, z$, $t = \text{Im} \, z$ for the complex co-ordinate $z \in \mathbb{C}$, and one takes a domain-dependent Hamiltonian $H : S \times X \to \mathbb{R}$. The lemma below appeared in e.g. [33 Lemma 4.2].

**Lemma 4.2 (Positivity of intersections).** Suppose that $J$ preserves $\Sigma$, and $X_{H(s,t)}|\Sigma$ is tangent to $\Sigma$ for all $s,t$. Let $u : S \to U(\Sigma)$ be a solution of (4.10) such that $u(S) \cap \Sigma \neq \emptyset$ and $u(\partial S) \cap \Sigma = \emptyset$, then $u(S)$ has positive intersection number with $\Sigma$.

By the intersection number, we mean the following. Given that $u(\partial S)$ is disjoint from $\Sigma$, it produces a well-defined class $[u(S)] \in H_2(U(\Sigma), \partial U(\Sigma); \mathbb{Z})$. We consider the intersection number between $[u(S)]$ and $[\Sigma] \in H_{2n-2}(U(\Sigma); \mathbb{Z})$.

**Proof.** By Gromov’s trick [41], solutions of (4.10) lift to $\tilde{J}$-holomorphic curves in $S \times U(\Sigma)$ which are sections of $S \times U(\Sigma) \to S$. The relevant almost complex structure $\tilde{J}$ is given pointwise by the following matrix, using the split basis for $T(S \times U(\Sigma)) = \mathbb{C} \oplus TU(\Sigma)$. We denote $H = H(s,t)$ for brevity.

\[
\tilde{J} = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
JX_H & X_H & J
\end{pmatrix}
\]

It is clear under our hypotheses on $J$ and $X_H$ that $\tilde{J}$ preserves the hypersurface $S \times \Sigma$. Therefore all $\tilde{J}$-holomorphic sections of $S \times U(\Sigma) \to S$ have positive intersection with $S \times \Sigma$ by Cieliebak and Mohnke [22], which is equivalent to the desired statement. \qed

Maps from cylinders of finite energy solving Floer’s equation converge to periodic Hamiltonian orbits (at least when the orbits are non-degenerate). If the orbit in question is a constant orbit in $\Sigma$, it makes sense to speak of the intersection sign of the compactified curve with $\Sigma$ at that constant orbit. The occurring intersection numbers have recently been analysed by Seidel [56], and we will use a slight reformulation of his result.

Let $S$ be one of the two domains:

\[S_- = (-\infty, R] \times S^1, \quad S_+ = [R, +\infty) \times S^1,\]

where $R \in \mathbb{R}$. We use the coordinates $(s,t) \in S$ where $t \in S^1$ and $s \in (-\infty, R]$ or $[R, +\infty)$. This time, we are interested in Floer’s equation with autonomous
Hamiltonian:

\[ \partial_s u + J \partial_t u = J X_H, \]

where \( H : U(\Sigma) \to \mathbb{R} \). Since we are only interested in the behaviour of Floer solutions near \( \Sigma \), it suffices to have \( H \) defined on \( U(\Sigma) \).

**Lemma 4.3** (Intersection at asymptotics [56, Equation (7.22)]). Let \( p \in \Sigma \) be a Morse critical point of \( H \), and assume that there is a chart for \( X \) at \( p \) mapping a neighbourhood of \( p \) to a neighbourhood of the origin in \( \mathbb{R}^{2n-2} \times \mathbb{R}^2 \) with the following properties. It takes \( \Sigma \) to \( \mathbb{R}^{2n-2} \times \{0\} \), \( \omega \) to the standard form, \( J \) to a split complex structure constant in the \( \mathbb{R}^2 \)-direction, and \( H \) to the following form:

\[ H = H_\Sigma(x_1, \ldots, x_{2n-2}) - \alpha (x_{2n-1}^2 + x_{2n}^2) \]

for some \( \alpha \in \mathbb{R} \).

Let \( S \) be one of the two domains: \( S_+ \) or \( S_- \), and \( u : S \to U(\Sigma) \) be a solution of (4.11) asymptotic to \( p \) as \( s \to \pm \infty \), seen as a constant periodic orbit. Assume that \( u(\partial S) \cap \Sigma = \emptyset \), then the intersection number \([u(S)] \cdot [\Sigma]\) is greater than or equal to:

(i) \( \lfloor \alpha \rfloor + 1 \) if \( S = S_+ \), or

(ii) \( -\lfloor \alpha \rfloor \) if \( S = S_- \).

In the above lemma, \( \overline{u(S)} = u(S) \cup \{p\} \) is the closure of \( u(S) \). Then \( p \) is, by hypothesis, an intersection point between \( u(S) \) and \( \Sigma \). The proof of the lemma is quite straightforward, because the splitting assumption makes Floer’s equation completely standard in the normal \( \mathbb{R}^2 \)-direction responsible for the intersection multiplicity, and that equation can be explicitly solved.

**Remark 4.1.** To get familiar with the lemma, it is helpful to consider the case of flowlines. A \( t \)-independent Floer solution \( u : S_{\pm\infty} \to X \) is a gradient flowline of \( H \), flowing down in the direction \( s \to +\infty \). Suppose \( \alpha > 0 \), then \( H \) is decreasing in the direction normal to \( \Sigma \). Hence there exist downward gradient flowlines flowing away from \( \Sigma \) (i.e. asymptotic to \( \Sigma \) at the negative end), but no flowlines flowing towards \( \Sigma \) (i.e. asymptotic to \( \Sigma \) at the positive end). The former flowlines, which exist, obviously have intersection number 0 with \( \Sigma \) showing that the bound in Lemma 4.3 (ii) is optimal.

Finally, let us explain how to find \( J \) and \( H \) that satisfy the conditions of Lemmas 4.2 and 4.3. Let \( \Sigma \subset (X, \omega) \) be a Donaldson hypersurface given as the vanishing set of an almost holomorphic section of a complex line bundle on \( X \). Let \( \mathcal{L} \to \Sigma \) be the restriction of that line bundle to \( \Sigma \). Fix a Hermitian metric on \( \mathcal{L} \), and a connection \( \nabla \) with curvature \(-2\pi i d\omega|_\Sigma\). The total space \( \mathcal{L} \) carries a canonical symplectic form considered e.g. by Biran [9], given by

\[ \omega_0 = \pi^* \omega|_\Sigma + d(r^2 \alpha^\nabla) \]

where \( \pi : \mathcal{L} \to \Sigma \) is the projection, \( \alpha^\nabla \) is the circular 1-form associated with the connection \( \nabla \), and \( r \) is the fibrewise norm. Let \( E_\mathcal{L} \) be the radius-\( \epsilon \) neighbourhood of the zero-section of \( \mathcal{L} \), for some small fixed \( \epsilon \). By the symplectic neighbourhood theorem, there is a neighbourhood \( U(\Sigma) \subset X \) which is symplectomorphic \((E_\mathcal{L}, \omega_0)\).

Fix a function \( H_\Sigma : \Sigma \to \mathbb{R} \). Consider the autonomous Hamiltonian

\[ H = l + H_\Sigma \circ \pi - \alpha r^2 : E_\mathcal{L} \to \mathbb{R}, \]
where \( l \in \mathbb{R} \) is some constant and \( \pi : E_L \to \Sigma \) is the projection. Let \( J \) be the almost complex symplectic which is the \( \nabla \)-lift of an almost complex structure on \( \Sigma \) to \( E_L \); then \( J \) preserves \( \Sigma \). Or standing setup will be to use the parameter

\[
0 < \alpha < 1, \quad \text{equivalently, } \lfloor \alpha \rfloor = 0.
\]

The chosen \( J, H \) satisfy the conditions of Lemma 4.2 but not Lemma 4.3 since \( \nabla \) has non-zero curvature, it is not possible to bring all of \( J, H, \omega \) to a split form specified in Lemma 4.3 in a neighbourhood of a point. However, one may homotope \( \nabla \) to a flat connection over a neighbourhood of each critical point of \( H_\Sigma \), assuming this finite set of points has been chosen in advance. Then (4.12) gives another symplectic form on \( E_L \) which is diffeomorphic to the standard one, by Moser’s lemma. After the connection is modified this way, the above choice of \( H, J \) will satisfy the conditions of Lemmas 4.2, 4.3. We remind that the constant \( \alpha \) affects the conclusion of Lemma 4.3 and that we are using \( 0 < \alpha < 1 \).

**Remark 4.2.** It is likely that Lemma 4.3 is true for all Hamiltonians of the form (4.13) without assuming that \( \nabla \) is flat near the critical points.

**Remark 4.3.** In Seidel’s setup of [56], the normal bundle to \( \Sigma \) is trivial. Rephrasing [56, Setup 8.2] in our language, he chooses the globally flat \( \nabla \) which ensures the splitting as in Lemma 4.3 near all points of \( \Sigma \).

### 4.6. s-shaped Hamiltonians

We are going to introduce the main class of Hamiltonians that will be used in the proof of Theorem 1.1. Let \( X \) be a symplectic manifold and \( S \subset X \) a contact type hypersurface. A small neighbourhood \( U(S) \) of \( S \) admits a Liouville vector field whose flow identifies \( U(S) \) with \([1 - \delta, 1 + \delta] \times S\). We call \( U(S) \) equipped with such an identification a *Liouville collar*, and the parameter \( r \in [1 - \delta, 1 + \delta] \) the *radial co-ordinate* on the Liouville collar. The original hypersurface \( S \) is identified with the middle of the collar: \( \{1\} \times S \).

**Remark 4.4.** When proving Theorem 1.1, the specific setup will be to take \( S = \partial M \) the boundary of a Liouville domain inside \( X \). See Figure 4 (left).

![Figure 4](image-url)

**Figure 4.** Hamiltonians \( H_l \) of this shape will be used to prove Theorem 1.1. Here \( S = \partial M \).

Let \( X_- \cup X_+ = X \setminus U(S) \) be the two components of the complement of \( U(S) \); with \( X_- \) being the component with convex boundary \( [1 - \delta] \times S \). Take \( l \in \mathbb{R}_+ \) and define the unperturbed s-shaped Hamiltonian \( H^0_l : X \to \mathbb{R} \) by:

\[
\begin{align*}
H^0_l(x) &= 0 & \text{on } X_-,& \\
H^0_l(x) &= h_l(r(x)) & \text{for } x \in U(S),& \\
H^0_l(x) &= l & \text{on } X_+,&
\end{align*}
\]

(4.14)
Here \( r(x) \in [1 - \delta, 1 + \delta] \) is the radial co-ordinate of a point in the collar, and \( h_l(r) : [1 - \delta, 1 + \delta] \to \mathbb{R} \) is a function of the collar co-ordinate whose shape is shown in Figure 4 (right), taking \( S = \partial M \). The parameter \( l \) is its maximum value. Note that \( \delta \) is another parameter of the construction, but it will be fixed throughout so we do not include it in the notation. Finally, let

\[
(4.15) 
H_l : X \times S^1 \to \mathbb{R}
\]

be a \( t \)-dependent perturbation if \( H^0_l \) with the following properties:

(i) the perturbation turns \( H_l \) into a Morse function on \( X_- \) and \( X_+ \);

(ii) near \( \Sigma, H_l \) has the form \( (4.13) \) described in Subsection 4.5, so that it satisfies Lemmas 4.2 and 4.3 for some tame \( J \) preserving \( \Sigma \). We shall only be using almost complex structures with this property. We also require that the function \( H_\Sigma \) from \( (4.13) \) is \( C^2 \)-small;

(iii) optionally, we may use a \( t \)-dependent perturbation (\( t \in S^1 \)) in the collar region. If we choose to do so, we perform the perturbation only in the subregion \([1 - \delta, 1] \times S \) of the collar \( U(S) \subset X \); elsewhere \( H_l \) is independent of \( t \).

We call \( l \) the height of \( H_l \). Denote \( H_l = H_l \) for brevity. We use Figure 4 to depict this perturbed Hamiltonian as well. The 1-periodic orbits of \( X_H \) are divided into the following types, using the standard notation (see e.g. [24]):

i: constant orbits in \( X_- \);

ii: orbits in \([1 - \delta, 1] \times S \) arising in the region where \( h''_l > 0 \), corresponding to Reeb orbits in \( S \).

iii: orbits in \([1, 1 + \delta] \times S \) arising in the region where \( h''_l < 0 \), corresponding to Reeb orbits in \( S \).

iv: constant orbits in \( X_+ \) lying away from \( \Sigma \);

ivb: constant orbits in \( X_+ \) lying in \( \Sigma \). Such always exist by our choice of the Hamiltonian, see \( (4.13) \).

Note that the constant orbits i, iv, and ivb are Morse, and therefore non-degenerate.

A comment about time-dependent versus autonomous Hamiltonians is due. Depending on whether one uses the optional time-dependent perturbation appearing above, the type ii orbits will either be fully non-degenerate (two perturbed Hamiltonian orbits corresponding to a Reeb orbit), or will come in \( S^1 \)-families, where the \( S^1 \) action is the rotation in \( t \). In Section 5 we shall focus on the perturbed setting but the arguments work equally well in the autonomous setting, after the usual adjustments following the \( S^1 \)-Morse-Bott framework for symplectic cohomology of Bourgeois and Oancea [15]. Later in Section 6 we will need the autonomous framework to perform a computation, and we will mention the relevant adjustments therein.

Let us now record a special case of a lemma due to Bourgeois and Oancea [15, pp. 654-655]; see [24, Lemma 2.3] for a detailed discussion of it.

**Lemma 4.4** (Asymptotic behaviour). Let \( S_+ \) be the domain \( [R, +\infty) \times S^1 \) and \( u : S_+ \to X \) a solution to Floer’s equation \((4.10)\) with the Hamiltonian \( H = H_l \) as above. Assume that \( u \) is asymptotic, as \( s \to +\infty \), to a type iii orbit \( \gamma \) of \( H \) lying in \( \{1 + r_0 \} \times S \), for some \( 0 < r_0 < \delta \). Then for \( s \gg 0 \), \( u(s, t) \in (1 + r_0, 1 + \delta] \times S \). □

**Remark 4.5.** We do not want to perturb \( H_{S,l} \) in a \( t \)-dependent in the region \([1, 1 + \delta] \times S \) precisely to allow us to refer to [15] in a most straightforward way, as
this reference considers autonomous Hamiltonians. However, the same result also holds for small $t$-dependent perturbations of the Hamiltonian by the Floer-Gromov compactness of [15]. We actually learned Lemma 4.4 from [24] which applies it in the perturbed context.

5. Proof

This section contains the proofs of Theorems 1.1 and Theorem 2.1 as well as the definition of the Borman-Sheridan class and higher deformation classes $D_k$.

5.1. Stabilising divisor. Our first step is quite standard; it is inspired by the Auroux-Kontsevich-Seidel lemma [7, Section 6] and the technique of stabilising divisors due to Cieliebak and Mohnke [22]. Fix a tame almost complex structure $J$ such that $\Sigma$ is a $J$-complex hypersurface. Consider two disk counting problems.

The original count we are interested in comes from (4.4):

\[ m_0 = \#_{\rho(L)} M_0 = W_L(\rho). \]

Recall that $M_0$ consists of unparametrised holomorphic Maslov index 2 disks with boundary on $L$, and passing through a fixed point $pt \in L$. One introduces another number:

\[ m = \#_{\rho} M \]

counting parametrised holomorphic Maslov index 2 maps $u: (D, \partial D) \to (X, L)$ such that $u(1) = pt$ is a fixed point on $L$, and $u(0) \in \Sigma$. The count is again weighted using $\rho$. Denote by $M$ the 0-dimensional moduli space just introduced. Here $D$ is the unit disk in $\mathbb{C}$, $1 \in D$ is the fixed point on its boundary, and $0 \in D$ is the fixed point in the interior. Our first claim is that

\[ m = dm_0. \]

Recall that the natural orientations (signs) on $M_0$ and $M$ arise from regarding them as fibre products:

\[ M_0 = (M_{1,0}(2))^{#_{ev_1}(pt)}, \quad M = (M_{1,1}(2))^{#_{ev_1 \times ev_2}(pt \times \Sigma)} \]

where $M_{1,0}(2)$ resp. $M_{1,1}(2)$ are the moduli spaces of Maslov index 2 disks with one boundary marked point, resp. one boundary and one interior marked points; and $ev_1, ev_2$ is the evaluation at the boundary resp. interior marked point. Now, combining Equation (4.9) with positivity of intersections and transversality for generic $J$, one sees that the image $u(D)$, $u \in M_0$, geometrically intersects $\Sigma$ at exactly $d$ points. If one marks the preimage of any such point $p$ and find the unique reparametrisation $\tilde{u}$ of $u$ such that $\tilde{u}(1) = pt$, $\tilde{u}(0) = p$, one obtains an element $\tilde{u} \in M$. Therefore the converse forgetful map $M \to M_0$ is $d$-to-one. Again by positivity of intersections, this map preserves orientations; hence Equation (5.1) follows. For the rest of the proof, we shall work with $M$ instead of $M_0$.

5.2. Domain-stretching. Let $M \subset X \setminus \Sigma$ be a Liouville subdomain. Fix the height parameter $l \in \mathbb{R}_+$. Let

\[ H_l: X \times S^1 \to \mathbb{R} \]

be an s-shaped Hamiltonian of height $l$ described in Subsection 4.6, see (4.14), (4.15), where $\partial M$ is used as the contact type hypersurface $S$.

Our goal is to introduce, for each fixed $l$, a sequence of domain-dependent Hamiltonian perturbations of the holomorphic equation for the Maslov index 2 disks
appearing in the definition of $\mathcal{M}$, by inserting the Hamiltonian $H_l$ as a perturbation over a sequence of annuli exhausting the unit disk $D$. The sequence will be parametrised by the integers $n \to \infty$. Introducing this sequence of perturbations can be called *domain-stretching*, by analogy with neck-stretching in the SFT sense. It is important to keep in mind the following contrast with SFT neck-stretching: while the latter procedure changes the holomorphic equation by modifying the almost complex structure in the *target* $X$, we modify the equations over the *domain*, the unit disk. Domain-stretching is part of the standard Floer-theoretic toolbox: for example, it is used to prove composition rules for continuation maps in Floer cohomology (in which case the domain is the cylinder rather than the disk).

It is helpful to consider, instead of the unit disk $D$, a differently parametrised disk

$$D_n := A_n \cup B$$

where

$$A_n = S^1 \times [0, n]$$

is an annulus and $B$ is a unit disk capping the annulus $A_n$ from the right side, see Figure 5. On $A_n$, we introduce the following standard co-ordinates: $t \in S^1$ and $s \in [0, n]$. We extend them to co-ordinates on $B$ using the standard vector fields $\partial_s, \partial_t$ shown in Figure 5. Note that $\partial_s, \partial_t$ vanish at one point on $B$. Denote this point by $0 \in B$; also, denote the point $\{s = t = 0\}$ in $A_n$ by $1 \in A_n$ (this notation is unusual, but it is consistent with the boundary point $1 \in D$ used in the definition of $\mathcal{M}$ above).

Consider the complex structure on the disk $D_n$ glued from the standard complex structure on $A_n$ (which depends on $n$) and the standard complex structure on $B$ (which does not). The resulting complex structure on $D_n$ is of course biholomorphic to the unit disk $D$, but we will use the presentation $D_n = A_n \cup B$ to introduce domain-dependent perturbations.

We now define a sequence of domain-dependent Hamiltonians

$$H_{n,l} : D_n \times X \to \mathbb{R}.$$  

We introduce them separately over each of the two pieces $A_n$ and $B$; this means that we specify the restrictions $H_{|A_n \times X}$ and $H_{|B \times X}$.

**$A_n$**: Set

$$H_{n,l}(t, s, x) = H_l(t, x) : S^1 \times [0, n] \times X \to \mathbb{R}.$$  

Here $H_l$ is the $s$-shaped Hamiltonian of height $l$ introduced in Subsection 4.6, where we use the parameter $\alpha < 1$. Observe that $H_{n,l}$ is an $s$-independent Hamiltonian in this region.

**$B$**: Set $H_{n,l} \equiv 0$ near the point $0 \in B$ (recall this is the point where $\partial_s, \partial_t$ vanish) and $H_{n,l} \equiv H_l$ near $\partial B$. Over the sub-annulus of $B$ shown in light shade in Figure 5, $H_{n,l}$ interpolates between $H_l$ and zero in such a way that $\partial_s H_{n,l} \leq 0$ everywhere.

We make the following additional provisions. We require that $X_{H_{n,l}(s,t,x)}|\Sigma$ is small and is always tangent to $\Sigma$. For convenience we also assume that as one moves from left to right over the interpolation region (the light-shaded annulus), $H_{n,l}$ is first given near $\Sigma$ by varying the constant $l$ from (4.13) from the given height parameter $l$ to a small constant $\epsilon$ keeping the rest of the data in (4.13) fixed, and is subsequently homotoped monotonously to zero.
Figure 5 gives an impression of how the function $H_{n,l}$ looks like. It shows a movie of functions $H_{n,l}(s,t,\cdot): X \to \mathbb{R}$ for various values of $s$ (the dependence on $t$ may be small, and does not matter).

![Figure 5. Domain-dependent Hamiltonians $H_{n,l}$ used in the domain-stretching sequence.](image)

We make a choice of $H_{n,l}$ for each $n,l$ and arrange that $H_{n,l} \leq H_{n,l'}$ for $l \leq l'$, pointwise on $D_n \times X$. In particular, over the region $A_n$ we are using $s$-shaped Hamiltonians from Subsection 4.6 satisfying $H_l \leq H_{l'}$. For now, we continue to work with a fixed $l$.

One identifies $D_n$ conformally with the unit disk $D$, so that $(s,t)$ become the standard polar coordinates on $D$, and the respective points $0,1$ in $D_n$ and $D$ are identified. This uniformisation allows to treat our Hamiltonians as being defined on the same domain, $H_{n,l}: D \times X \to \mathbb{R}$.

Consider the following perturbed holomorphic equation for $u$: $(D,\partial D) \to (X,L)$:

\[
\partial_s u + J\partial_t u = JX_{H_{n,l}}
\]

where $X_{H_{n,l}}$ is the Hamiltonian vector field with respect to the variable in $X$. The vector fields $\partial_s, \partial_t$ vanish at the origin $0 \in D_n = D$, so Equation (5.2) does not make sense as written over that point. However, because $H_{n,l} \equiv 0$ near the origin, the equation simply restricts to the unperturbed $J$-holomorphic equation near the origin, so extends over it.

In what follows, we use an almost complex structure which coincides with near $\Sigma$ with the one used in Subsection 4.5, and is generic otherwise.

Let $\mathcal{M}_{n,l}$ be the moduli space of Maslov index 2 maps solving Equation (5.2) which additionally satisfy $u(1) = pt \in L$, $u(0) \in \Sigma$, where $pt \in L$ is a fixed point. These spaces are 0-dimensional and regular (there is obviously enough freedom to ensure transversality; for example, by perturbing $J$ near $L$).

A homotopy from $X_{H_{n,l}}$ to zero in Equation (5.2) proves that for each $n,l$:

\[
\#_p\mathcal{M}_{n,l} = \#_p\mathcal{M} = m.
\]

(Disk or sphere bubbling is excluded because $L$ is monotone and the disks have Maslov index 2, therefore have the lowest possible symplectic area.)

5.3. **Breaking and gluing.** Recall that the Hamiltonian $H_{n,l}$ restricts to the $s$-independent Hamiltonian $H_l$ in the region $A_n \subset D_n$ of the domain. This is a setting to which the standard Floer-Gromov compactness theorem applies. It states that solutions in $\mathcal{M}_{n,l}$ converge, as $l$ is fixed and $n \to \infty$, to broken curves like the one shown in Figure 6. The breakings happen along 1-periodic Hamiltonian orbits of $H_l$. 

Figure 6. A broken configuration consists of the (a)- and the (b)-curve, which carry the depicted Hamiltonian perturbations $H_l^{(a)}$, $H_l^{(b)}$.

The broken curve contains two main parts (see Figure 6): we call them the (a)-curve and the (b)-curve. In addition, the broken curve may contain: cylinders solving Floer’s differential equation with respect to $H_l$ inserted between the (a)- and (b)-curves, and disk or sphere bubbles, as shown in Figure 7. Some sphere bubbles may be contained inside $\Sigma$, because $J$ preserves $\Sigma$. The claim is that these extra parts cannot appear.

Figure 7. A general Floer-Gromov limit.

Lemma 5.1. When $l$ is large enough (greater than the area of a Maslov index 2 disk with boundary on $L$), any broken curve which is the Floer-Gromov limit of curves in $\mathcal{M}_{n,l}$ is composed of only two parts: the (a)- and the (b)-curve.

We shall give a proof of Lemma 5.1 later in this section, because the proof will re-use arguments appearing in the next subsection. So we proceed assuming Lemma 5.1.

Let us discuss the (a)- and the (b)-curves in more detail. The domain of the (a)-curve is $[0,+\infty) \times S^1$. The domain of the (b)-curve is bi-holomorphic to $\mathbb{C}$, but it is more convenient to look at it as on $((-\infty,0] \times S^1)$.

Figure 6. A broken configuration consists of the (a)- and the (b)-curve, which carry the depicted Hamiltonian perturbations $H_l^{(a)}$, $H_l^{(b)}$.

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where $B$ is the capping disk appearing earlier. The (a)- and (b)-curves solve Floer’s equation (4.10) with respect to the Hamiltonians

$$H_l^{(a)}: [0,+\infty) \times S^1 \times X \to \mathbb{R} \quad \text{and} \quad H_l^{(b)}: ((-\infty,0] \times S^1) \cup B \times X \to \mathbb{R}.$$ 

given by:

$$H_l^{(a)}(s,t,x) \equiv H_l(t,x) \quad \text{(the $s$-independent $s$-shaped Hamiltonian)};$$

$$H_l^{(b)} \quad \text{is stitched using $H_l$ over $(-\infty,0] \times S^1$, and $H_n,l|_B$ over $B$.}$$

Both the (a)- and the (b)-curve must be asymptotic to the same periodic orbit which are denoted by:

$$\gamma, \quad \text{a 1-periodic orbit of $H_l$.}$$
(For type II and III orbits, whenever they are autonomous, the (a)- and (b)-curve may converge to different parametrisations of $\gamma$). The incidence conditions $u(1) = pt \in L$ and $u(0) \in \Sigma$ must be met by the new curves; now the marked point 0 belongs to the (b)-curve and the marked point 1 belongs to the (a)-curve as shown in Figure 6.

**Proposition 5.2.** For each $l$, the count of the configurations consisting of an (a)-curve and a (b)-curve sharing an asymptotic orbit $\gamma$, equals

$$\frac{1}{d}W_L(\rho),$$

if the count of the (a)-curves is weighted using the local system $\rho$.

**Proof.** By Gromov-Floer compactness and Floer’s gluing theorems, the count of such configurations equals $\#_{\rho} M_{n,l}$ for large enough $n$. Then the desired equality follows from (5.3) and (5.1). In the $S^1$-Morse-Bott case, the gluing theorem is due to [15].

5.4. **Ruling out type ivb orbits.** Recall the discussion of the periodic orbits of the $s$-shaped Hamiltonian $H_t$ in Subsection 4.6; they come in groups I, II, III, IvA, ivb. Our aim is to show that an orbit $\gamma$ arising from a broken curve as above must necessarily be a type I or II orbit when $l$ is large enough. Below is a restatement of the hypothesis that the embedding $M \subset X \setminus \Sigma$ is grading-admissible.

**Lemma 5.3.** The image of the intersection pairing $-\cdot [\Sigma] : H_2(X,M;\mathbb{Z}) \to \mathbb{Z}$ is contained in $d\mathbb{Z}$.

**Proof.** Let $\eta$ be the natural trivialisation of $(K_{X \setminus \Sigma})^d$. The obstruction $o$ to finding a $d$th root of $\eta$ over $M$ lies in $H^1(M;\mathbb{Z}/d\mathbb{Z})$, see e.g. [17] Section 2.2. Consider a 2-chain $B$ in $(X,M)$. The obstruction $o(b) \in \mathbb{Z}/d\mathbb{Z}$ evaluated on $\partial B$ equals the intersection number $B \cdot [\Sigma] \mod d$. We are given that $o(b)$ vanishes, which implies the lemma.

In what follows, by a *limiting orbit* $\gamma$ we mean a periodic orbit of $H_t$ arising as the common asymptotic of some broken configuration of an (a)- and a (b)-curve which is the Floer-Gromov limit of a sequence of solutions in $M_{n,l}$ as $n \to +\infty$.

**Lemma 5.4.** When $l$ is greater than the area of a Maslov index 2 disk with boundary on $L$, a limiting orbit $\gamma$ cannot be of type ivb.

**Proof.** Because type ivb orbits are constant, the images of the (a)- and the (b)-curve can be compactified by adding the corresponding asymptotic point. These two curves therefore define homology classes which we call $A \in H_2(X,L)$ and $B \in H_2(X)$, for the (a)- and (b)-curve respectively. (We use coefficients in $\mathbb{Z}$.) The (b)-curve may be entirely contained in $\Sigma$, because $J$ preserves $\Sigma$ and the Hamiltonian vector field of $H_t^{(b)}$ is always tangent to $\Sigma$. Consider the two cases separately; see Figure 8.

— If the (b)-curve is contained in $\Sigma$, then the Hamiltonian perturbation in its equation is small, as the function $H_\Sigma$ from (4.13) is taken to be small. This means that the image of the (b)-curve is close to being $J$-holomorphic; it follows that $\omega(B) \geq 0$ (a precise argument appears in Lemma 5.5 below). Therefore $c_1(B) \geq 0$ by monotonicity, where the Chern class is computed inside $X$, not
Σ. Observe that the (b)-curve may be constant, in which case \( B = 0 \). It follows that
\[
B \cdot \langle \Sigma \rangle \geq 0,
\]
since \( \Sigma \) is dual to a positive multiple of \( c_1(X) \).

— If the (b)-curve is not contained inside \( \Sigma \), note that it intersects \( \Sigma \) at least once, by the incidence condition \( u(0) \in \Sigma \). Due to positivity of intersections, using both Lemmas 4.2 and 4.3 with \( \lfloor \alpha \rfloor = 0 \), one obtains that
\[
B \cdot \langle \Sigma \rangle \geq 1.
\]

**Figure 8.** When \( \gamma \) is a type ivb orbit, the (b)-curve may or may not fall entirely within \( \Sigma \).

Next, recall that
\[
(A + B) \cdot \langle \Sigma \rangle = d
\]
because \( A + B \) is homologous to a Maslov index 2 disk \( D \) one started with, see (4.9). In both cases, it follows that
\[
A \cdot \langle \Sigma \rangle \leq d,
\]
or equivalently \( \mu(A) \leq 2 \). Therefore
\[
\omega(A) \leq 2\lambda
\]
where \( \lambda \) is the monotonicity constant. Now one uses a version of the standard energy estimate, see e.g. [49, 2.4], re-cast in a slightly different fashion. Let
\[
u(s, t) : [0, +\infty) \times S^1 \to X
\]
be an (a)-curve. Denote by \( \gamma' \subset L \) the boundary loop of \( u \), i.e. \( u(0, t) \). Recall that the (a)-curve solves the \( s \)-independent Floer’s equation with the Hamiltonian \( H_l = H_l^{(a)} \). Let \( X \) be its Hamiltonian vector field. Then one has:
\[
0 \leq E(u) = \int |\partial_s u|^2 \, ds \wedge dt
= \int \omega(\partial_s u, \partial_t u - X) \, ds \wedge dt
= \omega(A) - \int \omega(\partial_s u, X) \, ds \wedge dt
= \omega(A) - \int dH_l(\partial_s u) \, ds \wedge dt
= \omega(A) - \int_\gamma H_l(t) + \int_\gamma H_l(t).
\]
(5.5)

The last step uses the fact that \( H_l \) is \( s \)-independent. Recall that \( \gamma \subset \Sigma \) is the constant type ivb periodic orbit. By construction, \( H_l|_{\Sigma} \) is a perturbation of the constant function equal to \( l \), see (4.13) and Subsection 4.6 where \( H_l \) was defined. Therefore:
\[
\int_\gamma H_l(t) \geq l - \epsilon
\]
for a small \( \epsilon \). Also by construction, \( H_l \) is a perturbation of the zero function in \( X_- \) (the region of \( M \) away from the collar), and one can assume \( L \subset X_- \). So
\[
\int_\gamma H_l(t) \leq \epsilon.
\]
Putting the estimates together, one obtains:

\[ 0 \leq 2\lambda - l + 2\epsilon. \]

For \( l \) larger than \( 2\lambda + 2\epsilon \), this is a contradiction. \( \Box \)

**Lemma 5.5.** Suppose the \((b)\)-curve is contained inside \( \Sigma \), then it defines a class \( B \in H_2(X) \) such that \( \omega(B) \geq 0 \).

**Proof.** The argument uses an energy estimate similar to (5.5). Let \( u \) be such a \((b)\)-curve defining a class \( B \in H_2(X) \). The domain of \( u \) is bi-holomorphic to \( \mathbb{C} \); let us puncture the point 0 in the domain and reparametrise the resulting domain to \( \mathbb{R} \times S^1 \) straightening the co-ordinate vector fields \( \partial_s, \partial_t \). Recall that \( u \) solves Floer’s equation with an \( s \)-dependent Hamiltonian \( H_l^{(b)} \) as in (5.4). For the purpose of this remark, denote

\[ H = H_l^{(b)}. \]

Then the type ivb orbit \( \gamma \) is the \( s \to -\infty \) asymptotic of \( u \), and the point \( q = u(0) \in \Sigma \) is the \( s \to +\infty \) asymptotic (seen as a removable singularity). Accounting for the \( s \)-dependence, the estimate analogous to (5.5) is:

\[ 0 \leq E(u) = \int_{\mathbb{R} \times S^1} |\partial_s u|^2 ds \wedge dt = \ldots = \omega(B) - \int H(\partial_s u) ds \wedge dt = \omega(B) - \int_\gamma H(-\infty, t) + \int_q H(+\infty, t) + \int (\partial_s H) ds \wedge dt \]

Recall that \( \partial_s H \leq 0 \), and by the arrangements made in Subsection 5.2 there is a sub-annulus in the domain of the \((b)\)-curve where \( \partial_s H \mid \Sigma \) is constant with respect to the variables in \( \Sigma \): it only depends on \( s \), and integrates to \( \epsilon - l \) over the sub-annulus. This, together with the monotonicity of \( H \), implies that

\[ \int (\partial_s H) ds \wedge dt \leq \epsilon - l. \]

By the construction of \( H \) one also has:

\[ \int_\gamma H(-\infty, t) = \int_\gamma H_l(t) \geq l - \epsilon, \quad \int_q H(+\infty, t) = 0. \]

The estimate becomes

\[ 0 \leq \omega(B) - (l - \epsilon) + (\epsilon - l) = \omega(B) + 2\epsilon. \]

For sufficiently small \( \epsilon \) it follows that \( \omega(B) \geq 0 \) since the areas are discrete by monotonicity. \( \Box \)

**Remark 5.1.** The only case in the above proof which is ruled out by crucially using the condition that \( l \) is sufficiently large (larger than the area of a Maslov index 2 disk) is the case when the \((b)\)-curve is a constant curve with zero area, having a constant type ivb orbit asymptotic.

Indeed, revisiting the proof above one sees that otherwise \( B \cdot [\Sigma] \geq 1 \), so

\[ B \cdot [\Sigma] \geq d \]

because \( [\Sigma] \) is divisible by \( d \). Applying positivity of intersections—Lemmas 4.2 and 4.3 to the \((a)\)-curve, one obtains \( A \cdot [\Sigma] \geq 1 \). Indeed, the contribution to the intersection number from the constant asymptotic \( \gamma \) is at least \( |\alpha| + 1 = 1 \). Recalling that \( (A + B) \cdot [\Sigma] = d \), one gets a contradiction without using the fact that \( l \) is large.
5.5. **No extra bubbles.** We rewind to prove Lemma 5.1 which asserts that a broken configuration cannot have any other components (disk or sphere bubbles, Floer cylinders) except for the (a)- and the (b)-curve.

Proof of Lemma 5.7 The disk bubbles must have Maslov index $\geq 2$ by monotonicity, therefore have intersection number at least $d$ with $[\Sigma]$ by (4.9). The sphere bubbles, whether or not contained in $\Sigma$, also have homological intersection at least $d$ with $[\Sigma]$. This is obvious for the bubbles not in $\Sigma$, and true for the bubbles in $\Sigma$ for the following reason: they have positive area, therefore have positive Chern class in $X$, but $\Sigma$ is dual to $dc_1(X)$. So we record that if there is at least one disk or sphere bubble, the rest of the curve has homological intersection number $\leq 0$ with $\Sigma$. If there are no disk or sphere bubbles, the homological intersection is still $d$.

Let us forget the disk and sphere bubbles (if they exist) and look at the rest of the broken curve. It is composed of the (a)-curve, the (b)-curve and a string of Floer cylinders between them, attached to each other along asymptotic 1-periodic orbits of $H_1$.

---

**Case 1. There is at least one asymptotic orbit which has type ivb.** In this case some Floer cylinders and/or the (b)-curve may be contained in $\Sigma$, see Figure 9 where the curves are represented by segments. We shall arrive at a contradiction independently on whether there were any additional disk or sphere bubbles, and the argument will be an expansion on the proof of Lemma 5.4.

As a general observation, those Floer cylinders which are contained in $\Sigma$ have two type ivb asymptotics and compactify to closed cycles $A \in H_2(X)$. Such cycles have non-negative area, by a similar argument as was used to prove (see Lemmas 5.4, 5.5) that the (b)-curves contained in $\Sigma$ have non-negative area. Therefore their classes satisfy

\[ A \cdot [\Sigma] \geq 0 \quad (A = \text{Floer cylinder in } \Sigma) \tag{5.6} \]

Next, we call a connected union of Floer cylinders, none of which is contained in $\Sigma$, beginning and ending at a type ivb asymptotic a cylinder group. See the circled groups in Figure 9. Compactifying by the type ivb asymptotics, one sees that a cylinder group defines a class in $A \in H_2(X)$. We claim that the class of a cylinder group satisfies

\[ A \cdot [\Sigma] \geq d \quad (A = \text{cylinder group}) \tag{5.7} \]

It suffices to show that $A \cdot [\Sigma] \geq 1$, because $[\Sigma]$ is divisible by $d$. And indeed, the contribution from the beginning and the ending asymptotic of a group to $A \cdot [\Sigma]$ is
in total at least 
\[ |\alpha| + 1 - |\alpha| = 1 \]
by Lemma 4.3 (since one asymptotic is positive and the other one is negative), regardless of \( \alpha \). The contribution from any other geometric intersection is positive by Lemma 4.2.

**Case 1i.** The \((b)\)-curve is contained in \( \Sigma \). In this case we have shown in Lemma 5.5 that the class \( A \in H_2(X) \) of the \((b)\)-curve satisfies
\[(5.8) \quad A \cdot [\Sigma] \geq 0 \quad (A = \text{\((b)\)-curve in } \Sigma)\]

**Case 1ii.** The \((b)\)-curve is not contained in \( \Sigma \). Consider the last group of curves not contained in \( \Sigma \), beginning with a negative type ivb orbit and ending with the \((b)\)-curve; see Figure 9. This gives a cycle \( A \in H_2(X) \) such that
\[(5.9) \quad A \cdot [\Sigma] \geq d \quad (A = \text{group with the } (b)\text{-curve}).\]

Indeed the intersection at the negative type ivb orbit at least \(-|\alpha| = 0\), by Lemma 4.3 and our arrangement about \( \alpha \). But we also have a strictly positive intersection due to the incidence condition for the \((b)\)-curve, so the desired estimate follows.

**Case 1, common conclusion.** Consider the remaining group of curves beginning with the \((a)\)-curve and ending at the first positive type ivb asymptotic; see Figure 9. We call it an \((a)\)-group and it defines a relative homology class \( A \in H_2(X, L) \). Recall that the total intersection of all punctured curves with \( \Sigma \) was at most \( d \), and the intersections of the curves not in the \((a)\)-group are estimated by (5.6), (5.7), (5.8) and (5.9). So for the class \( A \) of an \((a)\)-group one has
\[ A \cdot [\Sigma] \leq d \quad (A = \text{\((a)\)-group})\]
consequently, \( \mu(A) \leq 0 \) and
\[ \omega(A) \leq 2\lambda. \]

Then one uses an analogue of (5.5) to get a contradiction whenever \( l \geq 2\lambda \). (This holds regardless of whether one had to forget any disk or sphere bubbles in the beginning.) Note that one has to apply estimate (5.5) to several curves comprising the group and add them up; this obvious detail is left to the reader.

**Case 2.** There are no type ivb asymptotics. In this case each broken curve in the broken configuration has non-negative intersection number with \( \Sigma \) on its own. Now, we claim there can be no disk or sphere bubbles: indeed, after forgetting them, the rest of the curve would have intersection \( \leq 0 \) with \( \Sigma \); on the other hand, the latter intersection number is at least 1 due to the incidence condition of the \((b)\)-curve with \( \Sigma \). Floer cylinders are ruled by a simple argument involving the dimensions of moduli spaces, using the fact the curves are not contained in \( \Sigma \) and are therefore generically regular.

\[ \square \]

5.6. Ruling out type iVa orbits.

**Lemma 5.6.** The \((a)\)-curve of the broken configuration is disjoint from \( \Sigma \).

**Proof.** Suppose first that the asymptotic orbit \( \gamma \) has type iVa or i: then it is constant. Consider the homology classes \( A \in H_2(X, L) \) and \( B \in H_2(X) \) of the \((a)\)- and the \((b)\)-curve. In this case the \((b)\)-curve is automatically not contained in \( \Sigma \) (because \( \gamma \) is not), so the whole configuration looks like in Figure 10. Arguing as before, one obtains \( B \cdot [\Sigma] \geq d \) and therefore \( A \cdot [\Sigma] = 0 \). By positivity of intersections (Lemma 4.2), this means that the \((a)\)-curve is disjoint from \( \Sigma \).
Suppose that $\gamma$ has type II or III; then it belongs to the collar $[1-\delta, 1+\delta] \times \partial M$. Denote

$$N' = X \setminus (M \cup ([1-\delta, 1+\delta] \times \partial M)), $$

which is diffeomorphic to $N = X \setminus M$. Then the images of the (a)-curve and (b)-curves define relative homology classes $A, B \in H_2(N', \partial N')$.

We know that $B \cdot [\Sigma] \geq 1$ by positivity of intersections (Lemma 4.2) and the fact that it intersects $\Sigma$ at least once by the incidence condition $u(0) \in \Sigma$ for the (b)-curve. Then by Lemma 5.3, $B \cdot [\Sigma] \geq d$. As above, $(A + B) \cdot [\Sigma] = d$ so $A \cdot [\Sigma] = 0$ which implies the result. □

**Lemma 5.7.** For sufficiently large $l$, a limiting orbit $\gamma$ cannot be of type IVa.

**Proof.** Let $\theta$ be the Liouville form on $X \setminus \Sigma$ making $L$ exact. By definition, the action of a loop $\gamma: S^1 \to X \setminus \Sigma$ is:

$$A(\gamma) = -\int_{S^1} \gamma^* \theta + \int_{S^1} H \circ \gamma.$$ 

When $\gamma$ is a periodic orbit of $H_l$ of type IVa one has:

$$A(\gamma) \geq l - \epsilon.$$ 

This is because $\gamma^* \theta = 0$ for a constant orbit, and $H_l(\gamma)$ is by construction close to $l$ in the region containing type III orbits.

Looking at the (a)-curve, let $\gamma' \subset L$ be its boundary loop. Then

$$A(\gamma') \leq \epsilon,$$

given that $\theta|_L = 0$ (by exactness) and $H_l|_L$ is small (by construction); compare with the proof of Lemma 5.4. However, since the Floer equation for the (a)-curve is $s$-independent, it must hold that

$$A(\gamma') \geq A(\gamma).$$

This gives a contradiction. □

**Remark 5.2.** At this point, we know that $\gamma$ must be of type II or III, for $l$ greater than the area of a Maslov index 2 disk. In fact, the only place where this condition is needed is discussed in Remark 5.1; all other arguments work for a small $l$ as well.

It is instructive to understand what happens when one takes the height $l$ for $H_l$ (see Figure 3) to be very small, and keep $\alpha$ a small positive number: $[\alpha] = 0$. This way one can arrange $H_l$ to have so small slopes uniformly that it acquires no 1-periodic orbits of type II, III at all. Revisiting the above proofs in this case (see Remark 5.1), one concludes that a limiting orbit $\gamma$ of the broken curve must be of type IVb, and moreover the (b)-curve must be constant.
The upshot is that when \( l \) is small, the breaking of Maslov 2 disks into the (a)-curve and the (b)-curve is essentially trivial, in the sense that the (b)-curve is constant and the (a)-curve ‘looks like the initial disk’. We learn that domain-stretching does not achieve anything structurally useful in this case.

5.7. Ruling out type III orbits. The next step is to show that type III orbits also cannot arise as an asymptotic orbit of a broken configuration of an (a)- and a (b)-curve.

Lemma 5.8. A limiting orbit \( \gamma \) cannot be of type III.

**Proof.** It is enough to look at the (a)-curve. Lemma 4.4 says that the (a)-curve near \( \gamma \) ‘goes towards the right’ in the collar co-ordinate, i.e. towards \( X_+ \), which is concave. It contradicts the no-escape lemma whose particular case is summarised below. □

Remark 5.3. We learned this way of combining the no-escape lemma with Lemma 4.4 from [24, Section 2.3]. The same argument is essentially used in [36, Proposition 4.4] and [39, Proposition 2.11].

The no-escape lemma, whose particular case appears below, is due to Abouzaid and Seidel [4, Lemmas 7.2 and 7.4]; see also [49, Lemma 19.5] and [24, Lemma 2.2]. It generalises the maximum principle for Floer solutions: while the maximum principle only applies inside a Liouville collar (or the symplectisation) of a contact manifold, the no-escape lemma allows an arbitrary Liouville cobordism instead of a collar.

Lemma 5.9 (No escape lemma). Let \( u : [0, +\infty) \times S^1 \to X \setminus \Sigma \) solve the Floer equation for the (a)-curve, with boundary on \( L \) and asymptotic orbit \( \gamma \). Observe the assumption that \( u \) avoids \( \Sigma \).

If \( \gamma \) is of type I or II then its image is contained in \( M \); if \( \gamma \) is of type III then its image is contained in \( M \cup [1, 1 + r_0] \times \partial M \) where \( \gamma \subset \{1 + r_0\} \times \partial M \), \( r_0 < \delta \), see Figure 4.

**Proof.** Recall that \( H_l \) is the \( s \)-shaped Hamiltonian used in the Floer equation for the (a)-curve. First, we explain why the image of \( u \) must lie inside \( M \cup (1, 1+\delta] \times \partial M \). Observe that both \( L \) and \( \gamma \) lie in this domain. Moreover, \( H_l \) is close to a constant function near \( \{1 + \delta\} \times \partial M \). Let us introduce the following additional requirement on the profile of \( H_l \), recall Figure 4. Recall that \( H_l \) is a function of the collar co-ordinate on \( [1-\delta, 1+\delta] \times \partial M \). We require that there exists a number \( \delta_0 \) close to \( \delta \) such that \( H_l \) is a linear function of small slope in a neighbourhood of \( \{1 + \delta_0\} \times \partial M \), and that the slopes of \( H_l \) are small uniformly on \( [1 + \delta_0, 1+\delta] \times \partial M \) and \( H_l \) has no periodic orbits in that region. See Figure 11 for a rough sketch. It follows that \( \gamma \) necessarily lies ‘to the left’ of \( \{1 + \delta_0\} \times \partial M \).

Assume that the (a)-curve enters the region \( [1 + \delta_0, 1+\delta] \times \partial M \). To get a contradiction, one can apply the no-escape lemma as stated in [49, Lemma 19.3 and Remark 19.4(2)], using the contact hypersurface \( \{1 + \delta_0\} \times \partial M \). The lemma requires that the Hamiltonian be linear in the collar co-ordinate, which has been arranged. It follows that the (a)-curve is contained in \( M \cup [1, 1+\delta_0] \times \partial M \).

Next, the inclusion into the smaller subdomain \( M \) or \( M \cup [1, 1 + r_0] \) follows from the maximum principle which, roughly, says that \( u \) cannot extend inside the collar \( [1-\delta, 1+\delta] \times \partial M \) ‘to the right’ of all of its asymptotic or boundary conditions. The
maximum principle applies whenever $H$ has the form $h(r)$ on the collar, $h'(r) \geq 0$ [49, Lemma 19.1], which certainly holds in our case.

Formally speaking, the above linearity condition for $H_l$ near $\{1+\delta_0\} \times \partial M$ should have been mentioned in the initial setup of $H_l$ in Subsection 4.6. However, because this is a minor technicality and there are alternative arguments (see the remark below) which do not require the linearity, we decided to keep the extra condition within this proof. □

![Figure 11. The s-shaped Hamiltonian $H_l$ near the collar, with an additional linear part of small slope.](image)

**Remark 5.4.** The two-step proof above seems like a necessary technicality, as one cannot apply the no-escape lemma directly to the hypersurface $\{1+r_0\} \times \partial M$. The reason is that our Hamiltonian $H_l = h(r)$ depends on the radial co-ordinate non-linearly inside the collar $[1-\delta, 1+\delta] \times \partial M$. The no-escape lemma in this setting still exists [49, Lemma 19.5], but it requires the condition that

$$-r_0h'(r_0) + h(r_0) \leq 0$$

which is not necessarily satisfied in our case. In the above proof, one of the possible workarounds was used. Alternatively, one can scale the Liouville form near the collar, which results in scaling $r_0$, to achieve inequality (5.10). A third solution is to notice that $-r_0h'(r_0) + h(r_0) = A(\gamma)$, and if $A(\gamma) > 0$ the (a)-curve cannot exist, as one must have $\epsilon \geq A(\gamma') \geq A(\gamma)$, see the proof of Lemma 5.7.

**Corollary 5.10.** A limiting orbit $\gamma$ either has type I or type II. The image of the (a)-curve of any broken configuration lies in $M$. □

5.8. **The degree is zero.** Recall that $\Sigma \subset X$ is a Donaldson divisor of degree $d$. Let $\zeta$ be a trivialisation of $K_M$ such that $\zeta^d$ is the natural trivialisation of $(K_X|_{\Sigma})^d$; it exists by the hypothesis of graded-admissibility. Let $(B, \partial B) \subset (X, M)$ be a 2-chain with boundary. By construction, one has the following identity between its relative Chern class with respect to $\zeta$, and the intersection number with $\Sigma$:

$$c_1(B, K_X|_B, \zeta|_{\partial B}) = \frac{1}{d}[B] \cdot [\Sigma].$$

**Lemma 5.11.** A limiting orbit $\gamma$ has degree zero in the above trivialisation of $K_M$.

**Proof.** Let $|\gamma| \in \mathbb{Z}$ be the degree. Our grading conventions (explained in the introduction) are such that the dimension of the moduli space of (a)-curves without the boundary point constraint $u(0) = pt \in L$ equals

$$n - |\gamma|,$$

using the fact that the (a)-curves are contained in $M$. The moduli space of the (b)-curves without the interior point constraint $u(1) \in \Sigma$ is

$$|\gamma| + 2,$$
where \( 2 \) is twice the relative Chern number from \([5.11]\), using the fact that the \((b)\)-curve has intersection number \(d\) with \(\Sigma\). Adding the incidence conditions and using regularity, one arrives at the conditions:

\[
    n - |\gamma| \geq n, \quad |\gamma| + 2 \geq 2.
\]

It follows that \(|\gamma| = 0\). \(\Box\)

5.9. The Borman-Sheridan class. Let \(CF^*(H_l)\) be the Floer complex generated by all periodic orbits of \(H_l\); it carries the usual Floer differential. Denote by

\[
    CF^*_{i,II}(H_l) \subset CF^*(H_l)
\]

the subspace generated by periodic orbits of \(H_l\) having type \(i\) or \(II\). We define

\[
    BS_l \in CF^0_{i,II}(H_l)
\]

to be the chain counting the output asymptotics \(\gamma\) of all rigid maps \(u: C \to X\) which:

- solve Floer’s equation with the Hamiltonian perturbation \(H_l^{(b)}\) from \([5.4]\),
- satisfy the incidence condition \(u(0) \in \Sigma\),
- have homological intersection number \(d\) with \(\Sigma\),
- and have asymptotic orbit \(\gamma\) of type \(i\) or \(II\).

We call such maps the \((b)\)-curves. The last condition has to be added explicitly, and is not guaranteed otherwise. (Above, we only proved that \(\gamma\) must of type \(i\) or \(II\) for configurations arising from the breaking of a Maslov index 2 disk: this means, we were using the existence of an \((a)\)-curve as well.)

It follows as in the proof of Lemma \([5.11]\) that the outputs of the \((b)\)-curves have degree zero, so indeed \(BS_l \in CF^0_{i,II}(H_l)\).

Proposition 5.12. In our setting, the following holds.

(i) \(CH^*_{i,II}(H_l)\) is a complex with respect to the differential \(d^0\) counting only those Floer cylinders which run between the type \(i\) and \(II\) orbits, and moreover do not intersect \(\Sigma\).

(ii) For \(l' > l\), there are chain maps

\[
    c_{i,II}^0: CF^*_{i,II}(H_l) \to CF^*_{i,II}(H_{l'})
\]

counting only those continuation map solutions \(CF^*(H_l) \to CF^*(H_{l'})\) which run between the type \(i\) and \(II\) orbits, and moreover do not intersect \(\Sigma\). (We use interpolating Hamiltonians that are monotonely non-decreasing in \(s\).)

(iii) The direct limit of the cohomologies of \(CH^*_{i,II}(H_l)\) with respect to \(c_{i,II}^0\) is isomorphic to the symplectic cohomology of \(M\):

\[
    SH^*(M) \cong \lim_{l \to +\infty} HF^*_{i,II}(H_l).
\]

(iv) The elements \(BS_l \in CF^0_{i,II}(H_l)\) are \(d^0\)-closed, and their homology classes are respected by the maps \(c_{i,II}^0\). Therefore, they define an element

\[
    BS \in SH^0(M)
\]

which we call the Borman-Sheridan class. It depends both on \(M\) and its Liouville embedding into \(X \setminus \Sigma\), but is invariant of Liouville deformations of these data.
Proof. The proofs use same the ingredients that have already been used above, so we shall be brief.

For (i), one has to show that $(d_0^0)^2 = 0$. It is enough to prove that Fredholm index 1 Floer trajectories connecting type I, III orbits cannot break at type IV or type II orbits. The proof is analogous to Lemmas 5.4, 5.7, 5.8 also employing an analogue of Lemma 5.9.

For (ii), one needs to rule out the same types of breaking for the continuation maps $c_{l,l'}^0$. This follows the same steps as above, with some changes standard in Floer theory: for example, the estimate in Lemma 5.4 becomes

$$0 \leq \omega(A) = \int_{\gamma} H_l(t) + \int_{\gamma'} H_{l'}(t) + \int \partial_t H(s, t, u(s, t)) ds \wedge dt,$$

where $H|_{S^1 \times M}$ is the chosen continuation map homotopy between $H_l$ and $H_{l'}$ such that $\partial_s H \leq 0$. Given the latter inequality, the argument of Lemma 5.4 goes through.

Using (i) and (ii), it follows that the direct limit in (iii) makes sense. We claim that there is a chain level isomorphism:

$$HF^*_l(H_l) \cong HF^*_s(\hat{H}_l),$$

where $\hat{H}_l$ were introduced in (4.6) as the Hamiltonians computing $SH^*(M)$. First, recall that

$$H_l|_{S^1 \times M} \equiv \hat{H}_l,$$

and the periodic orbits of $H_l$ contained in $M$ are precisely the type I and II orbits. It follows that the generators of (5.12) coincide. Moreover, solutions contributing to $d^0_l$ in fact belong to $M$ by the no escape lemma (analogous to Lemma 5.9), so the differentials defining both sides of (5.12) also coincide. Compare with a similar argument in e.g. [33, Lemma 2.5].

Again by a version of the no escape lemma, (5.12) intertwines $c_{l,l'}^0$ with the standard continuation maps $CF^*(\hat{H}_l) \to CF^*(\hat{H}_{l'})$. And since

$$SH^*(M) = \lim_{l \to +\infty} HF^*(\hat{H}_l)$$

by definition, one immediately obtains (iii).

To prove (iv), consider the moduli space like that defining $BS_l$, but 1-dimensional rather than 0-dimensional. One checks that its boundary points correspond to $d^0_l(BS_l)$. To do so, following the above steps one shows that the 1-parameter families of curves break along orbits which are of type I or II, and the ‘left’ parts of the broken curves are confined to $M$ and compute the $d^0_l$-differential. □

5.10. Concluding the proof. Let us return to the preceding steps where we applied a domain-stretching procedure to Maslov index 2 holomorphic disks and ended up in a broken configuration consisting of the (a)- and (b)-curve. Let us look at the (a)-curves.

By Corollary 5.10, the (a)-curves are contained in $M$, and the equation they solve is precisely the equation for $CO_l$ (see Subsection 4.2) because:

$$H_l|_{S^1 \times M} \equiv \hat{H}_l,$$

where the two sides were defined in (4.15) and (4.6); see also (5.4). So by Proposition 5.2, under isomorphism (5.12) we obtain:

$$CO_l(BS_l) = \frac{1}{\delta} W_L(\rho) \cdot 1_L \in HF^*_M(L, L).$$
Passing to the limit $l \to +\infty$, it is seen that
\[ CO(\mathcal{BS}) = \frac{1}{3} W_L(\rho) \cdot 1_L \in HF^*_M(L, L). \]
This completes the proof of Theorem 1.1.

5.11. Higher deformation classes.

Proof of Theorem 2.1. The proof is entirely analogous to the proof of Theorem 1.1. The first step of the proof—introducing the interior marked point—is unnecessary since it is already built in the definition of higher disk potentials. Apply the same domain-stretching procedure; the (b)-curve inherits the tangency condition eating up the total homological intersection number $k$ with $\Sigma$. Repeating the above arguments, one finds that the limiting orbit $\gamma$ is of type $i$ or $ii$, and that the (a)-curve is confined to $M$ thus computing the closed-open map. We define the class $D_k$ to be the count of the outputs of the (b)-curves, namely we introduce
\[ (D_k)_l \in CF^0_{i,ii}(H_l) \]
to be the chain counting the output asymptotics $\gamma$ of all rigid curves $u : C \to X$ which

- solve Floer’s equation with the Hamiltonian perturbation $H_l^{(b)}$ from (5.4),
- satisfy order $k$ tangency condition at the point $u(0) \in \Sigma$,
- have homological intersection number $k$ with $\Sigma$,
- and have asymptotic orbit $\gamma$ of type $i$ or $ii$.

After this, the class $D_k \in SH^0(M)$ is defined analogously to Proposition 5.12, and the conclusion of the proof is similar. □

6. Anticanonical divisors and partial compactifications

This sections covers two remaining topics: the proof of Proposition 2.3 which computes of the Borman-Sheridan class (and higher deformation classes) in the complement of an anticanonical divisor; and a version of Theorem 1.1 in a setting where $\mathcal{X}$ is allowed to be non-compact.

6.1. Equivariance. We begin with an observation about the Borman-Sheridan class. Recall that the outputs of the (b)-curves appearing in the previous section which define the elements
\[ BS_l \in CF^0_{i,ii}(H_l), \]
are Hamiltonian orbits of $H_l$ of type $i$ and $ii$. The following lemma will be helpful soon.

Lemma 6.1. The element $BS_l$ is a linear combination of type $ii$ orbits, i.e. type $i$ orbits do not contribute to $BS$.

For a proof we shall assume that $H_l$ and $H_{n,l}$ used in Section 5 are autonomous Hamiltonians. Before proceeding, let us review how the moduli problem for the (a)- and the (b)-curves gets modified in the autonomous setting.

According to the standard modification of Floer theory in the $S^1$-Morse-Bott setting [15, 17], one takes the following generators to define the complex $CF^0_{i,ii}(H_l)$ when $H_l$ is autonomous. Each type $i$ (constant) orbit gives one generator; for each $S^1$-family of type $ii$ orbits we pick one parameterised orbit $\gamma$ and introduce two formal generators $\hat{\gamma}, \check{\gamma}$. (Our notation follows [17]; in [15] these would be $\gamma_p$ and $\gamma_q$.) It is natural to treat a constant orbit $\gamma$ as a $\hat{\gamma}$-orbit. The grading rule is
|\gamma| = |\hat{\gamma}| - 1, and for a constant orbit \gamma, the degree of |\hat{\gamma}| is the Morse index. After a non-autonomous perturbation of the Hamiltonian, we will see two type II periodic orbits corresponding to each \gamma, with degrees matching the ones of \hat{\gamma} and \check{\gamma}.

The moduli problems involving Floer solutions asymptotic to the periodic orbits of \( H_t \) acquire the following modification. Each cylindrical end of a curve needs to be equipped with an asymptotic marker. Having asymptotic \( \hat{\gamma} \) at a negative puncture or asymptotic \( \check{\gamma} \) at a positive puncture means that the asymptotic marker is required to go to the initial point \( \gamma(0) \) of the orbit \gamma, where \gamma has the parameterisation fixed in advance. In the two other scenarios, the asymptotic marker is unconstrained. These rules in particular concern the (a)- and the (b)-curves appearing in Section 5.

**Proof of Lemma 6.1.** Consider the \( S^1 \)-action on the domain of the (b)-curves rotating in \( t \); it fixes the origin which is required to pass through \( \Sigma \). To understand whether this gives an \( S^1 \)-action on the moduli space of (b)-curves defining the classes \( BS_t \), one must pay attention to the asymptotic markers; recall that the (b)-curves have a negative puncture. The answer is that there is an \( S^1 \)-action on the part of the moduli space whose outputs are type I orbits, or type II orbits of form \( \hat{\gamma} \). Indeed, for \( \hat{\gamma} \)-asymptotics the negative asymptotic marker is unconstrained, so the moduli problem is still satisfied after rotation. For type I orbits the asymptotic marker is required to go to \( \gamma(0) \) but since \gamma is constant it holds that \( \gamma(0) = \gamma(t) \) for all \( t \), and the moduli problem is again satisfied after rotation.

Consider those (b)-curves whose asymptotic is of type I. The \( S^1 \)-action on them is non-trivial unless a curve in question is independent of \( t \), in which case it is a flowline. But the (b)-curves cannot be flowlines because they are required to have intersection number 1 with \( \Sigma \), and this number is zero for a flowline. So the \( S^1 \)-action is non-trivial and the (b)-curves are not rigid. On the other hand, the (b)-curves contributing to \( BS_t \) must be rigid, which means that the (b)-curves with a type I asymptotic do not exist. \( \square \)

**Remark 6.1.** The non-existence of (b)-curves with type I asymptotic can be compared to the following argument. Consider such a curve; its asymptotic has degree 0 so it corresponds to the minimum \( p \) of a Morse function. In the adiabatic limit such a curve converges to a Chern number 1 sphere passing through \( p \) at a fixed point in the domain, and through \( \Sigma \) at another fixed point in the domain. But rather than being rigid, this problem is has virtual dimension \(-2\) by an easy dimension count: \( 2n + 2c_1 - 6 + 4 - 2n - 2 = -2 \). The adiabatic limit reveals the same extra \( S^1 \)-symmetry as that used in the proof above.

Recall that one can define \( S^1 \)-equivariant symplectic cohomology \( SH^*_eq(M) \) of a Liouville domain \( M \), see [51, 15, 16], which is a \( \mathbb{C}[u] \)-module fitting into an exact sequence

\[
\ldots \rightarrow H^*(M)[u] \rightarrow SH^*_eq(M) \rightarrow SH^*(M)[u] \rightarrow \ldots
\]

The above proof essentially shows the following proposition (which will not be used).

**Proposition 6.2.** The Borman-Sheridan class admits a lift to \( SH^*_eq(M) \). \( \square \)

6.2. The Borman-Sheridan class in the complement of an anticanonical divisor. We will now prove Proposition 2.3. Let \( \Sigma \) be a smooth anticanonical divisor (i.e. \( d = 1 \)), and \( M = X \setminus \Sigma \). We will show that \( BS = r \); the fact that
\( D_k = r^k \) is analogous. By Lemma 6.1 one can know that all asymptotics of the (b)-curves are of type II.

Pick an \( \epsilon \)-neighbourhood \( U(\Sigma) \) of \( \Sigma \) and choose the \( s \)-shaped Hamiltonian \( H_l \) whose growth happens within this neighbourhood, see Figure 12.

![Figure 12. An \( s \)-shaped Hamiltonian with growth in a neighbourhood of \( \Sigma \).](image)

The lemma below is a variation on the argument of Biran and Khanevsky [11, Proposition 5.0.2], with several major differences. As opposed to [11], one has the freedom to make \( U(\Sigma) \) as small as desired. Another difference is that while [11] work with purely holomorphic disks, our (b)-curves have a Hamiltonian perturbation, and we quote Bourgeois and Oancea [15] for the relevant neck-stretching procedure.

**Lemma 6.3.** In the chain model (2.4) for \( SH^0(X \setminus \Sigma) \), the (b)-curves are all asymptotic to the orbit \( r \); therefore the class \( BS \) is a multiple of \( r \).

Moreover, there is a choice of \( J \) and \( H_l \) such that the (b)-curves are entirely contained in a neighbourhood of \( \Sigma \).

**Proof.** We make the following arrangements: \( H_l \) is \( \epsilon \)-small over all of its type II orbits; these orbits are \( \epsilon \)-close to \( \Sigma \); and \( U(\Sigma) \) is itself a neighbourhood whose radius is of order \( \epsilon \); see Figure 12. We denote by \( o(\epsilon) \) any quantity such that \( o(\epsilon) \to 0 \) as \( \epsilon \to 0 \).

Recall that all type II degree 0 orbits of \( H_l \) have the form \( r^k \) where \( r \) is the simple Hamiltonian orbit appearing in the chain model (2.4). Let \( u: \mathbb{C} \to X \) be a (b)-curve and \( (B, \partial B) \subset X \) be its image, with boundary an asymptotic orbit \( \partial B = r^k \). Then:

\[
0 \leq E(u) = \int_{r^k} (-\theta + H_l) + 2\pi (\langle [B], [\Sigma] \rangle) = -2\pi k + o(\epsilon) + 2\pi.
\]

Here is an explanation of this formula. Above, \( 2\pi (\langle [B], [\Sigma] \rangle) \) stands for the residue of the Liouville 1-form \( \theta \) over \( B \) at the intersection points \( B \cap \Sigma \), which is proportional to the intersection number. Next we have used that this intersection number equals 1. Because \( \partial B = r^k \) is an orbit which is \( \epsilon \)-close to \( \Sigma \) and links it \( k \) times, one has:

\[
\int_{r^k} \theta = 2\pi k + o(\epsilon) \quad \text{and} \quad \int_{r^k} H_l = o(\epsilon).
\]

To justify (6.1), it remains to note that although the Hamiltonian perturbation on the (b)-curve is domain-dependent (5.4), it is non-increasing in variable \( s \) by construction. When \( \epsilon \) is small enough, (6.1) implies that \( k = 1 \) and

\[
E(u) = o(\epsilon).
\]

Recall that our Hamiltonian is \( C^1 \)-small near \( \partial U(\Sigma) \). One applies neck-stretching along \( \partial U(\Sigma) \) combined with Hamiltonian slowdown as explained in [15, Section 5 and Figure 2]. Briefly speaking, neck-stretching modifies the almost complex structure in a small collar neighbourhood of \( \partial U(\Sigma) \) by making the collar ‘long’, and
Hamiltonian slowdown modifies $H_l$ on that collar so that it has constant slope tending to zero. (The modification of $H_l$ stays within the class of s-shaped Hamiltonians introduced in Subsection 4.6.)

By the SFT compactness theorem [14], [15, Proof of Proposition 5, Step 1], if the (b)-curve is not eventually contained in $U(\Sigma)$ for the neck-stretching sequence of almost complex structures and $H_l$s, one gets a non-trivial holomorphic building in the SFT limit. Let $u_1 \subset U(\Sigma)$ be the part of the building containing the original asymptotic Hamiltonian orbit $r^k$—this part is now constrained to $U(\Sigma)$. Moreover, $u_1$ must contain at least one additional SFT-type puncture. Denote the asymptotic orbits of its SFT punctures by $\gamma_1, \ldots, \gamma_q$, and denote by $k_i \geq 1$ the multiplicity of $\gamma_i$. Then $\{r, \gamma_1, \ldots, \gamma_q\}$ is the full list of asymptotic orbits for $u_1$, the first being the initial Hamiltonian orbit; see Figure 13. By positivity of intersections and the fact that the intersection number before passing to the SFT limit was equal to 1, one has that

$$[u_1] \cdot [\Sigma] \leq 1.$$  

Apply the estimate similar to (6.1) to $u_1$ to arrive at a contradiction:

$$0 \leq E(u_1) = \int_r (-\theta + H_l) - \sum_i \int_{\gamma_i} \theta + 2\pi ([u_1] \cdot [\Sigma]) \leq -2\pi - 2\pi \sum_i k_i + 2\pi + o(\epsilon) \leq -2\pi + o(\epsilon).$$

The last inequality follows from the fact that $k_i \geq 1$. One concludes that when $\epsilon$ is small enough and $J$ is sufficiently stretched, the (b)-curve is contained inside $\Sigma$. □

Proof of Proposition 2.3. One needs to show that $\mathcal{BS} = r$ and $\mathcal{D}_k = r^k$; we will prove the former since the latter is analogous. Given the previous lemma, there are several ways of completing the argument by a modification of [15]. Our strategy is to re-run the proof of Theorem 1.1 where instead of holomorphic disks we stretch holomorphic planes in the neighbourhood of $\Sigma$ whose count is known to be 1.

Let $U(\Sigma)$ be the neighbourhood appearing above, and consider its completion by an infinite concave end: $M = ((-\infty, 0) \times \partial U(\Sigma)) \cup U(\Sigma)$; see [14] for the basic terminology. Pick a contact form at the negative boundary $\partial_{-\infty}M$ whose Reeb flow lifts the Hamiltonian flow of a Morse function $h_{\Sigma}$ on $\Sigma$ with unique minimum. This contact structure is a perturbation of the standard one whose Reeb flow is the 1-periodic rotation around $\Sigma$. Denote by $\hat{r}$ a parameterised Reeb orbit over the minimum of $h_{\Sigma}$ considered as a degree 0 element of the non-equivariant contact homology complex of $\partial_{-\infty}M$ [15]. (We consider this complex just as a vector space and not interested in its differential, whose definition in general can meet certain difficulties.)
Equip $M$ with an almost complex structure $J$ which is cylindrical near the negative end. Consider the moduli space $\mathcal{M}(\hat{r})$ of $J$-holomorphic planes $u: \mathbb{C} \to M$ asymptotic to $\hat{r}$ at infinity considered as a negative puncture, with an asymptotic marker matching the initial point of $\hat{r}$, such that $u(0) \in \Sigma$ and the total intersection number with $\Sigma$ equals 1. This is a rigid problem and the count of solutions in this moduli space equals 1. The last statement can be seen in two ways. If one worked with the unperturbed contact structure, one could have arranged that the projection $M \to \Sigma$ is holomorphic for some almost complex structure on $\Sigma$. In the perturbed case, one can ensure that the projection was holomorphic over the critical points of $h_\Sigma$, that is, over the planes $\mathcal{C} \subset M$ containing all Reeb orbits. Such a plane over the minimum of $h_\Sigma$ provides one solution in $\mathcal{M}(\hat{r})$ and one can show the uniqueness by an energy argument, noting that the solutions in $\mathcal{M}(\hat{r})$ must have small energy. An alternative argument is to glue the elements in $\mathcal{M}(\hat{r})$ to analogous holomorphic planes with a positive puncture asymptotic to $\hat{r}$ (this time modulo $S^1$-reparameterisation) and passing through a fixed point in $\Sigma$. The glued solutions become holomorphic curves in the projectivisation of the normal bundle to $\Sigma$. One homotops the glued almost complex structure to one for which the projection onto $\Sigma$ is everywhere holomorphic, after which the glued solutions become the unique fibre sphere.

The count of $\mathcal{M}(\hat{r})$ has been established. Now one runs the analogue of the proof of Theorem 1.1 for these curves. All arguments carry over; it must be additionally mentioned that SFT breaking at the negative end is impossible because any non-trivial SFT bubble projects to a sphere in $\Sigma$ with positive Chern number, making the part of the broken configuration in $M$ belong to a moduli space of virtual dimension $\leq -2$ which is generically empty. The resulting (b)-curves identically compute the Borman-Sheridan class as defined in Section 5. The (a)-curves do not compute the closed-open map, but instead the continuation map $\Psi$ of Bourgeois and Oancea [15] realising the isomorphism from symplectic to non-equivariant contact homology; the count of such (a)-curves equals 1. Therefore the count of (b)-curves also equals 1, which shows that $BS = r$. The same proof works for $D_k$ by considering the $k$-fold Reeb orbit $\hat{r}^k$ instead of $\hat{r}$. □

6.3. Partial compactifications. In this subsection we discuss a version of Theorem 1.1 in a class of situations where $X$ is allowed to be non-compact. A convenient setting is the following one. Let $Y$ be a compact Fano variety equipped with its monotone Kähler symplectic form and $\Sigma \subset Y$ be a normal crossings anticanonical divisor. Denote its irreducible components by $\Sigma = \cup_{i \in I} \Sigma_i$. Choose a subset of the irreducible components $J \subset I$ and let $X = Y \setminus (\cup_{i \in J} \Sigma_i)$. Suppose $L \subset Y$ is a monotone Lagrangian submanifold which is disjoint from $\Sigma$ and exact in $Y \setminus \Sigma$. Obviously, $L \subset X$ is also monotone and therefore there it has well-defined potentials in $X$ and in $Y$:

$W_{L,X}, W_{L,Y}$.

Consider a class $A \in H_2(Y, L; \mathbb{Z})$ such that $\mu(A) = 2$. Then by (4.9), $A \cdot [\Sigma] = 1$. 

(from SH to Lagrangian enumerative geometry 43)
Lemma 6.4. The potential $W_{L,X}$ can be obtained by computing all holomorphic disks in $Y$ that contribute to $W_{L,Y}$ and picking out those disks whose homology classes $A$ have the following property:

$$A \cdot [\Sigma_i] = 0 \quad \text{for all} \quad i \in J.$$  

Proof. Clearly, the potential $W_{L,X}$ counts those disks that contribute to $W_{L,Y}$ which are moreover disjoint from $\Sigma_i$ for all $i \in J$. It follows that the classes of such disks satisfy $A \cdot [\Sigma_i] = 0$. Conversely, if $A \cdot [\Sigma_i] = 0$, then the disks in this class are disjoint from $\Sigma_i$ by positivity of intersections, assuming the chosen almost complex structure preserves $\Sigma$. □

Theorem 6.5. Let $Y$, $\Sigma$ and $X$ be as above. Suppose $M \subset Y \setminus \Sigma$ is any Liouville subdomain. There exists a class $BS_{M,X} \in SH^0(M)$ with the following property. For any monotone $L \subset Y$ which is contained in $M$ and exact in $X \setminus \Sigma$ (automatically, $L$ is exact in $M$), and any local system $L = (L, \rho)$, one has:

$$CO(BS_{M,X}) = W_{L,X}(\rho)$$

where $CO: M \to HF^*_M(L, L)$ is the closed-open map.

Proof. Let $\tilde{\Sigma}$ be the smoothing of $\Sigma$, contained in a neighbourhood of $\Sigma$. There is an embedding of Liouville domains

$$Y \setminus \Sigma \subset Y \setminus \tilde{\Sigma},$$

therefore $M \subset Y \setminus \tilde{\Sigma}$ is also a Liouville subdomain. One may run the proof of Theorem 1.1 applied to $L \subset M \subset Y \setminus \tilde{\Sigma}$.

Among the Maslov index 2 disks with boundary on $L$, one picks out those whose homology classes satisfy (6.3) as a topological condition. After domain-stretching, the (a)-curves are confined to $M$ therefore have zero intersection with any $\Sigma_i$. So the (b)-curves still have the property that their homological intersection with $[\Sigma_i]$, $i \in J$, vanishes. We define $BS_{M,X}$ by counting the output asymptotics $\gamma$ of those (b)-curves defining the usual Borman-Sheridan class $BS$ for $M \subset X \setminus \tilde{\Sigma}$ which satisfy this extra topological condition. The rest of the proof remains unchanged. □

Example 6.2. Let $L \subset \mathbb{C}^2$ be the monotone Clifford torus which remains monotone after the compactification

$$X := \mathbb{C}^2(x, y) \subset Y := \mathbb{C}P^2(x, y, z).$$

The space $Y = \mathbb{C}P^2$ plays the role of the compact Fano variety. Consider the anticanonical divisor

$$\Sigma = \Sigma_1 \cup \Sigma_2 \subset Y$$

where $\Sigma_1 = \{z = 0\}$ is the line at infinity, so that

$$X = Y \setminus \Sigma_1,$$

and $\Sigma_2$ is a smooth conic, e.g. $\{xy = z^2\}$, so that

$$L \subset M := \mathbb{C}P^2 \setminus \Sigma = \mathbb{C}P^2 \setminus (\Sigma_1 \cup \Sigma_2)$$

is exact. Theorem 6.5 asserts that there is a class $BS_{M,X} \in SH^0(M)$,
which under the closed-open map onto $L$, with various local systems, computes the potential

$$W_{L,X}(u,v) = (u + 1)v$$

of the Clifford torus in $\mathbb{C}^2$, written in some basis $(u,v)$. Next one can consider the Chekanov torus $L' \subset M \subset \mathbb{C}P^2$ and repeat the story for it (see e.g. [46] and [53, Section 11] for details), with the upshot that applying the closed-open map onto $L' \subset M$ makes the same Borman-Sheridan class compute the potential of the Chekanov torus in $\mathbb{C}^2$,

$$W_{L',X}(u,v) = v^{-1}.$$

Pascaleff computed [45] that

$$SH^0(M) \cong \mathbb{C}[p,q,(1-pq)^{-1}],$$

and we shall show in [28] that

$$BS_{M,X} = p.$$

**Remark 6.3.** In view of the example above and the mirror-symmetric context explained in the introduction, it may be an interesting problem to look for relationships between the symplectic cohomology of the complement of a normal crossings divisor $SH^0(Y \setminus \Sigma)$, and the LG potentials of monotone tori in the partial compactifications $X$ of $Y \setminus \Sigma$. (One hopes for a statement in the spirit of Corollary 2.5, except that now the usual LG potentials may already give interesting relations in symplectic cohomology rings.)

**REFERENCES**

[1] A. Abbondandolo and M. Schwarz. On the Floer homology of cotangent bundles. *Comm. Pure Appl. Math.*, 59(2):254–316, 2006.

[2] A. Abbondandolo and M. Schwarz. Floer homology of cotangent bundles and the loop product. *Geom. Topol.*, 14:1569–1722, 2010.

[3] M. Abouzaid. Symplectic cohomology and Viterbo’s theorem. In *Free loop spaces in geometry and topology: including the monograph Symplectic cohomology and Viterbo’s theorem by Mohammed Abouzaid*, volume 24 of *IRMA Lect. Math. Theor. Phys.*, 2015.

[4] M. Abouzaid and P. Seidel. An open string analogue of Viterbo functoriality. *Geom. Topol.*, 14:627–718, 2010.

[5] M. Abouzaid and I. Smith. Exact Lagrangians in plumbings. *Geom. Funct. Anal.*, 22(4):785–831, 2012.

[6] H. Argüz. *Log geometric techniques for open invariants in Mirror Symmetry*. PhD thesis, University of Hamburg, 2016.

[7] D. Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor. *J. Gökova Geom. Topol.*, 1:51–91, 2007.

[8] D. Auroux, D. Gayet, and J.-P. Mohsen. Symplectic hypersurfaces in the complement of an isotropic submanifold. *Math. Ann.*, 321(4):739–754, 2001.

[9] P. Biran. Lagrangian barriers and symplectic embeddings. *Geom. Funct. Anal.*, 11(3):407–464, 2001.

[10] P. Biran and O. Cornea. Rigidity and uniruling for Lagrangian submanifolds. *Geom. Topol.*, 13:2881–2989, 2009.

[11] P. Biran and M. Khanevsky. A Floer-Gysin exact sequence for Lagrangian submanifolds. *Comment. Math. Helv.*, 88(4):899–952, 2013.

[12] M. S. Borman and N. Sheridan. In preparation, 2016.

[13] F. Bourgeois. A Morse-Bott approach to contact homology. In *Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*, volume 35 of *Fields Inst. Commun.*, pages 55–77. Amer. Math. Soc., Providence, RI, 2003.

[14] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in Symplectic Field Theory. *Geom. Topol.*, 7:799–888, 2003.
[15] F. Bourgeois and A. Oancea. Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces. *Duke Math. J.*, 146(1):71–174, 2009.

[16] F. Bourgeois and A. Oancea. $S^1$-equivariant symplectic homology and linearized contact homology. *Internat. Math. Res. Notices*, 2017(13):3849–3937, 2016.

[17] Frédéric Bourgeois, Tobias Ekholm, and Yasha Eliashberg. Effect of Legendrian surgery. *Geom. Topol.*, 16(1):301–389, 2012. With an appendix by Sheel Ganatra and Maksim Maydanskiy.

[18] F. Charest and C. Woodward. Floer trajectories and stabilizing divisors. *J. Fixed Point Theory Appl.*, 19(2):1165–1236, 2017.

[19] K. Cieliebak, A. Floer, and H. Hofer. Symplectic homology. II. A general construction. *Math. Z.*, 218(1):103–122, 1995.

[20] K. Cieliebak and U. Frauenfelder. Rabinowitz Floer homology and symplectic homology. *Ann. Sci. Éc. Norm. Supér.*, 43(6):957–1015, 2010.

[21] K. Cieliebak and J. Latschev. The role of string topology in symplectic field theory. In *New Perspectives and Challenges in Symplectic Field Theory*, volume 49 of *CRM Proc. Lecture Notes*, 2009.

[22] K. Cieliebak and K. Mohnke. Symplectic hypersurfaces and transversality in Gromov-Witten theory. *J. Symplectic Geom.*, 5(3):281–356, 2007.

[23] K. Cieliebak and K. Mohnke. Punctured holomorphic curves and Lagrangian embeddings. *arXiv:1411.1870*, 2014.

[24] K. Cieliebak and A. Oancea. Symplectic homology and the Eilenberg-Steenrod axioms. *arXiv:1511.00485*, 2015.

[25] L. Diogo. *Filtered Floer and Symplectic homology via Gromov-Witten theory*. PhD thesis, Stanford, 2012.

[26] L. Diogo, D. Tonkonog, R. Vianna, and W. Wu. In preparation. 2017.

[27] S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Diff. Geom.*, 44(4):666–705, 1996.

[28] T. Ekholm, G. Dimitroglou Rizell, and D. Tonkonog. In preparation. 2017.

[29] Tobias Ekholm and Yanki Lekili. Duality between Lagrangian and Legendrian invariants. *arXiv:1701.01284*, 2017. Preprint, *arXiv:1701.01284*.

[30] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to Symplectic Field Theory. In *Visions in Mathematics: GAFA 2000 Special volume, Part II*, pages 560–673, 2010.

[31] T. Etnyre and Y. Lekili. Koszul duality patterns in Floer theory. *Geom. Topol.*, 21:3313–3389, 2017.

[32] A. Floer and H. Hofer. Symplectic homology. I. Open sets in $C^n$. *Math. Z.*, 215(1):37–88, 1994.

[33] S. Ganatra and D. Pomerleano. A Log PSS morphism with applications to Lagrangian embeddings. *arXiv:1611.06849*, 2016.

[34] S. Ganatra and D. Pomerleano. Remarks on symplectic cohomology of smooth divisor complements. *Draft*, 2017.

[35] O. Lazarev. Contact manifolds with flexible fillings. *arXiv:1610.04837*, 2016.

[36] Y. Li. Koszul duality via suspending Lefschetz fibrations. *arXiv:1710.09186*, 2017.

[37] D. McDuff and D. A. Salamon. *J-Holomorphic Curves and Symplectic Topology*, volume 52 of *Amer. Math. Soc. Colloq. Publ.*, 2004.

[38] M. McLean. A spectral sequence for symplectic homology. *arXiv:1011.2478*, 2010.

[39] M. McLean. The growth rate of symplectic homology and affine varieties. *Geom. Funct. Anal.*, 22(2):369–442, 2012.
[44] Y.-G. Oh. Relative Floer and quantum cohomology and the symplectic topology of Lagrangian submanifolds. In Proceedings for the 1994 Symplectic Topology program, Publ. of the Newton Institute. Cambridge University Press, 1996.

[45] J. Pascaleff. On the symplectic cohomology of log Calabi-Yau surfaces. arXiv:1304.5298, 2013.

[46] J. Pascaleff. Floer cohomology in the mirror of the projective plane and a binodal cubic curve. Duke Math. J., 163(13):2427–2516, 2014.

[47] J. Pascaleff and D. Tonkonog. The wall-crossing formula and Lagrangian mutations. arXiv:1711.03209, 2017.

[48] A. Ritter. Deformations of symplectic cohomology and exact Lagrangians in ALE spaces. Geom. Funct. Anal., 23(10):779–816, 2010.

[49] A. Ritter. Topological quantum field theory structure on symplectic cohomology. J. Topol., 6(2):391–489, 2013.

[50] P. Seidel. Fukaya categories and deformations. In Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), pages 351–360, 2002.

[51] P. Seidel. A biased view of symplectic cohomology. Current Developments in Mathematics, 2006:211–253, 2008.

[52] P. Seidel. Homological Mirror Symmetry for the genus two curve. J. Algebraic Geom., 20:727–769, 2011.

[53] P. Seidel. Lectures on categorical dynamics. Author’s website, 2013.

[54] P. Seidel. Fukaya $A_\infty$-structures associated to Lefschetz fibrations. I. arXiv:1504.06317, 2015.

[55] P. Seidel. Homological Mirror Symmetry for the Quartic Surface, volume 236, number 1116 of Mem. Amer. Math. Soc. 2015.

[56] P. Seidel. Fukaya $A_\infty$-structures associated to Lefschetz fibrations. III. arXiv:1608.04012, 2016.

[57] N. Sheridan. On the homological mirror symmetry conjecture for pairs of pants. J. Differential Geom., 29(2):271–367, 2011.

[58] N. Sheridan. Homological Mirror Symmetry for Calabi-Yau hypersurfaces in projective space. Invent. Math., 199(1):1–186, 2015.

[59] N. Sheridan. On the Fukaya category of a Fano hypersurface in projective space. Publ. Math. Inst. Hautes Études Sci., 2016.

[60] I. Smith. Floer cohomology and pencils of quadrics. Invent. Math., 189(1):149–250, 2012.

[61] Z. Sylvan. On partially wrapped Fukaya categories. arXiv:1604.02540, 2016.

[62] D. Tonkonog. Commuting symplectomorphisms and Dehn twists in divisors. Geom. Topol., 19:3345–3403, 2015.

[63] R. Vianna. Infinitely many exotic monotone Lagrangian tori in $\mathbb{C}P^2$. J. Topol., 9(2):535–551, 2016.

[64] R Vianna. Infinitely many monotone Lagrangian tori in del Pezzo surfaces. arXiv:1602.03355, 2016.

[65] C. Viterbo. Functors and computations in Floer homology with applications. II. Preprint, 1996.

[66] C. Viterbo. Functors and computations in Floer homology with applications. I. Geom. Funct. Anal., 9(5):985–1033, 1999.

[67] C. Viterbo. Symplectic real algebraic geometry. Preprint, 2000.

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