HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS

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Abstract. Let \( f \) be analytic in the unit disk \( \mathbb{D} \) and normalized so that \( f(0) = 0 = f'(0) - 1 \). In this paper we give sharp bound of Hankel determinant of the second order for the class of analytic functions satisfying
\[
\left| \arg \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right| < \frac{\gamma \pi}{2} \quad (z \in \mathbb{D}),
\]
for \( 0 < \alpha < 1 \) and \( 0 < \gamma \leq 1 \).

1. Introduction and preliminaries

Let \( \mathcal{A} \) denote the family of all analytic functions in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) and satisfying the normalization \( f(0) = 0 = f'(0) - 1 \).

A function \( f \in \mathcal{A} \) is said to be strongly starlike of order \( \beta, 0 < \beta \leq 1 \) if, and only if,
\[
\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \beta \frac{\pi}{2} \quad (z \in \mathbb{D}).
\]
We denote this class by \( \mathcal{S}^\beta \). If \( \beta = 1 \), then \( \mathcal{S}^1 \equiv \mathcal{S}^\ast \) is the well-known class of starlike functions.

In [1] the author introduced the class \( \mathcal{U}(\alpha, \lambda) \) \( (0 < \alpha \text{ and } \lambda < 1) \) consisting of functions \( f \in \mathcal{A} \) for which we have
\[
\left| \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) - 1 \right| < \lambda \quad (z \in \mathbb{D}).
\]
In the same paper it is shown that \( \mathcal{U}(\alpha, \lambda) \subset \mathcal{S}^\ast \) if
\[
0 < \lambda \leq \frac{1 - \alpha}{\sqrt{(1 - \alpha)^2 + \alpha^2}}.
\]
The most valuable up to date results about this class can be found in Chapter 12 from [4].

In the paper [2] the author considered univalence of the class of functions \( f \in \mathcal{A} \) satisfying the condition
\[
\left| \arg \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right| < \frac{\gamma \pi}{2} \quad (z \in \mathbb{D})
\]
for \( 0 < \alpha < 1 \) and \( 0 < \gamma \leq 1 \), and proved the following theorem.

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Theorem A. Let $f \in A$, $0 < \alpha < \frac{2}{\pi}$ and let
\[
\left| \arg \left[ \left( \frac{z}{f(z)} \right)^{1+\alpha} f'(z) \right] \right| < \gamma_*(\alpha) \frac{\pi}{2} \quad (z \in \mathbb{D}),
\]
where
\[
\gamma_*(\alpha) = \frac{2}{\pi} \arctan \left( \sqrt{\frac{2}{\pi\alpha} - 1} \right) - \alpha \sqrt{\frac{2}{\pi\alpha} - 1}.
\]
Then $f \in S_\beta$, where
\[
\beta = \frac{2}{\pi} \arctan \sqrt{\frac{2}{\pi\alpha} - 1}.
\]

2. Main result

In this paper we will give the sharp estimate for Hankel determinant of the second order for the class of analytic functions $f \in A$ which satisfied the condition (1.1).

Definition 1. Let $f \in A$. Then the $q$th Hankel determinant of $f$ is defined for $q \geq 1$, and $n \geq 1$ by
\[
H_q(n) = \left| \begin{array}{cccc}
 a_n & a_{n+1} & \cdots & a_{n+q-1} \\
 a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{array} \right|.
\]

Thus, the second Hankel determinant is $H_2(2) = a_2a_4 - a_3^2$.

Theorem 1. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ belongs to the class $A$ and satisfy the condition (1.1). Then we have the next sharp estimation:
\[
|H_2(2)| = |a_2a_4 - a_3^2| \leq \left( \frac{2\gamma}{2 - \alpha} \right)^2,
\]
where $0 < \alpha < 2 - \sqrt{2}$ and $0 < \gamma \leq \frac{1}{3}(\alpha^2 - 4\alpha + 2)$.

Proof. We can write the condition (1.1) in the form
\[
\left( \frac{f(z)}{z} \right)^{-(1+\alpha)} f'(z) = \left( \frac{1 + \omega(z)}{1 - \omega(z)} \right)^\gamma = (1 + 2\omega(z) + 2\omega^2(z) + \cdots)^\gamma,
\]
where $\omega$ is analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$. If we denote by $L$ and $R$ left and right hand side of equality (2.1), then we have
\[
L = \left[ 1 - (1 + \alpha)(a_2z + \cdots) + \left( \frac{-(1+\alpha)}{2} \right)(a_2z + \cdots)^2 \\
+ \left( \frac{-(1+\alpha)}{3} \right)(a_2z + \cdots)^3 + \cdots \right] \cdot (1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \cdots)
\]
\[
R = (1 + 2\omega(z) + 2\omega^2(z) + \cdots)^\gamma.
\]

Thus, we have
\[
|L - R| \leq \left( \frac{2\gamma}{2 - \alpha} \right)^2.
\]
and if we put \( \omega(z) = c_1z + c_2z^2 + \cdots \):

\[
R = 1 + \gamma \left[ 2(c_1z + c_2z^2 + \cdots) + 2(c_1z + c_2z^2 + \cdots)^2 + \cdots \right] \\
+ \left( \frac{\gamma}{2} \right) \left[ 2(c_1z + c_2z^2 + \cdots) + 2(c_1z + c_2z^2 + \cdots)^2 + \cdots \right]^2 \\
+ \left( \frac{\gamma}{3} \right) \left[ 2(c_1z + c_2z^2 + \cdots) + 2(c_1z + c_2z^2 + \cdots)^2 + \cdots \right]^3 + \cdots.
\]

If we compare the coefficients on \( z, z^2, z^3 \) in \( L \) and \( R \), then, after some calculations, we obtain

\[
a_2 = \frac{2\gamma}{1-\alpha} c_1, \\
a_3 = \frac{2\gamma}{2-\alpha} c_2 + \frac{2(3-\alpha)\gamma^2}{(1-\alpha)^2(2-\alpha)} c_1^2, \\
a_4 = \frac{2\gamma}{3-\alpha} (c_3 + \mu c_1 c_2 + \nu c_1^3),
\]

where

\[
(2.3) \quad \mu = \mu(\alpha, \gamma) = \frac{2(5-\alpha)\gamma}{(1-\alpha)(2-\alpha)} \quad \text{and} \quad \nu = \nu(\alpha, \gamma) = \frac{1}{3} + \frac{2}{3} \frac{(\alpha^2 - 6\alpha + 17)\gamma^2}{(1-\alpha)^3(2-\alpha)}. 
\]

By using the relations (2.2) and (2.3), after some simple computations, we obtain

\[
H_2(2) = \frac{4\gamma^2}{(1-\alpha)(3-\alpha)} \left( c_1 c_3 + \mu_1 c_1^2 c_2 + \left( \frac{1}{3} - \nu_1 \right) c_1^4 - \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} c_2^2 \right),
\]

where

\[
\mu_1 = \frac{2\gamma}{(2-\alpha)^2}, \quad \nu_1 = \frac{(\alpha^2 - 10\alpha + 13)\gamma^2}{3(1-\alpha)^2(2-\alpha)^2},
\]

and from here

\[
|H_2(2)| \leq \frac{4\gamma^2}{(1-\alpha)(3-\alpha)} \left( |c_1||c_3| + \mu_1|c_1|^2|c_2| \\
+ \left| \frac{1}{3} - \nu_1 \right| |c_1|^4 + \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} |c_2|^2 \right).
\]

For the function \( \omega(z) = c_1z + c_2z^2 + \cdots \) (with \( |\omega(z)| < 1, \ z \in \mathbb{D} \)) the next relations is valid (see, for example [3] p.128, expression (13)):

\[
|c_1| \leq 1, \ |c_2| \leq 1 - |c_1|^2, \ |c_3(1 - |c_1|^2) + c_1 c_2| \leq (1 - |c_1|^2)^2 - |c_2|^2.
\]

We may suppose that \( \alpha_2 \geq 0 \), which implies that \( c_1 \geq 0 \) and instead of relations (2.5) we have the next relations

\[
0 \leq c_1 \leq 1, \ |c_2| \leq 1 - c_1^2, \ |c_3| \leq 1 - c_1^3 - \frac{|c_2|^2}{1 + c_1}.
\]

By using (2.6) for \( c_1 \) and \( c_3 \), from (2.4) we have

\[
|H_2(2)| \leq \frac{4\gamma^2}{(1-\alpha)(3-\alpha)} \left[ c_1(1 - c_1^2) + \left( \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} - \frac{c_1}{1 + c_1} \right) |c_2|^2 \\
+ \mu_1 c_1^2 |c_2| + \left| \frac{1}{3} - \nu_1 \right| c_1^4 \right].
\]

\[
(2.7)
\]
Since for $0 < \alpha < 2 - \sqrt{2}$ we have \(\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} \geq \frac{1}{2} \geq \frac{\alpha}{1+c_1}\), then by using \(|c_2| \leq 1 - c_1^2\), from (2.7) after some calculations we obtain

\[
|H_2(2)| \leq \frac{4\gamma^2}{(1-\alpha)(3-\alpha)}F(c_1),
\]

where

\[
F(c_1) = \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} + Ac_1^2 + Bc_1^4,
\]

where

\[
A = \frac{2\gamma - (\alpha^2 - 4\alpha + 2)}{(2-\alpha)^2}, \quad B = \left| \frac{1}{3} - \nu_1 \right| - \frac{2\gamma + 1}{(2-\alpha)^2}.
\]

Further, by using the assumptions of the theorem that $0 < \alpha < 2 - \sqrt{2}$ and $0 < \gamma \leq \frac{1}{2}(\alpha^2 - 4\alpha + 2)$, we easily conclude that $A \leq 0$, while

\[
0 < \nu_1 = \frac{(\alpha^2 - 10\alpha + 13)\gamma^2}{3(1-\alpha)^2(2-\alpha)^2} \leq \frac{(\alpha^2 - 10\alpha + 13)(\alpha^2 - 4\alpha + 2)^2}{12(1-\alpha)^2(2-\alpha)^2} < \frac{13}{12}.
\]

If we have that $B \leq 0$, then from (2.9) we obtain that

\[
F(c_1) \leq \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2},
\]

and if $B > 0$, then

\[
F(c_1) \leq \max\{F(0), F(1)\} = \max\left\{ \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2}, \left| \frac{1}{3} - \nu_1 \right| \right\} = \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2},
\]

since

\[
\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} > \left| \frac{1}{3} - \nu_1 \right|
\]

when $0 < \alpha < 2 - \sqrt{2}$ and $0 < \gamma \leq \frac{1}{2}(\alpha^2 - 4\alpha + 2)$ (proven later). It means that in both cases we have that

\[
F(c_1) \leq \frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2},
\]

which by (2.8) implies

\[
|H_2(2)| \leq \left( \frac{2\gamma}{2-\alpha} \right)^2.
\]

We need to prove the inequality (2.10) for appropriate $\alpha$ and $\gamma$. First, if $\frac{1}{3} - \nu \leq 0$, i.e. if $0 < \nu_1 \leq \frac{1}{3}$, then, since $0 < \alpha < 2 - \sqrt{2}$, we have

\[
\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} > \frac{1}{2} > \frac{1}{3} - \nu_1,
\]

which implies that (2.10) is true. In case $\nu_1 > \frac{1}{3}$, we have that inequality (2.10) is equivalent to

\[
\frac{(1-\alpha)(3-\alpha)}{(2-\alpha)^2} > \frac{(\alpha^2 - 10\alpha + 13)\gamma^2}{3(1-\alpha)^2(2-\alpha)^2} - \frac{1}{3}.
\]

The last inequality is equivalent with

\[
\gamma^2 < \frac{(1-\alpha)^2(4\alpha^2 - 16\alpha + 13)}{\alpha^2 - 10\alpha + 13}.
\]
Since for $0 < \alpha < 2 - \sqrt{2}$ we have $\gamma \leq \frac{1}{2}(\alpha^2 - 4\alpha + 2)$, then for such $\alpha$ we have

$$\gamma^2 \leq \frac{1}{4}(\alpha^2 - 4\alpha + 2)^2$$

and from (2.10) it is sufficient to prove that

$$\frac{1}{4}(\alpha^2 - 4\alpha + 2)^2 \leq \frac{(1 - \alpha^2)(4\alpha^2 - 16\alpha + 13)}{\alpha^2 - 10\alpha + 13}$$

for $0 < \alpha < 2 - \sqrt{2}$. The inequality (2.11) is equivalent to

$$\phi_1(t) := \phi(2 - \sqrt{2} + t) = \frac{1}{4}(2 + t)[30 + 19t - t^2 - (20 + 6t)\sqrt{2 + t}]$$

The function $\phi_1$ is continuous function in the interval $[0,2]$. It is easily to check that

$$\phi_1'(t) = \frac{1}{4}[68 + 34t - 3t^2 - (42 + 15t)\sqrt{2 + t}]$$

and

$$\phi_1''(t) = \frac{1}{8}[68 - 12t - 45\sqrt{2 + t} - \frac{12}{\sqrt{2 + t}}].$$

In $\phi_1''$, the second and the third expression reach their minimum on the segment $[0,2]$ for $t = 0$, while the last expression for $t = 2$. Thus

$$\phi_1''(t) < \frac{1}{8}\left(68 - 12 \cdot 0 - 45\sqrt{2 + 0} - \frac{12}{\sqrt{2 + 2}}\right) = \frac{1}{8}(62 - 45\sqrt{2}) = -0.20 \ldots < 0,$$

i.e., $\phi_1'$ is an decreasing function from $\phi_1'(0) = 17 - 10.5\sqrt{2} = 2.15 \ldots > 0$ to $\phi_1'(2) = -5 < 0$, which implies that the function $\phi$ attains its maximum in the interval $(0,2)$, so that

$$\phi_1(t) \geq \min\{\phi_1(0), \phi_1(2)\} = \min\{15 - 10\sqrt{2}, 0\} = 0.$$  

This means that the inequality given by (2.12) is true.

The result of Theorem II is the best possible as the functions $f_2$, defined with

$$\left(\frac{z}{f_2(z)}\right)^{1+\alpha} f_2'(z) = \left(\frac{1 + z^2}{1 - z^2}\right)^\gamma$$

shows. In this case we have that $c_2 = 1$, $c_j = 0$ when $j \neq 2$, and consequently, $a_2 = a_4 = 0$, $a_3 = \frac{2}{\sqrt{\alpha}}$ and $H_2(2) = a_2a_4 - a_3^2 = -\frac{4\alpha^2}{(2-\alpha)^2}$.
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