On the nonexistence of time dependent global weak solutions to the compressible Navier-Stokes equations

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Abstract
In this paper we prove the nonexistence of global weak solutions to the compressible Navier-Stokes equations for the isentropic gas in \( \mathbb{R}^N \), \( N \geq 3 \), where the pressure law given by \( p(\rho) = a\rho^\gamma \), \( a > 0, 1 < \gamma \leq \frac{N}{4} + \frac{1}{2} \). In this case if the initial data satisfies \( \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x \ dx > 0 \), then there exists no finite energy global weak solution which satisfies the integrability conditions \( \rho|v|^2 \in L_{\text{loc}}^1(0, \infty; L^1(\mathbb{R}^N)) \) and \( v \in L_{\text{loc}}^1(0, \infty; L^{N\frac{N}{N-1}}(\mathbb{R}^N)) \).

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1 Introduction and Main Theorems
We are concerned on the compressible Navier-Stokes equations on \( \mathbb{R}^N \), \( N \geq 1 \).

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho v) &= 0, \\
\partial_t (\rho v) + \text{div}(\rho v \otimes v) &= -\nabla p + \mu \Delta v + (\mu + \lambda)\nabla \text{div} v + f, \\
p(\rho) &= a\rho^\gamma, \quad \gamma > 1, \ a > 0
\end{aligned}
\]
The system (NS) describes isentropic viscous compressible gas flows, and \( \rho, v, p \) and \( f \) denote the density, velocity, pressure and the external force respectively. We treat the viscous case \( \mu > 0, \lambda > 0 \). For the surveys of the known mathematical theories of the system (NS) we refer to [1][2][5]. In this paper we are concerned on the case where \( 1 < \gamma \leq N/4 + 1/2, \) and \( N \geq 3 \). We note that for this range of \( \gamma, N \) there exists no known existence results for the global weak solutions (see [4][2] for previous global existence results of the weak solutions). In this case of our aim is to prove nonexistence of global weak solutions to (NS) satisfying suitable integrability conditions for the solutions and for some class of initial data. This result can be regarded as a time dependent generalization of the corresponding stationary results obtained previously in [1]. The finite energy weak solution of (NS) is defined as follows.

**Definition 1.1** Let \((\rho_0, v_0)\) satisfies

\[
0 \leq \rho_0 \in L^1_{\text{loc}}(\mathbb{R}^N), \quad \rho_0v_0 \in L^1_{\text{loc}}(\mathbb{R}^N), \quad \rho_0^\gamma + \rho_0 |v_0|^2 \in L^1(\mathbb{R}^N). \tag{1.1}
\]

We say that the pair \((\rho, v)\) is a finite energy weak solution of the system (NS) with the initial data \((\rho, v)\) if it satisfies

\[
0 \leq \rho \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, \infty)), \quad \rho v \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, \infty)), \tag{1.2}
\]

\[
\rho^\gamma + |v|^2 \in L^\infty_{\text{loc}}(0, \infty; L^1(\mathbb{R}^N)), \quad v \in L^2_{\text{loc}}(0, \infty; H(\mathbb{R}^N)), \tag{1.3}
\]

and satisfies (NS) in the sense of distribution, namely

\[
\xi(0) \int_{\mathbb{R}^N} \rho_0(x)\psi(x)dx + \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)\psi(x)\xi'(t)dxdt
\]

\[
+ \int_0^\infty \int_{\mathbb{R}^N} \rho v(x, t) \cdot \nabla \psi(x)\xi(t)dx = 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}^N), \xi \in C_0^1([0, \infty)), \tag{1.4}
\]

\[
\xi(0) \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot \phi(x)dx + \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)v(x, t) \cdot \phi(x)\xi'(t)dxdt
\]

\[
+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)v(x, t) \otimes v(x, t) : \nabla \phi(x)\xi(t)dxdt
\]

\[
= -\int_0^\infty \int_{\mathbb{R}^N} p(x, t) \text{div} \phi(x)\xi(t)dxdt - \mu \int_0^\infty \int_{\mathbb{R}^N} v(x, t) \cdot \Delta \phi(x)\xi(t)dxdt
\]

\[
-(\mu + \lambda) \int_0^\infty \int_{\mathbb{R}^N} v(x, t) \cdot \nabla \text{div} \phi(x)\xi(t)dxdt - \int_0^\infty \int_{\mathbb{R}^N} f \cdot \phi(x)\xi(t)dxdt
\]

\[
\forall \phi \in \left[ C_0^\infty(\mathbb{R}^N) \right]^N, \xi \in C_0^1([0, \infty)). \tag{1.5}
\]
\[ p(\rho) = a\rho^\gamma \quad \text{almost everywhere in } \mathbb{R}^N \times (0, \infty), \quad (1.6) \]

Finally, we impose the energy inequality of the following form:

\[
\frac{d}{dt} \mathcal{E}(t) + \int_{\mathbb{R}^N} (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2)dx \leq 0 \quad \text{for almost every } t \geq 0,
\]

where \[ \mathcal{E}(t) := \int_{\mathbb{R}^N} \left[ \frac{1}{2} \rho |v|^2 + \frac{a\rho^\gamma}{\gamma - 1} \right] dx. \quad (1.7) \]

In the above definition we followed closely [5]. The following is our main theorem.

**Theorem 1.1** Let \( N \geq 3, 1 < \gamma \leq \frac{N}{4} + \frac{1}{2}, \mu > 0, \mu + \lambda > 0, \) and the external force \( f \in \left[ L^1_{\text{loc}}(\mathbb{R}^N \times [0, \infty)) \right]^N \) with \( \text{div } f = 0 \) is given. Let the initial data \((\rho_0, v_0)\) satisfy

\[
\mathcal{E}(0) < \infty, \quad \int_{\mathbb{R}^N} \rho_0(x)|v_0(x)||x|dx < \infty, \quad (1.8)
\]

\[
\int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x dx > 0. \quad (1.9)
\]

Then, there exists no finite energy global weak solution to (NS) such that

\[
\rho |x|^2 \in L^1_{\text{loc}}(0, \infty; L^1(\mathbb{R}^N)), \quad v \in L^1_{\text{loc}}(0, \infty; L^{N-1}(\mathbb{R}^N)). \quad (1.10)
\]

**Remark 1.1** An immediate consequence of the above theorem is that for initial data \((\rho_0, v_0)\) satisfying (1.8) and (1.9) there exists \( T_* < \infty \) such that

\[
\int_0^{T_*} \int_{\mathbb{R}^N} \left[ \rho(x,t)|x|^2 + |v(x,t)|^{\frac{N}{N-1}} \right] dxdt = \infty,
\]

where \((\rho(x,t), v(x,t))\) is a local in time (classical or weak) solution on the time interval \([0, T_*]\), if (NS) is at least locally well-posed in the functional setting of Definition 1.1 for the case considered above.

### 2 Proof of Theorem 1.1

In order to prove Theorem 1.1 we shall use the following lemma, which is proved in [3].
Lemma 2.1 Suppose \((\rho, v)\) is a finite energy global weak solution to \((NS)\) with the setting given by Theorem 1.1, then

\[
\int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)(1 + |x|^2)^{\frac{N+2}{4}} dx dt \leq C \mathcal{E}(0).
\] (2.1)

Since \(\frac{N+2}{4} \geq 1\) in our setting of Theorem 1.2, one immediate consequence of (2.1) is the following fact

\[
\lim_{\tau \to \infty} \int_\tau^{2\tau} \int_{\mathbb{R}^N} \frac{\rho(x,t)|x|^2}{1 + t^2} dx dt = 0.
\] (2.2)

Indeed, using (2.1), we deduce

\[
\lim_{\tau \to \infty} \int_\tau^{2\tau} \int_{\mathbb{R}^N} \rho(x,t)|x|^2 dx dt \leq \lim_{\tau \to \infty} \int_\tau^{2\tau} \int_{\mathbb{R}^N} \rho(x,t)(1 + |x|^2)^{\frac{N+2}{4}} dx dt = 0,
\]

where the last step follows from the dominated convergence theorem.

Proof of Theorem 1.2 Suppose there exists a global weak solution \((\rho, v)\) satisfying (1.4)-(1.7). Let us consider a radial cut-off function \(\sigma \in C_0^\infty(\mathbb{R}^N)\) such that

\[
\sigma(|x|) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 2,
\end{cases}
\] (2.3)

and \(0 \leq \sigma(x) \leq 1\) for \(1 < |x| < 2\). For \(R > 0\) we define

\[
\varphi_R(x) = \frac{1}{2} |x|^2 \sigma \left( \frac{|x|}{R} \right) = \frac{1}{2} |x|^2 \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).
\] (2.4)

We also introduce temporal cut-off function \(\eta \in C_0^\infty([0, \infty))\) as follows.

\[
\eta(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
0 & \text{if } t > 2,
\end{cases}
\] (2.5)

and for all \(\tau > 0\) we set

\[
\eta_\tau(t) = \eta \left( \frac{t}{\tau} \right).
\] (2.6)
Substituting $\phi(x) = \nabla \varphi_R(x)$, $\xi(t) = \eta_r(t)$ into (1.3), we obtain

$$0 = \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x\sigma_R(|x|)dx + \frac{1}{2R} \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x|x|\sigma'(\frac{|x|}{R})dx$$

$$+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)v(x,t) \cdot \nabla \varphi_R(x)\eta_r'(t)d\tau d\sigma$$

$$+ \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2\sigma_R(|x|)\eta_r(t)d\tau d\sigma$$

$$+ \frac{1}{2R} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)\sigma'(\frac{|x|}{R}) \frac{(v(x,t) \cdot x)^2}{|x|} \eta_r(t)d\tau d\sigma$$

$$+ \frac{1}{2R} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|(v(x,t) \cdot x)^2\sigma''(\frac{|x|}{R}) \eta_r(t)d\tau d\sigma$$

$$+ \frac{1}{2R^2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)(v(x,t) \cdot x)^2\sigma''(\frac{|x|}{R}) \eta_r(t)d\tau d\sigma$$

$$+ N \int_0^\infty \int_{\mathbb{R}^N} p(x,t)\sigma_R(|x|)\eta_r(t)d\tau d\sigma$$

$$+ \frac{2}{R} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)|x|\sigma'(\frac{|x|}{R}) \eta_r(t)d\tau d\sigma$$

$$+ \frac{N - 1}{2R} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)|x|\sigma'(\frac{|x|}{R}) \eta_r(t)d\tau d\sigma$$

$$+ \frac{1}{2R^2} \int_0^\infty \int_{\mathbb{R}^N} p(x,t)|x|^2\sigma''(\frac{|x|}{R}) \eta_r(t)d\tau d\sigma$$

$$+ (2\mu + \lambda) \int_0^\infty \int_{\mathbb{R}^N} v \cdot \nabla \Delta(|x|)\sigma(\frac{|x|}{R}) \eta_r(t)d\tau d\sigma,$$

$$:= I_1 + \cdots + I_{12}. \quad (2.7)$$

On the other hand, substituting $\phi(x) = \nabla \varphi_R(x)$, $\xi(t) = \eta_r'(t)$ into (1.4), we find that (note that $\xi(0) = \eta_r'(0) = 0$)

$$I_3 = \int_0^\infty \int_{\mathbb{R}^N} \rho v(x,t) \cdot \nabla \varphi_R(x)\eta_r'(t)d\tau d\sigma$$

$$= - \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)\varphi_R(x)\eta_r''(t)d\tau d\sigma$$

$$= - \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|x|^2\sigma_R(|x|)\eta_r''(t)d\tau d\sigma$$

$$\rightarrow - \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|x|^2\eta_r''(t)d\tau d\sigma \quad (2.8)$$
as $R \to \infty$ by the dominated convergence theorem. We also have
\[ I_4 \to \int_0^\infty \int_{\mathbb{R}^N} \rho(x,t)|v(x,t)|^2 \eta_r(t) \, dx \, dt \]  
(2.9)
as $R \to \infty$. Similarly,
\[ I_1 \to \int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x \, dx, \]  
(2.10)
and
\[ I_8 \to N \int_0^\infty \int_{\mathbb{R}^N} p(x,t) \eta_r(t) \, dx \, dt \]  
(2.11)
as $R \to \infty$. For $I_5, I_6$ we estimate
\[ |I_5| + |I_6| \leq \int_0^{2\tau} \int_{R<|x|<2R} \rho(x,t)|v(x,t)|^2 \left| \sigma' \left( \frac{|x|}{R} \right) \right| \frac{|x|}{R} \, dx \, dt \]
\[ \leq 2 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x)|v(x,t)|^2 \, dx \, dt \to 0 \]  
(2.12)
as $R \to \infty$. Similarly
\[ |I_2| \leq \int_{R<|x|<2R} \rho_0(x)|x| \, dx \to 0, \]  
(2.13)
and
\[ |I_7| \leq \frac{1}{2} \int_0^{2\tau} \int_{R<|x|<2R} \frac{|x|^2}{R^2} \rho(x,t)|v(x,t)|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \, dx \, dt \]
\[ \leq 2 \sup_{1<s<2} |\sigma''(s)| \int_0^{2\tau} \int_{R<|x|<2R} \rho(x,t)|v(x,t)|^2 \, dx \to 0 \]  
(2.14)
as $R \to \infty$. The estimates for $I_9, I_{10}$ and $I_{11}$ are similar to the above, and we find
\[ |I_9| \leq 2 \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)||x| \left| \sigma' \left( \frac{|x|}{R} \right) \right| \, dx \, dt \]
\[ \leq 4 \sup_{1<s<2} |\sigma'(s)| \int_0^{2\tau} \int_{R<|x|<2R} |p(x,t)| \, dx \, dt \to 0, \]  
(2.15)
\[ |I_{10}| \leq \frac{N - 1}{2R} \int_0^{2\tau} \int_{R <|x| < 2R} |p(x, t)||x| \left| \sigma' \left( \frac{|x|}{R} \right) \right| \, dx \, dt \]
\[ \leq (N - 1) \sup_{1 < s < 2} |\sigma'(s)| \int_0^{2\tau} \int_{R <|x| < 2R} |p(x, t)| \, dx \, dt \to 0, \quad (2.16) \]

and
\[ |I_{11}| \leq \frac{1}{2R^2} \int_0^{2\tau} \int_{\mathbb{R}^N} |p(x, t)||x|^2 \left| \sigma'' \left( \frac{|x|}{R} \right) \right| \, dx \, dt \]
\[ \leq 2 \sup_{1 < s < 2} |\sigma''(s)| \int_0^{2\tau} \int_{R <|x| < 2R} |p(x, t)| \, dx \, dt \to 0 \quad (2.17) \]

as \( R \to \infty \) respectively. Now we show the vanishing of the viscosity term as \( R \to \infty \). This follows from the estimates,
\[ |I_{12}| = (2\mu + \lambda) \left| \int_0^\infty \int_{\mathbb{R}^N} v \cdot \nabla \Delta (|x|^2 \sigma \left( \frac{|x|}{R} \right)) \eta_{\tau}(t) \, dx \, dt \right| \]
\[ \leq (2\mu + \lambda) \left| \int_0^\infty \int_{\mathbb{R}^N} (N + 5) \left[ \frac{(v \cdot x)}{R|x|} \sigma' \left( \frac{|x|}{R} \right) + \frac{(v \cdot x)}{R^2 \sigma''} \left( \frac{|x|}{R} \right) \right] \eta_{\tau}(t) \, dx \, dt \right| \]
\[ + (2\mu + \lambda) \left| \int_0^\infty \int_{\mathbb{R}^N} \frac{|x|}{R^2} \sigma''' \left( \frac{|x|}{R} \right) \eta_{\tau}(t) \, dx \, dt \right| \]
\[ \leq \frac{C}{R} \int_0^{2\tau} \int_{R \leq |x| \leq 2R} |v(x, t)| \, dx \, dt \]
\[ \leq C \int_0^{2\tau} \left( \int_{R \leq |x| \leq 2R} |v(x, t)|^{\frac{N}{N-1}} \, dx \right)^{\frac{N-1}{N}} \, dt \to 0 \quad (2.18) \]

as \( R \to \infty \). In summary, passing \( R \to \infty \) in \( (2.7) \), we obtain
\[ \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \eta_{\tau}'(t) \, dx \, dt = \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|v(x, t)|^2 \eta_{\tau}(t) \, dx \, dt \]
\[ + N \int_0^\infty \int_{\mathbb{R}^N} p(x, t) \eta_{\tau}(t) \, dx \, dt + \int_{\mathbb{R}^N} \rho_0(x) v_0(x) \cdot x \, dx, \quad (2.19) \]

and therefore
\[ \int_{\mathbb{R}^N} \rho_0(x) v_0(x) \cdot x \, dx \leq \frac{1}{2} \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \eta_{\tau}'(t) \, dx \, dt \quad (2.20) \]
for any $\tau > 0$. By (2.2) we have

$$\left| \int_0^\infty \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \eta''(t)dxdt \right| \leq \frac{1}{\tau^2} \int_{\tau}^{2\tau} \int_{\mathbb{R}^N} \rho(x, t)|x|^2 \left| \eta'' \left( \frac{t}{\tau} \right) \right| dxdt$$

$$\leq \frac{1 + 4\tau^2}{\tau^2} \sup_{1 < s < 2} |\eta''(s)| \int_{\tau}^{2\tau} \int_{\mathbb{R}^N} \rho(x, t)|x|^2 dxdt \to 0 \quad (2.21)$$

as $\tau \to \infty$. Combining (2.21) with (2.20), we obtain the following necessary condition for the global existence of weak solution,

$$\int_{\mathbb{R}^N} \rho_0(x)v_0(x) \cdot x \, dx \leq 0. \quad (2.22)$$

This proves the theorem. □

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