Twisted Vertex Operators and $A$-$D$-$E$
Representations

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Abstract
In this paper, we study the algebra of twisted vertex operators over an even integral $\mathbb{Z}_2$-lattice, and give a kind of systematic construction of fundamental representations for affine Lie algebras of type $A$, $D$, $E$ with their irreducible decompositions.

1 Introduction
In this paper, we construct a family of twisted vertex operators associated to an even integral $\mathbb{Z}_2$-lattice which, together with the action of the corresponding Heisenberg algebra, is invariant under the commutation relations.

For twisted or non-twisted affinization of a finite-dimensional simple simply-laced Lie algebra $\mathfrak{g}$, construction of fundamental representations is given associated to conjugacy classes in the Weyl group of $\mathfrak{g}$ in [7], and associated to automorphisms of $\mathfrak{g}$ in [9]. When the $\mathbb{Z}_2$-lattice is the root $\mathbb{Z}_2$-lattice of $\mathfrak{g}$, our representation is just the one associated to the automorphism of $\mathfrak{g}$ which is $(-1)$ times the identity transformation on the Cartan subalgebra, and so the construction corresponding to the longest element in the Weyl group in particular when $\mathfrak{g}$ is a simple Lie algebra of type $D_{2m}$, $E_7$ or $E_8$. We make a detail analysis on the structure of our representation and give its irreducible decomposition explicitly.

This work was motivated by the recent intensive research of M. Noumi and Y. Yamada et al. on the Painlevé VI equation and its Lie algebraic interpretation. The author would like to express hearty thanks to Professor M. Noumi for private communication and explanation on his works at the International Workshop on Integrable Models, Combinatorics and Representation Theory held in Kyoto on August 2001, and to Professor E. Date for kind information on $E_8$. 
2 Twisted vertex operators

Given a positive integer $n$, we consider $\mathbb{C}^n$ with a non-degenerate symmetric bilinear form $(\ |\ )$ defined by

$$ (\lambda|\mu) := \sum_{j=1}^{n} \lambda_j \mu_j $$

for $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$.

Let us consider the space $\mathbb{C}[x_r^{(j)}; j = 1, \ldots, n, r \in \mathbb{N}_{\text{odd}}]$. For $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$, we define operators $U_+^\lambda(z)$, $U_\lambda(z)$ and $U_{\lambda;\mu}(z, w)$ on this space as follows:

$$ U_+^\lambda(z) := \exp \left( -\sum_{j=1}^{n} \sum_{r \in \mathbb{N}_{\text{odd}}} \lambda_j \frac{\partial}{\partial x_r^{(j)}} z^{-r} \right), $$

$$ U_\lambda(z) := U_+^\lambda(z)U_\mu^\lambda(z), $$

$$ U_{\lambda;\mu}(z, w) := U_+^\lambda(z)U_\mu^\lambda(z)U_\lambda^\mu(z) $$

and

$$ U_{\lambda;\mu}(z, w) := U_+^\lambda(z)U_\mu^\lambda(w)U_\lambda^\mu(z) $$

and

$$ U_{\lambda;\mu}(z, w) := U_+^\lambda(z)U_\mu^\lambda(z)U_\lambda^\mu(z) $$

From this definition, it follows that

$$ U_\lambda(-z) = U_{-\lambda}(z), $$

$$ U_{\lambda;\mu}(-z, w) = U_{-\lambda;\mu}(z, w), $$

and that

$$ U_{\lambda;\lambda}(z, -z) = 1 := \text{the identity operator}, $$

$$ \frac{\partial}{\partial z} U_{\lambda;\lambda}(z, w) = \sum_{j=1}^{n} \lambda_j \left( \sum_{r \in \mathbb{N}_{\text{odd}}} r x_r^{(j)} z^{-r-1} \right) U_{\lambda;\lambda}(z, w) $$

$$ + U_{\lambda;\lambda}(z, w) \sum_{j=1}^{n} \lambda_j \left( \sum_{r \in \mathbb{N}_{\text{odd}}} z^{-r-1} \frac{\partial}{\partial x_r^{(j)}} \right). $$

Since

$$ U_{\lambda;\mu}(z, z) = U_{\lambda+\mu}(z) \quad \text{and} \quad U_{\lambda;\mu}(-z, z) = U_{-\lambda+\mu}(z), $$

$$ U_{\lambda;\lambda}(z, -z) = 1. $$
the constant terms of the Taylor series expansion of \( U_{\lambda,\mu}(z, w) \) around \( z = \pm w \) are given as follows:

\[
\begin{align*}
U_{\lambda,\mu}(z, w) &= U_{\lambda+\mu}(w) + O(z - w) \quad \text{around } z = w, \\
U_{\lambda,\mu}(z, w) &= U_{-\lambda+\mu}(w) + O(z + w) \quad \text{around } z = -w. 
\end{align*}
\] (2.5)

For \( j \in \{1, \cdots, n\} \) and \( r \in \mathbb{N}_{\text{odd}} \), we put

\[
a_{r}^{(j)} := \frac{\partial}{\partial x_{r}^{(j)}}, \quad a_{-r}^{(j)} := r x_{r}^{(j)},
\] (2.6a)

and consider the fields

\[
a^{(j)}(z) := \sum_{r \in \mathbb{Z}_{\text{odd}}} a_{r}^{(j)} z^{-r-1}
\] (2.6b)

for \( j = 1, \cdots, n \). Notice that

\[
a^{(j)}(-z) = a^{(j)}(z) \quad \text{for } j = 1, \cdots, n.
\] (2.7)

Then from (2.4), one has

\[
\frac{\partial}{\partial z} U_{\lambda,\lambda}(z, w) \bigg|_{z=-w} = \sum_{j=1}^{n} \lambda_{j} a^{(j)}(w).
\] (2.8)

**Lemma 2.1.** Let \( \lambda \in \mathbb{C}^{n} \). Then the Taylor series expansions of \( U_{\lambda,\pm\lambda}(z, w) \) around \( z = \mp w \) are given as follows:

1) \( U_{\lambda,\lambda}(z, w) = 1 + (z + w) \sum_{j=1}^{n} \lambda_{j} a^{(j)}(w) + O((z + w)^{2}), \)

2) \( U_{\lambda,-\lambda}(z, w) = 1 + (z - w) \sum_{j=1}^{n} \lambda_{j} a^{(j)}(w) + O((z - w)^{2}). \)

**Proof.** To prove 1), we compute the Taylor series expansion of \( U_{\lambda,\lambda}(z, w) \) around \( z = -w \), by using (2.4) and (2.8) as follows:

\[
U_{\lambda,\lambda}(z, w) = U_{\lambda,\lambda}(-w, w) + \left. \frac{\partial}{\partial z} U_{\lambda,\lambda}(z, w) \right|_{z=-w} + O((z + w)^{2})
\]

\[
= 1 + \sum_{j=1}^{n} \lambda_{j} a^{(j)}(w) + O((z + w)^{2}).
\]

2) follows from 1) and (2.3) and (2.7). \( \square \)

**Lemma 2.2.** For \( \lambda, \mu \in \mathbb{C}^{n} \) satisfying \( (\lambda|\mu) \in 2\mathbb{Z} \),
1) \( U^+_{\lambda}(z)U^-_{\mu}(w) = \iota_{z,w} \left( \left( \frac{z-w}{z+w} \right)^{\frac{\lambda|\mu}{2}} \right) U^-_{\mu}(w)U^+_{\lambda}(z), \)

2) \( U_{\lambda}(z)U_{\mu}(w) = \iota_{z,w} \left( \left( \frac{z-w}{z+w} \right)^{\frac{\lambda|\mu}{2}} \right) U_{\lambda\mu}(z, w), \)

where \( \iota_{z,w} \) means the expansion into the Taylor series in the domain \(|z| > |w|\).

**Proof.** For the proof of this lemma, we first notice the commutation relation of operators in one variable \( x \):

\[
e^a \frac{d}{dx} \circ e^b x = e^{ab} e^b x \circ e^{a} \frac{d}{dx}
\]

for \( a, b \in \mathbb{C} \), which is easily seen from \( e^a \frac{d}{dx} f(x) = f(x + a) \). Using this, one has

\[
U^+_{\lambda}(z)U^-_{\mu}(w) = \left( \prod_{j=1}^{\infty} e^{-\lambda_j \mu_j} \frac{z-r w^r}{r} \right) U^-_{\mu}(w)U^+_{\lambda}(z)
\]

\[
= \iota_{z,w} \left( \left( \frac{z-w}{z+w} \right)^{\frac{\lambda|\mu}{2}} \right) U^-_{\mu}(w)U^+_{\lambda}(z).
\]

Then, since \( (\lambda|\mu) \in 2\mathbb{Z} \) by assumption, one has

\[
e^{-(\lambda|\mu) \sum_{r \in \mathbb{Z}_{\text{odd}}} \frac{z-r w^r}{r}} = \exp \left\{ -(\lambda|\mu) \left( \sum_{r=1}^{\infty} \frac{z-r w^r}{r} - \sum_{r=1}^{\infty} \frac{z-2r w^{2r}}{2r} \right) \right\}
\]

\[
= \iota_{z,w} \left( \frac{1-w^2}{(1-w^2)^2} \right)^{-(\lambda|\mu)/2}
\]

\[
= \iota_{z,w} \left( \frac{z+w}{z-w} \right)^{-(\lambda|\mu)/2},
\]

proving 1). 2) follows from 1). \( \square \)

Note that the vertex operator \( U_{\lambda}(z) \) satisfies

\[
[a_r^{(j)}, U_{\lambda}(z)] = \lambda_j z^r U_{\lambda}(z), \tag{2.9a}
\]

from which one deduces

\[
[a_r^{(j)}, U_{\lambda}(w)] = \lambda_j \sum_{r \in \mathbb{Z}_{\text{odd}}} z^{-r-1} w^r \cdot U_{\lambda}(w). \tag{2.9b}
\]

Using the \( \delta \)-function defined by

\[
\delta(z-w) := \sum_{r \in \mathbb{Z}} z^{-r-1} w^r = \sum_{r \geq 0} z^{-r-1} w^r + \sum_{r < 0} z^{-r-1} w^r
\]

\[
= (\iota_{z,w} - \iota_{w,z}) \left( \frac{1}{z-w} \right), \tag{2.10}
\]
the formula (2.9b) is rewritten as
\[ [\alpha^{(j)}(z), U_{\lambda}(w)] = \frac{\lambda_j}{2} \{ \delta(z - w) - \delta(z + w) \} U_{\lambda}(w), \] (2.11)
since
\[
\sum_{r \in \mathbb{Z}_{\text{odd}}} z^{-r-1}w^r = \sum_{r \in \mathbb{Z}} z^{-r-1}w^r - \sum_{r \in \mathbb{Z}} z^{-2r-1}w^{2r}
= (t_{z,w} - t_{w,z}) \left( \frac{1}{z - w} - \frac{z}{z^2 - w^2} \right) = (t_{z,w} - t_{w,z}) \left( \frac{w}{z^2 - w^2} \right). \] (2.12a)
Note also that
\[
\sum_{r \in \mathbb{Z}} (-1)^r z^{-r-1}w^r = (t_{z,w} - t_{w,z}) \left( \frac{1}{z + w} \right), \] (2.12b)
\[
\sum_{r \in \mathbb{Z}} rz^{-r-1}w^r = (t_{z,w} - t_{w,z}) \left( \frac{w}{(z - w)^2} \right), \] (2.12c)
\[
\sum_{r \in \mathbb{Z}} (-1)^r rz^{-r-1}w^r = (t_{z,w} - t_{w,z}) \left( \frac{-w}{(z + w)^2} \right). \] (2.12d)

Let \( \mathfrak{h}_R \) be an \( n \)-dimensional real vector space equipped with a positive definite symmetric bilinear form \(( \ | \ )\), and \( Q \) be an even integral \( \mathbb{Z} \)-lattice in \( \mathfrak{h}_R \) of rank \( n \). We put
\[ \Delta := \{ \alpha \in Q ; \ (\alpha|\alpha) = 2 \}, \]
and decompose \( \Delta \) as \( \Delta = \Delta_+ \cup \Delta_- \). Let
\[ \nu : Q \times Q \rightarrow \{ \pm 1 \} \]
be an asymmetry function (cf. [1] §7.8 and [2] §5.5); namely a bi-multiplicative function satisfying the conditions
\[ \nu(\alpha, \alpha) = (-1)^{(\alpha|\alpha)} \]
\[ \nu(\alpha, \beta) = (-1)^{(\alpha|\beta)} \nu(\beta, \alpha) \]
for \( \alpha, \beta \in Q \). Let \( \mathbf{C}(Q/2Q) \) be the associative algebra spanned over \( \{ e^\alpha \}_{\alpha \in Q/2Q} \) with the usual non-twisted commutative multiplication, namely
\[ e^\alpha e^\beta := e^{\alpha + \beta}. \] (2.13)
We put
\[ V := \mathbf{C}(Q/2Q) \otimes \mathbf{C}[x_{ij}(r) ; j = 1, \cdots, n, \ r \in \mathbb{N}_{\text{odd}}]. \] (2.14)
Let \( h \) be the complexification of \( h_\mathbb{R} \). We extend \( \langle \cdot, \cdot \rangle \) to the symmetric bilinear form on \( h \) and fix an orthonormal basis \( \{ S_1, \cdots, S_n \} \) of \( h \). For \( \alpha \in \Delta \), we put
\[
\alpha(S) := \langle (\alpha|S_1), \cdots, (\alpha|S_n) \rangle \in \mathbb{C}^n,
\]
and consider the operator
\[
\Gamma(z) := \frac{1}{2} e^{\alpha \nu(\alpha, \cdot)} \otimes U_{\sqrt{\mathfrak{h}(S)}}(z)
\]
(2.16)

namely
\[
\Gamma(z)(e^\gamma \otimes f) := \frac{1}{2} e^{\nu(\alpha, \gamma)} e^{\alpha + \gamma} \otimes U_{(\sqrt{\mathfrak{h}(S_1)}, \cdots, \sqrt{\mathfrak{h}(S_n)})}(z) f
\]
(2.17)
for \( e^\gamma \otimes f \in V \). Notice that
\[
\Gamma_-(z) = \Gamma(-z)
\]
(2.18)
by (2.3).

**Lemma 2.3.** For \( \alpha, \beta \in Q \), the following formula holds:
\[
\Gamma_\alpha(z) \Gamma_\beta(w)(e^\gamma \otimes f) = \frac{1}{4} \nu(\alpha, \beta) \nu(\alpha + \beta, \gamma) e^{\alpha + \beta + \gamma} \otimes \nu_{z,w} \left( \frac{z-w}{z+w} \right)^{\alpha|\beta} U_{\sqrt{\mathfrak{h}(S)}}(z) U_{\sqrt{\mathfrak{h}(S)}}(w) f.
\]

**Proof.** For \( e^\gamma \otimes f \in V \), we have
\[
\Gamma_\alpha(z) \Gamma_\beta(w)(e^\gamma \otimes f) = \frac{1}{2} \nu(\beta, \gamma) \nu(\alpha + \beta, \gamma) e^{\alpha + \beta + \gamma} \otimes U_{\sqrt{\mathfrak{h}(S)}}(z) U_{\sqrt{\mathfrak{h}(S)}}(w) f.
\]
Then using Lemma 2.2 proves the lemma. \( \square \)

**Theorem 2.1.** For \( \alpha, \beta \in Q \), the commutators of vertex operators \( \Gamma_\alpha(z) \) and \( \Gamma_\beta(w) \) are given by the following formulas:

1) If \( \langle \alpha|\alpha \rangle = 2 \), then
\[
[\Gamma_\alpha(z), \Gamma_\alpha(w)] = (\iota_{z,w} - \iota_{w,z}) \left( \frac{w}{z+w} - \frac{w^2}{(z+w)^2} - \frac{\sqrt{2}w^2}{z+w} \sum_{j=1}^n (\alpha|S_j)a^{(j)}(w) \right).
\]
2) If \( \langle \alpha|\beta \rangle = 1 \), then
\[
[\Gamma_\alpha(z), \Gamma_\beta(w)] = -\nu(\alpha, \beta) (\iota_{z,w} - \iota_{w,z}) \left( \frac{w}{z+w} \right) \Gamma_{-\alpha + \beta}(w).
\]
3) If \((\alpha|\beta) = -1\), then
\[
[\Gamma_\alpha(z), \Gamma_\beta(w)] = \nu(\alpha, \beta)(t_{z,w} - t_{w,z}) \left( \frac{w}{z-w} \right) \Gamma_{\alpha+\beta}(w).
\]

4) If \((\alpha|\beta) = 0\), then \([\Gamma_\alpha(z), \Gamma_\beta(w)] = 0\).

Proof. 1) Letting \(\beta = \alpha\) in Lemma 2.3 and using \(\nu(\alpha, \alpha) = -1\), one has
\[
\Gamma_\alpha(z)\Gamma_\alpha(w)(e^\gamma \otimes f) = -\frac{1}{4}e^\gamma \otimes t_{z,w} \left( \frac{z-w}{z+w} \right)^2 U_{\sqrt{\mathfrak{A}_0(S);\sqrt{\mathfrak{A}_0(S)}}(z, w)f).
\]
Then, exchanging \(z\) and \(w\), this gives
\[
\Gamma_\alpha(w)\Gamma_\alpha(z)(e^\gamma \otimes f) = -\frac{1}{4}e^\gamma \otimes t_{w,z} \left( \frac{w-z}{z+w} \right)^2 U_{\sqrt{\mathfrak{A}_0(S);\sqrt{\mathfrak{A}_0(S)}}(w, z)f}.
\]

From these two equations, one has
\[
[\Gamma_\alpha(z), \Gamma_\alpha(w)](e^\gamma \otimes f) = -\frac{1}{4}e^\gamma \otimes (t_{z,w} - t_{w,z}) \left( \frac{z-w}{z+w} \right)^2 U_{\sqrt{\mathfrak{A}_0(S);\sqrt{\mathfrak{A}_0(S)}}(z, w)f}.
\] (2.19)
Notice that
\[
\left( \frac{z-w}{z+w} \right)^2 = \left( 1 - \frac{2w}{z+w} \right)^2 = 1 - \frac{4w}{z+w} + \frac{4w^2}{(z+w)^2},
\]
and so
\[
(t_{z,w} - t_{w,z}) \left( \frac{z-w}{z+w} \right)^2 = (t_{z,w} - t_{w,z}) \left( -\frac{4w}{z+w} + \frac{4w^2}{(z+w)^2} \right).
\]

Then, using Lemma 2.1.1), the formula (2.19) is rewritten as follows:
\[
[\Gamma_\alpha(z), \Gamma_\alpha(w)]
= (t_{z,w} - t_{w,z}) \left( \frac{w}{z+w} - \frac{w^2}{(z+w)^2} \right)
\times \left\{ 1 + (z+w)\sqrt{2} \sum_{j=1}^{n} (\alpha|S_j)a^{(j)}(w) + O((z+w)^2) \right\}
= (t_{z,w} - t_{w,z}) \left( \frac{w}{z+w} - \frac{w^2}{(z+w)^2} - \sqrt{2}w^2 \sum_{j=1}^{n} (\alpha|S_j)a^{(j)}(w) \right),
\]
proving 1).
2) Applying Lemma 2.3 to the case \((\alpha|\beta) = 1\), one has

\[
\Gamma_\alpha(z)\Gamma_\beta(w)(e^z \otimes f) = \frac{1}{4} \nu(\alpha, \beta) \nu(\alpha + \beta, \gamma) e^{\alpha+\beta+\gamma} \otimes \iota_{z,w} \left( \frac{z-w}{z+w} \right) U_{\sqrt{\Gamma_\alpha(S);\sqrt{\Gamma_\beta(S)}}}(z,w)f
\]

and, exchanging \(z \leftrightarrow w\) and \(\alpha \leftrightarrow \beta\), also

\[
\Gamma_\beta(w)\Gamma_\alpha(z)(e^z \otimes f) = \frac{1}{4} \nu(\beta, \alpha) \nu(\alpha + \beta, \gamma) e^{\alpha+\beta+\gamma} \otimes \iota_{w,z} \left( \frac{w-z}{w+z} \right) U_{\sqrt{\Gamma_\alpha(S);\sqrt{\Gamma_\beta(S)}}}(w,z)f
\]

since \(\nu(\beta, \alpha) = -\nu(\alpha, \beta)\). Then, from these two equations, one has

\[
\frac{z-w}{z+w} = 1 - \frac{2w}{z+w}
\]

and so

\[
\left( \iota_{z,w} - \iota_{w,z} \right) \left( \frac{z-w}{z+w} \right) = \left( \iota_{z,w} - \iota_{w,z} \right) \left( \frac{-2w}{z+w} \right).
\]

Then, by (2.5), the formula (2.20) is rewritten as follows:

\[
\Gamma_\alpha(z)\Gamma_\beta(w) \Gamma_\beta(w) \Gamma_\alpha(z) \Gamma_\beta(w) (e^z \otimes f) = \frac{-1}{2} \nu(\alpha, \beta) \nu(\alpha + \beta, \gamma) e^{\alpha+\beta+\gamma} \otimes \iota_{z,w} \left( \frac{w}{z+w} \right) U_{\sqrt{\Gamma_\alpha(S);\sqrt{\Gamma_\beta(S)}}}(w)f
\]

proving 2).

3) follows from 2) and (2.18) since

\[
\Gamma_\alpha(z)\Gamma_\beta(w) = \Gamma_\alpha(z), \Gamma_{-\beta}(-w)
\]

4) follows from Lemma 2.3 and (2.13).
Note that formulas in the above theorem are written, in the terminology of usual operator products, as follows:

**Corollary 2.1.** For \( \alpha, \beta \in Q \), the operator products of vertex operators \( \Gamma_\alpha(z) \) and \( \Gamma_\beta(w) \) are given by the following formulas:

1) If \( (\alpha | \alpha) = 2 \), then
   \[
   \Gamma_\alpha(z)\Gamma_\alpha(-w) \sim \frac{-w}{z-w} - \frac{w^2}{(z-w)^2} - \frac{\sqrt{2}w^2}{z-w} \sum_{j=1}^{n} (\alpha | S_j) a^{(j)}(w).
   \]

2) If \( (\alpha | \beta) = 1 \), then
   \[
   \Gamma_\alpha(z)\Gamma_\beta(-w) \sim \nu(\alpha, \beta) \cdot \Gamma_{\alpha-\beta}(w).
   \]

3) If \( (\alpha | \beta) = -1 \), then
   \[
   \Gamma_\alpha(z)\Gamma_\beta(w) \sim \nu(\alpha, \beta) \cdot \Gamma_{\alpha+\beta}(w).
   \]

4) If \( (\alpha | \beta) = 0 \), then
   \[
   \Gamma_\alpha(z)\Gamma_\beta(w) \sim 0.
   \]

### 3 Twisted affinization of simply-laced Lie algebras

In this section, we assume that \( \mathfrak{g} \) is a finite-dimensional simple Lie algebra of rank \( n \) with a symmetric Cartan matrix. Fix a cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \), and let \( \Delta \) be the set of all roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) and \( Q \) be the root lattice. Let \( (\ , \ ) \) be the invariant bilinear form on \( \mathfrak{g} \) normalized by \( (\alpha | \alpha) = 2 \) for all \( \alpha \in \Delta \). For each root \( \alpha \), let \( \mathfrak{g}_\alpha \) denote the root space of \( \alpha \). It is known (cf. [4], §7.8) that, given an asymmetry function \( \nu : Q \times Q \to \{\pm 1\} \), one can choose root vectors \( X_\alpha \in \mathfrak{g}_\alpha \) satisfying the condition

\[
[X_\alpha, X_\beta] = \begin{cases} 
\nu(\alpha, \beta)X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta \\
-H_\alpha & \text{if } \alpha + \beta = 0 
\end{cases}
\]  

(3.1a)

for all \( \alpha, \beta \in \Delta \), where \( H_\alpha \) is the element in \( \mathfrak{h} \) corresponding to \( \alpha \) under the natural identification of \( \mathfrak{h} \) with its dual space \( \mathfrak{h}^* \) via the inner product \( (\ , \ ) \). Notice that this condition means

\[
(X_\alpha, X_{-\alpha}) = -1 \quad \text{for all } \alpha \in \Delta.
\]

(3.1b)

Let \( \sigma \) be the automorphism of \( \mathfrak{g} \) such that

\[
\begin{align*}
\sigma(H) &= -H \quad \text{for all } H \in \mathfrak{h}, \\
\sigma(X_\alpha) &= X_{-\alpha} \quad \text{for all } \alpha \in \Delta.
\end{align*}
\]

(3.2)

We put

\[
\begin{align*}
\mathfrak{g}_0 &:= \{ X \in \mathfrak{g} : \sigma(X) = X \} \\
\mathfrak{g}_1 &:= \{ X \in \mathfrak{g} : \sigma(X) = -X \}.
\end{align*}
\]
and consider the affine Lie algebra

\[ \hat{\mathfrak{g}}(\sigma) := \left( \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{j \mod 2} \otimes t^j \right) \oplus \mathbb{C}K \oplus \mathbb{C}d \]

with the Lie bracket

\[ [X \otimes t^j, Y \otimes t^k] := [X, Y] \otimes t^{j+k} + \frac{j}{2}(X)Y + \frac{k}{2}(Y)X \delta_{j+k,0} K, \]

\[ [d, X \otimes t^j] := jX \otimes t^j, \]

\[ [K, \hat{\mathfrak{g}}(\sigma)] := \{0\}, \]

for \( j, k \in \mathbb{Z} \) and \( X \in \mathfrak{g}_j \) and \( Y \in \mathfrak{g}_k \).

For each \( \alpha \in \Delta \) and \( H \in \mathfrak{h} \), we define the fields

\[ \tilde{X}_\alpha(z) := \sum_{j \in \mathbb{Z}} (X_\alpha + (-1)^j X_{-\alpha}) z^{-j}, \]

\[ H(z) := \sum_{j \in \mathbb{Z}_{\text{odd}}} H_j z^{-j-1}, \]

where \( X_{(j)} := X \otimes t^j \) for \( X \in \mathfrak{g} \) and \( j \in \mathbb{Z} \) as usual. Note that

\[ \tilde{X}_{-\alpha}(z) = \tilde{X}_\alpha(-z) \quad \text{and} \quad H(-z) = H(z). \]

Then

**Lemma 3.1.** Let \( \alpha, \beta \in \Delta \) and \( H \in \mathfrak{h} \). Then

1) \[ [H(z), \tilde{X}_\alpha(w)] = \frac{\alpha(H)}{2} \tilde{X}_\alpha(w) \{ \delta(z-w) - \delta(z+w) \}, \]

2) \[ [\tilde{X}_\alpha(z), \tilde{X}_\beta(w)] = (t_{z,w} - t_{w,z}) \left( \frac{w}{z+w} - \frac{w^2}{(z+w)^2} \right) K + (t_{z,w} - t_{w,z}) \left( \frac{-2w^2}{z+w} \right) H_\alpha(w). \]

3) If \( \alpha \neq \pm \beta \), then

\[ [\tilde{X}_\alpha(z), \tilde{X}_\beta(w)] = \nu(\alpha, \beta) (t_{z,w} - t_{w,z}) \left( \frac{w}{z-w} \right) \tilde{X}_{\alpha+\beta}(w) \]

\[ - \nu(\alpha, \beta) (t_{z,w} - t_{w,z}) \left( \frac{w}{z+w} \right) \tilde{X}_{-\alpha+\beta}(w). \]

**Proof.** 1) is shown as follows:

\[ [H(z), \tilde{X}_\alpha(w)] = \sum_{j \in \mathbb{Z}_{\text{odd}}} [H \otimes t^j, (X_\alpha + (-1)^k X_{-\alpha}) \otimes t^k] z^{-j-1} w^{-k} \]
by (2.12a), proving 1).

For the proof of 2) and 3), we first notice the following:

\[
\begin{aligned}
&\{X_\alpha(z), \tilde{X}_\beta(w)\} \\
= &\sum_{j,k \in \mathbb{Z}} [(X_\alpha + (-1)^j X_{-\alpha}) \otimes t^j, (X_\beta + (-1)^k X_{-\beta}) \otimes t^k] z^{-j} w^{-k} \\
= &\sum_{j,k \in \mathbb{Z}} [X_\alpha + (-1)^j X_{-\alpha}, X_\beta + (-1)^k X_{-\beta}] \otimes t^j t^k z^{-j} w^{-k} \\
&+ K \sum_{j \in \mathbb{Z}} \frac{j}{2} (X_\alpha + (-1)^j X_{-\alpha}) \otimes (X_\beta + (-1)^j X_{-\beta}) z^{-j} w^j. \\
\end{aligned}
\]

(3.4)

Let us consider the case when \(\alpha = \beta\). Since

\[
\begin{aligned}
&[X_\alpha + (-1)^j X_{-\alpha}, X_\alpha + (-1)^j X_{-\alpha}] \\
= &(-1)^j [X_\alpha, X_{-\alpha}] + (-1)^j [X_{-\alpha}, X_\alpha] \\
= &-(-1)^k H_\alpha + (-1)^j H_\alpha = \begin{cases} 2(-1)^j H_\alpha & \text{if } j + k \text{ is odd} \\ 0 & \text{if } j + k \text{ is even}, \end{cases}
\end{aligned}
\]

and

\[
(X_\alpha + (-1)^j X_{-\alpha}) (X_\alpha + (-1)^j X_{-\alpha}) = -2(-1)^j
\]

by (3.1b), the formula (3.4) gives

\[
\begin{aligned}
&[\tilde{X}_\alpha(z), \tilde{X}_\alpha(w)] \\
= &2 \sum_{j,k \in \mathbb{Z} \atop j+k=\text{odd}} (-1)^j H_\alpha \otimes t^j t^k z^{-j} w^{-k} - K \sum_{j \in \mathbb{Z}} (-1)^j j z^{-j} w^j \\
= &2 \sum_{j,k \in \mathbb{Z} \atop j+k=\text{odd}} H_\alpha \otimes t^j t^k w^{-j-k} \cdot (-1)^j z^{-j} w^j - K \sum_{j \in \mathbb{Z}} (-1)^j j z^{-j} w^j \\
= &2 H_\alpha(w) \sum_{j \in \mathbb{Z}} (-1)^j z^{-j} w^j - K \sum_{j \in \mathbb{Z}} (-1)^j j z^{-j} w^j \\
= &2 w H_\alpha(w) (t_{z,w} - t_{w,z}) \left(\frac{z}{z + w}\right) - K (t_{z,w} - t_{w,z}) \left(\frac{-z w}{(z + w)^2}\right) \\
= &2 w H_\alpha(w) (t_{z,w} - t_{w,z}) \left(\frac{w}{z + w}\right) - K (t_{z,w} - t_{w,z}) \left(\frac{-w}{z + w} + \frac{w^2}{(z + w)^2}\right)
\end{aligned}
\]

(3.5)
by (2.12b) and (2.12d), proving 2).

In the case $\alpha \neq \pm \beta$, the second term in (3.4) vanishes, so we have

\[
\begin{align*}
\tilde{X}_\alpha(z) + \tilde{X}_\beta(w) &= \nu(\alpha, \beta) \sum_{j,k \in \mathbb{Z}} \{X_{\alpha+\beta} + (-1)^j X_{-\alpha-\beta}\} \otimes \tau^{j+k} z^{-j} w^{-k} \\
&\quad + \nu(\alpha, \beta) \sum_{j,k \in \mathbb{Z}} \{(-1)^k X_{\alpha-\beta} + (-1)^j X_{-\alpha+\beta}\} \otimes \tau^{j+k} z^{-j} w^{-k} \\
&= \nu(\alpha, \beta) \sum_{j,k \in \mathbb{Z}} \{X_{\alpha+\beta} + (-1)^j X_{-\alpha-\beta}\} \otimes \tau^{j+k} w^{-j-k} z^{-j} w^j \\
&\quad + \nu(\alpha, \beta) \sum_{j,k \in \mathbb{Z}} (-1)^j \{(-1)^k X_{\alpha-\beta} + X_{-\alpha+\beta}\} \otimes \tau^{j+k} w^{-j-k} z^{-j} w^j \\
&= \nu(\alpha, \beta) \tilde{X}_{\alpha+\beta}(w) \sum_{j \in \mathbb{Z}} z^{-j} w^j + \nu(\alpha, \beta) \tilde{X}_{-\alpha+\beta}(w) \sum_{j \in \mathbb{Z}} (-1)^j z^{-j} w^j \\
&= \nu(\alpha, \beta) \tilde{X}_{\alpha+\beta}(w) (\ell_{z,w} - \ell_{w,z}) \left( \frac{z}{z-w} \right) \\
&\quad + \nu(\alpha, \beta) \tilde{X}_{-\alpha+\beta}(w) (\ell_{z,w} - \ell_{w,z}) \left( \frac{z}{z+w} \right),
\end{align*}
\]

proving 3). \hfill \blacksquare

Noticing that, for $\alpha, \beta \in \Delta$ such that $\alpha \neq \pm \beta$,

\[
\alpha + \beta \in \Delta \iff (\alpha|\beta) = -1 \\
\alpha - \beta \in \Delta \iff (\alpha|\beta) = 1 \\
\alpha \pm \beta \notin \Delta \iff (\alpha|\beta) = 0
\]

and comparing Lemma 3.1 with Theorem 2.4, we obtain

**Theorem 3.1.** Let \( \{S_j\}_{j=1,\ldots,n} \) be an orthonormal basis of \( \mathfrak{h} \). Then the map

\[
\pi : \tilde{\mathfrak{g}}(\sigma) \rightarrow \text{End}(V)
\]

defined by

\[
\begin{align*}
\tilde{X}_\alpha(z) &\rightarrow \Gamma_\alpha(z) \quad (\forall \alpha \in \Delta) \\
H(z) &\rightarrow \frac{1}{\sqrt{2}} \sum_{j=1}^n (H|S_j)a^{(j)}(z) \quad (\forall H \in \mathfrak{h}) \\
K &\rightarrow 1 := \text{the identity operator} \\
d &\rightarrow -L_0 := -\sum_{r \in \mathbb{N}_{n\text{odd}}} \sum_{s=1}^n r_x^{(j)} \frac{\partial}{\partial x_r^{(j)}}
\end{align*}
\]

is a representation of \( \tilde{\mathfrak{g}}(\sigma) \).

This representation is not irreducible but a sum of finite numbers of fundamental representations. In the next section, we study its structure and give its irreducible decomposition. Since the action of \( \tilde{\mathfrak{g}}(\sigma) \) contains all \( a^{(j)}_r \)'s, all singular vectors belong to the subspace \( \mathbb{C}\{Q/2Q\} \otimes 1 \) of \( V \), which we simply denote
by \( \mathbb{C}\{Q/2Q\} \). So, in order to get the irreducible decomposition of this representation, one needs only to find out singular vectors in the space \( \mathbb{C}\{Q/2Q\} \).

We note also that the transformation \( \sigma \) is the longest element in the Weyl group of \( g \) if \( g \) is of type \( D_n \) (\( n \) : even) or \( E_7 \) or \( E_8 \). In these cases, \( \hat{g}(\sigma) \) is a non-twisted affine Lie algebra and the representation \( \pi \) is a realization of its fundamental representation associated to the longest element in the Weyl group (cf. [7]). Otherwise, \( \hat{g}(\sigma) \) is a twisted affine algebra.

4 Irreducible decomposition for \( A-D-E \) representations

Let \( \Pi = \{\alpha_1, \cdots, \alpha_n\} \) denote the set of simple roots of a finite-dimensional simple Lie algebra \( g \) with a symmetric Cartan matrix. Then an asymmetry function \( \nu \) is determined by \( \nu(\alpha_j, \alpha_k) \) \((1 \leq j, k \leq n)\) by its bi-multiplicative property. Then the Dynkin diagram of \( \Pi \) with orientation corresponds to \( \nu \) as follows:

\[
\nu(\alpha_j, \alpha_k) = \begin{cases} 
1 & \text{if } \alpha_j \to \alpha_k \text{ or } \alpha_j \text{ is not connected with } \alpha_k \\
-1 & \text{if } j = k \text{ or } \alpha_j \to \alpha_k 
\end{cases}
\]

For each \( \alpha \in Q \), we define the operator \( \hat{X}_\alpha \) acting on the space \( \mathbb{C}\{Q/2Q\} \) by

\[
\hat{X}_\alpha(e^\gamma) := \frac{\nu(\alpha, \gamma)}{2} e^{\alpha+\gamma}.
\]  

(4.1)

In view of (2.17) and Theorem 3.1, one sees that, when \( \alpha \) is a root, this operator \( \hat{X}_\alpha \) is just the action of \( \tilde{X}_\alpha(z) \) to the \( \mathbb{C}\{Q/2Q\} \)-component, or more exactly

\[
\hat{X}_\alpha = (\tilde{X}_\alpha)_0 \text{ on } \mathbb{C}\{Q/2Q\} = \text{ the action of } X_\alpha + X_{-\alpha} \text{ on } \mathbb{C}\{Q/2Q\}.
\]

For \( c_1, \cdots, c_n \in \{\pm 1\} \), we put

\[
v(c_1, \cdots, c_n) := \prod_{j=1}^n (1 + ic_j e^{\alpha_j}) \in \mathbb{C}\{Q/2Q\}.
\]  

(4.2)

Then the collection of these elements \( \{v(c_1, \cdots, c_n)\}_{c_1, \cdots, c_n \in \{\pm 1\}} \) forms a basis of \( \mathbb{C}\{Q/2Q\} \), and the action of \( \hat{X}_{\alpha_j} \) on \( \mathbb{C}\{Q/2Q\} \) is described in terms of this basis as follows:

**Lemma 4.1.** Let \( 1 \leq j \leq n \) and put

\[
\{k_1, \cdots, k_s\} := \{1 \leq k \leq n : k \neq j \text{ and } \nu(\alpha_j, \alpha_k) = -1\}.
\]

Then

\[
2\hat{X}_{\alpha_j}v(c_1, \cdots, c_n) = -ic_j v(c_1, \cdots, -c_k, \cdots, -c_{k_s}, \cdots, c_n).
\]
Proof. For the proof of this lemma, we notice that

\( i \) \quad 2\hat{X}_{\alpha_j}(1 + ic_j e^{\alpha_j}) = -ic_j(1 + ic_j e^{\alpha_j}), \)

\( ii \) \quad \hat{X}_{\alpha_j}((1 + ic_k e^{\alpha_k})u) = (1 + iv(\alpha_j, \alpha_k)c_k e^{\alpha_k}) \cdot \hat{X}_{\alpha_j}u \)

if \( k \neq j \) and \( u \in \mathbb{C}\{Q/2Q\} \).

Actually (i) holds since

\[
2\hat{X}_{\alpha_j}(1 + ic_j e^{\alpha_j}) = e^{\alpha_j} + ic_j \nu(\alpha_j, \alpha_j)e^{2\alpha_j} = e^{\alpha_j} - ic_j(1 + ic_j e^{\alpha_j} + 1),
\]

and (ii) is shown as follows:

\[
\hat{X}_{\alpha_j}((1 + ic_k e^{\alpha_k})u) = \hat{X}_{\alpha_j}(u + ic_k e^{\alpha_k}u) = \hat{X}_{\alpha_j}u + ic_k \hat{X}_{\alpha_j}(e^{\alpha_k}u) = \hat{X}_{\alpha_j}u + ic_k \nu(\alpha_j, \alpha_k)e^{\alpha_k} \cdot \hat{X}_{\alpha_j}u.
\]

Then, by the successive use of (ii), one has

\[
2\hat{X}_{\alpha_j}\left(\prod_{k=1}^{n}(1 + ic_k e^{\alpha_k})\right) = 2\hat{X}_{\alpha_j}\left(\prod_{k \neq j}(1 + ic_k e^{\alpha_k}) \cdot (1 + ic_j e^{\alpha_j})\right)
\]

\[
= \prod_{k \neq j}(1 + iv(\alpha_j, \alpha_k)c_k e^{\alpha_k}) \cdot 2\hat{X}_{\alpha_j}(1 + ic_j e^{\alpha_j})
\]

\[
= \prod_{k \neq j}(1 + iv(\alpha_j, \alpha_k)c_k e^{\alpha_k}) \cdot (-ic_j)(1 + ic_j e^{\alpha_j})
\]

proving the lemma. \( \square \)

We note also that

\[
2\hat{X}_{\alpha+\beta} = \nu(\alpha, \beta)(2\hat{X}_\alpha)(2\hat{X}_\beta) \quad \text{for } \alpha, \beta \in Q, \quad (4.3)
\]

since

\[
\hat{X}_{\alpha+\beta}(e^\gamma) = \frac{1}{2}\nu(\alpha + \beta, \gamma)e^{\alpha+\beta+\gamma}
\]

and

\[
\hat{X}_\alpha\left(\hat{X}_\beta(e^\gamma)\right) = \frac{1}{2}\nu(\beta, \gamma)\hat{X}_\alpha(e^{\beta+\gamma}) = \frac{1}{4}\nu(\beta, \gamma)\nu(\alpha, \beta + \gamma)e^{\alpha+\beta+\gamma}
\]

\[
= \frac{1}{4}\nu(\alpha, \beta)\nu(\alpha + \beta, \gamma)e^{\alpha+\beta+\gamma}.
\]

From this formula and Lemma 4.1, one obtains the following:

Lemma 4.2. 1) Let \( 1 \leq j \leq p - 1 \) and \( p \leq n - 2 \) in the following diagram

\[
\alpha_1 \cdots \cdots \cdots \alpha_p \alpha_{p+1} \cdots \cdots \alpha_{n-1}
\]

\[
\alpha_n
\]
So one may write
\[ Y \sim \text{field} \ n \text{ is even or odd:} \]

4.1 The case \( D_n \)

For \( D_n \), we consider the following orientation of Dynkin diagram according as \( n \) is even or odd:

\[ D_{2m} : \]

\[ \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_{2m-4} \alpha_{2m-3} \alpha_{2m-2} \alpha_{2m-1} \]

\[ \alpha_{2m} \]

Then
\[ 2 \tilde{X}_{\alpha_j + \alpha_{p+1}} \left( (1 + ic_j e^{\alpha_j})(1 + ic_{p+1} e^{\alpha_{p+1}})(1 + ic_2 e^{\alpha_2}) \right) = ic_j c_{p+1} c_n (1 + ic_j e^{\alpha_j})(1 + ic_{p+1} e^{\alpha_{p+1}})(1 + ic_2 e^{\alpha_2}). \]

2) Let \( 1 \leq j \leq n - 4 \) in the following diagram

Then
\[ 2 \tilde{X}_{\alpha_j + \alpha_{p+1}} \left( (1 + ic_j e^{\alpha_j})(1 + ic_{n-2} e^{\alpha_{n-2}})(1 + ic_{n-1} e^{\alpha_{n-1}})(1 + ic_n e^{\alpha_n}) \right) = ic_j c_{n-1} c_n (1 + ic_j e^{\alpha_j})(1 + ic_{n-2} e^{\alpha_{n-2}})(1 + ic_{n-1} e^{\alpha_{n-1}})(1 + ic_n e^{\alpha_n}). \]

We put
\[ Y_\alpha := X_\alpha + X_{-\alpha} \quad \text{for } \alpha \in \Delta. \]

Then \( Y_\alpha \) is an element in \( \mathfrak{g}_0 \cong \mathfrak{g}_0 \otimes \mathfrak{t}^0 \subset \mathfrak{g}^*(\sigma) \) and, by the definition (3.3) of field \( \tilde{X}_\alpha(z) \), the action of \( Y_\alpha \)'s on \( \mathbb{C}\{Q/2Q\} \) is just equal to \( \tilde{X}_\alpha \); namely

the action of \( Y_\alpha \) on \( \mathbb{C}\{Q/2Q\} = \tilde{X}_\alpha. \)

So one may write \( Y_\alpha v \) in place of \( \tilde{X}_\alpha v \) for \( \alpha \in \Delta \) and \( v \in \mathbb{C}\{Q/2Q\}. \)
Proposition 4.1.  
1) In the case $n = 2m$:

(i) $2Y_{\alpha_{2j-1}}v(c_1, \cdots, c_{2m}) = -ic_{2j-1}v(c_1, \cdots, c_{2m})$  
(1 $\leq j \leq m$),

(ii) $2Y_{\alpha_{2j}}v(c_1, \cdots, c_{2m})$

$$
= \begin{cases} 
-ic_{2j}v(c_1, \cdots, -c_{2j-1}, c_{2j}, c_{2j+1}, \cdots, c_{2m}) & (1 \leq j \leq m - 2) \\
-ic_{2m-2}v(c_1, \cdots, -c_{2m-3}, c_{2m-2}, -c_{2m-1}) & (j = m - 1) \\
-ic_{2m}v(c_1, \cdots, c_{2m}) & (j = 2m),
\end{cases}
$$

(iii) $2Y_{\alpha_{2j-1}+2(\alpha_{2j}+\cdots+\alpha_{2m-2})+\alpha_{2m-1}+\alpha_{2m}}v(c_1, \cdots, c_{2m})$

$$
= ic_{2j-1}c_{2m-1}c_{2m}v(c_1, \cdots, c_{2m})
$$  
(1 $\leq j \leq m - 1$).

2) In the case $n = 2m + 1$:

(i) $2Y_{\alpha_{2j-1}}v(c_1, \cdots, c_{2m+1})$

$$
= \begin{cases} 
-ic_{2j-1}v(c_1, \cdots, c_{2m+1}) & (1 \leq j \leq m) \\
-ic_{2m+1}v(c_1, \cdots, c_{2m-2}, -c_{2m-1}, c_{2m}) & (j = m + 1),
\end{cases}
$$

(ii) $2Y_{\alpha_{2j}}v(c_1, \cdots, c_{2m+1})$

$$
= \begin{cases} 
-ic_{2j}v(c_1, \cdots, -c_{2j-1}, c_{2j}, c_{2j+1}, \cdots, c_{2m+2}, c_{2m+1}) & (1 \leq j \leq m - 1) \\
-ic_{2m}v(c_1, \cdots, c_{2m-2}, -c_{2m-1}, c_{2m}) & (j = m),
\end{cases}
$$

(iii) $2Y_{\alpha_{2j-1}+2(\alpha_{2j}+\cdots+\alpha_{2m-1})+\alpha_{2m}+\alpha_{2m+1}}v(c_1, \cdots, c_{2m+1})$

$$
= ic_{2j-1}c_{2m}c_{2m+1}v(c_1, \cdots, c_{2m+1})
$$  
(1 $\leq j \leq m - 1$),

(iv) $2Y_{\alpha_{2m-1}+\alpha_{2m}+\alpha_{2m+1}}v(c_1, \cdots, c_{2m+1})$

$$
= ic_{2m-1}c_{2m}c_{2m+1}v(c_1, \cdots, c_{2m+1}).
$$
For an explicit description of Chevalley generators of \( \tilde{\mathfrak{g}}(\sigma) \), we define elements \( Z_{j,k} \) and \( Z'_{j,k} \) in \( \mathfrak{g}_0 \) for \( 1 \leq j \leq k \leq n-1 \) as follows:

\[
Z_{j,k} := \begin{cases} 
Y_{\alpha_j + \cdots + \alpha_k} + Y_{\alpha_j + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n} & (k \leq n-3) \\
Y_{\alpha_j + \cdots + \alpha_{n-2}} + Y_{\alpha_j + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n} & (k = n-2) \\
Y_{\alpha_j + \cdots + \alpha_{n-2} + \alpha_{n-1} - Y_{\alpha_j + \cdots + \alpha_{n-2} + \alpha_{n}} & (j < k = n-1) \\
Y_{\alpha_{n-1} - Y_{\alpha_n}} & (j = k = n-1)
\end{cases}
\]

and

\[
Z'_{j,k} := \begin{cases} 
Y_{\alpha_j + \cdots + \alpha_k} - Y_{\alpha_j + \cdots + \alpha_k + 2(\alpha_{k+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n} & (k \leq n-3) \\
Y_{\alpha_j + \cdots + \alpha_{n-2}} - Y_{\alpha_j + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n} & (k = n-2) \\
Y_{\alpha_j + \cdots + \alpha_{n-2} + \alpha_{n-1} + Y_{\alpha_j + \cdots + \alpha_{n-2} + \alpha_{n}} & (j < k = n-1) \\
Y_{\alpha_{n-1} + Y_{\alpha_n}} & (j = k = n-1).
\end{cases}
\]

For simplicity, we write \( Z_j := Z_{j,j} \) and \( Z'_j := Z'_{j,j} \). Then, by an easy calculation, one can check the following:

**Lemma 4.3.**

1) \( [Z_{j,k}, Z'_{r,s}] = 0 \) for all \( j, k, r, s \).

2) (i) \( [Z_{j,r}, Z_{s,s}] = \begin{cases} 
-2\nu(\alpha_r, \alpha_{r+1})Z_{r+1,s} & (r < s), \\
2\nu(\alpha_s, \alpha_{s+1})Z_{s+1,r} & (s < r)
\end{cases} \)

(ii) \( [Z'_{j,r}, Z'_{s,s}] = \begin{cases} 
-2\nu(\alpha_r, \alpha_{r+1})Z'_{r+1,s} & (r < s), \\
2\nu(\alpha_s, \alpha_{s+1})Z'_{s+1,r} & (s < r)
\end{cases} \)

3) (i) \( [Z_{j,r}, Z_{k,r}] = \begin{cases} 
-2\nu(\alpha_k - 1, \alpha_k)Z_{j,k-1} & (j < k), \\
2\nu(\alpha_{j-1}, \alpha_j)Z_{k,j-1} & (k < j)
\end{cases} \)

(ii) \( [Z'_{j,r}, Z'_{k,r}] = \begin{cases} 
-2\nu(\alpha_k - 1, \alpha_k)Z'_{j,k-1} & (j < k), \\
2\nu(\alpha_{j-1}, \alpha_j)Z'_{k,j-1} & (k < j)
\end{cases} \)

4) (i) \( [Z_{j,k-1}, Z_{k,s}] = 2\nu(\alpha_k - 1, \alpha_k)Z_{j,s} \)

(ii) \( [Z'_{j,k-1}, Z'_{k,s}] = 2\nu(\alpha_k - 1, \alpha_k)Z'_{j,s} \)

5) (i) \( [Z_{j,r}, Z_{k,j-1}] = -2\nu(\alpha_j - 1, \alpha_j)Z_{k,r} \)

(ii) \( [Z'_{j,r}, Z'_{k,j-1}] = -2\nu(\alpha_j - 1, \alpha_j)Z'_{k,r} \)

For \( \mathfrak{g} = D_n = so(2n) \), the \( \sigma \)-fixed subalgebra \( \mathfrak{g}_0 \) is \( so(n) \oplus so(n) \), which is a semisimple Lie algebra of type \( D_m \oplus D_m \) if \( n = 2m \) and \( B_m \oplus B_m \) if \( n = 2m+1 \), where \( D_2 := A_1 \oplus A_1 \). We define elements \( \tilde{e}_j, \tilde{f}_j, \tilde{h}_j \) for \( 0 \leq j \leq m \) and \( \tilde{e}'_j, \tilde{f}'_j, \tilde{h}'_j \) for \( 1 \leq j \leq m \) in \( \tilde{\mathfrak{g}}(\sigma) \) as follows:

In the case \( n = 2m \):

\[
\begin{align*}
\tilde{e}_j & := \frac{1}{2}\{Z_{2j-1,2j} - Z_{2j,2j+1} - iZ_{2j-1,2j+1} + iZ_{2j}\} \\
\tilde{e}_m & := \frac{1}{2}\{Z_{2m-3,2m-2} + Z_{2m-2,2m-1} + iZ_{2m-3,2m-1} - iZ_{2m-2}\} \\
\tilde{e}'_j & := \frac{1}{2}\{Z_{2j-1,2j} - Z_{2j,2j+1} - iZ_{2j-1,2j+1} + iZ_{2j}\} \\
\tilde{e}'_m & := \frac{1}{2}\{Z_{2m-3,2m-2} + Z_{2m-2,2m-1} + iZ_{2m-3,2m-1} - iZ_{2m-2}\}
\end{align*}
\]
In the case \( n = 2m + 1 \):

\[
\begin{align*}
\tilde{f}_j &:= \frac{1}{2} \{ -Z_{2j-1,2j} - Z_{2j,2j+1} - iZ_{2j-1,2j+1} - iZ_{2j} \} \\
(1 \leq j \leq m - 1) \\
\tilde{f}_m &:= \frac{1}{2} \{ -Z_{2m-3,2m-2} - Z_{2m-2,2m-1} + iZ_{2m-3,2m-1} - iZ_{2m-2} \} \\
\tilde{f}_j' &:= \frac{1}{2} \{ -Z_{2j-1,2j} + Z_{2j,2j+1} - iZ_{2j-1,2j+1} - iZ_{2j} \} \\
(1 \leq j \leq m - 1) \\
\tilde{f}_m' &:= \frac{1}{2} \{ -Z_{2m-3,2m-2} + Z_{2m-2,2m-1} + iZ_{2m-3,2m-1} - iZ_{2m-2} \} \\
\tilde{h}_j &:= \frac{1}{2} \{ Z_{2j-1} - Z_{2j+1} \} \\
(1 \leq j \leq m - 1) \\
\tilde{h}_m &:= \frac{1}{2} \{ Z_{2m-3} + Z_{2m-1} \} \\
\tilde{h}_j' &:= \frac{1}{2} \{ Z_{2j-1} - Z_{2j+1} \} \\
(1 \leq j \leq m - 1) \\
\tilde{h}_m' &:= \frac{1}{2} \{ Z_{2m-3} + Z_{2m-1} \} \\
\tilde{e}_0 &:= \frac{1}{2} \{ i(X_{\alpha_1} - X_{\alpha_1}) + H_{\alpha_1} \} \otimes t \\
\tilde{f}_0 &:= \frac{1}{2} \{ -i(X_{\alpha_1} - X_{\alpha_1}) + H_{\alpha_1} \} \otimes t^{-1} \\
\tilde{h}_0 &:= -iY_{\alpha_1} + \frac{k}{2}.
\end{align*}
\]

Then, by Lemma 4.3, one can easily check that these elements satisfy the conditions of Chevalley generators for the following Dynkin diagrams according as \( n = 2m \) or \( n = 2m + 1 \), letting \( \tilde{e}_j \) (resp. \( \tilde{e}_j' \)) be a root vector of a simple root \( \tilde{\alpha}_j \) (resp. \( \tilde{\alpha}_j' \)): 

\[
D_{2m}^{(1)}: \quad \tilde{\alpha}_{m-1} \tilde{\alpha}_{m-2} \cdots \tilde{\alpha}_1 \tilde{\alpha}_0 \tilde{\alpha}_1' \tilde{\alpha}_2' \cdots \tilde{\alpha}_{m-2}' \tilde{\alpha}_{m-1}'
\]

\[\alpha_m\]

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Theorem 4.1. Let $v = (c_1, \cdots, c_n)$ where $c_1, \cdots, c_n \in \{\pm 1\}$. Then

1) In the case $n = 2m$,

\[
\begin{align*}
\tilde{h}_j v &= c_{2j-1}(1 - c_{2j-1}c_{2j+1})(1 - c_{2m-1}c_{2m})v \quad (1 \leq j \leq m-1) \\
\tilde{h}_m v &= c_{2m-3}(1 + c_{2m-3}c_{2m-1})(1 - c_{2m-1}c_{2m})v \\
\tilde{h}_j' v &= c_{2j-1}(1 - c_{2j-1}c_{2j+1})(1 + c_{2m-1}c_{2m})v \quad (1 \leq j \leq m-1) \\
\tilde{h}_m' v &= c_{2m-3}(1 + c_{2m-3}c_{2m-1})(1 + c_{2m-1}c_{2m})v \\
2h_0 v &= (1 - c_1)v.
\end{align*}
\]

2) In the case $n = 2m+1$,

\[
\begin{align*}
\tilde{h}_j v &= c_{2j-1}(1 - c_{2j-1}c_{2j+1})(1 - c_{2m}c_{2m+1})v \quad (1 \leq j \leq m-1) \\
2\tilde{h}_m v &= c_{2m-1}(1 - c_{2m}c_{2m+1})v \\
\tilde{h}_j' v &= c_{2j-1}(1 - c_{2j-1}c_{2j+1})(1 + c_{2m}c_{2m+1})v \quad (1 \leq j \leq m-1) \\
2\tilde{h}_m' v &= c_{2m-1}(1 + c_{2m}c_{2m+1})v \\
2h_0 v &= (1 - c_1)v.
\end{align*}
\]

Notice that, for a level one representation of a simply-laced algebra, a singular vector is characterized as an eigenvector of all $\tilde{h}_j$'s and $\tilde{h}_j'$'s with non-negative integral eigenvalues. From this, singular vectors are easily obtained by Proposition 4.2. And then, all other elements belonging to the invariant subspace spanned by a singular vector are obtained from Proposition 4.1. The calculation is straightforward and the result is stated as follows:

Theorem 4.1. Let $v = (c_1, \cdots, c_n)$ where $c_1, \cdots, c_n \in \{\pm 1\}$. Then

1) In the case $n = 2m$,

(i) $v$ is a singular vector if and only if $c_{2j-1} = 1$ for $j = 1, 2, \cdots, m-1$. And then the weight of $v$ is determined by $(c_{2m-1}, c_{2m})$ as is shown in the following table:

| singular vector | weight |
|-----------------|--------|
| $(1, c_2, 1, c_4, \cdots, 1, \ c_{2m-2}, 1)$ | $\tilde{\Lambda}_m$ |
| $(1, c_2, 1, c_4, \cdots, 1, \ c_{2m-2}, -1)$ | $\tilde{\Lambda}_m$ |
| $(1, c_2, 1, c_4, \cdots, 1, \ c_{2m-2}, -1)$ | $\tilde{\Lambda}_{m-1}$ |
| $(1, c_2, 1, c_4, \cdots, 1, \ c_{2m-2}, -1)$ | $\tilde{\Lambda}_{m-1}$ |
for any choice of \( c_2, c_4, \cdots, c_{2m-2} \in \{\pm 1\} \).

(ii) Given a singular vector

\[
v_0 := v \left( 1, c_2, 1, c_4, \cdots, 1, c_{2m-2}, c_{2m-1} \right),
\]

the \( \mathbb{C} \)-linear span of all elements

\[
v \left( b_1, c_2, b_3, c_4, \cdots, b_{2m-3}, c_{2m-2}, c_{2m-1} \right)
\]

satisfying the conditions

(a) \( b_j \in \{\pm 1\} \) for all \( j \),
(b) \( b_{2m-1} b_{2m} = c_{2m-1} c_{2m} \),
(c) \( \prod_{1 \leq j \leq 2m-1 \atop j \text{ odd}} b_j = c_{2m-1} \),

tensored with \( \mathbb{C}[x_r^\ell] ; 1 \leq j \leq 2m, \; r \in \mathbb{N}_{\text{odd}} \) is the irreducible \( D_{2m}^{(1)} \)-module with the highest weight vector \( v_0 \).

2) In the case \( n = 2m + 1 \),

(i) \( v \) is a singular vector if and only if \( c_{2j-1} = 1 \) for \( j = 1, 2, \cdots, m \). And then the weight of \( v \) is determined by \( (c_{2m}, c_{2m+1}) \) as is shown in the following table:

| singular vector | weight |
|-----------------|--------|
| \( v \left( 1, c_2, 1, c_4, \cdots, 1, c_{2m-2}, 1, \pm 1 \right) \) | \( \Lambda'_m \) |
| \( v \left( 1, c_2, 1, c_4, \cdots, 1, c_{2m-2}, 1, \mp 1 \right) \) | \( \Lambda_m \) |

for any choice of \( c_2, c_4, \cdots, c_{2m-2} \in \{\pm 1\} \).

(ii) Given a singular vector

\[
v_0 := v \left( 1, c_2, 1, c_4, 1, \cdots, 1, c_{2m-2}, c_{2m}, c_{2m+1} \right),
\]

the \( \mathbb{C} \)-linear span of all elements

\[
v \left( b_1, c_2, b_3, c_4, \cdots, b_{2m-3}, c_{2m-2}, c_{2m-1}, c_{2m+1} \right),
\]

where \( b_j \in \{\pm 1\} \) for all \( j \), tensored with \( \mathbb{C}[x_r^\ell] ; 1 \leq j \leq 2m + 1, \; r \in \mathbb{N}_{\text{odd}} \) is the irreducible \( D_{2m+1}^{(2)} \)-module with the highest weight vector \( v_0 \).
In the case $D_n$, one can choose an orthonormal basis $\varepsilon_1, \cdots, \varepsilon_n$ of $\mathfrak{h}^*$ such that
\[
\alpha_j = \varepsilon_j - \varepsilon_{j+1} \quad (1 \leq j \leq n-1) \quad \text{and} \quad \alpha_n = \varepsilon_{n-1} + \varepsilon_n,
\]
and let $S_1, \cdots, S_n$ be its dual basis of $\mathfrak{h}$. Then the set of positive roots is given by
\[
\Delta_+ = \{ \varepsilon_j \pm \varepsilon_k : 1 \leq j < k \leq n \},
\]
and, for each $\varepsilon_j \pm \varepsilon_k$, the operator $\Gamma_{\varepsilon_j \pm \varepsilon_k}(z)$ is written as follows:
\[
\Gamma_{\varepsilon_j \pm \varepsilon_k}(z) = \frac{1}{2} e^{\varepsilon_j \pm \varepsilon_k} \nu(\varepsilon_j \pm \varepsilon_k, \cdot) \tilde{U}_{\varepsilon_j \pm \varepsilon_k}(z)
\]
where
\[
\tilde{U}_{\varepsilon_j \pm \varepsilon_k}(z) := \exp \left( \sqrt{2} \sum_{r \in \mathbb{N}_{\text{odd}}} (x_r^{(j)} \pm x_r^{(k)}) z^r \right) \times \exp \left( -\sqrt{2} \sum_{r \in \mathbb{N}_{\text{odd}}} \left( \frac{\partial}{\partial x_r^{(j)}} \pm \frac{\partial}{\partial x_r^{(k)}} \right) \frac{z^{-r}}{r} \right).
\]

**Example 4.1.** We consider $D_4$ with the oriented Dynkin diagram

```
α₁ --α₂ --α₃
     \ \ \ \α₄
```

We write simply $(m_1 m_2 m_3 m_4)$ for $\alpha = \sum_{j=1}^4 m_j \alpha_j$. Fix $c_1, c_2, c_3, c_4 \in \{\pm 1\}$, and put
\[
v := v(c_1, c_2, c_3, c_4) \quad \text{and} \quad v' := v(-c_1, c_2, -c_3, -c_4).
\]

Then, by Lemma 4.1, one sees that the $\mathbb{C}$-linear span of $v$ and $v'$ is invariant under the action of $\tilde{X}_\alpha$’s and so $\tilde{X}_\alpha$ for all $\alpha \in \Delta_+$. To write down the action of $\tilde{X}_\alpha$ on the space $\mathbb{C}v \oplus \mathbb{C}v'$ explicitly, we recall the Pauli’s spin matrices
\[
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and divide the set of positive roots into the disjoint union of three parts:
\[
\Delta_+^{(1)} := \{ \varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_3 \},
\Delta_+^{(2)} := \{ \varepsilon_1 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_4 \},
\Delta_+^{(3)} := \{ \varepsilon_1 \pm \varepsilon_2, \varepsilon_3 \pm \varepsilon_4 \}.
\]

Then this decomposition $\Delta_+ = \Delta_+^{(1)} \cup \Delta_+^{(2)} \cup \Delta_+^{(3)}$ has the following properties:
(i) \( \alpha, \beta \in \Delta^{(i)} \implies \alpha \pm \beta \notin \Delta. \)

(ii) For \( \{i, j, k\} = \{1, 2, 3\}, \)
\[ \alpha \in \Delta^{(i)}_+, \beta \in \Delta^{(j)}_+ \text{ and } \alpha + \beta \in \Delta \implies \alpha + \beta \in \Delta^{(k)}_+. \]

For \( \alpha \in \Delta_+ \), \( \Hat{X}_\alpha v \) and \( \Hat{X}_\alpha v' \) are computed by using Lemma 4.1 and (4.3), and are explicitly written as follows:

\[
\Delta^{(1)}_+ : \begin{cases} 
2 \Hat{X}^{(010)} \to 2 \Hat{X}_{e_2-e_3} = ic_1 \sigma_1 \\
2 \Hat{X}^{(111)} \to 2 \Hat{X}_{e_2-e_4} = ic_1 c_3 \sigma_1 \\
2 \Hat{X}^{(110)} \to 2 \Hat{X}_{e_1+e_4} = ic_1 c_4 \sigma_1 \\
2 \Hat{X}^{(011)} \to 2 \Hat{X}_{e_2+e_3} = ic_3 \sigma_1 \\
2 \Hat{X}^{(111)} \to 2 \Hat{X}_{e_1-e_3} = -ic_1 c_2 \sigma_2 \\
2 \Hat{X}^{(011)} \to 2 \Hat{X}_{e_2-e_4} = -ic_2 \sigma_2 \\
2 \Hat{X}^{(101)} \to 2 \Hat{X}_{e_2+e_4} = -ic_2 c_4 \sigma_2 \\
2 \Hat{X}^{(111)} \to 2 \Hat{X}_{e_1+e_3} = -ic_1 c_3 c_4 \sigma_2 \\
2 \Hat{X}^{(100)} \to 2 \Hat{X}_{e_1-e_2} = -ic_1 \sigma_3 \\
2 \Hat{X}^{(001)} \to 2 \Hat{X}_{e_3-e_4} = -ic_3 \sigma_3 \\
2 \Hat{X}^{(000)} \to 2 \Hat{X}_{e_3+e_4} = -ic_4 \sigma_3 \\
2 \Hat{X}^{(111)} \to 2 \Hat{X}_{e_1+e_2} = -ic_1 c_3 c_4 \sigma_3 .
\]

Then, in particular letting \( (c_1, c_2, c_3, c_4) := (-1, 1, -1, -1) \), one obtains the irreducible representation \( \pi \) of \( D_4^{(1)} \) on the space

\[
\tilde{V} := \left( v \otimes C[x_r^{(j)} ; 1 \leq j \leq 4, r \in N_{\text{odd}}] \right) \oplus \left( v' \otimes C[x_r^{(j)} ; 1 \leq j \leq 4, r \in N_{\text{odd}}] \right) \\
\cong C[x_r^{(j)} ; 1 \leq j \leq 4, r \in N_{\text{odd}}} \oplus C[x_r^{(j)} ; 1 \leq j \leq 4, r \in N_{\text{odd}}]
\]
as follows:

\[
\pi : \begin{cases} 
S_j \otimes t^r & \mapsto \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_r^{(j)}} \cdot \sigma_0 \\
S_j \otimes t^{-r} & \mapsto \frac{1}{\sqrt{2}} x_r^{(j)} \sigma_0 
\end{cases} \quad (j = 1, 2, 3, 4; r \in N_{\text{odd}})
\]

and

\[
\pi : \begin{cases} 
\Hat{X}_{e_j \pm e_k} (z) & \mapsto \frac{i}{2} \Hat{U}_{e_j \pm e_k} (z) \cdot \sigma_p \quad \text{(if } e_j \pm e_k \in \Delta^{(p)}_+), \\
d & \mapsto -\sum_{j=1}^{4} \sum_{r \in N_{\text{odd}}} r x_r^{(j)} \frac{\partial}{\partial x_r^{(j)}} \\
K & \mapsto 1 \quad (= \text{ the identity operator})
\end{cases}
\]

where we put \( \sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).
4.2 The case $A_n$

For $A_n$, we consider the following orientation of Dynkin diagram according as $n$ is odd or even:

$A_{2m-1}$:

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_{2m-3} & \alpha_{2m-2} & \alpha_{2m-1} \\
\end{array}
\]

$A_{2m}$:

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_{2m-2} & \alpha_{2m-1} & \alpha_{2m} \\
\end{array}
\]

Lemma 4.4. Let $n = 2m$ or $2m-1$ in $A_n$. Then the action of $Y_{\alpha_j}$ to an element $v := v(c_1, \cdots, c_n)$ ($c_1, \cdots, c_n \in \{\pm 1\}$) is given as follows:

1) $2Y_{\alpha_{2j-1}} v(c_1, \cdots, c_n) = -ic_{2j-1} v(c_1, \cdots, c_n)$ \hspace{1cm} $(1 \leq j \leq m)$,

2) $2Y_{\alpha_{2j}} v(c_1, \cdots, c_n)$ \hspace{1cm} \[
\begin{align*}
&= \begin{cases} 
-ic_{2j} v(c_1, \cdots, c_{2j-1}, c_{2j}, c_{2j+1}, \cdots, c_n) & \text{if } 2j < n \\
-ic_{2m} v(c_1, \cdots, c_{2m-2}, c_{2m-1}, c_{2m}) & \text{if } 2j = n = 2m.
\end{cases}
\end{align*}
\]

For $g = A_n = sl(n+1, \mathbb{C})$, the $\sigma$-fixed subalgebra $g_0$ is $so(n+1, \mathbb{C})$, which is a simple Lie algebra of type $D_m$ if $n = 2m - 1$ and $B_m$ if $n = 2m$. For $1 \leq j \leq k \leq n$ we put

\[
Y_{j,k} := Y_{\alpha_j + \cdots + \alpha_k} \quad \text{and} \quad Y_j := Y_{j,j} = Y_{\alpha_j},
\]

and define elements $\tilde{e}_j, \tilde{f}_j, \tilde{h}_j$ ($0 \leq j \leq m$) in $\tilde{g}(\sigma)$ as follows:

In the case $n = 2m - 1$:

\[
\begin{align*}
\tilde{e}_j &:= \frac{1}{2} \{ Y_{2j-1,2j} - Y_{2j,2j+1} - iY_{2j-1,2j+1} - iY_{2j} \} \quad (1 \leq j \leq m - 1) \\
\tilde{e}_m &:= \frac{1}{2} \{ Y_{2m-3,2m-2} + Y_{2m-2,2m-1} + iY_{2m-3,2m-1} - iY_{2m-2} \} \\
\tilde{f}_j &:= \frac{1}{2} \{ -Y_{2j-1,2j} + Y_{2j,2j+1} - iY_{2j-1,2j+1} - iY_{2j} \} \quad (1 \leq j \leq m - 1) \\
\tilde{f}_m &:= \frac{1}{2} \{ -Y_{2m-3,2m-2} - Y_{2m-2,2m-1} + iY_{2m-3,2m-1} - iY_{2m-2} \} \\
\tilde{h}_j &:= i \{ Y_{2j-1,2j+1} \} \quad (1 \leq j \leq m - 1) \\
\tilde{h}_m &:= i \{ Y_{2m-3} + Y_{2m-1} \} \\
\tilde{e}_0 &:= \frac{1}{2} \{ i(X_{\alpha_1} - X_{-\alpha_1}) + H_{\alpha_1} \} \otimes t \\
\tilde{f}_0 &:= \frac{1}{2} \{ -i(X_{\alpha_1} - X_{-\alpha_1}) + H_{\alpha_1} \} \otimes t^{-1} \\
\tilde{h}_0 &:= -iY_{\alpha_1} + \frac{g}{2}.
\end{align*}
\]

In the case $n = 2m$:

\[
\begin{align*}
\tilde{e}_j &:= \frac{1}{2} \{ Y_{2j-1,2j} - Y_{2j,2j+1} - iY_{2j-1,2j+1} - iY_{2j} \} \quad (1 \leq j \leq m - 1) \\
\tilde{e}_m &:= Y_{2m-2,2m-1} - iY_{2m-2}
\end{align*}
\]
Theorem 4.2. Let the fundamental integral form corresponding to the simple coroots system \( \tilde{\alpha} \). The action of Proposition 4.3. Lemma 4.4 as follows:

\[
\begin{align*}
\tilde{f}_j &:= \frac{1}{2}\{ -Y_{2j-1,2j} + Y_{2j,2j+1} - iY_{2j-1,2j+1} - iY_{2j} \} \\
(1 \leq j \leq m-1) \\
\tilde{f}_m &:= -Y_{2m-2,2m-1} - iY_{2m-2} \\
\tilde{h}_j &:= i\{ Y_{2j-1} - Y_{2j+1} \} \\
(1 \leq j \leq m-1) \\
\tilde{h}_m &:= 2iY_{2m-1} \\
\tilde{e}_0 &:= \frac{1}{2}\{ i(X_{\alpha_1} - X_{-\alpha_1}) + H_{\alpha_1} \} \otimes t \\
\tilde{f}_0 &:= \frac{1}{2}\{ -i(X_{\alpha_1} - X_{-\alpha_1}) + H_{\alpha_1} \} \otimes t^{-1} \\
\tilde{h}_0 &:= -iY_{\alpha_1} + \frac{K}{2}.
\end{align*}
\]

Then one can easily check that these elements satisfy the conditions of Chevalley generators for the following Dynkin diagrams according as \( n = 2m-1 \) or \( n = 2m \), letting \( \tilde{e}_j \) be a root vector of a simple root \( \tilde{\alpha}_j \):

\[
A^{(2)}_{2m-1} : \quad \tilde{\alpha}_0 \quad \tilde{\alpha}_1 \quad \tilde{\alpha}_2 \quad \cdots \quad \tilde{\alpha}_{m-3} \quad \tilde{\alpha}_{m-2} \quad \tilde{\alpha}_{m-1} \quad \tilde{\alpha}_m
\]

\[
A^{(2)}_{2m} : \quad \tilde{\alpha}_0 \quad \tilde{\alpha}_1 \quad \tilde{\alpha}_2 \quad \cdots \quad \tilde{\alpha}_{m-2} \quad \tilde{\alpha}_{m-1} \quad \tilde{\alpha}_m
\]

The action of \( \tilde{h}_j \) on the basis \( v(c_1, \cdots, c_n) \) of \( \mathbb{C}\{Q/2Q\} \) is calculated from Lemma 4.4 as follows:

**Proposition 4.3.** Let \( v := v(c_1, \cdots, c_n) \) where \( c_1, \cdots, c_n \in \{ \pm 1 \} \).

1) In the case \( n = 2m-1, \)

\[
2\tilde{h}_j v = \begin{cases} 
(1 - c_1)v & (j = 0), \\
(c_{2j-1} - 1 - c_{2j-1}c_{2j+1})v & (1 \leq j \leq m-1), \\
(c_{2m-3} + c_{2m-3}c_{2m-1})v & (j = m).
\end{cases}
\]

2) In the case \( n = 2m, \)

\[
2\tilde{h}_j v = \begin{cases} 
(1 - c_1)v & (j = 0), \\
(c_{2j-1} - 1 - c_{2j-1}c_{2j+1})v & (1 \leq j \leq m-1), \\
2c_{2m-1}v & (j = m).
\end{cases}
\]

From this proposition one obtains the following theorem, where \( \tilde{\Lambda}_j \)'s are the fundamental integral form corresponding to the simple coroots system \( \tilde{h}_j \).

**Theorem 4.2.** Let \( v := v(c_1, \cdots, c_n) \) where \( c_1, \cdots, c_n \in \{ \pm 1 \} \).

1) In the case \( n = 2m-1, \)
(i) \( v \) is a singular vector if and only if \( c_{2j-1} = 1 \) for \( j = 1, 2, \ldots, m-1 \). And then the weight of \( v \) is determined by \( c_{2m-1} \) as follows:

| singular vector | weight |
|-----------------|--------|
| \( v(1, c_2, 1, c_4, 1, c_6, \cdots, 1, c_{2m-2}, 1) \) | \( \tilde{\Lambda}_m \) |
| \( v(1, c_2, 1, c_4, 1, c_6, \cdots, 1, c_{2m-2}, -1) \) | \( \tilde{\Lambda}_{m-1} \) |

for any choice of \( c_2, c_4, \ldots, c_{2m-2} \in \{\pm 1\} \).

(ii) Given a singular vector

\[ v_0 := v(1, c_2, 1, c_4, 1, c_6, \cdots, 1, c_{2m-2}, c_{2m-1}) \]

the \( C \)-linear span of all elements

\[ v(b_1, c_2, b_3, c_4, b_5, c_6, \cdots, b_{2m-3}, c_{2m-2}, b_{2m-1}) \]

satisfying the conditions

(a) \( b_j \in \{\pm 1\} \) for all \( j \)

(b) \[ \prod_{1 \leq j \leq 2m-1, j = \text{odd}} b_j = c_{2m-1} \]

tensored with \( C[x^{(j)}_r; 1 \leq j \leq 2m-1, r \in \mathbb{N}_{\text{odd}}] \) is the irreducible \( A_{2m-1}^{(2)} \)-module with the highest weight vector \( v_0 \).

2) In the case \( n = 2m \),

(i) \( v \) is a singular vector if and only if \( c_{2j-1} = 1 \) for \( j = 1, 2, \ldots, m \). And then the weight of

\[ v(1, c_2, 1, c_4, 1, c_6, \cdots, 1, c_{2m-2}, 1, c_{2m}) \]

is \( \tilde{\Lambda}_m \) for any choice of \( c_2, c_4, \ldots, c_{2m} \in \{\pm 1\} \).

(ii) Given a singular vector

\[ v_0 := v(1, c_2, 1, c_4, 1, c_6, \cdots, 1, c_{2m-2}, 1, c_{2m}) \]

the \( C \)-linear span of all elements

\[ v(b_1, c_2, b_3, c_4, b_5, c_6, \cdots, b_{2m-3}, c_{2m-2}, b_{2m-1}, c_{2m}) \]

where \( b_j \in \{\pm 1\} \) for all \( j \), tensored with \( C[x^{(j)}_r; 1 \leq j \leq 2m, r \in \mathbb{N}_{\text{odd}}] \) is the irreducible \( A_{2m}^{(2)} \)-module with the highest weight vector \( v_0 \).
4.3 The case $E_n \ (n = 6, 7, 8)$

For $E_n$, we consider the following orientation of Dynkin diagrams:

\[ E_6 : \]
\[
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

\[ E_7 : \]
\[
\begin{array}{ccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}
\]

\[ E_8 : \]
\[
\begin{array}{ccccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\end{array}
\]

Then, by Lemma 4.1, one easily sees the following:

**Lemma 4.5.** Let $v = v(c_1, \cdots, c_n)$ for $E_n \ (n = 6, 7, 8)$ with the orientation of the Dynkin diagram as above. Then $\tilde{X}_{\alpha_j}v$ for $j = 1, \cdots, n$ are given as follows:

1) In the case $E_6$,

\[
2\tilde{X}_{\alpha_j}v = \begin{cases}
-ic_jv & (j = 2, 4, 6) \\
-ic_1v(c_1, -c_2, c_3, c_4, c_5) & (j = 1) \\
-ic_3v(c_1, -c_2, c_3, -c_4, c_5) & (j = 3) \\
-ic_5v(c_1, c_2, c_3, -c_4, c_5) & (j = 5).
\end{cases}
\]

2) In the case $E_7$,

\[
2\tilde{X}_{\alpha_j}v = \begin{cases}
-ic_jv & (j = 2, 4, 6, 7) \\
-ic_1v(c_1, -c_2, c_3, c_4, c_5, c_6) & (j = 1) \\
-ic_3v(c_1, -c_2, c_4, -c_4, c_5, c_6) & (j = 3) \\
-ic_5v(c_1, c_2, c_3, -c_4, c_5, -c_5) & (j = 5).
\end{cases}
\]
In the case $E_8$, 

$$2\tilde{X}_\alpha v = \begin{cases} 
-ic_jv & \left( j = 2, 4, 6, 8 \right) \\
-i_{c_1}v & \left( j = 1 \right) \\
-ic_3v & \left( j = 3 \right) \\
-ic_5v & \left( j = 5 \right) \\
-ic_7v & \left( j = 7 \right).
\end{cases}$$

From this lemma, one sees that, for example in the case $E_7$, $c_1, c_3, c_5$ and $c_4c_6c_7$ are unchanged under the action of $\tilde{X}_\alpha$'s. Since $\tilde{X}_\alpha$ is the action of the field $\tilde{X}_\alpha(z)$ to the $C\{Q/2Q\}$-component, one obtains the following:

**Theorem 4.3.**  
1) In the case $E_6$, for an arbitrary choice of $c_1, c_3, c_5 \in \{\pm 1\}$, the $C$-linear span of 

$$\left\{ v \left( \begin{array}{c} c_1, b_2, c_3, b_4, c_5 \\ b_6 \end{array} \right) ; b_2, b_4, b_6 \in \{\pm 1\} \right\}$$

tensored with $C[x_i^{(j)}; 1 \leq j \leq 6, r \in N_{\text{odd}}]$ is an irreducible $E_6^{(2)}$-module of level one.

2) In the case $E_7$, for an arbitrary choice of $c_1, c_3, c_5, c_7 \in \{\pm 1\}$, the $C$-linear span of 

$$\left\{ v \left( \begin{array}{c} c_1, b_2, c_3, b_4, c_5, b_6 \\ b_7 \end{array} \right) ; (i) \quad b_2, b_4, b_6, b_7 \in \{\pm 1\} \right\}$$

tensored with $C[x_i^{(j)}; 1 \leq j \leq 7, r \in N_{\text{odd}}]$ is an irreducible $E_7^{(1)}$-module of level one.

3) In the case $E_8$, for an arbitrary choice of $c_1, c_3, c_5, c_7 \in \{\pm 1\}$, the $C$-linear span of 

$$\left\{ v \left( \begin{array}{c} c_1, b_2, c_3, b_4, c_5, b_6, c_7 \\ b_8 \end{array} \right) ; b_2, b_4, b_6, b_8 \in \{\pm 1\} \right\}$$

tensored with $C[x_i^{(j)}; 1 \leq j \leq 8, r \in N_{\text{odd}}]$ is an irreducible $E_8^{(1)}$-module of level one.

The non-irreducibility of the representation space $C\{Q/2Q\} \otimes C[x_i^{(j)}]$ may suggest the existence of still bigger symmetry or the action of some group.

From the asymptotics of characters of integrable representations given in [3], one sees that the basic $E_8^{(1)}$-module decomposes into the sum of two fundamental
representations of $D_8^{(1)}$. Actually in the case 3) of the above theorem, the $C$-
linear space of
$$\left\{ v \left( c_1, b_2, c_3, b_4, c_5, b_6, c_7 \right) \right\};
\begin{align*}
& (i) \quad b_2, b_4, b_6, b_8 \in \{\pm 1\} \\
& (ii) \quad b_6 b_8 = c
\end{align*}$$
tensored with $C[x_{\ell}]$ is $D_8^{(1)}$-stable and $D_8^{(1)}$-irreducible for $c_1, c_3, c_5, c_7, c \in \{\pm 1\}$ with respect to the canonical inclusion of $D_8^{(1)}$ into $E_8^{(1)}$.

In concluding this note, we remark that the above construction gives rise to
the product expression of specialized character (cf. [4] §10.8) of fundamental representations with respect to a particular specialization. We consider an affine Lie algebra with the simple root system $\{\alpha_0, \ldots, \alpha_{\ell}\}$, following the enumeration of simple roots from §4.8 of [4]. Fix a number $s \in \{0, \ldots, \ell\}$, and consider the specialization
$$e^{-\alpha_j} \mapsto 1 \quad (j \neq s), \quad e^{-\alpha_s} \mapsto q,$$
which induces an algebra homomorphism
$$F_s : C[[e^{-\alpha_0}, \ldots, e^{-\alpha_\ell}]] \to C[[q]]$$
of the associative rings of formal power series. Then, from Theorems 4.1, 4.2 and 4.3, we obtain the following expression of the specialized character $F_s(e^{-\Lambda \text{ch}_\Lambda})$
for a level one dominant integral form $\Lambda$ and a specially chosen index $s$:

**Corollary 4.1.**

| affine algebra | $s$ (special index) | specialized character $F_s(e^{-\Lambda \text{ch}_\Lambda})$ |
|----------------|---------------------|--------------------------------------------------|
| $A_{2\ell-1}^{(2)}$ | $\ell$ | $2^{\ell-1} \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^{2^{\ell-1}}$ |
| $A_{2\ell}^{(2)}$ | $\ell$ | $2^\ell \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^2$ |
| $D_\ell^{(1)}$ (\ell: even) | $\ell/2$ | $2^{\ell/2-1} \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^\ell$ |
| $D_{\ell+1}^{(2)}$ (\ell: even) | $\ell/2$ | $2^{\ell/2} \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^{\ell+1}$ |
| $E_6^{(2)}$ | 4 | $2^3 \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^6$ |
| $E_7^{(1)}$ | 7 | $2^3 \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^7$ |
| $E_8^{(1)}$ | 7 | $2^4 \left( \frac{\varphi(q^2)}{\varphi(q)} \right)^8$ |

where $\Lambda$ is a dominant integral form of level one and $\varphi(q) := \prod_{n=1}^{\infty} (1 - q^n)$.  

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Actually this corollary holds because \( \deg(e^{\alpha}) = 0 \) (for all \( \alpha \in Q/2Q \)) and \( \deg(x^{(j)}) = r \) with respect to this specialization.

We remark that this result is, of course, in coincidence with the product expression of characters given in \( \S 2.3 \) of [11]:

\[
\begin{align*}
\text{ch}(\Lambda_0, A^{(2)}_{2\ell}) &= e^{\Lambda_0} \prod_{j=1}^{\infty} \prod_{\alpha \in \Delta_{\ell,+}} \left( 1 + e^{\pm \frac{j}{2} e^{-(j-\frac{1}{2})\delta}} \right), \\
\text{ch}(\Lambda_0, D^{(2)}_{\ell+1}) &= e^{\Lambda_0} \prod_{j=1}^{\infty} \prod_{\alpha \in \Delta_{\ell,+}} \prod_{\alpha \in \Delta_{\ell,+}} \left( 1 + e^{-j\delta} \right) \left( 1 + e^{-j\delta+\alpha} \right)(1 + e^{-(j-1)\delta-\alpha}),
\end{align*}
\]

where \( \Delta_{\ell,+} \) (resp. \( \Delta_{\ell,+} \)) is the set of all positive long (resp. short) roots of the finite-dimensional Lie algebra with the simple root system \( \{\alpha_1, \ldots, \alpha_\ell\} \). One easily sees that the specialization of these characters gives the same formulas with the above corollary for \( A^{(2)}_{2\ell} \) and \( D^{(2)}_{\ell+1} \), where \( e^{-\delta} = q^2 \) in our specialization since the coefficient of \( \alpha_s \) in the primitive imaginary root \( \delta \) is equal to 2.

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