VOLUME COLLAPSED THREE-MANIFOLDS WITH A LOWER CURVATURE BOUND

TAKASHI SHIOYA AND TAKAO YAMAGUCHI

Abstract. In this paper we determine the topology of three-dimensional closed orientable Riemannian manifolds with a uniform lower bound of sectional curvature whose volume is sufficiently small.

1. Introduction

As a continuation of our investigation [9] of collapsing three-manifolds with a lower curvature bound and an upper diameter bound, we study the topology of a three-dimensional closed Riemannian manifold with a lower curvature bound whose volume is sufficiently small, where we assume no upper diameter bound.

A closed three-manifold is called a graph manifold if it is a finite gluing of Seifert fibered spaces along their boundary tori.

Theorem 1.1. There exist small positive numbers $\epsilon_0$ and $\delta_0$ such that if a closed orientable three-manifold $M$ has a Riemannian metric with sectional curvature $K \geq -1$ and $\text{vol}(M) < \epsilon_0$, then one of the following holds:

1. $M$ is homeomorphic to a graph manifold;
2. $\text{diam}(M) < \delta_0$ and $M$ has finite fundamental group.

It was shown in [9] that in the case of (2) in Theorem 1.1, $M$ is homeomorphic to an Alexandrov space with nonnegative curvature.

Theorem 1.1 determines the possible topological type of $M$ if $M$ has not so small diameter. In fact, from [2], every three-dimensional graph manifold $M$ has a Riemannian metric $g_e$ with sectional curvature $|K_{g_e}| \leq 1$, $\text{diam}(M, g_e) \geq \delta_0$ and $\text{vol}(M, g_e) < \epsilon$ for each $\epsilon > 0$.

In the bounded curvature case, it follows essentially from [3] that if a closed three-manifold has a Riemannian metric with $|K| \leq 1$ whose volume is sufficiently small, then it is a graph manifold.

The strategy of our proof is as follows: We assume $M$ has large diameter which is the essential case. Applying our previous work [9], we obtain a local fiber structure on a neighborhood $B_p$ of each point.

2000 Mathematics Subject Classification. Primary 53C20, 53C23; Secondary 57N10, 57M99.

Key words and phrases. the Gromov-Hausdorff convergence, Alexandrov spaces, topology of three-manifolds, graph manifolds.
$p \in M$ over a metric ball $X_p$ in some Alexandrov space with curvature bounded below, where $\text{dim } X_p \in \{1, 2\}$. If $\text{dim } X_p = 2$, we have a local $S^1$-action on $B_p$. If $\text{dim } X_p = 1$, we have a (singular) sphere or torus bundle structure on $B_p$ over the closed interval $X_p$, and $B_p$ is homeomorphic to one of six compact three-manifolds, which will be called cylindrical if it is homeomorphic to either $S^2 \times I$ or $T^2 \times I$, or cylindrical with a cap if it is homeomorphic to $D^3$, $P^2 \tilde{\times} I$, $S^1 \times D^2$ or $K^2 \tilde{\times} I$, where $\tilde{\times}$ indicates the twisted product. Using those local data, we decompose $M$ into two parts as $M = U_1 \cup \hat{U}_2$, where $U_1$ is a closed domain which looks one-dimensional and $\hat{U}_2$ is one which looks two-dimensional, in the Gromov-Hausdorff sense. More precisely, $U_1$ is defined as the union of all $B_p$ with area of $X_p$ sufficiently small. Applying the critical point theory for distance functions, we conclude that each component of $U_1$ is either cylindrical or cylindrical with a cap. We shall construct a local $S^1$-action on the remaining piece $\hat{U}_2$, from which a graph manifold structure on $M$ is obtained. To do this, we need a gluing procedure, which is the main part of the present paper. To make the gluing procedure explicit and clear, we give quantitative descriptions of the local fibering $B_p \to X_p$ using the geometric properties of fibers in [10] and [11] over a regular part of $X_p$. Here the notion of strain radius comes in to control the behavior of the regular fibers. This forces us to obtain a sort of compactness of the set of regular parts of $X_p$’s with $B_p$ meeting $\hat{U}_2$. This is the reason why in the decomposition $M = U_1 \cup \hat{U}_2$ a neighborhood $B_p$ is included in the one-dimensional part $U_1$ even if $\text{dim } X_p = 2$ when $X_p$ has a small area.

The organization of this paper is as follows: In Section 2 we first establish a uniform lower bound on the strain radii of regular parts of $X_p$’s with $B_p$ meeting $\hat{U}_2$, and then provide some basic properties of Alexandrov surfaces to show that there are a lot of possibilities for the choices of the metric ball $X_p$ with boundary having nice geometric properties. In Section 3 we describe the geometry and topology of the local fibering $B_p \to X_p$ in detail. In Section 4 using these fiber structure we have the decomposition $M = U_1 \cup \hat{U}_2$ and determine the topology of $U_1$. In Section 5 we provide a preliminary gluing argument for the construction of local $S^1$-action on $\hat{U}_2$. The gluing procedure is completed in Section 6. In Section 7 we discuss a thick-thin decomposition of a closed orientable Riemannian three-manifold with a lower curvature bound.

An announcement in Perelman’s paper [7] has recently been come to our attention. He claims that if a three-manifold collapses under a local lower sectional curvature bound, then it is a graph manifold (Theorem 7.4). This result also follows from the argument in our Theorem 1.1 without the extra assumption (3) there, since our gluing argument in
Sections 5 and 6 is only local (see also Section 3). The authors do not know his proof of the statement above, up to now.

Acknowledgment. The authors would like to thank Grisha Perelman for correcting our claim in the first draft, on the relation between his Theorem 7.4 and our result, as above. The second author would like to thank John Morgan and Xiaochun Rong for the discussion at the American Institute of Mathematics, Palo Alto.

2. Strain radii and geometry of Alexandrov surfaces

We discuss some basic properties of strain radii and metric balls in Alexandrov surfaces with curvature bounded below. See [1] for general facts on Alexandrov spaces.

Let $X$ be an $m$-dimensional complete Alexandrov space with curvature bounded below, say, curvature $\geq -1$. For two points $x, y$ in $X$, a minimal geodesic joining $x$ to $y$ is denoted by $xy$. The angle between minimal geodesics $xy$ and $xz$ is denoted by $\angle yxz$. For a geodesic triangle $\Delta xyz$ in $X$ with vertices $x, y$ and $z$, we denote by $\hat{\Delta}xyz$ the corresponding angle at $\hat{y}$ of a comparison triangle $\Delta \hat{y}\hat{y}\hat{z}$ for $\Delta xyz$ in the hyperbolic plane of constant curvature $-1$.

For $\delta > 0$, the $\delta$-regular set $R_\delta(X)$ is defined as the set of points $p \in X$ such that there exists $m$ pairs of points, $(a_i, b_i)$, $1 \leq i \leq m$, called a $\delta$-strainer at $p$, such that

$$\hat{\angle}a_ipb_i > \pi - \delta, \quad \hat{\angle}a_ipa_j > \pi/2 - \delta,$$

$$\hat{\angle}b_ipb_j > \pi/2 - \delta, \quad \hat{\angle}a_ipb_j > \pi/2 - \delta,$$

for every $i \neq j$. The number $\min \{d(a_i, p), d(b_i, p) \mid 1 \leq i \leq m\}$ is called the length of the strainer. The $\delta$-strain radius at $p$, denoted by $\delta$-str. rad($p$), is defined as the supremum of such $r > 0$ that there exists a $\delta$-strainer at $p$ of length $r$. For a closed domain $D$ of $R_\delta(X)$, the $\delta$-strain radius of $D$, denoted by $\delta$-str. rad($D$), is defined as the infimum of $\delta$-str. rad($p$) when $p$ runs over $D$. It should be noted that the notion of strain radius is a natural generalization of that of injectivity radius for Riemannian manifolds.

For $1 \leq n \leq m$, an $(n, \delta)$-strainer at $p$ is defined by $n$ pairs of points, $\{(a_i, b_i)\}$, satisfying the same inequalities as above.

For a subset $C$ of $X$, we denote by $B(C, r)$ or $B(C, r; X)$ the closed metric $r$-ball around $C$ and by $S(C, r)$ or $S(C, r; M)$ the metric $r$-sphere around $C$. For $r < R$, $A(C; r, R)$ denotes the closure of $B(C, R) - B(C, r)$.

Lemma 2.1. For any $m, a > 0, d > 0, r > 0$ and $\delta > 0$, there exists a positive number $s = s_m(a, d, r, \delta)$ such that if $B$ is a metric ball in an $m$-dimensional complete Alexandrov space $X$ with curvature $\geq -1$ satisfying

$$\text{area}(B) \geq a, \quad \text{diam}(B) \leq d,$$

then $B$ contains a $\delta$-strainer with length $\leq s$.
then the closure $D$ of $B - B(S_\delta(X), r)$ has a definite lower bound for the strain radius:

\begin{equation}
\delta\text{-str. rad}(D) \geq s.
\end{equation}

**Proof.** Certainly we have a positive number $s_X$ depending on $X$ with $\delta\text{-str. rad}(D) \geq s_X$. Since the set of all isometry classes of $m$-dimensional compact Alexandrov spaces satisfying (2.1) is compact with respect to the Gromov-Hausdorff distance, this provides a uniform positive lower bound $s_m(a, d, r, \delta)$ for all $s_X$. □

Thus the domain $D$ has “bounded geometry” in the sense of (2.2). This elementary fact is important in our gluing argument in Section 5.

The complement $S_\delta(X) := X - R_\delta(X)$ is called the $\delta$-singular set. Setting $S_\delta(\text{int } X) := S_\delta(X) \cap \text{int } X$, we note that $S_\delta(X) = S_\delta(\text{int } X) \cup \partial X$. Let $ES(\text{int } X)$ denote the essential singular set of $\text{int } X$, i.e., the set of points $p \in \text{int } X$ with radius

$$\text{rad}(\Sigma_p) := \min_{\eta \in \Sigma_p} \max_{\xi \in \Sigma_p} d(\xi, \eta) \leq \pi/2.$$ 

From now on, we assume $m = 2$. Then $X$ is known to be a topological two-manifold possibly with boundary. Moreover $S_\delta(\text{int } X)$ is discrete for any $\delta > 0$.

**Lemma 2.2.** For any $p \in X$, $\delta > 0$ and $D > 0$, the number of elements of $S_\delta(\text{int } X) \cap B(p, D)$ has a uniform upper bound $\text{Const}(\delta, D)$.

In particular we have

$$\#(ES(\text{int } X) \cap B(p, D)) \leq \text{Const}(D).$$

This follows from an argument similar to Corollary 14.3 of [9], and hence the proof is omitted.

For every $p \in X$, $d_p$ denotes the distance function from $p$. $d_p$ is called regular at a point $q \neq p$ if there exists a $\xi \in \Sigma_q$ such that the directional derivatives of $d_p$ satisfies $d'_p(\xi) > 0$.

**Lemma 2.3 ([8]).** For a fixed $p \in X$, there exists a set $\mathcal{E} \subset (0, \infty)$ of measure zero such that for every $t \in (0, \infty) - \mathcal{E}$

(1) $t$ is a regular value of $d_p$;
(2) $B(p, t)$ is a topological manifold with (possibly empty) rectifiable boundary.

As a consequence of Lemmas 2.2 and 2.3 we have

**Corollary 2.4.** There exists a positive number $\sigma = \sigma(\delta)$ satisfying the following: For every Alexandrov surface $X$ as above with $\text{diam}(X) > 2$, for every $p \in X$ and for every $t \in [1/2, 1]$, there exists $\rho \in (t - 10^{-2}, t + 10^{-2})$ such that

(1) $B(p, \rho)$ is a topological manifold;
(2) $B(S(p, \rho), \sigma) \cap \text{int } X \subset R_\delta(X)$;
(3) $B(S(p, \rho), \sigma)$ is homeomorphic to $S(p, \rho) \times (0, 1)$. 

4
Proof. The existence of $\sigma$ satisfying (1) and (2) above is immediate from Lemmas 2.2 and 2.3. For (3), it suffices to prove that for every $R > 0$ there exists a positive constant $\text{const}(R)$ such that if $B(p, R)$ is a topological manifold, then the Euler number satisfies $\chi(B(p, R)) \geq -\text{const}(R)$. Suppose this does not hold. Then we have a sequence $(X_i, p_i)$ of pointed Alexandrov surfaces with curvature $\geq -1$ with uniformly bounded $R_i$ such that

1. $B(p_i, R_i)$ is a topological manifold;
2. $\chi(B(p_i, R_i)) \to -\infty$.

We may assume that $(X_i, p_i)$ converges to a pointed Alexandrov surface $(X, p)$. If $\dim X = 1$, it is not hard to see that $B(p_i, R_i)$ is either a cylinder or a Möbius band. If $\dim X = 2$, then take an $R$ with $R_i \leq R$ for every $i$ and choose a regular value $S$ of $d_p$ with $S > R$ such that $B(p, S)$ is a topological manifold. Then $B(p_i, S)$ is homeomorphic to $B(p, S)$ by the stability result (see [6]). This is a contradiction. □

In what follows, we let $\delta^* := \delta$, which is a sufficiently small positive number determined later on in (5.2). We also denote the constant $\sigma$ given in Corollary 2.4 by

\begin{equation}
\sigma^* := \sigma(\delta^*).
\end{equation}

3. Local structure

A local $S^1$-action $\psi$ on a three-manifold $M$ possibly with boundary consists of an open covering $\{U_\alpha\}$ of $M$ and a nontrivial $S^1$-action $\psi_\alpha$ on each $U_\alpha$ such that both the actions $\psi_\alpha$ and $\psi_\beta$ coincide up to orientation on the intersection $U_\alpha \cap U_\beta$. Let $X := M/S^1$, and $\pi : M \to X$ be the projection. $X$ is a topological two-manifold (see [5] for instance).

Set

\begin{equation}
\partial_0 X := \pi(\partial M), \quad \partial_* X := \overline{\partial X} - \partial_0 X.
\end{equation}

The fixed point set of $\psi$ coincides with $\partial_* X$.

Lemma 3.1. If a compact three-manifold $M$ admits a local $S^1$-action with no singular orbits on $\partial M$, then it is a graph manifold.

Proof. Note that each component $C$ of $\partial_* X$ is a circle. Take a small collar neighborhood $E(C)$ of $C$ in $X$. Then $N(C) := \pi^{-1}(E(C))$ is a solid torus. Setting

\[X_0 := X - \bigcup_C \text{int} N(C), \quad M_0 := \pi^{-1}(X_0),\]

we have the decomposition

\[M = M_0 \cup \left( \bigcup C N(C) \right),\]

where $C$ runs over all the components of $\partial_* X$. Since $M_0$ is a Seifert fibered space over $X_0$, $M$ is certainly a graph manifold. □
In what follows, let $M$ denote an orientable closed Riemannian manifold of dimension three satisfying

\begin{equation}
K \geq -1, \quad \text{vol}(M) < \epsilon. \tag{3.2}
\end{equation}

We shall determine the geometry and topology of local neighborhoods of $M$. First we recall the topological structure result for such an $M$ when it has uniformly bounded diameter.

**Theorem 3.2** (\cite{9}). For a given $D > 0$, there exists a positive constant $\epsilon(D) > 0$ satisfying the following: If $M$ satisfies $\text{diam}(M) \leq D$ and $\text{vol}(M) < \epsilon(D)$, then there exists a (possibly singular) fibration $f : M \to X$, where $X$ is a compact Alexandrov space with curvature $\geq -1$ and $\text{dim } X \leq 2$. The fiber structure of $M$ can be described in more detail as follows:

1. If $\text{dim } X = 2$, then $f$ is defined by a local $S^1$-action on $M$ with a possible exceptional orbit over a point in $\text{ES}(\text{int } X)$.
2. Let $\text{dim } X = 1$. If $X$ is a circle, then $M$ is either a sphere-bundle or a torus-bundle over $X$. If $X$ is a closed interval, then $M$ is a gluing of $U$ and $V$ along their boundaries, where $U$ and $V$ are ones of $D^3$ and $P^2 \times I$ or ones of $S^1 \times D^2$ and $K^2 \times I$.
3. If $\text{dim } X = 0$, then a finite cover of $M$ is homeomorphic to $S^1 \times S^2$, $T^3$, a nilmanifold or a simply connected Alexandrov space with nonnegative curvature.

The case when $X$ is a circle was proved in \cite{10}, and the essential part of the case of $\text{dim } X = 0$ was proved in \cite{4}.

By Lemma 3.1, a three-manifold $M$ satisfying one of the conclusions in Theorem 3.2 is a graph manifold except the case when $M$ has finite fundamental group and $\text{dim } X = 0$. We also obtain some universal positive constants $\delta_0$ and $\epsilon_0$ such that if $M$ satisfies $\text{diam}(M) < \delta_0$ and $\text{vol}(M) < \epsilon_0$, then $M$ is homeomorphic to one of the spaces in Theorem 3.2 (3). Thus Theorem 1.1 certainly holds in the bounded diameter case. Therefore from now we assume that $M$ has large diameter:

\begin{equation}
\text{diam}(M) \gg 1. \tag{3.3}
\end{equation}

We now determine the topology of a local neighborhood of each point of $M$.

A submersion $f : M \to N$ between Riemannian manifolds is called an $\epsilon$-almost Riemannian submersion if

1. the diameter of every fiber of $f$ is less than $\epsilon$;
2. for every point $p \in M$ and every tangent vector $\xi$ at $p$ that is normal to the fiber $f^{-1}(f(p))$,

\[ \left| \frac{|df(\xi)|}{|\xi|} - 1 \right| < \epsilon. \]
Note that an $\epsilon$-almost Riemannian submersion is a fiber bundle map since it is proper.

We denote by $\tau(\epsilon)$ (resp. $\tau(r|\epsilon)$) a function of $\epsilon$ (resp. of $r$ and $\epsilon$) with $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$ (resp. $\lim_{\epsilon \to 0} \tau(r|\epsilon) = 0$ for each fixed $r$).

The following is a quantitative version of Theorem 3.2 (2) (see [10]).

**Corollary 3.3.** There exists a positive number $\epsilon_1^*$ such that if a closed three-manifold $M$ with $K \geq -1$ satisfies $d_{GH}(M,I) < \epsilon_1^*$ for some closed interval $I$ of length $\geq 1/2$, then there exists a singular fibration $f : M \to I$ as in Theorem 3.2 such that

1. the diameter of every fiber of $f$ is less than $\tau(\epsilon_1^*)$;
2. the restriction of $f$ to $I_r$ is a $\tau(r|\epsilon_1^*)$-almost Riemannian submersion, where $r > 0$ and $I_r := \{ x \in I \mid d(x, \partial I) \geq r \}$.

Later we shall take $\epsilon_1^*$ such as $\epsilon_1^* \ll \sigma^*$ (see (4.5)). The final choice of $\epsilon_1^*$ will be determined at the end of Section 5.

Let $a^*$ be a positive number such that if $B$ is a metric $\rho$-ball with $1/10 \leq \rho \leq 1$ in a complete Alexandrov surface $X$ with curvature $\geq -1$ and $\text{area}(B) < a^*$, then

$$(3.4) \quad d_{GH}(B, I) < \epsilon_1^*/2,$$

for some closed interval $I$.

A surjective map $f : M \to X$ between Alexandrov spaces is called an $\epsilon$-almost Lipschitz submersion if

1. the diameter of every fiber of $f$ is less than $\epsilon$;
2. for every $p, q \in M$, if $\theta$ is the infimum of $\angle qpx$ when $x$ runs over $f^{-1}(f(p))$, then

$$\left| \frac{d(f(p), f(q)) - \sin \theta}{d(p, q)} \right| < \epsilon.$$

Remark that the notion of $\epsilon$-almost Lipschitz submersion is a generalization of $\epsilon$-almost Riemannian submersion. The following result was proved in Theorem 0.2 of [11] (see also Theorem 2.2 of [9]).

**Theorem 3.4 ([11]).** For given $m$ and $s > 0$ there exists $\nu > 0$ satisfying the following: Let $X$ be an $m$-dimensional complete Alexandrov space with curvature $\geq -1$ and with $\delta^*$-str. rad$(X) \geq s$. Then if the Gromov-Hausdorff distance between $X$ and a complete Riemannian manifold $M$ with $K \geq -1$ is less than $\nu$, then there exists a $(\tau(\delta^*) + \tau(s|\nu))$-almost Lipschitz submersion $f : M \to X$ which is a locally trivial bundle map.

The following is a localized and quantitative version of Theorem 3.2.

**Theorem 3.5.** For every $r > 0$, there exists a positive constant $\epsilon_0 = \epsilon_0(a^*, r, \delta^*)$ satisfying the following: For every $M$ satisfying (3.2) and (3.3) with $\epsilon \leq \epsilon_0$ and for every $p \in M$, there exist closed domains $B_p$
and \( \hat{B}_p \) around \( p \) and a pointed complete Alexandrov space \((X,x_0)\) with curvature \( \geq -1 \) and \( \dim X \in \{1,2\} \) such that

1. \( B_p \) and \( \hat{B}_p \) are small perturbations of metric balls around \( p \) and,
   \[ B(p,1/2) \subset B_p \subset \hat{B}_p \subset B(p,1), \quad \hat{B}_p - \text{int} B_p \simeq \partial B_p \times I; \]
2. \( B_p \) and \( \hat{B}_p \) have fiber structures over concentric metric balls \( X_p \subset \hat{X}_p \) in \( X \) around \( x_0 \);
3. \( B(x_0,1/2) \subset X_p \subset \hat{X}_p \subset B(x_0,1), \quad \overline{X_p - X_p} \simeq \partial_0 X_p \times I, \)
   where \( \partial_0 \) denotes the topological boundary.

Moreover the fiber structure on \( B_p \) in (2) can be described as follows. Let \( \pi_p : (\hat{B}_p,B_p) \to (\hat{X}_p,X_p) \) be the fiber projection, and let
\[ \partial_s X_p := \partial X_p - \partial_0 X_p. \]

be defined as in (3.1).

Case (A) \( \dim \hat{X}_p = 1 \). (\( \hat{X}_p \) is a closed interval \( I \) in this case).

(a) \( d_{GH}(\hat{B}_p,\hat{X}_p) < \epsilon^*_s \), and the diameter of every fiber of \( \pi_p \) is less than \( \tau(\epsilon^*_s) \);
(b) The restriction of \( \pi_p \) to \( I \) is a \( \tau(\epsilon^*_s) \)-almost Riemannian submersion;
(c) If \( \partial_s X_p \) is empty, then \( B_p \) is homeomorphic to either \( I \times S^2 \) or \( I \times T^2 \);
(d) If \( \partial_s X_p \) is nonempty, then \( B_p \) is homeomorphic to one of \( D^3 \), \( P^2 \times I \), \( S^1 \times D^2 \) and \( K^2 \times I \).

Case (B) \( \dim \hat{X}_p = 2 \).

(a) \( d_{GH}(\hat{B}_p,\hat{X}_p) < \tau(\epsilon) \), and the length of every fiber of \( \pi_p \) is less than \( \tau(\epsilon) \);
(b) \( B(B_p,\sigma^*) \subset \hat{B}_p \subset B(B_p,2\sigma^*) \);
(c) \( B(\partial X_p,2\sigma^*) \cap \text{int} X \subset R_{\delta^*}(X) \);
(d) \( B(\partial_0 X_p,2\sigma^*) \) is homeomorphic to \( \partial_0 X_p \times (0,1) \);
(e) \( D := \hat{X}_p - B(S_{\delta^*}(X),r) \) satisfies
   (i) \( \delta^*-\text{str. rad}(D) \geq s \), where \( s = s_2(\alpha^*,1,r,\delta^*) \) is the constant as in Lemma 2.2;
   (ii) the restriction of \( \pi_p \) to \( D \) is \( (\tau(\delta^*)+\tau(s|\epsilon)) \)-almost Lipschitz submersion which is an \( S^1 \)-bundle;
(f) \( \pi_p \) gives a local \( S^1 \)-action on \( \hat{B}_p \) whose fixed point set corresponds to \( \partial_s \hat{X}_p \), where there is a possible exceptional fiber over a point \( x \in \hat{X}_p \) only when \( x \in E\text{int} X \).

Proof. Suppose the theorem does not hold. Then there exist sequences \( \epsilon_i \to 0 \) and \( M_i \) satisfying (3.2) for \( \epsilon_i \) and (3.3) such that for some \( p_i \in M_i \), \( B(p_i,1) \) does not contain closed domains satisfying the above conclusion. Passing to a subsequence if necessary, we may assume that \( (M_i,p_i) \) converges to a pointed complete Alexandrov space \((X,x_0)\) with
curvature $\geq -1$. Observe $1 \leq \dim X \leq 2$. In view of Corollary it is possible to take metric balls $Y \subset \hat{Y}$ of $X$ around $x_0$ which are topological manifolds, satisfying

1. $B(x_0, 1/2) \subset Y \subset B(Y, \sigma^*) \subset \hat{Y} \subset B(Y, 2\sigma^*) \subset B(x_0, 1)$;
2. $B(\partial_0 Y, 2\sigma^*) \cap \text{int} X \subset R_{\ast}(X)$;
3. $B(\partial_0 Y, 2\sigma^*)$ is contained in a neighborhood of $\partial_0 Y$ homeomorphic to $\partial_0 Y \times (0, 1)$;
4. $\hat{Y} - Y \simeq \partial_0 Y \times I$.

If $\text{area}(\hat{Y}) < a^*$, then $(\hat{Y}, Y)$ are Gromov-Hausdorff close to some closed intervals $(I, I)$, and we put $\hat{X}_p := \hat{I}, X_p := I$ in this case. If $\text{area}(\hat{Y}) \geq a^*$, then we put $\hat{X}_p := \hat{Y}$ and $X_p := Y$. By Theorem and Corollary together with Lemma we obtain closed domains $B_i$ and $\hat{B}_i$ with $B(p_i, 1/2) \subset B_i \subset \hat{B}_i \subset B(p_i, 1)$ such that $B_i$ and $\hat{B}_i$ are fiber spaces over $Y$ and $\hat{Y}$ respectively as described above satisfying all the conclusions, which is a contradiction. 

\[ \square \]

**Remark 3.6.** (1) The several geometric properties of $B_p \subset \hat{B}_p$ and $X_p \subset \hat{X}_p$ in Theorem will be needed in the gluing argument later on.

(2) We will also need to consider a slight deformation of $X_p \subset \hat{X}_p$ according to requirements.

From now on, we put

\[ r := \sigma^*/100, \quad s := s_2(a^*, 1, r, \delta^*). \]

Then the constant $\epsilon_0 = \epsilon_0(a^*, r, \delta^*)$ in Theorem will become universal (see the end of Section 5).

We now recall basic geometric properties of the regular fibers of $\pi_p$.

**Definition 3.7.** For a point $p \in M$ suppose that there is a $(2, \delta^*/2)$-strainer $\{ (a_j, b_j) \}$ at $p$ of length $\geq s/2$. Then the subspace of the tangent space at $p$ generated by the directions of minimal geodesics joining $p$ to $a_1$ and $a_2$ is called a \textit{horizontal subspace} at $p$. Let a small circle $F$ in $M$ be given in such a way that for every $p \in F$ there is a $(2, \delta^*/2)$-strainer at $p$ of length $\geq s/2$. For $\tau > 0$, $F$ is called \textit{\tau-perpendicular to horizontal subspaces} if for each point $p \in F$, the angle $\theta$ between $F$ and every horizontal subspace at $p$ satisfies

\[ |\theta - \pi/2| < \tau. \]

**Lemma 3.8 ([10], [11].)** Let $\pi_p : \hat{B}_p \to \hat{X}_p$ and $D \subset \hat{X}_p$ (resp. $I_r \subset \hat{X}_p$) be as in Theorem. For every $x \in D$ (resp. $x \in I_r$) and $q \in \pi_{\hat{X}_p}^{-1}(x)$, the following holds:

1. For every $q'$ with $d(q, q') \geq r$, the angle $\theta$ between $\pi_{\hat{X}_p}^{-1}(x)$ and every minimal geodesic joining $q$ to $q'$ satisfies

\[ |\theta - \pi/2| < \tau(\delta^*) + \tau(s|\epsilon|) \quad (\text{resp. } \tau(r|\epsilon_1^r)). \]
(2) The fiber $\pi_p^{-1}(x)$ is $(\tau(\delta^*) + \tau(s|e))$-perpendicular to horizontal subspaces.

4. Decomposition

Let $M$ satisfy (3.2) with $\epsilon \leq \epsilon_0$. Take points $p_1, p_2, \ldots$, of $M$ such that the collections $\{B_{p_i}\}$ and $\{\hat{B}_{p_i}\}$ given by Theorem 3.5 are finite coverings of $M$. We may assume that

\begin{align}
&d(p_i, p_j) \geq 1/10 \quad \text{for every } i \neq j; \\
&\{B_{p_i}(1/10)\} \text{ covers } M,
\end{align}

where $B_{p_i}(1/10) := \{x \in B_{p_i} | d(x, \partial B_{p_i}) \geq 1/10\}$. By the Bishop-Gromov volume comparison theorem, we may assume that the maximal number of $\hat{B}_{p_i}$’s having nonempty intersection is uniformly bounded above by a universal constant $Q$ not depending on $M$. Let $X_{p_i} \subset \hat{X}_{p_i}$ be chosen as in Theorem 3.5 for $B_{p_i} \subset \hat{B}_{p_i}$.

For simplicity, we put

$$B_i := B_{p_i}, \quad \hat{B}_i := \hat{B}_{p_i}, \quad X_i := X_{p_i}, \quad \hat{X}_i := \hat{X}_{p_i}.$$ 

If $\dim X_i = 2$, then there exists a local $S^1$-action $\psi_i$ on $B_i$ such that $B_i/\psi_i \simeq X_i$, where $\pi_i := \pi_{p_i} : \hat{B}_i \to \hat{X}_i$ is the projection.

For each $j \in \{1, 2\}$, let $I_j$ denote the set of all $i$ with $\dim X_i = j$, and consider

$$U_j := \bigcup_{i \in I_j} B_i.$$ 

Let $\mathcal{B}_j := \{B_i | \dim X_i = j\}, \ j \in \{1, 2\}$. By Theorem 3.5, each element of $\mathcal{B}_1$ is either cylindrical or cylindrical with a cap (see Introduction).

**Lemma 4.1.** Each component of $U_1$ is homeomorphic to one of $D^3$, $P^2 \times I$, $S^2 \times I$, $S^1 \times D^2$, $K^2 \times I$ and $T^2 \times I$ unless $U_1 = M$.

If $U_1 = M$, then $M$ is homeomorphic to one of the spaces in Theorem 3.5 (2).

**Proof.** Slightly enlarging closed domains $B_i$ in $\mathcal{B}_1$ if necessary, we may assume that any two $\hat{B}_i, \hat{B}_j$ in $\mathcal{B}_1$ has intersection $\hat{B}_i \cap \hat{B}_j$ which is either empty or else having diameter $> \sigma^*$. Since $U_1$ is a part of $M$ which looks one-dimensional in the Gromov-Hausdorff sense, it follows from (1.1) that $U_1$ is a manifold. Suppose that $\hat{B}_i \cap \hat{B}_j$ is nonempty for two domains $\hat{B}_i, \hat{B}_j$ in $\mathcal{B}_1$. In the argument below, we may assume that $\hat{B}_i \not\subset \hat{B}_j$ and $\hat{B}_j \not\subset \hat{B}_i$. Since $\hat{B}_i$ and $\hat{B}_j$ are either cylindrical or cylindrical with a cap, it follows from (3.3) that at least one of $\hat{B}_i$ and $\hat{B}_j$, say $\hat{B}_j$, has disconnected boundary. Consider the distance function $d_{p_i}$, where $p_i$ is the reference point of $\hat{B}_i$. Letting $F$ denote $S^2$ or $T^2$, we know that $\hat{B}_i$ and $\hat{B}_j$ have $F$-fiber structures over $I$, which is singular at the top of the cap. By Lemma 3.8 one can construct a gradient-like vector field $V_i$ for $d_{p_i}$ on a neighborhood of $\overline{\hat{B}_j - \hat{B}_i}$ whose flow...
curves are transversal to every fiber of $B_j$ lying on a neighborhood of $B_j - B_i$. In view of (3.3), it follows that $B_i \cup B_j$ is homomorphic to $B_i$. Repeating the argument finitely many times, we obtain the conclusion of the lemma. □

From now on, we assume $U_1 \neq M$, and consider the decomposition of $M$

$$M = U_1 \cup \hat{U}_2,$$

where $\hat{U}_2$ denotes the closure of $M - U_1$.

For every fixed component $L$ of $\partial U_1$, there exists a unique $B_L \in \mathcal{B}_1$ such that a component of $\partial B_L$ coincides with $L$. Let $B^L_1, \ldots, B^L_n$ denote the set of all elements of $\mathcal{B}_2$ such that $B^L_i(1/10)$ meets $L$, $1 \leq i \leq n$. Since $\text{diam}(L) < \tau(\epsilon^*_i)$, $B^L_i$ contains $L$. Let $L_i$ be the unique component of $\partial B^L_i$ meeting $B_L$. Since the domain bounded by $L$ and $L_i$ is one-dimensional in the Gromov-Hausdorff sense, it follows from the curvature condition that $B^L_1, \ldots, B^L_n$ lie in a linear order and for every $1 \leq i \neq j \leq n$

$$d(\partial B^L_i, \partial B^L_j) \geq 1/20,$$

and we may assume that

$$B^L_n \supset L_i.$$

We denote by $\hat{U}_2$ the union of all $B_j$ in $\mathcal{B}_2$ which does not intersect $U_1$. Obviously we have

$$U_2 = \hat{U}_2 \cup \left( \bigcup_L (B^L_1 \cup \cdots \cup B^L_n) \right),$$

where $L$ runs over all the components of $\partial U_1$.

**Lemma 4.2.** $U_2$ is homeomorphic to $\hat{U}_2$.

*Proof.* Fix a component $L$ of $\partial U_1$ again. Let $\tilde{B}^L_n \supset B^L_n, \tilde{X}^L_n \supset X^L_n$ and $\pi^L_n : (\tilde{B}^L_n, \tilde{X}^L_n) \to (X^L_n, X^L_n)$ be the orbit projection as in Theorem 3.5. Let $L_n$ denote the component of $\partial \tilde{B}^L_n$ corresponding to $L_n$, and let $U$ be the domain bounded by $L$ and $L_n$. Take a point $x \in \tilde{U}_2$ with $d(x, L) \geq 1$ and a point $y \in \tilde{L}_n$. Since every fiber of $\pi^L_n$ meeting $U$ has diameter $< \tau(\epsilon^*_i)$, it follows that for every $z \in U$

$$\angle xyz > \pi - \tau(\sigma^*|\epsilon^*_i).$$

Let $V$ be a gradient-like vector field for $d_x$ defined on a neighborhood of $U$.

**Assertion 4.3.** The flow curves of $V$ are transversal to both $L$ and $L_n$.

*Proof.* Since the transversality to $L$ is immediate from Lemma 3.8 (1), it suffices to check the transversality to $L_n$. For every $p \in L_n$, let $\phi_p(t)$ be the flow curve of $V$ with $\phi_p(0) = p$. Put $q := \exp_p \sigma^*V(p)$,
\[ \bar{p} := \pi_n^L(p) \text{ and denote by } \bar{\xi} \text{ the direction at } \bar{p} \text{ defined by a minimal geodesic to } \pi_n^L(q). \text{ In a way similar to Lemma 4.6 of [11], we have } \\
\quad \quad \quad \quad d(\pi_n^L(\phi_p(t)), \exp_{\bar{p}} t\bar{\xi}) < t(\tau(\delta^*) + \tau(s|\epsilon)), \]
for every sufficiently small \( t > 0. \) This implies that \( \phi_p(t) \) makes an angle with \( L_n \) uniformly bounded away from zero. \( \square \)

Now in view of (4.4), Assertion 4.3 implies \( U_2 \simeq \hat{U}_2. \) \( \square \)

The proof of the following lemma is deferred to Sections 5 and 6.

**Lemma 4.4.** There exists a local \( S^1 \)-action defined on \( U_2, \) and hence on \( \hat{U}_2. \)

**Proof of Theorem 1.1 assuming Lemma 4.4.** Note that each component of \( \partial \hat{U}_2 \) is homeomorphic to \( S^2 \) or \( T^2. \) For each component \( L \) of \( \partial \hat{U}_2, \) let \( W(L) \) be the component of \( U_1 \) containing \( L. \) Suppose first that \( L \) is homeomorphic to \( S^2. \) Then one of the following holds:

1. \( \partial W(L) \) is connected and \( W(L) \) is homeomorphic to either \( D^3 \) or \( P^2 \times I; \)
2. \( W(L) \) is homeomorphic to \( S^2 \times I, \) and the other component of \( \partial W(L) \) is another component of \( \hat{U}_2. \)

Now consider the union
\[ V := \hat{U}_2 \cup \left( \bigcup_L W(L) \right), \]
where \( L \) runs over all the components of \( \hat{U}_2 \) homeomorphic to \( S^2. \) Since an \( S^1 \)-action on \( S^2 \) is essentially by rotation, the local \( S^1 \) action on \( \hat{U}_2 \) extends to a local \( S^1 \)-action on \( V \) such that the orbit space \( W(L)/S^1 \) is a disk whose singular locus is one of

1. an interval on the boundary of \( W(L)/S^1 \) (the case of \( W(L) \simeq D^3) ; \)
2. the union of an interval on the boundary of \( W(L)/S^1 \) and a point in \( \text{int } W(L)/S^1 \) of type \( (2, 1) \)-singularity (the case of \( W(L) \simeq P^2 \times I; \)
3. the disjoint union of two intervals on the boundary of \( W(L)/S^1 \) (the case of \( W(L) \simeq S^2 \times I). \)

Note also that each component of \( \partial V \) is homeomorphic to \( T^2 \) and having no singular orbits. Therefore Lemma 3.1 implies that \( V \) is a graph manifold. From construction, for each component \( L \) of \( \partial V, \) one of the following holds:

a) \( \partial W(L) \) is connected and \( W(L) \) is homeomorphic to either \( S^1 \times D^2 \) or \( K^2 \times I; \)
b) \( W(L) \) is homeomorphic to \( T^2 \times I, \) and the other component of \( \partial W(L) \) is another component of \( \partial V. \)

Thus \( M \) is a graph manifold. \( \square \)
5. Gluing

Let $B \subset \hat{B}$ be closed domains in a closed orientable three-manifold $M$ with sectional curvature $K \geq -1$, and let $X \subset \hat{X}$ be concentric closed metric balls of radii $t < \hat{t}$ in a two-dimensional complete Alexandrov space $Z$ with curvature $\geq -1$. Assume that

1. the Gromov-Hausdorff distance between $\hat{B}$ and $\hat{X}$ (resp. $B$ and $X$) is sufficiently small;
2. $X \subset \hat{X}$ satisfy the conclusion of Case (B) in Theorem 5.5;
3. area$(X) \geq a^*$;
4. $1/10 \leq t \leq t + s \leq \hat{t} \leq 1$,

where $s$ is as in (3.3). Let $D := X - B(S_{\delta^*}(Z), r)$, $\hat{D} := \hat{X} - B(S_{\delta^*}(Z), r)$. Note that $\delta^*$-str. rad$(\hat{D}) \geq s$. Applying Theorem 3.4 we have closed domains $\hat{N}$ and $N$ of $\hat{B}$ and $B$ respectively, and an almost Lipschitz submersion $\pi : (\hat{N}, N) \to (\hat{D}, D)$, which is an $S^1$-bundle.

First we need to establish the uniform boundedness of length ratio for the fibers of $\pi : N \to D$.

**Lemma 5.1.** There exists a $\zeta = \zeta(a^*, s^*, \delta^*) > 0$ such that the following holds: Suppose that

1. for every $x \in \hat{D}$ the length $\ell(x)$ of the fiber $\pi^{-1}(x)$ is less than $\zeta$;
2. for every $x \in D$ and $p \in \pi^{-1}(x)$, letting $\theta(p)$ denote the angle between $\pi^{-1}(x)$ and a horizontal subspace at $p$, we have $|\theta(p) - \pi/2| < \zeta$.

Then

$$c^{-1} < \frac{\ell(x)}{\ell(y)} < c,$$

for every $x, y \in D$, where $c = c(a^*, s)$ is a uniform positive constant.

**Proof.** Suppose the lemma does not hold. Then we have a sequence $\pi_i : (\hat{N}_i, N_i) \to (\hat{D}_i, D_i)$ of $S^1$-bundles satisfying the assumptions of the lemma for $\zeta_i$ with $\lim \zeta_i = 0$ such that $\frac{\ell(x_i)}{\ell(y_i)} \to \infty$, where $\pi_i^{-1}(x_i)$ (resp. $\pi_i^{-1}(x_i)$) has the maximal (resp. minimal) length among all the fibers of $\pi_i$ over $D_i$. Note that

$$N_i \approx \begin{cases} D_i \times S^1 & \text{if } D_i \text{ is orientable}, \\ D_i \times S^1 & \text{if } D_i \text{ is non-orientable}. \end{cases}$$

Take a finite covering $(\hat{E}_i, E_i) \to (\hat{N}_i, N_i)$ along fibers such that the length $\hat{\ell}(x_i)$ of $\hat{\pi}_i^{-1}(x_i)$ satisfies $1 < \hat{\ell}(x_i) < 2$, where $\hat{\pi}_i : \hat{E}_i \to \hat{D}_i$ is the natural projection. We may assume that $\hat{\pi}_i : (\hat{E}_i, E_i) \to (\hat{D}_i, D_i)$ converges to a Lipschitz map $\hat{\pi} : (E, E) \to (\hat{D}, D)$. Note that $D \subset \hat{D}$ are closed domain in some complete Alexandrov surface with curvature $\geq -1$ satisfying $\delta^*$-str. rad$(\hat{D}) \geq s$. Let $x$ be the limit of $x_i$, and
We also assume that \( \hat{F} = \pi^{-1}(x) \). Choose a \( \delta^* \)-strainer \( \{(a_j, b_j)\}_{j=1,2} \) at \( x \) of length \( s \). For every \( p \in F \), one can take \( q_j \in \pi^{-1}(a_j) \) and \( r_j \in \pi^{-1}(b_j) \) such that \( \{(q_j, r_j)\} \) is a strainer at \( p \) and that \( pq_j \) and \( pr_j \) are almost perpendicular to \( F \). Since \( F \) has a positive diameter, this implies \( \dim \hat{E} = 3 \). Thus \( \hat{E}_i \) does not collapse. Since \( E \subseteq R_{3\delta^*}(\hat{E}) \), it follows from Theorem 3.41 that \( E_i \) is almost isometric to \( E \) for large \( i \) in the sense that there is a bi-Lipschitz homeomorphism \( f_i : E_i \to E \) such that the Lipschitz constants of \( f_i \) and \( f_i^{-1} \) are close to one. Note that the length of shortest nonzero homotopic loops in \( E \) has a definite positive lower bound, while the length of \( \hat{\pi}_i^{-1}(y_i) \) converges to zero. This is a contradiction. \( \Box \)

Let \( \pi : (\hat{B}, B) \to (\hat{X}, X) \), \( (\hat{D}, D) \subseteq (\hat{B}, B) \), \( \pi : (\hat{N}, N) \to (\hat{D}, D) \) and \( \hat{B} \subseteq Z \) be as in Lemma 5.1. Let \( N_1 \) be the set of points \( p \) of \( B \) such that there is a \( (2, 2\delta^*) \)-strainer of length \( \geq s/2 \). Note \( N \subseteq N_1 \).

**Lemma 5.2.** Every fiber \( F \) of \( \pi \) contained in \( N_1 \) is \((\tau(2\delta^*) + \tau(s/2|\epsilon|))\)-perpendicular to horizontal subspaces.

**Proof.** It is obvious that \( \pi(N_1) \) is contained in \( R_{3\delta^*}(Z) \) and having \( 3\delta^* \)-strainer of length \( \geq s/3 \). Thus Theorem 3.41 together with Lemma 3.38 (2) yields the conclusion. \( \Box \)

Next we consider a gluing situation. Let \( B \) and \( B' \subseteq \hat{B}' \) be closed domains in \( M \), and let \( \pi : B \to X \) and \( \pi' : (\hat{B}', B') \to (\hat{X}', X') \) be the orbit maps of local \( S^1 \)-actions, where we assume that \( X \) is only a topological two-manifold. Here we consider \( \pi : B \to X \) as a result of gluing of several local \( S^1 \)-actions \( \{\pi_i : B_i \to X_i\} \). Note those \( S^1 \)-actions are neither isometric nor our gluing will be through isometric actions. This is the reason why we assume \( X \) to be only a topological two-manifold. On the other hand, we assume \( (\hat{X}', X'), (\hat{D}', D') \) and \( \pi' : (\hat{N}', N') \to (\hat{D}', D') \) are as in Lemma 5.1, where \( D' := X' - B(S_{3\delta^*}(Z'), r) \), \( \hat{D}' := \hat{X}' - B(\hat{S}_{3\delta^*}(Z'), r) \), and \( Z' \) is a two-dimensional complete Alexandrov space with curvature \( \geq -1 \) containing \( (\hat{X}', X') \)

Let \( N_1 \subseteq B \), \( N_1' \subseteq B' \) be defined as in Lemma 5.2 and suppose \( B \cap B' \) is nonempty.

In the sequel, for a closed domain \( A \) of \( M \) we denote by \( A \) a small perturbation of \( A \).

**Lemma 5.3.** For a given positive number \( \nu \) there exist \( \delta^* = \delta^*(\nu) > 0 \) and \( \zeta = \zeta(\nu) > 0 \) such that the following holds: Let \( \pi : B \to X \), \( \pi' : (\hat{B}', B') \to (\hat{X}', X') \) be as above satisfying

(a) every fiber \( F \) of \( \pi \) contained in \( N_1 \) and \( F' \) of \( \pi' \) contained in \( N' \) are \( \zeta \)-perpendicular to horizontal subspaces;

(b) any two orbits of \( \pi \) in \( N \) with distance \( \leq 1 \) have length ratio uniformly bounded as in Lemma 5.1.

We also assume that

\[
(5.1) \quad \pi'(\partial B' \cap B) \subseteq D'.
\]
Then we have a local \( S^1 \)-action \( \psi'' \) on a small perturbation \( B \cup B' \) of \( B \cup B' \) and a topological two-manifold \( X'' \) with the orbit projection \( \pi'': B \cup B' \to X'' \) satisfying

1. \( B \cup B' \) is a manifold with boundary, and
   \[
   (B \cup B')(10\ell') \subset B \cup B' \subset B(B \cup B', 10\ell'),
   \]
   where \( \ell' \) denotes the maximal length of fibers of \( \pi' \) meeting \( \partial B' \cap B \), and \( (B \cup B')(10\ell') = \{ x \in B \cup B' \mid d(x, \partial(B \cup B')) \geq 10\ell' \} \);
2. \( \psi'' = \begin{cases} 
\psi & \text{on } B \cup B' - B(B', 10\ell') \\
\psi' & \text{on } B';
\end{cases} \)
3. each orbit of \( \psi'' \) has length < 2\( \ell'' \), where \( \ell'' \) denotes the maximal length of all fibers of \( \pi \) and \( \pi' \) intersecting \( 10\ell'^{-}\)-neighborhood of \( \partial B' \cap B \);
4. every fiber of \( \pi'' \) contained in \( N_1'' \) is \( \nu \)-perpendicular to horizontal subspaces, where \( N_1'' \) is the set of points \( p \) of \( B \cup B' \) such that there is a \((2,2\delta^*)\)-strainer at \( p \) of length \( \geq s/2 \).

Remark 5.4. Under the situation of Lemma 5.3 if both \((\tilde{B}, B)\) and \((\tilde{B}', B')\) are as in Lemma 5.1 then by Lemmas 5.1 and 5.2 \( B \to X \) and \( \pi': (\tilde{B}', B') \to (X', X') \) satisfy the assumptions of Lemma 5.3 except (5.1) if \( \tau(2\delta^*) + \tau(s/2|\epsilon) < \zeta \) which is realized by \( \delta^* \ll 1 \) and \( \epsilon \ll \delta^* \). Note that the proof of Lemma 5.1 goes through for \( N_1 \) as well in place of \( N \).

For the proof of Lemma 5.3 we need a sublemma.

Let \( A \) be a small neighborhood of \( \pi'(\partial B' \cap B) \) in \( \pi'(\partial B') \), and let \( C \) be the closure of the intersection of \( \text{int } X \) with the boundary of the \( 10\ell'^{-}\)-neighborhood of \( \pi(B \cap B') \). Slightly perturbing \( A \) and \( C \) if necessary, we may assume that both are one-manifolds.

Fix any \( x, \hat{x} \in A \) with \( 10\ell' \leq d(x, \hat{x}) \leq 20\ell' \). Taking a nearest point \( z \) of \( \pi'((\pi^{-1}(C)) \) from \( x \), choose any point \( y \in \pi((\pi')^{-1}(z)) \). Similarly we choose \( \hat{y} \in C \) for \( \hat{x} \). Put \( F := (\pi')^{-1}(x), F' := (\pi')^{-1}(\hat{x}), F := (\pi)^{-1}(y) \) and \( \hat{F} := (\pi)^{-1}(\hat{y}) \).

Sublemma 5.5. Under the situation above, there exist \( \delta^* = \delta^*(\nu) > 0 \) and \( \zeta = \zeta(\nu) > 0 \) satisfying the following:
1. \[
\left| \frac{\ell(F)}{\ell(F')} - 1 \right| < \nu,
\]
   where \( \ell(F) \) denotes the length of \( F \).
2. There exists an annulus \( E \) (resp. \( \hat{E} \)) in \( M \) equipped with an \( S^1 \)-fiber structure via a bi-Lipschitz homeomorphism \( h: [0,1] \times S^1 \to E \) (resp. \( \hat{h}: [0,1] \times S^1 \to \hat{E} \)) such that
(a) \( F = h(0 \times S^1) \) and \( F' = h(1 \times S^1) \) (resp. \( \hat{F} = \hat{h}(0 \times S^1) \) and \( \hat{F}' = \hat{h}(1 \times S^1) \));
(b) for each \( t \in [0, 1] \), \( h(t \times S^1) \) (resp. \( \hat{h}(t \times S^1) \)) is \( \nu \)-perpendicular to horizontal subspaces.

(3) Let \([x, \hat{x}]\) and \([y, \hat{y}]\) be the subarcs of \( A \) and \( C \) respectively, and let \( T \) be the union of \((\pi')^{-1}([x, \hat{x}])\), \( \pi^{-1}([y, \hat{y}]) \), \( E \) and \( \hat{E} \). Then the domain \( D \) bounded by \( T \) has an \( S^1 \)-fiber structure via a bi-Lipschitz homeomorphism \( k: D^2 \times S^1 \to D \) such that
- (a) for each \( x \in \partial D^2 \), \( k(x \times S^1) \) coincides with a fiber on \( T \);
- (b) for each \( x \in D^2 \), \( k(x \times S^1) \) is \( \nu \)-perpendicular to horizontal subspaces.

\[ B B' \]

\[ \pi^{-1}(C) \]

\[ B' \]

\[ \pi'(A) \]

\[ F \]

\[ E \]

\[ F' \]

\[ \hat{F} \]

\[ \hat{F}' \]

\[ C \]

\[ A \]

\[ X \]

\[ X' \]

\[ y \]

\[ y' \]

\[ x \]

\[ x' \]

\[ \hat{y} \]

\[ \hat{y}' \]

\[ \hat{x} \]

\[ \hat{x}' \]

\[ \text{Figure 1.} \]

**Proof.** We prove it by contradiction. If the conclusion does not hold, we would have a sequence of closed three-manifolds \( M_i \) with \( K \geq -1 \) for which there are \( \pi_i: \hat{B}_i \to X_i \), \( \pi'_i: (\hat{B}'_i, B'_i) \to (\hat{X}'_i, X'_i) \) satisfying the assumptions of Sublemma 5.3 for \( \delta_i \to 0 \) and \( \zeta_i \to 0 \), but not
satisfying the conclusions for $\zeta$. Let $A_i \subset \partial X_i'$, $C_i \subset X_i$, $x_i, \hat{x}_i \in A_i$ and $y_i, \hat{y}_i \in C_i$ be defined as above. In particular, $10\ell'_i \leq d(x_i, \hat{x}_i) \leq 20\ell'_i$, where $\ell'_i$ is defined in a way similar to $\ell'$ (see Lemma 5.3 (1)). Put $F'_i := (\pi'_i)^{-1}(x_i)$, $\hat{F}'_i := (\pi'_i)^{-1}(\hat{x}_i)$, $F_i := (\pi_i)^{-1}(y_i)$ and $\hat{F}_i := (\pi_i)^{-1}(\hat{y}_i)$. Let $\ell_i(y_i)$ and $\ell'_i(x_i)$ denote the length of $F_i$ and $F'_i$ respectively. Passing to a subsequence, we may assume that $\frac{1}{\ell_i(x_i)} M_i \rightarrow x_i$ converges to a pointed space $(W, w_0)$, where $W$ is a complete Alexandrov space with nonnegative curvature. From assumption, we see that $W$ is actually isometric to $\mathbb{R}^2 \times S^1_1$, where $S^1_1$ denotes the circle of length 1. Thus for any fixed $R \gg 1$, $B(x_i, R; \frac{1}{\ell_i(x_i)} M_i)$ is almost isometric to $B(w_0, R)$. This together with the condition (a) of Lemma 5.3 implies that $\ell'_i(x_i)$ and the length of any orbit of $\pi_i$ nearby $(\pi'_i)^{-1}(x_i)$ are comparable in the sense of Lemma 5.1. Then by (5.3), $\ell'_i(x_i)$ and $\ell_i(y_i)$ are comparable.

The above convergence then yields $\frac{\ell_i(y_i)}{\ell'_i(x_i)} \rightarrow 1$, which proves (1).

Let $F, \hat{F}, F'$ and $\hat{F}'$ be the limits of $F_i$, $\hat{F}_i$, $F'_i$ and $\hat{F}'_i$ respectively under the above convergence. Since $F$ and $F'$ (resp. $\hat{F}$ and $\hat{F}'$) can be joined by one-parameter family of parallel circles, say $E$ (resp. say $\hat{E}$) of length 1, $F_i$ and $F'_i$ can be joined by one-parameter family, say $E_i$ (resp. say $\hat{E}_i$) of circles each of which is $\nu$-perpendicular to horizontal subspaces for sufficiently large $i$. Let $\varphi_i : B(w_0, R) \rightarrow B(x_i, \hat{x}_i)$ be an almost isometry. Note that the closed domain $D_i$ bounded by $\pi_i^{-1}([y_i, \hat{y}_i]), (\pi_i)^{-1}([x_i, \hat{x}_i])$ and $E_i$ and $\hat{E}_i$ is mapped via $\varphi_i^{-1}$ onto a domain $D$ bounded by $E, \hat{E}, \pi^{-1}([x, \hat{x}])$ and $\pi^{-1}([y, \hat{y}])$. Note that $\varphi_i$ maps horizontal subspaces to horizontal subspaces (see [10] for the details). Since $D$ is isometric to a product $H \times S^1_1$ for a rectangle $H$, this gives a compatible $S^1$-fiber structure on $D_i$, each of whose fibers is $\nu$-perpendicular to horizontal subspaces. This is a contradiction. $\square$

**Proof of Lemma 5.3.** We shall carry out the required gluing procedure on each component, say $U$, of $B \cap B'$. Let $A_0$ be any component of $A \cap \pi(U)$, and take consecutive points $x_1, \ldots, x_N$ of $A_0$ with $10\ell' \leq d(x_{\alpha}, x_{\alpha+1}) \leq 20\ell' \rightarrow \delta$ for each $1 \leq \alpha \leq N - 1$.

First consider

Case (A) $\partial B$ does not meet $\partial B'$ on $U$.

In this case, both $A_0$ and the component $C_0$ of $C$ corresponding to $A_0$ are circles. Applying Sublemma 5.3 to $x := x_{\alpha}$ and $\hat{x} := x_{\alpha+1}$, we obtain a closed domain $D$ bounded by $\pi^{-1}(C_0)$ and $(\pi')^{-1}(A_0)$ having an $S^1$-bundle structure via a bi-Lipschitz homeomorphism $k : (I \times S^1) \times S^1 \rightarrow D$ such that

1. for each $x \in \partial I \times S^1$, $k(x \times S^1)$ coincides with a fiber on $\pi^{-1}(C_0) \cup (\pi')^{-1}(A_0)$;
2. for each $x \in I \times S^1$, $k(x \times S^1)$ is $\nu$-perpendicular to horizontal subspaces.
Figure 2. Case (A)

**Case (B)** \( \partial B \) meets \( \partial B' \) on \( U \).

In this case \( A_0 \) is an arc. In a way similar to Case (A), we apply Sub-lemma 5.5 to obtain a closed domain \( D \) bounded by \( \pi^{-1}(C_0) \), \( (\pi')^{-1}(A_0) \) and two annuli joining \( (\pi')^{-1}(\partial A_0) \) and \( (\pi')^{-1}(\partial C_0) \) which has an \( S^1 \)-bundle structure via a bi-Lipschitz homeomorphism \( k : I^2 \times S^1 \to D \) such that

1. for each \( x \in \partial I^2 \), \( k(x \times S^1) \) coincides with a fiber on \( \partial D \);
2. for each \( x \in I^2 \), \( k(x \times S^1) \) is \( \nu \)-perpendicular to horizontal subspaces.

Thus we obtain the conclusion of Lemma 5.3.

Let \( \nu > 0 \) be sufficiently small like \( \nu = 10^{-10} \), and let \( \delta^* = \delta^*(\nu) \) and \( \zeta = \zeta(\nu) \) be the constants given in Lemma 5.3. Letting \( Q \) be the positive integer in Section 4 and setting

\[
(5.2) \quad \zeta^* := \left( \zeta(\zeta(\cdots(\zeta(\nu))\cdots)) \right), \quad \delta^* := \delta^*(\zeta^*),
\]

we choose \( \epsilon \) in (3.2) satisfying

\[
(5.3) \quad \tau(2\delta^*) + \tau(s/2|\epsilon) < \zeta^*,
\]
where $r$ and $\sigma^*$ are defined as in (3.5) and (2.3). Note that $\zeta^*$, $\delta^*$, $\sigma^*$ and $s$ are universal constants. We also choose a universal constant $\epsilon_1^*$ in Corollary 3.3 like $\epsilon_1^* \ll \sigma^*$.

6. Proof of Lemma 4.4

In this section, we shall prove Lemma 4.4. $d_H^M$ denotes the Hausdorff distance in $M$.

Put

$$B_{i_1, \ldots, i_k} := B_{i_1} \cup \cdots \cup B_{i_k},$$

for $i_1, \ldots, i_k \in \mathcal{I}_2$.

Assertion 6.1. We assume that $B_{i_j}(1/10)$ meets $B_{i_1, \ldots, i_{j-1}}$ for every $2 \leq j \leq k$. Then there exist a local $S^1$-action $\psi_{i_1, \ldots, i_k}$ on $B_{i_1, \ldots, i_k}$ and a topological two-manifold $X_{i_1, \ldots, i_k}$ with the orbit projection $\pi_{i_1, \ldots, i_k} : B_{i_1, \ldots, i_k} \to X_{i_1, \ldots, i_k}$ satisfying the following:
is a manifold with boundary, and satisfying Assertion 6.1 has been constructed. For simplicity, we set
\[ \psi_{i_1,\ldots,i_k} = \begin{cases} \psi_{i_1,\ldots,i_{k-1}} & \text{on } B_{i_1} \cup \ldots \cup B_{i_{k-1}} - B_{i_k} \\ \psi_{i_k} & \text{on } B_{i_k}; \end{cases} \]

(2) \( B_{i_1,\ldots,i_k} \) is a manifold with boundary, and \( d_H^N(B_{i_1,\ldots,i_k}, B_{i_1,\ldots,i_k}) < \tau(e) \).

(3) Each orbit of \( \psi_{i_1,\ldots,i_k} \) has diameter < \( \tau(e) \);

(4) There are no singular orbits of \( \psi_{i_1,\ldots,i_k} \) over \( \partial X_{i_1,\ldots,i_k} - \partial_X X_{i_1,\ldots,i_k} \),

where \( \partial X_{i_1,\ldots,i_k} \) is defined as in (3.1).

(5) Let \( N_{i_1,\ldots,i_k;1} \subset B_{i_1,\ldots,i_k} \) be the set of points \( p \in B_{i_1,\ldots,i_k} \) such that

there is a \( (2,2\delta^*) \)-strainer at \( p \) of length \( s/2 \). Then every fiber \( F \) of \( \psi_{i_1,\ldots,i_k} \) contained in \( N_{i_1,\ldots,i_k;1} \) is \( (Q-n) \)-perpendicular to horizontal subspaces, where

\[
(\xi^{-n}(\nu) := \xi(\cdots(\xi(\nu)\cdots)), \text{ and } n \text{ denotes the number of } B_i \text{ containing } F. \]

Proof. We prove it by induction. Assertion (6.1) certainly holds for \( k = 1 \) by Theorem 3.5, Lemmas 3.8(2) and 5.2. Assume that a local \( S^1 \)-action \( \psi_{i_1,\ldots,i_{k-1}} \) on \( B_{i_1,\ldots,i_{k-1}} \) satisfying Assertion (6.1) has been constructed. For simplicity, we set
\[
B := B_{i_1,\ldots,i_{k-1}}, \quad X := X_{i_1,\ldots,i_{k-1}}, \quad \hat{B} := \hat{B}_{i_k}, \quad B' := B_{i_k}, \quad \hat{X} := \hat{X}_{i_k}, \quad X' := X_{i_k}, \quad \psi := \psi_{i_1,\ldots,i_{k-1}}, \quad \psi' := \psi_{i_k}, \quad \pi := \pi_{i_1,\ldots,i_{k-1}}, \quad \pi' := \pi_{i_k}. \]

We shall carry out a gluing procedure on each component, say \( U \), of \( B \cap B' \) using Lemma 5.3. \( \partial_X \) and \( \partial_X' \) are defined as in (3.1). Let \( A \subset \partial_X X', A_0, C \subset X \) and \( C_0 \) be defined as in the proof of Lemma 5.3.

First consider the case when \( A_0 \) does not meet \( B(S^o_\delta(Z'), r) \), where \( Z' \) is the Alexandrov space containing \( X' \), which implies \( (\pi')^{-1}(A_0) \subset N' \). Thus every \( \pi' \)-fiber in \( (\pi')^{-1}(A_0) \) is \( \xi^{-n}(\nu) \)-perpendicular to horizontal subspaces. Since \( \pi^{-1}(C_0) \) is close to \( (\pi')^{-1}(A_0) \), we obtain that \( \pi^{-1}(C_0) \subset N_{i_1,\ldots,i_{k-1};1} \). Condition (5) of Assertion (6.1) for \( \psi_{i_1,\ldots,i_{k-1}} \) then implies that every \( \pi \)-fiber in \( (\pi)^{-1}(C_0) \) is \( \xi^{-n+1}(\nu) \)-perpendicular to horizontal subspaces. Therefore we can apply Lemma 5.3 to get the required gluing of \( \psi \) and \( \psi' \) on a neighborhood joining \( (\pi')^{-1}(A_0) \) and \( \pi^{-1}(C_0) \).

Next consider the other case when \( A_0 \) meets \( B(S^o_\delta(Z'), r) \). In this case it follows from the condition (c) of Case (B) of Theorem 3.5 that an endpoint of \( A_0 \) must be contained in \( B(\partial_X X', r) \).

Take \( x, \hat{x} \in A_0 \) such that the subarc \( \{x, \hat{x}\} \) of \( A_0 \) is contained in \( X' - B(S^o_\delta(Z'), r) \) and \( A_0 - [x, \hat{x}] \) as well as \( x \) is contained in \( B(\partial X', r) \).
The case \( \hat{x} \in B(\partial_x X', r) \) may happen. Let \( y, \hat{y} \in C_0 \) be defined as in the proof of Lemma 5.3 for \( x, \hat{x} \). Applying (the proof of) Lemma 5.3 we have a domain \( D \) bounded by \( (\pi')^{-1}([x, \hat{x}]), \pi^{-1}([y, \hat{y}]) \) and two annuli, say \( E \) and \( \hat{E} \), having a compatible \( S^1 \)-fiber structure via a bi-Lipschitz homeomorphism \( k : D^2 \times S^1 \rightarrow D \) such that for each \( x \in D^2 \), \( k(x \times S^1) \) is \( \zeta^{Q-n}(\nu) \)-perpendicular to horizontal subspaces. Note that each component, say \( W \), of the domain bounded by \( (\pi')^{-1}(A_0 - [x, \hat{x}]), \pi^{-1}(C_0 - [y, \hat{y}]) \) and \( E \) (and possibly \( \hat{E} \)) is a three-disk. Therefore one can put a compatible structure of \( S^1 \)-action on \( W \) all of whose orbit has diameter < \( \tau(\epsilon) \).

From the gluing constructions above, we obtain \( B_{i_1, \ldots, i_k} \) and a local \( S^1 \)-action \( \psi_{i_1, \ldots, i_k} \) on it satisfying the conclusion of the assertion. This completes the proof of Assertion 6.1.

From (4.4) and the gluing argument used in the proof of Assertion 6.1 it is now obvious that \( U_2 \simeq U_2 \). Thus we have completed the proof of Lemma 4.4.

7. THICK-THIN DECOMPOSITION

For a three-manifold \( N \), we denote by \( \text{Cap}(N) \) a three-manifold obtained by gluing of \( N \) and some copies of \( D^3 \) along all the sphere-components of \( \partial N \).

**Theorem 7.1.** If \( M \) is a closed orientable Riemannian three-manifold with sectional curvature \( K \geq -1 \), then we have a decomposition

\[
M = M_{\text{thick}} \cup M_{\text{thin}}.
\]
satisfying the following: Let $\epsilon_0$ and $\delta_0$ be positive numbers given Theorem 1.1.

1. For every $p \in M_{\text{thick}}$, $\text{vol}(B(p, 1)) \geq \epsilon_0/2$.
2. $\text{Cap}(M_{\text{thin}})$ is homeomorphic to a graph manifold if $\text{diam}(M) \geq \delta_0$.

Roughly speaking, $M_{\text{thin}}$ is a piece of $M$ which collapses.

Theorem 7.1 is closely related with a result in [7], where a thick-thin decomposition of a closed three-manifold in connection with Ricci flow is announced.

Proof of Theorem 7.1. If $\text{diam}(M) < \delta_0$, then we put $M_{\text{thin}} := M$. For the proof of Theorem 7.1, we may assume $\text{diam}(M) \geq \delta_0$. For every point $p \in M$, one of the following holds:

1. $\text{vol}(B(p, 1)) \geq \epsilon_0$. In this case, we put $B_p := B(p, 1)$ and $X_p := B(p, 1)$.
2. $\text{vol}(B(p, 1)) < \epsilon_0$. In this case, by Theorem 3.3 there exist a small perturbation $B_p$ of $B(p, 1)$ and a metric ball $X_p$ in some complete Alexandrov space $X$ with curvature $\geq -1$ and $1 \leq \text{dim } X \leq 2$ such that $B_p$ has a fiber structure over $X_p$.

Take points $p_1, p_2, \ldots$, of $M$ as in Section 4 such that the collection $\{B_{p_i}\}$ given as above is a covering of $M$. Let $X_{p_i}$ be also chosen as above. For simplicity, we put

$$B_i := B_{p_i}, \quad X_i := X_{p_i}.$$ 

If $\text{dim } X_i = 2$, then there exists a local $S^1$-action $\psi_i$ on $B_i$ such that $B_i/\psi_i \simeq X_i$. For each $1 \leq j \leq 3$, let $I_j$ denote the set of all $i$ with $\text{dim } X_i = j$, and consider

$$U_j := \bigcup_{i \in I_j} B_i.$$ 

By Lemma 4.1 each component of $U_1$ is either cylindrical or cylindrical with a cap. By Lemma 4.3 we have a small perturbation $U_2$ of $U_2$ on which one can construct a local $S^1$-action. Note that each component of $M_{\text{thin}} := U_1 \cup U_2$ is homeomorphic to $S^2$ or $T^2$. The argument in Section 4 shows that $\text{Cap}(M_{\text{thin}})$ is homeomorphic to a graph manifold. Now it is obvious that for any point $p$ in $M_{\text{thick}} := \overline{M} - M_{\text{thin}}$, $\text{vol}(B(p, 1)) \geq \epsilon_0/2$. \hfill \qed

8. Appendix: Collapsing under a local lower curvature bound

In this appendix, we give a short description about collapsing three-manifold under a local lower curvature bound, which is discussed in [7].
For a positive number $\epsilon$, a closed Riemannian $n$-manifold $(M, g)$ is called $\epsilon$-collapsed under a local lower curvature bound if for each $x \in M$, there exists a $\rho$, $0 < \rho \leq \text{diam}(M, g)$, with

\[
\text{vol } B(x, \rho) \leq \epsilon \rho^n, \quad K \geq -\rho^{-2} \quad \text{on } B(x, \rho).
\]

Theorem 1.1 extends to the following:

**Theorem 8.1** (Theorem 7.4 [7]). Let $\epsilon_0$ be a positive number given in Theorem 1.1. If a closed orientable Riemannian three-manifold $(M, g)$ is $\epsilon_0$-collapsed under a local lower curvature bound, then it is homeomorphic to a graph manifold.

Let $\rho(x)$, $0 < \rho(x) \leq \text{diam}(M, g)$, be the supremum of $\rho > 0$ satisfying (8.1). By Theorem 3.5, a small perturbation $B_x$ of $B(x, \rho(x))$ has a singular fibration over a metric ball in some Alexandrov space with curvature $\geq -1$ and dimension one or two. Choose a covering $\{B_{x_i}\}$ of $M$ such that $\{B(x_i, \rho(x_i)/10)\}$ is a maximal disjoint family. Let $U_1$, $U_2$ and $M = U_1 \cup \hat{U}_2$ be as in Section 4. In a way similar to Lemma 1.1, one can prove that each component of $U_1$ is either cylindrical or cylindrical with a cap.

Let $B_2$ be defined as in Section 4.

**Lemma 8.2.** Suppose $B_x$ and $B_y$ in $B_2$ satisfy that $B(x, \rho(x)/4)$ meets $B(y, \rho(y)/4)$. Then

\[
C^{-1} \leq \frac{\rho(x)}{\rho(y)} \leq C,
\]

for some universal positive number $C$.

**Proof.** Assuming $\rho(x) < \rho(y)$, we put $\rho(y) = R\rho(x)$. By triangle inequality, $B(x, R_1 \rho(x)) \subset B(y, \rho(y))$, where $R_1 := R/2$, which implies that

\[
K \geq -(R_1 \rho(x))^{-2} \quad \text{on } B(x, R_1 \rho(x)).
\]

Next we show

\[
\text{vol } B(x, R_1 \rho(x)) \leq \epsilon_0(R_1 \rho(x))^3
\]

if $R_1$ is larger than some uniform positive constan. In view of the maximality of $\rho(x)$, (8.2) and (8.3) yield the conclusion of the lemma.

First note that large parts of $B_x$ and $B_y$ have $S^1$-fiber structures. Let $\ell$ denote the length of a regular circle fiber $F$ contained in $B(x, \rho(x)/4) \cap B(y, \rho(y)/4)$. Let $g_x := \rho(x)^{-2} g$. Since $B(y, \rho(y)/8) \subset B(x, R_1 \rho(x)) \subset B(y, \rho(y))$, Lemma 1.1 implies

\[
C_1^{-1} \frac{\ell}{\rho(y)} \leq \text{vol}_{g_y} B(x, R_1 \rho(x)) \leq C_1 \frac{\ell}{\rho(y)},
\]

for some uniform positive number $C_1$. It follows that

\[
4C_1^{-1} R_1^2 \rho(x)^2 \leq \text{vol}_{g_y} B(x, R_1 \rho(x)) \leq 4C_1 \ell R_1^2 \rho(x)^2.
\]
Hence (8.3) holds if $4C_1 \ell < \epsilon_0 R_1 \rho(x)$. On the other hand, it follows from the assumption and Lemma 5.1 that
\[ C_1^{-1} \frac{\ell}{\rho(x)} \leq \text{vol}_{g_2} B(x, \rho(x)) < \epsilon_0. \]
Therefore we obtain (8.3) for $R_1 > 4C_1^2$.

We conclude that the constant $C$ in the lemma is given by $C = 8C_1^2$. □

Lemma 8.2 together with the Bishop-Gromov comparison theorem yields a uniform upper bound on the maximal number of intersections among the metric balls in $B_2$. Therefore our local gluing argument in Sections 5 and 6 goes through the present context as well to complete the proof of Theorem 8.1.

References

1. Yu. Burago, M. Gromov, and G. Perel’man, A. D. Aleksandrov spaces with curvatures bounded below, Uspekhi Mat. Nauk 47 (1992), no. 2(284), 3–51, 222, translation in Russian Math. Surveys 47 (1992), no. 2, 1–58.
2. J. Cheeger and M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded I, J. Differential Geom. 23 (1986), no. 3, 309–346.
3. ———, Collapsing Riemannian manifolds while keeping their curvature bounded II, J. Differential Geom. 32 (1990), no. 1, 269–298.
4. K. Fukaya and T. Yamaguchi, The fundamental groups of almost nonnegatively curved manifolds, Ann. of Math. (2) 136 (1992), 253–333.
5. P. Orlik, Seifert manifolds, Lecture Notes in Math., no. 291, Springer-Verlag, Berlin-New York, 1972.
6. G. Perelman, A. D. Alexandrov’s spaces with curvatures bounded from below II, preprint.
7. ———, Ricci flow with surgery on three-manifolds, arXiv:math. DG / 0303109.
8. K. Shiohama and M. Tanaka, Cut loci and distance spheres on Alexandrov surfaces, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 531–559.
9. T. Shioya and T. Yamaguchi, Collapsing three-manifolds under a lower curvature bound, J. Differential Geom. 56 (2000), no. 1, 1–66.
10. T. Yamaguchi, Collapsing and pinching under a lower curvature bound, Ann. of Math. (2) 133 (1991), 317–357.
11. ———, A convergence theorem in the geometry of Alexandrov spaces, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 601–642.

Mathematical Institute, Tohoku University, Sendai 980-8578, JAPAN

Institute of Mathematics, University of Tsukuba, Tsukuba 305-8571, JAPAN

E-mail address, T. Shioya: shioya@math.tohoku.ac.jp
E-mail address, T. Yamaguchi: takao@math.tsukuba.ac.jp

24