Integral Equations of Fields on the Rotating Black Hole

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Abstract

It is known that the radial equation of the massless fields with spin around Kerr black holes cannot be solved by special functions. Recently, the analytic solution was obtained by use of the expansion in terms of the special functions and various astrophysical application have been discussed. It was pointed out that the coefficients of the expansion by the confluent hypergeometric functions are identical to those of the expansion by the hypergeometric functions. We explain the reason of this fact by using the integral equations of the radial equation. It is shown that the kernel of the equation can be written by the product of confluent hypergeometric functions. The integral equation transforms the expansion in terms of the confluent hypergeometric functions to that of the hypergeometric functions and vice versa, which explains the reason why the expansion coefficients are universal.

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1 Introduction

One of the remarkable features of the Kerr black hole, which is known to be unique black hole solution with no electro-magnetic charge, is that the gauge invariant perturbation of the spin 0, the 1/2 fields, the electromagnetic fields, the gravitino and the gravity can be treated by separable equations called Teukolsky equations. It should be noted that Whiting provided an analytic proof of the stability of the Kerr black holes by using these equations. Unfortunately, Teukolsky equation cannot be solved by any popular special functions since the equation has two regular singular points at horizons and one irregular singular point at infinity. In order to determine the physical quantities such as the absorption probabilities of black holes, we usually had to use numerical integration of the equation. However, for the purpose of investigating the property of the scattering data, it seems useful to provide analytic solutions as has been achieved for Coulomb scattering. These may be used as starting points of the perturbation for the quantum corrections. One of physical application is the absorption coefficients of the black holes. Recently, the analytic methods have been getting a powerful tool even in the field of the gravitational wave astrophysics.

The technique for analyzing this type of equation is quite old because a equation similar to Teukolsky equation appeared when we consider the wave function in terms of spheroidal coordinates, which is known to be spheroidal wave equation. The solution of the equation can be expanded either by the hypergeometric functions or by the confluent hypergeometric functions. Therefore it is natural that similar expansions has been considered even for Teukolsky equation.

Regarding the structure of the expansion coefficients of Teukolsky equation, an interesting observation was given in Refs. who showed that the expansion coefficients in terms of the hypergeometric functions are identical to those by the confluent hypergeometric functions up to re-definition of the coefficients. This universality of the expansion coefficients appeared even in the case of the spheroidal wave functions. The wave
function of three dimensional flat space is separable in the spheroidal coordinates. The radial equation and the angular equation can be identified by some change of variables, which can be identified as a single spheroidal wave equation. The angular equation is usually expanded by Legendre functions whereas the radial equation is expanded by Bessel functions. It turns out that the expansion coefficients are identical up to some normalization factor. The reason was explained by use of the fact that the angle equation and the radial equation of spheroidal wave equation are connected by an integral equation, because of which the coefficients are identical up to normalization factor\[13\]. This fact indicates that there might be some transformation which connects these expansions even for Teukolsky equation. However we cannot obtain the integral equation by a simple generalization of the spheroidal functions because the construction of the kernel in the case of spheroidal functions crucially depends on the fact that the original space is just a three dimensional flat space. In the case of Teukolsky equation, we cannot construct any simple system even if we associate any angle variables. Therefore, the above construction cannot be applied. Instead of considering an analog of such spheroidal wave functions, we treat the equation as an analog of Heun’s equation\[16\]. Heun’s equation\[16\] has four definite singular points in the equation and Teukolsky equation can be considered as a confluent limit of Heun’s equation\[17\]. We will use a principle used for the construction of the integral equation of the Heun’s equation\[18\]. By constructing the integral equation for the Teukolsky equation, which can be regarded as a confluent analog of the kernel of the Heun’s’ equation, we will show that the expansion coefficients are universal.

In the next section, we review the analytic expansion of the Teukolsky equation. There are basically three type of expansions which cannot be connected by the analytic continuation.

In section 3, we will construct a integral equation of the Teukolsky equation. It will be shown that the integral kernel can be written by a product of the confluent hypergeometric functions.

In section 4, it will be shown that the these expansions can be connected by the integral transformation, which implies that the expansion coefficients are universal.
Section 5 is devoted to some discussions.

2 Analytic expansions of Teukolsky equation

In the Boyer-Lindquist coordinates and in units such that \( c = G = 1 \), the Kerr metric is written as

\[
\begin{align*}
    ds^2 &= \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \left(\frac{4Mar \sin^2 \theta}{\Sigma}\right) dt d\phi - \left(\frac{\Sigma}{\Delta}\right) dr^2 \\
    &\quad - \Sigma d\theta^2 - \sin^2 \theta \left(\frac{r^2 + a^2}{\Sigma}\sin^2 \theta\right) d\phi^2,
\end{align*}
\]  

(2.1)

where \( M \) is the mass of the black hole, \( aM \) its angular momentum, \( \Sigma = r^2 + a^2 \cos^2 \theta \), and \( \Delta = r^2 + a^2 - 2Mr \). Klein-Gordon equation for massless fields \( \psi \) in a Kerr black hole background can be separated by setting \( \psi = e^{-i\omega t} e^{im\phi} S^m_l(\theta) R(r) \), and radial equation is given by

\[
\Delta^{-s} \frac{d}{dr} \left(\Delta^{s+1} \frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r - M)K}{\Delta} + 4is\omega r - \lambda\right) R = 0,
\]  

(2.2)

where \( s \) is a parameter called spin weight of the field, and \( \lambda = E - 2am\omega + a^2\omega^2 - s(s + 1) \), and \( E \) is eigenvalue of spheroidal harmonics \( S^m_l(\theta) \). This equation has two regular singularities at \( r = r_\pm = M \pm \sqrt{M^2 - a^2} = M \pm p \) and an irregular singularity at \( r = \infty \). Setting the new variable \( z = (r_+ - r)/2p \), radial equation becomes

\[
\begin{align*}
    z^2(z - 1)\frac{\partial^2 R}{\partial z^2} + (s + 1)z(z - 1)(2z - 1)\frac{\partial R}{\partial z} \\
    + \left[\frac{K^2}{4p^2} + \frac{isK(2z - 1)}{2p} - z(z - 1)(8is\omega pz - 4is\omega r_+ + \lambda)\right] R = 0,
\end{align*}
\]  

(2.3)

where \( s \) takes integer or half integer value and \( K = (4p^2z^2 - 4pr_+z + 2Mr_+)\omega - am \). The analytic solution of this equation is obtained not by any special functions but by expansions in terms of special functions such as the hypergeometric functions and the confluent hypergeometric functions.

If we use the expansion in terms of the hypergeometric functions, we have two independent solutions of equation (2.3). Introducing the slightly modified hypergeometric
function $P(a; b; c; z)$

$$P(a, b; c; z) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(c)} F(a, b; c; z),$$

(2.4)

where $F(a, b; c; z)$ is the hypergeometric function, we can express two solutions around $z = 0$ in the form

$$R_{1}^{\nu} = e^{i\sigma z}(-z)^{\rho}(1 - z)^{\delta} \sum_{k=-\infty}^{\infty} h_{k}^{1\nu} P(a_{k}, b_{k}; c; z),$$

$$R_{2}^{\nu} = e^{i\sigma z}(-z)^{\rho+1-c}(1 - z)^{\delta} \sum_{k=-\infty}^{\infty} h_{k}^{2\nu} P(a_{k} - c + 1, b_{k} - c + 1; 2 - c; z),$$

(2.5)

where

$$\rho = -s - i\omega M - i\tau, \quad \delta = i\omega M - i\tau, \quad \sigma = 2p\omega,$$

(2.6)

and

$$\tau = (2\omega M^2 - am)/2p, \quad \text{and}$$

$$a_{k} = \rho + \delta + s - k - \nu, \quad b_{k} = \rho + \delta + s + k + \nu + 1, \quad c = 2\rho + s + 1.$$  

(2.7)

By inserting the expression (2.5) into (2.3), we find that the expansion coefficients $h_{k}^{1\nu}$ and $h_{k}^{2\nu}$ should satisfy the same recursion relations;

$$\alpha_{k}h_{k+1}^{\nu} + \beta_{k}h_{k}^{\nu} + \gamma_{k}h_{k-1}^{\nu} = 0,$$

(2.8)

$$\alpha_{k} = \frac{i\sigma(s - 2i\omega M + k + \nu + 1)(-2i\tau + k + \nu + 1)(s + 2i\omega M + k + \nu + 1)}{(2k + 2\nu + 3)(k + \nu + 1)},$$

$$\beta_{k} = 8\omega^2 M^2 - a^2 \omega^2 - E + (k + \nu)(k + \nu + 1) + \frac{2\sigma\tau(2\omega M + is)(2\omega M - is)}{(k + \nu)(k + \nu + 1)},$$

(2.9)

$$\gamma_{k} = \frac{i\sigma(s - 2i\omega M - k - \nu)(-2i\tau - k - \nu)(s + 2i\omega M - k - \nu)}{(2k + 2\nu - 1)(k + \nu)}.$$

The parameter $\nu$ can be obtained by the convergence of the recursion relation$^{7, 8}$ or by using the post-newtonian principle$^8$.

Note that the solutions around $z = 1$, which represents the inner horizon, can be obtained by the analytic continuation of the hypergeometric functions. Two independent solutions around $z = 1$ are

$$R_{3}^{\nu} = e^{i\sigma z}(-z)^{\rho}(1 - z)^{\delta} \sum_{k=-\infty}^{\infty} \frac{h_{k}^{3\nu}}{\Gamma(c - a_{k})\Gamma(c - b_{k})} P(a_{k}, b_{k}; a_{k} + b_{k} - c + 1; 1 - z),$$

(2.10)
\[ R_4^\nu = e^{i\sigma z} (-z)^\delta (1 - z)^\rho \]
\[ \times \sum_{k=-\infty}^{\infty} \frac{h_k^{4\nu}(1 - z)^{c-a_k-b_k}}{\Gamma(c - a_k)\Gamma(c - b_k)} P(c - a_k, c - b_k; c - a_k - b_k + 1; 1 - z). \]  
\[ (2.10) \]

Since these expansions are expressed as linear combinations of \( R_1^\nu \) and \( R_2^\nu \) due to the analytic continuation, the recursion relations for \( h_k^{3\nu} \) and \( h_k^{4\nu} \) are identical to those of \( h_k^1 \) and \( h_k^2 \).

Other type of solutions are given by the expansion in terms of the confluent hypergeometric functions. Introducing the new variable \( \rho = -2\omega pz \), one type of the expansions is given by [7, 8]

\[ R_1^\nu = \rho^\delta (\rho + 2\omega p)^{-\delta - s} \sum_{k=-\infty}^{\infty} g_k^{1\nu} G_k^{\nu}(\eta, \rho), \]
\[ R_2^\nu = \rho^\delta (\rho + 2\omega p)^{-\delta - s} \sum_{k=-\infty}^{\infty} g_k^{2\nu} G_{-k-\nu-1}(\eta, \rho), \]  
\[ (2.11) \]

where \( \eta = -(2\omega M + is) \) and \( \delta = i\omega M - i\tau \), and \( G_i(\eta, \rho) \) satisfies the Coulomb type equation

\[ \rho \frac{\partial^2}{\partial \rho^2} G_i(\eta, \rho) + 2 \frac{\partial}{\partial \rho} G_i(\eta, \rho) + \left( \rho - 2\eta - \frac{l(l+1)}{\rho} \right) G_i(\eta, \rho) = 0, \]  
\[ (2.12) \]

which is expressed in terms of Kummer’s confluent hypergeometric function \( \Phi(a; b; \rho) \) as

\[ G_i(\eta, \rho) = \frac{\Gamma(l+1+i\eta)}{e^{\pi\eta/2}\Gamma(2l+2)} e^{-i\eta(2i\rho)} \Phi(l+1-i\eta; 2l+2; 2i\rho). \]  
\[ (2.13) \]

By inserting (2.11) into the original equation, we find that the expansion coefficients \( g_k^{1\nu} \) and \( g_k^{2\nu} \) should satisfy the following recursion relations;

\[ \alpha'_k g_{k+1}^{\nu} + \beta'_k g_k^{\nu} + \gamma'_k g_{k-1}^{\nu} = 0, \]  
\[ (2.14) \]

\[ \alpha'_k = \alpha_k, \]
\[ \beta'_k = \beta_k, \]  
\[ (2.15) \]
\[ \gamma'_k = \gamma_k, \]

which is identical to (2.8).
Moreover, if we set $\chi = 2\omega p(1 - z)$, which is the expansion in terms of the inner horizon, another expansion can be obtained in the form

$$
R^\nu_3 = \chi^\epsilon (\chi - 2\omega p)^{-s - \epsilon} \sum_{k=-\infty}^{\infty} \frac{g_k^{3\nu}}{\Gamma(c - a_k)\Gamma(c - b_k)} G_{k+\nu}(\eta, \chi),
$$

$$
R^\nu_4 = \chi^\epsilon (\chi - 2\omega p)^{-s - \epsilon} \sum_{k=-\infty}^{\infty} \frac{g_k^{4\nu}}{\Gamma(c - a_k)\Gamma(c - b_k)} G_{-k-\nu-1}(\eta, \chi),
$$

(2.16)

where $\epsilon = -s - i\omega M - i\tau$. It turns out that the recursion relation for $g_k^{3\nu}$ and $g_k^{4\nu}$ are again identical to those of $g_k^{1\nu}$ and $g_k^{2\nu}$ and $h_k^\nu$. In this case, we cannot explain the reason of having identical recursion relations simply from the analytic continuation on the contrary to the case of hypergeometric functions.

In the above construction, the coefficients $g_k^\nu$ and $h_k^\nu$ are identical up to normalization factor as

$$
g_k^\nu \sim h_k^\nu.
$$

(2.17)

Since $h_k^\nu$, $g_k^\nu$ are expansion coefficients in terms of different kinds of special functions, such as the hypergeometric function and the confluent hypergeometric function, which have different analytic properties and different asymptotic behaviors, it is not trivial why the relation (2.17) holds. Moreover, the identification of the coefficients $g_k^{1\nu}$ and $g_k^{3\nu}$ is also the subject which cannot be explained by any analytic continuation of the special functions. In the next section, we are going to explain the reason by use of the integral equations.

### 3 Integral equation

We are going to construct the integral equation in terms of $R(z)$ which satisfies the equation (2.3);

$$
M_z R(z) \equiv z(z - 1) \left\{ \frac{\partial^2 R(z)}{\partial z^2} + (s + 1) \left( \frac{1}{z} + \frac{1}{z - 1} \right) \frac{\partial R(z)}{\partial z} \right\}
$$

(3.1)
\[ K^2 \left[ \frac{1}{4p^2z(z-1)} + \frac{isK(2z-1)}{2pz(z-1)} - 8is\omega p z + 4is\omega r + \lambda \right] R(z) = 0. \quad (3.2) \]

The integral transformation which maps one solution to other solution is given by

\[ R'(x) = \int_C y^s(y-1)^s K(x,y) R(y) dy, \quad (3.3) \]

where the function \( R'(x) \) is also a solution of equation \((2.3)\) if the kernel satisfy the condition

\[ (M_x - M_y)K(x,y) = 0, \quad (3.4) \]

and the surface term of the integral

\[ y^{s+1}(y-1)^{s+1} \left\{ \frac{\partial K(x,y)}{\partial y} R(y) - K(x,y) \frac{\partial R(y)}{\partial y} \right\} \quad (3.5) \]

vanishes at the end of \( C \). To find the kernel, we set new variables as

\[ \xi = xy, \quad \zeta = (x-1)(y-1). \quad (3.6) \]

Then the equation \((3.4)\) becomes

\[ \xi \frac{\partial^2 K}{\partial \xi^2} + (s+1) \frac{\partial K}{\partial \xi} + \left\{ 4p^2\omega^2 \xi - 8pM\omega^2 - 4is\omega p + \frac{(M\omega + \tau)(M\omega + \tau - is)}{\xi} \right\} K \quad (3.7) \]

\[ - \zeta \frac{\partial^2 K}{\partial \zeta^2} - (s+1) \frac{\partial K}{\partial \zeta} - \left\{ 4p^2\omega^2 \zeta + \frac{(M\omega - \tau)(M\omega - \tau - is)}{\zeta} \right\} K = 0. \]

Therefore we can separate the variable, and we put \( K(x,y) = P(\xi)Q(\zeta) \) so that \( P(\xi) \)

and \( Q(\zeta) \) satisfy the equations

\[ \xi \frac{\partial^2 P(\xi)}{\partial \xi^2} + (s+1) \frac{\partial P(\xi)}{\partial \xi} \]

\[ + \left\{ 4p^2\omega^2 \xi - 8pM\omega^2 - 4is\omega p + \frac{(M\omega + \tau)(M\omega + \tau - is)}{\xi} \right\} P(\xi) = \lambda P(\xi), \quad (3.8) \]

\[ \zeta \frac{\partial^2 Q(\zeta)}{\partial \zeta^2} + (s+1) \frac{\partial Q(\zeta)}{\partial \zeta} + \left\{ 4p^2\omega^2 \zeta + \frac{(M\omega - \tau)(M\omega - \tau - is)}{\zeta} \right\} Q(\zeta) = \lambda Q(\zeta), \quad (3.9) \]
where \( \lambda \) is a separation constant. Each equation has two independent solutions and these are expressed by using Kummer’s confluent hypergeometric functions as

\[
    P_1(\xi) = e^{i\sigma \xi} \xi^\rho \Phi \left( \frac{2\rho + s + 1}{2} - \frac{E + \lambda}{2i\sigma}; 2\rho + s + 1; -2i\sigma \xi \right),
\]

\[
    P_2(\xi) = e^{i\sigma \xi} \xi^{-\rho-s} \Phi \left( -\frac{2\rho + s - 1}{2} - \frac{E + \lambda}{2i\sigma}; 1 - 2\rho - s; -2i\sigma \xi \right),
\]

\[
    Q_1(\zeta) = e^{i\sigma \zeta} \zeta^\delta \Phi \left( \frac{2\delta + s + 1}{2} - \frac{\lambda}{2i\sigma}; 2\delta + s + 1; -2i\sigma \zeta \right),
\]

\[
    Q_2(\zeta) = e^{i\sigma \zeta} \zeta^{-\delta-s} \Phi \left( -\frac{2\delta + s - 1}{2} - \frac{\lambda}{2i\sigma}; 1 - 2\delta - s; -2i\sigma \zeta \right),
\]

where \( \rho = -s - iM\omega - i\tau \), \( \delta = iM\omega - i\tau \), \( \sigma = 2p\omega \) and \( E = 8M\omega^2 p + 4is\omega p \). We can thus solve the kernel equation in a general form. Namely, the kernel \( K(x, y) \) is expressed as the simple product of these solutions, or as the product of some linear combinations of these solutions such as Whittaker’s function. Since \( \lambda \) is an arbitrary constant, we choose \( \lambda \) in such a way that the kernel becomes a simple form in order to evaluate the integral transformation easily. The integral equation maps a solution of Teukolsky equation to another solution. Therefore, we can transform a analytic expansion to other expansions by various choice of the kernel. In the next section, we are going to perform the integral transformation of such various expansions of equation (2.3).

4 Integral transformations

First of all, we consider the transformation from the hypergeometric expansions (2.3) to Coulomb expansions (2.11) around \( z = 0 \); we take \( R_1^\nu(y) \) in (2.3) as \( R(y) \)

\[
    R_1^\nu = e^{i\sigma z} (-y)^\rho (1-y)^\delta \sum_{k=-\infty}^\infty h_k^{\nu} P(a_k, b_k; c; y). \tag{4.1}
\]

A suitable choice of the kernel for this transformation is as follows. We choose \( P(\xi) \) in the kernel by using Whittaker’s function \( M_{\kappa, \mu}(z) \) as

\[
    P(\xi) = \xi^{-(s+1)/2} M_{\kappa, \mu}(-2i\sigma \xi), \tag{4.2}
\]
where \( \kappa = (E + \lambda)/2i\sigma \) and \( \mu = \rho + s/2 \). In order to eliminate the factor \((1 - y)\) in the integral equation, we take \( Q_2(\zeta) \) in (3.11) for \( Q(\zeta) \). For the convergence of the integral, we take \( \lambda = i\sigma(-2\delta - s + 1) \) so that \( Q(\zeta) \) becomes

\[
Q(\zeta) = e^{i\sigma\zeta}e^{\delta - s}.
\] (4.3)

Then the integral equation is given by

\[
R_1'(x) = (\text{const.}) x^{-(s+1)/2} (x - 1)^{-\delta - s} e^{-i\sigma(x-1)} \\
\times \int C e^{i\sigma xy} y^{-(s+1)/2 + \rho + s} M_{\kappa,\mu}(-2i\sigma xy) \sum_{k=-\infty}^{\infty} h_k^{\nu} P(a_k, b_k; c; y) dy.
\] (4.4)

By taking the region of the integration as the interval from 0 to \(+\infty\) and using the formulae in [19], we can evaluate the integral as

\[
R_1'(\rho) = (\text{const.}) \rho^\delta (\rho + 2\omega p)^{-\delta - s} \sum_{k=-\infty}^{\infty} h_k^{\nu} \\
\times \{ C_1 \Gamma(2k + 2\nu + 1) \Gamma(-2k - 2\nu) G_{-k-\nu-1}(\eta, \rho) \\
+ C_2 \Gamma(-2k - 2\nu - 1) \Gamma(2k + 2\nu + 2) G_{k+\nu}(\eta, \rho) \},
\] (4.5)

where \( C_1, C_2 \) are constants which are independent of \( k \). Note that if we start with \( R_2'' \), we obtain the same result as the equation (4.5) after the integral transformation by using an appropriate kernel.

Note that the solution \( R_1'(\rho) \) consists of two independent solutions which are expanded in terms of Coulomb type functions and they are identical to the solution (2.11). Thus two kinds of expansions are connected by the integral transformation. Recognizing the coefficients of \( G_{\nu+k}(\eta, \rho) \), \( G_{-k-\nu-1}(\eta, \rho) \) as \( g_k^{\nu} \), \( g_k^{2\nu} \) respectively, the relation between \( h_k^{\nu} \) and \( g_k^{\nu} \), \( g_k^{2\nu} \) is

\[
g_k^{\nu} \sim \Gamma(2k + 2\nu + 1) \Gamma(-2k - 2\nu) h_k^{\nu} \sim h_k^{\nu},
\] (4.6)

\[
g_k^{2\nu} \sim \Gamma(-2k - 2\nu - 1) \Gamma(2k + 2\nu + 2) h_k^{\nu} \sim h_k^{\nu},
\] (4.7)

Therefore the fact that coefficients \( h_k^{\nu} \), \( g_k^{\nu} \) of different kinds of expansions satisfy the same recursion relations can be understood quite naturally.
Let us consider the inverse transformation. In this case, we start with a slightly modified form of $R^\nu_1(\rho)$ in (2.11) as $R(y)$

$$R^\nu_1(\rho) = \rho^\delta (\rho + 2\omega p)^{-\delta - s} \sum_{k=-\infty}^{\infty} g^1_k \nu G_{k+\nu}(\eta, \rho),$$

$$= \rho^\delta (\rho + 2\omega p)^{-\delta - s} \sum_{k=-\infty}^{\infty} g^1_k \nu \frac{\Gamma(k + \nu + 1 + i\eta)(2i)^{k+\nu+1}}{\Gamma(2k + 2\nu + 2)\rho} M_{\eta, k+\nu+1/2}(2i\rho). \quad (4.8)$$

Our choice of the kernel is as follows. In order to eliminate the factor $(y - 1)$, we take $Q_1(\zeta)$ with $\lambda = -i\sigma(2\delta + s + 1)$ so that

$$Q(\zeta) = e^{-i\sigma\zeta} \zeta^\delta. \quad (4.9)$$

For the convergence of the integral, we combine $P_1(\xi), P_2(\xi)$ into the form

$$P(\xi) = \xi^{-(s+1)/2} W_{-\kappa, \mu}(2i\sigma\xi), \quad (4.10)$$

where $W_{-\kappa, \mu}(z)$ is Whittaker’s function. Then the integral equation becomes

$$R_1'(x) = (\text{const.}) x^{-(s+1)/2} (x - 1)^\delta e^{i\sigma(x-1)}$$

$$\times \int C y^{s-(s+1)/2+\delta-1} e^{-i\sigma(x-1)y} W_{-\kappa, \mu}(2i\sigma xy)$$

$$\times \sum_{k=-\infty}^{\infty} g^1_k \nu \frac{\Gamma(k + \nu + 1 + i\eta)(2i)^{k+\nu+1}}{\Gamma(2k + 2\nu + 2)\rho} M_{\eta, k+\nu+1/2}(2i\rho) dy. \quad (4.11)$$

By taking the region of the integration as the interval 0 to infinity and using the formulae in [20], we can evaluate the integral. The solution is

$$R_1'(x) = (\text{const.}) x^\delta (1 - x)^{\delta} e^{i\sigma x} \sum_{k=-\infty}^{\infty} g^1_k \nu \left\{ C_1 P(a_k, b_k; c; x) + C_2 x^{1-c} P(a_k - c + 1, b_k - c + 1; 2 - c; x) \right\}. \quad (4.12)$$

Thus we could perform the inverse transformation, which completes the relation between two expansion around $z = 0$. The relation between expansion coefficients are

$$h^1_k \sim h^2_k \sim g^1_k \sim g^2_k. \quad (4.13)$$
We next consider the relation between expansions around \( z = 1 \). As before, we use \( R_3^\nu(\chi) \) in (2.16) as \( R(y) \) in the modified form

\[
R_3^\nu(\chi) = \chi^\epsilon (\chi - 2\omega p)^{-\epsilon-s} \sum_{k=-\infty}^{\infty} \frac{g_k^{3\nu}}{\Gamma(c - a_k)\Gamma(c - b_k)} G_{k+\nu}(\eta, \chi),
\]

\[
= \chi^{\epsilon-1} (\chi - 2\omega p)^{-\epsilon-s} \times \sum_{k=-\infty}^{\infty} \frac{g_k^{3\nu}\Gamma(k + \nu + 1 + i\eta)(2i\sigma)^{k+\nu+1}}{\Gamma(c - a_k)\Gamma(c - b_k)\Gamma(2k + 2\nu + 2)} M_{\eta, k+\nu+1/2}(2i\chi),
\]

where \( \chi = 2\omega p(1 - y) \) and \( \epsilon = -s - i\omega M - i\tau \). In this case, the role of \( y \) and \( 1 - y \) are interchanged in the integral equation. In order to eliminate the factor \( y \), we take \( P(\xi) \) as

\[
P(\xi) = \xi^\epsilon e^{i\sigma \xi},
\]

where we set \( \lambda = i\sigma(2\epsilon + s + 1) - E \). For the convergence of the integral, we take \( Q(\zeta) \) as

\[
Q(\zeta) = \zeta^{-(s+1)/2} W_{\kappa', \mu'}(-2i\sigma \zeta),
\]

where \( \kappa' = \lambda/2i\sigma, \mu' = \delta + s/2 \). Then the integral transformation becomes

\[
R_3^\nu = (\text{const.}) x^\epsilon (1 - x)^{-(s+1)/2} e^{i\sigma x}
\times \int_{\mathcal{C}} (1 - y)^{s-(s+1)/2+\epsilon-1} e^{i\sigma x(y-1)} W_{\kappa', \mu'}(-2i\sigma(1 - x)(1 - y))
\times \sum_{k=-\infty}^{\infty} \frac{g_k^{3\nu}\Gamma(k + \nu + 1 + i\eta)(2i\sigma)^{k+\nu}}{\Gamma(c - a_k)\Gamma(c - b_k)\Gamma(2k + 2\nu + 2)} M_{\eta, k+\nu+1/2}(2i\sigma(1 - y)) dy.
\]

By considering the integral region as the interval \( 0 \) to infinity, and by using the formulae in [20], we obtain the solution of the equation as

\[
R_3^\nu = (\text{const.}) x^\epsilon (1 - x)^{\delta} e^{i\sigma x} \sum_{k=-\infty}^{\infty} g_k^{3\nu}
\left\{ \frac{C_1}{\Gamma(c - a_k)\Gamma(c - b_k)} P(a_k, b_k; a_k + b_k - c + 1; 1 - x) + \frac{C_2(1 - x)^{c - a_k - b_k}}{\Gamma(c - a_k)\Gamma(c - b_k)} P(c - a_k, c - b_k; c - a_k + b_k + 1; 1 - x) \right\}.
\]

Note that the same result holds if we start with \( R_4^\nu(\chi) \).

\( R_3^\nu \) consists of two independent solution which are expanded in terms of hypergeometric functions around \( z = 1 \). Thus two kinds of expansion around \( z = 1 \) are connected.
by the integral transformation. The relation between expansion coefficients are

\[ g_k^{3\nu} \sim g_k^{4\nu} \sim h_k^{3\nu} \sim h_k^{4\nu}. \] (4.19)

We thus find that all the coefficients are connected by the analytic continuation of the wave function and the integral transformation which maps the solution of Teukolsky equation to other solutions.

5 Conclusion

We have constructed the integral equation in terms of equation (2.3), and solve the kernel in the general form as the product of special functions. By various choice of the kernel, we have performed the integral transformations which connect various expansions of equation (2.3). In all regions of \( z \), we can relate these expansions one another by the integral transformation, and by the analytic continuation of hypergeometric functions. As a consequence, the coefficients \( g_k^{i\nu} \) of Coulomb type expansions satisfy the same recusion relations of the coefficients \( h_k^{i\nu} \) of the hypergeometric expansions, and they are identical up to normalization factors.

In any case of integral transformations, the kernel is expressed as the product of the confluent hypergeometric functions. It is this property that makes us possible to perform the integral transformation.

Let us consider the case of the spheroidal equation. The equation (3.4) is just the wave equation in the spheroidal coordinates where the variable \( x \) is the radial coordinate and \( y \) is the coordinate representing the angle [13]. In other words, \( x \) and \( y \) can be treated as dual coordinates. It was quite easy to obtain the integral equation of spheroidal wave functions because we know other type of separable coordinates in flat space. On the other hand, in the case of Teukolsky equation, we have dealt with dual coordinates of the radial function which are not the spheroidal coordinates in Kerr geometry for the construction of integral equation. The existence of other separable variable (3.9) seems
to show some kind of symmetry of the space-time in a more wider geometry. Further study on this point of view seems interesting.

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