Continuous and discrete dynamical Schrödinger systems: explicit solutions

To cite this article: B Fritzsche et al 2018 J. Phys. A: Math. Theor. 51 015202

View the article online for updates and enhancements.

Related content
- Dynamics of electrons and explicit solutions of Dirac–Weyl systems
  Alexander Sakhnovich
- Harmonic Maps and GBDT
  Alexander Sakhnovich
- Second harmonic generation
  Alexander Sakhnovich

Recent citations
- GBDT and algebro-geometric approaches to explicit solutions and wave functions for nonlocal NLS
  J Michor and A L Sakhnovich
Continuous and discrete dynamical Schrödinger systems: explicit solutions

B Fritzsche\(^1\), B Kirstein\(^1\), I Ya Roitberg\(^1\) and A L Sakhnovich\(^{2,3}\)

\(^1\) Fakultät für Mathematik und Informatik, Universität Leipzig, Augustusplatz 10, D-04009 Leipzig, Germany
\(^2\) Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria
E-mail: Bernd.Fritzsche\text{\textregistered}math.uni-leipzig.de, innaroitberg\text{\textregistered}gmail.com, Bernd.Kirstein\text{\textregistered}math.uni-leipzig.de and oleksandr.sakhnovych\text{\textregistered}univie.ac.at

Received 17 July 2017, revised 11 October 2017
Accepted for publication 2 November 2017
Published 1 December 2017

Abstract
We consider continuous and discrete Schrödinger systems with self-adjoint matrix potentials and with additional dependence on time (i.e. dynamical Schrödinger systems). Transformed and explicit solutions are constructed using our generalized (GBDT) version of the Bäcklund–Darboux transformation. Asymptotic expansions of these solutions in time are of interest.

Keywords: Schrödinger equation, dynamical system, Jacobi matrix, Bäcklund–Darboux transformation, explicit solution, asymptotic expansion

1. Introduction

Dynamical Dirac and Schrödinger systems play an essential role in mathematical physics and are actively studied, especially in the recent years (see, e.g. \cite{2, 3, 5, 9, 10, 20, 24, 31, 32, 42} and numerous references therein). Continuous dynamical Schrödinger system has the form:

\[
i \frac{\partial}{\partial t} \psi(x,t) = (H \psi)(x,t), \quad H := -\frac{\partial^2}{\partial x^2} + u(x) \quad (u = u^*),
\]

where \(u\) is an \(h \times h\) matrix function, \(h \in \mathbb{N}\), and \(\mathbb{N}\) is the set of natural numbers. The matrix function \(u\) is called the potential of (1.1) and this potential does not depend on \(t\) in our case.

\(^3\) Author to whom any correspondence should be addressed.

Original content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
In discrete dynamical Schrödinger system we use Jacobi matrices $\mathcal{J}$ instead of $H$ since Jacobi operators ‘can be viewed as the discrete analogue of Sturm–Liouville operators’ [41, Preface]. The corresponding system is given by the formula:

$$i \left( \frac{\partial}{\partial t} \Psi \right) (t) = \mathcal{J} \Psi(t),$$  \hspace{1cm} (1.2)

where $\mathcal{J}$ is a semi-infinite block Jacobi matrix and $\Psi$ is a block vector

$$\mathcal{J} = \begin{bmatrix}
    b_1 & a_1 & 0 & 0 & 0 & \ldots \\
    c_2 & b_2 & a_2 & 0 & 0 & \ldots \\
    0 & c_3 & b_3 & a_3 & 0 & \ldots \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \ldots \\
\end{bmatrix}, \quad \Psi(t) = \begin{bmatrix}
    \psi_1(t) \\
    \psi_2(t) \\
    \psi_3(t) \\
    \vdots \\
\end{bmatrix}.$$  \hspace{1cm} (1.3)

Here, the blocks $a_k$, $b_k$ and $c_k$ are $h \times h$ matrices and $c_{k+1} = a_k^*$ ($k \geqslant 1$).

Explicit solutions of dynamical systems are important as models and examples and they are also essential in applications. Various explicit solutions of time-independent systems were constructed using commutation methods [8, 13, 14, 21] and several versions of Bäcklund–Darboux transformations. Bäcklund–Darboux transformation is a well-known tool in the spectral theory and theory of explicit solutions. The original equation, which was studied by Darboux, is the Schrödinger equation

$$-y''(x, \lambda) + u(x)y(x, \lambda) = \lambda y(x, \lambda) \quad (u = u^*),$$  \hspace{1cm} (1.4)

Later, and especially in the last 40 years, this transformation was greatly modified, generalized and applied to a variety of linear and nonlinear equations (see, e.g. [7, 16, 22, 23, 34]).

It was shown recently in [32, 33] that the GBDT version of Bäcklund–Darboux transformation (for GBDT see [25–29, 34] and references therein) may be successfully applied to the construction of explicit solutions of dynamical systems as well.

In the present paper, we consider the important case of continuous dynamical Schrödinger system and a more difficult case of discrete system (i.e. system (1.2)). Some preliminaries are presented in section 2, continuous dynamical Schrödinger system is dealt with in section 3 and system (1.2) is considered in section 4.

The dependence of our solutions of (1.1) and (1.2) on time is described by the factor $e^{iA}$, where $A$ is a parameter matrix (generalized eigenvalue) of the GBDT transformation. Since $A$ is not necessarily self-adjoint and may have Jordan cells of different orders, the asymptotic expansion of our solutions of (1.1) and (1.2) essentially differs (see remark 3.4) from the classical Jensen-Kato formulas (see [18] as well as further references in [9, 10]).

As usual, $\mathbb{R}$ denotes the set of real values, $\mathbb{N}$ is the set of natural numbers, and the complex plane is denoted by $\mathbb{C}$. Notation $\mathcal{S}^*$ stands for the matrix which is the conjugate transpose of $\mathcal{S}$, we write $\mathcal{S} > 0$ when $\mathcal{S}$ is a positive-definite matrix, and $I_m$ stands for the $m \times m$ identity matrix. The notation $J = \text{diag}\{J_1, J_2, \ldots\}$ means that $J$ is a diagonal or block diagonal matrix with the entries (or block entries) $J_1, J_2$ and so on.

2. GBDT: preliminaries

GBDT (generalized Bäcklund–Darboux transformation) was first introduced in [25], and a more general version of GBDT for first order systems rationally depending on the spectral parameter (in particular, for systems of the form $w' = G(x, \lambda)w$, $G(x, \lambda) = -\sum_{k=-r}^{\infty} \lambda^k q_k(x)$) was treated in [26, 34] (see also some references therein). First order system
where \( w(x, \lambda) \) takes values in \( \mathbb{C}^{n} \) (\( m := 2h \)) and \( m \times m \) coefficients \( q_{1} \) and \( q_{2} \) have the form

\[
q_{1} = \begin{bmatrix}
0 & 0 \\
J_{h} & 0
\end{bmatrix}, \quad q_{0}(x) = - \begin{bmatrix}
0 & I_{h} \\
u(x) & 0
\end{bmatrix}, \quad u(x) = u(x)^{*},
\]

is equivalent to the matrix Schrödinger equation (1.4) with a self-adjoint \( h \times h \) potential \( u(x) \).

In the present section ‘GBDT: Preliminaries’, we show basic GBDT results for this system.

The presented GBDT technique is applied (see, e.g. [11, 12, 25, 26, 28, 34]) to the explicit solving of direct and inverse spectral problems and of nonlinear integrable equations. It is used for the construction of explicit solutions of important dynamical systems in [32, 33]. Here, the results of this section are used for the construction of explicit solutions of the dynamical and stationary Schrödinger equations (1.1) and (1.4), respectively, in the next section.

**Remark 2.1.** We consider systems (1.1) and (2.1) on finite or infinite intervals \( \mathcal{I} \), that is, we assume that \( x \in \mathcal{I} \). Without loss of generality we assume also that \( 0 \in \mathcal{I} \) and speak later about parameter matrices \( S(0) \) and \( \Pi(0) \) instead of \( S(x_{0}) \) and \( \Pi(x_{0}) \) for some fixed \( x_{0} \in \mathcal{I} \). The case of the semiaxis \( \mathcal{I} = [0, \infty) \) is studied in this paper in greater detail.

In general, GBDT is determined by the choice of 5 parameter matrices (this case was treated in [27], where \( u \) was not necessarily self-adjoint). However, relations (2.2) (including \( u = u^{*} \)) imply additional equalities:

\[
q_{k}^{*} = - j q_{j} j^{*} \quad (k = 0, 1), \quad j := \begin{bmatrix}
0 & I_{h} \\
-I_{h} & 0
\end{bmatrix}, \quad j^{*} = j^{-1} = -j.
\]

Thus the conditions of proposition 1.4 from [26] are fulfilled, and we may use this proposition and some formulas from its proof. The following statements in this section are particular cases of [27, theorem 2.1] (or [26, theorem 1.2]) completed by [26, proposition 1.4].

Hence, in the present case we use 3 parameter matrices. More precisely, we choose some initial system (2.1) (or, equivalently, the initial potential \( u = u^{*} \) of Schrödinger equation (1.4)) and fix \( n \in \mathbb{N} \). Then, we fix \( n \times n \) matrices \( A \) and \( S(0) = S(0)^{*} \), and an \( n \times m \) \( (m = 2h) \) matrix \( \Pi(0) \) such that the following matrix identity holds:

\[
A S(0) - S(0) A^{*} = \Pi(0) j \Pi(0)^{*}.
\]

Suppose that such parameter matrices are fixed and that the potential \( u(x) \) is locally summable on \( \mathcal{I} \). Now, we can introduce matrix functions \( \Pi(x) \) and \( S(x) \) with the values \( \Pi(0) \) and \( S(0) \) at \( x = 0 \) as the solutions of the linear differential equations

\[
\Pi' = A \Pi q_{1} + \Pi q_{0}, \quad S' = \Pi q_{1} j \Pi^{*},
\]

where \( q_{1} \) and \( q_{0} \) are given by (2.2), and so \( q_{ij} = (q_{ij})^{*} \). Thus, in view of \( S(0) = S(0)^{*} \), we have

\[
S(x) = S(x)^{*}.
\]

Notice that equations (2.5) are constructed in such a way that the identity

\[
A S(x) - S(x) A^{*} = \Pi(x) j \Pi(x)^{*}
\]

follows (for all \( x \in \mathcal{I} \) from (2.4) and (2.5). (The relation is obtained by the direct differentiation of the both sides of (2.7)). Assuming that \( \det S(x) \neq 0 \) we can define a matrix function

\[
w_{A}(x, \lambda) = I_{m} - j \Pi(x)^{*} S(x)^{-1} (A - \lambda I_{m})^{-1} \Pi(x),
\]
where \( \lambda \not\in \sigma(A) \) (\( \sigma \) means spectrum).

**Theorem 2.2.** Suppose that the relation (2.4) is valid, and matrix functions \( \Pi(x) \) and \( S(x) \) satisfy equation (2.5) where (2.2) holds. Then, in the points of invertibility of \( S(x) \), the matrix function \( w_A(x, \lambda) \) satisfies the system

\[
\begin{align*}
\dot{w}_A(x, \lambda) &= \tilde{G}(x, \lambda)w_A(x, \lambda) - w_A(x, \lambda)G(x, \lambda), \\
\tilde{G}(x, \lambda) &= -\lambda q_1(x) - \tilde{q}_0(x),
\end{align*}
\]

where the coefficient \( \tilde{q}_0(x) \) is given by the formula

\[
\tilde{q}_0 = q_0 - (q_1 X - jX q_1), \quad X(x) := \Pi(x)^* S(x)^{-1} \Pi(x).
\]  

**Remark 2.3.** Formulas (2.2) and (2.5) yield

\[
q_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & I_h \end{bmatrix}, \quad S'(x) \geq 0,
\]

and so \( S(x) > 0 \) for \( x \geq 0 \) under additional condition \( S(0) > 0 \). In particular, the condition of invertibility of \( S(x) \) from theorem 2.2 is fulfilled automatically when \( I = (0, \infty) \) and \( S(0) > 0 \).

The matrix functions \( S(x)^{-1}, X(x) \) and \( w_A(x, \lambda) \) are well-defined in this case.

According to theorem 2.2, the multiplication by \( w_A \) transforms each solution \( w \) of (2.1) into the solution \( \tilde{w} = w_A \tilde{w} \) of the system \( \tilde{w}' = \tilde{G}\tilde{w} \) with the coefficients of \( \tilde{G} \) given by (2.10) and (2.11). This transformation of the solutions \( w \) and coefficients \( q_k \) is called GBDT. Matrix function \( w_A \) is the so-called Darboux matrix. The right hand side of (2.8) (with the additional property (2.7) and \( x \) fixed) has the form of the Lev Sakhnovich transfer matrix function [34, 35, 37].

Under the conditions of theorem 2.2 we have also

\[
(\Pi^* S^{-1})'(x) = q_1^* \Pi(x)^* S(x)^{-1} A + \tilde{q}_0(x)^* \Pi(x)^* S(x)^{-1}.
\]  

Clearly, the definition (2.11) of \( X(x) \) and formula (2.6) imply that

\[
X(x) = X(x)^*.
\]

3. **Explicit solutions of the dynamical system (1.1) and GBDT of the matrix Schrödinger equation**

1. Let us write down the coefficient \( \tilde{q}_0 \) of the transformed system in the block form. We partition \( \Pi \) into two \( h \times h \) blocks and partition \( X \) introduced in (2.11) into four \( h \times h \) blocks:

\[
\Pi = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}, \quad X = \{X_{ij}\}_{i,j=1}^2, \quad X_{ij} = \Phi_i^* S^{-1} \Phi_j.
\]

Thus, \( \tilde{q}_0 \) in theorem 2.2 (see (2.11)) has the form
In order to rewrite (2.13) in a more convenient form, we shall need also the block representation of $\Pi^*S^{-1}$, $q^*_1$ and $\tilde{q}_0^*$:

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \Pi^*S^{-1}, \quad q^*_1 = \begin{bmatrix} 0 \\ I_h \\ u \end{bmatrix}, \quad \tilde{q}_0^* = -\begin{bmatrix} -X_{22} & u + X_{12} + X_{21} \\ I_h & X_{22} \end{bmatrix}.$$

which follows from (2.2), (2.14) and (3.2). Now, (2.13) takes the form

$$z_1' = z_2A + X_{22}z_1 - (u + X_{12} + X_{21})z_2, \quad z_2' = -z_1 - X_{22}z_2.$$  

Differentiating the second equality in (3.5) (and taking into account the first equality), we obtain

$$z_2'' = -z_2A + (u + X_{12} + X_{21} - X_{22} - X_{22}^2)z_2.$$  

which is a transformed continuous dynamical Schrödinger system (3.9).

**Theorem 3.1.** Let the $h \times h$ potential $u(x) = u(x)^*$ of the initial dynamical Schrödinger system (1.1) be locally summable on $I$, and let the parameter matrices $A$, $S(0) = S(0)^*$ and $\Pi(0)$ be chosen so that $AS(0)-S(0)A^* = \Pi(0)\Pi(0)^*$. Introduce $\Pi(x)$ and $S(x)$ using relations

$$\Pi'(x) = A\Pi(x)q_1 + \Pi(x)q_0(x), \quad S'(x) = \Pi(x)q_1^*\Pi(x)^*,$$

where $q_1 = \begin{bmatrix} 0 \\ I_h \\ 0 \end{bmatrix}$, $q_0(x) = -\begin{bmatrix} 0 \\ I_h \\ u \end{bmatrix}$. Then, in the points of invertibility of $S(x)$, the matrix function

$$\tilde{\psi}(x,t) = \begin{bmatrix} 0 \\ I_h \end{bmatrix} \Pi(x)^*S(x)^{-1}e^{-itA}$$

satisfies the transformed continuous dynamical Schrödinger system

$$i\frac{\partial}{\partial t} \tilde{\psi}(x,t) = (\tilde{H}\tilde{\psi})(x,t), \quad \tilde{H} := -\frac{\partial^2}{\partial x^2} + \tilde{u}(x),$$

where $\tilde{u} = \tilde{u}^*$ is given by the formula

$$\tilde{u}(x) = u(x) + 2(X_{12}(x) + X_{21}(x) + X_{22}(x)^2),$$

and $X_{ij}$ are the blocks of $X = \Pi^*S^{-1}\Pi$. 


Proof. Taking into account the equality $X = \Pi^*S^{-1}\Pi$ and formulas (2.13), (3.4) and (3.7), we calculate $X_{22}^*:\]

\begin{align*}
X_{22}^* &= \begin{bmatrix} 0 & I_0 \end{bmatrix} (\Pi^*S^{-1})^*/\Pi \begin{bmatrix} 0 & I_0 \end{bmatrix} \Pi^*S^{-1}\Pi' \begin{bmatrix} 0 \\
0 & I_0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & I_0 \end{bmatrix} q_0^*\Pi^*S^{-1}\Pi \begin{bmatrix} 0 \\
0 & I_0 \end{bmatrix} - \begin{bmatrix} 0 & I_0 \end{bmatrix} \Pi^*S^{-1}\Pi \begin{bmatrix} 0 \\
0 & I_0 \end{bmatrix} \\
&= -X_{12} - X_{22}^* - X_{21}. \tag{3.11}
\end{align*}

In view of (3.11) (and definition (3.10) of $\tilde{u}$), we rewrite (3.6) in the form

\begin{align*}
z_2(x)A &= -z_2''(x) + (u(x) + 2(X_{12}(x) + X_{21}(x) + X_{22}(x)^2))z_2(x) \\
&= -z_2''(x) + \tilde{u}(x)z_2(x). \tag{3.12}
\end{align*}

According to (3.3), we have $z_2 = \begin{bmatrix} 0 & I_0 \end{bmatrix} \Pi^*S^{-1}$. Therefore, (3.8) and (3.12) imply (3.9). We also note that $\tilde{u} = u^*$ is immediate from $u = u^*$ and from formulas (2.14) and (3.10). ■

Remark 3.2. Under conditions $S(0) > 0$ and $x \in [0, \infty)$, the matrix function $S(x)$ is invertible (recall remark 2.3). Thus, the matrix functions $S(x)^{-1}$, $X(x)$, $\tilde{u}(x)$ and $\tilde{v}(x)$ (considered in theorem 3.1) are well-defined under these conditions.

2. If the conditions of theorem 3.1 and remark 3.2 are valid, we obtain the following corollary.

Corollary 3.3. Consider dynamical Schrödinger equations on $I = [0, \infty)$, let the conditions of theorem 3.1 hold, and assume that $S(0) > 0$. Then, the columns of the matrix function $\begin{bmatrix} 0 & I_0 \end{bmatrix} \Pi(x)^*S(x)^{-1}$ belong to $L^2([0, \infty))$ (i.e. these columns are squarely summable) and the solutions $\tilde{v}(x,t)g$ ($g \in \mathbb{C}^b$) of system (3.9) belong to $L^2([0, \infty))$ for each fixed $t$.

Proof. In view of the second equality in (3.7) and the first equality in (2.12), we have

\begin{equation}
(S(x)^{-1})' = -S(x)^{-1}\Pi(x)^*S(x)^{-1}. \tag{3.13}
\end{equation}

Formula (3.13) implies that

\begin{equation}
\int_0^t \left( \begin{bmatrix} 0 & I_0 \end{bmatrix} \Pi(x)^*S(x)^{-1} \right)^* \begin{bmatrix} 0 & I_0 \end{bmatrix} \Pi(x)^*S(x)^{-1} dx = S(0)^{-1} - S(t)^{-1} < S(0)^{-1}, \tag{3.14}
\end{equation}

which proves the corollary. ■

Remark 3.4. For the study of the dependence on time of the solutions (3.8) and (4.16) of the discrete and continuous, respectively, Schrödinger equations, one may use the representation of $A$ in Jordan normal form:

\begin{align*}
A &= U J U^{-1}, \\
J &= \text{diag}\{J_1, J_2, \ldots, J_N\}, \\
J_i &= \lambda_i I_n + K_i, \\
K_i &= \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix}. \tag{3.15}
\end{align*}
The Jordan representation above yields the equality
\[
e^{-i\lambda t} = \text{U} \text{diag} \left\{ e^{-i\lambda_1 t} \sum_{k=0}^{n_1-1} \frac{(-i\lambda_1)^k}{k!}, \ldots, e^{-i\lambda_n t} \sum_{k=0}^{n_n-1} \frac{(-i\lambda_n)^k}{k!} \right\} \text{U}^{-1}. \tag{3.16}
\]

Taking into account formula (3.8), corollary 3.3 and representation (3.16), we see that the following asymptotics is valid generically:
\[
\|\tilde{\psi}(x,t)\| = C_{\pm} |g| e^{\tau_{\pm} t} |t|^{\tau_{\pm}} (1 + O(1/t)) \quad \text{for} \quad t \to \pm \infty,
\]
where \(\| \cdot \|\) is the norm in \(L^2_\mathbb{R}(0,\infty), g \in \mathbb{C}^b, \tau_+ = \max_{1 \leq i \leq N} \Im(\lambda_i), \)
\[
\tau_- = \min_{1 \leq i \leq N} \Im(\lambda_i), r_{\pm} = \max_{\Im(\lambda_i) = \tau_{\pm}} (n_1 i - 1).
\]

We note that in a different way the Jordan structure of \(A\) was used in [28] to study (and explain) an interesting multi-lump phenomena discovered in [1].

3. Using considerations similar to those in paragraph 1 of this section, we construct GBDT for matrix Schrödinger equation (1.4). Solution \(w\) of system (2.1) with the coefficients given by (2.2) can be written down in the block form: \(w = \begin{bmatrix} y & \hat{y} \end{bmatrix} (y, \hat{y} \in \mathbb{C}^b).\) Hence, we rewrite (2.1) as
\[
y'(x, \lambda) = \hat{y}(x, \lambda), \quad \hat{y}'(x, \lambda) = -\lambda \hat{y}(x, \lambda) + u(x)y(x, \lambda),
\]
that is, (1.4) is fulfilled. So system (2.1), (2.2) is equivalent to the Schrödinger equation (1.4). The following proposition is a corollary of theorem 2.2.

**Proposition 3.5.** Let a vector function \(y(x, \lambda)\) satisfy the Schrödinger equation (1.4), where the \(h \times h\) potential \(u = u^*\) is locally summable on \(\mathcal{L}\), and assume that the conditions of theorem 2.2 hold. Then, the vector function
\[
\tilde{\hat{y}}(x, \lambda) = \begin{bmatrix} y & \hat{y} \end{bmatrix}, \quad \tilde{\hat{w}}(x, \lambda) := \hat{w}_1(x, \lambda)w(x, \lambda), \tag{3.17}
\]
with \(w = \begin{bmatrix} y & \hat{y} \end{bmatrix}\) satisfies the matrix Schrödinger equation
\[
-\tilde{\hat{y}}''(x, \lambda) + \bar{u}(x)\tilde{\hat{y}}(x, \lambda) = \lambda \tilde{\hat{y}}(x, \lambda), \tag{3.18}
\]
where \(\bar{u} = \bar{u}^*\) is given by the formula (3.10).

**Proof.** According to theorem 2.2, we have \(\hat{w}' = \tilde{G}\tilde{\hat{w}}.\) We rewrite this equation in terms of the blocks \(\tilde{y}\) and \(\hat{y} : = \begin{bmatrix} 0 & \hat{w} \end{bmatrix}\) of \(\tilde{w}:
\[
\tilde{y}' = -X_{22}\tilde{y} + \hat{y}, \quad \hat{y}' = -\lambda \hat{y} + (u + X_{12} + X_{21})\hat{y} + X_{22}\hat{y}. \tag{3.19}
\]
Differently \(\tilde{y}\) in the first equation above and using the second equation, we obtain
\[
\tilde{y}'' = -\lambda \tilde{y} + (u + X_{12} + X_{21} + X_{22}^2 - X_{22}'\hat{y}) \hat{y}. \tag{3.19}
\]
Relation (3.18) is immediate from (3.10) and (3.19).
Instead of the Schrödinger equation (3.18) one can talk about the transformed Schrödinger operator $\tilde{H} = -\frac{d^2}{dx^2} + \tilde{u}$ with a properly defined domain.

4. If $\Pi$ and $S$ are known explicitly, then representations (3.8) and (3.17) provide explicit solutions of Schrödinger systems (3.9) and (3.18), respectively. In particular, $\Pi$ and $S$ are easily constructed in the case $u \equiv 0$ (see [15]). For this purpose we partition $\Pi$ into $n \times h$ blocks: $\Pi = [\Lambda_1 \Lambda_2]$. Then, the first equation in (3.7) takes (for $u \equiv 0$) the form

$$\Lambda'_1 = A \Lambda_2, \quad \Lambda'_2 = -\Lambda_1.$$ (3.20)

Remark 3.6. When $u \equiv 0$, then $\Pi(x)$ in theorems 2.2 and 3.1 and in proposition 3.5 is given by the formulas $\Pi(x) = [\Lambda_1(x) \Lambda_2(x)]$ and

$$\begin{bmatrix} \Lambda_1(x) \\ \Lambda_2(x) \end{bmatrix} = e^{xA} \begin{bmatrix} \Lambda_1(0) \\ \Lambda_2(0) \end{bmatrix}, \quad A := \begin{bmatrix} 0 & A \\ -I_h & 0 \end{bmatrix},$$ (3.21)

which is immediate from (3.20). According to (3.7), the matrix function $S(x)$ is given by the formula

$$S(x) = S(0) + \int_0^x \Lambda_2(\eta) \Lambda_2(\eta)^* d\eta.$$ (3.22)

Recall that we know (that is, we choose) parameter matrices $A$, $S(0)$ and $\Pi(0)$ or, equivalently, $A$, $S(0)$ and $\Lambda_k(0)$ ($k = 1, 2$) which determine GBDT-transformation.

4. Discrete dynamical Schrödinger system

1. GBDT (generalized Bäcklund–Darboux transformation) was applied to important linear and nonlinear discrete systems in [11, 12, 19, 29, 34]. In particular, discrete canonical systems and non-Abelian Toda lattices were studied in [29]. Jacobi matrices corresponding to explicit solutions of matrix Toda lattices were considered in [29, appendix]. Using some modification of the results from [29, appendix], we construct here explicit solutions of discrete dynamical Schrödinger systems. We present also direct proofs of the corresponding modified results from [29, appendix], whereas in [29, appendix] several essential facts are proved indirectly (via the theory of discrete canonical systems developed in the previous sections of [29]) and some details of the proofs are omitted.

We start with introducing generalized Bäcklund–Darboux transformation (GBDT) of block Jacobi matrices. This GBDT is essential for the construction of the explicit solutions of non-Abelian Toda lattices [29] and of the explicit solutions (4.16) of the discrete dynamical system (1.2).

Suppose that the sets of $h \times h$ matrices $\{C(k)\}_{k>0}$ and $\{Q(k)\}_{k>0}$ such that

$$C(k)Q(k)^* = Q(k)C(k) \quad (k > 0), \quad C(k) > 0 \quad (k > 0),$$ (4.1)

are given. The corresponding initial Jacobi matrix is introduced by the relations
\[ \mathcal{J} = \begin{bmatrix} b_1 & a_1 & 0 & 0 & 0 & \ldots \\ c_2 & b_2 & a_2 & 0 & 0 & \ldots \\ 0 & c_3 & b_3 & a_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \end{bmatrix}, \quad c_k = a_{k-1}^*, \quad (4.2) \]

\[ a_k = -iC(k)^{-1}\frac{1}{2}C(k+1)^{1/2}, \quad b_k = C(k)^{-1/2} \tilde{Q}(k)C(k)^{1/2}, \quad (4.3) \]

where \( k > 0 \) and (according to (4.1) and (4.3)) \( b_k = b_k^* \).

Recall that GBDT is determined by three parameter matrices. Thus, we fix \( n > 0, \) two \( n \times n \) parameter matrices \( A \) and \( S_0 > 0 \) and an \( n \times m \) \( (m = 2h) \) parameter matrix \( \Pi_0 \) such that

\[ AS_0 - S_0A^* = i\Pi_0\Pi_0^*, \quad j = \begin{bmatrix} 0 & I_h \\ I_h & 0 \end{bmatrix}. \quad (4.4) \]

Everywhere in this section \( j \) is given by the second equality in (4.4). Introduce matrices \( \Pi_k \) and \( S_k \) for \( k > 0 \) by the recursions

\[ \Pi_k = \Pi_{k-1}\xi(k)^{-1} - iA\Pi_{k-1}P, \quad S_k = S_{k-1} + \Pi_{k-1}\zeta(k)\Pi_{k-1}^*, \quad (4.5) \]

where

\[ \xi(k) = \begin{bmatrix} -i\tilde{Q}(k) \\ C(k)^{-1} \end{bmatrix}, \quad \zeta(k) = \begin{bmatrix} 0 & 0 \\ 0 & C(k)^{-1} \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 \\ 0 & I_h \end{bmatrix}. \quad (4.6) \]

The following properties easily follow from (4.1) and (4.6): \( jPj = I_m - P, \)

\[ \xi(k)\xi(k)^* = \xi(k)^*\xi(k) = j, \quad P\xi(k)j = \zeta(k), \quad PjP = 0. \quad (4.7) \]

Therefore, taking adjoints of both parts of the first equality in (4.5) (and multiplying the result by \( i^2j \)) we obtain an equivalent to this equality relation

\[ i^2 j\Pi_k^* = i\xi(k)(i^{k-1}j\Pi_{k-1}^*) - (I_m - P)(i^{k-1}j\Pi_{k-1}^*)A^* \quad (m = 2h). \quad (4.8) \]

**Remark 4.1.** Setting in (4.8) \( i^2j\Pi_k^* = W(k) \) and \( A^* = z \), we obtain an auxiliary linear system (10.1.9) from [36] for the matrix Toda chain, which explains the choice of the equation on \( \Pi_k \) in (4.5). Namely, we see that this equation is a generalized auxiliary system for Toda chain with the generalized eigenvalue \( A \).

Since \( S_0 > 0 \) and \( C(k) > 0 \), the second equality in (4.5) yields \( S_k > 0 \) for \( k > 0 \). Setting

\[ X(k) = \{X_{\nu}(k)\}^2_{\nu=1} = \Pi_k^*S_k^{-1}\Pi_k \quad (k \geq 0), \quad (4.9) \]

we define the transformed matrices \( \tilde{C}(k) \) and \( \tilde{Q}(k) \) via relations

\[ \tilde{C}(k) = C(k) + X_{21}(k-1) \quad (k > 0), \quad (4.10) \]

\[ \tilde{Q}(k) = Q(k) + i(X_{21}(k-1) - X_{21}(k)) \quad (k > 0). \quad (4.11) \]

Clearly \( X(k) \geq 0 \) for \( k \geq 0, \) and so \( \tilde{C}(k) > 0 \) for \( k > 0. \) Then, the transformation (GBDT) \( \tilde{\mathcal{J}} \) of the block Jacobi matrix \( \mathcal{J} \) is defined by the equalities

\[ \tilde{\mathcal{J}} = \begin{bmatrix} b_1 & \tilde{a}_1 & 0 & 0 & 0 & \ldots \\ \tilde{c}_2 & b_2 & \tilde{a}_2 & 0 & 0 & \ldots \\ 0 & \tilde{c}_3 & b_3 & \tilde{a}_3 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \end{bmatrix}, \quad \tilde{c}_k = \tilde{a}_{k-1}^*, \quad (4.12) \]
\[ \tilde{a}_k = -i \tilde{c}(k)^{-1/2} \tilde{C}(k+1)^{1/2}, \quad \tilde{b}_k = \tilde{c}(k)^{-1/2} \tilde{Q}(k) \tilde{C}(k)^{1/2}. \] (4.13)

We note that formulas (4.12) and (4.13) coincide (after removal of tildes) with the formulas (4.2) and (4.3) which define \( \mathcal{F} \).

According to [29, appendix], we have \( \tilde{b}_k = \tilde{b}_k^* \). Slightly modifying the proof of [29, theorem A.1], one may derive that under condition

\[ [J_h \quad 0] \Pi_k^* S_0^{-1} = 0 \] (4.14)

we have

\[ \mathcal{F} Y = YA \quad (Y = \{ y_k \}_{k \geq 1}, \quad y_k := [0 \tilde{C}(k)^{-1/2}] \Pi_{k-1}^* S_{k-1}^{-1}). \] (4.15)

**Theorem 4.2.** Suppose that Jacobi matrix \( \mathcal{F} \) is given by the formulas (4.12) and (4.13), that relations (4.1), (4.4) and (4.14) are valid, and that the matrices \( \tilde{C}(k) \) and \( \tilde{Q}(k) \) in (4.13) are given by (4.5)–(4.11).

Then the block vector function

\[ \Psi(t) = Ye^{-iAt}, \] (4.16)

where \( Y \) is introduced in (4.15), satisfies the discrete dynamical Schrödinger system

\[ i \frac{\partial}{\partial \tau} \Psi (t) = \mathcal{F} \Psi (t). \]

Theorem 4.2 is immediate from (4.15) and it remains to prove (4.15). More precisely, we prove the following theorem.

**Theorem 4.3.** Suppose that the relations (4.1) and (4.4) (where \( S_0 > 0 \)) are valid. Then the matrices \( \tilde{C}(k) \) and \( \tilde{Q}(k) \) given by (4.5)–(4.11) are well-defined and satisfy relations

\[ \tilde{C}(k)\tilde{Q}(k)^* = \tilde{Q}(k)\tilde{C}(k) \quad (k > 0), \quad \tilde{C}(k) > 0 \quad (k > 0). \] (4.17)

We also have \( \tilde{b}_k = \tilde{b}_k^* \) for the matrices \( \tilde{b}_k \) in (4.13).

If, in addition, (4.14) holds, then the matrix \( \mathcal{F} \) of the form (4.12), (4.13) satisfies (4.15).

**Proof.**

Step 1. Taking into account the inequalities \( S_0 > 0, \ C(k) > 0 \) and relations (4.5), (4.10), we explained already that \( S_k > 0 \ (k \geq 0) \) and that \( \tilde{C}(k) > 0 \ (k > 0) \). Therefore, the matrices \( \tilde{Q}(k) \) and \( \tilde{C}(k) \) are well-defined, the inequality for \( \tilde{C}(k) \) in (4.17) is valid, \( \tilde{C}(k) \) is invertible, and \( \mathcal{F} \) is also well-defined.

Using (4.4) we show by induction that the matrix identity

\[ AS_k - S_k A^* = i \Pi_k \sqrt{\Pi_k^*} \] (4.18)

holds for all \( k \geq 0 \). Namely, let us assume that the identity

\[ AS_{k-1} - S_{k-1} A^* = i \Pi_{k-1} \sqrt{\Pi_{k-1}^*} \]

is valid for some \( k > 0 \). Then, in view of the second equality in (4.5) we have

\[ AS_k - S_k A^* = AS_{k-1} - S_{k-1} A^* + A \Pi_{k-1} \zeta(k) \Pi_{k-1}^* - \Pi_{k-1} \zeta(k) \Pi_{k-1}^* A^* \]
\[ = i \Pi_{k-1} \sqrt{\Pi_{k-1}^*} + A \Pi_{k-1} \zeta(k) \Pi_{k-1}^* - \Pi_{k-1} \zeta(k) \Pi_{k-1}^* A^*. \] (4.19)
On the other hand, the first equality in (4.5) and relations (4.7) imply that
\[ \imath \Pi_k \Pi_k^* = \imath \Pi_{k-1} \Pi_{k-1}^* + A \Pi_{k-1} \xi(k) \Pi_{k-1}^* - \Pi_{k-1} \xi(k) \Pi_{k-1}^* A^*. \] (4.20)
(Here we used also the equality \( \xi(k)^{-1} = j \xi(k)^* j \), which is immediate from (4.7).)

Comparing (4.19) and (4.20) we obtain (4.18).

Step 2. Next, we prove the equality
\[ \Pi_k^* S_k^{-1} = \imath P \Pi_{k-1}^* S_{k-1}^{-1} A + j \tilde{\xi}(k) j \Pi_{k-1}^* S_{k-1}^{-1}, \] (4.21)

\[ \tilde{\xi}(k) := \begin{bmatrix} -\imath \tilde{Q}(k) & \tilde{C}(k) \\ \tilde{C}(k) & 0 \end{bmatrix}, \quad \tilde{C}(k) = C(k)^{-1} - X_{11}(k). \] (4.22)

Indeed, taking into account the second relation in (4.5), the equality
\[ j \xi(k)^* P = \xi(k), \]
which is immediate from (4.7), and the equalities
\[ j (I_m - P) j = P, \quad P (I_m - P) = 0, \]
we derive
\[ S_k^{-1} - S_{k-1}^{-1} = -S_k^{-1} \Pi_{k-1} \xi(k) \Pi_{k-1}^* S_{k-1}^{-1} \]
\[ = -S_k^{-1} (\Pi_{k-1} j \xi(k)^* j - \imath A \Pi_{k-1} P) (I_m - P) j \Pi_{k-1}^* S_{k-1}^{-1}. \] (4.23)

In view of the first relation in (4.5) and the equality \( j \xi(k)^* j = \xi(k)^{-1} \), we rewrite (4.23) in the form
\[ S_k^{-1} - S_{k-1}^{-1} = -S_k^{-1} \Pi_k (I_m - P) j \Pi_k^* S_{k-1}^{-1}. \] (4.24)

Multiplying both sides of (4.24) by \( \Pi_k^* \) from the left and using again the first relation in (4.5), we see that
\[ \Pi_k^* S_k^{-1} = \Pi_k^* S_{k-1}^{-1} - X(k) (I_m - P) j \Pi_{k-1}^* S_{k-1}^{-1} \]
\[ = j \xi(k) j \Pi_{k-1}^* S_{k-1}^{-1} + i \Pi_k^* A^* S_{k-1}^{-1} - X(k) (I_m - P) j \Pi_{k-1}^* S_{k-1}^{-1}. \] (4.25)

Substituting (into (4.18)) \( k - 1 \) instead of \( k \), we rewrite the result in the form
\[ A^* S_{k-1}^{-1} = S_{k-1}^{-1} A - i S_{k-1}^{-1} \Pi_k - j \Pi_{k-1}^* S_{k-1}^{-1}. \] (4.26)

After substitution of (4.26) into (4.25), we obtain
\[ \Pi_k^* S_k^{-1} = i \Pi_{k-1}^* S_{k-1}^{-1} A + j \xi(k) j \Pi_{k-1}^* S_{k-1}^{-1} + PX(k - 1) j \Pi_{k-1}^* S_{k-1}^{-1} \]
\[ - X(k) (I_m - P) j \Pi_{k-1}^* S_{k-1}^{-1}. \] (4.27)

Equality (4.21) follows from (4.27).

Step 3. Recall that \( \xi(k) \) is \( j \)-unitary, that is, the relation \( \xi(k) j \xi(k)^* = j \) (or, equivalently, \( \xi(k)^* j \xi(k) = j \)) in (4.7) holds. It is important to show that the transformed matrix \( \tilde{\xi}(k) \)
introduced in (4.22) is \( j \)-unitary as well. Taking into account (4.10) and (4.11) we rewrite (4.22) in the form
\[ \tilde{\xi}(k) = \xi(k) - j X(k) (I_m - P) + j PX(k - 1). \] (4.28)
Assuming that $\det A \neq 0$, we prove the equality
\begin{equation}
\tilde{\xi}(k) = \hat{\nu}(k)\xi(k)\hat{\nu}(k - 1)^{-1}, \quad \hat{\nu}(k) := I_m - \text{i}\Pi_k^* S_k^{-1}A^{-1}\Pi_k.
\end{equation}
(4.29)

We note that $\hat{\nu}(k) = w_A(k, 0)$, where $w_A(k, \lambda) = I_m - \text{i}\Pi_k^* S_k^{-1}(A - \lambda I_n)^{-1}\Pi_k$ (with matrices $A, \Pi_k, S_k$ satisfying (4.18)) is the transfer matrix function in Lev Sakhnovich form. (Compare with $w_A$ in (2.8).) According to [35] (see also [34, p 24]) we have $w_A(k, \lambda)jw_A(k, X)^* = j$, and so the matrices $\hat{\nu}(k) (k \geq 0)$ are $j$-unitary. Thus, (4.29) implies that $\tilde{\xi}(k)$ is $j$-unitary. If $\text{det}(A - \lambda I_n) \neq 0$ for small values $\lambda = X$ and approximate $\hat{\xi}(k)$ with the $j$-unitary matrices $\hat{\xi}(k)$ corresponding to the GBDT-generating triples $A - \lambda I_n, S_0, \Pi_0$. Hence, the equality (4.29) for the case $\text{det} A \neq 0$ yields the $j$-unitarity property
\begin{equation}
\tilde{\xi}(k)\tilde{\xi}(k)^* = \hat{\xi}(k)^*\hat{\xi}(k) = j
\end{equation}
(4.30)
without restriction on $\det A$. It remains to show that (4.29) is valid.

Indeed, let us rewrite (4.29) in the form
\begin{equation}
\xi(k)(I_m - \text{i}\Pi_k^* S_k^{-1}A^{-1}\Pi_{k-1}) = (I_m - \text{i}\Pi_k^* S_k^{-1}A^{-1}\Pi_k)\xi(k).
\end{equation}
(4.31)

Using the first relation in (4.5), we rewrite (4.31) in another equivalent form
\begin{equation}
\tilde{\xi}(k) - \text{i}\tilde{\xi}(k)j\Pi_{k-1}^* S_{k-1}^{-1}A^{-1}\Pi_{k-1} = \xi(k) - \text{i}\Pi_k^* S_k^{-1}A^{-1}\Pi_{k-1} - j\Pi_k^* S_k^{-1}\Pi_{k-1}P\xi(k).
\end{equation}
(4.32)

Substituting the expression for $\Pi_k^* S_k^{-1}$ from (4.21) into the second right hand term in (4.32), we obtain
\begin{equation}
\tilde{\xi}(k) - \text{i}\tilde{\xi}(k)j\Pi_{k-1}^* S_{k-1}^{-1}A^{-1}\Pi_{k-1} = \xi(k) + j\Pi_k^* S_k^{-1}\Pi_{k-1}P\xi(k).
\end{equation}
(4.33)

Therefore, using (4.28) and canceling similar terms we derive
\begin{equation}
-jX(k)(I_m - P) + jPX(k - 1) = j\Pi_k^* S_k^{-1}\Pi_{k-1}P\xi(k).
\end{equation}
Recalling that by definition $X(k - 1) = \Pi_k^* S_k^{-1}\Pi_{k-1}$ we further simplify the equality above, and so the following relation:
\begin{equation}
-j\Pi_k^* S_k^{-1}\Pi_k(I_m - P) = -j\Pi_k^* S_k^{-1}\Pi_{k-1}P\xi(k)
\end{equation}
(4.34)
is equivalent to (4.29). In view of (4.5) and (4.6), we have
\begin{equation}
\Pi_k(I_m - P) = \Pi_{k-1}P\xi(k),
\end{equation}
which proves (4.34). Thus, (4.29) is proved as well, and hence (4.30) holds.

In particular, (4.30) yields
\begin{equation}
(I_m - P)\tilde{\xi}(k)\tilde{\xi}(k)^* = 0, \quad P\tilde{\xi}(k)\tilde{\xi}(k)^* = I_h.
\end{equation}
(4.35)

According to (4.22), formula (4.35) is equivalent to the equalities
\begin{equation}
\tilde{C}(k)\tilde{Q}(k)^* - \tilde{Q}(k)\tilde{C}(k) = 0, \quad \tilde{C}(k) = C(k)^{-1}.
\end{equation}
(4.36)
Therefore, the first equality in (4.17) is valid, and we also rewrite (4.22) in the form
\[
\tilde{\xi}(k) = \begin{bmatrix} -i\tilde{Q}(k) \\ \tilde{C}(k)^{-1} \\ 0 \end{bmatrix}.
\] (4.37)

Comparing (4.37) and the first equality in (4.6), we see that the representations of \(\tilde{\xi}(k)\) and \(\xi(k)\) differ only by tildes in the notations.

The equality \(\tilde{b}_k = b^*_k\) (for \(\tilde{b}_k\) given by (4.13)) is immediate from the first relation in (4.17).

Step 4. Finally, let us prove (4.15). Using equalities (4.13) and the definition of \(Y\) in (4.15), we obtain
\[
\tilde{a}_k y_{k+1} = -i\tilde{C}(k)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_k S^{-1}_{k-1},
\] (4.38)
\[
\tilde{b}_k y_{k} = \tilde{C}(k)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_k S^{-1}_{k-1}.
\] (4.39)

Relations (4.21), (4.22) and (4.38), (4.39) imply that
\[
\tilde{a}_k y_{k+1} + \tilde{b}_k y_{k} = \tilde{C}(k)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_k S^{-1}_{k-1} A - i\tilde{C}(k)^{\frac{1}{2}} \begin{bmatrix} I_h & 0 \end{bmatrix} \Pi^*_k S^{-1}_{k-1}.
\] (4.40)

In particular, taking into account (4.14), we derive
\[
\tilde{b}_1 y_1 + \tilde{a}_1 y_2 = \tilde{C}(1)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_0 S^{-1}_0 A = y_1 A.
\] (4.41)

From (4.21) and (4.37) we see that
\[
\begin{bmatrix} I_h & 0 \end{bmatrix} \Pi^*_k S^{-1}_{k-1} = \tilde{C}(k-1)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_k S^{-1}_{k-2} (k > 1).
\] (4.42)

According to (4.13) and (4.42) we have
\[
\tilde{a}^*_k y_{k-1} = i\tilde{C}(k)^{\frac{1}{2}} \tilde{C}(k-1)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_k S^{-1}_{k-2} = i\tilde{C}(k)^{\frac{1}{2}} \begin{bmatrix} I_h & 0 \end{bmatrix} \Pi^*_k S^{-1}_{k-1}.
\] (4.43)

Now, formulas (4.40) and (4.43) yield for \(k > 1\) that
\[
\tilde{a}^*_k y_{k-1} + \tilde{b}_k y_{k} + \tilde{a}_k y_{k+1} = \tilde{C}(k)^{-1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Pi^*_k S^{-1}_{k-1} A = y_k A.
\] (4.44)

Equalities (4.41) and (4.44) imply (4.15).

5. Conclusion

Darboux transformations for the matrix Schrödinger operators in several dimensions have been actively studied (in particular, in the important works by Suzko and collaborators) [6, 30, 38–40]. In the papers mentioned above, the potentials depended on time. The special case of dynamical Dirac and Schrödinger equations, where the potential does not depend on time, is of physical interest (for instance, for the case of 2D-materials) and is studied in [2, 4, 17, 32] (see also further references therein). In this special case, GBDT enables us to construct wide classes of solutions (of the form (3.8)) of the dynamical Schrödinger equation (3.9).
The asymptotics of these solutions is discussed in remark 3.4. The GBDT for the discrete case, that is for the equation (1.2), is in some respects easier (less constraints) but the formulas look more complicated. However, the representation (4.16) of the solution has basically the same form as in the continuous case. A more detailed study of the obtained solutions and application of the approach to other dynamical systems would be of interest, and we plan some further work in this direction.

Acknowledgments

A L Sakhnovich is grateful to S Avdonin for a fruitful discussion. The research of A L Sakhnovich was supported by the Austrian Science Fund (FWF) under Grant No. P29177.

ORCID iDs

A L Sakhnovich https://orcid.org/0000-0002-1313-3895

References

[1] Ablowitz M J, Chakravarty S, Trubatch A D and Villarroel J 2000 A novel class of solutions of the non-stationary Schrödinger and the Kadomtsev–Petviashvili I equations Phys. Lett. A 267 132–46
[2] Avdonin S, Mikhailov V and Ramdani K 2014 Reconstructing the potential for the 1D Schrödinger equation from boundary measurements IMA J. Math. Control Inform. 31 137–50
[3] Belishev M and Mikhailov V 2014 Inverse problem for a 1D dynamical Dirac system (BC-method) Inverse Problems 30 125013
[4] Bellassoued M 2017 Stable determination of coefficients in the dynamical Schrödinger equation in a magnetic field Inverse Problems 33 055009
[5] Boiti M, Pempinelli F and Pogrebkov A K 2006 On the extended resolvent of the nonstationary Schrödinger operator for a Darboux transformed potential J. Phys. A: Math. Gen. 39 1877–98
[6] Chalykh O A and Oblomkov A A 2000 Harmonic oscillator and Darboux transformations in many dimensions Phys. Lett. A 267 256–64
[7] Cieslinski J L 2009 Algebraic construction of the Darboux matrix revisited J. Phys. A: Math. Theor. 42 404003
[8] Deift P A 1978 Applications of a commutation formula Duke Math. J. 45 267–310
[9] Egorova I E, Kopylova E A, Marchenko V A and Teschl G 2016 Dispersion estimates for one-dimensional Schrödinger and Klein–Gordon equations revisited Russ. Math. Surv. 71 391–415
[10] Egorova I E, Kopylova E A and Teschl G 2015 Dispersion estimates for 1D discrete Schrödinger and wave equations J. Spectr. Theory 5 663–96
[11] Fritzsch B, Kaashoek M A, Kirstein B and Sakhnovich A L 2016 Skewselfadjoint Dirac systems with rational rectangular Weyl functions: explicit solutions of direct and inverse problems and integrable wave equations Math. Nachr. 289 1792–819
[12] Fritzsch B, Kirstein B, Roitberg I and Sakhnovich A L 2008 Weyl matrix functions and inverse problems for discrete Dirac type self-adjoint system: explicit and general solutions Oper. Matrices 2 201–31
[13] Gesztesy F 1993 A complete spectral characterization of the double commutation method J. Funct. Anal. 117 401–46
[14] Gesztesy F and Teschl G 1996 On the double commutation method Proc. Am. Math. Soc. 124 1831–40
[15] Gohberg I, Kaashoek M A and Sakhnovich A L 1998 Sturm–Liouville systems with rational Weyl functions: explicit formulas and applications Integral Equ. Oper. Theory 30 338–77
[16] Gu C H, Hu H and Zhou Z 2005 Darboux Transformations in Integrable Systems: Theory and their Applications to Geometry (Mathematical Physics Studies vol 26) (Dordrecht: Springer)
[17] Hartmann R and Portnoi M E 2014 Quasi-exact solution to the Dirac equation for the hyperbolic-secant potential Phys. Rev. A 89 012101
[18] Jensen A and Kato T 1979 Spectral properties of Schrödinger operators and time-decay of the wave functions Duke Math. J. 46 583–611
[19] Kaashoek M A and Sakhnovich A L 2005 Discrete skew self-adjoint canonical system and the isotropic Heisenberg magnet model J. Funct. Anal. 228 207–33
[20] Kopylova E A and Teschl G 2016 Dispersion estimates for 1D discrete Dirac equations J. Math. Anal. Appl. 434 191–208
[21] Kostenko A, Sakhnovich A L and Teschl G 2012 Commutation methods for Schrödinger operators with strongly singular potentials Math. Nachr. 285 392–410
[22] Marchenko V A 1988 Nonlinear Equations and Operator Algebras (Mathematics and its Applications (Soviet Series) vol 17) (Dordrecht: D Reidel)
[23] Matveev V B and Salle M A 1991 Darboux Transformations and Solitons (Berlin: Springer)
[24] Prado R A and de Oliveira C R 2012 Sparse 1D discrete Dirac operators I: quantum transport J. Math. Anal. Appl. 385 947–60
[25] Sakhnovich A L 1994 Dressing procedure for solutions of nonlinear equations and the method of operator identities Inverse problems 10 699–710
[26] Sakhnovich A L 2001 Generalized Bäcklund–Darboux transformation: spectral properties and nonlinear equations J. Math. Anal. Appl. 262 274–306
[27] Sakhnovich A L 2003 Non-Hermitian matrix Schrödinger equation: Bäcklund–Darboux transformation, Weyl functions and PT-symmetry J. Phys. A: Math. Gen. 36 7789–802
[28] Sakhnovich A L 2003 Matrix Kadomtsev–Petviashvili equation: matrix identities and explicit non-singular solutions J. Phys. A: Math. Gen. 36 5023–33
[29] Sakhnovich A L 2007 Discrete canonical system and non-Abelian Toda lattice: Bäcklund–Darboux transformation and Weyl functions Math. Nachr. 280 631–53
[30] Sakhnovich A L 2011 The time-dependent Schrödinger equation of dimension $k + 1$: explicit and rational solutions via GBDT and multinodes J. Phys. A: Math. Theor. 44 475201
[31] Sakhnovich A L 2015 Dynamical and spectral Dirac systems: response function and inverse problems J. Math. Phys. 56 112702
[32] Sakhnovich A L 2017 Dynamics of electrons and explicit solutions of Dirac–Weyl systems J. Phys. A: Math. Theor. 50 115201
[33] Sakhnovich A L 2017 Dynamical canonical systems and their explicit solutions Discrete Continuous Dyn. Syst. A 37 1679–89
[34] Sakhnovich A L, Sakhnovich L A and Roitberg I 2013 Inverse Problems and Nonlinear Evolution Equations. Solutions, Darboux Matrices and Weyl–Titchmarsh Functions (De Gruyter Studies in Mathematics vol 47) (Berlin: De Gruyter)
[35] Sakhnovich L A 1976 On the factorization of the transfer matrix function Sov. Math. Dokl. 17 203–7
[36] Sakhnovich L A 1997 Interpolation Theory and its Applications (Mathematics and its Applications vol 428) (Dordrecht: Kluwer)
[37] Sakhnovich L A 1999 Spectral Theory of Canonical Differential Systems. Method of Operator Identities (Operator Theory, Advances and Applications vol 107) (Basel: Birkhäuser)
[38] Schulze-Halberg A, Pozdeeva E and Suzko A 2009 Explicit Darboux transformations of arbitrary order for generalized time-dependent Schrödinger equations J. Phys. A: Math. Theor. 42 115211
[39] Suzko A A 2003 Exactly solvable models with time-dependent potentials Phys. Lett. A 308 267–79
[40] Suzko A A and Schulze-Halberg A 2009 Darboux transformations and supersymmetry for the generalized Schrödinger equations in (1 + 1) dimensions J. Phys. A: Math. Theor. 42 295203
[41] Teschl G 2000 Jacobi Operators and Completely Integrable Nonlinear Lattices (Mathematical Surveys and Monographs vol 72) (Providence, RI: American Mathematical Society)
[42] Teschl G 2012 Ordinary Differential Equations and Dynamical Systems (Graduate Studies in Mathematics vol 140) (Providence, RI: American Mathematical Society)