A NOTE ON PRODUCT OF MEASURES

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Abstract. A slight modification to Halmos’ definition of product of measures yields a uniquely characterized associative product. The operation applies to arbitrary (not necessarily \(\sigma\)-finite) measures and is consistent with the Fubini–Tonelli theorem.

1. Introduction

In elementary context, it is a generally accepted convention that any reasonable product \(\mu \otimes \nu\) of two measures defined in measure spaces, say \((S, \mathcal{G})\) and \((T, \mathcal{T})\), takes advantage of some \(\sigma\)-finiteness assumptions. One of the most general elementary definitions, proposed by Halmos in \([H]\), assumes that \(\mathcal{G}\) and \(\mathcal{T}\) are \(\sigma\)-rings, and requires \(\mu\) and \(\nu\) to be \(\sigma\)-additive \(\sigma\)-finite measures on the rings. Keeping in mind that there exist various refined and elaborated generalizations of the product of measures (see e.g. \([F]\)), I would like to note that the Halmos’ approach can be easily and successfully applied to arbitrary measures. While being still important, \(\sigma\)-finiteness is no longer an assumption – it becomes the border between a computable (numeric) part and a declarative, purely infinite part of any measure.

2. The product

Definition 2.1. By a measurable space we shall mean any pair \((S, \mathcal{G})\) composed of a nonempty set \(S\) and a \(\sigma\)-ring \(\mathcal{G}\) of subsets of \(S\). An extended \(\sigma\)-additive real function \(\mu : \mathcal{G} \to [0, \infty]\) will be called a measure in the measurable space if \(\mu(\emptyset) = 0\). A triple \((S, \mathcal{G}, \mu)\) is a measure space if \(\mu\) is a measure in \((S, \mathcal{G})\).

We recall that a family \(\mathcal{C}\) of sets is a \(\sigma\)-ring if it is closed under countable unions and if \(A \setminus B \in \mathcal{C}\) whenever \(A, B \in \mathcal{C}\).

Any measure \(\mu\) in \((S, \mathcal{G})\) distinguishes the family of sets of finite measure, \(\mathcal{G}_\mu^f = \{A \in \mathcal{G}; \mu(A) < \infty\}\), as well as the \(\sigma\)-ring \(\mathcal{G}_\mu\) of \(\sigma\)-finite sets. Precisely, \(\mathcal{G}_\mu\) consists of all the unions of countable subsets of \(\mathcal{G}_\mu^f\), and is the smallest \(\sigma\)-ring containing all sets of finite measure.

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Definition 2.2. For an arbitrary measure space \((S, \mathcal{G}, \mu)\) the restriction of \(\mu\) to the \(\sigma\)-ring \(\mathcal{G}_\mu^\sigma\) will be called the \(\sigma\)-finite component of the measure and will be denoted by \(\mu^\sigma\). The corresponding triple \((S, \mathcal{G}_\mu^\sigma, \mu^\sigma)\) is the \(\sigma\)-finite component of the measure space.

The component is a \(\sigma\)-finite measure in terminology used by Halmos. In fact, Halmos calls a measure \(\sigma\)-finite if and only if its domain is generated by sets of finite measure.

The pair composed of \(\mathcal{G}\) and \(\mathcal{G}' = \mathcal{G}_\mu^\sigma\) has an important property

\[
\mathcal{G}' \subset \mathcal{G} \quad \text{and} \quad \forall_{A \in \mathcal{G}} \forall_{B \in \mathcal{G}'} A \subset B \implies A \in \mathcal{G}'.
\]

**Proposition 2.3.** Given any pair \((\mathcal{G}, \mathcal{G}')\) of \(\sigma\)-rings having the simple extension property \([2.1]\), for every measure \(\mu': \mathcal{G}' \to [0, \infty]\) the extension \(\mu\) of \(\mu'\) such that \(\mu(A) = \infty\), for \(A \in \mathcal{G} \setminus \mathcal{G}'\), is a measure.

**Proof.** For any equality of the form \(C = \bigcup_n C_n\), where \(C_n \in \mathcal{G}\) as \(n \in \mathbb{N}\), one has \(C \in \mathcal{G}'\) if and only if every summand is in \(\mathcal{G}'\). \(\square\)

For any family of sets \(\mathcal{C}\) we shall denote by \(\sigma(\mathcal{C})\) the \(\sigma\)-ring generated by the family, i.e. the smallest \(\sigma\)-ring containing \(\mathcal{C}\). Obviously, one has \(\mathcal{G}_\mu^\sigma = \sigma(\mathcal{G}_\mu^\mu)\). An analogous notion of \(\sigma\)-algebra is relative and depends on the space that is a fixed set, say \(S\), such that \(\bigcup \mathcal{C} \subset S\). The \(\sigma\)-ring is a \(\sigma\)-algebra if and only if contains the space \(S\). Modifying a classical notation and making it more precise we set

\[
\sigma_S(\mathcal{C}) := \sigma(\mathcal{C} \cup \{S\})
\]

for the \(\sigma\)-algebra generated by \(\mathcal{C}\). Any function \(f: S \to \mathbb{R}\) is measurable with respect to a \(\sigma\)-ring \(\mathcal{G}\) if \(f\) is \(\sigma_S(\mathcal{G})\)-measurable and \(\{x; f(x) \neq 0\} \in \mathcal{G}\).

Obviously, \(\sigma_S(\mathcal{G}) = \mathcal{G} \cup \{S \setminus A; A \in \mathcal{G}\}\).

We recall that for any two \(\sigma\)-rings \(\mathcal{G}\) and \(\mathfrak{T}\) the \(\sigma\)-ring

\[
\mathcal{G} \otimes \mathfrak{T} = \sigma(\{A \times B; A \in \mathcal{G}, B \in \mathfrak{T}\})
\]

is called the product of the \(\sigma\)-rings. Clearly, the product is a \(\sigma\)-algebra if and only if both \(\mathcal{G}\) and \(\mathfrak{T}\) are \(\sigma\)-algebras. By the product \((S, \mathcal{G}) \times (T, \mathfrak{T})\) of two measurable spaces we mean the product space \(S \times T\) equipped with the product \(\sigma\)-ring \(\mathcal{G} \otimes \mathfrak{T}\). The product of measurable spaces is associative.

Let us consider arbitrary measure spaces \((S, \mathcal{G}, \mu)\) and \((T, \mathfrak{T}, \nu)\).

**Theorem 2.4** (Product of \(\sigma\)-finite measures, see \([H]\)). If the \(\sigma\)-rings \(\mathcal{G}\) and \(\mathfrak{T}\) are generated by sets of finite measure then there exists a unique measure \(\mu \otimes \nu: \mathcal{G} \otimes \mathfrak{T} \to [0, \infty]\) such that

\[
(\mu \otimes \nu)(A \times B) = \mu(A) \cdot \nu(B) \quad \text{for} \ A \in \mathcal{G}, B \in \mathfrak{T}.
\]
Remark. Although the proof presented in [H] uses the Lebesgue integral, there exist more direct proofs (see e.g. [D]) which concentrate on the case when both measures are finite. By uniqueness, the construction is then extended to a consistent family of measures on arbitrary products $S' \times T' \subset S \times T$ of $\sigma-$finite measurable sets $S' \in \mathcal{G}, T' \in \mathcal{I}$.

Without any assumption on the measures we claim what follows.

**Corollary 2.5.** There exists a unique measure $\mu \otimes \nu$ in $(S \times T, \mathcal{G} \otimes \mathcal{I})$ such that

$$\sigma - \text{finite component is the product of } \sigma - \text{finite components of } \mu \text{ and } \nu. \text{ The product of measures } \mu \otimes \nu \text{ is the only measure in the product of measurable spaces which has the following two properties:}$$

(i) $(\mu \otimes \nu)(A \times B) = \mu(A) \cdot \nu(B)$ for any $A \in \mathcal{G}^f, B \in \mathcal{I}^f$.

(ii) The $\sigma-$ring $(\mathcal{G} \otimes \mathcal{I})^\sigma_{\mu \otimes \nu}$ of all $\sigma-$finite sets in $\mathcal{G} \otimes \mathcal{I}$ is generated by the family \{ $A \times B; A \in \mathcal{G}^f, B \in \mathcal{I}^f$ \}.

**Proof.** According to lemma 3.2, the pair $(\mathcal{G} \otimes \mathcal{I}, \mathcal{G}^\sigma_\mu \otimes \mathcal{I}^\sigma_\nu)$ has the simple extension property. Thus the measure $\mu^\sigma \otimes \nu^\sigma$ is uniquely extendible to a measure in $(S \times T, \mathcal{G} \otimes \mathcal{I})$ which has no more sets of finite measure. □

In view of associativity of the product of $\sigma-$finite measures, equality (2.2) gives rise to

**Corollary 2.6.** The above product of arbitrary measures is associative. □

**Remark.** The product measure $\mu \otimes \nu$ can be obtained via the Caratheodory formalism, if one starts with the semi-ring of ”rectangles” $A \times B$ and the function $A \times B \mapsto \mu(A) \cdot \nu(B)$, for $A$ and $B$ of finite measure – as in (i). However, if at least one of sets $A \in \mathcal{G}, B \in \mathcal{I}$ is \textit{not $\sigma-$finite} and the other set is nonempty then $(\mu \otimes \nu)(A \times B) = \infty$, while the product $\mu(A) \cdot \nu(B)$ equals either $\infty$ or 0 (according to the \textit{axiom } $\infty \cdot 0 = 0$).

**Example 2.7.** In the Borel measurable space $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ on the real line the product of the Lebesgue measure $\ell$ and the counting measure $\delta$ is a Borel measure on $\mathbb{R}^2$. Any Borel set $B \subset \mathbb{R}^2$ is $\sigma-$finite with respect $\ell \otimes \delta$ if and only if it is of the form $B = \bigcup_{n \in \mathbb{N}} A_n \times \{a_n\}$, where $A_n \in \mathcal{B}_\mathbb{R}$ and $a_n \in \mathbb{R}$ for $n \in \mathbb{N}$.

In order to deal with Lebesgue integrals, we propose the following
Definition 2.8. For an arbitrary measure space \((S, \mathcal{G}, \mu)\) by a Lebesgue integral with respect to the measure we mean the only non-negative linear functional \(\int d\mu: \mathcal{I}(S, \mu) \to \mathbb{R}, f \mapsto \int f d\mu\), where the linear space \(\mathcal{I}(S, \mu)\) consists of finite \(\mathcal{G}\)-measurable real functions on \(S\), and

(i) for any set \(A \in \mathcal{G}\), one has \(1_A \in \mathcal{I}(S, \mu) \iff \mu(A) < \infty\), and if \(A \in \mathcal{G}_\mu\), then \(\int 1_A d\mu = \mu(A)\);

(ii) for every non-decreasing sequence \((f_n)_{n \in \mathbb{N}}\) bounded at each point of \(S\), if the sequence \((\int f_n d\mu)_{n \in \mathbb{N}}\) is bounded then the pointwise limit \(f = \lim_{n \to \infty} f_n\) is an element of \(\mathcal{I}(S, \mu)\), and one has
\[
\int f d\mu = \lim_{n \to \infty} \int f_n d\mu.
\]

(iii) if \(f \in \mathcal{I}(S, \mu)\) then also \(|f|, \min(f, 1) \in \mathcal{I}(S, \mu)\);

Finite \(\mathcal{G}\)-measurable functions which are elements of \(\mathcal{I}(S, \mu)\) are called \(\mu\)-integrable.

Remark. Properties (i)–(ii) are well-known to characterize the Lebesgue integral as the linear functional having the smallest domain, and condition (iii) assures that the space \(\mathcal{I}(S, \mathcal{G})\) of integrable functions is not bigger. In fact, property (iii) means that the space of \(\mu\)-integrable functions is a Stone lattice. Together with the other properties, the Stone condition \(\min(f, 1) \in \mathcal{I}(S, \mu)\) is equivalent to the assertion \(\{x; f(x) \neq 0\} \in \mathcal{G}_\sigma^\mu\) for \(f \in \mathcal{I}(S, \mu)\), and is superfluous if the measure space is \(\sigma\)-finite.

Corollary 2.9. The Lebesgue integral with respect to an arbitrary measure \(\mu\) equals \(\int d\mu\) i.e. integrability as well as the integral depend on the \(\sigma\)-finite component \(\mu^\sigma\) only. \(\square\)

Given any measurable space \((S, \mathcal{G})\), let \(\mathcal{M}^+(S, \mathcal{G})\) stand for the cone of nonnegative extended real-valued \(\mathcal{G}\)-measurable functions on \(S\).

Definition 2.10. By an extended Lebesgue integral in an arbitrary measure space \((S, \mathcal{G}, \mu)\) we mean the only non-decreasing function \(\int d\mu: \mathcal{M}^+(S, \mathcal{G}) \to [0, \infty]\), equal to the integral \(\int d\mu: \mathcal{I}(T, \mu) \to \mathbb{R}\) on non-negative finite integrable functions, and such that

(i) for any set \(A \in \mathcal{G}\), one has \(\int 1_A d\mu = \mu(A)\);

(ii) for every non-decreasing sequence \((f_n)_{n \in \mathbb{N}}\) in \(\mathcal{M}^+(S, \mathcal{G})\), the following equality
\[
\int \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} \int f_n d\mu
\]
holds true.
An extended real-valued $\mathcal{G}$–measurable function $f$ on $S$ is called $\mu$–
integrable if $\int |f| \, d\mu < \infty$.

As a complement to corollary 2.9 we get

**Corollary 2.11.** For any $f \in \mathcal{M}^+(S, \mathcal{G})$ the function is integrable if and only if there exists a finite integrable $g \in \mathcal{I}(T, \mu)$ such that $f = g$ almost everywhere, i.e. $\mu(\{x; \ f(x) \neq g(x)\}) = 0$.

Every $\mu$–integrable function is $\mathcal{G}_\mu^\sigma$–measurable and $\mu^\sigma$–integrable. □

Classical expositions of the Lebesgue integral in $(S, \mathcal{G}, \mu)$ take advantage of a Daniell–Stone formalism and start from an extension of the assignment $1_A \mapsto \mu(A)$, for $A \in \mathcal{G}_\mu^f$, to a unique linear functional $\tilde{\mu}: \mathcal{P}_\mu \to \mathbb{R}$ associated with the measure. The domain $\mathcal{P}_\mu$ is (algebraically) generated by the characteristic functions, and consists of – so called – simple functions. The respective Daniell–Stone integral $\int d\tilde{\mu}$ is well-known to be a completion of the Lebesgue integral $\int d\mu$.

In the case of two arbitrary measure spaces, $(S, \mathcal{G}, \mu)$ and $(T, \mathcal{I}, \nu)$, the tensor product $\mathcal{P}_\mu \otimes \mathcal{P}_\nu$ is naturally isomorphic to a linear subspace of $\mathcal{G} \otimes \mathcal{I}$–measurable functions on $S \times T$, and the respective associated functionals yield a functional $\tilde{\mu} \otimes \tilde{\nu}: \mathcal{P}_\mu \otimes \mathcal{P}_\nu \to \mathbb{R}$ such that

$$(\tilde{\mu} \otimes \tilde{\nu})(f) = \tilde{\mu}(s \mapsto \tilde{\nu}(f(s, \cdot))) = \tilde{\nu}(t \mapsto \tilde{\mu}(f(\cdot, t))),$$

for any $f \in \mathcal{P}_\mu \otimes \mathcal{P}_\nu$. Turning back to the examined product of measures, we are now about to formulate and prove

**Main Theorem 2.12.** Let $(S, \mathcal{G}, \mu)$ and $(T, \mathcal{I}, \nu)$ be any measure spaces.

(i) The Lebesgue integral $\int d(\mu \otimes \nu)$ is equal to the Daniell–Stone integral $\int d(\tilde{\mu} \otimes \tilde{\nu})$ – restricted to $\mathcal{G} \otimes \mathcal{I}$–measurable functions.

(ii) Fubini: For any $\mu \otimes \nu$–integrable function $f: S \times T \to [-\infty, \infty]$, one has

$$\int f \, d(\mu \otimes \nu) = \int \left( s \mapsto \int f(s, \cdot) \, d\nu \right) \, d\mu$$

$$= \int \left( t \mapsto \int f(\cdot, t) \, d\mu \right) \, d\nu,$$

where the integrands on the right are integrable for almost every $s$ and $t$, respectively.

(iii) Tonelli: Equalities (2.3) remain valid if $f \in \mathcal{M}^+(S \times T, \mathcal{G} \otimes \mathcal{I})$ is $\mathcal{G}^\sigma_\mu \otimes \mathcal{I}^\sigma_\nu$–measurable, i.e. such that the set $\{x; \ f(x) \neq 0\}$ is $\sigma$–finite. Finite value of any of the three sides of (2.3) assures then $(\mu \otimes \nu)$–integrability of $f$. 
Proof. (i) The space $\mathcal{P}_\mu \otimes \mathcal{P}_\nu$ is a Stone lattice, so the Stone theorem (see e.g. [5]) assures that the Daniell–Stone integral $\int d(\tilde{\mu} \otimes \tilde{\nu})$ is equal to the Lebesgue integral with respect to a measure, say $\lambda$, and the corresponding $\sigma$–finite sets form the $\sigma$–ring $\mathcal{G}_\mu^\sigma \otimes \mathcal{G}_\nu^\sigma$. Since the product $\mu^\sigma \otimes \nu^\sigma$ and the $\sigma$–finite component $\lambda^\sigma$ of $\lambda$ are both defined on the same $\sigma$–ring and are equal on the $\pi$–system $\{A \times B; A \in \mathcal{G}_\mu^i, B \in \mathcal{G}_\nu^i\}$, they are equal – and so $\lambda = \mu \otimes \nu$.

Assertions (ii)–(iii) follow from corollaries 2.9, 2.11 and the classical $\sigma$–finite variant of the Fubini–Tonelli theorem. \hfill \Box

3. Technical lemmas

For any family of sets $\mathcal{C}$ and a set $S$ let $\mathcal{C}|S := \{A \cap S; A \in \mathcal{C}\}$ denote a restriction of the family (to $S$). Any restriction of a $\sigma$–ring remains a $\sigma$–ring. Basic properties of the operation are recalled in

Lemma 3.1. (i) $\sigma(\mathcal{C}|S) = \sigma(\mathcal{C})|S$.
(ii) For any $n \in \mathbb{N}$ and sets $S_1, \ldots, S_n$,

$$\bigotimes_{i \leq n}(\mathcal{G}_i|S_i) = (\bigotimes_{i \leq n}\mathcal{G}_i)|S_1 \times \cdots \times S_n$$

whenever $\mathcal{G}_i, i \leq n$, are arbitrary $\sigma$–rings. \hfill \Box

Lemma 3.2. Let $(S_i, \mathcal{G}_i, \mu_i), i \leq n$, be any finite sequence of measure spaces. Then one has

$$(3.1) \quad \bigotimes_{i \leq n}\mathcal{G}_i^\sigma = \sigma\{A_1 \times \cdots \times A_n; \forall i \leq n \mu_i(A_i) < \infty\},$$

where $\mathcal{G}_i^\sigma := (\mathcal{G}_i)^\sigma$ for $i \leq n$. Furthermore, the $\sigma$–ring $(3.1)$ is composed of all the measurable sets $C \in \bigotimes_{i \leq n}\mathcal{G}_i$ such that

$$(3.2) \quad C \subset S'_1 \times \cdots \times S'_n, \text{ for some } S'_i \in \mathcal{G}_i^\sigma, \ i = 1, \ldots, n.$$

Proof. Equality $(3.1)$ is a simple consequence of the notion of $\sigma$–finiteness. The family of sets $C$ satisfying $(3.2)$ is a $\sigma$–ring, and thus contains the product $\bigotimes_{i \leq n}\mathcal{G}_i^\sigma$. In order to prove the reverse inclusion we consider any $\sigma$–finite sets $S'_i \in \mathcal{G}_i^\sigma, i \leq n$, and an arbitrary $\bigoplus_{i \leq n}\mathcal{G}_i$–measurable subset $C \subset S'_1 \times \cdots \times S'_n$. Such a $C$ is an element of a $\sigma$–ring

$$\bigotimes_{i \leq n} \mathcal{G}_i|S'_1 \times \cdots \times S'_n = \bigotimes_{i \leq n} (\mathcal{G}_i|S'_i)$$

$= \sigma\{A_1 \times \cdots \times A_n; \forall i \leq n A_i \in \mathcal{G}_i|S'_i\}$

$\subset \sigma\{A_1 \times \cdots \times A_n; \forall i \leq n A_i \in \mathcal{G}_i^\sigma\} = \bigotimes_{i \leq n}\mathcal{G}_i^\sigma$.

\hfill \Box
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