Supersymmetry in Stochastic Quantization Method and Field-Dependent Kernel

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We define a discretized Langevin equation in Stratonovich-type interpretation. We show that a generating functional with a field-dependent kernel can be written in mid-point prescription only when we calculate in the interpretation. Moreover we investigate whether supersymmetry of the stochastic action with field-dependent kernel exists or not.

§ 1. Introduction

The stochastic quantization method (SQM) was first proposed by Parisi and Wu as an alternative quantization method in 1981. SQM can be applied to gauge theories without the gauge fixing procedure, i.e., without Faddeev-Popov ghost fields. Instead of introducing ghost field, the method produces the same contribution as the path-integral quantization method (PIQM). This fact was already confirmed perturbatively for Yang-Mills fields and for non-Abelian anti-symmetric tensor fields.

SQM has a powerful tool, "kernel", which, among others, gives new regularization schemes. Kernel is also introduced for systems including massless fermion. Moreover, "field-dependent" kernel is introduced for systems including graviton, systems with spontaneously broken symmetry, and bottomless systems. On the other hand, it is well-known that theories quantized stochastically display supersymmetry. So my question is whether SQM with field-dependent kernel has supersymmetry or not. In this paper we show that SQM with field-dependent kernel has supersymmetry as well as the one without kernel. While Ref. 12) showed that stochastic action with field-dependent kernel was invariant by operation with the supersymmetry generator $Q$, it did not show that the action was invariant by operation with $\bar{Q}$ and could be described in superfield formalism. Besides, boundary condition of generating functionals was not discussed in Ref. 12).

When we construct the stochastic action, Ito or Stratonovich calculus can be used. Leibnitz rule, which is indispensable to supersymmetry, can be used in the Stratonovich calculus, but cannot be used in Ito calculus. In this paper, we introduce a discretized Langevin equation with a field-dependent kernel in order to construct generating functional. Here we may use the discretized Langevin equation written in the Stratonovich-related interpretation, but we use the one in Stratonovich-type interpretation defined in this paper. When a field-dependent kernel is introduced, the stochastic action constructed from the discretized Langevin equation in the Stratonovich-type interpretation corresponds to the one in mid-point prescription.

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*) The definitions of $Q$, $\bar{Q}$ are given in this paper.
but the action constructed from the equation in Stratonovich-related interpretation does not. This is why we defined the Stratonovich-type interpretation.

§ 2. Boundary condition of generating functional

First, we take up a system with variables \( q(x) \) and the classical action \( S(q) \) in \( n \)-dimensional space-time. To quantize the system stochastically, we consider the fictitious time interval \([-T, T]\) and divide the interval to \( 2N \) segments with space \( T/N \). We shall let \( T \) tend to infinity later. Besides, we define the discretized Langevin equation with the field-dependent kernel \( K(\bar{q}_i) \),

\[
dq_i(x) = q_{i+1} - q_i = -K(\bar{q}_i)\frac{\delta S}{\delta \bar{q}_i}dt + R_i\frac{\delta R_i}{\delta \bar{q}_i}dt + R_idW_i, \\
i = 1, 2, \ldots, N, \quad -T = t_{-N} < t_{-N+1} < \cdots < t_N = T, \\
q_i(x) = q(x, t_i), \quad \bar{q}_i = \frac{1}{2}(q_{i+1} + q_i), \quad R_i = R(\bar{q}_i), \\
K(\bar{q}_i) = R^2(\bar{q}_i), \quad dW_i(x) = W(x, t_{i+1}) - W(x, t_i),
\]

(1)

where \( dt = T/N \) and we assume the kernel \( K(\bar{q}_i) \) is positive-definite. \( W(x, t_i) \) is a Wiener process defined by the following correlation,

\[
\langle (W(x, t_i) - W(x, t_h))(W(y, t_h) - W(y, t_i)) \rangle = 2(t_i - t_h)\delta^n(x - y), \\
i > j > k > l,
\]

\[
\langle \cdots \rangle = \int D(dW_i)\langle \cdots \rangle \exp \left[ -\frac{1}{4} \int d^n x \sum_{i=-N}^{N} \left( \frac{dW_i}{dt} \right)^2 dt \right].
\]

(2)

The expression of Eq. (1) is different from the ordinary one,

\[
dq_i = -\frac{1}{2} (X(q_{i+1}) + X(q_i))dt + \frac{1}{2} (R(q_{i+1}) + R(q_i))dW_i, \\
X(q) = K(q)\frac{\delta S(q)}{\delta q} + R(q)\frac{\delta R(q)}{\delta q},
\]

in Stratonovich-related interpretation, but we want to regard Eq. (1) as the expression in Stratonovich-type interpretation. The advantage of the interpretation is shown later.

Now we are able to show that Leibnitz rule can be used in the Stratonovich-type interpretation in the continuum limit \( dt \rightarrow 0 \) as follows. We shall calculate the expectation value

\[
\int d^n y \langle \frac{\delta f(\bar{q}_i)}{\delta \bar{q}_i(y)} dq_i(y) \rangle = \int d^n y \left\{ \frac{\delta f(\bar{q}_i)}{\delta \bar{q}_i(y)} \left( -K(\bar{q}_i)\frac{\delta S(\bar{q}_i)}{\delta \bar{q}_i(y)} + R_i\frac{\delta R_i}{\delta \bar{q}_i(y)} \right)dt + R_idW_i(y) \right\}
\]

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\[ = \int d^n y \left\{ \frac{\delta f(q_i)}{\delta q_i(y)} \left( -K(q_i) \frac{\delta S(q_i)}{\delta q_i(y)} + 2R(q_i) \frac{\delta R}{\delta q_i(y)} \right) dt + K(q_i) \frac{\delta^2 f(q_i)}{\delta q_i^2(y)} dt \right\}, \quad (3) \]

where \( f(q_i) \) is some function of \( q_i \). In second equality we use the fact that \( <q_i dW_i> = 0, <q_{i+1} dW_i> = <2R(q_i) dt> + O(dt^2) \). The final expression in Eq. (3) is equivalent to the value

\[ \int d^n y \left\{ \frac{\delta f(q_i)}{\delta q_i(y)} \left( -K(q_i) \frac{\delta S(q_i)}{\delta q_i(y)} + 2R(q_i) \frac{\delta R(q_i)}{\delta q_i(y)} \right) dt + K(q_i) \frac{\delta^2 f(q_i)}{\delta q_i^2(y)} dt \right\}, \quad (4) \]

up to \( dt \) because \( \bar{q}_i = q_i + (1/2) dq_i \) and \( dW_i \) can be regarded as \( \sqrt{dt} \) term. The expression coincides with \( <df(q_i)> = <f(q_{i+1}) - f(q_i)> \) in Ito calculus because the Langevin equation in Ito calculus is

\[ dq_i = -K(q_i) \frac{\delta S(q_i)}{\delta q_i} dt + 2R(q_i) \frac{\delta R(q_i)}{\delta q_i} dt + R(q_i) dW_i. \quad (5) \]

Equations (3) and (4) mean

\[ <df(q_i)> = \int d^n y \left\{ \frac{\delta f(q_i)}{\delta q_i(y)} dq_i(y) \right\} + O(dt^2). \quad (6) \]

Therefore, Leibnitz rule can be used in continuum limit of the Stratonovich-type interpretation.

Next, let us introduce the generating functional,

\[ Z[J] = \int D(dW_i) \exp \left[ -\int d^n x \sum_{i=1}^{N} \left\{ \frac{1}{4} \left( \frac{dW_i}{dt} \right)^2 + J_i(x) q_i^w(x) \right\} dt \right] \]

\[ = \int Dq_i \det(M_{ij}) \times \exp \left[ -\int d^n x \sum_{i=1}^{N} \left\{ \frac{1}{4} \left( \frac{dq_i}{dt} + K \frac{\delta S(q_i)}{\delta q_i} - R \frac{\delta R}{\delta q_i} \right) \right\} + J_i(x) q_i(x) \right] dt, \quad (7) \]

where \( q_i^w \) is a solution of Eq. (1). Another expression of the generating functional is

\[ \tilde{Z}[J] = \int Dq Dp \overline{CD} e^{-A + \int d^n x d\bar{q}(x,t) \bar{q}(x,t)}, \]

\[ A = \int d^n x \int_{-T}^{T} dt \left\{ pK(q) \bar{q} - i p \left( \frac{\partial}{\partial q} + K \frac{\delta S}{\delta q} - R \frac{\delta R}{\delta q} \right) \right\} - \overline{C} R(q) \frac{\partial}{\partial \bar{q}} R^{-1} \left( \frac{\partial}{\partial \bar{q}} + K \frac{\delta S}{\delta \bar{q}} - R \frac{\delta R}{\delta \bar{q}} \right) C, \quad (8) \]

in continuum limit. Here the auxiliary field \( p \) and Grassmannian variables \( \bar{C}, C \) are introduced. In order to assert that \( Z = \tilde{Z} \), we need to prove

\[ \lim_{dt \to 0} \det(M_{ij}) = I_{term}, \quad (9) \]

where
\[
I_{\text{termi}} = \int Dp DCD \bar{C} e^{-A_f},
\]
\[
A_f = \int d^n x \int_{-T}^T dt \left[ \left( p - \frac{i}{2} \left( \dot{q} + K \frac{\delta S}{\delta q} - R \frac{\delta R}{\delta q} \right) K^{-1} R \right)^2 \right.
\]
\[
- \bar{C} R(q) \frac{\partial}{\partial q} R^{-1} \left( \dot{q} + K \frac{\delta S}{\delta q} - R \frac{\delta R}{\delta q} \right) C
\]
\]
(10)

In order to calculate \( \det(M_{\nu}) \) and \( I_{\text{termi}} \), we take the twisted boundary condition,\(^{11}\)
\[
q(-T) = e^{-i\nu} q(T), \quad C(-T) = e^{-i\nu} C(T),
\]
\[
e^{-i\nu} \bar{C}(-T) = \bar{C}(T), \quad e^{-i\nu} p(-T) = p(T).
\]
(11)

The boundary condition for \( \nu = 0 \) corresponds to the periodic one,
\[
q(-T) = q(T), \quad C(-T) = C(T), \quad \bar{C}(-T) = \bar{C}(T), \quad p(-T) = p(T),
\]
(12)
or for \( \nu = -\infty \) the causal and anti-causal one,
\[
q(-T) = 0, \quad C(-T) = 0, \quad \bar{C}(T) = 0, \quad p(T) = 0.
\]
(13)

In this paper, we choose \( \nu \) as \( Re(\nu) = 0 \) because \( q \) is a real variable. We can calculate \( \det(M_{\nu}) \) as
\[
\det(M_{\nu}) = \prod_{i=-N}^{N} R_i^{-1}
\]
\[
\times \det \left( \begin{array}{ccc}
1 + G_{-Ndt} & 1 + G_{-N+1dt} & e^{-i\nu}(-1 + G_{-Ndt}) \\
-1 + G_{-N+1dt} & 1 + G_{-N+2dt} & \\
& \ddots & \ddots & \ddots
\end{array} \right),
\]
(14)

The Stratonovich-type interpretation has the advantage that the \( \det(M_{\nu}) \) is expressed in the mid-point prescription, i.e.,
\[
\det(M_{\nu}) = \prod_{i=-N}^{N} g \left( \frac{q_{i+1} + q_i}{2}, q_{i-1} - q_i \right),
\]
(15)

where \( g \) is some function. In ordinary Stratonovich interpretation, \( \det(M_{\nu}) \) cannot be expressed in the prescription in case that kernel is field-dependent. By straightforward calculation, the determinant is
\[
\lim_{dt \to 0} \det(M_{\nu}) = \prod_{t=-T}^{T} R^{-1}(t) \left\{ e^{\int_{t}^{T} dt dt_{xG(t)} - e^{-\int_{-T}^{0} dt dt_{xG(t)} - i\nu}} \right\}
\]
\[
= \prod_{t} R^{-1}(t) \sinh \left\{ \int_{-T}^{T} dt dt^n xG(t) + \frac{1}{2} i\nu \right\} \times \text{Constant}.
\]
(16)

Next \( I_{\text{termi}} \) can be also calculated as
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\[ I_{\text{term}} = \prod_t R^{-1}(t) \sinh \left[ \left\{ \frac{1}{2} iv + \int_t^\tau d^n x dt G(t) \right\} \right], \]

(17)

by following Ref. 18). From Eqs. (16) and (17), we have proved that \( Z \) is equivalent to \( \tilde{Z} \) when the boundary condition (11) is taken.

§ 3. Supersymmetry and field-dependent kernel

The expression of \( A \) in (8) is rather complicated and it is difficult to recognize whether the stochastic action with field-dependent kernel has supersymmetry or not. In fact, \( A \) is not invariant under the supersymmetric transformations under which the stochastic action without kernel is invariant.

Here, we consider the change of variables

\[ q' = \int_0^T dq R^{-1}(q), \quad p' = R(q) p, \quad \bar{C}' = \bar{C} R(q), \quad C' = R^{-1}(q) C. \]

(18)

This leads to

\[
\tilde{Z}[J] = \int Dp' Dq' D\bar{C}' DC' e^{-A' + \int d^p x dt \{ p' - \frac{1}{2} \bar{C}' R^{-1} \frac{\partial}{\partial q'} ( R^{-1} \bar{C}' ) \}},
\]

\[
A' = \int d^n x \int_T^0 dt \left\{ p'^2 - i p' \left( \frac{\partial S}{\partial q'} R^{-1} \frac{\partial}{\partial q'} \right) - \bar{C}' \frac{\partial}{\partial q'} \left( \frac{\partial S}{\partial q'} R^{-1} \bar{C}' \right) C' \right\}.
\]

(19)

The periodic boundary condition (12) can be also expressed in terms of new variables as

\[
q'(-T) = q'(T), \quad C'(-T) = C'(T), \quad \bar{C}'(-T) = \bar{C}'(T), \quad p'(-T) = p'(T),
\]

(20)

i.e., periodic boundary condition. For \( \nu = -i \infty \), the boundary condition (13) can be also expressed as

\[
q'(-T) = 0, \quad C'(-T) = 0, \quad \bar{C}'(T) = 0, \quad p'(T) = 0,
\]

(21)

i.e., causal and anti-causal boundary condition, where we choose the integral constant as \( \int R^{-1} dq |_{q=0} = 0 \).

The stochastic action \( A' \) with the boundary condition (20) is invariant under the super-transformations,

\[
\delta q' = \varepsilon C', \quad \delta \bar{C}' = -i \varepsilon p', \quad \delta C' = 0, \quad \delta p' = 0,
\]

(22)

and

\[
\delta q' = \bar{C}' \varepsilon, \quad \delta \bar{C}' = 0, \quad \delta C' = -i \varepsilon p' - \varepsilon q', \quad \delta p' = i \bar{C}' \varepsilon.
\]

(23)

On the other hand, \( A' \) with (21) is not invariant under the transformation (23) as well as the case of no kernel.\(^{11}\)

In terms of original variables \( q, C, \bar{C} \), and \( p \), the transformation (22) is expressed as
\[
\delta q = \bar{c}C, \quad \delta \bar{c} = \bar{c} \frac{\partial R}{\partial q} R^{-1} C - i \bar{c} p, \quad \delta C = - R \frac{\partial R^{-1}}{\partial q} \bar{c} C, \quad \delta p = - R^{-1} \frac{\partial R}{\partial q} \bar{c} C p,
\]
(24)

and the transformation (23) is
\[
\delta q = \bar{C} K(q) \epsilon, \quad \delta \bar{C} = 0,
\]
\[
\delta C = - i e K(q) p - \epsilon q + \bar{C} e \frac{\partial R(q)}{\partial q} R(q) C,
\]
\[
\delta p = i \bar{c} \epsilon + i \bar{C} \frac{\partial R(q)}{\partial q} R^{-1} \bar{q} \epsilon - \bar{C} e \frac{\partial R(q)}{\partial q} R(q) p.
\]
(25)

Finally, we express the stochastic action in terms of the superfield \( \Phi' \) as
\[
A' = - \int d^n x d^2 \theta \int_{-T}^T dt \{ \bar{D}_s \Phi' D_s \Phi' + L(q(\Phi')) - \delta^n(0) \ln R(q(\Phi')) \},
\]
\[
D_s = \partial_s - \theta \partial_t, \quad \bar{D}_s = \partial_s, \quad \Phi' = q' + \bar{C} \epsilon + \bar{C}' \theta - i \bar{\theta} \theta p',
\]
(26)

where \( \theta, \bar{\theta} \) are Grassmannian variables. As discussed above, the expression is invariant under operation with \( Q(\equiv \partial_s) \) and \( \bar{Q}(\equiv \bar{\partial}_s + \partial_t) \) for \( \nu = 0 \) and invariant under the operation with only \( Q \) for \( \nu = -i \infty \).

§ 4. Summary

We defined the discretized Langevin equation (1) with field-dependent kernel in Stratonovich-type interpretation in which Leibnitz rule can be used. When the field-dependent kernel is introduced, only the generating functional constructed in the Stratonovich-type interpretation can be expressed in the mid-point prescription. If one takes other interpretation, for example,
\[
q_i = a q_{i+1} + (1-a) q_i, \quad 0 < a < 1, \quad a = \frac{1}{2},
\]
in (1), the stochastic action given in the definition does not expressed in the mid-point prescription.

Besides, we showed that the generating functional (7) constructed from Wiener process distribution is equivalent to the continuous one (8) when we take the twisted boundary condition (11).

Moreover, we showed that the stochastic action with the periodic boundary condition (12) is invariant under the super-transformations (22) and (23) when field-dependent kernel is introduced. The stochastic action with the causal and anticausal boundary condition (13) is not invariant under the super-transformation (23) as well as the one without kernel. We also showed that the stochastic action can be also described in terms of superfield.
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