Chiral Phase Structure at Finite Temperature and Density in Einstein Universe

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The gravitational effect on the chiral phase structure at finite temperature and density is investigated in the framework of Nambu–Jona-Lasinio model in $D$-dimensional ultrastatic Einstein universe. In mean field approximation, the thermodynamic potential and the gap equation determining the curvature, temperature and density dependence of the chiral condensate are analytically derived. In the sense of chiral symmetry restoration and the order of the phase transition, the scalar curvature of the space-time plays similar role as the temperature.

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I. INTRODUCTION

The phase structure of quantum field theory, especially quantum chromodynamics (QCD), was widely investigated in flat space-time in the last two decades. It is quite interesting to make studies of how the phase structure changes in the circumstance of very compact stars and early universe where not only the temperature and density effect but also the gravitational and gauge field effects cannot be neglected. The QCD phase transitions that probably happen in these systems include mainly the restoration of spontaneously broken chiral symmetry and the deconfinement from hadron gas to quark matter and vice versa. They are closely related to the hadrosynthesis at the early universe. We focus on the chiral phase transition in this paper.

Four fermion models (and gauged four fermion models) are often applied to describe the dynamical symmetry breaking in QCD, electroweak theory and grand unified theory. One of the most frequently used four fermion models is the Nambu–Jona-Lasinio (NJL) model. The chiral symmetry of this model is spontaneously broken according to the emergence of a vacuum condensate of the composite field of fermions, $\langle \bar{\psi} \psi \rangle \neq 0$, and the fermion mass is proportional to the condensate. The NJL model is particularly convenient for the investigation of chiral symmetry when some external conditions, like temperature, chemical potential, non-trivial topology, external gauge field and gravity field are taken into account. Such conditions need to be considered in the study of high energy heavy ion collisions where the temperature and baryon density are high enough, compact stars like neutron stars and quark stars where the baryon density may be ten times the normal nuclear matter density and the strong gravity fields and external magnetic field cannot be neglected, and early universe where the gravity is certainly important and the temperature is extremely high. Because of the non-perturbative phenomena around the phase transition and the fermion sign problem in lattice QCD at finite chemical potential, the NJL model has been widely considered as a low energy effective theory of QCD in the study of chiral phase transition in flat space-time.

The chiral symmetry breaking in four fermion models in curved space-time at zero temperature and chemical potential was investigated in recent years. It was found that when the curvature of space-time is sufficiently large, there will be no more spontaneous chiral symmetry breaking in the vacuum. Therefore, in the sense of chiral symmetry restoration, the curvature $R$ behaves like temperature $T$. This phenomenon can be clearly seen in the discussion with both curvature and temperature effects. With the method of weak curvature expansion for fermion propagator, the chiral phase structure of Gross-Neveu model in 2+1 dimensional space-time is studied at finite chemical potential $\mu$ but zero temperature (14), the phase diagram in the $R - \mu$ plane is similar to the diagram in the $T - \mu$ plane in flat space-time. In the four dimensional curved NJL model at finite temperature and chemical potential, the weak curvature approximation leads to a first-order chiral phase transition.

A natural question we ask is if we can use the method of weak curvature expansion to treat the chiral properties around the critical curvature $R_c$, when $R_c$ is not small. In this paper, we study how the temperature, chemical potential and external gravity influence the chiral symmetry of the NJL model in different dimensions. We take the Einstein universe as the background space-time, in which we can analytically calculate the thermodynamic potential without using the weak curvature approximation in mean field approximation. In flat space-time, the chemical potential effect is very different from the temperature effect on the chiral phase transition. For instance, the transition is of first order at high chemical potential, while it is of second order at high temperature. We want to know how the chemical potential affects the chiral phase transition in highly curved space-time.

The paper is organized as follows. In Section II we review the NJL model in curved space-time at finite temperature and chemical potential and obtain the thermodynamic potential in mean field approximation. In Section III we analytically derive the thermodynamic potential and gap equation for the chiral condensate in the Einstein universe. The phase diagrams in different planes are shown and discussed in Section IV. We summarize in Section V.

II. THE CURVED NJL MODEL IN MEAN FIELD APPROXIMATION

In $D$-dimensional curved space-time with signature $(+, -, -, -, \cdots)$, the action $S[\psi, \bar{\psi}]$ of the $SU(N)$ NJL
model takes the form of
\[ S = \int d^D x \sqrt{|g|} \left[ -i \gamma^\mu \nabla_\mu \psi + G \left( (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2 \right) \right], \]
where \( g \) is the determinant of the metric tensor \( g_{\mu \nu}(x) \), \( \psi(x) \) represents a fermion field in a \( D \)-dimensional internal space, \( \bar{\psi} = \psi^\dagger 0 \) is the corresponding Pauli conjugate spinor, the index with a hat denotes a standard Dirac matrix defined in Minkowskian space-time, \( G \) is the four fermion coupling constant, \( \gamma^\mu(x) \) is the Dirac matrix in curved space-time satisfying \( \{ \gamma_\mu, \gamma_\nu \} = g_{\mu \nu} \), \( \nabla_\mu \) is the covariant derivative with the chemical potential parameter \( \mu \), \( \nabla_\mu = \nabla^\nu - i \mu \partial_\nu \) with \( \Gamma_\nu(x) \) being the spinor connection \( \{ \gamma_\mu, \gamma_\nu \} \) being the spinor connection, and \( \gamma^5 \) is defined as the same as in flat space-time, it is independent of the space-time coordinates. The model has a global \( SU(N) \otimes U(1)_L \otimes U(1)_R \) symmetry at \( \mu = 0 \).

It is convenient to introduce the tetrads formulism \( [16, 20, 21, 22] \) to connect the definitions in curved and flat space-times. Let \( e^a_\mu(x) \) be the tetrads defined by \( g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu \), where \( \eta_{ab} \) is the Minkowskian metric, we can rewrite \( \gamma^\mu = \gamma^a e^a_\mu \), and derive \( \Gamma_\mu = \frac{1}{8} \epsilon^{a b c} e^a_\mu \left[ \gamma_a, \gamma_b \right] \) with \( e^{\mu b}_a = \partial_\mu e^b_\nu + \Gamma^\nu_\mu e^b_\nu \) under the requirement \( \nabla_\mu \gamma^\nu = \nabla^\nu \gamma^\mu = 0 \), where \( e^a_\mu \) is the inverse of \( e^a_\mu \).

The essential quantity characterizing a system in a grand canonical ensemble can be taken to be the partition function \( Z \). It can be expressed in terms of the action of the system as
\[ Z = \int \langle \bar{\psi} \rangle [d\bar{\psi}] e^{iS[\bar{\psi}, \psi]}, \]
where \( S \) is the effective action
\[ S_{\text{eff}}[\sigma, \pi] = \int d^D x \sqrt{|g|} \left[ -\frac{N}{4G} \left( \sigma^2 + \pi^2 \right) \right] - i \ln \Det \left[ i\gamma^\mu \nabla_\mu - (\sigma + i\gamma^5 \pi) \right]. \]

The thermal expectation value \( \sigma \) is the order parameter of the chiral phase transition. The thermodynamic potential \( \Omega \) is a function of \( T \) and \( \mu \) with \( \sigma \) initially an undetermined parameter. In the spirit of thermodynamics, the physical system is described only by \( T \) and \( \mu \). The order parameter as a function of \( T \) and \( \mu \) is determined by the minimum of the thermodynamic potential \( [23, 26] \)
\[ \frac{\partial \Omega}{\partial \sigma} = 0, \quad \frac{\partial^2 \Omega}{\partial \sigma^2} \geq 0. \]

Since we did not consider the initial fermion mass in the action \( [1] \), the effective fermion mass produced through the spontaneous chiral symmetry breaking is just the chiral condensate, \( m = \sigma \).

The measurable bulk quantities like pressure \( p \), entropy density \( s \), fermion number density \( n \), and energy density \( \epsilon \) are related to \( \Omega \) by
\[ p = -\Omega, \quad s = -\frac{\partial \Omega}{\partial T}|_{\mu}, \quad n = -\frac{\partial \Omega}{\partial \mu}|_{T}, \quad \epsilon = -p + Ts + \mu n. \]

Now we come back to the calculation of the mean field thermodynamic potential \( [7] \). In ultrastatic space-time with line element \( ds^2 = dt^2 - g_{ij}(x)dx^i dx^j \), it is easy to show that the zero component \( \Gamma_0 \) of the spinor connection is zero, and we have \( g^{\mu \nu} \nabla_\mu \nabla_\nu = \partial^2 \partial t^2 - \nabla^2 \) with \( \nabla^2 = g^{ij} \nabla_i \nabla_j \) being the spinor Laplacian defined in the space manifold \( \Sigma \). Taking into account the clearly physical meaning of chemical potential, since the number density is still conserved in curved space-times, \( \partial / \partial t \int d^D-1 \sqrt{|g|} \bar{\psi} \gamma^\mu \psi = 0 \). We will discuss only the ultrastatic universe \( R \otimes \Sigma \) below with \( \Sigma \) being the \( (D - 1) \) dimensional space manifold.

With the known partition function, we obtain the thermodynamic potential \( \Omega \) as a function of temperature and chemical potential,
\[ \Omega(T, \mu) = -\frac{T}{V} \ln Z(T, \mu), \]
where \( V = \int d^D-1 \sqrt{|g|} \) is the volume of the space manifold \( \Sigma \).

In mean field approximation which is just the \( O(1) \) order in the large \( N \) expansion of fermion propagator, the fields \( \sigma \) and \( \pi \) are replaced by their thermal averages \( \langle \sigma \rangle \) and \( \langle \pi \rangle \). To simplify the notations, we express in the following the averages \( \langle \sigma \rangle \) and \( \langle \pi \rangle \) by \( \sigma \) and \( \pi \), respectively, without making confusion in the mean field approximation. With the short notations, the thermodynamic function can be simplified as
\[ \Omega = \frac{N}{4G} \left( \sigma^2 + \pi^2 \right) - \frac{T}{V} \ln \Det \left[ i\gamma^\mu \nabla_\mu - (\sigma + i\gamma^5 \pi) \right]. \]

If we do not consider the chiral anomaly induced by gravity, the thermodynamic potential is invariant under chiral transformation, and we can set \( \pi = 0 \) without loss of generality. In this case, we have
\[ \Omega = \frac{N\sigma^2}{4G} - \frac{T}{V} \ln \Det \left[ i\gamma^\mu \nabla_\mu - \sigma \right]. \]
relation \( \gamma^{\mu} \gamma^{\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu = g^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{1}{4} R - \mu^2 - 2i\mu \nabla_0 \), we have

\[
2 \ln \text{Det} \left[ i\gamma^{\mu} \tilde{\nabla}_\mu - \sigma \right] = \ln \text{Det} \left[ i\gamma^{\mu} \tilde{\nabla}_\mu - \sigma \right] + \ln \text{Det} \left[ \gamma^5 (i\gamma^{\mu} \tilde{\nabla}_\mu - \sigma) \gamma^5 \right] = \ln \text{Det} \left[ \left( \frac{\partial}{\partial t} - i\mu \right)^2 - \nabla^2 + \frac{R}{4} + \sigma^2 \right].
\]

(10)

In order to complete the calculation, we should solve the eigenequation of the operator in the last square bracket,

\[
\left( \frac{\partial}{\partial t} - i\mu \right)^2 - \nabla^2 + \frac{R}{4} + \sigma^2 \Psi_k = \lambda_k \Psi_k,
\]

(11)

where \( k \) stands for a set of complete quantum numbers describing the eigenvalue \( \lambda \) and eigenstate \( \Psi \). By separating the \( t \) and \( \vec{x} \) dependence of \( \Psi \), the \( \vec{x} \) dependent part \( \phi_l(\vec{x}) \) satisfies the corresponding eigenequation

\[
-\nabla^2 + \frac{R}{4} \phi_l = \omega_l^2 \phi_l
\]

(12)

in the space manifold and the normalization condition

\[
\int d^{D-1} \vec{x} \sqrt{|g|} \phi_l(\vec{x}) \phi_{l'}(\vec{x}) = \delta_{ll'}.
\]

(13)

If the solutions \( \omega_l^2 \) and \( \phi_l(\vec{x}) \) of the eigenequation are known, we may take

\[
\lambda_k = \omega_l^2 + \sigma^2 - (p_0 - \mu)^2,
\]

\[
\Psi_k(t, \vec{x}) = e^{ip_0 t} \phi_l(\vec{x}),
\]

(14)

where \( p_0 = i(2n + 1)\pi/\beta \) is the fermion frequency with \( n = 0, 1, 2, \ldots \), since the eigenfunction \( \Psi_k \) must be an antiperiodic function of the period \( \beta \).

After the frequency summation over \( n \), we finally obtain the thermodynamic potential in mean field approximation and high temperature expansion are widely used. However, around the phase transition, the interaction among the constituents of the system should be very strong and those methods based on perturbative expansion are in principle not suitable for the study of the phase structure. In fact, the eigenequation \( \square \) can be exactly solved in some special universes, for example, the static Einstein universe. The \( D \)-dimensional Einstein universe is represented by the line element

\[
ds^2 = dt^2 - a^2(\text{d}r^2 + \sin^2 \theta d\Omega_{D-2})
\]

(16)

defined with the topology \( \mathbb{R} \otimes S^{D-1} \), where \( a \) is related to the curvature, \( R = (D - 1)(D - 2)a^{-2} \). With the eigenvalues and the soft cutoff \( \Lambda \) of the spinor Laplacian on the sphere \( S^{D-1} \), the degeneracy \( d_l \), the eigenvalue \( \omega_l \) and the volume \( V \) can be expressed in terms of \( l \) and \( R \) as,

\[
d_l = \frac{2((D+1)/2)!}{l!(D-1)!}, \quad \omega^2_l = b^2_l R,
\]

\[
b^2_l = \frac{(2l + D - 1)^2}{4(D-1)(D-2)} + \frac{1}{4}, \quad l = 0, 1, 2, \ldots,
\]

\[
V(R) = \frac{2\pi^{D/2}a^{D-1}}{\Gamma(D/2)}.
\]

(17)

where \( \lfloor x \rfloor \) is the floor function of \( x \), and the summation over the eigenvalues in the mean field thermodynamic potential becomes now explicit,

\[
\Omega = \frac{N\sigma^2}{4G} - \frac{N}{V} \sum_{l=0}^{\infty} d_l e^{-\frac{E_l}{T}} \left( E_l + T \ln \left( 1 + e^{-\frac{E_l}{T}} \right) \right)
\]

+ \( T \ln \left( 1 + e^{-\frac{E_l}{T}} \right) \).

(18)

Because of the contact interaction among the particles, the NJL model is non-renormalizable when \( D \geq 4 \), and in general it is necessary to introduce a regulator that serves as a length scale in the problem. That is the reason why we have introduced a soft cutoff factor \( e^{-\omega_l/\Lambda} \) in the summation over \( l \). Otherwise, the first term in the square bracket which is the contribution from the vacuum will be divergent.

There are two parameters in the model, the coupling constant \( G \) and the soft cutoff \( \Lambda \). In the case of flat space-time, they can be fixed by fitting some observable quantities in the vacuum, for instance, the pion mass and the pion decay constant. Since we are not familiar with the particle properties in curved spaces, we are in principle not able to determine the two parameters in the Einstein universe. However, what we want to do in this paper is a general study of the gravitational effect on the chiral phase structure, without considering a specific system. We focus only on the qualitative dependence of the phase structure on the gravity, and do not care very much the precise critical values of the curvature, temperature, and chemical potential. To this end we scale the thermodynamic potential by the soft cutoff \( \Lambda \).
\[
\frac{\Omega}{\Lambda^D} = \frac{N(\sigma/\Lambda)^2}{4\Lambda^{D-2}G} - \frac{N}{\Lambda^{D-1}V} \sum_{l=0}^{\infty} d_l e^{-\omega_l} \left[ \frac{E_l}{\Lambda} + \frac{T}{\Lambda} \ln \left( 1 + e^{-\frac{E_l/\Lambda - \mu/\Lambda}{T/\Lambda}} \right) \right].
\]

(19)

Without causing confusion, we represent the scaled dimensionless quantities \(\Omega/\Lambda^D\), \(\sigma/\Lambda\), \(\Lambda^{D-2}G\), \(\Lambda^{D-1}V\), \(R/\Lambda^2\), \(T/\Lambda\), \(\mu/\Lambda\), \(\omega_l/\Lambda\) still by the corresponding quantities \(\Omega\), \(\sigma\), \(G\), \(V\), \(R\), \(T\), \(\mu\), \(\omega_l\), and rewrite (19) in the dimensionless form,

\[
\Omega = \frac{N\sigma^2}{4G} - \frac{N}{V} \sum_{l=0}^{\infty} d_l e^{-\omega_l} \left[ E_l + T \ln \left( 1 + e^{-\frac{E_l-\mu}{T}} \right) + T \ln \left( 1 + e^{-\frac{E_l+\mu}{T}} \right) \right].
\]

(20)

Now only one dimensionless parameter, the coupling constant \(G\) appears in the model.

Calculating the first order derivative of the thermodynamic potential (20) with respect to the condensate \(\sigma\), we obtain the gap equation which determines the \(T\)–, \(\mu\)– and \(R\)–dependence of the condensate,

\[
\sigma [1 - 2GI(T, \mu, R, \sigma)] = 0,
\]

(21)

with the function \(I\) defined as

\[
I = \frac{1}{V} \sum_{l=0}^{\infty} E_l e^{-\omega_l} \left( 1 - f(E_l + \mu) - f(E_l - \mu) \right),
\]

(22)

where \(f(x) = 1/(e^{x/T} + 1)\) is the Fermi-Dirac distribution function. The trivial solution \(\sigma = 0\) of the gap equation describes the symmetry restoration phase, and the other solution \(\sigma \neq 0\) corresponds to the symmetry breaking phase. It is necessary to note that the solution of the gap equation is not guaranteed to be the physical condensate, we should check for each solution if it is the position of the minimum thermodynamic potential.

What happens when the curvature tends to zero? While the line element (16) of the Einstein space-time can not return to the line element of Minkowskian space-time when \(R \to 0\) or \(a \to \infty\), because the two space-times have different topologies, the thermodynamic potential in the limit \(R \to 0\) becomes the one in the flat space-time. In the limit of \(a \to \infty\), we have

\[
\omega_l \to a^{-1} l, \quad d\omega_l = \omega_{l+1} - \omega_l \to a^{-1},
\]

\[
d_l \to \frac{a^{D+2} \omega_l^{D-2}}{(D-2)!},
\]

(23)

and the thermodynamic potential is reduced to

\[
\Omega \to \frac{N\sigma^2}{4G} - \frac{N2^{[(D-1)/2]}\Gamma(D/2)}{\pi^{D/2}\Gamma(D-1)} \int_0^{\infty} dp p^{D-2} e^{-p} \left[ E_\rho + T \ln \left( 1 + e^{-\frac{E_\rho-\mu}{T}} \right) + T \ln \left( 1 + e^{-\frac{E_\rho+\mu}{T}} \right) \right],
\]

(24)

where we have replaced the integrated variable \(\omega_l\) by \(p\) which could be considered as the fermion momentum. This is exactly the familiar formula for the thermodynamic potential of the NJL model in \(D\)-dimensional Minkowskian space-time.

Before discussing the chiral properties at finite temperature and density, we first consider the pure gravitational effect. In the curved vacuum with \(T = \mu = 0\), the thermodynamic potential is reduced to

\[
\Omega = \frac{N\sigma^2}{4G} - \frac{N}{V} \sum_{l=0}^{\infty} d_l E_l e^{-\omega_l},
\]

(25)

and correspondingly, the gap equation becomes

\[
0 = \sigma \left[ 1 - \frac{2G}{V} \sum_{l=0}^{\infty} \frac{d_l}{E_l} e^{-\omega_l} \right].
\]

(26)

To have chiral symmetry breaking \(\sigma \neq 0\) in the vacuum, the coupling constant \(G\) must exceed the critical value \(G_c\) which is determined by

\[
1 = \frac{2G_c}{V} \sum_{l=0}^{\infty} \frac{d_l}{\omega_l} e^{-\omega_l}.
\]

(27)

The critical coupling constant \(G_c\) as a function of the curvature \(R\) is shown in Fig 10 for the dimension \(D = 4, 5\) and 10. As \(R\) increases, \(G_c\) grows monotonously, reflecting the fact that the broken chiral symmetry can be restored by finite curvature effect. To keep chiral symmetry breaking in curved vacuum with finite \(R\), it needs stronger interaction than that in flat vacuum. However, this curvature effect is gradually washed out when the dimension of the space-time increases. From Fig 11 the effect is almost fully cancelled in the case of \(D = 10\). To have a remarkable curvature effect, we consider in the following only the cases with \(D = 4\) and 5. The initial value of \(G_c\) at \(R = 0\) can be analytically derived by taking the limit \(a \to \infty\) in Eq. (27),

\[
G_c(R = 0) = \frac{(D-2)\pi^{D/2} 2^{-(D+1)/2}}{\Gamma(D/2)}.
\]

(28)
the gap equation \[ \sigma \left[ 1 - 2 G \sum_{l=0}^{\infty} \frac{d_l}{E_l} e^{-\omega_l} \theta(E_l - \mu) \right] = 0, \quad (29) \]

where \( \theta(x) \) is the step function. Since \( E_l \) grows monotonously with increasing \( l \), the nonzero chiral condensate is just a constant \( \sigma_0 \) in the region

\[ \mu \leq \mu_0 = E_0 = \sqrt{\omega_0^2 + \sigma_0^2} \quad (30) \]

with \( \sigma_0 \) determined by

\[ 1 - 2 G \sum_{l=0}^{\infty} \frac{d_l}{E_l} e^{-\omega_l} = 0. \quad (31) \]

However, as we mentioned above, the solution \( \mu_0 \) of the gap equation is probably not the critical chemical potential \( \mu_c \) for chiral phase transition. We should check the minimum of the thermodynamic potential

\[ \Omega(\mu, R, \sigma) = \frac{\sigma^2}{4G} - \frac{1}{V} \sum_{l=0}^{\infty} d_l e^{-\omega_l} [\mu + (E_l - \mu) \theta(E_l - \mu)], \quad (32) \]

and determine \( \mu_c \) by the condition

\[ \Omega(\mu_c, R, \sigma_0) = \Omega(\mu_c, R, 0). \quad (33) \]

At \( R = 0 \), we have \( \mu_c = 2.2 < \mu_0 = 2.9 \) for \( D = 4 \) and \( \mu_c = 3.5 < \mu_0 = 4.7 \) for \( D = 5 \). At fixed \( R \), the physical condensate \( \sigma \) is simply a step function of \( \mu \),

\[ \sigma(\mu) = \sigma_0 \theta(\mu_c - \mu). \quad (34) \]

We show the chemical potential dependence of \( \sigma \) in Fig.\( 3 \) at \( T = 0 \) and for \( R = 0 \) and 3. The critical chemical potential drops down with increasing curvature and finally reaches \( \sqrt{\omega_0^2(R_c) + \sigma_0^2(R_c)} \) at the critical curvature \( R_c \). It is easy to see that \( R_c \) satisfies the condition

\[ 1 - 2 G \sum_{l=0}^{\infty} \frac{d_l}{\omega_l} e^{-\omega_l} = 0. \quad (35) \]

At \( D = 4 \) and 5, the critical curvature \( R_c \) are, respectively, 4.7 and 8.5. Above \( R_c \) there exists no more chiral breaking phase and the condensate keeps zero at any chemical potential.

The chiral condensate at finite temperature is shown in Fig.\( 4 \) at fixed \( R \). Similar to the case in flat spacetime, the temperature effect leads to chiral symmetry restoration. When \( \mu = 0 \), the chiral condensate decreases with increasing temperature and finally reaches zero at the critical value \( T_c \) determined by

\[ 1 - 2 G \sum_{l=0}^{\infty} \frac{d_l}{\omega_l} e^{-\omega_l} (1 - 2 f(\omega_l)) = 0. \quad (36) \]

Above \( T_c \) the chiral breaking phase disappears at any chemical potential.

The phase diagram in the \( T - \mu \) plane at fixed \( R \) is shown in Fig.\( 5 \). It is very similar to the familiar phase structure in flat space-time. Each phase transition line separates the chiral breaking phase below the line and
the chiral restoration phase above the line. The chiral breaking region is gradually suppressed with increasing curvature and finally disappears at the critical value $R_c$. The dashed and solid lines represent, respectively, the second and first order chiral phase transitions, and the dots indicate the tricritical points which link the corresponding second and first order transitions.

The phase diagrams in $R - \mu$ and $R - T$ planes are shown, respectively, in Figs. 6 and 7. Again the dashed and solid lines mean the second and first order phase transitions, and the dots indicate the tricritical points. Just as we expected, at fixed curvature, the phase...
transition becomes more and more easy when the temperature or chemical potential increases. The straight dashed lines in the $R - \mu$ plane at $T = 0$ reflect the fact that the Einstein universe is compact and in turn the lowest eigenvalue $\omega_0$ of equation (12) is not zero. The toothed lines in Fig. 6 at $T = 0$ come from the step function in equation (29) which makes the low limit of the summation jump up when $\mu$ increases. From the phase diagrams in Figs. 5, 6 and 7, the second order phase transition happens at high $T$ or high $R$ or low $\mu$, while the first order phase transition occurs at low $T$ or low $R$ or high $\mu$. While finite values of $R, T$ and $\mu$ can all lead to a chiral phase transition, the order behavior of the phase transition induced by curvature effect is similar to that by temperature effect, but different from that by chemical potential effect. Therefore, in the sense of chiral restoration, the gravitational effect is more like the temperature effect.

We now discuss the critical properties for the second order chiral phase transition. Expanding the gap equation for $\sigma$ in the chiral breaking phase

$$1 - 2GI(T, \mu, R, \sigma) = 0$$

around the critical value $R_c$ at fixed $T$ and $\mu$,

$$a \frac{R - R_c}{R_c} + b\sigma^2 + c\sigma^4 = 0$$

with the coefficients $a, b$ and $c$ defined as

$$a(T, \mu, R_c) = R_c \frac{\partial I}{\partial R} \bigg|_{\sigma=0}$$

$$b(T, \mu, R_c) = \frac{\partial I}{\partial \sigma^2} \bigg|_{\sigma=0},$$

$$c(T, \mu, R_c) = \frac{1}{2} \frac{\partial^2 I}{\partial (\sigma^2)^2} \bigg|_{\sigma=0}$$

at the critical point, where we have considered the fact that $I$ is a function of $\sigma^2$ and neglected the higher orders of $\sigma^2$.

The critical condensate $\sigma_c$ is then determined in the limit $R \to R_c$,

$$\sigma_c^2 (b + c\sigma_c^2) = 0.$$  

For a second order phase transition, there should be only one solution $\sigma_c = 0$, the coefficient $c$ must vanish. Therefore, around the critical point, $\sigma$ behaves as

$$\sigma(T, \mu, R) = \sqrt{\frac{a}{b}} \left| \frac{R - R_c}{R_c} \right|^{1/2},$$

which means the critical exponent $\beta_R = 1/2$.

For a first order phase transition, however, there should be two values of $\sigma_c$ at the critical point, one is $\sigma_c = 0$, and the other is $\sigma_c \neq 0$. From (40), the nonzero condensate is $\sigma_c = \sqrt{-b/c}$. Approaching to the tricritical point which is the end of the first order phase transition and the beginning point of the second order phase transition, the nonzero $\sigma_c \to 0$, we have therefore

$$b(T, \mu, R_c) = 0$$

at the tricritical point. In this case, $\sigma$ around the tricritical point behaves as

$$\sigma(T, \mu, R) = \sqrt{\frac{a}{c}} \left| \frac{R - R_c}{R_c} \right|^{1/4},$$

which means $\beta_R = 1/4$ at the tricritical point. Obviously, the discussion above for curvature is also valid for temperature, we have $\beta_T = 1/2$ for the second order phase transition and $\beta_T = 1/4$ at the tricritical point.

The location $(T_c, \mu_c)$ of the tricritical point at fixed $R$ is determined through its definition (12) together with the gap equation,

$$1 - 2GI(T_c, \mu_c, R_0, 0) = 0,$$

$$b(T_c, \mu_c, R) = 0.$$  

Fig. 8 shows the tricritical $T_c$ and $\mu_c$ as functions of $R$. $T_c$ ends at zero, while $\mu_c$ ends at $\sqrt{\omega_0^2(R_c) + \sigma_c^4(R_c)}$.

V. SUMMARY

We have investigated the gravitational effects on the chiral properties at finite temperature and density in the framework of the NJL model in the Einstein universe. In the mean field approximation, the thermodynamic potential and the gap equation to determine the chiral condensate are analytically derived without any further approximation, which allows us to reasonably study the chiral symmetry around the phase transition where any perturbative expansion is in principle not valid.

If we take the curvature $R$ of the space-time which describes the gravitational effect in the Einstein universe as an external parameter, the role it plays in the
The chiral phase transition is very much like the effect of temperature. Both finite $R$ and $T$ result in chiral phase transitions and the behavior of the order of the transition is similar. However, this gravitational effect on the phase transition will be washed out when the dimension of the universe is high enough.

The chiral phase diagram in the $T-\mu$ plane is similar to the one in flat space-time where the thermodynamics can be obtained from that in the Einstein universe by taking the limit $R \to 0$. However, the compact property of the Einstein universe makes the minimum critical chemical potential $\mu_c$ at $T = 0$ be finite but not zero, which leads to a flat roof structure in the phase diagram in the $R-\mu$ plane at $T = 0$. From the definition of the tricritical point, we determined its location in the $T-\mu$, $R-\mu$ and $R-T$ planes.

It should be noted here that in the problem of the early universe the effects of external gauge field on the chiral phase transition are important. These effects have been studied in [17, 18, 19], and our formulation permits the inclusion of these effects.

The change in the chiral properties induced by finite curvature effects may be useful for the investigation of compact stars where the gravitational effects can not be neglected. A natural extension of the study is to discuss the di-fermion condensate in curved space-time, which is significant for the study of color superconductivity, since it may exist in the core of compact stars.

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