MULTIPLE POSITIVE SOLUTIONS FOR A
SCHRÖDINGER-POISSON-SLATER SYSTEM

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ABSTRACT. In this paper we investigate the existence of positive solutions to the following Schrödinger-Poisson-Slater system

\[-\Delta u + u + \lambda \phi u = |u|^{p-2}u \quad \text{in } \Omega\]

\[-\Delta \phi = u^2 \quad \text{in } \Omega\]

\[u = \phi = 0 \quad \text{on } \partial \Omega\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^3\), \(\lambda\) is a fixed positive parameter and \(p < 2^* = \frac{2N}{N-2}\). We prove that if \(p\) is “near” the critical Sobolev exponent \(2^*\), then the number of positive solutions is greater than the Lusternik-Schnirelmann category of \(\Omega\).

1. INTRODUCTION

In [4, 5] Benci and Cerami proved a result on the number of positive solutions of the following problem

\[-\Delta u + u = |u|^{p-2}u \quad \text{in } \Omega\]

\[u = 0 \quad \text{on } \partial \Omega\]

where \(\Omega \subset \mathbb{R}^N\) is a smooth and bounded domain, \(N \geq 3\) and \(p < 2^* = \frac{2N}{N-2}\), the critical Sobolev exponent for the embedding of \(H_0^1(\Omega)\) in \(L^p(\Omega)\). In particular they ask how the number of positive solutions depends on the topology of \(\Omega\). The core of their results is that if \(\Omega\) is “topologically rich” then there are many solutions as soon as the nonlinearity acts strongly on the equation. For problem (1) this happens when \(p\) is near \(2^*\); indeed they prove the following result

Theorem 1.1. There exists a \(\bar{p} \in (2, 2^*)\) such that for every \(p \in [\bar{p}, 2^*)\) problem (1) has (at least) \(\text{cat}_{\bar{p}}(\Omega) + 1\) positive solutions.

Hereafter \(\text{cat}\) is the Lusternik-Schnirelmann category (see e.g. [16]).

They prove Theorem 1.1 by variational methods looking for the solutions as critical points of an energy functional restricted to a suitable manifold on which it is bounded from below. Then, since the Palais-Smale condition (see below for the definition) is satisfied the main effort is to find a sublevel

2000 Mathematics Subject Classification. 35J50, 55M30, 74G35.

Key words and phrases. Schrödinger-Poisson system, Lusternik-Schnirelmann category, multiplicity result.

The author is supported by J. Andalucía - FQM 116 and by M.I.U.R. - P.R.I.N. “Metodi variazionali e topologici nello studio di fenomeni nonlineari"
of the functional with a non-zero category, let us say \( k \); in these conditions
the Ljusternik-Schnirelmann theory would give the existence of at least \( k \)
critical points. By introducing the barycenter map, they are able to find
sublevels with category greater then the category of \( \Omega \) and so the existence
of at least \( \text{cat}_{\Omega}(\Omega) \) critical points is ensured. Actually this is done in [4]
while the existence of another solution is proved in [3].

Another approach with the Morse theory has been used in [6] for more
general nonlinearity than \(|u|^{p-2}u\).

We need to recall that problems like (1), in bounded or exterior domain,
even with the critical exponent and with a control parameter \( \varepsilon > 0 \) have been
object of wide investigation. Also the concentration (blow-up) of solutions in
specific points of the domain \( \Omega \) when the parameter tends to zero is studied:
we limit ourselves to citing [10, 15, 19, 22, 29] and the references therein.

The aim of this paper is to prove an analogous result of Theorem 1.1 for
the Schrödinger-Poisson-Slater system:

\[
\begin{aligned}
&-\Delta u + \omega u + \lambda \phi u = |u|^{p-2}u & \text{in } \Omega, \\
&-\Delta \phi = u^2 & \text{in } \Omega, \\
&u = \phi = 0 & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a (smooth and) bounded domain in \( \mathbb{R}^3 \), \( p \in (2, 2^*) \), \( \omega > 0 \) and \( \lambda \)
is a positive fixed parameter. It is assumed \( \text{cat}_{\Omega}(\Omega) > 1 \).

This system appears studying the nonlinear Schrödinger equation

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_x \psi + |\psi|^{p-2} \psi
\]

which describes quantum (non relativistic) particles interacting with the
electromagnetic field generated by the motion. Here \( \psi = \psi(x, t) \) is a complex
valued function and \( \hbar, m > 0 \) are interpreted respectively as the normalized
Plank constant and the mass of the particle. However, since they have no
role in our analysis, we set \( \hbar = 1 \) and \( m = 1/2 \). A model for the interaction
between matter and electromagnetic field is provided by the abelian gauge
theories but can also be derived by the Slater approach to the Hartree-Fock
model. Without entering in details (the reader interested is refereed e.g. to
[8, 26]), if \( \phi(x, t) \) and \( A(x, t) \) denote the gauge potentials of the e.m. field,
the search of stationary solutions, namely solutions \( \psi \) of the form

\[
\psi(x, t) = u(x)e^{i\omega t} \quad u(x) \in \mathbb{R}, \quad \omega > 0,
\]

in the purely electrostatic case

\[
\phi = \phi(x) \quad \text{and} \quad A = 0,
\]
leads exactly to the system we want to study. The boundary conditions
\( u = \phi = 0 \) on \( \partial \Omega \) mean that the particle is constraint to live in \( \Omega \). In the
following, referring to (2) we will assume for simplicity \( \omega = 1 \).

Problem (2) contains two kinds of nonlinearities: the first one is \( \phi u \) and
concerns the interaction with the electric field. This nonlinear term is non-
local since the electrostatic potential \( \phi \) depends also on the wave function to
which is related by the Poisson equation $-\Delta \phi = |\psi|^2 = u^2$. The second non-linearity is $|u|^{p-2}u$. This one contains the Slater correction term $C_S |u|^{2/3}u$, where $C_S$ is the Slater constant and depends on the particles considered (for more details see [9, 24]). Physically speaking, the local nonlinearity $|u|^{p-2}u$ represents the interaction among many particles and is in competition with the intrinsic nonlinearity of the system $\phi u$.

Motivated by some perturbation results (see e.g. [13, 23] in which the case with $\Omega = \mathbb{R}^3$ and $\lambda \to 0^+$ is considered), we have introduced the parameter $\lambda > 0$ which takes a role also in a bounded domain, at least for small values of $p$.

Because of its importance in many different physical frameworks, the Schrödinger-Poisson-Slater system (sometimes called Schrödinger-Maxwell system) has been extensively studied in the past years: besides the results on bounded domains (see e.g. [7, 20, 21, 25]), there are also many papers on $\mathbb{R}^3$ which treat different aspects of the SPS system, even with an additional external and fixed potential $V(x)$. In particular ground states, radially and non-radially solutions or semiclassical limit and concentration of solutions are studied, see e.g. [2, 3, 11, 12, 14, 17, 18, 24, 28].

We approach problem (2) by variational methods: the weak solutions are characterized as critical points of a $C^1$ functional $I = I(u)$ defined on the Sobolev space $H^1_0(\Omega)$ or a suitable submanifold (see below). A fundamental tool to apply variational techniques is the so-called Palais-Smale condition (PS for brevity): every sequence $\{u_n\}$ such that

$$\{I(u_n)\} \text{ is bounded and } I'(u_n) \to 0 \text{ in } H^{-1}(\Omega),$$

admits a converging subsequence. Sequences which satisfy (3) are called Palais-Smale sequences.

Now, it is known that when $p \in (4, 2^*)$ the PS condition holds (see e.g. [21]), hence we have hope to apply classical theorems of LS theory in the same spirit of [4] and [5], to find critical points of $I$; indeed we get the following result

**Theorem 1.2.** There exists a $\bar{p} \in (4, 2^*)$ such that for every $p \in [\bar{p}, 2^*)$ problem (2) has at least $\text{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$ positive solutions.

It is understood that $\bar{p}$ does not depend on the “strength” of the interaction $\lambda$. We remark that the weak solutions found by means of the variational method are indeed classical solutions, by standard regularity results.

To prove the theorem we use the general ideas of Benci and Cerami adapting their arguments to our problem which contains also the coupling term $\phi u$.

The paper is organized as follow: in the next Section we fix the notations and recall some useful facts. Sections 3 and 4 are devoted to the functional setting and to introduce the ingredients which allow us to use the abstract theory of Ljusternik-Schnirelmann. Finally the proof of Theorem 1.2 is completed in Section 5.
Acknowledgment. The author wishes to thank David Ruiz for helpful discussions on the matter and prof. G. Cerami for bringing to his attention paper [5].

2. Some notations and preliminaries

Without loss of generality we assume in all the paper $0 \in \Omega$. We denote by $| \cdot |_{L^p(A)}$ the $L^p$-norm of a function defined on the domain $A$. If the domain is specified (usually $\Omega$) or if there is no confusion, we use the notation $| \cdot |_p$. Moreover let $H^1_0(\Omega)$ be the usual Sobolev space with (squared) norm $\|u\|_2^2 = |\nabla u|^2 + |u|^2$ and dual $H^{-1}(\Omega)$.

We use $B_r(y)$ for the closed ball of radius $r > 0$ centered in $y$. If $y = 0$ we simply write $B_r$.

The letter $c$ will be used indiscriminately to denote a suitable positive constant whose value may change from line to line and we will use $o(1)$ for a quantity which goes to zero.

Finally, in view of our Theorem 1.2, from now on we assume $p > 4$. Other notations will be introduced in Section 4.

First of all, let $\phi_u \in H^1_0(\Omega)$ be the unique (and positive) solution of $-\Delta \phi = u^2$ and $\phi = 0$ on $\partial \Omega$ and let us recall the following properties that will be repeatedly used (for a proof see e.g. [24]):

- for any $\alpha, \beta \geq 0, t > 0$ let $u_t(\cdot) = t^\alpha u(t^\beta(\cdot))$. Then $\phi_{u_t}(\cdot) = t^{2(\alpha - \beta)} \phi_u(t^\beta(\cdot))$;
- $u_n \rightharpoonup u$ in $H^1_0(\Omega) \implies \int_\Omega \phi_{u_n} u_n^2 dx \to \int_\Omega \phi_u u^2 dx$;
- $|\nabla \phi_u|_2 \leq c|\nabla u|^2$ for some constant $c > 0$;
- $\int_\Omega |\nabla \phi_u|^2 dx = \int_\Omega \phi_u u^2 dx$.

The functional associated to (2) is

$$I_p(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2) \, dx + \frac{\lambda}{4} \int_\Omega \phi_u u^2 \, dx - \frac{1}{p} \int_\Omega |u|^p \, dx$$

and its critical points are the solutions of the system (see e.g. [12]). However the functional is unbounded from above and from below on $H^1_0(\Omega)$. The idea is to restrict the functional to a suitable manifold on which this unboundedness is removed.

In [5] the authors deal with $E(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|^p$ and to overcome the unboundedness they introduce the constraint

$$V_p = \{ u \in H^1_0(\Omega) : |u|^p = 1 \}.$$

On $V_p$ the functional $E$ is bounded from below (achieves its minimum), satisfies the PS condition and the classical LS theory applies. This gives constraint critical points and Lagrange multipliers appear in the right hand
side of the equation in (1). Finally, “stretching” the multipliers one gets solutions of (1).

In our case the constraint $V_p$ is not a good choice although $I_p$ would have a minimum on $V_p$. This is due to a different degree of homogeneity of the added term $\lambda \phi u u$; indeed it is easy to see that there is no way to eliminate the Lagrange multiplier once it appears. We study the functional (4) on a natural constraint and in this case the Nehari manifold works well.

3. The Nehari manifold

In this section we recall some known facts about the Nehari manifold that will be used throughout the paper.

The Nehari manifold associated to (4) is defined by

$$\mathcal{N}_p = \{ u \in H_0^1(\Omega) \setminus \{0\} : G_p(u) = 0 \}$$

where

$$G_p(u) := I_p'(u)[u] = \|u\|^2 + \lambda \int_\Omega \phi u u^2 dx - |u|^p.$$

On $\mathcal{N}_p$ the functional (4) has the form

$$(5) \quad I_p(u) = \frac{p-2}{2p} \|u\|^2 + \frac{\lambda(p-4)}{4p} \int_\Omega \phi u u^2 dx.$$

Sometimes we will refer to (5) as the constraint functional, also denoted with $I_p|_{\mathcal{N}_p}$.

In the next Lemma we recall the basic properties of the Nehari manifold.

Lemma 3.1. We have

1. $\mathcal{N}_p$ is a $C^1$ manifold,
2. there exists $c > 0$ such that for every $u \in \mathcal{N}_p : c \leq \|u\|$, 
3. for every $u \neq 0$ there exists a unique $t > 0$ such that $tu \in \mathcal{N}_p$, 
4. the following equalities are true

$$m_p = \inf_{u \neq 0} \max_{t > 0} I_p(tu) = \inf_{g \in \Gamma_p} \max_{t \in [0,1]} I_p(g(t))$$

where

$$\Gamma_p = \{ g \in C([0,1]; H_0^1(\Omega)) : g(0) = 0, I_p(g(1)) \leq 0, g(1) \neq 0 \}.$$

Then recalling that $p > 4$, we have

$$m_p := \inf_{u \in \mathcal{N}_p} I_p(u) > 0.$$

Moreover the manifold $\mathcal{N}_p$ is a natural constraint for $I_p$ (given by (1)) in the sense that any $u \in \mathcal{N}_p$ critical point of $I_p|_{\mathcal{N}_p}$ is also a critical point for the free functional $I_p$ (for a proof of these facts, see e.g. Section 6.4 in [1]).

Hence the (constraint) critical points we find are solutions of our problem since no Lagrange multipliers appear.

The Nehari manifold well-behaves with respect to the PS sequences:
Lemma 3.2. Let \( \{u_n\} \subset \mathcal{N}_p \) be a PS sequence for \( I_p|_{\mathcal{N}_p} \). Then it is a PS sequence for the free functional \( I_p \) on \( H_0^1(\Omega) \).

Proof. By definition, \( \{u_n\} \subset \mathcal{N}_p \), \( I_p|_{\mathcal{N}_p}(u_n) \) is bounded and there exist Lagrange multipliers \( \{\mu_n\} \subset \mathbb{R} \) such that \( (I_p|_{\mathcal{N}_p})'(u_n) = I_p'(u_n) - \mu_n G_p(u_n) \to 0 \) in \( H^{-1}(\Omega) \). Then recalling the definition of \( \mathcal{N}_p \) we have

\[
(I_p|_{\mathcal{N}_p})'(u_n)[u_n] = \mu_n G_p(u_n)[u_n] \to 0.
\]

Since \( G_p'(u_n)[u_n] \neq 0 \) it follows that the sequence of multipliers vanishes and

\[
I_p'(u_n) = (I_p|_{\mathcal{N}_p})'(u_n) + \mu_n G_p'(u_n) \to 0.
\]

\[\square\]

As we have already anticipated, for \( p \in (4, 2^*) \) it is known that the free functional \( I_p \) given by (4) satisfies the PS condition on \( H_0^1(\Omega) \) (see e.g. [21]). The fact that the PS condition follows also for the functional restricted to \( \mathcal{N}_p \) is standard.

In the following we will deal always with the restricted functional on the Nehari manifold; this will be denoted simply with \( I_p \).

As a consequence of the PS condition we deduce that

\[
\forall p \in (4, 2^*) : \quad m_p = \min_{\mathcal{N}_p} I_p = I_p(u_p),
\]

i.e. \( m_p \) is achieved on a function, hereafter denoted with \( u_p \), in \( \mathcal{N}_p \). Since \( u_p \) minimizes the energy \( I_p \), it will be called a ground state.

Observe that the sequence of minimizers \( \{u_p\}_{p \in (4, 2^*)} \) is bounded away from zero; indeed, since \( u_p \in \mathcal{N}_p \),

\[
\|u_p\|^2 \leq |u_p|^p \leq C \|u_p\|^p
\]

where \( C \) is a positive constant which can be made independent of \( p \). Hence

\[
\exists c > 0 \quad \text{ s.t. } \forall p \in (4, 2^*) : \quad 0 < c \leq \|u_p\|.
\]

Remark 3.3. Turning back to (6), we have that \( \{|u_p|^p\}_{p \in (4, 2^*)} \) is bounded away from zero. Moreover, denoting with \( |\Omega| \) the Lebesgue measure of \( \Omega \), by the Hölder inequality,

\[
|u_p| \leq |\Omega|^{\frac{2^*-p}{2^*p}} |u_p|^{2^*}
\]

and so also \( \{|u_p|^2^*\}_{p \in (4, 2^*)} \) is bounded away from zero.

Clearly, all we have stated until now is true also in the case \( \lambda = 0 \). Moreover also the case \( p = 2^* \) is covered for those results which do not require compactness (in particular Lemma [3.1] and [3.2]).
3.1. The limit cases. We consider in this subsection two limit cases related to (2). Our intent is to evaluate the limit of the sequence \( \{ m_p \} \) when \( p \to 2^* \).

The first case is the critical problem. Let us introduce the functional

\[
I_* (u) = \frac{1}{2} \| u \|^2 - \frac{1}{2^*} |u|_{2^*}^{2^*}
\]

whose critical points are the solutions of

\[
\begin{cases}
- \Delta u + u = |u|^{2^* - 2} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

It is known that the lack of compactness of the embedding of \( H_0^1(\Omega) \) in \( L^{2^*} (\Omega) \) implies that \( I_* \) does not satisfy the PS condition at every level. This is due to the invariance with respect to the conformal scaling

\[
u(\cdot) \mapsto u_R(\cdot) := R^{1/2} u(R(\cdot)) \quad (R > 1)
\]

which leaves invariant the \( L^2 \) norm of the gradient and the \( L^{2^*} \) norm, i.e.

\[
\| \nabla u_R \|_2^2 = \| \nabla u \|_2^2 \quad \text{and} \quad |u_R|^{2^*}_{2^*} = |u|^{2^*}_{2^*}.
\]

As a consequence, if

\[
\mathcal{N}_* = \{ u \in H_0^1(\Omega) : G_*(u) = 0 \}, \quad G_*(u) = \| u \|^2 - |u|_{2^*}^{2^*},
\]

is the Nehari manifold associated, it can be proved that

\[
m_* := \inf_{\mathcal{N}_*} I_* \quad \text{is not achieved.}
\]

The following lemma is known but for the sake of completeness we give the proof.

Lemma 3.4. There holds

\[
m_* = \frac{1}{3} S^{3/2}
\]

where \( S = \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\| u \|^2}{|u|_{2^*}^{2^*}} \) is the best Sobolev constant.

Proof. This is indeed an easy computation. First observe that for \( A, B > 0 \) it results

\[
\max_{t > 0} \left\{ \frac{t^2}{2} A - \frac{t^{2^*}}{2^*} B \right\} = \frac{1}{3} \left( \frac{A}{B^{1/3}} \right)^{3/2}.
\]

Then

\[
m_* = \inf_{u \neq 0} \max_{t > 0} I_*(t u) = \frac{1}{3} \left( \inf_{u \neq 0} \frac{\| u \|^2}{|u|_{2^*}^{2^*}} \right)^{3/2} = \frac{1}{3} S^{3/2}.
\]

The value \( m_* \) turns out to be an upper bound for the sequence of ground states levels \( \{ m_p \} \). Before to prove this, let us observe that, as easy computations show:

1) \( |u_R|_p^p = R^{p - 2^*} |u|_p^p \).
2) \( \int_\Omega \phi_{u_R} u_R^2 \, dx = R^{-3} \int_\Omega \phi_u u^2 \, dx. \)

**Lemma 3.5.** We have

\[
\limsup_{p \to 2^*} m_p \leq m_*.
\]

**Proof.** Fix \( \varepsilon > 0 \). By definition of \( m_* \) there exists \( u \in \mathcal{N}_* \) such that

\[
I_*(u) = \frac{1}{2} \| u \|^2 - \frac{1}{2_*} |u|_{2_*}^{2_*} = \frac{1}{3} \| u \|^2 < m_* + \frac{\varepsilon}{2}.
\]

For \( R > 1 \) (to be specified later), we have

\[
I_*(u_R) = \frac{1}{2} |\nabla u|_2^2 + \frac{1}{2R^2} |u|_2^2 - \frac{1}{2_*} |u|_{2_*}^{2_*} < m_* + \frac{\varepsilon}{2}.
\]

Now consider, for any \( p \in (4, 2^*) \), the unique positive value \( t_p \) such that \( t_p u_R \in \mathcal{N}_p \).

By definition, \( t_p \) satisfies

\[
\| t_p u_R \|^2 + \lambda t_p^4 \int_\Omega \phi_{u_R} u_R^2 \, dx = |t_p u_R|^p
\]

from which we deduce:

- \( \{t_p\}_{p \in (4, 2^*)} \) is bounded away from zero.

Indeed by \([9]\) and the embedding of \( L^p \) in \( H^1_0 \) we get \( \| t_p u_R \|^2 \leq C \| t_p u_R \|^p \) so \( \| t_p u_R \|^2 \geq c \) and finally \( t_p^2 \geq \frac{c}{\| u_R \|^2} \geq \frac{c}{\| u \|^2} > 0 \).

- \( \{t_p\}_{p \in (4, 2^*)} \) is bounded above.

Indeed

\[
\frac{\| u_R \|^2}{t_p^2} + \lambda \int_\Omega \phi_{u_R} u_R^2 \, dx = t_p^p |u_R|^p
\]

and, by the continuity of the map \( p \mapsto |u_R|^p \), it is readily seen that if \( t_p \) tends to \( +\infty \) we get a contradiction.

So we may assume that \( \lim_{p \to 2^*} t_p = t_* \) and passing to the limit in \([9]\) we get

\[
t_*^2 |\nabla u|_2^2 + \frac{t_*^2}{R^2} |u|_2^2 + \lambda \frac{t_*^4}{R^3} \int_\Omega \phi_u u^2 \, dx = t_*^{2_*} |u|_{2_*}^{2_*}.
\]

or equivalently,

\[
(t_*^{2_*} - t_*^2) |\nabla u|_2^2 = \frac{t_*^2}{R^2} |u|_2^2 + \lambda \frac{t_*^4}{R^3} \int_\Omega \phi_u u^2 \, dx - t_*^{2_*} |u|_{2_*}^{2_*}.
\]

Now if \( R \) is chosen sufficiently large, the r.h.s. above is negative and we deduce

\[
t_* < 1.
\]
Furthermore
\[
I_p(t_p u_R) = \frac{p-2}{2p} \|t_p u_R\|^2 + \lambda \frac{p-4}{4p} t_p^4 \int_\Omega \phi u_R u_R^2 dx
\]
\[
= \frac{p-2}{2p} t_p^2 |\nabla u_R|^2 + \frac{p-2}{2p} \frac{t_p^2}{R^2} |u_R|^2 + \lambda \frac{p-4}{4p} \frac{t_p^4}{R^3} \int_\Omega \phi u u^2 dx
\]
and passing to the limit for \( p \to 2^* \), taking advantage of (10),
\[
\lim_{p \to 2^*} I_p(t_p u_R) = \frac{1}{3} t_p^2 |\nabla u_R|^2 + \frac{1}{3} \frac{t_p^2}{R^2} |u_R|^2 + \lambda \frac{t_p^4}{12R^3} \int_\Omega \phi u^2 dx
\]
\[< \frac{1}{3} \|u\|^2 + \frac{\lambda}{12R^3} \int_\Omega \phi u^2 dx.\]
Lastly, if \( R \) is such that \( \lambda \frac{1}{12R^3} \int_\Omega \phi u^2 dx < \varepsilon/2 \) we get, using (8)
\[
\limsup_{p \to 2^*} m_p \leq \lim_{p \to 2^*} I_p(t_p u_R) < \frac{1}{3} \|u\|^2 + \frac{\varepsilon}{2} < m_* + \varepsilon
\]
which concludes the proof since \( \varepsilon \) is arbitrary.

Note that by (5), the boundedness of \( \{m_p\}_{p \in (4,2^*)} \) implies the boundedness of the ground state solutions, namely
\[
(11) \quad \exists c > 0 \text{ such that } \forall p \in (4,2^*) : \|u_p\| \leq c.
\]
We need now a technical lemma.

**Lemma 3.6.** Let \( p \in (4,2^*) \) and \( t_p > 0 \) the unique value such that \( t_p u_p \in \mathcal{N}_* \). Then
\[
\limsup_{p \to 2^*} t_p \leq 1.
\]

**Proof.** By definition of \( \mathcal{N}_* \), \( t_p \) satisfies
\[
t_p^{2^*} |u_p|^{2^*} = t_p^2 \|u_p\|^2
\]
and using that \( u_p \in \mathcal{N}_* \) and the Hölder inequality we get
\[
(12) \quad t_p^{2^*-2} = \frac{|u_p|^p}{|u_p|^{2^*}} - \lambda \frac{1}{\|u_p\|^{2^*}} \int_\Omega \phi u u^2 dx \leq \frac{|u_p|^p}{|u_p|^{2^*}} \leq \left| \frac{\Omega}{|u_p|^{2^*}} \right|^{\frac{2^*-p}{2^*-p}}.
\]
By the embedding \( L^{2^*}(\Omega) \hookrightarrow H^1_0(\Omega) \) and (11) we deduce that the sequence \( \{|u_p|^{2^*}\}_{p \in (4,2^*)} \) is bounded. Moreover recalling Remark 3.3 we have that it is also bounded away from zero. So the conclusion follows by (12) since
\[
\lim_{p \to 2^*} \left| \frac{\Omega}{|u_p|^{2^*}} \right|^{\frac{2^*-p}{2^*-p}} = 1.
\]

**Remark 3.7.** Again note that Proposition 3.5, (11) and Lemma 3.6 hold also for problem (2) with \( \lambda = 0 \).
The other limit case we consider is that related to problem (1), namely setting \( \lambda = 0 \) in (2).

For any \( p \in (4, 2^*) \) let \( \tilde{I}_p(u) = \frac{1}{2} \| u \|^2 - \frac{1}{p} |u|^p \) be the functional on \( H^1_0(\Omega) \) whose critical points solve

\[
\begin{cases}
-\Delta u + u = |u|^{p-2} u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

As usual, we can define \( \tilde{N}_p = \{ u \in H^1_0(\Omega) \setminus \{0\} : \| u \|^2 = |u|^p \} \) on which the functional is \( \tilde{I}_p(u) = \frac{p-2}{2p} \| u \|^2 \) and we denote with

\[
\tilde{m}_p := \min_{\tilde{N}_p} \tilde{I}_p = \tilde{I}_p(\tilde{u}_p).
\]

By Remark 3.7 we have

\[(13) \quad \{\| \tilde{u}_p \|\}_{p \in (4, 2^*)} \text{ is bounded.}\]

Moreover, if \( t_p > 0 \) is such that \( t_p u_p \in \tilde{N}_p \), by (6) we get \( t_p^{p-2} = \frac{\| u_p \|^2}{|u_p|^p} \leq 1 \) and so

\[
\tilde{m}_p \leq \tilde{I}_p(t_p u_p) = \frac{p-2}{2p} t_p^2 \| u_p \|^2 \leq \frac{p-2}{2p} \| u_p \|^2 < I_p(u_p).
\]

This means

\[(14) \quad \tilde{m}_p < m_p.\]

Now we are ready to compute the limit of \( m_p \) when \( p \) tends to \( 2^* \).

**Proposition 3.8.** For any bounded domain we have

\[
\lim_{p \to 2^*} m_p = m_*.
\]

**Proof.** By (14) and Lemma 3.5 it is sufficient to prove that

\[
m_* \leq \liminf_{p \to 2^*} \tilde{m}_p.
\]

Let \( t_p > 0 \) the unique value such that \( t_p \tilde{u}_p \in \tilde{N}_p \). Applying Lemma 3.6 (with \( \lambda = 0 \)) we know

\[
\limsup_{p \to 2^*} t_p \leq 1.
\]

Finally, using (13) we derive

\[
m_* \leq I_*(t_p \tilde{u}_p) = \left( \frac{1}{2} - \frac{1}{2^*} \right) t_p^2 \| \tilde{u}_p \|^2
= \tilde{I}_p(\tilde{u}_p) t_p^2 + \left( \frac{1}{p} - \frac{1}{2^*} \right) \| \tilde{u}_p \|^2 t_p^2
= \tilde{m}_p t_p^2 + o(1)
\]

where \( o(1) \to 0 \) for \( p \to 2^* \). Hence the conclusion follows.
4. The barycenter map

In this section we introduce the barycenter map that will allow us to compare the topology of $\Omega$ with the topology of suitable sublevels of $I_p$; precisely sublevels with energy near the minimum level $m_p$.

Before to proceed, some other notations are in order. For $u \in H^1(\mathbb{R}^3)$ with compact support, let us denote with the same symbol $u$ its trivial extension out of $\text{supp } u$. The barycenter of $u$ (see [4]) is defined as

$$\beta(u) = \frac{\int_{\mathbb{R}^3} x |\nabla u|^2 dx}{\int_{\mathbb{R}^3} |\nabla u|^2 dx}.$$

From now on, we fix $r > 0$ a radius sufficiently small such that $B_r \subset \Omega$ and the sets

$$\Omega^+_r = \{x \in \mathbb{R}^3 : d(x, \Omega) \leq r\}$$

$$\Omega^-_r = \{x \in \Omega : d(x, \partial \Omega) \geq r\}$$

are homotopically equivalent to $\Omega$. In particular we denote by

$$h : \Omega^+_r \to \Omega^-_r$$

the homotopic equivalence map such that $h|_{\Omega^-_r}$ is the identity.

Let us introduce the space $D^{1,2}(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2\}$ which can also be characterized as the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the (squared) norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx.$$

A function in $H^1_0(\Omega)$ can be thought as an element of $D^{1,2}(\mathbb{R}^3)$.

The following “global compactness” result is taken from Struwe (see Theorem 3.1 of [27]) and will be useful to study the behavior of the PS sequences for the limit functional $I_*(u) = \frac{1}{2} \|u\|^2 - \frac{1}{2^*} |u|^{2^*}$.

**Theorem 4.1.** Let $\{v_n\}$ be a PS sequence for $I_*$ in $H^1_0(\Omega)$. Then there exist a number $k \in \mathbb{N}_0$, sequences of points $\{x_j^n\} \subset \Omega$ and sequences of radii $\{R_j^n\}$ ($1 \leq j \leq k$) with $R_j^n \to +\infty$ for $n \to +\infty$, there exist a positive solution $v \in H^1_0(\Omega)$ of (7) and non trivial solutions $v_j \in D^{1,2}(\mathbb{R}^3)$ ($1 \leq j \leq k$) of

$$-\Delta u = |u|^{2^*-2} \quad \text{in } \mathbb{R}^3,$$

such that, a (relabeled) subsequence $\{v_n\}$ satisfies

$$v_n - v - \sum_{j=1}^k v_j^{R_n} (\cdot - x_j^n) \to 0 \quad \text{in } D^{1,2}(\mathbb{R}^3),$$

$$I_*(v_n) \to I_*(v) + \sum_{j=1}^k \hat{I}(u^j)$$

where $\hat{I} : H^1_0(\mathbb{R}^3) \to \mathbb{R}$ is given by

$$\hat{I}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^3} |u|^{2^*} dx.$$
Basically the theorem states that if the PS condition fails, it is due to the solutions of (16). For what concerns $\hat{I}$, it is known that it achieves its minimum on functions of type

\begin{equation}
U_R(x-a) = \frac{(3R^2)^{1/4}}{(R^2 + |x-a|^2)^{1/2}} \quad R > 0, a \in \mathbb{R}^3
\end{equation}

and the minimum value is exactly $\hat{I}(U_R(\cdot-a)) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla U|^2 dx = m_*$, namely the infimum of $I^*$. On the other hand, the value of $\hat{I}$ on solutions of (16) which do not belong to the family (17) is greater than $2m_*$. As a consequence, if the sequence $\{v_n\}$ of Theorem 4.1 is a PS sequence for $I^*$ at level $m^*$, we deduce $I^*(v) = 0$, $k = 1$ and $v^1 = U$. Furthermore, since $v$ is a solution of (7) and $I^*$ is positive on the solutions, necessarily $v = 0$ and so Theorem 4.1 gives

$$v_n - U_{R_n}(\cdot - x_n) \to 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^3).$$

Thanks to the previous theorem we can prove that, roughly speaking, if $p$ is near the critical exponent $2^*$, the functions with barycenter outside $\Omega$ have an energy away from the ground state level $m_p$.

**Proposition 4.2.** There exists $\varepsilon > 0$ such that if $p \in (2^* - \varepsilon, 2^*)$, it follows

$$u \in N_p \quad \text{and} \quad I_p(u) < m_p + \varepsilon \implies \beta(u) \in \Omega^+_p.$$

**Proof.** We argue by contradiction. Assume that there exist sequences $\varepsilon_n \to 0, p_n \to 2^*$ and $u_n \in N_{p_n}$ such that

\begin{equation}
I_{p_n}(u_n) \leq m_{p_n} + \varepsilon_n \quad \text{and} \quad \beta(u_n) \notin \Omega^*_p.
\end{equation}

Then, by Proposition 3.3

\begin{equation}
I_{p_n}(u_n) \to m^*
\end{equation}

and $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Let $t_n > 0$ such that $t_n u_n \in N_s$. By Lemma 3.6 we may assume (up to subsequence) that $t_n \to 1$ and we evaluate

\begin{align*}
I_{p_n}(u_n) - I_s(t_n u_n) &= \left(\frac{1}{2} - \frac{1}{p_n}\right) \|u_n\|^2 + \lambda \frac{p_n-4}{4p_n} \int_{\Omega} \phi_{u_n} u_n^2 dx - \left(\frac{1}{2} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \\
&\geq \left(\frac{1}{2} - \frac{1}{p_n}\right) \|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \\
&= \left(\frac{1}{2} - \frac{1}{p_n}\right) \|u_n\|^2 \left(1 - t_n^2\right) - \left(\frac{1}{p_n} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \\
&= o(1)
\end{align*}

which gives

$$m_* \leq I_s(t_n u_n) \leq I_{p_n}(u_n) + o(1).$$
By [19], $I_*(t_n u_n) \to m_*$ for $n \to +\infty$. The Ekeland’s variational principle implies that there exist $\{v_n\} \subset N_*$ and $\{\mu_n\} \subset \mathbb{R}$ such that

$$
\|t_n u_n - v_n\| \to 0 \\
I_*(v_n) = \frac{1}{3}\|v_n\|^2 \to m_* \\
I'_*(v_n) - \mu_n G'_*(v_n) \to 0
$$

and Lemma 3.2 (in the case $\lambda = 0$) ensures that $\{v_n\}$ is a PS sequence for the free functional $I_*$ at level $m_*$. By the remarks after Theorem 4.1, $v_n = U_{R_n}(\cdot - x_n) + w_n$ with a remainder $w_n$ such that $\|w_n\|_{D^{1,2}(\mathbb{R}^3)} \to 0$. It is clear that $t_n u_n = v_n + t_n u_n - v_n$; so, renaming the remainder again $w_n$, we have $t_n u_n = U_{R_n}(\cdot - x_n) + w_n$.

Now writing $x \in \mathbb{R}^3$ as $x = (x^1, x^2, x^3)$, the $i-$th coordinate of the barycenter of $u_n$ satisfies

$$
\beta(u_n)^i \|t_n u_n\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} x^i |\nabla U_{R_n}(x - x_n)|^2 \, dx \\
+ \int_{\mathbb{R}^3} x^i |\nabla w_n(x)|^2 \, dx + 2 \int_{\mathbb{R}^3} x^i \nabla U_{R_n}(x - x_n) \nabla w_n(x) \, dx.
$$

The aim is to localize the sequence of barycenters, so we pass to the limit in the above expression evaluating $\|t_n u_n\|^2_{D^{1,2}(\mathbb{R}^3)}$ and the right hand side.

First,

$$
\|t_n u_n\|^2_{D^{1,2}(\mathbb{R}^3)} = \|U\|^2_{D^{1,2}(\mathbb{R}^3)} + o(1)
$$

and simple computations show that

$$
\int_{\mathbb{R}^3} x^i |\nabla U_{R_n}(x - x_n)|^2 \, dx = \\
\frac{1}{R_n} \int_{\mathbb{R}^3} y^i |\nabla U(y)|^2 \, dy + x_n^i \int_{\mathbb{R}^3} |\nabla U(y)|^2 \, dy.
$$

Moreover, since $v_n$ are supported in $\Omega$, there holds $U_{R_n}(\cdot - x_n) = -w_n$ on $\mathbb{R}^3 \setminus \Omega$ and we evaluate

$$
\int_{\mathbb{R}^3} x^i |\nabla w_n(x)|^2 \, dx = \int_{\Omega} x^i |\nabla w_n(x)|^2 \, dx + A_n
$$
where

\[ A_n = \int_{\mathbb{R}^3 \setminus \Omega} x^i |\nabla w_n(x)|^2 \, dx \]

\[ = \int_{\mathbb{R}^3 \setminus \Omega} x^i R_n |\nabla U(R_n(x - x_n))|^2 \, dx \]

\[ = \int_{\mathbb{R}^3 \setminus R_n(\Omega - x_n)} \left( \frac{y^i}{R_n} + x^i_n \right) |\nabla U(y)|^2 \, dy \]

\[ = \frac{1}{R_n} \int_{\mathbb{R}^3 \setminus R_n(\Omega - x_n)} y^i |\nabla U(y)|^2 \, dy + x^i_n \int_{\mathbb{R}^3 \setminus R_n(\Omega - x_n)} |\nabla U(y)|^2 \, dy = o(1). \]

As a consequence,

\[ \int_{\mathbb{R}^3} x^i |\nabla w_n(x)|^2 \, dx = \int_{\Omega} x^i |\nabla w_n(x)|^2 \, dx + o(1) = o(1). \]

The last term in (20) is estimated as

\[ \int_{\mathbb{R}^3} x^i \nabla U_{R_n}(x - x_n) \nabla w_n(x) \, dx = \int_{\Omega} x^i \nabla U_{R_n}(x - x_n) \nabla w_n(x) \, dx - A_n \]

\[ \leq c \left( \int_{\Omega} |\nabla U_{R_n}(x - x_n)|^2 \, dx \right)^{1/2} \left( \int_{\Omega} |\nabla w_n|^2 \, dx \right)^{1/2} - A_n \]

with \( A_n \) defined as before and then,

\[ \int_{\mathbb{R}^3} x^i \nabla U_{R_n}(x - x_n) \nabla w_n(x) \, dx = o(1). \]

Putting together (21), (22), (23) and (24) by (20) we deduce

\[ \beta(u_n)^i = \frac{x^i_n \int_{\mathbb{R}^3} |\nabla U(y)|^2 \, dy + o(1)}{\|U\|_{D^{1,2}(\mathbb{R}^3)}^2 + o(1)}. \]

Since \( \{x_n\} \subset \Omega \), (25) implies that definitively \( \beta(u_n) \in \bar{\Omega} \) which is in contrast with (18) and proves the proposition.

5. Proof of Theorem 1.2

Here we complete the proof of our theorem but first we need a slight modification to the previous notations. We add a subscript \( r \) (\( r > 0 \) and small as before) to denote the same quantities defined in the previous sections when the domain \( \Omega \) is replaced by \( B_r \); namely integrals are taken on \( B_r \) and norms are taken for functional spaces defined on \( B_r \). Hence

\[ N_{p,r} = \left\{ u \in H^1_0(B_r) : \|u\|^2_{H^1_0(B_r)} + \lambda \int_{B_r} \phi_u u^2 \, dx = |u|^p_{L^p(B_r)} \right\} \]

and, for \( u \in N_{p,r} \)

\[ I_{p,r}(u) = \frac{p - 2}{2p} \|u\|^2_{H^1_0(B_r)} + \lambda \frac{p - 4}{4p} \int_{B_r} \phi_u u^2 \, dx, \]
Moreover let
\[ I_{p}^{m_{p,r}} = \{ u \in \mathcal{N}_{p} : I_{p}(u) \leq m_{p,r} \} \]
which is non vacuous since \( m_{p} < m_{p,r} \).

Define also, for \( p \in (4, 2^{*}) \) the map \( \Psi_{p,r} : \Omega_{r}^{-} \to \mathcal{N}_{p} \) such that
\[ \Psi_{p,r}(y)(x) = \begin{cases} 
    u_{p,r}(|x-y|) & \text{if } x \in B_r(y) \\
    0 & \text{if } x \in \Omega \setminus B_r(y)
\end{cases} \]
and note that we have
\[ \beta(\Psi_{p,r}(y)) = y \quad \text{and} \quad \Psi_{r,p}(y) \in I_{p}^{m_{p,r}}. \]

Moreover, since \( m_{p} + k_{p} = m_{p,r} \) where \( k_{p} > 0 \) and tends to zero if \( p \to 2^{*} \) (see Proposition 3.8), in correspondence of \( \varepsilon > 0 \) provided by Proposition 4.2, there exists a \( \bar{p} \in (4, 2^{*}) \) such that for every \( p \in [\bar{p}, 2^{*}] \) it results \( k_{p} < \varepsilon \); so if \( u \in I_{p}^{m_{p,r}} \) we have
\[ I_{p}(u) \leq m_{p,r} < m_{p} + \varepsilon, \]
at least for \( p \) near \( 2^{*} \). Hence the following maps are well-defined:
\[ \Omega_{r}^{-} \xrightarrow{\Psi_{p,r}} I_{p}^{m_{p,r}} \xrightarrow{h \circ \beta} \Omega_{r}^{-} \]
where \( h \) is given by [15]. Since the composite map \( h \circ \beta \circ \Psi_{p,r} \) is the identity of \( \Omega_{r}^{-} \), by a property of the category, the sublevel \( I_{p}^{m_{p,r}} \) “dominates” the set \( \Omega_{r}^{-} \) in the sense that
\[ \text{cat}_{I_{p}^{m_{p,r}}}(\Omega_{r}^{-}) \geq \text{cat}_{\Omega_{r}^{-}}(\Omega_{r}^{-}) \]
(see e.g. [16]) and our choice of \( r \) gives \( \text{cat}_{\Omega_{r}^{-}}(\Omega_{r}^{-}) = \text{cat}_{\Omega}(\Omega_{r}^{+}) \). In conclusion, we have found a sublevel of \( I_{p} \) on \( \mathcal{N}_{p} \) with category greater than \( \text{cat}_{\Omega}(\Omega_{r}^{+}) \). Since, as we have already said, the PS condition is verified on \( \mathcal{N}_{p} \), applying the Lusternik-Schnirelmann theory we get the existence of at least \( \text{cat}_{\Omega}(\Omega_{r}^{+}) \) critical points for \( I_{p} \) on the manifold \( \mathcal{N}_{p} \) which give rise to solutions of \( (2) \).

The existence of another solution is obtained with the same arguments of [5]. Since by hypothesis \( \Omega \) is not contractible in itself, by the choice of \( r \) it results \( \text{cat}_{\Omega_{r}^{-}}(\Omega_{r}^{-}) > 1 \), namely \( \Omega_{r}^{-} \) is not contractible in \( \Omega_{r}^{+} \). We claim now that the set \( \Psi_{p,r}(\Omega_{r}^{-}) \) can not be contractible in \( I_{p}^{m_{p,r}} \). Indeed, assume by contradiction that \( \text{cat}_{I_{p}^{m_{p,r}}}(\Psi_{p,r}(\Omega_{r}^{-})) = 1 \): this means that there exists a map \( \mathcal{H} \in C([0, 1] \times \Psi_{p,r}(\Omega_{r}^{-}); I_{p}^{m_{p,r}}) \) such that
\[ \mathcal{H}(0, u) = u \quad \forall u \in \Psi_{p,r}(\Omega_{r}^{-}) \quad \text{and} \]
\[ \exists w \in I_{p}^{m_{p,r}} : \mathcal{H}(1, u) = w \quad \forall u \in \Psi_{p,r}(\Omega_{r}^{-}). \]
Then \( F = \Psi_{p,r}^{-1}(\mathcal{H}(\Omega_{r}^{-})) \) is closed, contains \( \Omega_{r}^{-} \) and is contractible in \( \Omega_{r}^{+} \) as we can see by defining the map
\[ G(t, x) = \begin{cases} 
    \beta(\Psi_{r,p}(x)) & \text{if } 0 \leq t \leq 1/2, \\
    \beta(\mathcal{H}(2t - 1, \Psi_{p,r}(x))) & \text{if } 1/2 \leq t \leq 1.
\end{cases} \]
Then also $\Omega_r^-$ would be contractible in $\Omega_r^+$ giving a contradiction.

On the other hand we can choose a function $z \in N_p \setminus \Psi_{p,r}(\Omega_r^-)$ so that the cone
$$
\mathcal{C} = \left\{ \theta z + (1-\theta)u : u \in \Psi_{p,r}(\Omega_r^-), \theta \in [0,1] \right\}
$$
is compact and contractible in $H^1_0(\Omega)$ and $0 \notin \mathcal{C}$. Denoting with $t_u$ the unique positive number provided by Lemma 3.1, it follows that if we set
$$
\hat{\mathcal{C}} = \{ t_u u : u \in \mathcal{C} \}, \quad M_p = \max_{\hat{\mathcal{C}}} I_p
$$
then $\hat{\mathcal{C}}$ is contractible in $I_p^{M_p}$ and $M_p > m_{p,r}$. As a consequence also $\Psi_{p,r}(\Omega_r^-)$ is contractible in $I_p^{M_p}$.

Summing up, the set $\Psi_{p,r}(\Omega_r^-)$ is contractible in $I_p^{M_p}$ and not in $I_p^{m_{p,r}}$. Since the PS condition is satisfied we deduce the existence of another critical point with critical level between $m_{p,r}$ and $M_p$.

It remains to prove that these solutions are positive. Note that we can apply all the previous machinery replacing the functional (4) with
$$
I_p^+(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) dx + \frac{\lambda}{4} \int_{\Omega} \phi u^2 dx - \frac{1}{p} \int_{\Omega} (u^+)^p dx
$$
obtaining again at least $\text{cat}_{\Omega}(\Omega)$ nontrivial solutions. Finally the maximum principle ensures that these solutions are positive, hence they solve (2).

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