INEQUALITIES FOR EIGENVALUES OF FOURTH ORDER
ELLIPTIC OPERATORS IN DIVERGENCE FORM ON
RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study eigenvalue of linear fourth order elliptic
operators in divergence form with Dirichlet boundary condition on a bounded
domain in a compact Riemannian manifolds with boundary (possibly empty)
and find a general inequality for them. As an application, by using this in-
equality, we study eigenvalues of this operator on compact domains of complete
submanifolds in a Euclidean space.

1. Introduction

In this paper, let \((M,\langle ,\rangle)\) be an \(n\)-dimensional complete Riemannian manifold
and let \(\Omega \subset M\) be a bounded connected domain with smooth boundary \(\partial \Omega\) in \(M\).
Denote by \(\Delta\) the Beltrami-Laplace operator on \(M\). The study of the spectrum
of geometric operator is an important topic and many works have been done in
this area. The clamped plate problem or the Dirichlet biharmonic operator for a
connected bounded domain \(\Omega \subset \mathbb{R}^n\) is given by
\[
\begin{align*}
\Delta^2 u &= \lambda u, \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]
where \(\nu\) is the outward unit normal vector field of \(\partial \Omega\). Suppose that \(\{\lambda_i\}_{i=1}^{\infty}\)
is eigenvalues of the problem (1.1). Payne et al proved in paper [4] that
\[
\lambda_{k+1} - \lambda_k \leq \frac{8(n+2)}{n^2} \frac{1}{k} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots
\]
In 1984, Hile and Yeh [2] generalized and showed
\[
\sum_{i=1}^{k} \frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1} - \lambda_i} \geq \frac{n^2 k^2}{8(n+2)} \left( \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^{-\frac{1}{2}}, \quad k = 1, 2, \ldots
\]
In 1990, Hook [3] obtained the following inequality
\[
\frac{n^2 k^2}{8(n+2)} \leq \left( \sum_{i=1}^{k} \lambda_i^{\frac{1}{2}} \right) \left( \frac{1}{k} \sum_{i=1}^{k} \frac{\lambda_i^{\frac{1}{2}}}{\lambda_{k+1} - \lambda_i} \right), \quad k = 1, 2, \ldots
\]
In 2006, Cheng and Yang [1] obtained the inequality
\[
\lambda_{k+1} - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \leq \left( \frac{8(n+2)}{n^2} \right)^{\frac{1}{2}} \frac{1}{k} \sum_{i=1}^{k} \left( \lambda_i (\lambda_{k+1} - \lambda_i) \right)^{\frac{1}{2}}, \quad k = 1, 2, \ldots
\]

2010 Mathematics Subject Classification. 35P15, 35J93, 53C42.
Key words and phrases. Eigenvalue, Elliptic operator, Immersion.
In 2007, Wang and Xia [3], proved universal bounds for eigenvalues of the biharmonic operator on Riemannian manifolds, for instance they showed that, when \( \Omega \) is a compact domain in \( \mathbb{R}^n \), then

\[
\lambda_{k+1} \leq \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \left\{ \frac{64}{n^2 k^2} \left( \sum_{i=1}^{k} \lambda_i^2 \right) \left( \sum_{i=1}^{k} \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right) - \frac{1}{k} \sum_{i=1}^{k} \left( \lambda_i - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^2 \right\}^{1/2}, \quad k = 1, 2, \ldots
\]

The aim of the present work is to study the eigenvalues of linear fourth order elliptic operator in divergence form on Riemannian manifolds. In special case, this operator is the biharmonic operator. We prove some general inequalities for them. By using these inequalities, we obtain, when \( \Omega \) is a compact domains of complete submanifolds in a Euclidean space.

Let \( T \) be symmetric positive definite \((1,1)\)-tensor on \( M \) and \( \Omega \subset M \) be a compact domain with smooth boundary \( \partial \Omega \) in \( M \). We will studying the eigenvalue problem

\[
\begin{align*}
\mathcal{L}^2 u &= \lambda u, \quad \text{in } \Omega \\
u &= \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \mathcal{L} u = \text{div}(T(\nabla u)) \) and \( \nabla \) is the gradient operator of \( M \). If \( T \) be the identity tensor the \( \mathcal{L} = \Delta \). The main results of this paper are as follow

**Theorem 1.1.** Let \( \Omega \) be a domain in an \( n \)-dimensional complete Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) isometrically immersed in \( \mathbb{R}^m \), \( \lambda_i \) be the eigenvalue of \((1.2)\) and \( u_i \) be the corresponding orthonormal real-valued eigenfunction, that is

\[
\begin{align*}
\mathcal{L}^2 u_i &= \lambda_i u_i, \quad \text{in } \Omega \\
u_i &= \frac{\partial u_i}{\partial \nu} = 0, \quad \text{on } \partial \Omega \\
\int_{\Omega} u_i u_j dm &= \delta_{ij} \quad \forall i, j = 1, 2, \ldots
\end{align*}
\]

Then for any positive constant \( \delta \) and any positive integer \( k \), we have

\[
(1.4) \quad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} u_i^2 \text{tr}(T) dm \leq \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 A_i + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 B_i
\]

where

\[
A_i = 2 \int_{\Omega} u_i \mathcal{L} u_i \langle \text{tr}(\alpha \circ T) + \text{tr}(\nabla T), I \rangle dm + 2 \int_{\Omega} u_i (\nabla \mathcal{L} u_i, I) dm \\
+ \int_{\Omega} u_i^2 \langle |\text{tr}(\alpha \circ T)|^2 + |\text{tr}(\nabla T)|^2 \rangle dm + 4 \int_{\Omega} u_i (\nabla u_i, \nabla u_i) dm \\
+ 4 \int_{\Omega} |\nabla u_i|^2 dm + 2 \int_{\Omega} \mathcal{L} u_i (\nabla u_i, I) dm,
\]

and

\[
B_i = \int_{\Omega} \left\{ |T(\nabla u_i)|^2 + u_i \langle T(\nabla u_i), \text{tr}(\nabla T) \rangle + \frac{u_i^2}{4} \langle |\text{tr}(\alpha \circ T)|^2 + |\text{tr}(\nabla T)|^2 \rangle \right\} dm,
\]

where \( I(x) = (x_1, \ldots, x_m) \) for any \( x = (x_1, \ldots, x_m) \in \mathbb{R}^n, ||f||^2 = \int_{\Omega} f^2 dm, \) \( dm \) is the volume form on \( \Omega \), \( \alpha \) is the fundamental form of \( M \) and \( \alpha \circ T = \alpha(T(\cdot), \cdot) \).
Theorem 1.2. Let \( \Omega \) be a domain in an \( n \)-dimensional complete Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) isometrically immersed in \( \mathbb{R}^m \). \( \lambda_i \) be the \( i \)th eigenvalue of \( (1.3) \) and \( u_i \) be the corresponding orthonormal real-valued eigenfunction. Then for any positive constant \( \delta \) and any positive integer \( k \), we have

\[
\sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^2 \int_{\Omega} u_i^2 \text{tr}(T) \, dm \leq \delta \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^2 C_i + \frac{1}{\delta} \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right) D_i
\]

where

\[
C_i = 2 \sqrt{m - n} S_0 T_0 I_0 \lambda_i^{\frac{1}{2}} + I_0 \| T(\nabla L u_i) \|_{L^2(\Omega)} + (m - n) S_0^2 T_0^2 + T_0^2 + 4 T_0 \| T(\nabla u_i) \|_{L^2(\Omega)} + 4 \| T(\nabla u_i) \|_{L^2(\Omega)}^2 + 2 \lambda_i I_0 \| T(\nabla u_i) \|_{L^2(\Omega)}
\]

and

\[
D_i = \| T(\nabla u_i) \|_{L^2(\Omega)} + T_0 \| T(\nabla u_i) \|_{L^2(\Omega)} + \frac{1}{4} (m - n) S_0^2 T_0^2 + T_0^2
\]

where \( S_0 = \max \{ \sup_{\Omega} |S_{e_k}| : k = n + 1, \ldots, m \} \), \( S_{e_k} \) is the Weingarten operator of the immersion with respect to \( e_k \), \( T_0 = \sup_{\Omega} |T| \), \( T_0 = \sup_{\Omega} |\text{tr}(\nabla T)| \) and \( I_0 = \sup_{\Omega} |I| \).

Theorem 1.3. Let \( \Omega \) be a domain in an \( n \)-dimensional complete Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) isometrically immersed in \( \mathbb{R}^m \) with mean curvature \( H \) and \( \lambda_i \) be the \( i \)th eigenvalue of biharmonic operator, that is

\[
\begin{align*}
\Delta^2 u_i &= \lambda_i u_i, \quad \text{in } \Omega \\
u_i &= \frac{\partial u_i}{\partial \nu} = 0, \quad \text{on } \partial \Omega \\
\int_{\Omega} u_i u_j \, dm = \delta_{ij}, \quad \forall i, j = 1, 2, \ldots
\end{align*}
\]

Then for any positive constant \( \delta \) and any positive integer \( k \), we have

\[
\sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^2 \leq \delta \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right)^2 \left( 2 n H_0 I_0 \lambda_i^{\frac{1}{2}} + n^2 H_0^2 + 4 \lambda_i^{\frac{1}{2}} \right)
\]

\[
+ \frac{1}{\delta} \sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right) (\lambda_i^{\frac{1}{2}} + \frac{1}{4} n^2 H_0^2)
\]

where \( H_0 = \sup_{\Omega} |H| \) and \( I_0 = \sup_{\Omega} |I| \).

Corollary 1.4. Let \( \Omega \) be a domain in an \( n \)-dimensional complete minimal Riemannian submanifold \((M, \langle \cdot, \cdot \rangle)\) in \( \mathbb{R}^m \) and \( \lambda_i \) be the \( i \)th eigenvalue of biharmonic operator. Then for any positive integer \( k \), we have

\[
\lambda_{k+1} \leq \frac{1}{2k} \left( 2 + \frac{1}{n^2} \right) \sum_{i=1}^{k} \lambda_i
\]

\[
+ \left\{ \frac{1}{k^2} (1 + \frac{8}{n^2})^2 \left( \sum_{i=1}^{k} \lambda_i \right)^2 - \frac{1}{k} \left( 1 + \frac{16}{n^2} \right) \sum_{i=1}^{k} \lambda_i^2 \right\} \cdot \frac{1}{\delta},
\]

and

\[
\lambda_2 \leq \left( 1 + \frac{17}{2n^2} \right) \lambda_1.
\]
In this section, we describe the necessary tools about tensor $T$ and problem (1.2) which enable us to prove our results. Throughout the paper, for any vector fields $X, Y$, we denote $\langle T(X), Y \rangle$ with $T(X, Y)$. For any $u, v \in C^\infty(\Omega)$, straightforward computation implies that

$$L(uv) = vLu + uLv + 2T(\nabla u, \nabla v).$$

Let $d\mu$ be the volume form on the boundary induced by the outward normal vector field $\nu$ on $\partial \Omega$. The divergence theorem for operator $L$ as follows

$$\int_{\Omega} Lu \, dm = \int_{\partial \Omega} T(\nabla u, \nu) \, d\mu,$$

then the integration by parts yields

$$\int_{\Omega} vL u \, dm = - \int_{\Omega} T(\nabla u, \nabla v) \, dm + \int_{\partial \Omega} v T(\nabla u, \nu) \, d\mu.$$

Hence, the operators $L$ and $L^2$ are self-adjoint operator in the space of all function in $L^2(\Omega, dm)$ that vanish on $\partial \Omega$. Therefore the eigenvalues of problem (1.2) are real and discrete.

**Proposition 2.1.** Let $\Omega$ be a domain in an $n$-dimensional complete Riemannian manifold $(M, (\cdot, \cdot))$, $\lambda_i$ be the $i$th eigenvalue of (1.2) and $u_i$ be the corresponding orthonormal real-valued eigenfunction. Then for any $h \in C^4(\Omega) \cup C^3(\partial \Omega)$ and any positive integer $k$, we have

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) ||p_i||^2,$$

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 v_i \leq \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i + \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) ||T(\nabla h, \nabla u_i) + \frac{u_i L h}{2}||^2,$$

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 v_i \leq \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) ||p_i||^2 + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) ||T(\nabla h, \nabla u_i) + \frac{u_i L h}{2}||^2,$$

where $\delta$ is any positive constant,

$$w_i = \int_{\Omega} hu_i p_i \, dm,$$

$$p_i = \frac{L h L u_i + 2T(\nabla h, \nabla L u_i) + L(u_i L h) + 2LT(\nabla h, \nabla u_i)}{2},$$

$$v_i = \int_{\Omega} u_i^2 T(\nabla h, \nabla h) \, dm.$$

**Proof.** For each $i$, $1 \leq i \leq k$, consider the functions $\phi_i : \Omega \rightarrow \mathbb{R}$ given by

$$\phi_i = hu_i - \sum_{j=1}^{k} a_{ij} u_j$$

$$\phi_i = \int_{\Omega} h u_i - \sum_{j=1}^{k} a_{ij} u_j \, dm.$$
where \( a_{ij} = \int_{\Omega} hu_i u_j \, dm \). We have \( \phi_i|_{\partial\Omega} = \frac{\partial\phi_i}{\partial n}|_{\partial\Omega} = 0 \) and
\[
\int_{\Omega} \phi_i u_r \, dm = \int_{\Omega} hu_i u_r \, dm - \sum_{j=1}^{k} a_{ij} \int_{\Omega} u_r u_j \, dm = 0, \quad \forall i, r = 1, 2, ..., k.
\]

Then by the inequality of Rayleigh-Ritz, we get
\begin{equation}
\lambda_{k+1} \leq \frac{\int_{\Omega} \phi_i L^2 \phi_i \, dm}{\int_{\Omega} \phi_i^2 \, dm}, \quad \forall i = 1, 2, ..., k.
\end{equation}

Since
\[
L \phi_i = L(h u_i) - \sum_{j=1}^{k} a_{ij} L u_j = h L u_i + u_i L h + 2 \langle \nabla h, \nabla u_i \rangle - \sum_{j=1}^{k} a_{ij} L u_j,
\]
we obtain
\[
L^2 \phi_i = L h L u_i + \lambda_i h u_i + 2 \langle \nabla h, \nabla L u_i \rangle + L(u_i L h) + 2 L T(\nabla h, \nabla u_i) - \sum_{j=1}^{k} a_{ij} \lambda_j u_j,
\]

therefore we get
\begin{equation}
\int_{\Omega} \phi_i L^2 \phi_i \, dm = \lambda_i ||\phi_i||^2 + \int_{\Omega} hu_i p_i \, dm - \sum_{j=1}^{k} a_{ij} r_{ij},
\end{equation}
where \( r_{ij} = \int_{\Omega} p_i u_j \, dm \) and
\[
p_i = L h L u_i + 2 \langle \nabla h, \nabla L u_i \rangle + L(u_i L h) + 2 L T(\nabla h, \nabla u_i).
\]

Using integration by parts, we deduce that
\begin{equation}
\int_{\Omega} u_j L T(\nabla h, \nabla u_i) \, dm + \int_{\Omega} u_j T(\nabla h, \nabla L u_i) \, dm
= - \int_{\Omega} T(\nabla u_j, \nabla T(\nabla h, \nabla u_i)) \, dm - \int_{\Omega} \text{div}(u_j T \nabla h) L u_i \, dm
= \int_{\Omega} L u_j T(\nabla h, \nabla u_i) \, dm - \int_{\Omega} L u_i T(\nabla h, \nabla u_j) \, dm - \int_{\Omega} u_j L h L u_i \, dm.
\end{equation}

On the other hand
\begin{equation}
\int_{\Omega} L u_j T(\nabla h, \nabla u_i) \, dm - \int_{\Omega} L u_i T(\nabla h, \nabla u_j) \, dm
= - \int_{\Omega} h \text{div}(L u_j T \nabla u_i) + \int_{\Omega} h \text{div}(L u_i T \nabla u_j)
= - \int_{\Omega} \langle h \nabla L u_j, T \nabla u_i \rangle \, dm + \int_{\Omega} \langle h \nabla L u_i, T \nabla u_j \rangle \, dm
= \int_{\Omega} u_i \text{div}(h T \nabla u_j) \, dm - \int_{\Omega} u_j \text{div}(h T \nabla u_i) \, dm
= \int_{\Omega} (u_i h L^2 u_j - u_j h L^2 u_i) \, dm + \int_{\Omega} (\langle u_i T \nabla h, \nabla L u_j \rangle - \langle u_j T \nabla h, \nabla L u_i \rangle) \, dm
= (\lambda_j - \lambda_i) a_{ij} - \int_{\Omega} L u_j T(\nabla h, \nabla u_i) \, dm + \int_{\Omega} L u_i T(\nabla h, \nabla u_j) \, dm
- \int_{\Omega} u_i L u_j L h \, dm + \int_{\Omega} u_j L u_i L h \, dm,
\end{equation}
which implies that
\begin{equation}
2 \int_{\Omega} \mathcal{L} u_j (\nabla h, \nabla u_i) \, dm - 2 \int_{\Omega} \mathcal{L} u_i (\nabla h, \nabla u_j) \, dm \\
= (\lambda_j - \lambda_i) a_{ij} - \int_{\Omega} u_i \mathcal{L} u_j h \, dm + \int_{\Omega} u_j \mathcal{L} u_i h \, dm.
\end{equation}
Substituting (2.7) into (2.9), we have
\begin{equation}
2 \int_{\Omega} u_j \mathcal{L} (\nabla h, \nabla u_i) \, dm + 2 \int_{\Omega} u_i \mathcal{L} (\nabla h, \nabla u_j) \, dm \\
= (\lambda_j - \lambda_i) a_{ij} - \int_{\Omega} u_i \mathcal{L} u_j h \, dm - \int_{\Omega} u_j \mathcal{L} u_i h \, dm.
\end{equation}
Moreover
\begin{equation}
\int_{\Omega} u_j \mathcal{L} (u_i \mathcal{L} h) \, dm = \int_{\Omega} u_i \mathcal{L} u_j \mathcal{L} h \, dm.
\end{equation}
Combining (2.10), (2.11) and \(r_{ij} = \int_{\Omega} p_i u_j \, dm\), we can write
\begin{equation}
r_{ij} = (\lambda_j - \lambda_i) a_{ij}.
\end{equation}
It follows from (2.5), (2.6) and (2.7) that
\begin{equation}
(\lambda_{k+1} - \lambda_i) ||\phi_i||^2 \leq \int_{\Omega} \phi_i \mathcal{L}^2 \phi_i \, dm - \lambda_i ||\phi_i||^2 \\
\overset{\text{by} \int_{\Omega} \phi_i u_j \, dm = 0}\leq \int_{\Omega} \phi_i p_i \, dm = w_i + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2,
\end{equation}
where \(w_i = \int_{\Omega} h u_i p_i \, dm\). We use that \(\int_{\Omega} \phi_i u_j \, dm = 0\) again to get
\begin{equation}
(\lambda_{k+1} - \lambda_i) \left( \int_{\Omega} \phi_i p_i \, dm \right)^2 = (\lambda_{k+1} - \lambda_i) \left( \int_{\Omega} \phi_i (p_i - \sum_{j=1}^{k} r_{ij} u_j) \, dm \right)^2 \\
\overset{\text{multiplying by} \int_{\Omega} \phi_i u_j \, dm = 0 \text{ again}}\leq (\lambda_{k+1} - \lambda_i) ||\phi_i||^2 \left( ||p_i||^2 - \sum_{j=1}^{k} r_{ij}^2 \right) \\
\overset{\text{by} \int_{\Omega} \phi_i p_i \, dm \leq ||p_i||^2 - \sum_{j=1}^{k} r_{ij}^2, \text{ we obtain}}\leq \left( \int_{\Omega} \phi_i p_i \, dm \right) \left( ||p_i||^2 - \sum_{j=1}^{k} r_{ij}^2 \right),
\end{equation}
this implies that
\begin{equation}
(\lambda_{k+1} - \lambda_i) \left( \int_{\Omega} \phi_i p_i \, dm \right) \leq ||p_i||^2 - \sum_{j=1}^{k} r_{ij}^2.
\end{equation}
Multiplying (2.15) by \((\lambda_{k+1} - \lambda_i)\) and summing on \(i\) from 1 to \(k\), we obtain
\begin{equation}
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} \phi_i p_i \, dm \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)||p_i||^2 - \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)^2 a_{ij}^2.
\end{equation}
Multiplying (2.13) by \((\lambda_{k+1} - \lambda_i)^2\), summing on \(i\) from 1 to \(k\) and \(a_{ij} = a_{ji}\), we infer
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} \phi_i p_i \, dm = \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i + \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j) a_{ij}^2
\]
\[
= \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i - \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j)^2 a_{ij}^2,
\]
then
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)||p_i||^2,
\]
which shows that (2.14) is true. In order to prove (2.2), we set
\[
\delta_i \leq \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i - \lambda_j)^2 a_{ij}^2.
\]
Observe that
\[
b_{ij} = \int_{\Omega} u_j \left( T(\nabla h, \nabla u_i) + \frac{u_i}{2} \mathcal{L}h \right) \, dm.
\]
and
\[
-2 \int_{\Omega} \phi_i \left( T(\nabla h, \nabla u_i) + \frac{u_i}{2} \mathcal{L}h \right) \, dm = v_i + 2 \sum_{j=1}^{k} a_{ij} b_{ij},
\]
where
\[
v_i = \int_{\Omega} \left( -hu_i^2 \mathcal{L}h - 2hu_i T(\nabla h, \nabla u_i) \right) \, dm = \int_{\Omega} u_i^2 T(\nabla h, \nabla h) \, dm.
\]
Since for any positive constant \(\delta\) and for all \(x, y \in \mathbb{R}\) we have \(-2xy \leq \delta x^2 + \frac{y^2}{\delta}\), then multiplying (2.20) by \((\lambda_{k+1} - \lambda_i)^2\), for any positive constant \(\delta\), we get
\[
(\lambda_{k+1} - \lambda_i)^2 (v_i + 2 \sum_{j=1}^{k} a_{ij} b_{ij})
\]
\[
= (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} (-2\phi_i)(T(\nabla h, \nabla u_i) + \frac{u_i}{2} \mathcal{L}h - \sum_{j=1}^{k} b_{ij} u_j) \, dm
\]
\[
\leq \delta (\lambda_{k+1} - \lambda_i)^3 ||\phi_i||^2 + \frac{1}{\delta} (\lambda_{k+1} - \lambda_i) \int_{\Omega} (T(\nabla h, \nabla u_i) + \frac{u_i}{2} \mathcal{L}h - \sum_{j=1}^{k} b_{ij} u_j) \, dm
\]
\[
\leq \delta (\lambda_{k+1} - \lambda_i)^3 ||\phi_i||^2 + \frac{1}{\delta} (\lambda_{k+1} - \lambda_i)(||T(\nabla h, \nabla u_i) + \frac{u_i}{2} \mathcal{L}h||^2 - \sum_{j=1}^{k} b_{ij}^2),
\]
hence (2.13) implies that

\[(\lambda_{k+1} - \lambda_i)^2(v_i + 2 \sum_{j=1}^{k} a_{ij} b_{ij}) \leq \delta(\lambda_{k+1} - \lambda_i)^2(w_i + \sum_{j=1}^{k} (\lambda_i - \lambda_j)a_{ij}^2) + \frac{1}{\delta}(\lambda_{k+1} - \lambda_i)(||T(\nabla h, \nabla u_i) + \frac{u_i}{2}\mathcal{L}h||^2 - \sum_{j=1}^{k} b_{ij}^2).\]

Now, summing over \(i\) from 1 to \(k\), \(a_{ij} = a_{ij}\) and \(b_{ij} = -b_{ji}\) we conclude that

\[\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 v_i \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i + \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_i - \lambda_j)a_{ij}^2 + \frac{1}{\delta} \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)b_{ij}^2,\]

which gives

\[(2.21) \quad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 v_i \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)||T(\nabla h, \nabla u_i) + \frac{u_i}{2}\mathcal{L}h||^2 - \sum_{i,j=1}^{k} (\lambda_{k+1} - \lambda_i)b_{ij}^2,\]

Thus (2.2) is true. Substituting (2.18) into (2.21) complete the proof of the proposition. \(\square\)

Proof of Theorem 1.1 Let \(x_1, ..., x_n\) be the standard Euclidean coordinate of \(\mathbb{R}^m\), \(\nabla\) be the Canonical connection of \(\mathbb{R}^m\) and \(\{e_1, ..., e_m\}\) be a local orthonormal geodesic frame in \(p \in M\) adapted to \(M\), then

\[\nabla x_r = \sum_{i=1}^{n} e_i(x_r)e_i + \sum_{i=n+1}^{m} e_i(x_r)e_i, \quad e_r = \nabla x_r + (\nabla x_r)^\perp.\]

Therefore

\[\sum_{r=1}^{m} T(\nabla x_r, \nabla u_i) = \sum_{r=1}^{m} (\nabla x_r, T(\nabla u_i)) = \sum_{r=1}^{m} (e_r - (\nabla x_r)^\perp, T(\nabla u_i)) = \sum_{r=1}^{m} (e_r, T(\nabla u_i)),\]

and

\[(2.22) \quad \sum_{r=1}^{m} T(\nabla x_r, \nabla u_i)^2 = \sum_{r=1}^{m} (e_r, T(\nabla u_i)) = ||T(\nabla u_i)||^2.\]
Also, we have
\[ (2.23) \quad \sum_{r=1}^{m} T(\nabla x_r, \nabla x_r) = \sum_{r=1}^{m} \langle \varepsilon_r, T(\nabla x_r) \rangle = \sum_{r=1}^{m} (T(\varepsilon_r), \nabla x_r) = \sum_{r=1}^{m} (T(\varepsilon_r), \varepsilon_r) = tr(T). \]

For \( x = (x_1, ..., x_n) \), we compute
\[ \text{div}(T(\nabla x)) := \left( \text{div}(T(\nabla x_1)), ..., \text{div}(T(\nabla x_m)) \right) \]
\[ (2.24) \quad = \left( \sum_{i=1}^{n} e_i \langle T(\nabla x_1), e_i \rangle, ..., \sum_{i=1}^{n} e_i \langle T(\nabla x_m), e_i \rangle \right) \]
\[ = \sum_{i,j=1}^{n} \left( e_i e_j (x_1) \langle T(e_j), e_i \rangle, ..., e_i e_j (x_m) \langle T(e_j), e_i \rangle \right) \]
\[ + \sum_{i,j=1}^{n} \left( e_j (x_1) \langle \nabla e_i T(e_j), e_i \rangle, ..., e_j (x_m) \langle \nabla e_i T(e_j), e_i \rangle \right) \]
\[ = \sum_{i,j=1}^{n} (T(e_j), e_i) \nabla e_i e_j (x) + \sum_{i,j=1}^{n} \langle \nabla e_i T(e_j), e_i \rangle e_j (x) \]
\[ = \sum_{i,j=1}^{n} (T(e_j), e_i) \alpha(e_i, e_j) (x) + \sum_{i,j=1}^{n} \langle \nabla e_i T(e_j), e_j \rangle e_j (x) \]
\[ = \sum_{j=1}^{n} \alpha(T(e_j), e_j) (x) + \sum_{j=1}^{n} \nabla e_i T(e_i)(x) = tr(\alpha \circ T)(x) + tr(\nabla T)(x), \]
hence
\[ (2.25) \quad \sum_{r=1}^{m} (\text{div}(T(\nabla x_r)))^2 = ||\text{div}(T(\nabla x))||^2 = ||tr(\alpha \circ T)||^2 + ||tr(\nabla T)||^2, \]
and
\[ \sum_{r=1}^{m} \text{div}(T(\nabla x_r))T(\nabla x_r, \nabla u_i) = \sum_{r=1}^{m} \text{div}(T(\nabla x_r))T(\nabla u_i)(x_r) \]
\[ = \langle \text{div}(T(\nabla x)), T(\nabla u_i) \rangle = \langle tr(\nabla T), T(\nabla u_i) \rangle, \]
where \( \alpha \circ T = \alpha(T(\cdot, \cdot)) \in \mathcal{X}(M)^\perp \). By taking \( h = x_r \) in (2.23) we can write
\[ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 v_i \leq \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i \]
\[ + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)||T(\nabla x_r, \nabla u_i) + \frac{u_i \xi_{x_r}}{2}||^2. \]
where

\[ w_i = \int_{\Omega} x_r u_i p_i \, dm, \]
\[ p_i = \mathcal{L}x_r \mathcal{L}u_i + 2T(\nabla x_r, \nabla \mathcal{L}u_i) + \mathcal{L}(u_i \mathcal{L}h) + 2\mathcal{L}T(\nabla x_r, \nabla u_i), \]
\[ v_i = \int_{\Omega} u^2_i T(\nabla x_r, \nabla x_r) \, dm. \]

Summing over \( r \), we have

\[
\sum_{r=1}^{m} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 v_i \leq \delta \sum_{r=1}^{m} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 w_i
\]
\[
+ \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \sum_{r=1}^{m} \| T(\nabla x_r, \nabla u_i) + \frac{u_i \mathcal{L}x_r}{2} \|^2,
\]

and from (2.23) we get

(2.29) \[
\sum_{r=1}^{m} v_i = \sum_{r=1}^{m} \int_{\Omega} u^2_i T(\nabla x_r, \nabla x_r) \, dm = \int_{\Omega} u^2_i \text{tr}(T) \, dm.
\]

Also (2.26) and (2.24) imply that

(2.30) \[
\sum_{r=1}^{m} \int_{\Omega} x_r u_i \mathcal{L}x_r \mathcal{L}u_i \, dm = \int_{\Omega} u_i \mathcal{L}u_i \langle \text{div} T(\nabla x), x \rangle \, dm
\]
\[
= \int_{\Omega} u_i \mathcal{L}u_i \langle \text{tr}(\alpha \circ T) + \text{tr}(\nabla T), I \rangle \, dm,
\]

(2.31) \[
\sum_{r=1}^{m} \int_{\Omega} x_r u_i \mathcal{L}(u_i \mathcal{L}x_r) \, dm
\]
\[
= \sum_{r=1}^{m} \int_{\Omega} \mathcal{L}(x_r u_i) u_i \mathcal{L}x_r \, dm
\]
\[
= \sum_{r=1}^{m} \int_{\Omega} (x_r u_i \mathcal{L}u_i \mathcal{L}x_r + u^2_i \mathcal{L}x_r)^2 + 2T(\nabla u_i, \nabla x_r)u_i \mathcal{L}x_r) \, dm
\]
\[
= \int_{\Omega} u_i \mathcal{L}u_i \langle \text{tr}(\alpha \circ T) + \text{tr}(\nabla T), I \rangle \, dm + \int_{\Omega} u^2_i \langle \| \text{tr}(\alpha \circ T) \|^2 + \| \text{tr}(\nabla T) \|^2 \rangle \, dm
\]
\[
+ 2 \int_{\Omega} u_i \langle T(\nabla u_i), \text{tr}(\nabla T) \rangle \, dm,
\]
and

\[(2.32) \sum_{r=1}^{m} \int_{\Omega} 2x_r u_i \mathcal{L}(\nabla x_r, \nabla u_i) \, dm = 2 \sum_{r=1}^{m} \int_{\Omega} \mathcal{L}(x_r u_i) T(\nabla x_r, \nabla u_i) \, dm = 2 \sum_{r=1}^{m} \int_{\Omega} x_r \mathcal{L}_r T(\nabla x_r, \nabla u_i) \, dm + 2 \sum_{r=1}^{m} \int_{\Omega} x_r \mathcal{L}_r T(\nabla x_r, \nabla u_i) \, dm + 2 \sum_{r=1}^{m} \int_{\Omega} T(\nabla x_r, \nabla u_i)^2 \, dm = 2 \int_{\Omega} u_i \langle T(\nabla u_i), tr(\nabla T) \rangle \, dm + 4 \int_{\Omega} |T(\nabla u_i)|^2 \, dm + 2 \int_{\Omega} \mathcal{L} u_i \langle T(\nabla u_i), I \rangle \, dm \]

\[(2.33) \sum_{r=1}^{m} \int_{\Omega} 2x_r u_i T(\nabla x_r, \nabla \mathcal{L} u_i) \, dm = 2 \int_{\Omega} u_i \langle T(\nabla \mathcal{L} u_i), I \rangle \, dm.\]

Thus

\[(2.34) \sum_{r=1}^{m} u_i = 2 \int_{\Omega} u_i \mathcal{L} u_i \langle tr(\alpha \circ T) + tr(\nabla T), I \rangle \, dm + 2 \int_{\Omega} u_i \langle T(\nabla \mathcal{L} u_i), I \rangle \, dm + \int_{\Omega} u_i^2 (|tr(\alpha \circ T)|^2 + |tr(\nabla T)|^2) \, dm + 4 \int_{\Omega} u_i \langle T(\nabla u_i), tr(\nabla T) \rangle \, dm + 4 \int_{\Omega} |T(\nabla u_i)|^2 \, dm + 2 \int_{\Omega} \mathcal{L} u_i \langle T(\nabla u_i), I \rangle \, dm \]

and

\[(2.35) \sum_{r=1}^{m} |T(\nabla x_r, \nabla u_i) + \frac{u_i \mathcal{L} x_r}{2}|^2 \]

\[= \sum_{r=1}^{m} \int_{\Omega} \left( |T(\nabla x_r, \nabla u_i)|^2 + u_i \mathcal{L} x_r T(\nabla x_r, \nabla u_i) + \frac{1}{4} u_i^2 (\mathcal{L} x_r)^2 \right) \, dm \]

\[= \int_{\Omega} \left( |T(\nabla u_i)|^2 + u_i \langle T(\nabla u_i), tr(\nabla T) \rangle + \frac{1}{4} u_i^2 (|tr(\alpha \circ T)|^2 + |tr(\nabla T)|^2) \right) \, dm.\]

Substituting \((2.29), (2.34)\) and \((2.35)\) into \((2.27)\) we complete the proof of the theorem. \(\square\)
Proof of Theorem 1.3] Let $S_{e_i}$ be the Weingarten operator of the immersion with respect to $e_i$. Then
\[
||tr(\alpha \circ T)||^2 = || \sum_{i=1}^{n} \alpha(Te_i, e_i) ||^2 = || \sum_{i=1}^{n} \sum_{k=n+1}^{m} \langle \alpha(Te_i, e_i), e_k \rangle e_k ||^2 \\
= || \sum_{k=n+1}^{m} (\sum_{i=1}^{n} \langle S_{e_k} e_i, Te_i \rangle) e_k ||^2 = || \sum_{k=n+1}^{m} \langle S_{e_k}, T \rangle e_k ||^2 \\
\leq \sum_{k=n+1}^{m} ||(S_{e_k}, T)||^2 \sum_{k=n+1}^{m} ||e_k||^2 \leq \sum_{k=n+1}^{m} ||S_{e_k}||^2 ||T||^2 \sum_{k=n+1}^{m} ||e_k||^2 \\
\leq (m-n)S_0^2 T_s,
\]
where $S_0 = \max\{\sup_{\Omega} |S_{e_k}| : k = n+1, ..., m\}$ and $T_s = \sup_{\Omega} |T|$. If $T_0 = \sup_{\Omega} |tr(\nabla T)|$ and $I_0 = \sup_{\Omega} |I|$ then
\[
\int_{\Omega} u_i(T(\nabla u_i), tr(\nabla T)) \, dm \leq \left( \int_{\Omega} u_i^2 \, dm \right)^\frac{1}{2} \left( \int_{\Omega} |T(\nabla u_i)|^2 |tr(\nabla T)|^2 \, dm \right)^\frac{1}{2} \\
\leq T_0 ||T(\nabla u_i)||_{L^2(\Omega)}, \tag{2.36}
\]
and
\[
\int_{\Omega} u_i L u_i(tr(\alpha \circ T)+tr(\nabla T), I) \, dm \\
= \int_{\Omega} u_i L u_i(tr(\alpha \circ T), I) \, dm + \int_{\Omega} u_i L u_i(tr(\nabla T), I) \, dm \\
\leq \left( \int_{\Omega} u_i^2 \, dm \right)^\frac{1}{2} \left( \int_{\Omega} (L u_i)^2 ||tr(\alpha \circ T)||^2 |I|^2 \, dm \right)^\frac{1}{2} \\
+ \left( \int_{\Omega} u_i^2 \, dm \right)^\frac{1}{2} \left( \int_{\Omega} (L u_i)^2 |tr(\nabla T)|^2 |I|^2 \, dm \right)^\frac{1}{2} \\
\leq (\sqrt{n-2}S_0T_s + T_0) I_0 \lambda_i^{\frac{3}{2}} \tag{2.37}
\]
and
\[
\int_{\Omega} u_i(T(\nabla L u_i), I) \, dm \leq \left( \int_{\Omega} u_i^2 \, dm \right)^\frac{1}{2} \left( \int_{\Omega} |T(\nabla L u_i)|^2 |I|^2 \, dm \right)^\frac{1}{2} \\
\leq I_0 ||T(\nabla L u_i)||_{L^2(\Omega)}. \tag{2.38}
\]
Also, we have
\[
\int_{\Omega} u_i^2(||tr(\alpha \circ T)||^2 + |tr(\nabla T)|^2) \, dm \leq (m-n)S_0^2 T_s^2 + T_0^2
\]
and
\[
\int_{\Omega} L u_i(T(\nabla u_i), I) \, dm \leq \left( \int_{\Omega} (L u_i)^2 \, dm \right)^\frac{1}{2} \left( \int_{\Omega} |T(\nabla u_i)|^2 |I|^2 \, dm \right)^\frac{1}{2} \\
\leq \lambda_i I_0 ||T(\nabla u_i)||_{L^2(\Omega)}. \tag{2.39}
\]
By setting
\[
C_i = 2(\sqrt{n-2}S_0T_s + T_0) I_0 \lambda_i^{\frac{3}{2}} + I_0 ||T(\nabla L u_i)||_{L^2(\Omega)} + (m-n)S_0^2 T_s^2 + T_0^2 \\
+ 4T_0 ||T(\nabla u_i)||_{L^2(\Omega)} + 4 ||T(\nabla u_i)||_{L^2(\Omega)} + 2\lambda_i I_0 ||T(\nabla u_i)||_{L^2(\Omega)},
\]
and

\[ D_i = \|T(\nabla u_i)\|_{L^2(\Omega)} + T_0 \|T(\nabla u_i)\|_{L^2(\Omega)} + \frac{1}{4}(m-n)S_0^2 T^2 + T_0^2 \]

we get \( A_i \leq C_i \) and \( B_i \leq D_i \). Substituting these inequalities into Theorem 1.1 we complete the proof of the Theorem.

\[ \Box \]

**Proof of Theorem 1.3.** Taking \( T \) equal to identity in Theorem 1, we obtain

\[ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 E_i + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) F_i, \]

where

\[ E_i = 2 \int_{\Omega} u_i \Delta u_i (nH, I) \, dm + 2 \int_{\Omega} u_i (\nabla \Delta u_i, I) \, dm \]

\[ + \int_{\Omega} u_i^2 n^2 \|H\|^2 \, dm + 4 \int_{\Omega} |\nabla u_i|^2 \, dm + 2 \int_{\Omega} \Delta u_i (\nabla u_i, I) \, dm, \]

and

\[ F_i = \int_{\Omega} \left\{ |\nabla u_i|^2 + \frac{u_i^2}{4} n^2 \|H\|^2 \right\} \, dm. \]

On the other hand, we have

\[ \int_{\Omega} u_i (\nabla \Delta u_i, I) \, dm = - \int_{\Omega} \Delta u_i (\nabla u_i, I) \, dm \]

and

\[ \int_{\Omega} |\nabla u_i|^2 \, dm = \int_{\Omega} (\nabla u_i, \nabla u_i) \, dm = - \int_{\Omega} u_i \Delta u_i \, dm \]

\[ \leq \int_{\Omega} |u_i| |\Delta u_i| \, dm \leq \left( \int_{\Omega} u_i^2 \, dm \right)^{\frac{1}{2}} \left( \int_{\Omega} (\Delta u_i)^2 \, dm \right)^{\frac{1}{2}} \leq \lambda_i^\frac{1}{2}. \]

Hence

\[ E_i \leq 2nH_0 I_0 \lambda_i^\frac{1}{2} + n^2 H_0^2 + 4\lambda_i^\frac{1}{2} \]

and

\[ F_i \leq \lambda_i^\frac{1}{4} + \frac{1}{4} n^2 H_0^2 \]

where \( H_0 = \sup_{\Omega} |H| \) and \( I_0 = \sup_{\Omega} |I| \). Substituting these inequality into (2.40) we complete the proof of the theorem.

\[ \Box \]

**Proof of Corollary 1.4.** For a minimal hypersurface we have \( H = 0 \), therefore Theorem 3 results that

\[ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4\delta \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \lambda_i^\frac{1}{4} + \frac{1}{\delta} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^\frac{1}{4}. \]

Taking

\[ \delta = \left\{ \frac{\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^\frac{1}{4}}{4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \lambda_i} \right\}^\frac{1}{4}, \]

we get

\[ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \left\{ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \lambda_i^\frac{1}{4} + \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^\frac{1}{4} \right\}^\frac{1}{4}. \]
On the other hand
\[ \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \lambda_i^2 \right) \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i^2 \right) \leq \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \right) \left( \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i \right). \]

It and (2.42) imply that
\[ (2.43) \quad \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{16}{n^2} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i \]
solving this quadratic polynomial of \( \lambda_{k+1} \), we obtain (1.7) and (1.8). \( \square \)

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