The Perfect State Transfer Graph Limbo

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Perfect state transfer between qubits on a network, described by a uniformly coupled graph, has advantages over an engineered chain, such as much faster transfer times (that do not scale with the distance between the input and output vertices). The trade-off seems to be the need for a large number of vertices in the graph. Existing results are exponential in the transfer distance. Hence the question, “How low can you go?”. In this paper, we present reductions in the vertex count required, although the overall scaling with transfer distance remains exponential.

The task of perfect state transfer [1–3], wherein an unknown quantum state $|\psi\rangle$ is transported between two distant locations within a quantum computer by a fixed, time invariant Hamiltonian evolution, was originally intended as a plausible alternative for achieving that task without the vast control overheads of other implementations, such as those derived from the quantum circuit model. A huge range of different assumptions and paradigms have been tried. Two of the major classes of solutions are engineered chains [2, 3] and uniform networks [4–6]. Engineered chains achieve arbitrary transfer distances with a linearly scaling number of qubits, and a linearly scaling transfer time, with the challenge of having to fix the coupling strength between each neighbouring spin to a particular value. There are also some beautiful symmetries (based on the Jordan-Wigner transformation [7, 8]) that facilitate a whole host of interesting results, such as [3, 9, 10]. Uniform networks, meanwhile, still achieve an arbitrary transfer distance, with the advantage of a fixed transfer time but at the cost of requiring more spins. Existing solutions require exponentially many qubits. Can these uniformly coupled networks genuinely be considered as offering a plausible option for state transfer? In this paper, we suggest that the answer is negative for arbitrary transfer distances. By imposing that the coupling network of spins is embeddable in a two- or three-dimensional geometry, the degree of the connectivity is limited. We improve consequential limits on the transfer distance [6]. We then consider the number of qubits required to achieve transfer over a given distance. While we substantially reduce the exponent compared to existing families of transfer graphs, we are unable to address the question of whether a sub-exponential scaling is possible.

I. INTRODUCTION

Consider a graph $G$ of $N$ vertices, $V$, and edges $E$. Qubits placed at each vertex interact by a Hamiltonian that is specified by the edges of the graph:

$$H = \frac{1}{2} \sum_{(n,m) \in E} (X_n X_m + Y_n Y_m),$$

where $X_n$ is the Pauli-$X$ matrix applied to the qubit on vertex $n$. This Hamiltonian has several symmetries. Of primary importance is the commutation relation

$$[H, \sum_{n \in V} Z_n] = 0,$$

demonstrating that the Hamiltonian decomposes into subspaces based on the eigenvalues of the total spin operator $\sum_n Z_n$. In other words, the basis elements $|x\rangle$ for $x \in \{0,1\}^N$ are partitioned by the Hamming weight of $x$. In particular, the one-excitation subspace of $H$ is just represented by the adjacency matrix $A$ of the graph. Other Hamiltonian models have been considered (such as the Heisenberg model), which make connections to other graph descriptions such as the Laplacian or signless Laplacian. We are not considering those here.

We are interested in the task of perfect state transfer between a pair of qubits $a$ and $b$: an unknown state $|\psi\rangle$ starts on qubit $a$, with all other qubits in the state $|0\rangle$. After evolution by some fixed time $t_0$ under $H$, the state $|\psi\rangle$ is required to arrive on qubit $b$ (up to some known relative phase gate that can be corrected for later). Since $|0\rangle \otimes N$ is an eigenstate of $H$, it suffices to show that

$$| \langle a | e^{-iA t_0} | b \rangle | = 1.$$  

In this context, perfect transfer is achieved not through the engineering of coupling strengths, as is usually the case for one-dimensional chains [11–13], but by judicious choice of the graph itself. In many ways, this is far more challenging because there are only discrete choices rather than a continuum of options.

Our aim is to understand when, or even if, perfect transfer on uniform graphs is physically interesting, i.e. whether there are realistic experimental prospects. Our main criterion for this is to transferring a state over as large a distance as possible, as simply as possible. For example, given that the path $P_2$ can give a perfect transfer over a distance 1, we see any other graph giving transfer distance 1 (such as integral circulant graphs [12, 13] as
more complicated; they necessarily involve more vertices and a higher degree on at least some vertices, neither of which is desirable. It is not as if such complication can be mitigated by other features such as routing – after twice the perfect transfer time, there is always a perfect revival on the input site, so the state cannot be passed around multiple different sites, giving different users access to the transferring state.

In this paper, we provide two results. The first is a tightening of a known relationship between the maximum degree of the graph and its transfer distance. This suggests that any realistic graph implementation (determined by local couplings in a two- or three-dimensional structure) will only be able to transfer over very short distances. The second considers the scaling of the number of vertices with the transfer distance. We are able to significantly reduce the best-known exponent on the scaling of the number of vertices, but that relation nevertheless remains an exponential trade-off between transfer distance and vertex number.

II. THE DEGREE-DISTANCE BOUND

We shall denote the maximum transfer distance of a graph by \( D \), its maximum degree by \( d \), and its vertex count by \( N \). Our aim is, for a particular \( D \), to minimise the parameters \( d \) and \( N \), and see how they might be traded off between each other. The conditions for perfect state transfer are well characterised \[5, 14\]

**Lemma 1.** Let \( \Phi_a \) be the set of eigenvalues of \( A \) for which the eigenvector has support on \( a \). Vertex \( a \) exhibits perfect revivals iff its eigenvalues \( \lambda_n \in \Phi_a \) take the form

\[
\lambda_n = \frac{1}{2} \left( \alpha + \beta_n \sqrt{\Delta} \right),
\]

(1)

where \( \alpha \), \( \{\beta_n\} \) and \( \Delta \) are integers. The values \( \left\{\frac{\beta_n - \beta_m}{2}\right\} \) are coprime integers \( \gcd \left( \frac{\beta_n - \beta_m}{2} \right) = 1 \).

The most complete proof is in Theorem 6.1 of \[13\], although, there, \( \Delta \) is taken to be square-free. We have instead incorporated additional factors inside \( \Delta \) in order to fix the GCD condition \[13\].

**Lemma 2.** In order to achieve perfect state transfer between qubits \( a \) and \( b \), the graph must satisfy

\[
\langle \lambda_n | a \rangle = \pm \langle \lambda_n | b \rangle
\]

for all \( \lambda_n \in \Phi_a \), and the parity of the integer \( \frac{\beta_n - \beta_m}{2} \) must match the sign \( \frac{\langle \lambda_m | a \rangle \langle \lambda_n | b \rangle - \langle \lambda_m | b \rangle \langle \lambda_n | a \rangle}{\langle \lambda_m | b \rangle \langle \lambda_n | b \rangle} \). Perfect transfer occurs at all times that are odd multiples of \( \pi/\sqrt{\Delta} \), with perfect revivals on the input at all even multiples.

The maximum degree of a graph is a particularly limiting parameter for physical realisation. If we imagine a physical realisation in which only local spins can interact directly, then embedding in a two or three-dimensional geometry places severe constraints on the number of vertices that can interact; on the order of 4 in two-dimensions or 6 in three-dimensions. This in turn constrains the distance of transfer:

**Lemma 3.** For a given \( \Delta \), \( D \leq 2d/\sqrt{\Delta} \).

*Proof.* This originates from \[3\]. To summarise in our slightly different notation, apply Gershgorin’s circle theorem to the adjacency matrix \( A \): each diagonal element is 0, and the off-diagonals total to no more than \( d \), so all eigenvalues are bounded between \( \pm d \). Since every distinct eigenvalue must be separated by at least \( \sqrt{\Delta} \), there are no more than \( 2d/\sqrt{\Delta} + 1 \) distinct eigenvalues, and \( |\Phi_a| \leq 2d/\sqrt{\Delta} + 1 \). \[14\] makes the argument that if the eccentricity of \( a \), the maximum distance of any vertex from \( a \), is \( \varepsilon_a \), then \{\( A^k | a \rangle \}_{k=0}^{\varepsilon_a} \) must form a basis, which can be no larger than \( |\Phi_a| \). Thus, \( D \leq \varepsilon_a \leq |\Phi_a| - 1 \leq 2d/\sqrt{\Delta} \). \( \square \)

The above proof applies to all graphs with perfect revivals, and there are no known perfect transfer graphs for which \( D > d \), while there are examples of graphs with perfect revivals that exceed that bound. Some examples are depicted in Fig. 1. Our first task is to tighten this bound by specifically imposing the properties of perfect transfer graphs instead of just perfect revival graphs. However, we shall do this in a slightly restricted setting, wherein we assume that the vertex \( a \) that we transfer from is spectrally extremal, i.e. the number of eigenvectors with non-trivial support on \( a \) is \( |\Phi_a| = \varepsilon_a + 1 = D + 1 \). One would certainly anticipate that this would be the scenario to give maximum transfer distance (but it is an assumption nevertheless). To make progress in excluding certain values of \( \Delta \), we need an extra result first:

**Lemma 4.** For an ordered set of integers, \( \Lambda \), of size \( |\Lambda| = N > 3 \) such that the \( n \)th value has parity \((-1)^{n+1}\), \( R \) is rational, and the denominator of \( R \) (in its simplest form)
possible form) contains a factor of 2, where
\[ R = \frac{1}{\sum_{\lambda \in \Lambda} \prod_{\mu \in \Lambda \setminus \lambda} (\lambda - \mu)}. \]

Proof. As it is only the eigenvalue differences that are relevant, we take the smallest value in \( \Lambda \) to be 0, and the largest to be \( \lambda_{\text{max}} \). Let \( \Lambda \) contain all the integers between 0 and \( \lambda_{\text{max}} \) that are not contained within \( \Lambda \). It is vital to note that these occur in consecutive pairs to maintain the parity condition. We have that
\[ R = \frac{1}{\sum_{n=0}^{\lambda_{\text{max}}} \prod_{\mu \in \Lambda} (\mu - n)}. \]

If it were the case that the members of \( \Lambda \) did not occur in consecutive pairs, there would be an additional sign on each term in the sum given by \((-1)^{|\{\mu \in \Lambda : \mu < \lambda\}|}\), but we avoid this complication which would invalidate the rest of the proof. Next, note that we can extend the sum from \( \Lambda \) to the entire set of integers 0 to \( \lambda_{\text{max}} \) since the additional terms from \( \Lambda \) are all multiplied by zero. Thus,
\[ R = \frac{1}{\sum_{n=0}^{\lambda_{\text{max}}} \prod_{\mu \in \Lambda} (\mu - n)}. \]

We rewrite this as
\[ R = \frac{\lambda_{\text{max}}!}{\sum_{n=0}^{\lambda_{\text{max}}} \prod_{\mu \in \Lambda} (\mu - n)}. \]

Next, we express \( \prod_{\mu \in \Lambda} (\mu - n) \) in the form
\[ \sum_{q=0}^{[\lambda]} f(q)(\Lambda) \frac{n!}{(n-q)!}. \]

The precise form of \( f(q)(\Lambda) \) is irrelevant, but it follows by induction (adding pairs of integers to \( \Lambda \) of \( \mu \) and \( \mu + 1 \)) that these are integers. This lets us write
\[ R = \frac{\lambda_{\text{max}}!}{\sum_{q=0}^{[\lambda]} f(q)(\Lambda) \frac{\lambda_{\text{max}}-q}{n!} \sum_{n=0}^{\lambda_{\text{max}}-q} \prod_{\mu \in \Lambda} (\mu - n)}. \]

This reduces to
\[ R = \frac{1}{\sum_{q=0}^{[\lambda]} f(q)(\Lambda) \frac{\lambda_{\text{max}}-q}{n!} \prod_{\mu \in \Lambda} (\mu - n)}. \]

Consider each term \( 2^x/x! \). How many factors of 2 does \( x! \) contain? There are \( \lfloor x/2 \rfloor \) even numbers, \( \lfloor x/4 \rfloor \) that are divisible by 4 (each contributing an extra factor of 2), \( \lfloor x/8 \rfloor \) that are divisible by 8, and so on. Hence, the number of factors of 2 is no more than
\[ \frac{x}{2} + \frac{x}{4} + \ldots + 1 = \frac{x}{2} \sum_{n=0}^{\infty} 2^{-n} - \sum_{n=1}^{\infty} 2^{-n} = x - 1, \]
so there are no more than \( x - 1 \) factors of 2, and not all of the terms in \( 2^x/x! \) cancel. Hence, every term in the sum over \( q \) in Equation 2 has an even numerator. Thus, \( R \) has a denominator containing a factor of 2.

Theorem 1. For spectrally extremal perfect state transfer, \( \Delta \) must be even, and if \( \Delta \mod 4 = 2 \), the transfer distance \( D \) must be even.

Proof. Due to Theorem 1 in [15], the eigenvector elements satisfy
\[ |\langle a|\lambda \rangle|^2 = \frac{k}{\prod_{\mu \in \Phi_a}(\lambda - \mu)}, \]
for all \( \lambda \in \Phi_a \), where \( k \) is the number of paths of length \( \varepsilon_a \) from \( a \) to \( b \) (deriving from the factor that \( |\langle a|H^n|b \rangle| = 0 \) if \( n < \varepsilon_a \) and \( |\langle a|H^{\varepsilon_a}|b \rangle| = k \), and imposing strong cospectrality: \( |\langle \lambda|a \rangle| = \pm |\langle \lambda|b \rangle| \)). Moreover, the ordered values \( \lambda - \mu \) must be alternately odd and even multiples of \( \sqrt{\Delta} \). If we impose that \( \sum_{\lambda \in \Phi_a} |\langle a|\lambda \rangle|^2 = 1 \), and take the standard form of the eigenvalues, Eq. (1), then
\[ k = \frac{\Delta^{\frac{|\Phi_a|}{2} - 1}}{\left| \sum_{n} \prod_{\mu \notin \Phi_a} \prod_{\mu \in \Phi_a} (\frac{\lambda_{\text{max}} - q}{n!}) \right|}, \]
which must be an integer. We know by Lemma 4 that the right-hand side is of the form \( \Delta^{\frac{|\Phi_a|}{2} - 1} - R \) where \( R \) has a factor of 2 in the denominator (recall, it is the values \( \lambda_{\text{max}} - q \) that must be alternately odd and even integers). In order to cancel that factor of 2, it must be created by the \( \Delta \) term. Either \( \Delta \) is divisible by four, or it’s divisible by two and the transfer distance is even.

By eliminating the \( \Delta = 1 \) case, we have tightened the degree-distance relation provided in Lemma 3 putting a physically realistic option further from reach. The family of perfect state transfer graphs created from the hypercube of \( P_2 \) (of dimension \( D \)) saturate this bound: \( \Delta = 4 \) and \( D = d \). This family requires a number of vertices \( N = 2^D \). Better performance can be found for the family of graphs created from the hypercube of \( P_3 \), but only for even distances since \( \Delta = 2 \); \( D = d \) and \( N = 3^{D/2} \). We do not know if it is possible to saturate the \( D \leq \sqrt{2d} \) bound when \( \Delta = 2 \), or whether the true bound is indeed \( D \leq d \) (we suspect the latter).

Note that this study puts particular emphasis on the ‘standard’ perfect state transfer engineered chain solutions [2]: for extremal transfer where \( |\Phi_a| = D + 1 \) and \( \Delta = 4 \), all systems have a graph quotient corresponding to an engineered perfect state transfer chain, and this is the only chain that can saturate the \( D = d \) limit.

III. VERTEX NUMBER

We now turn our attention to minimising the total number of vertices in a graph of fixed transfer distance
An important result here, originally due to [4], is that if two graphs $G_1$ and $G_2$ achieve perfect transfer in the same time (i.e. have the same value of $\Delta$) over distances $D_1$ and $D_2$, then their Cartesian product $G_1 \square G_2$ achieves perfect transfer in that time, over a distance $D_1 + D_2$. Hence, any graph $G$ that achieves perfect transfer over distance $D$ determines a family $G^{\square k}$ with transfer distance $kD$ (and a number of vertices $N^k$). Our measure of success must, therefore, be the exponent. We refer to this as the efficiency, $\eta$:

$$\eta = \frac{1}{D} \log_2(N).$$

As a point of reference, the $P_2$ hypercubes all have $\eta = 1$, and the $P_3$ hypercubes have $\eta = \frac{1}{2} \log_2(3) \approx 0.792$. Smaller is better. Prior to this work, this was the best known efficiency. Our main tool in improving this efficiency is the concept of the partitioned graph and how, for a fixed partitioning, there is a manipulation rule that can trade number of vertices for number of edges without changing the ability for state transfer.

A. The Partitioned Graph

**Definition 1.** An equitable distance partitioning of a graph $G$ comprises distinct sets of vertices $\{V_i\}$ (called nodes) such that

- **Node $V_0$ comprises a single vertex; the input vertex.**
- **All vertices in node $V_i$ are equidistant from $V_0$.**
- **For any pair of nodes, $i$, $j$, every vertex in $V_i$ connects to the same number of vertices in $V_j$.**
- **No edges join vertices in the same node.**

The 'Partitioned Graph' is an equitable distance partition of the graph. In this, we partition vertices into equivalent sets, which we call a node. The following diagrammatic representation specifies a graph in which one set of $N_1$ vertices, each with degree $d_1$, connects to vertices in the other set, of which there are $N_2$, each with degree $d_2$:

A required consistency condition is that $N_1d_1 = N_2d_2$. There is also a restriction that $d_1 \leq N_2$ and $d_2 \leq N_1$.

Partitioned graphs have a natural graph quotient, produced by working in the subspace comprising the uniform superposition across all vertices within individual node. Each node represents a vertex in the quotient graph, and each edge has an effective coupling strength $\sqrt{d_1d_2}$. For perfect state transfer, we identify the input and output vertices as separate nodes (with occupancy 1). Perfect transfer between these two vertices in the graph quotient implies perfect transfer in the original graph.

B. A Node-Preserving Manipulation Rule for Partitioned Graphs

There is a particularly simple manipulation rule for partitioned graphs, which preserves the partitioning. For any node with an occupancy $N_1$ that is divisible by 4, such that every neighbouring node has a corresponding degree that is even, we can perform the replacement

![Diagram of node-preserving manipulation rule for partitioned graphs](image)

The consistency condition is still fulfilled, and the effective coupling in the quotient graph is unchanged. As such, the ability of the graph to perform state transfer between the extremal vertices is unchanged.

There are several variants of this manipulation rule. For example, one could reduce $N_1$ by any square factor $n^2$ long as the degrees on the adjoining nodes are divisible by $n$. However, in practice we have found the $n = 2$ is by far the most prevalent. Equally, while we usually think about the manipulation rule for the purposes of decreasing the vertex count, there are instances where increasing it is beneficial. For example,

![Diagram of node-preserving manipulation rule for partitioned graphs](image)

We could not, initially, apply the rule on the second node because the degree on the first node was too high, but by first applying the manipulation rule in reverse on the first node, this problem was circumvented, and there was a net reduction in the vertex count from 17 to 8.

Lastly, this manipulation rule can be applied to subgraphs of nodes – the occupancy of every node in the subgraph is reduced by the factor of 4, while all internal edges retain the same degrees, and edges joining the subgraph to the rest of the graph have degrees that change in the same way as for the basic rule. This allows us to significantly reduce the conditions under which the rule can be applied – if the occupation of a node is divisible by 4, then since $N_1d_1$ is divisible by 4, each of the neighbouring nodes has an occupation that is divisible by 4 (in which case, we can expand the subgraph to include that node) or the degree of the connected node is divisible by 2, and hence the basic version of the rule is applicable.

For example, consider the $D$-dimensional hypercube of $P_2$. Every vertex at a given distance from the input vertex is equivalent, and can thus be partitioned into a single node (the distance partition). One thus has a chain-like structure of $D + 1$ nodes, where the node occupancies are $\binom{D}{n}$ for $n = 0, \ldots, D$. The quotient graph is exactly the standard perfect state transfer chain with engineered couplings $\sqrt{(n+1)(D-n)}$. This is exactly the construction originally performed in [4]. For $D = 3$, this is depicted as:

![Diagram of node-preserving manipulation rule for partitioned graphs](image)
While this example does not demonstrate any gain under the manipulation rules, $D = 6$ does:

The most impressive gains that we found for this construction are for $D = 16$, where we reduce from $2^{16}$ vertices to just $8874$ ($\eta = 0.820$). Sec. [II] will demonstrate further gains by using different partitions.

C. The $\Delta$-doubling Lift

In order to reduce the number of vertices as much as possible, it seems sensible to start from the families with smallest total vertex number: the $P_3$ hypercubes. However, since these have $\Delta = 2$, they can only transfer over even distances. If we want odd distances, then we need to move to $\Delta = 4$. Thankfully, there is a connection. We can move from any graph with a bipartite node structure and a given value of $\Delta$ to a graph with a value of $2\Delta$, using the same node structure.

If the partitioned structure is bipartite, then to every node we can assign a label 0 or 1 based on the parity of distance from the input vertex. The input and output vertices can be labelled 0 without loss of generality. Our $\Delta$-doubling lift simply changes the node occupancy of a node with label 1 by doubling it, while degrees on the nodes with label 0 are doubled. This maintains the consistency condition while the effect in the graph quotient is to change every edge by a factor of $\sqrt{2}$. Hence all the eigenvalues are multiplied by $\sqrt{2}$, and $\Delta$ is thus doubled. This is very similar to a trick used in [10].

It must also be emphasised that the $\Delta$-doubling lift is highly compatible with the manipulation rule of Sec. [IIIB] for further reductions. For nodes with the 1 label, the neighbouring degrees are even due to the construction, and the node occupancy is already even, so it only needs an extra factor of 2 to be applicable (modulo some additional care with the conditions on the degree).

D. Node Splitting

While we have specified one manipulation rule that preserves the node structure, other manipulation rules can be formulated which, for example, show how a single node can be broken into several smaller ones, under limited conditions. (Technically, the method we are about to present contains the previous manipulation rule.)

Consider a node with occupation $N$, and $p$ neighbouring nodes with occupations $M_j$ for $j = 1, \ldots, p$, and coupled (in the graph quotient) by coupling strength $J_j$. We split the single node into a set of $k$ nodes (the value of $k$ to be determined), each with occupancy $N_i$. The degrees of each node when connecting to the $p$ neighbours are $\{d_j\}$. Our consistency condition is as before: $N_i d_j$ should be divisible by $M_j$, and the effective coupling strength of the graph quotient is correctly maintained if

$$M_j J_j^2 = \sum_i N_i d_j^2$$

We also need to ensure that a uniform superposition over all of the vertices in any of the neighbouring nodes hops onto the same state across the new set of nodes. This is achieved by ensuring that

$$d_j = \alpha_j d_1$$

for all $i, j$. We can determine the values of the $\alpha_j$,

$$\alpha_j = \sqrt{\frac{M_k J_j^2}{M_1 J_1^2}}.$$ 

This number is rational, and we refer to it in its lowest possible form as $\alpha_j = \frac{n_j}{m_j}$. For this to work, $d_1$ must be divisible by all the $d_j$. Also, to achieve the consistency condition that $N_i d_j$ should be divisible by $M_j$, we need $N_i d_1$ to be divisible by $M_j d_j / \text{GCD}(M_j, n_j)$ for all $j = 2, 3, \ldots, p$ (as well as $M_1$). Subject to these constraints, our aim is to minimise $\sum_i N_i$.

The simplest example of this in practice is the $D = 4$ transfer case, starting from the $P_2$ hypercube construction. We have the partitioned graph:

We aim to replace the central node with a set of $k$ nodes. By symmetry, we can take $p = 1$. Thus, we only have to find $\min \sum_i N_i$ subject to $\sum_i N_i d^2 = 24$ and $N_i d^2$ divisible by 4. Of course, the original values still work $(k = 1, N_1 = 6, d^1 = 2)$, but there is another solution: $k = 2, N_1 = 1, d^1 = 4, N_2 = 2, d^2 = 2$. This reduces the vertex count ($N_1 + N_2 = 3 < 6$), and returns the $D = 4$ graph of 13 vertices originally due to Coutinho [15].
One way to understand how this example works is that our partitioned graph is entirely agnostic to the specific way that vertices in the two nodes are connected up. This is what [4] referred to as a “scrambled hypercube”. We can choose to group these so that there’s some extra structure (for the simple case below, it doesn’t matter how you scramble it).

In the central column, we can partition the nodes, merging together the top and bottom vertices in one node (because, between them, they have degree 2 going each way, and the neighbouring node has degree 1), and the other vertices in a single node,

Applying our manipulation rule to the lower node yields:

\textbf{Lemma 5.} For \(\Delta = 4\), transfer over distances 4 and 5, the example of Coutinho, and its Cartesian product with \(P_2\), are the graphs with the minimum number of vertices such that the graph still has a graph quotient coinciding with the standard perfect state transfer chain.

\textit{Proof.} We start from the engineered chain that we want as the graph quotient, knowing that the first and last vertices must correspond to single qubits. For that to be possible, the second and penultimate qubits must correspond to a number of vertices in the graph equal to the transfer distance. These statements apply to all possible distances. Now we must work out what the possible correspondences are for the remaining column(s). Due to symmetry, both cases effectively only have one column to take care of. We consider the coupling between adjacent columns. Let \(\tilde{A} \in \{0, 1\}^{k \times D}\), where \(k\) is to be determined. We require that \(\tilde{A}^T \tilde{A}\) has an all-ones eigenvector with eigenvalue \(J^2\), where \(J = \sqrt{2(D - 1)}\) is the appropriate coupling strength in the quotient graph. It is straightforward to check by brute force that for \(D = 4\), \(k < 3\) is not admissible. Similarly for \(D = 5\), \(k < 7\) is not admissible. The smallest remaining values of \(k\) match the given constructions. \(\square\)

IV. \textbf{The \(P_3\) Hypercube Partition}

\textbf{A. A First Example}

Our goal is to reduce the vertex count by considering other equitable distance partitions. In particular, we will take inspiration from the structure of the \(P_3\) hypercubes (dimension \(D/2\) for even \(D\)). We will start by explicitly considering the simplest such case, where \(D = 4\).

Let us take the bottom left-hand corner as the input vertex, and the top right-hand corner as the output vertex. The vertices at distance 1 from the input are all equivalent, as are those at distance 3. At distance 2, the central vertex is distinct from the two corner vertices. Thus, we have a partitioning of the form

There are no advantages to be gained by applying manipulation rules. However, we can now apply the \(\Delta\)-doubling lift to get the graph due to Coutinho [15].

\textbf{B. General Case}

We shall now describe the general version of this construction. For a \(D/2\)-dimensional hypercube of \(P_3\), each vertex can be labelled by a value \(x \in \{0, 1, 2\}^{D/2}\), where \((0, 0, \ldots, 0)\) is the input vertex and \((2, 2, \ldots, 2)\) is the output vertex. The distance from the input vertex is given by \(w_x = \sum_{i=1}^{D/2} x_i\). We proceed by observing that all vertices with the same number of 0s in their label, \(n_0\), and
2, \( n_2 \), are equivalent and can thus be grouped together in a node of the partition. There are

\[
\frac{D_1^2}{n_0! n_2! (D/2 - n_0 - n_2)!}
\]  

(3)

such vertices in each node. These have clearly identifiable connectivities. For example, every vertex in the \((n_0, n_2)\) node connects to \(n_0\) vertices in the \((n_0 - 1, n_2)\) node (as there are \(n_0\) choices of which 0 to turn into a 1), and \(\frac{D^2}{2} - n_0 - n_2\) vertices in the \((n_0, n_2 + 1)\) node (changing a 1 into a 2). We thus make the observation that the nodes form a graph that corresponds to half a square lattice, and along each row and column, the corresponding graph quotient has coupling strengths equal to those of the perfect transfer chain of the appropriate length! To demonstrate this on a modestly sized graph, consider \(D = 10\). The node structure is

![Diagram of a graph](image)

The vertices with occupancy 20 can be reduced to 5 by the manipulation rule. To the side, the graph quotient is depicted. The effective coupling strengths are shown for the horizontal couplings. Nodes are arranged such that \(n_0\) decreases vertically, and \(n_2\) increases horizontally, meaning that state transfer is from bottom left \((n_0, n_2) = (5, 0)\) to top right, \((0, 5)\). The \(\Delta = 2\) graph contains 198 vertices, while the \(\Delta = 4\) version requires 199 vertices.

To have the best chance of the manipulation rule being applicable, we want multiple factors of 2 to appear in the node occupancies given by Eq. \((3)\). If \(D\) is a power of 2, there are only 6 nodes whose occupancies are not divisible by 4: \(n_0, n_2 = (D/2, 0), (0, D/2), (0, 0), (D/4, 0), (0, D/4), (D/4, D/4)\). Modulo a few edge effects due to the need for even degrees, we can therefore apply the grouped-node version of the manipulation rule on almost the entire graph! Moreover, if either \(n_0\) or \(n_2\) is odd, the final occupancy is still even, which is perfect for the \(\Delta\)-doubling lift! With some additional manual manipulation, we have achieved particularly drastic reductions for the \(D = 16, 32\) cases. The distance 32, \(\Delta = 2\) transfer graph has 680913 vertices, an efficiency of \(\eta = 0.606\), and is depicted in Fig. [2]. The \(\Delta\)-doubling lift yields a \(\Delta = 4\) version with 830895 vertices, an efficiency of \(\eta = 0.615\).

Applying the node-splitting technique to the \(D = 32\) perfect transfer graph of Fig. [2] yields no improvement. For the \(\Delta = 4\) case, there is some minor improvement as the nodes at \((n_0, n_2) = (0, 8), (8, 0)\) and \((8, 8)\), each with node occupancy 1430, can be replaced by two nodes with occupancies 286 and 130. The total vertex count is therefore reduced to 827853, efficiency \(\eta = 0.614\).

C. Sequential Improvements

The Cartesian product \(G^{\Box k}\) of \(G\) gives transfer over distances \(kD\) with identical efficiency. However, can we do better? As we increase the transfer distance, can we reduce the vertex number compared to \(|G|^k\)? We will now give one mechanism that provides this, and estimate the revised efficiency that it gives.

For a set of nodes labelled by \(i\) that make up graph \(G\), \(G^{\Box 2}\) has a corresponding set of nodes labelled by \((i, j)\). Provided \(i \neq j\), the nodes \((i, j)\) and \((j, i)\) are equivalent and can be replaced by a single node with double the occupancy. Nodes \((i, i)\) retain the same occupancy but their degrees double. If we repeat this to get \(G^{\Box 4}\), but with the new node structure, then the vast majority of nodes now contain a factor of 4 in their occupancies. We apply the manipulation rule on most of these vertices, reducing the vertex number. If we assume that the efficiency of \(G\), with transfer distance \(D\) was \(\eta_D\), then the new efficiency is approximately

\[
\eta_{4D} \gtrsim \frac{1}{4D} \log_2 \frac{2^{4D\eta_D}}{4} = \eta_D - \frac{1}{2D}.
\]

Then we repeat this to give

\[
\eta_{4^qD} \gtrsim \eta_D - \frac{2}{D} \sum_{k=1}^{q} \frac{1}{4^k}.
\]

In the large \(q\) limit, we have

\[
\eta \to \eta_D - \frac{2}{3D}.
\]

This lets us estimate achievable efficiencies of about \(\eta = 0.584\) for even \(D\) and \(\eta = 0.594\) for odd \(D\), starting from the \(D = 32\) case.

V. CONCLUSIONS & FUTURE WORK

We can achieve perfect state transfer over distance \(D\) with a vertex count that scales as \(2^{n_0 D}\) where \(\eta = 0.606\) if \(D\) is even, and \(\eta = 0.614\) if \(D\) is odd. We have estimated that better efficiencies are achievable in a relatively straightforward manner, but for much larger graphs. We also know that if we wish to transfer over an odd distance, \(D \leq d\), which is severely limiting for practical applications.

The big question that remains open is whether or not an exponential scaling in the number of vertices is necessary and, if so, can one bound from below the exponent?
FIG. 2. The $\Delta = 2$, distance 32 transfer graph. These numbers are the number of vertices. Connections are with nearest neighbours of the grid, with degrees determined by the effective coupling strength of the edge: $J = \sqrt{n(k-n)}$ between vertices $n, n+1$ in a row/column of $k$ nodes: the product of the in and out degrees along an edge is equal to $J^2$, and, for a given edge, the total number of incoming connections is equal to the total number of outgoing ones.

The best known lower bounds on vertex number are polynomial in $D$ [14]. In fact, those bounds can be slightly improved in the present context of extremal transfer:

$$2m = \text{Tr}(H^2) \geq \sum_{\lambda \in \Phi_a} \lambda^2 \geq \min_q \sum_{n=1}^{N} \left(\frac{2\pi n}{\tau} - q\right)^2$$

where $m$ is the number of edges in the graph, and $2t_0 = \tau \leq 2\pi$ is the perfect revival time. We know that every eigenvalue is separated by an integer multiple of $2\pi/\tau$, and $q$ is just an offset that we can optimise over. Hence,

$$2m \geq \frac{N}{12}(N^2 - 1).$$

Clearly these are too weak because there are already large gaps between those bounds and the transfer graphs of distance 1 to 5 that we know to be optimal. One clear challenge is that existing lower bound techniques only take into account the space $\Phi_a$. Since we achieve effective coupling strengths other than unity in the quotient graph by having node occupancies larger than 1, there is generally a large space outside $\Phi_a$ whose size we currently have no way of estimating.

It helps to illustrate the difficulty in resolving the polynomial/exponential divide in vertex number by considering a family of graphs due to Stevanović [17]. These graphs do not exhibit perfect transfer, only because instead of every neighbouring gap in eigenvalues being an odd integer, there is a single gap that is an even integer. Otherwise, perfect transfer would be possible, and only requires a number of vertices that is quadratic in $D$. In our graph partition terminology, an example (from which
the family may readily be extrapolated by looking at the
pattern of alternate nodes) is

Furthermore, we have concentrated exclusively on
graphs whose ultimate quotient is just the perfect state
transfer chain with engineered couplings $\sqrt{n(N-n)}$.
There are myriad other analytic options \[18\]. The ba-
sic constructions that yield these will typically involve
more vertices, but it may be the case that there are mul-
tiple common factors, leaving them more amenable to the
reduction techniques investigated here. We may gain in-
sight here as to what properties of chains might be useful.
In particular, it is worth observing that the Stevanović
family of graphs defined above are equivalent to chains
with coupling strengths that have significant factors in
common between consecutive edges, and we believe that
this is responsible for facilitating such a massive reduc-
tion in vertex number. However, our experiments to try
and replicate aspects of this structure have all resulted
in graphs that do not possess perfect state transfer.

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