THE 3D LIQUID CRYSTAL SYSTEM WITH CANNONE TYPE INITIAL DATA AND LARGE VERTICAL VELOCITY

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Abstract. The hydrodynamic theory of the nematic liquid crystals was established by Ericksen [4] and Leslie [8]. In this paper, based on a new technique, we obtain global well-posedness to a simplified model introduced by Lin [9] in the critical Besov space with Cannone type initial data and large vertical velocity, which improves the main result in [15]. In addition, the small condition on \( u_0 \) is independent of another small condition on \( d_0 - \bar{d}_0 \), which is quite different from the previous works [15, 16].

1. Introduction. In this paper, we are concerned with the 3D nematic liquid crystal system given by

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p - \Delta u &= -\nabla \cdot (\nabla d \odot \nabla d), \\
\partial_t d + u \cdot \nabla d &= \Delta d + |\nabla d|^2 d, \\
\nabla \cdot u &= 0, \\
u(0, x) &= u_0(x), \quad d(0, x) = d_0(x), \\
|d_0(x)| &= 1, \quad \lim_{|x| \to \infty} (u, d) &= (0, \bar{d}_0)
\end{align*}
\]

where \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^3\), \( p, u \) stand for scalar pressure and velocity, respectively, \( d \) is the unknown macroscopic continuum molecule orientation of the nematic liquid crystal flow. The notation \( \nabla d \odot \nabla d \) is the \( 3 \times 3 \) matrix whose \((i, j)\)-th entry is given by \( \partial_i d \cdot \partial_j d \), and \( \bar{d}_0 \) is a constant vector with \(|\bar{d}_0| = 1\).

We often see the liquid crystal state as an intermediate state between liquid and solid. Although there are three main types of liquid crystals: nematic, smectic and cholesteric, the nematic phase appears to be the most common one. The molecules in the nematic liquid crystals do not exhibit any potential order, however, they have long-range orientation order. The hydrodynamic theory of the nematic liquid crystals had been studied by Ericksen [4] and Leslie [8]. In both 2D and 3D, Lin-Liu [10] obtained the global weak solution to an approximate model, which can be given by replacing \(|\nabla d|^2 d \) by \(-\nabla \cdot (|d|^2 - 1)^2\). The initial condition was \( u_0 \in L^2 \) and \( d_0 \in H^1 \) in that paper. Provided the initial data satisfied \( u_0 \in H^1 \) and \( d_0 \in H^2 \), they also obtained a unique global classical solution in 2D, and in 3D with additional large viscosity. We refer to [11] for the result concerning the partial regularity of suitable weak solution.

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there exist two positive constants $C$ and $\alpha$
Assume that Theorem 1.1. proved a new global well-posedness with large initial vertical velocity. Precisely,
By applying the idea dealing with 3D Navier-Stokes equations, Liu et al. [15] to the upper hemisphere. We refer to [13] for the uniqueness of weak solution and [5] [15, 16, 17] for the studies of small data solution.
Denote $\mathcal{D} = d - \delta_0$ with $\mathcal{D}_0 = d_0 - \delta_0$, then we can rewrite system (1) as

\[
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla p - \Delta u = -\nabla \cdot (\mathcal{D} \odot \mathcal{D}), \\
\partial_t \mathcal{D} + u \cdot \nabla \mathcal{D} = \Delta \mathcal{D} + |\nabla \mathcal{D}|^2 \mathcal{D} + |\nabla \mathcal{D}|^2 \delta_0, \\
\nabla \cdot u = 0, \\
u(0, x) = u_0(x), \quad \mathcal{D}(0, x) = \mathcal{D}_0(x),
\end{cases}
\]

By using Littlewood-Paley Theory, [5] obtained local well-posedness for (2) with the initial data satisfying $(u_0, d_0 - \delta_0) \in B^{\frac{1}{2}, 1}_{2, 1}(\mathbb{R}^3) \times B^{\frac{3}{2}, 1}_{2, 1}(\mathbb{R}^3)$ and global well-posedness with the assumption of the small initial data, i.e.,

\[\|u_0\|_{B^{\frac{1}{2}, 1}_{2, 1}} + \|d_0 - \delta_0\|_{B^{\frac{3}{2}, 1}_{2, 1}} < \epsilon.\]

By applying the idea dealing with 3D Navier-Stokes equations, Liu et al. [15] proved a new global well-posedness with large initial vertical velocity. Precisely, they assumed

\[(\|d_0 - \delta_0\|_{B^{\frac{1}{2}, 1}_{2, 1}} + \|u_0\|_{B^{\frac{1}{2}, 1}_{2, 1}}) \exp \{C\|u_0\|^2_{B^{\frac{1}{2}, 1}_{2, 1}}\} < \epsilon,
\]

which was generalized to the $L^p$ framework in [16]. In fact, when $(u_0, d_0 - \delta_0) \in B^{\frac{1}{2}-\frac{1}{p}, 1}_{r, 1}(\mathbb{R}^3) \times B^{\frac{3}{2}-\frac{1}{p}, 1}_{r, 1}(\mathbb{R}^3)$, they obtained the local well-posedness with $(r, q)$ satisfying

\[(r, q) \in (1, \infty) \times (1, \infty), \quad -\min\{\frac{1}{3}, \frac{1}{2r}\} \leq \frac{1}{q} - \frac{1}{r} \leq \frac{1}{3}.\]

Under the small initial condition

\[(\|d_0 - \delta_0\|_{B^{\frac{1}{2}, 1}_{r, 1}} + \|u_0\|_{B^{\frac{1}{2}, 1}_{r, 1}}) \exp \{C\|u_0\|^2_{B^{\frac{1}{2}, -1}_{r, 1}}\} < \epsilon
\]

with $(r, q)$ satisfying

\[(r, q) \in (1, 6) \times (1, \infty), \quad -\min\{\frac{1}{3}, \frac{1}{2r}\} \leq \frac{1}{q} - \frac{1}{r} \leq \frac{1}{3},\]

they also showed the global well-posedness. In these works, [15, 16, 5], the viscosity parameters $\nu$ and $\mu$ occur in (2) and the small conditions are depend on these parameters. In the present work, we have set $\nu = \mu = 1$ for simplicity.

Now, we state our first result.

**Theorem 1.1.** Assume that $(u_0, d_0) \in \dot{B}^{\frac{1}{2}}_{2, 1}(\mathbb{R}^3) \times \dot{B}^{\frac{3}{2}}_{2, 1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. If there exist two positive constants $C$ and $\epsilon$ such that

\[\|d_0 - \delta_0\|_{\dot{B}^{\frac{1}{2}}_{2, 1}} \exp \{C\|u_0\|^2_{\dot{B}^{\frac{1}{2}, -1}_{2, 1}}\} + \|u_0\|_{\dot{B}^{\frac{3}{2}, -1}_{r, \infty}} \leq \epsilon, \quad 2 \leq p < \infty,
\]

then (2) has a unique global solution $(u, d)$ satisfying

\[u \in \dot{C}(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2, 1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2, 1}(\mathbb{R}^3)),
\]

and

\[d - \delta_0 \in \dot{C}(\mathbb{R}^+; \dot{B}^{\frac{1}{2}}_{2, 1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2, 1}(\mathbb{R}^3)).\]
Let us emphasize that both [15] and [16] had the large initial vertical velocity. Our main work devotes to obtaining this kind of result.

**Theorem 1.2.** Assume that \((u_0, d_0) \in \dot{B}^{\frac{2}{p}}_{2,1}(\mathbb{R}^3) \times \dot{B}^{\frac{2}{p}}_{2,1}(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = 0\). If there exist two positive constants \(C\) and \(\epsilon\) such that
\[
\|d_0 - \tilde{d}_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}} \exp\{C\|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^2\} + \|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^{\frac{3}{2} - \frac{3}{p}} + \|u_0\|_{\dot{B}^{\frac{2}{r}, \infty}_{p, \infty}}^\alpha \|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^{1 - \frac{3}{r}} < \epsilon, \quad 2 \leq p < 6,
\]
for some \(\alpha \in (0, 1)\) depending on \(p\), then (2) has a unique global solution \((u, d)\) satisfying
\[
u \in \dot{C}(\mathbb{R}^+; \dot{B}^{\frac{4}{5}}_{2,1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)),
\]
and
\[
d - \tilde{d}_0 \in \dot{C}(\mathbb{R}^+; \dot{B}^{\frac{2}{q}}_{2,1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{B}^{\frac{2}{r}}_{2,1}(\mathbb{R}^3)).
\]

**Remark 1.3.** (1) It is a trivial process to establish global well-posedness for (2) in general Besov space, i.e., the initial data satisfies
\[
\|d_0 - \tilde{d}_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}} \exp\{C\|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^2\} + \|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^{\frac{3}{2} - \frac{3}{p}} < \epsilon, \quad r \leq p < \infty,
\]
where \((r, q)\) meets (3). (2) One can also get a generalization of Theorem 1.2 via using our idea. In fact, we can get global solution under
\[
\|d_0 - \tilde{d}_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}} \exp\{C\|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^2\} + \|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^{\frac{3}{2} - \frac{3}{p}} + \|u_0\|_{\dot{B}^{\frac{2}{r}, \infty}_{p, \infty}}^\alpha \|u_0\|_{\dot{B}^{\frac{2}{q}, 1}_{2,1}}^{1 - \frac{3}{r}} < \epsilon, \quad r \leq p < 6,
\]
where \((r, q)\) meets (4), which improves the result in [16]. (3) It is worth pointing out that the index \(p\) in (7) and (8) is independent of \(q\), which means the small condition on \(u_0\) is independent of another small condition on \(d_0 - \tilde{d}_0\). This is very different from [15, 16]. Furthermore, our results allow large \(Lip\) norm of the velocity. (4) Although global well-posedness for (1) with the Koch-Tartar type data (see [2]) was obtained in [17], we ensure that it is difficult to get large vertical velocity under this data.

Let us show the idea. To avoid the coupling between \(u\) and \(D\), we shall split the Navier-Stokes equations from [11]. Indeed, we suffice to obtain global well-posedness for the following two new systems:
\[
\begin{cases}
\partial_t V + V \cdot \nabla V - \Delta V + \nabla p_1 = 0, \\
\nabla \cdot V = 0, \\
V(0, x) = u_0(x)
\end{cases}
\]
and
\[
\begin{cases}
\partial_t W + V \cdot \nabla W + W \cdot (V + W) - \Delta W + \nabla p_2 = -\nabla \cdot (\nabla D \odot \nabla D), \\
\partial_t D + (V + W) \cdot \nabla D - \Delta D = |\nabla D|^2 D + |\nabla D|^2 \tilde{d}_0, \\
\nabla \cdot W = 0, \\
W(0, x) = 0, \quad D(0, x) = D_0(x),
\end{cases}
\]
where (9) is the well-known Navier-Stokes equations which admits global solution with Cannone type data \(\dot{B}^{\frac{2}{r}, -1}_{r, \infty}\) (see [2]). By combining with some product estimates and blow-up criteria, we can get that the solution for (9) satisfies
\[
\|V\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|V\|_{L^1_t(B^{\frac{2}{3}}_{2,1})} < \infty.
\]
Let and lemmas. 

2. Preliminaries. 

Finally, we denote \( L^p \) space different lines. We use \( \| \cdot \|_p \) for \( p \in [1, \infty) \). The uniform constant \( C \) may be different on different lines. We use \( \| f \|_p \) to denote the \( L^p(\mathbb{R}^3) \) norm of \( f \). We denote \( L^p_r([0, t]; X) \) the space \( L^p([0, t]; X) \) endowed with the norm \( \| f \|_{L^p_r([0, t]; X)} = \sup_{t \in [0, T]} \| f(t) \|_p \). We denote \( (\cdot)^{(\cdot)} \) will be a generic element of \( L^1(\mathbb{Z}) \), i.e., \( \sum_{j \in \mathbb{Z}} c_j \leq 1 \). Finally, we denote \( \mathcal{L}_1^{p,q}(\mathbb{R}^3) \) the space \( L^p([0, t]; L^q(\mathbb{R}^3)) \). We denote \( \mathcal{L}_1^{p,q}(\mathbb{R}^3) \) the space \( L^p([0, t]; L^q(\mathbb{R}^3)) \). 

Notations. For \( A, B \) two operator, we denote by \( [A, B] = AB - BA \) the commutator between \( A \) and \( B \). In some places of this paper, we may use \( L^p \) and \( \dot{B}^s_{p,r} \) to stand for \( L^p(\mathbb{R}^3) \) and \( \dot{B}^s_{p,r}(\mathbb{R}^3) \), respectively. The uniform constant \( C \) may be different on different lines. We use \( \| f \|_p \) to denote the \( L^p(\mathbb{R}^3) \) norm of \( f \). We denote \( \mathcal{L}_1^{p,q}(\mathbb{R}^3) \) the space \( L^p([0, t]; \mathcal{L}_1^{p,q}(\mathbb{R}^3)) \) \( \mathcal{L}_1^{p,q}(\mathbb{R}^3) \). 

In section 2, we provide some definitions of spaces and several lemmas. The third section and the fourth section devote to the proof of Theorem 2.1 and Theorem 2.2 respectively. We provide some estimates in the Appendix. 

Let us complete this section by describing the notations we shall use in this paper.

Notations. For \( A, B \) two operator, we denote by \( [A, B] = AB - BA \) the commutator between \( A \) and \( B \). In some places of this paper, we may use \( L^p \) and \( \dot{B}^s_{p,r} \) to stand for \( L^p(\mathbb{R}^3) \) and \( \dot{B}^s_{p,r}(\mathbb{R}^3) \), respectively. The uniform constant \( C \) may be different on different lines. We use \( \| f \|_p \) to denote the \( L^p(\mathbb{R}^3) \) norm of \( f \). We denote \( \mathcal{L}_1^{p,q}(\mathbb{R}^3) \) the space \( L^p([0, t]; \mathcal{L}_1^{p,q}(\mathbb{R}^3)) \). 

2. Preliminaries. In this section, we give some necessary definitions, propositions and lemmas.

The Fourier transform is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx.
\]

Let \( \mathcal{C} = \{ \xi \in \mathbb{R}^3, \frac{3}{4} \leq \xi \leq \frac{8}{3} \} \). Choose a nonnegative smooth radial function \( \varphi \) supported in \( \mathcal{C} \) such that
\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.
\]

We denote \( \varphi_j = \varphi(2^{-j} \xi) \) and \( h = \mathfrak{F}^{-1} \varphi \), where \( \mathfrak{F}^{-1} \) stands for the inverse Fourier transform. Then the dyadic blocks \( \Delta_j \) and \( S_j \) can be defined as follows
\[
\Delta_j f = \varphi(2^{-j} D) f = 2^{2j} \int_{\mathbb{R}^3} h(2^j y) f(x - y) dy, \quad S_j f = \sum_{k \leq j - 1} \Delta_k f.
\]

One easily verifies that with our choice of \( \varphi \)
\[
\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \text{ if } |j - k| \geq 5.
\]

Let us recall the definition of the Besov space.

Definition 2.1. Let \( s \in \mathbb{R}, (p, q) \in [1, \infty]^2 \), the homogeneous Besov space \( \dot{B}^s_{p,q}(\mathbb{R}^3) \) is defined by
\[
\dot{B}^s_{p,q}(\mathbb{R}^3) = \{ f \in \mathfrak{S}'(\mathbb{R}^3); \| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} < \infty \},
\]
where
\[
\| f \|_{\dot{B}^s_{p,q}(\mathbb{R}^3)} = \begin{cases} 
\sum_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j f \|_{L^p(\mathbb{R}^3)}^{q}, & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j f \|_{L^p(\mathbb{R}^3)}, & \text{for } q = \infty,
\end{cases}
\]
and \( \mathfrak{S}'(\mathbb{R}^3) \) denotes the dual space of \( \mathfrak{S}(\mathbb{R}^3) = \{ f \in \mathcal{S}(\mathbb{R}^3); \partial^\alpha f(0) = 0; \forall \alpha \in \mathbb{N}^3 \} \) and can be identified by the quotient space of \( \mathfrak{S}' / \mathcal{P} \) with the polynomials space \( \mathcal{P} \).
The norm of the space \( \tilde{L}^{r_1}_{t} (\tilde{B}^{s}_{p,r}) \) is defined by
\[
\| f \|_{\tilde{L}^{r_1}_{t} (\tilde{B}^{s}_{p,r})} := \| 2^{j r_1} |\Delta_j f \|_{L^{r_1}_{t} L^{p}_{r}} \|_{c_r(2^j)}.
\]
for \( f \in \tilde{C}(0, t; \tilde{B}^{s}_{p,r}) \) means \( f \in \tilde{L}^{\infty}_{t} (\tilde{B}^{s}_{p,r}) \) and \( \| f(t) \|_{\tilde{B}^{s}_{p,r}} \) is continuous in time.

Lemma 2.2. \([1]\) Let \( s > 0, 1 \leq p, r \leq \infty \), then
\[
\| fg \|_{\tilde{B}^{s}_{p,1}(\mathbb{R}^d)} \leq C \left\{ \| f \|_{L^{p_1}(\mathbb{R}^d)} \| g \|_{\tilde{B}^{s}_{p_2,1}(\mathbb{R}^d)} + \| g \|_{L^{r_1}(\mathbb{R}^d)} \| g \|_{\tilde{B}^{s}_{r_2,1}(\mathbb{R}^d)} \right\},
\]
where \( 1 \leq p_1, r_1 \leq \infty \) such that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2} \).
Due to (12), we can get
\[
\| fg \|_{\tilde{B}^{s}_{p,1}} \leq C \| f \|_{\tilde{B}^{s}_{p,1}} \| g \|_{\tilde{B}^{s}_{p,1}},
\]
which is also used in our proof.

The following proposition and lemma provide Bernstein type inequalities and standard commutator estimate in \( d \) dimensions.

Proposition 2.3. \([1]\) Let \( 1 \leq p \leq q \leq \infty \). Then for any \( \beta, \gamma \in (\mathbb{N} \cup \{0\})^d \), there exists a constant \( C \) independent of \( f, j \) such that
1) If \( f \) satisfies
\[
\text{supp} \, \hat{f} \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq K2^j \},
\]
then
\[
\| \partial^\gamma f \|_{L^{q}(\mathbb{R}^d)} \leq C 2^{j |\gamma| + j d(\frac{1}{p} - \frac{1}{r})} \| f \|_{L^{p}(\mathbb{R}^d)}.
\]
2) If \( f \) satisfies
\[
\text{supp} \, \hat{f} \subset \{ \xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j \}
\]
then
\[
\| f \|_{L^{p}(\mathbb{R}^d)} \leq C 2^{-j |\gamma|} \sup_{|\beta|=|\gamma|} \| \partial^\beta f \|_{L^{p}(\mathbb{R}^d)}.
\]

Lemma 2.4. \([1]\) Let \( \theta \) be a \( C^1 \) function on \( \mathbb{R}^d \) such that \( (1 + |\cdot|) \hat{\theta} \in L^1(\mathbb{R}^d) \). There exists a constant \( C \) such that for any Lipschitz function \( \alpha \) with gradient in \( L^p(\mathbb{R}^d) \), and any function \( b \) in \( L^q(\mathbb{R}^d) \), we have for any positive \( \lambda \),
\[
\| \theta(\lambda^{-1} D), \alpha \|_{L^r(\mathbb{R}^d)} \leq C \lambda \| \nabla \alpha \|_{L^p(\mathbb{R}^d)} \| b \|_{L^q(\mathbb{R}^d)}, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}.
\]

Lemma 2.5. \([3]\) Let \( 1 < p < \infty \), \( \text{supp} \tilde{\alpha} \subset \{ R_1 < |\xi| < R_2 \} \). There exists a constant \( c \) depending on \( \frac{R_2}{R_1} \) and such that
\[
\frac{c R_2^2}{p^2} \int_{\mathbb{R}^d} |u|^p dx - \frac{1}{p - 1} \int_{\mathbb{R}^d} \Delta u |u|^{p-2} u dx.
\]
Let us introduce the homogeneous Bony’s decomposition,
\[
uv = T_u v + T_v u + R(u, v),
\]
where
\[
T_u v = \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad T_v u = \sum_{j \in \mathbb{Z}} \Delta_j u S_{j-1} v, \quad R(u, v) = \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v,
\]
here \( \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1} \).
3. **Proof of Theorem 1.1.** In this section, we give the proof of Theorem 1.1. As the mention in section 1, we shall consider two new systems (9) and (10). Now, we begin the proof.

**Step 1. Local well-posedness and regularity criterion for (9).** One can easily get the local well-posedness of the solution for (9), that is, there exists a \( T^{\star} > 0 \) such that (9) admits a unique solution \( V \) satisfying
\[
V \in \tilde{C}(\[0, T^{\star}\); \dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)) \cap L^1(\[0, T^{\star}\); \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)),
\]
see [5] for the details. Next, we will see that the condition
\[
\|V\|_{\tilde{L}^2_t(\dot{B}^{\frac{3}{p}}_{p,\infty})} \leq K_1 \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,\infty}}, \quad \forall t \in (0, T^{\star}),
\]
(15)
where \( p \in [1, \infty] \) and \( K_1 \) is a constant, can yield global regularity.

Applying the operator \( \Delta_j \) to the both sides of (9), taking the \( L^2 \) product with \( \Delta_j V \), and using the Bernstein’s inequality, one gets
\[
\frac{1}{2} \frac{d}{dt} \|\Delta_j V\|_2^2 + c2^j \|\Delta_j V\|_2^2 \leq \|[\Delta_j, V \cdot \nabla]V\|_2 \|\Delta_j V\|_2,
\]
where we have used
\[
\int (V \cdot \nabla) \Delta_j V \cdot \Delta_j V dx = 0.
\]
Dividing by \( \|\Delta_j V\|_2 \), and integrating over \((0, t)\), we can obtain
\[
\|\Delta_j V(t)\|_2 + c2^j \int_0^t \|\Delta_j V\|_2 d\tau \leq \|\Delta_j u_0\|_2 + \int_0^t \|[\Delta_j, V \cdot \nabla]V\|_2 d\tau.
\]
(16)
By Bony’s decomposition, we have
\[
\|\Delta_j, V \cdot \nabla]V\|_2 \leq \sum_{|k-j| \leq 4} \|\Delta_j, S_{k-1} V \cdot \nabla]\Delta_k V\|_2
\]
\[
+ \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k V \cdot \nabla S_{k-1} V)\|_2
\]
\[
+ \sum_{k \geq j-2} \|\Delta_k V \cdot \nabla \Delta_j S_{k+2} V\|_2
\]
\[
+ \sum_{k \geq j-3} \|\Delta_j (\Delta_k V \cdot \nabla \Delta_k V)\|_2
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]

For \( I_1 \) and \( I_2 \). By commutator estimate [13], Bernstein’s inequality and Young’s inequality for series,
\[
I_1 + I_2 \leq C \sum_{|k-j| \leq 4} \|\nabla S_{k-1} V\|_\infty \|\Delta_k V\|_2
\]
\[
\leq C2^j \|\Delta_j V\|_2 \sup_{k \in \mathbb{Z}} 2^{-k} \|\nabla S_{k-1} V\|_\infty
\]
\[
\leq C2^j \|\Delta_j V\|_2 \sup_{k \in \mathbb{Z}} \sum_{k' \leq k-2} 2^{k'-k} \|\Delta_{k'} V\|_\infty
\]
\[
\leq Cc_j 2^{-\frac{j}{2}} \|V\|_{\dot{B}^{\frac{3}{2}}_{2,1}} \|V\|_{\dot{B}^{\frac{3}{p},\infty}}.
\]
Using (15), setting $\epsilon$ small enough such that $\epsilon^2 < \frac{1}{2}$, then

$$I_3 \leq C\|\nabla \Delta_j V\|_\infty \sum_{k \geq j+2} \|\Delta_k V\|_2$$

$$\leq C\|\nabla \Delta_j V\|_\infty \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j+2} 2^{-\frac{j}{2}k} c_k$$

$$\leq C 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j+2} 2^{\frac{j}{2}(j-k)} c_k$$

$$\leq C 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j+2} 2^{\frac{j}{2}(j-k)} c_k$$

$$\leq C c_j 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j+2} 2^{\frac{j}{2}(j-k)} c_k$$

$$\leq C 2^j \sum_{k \geq j-3} \|\Delta_k V\|_2 \|[\Delta_k V]_\infty\|$$

$$\leq C 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j-3} 2^{\frac{j}{2}(j-k)} 2^{\frac{j}{2}k} \|[\Delta_k V]_2\|$$

$$\leq C 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j-3} 2^{\frac{j}{2}(j-k)} c_k$$

$$\leq C c_j 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j-3} 2^{\frac{j}{2}(j-k)} c_k$$

So

$$\|[\Delta_j, V \cdot \nabla] V\|_2 \leq C c_j 2^{-\frac{j}{2}} \|[V]_{B^{3/2}_{2,1}}\| \sum_{k \geq j-3} 2^{\frac{j}{2}(j-k)} c_k$$

In a way, one can also get

$$\|[\Delta_j, V \cdot \nabla] V\|_{L^1(0, t; L^2)} \leq C c_j 2^{-\frac{j}{2}} \|[V]_{L^2(0, t; B^{3/2}_{2,1})}\| \|[V]_{L^2(0, t; B^{3/2}_{2,1})}\|$$

Plugging this estimate into (10), multiplying by $2^{\frac{j}{2}}$ and using interpolation inequality,

$$\|[V]_{L^\infty(0, t; B^{3/2}_{2,1})} + c\|[V]_{L^1(0, t; B^{3/2}_{2,1})}\| \leq \|[u_0]_{B^{3/2}_{2,1}} + C\|[V]_{L^2(0, t; B^{3/2}_{2,1})}\| \|[V]_{L^2(0, t; B^{3/2}_{2,1})}\|$$

Using (15), setting $\epsilon$ in (5) small enough such that $C K_1 \epsilon < \frac{1}{2}$, then

$$\|[V]_{L^\infty(0, t; B^{3/2}_{2,1})} + c\|[V]_{L^1(0, t; B^{3/2}_{2,1})}\| \leq 2\|[u_0]_{B^{3/2}_{2,1}}\|.$$

Hence, we have proved global regularity under the condition (15).

**Step 2. Global solution to (9).** Thanks to step 1, we suffice to prove (15) holds for $p \in [2, \infty)$. Denote

$$\bar{T} := \sup\{t \in (0, T^*_1) : \|[V]_{L^\infty(B^{3/2}_{2,1})} < \eta_0\}, p \in [2, \infty),$$

where $\eta_0$ fixed later is a sufficiently small constant. We assume $\bar{T} < T^*_1$.

Using (14) and Bernstein’s inequality, then we have

$$\frac{d}{dt}\|[\Delta_j V]_p + c2^{2j}\|[\Delta_j V]_p \leq C\|[\Delta_j (V \cdot \nabla V)]_p \leq C 2^j\|[\Delta_j (V \otimes V)]_p.$$
Integrating in time, multiplying by \(2^{j(\frac{3}{2}-1)}\), we can obtain
\[
\|V\|_{L^\infty_t(B^{\frac{3}{2}}_p,\infty)} + c\|V\|_{L^1_t(B^{\frac{3}{2}+1}_p,\infty)} \leq \|u_0\|_{L^\infty_t(B^{\frac{3}{2}}_p,\infty)} + C\|V \otimes V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)}.
\]
By Bony’s decomposition,
\[
\|V \otimes V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)} \leq \sup_{j \in \mathbb{Z}} \sum_{j' \leq k \leq 2j} \|\Delta_j(S_{k-1}V \otimes \Delta_k V)\|_p
\]
\[
+ \|\Delta_j(\Delta_k V \otimes S_{k-1}V)\|_p
\]
\[
+ \sum_{k \geq j-3} \|\Delta_j(\Delta_k V \otimes \Delta_k V)\|_p \, d\tau
\]
\[= L_1 + L_2 + L_3.
\]
For \(L_1\) and \(L_2\). By Bernstein’s inequality,
\[
L_1 + L_2 \leq C \sup_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j(k-j)} \|\Delta_k V \otimes \Delta_k V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)}
\]
\[
\leq C \sup_{j \in \mathbb{Z}} 2^{j} \|\Delta_k V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)} \|S_{k-1}V\|_{L^\infty_t(B^{\frac{3}{2}}_p,\infty)}
\]
\[
\leq C \|V\|_{L^\infty_t(B^{\frac{3}{2}+1}_p,\infty)} \sup_{j \in \mathbb{Z}} 2^{2j} \|\Delta_k V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)} \sum_{k \leq k \leq 2j} 2^{k'}
\]
\[
\leq C \|V\|_{L^\infty_t(B^{\frac{3}{2}+1}_p,\infty)} \|V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)}.
\]
For \(L_3\), by Young’s inequality for series and Bernstein’s inequality, we have
\[
L_3 \leq C \sum_{j \in \mathbb{Z}} \sum_{k \geq j-3} 2^{j(k-j)} \|\Delta_k V \otimes \Delta_k V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)}
\]
\[
\leq C \sup_{j \in \mathbb{Z}} 2^{j} \|\Delta_k V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)} \|\Delta_k V\|_{L^\infty_t(B^{\frac{3}{2}}_p,\infty)}
\]
\[
\leq C \|V\|_{L^\infty_t(B^{\frac{3}{2}+1}_p,\infty)} \|V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)}.
\]
Combining with the above three estimates, one can obtain
\[
\|V \otimes V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)} \leq C \|V\|_{L^\infty_t(B^{\frac{3}{2}+1}_p,\infty)} \|V\|_{L^1_t(B^{\frac{3}{2}+1}_p,\infty)}.
\]
So
\[
\|V\|_{L^\infty_t(B^{\frac{3}{2}}_p,\infty)} + c\|V\|_{L^1_t(B^{\frac{3}{2}+1}_p,\infty)} \leq \|u_0\|_{L^\infty_t(B^{\frac{3}{2}}_p,\infty)} + C\|V \otimes V\|_{L^1_t(B^{\frac{3}{2}}_p,\infty)}.
\]
Choosing \(\eta_0 = \frac{\varepsilon}{2C}\) in \((17)\) such that \(C\eta_0 < \frac{\varepsilon}{2}\), then we can get
\[
\|V\|_{L^\infty_t(B^{\frac{3}{2}-1}_p,\infty)} + \frac{\varepsilon}{2}\|V\|_{L^1_t(B^{\frac{3}{2}+1}_p,\infty)} \leq \|u_0\|_{B^{\frac{3}{2}-1}_p,\infty}.
\]
Setting \(\varepsilon = \frac{\eta}{2}\), we can get
\[
\|V(t)\|_{B^{\frac{3}{2}-1}_p,\infty} \leq \|u_0\|_{B^{\frac{3}{2}-1}_p,\infty} < \frac{\eta_0}{2}.
\]
Thus we can get a contradiction with the previous assumption by continuous arguments. Hence, \(T = T^*_T\). Namely, we can get \((18)\) holds for all \(t \in (0, T^*_T)\). Combining
with interpolation inequality
\[ \|V\|_{L^p_t(B^{\frac{3}{2}}_{p,\infty})} \leq |V|^{\frac{1}{2}}_{L^p_t(B^{\frac{3}{2}}_{p,\infty})} \cdot \|V\|^{\frac{1}{2}}_{L^p_t(B^{\frac{3}{2}+1}_{p,\infty})} \leq C_1 \|u_0\|_{B^{\frac{3}{2}+1}_{p,\infty}}, \]
we can get (\ref{15}) with \( K_1 = C_1 \). It means that we have proved (\ref{9}) has a unique global solution satisfying
\[ \|V\|_{L^\infty_t(0,t;B^{\frac{3}{2}}_{2,1})} + c \|V\|_{L^1_t(0,t;B^{\frac{5}{2}}_{2,1})} \leq 2 \|u_0\|_{B^{\frac{3}{2}}_{2,1}}, \forall t > 0. \tag{19} \]

**Step 3. Global solution to (10).** Thanks to the global bound (\ref{19}) of \( V \), one can easily get there exists \( T_2^* > 0 \) such that (\ref{10}) admits a unique solution \((W,D)\) on \((0,T_2^*)\) satisfying
\[ W \in \tilde{C}([0,T_2^*);B^{\frac{7}{2}}_{2,1}) \cap L^1([0,T_2^*);B^{\frac{5}{2}}_{2,1}) \]
and
\[ D \in \tilde{C}([0,T_2^*);B^{\frac{7}{2}}_{2,1}) \cap L^1([0,T_2^*);B^{\frac{5}{2}}_{2,1}). \]
To get the global solution of (\ref{10}), we suffice to prove for all \( t \in (0,T_2^*) \) there holds
\[ \|W\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \int_0^t \|W\|_{B^{\frac{5}{2}}_{2,1}} \, d\tau < \infty \tag{20} \]
and
\[ \|D\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \int_0^t \|D\|_{B^{\frac{5}{2}}_{2,1}} \, d\tau < \infty. \tag{21} \]
As the previous steps, by Bernstein’s inequality, we have
\[ \|W\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + c \|W\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} \leq C \int_0^t \left( \|(V + W) \cdot \nabla W\|_{B^{\frac{1}{2}}_{2,1}} + \|\nabla D \circ \nabla D\|_{B^{\frac{3}{2}}_{2,1}} \right) \, d\tau, \]
\[ \|D\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + c \|D\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} \leq \|D_0\|_{B^{\frac{3}{2}}_{2,1}} + C \int_0^t \left( \|(V + W) \cdot \nabla D\|_{B^{\frac{1}{2}}_{2,1}} + \|\nabla D\|^{\frac{2}{3}}_{B^{\frac{3}{2}}_{2,1}} \right) \, d\tau. \]
By (\ref{12}), \( B^{\frac{3}{2}}_{2,1} \to L^\infty \) and Bernstein’s inequality, we have
\[ \|(V + W) \cdot \nabla W\|_{B^{\frac{1}{2}}_{2,1}} \leq C \|(V + W) \otimes W\|_{B^{\frac{1}{2}}_{2,1}} \leq C \|(V,W)\|_{B^{\frac{3}{2}}_{2,1}} \|W\|_{B^{\frac{3}{2}}_{2,1}}, \]
\[ \|\nabla D \circ \nabla D\|_{B^{\frac{3}{2}}_{2,1}} \leq C \|\nabla D\|_{B^{\frac{3}{2}}_{2,1}} \leq C \|D\|_{B^{\frac{3}{2}}_{2,1}}^2, \]
\[ \|(V + W) \cdot \nabla D\|_{B^{\frac{1}{2}}_{2,1}} \leq C \|(V,W)\|_{B^{\frac{3}{2}}_{2,1}} \|\nabla D\|_{B^{\frac{3}{2}}_{2,1}} \leq C \|(V,W)\|_{B^{\frac{3}{2}}_{2,1}} \|D\|_{B^{\frac{3}{2}}_{2,1}}. \]
Combining with the above estimates and using interpolation inequality, we obtain
\[ \|W\|_{L^\infty_t(B^{\frac{1}{2}}_{2,1})} + \|D\|_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + c \int_0^t \left( \|W\|_{B^{\frac{5}{2}}_{2,1}} + \|D\|_{B^{\frac{5}{2}}_{2,1}} \right) \, d\tau \leq \|D_0\|_{B^{\frac{3}{2}}_{2,1}} + C \int_0^t \|(V,W)\|_{B^{\frac{3}{2}}_{2,1}} \left( \|W\|_{B^{\frac{5}{2}}_{2,1}} + \|D\|_{B^{\frac{5}{2}}_{2,1}} \right) \, d\tau \]
\[ + C \int_0^t (1 + \|D\|_{B^{\frac{3}{2}}_{2,1}}) \|D\|_{B^{\frac{5}{2}}_{2,1}} \, d\tau. \]
where we have used the result in step 2, then for all $t \in (0, \bar{T})$, we have
\[
\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)} + \frac{c}{2} \int_0^t (\|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7}) d\tau
\]
\[
+ C \int_0^t \|V\|_{B_{R_1}^{\frac{3}{2}}} (\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)}) d\tau + \frac{c}{2} \int_0^t (\|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7}) d\tau.
\]
Denote
\[
\bar{T} = \sup \{ t \in (0, T^*_2) : \|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)} < \eta_1 \},
\]
where $\eta_1 > 0$ fixed later is a small constant. We assume that $T < T^*_2$. For all $t \in (0, \bar{T})$, we have
\[
\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)} + \frac{c}{2} \int_0^t (\|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7}) d\tau
\]
\[
+ C \int_0^t \|V\|_{B_{R_1}^{\frac{3}{2}}} (\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)}) d\tau
\]
\[
\leq \|D_0\|_{B_{R_1}^5} + C \int_0^t \|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7} d\tau + \frac{c}{2} \int_0^t (\|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7}) d\tau
\]
\[
+ C \int_0^t \|V\|_{B_{R_1}^{\frac{3}{2}}} (\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)}) d\tau
\]
\[
\leq \|D_0\|_{B_{R_1}^5} + C \eta_1 \int_0^t \|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7} d\tau + C \eta_1 \int_0^t \|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7} d\tau
\]
Choosing $\eta_1 = \frac{c}{4\bar{T}}$, and applying Gronwall’s lemma yields
\[
\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)} + \frac{c}{4} \int_0^t (\|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7}) d\tau
\]
\[
\leq \|D_0\|_{B_{R_1}^5} \exp \{ C \int_0^t \|V\|_{B_{R_1}^{\frac{3}{2}}} d\tau \}.
\]
Thanks to the estimate
\[
\int_0^t \|V\|_{B_{R_1}^{\frac{3}{2}}} d\tau \leq C \|V\|_{L_t^\infty(B_{R_1}^1)} \int_0^t \|V\|_{B_{R_1}^{\frac{3}{2}}} d\tau \leq C_1 \|u_0\|_{B_{R_1}^5},
\]
where we have use the result in step 2, then for all $t \in (0, \bar{T})$,
\[
\|W\|_{L_t^\infty(B_{R_1}^1)} + \|D\|_{L_t^\infty(B_{R_1}^1)} + \frac{c}{4} \int_0^t (\|W\|_{B_{R_1}^5} + \|D\|_{B_{R_1}^7}) d\tau
\]
\[
\leq \|D_0\|_{B_{R_1}^5} \exp \{ C_1 \|u_0\|_{B_{R_1}^5} \}.
\]
By setting $\varepsilon = \frac{c}{4\bar{T}}$ in (6), we can obtain a contradiction with the previous assumption by continuous arguments. Thus, we can get $\bar{T} = T^*_2$. So we have proved (20) and (21).

**Step 4. Global solution to (2).** Following the way in [5], one can get the local well-posedness of the solution $(u, D)$ for (2). In addition, we can see $(V + W, D)$ is also a solution to (2). Due to the uniqueness, there holds $u = W + V$. Thanks to the previous steps, we can obtain the global bound of $(u, D)$, which completes the proof of Theorem 1.1.
4. Proof of Theorem 1.2. In this section, we prove Theorem 1.2. The procedure is similar to the previous section, but we need a new regularity criterion to get global well-posedness for (9), which is given in the following lemma.

Lemma 4.1. Let $u_0 \in \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. There exists a unique solution $V$ to (7) in the space

$$\tilde{C}([0, T^*_1); \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)) \cap L^1([0, T^*_1); \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)),$$

where $T^*_1$ is the lifespan. If there holds

$$\|V^h\|_{L^\infty_t(B^{\frac{5}{2}-1}_{p,2})} + \|V^h\|_{L^1_t(B^{\frac{5}{2}+1}_{p,2})} < \epsilon_0, \quad \forall \, t \in (0, T^*_1), \quad (23)$$

where $p \in [2, 6)$ and $\epsilon_0$ is a small enough constant, then $T^*_1 = \infty$.

Proof of Lemma 4.1. Repeating step 1 in section 3 gives

$$\|V\|_{L^\infty_t(B^{\frac{5}{2}}_{2,1})} + c\|V\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} \leq \|u_0\|_{B^{\frac{5}{2}}_{2,1}} + \sum_{j \in \mathbb{Z}} 2^{\frac{5}{2}j} \|\Delta_j (V \cdot \nabla V)\|_{L^1_t(L^2)}$$

$$\leq \|u_0\|_{B^{\frac{5}{2}}_{2,1}} + \sum_{j \in \mathbb{Z}} 2^{\frac{5}{2}j} \{\|\Delta_j (V^h \cdot \nabla h)\|_{L^1_t(L^2)} + \|\Delta_j (V^h \nabla h)\|_{L^1_t(L^2)}\}$$

$$= \|u_0\|_{B^{\frac{5}{2}}_{2,1}} + L_1 + L_2 + L_3.$$

Here we have split $V \cdot \nabla V$ into three terms and applied divergence free condition. Thanks to

$$L_1, L_2, L_3 \leq C(\|V^h\|_{L^\infty_t(B^{\frac{5}{2}-1}_{p,2})} \|V\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} + \|V^h\|_{L^1_t(B^{\frac{5}{2}+1}_{p,2})} \|V\|_{L^\infty_t(B^{\frac{5}{2}}_{2,1})}), \quad (24)$$

the proof of which will be given in the Appendix, we have

$$\|V\|_{L^\infty_t(B^{\frac{5}{2}}_{2,1})} + c\|V\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} \leq \|u_0\|_{B^{\frac{5}{2}}_{2,1}} + C(\|V^h\|_{L^\infty_t(B^{\frac{5}{2}-1}_{p,2})} + \|V^h\|_{L^1_t(B^{\frac{5}{2}+1}_{p,2})})$$

$$\times (\|V\|_{L^\infty_t(B^{\frac{5}{2}}_{2,1})} + c\|V\|_{L^1_t(B^{\frac{5}{2}}_{2,1})})$$

Setting $\epsilon_0 C < \frac{1}{2}$, then we have

$$\|V\|_{L^\infty_t(B^{\frac{5}{2}}_{2,1})} + c\|V\|_{L^1_t(B^{\frac{5}{2}}_{2,1})} \leq 2\|u_0\|_{B^{\frac{5}{2}}_{2,1}}.$$

So we have obtained global regularity under (23).

Now, we begin the proof of Theorem 1.2. It suffices to prove that (23) holds when $2 \leq p < 6$. In fact, once (23) holds, (9) has a unique global solution, and then following the procedure in section 3 line by line can yield the Theorem 1.2.

Step 1. The estimate for $V^3$. One can get $V^3$ satisfies

$$\partial_t V^3 + V \cdot \nabla V^3 - \Delta V^3 + \partial_3 p = 0.$$

As the procedure in section 3, we can obtain

$$\|V^3\|_{L^\infty_t(B^{\frac{5}{2}-1}_{p,2})} + c\|V^3\|_{L^1_t(B^{\frac{5}{2}+1}_{p,2})}$$

$$\leq \|u_0^3\|_{B^{\frac{5}{2}-1}_{p,2}} + C(\|V \cdot \nabla V^3\|_{L^1_t(B^{\frac{5}{2}-1}_{p,2})} + \|\partial_3 p\|_{L^1_t(B^{\frac{5}{2}-1}_{p,2})}) \quad (25)$$

$$= \|u_0^3\|_{B^{\frac{5}{2}-1}_{p,2}} + C(I_1 + I_2).$$
Let us split $I'_1$ into two terms:

$$I'_1 = I_{11} + I_{12},$$

where $I_{11} = \|V^h \cdot \nabla_h V^3\|_{L_1(B_{\frac{3}{p}+1})}$ and $I_{12} = \|V^3 \partial_3 V^3\|_{L_1(B_{\frac{3}{p}+1})}$. Using Bony's decomposition, we have

$$I_{11} \leq \sup_{j \in \mathbb{Z}} 2^{k(\frac{3}{p}-1)} \int_0^t \sum_{|k-j| \leq 4} \|\Delta_j (S_{k-1} V^h \cdot \nabla_h \Delta_k V^3)\|_p$$

$$+ \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k V^h \cdot \nabla_h S_{k-1} V^3)\|_p$$

$$+ \sum_{k \geq j-3} \|\Delta_j (\Delta_k V^h \cdot \nabla_h \tilde{\Delta}_k V^3)\|_p d\tau$$

$$= L_{11} + L_{12} + L_{13}.$$
Collecting the above estimates, one can obtain
\[
I'_1 \leq C\|V^h\|_{L_t^3(B_{p,\infty}^\frac{3}{2})} \|V^3\|_{L_t^3(B_{p,\infty}^\frac{3}{2})} + C(\|V^3\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + \|V^h\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}-1)} \|V^3\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)}).
\]

For \(I'_2\), let recall that \(p\) can be given by
\[
p = (-\Delta)^{-1} \nabla \cdot (V \cdot \nabla V) = (-\Delta)^{-1} \partial_i \partial_j (V^j V^i)
\]
\[
= \sum_{i,j=1,2} (-\Delta)^{-1} \partial_i \partial_j (V^j V^i) + 2 \sum_{i=1,2} (-\Delta)^{-1} \partial_i \partial_3 (V^i V^3)
\]
\[
- \sum_{i=1,2} (-\Delta)^{-1} \partial_3 (\partial_i V^i V^3),
\]
and then
\[
I'_2 \leq C(\|V^h V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2})} + \|V^h V^3\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + \|V^3 V_h \cdot V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}-1)} ) = L_{21} + L_{22} + L_{23}.
\]

Following the estimate in step 2 of the section, one can get
\[
L_{21} + L_{22} \leq C(\|V^h\|_{L_{t}^\infty(B_{p,\infty}^\frac{3}{2}-1)} \|V^h\|_{L_t^3(B_{p,\infty}^\frac{3}{2})} + \|V^3\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + \|V^3\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}-1)}).
\]

For \(L_{23}\), it follows from using the procedure as the estimate of \(I_{11}\) and \(p < 6\) that
\[
L_{23} \leq C \|V^3\|_{L_{t}^\infty(B_{p,\infty}^\frac{3}{2}-1)} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + C \|V^h\|_{L_t^2(B_{p,\infty}^\frac{3}{2}}) \|V^3\|_{L_t^2(B_{p,\infty}^\frac{3}{2})}.
\]

As a consequence,
\[
I'_2 \leq C(\|V^h\|_{L_{t}^\infty(B_{p,\infty}^\frac{3}{2}-1)} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2})} + \|V^3\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + \|V^3\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}-1)}).
\]

Denote
\[
a(t) = \|V^3\|_{L_{t}^\infty(B_{p,\infty}^\frac{3}{2}-1)} + c\|V^3\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)}.
\]

Inserting the estimates of \(I'_1\) and \(I'_2\) into (25) yields
\[
a(t) \leq u_0^3 \|B_{p,\infty}^{\frac{3}{2}-1}\| + C(\|V^h\|_{L_{t}^\infty(B_{p,\infty}^\frac{3}{2})} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + \|V^3\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} + \|V^3\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)} \|V^h\|_{L_t^\infty(B_{p,\infty}^\frac{3}{2}-1)} + \|V^h\|_{L_t^2(B_{p,\infty}^\frac{3}{2}}) \|V^3\|_{L_t^2(B_{p,\infty}^\frac{3}{2})}.
\]

Denote
\[
b(t) = \|V^h\|_{L_{t}^\infty(B_{p,\infty}^\frac{3}{2}-1)} + c\|V^h\|_{L_t^1(B_{p,\infty}^\frac{3}{2}+1)},
\]
and
\[
\bar{T} := \sup\{t \in (0, T_1) : b(t) \leq \eta_2\},
\]
Choosing Theorem 1.2. thus, Hence, we can get (23) holds with

We need to bound four terms on the right hand of (28), while the first one can be bounded easily. In fact, one can get

Choosing \( \eta_2 < \frac{1}{2C} \), then we can obtain

Step 2. The estimate for \( V^h \). From (9), one can get \( V^h \) satisfies

We can obtain

We need to bound four terms on the right hand of (28), while the first one can be bounded easily. In fact, one can get

Thanks to Lemma 5.1, we have

Thus we can obtain

Choosing \( \eta_2 = 4\|u_0^h\|_{B_{t_r}^{\frac{3}{2}-1}} \) and setting \( \epsilon \) in (6) small enough such that

thus

This yields a contradiction with the previous assumption by continuous arguments. Hence, \( \bar{T} = T^*_T \), which implies that

So we can get (23) holds with \( \epsilon_0 = (4 + \frac{3}{2})\|u_0^h\|_{B_{t_r}^{\frac{3}{2}-1}} \). This completes the proof of Theorem 1.2.
5. Appendix.

Proof of (24). Since the derivative $\partial_h$ and $\partial_3$ play the same role, it suffices to prove
\[
\|\Delta_j(V^h \cdot \nabla_h V)\|_2 + \|\Delta_j(V^3 \partial_3 V^h)\|_2 
\leq Cc_j 2^{-\frac{3}{2}}(\|V^h\|_{B^{rac{3}{2}}_{2,1}} \|V\|_{B^{rac{3}{2}}_{2,1}} + \|V^h\|_{B^{rac{3}{2}}_{2,1}} \|V\|_{B^{rac{3}{2}}_{2,1}}).
\] (29)

Applying Bony's decomposition implies
\[
\|\Delta_j(V^h \cdot \nabla_h V)\|_2 \leq \sum_{|k-j| \leq 4} \|\Delta_j(S_{k-1} V^h \cdot \nabla_h \Delta_k V)\|_2
\]
\[+ \sum_{|k-j| \leq 4} \|\Delta_j(\Delta_k V^h \cdot \nabla_h S_{k-1} V)\|_2
\]
\[+ \sum_{k \geq j-3} \|\Delta_j(\Delta_k V^h \cdot \nabla_h \tilde{\Delta}_k V)\|_2
\]
\[= II_1 + II_2 + II_3.
\]

By Bernstein's inequality,
\[
II_1 \leq \sum_{|k-j| \leq 4} \|\nabla_h \Delta_k V\|_2 \sum_{k' \leq k-2} \|\Delta_{k'} V^h\|_{\infty}
\]
\[\leq C \|V^h\|_{B^{rac{3}{2}}_{2,\infty}} \sum_{|k-j| \leq 4} 2^k \|\nabla_h \Delta_k V\|_2
\]
\[\leq C 2^{-\frac{3}{2}} (\|V^h\|_{B^{rac{3}{2}}_{2,1}} \|V\|_{B^{rac{3}{2}}_{2,1}}) \sum_{|k-j| \leq 4} 2^\frac{3}{2} (j-k) c_k
\]
\[\leq Cc_j 2^{-\frac{3}{2}} (\|V^h\|_{B^{rac{3}{2}}_{2,1}} \|V\|_{B^{rac{3}{2}}_{2,1}}).
\]

Similarly, we can get
\[
II_2 \leq C \sum_{|k-j| \leq 4} \|\Delta_k V^h\|_{\infty} \sum_{k' \leq k-2} \|\nabla_h \Delta_k V\|_2
\]
\[\leq C \|\nabla V\|_{B^{rac{1}{2}}_{2,1}} \sum_{|k-j| \leq 4} \|\Delta_k V^h\|_{\infty} \sum_{k' \leq k-2} 2^{\frac{3}{2} k'} c_k
\]
\[\leq C \|V\|_{B^{rac{1}{2}}_{2,1}} \|V^h\|_{B^{rac{3}{2}}_{2,1}} \sum_{|k-j| \leq 4} 2^{-\frac{3}{2} k} c_k
\]
\[\leq Cc_j 2^{-\frac{3}{2}} (\|V^h\|_{B^{rac{3}{2}}_{2,1}} \|V\|_{B^{rac{3}{2}}_{2,1}}).
\]

\[
II_3 \leq C \sum_{k \geq j-3} \|\Delta_k V^h\|_{\infty} \|\nabla_h \tilde{\Delta}_k V\|_2
\]
\[\leq C \|V\|_{B^{rac{1}{2}}_{2,1}} \|V^h\|_{B^{rac{3}{2}}_{2,1}} \sum_{k \geq j-3} 2^{-\frac{3}{2} k} c_k
\]
\[\leq Cc_j 2^{-\frac{3}{2}} (\|V^h\|_{B^{rac{3}{2}}_{2,1}} \|V\|_{B^{rac{3}{2}}_{2,1}}).
\]

Combining with the above three estimates can lead the desired bound of $\|\Delta_j(V^h \cdot \nabla_h V)\|_2$. Next, let us see the estimate concerning $\|\Delta_j(V^3 \partial_3 V^h)\|_2$. Applying Bony's
decomposition again, one has
\[
\|\Delta_j (V^3 \partial_3 V^h)\|_2 \leq \sum_{|k-j| \leq 4} \|\Delta_j (S_{k-1} V^3 \partial_3 \Delta_k V^h)\|_2 \\
+ \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k V^3 \partial_3 S_{k-1} V^h)\|_2 \\
+ \sum_{k \geq j-3} \|\Delta_j (\Delta_k V^3 \partial_3 \Delta_k V^h)\|_2 \\
= II'_1 + II'_2 + II'_3.
\]
By using the similar arguments, we have
\[
II'_1 \leq C \sum_{|k-j| \leq 4} \|S_{k-1} V^3\|_{B_p^{\frac{3}{2}}} \|\partial_3 \Delta_k V^h\|_p \\
\leq C \|V^h\|_{B_p^{\frac{3}{2}+1}} \sum_{|k-j| \leq 4} 2^{-\frac{j}{2}} k \sum_{k' \leq k-2} 2^{\frac{j}{2}} \|\Delta_k V^3\|_2 \\
\leq C \|V^h\|_{B_p^{\frac{3}{2}+1}} \sum_{|k-j| \leq 4} 2^{-\frac{j}{2}} k \sum_{k' \leq k-2} 2^{\frac{j}{2}} \|\Delta_k V^3\|_2 \\
\leq C C_j 2^{-\frac{j}{2}} \|V^h\|_{B_p^{\frac{3}{2}+1}} \|V^3\|_{B_p^{\frac{3}{2}+1}},
\]
\[
II'_2 \leq C \sum_{|k-j| \leq 4} \|\Delta_k V^3\|_2 \sum_{k' \leq k-2} 2^k \|\Delta_k V^h\|_\infty \\
\leq C \|V^h\|_{B_p^{\frac{3}{2}+1}} \sum_{|k-j| \leq 4} 2^{2k} \|\Delta_k V^3\|_2 \\
\leq C \|V^h\|_{B_p^{\frac{3}{2}+1}} \|V^3\|_{B_p^{\frac{3}{2}+1}} \sum_{|k-j| \leq 4} 2^{-\frac{j}{2}} k c_k \\
\leq C C_j 2^{-\frac{j}{2}} \|V^h\|_{B_p^{\frac{3}{2}+1}} \|V^3\|_{B_p^{\frac{3}{2}+1}},
\]
\[
II'_3 \leq C \sum_{k \geq j-3} \|\Delta_k V^3\|_2 \|\partial_3 \Delta_k V^h\|_\infty \\
\leq C \|V^h\|_{B_p^{\frac{3}{2}+1}} \|V^3\|_{B_p^{\frac{3}{2}+1}} \sum_{k \geq j-3} 2^{-\frac{j}{2}} k c_k \\
\leq C C_j 2^{-\frac{j}{2}} \|V^h\|_{B_p^{\frac{3}{2}+1}} \|V^3\|_{B_p^{\frac{3}{2}+1}}.
\]
So we can obtain the desired estimate of \(\|\Delta_j (V^3 \partial_3 V^h)\|_2\), and then \([29]\) has been proved.

Denote
\[
b_1(t) = \|V^h\|_{L^p(B_p^{\frac{3}{2}+1})}, \quad b_2(t) = \|V^h\|_{L^1(B_p^{\frac{3}{2}+1})}, \\
a_1(t) = \|V^3\|_{L^p(B_p^{\frac{3}{2}+1})}, \quad a_2(t) = \|V^3\|_{L^1(B_p^{\frac{3}{2}+1})}.
\]
So \(a(t)\) and \(b(t)\) are nearly equal to \(a_1(t) + a_2(t)\) and \(b_1(t) + b_2(t)\), respectively.
Lemma 5.1. Let $p \in [2, 6)$, there hold
\[
\|V^3 \partial_3 V^h\|_{L^1_t(B^\frac{3}{p-1}_{p, \infty})} + \|V^3 \text{div}_h V^h\|_{L^1_t(B^\frac{3}{p-1}_{p, \infty})} + \|\nabla_h V^3\|_{L^1_t(B^\frac{3}{p-1}_{p, \infty})} \\
\leq C a(t)^{1-\alpha} \{b_1(t)^\alpha b_2(t) + b_2(t)^\alpha b_1(t)\}
\]
for some $\alpha \in (0, 1)$.

Proof. The lemma is a variant of Lemma 3.2 in \[18\]. We split it into two cases: $p \in [2, 5)$ and $p \in [5, 6)$. We only show the details of the case $p \in [2, 5)$, since the difference between the two cases have been given in \[18\]. By Bony’s decomposition,
\[
\|\Delta_j (V^3 \partial_3 V^h)\|_{L^1_t L^p} \leq \sum_{|k-j| \leq 4} \|\Delta_j (S_{k-1} V^3 \partial_3 \Delta_k V^h)\|_{L^1_t L^p} \\
+ \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k V^3 \partial_3 S_{k-1} V^h)\|_{L^1_t L^p} \\
+ \sum_{k \geq j-3} \|\Delta_j (\Delta_k V^3 \partial_3 \tilde{\Delta}_k V^h)\|_{L^1_t L^p} \\
= I_1 + I_2 + I_3.
\]
For $I_1$, by Bernstein’s inequality, and interpolation inequality
\[
\|V^3\|_{L^\infty} \leq \|V^3\|_{L^\frac{1}{p}}^{\frac{1}{p}} \|\partial_3 V^3\|_{L^\infty}^{\frac{1}{p}} = \|V^3\|_{L^\frac{1}{p}}^{\frac{1}{p}} \|\text{div}_h V^h\|_{L^\frac{1}{p}},
\]
one gets
\[
I_1 \leq \sum_{|k-j| \leq 4} \|\partial_3 \Delta_k V^h\|_{L^1_t L^p} \sum_{k' \leq k-2} \|\Delta_k' V^3\|_{L^\infty_t L^2} \\
\leq \sum_{|k-j| \leq 4} \|\partial_3 \Delta_k V^h\|_{L^1_t L^p} \sum_{k' \leq k-2} \|\Delta_k' V^3\|_{L^\infty_t L^2} \|\text{div}_h \Delta_k V^h\|_{L^\infty_t L^\frac{1}{p}}^{\frac{1}{p}} \\
\leq C b_2(t) a_1(t)^{1-\frac{1}{p}} b_1(t)^{\frac{1}{p}} \sum_{|k-j| \leq 4} \sum_{k' \leq k-2} 2^{-k^{\frac{1}{p}}} \sum_{k' \leq k-2} 2^{k'} \\
\leq C 2^{j(1-\frac{1}{p})} a_1(t)^{1-\frac{1}{p}} b_2(t)^{\frac{1}{p}} b_1(t)^{\frac{1}{p}},
\]
here we have used Bernstein’s inequality in two dimensions and Minkowski inequality. For $I_2$, using Bernstein’s inequality and \[31\],
\[
I_2 \leq \sum_{|k-j| \leq 4} \|\Delta_k V^3\|_{L^1_t L^p(L^\infty_t)} \sum_{k' \leq j-2} \|\partial_3 \Delta_k' V^h\|_{L^\infty_t L^\frac{1}{p}} \\
\leq \sum_{|k-j| \leq 4} \|\Delta_k V^3\|_{L^1_t L^p(L^\infty_t)} \|\text{div}_h \Delta_k V^h\|_{L^\infty_t L^\frac{1}{p}} \sum_{k' \leq j-2} \|\partial_3 \Delta_k' V^h\|_{L^\infty_t L^\frac{1}{p}} \\
\leq a_2(t)^{1-\frac{1}{p}} b_2(t)^{\frac{1}{p}} b_1(t) \sum_{|k-j| \leq 4} \sum_{k' \leq j-2} 2^{-k^{\frac{1}{p}}(2-\frac{1}{p})} \\
\leq C 2^{j(1-\frac{1}{p})} a_2(t)^{1-\frac{1}{p}} b_2(t)^{\frac{1}{p}} b_1(t).
\]
For $I_3$, applying Bernstein’s inequality and \[31\] again,
\[
I_3 \leq C 2^{j\frac{2}{p}} \sum_{k \geq j-3} \|\Delta_j (\Delta_k V^3 \tilde{\Delta}_k \partial_3 V^h)\|_{L^1_t L^p(L^\frac{1}{p})}
\]
\[ \leq C 2^j \frac{2}{p} \sum_{k \geq j-3} \| \Delta_k V^3 \|_{L^1_t L^\infty_x (L^p)} \| \Delta_k \partial_3 V^h \|_{L^\infty_t L^p_x (L^p_x)} \]
\[ \leq C 2^j \frac{2}{p} \sum_{k \geq j-3} \| \Delta_k V^3 \|_{L^1_t L^\infty_x (L^p)} \| \Delta_k \partial_3 V^h \|_{L^\infty_t L^p_x (L^p_x)} \]
\[ \leq C 2^j \frac{2}{p} \sum_{k \geq j-3} \| \Delta_k V^3 \|_{L^1_t L^\infty_x (L^p)} \| \Delta_k \partial_3 V^h \|_{L^\infty_t L^p_x (L^p_x)} \]
\[ \leq C 2^j \frac{2}{p} \sum_{k \geq j-3} \| \Delta_k V^3 \|_{L^1_t L^\infty_x (L^p)} \| \Delta_k \partial_3 V^h \|_{L^\infty_t L^p_x (L^p_x)} \]
\[ \leq C 2^j \frac{2}{p} a_2(t) \left( \sum_{k \geq j-3} 2^{k(1 - \frac{2}{p})} \right) \]
\[ \leq C 2^{j(1 - \frac{2}{p})} a_2(t) \frac{1}{p} \sum_{k \geq j-3} 2^{k(1 - \frac{2}{p})} b_1(t). \]

So we have

\[ \| V^3 \partial_3 V^h \|_{L^\frac{2}{p} (\mathbb{R}^n \times \mathbb{R}_+)} \leq C a(t)^{1 - \alpha} \left\{ b_1(t)^\alpha b_2(t) + b_2(t)^\alpha b_1(t) \right\}. \]

Similarly, we also have

\[ \| V^3 \text{div}_h V^h \|_{L^\frac{2}{p} (\mathbb{R}^n \times \mathbb{R}_+)} \leq C a(t)^{1 - \alpha} \left\{ b_1(t)^\alpha b_2(t) + b_2(t)^\alpha b_1(t) \right\}. \]

For the third term on the left hand side of (30), we can get a similar bound by modifying the proof of the first two terms slightly. Hence, (30) with \( \alpha = \frac{1}{p} \) can be proved.

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