QED symmetries in real-time thermal field theory.

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Abstract

We study the discrete and gauge symmetries of Quantum Electrodynamics at finite temperature within the the real-time formalism. The gauge invariance of the complete generating functional leads to the finite temperature Ward identities. These Ward identities relate the eight vertex functions to the elements of the self-energy matrix. Combining the relations obtained from the $Z_2$ and the gauge symmetries of the theory we find that only one out of eight longitudinal vertex functions is independent. As a consequence of the Ward identities it is shown that some elements of the vertex function are singular when the photon momentum goes to zero.

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I. INTRODUCTION

Quantum field theory at finite temperature (QFFT) has been a subject of great current interest for a variety of physical contexts [1]. Two main formulations of QFFT can be distinguished. In the imaginary-time formalism (ITF) [2], the continuous energy variables $k_0$ are replaced by the discrete Matsubara frequencies $\omega_n$, and loop integrals over $k_0$ are transformed into discrete sums. To obtain a graph with real external energies requires a non-trivial analytic continuation. On the other hand, the real-time formalism (RTF) allows the computation of dynamical quantities directly as functions of continuous real frequencies. In addition, it is possible to formulate the theory in a covariant way [1]. The price to paid is that the Feynman rules are now more complicated than those at zero temperature. The number of degrees of freedom is doubled and hence the propagator acquires a matrix structure. The RTF has been discussed using a canonical operator formulation [3] and also using the path integral method [4]. Here, we shall mainly consider the diagrammatic method based in the path integral representation of the generating functional.

One would expect physical quantities to be independent of the formalism used to calculate them. This, however, has been an issue that has generated a longstanding controversy. For example, the amputated $n$-point functions at finite temperature appear to lead to different results when calculated in the two formalism’s. Nevertheless, it has been shown that within the real-time formalism one can consider a sum of graphs, motivated by causality arguments, which at least for the two- and three-point functions agree with the corresponding analytically continued imaginary-time results [5–7]. Thus, whereas single graphs may differ in the two formalism’s, there exist quantities that agree when calculated by either method.

In this work we are interested in studying the discrete and gauge symmetries of QED at finite temperature within the real-time formalism. It is commonly believed that gauge invariance is not affected by the temperature [3]. There exist, of course, some previous studies on the subject [8,9]; in particular, in the case of gauge theories at high temperature it has been shown that the hard thermal loops and the tree amplitudes obey the same Ward
identities \cite{10}. That Ward identities may be carried over to finite temperature without any modification, except for the fact that the matrix elements depend on the temperature, is probably obvious in the ITF. However in the RTF, where there are eight different vertex, the structure of the Ward identities is far from obvious. To our knowledge an explicit derivation of the Ward identities in the RTF has not appeared in the literature. The purpose of this article is to provide a systematic derivation based on the path integral method. In particular, we consider the Ward identities that relate the 2- and 3- point Green functions.

Section \textsection II is devoted to the study of the effective two component thermal formulation of QED within the path integral formalism’s. Along the paper we stress the fact that the thermal information in the action is contained in the convergence factor $\epsilon$ that is inserted through boundary conditions. We also discuss the discrete symmetries of the effective $QED$ action. In section \textsection II the subject of the gauge invariance of the theory is examined. Ward identities are obtained by demanding the complete generating functional be gauge invariant. When applied to the photon propagator, these identities imply that any component of the photon polarization tensor is transverse to any order of perturbation theory. Furthermore, the Ward identities impose several relations among the vertex functions and the elements of the self-energy matrix. We exploit these relations to prove that only one out of eight components of the longitudinal part of the vertex functions is independent. We also show that the Ward identities at finite temperature demand that some of the vertex functions have to be singular when the photon momentum goes to zero. Finally, the general results are illustrated by a one-loop calculation.

\section{II. THERMAL QED}

Let us consider a gas of electrons and photons at finite temperature. The system is described by the $QED$ Lagrangian

$$
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi} \gamma^\mu (\partial_\mu + ie A_\mu) \psi - m \bar{\psi} \psi - \frac{\xi}{2} (\partial \cdot A)^2,
$$

(2.1)
which includes the gauge fixing term. The path integral representation of the generating functional can be written as

\[ Z \left[ J_\mu, \eta, \bar{\eta} \right] = \int [DA_\mu D\psi D\bar{\psi}] \exp \left\{ i \int_C d\tau d\vec{x} \left( \mathcal{L} + \bar{\psi} \eta + \bar{\eta} \psi + J_\nu A_\nu \right) \right\}, \quad (2.2) \]

where the integration over \( \tau \) is along a monotonically descendent contour \( C \) in the complex time-plane. The contour has end points \( \tau_i \) and \( \tau_i - i\beta \), with \( \beta \) denoting the inverse of the temperature. The fields in (2.2) satisfy the boundary conditions \( A_\nu(\tau_i) = A_\nu(\tau_i - i\beta) \), and \( \psi(\tau_i) = -e^{i\mu} \psi(\tau_i - i\beta) \), where \( \mu \) is the chemical potential. Except for the gauge-fixing and the source terms the action in (2.2) is gauge invariant. Indeed, the first three terms in \( \mathcal{L} \) are invariant under the gauge transformation

\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad \psi \rightarrow \exp \left\{ -i\epsilon \Lambda \right\} \psi, \quad (2.3) \]

with the time derivative taken along \( C \).

Within the RTF the curve \( C \) is chosen in such a way that includes the real time axis. A particular contour that has been extensively used is one of the form \( C = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \). The segment \( C_1 \) starts at \( \tau_i = -T \) and runs along the real time axis up to \( +T \). \( C_2 \) continues along the imaginary axis from \( +T \) to \( +T - i\sigma \beta \), \( C_3 \) goes from \( +T - i\sigma \beta \) to \( -T - i\sigma \beta \), and \( C_4 \) runs from \( -T - i\sigma \beta \) to \( -T - i\beta \). At some stage, the limit \( T \rightarrow \infty \) is taken to produce the standard RTF. The contour parameter \( \sigma \) can take any value within the interval \((0, 1)\). Here we shall adopt the symmetrical value \( \sigma = 1/2 \). Of course, physical results should not depend on \( \sigma \).

Following the method of Niemi and Semenoff [4] \( Z \) can be recast as an integral over fields evaluated at real time, but with the number of degrees of freedom effectively doubled. The fields lying on \( C_1 \) (type-1 field) are referred as the physical field, while the (type-2) fields associated with the section \( C_2 \) of the contour are named as (thermal) ghost fields. In terms of these fields the generating functional takes the form [11]:

\[ Z \left[ J_\mu^a, \eta^a, \bar{\eta}^a \right] = \int [DA_\mu^a D\psi_a D\bar{\psi}_a] \exp \left\{ iS_{eff} \right\}, \quad (2.4) \]
where the effective action

\[ S_{\text{eff}} = \int \left( \mathcal{L}[1, 2] + \bar{\psi}_a \eta_a + \bar{\eta}_a \psi_a + J_{\nu a} A^\nu_a \right) \, dx, \tag{2.5} \]

has been written in terms of the finite-temperature Lagrangian \( \mathcal{L}[1, 2] = \mathcal{L}_\psi + \mathcal{L}_A + \mathcal{L}_{\text{int}} \), with

\[
\begin{align*}
\mathcal{L}_\psi &= \bar{\psi}_a \left[ S^{(0)} \right]^{-1}_{ab} \psi_b, \\
\mathcal{L}_A &= \frac{1}{2} A^\mu_a \left[ D^{(0)}_{\mu\nu} \right]^{-1}_{ab} A^\nu_b, \\
\mathcal{L}_{\text{int}} &= - \left[ e \bar{\psi}_1 \gamma_\mu \psi_1 A^\mu_1 - e \bar{\psi}_2 \gamma_\mu \psi_2 A^\mu_2 \right].
\end{align*}
\tag{2.6}
\]

In these equations a summation over the repeated index \( a = 1, 2 \) is understood, while \( S^{(0)}_{ab} \) and \( \left[ D^{(0)}_{\mu\nu} \right]_{ab} \) denote the matrix elements of the bare thermal propagators. At the tree level, there are only two kinds of vertices, the type-1 and the type-2 vertices, that differ by a sign: \( \Gamma_{111} = -\Gamma_{222} = \gamma^\mu \). The physical Green functions can be expressed as a sum of Feynman diagrams, with the type-1 fields appearing on the external legs; the type-2 field only occur in the internal lines of the diagrams.

In momentum space the inverse of the fermion propagator can be worked out as

\[
\left[ S^{(0)}(p) \right]^{-1} = (\not{p} - m) \sigma_3 + i \epsilon \left[ V^{-1}(p_0) \right]^2,
\tag{2.7}
\]

where \( \sigma_3 \) is the diagonal Pauli matrix and

\[
V = \begin{pmatrix}
\cos \varphi_p & -\epsilon(p_0) e^{3\mu/2} \sin \varphi_p \\
\epsilon(p_0) e^{-\beta \mu/2} \sin \varphi_p & \cos \varphi_p
\end{pmatrix}, \quad \cos \varphi_p = \frac{\theta(p_0) e^{x/4} + \theta(-p_0) e^{-x/4}}{\sqrt{e^{x/2} + e^{-x/2}}},
\tag{2.8}
\]

with \( x = \beta(p_0 - \mu) \). For the photon the result is

\[
\left[ D^{(0)}_{\mu\nu}(k) \right]^{-1} = \left[ -k^2 g_{\mu\nu} + (1 - \xi) k_\mu k_\nu \right] \sigma_3 - i \epsilon g_{\mu\nu} \left[ U^{-1}(k_0) \right]^2,
\tag{2.9}
\]

with

\[
U = \begin{pmatrix}
\cosh \theta_k & \sin \theta_k \\
\sinh \theta_k & \cosh \theta_k
\end{pmatrix}, \quad \cosh^2 \theta_k = \frac{1}{1 - \exp^{-\beta |k_0|}}.
\tag{2.10}
\]
We have kept $\epsilon$ finite in order to define the theory properly [1]. At zero temperature $\epsilon$ is included as a convergence factor, so the path integral in Minkowski space is well defined. At finite temperature the procedure becomes essential, not only for convergence, but also to keep trace of the temperature dependence, which as seen from the previous expression appears only in the $\epsilon$-terms. Because of these contributions, the action is non-local in the time variable. In fact, the substitution of (2.7) in (2.6) leads to the following expression for the fermion part of the action in configuration space

$$L_\psi = \int dx_0dy_0d\vec{x} \tilde{\psi}_a(x_0, \vec{x}) \left[ \delta(x_0 - y_0) (i\not\partial - m) \sigma_3 + i\epsilon M(x_0 - y_0) \right]_{ab} \psi_b(y_0, \vec{x)},$$

(2.11)

where

$$M_{ab}(x_0 - y_0) = \int \frac{dk_0}{2\pi} \left[ V^{-1}(k_0) \right]_{ab} e^{-ik_0(x_0 - y_0)}.$$

(2.12)

Similarly, for the gauge-field we obtain

$$L_A = \int dx_0dy_0d\vec{x} A^\mu_a(x_0, \vec{x}) \times$$

$$\frac{1}{2} \left[ \delta(x_0 - y_0) \left( g_{\mu\nu} \Box + (\xi - 1)\partial_\mu \partial_\nu \right) \sigma_3 + i\epsilon g_{\mu\nu} N(x_0 - y_0) \right]_{ab} A^\nu_b(y_0, \vec{x}),$$

(2.13)

with

$$N_{ab}(x_0 - y_0) = \int \frac{dk_0}{2\pi} \left[ U^{-1}(k_0) \right]_{ab} e^{-ik_0(x_0 - y_0)}.$$

(2.14)

In the zero temperature limit $N = M = I\delta(x_0 - y_0)$, where $I$ is the identity matrix. Then, the physical and ghost fields uncouple and the time-locality of the action is restored.

In reference [4] it was pointed out that the scalar theory at finite temperature has a $Z_2$ symmetry under the interchange of type-1 and type-2 fields. Here we exploit the $Z_2$ symmetry for thermal QED to reduce the number of independent 2- and 3-point vertex functions. As usual we introduce the generating functional of one particle irreducible graphs

$$\Gamma[\tilde{\Psi}_a, \Psi_a, A^\mu_a] = W[\bar{\eta}_a, \eta_a, J^\mu_a] - \int \left( \bar{\eta}_a \Psi^a + \tilde{\Psi}_a \eta^a + J^\mu_a A^a_{\mu} \right),$$

(2.15)
where \( W[\eta_a, \eta_a, J^\mu_a] = -i \ln Z[\eta_a, \eta_a, J^\mu_a] \) is the connected generating functional and

\[
\frac{\delta W}{\delta J^\mu_a} = A^\mu_a, \quad \frac{\delta \Gamma}{\delta A^\mu_a} = -J^\mu_a, \\
\frac{\delta W}{\delta \eta^a} = \psi^a, \quad \frac{\delta \Gamma}{\delta \psi^a} = -\eta^a, \\
\frac{\delta W}{\delta \bar{\eta}^a} = \bar{\psi}^a, \quad \frac{\delta \Gamma}{\delta \bar{\psi}^a} = -\eta^a.
\]

(2.16)

In Eqs. (2.15) and (2.16), \( \Psi^a, \bar{\Psi}^a, \Psi_a, \) and \( A^\mu_a \) refer to the classical fields, but for simplicity we have denoted them with the same notation than the one used for the fields appearing in the path integral. It is convenient to write the fields as two components spinors

\[
A^\mu = \begin{pmatrix} A^\mu_1 \\ A^\mu_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\Psi} = (\bar{\psi}_1 \quad \bar{\psi}_2).
\]

(2.17)

In terms of them the \( Z_2 \) symmetry for \( \Gamma \) is realized as follows:

\[
\Gamma[\bar{\Psi}, \Psi, A^\mu] = -\Gamma^*[\bar{\Psi}^* \bar{\sigma}, \bar{\sigma} \Psi^*, \sigma_1 A^\mu],
\]

(2.18)

where \( \sigma_1 \) is the Pauli matrix and

\[
\bar{\sigma} = \begin{pmatrix} 0 & ie^{\beta\mu/2} \\ -ie^{-\beta\mu/2} & 0 \end{pmatrix}.
\]

(2.19)

As is well known, the fermion 2-point function \( \delta^2 \Gamma/\delta \bar{\psi} \delta \psi \) can be related to the inverse of the exact propagator. The explicit relation in momentum space is

\[
(2\pi)^4 \delta(p - p')iS^{-1}_a(p) = \int dx dy e^{i(p'x - px)} \frac{\delta^2 \Gamma}{\delta \bar{\psi}_a(x) \delta \psi_b(y)}.
\]

(2.20)

Taking into account that \( S^{-1}(p) = [S^{(0)}(p)]^{-1} - \Sigma(p) \), the \( Z_2 \) symmetry (2.18) imply that the elements of the fermion self-energy matrix \( \Sigma \) are related as follows: \( \Sigma_{11}(p) = -\Sigma^*_2(p) \), \( \Sigma_{12}(p) = e^{\beta\mu} \Sigma^*_2(p) \). An additional relation among the elements of \( \Sigma \) is obtained from the circling relation [6]:

\[
\Sigma_{11}(p) + e^{-\beta p_0/2} \Sigma_{12}(p) + e^{\beta p_0/2} \Sigma_{21}(p) + \Sigma_{22}(p) = 0.
\]

(2.21)
The previous relations taken together imply that only one component of the matrix self-energy is independent. Therefore, for example, knowing $\Sigma_{11}$ is enough to determine the other components of $\Sigma$. In particular, for the retarded self-energy, $\Sigma_R(p) = \Sigma_{11}(p) + e^{-\beta p_0/2}\Sigma_{12}(p)$, we have:

\[
\begin{align*}
\text{Re } \Sigma_R(p) &= \text{Re } \Sigma_{11}(p) \\
\text{Im } \Sigma_R(p) &= \epsilon(p_0) \sec(2\varphi_p) \text{Im } \Sigma_{11}(p) = -ie^{-\beta\mu/2} \csc(2\varphi_p) \Sigma_{12}(p) .
\end{align*}
\]

(2.22)

In practice it is simpler to calculate $\text{Im } \Sigma_R$ in terms of the $\Sigma_{12}$, which is a purely imaginary quantity. For this reason, and also for future comparison with the result for the 3-point function (see Eq. (3.16)), we have written the second equality for the imaginary part of $\Sigma_R$.

We now turn the attention on the proper vertex function. Within the RTF there are eight connected three point functions $\Gamma^\mu_{abc}(q,p)$, which are defined by

\[
\begin{align*}
\text{ie } (2\pi)^4 \delta(p' - p - q) \Gamma^\mu_{abc}(q,p) &= \int dxdydz e^{i(p'z - pq - qx)} \frac{\delta^3 \Gamma}{\delta \bar{\psi}_c(z) \delta \psi_b(y) \delta A_\mu^a(x)} .
\end{align*}
\]

(2.23)

The first index corresponds to the photon with momentum $q$, while the second and the third indices refer to the incoming fermion with momentum $p$ and the outgoing fermion with momentum $p + q$, respectively. Not all of these complex functions are independent. The $Z_2$ symmetry of the theory (2.18) leads to the following relations:

\[
\begin{align*}
\Gamma^\mu_{222}(q,p) &= -[\Gamma^\mu_{111}(q,p)]^* , \\
\Gamma^\mu_{221}(q,p) &= e^{\beta\mu} [\Gamma^\mu_{112}(q,p)]^* , \\
\Gamma^\mu_{212}(q,p) &= e^{-\beta\mu} [\Gamma^\mu_{121}(q,p)]^* , \\
\Gamma^\mu_{122}(q,p) &= -[\Gamma^\mu_{211}(q,p)]^* .
\end{align*}
\]

(2.24)

The circling equation of Ref. [6] adapted to the vertex function of QED introduces a further relation among the various $\Gamma^\mu_{abc}$, namely

\[
\begin{align*}
\Gamma^\mu_{111} + e^{-\beta q_0/2}\Gamma^\mu_{211} + e^{-\beta p_0/2}\Gamma^\mu_{121} + e^{-\beta r_0/2}\Gamma^\mu_{221} \\
+ e^{\beta r_0/2}\Gamma^\mu_{112} + e^{\beta p_0/2}\Gamma^\mu_{212} + e^{\beta q_0/2}\Gamma^\mu_{122} + \Gamma^\mu_{222} &= 0 ,
\end{align*}
\]

(2.25)
where \( r_0 = p_0 + q_0 \). Equations (2.24) and (2.23) represent five complex constrictions and consequently, one is left with only three independent 3-point functions. It is convenient to choose them in the following way [6]:

\[
\begin{align*}
\Gamma^{\mu}_{R1}(q,p) &= \Gamma^{\mu}_{111} + e^{-\beta p_0/2}\Gamma^{\mu}_{121} + e^{\beta r_0/2}\Gamma^{\mu}_{112} + e^{\beta q_0/2}\Gamma^{\mu}_{122}, \\
\Gamma^{\mu}_{R2}(q,p) &= \Gamma^{\mu}_{111} + e^{-\beta q_0/2}\Gamma^{\mu}_{211} + e^{\beta r_0/2}\Gamma^{\mu}_{112} + e^{\beta p_0/2}\Gamma^{\mu}_{212}, \\
\Gamma^{\mu}_{R3}(q,p) &= \Gamma^{\mu}_{111} + e^{-\beta q_0/2}\Gamma^{\mu}_{211} + e^{-\beta p_0/2}\Gamma^{\mu}_{121} + e^{-\beta r_0/2}\Gamma^{\mu}_{221}. 
\end{align*}
\] (2.26)

These are the retarded functions with, respectively, \( t_1, t_2, \) and \( t_3 \) as the largest time, their complex conjugate being the corresponding advanced functions. In the next section we shall see how the Ward identities lead to a further reduction in the number of independent 3-point functions.

III. GAUGE INVARIANCE AND WARD IDENTITIES

As stated in section II, at finite temperature the Lagrangian (without gauge-fixing) with support in the complex contour \( C \) is gauge invariant. However, the original theory has been replaced by an effective one with two components, and we must ask about the gauge-invariance properties of this effective theory, as well as of their consequences. To investigate this point, let us consider the following independent gauge transformations on the type-1 and type-2 fields:

\[
\begin{align*}
A^{\mu}_a &\rightarrow A^{\mu}_a + \partial^{\mu}\Lambda_a, \\
\psi_a &\rightarrow \exp(-ie\Lambda_a)\psi_a, 
\end{align*}
\] (3.1)

where \( \Lambda_a(x) \) (\( a = 1, 2 \)) are arbitrary functions of \( x \), that in what follows we assume to be infinitesimal. Under such transformation, the finite temperature effective action \( S_{eff} \) in (2.5) is not invariant; the origin is threefold: The same as in the zero temperature case, the gauge fixing term and the source term are changed; in addition, the temperature dependent contributions, contained in the \( \epsilon \)-terms, are also gauge dependent. Accordingly,
it is convenient to decompose the variation of the effective action into two parts and write

$$\delta S_{\text{eff}} = \delta S_0 + \delta S_T.$$ 

$\delta S_0$ arises from the variation of the gauge fixing and source terms, and using (2.6), (2.11) and (2.13) it can be worked out as

$$\delta S_0 \left[ \bar{\Psi}_a, \Psi_a, A^\mu_a \right] = \int dx \left[ -\xi \epsilon_a (\partial \cdot A_a) \Box + J^\mu_a \partial_\mu - ie \left( \bar{\eta}_a \psi_a - \bar{\psi}_a \eta_a \right) \right] \Lambda_a, \quad (3.2)$$

with $\epsilon_1 = 1, \epsilon_2 = -1$. The quantity $\delta S_T$ is obtained from the $\epsilon$-terms in (2.11) and (2.13).

The result is

$$\delta S_T \left[ \bar{\Psi}_a, \Psi_a, A^\mu_a \right] = i\epsilon \int d\vec{x}dx_0dy_0 \left\{ -\Lambda_a(x_0) \frac{\partial}{\partial x^\mu} \left[ N_{ab}(x_0 - y_0)A^\mu_b(y_0) \right] 
+ i\epsilon \left[ \Lambda_a(x_0) - \Lambda_b(y_0) \right] \bar{\psi}_a(x_0, \vec{x})M_{ab}(x_0 - y_0)\psi_b(y_0, \vec{x}) \right\}. \quad (3.3)$$

Although $S_{\text{eff}}$ varies with the gauge transformations, the physical consequences of the theory, expressed in terms of Green’s functions, should not depend on the gauge. Thus, the generating functional $Z$ must be gauge invariant and this non-trivial requirement leads to the Ward identities. Combining the previous results with Eq. (2.4) the gauge invariance of $Z$ implies

$$\left\{ \delta S_0 \left[ \delta \frac{\delta}{\delta \eta_a}, \delta \frac{\delta}{\delta \bar{\eta}_a}, \delta \frac{\delta}{\delta J^\mu_a} \right] + \delta S_T \left[ \delta \frac{\delta}{\delta \eta_a}, \delta \frac{\delta}{\delta \bar{\eta}_a}, \delta \frac{\delta}{\delta J^\mu_a} \right] \right\} Z[J^\mu, \eta, \bar{\eta}] = 0. \quad (3.4)$$

By means of Eqs. (2.14), (2.16) and using the results in (3.2) and (3.3) we convert (3.4) into a condition for $\Gamma$

$$-\xi \epsilon_a \Box \partial \cdot A^a(x) + \partial^\mu \frac{\delta \Gamma}{\delta A^\mu_a(x)} - ie \left( \psi_a(x) \frac{\delta \Gamma}{\delta \psi_a(x)} - \bar{\psi}_a(x) \frac{\delta \Gamma}{\delta \bar{\psi}_a(x)} \right)$$

$$+ i\epsilon \int dy_0 \left[ i\bar{\psi}_a(x_0)M_{ab}(x_0 - y_0)\psi_b(y_0) + M_{ab}(x_0 - y_0) \frac{\partial \psi_b(y_0)}{\partial \eta_a(y_0)} - (a \leftrightarrow b, x_0 \leftrightarrow y_0) \right]$$

$$- i\epsilon \int dy_0 \frac{\partial}{\partial x^\mu} \left[ N_{ab}(x_0 - y_0)A^\mu_b(y_0, \vec{x}) \right] = 0, \quad (3.5)$$

where we used that the gauge functions $\Lambda_a(x)$ are arbitrary, and no summation over the index $a$ is implied.

Equation (3.3) expresses the general content of the Ward identities in the real-time formalism. Repeated differentiation of it, at $A^\mu_a = \bar{\psi}_a = \psi_a = 0$, generates relations between the
one particle irreducible Green functions that are consequence of the gauge invariance of the theory. First we differentiate (3.5) with respect to $A_\mu(x)$ and using the fact that $\delta \Gamma / \delta A_\mu \delta A_\nu$ is the inverse of the full photon propagator $D_{\mu\nu}$, the following result (in momentum space) is obtained

$$q^\mu D_{\mu\nu}^{-1}(q) = -\xi_3 q_\nu q^2 - i \epsilon q_\nu \left[ U^{-1}(q_0) \right]^2.$$ (3.6)

This relation for the full inverse photon propagator holds to all orders in perturbation theory. The effect of radiative corrections is to add the negative of the self-energy $\Pi_{\mu\nu}$ to the inverse of the bare propagator $[D_{\mu\nu}^0]^{-1}$. From the expression in (2.9) it is easy to check that (3.6) is satisfied by $[D_{\mu\nu}^0]^{-1}$. Consequently we get

$$q^\mu \Pi_{\mu\nu}(q) = 0.$$ (3.7)

In a recent work, the transversality of the polarization tensor at finite temperature has been corroborated by an explicitly one-loop calculation [12]. As we proved here, this property remains valid to all orders in perturbation theory.

As a second application of the identity (3.5), we act on it with $\delta^2 / \delta \bar{\psi}(x) \delta \psi(y)$ and put $A_\mu^a = \bar{\psi}_a = \psi_a = 0$. Using the definitions of the proper vertex function (2.23) and the full inverse fermion propagator (2.20), we arrive to

$$q_\mu \Gamma_{abc}^\mu(q,p) = \delta_{ab} \left[ S_{ca}^{-1}(p+q) - i \epsilon \left[ V^{-1}(p_0 + q_0) \right]_{ca}^2 \right] - \delta_{ac} \left[ S_{ab}^{-1}(p) - i \epsilon \left[ V^{-1}(p_0) \right]_{ab}^2 \right].$$ (3.8)

These are the Ward identities that, in the RTF, relate the components of the full electron propagator to the vertex functions. Equation (3.8) represents eight relations, one for each element of $\Gamma_{abc}^\mu$. They include explicit temperature contributions due to the presence of the $\epsilon$-terms. At zero temperature, the matrix $V(p_0)$ reduces to the identity, the $\epsilon$ dependent terms cancel, and (3.8) decompose in two independent relations, one for the type-1 fields and another for the type-2 fields.

If we substitute the inverse bare propagator (2.7), Eq. (3.8) simplifies to

$$q_\mu \Gamma_{abc}^\mu(q,p) = q_\mu \gamma^\mu \epsilon_a \delta_{ab} \delta_{ca},$$ (3.9)
which are trivially satisfied, because the only vertices that do not vanish at the tree level are \( \Gamma_{111}^\mu = -\Gamma_{222}^\mu = \gamma^\mu \). According to this result, it is convenient to write

\[
\Gamma_{abc}^\mu(q, p) = \gamma^\mu \epsilon_a \delta_{ab} \delta_{ca} + \Lambda_{abc}^\mu(q, p), \tag{3.10}
\]

where \( \Lambda_{abc}^\mu(q, p) \) represents the radiative corrections to the vertex function. In terms of the fermion self-energy matrix \( \Sigma(p) \), the inverse of the complete fermion propagator can be written as \( S^{-1}(p) = [S^{(0)}(p)]^{-1} - \Sigma(p) \). This relation combined with Eqs. (3.8)-(3.10) implies that

\[
q_\mu \Lambda_{abc}^\mu(q, p) = -[\delta_{ab} \Sigma_{ca}(p + q) - \delta_{ac} \Sigma_{ab}(p)]. \tag{3.11}
\]

Explicitly, these relations are

\[
\begin{align*}
q_\mu \Lambda_{111}^\mu(q, p) &= -[\Sigma_{11}(p + q) - \Sigma_{11}(p)], \\
q_\mu \Lambda_{112}^\mu(q, p) &= -\Sigma_{21}(p + q), \\
q_\mu \Lambda_{121}^\mu(q, p) &= \Sigma_{12}(p), \\
q_\mu \Lambda_{122}^\mu(q, p) &= 0, \\
q_\mu \Lambda_{211}^\mu(q, p) &= 0, \\
q_\mu \Lambda_{212}^\mu(q, p) &= \Sigma_{21}(p), \\
q_\mu \Lambda_{221}^\mu(q, p) &= -\Sigma_{12}(p + q), \\
q_\mu \Lambda_{222}^\mu(q, p) &= -[\Sigma_{22}(p + q) - \Sigma_{22}(p)]. \tag{3.12}
\end{align*}
\]

Several important results can be read from the above identities. First, the vertex functions with equal thermal fermion indices but a different photon index (i.e. \( \Lambda_{122}^\mu \) and \( \Lambda_{211}^\mu \)) are transverse. Next, in the limit \( q_\mu \to 0 \) the diagonal vertex functions \( \Lambda_{111}^\mu(0, p) \) and \( \Lambda_{222}^\mu(0, p) \) may be determined as follows in terms of the diagonal components of the self-energy:

\[
\begin{align*}
\Lambda_{111}^\mu(0, p) &= -\frac{\partial \Sigma_{11}(p)}{\partial p_\mu}, \\
\Lambda_{222}^\mu(0, p) &= -\frac{\partial \Sigma_{22}(p)}{\partial p_\mu}. \tag{3.13}
\end{align*}
\]
Furthermore, comparing the second (third) line with the sixth (seventh) line in (3.12) we see that the relations

\[
\Lambda_{112}^\mu(q,p) = -\Lambda_{212}^\mu(q,p + q), \\
\Lambda_{221}^\mu(q,p) = -\Lambda_{121}^\mu(q,p + q),
\]

are satisfied by the longitudinal components of the corresponding elements of \(\Lambda_{\mu abc}^\mu\). Notice that the fermion momentum in the right side is shifted from \(p\) to \(p + q\). The four vertex functions in the last equations are imaginary since \(\Sigma_{12}\) and \(\Sigma_{21}\) in Eqs (3.12) are imaginary quantities.

Equation (3.12) implies that, either the non-diagonal elements of \(\Sigma\) vanish or that the vertex functions with different values for the fermion thermal index have a singularity at \(q^\mu = 0\). Since we know that the self-energy is in general non-diagonal, we conclude that the four vertex function in (3.14) are singular in the \(q_\mu \to 0\) limit. Later on, this fact will be illustrated within a one loop-calculation.

The foregoing relations can be applied to simplify the longitudinal part of the effective casual vertex defined in (2.26). First, if we combine the circling equation (2.25) with (2.24) and (3.14) we can derive the following relation

\[
\text{Im} \Lambda_{111}^\mu(q,p) = ie^{-\beta_\mu/2} [\csc(2\varphi_{p+q})\Lambda_{121}^\mu(q,p + q) - \csc(2\varphi_p)\Lambda_{121}^\mu(q,p)],
\]

for the longitudinal components. Using these results we then find that the retarded vertex functions in Eq. (2.26) can be written as

\[
\text{Re} \Lambda_{R1}^\mu(q,p) = \text{Re} \Lambda_{R2}^\mu(q,p) = \text{Re} \Lambda_{R3}^\mu(q,p) = \text{Re} \Lambda_{111}^\mu(q,p), \\
\text{Im} \Lambda_{R1}^\mu(q,p) = \frac{1}{i} e^{-\beta_\mu/2} [\csc(2\varphi_{p+q})\Lambda_{121}^\mu(q,p + q) + \csc(2\varphi_p)\Lambda_{121}^\mu(q,p)], \\
\text{Im} \Lambda_{R2}^\mu(q,p) = -\text{Im} \Lambda_{R3}^\mu(q,p) = \frac{1}{i} e^{-\beta_\mu/2} [\csc(2\varphi_{p+q})\Lambda_{121}^\mu(q,p + q) - \csc(2\varphi_p)\Lambda_{121}^\mu(q,p)].
\]

These equations are very useful because they show that we need only know one (complex) of eight vertex functions \(\Lambda_{abc}^\mu\) to determine the retarded vertex function of QED. In fact,
according to (3.16) we can choose to calculate $Re \Lambda_{111}^{\mu}(q,p)$ and $Im \Lambda_{121}^{\mu}(q,p)$ to determine the other components of $\Lambda_{abc}^{\mu}$ as well as the three retarded vertex functions. Although, $\Lambda_{121}^{\mu}(q,p)$ is singular for $q_{\mu} \to 0$, we notice that this singularity is cancelled for $\Lambda_{R2}^{\mu}(q,p)$ and $\Lambda_{R3}^{\mu}(q,p)$. Consequently, only the imaginary part of $\Lambda_{R1}^{\mu}(q,p)$ presents a singularity at $q_{\mu} \to 0$.

Referring back to Eqs. (2.22) we see that the real and imaginary parts of $\Sigma_{R}$ are obtained from $Re \Sigma_{11}$ and $\Sigma_{12}$ respectively. Here, we have derived an analogous result for the retarded vertex functions. Indeed, in (3.16) the real and imaginary parts of the retarded vertex functions are expressed in terms of $Re \Lambda_{111}$ and $\Lambda_{121}$ respectively. However, it should be stressed that whereas the result for the fermion self-energy holds in any theory, Eq. (3.16) ensues from the gauge invariance of the theory.

Finally, combining (2.26) with (3.12) the Ward identities for the retarded vertex functions read

$$
q_{\mu} \Lambda_{R1}^{\mu}(q,p) = - [\Sigma_{A}(p + q) - \Sigma_{R}(p)] ,
$$

$$
q_{\mu} \Lambda_{R3}^{\mu}(q,p) = q_{\mu} \Lambda_{R2}^{\mu}(q,p) = - [\Sigma_{R}(p + q) - \Sigma_{R}(p)] ,
$$

(3.17)

where $\Sigma_{A} = \Sigma_{R}^{*}$ is the advanced self-energy. Thus, we notice that the vertex function with the largest time corresponding to the out-going fermion is only related to the retarded self-energy. On the contrary, if the photon vertex has the largest time, then the Ward identity relates the vertex to a combination of the advanced and retarded self-energies. These equations are consistent with the fact that only $\Lambda_{R1}^{\mu}(q,p)$ is singular at $q_{\mu} = 0$.

Now, we explicitly verify that Ward identities are satisfied to the one-loop order. To this order, the contribution to the vertex function, represented by diagram (a) in Fig. ??, is

$$
-i e \Lambda_{abc}^{\mu}(q,p) = - (ie)^{3} \epsilon_{a} \epsilon_{b} \epsilon_{c} \int \frac{d^{4}k}{(2\pi)^{4}} \frac{1}{i} \left[ D^{(0)}_{\rho \sigma} \right]_{bc} (k) \gamma_{\sigma} i S_{ab}^{(0)}(p + q - k) \gamma_{\rho} .
$$

(3.18)

The fermion and photon propagators are the bare propagators in Eqs. (2.7) and (2.9), respectively.
Similarly, the first contribution to the self-energy is shown in diagram (b) of Fig. ?? and reads

$$-i\Sigma_{ab}(q,p) = -(ie)^2\epsilon_a\epsilon_b \int \frac{d^4k}{(2\pi)^4} \frac{1}{i} \left[ D^{(0)} \right]_{ba}^{\rho\sigma}(k) \gamma_\sigma i S^{(0)}_{ba}(\slashed{p} - k) \gamma_\rho. \quad (3.19)$$

(a) The vertex $\Lambda^\mu_{abc}(q,p)$ to the one-loop order. (b) Fermion self-energy to the one-loop order.

The bare fermion propagator in (2.7) satisfies the relation

$$S_{ab}^{(0)}(p) (\slashed{p} - m) = \epsilon_a\delta_{ab} + O \left\{ \epsilon^2, \frac{\epsilon (p^2 - m^2)}{(p^2 - m^2)^2 + \epsilon^2} \right\} \approx \epsilon_a\delta_{ab}. \quad (3.20)$$

Here we can safely take the limit $\epsilon \to 0$, obtaining well defined results. These are all that we need. If we contract the vertex in (3.18) with $q_\mu$, making the replacement $\slashed{q} = (\slashed{q} + \slashed{p} - \slashed{k} - m) - (\slashed{p} - \slashed{k} - m)$ and using Eqs. (3.20) and (3.19) we immediately obtain the result given in Eq. (3.11). Hence, the Ward identities in (3.11 and 3.12) are indeed verified at the one loop-order.

Finally, we turn the attention to the origin of the singular behavior of the vertex functions $\Lambda_{abc}(q,p); b \neq c$ in the limit $q_\mu \to 0$. Let us consider for example the function $\Lambda_{121}(q,p)$,
an easy algebraic manipulation of Eq. \( 3.18 \) shows that the loop integral contains, among others, the factors

\[
\sin^2 \varphi_{p+q-k} \Delta^*(p + q - k) \Delta(p - k) - \cos^2 \varphi_{p+q-k} \Delta(p + q - k) \Delta^*(p - k)
\]  

(3.21)

where \( \Delta(p) = 1/(p^2 - m^2 + i\epsilon) \) is the Feynman propagator. Then, it is obvious that for any value \( q_\mu \neq 0 \) the integral is free of pinch singularities, or \( \Delta^*(k)\Delta(k) \) terms. However, in the \( q_\mu \to 0 \) limit the integral develops a pinch singularity. This is the origin, at the one-loop level, of the singular behavior of the vertex function. Nevertheless, it must be stressed that we have proved that the singular behavior should remain valid to all orders in perturbation theory. A detailed calculation of the one-loop vertex functions \( \Lambda_{abc}(q,p) \) with the corresponding analysis of the structure of the singularity at \( q_\mu = 0 \) will be presented elsewhere [13].

**IV. CONCLUSIONS**

We have studied the discrete and gauge symmetries of Quantum Electrodynamics at finite temperature within the the real-time formalism. In the path integral representation of the generating functional the thermal information is inserted through boundary conditions, which are taken into account in the \( \epsilon \) factor. The presence of this convergence factor leads to an action that is non-local in time, besides being gauge dependent. By demanding the generating functional to be gauge independent we obtain the Ward identities, which relate the eight vertex functions to the elements of the self-energy matrix. Combining the relations obtained from the \( Z_2 \) symmetry and the gauge symmetry we find that (for the longitudinal part) only one out of eight vertex functions is independent. In addition, the retarded vertex functions obey relations similar to those known for the retarded self-energy.

As a consequence of the Ward identities, the vertex functions \( \Lambda_{abc}^\mu \) with \( b \neq c \) are found to be singular when the photon momentum goes to zero. The zero-momentum limit of Feynman amplitudes at finite temperature has generated much discussion. In particular,
it is known that for the thermal self-energy calculated in perturbation theory, the limits \( p_0 \to 0 \) and \(|\vec{p}| \to 0\) are not interchangeable. It has been argued that the nonanalyticity of the self-energy at \( p_\mu = 0 \) should be expected on physical grounds. In particular Weldon \[14\] presented various methods to prove that the self-energy for scalar bosons has a branch cut in \( p_0 \) and consequently the \( p_0 \to 0 \) and \(|\vec{p}| \to 0\) limits do not commute. On the other hand, various attempts have been made to improve the usual perturbation expansion in such a way that the self energy becomes analytic at \( p_\mu = 0 \) \[15\]. Although our results refer to the 3-point function of QED, they may be relevant in the understanding of the zero-momentum limit of Feynman amplitudes at finite temperature. The singularity of the QED vertex function at zero momentum occurs because of the Ward identities, and therefore is not a consequence of the approximation made in a perturbative calculation. However, it should be stressed that in this paper the singularity in the QED vertex functions is predicted based on the Ward identities, whereas this kind of singular behavior is known to exist even for theories without gauge invariance.

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