THE CONFINING BRANCH OF $QCD$

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ABSTRACT

We show that, as a consequence of a physical interpretation based on the Abelian projection and on the $QCD$ string, four-dimensional $QCD$ confines the electric flux if and only if the functional integral in the fiberwise-dual variables admits a hyper-Kahler reduction under the action of the gauge group.

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1 Introduction

It has been conjectured long ago that quark confinement in $QCD$ would arise because the electric flux lines are squeezed into long flux tubes by the condensation of the magnetic charge [1, 2]. This mechanism of confinement is known as the dual Meissner effect, since it is dual, in the sense of electric/magnetic duality, to the confinement of magnetic fluxes, that arises in a type-two superconductor, because of the condensation of the electric charge [3].

Since the electric flux lines would span two-dimensional surfaces embedded into the four-dimensional space-time, the dual Meissner effect leads to an effective theory of $QCD$ in terms of closed strings [4].

The absolute confinement of the electric flux requires indeed that the flux line cannot break into an open string.

More analytically, the string functional integral would arise as the string solution of the Migdal-Makeenko equation [4, 5] in the large-$N$ limit of $QCD$ [7, 8]. This is the string program, that has received recently a new revival, most notably because of the implementation in the string setting of the zigzag symmetry [4].

A distinctive feature of the string program is that the existence of the string is assumed as an ansatz for the solution of the Migdal-Makeenko equation. In fact it is difficult to see how the strings would arise directly in terms of the functional integral over the four-dimensional gauge connections.

A remarkable achievement in this direction was the representation of the partition function of the two-dimensional gauge theory as a sum over branched coverings of the two-dimensional space-time [9]. These coverings are interpreted as the string world sheets, giving evidence in favour of the string solution of pure gauge theories.

Yet this representation is obtained from the exact result for the two-dimensional partition function, without a direct link to gauge configurations in the functional integral.

In an unrelated development, the cotangent bundle of the moduli of holomorphic bundles on a Riemann surface has played a key role in the Seiberg-Witten solution of the Coulomb branch of some four-dimensional supersymmetric gauge theories [10].

Branched covers appear in these solutions as the spectral curves of the charac-
teristic equation associated to a holomorphic one-form that labels cotangent directions to the moduli of holomorphic bundles [1]. The pre-potential, the unique holomorphic function that determines the low-energy effective action of the supersymmetric theory, is constructed by means of the spectral curve.

More precisely, certain submanifolds of the cotangent bundle, that correspond to moduli of representations of the fundamental group of the underlying two-dimensional base manifold, admit an integrable fibration by Jacobians of branched coverings of the base two-dimensional manifold, the Hitchin fibration, that in turn is equivalent to assign the pre-potential.

While no direct link to physical four-dimensional fields may be attributed to these coverings in the framework of the Seiberg-Witten solution, a link to the string program would possibly arise, if the cotangent bundle of unitary connections in two-dimensions could be embedded into the four-dimensional $QCD$ functional integral. Such an embedding was found in [12].

It was found there that the correct variables to define this embedding are neither the four-dimensional gauge connections, $A$, nor their dual variables, $A^D$, but a partial mixing of them, that correspond to a partial or fiberwise duality transformation [12].

The coordinates of the cotangent bundle of unitary connections, $T^*A$, appear naturally as the shift $A^D = A + \Psi$ is performed for two dualized polarizations among the four components of the four-dimensional gauge connection.

In addition, it was found in [13], that there is a dense embedding into the $QCD$ functional integral, of an elliptic fibration of the moduli space of parabolic $K(D)$ pairs into (an elliptic fibration of) the quotient of the cotangent bundle by the action of the gauge group.

The last space admits a Hitchin fibration by the moduli of line bundles over branched spectral covers, thus giving a dense embedding of these objects into the $QCD$ functional integral.

While in [13] the integrability properties of the Hitchin fibration were used to reduce the problem of computing the functional integral in the large-$N$ limit to the evaluation of the saddle-point of a certain effective action that contains the Jacobian of the change of variables to the collective field of the Hitchin fibration, in this paper we shall address the following, more qualitative issue, that relates to the string program.

What is the locus in the functional integral of the confining branch of $QCD$, that is, what is the locus in the moduli space of parabolic $K(D)$ pairs, whose
image by the Hitchin map contains only Riemann surfaces spanned by closed strings?

A partial answer to this question was given in [14]. In [14] a physical interpretation of the occurrence of Hitchin bundles in the fiberwise dual functional integral was given, in the light of 't Hooft concept of Abelian projection [15]. This interpretation identifies the branch points of the spectral covers as magnetic monopoles and the parabolic points as electric charges. Since confinement requires magnetic condensation and 't Hooft alternative excludes electric condensation, the confining branch is the locus, in the parabolic $K(D)$ pairs, whose image by the Hitchin map has no parabolic singularity on the spectral cover [14], a not completely trivial condition.

It should be noticed that this idea is in complete analogy with the two-dimensional case [9], in which the partition function is localized on branched coverings of the base compact space-time, without parabolic points. In fact the occurrence of parabolic points would imply the presence in the vacuum to vacuum amplitudes of string diagrams with the topology of open strings, a situation that it is appropriate to the Coulomb rather than the confinement phase. This last statement may be exemplified thinking to a sphere with two parabolic points as a topological cylinder, a vacuum diagram of an open string theory.

We will find in this paper that the confinement locus is characterized precisely by the condition that the residues of the Higgs current, $\Psi$, on the parabolic divisor be nilpotent.

This condition turns out to be equivalent to the existence of a (dense in the large-$N$ limit) hyper-Kahler reduction of the cotangent bundle of unitary connections under the action of the gauge group. The confining branch of $QCD$ is, therefore, the hyper-Kahler locus of the Hitchin fibration of parabolic bundles, embedded in the $QCD$ path integral as prescribed by fiberwise duality. On the other side, this is precisely the locus for which spectral covers with the topology of closed string diagrams, but not open ones, occur in the functional integral. The dual mechanism of superconductivity and the string interpretation are therefore compatible, as it should be, and as it has been for long time believed [4].

One more comment. It is a rather strange fact that the same or analogue objects, that are used to construct the Seiberg-Witten solution of four-dimensional SUSY theories in the Coulomb branch, appear here as giving
rise to a physical string interpretation of the $QCD$ functional integral, with an associated hyper-Kahler structure but no supersymmetry.

In fact we think that the explanation of this fact has much to do with duality as opposed to supersymmetry.

The Seiberg-Witten solution starts from supersymmetry, through the structure theorem for the low-energy effective action, as determined by the pre-potential, and ends up with a non-linear geometric realization of the Abelian electric magnetic/duality of the effective theory in the Coulomb branch, in terms of a Legendre transformation of the pre-potential \[10\].

We start instead from the non-Abelian duality of the microscopic theory, as defined by the functional integral, to gain, by means of fiberwise duality and the embedding of parabolic bundles, control over the large-$N$ limit \[13\] and a mathematical realization of the dual Meissner effect \[14\] at the same time.

2 The nilpotent condition

In this section we show that the spectral covers that are in the image by the Hitchin map of parabolic $K(D)$ pairs have no parabolic divisor if and only if the levels of the non-hermitian moment maps are nilpotent on each point of the parabolic divisor.

This in turn is a necessary and sufficient condition for the moduli space of parabolic $K(D)$ pairs to admit a hyper-Kahler structure. In \[10, 13\] a special name was used to characterize this closed subspace: parabolic Higgs bundles.

In any case the confinement criterium of this paper explains the physical meaning of the hyper-Kahler structure, a mathematical condition whose meaning was suspected to be physically relevant but not elucidated in \[13\]. Indeed, there it was argued that the two cases of the parabolic $K(D)$ pairs and of the parabolic Higgs bundles present equivalent difficulties from the point of view of solving the large-$N$ limit, in fact differing by contributions of order of $\frac{1}{N}$. We now argue that parabolic Higgs bundles correspond to the confining branch of $QCD$ in the fiberwise-dual variables.

The functional integral for $QCD$ in \[13\] is defined in terms of the variables $(A_z, A_{\bar{z}}, \Psi_z, \Psi_{\bar{z}})$, obtained by means of a fiberwise duality transformation from $(A_z, A_{\bar{z}}, A_u, A_{\bar{u}})$, where $(z, \bar{z}, u, \bar{u})$ are the complex coordinates on the
product of two two-dimensional tori, over which the theory is defined.

\((A_z, A_{\bar{z}}, \Psi_z, \Psi_{\bar{z}})\) define the coordinates of an elliptic fibration of \(T^*A\), the cotangent bundle of unitary connections on the \((z, \bar{z})\) torus with the \((u, \bar{u})\) torus as a base.

The set of pairs \((A, \Psi)\) that are solutions of the following differential equations (elliptically fibered over the \((u, \bar{u})\) torus) is embedded into the space of parabolic \(K(D)\) pairs [14, 13]:

\[
F_A - i\Psi \wedge \Psi = \frac{1}{|D|} \sum_p \mu_0 \delta_p dz \wedge d\bar{z}
\]

\[
\partial_A \psi = \frac{1}{|D|} \sum_p \mu_\rho \delta_p d\bar{z} \wedge d\bar{z}
\]

\[
\partial_{\bar{A}} \bar{\psi} = \frac{1}{|D|} \sum_p \bar{\mu}_p \delta_p dz \wedge dz
\]

(1)

where \(\delta_p\) is the two-dimensional delta-function localized at \(z_p\) and \((\mu_0, \mu_p, \bar{\mu}_p)\) are the set of levels for the moment maps [13]. The space of parabolic \(K(D)\) pairs consists of a parabolic bundle with a holomorphic connection \(\bar{\partial}_A\) and a parabolic morphism \(\psi\). Eq.(1) defines a dense stratification of the functional integral over \(T^*A\) because the set of levels is dense everywhere in function space, in the sense of the distributions, as the divisor \(D\) gets larger and larger.

According to Hitchin [17], there is a Hitchin fibration of parabolic \(K(D)\) pairs, defined by U(1) bundles over the following spectral cover:

\[
Det(\lambda I - \Psi_z) = 0
\]

(2)

The spectral cover depends only from the eigenvalues of \(\Psi_z\). The condition that the spectral cover has no parabolic point is therefore the condition that the eigenvalues of \(\Psi_z\) have no poles. We notice that the residues of the poles of \(\Psi_z\) are determined by the levels of the non-hermitian moment maps. In fact \(\Psi_z\) can be made meromorphic with residue at the point \(p\) conjugated to the level \(\mu_p\) by means of a gauge transformation \(G\) in the complexification of the gauge group, that gauges to zero the connection \(\bar{A}_z\), fiberwise:

\[
\partial \psi - \frac{1}{|D|} \sum_p G\mu_p G^{-1} \delta_p dz \wedge d\bar{z} = 0
\]

\[
\partial \bar{\psi} - \frac{1}{|D|} \sum_p \bar{G}^{-1} \bar{\mu}_p \bar{G} \delta_p d\bar{z} \wedge dz = 0
\]

(3)
From this equation it follows that the residues of the eigenvalues of $\Psi z$ are proportional to the eigenvalues of $\mu_p$. If the eigenvalues of $\psi$ have no poles on the covering, $\mu_p$ must have zero eigenvalues and therefore must be nilpotent and vice versa, that is the conclusion looked forward.

There is however an apparent puzzle. Though the eigenvalues of $\psi$ cannot have poles on the covering if the levels of the non-hermitian moment maps are nilpotent, the traces of powers of $\Psi z$, that are expressed through symmetric polynomials in the eigenvalues, certainly are meromorphic functions on the torus. How can this happen if the eigenvalues of $\psi$ have no poles on the covering? The answer is the following, as we have found by a direct check in the $SU(2)$ case. If $\mu_p$ is nilpotent, the eigenvalues of $\Psi z$ have singularities that are not parabolic but that look in the coordinates of the $z$ torus branched singularities, for example $z^{-\frac{1}{2}}$. However we should remind the reader that the eigenvalues of $\psi$ are really differentials on the covering. Therefore $z^{-\frac{1}{2}}$ should be really interpreted as $z^{-\frac{1}{2}}dz$, that is $d(z^{1/2})$, that is, in fact, smooth on a simply branched covering. There is no singularity on the covering.

Yet, symmetric powers of the eigenvalues of $\Psi z$ may have meromorphic singularities on the torus.

It remains to show that if the residue of $\psi$ is nilpotent the quotient is hyper-Kähler. This is a known result \cite{16}. This concludes our proof.

In fact a slightly stronger statement holds. If the residues of the Higgs field are nilpotent, Eq.\((1)\) can be interpreted as the vanishing condition for the moment maps of the action of the compact $SU(N)$ gauge group on the pair $(A, \Psi)$ and on the cotangent space of flags \cite{18}. The quotient under the action of the compact gauge group of the set:

\[
F_A - i\Psi \wedge \Psi - \frac{1}{|D|} \sum_p \mu_p^0 \delta_p dz \wedge d\bar{z} = 0
\]

\[
\bar{\partial}_A \psi - \frac{1}{|D|} \sum_p n_p \delta_p dz \wedge d\bar{z} = 0
\]

\[
\partial_A \bar{\psi} - \frac{1}{|D|} \sum_p \bar{n}_p \delta_p d\bar{z} \wedge dz = 0
\]

with fixed eigenvalues of the hermitian moment map is, by a general result \cite{13}, the same as the quotient defined by the complex moment maps:

\[
\bar{\partial}_A \psi - \frac{1}{|D|} \sum_p n_p \delta_p dz \wedge d\bar{z} = 0
\]
\[ \partial_A \bar{\psi} - \frac{1}{|D|} \sum_p \bar{n}_p \delta_p d \bar{z} \wedge dz = 0 \] (5)

under the action of the complexification of the gauge group.

3 Conclusions

Our conclusion is that if QCD confines the electric charge the functional integral in the fiberwise dual-variables defined in [12, 13] must be localized on the hyper-Kahler locus of parabolic \( K(D) \) pairs, the parabolic Higgs bundles. This space is characterized by a nilpotent residue of the Higgs current. These are precisely the parabolic \( K(D) \) pairs whose image by the Hitchin map contain spectral covers arbitrarily branched, but with no parabolic points. The physical interpretation is that there is a monopole condensate in the vacuum but no electric condensate and only closed electric strings occur into vacuum to vacuum diagrams.

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