BOGOMOLOV’S INEQUALITY FOR PRODUCT TYPE VARIETIES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We prove Bogomolov’s inequality for semistable sheaves on product type varieties in arbitrary characteristic. This gives the first examples of varieties with positive Kodaira dimension in positive characteristic on which Bogomolov’s inequality holds for semistable sheaves of any rank. The key ingredient in the proof is a high rank generalization of the slope inequality established by Xiao and Cornalba-Harris. This Bogomolov’s inequality is applied to study the positivity of linear systems and semistable sheaves and construct Bridgeland stability conditions on product type surfaces in positive characteristic. We also give some new counterexamples to Bogomolov’s inequality and pose some open questions.

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1. Introduction

Throughout this paper, we fix an algebraically closed field $k$ of arbitrary characteristic. Let $X$ be a smooth projective variety defined over $k$ with $\dim X \geq 2$, and let $H$ be an ample divisor on $X$. The famous Bogomolov’s inequality says that if $\text{char}(k) = 0$, then

$$\Delta(E)H^{\dim X-2} = (\text{ch}^1_1(E) - 2\text{ch}_0(E)\text{ch}_2(E))H^{\dim X-2} \geq 0,$$

for any $\mu_H$-semistable sheaf $E$ on $X$. It was proved by Bogomolov [4] when $\dim X = 2$, and it can be easily generalized to higher dimensional case by the Mehta-Ramanathan restriction theorem.

In the case of $\text{char}(k) > 0$, Langer [18] proved that the same inequality holds for strongly $\mu_H$-semistable sheaves. Mehta and Ramanathan [20] showed that if
X satisfies $\mu_H^+(\Omega_X^1) \leq 0$, then all $\mu_H$-semistable sheaves on $X$ are strongly $\mu_H$-semistable. Thus Bogomolov’s inequality holds on such an $X$. One notices that the Kodaira dimension of this $X$ is non-positive.

In general it is well known that Bogomolov’s inequality fails for semistable sheaves in positive characteristic. And it is only known to be held for semistable sheaves of small rank on some special varieties. For example, Shepherd-Barron [27] proved that Bogomolov’s inequality holds for rank two semistable sheaves on surfaces which are neither quasi-elliptic with $\kappa(X) = 1$ nor of general type, and Langer [19] showed this inequality holds for any semistable sheaf $E$ with $\text{rk}E \leq \text{char}(k)$ on a variety which can be lifted to the ring of Witt vectors of length 2. However, in this paper, we prove that Bogomolov’s inequality holds for semistable sheaves of arbitrary rank on product type varieties in any characteristic.

**Definition 1.1.** Let $X$ be a smooth projective variety defined over $k$.

1. We say that $X$ is a product type variety if there exist smooth projective curves $C_1, \cdots, C_n$ defined over $k$ and a finite separable surjective morphism $f : C_1 \times \cdots \times C_n \to X$.
2. A divisor $H$ on the product type variety $X$ is called a product type ample divisor if $f^*H$ can be written as $f^*H = p_1^*A_1 + \cdots + p_n^*A_n$ for some ample divisors $A_i$ on $C_i$, where $p_i : C_1 \times \cdots \times C_n \to C_i$ is the projection for $i = 1, \cdots, n$.

A simple example of product type varieties is the symmetric product of a curve. The varieties isogenous to a product of curves introduced by Catanese [8] are other important examples. See also [2, 10] for a huge number of interesting examples called product-quotient varieties. The product-quotient varieties are our product type varieties if they are smooth. Our main results are the following two theorems.

**Theorem 1.2.** Let $X$ be a product type variety of dimension $n$ and $H$ a product type ample divisor on $X$. Then for any $\mu_H$-semistable sheaf $E$ on $X$, we have $H^{n-2}\Delta(E) \geq 0$.

The product type assumption on the ample divisor can be dropped when $n = 2$:

**Theorem 1.3.** Let $S$ be a product type surface and $H$ any ample divisor on $S$. Then for any $\mu_H$-semistable sheaf $E$ on $S$, we have $\Delta(E) \geq 0$.

In characteristic zero there are several proofs of Bogomolov’s inequality. The first proof is due to Bogomolov [4]. The key ingredient in his proof is that the tensor power of a semistable vector bundle is still semistable. The second one is given by Gieseker [12] using reduction mod $p$ and estimating the dimension of the space of sections of the Frobenius pull back of a semistable sheaf. The third proof is transcendental, using the Kobayashi-Hitchin correspondence between the polystability and the existence of Hermite-Einstein metric (see [17]).

Unfortunately, all these proofs do not work in positive characteristic. And our proof of the above theorems is totally different from theirs. The strategy of the proof is the following. Firstly, we use a result of [26] to show the equivalence between semistability and Hilbert stability for locally free sheaves on curves. Then we generalize the method of Cornalba-Harris [9] to prove a high rank slope inequality for relative semistable sheaves on a fibration (Theorem 4.2). This inequality implies that Moriwaki’s relative Bogomolov’s inequality [21] holds for trivial fibrations in
any characteristic (Corollary 4.4). By the techniques of the changes of polarizations in [15, Appendix 4.C], one obtains our main theorems. We exhibit the strategy of our proof in the following chain of implications.

\[
\text{Semistability} \implies \text{Hilbert stability} \implies \text{High rank slope inequality} \implies \text{Relative Bogomolov’s inequality} \implies \text{Bogomolov’s inequality}
\]

**Applications and open questions.** Bogomolov’s inequality has many interesting applications, such as the positivity of adjoint linear systems (see [25, 3]), and vanishing theorems for semistable sheaves (see [29]). By Theorem 1.3, all these related results automatically hold for product type surfaces in positive characteristic (see Theorem 6.1 and 6.2).

The authors in [6, 1] showed that Bogomolov’s inequality for semistable sheaves of any rank can be used to construct Bridgeland stability conditions on surfaces. Hence Bridgeland stability condition always exists on surfaces in characteristic zero. It is natural to ask:

**Question 1.4.** For a smooth projective surface in positive characteristic, is there any Bridgeland stability condition on it?

Theorem 1.3 can give an affirmative answer of this question for product type surfaces (see Theorem 6.5). They are the first examples of Bridgeland stability conditions on some surfaces with positive Kodaira dimension in positive characteristic. Because of the existence of counterexamples to Bogomolov’s inequality, it seems that a positive answer of Question 1.4 needs a completely new construction. In [30], the author also uses Theorem 1.2 to prove the existence of Bridgeland stability conditions on some threefolds of general type.

Theorem 1.2 and 1.3 inspire us to construct some new counterexamples to Bogomolov’s inequality (Theorem 7.1). Langer’s [19, Theorem 1] and our counterexamples lead us to pose the below conjecture:

**Conjecture 1.5.** Let \( S \) be a smooth projective minimal surface defined over \( k \) with \( c_1(S)^2 \leq 2c_2(S) \). Let \( H \) be an ample divisor on \( S \). If \( S \) can be lifted to the ring \( W_2(k) \) of Witt vectors of length 2, then \( \Delta(E) \geq 0 \) for any \( \mu_H \)-semistable torsion free sheaf \( E \).

We notice the condition \( c_1^2 \leq 2c_2 \) in Conjecture 1.5 is satisfied for surfaces isogenous to a product of curves.

**Organization of the paper.** Our paper is organized as follows. In Section 2 we review basic notions and properties of the classical stability for coherent sheaves. Some results of Butler [7] and M. Teixidor [33] have been generalized to any characteristic which may be of interest in other contexts. Then in Section 3 we recall the definition and properties of Hilbert stability for locally free sheaves on curves. In Section 4 we show the high rank slope inequality for relative semistable sheaves (Theorem 4.2) and the relative Bogomolov’s inequality for trivial fibrations (Corollary 4.4). We prove Theorem 1.2 and 1.3 in Section 5. The applications of our main theorems will be discussed in Section 6 (Theorem 6.1, 6.2 and 6.5). In Section 7 we give new counterexamples to Bogomolov’s inequality in positive characteristic (Theorem 7.1).
Notation. We work over an algebraically closed field $k$ of arbitrary characteristic in this paper, and write $\text{char}(k)$ for its characteristic. Let $X$ be a smooth projective variety defined over $k$. We denote by $\text{D}^b(X)$ its bounded derived category of coherent sheaves and by $\kappa(X)$ its Kodaira dimension. $K_X$ and $\omega_X$ denote the canonical divisor and canonical sheaf of $X$, respectively. When $\dim X = 1$, we write $g(X)$ for the genus of the curve $X$. For a morphism $f : X \to Y$ of smooth varieties, we denote by $K_{X/Y}$ the relative canonical divisor $K_X - f^*K_Y$ of $f$. For a triangulated category $\mathcal{D}$, we write $\text{K}(\mathcal{D})$ for the Grothendieck group of $\mathcal{D}$.

We write $\text{ch}(E)$ and $c(E)$ for the Chern character and Chern class of a complex $E \in \text{D}^b(X)$, respectively. We also write $H^j(F)$ ($j \in \mathbb{Z}_{\geq 0}$) for the cohomology groups of a sheaf $F \in \text{Coh}(X)$ and $h^j(F)$ for the dimension of $H^j(F)$. For a sheaf $G \in \text{Coh}(X)$, we denote by $G^* := \text{Hom}(G, \mathcal{O}_X)$ the dual sheaf of $G$ and by $\Delta(G) := \text{ch}_2^2(G) - 2 \text{ch}_0(G) \text{ch}_2(G)$ the discriminant of $G$. Given a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\mathbb{R}z$ and $3z$, respectively.

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2. Slope stability for sheaves

In this section, we will review some basic properties of slope stability for coherent sheaves, and generalize the tensor product theorem and Butler’s theorem to arbitrary characteristic.

Let $X$ be a smooth projective variety with $\dim X = n$ defined over $k$. Let us fix a collection of nef divisors $H_1, \ldots, H_{n-1}$ on $X$. We define the slope $\mu_{H_1, \ldots, H_{n-1}}$ of a coherent sheaf $E \in \text{Coh}(X)$ by

$$
\mu_{H_1, \ldots, H_{n-1}}(E) = \begin{cases} 
+\infty, & \text{if } \text{rk}(E) = 0, \\
\frac{H_1 \cdots H_{n-1} \text{ch}_1(E)}{\text{rk}(E)}, & \text{otherwise.}
\end{cases}
$$

We write $\mu$ for $\mu_{H_1, \ldots, H_{n-1}}$ if there is no confusion. When $H_1 = \cdots = H_{n-1} = H$, we also write $\mu_H$ for $\mu_{H_1, \ldots, H_{n-1}}$.

Definition 2.1. A coherent sheaf $E$ on $X$ is $\mu$-(semi)stable (or (semi)stable) if, for all non-zero subsheaves $F \to E$, we have

$$
\mu(F) < (\leq) \mu(E/F).
$$

Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to $\mu$-stability exist in $\text{Coh}(X)$: given a non-zero sheaf $E \in \text{Coh}(X)$, there is a filtration

$$
0 = E_0 \subset E_1 \subset \cdots \subset E_m = E
$$

such that: $G_i := E_i/E_{i-1}$ is $\mu$-semistable, and $\mu(G_1) > \cdots > \mu(G_m)$. We set $\mu^+(E) := \mu(G_1)$ and $\mu^-(E) := \mu(G_m)$.

Lemma 2.2. If $E$ and $F$ are coherent sheaves on $X$ with $\mu^-(E) > \mu^+(F)$, then $\text{Hom}(E,F) = 0$.

Proof. See [15, Lemma 1.3.3].
2.1. **Tensor product theorem.** Let \( C \) be a smooth projective curve defined over \( k \). It is well known that if \( \text{char}(k) = 0 \), then

\[
\mu^+(E \otimes F) = \mu^+(E) + \mu^+(F),
\]

for any locally free sheaves \( E \) and \( F \) on \( C \) ([17] Lemma 2.5). This result fails when \( \text{char}(k) > 0 \) (see [11] for counterexamples). However, we have the following theorem:

**Theorem 2.3.** Let \( E \) and \( F \) be locally free sheaves on \( C \). Then

1. \( 0 \leq \mu^+(E \otimes F) - \mu^+(E) - \mu^+(F) \leq g(C) \);
2. \(-g(C) \leq \mu^-(E \otimes F) - \mu^-(E) - \mu^-(F) \leq 0 \);
3. \( \mu^-(\otimes^m E) \geq m\mu^-(E) - (m - 1)g(C) \), for any positive integer \( m \).

**Proof.** It is clear that \( \mu^+(E \otimes F) \geq \mu^+(E) + \mu^+(F) \). One needs to show

\[
\mu^+(E \otimes F) \leq \mu^+(E) + \mu^+(F) + g(C).
\]

Set \( l \) be the smallest integer that is greater than or equal to \(-\mu^+(E) - \mu^+(F) - 1\). Take a line bundle \( L \) on \( C \) with \( \text{deg} L = l \). One sees that

\[
-1 \leq \mu^+(E) + \mu^+(F) + l = \mu^+(E) + \mu^+(F \otimes L) < 0.
\]

We will prove that \( \mu^+(E \otimes F \otimes L) \leq g(C) - 1 \). Let \( G \) be a subsheaf of \( E \otimes F \otimes L \) such that \( \mu(G) = \mu^+(E \otimes F \otimes L) \). By the Riemann-Roch theorem, it follows that \( h^0(G) \geq \text{deg} G - \text{rk} G(g(C) - 1) \). Hence, if \( \mu^+(E \otimes F \otimes L) > g(C) - 1 \), we have

\[
\frac{h^0(G)}{\text{rk} G} \geq \mu(G) - g(C) + 1 > 0.
\]

This implies

\[
\text{hom}(E^*, F \otimes L) = h^0(E \otimes F \otimes L) \geq h^0(G) > 0.
\]

By Lemma 2.2 one obtains \( \mu^-(E^*) \leq \mu^+(F \otimes L) \). Then \( \mu^+(E) + \mu^+(F \otimes L) \geq 0 \), since \( \mu^+(E) = -\mu^-(E^*) \). It contradicts (2.1). Thus

\[
\mu^+(E \otimes F \otimes L) \leq g(C) - 1 \leq g(C) + \mu^+(E) + \mu^+(F \otimes L).
\]

This implies the first conclusion.

Since \( \mu^-(E) = -\mu^+(E^*) \), \( \mu^-(F) = -\mu^+(F^*) \) and \( \mu^-(E \otimes F) = -\mu^+(E^* \otimes F^*) \), one can immediately get (2) from (1). The assertion (3) follows from (2) by induction on \( m \). \( \square \)

2.2. **Butler’s theorem.** The following is a generalization of [7] Theorem 2.1] to arbitrary characteristic.

**Theorem 2.4.** Let \( E \) and \( F \) be locally free sheaves on a smooth projective curve \( C \) defined over \( k \). Assume \( \mu^-(E) \geq 3g(C) \) and \( \mu^-(F) \geq 3g(C) \). Then the multiplication map

\[
H^0(E) \otimes H^0(F) \to H^0(E \otimes F).
\]

is surjective.

**Proof.** Since \( \mu^-(E) \geq 3g(C) \), by [7] Lemma 1.12], one sees that \( E \) is generated by global sections. Hence the evaluation map of \( E \) determines an exact sequence:

\[
0 \to M_E \to H^0(E) \otimes \mathcal{O}_C \to E \to 0.
\]
From [7, Corollary 1.3], it follows that
\[ \mu^-(M_E) \geq \frac{-\mu^-(E)}{\mu^-(E) - g(C)} \geq -\frac{3}{2}. \]

By Lemma 2.3 we have
\[ \mu^-(M_E \otimes F) \geq 2g(C) - \frac{3}{2} > 2g(C) - 2. \]

This implies
\[ h^1(M_E \otimes F) = \text{hom}(M_E \otimes F, \omega_C) = 0. \]

Tensoring sequence (2.2) by $F$ and taking cohomology proves the theorem. □

From this, we can deduce the following generalization of a result in [33].

**Theorem 2.5.** Let $E$ be a semistable locally free sheaf of rank $r$ on a smooth projective curve $C$ defined over $k$. If $\mu(E) \geq 3g(C)$, then the map
\[ \wedge^r H^0(E) \to H^0(\wedge^r E) \]
is surjective.

**Proof.** From Theorem 2.3 and 2.4 one infers that the map
\[ \otimes^r H^0(E) \to H^0(\otimes^r E) \]
is surjective. On the other hand, the canonical map $\tau: \otimes^r E \to \wedge^r E$ gives rise to an exact Koszul complex:
\[ \cdots \to \wedge^2(\otimes^r E) \otimes \wedge^r E^* \to \otimes^r E \xrightarrow{\tau} \wedge^r E \to 0. \]

Since $\wedge^2(\otimes^r E) \otimes \wedge^r E^*$ is a quotient of $(\otimes^{2r} E) \otimes \wedge^r E^*$, one sees that
\[
\mu^-(\wedge^2(\otimes^r E) \otimes \wedge^r E^*) \begin{cases} 
geq \mu^-(\otimes^{2r} E) - \deg E \\ \geq 2r\mu^-(E) - (2r - 1)g(C) - \deg E \\ = r\mu(E) - (2r - 1)g(C) \\ \geq (r + 1)g(C) \\ \geq 2g(C). \end{cases}
\]

This infers $H^1(\wedge^2(\otimes^r E) \otimes \wedge^r E^*) = 0$. Thus $H^1(\ker \tau) = 0$ and the map
\[ H^0(\otimes^r E) \to H^0(\wedge^r E) \]
is surjective. It follows that the composite map
\[ \otimes^r H^0(E) \to H^0(\wedge^r E) \]
is also surjective. By the universal property of the exterior algebra, we obtain the desired surjection. □

**Remark 2.6.** One sees that the bound $3g(C)$ in Theorem 2.4 can be slightly improved by its proof. But we don’t need this.
3. Hilbert stability for locally free sheaves

Throughout this section, we let $E$ be a semistable locally free sheaf on a smooth projective curve $C$ defined over $k$ with $\text{deg } E = d$ and $\text{rk } E = r$. We further assume that $\mu(E) \geq 3g(C)$ and set $V = H^0(E)$. We will recall the definition and some basic properties of Hilbert stability for such an $E$ on $C$.

By our assumptions, one sees that the evaluation map $V \otimes O_C \rightarrow E$ is surjective, and it defines a morphism $f : C \rightarrow G(V, r)$, here $G(V, r)$ is the Grassmannian of $r$ dimensional quotients of $V$. Let $p : G(V, r) \hookrightarrow \mathbb{P}(\wedge^r V)$ be the Plücker embedding, where $\mathbb{P}(\wedge^r V)$ is the projective space of 1 dimensional quotients of $\wedge^r V$.

**Lemma 3.1.** The morphisms $f$ and $p \circ f$ are embeddings.

**Proof.** Note that the morphism $p \circ f : C \rightarrow \mathbb{P}(\wedge^r V)$ is defined by the surjection $\wedge^r V \otimes O_C \rightarrow \wedge^r E$.

Since $\mu(E) \geq 3g(C)$, one sees $\wedge^r E$ is a very ample line bundle, and the map $\wedge^r V \rightarrow H^0(\wedge^r E)$ is surjective by Theorem 2.5. We deduce that $p \circ f$ is an embedding. Thus $f$ is an embedding. □

We let $I_C$ be the ideal sheaf of $C$ in $\mathbb{P}(\wedge^r V)$. Since $H^1(I_C(m)) = 0$ for $m$ large enough, from the short exact sequence

$$0 \rightarrow I_C(m) \rightarrow O_{\mathbb{P}(\wedge^r V)}(m) \rightarrow O_C(m) \rightarrow 0,$$

one sees that the map $H^0(O_{\mathbb{P}(\wedge^r V)}(m)) \rightarrow H^0(O_C(m))$ is surjective. Hence we obtain a surjection

$$\psi_m : S^m(\wedge^r V) \rightarrow H^0((\det E)^{\otimes m}).$$

Let $P(m) = h^0((\det E)^{\otimes m}) = dm - g(C) + 1$. One final obtains a map

$$\varphi_m : \wedge^{P(m)} S^m(\wedge^r V) \rightarrow \wedge^{P(m)} H^0((\det E)^{\otimes m}) \cong k.$$

It gives a point

$$[\varphi_m] \in \mathbb{P}\left(\wedge^{P(m)} S^m(\wedge^r V)\right)$$

**Definition 3.2.** We say $(C, E)$ is $m$-Hilbert stable (resp., semistable) if the point $[\varphi_m]$ is stable (resp., semistable) under the induced action of $SL(V)$, i.e., $[\varphi_m]$ has closed orbit and finite stabilizer (resp., 0 is not in the closure of the orbit of $[\varphi_m]$).

We say $(C, E)$ is Hilbert stable (resp., semistable) if it is $m$-Hilbert stable (resp., semistable) for all $m$ sufficiently large.

Recall that a necessary and sufficient condition for the semistability of $[\varphi_m]$ is the existence of a $SL(V)$-invariant non-constant homogeneous polynomial

$$h \in S^N \left(\wedge^{P(m)} S^m(\wedge^r V)\right)$$

such that $(S^N \varphi_m)(h) \neq 0$. 

**Theorem 3.3.** There is a constant $d_0 = d_0(r, g(C))$ so that for each $d \geq d_0$, there exists a constant $m_0 = m_0(d, r, g(C))$ such that if $m \geq m_0$, then $(C, E)$ is $m$-Hilbert stable (resp., semistable) if and only if $E$ is stable (resp., semistable).

**Proof.** The required conclusion was first proved for rank 2 by Gieseker and Morrison in [13, Theorem 1.1] (see also [34] for a different proof). For general rank, one of the implications was proved in [35, Proposition 2.2], and the equivalence was given by Schmitt [26] in characteristic zero. The characteristic 0 assumption in Schmitt’s proof is only used in [26, Corollary] which has been generated to arbitrary characteristic case in Theorem [26]. Hence Schmitt’s proof works in any characteristic. \hfill \Box

4. High rank slope inequalities

In this section, we will prove the high rank slope inequality and relative Bogomolov’s inequality for a trivial fibration. Throughout this section, we let $\pi : X \to Y$ be a flat and surjective morphism of projective varieties over $k$ with $\dim X = n$ and $\dim Y = n - 1$. Let $y$ be a general point of $Y$. We further assume that the general fiber $X_y := X \times_Y \text{Spec}(k(y))$ is a connected smooth curve of genus $g$. For a sheaf $E$ on $X$, we write $E_y$ for the restriction of $E$ to $X_y$.

The following is a high rank generalization of [9, Theorem 1.1] (see also [28]).

**Theorem 4.1.** Let $E$ be a torsion free sheaf on $X$ such that $E_y$ is semistable and $\mu(E_y) \geq 3g$. Suppose that $(X_y, E_y)$ is $m$-Hilbert semistable for some positive integer $m$, and both $\pi_*E$ and $\pi_*(\det E)^m$ are locally free. Set $\det E = L$, $\rk E = r$ and $P(m) = \rk \pi_*(L^m)$. Let $D_m(E)$ be the line bundle

$$ (\det \pi_*(L^m))^{\rk \pi_*E} \otimes (\det \pi_*(E))^{-P(m)r}. $$

Then there is a positive integer $N$ such that $(D_m(E))^N$ is effective.

**Proof.** We consider the natural morphism $\gamma^m : S^m(\wedge^r \pi_*E) \to \pi_*(L^m)$. By our assumptions, one sees that the fibre of $\gamma^m$ at $y$,

$$ \gamma^m_{y} : S^m(\wedge^r H^0(E_y)) \to H^0(L^m_y), $$

is surjective. Set $V = H^0(E_y)$. Since $(X_y, E_y)$ is $m$-Hilbert semistable, there exists a $SL(V)$-invariant non-constant degree $N_0$ homogeneous polynomial

$$ f \in S^{N_0} \left( \bigwedge^{P(m)} S^m(\wedge^r V) \right) $$

such that

$$ 0 \neq \left( S^{N_0} \bigwedge^{P(m)} \gamma^m_{y} \right)(f) \in \left( \det H^0(L^m_y) \right)^{N_0}. \quad (4.1) $$

We may assume that $\dim V$ divides $N_0$, and set $N_0 = N \dim V$. One sees that the one dimensional linear subspace $W$ generated by $f$ in $S^{N_0} \left( \bigwedge^{P(m)} S^m(\wedge^r V) \right)$ is invariant under the action of $GL(V)$.

Let

$$ \rho : GL(V) \to GL \left( \bigwedge^{P(m)} S^m(\wedge^r V) \right) $$

be the standard representation and $\sigma : GL(V) \to GL(W)$ the restriction representation from $S^{N_0}\rho$. Composing the transition functions of $\pi_*E$ with $\rho$ (resp., $\sigma$), one can construct a new locally free sheaf $(\pi_*E)_\rho$ (resp., $(\pi_*E)_{\sigma}$) and an injective morphism

$$(\pi_*E)_{\sigma} \hookrightarrow S^{N_0}(\pi_*E)_\rho.$$ 

Since $(\pi_*E)_\rho = \wedge^{P(m)}S^m(\wedge^r\pi_*E)$, composing this injection with $S^{N_0}\wedge^{P(m)}S^m_y$, we get a morphism $\tilde{\gamma} : (\pi_*E)_{\sigma} \to (\det \pi_*(L^m))^{N_0}$. From property (4.1) and our construction, it follows that $\tilde{\gamma}$ is non-zero. It remains to compute $(\pi_*E)_\sigma$ explicitly.

Take an element $A \in GL(V)$. We can write

$$A = (\det A)^{\frac{1}{\dim V}} B,$$

where $B \in SL(V)$. The action of $A$ on $f$ is given by the following:

$$\sigma(A)f = S^{N_0}\rho((\det A)^{\frac{1}{\dim V}} f(B))$$

$$= S^{N_0}\rho \left( (\det A)^{\frac{1}{\dim V}} \text{id}_V \right) (S^{N_0}\rho(B)f)$$

$$= S^{N_0}\rho \left( (\det A)^{\frac{1}{\dim V}} \text{id}_V \right) f$$

$$= (\det A)^{\frac{N_0P(m)mr}{\dim V}} f$$

This implies that $(\pi_*E)_\sigma = (\det \pi_*E)^{N_0P(m)mr}$. Hence the line bundle

$$(\det \pi_*(L^m))^{N_0} \otimes (\det \pi_*E)^{-N_0P(m)mr} = \left((\det \pi_*(L^m))^{\dim V} \otimes (\det \pi_*E)^{-P(m)mr}\right)^N$$

is effective. \qed

From Theorem 4.1, we can deduce the following slope inequality for relative semistable sheaves on $X$. The original slope inequality is proved by Xiao [37] for the relative canonical sheaf of a surface fibration in characteristic zero and independently by Cornalba-Harris [9] for semi-stable fibrations. Stoppino [28] showed that the method of Cornalba-Harris still works for non-semistable fibrations. See also [22, 31] for the slope inequality in positive characteristic.

Theorem 4.2. Assume that $X$ and $Y$ are smooth. Let $E$ be a torsion free sheaf of rank $r$ on $X$ such that $E_y$ is semistable. Let $A_1, \cdots, A_{n-2}$ be ample divisors on $Y$. Then there exists an integer $d_0$ such that if $\deg E_y \geq d_0$, we have

$$\pi^*(A_1 \cdots A_{n-2})c_1^2(E) \geq \frac{2r \deg E_y}{h^0(E_y)} A_1 \cdots A_{n-2}c_1(\pi_*E).$$

Proof. By Theorem 4.1 one sees that there is an integer $d_0 \geq 3gr$ so that $(X_Y, E_y)$ is $m$-Hilbert semistable when $\deg E_y \geq d_0$ and $m$ large enough. Therefore, from Theorem 4.1 we obtain an effective line bundle

$$\left( (\det \pi_*(L^m))^{r \pi_*(E)} \otimes (\det \pi_*E)^{-r \pi_*(L^m)mr} \right)^N,$$

here $N$ is an positive integer and $L = \det E$. This implies

$$A_1 \cdots A_{n-2} \left( r\pi_*(E)c_1(\pi_*(L^m)) - r \pi_*(L^m)mrc_1(\pi_*E) \right) \geq 0.$$
On the other hand, by the Grothendieck-Riemann-Roch theorem, one has the following formula (see [22, Lemma 2.3] for example):

\[
c_1(\pi_*(L^m)) = \frac{\pi_*(c_2^2(E))}{2}m^2 + Z_1m + Z_0,
\]

here \( Z_1 \) and \( Z_0 \) are \( \mathbb{Q} \)-divisors of \( Y \). Moreover, some simple computations show that

\[
\text{rk } \pi_* (L^m) = h^0(L^m) = m \deg L_y - g + 1
\]

and \( \text{rk}(\pi_* E) = h^0(E_y) \). Substituting these equations into (4.2) and letting \( m \to +\infty \), we obtain the desired inequality. □

**Corollary 4.3.** Assume that \( X \) and \( Y \) are smooth. Let \( H \) be a \( \pi \)-relatively ample divisor on \( X \), \( A_1, \ldots, A_{n-2} \) ample divisors on \( Y \) and \( E \) a rank \( r \) torsion free sheaf on \( X \). Suppose \( E_y \) is semistable. Then we have

\[
\pi^*(A_1 \cdots A_{n-2}) (\deg H_y)HK_{X/Y} - (g - 1)H^2 \geq 0.
\]

If the equality holds, then

\[
\pi^*(A_1 \cdots A_{n-2}) \Delta(E) \geq \pi^*(A_1 \cdots A_{n-2}) \left( \frac{r^2}{6}(c_1^2(X) + c_2(X)) - rc_1(K_{X/Y}) \right)
\]

\[
- \frac{r}{\deg H_y} \pi^*(A_1 \cdots A_{n-2}) \left( \deg E_y)HK_{X/Y} - (2g - 2)Hc_1(E) \right)
\]

\[
- r^2(g - 1)A_1 \cdots A_{n-2}K_Y.
\]

**Proof.** Let \( S_m \) be the divisor class

\[
h^0(E_y(mH))\pi_*c_1^2(E(mH)) - 2r\deg(E_y(mH))c_1(\pi_*E(mH)).
\]

Applying Theorem 4.2 for \( E(mH) \), we have \( A_1 \cdots A_{n-2}S_m \geq 0 \) for \( m \gg 0 \). It remains to understand the terms in \( S_m \) explicitly.

From the Grothendieck-Riemann-Roch theorem, it follows that

\[
\text{ch}(\pi_*E(mH)) \text{td}(Y) = \pi_* \left( \text{ch}(E(mH)) \text{td}(X) \right).
\]

This implies

\[
\text{ch}_1(\pi_*E(mH)) + \frac{1}{2} \text{ch}_0(\pi_*E(mH))c_1(Y)
\]

\[
= \pi_* \left( \text{ch}_2(E(mH)) + \frac{1}{2} \text{ch}_1(E(mH))c_1(X) + \frac{r}{12}(c_1^2(X) + c_2(X)) \right).
\]

We now compute the terms of the above equation:

\[
\text{ch}_1(E(mH)) = \text{ch}_1(E) + rmH;
\]

\[
\text{ch}_2(E(mH)) = \text{ch}_2(E) + mH \text{ch}_1(E) + \frac{1}{2} rm^2H^2;
\]

\[
\text{ch}_0(\pi_*E(mH)) = \chi(E_y(mH))
\]

\[
= \deg E_y + rm \deg H_y - r(g - 1).
\]
It follows that
\[
\text{ch}_1(\pi_*(E(mH))) = \pi_* \left( \frac{r}{12} (c_1^2(X) + c_2(X)) + \frac{1}{2} c_1(E) c_1(X) + \text{ch}_2(E) \right) \\
+ \frac{r}{2} (g - 1) - \frac{\text{deg } E_y}{2} c_1(Y) + \frac{r \text{deg } H^2}{2} m^2 \\
+ \pi_* \left( H c_1(E) + \frac{r}{2} H c_1(X) - \frac{r \text{deg } H_y}{2} c_1(Y) \right) m.
\]

Since
\[
\pi_*(H c_1(X)) - \text{deg } H_y c_1(Y) = \pi_*(H c_1(X)) - \pi_* \left( H \pi^* c_1(Y) \right) \\
= -\pi_*(H K_{X/Y})
\]
and
\[
\pi_*(c_1(E) c_1(X)) - \text{deg } E_y c_1(Y) = \pi_*(c_1(E) c_1(X)) - \pi_* \left( c_1(E) \pi^* c_1(Y) \right) \\
= -\pi_* \left( c_1(E) K_{X/Y} \right),
\]
one obtains
\[
\text{ch}_1(\pi_*(E(mH))) = \pi_* \left[ \frac{r H^2}{2} m^2 + H (c_1(E) - \frac{r}{2} K_{X/Y}) m + \text{ch}_2(E) \right] \\
- \frac{1}{2} c_1(E) K_{X/Y} + \frac{r}{12} (c_1^2(X) + c_2(X)) - \frac{r (g - 1)}{2} K_Y.
\]

Some other simple computations show that
\[
h^0(E_y(mH)) = \text{deg } E_y + rm \text{deg } H_y - r(g - 1);
\]
\[
c_1^2(E(mH)) = c_1^2(E) + 2 rm H c_1(E) + r^2 m^2 H^2;
\]
\[
\text{deg } (E_y(mH)) = \text{deg } E_y + rm \text{deg } H_y.
\]
Substituting these equations into the expression for \( S_m \), one has
\[
S_m = m^2 r^3 \pi_* \left( (\text{deg } H_y) H K_{X/Y} - (g - 1) H^2 \right) \\
+ mr \text{deg } H_y \pi_* \left( c_1^2(E) - 2 r \text{ch}_2(E) + r c_1(E) K_{X/Y} - \frac{r^2}{6} (c_1^2(X) + c_2(X)) \right) \\
+ mr^3 (g - 1) (\text{deg } H_y) K_Y + mr^2 \pi_* \left( (\text{deg } E_y) H K_{X/Y} - (2g - 2) H c_1(E) \right) \\
+ Z,
\]
where \( Z \) is a \( \mathbb{Q} \)-divisor on \( Y \) which is independent of \( m \). From the positivity of \( A_1 \cdots A_{n-2} S_m \) for \( m \gg 0 \), one sees that
\[
A_1 \cdots A_{n-2} \pi_* \left( (\text{deg } H_y) H K_{X/Y} - (g - 1) H^2 \right) \geq 0.
\]
If the equality holds, we obtain the positivity of the coefficient of \( m \) in \( A_1 \cdots A_{n-2} S_m \). Hence we are done!

From Corollary 4.3 one can deduce relative Bogomolov’s inequality [21] for trivial fibrations in any characteristic. We let \( C \) be a smooth projective curve of genus \( g \) defined over \( k \).
Corollary 4.4. Assume $Y$ is smooth, and $X = C \times Y$ is a product of $C$ and $Y$ with projections $\pi : X \to Y$ and $p : X \to C$. Let $A_1, \ldots, A_{n-2}$ be ample divisors on $Y$ and $E$ a torsion free sheaf on $X$. If $E_y$ is semistable, then

$$\pi^*(A_1 \cdots A_{n-2}) \Delta(E) \geq 0.$$ 

Proof. By our assumptions, one sees that $K_X = p^*K_C + \pi^*K_Y$, $K_{X/Y} = p^*K_C$ and $c_2(X) = \pi^*c_2(Y) + p^*K_C \cdot \pi^*K_Y$. Hence

$$\pi_*(c_1^2(X) + c_2(X)) = 3\pi_*\left(p^*K_C \cdot \pi^*K_Y \right) = 6(g-1)K_Y.$$ 

Let $c$ be a point of $C$ and let $H = p^*c$. Then one deduces that $\deg H_y = 1$, $H^2 = 0$ and $HK_{X/Y} = 0$. Thus

$$\pi^*(A_1 \cdots A_{n-2})\left((\deg H_y)HK_{X/Y} - (g-1)H^2 \right) = 0.$$ 

By Corollary 4.3 one sees that the inequality holds. On the other hand, we have

$$A_1 \cdots A_{n-2}\pi_*\left(\left((2g-2)H - K_{X/Y}\right)c_1(E)\right) = A_1 \cdots A_{n-2}\pi_*\left(\left((2g-2)p^*c - p^*K_C\right)c_1(E)\right) = 0.$$ 

Therefore, the right hand side of the inequality is zero in our situation. It follows that $\pi^*(A_1 \cdots A_{n-2}) \Delta(E) \geq 0$. \qed

5. Proof of the main theorems

The aim of this section is to prove our main theorems from relative Bogomolov’s inequality in Corollary 4.4.

Theorem 5.1. Let $X = C \times Y$ be the product of a smooth curve $C$ and a $n - 1$ dimensional smooth projective variety $Y$ defined over $k$ with projections $\pi : X \to Y$ and $p : X \to C$. Let $A_1, \ldots, A_{n-2}$ be ample divisors on $Y$, $H$ an ample divisor on $X$ and $E$ a torsion free sheaf on $X$. If $E$ is $\mu_{\pi^*A_1, \ldots, \pi^*A_{n-2}, H}$-semistable, then

$$\pi^*(A_1 \cdots A_{n-2}) \Delta(E) \geq 0.$$ 

Proof. The proof is by induction on the rank of $E$. For rank 1 the assertion is obvious. Assume that the theorem holds for every semistable sheaf of rank less than $r$ and $E$ is of rank $r$.

Let $K(Y)$ be the function field of $Y$ and $y$ be a general point of $Y$. Denote the generic fibre $X \times_Y \text{Spec}(K(Y))$ by $X_y$. Let $E_\eta$ be the restriction of $E$ to $X_\eta$ and $E_y$ the restriction of $E$ to the general fibre $X_y$ of $\pi$. If $E$ is $\mu_{\pi^*A_1, \ldots, \pi^*A_{n-2}, \pi^*A_1}$-semistable, then $E_y$ is semistable. By the openness of semistability (see Proposition 2.3.1), one sees that $E_y$ is semistable. Hence Corollary 4.4 implies

$$\pi^*(A_1 \cdots A_{n-2}) \Delta(E) \geq 0.$$ 

Now we assume that $E$ is not $\mu_{\pi^*A_1, \ldots, \pi^*A_{n-2}, \pi^*A_1}$-semistable. By Lemma 4.C.5 there is a non-negative rational number $t$ and a saturated subsheaf $E_0 \subset E$ with $\text{rk} E_0 = t_0$ such that

$$\mu_{\pi^*A_1, \ldots, \pi^*A_{n-2}, \pi^*A_1}(E_0) > \mu_{\pi^*A_1, \ldots, \pi^*A_{n-2}, \pi^*A_1}(E),$$

\footnote{This lemma is stated for the surface case, but its proof still works for our situation.}
and $E$ and $E_0$ are $\mu_{\pi^*A_1,\ldots,\pi^*A_{n-2},H_\tau}$-semistable of the same slope, where

$$H_\tau = H + t\pi^*A_1.$$ 

Since $E_0$ is saturated, one sees that $E_1 = E/E_0$ is torsion free and $\mu_{\pi^*A_1,\ldots,\pi^*A_{n-2},H_\tau}$-semistable of rank $r_1 = r - r_0$. Set $\xi = r_0c_1(E_0) - r_0c_1(E)$. Then

$$\pi^*(A_1 \cdots A_{n-2})H_\tau \xi = 0.$$ 

It follows from the Hodge index theorem that

$$\pi^*(A_1 \cdots A_{n-2})\xi^2 \leq 0.$$ 

Moreover, the following identity holds:

$$\pi^*(A_1 \cdots A_{n-2})\left(\Delta(E) - \frac{r}{r_0} \Delta(E_0) - \frac{r}{r_1} \Delta(E_1)\right) = -\frac{\pi^*(A_1 \cdots A_{n-2})\xi^2}{r_0 r_1} \geq 0.$$ 

By our induction assumption, one has $\pi^*(A_1 \cdots A_{n-2})\Delta(E) \geq 0$ for $i = 0, 1$ and therefore

$$\pi^*(A_1 \cdots A_{n-2})\Delta(E) \geq 0.$$ 

By this, we immediately deduce Theorem 1.3.

**Corollary 5.2 (Theorem 1.3).** Let $H$ be an ample divisor on a product type surface $S$. Then for any $\mu_H$-semistable sheaf $E$ on $S$, we have $\Delta(E) \geq 0$.

**Proof.** Let $f : C_1 \times C_2 \to S$ be the finite separable morphism associated with $S$, where $C_1$ and $C_2$ are curves. It turns out that $f^*E$ is $\mu_{f^*H}$-semistable. It follows from Theorem 5.1 that $\Delta(f^*E) \geq 0$. Thus $\Delta(E) \geq 0$. $\square$

Theorem 1.3 can be generalized to high dimensional product type varieties. Let $X_n = C_1 \times \cdots \times C_n$ be a product of smooth projective curves defined over $k$ with projections $p_i : X_n \to C_i$ for $i = 1, \ldots, n$. The following theorem is more general than Theorem 1.2.

**Theorem 5.3.** Let $X$ be a product type variety with respect to a finite separable morphism $f : X_n \to X$. Let $H_1, \ldots, H_{n-1}$ be product type ample divisors on $X$. Then for any $\mu_{H_1,\ldots,H_{n-1}}$-semistable sheaf $E$ on $X$, we have

$$H_2 \cdots H_{n-1}\Delta(E) \geq 0.$$ 

**Proof.** Since the semistability is invariant under the pull back of $f$, we can assume that $X = X_n$ and $f$ is the identity.

For $n \geq 2$, let $P(r, n)$ denote the statement: for any $n-1$ product type ample divisors $H_1, \ldots, H_{n-1}$ on $X_n$, one has

$$H_2 \cdots H_{n-1}\Delta(E) \geq 0$$

for any $\mu_{H_1,\ldots,H_{n-1}}$-semistable sheaf $E$ of rank $\leq r$ on $X_n$.

Obviously, by Theorem 1.3, $P(r, 2)$ and $P(1, n)$ hold for any positive integers $n \geq 2$ and $r$. By induction one sees that if $P(r-1, n)$ and $P(r, n-1)$ imply $P(r, n)$, then $P(r, n)$ holds for any positive integers $n \geq 2$ and $r$.

Now we assume $P(r-1, n)$ and $P(r, n-1)$ hold. We let $H_j, \ldots, H_{n-1}$ be product type ample divisors on $X_n$ and $F_j$ the general fiber of $p_j : X_n \to C_j$. Let $E$ be a $\mu_{H_1,\ldots,H_{n-1}}$-semistable sheaf of rank $r$ on $X_n$. If $E$ is not $\mu_{F_j, H_2,\ldots,H_{n-1}}$-semistable
for some $1 \leq j \leq n$, then similar to the proof of Theorem 6.1.1 there is a non-negative rational number $t$ and a saturated subsheaf $E_0 \subset E$ with $\text{rk} E_0 = r_0$ such that
\[
\mu_{F_j,H_2,\ldots,H_{n-1}}(E_0) > \mu_{F_j,H_2,\ldots,H_{n-1}}(E),
\]
and $E$ and $E_0$ are $\mu_{H_1,H_2,\ldots,H_{n-1}}$-semistable of the same slope, where
\[
H_t = H_1 + tF_j.
\]
Since $E_0$ is saturated, one sees that $E_1 = E/E_0$ is torsion free and $\mu_{H_1,H_2,\ldots,H_{n-1}}$-semistable of rank $r_1 = r - r_0$. Set $\xi = r c_1(E_0) - r_0 c_1(E)$. Then
\[
H_2 \cdots H_{n-1} H_t \xi = 0.
\]
It follows from the Hodge index theorem that
\[
H_2 \cdots H_{n-1} \xi^2 \leq 0.
\]
Moreover, the following identity holds:
\[
H_2 \cdots H_{n-1} \left( \Delta(E) - \frac{r}{r_0} \Delta(E_0) - \frac{r}{r_1} \Delta(E_1) \right) = - \frac{H_2 \cdots H_{n-1} \xi^2}{r_0 r_1} \geq 0.
\]
From our induction assumptions, it follows that $H_2 \cdots H_{n-1} \Delta(E_i) \geq 0$ for $i = 0, 1$ and therefore
\[
H_2 \cdots H_{n-1} \Delta(E) \geq 0.
\]
Now we assume that $E$ is $\mu_{F_j,H_2,\ldots,H_{n-1}}$-semistable for any $1 \leq j \leq n$, then by the openness of semistability, one sees that $E|_{F_j}$ is $\mu_{H'_2,\ldots,H'_{n-1}}$-semistable, here $H_i' = H_i|_{F_j}$, $i = 2, \ldots, n-1$. Since
\[
F_j \cong C_1 \times \cdots \times C_{j-1} \times C_{j+1} \times \cdots \times C_n,
\]
by the induction assumptions, we have $H'_2 \cdots H'_{n-1} \Delta(E|_{F_j}) \geq 0$, i.e.,
\[
H_2 H_3 \cdots H_{n-1} \Delta(E) \geq 0,
\]
for $1 \leq j \leq n$. Since $H_2$ is of product type, we can write
\[
H_2 = p_{1}^{b_1} B_1 + \cdots + p_{n}^{b_n} B_n,
\]
where $B_i$ is a divisor on $C_i$ with $\text{deg} B_i = b_i > 0$. Then one concludes
\[
H_2 H_3 \cdots H_{n-1} \Delta(E) = \sum_{j=1}^{n} (b_j F_j) H_3 \cdots H_{n-1} \Delta(E) \geq 0.
\]
Thus we are done!

\[\square\]

6. Applications of Bogomolov’s inequality

In this section we exhibit the applications of Theorem 1.3 to the positivity of linear systems and torsion free sheaves and the construction of Bridgeland stability conditions on product type surfaces in positive characteristic. We always let $S$ be a product type surface defined over $k$ in this section.

6.1. Adjoint linear systems.

Theorem 6.1. Let $d \geq 1$ be an integer and $L$ be a nef divisor on $S$ with $L^2 > 4d$. If $|K_S + L|$ is not $(d-1)$-very ample, then there exists a curve $D$ on $S$ such that
\[
LD - d \leq D^2 < \frac{1}{2} LD < d.
\]

Proof. By Theorem 1.3, the proof is the same as that of [25] and [8]. \[\square\]
6.2. Positivity of semistable sheaves. Let $H$ be an ample divisor and $E$ a $\mu_H$-semistable torsion free sheaf on $S$ with $\text{rk} E \geq 2$. We define the generalized discriminant of $E$ to be

$$\Omega_H(E) := (H \text{ch}_1(E))^2 - 2H^2 \text{ch}_0(E) \text{ch}_2(E).$$

By Theorem 1.3 and Hodge index theorem, one sees that $\Omega_H(E) \geq 0$.

Theorem 6.2. If $l > (\Omega_H(E) - \mu_H(E))/H^2$, then we have $H^1(KS + lH) = 0$, and $E(KS + lH)$ is generated by global sections if $l > 2 \text{rk} E + (\Omega_H(E) - \mu_H(E))/H^2$.

Proof. The first assertion is just [29, Corollary 1.8]. For the second assertion, we consider the short exact sequence

$$0 \to K \to E \xrightarrow{f} O_x \to 0,$$

where $x$ is a point in $S$, $f$ is any surjection and $K = \ker f$. One sees that $K$ is also $\mu_H$-semistable and

$$\Omega_H(E) = \Omega_H(K) - 2H^2 \text{rk} E.$$

Hence by the first assertion, we conclude that $H^1(K(KS + lH)) = 0$ if

$$l > 2 \text{rk} E + (\Omega_H(E) - \mu_H(E))/H^2.$$

This implies the induced map $H^0(E(KS + lH)) \to H^0(O_x)$ is surjective for any $x \in S$ and any surjection $f : E \to O_x$. Therefore $E(KS + lH)$ is generated by global sections. \qed

6.3. Bridgeland stability conditions on surfaces. The notion of Bridgeland stability condition was introduced in [5]. In recent years, this stability condition has drawn a lot of attentions, and has been investigated intensively. Let $X$ be a smooth projective variety defined over $k$ with $\text{dim} X = n$.

Definition 6.3. A Bridgeland stability condition on $X$ is a pair $\sigma = (Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded $t$-structure on $D^b(X)$, and $Z : K(D^b(X)) \to \mathbb{C}$ is a group homomorphism (called central charge) such that

- $Z$ satisfies the following positivity property for any non-zero $E \in \mathcal{A}$:

  $$Z(E) \in \{re^{i\pi\phi} : r > 0, 0 < \phi \leq 1\}.$$

- Every non-zero object in $\mathcal{A}$ has a Harder-Narasimhan filtration in $\mathcal{A}$ with respect to $\nu_Z$-stability, here the slope $\nu_Z$ of an object $E \in \mathcal{A}$ is defined by

  $$\nu_Z(E) = \begin{cases} +\infty, & \text{if } \Im Z(E) = 0, \\ \frac{\Im Z(E)}{\Re Z(E)}, & \text{otherwise}. \end{cases}$$

We now review the construction of Bridgeland stability condition in [6,1]. For a fixed $\mathbb{Q}$-divisor $D$ on $X$, we define the twisted Chern character $\text{ch}_D^\mathbb{C} = e^{-D} \text{ch}$. More explicitly, we have

$$\begin{align*}
\text{ch}_0^D &= \text{ch}_0 = \text{rk} \\
\text{ch}_1^D &= \text{ch}_1 - D \text{ch}_0 \\
\text{ch}_2^D &= \text{ch}_2 - D \text{ch}_1 + \frac{D^2}{2} \text{ch}_0 \\
\text{ch}_3^D &= \text{ch}_3 - D \text{ch}_2 + \frac{D^2}{2} \text{ch}_1 - \frac{D^3}{6} \text{ch}_0.
\end{align*}$$
We define the twisted slope $\mu_{H,D}$ of a coherent sheaf $E \in \text{Coh}(X)$ by
\[
\mu_{H,D}(E) = \begin{cases} 
+\infty, & \text{if } \text{ch}_0^D(E) = 0, \\
\frac{H^n - \text{ch}_0^D(E)}{H^n \text{ch}_0^D(E)}, & \text{otherwise}.
\end{cases}
\]

Similarly as Definition 2.1 one can define the stability for sheaves with respect to $\mu_{H,D}$. Let $\alpha > 0$ and $\beta$ be two real numbers. There exists a torsion pair $(\mathcal{T}_{\beta H+D}, \mathcal{F}_{\beta H+D})$ in $\text{Coh}(X)$ defined as follows:
\[
\mathcal{T}_{\beta H+D} = \{ E \in \text{Coh}(X) : \mu_{H,\beta H+D}^-(E) > 0 \}
\]
\[
\mathcal{F}_{\beta H+D} = \{ E \in \text{Coh}(X) : \mu_{H,\beta H+D}^+(E) \leq 0 \}
\]
Equivalently, $\mathcal{T}_{\beta H+D}$ and $\mathcal{F}_{\beta H+D}$ are the extension-closed subcategories of $\text{Coh}(X)$ generated by $\mu_{H,\beta H+D}$-stable sheaves of positive and non-positive slope, respectively.

**Definition 6.4.** We let $\text{Coh}^{\beta H+D}(X) \subset \text{D}^b(X)$ be the extension-closure $\text{Coh}^{\beta H+D}(X) = \langle \mathcal{T}_{\beta H+D}, \mathcal{F}_{\beta H+D}[1] \rangle$.

By the general theory of torsion pairs and tilting [13], $\text{Coh}^{\beta H+D}(X)$ is the heart of a bounded $t$-structure on $\text{D}^b(X)$; in particular, it is an abelian category. Consider the following central charge
\[
Z_{\alpha,\beta}(E) = H^n - 2 \left( \frac{\alpha^2 H^2}{2} \text{ch}_0^{\beta H+D}(E) - \text{ch}_1^{\beta H+D}(E) + iH \text{ch}_1^{\beta H+D}(E) \right).
\]

**Theorem 6.5.** Let $X$ be a smooth projective surface defined over $k$. If $\mu_{H}^+(\Omega^1_X) \leq 0$ or $X$ is of product type, then for any $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, $\sigma_{\alpha,\beta} = (Z_{\alpha,\beta}, \text{Coh}^{\beta H+D}(X))$ is a Bridgeland stability condition.

**Proof.** The assumption on $X$ guarantees that Bogomolov’s inequality holds on it. Hence the required assertion is proved in [9, 11].

7. COUNTEREXAMPLES TO BOGOMOLOV’S INEQUALITY

In this section we exhibit some new counterexamples to Bogomolov’s inequality in positive characteristic. In [24, 23], the authors constructed some surfaces in positive characteristic with $c_2^2 > 0$ and $c_2 < 0$ on which Kodaira’s vanishing fails. These surfaces give rise to rank two semistable sheaves violating Bogomolov’s inequality.

We now construct some high rank counterexamples to Bogomolov’s inequality. Let $X$ be a smooth projective surface defined over $k$ and $H$ an ample divisor on $X$. Assume that $\text{char}(k) = p > 0$. Denote by $F$ the absolute Frobenius morphism of $X$.

**Theorem 7.1.** Assume that $K_X H > 0$, $K_X^2 > 2c_2(X)$ and $\Omega^1_X$ is $\mu^H$-semistable. Let $\mathcal{L}$ be a line bundle on $X$. Then $F_* \mathcal{L}$ is $\mu^H$-semistable but $\Delta(F_* \mathcal{L}) < 0$.

**Proof.** By [10, Theorem 5.1], one sees that $F_* \mathcal{L}$ is $\mu^H$-semistable (see also [32]). It remains to compute $\Delta(F_* \mathcal{L})$ explicitly.

From the Grothendieck-Riemann-Roch theorem, it follows that
\[
\text{ch}(F_* \mathcal{L}) \text{td}(X) = F_*(\text{ch} \mathcal{L} \text{td}(X)).
\]
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Since \( td(X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) \), the above equation implies
\[
\frac{1}{2} \operatorname{ch}_0(F_*L)c_1 + \operatorname{ch}_1(F_*L) = F_*\left( \frac{c_1}{2} + c_1(L) \right) = p\left( \frac{c_1}{2} + c_1(L) \right)
\]
and
\[
\frac{c_1^2 + c_2}{12} \operatorname{ch}_0(F_*L) + \frac{c_1}{2} \operatorname{ch}_1(F_*L) + \operatorname{ch}_2(F_*L) = \frac{c_1^2 + c_2}{12} + \frac{c_1}{2}c_1(L) + \operatorname{ch}_2(L).
\]
A simple computation shows
\[
\operatorname{ch}_0(F_*L) = p^2, \quad \operatorname{ch}_1(F_*L) = \frac{p^2 - p}{2}K_X + pc_1(L)
\]
and
\[
\operatorname{ch}_2(F_*L) = \frac{1}{12}(K_X^2 + c_2) + \frac{p^2 - p}{4}K_X^2 + \frac{p - 1}{2}K_Xc_1(L) + \frac{c_2^2(L)}{2}.
\]
Therefore one concludes that
\[
\Delta(F_*L) = \frac{p^4 - p^2}{12}(2c_2 - K_X^2) < 0.
\]

To get a surface satisfying the hypotheses of Theorem 7.1, let \( S \) be a smooth complex projective surface with ample \( K_S \) and \( K_S^2 > 2c_2(S) \). It is well known that \( \Omega_S^1 \) is \( \mu_{K_S} \)-stable (see [36] for example). By the standard spreading out technique, we have a subring \( R \subset \mathbb{C} \), finitely generated over \( \mathbb{Z} \), and a scheme \( \pi : S_R \to \text{Spec } R \) so that \( \pi \) is smooth and projective and \( S = S_R \times \mathbb{C} \). By the openness of ampleness and stability, one sees that \( K_{S_m} \) is ample and \( \Omega_{S,m}^1 \) is \( \mu_{K_{S,m}} \)-semistable for a general maximal ideal \( m \in \text{Spec } R \), where \( S_m = S_R \times \mathbb{C} \) (the geometric fibre of \( \pi \) over \( m \)). One notes that \( \text{char}(R/m) > 0 \). Hence \( S_m \) satisfies the hypotheses of Theorem 7.1.

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