On the Uniqueness of the effective Lagrangian for N= 2 SQCD

M. Magro, L.O’Raifeartaigh and I. Sachs
Dublin Institute for Advanced Studies,
10 Burlington Road, Dublin 4, Ireland.

Abstract

The low energy effective Lagrangian for $N=2 SU(2)$ supersymmetric Yang-Mills theory coupled to $N_F < 4$ massless matter fields is derived from the BPS mass formula using asymptotic freedom and assuming that the number of strong coupling singularities is finite.

1 Introduction

Over the last three years important progress has been made on the problem of strongly coupled Yang-Mills theory with extended supersymmetry, pioneered by the work of Seiberg and Witten [1, 2] for an $SU(2)$ gauge group. Assuming duality, in the sense that for strong coupling the low energy effective action is dominated by magnetic monopoles, they obtained an exact result for the low energy effective Lagrangian. Extensions to higher groups were later proposed [3]. The assumption of the monopole dominance is self-consistent and well motivated on physical grounds. Nevertheless one expects this to be a consequence of rather more fundamental properties of the theory. Apart from clarifying the underlying structure of the solutions obtained so far investigation of this problem should be particularly interesting in view of possible generalizations.

In the case of $N=2$ Yang-Mills theory without matter results in this direction have been reported previously. In particular the Seiberg-Witten solution [1] was re-derived in [4] under the assumption that the moduli space of inequivalent vacua was of genus zero and that it was parameterized by the Seiberg-Witten parameter $u$. More recently [5] it has been shown neither of these assumptions is necessary. The same result can already be derived from the BPS-mass formula and the assumption that the moduli space is simply the space of inequivalent (complex) couplings $\tau$. The central ingredient in this analysis is the so-called maximal equivalence group $G$ whose elements relate equivalent vacua. The equivalence is with respect to the mass-spectrum of the full tower of BPS-saturated states (not the subset of stable BPS-states). That $G$ is not trivial follows on general grounds from the periodicity of the $\theta$-vacuum on one hand and the anomalous electric charge of the monopoles induced by the $\theta$-angle on the other. At the same time $G \subset U(1) \times PSL(2, \mathbb{Z})$ due to the particular form of the BPS-formula [1].

In this paper we consider the case where the $N=2$ YM-theory is coupled to $N_F < 4$ massless matter hypermultiplets [2] in the fundamental representation of $SU(2)$. $N_F < 4$ is required by asymptotic freedom which we shall use throughout. The moduli space is parameterized by a possibly multiple covering of the fundamental domain of $G$ in the
τ-plane. Thus τ is not necessarily a uniformizing parameter. A further ingredient for our strong-coupling analysis is the asymptotic freedom which requires that the mass of the lightest charged field be finite in the non-perturbative regime. It turns out that the $SL(2,\mathbb{Z})$ property and the finite mass constraint are almost, but not quite strong enough to determine the low-energy effective action uniquely. Further constraints are then obtained by adding a further observable to the system. This observable is provided by the superconformal anomaly [6]. We then show that the finite mass constraint [5] plus the condition that the anomaly be a $U(1)$-section under $G$ determine the set of possible fundamental domains for the coupling $\tau$. For $N_F = 0,1,2$ there is a unique domain, which corresponds to the S-W-solution for these cases. For $N_F = 3$ there are several different domains, but only for one of them is the superconformal anomaly automatically a $U(1)$-section. The corresponding solution is precisely the S-W-solution. For all the other domains the conditions that the anomaly be a $U(1)$-section leads to a formally overdetermined system. Whether all of them are actually overdetermined cannot be decided from the existing mathematical literature. It is not even clear whether these domains correspond to equivalence groups.

The role played here by the superconformal anomaly is reminiscent of that played by conformal- or axial anomalies to obtain exact results in 2-dimensional models. The superconformal anomaly has previously been used to compute certain $N = 2$ supersymmetric Green functions [6]. Here we show that, in conjunction with the $SL(2,\mathbb{Z})$-structure, the superconformal anomaly determines the low-energy effective Lagrangian and completely parametrises the quantum moduli space. What distinguishes the 4-dimensional model is that the non-perturbative contributions are not computable by other means (except for 1 and 2 instantons contributions [8, 9, 10, 11, 12]).

The plan of this paper is as follows. In section 2 we review the properties of $N = 2$ Yang-Mills theory coupled to matter. In section 3 we establish the necessary ingredients for the non-perturbative analysis in the later sections and recall the general construction of solutions [5]. In section 4 we give a quick derivation of the anomalous superconformal Ward-identity and obtain the constraints it imposes on the general solution. In section 5 we obtain the unique solutions for $N_F = 0,1$ and 2, the SW-solution for $N_F = 3$ and discuss the over determination problem for the other possible candidates. For the purpose of comparison with other papers [13, 14] it is shown in the appendix how to obtain the second order differential equations satisfied by our solutions.

## 2 Review of $N = 2$ YM-Theory with Matter

We start with a review of $N = 2$ YM-theory with massless hypermultiplets in the fundamental representation. In addition to the canonical $N = 1$ kinetic terms and minimal gauge coupling for all the fields, $N = 2$-supersymmetry requires a superpotential

$$W = \sqrt{2} \tilde{M}_i A M^i,$$

where $A$ is the chiral multiplet in the $N = 2$ Yang-Mills multiplet and $M_i; \ i = 1, \ldots, N_F$ are the matter hypermultiplets. The flat directions in the potential for the scalar component $\phi$ of $A$ survive in the presence of hypermultiplets. More precisely, the
potential vanishes for constant $\phi$ taking its value in the Cartan subalgebra of the gauge group $G$. In what follows we take $G=SU(2)$. For $\phi \neq 0$, the Higgs mechanism breaks the gauge symmetry spontaneously to $U(1)$, inducing a mass term for all charged fields including the hypermultiplets as follows from (1). The theory is then in the Coulomb branch \[2\]. In this regime the most general low energy effective Lagrangian has a $F$-term of the form \[15, 16\]

$$\Gamma[A] = \frac{1}{4\pi} \text{Im} \int d^4x d^2\theta_1 d^2\theta_2 \mathcal{F}(A),$$

where the prepotential $\mathcal{F}$, to be determined, is the result of integrating out the massive fields. In $N=1$ notation (2) becomes

$$\Gamma[A, W_\alpha] = \frac{1}{4\pi} \text{Im} \text{Tr} \int d^4x \left\{ \int d^2\theta d^2\bar{\theta} \left( A_D\bar{A} - \bar{A}_D A \right) + \frac{1}{4} \int d^2\theta \tau(A) W^\alpha W_\alpha \right\},$$

where $A = \phi + \ldots$ and $W_\alpha$ are the chiral- and vector $N=1$ superfields respectively. Furthermore

$$A_D = \frac{1}{2} \mathcal{F}'(A) \quad \text{and} \quad \tau(A) = \frac{1}{2} \mathcal{F}''(A) = A'_D(A).$$

The normalisations for $N_F=0$ \[3\] and $N_F > 0$ \[4\] differ by a factor of 2. Throughout this paper we take $N_F > 0$ unless explicitly stated otherwise. Since $\tau(\phi)$ is the coefficient of the kinetic term in (3) its imaginary part must be positive. The real part of $\tau$ plays the role of an effective $\theta$-angle: $\text{Re} \tau = \frac{\theta}{2\pi}$. Thus a shift of $\theta$ by $2\pi$ corresponds to $\tau \mapsto T^2(\tau) = \tau + 2$. Consequently, the observables of the low energy effective theory are invariant under $\{\tau \mapsto T^{2n}(\tau), n \in \mathbb{Z}\}$. This is therefore the case, in particular for the mass of BPS-saturated states which, for consistency with the SUSY-algebra, must be proportional to the central charge \[17\]

$$M = \sqrt{2}|Z|.$$  

At the classical level the theory is parameterized by the two real parameters $|\phi^3|$ and $\text{Im}(\tau)$. The moduli space of inequivalent vacua is therefore 2-dimensional. The central charge $Z$ is given by \[17\]

$$Z = |\phi^3| n_e + \tau n_m$$

where the integers $n_e$ and $n_m$ label electric and magnetic charges respectively. In the quantum theory the mass of the lightest charged field $m$ sets the scale for the low energy coupling. For $N_F < 4$ the theory is asymptotically free and hence perturbation theory is valid as long as $m > \Lambda$. In particular at the semiclassical level we then have ($\Lambda = 1$) \[18\]

$$\tau(a) = \frac{i}{\pi} (4 - N_F) \log(a) + c \quad \text{where} \quad a = \frac{1}{2} \langle \phi^3 \rangle$$

and $c$ depends on the renormalisation scheme adopted. Higher loop perturbative corrections to the running coupling are absent \[19\]. As explained in \[17, 20\] the topological
nature of the mass spectrum guarantees that the structure for the BPS-spectrum is the same at the quantum level. This implies in particular the linearity in \((n_e, n_m)\). The coefficients, however, can be modified. Computing the BPS-bounds from the gauge-invariant extension \(F(\sqrt{A^a A^a})\) of the low energy effective action, it has been argued in \([1, 20]\), that the quantum corrected BPS-formula is given by

\[
Z = an_e + a_D n_m,
\]

where \(a_D = \frac{1}{2}F'(a)\). Alternatively one can compute the centre of the low-energy effective theory from

\[
\{Q^{\text{eff}}, \bar{Q}^{\text{eff}}\} = i(P^{\text{eff}} - Z^{\text{eff}}),
\]

where \(Q^{\text{eff}}\) are the supercharges obtained from the low energy effective Lagrangian \(F(\sqrt{A^a A^a})\). One has \([21]\)

\[
\begin{align*}
Q^{\text{eff}} &= \frac{1}{4\pi} \int d^3 x (X + iY)^a \left[\text{Im}(F_{ab})\right]^{-1} \bar{\psi}^b + O(\psi^2), \\
\bar{Q}^{\text{eff}} &= \pi^a_{\bar{\psi}} (X + iY)^\dagger a + O(\psi^2),
\end{align*}
\]

where \(O(\psi^2)\) corresponds to terms in the Hamiltonian which vanish when the fermion fields are set to zero, and

\[
X^a = \gamma^i \gamma^5 B_i - i4\pi \gamma^i \left[\text{Im}(F_{ab})\right]^{-1} \pi_i^b + (\frac{1}{2}) \left[\text{Im}(F_{ab})\right]^{-1} C^b \gamma^0 \gamma^5,
\]

\[
Y^a = \left[\text{Im}(F_{ab})\right]^{-1} \pi_i^b + \gamma^i \gamma^0 D_i \Phi.
\]

\[
\Phi^a = \text{Re} \phi^a + i\gamma^5 \text{Im} \phi^a,
\]

\[
\pi_i^a = \frac{1}{4\pi} \left[\text{Im}(F_{ab})\right] E^b
\]

and \(C^a = [\phi_D, \phi^d]^a\). Computing the Poisson brackets of the effective supercharges one then again finds the result \([8]\). This shows that independent of the actual form of the function \(F\) appearing in the effective action, the centre is given by \((8)\). Comparison with the results in \([20]\) furthermore shows that the mass of the BPS-states indeed saturates the inequality \(M \geq |Z|\) also at the quantum level, as anticipated in \([17, 1]\). The details of this calculation will be presented elsewhere \([21]\). To summarize, the low energy effective theory is parameterised by either of the complex parameters \(a\) with values in \(C\), or \(\tau\) which takes any value in the upper half plane \(H\). In particular, the space of inequivalent vacua, or moduli space, \(\mathcal{M}\) is 1-complex dimensional.

### 3 Non-Perturbative Contributions

In addition to the perturbative corrections, reviewed in the last section, the low-energy effective action receives non-perturbative corrections due to topologically non-trivial contributions. The problem is then to determine \(F(a)\) or equivalently \(\tau(a)\). We formulate the problem such as to make maximal use of the analyticity of \(\tau(a)\) (which reflects the chirality of the supersymmetry algebra if derivatives are neglected) and the underlying \(SL(2,\mathbb{Z})\)-structure. Specifically our analysis uses the following properties:
(a) The effective coupling constant \( \tau = \frac{\theta_{\text{eff}}}{\pi} + i 8\pi \frac{g_{\text{eff}}}{2} \) takes all values in the upper half plane \( H \).

(b) The BPS-spectrum \( M \) is a single valued function on the moduli space, \( M = M(P), \ P \in \mathcal{M} \).

(c) The mass \( m \) of the lightest charged field (possibly composite) is finite except in the asymptotically free region.

(d) The superconformal anomaly \( u \) is a \( U(1) \) section under transformations of the equivalence group \( G \).

(e) The set of singular points of \( \mathcal{M} \) is finite.

The condition (c) reflects the asymptotic freedom of the underlying non-abelian theory. As for (d) this condition will become clear in section 4. Finally, the condition (e) is a technical assumption, which might not be necessary.

At the quantum level the quantities \( a, a_D \) and \( \tau \) will be transcendental functions of each other. This can be seen already from the 1-loop correction \([7]\). Correspondingly these functions have non-trivial Riemann surfaces. This multivaluedness leads then to identifications in the space of vacua, that is, certain points in the space of vacua are physically equivalent. To reduce this degeneracy we introduce the maximal equivalence group \( G \) defined by its (linear) action on the vector \( a = (a_D, a) \). The elements of \( G \) are the transformations which identify all different \( a \) corresponding to equivalent physical vacua \( P \). Invariance of the BPS-spectrum \( M \) then implies that \( G \subset U(1) \times PSL(2, \mathbb{Z}) \). On the other hand the conditions \( \text{Im} \tau \geq 0 \) and

\[
\tau(a) = \frac{da_D}{da}, \tag{12}
\]

imply that \( G \) is represented on \( \tau \) by a subgroup \( G \) of \( PSL(2, \mathbb{Z}) \), the projective modular transformations. \( G \) is not trivial since \( T^2 \in G \). Note that the definition of the maximal equivalence group given here is slightly more general than that of \([3]\) for pure YM-theory. As we will see below, this more general setting does not lead to new solutions in that case.

The moduli space is then in \( 1-1 \) correspondence with a possibly multiple covering \( D \) of \( H/G \). We can therefore parameterise \( \mathcal{M} \) by the upper half plane \( H \) by means of the Fuchsian map

\[
\tau : H \rightarrow D \rightarrow \mathbb{H} \quad \tau(z) = \frac{y_1(z)}{y_2(z)} \quad \text{where} \quad y'' + Qy = 0 \tag{13}
\]

and \( 2Q(z) = \{ \tau, \bar{z} \} \) is the Schwarzian of \( \tau \). While the function \( \tau(z) \) is normally complicated, \( Q(z) \) has the simple form \([22]\).

\[\uparrow\]Up to the extra \( U(1) \)-factor the same condition was obtained in \([1]\) (see paragraph below eqn. (4.7)).
\[ Q(z) = \frac{1}{2} \sum_{i=1}^{n} \left[ \frac{1}{2} \frac{1 - \alpha_i^2}{(z - a_i)^2} + \frac{\beta_i}{z - a_i} \right], \quad (14) \]

where \( n+1 \) is the number of edges of \( D \), the \( a_i \)'s are the points on the real axis into which the corners of the polygon \( D \) are mapped and the \( \pi \alpha_i \in [0, \pi) \) the interior angles of \( D \). Little is known about the geometrical interpretation of the *accessory parameters* \( \beta_i \) for a general polygon \([22, 23]\). This is the origin of the 'technical problem' for \( N_F = 3 \) anticipated in the introduction. In (14) we have chosen to map the weak-coupling singularity \( \tau = i\infty \) to infinity in the \( z \)-plane. We then have \([22]\)

\[ Q(z) \simeq \frac{1}{4z^2} \quad \text{for} \quad z \to \infty. \quad (15) \]

For later use we define the index \( \mu \) of \( D \) by the ratio

\[ \mu = \frac{\text{Area}(D)}{\text{Area}(D_0)}, \quad (16) \]

where the area is defined with respect to the Poincaré metric \([24, 25]\) on the \( \tau \) plane and \( D_0 \) is the fundamental domain of \( \text{PSL}(2, \mathbb{Z}) \). The index is then related to the angles \( \alpha_i \) of \( D \) by

\[ \sum_i (1 - \alpha_i) = \frac{\mu}{3} + 1. \quad (17) \]

As explained in \([5]\), for a given equivalence group \( G \) the general solution for the section \( a \) is given by

\[ a = f' y - f y' = W(f, y), \quad (18) \]

where \( f \) is a \( U(1) \)-section under equivalence transformations, in order to leave the mass spectrum invariant, \( y \) is as in (13) and \( W(f, y) \) is the Wronskian of \( f \) and \( y \). To complete the construction we need to match the boundary conditions with those given by the semiclassical contribution \([7]\). It follows from (13) and (18) that

\[ y(z) \simeq z^\frac{1}{2} (c \ln(z), 1) \quad \text{for} \quad z \to \infty. \quad (19) \]

The constant \( c \) is constrained by the \( \theta \)-vacuum symmetry explained above. In the presence of matter multiplets this symmetry is in fact enhanced. This is due to the absence of odd-instanton contributions \([3]\) enlarging the minimal symmetry \( \theta \to \theta + 2\pi \) to \( \theta \to \theta + \pi \) or equivalently \( \tau \to T(\tau) = \tau + 1 \). Therefore we identify \( \tau \) and \( \tau + 1 \) and hence

\[ \tau(z) \to \frac{i}{\pi} \ln(z) \quad \text{for} \quad z \to \infty. \quad (20) \]

Comparing (20) with the semiclassical result \([7]\) we then obtain for \( z \to \infty \)
This completes the construction of $a$ and $a_D$ for a given equivalence group. Without further constraints the non-perturbative contributions are then not uniquely determined. Indeed any Fuchsian function $\tau(z)$ mapping $H$ into a fundamental domain of $G$ and any $U(1)$-section $f(z)$ satisfying the boundary condition (21) is a solution. This freedom is partly removed by the finite mass condition (c). As explained in [5] this constraint leads to a lower bound for the exponents $r_z$ of $f$ at its singularities. More precisely

$$\begin{aligned}
 r_z = \frac{1}{2}(1 - \alpha_i) & \quad \text{or} \quad r_z \geq \frac{1}{2}(1 + \alpha_i) \quad \text{for} \quad z = a_i, \\
 r_z \geq 1 & \quad \text{for} \quad z \neq a_i.
\end{aligned}$$  

(22)

From (22) we obtain in particular the necessary condition

$$\begin{aligned}
 r_z \geq \frac{1}{2}(1 - \alpha_i) & \quad \text{for} \quad z = a_i, \\
 r_z \geq 1 & \quad \text{for} \quad z \neq a_i.
\end{aligned}$$  

(23)

On the other hand, in order to be well defined on the fundamental domain $D$ the exponents $r_z$ must satisfy the total residue condition

$$\sum_{z \text{ interior}} r_z + \frac{1}{2} \sum_{z \in \mathbb{R}} r_z + \frac{1}{2} r_\infty = 0.$$  

(24)

Substituting the lower bound (23) into (24) and using (17) and (21) we find that all the singularities of $f$ coincide with those of $Q$ and that the area of $D$ is bounded by

$$\mu \leq \frac{6}{4 - N_F}.$$  

(25)

Repeating the same analysis for $N_F = 0$ leads to

$$\mu \leq 3.$$  

(26)

To summarize, the finite mass condition, which reflects the asymptotic freedom of the underlying non-abelian theory considerably reduces the set of admissible equivalence groups. However, a finite set of polygons remains and each of them is a candidate for a possible solution of the low energy effective action. In order to impose further constraints we need to extend the set of observables of the low energy theory. The superconformal anomaly introduced in the next section provides this observable.

4 The Superconformal Anomaly

The BPS-formula used in the previous section relates $a$ and $F'(a)$ to a physical observable. The invariance of the BPS-spectrum under equivalence transformations is responsible for the $PSL(2, \mathbb{Z})$-structure which played an important role in determining the most general equivalence group in the previous sections. The purpose of this section
is to relate yet another physical object to $a$ and $F(a)$. This will be the superconformal anomaly. The invariance of the anomaly will then be used to further constrain the equivalence group.

4.1 The Superconformal Ward-Identity

Following [6] we first obtain a relation between the low energy effective action and the superconformal anomaly. The Ward identity is most easily derived in $N=1$ superspace. Using the invariance of the classical action the Schwinger functional satisfies the formal identity for an arbitrary superconformal transformation

$$\int d\mu(a,v) \ e^S[H_{\alpha\bar{\alpha},a,w_a}+(J,a+\delta a)] = \int d\mu(a,v) \ e^{S[H_{\alpha\bar{\alpha},a,w_a}+(J,a)]}$$  \hfill (27)

where $a, v$ are the $N=1$ integration variables and $H_{\alpha\bar{\alpha}}$ is the supergravity prepotential coupling to the supercurrent $j_{\alpha\bar{\alpha}}$. Expanding both sides of the equality to the first order and using $J = \frac{\delta \Gamma[A]}{\delta A}$, we obtain

$$\int d^4x d^2\theta d^2\bar{\theta} \ \frac{\delta \Gamma[A]}{\delta A} \langle \delta A \rangle = \int d^4x d^2\theta d^2\bar{\theta} \ \delta H^{\alpha\bar{\alpha}} \langle j_{\alpha\bar{\alpha}} \rangle.$$  \hfill (29)

The $N=1$ fields and the supergravity prepotential transform under an arbitrary transformation $\delta A = \bar{D}^2 L^\alpha D_\alpha A - q(\bar{D}^2 D_\alpha L^\alpha)A$ and $\delta H^{\alpha\bar{\alpha}} = (D^\alpha L^{\bar{\alpha}} - D^{\bar{\alpha}} L^\alpha)$, respectively, where $L^\alpha$ is the parameter superfield [25] and $q$ is the $R$-weight of $A$. We now consider a global axial transformation $\bar{D}^2 D_\alpha L^\alpha = i\Delta$, where $\Delta$ is a real parameter. From (30) and (3) we then obtain for the variation of the low energy effective action

$$\delta_\Delta \Gamma[A] = -\frac{iq}{4\pi} \Delta \int d^4x d^2\theta d^2\bar{\theta} \ (F''(A)A\bar{A} - \bar{F}'(A)\bar{A} + \text{h.c.}) \quad \text{(31)}$$

On the other hand we have from (29,30)

$$\delta_\Delta \Gamma[A] = \int d^4x d^2\theta d^2\bar{\theta} \ (L^\alpha D^{\bar{\alpha}} - L^{\bar{\alpha}} D^\alpha)j_{\alpha\bar{\alpha}} \quad \text{(32)}$$

where the last equality uses the anomaly equation [26, 27]

$$D^{\bar{\alpha}} j_{\alpha\bar{\alpha}} = D_\alpha S \quad \text{(33)}$$

with $S$ a chiral superfield ($D_\alpha S = 0$). Rewriting the result (32) in $N=2$ formulation we then get
\[ \text{Re} \int d^4x d^4\theta \left( \mathcal{F}(\mathcal{A}) - \frac{1}{2} \mathcal{F}'(\mathcal{A}) \mathcal{A} \right) = c \text{Re} \int d^4x d^4\theta \ i \mathcal{U}, \]  

(34)

where \( c \) is a constant and \( \mathcal{U} \) is the \( N=2 \) anomaly multiplet. This is the sought relation between the prepotential and the super conformal anomaly \( \mathcal{U} \). The same relation has been derived in [29] between \( \mathcal{F}(a) \) and the Seiberg-Witten parameter \( u \), assuming the Seiberg-Witten solution, for pure YM-theory. The derivation of the relation (34) is similar to that in [6]. It does not make any assumption on the form of \( \mathcal{F} \) and furthermore establishes the physical nature of the Seiberg-Witten parameter \( u \).

### 4.2 Constraints on the Accessory Parameters

For our purpose the precise form of the anomaly is not important. We will only make use of the fact that \( u = \mathcal{U}|_{\theta = 0} \) is a low energy observable and therefore a \( U(1) \)-section under equivalence transformations (the phase of \( u \) is not observable as it can be absorbed in a redefinition of the superspace coordinate \( \theta_\alpha \). In the perturbative limit we have from (7), (15) and (20)

\[ u(z) \propto z^{\frac{4}{4-N_F}}, \]  

(35)

where we have used \( u \propto a^2 \) in that limit. Now, using

\[ \frac{du}{dz} = ci \frac{d}{dz} \left( \mathcal{F}(a) - \frac{1}{2} \mathcal{F}'(a) a \right) = ci \mathcal{W}(a_D, a) z, \]  

(36)

where \( \mathcal{W}(a_D, a) z \) is the Wronskian of \( a_D \) and \( a \), we obtain from (18)

\[ \frac{du}{dz} = \frac{4}{4-N_F} f^2(z) h(z), \]  

(37)

The form of (37) suggests the Ansatz

\[ u(z) = \frac{4}{4-N_F} f^2(z) h(z), \]  

(38)

where \( h \) is a \( U(1) \)-section. The boundary conditions (15) and (21) require for \( h \) the asymptotic behaviour

\[ h(z) \simeq \frac{1}{2} \frac{1}{4-N_F} \frac{1}{z} \text{ for large } z. \]  

(39)

Substitution of (38) into (37) then leads to

\[ 2 \frac{f'}{f} h + h' = \frac{f''}{f} + Q \equiv \tilde{Q}. \]  

(40)

By construction \( \tilde{Q} \) has at most simple and double poles. At a double pole \( a_i \), say of

\footnote{This statement is equivalent to the fact that in \( N=1 \)-language the anomaly is of the form \( \text{Tr} AA + cFF \) which is clearly independent of the phase of \( \phi \).}
\( \tilde{Q}, h \) then has a simple pole\(^3\). The residue \( h_i \) is then obtained from (37), i.e.

\[
h_i = \frac{1}{2} \frac{r_i^2 - r + \frac{1}{4}(1 - \alpha_i^2)}{r_i - \frac{1}{2}}.
\]  (41)

The residue \( q_i \) of \( \tilde{Q} \) at this point is then given in turn by

\[
q_i = 2r_i \left( h(z) - \frac{h_i}{z - a_i} \right) \big|_{z=a_i} + 2h_i \left( \frac{f'(z)}{f(z)} - \frac{r_i}{z - a_i} \right) \big|_{z=a_i}.
\]  (42)

The only other possibility for \( h \) to be singular is if \( \tilde{Q} \) has a simple pole at \( a_i \); this happens if and only if \( r_i = \frac{1}{2}(1 \pm \alpha_i) \). Cancellation of the double poles on the left hand side of (40) then requires that \( \alpha_i = 0 \). If \( \alpha_i \neq 0 \), then \( h \) must be regular at this point and (40) requires

\[
\beta_i = 4r_i h(a_i) - 4r_i \left( \frac{f'}{f} - \frac{r_i}{z - a_i} \right) \big|_{z=a_i}.
\]  (43)

To summarize, unless \( r_i = \frac{1}{2} \) for all \( i \), the superconformal Ward-identity which requires \( u \) to be a \( U(1) \)-section, leads to extra relations between the \( a_i, \alpha_i \) and the accessory parameters \( \beta_i \) of \( Q \). On the other hand we know from the theory of Fuchsian maps that there is only a discrete set of parameters \( (\alpha_i, \beta_i, a_i) \) for which the upper half plane \( H \) is mapped into an \( \text{PSL}(2, \mathbb{Z}) \) polygon. Furthermore, for a given polygon these parameters are uniquely fixed once the origin and the scale in \( H \) are chosen. We therefore conclude that

\textbf{Unless all the angles} \( \alpha_i \) \textbf{are zero the system is over-determined by the requirement that} \( u \) \textbf{be a} \( \text{U}(1) \)-section under equivalence transformations.}

Unfortunately the accessory parameters \( \beta_i \) of \( Q \) are known explicitly only for some simple polygons \([22]\). This forces us to proceed in a somewhat roundabout way: For \( \mu = 1 \) the only polygon is the first domain in Fig. 1. The Schwarzian for this polygon is given by

\[
Q(z) = \frac{2}{9(z-1)^2} + \frac{2}{9(z+1)^2} - \frac{7}{36(z^2-1)}.
\]  (44)

Recalling that \( T \in \mathcal{G} \), one finds that this polygon is compatible with a \( f \)-section only for \( N_F = 3 \) in which case

\[
f(z) \propto (z^2 - 1)^{3/4} \quad \text{and} \quad \frac{du}{dz} \propto \frac{36z^2 - 31}{9(z^2 - 1)^{7/2}}.
\]  (45)

\[\text{From (11) and (12) we obtain } \beta_1 = -\beta_2 = -11/12 \text{ which is incompatible with (14). Additionally one can directly integrate (45) to:}\]

\[\text{[3] Although a second order pole would at first sight be compatible with the boundary condition (21) for } N_F = 3, \text{ explicit inspection of all possible polygons shows that this possibility is in fact excluded.}\]
\[ u(z) \propto 2z \sqrt{z^2 - 1} - \frac{13}{9} \log(z + \sqrt{z^2 - 1}). \]  \hspace{1cm} (46)

The presence of the log term shows that \( u \) is not a \( U(1) \)-section. Therefore \( \mu = 1 \) never corresponds to a solution.

Explicit inspection of all possible polygons with \( 2 \leq \mu \leq 6 \) and their corresponding \( f \)-sections, taking into account the identifications due to \( T \in \mathcal{G} \) and the boundary conditions, reveals that the exponents always satisfy \( r_i = \frac{1}{2}(1 - \alpha_i) \) with exceptions only for \( \alpha_i = 0 \). At these points, \((41)\) becomes simply
\[ h_i = \frac{1}{2}(r_i - \frac{1}{2}). \]  \hspace{1cm} (47)

If \( n_0 \) is the number of zero angles of the polygon \( D \), the total residue condition \((24)\) together with \((17)\) implies
\[ \frac{\mu}{6} - \frac{1}{4 - N_F} = \frac{n_0}{2} - \sum_{i=1}^{n_0} r_i. \]  \hspace{1cm} (48)

Suppose now that \( r_i \neq \frac{1}{2} \) for all zero angles. Substituting \((47)\) and \((33)\) into \((48)\) we then find \( \mu = 0 \) which is a contradiction. We therefore conclude that there is at least one zero angle for which \( r = \frac{1}{2} \). In addition it follows from \((18)\) that for \( \mu < 6/4 - N_F \), any admissible polygon has at least two zero angles. We finish this section noting from \((41)\) that \( Q \), which carries the underlying \( SL(2, \mathbb{Z}) \) structure can be constructed from the two \( U(1) \)-sections \( f \) and \( u \).

5 Solution of the Model

We consider the cases \( N_F = 1, 2, 3 \) individually.

\( N_F = 1 \)

From \((23)\) we have \( \mu \leq 2 \). Since \( \mu = 1 \) has been excluded in the previous section, the only possibility is \( \mu = 2 \) which corresponds to a double covering of the fundamental domain \( D_0 \) of \( PSL(2, \mathbb{Z}) \) (see Fig. 1).

This domain has angles \( \alpha_1 = 0 \) and \( \alpha_2 = \alpha_3 = \frac{2}{3} \). The Schwarzian for this polygon is easily found to be
\[ Q(z) = \frac{1}{4z^2} + \frac{5}{36(z - 1)^2} + \frac{5}{36(z + 1)^2} - \frac{5}{18(z^2 - 1)}. \]  \hspace{1cm} (49)

There is a unique section \( f \) compatible with \((24)\) and \((22)\). It is given by
\[ f(z) = f_0 z^{1/2}(z^2 - 1)^{1/6}, \]  \hspace{1cm} (50)

where \( f_0 \) is a constant. The equivalence group of this solution is a subgroup of index 2
of $U(1) \times PSL(2, \mathbb{Z})$. Its elements contain an even number of $S$-generators of $PSL(2, \mathbb{Z})$. Recall that the representation of $G$ on $\tau$ is by a subgroup of $PSL(2, \mathbb{Z})$ for which the relation $(TS)^3 = 1$ holds. Therefore the representation on $\tau$ of $G$ generates all of $PSL(2, \mathbb{Z})$. This explains the question raised in [30] concerning this solution. It also shows why it is important to define the equivalence group by its action on $a$ rather than $\tau$.

Using (49), (50) and integrating (37) we then have $u - u_0 \propto (z^2 - 1)^{1/3}$ where $u_0$ is a constant of integration. Proceeding as explained in [4] and in the appendix, one then easily shows that for $u_0 = 0$, $a$ satisfies precisely the differential equation corresponding to the Seiberg-Witten solution [14]. Note that there is an ambiguity in the relation between $u$ and $z$ due to the integration constant $u_0$. It is constrained to zero, either by an explicit 2-instanton computation [8, 11, 12] or by imposing invariance under the discrete $R$-transformations $\phi \rightarrow e^{2\pi i/3}\phi$.

$N_F = 2$

This case is identical to $N_F = 0$ once the normalisation are chosen appropriately [3]. We have $\mu \leq 3$. From the general discussion presented above, the two domains with $\mu = 1, 2$ are however excluded as they do not have enough zero angles. For $\mu = 3$ the only polygons $D$ are fundamental domains for $\Gamma_0(2)$ and without restricting the generality we can choose one which has only zero angles [5]. Hence the necessary condition (23) is at the same time sufficient and therefore the superconformal anomaly imposes no further constraints. Consequently the unique solution for $N_F = 2$ has $\mu = 3$. The equivalence group is $\Gamma_0(2)$. Again this solution is precisely that proposed by Seiberg and Witten [4]. The proof of this is identical to that presented in section 5 of [4]. The above analysis also shows that the more general definition of the equivalence group adopted here does not lead to new solutions for $N_F = 0$, as anticipated in the introduction.

$N_F = 3$

This case is more involved. $\mu = 1$ and 2 are excluded from the previous discussion. $\mu = 3$ can be shown to be incompatible with (48). For $\mu = 4$ and 5 we found one and three admissible polygons with 2 zero angles respectively. For $\mu = 6$ there is a whole set of polygons among which there is precisely one with only zero angles ($n = 3$),
and which is a fundamental domain of $\Gamma_0(4)$. In that case again, the superconformal Ward-identity implies no new constraint and therefore this last polygon is a solution. This polygon has for Schwarzian $Q$,

$$Q(z) = \frac{1}{4z^2} + \frac{1}{4(z-1)^2} + \frac{1}{4(z+1)^2} - \frac{1}{2(z^2-1)},$$

(51)

and a unique $U(1)$-section $f$

$$f(z) \propto z^{1/2}(z^2 - 1)^{1/2}.$$ (52)

Proceeding as before we find $u' = \text{const}$. Again this solution is identical to the solution proposed by Seiberg and Witten modulo the undetermined integration constant in the relation between $u$ and $z$ whose value can only be fixed by explicit 2-instanton computations [12].

For all the other candidates the Ward-identity does lead to extra constraints and the system is therefore over-determined. We therefore expect these to be ruled out. However we are not able, presently to show this explicitly. Indeed, although for all these candidates we can write down a closed form for the Schwarzian $Q$, we have been unable to decide whether $Q$ corresponds to an $PSL(2, \mathbb{Z})$-map, due to the lack of information about the accessory parameters $\beta_i$ in the Fuchsian maps [22].

Remark: At this point a remark about possible candidates for alternative solutions in the framework of [2] is of order. In [31] it was noted that in addition to the solutions proposed in [2] the curves

$$y^2 = x^2(x-u) - \Lambda_2^4(x - cu) \quad \text{for} \quad N_F = 2,$$

(53)

where $c = 1/9$, and

$$y^2 = x^2(x-u) - \Lambda_3^2u^2 \quad \text{for} \quad N_F = 3$$

(54)

satisfy the conditions i) to iv) given in section 11.3 of [2]. A careful analysis shows however that the curves (53) and (54) have unstable singularities (i.e. three roots coincide at some singular points in the $u$-plane). A possibly related property of these curves is that the holomorphic 1-form $\lambda$ defined by

$$\partial_u \lambda = \frac{\sqrt{2}}{8\pi} dx \frac{y}{y} + dw,$$

(55)

where $w$ is a mereomorphic function, has non-vanishing residues. Indeed integrating (53) we get

$$\lambda = 2u dx y + 2x^2 - \frac{\Lambda_2^4}{\Lambda_3^2c - x^2} x dx y \quad \text{for} \quad N_F = 2,$$

(56)

$$\lambda = \frac{\Lambda_3^2u}{4\Lambda_3^2 + x} dx \frac{y}{y} + \frac{2u - x}{4\Lambda_3^2 + x} x dx y \quad \text{for} \quad N_F = 3.$$
Residues are however incompatible with two-dimensional monodromies. We conclude therefore that the curves $[53,54]$ do not lead to new solutions.

6 Conclusions

Using supersymmetry, asymptotic freedom and assuming a finite number of strong-coupling singularities we have derived the low energy effective Lagrangians for, $N = 2$ supersymmetric $QCD$ with $N_F < 3$ massless hypermultiplets. For $N_F = 3$ the argument is only almost complete. The problem however is not of a principle kind, but due to our lack of information about the accessory parameters for Fuchsian maps. An important role is played by the superconformal anomaly. It puts non-trivial constraints on an otherwise degenerate system in such a way that the solution becomes unique.

The low energy Lagrangians we found are (up to a constant of integration) identical with those of Seiberg and Witten [2]. It should be stressed however that we do not make any assumption concerning the role of the monopoles as in [2] nor do we assume that the superconformal anomaly parameterizes the moduli space [1]. Rather we have shown that these two properties are consequences of more fundamental properties namely the extended supersymmetry and the underlying $PSL(2, Z)$-structure of the theory. Although the finite mass condition, which reflects the asymptotic freedom of the underlying non-abelian theory, was used in the present derivation it is conceivable that this is itself a consequence of the superconformal Ward-identity. A more detailed analysis of the superconformal Ward-identity along the lines presented in section 4.2 might answer this question.

Our results also clarify the 'exceptional' role of $N_F = 1$ observed in [30].

An interesting technical observation is that for cases discussed so far the upper bound for the index $\mu$ is always saturated. We believe that the reason for this originates in the condition that the superconformal anomaly be a $U(1)$-section, but we have not been able to produce a general argument for this conjecture so far.

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Appendix A. Differential Equations

The simplest way to exhibit the equivalence between the solutions obtained in section 4 and the corresponding Seiberg-Witten solutions is to write down their differential equations. First we fix the multiplicative constants appearing in $f$ and $y$ and then obtain the differential equation satisfied by $a$. Without restricting the generality we may take $y$ as in (19); the other constant $f_0$ introduced in (21) is then determined as follows. The perturbative result for $\tau(a)$ has been derived in section 2.1 of [31]:
\[ \tau(a) \simeq \frac{i}{\pi} \log \left( \frac{a^{4-N_F}}{a_0} \right), \quad a_0 = 2^{N_F-12} \Lambda_{N_F}^{4-N_F} \]  
(A.1)

where \( \Lambda_{N_F} \) is the dynamical scale in the Pauli-Villars scheme. On the other hand, \( \tau(z) \simeq \frac{i}{\pi} \log(\tau_0 z) \) where \( \tau_0 \) depends on \( N_F \). Thus,

\[ a(z) \simeq (a_0 \tau_0 z)^{1-N_F}. \]  
(A.2)

It follows from (18), (19) and (A.2) that \( f_0 = (4 - N_F)^{(a_0 \tau_0)^{1-N_F}}. \)

Since the procedure is the same in all cases we write explicitly the differential equation satisfied by \( a \) only for \( N_F = 1 \). \( Q \) and \( f \) are then given by (19) and (30) respectively. The constant \( \tau_0 \) is then determined noting that \( z \) is related to the \( PSL(2, \mathbb{Z}) \)-modular function \( J \) by \( J = -\frac{(z^2 - 1)^2}{4z^2} \) and by using the asymptotic limit for \( \tau(J) \) [32]. With this we then obtain from (13), (13) and (37) the sought differential equation for \( a \):

\[ a_{uu} + \frac{1}{4} \frac{u - u_0}{(u - u_0)^3} + \frac{27\Lambda_1}{16} a = 0. \]  
(A.3)

Eqn. (A.3) is precisely the differential equation satisfied by the Seiberg-Witten Ansatz [14] provided that \( u_0 = 0 \) and that the scale \( \Lambda_{SW}^1 \) used by Seiberg and Witten is related to the Pauli-Villars scale by \( (\Lambda_{SW}^1)^3 = 4\Lambda_1^3 \). In particular we recover in this way the result of [31, 8]. The other cases \( N_F = 0, 2 \) and 3 can be done in the same way.

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