MODEL CATEGORIES STRUCTURES FROM RIGID OBJECTS IN EXACT CATEGORIES

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ABSTRACT. Let $E$ be a weakly idempotent complete exact category with enough injective and projective objects. Assume that $M \subseteq E$ is a rigid, contravariantly finite subcategory of $E$ containing all the injective and projective objects, and stable under taking direct sums and summands. In this paper, $E$ is equipped with the structure of a prefibration category with cofibrant replacements. As a corollary, we show, using the results of Demonet and Liu in [DL13], that the category of finite presentation modules on the costable category $\overline{M}$ is a localization of $E$. We also deduce that $E \to \text{mod,} \overline{M}$ admits a calculus of fractions up to homotopy. These two corollaries are analogues for exact categories of results of Buan and Marsh in [BM13], [BM12] (see also [Be13]) that hold for triangulated categories.

If $E$ is a Frobenius exact category, we enhance its structure of prefibration category to the structure of a model category (see the article of Palu in [Pal14] for the case of triangulated categories). This last result applies in particular when $E$ is any of the Hom-finite Frobenius categories appearing in relation to cluster algebras.

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Homotopical algebra was first introduced by Quillen in [Qui67]. It appears in many subjects, such as geometry (for example the study of $G$-spaces in [DK85], spectra in [BF78]) or algebra (the study of commutative rings in [Qui70] or simplicial groups in [Qui69]). Its aim was to axiomatize the structure of the category of topological spaces that is relevant for doing homotopy, thus allowing to make use of different categories (such as sSet) and to compare different homotopy theories. This axiomatization not only covers homotopy, but also homological algebra, whence the name of homotopical algebra.

In a seemingly unrelated direction, some subquotients of triangulated categories play a specific role in the theory of additive categorification of cluster algebras. Let $T$ be a cluster-tilting object in a cluster category $C$ with shift functor $\Sigma$. Buan, Marsh and Reiten proved in [BMR08] that the ideal quotient $C/(\Sigma T)$ is equivalent to the category $\text{modEnd}_C(T)^{op}$ of finitely presented modules over the endomorphism algebra of $T$. This result was then generalized by Keller and Reiten in [KR08] (to the case of $m$-cluster-tilting objects), Koenig and Zhu in [KZ08] (to the case of triangulated categories without any Calabi-Yau assumption) and Iyama and Yoshino in [IY08]: If $R$ is a rigid object in a Hom-finite triangulated category with a Serre functor, let $R*\Sigma R$ be the full subcategory of $C$ whose objects $X$ appear in some triangle $R_1 \to R_0 \to X \to \Sigma R_1$, with $R_1, R_0 \in \text{add}R$. Then the functor $C(R, -)$ induces an equivalence of categories

$$R/(\Sigma R) \to \text{modEnd}_C(R)^{op}.$$  

In [BM13], Buan and Marsh proved that $R/(\Sigma R)$ can be described as a localization of $C$ (at the class of all morphisms $f$ such that $C(R, f)$ is an isomorphism in $\text{modEnd}_C(R)^{op}$). They further proved in [BM12], that this localization admits a calculus of fractions at the level of the category $C/(\Sigma R)$.

Finally, Palu in [Pal14] showed that there was almost a model structure on $C$ such that the homotopy category of $C$ was the localization. To be accurate, he showed that $R*\Sigma R/(\Sigma R)$ was equivalent to the category of cofibrant and fibrant objects up to homotopy.

Our aim in this paper is to prove a similar result for exact categories. In doing so, we provide a conceptual explanation for the existence of the calculus of fractions. Demonet and Liu in [DL13] have shown a theorem similar to that of the first result for triangulated categories: Let $\mathcal{E}$ be an exact category with enough projective and injective objects. Let

$$\text{pr}M = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

and

$$\text{tr}M = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \text{Inj}, 0 \to M \to I \to X \to 0\}.$$  

Demonet and Liu proved that there is an equivalence of categories

$$\text{pr}M/\text{tr}M \to \text{mod} \mathcal{M}$$

via the functor

$$G : \mathcal{E} \to \text{Mod} \mathcal{M}, X \mapsto \mathcal{E}(-,X)/\mathcal{M}.$$  

This shows a result similar to that of the first one in the case of triangulated categories. However, there is no known description of this subquotient as a localization of $\mathcal{E}$. The
aim of the paper is to give one in the case where \( \mathcal{E} \) is an exact, then a Frobenius category. We are going to show that, under some technical assumptions the localization of some class of weak equivalences is equivalent to \( \text{mod } \overline{\mathcal{M}} \), and that there exists a (homotopy) calculus of fractions.

More precisely, we show the following results: we start with the particular case of a Frobenius category

**Theorem 0.1.** Let \( \mathcal{E} \) be a Frobenius category. Let \( \mathcal{M} \) be a full subcategory of \( \mathcal{E} \) containing the injectives, and contravariantly finite and let \( \mathcal{W} \) be the class of those morphisms whose image under the functor \( G \) is an isomorphism. Then, there exist two classes of morphisms, \( \mathcal{F}_{\text{ib}} \) and \( \mathcal{C}_{\text{of}} \), forming a model structure on \( \mathcal{E} \). More precisely, all the objects are fibrant, and an object is cofibrant if and only if it belongs to \( \text{pr} \mathcal{M} \).

We note however that one of the factorizations only exists for morphisms with cofibrant domain.

These results apply to the following classes of examples: Geiss, Leclerc and Schröer studied module categories over the preprojective algebras in the Dynkin case in [GLS06], in the non-Dynkin case in [GLS08a] and a nice survey can be found in [GLS08b].

Another class of examples is given by the categories of Cohen-Macaulay modules over simple hypersurface singularities studied by Burban, Iyama, Keller and Reiten in [BKR08] (their stable categories also appear in the article of Buan, Palu and Reiten, [BPR16]).

As a consequence, results due to Quillen in [Qui67] can be applied in order to obtain the following:

**Corollary 0.2.** Let \( \mathcal{E} \) be a weakly idempotent Frobenius category. Let \( \mathcal{M} \) be a subcategory of \( \mathcal{E} \) which is rigid and contravariantly finite, containing all the injective objects.

Let \( \sim \) be the homotopy relation on \( \mathcal{E} \) given by the model structure of theorem 0.1. Then two morphisms \( f \) and \( g \) are homotopic if and only if \( f - g \) factorizes through \( \overline{\mathcal{O}} \mathcal{M} \). Let \( \text{Ho} \mathcal{E} \) be the localization of the quasi-isomorphisms of \( \mathcal{E} \) at the class \( \mathcal{W} \) of weak equivalences. Then there is an equivalence of categories

\[
\text{pr}\mathcal{M}/(\overline{\mathcal{O}} \mathcal{M}) \simeq \text{Ho} \mathcal{E}.
\]

From the equivalence of categories of Demonet and Liu in [DL13], there is an equivalence of categories \( \text{Ho} \mathcal{E} \to \text{mod } \overline{\mathcal{M}} \).

Then we generalized these results to exact categories. Unfortunately, they do not have a model structure. They actually have a prefibration (or dually a precofibration) structure.

**Theorem 0.3.** Let \( \mathcal{E} \) be a weakly idempotent complete exact category with enough injective objects. Assume that \( \mathcal{M} \subseteq \mathcal{E} \) is a rigid, contravariantly finite subcategory of \( \mathcal{E} \) containing all the injective objects, and stable under taking direct sums and summands.

Then there exist two classes of morphisms \( \mathcal{F}_{\text{ib}} \) and \( \mathcal{W} \) (the same weak equivalences as in the Frobenius case) such that \( (\mathcal{E}, \mathcal{F}_{\text{ib}}, \mathcal{W}) \) has a prefibration structure in the sense of Anderson-Brown-Cisinski (see [RB06] for a deeper study). All the objects are fibrant, and an object is cofibrant if and only if it belongs to \( \text{pr} \mathcal{M} \). Moreover, under the assumption that \( \mathcal{M} \) contains the projective objects, there exist some cofibrant replacements for any object of \( \mathcal{E} \).
This theorem is sufficient to prove corollary 0.2 in the case where $E$ is only equipped with an exact structure.

**Corollary 0.4.** We deduce from this that there is a homotopy calculus of fractions in the sense of Radulescu-Banu in [RB06].

With this result, Radulescu-Banu gives a theoretical interpretation to the results of Buan and Marsh in [BM12].

The paper is organised as follows: In the first part, we get interested in the exact case. Section 1 is devoted to the study of $\text{pr}_M$. Then in section 2, we focus on weak equivalences and fibrations. In section 3 we set the factorization. Section 4 is devoted to the cofibrant replacements and the homotopy relation. The two last sections of this part consist in the proof the the prefibration structure (theorem 0.3) and the theorem of Quillen (0.2).

In the second part, we take the particular case of Frobenius categories. In section 1, we give additional properties of $\text{pr}_M$ that hold in this more specific setup. Then, in section 2, we show a characterization of weak equivalences. Section 3 deals with fibrations and cofibrations, and in section 4, we find the second factorization. Finally, in section 5, we show the model structure on the Frobenius category.

### 1. Preliminaries

1.1. **Conventions and results on exact categories.** In this section, we choose as a reference the article of Bulher [Büh10].

**Definition 1.1.** Let $E$ be an additive category. A short exact sequence $(i, p)$ in $E$ is a pair of composable morphisms $A \xrightarrow{i} A' \xrightarrow{p} A''$ such that $i$ is a kernel of $p$ and $p$ is a cokernel of $i$. If a class $C$ of short exact sequences on $E$ is fixed, an inflation is a morphism $i$ for which there exists a morphism $p$ such that $(i, p) \in C$. Deflations are defined dually.

An exact structure on $E$ is a class $C$ of short exact sequences, which is closed under isomorphisms and satisfies the following axioms:

- **E0:** The identity morphism is an inflation.
- **E0':** The identity morphism is a deflation.
- **E1:** The class of inflations is closed under composition.
- **E1':** The class of deflations is closed under composition.
- **E2:** The push-out of an inflation along an arbitrary morphism exists and yields an inflation.
- **E2':** The pull-back of a deflation along an arbitrary morphism exists and yields a deflation.

**Definition 1.2.** We say that $E$ has enough injective objects if any object $A$ of $E$ appears in a short exact sequence $A \rightarrow I \rightarrow B$, where $I$ is injective. Dually, we define what "having enough projective" means.

It is said that $E$ is a Frobenius category if there are enough projective and injective objects, and if they coincide.
Lemma 1.3 (Buhler, [Büh10]). Let

\[ \begin{array}{ccc}
A \rightarrow & B \\
\downarrow a & \downarrow b \\
C \rightarrow & D
\end{array} \]

be a commutative square, where the horizontal arrows are inflations. The following assertions are equivalent:

- The square is a pushout
- The sequence \( A \rightarrow B \oplus C \rightarrow D \) is short exact
- The square is bicartesian, that is, a pushout and a pullback

Lemma 1.4 (Demonet-Liu, [DL13]). If \( 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \) and \( 0 \rightarrow Y \rightarrow W \rightarrow V \rightarrow 0 \) are two short exact sequences, then there is a commutative diagram of short exact sequences where the upper-right square is both a push-out and a pull-back:

\[ \begin{array}{cc}
0 & 0 \\
\downarrow & \downarrow \\
0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \\
\downarrow & \downarrow \\
0 \rightarrow X \rightarrow W \rightarrow U \rightarrow 0 \\
\downarrow & \downarrow \\
V \rightarrow V \\
\downarrow & \downarrow \\
0 & 0
\end{array} \]

1.2. **Weakly idempotent complete category.** In this part, we explain the notion of weakly idempotent complete category, a concept we need all throughout the paper.

**Definition 1.5.** Let \( C \) be an additive category. Then \( C \) is said to be weakly idempotent complete when any retraction has a kernel (which is equivalent to the fact that any section has a cokernel).

We also have the following characterization:

**Theorem 1.6 ([Büh10]).** If \( C \) is exact, then the category \( C \) is weakly idempotent complete if and only if any retraction is a deflation (or equivalently, any section is an inflation).

**Corollary 1.7 ([Büh10]).** If \( C \) is exact, then the category \( C \) is weakly idempotent complete if and only if:

- for two composable morphisms \( f \) and \( g \), if \( g \circ f \) is a deflation, then so is \( g \).

We will freely make use of this latter characterization throughout the paper.
1.3. An equivalence of categories. We recall the definition of a rigid subcategory.

**Definition 1.8.** Let $\mathcal{E}$ be an exact category. Then, a subcategory $\mathcal{M}$ of $\mathcal{E}$ is said to be rigid if $\text{Ext}^1_{\mathcal{E}}(\mathcal{M}, \mathcal{M})$ is zero.

Demonet and Liu showed in [DL13] that there is an equivalence of category that we will try and describe as a localization.

**Theorem 1.9 ([DL13]).** Let $\mathcal{E}$ be an exact category. Let $\mathcal{M}$ be a full rigid subcategory which contains the injective objects. Let

$$\text{pr}_\mathcal{M} = \{X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0\}$$

and

$$\emptyset \mathcal{M} = \{X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \text{Inj}, 0 \to M \to I \to X \to 0\}.$$

Let $G$ be the following functor:

$$G : \mathcal{E} \to \text{Mod}\overline{\mathcal{M}}$$

$$X \mapsto \mathcal{E}(\cdot, X)[\mathcal{M}]$$

Then, the functor $G$ induces the following equivalence:

$$\text{pr}_\mathcal{M}/\emptyset \mathcal{M} \simeq \text{mod}\overline{\mathcal{M}}.$$

1.4. Model categories. In this part, we recall the definition of a model category due to Quillen in [Qui67]. For a gentle introduction to model categories, see Dwyer and Spalinski in [DS95]. For a deeper study, see Hovey in [Hov99].

**Definition 1.10.** Let $\mathcal{C}$ be a category equipped with a class $W$ of morphisms called weak-equivalences. It is said that there is a model structure on $\mathcal{C}$ when there exist three classes of morphisms called $(W, \text{Fib}, \text{Cof})$ such that:

- **MC1:** Finite limits and colimits exist in $\mathcal{C}$.
- **MC2:** $W$ has the "two-out-of-three" property, it means that, for two composable maps $g$ and $f$, if two of the three maps $f$, $g$ or $gf$ are in $W$, so is the third.
- **MC3:** The three classes $\text{Fib}, \text{Cof}$ and $W$ are stable under retracts.
- **MC4:** For a commutative diagram

\[
\begin{align*}
A \xrightarrow{a} X \\
\downarrow^h \quad \downarrow^f \\
B \xrightarrow{b} Y
\end{align*}
\]

a lift exists if: either $h$ is a cofibration and $f$ is an acyclic fibration (it means that $f \in \text{Fib} \cap W$) or $h$ is an acyclic cofibration and $f$ is a fibration.

- **MC5:** Any morphism can be factored in two ways: as a cofibration followed by an acyclic fibration, and as an acyclic cofibration followed by a fibration.

1.5. The homotopy category. In his book, Quillen proved that the localization $\text{Ho} \mathcal{C}$ of a model category $\mathcal{C}$ obtained by inverting all the weak equivalences is equivalent to the quotient category $\mathcal{C}_{\text{cj}}/ \sim$ of the cofibrant and fibrant objects by the homotopy equivalence. This proves that the localization is indeed a category. For a detailed proof of this result, see also Hovey in [Hov99 1.2.9].
Lemma 1.11 (Quillen). Let $\mathcal{C}$ be a model category. If $X$ and $Y$ are fibrant and cofibrant, the homotopy relation is an equivalence relation on $\mathcal{C}(X,Y)$, compatible with the composition.

The consequence is that the category of fibrant and cofibrant objects up to homotopy is well-defined.

Theorem 1.12 ([Qui67]). Let $\mathcal{C}$ be a model category. Let $\mathcal{C}_{cf}$ be the full subcategory of fibrant cofibrant objects. Let $\sim$ be the homotopy relation. Let $\text{Ho} \mathcal{C}$ be the localization of $\mathcal{C}$ obtained by inverting all the weak equivalences. Then we have $\text{Ho} \mathcal{C} \cong \mathcal{C}_{cf}/\sim$.

Remark 1.13. In his proof, Hovey only needs the existence of the second factorization on cofibrant objects. So, as we are only going to show this factorization on cofibrant objects, this is sufficient to use this theorem.

1.6. Constructing model structures. There is a theorem we can use in order to show that a category is equipped with a model structure. But let first recall a definition.

Definition 1.14. Let $f : A \to B$ and $g : X \to Y$ be two morphisms in a category $\mathcal{C}$. We say that $f \Box g$ if, for any commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow{a} & & \downarrow{g} \\
B & \xrightarrow{b} & Y \\
\end{array}
\]

there exists an arrow from $B$ to $X$ such that both triangles commute.

If $A$ is a class of morphisms, we write

$\Box A = \{f \text{ morphism of } \mathcal{C} \text{ such that } f \Box g, \ \forall g \in A\}$.

Dually, we write

$A^\Box = \{g \text{ morphism of } \mathcal{C} \text{ such that } f \Box g, \ \forall f \in A\}$.

Theorem 1.15 ([Hov99]). Let $\mathcal{C}$ be a category. Let $W$, $\text{Fib}$, $\text{Cof}$ be three classes of morphisms, where there exists $I$ (respectively $J$) such that $\text{Fib} = J^\Box$ (respectively $\text{Cof} = \Box I$). Then there is a model structure on $\mathcal{C}$ if and only if:

(i) The class of morphisms $W$ has the "two-out-of-three" property, and is stable under retracts.

(ii) We have $J^\Box \cap W = I^\Box$.

(iii) We have $\Box (J^\Box) \subseteq W \cap \Box (I^\Box)$.

(iv) Any morphism can be factored through a weak cofibration followed by a fibration.

Any morphism can be factored through a cofibration followed by a weak fibration.

Remark 1.16. The previous theorem is usually stated in a slightly different manner. Factorizations are not required to be given a priori; instead, one requires that the domains of $I$ (respectively $J$) be small relative to $I$-cell (respectively $J$-cell), where, for a class of morphisms $A$, $A$-cell is the collection of $A$-cell complexes, which is a transfinite
composition of pushouts of elements of $A$. The factorizations are then given by the small object argument.

Here, we cannot apply the small object argument, because, in most of our examples (for example, see [BIKR08]), we do not have all the colimits. We therefore explicitly compute the factorizations without using the small object argument.

1.7. About a weak model structure on triangulated categories. In his paper [Pal14], Palu has shown that any covariantly finite rigid subcategory of a triangulated category gives rise to a weak model structure. This inspires the idea of the proof in this article, but all proofs are independent.

**Theorem 1.17** (Palu, [Pal14]). Let $C$ be a triangulated category with a covariantly finite, rigid subcategory $T$.

There is a left-weak model category structure on $C$, where:

- All objects are fibrant.
- An object is cofibrant if and only if it belongs to $T \ast \Sigma T$.
- Weak equivalences $W$ compose the class of morphisms $X \to Y$ such that, for any triangle $Z \to X \to Y \to \Sigma Z$, both other morphisms belong to the ideal $(T^\perp)$.
- Two morphisms are homotopic if and only if their difference factors through $\text{add} T$.

It means that $C$ contains three classes of morphisms $W, \text{Fib}, \text{Cof}$ such that:

1. Pullbacks of trivial fibrations along deflations exist and are trivial fibrations.
2. The class $W$ has the "two-out-of-three" property.
3. The three classes $W, \text{Fib}, \text{Cof}$ contain all identities, are stable under retracts and composition.
4. We have $W \cap \text{Cof} \subseteq \square \text{Fib}$ and $\text{Cof} \subseteq \square (W \cap \text{Fib})$
5. Any morphism can be factored through a weak cofibration followed by a fibration. Any morphism with cofibrant domain can be factored through a cofibration followed by a weak fibration.

1.8. Prefibration categories. In his paper [RB06], Radulescu-Banu works on prefibration categories.

**Definition 1.18.** An ABC prefibration category (ABC stands for Anderson-Brown-Cisinski) consists of a category $E$, with two classes of maps, the fibrations $\text{Fib}$ and the weak equivalences $W$ satisfying the following axioms:

**F1:** $E$ has a final object which is fibrant. Fibrations are stable under composition. All isomorphisms are weak equivalences, and all isomorphisms with fibrant codomain are trivial fibration (it means fibrations which are also weak equivalences).

**F2:** $W$ has the "two-out-of-three" property, it means that, for two composable maps $g$ and $f$, if two of the three maps $f, g$ or $gf$ are in $W$, so is the third.

**F3:** Both classes $\text{Fib}$ and $W$ are stable under retracts.

**F4:** For a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{h} & & \downarrow{f} \\
B & \xrightarrow{b} & Y
\end{array}
\]
the pullback exists in \( \mathcal{E} \). Moreover, if \( f \) is a fibration (respectively a trivial fibration), then \( h \) is a fibration (respectively a trivial fibration).

**F5**: Any morphism \( f : A \to B \), with \( B \) fibrant, can be factored through a weak equivalence followed by a fibration.

He also shows the useful theorem:

**Theorem 1.19** (Radulescu-Banu, [RB06], Theorem 6.4.2). If \( \mathcal{E}, \mathcal{W} \) is a category pair satisfying the following:

1. The two-of-the-three axiom.
2. For a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{h} & & \downarrow{f} \\
B & \xrightarrow{b} & Y
\end{array}
\]

the pullback exists in \( \mathcal{E} \). Moreover, if \( f \) is a weak equivalence, then \( h \) is a weak equivalence.

3. For any maps \( A \xrightarrow{f} B \xrightarrow{t} B' \) with \( t \in \mathcal{W} \), and \( t \circ f = t \circ g \), there exists \( t' : A' \to A \in \mathcal{W} \) such that \( f \circ t' = g \circ t' \).

Then, we have the following results:

1. Each map \( h : A \to B \) in the homotopy category can be written as a right fraction \( f \circ s^{-1} \), with \( s \in \mathcal{W} \).
2. Two fractions \( f \circ s^{-1} \) and \( g \circ s^{-1} \) are equal in the homotopy category if and only if there exist weak equivalences \( s', t' \), as in the diagram below, and such that \( s \circ s' = t \circ t' \) and \( f \circ s' = g \circ t' \):

1.9. **Some examples of such categories.** As it has been said in the introduction, there are several examples of applications of this article.

**Definition 1.20.** Let \( A \) be a finite-dimensional algebra over an algebraically closed field \( K \). Let \( Q = (Q_0, Q_1, s, t) \) be a Dynkin quiver. Let \( \overline{Q} \) be the quiver obtained from \( Q \) by adding, for each arrow \( \alpha \) of \( Q \) from \( i \) to \( j \), an arrow \( \alpha^* \) from \( j \) to \( i \). Let \( K\overline{Q} \) be the path algebra over \( \overline{Q} \). Then the preprojective algebra associated with \( Q \) is defined as

\[
\Lambda_Q = K\overline{Q}/T
\]
where \( I \) is the ideal generated by the element
\[ \sum_{\alpha \in Q_1} (\alpha^* \circ \alpha - \alpha \circ \alpha^*). \]

We define by \( \pi_Q \) the restriction functor \( \text{mod} \Lambda \to \text{mod} KQ \).

**Lemma 1.21.** Let \( Q \) and \( Q' \) be two quivers of the same Dynkin type. Then \( \Lambda_Q \) is isomorphic to \( \Lambda_{Q'} \).

In the Dynkin case, the category \( \text{mod} \Lambda \) is Frobenius, Hom-finite, stably 2-Calabi-Yau and has cluster-tilting objects.

**Definition 1.22.** Let \( Q \) be a finite connected quiver with \( n \) vertices and without oriented cycles, and let \( \Lambda \) be its associated preprojective algebra. As \( KQ \) is a subalgebra of \( \Lambda \), we can introduce \( \pi_Q : \text{mod}(\Lambda) \to \text{mod}(KQ) \) as the restriction functor.

Let \( M \) be a \( KQ \)-module. Then \( M \) is called terminal if the following hold:

- \( M \) is preinjective
- If \( X \) is an indecomposable \( KQ \)-module, with \( \text{Hom}_{KQ}(M,X) \neq 0 \), then \( X \in \text{add} M \).
- Any indecomposable injective \( KQ \)-module belongs to \( \text{add} M \).

**Theorem 1.23** (Geiss, Leclerc, Shröer, [GLS08a], Theorem 2.1). Let \( \mathcal{C}_M \) be the subcategory of all \( \Lambda \)-modules whose image under \( \pi_Q \) belongs to \( \text{add} M \). Then the following holds:

- The category \( \mathcal{C}_M \) is Hom-finite.
- The category \( \mathcal{C}_M \) is Frobenius with \( n \) indecomposable projective objects.
- The stable category of \( \mathcal{C}_M \) is 2-Calabi-Yau.

We can consequently apply the results of part II.

**Part 1. The case of exact categories**

2. Study of the properties of \( \text{pr}M \)

We show some preliminary lemmas which will be used in section 6 in order to associate a prefibration structure with a given rigid subcategory.

We note that Lemma 1.1, which is an analogue of lemma 3.3 shown by Buan and Marsh in [BM13] for exact categories, might be of independent interest.

Let \( \mathcal{E} \) be a weakly idempotent complete exact category with enough injective and projective objects. Assume that \( \mathcal{M} \subseteq \mathcal{E} \) is a rigid, contravariantly finite subcategory of \( \mathcal{E} \) containing all the injective objects, and stable under taking direct sums and summands.

Let
\[ \text{pr}\mathcal{M} = \{ X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \to M_1 \to M_0 \to X \to 0 \} \]
and
\[ \text{pr}\mathcal{M} = \{ X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \text{Inj}, 0 \to M \to I \to X \to 0 \}. \]

Let
\[ G : \mathcal{E} \to \text{Mod}\overline{\mathcal{M}} \]
\[ X \mapsto \mathcal{E}(\text{--},X)/\overline{\mathcal{M}} \]
which induces the following equivalence of categories
\[ \text{pr}\mathcal{M}/\mathcal{M'} \simeq \text{mod } \mathcal{M}. \]

For more details, see the article of Demonet and Liu, [DL13].

In the following lemma, we prove that if \( \mathcal{M} \) is contravariantly finite, then so is \( \text{pr}\mathcal{M} \), provided that \( \mathcal{M} \) also contains all projective objects.

This lemma tells us that we can replace each object by a cofibrant replacement.

**Lemma 2.1.** Assume that the rigid subcategory \( \mathcal{M} \), contains all injectives and all projectives. Then, for any \( X \in \mathcal{E} \), there exist \( A \in \text{pr}\mathcal{M} \) and a right \( \text{pr}\mathcal{M} \) approximation \( A \to X \).

**Proof.** Let \( X \in \mathcal{E} \). Let \( M_0 \to X \) be an \( \mathcal{M} \)-approximation of \( X \). Since \( \mathcal{E} \) is weakly idempotent complete, with enough projectives and \( \mathcal{M} \) contains the projective objects, the morphism \( M_0 \to X \) is a deflation. Let
\[
0 \to K_0 \to M_0 \to X \to 0
\]
be the associated short exact sequence. Similarly, let
\[
0 \to K_1 \to M_1 \to K_0 \to 0
\]
be a short exact sequence coming from an \( \mathcal{M} \)-approximation of \( K_0 \). Then we have the following diagram:

\[
\begin{array}{ccc}
0 & \to & K_0 \\
\downarrow & & \downarrow \\
K_1 & \to & M_1 \\
\downarrow^\beta & & \downarrow^\alpha \\
M_0 & \to & X \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

We have \( a \circ b \circ \alpha = 0 \). Let \( A \) be the push-out of the square:

\[
\begin{array}{ccc}
M_1 & \to & M_0 \\
\downarrow & & \downarrow \\
IM_1 & \to & A \\
\downarrow & & \downarrow \\
0 & \to & X
\end{array}
\]

Then there exists a morphism \( \varphi : A \to X \) such that \( \varphi \circ r = a \) and the other triangle commutes.
We have $A \in \text{pr}M$. Indeed, we have the following short exact sequence:

$$0 \rightarrow M_1 \rightarrow I_{M_1} \oplus M_0 \rightarrow A \rightarrow 0.$$ 

Moreover, $A \rightarrow X$ is an approximation. Indeed, let $i : B \rightarrow X$ be a morphism, with $B \in \text{pr}M$. Let us show that there exists $B \rightarrow A$ which makes the triangle commute.

Let

$$0 \rightarrow M'_1 \rightarrow M'_0 \rightarrow B \rightarrow 0$$

be a short exact sequence with $M_0, M_1 \in M$. We have the following diagram:

As $a$ is an $M$-approximation, then there exists a morphism $j : M'_0 \rightarrow M_0$ which makes the lower-right square commute. Then, there exists a morphism $M'_1 \rightarrow K_0$ which makes it a morphism of short exact sequences.

As $M_1 \rightarrow K_0$ is an $M$-approximation, then there exists $\delta : M'_1 \rightarrow M_1$ which makes the upper triangle commute. Since $I_{M_1}$ is injective and $M'_1 \rightarrow M'_0$ is an inflation, there exists a morphism $\varepsilon : M'_0 \rightarrow I_{M_1}$ which leads to a morphism of short exact sequences.

All the conditions are required to build a morphism $k : B \rightarrow A$ such that $l \circ k = i + 0$ since $I_{M_1} \rightarrow A \rightarrow X = 0$. Then we have shown the result. \qed

### 3. Weak equivalences and fibrations

**Definition 3.1.** We recall that $G$ is the functor

$$G : \mathcal{E} \rightarrow \text{Mod}(\mathcal{M})$$

$$X \mapsto \mathcal{E}(-, X)[\mathcal{M}]$$

We call by $\mathcal{W}$, the weak equivalences, the class of morphisms $f$ for which $Gf$ is an isomorphism.

**Definition 3.2.** Let $f : X \rightarrow Y$ and $g : A \rightarrow B$ be two morphisms. We say that $f \square g$ when, for any commutative square

there exists a morphism $B \rightarrow X$ such that both triangles commute. For a class of morphisms $\mathcal{A}$, we call by

$$\mathcal{A} \square = \{ g, \forall f \in \mathcal{A}, f \square g \} \text{ and } \square \mathcal{A} = \{ f, \forall g \in \mathcal{A}, f \square g \}.$$
Let
\[ J = \{ f : 0 \to \mathcal{U}M, M \in \mathcal{M} \}. \]
The morphisms of \( J^\square \) are called fibrations and compose the class \( \mathcal{F}ib \).

The next lemma shows that the \( \mathcal{pr}\mathcal{M} \)-approximation constructed in lemma 2.1 is actually a weak equivalence. This permits to take cofibrant replacements as we will see later.

**Lemma 3.3.** Let \( h : A \to X \) be the \( \mathcal{pr}\mathcal{M} \)-approximation constructed in lemma 2.1. Then \( h \in J^\square \cap W \).

**Proof.** We first show that \( h \in J^\square \). Let \( \mathcal{U}M \to X \) be a morphism. Since \( \mathcal{U}M \in \mathcal{pr}\mathcal{M} \), and \( A \to X \) is a \( \mathcal{pr}\mathcal{M} \)-approximation, there automatically exists a lift as wanted.

Next, we have to show that the morphism \( E(-,A)|M \to E(-,X)|M \) is an isomorphism. It is surjective, since if we take \( M \to X \) a morphism, since \( M \in \mathcal{pr}\mathcal{M} \), there exists a lift as we have seen in order to show that \( h \) is a fibration.

Then, if \( a : M \to A \) is a morphism such that \( h \circ a = 0 \) (we will see later the case where this morphism factorizes through an injective module). Using the same notations as in lemma 2.1 as \( M \) is rigid, there exists \( \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} : M \to I_1 \oplus M_0 \) such that \( \pi \circ b_1 + r \circ b_2 = a \) where \( \pi : I_1 \to A \). As \( \alpha : M_1 \to K_0 \) is an \( \mathcal{M} \)-approximation, there exists \( c : M \to M_1 \) such that \( b \circ \alpha \circ c = b_2 \). Then we have
\[ a = \pi \circ b_1 + r \circ b_2 \]
so
\[ a = \pi \circ b_1 + r \circ b \circ \alpha \circ c. \]
By the pushout of lemma 2.1, \( r \circ b \circ \alpha = \pi \circ \iota_1 \) where \( \iota_1 : M_1 \to I_1 \). Then
\[ a = \pi \circ b_1 + \pi \circ \iota_1 \circ c. \]
This shows that \( a \) factorizes through an injective module.

Finally, if we suppose that \( h \circ a \) factorizes through an injective \( J \), for example \( h \circ a = \mu \circ \nu \), as \( J \in \mathcal{pr}\mathcal{M} \), there exists \( \hat{a} \) such that \( h \circ \hat{a} = \mu \). We then proceed as above with the morphism \( a - \hat{a} \circ \nu \). This finishes to show the result. \( \square \)

We recall that trivial fibrations are those morphisms that are both fibrations and weak equivalences.

**Lemma 3.4.** Let \( f : X \to Y \) be a trivial fibration. If \( \alpha \) is a morphism from an element \( M \) of \( \mathcal{M} \) to \( Y \), then there exists \( \beta : M \to X \) such that \( f \circ \beta = \alpha \).

**Proof.** As \( f \) is a weak equivalence, there exists \( \tilde{\beta} : M \to X \), \( \iota_M : M \to I_M \) and \( \gamma : I_M \to Y \) such that \( \alpha + \gamma \circ \iota_M = f \circ \tilde{\beta} \). As \( f \in J^\square \) and \( I_M \in \mathcal{U}\mathcal{M} \), there exists \( \delta : I_M \to X \) such that \( f \circ \delta = \gamma \). Then we have
\[ \alpha = f \circ (\tilde{\beta} - \delta \circ \iota_M). \]
\( \square \)

**Lemma 3.5.** Let \( f : X \to Y \) be a trivial fibration. Then it is automatically a deflation.
Proof. Let \( P_Y \) be a projective cover of \( Y \). As \( P_Y \in \mathcal{M} \), and \( f \) is a trivial fibration, from the previous lemma, there exists a lift from \( P_Y \to X \). From Buhler in [Büh10, Proposition 7.6, (ii)], as \( P_Y \to Y \) is a deflation, then \( f \) is a deflation. \( \square \)

4. Factorization

Let us now show a characterization of the morphisms of \( \square(J^\square) \).

**Lemma 4.1.** Suppose that \( \mathcal{U}M \to Y \) is a right \( \mathcal{U}M \)-approximation. A morphism \( f : X \to Y \) is in \( \square(J^\square) \) if and only if it is a retract of the canonical injection \( X \to X \oplus \mathcal{U}M \).

**Proof.** Let \( f : X \to Y \in \square(J^\square) \). Let \( \alpha : \mathcal{U}M \to Y \) be a \( \mathcal{U}M \)-approximation of \( Y \). Then, we have the following commutative square:

\[
\begin{array}{ccc}
X & \xrightarrow{(1)} & X \oplus \mathcal{U}M \\
\downarrow f & & \downarrow (f \alpha) \\
Y & \xrightarrow{(s)} & Y
\end{array}
\]

The morphism \( (f \alpha) \in J^\square \). Indeed, if \( \mathcal{U}M' \to Y \) is a morphism, as \( \alpha \) is a \( \mathcal{U}M \)-approximation, there exist a lift as wanted (which is zero on \( X \)). As \( f \in \square(J^\square) \), there exist \( s : Y \to X \oplus \mathcal{U}M \) which makes both triangles commute. Then, \( f \) is a retract of the canonical injection \( X \to X \oplus \mathcal{U}M \).

Conversely, it is well-known that the lifting property \( \square \) is stable under retract. \( \square \)

**Lemma 4.2.** Under the assumption that there exist some \( \mathcal{U}M \)-approximations, any morphism can be factorized through a morphism in \( \square(J^\square) \) followed by a morphism in \( J^\square \).

**Proof.** Let \( f : X \to Y \) be a morphism. It factorizes through \( X \to X \oplus \mathcal{U}M \to Y \) by \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( (f \alpha) \), where \( \alpha \) is a \( \mathcal{U}M \)-approximation (this morphism is a retract of itself). The first morphism is in \( \square(J^\square) \). As \( (f \alpha) \) is a \( \mathcal{U}M \)-approximation, it satisfies the lifting property of \( J^\square \). Then \( (f \alpha) \in J^\square \).

5. Cofibrant objects and homotopy

5.1. Cofibrant objects. Cofibrant objects are objects which have a lift along trivial fibrations.

In this subsection, we characterize fibrant and cofibrant objects.

**Lemma 5.1.** Any object is fibrant.

**Proof.** For any \( X \in \mathcal{E} \), the map \( X \to 0 \) is a fibration. \( \square \)

**Lemma 5.2.** Suppose that the subcategory \( \mathcal{M} \) contains the projective objects. Let \( C \in \mathcal{E} \). Then \( C \) is cofibrant if and only if \( C \in \text{pr} \mathcal{M} \).
Proof. Let $C \in \text{pr}\mathcal{M}$. We introduce $0 \to M_1 \xrightarrow{h'} M_0 \xrightarrow{h} C \to 0$. Let $f : X \to Y$ be a trivial fibration and $b : C \to Y$. As $f \in W$, $Gf$ is an isomorphism, and there exists from lemma 3.4 a morphism $a : M_0 \to X$ such that $f \circ a = b \circ h$.

Since $\mathcal{M}$ contains all the projective objects, the lemma 3.5 shows that $f$ is a deflation. Let $k : K \to X$ be the kernel of $f$. We then have a morphism of short exact sequences:

$$
\begin{array}{cccc}
M_1 & \xrightarrow{c} & K & \\
h' & & k & \\
\downarrow & & \downarrow & \\
M_0 & \xrightarrow{a} & X & \\
\downarrow & & \downarrow & f \\
C & \xrightarrow{b} & Y & \\
\end{array}
$$

As $k \in \mathcal{M}^1$, there exists $I$ an injective object, $\alpha : M_1 \to I$ and $\beta : I \to X$ such that $k \circ c = \beta \circ \alpha$.

As $h'$ is an inflation, there exists $\beta' : M_0 \to I$ such that $\beta' \circ h' = \alpha$.

$$
\begin{array}{cccc}
M_1 & \xrightarrow{c} & K & \\
h' & \downarrow & \downarrow & \\
\alpha & \downarrow & \downarrow & \beta \\
\downarrow & & \downarrow & \\
M_0 & \xrightarrow{a} & X & \\
\downarrow & & \downarrow & f \\
C & \xrightarrow{b} & Y & \\
\end{array}
$$

So,

$$h' \circ a = \beta \circ \beta' \circ h'$$

and there exists $\gamma : C \to X$ such that $\gamma \circ h = \beta \circ \beta' + a$.

Then,

$$f \circ \gamma \circ h = f \circ a + f \circ \beta \circ \beta'$$

and

$$(f \circ \gamma - b) \circ h = f \circ \beta \circ \beta'.$$
Then, in $\text{mod}\tilde{M}$, we have the good lifting. By Demonet and Liu in [DL13], we have that $\text{mod}\tilde{M} \simeq \text{pr}M/\mathfrak{M}$.

As $C \in \text{pr}M$, there exists $M \in M$ such that the morphism $b$ factorizes through $\mathfrak{M}M$, let us say

\[
\begin{array}{c}
C \\
\delta
\end{array} \xrightarrow{b} \begin{array}{c}
Y \\
\varepsilon
\end{array}
\]

such that $\varepsilon \circ \delta = b$. As $f \in J$, there exists $\iota : \mathfrak{M}M \to X$ such that $f \circ \iota = \varepsilon$. Then $\iota \circ \delta : C \to X$ is the good candidate to lift $b$, it means that

$f \circ \iota \circ \delta = \varepsilon \circ \delta = b$.

On the other hand, let $X$ be a cofibrant object. We can say that the $\text{pr}M$-approximation is a trivial fibration from lemma 3.3, then a retraction, and $X$ is a direct summand of $A$.

Let $0 \to Y \to M_0 \to X \to 0$ be an $M$-approximation of $X$. We have the following diagram:

5.2. Homotopy. We define here cylinder objects and left homotopies.

**Definition 5.3.** Let $X \in E$. A cylinder object for $X$ is a factorization of the morphism $\nabla : X \oplus X \to X$ (which is the identity on each copy of $X$) through $X'$, where $X' \to X$ is a weak equivalence.

Let $f, g : X \to Y$ be two morphisms. A left homotopy from $f$ to $g$ is a morphism $h : X' \to Y$, where $X'$ is a cylinder object for $X$, such that $h \circ (\nabla_1 \nabla_2) = (f \ g)$ where $(\nabla_1 \nabla_2)$ is the morphism $X \oplus X \to X'$ in the factorization of $\nabla$.

Dually, we can define path objects and right homotopies.

**Definition 5.4.** Let $Y \in E$. A path object for $Y$ is a factorization of the morphism $\Delta : Y \to Y \oplus Y$ (which is the identity on each copy of $Y$) through $Y'$, where $Y \to Y'$ is a weak equivalence.

Let $f, g : X \to Y$ be two morphisms. A right homotopy from $f$ to $g$ is a morphism $k : X \to Y'$, where $Y'$ is a path object for $Y$, such that $(\Delta_1 \Delta_2) \circ k = \begin{pmatrix} f \\ g \end{pmatrix}$ where $(\Delta_1 \Delta_2)$ is the morphism $Y' \to Y$ in the factorization of $\Delta$.

**Lemma 5.5.** For two morphisms $f$ and $g$ from an object $X$ to $Y$, $f$ and $g$ are homotopic if and only if $f - g$ factorizes through $\mathfrak{M}^\perp$.

**Proof.** We begin by noting this fact: in the next diagram, a factorization of $\Delta$ is a path object if and only if it is isomorphic to $Y \xrightarrow{(1 \ 0)} Y \oplus V \xrightarrow{(\Delta_1 \Delta_2)} Y \oplus Y$ for a $V \in \mathfrak{M}^\perp$.

Indeed, let $Y \xrightarrow{r} Y' \xrightarrow{\iota} Y \oplus Y$ be a factorization of $\Delta$. As $r$ is a section, it is isomorphic to some $Y \to Y \oplus V$. Then $r \in W$ if and only if $V \in \mathfrak{M}^\perp$. 

\[\mathfrak{M}^\perp\]
Now, let us suppose that $f - g$ factorizes in the following way:

\[
\begin{array}{ccc}
X & \overset{f - g}{\longrightarrow} & Y \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A & & \\
\end{array}
\]

The object $Y \oplus A$ is a path object. Then we have

\[
\begin{array}{ccc}
X & \overset{(g \ \alpha)}{\longrightarrow} & Y \oplus A = Y' \overset{(1 \ 0)}{\longrightarrow} Y \\
\downarrow{(f \ \beta)} & & \downarrow{\Delta} \\
Y \oplus Y & & \\
\end{array}
\]

and the morphism \( \begin{pmatrix} 1 & 1 \\ \beta & 0 \end{pmatrix} \) gives a homotopy between $f$ and $g$. Moreover, $s \circ a = \text{Id}$, thus $s \simeq Y \oplus A \to Y$ and $s \in \mathcal{W}$, so $A \in \mathcal{M}^\perp$. Then $f$ is homotopic to $g$.

Conversely, if $f$ is homotopic to $g$, then there exists $k : X \to Y'$ such that \( \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix} \circ k = \begin{pmatrix} 1 & 1 \\ \beta & 0 \end{pmatrix} \). Then, $f - g = (\Delta_1 - \Delta_2) \circ k$. As $Y' = Y \oplus A$, and $A \in \mathcal{M}^\perp$, this finishes to show the result. \(\square\)

**Remark 5.6.** The notions of left and right homotopy are the same for the cofibrant objects.

The following lemma is a corollary of the theorem of Demonet and Liu in [DL13]. However, we give a direct proof here.

**Lemma 5.7.** With the above notations, we have

\[ \text{pr}_M \cap \mathcal{M}^\perp \simeq \mathcal{U}M. \]

**Proof.** The indirect inclusion is obvious since $\mathcal{M}$ is rigid. Now, let $X \in \text{pr}_M \cap \mathcal{M}^\perp$. As $X \in \text{pr}_M$, we have a short exact sequence

\[ 0 \to M_1 \to M_0 \to X \to 0, \]

with $M_1, M_0 \in \mathcal{M}$. We have the following diagram:

\[
\begin{array}{ccc}
M_1 & \longrightarrow & M_0 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
M_1 & \longrightarrow & X \\
\downarrow{\psi} & & \downarrow{k} \\
\mathcal{U}M_0 & \longrightarrow & \mathcal{U}M_1 \\
\end{array}
\]

In order to show that $X \in \mathcal{U}M$, we will show that the short exact sequence

\[ 0 \to X \to \mathcal{U}M_1 \to \mathcal{U}M_0 \to 0 \]
splits. As $X \in \overline{\mathcal{M}}$, the morphism $c : M_0 \to X$ factorizes through $I$. We call by $\alpha : M_0 \to I$ and $\beta : I \to X$ such that $\beta \circ \alpha = c$. Let $k : X \to \overline{\mathcal{M}}_1$ and $h : I \to \overline{\mathcal{M}}_1$. Then,

$$k \circ \beta \circ \alpha = k \circ c = h \circ \alpha.$$  

Then, $h - k \circ \beta$ factorizes through $\overline{\mathcal{M}}_0$. Let us call $\varphi : \overline{\mathcal{M}}_0 \to \overline{\mathcal{M}}_1$ and $\psi : I \to \overline{\mathcal{M}}_0$. Then, we have

$$h = k \circ \beta + \varphi \circ \psi.$$  

Let $\xi : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_0$ in the short exact sequence (see diagram for sake of clarity). Then we have

$$\xi \circ \varphi \circ \psi = \xi \circ (h - k \circ \beta) = \xi \circ h - \xi \circ k \circ \beta = \psi - 0.$$  

As $\psi$ is a deflation, we can conclude that $\xi \circ \varphi = 1$. Then, there is a section to the short exact sequence and $X \in \overline{\mathcal{M}}$ (which is stable under direct summands since $\mathcal{M}$ is stable under direct summands. \qed

6. Prefibration structures from rigid subcategories

In this section, we show that an exact category $\mathcal{E}$ is nearly equipped with a structure of prefibration category in the sense of Anderson-Brown-Cisinski (for more details, see the book of Radulescu-Banu, \cite{RB06}).

We recall that $G : \mathcal{E} \to \text{Mod}\overline{\mathcal{M}}$ induces the following equivalence of categories from Demonet and Liu in \cite{DL13}

$$\text{pr}\mathcal{M}/\overline{\mathcal{M}} \simeq \text{mod}\overline{\mathcal{M}}.$$

**Theorem 6.1.** Let $\mathcal{E}$ be a weakly idempotent complete exact category with enough injective objects. Assume that $\mathcal{M} \subseteq \mathcal{E}$ is a rigid, contravariantly finite subcategory of $\mathcal{E}$ containing all the injective objects, and stable under taking direct sums and summands. Suppose moreover that $\overline{\mathcal{M}}$ is contravariantly finite. Let

$$J = \{ f : 0 \to \overline{\mathcal{M}}, M \in \mathcal{M} \}$$

and $J\square$ be the class of fibrations. Let $\mathcal{W}$ be the class of morphisms whose image under functor $G$ are isomorphisms.

Then $\mathcal{E}$ is almost equipped with a structure of prefibration category, it means that there exist two spaces of morphisms, $\mathcal{W}$, the weak equivalences, and $\text{Fib}$ the fibrations, such that:

(i) The space $\mathcal{W}$ is stable under retracts and satisfies the two out of three axiom.

(ii) The space $\text{Fib}$ is stable under composition, and all the isomorphisms are fibrations.

(iii) Pullbacks exist along fibrations, and the pull-back of a fibration is a fibration.

Moreover, if $0 \to A \to B \to Y \to 0$ is a short exact sequence, if $G_l$ is a monomorphism and $f : X \to Y$ is a trivial fibration ($f \in \text{Fib} \cap \mathcal{W}$), then $h$ defined by the following pull-back is a trivial fibration.
(iv) There exist path objects such that for any object $X$, the diagonal map $X \to X \oplus X$ can be factorized through $X'$, where the first morphism is a weak equivalence.

(v) For any object $B$, the morphism $0 \to B$ is a fibration.

Let us first show the following lemma:

**Lemma 6.2.** Let

\[
\begin{array}{ccc}
E & \overset{a}{\longrightarrow} & X \\
h \downarrow & \downarrow f & \\
A & \overset{i}{\longrightarrow} & B \\
\end{array}
\]

where $(i, p)$ is a short exact sequence, $G_i$ monomorphism and $f \in J^{\square} \cap \mathcal{W}$. Then $h \in J^{\square}$.

**Proof.** First, we have, without using the fact that $G_i$ is a monomorphism, that $h \in J^{\square}$. Indeed, let

\[
\begin{array}{ccc}
E & \overset{a}{\longrightarrow} & X \\
h \downarrow & \downarrow f & \\
B & \overset{p}{\longrightarrow} & Y
\end{array}
\]

be a commutative square, with $f \in J^{\square}$. Let us show that $h \in J^{\square}$. Let $M \in \mathcal{M}$ and $b : \mathcal{U}M \to B$. As $f \in J^{\square}$, there exists $\alpha : \mathcal{U}M \to X$ such that $f \circ \alpha = p \circ b$. We use the pullback property to claim that there exists $\varphi : \mathcal{U}M \to E$ such that $h \circ \varphi = b$. Then $h \in J^{\square}$. Second, let us show that $Gh$ is a monomorphism. As $h$ is a deflation (since it belongs to $J^{\square}$, and from lemma 3.5), the morphisms $f$ and $h$ have the same kernel. Then we have the following diagram:

\[
\begin{array}{ccc}
K & \overset{K}{\longrightarrow} \\
\downarrow & \downarrow & \\
A & \overset{E}{\longrightarrow} & X \\
\downarrow h & \downarrow & \\
A & \overset{i}{\longrightarrow} & B \\
\downarrow & \downarrow & \\
B & \overset{p}{\longrightarrow} & Y
\end{array}
\]

We have a short exact sequence $0 \to A \to E \to X \to 0$. Let us show that $Gh$ is a monomorphism. Let $\beta : D \to GE$ such that $Gh \circ \beta = 0$. We show that $\beta = 0$. We have $Gp \circ Gh \circ \beta = 0$. Then

\[G(p \circ h) \circ \beta = 0\]
and
\[ G(f \circ a) \circ \beta = 0. \]
Thus, we have
\[ Gf \circ Ga \circ \beta = 0. \]
As \( Gf \) is a monomorphism since \( f \in W \), we have
\[ Ga \circ \beta = 0. \]
The fact that \( G \) is left exact shows that there exists \( c : D \to GA \) such that
\[ \beta = Gk \circ c. \]
Moreover, by hypothesis
\[ Gi \circ c = Gh \circ \beta = 0. \]
As \( Gi \) is a monomorphism, \( c = 0 \) and from \( \square \) we have \( \beta = 0 \). This shows that \( Gh \) is a monomorphism.

Now we show that \( Gh \) is an epimorphism. Let \( b : M \to B \) be a morphism, where \( M \in \mathcal{M} \). Then we have \( p \circ b : M \to Y \). From lemma \( \ref{lemma} \) there exists \( \alpha : M \to X \) such that
\[ f \circ \alpha = p \circ h. \]
Then, from the pullback property, there exists \( \varphi : M \to E \) such that \( h \circ \varphi = b \) and this shows that \( Gh \) is an epimorphism. \qed

\textit{Proof of the theorem.} Some of the properties are trivial because of the results on Frobenius categories which did not use the Frobenius condition.

(i) Let \( f \) be a retract of \( w \in W \). Let us show that \( Gf \) is an isomorphism. We have:

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X \xrightarrow{w} Y \xrightarrow{s} B \\
\end{array}
\]
As \( Gf \) is a retract of an isomorphism, it is itself an isomorphism. Indeed,

\[
\begin{array}{c}
GA \xrightarrow{Gf} GB \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
GX \xrightarrow{\beta} GY \xrightarrow{\gamma} GB \\
\end{array}
\]
The morphism \( \beta \circ Gw^{-1} \circ \gamma \) is an inverse of \( Gf \).
Then \( Gf \) is an isomorphism.
Now let us show that \( W \) has the 2-out-of-3 property.
- If \( f, g \in W \). Then \( G(f \circ g) = G(f) \circ G(g) \), and we have that \( f \circ g \in W \).
• If $f, f \circ g \in W$. Then $(G(f \circ g))^{-1} \circ G(f \circ g) = \text{Id}$. So $(G(f \circ g))^{-1} \circ G(f) \circ G(g) = \text{Id}$. Let $G(g)^{-1}$ be the morphism $(G(f \circ g))^{-1} \circ G(f)$. It is a left inverse. Let us show that it is also a right inverse:

$G(f \circ g) \circ (G(f \circ g))^{-1} = \text{Id}$. Then $G(f) \circ G(g) \circ (G(f \circ g))^{-1} = \text{Id}$, and $G(g) \circ (G(f \circ g))^{-1} \circ G(f) = \text{Id}$ and it is indeed a right inverse.

• If $g, f \circ g \in W$. Then we use the same hint to show that $Gf$ is an isomorphism.

(ii) This result is easy, because fibrations are defined by a lifting property.

(iii) Fibrations are deflations (this is because $E$ is weakly idempotent complete, see lemma 3.5), then pullbacks exist along fibrations. The rest of the item is the previous lemma.

(iv) We have the factorization from lemma 4.2. It means that any morphism can be factorised through a morphism of $\square(J\square)$ followed by a morphism of $J\square$. Indeed, it only uses the fact that there exist some $\square\mathcal{M}$-approximations, which we suppose in the hypotheses of this theorem. Then for any $X$, the diagonal $X \rightarrow X \oplus X$ can be factorised $X \rightarrow X' \rightarrow X \oplus X$, where the first morphism is a weak equivalence (we have seen that the morphisms of $\square(J\square)$ are weak equivalences from lemma 4.1).

(v) By definition of $J$, any object is fibrant, then $0 \rightarrow B$ is a fibration for any object $B$ of $E$.

\[ \square \]

7. Theorem of Quillen

We recall that the objects of $\operatorname{pr}\mathcal{M}$ are exactly the same as cofibrant objects (defined here by satisfying the lifting property along trivial fibrations).

Now, we show that the theorem of Quillen is satisfied. We note that we still have the same notion of homotopy.

**Theorem 7.1.** Let $E$ be a weakly idempotent complete exact category with enough injective objects. Assume that $\mathcal{M} \subseteq E$ is a rigid, contravariantly finite subcategory of $E$ containing all the injective objects, and stable under taking direct sums and summands. Suppose moreover that $\mathcal{U}\mathcal{M}$ is contravariantly finite. Let

$$\operatorname{pr}\mathcal{M} = \{ X \in E, \exists M_1, M_0 \in \mathcal{M}, 0 \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0 \}$$

and

$$\mathcal{U}\mathcal{M} = \{ X \in E, \exists M \in \mathcal{M}, I \in \operatorname{Inj}, 0 \rightarrow M \rightarrow I \rightarrow X \rightarrow 0 \}.$$ 

Let $H_0 E$ be the localization of $E$ at the class $W$ of weak equivalences. Let $\operatorname{mod} \mathcal{M}$ be the category of finitely presented $\mathcal{M}$ modules. There is an equivalence of categories

$$H_0 E \simeq \operatorname{mod} \mathcal{M}.$$ 

**Remark 7.2.** In order to prove this theorem, we need the following lemmas. The proof of these is well-known, but we give here some details in order to show that the restriction of (iii) in theorem does not affect the results. Then, some parts of the proofs simplify due to the particular shape of the relation of homotopy.
Lemma 7.3. Let \( A, B, X \) be three cofibrant objects. Let \( F \) be the functor defined by
\[
F : A \mapsto \mathcal{E}(A, X)/\sim
\]
on objects, and, if \( f : A \to B \),
\[
Ff : \mathcal{E}(B, X)/\sim \mapsto \mathcal{E}(A, X)/\sim.
\]
Suppose moreover that \( f \) is a weak equivalence. Then \( F \) induces an isomorphism
\[
Ff : \mathcal{E}(B, X)/\sim \mapsto \mathcal{E}(A, X)/\sim
\]
where \( \sim \) is the right homotopy relation (which is also left taken into account the definition of the homotopy).

Proof. This functor is well-defined because the relation \( \sim \) behaves well with the right composition. Let now \( f : A \to B \). We factor \( f \) through a morphism \( g : A \to C \in \square(J\square) \) followed by a morphism \( p : C \to B \in J\square \). As \( g \in W \) and \( f \in W \), then by the two out of three property, \( p \) is also a weak equivalence. In addition, \( B \) is cofibrant, then there exists \( w : B \to C \) such that \( p \circ w = 1 \).

Then, we have
\[
1 = F(1) = F(p \circ w) = F(p) \circ F(w).
\]
Then \( F(p) \in W \). This shows by the two out of three property, that \( F(f) \in W \), and \( Ff \) is surjective.

Let us now show that it is injective. If \( \alpha : B \to X \) is such that \( F\alpha \sim 0 \), then \( \alpha \circ f \sim 0 \). As \( A \) is cofibrant, we have \( \alpha \circ f \in \mathcal{M} \) (from the equivalence of Demonet and Liu in [DL13]). This shows that \( G(\alpha \circ f) = 0 \) since \( \mathcal{M} \) is rigid. But \( f \in W \), then \( Gf \) is an isomorphism, then \( G\alpha = 0 \). As \( B \) is cofibrant, \( \alpha \in \mathcal{M} \). Then \( \alpha \sim 0 \). Then \( Ff \) is injective and this shows the lemma.

Now we can show the following result:

Lemma 7.4. If \( A \) and \( B \) are two cofibrant objects, then
\[
\text{Ho } \mathcal{E}(A, B) = \mathcal{E}(A, B)/\sim.
\]

Proof. Let \( A \) and \( B \) be two cofibrant objects.

Step 1: We show that, \( f : A \to B \) is a weak equivalence if and only if it is a homotopy equivalence.

Suppose \( f \) is a weak equivalence. We use the previous lemma with \( X = A \). From the surjectivity of \( Ff \), there exists \( g : B \to A \) such that \( g \circ f \sim 1 \). Then, \( f \circ g \circ f \sim f \). Now we apply the result to \( X = B \) and have that
\[
F(f \circ g) = Ff(1).
\]
However, \( Ff \) is injective, then \( f \circ g \sim 1 \) and \( f \) is a homotopy equivalence.

Suppose now that \( f \) is a homotopy equivalence. Let \( f' \) be a homotopy inverse for \( f \). We have \( f \circ f' \sim 1 \) and \( f' \circ f \sim 1 \). Then \( f \circ f' - 1 \in \mathcal{M} \) and \( f' \circ f - 1 \in \mathcal{M} \). Then \( G(f \circ f') = 1 \) and \( G(f' \circ f) = 1 \). Then \( Gf \) is an isomorphism and \( f \in W \).

This shows that \( f = g \) in \( \text{Ho}\mathcal{E}(A, B) \) if and only if \( f \sim g \).

Step 2: We now show the surjectivity of \( \mathcal{E}(A, B) \to \text{Ho}(A, B) \). We are going to use the book of Radulescu (see [RB06, Theorem 6.4.2]). Let us check the hypothesis with
the pair of categories \((\mathcal{E}/\{f, Gf = 0\}, \mathcal{W})\), where \(\mathcal{W}\) is the image of \(\mathcal{W}\) in the quotient of \(\mathcal{E}\) by \(\{f, Gf = 0\}\).

- The two out of three property is automatically checked.
- If we have a pair of morphisms

\[
\begin{array}{ccc}
A' & \downarrow a \\
B & \overset{p}{\rightarrow} & A \\
\end{array}
\]

such that \(a \in \mathcal{W}\), then there exists \(B', h : B' \rightarrow B\) and \(k : B' \rightarrow A'\) such that the following square commutes:

\[
\begin{array}{ccc}
B' & \overset{k}{\rightarrow} & A' \\
\downarrow h & & \downarrow a \\
B & \overset{p}{\rightarrow} & A \\
\end{array}
\]

Indeed, we introduce the factorization of \(A' \rightarrow A\) by \(b \in \square (J^\Box)\) followed by \(c \in J^\Box\). As \(a, b \in \mathcal{W}\), we also have \(c \in \mathcal{W}\). Then \(c\) is a trivial fibration. Let \(B'\) be a pr\(\mathcal{M}\)-approximation of \(B\). We lift the morphism \(p \circ h\) to \(c\), let us say \((h_1, h_2) : B' \rightarrow A' \oplus \mathcal{U}M\) (which is the shape of the factor in the factorization we have). Then \(h_1\) is a lift from \(B'\) to \(A'\). We can check that the square commutes and \(h \in \mathcal{W}\).

- Suppose that we have

\[
\begin{array}{ccc}
A & \overset{f}{\rightarrow} & B \\
\downarrow g & & \downarrow t \\
B' & \rightarrow & B \\
\end{array}
\]

with \(t \in \mathcal{W}\) and \(t \circ f = t \circ g\). Then \(Gf = Gg\), so, if \(t' : A' \rightarrow A \in \mathcal{W}\), we have \(G(f \circ t') = G(g \circ t')\), then \(f \circ t' - g \circ t' \in \mathcal{M}^\perp\). Then \(f \circ t' = g \circ t'\) in \(\mathcal{E}/\{f, Gf = 0\}\).

We can now apply the theorem of Radulescu 6.4.2 and any morphism in \(\text{Ho}(\mathcal{E}(A, B))\) can be written as \(f \circ s^{-1}\) with \(s \in \mathcal{W}\). As \(A\) and \(B\) are cofibrant, we factor through \(A' \rightarrow A\) in this way:

\[
\begin{array}{ccc}
A' & \overset{f}{\rightarrow} & B' \\
\downarrow s & & \downarrow (f, 0) \\
A' \oplus \mathcal{U}M & \rightarrow & B \\
\downarrow \alpha \in J^\Box & & \downarrow a \\
A & \rightarrow & B \\
\end{array}
\]

As \(s \in \mathcal{W}\), then \(\alpha \in J^\Box \cap \mathcal{W}\). Then we can lift \(A' \oplus \mathcal{U}M\) to \(B\) by \((f, 0)\).

From the theorem of Radulescu, if \(\alpha \in \text{Ho}(A, B)\), then there exists \(s \in \mathcal{W}\) and \(f \in \mathcal{E}(A, B)\) such that

\[
\alpha = f \circ s^{-1}.
\]
Then $\alpha = \overline{\alpha}$ (since $\overline{\alpha} \circ \overline{s} = \overline{f}$).

Then we have shown the surjectivity and then the lemma.

Proof of the theorem of Quillen. First of all, the functor is well-defined, since

$$0 \to \overline{0}M \in W$$

for any $M \in M$ (because $\text{Ext}^1(-, M)|M = 0$ implies $\overline{\text{Ext}}^1(-, \overline{0}M)|M = 0$).

Then, the functor is essentially surjective, since there exist some $pr_M$-approximations, which are weak equivalences.

Next, from lemma 7.4, we have that $f = g$ in $\text{Ho}(A, B)$ if and only if $f \sim g$ which immediately shows that the functor is faithful.

Finally, the functor is full, from the surjectivity of lemma 7.4.

The theorem of Demonet and Liu in [DL13] finishes to show the result.
Part 2. The particular case of Frobenius categories

We recall the notations:

Let $\mathcal{E}$ be a Frobenius category. Let $\mathcal{M}$ be a full rigid subcategory which contains the injective objects. Let

\[ \text{pr}\mathcal{M} = \{ X \in \mathcal{E}, \exists M_1, M_0 \in \mathcal{M}, 0 \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0 \} \]

and

\[ \text{℄}\mathcal{M} = \{ X \in \mathcal{E}, \exists M \in \mathcal{M}, I \in \text{Inj}, 0 \rightarrow M \rightarrow I \rightarrow X \rightarrow 0 \}. \]

Let

\[ G : \mathcal{E} \rightarrow \text{Mod}\overline{\mathcal{M}} \]
\[ X \mapsto \overline{\mathcal{E}}(\cdot, X)/\mathcal{M} \]

which induces the following equivalence of categories

\[ \text{pr}\mathcal{M}/\text{℄}\mathcal{M} \simeq \text{mod}\overline{\mathcal{M}}. \]

For more details, see the article of Demonet and Liu, [DL13].

Let $\mathcal{M}$ is a subcategory of $\mathcal{E}$ which is contravariantly finite. As the injective objects are also projective, note that $\mathcal{M}$ contains the projective objects.

8. A deeper study of $\text{pr}\mathcal{M}$

We have this first lemma:

**Lemma 8.1.** Let $X \in \mathcal{E}$. If there is an $\mathcal{M}$-approximation of $\Omega X$, then there exists a $\text{℄}\mathcal{M}$-approximation of $X$. In particular, if $\mathcal{M}$ is contravariantly finite, then so is $\text{℄}\mathcal{M}$.

**Proof.** Let $a : M \rightarrow \Omega X$ be an $\mathcal{M}$-approximation of $\Omega X$. We can introduce $Y$ as the push-out of the pair of morphisms:

\[
\begin{array}{ccc}
M & \xrightarrow{a} & \Omega X \\
\downarrow \iota_M & & \downarrow c \\
I\mathcal{M} & \xrightarrow{b} & Y
\end{array}
\]

in particular

\[ c \circ a = b \circ \iota_M. \]

Let

\[
0 \rightarrow \Omega X \xrightarrow{i_X} P_X \xrightarrow{\pi_X} X \rightarrow 0
\]

be a short exact sequence expressing the fact that $\Omega X$ is a sisygy of $X$. We then have the following diagram:
Note that we have

\[(3)\quad e \circ b = \pi_M.\]

As \(\iota_X \circ a\) factorizes through the module \(I_M\), there exists \(\alpha : Y \to P_X\) such that

\[(4)\quad \iota_X = \alpha \circ c \quad \text{and} \quad \alpha \circ b = d.\]

Then there exists \(\beta\), which makes a morphism of short exact sequences:

\[(5)\quad \pi_X \circ \alpha = \beta \circ e\]

Then we have a short exact sequence

\[0 \to Y \to P_X \oplus \Omega M \to X \to 0.\]

We now show that \((\pi_X, \beta)\) is a \(\Omega M\)-approximation (we note that, \(E\) being Frobenius, \(\Omega M\) contains the projective modules).

Let \(\gamma : \Omega N \to X\) be a morphism. As \(I_N\) is also projective, there exists \(\delta : I_N \to P_X\) such that

\[(6)\quad \pi_X \circ \delta = \gamma \circ \pi_N\]
We complete into a morphism of short exact sequences by \( \varepsilon \), it means that
\[
\iota_X \circ \varepsilon = \delta \circ \iota_N. \tag{7}
\]
As \( a : M \to \Omega X \) is an \( \mathcal{M} \)-approximation, there exists \( \zeta : N \to M \) such that
\[
a \circ \zeta = \varepsilon. \tag{8}
\]
As \( I_M \) is injective, and \( \iota_N \) is an inflation, there exists \( \eta : I_N \to I_M \) such that
\[
\eta \circ \iota_N = \iota_M \circ \zeta. \tag{9}
\]
Then there exists \( \theta : \mathcal{U}N \to \mathcal{U}M \) which makes a morphism of short exact sequences:
\[
\theta \circ \pi_N = \pi_M \circ \eta. \tag{10}
\]
Then we have:
\[
\delta \circ \iota_N = \iota_X \circ \varepsilon \tag{11}
\]
\[
= \iota_X \circ a \circ \zeta \tag{b}
\]
\[
= \alpha \circ c \circ a \circ \zeta \tag{9}
\]
\[
= \alpha \circ b \circ \iota_M \circ \zeta \tag{2}
\]
\[
= \alpha \circ b \circ \eta \circ \iota_N \tag{b}
\]
Then, the morphism \( \alpha \circ b \circ \eta = \delta \) factorizes through \( \pi_N \).
Thus, there exists \( \kappa : \mathcal{U}N \to PX \) such that
\[
k \circ \pi_N = \delta - \alpha \circ b \circ \eta. \tag{12}
\]
Let us finally show that the morphism \( \begin{pmatrix} \kappa \\ \theta \end{pmatrix} \) satisfies \( (\pi_X \beta) \begin{pmatrix} \kappa \\ \theta \end{pmatrix} = \gamma \).
As \( \pi_N \) is a deflation, is suffices to show that
\[
\pi_X \circ \kappa \circ \pi_N + \beta \circ \theta \circ \pi_N = \gamma \circ \pi_N.
\]
We have:
\[
\begin{align*}
\pi_X \circ \kappa \circ \pi_N + \beta \circ \theta \circ \pi_N &= \pi_X (\delta - \alpha \circ b \circ \eta) + \beta \circ \theta \circ \pi_N \quad \text{by} \ 12 \\
&= \pi_X (\delta - \alpha \circ b \circ \eta) + \beta \circ \pi_M \circ \eta \quad \text{by} \ 10 \\
&= \gamma \circ \pi_N - \pi_X \circ \alpha \circ b \circ \eta + \beta \circ \pi_M \circ \eta \quad \text{by} \ 8 \\
&= \gamma \circ \pi_N - \beta \circ e \circ b \circ \eta + \beta \circ \pi_M \circ \eta \quad \text{by} \ 5 \\
&= \gamma \circ \pi_N - \beta \circ \pi_M \circ \eta + \beta \circ \pi_M \circ \eta \quad \text{by} \ 3 \\
&= \gamma \circ \pi_N.
\end{align*}
\]

\[
\square
\]

**Definition 8.2.** We call by \( \copr \mathcal{O} \mathcal{M} \) the class of objects \( X \in \mathcal{E} \) such that there exist \( M, M' \in \mathcal{M} \) such that \( 0 \to X \to \mathcal{O} \mathcal{M} \to \mathcal{O} \mathcal{M}' \to 0 \) is a short exact sequence.

\( \copr \mathcal{O} \mathcal{M} = \{ X \in \mathcal{E}, 0 \to X \to \mathcal{O} \mathcal{M} \to \mathcal{O} \mathcal{M}' \to 0 \} \).

**Lemma 8.3.** We have the following equality:

\( \copr \mathcal{O} \mathcal{M} = \pr \mathcal{M} \).

**Proof.** Let \( C \in \copr \mathcal{O} \mathcal{M} \). Let us show that we then have a short exact sequence \( 0 \to M_1 \to M' \to C \to 0 \). We have the diagram:

\[
\begin{array}{ccc}
M_1 & \longrightarrow & \mathcal{O} \mathcal{M}_0 \\
\downarrow & & \downarrow \\
M_1 & \longrightarrow & I \\
\downarrow & & \downarrow \\
\mathcal{O} \mathcal{M}_0 & \longrightarrow & \mathcal{O} \mathcal{M}_0 \\
\end{array}
\]

We have \( \Omega \mathcal{O} \mathcal{M} \simeq M' \). Indeed, we have a two-ways short exact sequence

\[
\begin{array}{ccc}
M & \longrightarrow & I \\
\downarrow & & \downarrow \\
\Omega \mathcal{O} \mathcal{M} & \longrightarrow & \mathcal{O} \mathcal{M} \\
\end{array}
\]

Then \( \Omega \mathcal{O} \mathcal{M} \simeq M \oplus I \in \mathcal{M} \). The other inclusion is proved in part 1, lemma 5.7.

**Lemma 8.4.** The class \( \mathcal{O} \mathcal{M} \) is rigid.

**Proof.** Let \( M, M' \in \mathcal{M} \). We show that a morphism \( f : \mathcal{O} \mathcal{M} \to \mathcal{O}^2 M' \) is zero. Let us introduce the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{b} & \mathcal{O} \mathcal{M}' \\
\downarrow \iota_M & & \downarrow \iota_{\mathcal{O} \mathcal{M}'} \\
I_M & \xrightarrow{g} & I_{\mathcal{O} \mathcal{M}'} \\
\downarrow \pi_M & & \downarrow \pi_{\mathcal{O} \mathcal{M}'} \\
\mathcal{O} \mathcal{M} & \xrightarrow{f} & \mathcal{O}^2 M' \\
\end{array}
\]

We have:

\[
\begin{align*}
\pi_X \circ \kappa \circ \pi_N + \beta \circ \theta \circ \pi_N &= \pi_X (\delta - \alpha \circ b \circ \eta) + \beta \circ \theta \circ \pi_N \\
&= \pi_X (\delta - \alpha \circ b \circ \eta) + \beta \circ \pi_M \circ \eta \\
&= \gamma \circ \pi_N - \pi_X \circ \alpha \circ b \circ \eta + \beta \circ \pi_M \circ \eta \\
&= \gamma \circ \pi_N - \beta \circ e \circ b \circ \eta + \beta \circ \pi_M \circ \eta \\
&= \gamma \circ \pi_N - \beta \circ \pi_M \circ \eta + \beta \circ \pi_M \circ \eta \\
&= \gamma \circ \pi_N.
\end{align*}
\]
As $I_M$ is also projective and $\pi_{\mathcal{U}M'}$ is a deflation, there exists $g$ which makes the lower square commute. Then, there exists $h$ such that the upper square commutes. Now, $h$ factorizes through an injective module since $\mathcal{M}$ is rigid. Let us say that there exists $\alpha : I_M \to \mathcal{U}M'$ such that $\alpha \circ \iota_M = h$. Then, there exists $\beta : \mathcal{U}M \to I_{\mathcal{U}M'}$ such that $g = \iota_{\mathcal{U}M'} \circ \alpha + \beta \circ \pi_M$. Then,

$$f \circ \pi_M = \pi_{\mathcal{U}M'} \circ g = \pi_{\mathcal{U}M'} \circ (\iota_{\mathcal{U}M'} \circ \alpha + \beta \circ \pi_M) = \pi_{\mathcal{U}M'} \circ \beta \circ \pi_M$$

As $\pi_M$ is a deflation, this shows that $f = \pi_{\mathcal{U}M'} \circ \beta$ and thus factorizes through an injective. Then $\mathcal{U}M$ is rigid.

**Lemma 8.5.** Let $A, C$ be objects of $\mathcal{E}$. Let $M \in \mathcal{M}$. Suppose that we have a short exact sequence

$$0 \to A \to C \to \mathcal{U}M \to 0.$$

Then $C \in \text{pr}\mathcal{M}$ if and only if $A \in \text{pr}\mathcal{M}$.

**Proof.** First, suppose that $A \in \text{pr}\mathcal{M}$. Then, we have the following diagram:

\[
\begin{array}{ccc}
A & \to & \mathcal{U}M_1 \\
\downarrow & & \downarrow \\
C & \to & E \\
\downarrow & & \downarrow \\
\mathcal{U}M_2 & \to & \mathcal{U}M_2
\end{array}
\]

Where $E$ is the pullback of the morphisms $A \to \mathcal{U}M_1$ and $A \to C$. Then there is a section to the short exact sequence

$$0 \to \mathcal{U}M_1 \to E \to \mathcal{U}M_2 \to 0,$$

and $E \cong \mathcal{U}M_1 \oplus \mathcal{U}M_2 \cong \mathcal{U}M$. Then we have a short exact sequence $0 \to C \to \mathcal{U}M \to \mathcal{U}M_0 \to 0$. Then the previous lemma concludes this implication.

Conversely, if $C \in \text{pr}\mathcal{M}$. Then, by the previous lemma, $C \in \text{copr}\mathcal{U}\mathcal{M}$, so there exists a short exact sequence

$$0 \to C \to \mathcal{U}M^0 \to \mathcal{U}M^1 \to 0.$$
Remark 8.6. We have the dual lemma: If there is a short exact sequence $0 \to M \to C \to A \to 0$, then $A \in \text{pr}M$ if and only if $C \in \text{pr}M$.

9. Weak equivalences

We start with a definition.

Definition 9.1. Let $\mathcal{A}$ be a subcategory of $\mathcal{E}$. Let $g : Y \to Z$ be a morphism. We say that $g \in \overline{\mathcal{A}}^\perp$ if, for any morphism $b : \overline{\mathcal{A}} \to Y$, with $A \in \mathcal{A}$, the morphism $g \circ b$ factorizes through an injective module.

Remark 9.2. We note that $\overline{\mathcal{A}}^\perp$ is not an ideal.

This lemma characterizes weak equivalences.

Lemma 9.3. Let $X, Y \in \mathcal{E}$. Let $f : X \to Y$. Then $Gf$ is an isomorphism if and only if, for

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \iota_X \quad \quad \downarrow g \quad \quad \\
I_X \xrightarrow{u} Z \\
\downarrow \pi_X \\
\overline{X} \quad \quad \overline{X}
\end{array}
\]

we have $I_X \oplus Y \xrightarrow{[u \quad g]} Z \in \overline{\mathcal{M}}^\perp$ and, for

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow g \\
Z \xrightarrow{\overline{P}_Y} Y \\
\downarrow \overline{Y} \\
\overline{Y} \quad \quad \overline{Y}
\end{array}
\]

we have $Z \xrightarrow{X \oplus P_Y} \overline{\mathcal{M}}^\perp$.

Proof. If $Gf$ is an isomorphism. Let

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\downarrow \iota_X \quad \quad \downarrow g \\
I_X \xrightarrow{u} Z \\
\downarrow \pi_X \\
\overline{X} \quad \quad \overline{X}
\end{array}
\]
be the diagram which induces the short exact sequence

\[ 0 \rightarrow X \rightarrow I_X \oplus Y^{(u, g)} \rightarrow Z \rightarrow 0. \]

Let us show that \((u, \bar{g}) \in \mathcal{M}^\perp\). Since \(u : I_X \rightarrow Z\), with \(I_X\) injective we already have that \(E(M, u) = 0\). Let

\[ E(M, g) : E(M, Y) \rightarrow E(M, Z) \]

\[ h : M \in \mathcal{M} \rightarrow Y \mapsto g \circ h. \]

As \(Gf\) is an isomorphism, there exists \(h' \in E(M, X)\) such that \(f \circ \bar{h}' = \bar{h}\). Then

\[ \bar{h}' \rightarrow f \circ \bar{h}' = \bar{h} \rightarrow g \circ f \circ \bar{h}' = g \circ h = 0, \]

and \(E(M, g) = 0\).

In the same way, let

\[ X \xrightarrow{f} Y, \]

\[ Z \xrightarrow{P_Y} \bar{Y}, \]

\[ \bar{Y} = \bar{Y} \]

be the diagram which induces the short exact sequence

\[ 0 \rightarrow Z \rightarrow X \oplus P_Y \rightarrow Y \rightarrow 0. \]

Let us show that \(\left( \frac{\bar{g}}{\bar{u}} \right) \in \mathcal{M}\). We already have that \(E(M, \bar{u}) = 0\). Let

\[ E(M, \bar{g}) : E(M, Z) \rightarrow E(M, X) \]

\[ h : M \in \mathcal{M} \rightarrow Z \mapsto \bar{g} \circ h. \]

The map \(\bar{g} \circ h\) is sent on 0 by \(Gf\). As \(Gf\) is an isomorphism, we have that \(\bar{g} \circ h = 0\).

Conversely, let us suppose that \(I_X \oplus Y^{(u \bar{g})} Z \in \mathcal{M}^\perp\) and \(Z \rightarrow X \oplus P_Y \in \mathcal{M}^\perp\).

Let us introduce the short exact sequences

\[ 0 \rightarrow X \rightarrow I_X \oplus Y^{(u \bar{g})} Z \rightarrow 0 \]

and

\[ 0 \rightarrow Z \rightarrow X \oplus P_Y \rightarrow Y \rightarrow 0 \]

We take the image of equation (14) by the functor \(G = E(A, -)\) for an \(A \in \mathcal{M}\). As \(G\) is left exact, we have the following long exact sequence:

\[ 0 \rightarrow E(A, X) \rightarrow E(A, Y \oplus I_A) \rightarrow E(A, Z) \rightarrow \text{Ext}^1(A, X) \rightarrow \cdots \]

If we factorize through the injective modules, then we get:

\[ \cdots \rightarrow \text{Ext}^1(A, Z) \rightarrow E(A, X) \rightarrow E(A, Y) \rightarrow E(A, Z) \rightarrow \text{Ext}^1(A, X) \rightarrow \cdots \]
where $\text{Ext}$ corresponds to the extension morphisms in the category $\mathcal{E}$. Moreover, we have $\mathcal{E}(A, g) = 0$ because $A \in \mathcal{M}$ and $g \in \mathcal{M}^\perp$. Then $Gf$ is an epimorphism.

Now let us apply the functor $G$ to equation \[15\] and factorize through projective modules. We then have:

$$\cdots \to \text{Ext}^{-1}(A, Y) \to \mathcal{E}(A, Z) \to \mathcal{E}(A, X) \to \mathcal{E}(A, Y) \to \text{Ext}^1(A, Z) \to \cdots$$

We have $\mathcal{E}(A, \tilde{g}) = 0$ because $A \in \mathcal{M}$ and $\tilde{g} \in \mathcal{M}^\perp$. Then $Gf$ is a monomorphism, thus an isomorphism. □

10. Fibrations and cofibrations

Let us recall the notations:

Let $W$ be the class of morphisms $f$ such that $Gf$ is an isomorphism. Such morphisms are called weak equivalences. Let

$$J = \{ f : 0 \to \mathcal{M}, M \in \mathcal{M} \}$$

and

$$I = \{ f : M_0 \to X \oplus I_0, X \in \text{prM} \} \cup \{ 0 \to M, M \in \mathcal{M} \}$$

where $0 \to M_1 \to M_0 \to X \to 0$ is a short exact sequence and $I_0$ appears in the short exact sequence $0 \to M_0 \to I_0 \to \mathcal{M}_0$.

The morphisms of $J^\square$ are called fibrations and compose the class $\text{Fib}$. The following lemma characterizes fibrations. We recall that $g \in \mathcal{M}^\perp$ if, for any morphism $b : \mathcal{M} \to Y$, with $M \in \mathcal{M}$, the morphism $g \circ b$ factorizes through an injective module.

**Definition 10.1.** For a morphism $f : X \to Y$, we call a cone of $f$, any morphism $g$, obtained from a push-out of an inflation $X \to I$ for some injective $I$ along $f$. Then, we notice that $g$ is unique up to isomorphism.

**Lemma 10.2.** Suppose that $\mathcal{E}$ is a weakly idempotent complete category (see the article of Bulher, [Büh10]). Then $f$ is a fibration if and only if $f$ is a deflation, and $g \in \mathcal{M}^\perp$, where $g$ is a cone of $f$, defined by

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow \scriptstyle{\iota_X} & & \downarrow \scriptstyle{g} \\
\mathcal{M} & \xrightarrow{u} & Z \\
\downarrow \scriptstyle{\pi_M} & & \downarrow \\
\mathcal{M} & \xrightarrow{b} & \mathcal{M}^\perp
\end{array}$$

Proof. Let $f : X \to Y$ be a deflation. Let $g$ be a cone of $f$. Suppose that $g \in \mathcal{M}^\perp$. Let us show that $f \in J^\square$. Let $b : \mathcal{M} \to Y$, where $\mathcal{M} \in \mathcal{M}$. As $(g \circ b) \in \mathcal{M}^\perp$, then the morphism $(g \circ b) \circ (b)$ factorizes through an injective. We have

$$(g \circ b) \begin{pmatrix} b - \alpha_1 \circ \iota \\ 0 - \alpha_2 \circ \iota \end{pmatrix} = 0.$$
We have the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{M} \\
\downarrow \gamma \\
I \\
\downarrow \phi \\
Z
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
X \\
\downarrow \beta \\
Y \oplus I_X \\
\downarrow (g u) \\
0
\end{array}
\end{array}
\end{array}
\]

Then, there exists \( \beta : \mathcal{M} \rightarrow X \) such that

\[
\begin{pmatrix}
  b - \alpha_1 \circ \iota \\
  \alpha_2 \circ \iota
\end{pmatrix} = \beta \circ \begin{pmatrix}
  f \\
  \iota_X
\end{pmatrix}.
\]

As \( I \) is also projective since \( \mathcal{E} \) is a Frobenius category, and \( f \) is a deflation, there exists \( \tilde{\alpha}_1 : I \rightarrow X \) such that

\[
\alpha_1 = f \circ \tilde{\alpha}_1.
\]

Then, we have

\[
f \circ (\beta - \tilde{\alpha}_1 \circ \iota) = b
\]

and this shows the first implication.

On the other hand, let \( g \) be a cone of \( f \). If \( f \in \square X : X \rightarrow Y \), then it is a deflation. Indeed, let \( \pi_Y : P_Y \rightarrow Y \) be a projective pre-cover of \( Y \) (we can equivalently introduce a \( \mathcal{M} \)-approximation of \( Y \)). There exists a morphism \( \varphi : P_Y \rightarrow X \) which make the diagram commutative. As \( f \circ \varphi = \pi_Y \) is a deflation and \( \mathcal{E} \) is a weakly idempotent complete category, from the article of Bulher, \([\text{Büh10, Theorem 7.6}]\), then \( f \) is also a deflation. Let us now show that \( g \in \mathcal{M}^\perp \). Let \( h : B \rightarrow Y \), where \( B \in \mathcal{M} \). We have to show that \( g \circ h = 0 \). As \( f \in \square X \), there exists \( \varphi : B \rightarrow X \) such that \( f \circ \varphi = h \). Then \( g \circ h = g \circ f \circ \varphi \). As we know \( g \circ f \) factorize through an injective, and so \( g \circ h \). Then \( g \in \mathcal{M}^\perp \).

\[\square\]

11. Factorizations

In addition to the first factorization, we also have a second factorization:

**Lemma 11.1.** Any morphism \( f : X \rightarrow Y \), where \( X \in \text{pr}\mathcal{M} \) can be factorized through a morphism in \( \square (I^\square) \) followed by a morphism in \( I^\square \).

**Proof.** Let us show that any morphism \( f : X \rightarrow Y \), where \( X \in \text{pr}\mathcal{M} \) can be factorized through a morphism in \( \square (I^\square) \) followed by a morphism in \( I^\square \).

We are going to use item (iv) in the next theorem (whose proof does not need this lemma) We have made a proof of point (iv) lower for sake of clarity (in order to have a one-piece-proof for the main theorem).

Let \( f : X \rightarrow Y \) be a morphism, such that \( X \in \text{pr}\mathcal{M} \). Let

\[
0 \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0
\]
be a short exact sequence, and let \( c : M_0 \to X \). From lemma [2.1] there exist \( a : A \to Y \) a pr\( \mathcal{M} \)-approximation, with \( A \in \text{pr}\mathcal{M} \). Let \( \varepsilon : X \to \varnothing M_1 \) be the induced morphism in the short exact sequence

\[
0 \to M_0 \to X \oplus I_{M_0} \to \varnothing M_1 \to 0.
\]

As \( A \to Y \) is a pr\( \mathcal{M} \)-approximation, and \( X \in \text{pr}\mathcal{M} \), there exists \( r : X \to A \) such that

\[
f = a \circ r.
\]

We are going to show that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\alpha} & \quad & \quad \uparrow{(a, 0)} \\
A \oplus \varnothing M_1 & \quad & \quad \quad
\end{array}
\]

is the good factorization.

It is immediate that \((a, 0)\) is a trivial fibration. So it is an element of \( I^\square \). Let us show that \( \left( \begin{array}{c}
\varepsilon \\
r
\end{array} \right) \in \square (I^\square) \). We introduce the following commutative square:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & U \\
\downarrow{\langle \varepsilon \rangle} & & \downarrow{h} \\
A \oplus \varnothing M_1 & \xrightarrow{(\beta_1, \beta_2)} & V
\end{array}
\]

with \( h \in I^\square \).

The object \( A \oplus \varnothing M_1 \) is in \( \text{pr}\mathcal{M} \), then from lemma [5.2] part 1, it is cofibrant. Then, there exists a morphism \((\varphi_1, \varphi_2) : A \oplus \varnothing M_1 \to U\) such that

\[
(16) \quad h \circ \varphi_1 = \beta_1 \text{ and } h \circ \varphi_2 = \beta_2.
\]

Unfortunately, this morphism is not the good candidate, because it does not make the upper triangle commute. We have to modify it.

As \( h \in J^\square \cap W \), then it is a deflation, we introduce its kernel \( u : K \to X \). There exists \( \gamma : X \to K \) such that

\[
(17) \quad \alpha = u \circ \gamma + \varphi_1 \circ r + \varphi_2 \circ \varepsilon.
\]

Moreover, we know that \( u \in M^\perp \). So the morphism \( u \circ \gamma \circ c \) factorizes through an injective module, let us say \( I_{M_1} \). Then we have the following diagram:
From the push-out property, there exists a morphism $a_2 : \overline{U}M_1 \to C$ such that
\begin{equation}
 u \circ \gamma = a_2 \circ \varepsilon.
\end{equation}

Then, from equations \ref{eq:17} and \ref{eq:18} we have $(\varphi_1 \varphi_1 + a_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} = \alpha$.

Unfortunately again, this morphism is not the good candidate. Now, the upper triangle commutes, but we have lost the commutativity of the lower one. We have to modify it one last time.

Moreover, the morphism $\begin{pmatrix} r \\ \varepsilon \end{pmatrix}$ is an inflation. We can introduce its cokernel $C$. Let $(c_1, c_2) : A \oplus \overline{U}M_1 \to C$. We have
\begin{equation}
 h \circ (\varphi_1 \varphi_2 + a_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} = h \circ \alpha = (\beta_1 \beta_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix}.
\end{equation}

Then the morphism $(\varphi_1 \varphi_2 + a_2) - (\beta_1 \beta_2)$ factorizes through the cokernel $C$ of $\begin{pmatrix} r \\ \varepsilon \end{pmatrix}$.

Then there exists $b : C \to V$ such that
\begin{equation}
 (\beta_1 \beta_2) = h \circ (\varphi_1 \varphi_2 + a_2) - b \circ (c_1, c_2).
\end{equation}

Moreover, $C \in \text{prM}$ from lemma \ref{lem:8.5}. Then it is cofibrant, as $h \in J^{\square} \cap W$, there exists $d : C \to U$ such that
\begin{equation}
 h \circ d = b.
\end{equation}

We now have enough information in order to choose the good candidate for the morphism $A \oplus \overline{U}M_1 \to U$ which makes both triangles commute. It is:
\begin{equation}
 (\varphi_1 - d \circ c_1 \varphi_2 + a_2 - d \circ c_2).
\end{equation}

Indeed, for the upper triangle:
\begin{equation}
 (\varphi_1 - d \circ c_1 \varphi_2 + a_2 - d \circ c_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} = \varphi_1 \circ r - d \circ c_1 \circ r + \varphi_2 \circ \varepsilon + a_2 \circ \varepsilon - d \circ c_2 \circ \varepsilon
\begin{align*}
 &= \alpha - u \circ \gamma - \varphi_2 \circ \varepsilon - d \circ c_1 \circ r + \varphi_2 \circ \varepsilon + a_2 \circ \varepsilon - d \circ c_2 \circ \varepsilon \quad \text{from } \ref{eq:17} \\
 &= \alpha - a_2 \circ \varepsilon - d \circ c_1 \circ r + a_2 \circ \varepsilon - d \circ c_2 \circ \varepsilon \quad \text{from } \ref{eq:18} \\
 &= \alpha - d \circ (c_1, c_2) \circ \begin{pmatrix} r \\ \varepsilon \end{pmatrix} \\
 &= \alpha.
\end{align*}

The upper triangle commutes.

Now let us show the commutativity of the lower triangle:
\begin{equation}
 h \circ (\varphi_1 - d \circ c_1 \varphi_2 + a_2 - d \circ c_2) = (h \circ \varphi_1 - h \circ d \circ c_1 \ h \circ \varphi_2 + h \circ a_2 - h \circ d \circ c_2)
\end{equation}
\begin{align*}
 &= (h \circ \varphi_1 - b \circ c_1 \ h \circ \varphi_2 + h \circ a_2 - b \circ c_2) \quad \text{from } \ref{eq:20} \\
 &= (h \circ \varphi_1 + \beta_1 - h \circ \varphi_1 \beta_2 + b \circ c_2 - c \circ c_2) \quad \text{from } \ref{eq:19} \\
 &= (\beta_1 \beta_2)
\end{align*}
Then, the lower triangle commutes, so \( \left( \begin{array}{c} r \\ \varepsilon \end{array} \right) \in \square \left( \begin{array}{c} I \\ \square \end{array} \right) \), and we have the factorization when the domain of the morphism is cofibrant. \( \square \)

### 12. The almost model structure on Frobenius categories

We are going to show the following theorem:

**Theorem 12.1.** Let \( E \) be a weakly idempotent complete Frobenius category. Assume that \( M \subseteq E \) is a rigid, contravariantly finite subcategory of \( E \) containing all the injective objects, and stable under taking direct sums and summands.

Let

\[
J = \{ f : 0 \to \emptyset M, M \in \mathcal{M} \}
\]

and

\[
I = \{ f : M_0 \to X \oplus I_0, X \in \text{pr}\mathcal{M} \} \cup \{ 0 \to M, M \in \mathcal{M} \}.
\]

Let \( G \) be the functor

\[
G : E \to \text{Mod}\overline{\mathcal{M}}
\]

\[
X \mapsto \overline{E(-,X)}/\overline{\mathcal{M}}
\]

Let

\[
W = \{ f, Gf \text{ is an isomorphism} \}.
\]

Let \( J^\square \) be the class of fibrations. The cofibrations are given by the left-lifting property from trivial fibrations. Then, \((\text{Fib}, \text{Cof}, W)\) nearly form a model structure for the category \( E \). Indeed, the second factorization is found only when the domain is cofibrant.

**Proof.** We show the conditions required in order to apply theorem 1.15 of Hovey in [Hov99, Theorem 2.1.19](#).

(i) \( W \) is stable under retracts.

Let \( f \) be a retract of \( w \in W \). Let us show that \( Gf \) is an isomorphism. We have:

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}} & A \\
\downarrow f & & \downarrow f \\
B & \xleftarrow{\text{Id}} & B \\
\end{array}
\]

As \( Gf \) is a retract of an isomorphism, it is itself an isomorphism. Indeed,

\[
\begin{array}{ccc}
Gf & \xrightarrow{\text{Id}} & Gf \\
\downarrow Gw^{-1} & \xleftarrow{\beta} & \downarrow Gw \\
GB & \xrightarrow{\gamma} & GB \\
\end{array}
\]

The morphism \( \beta \circ Gw^{-1} \circ \gamma \) is an inverse of \( Gf \).

Then \( Gf \) is an isomorphism.
(ii) "2 out of 3" property.
- If \( f, g \in \mathcal{W} \). Then \( G(f \circ g) = G(f) \circ G(g) \), and we have that \( f \circ g \in \mathcal{W} \).
- If \( f, f \circ g \in \mathcal{W} \). Then \((G(f \circ g))^{-1} \circ G(f) \circ G(g) = \text{Id} \). Let \( (G(f \circ g))^{-1} \circ G(f) \) be the morphism \( G(f \circ g)^{-1} \circ G(f) \). It is a left inverse. Let us show that it is also a right inverse:
  \[
  G(f \circ g) \circ (G(f \circ g)^{-1} \circ G(f)) = \text{Id}.
  \]
  So \( G(f \circ g)^{-1} \circ G(f) = \text{Id} \). Let \( G(f \circ g)^{-1} \circ G(f) \) be the morphism \( G(f \circ g)^{-1} \circ G(f) \). It is a left inverse.
- If \( g, f \circ g \in \mathcal{W} \). Then \( G(f \circ g) \) is an isomorphism.

(iii) \( \square(J) \subseteq \mathcal{W} \cap \square(I) \).
As \( J \subseteq I \), we automatically have that \( \square(J) \subseteq \square(I) \). From lemma \ref{lemma} it is immediate that any morphism of \( \square(J) \) is a weak equivalence.

(iv) \( I = J \cap \mathcal{W} \).
First, let us show that \( J \cap \mathcal{W} \subseteq I \). Let \( f \in J \cap \mathcal{W} \). We introduce the following commutative square:

\[
\begin{array}{ccc}
M_0 & \xrightarrow{a} & X \\
\downarrow^{(h_{0})} & & \downarrow^{f} \\
A \oplus I_0 & \xrightarrow{(b_1, b_2)} & Y
\end{array}
\]

Then we have

\[
(23) \quad b_1 \circ h + b_2 \circ \iota_0 = f \circ a.
\]

As \( f \) is a deflation, there exists \( K \) such that \( 0 \rightarrow K \rightarrow X \rightarrow Y \rightarrow 0 \) is a short exact sequence. If we consider the diagram:

\[
\begin{array}{ccc}
K & \xrightarrow{h} & K \\
\downarrow & & \downarrow \\
K \oplus I_0 & \xrightarrow{h_{0}} & X \\
\downarrow & & \downarrow \\
I_0 & \xrightarrow{\iota_0} & Y
\end{array}
\]

as \( f \) is a deflation, the morphism \( K \oplus I_0 \rightarrow I_0 \) is a deflation. They moreover have the same kernel (as \( K \oplus I_0 \) is a pullback), thus the short exact sequence

\[
0 \rightarrow K \oplus I_0 \rightarrow K \oplus I_0 \rightarrow Y \rightarrow 0
\]

exists. As \( (h_{0}) \in I \), then there exist \( M_1, M_0 \in \mathcal{M} \) such that

\[
0 \rightarrow M_1 \rightarrow M_0 \rightarrow A \rightarrow 0
\]
is a short exact sequence. We build the following morphism of short exact sequences:

\[
\begin{array}{cccc}
0 & 0 \\
M_1 & K \oplus I_0 \\
k & (\beta_1, \beta_2) \\
M_0 & X \oplus I_0 \\
h & (f - b_2) \\
A & Y \\
0 & 0 \\
\end{array}
\]

From equation 23, we have that

\[b_1 \circ h = (f - b_2) \circ (a_{i_0})\].

Then, there exists \( \begin{pmatrix} c \\ \mu \end{pmatrix} \) such that

\[
\begin{pmatrix} a \circ k \\ i_0 \circ k \end{pmatrix} = \begin{pmatrix} \tilde{g} \circ c + \alpha \circ \mu \\ m u \end{pmatrix}.
\]

As \( \tilde{g} \in M^1 \), the morphism \( M_1 \rightarrow K \rightarrow X \) factorizes through an injective. We can suppose without loss of generality, that it factorizes through \( I_0 \). Then we have:

\[
\begin{array}{cccc}
M_1 & K \oplus I_0 \\
k & I_0 \\
M_0 & X \oplus I_0 \\
\end{array}
\]

with

\[
\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \circ \alpha_A = \begin{pmatrix} a \circ k \\ i_0 \circ k \end{pmatrix}.
\]
As $I_0$ is injective, and $k$ is an inflation, we can lift $\alpha_A$ to $\iota_0 : M_0 \rightarrow I_0$ such that $\iota_0 \circ k = \alpha_A$. Then,

$$
\begin{array}{c}
M_0 \xrightarrow{(\alpha_0)} X \oplus I_0 \\
\downarrow \iota_0 \quad \downarrow (\beta_1 \beta_2) \\
I_0 \quad \downarrow (f - b_2) \\
A \xrightarrow{b_1} Y
\end{array}
$$

Then, there exists $\left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) : A \rightarrow X \oplus I_0$ such that

$$
\left( \begin{array}{c} a \\ \iota_0 \end{array} \right) = \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right) \circ \iota_0 + \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) \circ h.
$$

Then, we have a morphism $A \oplus I_0 \rightarrow X$ which makes the upper triangle commute:

$$
\begin{array}{c}
M_0 \xrightarrow{(h \ i_0)} X \\
\downarrow (h \ i_0) \quad \downarrow f \\
A \oplus I_0 \xrightarrow{(b_1 \ b_2)} Y
\end{array}
$$

We have

$$
(\alpha_1 \beta_1) \circ (h \ i_0) = \alpha_1 \circ h + \beta_1 \circ \iota_0 = a
$$

from equation 24.

The morphism $M_0 \rightarrow A \oplus I_0$ is an inflation. Let us introduce its cokernel $C$. Then we have the push-out:

$$
\begin{array}{c}
M_0 \xrightarrow{h} A \\
\downarrow \gamma_1 \quad \downarrow f \circ \alpha_1 - b_1 \\
I_0 \xrightarrow{\gamma_2} C \xrightarrow{f \circ \beta_1 - b_2} Y
\end{array}
$$

Then there exists a unique $\psi : C \rightarrow Y$ such that:

$$
(25) \quad \psi \circ \gamma_2 = f \circ \beta_1 - b_2
$$

and

$$
(26) \quad \psi \circ \gamma_1 = f \circ \alpha_1 - b_1
$$

The push-out $C$ is exactly the cokernel of the morphism $M_1 \rightarrow M_0 \rightarrow I_0$. Then $C \cong \overline{\Omega} M_1$. As $f \in J^2$, if we still denote by $\psi$ the morphism from $\overline{\Omega} M_1$ to $Y$, 

...
then there exists $\zeta: UM_1 \to X$ such that $f \circ \zeta = \psi$. Then the equations 25 and 26 give respectively

$$f \circ \zeta \circ \gamma_1 = f \circ \alpha_1 - b_1$$

so

$$f \circ (\alpha_1 - \zeta \circ \gamma_1) = b_1$$

and

$$f \circ \zeta \circ \gamma_2 = f \circ \beta_1 - b_2$$

then

$$f \circ (\beta_1 - \zeta \circ \gamma_2) = b_2$$

and, the morphism

$$(\alpha_1 - \zeta \circ \gamma_1 \beta_1 - \zeta \circ \gamma_2): A \oplus I_0 \to X$$

make both triangles commute.

Now, let us show the inverse inclusion. We have immediately $I \subseteq J$ since $J \subseteq I$. Let $f \in I$. Let us show that, following the natural notations, $g \in \overline{M}$ and $\tilde{g} \in \overline{M}$. First, let $b: M \to Y$ be a morphism, with $M \in \mathcal{M}$. As $f \in I$, there exists $\varphi: M \to X$ such that $f \circ \varphi = b$ then $g \circ b = g \circ f \circ \varphi$ and this morphism factorizes through an injective. This shows that $g \in \overline{M}$. Second, Let $K$ be the kernel of $f$. Let $b: M \to K$. Let $N$ be an $\mathcal{M}$-approximation of $X$. We denote it by $a: N \to X$. As $a$ is an $\mathcal{M}$-approximation, there exists $h: M \to N$ such that $\tilde{g} \circ b = a \circ h$. Let $\iota_M: M \to I_M$ be the canonical injection. As $\left(\begin{smallmatrix} h \\ \iota_M \end{smallmatrix}\right)$ is an inflation, we introduce $C$ its cokernel. Let $(k u): N \oplus I_M \to C$. If we put $(a 0): N \oplus I_M \to X$, then there exists $r: C \to Y$ such that there is a morphism of short exact sequences:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
M & \xrightarrow{b} & K \\
\downarrow^{(h \ \iota_M)} & & \downarrow \tilde{g} \\
N \oplus I_M & \xrightarrow{(a \ 0)} & X \\
\downarrow^{(k \ u)} & \downarrow^{f} & \downarrow \\
C & \xrightarrow{r} & Y \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

We are going to build an element of $I$ from $N \oplus I_M \to C$. We add $J$ the injective envelope of $N \oplus I_M$ ($J = I_M \oplus I_N$). Then the morphism

$$\left(\begin{smallmatrix} k \\ u \\ \iota_1 \ \iota_2 \end{smallmatrix}\right): N \oplus I_M \to C \oplus J$$
is in $I$. Then the following square is commutative

$$
\begin{array}{ccc}
N \oplus I_M & \rightarrow & X \\
\downarrow & & \downarrow \\
C \oplus J & \rightarrow & Y
\end{array}
$$

and as $f \in I\Box$, there exist $(\varphi_1 \varphi_2) : C \oplus J \rightarrow X$ such that $f \circ \varphi_1 = r$ and $f \circ \varphi_2 = 0$ and

$$(\varphi_1 \circ k + \varphi_2 \circ \iota_1 \varphi_1 \circ \iota_2) = (a \ 0).$$

Then we have

$$
\tilde{g} \circ b = a \circ h
$$

$$
= (a \ 0) \circ \left(\begin{array}{c}
h \\
\iota_M
\end{array}\right)
$$

$$
= (\varphi_1 \varphi_2) \circ \left(\begin{array}{cc}
k & u \\
\iota_1 & \iota_2
\end{array}\right) \circ \left(\begin{array}{c}
h \circ \iota_M
\end{array}\right)
$$

$$
= (\varphi_1 \varphi_2) \circ \left(\begin{array}{c}
0 \\
\iota_1 \circ h + \iota_2 \circ \iota_M
\end{array}\right)
$$

(27)

This shows that $\tilde{g} \circ b$ factorizes through an injective, thus $\tilde{g} \in \mathcal{M}^\perp$.

Then we have shown that $I\Box = J\Box \cap \mathcal{W}$.

(v) The first required factorization is exactly corollary [4.2], part 1.

(vi) The second required factorization is exactly lemma [11.1].

As a consequence, we have directly the theorem of Quillen for Frobenius categories.

**Theorem 12.2.** Let $\text{Ho } \mathcal{E}$ be the localization of the quasi-isomorphisms of $\mathcal{E}$ at the class $\mathcal{W}$. Let $\text{mod } \mathcal{M}$ be the class of $\mathcal{M}$-modules. There is an equivalence of categories

$$
\text{Ho } \mathcal{E} \simeq \text{mod } \mathcal{M}.
$$

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