Topological supergravity structure 
of non-critical superstring theories

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ABSTRACT

We obtain a bosonization prescription that allows to represent the energy-momentum tensor and supersymmetry generators of non-critical superstring theories with minimal matter as those of topological supergravity. Superstrings with \( N = 1 \) and \( N = 2 \) world-sheet supersymmetry are considered. The topological symmetry associated with the topological supergravity representation is studied. It is shown, in particular, that the compatibility of this topological structure with the supersymmetry enhances the superconformal symmetry of the models concerned.

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One of the main motivations to study topological field theories was the conjecture that (super)string theory may have a topological phase [1,2]. For this and other reasons a lot of research effort has been directed to the study of topological structures in non-critical string theories [3 - 9]. The hope is that the study of the topological aspects of string theory might provide some clues that could help to understand the underlying symmetry structure of (super)string theory.

Topological (super)gravity is believed to play an important role in the hypothetical unbroken phase of (super)string theories. It is thus important to identify the topological (super) gravity aspects of (super)string theory. In ref. [10] the bosonic non-critical string theory with minimal matter was mapped to the free field system introduced in ref. [11] to represent topological gravity. This mapping is achieved by means of a convenient bosonization, generalizing the one in ref. [12], in which the matter and Liouville fields are combined to form an odd copy of the ghost sector, i.e. a spin-two pair of commuting fields. The resulting system is a topological ghost model, that has a natural topological BRST symmetry. It is the purpose of this paper to find a similar mapping for the superstring case. Two different cases will be analyzed: the $N = 1$ Neveu-Schwarz-Ramond (NSR) and the $N = 2$ non-critical superstrings.

Let us consider first the $N = 1$ NSR non-critical superstring [13, 14, 15]. The components of the matter and Liouville supermultiplets will be denoted by $(\phi_M, \psi_M)$ and $(\phi_L, \psi_L)$ respectively, where $\phi_M$ and $\phi_L$ are scalar fields and $\psi_M$ and $\psi_L$ are Majorana fermions. The ghost sector of the theory contains the anticommuting ghosts $b$ and $c$ with conformal weights $2$ and $-1$ together with the bosonic ghosts $\beta$ and $\gamma$ with conformal weights $3/2$ and $-1/2$. We will concentrate ourselves in the analysis of the holomorphic sector of the theory. The basic operator product expansions (OPE’s) of the fields will be taken as:

$$\phi_M(z) \phi_M(w) \sim \phi_L(z) \phi_L(w) \sim -\log(z - w)$$
$$\psi_M(z) \psi_M(w) \sim \psi_L(z) \psi_L(w) \sim b(z)c(w) \sim \beta(z)\gamma(w) \sim \frac{1}{z - w}.$$  \(1\)

The energy-momentum tensor $T$ of the theory can be represented as $T = T^M +$
\[ T^M = -\frac{1}{2} (\partial \phi_M)^2 + i Q_M \partial^2 \phi_M - \frac{1}{2} \psi_M \partial \psi_M \]
\[ T^L = -\frac{1}{2} (\partial \phi_L)^2 + i Q_L \partial^2 \phi_L - \frac{1}{2} \psi_L \partial \psi_L \]
\[ T^{gh} = -2b \partial c - \partial bc + \frac{3}{2} \beta \partial \gamma + \frac{1}{2} \partial \beta \gamma. \]

In eq. (2) we have adopted a Coulomb gas representation of the matter and Liouville sectors. \( Q_M \) and \( Q_L \) parametrize the matter and Liouville background charges. Since the total central charge must vanish and the central charge of the ghost sector is \(-15\), these quantities must satisfy the constraint \( Q_M^2 + Q_L^2 = -1 \).

In eq. (2), as well as in the following, the products of fields should be understood as normal-ordered. The \( N = 1 \) supersymmetry of this system is a consequence of the fact that there exists a dimension-3/2 fermionic field (denoted by \( T_F \)) that verifies the basic OPE of the \( N = 1 \) superconformal algebra:

\[ T_F(z) T_F(w) \sim \frac{1}{2} \frac{T(w)}{z-w}. \]

Given the representation (2) for \( T \), it is easy to find the expression for \( T_F \). Indeed one can check that the operator

\[ T_F = i \frac{1}{2} \partial \phi_M \psi_M + i \frac{1}{2} \partial \phi_L \psi_L + Q_L \partial \psi_L + Q_M \partial \psi_M + \frac{1}{2} \gamma b + \frac{3}{2} \partial c \beta + c \partial \beta, \]

satisfies eq. (3). Under the action of \( T_F \) the fields of the theory are classified in supersymmetry doublets. One of such doublets \((X,Y)\) is constituted by two operators \( X \) and \( Y \) of opposite statistics and whose conformal weights differ by \( 1/2 \) \( (\Delta_Y = \Delta_X + \frac{1}{2}) \). The action of \( T_F \) on \((X,Y)\) is given by:

\[ T_F(z) X(w) \sim \frac{1}{2} \frac{Y(w)}{z-w} \]
\[ T_F(z) Y(w) \sim \frac{\Delta_X X(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial X(w)}{z-w}. \]

The ghost fields \( b, c, \beta \) and \( \gamma \) can be accommodated in two of such \( N = 1 \) doublets.
Indeed one may check that eq. (5) is satisfied for \( X = c \) and \( Y = \gamma \) whereas the antighosts \( \beta \) and \( b \) verify (5) with \( X = \beta \) and \( Y = -b \). Moreover, the matter and Liouville fields also form two supersymmetry doublets, which are however anomalous due to the existence of the background charges \( Q_M \) and \( Q_L \).

As it was pointed out above, the purpose of this paper is to find a mapping from non-critical superstring theories to topological ghost systems. Essentially we want to show that one can combine the matter and Liouville fields in such a way that they can be regarded as a topological copy of the superstring ghost sector. By a topological copy we mean a series of fields having the same conformal dimensions as the ghosts \((b,c)\) and \((\beta, \gamma)\) but obeying opposite statistics. Accordingly we want to represent the matter + Liouville sector of the superstring in terms of two pairs of fields \((B,C)\) and \((\mathcal{B}, \Gamma)\). The fields \(B\) and \(C\) (\(\mathcal{B}\) and \(\Gamma\)) are bosonic (fermionic) and they have the same conformal dimensions as their lower case counterparts (i.e. \(\Delta_B = 2, \Delta_C = -1, \Delta_B = 3/2\) and \(\Delta_\Gamma = -1/2\)). The resulting topological ghost system is nothing but the free field representation of topological supergravity [16], which is the supersymmetric generalization of the ghost system used in ref. [11] to represent topological gravity.

Supersymmetry will play a central role in our approach and thus it is natural to require that its generator \(T_F\) should be realized locally in terms of the new fields \(B, C, \mathcal{B}\) and \(\Gamma\). The best way to achieve this locality requirement is by demanding that \(B, C, \mathcal{B}\) and \(\Gamma\) belong to supersymmetry doublets. Taking into account the conformal weights and statistics of these fields, the only possible doublets that one can form with them are \((C, \Gamma)\) and \((B, B)\). Let us adopt a vector notation for the fields of the matter and Liouville sectors. We shall assemble the scalar and Majorana fields in a two component vector, i.e. we define \(\vec{\phi} = (\phi_M, \phi_L)\) and \(\vec{\psi} = (\psi_M, \psi_L)\). Similarly the background charges \(Q_M\) and \(Q_L\) will be considered as the two components of the vector \(\vec{Q}\) (i.e. \(\vec{Q} = (Q_M, Q_L)\)).

In order to represent the \((C, \Gamma)\) and \((\mathcal{B}, B)\) doublets in terms of the fields \(\vec{\phi}\) and \(\vec{\psi}\) we follow the manifest supersymmetric bosonization of refs. [17, 18]. Let
us first apply this bosonization formalism to the $(C, \Gamma)$ doublet. The field $C$ in this approach is represented as an exponential of $\vec{\phi}$, i.e. as an expression of the form $C = e^{\vec{\mu} \cdot \vec{\phi}}$ with $\vec{\mu}$ a complex constant numerical vector. Once the form of $C$ is known, its supersymmetric partner $\Gamma$ is obtained by acting with $T_F$ and comparing the resulting expression with the first equation in (5). After a simple calculation one gets:

$$C = e^{\vec{\mu} \cdot \vec{\phi}} \quad \Gamma = -i\vec{\mu} \cdot \vec{\psi} e^{\vec{\mu} \cdot \vec{\phi}}. \quad (6)$$

The $(B, B)$ form a doublet which is conjugate to $(C, \Gamma)$. This means that the only non-vanishing OPE’s among these fields are:

$$B(z) \Gamma(w) \sim B(z) C(w) \sim \frac{1}{z-w}. \quad (7)$$

By inspecting the form of $\Gamma$ one realizes that there is a chance to reproduce the correct OPE $B(z) \Gamma(w)$ only when $B$ and $\Gamma$ are given by a similar expression. Accordingly we consider the following ansatz for $B$ and $B$:

$$B = \vec{\rho} \cdot \vec{\psi} e^{-\vec{\mu} \cdot \vec{\phi}} \quad B = i[\vec{\mu} \cdot \vec{\psi} \vec{\rho} \cdot \vec{\psi} + \vec{\rho} \cdot \partial \vec{\phi}] e^{-\vec{\mu} \cdot \vec{\phi}}, \quad (8)$$

where $\vec{\rho}$ is a new numerical vector. Notice that the expression of $B$ can be obtained by acting with $T_F$ on $B$. Actually the OPE’s of eqs. (5) and (7) impose conditions on the inner products of the vectors $\vec{\mu}$, $\vec{\rho}$ and $\vec{Q}$. These conditions are:

$$\vec{\mu}^2 = \vec{\rho}^2 = 0 \quad \vec{\rho} \cdot \vec{\mu} = i \quad \vec{Q} \cdot \vec{\mu} = i \quad \vec{Q} \cdot \vec{\rho} = -\frac{1}{2}. \quad (9)$$

Eq. (9) can be easily solved for $\vec{\mu}$ and $\vec{\rho}$. In order to find this solution, let us parametrize the background charges $Q_M$ and $Q_L$ as:

$$Q_M = \frac{1}{2} \left( \frac{1}{\lambda} - \lambda \right) \quad Q_L = \frac{i}{2} (\lambda + \frac{1}{\lambda}), \quad (10)$$

where $\lambda$ is a constant. Notice that the condition $Q_M^2 + Q_L^2 = -1$ is automatically satisfied. The parameter $\lambda$ is related to the central charge of the matter sector.
For example, for $(p, q)$ minimal superconformal matter $\lambda = \sqrt{\frac{q}{p}}$. In terms of $\lambda$ the solution of eq. (9) is:

$$\vec{\mu} = (i\lambda, \lambda) \quad \vec{\rho} = \left( \frac{1}{2\lambda}, \frac{i}{2\lambda} \right).$$

(11)

Using these values of $\vec{\mu}$ and $\vec{\rho}$ in our bosonization formulas (eqs. (6) and (8)), we can get the explicit expressions of the fields $B, C, B$ and $\Gamma$. The result is:

$$B = i[-\psi_M \psi_L + \frac{1}{2\lambda} (\partial \phi_M + i \partial \phi_L)] e^{-i\lambda(\phi_M - i\phi_L)}$$

$$C = e^{i\lambda(\phi_M - i\phi_L)}$$

$$B = \frac{1}{2\lambda} (\psi_M + i \psi_L) e^{-i\lambda(\phi_M - i\phi_L)}$$

$$\Gamma = \lambda(\psi_M - i \psi_L) e^{i\lambda(\phi_M - i\phi_L)}.$$  

(12)

It is now easy to express $T$ and $T_F$ in terms of the new fields. First of all it is straightforward to check that the matter and Liouville contributions to the energy-momentum tensor are given by:

$$T^M + T^L = 2B \partial C + \partial BC - \frac{3}{2} B \partial \Gamma - \frac{1}{2} \partial B \Gamma,$$

(13)

Moreover, the supersymmetry generator $T_F$ can be written as:

$$T_F = \frac{1}{2} \Gamma B + \frac{3}{2} \partial C B + C \partial B + \frac{1}{2} \gamma b + \frac{3}{2} \partial c \beta + c \partial \beta.$$  

(14)

A simple comparison of the ghost and matter+Liouville contributions to $T$ and $T_F$ reveals the existence of a clear symmetry relating them. This symmetry is in fact a
topological BRST symmetry as we shall check below. The current associated with the generator of this symmetry can be taken as:

$$Q = bC - \beta \Gamma.$$  \hspace{1cm} (15)

It is a simple exercise to verify that $T$ and $T_F$ are BRST invariant $i.e.$ that they are invariant under the action of the zero-mode of $Q$. Actually it is interesting to point out that the relative coefficient of the two terms of $Q$ (including the sign) is uniquely fixed by the BRST invariance of $T_F$, which is a condition that we must require in order to have a topological symmetry compatible with the supersymmetry of the model. The BRST current $Q$ endows the NSR superstring with the structure of a topological Conformal Field Theory. In such theories the energy-momentum tensor $T$ is $Q$-exact and its BRST ancestor is usually denoted by $G$. The local relation between $G$ and $T$ is determined by the OPE:

$$Q(z) G(w) \sim \frac{d}{(z-w)^3} + \frac{R(w)}{(z-w)^2} + \frac{T(w)}{z-w},$$  \hspace{1cm} (16)

where $d$ is a c-number anomaly and $R$ is a $U(1)$ current. In our case, $i.e.$ when $Q$ is given by eq. (15), $d = -1$ while $G$ and $R$ are:

$$G = c \partial B + 2c \partial C - \frac{1}{2} \gamma \partial B - \frac{3}{2} \partial \gamma B$$

$$R = cb + 2BC + \frac{1}{2} \beta \gamma + \frac{3}{2} \Gamma B,$$  \hspace{1cm} (17)

It is easy to check that $T$, $G$, $Q$ and $R$ close a topologically twisted $N = 2$ superconformal algebra [19, 20]. Let us now try to find the operator algebra closed by $T_F$ and $T$, $G$, $Q$ and $R$. This algebra determines how the topological symmetry and the supersymmetry are interrelated. It is interesting to remember here that the $U(1)$ current $R$ defines a grading which is characteristic of the topological symmetry we are dealing with. In fact all the generators of the topological algebra should have a well-defined $R$-charge. From the explicit expression of $R$ in eq. (17)
it follows that the $R$-charges of the fields $b$, $c$, $\beta$, $\gamma$, $B$, $C$, $B$ and $\Gamma$ are respectively $-1$, $+1$, $-1/2$, $+1/2$, $-2$, $+2$, $-3/2$ and $+3/2$. By a simple counting one easily concludes that there are two types of terms in $T_F$ with $R$-charges $\pm 1/2$. Actually the topological $U(1)$ current $R$ induces a splitting of $T_F$ in two pieces. Let us write $T_F = \frac{1}{2}(T^+_F + T^-_F)$, where $T^+_F$ and $T^-_F$ have $R$-charges $+1/2$ and $-1/2$ respectively and whose explicit expressions are given by:

$$T^+_F = 3\partial CB + 2C\partial B + 3\partial c\beta + 2c\partial \beta \quad \quad T^-_F = \Gamma B + \gamma b \quad (18)$$

It is interesting to notice that the splitting of eq. (18) is quite special. In fact $T^\pm_F$ are the generators of an $N = 2$ superconformal algebra. This fact may be verified by computing the operator algebra closed by $T^\pm_F$. One of the relations in this algebra is:

$$T^+_F(z) T^-_F(w) \sim \frac{J(w)}{(z-w)^2} + \frac{T(w) + \frac{1}{2} \partial J(w)}{z-w}, \quad (19)$$

where $J$ is the $U(1)$ current associated to the $N = 2$ superconformal algebra which is given by:

$$J = -2cb - 2BC - 3\beta\gamma - 3\Gamma B. \quad (20)$$

We thus see that the original $N = 1$ supersymmetry of the string is promoted to an $N = 2$ superconformal symmetry. This enhancement of the supersymmetry is essential in order to make topological symmetry and supersymmetry compatible. In fact, the hidden $N = 2$ superconformal symmetry of the NSR ghost sector was found some time ago in ref. [21]. The topological symmetry exhibited by our bosonization extends this $N = 2$ supersymmetry to the matter+Liouville sector.

Under the $N = 2$ supersymmetry just uncovered, it should be possible to arrange all operators of the theory in $N = 2$ supermultiplets. In general such a supermultiplet is composed by four fields $(X, Y^+, Y^-, Z)$ whose conformal weights
are related as $\Delta Y^\pm = \Delta X + 1/2$ and $\Delta Z = \Delta X + 1$. The action of $T^\pm_F$ on the fields of the supermultiplet is given by:

$$
T^\pm_F(z) X(w) \sim \pm \frac{Y^\pm(w)}{z-w}
$$

$$
T^+_F(z) Y^+(w) \sim T^-_F(z) Y^-(w) \sim 0
$$

$$
T^\pm_F(z) Y^\mp(w) \sim \pm \frac{\Delta X X(w)}{(z-w)^2} + \frac{Z(w) \pm \frac{1}{2} \partial X(w)}{z-w}
$$

$$
T^\pm_F(z) Z(w) \sim \frac{\Delta Y^\pm Y^\pm(w)}{(z-w)^2} + \frac{1}{2} \frac{\partial Y^\pm(w)}{z-w}.
$$

An example of such an $N = 2$ multiplet is the one constituted by the generators of the $N = 2$ superconformal algebra $(J,T^+_F, T^-_F, T)$. The BRST ancestor of $T$, which we called $G$ in eq. (17), also belongs to an $N = 2$ multiplet. Let us denote the members of this multiplet as $(g, G^+_B, G^-_B, G)$. The operators $g$ and $G$ are fermionic whereas $G^+_B$ and $G^-_B$ are bosonic. Their conformal weights are $\Delta_g = 1$, $\Delta_{G^+_B} = 3/2$ and $\Delta_G = 2$. Using eqs. (17) and (18) one can easily obtain the expressions of $g$ and $G^\pm_B$. The result is:

$$
g = 3B \gamma - 2cB \quad \quad G^+_B = -3cB - 2cB \quad \quad G^-_B = B \gamma. \quad (22)
$$

Acting the with BRST current $Q$ on $(g, G^+_B, G^-_B, G)$ one generates $(J, T^+_F, T^-_F, T)$ as the residue of the single pole singularity. The action of $Q$ on $G$ is displayed in eq. (16), whereas the action of $Q$ on $g$ and $G^\pm_B$ is given by:

$$
Q(z) g(w) \sim - \frac{1}{(z-w)^2} + \frac{J(w)}{z-w}
$$

$$
Q(z) G^+_B(w) \sim - \frac{R_F(w)}{(z-w)^2} - \frac{T^+_F(w)}{z-w}
$$

$$
Q(z) G^-_B(w) \sim - \frac{T^-_F(w)}{z-w}.
$$

Notice that, in particular, eq. (23) shows that $J$ and $T^\pm_F$ are BRST-exact operators.
In eq. (23) $R_F$ is a dimension-1/2 fermionic current whose explicit expression is:

$$R_F = 2\beta c + 3BC. \tag{24}$$

Supermultiplets with four fields are not the only possibility to arrange the operators of a theory with $N = 2$ supersymmetry. We can also have chiral multiplets constituted by two fields $(X, Y^-)$ that under the action of $T_F^\pm$ behave as follows:

$$T_F^-(z) X(w) \sim \frac{Y^-(w)}{z-w}$$
$$T_F^+(z) X(w) \sim T_F^-(z) Y^-(w) \sim 0$$
$$T_F^+(z) Y^-(w) \sim \frac{2\Delta X(w)}{(z-w)^2} + \frac{\partial X(w)}{z-w} \tag{25}$$

The ghost fields $(c, \gamma)$ and the corresponding antighosts $(\beta, -b)$, together with their topological partners $(C, \Gamma)$ and $(B, B)$, are examples of chiral multiplets. It turns out that the BRST current $Q$ also belongs to a chiral multiplet. It may be checked that there exists a bosonic dimension-1/2 operator $Q_B$ such that the fields $(Q_B, Q)$ satisfy eq. (25). This supersymmetric partner of the BRST current is:

$$Q_B = -C\beta. \tag{26}$$

There is still another chiral multiplet formed by the fermionic operator $R_F$ and the particular combination $R - \frac{1}{2}J$ of the $U(1)$ currents $R$ and $J$. However the action of $T_F^\pm$ on $(R_F, R - \frac{1}{2}J)$ is anomalous (recall that $R$ is anomalous under the action of $T$). In fact one can check that instead of the first equation of (25) with $X = R_F$ and $Y^- = R - \frac{1}{2}J$ one has:

$$T_F^-(z) R_F(w) \sim \frac{1}{(z-w)^2} + \frac{R(w) - \frac{1}{2}J(w)}{z-w} \tag{27}$$

These two chiral multiplets are connected by means of the BRST current $Q$. Indeed one can verify that the operators $(Q_B, Q)$ can be obtained by acting with $Q$ on the
fields \((R_F, R - \frac{1}{2}J)\). One has, for example, that:

\[
Q(z)R_F(w) \sim \frac{Q_B(w)}{z-w}. \tag{28}
\]

The twelve operators introduced so far close an algebra which has a topologically twisted \(N = 2\) algebra (obeyed by \(T, G, Q\) and \(R\)) and an untwisted \(N = 2\) superconformal algebra (generated by \(T, T_F^+, T_F^-\) and \(J\)) as subalgebras. The compatibility between the topological and superconformal symmetries is reflected in the fact that all the generators of the algebra have well-defined charges with respect to the two \(U(1)\) currents \(R\) and \(J\) of these two types of symmetries. In fact one can verify that the operators \(J, T_F^+, T_F^-, T, g, G_B^+, G_B^-, G, Q_B, Q, R_F\) and \(R\) have \(R\)-charges 0, \(1/2\), \(-1/2\), 0, \(-1\), \(-3/2\), \(-1\), \(3/2\), 1, \(1/2\) and 0 respectively, whereas the \(J\)-charges of these same fields are 0, \(-1\), 0, 0, \(+1\), \(-1\), 0, \(+1\), 0, \(+1\) and 0. The OPE’s of \(R\) and \(J\) with the generators of the algebra always contain a single pole singularity dictated by these charges. In some of these OPE’s, however, higher order poles could appear. Two examples of this anomalous behaviour under the action of the \(R\) and \(J\) currents are:

\[
R(z) T_F^+(w) \sim \frac{R_F(w)}{(z-w)^2} + \frac{1}{2} \frac{T_F^+(w)}{z-w} \tag{29}
\]

\[
J(z) G(w) \sim \frac{g(w)}{(z-w)^2}.
\]

Other interesting OPE’s of the algebra are:

\[
G(z) R_F(w) \sim - \frac{G_B^+(w)}{z-w}
\]

\[
G(z) Q_B(w) \sim \frac{1}{2} \frac{R_F(w)}{(z-w)^2} + \frac{\partial R_F(w) - T_F^+(w)}{z-w}
\]

\[
G_B^+(z) R_F(w) \sim G_B^+(z) Q_B(w) \sim 0 \tag{30}
\]

\[
G_B^-(z) R_F(w) \sim \frac{1}{2} \frac{g(w)}{z-w}
\]

\[
G_B^-(z) Q_B(w) \sim \frac{1}{(z-w)^2} + \frac{R(w) + \frac{1}{2}J(w)}{z-w}.
\]

It would be interesting to relate the topological algebra just described to some
superconformal symmetry. Usually, in order to relate topological and superconformal algebras one has to perform a twisting in their generators. It turns out that our topological algebra can be related to a small $N = 4$ superconformal symmetry [22]. This $N = 4$ supersymmetry can be regarded as two $N = 2$ supersymmetries that share the same $U(1)$ current. The generators of the two $N = 2$ supersymmetries will be denoted by $G_i^\pm$ for $i = 1, 2$ whereas the common $U(1)$ current will be called $J_0$. The mixing of the two $N = 2$ supersymmetries requires the introduction of two extra currents (denoted by $J^{++}$ and $J^{--}$) which, together with $J_0$, close an affine $SU(2)$ algebra ($J_0$ is the Cartan generator of this algebra). Denoting the twisted energy-momentum tensor by $\tilde{T}$, the $N = 4$ generators are:

$$
\begin{align*}
G_1^+ &= T_F^+ - \partial R_F \\
G_1^- &= T_F^- \\
G_2^+ &= Q \\
G_2^- &= G + \frac{1}{2} \partial g \\
J_0 &= R + \frac{1}{2} J \\
J^{++} &= Q_B \\
J^{--} &= -G_B^- \\
\tilde{T} &= T - \frac{1}{2} \partial (R - \frac{1}{2} J).
\end{align*}
$$

Notice that the $N = 4$ algebra has fewer generators than our topological algebra (eight versus twelve). The central charge of $\tilde{T}$ is $-6$. This is the Virasoro anomaly of an $N = 2$ ghost multiplet. In fact the conformal weights with respect to $\tilde{T}$ of the fields $(b, c)$, $(\beta, \gamma)$, $(B, C)$ and $(\mathcal{B}, \Gamma)$ are $(1, 0)$, $(1/2, 1/2)$, $(1/2, 1/2)$ and $(0, 1)$ respectively. Given these conformal dimensions one can form two conjugate $N = 2$ multiplets: $(c, -C, \gamma, -\Gamma)$ and $(\mathcal{B}, \beta, B, -b)$. It can be shown that, arranged in this way, these fields behave as in eq. (21) with respect to the superconformal generators $G_2^+ + \frac{1}{2} G_1^+$ and $G_1^- + \frac{1}{2} G_2^-$. This means that we have embedded the $N = 1$ NSR string in an $N = 2$ ghost system.

Let us now consider the $N = 2$ non-critical superstring [14]. In this case the matter and Liouville sectors can be regarded as the complexification of the $N = 1$ NSR superstring. Therefore, along with the bosonic and fermionic fields $\vec{\phi} = (\phi_M, \phi_L)$ and $\vec{\psi} = (\psi_M, \psi_L)$, we must deal with their hermitian conjugates $\vec{\phi}^\dagger = (\phi_M^\dagger, \phi_L^\dagger)$ and $\vec{\psi}^\dagger = (\psi_M^\dagger, \psi_L^\dagger)$. The ghost sector now contains a pair of anticommuting fields $(\hat{b}, \hat{c})$ with conformal weights $(1, 0)$, two commuting pairs
(β⁺, γ⁻) and (β⁻, γ⁺) with conformal weights (3/2, −1/2) and a pair (b, c) of anti-commuting ghosts with conformal weights (2, −1). The basic OPE's among these fields are:

\[
\begin{align*}
\phi_M(z) &\phi_M^\dagger(w) \sim \phi_L(z) \phi_L^\dagger(w) \sim -\log(z - w) \\
\psi_M(z) &\psi_M^\dagger(w) \sim \psi_L(z) \psi_L^\dagger(w) \sim \frac{1}{z - w} \\
\tilde{b}(z) &\tilde{c}(w) \sim \beta_+(z) \gamma_-(w) \sim \beta_-(z) \gamma_+(w) \sim b(z) c(w) \sim \frac{1}{z - w}.
\end{align*}
\] (32)

The generators of the \( N = 2 \) world-sheet superconformal symmetry in the matter+Liouville sector are given by:

\[
\begin{align*}
T^M + T^L &= -\partial \vec{\phi} \cdot \partial \vec{\phi}^\dagger + i \vec{Q} \cdot (\partial^2 \vec{\phi} + \partial^2 \vec{\phi}^\dagger) - \frac{1}{2} \vec{\psi} \cdot \partial \vec{\psi}^\dagger - \frac{1}{2} \vec{\psi}^\dagger \cdot \partial \vec{\psi} \\
T_{F,+}^{M,+} + T_{F,-}^{L,+} &= i \partial \vec{\phi} \cdot \vec{\psi}^\dagger + 2 \vec{Q} \cdot \partial \vec{\psi}^\dagger \\
T_{F,+}^{M,-} + T_{F,-}^{L,-} &= i \partial \vec{\phi}^\dagger \cdot \vec{\psi} + 2 \vec{Q} \cdot \partial \vec{\psi} \\
J^M + J^L &= \vec{\psi}^\dagger \cdot \vec{\psi} + 2i \vec{Q} \cdot \partial \vec{\phi} - 2i \vec{Q} \cdot \partial \vec{\phi}^\dagger,
\end{align*}
\] (33)

whereas in the ghost sector the \( N = 2 \) supersymmetry is generated by:

\[
\begin{align*}
T^{\text{gh}} &= -2 b \partial c - dc - \tilde{b} \partial \tilde{c} + \frac{3}{2} \beta_+ \partial \gamma_- + \frac{1}{2} \partial \beta_+ \gamma_- + \frac{3}{2} \beta_- \partial \gamma_+ + \frac{1}{2} \partial \beta_- \gamma_+ \\
T_{F}^{\text{gh},+} &= \mp \beta \gamma_{\pm} + \partial \gamma_{\pm} \tilde{b} + \frac{1}{2} \gamma_{\pm} \partial \tilde{b} \pm \frac{3}{2} \beta_{\pm} \partial c \pm \partial \beta_{\pm} c + \tilde{c} \beta_{\pm} \\
J^{\text{gh}} &= \gamma_+ \beta_- - \beta_+ \gamma_- \quad - \partial (bc).
\end{align*}
\] (34)

The \( N = 2 \) supersymmetry classifies the ghost fields in multiplets. Indeed, one can check that eq. (21) is satisfied for \((X, Y^+, Y^-, Z) = (c, \gamma_+, \gamma_-, \tilde{c})\). Similarly the \( N = 2 \) antighost multiplet is \((X, Y^+, Y^-, Z) = (\tilde{b}, -\beta_+, \beta_-, b)\). Notice that now the ghost central charge is equal to −6. This means that the vector \( \vec{Q} = (Q_M, Q_L) \) in eq. (33) must satisfy \( \vec{Q}^2 = Q_M^2 + Q_L^2 = 0 \). We shall not consider here the case \( \vec{Q} = 0 \), which corresponds to the critical \( N = 2 \) string. In fact, as we did for the
\[ N = 1 \text{ case in eq. (10), we shall now parametrize } Q_M \text{ and } Q_L \text{ as follows:} \]

\[ Q_M = \frac{1}{4\lambda} \quad Q_L = \frac{i}{4\lambda}. \quad (35) \]

In analogy with what we have done in the \( N = 1 \) NSR string, we want to show how the matter+Liouville sector of the \( N = 2 \) superstring can be represented as a topological copy of the ghost sector. Accordingly, let us consider two pairs \((\tilde{B}, \tilde{C})\) and \((B, C)\) of commuting fields with conformal weights \((1, 0)\) and \((2, -1)\) respectively together with another two pairs of anticommuting fields \((\mathcal{B}^+, \Gamma^-)\) and \((\mathcal{B}^-, \Gamma^+)\) each of which having conformal dimensions \((3/2, -1/2)\). The OPE’s among these fields will be taken to be:

\[ \tilde{B}(z)\tilde{C}(w) \sim \mathcal{B}^+(z)\Gamma^-(w) \sim \mathcal{B}^-(z)\Gamma^+(w) \sim B(z)C(w) \sim \frac{1}{z-w}. \quad (36) \]

In order to extract these new fields from the matter+Liouville sector, the realization of the \( N = 2 \) symmetry of the model in terms of them will be our guiding principle. In complete parallel with the ghost sector, we shall distribute the new fields in two \( N = 2 \) multiplets. The content of one of these multiplets will be \((X, Y^+, Y^-, Z) = (C, \Gamma^+, \Gamma^-, \tilde{C})\) while its conjugate multiplet will be formed by \((\tilde{B}, \mathcal{B}^+, -\mathcal{B}^-, B)\).

Let us adopt the following ansatz for the fields \(C\) and \(\tilde{B}\) \((i.e.\ for the components of the multiplets with the lowest dimension)\):

\[ C = e^{\mu \cdot \phi + \mu^\dagger \cdot \phi^\dagger} \quad \tilde{B} = (\tilde{\rho} \cdot \tilde{\phi} + \tilde{\rho}^\dagger \cdot \tilde{\phi}^\dagger) e^{-\mu \cdot \phi - \mu^\dagger \cdot \phi^\dagger}. \quad (37) \]

In eq. (37) \(\mu, \mu^\dagger, \tilde{\rho} \text{ and } \tilde{\rho}^\dagger\) are numerical vectors to be determined. Requiring the new fields to behave as in eqs. (21) and (36), one gets the form of the remaining members of the multiplets, together with many conditions that the numerical vectors must satisfy. These conditions are enough to determine \(\mu, \mu^\dagger, \tilde{\rho} \text{ and } \tilde{\rho}^\dagger\). Using the parametrization of \(Q_M\) and \(Q_L\) given in eq. (35), the final expressions
of the fields \((C, \Gamma^+, \Gamma^-, \tilde{C})\) are:

\[
C = e^{i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}
\]

\[
\Gamma^+ = -\lambda(\psi_M^\dagger - i\psi_L^\dagger) e^{i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}
\]

\[
\Gamma^- = \lambda(\psi_M - i\psi_L) e^{i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}
\]

\[
\tilde{C} = [\lambda^2(\psi_M^\dagger - i\psi_L^\dagger)(\psi_M - i\psi_L) + \frac{i\lambda}{2}(\partial\phi_M - \partial\phi_M^\dagger - i\partial\phi_L + i\partial\phi_L^\dagger)] e^{i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)},
\]

while the conjugate fields are given by:

\[
\tilde{B} = \frac{i}{2\lambda}(\phi_M - \phi_M^\dagger + i\phi_L - i\phi_L^\dagger) e^{-i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}
\]

\[
B^+ = \left[\frac{1}{2\lambda}(\psi_M^\dagger + i\psi_L^\dagger) + \frac{i}{2}(\psi_M - i\psi_L)(\phi_M - \phi_M^\dagger + i\phi_L - i\phi_L^\dagger)\right] e^{-i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}
\]

\[
B^- = \left[\frac{1}{2\lambda}(\psi_M + i\psi_L) + \frac{i}{2}(\psi_M - i\psi_L)(\phi_M - \phi_M^\dagger + i\phi_L - i\phi_L^\dagger)\right] e^{-i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}
\]

\[
B = \left[i\frac{\lambda}{2}(\psi_M^\dagger - i\psi_L^\dagger)(\psi_M - i\psi_L)(\phi_M - \phi_M^\dagger + i\phi_L - i\phi_L^\dagger)
- \frac{i}{4\lambda}(\partial\phi_M + \partial\phi_M^\dagger + i\partial\phi_L + i\partial\phi_L^\dagger)
+ \frac{1}{4}(\partial\phi_M - \partial\phi_M^\dagger - i\partial\phi_L + i\partial\phi_L^\dagger)(\phi_M - \phi_M^\dagger + i\phi_L - i\phi_L^\dagger)\right] e^{-i\lambda(\phi_M + \phi_M^\dagger - i\phi_L - i\phi_L^\dagger)}.
\]

(39)

It is straightforward, although in some cases tedious, to prove that the matter+Liouville contributions to \(T, T_{F}^\pm\) and \(J\) can be written as:

\[
T^M + T^L = 2B\partial C + \partial BC - \frac{3}{2}B^+\partial\Gamma^- - \frac{3}{2}B^-\partial\Gamma^+ - \frac{1}{2}\partial B^+\Gamma^- - \frac{1}{2}\partial B^-\Gamma^+ + \tilde{B}\partial\tilde{C}
\]

\[
T_F^{M,\pm} + T_F^{L,\pm} = \mp B\Gamma^\pm + \partial\Gamma^\pm\tilde{B} + \Gamma^\pm\partial\tilde{B} \pm \frac{3}{2}B^\pm\partial C \pm \partial B^\pm C + \tilde{C}B^\pm
\]

\[
J^M + J^L = B^+\Gamma^- - B^-\Gamma^+ + \partial(\tilde{B}\tilde{C}).
\]

(40)

As in the \(N = 1\) case, the topological symmetry relating the ghost and matter+Liouville sectors is now evident. We can take the current of its generator as:

\[
Q = bC + \tilde{b}\tilde{C} - \beta_+\Gamma^- - \beta_-\Gamma^+.
\]

(41)

Moreover, it is also possible to fulfill eq. (16) with \(d = 0\) where \(G\) and \(R\) are given
by:

\[ G = c \partial B + 2 \partial c B - \frac{1}{2} \gamma_+ \partial B^- - \frac{3}{2} \partial \gamma_+ B^- - \frac{1}{2} \gamma_- \partial B^+ - \frac{3}{2} \partial \gamma_- B^+ \]
\[ R = \bar{c} b + 2 B C + \frac{1}{2} \beta_+ \gamma_- + \frac{1}{2} \beta_- \gamma_+ - \frac{3}{2} B^+ \Gamma^- - \frac{3}{2} B^- \Gamma^+ + \bar{B} \bar{C}. \]  

(42)

Interestingly, with respect to the \( R \) current of eq. (42), the \( N = 2 \) supersymmetry generators of eqs. (34) and (40) split in a way very similar to the \( N = 1 \) case. Indeed one can easily check that the terms in \( T_F^{\pm} \) that contain derivatives of the fields have \( R \)-charge \( +1/2 \) whereas those without derivatives have \( R \)-charge \( -1/2 \). Therefore, writing \( T_F^{++} = T_F^{++} + T_F^{+-} \) and \( T_F^{--} = T_F^{-+} + T_F^{--} \), where \( T_F^{++} (T_F^{--}) \) have \( R \)-charge \( +1/2 \) (\( -1/2 \)), one has:

\[ T_F^{++} = \partial \Gamma^\pm \bar{B} + \frac{1}{2} \Gamma^\pm \partial \bar{B} \pm \frac{3}{2} B^\pm \partial C \pm \partial B^\pm C \pm \partial \gamma^\pm \bar{b} \pm \frac{1}{2} \gamma^\pm \partial \bar{b} \pm \frac{3}{2} \beta^\pm \partial c \pm \partial \beta^\pm c \]
\[ T_F^{+-} = \mp B \Gamma^\pm + \bar{C} B^\pm \mp b \gamma^\pm + \bar{c} \beta^\pm. \]  

(43)

Moreover, the \( N = 2 \) current \( J \) splits in a similar way. It can be checked that \( T_F^{\pm\pm} \) close a superconformal algebra with twelve generators. We have thus obtained an extension of the supersymmetry induced by the topological symmetry for the \( N = 2 \) string, which is completely similar to the splitting found for the \( N = 1 \) NSR string. Actually, one can regard this supersymmetry as the hidden \( N = 4 \) supersymmetry of the ghost sector of the \( N = 2 \) superstring. We will not attempt to study this supersymmetry here. Let us only mention that, curiously, it can also be related to the small \( N = 4 \) superconformal algebra.

In conclusion, we have found a bosonization that allows to give a topological supergravity representation of the energy-momentum tensor and supersymmetry generators of the non-critical superstrings with minimal matter. The compatibility of the topological symmetry and supersymmetry induces a “doubling” of the latter. Many aspects of this bosonization remain to be explored. It would be interesting, for example, to characterize the physical states of the superstring that can be obtained as local excitations of our topological supergravity fields. In view
of the results of [10] for the bosonic string, we expect to recover in this supergravity representation only a part of the physical state spectrum of the non-critical superstrings. We intend to study this and other related issues in the future.

Acknowledgements:

We would like to thank I.P. Ennes, J. M. F. Labastida and P.M. Llatas for helpful discussions. This work was partially supported by DGICYT under grant PB 93-0344, by CICYT under grant AEN 94-0928 and by a Spanish Ministry of Education (MEC) fellowship for one of us (S. R.).

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