Uniqueness and non–uniqueness of prescribed mass NLS ground states on metric graphs

Simone Dovetta
simone.dovetta@uniroma1.it
Dipartimento di Scienze di Base ed Applicate per l’Ingegneria
Università degli Studi di Roma "La Sapienza"

Variational Methods and Equations on Graphs
8th European Congress of Mathematics
June 23, 2021
Given a metric graph $G$, consider the NLS energy functional

$$E(u) = \frac{1}{2} \int_G |u'|^2 \, dx - \frac{1}{p} \int_G |u|^p \, dx$$

subject to the mass constraint

$$\int_G |u|^2 \, dx = \mu \quad (\mu > 0, \text{prescribed}).$$
Given a metric graph $G$, consider the NLS energy functional

$$E(u) = \frac{1}{2} \int_G |u'|^2 \, dx - \frac{1}{p} \int_G |u|^p \, dx$$

subject to the mass constraint

$$\int_G |u|^2 \, dx = \mu \quad (\mu > 0, \text{ prescribed}).$$

A minimizer of $E$ among functions fulfilling the mass constraint is called ground state.
Given a metric graph $\mathcal{G}$, consider the NLS energy functional

$$E(u) = \frac{1}{2} \int_{\mathcal{G}} |u'|^2 \, dx - \frac{1}{p} \int_{\mathcal{G}} |u|^p \, dx$$

subject to the mass constraint

$$\int_{\mathcal{G}} |u|^2 \, dx = \mu \quad (\mu > 0, \text{prescribed}).$$

A minimizer of $E$ among functions fulfilling the mass constraint is called ground state.

**Problem.** Are ground states at fixed mass unique?
Physical motivations

Context: propagation of signals along branched structures.

- Metric graphs provide one-dimensional approximations for constrained dynamics in which transverse dimensions are negligible compared to longitudinal ones.
Physical motivations

Context: propagation of signals along branched structures.

- Metric graphs provide one-dimensional approximations for constrained dynamics in which transverse dimensions are negligible compared to longitudinal ones.

- Bose–Einstein condensates
- Spectrum of valence electrons in organic molecules
- Nanotechnologies (circuits of quantum wires)
- Spectra of electromagnetic waves in thin dielectrics
- Thin acoustic waveguides
- And more...
A metric graph $\mathcal{G}$ is a connected network made up of intervals (bounded or unbounded), joined together at their endpoints, according to a given topology.
A metric graph $G$ is a connected network made up of intervals (bounded or unbounded), joined together at their endpoints, according to a given topology.

Any bounded edge $e$ is identified with an interval $[0, \ell_e]$, any unbounded one with a half–line $[0, +\infty)$. 
A metric graph $G$ is a connected network made up of intervals (bounded or unbounded), joined together at their endpoints, according to a given topology.

- Any bounded edge $e$ is identified with an interval $[0, \ell_e]$, any unbounded one with a half-line $[0, +\infty)$.
- With the shortest-path distance, we obtain a metric space $G$. 
A metric graph $G$ is a connected network made up of intervals (bounded or unbounded), joined together at their endpoints, according to a given topology.

- Any bounded edge $e$ is identified with an interval $[0, \ell_e]$, any unbounded one with a half–line $[0, +\infty)$.
- With the shortest-path distance, we obtain a metric space $G$.
- The spaces $L^p(G)$ are defined in the usual way, with Lebesgue measure on every edge.
The Sobolev space $H^1(G)$ is defined as follows

\[ u \in H^1(G) \iff \begin{cases} u \in H^1(e) & \text{for every edge } e \text{ of } G \\ u : G \to \mathbb{R} & \text{is continuous on } G \end{cases} \]
The Sobolev space $H^1(G)$ is defined as follows

\[ u \in H^1(G) \iff \begin{cases} u \in H^1(e) & \text{for every edge } e \text{ of } G \\ u : G \to \mathbb{R} & \text{is continuous on } G \end{cases} \]

Here is what a typical $H^1(G)$ function looks like:
Existence/non–existence of **ground states** has been widely investigated on various graphs.

**Compact graphs**

**Graphs with half–lines**

**Periodic graphs**

**Infinite trees**
What about uniqueness?

Let \( u \) be a ground state at mass \( \mu \). Then there exists \( \lambda \in \mathbb{R} \) so that \( u \) solves on \( G \) the stationary NLS equation

\[
  u'' + |u|^{p-2}u = \lambda u. 
\]  

(1)
What about uniqueness?

Let $u$ be a ground state at mass $\mu$. Then there exists $\lambda \in \mathbb{R}$ so that $u$ solves on $\mathcal{G}$ the stationary NLS equation

$$u'' + |u|^{p-2}u = \lambda u. \quad (1)$$

**Note.** In principle, $\lambda = \lambda(u)$ depends on $u$. 
Let $u$ be a ground state at mass $\mu$. Then there exists $\lambda \in \mathbb{R}$ so that $u$ solves on $G$ the stationary NLS equation

$$u'' + |u|^{p-2} u = \lambda u.$$  \hfill (1)

**Note.** In principle, $\lambda = \lambda(u)$ depends on $u$.

It is a two-level problem:

1. do ground states at the same mass $\mu$ share the same $\lambda$?
2. given $\lambda$, is there a unique solution to the stationary NLS (1)?
Let $u$ be a ground state at mass $\mu$. Then there exists $\lambda \in \mathbb{R}$ so that $u$ solves on $G$ the stationary NLS equation

$$u'' + |u|^{p-2}u = \lambda u. \quad (1)$$

**Note.** In principle, $\lambda = \lambda(u)$ depends on $u$.

It is a two–level problem:

1. do ground states at the same mass $\mu$ share the same $\lambda$?
2. given $\lambda$, is there a unique solution to the stationary NLS (1)?

Main difficulties: nonlinearity, mass constraint, general domains
Uniqueness is known to be challenging also on domains $\Omega$ in $\mathbb{R}^n$.\[\Delta u + |u|^{p-2}u = \lambda u\]
Uniqueness is known to be challenging also on domains \( \Omega \) in \( \mathbb{R}^n \).

Given \( \lambda > 0 \), uniqueness of the positive solution to

\[
\Delta u + |u|^{p-2}u = \lambda u
\]

has been proved on the ball and in few other cases.
Uniqueness is known to be challenging also on domains $\Omega$ in $\mathbb{R}^n$.

Given $\lambda > 0$, uniqueness of the positive solution to

$$
\Delta u + |u|^{p-2}u = \lambda u
$$

has been proved on the ball and in few other cases.

Conversely, non–uniqueness is known to hold on dumbbell domains...
Uniqueness is known to be challenging also on domains $\Omega$ in $\mathbb{R}^n$.

Given $\lambda > 0$, uniqueness of the positive solution to

$$\Delta u + |u|^{p-2}u = \lambda u$$

has been proved on the ball and in few other cases.

Conversely, non–uniqueness is known to hold on dumbbell domains

Open conjecture. If $\Omega$ is bounded and convex and $2 < p \leq 2^*$, then whenever a positive solution exists, it is unique.
Let $G$ be a given metric graph, $p \in (2, 6]$ and $J \subset \mathbb{R}^+$ be an interval so that ground states at mass $\mu$ on $G$ exist for every $\mu \in J$. 

Theorem (D., Serra, Tilli, Adv. Math. '20)

For all but at most countably many $\mu \in J$:  

$\lambda = \lambda(\mu)$; 

$\lambda(\mu)$ is a strictly increasing function of $\mu$; 

$E(\mu) := \inf_{u \in H^1_\mu(G)} E(u)$ is differentiable at $\mu$ and $E'(\mu) = -\frac{1}{2} \lambda \mu$. 

Note. The proof relies on the minimality of ground states only. 

Wide range of applications: subcritical/critical powers, general metric graphs, general domains in $\mathbb{R}^n$. 
Main results I: uniqueness of $\lambda$

Let $G$ be a given metric graph, $p \in (2, 6]$ and $J \subset \mathbb{R}^+$ be an interval so that ground states at mass $\mu$ on $G$ exist for every $\mu \in J$.

Theorem (D., Serra, Tilli, Adv. Math. '20)

For all but at most countably many $\mu \in J$:

- ground states at mass $\mu$ share the same $\lambda = \lambda(\mu)$;
- $\lambda(\mu)$ is a strictly increasing function of $\mu$;
- $E(\mu) := \inf_{u \in H^1_{\mu}(G)} E(u)$ is differentiable at $\mu$ and

$$E'(\mu) = -\frac{1}{2} \lambda \mu.$$
Let $\mathcal{G}$ be a given metric graph, $p \in (2, 6]$ and $J \subset \mathbb{R}^+$ be an interval so that ground states at mass $\mu$ on $\mathcal{G}$ exist for every $\mu \in J$.

**Theorem (D., Serra, Tilli, Adv. Math. '20)**

For all but at most countably many $\mu \in J$:

- ground states at mass $\mu$ share the same $\lambda = \lambda(\mu)$;
- $\lambda(\mu)$ is a strictly increasing function of $\mu$;
- $\mathcal{E}(\mu) := \inf_{u \in H^1_{\mu}(\mathcal{G})} E(u)$ is differentiable at $\mu$ and

$$\mathcal{E}'(\mu) = -\frac{1}{2} \lambda \mu.$$

**Note.** The proof relies on the minimality of ground states only.
Let $\mathcal{G}$ be a given metric graph, $p \in (2, 6]$ and $J \subset \mathbb{R}^+$ be an interval so that ground states at mass $\mu$ on $\mathcal{G}$ exist for every $\mu \in J$.

**Theorem (D., Serra, Tilli, Adv. Math. '20)**

For all but at most countably many $\mu \in J$:

- ground states at mass $\mu$ share the same $\lambda = \lambda(\mu)$;
- $\lambda(\mu)$ is a strictly increasing function of $\mu$;
- $\mathcal{E}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} E(u)$ is differentiable at $\mu$ and

$$\mathcal{E}'(\mu) = -\frac{1}{2} \lambda \mu.$$

**Note.** The proof relies on the minimality of ground states only.

Wide range of applications: subcritical/critical powers, general metric graphs, general domains in $\mathbb{R}^n$. 
The dependence of $\lambda$ on $\mu$ may be either $0$ or $\lambda = \mu$.

Two natural questions:
- Can we prove uniqueness of ground states at masses where $\lambda$ is unique?
- Can we get rid in general of the possible at most countable set of masses where $\lambda$ may not be unique?
The dependence of $\lambda$ on $\mu$ may be either $0 < \lambda \mu$ or $0 \lambda \mu$. Two natural questions:
can we prove uniqueness of ground states at masses where $\lambda$ is unique?
can we get rid in general of the possible at most countable set of masses where $\lambda$ may not be unique?
The dependence of $\lambda$ on $\mu$ may be either

\begin{align*}
&\lambda \to 0, \\
&\lambda \to 0
\end{align*}

or

Two natural questions:

- can we prove uniqueness of ground states at masses where $\lambda$ is unique?
- can we get rid in general of the possible at most countable set of masses where $\lambda$ may not be unique?
Main results II: uniqueness of ground states

Theorem (D., Serra, Tilli, Adv. Math. ’20)

Let $p \in (2, 6]$ and $G$ be either a graph with a pendant and $N$ half–lines or a tadpole.
Theorem (D., Serra, Tilli, Adv. Math. ’20)

Let $p \in (2, 6]$ and $G$ be either a graph with a pendant and $N$ half–lines or a tadpole. Then, for all but at most countably many masses, the ground state at fixed mass is unique.
Main results II: uniqueness of ground states

Theorem (D., Serra, Tilli, Adv. Math. ’20)

Let $p \in (2, 6]$ and $G$ be either a graph with a pendant and $N$ half-lines or a tadpole. Then, for all but at most countably many masses, the ground state at fixed mass is unique.

Idea of the proof:
- same $\lambda$ for all but at most countably many $\mu$;
Main results II: uniqueness of ground states

**Theorem (D., Serra, Tilli, Adv. Math. ’20)**

Let \( p \in (2, 6] \) and \( G \) be either a graph with a pendant and \( N \) half–lines or a tadpole. Then, for all but at most countably many masses, the ground state at fixed mass is unique.

**Idea of the proof:**
- same \( \lambda \) for all but at most countably many \( \mu \);
- ODE methods to prove uniqueness of positive solutions to
  \[
  u'' + |u|^{p-2}u = \lambda u.
  \]
Main results III: non–uniqueness

Theorem (D., Serra, Tilli, Adv. Math. ’20)

Let $p \in (2, 6)$. For every $\mu > 0$ there exist a graph $G$ that admits two ground states at mass $\mu$ with different $\lambda$. 
Main results III: non-uniqueness

Theorem (D., Serra, Tilli, Adv. Math. ’20)

Let $p \in (2, 6)$. For every $\mu > 0$ there exist a graph $\mathcal{G}$ that admits two ground states at mass $\mu$ with different $\lambda$.

Our theorem on the uniqueness of $\lambda$ is sharp: a priori, the at most countable set of masses where it may fail cannot be removed.
Theorem (D., Serra, Tilli, Adv. Math. '20)

Let \( p \in (2, 6) \). For every \( \mu > 0 \) there exist a graph \( \mathcal{G} \) that admits two ground states at mass \( \mu \) with different \( \lambda \).

Our theorem on the uniqueness of \( \lambda \) is sharp: a priori, the at most countable set of masses where it may fail cannot be removed.

Idea of the proof: given \( p \) and \( \mu \), we calibrate the graph...
For every $r$ large enough, we construct two critical points of $E$: 
For every $r$ large enough, we construct two critical points of $E$: one concentrates on the bounded edge
For every $r$ large enough, we construct two critical points of $E$: one concentrates on the bounded edge

the other concentrates on the self–loop
There exist: large enough sufficiently close to a given value so that both $u_1$ and $u_2$ are ground states at mass $\mu$.

$E(u_1) = E(\mu) = E(u_2)$

but $\lambda(u_1) \neq \lambda(u_2)$. 
There exist:

- \( s \) large enough
- \( t \) sufficiently close to a given value \( \bar{t} \)
There exist:

- s large enough
- t sufficiently close to a given value \( \bar{t} \)

so that both \( u_1 \) and \( u_2 \) are ground states at mass \( \mu \)

\[ E(u_1) = \mathcal{E}(\mu) = E(u_2) \]
There exist:

- $s$ large enough
- $t$ sufficiently close to a given value $\bar{t}$

so that both $u_1$ and $u_2$ are ground states at mass $\mu$

$$E(u_1) = \mathcal{E}(\mu) = E(u_2)$$

but

$$\lambda(u_1) \neq \lambda(u_2).$$
Thank you!