Integrability of three dimensional models: cubic equations

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We extend basic properties of two dimensional integrable models within the Algebraic Bethe Ansatz approach to 2+1 dimensions and formulate the sufficient conditions for the commutativity of transfer matrices of different spectral parameters, in analogy with Yang-Baxter or tetrahedron equations. The basic ingredient of our models is the R-matrix, which describes the scattering of a pair of particles over another pair of particles, the quark-anti-quark (meson) scattering on another quark-anti-quark state. We show that the Kitaev model belongs to this class of models and its R-matrix fulfills well-defined equations for integrability.

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The importance of 2D integrable models in modern physics is hard to overestimate. Being initially an attractive tool in mathematical physics they became an important technique in low dimensional condensed matter physics, capable to reveal non-perturbative aspects in many body systems with great potential of applications. The basic constituent of 2D integrable systems is the commutativity of the evolution operators, the transfer matrices of the models of different spectral parameters. This property is equivalent to the existence of as many integrals of motion as number of degrees of freedom of the model. It appears, that commutativity of transfer matrices can be ensured by the Yang-Baxter (YB) equations for the R-matrix and the integrability of the model is associated with the existence of the solution of YB-equations.

Since the 80s of last century there was a natural desire to extend the idea of integrability to three dimensions, which resulted in a formulation of the so-called tetrahedron equation by Zamolodchikov. The tetrahedron equations (ZTE) were studied and several solutions have been found until now. However, earlier solutions either contained negative Boltzmann weights or were slight deformations of models describing free particles. Only in a recent work non-negative solutions of ZTE were obtained in a vertex formulation, and these matrices can be served as Boltzmann weights for a 3D solvable model with infinite number of discrete spins attached to the edges of the cubic lattice. In this sense it is remarkable to note that among the general solutions obtained in this paper it is also possible to detect R-matrices with real and non-negative entries which can be considered as Boltzmann weights in the context of the 3D statistical solvable models with 1/2-spins attached to the vertexes of 3D cubic lattice.

Although initially the tetrahedron equations were formulated for the scattering matrix S of three infinitely long straight strings in a context of 3D integrability they can also be regarded as weight functions for statistical models. In a Bethe Ansatz formulation of 3D models their 2D transfer matrices of the quantum states on a plane can be constructed via three particle R-matrix, which, as an operator, acts on a tensorial cube of linear space $V$, i.e. $R: V \otimes V \otimes V \to V \otimes V \otimes V$. 

Another approach to 3D integrability based on Frenkel-Moore simplex equations also uses three-state R-matrices. They are higher dimensional extension of quantum Yang-Baxter equations without spectral parameters. However these equations are less examined.

Motivated by the desire to extend the integrability conditions in 3D to other formulations we consider a new kind of equations with the R-matrices acting on a quartic tensorial power of linear spaces $V$.

\[
\hat{R}_{1234} : V_1 \otimes V_2 \otimes V_3 \otimes V_4 \to V_1 \otimes V_2 \otimes V_3 \otimes V_4, \quad (0.1)
\]

which can be represented graphically as in Fig.1a. The R-matrix can be represented also in the form displayed in Fig.1b, where the final spaces are permuted $(V_1$ and $V_2$ with $V_3$ and $V_4$, respectively): $R_{1234} = \hat{R}_{1234} \hat{P}_{13} \hat{P}_{24}$. Explicitly it can be written as follows

\[
R^{\alpha_1 \beta_2 \gamma_3 \delta_4}_{\alpha_2 \alpha_3 \alpha_4} = \hat{R}^{\beta_1 \gamma_2 \delta_3 \alpha_4}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}. \quad (0.2)
\]

Identifying the space $V_1 \otimes V_2$ and $V_3 \otimes V_4$ with the quantum spaces of quark-anti-quark pairs connected with a string one can regard this R-matrix as a transfer matrix for a pair of scattering mesons. Within a terminology used in the algebraic Bethe Ansatz for 1+1 integrable models this R-matrix can be viewed also as a matrix, which has two quantum states and two auxiliary states.

The space of quantum states $\Phi_{t} = \otimes_{(n,m) \in \mathbb{L}} V_{n,m}$ of the system on a plane is defined by a direct product of linear spaces $V_{n,m}$ of quantum states on each site $(n, m)$ of the lattice $\mathbb{L}$ (see Fig.2a). We fix periodic boundary conditions on both directions: $V_{n,m+L} = V_{n,m}$ and $V_{n,L,m} = V_{n,m}$. The time evolution of this state is determined by the action of the operator/transfer matrix $T$: $\Phi_{t+1} = \Phi_{t} T$, which is a product of local evolution opera-
tors, R-matrices as follows. First we fix a chess like structure of squares on a lattice $L$ and associate to each of the black squares a R-matrix $R_{(n+1,m)/(n+1,m+1)}$, which acts on a product of four spaces at the sites. In this way the whole transfer matrix becomes

$$T = Tr\Pi^{L L}_{m=1} R_{(2n+1,2m)}(2n+2,m+1)(2n+1,2m+1)$$

where the Trace is taken over states on boundaries. The indices of the R-matrices in the first and second lines of this product just ensure chess like ordering of their action. In Fig. 2b we present this product graphically. First we identify the second pair of states $(2n+1,2m+1)$ (in first row) and $(2n+2,m+1)$ (in second row) of R-matrices with the corresponding links on the lattice. Then we rotate the box of the R-matrix by $\pi/4$ in order to ensure the correct order for their action in a product. In the same way we define the second list of the transfer matrix, which will act in the order $T_B T_A$. Fig. 2c presents a vertical 2D cut of two lists of the product $T_B T_A$ drawn from the side. The $\pi/4$ rotated lines mark the spaces $V_{n,m}$ attached to sites $(n,m)$ of the lattice. Though transfer matrix (0.3) is written in $\bar{R}$ formalism, it can easily be converted to the product of R-matrices.

The arrangement of R-matrices in the first row (first plane of the transfer matrix $T_B$) acts on the sites of dark squares of the lattice while R-matrices in the second row (second plane of the transfer matrix $T_A$) act on the sites of the white squares.

Being an evolution operator the transfer matrix should be linked to time. According to the general prescription the transfer matrix $T(u)$ is a function of the so-called spectral parameter $u$ and the linear term $H_1$ in its expansion $T(u) = \sum_u u^k H_k$ defines the Hamiltonian of the model, while the partition function is $Z = Tr T^N$. Integrable models should have as many integrals of motion, as degrees of freedom. This property may be reached by considering two planes of transfer matrices with different spectral parameters, $T(u)$ and $T(v)$ and demanding their commutativity $[T(u), T(v)] = 0$, or equivalently demanding the commutativity of the coefficients $[H_r, H_s] = 0$ of the expansion. This means, that all $H_r, r > 1$ are integrals of motion. In 2D integrable models the sufficient conditions for commutativity of transfer matrices are determined by the corresponding YB-equations.

In order to obtain the analog of the YB equations, which will ensure the commutativity of transfer matrices (0.3) we use the so-called railway construction. Let us cut horizontally two planes of the R-matrix product of two transfer matrices (on Fig. 2b) we present a product of R-matrices for one transfer matrix plane) into two parts and substitute in between the identity

$$\Pi^{L}_{m=1} id_{(2n+1,m)} id_{(2n,m)}$$

which maps two chains of sites, $(2n,m), m = 1 \cdots L$ and $(2n+1,m), m = 1 \cdots L + 1$, into itself. The Trace have to be taken by identifying spaces $1$ and $L+1$. In this expression we have introduced another set of R-matrices, called intertwiners, which will be specified below. For further convenience we distinguish $\bar{R}_{(2n+1,m)/(2n+1,m+1)}(2n,m)(2n+1,m+1)$ matrices for even and odd values of $m$ marking them as $\bar{R}_3$ and $\bar{R}_4$ respectively. In the left side of Fig. 3 we present one half of the plane of R-matrices together with an inserted chain of $\bar{R}_3 \bar{R}_4$ as intertwiners. The chain of intertwiners can also be written by $\bar{R}$-matrices.

Now let us suggest, that the product of these intertwiners with the first double chain of $\bar{R}$-matrices from the product of two planes of transfer matrices is equal to the product of the same operators written in opposite
The commutativity of the Kitaev model is trivially clear from the very beginning since all terms in the Hamiltonian defined on white and dark plaquettes commute with each other. The latter indicates, that the number of integrals of motion of the model coincides with its degrees of freedom. However, in this paper we are aiming to show, that one can develop 3D Algebraic Bethe Ansatz approach in such a way, that the Kitaev model will automatically be integrable. Namely, we will show now, that $R_A$ and $R_B$-matrices of the Kitaev's model fulfill Eq. 0.6. The explicit form of Eq. 0.6 by use of indices according to the definition in Fig. 1a reads

$$
R^4_{\alpha_5\alpha_2\alpha_3\beta_1\beta_2\beta_3\beta_4} \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6 R^3_{\alpha_4\alpha_1\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7\beta_8\beta_9} (v) R^2_{\alpha_3\alpha_2\beta_1\beta_2\beta_3} (v) R^1_{\alpha_3\alpha_2\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 \gamma_7\gamma_8\gamma_9} (u) \delta_{\alpha_7} =
R^4_{\alpha_5\alpha_2\alpha_3\beta_1\beta_2\beta_3\beta_4} (u) R^3_{\alpha_4\alpha_1\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6\beta_7\beta_8\beta_9} (v) R^2_{\alpha_3\alpha_2\beta_1\beta_2\beta_3} (v) R^1_{\alpha_3\alpha_2\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 \gamma_7\gamma_8\gamma_9} (u) \delta_{\alpha_7}
$$

(0.8)

where $R^1 (u) = R^A (u)$ and $R^2 (v) = R^B (v)$. It appears, that the intertwiners

$$
R^4 = R^A^{-1} (u), \quad R^3 = R^B (v)
$$

(0.9)

where $R^A^{-1} (u) = 1 \otimes 1 \otimes 1 \otimes 1 - u \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x$.
fulfill the cubic equations (0.8) for any parameters $u$ and $v$. This can be directly checked both, by a computer algebra program and analytically. The commutativity of transfer matrices $T_A(u)$ with $T_A(v)$ and $T_B(u)$ with $T_B(v)$ is trivial in the Kitaev model.  

**Summary.** We have formulated a class of three dimensional models defined by the $R$-matrix of the scattering of a two particle state on another two particle state, i.e. a meson-meson type scattering. We derived a set of equations for these $R$-matrices, which are a sufficient conditions for the commutativity of the transfer matrices with different spectral parameters. These equations differ from the tetrahedron equations, which also ensure the integrability of 3D models, but are based on the $R$-matrix of 3 particle scatterings. Our set of equations will be reduced to tetrahedron type of equations by considering the two auxiliary spaces in the $R$-matrix as one (fusion) and replacing it by one thick line. We showed that the Kitaev model\textsuperscript{12} belongs to this class of integrable models.

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