ON THE STRONG-TO-STRONG INTERACTION CASE FOR DOUBLY NONLOCAL CAHN-HILLIARD EQUATIONS

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ABSTRACT. We consider a doubly nonlocal Cahn-Hilliard equation for the non-local phase-separation of a two-component material in a bounded domain in the case when mass transport exhibits non-Fickian behavior. Such equations are important for phase-segregation phenomena that exhibit non-standard (anomalous) behaviors. Recently, four different cases were proposed to handle this important equation and the two levels of nonlocality and interaction that are present in the equation. The so-called strong-to-weak interaction case (when one kernel is integrable in some sense while the other is not) was investigated recently for the doubly nonlocal parabolic equation with a regular polynomial potential. In this contribution, we address the so-called strong-to-strong interaction case when both kernels are strongly singular and non-integrable in a suitable sense. We establish well-posedness results along with some regularity and long-time results in terms of finite dimensional global attractors.

1. Introduction. Phase-separation in binary materials is considered a central problem in materials science. In this, the Cahn-Hilliard equation (CHE) finds itself center-stage in a three-act “play”. The first act starts in the late 1950’s with an elegant phenomenological derivation of the equation by Cahn and Hilliard [3]. Since then it has found applications in other important physical phenomena, in image processing, fluid dynamics, population dynamics, and the list goes on. The basic form of CHE is very well-known to the scientific community; it reads as a parabolic equation that is fourth-order in space and the corresponding boundary value problem in a bounded domain requires an addition of two (usually no-flux) boundary conditions. Rather than giving a full account of the enormous scientific literature, we refer the reader to [18] where a complete description of the most up-to-date analytical and numerical studies has been undertaken in detail for the standard Cahn-Hilliard equation (CHE). The second act is the most important and brings us closer to the mid 1990’s when Giacomin and Lebowitz [11] gave a rigorous physical derivation of the Cahn-Hilliard equation, by starting from a microscopic model for a lattice gas with long-range Kac smooth potentials \( J \in C^2(\mathbb{R}^n), J(x) = J(-x) \). This is the nonlocal Cahn-Hilliard equation; it is more general than the classical CHE proposed earlier by Cahn and Hilliard [3], in particular, since at least formally the CHE is a “local first-order” approximation of the nonlocal CHE in a suitable...
sense [11](cf. also [1, 2, 9, 10]). In the case of constant mobility, the nonlocal CHE reads as the parabolic equation:
\[ \partial_t \varphi = \Delta X \mu, \mu = B(\varphi) + F'(\varphi), \text{ in } X \times (0, \infty), \] (1.1)

where
\[ B(\varphi) \overset{\text{def}}{=} \text{P.V.} \int_X J(x-y)(\varphi(x)-\varphi(y))\,m(dy). \]

Generally, \( X \) may be a locally compact metric space and \( m \) a Radon measure on \( X \) such that \( \text{supp}(m) = X \). Moreover, P.V. stands for principal value such that \( B(\varphi) \) is understood as
\[ \lim_{\varepsilon \to 0^+} \int_{X \setminus B_\varepsilon(x)} J(x-y)(\varphi(x)-\varphi(y))\,m(dy) \]
whenever the limit exists. Here \( \varphi \in [-1, 1] \) is the relative difference of the two phases (or the concentration of one phase if \( \varphi \in [0, 1] \)), \( \mu \) is called the chemical potential and is determined as the Fréchet derivative of the free (nonlocal) energy
\[ E_{\text{nonloc}}(\varphi) = \int_X \int_X J(x-y)|\varphi(x)-\varphi(y)|^2m(dx)\,m(dy) + \int_X F(\varphi)m(dx). \] (1.2)

Notice that in the theory of Giacomin and Lebowitz [11], it is assumed that, at least minimally \( J \in L^1(X) \), and so
\[ B(\varphi) = a(x)\varphi - J \ast \varphi, \]
if we set
\[ (J \ast \varphi)(x) := \int_X J(x-y)\varphi(y)m(dy), \quad a(x) := \int_X J(x-y)m(dy). \]

In the framework of phase transition, two important classes of potentials \( F \) have been widely used so far: (i) \( F \) is a singular logarithmic potential defined over \([-1, 1]\), which penalizes any values outside the interval \([-1, 1]\), and (ii) \( F \) is a regular polynomial potential defined all over \( \mathbb{R} \), typically, the double-well function \( F(s) = \theta s^4 - \theta_s s^2, \quad 0 < \theta < \theta_s \). An essential feature of (1.1) is that each material component of the two-phase mixture is actually preserved over time. More precisely,
\[ \int_X \varphi(t,x)\,m(dx) = \int_X \varphi(0,x)\,m(dx), \] (1.3)
provided, say, that \( X \) is a bounded Lipschitz domain in \( \mathbb{R}^n \) with non-empty boundary \( \partial X \), such that the chemical potential \( \mu \) obeys the equation
\[ \partial_t \mu = 0 \text{ on } \partial X \times (0, \infty). \] (1.4)

In the case \( X \) is a compact manifold without boundary \( (\partial X = \emptyset) \), (1.4) is no longer required for (1.3) to hold.

Recently in its third act, the scientific community had began to develop, as complete as possible, analytical studies concerning (1.1)-(1.4). For well-posedness we may quote [1, 2, 9, 11], and [10, 15, 16] for the long-term behavior of solutions in terms of finite-dimensional attractors and convergence to steady states as time goes to infinity. We also refer the reader to [8] for an extensive account of the pertinent scientific literature dealing with the nonlocal Cahn-Hilliard equation (1.1)-(1.4).

We observe that the nonlocal CHE is deterministic in nature since mass transport obeys Fick’s law of diffusion (cf. the first equation of (1.1)). However, it was recently observed in Gal [8] that when the physical phenomena takes place in a heterogeneous environment \( X \), we may need to deal with anomalous mass transport or diffusion.
In this case the non-Fickian behavior of the chemical potential may manifest itself through a nonlocal formulation such as,
\[ \partial_t \varphi = -A(\mu), \quad \mu = B(\varphi) + F'(\varphi), \quad \text{in } X \times (0, \infty). \] (1.5)
Here, the nonlocal operator \( A \) is defined as
\[ A(\mu) = \text{P.V.} \int_X K(x-y)(\mu(y) - \mu(x)) \, m(dy), \] (1.6)
which needs to be understood in a principal value sense if \( K(x) = K(-x) \) and \( K \) is non-smooth, say if \( K \notin L^1(X) \). In Gal [8], (1.5) is dubbed as the doubly nonlocal Cahn-Hilliard equation; it was also observed that (1.5) occurs as a generalization of the nonlocal CHE (1.1). A further classification of the physical relevant cases with respect to the heterogeneous pair \((K,J)\) for (1.5) was also given in [8]. In the case when \( X \subset \mathbb{R}^n \) is a bounded domain with non-empty Lipschitz continuous boundary \( \partial X \), we recall them as follows:

1. The strong-to-weak interaction case: \( K \notin L^1_{\text{loc}}(\mathbb{R}^n) \) and \( J \in L^1_{\text{loc}}(\mathbb{R}^n) \);
2. The strong-to-strong interaction case: both \( K \notin L^1_{\text{loc}}(\mathbb{R}^n) \);
3. The weak-to-strong interaction case: \( K \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( J \notin L^1_{\text{loc}}(\mathbb{R}^n) \);
4. The weak-to-weak interaction case: both \( K,J \in L^1_{\text{loc}}(\mathbb{R}^n) \).

Furthermore, the strong-to-weak interaction case (1) for (1.5) was settled completely in [8] when \( X \subset \mathbb{R}^n \) is a bounded domain with Lipschitz continuous (non-empty) boundary \( \partial X \) and both \( K, J \) are symmetric. In particular, we gave a unified analysis to establish sharp results in terms of existence, regularity and stability (with respect to the initial data) of properly-defined solutions. There we have also discussed and derived sufficient conditions for problem (1.5) to possess finite dimensional global and exponential attractors, and for solutions to eventually convergence to single steady states as time goes to infinity. In this contribution, we aim to continue that investigation by addressing the next difficult case (2). To treat this case rigorously and to develop a complete theory for well-posedness and long-term behavior of solutions we will require new methods to solve it. In fact, in our attempt to solve case (2) we have stumbled upon further generalizations of (1.5) that include some cases that were not covered before in the scientific literature. This will become evident to the reader in the subsequent sections (see also Tables 1, 2). Our main goal is then to develop well-posedness and long-time dynamical results for the equation (1.5) associated with a general class of self-adjoint (nonnegative) operators \( A, B \), and then subsequently recover results for the case (2) of interest as a particular case. In doing so, we shall assume that \((X,m)\) is also a \( \sigma \)-finite measure space. Furthermore, using specific, easy to check, assumptions about the pair \((X,m)\), and the local and/or nonlocal operators \( A, B \), we will be then introduced to specific models, for which our results are sharp (cf. Subsection 2.2). We emphasize that our assumptions on \( A, B \) and/or \((X,m)\) we employ are of general character, and as a result do not require a specific form as suggested by the examples in Tables 1, 2 (cf. Section 3); this abstraction allows to recover the doubly nonlocal Cahn-Hilliard equation (1.5), the classical Cahn-Hilliard equation and the nonlocal Cahn-Hilliard equation (1.1) among all of the existing phase-transition models, as well as to represent a much larger family of nonlocal models, of the form (1.5), that have not been explicitly studied anywhere in detail. For clarity, we list in Tables 1, 2 the corresponding operators and the special cases covered by our theory.
examples of operators $A, B$ are provided in Section 2 and include also local and nonlocal operators defined over compact manifolds $X$ without boundary. We first denote by $-\Delta_{X,N}$ the self-adjoint nonnegative operator that is the realization of the Neumann Laplacian $-\Delta_X$ with the no-flux boundary condition (1.4), cf. Subsection 2.2. To briefly define the so-called regional fractional Laplacian operator, let us fix \( s \in (1/2, 1) \), and set
\[
\mathcal{L}(X) := \{ u : X \to \mathbb{R} \text{ measurable, } \int_X \frac{|u(x)|}{(1 + |x|)^{n+2s}} m(dx) < \infty \}.
\]
For $u \in \mathcal{L}(X)$, we define the regional fractional Laplacian $(-\Delta)^s_X$ by the formula
\[
(-\Delta)^s_X u(x) = \text{P.V.} C_{n,s} \int_X \frac{(u(x) - u(y))}{|x-y|^{n+2s}} m(dy), \quad x \in X,
\] with a given normalized constant $C_{n,s}$ (see Subsection 2.2). Associated with formula (1.7) and a smooth function $u \in C^1(X)$, one may further define the so-called fractional normal derivative $N^2_{-2s} u$ (cf. [14]). It turns out that a Green type formula involving the fractional normal derivative holds for the regional fractional Laplace operator $(-\Delta)^s_X$ [14], see Subsection 2.2. This can be used to define a self-adjoint nonnegative operator $(-\Delta)^s_{X,N}$, that is the realization of $(-\Delta)^s_X$ with the fractional Neumann-type boundary condition $N^2_{-2s} u = 0$ on $\partial X$ (see [22]; cf. also [12]). To the best of our knowledge, the second model in Table 2 was also investigated in [1] in the case when $F$ belongs to the class (i) defined earlier. Nevertheless it is worthwhile to mention that we generalize even results for the classical Cahn-Hilliard equation (see Table 1) which generally assumed a scenario in which $X$ is quite smooth. This is no longer required in our framework, and besides, it also allows to treat important cases of phase segregation phenomena beyond the ones usually found in the scientific literature. In particular, all of our results on well-posedness, regularity and long-time behavior in terms of finite dimensional global attractors apply to these models in Table 1 and Table 2 respectively. Otherwise, our assumptions on the operators $A, B$ (according to Section 2.2) allow us to deduce comparable results for other (“local-nonlocal” or “doubly nonlocal”) phase segregation models that have never been considered before.

Table 1. $X \subset \mathbb{R}^n$ is a bounded domain with Lipschitz continuous boundary $\partial X$. The general model covered is the mass-conserved one given by (1.5) with the following choices of operators $A, B$. A physically relevant choice that satisfies our assumptions is the double-well potential $F(s) = \theta s^4 - \theta_c s^2$, $0 < \theta < \theta_c$.

| Model | Classical CHE | Doubly nonlocal CHE, case (2) |
|-------|---------------|---------------------------------|
| $A$   | $-\Delta_{X,N}$ | $(-\Delta)^{s_1}_{X,N}$, $s_1 \in (1/2, 1)$ |
| $B$   | $-\Delta_{X,N}$ | $(-\Delta)^{s_2}_{X,N}$, $s_2 \in (1/2, 1)$ |

Table 2. The information is the same as in Table 1.

| Model | CHE: anomalous transport | CHE: nonlocal strong energy |
|-------|--------------------------|-----------------------------|
| $A$   | $(-\Delta)^{s}_{X,N}$, $s \in (1/2, 1)$ | $-\Delta_{X,N}$ |
| $B$   | $-\Delta_{X,N}$ | $(-\Delta)^{s}_{X,N}$, $s \in (1/2, 1)$ |
The remainder of the paper is structured as follows. In Section 2, we define
the functional framework needed for our approach, explain its basic properties (see
Subsection 2.1) and provide many examples of nonlocal operators (see Subsection
2.2). Section 3 is dedicated to the regularity of weak solutions as well as the existence
of strong solutions (Subsection 3.1) while Subsection 3.2 is devoted to the long-term
behavior of weak solutions for (1.5) in terms of finite dimensional attractors. In the
final Subsection 3.3, we briefly address the problem in the case when $F$ is a singular
logarithmic potential.

2. **Functional framework.** Subsection 2.1 of this section contains a comprehen-
sive account of the theory of (symmetric) Dirichlet forms and then further develops
some general (useful) properties that will be needed in the subsequent sections. In
Subsection 2.2, actual examples will be presented to emphasize the importance of
this analytic theory in the general Cahn-Hilliard theory of nonlocal phase-transition.

2.1. **Dirichlet forms and Markovian semigroups.** We introduce the notion of
Dirichlet form on an $L^2$-type space (see [6, Chapter 1]). To this end, let $X$ be a
locally compact metric space and $m$ a Radon measure on $X$ such that $\text{supp}(m) = X$
Let $L^2(X, m)$ be the real Hilbert space with inner product $(\cdot, \cdot)$ and let $E_A$
with domain $D(E_A) =: V_A$ be a bilinear form on $L^2(X) = L^2(X, m)$.

**Definition 2.1.** The form $E_A$ is said to be a Dirichlet form if the following condi-
tions hold:

(a) $E_A : V_A \times V_A \to \mathbb{R}$, where the domain $D(E_A) = V_A$ of the form is a dense
linear subspace of $L^2(X)$.

(b) $E_A(u, v) = E_A(v, u)$, $E_A(u + v, w) = E_A(u, w) + E_A(v, w)$, $E_A(au, v) = aE_A$
$(u, v)$ and $E_A(u, u) \geq 0$, for all $u, v, w \in V_A$ and $a \in \mathbb{R}$.

(c) Let $\lambda > 0$ and define $E_{A, \lambda}(u, v) = E_A(u, v) + \lambda (u, v)$, for $u, v \in D(E_{A, \lambda}) = V_A$.
The form $E_A$ is closed, that is, if $u_n \in V_A$ with

$$E_{A, \lambda}(u_n - u_m, u_n - u_m) \to 0 \text{ as } n, m \to \infty,$$

then there exists $u \in V_A$ such that

$$E_{A, \lambda}(u_n - u, u_n - u) \to 0 \text{ as } n \to \infty.$$

(d) For each $\epsilon > 0$ there exists a function $\phi_\epsilon : \mathbb{R} \to \mathbb{R}$, such that $\phi_\epsilon \in C^\infty(\mathbb{R})$,
$\phi_\epsilon(t) = t$, for $t \in [0, 1]$, $-\epsilon \leq \phi_\epsilon(t) \leq 1 + \epsilon$, for all $t \in \mathbb{R}$, $0 \leq \phi_\epsilon(t) -
\phi_\epsilon(\tau) \leq t - \tau$, whenever $\tau < t$, such that $u \in V_A$ implies $\phi_\epsilon(u) \in V_A$ and
$E_A(\phi_\epsilon(u), \phi_\epsilon(u)) \leq E_A(u, u)$.

**Remark 2.2.** We make the following important remarks.

- Clearly, $D(E_A) = V_A$ is a real Hilbert space with inner product $E_{A, \lambda}(u, u)$
for each $\lambda > 0$. We recall that a form $E_A$ which satisfies (a)-(c) is closed and
symmetric. If $E_A$ also satisfies (d), then it is said to be a Markovian form.

- When $E_A$ is closed, (d) is equivalent to the following more simple condition:

(d') $u \in V_A$, $v$ is a normal contraction of $u \Rightarrow v \in V_A$: $E_A(v, v) \leq E_A(u, u)$. We
call $v \in L^2(X)$ a normal contraction of $u \in L^2(X)$ if some Borel version
of $v$ is a normal contraction of some Borel version of $u \in L^2(X)$, that is,
$|v(x)| \leq |u(x)|$, for all $x \in X$, and

$$|v(x) - v(y)| \leq |u(x) - u(y)|, \text{ for all } x, y \in X.$$
It is well-known that there is a one-to-one correspondence between the family
of closed symmetric forms $E_A$ on $L^2(X)$ and the family of non-negative (definite)
self-adjoint operators $A$ on $L^2(X)$ in the following sense:

$$
\begin{cases}
V_A = D\left( A^{1/2} \right), \ D(A) \subset D(E_A), \\ E_A(u,v) = (Au,v), \ u \in D(A), \ v \in V_A.
\end{cases}
$$

(2.1)

From now on, we shall refer to $(E_A, V_A)$ as a Dirichlet space whenever $E_A$ is a
Dirichlet form on $D(E_A) = V_A$ in the sense of Definition 2.1.

We introduce some further terminology. Let $Y, Z$ be two Banach spaces endowed
with norms $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. We denote by $Y \rightarrow Z$ if $Y \subseteq Z$
and there exists a constant $C > 0$ such that $\|u\|_Z \leq C \|u\|_Y$, for $u \in Y \subseteq Z$. In particular,
this means that the injection of $Y$ into $Z$ is continuous. In addition, if the injection
is also compact we shall denote it by $Y \hookrightarrow Z$. By the dual $Y^*$ of $Y$, we think of
$Y^*$ as the set of all (continuous) linear functionals on $Y$. When equipped with the
operator norm $\|\cdot\|_{Y^*}$, $Y^*$ is also a Banach space.

**Proposition 2.3.** Let $Y, Z$ be Banach spaces such that $Y \rightarrow Z$ and $Z \hookrightarrow
L^2(X, m)$. Then for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that,
for all $u \in Y \subseteq Z$,

$$
\|u\|^2_{L^2(X)} \leq \varepsilon \|u\|^2_Z + C(\varepsilon) \|u\|^2_Y.
$$

**Proof.** By assumption $Y \rightarrow Z \hookrightarrow L^2(X)$ and so by duality, $Z \hookrightarrow L^2(X) \rightarrow Z^* \rightarrow Y^*$. The statement is then a simple consequence of Ehrling’s lemma. \qed

At this point it is also useful to recall the following result (see [17]).

**Lemma 2.4.** Let $\Xi : [k_0, \infty) \rightarrow \mathbb{R}$ be a nonnegative, non-increasing function such
that there are positive constants $c, \nu > 0$ and $\delta > 1$ such that

$$
\Xi(j) \leq c(j-k)^{-\nu} \Xi(k)^{\delta}, \quad \forall \ j > k \geq k_0 \geq 0.
$$

Then $\Xi(k_0 + K) = 0$ with $K = c^{1/\nu} \Xi(k_0)^{(\delta - 1)/\nu} 2^{\delta - 1}.$

The following result is also basic but we choose to give a proof for the sake of
completeness. However, we note that the conclusion of this statement can be also
checked directly in specific cases.

**Proposition 2.5.** Let $E_A$ be a given Dirichlet (or Markovian) form in the sense of
Definition 2.1 such that $E_A(k, v) \geq 0$, for any $0 \leq v \in V_A$ and any constant
(function) $0 \leq k \in V_A$. Let now $u \in V_A$. Then $u_k := (|u| - k)^+ \text{sgn}(u) \in V_A$ and we have

$$
E_A(u_k, u_k) \leq E_A(u, u_k), \text{ for any } k \geq 0.
$$

(2.2)

**Proof.** To prove the first claim, i.e., $u_k \in V_A$ it suffices to show that both $|u|$, $u^+ = \max\{u, 0\} \in V_A$ for as long as $u \in V_A$. Indeed, since $u \in V_A$ and $|u|$ is a
normal contraction of $u$, it follows by Definition 2.1 property (d), that $|u| \in V_A$.
In this case, $|u|$ is also a normal contraction of $u^+$ and so by the same property,
$u^+ \in V_A$. We now need to show the second claim (2.2). Define two sets $A_k :=
\{x \in X : |u(x)| > k\}$ and $B_k := X \setminus A_k$ and observe that $u_k = (|u| - k) \text{sgn}(u)$
on $A_k$, while $u_k = 0$ on $B_k$, for each $k \geq 0$. We further split $A_k = A_k^1 \cup A_k^2$,\nwhere $A_k^1 := \{x \in A_k : u > 0\}$ and $A_k^2 := \{x \in A_k : u < 0\}$. We easily see that
$u_k = u - k > 0$ on $A_k^1$ and $u_k = u + k < 0$ on $A_k^2$. Next, $E_A(u - u_k, u_k) = 0$ on $B_k$,\n
while on $A_k^1$ we have $\mathcal{E}_A (u - u_k, u_k) = \mathcal{E}_A (k, u - k) \geq 0$ owing to $k \geq 0$, $u - k > 0$. On $A_k^2$ we have

$$\mathcal{E}_A (u - u_k, u_k) = \mathcal{E}_A (-k, u + k) = \mathcal{E}_A (k, -(u + k)) \geq 0$$

since $k \geq 0$ and $u + k < 0$. The proof is finished.

Our next goal is to investigate a nonlinear elliptic problem associated with the Dirichlet form $\mathcal{E}_A$ as follows:

$$Au (x) + f (u (x)) = h (x), \quad x \in X,$$

(2.3)

where $h \in L^s (X, m)$ for some $s > 1$. Here, $f \in C^1 (\mathbb{R})$ is a nonlinear function which satisfies suitable assumptions (see below).

**Definition 2.6.** We say that $u$ is a bounded generalized solution of (2.3) if $u \in V_A \cap L^\infty (X)$ and

$$\mathcal{E}_A (u, v) + \int_X f (u (x)) v (x) \, m (dx) = \int_X h (x) v (x) \, m (dx),$$

for all $v \in V_A$.

**Theorem 2.7.** Let the assumptions of Proposition [2.5] be satisfied and suppose that the Dirichlet form $\mathcal{E}_A$ is also coercive in the following sense:

$$\mathcal{E}_A (u, u) \geq \beta_0 \|u\|_{V_A}^2 - \lambda \|u\|_{L^2 (X)}^2, \quad u \in V_A,$$

(2.4)

for some $\beta_0 > 0$, $\lambda \in [0, \alpha_0]$ with $\alpha_0 > 0$ as in (2.5). Assume also the following conditions:

- $V_A \to L^{2q_A} (X, m)$, for some $q_A > 1$ and $V_A \subseteq L^2 (X)$.
- There exist $\alpha_0 > 0$, $\alpha_1, \alpha_2 \geq 0$ such that, for all $t \in \mathbb{R}$, $|t| \geq t_0$, for some $t_0 > 0$,

$$f (t) t \geq \alpha_0 t^2 - \alpha_1, \quad f' (t) \geq -\alpha_2.$$  

(2.5)

Then problem (2.3) has at least one bounded solution in the sense of Definition 2.6 provided that

$$h \in L^s (X, m) \quad \text{with} \quad s > \frac{q_A}{q_A - 1}.$$  

Moreover, the following estimate holds:

$$\|u\|_{L^\infty (X)} \leq C \left( 1 + \|h\|_{L^\infty (X)} \right),$$  

(2.6)

for some constant $C > 0$ independent of $u$ and $h$.

**Corollary 2.8.** Under the assumptions of Theorem 2.7, there exists a strong solution of (2.3) provided that $h \in L^s (X) \cap L^2 (X)$, i.e., $u \in D (A) \cap L^\infty (X)$ such that

$$\|Au\|_{L^2 (X)} \leq Q \left( 1 + \|h\|_{L^\infty (X) \cap L^2 (X)} \right),$$

for some function $Q > 0$ independent of $u$ and $h$.

**Proof of Theorem 2.7.** We first approximate the nonlinear function $f$ by a globally Lipschitz function $f_\varepsilon$, $\varepsilon > 0$, given by $f_\varepsilon (t) = f (t)$, $t \in [-\varepsilon^{-1}, \varepsilon^{-1}]$, $f_\varepsilon (t) \equiv f (-\varepsilon^{-1}) + f' (-\varepsilon^{-1}) (t + \varepsilon^{-1})$ for $t < -\varepsilon^{-1}$ and $f_\varepsilon (t) \equiv f (\varepsilon^{-1}) + f' (\varepsilon^{-1}) (t - \varepsilon^{-1})$ for $t > \varepsilon^{-1}$. Clearly, $f_\varepsilon \in C^1 (\mathbb{R})$ and $\left| f'_\varepsilon (t) \right| \leq C_\varepsilon$, for all $t \in \mathbb{R}$, with a constant.
\(C_\varepsilon > 0\) that may generally explode as \(\varepsilon \to 0^+\). On the other hand, assumption \([2.5]\) yields, for \(\varepsilon \in (0, \varepsilon_0]\) and for some sufficiently small \(\varepsilon_0 = \varepsilon_0(0, \alpha_0)\), that
\[
f_\varepsilon(t)t \geq \alpha_0 t^2 - \alpha_1 \quad \text{and} \quad f_\varepsilon(t)\text{sgn}(t) \geq \alpha_0 |t| - \frac{\alpha_1}{|t|} \geq \pi := -\alpha_1 t_0^{-1},
\]
for all \(|t| \geq t_0 > 0\). Next, we replace \(f\) by \(f_\varepsilon\) into \([2.3]\) and consider the corresponding approximate problem \(A u_\varepsilon + f_\varepsilon(u_\varepsilon) = h\) in \(X\), for which we first establish the existence of at least one solution \(u_\varepsilon \in V_A\). In a second step, we shall derive bounds for the solution \(u_\varepsilon\) that are uniform in \(\varepsilon > 0\) so that we can pass to the limit as \(\varepsilon \to 0\) to arrive at the final conclusion of the theorem. For practical purposes, in this proof \(C > 0\) denotes a positive constant that is independent of \(\varepsilon\). Such a constant may vary even from line to line and its further dependence on other parameters shall be pointed out as needed.

In order to prove the existence of at least one solution \(u = u_\varepsilon\) to \(A u + f_\varepsilon(u) = h\) in \(X\), let \(B_\varepsilon\) be the mapping defined for \(u, v \in V_A\) by
\[
B_\varepsilon(u, v) := \mathcal{E}_{A, \lambda}(u, v) + \mathcal{C}_{\varepsilon, \lambda}(u, v), \quad \lambda \geq 0,
\]
where
\[
\mathcal{E}_{A, \lambda}(u, v) = \mathcal{E}(u, v) + \lambda (u, v),
\]
\[
\mathcal{C}_{\varepsilon, \lambda}(u, v) := \int_X (f_\varepsilon(u(x)) - \lambda u(x)) v(x) m(dx).
\]
Note that \([2.8]\) is well defined since \(f_\varepsilon - \lambda u \in C(R)\), for each \(\varepsilon > 0\) and \(\lambda \geq 0\). The mapping \(v \mapsto B_\varepsilon(u, v)\) is also linear on \(V_A\) since both \(\mathcal{E}_{A, \lambda}\) and \(\mathcal{C}_{\varepsilon, \lambda}\) are. Clearly, \(|\mathcal{E}_{A, \lambda}(u, v)| \leq C_\lambda \|u\|_{V_A} \|v\|_{V_A}\), hence, for each \(\varepsilon > 0\) the mapping \(v \mapsto \mathcal{E}_{A, \lambda}(u, v)\) is also continuous on \(V_A\). Therefore, there exists an operator \(\bar{A}_\lambda : V_A \to V_A^*\) such that
\[
\langle \bar{A}_\lambda u, v \rangle = \mathcal{E}_{A, \lambda}(u, v), \quad \text{for every } v \in V_A,
\]
where \(\langle \cdot, \cdot \rangle\) denotes the duality between \(V_A\) and \(V_A^*\). Moreover, \(\bar{A}_\lambda\) maps bounded subsets of \(V_A\) into bounded subsets of \(V_A^*\) and \(\bar{A}_\lambda\) is coercive by \([2.4]\):
\[
\lim_{\|u\|_{V_A} \to \infty} \frac{\mathcal{E}_{A, \lambda}(u, u)}{\|u\|_{V_A}} = +\infty.
\]
Thus, by a classical result due to Lions and Leray (see, e.g., [23 Chapter 27]) together with the fact that the embedding \(V_A \to L^2(X)\) is compact, we deduce that the operator \(\bar{A}_\lambda\) is also surjective; in particular, \(\bar{A}_\lambda : V_A \to V_A^*\) is (pseudo)monotone, coercive, continuous and bounded, and for every \(h \in V_A^*\), there exists \(u = u_\varepsilon \in V_A\) such that \(\bar{A}_\lambda(u) = h\), or equivalently, \(\mathcal{E}_{A, \lambda}(u, v) = \langle \bar{A}_\lambda(u), v \rangle = \langle h, v \rangle\), for every \(v \in V_A\). The assumption on \(s > 2q_A/(2q_A - 2) > 2q_A/(2q_A - 1)\) implies that \(h \in L^s(X) \to V_A^*\). Since \(|f_\varepsilon(t) - \lambda t| \leq C_{\varepsilon, \lambda} (1 + |t|)\) for \(\lambda, \varepsilon > 0\), it also holds \(|\mathcal{C}_{\varepsilon, \lambda}(u, v)| \leq C_{\varepsilon, \lambda} (1 + \|u\|_{V_A}) \|v\|_{V_A}\). Then there exists an operator \(\tilde{C}_{\varepsilon, \lambda} : V_A \to V_A^*\) such that \(\langle \tilde{C}_{\varepsilon, \lambda} u, v \rangle = \mathcal{C}_{\varepsilon, \lambda}(u, v),\) for all \(u, v \in V_A\). The operator \(\tilde{C}_{\varepsilon, \lambda} : V_A \to V_A^*\) is also strongly continuous owing to the compactness of the embedding \(V_A \hookrightarrow L^2(X)\). To make a nervous reader happy, it suffices to show that if \(u_n \to u\) in \(V_A\) then \(\mathcal{C}_{\varepsilon, \lambda}(u_n - u, v) \to 0\) as \(n \to \infty\); but this is immediate owing to the mean value theorem for \(f \in C^1\) and the fact that \(f_\varepsilon \in L^\infty(R)\). Thus, the generalized problem for \(A u_\varepsilon + f_\varepsilon(u_\varepsilon) = h\) can be written as \(\langle B_\varepsilon(u_\varepsilon), v \rangle = \langle h, v \rangle\), for all \(v \in V_A\), where \(B_\varepsilon = \bar{A}_\lambda + \tilde{C}_{\varepsilon, \lambda} : V_A \to V_A^*\) is bounded, continuous and pseudo-monotone, as the
we find for coercive (indeed, \(23, \text{Proposition 27.6}\)). By (2.7), if \(0 \leq |v| < \lambda\) holds for all \(v \in V_A\). Here in (2.11), we have set \(f_{\lambda}(t) := f(t) - \lambda t\) and assumed \(\lambda \in [0, \alpha_0]\) with \(\alpha_0 > 0\) as in (2.7).

Our subsequent goal is to show that \(u_{\varepsilon} \in L^\infty(X)\) uniformly with respect to \(\varepsilon > 0\). For \(u_{\varepsilon} \in V_A\) and \(k \geq k_0 := t_0\), we define \(u_{\varepsilon,k} := (|u_{\varepsilon} - k|)^+\text{sgn}(u_{\varepsilon})\) and \(A_k := \{x \in X : |u_{\varepsilon}(x)| > k\}\). Clearly, \(u_{\varepsilon,k} \in V_A\) with \(u_{\varepsilon,k} = 0\) on \(X \setminus A_k\) by Proposition 2.5. Taking \(v = u_{\varepsilon,k}\) as a test function in (2.11) and exploiting (2.2), we find

\[
E_{A,\lambda}(u_{\varepsilon,k}, u_{\varepsilon,k}) + \int_{X \cap A_k} \bar{\alpha}|u_{\varepsilon,k}|m(dx) \leq E_{A,\lambda}(u_{\varepsilon,k}, u_{\varepsilon,k})
\]

(2.12)

owing to the uniform bound (2.7). Indeed, observe that \(f_{\varepsilon,\lambda}(t) \text{sgn}(t) \geq \bar{\sigma}\), for \(|t| \geq t_0\). From (2.12), Holder’s inequality and the fact that \(V_A \rightarrow L^{2q_A}(X)\), it follows

\[
E_{A,\lambda}(u_{\varepsilon,k}, u_{\varepsilon,k}) \leq \int_{X \cap A_k} (\|v\| + \|\bar{v}\|)u_{\varepsilon,k}|m(dx)
\]

(2.13)

\[
\leq C\|u_{\varepsilon,k}\|_{L^{2q_A}(X)}\|v\|_{L^\infty(X)} + \|\bar{v}\|_{L^\infty(X)}\|\chi_{A_k \cap X}\|_{L^p(X)}
\]

\[
\leq C\|u_{\varepsilon,k}\|_{V_A}\|v\|_{L^\infty(X)} + \|\bar{v}\|_{L^\infty(X)}\|\chi_{A_k}\|_{L^p(X)}
\]

for some constant \(C > 0\) independent of \(\varepsilon\); here, \(s, p > 1\) are such that \(1/s + 1/p = 1\). We further recall that by assumption (2.4) we have for every \(k \geq k_0\),

\[
\beta_0\|u_{\varepsilon,k}\|_{V_A} \leq \varepsilon\|u_{\varepsilon,k}\|_{V_A} \leq C\|h\|\|\bar{\alpha}\|_{L^\infty(X)}\|\chi_{A_k}\|_{L^p(X)}
\]

(2.14)

The first inequality on the left-hand side of (2.14) is a consequence of the embedding \(V_A \rightarrow L^{2q_A}(X)\). Let now \(h > k\) and observe that \(A_h \subset A_k\), and on \(A_h\) we have \(|u_{\varepsilon,k}| \geq h - k\). This implies from (2.14) that

\[
\|h - k\|_{L^{2q_A}(X)} \leq C\|h\|\|\bar{\alpha}\|_{L^\infty(X)}\|\chi_{A_k}\|_{L^p(X)}
\]

so that

\[
\|\chi_{A_k}\|_{L^{2q_A}(X)} \leq C(h - k)^{-1}\|h\|\|\bar{\alpha}\|_{L^\infty(X)}\|\chi_{A_k}\|_{L^p(X)}
\]

(2.15)

Since \(1/p = 1 - 1/(2q_A) - 1/s > 1/(2q_A)\) owing to the fact that \(s > q_A/(q_A - 1)\), we have \(p < 2q_A\), and therefore, \(p := 2q_A/p > 1\). Holder’s inequality combined once again with (2.15) gives

\[
\|\chi_{A_k}\|_{L^{2q_A}(X)} \leq C(h - k)^{-1}\|h\|\|\bar{\alpha}\|_{L^\infty(X)}\|\chi_{A_k}\|_{L^{2q_A}(X)}
\]

(2.16)
The application of Lemma 2.4 to (2.16) with \( \Xi(h) := \| \chi_{A_h} \|_{L^{2,q_A}(X)} \) yields the existence of a constant \( C > 0 \) that is independent \( \varepsilon > 0 \), such that

\[
\| \chi_{A_h} \|_{L^{2,q_A}(X)} = 0 \quad \text{with } K := C\| h \|_{L^\infty(X)}.
\]

This yields that \( |u_\varepsilon(x)| \leq K \), a.e. \( x \in X \) and thus the desired uniform bound in \( L^\infty(X) \). Summing up, we have obtained

\[
\| u_\varepsilon \|_{L^\infty(X)} \leq C\| h \|_{L^\infty(X)} + |\bar{\alpha}| \| \cdot \|_{L^\infty(X)},
\]

for some constant \( C > 0 \) independent of \( \varepsilon, h \) and \( u_\varepsilon \). Finally, we can combine (2.17) with the coercivity estimate for \( \mathcal{E}_{A,\lambda}(u_\varepsilon, u_\varepsilon) \geq \beta \| u_\varepsilon \|_{V_A}^2 \) and (2.11) to deduce

\[
\| u_\varepsilon \|_{V_A} \leq C.
\]

By taking a subsequence if necessary, from (2.17), (2.18) we may infer that that there exists a function \( u \in V_A \cap L^\infty(X) \) such that as \( \varepsilon \to 0^+ \),

\[
\begin{align*}
&u_\varepsilon \to u \text{ weakly star in } L^\infty(X,m), \\
&u_\varepsilon \to u \text{ weakly in } V_A,
\end{align*}
\]

and

\[
u_\varepsilon \to u \text{ strongly in } L^2(X,m),
\]

due to \( V_A \overset{\text{d}}{\to} L^2(X) \). The continuity of \( f_\varepsilon \in C(\mathbb{R}) \), together with the strong convergence (2.21), implies by a standard argument that \( f_\varepsilon(u_\varepsilon) \to f(u) \) a.e. in \( X \) and \( f_\varepsilon(u_\varepsilon) \to f(u) \) weakly in \( L^2(X) \) by virtue of the (weak) Lebesgue dominated convergence theorem. Passage to the limit as \( \varepsilon \to 0 \) in \( \mathcal{E}_{A,\lambda}(u_\varepsilon, v) \), for every \( v \in V_A \), is standard owing to the properties of the Dirichlet form \( \mathcal{E}_A \). Therefore, we have obtained the desired bounded solution \( u \in V_A \cap L^\infty(X) \) in the sense of Definition 2.6. The proof is finished.

Finally, we state and prove a very important theorem about the nonnegative self-adjoint operator \( A \) that is in one-to-one correspondence to the Dirichlet form \( \mathcal{E}_A \) (see (2.1)).

**Theorem 2.9.** Let \( A \) be the operator associated with the Dirichlet space \( (\mathcal{E}_A, V_A) \) and suppose \( (X,m) \) is a \( \sigma \)-finite measure space. Assume \( V_A \to L^{2,q_A}(X,m) \), for some \( q_A > 1 \) and \( V_A \overset{\text{d}}{\to} L^2(X) \). Then the following assertions hold:

1. The operator \( -A \) generates a submarkovian semigroup \( (e^{-tA})_{t \geq 0} \) on \( L^2(X) \). The semigroup can be extended to a contraction semigroup on \( L^p(X,m) \) for every \( p \in [1,\infty] \), and each semigroup is strongly continuous if \( p \in [1,\infty] \) and bounded analytic if \( p \in (1,\infty) \). Each such semigroup on \( L^p(X) \) is compact for every \( p \in [1,\infty] \).

2. The operator \( A \) has a compact resolvent, and hence has a discrete spectrum. The spectrum of \( A \) is an increasing sequence of real numbers \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots \), that converges to \( +\infty \). The semigroup \( (e^{-tA})_{t \geq 0} \) is also ultra-contractive in the following sense:

\[
\| e^{-tA} \|_{L(L^2(X);L^\infty(X))} \leq C t^{-\frac{2q_A}{q_A - 1}} e^t, \quad d_A = \frac{2q_A}{q_A - 1},
\]

for all \( t > 0 \), for some \( C > 0 \) independent of \( t \).

3. If \( u_n \) is an eigenfunction associated with \( \lambda_n \), then \( u_n \in D(A) \cap L^\infty(X) \).

4. For \( \theta \in (0,1] \), the embedding \( D\left(A^\theta \right) \to L^\infty(X) \) holds provided that

\[
\theta > \frac{q_A}{2(q_A - 1)}.
\]
Proof. The first two conclusions are more or less consequences of the abstract theories of [6] and [5]. In particular, since $\mathcal{E}_A$ is a Dirichlet form the operator $-A$ generates a submarkovian semigroup $(e^{-tA})_{t \geq 0}$ on $L^2(X)$ which is also analytic (see [6, Theorem 1.4.1]). It follows from [5, Theorem 1.4.1] that the semigroup can be extended to a contraction semigroup on $L^p(X)$ for every $p \in [1, \infty]$, and each semigroup is strongly continuous if $p \in [1, \infty]$ and bounded analytic if $p \in (1, \infty)$. The embedding $V_A \to L^{2q_A}(X, m)$ yields for every $\lambda > 0$,

$$C_\lambda \|u\|_{L^{2q_A}(X)} \leq \mathcal{E}_A (u, u) + \lambda \|u\|_{L^2(X)}^2, \quad u \in V_A,$$

(2.23)

for some $C_\lambda > 0$. Furthermore, since $V_A \hookrightarrow L^2(X)$ the operator $A$ has a compact resolvent. By [3, Corollary 2.4.3], the estimate (2.23) also implies that the semigroup $(e^{-tA})_{t \geq 0}$ is ultra-contractive in the sense of (2.22). Since supp$(m) = X$ and $m(X) < \infty$, the compactness of the semigroup on $L^2(X)$ together with the ultra-contractivity estimate (2.22) implies that the semigroup on $L^p(X)$ is compact for every $p \in [1, \infty]$ (see, e.g., [5, Theorem 1.6.4]). We briefly explain how to get the last two conclusions of the theorem. We first observe that if $u_n \in D(A)$ is an eigenfunction associated with $\lambda_n$, i.e., $Au_n = \lambda_n u_n$. Let $0 < \upsilon \in \rho(-A)$, the resolvent set of $-A$, such that $\upsilon I + A$ is invertible. Then $u_n = (\upsilon I + A)^{-1} (\lambda_n + \upsilon) u_n$ and exploiting the following representation

$$(\upsilon I + A)^{-1} f = \int_0^\infty e^{-\upsilon t} e^{-tA} f dt, \quad f \in L^2(X),$$

it follows owing to (2.22) that

$$\|u_n\|_{L^\infty(X)} \leq C (\lambda_n + \upsilon) \|u_n\|_{L^2(X)} < \infty,$$

which is the desired conclusion. Since $\upsilon I + A$ is invertible we have that the $L^2(X)$-norm of $(\upsilon I + A)^{\theta}$ defines an equivalent norm on $D(A^\theta)$. Besides, for every $f \in L^2(X)$,

$$(\upsilon I + A)^{-\theta} f = \frac{1}{\Gamma(\theta)} \int_0^\infty t^{\theta-1} e^{-\upsilon t} e^{-tA} f dt, \quad \upsilon > 0.$$ 

Using (2.22) for $t \in (0, 1)$ and the contractivity of $e^{-tA}$ for $t > 1$, for $u \in D(A^\theta)$, we deduce

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{D(A^\theta)} \int_1^\infty t^{-\frac{d_A}{4}+\theta-1} dt + C \|u\|_{D(A^\theta)} \int_1^\infty e^{-t} dt.$$ 

The first integral is finite if and only if $\theta > d_A/4$; hence, the fourth conclusion of the theorem is proven as well.

Assume that $(X, m)$ is a $\sigma$-finite measure space and $V_A \hookrightarrow L^2(X)$. We conclude this subsection with a Poincare-Wirtinger like inequality in the space $V_A$ in the case when $\lambda_1 = 0$ is an eigenvalue of $A$ (and so $1 \in D(A)$ is an eigenfunction). Then, it holds

$$\int_X |u(x) - \langle u \rangle_X|^2 m(dx) \leq C_{A,X} \mathcal{E}_A (u, u), \quad u \in V_A,$$

(2.24)

for some $C_{A,X} > 0$ independent of $u$. Here, we have set

$$\langle u \rangle_X = \frac{1}{m(X)} \int_X u(x) m(dx).$$

The general strategy of proof is based on a contradiction argument and the compactness of the embedding $V_A \hookrightarrow L^2(X)$, so we choose the omit the details. Alternatively, (2.24) follows also as a special case of [24, Lemma 4.3.1]. As a consequence,
we observe that both
\[ \mathcal{E}_A(u, u) + \langle u \rangle^2_X \approx \|u\|_{V_A}^2 \quad \text{and} \quad \|A^{-1/2}(u - \langle u \rangle_X)\|_{L^2(X)}^2 + \langle u \rangle^2_X \approx \|u\|_{V_A^*}^2 \]
define also equivalent norms for \(V_A\) and its dual \(V_A^*\), respectively.

2.2. Examples of Dirichlet forms. In this section, we assume that \(X = \Omega\) is a bounded domain in \(\mathbb{R}^n, n \geq 1\) with Lipschitz continuous boundary \(\partial \Omega\), where \(m\) is the usual Lebesgue measure on \(\Omega\). We shall apply the statement of Theorem 2.9 to a variety of operators and the associated Dirichlet forms.

Example 2.10. Define the (weak) Neumann Laplacian \(A_{\Delta_N} u = -\Delta u\) with domain given by
\[ D(A_{\Delta_N}) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega), \partial_n u = 0 \text{ on } \partial \Omega\}. \tag{2.25} \]
The boundary condition in (2.25) is understood in the following (variational) sense:
\[ \int_{\Omega} A_{\Delta_N} u v dx = \int_{\Omega} \nabla u \cdot \nabla v dx = \mathcal{E}_{A_{\Delta_N}}(u, v), \quad \text{for all } v \in H^1(\Omega), \tag{2.26} \]
whenever \(\Delta u \in L^2(\Omega)\) and \(u \in H^1(\Omega)\). It turns out that \(E_A, A = A_{\Delta_N}\), is a Dirichlet form in the sense of Definition 2.1 with \(V_A = H^1(\Omega)\) (see [5,6]) and all the results (i.e., Theorem 2.7 and Theorem 2.9) of the previous section are indeed applicable. More precisely, we have \(V_A \to L^{\frac{2n}{n-2}}(\Omega)\), provided that \(n > 2\), and \(V_A \to L^r(\Omega), \) for any \(r \in (2, \infty)\), in the case \(n \leq 2\). It holds \(q_{AN} = n/(n-2)\) or \(q_{AN} = r/2\) according to whether \(n > 2\) or \(n \leq 2\), respectively. In particular, the solution of the elliptic problem (2.3) enjoys all the properties stated by Theorem 2.7 provided that \(h \in L^s(\Omega)\) with \(s > \frac{n}{2}\) if \(n > 2\) and with \(s \geq 1 + \delta_r\) if \(n \leq 2\), for some (arbitrarily) small \(\delta_r > 0\). Finally, \(A = A_{\Delta_N}\) enjoys all the properties stated in Theorem 2.9 and in particular, since \(q_A = n/(n-2)\) the embedding \(D(A) \to L^\infty(\Omega)\) holds provided that \(n < 4\). The latter is also known to be optimal with respect to the assumption on \(\Omega\) (see, for instance, [5]).

Next, we give examples of Dirichlet forms that are associated with proper fractional Laplace operators with various boundary conditions, that were introduced recently in [12,14,22] (cf. also [13]). To this end, suppose that we are given a (symmetric) kernel \(K(x, y) = J(y, x)\), where \(J : \Omega \times \Omega \to \mathbb{R}\) satisfies the following condition:
\[ c_\Omega \leq J(x, y) |x-y|^{n+2s} \leq C_\Omega, \quad \text{for all } x, y \in \Omega, x \neq y, \tag{2.27} \]
for two constants \(0 < c_\Omega \leq C_\Omega\) depending only on \(\Omega\) and \(J\). Here, we have let \(s \in (1/2, 1)\) and then put
\[ \mathcal{E}_A(u, v) = \int_{\Omega} \int_{\Omega} J(x, y) (u(x) - u(y)) (v(x) - v(y)) dy dx, \tag{2.28} \]
with domain \(D(\mathcal{E}_A) = V_A = H^s(\Omega), s \in (1/2, 1),\) with the latter being dense and compactly contained in \(L^2(\Omega)\).

Proposition 2.11. \(\mathcal{E}_A\) is a Dirichlet form in the sense of Definition 2.1. It also satisfies the conclusion of Proposition 2.9.

Proof. Most of the properties in Definition 2.1 can be checked directly for the form \(\mathcal{E}_A\). We show that for a Borel version of \(u \in L^2(\Omega)\), that satisfies \(u = 0\) a.e., it
holds \( E_A(u, u) = 0 \). For this, take a compact subset \( \Omega_c \subset \Omega \) and define the sets

\[ \Omega^\varepsilon_c := \{ (x, y) \in \Omega_c \times \Omega_c : |x - y| > \varepsilon \}, \quad \varepsilon > 0. \]

We have

\[
\int \int_{\Omega^\varepsilon_c} J(x, y) (u(x) - u(y))^2 \, dydx \leq 2 \int \int_{\Omega^\varepsilon_c} J(x, y) \left( u(x)^2 + u(y)^2 \right) \, dydx
\]

\[
= 4 \int \int_{\Omega^\varepsilon_c} J(x, y) u(x)^2 \, dydx
\]

\[
\leq \frac{4C\varepsilon}{c^{n+2s}} \int \int_{\Omega^\varepsilon_c} u(x)^2 \, dydx
\]

\[
\leq \frac{4C}{c^{n+2s}} \int \int_{\Omega} u(x)^2 \, dydx = 0,
\]

on account of \( \text{(2.27)} \), \( u = 0 \) a.e in \( \Omega \), and the symmetry of \( J \). In particular, we have

\[
\int \int_{\Omega^\varepsilon_c} J(x, y) (u(x) - u(y))^2 \, dydx \leq 0
\]

so that passing to the limit first as \( \varepsilon \to 0^+ \) and then as \( \Omega_c \searrow \Omega \), we get \( E_A(u, u) = 0 \).

Next, we show that \( E_A \) is closed. Let \( u_n \in D(E_A) = V_A \) such that

\[ E_A(u_n - u_m, u_n - u_m) \to 0, \quad \text{as} \ m, n \to \infty. \]

Owing to \( V_A \hookrightarrow L^2(\Omega) \), \( u_n \) converges strongly to some function \( u \in V_A \) in \( L^2(\Omega) \), along a proper subsequence \( u_{n_k} \) (which has a limit \( u(x) \) a.e. in \( \Omega \)). Then

\[ E_A(u - u_m, u - u_m) \]

\[
= \int \int_{\Omega} J(x, y) \lim_{k \to \infty} \left( (u_{n_k}(x) - u_{n_k}(y)) - (u_m(x) - u_m(y)) \right)^2 \, dydx
\]

\[
\leq \liminf_{k \to \infty} \int \int_{\Omega} J(x, y) \left( (u_{n_k}(x) - u_{n_k}(y)) - (u_m(x) - u_m(y)) \right)^2 \, dydx
\]

\[
= \liminf_{k \to \infty} E_A(u_{n_k} - u_m, u_{n_k} - u_m),
\]

by Fatou’s lemma. Passing now to the limit as \( m \to \infty \) on both sides of the preceding inequality, we deduce \( E_A(u - u_m, u - u_m) \to 0 \) as \( m \to \infty \), which shows that \( u_m \to u \) strongly in the \( V_A \)-metric. Finally, it is immediate to check condition (\( a' \)) of Remark 2.2. Thus, \( E_A \) is a Dirichlet form. Finally, it also immediate to see that \( E_A(k, u) = 0 \), for \( 0 \leq u \in V_A \) and any constant \( k \geq 0 \). Then the conclusion of Proposition 2.11 is also verified. The proof is finished.

**Example 2.12.** Let now \( A \) be the (nonnegative) self-adjoint operator that is found into a one-to-one correspondence via \( \text{(2.1)} \) to the Dirichlet space \( (E_A, V_A) \), as given by \( \text{(2.28)} \). The statement of Theorem 2.9 is applicable to \( A \). Observe that \( V_A \hookrightarrow L^{n/2s}(\Omega) \) whenever \( n > 2s \), while \( V_A \hookrightarrow L^r(\Omega) \) in the case \( n \leq 2s \), we have \( q_A = n/(n - 2s) \) for \( n > 2s \), as well as, \( q_A = r/2 \) whenever \( n \leq 2s \). In particular, the embedding \( D(A) \hookrightarrow L^\infty(\Omega) \) is also verified provided that \( n < 4s \). The statements of Theorem 2.7 and Corollary 2.8 also apply to the elliptic boundary value problem \( \text{(2.3)} \) associated with the operator \( A \).

Unfortunately, under the general assumptions of Proposition 2.11 there is no further explicit characterization of the operator \( A \) (of Example 2.12) and its domain \( D(A) \) in the literature. However, in some special situations when the kernel \( J \) is
more explicit such characterizations can be proven (see [12, 14, 22]). In order to recall this case in detail, let us fix $s \in (0, 1)$, and set

$$\mathcal{L}(\Omega) := \{ u : \Omega \to \mathbb{R} \text{ measurable, } \int_{\Omega} \frac{|u(x)|}{(1 + |x|)^{n + 2s}} dx < \infty \}.$$ 

We have $\mathcal{L}(\Omega) \neq \emptyset$ since it contains $L^\infty(\Omega)$ and also smooth functions with compact support in $\Omega$. Then, for $u \in \mathcal{L}(\Omega)$, we define the regional fractional Laplacian $(-\Delta)^s_{\Omega} u$ by the formula

$$(-\Delta)^s_{\Omega} u(x) = \lim_{\varepsilon \downarrow 0} C_{n,s} \int_{\{|y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \quad x \in \Omega,$$ 

provided that the limit exists. The function $C_{n,s}$ has been introduced in [14]. If $0 < \alpha < 2$, then we always assume that $\Delta u \in \mathcal{L}(\Omega)$ and also smooth functions with compact support in $\Omega$.

Definition 2.13. For $0 \leq \alpha < 2$, $u \in C^1(\Omega)$ and $z \in \partial \Omega$, we define the operator $\mathcal{N}^\alpha$ on $\partial \Omega$ by

$$\mathcal{N}^\alpha u(z) := -\lim_{t \downarrow 0} \frac{d u(z + t \vec{n}(z))}{d t} t^\alpha,$$ 

provided that the limit exists.

The operator $\mathcal{N}^\alpha$ has been introduced in [14]. If $0 < s := 1 - \frac{\alpha}{2} < 1$ then we can let $\alpha = 2 - 2s$. The function $\mathcal{N}^{2-2s} u$ is called the fractional normal derivative associated with a smooth function $u \in C^1(\Omega)$ (see [14]). We remark:

- If $\alpha = 0$, then $\mathcal{N}^0 u(z) = -\nabla u \cdot \vec{n}(z) = \frac{\partial u(z)}{\partial \nu}$ for every $u \in C^1(\Omega)$ and $z \in \partial \Omega$.
- If $0 < \alpha < 2$, then $\mathcal{N}^\alpha u(z) = 0$ for every $u \in C^1(\Omega)$ and $z \in \partial \Omega$.

Next, let $\beta > 0$. By [14, p.294], there exist a real number $\delta > 0$ (depending only on $\Omega$) and a function $h_\beta \in C^2(\Omega)$ (depending on $\Omega$ and $\beta$) such that

$$h_\beta(x) = \begin{cases} \rho(x)^{\beta - 1}, & \forall x \in \Omega_\delta, \text{ when } \beta \in (0, 1) \cup (1, \infty); \\ \ln(\rho(x)), & \forall x \in \Omega_\delta, \text{ when } \beta = 1. \end{cases}$$

For $\beta > 0$, define the space

$$C^2_\beta(\Omega) = \{ u : u(x) = f(x) h_\beta(x) + g(x), \forall x \in \Omega, \text{ for some } f, g \in C^2(\Omega) \}.$$ When $\beta > 1$, we always assume that $u \in C^2_\beta(\Omega)$ is defined on $\Omega$ by continuous extension. The following explicit representation of the operator $\mathcal{N}^\alpha$ is taken from [22, Lemma 6.3] (cf. also [14]).

Lemma 2.14. Let $1 < \beta \leq 2$, $u := fh_\beta + g \in C^2_\beta(\Omega)$ be the representation of $u$. Let $u_0 := fh_\beta$ so that $u = u_0 + g$. Then the following assertions hold.

1. If $\beta \in (1, 2)$, then for $z \in \partial \Omega$,

$$\mathcal{N}^{2-\beta} u(z) = (1 - \beta) \lim_{\Omega \ni x \to z} \frac{u(x) - u(z)}{\rho(x)^{\beta - 1}} = (1 - \beta) \lim_{\Omega \ni x \to z} \frac{u_0(x)}{\rho(x)^{\beta - 1}}.$$ 

2. If $\beta = 2$, then for $z \in \partial \Omega$,

$$\mathcal{N}^0 u(z) = -\lim_{\Omega \ni x \to z} \frac{u(x) - u(z)}{\rho(x)}.$$ 

(2.32)
Next, let
\[ C_s := \frac{C_{1,s}}{2s(2s-1)} \int_0^\infty \frac{\tau - 1|1-2s| - (\tau \vee 1)^{1-2s}}{\tau^{2-2s}} d\tau, \quad \frac{1}{2} < s < 1, \]
and let the constant \( B_{n,s} \) be such that
\[ \frac{C_{1,s}}{C_n,s} B_{n,s} := \begin{cases} C_s & \text{if } n = 1 \\ \frac{2\pi^{n-1}}{\Gamma(\frac{n-1}{2})} \int_0^{\pi/2} \cos^s(\theta) \sin^{n-2}(\theta) d\theta, & \text{if } n \geq 2. \end{cases} \]
We have the following fractional Green type formula for the regional fractional Laplace operator \((-\Delta)^s_{\Omega}\) (see [14]).

**Theorem 2.15.** Let \( 1/2 < s < 1 \) and let \((-\Delta)^s_{\Omega}\) be the nonlocal operator defined in (2.29). Then, for every \( u := fh_{2s} + g = u_0 + g \in C^2_{2s}(\Omega) \) and \( v \in W^{s,2}(\Omega) \),
\[
\int_{\Omega} v(x)(-\Delta)^s_{\Omega} u(x) dx = \frac{1}{2} C_n,s \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dxdy - B_{n,s} \int_{\partial\Omega} v(x) N^{2-2s} u(x) d\sigma.
\]
In this instance, for \( u \in C^2_{2s}(\Omega) \) the function \( B_{n,s} N^{2-2s} u \) is called the fractional normal derivative of \( u \) in direction of the outer normal vector.

Let us now return to the case when \( \Omega \subset \mathbb{R}^n \) is a bounded domain with Lipschitz continuous boundary \( \partial \Omega \). We may consider the bilinear symmetric closed form \( \mathcal{E}_N \) with domain \( D(\mathcal{E}_N) = H^s(\Omega) \) and given for \( u, v \in H^s(\Omega) \) by
\[
\mathcal{E}_N(u,v) = \frac{C_n,s}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dxdy.
\]
Since \( H^s(\Omega) = H^s_0(\Omega) \) for all \( 0 < s \leq 1/2 \), we shall further assume that \( 1/2 < s < 1 \) in order to have a nontrivial boundary “trace” \( u|_{\partial\Omega} \neq 0 \), and boundary condition for \( u \) (see, for instance, [22] for further details). Note once more that \( \mathcal{E}_N \) is a special case of (2.28). According to Proposition 2.11 (cf. also [22]), \( \mathcal{E}_N \) is a Dirichlet form on \( V_A = H^s(\Omega) \), while \( A = A_N \) is the closed linear self-adjoint operator which can be defined in the sense of (2.1). We may call \( A_N \) as the realization of the regional fractional Laplace operator \((-\Delta)^s_{\Omega}\) on \( L^2(\Omega) \) with the fractional Neumann type boundary conditions \( B_{n,s} N^{2-2s} u = 0 \) on \( \partial\Omega \). Indeed, we have the following explicit description of the operator \( A_N \) proven in [22] Proposition 6.1] by virtue of Theorem 2.15.

**Proposition 2.16.** If \( \Omega \) is a bounded open set of class \( C^{1,1} \), then
\[
D(\mathcal{E}_N) \cap C^2_{2s}(\Omega) = \{ u \in C^2_{2s}(\Omega), N^{2-2s} u = 0 \text{ on } \partial\Omega \}, \quad A_N u = (-\Delta)^s_{\Omega}.
\]

**Remark 2.17.** All the considerations of Example 2.12 apply to the operator \( A_N \) (recall that \( s \in (1/2, 1) \)).

We conclude this section with some important examples of Dirichlet forms defined over compact manifolds without boundary.

**Example 2.18.** Let \( X = M \) be a compact Lipschitz surface in \( \mathbb{R}^{n-1} \) \((n \geq 2)\) without boundary, and let \( S(x) \) stand for the usual Lebesgue (surface) measure on \( M \). Define the Laplace-Beltrami operator \( A_{LB} u = -\Delta_M u \) with domain given by
\[
D(A_{LB}) = \{ u \in H^1(M) : \Delta_M u \in L^2(M) \}.
\]
We observe that the integration by parts formula holds:
\[
\int_M A_{LB} u v dS = \int_M \nabla_M u \cdot \nabla_M v dS = \mathcal{E}_{A_{LB}}(u,v), \text{ for all } v \in H^1(M),
\]
whenever \(\Delta_M u \in L^2(M)\) and \(u \in H^1(M)\). It turns out that \(\mathcal{E}_A, A = A_{LB}\), is a Dirichlet form in the sense of Definition 2.1 and \(V_A = H^1(M)\) and all the results of the previous section are indeed applicable. More precisely, we have \(V_A \in L^{(2n-1)/n-1}(M)\), provided that \(n > 3\), and \(V_A \in L^r(M)\), for any \(r \in (2,\infty)\), in the case \(n \leq 3\). It holds \(q_{A_{LB}} = (n-1)/(n-3)\) or \(q_{A_{LB}} = r/2\) according to whether \(n > 3\) or \(n \leq 3\), respectively. In particular, the solution of the elliptic problem (2.3) enjoys all the properties stated in Theorem 2.7 and in particular, the embedding \(D(A_{LB}) \to L^\infty(M)\) holds provided that \(n < 5\).

**Example 2.19.** With reference to Example 2.1\(^8\) let \(0 < l < 1\) and consider a fractional version of the Laplace-Beltrami type operator, as
\[
(-\Delta)^l_M u = C_{n-1,l} \text{P.V.} \int_M \frac{u(x) - u(y)}{|x-y|^{n+2l}} dS(y).
\]
For the closed bilinear symmetric form
\[
\mathcal{E}_{A^l_{LB}}(u,v) = \frac{C_{n-1,l}}{2} \int_M \int_M \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2l}} dS(x) dS(y),
\]
we observe that
\[
\mathcal{E}_{A^l_{LB}}(u,v) = (A^l_{LB} u, v), \text{ for all } u \in D(A^l_{LB}), v \in H^1(M).
\]
As in the proof of Proposition 2.11 it turns out that \(\mathcal{E}_A, A = A^l_{LB}\) is Dirichlet form for \(l \in (0,1)\) with \(q_A = (n-1)/(n-1-2l)\) such that \(V_A = H^1(M) \to L^{2/q_A}(M)\) if \(n > 1 + 2l\) and \(V_A \to L^r(M)\) for any \(r \in (2,\infty)\) provided that \(n \leq 1 + 2l\). Finally, we have \([13]\) that
\[
D(A^l_{LB}) = \{u \in H^1(M), (-\Delta)^l_M u \in L^2(M)\}, \ A^l_{LB} u = (-\Delta)^l_M u.
\]
and all the results of the previous section are applicable to the operator \(A^l_{LB}\). In particular, \(D(A^l_{LB}) \to L^\infty(M)\) provided that \(n < 4l + 1\).

3. **The abstract parabolic problem.** Let us first introduce the initial value problem that we wish to investigate. In this section, assume \((X,m)\) is \(\sigma\)-finite measure space. Consider
\[
\begin{align*}
\partial_t u(t) + A \mu(t) &= 0, \text{ in } X \times (0,T) \\
Bu(t) + f(u(t)) &= \mu(t), \text{ in } X \times (0,T),
\end{align*}
\]
subject to the initial condition
\[
u(t=0) = u_0 \text{ in } L^2(X).
\]
Clearly, \([3.1]-[3.2]\) is synonymous with an abstract version of the Cahn-Hilliard equation
\[
\partial_t u(t) + A(Bu(t) + f(u(t))) = 0,
\]
for suitable choices of the operators \(A, B\). In particular, when both \(A\) and \(B\) are appropriate “fractional” diffusion operators in \([3.1]-[3.2]\). the strong character of interaction is reflected by means of the term \(A(Bu)\) in \([3.4]\); we also point out that
Definition 3.2. Let $u$ be a solution if

$$
V_A \rightarrow L^{2q_A} (X) \text{ and } V_B \rightarrow L^{2q_B} (X).
$$

Finally, assume that $\lambda_1 = 0$ is an eigenvalue of $A$ (i.e., $0 \in \sigma_d (A)$), and suppose that constant (in $x \in X$) functions belong to $D (B)$ (i.e., $1 \in D (B)$).

We note that the latter condition $1 \in D (B)$ is actually not a restriction but in fact the most interesting case to investigate, while the former one ($0 \in \sigma_d (A)$) is necessary in order to have the following property for (3.1)-(3.2):

$$
\partial_t \langle u (t) \rangle_X = 0 \Rightarrow \langle u (t) \rangle_X = \langle u_0 \rangle_X, \text{ for all } t \geq 0. \tag{3.5}
$$

As far as the nonlinear potential $f = F'$ is concerned, we make the following assumptions:

(HF-1) $F \in C^2 (\mathbb{R})$ such that $\lim_{|s| \to \infty} F (s) = \infty$ and for some $c_F, c_1 > 0, c_2 \geq 0$,

$$
f (s) s \geq c_1 s^2 - c_2 \text{ and } f' (s) = F'' (s) \geq -c_F, \text{ for all } s \in \mathbb{R}.
$$

(HF-2) There exists a constant $c_f > 0$ and $p \in (1, 2]$ such that

$$
|f (s)|^p \leq c_f (|F (s)| + 1), \text{ for all } s \in \mathbb{R}.
$$

(HF-3) There exist $C_1 > 0, C_2 \geq 0$ and $p \in (1, 2]$ such that

$$
F (s) \geq C_1 |s|^{p/(p-1)} - C_2, \text{ for all } s \in \mathbb{R}.
$$

Remark 3.1. The following remarks are in order.

- (HF-1) implies that $F$ is a quadratic perturbation of some strictly convex function; (HF-2) is satisfied by potentials of arbitrary polynomial growth of order $p = p/ (p-1)$. For instance, the double-well potential $F (s) = \theta s^4 - \sigma s^2$ with $0 < \theta < \sigma$, satisfies both (HF-2) and (HF-3) with $p = 4/3$.
- Under assumption (HA-B), we note that the whole statement of Theorem 2.9 is in full force. In particular, $A$ and $B$ can be any of the operators as suggested by Examples 2.10, 2.12, 2.18 and 2.19.

3.1. Well-posedness: Weak and strong solutions. Let us define the (weak) energy space

$$
Y_M = \left\{ u \in V_B : F (u) \in L^1 (X), |\langle u \rangle_X| \leq M \right\},
$$

for some given $M > 0$, and equip $Y_M$ with the following metric

$$
d (u_1, u_2) = \|u_1 - u_2\|_V + \left| \int_X F (u_1) - F (u_2) m (dx) \right|^{1/2}.
$$

Our definition of a weak solution for the abstract boundary value problem (3.1)-(3.3) is as follows.

Definition 3.2. Let $u_0 \in Y_M$ and $0 < T < +\infty$ be given. We say $u$ is a weak solution if
Note that by Theorem 2.9, Remark 3.6. (3.3) in the sense of Definition 3.4.

Then there exists at least one strong solution \( u \in L^\infty(0,T;V_M) \), \( \partial_t u \in L^2(0,T;V_A^*) \), \( \mu \in L^2(0,T;V_B) \), \( F(u) \in L^\infty(0,T;L^1(X)) \), \( f(u) \in L^\infty(0,T;L^p(X)) \) \( u \in V_B \cap L^\infty(\Omega), \omega \in V_A \), a.e. \( t \in (0,T) \) we have
\[
\langle \partial_t u(t), \omega \rangle + E_A(\mu(t), \omega) = 0,
\]
\[
E_B(u(t), v) + \langle f(u(t)), v \rangle = \langle \mu(t), v \rangle.
\]

We have \( u(0) = u_0 \) in \( X \).

Remark 3.3. Setting \( \omega \equiv 1 \) in (3.8), by (HA-B) we deduce the conservation of mass (3.5). By the second of (3.6), the initial condition \( u(0) = u_0 \) makes sense at least in a weak sense as \( u \in C([0,T];V_A^*) \) (after, modifying, possibly, on a set of zero Lebesgue measure in \([0,T]\)). However, when \( V_B \to V_A, u \in C([0,T];L^2(X)) \) owing to the first two of (3.6), on the other hand, in general, as \( u \in L^\infty(0,T;V_B) \) by (3.6), and \( V_B \to L^2(X), u(t) \in L^2(X) \) is uniquely determined for any \( t \in [0,T] \). In (3.9), \( \langle \cdot, \cdot \rangle \) is understood as the duality map between \( L^p(\Omega) \) and \( L^p(\Omega) \), \( p = p/(p-1) \).

We also define what we mean by a strong solution. To this end, we define a strong energy space
\[
Z_M = \{ u \in D(B) : \mu_0 \in V_A, \quad |\langle u \rangle_X| \leq M \}
\]
and endow it with norm (with respect to the pair \( (u, \mu_0) \in D(B) \times V_A \)),
\[
\| u \|^2_{Z_M} = \| u \|^2_{D(B)} + \| \mu_0 \|^2_{V_A},
\]
where \( \mu_0 \) is computed via the equation
\[
\mu_0 = Bu_0 + f(u_0) \quad \text{in} \quad X.
\]

Here we have also equipped \( D(B) \) with its graph norm.

Definition 3.4. Let \( 0 < T < +\infty \) be given. We say \( u \) is a strong solution of (3.1)-(3.3) if it is a weak solution in the sense of Definition 3.2 and \( u, \mu \) satisfy
\[
u \in L^\infty(0,T;D(B) \cap L^\infty(X)), \quad \partial_t u \in L^2(0,T;V_B),
\]
\[
\mu \in L^\infty(0,T;V_A) \cap L^2(0,T;D(A)).
\]

In particular, for the strong solution we have \( \partial_t u = -A\mu \), a.e. in \( X \times (0,T) \) and \( \mu = Bu + f(u) \), a.e. in \( X \times (0,T) \).

We first prove the existence of a strong energy solution to problem (3.1)-(3.3) by passing to the limit as \( \epsilon, \alpha \to 0 \) in a regularized version of the system (see below). The latter possesses a sufficiently smooth solution \( u_{\epsilon,\alpha} \). After that we will derive additional uniform estimates for the solutions \( u_{\epsilon,\alpha} \) as \( \epsilon, \alpha \to 0 \) in order to pass to the limit.

Theorem 3.5. Assume (HA-B) and (Hf-1) and let \( u_0 \in Z_M \) for some \( q_B > 2 \). Then there exists at least one strong solution \( u \in L^\infty(0,T;Z_M) \) to problem (3.1)-(3.3) in the sense of Definition 3.4.

Remark 3.6. Note that by Theorem 2.9, \( q_B > 2 \) is equivalent to having \( D(B) \to L^\infty(X) \), and therefore for any \( u \in L^\infty(0,T;Z_M) \), the nonlinear term \( f(u) \in L^\infty(0,T;L^\infty(X)) \) is well-defined.
In fact, from (3.17), we have
\[ \mu(\text{lower}) \text{ also yields that} \]
\[ \mu(\text{lower}) \]
\[ \text{such that} \]
\[ \limsup_{\epsilon \to 0} F_{\epsilon}(s) = F(s), \tag{3.12} \]

such that
\[ F_{\epsilon}''(s) \geq -c_F \text{ and } f_{\epsilon}(s) s \geq c_1 s^2 - c_2, \tag{3.13} \]

for all \( s \in \mathbb{R} \), owing to assumption (HF-1) (cf. also Remark 3.1). We then insert an additional term \( \alpha \partial_t u, \alpha > 0 \), into (3.2). More precisely, our approximated problem \( P_{\epsilon,\alpha}, \epsilon, \alpha \in (0, 1) \), consists in finding a function \( u_{\epsilon,\alpha} \in H^1(0,T;L^2(X)) \), in the regularity class of (3.10)-(3.11), to the abstract problem
\[ \partial_t u_{\epsilon,\alpha}(t) + A\mu_{\epsilon,\alpha}(t) = 0, \text{ in } X \times (0,T) \]
\[ Bu_{\epsilon,\alpha}(t) + \alpha \partial_t u_{\epsilon,\alpha}(t) + f_{\epsilon}(u_{\epsilon,\alpha}(t)) = \mu_{\epsilon,\alpha}(t), \text{ in } X \times (0,T), \tag{3.15} \]

subject to the initial condition
\[ u_{\epsilon,\alpha}(0) = u_0. \tag{3.16} \]

We also prepare a fixed initial datum \( \mu_{\epsilon,\alpha}(0) \in L^2(X) \) that solves
\[ \mu_{\epsilon,\alpha}(0) + \alpha A\mu_{\epsilon,\alpha}(0) = \mu_0 := Bu_0 + f(u_0) \text{ in } L^2(X) \], \tag{3.17} \]

whenever it is assumed that \( u_0 \subset D(Z_M) \subset D(B) \to L^\infty(X) \). Observe that the latter also yields that \( \mu_{\epsilon,\alpha}(0) \in V_A \) uniformly with respect to \( \epsilon, \alpha > 0 \) as \( \mu_0 \in V_A \). In fact, from (3.17), we have
\[ \mathcal{E}_A(\mu_{\epsilon,\alpha}(0), \mu_{\epsilon,\alpha}(0)) + \alpha \|A\mu_{\epsilon,\alpha}(0)\|^2_{L^2(X)} \]
\[ = \mathcal{E}_A(\mu_0, \mu_{\epsilon,\alpha}(0)) \]
\[ \leq \frac{1}{2} \mathcal{E}_A(\mu_0, \mu_0) + \frac{1}{2} \mathcal{E}_A(\mu_{\epsilon,\alpha}(0), \mu_{\epsilon,\alpha}(0)) \]

so that \( \mathcal{E}_A(\mu_{\epsilon,\alpha}(0), \mu_{\epsilon,\alpha}(0)) < \infty \), uniformly in \( (\epsilon, \alpha) \) provided that \( \mu_0 \in V_A \). Observe in (3.17) that also \( \langle \mu_{\epsilon,\alpha}(0) \rangle_X = \langle \mu_0 \rangle_X \) and so then, clearly,
\[ \|\mu_{\epsilon,\alpha}(0)\|^2_{L^2(X)} + \alpha \|A\mu_{\epsilon,\alpha}(0)\|^2_{L^2(X)} \leq C \|u_0\|^2_{Z_M}, \tag{3.18} \]

for some \( C > 0 \) independent of \( \alpha, \epsilon \). For \( u_0 \in D(Z_M) \), the existence of at least one strong solution to problem \( P_{\epsilon,\alpha} \), in the sense of Definition 3.4, follows from the application of a backward Euler scheme [21, Theorem 2.9 and Remark 2.1].

**Step 2.** Now, we derive uniform estimates for \( u_{\epsilon,\alpha} \), with respect to \( \epsilon, \alpha \in (0, 1) \). We point out that all test functions used in this step are admissible in working with (3.14)-(3.15) on account of the regularity of \( u_{\epsilon,\alpha} \) and \( \mu_{\epsilon,\alpha} \). For practical purposes, in this proof \( C, Q > 0 \) denote a positive constant and a function that are independent of \( \alpha, \epsilon \) and \( u \). Such parameters may vary even from line to line. Further dependencies of the constants and functions on other parameters will be pointed out as needed.

We begin by testing (3.14) in \( L^2(X) \) with \( \mu_{\epsilon,\alpha} \). Then we test (3.15) by \( \partial_t u_{\epsilon,\alpha} \) in \( L^2(X) \). Taking the sum of all equations that we obtain, we get
\[ \frac{d}{dt} E_{\epsilon,\alpha}(t) + \alpha \int_X |\partial_t u_{\epsilon,\alpha}(t)|^2 m(dx) + \mathcal{E}_A(\mu_{\epsilon,\alpha}(t), \mu_{\epsilon,\alpha}(t)) = 0, \tag{3.19} \]
for all $t \geq 0$, where we have set

$$E_{\epsilon,\alpha}(t) = E_B \left( (u_{\epsilon,\alpha}(t), u_{\epsilon,\alpha}(t)) \right) + \int_X F_\epsilon(u_{\epsilon,\alpha}(t)) \, m(dx).$$

(3.20)

According to (3.16), $u_{\epsilon,\alpha}(0) = u_0 \in D_B \cap L^\infty(X)$ uniformly in $(\epsilon, \alpha)$, so that $E_{\epsilon,\alpha}(0) \leq C$ uniformly in $(\epsilon, \alpha)$, also owing to (3.12). Indeed,

$$\int_X F_\epsilon(u_0) \, m(dx) \leq C.$$

Thus integrating (3.19) over $(0, T)$, we deduce

$$u_{\epsilon,\alpha} \text{ is uniformly bounded in } L^\infty(0, T; V_B),$$

(3.21)

$$F_\epsilon(u_{\epsilon,\alpha}) \text{ is uniformly bounded in } L^\infty(0, T; L^1(X)),$$

(3.22)

$$\alpha^{1/2} \partial_t u_{\epsilon,\alpha} \text{ is uniformly bounded in } L^2(0, T; L^2(X)),$$

(3.23)

$$E_A(\mu_{\epsilon,\alpha}, \mu_{\epsilon,\alpha}) \text{ is uniformly bounded in } L^1(0, T).$$

(3.24)

Here we also recalled that $\langle u_{\epsilon,\alpha}(t) \rangle_X = \langle u_0 \rangle_X$, for all $t \geq 0$. In order to deduce higher-order estimates for $u_{\epsilon,\alpha}$ we need to differentiate the equations (3.14)–(3.15) with respect to $t$, and set $v_{\epsilon,\alpha} = \partial_t u_{\epsilon,\alpha}$, $\omega_{\epsilon,\alpha} = \partial_t \mu_{\epsilon,\alpha}$. Recall that by (3.5), $\langle v_{\epsilon,\alpha}(t) \rangle_X = 0$, for all $t > 0$. Then we have

$$\partial_t v_{\epsilon,\alpha}(t) + A \omega_{\epsilon,\alpha}(t) = 0,$$

(3.25)

$$B v_{\epsilon,\alpha} + f_\epsilon(u_{\epsilon,\alpha}) v_{\epsilon,\alpha} + a \partial_t v_{\epsilon,\alpha} = \omega_{\epsilon,\alpha},$$

(3.26)

subject to the initial condition

$$v_{\epsilon,\alpha}|_{t=0} = -A \mu_{\epsilon,\alpha}(0).$$

(3.27)

Testing now equation (3.25) in $L^2(X)$ by $A^{-1} v_{\epsilon,\alpha}$, and equation (3.26) by $v_{\epsilon,\alpha}$ in $L^2(X)$, then taking the sum of the equations we obtain, we get

$$\frac{d}{dt} \left( \alpha \| v_{\epsilon,\alpha} \|^2_{L^2(X)} + \| v_{\epsilon,\alpha} \|^2_{V_A^2} \right) + 2 E_B(v_{\epsilon,\alpha}, v_{\epsilon,\alpha}) = -2(f_\epsilon(u_{\epsilon,\alpha}) v_{\epsilon,\alpha}, v_{\epsilon,\alpha}).$$

(3.28)

Since by (3.13), $f_\epsilon(u_{\epsilon,\alpha}) \geq -c_\epsilon$, we immediately arrive at the inequality

$$\frac{d}{dt} \left( \alpha \| v_{\epsilon,\alpha} \|^2_{L^2(X)} + \| v_{\epsilon,\alpha} \|^2_{V_A^2} \right) + 2 C \| v_{\epsilon,\alpha} \|^2_{V_\delta} \leq 2 c_\epsilon \| v_{\epsilon,\alpha} \|^2_{L^2(X)},$$

(3.29)

owing to the Poincaré-Wirtinger inequality $\| v \|^2_{V_\delta} \approx E_B(v, v) + \langle v \rangle_X^2 \geq C_B \| v \|^2_{L^2(X)}$, $C_B > 0$ (of course, now $\langle v_{\epsilon,\alpha} \rangle_X = 0$). We have the following cases: either (a) $V_A \to V_B$ or (b) $V_B \to V_A$ (of course, $V_A = V_B$ is trivial).

- In the second case (b), (3.29) and the interpolation inequality $\| v \|^2_{L^2(X)} \leq C \| v \|^2_{V_A^2} \| v \|^2_{V_A^2}$ yield

$$\frac{d}{dt} \left( \alpha \| v_{\epsilon,\alpha} \|^2_{L^2(X)} + \| v_{\epsilon,\alpha} \|^2_{V_A^2} \right) + 2 C \| v_{\epsilon,\alpha} \|^2_{V_A} \leq C \| v_{\epsilon,\alpha} \|^2_{V_A} \| v_{\epsilon,\alpha} \|^2_{V_A^2}$$

(3.30)

$$\leq \delta \| v_{\epsilon,\alpha} \|^2_{V_A} + \frac{C}{4 \delta} \| v_{\epsilon,\alpha} \|^2_{V_A^2},$$

for every $\delta \in (0, 2C)$. Consequently, by the Gronwall inequality we have

$$\alpha \| v_{\epsilon,\alpha}(t) \|^2_{V_A} + \| v_{\epsilon,\alpha}(t) \|^2_{V_A^2} + \int_0^t \| v_{\epsilon,\alpha}(s) \|^2_{V_A^2} \, ds$$

(3.31)

$$\leq C e^{Ct} \left( \alpha \| v_{\epsilon,\alpha}(0) \|^2_{V_A} + \| v_{\epsilon,\alpha}(0) \|^2_{V_A^2} \right).$$
for all \( t > 0 \). Integrating (3.29) once again over \((0, t)\), \( t > 0 \), and using the fact that \( V_A \to L^2(X) \), we get from (3.31),

\[
\int_0^t \|v_{\epsilon, \alpha}(s)\|_{V_B}^2 \, ds \leq C e^{Ct} \left( \alpha \|v_{\epsilon, \alpha}(0)\|_{L^2}^2 + \|v_{\epsilon, \alpha}(0)\|_{V_A}^2 \right). 
\]  

(3.32)

- The remaining case (a) can be settled as follows. Since \( V_B \to V_A \) and \( V_A \to L^2(X) \), by Proposition 2.3 we have

\[
\|v_{\epsilon, \alpha}\|_{L^2(X)} \leq \epsilon \|v_{\epsilon, \alpha}\|_{V_B} + C (\epsilon) \|v_{\epsilon, \alpha}\|_{V_A}.
\]

for any \( \epsilon > 0 \). Inserting the foregoing inequality into (3.29), for a sufficiently small \( \epsilon \in (0, 2C) \), we get

\[
\frac{d}{dt} (\alpha \|v_{\epsilon, \alpha}\|^2_{L^2(X)} + \|v_{\epsilon, \alpha}\|^2_{V_A}) + C \|v_{\epsilon, \alpha}\|^2_{V_B} \leq C \|v_{\epsilon, \alpha}\|^2_{V_A}.
\]

(3.33)

The application of Gronwall inequality yields once again (3.31)-(3.32). By virtue of (3.14)-(3.15), we note that (3.31) also automatically gives

\[
\alpha \|A \mu_{\epsilon, \alpha}(t)\|^2_{L^2(X)} + \mathcal{E}_A (\mu_{\epsilon, \alpha}(t), \mu_{\epsilon, \alpha}(t)) \leq Q (\|u_0\|_{Z_M}) e^{Ct},
\]

(3.34)

as well as

\[
\alpha \|\partial_t u_{\epsilon, \alpha}(t)\|^2_{L^2(X)} + \|\partial_t u_{\epsilon, \alpha}(t)\|^2_{V_A} + \int_0^t \|\partial_t u_{\epsilon, \alpha}(s)\|^2_{V_B} \, ds
\]

\[
\leq Q (\|u_0\|_{Z_M}) e^{Ct},
\]

(3.35)

where in the right-hand side of inequality (3.31) we have actually inserted (3.18).

In order to derive an uniform bound for \( \mu_{\epsilon, \alpha} \in L^\infty(0, T; V_A) \) we test (3.15) by \( \eta_{\epsilon, \alpha} := u_{\epsilon, \alpha} - \langle u_0 \rangle_X \) and recall that \( \langle u_{\epsilon, \alpha} \rangle_X = \langle u_0 \rangle_X \) (clearly, then \( \langle \eta_{\epsilon, \alpha} \rangle_X = 0 \). We find

\[
\mathcal{E}_B (\eta_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}) + (f_{\epsilon}(u_{\epsilon, \alpha}), \eta_{\epsilon, \alpha}) + \alpha (\partial_t u_{\epsilon, \alpha}, \eta_{\epsilon, \alpha})
\]

\[
= (\mu_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}) - (B \langle u_0 \rangle_X, \eta_{\epsilon, \alpha}).
\]

(3.36)

Since \( (\mu_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}) = (A \mu_{\epsilon, \alpha}, A^{-1} \eta_{\epsilon, \alpha}) = (A^{1/2} \mu_{\epsilon, \alpha}, A^{-1/2} \eta_{\epsilon, \alpha}) \), the first term on the right-hand side of (3.36) can be estimated in terms of

\[
C \mathcal{E}_A (\mu_{\epsilon, \alpha}, \mu_{\epsilon, \alpha})^{1/2} \|\eta_{\epsilon, \alpha}\|_{V_A}^{1/2} \leq C \varepsilon \mathcal{E}_A (\mu_{\epsilon, \alpha}, \mu_{\epsilon, \alpha}) + \varepsilon \|\eta_{\epsilon, \alpha}\|_{L^2(X)}
\]

\[
\leq C \mathcal{E}_A (\mu_{\epsilon, \alpha}, \mu_{\epsilon, \alpha}) + \varepsilon \mathcal{E}_B (\eta_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}),
\]

for any \( \varepsilon > 0 \). Clearly, as \( 1 \in D(B) \) by assumption, we have

\[
|(B \langle u_0 \rangle_X, \eta_{\epsilon, \alpha})| \leq C \|\eta_{\epsilon, \alpha}\|_{L^2(X)} \leq C \varepsilon + \varepsilon \mathcal{E}_B (\eta_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}).
\]

Furthermore, since

\[
\alpha (\partial_t u_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}) \leq \varepsilon \mathcal{E}_B (\eta_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}) + C \varepsilon \|\partial_t u_{\epsilon, \alpha}\|_{L^2(X)}
\]

from (3.36) together with (3.34)-(3.35), as well as the uniform estimate (3.18) on \( \mu_{\epsilon, \alpha}(0) \), we get

\[
\mathcal{E}_B (\eta_{\epsilon, \alpha}, \eta_{\epsilon, \alpha}) + (f_{\epsilon}(u_{\epsilon, \alpha}), \eta_{\epsilon, \alpha}) \leq Q (\|u_0\|_{Z_M}) e^{Ct}, \text{ for all } t > 0.
\]

(3.37)

Assumption (H1), together with (3.13), allow us to further observe that

\[
f_{\epsilon}(s) (s - \langle u_0 \rangle_X) \geq \frac{1}{2} |f_{\epsilon}(s)(1 + |s|) - C_M (1 + s^2)|,
\]
for some $C_M = C_M (f, g, \langle \varphi_0 \rangle) > 0$ independent of $(\epsilon, \alpha)$. Thus, (3.37) allows one to deduce
\[ \| f_\epsilon (u_{\epsilon, \alpha} (t)) \|_{L^1 (X)} \leq Q (\| u_0 \|_{Z_M}) e^{C t}, \tag{3.38} \]
exploiting once again the regularity estimate (3.21). Next, we infer from (3.15) that
\[ \langle \mu_{\epsilon, \alpha} (t) \rangle_X = \langle f_\epsilon (u_{\epsilon, \alpha} (t)) \rangle_X + \frac{1}{m (X)} (B (1), u_{\epsilon, \alpha} (t)), \tag{3.39} \]
owing to the fact that $\langle \partial_t u_{\epsilon, \alpha} (t) \rangle_X = 0$ and the self-adjointness of $B$. Thus, since all the terms on the right-hand side of (3.39) are already estimated in (3.38), and since $1 \in D (B)$, we further infer by virtue of (3.21) that
\[ \left\| \langle \mu_{\epsilon, \alpha} (t) \rangle_X \right\| \leq Q (\| u_0 \|_{Z_M}) e^{C t}. \tag{3.40} \]
Consequently, from equation (3.14) we deduce
\[ \partial_t u_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty (0, T; V^*_A). \tag{3.41} \]
Collecting now (3.40), (3.34) together with (3.35), we deduce that
\[ \mu_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty (0, T; V_A), \tag{3.42} \]
\[ \sqrt{\alpha} \mu_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty (0, T; D (A)), \tag{3.43} \]
\[ \alpha^{1/2} \partial_t u_{\epsilon, \alpha} \text{ is uniformly bounded in } L^\infty (0, T; L^2 (X)). \tag{3.44} \]
In order to deduce the uniform $L^\infty (X)$-estimate for $u_{\epsilon, \alpha}$, we exploit the “elliptic” regularity result of Theorem 2.7 (owing to the fact that $s = 2 > q_B/(q_B - 1) \Leftrightarrow q_B > 2$) for the nonlinear problem:
\[ B u_{\epsilon, \alpha} + f_\epsilon (u_{\epsilon, \alpha}) = h_{\epsilon, \alpha} := \mu_{\epsilon, \alpha} - \alpha \partial_t u_{\epsilon, \alpha}, \text{ m-a.e in } X, \tag{3.45} \]
where $\alpha \in (0, 1)$. Indeed, the preceding estimates (3.42)-(3.44) allow us to deduce
\[ \| h_{\epsilon, \alpha} (t) \|_{L^2 (X)} \leq Q (\| u_0 \|_{Z_M}) e^{C t}. \tag{3.46} \]
Therefore, by (3.13) and (3.46) we can infer
\[ \| u_{\epsilon, \alpha} (t) \|_{L^\infty (X)} \leq Q (\| u_0 \|_{Z_M}) e^{C t}. \tag{3.47} \]
Finally, having obtained (3.47) we can interpret the nonlinearity $f_\epsilon (u_{\epsilon, \alpha})$ in (3.45) as a bounded external force (note that we may assume that $|f_\epsilon (s)| \leq |f (s)|$, for sufficiently large $|s| \geq s_0$, as both $f_\epsilon, f$ are linear perturbations of some strictly monotone functions). We deduce according to the application of Corollary 2.8,
\[ \| u_{\epsilon, \alpha} (t) \|_{D (B)} \leq Q (\| u_0 \|_{Z_M}) e^{C t}, \tag{3.48} \]
for some positive function $Q$ independent of $(\epsilon, \alpha)$. Consequently, collecting (3.48), (3.34) and (3.40) we derive
\[ \| u_{\epsilon, \alpha} (t) \|_{Z_M} \leq Q (\| u_0 \|_{Z_M}) e^{C t}. \tag{3.49} \]
**Step 3.** In this final step, we can finally pass to the limit as $(\epsilon, \alpha) \to (0, 0)$ in the sequence of solutions $u_{\epsilon, \alpha}$ satisfying the approximated problem $P_{\epsilon, \alpha}$. Recalling all
Arguing as in the proof of (3.19)-(3.20), we find

\[ L \text{ one weak solution in the sense of Definition 3.2, which in addition belongs to } \mathbb{q}(HA-B) \text{ with Theorem 3.7.} \]

Let the foregoing estimates, we deduce, up to subsequences, that

\[ \partial_t u_{\epsilon,\alpha} \rightarrow \partial_t u \quad \text{weakly in } L^2(0, T; V_B), \]

(3.50)

\[ u_{\epsilon,\alpha} \rightarrow u \quad \text{weakly-* in } L^\infty(0, T; D(B)), \]

(3.51)

\[ \mu_{\epsilon,\alpha} \rightarrow \mu \quad \text{weakly-* in } L^\infty(0, T; V_A), \]

(3.52)

\[ \alpha^{1/2} \partial_t u_{\epsilon,\alpha} \rightarrow 0, \quad \text{strongly in } L^2(0, T; L^2(X)), \]

(3.53)

\[ \partial_t u_{\epsilon,\alpha} \rightarrow \partial_t u, \quad \text{weakly in } L^2(0, T; V_A^*). \]

(3.54)

On the other hand, due to (3.50)-(3.51) and classical compactness theorem of Aubin-Lions-Sobolev, we also have

\[ u_{\epsilon,\alpha} \rightarrow u \quad \text{strongly in } C \left([0, T]; L^2(X)\right). \]

(3.55)

On account of (3.55), (3.51), \( C^1 \ni f_\epsilon \rightarrow f \) on compact subsets of \( \mathbb{R} \), it is not difficult to show

\[ f_\epsilon (u_{\epsilon,\alpha}) \rightarrow f (u) \quad \text{strongly in } C \left([0, T]; L^2(X)\right). \]

(3.56)

We now have all the ingredients to deduce that \( u \) is a strong solution to (3.1)-(3.3) on \((0, T)\), for any \( T > 0 \). Indeed, we can pass to the limit in the approximating problem for \( u_{\epsilon,\alpha} \), and find that \( u \) is a strong solution such that \( u \in L^\infty(0, T; Z_M) \).

Since \( u \in H^1(0, T; V_B) \subset H^1(0, T; L^2(X)) \) by (3.50), it also follows that \( \mu \in L^2(0, T; D(A)) \). Finally, we recover the continuity property \( u(0) = u_0 \in X \) by means of a standard argument (note for instance (3.55)). The proof is finished. \( \square \)

We have the following existence of weak solutions for problem (3.1)-(3.3).

**Theorem 3.7.** Let \( F \) satisfy the assumptions (Hf-1), (Hf-2), (Hf-3) and assume (HA-B) with \( q_B > 2 \). For every initial datum \( u_0 \in Y_M \), there exists at least one weak solution in the sense of Definition 3.2 which in addition belongs to \( L^\infty(0, T; L^p(\Omega)) \), for any \( T > 0 \).

**Proof.** Choose a smooth sequence of data \( u_{0n} \in Z_M \) such that \( u_{0n} \rightarrow u_0 \in Y_M \), i.e., \( u_{0n} \rightarrow u_0 \) in \( V_B \) and \( F(u_{0n}) \rightarrow F(u_0) \) in \( L^1(X) \) (here we can use implicitly that \( D(B) \) is dense in \( V_B \)). The corresponding strong solution \( u_n \) exists by Theorem 3.5.

Arguing as in the the proof of (3.19)-(3.20), we find

\[ \frac{d}{dt} E_n(t) + E_A(\mu_n(t), \mu_n(t)) = 0, \]

(3.57)

for all \( t \geq 0 \), where we have set

\[ E_n(t) = E_B(u_n(t), u_n(t)) + \int_X F(u_n(t)) \, m(dx). \]

(3.58)

As before, owing to the fact that \( F \) is a quadratic perturbation of a strictly convex functions (see (Hf-1)), we first observe

\[ \limsup_{n \rightarrow \infty} \int_\Omega F(u_{0n}) \, dx \leq \int_\Omega F(u_0) \, dx. \]

Integrating now (3.57) over the time interval \((0, T)\), for any \( T > 0 \) we deduce

\[ u_n \text{ is uniformly bounded in } L^\infty(0, T; V_B), \]

(3.59)

\[ F(u_n) \text{ is uniformly bounded in } L^\infty(0, T; L^1(X)), \]

(3.60)

\[ E_A(\mu_n, \mu_n) \text{ is uniformly bounded in } L^1(0, T). \]

(3.61)
Furthermore, we can exploit these uniform estimates and assumption (Hf-2), to infer from (3.60) that
\[ f(u_n) \text{ is uniformly bounded in } L^\infty (0, T; L^p(X)) \] (3.62)
and
\[ u_n \text{ is uniformly bounded in } L^\infty (0, T; L^\overline{p}(X)), \quad \overline{p} = p/(p - 1). \] (3.63)
Next, let us recall the identity (3.39) which actually holds by any smooth solution \( u_n \),
\[ \langle \mu_n(t) \rangle_X = (f(u_n(t)))_X + \frac{1}{m(X)} (B(1), u_n(t)). \] (3.64)
By virtue of estimates (3.60), (3.59) and (3.62) we then get \( \langle \mu_n \rangle_X \in L^2(0, T) \) uniformly in \( n \). Together with (3.61), we can confirm that
\[ \mu_n \text{ is uniformly bounded in } L^2(0, T; V_A) \] (3.65)
and, thus by a comparison argument in the equation \( \partial_t u_n = -A\mu_n \), also
\[ \partial_t u_n \text{ is uniformly bounded in } L^2(0, T; V_A^*) \] (3.66)
We have the following cases:
- \( (a) \) \( V_B \to V_A \), whence \( u_n \in L^\infty (0, T; V_B) \subset L^\infty (0, T; V_A) \) uniformly. Recall that \( V_A \hookrightarrow L^2(X) \). The foregoing estimates (3.59)-(3.66) and on account of the compact embedding
\[ L^\infty (0, T; V_A) \cap H^1 (0, T; V_A^*) \hookrightarrow C ([0, T]; L^2(X)), \] (3.67)
such that, as \( n \to \infty \),
\[ \begin{cases} 
\partial_t u_n \to \partial_t u \text{ weakly in } L^2(0, T; V_A^*), \\
u_n \to u \text{ weakly-* in } L^\infty (0, T; V_B), \\
\mu_n \to \mu \text{ weakly in } L^2(0, T; V_A), \\
u_n \to u \text{ strongly in } C ([0, T]; L^2(X)). 
\end{cases} \] (3.68)
In particular, the last convergence together with the continuity of \( f \) and \( f(u_n) \to f(u) \) weakly in \( L^p(0, T; L^p(X)) \). (3.69)
Thus, on account of (3.69) we can pass to the limit as \( n \) goes to infinity in the problem satisfied by the strong solution \( u_n \). We easily obtain that the limit function in (3.67) is the desired weak solution in the sense of Definition 3.2.
- \( (b) \) \( V_A \to V_B \), and so by duality and compactness \( V_B \hookrightarrow L^2(X) \to V_B^* \to V_A^* \).
Also in this case, the compactness of the embedding
\[ L^\infty (0, T; V_B) \cap H^1 (0, T; V_A^*) \hookrightarrow C ([0, T]; L^2(X)) \]
yields the desired properties in (3.68), as well as (3.67), (3.69). Therefore, the argument leading to the desired final conclusion is the same. This concludes the proof.

We now verify the stability of solutions of problem (3.1)-(3.3) with respect to the initial data in \( Y_M \).
Theorem 3.8. Let $u_1$ be any two weak solutions in the sense of Definition 3.2, corresponding to the initial data $u_i(0)$, $i = 1, 2$, with $\langle u_1(0) \rangle_X = \langle u_2(0) \rangle_X$. Assume (Hf-1) and that $u_i \in L^\infty(0, T; L^p(X))$ for any $T > 0$. Then the following estimate holds:

$$
\|u_1(t) - u_2(t)\|_{V_A}^2 + \int_0^t \|u_1(s) - u_2(s)\|_{V_B}^2 \, ds \leq C e^{Ct} \|u_1(0) - u_2(0)\|_{V_A}^2,
$$

(3.70)

where $C > 0$ is independent of $u_i$, time and the initial data.

Proof. Set $(\bar{\mu}, \bar{\pi})$ such that $\bar{\mu} = u_1 - u_2$. Then $(\bar{\mu}, \bar{\pi})$ satisfies the system

$$
\partial_t \bar{\pi} + A \bar{\pi} = 0,
$$

(3.71)

$$
B \bar{\pi} + f(u_1) - f(u_2) = \bar{\mu},
$$

(3.72)

subject to the initial condition

$$
\bar{\pi}(0) = u_1(0) - u_2(0).
$$

(3.73)

By assumption, we also have $(\bar{\pi}(0))_X = 0$ and so $(\bar{\pi}(t))_X \equiv 0$, as well. Testing equation (3.71) by $A^{-1}(\bar{\pi})$ and equation (3.72) by $\bar{\pi}$, we deduce

$$
\frac{d}{dt} \|\bar{\pi}(t)\|_{V_A}^2 + 2\mathcal{E}_B(\bar{\pi}(t), \bar{\pi}(t)) = -2( f(u_1(t)) - f(u_2(t)), \bar{\pi}(t)).
$$

(3.74)

Recalling the second condition of (Hf-1), we get

$$
\frac{d}{dt} \|\bar{\pi}(t)\|_{V_A}^2 + 2C \|\bar{\pi}(t)\|_{V_B}^2 \leq 2c_F \|\bar{\pi}(t)\|_{L^2(X)}^2,
$$

(3.75)

for some constant $C = C_B > 0$ depending only on $X$ and $B$. We can now estimate the term on the right-hand side of (3.75) exactly as in the cases (a)-(b) of the proof of Theorem 3.5 (see (3.30)-(3.34)) and then apply the Gronwall inequality. The proof is finished.

Remark 3.9. The assumption $q_B > 2$ on the operator $B$ in the statement of Theorem 3.7 is inessential. Indeed, in the proof of Theorem 3.7 we can deal instead with a class of strong solutions $u_\varepsilon$ that satisfy the following parabolic problem:

$$
\partial_t u_\varepsilon + A u_\varepsilon = 0, \varepsilon Du_\varepsilon + Bu_\varepsilon + f(u_\varepsilon) = \mu_\varepsilon, \varepsilon > 0,
$$

(3.76)

for some operator $D$ which satisfies (HA-B) with $q_D > 2$, such that $D(D) \subseteq D(B)$ and $0 \in \sigma_d(D)$ (assuming also for simplicity that $0 \in \sigma_d(B)$). The existence of a strong solution $u_\varepsilon$ to (3.76) can be handled exactly as in the proof of Theorem 3.5. Then one may pass to the limit as $\varepsilon \to 0$ in (3.70) to obtain the desired weak solution without that assumption. This procedure may be tailored according to the specific situation one is interested in.

We omit the simple (repetitive) details, but leave them to the interested reader and conclude this subsection with a result that combines all the previous statements.

Corollary 3.10. Let $F$ satisfy the assumptions (Hf-1), (Hf-2), (Hf-3) and assume (HA-B) with $q_A, q_B > 1$. For every initial datum $u_0 \in Y_M$, there exists a unique weak solution in the sense of Definition 3.2 which additionally belongs to $L^\infty(0, T; L^p(\Omega))$, for any $T > 0$. 

3.2. Finite dimensional attractors. In this subsection, we shall establish the existence of an exponential attractor for problem \((3.1) - (3.3)\). Our first result is concerned with a dissipative estimate enjoyed by the (unique) weak solution of Theorem 3.7. We require an additional (compare to the second of (Hf-1)) weak assumption on \(F\) as follows.

(Hf-4) There exist sufficiently large constants \(C_1 > 0, C_2 \geq 0\) with \(C_1 \geq c_F + C_{A,X}\) such that

\[
F(s) \geq C_1 s^2 - C_2, \text{ for all } s \in \mathbb{R}.
\]

Here, \(C_{A,X} > 0\) is the best Poincare-Wirtinger constant in the inequality

\[
\|u - \langle u \rangle_X\|_{L^2(X)}^2 \leq C_{A,X} \mathcal{E}_A(u,u), \ u \in V_A,
\]

and \(c_F > 0\) is the constant from (Hf-1).

Let us set

\[
E(u) := \mathcal{E}_B(u,u) + \int_X F(u) m(dx).
\]

In the rest of the (sub)section, we shall proceed formally when performing asymptotic estimates, but recall that the arguments can always be made rigorous by choosing to work with strong solutions (see Theorem 3.5 and Remark 3.9). The clean estimates follow by a standard passage to the limit in these uniform estimates for the strong solutions.

**Theorem 3.11.** Let the assumptions of Corollary \(3.10\), and further assume (Hf-4). Then the following dissipative estimate holds:

\[
E(u(t)) \leq E(u_0) e^{-2t} + 2L(M), \quad \forall t \geq 0,
\]

where \(M > 0\) is such that \(\|u_0\|_{Y_M} \leq M\) and \(L = L(M) > 0\) is a constant which is independent of the initial data and time. In particular, there exists a time \(t_\ast = t_\ast (\|u_0\|_{Y_M}) > 0\), such that

\[
\sup_{t \geq t_\ast} E(u(t)) \leq 3L(M).
\]

**Proof.** We first note that \((3.78)\) is an immediate consequence of \((3.77)\). To show \((3.77)\) we can proceed formally. In particular, each weak (and strong) solution satisfies the following energy identity

\[
\frac{d}{dt} E(u(t)) + \mathcal{E}_A(\mu(t), \mu(t)) = 0,
\]

for all \(t \geq 0\). Next, let us test \((3.2)\) with \(u\). We obtain

\[
(\mu, u) = \mathcal{E}_B(u,u) + (f(u), u).
\]

By the assumption (Hf-1) owing to the fact that \(F\) is a quadratic perturbation of a strictly convex function, for all \(s \in \mathbb{R}\) we have

\[
f(s)s \geq F(s) - \frac{c_F s^2}{2} - F(0).
\]

Therefore, from \((3.80)\) we get

\[
(\mu, u) \geq \mathcal{E}_B(u,u) + \int_X F(u)m(dx) - \frac{c_F}{2} \|u\|_{L^2(X)}^2 - F(0) |X|.
\]
On the other hand, we can exploit the Poincare-like inequality \( \| \mu - \langle \mu \rangle_X \|_{L^2(X)}^2 \leq C_{A,X} \mathcal{E}_A (\mu, \mu) \), and the conservation of mass \( \langle u \rangle_X = \langle u_0 \rangle_X \), to observe that

\[
(\mu, u) = (\mu - \bar{\mu}, u) \leq C_{A,X}^{1/2} \mathcal{E}_A^{1/2} (\mu, \mu) \| u \|_{L^2(X)},
\]

assuming for simplicity (for now) that \( \langle u_0 \rangle_X = \langle u \rangle_X = 0 \). Thus, by virtue of assumption (HF-4) we rewrite (3.81) and continue to estimate as follows:

\[
C_{A,X}^{1/2} \mathcal{E}_A^{1/2} (\mu, \mu) \| u \|_{L^2(X)} \geq \mathcal{E}_B (u, u) + \int_X F(u)m (dx)
- \frac{c_F}{2} \| u \|_{L^2(X)}^2 - F(0) |X| \\
\geq \frac{1}{2} \left( \mathcal{E}_B (u, u) + \int_X F(u)m (dx) \right) \\
+ \frac{1}{2} \int_X F(u)m (dx) - \frac{c_F}{2} \| u \|_{L^2(X)}^2 - F(0) |X| \\
\geq \frac{1}{2} \mathcal{E}_B (u, u) + \frac{C_1}{2} \| u \|_{L^2(X)}^2 \\
- \frac{c_F}{2} \| u \|_{L^2(X)}^2 - F(0) |X|.
\]

Application of Young’s inequality on the left-hand side of (3.82) yields

\[
\frac{1}{2} \mathcal{E}_B (u, u) + \frac{C_1}{2} \| u \|_{L^2(X)}^2 - \frac{c_F}{2} \| u \|_{L^2(X)}^2 - F(0) |X| \\
\leq \frac{1}{2} \mathcal{E}_A (\mu, \mu) + \frac{C_{A,X}}{2} \| u \|_{L^2(X)}^2
\]

and so we easily deduce

\[
\frac{1}{2} \mathcal{E}_B (u, u) \leq \frac{1}{2} \mathcal{E}_A (\mu, \mu) + L_*,
\]

for some constant \( L_* > 0 \) which depends only on \( C_1, F(0) \) and \( m(X) = |X| < \infty \). It follows by virtue of (3.83) and the energy identity (3.79) that we have

\[
\frac{d}{dt} E(u(t)) + E(u(t)) \leq 2L_*,
\]

for all \( t \geq 0 \). By means of the Gronwall inequality we then obtain

\[
E(u(t)) \leq E(u(0)) e^{-t} + 2L_*.
\]

If \( \langle u_0 \rangle_X \neq 0 \) is such that \( \langle u \rangle_X = \langle u_0 \rangle_X \in [-M, M] \) for some \( M > 0 \), observe that if \( u \) is a weak solution with initial datum \( u_0 \) for the problem (3.1)-(3.3) with potential \( F \) then for \( \tilde{u} = u - \langle u_0 \rangle_X \), \( \tilde{u} \) is a weak solution with initial datum \( u(0) = u_0 - \langle u_0 \rangle_X \) for the same problem with potential \( \tilde{F}(s) := F(s + \langle u_0 \rangle_X) - F(\langle u_0 \rangle_X) \). Since now \( \langle \tilde{u} \rangle_X = 0 \), we can employ the dissipative estimate (3.85) for the solution \( \tilde{u} \) and easily arrive at the final inequality (3.77), for some \( L(M) > 0 \), owing to the fact that \( |\langle u_0 \rangle_X| \leq M \). The proof is finished. \( \square \)

**Theorem 3.12.** Let the assumptions of Theorem 3.11 be satisfied. There exists a constant \( R = R(M) > 0 \), independent of time and the initial data, such that

\[
\sup_{t \geq t_* + 1} \left( \| u(t) \|_{L^M}^2 + \int_t^{t+1} \| \partial_t u(s) \|_{V}^2 \, ds \right) \leq R.
\]

Here \( t_* > 0 \) is the same as in the statement of Theorem 3.11.
Proof. As usual, in this proof $C, Q > 0$ will denote a constant and function which are independent of $u$, time and the initial data. Let $t_* > 0$ be the entering time from \[(3.78)\]. For $t \geq t_*$, from \[(3.79)\] it follows that
\[
\frac{d}{dt} \left[(t - t_*) E(u(t)) + (t - t_*) \mathcal{E}_A(\mu(t), \mu(t)) \right] = E(u(t)).
\]
Integrating the foregoing identity over $(t_*, T)$, with any $T > t_*$, in light of \[(3.78)\] we obtain
\[
\int_{t_*}^{T} (t - t_*) \mathcal{E}_A(\mu(t), \mu(t)) \, dt \leq L(M) (T - t_*). \tag{3.87}
\]
Moreover, the energy identity \[(3.79)\] also gives in light of \[(3.78)\] that
\[
\int_{t_*}^{T} \mathcal{E}_A(\mu(t), \mu(t)) \, dt \leq L(M). \tag{3.88}
\]
Next, we recall \[(3.28)-(3.29)\] contained in the proof of Theorem 3.5. Both of these actually hold by any strong solution (of course, now $\alpha = 0$). In particular, for all $t \geq t_*$ we have
\[
\frac{d}{dt} \| \partial_t u(t) \|^2_{V_A} + 2C_B \| \partial_t u(t) \|^2_{V_B} = -2(f'(u(t)) \partial_t u(t), \partial_t u(t)) \tag{3.89}
\]
\[
\leq 2c_F \| \partial_t u(t) \|_{L^2(\Omega)}^2.
\]
for some constant $C_B > 0$ which depends only on $B$ and $m(\Omega) < \infty$. We once again need to deal with the cases when either (a) $V_A \to V_B$ or (b) $V_B \to V_A$.

- (a) Inequality \[(3.89)\] yields in light of the same argument of \[(3.30)\] that
\[
\frac{d}{dt} \| \partial_t u(t) \|^2_{V_A} + C_B \| \partial_t u(t) \|^2_{V_B} \leq C \| \partial_t u(t) \|_{V_A}^2. \tag{3.90}
\]
Multiplying both sides of \[(3.90)\] by $(t - t_*)$ and integrating the resulting inequality over $(T, t_*)$, with any $T \geq t_* + 1$, we deduce
\[
(T - t_*) \| \partial_t u(T) \|^2_{V_A} + C_B \int_{t_*}^{T} (t - t_*) \| \partial_t u(t) \|^2_{V_B} \, dt \tag{3.91}
\]
\[
\leq C \int_{t_*}^{T} [(t - t_*) + 1] \mathcal{E}_A(\mu(t), \mu(t)) \, dt
\]
\[
\leq L(M) (T - t_* + 1),
\]
on account of estimates \[(3.87)-(3.88)\] and the fact that $\| \partial_t u(t) \|_{V_A}^2 = \mathcal{E}_A(\mu, \mu)$. In particular, \[(3.91)\] yields that
\[
\sup_{t \geq t_* + 1} \left( \| \partial_t u(t) \|_{V_A}^2 + \mathcal{E}_A(\mu(t), \mu(t)) \right) \leq 2L(M). \tag{3.92}
\]
It remains to note that integrating \[(3.90)\] over $(t, t + 1)$, and using \[(3.92)\] we also obtain
\[
\sup_{t \geq t_* + 1} \int_{t}^{t+1} \| \partial_t u(t) \|^2_{V_B} \, ds \leq 4L(M). \tag{3.93}
\]
- (b) The case $V_B \to V_A$ may be argued exactly as in \[(3.33)\]. This implies once again that \[(3.90)\] holds, and so both \[(3.92)-(3.93)\] can be verified as well.
Next, we exploit the fact that \( u \in L^\infty (t_*, \infty; V_B) \) uniformly with respect to time and the initial data (see \( \text{(3.78)} \)) to observe that \( f(u) = F^\prime (u) \in L^\infty (t_*, \infty; L^1(X)) \), uniformly in time and the initial data by virtue of the assumptions on \( F \) (see \( \text{(Hf-2)}, \text{(Hf-3)} \)). Recalling from \( \text{(3.39)} \)

\[
\langle \mu (t) \rangle_X = \langle f(u(t)) \rangle_X + \frac{1}{m(X)} (B(1), u(t)),
\]

by application of \( \text{(3.92)} \) we find that \( \langle \mu \rangle_X \in L^\infty (t_*, 1, \infty) \) uniformly in time and the initial data. Together with \( \text{(3.92)} \) and the Poincare-Wirtinger inequality \( \text{(2.24)} \) this yields

\[
\sup_{t \geq t_*, +1} \| \mu (t) \|_{V_A}^2 \leq Q(M), \tag{3.94}
\]

for some function \( Q > 0 \) independent of \( u, t \), and the initial data. In order to deduce the uniform \( L^\infty (X) \)-estimate for \( u(t) \), we can apply Theorem \( \text{2.7} \) to problem \( Bu(t) + f(u(t)) = \mu(t) \), a.e. in \( X \times (t_*, 1, \infty) \), to obtain by virtue of \( \text{(3.86)} \) and \( \text{(3.94)} \) that

\[
\sup_{t \geq t_*, +1} \| u(t) \|_{L^\infty(X)}^2 \leq Q(M). \tag{3.95}
\]

Note that in this instance, \( h := \mu \in V_A \to L^{2q_A} (X) \) and so \( h \in L^s (X) \) with \( s = 2q_A > q_A/(q_A - 1) \) for as long as \( q_A > 1 \) (the latter holds by assumption \( \text{(HA-B)} \)). It remains to exploit \( \text{(3.95)} \) and the statement of Corollary \( \text{2.8} \) with a nonlinearity which is to be interpreted as a bounded external force. We immediately arrive at

\[
\sup_{t \geq t_*, +1} \| u(t) \|_{L^2(B)}^2 \leq Q(M). \tag{3.96}
\]

Finally, collecting \( \text{(3.93)}, \text{(3.96)} \) and \( \text{(3.94)} \) we have obtained \( \text{(3.86)} \), which completes the proof.

According to Corollary \( \text{3.10} \), problem \( \text{(3.1)-(3.3)} \) generates a dissipative semigroup \( S(t) : Y_M \to Y_M \), given by \( S(t) u_0 = u(t) \), where \( u(t) \) is the unique weak solution of \( \text{(3.1)-(3.3)} \). Our main result of this section is the following.

**Theorem 3.13.** Let the assumptions of Theorem \( \text{3.12} \) be satisfied for some \( F \in C^3(\mathbb{R}) \). Further assume

\[
V_B \to L^{\frac{2q_B}{q_B - 1}} (X), \text{ for some } q_B > 1. \tag{3.97}
\]

Then the dynamical system \((S(t), Y_M)\) admits an exponential attractor \( M \) bounded in \( Z_M \) in the following sense:

(i) The sets \( M \) are positively invariant with respect to the semigroup \( S(t) \), that is, \( S(t) M \subseteq M \), for all \( t \geq 0 \).

(ii) The fractal dimension of the sets \( M \) is finite, that is, \( \dim_F (M, V_B) \leq c < \infty \), where \( c > 0 \) can be computed explicitly.

(iii) Each \( M \) attracts exponentially any bounded subset of \( Y_M \), that is, there exist a positive constant \( \rho \) and a monotone nonnegative function \( Q \), such that, for every bounded subset \( B \) of \( Y_M \), we have

\[
dist_{V_B} (S(t) B, M) \leq Q(\| B \|_{Y_M}) e^{-\rho t},
\]

where \( dist_{V^*} (X, Y) := \sup_{x \in X} \inf_{y \in Y} \| x - y \|_{V^*} \) is the Hausdorff semi-distance.

In order to prove this result we report the following basic abstract result on the existence of exponential attractors for discrete maps (see, for instance, \( \text{[18]} \)).
Theorem 3.14. Let \( X_1 \) and \( X_2 \) be two Banach spaces such that \( X_2 \) is compactly embedded in \( X_1 \). Let \( X_0 \) be a bounded subset of \( X_2 \) and consider a nonlinear map \( \Sigma : X_0 \to X_0 \) satisfying the smoothing property
\[
\| \Sigma (x_1) - \Sigma (x_2) \|_{X_2} \leq d \| x_1 - x_2 \|_{X_1},
\]
for all \( x_1, x_2 \in X_0 \), where \( d > 0 \) depends on \( X_0 \). Then the discrete dynamical system \( (X_0, \Sigma^n) \) possesses a discrete exponential attractor \( E^n_M \subset X_2 \), that is, a compact set in \( X_1 \) with finite fractal dimension such that
\[
\Sigma (E^n_M) \subset E^n_M, \quad \text{dist}_{X_1} (\Sigma^n (X_0), E^n_M) \leq d_X e^{-\rho_* n}, \quad n \in \mathbb{N},
\]
where \( d_X \) and \( \rho_* \) are positive constants independent of \( n \), with the former depending on \( X_0 \).

Let us now introduce a ball \( B_R \subset D(B) \) with sufficiently large radius \( R > 0 \) such that
\[
B_R := \left\{ u \in D(B) \cap L^\infty (X) : \| u \|_{D(B)+L^\infty(X)} \leq R, \  \| u \|_X \leq M \right\}.
\]
By Theorem 3.12, there exists a sufficiently large \( R_M > 0 \) such that the ball \( B := B_{R_M} \) will be absorbing for \( S (t) \) and \( S (t) B \subset B \) for all \( t \geq t_* + 2 \). Thus it suffices to construct the exponential attractor on the set \( B \). Next, we prove a series of elementary lemmas for which Theorem 3.12 proves essential. The first lemma is concerned with the Lipschitz-in-time regularity of the semigroup \( S (t) \).

Lemma 3.15. Let the assumptions of Theorem 3.13 be satisfied. There exists a constant \( C > 0 \) such that
\[
\| S(t)u_0 - S(\bar{t})u_0 \|_{V_\lambda^*} \leq C |t - \bar{t}|,
\]
for all \( t, \bar{t} > 0 \) and any \( u_0 \in \mathcal{B} \).

Proof. The claim follows from the basic equality
\[
S(t)u_0 - S(\bar{t})u_0 = \int_\bar{t}^t \partial_y (S(y))u_0 \, dy
\]
and estimate (3.92). \( \Box \)

The second result states the validity of the smoothing property for the semigroup \( S (t) \).

Lemma 3.16. Let the assumptions of Theorem 3.13 be satisfied. Indicate by \( u_i \), \( i = 1, 2 \), the solution to problem (3.1)-(3.3) which corresponds to the initial datum \( u_{i0} = u_i (0) \in \mathcal{B} \). Then the following estimate holds:
\[
\| u_1 (t) - u_2 (t) \|_{V_\lambda^*}^2 \leq C e^{C t} \| u_1 (0) - u_2 (0) \|_{L^2 (X)}^2,
\]
for all \( t \geq 1 \), and some positive constant \( C \) which only depends on \( \mathcal{B} \).

Proof. First, we note that due to estimate (3.86) we have
\[
\| u_i (t) \|_{D(B)} + \| u_i (t) \|_{L^\infty (X)} \leq R_0, \ i = 1, 2, \ t \geq 0
\]
and
\[
\int_0^t \| \partial_t u_i (s) \|_{V_\lambda^*}^2 \, ds \leq R_0 (1 + t), \ i = 1, 2, \ t \geq 0.
\]
Here the constant \( R_0 > 0 \) is independent of time and depends only on \( \mathbb{B} \). In this proof, the constant \( C = C ( R_0 ) > 0 \) will be independent of time and the initial data. As in the proof of Theorem 3.8 we generally have that \( \langle \pi ( t ) \rangle_X = \langle \pi ( 0 ) \rangle_X \neq 0 \). Thus, letting \( \overline{u} := \pi - M_0 \) where \( M_0 := \langle \pi ( 0 ) \rangle_X \) and then testing (3.105) by \( A^{-1} ( \overline{u} ) \) and (3.106) by \( \overline{u} \), we derive

\[
\frac{d}{dt} \| \overline{u} ( t ) \|^2_{V_A^\delta} + 2 \mathcal{E}_B ( u ( t ) , \overline{u} ( t ) ) = - 2 ( f ( u_1 ( t ) ) - f ( u_2 ( t ) ) , \overline{u} ( t ) ) ,
\]

which is the analogue of (3.74) in the case when \( \langle \pi ( 0 ) \rangle_X \neq 0 \). Since by (3.103) we have a uniform estimate on the \( L^\infty \)-norms of the solutions \( u_i \), we can estimate the term on the right-hand side of (3.107),

\[
( f ( u_1 ) - f ( u_2 ) , \overline{u} ) \leq C ( R_0 ) \left( M_0^2 + \| \overline{u} \|^2_{L^2 ( X )} \right).
\]

Since also for a strong solution \( u \in D ( B ) , \mathcal{E}_B ( u , \overline{u} ) = ( B \overline{u} , \overline{u} ) + ( B \langle \pi ( 0 ) \rangle_X , \overline{u} ) \), from (3.107) we get

\[
\frac{d}{dt} \| \pi ( t ) \|^2_{V_A^\delta} + C_B \| \pi ( t ) \|^2_{V_B} \leq C ( R_0 ) \left( M_0^2 + \| \overline{u} \|^2_{L^2 ( X )} \right) + 2 \mathcal{E}_B ( M_0 , \overline{u} ) \]

\[
\leq C ( R_0 ) \left( M_0^2 + \| \overline{u} \|^2_{L^2 ( X )} \right) + C_B , \delta ( X ) M_0^2 + \delta \mathcal{E}_B ( \overline{u} , \overline{u} ) ,
\]

for every \( \delta > 0 \). Arguing in (3.109) as before, according to whether (a) \( V_A \rightarrow V_B \) or (b) \( V_B \rightarrow V_A \) (see the proof of Theorem 3.5), by Gronwall inequality we get

\[
\| \pi ( t ) \|^2_{V_A^\delta} + C_B \int_0^t \| \pi ( s ) \|^2_{V_B} ds \leq C e^{C t} \left( M_0^2 + \| \pi ( 0 ) \|^2_{V_A^\delta} \right) \leq C e^{C t} \| \pi ( 0 ) \|^2_{L^2 ( X )} ,
\]

since \( M_0^2 \leq C_X \| \pi ( 0 ) \|^2_{L^2 ( X )} \) and \( L^2 ( X ) \rightarrow V_A^\delta \). Next, we test (3.105) by \( A^{-1} ( \partial_t \pi ) \) and (3.106) by \( \partial_t \pi \), and obtain after standard transformations (note that this time \( \langle \partial_t \pi \rangle_X = \langle \partial_t \overline{u} \rangle_X = 0 \)), that

\[
\frac{d}{dt} \mathcal{E}_B ( \pi ( t ) , \pi ( t ) ) + 2 \| \partial_t \pi ( t ) \|^2_{V_A^\delta} = 2 ( f ( u_1 ( t ) ) - f ( u_2 ( t ) ) , \partial_t \pi ( t ) ) .
\]

For the first term on the right-hand side, we can use (3.103) to find

\[
( f ( u_1 ) - f ( u_2 ) , \partial_t \pi ) \leq C \| \partial_t \pi \|^2_{L^2 ( X )} + C ( R_0 ) \| \pi \|^2_{L^2 ( X )} .
\]

Inserting now these estimates on the right-hand side of (3.111), we get

\[
\frac{d}{dt} \mathcal{E}_B ( \pi ( t ) , \pi ( t ) ) + 2 \| \partial_t \pi ( t ) \|^2_{V_A^\delta} \leq C \| \partial_t \pi \|^2_{L^2 ( X )} + C ( R_0 ) \| \pi \|^2_{L^2 ( X )} .
\]

We now need to control the last term on the right-hand side of (3.113). To this end, we differentiate both equations of (3.105)-(3.106) with respect to time, and set
The idea now is to add (3.113) to (3.118) and to absorb the term \( \| \partial_t u_2 (t) \|_{V_B}^2 \) from the right-hand side into the \( \delta_0 \)-term on the left-hand side of (3.118) for some sufficiently small \( \delta < \delta_0 \) (dealing with each of the cases \( V_A \to V_B \), \( V_B \to V_A \) separately, see Subsection 3.1). If we define

\[
\mathcal{F} (t) := \| \partial_t \bar{\pi} (t) \|_{V_A}^2 + \mathcal{E}_B (\bar{\pi} (t), \pi (t)) + \langle \bar{\pi} (t) \rangle_X^2,
\]

\[
\alpha (t) := C \left( 1 + \| \partial_t u_2 (t) \|_{V_B}^2 \right),
\]

\[
\kappa (t) := C \left( R_0 \right) \| \pi (t) \|_{L^2 (X)}^2,
\]

we thus deduce

\[
\frac{d}{dt} \mathcal{F} (t) + C (\delta_0) \| \partial_t \pi (t) \|_{V_A}^2 \leq \kappa (t) + C \| \partial_t u_2 (t) \|_{V_B}^2 \mathcal{F} (t),
\]

for all \( t > 0 \). Multiplying now both sides of (3.119) by \( t > 0 \), we have

\[
\frac{d}{dt} \left( t \mathcal{F} (t) \right) + C (\delta_0) t \| \partial_t \pi (t) \|_{V_A}^2 \leq C \left( t \| \partial_t u_2 (t) \|_{V_B}^2 + 1 \right) \mathcal{F} (t) + t \kappa (t).
\]
We note preliminarily that by virtue of (3.104) and (3.110) we have
\[ \int_0^t s \kappa(s) \, ds \leq t \int_0^t \kappa(s) \, ds \leq C t e^{C t} \| \bar{\mu}(0) \|^2_{L^2(X)}, \quad \int_0^t \alpha(s) \, ds \leq C (1 + t) \leq C e^{C t}. \]  
(3.121)
Integrating now (3.120) over \((0, T)\) with any \(T \geq 1\), we derive
\[ T F(T) \leq \int_0^T C \left( t \| \partial_t u_2(t) \|^2_{V_0} + 1 \right) F(t) \, dt + \int_0^T t \kappa(t) \, dt \]
\[ \leq T \int_0^T \alpha(t) F(t) \, dt + \int_0^T t \kappa(t) \, dt. \]
This entails, owing to the first of (3.121), that
\[ F(T) \leq \int_0^T \alpha(t) F(t) \, dt + C e^{C T} \| \bar{\mu}(0) \|^2_{L^2(X)}, \]  
for any \(T \geq 1\). \hspace{1cm} (3.122)
On the other hand, the application of Gronwall’s inequality to (3.122) yields
\[ F(T) \leq C e^{C T} \| \bar{\mu}(0) \|^2_{L^2(X)} + C \int_0^T e^{C t} \| \bar{\mu}(0) \|^2_{L^2(X)} \alpha(t) \exp \left( \int_t^T \alpha(s) \, ds \right) \, dt \]
\[ \leq C e^{C T} \| \bar{\mu}(0) \|^2_{L^2(X)} \]  
owing once again to (3.121). Since \(C \| \bar{\mu}(T) \|_{V_B}^2 \leq F(T)\) the claim (3.102) follows from (3.123) and this completes the proof. \hspace{1cm} \Box

**Proof of Theorem 3.13** We shall now exploit the statements of Lemmas 3.15 and 3.16 to finish the proof. First, we recall once again that \(B := B_{R_M}\) is an absorbing set for \(S(t)\) and \(S(t)B \subset B\) for \(t \geq t^4\), for some \(t^4 \geq 1\). Thus we can apply the statement of Theorem 3.14 to the map \(\Sigma = S(t^4)\). Indeed, setting \(X_0 = B\) the mapping \(\Sigma : X_0 \to X_0\) enjoys the smoothing property (3.102) owing to the fact that the embedding \(V_B \to L^2(X)\) is compact. Therefore Theorem 3.14 applies to \(\Sigma\) and there exists a compact set \(M^* \in X_0\) with finite fractal dimension (with respect to the metric topology of \(L^2(X)\)) that satisfies (3.99) and (3.100). Moreover, since \(B\) is a compact absorbing set for these semigroups in \(Y_M\) then these (discrete) attractors attract exponentially fast all the bounded subsets of \(Y_M\). In order to construct the attractor for the semigroup with continuous time, as usual we use the formula
\[ \mathcal{M} = \bigcup_{t \in [t^4, 2t^4]} S(t) M^*. \]
We now recall that the semigroup \(S(t)\) is Lipschitz continuous on \([t^4, 2t^4] \times B\) in the norm of \(V_A^s\). This was verified in Lemma 3.15 as well as in Theorem 3.8 (cf. also (3.110)). Thus, we can verify in a standard way that the exponential attractor \(M\) satisfies (i), (ii) and (iii) of Theorem 3.13 but with the norm of \(V_B\) being replaced by that of \(V_A^s\). In order to extend these properties to the required \(V_B\)-norm it suffices to recall that \(M\) is bounded in \(B\), the semigroup possesses a smoothing property from \(Y_M\) into \(Z_M\) and to argue as before according to the cases whether \(V_A \to V_B\) or \(V_B \to V_A\), as well as the fact that \(D(B) \to V_B \to L^2(X)\). The proof of Theorem 3.13 is finished. \hspace{1cm} \Box

As a consequence of Theorem 3.13 we may conclude this section with the following.
Corollary 3.17. Under the assumptions of Theorem 3.13, there exist the global attractor $\mathcal{G}$, bounded in $Z_{M}$, of fractal dimension such that $\dim_{F}(\mathcal{G}, V_{B}) < \infty$.

Remark 3.18. We note that assumption (3.97) of Theorem 3.13 is always satisfied if $q_{B} > 2$, owing to (HA-B) and the fact that $L^{2q_{B}}(X) \to L^{2q_{B}(q_{B}-1)}(X)$ in that case.

3.3. Remarks on the singular potential case. We conclude the paper with a well-posedness result for the doubly nonlocal problem (1.5) with a potential $F$ that may be logarithmic-like. More precisely, we suppose that $F \in C^{2}(-1,1) \cap C[-1,1]$ and $F = F^{1} \in C^{1}(-1,1)$ can be decomposed into $f = f_{0} + f_{1}$, with $f_{0}$ satisfying

$$\begin{align*}
\lim_{s \to \pm 1} f_{0}(s) &= \pm \infty, \\
\lim_{s \to \pm 1} f_{0}'(s) &= \infty,
\end{align*}$$

(3.124)

while $f_{1} \in C_{b}^{1}(\mathbb{R})$ is a globally Lipschitz-bounded function such that $f_{1}' \geq -\theta_{c}$. In applications, very often $f_{1}(s) = -\theta_{c}s$ and $f_{0}(s) = \theta \ln \left(\frac{1+|s|}{1-|s|}\right)$, for some $\theta_{c} > \theta > 0$.

We can prove the following.

Theorem 3.19. Let $f$ satisfy the above assumptions and assume (HA-B) holds for some operators $A,B$ with $q_{B} > 2$. Furthermore we assume that $B$ is given via (2.1) by means of the Dirichlet space $(\mathcal{E}_{B}, V_{B})$ introduced in Example 2.12. For every initial datum $u_{0} \in V_{B}$ with $F(u_{0}) \in L^{1}(X)$ and $(u_{0})_{X} \in (-1,1)$, there exists a unique (global) weak solution to problem (3.1)-(3.3). In particular, the weak solution $u$ satisfies $u(0) = u_{0}$ in $X$, such that

$$u \in L^{\infty}(0,T; V_{B}), \quad \partial_{t} u \in L^{2}(0,T; V_{A}^{*}), \quad \mu \in L^{2}(0,T; V_{B}),$$

$$F(u) \in L^{\infty}(0,T; L^{1}(X)), \quad f(u) \in L^{2}(0,T; L^{2}(X)),$$

and, for every $v \in V_{B}, \omega \in V_{A}$, a.e. $t \in (0,T)$ we have

$$\langle \partial_{t}u(t), \omega \rangle + \mathcal{E}_{A}(\mu(t), \omega) = 0,$$

(3.125)

$$\mathcal{E}_{B}(u(t), v) + \langle f(u(t)), v \rangle = \langle \mu(t), v \rangle.$$

(3.126)

Proof. The existence is based on the same construction exploited in the proof of Theorem 3.5 and Theorem 3.7. Indeed, we can construct smooth versions $f_{1,\epsilon} \in C^{2}(\mathbb{R})$ and a smooth monotone sequence $f_{0,\epsilon} \in C^{2}(\mathbb{R})$, approximating the singular part of the potential on compact subintervals $[-1+\epsilon, 1-\epsilon]$ of $(-1,1)$, such that $f_{\epsilon}(0) = 0$ and $f_{\epsilon} = f_{1,\epsilon} + f_{0,\epsilon}$ obeys the conditions (HF-1)-(HF-3), with a sufficiently large polynomial growth outside the interval $[-1+\epsilon, 1-\epsilon]$ as $\epsilon \to 0^{+}$ (see, for instance, [4] and the references therein). In particular, we have

$$|f_{0,\epsilon}(s)| \leq |f_{0}(s)|, \quad |F_{0,\epsilon}(s)| \leq |F_{0}(s)|, \quad \text{for any } s \in (-1,1),$$

(3.127)

and

$$\lim_{\epsilon \to 0^{+}} f_{0,\epsilon}(s) = f_{0}(s), \quad \lim_{\epsilon \to 0^{+}} F_{0,\epsilon}(s) = F_{0}(s), \quad s \in (-1,1).$$

(3.128)

Thus, the existence of a sequence of strong (smooth) solutions $u_{\epsilon, \alpha}$ to this approximated problem follows exactly as in the proof of Theorem 3.5 without any essential modifications. It remains to deduce some uniform in $(\epsilon, \alpha)$ a priori estimates for these solutions.

We first start by noting that the energy equality (3.19) together with the uniform bounds (3.21)-(3.24) hold on account of the fact that $F_{\epsilon}(s) = \int_{0}^{s} f_{\epsilon}(\zeta)d\zeta \geq -C_{F}$, uniformly in $s \in \mathbb{R}$, since $F_{0,\epsilon}(s) = \int_{0}^{s} f_{0,\epsilon}(\zeta)d\zeta$ is a convex potential on
\( \mathbb{R} \). We require additional uniform estimates with respect to \((\epsilon, \sigma)\). In what follows, we will exploit the following identity
\[
(\mu_{\epsilon, \alpha}, v) = (Bu_{\epsilon, \alpha}, v) + (f_{\epsilon \kappa}(u_{\epsilon, \alpha}), v) + (f_{1 \epsilon}(u_{\epsilon, \alpha}), v) + \alpha \left( \partial_t u_{\epsilon, \alpha}, v \right), \tag{3.129}
\]
which holds for all \( v \in V_B \).

**Step 1. (The \( L^2(0, T; V_A^c) \)-bound for \( \mu_{\epsilon, \alpha} \)).** Let \( \langle u_0 \rangle_X = M_0 \in (-1, 1) \), where \( u_0 \in V_B \), and recall that \( \langle u_{\epsilon, \alpha}(t) \rangle_X = \langle u_0 \rangle_X \) for all \( t \geq 0 \). Testing equation (3.125) with \( v = 1/m(X) \), we deduce
\[
(\mu_{\epsilon, \alpha}, 1/m(X)) = \left( B \partial_t u_{\epsilon, \alpha}, 1/m(X) \right) + \left( f_{\epsilon \kappa}(u_{\epsilon, \alpha}), 1/m(X) \right) + \left( f_{1 \epsilon}(u_{\epsilon, \alpha}), 1/m(X) \right) + \alpha \left( \partial_t u_{\epsilon, \alpha}, 1/m(X) \right). \tag{3.130}
\]

Since \( \mathcal{E}_A(\mu_{\epsilon, \alpha}; \mu_{\epsilon, \alpha}) \in L^1(0, T) \), uniformly by (3.24), in order to prove that \( \mu_{\epsilon, \alpha} \in L^2(0, T; V_A^c) \), it suffices to show that \( f_{\epsilon \kappa} \in L^2(0, T; L^1(X)) \), uniformly in \((\epsilon, \sigma)\) (indeed, the later bound is already valid for the smooth part \( f_1 \), by the assumption that \( f_1 \in C^1 \) and by the uniform bound (3.21)). We proceed as follows. We test the equation (3.125) with \( \omega = A^{-1} \left( \partial_t u_{\epsilon, \alpha} \right) \) to deduce the identity
\[
\| \partial_t u_{\epsilon, \alpha} \|^2_{V_A^c} + \left( A^{1/2} (\mu_{\epsilon, \alpha} - \langle u_{\epsilon, \alpha} \rangle_X), A^{-1/2} (\partial_t u_{\epsilon, \alpha}) \right) = 0. \tag{3.131}
\]
Then a standard energy estimate (as in Section 4, owing to Hölder, Poincare and Young’s inequalities, allows one to derive
\[
|\partial_t u_{\epsilon, \alpha}|^2_{V_A^c} \leq \delta \| \partial_t u_{\epsilon, \alpha} \|^2_{V_A^c} + C_3 \mathcal{E}_A(\mu_{\epsilon, \alpha}; \mu_{\epsilon, \alpha}),
\]
for every \( \delta > 0 \). Henceforth, it follows that
\[
\partial_t u_{\epsilon, \alpha} \in L^2(0, T; V_A^c) \text{ uniformly in } (\epsilon, \alpha), \tag{3.132}
\]
exactly as in (3.41). Next, we can test (3.129) by \( \pi_{\epsilon, \alpha} = u_{\epsilon, \alpha} - M_0 \) to obtain
\[
\left( \mu_{\epsilon, \alpha} - \langle u_{\epsilon, \alpha} \rangle_X, \pi_{\epsilon, \alpha} \right) = (Bu_{\epsilon, \alpha}, \pi_{\epsilon, \alpha}) + (f_{\epsilon \kappa}(u_{\epsilon, \alpha}), \pi_{\epsilon, \alpha}) + (f_{1 \epsilon}(u_{\epsilon, \alpha}), \pi_{\epsilon, \alpha}) + \alpha \left( \partial_t u_{\epsilon, \alpha}, \pi_{\epsilon, \alpha} \right). \tag{3.133}
\]
Owing now to the inequalities
\[
\begin{align*}
\int f_{\epsilon \kappa} (s) (s - m) &\geq \gamma |F_{\epsilon \kappa} (s)| - C_f, \\
\int f_{\epsilon \kappa} (s) (s - m) &\geq \gamma |f_{\epsilon \kappa} (s)| - C_f,
\end{align*}
\]
which hold for all \( s \in \mathbb{R} \), for some positive constants \( \gamma, C_f > 0 \) independent of \((\epsilon, \alpha) \in (0, 1]^2 \) (see [4] and the references therein), and the uniform estimates (3.21)-(3.24), (3.13), (3.131), we easily arrive at the uniform bound \( f_{\epsilon \kappa} \in L^1(0, T; L^1(X)) \) owing to the fact that \( 1 \in D(B) \). We note that, on the basis of (3.130), we also find that \( \langle \mu_{\epsilon, \alpha} \rangle_X \in L^1(0, T) \) uniformly in \((\epsilon, \alpha) \). Furthermore, we deduce from (3.132) a bound of the form
\[
\| \pi_{\epsilon, \alpha} \|^2_{V_B} + \gamma \| f_{\epsilon \kappa}(u_{\epsilon, \alpha}) \|_{L^1(X)} \leq C + \alpha^{1/2} \| \partial_t u_{\epsilon, \alpha} \|_{L^2(X)} \left( \alpha^{1/2} \| u_{\epsilon, \alpha} \|_{L^2(X)} \right) + C \mathcal{E}_A(\mu_{\epsilon, \alpha}; \mu_{\epsilon, \alpha})^{1/2} \| \pi_{\epsilon, \alpha} \|_{L^2(X)},
\]
for some constant \( C > 0 \) independent of \((\epsilon, \alpha) \). Here, we recall that \( \alpha \leq 1 \). Squaring both sides of this inequality and integrating over \((0, T)\), we immediately deduce that
\[
f_{\epsilon \kappa}(u_{\epsilon, \alpha}) \in L^2(0, T; L^1(X)) \text{ uniformly in } (\epsilon, \alpha). \tag{3.134}
\]
This allows to improve the regularity of $\mu_{\epsilon, \alpha}$ to $(\mu_{\epsilon, \alpha})_X \in L^2(0, T)$, exploiting (3.130) and estimates (3.21) once again. Finally, by the Poincaré’s inequality (2.24) we may conclude

$$\mu_{\epsilon, \alpha} \in L^2(0, T; V_A) \text{ uniformly in } (\epsilon, \alpha),$$

exactly as in the regular potential case (see (3.42)).

**Step 2. (The $L^2(0, T; L^2(X))$-bound for $f_{0\epsilon}$).** This time we take $v = f_{0\epsilon}(u_{\epsilon, \alpha})$ in (3.129), which is clearly an admissible test function on account of the regularity of $f_{0\epsilon}$ and $u_{\epsilon, \alpha}$. This multiplication yields

$$\left(\mu_{\epsilon, \alpha}, f_{0\epsilon}(u_{\epsilon, \alpha})\right) = (Bu_{\epsilon, \alpha}, f_{0\epsilon}(u_{\epsilon, \alpha})) + (f_{0\epsilon}(u_{\epsilon, \alpha}), f_{0\epsilon}(u_{\epsilon, \alpha})) + (f_{1\epsilon}(u_{\epsilon, \alpha}), f_{0\epsilon}(u_{\epsilon, \alpha})) + \alpha (\partial_t u_{\epsilon, \alpha}, f_{0\epsilon}(u_{\epsilon, \alpha})),$$

which together with the fact that $f'_{0\epsilon} \geq 0$ (note that, by assumption of the theorem, $(Bu_{\epsilon, \alpha}, f_{0\epsilon}(u_{\epsilon, \alpha})) = E_B(u_{\epsilon, \alpha}, f_{0\epsilon}(u_{\epsilon, \alpha})) \geq 0$) and elementary estimates give

$$\left\| f_{0\epsilon}(u_{\epsilon, \alpha}) \right\|_{L^2(X)}^2 \leq \eta \left\| f_{0\epsilon}(u_{\epsilon, \alpha}) \right\|_{L^2(X)}^2 + C_\eta \left( \left\| f_{1\epsilon}(u_{\epsilon, \alpha}) \right\|_{L^2(X)}^2 + \left\| \mu_{\epsilon, \alpha} \right\|_{L^2(X)}^2 + \alpha \left\| \partial_t u_{\epsilon, \alpha} \right\|_{L^2(X)}^2 \right),$$

for every $\eta > 0$. Choosing a sufficiently small $\eta < 1$, we further deduce by token of the uniform bounds (3.21), (3.24) together with (3.135) that

$$f_{0\epsilon}(u_{\epsilon, \alpha}) \in L^2(0, T; L^2(X)) \text{ uniformly in } (\epsilon, \alpha).$$

**Step 3. (Final argument).** With the bound (3.137) at our disposal, we can now argue verbatim by elliptic regularity as in the proof of Theorem 3.5 to deduce the uniform bound $u_{\epsilon, \alpha} \in L^2(0, T; D(B))$. This bound is also sufficient to pass to the limit strongly in the sequence $u_{\epsilon, \alpha}$ exactly as in the proof of Theorem 3.7. In particular, for the sequence $f_{\epsilon} = f_{0\epsilon} + f_{1\epsilon}$, instead we can show

$$\left\{ f_{1\epsilon}(u_{\epsilon, \alpha}) \rightarrow f_1(u) \text{ strongly in } L^2(0, T; L^2(X)) \right\},$$

$$f_{0\epsilon}(u_{\epsilon, \alpha}) \rightarrow \xi^* \text{ strongly in } L^2(0, T; L^2(X)),$$

as $(\epsilon, \alpha) \rightarrow (0, 0)$. The identification $\xi^* = f_0(u)$ a.e. in $X \times (0, T)$ can be easily proven by a standard monotonicity argument, exploiting as usual the identity (3.129) and the usual strong convergence properties. Thus, the existence of a weak solution can be deduced with only some minor inessential modifications. The uniqueness of the weak solution follows exactly as in the proof of Theorem 3.8 by noting that when $f_0(u) \in L^2(0, T; L^2(X))$ one no longer requires the regularity assumption $u \in L^\infty(0, T; L^p(X))$. This concludes the proof of the theorem.

It may be possible to prove additional regularity results for our doubly nonlocal problem in the case when $F$ is a singular logarithmic potential, but we feel that this case would deserve a separate and complete investigation. Our goal in Theorem 3.19 was merely to show that our framework can be easily extended to deal with this important case as well. In particular, our result in Theorem 3.19 stands for a clear generalization of some well-posedness results derived in [11] where specifically, $A = -\Delta_{\Omega, N}$, $B$ is the self-adjoint operator given by Example 2.12 (cf. Subsection 2.2).

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DOUBLY NONLOCAL CAHN-HILLIARD EQUATIONS

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