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Stability analysis and stabilization synthesis of singularly perturbed switched systems: An average dwell time approach

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Abstract: Stability analysis and stabilization synthesis problems of the singularly perturbed switched systems (SPSSs) are investigated in this paper. First, stability of the SPSS with stable subsystems is discussed by using an average dwell time approach with a full-order piecewise Lyapunov function. Then, the result is extended to the situation that not all the subsystems are Hurwitz stable. Furthermore, state feedback controllers are designed when all subsystems are stabilizable, and the design method is extended to the situation that not all subsystems are stabilizable. Finally, two examples are given to demonstrate the validity and effectiveness of the proposed methods.

Subjects: Automation Control; Control Engineering; Dynamical Control Systems

Keywords: singularly perturbed switched system; average dwell time; stability analysis; stabilization synthesis

1. Introduction

In the process of engineering system modeling, the existence of small parasitic parameters leads to the interaction of slow and fast phenomena in control systems, such as power systems, electro-mechanical systems and chemical reactions, which often exhibit multiple time scale behavior. High dimensionality and ill-conditioned numerical problems caused by these small parameters can not be avoided for the traditional control methods. In order to overcome these defections and improve control accuracy, singularly perturbed method is developed by constructing a state-space model with a small parameter $\varepsilon$, which determines the degree of separation between “fast” and “slow” dynamics of the system. Then, the system can be established as a singular perturbed system (SPS) (see Kokotovic, Khalil, & Reilly, 1986). In the past two decades, SPS has been studied by many researchers (e.g. Assawinchaichote, Nguang, & Shi, 2004; Del Vecchio & Slotine, 2013; Huang, Cai, & Zou, 2008;...
Park, 2013). The stability issue of SPS is more complex than common system because of the stability bound, which is referred as determining a parameter $\epsilon_0$ such that the SPS is stable for all $\epsilon \in (0, \epsilon_0]$. Stability bound estimation has attracted much attention for its importance in stability analysis of SPS (e.g. Cao & Schwartz, 2004; Chiou, Kung, & Li, 1999; Yang, Ma, Ma, & Wang, 2016; Yang, Sun, & Ma, 2013; Yang & Zhou, 2016).

Switched system, which is composed of a finite family of continuous-time or discrete-time subsystems with an appropriate switching rule, is an important type of hybrid systems. Each of the subsystems may have its own dynamic behavior during its activated period. In view of the importance of the switch control in engineering, the study of switched systems has attracted many research efforts (e.g. Branicky, 1998; Hespanha & Morse, 1999; Liberzon, 2003; Liberzon & Morse, 1999; Zhai, Hu, Yasuda, & Michel, 2001). In the review paper, see Liberzon and Morse (1999), the switch control focuses on the following three basic problems. (i) Find conditions such that the switched system is asymptotically stable for any switching signal. (ii) Identify classes of switching signals for which the switched system is asymptotically stable. (iii) Construct a switching signal that makes the switched system asymptotically stable. In the study of switched systems, several useful approaches have been applied, such as common Lyapunov function (CLF) approach (see Molchanov & Pyatnitskiy, 1989), switched quadratic Lyapunov function approach in Daafouz, Riedinger, and Jung (2002), multiple Lyapunov function approach, see Lin and Antsaklis (2009) and references therein, dwell time (average dwell time (ADT)) approach in Hespanha and Morse (1999). Meanwhile, the multiple Lyapunov function approaches and ADT approaches had been proved to be effective and flexible tools in the analysis and synthesis of switched systems, and will be used in this paper.

Recently, the study on singularly perturbed switched systems (SPSSs) has achieved some efforts for their applications in engineering. In Malloci, Daafouz, Iung, Bonidal, and Szczepanski (2009), when taking the physical relations linking the stands, the effects of the switchings on the system dynamics and the fact that the system treats products with different characteristics into account, the hot strip mill is formulated as a SPSS. However, for the hybrid property and multi-time-scale performance of the SPSSs, it is rare to find the research results. Considering the first problem proposed in Liberzon and Morse (1999), it has been proved that the stability of fast and slow switched subsystems can not guarantee the stability of the original SPSS under arbitrary switching, while sufficient conditions of coupling constrains were proposed in continuous-time and discrete-time, respectively (see Malloci, Daafouz, & Iung, 2009, 2010). Based on these results, a hot strip mill system was constructed as a SPSS model and a robust $H_2$ controller was designed in Malloci, Daafouz, Iung, and Bonidal (2010), controller design and stability analysis for SPSS with actuator saturation were considered in Ma, Wang, Zhou, and Yang (2016). Then, some researches related to the second question in Liberzon and Morse (1999) were achieved. A special class of SPSSs with time delay was considered under the assumption that the fast switched subsystems were stable, and stability analysis was derived with a dwell time approach. Above results are based on the decomposition of the SPSSs. The stabilization problem of SPSSs with actuator saturation was discussed by a dwell time approach without decomposing the system in Lian and Wang (2015). But all existing researches can not be utilized for non-standard SPSSs. To overcome this problem, in this paper, non-decomposition method is used by constructing a full order $\epsilon-$ dependent piecewise Lyapunov function such that proposed methods are available for both standard and non-standard SPSSs. Furthermore, stability bound of the SPSSs is considered.

In this paper, it is focused on the stability analysis and synthesis of SPSSs. The main contributions of this paper are as following: (i) Stability analysis of the SPSSs is discussed with an ADT approach when all the subsystems are supposed to be Hurwitz stable. (ii) Stability of SPSSs with stable and unstable subsystems are analyzed with ADT approach and the ration of total activation time of stable and unstable subsystems. (iii) Stabilization problem is discussed. State feedback controller designed method is proposed when all the subsystems are supposed to be stabilizable. (iv) Stabilization synthesis of the SPSSs with stabilizable and unstabilizable sunsystems is proposed.
The rest part of this paper is organized as follows. In Section 2, system description and preliminaries are presented. In Section 3, stability analysis is discussed, while stabilization of SPSSs is achieved in Section 4. Examples are given in Sections and concludes the whole paper.

**Notation:** Standard notations will be used. Throughout this paper, for real symmetric matrices $X$ and $Y$, the notation $X \leq Y$ (respectively, $X < Y$) means that the matrix $X - Y$ is negative semi-definite (respectively, negative definite). $I_n$ is the $n \times n$ identity matrix of order $n$. $\mathbb{R}_+$ and $\mathbb{R}_{n \times m}$ denote, respectively, the $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript $T$ stands for matrix transpose. $\lambda_n(Q)$ and $\lambda_n(Q)$ denote the maximal and minimal eigenvalues of a symmetric matrix $Q$, respectively.

### 2. Problem formulation and preliminaries

Consider the following linear time-invariant SPSS:

$$E(\epsilon)x(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$$

with $E(\epsilon) = \left[ \begin{array}{cc} 0 & \epsilon I_n \\ 0 & 0 \end{array} \right]$, $\epsilon$ represents the singular perturbation parameter, which is a small positive scalar. $x(t) = \left[ \begin{array}{c} x_1^T(t) \\ x_2^T(t) \end{array} \right] \in \mathbb{R}^n$ is the state vector with $x_1^T(t) \in \mathbb{R}^n$ and $x_2^T(t) \in \mathbb{R}^n$. $u(t) \in \mathbb{R}^m$ is the control input. $\sigma(t)$ donates a piecewise constant function of time, which is called switching signal, which takes values in a finite set $\mathcal{I} = \{1, 2, \ldots, N\}$, and $N \geq 1$ is the number of individual subsystems. For switching instants $t_i$, $i \in \mathcal{I}$, satisfying $0 < t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$, $\sigma(t)$ is continuous. When $t \in [t_i, t_{i+1})$, it means the $\sigma(t_i)$th subsystem is active, thus, $A_{\sigma(t_i)} = A_i \in \mathbb{R}^{n \times n}$, $B_{\sigma(t_i)} = B_i \in \mathbb{R}^{n \times m}$ are constant matrices of the $i$th subsystem.

Before proceeding, the following definitions and lemmas will be recalled.

**Definition 1** (Hespanha & Morse, 1999) For any switching signal $\sigma(t)$ and any $t_2 > t_1 \geq 0$, let $N_{\sigma(t)}$, denote the number of discontinuities $\sigma(t)$ in the interval $(t_1, t_2)$. We say that $\sigma(t)$ has the ADT property, if

$$N_{\sigma(t)} \leq N_0 + \frac{t_2 - t_1}{r_\sigma}, \quad N_0 \geq 0, \quad r_\sigma > 0$$

holds, where $N_0$ and $r_\sigma$ are called chatter bound and ADT, respectively. As commonly used in most literatures, we choose $N_0 = 0$.

**Remark 1** The parameter $N_0$ strictly limits the upper bound of switching times within an interval of length less than $r_\sigma$. For example, if $N_0 = 1$, then Definition 1 implies that (it cannot switch twice on any interval of length smaller than $r_\sigma$. Switching signals with this property are exactly the switching signals with dwell time $r_\sigma$. Note also that $N_0 = 0$ corresponds to the case of no switching, since (it cannot switch at all on any interval of length smaller than $r_\sigma$. The constant $N_0$ affects the overshoot bound for Lyapunov stability but otherwise does not change stability properties of the switched system, so $N_0$ is usually chosen as 0 for simplicity (see Liberzon, 2003; Zhang, Zhu, Shi, & Lu, 2016).

**Remark 2** Study results on time-controlled switched systems have been extended by recent works, such as minimum dwell time used in Briat (2014), stage mode-dependent dwell time proposed in Zhang, Zhuang, and Brozatz (2016) or the persistent dwell time (see Han, Ge, & Lee, 2010; Zhang, Zhuang, Shi, & Zhu, 2015). Therefore, these methods will be utilized in our future works on the SPSSs.

**Definition 2** (Liberzon, 2003) The equilibrium $x = 0$ of the switched system is globally uniformly exponentially stable (GUES) under certain switching signal $\sigma(t)$, if, in the continuous-time case, for any initial condition $x(t_0)$ (or $x(k_0)$ in the discrete-time case), there exist constants $M > 0, \alpha > 0, (0 < \beta < 1)$ such that the solution of the system satisfies $\|x(t)\| \leq Me^{-\alpha t-\epsilon} \|x(t_0)\|, \forall t \leq t_0$ with $\|x(k)\| \leq M \beta^{k-k_0} \|x(k_0)\|, \forall k \leq k_0$. 


Lemma 2.1 (Yang & Zhang, 2009) For a positive scalar \( \epsilon_0 \) and the symmetric matrices \( S_1 \) and \( S_2 \) of compatible dimensions, if the following inequalities hold,

\[
S_1 > 0, \\
S_1 + \epsilon_0 S_2 > 0,
\]

then

\[
S_1 + \epsilon S_2 > 0, \quad \forall \epsilon \in (0, \epsilon_0). 
\]

Lemma 2.2 (Yang & Zhang, 2009) For a positive scalar \( \epsilon_0 \) and matrices \( W_1 = W_1^T \), \( W_2 = W_2^T \) and \( W_3 \) of compatible dimensions, if the following inequalities hold,

\[
W_1 > 0, \\
\begin{bmatrix}
W_1 & \epsilon_0 W_3^T \\
\epsilon_0 W_3 & \epsilon_0 W_2
\end{bmatrix} > 0,
\]

then

\[
E(\epsilon) W(\epsilon) = W'(\epsilon) E(\epsilon), \quad \forall \epsilon \in (0, \epsilon_0),
\]

where

\[
W(\epsilon) = \begin{bmatrix}
W_1 & \epsilon W_3^T \\
\epsilon W_3 & \epsilon W_2
\end{bmatrix}.
\]

3. Stability analysis

In this section, SPSSs in form of (1) will be considered, let \( u(t) = 0 \). Stability of two classes of the autonomous SPSSs will be analyzed: (i) all the subsystems are Hurwitz stable; (ii) not all the subsystems are Hurwitz stable.

The achieved results are presented as following.

3.1. Stability with Hurwitz stable subsystems

In this subsection, all the subsystems are supposed to be Hurwitz stable.

Theorem 3.1 Given positive scalars \( \epsilon_0, \lambda > 0 \), \( \mu \geq 1 \), if there exist matrices \( W_{ij} = W_{ij}^T \) and \( W_{i3} \) of compatible dimensions, for any \( i, j \in I, i \neq j \) and \( \epsilon, \epsilon_0 \in (0, \epsilon_0) \) such that

\[
W_{ij} > 0, \\
\begin{bmatrix}
W_{ij} & \epsilon_0 W_{i3}^T \\
\epsilon_0 W_{i3} & \epsilon_0 W_{ij}
\end{bmatrix} > 0, \\
W_{ij} \leq \mu W_{ij}, \\
\begin{bmatrix}
W_{ij} & \epsilon_0 W_{i3}^T \\
\epsilon_0 W_{i3} & \epsilon_0 W_{ij}
\end{bmatrix} \leq \mu \begin{bmatrix}
W_{ij} & \epsilon_0 W_{i3}^T \\
\epsilon_0 W_{i3} & \epsilon_0 W_{ij}
\end{bmatrix},
\]

\[
A_i W_i(0) + W_i'(0) A_i^T < -\lambda W_i(0) E(0),
\]

\[
A_i W_i(\epsilon_0) + W_i'(\epsilon_0) A_i^T < -\lambda W_i(\epsilon_0) E(\epsilon_0)
\]
hold, where \( W_{i}(\varepsilon) = \begin{bmatrix} W_{i1} & \varepsilon W_{i3} \\ W_{i3} & W_{i2} \end{bmatrix} \).

Then, system (1) is exponentially stable under the switching law with ADT

\[
\tau_{\alpha} \geq \frac{\ln \mu}{\lambda}.
\]  

(8)

**Proof**  From (2) and (3), according to Lemma 2.1, we can get

\[
\begin{bmatrix} W_{i1} & \varepsilon W_{i3} \\ \varepsilon W_{i3} & W_{i2} \end{bmatrix} > 0,
\]

(9)

which, according to Lemma 2.2, implies

\[ E(\varepsilon)W_{i}(\varepsilon) = W_{i}'(\varepsilon)E(\varepsilon) > 0. \]

(10)

Define \( P_{j}(\varepsilon) = W_{j}^{-1}(\varepsilon) \), pre- and post-multiplying (10) by \( W_{j}^{-1}(\varepsilon) \) and its transpose, we have

\[ E(\varepsilon)P_{j}(\varepsilon) = P_{j}'(\varepsilon)E(\varepsilon) > 0. \]

(11)

From (4) and (5), it has

\[
\begin{bmatrix} W_{j1} & \varepsilon W_{j3} \\ \varepsilon W_{j3} & W_{j2} \end{bmatrix} \leq \mu \begin{bmatrix} W_{i1} & \varepsilon W_{i3} \\ \varepsilon W_{i3} & W_{i2} \end{bmatrix},
\]

(12)

which implies

\[ E(\varepsilon)W_{j}(\varepsilon) \leq \mu E(\varepsilon)W_{i}(\varepsilon). \]

(13)

Using Shur complement to (13), we can get

\[
\begin{bmatrix} -W_{j}^{-1}(\varepsilon)E^{-1}(\varepsilon) & I \\ I & -\mu E(\varepsilon)W_{i}(\varepsilon) \end{bmatrix} \leq 0.
\]

(14)

Pre- and post-multiplying (13) by \( \text{diag}(E(\varepsilon), I) \) and its transpose, it can get

\[
\begin{bmatrix} -E(\varepsilon)W_{j}^{-1}(\varepsilon) & E(\varepsilon) \\ E(\varepsilon) & -\mu E(\varepsilon)W_{i}(\varepsilon) \end{bmatrix} \leq 0,
\]

(15)

which is equivalent to

\[
\begin{bmatrix} -E(\varepsilon)P_{j}(\varepsilon) & E(\varepsilon) \\ E(\varepsilon) & -\mu E(\varepsilon)P_{j}(\varepsilon) \end{bmatrix} \leq 0.
\]

(16)

By Shur complement, we can get

\[-E(\varepsilon)P_{j}(\varepsilon) + \frac{1}{\mu}E(\varepsilon)P_{j}(\varepsilon) \leq 0,
\]

(17)

which implies

\[ \mu E(\varepsilon)P_{j}(\varepsilon) \geq E(\varepsilon)P_{j}(\varepsilon). \]

(18)

From (6) and (7), we have

\[ A_{i}W_{j}(\varepsilon) + W_{i}'(\varepsilon)A_{j}^{T} < -\lambda W_{i}'(\varepsilon)E(\varepsilon). \]

(19)
Pre- and post-multiplying (19) by $W^{-T}_i(\varepsilon)$ and its transpose, respectively, it has

$$P_i'(\varepsilon)A_i + A_i'P_i(\varepsilon) < -\lambda E(\varepsilon)P_i(\varepsilon).$$

(20)

Define an $\varepsilon$-dependent piecewise Lyapunov function

$$V(t) = V_{mt}(x(t)) = x^T(t)E(\varepsilon)P_{mt}(\varepsilon)x,$$

(21)

where $E(\varepsilon)P_{mt}(\varepsilon)$ is switched among $E(\varepsilon)P_i(\varepsilon), i \in I$ in accordance with the piecewise constant switching signal $\sigma(t)$.

Computing the derivative of $V(x(t))$ with respect to $t$ along the trajectories, we have

$$\dot{V}(x(t)) = 2x^T(t)E(\varepsilon)P_{i1}(\varepsilon)x(t)$$

$$= 2x^T(t)E(\varepsilon)P_i(\varepsilon)x(t)$$

$$= x^T(t)(P_i'(\varepsilon)A_i + A_i'P_i(\varepsilon))x(t).$$

According to (20), it has

$$\dot{V}(x(t)) \leq -\lambda V(x(t)).$$

(22)

From (18), (21) and (22), it can be obtained that the piecewise Lyapunov function has following properties

(1) Each $V(x(t)) = x^T(t)E(\varepsilon)P_{i1}(\varepsilon)x(t)$ is continuous and its derivative along the corresponding sub-system satisfies (22).

(2) There exist constant scalars $a > 0$, $b > 0$, such that

$$a ||x(t)|| \leq V(x(t)) \leq b ||x(t)||$$

with $a = \inf \lambda_{\infty}(E(\varepsilon)P_i(\varepsilon))$ and $b = \sup \lambda_{\infty}(E(\varepsilon)P_i(\varepsilon))$.

(3) There exists a constant scalar $\mu \geq 1$ such that (18) holds.

Using the differential inequality (22), we can obtain that

$$V(t) = V(x(t_0))e^{-\lambda t} \leq \mu V(x(t_0))e^{-\lambda t} \leq \mu V(x(t_0))e^{-\lambda t_{t-1}}e^{-\lambda t}$$

$$\leq \mu V(x(t_{t-1}))e^{-\lambda t_{t-1}}e^{-\lambda t}$$

$$\leq \vdots$$

$$\leq \mu V(x(0))e^{-\lambda t}$$

$$= \mu^{N_{mt}(0)}V(x(0))e^{-\lambda t}.$$ (24)

Now we consider the following ADT scheme

$$N_{mt}(0, t) \leq \frac{t}{\tau}, \tau = \frac{\ln \mu}{\lambda}.$$ (25)

Then, we can get that there always exists a positive scalar $\lambda' < \lambda$

$$V(t) \leq \mu^{N_{mt}(0)}V(x(0))e^{-\lambda' t} \leq V(x(0))e^{-(\lambda' - \lambda'\lambda)}$$

with $\lambda - \lambda' > 0$.

Then, according to (23), it follows that
\( a\|x(t)\|^2 \leq V(t) \leq e^{-\langle \beta - \alpha \rangle t} V(0), \)
\( \|x(t)\|^2 \leq \frac{a}{b} e^{-\langle \beta - \alpha \rangle t} \|x(t_0)\|^2, \)
\( \|x(t)\|^2 \leq \sqrt{\frac{a}{b}} e^{-\langle \beta - \alpha \rangle t} \|x(t_0)\|. \)

From Definition 2, the exponential stability of (1) can be demonstrated. \( \square \)

**Remark 3** Stability bound \( \epsilon_0 \) characterizes the robustness of system stability with respect to the perturbation parameter \( \epsilon \). When the singular perturbation parameter \( \epsilon \) is not exactly known, for example in some control systems \( \epsilon \) does not have a specific physic meaning, or the value of \( \epsilon \) is not precisely known, for example, in an electro-mechanical system, the value of \( \epsilon \) which denotes capacitance or inductance will drift, the stability of the system may not be guaranteed. So it is important to take the stability bound of SPSSs into consideration. The stability bound \( \epsilon_0 \) is given according to prior information, however, the estimation of \( \epsilon_0 \) can be obtained by a bisectional search algorithm (see Yang et al., 2016).

**Remark 4** Taking the singular perturbation property of the SPSSs into consideration, an \( \epsilon \)-dependent piecewise Lyapunov function is constructed, it implies that the ADT also takes \( \epsilon \) into consideration. The ADT is computed from the decay rate of Lyapunov function of the activated subsystem and the jump rate of Lyapunov functions of the adjacent two subsystems at the switch instant, as a result, the total energy of the SPSS keeps reducing such that the switched system is stabilized by the chosen ADT.

### 3.2. Stability with Hurwitz stable and unstable subsystems

In this subsection, SPSSs with both Hurwitz stable and unstable subsystems are taken onto consideration. Without loss of generality we suppose that the subsystems \( A_1, A_2, \ldots, A_p \) are Hurwitz unstable and the remaining subsystems \( A_{r+1}, A_{r+2}, \ldots, A_N \) are Hurwitz stable. Then, the stability analysis is given as following.

**Theorem 3.2** Given positive scalars \( a, \alpha, r > 0, \beta > 0, \delta < \alpha, \mu > 1 \), if there exist matrices \( W_{i,1} = W_{i,1}^T \)
\( W_{i,2} = W_{i,2}^T \) and \( W_{i,3} = W_{i,3}^T \) of compatible dimensions, for any \( i, j \in I, i \neq j \) and \( \epsilon \in (0, \epsilon_0) \) such that inequalities (2), (3), (4), (5) are satisfied, and

\[
A_i W_i(0) + W_i^T(0) A_i^T < -\alpha W_i^T(0) E(0), \quad i > r, 
\]

\[
A_i W_i(\epsilon_0) + W_i^T(\epsilon_0) A_i^T < -\alpha W_i^T(\epsilon_0) E(\epsilon_0), \quad i > r, 
\]

\[
A_i W_i(0) + W_i^T(0) A_i^T < \beta W_i^T(0) E(0), \quad 1 \leq i \leq r, 
\]

\[
A_i W_i(\epsilon_0) + W_i^T(\epsilon_0) A_i^T < \beta W_i^T(\epsilon_0) E(\epsilon_0), \quad 1 \leq i \leq r 
\]

hold, where \( W_i(\epsilon) = \begin{bmatrix} W_{i,1} & \epsilon W_{i,2} \\ W_{i,3} & W_{i,2} \end{bmatrix} \). Then, system (1) is exponentially stable under the switching law with ADT

\[
\tau_0 \geq \tau_0^* = \frac{\ln \mu}{\alpha}, 
\]

and the total activation time of unstable subsystems and stable subsystems are defined as \( T^+ \) and \( T^- \), respectively with

\[
\frac{T^+}{T^-} \geq \frac{\beta + \delta}{\alpha - \delta}. 
\]
Proof Similar to the proof of Theorem 3.1, we can get

\[ E(\varepsilon)P_i(\varepsilon) = P_i^T(\varepsilon)E(\varepsilon) > 0 \]  \hspace{1cm} (32)

with \( P_i(\varepsilon) = W_i^{-1}(\varepsilon) \) and

\[ \mu E(\varepsilon)P_j(\varepsilon) \geq E(\varepsilon)P_j(\varepsilon). \]  \hspace{1cm} (33)

Define an \( \varepsilon \)-dependent piecewise Lyapunov function

\[ V(t) = V_{m_t}(x(t)) = x^T E(\varepsilon)P_{m_t}(\varepsilon)x, \]  \hspace{1cm} (34)

where \( E(\varepsilon)P_{m_t}(\varepsilon) \) is switched among \( E(\varepsilon)P_i(\varepsilon), i = 1, 2, \ldots, N \) in accordance with the piecewise constant switching signal \( \sigma(t) \).

Computing the derivative of \( V_i(x(t)) \) with respect to \( t \) along the trajectories, we have

\[ \dot{V}_i(x(t)) = x^T(t) P_i^T(\varepsilon)A_i + A_i^T P_i(\varepsilon) x(t). \]  \hspace{1cm} (35)

For stable subsystems, for any \( t \in [t_i, t_{i+1}) \), we have

\[ \dot{V}_i(x(t)) = e^{-\alpha t - \varepsilon} V_i(x(t_i)). \]  \hspace{1cm} (36)

For unstable subsystems, for any \( t \in [t_i, t_{i+1}) \), we have

\[ \dot{V}_i(x(t)) = e^{\alpha t - \varepsilon} V_i(x(t_i)). \]  \hspace{1cm} (37)

The piecewise Lyapunov function has the following properties:

1. Each \( V_i(x(t)) = x^T(t) E(\varepsilon)P_i(\varepsilon)x(t) \) is continuous and its derivative satisfies

\[ V_i(x(t)) = \frac{\partial V_i}{\partial x} A_i x \leq \begin{cases} -\alpha V_i, & \pi > r, \\ \beta V_i, & 0 \leq i \leq r. \end{cases} \]  \hspace{1cm} (38)

2. There exist constant scalars \( a_2 > 0, a_1 > 0 \), and \( \alpha_2 \geq \alpha_1 \) such that

\[ a_1 \| x(t) \| \leq V_i(x(t)) \leq a_2 \| x(t) \| \]  \hspace{1cm} (39)

with \( a_1 = \inf \lambda_{m}(E(\varepsilon)P_i(\varepsilon)), \ a_2 = \sup \lambda_{m}(E(\varepsilon)P_j(\varepsilon)) \).

3. There exists a constant scalar \( \mu \geq 1 \) such that (18) holds. Then, according to (36) and (37), for any \( t > t_i \), it can get

which yields that

\[ V(t) \leq e^{-\alpha t - \varepsilon} V_i(x(t_i)), \]  \hspace{1cm} (40)

\[ V(t) \leq \mu e^{-\alpha t - \varepsilon} V_i(x(t_i)) \]

\[ \leq e^{-\alpha t - \varepsilon} \mu e^{\alpha t - \varepsilon} V_i(x(t_i)) \]

\[ \leq \mu e^{\alpha t - \varepsilon} V_i(x(0)) \]

\[ = \mu e^{\alpha t - \varepsilon} V_i(x(0)). \]  \hspace{1cm} (41)

According to (31), it has

\[ T^+ \beta - T^- \alpha \leq -\lambda T^- - \lambda T^+ \leq -\lambda t. \]  \hspace{1cm} (42)
From (30), we have

$$N(t) < \frac{\Delta t}{\ln \mu}. \quad (43)$$

Then, according to (41) and (42), it has

$$V(t) < \mu^{N(t)} e^{-\lambda t} V(x(0)). \quad (44)$$

According to (39) and the proof of (26), the exponentially stability of the switched system can be proved. \(\square\)

Remark 5  The main idea of ADT approach for SPSSs with Hurwitz stable and unstable subsystems is to keep the stable subsystems activated for a relatively long time, while keep the unstable subsystems activated for a relatively short time. Actually the stable subsystems play an important role in the stability of the whole SPSS, even if there is only one stable subsystem, the SPSS can be guaranteed stable with a well-designed ADT switching law. So controller can be designed to stabilize the stabilizable subsystems such that the whole SPSS can be stabilized with the designed ADT switching law, this will be shown in Section 4.

Remark 6  In Theorem 3.1, when \(\mu = 1\), the piecewise Lyapunov function will be a CLF, the ADT \(\ln \mu = 0\), which shows the performance of arbitrary switch. However, there does not exist CLF because of the existence of unstable subsystems in Theorem 3.2.

4. Stabilization of SPSSs

In this section, the stabilization problems of system (1) are taken into consideration. State feedback controllers will be designed.

4.1. Stabilization with stabilizable subsystems

In this subsection, an \(\epsilon\)-dependent stabilizing controller design method is proposed by an ADT approach under the assumption that all the subsystems are stabilizable.

**Theorem 4.1**  Given positive scalars \(\epsilon_0 > 0\), \(\mu \geq 1\), if there exist matrices \(W_{1j}, W_{2j}, W_{3j}\) and \(Y_j\) of compatible dimensions, for any \(i, j \in I, i \neq j\) and \(\epsilon \in (0, \epsilon_0]\) such that inequalities (2), (3), (4), (5) are satisfied, and

$$A_i W_i(0) + W_i^T(0) A_i^T + B_i Y_i^T + Y_i B_i^T < -\lambda W_i(0) E(0), \quad (45)$$

$$A_j W_j(\epsilon_0) + W_j^T(\epsilon_0) A_j^T + B_j Y_j^T + Y_j B_j^T < -\lambda W_j(\epsilon_0) E(\epsilon_0) \quad (46)$$

hold, where \(W_j(\epsilon) = \begin{bmatrix} W_{1j} & \epsilon W_{2j} \\ W_{3j} & W_{2j} \end{bmatrix} \).

Then, system (1) is exponentially stable under the switching law with ADT

$$\tau_\epsilon \geq \frac{\ln \mu}{\lambda}. \quad (47)$$

The \(\epsilon\)-dependent state feedback controller is designed as

$$K_j(\epsilon) = \frac{Y_j W_j^{-1}(\epsilon)}{2}. \quad (48)$$

**Proof**  From (45) and (46), \(\forall \epsilon \in (0, \epsilon_0]\), we have

$$A_i W_i(\epsilon) + W_i^T(\epsilon) A_i^T + B_i Y_i^T + Y_i B_i^T < -\lambda W_i(\epsilon) E(\epsilon). \quad (49)$$

Pre- and post-multiplying (49) by \(W_i^{-1}(\epsilon)\) and its transpose, respectively, it has

$$P_i(\epsilon) A_i + A_i^T P_i(\epsilon) + B_i K_i + K_i B_i^T P_i(\epsilon) < -\lambda E(\epsilon) P_i(\epsilon), \quad (50)$$

Page 9 of 17
where \(P_i(\epsilon) = W_i^{-1}(\epsilon)\).

Define an \(\epsilon\)-dependent piecewise Lyapunov function
\[
V_i(x(t)) = x^T(t)E(\epsilon)P_i(\epsilon)x(t).
\] (51)

Computing the derivative of \(V_i(x(t))\) with respect to \(t\) along the trajectories, we have
\[
\dot{V}_i(x(t)) = x^T(t)(P_i'(\epsilon)A_i + A_i^T P_i(\epsilon) + P_i'(\epsilon)B_iK_i + K_i^T B_i^T P_i(\epsilon))x(t).
\] (52)

According to (50) and (52), it has
\[
\dot{V}_i(x(t)) \leq -\lambda V_i(x(t)).
\] (53)

The following proof is similar to the proof of Theorem 1, the globally exponential stability of the SPSS can be guaranteed. \(\square\)

**Remark 7** The state feedback controller \(K_i(\epsilon) = Y_i W_i^{-1}(\epsilon)\) depends on the singular perturbation parameter \(\epsilon\) such that a higher control accuracy can be guaranteed. Furthermore, \(K_i(\epsilon)\) is well-defined for any \(\epsilon \in (0, \epsilon_0]\) and robust with respect to \(\epsilon\). When the singular perturbation parameter in practical systems changes within given stability bound, such as value of capacitance drift, the controller can still be valid. In addition, when \(\epsilon \to 0\), the controller can be reduced as \(\lim_{\epsilon \to 0} K_i(\epsilon) = Y_i W_i^{-1}(0)\), which is independent of \(\epsilon\).

### 4.2. Stabilization with stabilizable and unstabilizable subsystems

When the stabilization problem is considered, there may be subsystems that cannot be stabilized, for example, failure of the actuator or controller. Without loss of generality, we suppose that the subsystems \(A_1, A_2, \ldots, A_r\) are unstabilizable and the remaining subsystems \(A_{r+1}, A_{r+2}, \ldots, A_N\) are stabilizable. Thus, Theorem 3.2 can be extended as following.

**Theorem 4.2** Given positive scalars \(\varepsilon, \alpha > 0, \beta > 0, 0 < \lambda < \alpha, \mu > 1\), if there exist matrices \(W_{i1} = W_{i1}^T\), \(W_{i2} = W_{i2}^T\) with \(\varepsilon < \lambda < \alpha\) such that inequalities (2), (3), (4), (5) are satisfied, and

\[
A_i W_i(0) + W_i'(0) A_i^T + B_i Y_i^T + Y_i B_i^T < -\alpha W_i(0) E(0), \quad i > r, \tag{54}
\]

\[
A_i W_i(\varepsilon_0) + W_i'(\varepsilon_0) A_i^T + B_i Y_i^T + Y_i B_i^T < -\alpha W_i(\varepsilon_0) E(\varepsilon_0), \quad i > r, \tag{55}
\]

\[
A_i W_i(0) + W_i'(0) A_i^T < \beta W_i(0) E(0), \quad 1 \leq i \leq r, \tag{56}
\]

\[
A_i W_i(\varepsilon_0) + W_i'(\varepsilon_0) A_i^T < \beta W_i(\varepsilon_0) E(\varepsilon_0), \quad 1 \leq i \leq r \tag{57}
\]

hold, where \(W_i(\epsilon) = \begin{bmatrix} W_{i1} & \varepsilon W_{i2} \\ W_{i3} & W_{i4} \end{bmatrix}\).

The \(\epsilon\)-dependent state feedback controller is designed as
\[
K_i(\epsilon) = Y_i W_i^{-1}(\epsilon). \tag{58}
\]

Then, SPSS (1) is exponentially stable under the switching law with ADT
\[
\alpha > \alpha^* \geq \frac{\ln \mu}{\lambda}. \tag{59}
\]
and the total activation time of unstable subsystems and stable subsystems are defined as $T^+$ and $T^-$, respectively, with (31) holds.

Proof Define a piecewise $\epsilon$-dependent Lyapunov function

$$V_{\epsilon}(x(t)) = x^T(t)P_{\epsilon}(x(t))x(t).$$  

The other proof steps are similar to Theorems 3.2 and 4.1.

Remark 8 The unstabilizable subsystems are often caused by the failure of controllers, but the whole SPSS can still be stabilized by the designed switching law with ADT. The stabilization of the whole SPSS can be achieved by design controllers to stabilize the stabilizable subsystems with well-designed ADT and ratio of total activation time of stabilized and unstabilized subsystems.

Remark 9 Dwell time approach is a special class of ADT approach, ADT approach is more general because it allows some subsystems occasionally have a smaller dwell time between switching while this does not occur too frequently (see Lin & Antsaklis, 2009). In Lian and Wang (2015), SPSSs with stabilizable and unstabilizable subsystems subject to actuator saturation is discussed. The ratio of $\frac{T^-}{T^+}$ can only be guaranteed in each individual interval but not in the entire activated interval and dwell time approach is used, otherwise the state trajectory escaping the invariant set. It can be regarded as a special situation of Theorem 4.2.

Remark 10 In this paper, we focus on the switching law design for SPSSs under the assumption that stability bound is given with priori information. In order to improve the robustness of the system, the bisectional search algorithm will be considered in our future work to determine the estimation of the stability bound.

5. Example

In this section, two examples are given to illustrate the validity of our theoretical results of SPSS.

5.1. Example 1

Consider a SPSS with both stable and unstable subsystems in form of (1) with

$$A_1 = \begin{bmatrix} 0.25 & 1.15 \\ 0.6 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.8 & 1.0 \\ 0.4 & -1.5 \end{bmatrix}.$$  

Given $\epsilon_0 = 0.01$, and choose $\epsilon = 0.01 \in (0, \epsilon_0]$. Then, $E(\epsilon) = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}$ we can calculate the eigenvalues of each subsystem. Then, we can see that subsystem 1 is unstable and subsystem 2 is stable.

Choose

$$\alpha = 1, \beta = 1.4, \mu = 1.2, \lambda = 0.75,$$

it can be calculated that

$$\frac{T^-}{T^+} \geq \frac{\beta + \lambda}{\alpha - \lambda} = 3.8,$$

and the ADT is

$$\tau_0 \geq \tau^* \geq \frac{\ln u}{\lambda} = 0.24s.$$  

So let $\tau_0 = 1.5s$, $\frac{T^-}{T^+} = 4$, then we get the results as shown in the figures with the initial condition

$$x_0 = \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}.$$
Solving the LMIs in Theorem 3.2, we can get
\[ W_{11} = 503.2832, \quad W_{12} = 133.7961, \quad W_{13} = 160.3028, \]
\[ W_{21} = 495.6741, \quad W_{22} = 249.7755, \quad W_{23} = 162.7638. \]

The switching signal is shown in Figure 1.

Figure 2 shows the state trajectory of the SPSS with initial condition \( x_0 = [1.5 \quad -1.5]^T \). From this figure we can see that the state trajectory of the SPSS converges to the origin with the designed ADT and ratio of total activation time of unstable subsystems and stable subsystems.

From Figure 3, we can see the state responses of the SPSS, both \( x_1 \) starting from 1.5 and \( x_2 \) starting from \(-1.5\) converge to 0, furthermore, the state response of \( x_2 \) is much more faster than \( x_1 \), thus the singular perturbation property of the SPSS is shown.
Remark 11  The arbitrary switching method proposed in Malloci, Daafouz, and Jung (2009) can not be used in this case since the SPSS includes unstable subsystem.

Remark 12  The dwell time method proposed in Lian and Wang (2015) considered SPSSs subject to actuator saturation with stable and unstable subsystems, as a result, dwell time of each subsystem needs to be large than \( \tau_0 \). From Figure 1, we can see that the second dwell interval of subsystem 1 is less than \( \tau_0 \) but the ADT during the total time interval is more than \( \tau_0 \). Therefor, the SPSS is still stable, which shows the advantages of the proposed method in conservativeness reduction.

5.2. Example 2
This subsection will illustrate the proposed approaches by introducing a morphing aircraft model, which is constructed in Yin, Lu, and He (2013). The morphing process is as following: Keep a horizontal flight at the speed of 0.1 Maher without sideslipping. After changing the length wings for serval times, the aircraft can still keep the flight attitude and speed.

Then, a nonlinear model was established in the air path axis system:

\[
\begin{align*}
\dot{x}_1 &= T \cos \alpha - D - mg \sin(\theta - \alpha), \\
\dot{x}_2 &= -T \sin \alpha - L + mVq + mg \cos(\theta - \alpha), \\
\dot{\theta} &= q, \\
I_y \dot{q} &= M, \\
\dot{h} &= V \sin(\theta - \alpha),
\end{align*}
\]

(61)

where \( \theta \) and \( q \) stand for the angle of pitch and rate of pitch, respectively, \( I_y \) denotes the rotary inertia of the Y axis, \( L \) stands for the lift, \( M \) is the pitching moment and \( T \) is the thrust.

Basic parameters of the morphing aircraft are shown in Table 1, where \( m \) is the mass of the aircraft, \( S_w \) is reference area of the wings, \( c_{\lambda} \) is the geometric mean chord length of the wings, \( T_{\lambda} \) denotes the thrust coefficient and \( \rho \) denotes density of atmosphere.

By using the Jacobian linearization approach, it yields to a linear parameter varying (LPV) model with the definition of the rate of deformation \( \xi = \frac{\Delta b}{b}, \xi \in (0, 1) \), where \( \Delta b \) and \( b \) stand for the deformation amount and the original length of the aircraft, respectively. The aircraft model is...
where the state \( x(t) = [\Delta V \Delta \alpha \Delta \theta \Delta q \Delta h]^T \), \( \Delta V \) is the change of speed, \( \Delta \alpha \) is the change of angle of incidence, \( \Delta \theta \) is the change of the angle of pitch, \( \Delta q \) is the change of the rate of pitch and \( \Delta h \) is the change of height. The control input is \( u = [\Delta \delta_e \Delta \delta_t]^T \), \( \Delta \delta_e \) is the change of the angle of incidence, \( \Delta \delta_t \) is the change of throttle opening. And

\[
\dot{x}(t) = A(\xi)x(t) + Bu(t),
\]

where

\[
A(\xi) = \begin{bmatrix}
-0.0257\xi - 0.0514 & -0.5711\xi + 4.5398 & -9.8000 & 0 & 0 \\
-0.0002\xi - 0.0173 & -1.3845\xi - 1.4272 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & -5.7888\xi - 7.6609 & 0 & 0 & 0 \\
0 & -33.4 & 33.4 & 0 & 0 \\
\end{bmatrix} 
\]

\[
B = \begin{bmatrix}
0 & 0.0331 \\
-0.0776 & 0 \\
0 & 0 \\
-0.0132 & 0 \\
0 & 0 \\
\end{bmatrix} 
\]

Figure 4. Switching signal.

### Table 1. Basic parameters of the morphing aircraft

| Notation | Value | Units |
|----------|-------|-------|
| \( m \)  | 1,247 | kg    |
| \( S_w \) | 17.1  | m²    |
| \( c_A \) | 1.737 |       |
| \( I_r \) | 4,067.45 | kg · m² |
| \( T_s \) | 41.3  | N/%   |
| \( \rho \) | 1.0555 | kg/m³ |

Figure 4. Switching signal.
Because the morphing time is short, we choose two operating points as $\xi = 0.4$ and $\xi = 1$, the LPV model can be regarded as a SPSS. While the height does not change during the morphing, so we can get two subsystems as following by ignoring the change of height:

when $\xi = 0.4$,

$$A_1 = \begin{bmatrix}
-0.0411 & 4.3114 & -9.8000 & 0 \\
-0.0174 & -1.9810 & 0 & 1.0000 \\
0 & 0 & 0 & 0.003 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

$$B_1 = \begin{bmatrix}
0 \\
-0.0776 \\
0 \\
-0.0132
\end{bmatrix}.$$

when $\xi = 1$,

$$A_2 = \begin{bmatrix}
-0.0257 & 3.9687 & -9.8000 & 0 \\
-0.0175 & -2.8117 & 0 & 1.0000 \\
0 & 0 & 0 & 0.003 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

$$B_1 = \begin{bmatrix}
0 \\
-0.0776 \\
0 \\
-0.0132
\end{bmatrix}.$$

with

$$x = [\Delta V \, \Delta \alpha \, \Delta \theta \, \Delta q].$$

Then, given $\epsilon_0 = 0.003$, $\mu = 2$, $\lambda = 1.5$, initial condition $x_0 = [33.4 \, 0 \, 0 \, 0]^T$, choose $\epsilon = 0.003 \in (0, \epsilon_0]$, by solving the LMIs in Theorem 4.1, we can get
The ADT is $r_0 = 0.46$ s, so let the switching interval as 1.5, and 3 s, we can get the results as following.

Figure 4 shows the switching of the morphing process, it can be seen that the wings of the aircraft changes from $\xi = 0.4$ to $\xi = 1$, then changes back to $\xi = 0.4$ at $t = 1.5$ s and $t = 3$ s.

From Figure 5, after the morphing processes, the aircraft can maintain the same speed as before morphing and keep a smooth flight with the proposed state feedback controller.

**Remark 13** From the subsystems, we can see that the SPSS is a nonstandard SPSS, which can not be decomposed into fast and slow subsystems. Thus, methods proposed in Alwan, Liu, and Ingalls (2008) can not be used, which shows the advantages of methods proposed in this paper.

### 6. Conclusions

In this paper, the problems of stability analysis of SPSS are proposed with both stable and unstable subsystems with an ADT approach. And state feedback controller design methods are achieved for SPSS with both stabilizable and unstabilizable subsystems. The $\varepsilon$-dependent controller and time-controlled switching law design methods take full consideration of the singular perturbed property of SPSSs by constructing a full-order $\varepsilon$-dependent Lyapunov function, which can be used in both standard and nonstandard SPSSs. In addition, numerical and morphing aircraft examples are presented such that the effectiveness and advantages of the proposed approach are demonstrated.

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