GRAVITATIONAL FIELD OF
A MOVING SPINNING POINT PARTICLE

JAEGU KIM*

Department of Physics, Kangwon National University, Chunchon 200–701 Korea

ABSTRACT. The gravitational and electromagnetic fields of a moving charged spinning point particle are obtained in the Lorentz covariant form by transforming the Kerr–Newman solution in Boyer–Lindquist coordinates to the one in the coordinate system which resembles the isotropic coordinates and then covariantizing it. It is shown that the general relativistic proper time at the location of the particle is the same as the special relativistic one and the gravitational and electromagnetic self forces vanish.

PACS number: 04.20.Jb

1. Introduction

In the previous paper [1] we developed a Lorentz covariant method of solving the Einstein equations and obtained the gravitational field of a moving point particle for both uncharged and charged cases. It is shown that the general relativistic proper time at the location of the particle is the same as the special relativistic one and the gravitational and electromagnetic self forces vanish. Thus, the solution consistently describes the gravitational and electromagnetic fields of a charged point particle. We have seen that the isotropic coordinates played a crucial role in the description of the source as a point particle.

In this paper we will consider the gravitational and electromagnetic fields of a moving charged spinning point particle. For a spinless point particle we can express the metric tensor simply in terms of a Lorentz invariant quantity $R\cdot u$ and two Lorentz covariant tensors $\eta_{\mu\nu}$ and $u_\mu u_\nu$. But we cannot expect this kind of simplicity for a spinning point particle. For a spinning point particle moving with constant velocity, we can introduce one more Lorentz four vector $s^\mu$ which is a four vector generalization of spin angular momentum per unit mass. Then the Lorentz covariant quantities which can be used to make an ansatz for the metric tensor are $\eta_{\mu\nu}$, $\varepsilon_{\mu\nu\sigma\lambda}$, $R^\mu$, $u^\mu$, and $s^\mu$, from which one can construct two Lorentz invariant quantities and ten symmetric Lorentz covariant tensors of rank two. Thus, the most general expression for the metric tensor is so complicated that it is almost impossible to calculate the Ricci tensor and solve the Einstein–Maxwell equations directly. To avoid this complication we reverse the procedure described in [1] instead of solving the Einstein–Maxwell equations directly. We start with the Kerr–Newman solution

*jaegukim@cc.kangwon.ac.kr
in Boyer–Lindquist coordinates and transform the coordinate system to the one which resembles the isotropic coordinates. Then we let the particle move with constant velocity and obtain the metric tensor and the four vector potential by covariantizing them. Using computer, we confirm that the metric tensor and the four vector potential satisfy the Einstein–Maxwell equations by substituting them into the field equations. Finally, we check the consistency of the solution and its implications. We will use the same notational conventions given in [1].

2. Metric tensor and four vector potential

The line element and the potential one form for a source of mass $m$, spin angular momentum $J = ms$ and charge $q$ is given by the Kerr–Newman solution in Boyer–Lindquist coordinates [2] as

$$ds^2 = -\left[1 - \frac{(2m\rho - q^2)}{\Sigma}\right]dt^2 - \frac{2(2m\rho - q^2)s\sin^2\theta}{\Sigma}dt\,d\phi + \frac{\Sigma}{\Delta}d\rho^2$$

$$+ \Sigma\,d\theta^2 + \left[\rho^2 + s^2 + \frac{(2m\rho - q^2)s^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\phi^2,$$

(2.1a)

$$A_\mu\,dx^\mu = -\frac{q\rho}{\Sigma}(dt - s\sin^2\theta\,d\phi),$$

(2.1b)

where the source is spinning in the $\phi$ direction and

$$\Delta = \rho^2 - 2m\rho + s^2 + q^2,$$

(2.2a)

$$\Sigma = \rho^2 + s^2\cos^2\theta.$$  

(2.2b)

Recalling that the isotropic coordinates played a crucial role in the description of the source as a point particle [1], we transform the coordinate system by introducing a new radius $r$ such that the transformed metric reduces to the Schwarzschild metric in isotropic coordinates when $s = q = 0$

$$\rho = f(r).$$

(2.3)

The condition for isotropy is

$$f'^2 = \frac{\Delta}{r^2},$$

(2.4)

and its solution is found to be

$$\rho = \left(1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4\rho^2}\right)r.$$  

(2.5)

Using this transformation rule, we have
\[ ds^2 = -(1 - 2\Phi)dt^2 - 4s\Phi\sin^2 \theta \, dt \, d\phi \]
\[ + \left[ \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right)^2 + \frac{s^2\cos^2 \theta}{r^2} \right] (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2) \]
\[ + (1 + 2\Phi) s^2 \sin^4 \theta \, d\phi^2, \] (2.6a)

\[ A_\mu \, dx^\mu = \frac{q}{r} \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right) \]
\[ \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right)^2 + \frac{s^2\cos^2 \theta}{r^2} \]
\[ (-dt + s\sin^2 \theta \, d\phi), \] (2.6b)

where
\[ \Phi = \frac{m}{r} \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right) - \frac{q^2}{2r^2} \]
\[ \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right)^2 + \frac{s^2\cos^2 \theta}{r^2} \] (2.7)

From now on we will refer to the coordinate system in (2.6) as the isotropic coordinate system. Next we transform the spherical coordinates \((r, \theta, \phi)\) to the Cartesian coordinates \((x^1, x^2, x^3)\)

\[ x^1 = r \sin \theta \cos \phi, \] (2.8a)
\[ x^2 = r \sin \theta \sin \phi, \] (2.8b)
\[ x^3 = r \cos \theta. \] (2.8c)

Then the quantities in (2.6) can be written as

\[ s \cos \theta = \frac{s \cdot r}{r}, \] (2.9a)
\[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 = \eta_{ij} \, dx^i \, dx^j, \] (2.9b)
\[ sr^2 \sin^2 \theta \, d\phi = \varepsilon_{ijk} \, x^i \, dx^j \cdot \] (2.9c)

Now let the source move with constant four velocity \(u^\mu\). Then we can obtain new line element and potential one form by covariantizing the quantities in (2.6) as follows:

\[ r \rightarrow -R \cdot u, \] (2.10a)
\[ s \cos \theta \rightarrow -\frac{R \cdot s}{R \cdot u}, \] (2.10b)
\[ dt \rightarrow -u_\mu \, dx^\mu, \] (2.10c)
\[ dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \rightarrow (\eta_{\mu\nu} + u_\mu u_\nu) \, dx^\mu \, dx^\nu, \] (2.10d)
\[ sr^2 \sin^2 \theta \, d\phi \rightarrow \varepsilon_{\mu\nu\sigma\tau} u^\mu s^\nu R^\sigma \, dx^\tau, \] (2.10e)

where \(s^\mu\) is four spin angular momentum vector per unit mass

\[ s^\mu = (s \cdot u, s + \frac{(s \cdot u) u}{\gamma + 1}). \] (2.11)
Here $s$ is the spin angular momentum vector per unit mass in the rest frame of the source. From the new line element and potential one form we construct the metric tensor and the four vector potential in the Lorentz covariant form

\[
g_{\mu\nu} = e^A[\eta_{\mu\nu} + (1 - e^B)u_\mu u_\nu] - \frac{(1 - e^{A+B})}{(R \cdot u)^2}(u_\mu k_\nu + u_\nu k_\mu) + \frac{(2 - e^{A+B})}{(R \cdot u)^4}k_\mu k_\nu, \tag{2.12a}
\]

\[
A_\mu = -\frac{q}{R \cdot u} \left[ 1 - \frac{m}{R \cdot u} + \frac{m^2 - s^2 - q^2}{4(R \cdot u)^2} \left[ u_\mu - \frac{k_\mu}{(R \cdot u)^2} \right] \right], \tag{2.12b}
\]

where

\[
e^A = \left[ 1 - \frac{m}{R \cdot u} + \frac{m^2 - s^2 - q^2}{4(R \cdot u)^2} \right]^2 + \frac{(R \cdot s)^2}{(R \cdot u)^4}, \tag{2.13a}
\]

\[
e^{A+B} = \left[ 1 - \frac{m^2 - s^2 - q^2}{4(R \cdot u)^2} \right]^2 + \frac{(R \cdot s)^2}{(R \cdot u)^4} - \frac{s^2}{(R \cdot u)^2} \left[ 1 - \frac{m}{R \cdot u} + \frac{m^2 - s^2 - q^2}{4(R \cdot u)^2} \right]^2 + \frac{(R \cdot s)^2}{(R \cdot u)^4}, \tag{2.13b}
\]

\[
k_\mu = \varepsilon_{\mu\alpha\beta\gamma}u^\alpha s^\beta R^\gamma. \tag{2.13c}
\]

Note that the metric tensor and the four vector potential (2.12) reduce to those of spinless case [1] when $s$ is zero. Since the vector $k_\mu$ satisfies

\[
k_\mu k_\nu = [s^2(R \cdot u)^2 - (R \cdot s)^2]\eta_{\mu\nu} - s^2 R_\mu R_\nu - s^2(R \cdot u)(R_\mu u_\nu + R_\nu u_\mu) + (R \cdot s)(R_\mu s_\nu + R_\nu s_\mu) - (R \cdot s)^2 u_\mu u_\nu + (R \cdot u)(R \cdot s)(u_\mu s_\nu + u_\nu s_\mu) - (R \cdot u)^2 s_\mu s_\nu, \tag{2.14}
\]

we could replace the product $k_\mu k_\nu$ in (2.12a) by the expression on the right hand side of (2.14), but we will retain the expression (2.12a) for simplicity. Calculating the determinant of the metric tensor and the contravariant metric tensor, we have

\[
g = -e^{2A+C}, \tag{2.15a}
\]

\[
g^{\mu\nu} = e^{-A}\eta^{\mu\nu} + e^{-A}[(1 - e^{-B}) + (1 - e^{A+B})^2(e^{-B} - e^{2A-C})]u^\mu u^\nu + \frac{e^{-C}(1 - e^{A+B})}{(R \cdot u)^2}(u_\mu k_\nu + u_\nu k_\mu) - \frac{e^{-A-C}}{(R \cdot u)^4}k_\mu k_\nu, \tag{2.15b}
\]

where
\[ e^C = e^{2A+B} + \frac{s^2}{(R\cdot u)^2} - \frac{(R\cdot s)^2}{(R\cdot u)^4} \]

\[ = \left[ 1 - \frac{m^2 - s^2 - q^2}{4(R\cdot u)^2} \right]^2. \quad (2.16) \]

3. Field equations

Instead of solving the Einstein–Maxwell directly we have constructed the metric tensor and the four vector potential by transforming the Kerr–Newman solution in Boyer–Lindquist coordinates to the one in the isotropic coordinates and then covariantizing it. If (2.12) satisfies the Einstein–Maxwell equations, it immediately implies that (2.1) also satisfies the Einstein–Maxwell equations. But the converse is not necessarily true. Hence we have to show explicitly that (2.12) satisfies the Einstein–Maxwell equations. However, due to the complexity of the metric tensor (2.12) the Christoffel symbol turns out to be extremely complicated. Moreover, since the Ricci tensor has terms of products of two Christoffel symbols, it is almost impossible to calculate the Ricci tensor by hand. Nevertheless, we confirmed that (2.12) satisfied the Einstein–Maxwell equations by using computer with the symbol manipulating language Mathematica [3].

4. Discussion

We have obtained the metric tensor and the four vector potential of a moving charged spinning point particle in the Lorentz covariant form. To check consistency of the solution with the equations of motion, we consider the gravitational and electromagnetic self forces and find

\[ F_{\lambda}^{\gamma}_{\text{grav}} = -m[\Gamma_{\mu\nu}^\lambda]_{R=0} u^\mu u^\nu = 0, \quad (4.1a) \]

\[ F_{\lambda}^{\gamma}_{\text{em}} = q[F_{\gamma}^{\lambda\sigma} u^\sigma]_{R=0} = 0. \quad (4.1b) \]

Hence the gravitational and electromagnetic self forces vanish. Next we consider the general relativistic proper time interval at the location of the particle. Except for the case \( m^2 = s^2 + q^2 \), we have

\[ d\tau = \frac{1}{\gamma} \left[ -g_{\mu\nu} u^\mu u^\nu \right]_{R=0}^{1/2} dt = \frac{1}{\gamma} dt, \quad (4.2) \]

i.e., it is the same as the special relativistic proper time interval. For the case \( m^2 = s^2 + q^2 \), it is indeterminate. In this case one can choose the special relativistic proper time as the Affine parameter of the trajectory of the particle. Therefore, the solution (2.12) is consistent with the equations of motion and properly describes the source as a spinning point particle.

As we have seen, the most natural coordinate system for a charged spinning point particle is the isotropic coordinates rather than the Boyer–Lindquist coordinates. The topology of base manifold is trivial in the isotropic coordinates. We now turn to the coordinate transformation (2.5) and consider the global topology in Boyer–Lindquist coordinates induced by it. We can classify the transformation (2.6) into three cases according to the value of \( m^2 - s^2 - q^2 \).
First, consider the case with \( m^2 > s^2 + q^2 \), which may hold for an astrophysical object after gravitational collapse. Due to the nonlinearity of the transformation (2.6) the region between \( \rho = m + (m^2 - s^2 - q^2)^{1/2} \) and \( \rho = \infty \) is mapped twice, while the region inside \( \rho = m + (m^2 - s^2 - q^2)^{1/2} \) is not mapped by any value of \( \rho \). Thus the transformation (2.6) induces a nontrivial topology in Boyer–Lindquist coordinates. To visualize the global topology of spacetime in Boyer–Lindquist coordinates we consider embedding of a spacelike hypersurface of constant time \( t = 0 \), with one degree of rotational freedom suppressed (\( \theta = \pi/2 \)). In Boyer–Lindquist coordinates this hypersurface corresponds to two distinct asymptotically flat hypersurfaces which are connected by the Einstein–Rosen bridge at \( \rho = m + (m^2 - s^2 - q^2)^{1/2} \), which is the same as the topology of the Schwarzschild geometry in Schwarzschild coordinates [1]. The location of the particle \( r = 0 \) is mapped to \( \rho = \infty \) where spacetime is flat.

Second, consider the case with \( m^2 = s^2 + q^2 \). Note that the minimum value of \( \rho \) in (2.5) is \( m \). Hence the region inside \( \rho = m \) in Boyer–Lindquist coordinates should be excised. The location of the particle \( r = 0 \) is mapped to \( \rho = m \).

Last, consider the case with \( m^2 < s^2 + q^2 \), which holds for elementary particles with spin or charge. For instance estimating the numerical values for these quantities for an electron, we have \( m^2 = 4.6 \times 10^{-115} \text{ m}^2 \), \( s^2 = 1.1 \times 10^{-25} \text{ m}^2 \), and \( q^2 = 1.9 \times 10^{-72} \text{ m}^2 \). Thus, \( s \) plays dominant role in the behavior of the gravitational field of the electron at short distance. We note that the metric tensor and the four vector potential has a ring singularity at \( r = [(s^2 + q^2)^{1/2} - m]/2 \) and \( \theta = \pi/2 \). In this case the lower limit of \( \rho \) should be extended to \( -\infty \) instead of 0 and the location of the particle \( r = 0 \) is mapped to \( \rho = -\infty \) where spacetime is flat.

To understand the short distance behavior of the gravitational and electromagnetic forces we consider a test particle of mass \( m_0 \) and charge \( q_0 \) near the static spinning point source described by (2.6). Since the gravitational and electromagnetic forces in general have a very complicated dependence on distance, we will consider a static test particle on the equatorial plane (\( \theta = \pi/2 \)) for simplicity. In this case both forces on the test particle point in the radial direction and turn out to be

\[
F_g = -m_0 \Gamma^r_{tt} = -m_0 m \left( \frac{1 - m^2 - s^2 - q^2}{4r^2} \right) \left( \frac{1 + \frac{m^2 - q^2}{mr} + \frac{m^2 - s^2 - q^2}{4r^2}}{1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2}} \right)^5 \approx \frac{q_0 q}{r^2} \left( \frac{1 - m^2 - s^2 - q^2}{4r^2} \right) \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right)^4, \tag{4.3a}
\]

\[
F_e = q_0 F^r_r = \frac{q_0 q}{r^2} \left( \frac{1 - m^2 - s^2 - q^2}{4r^2} \right) \left( 1 + \frac{m}{r} + \frac{m^2 - s^2 - q^2}{4r^2} \right)^4. \tag{4.3b}
\]

We note that \( F_g \) and \( F_e \) are inversely proportional to \( r^2 \) near \( r = \infty \). Now let the test particle move from \( r = \infty \) to \( r = 0 \) and consider the forces on it for three different cases separately.
First, consider the case with \( m^2 > s^2 + q^2 \). As the test particle approaches \( r = (m^2 - s^2 - q^2)^{1/2}/2 \), both \( F_g \) and \( F_e \) approach zero and reverse their directions when it passes \( r = (m^2 - s^2 - q^2)^{1/2}/2 \). Thus, at short distance the gravitational force becomes repulsive and the electrostatic force becomes attractive for the same type of charges and repulsive for opposite type of charges. Both \( F_g \) and \( F_e \) are proportional to \( r^4 \) near \( r = 0 \).

Second, consider the case with \( m^2 = s^2 + q^2 \). In this case \( F_g \) and \( F_e \) never change their directions and are proportional to \( r^2 \) near \( r = 0 \).

Last, consider the case with \( m^2 < s^2 + q^2 \). In this case both \( F_g \) and \( F_e \) are infinite at \( r = [(s^2 + q^2)^{1/2} - m]/2 \) due to the ring singularity and \( F_e \) never changes its direction. For \( q = 0 \) \( F_g \) never changes its direction, but for \( q \neq 0 \) \( F_g \) changes its direction twice at \( r = [(s^2 - q^2 + q^4/m^2)^{1/2} + q^2/m - m]/2 \) and at \( r = [(s^2 + q^2)^{1/2} - m]/2 \). Thus, for \( q \neq 0 \) the gravitational force becomes repulsive in the region between \( r = [(s^2 + q^2)^{1/2} - m]/2 \) and \( r = [(s^2 - q^2 + q^4/m^2)^{1/2} + q^2/m - m]/2 \). Both \( F_g \) and \( F_e \) are proportional to \( r^4 \) near \( r = 0 \).

We have seen that both the gravitational and electrostatic forces on the static test particle on the equatorial plane approach zero as the test particle approaches \( r = 0 \). Although the behavior of the gravitational and electromagnetic forces on the nonstatic test particle off the equatorial plane is very complicated, we also have confirmed that they both approach zero as the test particle approaches \( r = 0 \). Therefore the gravitational and electromagnetic forces are asymptotically free at short distance. This sheds a new light on the theory of quantum gravity, since quantum theory is less divergent than classical theory.

Acknowledgement

I would like to thank Professor Robert Finkelstein for reading the manuscript and comments and Professors Young Jik Ahn and Taejin Lee for helpful discussions.

References

[1] Kim J 1993 “Gravitational Field of a Moving Point Particle” gr–qc/9311027 (to be published)
[2] Boyer R H and Lindquist R W 1967 J. Math. Phys. 8 265
[3] Wolfram S 1991 “Mathematica: A System for Doing Mathematics by Computer” 2nd Ed. (Addison–Wesley)