Abstract. In this contribution we consider stochastic growth models in the Kardar–Parisi–Zhang universality class in 1 + 1 dimension. We discuss the large time distribution and processes and their dependence on the class of initial condition. This means that the scaling exponents do not uniquely determine the large time surface statistics, but one has to further divide into subclasses.

Some of the fluctuation laws were first discovered in random matrix models. Moreover, the limit process for curved limit shape turned out to show up in a dynamical version of Hermitian random matrices, but this analogy does not extend to the case of symmetric matrices. Therefore the connection between growth models and random matrices is only partial.

Keywords: exact results, stochastic particle dynamics (theory), random matrix theory and extensions

ArXiv ePrint: 1008.4853
1. Introduction

In this paper we discuss results in the Kardar–Parisi–Zhang (KPZ) universality class of stochastic growth models. We focus on the connections with random matrices occurring in the one-dimensional case. Consider a surface described by a height function \( x \mapsto h(x, t) \) with \( x \in \mathbb{R}^d \) denoting space and \( t \in \mathbb{R} \) being the time variable and subjected to a random dynamics. If the growth mechanism is local and there is a smoothing mechanism providing a deterministic macroscopic growth, then the macroscopic evolution of the interface will be governed by

\[
\frac{\partial h}{\partial t} = v(\nabla h)
\]

where \( u \mapsto v(u) \) is the macroscopic growth velocity as a function of the surface slope \( u \). In this context, we can also focus on a mesoscopic scale where the random nature of the dynamics is still visible. In the famous paper of Kardar et al [35], the smoothing mechanism is related with the surface tension and it takes the form \( \nu \Delta h \), while the local random dynamics enters as a space–time white noise \( \eta \). Moreover, the Taylor expansion of \( v \) for small slopes\(^1\) results in the KPZ equation\(^2\)

\[
\frac{\partial h(x,t)}{\partial t} = \nu \Delta h(x, t) + \frac{1}{2} \lambda (\nabla h(x, t))^2 + \eta(x, t),
\]

\(^1\) The order 0 and 1 in the Taylor expansion can be set to be zero by a simple change of (moving) frame.

\(^2\) In more than one dimension, \((\nabla h)^2\) should be replaced by \((\nabla h, C \nabla h)\) with \( C \) a matrix. Then one distinguishes the isotropic class, if all the eigenvalues of \( C \) have the same sign, and anisotropic class(es) otherwise. For instance, in \( d = 2 \) the surface fluctuations are very different: for anisotropic they are normally distributed in the \( \sqrt{\ln(t)} \) scale [38, 54] and the correlations are the ones of the massless free field [9,11]; for isotropic it is numerically known that the growth is as \( t^\alpha \) for some \( \alpha \simeq 0.240 \) [48].
From interacting particle systems to random matrices

where \( \lambda = v''(0) \neq 0 \) in the non-linear term is responsible for lateral spread of the surface and lack of time reversibility\(^3\).

From now on we consider the one-dimensional case, \( d = 1 \). Denote by \( h_{\text{ma}} \) the limit shape,

\[
h_{\text{ma}}(\xi) := \lim_{t \to \infty} \frac{h(\xi t, t)}{t}.
\]  

(3)

The fluctuation exponent is \( 1/3 \), the spatial correlation exponent is \( 2/3 \) [29,52]. This means that the height fluctuations grow in time as \( t^{1/3} \) and spatial correlations are \( t^{2/3} \), i.e., the rescaled height function at time \( t \) around the macroscopic position \( \xi \) (at which \( h_{\text{ma}} \) is smooth)\(^4\)

\[
h_{\text{resc}}(u) = \frac{h(\xi t + ut^{2/3}, t) - t h_{\text{ma}}(\xi + ut^{-1/3})}{t^{1/3}}
\]  

(4)

converges to a non-trivial limit process in the \( t \to \infty \) limit. Concerning correlations in space–time, it is known that along special directions the decorrelation occurs only on the macroscopic scale (i.e., with scaling exponent 1), while along any other direction the correlations are asymptotically like the spatial correlations at fixed time [17,24]. The special directions are the characteristic solutions of the PDE for the macroscopic height gradient. More precisely, denote by \( \xi \) and \( \tau \) the macroscopic variables for space and time. Also, let

\[
\bar{h}(\xi, \tau) := \lim_{t \to \infty} t^{-1}h(\xi t, \tau t) \quad \text{and} \quad u(\xi, \tau) := \partial \bar{h}(\xi, \tau)/\partial \xi.
\]  

(5)

Then, \( u \) satisfies the PDE

\[
\partial u/\partial \tau + a(u)\partial u/\partial \xi = 0 \quad \text{where} \quad a(u) = -\partial v(u)/\partial u
\]  

(6)

with \( v \) the macroscopic speed of growth\(^5\). The characteristic solutions of (6) are the trajectories satisfying \( \partial \xi/\partial \tau = a(u) \) and \( \partial u/\partial \tau = 0 \) (see e.g. [21,53] for more insights on characteristic solutions).

The question is therefore to determine the limit process

\[
u \mapsto \lim_{t \to \infty} h_{\text{resc}}(u) = ?
\]  

(7)

One might be tempted to think that the scaling exponents are enough to distinguish between classes of models and therefore that the result of our question is independent of the initial condition. However, as we will see, this is not true\(^6\).

To have an intuition about the relevance of the initial condition, consider the fluctuations of \( h(0,t) \) with (a) a deterministic initial condition, \( h(x,0) = 0 \) for all \( x \in \mathbb{R} \),

\(^3\) When \( \lambda = 0 \) we are in the Edwards–Wilkinson class [20] and the fluctuations are Gaussian with fluctuation exponent 1/4.

\(^4\) However, depending on the initial conditions, (2) can produce spikes in the macroscopic shape. If one looks at the surface gradient \( u = \nabla h \), the spikes of \( h \) correspond to shocks in \( u \), and it is known that the shock position fluctuates on a scale \( t^{1/2} \). For particular models, properties of the shocks have been analyzed, but mostly for stationary growth (see [15,18,22] and references therein).

\(^5\) In the asymmetric exclusion process explained below, one usually considers the particle density \( \rho \) instead of \( u \). This is, however, just a rotation of the frame, since they are simply related by \( u = 1 - 2\rho \). The PDE for \( \rho \) is the well known Burgers equation.

\(^6\) For other observables the relevance of the initial condition was observed already in [51].
and (b) a random but still macroscopically flat initial condition, \( h(x,0) \) a two-sided Brownian motion with \( h(0,0) = 0 \). For this case, the height function \( h(0,t) \) is correlated with a neighborhood of \( x = 0 \) of order \( t^{2/3} \). In this region (at time \( t = 0 \)) for (a) fluctuations are absent while in (b) the fluctuations on the initial condition are of order \( t^{1/3} \): this is the same scale as the fluctuations of \( h(0,t) \) and therefore the fluctuation laws for (a) and (b) will be different.

How should we proceed to answer to our question? Literally taken the KPZ equation (2) is ill defined, because locally one will see a Brownian motion and the problem comes from the square of a white noise (in the non-linear term). However it is possible to give a sense of a solution of the KPZ equation as shown in \([2,44]\), and this solution agrees with the one coming from discrete approximations/models (weakly asymmetric simple exclusion process \([7]\)). These works also provide an explicit solution of the finite time one-point distribution for an important initial condition, see \([43]\) for more explanations.

Another point of view is to see the KPZ equation as one of the models in the KPZ universality class of growth models. By universality it is expected that the limit processes do not depend on the model in the class (but they depend on the type of initial condition). From this perspective, we can take any of the models in the KPZ class and try to obtain the large time limit.

In the rest of the paper we consider one such model, the totally asymmetric simple exclusion process (TASEP) in which the asymptotic processes have been unraveled. Another model for which analogous results have been determined is the polynuclear growth (PNG) model\(^7\). In particular, we will discuss which limit distributions/processes also appear in random matrix theory and when the connection is only partial.

2. TASEP

The totally asymmetric simple exclusion process (TASEP) in continuous time is the simplest non-reversible interacting stochastic particle system. In TASEP particles are on the lattice of integers, \( \mathbb{Z} \), with at most one particle at each site (exclusion principle). The dynamics is defined as follows. Particles jump to the neighboring right site with rate 1 provided that the site is empty. Jumps are independent of each other and take place after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty, see figure 1.

Denote by \( \eta_x(t) \) the occupation variable of site \( x \in \mathbb{Z} \) at time \( t \geq 0 \), i.e., \( \eta_x(t) \) is 1 if there is a particle and 0 if the site is empty. TASEP configurations are in bijection with the surface profile defined by setting the origin \( h(0,0) = 0 \) and the discrete height

\(^7\) At least half of the limit results described below were first obtained for the PNG model.
gradient to be $1 - 2\eta_x(t)$. If we denote by $N_t$ the number of particles which have crossed the bond 0 to 1 during the time span $[0, t]$, then the height function is given by

$$h(x, t) = \begin{cases} 
2N_t + \sum_{y=1}^{x}(1 - 2\eta_y(t)), & \text{for } x \geq 1, \\
2N_t, & \text{for } x = 0, \\
2N_t - \sum_{y=x+1}^{0}(1 - 2\eta_y(t)), & \text{for } x \leq -1
\end{cases}$$

as illustrated in figure 2.

Let us verify that the TASEP belongs to the KPZ universality class. Under hydrodynamical scaling, the particle density $\rho$ evolves according to the Burgers equation $\partial_t \rho + \partial_x (\rho(1 - \rho)) = 0$. Thus, we have a deterministic limit shape. The second requirement, the locality of the growth dynamics, is obviously satisfied. Finally, the speed of growth $v$ of the interface is twice the current density, which is given by $\rho(1 - \rho)$. As the the gradient is $u = 1 - 2\rho$, it follows that $v(u) = (1 - u^2)/2$, which implies $v''(u) = -1 \neq 0$.

Now we discuss some of the large time results for the TASEP height function\(^8\). We consider two non-random initial conditions generating a curved and a flat macroscopic shape. The limit processes will be called the Airy$_2$ and Airy$_1$ processes, which are defined in the appendix.

### 2.1. TASEP with step initial condition

Consider the initial condition $\eta_x(0) = 1$ for $x \leq 0$ and $\eta_x(0) = 0$ for $x \geq 1$, see figure 3. This is called the step initial condition.

The macroscopic limit shape for this initial condition is a parabola continued by two straight lines:

$$h_{\text{ma}}(\xi) = \begin{cases} 
\frac{1}{2}(1 + \xi^2), & \text{for } |\xi| \leq 1, \\
|\xi|, & \text{for } |\xi| \geq 1.
\end{cases}$$

\(^8\) Most of the results have been first computed for particle positions and the statements described below are obtained by a geometric transformation.
From this we have the scaling

\[ h_t^{\text{resc}}(u) := \frac{h(2u(t/2)^{2/3}, t) - (t/2 + u^2(t/2)^{1/3})}{-(t/2)^{1/3}}. \]  

(10)

The large time results for the rescaled height function \( h_t^{\text{resc}} \) are the following. First, for the one-point distribution [31]

\[ \lim_{t \to \infty} P(h_t^{\text{resc}}(0) \leq s) = F_2(s), \]  

(11)

where \( F_2 \) is known as the GUE Tracy–Widom distribution, first discovered in random matrices [49] (see section 3). Moreover, concerning the joint distributions, it is proven [10, 14, 32] that (in the sense of finite-dimensional distribution)

\[ \lim_{t \to \infty} h_t^{\text{resc}}(u) = A_2(u), \]  

(12)

where \( A_2 \) is called the Airy \(_2\) process, first discovered in the PNG model by Prähofer and Spohn [41] (see the appendix). In particular, the Airy \(_2\) process is stationary, locally looks like a Brownian motion and has correlations decaying slowly: like \( u^{-2} \) (see figure 5).

By universality it is expected that the Airy \(_2\) process describes the large time surface statistics for initial conditions generating a smooth curved macroscopic shape for models in the KPZ class. This happens when the characteristic lines for space–time points on the curved limit shape go all together to a single point at time \( t = 0 \) (for the TASEP and PNG they are straight lines back to the origin).

2.2. TASEP with flat initial condition

The second type of non-random initial condition we discuss here is called the flat initial condition. In terms of TASEP particles, it is given by \( \eta_x(0) = 1 \) for \( x \) even and \( \eta_x(0) = 0 \) for \( x \) odd, see figure 4.

9 With respect to (4) we adjusted the coefficients to avoid having them in the asymptotic process.

10 In [32] the process was studied in a slightly different cut, but because of slow decorrelation [17, 24] the present result can be proven from it. Remark also that the convergence in [32] is in a stronger sense.

11 The height function of the TASEP in discrete time with parallel update and step initial condition is the same as the arctic line in the Aztec diamond for which the Airy\(_2\) process was obtained by Johansson in [33]. Extensions to process on a space-like path for the PNG model were made in [16]. The tagged particle problem was studied in [30], extension to space-like paths in [10, 13] and to any space–time paths except the characteristic line in [17, 24].

12 Also for random initial conditions as in the case of Bernoulli-\( \rho_- \) on \( \mathbb{Z}_- \) and Bernoulli-\( \rho_+ \) on \( \mathbb{Z}_+ \), \( \rho_- > \rho_+ \), see [6, 40].
Figure 4. Height configuration for deterministic and flat initial conditions.

The macroscopic limit shape is very simple, \( h_{\text{mac}}(\xi) = \frac{1}{2} \), so that the rescaled height function becomes

\[
 h_{\text{resc}}(u) := \frac{h(2u(t/2)^{2/3}, t) - t/2}{-(t/2)^{1/3}}. \tag{13}
\]

In the large time limit, the one-point distribution of \( h_{\text{resc}} \) is given by

\[
 \lim_{t \to \infty} \mathbb{P}(h_{\text{resc}}(0) \leq s) = F_1(2s), \tag{14}
\]

where \( F_1 \) is known as the GOE Tracy–Widom distribution, first discovered in random matrices [50] (see section 3). Moreover, as a process, it was discovered by Sasamoto [12,42] and it is proven that (in the sense of finite-dimensional distribution)

\[
 \lim_{t \to \infty} h_{\text{resc}}(u) = A_1(u), \tag{15}
\]

where \( A_1 \) is called the Airy\(_1\) process (see the appendix). In particular, the Airy\(_1\) process is stationary, it behaves locally like a Brownian motion, but unlike for the Airy\(_2\) process, the decorrelations decay super-exponentially fast (see figure 5), see the review [25] for more information and references.

The Airy\(_1\) process is expected to describe the large time surface behavior for non-random initial conditions generating a straight limit shape for models in the KPZ class. Unlike for the curved limit shape, the characteristic lines for space–time points for flat limit shape do not join at initial time. This fact is at the origin of (a) the different fluctuation behavior between curved and flat and (b) the difference between random flat and non-random flat.

2.3. TASEP with stationary initial condition

The only translation invariant stationary measures for the continuous time TASEP are Bernoulli product measures with parameter \( \rho, \rho \in [0,1] \), which is the density of particles [37]. The cases \( \rho \in \{0,1\} \) are degenerate and nothing happens, so consider a fixed \( \rho \in (0,1) \). The height function at time 0 is a two-sided random walk with \( h(0,0) = 0 \), \( \mathbb{P}(h(x+1,0) - h(x,0) = 1) = 1 - \rho \) and \( \mathbb{P}(h(x+1,0) - h(x,0) = -1) = \rho \).

Unlike the deterministic initial conditions we need to consider the regions where the height function is non-trivially correlated with \( h(0,0) \). The reason is that for the regions at time \( t \) which are correlated with \( h(at,0), a \neq 0 \), the dynamical fluctuations (of order \( t^{1/3} \)) are dominated by the fluctuations in the initial condition (of order \( t^{1/2} \)). The correlations in the TASEP are carried by second-class particles, which move with speed \( 1 - 2\rho \). As

---

13 For the geometric case corresponding to the discrete time TASEP this result was proven by Baik and Rains in [5].

14 For density \( 1/2 \), the characteristic lines are all lines parallel to the time axis. For density \( \rho \), they have the form \( x = x_0 + (1 - 2\rho)t \).
Figure 5. Covariance $g_2(u) = \text{Cov}(A_2(u), A_2(0))$ of the Airy$_2$ process (dashed line) and $g_1(u) = \text{Cov}(A_1(u), A_1(0))$ of the Airy$_1$ process (solid line). One clearly sees the difference of behavior: $g_2(u) \simeq 2u^{-2}$ for $u \gg 1$, while $g_1(u)$ goes to zero super-exponentially fast.

predicted by the KPZ scaling, the limit
\[
\lim_{t \to \infty} \frac{h((1-2\rho)t + ut^{2/3}, t) - (1 - 2\rho(1-\rho))t}{t^{1/3}}
\]
exists. The one-point distribution was derived in the PNG model with external source [4] and for the TASEP in [27] (correctly conjectured using universality in [40]). The extension to joint distributions is worked out in [3]. So far no connections with random matrices for this initial condition are known. Therefore we do not enter into further details.

3. Random matrices

We said that $F_1$ and $F_2$ appeared first in random matrices. We will explain this below and discuss whether the Airy processes also show up for random matrix models.

3.1. Hermitian matrices

The Gaussian unitary ensemble (GUE) of random matrices consists of Hermitian matrices $H$ of size $N \times N$ distributed according to the probability measure\(^{15}\)
\[
p^{\text{GUE}}(H) \, dH = \frac{1}{Z_N} \exp \left( -\frac{1}{2N} \text{Tr}(H^2) \right) \, dH,
\]

\(^{15}\) The scaling of this paper is such that the asymptotic density of eigenvalues remains bounded, the macroscopic variable is $N$ and the scaling exponents are easily compared with KPZ. In the literature there are another two standard normalization constants, which are just a rescaling of eigenvalues. The first one consists of replacing $1/(2N)$ by $N$: this is appropriate if one studies spectral properties, since in the large $N$ limit the spectrum remains bounded. The second is to replace $1/(2N)$ by $1$ so that the measure does not depend on $N$: this is most appropriate if one looks at eigenvalues’ minors.

doi:10.1088/1742-5468/2010/10/P10016
where \(dH = \prod_{i=1}^{N} dH_{i,i} \prod_{1 \leq i < j \leq N} d\text{Re}(H_{i,j}) d\text{Im}(H_{i,j})\) is the reference measure (and \(Z_N\) the normalization constant).

Denote by \(\lambda_{\text{GUE}, \max}^N\) the largest eigenvalue of an \(N \times N\) GUE matrix. Then Tracy and Widom in [49] proved that the asymptotic distribution of the (properly rescaled) largest eigenvalue is \(F_2\) (see figure 6):

\[
\lim_{N \to \infty} P(\frac{\lambda_{\text{GUE}, \max}^N - 2N}{N^{1/3}} \leq s) = F_2(s).
\]

The parallel between the GUE and the TASEP with step initial condition goes even further. In 1962 Dyson [19] introduced a matrix-valued Ornstein–Uhlenbeck process which is now called Dyson’s Brownian motion (DBM). For Hermitian matrices, GUE DBM is the stationary process on matrices \(H(t)\) whose evolution is governed by

\[
dH(t) = -\frac{1}{2N} H(t) \, dt + dB(t)
\]

where \(dB(t)\) is a (Hermitian) matrix-valued Brownian motion. More precisely, the entries \(B_{i,i}(t), 1 \leq i \leq N, \text{Re}(B_{i,j})(t)\) and \(\text{Im}(B_{i,j})(t), 1 \leq i < j \leq N,\) perform independent Brownian motions with variance \(t\) for diagonal terms and \(t/2\) for the remaining entries. Denote by \(\lambda_{\text{GUE}, \max}^N(t)\) the largest eigenvalue at time \(t\) (when started from the stationary measure (17)). Its evolution is, in the large \(N\) limit, governed by the Airy_2 process:

\[
\lim_{N \to \infty} \frac{\lambda_{\text{GUE}, \max}^N(2uN^{2/3}) - 2N}{N^{1/3}} = A_2(u).
\]
Thus we have seen that the connection between the GUE and the TASEP extends to the process\footnote{16}.}

\section{Symmetric matrices}

The Gaussian orthogonal ensemble (GOE) of random matrices consists of symmetric matrices $H$ of size $N \times N$ distributed according to

$$p^{\text{GOE}}(H) \, dH = \frac{1}{Z_N} \exp\left(-\frac{1}{4N} \text{Tr}(H^2)\right) \, dH,$$

where $dH = \prod_{1 \leq i \leq j \leq N} dH_{i,j}$ is the reference measure (and $Z_N$ the normalization constant).

Denote by $\lambda_{N,\text{max}}^{\text{GOE}}$ the largest eigenvalue of an $N \times N$ GOE matrix. The asymptotic distribution of the (properly rescaled) largest eigenvalue is $F_1$ (see figure 6) \cite{50}:

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{\lambda_{N,\text{max}}^{\text{GOE}} - 2N}{N^{1/3}} \leq s\right) = F_1(s).$$

DBM is defined also for symmetric matrices: GOE DBM is the stationary process on matrices $H(t)$ whose evolution is governed by

$$dH(t) = -\frac{1}{4N} H(t) \, dt + dB(t)$$

where $dB(t)$ is a symmetric matrix-valued Brownian motion. More precisely, the entries $B_{i,j}(t)$, $1 \leq i \leq j \leq N$, perform independent Brownian motions with variance $t$ for diagonal terms and $t/2$ for the remaining entries.

We consider now the evolution of the largest eigenvalue $\lambda_{N,\text{max}}^{\text{GOE}}(t)$ (when started from the stationary distribution (21)). By analogy with the GUE case, one might guess that in the large $N$ limit the limit process of a properly rescaled $\lambda_{N,\text{max}}^{\text{GOE}}(t)$ is the Airy$_1$ process. However, as shown numerically in \cite{8}, this is not the case. To see this, we considered the scaling (24) where the coefficients are chosen such that the variance at $u = 0$ is the same as the variance of the Airy$_1$ process, and the covariance for $|u| \ll 1$ coincides at first order with the covariance of $\mathcal{A}_1$. The large time limit of the rescaled largest eigenvalue is denoted by $\mathcal{B}_1$. Comparing the covariances as shown in figure 7 we conclude that $\mathcal{A}_1 \neq \mathcal{B}_1$:

$$\lim_{N \to \infty} \frac{\lambda_{N,\text{max}}^{\text{GOE}}(8uN^{2/3}) - 2N}{2N^{1/3}} =: \mathcal{B}_1(u) \neq \mathcal{A}_1(u).$$

Thus the connection between the GOE and the TASEP does not extend to multi-time distributions\footnote{17}.

\footnotetext[16]{This connection extends partially to the evolution of minors, see [1,26].}

\footnotetext[17]{The connection extends to the top eigenvalues too [23], but not still at a fixed time only.}

\textit{doi:10.1088/1742-5468/2010/10/P10016}
Figure 7. Log–log plot of the rescaled correlation functions for the GOE and the GUE. For $k = 1, 2$, we denote $g_k = \text{Cov}(A_k(u), A_k(0))$ and $f_{\text{GUE}}^N(u)$ (resp. $f_{\text{GOE}}^N(u)$) is the covariance of the GUE (resp. GOE) largest eigenvalues of an $N \times N$ matrix rescaled as in (20) (resp. (24)).

4. Conclusion

We saw that large time fluctuations in KPZ growth models depend on the initial conditions. By analyzing special models we determine the limiting distributions and processes, which by universality should be the same for the whole KPZ universality class. The conjecture [39] is that when the limit shape is curved one gets the Airy$_2$ process, when the limit shape is flat one has to further distinguish according to the roughness exponent $\alpha$ of the initial condition ($|h(x,0) - h(0,0)| \sim |x|^\alpha$). For $\alpha = 0$ one expects the Airy$_1$ process, for $\alpha = 1/2$ the result from the stationary initial condition. Finally, for $0 < \alpha < 1/2$, the characteristics should still be Airy$_1$ but there should be a region (away by of order $t^{1/(3\alpha)}$ from the characteristic lines) with a different (yet unknown) process, which depends on the exponent $\alpha$.

The two schemes in figure 8 summarize the main message of this contribution.

(a) The TASEP with step initial condition is a representant for models in the KPZ class with curved limit shape. For this model the large time limit distribution is $F_2$ and the limit process is the Airy$_2$ process, $A_2$. The GUE (with DBM dynamics) is a representant of random matrices with Hermitian symmetries and also for this model the $F_2$ distribution and the process $A_2$ arise in the limit of large matrices ($N \to \infty$).

(b) The TASEP with flat initial condition is one of the KPZ growth models with deterministic initial conditions and straight limit shape. The long time fluctuations are governed by the $F_1$ distributions and the limit process is the Airy$_1$ process. On the random matrix side, the GOE (with DBM dynamics) is a representant for symmetric random matrices and $F_1$ is the distribution of its rescaled largest eigenvalue.
as \( N \to \infty \). However, it is not the Airy_1 process which describes its dynamical extension.

The interested reader is referred to [28, 36] for more insights about random matrices and stochastic growth models. Therein a guide to the literature is provided. Also, two summer school lecture notes on the subject are available [34, 45].

On the experimental side, recently Takeuchi built up a very nice experimental setup in which the theoretical predictions (scaling exponents, one-point distributions and covariance) have been verified with very good agreement both for curved limit shape [47] and flat limit shape [46].

**Acknowledgments**

This work was supported by the DFG (German Research Foundation) through the SFB 611, project A12.

**Appendix. Definition of the limit processes**

In this short appendix we give the definitions of the Airy_1 and Airy_2 processes. More information about these processes like properties and references can be found for example in the reviews [25, 28].

**Definition 1.** The Airy_1 process \( A_1 \) is the process with \( m \)-point joint distributions at \( u_1 < u_2 < \cdots < u_m \) given by the Fredholm determinant

\[
P \left( \bigcap_{k=1}^{m} \{ A_1(u_k) \leq s_k \} \right) = \det \left( 1 - \chi s K_1 \chi_s \right)_{L^2([u_1, \ldots, u_m] \times \mathbb{R})} \tag{A.1}
\]
where \( \chi_s(u_k, x) = 1(x > s_k) \) and the kernel \( K_1 \) is given by

\[
K_1(u, s; u', s') = -\frac{1}{\sqrt{4\pi(u' - u)}} \exp\left(-\frac{(s' - s)^2}{4(u' - u)}\right) 1(u < u') + \text{Ai}(s + s' + (u' - u)^2) \exp\left((u' - u)(s + s') + \frac{2}{3}(u' - u)^3\right).
\]

**Definition 2.** The Airy \( A_2 \) process \( A_2 \) is the process with \( m \)-point joint distributions at \( u_1 < u_2 < \cdots < u_m \) given by the Fredholm determinant

\[
\mathbb{P}\left( \bigcap_{k=1}^m \{A_2(u_k) \leq s_k\} \right) = \det(1 - \chi_s K_2 \chi_s)_{L^2([u_1, \ldots, u_m] \times \mathbb{R})}
\]

where \( \chi_s(u_k, x) = 1(x > s_k) \) and \( K_2 \) is the extended Airy kernel given by

\[
K_2(u, s; u', s') = \left\{ \begin{array}{ll}
\int_{\mathbb{R}^+} d\lambda e^{(u'-u)\lambda} \text{Ai}(x + \lambda) \text{Ai}(y + \lambda), & u \geq u', \\
-\int_{\mathbb{R}^-} d\lambda e^{(u'-u)\lambda} \text{Ai}(x + \lambda) \text{Ai}(y + \lambda), & u < u'.
\end{array} \right.
\]

**References**

[1] Adler M, Nordenstam E and van Moerbeke P, *The Dyson Brownian minor process*, 2010 arXiv:1006.2956

[2] Amir G, Corwin I and Quastel J, *Probability distribution of the free energy of the continuum directed random polymer in \( 1+1 \) dimensions*, 2010 arXiv:1003.0443 Commun. Pure Appl. Math. at press

[3] Baik J and Rains E M, *Current fluctuations for TASEP: a proof of the Prähöfer–Spohn conjecture*, 2008 Commun. Math. Phys. 2000 J. Stat. Phys. 100 523

[4] Baik J and Rains E M, *Symmetrized random permutations*, 2001 Random Matrix Models and Their Applications vol 40 (Cambridge: Cambridge University Press) pp 1–19

[5] Bertini L and Giacomin G, *Stochastic Burgers and KPZ equations from particle system*, 1997 Commun. Math. Phys. 183 571

[6] Bornemann F, Ferrari P L and Prähöfer M, *The Airy\(_1\) process is not the limit of the largest eigenvalue in GOE matrix diffusion*, 2008 J. Stat. Phys. 133 405

[7] Borodin A and Ferrari P L, *Anisotropic growth of random surfaces in \( 2+1 \) dimensions*, 2008 arXiv:0804.3035

[8] Borodin A and Ferrari P L, *Large time asymptotics of growth models on space-like paths I: PushASEP*, 2008 Electron. J. Probab. 13 1380

[9] Borodin A and Ferrari P L, *Anisotropic KPZ growth in \( 2+1 \) dimensions: fluctuations and covariance structure*, 2009 J. Stat. Mech. P02009

[10] Borodin A, Ferrari P L, Prähöfer M and Sasamoto T, *Fluctuation properties of the TASEP with periodic initial configuration*, 2007 J. Stat. Phys. 129 1055

[11] Borodin A, Ferrari P L and Sasamoto T, *Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP*, 2008 Commun. Math. Phys. 283 417

[12] Borodin A, Ferrari P L and Sasamoto T, *Transition between Airy\(_1\) and Airy\(_2\) processes and TASEP fluctuations*, 2008 Commun. Pure Appl. Math. 61 1603

[13] Corwin I, Ferrari P L and Péché S, *Universality of slow decorrelation in KPZ models*, 2010 arXiv:1001.5345

[14] doi:10.1088/1742-5468/2010/10/P10016
From interacting particle systems to random matrices

[18] Derrida B, Janowsky S A, Lebowitz J L and Speer E R, Exact solution of the totally asymmetric simple exclusion process: shock profiles, 1993 J. Stat. Phys. 73 813
[19] Dyson F J, A Brownian-motion model for the eigenvalues of a random matrix, 1962 J. Math. Phys. 3 1191
[20] Edwards S F and Wilkinson D R, The surface statistics of a granular aggregate, 1982 Proc. R. Soc. A 381 17
[21] Evans L C, 1998 Partial Differential Equations, Graduate Studies in Mathematics, vol. 19 (Providence, RI: American Mathematical Society)
[22] Ferrari P A, Shock fluctuations in asymmetric simple exclusion, 1992 Probab. Theory Relat. Fields 91 81
[23] Ferrari P L, Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues, 2004 Commun. Math. Phys. 252 77
[24] Ferrari P L, Slow decorrelations in KPZ growth, 2008 J. Stat. Mech. P07022
[25] Ferrari P L, The universal \( \text{Airy}_1 \) and \( \text{Airy}_2 \) processes in the Totally Asymmetric Simple Exclusion Process, 2008 Integrable Systems and Random Matrices: In Honor of Percy Deift (Contemporary Math.) ed J Baik, T Kriecherbauer, L-C Li, K McLaughlin and C Tomei (Providence, RI: American Mathematical Society) pp 321–32
[26] Ferrari P L and Frings R, On the partial connection between random matrices and interacting particle systems, 2010 arXiv:1006.3946
[27] Ferrari P L and Spohn H, Scaling limit for the space–time covariance of the stationary totally asymmetric simple exclusion process, 2006 Commun. Math. Phys. 265 1
[28] Ferrari P L and Spohn H, Random growth models, 2010 arXiv:1003.0881
[29] Forster D, Nelson D R and Stephen M J, Slow decorrelations in KPZ growth, 1993 J. Stat. Phys. 71 621
[30] Imamura T and Sasamoto T, Dynamical properties of a tagged particle in the totally asymmetric simple exclusion process with the step-type initial condition, 2007 J. Stat. Phys. 128 799
[31] Johansson K, Shape fluctuations and random matrices, 2000 Commun. Math. Phys. 209 437
[32] Johansson K, Discrete polynuclear growth and determinantal processes, 2003 Commun. Math. Phys. 242 277
[33] Johansson K, The arctic circle boundary and the Airy process, 2005 Ann. Probab. 33 1
[34] Johansson K, Random matrices and determinantal processes, 2006 Mathematical Statistical Physics, Session LXXXIII (Lecture Notes of the Les Houches Summer School 2005) ed A Bovier, F Dunlop, A van Enter, F den Hollander and J Dalibard (Amsterdam: Elsevier Science) pp 1–56
[35] Kardar K, Parisi G and Zhang Y Z, Dynamic scaling of growing interfaces, 1986 Phys. Rev. Lett. 56 889
[36] Kriecherbauer T and Krug J, A pedestrian’s view on interacting particle systems, KPZ universality, and random matrices, 2010 J. Phys. A: Math. Theor. 43 403901
[37] Liggett T M, Coupling the simple exclusion process, 1976 Ann. Probab. 4 339
[38] Prüfer M and Spohn H, An exactly solved model of three dimensional surface growth in the anisotropic KPZ regime, 1997 J. Stat. Phys. 88 999
[39] Prüfer M and Spohn H, Universal distributions for growth processes in 1 + 1 dimensions and random matrices, 2000 Phys. Rev. Lett. 84 4882
[40] Prüfer M and Spohn H, Current fluctuations for the totally asymmetric simple exclusion process, 2002 In and Out of Equilibrium (Progress in Probability) ed V Sidoravicius (Basle: Birkhäuser)
[41] Prüfer M and Spohn H, Scale invariance of the PNG droplet and the Airy process, 2002 J. Stat. Phys. 108 1071
[42] Sasamoto T, Spatial correlations of the 1D KPZ surface on a flat substrate, 2005 J. Phys. A: Math. Gen. 38 L549
[43] Sasamoto T and Spohn H, The 1 + 1-dimensional Kardar–Parisi–Zhang equation and its universality class, 2010 Contribution to StatPhys24 special issue, in preparation
[44] Sasamoto T and Spohn H, Universality of the one-dimensional KPZ equation, 2010 arXiv:1002.1883
[45] Spohn H, Exact solutions for KPZ-type growth processes, random matrices, and equilibrium shapes of crystals, 2006 Physica A 369 71
[46] Takeuchi K, 2010 private communication
[47] Takeuchi K and Sano M, Growing interfaces of liquid crystal turbulence: universal scaling and fluctuations, 2010 Phys. Rev. Lett. 104 230601
[48] Tang L-H, Forrest B M and Wolf D E, Kinetic surface roughening. II. Hypercube stacking models, 1992 Phys. Rev. A 45 7162
[49] Tracy C A and Widom H, Level-spacing distributions and the Airy kernel, 1994 Commun. Math. Phys. 159 151

doi:10.1088/1742-5468/2010/10/P10016
[50] Tracy C A and Widom H, On orthogonal and symplectic matrix ensembles, 1996 Commun. Math. Phys. 177 727

[51] van Beijeren H, Fluctuations in the motions of mass and of patterns in one-dimensional driven diffusive systems, 1991 J. Stat. Phys. 63 47

[52] van Beijeren H, Kutner R and Spohn H, Excess noise for driven diffusive systems, 1985 Phys. Rev. Lett. 54 2026

[53] Varadhan S R S, Large deviations for the asymmetric simple exclusion process, 2004 Adv. Stud. Pure Math. 39 1

[54] Wolf D E, Kinetic roughening of vicinal surfaces, 1991 Phys. Rev. Lett. 67 1783