Computing lexicographically safe Nash equilibria in finite two-person games with tight game forms given by oracles

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Abstract

In 1975 the first author proved that every finite tight two-person game form $g$ is Nash-solvable, that is, for every payoffs $u$ and $w$ of two players the obtained normal form game $(g; u, w)$ has a Nash equilibrium (NE) in pure strategies. Several proofs of this theorem were obtained later. Here we strengthen the result and give a new proof, which is shorter than previous ones. We show that game $(g; u, w)$ has two types of NE, realized by a lexicographically safe (lexsafe) strategy of one player and some special best response of the other. The prove is constructive, we obtain a polynomial algorithm computing these lexsafe NE. This is trivial when game form $g$ is given explicitly. Yet, in applications $g$ is frequently realized by an oracle $O$ such that size of $g$ is exponential in the size $|O|$ of $O$. We assume that game form $g = g(O)$ generated by $O$ is tight and that an arbitrary $\pm 1$ game $(g; u^0, w^0)$ (in which payoffs $u^0$ and $w^0$ are zero-sum and take only values $\pm 1$) can be solved in time polynomial in $|O|$. These assumptions allow us to compute two (one for each player) lexsafe NE in time polynomial in $|O|$. These NE may coincide. We consider four types of oracles known in the literature and show that all four satisfy the above assumptions.

Keywords: Nash equilibrium, Nash-solvability, game form, tightness, deterministic graphical game structure, game in normal and in positional form, monotone bargaining, veto voting, Jordan game.

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1. Introduction

Here we outline main results. Precise definitions will be given later.

Consider a finite \( n \)-person game in normal form representing it as a pair \((g, u)\), where \( u \) is the payoff function of \( n \) players and \( g \) is the so-called game form. Respectively, the latter can be viewed as a game without payoffs, which are not given yet. Such approach is standard and convenient: game form \( g \) “is responsible” for structural properties of game \((g, u)\), which hold for any payoff \( u \). For example, game form \( g \) is called Nash-solvable if game \((g, u)\) has a Nash equilibrium (NE) in pure strategies for every payoff \( u \).

In 1950 Nash proved that every \( n \)-person normal form game has a NE in mixed strategies \cite{39, 40}. Yet, there are large families of games solvable in pure strategies, for example, finite \( n \)-person positional (graphical) games with perfect information. Its game structures \( \Gamma \) uniquely defines a finite \( n \)-person game form \( g(\Gamma) \) that is Nash-solvable. In Section 8.3 we consider this class of Nash-solvable game forms. Furthermore, we expand the set of outcomes including not only terminal positions but also other strongly connected components of the corresponding directed graph. Doing so, we also expand substantially the corresponding family of game forms, which remain Nash-solvable, but only in case of two players, \( n = 2 \).

Yet, perfect information is only sufficient but not necessary for Nash-solvability. A concept of tightness fits much better. This property is of algebraic nature. It was introduced in \cite{20, 21} and in the latter paper it was shown that a finite two-person game form is Nash-solvable if and only if it is tight. Note that already for \( n = 3 \) tightness is neither necessary nor sufficient for Nash-solvability. These results were obtained in \cite{21, 23}; several different proofs were given later \cite{3, 5, 11, 27, 30}. Tightness remains necessary (and, of course, sufficient) for Nash-solvability in the zero-sum case too. This result was obtained earlier: it follows easily from the so-called Bottleneck Extrema Theorem by Edmonds and Fulkerson \cite{12}; see also \cite{20}.

Here we suggest a new (and much simpler) proof of the general result. We introduce a concept of lexicographically safe (lexsafe) pure strategy of a player in a given game \((g, u)\). This is a refinement of the standard concept of a safe (maxmin) strategy that maximizes the worst possible outcome, while the lexsafe strategy realizes the lexicographical maximum of all possible outcomes. Thus, the lexsafe strategies are most safe, but may be not rational. (For comparison, recall that NE may be not Pareto optimal.) One can view this as a price of stability.

We prove that a NE appears whenever one player applies a lexsafe strategy, while the opponent chooses some special best response to it. Yet, if both players
choose their lexsafe strategies then the obtained pair may be not a NE.

Thus, there are two types of NE: lexsafe for one or for the other player. These NE may coincide. For example, it happens in the zero-sum case, or when the considered game has a unique NE.

By definition, lexsafe strategies of a player do not depend on the payoff of the opponent; the player may be just unaware of it. This is an interesting property important for applications.

In the proof given in \cite{21,23} the lexsafe strategies were implicitly constructed by an iterative algorithm increasing strategies in a lexicographical order. Here we suggest a simple polynomial algorithm searching for a lexsafe strategy of a player and for a corresponding NE. Such algorithm is obvious when a game form $g$ is given explicitly. Yet, in applications $g$ is frequently given by an oracle $\mathcal{O}$, which size may be logarithmic in the size of $g$. We assume that this oracle solves in polynomial time any two-person game $(g,u^0)$ in which payoff $u^0$ is zero-sum and takes only values $\pm 1$; oracle $\mathcal{O}$ tells us who wins and determines a winning strategy. Based on this assumption, we provide an algorithm computing a lexsafe NE in an arbitrary game $(g,u)$ in time polynomial in the size of $\mathcal{O}$.

In the last section we consider four examples of such oracles from different areas of game theory and show that all four satisfy the above assumption.

2. Basic definitions

2.1. Game forms

In this paper we consider finite, not necessarily zero-sum, normal form games of two players, Alice and Bob. They have finite sets of strategies $X$ and $Y$, respectively. A game form is a mapping $g : X \times Y \to \Omega$, where $\Omega$ is a finite set of outcomes. Several examples are given in Figure 1 where game forms are represented by tables with rows, columns, and entries labelled by $x \in X$, $y \in Y$, and $\omega \in \Omega$, respectively.

Mapping $g$ is assumed to be surjective, but not necessarily injective, that is, an outcome $\omega \in \Omega$ may occupy an arbitrary array in the table of $g$.

A pair of strategies $(x,y)$ is called a situation (term ”strategy profile” is also used in literature). Sets $g(x) = \{g(x,y) \mid y \in Y\}$ and $g(y) = \{g(x,y) \mid x \in X\}$ are called the supports of strategies $x \in X$ and $y \in Y$, respectively.

A strategy is called minimal if its support is not a proper superset of the support of any other strategy. For example, in $g_6$ the first strategies of Alice and Bob are minimal, while the second are not; in the remaining eight game forms all strategies are minimal. Moreover, any two strategies of a player have distinct supports, for every game form, except $g_7$. 
A situation \((x, y)\) is called simple if \(g(x) \cap g(y) = \{g(x, y)\}\). For example, all situations of game forms \(g_1, g_2, g_8, g_9\) are simple (such game forms are called rectangular); in contrast, no situation is simple in \(g_7\); in \(g_3\) all are simple, except three on the main diagonal; in \(g_4\) all are simple, except the central one; in \(g_6\) all are simple, except one with the outcome \(\omega_2\).

2.2. Payoffs and games in normal form

Payoffs of Alice and Bob are defined by real valued mappings \(u : \Omega \to \mathbb{R}\) and \(w : \Omega \to \mathbb{R}\), respectively. We assume that both players are maximizers. Triplet \((g; u, w)\) defines a finite two-person game in normal form, or just a game, for short. Game \((g; u, w)\) and payoffs \((u, w)\) are called:

- zero-sum if \(u + w = 0\), that is, \(u(\omega) + w(\omega) = 0\) for all \(\omega \in \Omega\);
- zero-sum \(\pm 1\) (or just \(\pm 1\), for short) if, in addition, functions \(u\) and \(w\) take only two values 1 and \(-1\).

Alternatively, a \(\pm 1\) payoff can be given by a partition \(\Omega = \Omega_A \cup \Omega_B\), where \(\Omega_A\) and \(\Omega_B\) are the outcomes preferred by Alice and by Bob, respectively:

\[ u(\omega) = 1, \quad w(\omega) = -1 \quad \text{for} \quad \omega \in \Omega_A \quad \text{and} \quad u(\omega) = -1, \quad w(\omega) = 1 \quad \text{for} \quad \omega \in \Omega_B. \]

For a \(\pm 1\) game notation \((g; \Omega_A, \Omega_B)\) will be used along with \((g; u, w)\).
2.3. Nash equilibria and saddle points

Given a game \((g; u, w)\), a situation \((x, y)\) of its game form \(g : X \times Y \to A\) is called a Nash equilibrium (NE) if

\[
u(g(x, y)) \geq u(g(x', y)), \forall x' \in X, \text{ and } w(g(x, y)) \geq w(g(x, y')), \forall y' \in Y;
\]

that is, if neither Alice nor Bob can profit replacing her/his strategy provided the opponent keeps his/her one unchanged, or in other words, if \(x\) is a best response for \(y\) and \(y\) is a best response for \(x\). Note that a best response may be not unique.

This concept of solution was introduced in 1950 by John Nash [39, 40]. In the zero-sum case, a NE is called a saddle point. The latter concept was known much earlier; the former one is its natural extension to the non-zero-sum case.

Recall that a zero-sum game \((g, u, w)\) has a saddle point if and only if

\[
\max \min = \max_{x \in X} \min_{\omega \in g(x)} u(\omega); \min \max = \min_{y \in Y} \max_{\omega \in g(y)} u(\omega);
\]

furthermore, \(\max \min < \min \max\) if and only if \((g; u, w)\) has no saddle point.

**Remark 1.** In [39, 40] solvability in mixed strategies is studies. In contrast, we restrict the players to their pure strategies. Such approach is considered, for example, in [1, 2, 3, 4, 9, 5, 7, 8, 14, 21, 23, 26, 27, 30, 32, 34, 36, 37, 42, 44, 46, 47].

2.4. Solvability of game forms

A game form \(g\) is called: (i) Nash-, (ii) zero-sum-, (iii) \(\pm 1\)-solvable if the corresponding game \((g; u, w)\) has a NE for (i) all, (ii) all zero-sum, (iii) all zero-sum \(\pm 1\) payoffs, respectively.

Implications \((i) \Rightarrow (ii) \Rightarrow (iii)\) are obvious. In fact, all three properties are equivalent [21, 23, 25]. For (ii) and (iii) it was shown earlier by Edmonds and Fulkerson [12]; see also [20]. The list of equivalent properties \((i), (ii), (iii)\) was extended in [21] as follows.

2.5. Replacing payoffs by preferences and eliminating ties

Given a game \((g; u, w)\), we can assume wlog that payoffs \(u : \Omega \to \mathbb{R}\) and \(w : \Omega \to \mathbb{R}\) have no ties. Indeed, one can get rid of all ties by arbitrarily small perturbations of values of \(u\) and \(w\). In accordance with definition, the set of NE will be either unchanged or reduced by such perturbations. We focus on Nash-solvability (in pure strategies), that is we study conditions that guarantee the
existence of a NE for arbitrary payoffs \( u \) and \( w \). Hence, we can wlog assume that both, \( u \) and \( w \), have no ties and replace them by linear orders \( \succ_A \) and \( \succ_B \) over the set of outcomes \( \Omega \), which are called the preferences of Alice and Bob, respectively. Thus, game \((g; u, w)\) can be replaced by \((g; \succ_A, \succ_B)\) and it is enough to study Nash-solvability of the latter. Although some NE of \((g; u, w)\) may disappear in \((g; \succ_A, \succ_B)\), yet, Nash-solvability holds or fails for both games simultaneously.

**Remark 2.** Above arguments would fail in the case of mixed strategies.

### 2.6. Tight game forms

Mappings \( \phi : X \rightarrow Y \) and \( \psi : Y \rightarrow X \) are called response strategies of Bob and Alice, respectively. The motivation for this name is clear: a player chooses his/her strategy as a function of a known strategy of the opponent. Standardly, \( gr(\phi) \) and \( gr(\psi) \) denote the graphs of mappings \( \phi \) and \( \psi \) in \( X \times Y \).

Game form \( g : X \times Y \rightarrow \Omega \) is called tight if

\[
\text{(l)} \quad g(gr(\phi)) \cap g(gr(\psi)) \neq \emptyset \quad \text{for any mappings} \quad \phi \quad \text{and} \quad \psi.
\]

It is not difficult to verify that in Figure 1 the first six game forms \((g_1 - g_6)\) are tight, while the last three \((g_7 - g_9)\) are not.

In [12, 20, 21, 23, 27] the reader can find several equivalent properties characterizing tightness. Here we recall some of them.

1. **(ll-A)** For every response strategy \( \phi : X \rightarrow Y \) there exists a strategy \( y \in Y \) such that \( g(y) \subseteq g(gr(\phi)) \).

2. **(ll-B)** For every response strategy \( \psi : Y \rightarrow X \) there exists a strategy \( x \in X \) such that \( g(x) \subseteq g(gr(\phi)) \).

It is not difficult to see that (l) and (ll-A) are equivalent [20, 23]. Then, by transposing \( g \), we conclude that (l) and (ll-B) are equivalent as well. Hence, all three properties are equivalent. One can verify this for nine examples \( g_1 - g_9 \).

Properties (ll-A) and (ll-B) show that playing a zero-sum game \((g; u, w)\) with a tight game form \( g \) Bob and Alice do not need non-trivial response strategies but can restrict themselves by the standard ones, that is, by \( Y \) and \( X \), respectively.

Given a game form \( g : X \times Y \rightarrow \Omega \), introduce on the ground set \( \Omega \) of the outcomes two multi-hypergraphs \( \mathcal{A} = \mathcal{A}(g) \) and \( \mathcal{B} = \mathcal{B}(g) \) whose edges are the supports of strategies of Alice and Bob, respectively:

\[
\mathcal{A}(g) = \{g(x) \mid x \in X\} \quad \text{and} \quad \mathcal{B}(g) = \{g(y) \mid y \in Y\}.
\]
Recall that distinct edges of a multi-hypergraph may contain one another or even coincide. Obviously, the edges of $A$ and $B$ pairwise intersect, that is, $g(x) \cap g(y) \neq \emptyset$ for all $x \in X$ and $y \in Y$. Furthermore, $g$ is tight if and only if

(iii) hypergraphs $A(g)$ and $B(g)$ are dual, that is, satisfy also the following two (equivalent) properties:

(iii-A) for every $\Omega_A \subseteq \Omega$ such that $\Omega_A \cap g(y) \neq \emptyset$ for all $y \in Y$ there exists an $x \in X$ such that $g(x) \subseteq \Omega_A$;

(iii-B) for every $\Omega_B \subseteq \Omega$ such that $\Omega_B \cap g(x) \neq \emptyset$ for all $x \in X$ there exists an $y \in Y$ such that $g(y) \subseteq \Omega_B$.

Remark 3. Verification of tightness of an explicitly given game form is an important open problem. No polynomial algorithm is known. A quasi-polynomial one was suggested in [13]; see also [29].

3. Tightness and solvability

Let us recall an old theorem.

Theorem 1. ([21, 23]) The following properties of a game form are equivalent:

(i) Nash-, (ii) zero-sum-, (iii) $\pm 1$-solvability, and (iv) tightness.

Proof. As we already mentioned, implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.

Also (iii) $\Rightarrow$ (iv) is easily seen. Indeed, assume for contradiction that a game form $g$ is not tight. Then, there exists a response strategies $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ of Bob and Alice such that $g(gr(\phi)) \cap g(gr(\psi)) = \emptyset$. Then, we can partition $\Omega$ into two sets of outcomes $\Omega_A$ and $\Omega_B$ (winning for Alice and Bob, respectively) in such a way that $g(gr(\phi)) \subseteq \Omega_B$ and $g(gr(\psi)) \subseteq \Omega_A$. (Note that for tight $g$ this would not be possible.) Then, $-1 = maxmin < minmax = 1$ in the obtained $\pm 1$ game $(g; \Omega_A, \Omega_B)$ and, hence, it has no saddle point.

The inverse implication (iii) $\Leftarrow$ (iv), (as well as (ii) $\Leftarrow$ (iv), which looks stronger) are proven similarly; see [12, 20]. Assume that a zero-sum game $(g; u, w)$ has no saddle point. Then, (iii) fails and $maxmin < minmax$. Consider arbitrary best response strategies $\phi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ of Bob and Alice, respectively. Obviously, $g(gr(\phi)) \cap g(gr(\psi)) = \emptyset$ and, thus, $g$ is not tight.

The last claim means that a tight game form is SP-solvable. Moreover, it has a simple SP situation in minimal strategies [23].

To finish the proof of the theorem it only remains to show implication (i) $\Leftarrow$ (iv); that is, tightness implies Nash-solvability. First, this was done in
4. Lexicographical theorem for dual multi-hypergraphs

4.1. Summary

Let \( \mathcal{A} \) and \( \mathcal{B} \) be an arbitrary pair of finite dual multi-hypergraphs on a common ground set \( \Omega \). Each of them may have embedded or equal edges. An edge is called *containment minimal* (or just *minimal*, for short) if it is not a strict superset of another edge. (Note that minimal edges may still be equal.)

If \( \mathcal{A} \) and \( \mathcal{B} \) are dual then

(j) \( A \cap B \neq \emptyset \) for every pair \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \);

(jj) if \( A \) is minimal then for every \( \omega \in A \) there exists a (minimal) \( B \in \mathcal{B} \) such that \( A \cap B = \{\omega\} \).

We will extend claim (jj) as follows. A linear order \( \succ \) over \( \Omega \) uniquely defines a lexicographic order \( \succ_\ell \) over the power set \( 2^\Omega \).

(jjj-A) Let \( A \) be a lexicographically maximal (lexmax) edge of \( \mathcal{A} \). Then, edge \( A \) is minimal in \( \mathcal{A} \) and for every \( \omega \in A \) there exists a (minimal) \( B \in \mathcal{B} \) such that \( A \cap B = \{\omega\} \) and \( \omega \succeq \omega' \) for each \( \omega' \in B \).

By swapping \( A, \mathcal{A} \) and \( B, \mathcal{B} \), we obtain the dual statement (jjj-B).

These two statements form the *lexicographical theorem* for dual multi-hypergraphs. To formulate it accurately, we will need a few definitions.

4.2. Lexicographical orders over the subsets.

A linear order \( \succ \) over a set \( \Omega \) uniquely determines a lexicographical order \( \succ_\ell \) over the power set \( 2^\Omega \) (of all subsets of \( \Omega \)) as follows. Roughly speaking, the more small elements are out of a set - the better it is. In particular, \( \Omega' \succ_\ell \Omega'' \) whenever \( \Omega' \subset \Omega'' \) and, hence, the empty set \( \emptyset \subset \Omega \) is the best in \( 2^\Omega \).

**Remark 4.** Also \( \{\omega'\} \succ_\ell \{\omega', \omega''\} \) for any \( \omega', \omega'' \in \Omega \) and order \( \succ \), although set \( \{\omega', \omega''\} \) gives a chance for a better outcome \( \omega'' \) if \( \omega' \prec \omega'' \); see game form \( g_6 \) in Figure 1 and subsection 6.2 for more detail.

More precisely, to compare two arbitrary subsets \( \Omega', \Omega'' \subseteq \Omega \) consider their symmetric difference \( \Delta = (\Omega' \setminus \Omega'') \cup (\Omega'' \setminus \Omega') \). Clearly, \( \Delta \neq \emptyset \) if and only if sets \( \Omega' \) and \( \Omega'' \) are distinct. Let \( \omega \) be the minimum with respect to \( \succ \) element in \( \Delta \). If \( \omega \in (\Omega' \setminus \Omega'') \) then \( \Omega'' \succ_\ell \Omega' \); if \( \omega \in (\Omega'' \setminus \Omega') \) then \( \Omega' \succ_\ell \Omega'' \).

We can reformulate this definition equivalently as follows. Without loss of generality (wlog), set \( \Omega = \{\omega_1, \ldots, \omega_p\} \) and assume that \( \omega_1 \prec \cdots \prec \omega_p \); assign
the negative weight \( w(\omega_i) = -2^{k-i} \) to every \( \omega_i \in \Omega \), and set \( w(S) = \sum_{\omega \in S} w(\omega) \) for each subset \( S \subseteq \Omega \). Then, \( \Omega' \succ_{\ell} \Omega'' \) if and only if \( w(\Omega') > w(\Omega'') \).

Denote by \( supp(S) \) the 0,1-vector \((s_1, \ldots, s_k)\), where \( s_i = 1 \) if and only if \( \omega_i \in S \). Then obviously, \( \Omega' \succ_{\ell} \Omega'' \) if and only if \( supp(\Omega') \) is less than \( supp(\Omega'') \) in the standard lexicographical order.

4.3. Dual multi-hypergraphs

Two finite multi-hypergraphs \( \mathcal{A} \) and \( \mathcal{B} \) on the common ground set \( \Omega \) are called dual if (j) holds: \( A \cap B \neq \emptyset \) for every pair \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), and also (jv-A) for each \( B^T \subseteq \Omega \) such that \( B^T \cap B \neq \emptyset \) for every \( B \in \mathcal{B} \) there exists an \( A \in \mathcal{A} \) such that \( A \subseteq B^T \).

If (j) and (jv-A) both hold we say that \( \mathcal{A} \) is dual to \( \mathcal{B} \) and use notation \( \mathcal{A} \equiv \mathcal{B} \). Swapping \( \mathcal{A}, \mathcal{A} \) and \( \mathcal{B}, \mathcal{B} \) in (jv-A) we obtain (jv-B) and an equivalent definition of duality, that is, (j) and (jv-A) hold if and only (j) and (jv-B) hold. In other words, \( \mathcal{A} \equiv \mathcal{B} \) if and only if \( \mathcal{B} \equiv \mathcal{A} \). So we just say that multi-hypergraphs \( \mathcal{A} \) and \( \mathcal{B} \) are dual.

**Remark 5.** Dual multi-hypergraphs have numerous applications and appear in different areas under different names, such as “clutters” and “blockers” [12] or DNFs and CNFs of monotone Boolean functions [10].

4.4. Lexicographical Theorem

Claims (j) and (jj) are well-known [10]. Actually, (j) is required by the definition of duality and (jj) is obvious. Indeed, if (jj) fails then edge \( A \) cannot be minimal, since its proper subset \( A \setminus \{\omega\} \) would still intersect all \( B \in \mathcal{B} \).

Our main result is statement (jjj-A). Fix an arbitrary order \( >_{\ell} \) over \( \Omega \) and find a lexmax edge \( A^L \in \mathcal{A} \), that is, one maximal with respect to the lexicographical order \( >_{\ell} \) over \( 2^\Omega \). Note that such \( A^L \) may be not unique but all lexmax edges are equal. The lexicographic theorem is formulated as follows:

**Theorem 2.** A lexmax edge \( A^L \) is minimal in \( \mathcal{A} \). Furthermore, for every \( \omega^* \in A^L \) there exists a (minimal) edge \( B^M \in \mathcal{B} \) such that \( A^L \cap B^M = \{\omega^*\} \) and \( \omega^* >_{\ell} \omega \) for each \( \omega \in B^M \setminus \{\omega^*\} \).

**Proof.** A lexmax edge must be minimal, since a set is strictly less than any its proper subset in order \( >_{\ell} \).

Assume for contradiction that there exists an \( \omega^* \in A^L \) such that for every (minimal) \( B \in \mathcal{B} \) satisfying (jj), \( B \cap A^L = \{\omega^*\} \), there exists an \( \omega \in B \) such that \( \omega >_{\ell} \omega^* \). Clearly, this assumption holds for every \( B^0 \in \mathcal{B} \) if it holds for each minimal \( B^0 \in \mathcal{B} \). Let us show that it contradicts the lexmaximality of \( A^L \). To do so partition all edges \( B \in \mathcal{B} \) into two types:
(a) there is an $\omega \in B \cap A^L$ distinct from $\omega^*$;
(b) $B \cap A^L = \{\omega^*\}$.

In case (b), by our assumption, there is an $\omega \in B$ such that $\omega \succ \omega^*$.

In both cases, (a) and (b), choose the specified $\omega$ from $B$, thus, getting a transversal $B^T$. By (jv-A), there exists an $A \in A$ such that $A \subseteq B^T$ and, hence, $A \succeq \ell B^T$. Furthermore, by construction, $B^T \succeq \ell A^L$. Indeed, $\omega^* \not\in B^T$ and it is replaced in $B^T$ by some larger elements, $\omega \succ \omega^*$, in case (b), while all other elements of $B^T$, if any, belong to $A^L \setminus \{\omega^*\}$, according to case (a).

Thus, by transitivity, $A \succ \ell A^L$. Yet, by assumption of the theorem, $A^L$ is a lexmax edge of $A$, which is a contradiction.

4.5. Sperner hypergraphs

A multi-hypergraph is called Sperner if no two of its distinct edges contain one another; in particular, they cannot be equal. In this case, we have a hypergraph rather than multi-hypergraph. For a multi-hypergraph there exists a unique dual Sperner hypergraph. If $A$ and $B$ are dual and Sperner then $A^{dd} = A$ and $B^{dd} = B$; furthermore $\cup_{A \in A} A = \cup_{B \in B} B = \Omega$. In general, for multi-hypergraphs, $\cup_{A \in A} A$ and $\cup_{B \in B} B$ may be different subsets of $\Omega$.

**Remark 6.** Here we assume that the reader is familiar with basic notions related to monotone Boolean functions, in particular, with DNFs and duality. An introduction can be found in [10, Sections 1, 3 and 4].

It is well known [10] that (dual) multi-hypergraphs are in one-to-one correspondence with (dual) monotone DNFs: (prime) implicants of the latter correspond to (minimal) edges of the former. Furthermore, Sperner hypergraphs correspond to irredundant DNFs. However, we do not restrict ourselves to this case. Although the lexicographical theorem would not lose much but its applications to Nash-solvability would.

5. Determining edges $A^L$ and $B^M$ of Theorem 2 in polynomial time.

5.1. Preliminaries

Edges $A$ and $B$ mentioned in (jjj-A) can be found in polynomial time. The problem is trivial when multi-hypergraphs $A$ and $B$ are given explicitly. We will solve it when only $A$ is given, and not explicitly, but by a polynomial containment oracle. For an arbitrary subset $\Omega_A \subseteq \Omega$ this oracle answers in polynomial time the question $Q(A, \Omega_A)$: whether $\Omega_A$ contains an edge $A \in A$.

By duality of $A$ and $B$, we have $A \not\subseteq \Omega_A$ for all $A \in A$ if and only if $B \subseteq \Omega_B = \Omega \setminus \Omega_A$ for some $B \in B$. In other words, question $Q(A, \Omega_A)$ is answered in the negative if and only if $Q(B, \Omega_B)$ is answered in the positive.
Thus, we do not need two separate oracles for $A$ and $B$; it is sufficient to have one, say, for $A$.

5.2. Determining a lexmax edge $A^L$ in polynomial time.

Recall that multi-hypergraph $A$ may contain several lexmax edges $A^L$, but they are all equal. Fix an arbitrary linear order $\succ$ over $\Omega$. Wlog we can assume that $\Omega = \{\omega_1, \ldots, \omega_p\}$ and $\omega_1 \prec \cdots \prec \omega_p$.

Step 1: Consider $\Omega_1^t = \{\omega_t, \ldots, \omega_p\}$ and, by asking question $Q(A, \Omega_1^t)$ for $t = 1, \ldots, p$, find the maximum $t_1$ for which the answer is still positive. Then, $\omega_{t_1}$ belongs to $A^L$, while $\omega_1, \ldots, \omega_{t_1-1}$ do not.

Step 2: Consider $\Omega_2^t = \{\omega_1, \omega_{t_1+t}, \ldots, \omega_p\}$ and, by asking question $Q(A, \Omega_2^t)$ for $t = 1, \ldots, p - t_1$, find the maximum $t_2$ for which the answer is still positive. Then, $\omega_{t_1}, \omega_{t_1+t_2} \in A^L$, while $\omega_t \not\in A^L$ for any other $t < t_1 + t_2$.

Step 3: Consider $\Omega_3^t = \{\omega_1, \omega_{t_1+t_2}, \omega_{t_1+t_2+t_3}, \ldots, \omega_p\}$ and, by asking question $Q(A, \Omega_3^t)$ for $t = 1, \ldots, p - (t_1 + t_2)$, find the maximum $t_3$ for which the answer is still positive. Then, $\omega_{t_1}, \omega_{t_1+t_2}, \omega_{t_1+t_2+t_3} \in A^L$, while $\omega_t \not\in A^L$ for any other $t < t_1 + t_2 + t_3$; etc.

This procedure will produce a lexmax edge $A^L$ in at most $p$ polynomial iterations. Note that on each step $i$ we can speed up the search of $t_i$ by applying dichotomy.

5.3. Determining an edge $B^M$ from Theorem 2

First, find a lexmax edge $A^L \in A$ and choose an arbitrary $\omega^* \in A^L$. We look for an edge $B^M \in B$ such that $A^L \cap B^M = \{\omega^*\}$ and $\omega^* \succ \omega$ for every $\omega \in B^M \setminus \{\omega^*\}$. In other "words",

$$B^M \subseteq \Omega_B = \Omega \setminus [(A^L \setminus \{\omega^*\}) \cup \{\omega \mid \omega \succ \omega^*\}].$$

By Theorem 2 such $B^M$ exists and, hence, the oracle answers $Q(B, \Omega_B)$ in the positive, or equivalently, $Q(A, \Omega \setminus \Omega_B)$ in the negative. We could take any $B^M \in B$ such that $B^M \subseteq \Omega_B$. Yet, multi-hypergraph $B$ is not given explicitly. To get $B^M$ we need "to minimize" $\Omega_B$. To do so, let us delete its elements one by one in some order until we obtain a minimum set $\Omega'_B$ for which the answer to $Q(A, \Omega \setminus \Omega'_B)$ is still negative, that is, answers to $Q(A, \Omega \setminus (\Omega'_B \setminus \{\omega\})$ become positive for every $\omega \in \Omega'_B$. Then, we set $B^M = \Omega'_B$. Again we can speed up the procedure by applying dichotomy.

Note that the above reduction procedure may be not unique, since we can eliminate elements of $\Omega \setminus \Omega_B$ in an arbitrary order. Thus, in contrast to $A^L$, there may be several not equal edges $B^M$ satisfying all conditions of Theorem 2.
6. Lexicographically safe NE in games with tight game forms

6.1. Summary

First, we apply Theorem 2 to finish the proof of Theorem 1. It remains to show that \((i) \iff (iv)\), that is, tightness implies Nash-solvability. In other words, a game \((g; u, w)\) has a NE for any payoffs \(u\) and \(w\) whenever game form \(g\) is tight. The proof is constructive: we will obtain two special types of NE.

Given \(g\) and \(u\), choose a lexmax strategy \(x \in X\) of Alice. By Theorem 2, there is a strategy \(y \in Y\) of Bob such that \((x, y)\) is a NE. By definition, \(y\) must be a best response to \(x\) such that \(x\) is also a best response to \(y\). By Theorem 2, the obtain situation \((x, y)\) is simple and both strategies, \(x\) and \(y\) are minimal. More precisely, \(x\) must be minimal, while \(y\) can be chosen minimal. These NE will be called lexsafe NE of Alice and the set of these NE will denoted by \(\text{NE-A}\). Similarly, we define a set \(\text{NE-B}\) of Bob’s lexsafe NE.

Remark 7. We assume that both players are maximizers and adjective “lexsafe” can be replaced by “lexmax”. If both players are minimizers then it can be replaced by “lexmin”. In the zero-sum case Alice is the maximizer, while Bob is the minimizer. Flexible term lexsafe may replace both, lexmax or lexmin.

Results of Section 5 provide a polynomial algorithm determining at least one NE from \(\text{NE-A}\) and at least one from \(\text{NE-B}\) (which may coincide) in a given game \((g; u, w)\) with a tight game form \(g\). This is trivial when \(g\) is given explicitly. Yet, the algorithm works when one of two multi-hypergraphs \(\mathcal{A}(g)\) or \(\mathcal{B}(g)\) is given by a polynomial containment oracle.

6.2. Lexicographically safe strategies of players

Given \(g\) and preference \(\succ_A\) of Alice, let us introduce the lexicographical preorder over Alice’s strategies \(x \in X\) as follows. Consider lexicographical order \(\succ^\ell_A\) over \(2^\Omega\) defined by the linear order \(\succ_A\) over \(\Omega\). The larger is the support \(g(x) \subseteq \Omega\) in order \(\succ^\ell_A\), the safer is strategy \(x\) for Alice, while strategies with the same support are equally safe. Alice’s strategies that maximize support \(g(x)\) in order \(\succ^\ell_A\) will be called her lexmax (or lexsafe) strategies.

In particular, all lexmax strategies have the same support.

Furthermore, a lexsafe strategy is minimal. Indeed, \(x\) is safer than \(x'\) whenever \(g(x) \subset g(x')\) and containment is strict.

Note also that Alice’s lexsafe strategies are defined by \(g\) and \(\succ_A\), while Bob’s preference \(\succ_B\) is irrelevant. Alice may be even unaware of it, which is important for applications.

Similarly, using \(y \in Y\) and \(\succ_B\) instead of \(x\) and \(\succ_A\), we define Bob’s lexsafe strategies. Respectively, they depend only on \(g\) and \(\succ_B\), while \(\succ_A\) is irrelevant.
The concept of a lexsafe strategy can be viewed as a refinement of the classical concept of a safe (maxmin) strategy. The latter optimizes the worst case scenario outcome, while lexsafe strategies optimize the whole set of outcomes in the lexicographical order defined above.

Thus, lexsafe strategies are safest, but sometimes may be not rational. For example, let $g(x) = \{\omega\}$, $g(x') = \{\omega, \omega'\}$ and $\omega \prec_A \omega'$. Then $x \succ_A x'$, although strategy $x'$ is better for Alice than $x$. Indeed, $x'$ gives her a chance to obtain the better outcome $\omega'$, while $x$ excludes $\omega'$ and ensures $\omega$; see Remark 5.

Consider a preference over $\Omega$ such that outcomes $\omega, \omega' \in \Omega$ are, respectively, the worst and the best outcomes for both Alice and Bob simultaneously. Consider a game form $g : X \times Y \to \Omega$ having two strategies $x^* \in X$ and $y^* \in Y$ such that $g(x, y) = \omega$ if and only if $x = x^*$ or $y = y^*$ and $g(x, y) = \omega'$ otherwise. Then, $x^*$ and $y^*$ are the only lexsafe strategies of Alice and Bob; furthermore, situation $(x^*, y^*)$ is a unique lexsafe NE, but outcome $g(x^*, y^*) = \omega$ is worse than $\omega'$ for both players. One can view this as a price of stability. For comparison, recall that NE may be not Pareto optimal.

6.3. Lexsafe Nash equilibria in games with tight game forms

Recall that game form $g : X \times Y \to \Omega$ is tight if and only if its hypergraphs $A = A(g) = \{g(x) \mid x \in X\}$ and $B = B(g) = \{g(y) \mid y \in Y\}$ are dual.

Given a game $(g; \succ_A, \succ_B)$ with a tight game form $g$, choose any lexsafe strategy $x^L$ of Alice. By Theorem 2 it is minimal. Let us show that there exists a strategy $y^M$ of Bob such that $(x^L, y^M)$ is a NE. By definition, $y^M$ is a best response to $x^L$, that is, $g(x^L, y^M) \succeq_B g(x^L, y)$ for any $y \in Y$. (Note, however, that the preference is not strict, because for some $y$ two outcomes may coincide: $g(x^L, y^M) = g(x^L, y)$.) Let us apply Theorem 2 setting

$$g(x^*) = A^L, \quad g(y^M) = B^M, \quad g(x^L, y^M) = \omega^*,$$

and conclude that there exists a (minimal) strategy $y^M$ such that $x^L$, in its turn, is a best response to $y^M$. Thus, $(x^L, y^M)$ is a NE. Theorem 1 is proven.

Moreover, we can strengthen it summarizing remarkable properties of the obtained NE. Recall that in Theorem 2 both edges $A^L$ and $B^M$ are minimal and $A^L \cap B^M = \{\omega^*\}$. Hence, for the obtained NE $(x^L, y^M)$ both strategies $x^L$ and $y^M$ are minimal and situation $(x^L, y^M)$ is simple, that is, $g(x^L) \cap g(y^M) = \{\omega^*\}$; see [23]. More precisely, $X^L$ must be minimal, while $Y^L$ can be chosen minimal.

Denote by $X^L$ the set of all lexmax strategies of Alice. By definition, they all have the same support. Let us fix $x^L \in X^L$ and denote by $Y^M(x^L)$ the set of all Bob’s best responses to $x^L$. In fact, $Y^M(x^L)$ does not depend on $x^L$ provided $x^L \in X^L$. Indeed, set $g(x^L)$ is unique, that is, the same for all
\[x^L \in X_L \text{ and } g(x^L) \cap g(y^M) = \omega^* \text{ for all } y^M \in Y^M(x^L) \text{ and for all } x^M \in X^L.\]

Hence, \(y^M(x^L)\) is a best response of Bob to each Alice’s lexmax strategy. Denote by \(y^M\) the set of all such best responses.

Thus, we obtain \(X^L \subseteq X \text{ and } Y^M \subseteq Y\) such that for any pair \(x^L \in X^L\) and \(y^M \in Y^M\) situation \((x^L, y^M)\) is simple, \(g(x^L, y^M) = \{\omega^*\}\), and \((x^L, y^M)\) is a NE, because \(X^L\) is a best response to \(Y^M\) and vice versa.

In other words, the direct product \(\text{NE-A} = (x^L \times y^M) \subseteq X \times Y\) consists of simple NE situations corresponding to the same outcome \(\omega^* \in \Omega\). Furthermore, all strategy of \(X^L\) are lexsafe and, hence, minimal, while \(Y^M\) contains minimal strategies. We will call \(\text{NE-A}\) the box of Alice’s lexsafe equilibria.

By construction, \(X^L\) depends only on Alice’s preference \(\succ_A\), while Bob’s preference \(\succ_B\) is irrelevant and Alice may be just unaware of it, which is important for applications. In contrast, \(Y^M\) is a set of some (special) Bob’s best responses to \(X^L\), which are the same for all \(x^L \in X^L\).

Swapping the players, we obtain the box of Bob’s lexsafe equilibria \(\text{NE-B} = (x^M \times y^L) \subseteq X \times Y\) with similar properties. Thus, we can strengthen Theorem 1 as follows:

**Theorem 3.** Every game \((g; \succ_A, \succ_B)\) with a tight game form \(g\) has two non-empty boxes of lexmax equilibria \(\text{NE-A} = X^L \times Y^M\) and \(\text{NE-B} = X^M \times Y^L\) of Alice and Bob satisfying the above properties.

Boxes \(\text{NE-A}\) and \(\text{NE-B}\) may intersect or even coincide. For example, this always happens in the sero-sum case. In this case \(X^L\) and \(X^M\) are maxmin strategies of Alice, while \(Y^M\) and \(Y^L\) are minmax strategies of Bob. More detail can be found in the first arXiv version of this paper [31]. \(\text{NE-A}\) and \(\text{NE-B}\) may be equal in the non-zero-sum case too. For example a game may have a unique NE.

### 6.4. A pair of lexsafe strategies of Alice and Bob may be not a NE

For example, consider tight game form \(g_1\) in Figure 1. Define preferences \(\succ_A\) and \(\succ_B\) such that \(\omega_2 \succ_A \omega_1 \succ_A \omega_3\) and \(\omega_2 \succ_B \omega_3\). It is easy to verify that \(x_1\) and \(y_1\) are lexsafe strategies of Alice and Bob, respectively. Yet, situation \((x_1, y_1)\) is not an NE. Alice can improve her result \(g_1(x_1, y_1) = \omega_1\) by switching to \(x_2\) and getting \(g(x_2, y_1) = \omega_2\). Thus, two lexsafe strategies, of Alice and Bob, do not form an NE. However, sets \(\text{NE-A}\) and \(\text{NE-B}\) are not empty, in accordance with Theorem 3 \(\text{NE-A} = \{(x_2, y_1)\}\) and \(\text{NE-B} = \{(x_1, y_2)\}\). The corresponding NE outcomes are \(\omega_1\) and \(\omega_2\), respectively.

Note that \(\omega_2\) is the best outcome for both players if \(\omega_2 \succ_B \omega_1\). In this case \(\text{NE-B}\) is not Pareto-optimal.
Remark 8. One could conjecture that each player prefers lexsafe NE of the opponent to his/her own. Such result would be similar to the analogous one from the matching theory; see, for example, [33]. There are two types of stable matchings given by the Gale-Shapley algorithm [16], depending on men propose to women or vice versa. Yet, this conjecture is disproved by the above example.

7. Computing lexsafe NE in polynomial time

If game form \( g : X \times Y \to \Omega \) is given explicitly then to find all its NE is simple: one can just consider all situations \((x, y) \in X \times Y\) one by one verifying Nash’s definition for each of them. Yet, in applications \( g \) is frequently given by an oracle \( O \) such that size of \( g \) is exponential in size \(|O|\) of this oracle. Then, the straightforward search for NE suggested above becomes not efficient. Four such oracles will be considered in the next section. The following three properties of oracle \( O \) will allow us to construct an algorithm computing two lexsafe NE (from NE-A and NE-B, respectively) for a given game \((g; \succ_A, \succ_B)\) with tight game form \( g = g(O) \) realized by \( O \), in time polynomial in \(|O|\).

(I) Oracle \( O \) contains explicitly all outcomes \( \Omega \) of \( g \).
(Yet, strategies \( x \in X \) and \( y \in Y \) are implicit in \( O \); moreover, \(|X|\) and \(|Y|\) may be exponential in \(|O|\).)
(II) The game form \( g = g(O) \) defined by \( O \) is tight.
(III) Every \( \pm 1 \) game \((g(O); \Omega_A, \Omega_B)\) can be solved in time polynomial in \(|O|\).

Requirement (III) needs a discussion. By tightness of \( g \), exactly one of the following two options holds:

(a) there exists \( x \in X \) with \( g(x) \subseteq \Omega_A \) (Alice wins);
(b) there exists \( y \in Y \) with \( g(y) \subseteq \Omega_B \) (Bob wins).

Note that (a) (respectively, (b)) holds if and only if the monotone Boolean function corresponding to multi-hypergraph \( A(g) \) (respectively, \( B(g) \) takes value 1; see Remark 6.

To solve a \( \pm 1 \) game we determine which option, (a) or (b), holds and output a winning strategy, \( x \) or \( y \), respectively.

To solve a \( \pm 1 \) game we determine which option, (a) or (b), holds and output a winning strategy, \( x \) or \( y \), respectively.

Note that it is possible to output a minimal winning strategy whenever (III) holds. Indeed, suppose Alice wins and we output her winning strategy \( x \), with \( g(x) \subseteq \Omega_A \). Reduce \( \Omega_A \) by one outcome \( \omega \) by moving it to \( \Omega_B \), solve the obtained \( \pm 1 \) game, and repeat the procedure for all \( \omega \in \Omega_A \). If Bob wins in all obtained games then \( x \) is already minimal. Otherwise we can move an outcome \( \omega \) from \( \Omega_A \) to \( \Omega_B \) and Alice still wins. Repeating, we obtain a minimal
winning strategy of Alice (in the original game) in at most $|\Omega_A|$ steps. We can speed up the above procedure using dichotomy. The same works for Bob.

Theorems 1–3 immediately imply the following statement.

**Theorem 4.** Given an oracle $\mathcal{O}$ satisfying requirements (I,II,III), a lexsafe NE of Alice (of Bob) exists and can be computed in time polynomial in $|\mathcal{O}|$. □

8. Examples of oracles

8.1. Summary

Here we consider four types of oracles known in the literature and verify that all four satisfy requirements (I, II, III).

In Section 8.3 we consider game forms corresponding to positional (graphical) game structures with perfect information, due to which Nash-solvability holds even in the $n$-person case. Yet, we substantially extend this class of game forms by modifying the set of outcomes. The standard approach assumes that the set of outcomes $\Omega$ is formed by the terminal vertices of the input directed graph $\Gamma$. Yet, Nash-solvability still holds if we extend $\Omega$ by redefining it as the set of all strongly connected components of $\Gamma$. Yet, in this case Nash-solvability holds only if we restrict ourselves to 2-person games.

In Section 8.4 we introduce so-called Jordan game forms in which Alice and Bob connect two pairs of opposite sides of the square. In Section 8.5 we consider monotone bargaining and in Section 8.6 veto voting schemes. In these three examples perfect information is not assumed, nevertheless requirements (I,II,III), tightness among them, hold.

Subsections 8.3 - 8.6 can be read in an arbitrary order.

8.2. Game forms and game correspondences

A *game correspondence* is defined as an arbitrary mapping $G : X \times Y \rightarrow 2^\Omega \setminus \{\emptyset\}$, that is, $G$ assigns a non-empty subset of outcomes to each situation.

Given $G$, define a game form $g \in G$, choosing an arbitrary outcome $g(x,y) \in G(x,y)$ for each situation $(x,y)$. Conversely, given a game form $g : X \times Y \rightarrow \Omega$, define a game correspondence $G$ setting $G(x,y) = g(x) \cap g(y)$. Then, obviously, $g \in G$.

By property (jj) of Section 4 if at least one $g^* \in G$ is tight then all $g \in G$ are tight. In this case $G$ is called *tight* too. Moreover, all $g \in G$ have the same Sperner reduced dual hypergraphs $\mathcal{A}^0(g)$ and $\mathcal{B}^0(g)$, same simple situations, and for any $u$ and $w$, the same sets of simple situations in NE-A and NE-B.
8.3. Deterministic graphical multi-stage game structures

Let $\Gamma = (V, E)$ be a directed graph (digraph) whose vertices and arcs are interpreted as positions and moves, respectively. Denote by $V_T$ the set of terminal positions (of out-degree zero) and by $V_A, V_B$ the sets of positions of positive out-degree controlled by Alice and Bob, respectively. We assume that $V = V_A \cup V_B \cup V_T$ is a partition of $V$.

A strategy $x \in X$ of Alice (respectively, $y \in Y$ of Bob) is a mapping that assigns to each position $v \in V_A$ (respectively, $v \in V_B$) a move from this position. An initial position $v_0 \in V_A \cup V_B$ is fixed. Each situation $(x, y)$ defines a unique walk in $\Gamma$ that begins in $v_0$ and then follows the decisions made by strategies $x$ and $y$. This walk $P(x, y)$ is called a play. Each play either terminates in $V_T$ or is infinite. In the latter case, it forms a “lasso”: first, an initial path, which may be empty, and then, a directed cycle (dicycle) repeated infinitely. (Indeed, since both players are restricted to their stationary strategies, a move may depend only on the current position but not on previous positions and/or moves. Hence, if a play visits a position twice then all further moves will be repeated as well.)

The (positional structure defined above can also be represented in normal form. We introduce a game form $g : X \times Y \to \Omega$, where, as before, $\Omega$ denotes a set of outcomes. Yet, there are several ways to define this set. One is to “merge” all infinite plays (lassos) and consider them as a single outcome $c$, thus, setting $\Omega = V_T \cup \{c\}$. This model was introduced by Washburn [47] and called deterministic graphical game structure (DGGS).

The following generalization was suggested in [27]. Digraph $\Gamma$ is called strongly connected if for any $v, v' \in V$ there is a directed path from $v$ to $v'$ (and, hence, from $v'$ to $v$, as well). By this definition, the union of two strongly connected digraphs with a common vertex is strongly connected. A vertex-inclusion-maximal strongly connected induced subgraph of $\Gamma$ is called its strongly connected component (SCC). In particular, each terminal position $v \in V_T$ is an SCC. It is both obvious and well-known that any digraph $\Gamma = (V, E)$ admits a unique decomposition into SCCs: $\Gamma^\omega = \Gamma[V^\omega] = (V^\omega, E^\omega)$ for $\omega \in \Omega$, where $\Omega$ is a set of indices. Furthermore, partition $V = \bigcup_{\omega \in \Omega} V^\omega$ can be constructed in time linear in the size of $\Gamma$, that is, in $(|V| + |E|)$.

Partitioning into SCCs has numerous applications; see [43, 45] for more details. One more application was suggested in [27]. For each $\omega \in \Omega$, contract the SCC $\Gamma^\omega$ into a single vertex $v^\omega$. Then, all edges of $E^\omega$ (including loops) disappear and we obtain an acyclic digraph $\Gamma^* = (\Omega, E^*)$. Set $\Omega$ can be treated as the set of outcomes. Each situation $(x, y)$ uniquely defines a play $P = P(x, y)$. This play either comes to a terminal $v \in V_T$ or forms a lasso. The cycle of this lasso is contained in an SCC $\omega$ of $\Gamma$. Each terminal is an
SCC as well. In both cases an SCC \( \omega \in \Omega \) is assigned to the play \( P(x,y) \). Thus, we obtain a game form \( g : X \times Y \to \Omega \), which is the normal form of the multi-stage DGGS (MSDGGS) defined by \( \Gamma \).

An SCC is called transient if it is not a terminal and contains no dicycles. Obviously, a transient SCC consists of a single vertex and no play can result in it. Thus, it is not an outcome. For example, \( \Omega = V_T \) in any acyclic digraph, while each remaining SCC is transient.

**Proposition 1.** In both cases, DGGS and MSDGGS, the corresponding oracles satisfy requirements (I, II, III).

**Proof.** Indeed, (I) holds since the SCCs, of a given digraph \( \Gamma \) can be generated in time linear in the size of \( \Gamma \).

Both requirements, (II) and (III), for both oracles, DGGS and MSDGGS, can be verified simultaneously. Consider the corresponding game forms \( g' \) and \( g \) and note that \( g' \) is obtained from \( g \) by merging some outcomes. Namely, all outcomes corresponding to the non-terminal SCCs are replaced by a single outcome \( c \). It is both obvious and well-known that merging outcomes respects tightness. Hence, it is enough to verify (II) and (III) for MSDGGSs.

By Theorem 1, to verify (II) it is sufficient to prove \( \pm 1 \) solvability. For DGGS it was done in [47]; see also [5, Section 3], [2], [8, Section 12]. This result was extended to MSDGGS in [27]. Furthermore, all proofs in [27] were constructive, the corresponding \( \pm 1 \) games were solved in time polynomial in the size of \( \Gamma \), which implies (III).

For reader’s convenience, we briefly sketch here the proof of (II,III) from [27]. Consider a \( \pm 1 \) game \((g; \Omega_A, \Omega_B)\) with game form \( g = g(\mathcal{O}) \) generated by a MSDGGS oracle \( \mathcal{O} \). We would like to apply Backward Induction, yet, digraph \( \Gamma \) may have dicycles. So we modify Backward Induction to make it work in presence of dicycles. Recall that \( \Omega \) is the set of SCCs of \( \Gamma \) and \( \Gamma^* = (\Omega, E^*) \) is acyclic. Consider an SCC \( \Gamma' = (V', E') \) in \( \Gamma \) that is not terminal, but each move \((v', v)\) from a position \( v' \in V' \) either ends in a terminal \( v \in V_T \), or stays in \( \Gamma' \), that is, \( v, v' \in V' \). Obviously, such a SCC exists. Note that it may be transient. In this case the standard Backward Induction is applicable.

Suppose that \( \Gamma' \) is not transient, in other words, it contains a dicycle. Wlog we can assume that \( \omega \in \Omega_A \), that is, Alice wins if the play cycles in \( \Gamma' \). Then, Bob wins in a position \( v' \in V' \) if and only if he can force the play to terminate in \( \Omega_B \), while Alice wins in all other positions of \( V' \). Note that it is not necessary for Alice to force the play to come to a terminal from \( \Omega_A \), if the play cycles in \( \Gamma' \) Alice wins as well. Thus, every position of \( \Omega' \) belongs either
to $\Omega_A$ or to $\Omega_B$. We make all these positions terminal, by eliminating all edges $E'$ of $\Gamma'$, and repeat until the initial position $v_0$ of $\Gamma$ is evaluated.

This procedure proves solvability of game form $g = g(\mathcal{O})$ (which is equivalent to its tightness (II), by Theorem 1), moreover, a $\pm 1$ game $(g; \Omega_A, \Omega_B)$ is solved in time linear in the size of $\mathcal{O} = \Gamma$ (which is (III)).

\textbf{Acyclic deterministic graphical game structures}

A game form is called \textit{rectangular} if all its situations are simple. It is shown in [22] that a game form $g$ is generated by a DGGS whose graph is a tree if and only if $g$ is tight and rectangular. Two examples, $\Gamma_1$ and $\Gamma_2$ generating game tight rectangular game forms $g_1$ and $g_2$ are given in Figure 2; see also Figure 1. More examples can be found in [25, Section 3], where the above characterization is extended to the $n$-person case.

Acyclic DGGS $\Gamma_1$ in Figure 2 generates game form $g_1$. Recall game $(g_1; u, w)$ from Section 6.4 with $u(\omega_2) > u(\omega_1) > u(\omega_3)$ and $w(\omega_2) > w(\omega_3)$. Note that the Backward Induction NE (see [15, 35], and also [26]) is NE-A and is not Pareto-optimal. In general, this NE may differ from both, NE-A and NE-B.

Acyclic DGGS $\Gamma_3$ in Figure 2 generates game form $g_6$; see also Figure 1. Clearly, in absence of dicycles in $\Gamma$, the concepts of DGGS and MSDGGS coincide. It is also clear that an acyclic DGGS is a special case of MSDGGS. Thus, properties (I, II, III) required from an oracle hold for both.

\textbf{Cyclic deterministic graphical game structures}

The outcomes of MSDGGS are all its non-transient SCCs. In particular, each terminal position is an outcome. Let us now assume that every simple
dicycle is a separate outcome (and each terminal remains an outcome as well). Such DGGSs, called cyclic, were studied in [8]; some special cases were considered earlier [17, 18, 19]. Cyclic DGGS can also serve as oracles generating game forms; see examples in [8, Figures 1 and 2]; compare examples 3 and 4 in [8, Figure 2] with game forms $g_4$ and $g_5$ in Figure 1.

Game forms generated by the cyclic DGGS may be not tight; see Figure 1 in [8]. In other words, property (II) fails for the corresponding oracles, in general. Yet, it holds in some important special cases.

A digraph $G = (V, E)$ is called symmetric if $(v, v') \in E$ whenever $(v', v) \in E$. Cyclic DGGS on symmetric digraphs are called symmetric. Symmetric Cyclic DGGS satisfying (II) are called solvable and explicitly characterized in [8, sections 1-3 and Theorems 1-3]. It follows from results of [8] that (III) also holds for solvable cyclic symmetric DGGS. Hence, Theorem 4 is applicable.

8.4. Jordan oracle; choosing Battlefields in Wonderland

Wonderland is a subset of the plane homeomorphic to the closed disc. Wlog, we can consider a square $Q$ with the sides $N, E, S, W$. Let us partition $Q$ into areas $\Omega = \{\omega_1, \ldots, \omega_p\}$ each of which is homeomorphic to the closed disc, too. Every two distinct areas $\omega_i, \omega_j \in \Omega$ are either disjoint or intersect in a set homeomorphic to a closed interval that contains more than one point. Equivalently, we can require that the borders of the areas in $Q$ form a regular graph of degree 3. (Note that four vertices of the square are not vertices of this graph.) Two examples are given in Figures 3 and 4.

Remark 9. Consider game form $g_5$ in Figure 1 and merge outcomes $\omega_5$ and $\omega_6$ in it getting $g'_5$. (This operation respects tightness). Note that $g'_5 \in G$, where $G$ is the game correspondence given in Figure 3. See also [8, Figure 4], where $g_5$ also appears as the normal form of a cyclic game form.

The following interpretation was suggested in [30]. Two players, Alice Tweedledee and Bob Tweedledum, agreed to have a battle. The next thing to do is to agree on a battlefield, which should be an area $\omega \in \Omega$. The strategies $x \in X$ of Alice are all (inclusion-minimal) subsets $x \subseteq \Omega$ connecting $W$ and $E$. Respectively, the strategies $y \in Y$ of Bob are all (inclusion-minimal) subsets $y \subseteq \Omega$ connecting $N$ and $S$.

Proposition 2. Any two such subsets $x$ and $y$ intersect.

Proof. It follows the Jordan curve theorem and the fact that all vertices in the square are of degree 3 (except its four corners, which are of degree 2). Note that $x$ and $y$ might be disjoint if we allow vertices of degree 4 or more. □
Intersection $x \cap y$ may contain several areas of $\Omega$. Thus, a game correspondence $G : X \times Y \to 2^{\Omega} \setminus \{\emptyset\}$ is defined.

**Proposition 3.** Game correspondence $G$ is tight.

Proof. Again, it follows from the Jordan curve theorem and the assumption that all vertices in the square are of degree 3. Choose an arbitrary $g \in G$ and consider a $\pm 1$ game $(g; \Omega_A, \Omega_B)$ determined by a partition $\Omega = \Omega_A \cup \Omega_B$. Then, from the following two options exactly one holds:

(a) areas from $\Omega_A$ connect W and E; (b) areas from $\Omega_B$ connect N and S.  

The above observations imply that Jordan oracle $\mathcal{O}$ satisfies requirements (I) and (II). It remains to verify (III),

**Proposition 4.** By using oracle $\mathcal{O}$, one can decide whether (a) or (b) holds and find corresponding $x$ or $y$, respectively, in time linear in $|\mathcal{O}|$.

Proof. Consider all areas from $\Omega_B$ boarding N, then add all areas from $\Omega_B$ boarding these areas, etc. Such iterations will stop in time linear in $|\mathcal{O}|$ either reaching S (then, obviously, (b) holds) or not (then (a) holds, again by the Jordan curve theorem). Moreover, in the first case we obtain a set of areas $y'$ from $\Omega_B$ connecting N and S; in the second case - a set of areas $x'$ from $\Omega_A$
connecting W and E. The former strategy $y'$ is obtained explicitly; the latter one, $x'$, is easy to construct. To do so, denote by $\Omega'_B$ the set of areas obtained in the course of iterations. It does not reach S. Hence, the areas from $\Omega_A$ that border $\Omega'_B$ connect W and E, by the Jordan curve theorem once more.

This case is realized in Figure 4: Alice wins.

Remark 10. It is not necessary to restrict ourselves by minimal strategies. In linear time we can reduce arbitrary strategy (set) $x'$ of Alice to an inclusion-minimal set $x$ connecting W and E, thus, getting minimal strategies of Alice. To do so, we eliminate areas from $x'$ one by one until (a) still holds. We require inclusion-minimality of subsets $x \in \Omega$ just to reduce the number of strategies (which may still remain exponential in $|O|$). Of course, the same is true for Bob’s strategies.

8.5. Monotone bargaining schemes

The following oracle was introduced in [30]. Two players, Alice and Bob, possess items $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, respectively. Both sets are ordered: $a_1 \prec \cdots \prec a_m$ and $b_1 \prec \cdots \prec b_n$. Both players know both orders.

The direct product $\Omega = A \times B = \{(a,b) \mid a \in A, b \in B\}$ is the set of outcomes.
Alice’s strategies are monotone non-decreasing mappings \( x : A \to B \) (that is, \( x(a) \geq x(a') \) whenever \( a > a' \)) showing that she is ready to exchange \( a \) for \( x(a) \) for any \( a \in A \). Similarly, Bob’s strategies are monotone non-decreasing mappings \( y : B \to A \) (that is, \( y(b) \geq y(b') \) whenever \( b > b' \)) showing that he is ready to exchange \( b \) for \( y(b) \) for any \( b \in B \).

It is not difficult to compute the numbers of strategies and outcomes:

\[
|X| = \binom{m+n-1}{m}, \quad |Y| = \binom{m+n-1}{n}, \quad |\Omega| = |A \times B| = mn. \tag{2}
\]

Given a situation \((x, y)\), an outcome \((a, b) \in \Omega\) is called a deal (in this situation) if \( x(a) = b \) and \( y(b) = a \). Denote by \( G(x, y) \subseteq \Omega \) the set of all deals in the situation \((x, y)\). We will show that \( G(x, y) \neq \emptyset \). Yet, \( G(x, y) \) may contain several deals.

This construction is called a monotone bargaining (MB) scheme. It can be viewed as an oracle \( O \) generating game correspondence \( G : X \times Y \to 2^\Omega \setminus \{\emptyset\} \). By (2), requirement (I) holds for \( O \).

Note that \( G = G_{m,n} \) is uniquely defined by \( m \) and \( n \). A game form \( g \in G \) is called an MB game form. For example, if \( m = n = 3 \) then \( |X| = |Y| = 3 \) and we obtain game form \( g_4 \) in Figure 1; game correspondence \( G(x, y) \) is given in [30, Figure 1a].

The following interpretation was suggested in [30]. Alice and Bob are dealers possessing the sets of objects \( A \) and \( B \), respectively, and a deal \((a, b) \in A \times B\) means that they exchange \( a \) and \( b \). They may be art-dealers, car dealers; or one of them may be just a buyer with a discrete budget. For example, \( A = \{a_1, \ldots, a_m\} \) and \( B = \{b_1, \ldots, b_n\} \) may be paintings or sculptures ordered in accordance with their age (not price or value).

To any pair of mappings \( x : A \to B \) and \( y : B \to A \) (not necessarily monotone non-decreasing) let us assign a bipartite digraph \( \Gamma = \Gamma(x, y) \) on the vertex-set \( A \cup B \) as follows: \([a, b]\) (respectively, \([b, a]\)) is an arc of \( \Gamma(x, y) \) whenever \( x(a) = b \) (respectively, \( y(b) = a \)).

Some visualization helps. Embed \( \Gamma(x, y) \) into a plane; putting ordered \( A \) and \( B \) in two parallel columns. Two arcs corresponding to \( x \) may have a common head, but not tail. Furthermore, they cannot cross if mapping \( x \) is monotone non-decreasing. Similarly for \( y \). By construction, digraph \( \Gamma \) is bipartite, with parts \( A \) and \( B \). Hence, every dicycle in \( \Gamma \) is even. There is an obvious one-to-one correspondence between the dicycles of length 2 in \( \Gamma(x, y) \) and the deals of \( G(x, y) \).

**Proposition 5.** For each situation \((x, y)\) its digraph \( \Gamma(x, y) \) contains at least one dicycle of length 2 (a deal) and cannot contain longer dicycles.
Proof. For any initial vertex \( v \in A \cup B \), strategies \( x \) and \( y \) uniquely define an infinite walk from \( v \), which is called a play. Since sets \( A \) and \( B \) are finite and there are no terminals, this play is a lasso: it consists of an initial directed path, which may be empty, and a dicycle \( C \) repeated infinitely. Furthermore, \( C \) must be a dicycle of length 2 whenever mappings \( x \) and \( y \) are monotone non-decreasing. Indeed, if \( C \) is longer than 2 then crossing arcs appear and, hence, either \( x \), or \( y \), or both are not monotone.

Consider a ±1 MB game \((g; \Omega_A, \Omega_B)\), where \( g = g(O) \) is an MB game form generated by an MB scheme \( O \). As we already mentioned, requirement (I) holds for \( O \). The following statement shows that (II) and (III) hold as well.

**Proposition 6.** Game form \( g = g(O) \) is tight and each ±1 MB game \((g; \Omega_A, \Omega_B)\) can be solved in time polynomial in \(|O| = mn\).

The first part was already proven in [30]. Yet, here we provide a much shorter proof.

Proof. For the sake of simplicity, we will slightly abuse notation writing that both directed edges \([a, b]\) and \([b, a]\) are in \( \Omega_A \) or in \( \Omega_B \) whenever the corresponding deal \((a, b)\) is in \( \Omega_A \) or in \( \Omega_B \), respectively.

Consider complete bipartite symmetric digraph \( \Gamma \) on \( m + n \) vertices \( A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\} \), and with \( 2mn \) directed edges \([a_i, b_j], [b_j, a_i] \mid i = 1, \ldots, m; j = 1, \ldots, n\} \). The following two statements are obvious:

(a) Alice wins if she has a monotone non-decreasing strategy \( x^* : A \to B \) such that \([a, x^*(a)] \in \Omega_A \) for all \( a \in A \).

(b) Bob wins if he has a monotone non-decreasing strategy \( y^* : B \to A \) such that \([b, y^*(b)] \in \Omega_B \) for all \( b \in B \).

Indeed, it is easily seen that \( x^* \) and \( y^* \) are the winning strategies of Alice and Bob, respectively. It is enough to show that \( g(x^*, y) \in \Omega_A \) for any \( y \in Y \).

Recall the proof of Proposition 5. Fix \( x^* \), choose an arbitrary \( y \in Y \), and consider the play \( P = P(x^*, y) \) beginning from an arbitrary initial position \( v \in A \cup B \). By Proposition 5, \( P \) is a lasso resulting in a 2-cycle \((a, b)\).

The corresponding deal \((a, b) \in \Omega_A \), in case (a), for any \( y \), by the choice of \( x^* \), and hence, Alice wins. Similarly, \( g(x, y^*) \in \Omega_B \) in case (b) for any \( x \in X \), by the choice of \( y^* \), and Bob wins.

Obviously, (a) and (b) cannot hold simultaneously, since otherwise \((a, b) \in \Omega_A \cap \Omega_B \), which is a contradiction, since \( \Omega = \Omega_A \cup \Omega_B \) is a partition.

Let us show that either (a) or (b) holds (in other words, \( g \) is tight, which implies (II)). The proof will be constructive: we obtain either \( x^* \) satisfying (a) or \( y^* \) satisfying (b) in time polynomial in \( mn \) (which in its turn, implies (III)).
We will construct a play $P$ by the following greedy iterative algorithm. Let $a^1 = a_1$ be an initial position of $P$. (We use superscripts to number iterations.)

If $[a^1, b) \in \Omega_B$ for all $b \in B$ then Bob wins. (His winning strategy $y^*$ is defined by: $y^*(b) = a^1$ for all $b \in B$. Then $[y^*(b), b) \in \Omega_B$ for all $b \in B$ and (b) holds.) Otherwise, denote by $b^1$ the (unique) minimal $b \in B$ such that $[a^1, b) \in \Omega_A$. Then, by definition, $[b^1, a^1) \in \Omega_A$ too. Furthermore, by this choice of $b^1$, we have: $[b, a^1) \in \Omega_B$ for all $b \prec b^1$, while $[b^1, a^1) \in \Omega_A$.

If $[b^1, a) \in \Omega_A$ for all $a \succeq a^1$ then Alice wins. (Her winning strategy $x^*$ is defined by: $x^*(a) = b^1$ for all $a \in A$. Then $[a, x^*(a)) \in \Omega_A$ for all $a \in A$.) Otherwise, denote by $a^2$ the (unique) minimal $a \in A$ such that $[b^1, a) \in \Omega_B$. Then, by definition, $[a^2, b^1) \in \Omega_B$ too. Furthermore, by the choice of $a^2$, we have: $[a, b^1) \in \Omega_A$ for all $a \prec a^2$, while $[a^2, b^1) \in \Omega_B$.

The general $k$-th step of this greedy recursion is as follows.

If $[a^k, b) \in \Omega_B$ for all $b \succeq b^{k-1}$ then Bob wins. (His winning strategy $y^*$ is defined by: $y^*(b) = a^i$ for each $b$ such that $b^i \succ b \succeq b^{i-1}$, for $i = 1, \ldots, k$, assuming conventionally that $b^0 \succ b^0$ holds for all $b \in B$).

Otherwise, denote by $b^k$ the (unique) minimal $b \in B$ such that $b \succ b^{k-1}$ and $[a^k, b) \in \Omega_A$. Then $[b^k, a^k) \in \Omega_A$ too.

Furthermore, by the choice of $b^k$, we have: $[b, a^k) \in \Omega_B$ for all $b$ such that $b^k \succ b \succeq b^{k-1}$, while $[b^k, a^k) \in \Omega_A$.

If $[b^k, a) \in \Omega_A$ for all $a \succeq a^k$ then Alice wins. (Her winning strategy $x^*$ is defined by: $x^*(a) = b^j$ for each $a$ such that $a^{j+1} \succ a \succeq a^j$, for $j = 1, \ldots, k$, assuming conventionally that $a^{k+1} \succ a$ holds for all $a \in A$.)

Otherwise, denote by $a^{k+1}$ the (unique) minimal $a \in A$ such that $[b^k, a) \in \Omega_B$. Then $[a^{k+1}, b^k) \in \Omega_B$, too.

Furthermore, by the choice of $a^{k+1}$, we have: $[a, b^k) \in \Omega_A$ for all $a$ such that $a^{k+1} \succ a \succeq a^k$, while $[a^{k+1}, b^k) \in \Omega_B$.

After each iteration $a^k$ (respectively, $b^k$) both Alice and Bob have winning moves in all positions $a \prec a^k$ and $b \preceq b^{k-1}$ (respectively, $a \preceq a^k$ and $b \prec b^k$). Since sets $A$ and $B$ are finite, the procedure will stop on some iteration either $a^{k^*} < a_m$ or $b^{k^*} < b_n$, indicating that Bob or, respectively, Alice wins.

Furthermore, we obtain his or her winning strategy in time linear in $mn$.

The following slightly different procedure can be applied too. First, we start looking for a winning strategy $x^*$ for Alice. Consider successively $a_1, a_2, \ldots$ and construct (again recursively and greedily) her monotone non-decreasing strategy $x^*$ as follows: $x^*(a_i) = b^i$ such that $[a_i, b^i) \in \Omega_A$, $b^i \succeq b^{i-1}$, and $b^i$ is the minimal element of $B$ satisfying these two properties. If this will work for all $i = 1, \ldots, m$ then Alice wins and we obtain her winning strategy $x^*$ satisfying (a). Otherwise, if the procedure stops on some $i < m$ (no required
exists for $a_i$) then Bob wins. His winning strategy $y^*$ satisfying (b) is defined as follows:

$$y^*(b) = a^i$$
for all $b$ such that $b^{i-1} \preceq b^i$, where $a_i$ is the smallest $a$ such that $x^*(a) = b^i$, for $i = 1, 2, \ldots$ By convention, $b^0 \prec b$ for all $b \in B$.

Thus, requirements (I,II,III) hold for the MB schemes and, hence, Theorem 4 is applicable.

8.6. Veto voting schemes

Two voters (players), Alice and Bob choose among candidates (options, outcomes) $\Omega = \{\omega_1, \ldots, \omega_p\}$. They are assigned some positive integer veto powers and given $\mu_A$ and $\mu_B$ veto cards, respectively. Each candidate $\omega \in \Omega$ is assigned an integer positive veto resistance $\lambda_\omega$. We assume that

$$\mu_A + \mu_B + 1 = \lambda_{\omega_1} + \cdots + \lambda_{\omega_p}. \quad (3)$$

A strategy of a voter is an arbitrary distribution of her/his veto cards among the candidates. Given a pair of strategies $x$ and $y$, a candidate $\omega \in \Omega$ who got at least $\lambda_\omega$ veto cards (from Alice and Bob together) is vetoed. From the set $G(x,y)$ of all not vetoed candidates one $g(x,y) \in G(x,y)$ is elected. By (3), $G(x,y) \neq \emptyset$. Thus, we obtain a veto voting (VV) scheme $O$, VV game form $g = g(O)$, and VV game correspondence $G = G(O)$; see, for example, [24],[38, Chapter 6],[41, Chapter 5] for more details.

By construction, VV schemes are oracles satisfying (I). For example, game form $g_3$ in Figure 1 corresponds to the VV scheme defined by

$$\mu_A = \mu_B = \lambda_{\omega_1} = \lambda_{\omega_2} = \lambda_{\omega_3} = 1.$$  

Let us show that requirements (II) and (III) also hold for VV schemes.

**Proposition 7.** Each game form $g$ defined by a VV scheme satisfying (3) is tight. Furthermore, every $\pm 1$ game $(g; \Omega_A, \Omega_B)$ can be solved in time linear in $|O| = \log(\mu_A\mu_B \prod_{\omega \in \Omega} \lambda_\omega)$.

**Proof.** To see this, consider a $\pm 1$ game $(g; \Omega_A, \Omega_B)$. By (3), from two options, (a) Alice can veto $\Omega_B$ and (b) Bob can veto $\Omega_A$, exactly one holds. Alice or Bob wins in case of (a) or (b), respectively. Given numbers $\mu_A, \mu_B$, and $\lambda_\omega, \omega \in \Omega$, one can decide in linear time whether (a) or (b) holds. In each case the winning strategy of Alice or Bob is straightforward: just veto all opponent’s candidates, $\Omega_B$ or $\Omega_A$, respectively.

Thus, the VV oracles satisfy (I,II,III) and Theorem 4 is applicable.
8.7. Tight game correspondences and forms of arbitrary monotone properties

The most general setting is defined as follows. Given a finite ground set $\Omega$, consider a family of its subsets $\mathcal{P} \subseteq 2^\Omega$. Standardly, we call $\mathcal{P}$ a property and say that a subset $\Omega' \subseteq \Omega$ satisfies $\mathcal{P}$ or not if $\Omega' \in \mathcal{P}$ or $\Omega' \notin \mathcal{P}$, respectively. Property $\mathcal{P}$ is called inclusion monotone non-decreasing (or simply monotone, for short) if $\Omega'' \in \mathcal{P}$ implies $\Omega' \in \mathcal{P}$ whenever $\Omega'' \subseteq \Omega' \subseteq \Omega$. We restrict ourselves to monotone properties.

Define the sets of strategies $X$ of Alice and $Y$ of Bob as follows:

- $x \in X$ is any (inclusion minimal) subset $\Omega_A \subseteq \Omega$ such that $\Omega_A \in \mathcal{P}$;
- $y \in Y$ is any (inclusion minimal) subset $\Omega_B \subseteq \Omega$ such that $\Omega \setminus \Omega_B \notin \mathcal{P}$.

The restriction in parenthesis does not matter, it can be waved or kept. In the latter case, sets $X$ and $Y$ are significantly reduced.

Define a game correspondence $G = G(\mathcal{P})$ by setting $G(x, y) = x \cap y$. It is both obvious and well-known that $G(x, y) \neq \emptyset$ for any $x \in X$, $y \in Y$ and, moreover, $G$ is tight. Hence, any game form $g \in G$ is tight too.

Thus, (I) and (II) hold automatically whenever a monotone property $\mathcal{P}$ is given by an oracle $O(\mathcal{P})$. Yet, (III) must be required in addition. In other words, $O(\mathcal{P})$ must be a polynomial membership oracle, which for a given subset $\Omega' \subseteq \Omega$, decides if $\Omega' \in \mathcal{P}$ in time polynomial in $|\Omega| + |O(\mathcal{P})|$.

It is easily seen that this general setting includes in particular all four examples of oracles given in this section before; see more examples in [6, 28].

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References

[1] N. Alon, M. Feldman, A. D. Procaccia, and M. Tennenholtz, A note on competitive diffusion through social networks, Information Processing Letters, 110:6 (2010) 221–225.

[2] D. Andersson, K. Hansen, P. Miltersen, and T. Sorensen, Deterministic graphical games, revisited, J. Logic and Computation. 22:2 (2012) 165-178. Preliminary version in Fourth Conference on Computability in Europe (CiE-08), Lecture Notes in Computer Science. 5028 (2008) 1–10.
[3] A.B. Barabas, R.E. Basko, and I.S. Menshikov, On an approach to analysis of conflict situations, Proc. Moscow Comput. Center 2 (1990) 13–20 (in Russian).

[4] E. Boros, O. Čepek, V. Gurvich, and K. Makino, Recognizing distributed approval voting forms and correspondences, http://arxiv.org/abs/2010.15730 (2020) 1–16, Annals of Operations Research, to appear.

[5] E. Boros and V. Gurvich, On Nash-solvability in pure strategies of finite games with perfect information which may have cycles, Math. Soc. Sciences 46 (2003) 207–241.

[6] E. Boros, V. Gurvich, K. Elbassioni, and L. Khachiyan, Generating dualbounded hypergraphs, Optimization Methods and Software 17:5 (2002) 749 – 781.

[7] E. Boros, V. Gurvich, K. Makino, and D. Papp, Acyclic, or totally tight, two-person game forms; a characterization and main properties; Discrete Math. 310:6-7 (2010) 1135–1151.

[8] E. Boros, V. Gurvich, K. Makino, and Wei Shao, Nash-solvabile two-person symmetric cycle game forms, Discrete Appl. Math. 159:15 (2011) 1461–1487.

[9] L. Bulteau, V. Froese, and N. Talmon, Multi-player diffusion games on graph classes, https://arxiv.org/abs/1412.2544 (2017) 1–21.

[10] Y. Crama and P. L. Hammer, Boolean functions: Theory, algorithms, and applications, Cambridge University Press, 2011.

[11] V.I. Danilov and A.I. Sotskov, Social Choice Mechanisms, Moscow Nauka 1991 (in Russian); English translation in Studies of Economic Design, Springer, Berlin-Heidelberg, 2002.

[12] J. Edmonds and D.R. Fulkerson, Bottleneck extrema, J. of Combinatorial Theory 8 (1970) 299–306.

[13] M. Fredman and L. Khachiyan, On the complexity of dualization of Monotone Disjunctive Normal Forms, J. Algorithms 21 (1996) 618–628.

[14] N. Fukuzono, T. Hanaka, H. Kiya, H. Ono, and R. Yamaguchi, Two-player competitive diffusion game: graph classes and the existence of a
Nash equilibrium, 46th International Conference on Current Trends in
Theory and Practice of Informatics, Limassol, Cyprus, Jan. 20–24, 2020,
Proceedings (2020) 627–635,

[15] D. Gale, A theory of N-person games with perfect information, Proc.
Natl. Acad. Sci. 39 (1953) 496–501.

[16] D. Gale and L.S. Shapley, College admissions and the stability of mar-
riage, The American Mathematical Monthly 69:1 (1962) 9–15.

[17] A.I. Gol’berg and V.A. Gurvich, Tight cyclic game forms, Russian Math.
Surveys, 46:2 (1991) 241–242.

[18] A.I. Gol’berg and V.A. Gurvich. Some properties of tight cyclic game
forms, Soviet. Math Dokl., 43:3 (1991) 898–903.

[19] A.I. Gol’berg and V.A. Gurvich, A tightness criterion for reciprocal bi-
partite cyclic game forms, Russian Acad. Sci. Dokl. Math. 45:2 (1992)
348–354.

[20] V. Gurvich, To theory of multi-step games, USSR Comput. Math. and
Math. Phys. 13:6 (1973) 143–161.

[21] V. Gurvich, Solution of positional games in pure strategies, USSR Com-
put. Math. and Math. Phys. 15:2 (1975) 74–87.

[22] V. Gurvich, On the normal form of positional games, Soviet Math Dokl.
25:3 (1982) 572–575.

[23] V. Gurvich, Equilibrium in pure strategies, Soviet Math. Dokl. 38:3 (1989)
597–602.

[24] V. Gurvich, War and Peace in veto voting, European J. of Operational
Research 185 (2008) 438–443.

[25] V. Gurvich, Decomposing complete edge-chromatic graphs and hyper-
graphs, revisited, Discrete Appl. Math. 157 (2009) 3069–3085.

[26] V. Gurvich, Generalizing Gale’s theorem on backward induction and dom-
ination of strategies, arXiv http://arxiv.org/abs/1711.11353 (2017) 12 pp.

[27] V. Gurvich, Backward induction in presence of cycles, Oxford Journal of
Logic and Computation 28:7 (2018) 1635–1646.
[28] V. Gurvich, Complexity of generation, Computer Science in Russia, XIII-th International Computer Science Symposium in Russia (CSR-13) Moscow, June 6–10, 2018; Lecture Notes in Computer Science LNCS 10846 (2018) 1–14.

[29] V. Gurvich and L. Khachiyan, On generating the irredundant conjunctive and disjunctive normal forms of monotone boolean functions, Discrete Appl. Math. 96-97 (1999) 363–373.

[30] V. Gurvich and G. Koshevoy, Monotone bargaining is Nash-solvable, Discrete Appl. Math. 250 (2018) 1–15.

[31] V. Gurvich and M. Naumova, Polynomial algorithms computing two lexicographically safe Nash equilibria in finite two-person games with tight game forms given by oracles; https://arxiv.org/abs/2108.05469; 2021.

[32] V. Gurvich and M. Naumova, Lexicographically maximal edges of dual hypergraphs and Nash-solvability of tight game forms; https://arxiv.org/abs/2204.10213; Published online: 19 October 2022, Annals of Math. and Artificial Intelligence.

[33] D. Gusfield and R. W. Irving, The stable marriage problem: Structure and Algorithms, MIT Press, 1989.

[34] T. Harks, M. Klimm, and J. Matuschke, Pure Nash equilibria in resource graph games, Journal of Artificial Intelligence Research (2021).

[35] H. Kuhn, Extensive games and the problem of information, in Contributions to the theory of games, Volume 2, Princeton (1953) 193–216.

[36] N.S. Kukushkin, Acyclicity of improvements in finite game forms, Int. J. Game Theory 40 (2011) 147–177.

[37] D. Monderer and L. S. Shapley, Potential games, Games and Economic behavior 14 (1996) 124–143.

[38] H. Moulin, The Strategy of social choice, Advanced textbooks in Economics, 18, North Holland Publishing Company, Amsterdam, New York, Oxford, 1983.

[39] J. Nash, Equilibrium points in n-person games, Proceedings of the National Academy of Sciences 36:1 (1950) 48–49.

[40] J. Nash, Non-cooperative games, Annals of Math. 54:2 (1951) 286–295.
[41] B. Peleg, Game theoretic analysis of voting in committees, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1984.

[42] R. Rosenthal, A class of games possessing pure-strategy Nash equilibria: Internat. J. Game Theory, 2:1 (1973) 65–67.

[43] M. Sharir, A strong-connectivity algorithm and its application in data flow analysis, Comput. Math. Appl. 7 (1981) 67–72.

[44] L. Small and O. Mason, Nash equilibria for competitive information diffusion on trees, Information Processing Letters 113:7 (2013) 217–219.

[45] R. E. Tarjan, Depth-first search and linear graph algorithms, SIAM J. Computing 1:2 (1972) 146–160.

[46] R. Takehara, M. Hachimori, and M. Shigeno, A comment on pure-strategy Nash equilibria in competitive diffusion games, Information Processing Letters 112:3 (2012) 59—60.

[47] A. R. Washburn, Deterministic graphical games, J. of Math. Analysis and Appl. 153 (1990) 84–96.