Topological Symmetries

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Abstract

We introduce the notion of a topological symmetry as a quantum mechanical symmetry involving a certain topological invariant. We obtain the underlying algebraic structure of the $\mathbb{Z}_2$-graded uniform topological symmetries of type (1, 1) and (2, 1). This leads to a novel derivation of the algebras of supersymmetry and $p = 2$ parasupersymmetry.

1 Introduction

Since Witten’s pioneering work on supersymmetric quantum mechanics \cite{Witten}, there has been a growing interest in supersymmetry and its applications \cite{Berezin, Berezin2, Barut}. The interest in supersymmetry has also motivated the development of various generalizations of supersymmetry. Most notable of these are parasupersymmetries \cite{Barut, Barut2, Barut3}, the $q$-deformed supersymmetries \cite{Dabrowski} and the fractional supersymmetries \cite{Finkelstein}.

One of the most intriguing aspects of supersymmetric quantum mechanics is its relationship with the Atiyah-Singer index theorem \cite{Atiyah}. This has already been noticed by Witten \cite{Witten} in early 1980’s and subsequently led to new proofs of this theorem \cite{Kapustin, Minwalla, Seiberg}. Supersymmetric proofs of the index theorem together with Witten’s supersymmetric derivation of the Morse inequalities \cite{Witten} and its impact on Floer’s theory \cite{Floer, Floer2} are among the greatest mathematical achievements of supersymmetric quantum mechanics.

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The effectiveness of supersymmetry in providing insight into some of the most fundamental results of differential geometry and topology suggests the study of the topological content of various generalizations of supersymmetry. The only known generalization of supersymmetry which exhibits similar topological properties is a certain type of \( p = 2 \) parasupersymmetries. The topological properties of \( p = 2 \) parasupersymmetry (PSUSY) has been extensively studied in Refs. [17, 18].

The purpose of the present article is to introduce a generalization of supersymmetry (SUSY) which shares its topological properties. We shall term such a symmetry a topological quantum mechanical symmetry or simply topological symmetry (TS).

The first step towards such a generalization is to recall the basic properties of the \( N = 1 \) supersymmetric quantum mechanics.\(^1\)

\( N = 1 \) supersymmetric quantum mechanics is specified by a \( \mathbb{Z}_2 \)-graded Hilbert space \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) and the superalgebra

\[
\begin{align*}
[H,Q] &= 0 , \\
H &= \frac{1}{2} \{Q,Q^\dagger\}, \\
Q^2 &= 0 ,
\end{align*}
\]

where \( H \) and \( Q \) are the Hamiltonian and the generator of the supersymmetry, respectively. The \( \mathbb{Z}_2 \)-grading of the Hilbert space is implemented using a chirality or parity operator \( \tau : \mathcal{H} \to \mathcal{H} \) satisfying

\[
\begin{align*}
\tau^2 &= 1 , \quad \tau^\dagger = \tau , \\
[H,\tau] &= 0 , \\
\{\tau,Q\} &= 0 .
\end{align*}
\]

The subspaces \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) are identified with the eigenspaces of \( \tau \),

\[
\mathcal{H}_\pm := \{|\psi\rangle \in \mathcal{H} \mid \tau |\psi\rangle = \pm |\psi\rangle \} .
\]

The elements of \( \mathcal{H}_\pm \) are said to have definite parity \( \pm \). An operator \( O \) acting on \( \mathcal{H} \) is said to have definite parity + or −, if it commutes or anticommutes with \( \tau \), respectively.

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\(^1\)We use the terminology in which \( N \) is half of the number of Hermitian generators. What we call \( N = 1 \) SUSY sometimes is referred to as the \( \mathcal{N} = 2 \) SUSY where \( \mathcal{N} \) is the number of Hermitian SUSY generators.
It is well-known that using Eqs. (1) – (3) one can obtain the degeneracy structure of a general $N = 1$ supersymmetric system \([1, 2, 17]\). These systems have a nonnegative spectrum and the eigenspaces corresponding to positive eigenvalues are spanned by pairs of eigenvectors of opposite parity. This particular degeneracy structure of supersymmetric systems is sufficient to show the topological invariance of the Witten index:

\[
\text{index}_W := n_0^+ - n_0^- ,
\]

\[
n_0^\pm := \text{number of zero energy states with parity } \pm .
\]

The degeneracy structure is obtained from the algebraic structure. Therefore, the situation may be described by the following diagram.

\[
\text{algebraic structure} \rightarrow \text{degeneracy structure} \rightarrow \text{topological invariants}
\]

The same analysis is valid for the case of the $p = 2$ PSUSY studied in Refs. \([17, 18]\).

The idea pursued in this article is to reverse the first arrow in (10). More specifically, we wish to

- find and postulate the type of degeneracy structures which lead to topological invariants such as the Witten index, and

- obtain the algebraic structure of symmetries which support this type of degeneracy structures.

\section{Topological Symmetries and Their Invariants}

**Definition 1:** Let $m_+$ and $m_-$ be two positive integers. Then a quantum mechanical symmetry is called a $\mathbb{Z}_2$-\textit{graded topological symmetry} (TS) of type $(m_+, m_-)$, if the following conditions are fulfilled.

1. The Hilbert space is $\mathbb{Z}_2$-graded. The $\mathbb{Z}_2$-grading is achieved via a parity operator $\tau$ satisfying Eqs. (3), i.e., $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_\pm$ are given by Eq. (7).

2. The energy spectrum of the system is nonnegative.

3. There is an energy eigenbasis consisting of definite parity state vectors, i.e., Eq. (3) holds.
4. For every positive energy eigenvalue $E$, there exists a positive integer $\lambda_E$ such that $E$ is $m_E := \lambda_E(m_+ + m_-)$ fold degenerate. Furthermore, the corresponding eigenspace is spanned by $\lambda_E m_-$ negative parity eigenvectors and $\lambda_E m_+$ positive parity eigenvectors.

Clearly, SUSY is an example of TS of type $(1, 1)$. Therefore, TSs are generalizations of the SUSY.

**Definition 2:** A $\mathbb{Z}_2$-graded topological symmetry is said to be *uniform*, if for all $E > 0$, $\lambda_E = 1$.

Given the above definition of a $\mathbb{Z}_2$-graded uniform topological symmetry (UTS), we can easily prove the following theorem.

**Theorem:** Consider a $\mathbb{Z}_2$-graded UTS of type $(m_-, m_+)$ and let $n_0^\pm$ denote the number of zero energy eigenstates of the system with parity $\pm$. Then the quantity

$$
\Delta_{(m_+, m_-)} := m_-n_0^+ - m_+n_0^-
$$

is a topological invariant, i.e., it is invariant under the continuous changes of the quantum system\(^2\) that do not destroy the UTS.

**Proof:** Suppose that under a continuous change of the system a zero energy state vector with positive parity is elevated to a positive energy level. This positive energy level must have $m_+$ positive parity states and $m_-$ negative parity states. Hence, the initial zero energy state must be accompanied by $(m_+ - 1)$ positive parity zero energy eigenstates and $m_-$ negative parity zero energy eigenstates. This implies that after the change $\Delta_{(m_+, m_-)}$ is given by

$$
\Delta_{(m_+, m_-)}^{\text{after}} = m_-(n_0^+ - m_+) - m_+(n_0^- - m_-) = m_-n_0^+ - m_+n_0^- = \Delta_{(m_+, m_-)}^{\text{before}}.
$$

Every possible change of the zero energy states is a combination of this particular change and its converse. Therefore, in general $\Delta_{(m_+, m_-)}$ remains invariant. □

\(^2\)These include continuous changes of the Hamiltonian and the boundary conditions.
The same proof is valid for the case of nonuniform $\mathbb{Z}_2$-graded TSs. Therefore, $\Delta_{\ell_+, \ell_-}$ is a topological invariant for any $\mathbb{Z}_2$-graded TS.

We shall next provide the basic framework for addressing the characterization problem for TSs. In this paper we shall only consider the uniform TSs. But our method applies to nonuniform TSs as well.

We shall demand TSs to have symmetry generators $Q_a$ (with $a = 1, 2, \cdots, N$) which have negative parity. In particular, we shall only consider the $N = 1$ UTSs where the label $a = 1$ can be dropped. In this case, Eq. (5) is valid.

Next we introduce the Hermitian symmetry generators,

$$
Q_1 := \frac{1}{\sqrt{2}}(Q + Q^\dagger), \quad \text{and} \quad Q_2 = - \frac{i}{\sqrt{2}}(Q - Q^\dagger).
$$

(12)

In view of Eqs. (1), (4), (6) and (12), we have

$$
[H, Q_i] = 0,
$$

(13)

$$
\{\tau, Q_i\} = 0,
$$

(14)

where $i \in \{1, 2\}$.

We can use Eq. (3) to construct an orthonormal basis of the Hilbert space in which $H$ and $\tau$ are diagonal. Our strategy will be to use the information on the degeneracy structure of the energy eigenspaces and Eqs. (1), (13), and (14) to obtain matrix representations of $Q_1$ and $Q_2$ in the energy eigenspaces $\mathcal{H}_E$ with eigenvalue $E > 0$. We shall denote the representation of an operator $O$ in the eigenspace $\mathcal{H}_E$ by $O^E$. Clearly for the Hamiltonian $H$, we have $H^E = EI_m$, where $I_m$ is the $m \times m$ identity matrix and $m := m_+ + m_-$. Finally, we shall try to use these representations to obtain the most general algebraic relations satisfied by $Q$ and $H$.

3 Uniform Topological Symmetries of Type $(1, 1)$

For the $\mathbb{Z}_2$-graded TS of type $(1, 1)$, the positive energy levels are doubly degenerate ($m = 2$). In a basis that diagonalizes $H$ and $\tau$, we have (up to a permutation of the basis vectors)

$$
\tau^E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

(15)
where $E > 0$ and we have used Eqs. (14). Next let us note that $Q_i^E$ are $2 \times 2$ Hermitian matrices satisfying (14). These conditions are sufficient to conclude that

$$Q_1^E = \begin{pmatrix} 0 & \mu^* \\ \mu & 0 \end{pmatrix} \quad \text{and} \quad Q_2^E = \begin{pmatrix} 0 & \nu^* \\ \nu & 0 \end{pmatrix},$$

where $\mu$ and $\nu$ are complex numbers.

Using the matrix representations (15) and (13), we can easily compute

$$Q^E := \frac{1}{\sqrt{2}} (Q_1^E + iQ_2^E) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mu^* + i\nu^* \\ \mu + i\nu & 0 \end{pmatrix},$$

$$\begin{align*}
(Q_1^E)^2 &= |\mu|^2 I_2, \\
(Q_2^E)^2 &= |\nu|^2 I_2, \\
(Q^E)^2 &= \frac{1}{2} \left[ |\mu|^2 - |\nu|^2 + i(\mu\nu^* + \mu^*\nu) \right] I_2 = -\det(Q^E)I_2,
\end{align*}$$

where ‘det’ stands for ‘determinant’. Next we introduce the Hermitian operators $M$, $K_1$ and $K_2$ which commute with $H$ and have the following representations in $H_E$ with $E > 0$.

$$M^E = |\mu|^2 I_2, \quad K_1^E = (|\mu|^2 - |\nu|^2)I_2, \quad K_2^E = (\mu\nu^* + \mu^*\nu)I_2.$$  (21)

In view of Eqs. (18) – (21), $M^E$, $K^E$, and $K_i^E$ commute with $Q_i^E$, $Q^E$ and $\tau^E$. Generalizing these equations to operator identities, we find

$$[M, Q_i] = [K_j, Q_i] = [M, K_j] = 0,$$  (22)

$$Q_1^2 = M,$$  (23)

$$Q_2^2 = M - K_1,$$  (24)

$$\{Q_1, Q_2\} = K_2.$$  (25)

We can also express Eqs. (22) – (25) in terms of $Q$. This yields

$$[M, Q] = [K, Q] = [M, K] = 0,$$  (26)

$$\frac{1}{2} \{Q, Q^\dagger\} = M - \frac{1}{2}(K + K^\dagger),$$  (27)

$$Q^2 = K,$$  (28)

where $K := (K_1 + iK_2)/2$.

Next let us note that under a linear transformations of the form:

$$Q_1 \rightarrow \tilde{Q}_1 := a Q_1 + b Q_2,$$

$$Q_2 \rightarrow \tilde{Q}_2 := c Q_1 + d Q_2,$$  (29)
the algebra (22) – (25) is left form-invariant. More specifically, the transformed generators \( \tilde{Q}_i \) satisfy the same algebra provided that the operators \( M \) and \( K_i \) are transformed according to

\[
M \rightarrow \tilde{M} := (a^2 + b^2)M - b^2 K_1 + ab K_2 ,
\]

\[
K_1 \rightarrow \tilde{K}_1 := (a^2 + b^2 - c^2 - d^2)M + (d^2 - b^2)K_1 + (ab - cd)K_2 ,
\]

\[
K_2 \rightarrow \tilde{K}_2 := 2(ac + bd)M - 2bd K_1 + (ad + bc)K_2 .
\]

This observation may be used to find a new set of negative parity symmetry generators \( \tilde{Q}_i \) which would satisfy the algebra (22) – (25) with \( K_i \) set to zero. The most general linear transformations (29) for which \( \tilde{K}_1 = \tilde{K}_2 = 0 \) are the ones satisfying (either of)

\[
\frac{a + ic}{b + id} = -\frac{K_2}{2M} \pm i \sqrt{1 - \frac{K_1}{M} - \frac{K_2^2}{4M^2}}.
\]

Using the energy eigenbasis in which Eqs. (21) hold, we can show that indeed the terms inside the square root in (33) add up to a positive operator. This is a necessary condition for the corresponding linear transformation to be invertible.

The above analysis shows that without loss of generality we can set \( K = 0 \) in Eqs. (26) – (28). This yields

\[
[M, Q] = 0 ,
\]

\[
\frac{1}{2} \{ Q, Q^\dagger \} = M ,
\]

\[
Q^2 = 0 .
\]

Since \( M \) has the same degeneracy structure as the Hamiltonian (at least in \( \mathcal{H} - \mathcal{H}_0 \)), we can write \( H \) as a function of \( M \). In particular, we can identify \( H \) with \( M \), in which case the algebra (34) – (36) reduces to the superalgebra (1) – (3). Therefore, the algebra of the \( \mathbb{Z}_2 \)-graded UTS of type (1,1) is the same as the algebra of SUSY. The above analysis may be viewed as a derivation of the superalgebra (1) – (3) from a set of basic principles, i.e., the definition of the \( \mathbb{Z}_2 \)-graded UTS of type (1,1).

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\( ^3 \)Here the coefficients \( a, b, c \) and \( d \) are assumed to be Hermitian operators commuting with \( H, Q_i, K_i \) and \( M \).

\( ^4 \)Strictly speaking most of the conclusions drawn in this article are based on the information obtained from the restriction of the relevant operators to \( \mathcal{H} - \mathcal{H}_0 \). We will however suppose that the same results are generally valid in \( \mathcal{H} \). This is consistent with the fact that we have not given the representations of \( M \) and \( K_i \) in \( \mathcal{H}_0 \).


4 Uniform Topological Symmetry of Type \((2, 1)\)

For the \(\mathbb{Z}_2\)-graded UTS of type \((2, 1)\), the positive energy levels are triply degenerate \((m = 3)\).

We will again work in a basis in which \(H\) and \(\tau\) are diagonal. Restricting to an eigenspace \(\mathcal{H}_E\) of \(H\) with \(E > 0\) and enforcing Eqs. (11) and (14), we can easily show that (up to permutations of the basis vectors of \(\mathcal{H}_E\))

\[
\tau^E = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad Q_1^E = \begin{pmatrix}
0 & 0 & \mu_1^* \\
0 & 0 & \mu_2^* \\
\mu_1 & \mu_2 & 0
\end{pmatrix}, \quad \tag{37}
\]

\[
Q_2^E = \begin{pmatrix}
0 & 0 & \nu_1^* \\
0 & 0 & \nu_2^* \\
\nu_1 & \nu_2 & 0
\end{pmatrix}, \quad Q^E = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & \mu_1^* + i\nu_1^* \\
0 & 0 & \mu_2^* + i\nu_2^* \\
\mu_1 + i\nu_1 & \mu_2 + i\nu_2 & 0
\end{pmatrix}. \quad \tag{38}
\]

Now in order to obtain the most general algebraic identities satisfied by these matrices we appeal to the Cayley-Hamilton theorem \([19]\). This theorem states that any \(m \times m\) matrix \(A\) satisfies its characteristic equation, \(P_A(x) = 0\), where \(P_A(x)\) is the characteristic polynomial for \(A\). It is not difficult to show that the characteristic polynomial for a \(3 \times 3\) matrix \(A\) of the form

\[
A = \begin{pmatrix}
0 & 0 & \alpha \\
0 & 0 & \beta \\
\gamma & \delta & 0
\end{pmatrix}
\]

is given by \(P_A(x) = x^3 - (\alpha\gamma + \beta\delta)x\). Using this equation and the identity \(P_A(A) = 0\) for \(Q_1^E\), \(Q_2^E\) and \(Q^E\), we find

\[
(Q_1^E)^3 = \sum_{j=1}^{2} |\mu_j|^2 Q_1^E, \quad \tag{39}
\]

\[
(Q_2^E)^3 = \sum_{j=1}^{2} |\nu_j|^2 Q_2^E, \quad \tag{40}
\]

\[
(Q^E)^3 = \frac{1}{2} \sum_{j=1}^{2} [ |\mu_j|^2 - |\nu_j|^2 + i(\mu_j\nu_j^* + \mu_j^*\nu_j) ] Q^E. \quad \tag{41}
\]

Next we introduce the Hermitian operators \(M\), \(K_1\), and \(K_2\) which commute with \(H\) and have the following matrix representations in \(\mathcal{H}_E\) with \(E > 0\).

\[
M^E = \sum_{j=1}^{2} |\mu_j|^2 I_3, \quad \tag{42}
\]

\[
K_1^E = \sum_{j=1}^{2} (|\mu_j|^2 - |\nu_j|^2) I_3, \quad \tag{43}
\]

\[
K_2^E = \sum_{j=1}^{2} (\mu_j\nu_j^* + \mu_j^*\nu_j) I_3. \quad \tag{44}
\]
Now setting $K := (K_1 + iK_2)/2$ and making use of Eqs. (12) - (14), we can generalize Eqs. (33) - (31) to the operator relations

\[
Q_1^3 = MQ_1, \quad (45)
\]
\[
Q_2^3 = (M - K_1)Q_2, \quad (46)
\]
\[
Q^3 = KQ. \quad (47)
\]

Eqs. (42) - (44) also suggest that the operators $M$ and $K_i$ commute among themselves and with $Q_i$. Furthermore, we can rewrite Eq. (47) in terms of $Q_1$ and $Q_2$. Simplifying the resulting equation using Eqs. (45) and (46) we obtain the following general algebra

\[
[M, Q_i] = [M, K_i] = [K_i, Q_j] = 0, \quad (48)
\]
\[
Q_1^3 = MQ_1, \quad (49)
\]
\[
Q_2^3 = (M - K_1)Q_2, \quad (50)
\]
\[
Q_2Q_1Q_2 + \{Q_1, Q_2^3\} = (M - K_1)Q_1 + K_2Q_2 , \quad (51)
\]
\[
Q_1Q_2Q_1 + \{Q_2, Q_1^3\} = MQ_2 + K_2Q_1 . \quad (52)
\]

We can show by direct computation that this algebra remains form-invariant under linear transformations (29) of $Q_i$. More remarkable is the fact that under such a transformation the operators $M$ and $K_i$ transform according to the same relations as in the case of UTS of type (1, 1), namely Eqs. (30) - (32). Therefore, we can always transform to a new set of symmetry generators for which $K_i = 0$. Setting $K_i = 0$ in Eqs. (48) - (52) and rewriting these equations in terms of $Q$, we find

\[
[M, Q] = 0, \quad (53)
\]
\[
\{Q^2, Q^\dagger\} + QQ^\dagger Q = 2MQ, \quad (54)
\]
\[
Q^3 = 0. \quad (55)
\]

The operator $M$ has the same degeneracy structure as the Hamiltonian. Therefore, we can identify $H$ with a function of $M$. In particular, we can set $H = M/2$. In this case the algebra (53) - (55) becomes identical to the algebra of $p = 2$ parasupersymmetry [5]. In other words, the algebra of the $\mathbb{Z}_2$-graded UTS of type (2, 1) is precisely the algebra of the $p = 2$
parasupersymmetry (of Rubakov and Spiridonov \cite{rubakov}). Again, the above analysis may be viewed as a derivation of the algebra of \(p = 2\) parasupersymmetry from a set of basic principles, i.e., the definition of the \(\mathbb{Z}_2\)-graded UTS of type \((2, 1)\).

As shown in Ref. \cite{ref17}, the algebra (53) – (55) does not imply the degeneracy structure of type \((2, 1)\) UTS. Therefore, the type \((2, 1)\) \(\mathbb{Z}_2\)-graded UTSs belong to a special class of symmetries whose generator \(Q\) satisfies Eqs. (53) – (55). A method for constructing the corresponding moduli spaces is given in Ref. \cite{ref18}.

5 Conclusion

In this article we have introduced a general notion of a topological symmetry. We have provided a simple framework for the study of the \(\mathbb{Z}_2\)-graded topological symmetries. We showed that the algebras of the \(\mathbb{Z}_2\)-graded UTS of order \((1, 1)\) and \((2, 1)\) are essentially the algebras of supersymmetric quantum mechanics and \(p = 2\) parasupersymmetric quantum mechanics, respectively.

By construction, topological symmetries involve a class of integer-valued topological invariants \(\Delta_{(m_+, m_-)}\). These are the analogues of the Witten index of supersymmetry and the parasupersymmetric topological invariant \cite{ref18}. The physical interpretation of these invariants is that they are a measure of the existence of zero-energy ground states. For the known cases, the latter is an indication of the exactness of symmetry. The mathematical interpretation of \(\Delta_{(m_+, m_-)}\) is not quite clear. For the known cases they are related to the analytic indices of Fredholm operators \cite{ref18}. The general case requires a detailed study of the algebraic structure of general topological symmetries. The algebras of \(\mathbb{Z}_2\)-graded TS of arbitrary type \((m_+, m_-)\) are currently under investigation.

Finally, one can easily generalize the definition of the \(\mathbb{Z}_2\)-graded topological symmetry of type \((m_+, m_-)\) to a \(\mathbb{Z}_n\)-graded topological symmetry of type \((m_1, m_2, \ldots, m_n)\). Such a system will have states with definite ‘color’ taking values in \(\{1, 2, \ldots, n\}\). The spectrum will be nonnegative. The positive energy eigenvalues \(E\) will be \(m_E = \lambda_E \sum_{\ell=1}^n m_\ell\) fold degenerate. The energy eigenspaces with positive eigenvalue will all have \(\lambda_E m_1\) states of color 1, \(\lambda_E m_2\) states of color 2, \(\ldots\), and \(\lambda_E m_n\) states of color \(n\). One may try to define topological invariants for these
more general topological symmetries. For example for a $\mathbb{Z}_3$-graded TS of type $(1,1,1)$, one can introduce the invariant $\Delta_{(1,1,1)} = (n_0^{(1)} - n_0^{(2)})^2 + (n_0^{(2)} - n_0^{(3)})^2 + (n_0^{(3)} - n_0^{(1)})^2$, where $n_0^{(\ell)}$ is the number of zero energy states of color $\ell$. The basic properties of $\mathbb{Z}_n$-graded topological symmetries will be explored in [20].

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