Spanning Trees of Connected $K_{1,t}$-free Graphs Whose Stems Have a Few Leaves

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Abstract
Let $T$ be a tree, a vertex of degree one is called a leaf. The set of leaves of $T$ is denoted by Leaf($T$). The subtree $T − \text{Leaf}(T)$ of $T$ is called the stem of $T$ and denoted by Stem($T$). In this paper, we give a sharp sufficient condition to show that a $K_{1,t}$-free graph has a spanning tree whose stem has a few leaves. By applying the main result, we give improvements of previous related results.

Keywords Spanning tree · $K_{1,t}$-free · Stem · Leaf

Mathematics Subject Classification 05C05 · 05C07 · 05C69

1 Introduction

In this paper, we only consider finite simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $\text{deg}_G(v)$ to denote the set of neighbors of $v$ and the degree of $v$ in $G$, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of $X$. We define $G − uv$ to be the graph obtained from $G$ by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from $G$ by adding an edge $uv$ between two non-adjacent vertices $u$ and $v$ of $G$. For two vertices $u$ and $v$ of $G$, the distance between $u$ and $v$ in $G$ is denoted by $d_G(u, v)$.

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For an integer \( m \geq 2 \), let \( \alpha^m(G) \) denote the number defined by
\[
\alpha^m(G) = \max\{|S| : S \subseteq V(G), d_G(x, y) \geq m \text{ for all distinct vertices } x, y \in S\}.
\]

For an integer \( p \geq 2 \), we define
\[
\sigma^m_p(G) = \min \left\{ \sum_{a \in S} \deg_G(a) : S \subseteq V(G), |S| = p, \right. \\
\left. d_G(x, y) \geq m \text{ for all distinct vertices } x, y \in S \right\}.
\]

For convenience, we define \( \sigma^m_p(G) = +\infty \) if \( \alpha^m(G) < p \). We note that \( \alpha^2(G) \) is often written \( \alpha(G) \), which is the independence number of \( G \), and \( \sigma^2_p(G) \) is often written \( \sigma_p(G) \), which is the minimum degree sum of \( p \) independent vertices.

Let \( T \) be a tree, a vertex of degree one is called a leaf. The set of leaves of \( T \) is denoted by \( \text{Leaf}(T) \). The subtree \( T - \text{Leaf}(T) \) of \( T \) is called the stem of \( T \) and is denoted by \( \text{Stem}(T) \). A tree having at most \( l \) leaves is called an \( l \)-ended tree and a stem having at most \( l \) leaves is called an \( l \)-ended stem. There are several well-known conditions (such as the independence number conditions and the degree sum conditions) ensuring that a graph \( G \) contains a spanning tree with a bounded number of leaves or branch vertices (see the survey paper [5] and the references cited therein for details). Win [7] obtained a sufficient condition related to the independence number for \( k \)-connected graphs, which confirms a conjecture of Las Vergnas [4]. Broersma and Tuinstra [1] gave a degree sum condition for a connected graph to contain a spanning tree with at most \( l \) leaves.

**Theorem 1.1** (Win [7]) Let \( k \geq 1 \) and \( l \geq 2 \) be integers and let \( G \) be a \( k \)-connected graph. If \( \alpha(G) \leq k + l - 1 \), then \( G \) has a spanning \( l \)-ended tree.

**Theorem 1.2** (Broerma and Tuinstra [1]) Let \( G \) be a connected graph and let \( l \geq 2 \) be an integer. If \( \sigma^2(G) \geq |G| - l + 1 \), then \( G \) has a spanning \( l \)-ended tree.

Recently, many researches are studied on spanning trees in connected graphs whose stems have a bounded number of leaves or branch vertices (see [2,3,6,8] for more details). We introduce here some results on spanning trees whose stems have a few leaves.

**Theorem 1.3** (Tsugaki and Zhang [6]) Let \( G \) be a connected graph and let \( l \geq 2 \) be an integer. If \( \sigma^3(G) \geq |G| - 2l + 1 \), then \( G \) has a spanning tree with \( l \)-ended stem.

**Theorem 1.4** (Kano and Yan [2]) Let \( G \) be a connected graph and let \( l \geq 2 \) be an integer. If \( \sigma^{l+1}(G) \geq |G| - l - 1 \), then \( G \) has a spanning tree with \( l \)-ended stem.

Moreover, for a positive integer \( t \geq 3 \), a graph \( G \) is said to be \( K_{1,t} \)-free graph if it contains no \( K_{1,t} \) as an induced subgraph. If \( t = 3 \), the \( K_{1,3} \)-free graph is also called the claw-free graph. Kano and Yan also gave the following result.
Theorem 1.5 (Kano and Yan [2]) Let $G$ be a connected claw-free graph and let $l \geq 2$ be an integer. If $\sigma_{l+1}(G) \geq |G| - 2l - 1$, then $G$ has a spanning tree with $l$-ended stem.

On the other hand, if the maximum degree of a graph $G$ is denoted by $\Delta(G)$, then $G$ is nothing but a $K_{1,t}$-free graph for all $t \geq \Delta(G) + 1$. Then, we may generalize the above theorems by studying the $K_{1,t}$-free graph. In this paper, we would like to study on the spanning tree of a $K_{1,t}$-free graph whose stem has a bounded number of leaves. Firstly, we want to prove the following result.

Theorem 1.6 For a positive integer $t \geq 3$, let $G$ be a connected $K_{1,t}$-free graph and let $l \geq 2$ ($l \neq t - 2$) be an integer. If

$$
\sigma_{l+1}^4(G) \geq |G| - \left\lfloor \frac{l(t - 1)}{t - 2} \right\rfloor - 1,
$$

then $G$ has a spanning tree with $l$-ended stem. Here, the notation $\lfloor r \rfloor$ stands for the biggest integer not exceed the real number $r$.

We also note that the reason why we consider $\sigma_{l+1}^4(G)$ is based on the following theorem of Kano and Yan. They proved that if a connected graph $G$ satisfies that $|S| \leq l$ for every $S \subseteq V(G)$ such that $d_G(x, y) \geq 4$ for all distinct vertices $x, y \in S$, then $G$ has a spanning tree with $l$-ended stem.

Theorem 1.7 (Kano and Yan [2]) Let $G$ be a connected graph and let $l \geq 2$ be an integer. If $\alpha^4(G) \leq l$, then $G$ has a spanning tree with $l$-ended stem.

By using Theorem 1.6 when $t = 3$, we have Theorem 1.5. Moreover, Theorem 1.6 is an improvement of Theorem 1.4 when we consider the positive integer $t$ big enough.

We now construct two examples to show that the conditions of Theorem 1.6 are sharp.

Let $t, k, m$ be integers such that $t \geq 3, k \geq 2, m \geq 1$ and let $l = k(t - 2)$. Let $D$ be a complete graph with $k + 1$ vertices $u_1, u_2, \ldots, u_{k+1}$. Let $D_1, D_2, \ldots, D_{k(t-2)+1}$ be copies of the graph $K_m$. Let $v_1, v_2, \ldots, v_{k(t-2)+1}$ be vertices which are not in $V(D) \cup V(D_1) \cup V(D_2) \cup \cdots \cup V(D_{k(t-2)+1})$. For each $i \in \{1, 2, \ldots, k\}$, join $u_i$ to all vertices of the graphs $D_{i-1}(t-2)+1, D_{i-1}(t-2)+2, \ldots, D_{i(t-2)}$ and join $u_{k+1}$ to all vertices of the graph $D_{k(t-2)+1}$. Join $v_j$ to all vertices of $D_j$ for all $j \in \{1, 2, \ldots, k(t-2) + 1\}$. Then, the resulting graph $G$ is a $K_{1,t}$-free graph (see Fig. 1).

Moreover, we have $|G| = k + 1 + (k(t - 2) + 1)(m + 1)$ and

$$
\sigma_{l+1}^4(G) = \sigma_{k(t-2)+1}^4(G) = \sum_{i=1}^{k(t-2)+1} \deg_G(v_i)
$$

$$
= (k(t - 2) + 1)m = |G| - k(t - 1) - 2 = |G| - \left\lfloor \frac{l(t - 1)}{t - 2} \right\rfloor - 2.
$$

But $G$ has no spanning tree with $l$-ended stem. Hence, the condition (1.1) is sharp.
On the other hand, when $l = t - 2$, let $E_i (1 \leq i \leq l + 1)$ be connected graphs. For each $i \in \{1, 2, \ldots, l + 1\}$, denote by $K_i$ the set of vertex $v$ in $V(E_i)$ such that $N_{E_i}(v) = V(E_i) - \{v\}$. Suppose that $K_i \neq \emptyset$ for all indices $1 \leq i \leq l + 1$. Let $u$ be a vertex which is not in $V(E_1) \cup V(E_2) \cup \cdots \cup V(E_{l+1})$. For each $i \in \{1, 2, \ldots, l + 1\}$, join $u$ to all vertices in $V(E_i) - K_i$ and some vertices in $K_i$ excepting at least one vertex in $K_i$. The resulting graph is denoted by $M$. We only consider the case $M$ is a $K_{1,t}$-free graph. By the definition of $M$, we may obtain that each subset $S \subseteq V(M)$ such that $|S| = l + 1$ and $d_M(x, y) \geq 4$ must contain one and only one vertex in $K_i$ for all $1 \leq i \leq l + 1$. Hence,

$$
\sigma_{t+1}^4(M) = \sum_{i=1}^{l+1} \deg_M(\text{one vertex } v_i \in K_i \text{ such that } v_i u \notin E(M)) = \sum_{i=1}^{l+1} (|E_i| - 1) = |M| - (l + 2) = |M| - \left\lfloor \frac{l(t - 1)}{t - 2} \right\rfloor - 1.
$$

But $M$ has no spanning tree with $l$-ended stem.

A natural question is whether we can find all graphs so that the claim of Theorem 1.6 is not correct in the case $l = t - 2$. We will give an answer for this question. In particular, we state the following theorem which is an improvement of Theorem 1.6.
**Theorem 1.8** For a positive integer \( t \geq 3 \), let \( G \) be a connected \( K_{1,t} \)-free graph and let \( l \geq 2 \) be an integer. If

\[
\sigma_{l+1}^4(G) \geq |G| - \left\lfloor \frac{l(t-1)}{t-2} \right\rfloor - 1,
\]

then \( G \) has a spanning tree with \( l \)-ended stem except for the case \( l = t-2 \) and \( G \) is isomorphic to a graph \( M \).

**Remark 1.9** Since \( \sigma_{l+1}^4(M) = |M| - \left\lfloor \frac{l(t-1)}{t-2} \right\rfloor - 1 \), it follows from Theorem 1.8 that if \( G \) is a connected \( K_{1,t} \)-free graph such that \( \sigma_{l+1}^4(G) \geq |G| - \left\lfloor \frac{l(t-1)}{t-2} \right\rfloor \), then \( G \) has a spanning tree with \( l \)-ended stem.

## 2 Proof of Theorem 1.8

We prove the theorem by contradiction. Suppose to the contrary that \( G \) contains no spanning tree with \( l \)-ended stem. Let \( T \) be a tree such that \( |\text{Leaf}(\text{Stem}(T))| \leq l \). Choose a tree \( T \) so that

\[(T1) \quad |T| \text{ is as large as possible, and}
(T2) \quad |\text{Leaf}(T)| \text{ is as large as possible, subject to (T1).}

By the maximality of \( T \), we have the following three claims. We note that their proofs are written in [2,8], but for the convenience of readers, we reintroduce them here.

**Claim 2.1** For every \( v \in V(G) - V(T) \), \( N_G(v) \subseteq \text{Leaf}(T) \cup (V(G) - V(T)) \).

Because \( G \) is connected and \( T \) is not a spanning tree of \( G \) and by Claim 2.1, there exist two vertices \( v_1 \in V(G) - V(T) \) and \( v_2 \in \text{Leaf}(T) \) such that \( v_1 v_2 \in E(G) \). We may obtain that \( \text{Stem}(T) \) has exactly \( l \) leaves. Indeed, otherwise we consider the tree \( T' = T + v_1 v_2 \). Then, \( T' \) has \( l \)-ended stem and \( |T'| > |T| \), this implies a contradiction with the maximality of \( T \). Let \( \{x_1, x_2, \ldots, x_l\} \) be the leaf set of \( \text{Stem}(T) \).

**Claim 2.2** For every \( x_i \) \((1 \leq i \leq l)\), there exists a vertex \( y_i \in \text{Leaf}(T) \) such that \( y_i \) is adjacent to \( x_i \) and \( N_G(y_i) \subseteq \text{Leaf}(T) \cup \{x_i\} \).

**Proof** By the definition of \( \text{Stem}(T) \), it is easy to see that for each leaf \( x \in \text{Leaf}(\text{Stem}(T)) \), there exists at least a vertex \( y \) in \( \text{Leaf}(T) \) such that \( y \) is adjacent to \( x \). Suppose that for some \( 1 \leq i \leq l \), each leaf \( y_{ij} \) of \( T \) adjacent to \( x_i \), is also adjacent to a vertex \( z_{ij} \in (V(\text{Stem}(T)) - \{x_i\}) \). Then, we consider \( T' \) to be the tree obtained from \( T \) by removing the edge \( y_{ij} x_i \) and adding the edge \( y_{ij} z_{ij} \). Hence, \( T' \) is a tree with \( l \)-end stem such that \( |T'| = |T| \) and \( \text{Leaf}(T') = \text{Leaf}(T) \cup \{x_i\} \), which contradicts the condition (T2). Therefore, for each \( x_i \), there exists a leaf \( y_i \in N_G(x_i) \) such that \( N_G(y_i) \cap (V(\text{Stem}(T)) - \{x_i\}) = \emptyset \). By the maximality of \( T \) we also see that \( N_G(y_i) \cap (V(G) - V(T)) = \emptyset \). The claim holds. \( \square \)
Claim 2.3 For any two distinct vertices $y, z \in \{v_1, y_1, y_2, \ldots, y_l\}$, $d_G(y, z) \geq 4$.

**Proof** First, we show that $d_G(v_1, y_j) \geq 4$ for every $1 \leq i \leq l$. Let $P_i$ be the shortest path connecting $v_1$ and $y_i$ in $G$. If all the vertices of $P_i$ between $v_1$ and $y_i$ are contained in $\text{Leaf}(T) \cup (V(G) - V(T)) \cup \{x_i\}$, add $P_i$ to $T$ (if $P_i$ passes through $x_i$, we just add the segment of $P_i$ between $v_1$ and $x_i$ and remove the edges of $T$ joining $V(P_i) \cap \text{Leaf}(T)$ to $V(\text{Stem}(T))$ except the edge $y_i x_i$. The resulting tree is denoted by $T'$. Then, $T'$ is a tree in $G$ with $l$-ended stem and $|T'| > |T|$, which contradicts to the maximality of $T$. So we conclude that for each $1 \leq j \leq l$, if $P$ is a shortest path connecting $v_1$ and $y_j$ in $G$, then $V(P) \cap (V(\text{Stem}(T)) - \{x_j\}) \neq \emptyset$.

Hence, we may choose the vertex $s$ in $V(\text{Stem}(T)) \cap V(P_i)$ such that it is nearest to $v_1$ in $P_i$. If $s = x_j$ for some $1 \leq j \leq l$, then we add the segment of $P_i$ between $v_1$ and $x_j$ (which is denoted by $Q$) to $T$ and remove the edges of $T$ joining $V(Q) \cap \text{Leaf}(T)$ to $V(\text{Stem}(T))$ except $x_j y_j$. Hence, the resulting tree has $l$-ended stem and its order is greater then $|T|$, contradicting the maximality of $T$. Thus, $s \in V(\text{Stem}(T)) - \{x_1, \ldots, x_l\}$. By Claims 2.1 and 2.2, we have $d_G(v_1, s) \geq 2, d_G(s, y_i) \geq 2$. Therefore, we conclude that $d_G(v_1, y_i) = |P_i| - 1 \geq d_G(v_1, s) + d_G(s, y_i) \geq 4$.

Next, we show that $d_G(y_i, y_j) \geq 4$ for all $1 \leq i < j \leq l$. Let $P_{ij}$ be the shortest path connecting $y_i$ and $y_j$ in $G$. We note that if $P_{ij}$ passes through $x_i$ (or $x_j$), then $y_i x_i \in E(P_{ij})$ (or $y_j x_j \in E(P_{ij})$), respectively. We consider the following two cases.

**Case 1.** All vertices of $P_{ij}$ between $y_i$ and $y_j$ are contained in $\text{Leaf}(T) \cup (V(G) - V(T)) \cup \{x_i, x_j\}$. Then, we add $P_{ij}$ to $T$ and remove the edges of $T$ joining $V(P_{ij}) \cap \text{Leaf}(T)$ to $V(\text{Stem}(T))$ except the edges $y_i x_i$ and $y_j x_j$. Hence, the resulting graph has exactly a cycle, which contains an edge $e$ of $\text{Stem}(T)$ incident with a branch vertex in $\text{Stem}(T)$. By removing the edge $e$ and by adding an edge $v_1 v_2$, we have a resulting tree $T'$ with $l$-ended stem of order greater than $|T|$, which contradicts the maximality of $T$. So we conclude that for every $1 \leq i < j \leq l$, if $P$ is a shortest path connecting $y_i$ and $y_j$ in $G$, then $V(P) \cap (V(\text{Stem}(T)) - \{x_i, x_j\}) \neq \emptyset$.

**Case 2.** There exists a vertex $s \in V(P_{ij}) \cap (V(\text{Stem}(T)) - \{x_i, x_j\})$. Then, $d_G(y_i, s) \geq 2, d_G(s, y_j) \geq 2$ by Claim 2.2. This concludes that $d_G(y_i, y_j) = |P_{ij}| - 1 \geq d_G(y_i, s) + d_G(s, y_j) \geq 4$.

So the assertion of the claim holds. \hfill \Box

Denote $Y = \{y_1, y_2, \ldots, y_l\}$. By Claims 2.1–2.3, we have

$$N_G(v_1) \subseteq (V(G) - V(T) - \{v_1\}) \cup (N_G(v_1) \cap (\text{Leaf}(T) - Y)),$$

$$\bigcup_{i=1}^{l} N_G(y_i) \subseteq (\text{Leaf}(T) - Y - N_G(v_1)) \cup \{x_1, \ldots, x_l\}.$$

Hence by setting $q = |N_G(v_1) \cap (\text{Leaf}(T) - Y)|$, we obtain

$$\deg_G(v_1) + \sum_{i=1}^{l} \deg_G(y_i) \leq (|G| - |T| - 1 + q) + (|\text{Leaf}(T)| - l - q) + l$$

$$= |G| - |\text{Stem}(T)| - 1.$$
On the other hand, by the assumption of Theorem 1.8, and by Claim 2.3, we have

\[ |G| - \left\lfloor \frac{l(t - 1)}{t - 2} \right\rfloor - 1 \leq \sigma_{l+1}^A(G) \leq \deg_G(v_1) + \sum_{i=1}^{l} \deg_G(y_i). \]

Therefore, we obtain \(|\text{Stem}(T)| \leq \left\lfloor \frac{l(t - 1)}{t - 2} \right\rfloor\). By combining with \(|\text{Leaf}(\text{Stem}(T))| = l\), we conclude that

\[ |\text{Stem}(\text{Stem}(T))| \leq \left\lfloor \frac{l}{t - 2} \right\rfloor. \tag{2.1} \]

Claim 2.4 \(N_G(v_2) \cap \{x_1, x_2, \ldots, x_l\} = \emptyset\).

**Proof** Suppose the assertion of the claim is false. Then, there exists some \(i \in \{1, \ldots, l\}\) such that \(v_2x_i \in E(G)\). Combining with the fact that \(v_2v_1 \in E(G)\) and \(x_iy_i \in E(G)\) (by Claim 2.2), we obtain that \(d_G(v_1, y_i) \leq 3\). This contradicts Claim 2.3. Claim 2.4 is proved.

Now, we complete the Proof of Theorem 1.8 by considering the following two steps.

**Step 1.** \(|\text{Stem}(\text{Stem}(T))| = 1\).

We assume that \(\text{Stem}(\text{Stem}(T)) = \{u\}\). By \(t \geq 3\) and \(|\text{Stem}(\text{Stem}(T))| \leq \left\lfloor \frac{l}{t - 2} \right\rfloor\), we obtain \(l \geq t - 2\). We consider the following two cases.

**Case 1.** \(l \geq t - 1\).

By combining with Claims 2.3 and 2.4, \(G\) induced a \(K_{1,t}\) subgraph with the vertex set \(\{u, x_1, x_2, \ldots, x_{t-1}, v_2\}\), this gives a contradiction.

**Case 2.** \(l = t - 2\). In this case, we will show that \(G\) is isomorphic to a graph \(M\).

For each \(X \subseteq V(G)\), we denoted by \(G[X]\) the subgraph of \(G\) induced by \(X\). For each \(j \in \{1, 2, \ldots, l\}\), we set \(E_j = G[(N_G(x_j) - \{u\}) \cup N_G(y_j)]\) and \(E_{l+1} = G[(V(G) - \bigcup_{i=1}^{t-1} V(E_i)) - \{u\})\). For each \(1 \leq j \leq l\), by the maximality of \(T\), we obtain that \(N_G(x_j) \subseteq V(T)\). Moreover, since \(d_G(y_i, y_j) \geq 4\) for all \(1 \leq i \neq j \leq l\), we obtain \(x_i, x_j \notin E(G)\). Hence, \(N_G(x_j) \subseteq V(T) - \{x_1, x_2, \ldots, x_l\}\). Then, combining with Claim 2.2 and the definition of \(E_j\), we conclude \(V(E_j) \subseteq \text{Leaf}(T) \cup \{x_j\}\) for each \(j \in \{1, 2, \ldots, l\}\). Hence, \(V(G) - V(T) \subseteq V(E_{l+1})\). On the other hand, since \(d_G(v_1, y_j) \geq 4\), it implies that \(v_2 \notin V(E_j)\) for all \(1 \leq j \leq l\). Hence \(v_2 \in V(E_{l+1})\) (see Fig. 2).

By using the same arguments in the proofs of Claim 2.3, we conclude again the following fact.

**Fact 1** For each \(1 \leq j \leq l\), if \(P\) is a shortest path connecting \(v_1\) and \(y_j\) in \(G\), then \(V(P) \cap (V(\text{Stem}(T)) - \{x_j\}) \neq \emptyset\), and for every \(1 \leq i < j \leq l\), if \(P\) is a shortest path connecting \(y_i\) and \(y_j\) in \(G\), then \(V(P) \cap (V(\text{Stem}(T)) - \{x_i, x_j\}) \neq \emptyset\).

We now give the following facts:

**Fact 2** For every \(1 \leq i < j \leq l+1\), then \(V(E_i) \cap V(E_j) = \emptyset\).
Proof By the definition of $E_{l+1}$ we obtain that $V(E_i) \cap V(E_{l+1}) = \emptyset$ for all $1 \leq i \leq l$.

Now, assume that there exists a vertex $x \in V(E_i) \cap V(E_j)$ for some $1 \leq i < j \leq l$. If $x \in N_G(x_i) \cap N_G(x_j)$, then $x \in \text{Leaf}(T)$. Consider the path $P$ in $G$ with its vertex set $\{y_i, x_i, x, x_j, y_j\}$. By combining with $d_G(y_i, y_j) \geq 4$, we obtain that $P$ is a shortest path connecting $y_i$ and $y_j$ in $G$. But $V(P) \cap (V(\text{Stem}(T)) - \{x_i, x_j\}) = \emptyset$, which contradicts Fact 1. Otherwise, without loss of generality, we may assume that $x \in N_G(y_i) \cap N_G(x_j)$ (or $x \in N_G(y_i) \cap N_G(y_j)$). Then, $d_G(y_i, y_j) \leq 3$ (or $d_G(y_i, y_j) \leq 2$ respectively), this contradicts Claim 2.3. Fact 2 is proved. \qed

Fact 3 For each $1 \leq i < j \leq l + 1$, if $x \in V(E_i), y \in V(E_j), (1 \leq i < j \leq l + 1)$ such that $xy \notin E(G)$.

Proof Suppose to the contrary that there exist two vertices $x \in V(E_i), y \in V(E_j), (1 \leq i < j \leq l + 1)$ such that $xy \in E(G)$.

Subcase 1. $1 \leq i < j \leq l$.

If $x \in N_G(y_i)$ and $y \in N_G(y_j)$, then $d_G(y_i, y_j) \leq 3$. This contradicts Claim 2.3.

If $x \in N_G(y_i)$ and $y \in N_G(x_j)$, we consider the path $P$ in $G$ with its vertex set $\{y_i, x, y, x_j, y_j\}$. By combining with $d_G(y_i, y_j) \geq 4$, we obtain $y \neq y_j$, and then $P$ is a shortest path connecting $y_i$ and $y_j$ in $G$. But $V(P) \cap (V(\text{Stem}(T)) - \{x_i, x_j\}) = \emptyset$, which contradicts Fact 1. By the same arguments, we also give a contradiction if $x \in N_G(x_i)$ and $y \in N_G(y_j)$.

If $x \in N_G(x_i)$ and $y \in N_G(x_j)$, remove the edges connecting $x$ and $y$ to $V(\text{Stem}(T))$ in $T$. After that add the edges $x_i x, xy, yx_j$ and $v_1 v_2$ and remove the edge $x_j u$. Then, the resulting tree $T'$ has $l$-ended stem and $|T'| > |T|$, this contradicts the maximality of $T$.

The subcase 1 is proved.

Subcase 2. $1 \leq i < j = l + 1$. 

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Firstly, we show that $N_G(v_1) \cap V(E_a) = \emptyset$ for all $1 \leq a \leq l$. Indeed, suppose to the contrary that there exists a vertex $z \in N_G(v_1) \cap V(E_a)$. If $z \in N_G(x_a)$, then we consider the path $P$ in $G$ with its vertex set $\{v_1, z, x_a, y_a\}$. This is a contradiction with $d_G(v_1, y_a) \geq 4$. Otherwise, $z \in N_G(y_a)$, then $d_G(v_1, y_a) \leq 2$. This also gives a contradiction with $d_G(v_1, y_a) \geq 4$. Therefore, we conclude that $N_G(v_1) \cap V(E_a) = \emptyset$ for all $1 \leq a \leq l$. In particular, we obtain $N_G(v_1) \subseteq V(E_{l+1}) - \{v_1\}$.

Secondly, we prove that $\text{deg}_G(y_a) = |E_a| - 1$ for all $1 \leq a \leq l$ and $\text{deg}_G(v_1) = |E_{l+1}| - 1$. Indeed, for each $1 \leq a \leq l$, by Claim 2.2 and the definition of $E_a$, we obtain $N_G(y_a) \subseteq V(E_a) - \{y_a\}$. By combining the assumptions of Theorem 1.8, Claim 2.3, Fact 2 and $N_G(v_1) \subseteq V(E_{l+1}) - \{v_1\}$, we have

$$|G| - \left\lceil \frac{l(t-1)}{t-2} \right\rceil - 1 \leq \sigma_{l+1}^4(G) \leq \sum_{a=1}^{l} \text{deg}_G(y_a) + \text{deg}_G(v_1)$$

$$\leq \sum_{a=1}^{l+1} (|E_a| - 1) = |G| - l - 2 = |G| - \left\lfloor \frac{l(t-1)}{t-2} \right\rfloor - 1.$$

Therefore, the equalities happen. Hence, $\text{deg}_G(y_a) = |E_a| - 1$ for every $1 \leq a \leq l$ and $\text{deg}_G(v_1) = |E_{l+1}| - 1$, in particular we obtain $N_G(v_1) = V(E_{l+1}) - \{v_1\}$.

Finally, since $x \in V(E_i)$ and $y \in V(E_{l+1})$ such that $xy \in E(G)$, then $y \neq v_1$ (by $N_G(v_1) \cap V(E_i) = \emptyset$) and $y v_1 \in E(G)$ (by $N_G(v_1) = V(E_{l+1}) - \{v_1\}$). If $x \in N_G(x_i)$, then we consider the path $P$ in $G$ with its vertex set $\{v_1, y, x, x_i, y_i\}$. Since $d_G(v_1, y_i) \geq 4$, this implies that $P$ is a shortest path connecting $v_1$ and $y_i$ in $G$. This is a contradiction with Fact 1. Otherwise, $x \in N_G(y_i)$, then $d_G(v_1, y_i) \leq 3$. This also gives a contradiction with $d_G(v_1, y_i) \geq 4$. Therefore, we obtain that $xy \notin E(G)$ for all $x \in V(E_i), y \in V(E_{l+1})$. This completes the proof of the subcase 2.

Therefore, Fact 3 holds. \qed

**Fact 4** For each $j \in \{1, 2, \ldots, l+1\}$, $E_j$ is connected. Moreover, for every $w \in V(E_j)$ such that $uw \notin E(G)$, then $N_G(w) = V(E_j) - \{w\}$ and $\text{deg}_G(w) = |E_j| - 1$.

**Proof** Set $y_{l+1} = v_1$. In the proof of subclaim 2 of Fact 3, we conclude that $\text{deg}_G(y_j) = |E_j| - 1$. This implies that $N_G(y_j) = V(E_j) - \{y_j\}$. Hence, $E_j$ is connected.

Now, by Fact 2, Fact 3, the definition of $E_a$ and $E_a$ is connected for all $a \in \{1, 2, \ldots, l+1\}$, the graph $G[V(G) - \{u\}]$ is disconnected and has $l+1$ components $E_1, \ldots, E_{l+1}$. Then, for every $1 \leq a < b \leq l+1$, if $P$ is a path connecting two vertices $x \in E_a$ and $y \in E_b$, then $P$ must pass through $u$. So for every $x \in E_a$, $y \in E_b$ such that $xy \notin E(G)$ and $yu \notin E(G)$, then $d_G(x, y) \geq 4$. In particular, $d_G(w, y_a) \geq 4$ for all $1 \leq a \leq l+1, a \neq j$. Moreover, by Fact 3 and $wu \notin E(G)$, we have $N_G(w) \subseteq V(E_j) - \{w\}$. Hence, by Facts 2 and 3 and the assumptions of Theorem 1.8, we obtain

$$|G| - \left\lfloor \frac{l(t-1)}{t-2} \right\rfloor - 1 \leq \sigma_{l+1}^4(G) \leq \text{deg}_G(w) + \sum_{a=1, a \neq j}^{l+1} \text{deg}_G(y_a)$$

$$\leq \sum_{a=1}^{l+1} (|E_a| - 1) = |G| - l - 2 = |G| - \left\lfloor \frac{l(t-1)}{t-2} \right\rfloor - 1.$$
Therefore, the equalities happen. So $\deg_G(w) = |E_j| - 1$, and we thus also obtain $N_G(w) = V(E_j) - \{w\}$. These complete the proof of Fact 4. \hfill \Box

For each $1 \leq i \leq l + 1$, denote by $K_i$ the set of vertex $w$ in $V(E_i)$ such that $N_{E_j}(w) = V(E_i) - \{w\}$. Then, for every $1 \leq i \leq l + 1$, $y_i \in K_i$ and in particular $K_i \neq \emptyset$. On the other hand, by Facts 3 and 4, we can see that if $uw \notin E(G)$, then $N_{E_j}(w) = N_G(w) = V(E_i) - \{w\}$. Hence, $w \in K_i$ and $u$ joins to all vertices in $V(E_i) - K_i$ for all $1 \leq i \leq l + 1$. Therefore, using the definitions of $E_j (1 \leq j \leq l + 1)$ and Facts 1–4, we obtain that $G$ is isomorphic to a graph $M$.

Hence, we conclude that if $l = t - 2$, then $G$ is isomorphic to a graph $M$.

**Step 2.** $|\text{Stem}(\text{Stem}(T))| \geq 2$.

By Claim 2.4, there exists a vertex $v_3 \in N_G(v_2) \cap V(\text{Stem}(\text{Stem}(T)))$.

Now, we conclude that $|N_T(v_3) \cap \{x_1, x_2, \ldots, x_l\}| < t - 2$. Indeed, otherwise, without loss of generality, we may assume $x_1, x_2, \ldots, x_{l-2} \in N_T(v_3)$. Since $|\text{Stem}(\text{Stem}(T))| \geq 2$, there exists $s \in V(\text{Stem}(\text{Stem}(T))) \cap N_T(v_3)$. We consider the subgraph with the vertex set $\{v_3, v_2, s, x_1, x_2, \ldots, x_{l-2}\}$ in $G$. By combining with Claim 2.4, the fact that $G$ is $K_{1,t}$-free and since $\{x_1, \ldots, x_{l-2}\}$ is an independent set by Claim 2.3, we have the following two cases.

**Case 1.** $sv_2 \in E(G)$. This implies that the tree $T' = T + sv_2 + v_2v_1 - sv_3$ has $l$-ended stem and $|T'| > |T|$, this contradicts to the maximality of $T$.

**Case 2.** $x_is \in E(G)$ for some $j \in \{1, \ldots, t - 2\}$. Then, we consider the tree $T' = T + x_is + v_2v_1 - sv_3$. Hence, $T'$ has $l$-ended stem and $|T'| > |T|$, this also contradicts to the maximality of $T$.

Therefore, $|N_T(v_3) \cap \{x_1, x_2, \ldots, x_l\}| < t - 2$.

Now, if $|N_T(u) \cap \{x_1, x_2, \ldots, x_l\}| \leq t - 2$ for all $u \in V(\text{Stem}(\text{Stem}(T))) - \{v_3\}$, then combining with $|N_T(v_3) \cap \{x_1, x_2, \ldots, x_l\}| < t - 2$, we have

$$l = |\text{Leaf}(\text{Stem}(T))| < (t - 2)|\text{Stem}(\text{Stem}(T))| + t - 2$$

$$= (t - 2)|\text{Stem}(\text{Stem}(T))|$$

$$\leq (t - 2)\left[\frac{l}{t - 2}\right] \leq l \text{ (by (2.1))}.$$  

This is a contradiction. Hence, there exists a vertex $u \in V(\text{Stem}(\text{Stem}(T)))$ such that $|N_T(u) \cap \{x_1, x_2, \ldots, x_l\}| \geq t - 1$. Without loss of generality, we may assume $x_1, x_2, \ldots, x_{l-1} \in N_T(u)$. Set $s \in V(\text{Stem}(\text{Stem}(T))) \cap N_T(u)$. Now, if $x_is \in E(G)$ for some $j \in \{1, \ldots, t - 1\}$, then we consider the tree $T' = T + x_is + v_2v_1 - su$. Hence, $T'$ has $l$-ended stem and $|T'| > |T|$, this also contradicts to the maximality of $T$. Hence, we obtain $x_is \notin E(G)$ for all $j \in \{1, \ldots, t - 1\}$. Then, $G$ induces a $K_{1,t}$ subgraph with vertex set $\{u, s, x_1, x_2, \ldots, x_{l-1}\}$. This gives a contradiction with the assumption of Theorem 1.8.

Therefore, we complete the Proof of Theorem 1.8.

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