V-stable tamed Euler schemes

Lukasz Szpruch*

Abstract

In this paper we introduce a family of numerical approximations for the stochastic differential equations (SDEs) with no globally Lipschitz coefficients. We show that for a given Lyapunov function $V : \mathbb{R}^d \rightarrow [1, \infty)$ we can construct a suitably tamed Euler scheme that has so called $V$-stability property of the original SDEs. $V$-stability condition plays a crucial role in numerous stability and integrability results for SDEs developed by Khasminski [6]. These results can be recovered by tamed-Euler scheme.

Key words: Lyapunov functions, Numerical Methods, Stability, Stochastic Differential Equation.

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1 Introduction

In this article we investigate, integrability, stability and convergence properties of numerical approximations for stochastic differential equations (SDEs) with non-Lipschitz drift and diffusion coefficients. This is a relatively new area of studies as the majority of research on numerical analysis for SDEs relies on, restrictive, global Lipschitz assumptions. It turns out that if one is interested in strong or numerically weak [2] approximations classical Euler schemes fail to converge. Even in the context of Monte Carlo evaluations of Lipschitz continuous functionals of the solution to the SDEs that relies on the weak approximations, divergence property of the Euler scheme prohibits from using efficient Multilevel Monte Carlo method [1, 5].

As a remedy, detailed analysis of the implicit schemes revealed that it is indeed possible to go beyond global Lipschitz restriction at the cost of solving an implicit equation at every iteration of the scheme [8, 7]. However, very recently it has been demonstrated that even explicit Euler scheme if appropriately tamed converge to the solution of SDEs with non-globally Lipschitz continuous coefficients, see [3, 4, 10, 9]. In this article we continue the research initiated in the above mentioned work by paying particular attention to the way explicit scheme can be tamed in order to resemble a similar property as the underlying system of SDEs.

*School of Mathematics, The University of Edinburgh, Edinburgh EH9 3JZ, UK (*szpruch@ed.ac.uk).
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a normal filtration \((\mathcal{F}_t)_{t \in [0,T]}\) and let \(w(t) = (w_1(t), \ldots, w_m(t))\) be a standard \((\mathcal{F}_t)_{t \in [0,T]}\)-Brownian motion. We consider SDE
\[
dx(t) = \mu(x(t))dt + \sigma(x(t))dw(t), \tag{1.1}
\]
where \(\mu = (\mu_1, \ldots, \mu_d) : \mathbb{R}^d \to \mathbb{R}^d\) and \(\sigma = (\sigma_{i,j})_{i \in \{1, \ldots, d\}, j \in \{1, \ldots, m\}} = (\sigma_s)_{s \in \{1, \ldots, m\}} : \mathbb{R}^d \to \mathbb{R}^{d \times m}\) are locally Lipschitz continuous. By a suitable choice of the Lyapunov function \(V : \mathbb{R}^d \to [1, \infty)\) we can, for example, establish existence, stability and moment bounds for the solution to the SDE \((1.1)\). In order to guarantee that \((1.1)\) has a well defined strong solution we assume that there exists a twice continuously differentiable function \(V : \mathbb{R}^d \to [1, \infty)\) such that
\[
L_{\mu,\sigma}V(x) := V'(x)\mu(x) + \frac{1}{2} \sum_{s=1}^m V''(x)(\sigma_s(x), \sigma_s(x))
= \langle \mu(x), (\nabla V)(x) \rangle + \frac{1}{2} \text{trace}(\sigma(x)\sigma(x)^T(\text{Hess}V)(x))
= \sum_{i=1}^d \left(\frac{\partial V}{\partial x_i}\right)(x)\mu_i(x) + \frac{1}{2} \sum_{i,j=1}^d \sum_{s=1}^m \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)(x)\sigma_{i,s}(x)\sigma_{j,s}(x) \leq \rho V(x) \quad \text{for } \rho \in \mathbb{R}. \tag{1.2}
\]
If the above condition holds, then the standard result from stochastic analysis ensures that
\[
\mathbb{E}[V(x(t))] \leq e^{\rho t}\mathbb{E}[V(x(0))] \quad \text{for } t \geq 0. \tag{1.3}
\]
We refer to the inequality \((1.3)\) as the V-stability property. Apart from the existence result, there are several immediate consequences of the V-stability property:

- **Moments bound:** If we further assume that there exists \(l \geq 1\), such that \(\|x\|^l \leq V(x)\) then inequality \((1.3)\) implies that \(\sup_{t \in [0,T]} \mathbb{E}[\|x(t)\|^l] < \infty\) (provided \(\mathbb{E}[V(x(0))] \leq \infty\)).
- **Stability:** If we further assume that \(\rho < 0\) then inequality \((1.3)\) implies that \(\mathbb{E}[V(x(t))] \leq e^{-\alpha t}\mathbb{E}[V(x(0))]\) for \(0 < \alpha < \rho\) (to see it apply Itô’s formula to the function \(e^{\alpha t}V(x)\)).

Then main goal of this paper is construct explicit numerical scheme that is V-stable in the sense of \((1.3)\). It has been proven in \cite{3} (Corollary 2.17) that in general condition \((1.3)\) can not be recovered if we use classical explicit Euler scheme. Nevertheless, in \cite{3, 4, 10}, authors showed, using ”rare events analysis”, that suitably named Euler schemes enjoy similar integrability properties as underlying SDEs. Unfortunately, methodology developed in \cite{3} in general does not allow to recover condition \((1.3)\). See also discussion in \cite{10} where the analysis similar to \cite{4} was carried out for Lyapunov function \(V(x) = 1 + \|x\|^p, \ p \geq 2\) and condition \((1.3)\) has not been recovered. It turns out that for a special case of Lyapunov function \(V = 1 + \|x\|^2\) (or \(V = 1 + \|x\|^p, \ p \geq 2\)) using drift implicit Euler scheme allows to obtain discrete time counterpart of \((1.3)\). \cite{8, 7}. At this point it is also worth to mention that using slightly different taming and allowing for non-Lipschitz condition only in the drift term it is possible to recover condition \((1.3)\) for the Lyapunov function \(V(x) = 1 + \|x\|^p, \ p \geq 2\), see \cite{9}.

A general tamed Euler approximation with time step \(0 \leq h \leq 1\) for the SDE \((1.1)\) has the following form
\[
X_{k+1} = X_k + \mu^h(X_k)h + \sigma^h(X_k)\Delta w_{k+1}, \tag{1.4}
\]
where \(\Delta w_{k+1} = w((k+1)h) - w(kh)\) and \(\mu^h \to \mu\) and \(\sigma^h \to \sigma\), as \(h \to 0\).\footnote{Precise definition of theses limits may vary applications.}
The main contribution of this paper is to show that if taming is appropriately customized to the system of SDEs under consideration then the V-stability property of SDEs carries on to the numerical scheme. From inequality (1.3) we can immediately deduce moments bounds and stability properties of the scheme. Then main difficulty in obtaining V-stability property of the scheme is the lack of a suitable extension of Itô formula in discrete time. However, suitable taming allows to control a remainder terms of Taylor expansion and therefore mimic results that can be obtained for continuous time SDEs.

In this paper we do not use a ”rare events approach” but further explore powerfulness of taming approach. We will show that taming should be adjusted to the problem under consideration. We keep assumption and notation similar as in [4] in order to allow a reader for easy comparison of the results.

The paper is structured as follows: Section 2 contains main results. In this section we answer the questions under what conditions on the coefficients of tamed Euler it has V-stable property. In section 3 we propose a concrete example of taming that satisfies assumptions imposed in Section 2. In section we conclude by establishing strong convergence of the tamed Euler scheme.

2 V-schemes

In this section we introduce V-stable schemes along with required assumptions and necessary notation. Following [4], we considered the following space of Lyapunov functions: Let \( n, d \in \mathbb{N} \), \( p \in (0, \infty) \) and

\[
C_p^n(\mathbb{R}^n, [1, \infty)) := \left\{ V \in C^{n-1}(\mathbb{R}^d, [1, \infty)) : \begin{array}{l}
V^{(n-1)} \text{ is locally Lipschitz continuous and} \\
\exists c_V > 0 \ \forall x \in \mathbb{R}^d \ \forall i \in \{1, 2, \ldots, n\} \text{ such that} \\
\|V^{(i)}(x)\|_{L^p(\mathbb{R}^d, \mathbb{R})} \leq c_V |V(x)|^{1-1/p}
\end{array} \right\}.
\]

Space \( C_p^n(\mathbb{R}^n, [1, \infty)) \) is rich enough to consider Lyapunov functions used to analyse many important SDEs (see [4] for more details). Once we fix a Lyapunov function \( V \in C_p^n(\mathbb{R}^n, [1, \infty)) \) it is convenient to express the growth condition of the coefficients of the SDE (2.1) in the following form

Assumption 2.1. Let \( \gamma_0, \gamma_1 > 0 \) and \( V \in C_p^n(\mathbb{R}^d, [1, \infty)) \). We assume that there exists a constant \( c > 0 \) such that

\[
|\mu(x)| \leq cV(x)^{\gamma_0/p} \quad \text{and} \quad |\sigma(x)| \leq cV(x)^{\gamma_1/p}, \quad x \in \mathbb{R}^d.
\]

Assumption 2.1 essentially imposes polynomial growth condition on the coefficients of the SDEs (2.1). Indeed, we may observe that if there exists \( L > 0 \) such that \( \|\mu(x)\| \leq L(1 + \|x\|^\gamma) \) then we can find \( c > 0 \) such that \( \|\mu(x)\| \leq cV(x)^{\gamma_0/p} \). The same applies to the diffusion coefficient. Expressing the bound in terms of Lyapunov function makes analysis more transparent.

We propose the following general taming of the coefficients in (1.4)

\[
\mu^h(x) := \frac{\mu(x)}{1 + G_\mu(x, h)}, \quad \sigma^h(x) := \frac{\sigma(x)}{1 + G_\sigma(x, h)}, \quad x \in \mathbb{R}^d,
\]
such that $0 \leq G_\mu(x, h), G_\sigma(x, h) \to 0$ as $h \to 0$. The $G_\mu(x, h)$ and $G_\sigma(x, h)$ can be customized for the problem under consideration.

**Motivational example** To clarify the idea of this paper without going into technical details let us consider a scalar SDE

$$dx(t) = \mu(x(t))dt + \sigma(x(t))dw(t)$$  \hfill (2.2)

and assume that coefficients satisfy the following condition

$$2x\mu(x) + |\sigma(x)|^2 \leq \rho(1 + |x|^2) \quad \forall x \in \mathbb{R}. \hfill (2.3)$$

Condition (2.3) corresponds to the special case of the Lyapunov function $V \in C^2_b(\mathbb{R}, (1, \infty))$ of the form $V(x) = 1 + |x|^2$ in (1.3) and immediately gives for all $t > 0$

$$\mathbb{E}[V(x(t))] \leq e^{\rho t}\mathbb{E}[V(x(0))]$$

Moreover, we assume that the coefficients in (2.2) satisfy Assumption (2.1). By squaring both sides of tamed Euler scheme

$$X_{k+1} = X_k + \mu^h(X_k)h + \sigma^h(X_k)\Delta w_{k+1},$$

we arrive at

$$\mathbb{E}_k[|X_{k+1}|^2] = |X_k|^2 + (2X_k\mu^h(X_k) + |\sigma^h(X_k)|^2)h + |\mu^h(X_k)|^2h^2,$$

were $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_k]$, with $\mathcal{F}_k$ Brownian filtration generated by $(X_s)_{0 \leq s \leq k}$. Now we are seeking a taming such that for $\tilde{\rho} > 0$

$$|\mu^h(x)|^2h \leq \tilde{\rho} V(x) \quad \forall x \in \mathbb{R}. \hfill (2.4)$$

and

$$2x\mu^h(x) + |\sigma^h(x)|^2 \leq \rho V(x) \quad \forall x \in \mathbb{R}. \hfill (2.5)$$

so that for any $k, N \in \mathbb{N}$, $k < N$,

$$\mathbb{E}_k[V(X_{k+1})] = V(X_k) + (\rho + \tilde{\rho})V(X_k)h \implies \mathbb{E}[V(X_N)] \leq e^{(\rho + \tilde{\rho})Nh}\mathbb{E}[V(X_0)].$$

Using taming (2.1), condition (2.5) holds provided

$$1 + G_\mu(x, h) \leq (1 + G_\sigma(x, h))^2.$$  

Therefore, we take $G_\sigma(x, h) = G_\mu(x, h)$. We then define $G_\mu(V(X), h) := \tau_0|V(x)|^{\tau_0/2}h^{\beta_0}$, with $\tau_0 = c/\sqrt{\rho}$, $k_0 = (\gamma_0 - 1)_+$ and $\beta_0 = 1/2$ so that

$$|\mu^h(x)|^2h_{1/2} = \frac{|\mu(x)|h_{1/2}}{1 + \tau_0|V(x)|^{\tau_0/2}} \leq \frac{c|V(x)|^{\tau_0/2}h_{1/2}}{1 + \tau_0|V(x)|^{\tau_0/2}h_{1/2}} \leq \sqrt{\rho} V(x)^{1/2},$$

as required.
2.1 Main Results

The difficulty with exploring powerfulness Lyapunov technique in the context of numerical schemes for SDEs is lack of discrete time Itô formula. However suitable tamed schemes allows to recover many of the classical results by appropriately controlling a remainder terms of Taylor expansions. This is a topic of Theorem \[\text{Theorem 2.2}\]

In the first part of this section we focus on the subspace of \(C^\infty_p(\mathbb{R}^d, [1, \infty))\) with \(p \in \mathbb{N}\) that we will denote by \(\hat{C}_p(\mathbb{R}^d, [1, \infty))\) (This class contains almost all examples of SDEs presented in \[\text{[1]}\]). As an example one may consider a very popular Lyapunov function \[\text{Example one may consider a very popular Lyapunov function}\]

For SDEs is lack of discrete time Itô formula. However suitable tamed schemes allows to recover

The difficulty with exploring powerfulness Lyapunov technique in the context of numerical schemes

**Theorem 2.2.** Let \(T \in (0, \infty), d, m \in \mathbb{N}, \) let \(\mu^h : \mathbb{R}^d \to \mathbb{R}^d, \sigma^h : \mathbb{R}^d \to \mathbb{R}^{d \times m}, \rho \in \mathbb{R}\) and \(V \in \hat{C}_p(\mathbb{R}^d, [1, \infty)), p \geq 3,\) be a function such that

\[L_{\mu^h, \sigma^h}V(x) \leq \rho V(x).\] (2.6)

Moreover, we consider tamed Euler scheme \[\text{Tamed Euler scheme}\] and assume that there exists constants \(c_\mu, c_\sigma > 0\) such that

\[\|\mu^h(x)\|_{L^2(\mathbb{R}^d)} h^{1/2} \leq c_\mu V(x)^{1/p} \quad \text{and} \quad \|\sigma^h(x)\|_{L^2(\mathbb{R}^d, \mathbb{R}^d)} h^{1/4} \leq c_\sigma V(x)^{1/p}.\] (2.7)

Then there exists a constant \(\hat{\rho} := \hat{\rho}(c_\mu, c_\sigma)\) such that

\[\mathbb{E}[V(X(t_k))] \leq e^{(\rho + \hat{\rho})kh}\mathbb{E}[V(X_0)] \quad \forall k \geq 0.\]

**Proof.** Since \(V \in \hat{C}_p(\mathbb{R}^d, [1, \infty))\) we can apply Taylor expansion to get

\[E_k[V(X_{k+1})] = V(X_k) + \mathbb{E}_k[\sum_{|\alpha| \leq p-1} \frac{\partial^\alpha V(X_k)}{\alpha!} ((X_{k+1} - X_k)^\alpha)]\] (2.8)

Take \(1 \leq s \leq p - 1.\) Using multi-index notation we have

\[\mathbb{E}_k[\sum_{|\alpha| = s} \frac{\partial^\alpha V(X_k)}{\alpha!} (X_{k+1} - X_k)^\alpha] = \mathbb{E}_k[\sum_{|\alpha| = s} \frac{\partial^\alpha V(X_k)}{\alpha!} \sum_{\nu \leq \alpha} \binom{\alpha}{\nu} (\mu^h(X_k)h)^{\alpha-\nu} (\sigma^h(X_k)\Delta w_{k+1})^\nu]\] (2.9)

which by multinomial theorem is equivalent to

\[\mathbb{E}_k[\sum_{|\alpha| = s} \frac{\partial^\alpha V(X_k)}{\alpha!} (X_{k+1} - X_k)^\alpha] = \frac{1}{s!} \mathbb{E}_k[\sum_{\beta_1 + \ldots + \beta_d = s} \binom{s}{\beta_1, \ldots, \beta_d} (\mu^h(X_k)^{\beta_1} \ldots (X_{d,k+1} - X_{d,k})^{\beta_d}) \frac{\partial^s}{\partial x_{\beta_1}^{\beta_1} \ldots \partial x_{\beta_d}^{\beta_d}} V(X_k)].\]
where, for \(i = 1, \ldots, d\),

\[
(X_{i,k+1} - X_{i,k})^{\beta_i} = (\mu_i(X_k) + \sum_{j=1}^{m} \sigma_{ij}(X_{k+1}) \Delta w_{j,k+1})^{\beta_i}
\]

\[
= \sum_{\gamma_0 + \cdots + \gamma_m = \beta_i} \left( \sum_{i=1}^{\gamma_0} \gamma_0^i (\mu_i(X_k))^{\gamma_0^i} \cdots \sigma_{im}(X_{k+1})^{\gamma_m} (\Delta w_{1,k+1})^{\gamma_1} \cdots (\Delta w_{m,k+1})^{\gamma_m} \right)\]

Therefore, it is easy to see that the first two terms of (2.8) are of the form

\[
\mathbb{E}_k \left[ \sum_{|\alpha|=1} \frac{\partial^\alpha V(X_k)}{\alpha!} (\mu^h(X_k)h + \sigma^h(X_k)\Delta w_{k+1})^\alpha \right] = V^{(1)}(x) \mu(x) h = \langle \mu(x), (\nabla V)(x) \rangle \bar{h},
\]

and

\[
\mathbb{E}_k \left[ \sum_{|\alpha|=2} \frac{\partial^\alpha V(X_k)}{\alpha!} (\mu^h(X_k)h + \sigma^h(X_k)\Delta w_{k+1})^\alpha \right]
\]

\[
= \frac{1}{2} \sum_{s=1}^{m} V''(X_k)(\sigma_s(X_k), \sigma_s(X_k))h + \frac{1}{2} V''(X_k)(\mu(X_k), \mu(X_k))h^2
\]

\[
= \frac{1}{2} \sum_{i,j=1}^{d} \sum_{s=1}^{m} \left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right)(X_k) \sigma_{i,s}(X_k) \sigma_{j,s}(X_k) h + \frac{1}{2} \sum_{i,j=1}^{d} \left( \frac{\partial^2 V}{\partial x_i \partial x_j} \right)(X_k) \mu_i(X_k) \mu_j(X_k) h^2.
\]

We can now analyse the third term of the expansion (2.8) with \(|\alpha| = 3\). First we notice due to marginality property of Brownian motion cases \(|\nu| = 1\) and \(|\nu| = 3\) are zero. Hence

\[
\mathbb{E}_k \left[ \sum_{|\alpha|=3} \frac{\partial^\alpha V(X_k)}{\alpha!} \sum_{\nu\leq\alpha} \left( \frac{\alpha}{\nu} \right) (\mu^h(X_k)h)^{\alpha-\nu} (\sigma^h(X_k)\Delta w_{k+1})^\nu \right]
\]

\[
= \mathbb{E}_k \left[ \sum_{|\alpha|=3} \frac{\partial^\alpha V(X_k)}{\alpha!} \sum_{\nu\leq\alpha} \left( \frac{\alpha}{\nu} \right) (\mu^h(X_k)h)^{\alpha-\nu} (\sigma^h(X_k)\Delta w_{k+1})^\nu \right]
\]

\[
+ \mathbb{E}_k \left[ \sum_{|\alpha|=3} \frac{\partial^\alpha V(X_k)}{\alpha!} \sum_{\nu\leq\alpha} \left( \frac{\alpha}{\nu} \right) (\mu^h(X_k)h)^{\alpha-\nu} (\sigma^h(X_k)\Delta w_{k+1})^\nu \right]
\]

\[
\leq \phi^3 \left( V^{(3)}(X_k) \right)_{\mathcal{L}^3([\mathbb{R}^d, \mathbb{R}])} \left( \left\| \mu^h(X_k) \right\|_{\mathcal{L}^3([\mathbb{R}^d])}^3 h^3 + \left\| \sigma^h(X_k) \right\|_{\mathcal{L}^1([\mathbb{R}^d])}^2 \right),
\]

where \(\phi^3\) is the constant such that

\[
0 \leq \phi^s \leq \frac{(d(m+1))^{s-1}}{s!} \quad s = 3, \ldots, p - 1.
\]

The upper bound of the above constant will become clear from the analysis of higher order terms.

Take \(3 \leq s \leq p - 1\). We will show that

\[
\mathbb{E}_k \left[ \sum_{|\alpha|=s} \frac{\partial^\alpha V(X_k)}{\alpha!} \sum_{\nu\leq\alpha} \left( \frac{\alpha}{\nu} \right) (\mu^h(X_k)h)^{\alpha-\nu} (\sigma^h(X_k)\Delta w_{k+1})^\nu \right]
\]

\[
\leq \phi^s \left( V^{(s)}(X_k) \right)_{\mathcal{L}^s([\mathbb{R}^d, \mathbb{R}])} \left( \left\| \mu^h(X_k) \right\|_{\mathcal{L}^s([\mathbb{R}^d])}^s h^s + \left\| \sigma^h(X_k) \right\|_{\mathcal{L}^s([\mathbb{R}^d, \mathbb{R}])}^s h^{s/2} \right), \tag{2.10}
\]
Indeed, by the multinomial theorem and Hölder’s inequality

\[
\mathbb{E}_k \left[ \sum_{|\alpha|=s} \frac{\partial^\alpha V(X_k)}{\alpha!} (X_{k+1} - X_k)^\alpha \right] \leq \left\| V^{(s)}(X_k) \right\|_{L^1(\mathbb{R}^d, \mathbb{R})} \mathbb{E}_k \left[ \sum_{|\alpha|=s} \frac{1}{\alpha!} |(X_{k+1} - X_k)^\alpha| \right]
\]

\[
= \left\| V^{(s)}(X_k) \right\|_{L^1(\mathbb{R}^d, \mathbb{R})} \frac{1}{s!} \mathbb{E}_k \left( \sum_{i=1}^d |X_{i,k+1} - X_{i,k}| \right)^s
\]

\[
\leq \left\| V^{(s)}(X_k) \right\|_{L^1(\mathbb{R}^d, \mathbb{R})} \frac{d^{s-1}}{s!} \sum_{i=1}^d \mathbb{E}_k |X_{i,k+1} - X_{i,k}|^s.
\]

Recall that

\[
|X_{i,k+1} - X_{i,k}|^s = |\mu_i(X_k) + \sum_{j=1}^m \sigma_{ij}(X_{k+1}) \Delta w_{j,k+1}|^s
\]

\[
\leq (m + 1)^{s-1} \left( |\mu_i(X_k)|^s h^s + \sum_{j=1}^m |\sigma_{ij}(X_{k+1}) \Delta w_{j,k+1}|^s \right)
\]

Hence

\[
\mathbb{E}_k \left[ \sum_{|\alpha|=s} \frac{\partial^\alpha V(X_k)}{\alpha!} (X_{k+1} - X_k)^\alpha \right]
\]

\[
\leq \left\| V^{(s)}(X_k) \right\|_{L^1(\mathbb{R}^d, \mathbb{R})} \frac{(d(m + 1))^{s-1}}{s!} \left( \left\| \mu^h(X_k) \right\|^s_{L^1(\mathbb{R}^d)} h^s + \left\| \sigma^h(X_k) \right\|^s_{L^1(\mathbb{R}^d, \mathbb{R})} h^{s/2} \right),
\]

which proves (2.10) with the required bound on constant \(c^s\).

Returning to (2.8) and using the above bounds, we obtain

\[
\mathbb{E}_k [V(X_{k+1})] = V(X_k) + \langle \mu(x), (\nabla V)(x) \rangle h + \frac{1}{2} \sum_{k=1}^m V''(X_k)(\sigma_k(X_k), \sigma_k(X_k)) h + R_{\mu^h, \sigma^h}(X_k)
\]

where

\[
R_{\mu^h, \sigma^h}(X_k) \leq \frac{1}{2} \left\| V^{(2)}(X_k) \right\|_{L^2(\mathbb{R}^d, \mathbb{R})} \left\| \mu(X_k) \right\|^2 h^2 + \phi^3 \left\| V^{(3)}(X_k) \right\|_{L^3(\mathbb{R}^d, \mathbb{R})} \left( \left\| \mu^h(X_k) \right\|^3 h^3 + \left\| \mu^h(X_k) \right\|^2 \left\| \sigma^h(X_k) \right\|^2 h^2 \right) + \sum_{s=1}^{p-1} \phi^s \left\| V^{(s)}(X_k) \right\|_{L^s(\mathbb{R}^d, \mathbb{R})} \left( \left\| \mu^h(X_k) \right\|^s h^s + \left\| \sigma^h(X_k) \right\|^s h^{s/2} \right).
\]

Now we show that if

\[
|\mu^h(x)| h^{1/2} \leq c_{\mu} V(x)^{1/p}, \quad \text{and} \quad |\sigma^h(x)| h^{1/4} \leq c_{\sigma} V(x)^{1/p}.
\]

Then there exists \(\tilde{\rho}(c_{\mu}, c_{\sigma}) > 0\) such that

\[
R_{\mu^h, \sigma^h}(X_k) \leq \tilde{\rho} V(X_k) h.
\]
First recall that there exists \( c_V > 0 \) for all \( x \in \mathbb{R}^d \) such that for \( i \in \{1, 2, \ldots, n\} \)
\[
\left\| V^{(i)}(x) \right\|_{L^i(\mathbb{R}^d, \mathbb{R})} \leq c_V |V(x)|^{1-i/p}
\]

Hence
\[
R_{\mu, \sigma} V(X_k) \leq \frac{1}{2} c_V |V(X_k)|^{1-2/p}(\|\mu(X_k)\| h^{1/2})^2 h
+ \phi^3 c_V |V(X_k)|^{1-3/p} \left( \left\| \mu h(X_k) \right\| h^{1/2} \right)^3 h^{3/2} + \left( \left\| \mu h(X_k) \right\| h^{1/2} \right)^2 \left( \left\| \sigma h(X_k) \right\| \cdot h^{1/4} \right)^2 h
+ \sum_{s=4}^{p-1} \phi^s c_V |V(X_k)|^{1-s/p} \left( \left\| \mu h(X_k) \right\| h^{1/2} \right)^s h^{s/2} + \left( \left\| \sigma h(X_k) \right\| \cdot h^{1/4} \right)^s h^{s/4}
\]

Which gives
\[
R_{\mu, \sigma} V(X_k) \leq \frac{1}{2} c_V |V(X_k)|^{1-2/p}(c_\mu^2 V(X_k)^2/p h
+ \phi^3 c_V |V(X_k)|^{1-3/p} \left( c_\mu^3 V(X_k)^3/p h^{3/2} + c_\mu V(x)^{1/p} c - \sigma^2 V(x)^{2/p} h \right)
+ \sum_{s=4}^{p-1} \phi^s c_V |V(X_k)|^{1-s/p} \left( \left( c_\mu^s V(X_k)^s/p h^{s/2} \right) + c_\sigma^s V(X_k)^s/p h^{s/4} \right)
= V(X_k) c_V \left( \frac{1}{2} c_\mu^2 + \phi^3 \left( c_\mu^3 h^{3/2} + c_\mu c_\sigma^2 \right) + \sum_{s=4}^{p-1} \phi^s \left( \left( c_\mu^s h^{s/2-1} + c_\sigma^s h^{s/4-1} \right) \right) ) h
\]

To complete the proof it is enough to choose \( \tilde{\rho} \) such that
\[
c_V \left( \frac{1}{2} c_\mu^2 + \phi^3 \left( c_\mu^3 h^{3/2} + c_\mu c_\sigma^2 \right) + \sum_{s=4}^{p-1} \phi^s \left( \left( c_\mu^s h^{s/2-1} + c_\sigma^s h^{s/4-1} \right) \right) \right) = \tilde{\rho}
\]
\[\text{(2.11)}\]

**Remark 2.3.** For the practical implementation we can take \( c_\mu = c_\sigma \geq 1 \) and since
\[
\sup_{3 \leq s \leq p-1} \phi^s \leq \frac{d(m+1)^{p-2}}{3!}
\]
we immediately have from (2.11) that
\[
c_V c_\mu^{p-1}(p-1) \frac{d(m+1)^{p-2}}{3!} = \tilde{\rho}.
\]

Therefore by a suitable choice of parameter \( c_\mu \) we can make \( \tilde{\rho} \) sufficiently small. This have important consequences in the stability analysis for the tamed numerical schemes where condition \[\text{(1.3)}\] holds with \( \rho < 0 \).

In the similar way we extend applicability of V-tamed schemes to Lyapunov function \( V \in C_p^0(\mathbb{R}^d, [1, \infty)) \), with \( p \geq n \geq 3 \). It turns out that the smoothness of the Lyapunov functions affects the rate of taming of the diffusion coefficient.

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Lemma 2.4. Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $\mu^h : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma^h : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $\rho \in \mathbb{R}$ and $V \in C_p^0(\mathbb{R}^d, [1, \infty))$, with $n, p \geq 3$, be a function such that $L_{\mu^h, \sigma^h} V(x) \leq \rho V(x)$. Moreover, we assume that there exists constants $c_\mu, c_\sigma > 0$ such that

$$
\left\| \mu^h(x) \right\|_{L^2(\mathbb{R}^d)} h^{\beta_0} \leq c_\mu V(x)^{1/p} \quad \text{and} \quad \left\| \sigma^h(x) \right\|_{L^2(\mathbb{R}^d)} h^{\beta_1} \leq c_\sigma V(x)^{1/p}, \quad (2.12)
$$

where $\beta_0 \leq 1/2$ and $\beta_1 \leq 1/2 - 1/\min \{n, 4\}$. Then there exists a constant $\hat{\rho} := \hat{\rho}(c_\mu, c_\sigma)$ such that

$$
\mathbb{E} \left[ V(X(t_k)) \right] \leq c^{(\rho + \hat{\rho})kh} \mathbb{E}[V(X_0)] \quad \forall k \geq 0.
$$

Proof. The proof is very similar to the proof of Theorem 2.4. We write

$$
V(X_{k+1}) = V(X_k) + \sum_{|\alpha| \leq n - 1} \frac{\partial^\alpha V(X_k)}{\alpha!} (X_{k+1} - X_k)^\alpha + n \sum_{|\alpha| = n} \frac{(X_{k+1} - X_k)^\alpha}{\alpha!} \int_0^1 (1 - t)^n - 1 \partial^\alpha V(X_k + t(X_{k+1} - X_k)) dt. \quad (2.13)
$$

It is therefore enough to look into a remainder term for $n \geq 3$

$$
\begin{align*}
&n \sum_{|\alpha| = n} \frac{(X_{k+1} - X_k)^\alpha}{\alpha!} \int_0^1 (1 - t)^n \partial^\alpha V(X_k + t(X_{k+1} - X_k)) dt \\
&\leq n \sum_{|\alpha| = n} \frac{|(X_{k+1} - X_k)^\alpha|}{\alpha!} \int_0^1 (1 - t)^{n-1} \left\| V^{(n)}(X_k + t(X_{k+1} - X_k)) \right\|_{L^{(n)}(\mathbb{R}^d, \mathbb{R})} dt \\
&\leq c_V n \sum_{|\alpha| = n} \frac{|(X_{k+1} - X_k)^\alpha|}{\alpha!} \int_0^1 (1 - t)^{n-1} |V(X_k + t(X_{k+1} - X_k)|^{1-n/p} dt.
\end{align*}
$$

By Lemma 2.12 in [4] we have

$$
V(x + y) \leq c_V^p 2^{p-1} \left( |V(x)| + \|y\|^p \right),
$$

that leads to

$$
|V(x + y)|^{(p-n)/p} \leq c_V^{p-n} 2^{(p-1)(p-n)/p} \left( |V(x)| + \|y\|^p \right)^{(p-n)/p}
\leq c_V^{p-n} 2^{(p-n)} \left( |V(x)|^{(p-n)/p} + \|y\|^{(p-n)} \right).
$$
Consequently

\[
\begin{align*}
\sum_{|\alpha|=n} & \frac{(X_{k+1} - X_k)^\alpha}{\alpha!} \int_0^1 (1-t)^{n-1} \partial^\alpha V(X_k + t(X_{k+1} - X_k)) dt \\
\leq & c_V n \sum_{|\alpha|=n} \frac{|(X_{k+1} - X_k)^\alpha|}{\alpha!} c_{V,n}^{p-n} (p-n) \left( |V(X_k)|^{(p-n)/p} + \|X_{k+1} - X_k\|^{(p-n)} \right) \\
\leq & \frac{n c_{V,n}^{p-n+1} 2^{p-n}}{(n-1)!} \left( \sum_{i=1}^d |X_{i,k+1} - X_{i,k}| \right)^n \left( |V(X_k)|^{(p-n)/p} + \|X_{k+1} - X_k\|^{(p-n)} \right) \\
\leq & c_V \psi \left( \left\| \mu^h(X_k) \right\|_{L^n(\mathbb{R}^d)} h^n + \left\| \sigma^h(X_k) \right\|_{L^n(\mathbb{R}^d)} h^{n/2} \left| V(X_k) \right|^{(p-n)/p} h_p + \left\| \sigma^h(X_k) \right\|_{L^p(\mathbb{R}^d)} h^{p/2} \right),
\end{align*}
\]

were by the similar argument as in Lemma 2.4.

\[
- \psi = \frac{(d(m+1))^{p-1} c_V^{p-n} 2^{p-n}}{(n-1)!}.
\]

Now we show that if

\[
\left| \mu^h(x) \right| h^{\beta_0} \leq c_\mu V(x)^{1/p} \quad \text{and} \quad \left| \sigma_k^h(x) \right| h^{\beta_1} \leq c_\sigma V(x)^{1/p} \quad \beta_0, \beta_1 > 0.
\]

Then, there exists \( \tilde{\rho}(c_\mu, c_\sigma) > 0 \) such that

\[
R_{\mu_\ast, \sigma_\ast} V(X_k) \leq \tilde{\rho} V(X_k) h
\]

where

\[
\begin{align*}
R_{\mu_\ast, \sigma_\ast} V(X_k) & \leq \frac{1}{2} c_V |V(X_k)|^{1-2/p} \left( \left\| \mu(X_k) \right\| h^{\beta_0} \right)^2 h^{1-2\beta_0} h \\
& + \phi^3 c_V |V(X_k)|^{1-3/p} \left( \left\| \mu^h(X_k) \right\| h^{\beta_0} \right)^3 h^{2-3\beta_0} + \left( \left\| \sigma^h(X_k) \right\| L_{(\mathbb{R}^m, \mathbb{R}^d)} h^{\beta_1} \right) \left( \left\| \sigma^h(X_k) \right\| L_{(\mathbb{R}^m, \mathbb{R}^d)} h^{\beta_1} \right) h^{1-2\beta_0-2\beta_1} h \\
& + c_V \tilde{\psi} \left( \left\| \mu^h(X_k) \right\|_{L^n(\mathbb{R}^d)} h^{\beta_0} \right)^n h^{1-\beta_0} + \left( \left\| \sigma^h(X_k) \right\|_{L^n(\mathbb{R}^d)} h^{\beta_1} \right)^n h^{1-\beta_0} \left| V(X_k) \right|^{(p-n)/p} h_p \\
& + c_V \tilde{\psi} \left( \left\| \mu^h(X_k) \right\|_{L^p(\mathbb{R}^d)} h^{\beta_0} \right)^p h^{1-\beta_0} + \left( \left\| \sigma^h(X_k) \right\|_{L^p(\mathbb{R}^d)} h^{\beta_1} \right)^p h^{1-\beta_0} \left| V(X_k) \right|^{(p-n)/p} h_p \leq \tilde{\rho} V(X_k) h.
\end{align*}
\]

Due to restriction imposed on \( \beta_0 \) and \( \beta_1 \) and the fact that \( h \leq 1 \),

\[
\tilde{\rho} = c_V \left( \frac{1}{2} c_\mu^2 + \phi^3 \left( c_\mu c_\sigma^2 h^{2-\beta_0} + c_\mu c_\sigma^2 \right) + \tilde{\psi}(c_\mu^2 + c_\sigma^2 + c_\mu + c_\sigma) \right).
\]

\[\square\]
2.2 Powers of Lyapunov function

For many of the SDEs if $V : \mathbb{R}^d \to [1, \infty)$ is a Lyapunov function in the sense of (1.2) then the function $\tilde{V}(\cdot) := V(\cdot)^q$, $q > 0$, also is a Lyapunov function. This has been shown in Lemma 2.14 and Corollary 2.15 in [4]. What is more if $V : \mathbb{R}^d \to [1, \infty)$ satisfy

$$LV_{\mu,\sigma}V(x) + \frac{(q-1)\|V'(x)\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R})}}{2V(x)} \leq \rho V(x), \forall x \in \mathbb{R}^d,$$

then solution to the SDE (1.1) has the following property

$$\mathbb{E}[\tilde{V}(x(t))] \leq e^{\rho t}\mathbb{E}[\tilde{V}(x(0))] \quad \text{for } t \geq 0.$$  (2.15)

As expected the same can be recovered for the appropriately tamed Euler scheme (1.4). Here we present a direct extension of Lemma 2.4 to the Lyapunov function $\tilde{V}(x) := V(x)^q$, $q > 0$ $x \in \mathbb{R}^d$.

**Lemma 2.5.** Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, $q \geq 1$, let $\mu^h : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma^h : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, $\rho \in \mathbb{R}$ and $V \in C_b^\infty(\mathbb{R}^d, [1, \infty))$, with $p \geq n \geq 3$, be a function such that

$$LV_{\mu^h,\sigma^h}V(x) + \frac{(q-1)\|V'(x)\sigma^h(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R})}}{2V(x)} \leq \rho V(x), \forall x \in \mathbb{R}^d.$$  (2.16)

Moreover, we assume that there exists constants $c_\mu, c_\sigma > 0$ such that

$$\|\mu^h(x)\|_{L^2(\mathbb{R}^d)} h^{\beta_0} \leq c_\mu V(x)^{1/p} \quad \text{and} \quad \|\sigma^h(x)\|_{L^2(\mathbb{R}^m,\mathbb{R}^d)} h^{\beta_1} \leq c_\sigma V(x)^{1/p},$$

where $\beta_0 \leq 1/2$ and $\beta_1 \leq 1/2 - 1/\min\{n, 4\}$. Then there exists a constant $\tilde{\rho} := \tilde{\rho}(c_\mu, c_\sigma)$ such that

$$
\mathbb{E} \left[ |V(X(t_k))|^q \right] \leq e^{(q\rho + \tilde{\rho})kh} \mathbb{E} \left[ |V(X_0)|^q \right] \quad \forall k \geq 0.
$$

**Proof.** The proof is almost identical the proof of Lemma 2.4 hence we only sketch the key steps. First observe that

$$\tilde{V}'(x)(v_1) = q(V(x))^{(q-1)}V'(x_1)(v_1).$$

and

$$\tilde{V}''(x)(v_1, v_2) = q(q-1)(V(x))^{q-2}V'(x)(v_1)V'(x)(v_2) + q(V(x))^{(q-1)}V''(x)(v_1, v_2).$$

Higher order derivatives can be easily calculated using general Leibniz rule. In particular it is easy to find a constant $c_q > 0$ such that

$$\left\|\tilde{V}^{(i)}(x)\right\|_{L^1(\mathbb{R}^n,\mathbb{R})} \leq c_q V(x)^{q-1} = c_q V(x)^{q-1}V^{1-1/p}.$$  

The above property allows us to estimate the reminder term of the expansion (2.13) in the same way (up to constants) as in proof of Lemma 2.3. The first two terms of the expansion (2.8) can be estimated as follows

$$
\mathbb{E}_k \left[ \sum_{|\alpha| = 1} \frac{\partial^\alpha \tilde{V}(X_k)}{\alpha!} (\mu^h(X_k)h + \sigma^h(X_k)\Delta w_{k+1})^\alpha \right] = q(V(X_k))^{q-1}V^{1}(x)\mu(x)h.
$$

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The results in the previous section give a general conditions for the coefficients of tamed Euler scheme \([1,3]\) to be \(V\)-stable in the sense of \([1,3]\). In this section be give a concrete example of possible taming that satisfy assumptions of Theorem 2.2 and Lemmas 2.4 and 2.5. We assume that there exists a Lyapunov function \(V : \mathbb{R}^d \to [1, \infty)\) that preserve condition \([1,2]\). Therefore we are seeking taming

\[
\mu^h(x) := \frac{\mu(x)}{1 + G_\mu(x, h)}, \quad \sigma^h(x) := \frac{\sigma(x)}{1 + G_\sigma(x, h)}, \quad x \in \mathbb{R}^d, \quad (3.1)
\]

where \(0 \leq G_\mu(V(X), h), G_\sigma(V(X), h) \to 0\) as \(h \to 0\), that preserve condition \([1,2]\). That is

\[
(L_{\mu, \sigma} V)(x) \leq \rho V(x) \implies (L_{\mu^h, \sigma^h} V)(x) \leq \rho V(x), \quad x \in \mathbb{R}^d. \quad (3.2)
\]

Moreover, for Theorem 2.2 and Lemmas 2.4 and 2.5 with \(n \geq 4\), to hold\(^2\) we also require that

\[
|\mu^h(x)| h^{1/2} \leq c_\mu V(x)^{1/p} \quad \text{and} \quad |\sigma^h(x)| h^{1/4} \leq c_\sigma V(x)^{1/p}. \quad (3.3)
\]

To verify condition \([3.2]\) with taming \([2.1]\) we require that

\[
L_{\mu^h, \sigma^h} V(x) := V'(x) \mu^h(x) + \frac{1}{2} \sum_{k=1}^m V''(x)(\sigma^h_k(x), \sigma^h_k(x))
= \frac{V'(x) \mu(x)}{1 + G_\mu(x, h)} + \frac{1}{2} \sum_{k=1}^m \frac{V''(x)(\sigma^h_k(x), \sigma^h_k(x))}{(1 + G_\sigma(x, h))^2} \leq \rho V(x).
\]

Hence, condition \([3.2]\) holds if either of the following conditions is satisfied:

i) in general case we require that

\[
(1 + G_\mu(x, h))^2 = (1 + G_\sigma(x, h))^2;
\]

ii) if \(V''(x)(\sigma^h_k(x), \sigma^h_k(x)) > 0\) for all \(x \in \mathbb{R}^d\) (which often will be the case for the type of Lyapunov functions considered in this paper), we require that

\[
(1 + G_\mu(x, h)) \leq (1 + G_\sigma(x, h))^2.
\]

\(^2\)The case \(n = 3\) in Lemmas 2.4 and 2.5 can be treated in a very similar way.
Let us focus on the more general case \( i \) and take 
\[
G_\mu(x, h) := 2\tau_1|V(x)|^\kappa_1/p h^{\beta_1} + \tau_1^2|V(x)|^{(2\kappa_1)/p} h^{2\beta_1}
\]
and 
\[
G_\sigma(x, h) := \tau_1|V(x)|^{\kappa_1/p} h^{\beta_1}.
\]
In order for (3.3) to hold we take \( \beta_1 = 1/4 \), \( \tau_1 \geq c/c_\sigma \) and \( k_1 \geq (\gamma_1 - 1)_+ \) so that
\[
|\sigma^h(x)|h^{1/4} = \frac{|\sigma(x)|h^{1/4}}{1 + \tau_1|V(x)|^{\kappa_1/p} h^{\beta_1}} \leq \frac{c|V(x)|^{\kappa_1/p} h^{1/4}}{1 + \tau_1|V(x)|^{\kappa_1/p} h^{\beta_1}} \leq c_\sigma V(x)^{1/p}.
\]
We also need to choose \( \tau_1^2 \geq c/c_\sigma \) and \( k_1 \geq (\gamma_0 - 1)_+/2 \) so that
\[
|\mu^h(x)|h^{1/2} \leq \frac{c|V(x)|^{\gamma_0/p} h^{1/2}}{1 + 2\tau_1|V(x)|^{\kappa_1/p} h^{1/4} + \tau_1^2|V(x)|^{(2\kappa_1)/p} h^{1/2}} \leq c_\mu V(x)^{1/p}.
\]
Therefore we choose
\[
\max \{ (\gamma_1 - 1)_+^+, (\gamma_0 - 1)_+/2 \} \leq k_1 \quad \text{and} \quad \max \{ \sqrt{\frac{c}{c_\mu}}, \frac{c}{c_\sigma} \} \leq \tau_1 \quad \forall x \in \mathbb{R}^d,
\]
which gives a required taming.

4 Strong Convergence

For completeness, in this section we explain how in general one may establish strong convergence of the tamed Euler scheme (1.4). We build on the strong convergence results for tamed Euler schemes established in [3] (see their Definition 3.1 and Corollary 3.12) and [10] (see the proof of their Lemma 3.2). Roughly, both results state that provided appropriate moment bound for the tamed Euler (1.4) holds and in addition one-step of tamed Euler converges to one-step of the standard Euler scheme, given by

\[
\tilde{X}_{k+1} = x + \mu(x)h + \sigma(x)\Delta w_{k+1}, \quad x \in \mathbb{R}^d,
\]
then the tamed Euler scheme (1.4) strongly converge to the solution of the SDE (1.1).

In the light of calculations in the previous section it is natural to assume the following growth condition on the taming functions

**Assumption 4.1.** Let \( \tau_0, \tau_1, \gamma_0, \gamma_1, \beta_0, \beta_1 > 0 \). We assume that

\[
G_\mu(V(X), h) \leq \tau_0|V(x)|^{\kappa_0/p} h^{\beta_0} \quad \text{and} \quad G_\sigma(V(X), h) \leq \tau_1|V(x)|^{\kappa_1/p} h^{\beta_1}.
\]

The difference of one-step Euler schemes (1.4) and (4.1) is given by
\[
\tilde{X}_{k+1} - X_{k+1} = \frac{\mu(x)G_\mu(x, h)}{1 + G_\mu(x, h)} h + \frac{\sigma(x)G_\sigma(x, h)}{1 + G_\sigma(x, h)} \Delta w_{k+1}.
\]

By Assumptions 2.1 and 4.1 there exists a constant \( c > 0 \) such that
\[
\|E[\tilde{X}_{k+1} - X_{k+1}]\| = \|\frac{\mu(x)G_\mu(x, h)}{1 + G_\mu(x, h)} h \leq \|\mu(x)G_\mu(x, h)\| h \leq cV(x)^{(\gamma_0 + \kappa_0)/p} h^{1+\beta_0}
\]

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and
\[
\mathbb{E}[\|\tilde{X}_{k+1} - X_{k+1}\|^2] = \left(\frac{\mu(x)G_\mu(x,h)}{1 + G_\mu(x,h)}\right)^2 h^2 + \left(\frac{\sigma(x)G_\sigma(x,h)}{1 + G_\sigma(x,h)}\right)^2 h
\]
\[
\leq \left(\frac{\mu(x)G_\mu(x,h)}{1 + G_\mu(x,h)}\right)^2 h^2 + \left(\frac{\sigma(x)G_\sigma(x,h)}{1 + G_\sigma(x,h)}\right)^2 h
\]
\[
\leq cV(x)^2(\gamma_0 + \kappa_0)/p h^{2(1 + \beta_0)} + V(x)^2(\gamma_1 + \kappa_1)/p h^{1 + 2\beta_1}.
\]

This immediately shows that tamed Euler scheme (1.4) is \((\mu, \sigma)\)-consistent in the sense of Definition 3.1 in [4]. We assume also that statement of Lemma 2.5 holds for a given \(q \geq 1\) and that Lyapunov function has the property that \(\|x\|^q \leq V(x)\). Then we have that for \(T > 0\)
\[
\lim_{h \to 0} \sup_{k=1,\ldots,\lceil T/h \rceil} \mathbb{E}[\|X_k\|^q] < \infty.
\]

Then together with Corollary 3.12 in [4] immediately allow us to conclude that
\[
\lim_{h \to 0} \sup_{k=1,\ldots,\lceil T/h \rceil} \mathbb{E}[\|x(kh) - X_k\|^p] = 0,
\]
for all \(p \in (0, lq)\).

We also would like to stress out that by imposing additional regularity conditions on coefficients of the SDEs (see [10]) it is relatively straightforward to prove the rate of the strong \(L^2\) convergence 1/4 of tamed Euler scheme [3] provided
\[
\lim_{h \to 0} \sup_{k=1,\ldots,\lceil T/h \rceil} \mathbb{E}[V(X_k)^2(\gamma_0 + \kappa_0)/p + V(X_k)^2(\gamma_1 + \kappa_1)/p] \leq \infty.
\]

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