Global Exponential Stability of a Delayed HIV Infection Model with a Nonlinear Incidence Rate

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1. Introduction

As is well-known, acquired immune deficiency syndrome (AIDS) has received widespread attention since its discovery. Initial models used to gain an insight into HIV immunology relied on systems of ordinary differential equations (see, e.g., [1–6]). In general, an underlying assumption in such an ODE model for HIV infection is that infection of cells by virions is instantaneous. In fact, in the real world, there may be intracellular delays in the viral infection and replication, and immune response processes. Furthermore, delay-differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could affect the stability of the systems and may lead to some complex dynamic behaviors such as oscillation, chaos, and instability [7, 8]. Hence, time delays have been incorporated into HIV infection models by some authors (see [9–16] and the references cited therein).

On the other hand, compared with bilinear incidence rate, nonlinear incidence rate usually has more complex properties. When the number of viruses and susceptible cells is large, the number of susceptible cells contacted by viruses per unit time is limited, so the nonlinear incidence rate should be adopted in this case [17]. Recently, under the assumption that there is a lag between the time target cells are contacted by the virus particles and the time the contacted cells become actively infected, Yuan et al. [18] proposed the following delay-differential system for HIV infection models with a nonlinear incidence rate:

\[
\begin{align*}
    x'(t) &= \mu - kx(t) - ax(t)f(v(t)), \\
    y'(t) &= ax e^{-mt}x(t-\tau)f(v(t-\tau)) - y(t) - \beta y(t)h(z(t)), \\
    v'(t) &= py(t) - dv(t), \\
    z'(t) &= \delta y(t) - qz(t),
\end{align*}
\]

(1)

where \(x(t), y(t), v(t), \) and \(z(t)\) denote \(CD4^+\) cells that are susceptible to infection, productively infected cells, virus, and the effector population of CTLs (cytotoxic T lymphocytes), respectively, at time \(t\); \(\mu\) represents the newly added susceptible cell, \(k\) represents its death rate constant, \(a\) is the infection rate constant, \(\gamma\) signifies the infected cell death rate constant, and \(\beta\) represents the killing rate constant of productively infected cell by CTLs; \(p\) denotes the rate constant of virus production by infected cell, and \(d\)
determines the clearance rate constant of virus; effectors are generated in the presence of infected cells at rate \( \delta y \) and die at rate constant \( \tau \) per cell; \( \tau > 0 \) represents lag between the time target cells are contacted by the virus particles and the time the contacted cells become actively infected, and \( e^{-mt} \) is assumed to describe the surviving rate constant of each target cell to get infected. The functions \( f(x) \) and \( h(x) \) represent the force of infection by the infective at density \( x \) and the force of CTLs to kill infected cells at density \( x \), respectively. Furthermore, the functions \( f(x) \) and \( h(x) \) are locally Lipschitz on \([0, \infty)\) and satisfy the following:

(A1) \( f(0) = 0 \), the derivatives \( f'(x) > 0 \) and \( f(x)/x \) \( (x) \leq 0 \) in \((0, \infty)\)

(A2) \( h(x) \geq 0 \) in \([0, \infty)\) 

Due to their biological relevance, all parameters in model (1) are positive. In fact, the first equation of (1) has more general form \( x' = n(x(t)) - g(x(t), v(t)) \), where \( n(x) \) is a general function that accounts for both production and turnover of healthy target cells \( [19, 20] \). Generally speaking, there are two main forms of this function. In addition to \( n(x(t)) = \lambda - dx(t) \) \([21, 22]\), another typical function appearing in the literature is \( n(x(t)) = \lambda - dx(t) + rx(t)[1 - (x(t)/K)] \), where \( \lambda, d, r, K \) are positive real numbers \([23, 24]\). In particular, if we allow \( r = 0 \), the second form of \( n(x(t)) \) will become the first one. In order to include the aforementioned two forms, we consider the following delay-differential system:

\[
\begin{align*}
    x'(t) &= \mu - kx(t) + px(t) \left[ 1 - \frac{x(t)}{M} \right] - ax(t) f (v(t)), \\
    y'(t) &= ae^{-mt} x(t - \tau) f (v(t - \tau)) - gy(t) - \beta y(t) h(z(t)), \\
    v'(t) &= py(t) - dv(t), \\
    z'(t) &= \delta y(t) - qz(t).
\end{align*}
\]

Here, \( 0 < \rho < k, \rho \) is the maximum proliferation rate of uninfected cells, and \( M \) is the maximum level of uninfected cell concentration in the body. The other parameters are positive and have similar meanings to those in system (1).

For the sake of convenience, we denote by \( \mathbb{R}^n \) the set of all \( n \)-dimensional real vectors. For any \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), we let \( |x| \) denote the absolute-value vector given by \( |x| = (|x_1|, |x_2|, \ldots, |x_n|) \) and define \( \|x\| = \max_{i \in [1, 2, \ldots, n]} |x_i| \). Let \( \mathbb{R}_+ \) denote nonnegative real number space, \( C = C([\tau, 0], \mathbb{R}) \) be the Banach space of continuous functions mapping the interval \([\tau, 0]\) into \( \mathbb{R} \) equipped with the usual supremum norm \( \|\cdot\| \), and let \( C_{+} = C([\tau, 0], \mathbb{R}_+) \). Set \( x_i(\theta) = x(t + \theta) \) for all \( \theta \in [\tau, 0] \).

From the biological meanings, the initial conditions associated with (2) are defined as follows:

\[
x(\theta) = \varphi(\theta), v(\theta) = \psi(\theta), \quad y(0), z(0) \in \mathbb{R}_+, \theta \in [\tau, 0], \quad \varphi, \psi \in C([\tau, 0], \mathbb{R}_+).
\]

Most recently, by using the characteristic equation and the Fluctuation lemma, Yuan et al. \([18]\) proved the following result:

**Theorem 1.** Let the basic reproduction number

\[
R_0 = \frac{\alpha p e^{-mt} f' (0)}{kyd} < 1.
\]

Then, the noninfected equilibrium \( E^0 = ((\mu/k), 0, 0, 0) \) of (1) is globally asymptotically stable.

However, Theorem 1 does not give us any information about the convergence rate, which is vitally important to the disease prevention and control in real-world applications of theoretical results on epidemic models. In fact, the known convergence rate means that the range of the population is predictable; that is to say, we can estimate the range of changes in the population within a given time range. In particular, since the exponential convergence rate reveals the variation range of population in different time periods, there have been extensive results on the problem of the exponential stability of epidemic models in the literature studies \([25–28]\). Now, a question naturally arises: under what conditions is the noninfected equilibrium \( (x^*, y^*, v^*, z^*) = (x^* (\rho), 0, 0, 0) \) of system (2) with initial conditions (3) are exponentially stable? Here,

\[
x^* (\rho) = \frac{M (\rho - k) + (M^2 (\rho - k)^2 + 4pM\mu)^{1/2}}{2p}, \quad \rho > 0; \quad x^* (0) = \frac{\mu}{R}, \quad \rho = 0.
\]

Motivated by the above discussions, the main purpose of this paper is to establish sufficient conditions for the
exponential stability of the noninfected equilibrium \((x^*, y^*, v^*, z^*) = (x^*, 0, 0, 0)\) of system (2). To the best of our knowledge, it is the first time to focus on the problem of the exponential stability for (2). In particular, a numerical example is provided to illustrate our theoretical results.

Similar to the reference [29], we give the definition of the exponential stability as follows.

**Definition 1.** Let \((x(t), y(t), v(t), z(t))\) be the solution of (2) with initial value conditions (3). If there exists a positive constant \(\lambda > 0\) such that
\[
\begin{align*}
&|x(t) - x^*| = O(e^{-\lambda t}), \\
&|y(t) - y^*| = O(e^{-\lambda t}), \\
&|v(t) - v^*| = O(e^{-\lambda t}), \\
&|z(t) - z^*| = O(e^{-\lambda t}),
\end{align*}
\]
as \(t \to +\infty\), then the noninfected equilibrium is said to be globally exponentially stable.

In the rest of the paper, we give our main result in Section 2. The theoretical result is illustrated with examples as well as numerical simulations in Section 3. Finally, conclusions are made in Section 4.

**2. Main Results**

By the fundamental theory of functional differential equations [30], we have that there is a unique solution \((x(t), y(t), v(t), z(t))\) satisfying system (2) with initial conditions (3). Furthermore, by a similar argument as that in Zhu et al. [31], it is not hard to show that every solution of (2) with initial value conditions (3) is nonnegative and bounded on \([0, \infty)\). From (2), we have
\[
x'(t) \leq \mu - (k - \rho)x(t),
\]
which implies
\[
x(t) \leq x(0)e^{-(k - \rho)t} + \frac{\mu(1 - e^{-(k - \rho)t})}{k - \rho} \\
= \frac{\mu}{k - \rho} + \left[ x(0) - \frac{\mu}{k - \rho} \right] e^{-(k - \rho)t},
\]
for all \(t \in (0, \infty)\).

In addition, from (A1), we can get
\[
f(\xi) \leq f'(0)\xi, \quad \xi \in (0, \infty).
\]

The following theorem is our main result.

**Theorem 2.** Assume that the following
\[
\alpha f' (0) < \min \left\{ \frac{k - \rho + (px^*/M) - (\rho u/M (k - \rho)) ye^{\mu t} (k - \rho)}{x^*} \right\}, \quad p < d, \delta < q,
\]
hold. Then, there exist three positive constants \(\lambda, K, \) and \(t_\varepsilon\) such that
\[
\begin{align*}
&|x(t) - x^*| \leq Ke^{-\lambda t}, \\
&|y(t) - y^*| \leq Ke^{-\lambda t}, \\
&|v(t) - v^*| \leq Ke^{-\lambda t}, \\
&|z(t) - z^*| \leq Ke^{-\lambda t},
\end{align*}
\]
for all \(t > t_\varepsilon\). Here, \((x^*, y^*, v^*, z^*) = (x^* (\rho), 0, 0, 0)\), and
\[
x^* (\rho) = \frac{M (\rho - k) + (M^2 (\rho - k)^2 + 4\rho M \mu)^{1/2}}{2\rho}, \quad \rho > 0; \; x^* (0) = \frac{\mu}{K} \rho = 0.
\]
Proof. Let

\[
X(t) = (x_1(t), x_2(t), x_3(t), x_4(t))
= (x(t) - x^*, y(t) - 0, v(t) - 0, z(t) - 0),
\]

then we can get

\[
\begin{align*}
\dot{x}_1(t) &= -kx_1(t) + px_1(t) - \frac{\rho}{M}x^* - \frac{\rho}{M}x(t)x_1(t) - ax_1(t)\alpha f(x_3(t)) - ax^* f(x_3(t)), \\
\dot{x}_2(t) &= ae^{-mt}x(t - t)f(x_3(t - t)) - \beta x_2(t)h(x_4(t)), \\
\dot{x}_3(t) &= px_2(t) - \delta x_3(t), \\
\dot{x}_4(t) &= \delta x_2(t) - qx_4(t),
\end{align*}
\]

for all \( t \geq t_0 \).

From (8) and (10), there exist \( t_* > 0 \) and \( \epsilon_0 > 0 \) such that

\[
\begin{align*}
x(t) &\leq \epsilon_0 + \frac{\mu}{k - \rho} \quad \text{for all} \quad t \in (t_*, \infty), \\
\epsilon_0 &< y,
\end{align*}
\]

(15)

Then, for any \( \epsilon \in (0, \epsilon_0] \), we can choose two positive constants \( \lambda \) and \( \eta \) such that

\[
\begin{align*}
\lambda &+ \frac{\rho}{M} \left( \frac{\mu}{k - \rho} + \epsilon \right) > \lambda e^{-\eta t} f' \left( 0 \right) - \left( k - \rho + \frac{\rho}{M}x^* \right) < -\eta < 0, \\
\lambda &+ \epsilon x^* f' \left( 0 \right) - \left( k - \rho + \frac{\rho}{M}x^* \right) - y < -\eta < 0, \\
\lambda &+ p - d < -\eta < 0, \\
\lambda &+ \delta - q < -\eta < 0.
\end{align*}
\]

(16)

\[
\|X\|_t = \max \left\{ \max_{s \in [t_0 - \tau t_1]} x_1(s), \max_{s \in [t_0 - \tau t_1]} x_2(s), \max_{s \in [t_0 - \tau t_1]} x_3(s), \max_{s \in [t_0 - \tau t_1]} x_4(s) \right\}.
\]

(18)
Consequently,
\[ \|X(t)\| < \|X(t_1)\| + \epsilon < K_p \left( \|X(t_1)\| + \epsilon \right) = K_p \left( \|X(t)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda t} . \] (19)

In the following, we will show
\[ \|X(t)\| < K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda t} , \quad \text{for all } t > t_1 . \] (20)

If not, one of the following four cases must occur.

Case I: there exists \( \theta_1 > 0 \) such that
\[
\begin{align*}
&\left\{ \begin{array}{l}
|X_1(\theta)| = K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} , \\
\|X(t)\| < K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda t} , \quad \text{for all } t \in [t_0 - \tau, \theta_1) .
\end{array} \right.
\end{align*}
\] (21)

Case II: there exists \( \theta_2 > 0 \) such that
\[
\begin{align*}
&\left\{ \begin{array}{l}
|X_2(\theta)| = K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} , \\
\|X(t)\| < K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda t} , \quad \text{for all } t \in [t_0 - \tau, \theta_2) .
\end{array} \right.
\end{align*}
\] (22)

Case III: there exists \( \theta_3 > 0 \) such that
\[
\begin{align*}
&\left\{ \begin{array}{l}
|X_3(\theta)| = K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} , \\
\|X(t)\| < K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda t} , \quad \text{for all } t \in [t_0 - \tau, \theta_3) .
\end{array} \right.
\end{align*}
\] (23)

Case IV: there exists \( \theta_4 > 0 \) such that
\[
\begin{align*}
&\left\{ \begin{array}{l}
|X_4(\theta)| = K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} , \\
\|X(t)\| < K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda t} , \quad \text{for all } t \in [t_0 - \tau, \theta_4) .
\end{array} \right.
\end{align*}
\] (24)

If Case I holds, in view of (9), (14), (16), (17) and (21), we have
\[
|X_1(\theta)| = e^{-\left( k - \frac{\rho}{M} x^* \right)(\theta - \tau_1)} \left| x_1(t_1) \right| + \int_{t_1}^{\theta_1} e^{-\left( k - \frac{\rho}{M} x^* \right)(\theta - \tau)} \left[ k - \frac{\rho}{M} x^* + \alpha f(x_3(s)) \right] ds \left[ - \frac{\rho}{M} x_1(x_3(s)) - ax^* f(x_3(s)) \right] dv
\]
\[
\leq e^{-\left( k - \frac{\rho}{M} x^* \right)(\theta - \tau_1)} \left| x_1(t_1) \right| + \int_{t_1}^{\theta_1} e^{-\left( k - \frac{\rho}{M} x^* \right)(\theta - \tau)} \left[ k - \frac{\rho}{M} x^* + \alpha f(x_3(s)) \right] ds \left[ - \frac{\rho}{M} x_1(x_3(s)) + ax^* f^*(0) x_3(s) \right] dv
\]
\[
\leq K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} \left[ 1 - \left( \frac{k - \frac{\rho}{M} x^*}{k - \frac{\rho}{M} x^* - \lambda} \right) \left[ k - \frac{\rho}{M} x^* - \lambda \right] \left( \frac{\rho}{M} x^* + \alpha f^*(0) x^* \right) \right]
\]
\[
\leq K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} \left[ 1 - \left( \frac{k - \frac{\rho}{M} x^*}{k - \frac{\rho}{M} x^* - \lambda} \right) \left[ k - \frac{\rho}{M} x^* - \lambda \right] \left( k - \frac{\rho}{M} x^* - \lambda \right) \right]
\]
\[
= K_p \left( \|X(t_1)\| + \epsilon \right) e^{\lambda t_1} e^{-\lambda \theta} .
\] (25)

which contradicts the first equation in (21). Hence, Case I could not hold.

If Case II holds, combining (A2), (9), (14), (16) and (17), and (22), we deduce that
\[ |x_2(\theta_2)| = \left| e^{-\int_{t_i}^{\theta_2} \gamma h(x_4(s)) \, ds} x_2(t_i) + \int_{t_i}^{\theta_2} e^{-\int_{\theta_2}^{\theta} \gamma h(x_4(s)) \, ds} \left[ ae^{-mt} x(v) f(x_3(v)) \right] \, dv \right| \]

\[ \leq e^{-\theta_2-t_i} |x_2(t_i)| + \int_{t_i}^{\theta_2} e^{-\theta_2-s} \left[ ae^{-mt} f(0) \left( \frac{\mu}{K-\rho} + \epsilon \right) \right] |x_3(v) - x_2(v)| \, dv \]

\[ = e^{-\theta_2-t_i}(\|X\| + \epsilon) \left\{ \frac{1}{K_p} e^{-\theta_2-t_i} (\gamma - \lambda) + \int_{t_i}^{\theta_2} e^{-\theta_2-s} \left( \gamma - \lambda \right) \, ds \right\} \]

\[ \leq K_p(\|X\| + \epsilon) e^{s_\theta} e^{-\theta_2-t_i} \left\{ 1 - \left( 1 - \frac{1}{K_p} \right) e^{-\theta_2-t_i} (\gamma - \lambda) \right\} \]

which contradicts the first equation in (22). Thus, (22) could not hold.

If Case III holds, together with (14) and (16), (23) yields

\[ |x_3(\theta_3)| = \left| e^{-\theta_3-t_i} x_3(t_i) + \int_{t_i}^{\theta_3} pX_3(s) e^{-\theta_3-s} \, ds \right| \]

\[ \leq e^{-\theta_3-t_i}(\|X\| + \epsilon) + \int_{t_i}^{\theta_3} p e^{-\theta_3-s} K_p(\|X\| + \epsilon) e^{s_\theta} e^{-\theta_2-t_i} \, ds \]

\[ \leq K_p(\|X\| + \epsilon) e^{s_\theta} e^{-\theta_2-t_i} \left\{ \frac{1}{K_p} e^{-\theta_2-t_i} (d-\lambda) + \int_{t_i}^{\theta_3} e^{-\theta_2-s} (d-\lambda) \, ds \right\} \]

\[ \leq K_p(\|X\| + \epsilon) e^{s_\theta} e^{-\theta_2-t_i} \left\{ 1 - \left( 1 - \frac{1}{K_p} \right) e^{-\theta_2-t_i} (d-\lambda) \right\} \]

\[ < K_p(\|X\| + \epsilon) e^{s_\theta} e^{-\theta_2-t_i}, \]
which contradicts the first equation in (23). Thus, (23) could not hold.

If Case IV holds, together with (14) and (16), (24) yields

\|x(t)\| \leq K \|x(t)\| e^{\lambda t}, \quad \text{for all } t > t_\varepsilon, \quad (29)

which proves (11), where \( K = K_\varepsilon \|x\| e^{\lambda t_\varepsilon} \). This completes the proof of Theorem 2.

According to the above discussion, it is easy to see that our result is also true when \( p = 0 \). Furthermore, compared with Theorem 1, we find that the significantly stronger conclusion of Theorem 2 is obtained with only slightly stricter conditions. \( \square \)

### 3. A Numerical Example

In this section, we will show the existence and global exponential stability of the noninfected equilibrium of system (2) by a numerical example.

Let \( \mu = 24, k = 0.3, \rho = 0.05, M = 80, \alpha = 0.01, m = 2, \tau = 1, \gamma = 0.25, \beta = 0.03, p = 0.02, d = 0.05, \delta = 0.04, q = 0.07, f(x) = 0.2x, \) and \( h(x) = x + 0.5 \sin x \). Then,

\[
x^*(\rho) = \frac{M(\rho - k) + (M^2(\rho - k)^2 + 4\rho M \mu)^{1/2}}{2\rho}, \quad \rho > 0; \quad x^*(0) = 80,
\]

\[
a f'(0) = 0.002 < 0.003 = \min \left\{ \frac{k - \rho + (px^*/M) - (\rho \mu/M(k - \rho)) - ye^{\mu x} (k - \rho)}{x^*}, \frac{\mu}{\mu} \right\},
\]

\[
p = 0.02 < d = 0.05,
\]

\[
\delta = 0.04 < q = 0.07,
\]

(31)

which is globally exponentially stable, and all solutions of system (31) converge exponentially to the noninfected equilibrium \((80, 0, 0, 0)\) with the exponential convergent rate \( \zeta \approx 0.002 \). This fact is verified by the numerical simulation in Figure 1.
Since the results in [3, 4, 19, 32, 33] give no opinions about the global exponential stability of the HIV infection models, it is clear that all the results in the above references cannot be applicable to prove the global exponential stability of system (31).

4. Conclusion

We have proved the global exponential stability of the noninfected equilibrium for a delayed HIV infection model with a nonlinear incidence rate. It is worth pointing out that the required conditions are simple and easy to verify. It is natural to ask whether our methods in this paper are available to study the global exponential stability of the infected equilibrium of the delayed HIV infection models. It is an issue worth our further study. In addition, as pointed out in the literature [34, 35], the further extension of the model is to consider the case with reaction-diffusion term, which is also the focus of our further research.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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