Primitive Cohomology of Real Degree Two on Compact Symplectic Manifolds

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Abstract: In this paper, we define the generalized Lejmi’s $P_J$ operator on a compact almost Kähler 2n-manifold. We get that $J$ is $C^\infty$-pure and full if dim ker $P_J = b^2 - 1$. Additionally, we investigate the relationship between $J$-anti-invariant cohomology introduced by T.-J. Li and W. Zhang and new symplectic cohomologies introduced by L.-S. Tseng and S.-T. Yau on a closed symplectic 4-manifold.

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1 Introduction

For an almost complex manifold $(M, J)$, T.-J. Li and W. Zhang [18] introduced subgroups, $H^+_J$ and $H^-_J$, of the real degree 2 de Rham cohomology group $H^2_{dR}(M, \mathbb{R})$, as the sets of cohomology classes which can be represented by $J$-invariant and $J$-anti-invariant real 2-forms, respectively. Let us denote by $h^+_J$ and $h^-_J$ the dimensions of $H^+_J$ and $H^-_J$, respectively.

It is interesting to consider whether or not the subgroups $H^+_J$ and $H^-_J$ induce a direct sum decomposition of $H^2_{dR}(M, \mathbb{R})$. In the case of direct sum decomposition, $J$ is said to be $C^\infty$ pure and full. This is known to be true for integrable almost complex structures $J$ which admit compatible Kähler metrics on compact manifolds of any dimension. In this case, the induced decomposition is nothing but the classical real Hodge-Dolbeault decomposition of $H^2_{dR}(M, \mathbb{R})$ (see [5]).

In dimension 4, T. Draghici, T.-J. Li and W. Zhang [9, Theorem 2.3] proved that on any closed almost complex 4-manifold $(M, J)$, $J$ is $C^\infty$ pure and full. Further in [10], they computed the subgroups $H^+_J$ and $H^-_J$ and their dimensions $h^+_J$ and $h^-_J$ for almost complex structures metric related to an integrable one.

In the fifth section of [16], Lejmi introduced the differential operator $P_J$ on a compact almost Kähler 4-manifold $(M, g, J, \omega)$,

$$P_J : \Omega^2_0 \rightarrow \Omega^2_0 \quad \psi \mapsto \frac{1}{2} \Delta_g \psi - \frac{1}{4} g(\Delta_g \psi, \omega) \psi.$$

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He proved that $P_J$ is a self-adjoint strongly elliptic linear operator of order 2. In this paper, we define the generalized operator $P_J$ on a compact almost Kähler 2n-manifold. We prove that $P_J$ is also a self-adjoint strongly elliptic linear operator on a compact almost Kähler manifold $(M, g, J, \omega)$ of dimension 2n.

**Proposition 2.3.** Suppose that $(M, g, J, \omega)$ is a closed almost Kähler 2n-manifold, then $\ker P_J = H^+_J \oplus H^+_{J,0}$ and the harmonic representatives of $H^+_J$ and $H^+_{J,0}$ are of pure degree, that is,

$$H^+_J \cong H^+_J, \quad H^+_{J,0} \cong H^+_{J,0}.$$ 

By studying the properties of $P_J$, we get that $J$ is $C^\infty$ pure and full when $\dim \ker P_J = b^2 - 1$.

**Theorem 2.5.** Suppose that $(M, g, J, \omega)$ is a closed almost Kähler 2n-manifold, if $\dim \ker P_J = b^2 - 1$, then $J$ is $C^\infty$ pure and full and

$$H^+_{dR}(M; \mathbb{R}) = H^+_J \oplus H^+_J = \text{Span}_\mathbb{R}\{\omega\} \oplus H^+_{J,0} \oplus H^+_J = H^{(1,0)}_\omega(M; \mathbb{R}) \oplus H^{(0,2)}_\omega(M; \mathbb{R}).$$

Moreover, $J$ is pure and full.

Recently, L.-S. Tseng and S.-T. Yau [22] introduced new cohomologies for a closed symplectic manifold $M$. On a compact symplectic manifold $(M, \omega)$ of dimension 2n, the symplectic star operator $\ast_s$ acts on a differential $k$-form $\alpha$ by

$$\alpha \wedge \ast_s \alpha' = (\omega^{-1})^k(\alpha, \alpha')\text{dvol} = \frac{1}{k!}(\omega^{-1})^{i_1 j_1} \cdots (\omega^{-1})^{i_k j_k} \alpha_{i_1 \cdots i_k} \alpha'_{j_1 \cdots j_k} \frac{\omega^k}{n!}$$

with repeated indices summed over. Note that $\ast_s \ast_s = 1$, which follows from Weil’s identity [14, 23]

$$\ast_s \frac{L^r}{r!} B_k = (-1)^{k(k+1)/2} \frac{L^{n-r-k}}{(n-r-k)!} B_k$$

for any primitive $k$-form $B_k$. Also by Weil’s identity, for any primitive $k$-form $B_k$, we can get

$$\ast_s \frac{L^r}{r!} B_k = (-1)^{k(k+1)/2} \frac{L^{n-r-k}}{(n-r-k)!} J(B_k),$$

where $J = \sum_{p,q} (\sqrt{-1})^{p-q} \Pi^{p,q}$ projects a $k$-form onto its $(p, q)$ parts times the multiplicative factor $(\sqrt{-1})^{p-q}$. The adjoint of the standard exterior derivative takes the form

$$d^\Lambda = (-1)^{k+1} \ast_s d \ast_s.$$ 

By using the properties $d^2 = (d\Lambda)^2 = 0$ and the anti-commutativity $dd^\Lambda = -d^\Lambda d$, Tseng and Yau [22] considered new symplectic cohomology groups $H^{k}_{d^2+d\Lambda}(M)$ and $H^{k}_{d^2d\Lambda}(M)$. They also proved that the space of $d + d\Lambda$-harmonic $k$-forms $H^{k}_{d^2+d\Lambda}(M)$ and the space of $dd^\Lambda$-harmonic $k$-forms $H^{k}_{d^2d\Lambda}(M)$ are finite dimensional and isomorphic to $H^{k}_{d^2}(M)$ and $H^{k}_{d^2d\Lambda}(M)$, respectively.
By considering the relationship between \( H^{-} (\cong H^{+} ) \) and symplectic cohomology groups on a closed almost Kähler 4-manifold, we obtain the following theorem.

**Theorem 3.2.** Suppose that \(( M, g, J, \omega )\) is a closed almost Kähler 2\( n\)-manifold, then

\[
\ker P_{J} = H^{-}_{d+\Lambda}(M) \cap H^{-}_{dd\Lambda}(M).
\]

If \( H^{-}_{d+\Lambda}(M) = H^{-}_{dd\Lambda}(M) \), then

\[
H^{2}_{d+\Lambda}(M) = H^{2}_{dd\Lambda}(M) = \text{Span}_{\mathbb{R}} \{ \omega \} \oplus H_{J}^{+} \oplus H_{J,0}^{-}.
\]

In particular, if \( n = 2 \),

\[
H^{2}_{d+\Lambda}(M) = H_{J}^{+} \oplus H_{g}^{-} \oplus ( H_{J}^{-} \oplus H_{g}^{-} )_{d+\Lambda}^{-\perp},
\]

\[
H^{2}_{dd\Lambda}(M) = H_{J}^{-} \oplus H_{g}^{+} \oplus ( H_{J}^{+} \oplus H_{g}^{+} )_{dd\Lambda}^{-\perp},
\]

\[
*_{g}( H_{J}^{+} \oplus H_{g}^{-} )_{d+\Lambda}^{-\perp} = ( H_{J}^{-} \oplus H_{g}^{+} )_{dd\Lambda}^{-\perp}.
\]

2 Primitive de Rham cohomology of degree two

An almost Kähler structure on a real manifold \( M \) of dimension \( 2n \) is given by a triple \(( g, J, \omega )\) of a Riemannian metric \( g \), an almost complex structure \( J \) and a symplectic form \( \omega \), which satisfies the compatibility relation

\[
g(\cdot, \cdot) = \omega(\cdot, J\cdot).
\]

(2.1)

We say that the almost complex structure \( J \) is \( \omega \) compatible if it induces a Riemannian metric via (2.1).

Suppose that \(( M, g, J, \omega )\) is a closed almost Kähler 2\( n\)-manifold. The almost complex structure \( J \) acts on the space \( \Omega^{2} \) of smooth 2-forms on \( M \) as an involution by

\[
\alpha \mapsto J \cdot \alpha, \quad \alpha \in \Omega^{2}(M).
\]

(2.2)

This gives the \( J \)-invariant, \( J \)-anti-invariant decomposition of 2-forms (see [6]):

\[
\Omega^{2} = \Omega_{J}^{+} \oplus \Omega_{J}^{-}, \quad \alpha = \alpha_{J}^{+} + \alpha_{J}^{-}
\]

as well as the splitting of corresponding vector bundles

\[
\Lambda^{2} = \Lambda_{J}^{+} \oplus \Lambda_{J}^{-}.
\]

(2.3)

Let \( Z^{2} \) denote the space of closed 2-forms on \( M \) and set

\[
Z_{J}^{+} \triangleq Z^{2} \cap \Omega_{J}^{+}, \quad Z_{J}^{-} \triangleq Z^{2} \cap \Omega_{J}^{-}.
\]

Define the \( J \)-invariant and \( J \)-anti-invariant cohomology subgroups \( H_{J}^{\pm} \) (see [18]) by

\[
H_{J}^{\pm} = \{ a \in H^{2}_{dd\Lambda}(M; \mathbb{R}) \mid \text{there exists } \alpha \in Z_{J}^{\pm} \text{ such that } a = [\alpha] \}.
\]
Let us denote by $h^+_J$ and $h^-_J$ the dimensions of $H^+_J$ and $H^-_J$, respectively. We say $J$ is $C^\infty$ pure if $H^+_J \cap H^-_J = \{0\}$, $C^\infty$ full if $H^+_J + H^-_J = H^2_{dR}(M, \mathbb{R})$, and $J$ is $C^\infty$ pure and full if

$$H^2_{dR}(M, \mathbb{R}) = H^+_J \oplus H^-_J.$$  

T. Draghici, T.-J. Li and W. Zhang have proved that for any closed almost complex 4-manifold $(M, J)$, $J$ is $C^\infty$ pure and full (see [11]).

On a smooth closed manifold $M$, the space $\Omega^*(M)$ of smooth forms is a vector space, and with $C^\infty$ topology, it is a Fréchet space. The space $\mathcal{E}_*(M)$ of currents is the topological dual space, which is also a Fréchet space (see [13, 19]). As a topological vector space, $\Omega^*(M)$ is reflexive, thus it is also the dual space of $\mathcal{E}_*(M)$. Denote the space of closed currents by $\mathcal{Z}_*(M)$ and the space of boundaries by $\mathcal{B}_*(M)$. On a closed almost complex manifold $(M, J)$, there is a natural action of $J$ on the space $\Omega^k(M) \triangleq \Omega^k(M) \otimes \mathbb{C}$, which induces a topological type decomposition

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)_\mathbb{C}.$$  

If $k$ is even, $J$ also acts on $\Omega^k(M)$ as an involution. Specifically, if $k = 2$, $J$ acts on $\Omega^2(M)$ as $[22]$, and decomposes it into the topological direct sum of the invariant part $\Omega^2_+M$ and the anti-invariant part $\Omega^2_-M$. In this case, the two decompositions are related in the following way:

$$\Omega^2_+M = \Omega^{1,1}(M)_\mathbb{R} \triangleq \Omega^{1,1}_J(M)_\mathbb{C} \cap \Omega^2(M),$$

$$\Omega^2_-M = \Omega^{2,0}(M)_\mathbb{R} \triangleq (\Omega^{2,0}_J(M)_\mathbb{C} \oplus \Omega^{2,0}_J(M)_\mathbb{C}) \cap \Omega^2(M).$$

For the space of real 2-currents, we have a similar decomposition

$$\mathcal{E}_2(M) = \mathcal{E}^I_1(M)_\mathbb{R} \oplus \mathcal{E}^I_2(M)_\mathbb{R},$$

and the corresponding subspaces of closed and boundary currents,

$$\mathcal{B}^I_1 \subset \mathcal{Z}^I_1 \subset \mathcal{E}^I_1(M)_\mathbb{R},$$

$$\mathcal{B}^I_2 \subset \mathcal{Z}^I_2 \subset \mathcal{E}^I_2(M)_\mathbb{R}.$$  

We note the dual space of $\mathcal{E}^I_1(M)_\mathbb{R}$ is $\mathcal{Z}^I_1(M)_\mathbb{R}$, and vice versa. Similarly, $\mathcal{E}^I_2(M)_\mathbb{R}$ is the dual space of $\mathcal{Z}^I_2(M)_\mathbb{R}$. If $S = (1, 1)$ or $(2, 0)$ (cf. [12] [18]), define

$$H^J_S(M)_\mathbb{R} = \frac{\mathcal{Z}^J_S}{\mathcal{B}^J_S}.$$  

$J$ is said to be pure if

$$\frac{\mathcal{Z}^J_1}{\mathcal{B}^J_1} \cap \frac{\mathcal{Z}^J_2}{\mathcal{B}^J_2} = 0.$$  

$J$ is said to be full if

$$\frac{\mathcal{Z}^J_2}{\mathcal{B}^J_2} = \frac{\mathcal{Z}^J_1}{\mathcal{B}^J_1} + \frac{\mathcal{Z}^J_2}{\mathcal{B}^J_2}.$$  

Therefore, an almost complex structure $J$ is pure and full if and only if

$$H^2(M; \mathbb{R}) = H^I_1(M)_\mathbb{R} \oplus H^J_{(2,0),(0,2)}(M)_\mathbb{R}, \quad (2.4)$$

which induces a topological type decomposition

$$\Omega^k(M)_\mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M)_\mathbb{C}. $$

If $k$ is even, $J$ also acts on $\Omega^k(M)_\mathbb{C}$ as an involution. Specifically, if $k = 2$, $J$ acts on $\Omega^2(M)$ as $[22]$, and decomposes it into the topological direct sum of the invariant part $\Omega^2_+M$ and the anti-invariant part $\Omega^2_-M$. In this case, the two decompositions are related in the following way:

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For the space of real 2-currents, we have a similar decomposition

$$\mathcal{E}_2(M) = \mathcal{E}^I_1(M)_\mathbb{R} \oplus \mathcal{E}^I_2(M)_\mathbb{R},$$

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$$\mathcal{B}^I_1 \subset \mathcal{Z}^I_1 \subset \mathcal{E}^I_1(M)_\mathbb{R},$$

$$\mathcal{B}^I_2 \subset \mathcal{Z}^I_2 \subset \mathcal{E}^I_2(M)_\mathbb{R}.$$  

We note the dual space of $\mathcal{E}^I_1(M)_\mathbb{R}$ is $\mathcal{Z}^I_1(M)_\mathbb{R}$, and vice versa. Similarly, $\mathcal{E}^I_2(M)_\mathbb{R}$ is the dual space of $\mathcal{Z}^I_2(M)_\mathbb{R}$. If $S = (1, 1)$ or $(2, 0)$ (cf. [12] [18]), define

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$J$ is said to be pure if

$$\frac{\mathcal{Z}^J_1}{\mathcal{B}^J_1} \cap \frac{\mathcal{Z}^J_2}{\mathcal{B}^J_2} = 0.$$  

$J$ is said to be full if

$$\frac{\mathcal{Z}^J_2}{\mathcal{B}^J_2} = \frac{\mathcal{Z}^J_1}{\mathcal{B}^J_1} + \frac{\mathcal{Z}^J_2}{\mathcal{B}^J_2}.$$  

Therefore, an almost complex structure $J$ is pure and full if and only if

$$H^2(M; \mathbb{R}) = H^I_1(M)_\mathbb{R} \oplus H^J_{(2,0),(0,2)}(M)_\mathbb{R}, \quad (2.4)$$

which induces a topological type decomposition

$$\Omega^k(M)_\mathbb{C} = \bigoplus_{p+q=k} \Omega^{p,q}(M)_\mathbb{C}. $$
where $H_2(M; \mathbb{R})$ is the 2-nd de Rham homology group.

In particular, if $(M, g, J, \omega)$ is a closed almost K"ahler 4-manifold, then the Hodge star operator $*_{g}$ gives the well-known self-dual, anti-self-dual decomposition of 2-forms:

$$\Omega^2 = \Omega^+_{g} \oplus \Omega^-_{g}$$

as well as the corresponding splitting of the bundles (see [6, 7])

$$\Lambda^2 = \Lambda^+_{g} \oplus \Lambda^-_{g}. \quad (2.5)$$

The Hodge-de Rham Laplacian commutes with $*_{g}$, so the decomposition (2.5) holds for the space $H^2_g$ of harmonic 2-forms as well. By Hodge theory, this induces cohomology decomposition by the metric $g$:

$$H^2_{dR} (M; \mathbb{R}) \cong H^2_g = H^+_{g} \oplus H^-_{g}. \quad (2.6)$$

One defines (see [7])

$$H^\pm_{g} = \{ \alpha \in H^2_{dR} (M; \mathbb{R}) | \alpha = [\alpha] \text{ for some } \alpha \in Z^\pm_{g} := Z^2 \cap \Omega^\pm_{g} \}.$$

It is easy to see that

$$H^\pm_{g} \cong Z^\pm_{g} = H^\pm_{g}$$

and (2.6) can be written as

$$H^2_{dR} (M; \mathbb{R}) = H^+_{g} \oplus H^-_{g}.$$

There are the following relations between the decompositions (2.3) and (2.5) on an almost K"ahler 4-manifold (cf. [6, 7]):

$$\Lambda^+_J = \langle \omega \rangle \oplus \Lambda^-_g,$$

$$\Lambda^+_g = \langle \omega \rangle \oplus \Lambda^-_J,$$

$$\Lambda^+_J \cap \Lambda^+_g = \langle \omega \rangle, \quad \Lambda^-_J \cap \Lambda^-_g = \{0\}.$$}

It is easy to see that $Z^-_J \subset \mathcal{H}^+_J$ and $H^-_g \subset H^+_J$. Let $b^2$, $b^+$ and $b^-$ be the second, the self-dual and the anti-self-dual Betti number of $M$, respectively. Thus $b^2 = b^+ + b^-$. It is easy to see that, for a closed almost K"ahler 4-manifold $(M, g, J, \omega)$, there hold (see [9, 10, 21]):

$$H^+_J \cong Z^-_J, \quad h^+_J + h^-_J = b^2, \quad h^+_J \geq b^- + 1, \quad 0 \leq h^-_J \leq b^+ - 1. \quad (2.7)$$

Suppose that $(M, g, J, \omega)$ is a closed almost K"ahler 2n-manifold. A differential k-form $B_k$ with $k \leq n$ is called primitive if $L^{n-k+1}B_k = 0$ (see [22, 24]). Here $L$ is the Lefschetz operator (see [4, 22, 24]) which is defined acting on a k-form $\Lambda_k \in \Omega^k (M)$ by

$$L(\Lambda_k) = \omega \wedge \Lambda_k.$$ 

Define the space of primitive k-forms by $\Omega^k_0$. Specifically,

$$\Omega^2_0 = \{ \alpha \in \Omega^2 : \omega^{n-1} \wedge \alpha = 0 \}.$$
Therefore, 
\[ \Omega^2 = \Omega^2_1 \oplus \Omega^2_0, \]
where \( \Omega^2_1 \equiv \{ f \omega : f \in C^\infty(M) \} \). It is easy to see that \( \Omega^2_J \subset \Omega^2_0 \). So we can get the following decomposition
\[ \Omega^2_0 = \Omega^+_J \oplus \Omega^-_J, \]
(2.8)
where \( \Omega^+_J \) is the space of the primitive \( J \)-invariant 2-forms. We consider the following second order linear differential operator on \( \Omega^2_0 \).
\[ P_J : \Omega^2_0 \to \Omega^2_0, \]
\[ \psi \mapsto \Delta g \psi - \frac{1}{n} g(\Delta g \psi, \omega), \]
where \( \Delta g \) is the Riemannian Laplacian with respect to the metric \( g(\cdot, \cdot) = \omega(\cdot, J \cdot) \) (here we use the convention \( g(\omega, \omega) = n \)).

**Lemma 2.1.** \( P_J \) is a self-adjoint strongly elliptic linear operator with kernel the primitive \( g \)-harmonic 2-forms.

**Proof.** We claim that
\[
(P_J - \Delta g)(\psi) = \frac{1}{n} [d^*(\nabla^g \psi, \omega)_g - (\nabla^g \psi, \nabla^g \omega)_g - (2 - \frac{4}{n - 2}) \text{Tr}(\omega \cdot \text{Ric} \cdot \psi) - \frac{2 s^g}{(n - 1)(n - 2)} (\psi, \omega)_g - W^g(\psi, \omega)]_g.
\]
where \( W^g \) is the Weyl tensor (see [3]), \( \nabla^g \) is the Levi-Civita connection, \( s^g \) is the Riemannian scalar curvature with respect to the metric \( g(\cdot, \cdot) \) and \( \text{Tr}(\omega \otimes \text{Ric} \otimes \psi) \). Indeed, by Weitzenböck-Bochner formula (see [3]),
\[
(\Delta g \psi - \text{Tr}(\nabla^g)^2 \psi, \omega)_g = (\Delta g \psi - d^*(\nabla^g \psi, \omega)_g
= \sum_{i,j,k,l} 2 R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,j,k,l} R_{ik} \psi_{ik} \omega_{lj}
= \sum_{i,j,k,l} 2 R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,j,k,l} R_{ik} \psi_{ki} \omega_{lj}
= \sum_{i,j,k,l} 2 R_{iklj} \psi_{kl} \omega_{ij} - \sum_{i,j,k,l} 2 R_{ik} \psi_{kl} \omega_{ij}.
\]
On the other hand,
\begin{equation*}
W^g(\omega, \psi) = (R - \frac{1}{n-2} \text{Ric} \circ g + \frac{s^g}{2(n-1)(n-2)} g \circ g) (\omega, \psi)
\end{equation*}
\begin{equation*}
= \sum_{i,j,k,l} \omega_{ij} \psi_{kl} R_{ijkl} - \sum_{i,j,k,l} \frac{1}{n-2} \omega_{ij} \psi_{kl} (R_{ij} g_{kl} + R_{jk} g_{il})
- R_{ikl} g_{jl} - R_{jkl} g_{ij} + \frac{s^g}{2(n-1)(n-2)} g \circ g (\omega, \psi)
\end{equation*}
\begin{equation*}
= \sum_{i,j,k,l} \omega_{ij} \psi_{kl} R_{ijkl} - \sum_{i,k,l} \frac{4}{n-2} \omega_{ik} \psi_{kl} R_{il} - \sum_{i,j} \frac{2s^g}{n-1}(n-2) \omega_{ij} \psi_{ij}
- \sum_{i,j} \frac{2s^g}{2(n-1)(n-2)} \omega_{ij} \psi_{ij}
\end{equation*}
\begin{equation*}
= \sum_{i,j,k,l} -2 R_{ikl} \psi_{kl} \omega_{ij} - \sum_{i,k,l} \frac{4}{n-2} R_{il} \psi_{kl} \omega_{ik}
- \sum_{i,j} \frac{2s^g}{2(n-1)(n-2)} \omega_{ij} \psi_{ij}.
\end{equation*}

Here we compute (2.9) and (2.10) under the local coordinates system \((x^1, x^2, \ldots, x^{2n})\). In addition, we suppose \((\omega_{ij})\) to be the local representation of \(\omega\). Similarly, \(\psi = (\psi_{ij})\), \(\text{Ric} = (R_{ij})\) and \(R = (R_{ijkl})\), where \(\text{Ric}\) is the Ricci curvature tensor and \(R\) is the Riemannian curvature tensor with respect to metric \(g\). By (2.9) and (2.10), we can get the above claim. It is easy to see that \(P_J - \Delta_g\) is a linear differential operator of order 1. So the operator \(P_J\) is a self-adjoint strongly elliptic linear operator of order 2.

It remains to prove that the kernel of \(P_J\) is the space of the primitive \(g\)-harmonic 2-forms, that is, \(\mathcal{H}^2_g \cap \Omega^2_0\). Clearly, \(\mathcal{H}^2_g \cap \Omega^2_0 \subset \ker P_J\). For any \(\psi \in \ker P_J\),
\begin{equation*}
0 = \int_M (P_J(\psi), \psi)_g dvol_g
= \int_M (\Delta_g \psi - \frac{1}{n} (\Delta_g \psi, \omega)_g \omega, \psi)_g
= \int_M (\Delta_g \psi, \psi)_g
= \int_M (d\psi, d\psi)_g + (d^* \psi, d^* \psi)_g.
\end{equation*}

Hence, \(d\psi = d^* \psi = 0\) and \(\psi\) is a primitive \(g\)-harmonic 2-form. So we get \(\ker P_J = \mathcal{H}^2_g \cap \Omega^2_0\). \(\square\)

**Remark 2.2.** Let \(J_t, t \in [0, 1]\) be a smooth family of \(\omega\)-compatible almost complex structures on \(M\), then \(\dim \ker P_{J_t}\) is an upper-semi-continuous function in \(t\), by a classical result of Kodaira and Morrow showing the upper-semi-continuity of the kernel of a family of elliptic differential operators (Theorem 4.3 in [15]).

We define \(\mathcal{H}^2_J\) to be the space of the harmonic \(J\)-anti-invariant 2-forms and \(\mathcal{H}^2_{J,0}\) to be the space of the harmonic primitive \(J\)-invariant 2-forms. Define the primitive
J-invariant cohomology subgroup $H^{+}_{J,0}$ by

$$H^{+}_{J,0} = \{ \alpha \in H^{2}(M; \mathbb{R}) \mid \text{there exists } \alpha \in \mathbb{Z}^{2} \cap \Omega^{+}_{J,0} \text{ such that } \alpha = [\alpha] \}.$$ 

**Proposition 2.3.** Suppose that $(M, g, J, \omega)$ is a closed almost Kähler 2n-manifold, then $\ker P_{J} = H^{+}_{J} \oplus H^{+}_{J,0}$ and the harmonic representatives of $H^{+}_{J}$ and $H^{+}_{J,0}$ are of pure degree, that is,

$$H^{+}_{J} \cong H^{-}_{J}, \ H^{+}_{J,0} \cong H^{+}_{J,0}.$$ 

**Proof.** For any $\alpha \in \ker P_{J}$, $\alpha$ is primitive and harmonic. So $d\alpha = 0, d_{g} \alpha = 0$ and $\alpha$ can be written as $\alpha = \beta_{\alpha} + \gamma_{\alpha}$, where $\beta_{\alpha} \in \Omega^{+}_{J}$ and $\gamma_{\alpha} \in \Omega^{+}_{J,0}$. By a direct computation, we get

$$*_g \alpha = \frac{L^{n-2}}{(n-2)!} (\beta_{\alpha} - \gamma_{\alpha})$$

and

$$\left( \frac{L^{n-2}}{(n-2)!} + *_{g} \right) \alpha = \frac{2L^{n-2}}{(n-2)!} \beta_{\alpha}.$$ 

Hence,

$$d \left( \frac{L^{n-2}}{(n-2)!} \beta_{\alpha} \right) = d \left( \frac{L^{n-2}}{(n-2)!} + *_{g} \right) \alpha = 0.$$ 

So we can get $d \beta_{\alpha} = 0$. Since $*_g \beta_{\alpha} = \frac{L^{n-2}}{(n-2)!} \beta_{\alpha}$, $d \,*_{g} \beta_{\alpha} = 0$. Therefore, $\beta_{\alpha} \in H^{+}_{J}$. Similarly,

$$\left( \frac{L^{n-2}}{(n-2)!} - *_{g} \right) \alpha = \frac{2L^{n-2}}{(n-2)!} \gamma_{\alpha}$$

and we can get $\gamma_{\alpha} \in H^{+}_{J,0}$. Thus, $\ker P_{J} = H^{+}_{J} \oplus H^{+}_{J,0}$.

For any $a = [\alpha] \in H^{+}_{J}$, $\alpha \in \mathbb{Z}^{2}$. By (1.1),

$$*_g \alpha = - \frac{L^{n-2}}{(n-2)!} \mathcal{J}(\alpha) = \frac{L^{n-2}}{(n-2)!} \alpha.$$ 

So $d_* \alpha = d \left( \frac{L^{n-2}}{(n-2)!} \right) \alpha = 0$ and $\alpha$ is a harmonic $J$-anti-invariant form, that is, $\alpha \in H^{+}_{J}$. Hence, $H^{+}_{J} \cong H^{-}_{J}$. Similarly, $H^{+}_{J,0} \cong H^{+}_{J,0}$. The harmonic representatives of $H^{+}_{J}$ and $H^{+}_{J,0}$ are of pure degree. \qed

**Remark 2.4.** In case $n = 2$, on a closed almost Kähler 4-manifold, Lejmi [10] proved that $P_{J}$ preserves the decomposition

$$\Omega^{0} = \Omega^{+}_{J,0} \oplus \Omega^{-}_{J}.$$ 

Furthermore, $P_{J}|_{\Omega^{+}_{J,0}}(\psi) = \Delta_{g} \psi$ and $P_{J}|_{\Omega^{-}_{J}}(\psi) = 2d_{J}^{*}d_{J} \psi$. He also pointed out that $P_{J}|_{\Omega^{-}_{J}}(\psi) = 2d_{J}^{*}d_{J} \psi$ is a self-adjoint strongly elliptic linear operator from $\Omega^{-}_{J}$ to $\Omega^{-}_{J}$ on a closed almost Kähler 4-manifold. It follows that the kernel of $P_{J}$ consists of primitive harmonic 2-forms which splits as anti-self-dual and $J$-anti-invariant ones. So he gets

$$\dim \ker P_{J} = b^{-} + h^{-}_{J}.$$ 

But when $n > 2$, by computing the principal symbol of $d_{J}^{*}d_{J}$, one finds that $d_{J}^{*}d_{J}$ is no longer a self-adjoint strongly elliptic linear operator. So we are not able to get any good properties about the $\dim \ker P_{J}$ in higher dimension.
In [1], D. Angella and A. Tomassini define

\[ H^{(r,s)}_\omega(M; \mathbb{R}) \triangleq \{ [L'] \beta \in H^{2r+s}_{dR}(M; \mathbb{R}) : \beta \in \Omega^n_0 \} \subseteq H^{2r+s}_{dR}(M; \mathbb{R}) \]

for \( r, s \in \mathbb{N} \). Obviously, for every \( k \in \mathbb{N} \), one has

\[ \sum_{2r+s=k} H^{(r,s)}_\omega(M; \mathbb{R}) \subseteq H^{2r+s}_{dR}(M; \mathbb{R}). \]

A natural question is that when the above inclusion is actually an equality, and when the sum is a direct sum. Fortunately, Angella and Tomassini have proved that

\[ H^2_{dR}(M; \mathbb{R}) = H^{(1,0)}_\omega(M; \mathbb{R}) \oplus H^{(0,2)}_\omega(M; \mathbb{R}) \]  \hspace{1cm} (2.11)

in [1]. Clearly, \( H^{(1,0)}_\omega(M; \mathbb{R}) \cong \text{Span}_\mathbb{K}\{\omega\} \) and \( \dim H^{(0,2)}_\omega(M; \mathbb{R}) = b^2 - 1 \). Here we want to emphasize that \( \ker P_J \cong H^{-}_J \oplus H^{+}_J \subseteq H^{(0,2)}_\omega(M; \mathbb{R}) \) and if \( \dim \ker P_J = b^2 - 1 \), then \( \ker P_J \cong H^{-}_J \oplus H^{+}_J = H^{(0,2)}_\omega(M; \mathbb{R}) \).

It is well known that on any closed almost complex 4-manifold \((M, J)\), \( J \) is \( C^\infty \) pure and full (see [9]). But we can not get this result in higher dimension. In [12], A. Fino and A. Tomassini showed the existence of a compact 6-dimensional nil-manifold with an almost complex structure which is not \( C^\infty \) pure, i.e., the intersection of \( H^-_J(M) \) and \( H^+_J(M) \) is non-empty. They also proved that on a compact almost complex 2n-manifold \((M, J)\), if \( J \) admits a compatible symplectic structure, then \( J \) is \( C^\infty \) pure. With this result and by studying the dimension of \( \dim \ker P_J \), we can get the following theorem.

**Theorem 2.5.** Suppose that \((M, g, J, \omega)\) is a closed almost Kähler 2n-manifold, if \( \dim \ker P_J = b^2 - 1 \), then \( J \) is \( C^\infty \) pure and full and

\[ H^2_{dR}(M; \mathbb{R}) = H^+_J \oplus H^-_J = \text{Span}_\mathbb{K}\{\omega\} \oplus H^{+}_J \oplus H^{-}_J = H^{(1,0)}_\omega(M; \mathbb{R}) \oplus H^{(0,2)}_\omega(M; \mathbb{R}). \]

Moreover, \( J \) is pure and full.

**Proof.** In [12], A. Fino and A. Tomassini showed that \( J \) is \( C^\infty \) pure if \((M, g, J, \omega)\) is a closed almost Kähler 2n-manifold. In [9], T. Draghici, T.-J. Li and W. Zhang proved the same result on a closed almost Kähler 2n-manifold. Next, we will prove that \( J \) is \( C^\infty \) full under the condition of \( \dim \ker P_J = b^2 - 1 \).

By proposition 2.3, if \( \dim \ker P_J = b^2 - 1 \), we get

\[ \dim H^-_J + \dim H^{+}_J = b^2 - 1. \]

Hence,

\[ b^2 = \dim H^-_J + (\dim H^{+}_J + 1). \]  \hspace{1cm} (2.12)

It is easy to see that

\[ H^{+}_J \oplus H^-_J \subseteq H^2_{dR}(M; \mathbb{R}), \]

and

\[ \text{Span}_\mathbb{K}\{\omega\} \oplus H^{+}_J \subseteq H^{+}_J. \]
So we will get
\[
 h^+_J + h^-_J \leq b^2 \tag{2.13}
\]
and
\[
 \dim H^+_J + 1 = \dim H^+_J + 1 \leq h^+_J. \tag{2.14}
\]
Therefore, by (2.12), (2.13) and (2.14),
\[
 h^+_J + h^-_J \leq b^2 = \dim H^-_J + (\dim H^+_J + 1) \leq h^+_J + h^-_J.
\]
Clearly, \( h^+_J + h^-_J = b^2 \) and \( \dim H^+_J + 1 = h^+_J \). So we can get
\[
 H^2_dR (M; \mathbb{R}) = H^+_J \oplus H^-_J
\]
and
\[
 \text{Span}_R \{ \omega \} \oplus H^+_{J,0} = H^+_J.
\]
Then we can get the following decompositions
\[
 H^2_dR (M; \mathbb{R}) = H^+_J \oplus H^-_J = \text{Span}_R \{ \omega \} \oplus H^+_{J,0} \oplus H^-_J = H^{(1,0)}_2 (M; \mathbb{R}) \oplus H^{(0,2)}_2 (M; \mathbb{R})
\]
and \( H^+_J \cong H^-_J \), where \( H^+_J \) is the space of the harmonic \( J \)-invariant \( 2 \)-forms. Of course, \( J \) is \( C^\infty \) pure and full.

We have proven that \( J \) is \( C^\infty \) pure and full. Then by Theorem 3.7 in [12], we can get that \( J \) is pure. In addition, in the above proof, we have gotten that \( H^-_J \cong H^-_J \) and \( H^+_J \cong H^+_J \) when \( \dim \ker P_J = b^2 - 1 \). Hence the harmonic representatives of \( H^2_dR (M; \mathbb{R}) \) are of pure degree (cf. [2]). Also by Theorem 3.7 in [12], we can get that \( J \) is pure and full. This completes the proof of the Theorem.

In the above theorem, \( \dim \ker P_J = b^2 - 1 \) is just the sufficient condition but not the necessary condition for \( J \)’s \( C^\infty \) pureness and fullness. Just by \( J \)’s \( C^\infty \) pureness and fullness, we cannot get \( \dim \ker P_J = b^2 - 1 \). Indeed, we have the following counter-example which is constructed by T. Draghici ([8]).

Example 2.6. (also cf. [2]) Let \( T^4 \) be the standard torus with coordinates \( \{ x^1, x^2, x^3, x^4 \} \).

Denote by \((g_0, J_0, \omega_0)\) be a standard flat Kähler structure on \( T^4 \), so \( (T^4, \omega_0) \) has the hard Lefschetz property (cf. [24]), that is, the map
\[
 H^k_dR (T^4; \mathbb{R}) \rightarrow H^{4-k}_dR (T^4; \mathbb{R}), \quad \alpha \mapsto [\omega_0]^{2-k} \wedge \alpha,
\]
is an isomorphism for all \( k \leq 2 \). We choose
\[
g_0 = \sum_i dx^i \otimes dx^i, \quad \omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4,
\]
so \( J_0 \) is given by
\[
 J_0 dx^1 = dx^2, \quad J_0 dx^2 = -dx^1, \quad J_0 dx^3 = dx^4, \quad J_0 dx^4 = -dx^3.
\]
Equivalently, \( J_0 \) may be given by specifying
\[
 \Lambda_{J_0} = \text{Span} \{ dx^1 \wedge dx^3 - dx^2 \wedge dx^4, dx^1 \wedge dx^4 + dx^2 \wedge dx^3 \}. 
\]
Consider the almost complex structure $J$ given by

$$Jdx^1 = m dx^2, \quad Jdx^2 = -\frac{1}{m}dx^1, \quad Jdx^3 = dx^4, \quad Jdx^4 = -dx^3,$$

where $m = m(x^2, x^4)$ is a positive periodic function on $x^2, x^4$ only. It is easy to see that

$$\Lambda^J = \text{Span}\{dx^1 \wedge dx^3 - m dx^2 \wedge dx^4, dx^1 \wedge dx^4 + mdx^2 \wedge dx^3\},$$

and $J$ is compatible with $\omega_0$.

We claim that in any solution $(A, B)$ of $(2.16)$, the torus. The condition that such a form to be closed is equivalent with

$$Jdx^1 = m dx^2, \quad Jdx^2 = -\frac{1}{m}dx^1, \quad Jdx^3 = dx^4, \quad Jdx^4 = -dx^3,$$

where $m = m(x^2, x^4)$ is a positive periodic function on $x^2, x^4$ only. It is easy to see that

$$\Lambda^J = \text{Span}\{dx^1 \wedge dx^3 - m dx^2 \wedge dx^4, dx^1 \wedge dx^4 + mdx^2 \wedge dx^3\},$$

and $J$ is compatible with $\omega_0$.

We claim that we can choose $m$ such that $h^J_\omega = 1$ (proved by T. Draghici in [8]).

Denote by $\psi_1 = dx^1 \wedge dx^3 - mdx^2 \wedge dx^4$ and $\psi_2 = dx^1 \wedge dx^4 + mdx^2 \wedge dx^3$. Note that $m = m(x^2, x^4)$, we have

$$d\psi_1 = 0, \quad d\psi_2 = m_4 dx^2 \wedge dx^3 \wedge dx^4 = (\log m)_4 dx^4 \wedge \psi_2. \quad (2.15)$$

Here we denote by $m_4 = (\partial m)/(\partial x^4)$ and $(\log m)_4 = (\partial (\log m))/(\partial x^4)$. The general $J$-anti-invariant form is written as $A\psi_1 + B\psi_2$, where $A, B$ are smooth functions on the torus. The condition that such a form to be closed is equivalent with

$$dA \wedge \psi_1 + (dB + B(\log m)_4 dx^4) \wedge \psi_2 = 0. \quad (2.16)$$

We claim that in any solution $(A, B)$ of $(2.16)$, $A$ must be a constant. To see this, taking the Hodge operator $*_g$ of both sides of the above equation we get

$$JdA = -(dB + B(\log m)_4 dx^4). \quad (2.17)$$

Taking one more differential, and then taking trace with respect to $\omega_0$ we get

$$\Delta A = B_3 (\log m)_4, \quad (2.18)$$

where $B_3 = (\partial B)/(\partial x^3)$. By $(2.17)$, we get $B_3 = A_4 := (\partial A)/(\partial x^4)$. Then $(2.18)$ evolves into

$$-\Delta A + A_4 (\log m)_4 = 0. \quad (2.19)$$

By the maximum principle, it follows that $A = \text{const}$. Plugging this back in $(2.17)$, we see that $B$ must satisfy

$$dB = -B(\log m)_4 dx^4.$$

It is easy to see that the compatibility relation of this system is

$$B(\log m)_4 2 \neq 0.$$

Thus, if $(\log m)_4 \neq 0$ (indeed, it will be sufficient that the locus of $(\log m)_4$ has zero Lebesgue measure.), the only solutions for $(2.16)$ are $A = \text{const}$ and $B = 0$, hence $h^J_\omega = 1$. We choose $m = e^{\sin 2\pi (x^2 + x^4)}$ to be a positive periodic function on $\mathbb{T}^4$ such that $(\log m)_4 = -4\pi^2 \sin 2\pi (x^2 + x^4)$ has zero Lebesgue measure. So on $(\mathbb{T}^4, g, J, \omega_0)$, $h^J_\omega$ is equal to 1. Of course, $J$ is $C^\infty$ pure and full (T. Draghici, T.-J. Li and W. Zhang proved in [9] that on any closed almost complex 4-manifold $(M, J)$,
J is $C^\infty$ pure and full.). Additionally, Lejmi [10] proved that, on a closed almost Kähler 4-manifold $(M, g, J, \omega)$, the kernel of $P_J$ consists of primitive harmonic 2-forms and dim $\ker P_J = b^+ + h_J^{-}$.

So on $(T^4, g, J, \omega_0)$, dim $\ker P_J = b^- + h_J^{-} = 4 < 5 = b^2 - 1$.

Let us denote by $e_i \triangleq dx^i$ and $e_{ij} \triangleq dx^i \wedge dx^j$. Please see the following table.

| $\mathbb{Z}_j^{+}$ | $\omega_0$, $e_{12} - e_{34}$, $e_{13} + e_{24}$, $e_{14} - e_{23}$ |
|-------------------|---------------------------------------------------------------|
| $\mathbb{Z}_j^{-}$ | $e_{13} - e_{24}$, $e_{14} + e_{23}$ |
| $\ker P_{J_0}$ | $e_{12} - e_{34}$, $e_{13} + e_{24}$, $e_{14} - e_{23}$, $e_{13} - e_{24}$, $e_{14} + e_{23}$ |
| $\mathcal{H}_{g_0}^{+}$ | $e_{12} - e_{34}$, $e_{13} + e_{24}$, $e_{14} - e_{23}$ |
| $\mathcal{H}_{g_0}^{-}$ | $\frac{1}{1-m} (e_{12} + e_{34} + e_{14} - me_{23})$ |
| $\mathcal{Z}_j$ | $e_{13} - me_{24}$ |
| $\ker P_{J_0}$ | $e_{12} - e_{34}$, $e_{13} + me_{24}$, $\frac{1}{1+m} (e_{12} - e_{34} + e_{14} - me_{23})$ |
| $\mathcal{H}_{g}^{+}$ | $\omega_0$, $e_{13} - me_{24}$ |
| $\mathcal{H}_{g}^{-}$ | $\frac{1}{1-m} (e_{12} - e_{34} + e_{14} - me_{23})$ |

Table 1. Bases for $\mathbb{Z}_j^{+}$, $\mathbb{Z}_j^{-}$, $\ker P_{J_0}$, etc. of $T^4$.

By the above example, we want to propose the following question:

**Question 2.7.** On any closed symplectic 4-manifold $(M, \omega)$, is there a $\omega$-compatible (or $\omega$-tame) almost complex structure $J$ such that $\dim \ker P_J = b^2 - 1$?

**Remark 2.8.** On any closed symplectic 4-manifold $(M, \omega)$, if an almost complex structure $J$ is tamed by symplectic form $\omega$ and $h_J^{-} = b^+ - 1$, then it implies that there exists the generalized $\delta\bar{\partial}$-lemma (cf. [11, 20]).

The above example $T^4$ admits a Kähler structure $(g_0, J_0, \omega_0)$. In Section 3, we will give a non-Kähler example $M^6(c)$ which is constructed by M. Fernández, V. Muñoz and J. A. Santisteban. They have proven that $M^6(c)$ does not admit any Kähler metric (cf. [11]). We will prove that there exists an almost complex structure $J$ on $M^6(c)$ such that $\dim \ker P_J = b^2 - 1$. Please see the Example 5.3.

## 3 Primitive symplectic cohomology of degree two

L.-S. Tseng and S.-T. Yau [22] considered new symplectic cohomology groups

$$H^k_{d+dA}(M) = \frac{\text{Ker}(d + dA) \cap \Omega^k(M)}{\text{Im}(dd^A) \cap \Omega^k(M)}.$$ 

and

$$H^k_{dA}(M) = \frac{\text{Ker}(dd^A) \cap \Omega^k(M)}{(\text{Im} d + \text{Im} dA) \cap \Omega^k(M)}$$

on a compact symplectic manifold $(M, \omega)$ of dimension $2n$. We denote the spaces of $d + dA$ harmonic $k$-forms and $ddA$ harmonic $k$-forms by $H^k_{d+dA}(M)$ and $H^k_{dA}(M)$,
respectively. For any almost Kähler triple \((g, J, \omega)\), a k-form \(\alpha \in \Omega^k(M)\) is said to be \(d + d^A\)-harmonic (see [22]) if
\[
d\alpha = d^A\alpha = 0 \quad \text{and} \quad (d^A)^*\alpha = 0,
\]
and \(dd^A\)-harmonic (see [22]) if
\[
d^*\alpha = (d^A)^*\alpha = 0 \quad \text{and} \quad dd^A\alpha = 0,
\]
where \(d^* = -* gdg, d^A = * gdg^* g\) and \((d^A)^* = (-1)^{k+1} * gdg^* g\). Tseng and Yau also proved that \(H^k_{d+d^A}(M)\) and \(H^k_{dd^A}(M)\) are finite dimensional and isomorphic to \(H^k_{\text{d+dd}}(M)\) and \(H^k_{\text{dd}}(M)\), respectively. Let \(H^k_{d+d^A}(M)\) and \(H^k_{dd^A}(M)\) denote the spaces of primitive \(d + d^A\) harmonic 2-forms and primitive \(dd^A\) harmonic 2-forms, respectively.

\[
H^k_{d+d^A}(M) \cong H^k_{d+d^A}(M) \cap \Omega^k_0, \quad H^k_{dd^A}(M) \cong H^k_{dd^A}(M) \cap \Omega^k_0.
\]

Hence, we can get the following decompositions
\[
H^2_{d+d^A}(M) = \text{Span}_\mathbb{R}\{\omega\} \oplus H^2_{d+d^A}(M),
\]
\[
H^2_{dd^A}(M) = \text{Span}_\mathbb{R}\{\omega\} \oplus H^2_{dd^A}(M).
\]

**Definition 3.1.** Let \((M, g, J, \omega)\) be a closed almost Kähler 4-manifold. Set
\[
(H^+_{d+\alpha})_{d+d^A} = \{\alpha \in H^2_{d+d^A} \mid \alpha = d^+_J \theta^1 + d^+_g \theta^2\}
\]
and
\[
(H^+_{d+\alpha})_{dd^A} = \{\alpha \in H^2_{dd^A} \mid \alpha = d^+_J \theta^1 + d^+_g \theta^2\},
\]
where \(d^+_J = P^-_J \circ d, d^+_g = P^-_g \circ d\) and \(\theta^1, \theta^2 \in \Omega^1(M)\). Here \(P^-_J\) is the projection from \(\Omega^2(M)\) to \(\Omega^1_J(M)\) and \(P^-_g\) is the projection from \(\Omega^2(M)\) to \(\Omega^1_g(M)\).

**Theorem 3.2.** Suppose that \((M, g, J, \omega)\) is a closed almost Kähler 2n-manifold, then
\[
\ker P^+_J = H^2_{d+d^A}(M) \cap H^2_{dd^A}(M).
\]
If \(H^2_{d+d^A}(M) = H^2_{dd^A}(M)\), then
\[
H^2_{d+d^A}(M) = H^2_{dd^A}(M) = \text{Span}_\mathbb{R}\{\omega\} \oplus H^-_{d+d^A} \oplus H^+_{d+d^A}.
\]
In particular, if \(n = 2\),
\[
H^2_{d+d^A}(M) = H^-_{d+d^A} \oplus (H^-_{d+d^A} \oplus H^-_{d+d^A})_{d+d^A}^+, \quad H^2_{dd^A}(M) = H^-_{dd^A} \oplus (H^-_{dd^A} \oplus H^-_{dd^A})_{dd^A}^+, \quad \ast_g (H^-_{d+d^A} \oplus H^-_{d+d^A})_{d+d^A}^+ = (H^-_{d+d^A} \oplus H^-_{d+d^A})_{dd^A}^+.
\]
Proof. Let us begin with the first assertion of the Theorem. We claim that \( \mathcal{H}_{\gamma}^- \oplus \mathcal{H}_{\gamma,0}^+ \subset \mathcal{H}_{d+d^\ast}(M) \) and \( \mathcal{H}_{\gamma}^- \oplus \mathcal{H}_{\gamma,0}^+ \subset \mathcal{H}_{d^\ast}(M) \). Indeed, for any \( \alpha = \beta + \gamma \in \mathcal{H}_{\gamma}^- \oplus \mathcal{H}_{\gamma,0}^+ \), \( \beta \in \mathcal{H}_{\gamma}^-, \gamma \in \mathcal{H}_{\gamma,0}^+ \). By Weil’s identity, we have \( \ast_s \alpha = -\frac{L^{n-2}}{(n-2)!} \alpha \). Then

\[
d \ast_s \alpha = -d \frac{L^{n-2}}{(n-2)!} \alpha = -\frac{L^{n-2}}{(n-2)!} d \alpha = 0.
\]

Hence, \( d^* \alpha = 0 \). Also by Weil’s identity, we have

\[
\ast_s \ast_g \alpha = \ast_s \ast_g \beta + \ast_s \ast_g \gamma = \ast_s \left( \frac{L^{n-2}}{(n-2)!} \beta - \frac{L^{n-2}}{(n-2)!} \gamma \right) = -\beta + \gamma.
\]

Hence, \( d \ast_g \alpha = d(-\beta + \gamma) = 0 \). So we get \( (d(d^*))^\ast \alpha = 0 \). Therefore, \( \alpha \in \mathcal{H}_{d+d^\ast}(M) \) and \( \mathcal{H}_{\gamma}^- \oplus \mathcal{H}_{\gamma,0}^+ \subset \mathcal{H}_{d+d^\ast}(M) \). It is similar for \( \mathcal{H}_{J}^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{d^\ast}(M) \). In particular, if \( n = 2 \), \( \mathcal{H}_{J,0}^+ = \mathcal{H}_g^- \), we can get \( \mathcal{H}_{J}^- \oplus \mathcal{H}_g^- \subset \mathcal{H}_{d+d^\ast}(M) \) and \( \mathcal{H}_{J}^- \oplus \mathcal{H}_g^- \subset \mathcal{H}_{d^\ast}(M) \). So we can get

\[
\mathcal{H}_{J}^- \oplus \mathcal{H}_{J,0}^+ \subset \mathcal{H}_{d+d^\ast}(M) \cap \mathcal{H}_{d^\ast}(M).
\]

For the other hand, it follows straightforwardly from the definitions of \( \mathcal{H}_{d+d^\ast}(M) \), \( \mathcal{H}_{d^\ast}(M) \) and ker \( P_J \). If \( \mathcal{H}_{d+d^\ast}(M) = \mathcal{H}_{d^\ast}(M) \) (i.e. \( \mathcal{H}_{d+d^\ast}^2(M) = \mathcal{H}_{d^\ast}^2(M) \)), then ker \( P_J = \mathcal{H}_{d+d^\ast}^2(M) = \mathcal{H}_{d^\ast}^2(M) \). Hence,

\[
\mathcal{H}_{d+d^\ast}^2(M) = \mathcal{H}_{d^\ast}^2(M) = \text{Span}_{\mathbb{R}}(\omega) \oplus \ker P_J = \text{Span}_{\mathbb{R}}(\omega) \oplus \mathcal{H}_{\gamma}^- \oplus \mathcal{H}_{J,0}^+.
\]

In the following, we suppose that \( (M, g, J, \omega) \) is a closed almost Kähler 4-manifold. Then it is easy to see that \( \Omega_\gamma^- = \Omega_{J,0}^+ \). So \( \Omega^2 \) can be written as

\[
\Omega^2 = \Omega_1^2 \oplus \Omega_0^2 = \Omega_1^2 \oplus \Omega_\gamma^- \oplus \Omega_J^-.
\]

For any \( \alpha \in \mathcal{H}_{d+d^\ast}^2 \), by the definition of \( \mathcal{H}_{d+d^\ast}^2 \),

\[
d \alpha = d^\chi \alpha = 0, \quad (d(d^\ast))^\ast \alpha = 0,
\]

where \( (d(d^\ast))^\ast = -s_g \ast d^\chi s_g \). It is clear that

\[
d \alpha = 0, \quad d \ast_s \alpha = 0.
\]

Hence

\[
\frac{1}{2} (1 + \ast_s) \alpha = 0.
\]

Since \( \frac{1}{2} (1 + \ast_s) \alpha \in \Omega_1^2 \), it can be written as

\[
\frac{1}{2} (1 + \ast_s) \alpha = f_\alpha \omega.
\]

Since \( d \omega = 0 \), we have \( d(f_\alpha \omega) = df_\alpha \wedge \omega = 0 \). It follows that \( f_\alpha = c_\alpha \) is a constant since \( \omega \) is nondegenerate.

Let

\[
\alpha_1 = \alpha - c_\alpha \omega = \frac{1}{2} (1 - \ast_s) \alpha \in \Omega_0^2 = \Omega_J^- \oplus \Omega_\gamma^-.
\]

Hence, \( \alpha_1 \) is still in \( \mathcal{H}_{d+d^\ast}^2 \) and \( \alpha_1 \) can be written as

\[
\alpha_1 = \alpha_{1,J} + \alpha_{1,g}.
\]
\[ \alpha_{1,j} = \beta_\alpha + d_j \theta^1_\alpha, \theta^1_\alpha = d^* \eta_\alpha \]
and
\[ \alpha_{1,g} = \gamma_\alpha + d_g \theta^2_\alpha, \theta^2_\alpha = d^* \xi_\alpha, \]
where \( \beta_\alpha \in \mathcal{Z}_J, \gamma_\alpha \in \mathcal{H}_g, \eta_\alpha \in \Omega_J^{-1} \) and \( \xi_\alpha \in \Omega_g^{-1} \). Then
\[ \alpha = c_\alpha \omega + \alpha_1 = c_\alpha \omega + \beta_\alpha + \gamma_\alpha + (d_j \theta^1_\alpha + d_g \theta^2_\alpha) \]
and \( (d_j \theta^1_\alpha + d_g \theta^2_\alpha) \in (\mathcal{H}_J \oplus \mathcal{H}_g)^{-1}_{d+d^*} \). So we can get that
\[ \mathcal{H}^2_{d+d^*} = \text{Span}_\mathbb{R} \{ \omega \} \oplus \mathcal{H}_J^{-1} \oplus \mathcal{H}_g^{-1} \oplus (\mathcal{H}_J \oplus \mathcal{H}_g)^{-1}_{d+d^*}. \]
It is easy to see that
\[ (\mathcal{H}_J^{-1} \oplus \mathcal{H}_g)^{-1}_{d+d^*} \subset \mathcal{H}^2_{d+d^*} \]
is just the orthogonal complement of \( \mathcal{H}_J^{-1} \oplus \mathcal{H}_g^{-1} \) in \( \mathcal{H}^2_{d+d^*} \) with respect to the cup product. Similarly, one has
\[ \mathcal{H}^2_{d+d^*} = \text{Span}_\mathbb{R} \{ \omega \} \oplus \mathcal{H}_J^{-1} \oplus \mathcal{H}_g^{-1} \oplus (\mathcal{H}_J \oplus \mathcal{H}_g)^{-1}_{d+d^*} \]
since \( *_{g} \mathcal{H}^2_{d+d^*} = \mathcal{H}^2_{d+d^*} \) and \( *_{g} \mathcal{H}^{-1}_{d+d^*} = \mathcal{H}^{-1}_{d+d^*} \) (see [22 Proposition 3.24]). This completes the proof of the Theorem.

It is helpful to have explicit examples showing clearly the differences between the different cohomologies and \( \ker P_J \) discussed above. For this we consider the following examples.

**Example 3.3.** (cf. [11]) Let \( G(c) \) be the connected completely solvable Lie group of dimension 5 consisting of matrices of the form
\[
\begin{pmatrix}
    e^{cz} & 0 & 0 & 0 & 0 \\
    0 & e^{-cz} & 0 & 0 & y_1 \\
    0 & 0 & e^{cz} & 0 & x_2 \\
    0 & 0 & 0 & e^{-cz} & y_2 \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
where \( x_i, y_i, z \in \mathbb{R} \) and \( c \) is a nonzero real number. Then a global system of coordinates \( x_1, y_1, x_2, y_2 \) and \( z \) for \( G(c) \) is given by \( x_1(a) = x_1, y_1(a) = y_1 \) and \( z(a) = z \). A standard calculation shows that a basis for the right invariant 1-forms on \( G(c) \) consists of
\[ \{ dx_1 - cx_1 dz, dy_1 - cy_1 dz, dx_2 - cx_2 dz, dy_2 - cy_2 dz, dz \}. \]
Alternatively, the Lie group \( G(c) \) may be described as a semidirect product \( G(c) = \mathbb{R} \ltimes \mathbb{R}^4 \), where \( \psi(z) \) is the linear transformation of \( \mathbb{R}^4 \) given by the matrix
\[
\begin{pmatrix}
    e^{cz} & 0 & 0 & 0 \\
    0 & e^{-cz} & 0 & 0 \\
    0 & 0 & e^{cz} & 0 \\
    0 & 0 & 0 & e^{-cz}
\end{pmatrix},
\]
(3.3)
for any $z \in \mathbb{R}$. Thus, $G(c)$ has a discrete subgroup $\Gamma(c) = \mathbb{Z} \times \mathbb{Z}^4$ such that the quotient space $G(c)/\Gamma(c)$ is compact. Therefore, the forms $dx_i - cx_i dz, dy_i - cy_i dz$ and $dz$ $(i = 1, 2)$ descend to 1-forms $\alpha_i, \beta_i$ and $\gamma$ $(i = 1, 2)$ on $G(c)/\Gamma(c)$.

M. Fernández, V. Muñoz and J. A. Santisteban considered the manifold $M^6(c) = G(c)/\Gamma(c) \times S^1$. Here, there are 1-forms $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma$ and $\eta$ on $M^6(c)$ such that

$$d\alpha_i = -c\alpha_i \wedge \gamma, \ d\beta_i = c\beta_i \wedge \gamma, \ d\gamma = d\eta = 0, \quad (3.4)$$

where $i = 1, 2$ and such that at each point of $M^6(c)$, $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \eta\}$ is a basis for the 1-forms on $M^6(c)$. Using Hattori's theorem, they compute the real cohomology of $M^6(c)$:

$$H^0(M^6(c)) = \langle 1 \rangle,$$

$$H^1(M^6(c)) = \langle [\gamma], [\eta] \rangle,$$

$$H^2(M^6(c)) = \langle [\alpha_1 \wedge \beta_1], [\alpha_1 \wedge \beta_2], [\alpha_2 \wedge \beta_1], [\alpha_2 \wedge \beta_2], [\gamma \wedge \eta] \rangle,$$

$$H^3(M^6(c)) = \langle [\alpha_1 \wedge \beta_1 \wedge \gamma], [\alpha_1 \wedge \beta_2 \wedge \gamma], [\alpha_2 \wedge \beta_1 \wedge \gamma], [\alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \eta], [\alpha_1 \wedge \beta_2 \wedge \eta], [\alpha_2 \wedge \beta_1 \wedge \eta], [\alpha_2 \wedge \beta_2 \wedge \eta] \rangle,$$

$$H^4(M^6(c)) = \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \eta], [\alpha_1 \wedge \beta_1 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_2 \wedge \beta_2 \wedge \eta] \rangle,$$

$$H^5(M^6(c)) = \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma], [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \eta] \rangle,$$

$$H^6(M^6(c)) = \langle [\alpha_1 \wedge \beta_1 \wedge \alpha_2 \wedge \beta_2 \wedge \gamma \wedge \eta] \rangle. \quad (3.5)$$

Therefore, the Betti number of $M^6(c)$ are

$$b^0 = 1, \quad b^1 = 2, \quad b^2 = 5, \quad b^3 = 8. \quad (3.6)$$

We denote by $(g, J, \omega)$ be an almost Kähler structure on $M^6(c)$, where we choose

$$g = \alpha_1 \otimes \alpha_1 + \beta_1 \otimes \beta_1 + \alpha_2 \otimes \alpha_2 + \beta_2 \otimes \beta_2 + \gamma \otimes \gamma + \eta \otimes \eta$$

and

$$\omega = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \gamma \wedge \eta.$$ 

So $J$ is given by

$$J\alpha_1 = \beta_1, \ J\alpha_2 = \beta_2, \ J\gamma = \eta.$$ 

It is clear that the maps

$$[\omega] : H^2_{dR}(M^6(c); \mathbb{R}) \to H^4_{dR}(M^6(c); \mathbb{R})$$

and

$$[\omega]^2 : H^1_{dR}(M^6(c); \mathbb{R}) \to H^5_{dR}(M^6(c); \mathbb{R})$$

are isomorphisms. Thus, $(M^6(c), \omega)$ satisfies the hard Lefschetz property. By simple calculation, we can get

$$Z^-_J = \text{Span}_\mathbb{R}\{\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1\}, \quad (3.7)$$

$$Z^+_J = \text{Span}_\mathbb{R}\{\alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_1, \ \alpha_1 \wedge \beta_1, \ \alpha_2 \wedge \beta_2, \ \gamma \wedge \eta\}, \quad (3.8)$$

$$\text{ker} \ P_J = \text{Span}_\mathbb{R}\{\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1, \ \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2, \ \gamma \wedge \eta, \ \alpha_1 \wedge \beta_1 - \gamma \wedge \eta, \ \alpha_2 \wedge \beta_2 - \gamma \wedge \eta\}. \quad (3.9)$$
Hence, \( \dim \ker P_J = 4 = b^2 - 1 \). Of course, \( J \) is \( C^\infty \) pure and full.

| \( H^2_{dR} \) | \( \alpha \land \beta_1, \alpha_1 \land \beta_2, \alpha_2 \land \beta_1, \alpha_2 \land \beta_2, \gamma \land \eta \) |
| --- | --- |
| \( \mathcal{Z}^+_j \) | \( \alpha \land \beta_1 \land \beta_2, \alpha_1 \land \beta_1, \alpha_2 \land \beta_2, \gamma \land \eta \) |
| \( \mathcal{Z}^-_j \) | \( \alpha \land \beta_1 \land \beta_2 - \alpha_2 \land \beta_1 \) |
| \( \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda} \) | \( \alpha_1 \land \beta_2 - \alpha_2 \land \beta_1, \alpha_1 \land \beta_2 + \alpha_2 \land \beta_1, \alpha_1 \land \beta_1 - \gamma \land \eta, \alpha_2 \land \beta_2 - \gamma \land \eta \) |
| \( \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda} \) | \( \alpha_1 \land \beta_2 - \alpha_2 \land \beta_1, \alpha_1 \land \beta_2 + \alpha_2 \land \beta_1, \alpha_1 \land \beta_1 - \gamma \land \eta, \alpha_2 \land \beta_2 - \gamma \land \eta \) |
| \( \ker P_J \) | \( \alpha \land \beta_2 - \alpha_2 \land \beta_1, \alpha_1 \land \beta_2 + \alpha_2 \land \beta_1, \alpha_1 \land \beta_1 - \gamma \land \eta, \alpha_2 \land \beta_2 - \gamma \land \eta \) |

**Table 2.** Bases for \( H^2_{dR}, \mathcal{Z}^+_j, \mathcal{Z}^-_j, \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda}, \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda} \) and \( \ker P_J \) of \( M^6(c) \).

By the above table, we can see that \( \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda}(M^6(c)) = \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda}(M^6(c)) \). So

\[
\mathcal{H}^2_{d+d^\Lambda}(M^6(c)) = \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda}(M^6(c)) = \text{Span}_\mathbb{R}\{\omega\} \oplus \mathcal{H}^\perp_J \oplus \mathcal{H}^{d+d^\Lambda}_{J,0}.
\]

**Proposition 3.4.** (cf. [17]) The manifold \( M^6(c) \) does not admit Kähler metric.

**Example 3.5.** (cf. [22]) Let \( M \) be the Kodaira-Thurston nilmanifold defined by taking \( \mathbb{R}^4 \) and modding out by the identification

\[
(x_1, x_2, x_3, x_4) \sim (x_1 + a, x_2 + b, x_3 + c, x_4 + d - bx_3),
\]

where \( a, b, c, d \in \mathbb{Z} \). The resulting manifold is a torus bundle over a torus with a basis of cotangent 1-forms given by

\[
e_1 = dx_1, \ e_2 = dx_2, \ e_3 = dx_3, \ e_4 = dx_4 + x_2 dx_3.
\]

It is well known that Kodaira-Thurston manifold admits no Kähler structure. We take the symplectic form to be

\[
\omega = e_1 \land e_2 + e_3 \land e_4.
\]

Consider the \( \omega \)-compatible almost complex structure \( J \) given by

\[
J(e_1) = e_2, \ J(e_2) = -e_1, \ J(e_3) = e_4, \ J(e_4) = -e_3.
\]

Let us denote the compatible metric by

\[
g(\cdot, \cdot) = \omega(\cdot, J \cdot) = e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3 + e_4 \otimes e_4.
\]

(\( g, J, \omega \)) is an almost Kähler structure but not Kähler since the almost complex structure \( J \) is not integrable. Please see the following table for the relationship between \( H^2_{dR}, H^2_{d+d^\Lambda}, H^2_{d+d^\Lambda} \) and \( \ker P_J \).

| \( H^2_{dR} \) | \( \omega, \ e_1 \land e_2 - e_3 \land e_4, \ e_1 \land e_3, \ e_2 \land e_4 \) |
| --- | --- |
| \( \mathcal{Z}^+_j \) | \( \omega, \ e_1 \land e_2 - e_3 \land e_4 \) |
| \( \mathcal{Z}^-_j \) | \( \omega, \ e_1 \land e_2 - e_3 \land e_4 \) |
| \( \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda} \) | \( \omega, \ e_1 \land e_2 - e_3 \land e_4, \ e_1 \land e_3, \ e_2 \land e_4 \) |
| \( \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda} \) | \( \omega, \ e_1 \land e_2 - e_3 \land e_4, \ e_1 \land e_3, \ e_2 \land e_4 \) |
| \( \ker P_J \) | \( e_1 \land e_2 - e_3 \land e_4 \) |

**Table 3.** Bases for \( H^2_{dR}, \mathcal{Z}^+_j, \mathcal{Z}^-_j, \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda}, \mathcal{H}^{d+d^\Lambda}_{d+d^\Lambda} \) and \( \ker P_J \) of \( M \).
By the above table, we can see that $(M,\omega)$ does not satisfy the hard Lefschetz property. It is easy to see that the dimension of $\ker P_J$ is equal to $b^2 - 1 = 3$.

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