GLOBAL WELL-POSEDNESS FOR FRACTIONAL SOBOLEV-GALPERN TYPE EQUATIONS

HUY TUAN NGUYEN AND NGUYEN ANH TUAN
Division of Applied Mathematics, Science and Technology Advanced Institute,
Van Lang University, Ho Chi Minh City, Vietnam
Faculty of Technology, Van Lang University, Ho Chi Minh City, Vietnam
CHAO YANG∗
College of Mathematical Sciences,
Harbin Engineering University, 150001, China

Abstract. This article is a comparative study on an initial-boundary value problem for a class of semilinear pseudo-parabolic equations with the fractional Caputo derivative, also called the fractional Sobolev-Galpern type equations. The purpose of this work is to reveal the influence of the degree of the source nonlinearity on the well-posedness of the solution. By considering four different types of nonlinearities, we derive the global well-posedness of mild solutions to the problem corresponding to the four cases of the nonlinear source terms. For the advection source function case, we apply a nontrivial limit technique for singular integral and some appropriate choices of weighted Banach space to prove the global existence result. For the gradient nonlinearity as a local Lipschitzian, we use the Cauchy sequence technique to show that the solution either exists globally in time or blows up at finite time. For the polynomial form nonlinearity, by assuming the smallness of the initial data we derive the global well-posed results. And for the case of exponential nonlinearity in two-dimensional space, we derive the global well-posedness by additionally using an Orlicz space.

1. Introduction. In the current work, we especially concern about the following time-fractional pseudo-parabolic equation

\[ \partial_t^\alpha u - \partial_t^\alpha \Delta u - \Delta u = H(u), \]  

(1)

wherein \( \Omega \) is a bounded domain of \( \mathbb{R}^d \) \( (d \in \mathbb{N}) \), \( u \) is an unknown function from \( \Omega \times (0, \infty) \) to \( \mathbb{R} \), \( H \) is the source function which is going to be defined in more details later. The notation \( \partial_t^\alpha \) is abbreviated for the time-fractional derivative of order \( \alpha \in (0, 1) \) in the Caputo sense, defined by

\[ \partial_t^\alpha u(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \partial_\tau u(\tau) d\tau, \]

provided that the right-hand side (RHS) of the above equation makes sense. In addition, for Equation (1), we assume that the boundary \( \partial \Omega \) of \( \Omega \) is sufficiently

2020 Mathematics Subject Classification. Primary: 35K20; Secondary: 35K58.
Key words and phrases. Fractional pseudo-parabolic, globally Lipschitz source, exponential nonlinearity, global well-posedness.
∗ Corresponding author: yangchao@hrbeu.edu.cn (Chao Yang).
Other authors: nguyenhuytuan@vlu.edu.vn (Huy Tuan Nguyen), nguyenanhtuan@vlu.edu.vn (Nguyen Anh Tuan).
smooth. Also, for Equation (1) we consider the homogeneous Dirichlet boundary condition
\[ u(x,t) = 0, \quad (x,t) \in \partial \Omega \times (0, \infty), \]  
and the following initial value condition
\[ u(x,t) = u_0(x), \quad (x,t) \in \Omega \times \{0\}, \]  
here, \( u_0 \) is an initial function which satisfies some specific assumptions in different theorems claimed in the present paper. In Equation (1), the operator \( (-\Delta) = A : \text{dom}(A) \subset L^2(\Omega) \rightarrow L^2(\Omega) \) is uniformly symmetric elliptic. As we know that, \( A \) owns a set of positive eigenvalues \( \{\theta_k\}_{k \in \mathbb{N}} \) whose elements organize an increasing sequence. Corresponding to this set of Dirichlet eigenvalues, there is a set of eigenfunctions \( \{\phi_k\}_{k \in \mathbb{N}} \) which forms a complete orthonormal of \( L^2(\Omega) \). Our main goal is to investigate the globally well-posed results for a mild solution of the initial-boundary value problem (1)-(3), and we shall introduce the background of our work in the following subsection.

1.1. Background of the problem. A generalized form of Equation (1) reflecting the theory of isotropic incompressible homogeneous fluid was first considered by Coleman and Walter in [12]. To investigate some non-steady fluid flows as the second-order fluids due to the effect of pressure, Ting in [22] considered a special type semilinear pseudo-parabolic equation in the following form
\[ a\partial_t u = b\Delta u + c\partial_t \Delta u + H(u). \]  
This equation along with a specified type of initial-boundary conditions describes a bounded flow whose external force affects its solid boundary. And such types of equations like (4) also appear in the study about thermodynamic temperature [10] and population recovery [19]. In one-dimensional space, by taking \( b = 0 \) and \( H(u) = -u_x - uu_x \) in (4), we have the regularized long-wave equation or the Benjamin–Bona–Mahony BBM equation, which was proposed by Benjamin, Bona, and Mahony in [4] with applications in the study of long waves propagation. The work [4] was extended by Amick, Bona, and Schonbek in [2] by investigating the large time behavior of solution to the Cauchy problem related to the one-dimensional case of (4) with \( H(u) = -u_x - uu_x \). This nonlinear pseudo-parabolic equation can be seen as an additional consideration of dissipation mechanisms for the BBM equation, also Celebi et.al. in [7] considered the generalized BBM equation (GBBM) in the form of
\[ \partial_t u - \partial_t \Delta u - b\Delta u + (\eta \cdot \nabla)u + \nabla \cdot F(u) = 0, \]  
where \( \eta \in \mathbb{R}^d \) is a constant vector, \( F(u) \) is a \( d \)-dimensional vector field, which received a lot of attentions in the PDEs forum [1, 18]. In addition, from [29, 26] and the references given there, the readers will find that Equation (4) can also be used in biological sciences, filtration theory or the study of semiconductors.

Given the representativeness and application value of this kind of mathematical model, it has attracted many mathematicians’ attention with many rich results. It is an impossible task to mention all of them, so we only make an overview of the works closely related to the research of this paper. In [21], Showalter and Ting considered an initial value problem for a generalized form of (4) as follows
\[ M\partial_t u + Lu = H, \]
here, $M, L$ are second-order differential operators independent of $t$, containing variable coefficients with some specific properties, $M$ is uniformly strongly elliptic on a bounded open set $\Omega \subset \mathbb{R}^d$. In the homogeneous case, based on properties of the Friedrichs extensions of $M$ and $L$ which can be obtained by the Lax-Milgram theorem, the authors constructed a group $\{E(t) : t \in \mathbb{R}\}$ which helps to get the unique existence of the weak solution. Also, thanks to the results that $H^1_0(\Omega) \cap H^p$ is invariant under effects of the group $\{E(t) : t \in \mathbb{R}\}$, the regularity of the solution was proved. The main results of this work also include the asymptotic behavior for the solution and the extended theories for the nonhomogeneous case. We also notice that the different forms of the nonlinearities in (4) are of special interest, and attract a lot of attention. In [29], Xu and Su considered a pseudo-parabolic equation with the well-known polynomial source

$$H(u) = u^p,$$

where $p \in (1, \infty)$ if $d = 1, 2$, or $p \in \left(1, \frac{d+2}{d-2}\right)$ if $d \geq 3$.

In consideration of the subcritical (resp. critical) case that the initial energy is less than (resp. equal to) the depth of potential well $J(u_0) < d$ (resp. $J(u_0) = d$) and the positive (resp. non-negative) value at $u_0$ of the Nehari functional, i.e., $I(u_0) > 0$ (resp. $I(u_0) \geq 0$), by using the Galerkin and the potential well theory, the authors proved the global existence, uniqueness and the asymptotic behavior of the solution. Otherwise, when $I(u_0) < 0$, the solution is proved to be blowing up at a finite time. Further for the arbitrarily initial energy, that is $J(u_0) > 0$, the comparison principle and variational methods are adopted to obtain the finite-time blow-up results. In [6], besides studying (4) with a source term satisfying growth conditions of polynomial type, Caraballo and his colleagues also considered the influence of external forces with some kind of delay. Namely, $H$ was given by

$$H(t, u) = f(t, u_t) + g(u),$$

where $f(t, u_t)$ is the time-dependent delay term caused by memory or hereditary characteristics and $g \in C^1(\mathbb{R})$ satisfying

$$\limsup_{|v| \to +\infty} \frac{g(v)}{v} \leq \frac{\theta_1}{6}$$

or

$$\limsup_{|v| \to +\infty} \frac{g(v)}{v} \leq 0$$

and

$$|g(v) - g(w)| \leq C|v - w| \left(1 + |v|^{p-1} + |w|^{p-1}\right), \quad p > 1, C > 0.$$

The above series of work has aroused great interest among colleagues. On the one hand, there are a large number of practical problems surrounding this type of model; on the other hand, there is a huge gap between the existing research and the many different variants that exist widely. Therefore, a large number of subsequent researches are rapidly developed around different variants of the model, including the couple form of a parabolic system [28], the pseudo-parabolic model (4) with the singular potential [17], a nonlocal form of (4) with the nonlinearity $H(u) = |u|^{p-1}u - \int_{\Omega} |u|^{p-1}udx$ [26]. It’s also necessary to consider the case where the nonlinear source function grows much faster than the polynomial level. In this case, the nonlinearity of exponential type is considered as an optimal alternative to the polynomial source. In [30] Zhu et.al. investigated Equation (4) with $H$ as an exponential nonlinearity. By the elliptic theory, the authors showed the local existence and uniqueness of the solution. And once again, the potential well theory was applied to prove that when the initial energy is low, this solution exists globally. A sufficient condition for a blowing-up solution without any limit of initial energy was also provided. The logarithmic forms of $H$ were concerned in [8, 9].
In [8], Chen and Tian studied the subcritical and critical energy cases for (4) with $H(u) = u \log |u|$. When $I(u_0) < 0$, the global existence and uniqueness of solutions were proved. In contrary, if $I(u_0) < 0$, unlike the polynomial cases mentioned above, the authors showed that the solution doesn’t blow up in finite time but at $+\infty$. The same topic was also concerned for an initial-boundary value problem for infinitely degenerate semilinear pseudo-parabolic equations with logarithmic nonlinearity in [9]. The global existence and the asymptotic behavior of the solutions were discussed for the cases of subcritical/critical initial energy, and the infinite time blow up was also showed. The existence of the solution was established to show the instability further.

The above work is a part of the representative work on the pseudo-parabolic equations, but in fact, there are many results on such mathematical models, and we obviously cannot list them all. Taking a glimpse of the whole leopard, we can still see that this type of model is widely and intensively concerned not only because of its physical and practical application background, but also because of its interesting mathematical phenomena. In particular, the above work has given us such an inspiration: the nonlinearity of the model dramatically affects the properties and behaviors of the solution. Roughly speaking, a weaker nonlinearity will ensure the global-in-time existence of the solution, while a stronger nonlinearity will cause the dynamic properties of the solution to be differentiated due to the scale of the initial data. We hope to systematically describe this phenomenon in a unified work, which is the original intention of this work. To achieve this goal, we have adopted the strategy of gradually strengthening the degree of the nonlinearity, that is, discussing the linear advective form inhomogeneous terms, and the inhomogeneous terms strengthened to the squared nonlinearity, the power-type nonlinearity, as well as the exponential nonlinear terms.

At the same time, we further expand our understanding of this type of problem from the perspective of the equation structure, that is, we consider the time-fractional derivative in the sense of Caputo for Equation (1). In recent decades, the fractional calculus has been proved that it is useful in application to many fields of science such as the study of Brownian motion [27], chemical stimuluses of an organism with memory effects [16], and waves in linear viscoelastic media [20]. The greatest motivation for considering Problem (1)-(3) comes from the fact that due to the non-local nature of the fractional differential-integral operators, the time-fractional pseudo-parabolic equation has not only provided both new insights into physical models but been also very mathematically interesting. In view of one of the the original ideas for the application of Equation (4), that is, the study of some fluid flows, it is natural to propose the fractional derivative in the investigation of some certain viscous fluids. In fact, many modified versions of (4) have been proposed in [3] and references given there. Because the fractional derivative will help us to capture the viscoelastic properties of the flow, the time-fractional pseudo-parabolic equations are useful for describing the behavior of some non-Newtonian fluids. In mathematical aspect, it is a hot stream to consider the time-fractional version of the classical mathematical models including the parabolic type equation [25], the time-space fractional Shrödinger equation with polynomial type nonlinearity [14], the time-fractional Navier-Stokes equations (FNS) [13], and also the time-fractional pseudo-parabolic equations [24]. Surprisingly in [13], it was shown that the order of time-fractional derivative influences the regularity not only in time variable but also in the spatial variable. In [24], a first attempt to explore the influence of the
degree of nonlinearity on the dynamic behavior of the solution was conducted by considering the logarithmic nonlinearity and globally Lipschitz nonlinearity. All of the above achievements in this direction pushed us to consider not only the influence of the degree of the nonlinearities but also the order of the time-fractional derivative on the behavior of the solution to the time-fractional pseudo-parabolic equations with four distinguished types of nonlinearities, in which the degree of nonlinearities increase gradually.

1.2. Structure of the work. To provide an overview of the present paper, we give the outline of the work, including some summaries of the mathematical contributions.

- Section 2 provides some basic knowledge about function spaces, special operators, the mild formula, and some linear estimates.
- In Section 3, we consider the source term $H$ as a gradient type. To prove the well-posedness results for $H$ in the advective form $H(u) = (\eta \cdot \nabla)u$, we apply the technique for singular integration developed in [11] to overcome the difficulties arising in finding the proper functional space and proving the convergence in such space for the constructed Picard sequence, without restriction on the time interval and the smallness assumptions on the initial data. Also, in this section, for $H(u) = (\eta \cdot \nabla)u + \nabla \cdot F(u)$, where $F$ is a vector field, we firstly prove the local-in-time existence, then this solution is extended to the one in some larger time interval. As a consequence, the mild solution is shown to be the global-in-time solution or finite time blow up solution.
- Section 4 states our investigation of Problem (1)-(3) with the polynomial nonlinearity and the exponential nonlinearity. Since the nonlinear estimates for $H$ in these cases require much more strict conditions for the parameters and dimension $d$, some smallness assumptions for $u_0$ are necessary for getting the global well-posedness. Also, for the power-type source term $H(u) = |u|^{p-1}u$, $p > 2$, the use of fractional Hilbert spaces and Sobolev embeddings is very beneficial. The nonlinearity of exponential type is even more difficult for us to control because of its rapid growth. Fortunately, by using the Orlicz space, we overcome this challenge and obtain the desired results.

2. Preliminaries. Entire this work, we always use the letter $I$ and the notation $\mathcal{T}$ to abbreviate, respectively, an interval of time and a time point in $[0, \infty)$.

2.1. Basic materials. We first establish some functional space concepts. Suppose that $X$ is a Banach space associated with the norm $\|\cdot\|_X$. We use the notation $C(I \to X)$ to denote the space of all continuous functions $w : I \to X$. If $I$ is compact, $C(I \to X)$ is a Banach space with the norm

$$\|w\|_{C(I \to X)} := \sup_{t \in I} \|w(t)\|_X < \infty.$$ 

Suppose that $\Xi : \mathbb{R} \to \mathbb{R}^+$ is a Young function, i.e., a convex function which is even, continuous on $[0, \infty)$, and satisfies

$$\lim_{z \to \infty} \frac{\Xi(z)}{z} = \infty \quad \text{and} \quad \lim_{z \to 0} \frac{\Xi(z)}{z} = 0.$$
Then, we define the Orlicz space \( L^\Xi(\Omega) \) as the space of all measurable functions \( w(x) \) such that
\[
\int_\Omega \Xi\left( \left| \frac{w(x)}{\kappa} \right| \right) dx < \infty \quad \text{for some } \kappa > 0.
\]
The space \( L^\Xi(\Omega) \) is a Banach space with respect to the Luxemburg norm
\[
\| w \|_{L^\Xi(\Omega)} := \inf \left\{ \kappa \in \mathbb{R} \mid \kappa > 0, \int_\Omega \Xi\left( \left| \frac{w(x)}{\kappa} \right| \right) dx \leq 1 \right\}.
\]

**Remark 1.** Note that, we can use the framework of Orlicz space to cover the definitions of some well-known Lebesgue spaces as below

1. Assume that \( \Xi(z) = z^p \) with \( 1 < p < \infty \). Then, \( L^\Xi(\Omega) \) is the usual Lebesgue space \( L^p(\Omega) \).
2. Let \( \Xi \) be a Young function whose value equals to 0 on \([-1, 1]\) and is not be bound outside \([-1, 1]\). Then, \( L^\Xi(\Omega) \) is the usual Lebesgue space \( L^\infty(\Omega) \).

**Remark 2.** From now on, we always use the symbols \( L^\Xi(\Omega) \) to indicate the Orlicz space with the Young function \( \Xi(z) = e^{z^2} - 1 \).

For the purpose of deriving main estimates for Problem (1)-(3) involving the exponential nonlinearity, we introduce the following lemma which can be found in [15, Lemma 2.1] about the relationship between the space \( L^\Xi(\Omega) \) and some usual Lebesgue spaces. For readers’ convenience, we only briefly present its proof.

**Lemma 2.1.** For any \( p \in [2, \infty) \), we have the estimate
\[
\| w \|_{L^p(\Omega)} \leq \left( \Gamma\left( \frac{p}{2} + 1 \right) \right)^{\frac{1}{p}} \| w \|_{L^\Xi(\Omega)}.
\]

**Proof.** For \( q \geq 1 \), using the properties of the exponential function and the Gamma function (see Definition 5.5), we get
\[
\frac{z^q}{\Gamma(q+1)} + 1 < e^z.
\]
Then, from the fact that \( \left\{ \kappa \in \mathbb{R} \mid \kappa > 0, \int_\Omega \Xi\left( |w(x)|\kappa^{-1} \right) dx \leq 1 \right\} = \left[ \| w \|_{L^\Xi(\Omega)}, \infty \right) \) and the monotone convergence theorem, we obtain
\[
\int_\Omega \left( \frac{|w(x)|}{\| w \|_{L^\Xi(\Omega)}} \right)^{2q} \frac{1}{\Gamma(q+1)} dx \leq \int_\Omega \Xi\left( \frac{|w(x)|}{\| w \|_{L^\Xi(\Omega)}} \right) dx \leq 1.
\]
Then, by choosing \( q = \frac{p}{2} \), we obtain the desired result. \( \square \)

We note that the scalar product on \( L^2(\Omega) \) between \( w, v \in L^2(\Omega) \) is given by
\[
\int_\Omega w(z)v(z)dz.
\]
Then, from the spectral decomposition of the operator \( A \), for any \( \nu \geq 0 \), we can define the following Hilbert scale space
\[
D^\nu(\Omega) := \left\{ w \in L^2(\Omega) \mid \| w \|^2_{D^\nu(\Omega)} = \sum_{k=1}^\infty \theta_k^\nu \left( \int_\Omega w(z)\phi_k(z)dz \right)^2 < \infty \right\}.
\]
In view of this setting, we define the space $D^{-\nu}(\Omega)$ by the dual space of $D^{\nu}(\Omega)$ with respect to the pairing $\langle \cdot, \cdot \rangle_*$, which is a Banach space equipped with the norm
\[
\|w\|_{D^{-\nu}(\Omega)} = \left( \sum_{k=1}^{\infty} \theta_k^{-\nu} \langle w, \phi_k \rangle_*^2 \right)^{\frac{1}{2}}.
\]

**Remark 3.** If $w \in L^2(\Omega)$ and $v \in D^{\nu}(\Omega)$, we have
\[
\langle w, v \rangle_* = \langle w, v \rangle := \int_{\Omega} w(z)v(z)dz.
\]

**Remark 4.** Based on [5, Section 3], we see that $D^{\nu}(\Omega)$ coincides with the Sobolev-Slobodecki space $W^{\nu,2}_0(\Omega)$ when $\nu \in (1/2, 1]$.

We next introduce the definition of the Mittag-Leffler function with two parameters, which is the generalization of the classical Mittag-Leffler function, as follows
\[
E_{\alpha,\zeta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \zeta)},
\]
where $\alpha$ is a positive real number and $\zeta$ is a complex constant. When $\zeta = 1$, we use the symbol $E_{\alpha}$ instead of $E_{\alpha,1}$, and refer to [20, Section 1] for the following equality
\[
E_{\alpha}(-z) := \int_0^\infty e^{-zr} W_\alpha(r)dr,
\]
where $W_\alpha$ is the M-Wright type function. Also, the following lemma is very useful to control the values of the Mittag-Leffler function.

**Lemma 2.2.** ([20, Theorem 1.6]) Let $\alpha \in (0,1)$, $\zeta$ be a real constant, and $\lambda \in \left(\frac{\pi \alpha}{2}, \pi \alpha\right)$. Then, there exists a positive constant $M$ such that
\[
\left|E_{\alpha,\zeta}(z)\right| \leq \frac{M}{1 + |z|},
\]
whenever $|z| \geq 0$ and $\lambda \leq |\arg(z)| \leq \pi$.

**Lemma 2.3.** ([24, Section 2]) Let $\alpha \in (0,1)$ and $a, t \in (0, \infty)$. Then, for any $a > 0$, we have
\[
\partial_t \left( E_{\alpha}(-at^\alpha) \right) = -at^{\alpha-1}E_{\alpha,a}(-at^\alpha)
\]
and
\[
\partial_t \left( t^\alpha E_{\alpha,a}(-at^\alpha) \right) = t^{\alpha-2}E_{\alpha,a-1}(-at^\alpha).
\]

### 2.2. Mild solution

For purpose of formulating a mild solution of Problem (1)-(3), the following lemma on the Laplace transform of the Caputo derivative operator plays an important role.

**Lemma 2.4.** Suppose that the Laplace transform and the Caputo derivative of order $\alpha$ of a function $w$ exist. Then, the equation below holds
\[
\hat{\partial_t^\alpha w}(s) = \mathcal{L}\{\partial_t^\alpha w\}(s) = s^\alpha \tilde{w}(s) - s^{\alpha-1}w(0),
\]
where $\mathcal{L}$ stands for the Laplace transform operator.
Integrating the both sides of Equation (1) with an arbitrary function \( \phi_k \) and then applying the Laplace transform, we have

\[
\mathcal{L}^{-1} \int_{\Omega} [I + \mathcal{A}] \tilde{u}(z, s) \phi_k(z) \, dz = \mathcal{L}^{-1} \int_{\Omega} \mathcal{A} \tilde{u}(z, s) \phi_k(z) \, dz
\]

This equation is equivalent to

\[
\int_{\Omega} \tilde{u}(z, s) \phi_k(z) \, dz = \frac{s^{-1}(1 + \theta_k)}{s^\alpha(1 + \theta_k) + \theta_k} \int_{\Omega} u_0(z) \phi_k(z) \, dz
\]

Then by the inverse Laplace transform, we obtain

\[
\int_{\Omega} u(z, t) \phi_k(z) \, dz = \int_{\Omega} E_\alpha \left( -\frac{\theta_k t^\alpha}{1 + \theta_k} \right) u_0(z) \phi_k(z) \, dz
\]

For purpose of simplifying notations, we set

\[
G_1(x, z, t) = \sum_{k=1}^{\infty} E_\alpha \left( -\frac{\theta_k t^\alpha}{1 + \theta_k} \right) \phi_k(x) \phi_k(z)
\]

and

\[
G_2(x, z, t) = \sum_{k=1}^{\infty} \frac{t^{\alpha-1}}{1 + \theta_k} E_{\alpha, \alpha} \left( -\frac{\theta_k t^\alpha}{1 + \theta_k} \right) \phi_k(x) \phi_k(z).
\]

For purpose of simplifying notations, we set

\[
\mathcal{S}(t) w(x) := \int_{\Omega} G_1(x, z, t) w(z) \, dz \quad \text{and} \quad \mathcal{R}(t) w(x) := \int_{\Omega} G_2(x, z, t) w(z) \, dz.
\]

Then, we can rewrite the formula for the mild solution in the following way

\[
u(x, t) = \mathcal{S}(t) u_0(x) + \int_0^t \mathcal{R}(t - \tau) H(u(x, \tau)) \, d\tau.
\]

From the standpoint of this formulation concept, we derive some fundamental linear estimate through the following lemma

**Lemma 2.5.** Let \( \alpha \in (0, 1) \), \( \nu, \mu \in [0, 1] \) and \( \nu^* \in [0, 2] \). Assume that \( w \in D^1(\Omega) \). Then, we can find positive constants \( C_1 \) and \( C_2 \) such that

\[
(i) \quad \| \mathcal{S}(t) \|_{\mathcal{L}(D^\nu(\Omega))} \leq C_1 t^{-\alpha \mu};
\]

\[
(ii) \quad \| \mathcal{R}(t) \|_{\mathcal{L}(D^{\nu - \nu^*}(\Omega), D^\nu(\Omega))} \leq C_2 t^{\alpha - 1}.
\]
Proof. (i) We consider three different cases of $\mu$ as follows to prove the conclusion (i).

**Case 1.** $\mu = 0$. For any $w = \sum_{k=1}^{\infty} \langle w, \phi_k \rangle \phi_k \in L^2(\Omega)$, we recall the following Parseval’s identity as below
\[
\|w\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} \langle w, \phi_k \rangle^2 .
\]

The following inequality follows immediately from the above formula and Lemma 2.2
\[
\| \mathcal{S}(t)w \|_{D^{\nu}(\Omega)}^2 = \sum_{k=1}^{\infty} \theta_k^{2\nu} \left( \frac{E_\alpha \left( -\theta_k t^\alpha \frac{1}{1+\theta_k} \right)}{1 + \theta_k t^{\alpha}} \right)^2 \langle w, \phi_k \rangle^2 
\leq \sum_{k=1}^{\infty} \theta_k^{2\nu} \left( \frac{M}{1 + \theta_k t^{\alpha}} \right)^2 \langle w, \phi_k \rangle^2 
\leq M^2 \|w\|_{D^{\nu}(\Omega)}^2.
\]

**Case 2.** $\mu = 1$. Then, for a given $w$ in $D^{\nu}(\Omega)$, Lemma 2.2 and Parseval’s identity show us
\[
\| \mathcal{S}(t)w \|_{D^{\nu}(\Omega)}^2 = \sum_{k=1}^{\infty} \theta_k^{2\nu} \left( \frac{E_\alpha \left( -\theta_k t^\alpha \frac{1}{1+\theta_k} \right)}{1 + \theta_k t^{\alpha}} \right)^2 \langle w, \phi_k \rangle^2 
\leq 2M^2 \sum_{k=1}^{\infty} \left( \theta_k^{2\nu} \left( \frac{1 + \theta_k^2}{\theta_k^{2\nu}} \right) \right)^2 \langle w, \phi_k \rangle^2 
\leq 2M^2 \left( 1 + \frac{1}{\theta_1^{2\nu}} \right) t^{-2\alpha} \|u_0\|_{D^{\nu}(\Omega)} .
\]

**Case 3.** $0 < \mu < 1$. Given $w \in D^{\nu}(\Omega)$, by analogous arguments as in above, we get
\[
\| \mathcal{S}(t)w \|_{D^{\nu}(\Omega)}^2 = \sum_{k=1}^{\infty} \theta_k^{2\nu} \left( E_\alpha \left( -\theta_k t^\alpha \frac{1}{1+\theta_k} \right) \right)^2 \langle w, \phi_k \rangle^2 .
\]

Thanks to the relationship between the Mittag-Leffler function and the M-Wright type function, the equality (5) becomes
\[
\| \mathcal{S}(t)w \|_{D^{\nu}(\Omega)}^2 = \sum_{k=1}^{\infty} \theta_k^{2\nu} \left( \int_0^\infty \exp \left( -r \theta_k t^\alpha \frac{1}{1+\theta_k} \right) W_\alpha(r)dr \right)^2 \langle w, \phi_k \rangle^2 
\leq \sum_{k=1}^{\infty} C^{2\nu} \left( \frac{\theta_k^{2\nu} \left( 1 + \theta_k^2 \right)^2}{\theta_k^{2\nu}} \right) \left( \int_0^\infty \theta_k^{2\nu} \left( \frac{1}{\theta_k^{2\nu}} \right) \right)^2 \langle w, \phi_k \rangle^2 ,
\]
wherein, we have used the fundamental inequality $e^{-z} \leq C \nu z^{-\nu}, \nu \in (0, 1), C_\nu > 0$. Then, Lemma 5.2 implies
\[
\| \mathcal{S}(t)w \|_{D^{\nu}(\Omega)} \leq \frac{2^{2\nu-1} C_\nu \left( \theta_1^{2\nu} + 1 \right)^\nu \Gamma(1 - \mu)}{\Gamma(1 - \alpha \mu)} \|w\|_{D^{\nu}(\Omega)} .
\]
(ii) We consider only the case $\nu^* = 1$, the other cases are similar. For given $w \in D^{\nu-1}(\Omega)$, we observe that
\[
\|\mathcal{R}(t)w\|_{D^{\nu}(\Omega)} \leq \sum_{k=1}^{\infty} \theta_k^{\nu} \left( E_{\alpha,\nu} \left( \frac{-\theta_k \tau}{1+\theta_k} \right) \right)^2 \langle w, \phi_k \rangle.
\]
From this, we can easily obtain the positive constant $C_2$ via the inequality below
\[
\|\mathcal{R}(t)w\|_{D^{\nu}(\Omega)} \leq M \frac{t^{\alpha - 1}}{\theta_k^2} \left( \sum_{k=1}^{\infty} \theta_k^{\nu-1} \langle w, \phi_k \rangle^2 \right)^{\frac{1}{2}},
\]
noting that we have applied Lemma 2.2 to get
\[
E_{\alpha,\nu} \left( \frac{-\theta_k \tau}{1+\theta_k} \right) \leq M \frac{|\eta|}{1 + |\eta|}. \]

The proof is completed.

3. The generalized BBM equation. In this section, we investigate the initial value problem involving the generalized BBM equation. More precisely, Equation (1) can be given in the exact form
\[
\partial_t^\alpha u - \partial_t^\alpha \Delta u - b \Delta u + (\eta \cdot \nabla) u + \nabla \cdot F(u) = 0,
\]
whereby, $\eta$ is a $d$-dimensional constant vector and $F$ is a $d$-dimensional vector field.

3.1. The time-fractional pseudo-parabolic equation with advection term. Throughout the current subsection, we assume that $F$ is the 0-vector in $H(u) = (\eta \cdot \nabla) u + \nabla \cdot F(u)$. Then, by the Cauchy–Schwarz inequality, for $u \in D^1(\Omega)$ we have
\[
\int_{\Omega} \left( \sum_{j=1}^{d} \eta_j u_{x_j}(x) \right)^2 \, dx \leq |\eta|^2 \int_{\Omega} \left( \sum_{j=1}^{d} u^2_{x_j}(x) \right) \, dx = |\eta|^2 \int_{\Omega} |\nabla u(x)|^2 \, dx.
\]

It means that we can find a positive constant $\mathcal{C}_1$ independent of $w, v \in D^1(\Omega)$ such that
\[
\|H(w) - H(v)\|_{L^2(\Omega)} \leq \mathcal{C}_1 \|w - v\|_{D^1(\Omega)}.
\]

Our main principle is the successive approximation in some reasonable Banach spaces. To this end, for any $a, \sigma > 0$ we denote by $Y = Y(a, \sigma)$, the space of all functions $w \in C \left( I \rightarrow D^1(\Omega) \right)$ satisfying
- $\sup_{t \in I \setminus \{0\}} t^a e^{-\sigma t} \|w(t)\|_{D^1(\Omega)} < \infty$,
- $w(x, 0) = u_0(x)$,

and construct a sequence $\{w_n\}_{n=1}^{\infty}$ inductively in the following way
\[
w_1(x, t) := \mathcal{S}(t)u_0(x),
\]
\[
w_{n+1}(x, t) := \mathcal{S}(t)u_0(x) + \int_0^t \mathcal{R}(t - \tau)H(w_n(x, \tau)) \, d\tau.
\]

Besides, we introduce some necessary lemmas to help us present the main points of the proof more clearly.
Lemma 3.1. ([11, Lemma 8]) Let $a, \sigma > 0$ and $m_1, m_2 > -1$ such that $m_1 + m_2 > -1$. Then,
\[
\lim_{\sigma \to \infty} \left( \sup_{t \in \mathbb{R} \setminus \{0\}} t^a \int_0^1 s^{m_1(1 - s)} t^{m_2} e^{-\sigma(t-s)} ds \right) = 0.
\]

Lemma 3.2. Presume that $u_0$ belongs to $D^1(\Omega)$ and $I = [0, \mathcal{T}]$. Then, $\{w_n\}_{n=1}^\infty$ forms a subset of $C(I \to D^1(\Omega))$.

Proof. Firstly, for any $t \geq 0$ and $\varepsilon > 0$, Lemma 2.3 makes the following formula hold
\[
\left| E_\alpha \left( -\frac{\theta_k(t+\varepsilon)^\alpha}{1+\theta_k} \right) - E_\alpha \left( -\frac{\theta_k t^\alpha}{1+\theta_k} \right) \right| = \left| \int_t^{t+\varepsilon} \frac{\theta_k r^{\alpha-1}}{1+\theta_k} E_{\alpha,\alpha} \left( -\frac{\theta_k r^\alpha}{1+\theta_k} \right) dr \right|.
\]
Then, for $u_0 \in D^1(\Omega)$, one deduces that
\[
\|w_1(t+\varepsilon) - w_1(t)\|_{D^1(\Omega)}^2 = \sum_{k=1}^{\infty} \theta_k \left| E_\alpha \left( -\frac{\theta_k(t+\varepsilon)^\alpha}{1+\theta_k} \right) - E_\alpha \left( -\frac{\theta_k t^\alpha}{1+\theta_k} \right) \right|^2 \langle u_0, \phi_k \rangle^2
\leq \sum_{k=1}^{\infty} \left| \int_t^{t+\varepsilon} \frac{\theta_k r^{\alpha-1}}{1+\theta_k} E_{\alpha,\alpha} \left( -\frac{\theta_k r^\alpha}{1+\theta_k} \right) dr \right|^2 \langle u_0, \phi_k \rangle^2
\leq \frac{M^2(t-\varepsilon)^\alpha - t^{\alpha}}{\alpha^2} \|u_0\|_{D^1(\Omega)}^2.
\]
One may obtain immediately for any $t \geq 0$ that
\[
\lim_{\varepsilon \to 0} \|w_1(t+\varepsilon) - w_1(t)\|_{D^1(\Omega)} = 0,
\]
which means that $w_1$ is continuous on $I$ with respect to the $D^1(\Omega)$ norm. On the other hand, for every $\varepsilon > 0$, we get
\[
\|w_2(\varepsilon) - w_1(\varepsilon)\|_{D^1(\Omega)}
\leq \int_0^{\varepsilon} \|\mathcal{A}(\varepsilon - \tau)H(w_n(x, \tau))\|_{D^1(\Omega)} d\tau
\leq \int_0^{\varepsilon} \left( \sum_{k=1}^{\infty} \frac{\theta_k(\varepsilon - \tau)^{2(\alpha-1)}}{(1+\theta_k)^2} E_{\alpha,\alpha} \left( \frac{\theta_k(\varepsilon - \tau)^\alpha}{1+\theta_k} \right) \langle H(w_1(\tau)), \phi_k \rangle \right)^2 d\tau.
\]
Using Lemma 2.5(ii) and the fact that $H(0) = 0$, we deduce
\[
\|w_2(\varepsilon) - w_1(\varepsilon)\|_{D^1(\Omega)} \leq C_2 \int_0^{\varepsilon} (\varepsilon - \tau)^{\alpha-1} \|H(w_1(\tau))\|_{L^2(\Omega)} d\tau
\leq C_1 C_2 \left( \int_0^{\varepsilon} (\varepsilon - \tau)^{\alpha-1} d\tau \right) \left( \sup_{t \in I} \|w_1(t)\|_{D^1(\Omega)} \right).
\]
This result along with (8) help us conclude that $w_2$ is continuous at $t = 0$ with respect to the $D^1(\Omega)$ norm. We now turn to consider the case when $t$ is positive.
For this purpose, we proceed with two following claims, which help us make the proof more clear.

**Claim 1.** For any \( t, \varepsilon > 0, k \in \mathbb{N} \), and \( w_1 \in C \left( I \to D^1(\Omega) \right) \), we can apply Lemma 2.3 as follows

\[
\mathcal{E}(k, t, \tau, \varepsilon) := \left| (t + \varepsilon - \tau)^{\alpha-1} E_{\alpha, \alpha} \left( -\frac{-\theta_k(t + \varepsilon - \tau)^{\alpha}}{1 + \theta_k} \right) - (t - \tau)^{\alpha-1} E_{\alpha, \alpha} \left( -\frac{-\theta_k(t - \tau)^{\alpha}}{1 + \theta_k} \right) \right|
\]

\[
= \left| \int_{t-\tau}^{t+\varepsilon-\tau} \frac{\theta_k^{\alpha-2}}{1 + \theta_k} E_{\alpha, \alpha-1} \left( -\frac{-\theta_k^{\alpha}}{1 + \theta_k} \right) \, d\tau \right|.
\]

**Lemma 2.2** implies that, for any \( \tau \in (0, t) \),

\[
\mathcal{E}(k, t, \tau, \varepsilon) \leq M(1 - \alpha)^{-1} \left[ (t + \varepsilon - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1} \right]. \tag{9}
\]

In view of this result, we infer the following estimate

\[
\int_0^t \left( \sum_{k=1}^\infty \frac{\theta_k \mathcal{E}(k, t, \tau, \varepsilon)}{(1 + \theta_k)^2} \langle H(w_1(\tau)), \phi_k \rangle^2 \right)^{\frac{1}{2}} \, d\tau \leq \frac{M\mathcal{E}_1}{(1 - \alpha)^{\frac{1}{2}}} \left( \int_0^t \left| (t + \varepsilon - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1} \right| \, d\tau \right) \left( \sup_{t \in I} \|w_1(t)\|_{D^1(\Omega)} \right).
\]

We notice that, for any \( \tau \in (0, t) \), we have

\[
\lim_{\varepsilon \to 0} \left( (t - \tau)^{\alpha-1} - (t + \varepsilon - \tau)^{\alpha-1} \right) = 0,
\]

\[
(t + \varepsilon - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1} \leq 2(t - \tau)^{\alpha-1}.
\]

Then, applying the dominated convergence theorem, the following limit holds

\[
\lim_{\varepsilon \to 0} \left( \int_0^t \left( \sum_{k=1}^\infty \frac{\theta_k \mathcal{E}(k, t, \tau, \varepsilon)}{(1 + \theta_k)^2} \langle H(w_1(\tau)), \phi_k \rangle^2 \right)^{\frac{1}{2}} \, d\tau \right) = 0.
\]

**Claim 2.** Using again Lemma 2.5(ii) and the fact that \( w_1 \in C \left( I \to D^1(\Omega) \right) \), we obtain

\[
\int_t^{t+\varepsilon} \| \mathcal{A}(t + \varepsilon - \tau) H(w_1(x, \tau)) \|_{D^1(\Omega)} \, d\tau \leq C_2 \int_t^{t+\varepsilon} (t + \varepsilon - \tau)^{\alpha-1} \|H(w_1(\tau))\|_{L^2(\Omega)} \, d\tau \leq \mathcal{E}_1 C_2^{\alpha-1} \varepsilon \left( \sup_{t \in I} \|w_1(t)\|_{D^1(\Omega)} \right).
\]

Thus, one has

\[
\lim_{\varepsilon \to 0} \left( \int_t^{t+\varepsilon} \| \mathcal{A}(t + \varepsilon - \tau) H(w_1(x, \tau)) \|_{D^1(\Omega)} \, d\tau \right) = 0.
\]

Now, the results of **Claim 1** and **Claim 2** along with the fact that \( w_1 \in C \left( I \to D^1(\Omega) \right) \) yield that \( w_2 \) is continuous at \( t > 0 \). Note that, we have already proved the continuity at 0 of \( w_2 \). Therefore, we can conclude that \( w_2 \in C \left( I \to D^1(\Omega) \right) \). Based on this result, the induction arguments show that \( w_{n+1} \) belongs to \( C \left( I \to D^1(\Omega) \right) \) whenever \( w_n \) is in \( C \left( I \to D^1(\Omega) \right) \), for every \( n \in \mathbb{N} \). The proof is completed. \( \square \)
Theorem 3.3. (Global existence) Assume that \( u_0 \) is an element of the space \( D^1(\Omega) \) and \( I = [0, T] \). Then, Problem (1)-(3) possesses at least one mild solution \( u \) in \( C(I \rightarrow D^1(\Omega)) \).

Proof. We first show that if \( u_0 \) belongs to \( D^1(\Omega) \) and \( I = [0, T] \), \( \{w_n\}_{n=1}^{\infty} \) is a subset of the space \( \mathcal{Y}(\alpha, \sigma) \), for some big-value \( \sigma \). If \( u_0 \in D^1(\Omega) \), Lemma 2.5(i) shows that

\[
\|w_1(t)\|_{D^1(\Omega)} = \|\mathcal{S}(t)u_0\|_{D^1(\Omega)} \leq C_1 t^{-\alpha} \|u_0\|_{D^1(\Omega)}.
\]

It thus follows that

\[
\sup_{t \in I \setminus \{0\}} t^\alpha e^{-\sigma t} \|w_1(t)\|_{D^1(\Omega)} < \infty. \tag{10}
\]

Next, we suppose that \( w_n \in \mathcal{Y}(\alpha, \sigma) \). Then, by using Lemma 2.5(ii), we have

\[
\begin{align*}
\|w_{n+1}(t) - w_1(t)\|_{D^1(\Omega)} & = \left\| \int_0^t \mathcal{A}(t - \tau)H(w_n(\tau))d\tau \right\|
\leq C_2 \int_0^t (t - \tau)^{\alpha - 1} \|H(w_n(\tau))\|_{L^2(\Omega)} d\tau
\leq \mathcal{C}_1 C_2 \int_0^t (t - \tau)^{\alpha - 1} \|w_n(\tau)\|_{D^1(\Omega)} d\tau.
\end{align*}
\]

Multiplying the both sides of the above by \( t^\alpha e^{-\sigma t}, t > 0 \), we obtain

\[
t^\alpha e^{-\sigma t} \|w_{n+1}(t) - w_1(t)\|_{D^1(\Omega)} \leq \mathcal{C}_1 C_2 Q(t, \alpha, \sigma) \left( \sup_{t \in I} t^\alpha e^{-\sigma t} \|w_n(t)\|_{D^1(\Omega)} \right), \tag{11}
\]

where we set

\[
Q(t, h, \sigma) := t^h \int_0^t (t - \tau)^{\alpha - 1} e^{-\sigma(t - \tau)} d\tau.
\]

By virtue of Lemma 3.1, (10), (11) and the triangle inequality, we deduce

\[
\sup_{t \in I \setminus \{0\}} t^\alpha e^{-\sigma t} \|w_{n+1}(t)\|_{D^1(\Omega)} < \infty \quad \text{for some large } \sigma.
\]

In addition, the use of Lemma 3.2 yields that \( w_n \) is continuous on \( I \), for every \( n \geq 1 \). It follows that \( w_{n+1} \in \mathcal{Y}(\alpha, \sigma) \). Furthermore, \( \{w_n\}_{n=1}^{\infty} \) can be proved to be a Cauchy sequence in \( \mathcal{Y}(\alpha, \sigma_0) \) for some \( \sigma_0 > 0 \). Let us verify this statement. Firstly, Lemma 3.1 ensures the existence of a sufficiently large constant \( \sigma_0 \) such that

\[
\sup_{t \in I \setminus \{0\}} Q(t, \alpha, \sigma_0) = \frac{3}{4\mathcal{C}_1 C_2}.
\]

Secondly, assume that \( w_{n-1}, w_n, w_{n+1} \in \mathcal{Y}(\alpha, \sigma_0), n \geq 2 \). Then, by the same arguments as above, we obtain

\[
\begin{align*}
\|w_{n+1}(t) - w_n(t)\|_{D^1(\Omega)} & \leq C_2 \int_0^t (t - \tau)^{\alpha - 1} \|H(w_n(\tau)) - H(w_{n-1}(\tau))\|_{L^2(\Omega)} d\tau \\
& \leq \mathcal{C}_1 C_2 \int_0^t (t - \tau)^{\alpha - 1} \|w_n(\tau) - w_{n-1}(\tau)\|_{D^1(\Omega)} d\tau.
\end{align*}
\]
It thus implies that
\[ t^\alpha e^{-\sigma_0 t} \|w_{n+1}(t) - w_n(t)\|_{D^1(\Omega)} \leq \frac{3}{4} \sup_{t \in I_1(0)} \left( t^\alpha e^{-\sigma_0 t} \|w_n(t) - w_{n-1}(t)\|_{D^1(\Omega)} \right). \]

From the standpoint of this result, for any \( n_2 > n_1 \geq 2 \), we find
\[ t^\alpha e^{-\sigma_0 t} \|w_{n_2}(t) - w_{n_1}(t)\|_{D^1(\Omega)} \leq \sum_{n=n_1}^{n_2} \left( \frac{3}{4} \right)^{n-1} \sup_{t \in I_1(0)} \left( t^\alpha e^{-\sigma_0 t} \|w_2(t) - w_1(t)\|_{D^1(\Omega)} \right). \]

(12)

Then, the use of geometric series helps us to declare that \( \{w_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathcal{Y}(\alpha, \sigma_0) \). Now, the completeness of \( \mathcal{Y}(\alpha, \sigma_0) \) provides a function \( u \) which is the limit of the sequence \( \{w_n\}_{n=1}^\infty \). Applying the dominated Lebesgue theorem, we have
\[ u = \lim_{n \to \infty} \left( \int_{\Omega} G_1(x, z, t) u_0(z) dz + \int_0^t \int_{\Omega} G_2(x, z, t - \tau) H(w_n(z, \tau)) d\tau dz \right) \]
\[ = \int_{\Omega} G_1(x, z, t) u_0(z) dz + \int_0^t \int_{\Omega} G_2(x, z, t - \tau) H(u(z, \tau)) d\tau dz. \]

Then, we can conclude that \( u \) is the mild solution to Problem (1)-(3). \( \square \)

**Remark 5.** The main feature of using the space \( \mathcal{Y} \) is to get us to the result (11). Then, Lemma 3.1 can be used to show the contractive arguments without any restrictions on \( \mathcal{S} \) or \( u_0 \).

**Theorem 3.4.** (Uniqueness and stability) Suppose that Problem (1)-(3) admits a mild solution \( u \) in \( \mathcal{Y}(\alpha, \sigma_0) \). Then, this solution is globally unique. Furthermore, it depends continuously on the initial data.

**Proof.** The uniqueness and the stability results of the mild solution can be obtained by Gönwall’s inequality. Indeed, assume that \( u, w \) are the milks solution of Problem (1)-(3) in \( \mathcal{Y}(\alpha, \sigma_0) \) corresponding to the initial data \( u_0, w_0 \in D^1(\Omega) \). Then, we have
\[ \|u(t) - w(t)\|_{D^1(\Omega)} \]
\[ \leq C_1 t^{-\alpha} \|u_0 - w_0\|_{D^1(\Omega)} \]
\[ + C_1 C_2 \int_0^t (t - \tau)^{\alpha - 1} \|u(\tau) - w(\tau)\|_{D^1(\Omega)} d\tau. \]

(13)

Since \( u, w \) are in the space \( \mathcal{Y}(\alpha, \sigma_0) \), then, Hölder’s inequality shows that
\[ \left( t^\alpha e^{-\sigma_0 t} \int_0^t (t - \tau)^{\alpha - 1} \|u(\tau) - w(\tau)\|_{D^1(\Omega)} d\tau \right)^2 \]
\[ \leq Q(t, 2\alpha, 2\sigma_0) \int_0^t (t - \tau)^{\alpha - 1} \left( \tau^\alpha e^{-\sigma_0 \tau} \|u(\tau) - v(\tau)\|_{D^1(\Omega)} \right)^2 d\tau. \]

Accordingly, the inequality (13) becomes
\[ \left( t^\alpha e^{-\sigma_0 t} \|u(t) - w(t)\|_{D^1(\Omega)} \right)^2 \]
\[ \leq \left( C_1 \|u_0 - w_0\|_{D^1(\Omega)} \right)^2 + \frac{3}{4} \int_0^t (t - \tau)^{\alpha - 1} \left( \tau^\alpha e^{-\sigma_0 \tau} \|u(\tau) - v(\tau)\|_{D^1(\Omega)} \right)^2 d\tau \]
We now apply the fractional Grönwall’s inequality (see Lemm 5.4) to get the following estimate

\[ t^\alpha e^{-\sigma t} \|u(t) - w(t)\|_{D^1(\Omega)} \leq C_1 \|u_0 - w_0\|_{D^1(\Omega)} E_{\alpha,1} \left( \frac{3\Gamma(\alpha)t^\alpha}{4} \right). \]

The proof is completed.

3.2. The non-trivial case of the vector field. In this subsection, we assume that the vector field \( F \) in \( H(u) = (\eta \nabla)u + \nabla F(u) \), is given by \( F(u) = (u^2, u^2, \ldots, u^2) \) and \( d \in \{2, 3, 4\} \). Then, we can check that \( H \) is a local Lipschitzian from \( D^1(\Omega) \) to \( D^{-1}(\Omega) \). To verify this statement, we first recall the embedding \( L^{\frac{d+2}{d-2}}(\Omega) \to D^{-1}(\Omega) \) in Lemma 5.3, which along with the Cauchy–Schwarz help us to find a positive constant \( C \) such that

\[ \|\nabla \cdot F(u)\|_{D^{-1}(\Omega)} \leq C \|\nabla u\|_{L^{\frac{d+2}{d-2}}(\Omega)} \leq C \|u\|_{L^1(\Omega)} \|\nabla u\|_{L^2(\Omega)}, \]

where the Hölder inequality with two conjugate indices \( \frac{d+2}{d} \) and \( \frac{d+2}{d-2} \) has been applied to get the second inequality. We now can easily check that

\[
\begin{cases}
D^1(\Omega) \to L^d(\Omega), & \text{if } d = 2, \\
D^1(\Omega) \to L^{\frac{d+2}{d-2}}(\Omega) \to L^d(\Omega), & \text{if } d \in \{3, 4\}.
\end{cases}
\]

Therefore, there exists a positive constant \( C'(d) \) from the above observations satisfying

\[ \|\nabla u\|_{D^{-1}(\Omega)} \leq C'(d) \|u\|^2_{D^1(\Omega)}. \]

As a consequence of the above estimates, for any \( u, v \in D^1(\Omega) \), we can find that

\[ \|\nabla \cdot F(u) - \nabla \cdot F(v)\|_{D^{-1}(\Omega)} \leq C''(d) \left( \|u\|_{D^1(\Omega)} + \|v\|_{D^1(\Omega)} \right) \|u - v\|_{D^1(\Omega)}, \]

here, \( C''(d) \) is independent on \( u, v \). In addition, from (7) and the fact that \( L^2(\Omega) \to D^{-1}(\Omega) \), for any \( u, v \in D^1(\Omega) \), we can find a constant \( \overline{C}_1 \) independent of \( u, v \) such that the following estimate holds

\[ \|H(u) - H(v)\|_{D^{-1}(\Omega)} \leq \overline{C}_1 \left( \|u\|_{D^1(\Omega)} + \|v\|_{D^1(\Omega)} + 1 \right) \|u - v\|_{D^1(\Omega)} \]  (14)

The main aim of this part is to investigate the well-posed results that either the mild solution exists globally or blows up in finite maximal time.

**Theorem 3.5.** (Local existence) Let \( \alpha \in (0, 1) \) and \( I = [0, T] \) with reasonable choice for \( \mathcal{I} \). If \( u_0 \in D^1(\Omega) \), Problem (1)-(3) possesses a locally unique mild solution.

**Proof.** Thanks to the assumption \( u_0 \in D^1(\Omega) \), Lemma 2.5(i) implies

\[ \|\mathcal{I}(t)u_0\|_{D^1(\Omega)} \leq C_1 \|u_0\|_{D^1(\Omega)}. \]  (15)

For any \( w, v \in C(I \to D^1(\Omega)) \), from Lemma 2.5(ii) and inequality (14) we have

\[
\begin{align*}
&\int_0^t \|\mathcal{I}(t - \tau)\left( H(w(\tau)) - H(v(\tau)) \right)\|_{D^1(\Omega)} \, d\tau \\
\leq& \int_0^t (t - \tau)^{\alpha - 1} \|H(w(\tau)) - H(v(\tau))\|_{D^{-1}(\Omega)} \, d\tau \\
\leq& \overline{C}_1 C_2 \int_0^t (t - \tau)^{\alpha - 1} \left( \|w(\tau)\|_{D^1(\Omega)} + \|v(\tau)\|_{D^1(\Omega)} + 1 \right) \|w(\tau) - v(\tau)\|_{D^1(\Omega)} \, d\tau.
\end{align*}
\]
This implies that
\[
\int_0^t \| \mathcal{R}(t - \tau) \left( H(w(\tau)) - H(v(\tau)) \right) \|_{D^1(\Omega)} \, d\tau \\
\leq \varepsilon_1 C_2 \alpha^{-1} \mathcal{F}^n \left( \| w \|_{C(I \to D^1(\Omega))} + \| v \|_{C(I \to D^1(\Omega))} + 1 \right) \| w - v \|_{C(I \to D^1(\Omega))}. \quad (16)
\]

Then, if we consider the Picard sequence
\[
w_1(x, t) := \mathcal{F}(t)u_0(x), \\
w_{n+1}(x, t) := \mathcal{F}(t)u_0(x) + \int_0^t \mathcal{R}(t - \tau) H(w_n(x, \tau)) \, d\tau,
\]
Inequality (15) immediately shows that \( w_1 \in C(I \to D^1(\Omega)) \). Besides, for \( w_n \in C(I \to D^1(\Omega)) \), Estimate (16) gives us the following result
\[
\| w_{n+1}(\tau) - w_1(\tau) \|_{D^1(\Omega)} \\
\leq \int_0^t \| \mathcal{R}(t - \tau) H(w_n(\tau)) \|_{D^1(\Omega)} \, d\tau \\
\leq \varepsilon_1 C_2 \alpha^{-1} \mathcal{F}^n \left( \| w_n \|_{C(I \to D^1(\Omega))} + \| w_1 \|_{C(I \to D^1(\Omega))} \right),
\]
i.e.,
\[
\| w_{n+1} - w_1 \|_{C(I \to D^1(\Omega))} \leq \varepsilon_1 C_2 \alpha^{-1} \mathcal{F}^n \left( \| w_n \|_{C(I \to D^1(\Omega))} + \| w_1 \|_{C(I \to D^1(\Omega))} \right).
\]
Then, we can conclude that \( \{ w_n \} \in C(I \to D^1(\Omega)) \). Furthermore, if \( \mathcal{F} \) is small enough, Estimate (16) infers for \( w_{n-1}, w_n, w_{n+1} \in C(I \to D^1(\Omega)) \) that
\[
\| w_{n+1} - w_n \|_{C(I \to D^1(\Omega))} \leq \frac{3}{4} \| w_n - w_{n-1} \|_{C(I \to D^1(\Omega))}.
\]
It means \( \{ w_n \} \) is a Cauchy sequence in \( C(I \to D^1(\Omega)) \). Consequently, there exists a locally unique mild solution \( u \) of Problem (1)-(3) in \( C(I \to D^1(\Omega)) \).

**Lemma 3.6.** Let \( u_0 \) be in \( D^1(\Omega) \). Suppose that \( u \) is a unique mild solution to Problem (1)-(3) on \([0, \mathcal{F}]\). Then this solution can be extended to \([0, \mathcal{F} + T]\), for some \( T > 0 \).

**Proof.** Assume that \( u \) is the unique mild solution of Problem (1)-(3) on \([0, \mathcal{F}]\) and \( I = [0, \mathcal{F} + T] \), where \( T > 0 \) will be chosen later. For a constant \( R > 0 \), we consider the space below
\[
E := \left\{ w \in C(I \to D^1(\Omega)) \mid w(t, \cdot) = u(t, \cdot), \quad \forall t \in [0, \mathcal{F}], \quad \| w(t, \cdot) - u(T, \cdot) \|_{D^1(\Omega)} \leq R, \quad \forall t \in [\mathcal{F}, \mathcal{F} + T] \right\}.
\]
We aim to show that Problem (1)-(3) has a unique mild solution that belongs to \( E \), then it is the extension of \( u \) to \( I \). To this end, we take an element \( \bar{w} \) of \( E \) and define the sequence \( \{ w_n \} \) as
\[
w_1(x, t) := \mathcal{F}(t)u_0(x) + \int_0^t \mathcal{R}(t - \tau) H(\bar{w}(x, \tau)) \, d\tau,
\]
\[
w_{n+1}(x, t) := \mathcal{F}(t)u_0(x) + \int_0^t \mathcal{R}(t - \tau) H(w_n(x, \tau)) \, d\tau.
\]
We can check easily for every \( t \in [0, T] \), that
\[
\|w_1(t) - u(\mathcal{T})\|_{D^1(\Omega)} = \|w_2(t) - u(\mathcal{T})\|_{D^1(\Omega)} = \ldots = \|w_n(t) - u(\mathcal{T})\|_{D^1(\Omega)}, \quad \forall x \in \Omega.
\]
On the contrary, when \( t \in [\mathcal{T}, \mathcal{T} + T] \), a simple calculation shows that
\[
\|w_1(t) - u(\mathcal{T})\|_{D^1(\Omega)} \leq \|\mathcal{F}(t)u_0 - \mathcal{F}(\mathcal{T})u_0\|_{D^1(\Omega)}
+ \int_{\mathcal{T}}^t \|\mathcal{A}(t-\tau)H(w(x, \tau))\|_{D^1(\Omega)} \, d\tau
+ \int_0^\mathcal{T} \|\mathcal{A}(t-\tau)H(u(x, \tau)) - \mathcal{A}(\mathcal{T} - \tau)H(u(x, \tau))\|_{D^1(\Omega)} \, d\tau.
\]
Similar to Lemma 3.2, we derive
\[
\|\mathcal{F}(t)u_0 - \mathcal{F}(\mathcal{T})u_0\|_{D^1(\Omega)} \leq \alpha^{-1} \mathcal{M}(t - \mathcal{T})^\alpha \|u_0\|_{D^1(\Omega)} \leq \alpha^{-1} \mathcal{M}T^\alpha \|u_0\|_{D^1(\Omega)}
\]
and
\[
\int_{\mathcal{T}}^t \|\mathcal{A}(t-\tau)H(w(x, \tau)) - \mathcal{A}(\mathcal{T} - \tau)H(u(x, \tau))\|_{D^1(\Omega)} \, d\tau
\leq \frac{\mathcal{M}_1 C_2 |t - \mathcal{T}|^\alpha}{\alpha (1 - \alpha) \theta_1^\alpha} \left( \|w\|^2_{C(I \to D^1(\Omega))} + \|u\|^2_{C(I \to D^1(\Omega))} \right).
\]
In addition, the following estimate also holds
\[
\int_{\mathcal{T}}^t \|\mathcal{A}(t-\tau)H(w(x, \tau))\|_{D^1(\Omega)} \, d\tau
\leq \mathcal{M}_1 C_2 \int_{\mathcal{T}}^t (t-\tau)^{\alpha - 1} \left( \|w\|^2_{D^1(\Omega)} + \|w\|_{D^1(\Omega)} \right) \, d\tau
\leq \frac{\mathcal{M}_1 C_2 (t - \mathcal{T})^\alpha}{\alpha} \left( \|w(\mathcal{T})\|_{D^1(\Omega)} + R \right)^2 + \left( \|w(\mathcal{T})\|_{D^1(\Omega)} + R \right)
\leq \frac{\mathcal{M}_1 C_2 T^\alpha}{\alpha} \left( \|w(\mathcal{T})\|_{D^1(\Omega)} + R \right)^2 + \left( \|w(\mathcal{T})\|_{D^1(\Omega)} + R \right),
\]
provided that for any \( w \in \mathcal{E} \), by the triangle inequality, the following estimate holds for any \( t \in [\mathcal{T}, \mathcal{T} + T] \)
\[
\|w(t)\|_{D^1(\Omega)} \leq \|u(\mathcal{T})\|_{D^1(\Omega)} + R.
\]
From the above estimates, by taking \( T \) small enough, one has
\[
\|u_1(t) - u(\mathcal{T})\|_{D^1(\Omega)} \leq R.
\]
It follows that \( w_1 \) is in \( \mathcal{E} \). The same arguments show that if \( w_n \in \mathcal{E} \) for \( n \geq 1 \), then \( w_{n+1} \) also belongs to \( \mathcal{E} \). To complete the proof, we have to show that \( \{w_n\}_{n=1}^\infty \) is a Cauchy sequence in \( \mathcal{E} \). Indeed, for \( w_{n-1}, w_n \in \mathcal{E} \) and \( t \geq \mathcal{T} \), we obtain
\[
\|w_{n+1}(t) - w_n(t)\|_{D^1(\Omega)}
\leq \int_{\mathcal{T}}^t \|\mathcal{A}(t-\tau)H(w_n(\tau)) - w_{n-1}(x, \tau)\|_{D^1(\Omega)} \, d\tau
\leq \int_{\mathcal{T}}^t \mathcal{M}_1 C_2 (t-\tau)^{\alpha - 1} \left( \|w_{n-1}(\tau)\|_{D^1(\Omega)} + R_{n-1}(t) \right) \, d\tau
\leq \frac{\mathcal{M}_1 C_2 T^\alpha}{\alpha} \left( \|u(\mathcal{T})\|_{D^1(\Omega)} + R + 1 \right) \|w_n - w_{n-1}\|_{C(I \to D^1(\Omega))},
\]
Hence, if $T$ is sufficiently small, we obtain

$$\|w_{n+1} - w_n\|_{C(I\to D^1(\Omega))} \leq \frac{3}{4}\|w_n - w_{n-1}\|_{C(I\to D^1(\Omega))},$$

the same arguments as (12) show that $\{w_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $E$. As we said before, $E$ is a Banach space, then we can find a limit function of $\{w_n\}_{n=1}^{\infty}$ which is the unique mild solution to Problem (1)-(3) on $[0, T + T]$. The proof is completed.

**Theorem 3.7.** (Global and finite time blowup solution) Let $u_0$ be in $D^1(\Omega)$ and $u$ be the unique mild solution to Problem (1)-(3). If we define

$$\mathcal{T}_{\text{max}} := \sup \left\{ \mathcal{T} > 0 \mid u \text{ exists uniquely on } [0, \mathcal{T}] \right\},$$

then either $\mathcal{T}_{\text{max}} = \infty$ or

$$\lim_{t \to \mathcal{T}_{\text{max}}} \|u(t)\|_{D^1(\Omega)} = \infty.$$

**Proof.** Assume that the maximal time point $\mathcal{T}_{\text{max}}$ is finite. Let us make a contradict assumption that there exists a finite constant $\mathcal{M}$ such that

$$\|u(t)\|_{D^1(\Omega)} \leq \mathcal{M}, \quad \forall t \in [0, \mathcal{T}_{\text{max}}]. \quad (17)$$

Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $I$ satisfying $t_n \xrightarrow{n \to \infty} \mathcal{T}_{\text{max}}$. We aim to show that $\{u(t_n)\}_{n=1}^{\infty} := \{u(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $D^1(\Omega)$. In fact, for $t_n < t_m$, we find that

$$\|u_m - u_n\|_{D^1(\Omega)} \leq \|\mathcal{T}(t_m)u_0 - \mathcal{T}(t_n)u_0\|_{D^1(\Omega)}$$

$$+ \int_{t_n}^{t_m} \|\mathcal{T}(t_m - \tau)H(u(\tau))\|_{D^1(\Omega)} \, d\tau$$

$$+ \int_{t_0}^{t_n} \|\mathcal{T}(t_m - \tau)H(u(\tau)) - \mathcal{T}(t_n - \tau)H(u(\tau))\|_{D^1(\Omega)} \, d\tau =: \sum_{j \in \{1, 2, 3\}} \mathcal{I}_j.$$

On one hand, since $u_0$ is in $D^1(\Omega)$, we have

$$\mathcal{I}_1 \leq \mathcal{M} (t_m - t_n)^\alpha \|u_0\|_{D^1(\Omega)} \leq \alpha^{-1} \mathcal{M} \left((t_m - \mathcal{T}_{\text{max}})^\alpha + (\mathcal{T}_{\text{max}} - t_n)^\alpha\right) \|u_0\|_{D^1(\Omega)}.$$

Then, for any $\varepsilon > 0$, we can find a number $n_1$ such that

$$\mathcal{I}_1 \leq \frac{\varepsilon}{3}, \quad \forall m, n \geq n_1.$$

On the other hand, the use of Lemma 2.5(ii) shows that

$$\mathcal{I}_2 \leq C_2 \int_{t_n}^{t_m} (t_m - \tau)^{\alpha - 1} \|H(u(\tau))\|_{D^{-1}(\Omega)} \, d\tau$$

$$\leq \mathcal{C}_2 C_2 \int_{t_n}^{t_m} (t_m - \tau)^{\alpha - 1} \left(\|u(\tau)\|^2_{D^1(\Omega)} + \|u(\tau)\|_{D^1(\Omega)}\right) \, d\tau$$

$$\leq \alpha^{-1} \left(\mathcal{M}^2 + \mathcal{M}\right) \mathcal{C}_2 C_2 \left((t_m - \mathcal{T}_{\text{max}})^\alpha + (\mathcal{T}_{\text{max}} - t_n)^\alpha\right).$$

Then, we can also find a number $n_2$ such that

$$\mathcal{I}_2 \leq \frac{\varepsilon}{3}, \quad \forall m, n \geq n_2.$$
Then, we can finish our proof by making the conclusion that provided $T_u$ for extending $u$

Thus we have extended the other Lipschitz cases of the source function.

4. The other Lipschitz cases of the source function. Throughout this section, we concern in the following form of the initial value problem (1)-(3)

$$
\begin{cases}
\partial_t^\alpha u - \partial_t^\alpha \Delta u - \Delta u = H(u), & \text{in } \Omega \times (0, \infty), \\
u = 0, & \text{on } \partial \Omega \times (0, \infty), \\
u = u_0, & \text{in } \Omega \times \{0\}.
\end{cases}
$$

In particular, our main results for a mild solution $u : [0, \infty) \to \Omega$ are going to revolve around the following two hypotheses for the source functions $H$.

(H₁) The nonlinearity of polynomial type

$H(u) = |u|^{p-1}u, \quad p > 2$. 
(H2) The nonlinearity of exponential type

\[ H(u) = u^3 e^{u^2}. \]

4.1. The polynomial nonlinearity.

Lemma 4.1. Let \( d \geq 2, \ p > 2, \) and \( \nu > 0 \) such that

\[ \nu < \min \left\{ 1, \frac{d + 2 - 2(p - 1)d}{2 - 4(p - 1)} \right\}. \]

Then, we can find an independent constant \( \mathcal{C}_2 > 0 \) such that

\[ \|H(u) - H(v)\|_{D^{\nu-1}(\Omega)} \leq \mathcal{C}_2 \left( \|u\|_{L^p(\Omega)}^{p-1} + \|v\|_{L^p(\Omega)}^{p-1} \right) \|u - v\|_{D^{\nu}(\Omega)}, \]

for every \( u, v \in D^\nu(\Omega). \)

Proof. Firstly, for any \( q \geq 1, \) Hölder’s inequality gives us the following estimate

\[ \|u|^{p-1}(u - v)\|_{L^q(\Omega)} \leq \|u - v\|_{L^2(\Omega)} \|u\|_{L^p(\Omega)}^{p-1}. \]

Then, if we choose \( q = \frac{2d}{\sigma - 2(p - 1)} > 1, \) Lemma 5.3 shows that the Lebesgue space \( L^q(\Omega) \) embeds into the Hilbert space \( D^{\nu-1}(\Omega) \). Also, by another use of Lemma 5.3 and the assumptions for \( d, p, \nu \) and \( \Omega, \) we have

\[ D^{\nu}(\Omega) \hookrightarrow L^{2(p-1)q}(\Omega) \hookrightarrow L^q(\Omega). \]

Combining the above result with the fact \( (a+b)^m \leq 2^{m-1}(a^m + b^m), \) \( a, b > 0, m \geq 1 \) and the triangle inequality, for any \( u, v \in D^\nu(\Omega) \) we have

\[ \|H(u) - H(v)\|_{D^{\nu-1}(\Omega)} \leq \mathcal{C}_2 \left( \|u\|_{D^{\nu}(\Omega)}^{p-1} + \|v\|_{D^{\nu}(\Omega)}^{p-1} \right) \|u - v\|_{D^{\nu}(\Omega)}, \]

where \( \mathcal{C}_2 \) is independent on \( u, v. \) The proof is completed. \( \square \)

Definition 4.2. Let \( I = [0, \tau] \) and \( \sigma, M > 0. \) Then, we define a subset of \( C(I \to D^\nu(\Omega)) \) as follows

\[ \mathcal{X} = \mathcal{X}(\sigma, M, \nu) := \left\{ w \in C(I \to D^\nu(\Omega)) \mid w(0) = u_0 \text{ and } \sup_{t \in I \setminus \{0\}} t^\sigma \|w(t)\|_{D^{\nu}(\Omega)} < M \right\}. \]

Remark 6. Observe that \( \mathcal{X} \) is a Banach space involving the norm

\[ \|w\|_{\mathcal{X}} = \sup_{t \in I \setminus \{0\}} t^\sigma \|w(t)\|_{D^{\nu}(\Omega)}. \]

Theorem 4.3. Let \( \alpha \in (0, 1) \) and \( p > 2 \) such that \( \frac{p}{p-1} < \alpha^{-1}. \) For a given datum \( u_0 \in D^\nu(\Omega) \) sufficiently small and \( I = [0, \tau]. \) Problem (1)-(3) admits a unique mild solution that belongs to the space \( \mathcal{X} = \mathcal{X} \left( \frac{\alpha}{p-1}, 2C_1 \|u_0\|_{D^\nu(\Omega)}, \nu \right). \)

Proof. In the same spirit with Section 3, we consider again the sequence \( \{w_n\}_{n=1}^\infty \) established by

\[ w_1(x, t) := \mathcal{S}(t)u_0(x), \]

\[ w_{n+1}(x, t) := \mathcal{S}(t)u_0(x) + \int_0^t \mathcal{A}(t - \tau)H(w_n(x, \tau))d\tau. \]

By analogous argument as Lemma 3.2, we can prove that \( \{w_n\}_{n=1}^\infty \) is a subset of \( C(I \to D^\nu(\Omega)). \) So, we need only to focus on deriving that \( \{w_n\}_{n=1}^\infty \) is a convergent
sequence in \( X = X \left( \alpha \mu, 2C_1 \|u_0\|_{D^{\nu}(\Omega)}, \nu \right) \), where \( \mu = \frac{1}{p-\nu} \). The Picard iteration process for this proof includes three main steps.

- **Step 1.** We show that \( w_1 \) is in \( X \).
  Indeed, since \( \mu \in (0, 1) \) and \( u_0 \in D^{\nu}(\Omega) \), thanks to Lemma 2.5(i), we obtain
  \[
  \|w_1(t)\|_{D^{\nu}(\Omega)} \leq C_1 t^{-\alpha \mu} \|u_0\|_{D^{\nu}(\Omega)},
  \]
  which implies
  \[
  t^{\alpha \mu} \|w_1(t)\|_{D^{\nu}(\Omega)} \leq C_1 \|u_0\|_{D^{\nu}(\Omega)}.
  \]

- **Step 2.** Assume that \( w_n \) belongs to \( X \) for any \( n \in \mathbb{N} \), then \( w_{n+1} \in X \).
  Indeed, Lemma 2.5(ii) provides us the following estimate
  \[
  \|w_{n+1}(t) - w_1(t)\|_{D^{\nu}(\Omega)} \leq C_2 \int_0^t (t - \tau)^{\alpha - 1} \|H(w_n(\tau))\|_{D^{\nu-1}(\Omega)} d\tau.
  \]  \tag{19}
  Using Lemma 4.1 with \( v = 0 \) and the fact that \( w_n \in X \), we have
  \[
  \|w_{n+1}(t) - w_1(t)\|_{D^{\nu}(\Omega)} \leq C_2 \int_0^t (t - \tau)^{\alpha - 1} \|w_n(\tau)\|_{D^{\nu}(\Omega)}^p d\tau \\
  \leq C_2 \int_0^t (t - \tau)^{\alpha - 1} (1 - p \alpha \mu) \|w_n\|_{X}^p d\tau \\
  \leq C_2 (2C_1 \|u_0\|_{D^{\nu}(\Omega)})^p.
  \]
  Also, from our assumptions, we find that
  \[
  p \alpha \mu < 1 \quad \text{and} \quad 1 - p \mu + \mu = 0,
  \]
  which tells that
  \[
  t^{\alpha \mu} \int_0^t (t - \tau)^{\alpha - 1} d\tau = \beta(\alpha, 1 - p \alpha \mu),
  \]
  where the Beta function is defined in Definition 5.5. From the standpoint of this result, we infer
  \[
  t^{\alpha \mu} \|w_{n+1}(t)\|_{D^{\nu}(\Omega)} \leq C_1 \|u_0\|_{D^{\nu}(\Omega)} + C_2 (2C_1 \|u_0\|_{D^{\nu}(\Omega)})^p \tag{20}
  \]
  Hence, the statement that \( w_{n+1} \in X \) follows from the small data \( u_0 \in D^{\nu}(\Omega) \), then Step 2 is completed.

- **Step 3.** We show that \( \{w_n\}_{n=1}^\infty \) is a Cauchy sequence.
  Suppose that \( w_n \) and \( w_{n-1} \) are in \( X \). Our techniques are not too different from Step 2. Indeed, we have
  \[
  \|w_{n+1}(t) - w_n(t)\|_{D^{\nu}(\Omega)} \\
  \leq C_2 \int_0^t (t - \tau)^{\alpha - 1} \left( \|w_n(\tau)\|_{D^{\nu}(\Omega)}^p + \|w_{n-1}(\tau)\|_{D^{\nu}(\Omega)}^p \right) \|w_n(\tau) - w_{n-1}(\tau)\|_{D^{\nu}(\Omega)} d\tau.
  \]  \tag{21}
Since \( \{w_n\}_{n=1}^{\infty} \in X \), multiplying the both sides of (21) by \( t^{\alpha\mu} \), we have
\[
\|w_{n+1}(t) - w_n(t)\|_{D^\nu(\Omega)} \leq 2C_2 \left( t^{\alpha\mu} \int_0^t (t - \tau)^{\alpha - 1} \tau^{\alpha\mu} d\tau \right) \left( 2C_1 \|u_0\|_{D^\nu(\Omega)} \right)^{p-1} \|w_n(t) - w_{n-1}(t)\|_X.
\]
Then, we can take the supremum over the interval \( I \setminus \{0\} \) on both sides of (22) to obtain
\[
\|w_{n+1}(t) - w_n(t)\|_X \leq 2C_2 \beta(\alpha, 1 - \rho\mu) \left( 2C_1 \|u_0\|_{D^\nu(\Omega)} \right)^{p-1} \|w_n(t) - w_{n-1}(t)\|_X.
\]
Therefore, if \( \|u_0\|_{D^\nu(\Omega)} \) is sufficiently small, \( \{w_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( X \).

From the above three steps, we can use again the arguments performed as in the proofs of Section 3 to find the limit function \( u \) of \( \{w_n\}_{n=1}^{\infty} \) which is the unique mild solution of Problem (1)-(3). The proof is completed. \( \square \)

4.2. Exponential nonlinearity. This subsection concerns the global well-posedness of Problem (1)-(3) with the source function \( H(u) = u^3e^{w^2} \). Here for any \( w, v \in \mathbb{R} \), we have
\[
|H(w) - H(v)| \leq \mathcal{C}_3 |w - v| \left( w^2 e^{w^2} + v^2 e^{v^2} \right),
\]
for some independent \( \mathcal{C}_3 > 0 \). We recall the following theorem in [23, Theorem 2] which is used to control the solution operator in the framework of Orlicz space.

**Theorem 4.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a cone domain. Then, \( W^{1,2}_0(\Omega) \) embeds continuously into the Orlicz space \( L^{\Xi}(\Omega) \).

**Lemma 4.5.** Let \( \alpha \in \left( 0, \frac{2}{3} \right) \), \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with sufficiently smooth boundary, and \( M \) be a finite constant. Suppose that \( I = [0, T] \) and \( w, v \) are functions in \( C(I \to L^{\Xi}(\Omega)) \) such that
\[
\max \left( \sup_{t \in I} \|w(t)\|_{L^{\Xi}(\Omega)}, \sup_{t \in I} \|w(t)\|_{L^{\Xi}(\Omega)}, \sup_{t \in I} \|w(t)\|_{L^{\Xi}(\Omega)}, \sup_{t \in I} \|w(t)\|_{L^{\Xi}(\Omega)} \right) < M.
\]
If \( M \) is small enough, we can derive the following inequalities
\[
\left\| \int_0^t \mathcal{A}(t - \tau) (H(w(\tau)) - H(v(\tau))) d\tau \right\|_{L^\Xi(\Omega)} \leq \frac{2C_2 \mathcal{C}_3 T^\alpha}{\alpha} \|w - v\|_{L^\Xi(\Omega)} M^2 \left( \sqrt{2} + \sqrt{6} (6M^2)^{\frac{1}{2}} \right) \tag{23}
\]
and
\[
\int_0^T \|w_{\epsilon}(t) - w(t)\|_{L^\Xi(\Omega)} dt \leq \frac{2C_2 \mathcal{C}_3 \|w - v\|_{L^\Xi(\Omega)} M^2 (\sqrt{2} + \sqrt{6} (6M^2)^{\frac{1}{2}})}{\left( \beta(\alpha, 1 - \frac{2\rho\mu}{\alpha}) \right)^{1/2}} \tag{24}
\]
Proof. Assume that \( \partial \Omega \) is sufficiently smooth. Then, for any \( w, v \in L^\infty(\Omega) \), we can use Theorem 4.4 to obtain

\[
\left\| \int_0^t \mathcal{R}(t - \tau) \left( H(w(\tau)) - H(v(\tau)) \right) d\tau \right\|_{L^\infty(\Omega)} \\
\leq \int_0^t \left\| \mathcal{R}(t - \tau) \left( H(w(\tau)) - H(v(\tau)) \right) \right\|_{D^\alpha(\Omega)} d\tau.
\]

Using Lemma 2.5, we have

\[
\left\| \int_0^t \mathcal{R}(t - \tau) \left( H(w(\tau)) - H(v(\tau)) \right) d\tau \right\|_{L^\infty(\Omega)} \\
\leq C_2 \int_0^t (t - \tau)^{\alpha-1} \left\| H(w(\tau)) - H(v(\tau)) \right\|_{L^2(\Omega)} d\tau.
\]

To continuous, we make a nonlinear estimate for the source function with \( w = w(t), v = v(t), t \in I \), as follows

\[
\parallel H(w) - H(v) \parallel_{L^2(\Omega)} \\
\leq \mathcal{E}_3 \parallel (w - v)(u^2 + v^2) \parallel_{L^2(\Omega)} + \mathcal{E}_3 \sum_{u \in \{w, v\}} \parallel w - v \parallel \parallel \exp \left( e^{u^2} - 1 \right) \parallel_{L^2(\Omega)} (26)
\]

\[
\leq \mathcal{E}_3 \sum_{u \in \{w, v\}} \left( \parallel w - v \parallel_{L^2(\Omega)} \parallel u \parallel_{L^2(\Omega)}^2 + \parallel w - v \parallel_{L^2(\Omega)} \parallel u \parallel_{L^2(\Omega)} \parallel e^{u^2} - 1 \parallel_{L^2(\Omega)} \right),
\]

here, we have used H"older’s inequality. If \( \parallel u \parallel_{L^2(\Omega)} < \left( \frac{1}{\mathcal{E}_3} \right)^{\frac{1}{2}} \), it follows from the techniques used in [15, Lemma 3.2] that

\[
\parallel e^{u^2} - 1 \parallel_{L^2(\Omega)}^6 \leq \int_\Omega \left( e^{6u^2(x)} - 1 \right) dx \\
\leq \int_\Omega \exp \left( \frac{6 \parallel u \parallel_{L^2(\Omega)}^2 u^2(x)}{\parallel u \parallel_{L^2(\Omega)}^2} \right) - 1 \right) dx
\]

\[
\leq 6 \parallel u \parallel_{L^2(\Omega)}^2,
\]

where we have used the fact that

\[
\left\{ \kappa \in \mathbb{R} \mid \kappa > 0, \int_\Omega \Xi \left( \frac{|w(x)|}{\kappa} \right) dx \leq 1 \right\} = \left[ \parallel w \parallel_{L^2(\Omega)}, \infty \right).
\]

According to (26), (27) and Lemma 2.1, we find that

\[
\parallel H(w) - H(v) \parallel_{L^2(\Omega)} \\
\leq \mathcal{E}_3 \sum_{u \in \{w, v\}} \parallel w - v \parallel_{L^2(\Omega)} \parallel u \parallel_{L^2(\Omega)}^2 \left( (\Gamma(3))^{\frac{1}{2}} + (\Gamma(4))^{\frac{1}{2}} \left( 6 \parallel u \parallel_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \right)
\]

\[
= \mathcal{E}_3 \sum_{u \in \{w, v\}} \parallel w - v \parallel_{L^2(\Omega)} \parallel u \parallel_{L^2(\Omega)}^2 \left( \sqrt{2} + \sqrt{6} \left( 6 \parallel u \parallel_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \right),
\]

here we have used the fact that \( \Gamma(n) = (n - 1)! \) for \( n \in \mathbb{N} \). Inclusion inequalities (25) and (28) allow us to deduce the inequality (23). Furthermore, by using (23) and

\[
t^\frac{\alpha}{2} \int_0^t (t - \tau)^{\alpha-1} \tau^{-\frac{3\alpha}{2}} d\tau = \beta \left( \alpha, 1 - \frac{3\alpha}{2} \right),
\]
This result yields that

\[ \text{Then, there exists a unique mild solution of Problem (1) in } C(I \rightarrow L^2(\Omega)). \]

Proof. We start by considering again the sequence \( \{w_n\}_{n=1}^{\infty} \) as in Theorem 4.3 and the following space

\[ U := \left\{ w \in C(I \rightarrow L^2(\Omega)) \mid \|w\|_U < 2C_1 \|u_0\|_{D^1(\Omega)} \right\}, \]

where the U-norm is given by

\[ \|w\|_U := \max \left\{ \sup_{t \in I} \|w\|_{L^2(\Omega)} , \sup_{t \in I} t^\frac{3}{2} \|w\|_{L^2(\Omega)} \right\}. \]

Using Theorem 4.4 and considering Lemma 2.5 in two cases: \( \mu = 0 \) and \( \mu = \frac{1}{2} \), we have

\[ \begin{cases} \|w_1(t)\|_{L^2(\Omega)} \leq \|\mathcal{A}(t)u_0\|_{D^1(\Omega)} \leq C_1 \|u_0\|_{D^1(\Omega)}, & \mu = 0, \ t \in I \\ t^\frac{3}{2} \|w_1(t)\|_{L^2(\Omega)} \leq t^\frac{3}{2} \|\mathcal{A}(t)u_0\|_{D^1(\Omega)} \leq C_1 \|u_0\|_{D^1(\Omega)}, & \mu = \frac{1}{2}, \ t \in I. \end{cases} \]

This result yields that \( w_1 \in U \). Next we assume that \( w_n \) is in \( U \), then we can show that \( w_{n+1} \) is also in \( U \) In fact, for given small data \( u_0 \), set

\[ \mathcal{A}_* := \left( \frac{32C_2{\mathcal{C}_3} \|u_0\|^2_{D^1(\Omega)} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|^2_{D^1(\Omega)})^\frac{1}{2} \right) \right)}{\alpha C_1} \right)^{\frac{1}{2}}. \]

On the one hand, when \( t \leq \mathcal{A}_* \), we apply Lemma 4.5 with \( v = 0 \) to find that

\[ \left\| \int_0^t \mathcal{A}(t-\tau)H(w_n(\tau))d\tau \right\|_{L^2(\Omega)} \]

\[ \leq \left( \int_0^t \mathcal{A}(t-\tau)H(w_n(\tau))d\tau \right)_{L^2(\Omega)} \]

\[ \leq \frac{2C_2{\mathcal{C}_3} \mathcal{A}_*^{\alpha}}{\alpha(2 \|u_0\|_{D^1(\Omega)})^3} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|^2_{D^1(\Omega)})^\frac{1}{2} \right) \right) \]

\[ = 2^{-\frac{3}{2}} \frac{C_1}{2} \|u_0\|_{D^1(\Omega)}. \]

On the other hand, if \( t \) is larger than \( \mathcal{A}_* \) defined above, we can deduce that

\[ \left\| \int_0^t \mathcal{A}(t-\tau)H(w_n(\tau))d\tau \right\|_{L^2(\Omega)} \]

\[ \leq \mathcal{A}_*^{-\frac{3}{2} \frac{alpha}{2}} \left( \int_0^t \mathcal{A}(t-\tau)H(w_n(\tau))d\tau \right)_{L^2(\Omega)} \]

\[ \leq \frac{2C_2{\mathcal{C}_3} \mathcal{A}_*^{-\frac{3}{2} \frac{alpha}{2}}}{\left( \beta \left( \alpha, 1 - \frac{3\alpha}{2} \right) \right)^{-1}} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|^2_{D^1(\Omega)})^\frac{1}{2} \right) \right) \]

\[ = 128 \|u_0\|^4_{D^1(\Omega)} \left( C_2{\mathcal{C}_3} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|^2_{D^1(\Omega)})^\frac{1}{2} \right) \right) \right)^{\frac{3}{2}} \]

\[ = \frac{128 \|u_0\|^4_{D^1(\Omega)} \left( C_2{\mathcal{C}_3} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|^2_{D^1(\Omega)})^\frac{1}{2} \right) \right) \right)^{\frac{3}{2}}}{\left( \beta \left( \alpha, 1 - \frac{3\alpha}{2} \right) \right)^{-1}}. \]
Combining all of the above arguments, whether \( t \leq \mathcal{T} \) or \( t > \mathcal{T} \), we can see that
\[
\left\| \int_0^t \mathcal{R}(t-\tau)H(w_n(\tau))d\tau \right\|_{L^2(\Omega)} \leq C_1 \|u_0\|_{D^1(\Omega)}.
\]  
(29)

In addition, apply again Lemma 4.5, for any \( w_n \in U \), we obtain
\[
t^2 \left\| \int_0^t \mathcal{R}(t-\tau)H(w_n(\tau))d\tau \right\|_{L^2(\Omega)} \leq \frac{2C_2\mathcal{C}_3(2C_1 \|u_0\|_{D^1(\Omega)})^3}{\left(\beta \left(1 - \frac{2\beta}{3\alpha}\right)\right)^{1/2}} \left(\sqrt{2} + \sqrt{6} \left(6(2C_1 \|u_0\|_{D^1(\Omega)})^2\right)^{\frac{1}{2}}\right).
\]  
(30)

Therefore, for any \( w_n \in U \), we deduce the following two claims.

**Claim 1.** Combining the conclusions that \( w_1 \in U \) and the inequality (30) gives us
\[
\|w_{n+1}(t)\|_{L^2(\Omega)} \leq \|\mathcal{R}(t)u_0\|_{L^2(\Omega)} + \left\| \int_0^t \mathcal{R}(t-\tau)H(w_n(\tau))d\tau \right\|_{L^2(\Omega)} \leq 2C_1 \|u_0\|_{D^1(\Omega)}.
\]

**Claim 2.** Similarly, since \( w_1 \in U \) and the estimate (31) holds, we have
\[
t^2 \|w_{n+1}(t)\|_{L^2(\Omega)} \leq t^2 \|\mathcal{R}(t)u_0\|_{L^2(\Omega)} + t^2 \left\| \int_0^t \mathcal{R}(t-\tau)H(w_n(\tau))d\tau \right\|_{L^2(\Omega)} \leq 2C_1 \|u_0\|_{D^1(\Omega)},
\]
as long as \( \|u_0\|_{D^1(\Omega)} \) is sufficiently small.

These claims show that for any \( t \in I \), \( \|w_{n+1}\| \) is less than or equals to \( 2C_1 \|u_0\|_{D^1(\Omega)} \). It means that \( w_{n+1} \) belongs to \( U \) whenever \( w_n \) is in \( U \), provided that the continuity of \( w_{n+1} \) is inferred by Lemma 3.2. The remaining work is to show that \( \{w_n\}_{n=1}^\infty \) is a Cauchy sequence with respect to the \( U \) norm. Based on this result, we can easily obtain the unique mild solution \( u \) of Problem (1)-(3) which is the limit function of the sequence \( \{w_n\}_{n=1}^\infty \). To this end, we take two elements of the sequence, \( w_{n-1}, w_n \in U \). Analogous to the way we find the estimate (30), we set
\[
\mathcal{T}_{**} := \left(\frac{64C_2\mathcal{C}_3 \|u_0\|_{D^1(\Omega)}^2 \left(\sqrt{2} + \sqrt{6} \left(6(2C_1 \|u_0\|_{D^1(\Omega)})^2\right)^{\frac{1}{2}}\right)}{\alpha C_1}\right)^{-\frac{1}{4}}.
\]

On the one hand, for any \( t > \mathcal{T}_{**} \), we have
\[
\left\| \int_0^t \mathcal{R}(t-\tau)(H(w_n(\tau)) - H(w_{n-1}(\tau)))d\tau \right\|_{L^2(\Omega)} \leq \int_{\mathcal{T}_{**}}^t \left\| \mathcal{R}(t-\tau)(H(w_n(\tau)) - H(w_{n-1}(\tau))) \right\|_{L^2(\Omega)} d\tau.
\]
As a result, if \( \|I\| \leq 2 \) and \( \alpha \leq 2 \), we can conclude that

\[
\left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|_{D^1(\Omega)})^2 \right)^{\frac{1}{2}} \right) \|w_n(t) - w_{n-1}(t)\|_U
\]

In view of Lemma 4.5, one finds that

\[
I_1 \leq \frac{2C_3 \mathcal{T}_*}{\alpha(2 \|u_0\|_{D^1(\Omega)})^2} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|_{D^1(\Omega)})^2 \right)^{\frac{1}{2}} \right) \|w_n(t) - w_{n-1}(t)\|_U
\]

\[
\leq \frac{1}{4} \|w_n(t) - w_{n-1}(t)\|_U.
\]

Applying Lemma 4.5 in the same way as in (29), we have

\[
I_2 \leq \frac{2C_3 \mathcal{T}_* \mathcal{T}_* \mathcal{T}_*}{\alpha(2 \|u_0\|_{D^1(\Omega)})^2} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|_{D^1(\Omega)})^2 \right)^{\frac{1}{2}} \right) \|w_n(t) - w_{n-1}(t)\|_U
\]

\[
= \frac{64 \|u_0\|_{D^1(\Omega)}^2 \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|_{D^1(\Omega)})^2 \right)^{\frac{1}{2}} \right)}{(\beta (\alpha, 1 - \frac{3\alpha}{2}))^{-1}} \|w_n(t) - w_{n-1}(t)\|_U
\]

\[
\leq \frac{1}{4} \|w_n(t) - w_{n-1}(t)\|_U,
\]

provided that \( \|u_0\|_{D^1(\Omega)} \) is sufficiently small. In addition, for any \( t \leq \mathcal{T}_* \), we note that

\[
\left\| \int_0^t H(t-\tau)(H(w_n(\tau)) - H(w_{n-1}(\tau))) d\tau \right\|_{L^2(\Omega)} \leq I_1 \leq I_1 + I_2.
\]

Wherefore, for any \( t \in I \) the following estimate holds

\[
\sup_{t \in I} \|w_{n+1}(t) - w_n(t)\|_{D^1(\Omega)} = \sup_{t \in I} \left\| \int_0^t H(t-\tau)(H(w_n(\tau)) - H(w_{n-1}(\tau))) d\tau \right\|_{L^2(\Omega)}
\]

\[
\leq I_1 + I_2 \leq \frac{1}{2} \|w_n(t) - w_{n-1}(t)\|_U.
\]

(32)

On the other hand, by replacing respectively \( w, v \) by \( w_n, w_{n-1} \in U \) in Lemma 4.5, the following inequality holds

\[
t^2 \left\| \int_0^t H(t-\tau)(H(w_n(\tau)) - H(w_{n-1}(\tau))) d\tau \right\|_{L^2(\Omega)} \leq \frac{2C_3(2 \|u_0\|_{D^1(\Omega)})^2}{(\beta (\alpha, 1 - \frac{3\alpha}{2}))^{-1}} \left( \sqrt{2} + \sqrt{6} \left( 6(2 \|u_0\|_{D^1(\Omega)})^2 \right)^{\frac{1}{2}} \right) \|w_n(t) - w_{n-1}(t)\|_U.
\]

As a result, if \( \|u_0\|_{D^1(\Omega)} \) is small enough, we deduce

\[
\sup_{t \in I \setminus \{0\}} t^2 \|w_{n+1}(t) - w_n(t)\|_{L^2(\Omega)} \leq \frac{1}{2} \|w_n(t) - w_{n-1}(t)\|_U.
\]

(33)

On account of (31), (32) and (33), we can conclude that

\[
\|w_{n+1}(t) - w_n(t)\|_U \leq \frac{1}{2} \|w_n(t) - w_{n-1}(t)\|_{L^2(\Omega)},
\]
for every \( w_n, w_{n-1} \in U \). It means that \( \{w_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( U \). The proof is completed. \( \square \)

**Remark 7.** We note that in the above proof, we consider only the case \( T > \max(\mathcal{T}_*, \mathcal{T}_{*+}) \). When, \( T < \min(\mathcal{T}_*, \mathcal{T}_{*+}) \), the proof is similar and easier, so we omit it here.

5. **Appendix.**

**Definition 5.1.** Let \( \alpha \in (0, 1) \). Then, the definition of the M-Wright type function \( \mathcal{W}_\alpha \) is given by

\[
\mathcal{W}_\alpha(r) := \sum_{k=0}^{\infty} \frac{r^k}{k! \Gamma(-\alpha k + 1 - \alpha)}.
\]

**Lemma 5.2.** ([13, Section 2]) For \( \alpha \in (0, 1) \) and \( \mu \in (-1, \infty) \), there holds

\[
\int_0^\infty r^\mu \mathcal{W}_\alpha(r) dr = \frac{\Gamma(1 + \mu)}{\Gamma(1 + \alpha \mu)}.
\]

**Lemma 5.3.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded domain with smooth boundary and \( p \in [1, \infty) \). Then,

(i) if \( 0 \leq \nu < \frac{d}{2} \) and \( 1 \leq p \leq \frac{2d}{d-2\nu} \), or \( \nu = \frac{d}{2} \) and \( 1 \leq p < \infty \), we have

\[
D^\nu(\Omega) \hookrightarrow L^p(\Omega);
\]

(ii) if \( -\frac{d}{2} < \nu \leq 0 \) and \( p \geq \frac{2d}{d-2\nu} \), we have

\[
D^\nu(\Omega) \hookleftarrow L^p(\Omega).
\]

**Lemma 5.4.** (Fractional Grönwall inequality) Let \( a, b \) be positive constants and \( \alpha \in (0, 1) \). Suppose that function \( w \in L^\infty(0, T] \) satisfies the following inequality

\[
w(t) \leq a + b \int_0^t (t - \tau)^{\alpha - 1} w(\tau) d\tau, \quad \text{for all } t \in (0, T],
\]

then, the result below is satisfied

\[
w(t) \leq a E_\alpha(b \Gamma(\alpha) t^\alpha).
\]

**Definition 5.5.** Let \( p, q > 0 \). Then, the Beta function and the Gamma function can be defined respectively by

\[
\beta(p, q) = \int_0^1 (1 - z)^{p-1} z^{q-1} dz \quad \text{and} \quad \Gamma(p) = \int_0^{\infty} z^{p-1} e^{-z} dz.
\]

**Acknowledgment.** The first and second authors are thankful to the Van Lang University.
REFERENCES

[1] J. Albert, Dispersion of low-energy waves for the generalized Benjamin-Bona-Mahony equation, *J. Differential Equations*, 63 (1986), 117–134.
[2] C. J. Amick, J. L. Bona and M. E. Schonbek, Decay of solutions of some nonlinear wave equations, *J. Differential Equations*, 81 (1989), 1–49.
[3] E. Bazhlekova, B. Jin, R. Lazarov and Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid, *Numer. Math.*, 131 (2015), 1–31.
[4] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in non-linear dispersive systems, *Philos. Trans. Roy. Soc. London Ser. A*, 272 (1972), 47–78.
[5] M. Bonforte, Y. Sire and J. L. Vázquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, *Discrete Contin. Dyn. Syst.*, 35 (2015), 5725–5767.
[6] T. Caraballo, A. M. M. Duran and F. Rivero, Asymptotic behaviour of a non-classical and non-autonomous diffusion equation containing some hereditary characteristic, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 1817–1833.
[7] A. O. Celebi, V. K. Kalantarov and M. Polat, Attractors for the generalized Benjamin-Bona-Mahony equation, *J. Differential Equations*, 157 (1999), 439–451.
[8] H. Chen and S. Tian, Initial boundary value problem for a class of semilinear pseudo-parabolic equations with logarithmic nonlinearity, *J. Differential Equations*, 258 (2015), 4421–4442.
[9] H. Chen and H. Xu, Global existence and blow-up of solutions for infinitely degenerate semilinear pseudo-parabolic equations with logarithmic nonlinearity, *Discrete Contin. Dyn. Syst.*, 39 (2019), 1185–1203.
[10] P. J. Chen and E. G. Morton, On a theory of heat conduction involving two temperatures, *Zeitschrift für Angewandte Mathematik und Physik*, 19 (1968), 614–627.
[11] Y. Chen, H. Gao, M. J. Garrido-Atienza and B. Schmalfuss, Pathwise solutions of SPDEs driven by Hölder-continuous integrators with exponent larger than 1/2 and random dynamical systems, *Discrete Contin. Dyn. Syst.*, 34 (2014), 79–98.
[12] B. D. Coleman and W. Noll, An approximation theorem for functionals, with applications in continuum mechanics, *Arch. Rational Mech. Anal.*, 6 (1960), 355–370.
[13] P. M. de Carvalho-Neto and G. Planas, Mild solutions to the time fractional Navier-Stokes equations in \( \mathbb{R}^N \), *J. Differential Equations*, 259 (2015), 2948–2980.
[14] R. Grande, Space-time fractional nonlinear Schrödinger equation, *SIAM J. Math. Anal.*, 51 (2019), 4172–4212.
[15] N. Ioku, The Cauchy problem for heat equations with exponential nonlinearity, *J. Differential Equations*, 251 (2011), 1172–1194.
[16] L. Li, J. G. Liu and L. Wang, Cauchy problems for Keller-Segel type time-space fractional diffusion equation, *J. Differential Equations*, 265 (2018), 1044–1096.
[17] W. Lian, W. Juan and R. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differential Equations*, 269 (2020), 4914–4959.
[18] L. A. Medeiros and G. P. Menzala, Existence and uniqueness for periodic solutions of the Benjamin-Bona-Mahony equation, *SIAM J. Math. Anal.*, 8 (1977), 792–799.
[19] V. Padrón, Effect of aggregation on population recovery modeled by a forward-backward pseudoparabolic equation, *Trans. Amer. Math. Soc.*, 356 (2004), 2739–2756.
[20] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Mathematics in Science and Engineering, 198. Academic Press, Inc., San Diego, CA, 1999.
[21] R. E. Showalter and T. W. Ting, Pseudoparabolic partial differential equations, *SIAM J. Math. Anal.*, 1 (1970), 1–26.
[22] T. W. Ting, Certain non-steady flows of second-order fluids, *Arch. Rational Mech. Anal.*, 14 (1963), 1–26.
[23] N. Trudinger, On imbeddings into orlicz spaces and some applications, *J. Math. Mech.*, 17 (1967), 473–483.
[24] N. H. Tuan, V. V. Au and R. Xu, Semilinear Caputo time-fractional pseudo-parabolic equations, *Commun. Pure Appl. Anal.*, 20 (2021), 583–621.
[25] N. H. Tuan and T. Caraballo, On initial and terminal value problems for fractional nonclassical diffusion equations, *Proc. Amer. Math. Soc.*, 149 (2021), 143–161.
[26] X. Wang and R. Xu, Global existence and finite time blowup for a nonlocal semilinear pseudo-parabolic equation, *Adv. Nonlinear Anal.*, 10 (2021), 261–288.

[27] Y. Xiao, Packing measure of the sample paths of fractional Brownian motion, *Trans. Amer. Math. Soc.*, 348 (1996), 3193–3213.

[28] R. Xu, W. Lian and Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.*, 63 (2020), 321–356.

[29] R. Xu and J. Su, Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, 264 (2013), 2732–2763.

[30] X. Zhu, F. Li and T. Rong, Global existence and blow up of solutions to a class of pseudo-parabolic equations with an exponential source, *Commun. Pure Appl. Anal.*, 14 (2015), 2465–2485.

Received August 2021; revised November 2021; early access January 2022.

E-mail address: nguyenhuytuan@vlu.edu.vn
E-mail address: nguyenanhtuan@vlu.edu.vn
E-mail address: yangchao@hrbeu.edu.cn