GEVREY REGULARITY OF MILD SOLUTIONS TO THE NON-CUTOFF BOLTZMANN EQUATION

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Abstract. In the paper, for the Cauchy problem on the non-cutoff Boltzmann equation in torus, we establish the global-in-time Gevrey smoothness in velocity and space variables for a class of low-regularity mild solutions near Maxwellians with the Gevrey index depending only on the angular singularity. This together with [24] provides a self-contained well-posedness theory for both existence and regularity of global solutions for initial data of low regularity in the framework of perturbations. For the proof we treat in a subtle way the commutator between the regularization operators and the Boltzmann collision operator involving rough coefficients, and this enables us to combine the classical Hörmander’s hypoelliptic techniques together with the global symbolic calculus established for the linearized Boltzmann operator so as to improve the regularity of solutions at positive time.

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1. Introduction

The global well-posedness, such as existence, uniqueness and regularity, for the nonlinear Boltzmann equation in the spatially inhomogeneous setting, is a fundamental mathematical problem in collisional kinetic theory. On one hand, for general initial data with finite mass, energy and entropy, it is well known that the global existence of appropriate renormalized weak solutions in $L^1$ was established first by DiPerna-Lions [21] under the Grad’s angular cutoff assumption and later by Alexandre-Villani [10] for the non-cutoff collision kernel with physically realistic long-range interactions even including the Coulomb potential, while both uniqueness and regularity of such general global solutions have remained largely open.

On the other hand, whenever initial data are close enough to Maxwellians in a certain sense, the theory of unique existence of global classical solutions is well investigated from different aspects. In the cutoff case, it was started first by Ukai [57] using the spectral analysis and later by Liu-Yu [50], Liu-Yang-Yu [49] and Guo [31, 32] using the energy method. In the non-cutoff case, AMUXY [5, 7, 8] and Gressman-Strain [29] independently constructed the unique global classical solutions near global Maxwellians in the whole space and in the torus domain, respectively. In those aforementioned works, in order to treat the nonlinearity in space variables, the high-order Sobolev space $H^\ell$ with $\ell$ large enough, for instance $\ell > 3/2$ in case of three space dimensions, was used to control the $L^{\infty}$ norm of the solutions through the Sobolev embedding. In the meantime, those theories are focused on the perturbation with the Gaussian tail in large velocity. Thus it has been a challenging task to look for an enlarged function space of the perturbed solutions either with the lower regularity in space and velocity variables or with the slower large-velocity decay such that the existence with the uniqueness principle still can be achieved. Indeed, a lot of great progresses have been made in these two directions for the cutoff collision kernel. In particular, we would mention that

- Guo [33] developed a robust $L^2 - L^{\infty}$ interplay approach for the cutoff Boltzmann equation in general bounded domains. Since then, there have existed extensive studies of formulation of singularity and regularity of such solutions induced by the boundary, see [35] and references therein. Note that an $L^2 \cap L^{\infty}_2$ solution for the case of the whole space was also constructed in [58] by the multiple Duhamel iterations.
- Gualdani-Mischler-Mouhot [30] developed a new general perturbation theory with the perturbation having the only polynomial tail in large velocity for the cutoff Boltzmann equation in torus, and it also produced many applications in kinetic theory, for instance, [12, 14, 15] and references therein.

It is extremely hard to extend those results in [30, 33] to the non-cutoff Boltzmann equation due to the difficulty arising from the angular singularity of the Boltzmann collision integral, whereas we may refer to [34, 43] and [14, 15] for the corresponding development in the context of the Landau equation with the explicit velocity diffusion property. Recently, for the non-cutoff Boltzmann equation in torus, Alonso-Morimoto-Sun-Yang [12] extent the result in [30] to construct the classical solutions with polynomial tails; see also the independent works [36] and [37]. Moreover, the same authors [11] established the unique existence of solutions in the $L^2 \cap L^{\infty}$ setting via the De Giorgi type argument (cf. [28]) with the help of a strong averaging lemma.

In the current work, we are devoted to studying the smoothness of a class of unique low-regularity solutions to the non-cutoff Boltzmann equation near global Maxwellians in torus. In fact, instead of directly using the $L^{\infty}$ space, motivated by the early works [25, 26, 51], the construction of solutions can be carried out in the Wiener algebra that is the space of all
integrable functions on the torus whose Fourier series are absolutely convergent, cf. [24]. The goal of this work is to further establish the Gevrey smoothness of the solution in both space and velocity variables uniformly for all positive time with the Gevrey index depending only on the angular singularity. This then provides a complete well-posedness Boltzmann theory for existence, uniqueness and regularity of global solutions with low-regularity initial data. The further review with focus on the regularity issue for the non-cutoff Boltzmann equation will be provided later on.

1.1. Boltzmann equation. The spatially inhomogeneous Boltzmann equation in torus reads as

$$\partial_t F + v \cdot \nabla_x F = Q(F,F).$$

(1.1)

Here, the unknown $F(t,x,v) \geq 0$ stands for the density distribution function of gas particles with position $x = (x_1,x_2,x_3) \in \mathbb{T}^3$ and velocity $v = (v_1,v_2,v_3) \in \mathbb{R}^3$ at time $t > 0$. The Boltzmann collision operator on the right hand side of (1.1) is bilinear and acts only on velocity variables, taking the form of

$$Q(G,F)(v) = \int_{\mathbb{S}^2} B(v-v_s,\sigma)|G(v'_s)F(v') - G(v_s)F(v)|\ d\sigma dv_s.$$  

In the above integrand the velocity pairs $(v',v'_s)$ and $(v,v_s)$ are given by the relation

$$\begin{cases} v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2}\sigma, \\ v'_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2}\sigma, \end{cases}$$

with $\sigma \in \mathbb{S}^2$, according to conservations of molecular momentum and energy before and after an elastic collision

$$v' + v'_s = v + v_s, \quad |v'|^2 + |v'_s|^2 = |v|^2 + |v_s|^2.$$  

Moreover, the cross section $B(v-v_s,\sigma)$ depends only on the relative speed $|v - v_s|$ and the deviation angle $\theta$ with $\cos \theta = \langle \frac{v - v_s}{|v - v_s|}, \sigma \rangle$. We assume that $B(v-v_s,\sigma)$ is supported without loss of generality on $0 \leq \theta \leq \pi/2$ such that $\cos \theta \geq 0$ and also assume that it takes the specific form

$$B(v-v_s,\sigma) = |v - v_s|^\gamma b(\cos \theta),$$  

(1.2)

where $|v - v_s|^\gamma$ is called the kinetic part with $-3 < \gamma \leq 1$, and $b(\cos \theta)$ is called the angular part satisfying that there are $C_b > 1$ and $0 < s < 1$ such that

$$\frac{1}{C_b \theta^{1+2s}} \leq \sin \theta b(\cos \theta) \leq \frac{C_b}{\theta^{1+2s}}, \quad \forall \theta \in (0, \frac{\pi}{2}],$$  

(1.3)

We are concerned with the solution to the Boltzmann equation (1.1) around the normalized global Maxwellian $\mu = \mu(v) = (2\pi)^{-3/2}e^{-|v|^2/2}$. Thus, let $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v)$, then the reformulated unknown $f = f(t,x,v)$ satisfies that

$$\partial_t f + v \cdot \nabla_x f - \mathcal{L} f = \Gamma(f,f),$$  

(1.4)

with the linearized collision operator $\mathcal{L}$ and the nonlinear collision operator $\Gamma(\cdot,\cdot)$ respectively given by

$$\mathcal{L} f = \mu^{-1/2}Q(\mu,\sqrt{\mu}f) + \mu^{-1/2}Q(\sqrt{\mu}f,\mu),$$  

(1.5)

and

$$\Gamma(g,h) = \mu^{-1/2}Q(\sqrt{\mu}g,\sqrt{\mu}h).$$  

(1.6)
Due to the fact that the Boltzmann collision term $Q(F; F)$ admits five collision invariants $1, v$ and $|v|^2$, a solution of (1.1) with suitable regularity and integrability in velocity has conservations of total mass, momentum and energy, so that for simplicity we always assume that $f(t, x, v)$ satisfies

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1, v, |v|^2) \sqrt{p} f(t, x, v) \, dv \, dx = 0$$ (1.7)

for any $t \geq 0$. In particular, (1.7) should be satisfied for all $t > 0$ if it holds true initially.

1.2. Norms, spaces and results. The linearized operator $L$ is self-adjoint and non-positive definite on $L^2_v$, satisfying that there is a constant $c > 0$ such that

$$- (L f, f)_{L^2_v} \geq c |f|^2_{D}$$ (1.8)

for any $f$ in $(\ker L)^\perp$. Here, the dissipative norm $| \cdot |_D$ can be characterized in two kinds of ways by either the triple norm $\| \cdot \|$ in [8] or the anisotropic norm $| \cdot |_{N^{\gamma,s}}$ in [29], respectively defined as

$$\|f\|^2 := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(v - v_*, \sigma) \mu_*(f - f')^2 \, d\sigma dv_*, $$

$$+ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(v - v_*, \sigma) f^2_\sigma (\sqrt{\mu} - \sqrt{\mu'})^2 \, d\sigma dv_*, $$

and

$$|f|^{2}_{N^{\gamma,s}} := \| \langle v \rangle^{s+\frac{3}{2}} f \|_{L^2_v}^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} (\langle v \rangle \langle v' \rangle)^{\frac{3}{2}+\frac{2s+1}{2}} (f' - f)^2 \, d(v,v')d\sigma dv' \leq 1 dv' ,$$

where we have used the standard notations $f' = f(v')$, $f_\sigma = f(v_*)$, $\mu' = \mu(v')$, $\mu_\sigma = \mu(v_*)$ for shorthand, $\langle \cdot \rangle = (1+|\cdot|^2)^{1/2}$, and the anisotropic metric $d(v,v') = \{|v-v'|^2 + \frac{1}{4}(|v|^2-|v'|^2)^2\}^{1/2}$. The third way to characterize $| \cdot |_D$, recently introduced in [2], is to use the norm $\|(a^{1/2})^w f\|_{L^2_v}$, where $(a^{1/2})^w$ stands for the Weyl quantization with symbol $a^{1/2}$. The definition of $a^{1/2}$ as well as some basic facts on the symbolic calculus will be given in Section 2 later on. Moreover, in terms of [29, (2.13)-(2.15)], [8, Proposition 2.1] and [2, Theorem 1.2], one has the equivalence of those norms as

$$|f|^2_{L^2_v} \sim \|f\|^2 \sim |f|^2_{N^{\gamma,s}} \sim \|(a^{1/2})^w f\|_{L^2_v}^2 \sim -(L f, f)_{L^2_v} + \| \langle v \rangle^\ell f \|_{L^2_v}^2$$ (1.9)

for any suitable function $f$ and for any $\ell \in \mathbb{R}$.

Throughout the paper we denote the Fourier transform of $f(t, x, v)$ with respect to space variable $x \in \mathbb{T}^3$ by

$$\hat{f}(t, k, v) = \mathcal{F}_x f(t, k, v) = \int_{\mathbb{T}^3} e^{-ik \cdot x} f(t, x, v) \, dx, \quad k \in \mathbb{Z}^3.$$

Then, to look for a solution $f = f(t, x, v)$, we define the mixed Lebesgue space $L^p_k L^q_T L^r_v$ with the norm

$$\|f\|_{L^p_k L^q_T L^r_v} := \left\{ \left( \int_{\mathbb{T}^3} \int_0^T \|\hat{f}(t, k, \cdot)\|_{L^q_v}^q \, dt \, d\Sigma(k) \right)^\frac{1}{q}, \quad q < \infty, \right.$$

$$\left. \left( \int_{\mathbb{T}^3} \left( \sup_{0 < t < T} \|\hat{f}(t, k, \cdot)\|_{L^q_v} \right)^p \, d\Sigma(k) \right)^\frac{1}{p}, \quad q = \infty, \right.$$

for $1 \leq p, r < \infty$ and $1 \leq q \leq \infty$, where for convenience we have used $d\Sigma(k)$ through the paper to denote the discrete measure on $\mathbb{Z}^3$, meaning that $\int_{\mathbb{T}^3} g(k) \, d\Sigma(k) = \sum_{k \in \mathbb{Z}^3} g(k)$ for
any summable function $g = g(k)$ on $\mathbb{Z}^3$. For simplicity we also denote the corresponding space $L^p_{t}L^q_{x} L^r_v$ with $0 \leq \tau \leq T$ whenever functions are restricted only to the time interval $\tau < t < T$ and hence we may write $L^p_{t}L^q_{x} L^r_v = L^p_{0}L^q_{0}L^r_v$ for $\tau = 0$. Correspondingly, for an initial datum $f_0 = f_0(x,v)$ that does not involve time variable, we define the space $L^p_{k}L^q_{v}$ with the norm

$$
\|f_0\|_{L^p_{k}L^q_{v}} := \left( \int_{\mathbb{Z}^3} \|\hat{f_0}(k,\cdot)\|_{L^q_v}^p d\Sigma(k) \right)^{\frac{1}{p}},
$$

and the norm of higher order in space variables

$$
\|f_0\|_{L^p_{k,m}L^q_{v}} := \left( \int_{\mathbb{Z}^3} \|(k)^m \|\hat{f_0}(k,\cdot)\|_{L^q_v}^p d\Sigma(k) \right)^{\frac{1}{p}},
$$

for $1 \leq p, r < \infty$ and $m \geq 0$.

Finally we also introduce the Gevrey space under consideration. We say that $f = f(x,v) \in \mathcal{G}^r(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ of index $r \geq 1$ if $f \in C^\infty(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ and there is a constant $C$ such that

$$
\|\partial_x^\alpha \partial_v^\beta f\|_{L^2_{x,v}} \leq C|\alpha| + |\beta| + 1 \cdot [(|\alpha| + |\beta|)!]^r, \quad \forall \alpha, \beta \in \mathbb{Z}^3_+.
$$

With the preparation of notations above, the main result of the paper is stated as follows.

**Theorem 1.1.** Assume (1.2) and (1.3) with $\gamma \geq 0$ and $0 < s < 1$. There are $\varepsilon_0 > 0$ and $C > 0$ such that if the initial datum $F_0$ for (1.1) has the form of $F_0(x,v) = \mu + \sqrt{\nu}f_0(x,v) \geq 0$ with $f_0 \in L^1_{k}L^2_{v}$ satisfying (1.7) and

$$
\|f_0\|_{L^1_{k}L^2_{v}} \leq \varepsilon_0, \quad \text{(1.10)}
$$

then the Cauchy problem on the non-cutoff Boltzmann equation (1.1) or (1.4) with initial data $F|_{t=0} = F_0$ admits a unique global smooth solution $F(t,x,v) = \mu + \sqrt{\nu}f(t,x,v) \geq 0$ with $f \in L^1_{k}L^\infty_{x}L^2_{v}$ for any $T > 0$ and $f(t,\cdot,\cdot) \in \mathcal{G}^{\frac{1+2s}{2s}}(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ for any $t > 0$ such that the following estimate holds true:

$$
\int_{\mathbb{Z}^3} \left( \sup_{t > 0} \phi(t)^{\frac{1+2s}{2s}}(m + |\beta|) \langle k \rangle^m \|\partial_v^\beta \hat{f}(t,k,\cdot)\|_{L^2_v} \right) d\Sigma(k) \leq C^{m + |\beta| + 1} [(m + |\beta|)!]^{\frac{1+2s}{2s}}, \quad \text{(1.11)}
$$

for any $m \in \mathbb{Z}_+$ and $\beta \in \mathbb{Z}^3_+$. where $\phi(t) := \min \{t, 1\}$. In particular, for any $t > 0$, it also holds that $F(t,\cdot,\cdot) \in \mathcal{G}^{\frac{1+2s}{2s}}(\mathbb{T}^3_x \times \mathbb{R}^3_v)$.

We remark that the global existence and uniqueness of mild solutions $f(t,x,v)$ in the low-regularity space $L^1_{k}L^\infty_{x}L^2_{v}$ have been obtained by [24]. This paper aims to establish without imposing any additional assumption on initial data the global smoothness of such low-regularity solutions in the sense that $f(t,\cdot,\cdot) \in \mathcal{G}^{(1+2s)/2s}$ for any $t > 0$ and the quantitative estimate (1.11) is satisfied globally in time.

### 1.3. Related literature on regularity.

For the Boltzmann collision operator without angular cutoff, the grazing collisions may induce the velocity diffusion similar to the case of the Landau operator. Due to this, the solution to the non-cutoff Boltzmann equation has a smoothing effect in velocity variables. The rigorous mathematical proof was first given by Desvillettes in [19] and [20] for the non-cutoff Kac equation and for the non-cutoff spatially homogeneous Boltzmann equation with Maxwell molecule potentials in two dimensions, respectively, where $C^\infty$ regularization is proved.
For general collision kernels with an angular singularity, the fundamental important work Alexandre-Desvillettes-Villani-Wennberg [1] found out that the Boltzmann operator behaves as the fractional Laplacian operator in velocity variables in the sense that
\[
F \mapsto -Q(G, F) \sim C_G(-\Delta_v)^s F + \text{lower order terms},
\]
where the fractional order diffusion property is local for any finite velocity \(|v| < R\) with \(R < \infty\) so that \(C_G\) may also depend on \(R\). For (1.12), the global nonlinear sharp version was resolved by Gressman-Strain [29], and the global linearized version was recently obtained by Alexandre-Li-Hérau [2]. Note that the proof of [2] is based on the multiplier method and the Wick quantization together with the careful analysis of the symbolic properties of the Weyl symbol of the Boltzmann collision operator.

Motivated by (1.12), similarly for treating the heat equation with the fractional Laplacian \((-\Delta_v)^s\), it has been expected that any weak solution of the fully nonlinear spatially homogeneous Boltzmann equation belongs to the Gevrey class \(G^{1/2s}(\mathbb{R}^3)\) at any positive time. The answer was confirmed by Barbaroux-Hundertmark-Ried-Vugalter [13] for the Maxwell molecule model. Readers may refer to [13, 27, 46, 47, 52, 53] and references therein for an almost complete list of literature with focus on the smoothness effect for the spatially homogeneous Boltzmann equation.

In the spatially inhomogeneous case, it is more difficult to treat the regularity problem due to the presence of the transport term so that the equation is degenerate in space variables. Series of works have been done by AMUXY [3, 4, 6] in perturbation framework under suitably strong assumptions on initial data in terms of the generalized uncertainty principle and the hypoelliptic regularisation basing on the complex multiplier estimates. Regarding the Gevrey regularity of solutions, inspired by [17, 18, 48, 54], it also can be conjectured that any finite-regularity solution of the spatially inhomogeneous Boltzmann equation belongs to the Gevrey class \(G^{1/2s}(\mathbb{R}^3_x \times \mathbb{R}^3_v)\) at any positive time. The conjecture was recently justified by Morimoto-Xu [55] with \(s = 1\) for the case of the Landau equation with the Maxwell molecule potentials. For the Boltzmann case, Chen-Hu-Li-Zhan [16] obtained the Gevrey regularity in \(G^{(1+2s)/2s}(\mathbb{R}^3_x \times \mathbb{R}^3_v)\) provided that the initial data belong to \(H^\ell_m(\mathbb{R}^3_x \times \mathbb{R}^3_v)\) with the Sobolev exponent \(\ell \geq 6\) and the order of velocity moments \(m\) large enough. It is still a problem to prove even the same Gevrey regularity as in [16] for the lower-regularity global solution near global Maxwellians.

In the end, we also mention extensive studies of the conditional regularity of solutions to the spatially inhomogeneous Boltzmann equation for general initial data in [39–42, 56] by Silvestre together with his collaborators. It would be interesting to develop a self-contained theory of both existence and regularity without any extra condition on solutions to the spatially inhomogeneous non-cutoff Boltzmann equation with initial data allowing to have possibly large oscillations in space variable, for instance, see [22, 23] in the cutoff case where the \(L^\infty\) norm can be arbitrarily large but the relative entropy is small enough.

1.4. **Strategy of the proof.** To clarify the argument roughly, we first explain how to use the time-weighted energy method as well as the derivative iteration technique to capture the Gevrey regularity basing on the following linear toy model with diffusions in both velocity and space variables
\[
\left(\partial_t + v \cdot \nabla_x + (-\Delta_x)^{\frac{2s}{1+2s}} + (-\Delta_v)^s\right) f = 0, \quad f|_{t=0} = f_0,
\]
where the velocity diffusion operator is consistent with that of the linearized Boltzmann operator as in (1.8) and (1.9) while the space diffusion operator is inspired by [2]. Let \( f \) be a smooth solution to the above Cauchy problem. We then may perform the energy estimates of \( \nabla_x^m f \) for any integer \( m \geq 0 \). Since we are concerned with the regularity of solutions, it is natural to introduce an auxiliary function of \( t \) that vanishes at \( t = 0 \) in order to overcome the singularity of \( \nabla_x^m f |_{t=0} \). Choosing \( t \mapsto t^{1+2s}m \) as the desired time weight function, an informal computation gives that

\[
\frac{1}{2} \frac{d}{dt} \left( t^{1+2s}m \| \nabla_x^m f \|_{L^2}^2 \right) + t^{1+2s}m (\Delta_x)^{1+2s} \nabla_x^m f \|_{L^2}^2 + t^{1+2s}m (\Delta_v)^{1+2s} \nabla_x^m f \|_{L^2}^2
\]

\[
= 1 + 2s \cdot 2m t^{1+2s} \nabla_x^m f \|_{L^2}\]

\[
\leq ct^{1+2s}m (\Delta_x)^{1+2s} \nabla_x^m f \|_{L^2}^2 + C_{\varepsilon} \cdot m^{1+2s} \cdot (m-1) \nabla_x^{m-1} f \|_{L^2}^2,
\]

for an arbitrary constant \( \varepsilon > 0 \), where the last inequality follows from the interpolation inequality. This ensures to conclude the Gevrey regularity in \( x \) by induction on \( m \) and then the Gevrey regularity in \( v \) in a similar way. Note that one can not expect a Gevrey index with respect to \( v \) variable better than \( x \) variable because the spatial derivatives have to be induced when making estimates on the commutator between the transport operator and \( \partial_v^m \).

We remark that the rigorous counterpart of the above informal calculation can be achieved by introducing some kind of regularization operators that commute with the diffusion parts.

Back to the nonlinear Boltzmann equation with non-cutoff potentials, we follow the similar strategy as for the above linear toy model. However, new difficulties arise from the non-trivial treatment of commutators between the collision part and the regularization operators

\[
(1 - \Delta_x)^{-1/2} \text{ and } (1 - \delta \Delta_v)^{-1}
\]

for \( \delta > 0 \) suitably small. To overcome these difficulties we will make use of the symbolic calculus developed in [2]. The argument here will be more subtle since the regularity iteration procedure begins with the solutions of quite low regularity from the existence theory in [24] corresponding to (1.11) with \( m = |\beta| = 0 \). To deal with the commutator between the collision operator and the regularization \( (1 - \delta \Delta_x)^{-1/2} \) or its Fourier counterpart \( (1 + \delta |k|^2)^{-1/2} \), we give in Lemma 2.6 a new elementary inequality

\[
\frac{\langle k \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \sum_{1 \leq j \leq m-1} \binom{m}{j} \langle k - \ell \rangle^j (\ell)^{m-j} + \frac{2 \langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}} + \frac{2 \langle \ell \rangle^m}{(1 + \delta |\ell|^2)^{1/2}},
\]

which is crucially used for iteration estimates on such low regularity solution. Moreover, to estimate the commutator between the collision operator and the regularization operator \( (1 - \delta \Delta_v)^{-1} \), one main view is to write

\[
(1 - \delta \Delta_v)^{-1} \Gamma(f, \partial_v^\beta f) = (1 - \delta \Delta_v)^{-1} \Gamma(f, (1 - \Delta \Delta_v)(1 - \delta \Delta_v)^{-1} \partial_v^\beta f),
\]

see (4.11) in the proof of Lemma 4.4, for instance. Such view is useful for dealing with the commutator by the Leibniz formula without involving any pseudo-differential calculus.

1.5. Arrangement of the paper. The rest of this paper is arranged as follows. In Section 2, we will list a few preliminary facts that will be used throughout the proof of Theorem 1.1. Section 3 and Section 4 are the key parts, devoted to proving the Gevrey regularity in space variables and velocity variables, respectively. In Section 5, we will complete the proof of Theorem 1.1. In the appendix Section 6, we will give some basic facts on the Weyl and Wick quantizations of symbol class.
2. Preliminaries

We list here a few preliminary facts that will be used throughout the paper. In the following discussion we denote by $\langle D_x \rangle^\theta$ for $\theta \in \mathbb{R}$ the Fourier multiplier in $x$ variable, that is,

$$\mathcal{F}_x(\langle D_x \rangle^\theta h)(k) = \langle k \rangle^\theta \mathcal{F}_x h(k).$$

Recall that $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\mathcal{F}_x$ stands for the partial Fourier transform in $x$ variable. Similarly, letting $\mathcal{F}_v$ be the partial Fourier transform in $v$ variable,

$$\mathcal{F}_v(\langle D_v \rangle^\theta h)(\eta) = \langle \eta \rangle^\theta \mathcal{F}_v h(\eta),$$

with $\eta \in \mathbb{R}^3$ being the Fourier dual variable of $v$.

We first recall the global symbolic calculus for the linearized Boltzmann operator that was established by [2]. Denote by $b^w$ and $b^{\text{Wick}}$, respectively, the Weyl and Wick quantizations of a symbol $b = b(v, \eta)$, with $\eta$ the Fourier dual variable of $v$. Note the symbols considered in this work are independent of $(x, k)$ variables and thus the corresponding Weyl or Wick quantizations are pseudo-differential operators acting only on $v$ variable. The basic properties of the quantization of symbols are listed in Appendix 6, and one may refer to [38, 45] for extensive discussions. We list two classes of symbols under consideration. One is

$$S(1, |dv|^2 + |d\eta|^2),$$

and the other is

$$S(\tilde{a}, |dv|^2 + |d\eta|^2).$$

Here and below $|dv|^2 + |d\eta|^2$ stands for the flat metric and

$$\tilde{a} = \tilde{a}(v, \eta) := \langle v \rangle^\gamma (1 + |v|^2 + |v \wedge \eta|^2 + |\eta|^2)^s, \quad (v, \eta) \in \mathbb{R}^6,$$

with $\gamma, s$ the numbers given in (1.2) and (1.3), and $v \wedge \eta$ the cross product, that is,

$$v \wedge \eta = (v_2\eta_3 - v_3\eta_2, v_3\eta_1 - v_1\eta_3, v_1\eta_2 - v_2\eta_1).$$

Recall that we say $b \in S(\tilde{a}, |dv|^2 + |d\eta|^2)$ if

$$|\partial^\alpha_v \partial^\beta_\eta b(v, \eta)| \leq C_{\alpha, \beta} \tilde{a}(v, \eta), \quad \forall \alpha, \beta \in \mathbb{Z}_+^3,$$

with $C_{\alpha, \beta}$ constants depending on $\alpha$ and $\beta$. Furthermore, by $b \in S(\tilde{a}, |dv|^2 + |d\eta|^2)$ uniformly with respect to a parameter $\tau$, it means that the constants $C_{\alpha, \beta}$ in the above estimate are independent of $\tau$. Similar things hold for the definition of $S(1, |dv|^2 + |d\eta|^2)$. An elementary property to be frequently used is the $L^2$ continuity theorem in the class $S(1, |dv|^2 + |d\eta|^2)$, saying (cf. [45, Theorem 2.5.1] for instance) that if $b \in S(1, |dv|^2 + |d\eta|^2)$ then there exists a constant $C$ such that

$$\|b^w h\|_{L^2} \leq C\|h\|_{L^2}, \quad \forall \ h \in L^2_v.$$  \hspace{1cm} (2.2)

Note that if one further assumes $b \in S(1, |dv|^2 + |d\eta|^2)$ uniformly with respect to a parameter $\tau$ then the constant $C$ in (2.2) will be independent of $\tau$. Let us also recall here the composition formula of the Weyl quantization. Let $M_j, j = 1, 2$, be two admissible weights for the flat metric $|dv|^2 + |d\eta|^2$, see Section 6 for the definition of admissible weights. If $b_j \in S(M_j, |dv|^2 + |d\eta|^2)$ then $b_1 b_2 \in S(M_1 M_2, |dv|^2 + |d\eta|^2)$ and we have the following composition formula for the Weyl quantization:

$$b^w_1 b^w_2 = (b_1 b_2)^w + q^w, \quad q \in S(M_1 M_2, |dv|^2 + |d\eta|^2).$$  \hspace{1cm} (2.3)
Moreover, if it additionally holds that \( \partial_\alpha \partial_\beta b_j \in S(\tilde{M}_j, |dv|^2 + |d\eta|^2) \subset S(M_j, |dv|^2 + |d\eta|^2) \) for any \( \alpha, \beta \in \mathbb{Z}_+^3 \) with \( |\alpha| + |\beta| = 1 \), then the symbol \( q \) in (2.3) satisfies

\[
q \in S(M_1 \tilde{M}_2, |dv|^2 + |d\eta|^2).
\]

This yields that the commutator between \( b_1^w \) and \( b_2^w \), denoted by \([b_1^w, b_2^w]\), is also a Weyl quantization of some symbol, that is,

\[
[b_1^w, b_2^w] = \tilde{q}, \quad \tilde{q} \in S(M_1 \tilde{M}_2, |dv|^2 + |d\eta|^2),
\]

provided that \( \partial_\alpha \partial_\beta b_j \in S(\tilde{M}_j, |dv|^2 + |d\eta|^2) \) for any \( \alpha, \beta \in \mathbb{Z}_+^3 \) with \( |\alpha| + |\beta| = 1 \). Recall that the commutator \([T_1, T_2]\) between two operators \( T_1 \) and \( T_2 \) is defined by

\[
[T_1, T_2] = T_1 T_2 - T_2 T_1.
\]

Now we are ready to state the symbolic calculus established by Alexandre-Hérau-Li [2].

**Proposition 2.1** (Proposition 1.4 and Lemma 4.3 of [2]). Suppose that the non-cutoff Boltzmann collision kernel satisfies (1.2) and (1.3) with \( 0 < s < 1 \) and \( \gamma > -3 \). Then the linearized collision operator \( \mathcal{L} \) defined by (1.5) can be written as

\[
\mathcal{L} = -a^w - \mathcal{R},
\]

such that the following properties are fulfilled by \( a \) and \( \mathcal{R} \).

(i) There exists a positive constant \( C \geq 1 \) such that

\[
C^{-1} \hat{a}(v, \eta) \leq a(v, \eta) \leq C \hat{a}(v, \eta), \quad \forall (v, \eta) \in \mathbb{R}^6,
\]

with \( \hat{a} \) defined by (2.1). Moreover \( a \in S(\hat{a}, |dv|^2 + |d\eta|^2) \).

(ii) As for the operator \( \mathcal{R} \) we have for any \( \varepsilon > 0 \),

\[
\| \mathcal{R} h \|_{L^2_{\varepsilon}} \leq \varepsilon \| a^w h \|_{L^2_{\varepsilon}} + C_{\varepsilon} \| \langle v \rangle^{2s+\gamma} h \|_{L^2_{\varepsilon}}, \quad \forall h \in \mathcal{S}(\mathbb{R}_v^3),
\]

with \( C_{\varepsilon} \) a constant depending on \( \varepsilon \). Here and below \( \mathcal{S}(\mathbb{R}_v^3) \) stands for the Schwartz space in \( \mathbb{R}_v^3 \).

(iii) The operators \( a^w \) and \( (a^{1/2})^w \) are invertible on \( L^2_{\varepsilon} \) and their inverses can be respectively written as

\[
(a^w)^{-1} = H_1(a^{-1})^w = (a^{-1})^w H_2
\]

and

\[
[(a^{1/2})^w]^{-1} = G_1(a^{-1/2})^w = (a^{-1/2})^w G_2,
\]

with \( H_j, G_j \) being bounded operators on \( L^2_{\varepsilon} \).

**Lemma 2.2.** Let \( b^w \) be the Weyl quantization of symbol

\[
b = b(v, \eta) \in S(1, |dv|^2 + |d\eta|^2).
\]

Then, for any \( h \in L^2_{\varepsilon} \) with \( (a^{1/2})^w h \in L^2_{\varepsilon} \), it holds that

\[
\| b^w(a^{1/2})^w h \|_{L^2_{\varepsilon}} + \| (a^{1/2})^w b^w h \|_{L^2_{\varepsilon}} + \| [(a^{1/2})^w, b^w] h \|_{L^2_{\varepsilon}} \leq C \| (a^{1/2})^w h \|_{L^2_{\varepsilon}}.
\]

**Proof.** It follows from (2.2) that

\[
\| b^w(a^{1/2})^w h \|_{L^2_{\varepsilon}} \leq C \| (a^{1/2})^w h \|_{L^2_{\varepsilon}}.
\]

Observe

\[
\| [(a^{1/2})^w, b^w] h \|_{L^2_{\varepsilon}} \leq \| b^w(a^{1/2})^w h \|_{L^2_{\varepsilon}} + \| [(a^{1/2})^w, b^w] h \|_{L^2_{\varepsilon}}.
\]
Then it remains to control the commutator term. In view of (2.4), we can write \([(a^{1/2})^w, b^w] = \tilde{q}^w\) for some \(\tilde{q} \in S(\tilde{a}^{1/2}, |dv|^2 + |d\eta|^2)\). Thus, using the composition formula (2.3) of Weyl quantization and the assertion (iii) in Proposition 2.1, we conclude that \(\tilde{q}^w [(a^{1/2})^w]^{-1}\) is bounded on \(L^2_v\). Consequently, writing that
\[
[(a^{1/2})^w, b^w] = [(a^{1/2})^w, b^w] [(a^{1/2})^w]^{-1} (a^{1/2})^w,
\]
one has
\[
\|[(a^{1/2})^w, b^w] h\|_{L^2_v} \leq C \| (a^{1/2})^w h \|_{L^2_v}.
\]
The proof of Lemma 2.2 is thus completed. \(\square\)

The second fact is concerned with the trilinear estimate, which says (cf. [9, theorem 1.2] or [29, theorem 2.1]) that there is a constant \(C\) such that
\[
\| (\Gamma(f, g), h) \|_{L^2_v} \leq C \| f \|_{L^2_v} \|[a^{1/2}]^w g\|_{L^2_v} \| (a^{1/2})^w h \|_{L^2_v}, \quad \forall f, g, h \in \mathcal{S}(\mathbb{R}^3),
\]
where the equivalence in (1.9) has been used and we also recall that \(\mathcal{S}(\mathbb{R}^3)\) stands for the Schwartz space in \(\mathbb{R}^3\). Furthermore, we mainly employ the counterpart of the above estimate after performing the partial Fourier transform in \(x\) variable. Precisely, recall that the Fourier transform in \(x\) variable for the nonlinear term \(\Gamma(f, g)\) in (1.6) is given by
\[
\hat{\Gamma}(\hat{f}, \hat{g})(k, v) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^2} B(v-u, \sigma) [\mu(v)\mu(u)]^{1/2} \left( [\hat{f}(u') \ast \hat{g}(v')](k) - [\hat{f}(u) \ast \hat{g}(v)](k) \right) d\sigma du,
\]
where the convolutions are taken with respect to the Fourier variable \(k \in \mathbb{Z}^3\):
\[
[\hat{f}(u) \ast \hat{g}(v)](k) := \int_{\mathbb{Z}^3} \hat{f}(k-\ell, u) \hat{g}(\ell, v) d\Sigma(\ell),
\]
for any velocities \(u, v \in \mathbb{R}^3\). Then, by [24, Lemma 3.2]), the following estimate
\[
\| (\hat{\Gamma}(\hat{f}(k), \hat{g}(k)), \hat{h}(k)) \|_{L^2_v} \leq C \int_{\mathbb{Z}^3} \| \hat{f}(k-\ell) \|_{L^2_v} \| \hat{g}(\ell) \|_{D} \| \hat{h}(k) \|_{D} d\Sigma(\ell),
\]
or equivalently
\[
\| (\hat{\Gamma}(\hat{f}(k), \hat{g}(k)), \hat{h}(k)) \|_{L^2_v} \leq C \int_{\mathbb{Z}^3} \| \hat{f}(k-\ell) \|_{L^2_v} \| (a^{1/2})^w \hat{g}(\ell) \|_{L^2_v} \| (a^{1/2})^w \hat{h}(k) \|_{L^2_v} d\Sigma(\ell),
\]
holds true for any \(k \in \mathbb{Z}^3\) and for any \(f, g, h \in L^1_k(\mathcal{S}(\mathbb{R}^3))\).

The following lemma and corollary will be used in the proof later on.

**Lemma 2.3.** For any \(f, g \in L^2_v\) with \((a^{1/2})^w g \in L^2_v\), it holds that \((a^{-1/2})^w \Gamma(f, g) \in L^2_v\) with
\[
\|(a^{-1/2})^w \Gamma(f, g)\|_{L^2_v} \leq C \| f \|_{L^2_v} \|(a^{1/2})^w g\|_{L^2_v}.
\]
Moreover, for any \(f, g \in L^1_k L^2_v\) such that \((a^{1/2})^w g \in L^1_k L^2_v\), it holds that
\[
\|(a^{-1/2})^w \Gamma(\hat{f}(k), \hat{g}(k))\|_{L^2_v} \leq C \int_{\mathbb{Z}^3} \| \hat{f}(k-\ell) \|_{L^2_v} \|(a^{1/2})^w \hat{g}(\ell)\|_{L^2_v} d\Sigma(\ell),
\]
for any \(k \in \mathbb{Z}^3\), where \(\hat{\Gamma}(\hat{f}(k), \hat{g}(k))\) given in (2.6) stands for the Fourier transform of \(\Gamma(f, g)\) in space variables.
Before giving the proof of Lemma 2.3, we notice that $\mathcal{L}h = \Gamma(\sqrt{\mu}, h) + \Gamma(h, \sqrt{\mu})$. Then, as an immediate consequence of the first assertion in Lemma 2.3, we have the following

**Corollary 2.4.** For any $h \in L^2_v$ such that $(a^{1/2})^w h \in L^2_v$, it holds that

$$\|(a^{1/2})^w \mathcal{L}h\|_{L^2_v} \leq C\|(a^{1/2})^w h\|_{L^2_v}. $$

**Proof of Lemma 2.3.** (a) We first claim that $(a^{1/2})^w \Gamma(f, g) \in L^2_v$ for any $f, g \in \mathcal{S}(\mathbb{R}^3_v)$ with the estimate

$$\|(a^{1/2})^w \Gamma(f, g)\|_{L^2_v} \leq C\|f\|_{L^2_v}\|(a^{1/2})^w g\|_{L^2_v}, \quad \forall f, g \in \mathcal{S}(\mathbb{R}^3_v).$$

In fact, observe that $(a^{1/2})^w$ is self-adjoint on $L^2_v$. Then, for any $h \in \mathcal{S}(\mathbb{R}^3_v)$, it follows from (2.5) that

$$\|(a^{1/2})^w \Gamma(f, g), h\|_{L^2_v} = \|(\Gamma(f, g), (a^{1/2})^w h)\|_{L^2_v} \leq C\|f\|_{L^2_v}\|(a^{1/2})^w g, h\|_{L^2_v} \leq C\|f\|_{L^2_v}\|(a^{1/2})^w g\|_{L^2_v}\|h\|_{L^2_v},$$

where we have used (2.3) and (2.2) as well as Proposition 2.1 in the last inequality. This together with the fact that the Schwartz space $\mathcal{S}(\mathbb{R}^3_v)$ is dense in $L^2_v$ give the desired claim.

(b) We then consider the case of $f \in L^2_v$ and $g \in \mathcal{S}(\mathbb{R}^3_v)$. In fact, using again the fact that $\mathcal{S}(\mathbb{R}^3_v)$ is dense in $L^2_v$, we can find a sequence of smooth functions $f_n \in \mathcal{S}(\mathbb{R}^3_v)$ such that $\|f_n - f\|_{L^2_v} \to 0$ as $n \to +\infty$. This together with (2.8) imply that $(a^{1/2})^w \Gamma(f_n, g)$ is a Cauchy sequence in $L^2_v$ with a limit denoted as $m \in L^2_v$. Next, we show that

$$m = (a^{1/2})^w \Gamma(f, g) \text{ in } L^2_v. $$

Indeed, since it holds that $(a^{1/2})^w \Gamma(f_n, g) \to m$ and $f_n \to f$ in $L^2_v$-norm, we are able to extract a subsequence $\{f_{n_j}\}_{j \geq 1}$ of $f_n$, such that $(a^{1/2})^w \Gamma(f_{n_j}, g) \to m$ and $f_{n_j} \to f$ pointwise a.e. in $\mathbb{R}^3$. As a result, by the Dominated Convergence Theorem, one has

$$(a^{1/2})^w \Gamma(f_{n_j}, g) \to (a^{1/2})^w \Gamma(f, g) \text{ pointwise a.e. in } \mathbb{R}^3_v,$$ which proves (2.9). Thus, $(a^{1/2})^w \Gamma(f, g)$ is the limit of $(a^{1/2})^w \Gamma(f_n, g)$ in $L^2_v$. Moreover, it follows from (2.8) that

$$\|(a^{1/2})^w \Gamma(f, g)\|_{L^2_v} \leq C\|f\|_{L^2_v}\|(a^{1/2})^w g\|_{L^2_v},$$

for any $f \in L^2_v$ and $g \in \mathcal{S}(\mathbb{R}^3_v)$.

(c) Finally we suppose $f \in L^2_v$ and $(a^{1/2})^w g \in L^2_v$. Then, using the density argument again, we can find a sequence of functions $G_n \in \mathcal{S}(\mathbb{R}^3_v)$ such that $\|G_n - (a^{1/2})^w g\|_{L^2_v} \to 0$ as $n \to +\infty$. Define

$$g_n := [\left[(a^{1/2})^w\right]^{-1} G_n] \in \mathcal{S}(\mathbb{R}^3_v), \quad n \geq 1.$$ Then, by virtue of (2.3) and (2.2), it follows from (2.10) that

$$\|(a^{1/2})^w \Gamma(f, g_n - g_m)\|_{L^2_v} \leq C\|f\|_{L^2_v}\|(a^{1/2})^w (g_n - g_m)\|_{L^2_v} \leq C\|f\|_{L^2_v}\|G_n - G_m\|_{L^2_v}.$$ This shows that $(a^{1/2})^w \Gamma(f, g_n)$ is a Cauchy sequence in $L^2_v$ with $(a^{1/2})^w \Gamma(f, g)$ as its limit in $L^2_v$ by using the same argument as above along with the fact that

$$\|g_n - g\|_{L^2_v} = \left[\left[(a^{1/2})^w\right]^{-1} \{G_n - (a^{1/2})^w g\}\right]_{L^2_v} \leq C\|G_n - (a^{1/2})^w g\|_{L^2_v} \to 0.$$
Moreover, it holds that
\[ \| (a^{-1/2})^w \Gamma(f, g) \|_{L^2_v} \leq C \| f \|_{L^2_v} \| (a^{1/2})^w g \|_{L^2_v}. \]
This has proved the first assertion in Lemma 2.3. The second one for the counterpart after performing the Fourier transform in space variables can be treated in the same way via (2.7) instead of (2.5). The proof of Lemma 2.3 is thus completed. \(\square\)

The following technical lemma will be frequently used in treating estimates on \(\Gamma(g, h)\).

**Lemma 2.5.** For an arbitrarily given integer \(j_0 \geq 1\), it holds that
\[
\int_{Z^3} \left[ \int_0^T \left( \int_{Z^3} \sum_{1 \leq j \leq j_0} \| \hat{f}_j(t, k - \ell) \|_{L^2_v} \| (a^{1/2})^w \hat{g}_j(t, \ell) \|_{L^2_v} d\Sigma(\ell) \right)^2 dt \right]^{1/2} d\Sigma(k) \\
\leq \sum_{1 \leq j \leq j_0} \left( \int_{Z^3} \left( \int_0^T \| \hat{f}_j(t, k) \|_{L^2_v} d\Sigma(k) \right) \int_{Z^3} \left( \int_0^T \| (a^{1/2})^w \hat{g}_j(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k), \tag{2.11} \right)
\]
for any \(f_j \in L^1_k L^\infty_v L^2_v\) and any \(g_j\) such that \((a^{1/2})^w g_j \in L^1_k L^2_v L^2_v\) with \(1 \leq j \leq j_0\).

**Proof.** Using the triangle inequality that \(\| \sum_{j=1}^{j_0} A_j \|_{L^2(0, T)} \leq \sum_{j=1}^{j_0} \| A_j \|_{L^2(0, T)}\) for a sequence of functions \(A_j\) in \(L^2(0, T)\), it suffices to prove that the desired estimate (2.11) is satisfied for \(j_0 = 1\). We then write \(f\) for \(f_1\) and likewise \(g\) for \(g_1\). Direct computations give that
\[
\int_{Z^3} \left( \int_0^T \left( \int_{Z^3} \| \hat{f}(t, k - \ell) \|_{L^2_v} \| (a^{1/2})^w \hat{g}(t, \ell) \|_{L^2_v} d\Sigma(\ell) \right)^2 dt \right)^{1/2} d\Sigma(k) \\
\leq \int_{Z^3} \left[ \int_0^T \| \hat{f}(t, k - \ell) \|_{L^2_v}^2 \| (a^{1/2})^w \hat{g}(t, \ell) \|_{L^2_v}^2 dt \right]^{1/2} d\Sigma(k) \\
\leq \int_{Z^3} \left[ \int_0^T \| \hat{f}(t, k - \ell) \|_{L^2_v} \left( \int_0^T \| (a^{1/2})^w \hat{g}(t, \ell) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(\ell) \right] d\Sigma(k) \\
= \left( \int_{Z^3} \sup_{0 < t < T} \| \hat{f}(t, k) \|_{L^2_v} d\Sigma(k) \right) \int_{Z^3} \left( \int_0^T \| (a^{1/2})^w \hat{g}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k), \tag{2.12} \right)
\]
where we have used Minkowski’s inequality and Fubini’s theorem in the first and last inequalities, respectively. The proof of Lemma 2.5 is completed. \(\square\)

The following Lemma gives an elementary inequality that will be essentially adopted to treat the iterative estimates for obtaining the Gevrey regularity in space variables.

**Lemma 2.6.** There is a generic constant \(C > 0\) such that for any \(m \geq 1\) the following estimate
\[
\frac{\langle k \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \sum_{1 \leq j \leq m-1} \binom{m}{j} \langle k - \ell \rangle^j \langle \ell \rangle^{m-j} + \frac{2 \langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}} + \frac{2 \langle \ell \rangle^m}{(1 + \delta |\ell|^2)^{1/2}} \tag{2.13} \]
holds for any \(k, \ell \in \mathbb{Z}^3\) and any \(0 < \delta < 1\), with the convention that the summation term over \(1 \leq j \leq m - 1\) on the right hand side disappears when \(m = 1\).

**Proof.** First note that the function \(|k| \mapsto \langle k \rangle^m/(1 + \delta |k|^2)^{1/2}\) is nondecreasing in \(|k|\) when \(0 < \delta < 1\) and \(m \geq 1\). Then, in case of \(|k| \leq |\ell|\), it is direct to see
\[
\frac{\langle k \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \frac{\langle \ell \rangle^m}{(1 + \delta |\ell|^2)^{1/2}},
\]
so (2.13) holds true. In case of $|k| > |\ell|$, using $\langle k \rangle \leq \langle k - \ell \rangle + \langle \ell \rangle$, it follows that

$$\langle k \rangle^m \leq (\langle k - \ell \rangle + \langle \ell \rangle)^m = \sum_{0 \leq j \leq m} \binom{m}{j} \langle k - \ell \rangle^j \langle \ell \rangle^{m-j},$$

so one has

$$\frac{\langle k \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \sum_{1 \leq j \leq m-1} \binom{m}{j} \frac{\langle k - \ell \rangle^j \langle \ell \rangle^{m-j}}{(1 + \delta |k|^2)^{1/2}} + \frac{\langle k - \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}} + \frac{\langle \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}}.$$  

Then, to show (2.13) it suffices to verify that

$$\frac{\langle k - \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}} + \frac{\langle \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \frac{2 \langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}} + \frac{2 \langle \ell \rangle^m}{(1 + \delta |\ell|^2)^{1/2}}$$

(2.14)

for any $k$ and $\ell$ with $|k| > |\ell|$. Indeed, we consider two cases $|k| \geq 2|\ell|$ and $|\ell| \leq |k| \leq 2|\ell|$ as follows. For $|k| \geq 2|\ell|$, it holds that

$$\frac{\langle \ell \rangle^m}{(1 + \delta |\ell|^2)^{1/2}} \leq \frac{\langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}} \leq \frac{\langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}},$$

and since $|k| \geq 2|\ell|$ implies $|k - \ell| \leq |k| + |\ell| \leq |k| + \frac{1}{2} |k| = \frac{3}{2} |k|$ as well, similarly one has

$$\frac{\langle k - \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \frac{\langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}} \leq \frac{3}{2} \frac{\langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}},$$

so (2.14) is satisfied in case of $|k| \geq 2|\ell|$. Similarly, for $|\ell| \leq |k| \leq 2|\ell|$ that gives $|k - \ell| \leq |k| + |\ell| \leq 2|k|$, one has

$$\frac{\langle \ell \rangle^m}{(1 + \delta |\ell|^2)^{1/2}} \leq \frac{\langle k - \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}}, \quad \frac{\langle k - \ell \rangle^m}{(1 + \delta |k|^2)^{1/2}} \leq \frac{2 \langle k - \ell \rangle^m}{(1 + \delta |k - \ell|^2)^{1/2}},$$

that yield (2.14) as well. Therefore this shows (2.14) and completes the proof of Lemma 2.6. \[\square\]

3. Gevrey smoothing effect in spatial variable

In this section we start to study the nonlinear Cauchy problem on the reformulated equation (1.4) supplemented with $f|_{t=0} = f_0$. To the end, for convenience we always assume $\gamma \geq 0$ and $0 < s < 1$ for the collision kernel (1.2) and (1.3). First of all, we state the existence result established in [24].

**Proposition 3.1.** There are $\varepsilon_0 > 0$ and $C_0 > 0$ such that if the initial datum $f_0$ is chosen such that $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geq 0$ with $f_0 \in L^1_k L^2_v$ satisfying (1.7) and $\|f_0\|_{L^1_k L^2_v} \leq \varepsilon_0$, then the Cauchy problem on the non-cutoff Boltzmann equation (1.4) with $f|_{t=0} = f_0$ admits a unique global mild solution $f(t, x, v)$ such that $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geq 0$ with $f \in L^1_k L^\infty_v L^2_t$ for any $T > 0$ satisfying the estimate

$$\|f\|_{L^1_k L^\infty_v L^2_t} + \|f^{(a^{1/2})} f\|_{L^1_k L^2_v L^2_t} \leq C_0 \|f_0\|_{L^1_k L^2_v}.$$  

(3.1)

The main goal of this section is to further prove the Gevrey smoothness in space variable $x$ for the obtained solution $f(t, x, v)$.
Theorem 3.2. Let $\varepsilon_0 > 0$ be further small, then there is a constant $\tilde{C}_0 > 0$, depending only on $s, \gamma, \varepsilon_0$ and the constant $C_0$ above, such that for any $0 < T < \infty$ and any integer $m \in \mathbb{Z}_+$, the solution $f(t, x, v)$ obtained in Proposition 3.1 satisfies

$$
\nabla_x^m f \in L^1_k L^\infty_{r,t} L^2_v, \quad \nabla_x^m f, (a^{1/2})^w \nabla_x f \in L^1_k L^2_{r,t} L^2_v
$$

(3.2)

for any small $\tau > 0$, with the quantitative estimate

$$
\int_{\mathbb{Z}^3} \langle k \rangle^m \left( \sup_{0 < t < T} \phi(t) \| \hat{f}(t, k) \|_{L^2_v} \right) d\Sigma(k) + \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k)
$$

$$
+ \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| (a^{1/2})^w \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq \tilde{C}_0^{m+1}(m!)^{\frac{1+2s}{2s}}.
$$

Here and below we have denoted $\phi(t) = \min\{t, 1\}$ and $s = \frac{1+2s}{2s}$ with $s$ the parameter given in (1.3).

Theorem 3.2 is just an immediate consequence of the following two propositions, by using induction on $m$.

Proposition 3.3 (Initial step for $m = 0, 1$). Under the same assumption as in Theorem 3.2, there is a constant $C_1$, depending only on $s, \gamma$ and the number $C_0$ in (3.1), such that (3.2) holds true for $m = 0, 1$ and the following estimate

$$
\int_{\mathbb{Z}^3} \langle k \rangle^m \left( \sup_{0 < t < T} \phi(t) \| \hat{f}(t, k) \|_{L^2_v} \right) d\Sigma(k) + \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k)
$$

$$
+ \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| (a^{1/2})^w \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq C_1 \| f_0 \|_{L^1_k L^2_v},
$$

is satisfied for $m = 0, 1$.

Proposition 3.4 (Inductive regularity). Let $f(t, x, v)$ satisfy the same conditions as in Theorem 3.2 and let $C_1$ be the constant constructed in the previous Proposition 3.3. Then there is a constant $\tilde{C}_0 \geq C_1$, depending only on $s, \gamma$ and the number $C_0$ in (3.1), such that for an integer $m \geq 2$, we have (3.2) as well as

$$
\int_{\mathbb{Z}^3} \langle k \rangle^m \left( \sup_{0 < t < T} \phi(t) \| \hat{f}(t, k) \|_{L^2_v} \right) d\Sigma(k) + \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k)
$$

$$
+ \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| (a^{1/2})^w \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq \tilde{C}_0^{m-1} [(m-1)!]^{\frac{1+2s}{2s}},
$$

provided that (3.2) together with the following estimate

$$
\int_{\mathbb{Z}^3} \langle k \rangle^m \left( \sup_{0 < t < T} \phi(t) \| \hat{f}(t, k) \|_{L^2_v} \right) d\Sigma(k) + \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k)
$$

$$
+ \int_{\mathbb{Z}^3} \langle k \rangle^m \left( \int_0^T \phi(t) 2^{km} \| (a^{1/2})^w \hat{f}(t, k) \|_{L^2_v} dt \right)^{1/2} d\Sigma(k)
$$

$$
\leq \begin{cases} 
\tilde{C}_0, & \text{if } n \leq 1, \\
\tilde{C}_0^{n-1} [(n-1)!]^{\frac{1+2s}{2s}}, & \text{if } n \geq 2,
\end{cases}
$$

(3.3)

hold true for any integer $n$ with $0 \leq n \leq m - 1$. 
The rest part of this section is devoted to proving Propositions 3.3 and 3.4. We remark that it suffices to consider only the case of $0 < t \leq 1$, where the main difficulty is to control the terms involving the large factor $t^{-1}$ that arises from the auxiliary function $\phi(t)$ introduced to overcome the singularity of $(k)^m \hat{f}(t, k)$ at $t = 0$. Once the regularity is achieved for $0 < t \leq 1$, the counterpart over $1 \leq t < T$ can be treated in a similar way as in the case of $0 < t \leq 1$ but with the simpler argument, since it is essentially the propagation of regularity from $t = 1$ to $1 \leq t < T$. As to be seen below, we will combine the subelliptic estimates with energy estimates to deal with the large factor $t^{-1}$ for $0 < t \leq 1$.

As discussed above we will focus on the following discussion the case of $0 < t \leq 1$ and hence we choose $\phi(t) = t$. For simplicity we will use the capital letter $C$ to denote some generic constants, that may vary from line to line and depend only on $\gamma, s$ and the number $C_0$ in (3.1), and moreover use $C_\varepsilon$ to denote some generic constants depending on a given number $0 < \varepsilon \ll 1$ additionally. Note these generic constants $C$ and $C_\varepsilon$ as below are independent of the derivative order related to $m$.

3.1. Regularization operators and uniform estimates. Note that we can not directly perform estimates for $(k)^m \hat{f}(t, k, v)$ due to its low regularity. So to begin with we introduce its regularization defined by

$$
\hat{f}_{m, \delta}(t, k, v) = t^{\varepsilon m}(1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{f}(t, k, v), \quad 0 < \delta \ll 1, \quad m \geq 0. \tag{3.4}
$$

Moreover, with each $k \in \mathbb{Z}^3$ we associate an operator

$$
\Lambda_{\delta} = \Lambda_{\delta, k} = (1 + \delta_1 |v|^2)^{-1-\gamma} (1 + \delta_1 |k|^2 - \delta_1 \Delta_v)^{-1}, \quad \delta_1 \ll 1, \tag{3.5}
$$

with $\gamma$ given in (1.2). Here and below we have written $\Lambda_{\delta, k}$ as $\Lambda_{\delta}$ to omit the dependence on $k$ for brevity.

If $m \geq 2$, then by the induction assumption (3.3) we see

$$
\sup_{0 < t \leq 1} \sup_{k \in \mathbb{Z}^3} t^{\varepsilon (m-1)} \langle k \rangle^{m-1} \| \hat{f}(t, k) \|_{L^2_k} \leq C \in (0, \infty). \tag{3.6}
$$

As a result, it holds that

$$
\| \hat{f}_{m, \delta}(t, k) \|_{L^2_k} \leq C_{\delta} t^{\varepsilon (m-1)} \langle k \rangle^{m-1} \| \hat{f}(t, k) \|_{L^2_k} \leq C_{\delta} \varepsilon, \quad \forall \; k \in \mathbb{Z}^3, \; \forall \; 0 < t \leq 1, \tag{3.7}
$$

for some constant $C_{\delta}$ depending on $\delta$. Similarly, one has

$$
\left( \int_0^1 \| (a^{1/2} \hat{f}_{m, \delta}(t, k) \|_{L^2_k}^2 dt \right)^{\frac{1}{2}} \leq C_{\delta}, \quad \forall \; k \in \mathbb{Z}^3. \tag{3.8}
$$

Note that the assertions (3.6) and (3.7) are also true for $0 \leq m \leq 1$ due to the condition (3.1).

Direct computation shows

$$
\left\{ \begin{array}{l}
|\partial_\alpha^\varepsilon (1 + \delta_1 |v|^2)^{-1-\gamma} | \leq C_\alpha (1 + \delta_1 |v|^2)^{-1-\gamma}, \\
|\partial_\eta^\beta (1 + \delta_1 |k|^2 + \delta_1 |\eta|^2)^{-1} | \leq C_\beta (1 + \delta_1 |\eta|^2)^{-1}, \\
\end{array} \right. \tag{3.9}
$$

$$
\forall \alpha, \beta \in \mathbb{Z}_+^3, \; \forall \alpha \in \mathbb{Z}_+^3 \text{ with } |\alpha| \geq 1,
$$

and moreover

$$
|\partial_\alpha^\varepsilon (1 + \delta_1 |v|^2)^{-1-\gamma} | + |\partial_\eta^\beta (1 + \delta_1 |k|^2 + \delta_1 |\eta|^2)^{-1} | \leq C_\alpha \delta_1^{1/2}, \quad \forall \; \alpha \in \mathbb{Z}_+^3 \tag{3.10}
$$
where $C_\alpha, C_\beta$ are constants depending only on $\alpha$ and $\beta$ respectively but not on $k, \delta_1$. Thus, by (2.3) we can write $\Lambda_{\delta_1}$ defined by (3.5) as

$$\Lambda_{\delta_1} = q^w$$

(3.8)

with $q \in S((1 + \delta_1 |v|^2)^{-1-\gamma}(1 + \delta_1 |\eta|^2)^{-1}, |dv|^2 + |d\eta|^2)$, and moreover for any $\alpha, \beta \in \mathbb{Z}_+$ with $|\alpha| + |\beta| = 1$ we have

$$q \in S(1, |dv|^2 + |d\eta|^2)$$

and $\delta_1^{-1/2} \partial^\alpha_v \partial^\beta_\eta q \in S(1, |dv|^2 + |d\eta|^2)$ uniformly w.r.t. $\delta_1$ and $k$. (3.9)

This with assertions (i) and (iii) in Proposition 2.1 as well as (2.3) give that

$$\|(a^{1/2})^w \Lambda_{\delta_1} h\|_L^w + \|\{a^{1/2} \Lambda_{\delta_1}^w h\|_L^w \leq C_{\delta_1} \|h\|_{L^2_v}, \quad \forall h \in L^2_v. \quad (3.10)$$

Now we list some uniform estimates for the regularization operator $\Lambda_{\delta_1}$ to be used frequently later. By uniform it means that the estimates presented below hold with constants independent of $\delta_1$ and $k$. It is clear to see that

$$\|\Lambda_{\delta_1} h\|_{L^2_v} \leq \|h\|_{L^2_v}, \quad \forall h \in L^2_v. \quad (3.11)$$

Moreover, combining (3.8) with (3.9), we apply Lemma 2.2 to conclude that for any $h \in L^2_v$ with $(a^{1/2})^w h \in L^2_v$,

$$\|\Lambda_{\delta_1} (a^{1/2})^w h\|_{L^2_v} + \|(a^{1/2})^w \Lambda_{\delta_1} h\|_{L^2_v} + \|[\Lambda_{\delta_1}, (a^{1/2})^w h]\|_{L^2_v} \leq C \|(a^{1/2})^w h\|_{L^2_v},$$

with $C$ independent of $\delta_1$ and $k$. Meanwhile, by (2.4) and the second assertion in (3.9), one has

$$\|[\Lambda_{\delta_1}, (a^{1/2})^w h]\|_{L^2_v} \leq C \delta_1^{1/2} \|(a^{1/2})^w h\|_{L^2_v}. \quad (3.12)$$

Note that the above estimate (3.12) still holds with $\Lambda_{\delta_1}$ replaced by its adjoint $\Lambda_{\delta_1}^*$ on $L^2_v$.

Next, we will perform estimates for the regularization $\Lambda_{\delta_1} \hat{f}_{m,\delta}$. To begin with we derive the equations solved by $\Lambda_{\delta_1} \hat{f}_{m,\delta}$. Observe

$$(\partial_t + iv \cdot k - \mathcal{L}) \hat{f} = \hat{\Gamma}(\hat{f}, \hat{f}), \quad (3.13)$$

and thus

$$(\partial_t + iv \cdot k) \Lambda_{\delta_1} \hat{f}_{m,\delta} - \Lambda_{\delta_1} \mathcal{L} \hat{f}_{m,\delta} = \Lambda_{\delta_1} t^m(1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f})$$

$$+ \zeta mt^{-1} \Lambda_{\delta_1} \hat{f}_{m,\delta} + i[v \cdot k, \Lambda_{\delta_1}] \hat{f}_{m,\delta}. \quad (3.14)$$

**Lemma 3.5 (Regularization).** Let $f$ satisfy the condition (3.1). Then for any $0 < t \leq T \leq 1$, it holds that

$$t^{-1} \Lambda_{\delta_1} \hat{f}_{m,\delta}, [v \cdot k, \Lambda_{\delta_1}] \hat{f}_{m,\delta}, (v \cdot k) \Lambda_{\delta_1} \hat{f}_{m,\delta} \in L^1_T L^\infty_v L^2_v$$

and

$$\partial_t \Lambda_{\delta_1} \hat{f}_{m,\delta}, \Lambda_{\delta_1} \mathcal{L} \hat{f}_{m,\delta}, \Lambda_{\delta_1} t^m(1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}) \in L^1_T L^2_v L^2_v$$

for $0 \leq m \leq 1$, and the above assertion still holds for $m \geq 2$ provided that the induction assumption (3.3) is fulfilled.

**Proof.** We first consider the case $m \geq 2$. Direct verification shows

$$\|(v \cdot k) \Lambda_{\delta_1} \hat{f}_{m,\delta}(t, k)\|_{L^2_v} \leq C_{\delta_1} \|\hat{f}_{m,\delta}(t, k)\|_{L^2_v} \leq C_{\delta_1} C_{\delta} t^{(m-1)} \langle k \rangle^{m-1} \|\hat{f}(t, k)\|_{L^2_v},$$

This with the induction assumption (3.3) gives

$$(v \cdot k) \Lambda_{\delta_1} \hat{f}_{m,\delta} \in L^1_T L^\infty_v L^2_v,$$
and likewise for $t^{-1}\Lambda_{\delta_1}\hat{f}_{m,\delta}$ and $[v \cdot k, \Lambda_{\delta_1}]\hat{f}_{m,\delta}$ by observing the facts that

$$t^{-1}\|\Lambda_{\delta_1}\hat{f}_{m,\delta}(t, k)\|_{L^2_v} \leq C_\delta t^{s(m-1)} \|\hat{f}(t, k)\|_{L^2_v} \leq C_\delta t^{s(m-1)} \|\hat{f}(t, k)\|_{L^2_v}$$

and that the commutator

$$[v \cdot k, \Lambda_{\delta_1}] = -2(1 + \delta_1 |v|^2)^{-1-\gamma}(1 + \delta_1 |k|^2 - \delta_1 \Delta_v)^{-2}\delta_1 k \cdot \nabla_v$$

(3.15)
is uniformly bounded on $L^2_v$ with respect to $k$ and $\delta_1$. We have proved the first assertion for $m \geq 2$. For the second one we apply Lemma 2.3 to obtain, by virtue of (3.10) as well as the assertion (iii) in Proposition 2.1,

$$\|\Lambda_{\delta_1}t^{km}(1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f})\|_{L^2_v} \leq C_\delta C_{\delta} t^{km} \int_{\mathbb{Z}^d} \langle k - \ell \rangle^{m-1} \|\hat{f}(k - \ell)\|_{L^2_v} \langle \|a^{1/2}w\hat{\Gamma}(\hat{f}, \hat{f})\|_{L^2_v} \rangle d\Sigma(\ell)$$

$$\leq C_m C_{\delta} t^{km} \int_{\mathbb{Z}^d} \|\hat{f}(k - \ell)\|_{L^2_v} \langle \|a^{1/2}w\hat{\Gamma}(\hat{f}, \hat{f})\|_{L^2_v} \rangle d\Sigma(\ell).$$

This, along with the induction assumption (3.3), enable us to repeat the calculation in (2.12) to conclude

$$\Lambda_{\delta_1}t^{km}(1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}) \in L^1_k L^2_v L^2_v,$$

and likewise for $\Lambda_{\delta_1}\mathcal{L}\hat{f}_{m,\delta}$. Combining the above assertions with (3.14) gives $\partial_t\Lambda_{\delta_1}\hat{f}_{m,\delta} \in L^1_k L^2_v L^2_v$. We have proved the conclusion as desired in Lemma 3.5 for $m \geq 2$, and the treatment for $0 \leq m \leq 1$ is straightforward by following the above argument. The proof is thus completed. \hfill \Box

3.2. Subelliptic estimate for regularized solutions. In this subsection we will derive a subelliptic estimate for $\Lambda_{\delta_1}\hat{f}_{m,\delta}$ that is defined by (3.4) and (3.5). Since the conditions for $m \leq 1$ and $m \geq 2$ may be different in the following argument, we introduce a uniform Assumption $\mathcal{H}_m$ for $m \geq 0$ that is defined as below.

**Definition 3.6** (Assumption $\mathcal{H}_m$). Let $m \geq \mathbb{Z}_+$ and let $f(t, x, v)$ be the global mild solution to (1.4). We say that $f$ satisfies the Assumption $\mathcal{H}_m$ if $f$ satisfies the estimate (3.1) when $m \leq 1$, and satisfies additionally the induction assumption (3.3) when $m \geq 2$.

**Proposition 3.7.** Let $f$ satisfy Assumption $\mathcal{H}_m$ above. Then the following estimates hold.

(i) It holds that

$$\int_{\mathbb{Z}^d} \langle k \rangle^{1+zs} \left( \int_0^1 \|\hat{f}(t, k)\|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C\|f_0\|_{L^1_k L^2_v}.$$

(ii) For $m = 1$, it holds that

$$\int_{\mathbb{Z}^d} \langle k \rangle^{1+zs} \left( \int_0^1 \|\hat{f}_{m,\delta}(t, k)\|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C\|f_0\|_{L^1_k L^2_v}$$

$$+ C \sup_{0 < t \leq 1} \|\hat{f}_{m,\delta}(t, k)\|_{L^2_v} d\Sigma(k) + C \int_{\mathbb{Z}^d} \left( \int_0^1 \|a^{1/2}\hat{f}_{m,\delta}(t, k)\|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k).$$
(iii) For \( m \geq 2 \), it holds that
\[
\int_{\mathbb{R}^3} \langle k \rangle^\frac{m-2}{2} \left(\int_0^1 \|\hat{f}_{m,\delta}(t, k)\|_{L^2_v}^2 dt\right)^{1/2} d\Sigma(k) \leq C \tilde{C}_0^{m-2} [(m-1)!]^{\frac{m-2}{2}}
\]
\[
+ C \int_{\mathbb{R}^3} \sup_{0<\ell \leq 1} \|\hat{f}_{m,\delta}(t, k)\|_{L^2_v} d\Sigma(k) + C \int_{\mathbb{R}^3} \left(\int_0^1 \|a^{1/2} w \hat{f}_{m,\delta}(t, k)\|_{L^2_v}^2 dt\right)^{1/2} d\Sigma(k).
\]
Here \( \tilde{C}_0 \) is the constant in the assumption (3.3).

We will follow the argument presented in [2] where the standard subelliptic estimate was established in \( L^2(\mathbb{T}^3_x) \). Here we will derive the estimates in the setting of \( L^1_k(\mathbb{T}^3_x) \) instead. Let \( \lambda^\text{Wick}_k \) be the Wick quantization of symbol \( \lambda_k \) (see Appendix 6 for the definition of Wick quantization), which is defined by, recalling \( \tilde{a} \) is given in (2.1),
\[
\lambda_k(v, \eta) = \frac{d_k(v, \eta)}{\tilde{a}(v, k)^{1+2\gamma}} \chi\left(\frac{\tilde{a}(v, \eta)}{\tilde{a}(v, k)^{1+2\gamma}}\right),
\]
with
\[
d_k(v, \eta) = \langle v \rangle^{\gamma} \left(1 + |v|^2 + |k|^2 + |v \land k|^2\right)^{s-1} \left(k \cdot \eta + (v \land k) \cdot (v \land \eta)\right)
\]
and \( \chi \in C_c^\infty(\mathbb{R} ; [0, 1]) \) a given cut-off function such that \( \chi = 1 \) on \( [-1, 1] \) and \( \text{supp} \chi \subset [-2, 2] \).

To obtain the subelliptic estimate it will rely on the following property linking \( d_k \) and \( \tilde{a} \) that
\[
\{d_k(v, \eta), v \cdot k\} = \tilde{a}(v, k) = \langle v \rangle^{2+\gamma} \left(1 + |v|^2 + |k|^2 + |v \land k|^2\right)^{s-1},
\]
where \( \{\cdot, \cdot\} \) is the Poisson bracket defined in (6.4). Observe by direct calculations that
\[
\left|\partial^\alpha_v \partial^\beta_\eta d_k(v, \eta)\right| \leq C_{\alpha, \beta} \tilde{a}(v, k)^{\frac{2s}{2s-1}} \tilde{a}(v, \eta)^{\frac{1}{2s-1}}, \quad \forall \alpha, \beta \in \mathbb{Z}_+^3,
\]
with \( C_{\alpha, \beta} \) constants depending only on \( \alpha, \beta \) but independent of \( k \), and it is clear to see that \( \tilde{a}(v, \eta) \leq 2\tilde{a}(v, k)^{1+2\gamma} \) on the support of
\[
(v, \eta) \mapsto \chi\left(\frac{\tilde{a}(v, \eta)}{\tilde{a}(v, k)^{1+2\gamma}}\right).
\]

Thus we can verify directly that
\[
\left|\partial^\alpha_v \partial^\beta_\eta \lambda_k(v, \eta)\right| \leq C_{\alpha, \beta}, \quad \forall \alpha, \beta \in \mathbb{Z}_+^3, \quad \forall (v, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3,
\]
with \( C_{\alpha, \beta} \) being constants depending only on \( \alpha, \beta \) but independent of \( k \). This gives
\[
\lambda_k \in S(1, |dv|^2 + |d\eta|^2) \text{ uniformly for } k.
\]
Here by uniformly for \( k \) we mean that the constants in (3.16) are independent of \( k \). Using the relationship (6.2) between the Wick and Weyl quantizations we can write
\[
\lambda_k^\text{Wick} = \tilde{\lambda}_k^w
\]
for some real-valued symbol \( \tilde{\lambda}_k \in S(1, |dv|^2 + |d\eta|^2) \) uniformly for \( k \). As a result, since any quantization \( b^w \) of real-valued symbol \( b \) is self-adjoint on \( L^2_k \) (cf. Appendix 6), so is \( \lambda_k^\text{Wick} \) in particular. Moreover, by (2.2) and Lemma 2.2, there exists a constant \( C_{\gamma, s} \) that depends only on \( \gamma \) and \( s \) but is independent of \( k \), such that for any \( h \in L^2(\mathbb{R}^3) \),
\[
\|\lambda_k^\text{Wick} h\|_{L^2_k} \leq C_{\gamma, s} \|h\|_{L^2_k},
\]
and moreover, with \((a^{1/2})^w h \in L^2_v\) additionally,
\[
\|\Lambda_k^{Wick} (a^{1/2})^w h \|_{L^2_v} \leq C_{s,\gamma} \| (a^{1/2})^w h \|_{L^2_v}.
\]

The main reason that we use the Wick quantization rather than the classical Weyl quantization is due to its positivity of the former; see (6.1) in Appendix 6.

In view of (3.11), (3.18) and Assumption \(\mathcal{H}_m\) in Definition 3.6, we see \(\Lambda_k^{Wick} \delta_1 \hat{f}_{m,\delta} \in L^1_v L^\infty_v L^2_v\), and thus combining this with Lemma 3.5, we are able to take the scalar \(L^2_v\)-product on both sides of (3.14) with \(\Lambda_k^{Wick} \delta_1 \hat{f}_{m,\delta}\), and then integrate the real parts of the resulting equation over \([t_1, t_2]\); this gives
\[
\int_{t_1}^{t_2} \text{Re} (i(v \cdot k) \Lambda_\delta \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} dt = \sum_{1 \leq p \leq 4} J_{m,p}
\]  
with
\[
J_{m,1} = \frac{1}{2} \langle \Lambda_\delta \hat{f}_{m,\delta}(t_1, k), \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta}(t_1, k) \rangle_{L^2_v},
J_{m,2} = \int_{t_1}^{t_2} \text{Re} (\Lambda_\delta \mathcal{L} \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} dt,
J_{m,3} = \int_{t_1}^{t_2} \text{Re} (\Lambda_\delta t^m (1 + \delta |k|^2)^{1/2} \langle k \rangle^m \hat{f}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} dt,
J_{m,4} = \int_{t_1}^{t_2} \text{Re} (\text{cmt}^{-1} \Lambda_\delta \hat{f}_{m,\delta} + i[v \cdot k, \Lambda_\delta] \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} dt.
\]
Here we have used the relation
\[
\text{Re} (\partial_t \Lambda_\delta \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} = \frac{1}{2} \frac{d}{dt} \langle \Lambda_\delta \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta} \rangle_{L^2_v}
\]
due to the fact that \(\lambda_k^{Wick}\) is self-adjoint on \(L^2_v\). As a result,
\[
\int_{\mathbb{Z}^3} \int_{t_1}^{t_2} \text{Re} (i(v \cdot k) \Lambda_\delta \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} dt \left| \frac{d}{d\Sigma(k)} \right| \leq C \sum_{1 \leq p \leq 4} \int_{\mathbb{Z}^3} |J_{m,p}| \frac{1}{2} d\Sigma(k).
\]  

We will proceed through the following lemmas to derive the lower and upper bounds respectively for the terms on the left and right hand sides of (3.20).

**Lemma 3.8.** Let \(m \in \mathbb{Z}_+\). Under Assumption \(\mathcal{H}_m\) given in Definition 3.6, it holds that, for any \(0 < t_1 < t_2 \leq 1\),
\[
\int_{\mathbb{Z}^3} \left| \frac{d}{d\Sigma(k)} \left( \int_{t_1}^{t_2} \| \Lambda_\delta \hat{f}_{m,\delta}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} \right| d\Sigma(k) \leq C \int_{\mathbb{Z}^3} \left( \int_{t_1}^{t_2} \text{Re} (i(v \cdot k) \Lambda_\delta \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} dt \right)^{1/2} d\Sigma(k) + C \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k).
\]

**Proof.** We follow the argument presented in [2]. Observe \(v \cdot k = (v \cdot k)^{Wick}\) by (6.2). Then, by (6.3) in Appendix 6,
\[
\text{Re} (i(v \cdot k) \Lambda_\delta \hat{f}_{m,\delta}, \lambda^{Wick} \Lambda_\delta \hat{f}_{m,\delta})_{L^2_v} = \left\{ \lambda_k, v \cdot k \right\}^{Wick} \Lambda_\delta \hat{f}_{m,\delta}, \Lambda_\delta \hat{f}_{m,\delta} \right\}_{L^2_v},
\]  
\[
(3.21)
\]
with the Poisson bracket \( \{ \cdot, \cdot \} \) defined by (6.4). Moreover, recalling
\[
\lambda_k(v, \eta) = \frac{d_k(v, \eta)}{\bar{a}(v, k)^{\frac{2s}{1+2s}}} \psi(v, \eta),
\]
with
\[
\psi(v, \eta) = \chi \left( \frac{\bar{a}(v, \eta)}{\bar{a}(v, k)^{\frac{2s}{1+2s}}} \right)
\]
and
\[
d_k(v, \eta) = \langle v \rangle^\gamma \left( 1 + |v|^2 + |k|^2 + |v \wedge k|^2 \right)^{s-1} (k \cdot \eta + (v \wedge k) \cdot (v \wedge \eta)),
\]
we compute directly
\[
\{ \lambda_k, v \cdot k \} = \bar{a}(v, k)^{\frac{1}{1+2s}} \psi - \frac{\langle v \rangle^{\gamma+2} \left( 1 + |v|^2 + |k|^2 + |v \wedge k|^2 \right)^{s-1}}{\bar{a}(v, k)^{\frac{2s}{1+2s}}} \psi + \frac{d_k(v, \eta)}{\bar{a}(v, k)^{\frac{2s}{1+2s}}} k \cdot \partial_\eta \psi
\]
\[
= \bar{a}(v, k)^{\frac{1}{1+2s}} - \bar{a}(v, k)^{\frac{1}{1+2s}} (1 - \psi) - \langle v \rangle^{\gamma+2} \left( 1 + |v|^2 + |k|^2 + |v \wedge k|^2 \right)^{s-1} \bar{a}(v, k)^{\frac{s-2}{1+2s}} \psi
\]
\[
+ d_k(v, \eta) \bar{a}(v, k)^{\frac{-2s}{1+2s}} k \cdot \partial_\eta \psi.
\]
This with (3.21) yield that
\[
\left( (\bar{a}(v, k)^{\frac{1}{1+2s}})^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v} = \text{Re} \left( i (v \cdot k) \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}
\]
\[
+ \left( (\bar{a}(v, k)^{\frac{1}{1+2s}} (1 - \psi))^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}
\]
\[
+ \left( \langle v \rangle^{\gamma+2} \left( 1 + |v|^2 + |k|^2 + |v \wedge k|^2 \right)^{s-1} \bar{a}(v, k)^{\frac{-2s}{1+2s}} \psi \right)^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}
\]
\[
+ \left( -d_k(v, \eta) \bar{a}(v, k)^{\frac{-2s}{1+2s}} k \cdot \partial_\eta \psi \right)^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}.
\]
Note that \( \bar{a}(v, k)^{\frac{1}{1+2s}} \leq \bar{a}(v, \eta) \) on the support of \( 1 - \psi \) with \( \psi \) defined by (3.22), and thus
\[
\bar{a}(v, k)^{\frac{1}{1+2s}} (1 - \psi) \leq \bar{a}(v, \eta).
\]
Then the positivity (6.1) of the Wick quantization gives
\[
\left( (\bar{a}(v, k)^{\frac{1}{1+2s}} (1 - \psi)) \right)^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v} \leq \left( (\bar{a}(v, \eta))^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}.
\]
Similarly, observing
\[
\langle v \rangle^{\gamma+2} \left( 1 + |v|^2 + |k|^2 + |v \wedge k|^2 \right)^{s-1} \bar{a}(v, k)^{\frac{-2s}{1+2s}} \psi \leq \langle v \rangle^{2s+\gamma} \leq C \bar{a}(v, \eta),
\]
and
\[
-d_k(v, \eta) \bar{a}(v, k)^{\frac{-2s}{1+2s}} k \cdot \partial_\eta \psi \leq C \bar{a}(v, \eta),
\]
we have
\[
\left( \langle v \rangle^{\gamma+2} \left( 1 + |v|^2 + |k|^2 + |v \wedge k|^2 \right)^{s-1} \bar{a}(v, k)^{\frac{-2s}{1+2s}} \psi \right)^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}
\]
\[
+ \left( -d_k(v, \eta) \bar{a}(v, k)^{\frac{-2s}{1+2s}} k \cdot \partial_\eta \psi \right)^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}
\]
\[
\leq C \left( (\bar{a}(v, \eta))^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta} \right)_{L^2_v}.
\]
We combine the above estimate with (3.24) and (3.23) to conclude
\[
\left(\left(\alpha(v, \kappa)\right)^{\frac{1}{1-2s}}\right)_{L_v^2} \leq \text{Re} \left( i (v \cdot k) \Lambda_{\delta_1} \hat{f}_{m,\delta}, \lambda_{k} \Lambda_{\delta_1} \hat{f}_{m,\delta}\right)_{L_v^2} + C \left(\left(\alpha(v, \eta)\right) \Lambda_{\delta_1} \hat{f}_{m,\delta}, \lambda_{k} \Lambda_{\delta_1} \hat{f}_{m,\delta}\right)_{L_v^2}.
\]
(3.25)

For the term on the left hand side, we have, by (6.2),
\[
\left(\alpha(v, k)\right)^{\frac{1}{1-2s}}_{L_v^2} = 8 \int \alpha(v - \tilde{v}, k) \left(\frac{1}{2\pi}\right)^{d-2}\left(e^{-2\pi((v^2 + \tilde{v}^2))}\right) d\tilde{v} d\eta \geq C^{-1} \alpha(v, k)^{1/(1+2s)} \geq C^{-1} \langle k \rangle^{\frac{2}{1+2s}},
\]
where the first inequality follows from direct calculations (cf. [2, p.61]), and the last inequality holds because of the definition (2.1) of \(\alpha\) by observing \( \gamma \geq 0\). On the other hand, for the last term on the right hand side of (3.25) we write
\[
\left(\alpha(v, \eta)\right)^{\text{Wick}} = \langle a^{1/2} \rangle^w \left(\left(\langle a^{1/2} \rangle^w\right)^{-1-1}(\alpha(v, \eta))^{\text{Wick}} \left(\left(\langle a^{1/2} \rangle^w\right)^{-1}(a^{1/2})^w,\right.ight.
\]
where the boundedness on \(L_v^2\) follows from the assertion (iii) in Proposition 2.1 as well as the composition formula (2.3) and the relationship (6.2) between the Wick and Weyl quantizations. This gives that
\[
\left(\left(\alpha(v, \eta)\right)^{\text{Wick}} \Lambda_{\delta_1} \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta}\right)_{L_v^2} \leq C \|\langle a^{1/2} \rangle^w \Lambda_{\delta_1} \hat{f}_{m,\delta}\|_{L_v^2}^2 \leq C \|\langle a^{1/2} \rangle^w \hat{f}_{m,\delta}\|_{L_v^2}^2,
\]
where in the last inequality we have used Lemma 2.2 in view of (3.8) and (3.9). As a result, combining the above inequalities with (3.25) we have
\[
\langle k \rangle^{\frac{2}{1+2s}} \int_{t_1}^{t_2} \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t, k)\|_{L_v^2}^2 dt \leq C \int_{t_1}^{t_2} \text{Re}(i(v \cdot k)\Lambda_{\delta_1} \hat{f}_{m,\delta}, \lambda_{k} \Lambda_{\delta_1} \hat{f}_{m,\delta})_{L_v^2} dt + C \int_{t_1}^{t_2} \|\langle a^{1/2} \rangle^w \hat{f}_{m,\delta}(t, k)\|_{L_v^2}^2 dt,
\]
which yields the desired estimate in Lemma 3.8. The proof is completed. \(\square\)

**Lemma 3.9.** Let \(J_{m,1}\) be defined in terms of (3.19), that is
\[
J_{m,1} = \frac{1}{2} \left(\left(\langle a^{1/2} \rangle^w \Lambda_{\delta_1} \hat{f}_{m,\delta}(t_1, k)\right)_{L_v^2} - \frac{1}{2} \left(\left(\langle a^{1/2} \rangle^w \Lambda_{\delta_1} \hat{f}_{m,\delta}(t_2, k)\right)_{L_v^2}\right)\right).
\]
Then, for any 0 < \(t_1 < t_2 \leq 1\) and any \(m \geq 0\), it holds that
\[
\int_{\mathbb{R}^3} |J_{m,1}|^{1/2} d\Sigma(k) \leq C \int_{\mathbb{R}^3} \|\hat{f}_{m,\delta}(t_1, k)\|_{L_v^2} d\Sigma(k) + C \int_{\mathbb{R}^3} \|\hat{f}_{m,\delta}(t_2, k)\|_{L_v^2} d\Sigma(k).
\]

**Proof.** This just follows from (3.11) and (3.18). \(\square\)

**Lemma 3.10.** With \(J_{m,2}\) defined in terms of (3.19), that is,
\[
J_{m,2} = \int_{t_1}^{t_2} \text{Re}(\Lambda_{\delta_1} \mathcal{L} \hat{f}_{m,\delta}, \lambda_{k} \Lambda_{\delta_1} \hat{f}_{m,\delta})_{L_v^2} dt,
\]
it holds that for any 0 < \(t_1 < t_2 \leq 1\) and any \(m \geq 0\),
\[
\int_{\mathbb{R}^3} |J_{m,2}|^{1/2} d\Sigma(k) \leq C \left(\int_0^1 \|\langle a^{1/2} \rangle^w \hat{f}_{m,\delta}(t, k)\|_{L_v^2}^2 dt\right)^{1/2} d\Sigma(k).
\]
Proof. Using the assertion (iii) in Proposition 2.1 gives that
\[ |J_{m,2}| \leq C \int_{t_1}^{t_2} \| (a^{-1/2})^w L \tilde{f}_{m,\delta} \|_{L^2} \| (a^{-1/2})^w \Lambda_{\delta}^m \Lambda_k^{Wick} \Lambda_{\delta} \tilde{f}_{m,\delta} \|_{L^2} dt \leq C \int_{0}^{1} \| (a^{-1/2})^w \tilde{f}_{m,\delta} \|_{L^2}^2 dt, \]
where we have denoted by \( \Lambda_{\delta}^m \) the adjoint operator of \( \Lambda_{\delta} \) on \( L^2_v \) and the last inequality follows from Corollary 2.4 and Lemma 2.2 in view of (3.9) and (3.17). Thus the proof of Lemma 3.10 is completed.

Lemma 3.11. Let \( m \in \mathbb{Z}_+ \) and let \( J_{m,3} \) be defined in terms of (3.19), that is,
\[ J_{m,3} = \int_{t_1}^{t_2} \text{Re} \left( \Lambda_{\delta}^m (1 + \delta |k|)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}) \right) \frac{\Lambda_k^{Wick} \Lambda_{\delta} \tilde{f}_{m,\delta}}{L^2_v} dt. \]
Suppose that \( f \) satisfies Assumption \( H_m \) given in Definition 3.6. Then, for any \( \varepsilon > 0 \), the following things hold.
(i) For \( m = 0 \), it holds that
\[ \int_{\mathbb{Z}^3} |J_{m,3}|^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^1_v L^2_v}. \]
(ii) For \( m = 1 \), it holds that
\[ \int_{\mathbb{Z}^3} |J_{m,3}|^{1/2} d\Sigma(k) \leq (\varepsilon + C\varepsilon^{-1}\varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_{0}^{1} \| (a^{-1/2})^w \tilde{f}_{m,\delta}(t,k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) \]
\[ + C\varepsilon^{-1}\varepsilon_0 \int_{\mathbb{Z}^3} \sup_{0 \leq t \leq 1} \| \tilde{f}_{m,\delta}(t,k) \|_{L^2_v} d\Sigma(k). \]
(iii) For \( m \geq 2 \), it holds that
\[ \int_{\mathbb{Z}^3} |J_{m,3}|^{1/2} d\Sigma(k) \leq (\varepsilon + C\varepsilon^{-1}\varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_{0}^{1} \| (a^{-1/2})^w \tilde{f}_{m,\delta}(t,k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \]
\[ + C\varepsilon^{-1}\varepsilon_0 \int_{\mathbb{Z}^3} \sup_{0 \leq t \leq 1} \| \tilde{f}_{m,\delta}(t,k) \|_{L^2_v} d\Sigma(k) + C\varepsilon^{-1} \tilde{C}_0^{m-2} [(m - 1)!]^{1/2} \]
with \( \tilde{C}_0 \) the constant in the induction assumption (3.3).

Proof. We first apply the assertion (iii) of Proposition 2.1 and then Lemmas 2.2 and 2.3, to compute
\[ |(\Lambda_{\delta}^m (1 + \delta |k|)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}) \right) \frac{\Lambda_k^{Wick} \Lambda_{\delta} \tilde{f}_{m,\delta}(k)}{L^2_v}| \]
\[ \leq C \| (a^{-1/2})^w \Lambda_{\delta}^m \Lambda_k^{Wick} \Lambda_{\delta} \tilde{f}_{m,\delta}(k) \|_{L^2_v} \times \left( \tau^{cm}(1 + \delta |k|)^{-1/2} \langle k \rangle^m (a^{-1/2})^w \hat{\Gamma}(\hat{f}, \hat{f}) \right)_{L^2_v} \]
\[ \leq C \| (a^{-1/2})^w \tilde{f}_{m,\delta}(k) \|_{L^2_v} \int_{\mathbb{Z}^3} \frac{\tau^{cm}(k)^m}{(1 + \delta |k|)^{1/2}} \| \hat{f}(k - \ell) \|_{L^2_v} \| (a^{-1/2})^w \hat{f}(\ell) \|_{L^2_v} d\Sigma(\ell). \]
This gives
\[ \int_{\mathbb{Z}^3} |J_{m,3}|^{1/2} d\Sigma(k) \leq C \int_{\mathbb{Z}^3} \left[ \int_{0}^{1} \| (a^{-1/2})^w \tilde{f}_{m,\delta}(t,k) \|_{L^2_v} \right]^{1/2} \left( \int_{\mathbb{Z}^3} \frac{\tau^{cm}(k)^m}{(1 + \delta |k|)^{1/2}} \| \hat{f}(t,k - \ell) \|_{L^2_v} \| (a^{-1/2})^w \hat{f}(\ell) \|_{L^2_v} d\Sigma(\ell) dt \right) \]
Hence it holds that
\[
\int_{\mathbb{R}^3} |J_{m,\delta}|^{1/2} d\Sigma(k) \leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \\
+ C\varepsilon^{-1} \int_{\mathbb{R}^3} \left[ \int_0^1 \left( \int_{\mathbb{R}^3} \frac{(\ell^m(k)^m)}{(1 + \delta |k|^2)} \| \hat{f}(t, k - \ell) \|_{L^2_v} \| (a^{1/2})^w \hat{f}(t, \ell) \|_{L^2_v} d\Sigma(\ell) \right) dt \right]^{1/2} d\Sigma(k).
\]

(3.26)

(a) We first consider the case when \( m = 0 \). In view of (3.26) the first assertion for \( m = 0 \) obviously holds as a result of (3.1) by observing that for \( m = 0 \) we have \( \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v} \leq \| (a^{1/2})^w \hat{f} \|_{L^2_v} \) and
\[
\int_{\mathbb{R}^3} \left[ \int_0^1 \left( \int_{\mathbb{R}^3} \frac{(\ell^m(k)^m)}{(1 + \delta |k|^2)} \| \hat{f}(t, k - \ell) \|_{L^2_v} \| (a^{1/2})^w \hat{f}(t, \ell) \|_{L^2_v} d\Sigma(\ell) \right) dt \right]^{1/2} d\Sigma(k) \\
\leq C \left( \int_{\mathbb{R}^3} \sup_{0 < t \leq 1} \| \hat{f}(t, k) \|_{L^2_v} d\Sigma(k) \right) \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k),
\]
where in the last inequality we have used Lemma 2.5.

(b) Next consider the case when \( m = 1 \). Applying Lemma 2.6 for \( m = 1 \) gives
\[
\frac{t^c(k)}{(1 + \delta |k|^2)^{1/2}} \leq \frac{2t^c(k - \ell)}{(1 + \delta |k - \ell|^2)^{1/2}} + \frac{2t^c(\ell)}{(1 + \delta |\ell|^2)^{1/2}}.
\]

Then, using again Lemma 2.5 and (3.1),
\[
\int_{\mathbb{R}^3} \left[ \int_0^1 \left( \int_{\mathbb{R}^3} \frac{t^c(k)}{(1 + \delta |k|^2)^{1/2}} \| \hat{f}(t, k - \ell) \|_{L^2_v} \| (a^{1/2})^w \hat{f}(t, \ell) \|_{L^2_v} d\Sigma(\ell) \right) dt \right]^{1/2} d\Sigma(k) \\
\leq C\varepsilon_0 \sup_{0 < t \leq 1} \| \hat{f}_{1,\delta}(t, k) \|_{L^2_v} d\Sigma(k) + C\varepsilon_0 \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{1,\delta}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k).
\]

Substituting the above estimate into (3.26) with \( m = 1 \), we obtain the second assertion for \( m = 1 \).

(c) It remains to consider the case when \( m = 2 \). In view of (3.26) the desired assertion follows from showing that
\[
\int_{\mathbb{R}^3} \left[ \int_0^1 \left( \int_{\mathbb{R}^3} \frac{(\ell^m(k)^m)}{(1 + \delta |k|^2)^{1/2}} \| \hat{f}(t, k - \ell) \|_{L^2_v} \| (a^{1/2})^w \hat{f}(t, \ell) \|_{L^2_v} d\Sigma(\ell) \right) dt \right]^{1/2} d\Sigma(k) \\
\leq C \tilde{C}_0^{-m-2} [(m - 1)!] \frac{\varepsilon_0}{\| J_{m,\delta} \|_{L^2_v}} + C\varepsilon_0 \int_{\mathbb{R}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m,\delta}(t, k) \|_{L^2_v} d\Sigma(k) \\
+ C\varepsilon_0 \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k).
\]

(3.27)
Indeed, we use Lemma 2.6 to get
\[
\int_{\mathbb{Z}_t^3} \frac{\langle k \rangle^m}{(1 + \delta \|k\|^2)^{1/2}} \| \hat{f}(k - \ell) \|_{L^2_\ell} \| (a^{1/2})^w \hat{f}(\ell) \|_{L^2_\ell} d\Sigma(\ell)
\]
\[
\leq C \sum_{j=1}^{m-1} \binom{m}{j} t^m \int_{\mathbb{Z}_t^3} \langle k - \ell \rangle^j \langle \ell \rangle^{m-j} \| \hat{f}(k - \ell) \|_{L^2_\ell} \| (a^{1/2})^w f(\ell) \|_{L^2_\ell} d\Sigma(\ell)
\]
\[
+ C t^m \int_{\mathbb{Z}_t^3} \langle k - \ell \rangle^m \| \hat{f}(k - \ell) \|_{L^2_\ell} \| (a^{1/2})^w \hat{f}(\ell) \|_{L^2_\ell} d\Sigma(\ell)
\]
Moreover, we may apply Lemma 2.5 for \( j_0 = m + 1 \), with
\[
(f_j, g_j) = (t^{c_j} \langle D_x \rangle^j f / j!, t^{c(m-j)} \langle D_x \rangle^{m-j} f / (m-j)!) \quad 1 \leq j \leq m - 1,
\]
and
\[
(f_m, g_m) = (t^{c^m} \langle D_x \rangle^m (\delta D_x)^{-1} f, f), \quad (f_{m+1}, g_{m+1}) = (f, t^{c^m} \langle D_x \rangle^m (\delta D_x)^{-1} f).
\]
These, together with (3.1), give that
\[
\int_{\mathbb{Z}^3} \left[ \int_0^1 \left( \int_{\mathbb{Z}_t^3} \frac{t^m \langle k \rangle^m}{(1 + \delta \|k\|^2)^{1/2}} \| \hat{f}(t, k - \ell) \|_{L^2_\ell} \| (a^{1/2})^w \hat{f}(t, \ell) \|_{L^2_\ell} d\Sigma(\ell) \right) dt \right]^{1/2} d\Sigma(k)
\]
\[
\leq C \sum_{j=1}^{m-1} \binom{m}{j} \int_{\mathbb{Z}^3} \langle k \rangle^j \sup_{0 < t \leq 1} t^c \| \hat{f}(t) \|_{L^2_\ell} d\Sigma(k) \int_{\mathbb{Z}^3} \langle k \rangle^{m-j} \left( \int_{0}^{1} t^{2c(m-j)} \| \hat{f}(t) \|_{L^2_\ell}^2 dt \right)^{1/2} d\Sigma(k)
\]
\[
+ C \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m,\delta}(t, k) \|_{L^2_\ell} d\Sigma(k) + C \int_{\mathbb{Z}^3} \left( \int_{0}^{1} \| (a^{1/2})^w \hat{f}_{m,\delta}(t, k) \|_{L^2_\ell}^2 dt \right)^{1/2} d\Sigma(k).
\]
Moreover, for the summation on the left hand side, we use the induction assumption (3.3) to compute
\[
\sum_{1 \leq j \leq m-1} \binom{m}{j} \int_{\mathbb{Z}^3} \langle k \rangle^j \sup_{0 < t \leq 1} t^c \| \hat{f}(t, k) \|_{L^2_\ell} d\Sigma(k)
\]
\[
\times \int_{\mathbb{Z}^3} \langle k \rangle^{m-j} \left( \int_{0}^{1} t^{2c(m-j)} \| (a^{1/2})^w \hat{f}(t, k) \|_{L^2_\ell}^2 dt \right)^{1/2} d\Sigma(k)
\]
\[
\leq \sum_{1 \leq j \leq m-1} \frac{m!}{j!(m-j)!} \frac{c^j}{1} [(j-1)!]^{1+\frac{\alpha}{2\beta}} \left( C_0^{m-j-1} [(m-j-1)!]^{1+\frac{\alpha}{2\beta}} \right)
\]
\[
\leq CC_0^{m-2} \sum_{1 \leq j \leq m-1} \frac{m!}{j!(m-j)!} [(j-1)!]^{\frac{1}{\delta}} [(m-j)!]^{\frac{1}{\delta}}
\]
\[
\leq CC_0^{m-2} \sum_{1 \leq j \leq m-1} \frac{m!}{j!(m-j)!} [(m-2)!]^{\frac{1}{\delta}}
\]
\[
\leq CC_0^{m-2} [(m-1)!]^{1+\frac{\alpha}{2\beta}} \sum_{1 \leq j \leq m-1} \frac{m}{j!(m-j)m^{\frac{1}{\delta}}} \leq C C_0^{m-2} [(m-1)!]^{1+\frac{\alpha}{2\beta}}, \quad (3.28)
\]
where the last inequality holds because of the fact that
\[
\sum_{1 \leq j \leq m-1} \frac{m}{j(m-j)m^{3/2}} = \left( \sum_{1 \leq j < (m-1)/2} + \sum_{(m-1)/2 \leq j \leq m-1} \right) \frac{m}{j(m-j)m^{3/2}} \\
\leq \frac{8}{m^{3/2}} \sum_{1 \leq j \leq m-1} \frac{1}{j} \leq C_s
\]
with $C_s$ a constant depending only on $s$. As a result, combining these inequalities we obtain the desired estimate (3.27). The proof of Lemma 3.11 is thus completed. \(\Box\)

**Lemma 3.12.** Let $m \in \mathbb{Z}_+$ and let $J_{m,4}$ be defined in terms of (3.19), that is,
\[
J_{m,4} = \int_{t_1}^{t_2} \operatorname{Re} \left( \frac{\epsilon_m}{\Lambda_{\delta,1}} \dot{f}_{m,\delta} + i [v \cdot k, \Lambda_{\delta,1}] \dot{f}_{m,\delta}, \lambda_k \Lambda_{\delta,1} \dot{f}_{m,\delta} \right)_{L^2_v} \, dt.
\]
Suppose that $f$ satisfies Assumption $\mathcal{H}_m$ given in Definition 3.6. Then for any $\bar{\epsilon} > 0$ the following estimates hold.
(i) For $m = 0$, it holds that
\[
\int_{\mathbb{Z}^3} |J_{m,4}|^{1/2} \, d\Sigma(k) \leq C \|f_0\|_{L^1_t L^2_v}.
\]
(ii) For $m = 1$, it holds that
\[
\int_{\mathbb{Z}^3} |J_{m,4}|^{1/2} \, d\Sigma(k) \leq \bar{\epsilon} \int_{\mathbb{Z}^3} \langle k \rangle^{\frac{7}{12}} \left( \int_0^1 \|\Lambda_{\delta,1} \dot{f}_{m,\delta}(t, k)\|_{L^2_v} \, dt \right)^{1/2} \, d\Sigma(k) \\
+ C_{\bar{\epsilon}} \int_{\mathbb{Z}^3} \langle k \rangle^{\frac{7}{12}} \left( \int_0^1 \|\Lambda_{\delta,1} \dot{f}_{m-1,\delta}(t, k)\|_{L^2_v} \, dt \right)^{1/2} \, d\Sigma(k).
\]
(iii) For $m \geq 2$, it holds that
\[
\int_{\mathbb{Z}^3} |J_{m,4}|^{1/2} \, d\Sigma(k) \leq \bar{\epsilon} \int_{\mathbb{Z}^3} \langle k \rangle^{\frac{7}{12}} \left( \int_0^1 \|\Lambda_{\delta,1} \dot{f}_{m,\delta}(t, k)\|_{L^2_v} \, dt \right)^{1/2} \, d\Sigma(k) \\
+ C_{\bar{\epsilon}} \bar{\epsilon}^{m-2} [(m-1)!]^{1/2}.
\]

**Proof.** We write
\[
\int_{\mathbb{Z}^3} |J_{m,4}|^{1/2} \, d\Sigma(k) \leq C \int_{\mathbb{Z}^3} \left( \int_0^1 \left| (mt^{-1} \Lambda_{\delta,1} \dot{f}_{m,\delta}, \lambda_k \Lambda_{\delta,1} \dot{f}_{m,\delta})_{L^2_v} \right| \, dt \right)^{1/2} \, d\Sigma(k) \\
+ C \int_{\mathbb{Z}^3} \left( \int_0^1 \left| [v \cdot k, \Lambda_{\delta,1}] \dot{f}_{m,\delta}, \lambda_k \Lambda_{\delta,1} \dot{f}_{m,\delta} \right|_{L^2_v} \, dt \right)^{1/2} \, d\Sigma(k).
\]
Using (3.18) gives
\[
\int_{\mathbb{Z}^3} \left( \int_0^1 \left| (mt^{-1} \Lambda_{\delta,1} \dot{f}_{m,\delta}, \lambda_k \Lambda_{\delta,1} \dot{f}_{m,\delta})_{L^2_v} \right| \, dt \right)^{1/2} \, d\Sigma(k) \\
\leq C \int_{\mathbb{Z}^3} \left( \int_0^1 \|mt^{-1} \Lambda_{\delta,1} \dot{f}_{m,\delta}(t, k)\|_{L^2_v}^2 \, dt \right)^{1/2} \, d\Sigma(k).
\]
It follows from (3.15) that $[v \cdot k, \Lambda_{\delta_1}]\Lambda_{\delta_1}^{-1}$ is bounded on $L^2_\omega$ uniformly with respect to $k$ and $\delta_1$. Thus, writing $[v \cdot k, \Lambda_{\delta_1}] = [v \cdot k, \Lambda_{\delta_1}]\Lambda_{\delta_1}^{-1}\Lambda_{\delta_1}$, we have

$$
\int_{\mathbb{R}^3} \left( \int_0^1 \left( \langle [v \cdot k, \Lambda_{\delta_1}]\hat{f}_{m,\delta}, \lambda_k^W \Lambda_{\delta_1} \hat{f}_{m,\delta} \rangle_{L^2_\omega} \right) dt \right)^{1/2} d\Sigma(k)
\leq C \int_{\mathbb{R}^3} \left( \int_0^1 \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k).
$$

Combining these estimates yields that

$$
\int_{\mathbb{R}^3} |J_{m,4}|^{1/2} d\Sigma(k) \leq C \int_{\mathbb{R}^3} \left( \int_0^3 m^{-1} \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k) \quad (3.29)
$$

for $m \geq 1$, and that

$$
\int_{\mathbb{R}^3} |J_{m,4}|^{1/2} d\Sigma(k) \leq C \int_{\mathbb{R}^3} \left( \int_0^1 \|\hat{f}_{m,\delta}\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k) \leq C \int_{\mathbb{R}^3} \left( \int_0^1 \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k)
$$

for $m = 0$. So, from (3.1), the assertion for $m = 0$ follows. It remains to consider the case of $m \geq 1$. We use Young inequality with an arbitrary parameter $\vartheta > 0$ that

$$
1 \leq \vartheta \langle k \rangle^{\frac{2+2s}{2s}} + \vartheta^{-\frac{1+2s}{2s}} \langle k \rangle^{-2+\frac{2s}{2s}}
$$

to obtain, choosing in particular $\vartheta = \tilde{\varepsilon}^2 t/m$ with arbitrarily small number $\tilde{\varepsilon} > 0$ and recalling $\varsigma = \frac{1+2s}{2s}$,

$$
mt^{-1} \leq \tilde{\varepsilon}^2 \langle k \rangle^{\frac{2s}{1+2s}} + \tilde{\varepsilon}^{-\frac{2(1+s)}{2s}} m^{\frac{4+2s}{2s}} t^{-2\varsigma} \langle k \rangle^{-2+\frac{2s}{1+2s}}.
$$

This with (3.29) yield

$$
\int_{\mathbb{R}^3} |J_{m,4}|^{1/2} d\Sigma(k) \leq \tilde{\varepsilon} \int_{\mathbb{R}^3} \langle k \rangle^{\frac{1}{1+2s}} \left( \int_0^1 \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k)
+ C\tilde{\varepsilon}^\varsigma \int_{\mathbb{R}^3} \langle k \rangle^{\frac{1}{1+2s}} \left( \int_0^1 \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k). \quad (3.30)
$$

Observe for $m \geq 1$,

$$
\langle k \rangle^{-2} t^{-2\varsigma} \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} = \|\Lambda_{\delta_1} \hat{f}_{m-1,\delta}(t,k)\|^2_{L^2_\omega} \leq t^{2s(m-1)} \langle k \rangle^{2(m-1)} \|\hat{f}(t,k)\|^2_{L^2_\omega}
$$
due to the definition (3.4) of $\hat{f}_{m,\delta}$. Thus we combine the above two inequalities to conclude

$$
\int_{\mathbb{R}^3} |J_{m,4}|^{1/2} d\Sigma(k) \leq \tilde{\varepsilon} \int_{\mathbb{R}^3} \langle k \rangle^{\frac{1}{1+2s}} \left( \int_0^1 \|\Lambda_{\delta_1} \hat{f}_{m,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k)
+ C\tilde{\varepsilon} \int_{\mathbb{R}^3} \langle k \rangle^{\frac{1}{1+2s}} \left( \int_0^1 \|\Lambda_{\delta_1} \hat{f}_{m-1,\delta}(t,k)\|^2_{L^2_\omega} dt \right)^{1/2} d\Sigma(k)
$$
for \( m = 1 \), and meanwhile
\[
\int_{\mathbb{R}^3} |J_{m, 1}|^{1/2} d\Sigma(k) \leq \tilde{\epsilon} \int_{\mathbb{R}^3} \langle k \rangle^{1+\frac{\alpha}{2}} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{m, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \\
+ C_{\tilde{\epsilon} m} \tilde{c}_0^{-2} \tilde{c}_m^{-2} [(m - 2)!]^{1/25}
\]
\[
\leq \tilde{\epsilon} \int_{\mathbb{R}^3} \langle k \rangle^{1+\frac{\alpha}{2}} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{m, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) + C_{\tilde{\epsilon} m} \tilde{c}_0^{-2} [(m - 1)!]^{1/25}
\]
for \( m \geq 2 \), where we have used the induction assumption (3.3) when \( m \geq 2 \). Thus, we have proved all the assertions in Lemma 3.12, completing the proof.

\( \square \)

**Ending the proof of Proposition 3.7.** (a) We prove the first assertion in Proposition 3.7 under the estimate (3.1). Combining the estimates for \( m = 0 \) in Lemmas 3.8-3.12 with (3.20) gives
\[
\int_{\mathbb{R}^3} \langle k \rangle^{1+\frac{\alpha}{2}} \left( \int_{t_1}^{t_2} \| \Lambda_{\delta} \hat{f}_{0, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^1_t L^2_v} + C \int_{\mathbb{R}^3} \| \hat{f}_{0, \delta}(t_1, k) \|_{L^2_v} d\Sigma(k)
\]
\[
+ C \int_{\mathbb{R}^3} \| \hat{f}_{0, \delta}(t_2, k) \|_{L^2_v} d\Sigma(k) + C \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^{1/2} \hat{f}_{0, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\]
\[
\leq C \| f_0 \|_{L^1_t L^2_v} + C \int_{\mathbb{R}^3} \sup_{0 < t \leq 1} \| \hat{f}(t, k) \|_{L^2_v} d\Sigma(k) + C \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^{1/2} \hat{f}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k),
\]
that further can be bounded by \( C \| f_0 \|_{L^1_t L^2_v} \), where we have used the estimate (3.1) by observing
\[
\| \hat{f}_{0, \delta} \|_{L^2_v} \leq \| \hat{f} \|_{L^2_v} \quad \| (a^{1/2}) \hat{f}_{0, \delta} \|_{L^2_v} \leq \| (a^{1/2}) \hat{f} \|_{L^2_v}.
\]
Letting \( t_1 \to 0 \) and \( t_2 \to 1 \), and further letting \( \delta_1, \delta \to 0 \), we conclude by the Fatou Lemma that
\[
\int_{\mathbb{R}^3} \langle k \rangle^{1+\frac{\alpha}{2}} \left( \int_0^1 \| \hat{f}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^1_t L^2_v}. \tag{3.31}
\]
We have proved the assertion (i) in Proposition 3.7.

(b) Next, we deal with the case of \( m = 1 \). Combining the estimates for \( m = 1 \) in Lemmas 3.8-3.12 with (3.20), we get, letting the constant \( \tilde{\epsilon} \) in Lemma 3.12 be small enough,
\[
\int_{\mathbb{R}^3} \langle k \rangle^{1+\frac{\alpha}{2}} \left( \int_{t_1}^{t_2} \| \Lambda_{\delta} \hat{f}_{1, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq C \int_{\mathbb{R}^3} \langle k \rangle^{1+\frac{\alpha}{2}} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{0, \delta} \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\]
\[
+ C \int_{\mathbb{R}^3} \sup_{0 < t \leq 1} \| \hat{f}_{1, \delta}(t, k) \|_{L^2_v} d\Sigma(k) + C \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^{1/2} \hat{f}_{1, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\]
\[
\leq C \| f_0 \|_{L^1_t L^2_v} + C \int_{\mathbb{R}^3} \sup_{0 < t \leq 1} \| \hat{f}_{1, \delta}(t, k) \|_{L^2_v} d\Sigma(k)
\]
\[
+ C \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^{1/2} \hat{f}_{1, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k),
\]
where in the last inequality we have used (3.31) since \( \| \Lambda_{\delta} \hat{f}_{0, \delta} \|_{L^2_v} \leq \| \hat{f} \|_{L^2_v} \). Thus, letting \( t_1 \to 0 \) and \( t_2 \to 1 \), and further letting \( \delta_1 \to 0 \), we obtain the assertion (ii) in Proposition 3.7.
(c) It remains to treat the case of $m \geq 2$. We combine the estimates for $m \geq 2$ in Lemmas 3.8-3.12 with the estimate (3.20), and then let the constant $\tilde{c}$ in Lemma 3.12 be small enough; this gives that
\[
\int_{\mathbb{Z}^3} \left( \int_{t_1}^{t_2} \| \Lambda_{\delta_1} \hat{f}_{m, \delta}(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq C \tilde{C}_0^{m-2} [(m - 1)]^{1+2\alpha} \\
+ C \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m, \delta}(t, k) \|_{L^2_v} d\Sigma(k) + C \int_{\mathbb{Z}^3} \left( \int_{0}^{1} \| (a^{1/2})^w \hat{f}_{m, \delta}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k).
\]
Similarly as above, letting $t_1 \to 0$ and $t_2 \to 1$ and then letting $\delta_1 \to 0$, we get the assertion (iii) as desired in Proposition 3.7. The proof of Proposition 3.7 is thus completed. \qed

3.3. Energy estimates for regularized solutions. This part is devoted to proving the following energy estimates for $\hat{f}_{m, \delta}$.

**Proposition 3.13.** Let $m \in \mathbb{Z}_+$ with $m \geq 1$ and let $f(t, x, v)$ satisfy the Assumption $\mathcal{H}_m$ given in Definition 3.6. The following estimates hold.

(i) For $m = 1$, it holds that
\[
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m, \delta}(t, k) \|_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_{0}^{1} \| (a^{1/2})^w \hat{f}_{m, \delta} \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^1_h L^2_v}.
\]

(ii) If $m \geq 2$, it holds that
\[
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m, \delta}(t, k) \|_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_{0}^{1} \| (a^{1/2})^w \hat{f}_{m, \delta} \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C \tilde{C}_0^{m-2} [(m - 1)]^{1+2\alpha},
\]
where $\tilde{C}_0$ is the constant in the induction assumption (3.3).

**Proof.** The argument of the proof is quite similar as in the previous Subsection 3.2, so we only sketch it for brevity.

Taking the $L^2_v$-product on both sides of (3.14) with $\Lambda_{\delta_1} \hat{f}_{m, \delta}$ and then integrating the real parts of the resulting equation over $[t_1, t_2]$ for any $0 < t_1 < t_2 \leq 1$, we have
\[
\frac{1}{2} \| \Lambda_{\delta_1} \hat{f}_{m, \delta}(t_2, k) \|^2_{L^2_v} + \int_{t_1}^{t_2} \text{Re}(\Lambda_{\delta_1} \mathcal{L} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta})_{L^2_v} dt \\
= \frac{1}{2} \| \Lambda_{\delta_1} \hat{f}_{m, \delta}(t_1, k) \|^2_{L^2_v} + \int_{t_1}^{t_2} \text{Re}( \Lambda_{\delta_1} t^{m}(1 + \delta |k|^2)^{-1/2} (k)^m \hat{f}(\tilde{f}, \tilde{f}), \Lambda_{\delta_1} \hat{f}_{m, \delta})_{L^2_v} dt \\
+ \int_{t_1}^{t_2} \text{Re}(\mathcal{S} t^{-1} \Lambda_{\delta_1} \hat{f}_{m, \delta} + i[v \cdot k, \Lambda_{\delta_1}] \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta})_{L^2_v} dt.
\]

This, along with
\[
\| (a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m, \delta} \|_{L^2_v}^2 \leq C(\mathcal{L} \Lambda_{\delta_1} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta})_{L^2_v} + C \| \Lambda_{\delta_1} \hat{f}_{m, \delta} \|_{L^2_v}^2 \\
\leq C \left( \text{Re}(\Lambda_{\delta_1} \mathcal{L} \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta})_{L^2_v} + \text{Re}(\mathcal{L}, \Lambda_{\delta_1}] \hat{f}_{m, \delta}, \Lambda_{\delta_1} \hat{f}_{m, \delta})_{L^2_v} \right) + C \| \Lambda_{\delta_1} \hat{f}_{m, \delta} \|_{L^2_v}^2
\]
due to (1.9), yield that

$$
\| \Lambda_{\delta_1} \hat{f}_{m,\delta}(t_2, k) \|_{L^2_0} d\Sigma(k) + \left( \int_{t_1}^{t_2} \| (a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0}^2 dt \right)^{1/2}
\leq C \| \Lambda_{\delta_1} \hat{f}_{m,\delta}(t_1, k) \|_{L^2_0} + C \left( \int_0^1 \| (a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0}^2 dt \right)^{1/2} + C \left( \int_0^1 \left( \| [\mathcal{L}, \Lambda_{\delta_1}] \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0} \right) dt \right)^{1/2}
+ C \left( \int_0^1 \left( \Lambda_{\delta_1} t^{m} (1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}), \Lambda_{\delta_1} \hat{f}_{m,\delta} \right)_{L^2_0} \right)^{1/2}
+ C \left( \int_0^1 \left( \langle \arrow k \delta t^{-1} \Lambda_{\delta_1} \hat{f}_{m,\delta} + i [v \cdot k, \Lambda_{\delta_1}] \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \rangle \right)_{L^2_0} \right)^{1/2}.
$$

Letting $t_1 \to 0$, observing that the first term on the left side vanishes because of (3.6), and then taking the supremum for $0 < t_2 \leq 1$, we conclude that after integrating for $k \in \mathbb{Z}^3$,

$$
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \Lambda_{\delta_1} \hat{f}_{m,\delta}(t, k) \|_{L^2_0} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0}^2 dt \right)^{1/2} d\Sigma(k)
\leq C \int_{\mathbb{Z}^3} \left( \int_0^1 \| \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0}^2 dt \right)^{1/2} d\Sigma(k) + C \int_{\mathbb{Z}^3} \left( \int_0^1 \left( \| [\mathcal{L}, \Lambda_{\delta_1}] \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0} \right) dt \right)^{1/2} d\Sigma(k)
+ C \int_{\mathbb{Z}^3} \left[ \int_0^1 \left( \Lambda_{\delta_1} t^{m} (1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}), \Lambda_{\delta_1} \hat{f}_{m,\delta} \right)_{L^2_0} \right]^{1/2} d\Sigma(k)
+ C \int_{\mathbb{Z}^3} \left[ \int_0^1 \left( \langle \arrow k \delta t^{-1} \Lambda_{\delta_1} \hat{f}_{m,\delta} + i [v \cdot k, \Lambda_{\delta_1}] \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \rangle \right)_{L^2_0} \right]^{1/2} d\Sigma(k).
$$

(a) We consider the case when $m \geq 2$. Just repeating the proof of Lemma 3.11, we have that for any $\varepsilon > 0$,

$$
\int_{\mathbb{Z}^3} \left[ \int_0^1 \left( \Lambda_{\delta_1} t^{m} (1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{\Gamma}(\hat{f}, \hat{f}), \Lambda_{\delta_1} \hat{f}_{m,\delta} \right)_{L^2_0} \right]^{1/2} d\Sigma(k)
\leq (\varepsilon + C\varepsilon^{-1} \varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta}(t, k) \|_{L^2_0}^2 dt \right)^{1/2} d\Sigma(k)
+ C\varepsilon^{-1} \varepsilon_0 \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m,\delta}(t, k) \|_{L^2_0} d\Sigma(k) + C\varepsilon^{-1} \varepsilon_0 \|(m - 1)! \|_{\mathbb{Z}^2}^{1/2}.
$$

Following the same argument as for proving Lemma 3.12 gives

$$
\int_{\mathbb{Z}^3} \left( \int_0^1 \| \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_0}^2 dt \right)^{1/2} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left[ \int_0^1 \left( \langle \arrow k \delta t^{-1} \Lambda_{\delta_1} \hat{f}_{m,\delta} + i [v \cdot k, \Lambda_{\delta_1}] \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \rangle \right)_{L^2_0} \right]^{1/2} d\Sigma(k)
\leq \varepsilon \int_{\mathbb{Z}^3} \langle \delta \rangle^{1/2} \left( \int_0^1 \| \Lambda_{\delta_1} \hat{f}_{m,\delta}(t, k) \|_{L^2_0}^2 dt \right)^{1/2} d\Sigma(k) + C\varepsilon^{-1} \varepsilon_0 \|(m - 1)! \|_{\mathbb{Z}^2}^{1/2},
$$
for any $\varepsilon > 0$. Now, we substitute these inequalities into (3.32) to obtain that for any $\varepsilon > 0$,

$$
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} ||\Lambda_{\delta_1} \hat{f}_{m,\delta}(t, k)||_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^1 \left( (a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m,\delta} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\leq (\varepsilon + C\varepsilon^{-1} \varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_0^1 ||(a^{1/2})^w \hat{f}_{m,\delta}(t, k)||_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\leq C \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 ||(a^{1/2})^w \hat{f}_{m,\delta}(t, k)||_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\leq C \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 \left( ||\Lambda_{\delta_1} \hat{f}_{m,\delta}||_{L^2_v} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\leq C \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 \left( ||\Lambda_{\delta_1} \hat{f}_{m,\delta}||_{L^2_v} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k).
$$

This with the assertion (iii) in Proposition 3.7 give that for any $\varepsilon > 0$,

$$
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} ||\Lambda_{\delta_1} \hat{f}_{m,\delta}(t, k)||_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^1 \left( (a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m,\delta} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\leq C \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 \left( ||\Lambda_{\delta_1} \hat{f}_{m,\delta}||_{L^2_v} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\leq C \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 \left( ||\Lambda_{\delta_1} \hat{f}_{m,\delta}||_{L^2_v} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k)
\leq C \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 \left( ||\Lambda_{\delta_1} \hat{f}_{m,\delta}||_{L^2_v} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k).
$$

Thus, letting $\delta_1 \rightarrow 0$ first and then choosing $\varepsilon > 0$ suitably small, we obtain the assertion (ii) in Proposition 3.13, provided that $\varepsilon_0$ is small enough and that

$$
\lim_{\delta_1 \rightarrow 0} \int_{\mathbb{Z}^3} \left( \int_0^1 \left( ||\Lambda_{\delta_1} \hat{f}_{m,\delta}||_{L^2_v} \right) ||_{L^2_v} dt \right)^{1/2} d\Sigma(k) = 0. \tag{3.33}
$$

It remains to prove (3.33). To do so we write

$$
[\mathcal{L}, \Lambda_{\delta_1}] = [\mathcal{L}, \Lambda_{\delta_1} - Id] = \mathcal{L} (\Lambda_{\delta_1} - Id) - (\Lambda_{\delta_1} - Id) \mathcal{L}
$$

with Id the identity operator on $L^2_v$. Moreover, applying Corollary 2.4 as well as the assertion (iii) in Proposition 2.1 gives

$$
\left| \left( \mathcal{L} (\Lambda_{\delta_1} - Id) \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \right)_{L^2_v} \right| \leq C \|(a^{1/2})^w (\Lambda_{\delta_1} - Id) \hat{f}_{m,\delta} \|_{L^2_v} \|(a^{1/2})^w \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_v}
\leq C \|(\Lambda_{\delta_1} - Id) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v} \|(a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v} + C \delta_1^{1/2} \|(a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2
$$

and

$$
\left| \left( (\Lambda_{\delta_1} - Id) \mathcal{L} \hat{f}_{m,\delta}, \Lambda_{\delta_1} \hat{f}_{m,\delta} \right)_{L^2_v} \right| \leq C \|(a^{1/2})^w (\Lambda_{\delta_1}^* - Id) \Lambda_{\delta_1} \hat{f}_{m,\delta} \|_{L^2_v} \|(a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}
\leq C \|(\Lambda_{\delta_1}^* - Id) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v} \|(a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v} + C \delta_1^{1/2} \|(a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2,$$
where we have used (3.12). Thus, we conclude
\[
\left( \int_0^1 \left( |(\mathcal{L}, \Lambda_{\delta_1}, \hat{f}_{m,\delta}, \Lambda_{\delta_1}, \hat{f}_{m,\delta})|_{L^2_v} \right)^2 dt \right)^{1/2} \\
\leq C \left( \int_0^1 \| (\Lambda_{\delta_1} - \text{Id}) (a^{1/2})^w f_{m,\delta} \|_{L^2_v}^2 dt \right)^{1/4} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \right)^{1/4} \\
+ C \left( \int_0^1 \| (\Lambda_{\delta_1}^* - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \right)^{1/4} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \right)^{1/4} \\
+ C \delta^{1/4} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \right)^{1/2}.
\] (3.34)

We claim that
\[
\lim_{\delta_1 \to 0} \int_0^1 \| (\Lambda_{\delta_1} - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt = \lim_{\delta_1 \to 0} \int_0^1 \| (\Lambda_{\delta_1}^* - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt = 0. \quad (3.35)
\]

In fact, by Fubini theorem,
\[
\int_{[0,1] \times \mathbb{R}^3} \left( |(\Lambda_{\delta_1} - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} |^2 dt dv = \int_0^1 \| (\Lambda_{\delta_1} - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \\
\leq \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt = \int_{[0,1] \times \mathbb{R}^3} \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \leq C_\delta
\]
with $C_\delta$ depending on $\delta$ but independent of $\delta_1$, where in the last inequality we have used (3.7). It then follows from the Dominated Convergence Theorem that
\[
\lim_{\delta_1 \to 0} \int_0^1 \| (\Lambda_{\delta_1} - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt = \lim_{\delta_1 \to 0} \int_{[0,1] \times \mathbb{R}^3} \| (\Lambda_{\delta_1}^* - \text{Id}) (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt = 0.
\]

It is likewise for $(\Lambda_{\delta_1}^* - \text{Id})$. Thus we have proved (3.35). Combining (3.34) and (3.35) yields
\[
\lim_{\delta_1 \to 0} \left( \int_0^1 \left( |(\mathcal{L}, \Lambda_{\delta_1}, \hat{f}_{m,\delta}, \Lambda_{\delta_1}, \hat{f}_{m,\delta})|_{L^2_v} \right)^2 dt \right)^{1/2} = 0,
\]
which along with the Dominated Convergence Theorem give the desired estimate (3.33) by observing that
\[
\int_{\mathbb{Z}^3} \left( \int_0^1 \left( |(\mathcal{L}, \Lambda_{\delta_1}, \hat{f}_{m,\delta}, \Lambda_{\delta_1}, \hat{f}_{m,\delta})|_{L^2_v} \right)^2 dt \right)^{1/2} d\Sigma(k) \leq C \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta} \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C_\delta.
\]

Therefore, the assertion (ii) in Proposition 3.13 is proved.

(b) It remains to consider the case when $m = 1$. In fact, repeating the proof of the assertion (ii) for $m = 1$ in Lemma 3.11 gives that for any $\varepsilon > 0$,
\[
\int_{\mathbb{Z}^3} \left[ \int_0^1 \left( |(\Lambda_{\delta_1} t^{m}(1 + \delta |k|^2)^{-1/2} \langle k \rangle^m \hat{f}(\hat{f}, \hat{f})(\Lambda_{\delta_1}, \hat{f}_{m,\delta})|_{L^2_v} \right)^2 dt \right]^{1/2} d\Sigma(k) \\
\leq (\varepsilon + C\varepsilon^{-1}\varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m,\delta}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \\
+ C\varepsilon^{-1}\varepsilon_0 \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{f}_{m,\delta}(t, k) \|_{L^2_v} d\Sigma(k).
Similarly for showing the assertion (ii) for \( m = 1 \) in Lemma 3.12 and by using the fact that 
\[ \| \Lambda_{\delta} \hat{f}_{0, \delta} \|_{L^2} \leq \| \hat{f} \|_{L^2} \], we have
\[
\int_{Z^3} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{m, \delta} \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) 
+ \int_{Z^3} \left[ \int_0^1 \left( \epsilon m t^{-1} \Lambda_{\delta} \hat{f}_{m, \delta} + iv \cdot k, \Lambda_{\delta} \hat{f}_{m, \delta}, \Lambda_{\delta} \hat{f}_{m, \delta} \right)_{L^2} dt \right]^{1/2} d\Sigma(k) 
\leq \varepsilon \int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{m, \delta}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| \hat{f}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) 
\leq \varepsilon \int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{m, \delta}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \| f_0 \|_{L^4_{-1} L^2},
\]
for any \( \varepsilon > 0 \), where in the last line we have used the assertion (i) in Proposition 3.7. Now, we substitute these inequalities into (3.32) to obtain that for any \( \varepsilon > 0 \),
\[
\int_{Z^3} \sup_{0 < t \leq 1} \| \Lambda_{\delta} \hat{f}_{m, \delta}(t, k) \|_{L^2} d\Sigma(k) + \int_{Z^3} \left( \int_0^1 \| (a^{1/2})^w \Lambda_{\delta} \hat{f}_{m, \delta} \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) 
\leq (\varepsilon + C_\varepsilon^{-1} \varepsilon) \int_{Z^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m, \delta}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) 
+ C_\varepsilon \int_{Z^3} \sup_{0 < t \leq 1} \| \hat{f}_{m, \delta}(t, k) \|_{L^2} d\Sigma(k) + C_\varepsilon \| f_0 \|_{L^4_{-1} L^2} 
+ \varepsilon \int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| \Lambda_{\delta} \hat{f}_{m, \delta}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) 
+ C \int_{Z^3} \left( \int_0^1 \| [(\mathcal{L}, \Lambda_{\delta} \hat{f}_{m, \delta}, \Lambda_{\delta} \hat{f}_{m, \delta})]_{L^2} \| dt \right)^{1/2} d\Sigma(k).
\]
The rest argument to obtain the assertion (i) in Proposition 3.13 is the same as that in the case of \( m \geq 2 \), so we omit it for brevity. The proof of Proposition 3.13 is thus completed. \( \square \)

3.4. Completing the proofs. Now we are ready to prove the main results in this section.

Proof of Proposition 3.3. Let \( f(t, x, v) \) be the global mild solution to (1.4) obtained in Proposition 3.1 satisfying the estimate (3.1). Then, by the assertions (i) in Propositions 3.7 and 3.13, we have
\[
\int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| \hat{f}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^4_{-1} L^2},
\]
and
\[
\int_{Z^3} \sup_{0 < t \leq 1} \| \hat{f}_{m, \delta}(t, k) \|_{L^2} d\Sigma(k) + \int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m, \delta} \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^4_{-1} L^2},
\]
with \( m = 1 \). These together with the assertion (ii) in Proposition 3.7 yield that with \( m = 1 \),
\[
\int_{Z^3} \sup_{0 < t \leq 1} \| \hat{f}_{m, \delta}(t, k) \|_{L^2} d\Sigma(k) + \int_{Z^3} \langle k \rangle^{1/2} \left( \int_0^1 \| \hat{f}_{m, \delta}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) 
+ \int_{Z^3} \left( \int_0^1 \| (a^{1/2})^w \hat{f}_{m, \delta}(t, k) \|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^4_{-1} L^2}.
\]
Combining the above estimate with (3.36) and (3.1) and letting the parameter δ above tend to 0, we conclude that for any m ∈ ℤ⁺ with 0 ≤ m ≤ 1,
\[
\int_{ℤ^3} \langle k \rangle^m \left( \sup_{0 < t \leq 1} t^m \| \hat{f}(t, k) \|_{L^2_v} \right) d\Sigma(k) + \int_{ℤ^3} \langle k \rangle^m \left( \int_0^1 t^m \| \hat{f}(t, k) \|_{L^2_v}^2 \right)^{1/2} d\Sigma(k) + \int_{ℤ^3} \langle k \rangle^m \left( \int_0^1 t^m \| (a^{1/2} w \hat{f}(t, k)) \|_{L^2_v}^2 \right)^{1/2} d\Sigma(k) \leq 4^{-1} C_1 \| f_0 \|_{L^1_k L^2_v},
\]
provided that the constant C₁ is chosen large enough. We then have proved Proposition 3.3 for 0 < t ≤ 1. As mentioned at the beginning, the treatment for 1 ≤ t < T is similar. By combining the L¹_k L²_v-norm of f|t=1 established in (3.37), we see that all the estimates in Subsections 3.2-3.3 are also true with 0 < t ≤ 1 therein replaced by 1 ≤ t < T. It follows that for T > 1,
\[
\int_{ℤ^3} \langle k \rangle^m \left( \sup_{1 < t < T} \| \hat{f}(t, k) \|_{L^2_v} \right) d\Sigma(k) + \int_{ℤ^3} \langle k \rangle^m \left( \int_1^T \| \hat{f}(t, k) \|_{L^2_v}^2 \right)^{1/2} d\Sigma(k) + \int_{ℤ^3} \langle k \rangle^m \left( \int_1^T \| (a^{1/2} w \hat{f}(t, k)) \|_{L^2_v}^2 \right)^{1/2} d\Sigma(k) \leq 2^{-1} C_1 \| f_0 \|_{L^1_k L^2_v}.
\]
This together with (3.37) give the desired result in Proposition 3.3. The proof is therefore completed.

Proof of Proposition 3.4. The estimate for 0 < t ≤ 1 just follows from the assertions for m ≥ 2 in Propositions 3.7 and 3.13. Following the similar argument and using the estimate for f|t=1, we then obtain the desired estimate for 1 ≤ t ≤ T. The proof of Proposition 3.4 is thus completed.

4. GEVREY REGULARITY IN VELOCITY VARIABLES

This section is devoted to proving the Gevrey smoothness in velocity variables. Compared with the result for space variables in the previous section, the main difference arises from the treatment of two commutators with one between the regularization operator and the collision operator and the other one between the velocity derivative and the transport operator.

Theorem 4.1. Let f(t, x, v) be the global mild solution to (1.4) obtained in Proposition 3.1 and all the results as in Theorem 3.2 be satisfied. Recall that C₀ is the constant constructed in Proposition 3.4. Let ε₀ > 0 be further small, then there is a positive constant C* ≥ (C₀ + 1)², depending only on s, γ in (1.2) and (1.3), such that for any 0 < T < ∞ and multi-index β with |β| ≥ 0, the solution f(t, x, v) satisfies
\[
\partial_v^\beta f \in L^1_k L^\infty_v L^2_{v'} T^1_k L^2_v; \quad (a^{1/2})^w \partial_v^\beta f \in L^1_k L^2_{v'} T^1_k L^2_v
\]
for any small τ > 0, with the quantitative estimate
\[
\int_{ℤ^3} \sup_{0 < t < T} \phi(t)^{1+|\beta|} \| \partial_v^\beta \hat{f}(t, k) \|_{L^2_v} d\Sigma(k) + \int_{ℤ^3} \left( \int_0^T \phi(t)^{2|\beta|} \| (a^{1/2})^w \partial_v^\beta \hat{f}(t, k) \|_{L^2_v}^2 \right)^{1/2} d\Sigma(k) \leq \tilde{C}_*^{1+|\beta|} \phi(t)^{1+2\varsigma}. \]
Here we recall that \( \phi(t) = \min\{t, 1\} \) and \( \varsigma = \frac{1+2\varsigma}{2\varsigma} \).
We will prove the above result by using induction on the order $|\beta|$ of velocity derivatives. Precisely, Theorem 4.1 follows from the following two propositions.

**Proposition 4.2 ($H^2_v$-regularity).** Under all the assumptions of Theorem 4.1, there is a positive constant $C_2 \leq 1$, depending only on $s$ and $\gamma$, such that for any $0 < T < \infty$ and for any $\beta \in \mathbb{Z}_+^3$ with $0 \leq |\beta| \leq 2$, we have (4.1) with any small $\tau > 0$ as well as the estimate

$$
\int_{\mathbb{R}^3} \sup_{0 < t < T} \phi(t) \| \partial_v^\beta \hat{f}(t, k) \|_{L^2_v} \mathrm{d}\Sigma(k)
+ \int_{\mathbb{R}^3} \left( \int_0^T \phi(t) \| (a^{1/2})^w \partial_v^\beta \hat{f}(t, k) \|_{L^2_v}^2 \mathrm{d}t \right)^{\frac{1}{2}} \mathrm{d}\Sigma(k) \leq C_2.
$$

**Proposition 4.3 (Inductive Gevrey regularity).** Suppose that all the assumptions of Theorem 4.1 hold. Let $C_2 \leq 1$ and $C_0$ be constants constructed in Proposition 4.2 and Proposition 3.4, respectively. Fix an arbitrary integer $m \geq 3$. Then, there is a positive constant $\tilde{C}_m \geq (1 + C_0)^2$, depending only on $s$ and $\gamma$ but independent of $m$, such that if for any $\beta \in \mathbb{Z}_+^3$ with $0 \leq |\beta| \leq m - 1$ (4.1) holds for any small $\tau > 0$ with the estimate

$$
\int_{\mathbb{R}^3} \sup_{0 < t < T} \phi(t) \| \partial_v^\beta \hat{f}(t) \|_{L^2_v} \mathrm{d}\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^T \phi(t) \| (a^{1/2})^w \partial_v^\beta \hat{f}(t) \|_{L^2_v}^2 \mathrm{d}t \right)^{1/2} \mathrm{d}\Sigma(k)
\leq \begin{cases} 
C_2, & \text{if } |\beta| \leq 2 \\
\tilde{C}_m^{[|\beta|]} \left( [(|\beta| - 1)]! \frac{1 + |\beta|}{2} \right), & \text{if } |\beta| \geq 3,
\end{cases}
$$

then we have the same thing for any $\beta \in \mathbb{Z}_+^3$ with $|\beta| = m$.

For convenience of presentation, we first focus on the proof of Proposition 4.3 in the following Subsection 4.1. The proof of Proposition 4.2 for $H^2_v$-regularity will be postponed to Subsection 4.2 since we will treat it in a similar way with the more straightforward argument.

### 4.1. Inductive Gevrey regularity

We are going to first give the proof of Proposition 4.3. As in the previous section for treating smoothness in space variables, we only consider the case of $0 < t \leq 1$ and perform estimates for the regularization

$$
\tilde{F}_{m, \delta}(t, k, v) = t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \hat{f}(t, k, v), \quad 0 < \delta \ll 1, \; m \geq 1.
$$

Here we are first concerned with the velocity derivative in the first component $v_1$ and the same thing can be done for $v_2$ and $v_3$ as well. Then, in view of (3.13) and recalling that $\Lambda_{\delta_1}$ is defined by (3.5), we have

$$
(\partial_t + iv \cdot k - \mathcal{L}) \Lambda_{\delta_1} \tilde{F}_{m, \delta} = \Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \hat{f}(\hat{f}, \hat{f})
+ [\Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m, \; \mathcal{L}] \hat{f} + \zeta m t^{-1} \Lambda_{\delta_1} \tilde{F}_{m, \delta} + [iv \cdot k, \; \Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m] \hat{f}.
$$

Note that similar results as in Lemma 3.5 also hold for $\tilde{F}_{m, \delta}$. This enables us to take the $L^2_v$ inner product of the above equation with $\tilde{F}_{m, \delta}$ and then integrate the resulting result over $[0, t]$ for any $0 < t \leq 1$, so as to obtain by following the argument in the previous Subsection.
3.3 that

\[
\int_{\mathbb{S}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k)
\leq C \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k)
\]

Lemma 4.4. Let all the assumptions of Proposition 4.3 hold. Then, assuming the induction assumption in Proposition 4.3, it holds that for any \( \varepsilon > 0 \),

\[
\int_{\mathbb{S}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k)
\leq \left( \varepsilon + C \varepsilon^{-1} \varepsilon_0 \right) \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k)
\]

Proof. Recall that \( \Gamma(g, h) = T(g, h, \mu^{1/2}) \) with

\[
T(g, h, w) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^2} B(v - v_s, \sigma) w(v, \sigma) \left( g(v') h(v) - g(v) h \right) d\sigma dv.
\] (4.5)

Since \( \Gamma \) is invariant under translation with respect to \( v \), one can see that the Leibniz formula in \( v \) can be applied to obtain

\[
\partial^m_{v_1} \hat{F}(\hat{f}, \hat{\tilde{f}}) = \sum_{j=0}^m \sum_{p=0}^j \binom{m}{j} \left( \frac{j}{p} \right) \tilde{T}(\partial^{m-j}_{v_1} \hat{f}, \partial^{j-p}_{v_1} \hat{\tilde{f}}, \partial^p_{v_1} \mu^{1/2}).
\]

Then,

\[
\int_{\mathbb{S}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k) \leq I + J + K,
\]

with

\[
I = \int_{\mathbb{S}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \tilde{T}(\hat{f}, \partial^m_{v_1} \hat{f}, \mu^{1/2}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k)
\]

and

\[
J = \sum_{p=1}^m \binom{m}{p} \int_{\mathbb{S}^3} \left( \int_0^1 \left| \left( \Lambda^{m\omega} - \delta \partial^2_v \right) \tilde{T}(\hat{f}, \partial^m_{v_1} \hat{f}, \mu^{1/2}), \Lambda^{m\omega} \hat{F}(\hat{f}, \hat{\tilde{f}}) \right| dt \right)^{1/2} d\Sigma(k),
\]

(4.7)
\[ J = \int_{\mathbb{Z}^3} \sum_{j=1}^{m-1} \sum_{p=0}^{j} \binom{m}{j} \binom{j}{p} \left( \int_0^1 \left| (\Lambda_{\delta i} t^{mc}) (1 - \delta \partial^2_{v_i})^{-1} \hat{T}(\partial^m_{v_i} \hat{f}, \partial^2_{v_i} \hat{f}, x^{-1} \mu^{1/2}), \Lambda_{\delta i} \hat{F}_{m, \delta} \right| dt \right)^{1/2} d\Sigma(k), \] (4.8)

and

\[ K = \int_{\mathbb{Z}^3} \left( \int_0^1 \left| (\Lambda_{\delta i} t^{mc}) (1 - \delta \partial^2_{v_i})^{-1} \hat{T}(\partial^m_{v_i} \hat{f}, \mu^{1/2}), \Lambda_{\delta i} \hat{F}_{m, \delta} \right| dt \right)^{1/2} d\Sigma(k). \] (4.9)

We estimate \( I, J \) and \( K \) as follows.

**Estimate on \( I \).** For \( I \) in (4.7), we claim that for any \( \varepsilon > 0 \),

\[ I \leq (\varepsilon + C \varepsilon^{-1} \varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_0^1 \left| (a^{1/2})_{w} \hat{F}_{m, \delta}(k) \right|^{2} d\Sigma(k) \right)^{1/2} d\Sigma(k) + C \varepsilon \hat{C}^m \leq [(m - 1)!]^{1/2} \varepsilon. \] (4.10)

In fact, in order to estimate the first term on the right hand side of \( I \), we use the fact that \( (1 - \delta \partial^2_{v_i})^{-1} (1 - \delta \partial^2_{v_i})^{-1} = 1 \) and then apply the Leibniz formula to write

\[ \hat{T}(\hat{f}, \partial^m_{v_i} \hat{f}, \mu^{1/2}) = \hat{T}(\hat{f}, (1 - \delta \partial^2_{v_i})^{-1} \partial^m_{v_i} \hat{f}, \mu^{1/2}) = (1 - \delta \partial^2_{v_i})^{-1} \hat{T}((1 - \delta \partial^2_{v_i})^{-1} \partial^m_{v_i} \hat{f}, \mu^{1/2}) + \delta \sum_{j=1}^{2} \sum_{p=0}^{j} \binom{j}{p} \hat{T}(\partial^j_{v_i} \hat{f}, \partial^2_{v_i} \hat{f}, (1 - \delta \partial^2_{v_i})^{-1} \partial^m_{v_i} \hat{f}, \partial^p_{v_i} \mu^{1/2}). \] (4.11)

As a result, recalling the definition (4.3) of \( \hat{F}_{m, \delta} \), one has

\[ \int_{\mathbb{Z}^3} \left( \int_0^1 \left| (\Lambda_{\delta i} t^{mc}) (1 - \delta \partial^2_{v_i})^{-1} \hat{T}(\hat{f}, \partial^m_{v_i} \hat{f}, \mu^{1/2}), \Lambda_{\delta i} \hat{F}_{m, \delta} \right| dt \right)^{1/2} d\Sigma(k) \leq C (I_1 + I_2) \] (4.12)

with

\[ I_1 = \int_{\mathbb{Z}^3} \left( \int_0^1 \left| (\Lambda_{\delta i} \hat{T}(\hat{f}, \hat{F}_{m, \delta}, \mu^{1/2}), \Lambda_{\delta i} \hat{F}_{m, \delta} \right| dt \right)^{1/2} d\Sigma(k), \]

and

\[ I_2 = \sum_{j=1}^{2} \sum_{p=0}^{j} \int_{\mathbb{Z}^3} \delta \left[ \int_0^1 \left| (\Lambda_{\delta i} t^{mc}) (1 - \delta \partial^2_{v_i})^{-1} \hat{T}(\partial^j_{v_i} \hat{f}, \partial^2_{v_i} \hat{f}, (1 - \delta \partial^2_{v_i})^{-1} \partial^m_{v_i} \hat{f}, \partial^p_{v_i} \mu^{1/2}), \Lambda_{\delta i} \hat{F}_{m, \delta} \right| dt \right]^{1/2} d\Sigma(k). \]
Note that $\Gamma(g,h) = T(g,h,\mu^{1/2})$. Thus, using Lemmas 2.3 and 2.5 as well as (3.1), we compute to have that for any $\varepsilon > 0$,

$$I_1 \leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2}w) \hat{F}_{m,\delta}(t,k) \|_{L^2_k}^2 \, dt \right)^{1/2} d\Sigma(k)$$

$$+ C\varepsilon^{-1} \int_{\mathbb{Z}^3} \sup_{0 \leq t \leq 1} \| \hat{f}(t,k) \|_{L^2_k} d\Sigma(k) \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2}w) \hat{F}_{m,\delta}(t,k) \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k)$$

$$\leq (\varepsilon + C\varepsilon^{-1}\varepsilon_0) \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2}w) \hat{F}_{m,\delta}(t,k) \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k). \tag{4.13}$$

Next we treat the term $I_2$. Notice that the terms $T(g,h,\partial^p v_1,\mu^{1/2})$ for $p \leq 2$ enjoy the same properties as those in Lemma 2.3 for $\Gamma(g,h) = T(g,h,\mu^{1/2})$. Consequently, using again Lemmas 2.3 and 2.5, it holds that for any $\varepsilon > 0$,

$$I_2 \leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2}w) (1 - \delta \partial^2_{v_1})^{-1} \Lambda^*_\delta \Lambda_{\delta_1} \hat{F}_{m,\delta}(k) \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k)$$

$$+ C\varepsilon^{-1} \sum_{1 \leq j \leq 2} \sum_{0 \leq p \leq j} \left[ \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \partial^{j-p}_{v_1} \hat{f} \|_{L^2_k} d\Sigma(k) \right]$$

$$\times \int_{\mathbb{Z}^3} \left( \int_0^1 t^{2c(m-j)} \| (a^{1/2}w) \delta \partial^{-j}_{v_1} (1 - \delta \partial^2_{v_1})^{-1} \partial^m_{v_1} \hat{f} \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k).$$

As for the first term on the right hand side, we use Lemma 2.2 to obtain

$$\| (a^{1/2}w) (1 - \delta \partial^2_{v_1})^{-1} \Lambda^*_\delta \Lambda_{\delta_1} \hat{F}_{m,\delta}(k) \|_{L^2_k} \leq \| (a^{1/2}w) \hat{F}_{m,\delta}(k) \|_{L^2_k}.$$ 

Meanwhile, for the second term, we notice that $\delta \partial^{-j}_{v_1} (1 - \delta \partial^2_{v_1})^{-1} \partial^m_{v_1}$ is the Weyl quantization of symbol $\delta \eta^{-j}_{v_1} (1 + \delta |\eta|^2)^{-1} \eta^j_{v_1}$, which belongs to $S(1,|dv|^2+|d\eta|^2)$ uniformly with respect to $\delta$. This with Lemma 2.2 yield that

$$\| (a^{1/2}w) \delta \partial^{-j}_{v_1} (1 - \delta \partial^2_{v_1})^{-1} \partial^m_{v_1} \hat{f} \|_{L^2_k} \leq \| (a^{1/2}w) \partial^{-j}_{v_1} \hat{f} \|_{L^2_k}.$$ 

Then, combining these inequalities gives

$$I_2 \leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2}w) \hat{F}_{m,\delta}(k) \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k)$$

$$+ C\varepsilon^{-1} \left( \sum_{|\beta| \leq 2} \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \partial^{|\beta|}_{v_1} \hat{f} \|_{L^2_k} d\Sigma(k) \right)$$

$$\times \sum_{1 \leq j \leq 2} \int_{\mathbb{Z}^3} \left( \int_0^1 t^{2c(m-j)} \| (a^{1/2}w) \partial^m_{v_1} \hat{f} \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k)$$

$$\leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2}w) \hat{F}_{m,\delta}(k) \|_{L^2_k}^2 \right)^{1/2} d\Sigma(k) + C\varepsilon^{m-2} \left[ (m-1)! \right]^{1/2}.$$
where the last inequality holds because of the induction assumption specified in Proposition 4.3. Now, we substitute the above estimate and \((4.13)\) into \((4.12)\) to conclude

\[
\int_{\mathbb{R}^3} \left(\int_0^1 \left| (\Lambda_{\delta_1} t^m c (1 - \delta \partial^2_{\nu_1})^{-1} \tilde{T}(\hat{f}, \partial^p_{\nu_1} \hat{f}, \partial^p_{\nu_1} \mu^2), \Lambda_{\delta_1} \tilde{F}_{m, \delta} \right)_{L^2} \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq (\varepsilon + C\varepsilon^{-1} \xi_0) \int_{\mathbb{R}^3} \left( \int_0^1 \left\| (a^{1/2})^w \tilde{F}_{m, \delta}(k) \right\|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \tilde{C}_*^{m-2} [(m - 1)!]^{\frac{1 + 2\delta}{2}}. \tag{4.14}
\]

We have completed the estimate on the first term on the right hand side of \(I\) given in \((4.7)\).

It remains to estimate the second term on the right hand side of \(I\). In fact, following the above argument for treating \(I_2\) and observing that

\[
\left\| \partial^p_{\nu_1} \mu^{1/2} \right\|_{L^2} \leq 8^{p+1} p!, \quad \forall \ p \geq 0,
\]

we have

\[
\sum_{p=1}^{m} \left( \begin{array}{c} m \\ p \end{array} \right) S^{p+1} p! \int_{\mathbb{R}^3} \left( \int_0^1 (a^{1/2})^w \partial^p_{\nu_1} \hat{f} \right|_{L^2}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \sum_{p=1}^{m-1} \frac{m!}{(m-p)!} 8^{p+1} \tilde{C}_*^{m-p-1} [(m-p-1)!]^{\frac{1 + 2\delta}{2}} + m! 8^{m+1} \int_{\mathbb{R}^3} \left( \int_0^1 (a^{1/2})^w \hat{f} \right|_{L^2}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \tilde{C}_*^{m-2} [(m-1)!]^{\frac{1 + 2\delta}{2}} + \sum_{p=1}^{m-1} \frac{m! [(m-p-1)!]^{\frac{1 + 2\delta}{2}}}{(m-p)! [(m-1)!]^{\frac{1 + 2\delta}{2}}} \tilde{C}_*^2 (8/\tilde{C}_*)^{p+1} + \varepsilon_0 m! 8^{m+1}
\]

\[
\leq C \tilde{C}_*^{m-2} [(m-1)!]^{\frac{1 + 2\delta}{2}},
\]

where the last inequality holds true since we have chosen the constant \(C_* > 0\) large enough. Thus, combining these inequalities, we have

\[
\sum_{p=1}^{m} \left( \begin{array}{c} m \\ p \end{array} \right) S^{p+1} p! \int_{\mathbb{R}^3} \left( \int_0^1 (\Lambda_{\delta_1} t^m c (1 - \delta \partial^2_{\nu_1})^{-1} \tilde{T}(\hat{f}, \partial^p_{\nu_1} \hat{f}, \partial^p_{\nu_1} \mu^2), \Lambda_{\delta_1} \tilde{F}_{m, \delta} \right)_{L^2} \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left\| (a^{1/2})^w \tilde{F}_{m, \delta}(k) \right\|_{L^2}^2 dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \tilde{C}_*^{m-2} [(m - 1)!]^{\frac{1 + 2\delta}{2}}.
\]

This with \((4.14)\) yield the desired upper bound \((4.10)\) of \(I\).
Estimate on $J$. Recall that $J$ is given in (4.8). Using the similar argument for treating $I$ above with slight modifications, we arrive at

$$J \leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \|(a^{1/2})^w \hat{F}_{m,\delta}(k)\|_{L^2_v}^2 \, dt \right)^{1/2} \, d\Sigma(k)$$

$$+ C \sum_{j=1}^{m-1} \sum_{p=0}^j \binom{m}{j} \binom{j}{p} 8^{p+1} p! \left( \int_{\mathbb{Z}^3} \sup_{0<t\leq 1} t^{(m-j)} \|\partial_{v_1}^{m-j} \hat{f}\|_{L^2_v} \, d\Sigma(k) \right)$$

$$\times \int_{\mathbb{Z}^3} \left( \int_0^1 t^{2c(j-p)} \|(a^{1/2})^w \partial_{v_1}^{j-p} \hat{f}\|_{L^2_v}^2 \, dt \right)^{1/2} \, d\Sigma(k).$$

Moreover, for any $1 \leq j \leq m-1$, it holds that

$$\sum_{p=0}^j \binom{j}{p} 8^{p+1} p! \int_{\mathbb{Z}^3} \left( \int_0^1 t^{2c(j-p)} \|(a^{1/2})^w \partial_{v_1}^{j-p} \hat{f}\|_{L^2_v} \, dt \right)^{1/2} \, d\Sigma(k)$$

$$\leq \varepsilon 8^{j+1} j! + \sum_{p=0}^{j-1} \frac{j!}{(j-p)!} 8^{p+1} \hat{C}^{j-p-1} [(j-p-1)!]^{1+2s} \leq C \hat{C}^{j-1} [(j-1)!]^{1+2s},$$

where in the second inequality we have used the induction assumption (4.2) specified in Proposition 4.3. As a result, combining the above inequality with the inductive assumption (4.2) and the same argument as for deriving (3.28), we have

$$\sum_{j=1}^{m-1} \sum_{p=0}^j \binom{m}{j} \binom{j}{p} 8^{p+1} p! \left( \int_{\mathbb{Z}^3} \sup_{0<t\leq 1} t^{(m-j)} \|\partial_{v_1}^{m-j} \hat{f}\|_{L^2_v} \, d\Sigma(k) \right)$$

$$\times \int_{\mathbb{Z}^3} \left( \int_0^1 t^{2c(j-p)} \|(a^{1/2})^w \partial_{v_1}^{j-p} \hat{f}\|_{L^2_v}^2 \, dt \right)^{1/2} \, d\Sigma(k)$$

$$\leq C \sum_{j=1}^{m-1} \frac{m!}{j!(m-j)!} \hat{C}^{m-j-1} [(m-j-1)!]^{1+2s} \hat{C}^{j-1} [(j-1)!]^{1+2s} \leq C \hat{C}^{m-2} [(m-1)!]^{1+2s}.$$}

Thus we conclude by combining these inequalities that

$$J \leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \|(a^{1/2})^w \hat{F}_{m,\delta}(k)\|_{L^2_v}^2 \, dt \right)^{1/2} \, d\Sigma(k)$$

$$+ C \varepsilon^{-1} \epsilon_0 \int_{\mathbb{Z}^3} \sup_{0<t\leq 1} \|\hat{F}_{m,\delta}(t,k)\|_{L^2_v} \, d\Sigma(k) + C \hat{C}^{m-2} [(m-1)!]^{1+2s}. \quad (4.15)$$

Estimate on $K$. Recall $K$ in (4.9). The estimate on $K$ is similar to that on $I$ shown before. In fact, we can follow the argument for proving (4.14) to conclude

$$K \leq \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \|(a^{1/2})^w \hat{F}_{m,\delta}(t,k)\|_{L^2_v}^2 \, dt \right)^{1/2} \, d\Sigma(k)$$

$$+ C \varepsilon^{-1} \epsilon_0 \int_{\mathbb{Z}^3} \sup_{0<t\leq 1} \|\hat{F}_{m,\delta}(t,k)\|_{L^2_v} \, d\Sigma(k) + C \hat{C}^{m-2} [(m-1)!]^{1+2s}. \quad (4.16)$$

The details are omitted for brevity.

Now, we substitute all the above estimates (4.10), (4.15) and (4.16) back to (4.6) and hence complete the proof of Lemma 4.4. \qed
Lemma 4.5. Let all the assumptions of Proposition 4.3 hold. Then, assuming the induction assumption specified in Proposition 4.3, it holds that for any $\varepsilon > 0$,

$$
\lim_{\delta_1 \to 0} \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( \Lambda_{\delta_1} t^{mc} (1 - \delta \partial_v^2) \right)^{-1} \partial_v^m, \ L \right| \hat{f}, \ \Lambda_{\delta_1} \hat{F}_{m, \delta} \right|_{L^2_x}^2 dt \right)^{1/2} d\Sigma(k)
$$

\[\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left( (a^{1/2})^w \hat{F}_{m, \delta}(t, k) \right)^2_{L^2} dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \tilde{C}^{-2m-2} [(m-1)! ]^{1+2s}.
\]

Proof. Observe that we have the following identity

$$
[\Lambda_{\delta_1} t^{mc} (1 - \delta \partial_v^2) \right)^{-1} \partial_v^m, \ L] = [\Lambda_{\delta_1}, \ L] t^{mc} (1 - \delta \partial_v^2) \right)^{-1} \partial_v^m + \Lambda_{\delta_1} t^{mc} [(1 - \delta \partial_v^2) \right)^{-1} \partial_v^m, \ L],
$$

where for the commutator in the last term we can further write

$$
[(1 - \delta \partial_v^2) \right)^{-1} \partial_v^m, \ L] \hat{f} = (1 - \delta \partial_v^2) \right)^{-1} \partial_v^m \Gamma(\hat{f}, \mu^{1/2}) - \Gamma((1 - \delta \partial_v^2) \right)^{-1} \partial_v^m \hat{f}, \mu^{1/2}) + (1 - \delta \partial_v^2) \right)^{-1} \partial_v^m \hat{f})
$$

This enables us to follow the argument in the proof of Lemma 4.4 so as to conclude

$$
\int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( \Lambda_{\delta_1} t^{mc} [(1 - \delta \partial_v^2) \right)^{-1} \partial_v^m, \ L] \hat{f}, \ \Lambda_{\delta_1} \hat{F}_{m, \delta} \right|_{L^2_x}^2 dt \right)^{1/2} d\Sigma(k)
$$

\[\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left( (a^{1/2})^w \hat{F}_{m, \delta}(t, k) \right)^2_{L^2} dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \tilde{C}^{-2m-2} [(m-1)! ]^{1+2s}.
\]

Meanwhile, similarly as for showing (3.33), we have

$$
\lim_{\delta_1 \to 0} \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( [\Lambda_{\delta_1}, \ L] t^{mc} (1 - \delta \partial_v^2) \right)^{-1} \partial_v^m \hat{f}, \ \Lambda_{\delta_1} \hat{F}_{m, \delta} \right|_{L^2_x}^2 dt \right)^{1/2} d\Sigma(k) = 0.
$$

Combining those estimates gives the conclusion of Lemma 4.5. Therefore, the proof is completed. \hfill \Box

Lemma 4.6. Let all the assumptions of Proposition 4.3 hold. Then, assuming the induction assumption specified in Proposition 4.3, it holds that for any $\varepsilon > 0$,

$$
\int_{\mathbb{R}^3} \left( \int_0^1 \left( t^{mc} \right)^{m-1} \left| \Lambda_{\delta_1} \hat{F}_{m, \delta}(t) \right|^2_{L^2_x} dt \right)^{1/2} d\Sigma(k)
$$

\[\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left( (a^{1/2})^w \hat{F}_{m, \delta} \right)^2_{L^2} dt \right)^{1/2} d\Sigma(k) + C_\varepsilon \tilde{C}^{-2m-2} [(m-1)! ]^{1+2s}.
\]

Proof. We start with the Sobolev interpolation inequality of the form

$$
\left\| \hat{F}_{m, \delta} \right\|_{L^2_x}^2 \leq \tau \left\| \langle D_v \rangle^{\frac{s}{s+2s}} \hat{F}_{m, \delta} \right\|_{L^2_x}^2 + \tau^{-\frac{s}{s+2s}} \left\| \langle D_v \rangle^{\frac{s}{s+2s}-1} \hat{F}_{m, \delta} \right\|_{L^2_x}^2,
$$

where for}
for any \( \tau > 0 \). In particular, choosing \( \tau = t\varepsilon^2/m \) for \( \varepsilon > 0 \) and recalling the notation \( \zeta = \frac{1+2s}{2s} \), it follows that

\[
\int_{\mathbb{R}^3} \left( \int_0^1 nt^{-1} \| \Lambda_{\delta_1} \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) \leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| \langle D_{v_1} \rangle_{1+2s} \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) 
\]

\[
+ C\varepsilon m^{\frac{1+2s}{4s}} \int_{\mathbb{R}^3} \left( \int_0^1 t^{-2c} \| \langle D_{v_1} \rangle_{1+2s} \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) 
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) 
\]

\[
+ C\varepsilon m^{\frac{1+2s}{4s}} \int_{\mathbb{R}^3} \left( \int_0^1 t^{-2c} \| (a^{1/2})^w \langle D_{v_1} \rangle_{1+2s} \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k),
\]

(4.17)

where in the last inequality we have used the assertion (iii) in Proposition 2.1. Moreover, in view of (4.3), we use the induction assumption (4.2) as well as Lemma 2.2 to compute

\[
m^{\frac{1+2s}{4s}} \int_{\mathbb{R}^3} \left( \int_0^1 t^{-2c} \| (a^{1/2})^w \langle D_{v_1} \rangle_{1+2s} \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) 
\]

\[
\leq Cm^{\frac{1+2s}{4s}} \int_{\mathbb{R}^3} \left( \int_0^1 t^{2(m-1)c} \| (a^{1/2})^w \partial_{v_1}^{m-1} \hat{f}(t,k) \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) \leq C\tilde{C}_s^{m-2} [(m-1)!]^{\frac{1+2s}{2s}}.
\]

Thus the desired estimate follows by combining those inequalities. The proof of Lemma 4.6 is then completed. \( \square \)

**Lemma 4.7.** Let all the assumptions of Proposition 4.3 hold. Then, assuming the induction assumption specified in Proposition 4.3, it holds that for any \( \varepsilon > 0 \),

\[
\int_{\mathbb{R}^3} \left( \int_0^1 \left( \left[ iv \cdot k, \Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \right] \hat{f}, \Lambda_{\delta_1} \hat{F}_{m,\delta} \right)_{L_0^2} \right)^{1/2} d\Sigma(k) 
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 t^{2cm} \| (a^{1/2})^w \hat{F}_{m,\delta} \|^2_{L_0^2} dt \right)^{\frac{1}{2}} d\Sigma(k) + C\varepsilon \tilde{C}_s^{m-2} [(m-1)!]^{\frac{1+2s}{2s}}.
\]

**Proof.** Note that

\[
\left[ iv \cdot k, \Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \right] 
\]

\[
= -i\Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} mk_1 \partial_{v_1}^{m-1} - 2i\Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-2}(\delta k_1 \partial_{v_1}) \partial_{v_1}^{m} 
\]

\[
- 2i(1 + \delta \| v \|^2)^{-1-\gamma} (1 + \delta \| k \|^2 - \delta_1 \Delta_{v_1})^{-2}(\delta_1 k \cdot \partial_{v_1}) t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m.
\]

This implies

\[
\| [iv \cdot k, \Lambda_{\delta_1} t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \hat{f}] \|_{L_0^2} \leq Cm \| \hat{f} \|_{L_0^2} \| (1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \|_{L_0^2} 
\]

\[
\leq Cm t^{mc} \| \hat{f} \|_{L_0^2} + Cm t^{mc}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^m \|_{L_0^2}
\]

\[
= Cm t^{mc} \| \hat{f} \|_{L_0^2} + Cm \| \hat{F}_{m,\delta} \|_{L_0^2}. \quad (4.18)
\]
Consequently, via the similar arguments as for deriving (3.30) and (4.17), we have

\[
\int_{\mathbb{Z}^3} \left( \int_0^1 \left( \left[ i v \cdot k, \Lambda_{\delta_1} t^{m} (1 - \delta \hat{\partial}^2 v_1) \right] \hat{f}, \Lambda_{\delta_1} \hat{F}_{m, \delta} \right)_{L^2} dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq C \int_{\mathbb{Z}^3} \left( \int_0^1 \left[ i v \cdot k, \Lambda_{\delta_1} t^{m} (1 - \delta \hat{\partial}^2 v_1) \right] \hat{f}, \Lambda_{\delta_1} \hat{F}_{m, \delta} \right)_{L^2} dt \, d\Sigma(k) + \varepsilon \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2} w \hat{F}_{m, \delta} )_{L^2} \right)^{1/2} d\Sigma(k)
\]

\[
+ C C_0 \cdot \left[ (m - 1)! \right]^{1/2} + C_\varepsilon C_{\delta_1}^{m-2} \left[ (m - 1)! \right]^{1/2} ,
\]

where we have used Proposition 3.4 with the constant \( C_0 \) therein and also we have chosen \( C_{\delta_1} \) such that \( C_{\delta_1} \geq (C_0 + 1)^2 \). The proof of Lemma 4.7 is thus completed.

**Ending the proof of Proposition 4.3.** We substitute all the estimates in Lemmas 4.4-4.7 into (4.4) and further let \( \delta_1 \to 0 \). It thus follows from the Fatou Lemma that

\[
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{F}_{m, \delta}(t, k) \|_{L^2}^2 d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2} w \hat{F}_{m, \delta}(t, k) )_{L^2} \right)^{1/2} d\Sigma(k)
\]

\[
\leq \varepsilon + C_\varepsilon^{-1} \varepsilon_0 \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2} w \hat{F}_{m, \delta}(t, k) )_{L^2} \right)^{1/2} d\Sigma(k)
\]

\[
+ C_\varepsilon^{-1} \varepsilon_0 \int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{F}_{m, \delta}(t, k) \|_{L^2}^2 d\Sigma(k) + C_\varepsilon C_{\delta_1}^{m-2} \left[ (m - 1)! \right]^{1/2} .
\]

In the above estimate, we let \( \varepsilon > 0 \) and further \( \varepsilon_0 > 0 \) be suitably small so that those integral terms on the right can be absorbed. Furthermore, letting \( \delta \to 0 \), in view of the definition (4.3) of \( \hat{F}_{m, \delta} \), it follows again from the Fatou Lemma that

\[
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \| \hat{F}_{m, \delta}(t, k) \|_{L^2}^2 d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2} w \hat{F}_{m, \delta}(t, k) )_{L^2} \right)^{1/2} d\Sigma(k)
\]

\[
\leq C C_{\delta_1}^{m-2} \left[ (m - 1)! \right]^{1/2} .
\]

Note that the above estimate still holds with \( \partial^m_{v_1} \) replaced by \( \partial^m_{v_1}, j = 2, 3 \). Thus for any \( \beta \in \mathbb{Z}_+^3 \) with \( |\beta| = m \geq 3 \), using the fact that

\[
\|\partial^\beta_v \hat{f} \|_{L^2} \leq C \sum_{1 \leq j \leq 3} \|\partial^m_{v_j} \hat{f} \|_{L^2},
\]

it holds that

\[
\int_{\mathbb{Z}^3} \sup_{0 < t \leq 1} \|\partial^\beta_v \hat{f} \|_{L^2} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^1 \| (a^{1/2} w \hat{F}_{m, \delta}(t, k) )_{L^2} \right)^{1/2} d\Sigma(k)
\]

\[
\leq C C_{\delta_1}^{m-2} \left[ (|\beta| - 1)! \right]^{1/2} \leq \frac{1}{4} C_{\delta_1}^{m-1} \left[ (|\beta| - 1)! \right]^{1/2} ,
\]

where we have chosen \( C_{\delta_1} > 4C \) in the second line. Thus, we have proved Proposition 4.3 for \( 0 < t \leq 1 \). The estimates for \( 1 \leq t \leq T \) are similar and here we omit them for brevity. The proof of Proposition 4.3 is therefore completed. \( \square \)
4.2. Low-order regularity. In this part we are going to show Proposition 4.2 for the low-order $H^2_v$-regularity. As mentioned before, the key argument is similar to that for the proof of Proposition 4.3 in the previous subsection. The main differences between them arise from the absence of the induction assumption (4.2). Instead we will prove the desired result for $|\beta| \leq 2$ starting from the global existence result of low-regularity solutions in Proposition 3.1, in particular the estimate (3.1).

To begin with we study the smoothness in $H^2_v$ for solutions to the following linear Cauchy problem with initial data in $L^1_kL^2_v$:

$$\left( \partial_t + v \cdot \nabla_x - L \right) h = \Gamma(g, h), \quad h(0, x, v) = f_0(x, v),$$

(4.19)

where $g$ is a given function satisfying certain conditions listed as below.

Proposition 4.8 (Linear problem: existence and spatial regularity). Let $f_0 \in L^1_kL^2_v$. There is a constant $c_0 > 0$ such that if $g$ satisfies

$$\int_{\mathbb{Z}^3} \sup_{0 < t < T} \| \hat{g}(t, k) \|_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^T \| (a^{1/2})^w \hat{g}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq c_0,$$

for any $T > 0$, then the linear Cauchy problem (4.19) admits a unique global-in-time mild solution $h \in L^1_kL^\infty_TL^2_v$ for any $T > 0$. Moreover, there is a constant $C_3 > 0$, depending only on the parameters $s, \gamma$ in (1.2) and (1.3) but independent of the constant $c_0$ above, such that it holds for any $T > 0$ that

$$\| h \|_{L^1_kL^\infty_TL^2_v} + \| (a^{1/2})^w h \|_{L^1_kL^2_TL^2_v} \leq C_3 \| f_0 \|_{L^1_kL^2_v}$$

(4.20)

and

$$\int_{\mathbb{Z}^3} \langle k \rangle^N \sup_{0 < t < T} \phi(t)^N \| \hat{h}(t, k) \|_{L^2_v} d\Sigma(k)$$

$$+ \int_{\mathbb{Z}^3} \langle k \rangle^N \left( \int_0^T \phi(t)^{2N} \| (a^{1/2})^w \hat{h}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C_3 \| f_0 \|_{L^1_kL^2_v},$$

(4.21)

with any integer $0 \leq N \leq 2$. Here we recall that $\phi(t) = \min\{t, 1\}$ and $\varsigma = \frac{1+2s}{2s}$.\hfill \Box

Proof. The existence and uniqueness of solutions satisfying (4.20) follow from the same strategy as in [24], where the corresponding results were established for the nonlinear rather than linear problem. And the spatial regularity (4.21) can be achieved similarly as in Proposition 3.3 by virtue of (4.20). We omit the details here for brevity.

Proposition 4.9 (Linear problem: $H^2_v$-smoothing effect). There are constants $\varepsilon_0 > 0$ and $C_4 > 0$ such that if $f_0 \in L^1_kL^2_v$ with

$$\| f_0 \|_{L^1_kL^2_v} \leq \varepsilon_0$$

(4.22)

holds and $g$ satisfies that for any $T > 0$ and for any $\beta \in \mathbb{Z}^3$, with $|\beta| \leq 2$,

$$\int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\varsigma|\beta|} \| \partial^\beta_v \hat{g}(t, k) \|_{L^2_v} d\Sigma(k)$$

$$+ \int_{\mathbb{Z}^3} \left( \int_0^T \phi(t)^{2\varsigma|\beta|} \| (a^{1/2})^w \partial^\beta_v \hat{g}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C_4 \| f_0 \|_{L^1_kL^2_v},$$

(4.23)
then the global solution \( h \in L^1_t L^\infty_x L^2_v \) to (4.19) constructed in Proposition 4.8 satisfies the estimates (4.20) and (4.21), and it further holds that for any \( T > 0 \) and any \( \beta \in \mathbb{Z}^3_+ \) with \( |\beta| \leq 2 \),

\[
\int_{\mathbb{R}^3} \sup_{0 < t < T} \phi(t)^{c_{|\beta|}} \| \partial_\delta^\beta \hat{h}(t, k) \|_{L^2_v}^2 d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^T \phi(t)^{2c_{|\beta|}} \| (a^{1/2})^w \partial_\nu \hat{h}(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) \leq C_4 \| f_0 \|_{L^1_t L^6_v}. \tag{4.24}
\]

As for the proof of Proposition 4.9 above, in terms of Proposition 4.8 it suffices to focus on the proof of (4.24) with \( |\beta| = 1 \) and 2. We proceed it through the following two lemmas.

**Lemma 4.10** (\(|\beta| = 2\)). Under the same conditions on \( f_0 \) and \( g \) as in Proposition 4.9, the estimate (4.24) holds for any \( \beta \in \mathbb{Z}^3_+ \) with \( |\beta| = 2 \), provided that \( \varepsilon_0 > 0 \) is small enough.

**Proof.** As in the previous discussions it suffices to consider the case of \( T \leq 1 \). In such case, one has \( \phi(t) = t \). We introduce the regularization \( \hat{h}_\delta \) by setting

\[
\hat{h}_\delta(t, k, v) = t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu^2 \hat{h}(t, k, v), \quad 0 < \delta \leq 1,
\]

and let \( \Lambda_{\delta t} \) be the regularization operator defined by (3.5). Observe

\[
(\partial_t + iv \cdot k) \Lambda_{\delta t} \hat{h}_\delta - \Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu \hat{h} = \Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu ^2 \hat{g}(\hat{g}, \hat{\hat{g}}) + 2\delta t^{-1} \Lambda_{\delta t} \hat{h}_\delta + [iv \cdot k, \Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu ^2 \hat{h}].
\]

Then, we perform the similar energy estimates for \( \Lambda_{\delta t} \hat{h}_\delta \) as in the previous parts, to get

\[
\int_{\mathbb{R}^3} \sup_{0 < t \leq 1} \| \Lambda_{\delta t} \hat{h}_\delta(t, k) \|_{L^2_v}^2 d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \Lambda_{\delta t} \hat{h}_\delta(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq C \int_{\mathbb{R}^3} \left( \int_0^1 \| (\partial_t + iv \cdot k) \Lambda_{\delta t} \hat{h}_\delta - \Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu ^2 \hat{g}(\hat{g}, \hat{h}), \Lambda_{\delta t} \hat{h}_\delta \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
+ C \int_{\mathbb{R}^3} \left( \int_0^1 \| [(\Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu ^2, \mathcal{L}] \hat{h}, \Lambda_{\delta t} \hat{h}_\delta \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
+ C \int_{\mathbb{R}^3} \left( \int_0^1 t^{-1} \| \Lambda_{\delta t} \hat{h}_\delta(t) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
+ C \int_{\mathbb{R}^3} \left( \int_0^1 \| [(iv \cdot k, \Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu ^2) \hat{h}, \Lambda_{\delta t} \hat{h}_\delta \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k). \tag{4.26}
\]

In what follows we proceed through five steps to derive the upper bound for those terms on the right hand side.

**Step 1.** We begin with the estimate of the first term on the right hand side of (4.26). We claim that

\[
\int_{\mathbb{R}^3} \left( \int_0^1 \| (\partial_t + iv \cdot k) \Lambda_{\delta t} \hat{h}_\delta - \Lambda_{\delta t} t^{2c}(1 - \delta \partial_\nu^2)^{-1} \partial_\nu ^2 \hat{g}(\hat{g}, \hat{h}), \Lambda_{\delta t} \hat{h}_\delta \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq (\varepsilon + CC_4 \varepsilon_0) \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta(t, k) \|_{L^2_v}^2 dt \right)^{1/2} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_t L^6_v}, \tag{4.27}
\]
where $C_3$ and $C_4$ are the constants given in (4.21) and (4.23), respectively. Indeed, we follow the similar argument as for proving Lemma 4.4 by letting $m = 2$ therein. Precisely, with the notation $\mathcal{T}(g, h, w)$ defined by (4.5), similar to obtain (4.6), we apply the Leibniz formula to write

$$\int_{\mathbb{R}^3} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \partial_{v_1}^2 \hat{T}(g, \hat{\delta}h, \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} \right| dt \right)^{1/2} d\Sigma(k) \leq \tilde{I} + \tilde{J} + \tilde{K},$$

(4.28)

with

$$\tilde{I} = \int_{\mathbb{R}^3} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} \right| dt \right)^{1/2} d\Sigma(k)$$

+ $$\sum_{p=1}^2 \left( \int_{\mathbb{R}^3} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} \right| dt \right)^{1/2} d\Sigma(k),$$

(4.29)

$$\tilde{J} = \int_{\mathbb{R}^3} \sum_{0 \leq p \leq 1} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} \right| dt \right)^{1/2} d\Sigma(k),$$

(4.30)

and

$$\tilde{K} = \int_{\mathbb{R}^3} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} \right| dt \right)^{1/2} d\Sigma(k).$$

(4.31)

We are going to estimate $\tilde{I}$, $\tilde{J}$ and $\tilde{K}$ as follows.

**Estimate on $\tilde{I}$.** We first consider the first term on the right hand side of $\tilde{I}$ in (4.29). By using the formula (4.11) for $m = 2$, repeating the same argument as to estimate $I_1$ in (4.12) and using the estimates (4.20) and (4.23), we have

$$\int_{\mathbb{R}^3} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} \right| dt \right)^{1/2} d\Sigma(k)$$

$$\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left| (a^{1/2} \omega \hat{h}_{\delta}(t, k)) L^2 dt \right)^{1/2} d\Sigma(k) + CC_4 \varepsilon^{-1} I_0 \int_{\mathbb{R}^3} \left( \int_0^1 \left| (a^{1/2} \omega \hat{h}_{\delta}) L^2 dt \right)^{1/2} d\Sigma(k)$$

$$+ CC_3 I_0^{-1} \varepsilon \| f_0 \|_{L^1 L^2} + C \sum_{0 \leq p \leq 1} \int_{\mathbb{R}^3} \left( \int_0^1 \delta \left| (\Lambda_{\delta_1} t^{2c}(1 - \delta \partial_{v_1}^2)^{-1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_{\delta})_{L^2} dt \right)^{1/2} d\Sigma(k),$$

where $\varepsilon > 0$ is an arbitrarily small constant. Furthermore, as for the last term on the right hand side of the above estimate, we use

$$\hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}) = \partial_{v_1} \hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}) - \hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}) - \hat{T}(g, \partial_{v_1}^2 \hat{h}, \partial_{v_1}^2 \mu^{1/2}),$$

where $\hat{T}$, $\partial_{v_1}^2 \hat{h}$, and $\partial_{v_1}^2 \mu^{1/2}$ are defined as in (4.6).
Next, we deal with the second term on the right hand side of $\tilde{\epsilon}$ to $\delta$ as in (4.29). By direct computations, it holds that

$$\sum_{0 \leq p \leq 1} \int_{\mathbb{Z}^3} \left[ \int_0^1 \delta \right. \\
 \times \left| (\Lambda_0, t^{2s}(1 - \delta \partial^2)_{v_1}^{-1}) \tilde{T}(\partial^0 \hat{g}, \partial v_1 (1 - \delta \partial^2)_{v_1}^{-1} \partial^2 h, \partial v_1 \mu^{\frac{1}{2}}), \Lambda_0 \hat{h}_\delta)_{L^2} \right| dt \left. \right| \right]^\frac{1}{2} d\Sigma(k) \leq \epsilon \int_{\mathbb{Z}^3} \left[ \int_0^1 \left\| (a^{1/2})^w \hat{h}_\delta(t,k) \right\|^2_{L^2} dt \right]^{1/2} d\Sigma(k) + CC_3 C_4 \epsilon^{-1} \epsilon_0 \| f_0 \|_{L^1_{L^2}} \\
 + CC_4 \epsilon^{-1} \epsilon_0 \int_{\mathbb{Z}^3} \left[ \int_0^1 \left\| (a^{1/2})^w \delta_{1/2}(1 - \delta \partial^2)_{v_1}^{-1} \partial^2 h \right\|^2_{L^2} dt \right]^{1/2} d\Sigma(k) \leq (\epsilon + CC_4 \epsilon^{-1} \epsilon_0) \int_{\mathbb{Z}^3} \left[ \int_0^1 \left\| (a^{1/2})^w \hat{h}_\delta(t,k) \right\|^2_{L^2} dt \right]^{1/2} d\Sigma(k) + CC_3 C_4 \epsilon^{-1} \epsilon_0 \| f_0 \|_{L^1_{L^2}},$$

where the last inequality holds true because

$$t^{2s}\left( (a^{1/2})^w \delta_{1/2}(1 - \delta \partial^2)_{v_1}^{-1} \partial^2 h \right)^2_{L^2} \leq \left\| (a^{1/2})^w \delta(1 - \delta \partial^2)_{v_1}^{-1} \partial^2 h \right\|^2_{L^2} \times \left\| (a^{1/2})^w \delta \right\|^2_{L^2} \leq C \left\| (a^{1/2})^w \hat{h}_\delta \right\|^2_{L^2} \| (a^{1/2})^w h \|_{L^2}.$$

Now, we combine the above estimates to conclude, for any $\epsilon > 0$,

$$\int_{\mathbb{Z}^3} \left[ \int_0^1 \left| (\Lambda_0, t^{2s}(1 - \delta \partial^2)_{v_1}^{-1}) \tilde{T}(\hat{g}, \partial v_1 \hat{h}, \mu^{1/2}), \Lambda_0 \hat{h}_\delta)_{L^2} \right| dt \right]^{1/2} d\Sigma(k) \leq (\epsilon + CC_4 \epsilon^{-1} \epsilon_0) \int_{\mathbb{Z}^3} \left[ \int_0^1 \left\| (a^{1/2})^w \hat{h}_\delta(t,k) \right\|^2_{L^2} dt \right]^{1/2} d\Sigma(k) + CC_3 C_4 \epsilon^{-1} \epsilon_0 \| f_0 \|_{L^1_{L^2}}.$$

(4.32)

Next, we deal with the second term on the right hand side of $\tilde{I}$ in (4.29). By direct computations, it holds that

$$\sum_{p=1}^2 \binom{2}{p} \int_{\mathbb{Z}^3} \left[ \int_0^1 \left| (\Lambda_0, t^{2s}(1 - \delta \partial^2)_{v_1}^{-1}) \tilde{T}(\hat{g}, \partial v_1 \hat{h}, \partial v_1 \mu^{\frac{1}{2}}), \Lambda_0 \hat{h}_\delta)_{L^2} \right| dt \right]^{1/2} d\Sigma(k) \leq \epsilon \int_{\mathbb{Z}^3} \left[ \int_0^1 \left\| (a^{1/2})^w \hat{h}_\delta(t,k) \right\|^2_{L^2} dt \right]^{1/2} d\Sigma(k) + CC_3 C_4 \epsilon^{-1} \epsilon_0 \| f_0 \|_{L^1_{L^2}} \\
+ C \int_{\mathbb{Z}^3} \left[ \int_0^1 \left| (\Lambda_0, t^{2s}(1 - \delta \partial^2)_{v_1}^{-1}) \tilde{T}(\hat{g}, \partial v_1 \hat{h}, \partial v_1 \mu^{\frac{1}{2}}), \Lambda_0 \hat{h}_\delta)_{L^2} \right| dt \right]^{1/2} d\Sigma(k).$$

(4.33)

To control the last term in the above inequality, we use the Leibniz formula to write, similar as in (4.11),

$$\tilde{T}(\hat{g}, \partial v_1 \hat{h}, \partial v_1 \mu^{\frac{1}{2}}) = (1 - \delta \partial^2)_{v_1}^{-1} \tilde{T}(\hat{g}, (1 - \delta \partial^2)_{v_1}^{-1} \partial v_1 \hat{h}, \partial v_1 \mu^{\frac{1}{2}}) \\
+ \delta \sum_{j=1}^2 \sum_{p=0}^j \binom{2}{j} \binom{j}{p} \hat{T}(\partial v_1^{-j} \hat{g}, \partial v_1^{-j}(1 - \delta \partial^2)_{v_1}^{-1} \partial v_1 \hat{h}, \partial v_1^{j+1} \mu^{1/2}).$$
As a result, we may repeat the same argument as for treating (4.11) with $m = 1$ by observing that the operator $\delta \partial \nu_j^2 (1 - \delta \partial \nu_j^{2 - 1} \partial \nu_j, 1 \leq j \leq 2$, is uniformly bounded on $L^2$ with respect to $\delta$. Therefore, by virtue of (4.20), (4.22) and (4.23), we conclude that

$$
\int_{\mathbb{Z}^2} \left( \int_0^1 \left| (\Lambda_{\delta t} t^{2\kappa} (1 - \delta \partial \nu_j^{2 - 1} \tilde{T}(\hat{g}, \partial \nu_j \hat{h}, \partial \nu_j, \mu) \cdot \Lambda_{\delta t} \hat{h}), \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) 
\leq \varepsilon \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (a^{1/2})^{w} \hat{h}(t, k) \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1 L^2}^2 
+ CC_4 \varepsilon^{-1} \varepsilon_0 \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (a^{1/2})^{w} \hat{h}(t, k) \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) 
\leq (\varepsilon + CC_4 \varepsilon^{-1} \varepsilon_0) \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (a^{1/2})^{w} \hat{h}(t, k) \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1 L^2}^2,
$$

where in the last inequality we have used the fact that

$$
\| (a^{1/2})^{w} \hat{h}(t, k) \|_{L^2}^2 \leq C \| (a^{1/2})^{w} \hat{h} \|_{L^2}^2 + C \| (a^{1/2})^{w} \hat{h} \|_{L^2}^2.
$$

Substituting (4.34) into (4.33), we obtain

$$
\sum_{p=1}^{2} \binom{2}{p} \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (\Lambda_{\delta t} t^{2\kappa} (1 - \delta \partial \nu_j^{2 - 1} \tilde{T}(\hat{g}, \partial \nu_j \hat{h}, \partial \nu_j, \mu) \cdot \Lambda_{\delta t} \hat{h}), \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) 
\leq (\varepsilon + CC_4 \varepsilon^{-1} \varepsilon_0) \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (a^{1/2})^{w} \hat{h}(t, k) \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1 L^2}^2.
$$

This together with (4.32) give the estimate on $I$ in (4.29) as

$$
I \leq (\varepsilon + CC_4 \varepsilon^{-1} \varepsilon_0) \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (a^{1/2})^{w} \hat{h}(t, k) \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1 L^2}^2;
$$

where $\varepsilon > 0$ is an arbitrarily small constant.

**Estimate on $J$.** Recall that $J$ is given in (4.30). Following a similar argument as that for proving (4.15), we can verify directly that

$$
J \leq \varepsilon \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (a^{1/2})^{w} \hat{h}(t, k) \right|_{L^2}^2 dt \right)^{\frac{1}{2}} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1 L^2}^2
+ \int_{\mathbb{Z}^2} \left( \int_0^1 \left| (\Lambda_{\delta t} t^{2\kappa} (1 - \delta \partial \nu_j^{2 - 1} \tilde{T}(\partial \nu_j \hat{g}, \partial \nu_j \hat{h}, \mu) \cdot \Lambda_{\delta t} \hat{h}), \right|_{L^2}^2 \right)^{\frac{1}{2}} d\Sigma(k).
$$
It remains to treat the last term in (4.35). Repeating the argument for proving (4.34) gives

\[
\int_{\mathbb{R}^3} \left( \int_0^1 \left| (\Lambda_{\delta_1} t^{2\varepsilon}(1 - \delta \partial^2_{v_1})^{-1} \hat{T}(\partial_{v_1} \hat{g}, \partial_{v_1} \hat{h}, \mu^{1/2}) \right| L_c^2 \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{h}_{\delta}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_k L^2_x}^2
\]

\[
+ CC_4 \varepsilon^{-1} \varepsilon_0 \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{t}(1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k)
\]

\[
+ C \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( \Lambda_{\delta_1} t^{2\varepsilon}(1 - \delta \partial^2_{v_1})^{-1} \hat{T}(\partial^2_{v_1} \hat{g}, (1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}, \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_\delta \right)_2 \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq (\varepsilon + CC_4 \varepsilon^{-1} \varepsilon_0) \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{h}_{\delta}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_k L^2_x}^2,
\]

where in the last inequality we have used the following two estimates that

\[
\|(a^{1/2})^w \tilde{t}(1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}\|_{L^2_x}^2 \leq C \|(a^{1/2})^w \tilde{h}_\delta\|_{L^2_x}^2 + C \|(a^{1/2})^w \tilde{h}\|_{L^2_x}^2,
\]

and that

\[
\int_{\mathbb{R}^3} \left( \int_0^1 \left| \left( \Lambda_{\delta_1} t^{2\varepsilon}(1 - \delta \partial^2_{v_1})^{-1} \hat{T}(\partial^2_{v_1} \hat{g}, (1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}, \mu^{1/2}), \Lambda_{\delta_1} \hat{h}_\delta \right)_2 \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{h}_{\delta}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_k L^2_x}^2,
\]

which follows from the formula that

\[
\delta \hat{T}(\partial^2_{v_1} \hat{g}, (1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}, \mu^{1/2}) = \delta^{1/2} \partial_{v_1} \hat{T}(\partial^2_{v_1} \hat{g}, \partial^2_{v_1} \hat{g} - (1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}, \mu^{1/2})
\]

\[
- \hat{T}(\partial^2_{v_1} \hat{g}, \partial(1 - \delta \partial^2_{v_1})^{-1} \partial^2_{v_1} \hat{h}, \mu^{1/2}) - \hat{T}(\partial^2_{v_1} \hat{g}, \partial(1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}, \partial_{v_1} \mu^{1/2}).
\]

Now, we substitute the above estimate into (4.35) to conclude

\[
\overline{\lim}_{\delta_1 \to 0} \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left[ \Lambda_{\delta_1} t^{2\varepsilon}(1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1}, \L \right] \hat{h}, \Lambda_{\delta_1} \hat{h}_\delta \right)_2 \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{h}_{\delta}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k) + CC_3 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_k L^2_x}^2,
\]

(4.36)

**Estimate on \( \tilde{K} \).** Recall that \( \tilde{K} \) is given in (4.31). Similar to (4.16), we have, using the estimates (4.22)-(4.23),

\[
\tilde{K} \leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{h}_{\delta}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k) + CC_3 C_4 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_k L^2_x}^2.
\]

Finally, we can substitute all the above estimates on \( \tilde{I}, \tilde{J} \) and \( \tilde{K} \) to (4.28) so as to conclude the desired estimate (4.27).

**Step 2.** In this step we treat the second term on the right hand side of (4.26). We claim that

\[
\lim_{\delta \to 0} \int_{\mathbb{R}^3} \left( \int_0^1 \left| \left[ \Lambda_{\delta} t^{2\varepsilon}(1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1}, \L \right] \hat{h}, \Lambda_{\delta} \hat{h}_\delta \right)_2 \right| dt \right)^{1/2} d\Sigma(k)
\]

\[
\leq \varepsilon \int_{\mathbb{R}^3} \left( \int_0^1 \left| \int_0^1 (a(1/2)^w \tilde{h}_{\delta}(t, k))_2^2 dt \right|^{1/2} d\Sigma(k) + CC_3 \varepsilon^{-1} \varepsilon_0 \| f_0 \|_{L^1_k L^2_x}^2,
\]

(4.36)
with the constant $C_3$ given in (4.21). Indeed, the above estimate follows from the similar arguments as those in the proof of Lemma 4.5 by letting $m = 2$ therein as well as in the previous step 1 with slight modifications. We omit the details for brevity.

**Step 3.** As for the third term on the right hand side of (4.26), we use the argument for proving Lemma 4.6 to get

$$\int_{\mathbb{R}^3} \left( \int_0^1 t^{-1} \| \Lambda_{\delta_t} \hat{h}_\delta(t) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)$$

$$\leq \epsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) + \int_{\mathbb{R}^3} \left( \int_0^1 t^{-\frac{3}{2}} \| \langle D_v \rangle^{\frac{3}{2}} \hat{h}_\delta \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)$$

$$\leq \epsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) + C_3 \epsilon^{-1} \| f_0 \|_{L^1_k L^2_v}, \quad (4.37)$$

where in the last inequality we have used the fact that, in view of the representation (4.25) of $\hat{h}_\delta$,

$$\int_{\mathbb{R}^3} \left( \int_0^1 t^{-\frac{3}{2}} \| (D_v) \hat{\delta} \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)$$

$$\leq \epsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) + C_3 \epsilon^{-1} \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k).$$

**Step 4.** It remains to deal with the last term on the right hand side of (4.26). Using (4.18) for $m = 2$ gives

$$||iv \cdot k, \Lambda_{\delta_t} t^2 (1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h}||_{L^2_v} \leq C t^2 \langle k \rangle \| (1 - \delta \partial^2_{v_1})^{-1} \partial_{v_1} \hat{h} \|_{L^2_v}$$

$$\leq C t^2 \langle k \rangle^2 \| \hat{h} \|_{L^2_v} + C \| \hat{h}_\delta \|_{L^2_v}.$$ Then, it holds that

$$\int_{\mathbb{R}^3} \left( \int_0^1 \| (\Lambda_{\delta_t} t^2 (1 - \delta \partial_{v_1})^{-1} \partial_{v_1}, \mathcal{L}) \hat{h}, \Lambda_{\delta_t} \hat{h}_\delta \|_{L^2_v} \right) dt \right)^{1/2} d\Sigma(k)$$

$$\leq C \int_{\mathbb{R}^3} \langle k \rangle^2 \left( \int_0^1 t^{4c} \| (a^{1/2})^w \hat{h}(t, v) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) + C \int_{\mathbb{R}^3} \left( \int_0^1 \| \hat{h}_\delta(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k)$$

$$\leq \epsilon \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) + C_3 \epsilon^{-1} \| f_0 \|_{L^1_k L^2_v}, \quad (4.38)$$

where in the last inequality we have used (4.21) and (4.37).

**Step 5.** Finally we substitute the estimates (4.27), (4.36), (4.37) and (4.38) into (4.26) and let $\delta_1 \to 0$. Choosing $\epsilon > 0$ small enough, it follows that

$$\int_{\mathbb{R}^3} \sup_{0 \leq t \leq 1} \| \hat{h}_\delta(t, k) \|_{L^2_v} d\Sigma(k)$$

$$+ \int_{\mathbb{R}^3} \left( \int_0^1 \| (a^{1/2})^w \hat{h}_\delta(t, k) \|^2_{L^2_v} dt \right)^{1/2} d\Sigma(k) \leq C_3 \epsilon_{40} \| f_0 \|_{L^1_k L^2_v} + C_3 \| f_0 \|_{L^1_k L^2_v}.$$
Moreover, letting $\delta \to 0$, we further obtain that, in view of the definition (4.25) of $\hat{h}_\delta$,

$$
\int \sup_{Z^3} T^{2c} \| \partial_v^2 \hat{h}(t, k) \|_{L^2_c}^2 d\Sigma(k) + \int \left( \int_0^1 T^{2c} \| (a^{1/2})^w \partial_v^2 \hat{h}(t, k) \|_{L^2_c}^2 dt \right)^{1/2} d\Sigma(k)
\leq \CC \epsilon_0 \| f_0 \|_{L^1_cL^2_c} + \CC \| f_0 \|_{L^1_cL^2_c}.
$$

Notice that the above estimate still holds with $\partial_v^2$ replaced by $\partial_v^2$ or $\partial_v^3$. Then, we conclude that for any $|\beta| = 2$,

$$
\int \sup_{Z^3} T^{2c} \| \partial_v^2 \hat{h}(t, k) \|_{L^2_c}^2 d\Sigma(k) + \int \left( \int_0^1 T^{2c} \| (a^{1/2})^w \partial_v^2 \hat{h}(t, k) \|_{L^2_c}^2 dt \right)^{1/2} d\Sigma(k)
\leq \CC \epsilon_0 \| f_0 \|_{L^1_cL^2_c} + \CC \| f_0 \|_{L^1_cL^2_c}.
$$

Meanwhile, the case of $T \geq 1$ can be treated in a similar way by combining the above estimate for $\hat{h}_{|t=1}$. Thus the desired estimate of Lemma 4.10 follows, provided that $\epsilon_0 > 0$ is small enough and $C_4$ is chosen large enough such that $C_4 > 4(C + 1)C_3$ with $C$ the constant in (4.39). The proof of Lemma 4.10 is completed. $\square$

Lemma 4.11 ($|\beta| = 1$). Under the same conditions on $f_0$ and $g$ as in Proposition 4.9, the estimate (4.24) holds for any $\beta \in Z^3_+$ with $|\beta| = 1$, provided that $\epsilon_0$ is small enough.

Proof. The desired estimate is an immediate consequence of (4.39) and (4.20) by observing that for $|\beta| = 1$,

$$
\| \phi(t)^c \partial_v^2 \hat{h} \|_{L^2_c}^2 \leq \frac{1}{2} \left( \| \phi(t)^c \partial_v^2 \hat{h} \|_{L^2_c}^2 + \| \hat{h} \|_{L^2_c}^2 \right)
$$

and

$$
\| \phi(t)^c (a^{1/2})^w \partial_v^2 \hat{h} \|_{L^2_c}^2 \leq C \left( \| \phi(t)^c (a^{1/2})^w \partial_v^2 \hat{h} \|_{L^2_c}^2 + \| (a^{1/2})^w \hat{h} \|_{L^2_c}^2 \right).
$$

We complete the proof of Lemma 4.11. $\square$

Proof of Proposition 4.9. It directly follows from (4.20) and Lemmas 4.10-4.11. $\square$

With Proposition 4.9, the rest is devoted to proving the main result on the $H^2_c$-smoothness of solutions to the Cauchy problem on the nonlinear Boltzmann equation.

Proof of Proposition 4.2. We consider the following iteration equations with $f^0 \equiv 0$,

$$
\partial_t f^n + v \cdot \partial_x f^n - \mathcal{L} f^{n+1} = \Gamma(f^{n-1}, f^n), \quad f^n|_{t=0} = f_0, \quad n \geq 1,
$$

where $f_0$ is the initial datum to the Boltzmann equation (1.4) satisfying the smallness condition (1.10). Then, Proposition 4.9 ensures the existence of $\{f^n\}_{n \geq 1}$ satisfying the estimate that, for any $T > 0$ and any $n \geq 1$,

$$
\int \sup_{Z^3} \phi(t)^{c|\beta|} \| \partial_v^2 \hat{f}^{n}(t, k) \|_{L^2_c} d\Sigma(k)
\leq \int \left( \int_0^T \phi(t)^{c|\beta|} \| (a^{1/2})^w \partial_v^2 \hat{f}^{n}(t, k) \|_{L^2_c}^2 dt \right)^{1/2} d\Sigma(k) \leq C \| f_0 \|_{L^1_cL^2_c}.
$$

Consider the difference

$$
w^n = f^{n+1} - f^n, \quad n \geq 1.$$
Then for any \( n \geq 1 \),
\[
\partial_t w^n + v \cdot \partial_x w^n - \mathcal{L} w^n = \Gamma(f^n, w^n) + \Gamma(w^{n-1}, f^n), \quad w^n|_{t=0} = 0,
\]
with \( w^0 = f^1 \). By virtue of (4.40) as well as the smallness of \( \|f_0\|_{L^1_k L^2_v} \), we follow the argument in the proof of Proposition 4.9 with minor modifications to obtain, for any \( \beta \in \mathbb{Z}^3_+ \) with \( |\beta| \leq 2 \),
\[
\int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{w}^n(t, k)\|_{L^2_v} d\Sigma(k) + \int_{\mathbb{Z}^3} \left( \int_0^T \phi(t)^{2\mid \beta \mid} \|\left( a^{1/2} w \partial_v^\beta \hat{w}^n(t, k)\right)^2 \|_{L^2_v} dt \right)^{\frac{1}{2}} d\Sigma(k)
\leq C \left( \int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{w}^{n-1}(t, k)\|_{L^2_v} d\Sigma(k) \right)
\times \left( \int_{\mathbb{Z}^3} \left( \int_0^T \phi(t)^{2\mid \beta \mid} \|\left( a^{1/2} w \partial_v^\beta \hat{w}(t, k)\right)^2 \|_{L^2_v} dt \right)^{\frac{1}{2}} d\Sigma(k) \right)
\leq CC_4 \|f_0\|_{L^1_k L^2_v} \int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{w}^{n-1}(t, k)\|_{L^2_v} d\Sigma(k),
\]
which implies that, for \( \|f_0\|_{L^1_k L^2_v} \) small enough,
\[
\int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{w}^n(t, k)\|_{L^2_v} d\Sigma(k) \leq \left( CC_4 \|f_0\|_{L^1_k L^2_v} \right)^{n+1} \leq 2^{-n-1}, \quad n \geq 1.
\]
Thus, combining the above estimates, it holds that
\[
\int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{w}^n(t, k)\|_{L^2_v} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^T \phi(t)^{2\mid \beta \mid} \|\left( a^{1/2} w \partial_v^\beta \hat{w}^n(t, k)\right)^2 \|_{L^2_v} dt \right)^{\frac{1}{2}} d\Sigma(k) \leq 2^{-n-1}.
\]
This implies that for any \( |\beta| \leq 2 \), \( \phi(t)^{\mid \beta \mid} \partial_v^\beta f^n \) and \( \phi(t)^{\mid \beta \mid} \left( a^{1/2} w \partial_v^\beta \hat{f}(t, k)\right) \) are the Cauchy sequences in \( L^1_0 L^\infty_k L^2_v \) and \( L^1_k L^\infty_v L^2_v \), respectively, with the limit solving the nonlinear Boltzmann equation (1.4) with initial datum \( f_0 \) and thus equal to \( f \) by the uniqueness of solutions in \( L^1_k L^\infty_v L^2_v \). Moreover, it follows from (4.40) that
\[
\int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{f}(t, k)\|_{L^2_v} d\Sigma(k)
+ \int_{\mathbb{Z}^3} \left( \int_0^T \phi(t)^{2\mid \beta \mid} \|\left( a^{1/2} w \partial_v^\beta \hat{f}(t, k)\right)^2 \|_{L^2_v} dt \right)^{\frac{1}{2}} d\Sigma(k) \leq C_4 \|f_0\|_{L^1_k L^2_v}.
\]
We then have proved the desired result in Proposition 4.2, completing the proof.

5. GEVREY SMOOTHING EFFECT IN SPACE AND VELOCITY VARIABLES

We are ready to prove the main result, Theorem 1.1, which is just an immediate consequence of Theorems 3.2 and 4.1. In fact, observe that for any \( m \in \mathbb{Z}_+ \) and any \( \beta \in \mathbb{Z}_+^3 \),
\[
\langle k \rangle^m \phi(t)^{\mid \beta \mid} \|\partial_v^\beta \hat{f}(t, k)\|_{L^2_v} \leq \langle k \rangle^m \phi(t)^{\mid \beta + \mid \beta \mid} \|\hat{f}(t, k)\|_{L^2_v}^{1/2} \|\partial_v^{2\beta} \hat{f}(t, k)\|_{L^2_v}^{1/2}.
\]
Thus, it holds that
\[
\int_{\mathbb{Z}^3} \langle k \rangle^m \sup_{0 < t < T} \phi(t)^{c(m+|\beta|)} \| \partial_v^\beta \hat{f}(t, k) \|_{L_0^2} d\Sigma(k)
\]
\[
\leq \left( \int_{\mathbb{Z}^3} \langle k \rangle^{2m} \sup_{0 < t < T} \phi(t)^{2m} \| \hat{f}(t, k) \|_{L_0^2} d\Sigma(k) \right)^{1/2} \left( \int_{\mathbb{Z}^3} \sup_{0 < t < T} \phi(t)^{2c|\beta|} \| \partial_v^{2\beta} \hat{f}(t, k) \|_{L_0^2} d\Sigma(k) \right)^{1/2}
\]
\[
\leq \left( \tilde{C}_0^2 m + [(2m)!]^{\frac{1+\alpha_0}{2}} \right)^{1/2} \left( \tilde{C}_s^2 |\beta| + [(2|\beta|)!]^{\frac{1+\alpha_0}{2}} \right)^{1/2}
\]
\[
\leq C_0^m + \tilde{C}_s^2 |\beta| + \frac{1}{2} \left( \frac{3}{2} \right) C_0^m + \tilde{C}_s^2 |\beta| + \frac{1}{2} \left( \frac{3}{2} \right) (m!)^{\frac{1+\alpha_0}{2 s}} (|\beta|!)^{\frac{1+\alpha_0}{2 s}}
\]
\[
\leq C_m^m + |\beta| + 1 \left( |\beta|!^{\frac{1+\alpha_0}{2 s}} \right)
\]
where in the second inequality we have used Theorems 3.2 and 4.1, the third inequality follows from the fact that \((m + n)! \leq 2^{m+n} m n!\), and the last inequality holds if we have chosen \(C = 2^{1+\alpha_0} (C_0 + \tilde{C}_s)\) with the constants \(C_0\) and \(\tilde{C}_s\) given in Theorem 3.2 and Theorem 4.1, respectively. This gives the desired estimate (1.11) in Theorem 1.1. In the end, we briefly explain \(f(t, \cdot, \cdot) \in C^{\frac{1+\alpha_0}{s}}(T^3 \times \mathbb{R}^3)\) for any positive time \(t > 0\) by referring to the direct computations as
\[
\sup_{0 < t < T} \phi(t)^{c(|\alpha| + |\beta|)} \| \partial_x^\alpha \partial_v^\beta f(t, x, v) \|_{L_0^2 \cdot L_0^2}
\]
\[
\leq \sup_{0 < t < T} \phi(t)^{c(|\alpha| + |\beta|)} \left( \int_{\mathbb{T}^3} \langle k \rangle^{2|\alpha|} \| \partial_v^\beta \hat{f}(t, k) \|_{L_0^2} d\Sigma(k) \right)^{1/2}
\]
\[
\leq \sup_{0 < t < T} \phi(t)^{c(|\alpha| + |\beta|)} \int_{\mathbb{T}^3} \langle k \rangle^{2|\alpha|} \| \partial_v^\beta \hat{f}(t, k) \|_{L_0^2} d\Sigma(k)
\]
\[
\leq \int_{\mathbb{T}^3} \langle k \rangle^{2|\alpha|} \sup_{0 < t < T} \phi(t)^{c(|\alpha| + |\beta|)} \| \partial_v^\beta \hat{f}(t, k) \|_{L_0^2} d\Sigma(k)
\]
\[
\leq C_{\alpha, \beta}^{|\alpha| + |\beta| + 1} \left( |\alpha| + |\beta|! \right)^{\frac{1+\alpha_0}{2 s}}.
\]
Then the proof of Theorem 1.1 is completed. \(\square\)

6. APPENDIX

We recall here some notations and basic facts of symbolic calculus, and refer to [38, Chapter 18] or [45] for detailed discussions on the pseudo-differential calculus.

We consider the flat metric \(|dv|^2 + |d\eta|^2\), and let \(M\) be an admissible weight function with respect to \(|dv|^2 + |d\eta|^2\), that is, the weight function \(M\) satisfies the following conditions:

(a) (slowly varying condition) there exists a constant \(\delta\) such that
\[ |X - Y| \leq \delta, \quad M(X) \approx M(Y), \quad \forall X, Y \in \mathbb{R}^6; \]

(b) (temperance) there exist two constants \(C\) and \(N\) such that
\[ M(X)/M(Y) \leq C |X - Y|^N, \quad \forall X, Y \in \mathbb{R}^6. \]

Considering symbols \(q(k, v, \eta)\) as a function of \((v, \eta)\) with parameter \(k\), we say that \(q \in S(M, |dv|^2 + |d\eta|^2)\) uniformly with respect to \(k\), if
\[ \left| \partial_v^\alpha \partial_\eta^\beta q(k, v, \eta) \right| \leq C_{\alpha, \beta} M, \quad \forall \alpha, \beta \in \mathbb{Z}_+^3, \forall v, \eta \in \mathbb{R}^3; \]
with $C_{\alpha, \beta}$ a constant depending only on $\alpha$ and $\beta$, but independent of $k$. The space $S(M, |dv|^2 + |d\eta|^2)$ endowed with the semi-norms
\[
\|q\|_{N; S(M, |dv|^2 + |d\eta|^2)} = \max_{0 \leq |\alpha| + |\beta| \leq N} \sup_{(v, \eta) \in \mathbb{R}^6} \left| M(v, \eta)^{-1} \partial_{v}^\alpha \partial_{\eta}^\beta q(v, \eta) \right|
\]
becomes a Fréchet space. Let $q \in \mathcal{S}'(\mathbb{R}^3 \times \mathbb{R}^3)$ be a tempered distribution and let $t \in \mathbb{R}$, the operator $\text{op}_t q$ is an operator from $\mathcal{S}'(\mathbb{R}^3)$ to $\mathcal{S}'(\mathbb{R}^3)$, whose Schwartz kernel $K_t$ is defined by the oscillatory integral:
\[
K_t(z, z') = (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(z - z') \cdot \zeta} q((1 - t)z + tz', \zeta)d\zeta.
\]
In particular we denote $q^w = \text{op}_{1/2} q$. Here $q^w$ is called the Weyl quantization of symbol $q$. Note that if $q \in S(1, |dv|^2 + |d\eta|^2)$ and $q$ is real-valued then $q^w$ is bounded and self-adjoint in $L^2_v$.

Finally let us recall some basic properties of the Wick quantization. The importance in studying the Wick quantization lies in the fact that positive symbols give rise to positive operators. We refer the readers to Lerner’s works [44, 45] and references therein for extensive presentations of this quantization and its applications in mathematics and mathematical physics.

Let $Y = (v, \eta)$ be a point in $\mathbb{R}^6$. The Wick quantization of a symbol $q$ is given by
\[
q^{\text{Wick}} = (2\pi)^{-3} \int_{\mathbb{R}^6} q(Y) \Pi_Y \, dY,
\]
where $\Pi_Y$ is the projector associated to the Gaussian $\varphi_Y$ which is defined by
\[
\varphi_Y(z) = \pi^{-3/4} e^{-\frac{1}{2}|z - v|^2} e^{iz \cdot \eta/2}, \quad z \in \mathbb{R}^3.
\]
The main property of the Wick quantization is its positivity, i.e.,
\[
q(v, \eta) \geq 0 \quad \text{for all } (v, \eta) \in \mathbb{R}^6 \implies q^{\text{Wick}} \geq 0. \tag{6.1}
\]
According to [45, Proposition 2.4.3], the Wick and Weyl quantizations of a symbol $q$ are linked by the following identities
\[
q^{\text{Wick}} = \left( q + 2^3 e^{-2\pi|z|^2} \right)^w = q^w + r^w \tag{6.2}
\]
with
\[
r(Y) = 2^3 \int_0^1 \int_{\mathbb{R}^6} (1 - \theta) q''(Y + \theta Z) Z^2 e^{-2\pi|Z|^2} \, dZ \, d\theta.
\]
As a result, $q^{\text{Wick}}$ is a bounded operator in $L^2_v$ if $q \in S(1, g)$, and $q^{\text{Wick}}$ is self-adjoint in $L^2_v$ if $q$ is real-valued. We also recall the following composition formula obtained in the proof of Proposition 3.4 in [44]
\[
q_1^{\text{Wick}} q_2^{\text{Wick}} = \left[ q_1 q_2 - q_1' q_2' + \frac{1}{4} \{ q_1, q_2 \} \right]^{\text{Wick}} + \mathcal{T}, \tag{6.3}
\]
with $\mathcal{T}$ being a bounded operator in $L^2(\mathbb{R}^{2n})$, where $q_1 \in L^\infty(\mathbb{R}^{2n})$ and $q_2$ is a smooth symbol whose derivatives of order $\geq 2$ are bounded on $\mathbb{R}^6$. The notation $\{ q_1, q_2 \}$ denotes the Poisson bracket defined by
\[
\{ q_1, q_2 \} = \frac{\partial q_1}{\partial \eta} \cdot \frac{\partial q_2}{\partial v} - \frac{\partial q_1}{\partial v} \cdot \frac{\partial q_2}{\partial \eta} \tag{6.4}
\]
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