Nonlinear sixth order models with nonsmooth solutions and monoton nonlinearity

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Abstract. The sixth-order nonlinear spectral problem with nonsmooth solutions is studied. It is proved that the set of non-negative values for which the nonlinear spectral problem has at least one non-trivial non-negative solution is nonempty and coincides with a certain interval. We use the pointwise approach proposed by Yu. V. Pokorny analyzing solutions to a boundary value problem. This approach shown to be effective in the study of the second-order problems. Based on the previously obtained estimates of the Green's function of the boundary-value problem, it was possible to show that the operator inverting the studied nonlinear problem, representable as a superposition of completely continuous and continuous operators, acts from the cone of non-negative continuous functions to a narrower set. The last fact allows us to prove the uniqueness of a solution of a nonlinear boundary value problem using the theory of spaces with a cone.

1. Introduction
The paper studies a nonlinear mathematical model, implemented as a boundary problem

\[
\begin{align*}
L_0 u &\equiv -\left( \frac{p(x)u^{(m)}_{x\mu}}{x\mu} \right)'' + \left( ru''_{xx} \right)'' - \left( gu_x \right)' + Q'u = \lambda F(x, u); \\
u(0) &\equiv u''(0) = u'''_{xx\mu}(0) = 0; \\
u(\ell) &\equiv u_x(\ell) = u'''_{xx\mu}(\ell) = 0,
\end{align*}
\]

with derivatives in measures.

Notice that a qualitative theory with nonsmooth solutions began to develop rapidly after the publication in 1999 of the work of Yu. V. Pokorny [1]. So there are monographs [2, 3], papers [4–7] in which thoroughly studied linear boundary value problems of the second order with derivatives in measures. The pointwise approach used in linear problems has been shown to be effective both in nonlinear problems [8, 9], in problems with discontinuous solutions [10–12], and in fourth-order boundary value problems [13–15], as well as in the regional problems of hyperbolic type [16, 17]. This efficiency is explained quite simply: when using derivatives in the measures, the equation becomes pointwise defined, and makes it possible to use qualitative methods for analyzing solutions, in contrast to theory of generalized functions. Indeed, when using the theory of Schwartz-Sobolev distributions according difficult problems are manifested. The first, problem is that only weak solvability can be established, hence equations are not suitable for applications. The second problem which has not yet been solved arises when multiply the generalized function by the discontinuous one. The third problem is that equations in generalized functions are the
equality of two functionals defined over the space of basic functions, and it is extremely difficult
to apply methods of qualitative analysis to such equations.

2. Required Conventions and Terms
We will consider a solution of (1) in the class $E$ of twice continuously differentiable functions $u(x)$
such as $u''(x)$ is $μ$-absolutely continuous on $[0; ℓ]$; $pu''''(x)$ is twice continuously differentiable;
$(pu''''(x))''(x)$ is $μ$-absolutely continuous on $[0; ℓ]$. At the points $ξ$ belonging to the set of
discontinuity points $μ(x)$, the equation in (1) is understood as the equality

$$-Δ (pu''''(x))'' + Δ (gu''''(x))' + F(x, u(ξ)) = λF(ξ, u(ξ)), $$

where $Δu(ξ)$ is a complete jump of the function $u(x)$ at the point $ξ$; $λ > 0$ is spectral parameter.
Denote by $Λ$ the set of positive values of $λ$, for each of which (1) has at least one solution.
We call a real number $λ$ an eigenvalue of problem (1) if, for this $λ$ system (1) has at least one
non-trivial solution.

The equation from (1) is given almost everywhere (in measure $μ$) on the next extension of
the interval $[0; ℓ]$. Let $S(μ)$ be a set of discontinuity points of the function $μ(x)$. We define
the metric $g(x; y) = |μ(x) - μ(y)|$ on the $J_μ = [0; ℓ] \setminus S(μ)$. The resulting metric space $(J_μ; μ)$ is not
complete. The standard completion leads (up to isomorphism) to the set $[0; ℓ]_S$, in which each
point $ξ ∈ S(μ)$ is replaced with a pair of eigenvalues $ξ - 0, ξ + 0$, which were previously limit values.
By inducing ordering from the original set, we arrive at the inequalities $x < ξ - 0 < ξ + 0 < y$
for all $x, y$ for which the inequalities $x < ξ < y$ hold in the initial segment.

The function $v(x)$ at the points $ξ - 0$ and $ξ + 0$ of the set $[0; ℓ]_S$ is defined by the limiting
values. For a function defined in this way, we retain the previous notation. The function defined
on this set becomes continuous in the sense of the metric $g(x; y)$.

The union of $[0; ℓ]_S$ and $S(μ)$ gives us the set $[0; ℓ]_μ$, in which each point $ξ ∈ S(μ)$ is replaced
by a triple of eigenelements $\{ξ - 0; ξ; ξ + 0\}$. We suppose that the equation is given on this set.

We assume that the functions $p(x)$, $r(x)$, $g(x)$ and $Q(x)$ are $μ$-absolutely continuous on
$[0; ℓ]_μ$, $\min_{x ∈ [0; ℓ]_μ} p(x) > 0, Q(x)$ does not decrease, and $F(x, u)$ satisfies the conditions of
Carathéodory, i.e.,

- $F(x, u)$ for almost all $x$ (with respect to the $μ$-measure) is defined and continuous in $u$;
- the function $F(x, u)$ is measurable in $x$ for every $u$;
- $|F(x, u)| ≤ m(x)$, where $m(x)$ is a $μ$-summable function on $[0; ℓ]_S$.

We call the homogeneous equation

$$- (pu''''(x))'' + (ru''''(x))'' - (gu''''(x))' + Q''''u = 0 $$

non-oscillating on $[0; ℓ]$ if any of its non-trivial solutions has at most five zeros with respect to
multiplicities.

We denote by $K$ the cone of non-negative functions on $[0; ℓ]$.

3. The main results
The following theorems are proved.

Theorem 1. Let the following conditions be satisfied:

- equation $Lu = 0$ does not oscillate on $[0; ℓ]_μ$;
- $f(x, 0) ≡ 0$;
- $f(x, 0) ≡ 0$;
• $f(x, u)$ does not decrease with respect to $x$ and for $u \geq 0$;
• $\frac{f(x, u)}{u}$ decreases by $u$, where $u > 0$;
• the superposition operator generated by the function $f(x, u_0(x)u)$ acts continuously from $C[0; \ell]$ to $L_{p, \mu}[0; \ell]$ for some $p \in (1; +\infty]$.

Then the set $\Lambda$ of non-negative values of $\lambda$ for which problem (1) has at least one non-trivial solution in $K$ has the following properties:

(i) the set $\Lambda$ is nonempty and coincides with some interval $(\lambda_0, \lambda_\infty)$, for $0 \leq \lambda_0 < \lambda_\infty \leq +\infty$;
(ii) each $\lambda \in \Lambda$ corresponds to only one solution $u(x, \lambda) \in K$ of problem (1), and
\[ \lim_{\lambda \to \lambda_0^+} \|u(\cdot, \lambda)\|_C = 0, \quad \lim_{\lambda \to \lambda_\infty^-} \|u(\cdot, \lambda)\|_C = \infty; \]
(iii) the function $u(x, \lambda)$ is monotone in $\lambda$, i.e., for all $x \in [0; \ell]$ the inequality $(\lambda_1 - \lambda_2)(u(x, \lambda_1) - u(x, \lambda_2)) \geq 0$ holds;
(iv) For each fixed $\lambda^* \in \Lambda$ and for any initial approximation $u_0(x)$, the iterative sequence $\{u_n(x)\}_{n=0}^\infty$, defined as a solution of the differential model
\[
\begin{cases}
Lu = \lambda^* f(x, u_{n-1}(x)), \\
u(0) = u''_0(0) = u'''_{xx0}(0) = 0, \\
u(\ell) = u''_{x}(\ell) = v'''_{x\ell}(\ell) = 0,
\end{cases}
\]
where $n = 1, 2, \ldots$, converges uniformly to $u(x, \lambda^*)$.

Proof. Since $f(x, u)$ does not decrease in $u$ for $u \geq 0$ and $f(x, 0) \equiv 0$, we have $f(x, u) \geq 0$ for all $x \in [0; \ell]$ and $u \geq 0$. From the decrease of $\frac{f(x, u)}{u}$ for $u > 0$ it follows that if mathematical model (1) is solvable in $K$ for some $\lambda > 0$, then its solution $u(x, \lambda)$ belongs to the interior of $K_{u_0}$, where $K_{u_0} = K \cap E_{u_0}$ and the space $E_{u_0}$ consists of functions $u(x) \in C[0; \ell]$ for which the norm $\sup_{0 < x < \ell} \|u(x)\|_{u_0(x)}$ is finite; $u_0(x) = Cx^2(\ell - x)^2$ for some $C > 0$.

Let us show nonemptiness of $\Lambda$. Notice that it is enough to prove that at least for some $\lambda > 0$ there is a nontrivial solution of the equation
\[ u = \lambda Au, \]
where
\[ A = \tilde{G}\tilde{F} \]
and
\[ (\tilde{G}f)(x) = \int_0^\ell \frac{G(x, s)}{u_0(x)} f(s) \, d\mu(s), \quad (\tilde{F}u)(x) = f(x, u_0(x)u(x)), \]
belonging to $K$. With respect to the conditions of the theorem the operator
\[ A_k^{(r)}v = \frac{kAv + v_0}{kAv + v_0}_C \cdot r, \]
where $v_0(x) \equiv 1$, $v(x) = \frac{u(x)}{u_0(x)}$, $r$ is a positive fixed number, $k = 1, 2, \ldots$, leaves the cone $K$ invariant. Moreover, since the norm in $C[0; \ell]$ is monotone, i.e., $\|kAv + v_0\|_C \geq \|v_0\|_C$ at $v(x) \geq 0$, and converts $K$ to the unit sphere $C[0; \ell]$, we have that $A_k^{(r)}$ is completely continuous on $K$. Therefore, $A_k^{(r)}$ leaves invariant the set of functions $v(x) \in K$ for which $\|u\|_C \leq 1$. It is easy to see that this set is boundedly convex and closed. Due to the Schauder principle, $A_k^{(r)}$ has a fixed point $v_k$, i.e.,
\[ A_k^{(r)}v_k = v_k. \]
Recalling the definitions of $A^{(r)}_k$, we can rewrite the last equality in the following form

$$r(Av_k)(x) + \frac{r}{k} = \chi v_k(x),$$

where

$$\chi = \left\| Av_k + \frac{1}{k} \right\|_{C}.$$

From the compactness of $A$ and the equality $\|v_k\|_{C} = r$ it follows that the sequences $\{\chi_k v_k\}$ and $\{\chi_k\}$ are compact. Let us show that the contrary, i.e., $\inf_k \chi_k > 0$. Suppose the contrary, i.e., $\inf_k \chi_k = 0$. From (2) it follows that $u_k(x)$ is positive for all $x \in [0; \ell]$. Thus, $m_k = \inf_{x \in [0; \ell]} v_k(x) > 0$. Since $\|v_k\|_{C} \leq r$, we have $m_k \leq r$. If $m_k = r$ for some $k$, then $v_k(x) \equiv r$ and

$$(Av_k)(x) = \left(\frac{\chi_k}{r} - \frac{1}{k}\right) v_k(x).$$

The proof of the nonemptiness of $\Lambda$ is completed on this (it should be noted that the assumption $m_k = r$ is not connected with the assumption $\inf_k \chi_k = 0$).

Hence, we can assume $m_k < r$. Since $f(x, u)$ decreases in $u$, we have $f(x, r) \leq \frac{1}{m_k} f(x, m_k)$ for all $x \in [0; \ell]$. The non-decreasing $f(x, u)$ in $u$ and the inequality $v_k(x) \geq m_k$ implies $f(x, v_k(x)) \geq f(x, m_k)$. Therefore,

$$f(x, r) \leq \frac{1}{m_k} f(x, v_k(x))$$

for all $x \in [0; \ell]$. Then, due to the positivity of the operator $\tilde{G}$, we have

$$\frac{1}{m_k} (Av_k)(x) - (Av_0)(x) = \tilde{G} \left(\frac{1}{m_k} f(x, v_k(x)) - f(x, r)\right) \geq 0.$$

Thus we have that the inequality

$$v_k(x) \geq \frac{rm_k}{\chi_k} (Av_0)(x)$$

holds for all $x \in [0; \ell]$. But

$$m_k \geq \frac{rm_k}{\chi_k} (Av_0)(x),$$

therefore, $\chi_k \geq r(Av_0)(x)$, and together with the assumption $\inf_k \chi_k = 0$ it means $\min_k (Av_0)(x) = 0$, which obviously cannot be. Thus, $\inf_k \chi_k > 0$.

As have been noted earlier, the sequences $\{\chi_k v_k\}$ and $\{\chi_k\}$ are compact. Since $\inf_k \chi_k > 0$, the sequence $\{v_k\}$ is also compact. Separating a convergent subsequence from $\{v_k\}$, and denoting by $\{\chi_k\}$ the sequence $\{\chi_{k_m}\}$, which converges to $\chi_0$, from (2) will have

$$rAv_{k_m} + \frac{r}{k_m} = \chi_{k_m} v_{k_m},$$

wherein $v_{k_m}(x) \Rightarrow w_0(x)$. Passing in (3) to the limit $m \to \infty$, we get $Aw_0 = \frac{\chi_0}{r} w_0$, and the nonemptiness of $\Lambda$ is proved.
Since $\|v_k\|_C = r$, the norm $\|u_0\|_C = r$. Since $r$ is an arbitrary positive number, this proves that for some $\lambda$ the equation $u = \lambda A$ has a solution in $K$ with the norm $r$, that is, the set of values of $\|u(\cdot, \lambda)\|_C$ on $\Lambda$ fills $(0, \infty)$.

Let us show that for each $\lambda \in \Lambda$ problem (1) has exactly one non-negative solution in $K$. Let $v(x)$ and $w(x)$ be different solutions from the $K$ of model (1) corresponding to some $\lambda \in \Lambda$. Obviously $\lambda > 0$. (It will be shown below that $\Lambda = (\lambda_0, \lambda_\infty)$). Since every solution $u(x)$ of differential model (1), satisfying the conditions of the theorem, belongs to the interior of the cone $K_{u_0}$, for some positive and finite $\alpha$ and $\beta$ the inequalities $\alpha \leq \frac{u(x)}{u_0(x)} \leq \beta$ hold for all $x \in (0, \ell)$. Therefore, the function $\frac{v(x)}{w(x)}$ is strictly positive on $(0, \ell)$. Without loss of generality, we can assume that the inequalities $\frac{v(x)}{w(x)} > 1$ are violated for some $x \in (0, \ell)$, because otherwise we change $v(x)$ and $w(x)$ in places. Then for the quantity $\chi = \inf_{x \in (0, \ell)} w(x)$ we have the double inequality $0 < \chi < 1$. Functions $\tilde{v}(x) = \frac{v(x)}{u_0(x)}$ and $\tilde{w}(x) = \frac{w(x)}{u_0(x)}$ satisfy the nonlinear integral equation

$$\tilde{u}(x) = \lambda \int_0^\ell \tilde{G}(x, s)f(s, u_0(s)\tilde{u}(s))d\mu(s),$$

where $\tilde{u}(x) = \frac{u(x)}{u_0(x)}$ and $\tilde{G}(x, s) = \frac{G(x, s)}{u_0(x)}$. Moreover, $\tilde{v}(x)$ and $\tilde{w}(x)$ are positive on $(0, \ell)$.

Since the function $\frac{f(x, u)}{u}$ decreases in $u$, $\frac{1}{\chi} f(x, \chi \tilde{w}(x)) \geq f(x, \tilde{w}(x))$ for all $x \in [0, \ell]$; $\tilde{w}(x) > 0$ $x \in (0, \ell)$, the last inequality is strict almost everywhere (in the sense of the measure $\mu$). The definition of $\chi$ implies $\tilde{v}(x) \geq \chi \tilde{w}(x)$, which, together with the non-decreasing $f(x, u)$ in $u$, gives us $f(x, \tilde{v}(x)) \geq f(x, \chi \tilde{w}(x))$ for all $x$ belonging to $[0, \ell]$. Then the function $w(x) = f(x, \tilde{v}(x)) - f(x, \chi \tilde{w}(x))$ is positive on the set of full $\mu$-measure from $[0, \ell]$.

Hence, in combination with the strong positivity of the integral operator $\tilde{G}(x, s)$, with the kernel $\tilde{G}(x, s) = \frac{G(x, s)}{u_0(x)}$, the inequality $\tilde{G}(w)(x) > 0$ holds on $[0, \ell]$, that is, $(\tilde{G}w)(x) \geq \tilde{\chi}_0$ with some positive $\tilde{\chi}_0$. The latter means that $\tilde{v}(x) - \chi \tilde{w}(x) = (\tilde{G}w)(x) \geq \lambda \tilde{\chi}_0$ for all $x \in [0, \ell]$. So,

$$\frac{\tilde{w}(x)}{w(x)} = \frac{v(x)}{w(x)} \geq \chi_0 + \frac{\lambda \chi_0}{\|w\|_C},$$

which contradicts to the definition of the number $\chi$.

Let us show the monotonicity of $u(x, \lambda)$ on $\Lambda$. Let $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_1 < \lambda_2$. Put $u_1(x) = \frac{u(x, \lambda_1)}{u_0(x)}$ and $u_2(x) = \frac{u(x, \lambda_2)}{u_0(x)}$. Let us prove that the value $m_0 = \inf_{0 < x < \ell} \frac{u_2(x)}{u_1(x)}$ is not less than one. Suppose the contrary: $m_0 < 1$. Then from the inequality $u_2(x) \geq m_0 u_1(x)$ it follows

$$\frac{1}{m_0} f(x, u_0(x)u_2(x)) \geq \frac{1}{m_0} f(x, m_0 u_0(x) u_1(x)) \geq f(x, u_0(x) u_1(x)).$$

We have

$$\frac{1}{m_0} u_2(x) = \frac{\lambda_2}{m_0} (Au_2)(x) \geq \frac{\lambda_2}{m_0} \frac{Au_1}{\lambda_1} = \frac{\lambda_2}{\lambda_1} u_1(x).$$

This implies the inequality $\frac{u_2(x)}{u_1(x)} \geq \frac{\lambda_2}{\lambda_1} m_0$ holds for all $x \in (0, \ell)$. According to the definition of $m_0$, we obtain that the inequality $\lambda_2 \leq \lambda_1$ holds, which contradicts to the assumption.

We have showed the connectivity $\Lambda$. Assume that $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 < \lambda_2$, $u(x, \lambda_1)$ and $u(x, \lambda_2)$ are solutions of the differential model (1) for $\lambda_1, \lambda_2$ respectively. Let us that show the inclusion
\[ [\lambda_1, \lambda_2] \subset \Lambda \text{ holds. Denote by } v_1(x) = \frac{u(x, \lambda_1)}{u_0(x)}, \ v_2(x) = \frac{u(x, \lambda_2)}{u_0(x)} \text{ and } v(x) = \frac{u(x)}{u_0(x)}. \] The monotone operator \( A_\lambda v = \lambda Av \) leaves invariant a bounded closed and convex set of functions

\[ \mathcal{M} = \{ v(x) \in C[0; \ell] \mid v_1(x) \leq v(x) \leq v_2(x) \}, \]

acting in \( C[0; \ell] \). The complete continuity of \( A \) implies the existence of a fixed point \( v_\lambda \) in \( \mathcal{M} \) for the operator \( A_\lambda; v_\lambda = A_\lambda v_\lambda \). The last equality means that \( v_\lambda = \lambda Av_\lambda \), that is, \( A \) is a connected subset of \( R^+ \).

Setting \( \lambda_0 = \inf \Lambda \) and \( \lambda_\infty = \sup \Lambda \), we will have \( \Lambda \subset [\lambda_0; \lambda_\infty] \). Let us show that \( \lambda_0 \notin \Lambda \). If this is not so, then for \( \lambda_0 \) there exists a solution \( u(x, \lambda_0) \) of model (1). As it was established earlier, for any other solution \( u(x, \lambda) \) of mathematical model (1) for \( \lambda \in \Lambda \) the inequality \( u(x, \lambda) \geq u(x, \lambda_0) \) holds for all \( x \in [0; \ell] \), that is, \( \| u(\cdot, \lambda) \|_C \geq \| u(\cdot, \lambda_0) \|_C > 0 \) and the function \( \| u(\cdot, \lambda) \|_C \) cannot have arbitrarily small values. It is similarly proved that \( \lambda_\infty \notin \Lambda \).

As shown earlier, the function \( \| u(\cdot, \lambda) \|_C \) fills the interval \( (0; +\infty) \) with its values. Therefore,

\[
\lim_{\lambda \to \lambda_0+0} \| u(\cdot, \lambda) \|_C = 0
\]

and

\[
\lim_{\lambda \to \lambda_\infty-0} \| u(\cdot, \lambda) \|_C = +\infty.
\]

Here we used the items (i), (iii) proved earlier and the monotonicity of the norm in \( C[0; \ell] \).

We have proved the last preposition. Assume that \( \lambda^* \in \Lambda \) and \( u(x, \lambda^*) \) is a corresponding solution of (1). Denote by \( v^*(x) = \frac{u(x, \lambda^*)}{u_0(x)} \). The function \( v^*(x) \) satisfies the equation \( v = \lambda^* Av \), that is, the identity \( v^* = \lambda^* Av^* \) holds. Let \( v_0(x) \) be an arbitrary non-negative continuous function on \( [0; \ell] \). Let us show that the sequence \( \{ v_n(x) \} \), where \( v_n(x) = \lambda^*(Av_{n-1})(x), n = 1, 2, \ldots \), converges uniformly to \( v^*(x) \).

Since the functions \( v_1(x) \) and \( v^*(x) \) are positive on \( [0; \ell] \), for some \( \alpha \) and \( \beta \) satisfying the inequalities \( \alpha < 1 < \beta \) takes place \( x \in [0; \ell] \)

\[
\alpha v^*(x) \leq v_1(x) = \lambda^*(Av_0)(x) \leq \beta v^*(x) = \varpi_1(x).
\]

From non-decreasing \( f(x, u) \) and decreasing \( \frac{1}{u} f(x, u) \) by \( u \) it follows

\[
(A\varpi_1)(x) \geq \alpha \lambda^*(Av^*)(x) = \alpha v^*(x) = \varpi_1(x)
\]

and

\[
\varpi_1(x) = \beta v^*(x) = \beta \lambda^*(Av^*)(x) \geq \lambda^*(A\varpi_1)(x).
\]

Therefore, the monotonicity of \( A \) and (4) imply the inequalities

\[
\varpi_1(x) \leq \lambda^*(A\varpi_1)(x) \leq \lambda^*(A\varpi_1)(x) = u_2(x) \leq \lambda^*(A\varpi_1)(x) \leq \varpi_1(x).
\]

For the sequences \( u_n+1(x) = \lambda^*(A\varpi_n)(x) \) and \( \varpi_{n+1}(x) = \lambda^*(A\varpi_n)(x) \) we have

\[
u_n \leq \varpi_{n+1} \leq u_n \leq \varpi_{n+1} \leq \varpi_n
\]

for all \( n = 1, 2, \ldots \). Then each sequence, being monotonic and bounded, due to the compactness of the operator \( A \), converges to a fixed point of the operator \( \lambda^* A \). Since \( v^*(x) \) is the only fixed point we have

\[
\lim_{n \to \infty} u_n = \lim_{n \to \infty} \varpi_n = v^*.
\]

Then the sequence \( v_n \), concluded between \( u_n \) and \( \varpi_n \), also must converge to \( v^* \). The theorem is proved.
For the case when \( f(x, u) \) is continuously differentiable in a neighborhood of zero (with respect to the variable \( u \)) and in a neighborhood of infinity (in the sense of the definition below), the interval \( \Lambda \) can be effectively indicated.

**Definition.** We will call a function \( f(x, u) \) is continuously differentiable in a neighborhood of infinity if there exists a function \( f'_\infty(x) \) such that \( \frac{f(x, u)}{u} \xrightarrow{u \to \infty} f'_\infty(x) \) for \( u \to +\infty \).

**Theorem 2.** The numbers \( \lambda_0 \) and \( \lambda_\infty \) are the minimum eigenvalues of spectral problems

\[
\begin{align*}
Lu &= \lambda f'(x, 0)u, \\
u(0) &= u'(0) = 0, \\
u(\ell) &= u_x'(\ell) = 0,
\end{align*}
\]

and

\[
\begin{align*}
Lu &= \lambda f'_{\infty}(x)u, \\
u(0) &= u'(0) = 0, \\
u(\ell) &= u_x'(\ell) = 0,
\end{align*}
\]

respectively.

Take the sequence \( \lambda_k \in \Lambda \) tending to \( \lambda_0 \). Then the sequence of solutions \( u(x, \lambda_n) \) of problem (1) corresponding to \( \lambda_n \) converges uniformly to zero. The functions \( v_n(x) = \frac{u(x, \lambda_n)}{w_0(x)} \) also converge to zero; the sequence \( w_n(x) = \frac{v_n(x)}{\|v_n\|_C} \) is compact, therefore, a convergent subsequence of \( w_{nk}(x) \) can be distinguished from it: \( w_{nk}(x) \xrightarrow{k \to \infty} w_0(x) \). Then the function \( w_0(x) \) satisfies the equation

\[
w_0(x) = \lambda_0 \int_0^\ell \tilde{G}(x, s)f'_\infty(s, 0)w_0(s) d\mu(s).
\]

Moreover, for the integral operator the number \( \frac{1}{\lambda_0} \) is the maximum eigenvalue.

The reasoning for \( \lambda_\infty \) is similar.

Let us study the situation, when the model

\[
\begin{align*}
\left( pu''_{\mu x} + (ru')_\mu + Q_\mu u = f(x, u) \quad (x \in [0; \ell])_\mu, \\
u(0) &= u'(0) = u'''_{xx}(0) = 0, \\
u(\ell) &= u_x'(\ell) = u'''_{xx}(\ell) = 0
\end{align*}
\]

obviously has one known solution and the question is whether another exists. We will consider the known solution, without loss of generality, to be zero, since we can make a functional change that shifts this solution to zero.

**Theorem 3.** Let the following conditions be satisfied:

1) the superposition operator generated by the function \( f(x, u) \) acts continuously from \( C[0; \ell] \) to \( L_{p,\mu}[0; \ell] \);
2) \( f(x, u) \geq 0 \) for all \( x \in [0; \ell] \) and \( u \geq 0 \);
3) the homogeneous equation \( Lu = 0 \) does not oscillate on \( [0; \ell]_\mu \);
4) for some \( R > 0 \) and any \( \lambda \in (0; 1) \) differential model

\[
\begin{align*}
Lu &= \lambda f(x, u), \\
u(0) &= u'(0) = u'''_{xx}(0) = 0, \\
u(\ell) &= u_x'(\ell) = u'''_{xx}(\ell) = 0,
\end{align*}
\]

does not have solutions \( u(x) \) such that

\[
\sup_{x \in [0; \ell]} \frac{u(x)}{u_0(x)} \geq R,
\]
where \( u_0(x) = Cx^2(\ell - x)^2 \) for some \( C > 0 \).

Then the problem (5) has at least one solution in \( K \).

Proof. Since \( Lu = 0 \) does not oscillate on \([0; \ell]_\mu\), the integral operator

\[
(Au)(x) = \int_0^\ell G(x, s)f(s, u(s)) \, d\mu(s)
\]

converts the cone \( K \) to a narrower set

\[
K(\tilde{u}_0) = u(x) \in C[0; \ell] \| u(x) \geq \tilde{u}_0(x) \| u \|_C, x \in [0; \ell]),
\]

where \( \tilde{u}_0(x) = Mu_0(x) \) for some \( M > 0 \). Moreover, it is completely continuous in \( C[0; \ell] \). Every fixed point (8) is a solution of differential model (5). Thus, the question of the existence of a fixed point in \( K \) for the operator \( \tilde{A} \) narrows to \( K(\tilde{u}_0) \).

Solvability (6) is equivalent to the solvability of the equation \( \lambda Au = u \) with operator (8). If, in addition, it turns out that the last equation has a solution \( u_1 \in K(\tilde{u}_0) \), satisfying for some \( R_0 > 0 \) the inequality \( \| u_1 \|_C \geq R_0 \). Then according to the definition of \( K(\tilde{u}_0) \) we have \( u_1(x) \geq R_0 \tilde{u}_0(x) \) for all \( x \in [0; \ell] \); where \( u_1(x) \) is a solution of problem (7) for \( R = M \cdot R_0 \). Therefore, the condition of the theorem on the absence of such solutions means that the equation \( \lambda Au = u \) for \( \lambda \in (0, 1) \) does not have solutions \( u_1(x) \) in \( K(\tilde{u}_0) \) such that \( \| u_1 \| \geq R = \frac{R_0}{M} \). Consider the operator \( \hat{A} \) on \( K(\tilde{u}_0) \):

\[
\hat{A}u = \begin{cases} 
Au & \text{if } \| u \|_C \leq \hat{R}, \\
A \left( \frac{\hat{R}}{\| u \|_C} u \right) & \text{if } \| u \|_C > \hat{R}
\end{cases}
\]

(9)

The operator \( \hat{A} \) is completely continuous on \( K(\tilde{u}_0) \) and converts \( K(\tilde{u}_0) \) to the bounded part, that is, \( \hat{A} \) leaves invariant the intersection of \( K(\tilde{u}_0) \) with a ball of some radius centered at zero. And since this intersection is convex, bounded, and closed, by the Schauder principle, \( \hat{A} \) has a fixed point \( \hat{u} \) in \( K(\tilde{u}_0) \): \( \hat{u} = \hat{A}\hat{u} \).

If we assume that \( \| \hat{u} \|_C > \hat{R} \), then \( \hat{u} = \hat{A}\hat{u} = A \left( \frac{\hat{R}}{\| u \|_C} u \right) \). Hence, setting \( \tilde{u} = \hat{R} \frac{u}{\| u \|_C} \), we have \( \| \tilde{u} \|_C = \hat{R} \). In other words, the problem (6) for \( \lambda = \frac{\hat{R}}{\| u \|_C} < 1 \) has the solution \( \tilde{u} \) satisfying the inequality \( \| \tilde{u} \|_C > \hat{R} \). Thus, the inequality \( \| \tilde{u} \|_C > \hat{R} \) is impossible, therefore, \( \| \tilde{u} \|_C \leq \hat{R} \). But then, by virtue of the definition \( \hat{A}, \hat{A}u = \hat{u} \). The theorem is proved.

**Theorem 4.** Let the following conditions be satisfied:

(i) \( f(x, 0) \equiv 0 \);

(ii) the homogeneous equation \( Lu = 0 \) does not oscillate on \([0; \ell]_\mu\);

(iii) \( f(x, u) \geq 0 \) for all \( x \in [0; \ell] \) and \( u \geq 0 \);

(iv) the superposition operator generated by the function \( f(x, u) \) acts continuously from \( C[0; \ell] \) to \( L_{p, \mu}[0; \ell]_\mu \) for some \( p \in (1; +\infty] \);

(v) for some \( 0 < r < R < \infty \) the differential model

\[
\begin{cases}
Lu = \lambda f(x, u), \\
u(0) = u_0'(0) = u_{xx}(0) = 0, \\
u(\ell) = u_0'(\ell) = u_{xx}(\ell) = 0,
\end{cases}
\]

for any \( \lambda \in (0; 1) \) does not have solutions satisfying the inequalities

\[
\tilde{u}_0(x) \cdot \| u \|_C \leq u(x) \leq r,
\]
and for some non-negative non-trivial function \( h(x) \in L_{1,\mu}[0;\ell] \) and for any \( \lambda > 0 \) differential model

\[
\begin{align*}
Lu &= \lambda f(x, u) + \lambda h, \\
u(0) &= u_x'(0) = u''_{xx}(0) = 0, \\
u(\ell) &= u_x'(\ell) = u''_{xx}(\ell) = 0,
\end{align*}
\]

does not have no solutions satisfying the inequality \( u(x) \geq R\tilde{u}_0(x) \).

Thus the problem

\[
\begin{align*}
Lu &= f(x, u), \\
u(0) &= u_x'(0) = u''_{xx}(0) = 0, \\
u(\ell) &= u_x'(\ell) = u''_{xx}(\ell) = 0,
\end{align*}
\]

has a non-trivial solution in \( K \).

Proof. Using the operator \( A \) on \( K \setminus \{\Theta\} \) we introduce the operator

\[
Bu = \|u\|^2_C A \left( \frac{u}{\|u\|^2_C} \right).
\]

If the operator \( B \) has a fixed point \( u^* \), then the element \( v^* = \frac{u^*}{\|u^*\|^2_C} \) gives the fixed point of the operator \( A \). Therefore, it suffices to show that the operator \( B \) has a fixed point in \( K \).

The operator \( B \) transfers \( K \setminus \{\Theta\} \) to \( K(\tilde{u}_0) \), and \( B \) is completely continuous in \( K \) outside a ball of any radius.

Using arguments similar to Theorem 3, we see that for the operator \( B \) on the set of elements \( K(\tilde{u}_0) \) with a large norm, \( \lambda Bu = u \) for \( \lambda \in (0,1) \), and on elements of a small norm from \( K(\tilde{u}_0) \) for any \( \lambda > 0 \) it cannot be satisfied \( u = Bu + \lambda h_0 \), where \( h_0(x) = \int_0^\ell G(x, s)h(s)\,d\mu(s) \). Therefore, the operator \( B \) has a fixed point in \( K(\tilde{u}_0) \). The theorem is proved.

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