Superreplication when trading at market indifference prices.

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Abstract

We study superreplication of European contingent claims in discrete time in a large trader model with market indifference prices recently proposed by Bank and Kramkov. We introduce a suitable notion of efficient friction in this framework, adopting a terminology introduced by Kabanov, Rasonyi, and Stricker in the context of models with proportional transaction costs. In our framework, efficient friction ensures that large positions of the investor may lead to large losses, a fact from which we derive the existence of superreplicating strategies. We illustrate that without this condition there may be no superreplicating strategy with minimal costs. In our main result, we establish efficient friction under a tail condition on the conditional distributions of the traded securities and under an asymptotic criterion on risk aversions of the market makers. Another result asserts that strict monotonicity of the conditional essential infima and suprema of the security prices is sufficient for efficient friction. We give examples that satisfy the assumptions in our conditions, which include non-degenerate finite sample space models as well as Levy processes and an affine stochastic volatility model of Barndorff-Nielsen-Shepard type.

Keywords: utility indifference prices, large investor, liquidity, superreplication, monotone exponential tails

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1 Introduction

The problem of superreplicating a contingent claim has been widely studied in Mathematical Finance. For frictionless diffusion models this stochastic control problem was first addressed by El Karoui and Quenez [12] and, for general semimartingales, by Kramkov [21]. Also models with market frictions have received a lot of attention. For markets with portfolio constraints such as the prohibition of short selling see, e.g., Cvitanić and Karatzas [9], Jouini and Kallal [17], Föllmer and Kramkov [13], Broadie et al. [5]. For markets with proportional transaction costs similarly far reaching investigations have resulted from the work of, e.g., Soner et al. [24], Cvitanić et al. [10], Kabanov and Stricker [20], Campi and Schachermayer [6], Guasoni et al. [16]. For a complete treatment and list of references, we refer to the book, Kabanov and Safarian [19]. Recently, superreplication has also been studied in nonlinear models capturing illiquidity effects by, e.g., Çetin et al. [8], Gökay and Soner [15], Possamai et al. [23], Dolinsky and Soner [11].

The present paper focuses on the nonlinear large investor model with market indifference prices developed in Bank and Kramkov [1, 2, 3]. By contrast to the model discussed in Çetin et al. [7], Çetin et al. [8], Gökay and Soner [15], Dolinsky and Soner [11], this model does not postulate a local cost term depending on the size of the current transaction which would be attributed to a temporary market impact. Instead, market indifference prices can be viewed as a way to specify systematically the permanent price impact of a transaction. While the impact is adverse in the sense that a large buy order will substantially drive up marginal prices, it is far from obvious when this impact actually results in an efficient friction, i.e., in real costs to the large investor. Indeed, if after a purchase no new information becomes available about the ultimate value of the traded securities, a subsequent sale of the same position will take place at the same marginal prices, now processed in reverse order. Hence, the sale will recover all the expenses incurred from the initial purchase leading to a ‘free roundtrip’. This is similar to a phenomenon already observed in multi-variate asset price models with proportional transaction costs where a suitable notion of efficient friction was introduced by Kabanov et al. [18] to ensure that trades actually incur costs.

With the same purpose in mind, we adopt this terminology for our framework despite the mentioned differences in the models and we thus say that,
essentially, a model with market indifference prices exhibits efficient friction if an investor engaging in ever larger positions may face ever higher losses with positive probability. A simple two-period example with a single market maker shows that without this weak condition superreplicating strategies may fail to exist. Our main result, Theorem 3.2, thus develops a readily verifiable criterion for efficient friction to hold. This criterion is based on tail conditions for both the conditional distributions of terminal values of traded securities and the asymptotic risk aversions of market makers. Corollary 3.3 then shows that efficient friction indeed implies the existence of superreplicating strategies. In models where conditional essential infima and suprema of security prices are attained on sets with positive probability, Theorem 4.1 shows that the strict monotonicity of these extrema is also sufficient for efficient friction, even without extra assumptions on the market makers’ asymptotic risk aversions. We also characterize when a binomial model with an exponential market maker is complete and find again that this depends on a monotonicity condition on conditional extrema. As examples where our tail condition on conditional distributions holds we consider Lévy processes and a Barndorff-Nielsen-Shepard model with stochastic volatility; see Section 5.

2 Problem formulation and motivation

Before we can properly formulate the superreplication problem we want to address in Section 2.2, we first have to introduce the modeling framework we are going to use.

2.1 Trading at market indifference prices

For our model we shall use a discrete-time version of the framework introduced in Bank and Kramkov [1, 2] which we shall outline briefly in this section for the reader’s convenience. Specifically, we consider a financial market where \( M \in \{1, 2, \ldots \} \) market makers quote prices for \( J \in \{1, 2, \ldots \} \) securities with respective random payoffs \( \psi = (\psi^1, \ldots, \psi^J) \) at time \( T \). For simplicity, the market makers have a common view of uncertainty which is described by a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, \mathbb{P})\) where \( \mathcal{F}_0 = \{\emptyset, \Omega\} \) up to \( \mathbb{P} \)-null sets; in particular \( \psi \in L^0(\mathcal{F}_T, \mathbb{R}^J) \). The market makers, however, may have different attitudes towards risk:

**Assumption 2.1.** Each market maker \( m = 1, \ldots, M \) has a utility function \( u_m : \mathbb{R} \to \mathbb{R} \) which is strictly concave, increasing, twice continuous differentiable with

\[
\lim_{x \uparrow \infty} u_m(x) = 0.
\]
Moreover, absolute risk aversion is bounded in the sense that

\[
\frac{1}{c} \leq a_m(x) \triangleq \frac{u''_m(x)}{u'_m(x)} \leq c, \quad x \in \mathbb{R}, \quad \text{for some } c > 0.
\]

The market makers shall be allowed to trade freely among themselves (e.g., using a complete OTC market). As a result, they always allocate their total wealth in a (conditionally) Pareto optimal way, i.e., such that no reallocation of wealth is possible which would leave one market maker better off and none of them worse off in terms of (conditional) expected utilities. It is well-known that mathematically such allocations are most conveniently described as the maximizers of the representative agent’s utility function

\[
r(v, x) \triangleq \sup_{x^1 + \ldots + x^M = x} \sum_{m=1}^M v_m u_m(x^m), \quad x \in \mathbb{R},
\]

where \( v \in (0, \infty)^M \) assigns weights to our market makers. For instance, if \( \alpha_0 = (\alpha^m_0)_{m=1,\ldots,M} \) denotes the Pareto optimal initial allocation of wealth among the market makers, there is a vector \( v_0 \in (0, \infty)^M \), unique up to scaling by a positive constant, such that

\[
v^m_0 u'_m(\alpha^m_0) = \partial_x r(v_0, \Sigma_0), \quad m = 1, \ldots, M,
\]

where

\[
\Sigma_0 \triangleq \sum_{m} \alpha^m_0
\]

denotes the market makers’ total initial endowment; see, e.g., Lemma 3.2 in [1].

More generally, if by time \( t = 0, \ldots, T \) the market makers have jointly acquired in addition \( x \in \mathbb{R} \) units of cash and \( q \in \mathbb{R}^J \) securities \( \psi \), the resulting total endowment

\[
\Sigma(x, q) \triangleq \Sigma_0 + x + \langle q, \psi \rangle
\]

will lead to the representative agent’s expected utility

\[
F_t(v, x, q) \triangleq \mathbb{E} [r(v, \Sigma(x, q)) | \mathcal{F}_t].
\]

Theorem 4.1 in [3] shows that for each \( t = 0, \ldots, T \), this is a saddle function of class \( C^2 \) in the variables

\[
(v, x, q) \in A \triangleq (0, \infty)^M \times \mathbb{R} \times \mathbb{R}^J,
\]

if, in addition to Assumption 2.1, we impose the following assumption:
Assumption 2.2. For any \( x \in \mathbb{R} \) and \( q \in \mathbb{R}^J \) there is an allocation \( \beta \in L^0(\mathcal{F}_T, \mathbb{R}^M) \) with total endowment \( \Sigma(x, q) \) such that

\[
\mathbb{E}[u_m(\beta^m)] > -\infty, \quad m = 1, 2, \ldots, M.
\]

Theorem 4.1 in [3] shows furthermore that also

\[
G_t(u, y, q) \equiv \sup_{v \in (0, \infty)^M} \inf_{x \in \mathbb{R}} \{\langle u, v \rangle + xy - F_t(v, x, q)\}
\]

is a twice continuously differentiable saddle function of the variables

\[
(u, v, q) \in B \equiv (-\infty, 0)^M \times (0, \infty) \times \mathbb{R}^J
\]

and that \( G_t \) and \( F_t \) are conjugate in the sense that conversely

\[
F_t(v, x, q) = \sup_{v \in (0, \infty)^M} \inf_{x \in \mathbb{R}} \{\langle u, v \rangle + xy - G_t(u, y, q)\}, \quad (v, x, q) \in A.
\]

In the sequel, we shall study how a single large investor can trade with the market makers in order to hedge against a liability \( H \) with maturity \( T \) by following a judiciously chosen dynamic strategy. So, let \( Q = (Q_t)_{t=1}^T \) denote the predictable positions the large investor will ask our market makers to hold in the marketed securities \( \psi \). We then have to describe the predictable cash balance \( X = (X_t)_{t=1}^T \) the market makers will ask for as compensation. As pointed out in Bank and Kramkov [1, 2] one natural possibility is to have \( X_t \in L^0(\mathcal{F}_{t-1}, \mathbb{R}) \) determined by utility indifference, i.e., by requiring that

\[
(3) \quad U_{t-1}^m = \mathbb{E}[u_m(\alpha_t^m)|\mathcal{F}_{t-1}], \quad m = 1, \ldots, M,
\]

where \( U_{t-1} = (U_{t-1}^m)_{m=1,\ldots,M} \in L^0(\mathcal{F}_{t-1}, (-\infty, 0)^M) \) records the market makers’ conditional expected utilities before the transaction and where \( \alpha_t = (\alpha_t^m)_{m=1,\ldots,M} \in L^0(\mathcal{F}_T, \mathbb{R}^M) \) denotes the (as it turns out) only Pareto optimal allocation of

\[
\Sigma_t \equiv \Sigma(X_t, Q_t) = \Sigma_0 + X_t + \langle Q_t, \psi \rangle
\]

for which the indifference relation (3) holds. In fact, as shown by Theorem 4.1 in [2] this cash balance is given by

\[
(4) \quad X_t = G_{t-1}(U_{t-1}, 1, Q_t)
\]

and the Pareto allocation is determined by the weights

\[
(5) \quad V_t = \partial_u G_{t-1}(U_{t-1}, 1, Q_t).
\]
Moreover, the market makers’ utilities at time $t = 1, \ldots, T$ are given by
\begin{equation}
U_t = \partial_t F_t(V_t, X_t, Q_t) = (\mathbb{E} [u_m(\alpha^m_t) | \mathcal{F}_t])_{m=1, \ldots, M}.
\end{equation}

Hence, for any strategy $Q$ the dynamics of the system are uniquely determined by the initial level of our market makers’ utilities and equations (4), (5), and (6).

It will turn out to be convenient to denote by $U^{s,u,Q} = (U^{s,u,Q}_t)_{t=s, \ldots, T}$, $X^{s,u,Q} = (X^{s,u,Q}_t)_{t=s+1, \ldots, T}$, and $V^{s,u,Q} = (V^{s,u,Q}_t)_{t=s+1, \ldots, T}$ the evolution of the system with (4), (5), and (6) when started at
\begin{equation}
U^s_{s,u,Q} \triangleq u \in L^0(\mathcal{F}_s, (-\infty, 0)^M)
\end{equation}
at some time $s \in \{0, \ldots, T\}$.

### 2.2 The superreplication problem of a large investor

With the market dynamics defined for any predictable strategy, we are now in a position to formulate the large investor’s superreplication problem. As usual we shall say that an initial capital $\pi$ suffices to superreplicate a contingent claim with payoff $H \in L^0(\mathcal{F}_T, \mathbb{R})$ at time $T$ if there is a strategy $Q$ which generates profits or losses $PL^Q_T$ by time $T$ such that
\begin{equation}
H \leq \pi + PL^Q_T.
\end{equation}

By construction of our market model, the large investor’s gains are the market makers’ joint losses and so (7) can be recast as the requirement that
\begin{equation}
X^{0,u_0,Q}_T + \langle Q_T, \psi \rangle \leq \pi - H
\end{equation}
where $u_0 = (\mathbb{E}u_m(\alpha^m_0))_{m=1, \ldots, M}$ denotes the initial expected utility levels for our market makers. So the superreplication price of $H$ turns out to be
\begin{equation}
\pi^H \triangleq \inf \{ \pi \in \mathbb{R} : (8) \text{ holds for some predictable } Q = (Q_t)_{t=1, \ldots, T} \}.
\end{equation}

**Remark 2.3.** As usual, relations such as (7) and (8) are tacitly understood in the $\mathbb{P}$-almost sure sense.

Obviously, if for some initial capital $\pi$ a strategy $Q$ can be found which replicates $H$, i.e., for which we obtain equality in (7) or, equivalently, (8), we have $\pi^H \leq \pi$. In fact, we would then have equality in this relation in arbitrage free, linear, discrete-time models typically considered in Mathematical Finance. It may thus be interesting to note that in the highly nonlinear model under investigation here this may very well not be the case. Indeed,
as pointed out in Remark 3.12 of Bank and Kramkov [2], there can be two strategies $Q$, $Q'$ which, starting from different initial capitals $x < x'$, give the large investor the same terminal wealth $x + PL_{T}^{Q} = x' + PL_{T}^{Q'} \triangleq H$ without this violating absence of arbitrage. As a result, in our nonlinear illiquid financial model pricing $H$ by replication may not be the appropriate concept to investigate. By contrast, the notion of superreplication clearly still makes sense in such a setting but it becomes an issue whether one can ensure existence of a strategy which superreplicates at the superreplication price $\pi^{H}$, i.e., whether the infimum in (9) is actually a minimum. This issue will be addressed by our main results.

3 Main results

The main goal of this paper is to identify readily verifiable conditions on the traded payoffs and the market makers’ risk preferences ensuring that any claim $H$ can be superreplicated starting from the superreplication price. This will be accomplished in Theorem 3.2 and its Corollary 3.3 below. Theorem 3.2 identifies conditions on the tails of both the payoff’s conditional distribution and of the market makers’ utilities that ensure a form of efficient friction to hold in our model. Corollary 3.3 then shows that in models with efficient friction optimal superreplication strategies exist.

3.1 Exponential tails decreasing in time

Our first condition ensures that the market makers’ assessment of the riskiness of the payoff $\psi$ may change sufficiently between any two trading periods. The counterexample given in Section 4.1 shows that a condition of this nature is in fact necessary even when we restrict ourselves to the particularly simple case of a single market maker with exponential utility.

To formulate our condition we introduce the following (partial) ordering relation between any two distributions $\mu$, $\nu$ on $(\mathbb{R}^{J}, \mathcal{B}(\mathbb{R}^{J}))$ with finite exponential moments:

\[
\mu \prec \nu \iff \lim_{|q| \uparrow \infty} \frac{\int \exp (\langle q, x \rangle) \mu(dx)}{\int \exp (\langle q, x \rangle) \nu(dx)} = 0.
\]

Thus $\mu \prec \nu$ will hold if the exponential tails of $\mu$ are dominated by those of $\nu$.

The condition we shall impose on the payoff profile $\psi$ amounts to the requirement that the conditional distributions of $\psi$ along the filtration $(\mathcal{F}_{t})_{t=0,\ldots,T}$,

\[
\nu_{t} \triangleq \mathbb{P} [\psi \in \cdot | \mathcal{F}_{t}], \quad t = 0,\ldots,T.
\]
have the potential to decrease at any time in the sense that

\[ \mathbb{P}[\nu_t < \nu_{t-1} | \mathcal{F}_{t-1}] > 0 \text{ for all } t = 1, \ldots, T. \]  

In Section 5 below we shall verify (12) if \( \psi \) is the value at time \( T \) of a Brownian motion or even of a Lévy process (monitored at discrete points in time). Similarly, one can actually consider terminal values of affine stock price models such as the Barndorff-Nielsen-Shepard style model presented in the same section.

### 3.2 Asymptotic risk aversion

The second condition we have to impose focuses on the market makers’ preferences. It essentially amounts to the requirement that their absolute risk aversion at \(-\infty\) stabilizes at a higher level than at \(+\infty\). For a generic strictly concave, increasing utility function \( u \in C^2(\mathbb{R}) \) this amounts to the condition that the absolute risk aversion \( a(x) \triangleq -u''(x)/u'(x) \) satisfies

\[ \int_{-\infty}^{0} |\bar{a} - a(x)| \, dx + \int_{0}^{\infty} |a - a(x)| \, dx < \infty \text{ for some } 0 < a \leq \bar{a} < \infty. \]  

It is easy to see that utility functions which are mixtures of exponential utilities of the form

\[ u(x) \triangleq -\int_{a}^{\bar{a}} \exp(-ax) \, \Upsilon(da), \quad x \in \mathbb{R}, \]  

satisfy condition (13), e.g., if the finite Borel measure \( \Upsilon \) charges both \( a \) and \( \bar{a} \). Note that the sup-convolution describing the representative agent’s utility (2) will inherit this property when each market maker’s utility satisfies it; see Lemma 3.5 below.

### 3.3 Efficient friction and existence of superreplication strategies

In their investigation of the superreplication problem under proportional transaction costs, Kabanov et al. \[18\] introduced a notion of efficient friction which ensured, essentially, that trading incurs costs. We adopt this idea and terminology for the purposes of our nonlinear model in the following definition.

**Definition 3.1.** A financial model in the framework \[1\] exhibits efficient friction, if for any time \( t = 1, \ldots, T \), for any choice of utility levels \( u^n \in \mathcal{F}_{t-1} \)
with \(-\infty < \inf_{m,n} u_m^n \leq \sup_{m,n} u_m^n < 0\), and for any sequence of strategies \(Q^n\) such that \(\{\lim_n |Q^n_t| = +\infty\}\) has positive probability, also the large investor’s losses \(X_{T}^{-1,u^n,Q^n} + \langle Q^n_T, \psi \rangle\) converge to \(+\infty\) in probability on a set with positive probability.

With this notion at hand, we are now in a position to state the main result of this paper which we will prove in Section 3.4:

**Theorem 3.2.** Let Assumptions 2.1 and 2.2 hold and assume that the market makers’ total initial endowment is of the form \(\Sigma_0 = \tilde{\Sigma}_0 + \langle q_0, \psi \rangle\) for some bounded random variable \(\tilde{\Sigma}_0\). Then our model exhibits efficient friction, if \(\psi\) exhibits potentially decreasing exponential tails in the sense that condition (12) holds and if, in addition, our market makers’ risk aversions stabilize at a higher level at \(-\infty\) than at \(+\infty\) in the sense that they satisfy (13) for constants \(0 < a_m \leq \pi^n < \infty\), \(m = 1, \ldots, M\).

As a corollary, let us note that indeed efficient friction ensures the existence of optimal superreplicating strategies.

**Corollary 3.3.** Let Assumptions 2.1 and 2.2 hold. If our model exhibits efficient friction, any contingent claim \(H\) can be superreplicated by an investment strategy \(Q^H\) with minimal initial capital \(\pi = \pi^H\) as in (9).

**Proof.** Recalling the convention that \(\inf \emptyset = +\infty\), we can assume that there are finite \(\pi^n\) and strategies \(Q^n\) such that

\[
\pi^n \downarrow \pi^H \quad \text{and} \quad X_{T}^{0,u_0,Q^n} + \langle Q^n_T, \psi \rangle \leq \pi^n - H, \text{a.s.} \quad n = 1, 2, \ldots.
\]

In particular, we have that the large investor’s losses are bounded from above uniformly in \(n\):

\[
\sup_{n=1,2,\ldots} \left\{ X_{T}^{0,u_0,Q^n} + \langle Q^n_T, \psi \rangle \right\} < \infty \text{ a.s.}
\]

We shall proceed inductively to construct a limiting strategy \(Q^H\) which superreplicates \(H\) starting with the minimal initial capital \(\pi^H\). In fact, for \(t = 1\) we can apply the efficient friction property (with \(u^n \triangleq u_0\), \(n = 1, 2, \ldots\)) to obtain that (15) rules out that \(|Q^n_T|\) explodes to \(+\infty\) with positive probability along any subsequence \(n'\). Hence, \(\sup_n |Q^n_T| < \infty\) and so we can choose a subsequence, again denoted by \(Q^n\), such that \(Q^n_T \triangleq \lim_n Q^n_T\) exists in \(L^0(\mathcal{F}_0, \mathbb{R})\) (see, e.g., Lemma 1.64 in Föllmer and Schied [14]). The continuity of our system dynamics specifying cash balances \(X_0^{0,u,Q^n}\) (cf. (4)), weights \(V_0^{0,u,Q^n}\) (cf. (5)), and utilities \(U_0^{0,u,Q^n}\) (cf. (6)) then ensures that the induced utilities \(U_1^{0,u,Q^n}\) are bounded away from zero and \(-\infty\). We
thus can now let $u^n \triangleq U_1^{0,u,Q^n}$, $n = 1, 2, \ldots$, and, observing that then 
$X_T^{n,0}Q^n = X_T^{1,u,Q^n}$, proceed successively in just the same way to construct a limiting $Q^n_H \in L^0(\mathcal{F}_t, \mathbb{R})$ for $t = 2$. The same reasoning applies for $t = 3, \ldots, T$.

Since along the way the superreplication property (14) is preserved for all the successive subsequences, the limiting strategy $Q^n_H$ will also superreplicate but only need the minimal initial capital $\pi^n_H$. The proof is accomplished. □

3.4 Proof of Theorem 3.2

The proof of Theorem 3.2 relies on three auxiliary results. The first shows that when the market makers’ risk aversion stabilize asymptotically the same is true for the representative agent’s risk aversion:

**Lemma 3.4.** Assume the market makers have utility functions $u_m$, $m = 1, \ldots, M$ satisfying Assumption 2.1 and suppose in addition that risk aversions $a_m(x) \triangleq -u_m''(x)/u_m'(x)$ stabilize at $\pm \infty$ in the sense that (13) holds for $u \triangleq u_m$ with respective constants $a = a_m$, $\bar{a} = \bar{a}_m$, $m = 1, \ldots, M$.

Then for any choice of $v \in (0, \infty)^M$ also the representative agent’s utility $r(v, \cdot)$ of (2) exhibits stabilizing risk aversion in the same sense namely at levels $\underline{a}$, $\overline{a}$ given by

$$\frac{1}{\underline{a}} = \sum_{m=1}^M \frac{1}{a_m} \quad \text{and} \quad \frac{1}{\overline{a}} = \sum_{m=1}^M \frac{1}{\bar{a}_m}.$$

**Proof.** Let $a_r(v, x) \triangleq -\partial_x^2 r(v, x)/\partial x r(v, x)$, $x \in \mathbb{R}$, denote the representative agent’s absolute risk aversion. It is straightforward to check (see, e.g., [1]) that

$$a_r(v, x) = \frac{1}{\sum_{m=1}^M \frac{1}{a_m(x_m(v, x))}},$$

where $(\hat{x}_m(v, x))_{m=1,\ldots,M}$ denotes the unique point in $\mathbb{R}^M$ at which the sup in (2) is attained.

Observing that the function $f(a_1, a_2, \ldots, a_m) \triangleq \frac{1}{\sum_{m=1}^M \frac{1}{a_m}}$ is Lipschitz continuous in $(a_m)_{m=1,\ldots,M} \in (0, \infty)^M$ with constant 1 for the $\| \cdot \|_1$-norm on $\mathbb{R}^M$ we first note that

$$|a_r(v, x) - \underline{a}| = \left| f(a_1(\hat{x}_1), \ldots, a_M(\hat{x}_M)) - f(\underline{a}_1, \ldots, \underline{a}_M) \right| \leq \sum_{m=1}^M |a_m(\hat{x}_m) - \underline{a}_m|.$$

From, e.g., Lemma 3.2 in [3] we obtain

$$\partial_x \hat{x}_m(v, x) = \frac{1/a_m(\hat{x}_m(v, x))}{\sum_{k=1}^M 1/a_k(\hat{x}_k(v, x))}, \quad m = 1, \ldots, M,$$
which is uniformly bounded away from 0 and 1. Hence,
\[
\int_0^\infty |a_m(\tilde{m}(v,x)) - a_m| \, dx = \int_0^\infty |a_m(y) - a_m| (\partial_x \tilde{m}(v,y))^{-1} \, dy < \infty
\]
since \(a_m\) satisfies condition (13). Together with the above Lipschitz estimate this yields that (13) holds for \(a_r(v,x)\) and \(\tilde{a}_r\). The argument for stabilization at \(-\infty\) is the same.

The second lemma will allow us to compare a utility function satisfying condition (13) with a sum of two exponential utilities:

**Lemma 3.5.** If a strictly concave, increasing utility function \(u \in C^2(\mathbb{R})\) with \(\lim_{x \to \infty} u(x) = 0\) satisfies condition (13), then there are constants \(C_1, C_2 > 0\) such that
\[
C_1 \left(-e^{-\alpha x} - e^{-\overline{\alpha} x}\right) \leq u(x) \leq C_2 \left(-e^{-\alpha x} - e^{-\overline{\alpha} x}\right), \quad x \in \mathbb{R}
\]
where \(\alpha\) and \(\overline{\alpha}\) are the same constants as in (13).

**Proof.** First, we note that
\[
(17) \quad u(x) = -u'(0) \int_x^\infty \exp \left(-\int_y^\infty a(z) \, dz\right) \, dy, \quad x \in \mathbb{R}.
\]
Hence, by L’Hopital’s rule,
\[
\lim_{x \to \infty} \frac{u(x)}{-e^{-\alpha x}} = \lim_{x \to \infty} \frac{u'(0) \int_x^\infty \exp \left(-\int_y^\infty a(z) \, dz\right) \, dy}{e^{-\alpha x}}
\]
\[
= \lim_{x \to \infty} \frac{u'(0) \exp \left(-\int_0^x a(z) \, dz\right)}{\alpha e^{-\alpha x}}
\]
\[
= \frac{u'(0)}{\alpha} \exp \left(\int_0^\infty (a - a(z)) \, dz\right) \in (0, \infty)
\]
due to the integrability condition (13). It follows that
\[
(18) \quad -c_2 e^{-\alpha x} \leq u(x) \leq -c_1 e^{-\alpha x}, \quad x > 0,
\]
for positive constants \(c_1\) and \(c_2\). In conjunction with \(0 < \alpha \leq \overline{\alpha}\) this yields
\[
\limsup_{x \to \infty} \frac{u(x)}{-e^{-\alpha x} - e^{-\overline{\alpha} x}} \leq \limsup_{x \to \infty} \frac{c_2 e^{-\alpha x}}{e^{-\alpha x} + e^{-\overline{\alpha} x}} < \infty
\]
as well as
\[
\liminf_{x \to \infty} \frac{u(x)}{-e^{-\alpha x} - e^{-\overline{\alpha} x}} \geq \liminf_{x \to \infty} \frac{c_1 e^{-\alpha x}}{e^{-\alpha x} + e^{-\overline{\alpha} x}} > 0.
\]
This shows that there exists \(C_2 > 0\) such that the right estimate of (16) holds true. The argument for finding \(C_1 > 0\) such that the left estimate holds is completely analogous. \(\square\)
The comparison with exponential utilities is also at the heart of the following technical result:

**Lemma 3.6.** Let \( u \in C^2(\mathbb{R}) \) be a strictly concave, increasing utility function with \( \lim_{x \to \pm \infty} u(x) = 0 \) whose risk aversion is bounded away from zero and infinity and satisfies condition (13) for some constants \( 0 < a \leq \overline{a} < \infty \). Define \( f(x) \equiv x - (-x)^{a/\overline{a}} \) for \( x \leq 0 \).

There exists a positive constant \( C > 0 \) depending only on the utility function \( u \) such that

\[
E[u(x + \Sigma) | \mathcal{F}_t] \geq C f \left( E[u(x + \Sigma) | \mathcal{F}_{t-1}] \frac{E[\exp(-a\Sigma) | \mathcal{F}_t]}{E[\exp(-a\Sigma) | \mathcal{F}_{t-1}]} \right)
\]

for any \( x \in \mathbb{R} \), any \( t = 1, \ldots, T \) and any random variable \( \Sigma \) with finite exponential moments \( E[\exp(-a\Sigma)] + E[\exp(-a\Sigma)] < \infty \).

**Proof.** Let \( t = 1, 2, \ldots, T \) be arbitrary. For given \( \Sigma \) with the above finite exponential moments and for any fixed \( x \in \mathbb{R} \), the growth estimate (16) of Lemma 3.5 allows us to define the \( \mathcal{F}_{t-1} \)-measurable random variable \( \xi \) by

\[
E[u(x + \Sigma) | \mathcal{F}_{t-1}] = C_2 E[-\exp(-a(\xi + \Sigma)) | \mathcal{F}_{t-1}],
\]

where \( C_2 > 0 \) is any constant such that (16) holds. By the same estimate we obtain

\[
C_2 E[-\exp(-a(\xi + \Sigma)) | \mathcal{F}_{t-1}] = E[u(x + \Sigma) | \mathcal{F}_{t-1}]
\]

\[
\leq C_2 E[-\exp(-a(\xi + \Sigma)) | \mathcal{F}_{t-1}],
\]

which implies that \( \xi \leq x \). We use this in the other part of estimate (16) to deduce

\[
E[u(x + \Sigma) | \mathcal{F}_t]
\]

\[
\geq C_1 E[-\exp(-a(\xi + \Sigma)) - \exp(-a(x + \Sigma)) | \mathcal{F}_t]
\]

\[
\geq C_1 E[-\exp(-a(\xi + \Sigma)) - \exp(-a(\xi + \Sigma)) | \mathcal{F}_t]
\]

\[
= C_1 \{ -\exp(-a\xi) E[\exp(-a\Sigma) | \mathcal{F}_t] - \exp(-a\xi) E[\exp(-a\Sigma) | \mathcal{F}_t] \}
\]

By definition of \( \xi \) we can write

\[
-\exp(-a\xi) = \frac{1}{C_2} \frac{E[u(x + \Sigma) | \mathcal{F}_{t-1}]}{E[\exp(-a\Sigma) | \mathcal{F}_{t-1}]}.
\]

This yields

\[
E[u(x + \Sigma) | \mathcal{F}_t] \geq \frac{C_1}{C_2} E[u(x + \Sigma) | \mathcal{F}_{t-1}] \frac{E[\exp(-a\Sigma) | \mathcal{F}_t]}{E[\exp(-a\Sigma) | \mathcal{F}_{t-1}]}
\]

\[
- C_1 \left( \frac{1}{C_2} \frac{E[u(x + \Sigma) | \mathcal{F}_{t-1}]}{E[\exp(-a\Sigma) | \mathcal{F}_{t-1}]} \right)^{a/\overline{a}} \frac{E[\exp(-a\Sigma) | \mathcal{F}_{t-1}]}{E[\exp(-a\Sigma) | \mathcal{F}_t]}
\]
and after applying Jensen’s inequality with the concave function $x \mapsto x^{2a/\pi}$ we see that there is a constant $C > 0$ such that

$$
\mathbb{E} \left[ u(x + \Sigma) \mid \mathcal{F}_t \right] \geq C \left\{ \frac{\mathbb{E} \left[ \exp(-\pi \Sigma) \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ \exp(-a \Sigma) \mid \mathcal{F}_{t-1} \right]} \right. \\
\left. - \left( - \frac{\mathbb{E} \left[ u(x + \Sigma) \mid \mathcal{F}_{t-1} \right]}{\mathbb{E} \left[ \exp(-a \Sigma) \mid \mathcal{F}_{t-1} \right]} \right)^{2a/\pi} \right\} \\
= Cf \left( \frac{\mathbb{E} \left[ \exp(-\pi \Sigma) \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ \exp(-a \Sigma) \mid \mathcal{F}_{t-1} \right]} \right)
$$

where $f$ is as defined in the assertion of our lemma.

We are finally in a position to give the

**Proof of Theorem 3.2.** Let $t \in \{1, \ldots, T\}$ and $Q^n, u^n, n = 1, 2, \ldots$ be as in Definition 3.1. For notational simplicity let us denote

$$U^n_s \triangleq U^{t-1, u^n, Q^n}_s, X^n_s \triangleq X^{t-1, u^n, Q^n}_s, V^n_s \triangleq V^{t-1, u^n, Q^n}_s, s = t, \ldots, T.$$  

We first recall that by Theorem 4.2 in [3] $G_{t-1}$ is contained in $\tilde{G}^2(c)$, a class of saddle functions introduced there. Property (G7) of these special saddle functions amounts in our context to

$$
\frac{1}{c} \leq -u^{m,n} V^{m,n}_t \leq c
$$

where $c$ is the bound on the market makers’ risk aversions occurring in (1). Thus, since by assumption $(-u^{m,n})_{n=1,2,\ldots}$ is bounded away from zero and $\infty$, so are the weights $V^{m,n}_t, n = 1, 2, \ldots$ for any $m = 1, \ldots, M$. As a consequence $\underline{V}_{t}^m \triangleq \inf_{n=1,2,\ldots} V^{m,n}_t > 0$ and $\overline{V}_{t}^m \triangleq \sup_{n=1,2,\ldots} V^{m,n}_t < \infty$ yield finite $\mathcal{F}_{t-1}$-measurable bounds on the initial weight of each market maker $m = 1, \ldots, M$.

Let us next argue that

$$
0 \geq \langle V^n_t, U^n_t \rangle \geq \mathbb{E} \left[ r(V^n_t, \Sigma(X^n_t, Q^n_t)) \mid \mathcal{F}_t \right] \\
\geq \mathbb{E} \left[ r(V^n_t, \Sigma(X^n_t, Q^n_t)) \mid \mathcal{F}_t \right] \\
\geq C_{\underline{V} t} f \left( \frac{\mathbb{E} \left[ \exp(-\pi \Sigma(0, Q^n_t)) \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ \exp(-a \Sigma(0, Q^n_t)) \mid \mathcal{F}_{t-1} \right]} \right)
$$

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for some random variable $C_{\mathcal{T}_t} > 0$. Indeed, the estimates in (21) are immediate from $0 < \sum_{m} V_{m, n}^t \leq V_{t, n}^m$ and $U_{r, n}^m < 0$, $m = 1, \ldots, M$. The identity in (21) holds because, by (6), $U_{r, n}^m$ is the vector of $\mathcal{F}_t$-conditional expected utilities that our market makers obtain when at time $t - 1$ the Pareto allocation of $\Sigma(X_{t, n}^n, Q_{t, n}^n)$ is formed given the weights $V_{t, n}^m$. Estimate (22) follows because the representative agent’s utility function $r(v, x)$ is decreasing in the weights $v$. Finally, (23) follows because our Lemmas 3.4 and 3.5 above allow us to apply Lemma 3.6 to the (random) utility function $u \triangleq r(\mathcal{V}_t, .)$ which provides us with the required random variable $C_{\mathcal{T}_t} > 0$.

Recalling that $\Sigma_0 = \Sigma_0 + \langle q_0, \psi \rangle$ with $q_0 \in \mathbb{R}^J$ and bounded $\Sigma_0$, we have the following estimate for the ratio in (23):

$$\frac{\mathbb{E}[\exp(-\langle \Sigma(0, Q_{t, n}^n) \rangle|\mathcal{F}_t)]}{\mathbb{E}[\exp(-\langle \Sigma(0, Q_{t, n}^n) \rangle)|\mathcal{F}_{t-1}]} \leq e^{-\langle \Sigma_0 \rangle|\mathcal{F}_0|\|0\|} \frac{\mathbb{E}[\exp(-\langle \Sigma(0, Q_{t, n}^n) \rangle)|\mathcal{F}_t]}{\mathbb{E}[\exp(-\langle \Sigma(0, Q_{t, n}^n) \rangle)|\mathcal{F}_{t-1}]}.$$  

By our assumption of decreasing exponential tails the latter ratio and, thus, also the former ratio converge to zero on $D_t \triangleq \{\nu_{t-1} < \nu_t\} \cap \{\lim_n |Q_{n, t}^n| = \infty\}$. This is a set with positive probability

$$\mathbb{P}[D_t] = \mathbb{E}\left[\mathbb{1}_{(\lim_n |Q_{n, t}^n| = \infty)} \mathbb{P}[\nu_{t-1} < \nu_t|\mathcal{F}_{t-1}]\right],$$

which is strictly positive by (12). We shall argue below that

$$0 \geq \mathbb{E}\left[r(\mathcal{V}_t, \Sigma(X_{t, n}^n, Q_{t, n}^n))|\mathcal{F}_{t-1}\right] \geq \langle \mathcal{V}_t, u^n \rangle, \ n = 1, 2, \ldots,$$

and so the first conditional expectation in (23) is bounded in $n$ by assumption on $(u^n)_{n=1,2,\ldots}$. As a consequence, the right side in (23) converges to $f(0) = 0$ on $D_t$ when $n \uparrow \infty$. This implies $\langle \mathcal{V}_t, U_{t, n}^m \rangle \to 0$ on this set. Since $\sum_{m} V_{m, n}^t > 0$, $m = 1, \ldots, M$, this yields that actually $U_{t, n}^m \to 0$ on $D_t$. Because $U_{t, n}^m$ is a martingale, we have $U_{t, n}^m = \mathbb{E}[U_{t, n}^m|\mathcal{F}_t]$ and so $U_{t, n}^m \to 0$ in probability on $D_t$. This, however, is equivalent to $X_{t, n}^m + \langle Q_{t, n}^m, \psi \rangle = \sum_{m=1}^{M} u_{m}^{-1}(U_{t, n}^{m, n}) - \Sigma_0 \to \infty$ in probability on $D_t$ which establishes the asserted efficient friction.

It remains to verify (24). For this, note that the Pareto allocation $\alpha_{t, n}^m$ of $\Sigma(X_{t, n}^n, Q_{t, n}^n)$ with weights $V_{t, n}^m$ gives us

$$r(\mathcal{V}_t, \Sigma(X_{t, n}^n, Q_{t, n}^n)) \geq \sum_{m=1}^{M} \mathcal{V}_{t, n}^m u_m(\alpha_{t, n}^{m, n}), \ n = 1, 2, \ldots.$$  

Our model’s utility indifference principle (3) yields

$$\mathbb{E}[u_m(\alpha_{t, n}^{m, n})|\mathcal{F}_{t-1}] = U_{t, n}^{m, n} = u_{m, n}, \ m = 1, \ldots, M, \ n = 1, 2, \ldots,$$

and so (24) follows by taking an $\mathcal{F}_{t-1}$-conditional expectation in the preceding estimate and recalling that $\mathcal{V}_t$ is $\mathcal{F}_{t-1}$-measurable. \qed
4 Ramifications

In this section we collect a few supplementary results and illustrations. We first illustrate in Section 4.1 that our tail condition on \( \psi \), (12), is necessary for the existence of optimal superreplicating strategies even in a simple binomial model with two periods and one exponential market maker. We furthermore show in Section 4.2 that under more stringent assumptions on \( \psi \), efficient friction holds even without the requirement on the market makers’ risk aversions. Section 4.3 is finally concerned with the special case of a general multi-period binomial model.

4.1 A binomial model where the superreplication price is not attained

We consider a two-period model with one asset and one market maker, where
\[ \Omega \triangleq \{-1, +1\}^2. \]
For \( \omega = (y_1, y_2) \in \Omega \), let \( Y_t(\omega) \triangleq y_t \) be the projection of \( \omega \) to its \( t \)-th component, \( t = 1, 2 \). The filtration \( (\mathcal{F}_t)_{t=0,1,2} \) is generated by \( Y = (Y_t)_{t=1,2} \). The distribution of \( Y \) is determined by
\[ P[Y_1 = +1] \triangleq p_1, \quad P[Y_2 = +1 | Y_1 = +1] \triangleq p_2, \quad P[Y_2 = +1 | Y_1 = -1] \triangleq p_3, \]
where \( p_1, p_2, p_3 \in (0, 1) \) with \( p_2 \neq p_3 \). Moreover, the single market maker’s utility function is given by \( u(x) \triangleq -e^{-\alpha x} \) for \( \alpha > 0 \), the initial endowment is \( \Sigma_0 \triangleq 0 \) and \( \psi \) is determined by
\[ \psi(Y_1, +1) \triangleq \psi^u, \quad \psi(Y_1, -1) \triangleq \psi^d \text{ with } \psi^u > \psi^d. \]

The specific form of exponential utility and predictability of the cash balance \( X^Q \) of the strategy \( Q \) yield that
\[ -1 = u(0) = \mathbb{E}[u(X_1^Q + Q_1 \psi)] = -\mathbb{E}[e^{-\alpha(X_1^Q + Q_1 \psi)}] = -e^{-\alpha X_1^Q} \mathbb{E}[e^{\alpha Q_1 \psi}]. \]
Thus
\[ X_1^Q = \frac{1}{\alpha} \log \left( \mathbb{E}[e^{\alpha Q_1 \psi}] \right) \]
and the utility level \( U_1^Q \) at time 1 of the strategy \( Q \) satisfies
\[ U_1^Q = -\frac{\mathbb{E}[e^{-\alpha Q_1 \psi} | \mathcal{F}_1]}{\mathbb{E}[e^{\alpha Q_1 \psi}]}. \]

By direct calculation, we find that
\[ U_1^Q = -\frac{A e^{-\alpha Q_1 (\psi^u - \psi^d)} + (1 - A)}{B e^{-\alpha Q_1 (\psi^u - \psi^d)} + (1 - B)} \text{ with } A \triangleq \begin{cases} p_2 & \text{if } Y_1 = +1 \\ p_3 & \text{if } Y_1 = -1 \end{cases} \]
and $B \triangleq p_1 p_2 + (1 - p_1) p_3$ in our model.

We consider taking ever larger positions at the first time step and liquidating our positions in the second time step. So let $(Q^n_t)_{n \in \mathbb{N}}$ be a sequence such that $Q^n_t \to \infty$ as $n \to \infty$. Then

$$U^n_1 \triangleq U^n_1 Q^n_1 \to U_1 = -\frac{1-A}{1-B} \text{ as } n \to \infty.$$ 

Since we liquidate our position after the first period, $Q^n_t = 0$ for every $n = 1, 2, \ldots$. This amounts to $U^n_1 = -e^{-\alpha X^n_2}$ and thus

$$\lim_n X^n_2 = X_2 \triangleq \frac{1}{\alpha} \log \left( -\frac{1}{U_1} \right) = \frac{1}{\alpha} \log \left( \frac{1-B}{1-A} \right),$$

because of predictability of $X$.

Next we shall show that the superreplicating cost of the claim $H \triangleq -X_2 \in \mathcal{F}_1$ is $\pi^H = 0$ and that there exists no strategy $Q$ that superreplicates with initial capital $\pi^H = 0$. Let us first show $\pi^H \leq 0$. Since $-X^n_2 \to H$, there exists a sequence of real numbers $\varepsilon_n \downarrow 0$ as $n \uparrow \infty$ and

$$\varepsilon_n - X^n_2 \geq H, \, n = 1, 2, \ldots.$$ 

Moreover, the strategy $(Q^n_t)_{t=1,2} = (Q^n_1, 0)$ yields the cash balance $X^n_2$ for all $n = 1, 2, \ldots$. Therefore, $\varepsilon_n \geq \pi^H$ for every such $n$ and by sending $\varepsilon_n \downarrow 0$, we obtain that $\pi^H \leq 0$. Thus if $\pi^H$ is not zero, then there exists $\varepsilon > 0$ and a superreplicating strategy $\tilde{Q}$ with cash balance $\tilde{X}$ such that

$$-\varepsilon - \tilde{X}_2 - \tilde{Q}_2 \psi \geq H.$$ 

Hence,

$$\tilde{X}_2 + \tilde{Q}_2 \psi < \varepsilon + \tilde{X}_2 + \tilde{Q}_2 \psi \leq -H.$$ 

Moreover, by construction of $H$, we have $\mathbb{E}[u(-H)] = u(0)$. These two observations lead to the contradiction

$$u(0) = \mathbb{E}[u(\tilde{X}_2 + \tilde{Q}_2 \psi)] < \mathbb{E}[u(-H)] = u(0).$$

It remains to show that $\pi^H = 0$ is not attained by any strategy $\tilde{Q}$. If $\tilde{Q}$ is a strategy that superreplicates with initial capital $\pi^H = 0$ and cash balance $\tilde{X}$, i.e.

$$-\tilde{X}_2 - \tilde{Q}_2 \psi \geq H;$$

then

$$\mathbb{E}[u(\tilde{X}_2 + \tilde{Q}_2 \psi)] = u(0) = \mathbb{E}[u(-H)].$$
yields that $\tilde{X}_2 + \tilde{Q}_2 \psi = -H = X_2$. Then since $X_2$ is $\mathcal{F}_1$ measurable,

$$E\left[u\left(\tilde{X}_2 + \tilde{Q}_2 \psi\right) | \mathcal{F}_1\right] = E[u(X_2) | \mathcal{F}_1] = u(X_2).$$

So $u(X_2)$ is the utility level at time 1 of the strategy $\tilde{Q}$, i.e. $U^{\tilde{Q}}_1 = u(X_2)$. However, by differentiating equation (25) we obtain

$$\frac{\partial}{\partial Q_1} U^{\tilde{Q}}_1 = \alpha (\psi^u - \psi^d) e^{-\alpha Q_1 (\psi^u - \psi^d)} \frac{A - B}{\left(B e^{-\alpha Q_1 (\psi^u - \psi^d)} + (1 - B)\right)^2},$$

where $A$ and $B$ are given as in (25). Without loss of generality assume that $p_2 > p_3$. Then, $U^{\tilde{Q}}_1$ is strictly increasing in $Q_1$ on the set $\{Y_1 = +1\}$ and strictly decreasing on $\{Y_1 = -1\}$. Since the utility level $u(X_2)$ is the limiting value of positions $Q^n_1$ tending to $+\infty$ and $U^{\tilde{Q}}_1$ is strictly monotone, there cannot exist a finite $\tilde{Q}_1$ with utility level $u(X_2)$, a contradiction.

### 4.2 Efficient friction when extremal payoffs may change any time

In this section we will consider the special case where only one security is marketed and so $\psi$ is a real-valued random variable. Let us denote by $\underline{\psi}$ its $\mathcal{F}_t$-measurable essential infimum of $\psi$ and by $\overline{\psi}$ its $\mathcal{F}_t$-measurable essential supremum, i.e.

$$\underline{\psi} \triangleq \text{ess sup} \{ \zeta \in \mathcal{F}_t : \zeta \leq \psi \} \quad \text{and} \quad \overline{\psi} \triangleq \text{ess inf} \{ \zeta \in \mathcal{F}_t : \zeta \geq \psi \}.$$

**Proposition 1.** A single security $\psi \in L^0(\mathbb{R})$ exhibits decreasing exponential tails in the sense of condition (12) if its conditional infima and suprema are potentially strictly monotone, i.e., if

$$P\left[\psi_{t-1} < \underline{\psi}, \overline{\psi}^{-1} > \overline{\psi} \mid \mathcal{F}_{t-1}\right] > 0, \quad t = 1, \ldots, T.$$ (26)

Moreover, if $|\Omega| < \infty$ or, more generally, if

$$P\left[\psi = \underline{\psi} \mid \mathcal{F}_t\right] > 0 \quad \text{and} \quad P\left[\psi = \overline{\psi} \mid \mathcal{F}_t\right] > 0, \quad t = 0, \ldots, T,$$ (27)

then conditions (12) and (26) are even equivalent.

**Proof.** Since

$$A \triangleq \left\{\psi_{t-1} < \psi, \overline{\psi}^{-1} > \psi\right\} = \bigcup_{\epsilon > 0} \left\{\psi_{t-1} + \epsilon < \underline{\psi}, \overline{\psi}^{-1} - \epsilon > \overline{\psi}\right\}$$

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the implication ‘(26) \implies (12)’ will follow if we show
\[
\left\{ \bar{\psi}_{t-1} + \varepsilon < \bar{\psi}, \ \bar{\psi} - \varepsilon > \bar{\psi} \right\} \subseteq \left\{ \lim_{|q| \to \infty} \frac{\mathbb{E}[e^{q\psi} | \mathcal{F}_t]}{\mathbb{E}[e^{q\psi} | \mathcal{F}_{t-1}]} = 0 \right\}
\]
for any \( \varepsilon > 0 \). To this end, note that because \( \bar{\psi}' \) is \( \mathcal{F}_{t-1} \)-measurable, we obtain the following estimate for any \( q > 0 \):
\[
\mathbb{E}[e^{q\psi} | \mathcal{F}_{t-1}] \geq \mathbb{E}\left[e^{q\psi} \mathbf{1}\{\psi > \bar{\psi}' - \varepsilon\} | \mathcal{F}_{t-1}\right]
\geq \exp\left(q\left(\bar{\psi}' - \varepsilon\right)\right) \mathbb{P}\left(\psi > \bar{\psi}' - \varepsilon | \mathcal{F}_{t-1}\right).
\]
Now \( \mathbb{P}\left(\psi > \bar{\psi}' - \varepsilon | \mathcal{F}_{t-1}\right) > 0 \) by definition of the conditional essential supremum \( \bar{\psi}' \).
So
\[
0 \leq \lim_{q \to \infty} \frac{\mathbb{E}[e^{q\psi} | \mathcal{F}_t]}{\mathbb{E}[e^{q\psi} | \mathcal{F}_{t-1}]} \leq \lim_{q \to \infty} \frac{\exp\left(q\left(\bar{\psi} - \bar{\psi}' + \varepsilon\right)\right)}{\mathbb{P}\left(\psi < \bar{\psi}' - \varepsilon | \mathcal{F}_{t-1}\right)} = 0
\]
on the set \( \left\{ \bar{\psi}_{t-1} + \varepsilon < \bar{\psi}, \ \bar{\psi}' - \varepsilon > \bar{\psi} \right\} \).

Similarly for \( q < 0 \), we use the estimate
\[
\mathbb{E}[e^{q\psi} | \mathcal{F}_{t-1}] \geq \mathbb{E}\left[e^{q\psi} \mathbf{1}\{\psi < \bar{\psi}' + \varepsilon\} | \mathcal{F}_{t-1}\right]
\geq \exp\left(q\left(\bar{\psi}' + \varepsilon\right)\right) \mathbb{P}\left(\psi < \bar{\psi}' + \varepsilon | \mathcal{F}_{t-1}\right)
\]
to deduce that
\[
0 \leq \lim_{q \to -\infty} \frac{\mathbb{E}[e^{q\psi} | \mathcal{F}_t]}{\mathbb{E}[e^{q\psi} | \mathcal{F}_{t-1}]} \leq \lim_{q \to -\infty} \frac{\exp\left(q\left(\bar{\psi} - \bar{\psi}' - \varepsilon\right)\right)}{\mathbb{P}\left(\psi < \bar{\psi}' + \varepsilon | \mathcal{F}_{t-1}\right)} = 0
\]
on the set \( \left\{ \bar{\psi}_{t-1} + \varepsilon < \bar{\psi}, \ \bar{\psi}' - \varepsilon > \bar{\psi} \right\} \), because \( \mathbb{P}\left(\psi < \bar{\psi}' + \varepsilon | \mathcal{F}_{t-1}\right) > 0 \). This establishes the first claim.
For the second claim, we note that given \((27)\) holds we obtain by dominated convergence

\[
\lim_{q \downarrow -\infty} \frac{\mathbb{E} \left[ e^{q\psi} \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ e^{q\psi} \mid \mathcal{F}_{t-1} \right]} = \lim_{q \downarrow -\infty} \frac{\mathbb{E} \left[ e^{q\psi} \left( 1_{\psi=\bar{\psi}_t} + 1_{\psi>\bar{\psi}_t} \right) \right]}{\mathbb{E} \left[ e^{q\psi} \left( 1_{\psi=\bar{\psi}_{t-1}} + 1_{\psi>\bar{\psi}_{t-1}} \right) \right]}
\]

\[
= \lim_{q \downarrow -\infty} \frac{e^{q\bar{\psi}_t} \left\{ \mathbb{P} \left[ \psi = \bar{\psi}_{t-1} \mid \mathcal{F}_t \right] + \mathbb{E} \left[ e^{q(\psi-\bar{\psi}_t)} 1_{\psi>\bar{\psi}_{t-1}} \mid \mathcal{F}_t \right] \right\}}{e^{q\bar{\psi}_{t-1}} \left\{ \mathbb{P} \left[ \psi = \bar{\psi}_{t-1} \mid \mathcal{F}_{t-1} \right] + \mathbb{E} \left[ e^{q(\psi-\bar{\psi}_{t-1})} 1_{\psi>\bar{\psi}_{t-1}} \mid \mathcal{F}_{t-1} \right] \right\}}
\]

\[
= \frac{\mathbb{P} \left[ \psi = \bar{\psi}_{t-1} \mid \mathcal{F}_t \right]}{\mathbb{P} \left[ \psi = \bar{\psi}_{t-1} \mid \mathcal{F}_{t-1} \right]}
\]

and, similarly,

\[
\lim_{q \uparrow +\infty} \frac{\mathbb{E} \left[ e^{q\psi} \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ e^{q\psi} \mid \mathcal{F}_{t-1} \right]} = \frac{\mathbb{P} \left[ \psi = \bar{\psi}_{t-1} \mid \mathcal{F}_t \right]}{\mathbb{P} \left[ \psi = \bar{\psi}_{t-1} \mid \mathcal{F}_{t-1} \right]}
\]

So, if the conditional supremum and infimum both change from time \(t-1\) to \(t\) with positive probability, the above ratios of conditional probabilities are zero along the limits as \(q \downarrow -\infty\) and \(q \uparrow +\infty\). This was to be shown. \(\square\)

Our next result illustrates that our assumption of stabilizing asymptotic risk aversions formulated in Theorem 3.2 is not needed for efficient friction to hold when essential suprema and infima are strictly monotone:

**Theorem 4.1.** Under Assumptions 2.1 and 2.2, a model with a single marketed security \(\psi\) satisfying condition (26) exhibits efficient friction.

**Remark 4.2.** As an application of this theorem, all non-degenerate trinomial and higher monomial models exhibit efficient friction.

**Proof.** We start with the same observations as in the proof of Theorem 3.2 and note that up to and including the estimate (22) all arguments hold true under the assumptions of the present theorem.

It thus suffices to identify, for \(t = 1, \ldots, T\), a set \(D_t \in \mathcal{F}_t\) with positive probability where

\[
\mathbb{E} \left[ r(V_t, \Sigma(X^n_t, Q^n_t)) \mid \mathcal{F}_t \right] \to 0.
\]

It then follows that \((V_t, U^n_t) \to 0\) on this set and, just as in the proof of Theorem 3.2, we can conclude that \(U^n_t \to 0\) and, thus, also \(U^n_T \to 0\) in probability on \(D_t\). As
before this is equivalent to $X^n_t + Q^n_t \psi \to \infty$ in probability on $D_t$, proving the asserted efficient friction.

By working with subsequences we can confine ourselves to the case where \{\lim_n Q^n_t = \pm \infty\} has positive probability. The argument on \{\lim_n Q^n_t = -\infty\} being similar, let us assume that the set \{\lim_n Q^n_t = +\infty\} has positive probability. Clearly, on \{Q^n_t > 0\} we have

$$0 \geq \mathbb{E} \left[ r(V_t, \Sigma(X^n_t, Q^n_t)) \right].$$

By condition (26) we can find an $\varepsilon > 0$ such that $D_t \triangleq \{\lim_n Q^n_t = +\infty\} \cap \{\psi_{t-1} + \varepsilon < \psi_t, \overline{\psi}_{t-1} - \varepsilon > \overline{\psi}_t\}$ has positive probability, since \{\lim_n Q^n_t = +\infty\} is $\mathcal{F}_{t-1}$-measurable and

$$D_t \uparrow \{\lim_n Q^n_t = +\infty\} \cap \{\psi_{t-1} < \psi_t, \overline{\psi}_{t-1} > \overline{\psi}_t\}$$

as $\varepsilon \downarrow 0$. We shall argue in Lemma 4.3 below that

$$\lim_n X^n_t/Q^n_t = -\psi_{t-1}$$

on \{\lim_n Q^n_t = +\infty\}.

On $D_t$, we can thus furthermore find a random variable $N^\varepsilon$ such that for $n > N^\varepsilon$ the above estimation can be continued by

$$\cdots \geq \mathbb{E} \left[ r(V_t, \Sigma_0 + Q^n_t(\psi_t + X^n_t/Q^n_t)) \right]$$

$$\geq \mathbb{E} \left[ r(V_t, \Sigma_0 + Q^n_t(\psi_t - \psi_{t-1} - \varepsilon/2)) \right]$$

$$\geq \mathbb{E} \left[ r(V_t, \Sigma_0 + Q^n_t\varepsilon/2) \right].$$

On $D_t$ the latter expression converges to zero by dominated convergence as required.

The following lemma establishes the asymptotics of cash balances for extreme long and short positions in $\psi$:

**Lemma 4.3.** Under Assumptions 2.1 and 2.2 we have

(28) \quad \lim_n X^n_t/Q^n_t = -\psi_{t-1} \quad \text{on} \quad \{\lim_n Q^n_t = +\infty\}

and

(29) \quad \lim_n X^n_t/Q^n_t = -\overline{\psi}_{t-1} \quad \text{on} \quad \{\lim_n Q^n_t = -\infty\}

for any sequence of strategies $(Q^n)_{n=1,2,...}$ with cash balances $(X^n)_{n=1,2,...}$ such that $(U^n_{t-1})_{n=1,2,...}$ is bounded away from 0 and $-\infty$. 

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Proof. The argument for (29) being similar, let us establish (28). To see that ‘≥’ holds, we note that

\[ (V^n_t, U^n_t) = \mathbb{E} [(V^n_t, U^n_t) | \mathcal{F}_{t-1}] = \mathbb{E} [r(V^n_t, \Sigma(X^n_t, Q^n_t)) | \mathcal{F}_{t-1}] \]

is bounded by assumption on \( u^n \triangleq U^n_{t-1} \) because (20) holds for our choice of \( V^n_{t}, n = 1, 2, \ldots \). Moreover, the last term is easily estimated for any \( A \in \mathcal{F}_T \):

\[
\mathbb{E} [r(V^n_t, \Sigma(X^n_t, Q^n_t)) | \mathcal{F}_{t-1}] \leq \mathbb{E} [r(V^n_t, \Sigma(X^n_t, Q^n_t))1_A | \mathcal{F}_{t-1}] \\
\leq \mathbb{E} [r(V^n_t, \Sigma_0) + \partial_x r(V^n_t, \Sigma_0)(X^n_t + Q^n_t \psi))1_A | \mathcal{F}_{t-1}] .
\]

On \{\lim_n Q^n_t = +\infty\} we thus can divide by the \( \mathcal{F}_{t-1} \)-measurable quantity \( Q^n_t \) in this series of inequalities and let \( n \uparrow \infty \) to deduce that

\[ 0 \leq \mathbb{E} \left[ \partial_x r(V^n_t, \Sigma_0)(\liminf_n X^n_t/Q^n_t + \psi)1_A | \mathcal{F}_{t-1} \right]. \]

As \( A \in \mathcal{F}_T \) is arbitrary, this implies that

\[ \liminf_n X^n_t/Q^n_t + \psi \geq 0, \]

i.e., \( -\psi_{t-1} \leq \liminf_n X^n_t/Q^n_t \), because \( X^n_t, Q^n_t \) are \( \mathcal{F}_{t-1} \)-measurable for all \( n = 1, 2, \ldots \).

On the other hand,

\[ (V^n_t, U^n_t) = \mathbb{E} [r(V^n_t, \Sigma(X^n_t, Q^n_t)) | \mathcal{F}_{t-1}] \]

is also bounded away from zero, again by assumption on \((u^n)_{n=1,2,\ldots}\) and (20). Moreover, on \{\( Q^n_t > 0 \)\},

\[ \mathbb{E} [r(V^n_t, \Sigma(X^n_t, Q^n_t)) | \mathcal{F}_{t-1}] \geq \mathbb{E} \left[ r(V^n_t, \Sigma_0 + X^n_t + Q^n_t \psi_{t-1}) | \mathcal{F}_{t-1} \right]. \]

Because \( X^n_t \) and \( Q^n_t \) are \( \mathcal{F}_{t-1} \)-measurable, \( n = 1, 2, \ldots \), this implies that \( \sup_n \{X^n_t + Q^n_t \psi_{t-1}\} < \infty \). On \{\( \lim_n Q^n_t = +\infty \)\} this yields \( \limsup_n X^n_t/Q^n_t + \psi_{t-1} \leq 0 \), proving ‘≤’ in (28).

\[ \square \]

4.3 (In)completeness of binomial models with exponential market makers

Next we consider a model with time horizon \( T \), one market maker \( M = 1 \) with exponential utility \( u(x) = -e^{-ax}, x \in \mathbb{R} \), one asset \( J = 1 \) and initial endowment \( \Sigma_0 \) satisfying \( \mathbb{E} [e^{-a\Sigma_0}] < \infty \) for the market maker. The model has a binomial structure, i.e., \( \Omega = \{ -1, +1 \}^T \), the filtration \((\mathcal{F}_t)_{t=1,\ldots,T}\) is generated by \( Y_t(\omega) \triangleq y_t \) for \( \omega = (y_1, \ldots, y_T) \in \Omega, t = 1, \ldots, T \), and we assume \( \mathbb{P}[\{\omega\}] > 0 \) for all \( \omega \in \Omega \). The payoff \( \psi \) can be any real-valued \( \mathcal{F}_T \)-measurable random variable.
Theorem 4.4. For a market maker with exponential utility, a binomial model as described above is complete if and only if at any time we see a new best lower bound for $\psi$ in one possible evolution to the next time period and a new best upper bound for $\psi$ in the other:

\begin{equation}
\Omega = \left\{ \psi_{t-1} < \psi_t \right\} \cup \left\{ \psi_{t-1} > \psi_t \right\}, \ t = 1, \ldots, T.
\end{equation}

Proof. By passing to $\mathbb{P}'$ with $d\mathbb{P}'/d\mathbb{P} = e^{-\alpha \Sigma_0}/\mathbb{E} e^{-\alpha \Sigma_0}$ we can assume $\Sigma_0 = 0$ without loss of generality. We will proceed through a number auxiliary claims:

Claim 1: $H \in \mathcal{F}_T$ is attainable by a strategy $Q$ with initial capital $\pi^H$ if and only if

\begin{equation}
\pi^H = \frac{1}{\alpha} \log \left( \mathbb{E} [e^{\alpha H}] \right)
\end{equation}

and $Q$ is a predictable process that satisfies

\begin{equation}
\frac{\mathbb{E} [e^{\alpha H} | \mathcal{F}_t]}{\mathbb{E} [e^{\alpha H} | \mathcal{F}_{t-1}]} = \frac{\mathbb{E} [e^{-\alpha Q_t \psi} | \mathcal{F}_t]}{\mathbb{E} [e^{-\alpha Q_t \psi} | \mathcal{F}_{t-1}]} \quad \text{on} \quad \left\{ Y_t = +1 \right\}, \ t = 1, 2, \ldots, T.
\end{equation}

Indeed, if $H$ is attainable by a strategy $Q$ with initial cost $\pi^H$, then

\[-1 = u(0) = \mathbb{E} [u(X_T + Q_T \psi)] = \mathbb{E} [u(\pi^H - H)] = -e^{-\alpha \pi^H} \mathbb{E} [e^{\alpha H}],\]

since $H = \pi - X_T - Q_T \psi$. Therefore, the price $\pi^H$ and the utility process $U$ satisfy

\begin{align*}
\pi^H &= \frac{1}{\alpha} \log \left( \mathbb{E} [e^{\alpha H}] \right), \\
\frac{U_t}{U_{t-1}} &= \frac{\mathbb{E} [u(\pi^H - H) | \mathcal{F}_t]}{\mathbb{E} [u(\pi^H - H) | \mathcal{F}_{t-1}]} = \frac{\mathbb{E} [e^{\alpha H} | \mathcal{F}_t]}{\mathbb{E} [e^{\alpha Q_t \psi} | \mathcal{F}_{t-1}]} \quad t = 1, 2, \ldots, T.
\end{align*}

On the other hand, by predictability of the associated cash process $X$,

\[U_{t-1} = \mathbb{E} [u(X_t + Q_t \psi) | \mathcal{F}_{t-1}] = -e^{-\alpha X_t} \mathbb{E} [e^{-\alpha Q_t \psi} | \mathcal{F}_{t-1}], \ t = 1, 2, \ldots, T\]

which implies that

\[\frac{U_t}{U_{t-1}} = \frac{\mathbb{E} [e^{-\alpha Q_t \psi} | \mathcal{F}_t]}{\mathbb{E} [e^{-\alpha Q_t \psi} | \mathcal{F}_{t-1}]}, \quad t = 1, 2, \ldots, T.\]

Hence, the strategy $Q$ satisfies (32).
Conversely, let $\pi^H$ be given by (31) and $Q$ be predictable with (32). Then, in fact, the identity in (32) holds on all of $\Omega = \{Y_t = +1\} \cup \{Y_t = -1\}$ because the random variable on either side has conditional expectation 1 given $\mathcal{F}_{t-1}$. Now let $U_0 = u(0)$ and let $U$ be the process defined recursively by

$$
\frac{U_t}{U_{t-1}} = \frac{\mathbb{E}[e^{\alpha H} | \mathcal{F}_t]}{\mathbb{E}[e^{\alpha H} | \mathcal{F}_{t-1}]} = \frac{\mathbb{E}[e^{-\alpha Q_t \psi} | \mathcal{F}_t]}{\mathbb{E}[e^{-\alpha Q_t \psi} | \mathcal{F}_{t-1}]} \quad t = 1, 2, \ldots, T.
$$

Then according to (31)

$$
\frac{U_T}{U_0} = \prod_{s=1}^{T} \frac{U_s}{U_{s-1}} = \frac{e^{\alpha H}}{\mathbb{E}[e^{\alpha H}]} = e^{-\alpha (\pi^H - H)}.
$$

On the other hand, $X_t$ defined by

$$
X_t = \frac{1}{\alpha} \log \left( -\mathbb{E}[e^{-\alpha Q_t \psi} | \mathcal{F}_{t-1}] \right), \quad t = 1, 2, \ldots, T,
$$

is the cash process of $Q$ with indirect utility $U$. In particular, for $t = T$, we have $U_T = u(0) e^{-\alpha(X_T + Q_T \psi)}$, and since $U_T = u(0) e^{-\alpha (\pi^H - H)}$ we get that

$$
\pi^H - X_T - Q_T \psi = H.
$$

Claim 2: As $H$ varies over all the $\mathcal{F}_T$-measurable random variables, the left side of (32) sweeps for fixed $t = 1, \ldots, T$ precisely across all the random variables $Z_t$ with $\mathbb{E}[Z_t | \mathcal{F}_{t-1}] = 1$ which are of the form $Z_t = f_t(Y_1, \ldots, Y_t)$ for some function $f_t : \{-1, +1\}^t \to (0, \infty)$ such that

$$
0 < f_t(Y_1, \ldots, Y_{t-1}, y_t) < 1/P[Y_t = y_t | \mathcal{F}_{t-1}], \quad y_t \in \{-1, +1\}.
$$

Indeed, this is readily checked from

$$
\mathbb{E}[Z_t | \mathcal{F}_{t-1}] = f_t(Y_1, \ldots, Y_{t-1}, +1) \mathbb{P}[Y_t = +1 | \mathcal{F}_{t-1}]
$$

$$
+ f_t(Y_1, \ldots, Y_{t-1}, -1) \mathbb{P}[Y_t = -1 | \mathcal{F}_{t-1}],
$$

Claim 3: As $Q_t$ varies over the $\mathcal{F}_{t-1}$-measurable random variables, the right side of (32) varies over the open interval bounded by $\mathcal{P}_t \triangleq \mathbb{P}_t^{\psi=\psi_{t-1} \mathcal{F}_t}$ and $\mathcal{T}_t \triangleq \mathbb{P}_t^{\psi=\psi_{t-1} \mathcal{F}_t}$. Indeed, this follows by the same argument as given in the proof for the second claim in Proposition 1.
In light of Claims 1–3, completeness holds if and only if for any \( t = 1, \ldots, T \) we have
\[
P_t \land \overline{P}_t = 0 \quad \text{and} \quad P_t \lor \overline{P}_t = 1 / \mathbb{P} \left[ Y_t = y_t | \mathcal{F}_{t-1} \right].
\]
It follows that completeness yields
\[
\{ P_t \leq \overline{P}_t \} \subseteq \{ P_t = 0 \} = \left\{ \mathbb{P} \left[ \psi = \psi_{t-1} \mid \mathcal{F}_t \right] = 0 \right\} = \left\{ \psi_{t-1} < \psi \right\}
\]
and, similarly,
\[
\{ P_t \geq \overline{P}_t \} \subseteq \{ P_t = 0 \} = \left\{ \mathbb{P} \left[ \psi = \overline{\psi}_{t-1} \mid \mathcal{F}_t \right] = 0 \right\} = \left\{ \overline{\psi}_{t-1} > \psi \right\}.
\]
In particular, completeness implies (30). Conversely, observe
\[
\left\{ \overline{\psi}_{t-1} > \psi \right\} \cup \left\{ \psi_{t-1} > \psi \right\} = \{ P_t = 0 \} \cup \{ P_t = 0 \}
\]
and thus (30) yields \( P_t \land \overline{P}_t = 0 \). To conclude we need to argue that \( P_t \lor \overline{P}_t(\omega) = 1 / \mathbb{P} \left[ Y_t = y_t \mid \mathcal{F}_{t-1} \right](\omega) \) for \( \omega = (y_1, \ldots, y_T) \in \Omega \). Without loss of generality, it is sufficient to prove for
\[
\omega \in \{ P_t > 0 \} = \left\{ \mathbb{P} \left[ \psi = \psi_{t-1} \mid \mathcal{F}_t \right] > 0 \right\} = \left\{ \psi = \psi_{t-1} \right\}
\]
we have in fact \( P_t(\omega) = 1 / \mathbb{P} \left[ Y_t = y_t \mid \mathcal{F}_{t-1} \right](\omega) \). We recall that
\[
\mathbb{P} \left[ \psi = \psi_{t-1} \mid \mathcal{F}_{t-1} \right] = \mathbb{E} \left[ \mathbb{P} \left[ \psi = \psi_{t-1} \mid \mathcal{F}_t \right] \mid \mathcal{F}_{t-1} \right].
\]
Thus \( P_t(\omega) = 1 / \mathbb{P} \left[ Y_t = y_t \mid \mathcal{F}_{t-1} \right](\omega) \) is equivalent to having
\[
\mathbb{P} \left[ \psi = \psi_{t-1} \mid \mathcal{F}_t \right](\omega') = 0
\]
where \( \omega' = (y_1', \ldots, y_T') \in \Omega \) is given by \( y_s' \triangleq y_s \) for \( s \neq t \) and \( y_t' \triangleq -y_t \). So assume for a contradiction \( \mathbb{P} \left[ \psi = \psi_{t-1} \mid \mathcal{F}_t \right](\omega') > 0 \). Then we have \( \overline{\psi}_t(\omega') = \overline{\psi}_{t-1}(\omega') \) in addition to \( \psi_t(\omega') = \psi_{t-1}(\omega') \). By (30), this implies \( \overline{\psi}_t(\omega) > \overline{\psi}_{t-1}(\omega) \) and \( \overline{\psi}_t(\omega') > \overline{\psi}_{t-1}(\omega') \) in contradiction to \( \overline{\psi}_{t-1}(\omega) = \overline{\psi}_{t-1}(\omega') = \overline{\psi}_t(\omega) \wedge \overline{\psi}_t(\omega') \). \( \square 

Remark 4.5. For the model with binomial lattice structure considered in this section, one can show by similar arguments that there exists a superreplicating strategy if
\[
\emptyset \neq \left\{ \psi_{t-1} < \psi \right\} \cup \left\{ \overline{\psi}_{t-1} > \psi \right\}, \quad t = 1, \ldots, T.
\]
Moreover, in Section 4.1 we provide an example, where superreplicating cost is not attained in the case
\[
\emptyset = \left\{ \psi_{t-1} < \psi \right\} \cup \left\{ \overline{\psi}_{t-1} > \psi \right\}.
\]
5 Examples of processes with decreasing exponential tails

To illustrate that our assumption of decreasing exponential tails is satisfied by typical models considered for stock prices, let us study a few examples.

5.1 Lévy processes

**Lemma 5.1.** Let \((X_t)_{t \geq 0}\) be a non-deterministic one-dimensional Lévy process with respect to the filtration \((\mathcal{F}_t)_{t \geq 0}\). Suppose that \(X_T\) has all exponential moments, i.e. for all \(q \in \mathbb{R}\), \(\mathbb{E}[e^{qX_T}] < \infty\). Then

\[
\psi \triangleq X_T
\]

exhibits decreasing exponential tails in the sense of condition (12).

**Proof.** Let us denote the Lévy triplet of \(X\) by \((b, c, \mu)\), where \(b \in \mathbb{R}\), \(c \geq 0\) and \(\mu\) is a Lévy measure on \(\mathbb{R}\) satisfying

\[
\mu(\{0\}) = 0, \quad \int_{\mathbb{R}} (1 \wedge |x|^2) \mu(dx) < \infty.
\]

We exclude the deterministic case where both \(c = 0\) and \(\mu = 0\) and we require that \(X_T\) has all exponential moments, i.e. for all \(q \in \mathbb{R}\), \(\mathbb{E}[e^{qX_T}] < \infty\). This is equivalent to stating

\[
\int_{\{|x| \geq 1\}} e^{qx} \mu(dx) < \infty, \quad q \in \mathbb{R}.
\]

Since \(X\) has homogeneous and independent increments we will have \(\nu_{t+h} \prec \nu_t\) for \(h > 0\) where \(\nu_s \triangleq \mathbb{P}[X_T \in \cdot | \mathcal{F}_s], s \in [0, T]\), if

\[
\frac{\mathbb{E}[e^{q\psi} | \mathcal{F}_{t+h}]}{\mathbb{E}[e^{q\psi} | \mathcal{F}_t]} = \exp(q(X_{t+h} - X_t) - hf(q)) \to 0 \text{ as } |q| \uparrow \infty
\]

where due to the Lévy-Khintchine formula

\[
f(q) = bq + \frac{1}{2}cq^2 + \int_{\mathbb{R}} \{e^{qx} - 1 - qx1_{\{|x|<1\}}\} \mu(dx).
\]

By considering first \(X\) and then \(-X\) we can confine ourselves to considering the limit as \(q \uparrow \infty\).
Case \(\mu((0, \infty)) > 0\): We shall show that \(\int_{\mathbb{R}} \{e^{qx} - 1 - qx1_{|x|<1}\} \mu(dx)\) converges to \(\infty\) exponentially fast as \(q \uparrow \infty\) which entails the same convergence for \(f(q)\) and thus proves (35).

Indeed, if \(\varepsilon \in (0, 1)\) is chosen such that \(\mu((\varepsilon, \infty)) > 0\) then

\[
\int_{(\varepsilon, \infty)} \{e^{qx} - 1 - qx1_{|x|<1}\} \mu(dx) \geq (e^{qx} - (1 + q)) \mu((\varepsilon, \infty))
\]

converges to \(\infty\) exponentially fast as \(q \uparrow \infty\). At the same time we can use the Taylor expansion

\[e^{qx} = 1 + qx + \frac{1}{2}z^2\]

for some \(z = z(qx)\) between 0 and \(qx\) to see first that

\[
\int_{(0,\varepsilon)} \{e^{qx} - 1 - qx1_{|x|<1}\} \mu(dx) \geq 0
\]

and second that

\[
\int_{(-\infty,0)} \{e^{qx} - 1 - qx1_{|x|<1}\} \mu(dx) \geq \int_{(-\infty,0)} qx1_{|x|\geq1} \mu(dx).
\]

Since the exponential moment condition (34) implies \(\int |x|1_{|x|\geq1} \mu(dx) < \infty\), the latter expression may diverge to \(-\infty\) at most linearly fast in \(q\).

The summation of (36), (37) and (38) thus proves the claimed exponential divergence of \(\int_{\mathbb{R}} \{e^{qx} - 1 - qx1_{|x|<1}\} \mu(dx)\) to \(\infty\).

Case \(\mu((0, \infty)) = 0\): In this case we infer from (38) that the integral in the definition of \(f(q)\) may converge to \(-\infty\) for \(q \uparrow \infty\) at most at the deterministic linear rate \(r \triangleq \int_{(-\infty,0)} |x|1_{|x|\geq1} \mu(dx) < \infty\). Hence, \(f(q)\) will diverge to \(+\infty\) quadratically in \(q\) if \(c > 0\) in which case (36) holds on \(\Omega\). If \(c = 0\), \(f(q)\) may converge to \(-\infty\) at most with linear speed \(|b| + r\). Hence, recalling that the argument for the previous case shows that the limit \(q \downarrow -\infty\) in (35) holds on \(\Omega\), it suffices to observe that (35) will hold for \(q \uparrow \infty\) on \(\{X_{t+h} - X_t - h(|b| + r) < 0\}\), a set with positive probability if \(c = 0\), \(\mu(0, \infty) = 0\) unless \(X\) is deterministic which we have ruled out from the start.

\[\square\]
5.2 A model of Barndorff-Nielsen Shephard-type

Clearly, Laplace transforms can be computed in many financial models and so our condition of decreasing exponential tails can be checked in more complex models as well. By way of illustration let us consider a stochastic volatility model in the style of Barndorff-Nielsen and Shephard \[4\].

Lemma 5.2. Let \( Z = (Z_t)_{t \geq 0} \) be a Lévy subordinator on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) with Lévy measure \( \mu \) such that

\[
\kappa(\theta) \triangleq \log \left( \mathbb{E} \left[ e^{\theta Z_1} \right] \right) = \int_{0}^{\infty} (e^{\theta y} - 1) \mu(dy) < \infty, \quad \theta \in \mathbb{R}
\]

and \( W = (W_t)_{t \geq 0} \) be an independent Brownian motion of \( Z \). Assume the payoff of the marketed claim is given by

\[\psi = X_T\]

where \((X_t)_{t \in [0,T]}\) follows the Barndorff-Nielsen Shephard dynamics

\[
\begin{align*}
    dX_t &= (m + \beta \sigma^2_t) dt + \sigma_t dW_t + \rho dZ_t, \\
    d\sigma^2_t &= -\lambda \sigma^2_t dt + dZ_t, \quad \sigma^2_0 > 0.
\end{align*}
\]

for some constants \( m, \beta \in \mathbb{R}, \lambda > 0 \) and \( \rho < 0 \). Then \( \psi \) exhibits decreasing exponential tails in the sense of condition \((12)\).

Proof. The Laplace transform \( \mathbb{E} \left[ e^{qX_T} \mid \mathcal{F}_t \right] \) is computed in Nicolato and Venardos \[22\]:

\[
\mathbb{E} \left[ e^{qX_T} \mid \mathcal{F}_t \right] = 
\exp \left( q (X_t + m (T - t)) + (q^2 + 2\beta q) \frac{\varepsilon(t,T)}{2} \sigma^2_t + \int_t^T \lambda \kappa (f(s,q)) ds \right),
\]

where

\[
\begin{align*}
    \varepsilon(s,T) &\triangleq \frac{1}{\lambda} \left( 1 - e^{-\lambda(T-s)} \right), \\
    f(s,q) &\triangleq \rho q + \frac{1}{2} (q^2 + 2\beta q) \varepsilon(s,T).
\end{align*}
\]

Moreover, according to Theorem 2 in Nicolato and Venardos \[22\], under condition \((39)\) \( \mathbb{E} \left[ e^{qX_T} \mid \mathcal{F}_t \right] < \infty \) for every \( q \in \mathbb{R} \), i.e. \( \psi = X_T \) has all exponential moments.
We claim that for any $h > 0$ we have $\nu_{t+h} < \nu_t$ where as before $\nu_s \triangleq \mathbb{P}[\psi \in \cdot | \mathcal{F}_s]$. From (40) it suffices to show

\begin{equation}
\lim_{|q| \to \infty} \left\{ q (X_{t+h} - X_t - mh) + (q^2 + 2\beta q) \left( \frac{\varepsilon(t+h,T)}{2} \sigma_{t+h}^2 - \frac{\varepsilon(t,T)}{2} \sigma_t^2 \right) - \int_t^{t+h} \lambda \kappa(f(s,q))ds \right\} = -\infty.
\end{equation}

To show this, we note that by Taylor expansion

$$
\kappa(f(s,q)) = \int_0^\infty \left( e^{f(s,q)y} - 1 \right) \mu(dy)
$$

$$
\geq \int_0^\infty \left( f(s,q)y + \frac{1}{2} f(s,q)^2 y^2 + \frac{1}{6} f(s,q)^3 y^3 \right) \mu(dy).
$$

For fixed $s \in [t, t+h]$ and $y > 0$ we obtain by direct calculation

$$
F(s,q,y) \triangleq f(s,q)y + \frac{1}{2} f(s,q)^2 y^2 + \frac{1}{6} f(s,q)^3 y^3
$$

$$
= \frac{1}{48} q^6 \varepsilon(s,T)^3 y^3 + P_5(s,q,y),
$$

where $P_5$ is a polynomial of order 5 in $q$. The coefficients of this polynomial $P_5$ are functions of $s$ and $y$, where the dependence of $P_5$ on $s$ is continuous. Moreover, since $\kappa(\theta) < \infty$ for all $\theta$, we have that

$$
\int_0^\infty y^n \mu(dy) < \infty, \quad n = 1, 2, \ldots.
$$

Hence,

$$
\lambda \int_t^{t+h} \kappa(f(s,q))ds \geq \lambda \int_t^{t+h} \int_0^\infty F(s,q,y) \mu(dy)ds
$$

and the latter expression is a polynomial of order 6 in $q$ with positive leading coefficient. Thus, the integral term dominates all other terms in (41) and we obtain the claimed convergence. \qed

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