On Specialization of a Program Model of Naive Pattern Matching in Strings
(Extended Abstract)

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Abstract. We have proved that for any pattern $p$ the tail recursive program model of naive pattern matching may be automatically specialized w.r.t. the pattern $p$ to a specialized version of the so-called KMP-algorithm, using the Higman-Kruskal relation that controls the unfolding/folding. Given an input string, the corresponding residual program finds the first occurrence of $p$ in the string in linear time on the string length. The current state of the automated program specialization art based on unfolding/folding is too weak in order to be able to reproduce the proof, done by hands, of the uniform property above, while it known before that program specialization is sometimes able to produce the KMP-algorithm for a few concrete static patterns.

Keywords: Program specialization · Supercompilation · Optimization · KMP-algorithm · Program verification

Proving uniform properties of program optimizers (or transformers) for various computational models is of fundamental value to our understanding of both compilation and computation. Here a property of a given optimizer is said to be uniform iff there are input static arguments of the program to be optimized s.t. the property holds for any of the arguments’ static values, while other input arguments may be dynamic. Thus a uniform task is posed w.r.t. a subset of input arguments when the task is supposed to be solved by an human interested in the uniform property of a program specializer. When one wants to pose the corresponding uniform task above to a program specializer rather than an human then the mentioned static arguments should be redeclared as dynamic ones. This paper concerns itself with solving a uniform task posed to an human since the modern program specialists are unable to solve the task. The author believes that we are still very far from proving non-trivial uniform properties of optimizers for realistic models of computation.

In this extended abstract, we report on a study of some uniform properties of a program specialization method known as Turchin’s supercompilation

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Namely, the properties of a supercompiler when it specializes a program model of naive pattern matching in strings w.r.t. the pattern.

One can apparently deem that any program analyzing character strings uses a predicate testing equality of such two strings as well as a function looking for the first occurrence of a given substring in an input string. Automated exploration of diverse program models of these two functions is an interesting, difficult, and practically important task.

The idea of studying program specialization methods by transforming the programs modeling naive pattern matching originates from Yo. Futamura and K. Nogi ([6], 1987). Here, by “naivety” of an algorithm is meant its natural essence not messed up by any thought, i.e., not incorporating some ingenuity. The authors experimented with the program model written in terms of the LISP language, which can be encoded, up to a morphism, in term rewriting systems [2] based on top-down pattern matching as follows.

\[ S \{ \text{-- Search -- Pattern matching} \]
\[ s_a : p, s_a : y = L(s_a : p, s_a : y, s_a : p, y); \]
\[ s_a : p, s_a : y = S(s_a : p, y); \]
\[ p, \text{Nil} = \text{F}; \}

\[ L \{ \text{-- Look for the first pattern symbol } s_a \text{ inside the string } \#y. \]
\[ s_a : p, s_a : y, q, z = L(p, y, q, z); \]
\[ s_a : p, s_a : y, q, z = S(q, z); \]
\[ s_a : p, \text{Nil}, q, z = S(q, z); \]
\[ \text{Nil, } y, q, z = \text{T}; \}

The worst-case time complexity of this program model is \( O(|p| \times |y|) \), where \( p \) and \( y \) are input pattern and string, respectively.

The specialization task of our interest is defined as follows.

\[ \mathcal{T}(P, x_{\pi_0}) \triangleq \text{Spec}(P, S(x_{\pi_0}, \#y_{\text{str}})), \]

where \( \text{Spec} \) is a program specializer, \( P \) stands for the program above, \( S \) is its entry function. We use the underlining sign to show encoded structures of the program to be specialized. This initial configuration takes a static pattern and dynamic string. A pointer moves from left to right along the string \( \#y_{\text{str}} \), looking for the first occurrence of \( s_a \) being the first letter of the input pattern \( x_{\pi_0} \). When such an occurrence is found the unscanned segment of \( \#y_{\text{str}} \) and the pattern are

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\[ ^1 \text{In the sequel the term parameter is used for dynamic variables, in order to stress that the parameter value is given but unknown to the transformer, while the variables are to be assigned. The parameters will be prefixed with the sign \#.} \]
saved as a backtracking point (the first rule of $S$), in order to continue the pattern matching in $s\overline{y}_{str}$ if a prefix of $s\overline{s}_{a \overline{y}_{str}}$ does not coincide with $x_{\pi_0}$ (see the second rule of $L$). The corresponding model is a tail recursive program. The predicate $S$ moves the pointer along the input string $s\overline{y}_{str}$. The predicate $L$ compares a prefix of $s\overline{s}_{a \overline{y}_{str}}$ with the pattern given in its first argument.

1 The Matiyasevich & Knuth-Morris-Pratt Algorithm

The residual program of the specialization task $\mathcal{S}(P, 'aaab')$ reported by Futamura and Nogi is not only a version of naive pattern matching specialized w.r.t. the pattern, but a specialized version of a searching algorithm solving the same task that was apparently discovered independently by Yu. V. Matiyasevich (1969, 1971 [1], 1973 [1]), and J. H. Morris and V. R. Pratt (1970, [12]) – published in 1977 by D. Knuth, J. Morris and V. Pratt [10]. See also [19].

The algorithm M-KMP in linear time on the pattern length $|p|$ generates firstly a function $f(q)$. Let a state defined by the scanned prefix $q$ of $p$ that we are looking for be given. Let $i$ stand for the pointer place of the observed occurrence of a letter unequal to the letter $d$ indexed with $|q| + 1$ in $p$. See Fig. 1. The function allows us, using the only program step, to move the pointer from the $i$-th index to the index $j(i) = i - f(q)$ along the string $s\overline{y}_{str}$, before which the pattern $p$ cannot have an occurrence. The function $f(q)$ is discovered from the structure of $p$. Then the direct search starts using $f(q)$.

The worst-case time complexity of the entire algorithm M-KMP is $O(|p| + |s\overline{y}_{str}|)$.

We name the $f$-function property described in the previous paragraph the primary property of this function. Let $l(q)$ be both a prefix and a suffix of the word $q$ then $f(\epsilon) = 0$, and if $q \neq \epsilon$: $f(q) = \max\{|l(q)| \mid l(q) \neq q\}$, where $\epsilon$ is the empty word. For example, for $p$
'aab': \( f(\epsilon) = 0, j(i) = i \), \( f('a') = 0, j(i) = i \),
\( f('aa') = 1, j(i) = i - 1 \); 

'ababa': \( f(\epsilon) = 0, j(i) = i \), \( f('a') = 0, j(i) = i \),
\( f('ab') = 0, j(i) = i \), \( f('aba') = 1, j(i) = i - 1 \),
\( f('abab') = 2, j(i) = i - 2 \).

An invariant of the algorithm second stage is that the scanned prefix of \( p \) and the unscanned prefix of \( y \) coincide. If the unscanned prefix of \( y \) coincides with \( p \) then we have found the first occurrence of \( p \) in \( y \). If the string has been finished not satisfying this property then it does not contain the input pattern as a substring. The multiple and overlapping occurrences of concrete substrings in the needed pattern cause the main difficulty in discovering the function \( f(q) \).

**Short history of the task considered:** The first Futamura-Nogi results above were obtained in 1987 by means of the generalized partial computation method [6] and published in 1988. Their results are substantially based on exploiting negative information about the parameterized program configurations used. The first paper reporting on successful partial evaluation experiments of specialization of the corresponding models appeared in 1989 [5], but in order to obtain desirable results of specializing the models w.r.t. three pattern samples including the pattern 'abcabcabcab' the paper authors were forced to disrupt the natural essence of naive pattern matching. Later the task of interest was popular enough, see for examples manual elaborating the idea above in the domain of finite trees [21], a report [17] on experiments done by means of partial deduction. Same technique is used in [18] in order to improve a number of examples of naive, nondeterministic program models to specialized versions of the M-KMP algorithm, among many others papers presenting a few specialization tasks resulting in similar residual programs. We have to point out to an interesting work [4] that calculates, by means of hands, an optimized version of a general naive matcher where the input pattern is dynamic, which results in the general M-KMP algorithm. That approach is based on an algebraic technique and uses not automated tricks including higher order relations decreasing the worst-case time complexity.

To our knowledge, excluding exhaustive computation for a number of concrete patterns, there was no attempt to describe an analysis of causes of unfold/fold specializing the naive pattern matching w.r.t. an arbitrary concrete pattern \( \pi_0 \), i.e., a static one, leading to generating the specialized version of the M-KMP algorithm, looking for \( \pi_0 \). Thus this report is the first one presenting results of such an analysis. Proving this uniform property of program specialization was remained a challenging task for a long time.

**Model and Result:** Throughout this paper we assume that the program model of naive pattern matching is fixed. It is the program \( P \) given in the introduction above. It is clear that the property of interest depends on both the program model and the program specializer. Thus this paper addresses a relation between the

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2 I.e., on an information described using the negation logical connective.
model and the specializer considered. We claim that for any given pattern \( \pi \) the supercompiler SCP4 [13,15] solving the task \( T(\mathcal{P}, \pi) \) above generates a residual program being a specialized version of the M-KMP algorithm and running in linear time on the input string length \( |\mathcal{y}_{\text{str}}| \) unknown to the specializer. The string \( \mathcal{y}_{\text{str}} \) is dynamic. So we have infinitely many specialization tasks, each of them corresponds to its own \( \pi \). We have manually proved the relation above uniformly over these tasks, i.e., actually over the pattern set. Given a program specializer \( \text{Spec} \), in order to automatically prove the relation above one should launch the specializer infinitely many times.

There is another specialization task uniting the tasks above. It is as follows
\[
\text{Spec}(\mathcal{P}, S(#\pi, #\mathcal{y}_{\text{str}})),
\]
where both \( #\pi \) and \( #\mathcal{y}_{\text{str}} \) stand for parameters ranging over dynamic patterns and dynamic strings, respectively. The current state of the automated program specialization art based on unfolding/folding is too weak in order to result in a specialized version of the M-KMP algorithm, when solving this task.

2 Preliminaries

We assume the reader to be familiar with the basics in program specialization. Let \( \mathcal{A} \) be a finite alphabet of letters, constants. Henceforth, \( \pi \) stands for a constant pattern unknown to the reader. See the footnote below. \( \xi_i \notin \mathcal{A} \) stands for the \( i \)-th unspecified letter of \( \pi \), \( 0 < i \leq |\pi| \). Here and below for any \( \alpha \in \mathcal{A}^* \) \( |\alpha| \) is the length of the word – the number of letters in \( \alpha \). \( \pi = \xi_1 \ldots \xi_i \ldots \xi_{|\pi|} \). \( \pi_i \) is the \( i \)-th nonempty suffix of \( \pi \) defined by \( \pi_0 = \pi \), for \( 0 < i < |\pi| \), \( 0 < |\pi_i| = |\pi| - i < |\pi| \), \( \omega_i \) is the \( i \)-th prefix of the word \( \pi_1 \) s.t. \( |\omega_i| = i - 1 \). So the following equalities hold for any \( i \) s.t. \( 0 < i < |\pi| \): \( \pi_1 = \omega_i \pi_i \), \( \pi_{(i-1)} = \xi_i \pi_i \). For example, \( \omega_1 = \varepsilon, \pi_{(|\pi| - 1)} = \xi_{|\pi|} \). In this notation the pattern first letter \( \xi_1 \) is an exclusive one, since it is appropriately treated by the program \( \mathcal{P} \). From the point of view of the following reasonings, the symbols \( \xi_i, \pi_j, \omega_k \) are meta-parameters\(^4\) meaning that their values are unknown to the reader, but fixed input data\(^3\) while their values are known to the specializer, where \( \xi_i \) ranges over \( \mathcal{A} \), \( \pi_j, \omega_k \) range over \( \mathcal{A}^* \). Sometimes we will briefly call them either letters or words respectively, by default, assuming that their values are unknown to us. A parameterized expression and a word are defined as follows. The ordinary parameters, i.e., without the meta prefix, were introduced above (see Page 2).

\(^3\) I.e., this single proof holds for each of the tasks, for any fixed pattern. This uniform proof (as well as property) concept is widely used in mathematics.

\(^4\) One may consider the symbols \( \xi_i, \pi_j, \omega_k \) as aliases for the pattern letters and segments. Note the value of \( \xi \) is in \( \mathcal{A} \), while the letter \( \xi \) itself is outside of \( \mathcal{A} \).
\[
\text{pexpr ::= } \text{Nil} \mid 'ν': \text{pexpr} \mid \text{parameter}: \text{pexpr} \mid \text{F(args)}
\]
\[
\text{args ::= } \text{pexpr} \mid \text{pexpr, args}
\]
\[
\text{word ::= } \text{Nil} \mid 'ν': \text{word}
\]

a Where \(ν \in \mathcal{A}\), \(\text{F}\) is a function name. We use a widely known abbreviation for the words, for example, ‘\text{abcda}’ stands for ‘\text{a}’;’\text{b}’;’\text{c}’;’\text{d}’;’\text{a}’:\text{Nil}’.

A configuration, i.e., a parameterized expression, containing a function call is said to be active, otherwise it is called passive.

**Definition 1.** Let a program \(P\) and its parameterized entry configuration \(\text{F(pexpr)}\) be given. Let \(T_n\) be a sequence of rooted directed trees\(^5\) that are defined recursively.

\(T_0 \triangleq \text{F(pexpr)}\). Given a tree \(T_n\), then \(T_{(n+1)}\) is the replacement result of every leaf of \(T_n\), labeled with an active configuration \([C]\), with the tree generated by the one-step unfolding of \([C]\). The sequence \(\{T_k\}_{k \geq 0}\) is said to be the complete unfolding tree of the pair \((P, \text{F(pexpr)})\) and denoted with \(\hat{P}_\text{F(pexpr)}\).

\(\hat{P}_\text{F(pexpr)}\) can be finite or infinite. \(\hat{P}_\text{F(pexpr)} \triangleq \{T_k\}_{k \geq 0}\) is said to be finite iff there exists \(k\) s.t. \(T_{(k+1)} = T_k\). For any \(k \in \mathbb{N}\), \(T_k\) is a partial computation tree. The sequence \(\hat{P}_\text{F(pexpr)}\) can be informally seen as \(\lim_{i \to \infty} T_i\). I.e., it can be considered as the infinite parallel unfolding of the pair \((P, \text{F(pexpr)})\). We omit the index of this tree if it is clear from the context of use. For example, all the following definitions make sense for any program \(P\) and any its entry configurations, so \(\hat{P}_\text{F(pexpr)}\) is shortened to \(\hat{P}\). We abuse notation and denote this “limit” by \(\hat{P}\). So \(\hat{P}\) also stands for a finite or infinite tree. It should be clear from context which definition is intended.

We call a node transient, if the one-step unfolding of the configuration labelling it produces the only edge outcoming from the node. Unless specified otherwise, we assume that all transient nodes are removed from the complete unfolding tree\(^6\).

Given a tree generated by the one-step unfolding of a configuration, a node in this tree is said to be a pivot node if it is the first node along a path starting at the tree root, and having at least two outcoming edges.

Paths generated by a single unfolding step are ordered. This order respects the order of the steps done by the machine meta-interpreting \(P\) and constructing the paths. The paths in \(\hat{P}\) are lexicographically ordered w.r.t. the following pairs: the name of the function being specialized; the order numbers of the rewriting rules corresponding to the current unfolding operation along the path considered. Henceforth, we use the order path terminology in \(\hat{P}\) corresponding to the lexicographical order, unless specified otherwise.

\(^5\) with edges going from the root.

\(^6\) Whenever a transient node is removed then its incoming and outcoming edges are replaced with a single edge labeled with the composition of the predicates labeling the removed edges.
Let $Q(t_i, t_j)$ stand for a formula of the form $t_i \neq t_j$, where every argument is either an $s$-parameter or a symbol. Here we are interested in the predicates $R(t_1, \ldots, t_n)$ being conjunctions of such elementary inequalities $Q(t_i, t_j)$. Such a predicate restricts domains of the parameters from its arguments.

Now we extend the configuration concept. See also [23][25][7].

**Definition 2.** A parameterized configuration is a pair of the form $(\text{pexpr}, R(p_1, \ldots, p_n))$, where $p_i$ are $s$-parameters. $R$ is a predicate specifying "negative information" restricting the domains of parameters $p_1, \ldots, p_n$ from pexpr.

**Definition 3.** Let a complete unfolding tree $\hat{P}$ and parameterized configurations $[C_1], [C_2]$ labeling nodes in $\hat{P}$ be given. We say $[C_1]$ covers $[C_2]$ if there is a renaming $\sigma$ of the $[C_1]$ parameters s.t. $\sigma(\text{pexpr}_1) = \text{pexpr}_2$ and the predicate $R_2(q_1, \ldots, q_k) \Rightarrow R_1(\sigma(p_1), \ldots, \sigma(p_n))$ is identically true.

**Definition 4.** Let a complete unfolding tree $\hat{P}$, a path $t$ starting at the $\hat{P}$ root, and parameterized configurations $[C_1], [C_2]$ along the path $t$ be given. We say a segment $\Delta_1$ of $t$ covers $[C_1]$ if there is a configuration $[C_0]$ along $\Delta_1$ that covers $[C_1]$.

Let $T$ be a subtree of $\hat{P}$, rooted in $[C_2]$. We say a segment $\Delta_2$ of the path $t$, consisting of $[C_2]$-ancestors, covers the subtree $T$ if along any infinite path $r_i$ starting at $[C_2]$ there is a configuration $[K_i]$ covered by $\Delta_2$.

The following lemmata relating to the naive pattern matching model given in the introduction are a part of our contribution. We have proved the statements below and Theorem \ref{thm:main} based on them, assuming, by default, that the unfolding/folding process is managed by the Higman-Kruskal relation [3][11] and other conditions (if given below).

**Lemma 1.** For any $\pi \in A^+$ the first $\hat{P}_{\#(\pi, s_{ystr})}$-path starting at the root ends at a leaf labeled with the passive configuration $T$ and all pivot configurations along this path generated by the supercompiler SCP4 [13][13] form the following finite sequence:

$$[S], [L_1], \ldots, [L_{(n-1)}],$$

where $n = |\pi|$, $[S]$ is the initial configuration, $[L_i] \triangleq L(\pi_1, s_{ystr}, \pi, \omega_1 \# s_{ystr} \#).$

The transient configuration $[L_n]$ follows this sequence of the active configurations along the first path; and for all $i, j \in \mathbb{N}$ s.t. $0 \leq i < j < n$ the inequality $|\pi_i| > |\pi_j|$ holds.

**Lemma 2.** For any $\pi \in A^+$ and for any infinite path $r$ that starts at the root of $\hat{P}_{\#(\pi, s_{ystr})}$ and goes through at least one configuration with a call of the function $\#$ there is a configuration $[S']$ of the form $(S(\pi, s_{a_s}, s_{ystr}), R(s_{a_s}))$ s.t. the root

\footnote{The sign $\#$ stands for the associative concatenation.}

\footnote{A path not including such a configuration corresponds to the input strings not containing the pattern’s first letter.}
of $\hat{P}_{S(\pi, y_{str})}$ is the only ancestor of $[S']$ with an $S$-call and it is a pivot. (See Fig. 2.)

**Fig. 2.** The complete unfolding tree $\hat{P}_{S(\pi, y_{str})}$. (Here and below the dashed and dotted arrows indicate segments of the paths, which may contain a number of edges, while each continuous arrow indicates the only edge.)

**Lemma 3.** For any $\pi \in A^+$ and for any infinite path $r$ that starts at the root of $\hat{P}_{S(\pi, y_{str})}$ and goes through at least one configuration of the form $\langle S(\pi, s_{str}), R(s_{str}) \rangle$, let $[S']$ be the first occurrence of such a configuration in $r$. See Fig. 3 below. Then the first pivot configuration after $[S']$ in the continuation of the path $r$ is a configuration $[C]$ of one of the forms:

1. if the predicate $R(\xi_1)$ is satisfiable then $[C]$ is of the form $L(\pi_1, s_{str}, \pi, s_{str})$ and this occurrence of $[C]$ is covered by one of its ancestors;
2. if the predicate $R(s_{str}) \land (s_{str} \neq \xi_1)$ is satisfiable then $[C]$ is an $[S']$-child of the form $S(\pi, s_{str})$.

**Fig. 3.** The complete unfolding tree $\hat{P}_{S(\pi, y_{str})}$.

3 Uniform Properties of the Complete Unfolding Tree of the Functional Program Model of the M-KMP Algorithm

In this section $P$ means the program considered in Introduction above. Let $t$ be a path starting at the $\hat{P}_{S(\pi, y_{str})}$-root, where $\pi \in A^+$. Let the path $t$ correspond to a value of $s_{str}$ of the form $\pi \oplus s_{str}$.
The theorem below states that for any π any nonempty prefix Ψ of t covers a subtree H of \( \hat{P}_{S(\pi, \#y_{str})} \) s.t. the H-root is a leaf of the tree that results from the last pivot configuration \([C] \in \Psi\) by means of the one-step unfolding and does not belong to the path t. See Fig. 4. The corresponding parameter remanings, the covering morphisms, depend on concrete configuration pairs – the covering and covered ones. Since the subtree H corresponds to the first failure, when a symbol of the string \#y_{str} does not meet a symbol of the pattern π, then the edge incoming in the H-root is labeled with a narrowing of the form \(\#y_{str} \rightarrow \#s_{b} \#y_{str}'\), this arrow should be read as “is of the form”, belonging to the path from the covering configuration to the covered one. The covering pivot L-configurations from \(\Psi\) include explicitly, i.e., as constant terms, the main invariant of the M-KMP algorithm.

3.1 The Main Contribution

We have proved the following theorem. See also Fig. 4.

**Theorem 1 (On Covering).** Let a word \(\pi \in A^+\) and \(\hat{P}_{S(\pi, \#y_{str})}\) be given. Let t be the first, the shortest, path from the \( \hat{P}_{S(\pi, \#y_{str})} \)-root to a leaf labeled with the configuration T, Ψ – a nonempty prefix of the path t, \([C] \) – the last pivot configuration in \(\Psi\). Then for any subtree H of \(\hat{P}_{S(\pi, \#y_{str})}\), rooted in a leaf of a tree resulted from \([C]\) by means of the one-step unfolding and not belonging to the path t, the prefix Ψ covers H. \[\triangle\]

![Fig. 4](image-url) The complete unfolding tree \(\hat{P}_{S(\pi, \#y_{str})}\).

The On-Covering Theorem means that for any \(\pi \in A^+\) no generalization happens during supercompilation of the task of interest and there are finitely many the configurations in \(\hat{P}_{S(\pi, \#y_{str})}\) modulo parameter renaming.

Furthermore, the renaming folding substitutions include neither constant, static data, nor repeated parameter, dynamic variable, which are necessary in order to generate an accumulator in the residual program, keeping the passed track needed for backtracking along the input string. For an example see the repeating occurrence of the variable y in the original program P, the rhs of the first rule of the function S. Since the residual function names are generated using the entire constant structure of the corresponding pivot configurations, removing
the used structures, the reader taking into account that the original program $P$ is a tail recursive may conclude that, actually, the rhs of any residual rewriting rule includes no constant data at all.

The above reasoning implies immediately that there is no backtracking in the residual program. The function $f$ defined in Sec. 4 is incorporated into the left-hand sides of the corresponding residual program running in $O(|x|y_{str})$ time.

**On computational complexity:** While the worst-case time complexity of the original program $P$ is $O(|x| \times |y_{str}|)$ the initial configuration $S(\pi_0, y_{str})$ of interest runs in linear time $O(|y_{str}|)$ for any $\pi_0 \in A^*$. The corresponding residual program also runs in linear time. From a theoretical point of view, such a result is almost nothing. Nevertheless the result we have presented above shows that supercompilation using the Higman-Kruskal relation transforms the tail recursive program model $P$ to a specialized version of the M-KMP algorithm. The naive algorithm differs meaningfully from the M-KMP algorithm and the last one is based on a quite nontrivial observation. See Sec. 1.

The formal structures of the two algorithms differ as well. The first one is tail recursive $S(p, y)$ while the second one is a composition of the form $M$-KMP($f(p), y$) based on call-by-value evaluation. The supercompiler using the Higman-Kruskal relation is able to recognize that for any fixed pattern $\pi_0$ the backtracking loops in computing $S(\pi_0, y)$ terminate and therefore can be completely unfolded. That in turn allows the supercompiler to noticeably improve the constant factor in the upper bound on the number of the interpretation steps of the residual program looking for well structured patterns. For the set of such a kind of patterns the constant factor does matter both in the practice of programming and programming-language theory.

**Future Work** It will be interesting to automatically generate some other efficient algorithms from naive program models solving the same tasks. For example, discovering periodicities in strings.

It would also be to interestingly investigate the average time complexity of the residual program of interest, which is more relevant to the practice as compared to the worst-case time complexity.

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9 For example the following pivot configuration

\[ L_{\mathcal{L}_1} : L('bcaca', 'abcabcaca', 'bca' \oplus y_{str}) \] will be transformed in the input format $F_c('bcaca', 'abcabcaca', 'bca' \oplus repeated-y_{str})(y_{str})$ of the residual function $F_c('bcaca', 'abcabcaca', 'bca' \oplus repeated-y_{str})$. Where the paired corner brackets stand for encoding their arg with a natural number.
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References

1. Matiyasevich Yu.V.: O raspoznovanii v realnoe vremya otnosheniya vkhodzh- 
deniya. Zap. nauchn. sem. LOMI 20, 104–114 (1971). (in Russian). English Translation: Matiyasevich, Yu.V.: Real-Time Recognition of the Inclusion Relation. In: Journal of Soviet Mathematics 1, 64–70, (1973). American Mathematical Society Translations. (The author gave a report on the contribution described in the paper in 1969, 15th May, at Leningrad seminar on constructive mathematics.) Available at https://link.springer.com/content/pdf/10.1007/BF01117471.pdf
2. Barendsen, E., Bethke, I., Heering, J., Kennaway, R., Klint, P., van Oostrom, V., van Raamsdonk, F., de Vries, F.J., Zantema, H.: Term Rewriting Systems. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press (2003)
3. Ben-Amram, A.M., Jones, N.D.: Computational Complexity via Programming Languages: Constant Factors do Matter. Acta Informatica 37, 83–120 (2000). https://doi.org/10.1007/s002360000035
4. Bird, R.S., Gibbons, J., Jones, G.: Formal Derivation of a Pattern Matching Algorithm. Science of Computer Programming 12(2), 93–104 (1989). https://doi.org/10.1016/0167-6423(89)90036-1
5. Consel, C., Danvy, O.: Partial Evaluation of Pattern Matching in Strings. Information Processing Letters 30(2), 79–86 (1989). https://doi.org/10.1016/0020-0190(89)90113-0
6. Futamura, Y., Nogi, K.: Generalized Partial Computation. In: the IFIP TC2 Work- shop. pp. 133–151. North-Holland Publishing Co., Amsterdam (1988)
7. Futamura, Y., Nogi, K., Takano, A.: Essence of Generalized Partial Computation. Theoretical Computer Science 90, 61–79 (1991). https://doi.org/10.1016/0304-3975(91)90299-H
8. Higman, G.: Ordering by Divisibility in Abstract Algebras. Proc. London Math. Soc. 2(7), 326–336 (1952). https://doi.org/10.1112/plms/s3-2.1.326
9. Jones, N.D.: Computability and Complexity from a Programming Perspective. The MIT Press (2000)
10. Knuth, D., Morris, J., Pratt, V.: Fast Pattern Matching in Strings. SIAM Journal on Computing 6(2), 323–350 (1977). https://doi.org/10.1137/0206024
11. Kruskal, J.: Well-Quasi-Ordering, the Tree Theorem, and Vazsonyi’s Conjecture. Trans. Amer. Math. Society 95, 210–225 (March 1960). https://doi.org/10.2307/1993287
12. Morris, J.H., Pratt, V.R.: A Linear Pattern-Matching Algorithm. Tech. Rep. 40, Computing Center, Univ. of California, Berkeley (1970)
13. Nemytykh, A.P.: The Supercompiler SCP4: General Structure. URSS, Moscow (2007), (Book in Russian)
14. Nemytykh, A.P., Pinchuk, V.A., Turchin, V.F.: A Self-Applicable Supercom- piler. In: PEPM’96. LNCS, vol. 1110, pp. 322–337. Springer-Verlag (1996). https://doi.org/10.1007/3-540-61580-6_16
15. Nemytykh, A.P., Turchin, V.F.: The Supercompiler SCP4: Sources, On-Line Demonstration. [online] (2000). http://www.botik.ru/pub/local/scp/refal5/
16. Nepeivoda, Antonina: The Model Supercompiler MSCP-A: Sources. [online].
http://refal.botik.ru/mscp/mscp-a_eng.html
17. Pettorossi, A., Proietti, M., Renault, S.: How to Extend Partial Deduction
to Derive the KMP String-Matching Algorithm from a Naive Specification (Poster
Abstract). p. 539. JICSLP ’96 (1996)
18. Pettorossi, A., Proietti, M., Renault, S.: Derivation of Efficient Logic Programs by
Specialization and Reduction of Nondeterminism. Higher-Order Symb.
Comput. 18, 121–210 (June 2005). https://doi.org/10.1007/s10990-005-7008-3
19. Shen, A.: Programming: Theorems and Tasks. MCCME, Moscow (2021), (Book in
Russian, 7th edition)
20. Slisenko, A.O.: Detection of Periodicities and String-Matching in Real Time. J.
Soviet Math. 22(3), 1316–1387 (1983). https://doi.org/10.1007/BF01084395
21. Smith, D.: Partial Evaluation of Pattern Matching in Constraint Logic Program-
ing Languages. In: 1991 ACM SIGPLAN Symposium on PEPM, PEPM'91. pp.
62–71. ACM (1991). https://doi.org/10.1145/115865.115873
22. Sørensen, M.: Turchin’s Supercompiler Revisited. Master’s thesis, Department of
Computer Science, University of Copenhagen (1994). DIKU-rapport 94/17
23. Turchin, V.: The language Refal – The Theory of Compilation and
Metasystem Analysis. Tech. Rep. 20, Courant Institute of Mathe-
matical Sciences, New York University (feb 1980), available at URL
https://pat.keldysh.ru/~roman/doc/Turchin/1980-Turchin--The_Language_REFAL--The_Theory_of_Compilation_and_Metasystem_Analysis.pdf
24. Turchin, V.F.: The Concept of a Supercompiler. ACM Transac-
tions on Programming Languages and Systems 8(3), 292–325 (1986).
https://doi.org/10.1145/5956.5957
25. Turchin, V.: The Basics of Metacomputation, (Chapter 3). Tech. rep., The
School “Metacomputation in the Language Refal”, Obninsk, Russia (July
11-23 1990), a chapter from an unpublished book, 63 p, Available at URL
https://pat.keldysh.ru/~roman/doc/Turchin/1990-Turchin--The_Basics_of_Metacomputation--Obninsk_ch3.pdf
26. Turchin, V.F.: Program Transformation with Metasystem
Transitions. Journal of Functional Programming 3(3), 283–313 (1993).
https://doi.org/10.1017/S0956796800000757
27. Turchin, V.F., Nemytykh, A.P.: Metavariabes: their Implementation and
Use in Program Transformation. Tech. Rep. TR 2005-012, The City Col-
lege of the City University of New York (1995). 34pp, Available at URL
http://refal.botik.ru/library/Turchin-Nemytykh-Metavaries_their_Implementation_and_Use_in_Progr