Cosmic String Loop Collapse in Full General Relativity

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We present the first fully general relativistic dynamical simulations of Abelian Higgs cosmic strings using 3+1D numerical relativity. Focusing on cosmic string loops, we show that they collapse due to their tension and can either (i) unwind and disperse or (ii) form a black hole, depending on their tension \( G \mu \) and initial radius. We show that these results can be predicted using an approximate formula derived using the hoop conjecture, and argue that it is independent of field interactions.

The recent detection of Gravitational Waves (GW) from black hole (BH) [1] binaries by the LIGO/VIRGO collaboration marked the start of a new era of observations. Beyond astrophysical objects such as BH and neutron stars, this paved the way for the use of GW to search directly for signatures of new physics. One of the key targets for this search are cosmic strings [2–4].

Cosmologically, cosmic strings networks naturally arise after a phase transition in the early universe, possibly during GUT symmetry breaking. More speculatively, string theory also suggests the presence of cosmological fundamental superstrings, especially through the mechanism of brane inflation [5, 6]. These networks may manifest themselves through several channels, such as imprints via lensing on the Cosmic Microwave Background [7] and possibly through the presence of a stochastic gravitational wave background. The latter in particular is recently searched for by the LIGO/VIRGO collaboration [8]. More intriguingly, one can search for localized coherent events of these strings, such as when the strings self-interact through the formation of sharp cusps or through the collisions of traveling kinks that are formed during the intercommutation (i.e. collisions) of cosmic strings.

Before this work, the two primary methods of modeling cosmic strings has been through solving the field theory equations in flat or expanding spacetime, or through an effective Nambu-Goto prescription with weak coupling to gravity (see e.g. [9]). In either case, by considering the stress-energy of a network of strings, one can then compute in the weak gravity limit a stochastic GW background [9, 10]. Local events such as the collisions of traveling kinks and cusps along the strings are expected to produce bursts of GW – these bursts events have been

\[ G_{\mu} = 1.6 \times 10^{-2} \] and initial radius \( R = 100 \ M_{\odot}^{-1} \). We use our results to put a bound on the production rate of planar cosmic strings loops as \( N \lesssim 10^{-2} \) Gpc\(^{-3}\) yr\(^{-1}\).

I. INTRODUCTION

The recent detection of Gravitational Waves (GW) from black hole (BH) [1] binaries by the LIGO/VIRGO collaboration marked the start of a new era of observations. Beyond astrophysical objects such as BH and neutron stars, this paved the way for the use of GW to search directly for signatures of new physics. One of the key targets for this search are cosmic strings [2–4].

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its integrated GW energy emitted from such a collapse. For the latter, we found that the total energy emitted in gravitational waves is \(0.5 \pm 0.2\%\) of the initial mass, which is in agreement with the bound of \(<29\%\). We will discuss direct detection prospects of such individual collapse events with GW detectors in section V.

II. ABELIAN HIGGS WITH GRAVITY

The action of the Abelian Higgs model minimally coupled to gravity

\[
S = S_{EH} - \int d^4x \sqrt{-g} \left[ (D_\mu \phi)^* (D^\mu \phi) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + V(\phi) \right],
\]

(1)

where \(S_{EH} = \int d^4x \sqrt{-g} (R/16\pi G)\), \(D_\mu\) is the covariant derivative \((\partial_\mu - ieA_\mu)\) with its \(U(1)\) gauge field \(A_\mu\), and \(V(\phi)\) is the potential of the complex scalar field \(\phi\) given by

\[
V(\phi) = \frac{1}{4} \lambda \left( |\phi|^2 - \eta^2 \right)^2, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

(2)

For simplicity, we set the charge \(e\) and the dimensionless coupling constant \(\lambda\) to obey the critical coupling limit

\[
\beta = \frac{\lambda}{2e^2} = 1,
\]

(3)

in which the Higgs and vector masses are identical and \(\mu\) simplifies to

\[
\mu = 2\pi\eta^2.
\]

(4)

As a check of our code, we numerically construct a fully relativistic infinite static string coupled to gravity and demonstrate that its evolution is indeed static and stable. The details of this construction can be found in Appendix B.

In this paper, we consider circular string loops. To construct the initial conditions, we define toroidal coordinates

\[
x = \cos \varphi (R + r \cos \theta), \quad y = \sin \varphi (R + r \cos \theta), \quad z = r \sin \theta,
\]

(5)

where \(R\) is the radius of the loop and choose the following ansatz for the field variables

\[
\phi = f(r)e^{in\theta}, \quad A_\theta = \frac{n\alpha(r)}{e},
\]

(6)

where \(n\) is the winding number of the string which is set to one throughout this paper. To construct the loop we use the profile \(f(r)\) from the static string \(^2\). After making the conformal metric ansatz

\[
\gamma_{ij} dx^i dx^j = \chi (dx^2 + dy^2 + dz^2),
\]

(7)

we solve the Hamiltonian constraint to obtain the conformal factor \(\chi\).

III. RESULTS

We simulate the collapse of circular loops, scanning through the initial condition parameter radius \(R\) and the model symmetry-breaking scale \(\eta\) (and hence string tension via eq. 4), in the critical coupling limit with \(e = 1\) and \(\lambda = 2\). The loop begins at rest but quickly accelerates to close to the speed of light due mainly to the string tension. We find this motion to be consistent with the Nambu-Goto action dynamics (see Appendix C)

\[
r = R \cos \frac{\tau}{R},
\]

(8)

up to \(r \sim \delta\) which is the thickness of the string given by

\[
\delta = \frac{1}{\eta \sqrt{\lambda}},
\]

(9)

and \(\tau\) is the time coordinate at spatial infinity. Depending on the choice of \(\mu\) and \(R\), there are two possible outcomes: (i) the string unwinds itself and the resulting radiation disperses or (ii) a BH forms.

\(^1\) We use the \(-++\) convention for the metric, and set \(\hbar = c = 1\) and \(M_{Pl} = 1/\sqrt{G}\).

\(^2\) See Appendix B for details.
This result can be predicted using the loop conjecture as follows. A BH forms if the loop mass $M_{\text{loop}} = 2\mu R$ is enclosed within a radius smaller than its Schwarzschild radius $2GM_{\text{loop}}$. In addition, the smallest volume in which a loop can be contained before the string unwinds has radius $\delta$, which sets the Schwarzschild radius the lower bound for BH formation to be $2GM_{\text{loop}} > \delta$, or

$$R > \sqrt{\frac{1}{8\pi\lambda}}(G\mu)^{-3/2}M_{\text{pl}}^{-1}.$$  \hspace{1cm} (10)

Moreover, as the minimum radius of a loop is $R = \delta$, we don’t expect dispersion cases for $G\mu > (4\pi)^{-1}$ and all loops will form BHs. We find this estimate to be a good predictor (see fig. 2), which suggests that black hole formation is broadly independent of field interactions.

If a black hole forms, the amount of initial mass that falls into the black hole depends on the initial radius $R$ for fixed $G\mu$, with the rest being radiated in either gravitational waves or matter.

We investigate whether this collapse is a Type I or Type II transition [20] by studying the mass of the black hole close to the critical radius. Supposing it is a Type II collapse and let $R_\star$ be the critical point such that $M_{\text{BH}}(R_\star) = 0$, one can compute the critical index $\gamma$ defined by

$$M_{\text{BH}} \propto (R - R_\star)^\gamma.$$  \hspace{1cm} (11)

The value assuming the theoretical prediction of eq. [10] $R_\star^{th} = \sqrt{1/8\pi\lambda(G\mu)^{-3/2}M_{\text{pl}}^{-1}}$, is $\gamma = 0.39$, see fig. [3]. However, in our simulations we observed $R_\star^{ob} > R_\star^{th}$, giving $\gamma = 0.17$, showing that $\gamma$ is highly dependent on the choice of the actual value of $R_\star$ – of which we are unable to identify with confidence due to the lack of computational resources. Therefore, we conclude that $\gamma = 0.28 \pm 0.11$.

In the subcritical limit where $2GM_{\text{loop}} < \delta$, the loop unwinds as it collapses, transferring all the mass into matter and gravitational radiation. If $R \gg \delta$ the velocity at unwinding is much larger than the escape velocity and all the energy is radiated away. However, if $R \sim \delta$, the velocity can be small enough so that instead of full dispersal the mass slowly decays at the center and a soliton might form.

**IV. GRAVITATIONAL WAVES FROM BLACK HOLE FORMATION**

We compute the gravitational waveform from the collapse of a loop with $G\mu = 1.6 \times 10^{-2}$ and $R = 100 M_{\text{pl}}^{-1}$ into a black hole, fig. [1]. Post formation of the apparent horizon, the waveform exhibits the characteristic quasinormal mode decay, with the dominant mode being the $l = 2, m = 0$ mode as usual. We found the integrated energy of the signal to be

$$\epsilon \equiv \frac{E_{\text{GW}}}{M_{\text{loop}}} = 0.5 \pm 0.2 \%.$$  \hspace{1cm} (12)

The error bars come primarily from the presence of the spurious modes from the initial data mixing in with the early part of the collapse (grey area in fig. [1]). Even though the velocity of the loop at collision is ultra-relativistic, $c \sim 0.99$, the GW production is strongly suppressed when compared to other ultra-relativistic events. For comparison, a boosted head-on black hole merger (14±3%) and relativistic fluid particle collapse (16±2%) radiates a much larger fraction of its total mass in gravitational waves [21, 22]. This suggests that the initial apparent horizon is very spherical – possibly due to the thickness of our strings when compared to the Schwarzschild radius, i.e. $2GM_{\text{loop}} \sim O(1) \times \delta$. In the limit of infinitesimally thin strings, the maximum GW production was calculated by Hawking to be 29% [17].

Hence, we believe that one can boost the efficiency by colliding thinner strings (i.e. $2GM_{\text{loop}} \gg \delta$) – in this limit the loop conjecture argument above suggests that a black hole will form before the loop has a chance to interact and unwind, thus it is possible that the GW emission will be larger via Hawking’s argument, though this has not been demonstrated numerically.

Finally, loops in general are generated non-circularly with many different oscillating stable configurations. Nevertheless, in the presence of gravity, we expect gravity to eventually win out, with roughly the timescale of their gravitational collapse to be the free-fall time-scale. In the final stages of collapse, we expect the tension to circularize the loops and thus our results should hold in general.
V. DISCUSSION AND DETECTION PROSPECTS

We have extracted the gravitational wave signal for the case $G\mu = 1.6 \times 10^{-2}$, and $R = 100 \, M_\odot^{-1}$ and found that the efficiency $\epsilon = 0.5 \pm 0.2 \%$ of the initial mass is radiated into gravitational waves. The QNM frequency of our GW waveform (fig. 1) is in the UV range and out of any current or future detectors. On the other hand, if we assume that our numerical results scale, we can ask whether we can detect suitably massive cosmic strings loops in current or future detectors. The two key parameters are (i) the frequency and (ii) the luminosity of the event, both which depend on the masses. The former constraints our loop parameter space to $2\pi R \approx M_{\text{detector}}$. We choose $M_{\text{detector}}$ such that its frequency lies at peak sensitivity of LIGO/VIRGO ($f \sim 100 \, Hz$). For the latter, the strain $h$ observed at a distance $d$ from a source of GWs is

$$
\left( \frac{h}{10^{-21}} \right) \sim \sqrt{\frac{E_{\text{GW}}}{3 \times 10^{-3}M_\odot}} \left( \frac{10 \, \text{Mpc}}{d} \right) .
$$

Cosmic strings loops are generated during the evolution of the string network when strings intercommute, although there is presently no consensus on the probability distribution of loops and their classification (see e.g. [23, 24]). Furthermore, it is not clear that all loops will collapse due to the presence of non-intersecting loop configurations and the uncertainty in their angular momentum loss mechanisms. Hence, we will take the agnostic view that only planar loops will collapse—assuming that planar loops will circularize as argued by [25]. Suppose then $N(R, z)$ is the co-moving production density rate of planar loops of radius $R$ at redshift $z$ (i.e. it has dimensions $[N(R, z)] = L^{-3}T^{-1}$), then the detection rate is given by

$$
\Gamma = \int_0^{z_d} 4\pi \left[ \int_0^2 \frac{dz'}{H(z')} \right]^2 \frac{N(r, z)dz}{H(z)} ,
$$

such that $z_d$ is the maximum range in redshift of the detector, which itself depends on the energy of the GW $E_{\text{GW}}$ emitted. Our numerical results eq. (12) suggest that $0.5\%$ of the total string loop mass is emitted, which is an order of magnitude smaller than that of the typical BH-BH mergers, translating to about a factor of 3 shorter in detectable distance $d$. For LIGO/VIRGO and ET, the maximum redshift range is then $z_d \sim 0.005$ and $z_d \sim 0.05$ respectively. In this limit, $\Gamma$ can be approximated as

$$
\Gamma \approx 3^{3/2} \left( \frac{R}{GM_\odot} \right)^{3/2} (G\mu)^{3/2} \left( \frac{10^{-19}}{h} \right)^3 \left( \frac{N(R, z)}{\text{Mpc}^{-3}} \right) .
$$

Clearly, $\Gamma$ depends linearly on $N(R, z)$, which itself depends on the cosmic string model and its network evolution, which at present is still being debated vigorously as mentioned above. For example, in [25], it was estimated that $N(R, z) \propto (G\mu)^{2R/s-4}$ where $s$ is the correlation length of the loop. Other estimates are given in [26, 27]. On the other hand, we can use the non-detection of such collapse events in the present LIGO/VIRGO to put a constraint on $N(R, z)$. For $G\mu \sim 10^{-10}$ which leads to solar system sized loops of $R \sim \mathcal{O}(100)$ a.u., this is $N(R, z) < 10^{-2} \, \text{Gpc}^{-3} \, \text{yr}^{-1}$, which is a lower detection rate than what is expected from BH mergers of $\mathcal{O}(10) \, \text{Gpc}^{-3} \, \text{yr}^{-1}$ [28].

Finally, we note that this is a conservative estimate since these solar system sized loops satisfy $R_{\text{BH}} \sim \mathcal{O}(10^3) \times \delta$ and hence are thin loops. In this limit, $\epsilon$ might be closer to 29 %, with a corresponding increase in $d$. We will numerically investigate the collapse of these thin loops in a future work.

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Appendix A: Numerical Methodology

1. Evolution Equations

In this work, we use GRCHombo, a multipurpose numerical relativity code [29] which solves the BSSN [30-32] formulation of the Einstein equation. The 4 dimensional spacetime metric is decomposed into a spatial metric on a 3 dimensional spatial hypersurface, $\gamma_{ij}$, and an extrinsic curvature $K_{ij}$, which are both evolved along a chosen local time coordinate $t$. The line element of the decomposition is

$$
ds^2 = -\alpha^2 \, dt^2 + \gamma_{ij}(dx^i + \beta^i \, dt)(dx^j + \beta^j \, dt) ,
$$

where $\alpha$ and $\beta^i$ are the lapse and shift, gauge parameters. These gauge parameters are specified on the initial hypersurface and then allowed to evolve using gauge-driver equations, in accordance with the puncture gauge
where the constants $\eta$ and $\mu$ are of order $1/M_{\text{ADM}}$ and unity respectively.

The induced metric is decomposed as

$$\gamma_{ij} = \frac{1}{\chi} \tilde{\gamma}_{ij} , \quad \text{det} \tilde{\gamma}_{ij} = 1 , \quad \chi = (\text{det} \gamma_{ij})^{-\frac{1}{2}} .$$

(A5)

The extrinsic curvature is decomposed into its trace, $K = \gamma^{ij} \tilde{K}_{ij},$ and its traceless part $\tilde{\gamma}^{ij} \tilde{A}_{ij} = 0$ as

$$K_{ij} = \frac{1}{\chi} \left( \tilde{A}_{ij} + \frac{1}{3} K \tilde{\gamma}_{ij} \right) .$$

(A6)

The conformal connections are $\tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}_{jk}$ where $\tilde{\gamma}_{jk}$ are the Christoffel symbols associated with the conformal metric $\tilde{\gamma}_{ij}.$ The evolution equations for the gravity sector of BSSN are then

$$\partial_t \gamma_{ij} = \frac{2}{3} \chi \partial_k \beta^k + \beta^k \partial_k \chi ,$$

(A7)

$$\partial_t \tilde{\gamma}_{ij} = -2 \alpha \tilde{A}_{ij} + \tilde{\gamma}_{ij} \partial_t \beta^k + \tilde{\gamma}_{jk} \partial_t \beta^k$$

$$- \frac{2}{3} \tilde{\gamma}_{ij} \partial_t \beta^k + \beta^k \partial_t \tilde{\gamma}_{ij} ,$$

(A8)

$$\partial_t K = -\gamma^{ij} D_i D_j \alpha + \alpha \left( \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right)$$

$$+ \beta^i \partial_t K + 4 \pi \alpha (\rho + S) ,$$

(A9)

$$\partial_t \tilde{A}_{ij} = \chi \left[ -D_i D_j \alpha + \alpha (R_{ij} - 8 \pi \alpha S_{ij}) \right]_{\text{TP}}$$

$$+ \alpha (K \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^l_j)$$

$$+ \tilde{A}_{ik} \partial_j \beta^k + \tilde{A}_{jk} \partial_i \beta^k$$

$$- \frac{2}{3} \tilde{A}_{ij} \partial_t \beta^k + \beta^k \partial_t \tilde{A}_{ij} ,$$

(A10)

$$\partial_t \tilde{\Gamma}^i = 2 \alpha \left( \tilde{\gamma}_{jk} \tilde{A}^{jk} - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K - \frac{2}{3} \tilde{A}^{ij} \partial_j \chi \right)$$

$$- 2 \tilde{A}^{ij} \partial_j \alpha + \beta^k \partial_k \tilde{\Gamma}^i$$

$$+ \tilde{\gamma}^{jk} \partial_j \partial_t \beta^i + \frac{1}{3} \tilde{\gamma}^{ij} \partial_i \partial_t \beta^k$$

$$+ \frac{2}{3} \tilde{\gamma}^i \partial_k \beta^k - \tilde{\gamma}^i \partial_k \beta^k - 16 \pi \alpha \tilde{\gamma}^{ij} S_j .$$

(A11)

Meanwhile, the matter part of the Lagrangian is

$$\mathcal{L}_m = -(D_{\mu} \phi)^* (D^\mu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi) ,$$

(A12)

which gives the evolution equations

$$- D_\mu D^\mu \phi + \frac{\partial V(\phi)}{\partial \phi} = 0 ,$$

(A13)

$$\nabla_\mu F^{\mu\nu} = -e J^\nu ,$$

(A14)

with

$$J^\nu = 2 \Im (\phi^* D^\nu \phi) , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

(A15)

We decompose these equations in 3+1 coordinates, following [35]. Furthermore, we impose the Lorenz condition

$$\nabla^\mu A_\mu = 0 .$$

(A16)

Using the projector

$$P^\mu_\nu = \delta^\mu_\nu + n_\mu n^\nu ,$$

(A17)

where $n^\mu$ is the normal to the hypersurface, the gauge field and current can further be decomposed into traverse and longitudinal components via

$$A_\mu = A_\mu + n_\mu A , \quad J_\mu = J_\mu + n_\mu J ,$$

(A18)

such that

$$A_\mu = P_\mu^\nu A_\nu \quad \text{and} \quad A = -n^\nu A_\nu , \quad J_\mu = P_\mu^\nu J_\nu \quad \text{and} \quad J = -n^\nu J_\nu .$$

(A19)

The electric and magnetic fields are defined as

$$E_\mu = P_\mu^\nu n^\rho F_{\nu\rho} ,$$

(A20)

$$B_\mu = P_\mu^\nu n^\rho (\star F_{\nu\rho}) ,$$

(A21)

where $(\star F_{\nu\rho})$ is the dual Maxwell tensor. Using the previous decomposition we rewrite the Maxwell tensor as

$$F_{\mu\nu} = n_\mu E_\nu - n_\nu E_\mu + \partial_\mu A_\nu - \partial_\nu A_\mu .$$

(A22)
Hamiltonian constraint violation
gauss constraint violation
apparent horizon forms

\[ \nu \]

where \( \tilde{\text{negligible throughout.}} \)

\section*{FIG. 5. \( L^2 \) norm of constraints:} Loop with \( G_\mu = 1.6 \times 10^{-2} \) and \( R = 100 \, M_\odot^{-1} \) remains stable throughout evolution, even after black hole formation. The initial Hamiltonian constraint is smaller than it can be maintained by the evolution scheme. The momentum constraints violation are negligible throughout.

In addition, eq. [A14] gives the Gauss constraint

\[ \nabla_i E^i = \epsilon J, \quad (A23) \]

where \( \nabla = P^\nu_{\nu} \nabla_\nu. \)

To ensure that numerical violation of eq. [A23] is kept to a minimum, we stabilise it by introducing an auxiliary damping variable \( Z \) [55, 57], resulting in the following modified evolution equations

\[ \partial_t E^i = \alpha (E_i - \epsilon J^i + \tilde{\nabla}_i A) - \bar{A} \nabla_i \alpha + \beta^i \partial_j E^i + \nabla^i \beta^j - \frac{1}{2} \overline{\nabla^i \beta^j}, \quad (A24) \]

\[ \partial_t A = -A^i \nabla_i \alpha + \alpha (K A - \nabla_i A^i - Z) + \beta^i \partial_j A_i, \quad (A25) \]

\[ \partial_t A_i = -\alpha (E_i + \tilde{\nabla}_i A) - \bar{A} \nabla_i \alpha + \beta^l \partial_j A_i + \partial_i \beta J, \quad (A26) \]

\[ \partial_t Z = \alpha (\nabla_i E^i - \epsilon J - \kappa Z) + \beta^i \partial_j Z. \quad (A27) \]

From fig. 3 we see the scheme is effective at stopping the growth of constraint violations.

Finally, we decompose the complex scalar field

\[ \phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2), \quad (A28) \]

and rewriting the matter equation with BSSN variables,

\[ \partial_t \phi = \alpha \Pi M, a + \beta^i \partial_i \phi_a, \quad (A29) \]

\[ \partial_t \Pi M, a = \beta^i \partial_i \Pi M, a + \alpha \partial_t \partial_j \phi_a + \partial_i \phi_a \partial^i \alpha \]

\[ + \alpha \left( K \Pi M, a - \gamma^{ij} T_{ij} \partial_k \phi_a + \frac{dV}{d\phi_a} \right) + \alpha (e^2 A_\mu A^\mu \phi_a \pm 2 e A_\mu \partial \phi_a + 1), \quad (A30) \]

\[ \partial_t E^i = \alpha K E^i + \epsilon a \gamma^{ij} J_f + \alpha \gamma^{ij} \partial_j Z \]

\[ + \frac{1}{2} \overline{\nabla^i \beta^j} (D_k \partial_j A_i + \overline{D_k \partial_i A_j}) \]

\[ + \alpha \left( \nabla_i E^i - E^i \partial_j \beta^i - \epsilon \alpha J^i \right), \quad (A31) \]

\[ \partial_t A = \alpha K A - \epsilon a \gamma^{ij} \partial_j A_i + \alpha \gamma^{ij} \partial_j A_i = \alpha Z \]

\[ + \frac{1}{2} \overline{\nabla^i \beta^j} (D_k \partial_j A_i + \overline{D_k \partial_i A_j}) \]

\[ \partial_t A_i = -\alpha \gamma^{-1} \gamma^{ij} E^j = \alpha \partial_t A - \partial \phi \alpha \]

\[ + \beta^l \partial_j A_i + \partial_i \beta A_j, \quad (A32) \]

\[ \partial_t Z = \alpha (\nabla_i E^i - 3 \alpha \epsilon E^i \partial_j \alpha - \epsilon \alpha J - \alpha \kappa Z + \beta^i \partial_j Z, \quad (A33) \]

where \( a \in \{1, 2\} \) and the second order Klein Gordon equation has been decomposed into two first order equations as usual. The stress energy tensor for Abelian Higgs is

\[ T_{\mu \nu} = D_{(\mu} \phi^* D_{\nu)} \phi + F_{\mu \nu} F^\alpha_{\nu} + g_{\mu \nu} L_m, \quad (A35) \]

and its various components are defined as

\[ \rho = n_a n_b T^{ab}, \quad S_i = -\gamma_{ia} n_b T^{ab}, \]

\[ S_{ij} = \gamma_{ia} \gamma_{jb} T^{ab}, \quad S = \gamma^{ij} S_{ij}. \quad (A36) \]

The Hamiltonian constraint

\[ H = R + K^2 - K_{ij} K^{ij} - 16 \pi \rho, \quad (A37) \]

the momentum constraint

\[ \mathcal{M}_i = D_i (\gamma_{ij} K - K_{ij}) - 8 \pi S_i, \quad (A38) \]

and the Gauss constraint

\[ Z = \nabla_i E^i - \epsilon J^i n^i, \quad (A39) \]

are monitored throughout the evolution to check the quality of our simulations (see fig. 5). Our boundary conditions are Dirichlet.

2. Initial Data

We set up the field as mentioned in the main text using toroidal coordinates (see fig. 7). Time symmetry is assumed for our initial data,

\[ K = 0, \quad A_{ij} = 0 \quad (A40) \]
which automatically fulfils the momentum constraint (eq. A38). In addition, we make a conformally flat\footnote{This is not the unique solution to the constraint equations given the initial field configuration. However, it is the most easily implemented, as more general initial conditions require much greater computational resources to find. Conformal flatness is also consistent with the fact that the spacetime is asymptotically Schwarzschild.} ansatz $\tilde{\gamma}_{ij}$,
\begin{equation}
\tilde{\gamma}_{ij} = \delta_{ij}, \tag{A41}
\end{equation}
and impose the metric to be identity in the center of the string, similar as the static string (see eq. B35). We find that doing so reduces possible excitations of the string. For the gravitational wave extraction, we impose the condition
\begin{equation}
\lim_{r \to \infty} \chi = 1. \tag{A42}
\end{equation}

We solve for $\chi$ using the Hamiltonian constraint eq. A37. We reduce the spatial dimension of the problem by using its cylindrical symmetry. This solution is then further relaxed to obtain the final solution, which is that of an excited cosmic string loop.

As shown in fig. 6, the relative Hamiltonian violation from our prescription is
\[ H_{\text{rel}} = \frac{H}{16\pi \rho_{\text{max}}} < 1\% . \]

3. Numerical Extraction of Signal

We extract the Penrose scalar $\Psi_4$ with tetrads proposed by [38]. Similarly as in black hole binaries, there is some non-physical radiation associated with the initial data, which in our case consists of a toroidal shell of artificial radiation resulting in two GW peaks before the physical signal. While such stray-GW can often be ignored as they quickly radiate away at light speed, in our case due to the rapid collapse of our loops at ultra-relativistic speeds, they cannot be ignored.

The first peak at $t_{\text{ret}} < 0$ is due to this initial radiation travelling opposite to the collapse and could be separated by increasing the loop radius so that the real signal takes longer. However, the second peak (first peak in fig. 1) results from the radiation which travels together with the collapsing loop at similar velocity, which always mixes with the real signal. In any case, increasing the loop radius would result in a cleaner signal but this is
computationally very expensive.
To estimate the GW energy we use the equation
\[
\frac{dE_{GW}}{dt} = \frac{r^2}{16\pi} \int_{S_r} \left| \int_{t_{in}}^{t_{out}} |\Psi_4 dt'| \right|^2 d\Omega ,
\]
(A43)
where \(S_r\) is a sphere of radius \(r\).
In the cases for which the cosmic string loop does not form a black hole, most of the matter will escape, typically at velocities close to the speed of light. This scalar and vector radiation overlaps the gravitational wave signal and due to its large mass might leave an imprint on \(r\Psi_4\), making the signal extraction problematic.

4. Numerics and Convergence Tests

In fig. 5 we show that the volume-averaged Hamiltonian constraint violation
\[
L^2(H) = \sqrt{\frac{1}{V} \int_V |H^2| dV} ,
\]
(A44)
where \(V\) is the box volume with the interior of the apparent horizon excised, is under control throughout the simulation.

We use the gradient conditions on \(\phi\) and \(\chi\) to tag cells for regridding. The precise criteria is chosen depending on the symmetry breaking scale \(\eta\) and the total mass of the system. The major distinction for the amount of resolution needed is whether GW are being extracted or not. To obtain a clean GW large boxes are needed to avoid the detection of reflections of the non-physical signal with the boundaries, which increases the cost of the simulation. We double checked that our signal in fig. 1 was consistent with a \(l = 2\) \(m = 0\) QNM within numerical error.

We tested the convergence of our simulations with a cosmic string loop of \(\eta = 0.05\) \((G\mu = 1.6 \times 10^{-2})\) and \(R = 100\) \(M_{pl}^{-1}\) by using a box of size \(L = 2048\) \(M_{pl}^{-1}\) in which we improved by a factor of 1.5 between all three resolutions. The convergence of \(r\Psi_4\) for low (\(\Delta x_{\text{min}} = 1.33\) \(M_{pl}^{-1}\)), medium (\(\Delta x_{\text{min}} = 1.00\) \(M_{pl}^{-1}\)) and high (\(\Delta x_{\text{min}} = 0.66\) \(M_{pl}^{-1}\)) resolutions is shown in fig. 8.

Appendix B: Abelian Higgs Code Test

To test the code, we compare the evolution of a simulation with a known semi-analytic case of the infinite static string. Given the symmetry of the problem we use polar coordinates
\[
x = r \cos(\theta) ,
\]
\[
y = r \sin(\theta) ,
\]
\[
z = z .
\]
\[\Phi = f(r)e^{in\theta} ,
\]
(0) \(\alpha(\infty) = 1 .
\]
\[
A(0) = 0 ,
\]
\[
B(0) = 0 ,
\]
(B5)
For the metric, the following ansatz is chosen
\[
ds^2 = -e^{A(r)} dt^2 + e^{B(r)}(dr^2 + r^2 d\theta^2) + e^{A(r)} dz^2 ,
\]
(B4)
where \(A(r)\) and \(B(r)\) are radial functions numerically determined. We impose the metric and its derivatives to be locally flat
\[
A(0) = 0 ,
\]
\[
B(0) = 0 ,
\]
\[\text{B.7}\]
\[\text{B.7}\]
iteratively as follows. We solve the Klein-Gordon equation (eq. B7) for fixed flat background, then use this solution to calculate the stress-energy tensor and retrieve the values of \(A(r)\) and \(B(r)\) via B6 to build a new metric. Plugging this back into the Klein-Gordon equation we find new profiles for the fields using the new metric as background. The solution converges quickly (within \(\sim 5\) iterations), see fig. 9 for the obtained profiles of \(f\) and \(\alpha\).

FIG. 9. Radial profile of \(\alpha\) and \(f\) for an infinite static string with gravity in the critical coupling limit \((e = 1, \lambda = 2)\) and \(\eta = 0.05\) \(M_{pl}\) \((G\mu = 1.6 \times 10^{-2})\).
Appendix C: Comparison with Nambu-Goto

Previous work showed that without gravity \cite{40} the Nambu-Goto (NG) action is still valid at relativistic speeds. However, a comparison between the two approaches, leads to consistent results with NG up to roughly the point where the string radius is close to the string thickness (see fig. \ref{fig:comparison}). To reduce gauge effects we use the time of static observer at the position of the string,

$$\tau = \int |\rho_{\text{max}}(\rho)| \, dt .$$  \hfill (C1)

Having shown that NG is a good approximation, we use it to estimate the velocity before unwinding, which we define as the point where the radius of the ring $R$ is equal to the thickness of the string $\delta$. We find

$$v_s = \sqrt{1 - \left(\frac{\delta}{R}\right)^2} ,$$  \hfill (C2)

which, for our simulations, gives results ranging from 0.97 $c$ to 0.99 $c$. In the case for which we extract the gravitational wave signal ($G\mu = 1.6 \times 10^{-2}$, $R = 100 \, M_{Pl}^{-1}$) we estimate a velocity of 0.99 $c$ before collision.

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