CONDENSATION, BOUNDARY CONDITIONS, AND EFFECTS OF SLOW SITES IN ZERO-RANGE SYSTEMS

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Abstract. We consider the space-time scaling limit of the particle mass in zero-range particle systems on a 1D discrete torus \( \mathbb{Z}/N\mathbb{Z} \) with a finite number of defects. We focus on two classes of increasing jump rates \( g \), when \( g(n) \sim n^\alpha \), for \( 0 < \alpha \leq 1 \), and when \( g \) is a bounded function. In such a model, a particle at a regular site \( k \) jumps equally likely to a neighbor with rate \( g(n) \), depending only on the number of particles \( n \) at \( k \). At a defect site \( k_{j,N} \), however, the jump rate is slowed down to \( \lambda_j^{-1}N^{-\beta_j}g(n) \) when \( g(n) \sim n^\alpha \), and to \( \lambda_j^{-1}g(n) \) when \( g \) is bounded. Here, \( N \) is a scaling parameter where the grid spacing is seen as \( 1/N \) and time is speeded up by \( N^2 \).

Starting from initial measures with \( O(N) \) relative entropy with respect to an invariant measure, we show the hydrodynamic limit and characterize boundary behaviors at the macroscopic defect sites \( x_j = \lim_{N \to \infty} k_{j,N}/N \), for all defect strengths. For rates \( g(n) \sim n^\alpha \), at critical or super-critical slow sites \( (\beta_j = \alpha \text{ or } \beta_j > \alpha) \), associated Dirichlet boundary conditions arise as a result of interactions with evolving atom masses or condensation at the defects. Differently, when \( g \) is bounded, at any slow site \( (\lambda_j > 1) \), we find the hydrodynamic density must be bounded above by a threshold value reflecting the strength of the defect. Moreover, due to interactions with masses of atoms stored at the slow sites, the associated boundary conditions bounce between being periodic and Dirichlet.

1. Introduction

The purpose of this article is to understand the macroscopic boundary conditions which arise in hydrodynamic scaling limits for the space-time ‘bulk’ mass evolution of zero-range processes with a finite number of ‘defects’, and also related effects of ‘condensation’ at these defect locations. Such an aim more broadly fits into the larger study of how macroscopic boundary conditions emerge from inhomogeneous microscopic interactions.

In this view, there has been much interesting work on one-dimensional (1D) exclusion models where a site or bond is ‘slowed’ down. In [12], [13], [14], in computing the hydrodynamic limit in symmetric systems, different boundary conditions from Dirichlet to Neumann, and also Robin have been derived; see also [10]. There are however only a few works with respect to different interactions, in particular zero-range systems, which do not limit the particle numbers at a location. Among these, [17] studies the hydrodynamic limit for a system of 1D symmetric independent particles moving in a (random) trap environment. In [19], with respect to a 1D totally asymmetric zero-range process, with bounded, increasing rate function \( g(\cdot) \), effects of a slow site and a slow particle are found when starting from a ‘flat’ initial measure. In [4] (see also [5]), with respect to a class of such 1D totally asymmetric zero-range systems, however with a nontrivial density of site disorders, hydrodynamics is

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shown with respect to an effective flux function, constant at supercritical densities; see also the conjectures in Section 3.3 in [4] about slow sites in asymmetric models.

We also mention that 1D systems with ‘reservoir’ boundaries have been studied; see [9] for a discussion of ‘hydrostatics’, and related references, with respect to exclusion models. In zero-range processes, ‘static’ reservoir effects are studied and dynamical conjectures are discussed in [14].

In this context, the general goal of our work is to consider the effect of a finite number of ‘slow’ sites in a class of 1D symmetric zero-range systems in a torus \( \mathbb{Z}/N\mathbb{Z} \). In such a model, particles would ‘condense’ on a defect site if jump rates from it are ‘slow’ enough. One would expect that for the particle continuum mass, different boundary conditions would result depending on how ‘slow’ the defect is. Our main results describe corresponding hydrodynamic limits, in terms of a nonlinear parabolic PDE and evolving point masses, with specified boundary conditions at defects reflecting different types of condensation. To our knowledge, these are the first results for a general class of zero-range interactions, with bounded and unbounded rate functions \( g \). In particular, the behaviors found differ depending on the structure of the rate \( g \). The proof method, in the scheme of the ‘entropy’ method, specifies local ‘replacements’ which may be useful in other problems.

By specifying the locations of the ‘slow’ defects in the system, we fix the macroscopically separated points where ‘condensation’ can occur. There seems to be little work on the dynamical structure in such systems. This is in contrast to the well-developed study of ‘condensation’, which ‘spontaneously’ forms at a random location by introducing more particles in a zero-range system with a bounded decreasing rate function than is allowed to equilibrate. See [2] for a discussion of both mechanisms with respect to canonical and grand canonical invariant measures of a bounded rate zero-range process with a single defect.

In passing, we mention, among the recent literature on the ‘spontaneous’ formation of ‘condensation’, [3] considers, with respect to a thermodynamic limit in 1D symmetric zero-range models, the evolution of the random ‘condensate’ in a certain time-scale. In [20], with respect to 1D asymmetric dynamics in a set of \( L \) fixed sites, motion of the ‘condensate’ is described. In [6], a ‘martingale problem’ approach for the condensate dynamics is developed. In [23], some partial results on a hydrodynamic limit is given. See also references therein in these papers for a more complete history of the subject. For related notions of ‘metastability’, see books and surveys [7], [21], [24].

1.1. Sketch of results. To describe our results, we consider zero-range processes on the torus \( T_N := \{0, 1, \ldots, N-1 \} \) where the site \( N \) (the right neighbor of \( N-1 \)) is identified as 0. The jump rate functions \( g : \mathbb{N}_0 \mapsto [0, \infty) \) focused upon satisfy \( \lim_{n \to \infty} g(n)/n^\alpha = 1 \) (to fix a time-scale) and \( \alpha \in [0, 1] \). Moreover, we classify the \( g \)’s into two families: (1) \( g(n) \sim n^\alpha \) when \( 0 < \alpha \leq 1 \), and (2) \( g \) is bounded. In both settings, we will assume also that \( g \) is a Lipschitz, increasing function, which rules out the case \( \alpha > 1 \). Such an assumption is convenient to bound the rate \( g \) in estimates, and also allows ‘attractive’ process couplings; see Section 3.2 for a discussion of the use of ‘attractive’ coupling.

In this process, at a regular site \( k \in T_N \), if there are \( n \) particles there, one of them leaves with rate \( g(n) \) to a neighbor, jumping either to \( k-1 \) or \( k+1 \), with equal probability. However, at a defect site \( k \), the departure rate is altered as follows: Let \( \lambda > 0 \) and \( \beta \in \mathbb{R} \). If there are \( n \) particles at the defect \( k \), one of them leaves at rate \( \frac{1}{\lambda n} g(n) \). We will say the rate is ‘slow’ when \( \beta > 0 \), or \( \beta = 0 \) and \( \lambda > 1 \).

The zero-range system tracks the evolution of the unlabeled particles on \( T_N \). We denote by \( \xi_t = \{\xi_t(k) : k \in T_N\} \) the configuration of the process, where \( \xi_t(k) \) is the number of
particles at site $k$ at time $t \geq 0$. Given the symmetric transitions, it will be useful to define also the speeded-up process $\eta_t = \xi_{N^{-2}}$. For this Markov system, there is a family of product (reversible) invariant measures $\mathcal{R}_c^N$, indexed by an interval of ‘density’ parameters $c$; see Section 2.1.

For the system with rate $g(n) \sim n^{\alpha}$, we will start the process from measures $\mu^N$, associated to an initial macroscopic measure $\pi_0$ on the unit torus $T = [0, 1)$, with $O(N)$ relative entropy with respect to an invariant distribution $\mathcal{R}_c^N$, and stochastically bounded by another invariant distribution $\mathcal{R}_c^N$. With respect to bounded rates $g$, $\mu^N$ satisfies a similar but slightly different criteria, since when $\|g\|_\infty = g_\infty < \infty$, there will be a finite effective critical density above which the invariant measure is not defined; see Condition 3.1.

Here, in the setting $g(n) \sim n^{\alpha}$, the initial profile $\pi_0$ will be in form
\[
\pi_0(dx) = \rho_0(x)dx + \sum_{j, \beta_j = \alpha} m_{0,j} \delta_{x_j}(dx),
\]
with a similar formulation in the $g$ bounded setting. Examples of suitable initial measures $\mu^N$ are given by local equilibrium product measures; see Section 3.3.

1.1.1. Rates $g(n) \sim n^{\alpha}$. To describe the main result in the setting $g(n) \sim n^{\alpha}$, it will be helpful to get a sense of the ‘condensation’ of particles, under an invariant measure $\mathcal{R}_c^N$. Typically at a slow site, (1) the number of particles will be $O(N^{\beta/\alpha})$ when $\beta > \alpha$, referred to as ‘super-slow’ or ‘super-critical’, (2) order $O(N)$ when $\beta = \alpha$, called ‘critical’, and (3) $o(N)$ when $\beta < \alpha$, named as ‘sub-critical’. So, with a finite number of slow sites, if one of the $k_{j,N}$’s is ‘super-slow’, there will be a superlinear $O(N^{\beta/\alpha})$ number of particles at that location. Whereas, when all the $\beta_j \leq \alpha$, there will be $O(N)$ particles on $T_N$. See Section 2.2 for precise statements.

Let $\mathcal{D}_{s,N}$ be the set of super-slow sites $k_{j,N}$, and let $\mathcal{D}_s$ be the corresponding set of continuum points $x_j \sim k_{j,N}/N$. Since, at these locations, the particle numbers are superlinear, we omit them in the definition of the empirical measure with respect to the diffusively scaled system:
\[
\pi_t^N = \frac{1}{N} \sum_{k \in T_N \setminus \mathcal{D}_{s,N}} \eta_t(k) \delta_{k/N}.
\]
By a hydrodynamic limit, we mean, with respect to test functions $G \in C(T)$, that $\langle G, \pi_t^N \rangle$ converges in probability to $\langle G, \pi_t \rangle$, where $\pi_t$ is a measure-valued weak solution of a specified macroscopic evolution.

Our first result (Theorem 4.1) is that three different behaviors and boundary conditions, at the macroscopic defect sites $x_j$, arise in the hydrodynamic limit depending on when $\beta_j > \alpha$, $\beta_j = \alpha$ and $\beta_j < \alpha$. As might be suspected, when $\beta_j < \alpha$, the defect is not ‘slow’ enough to be seen in the continuum limit. However, defects where $\beta_j \geq \alpha$ on the other hand do register. To describe the bulk mass macroscopic flow, segregate the unit torus $T$ into intervals with endpoints $x_j$ corresponding to $\beta_j \geq \alpha$. In the interior of each interval, the hydrodynamic limit is described by a nonlinear heat equation
\[
\partial_t \rho = \partial_{xx} \Psi(\rho).
\]
Here, the ‘fugacity’ function $\Psi$ is a continuum homogenization of the microscopic rate $g$ (cf. (2.3)). We note this equation also arises in the context of the zero-range system without disorder. (cf. Chapter 4 [18]).

However, at a defect point $x_j$ where $\beta_j = \alpha$, an atom evolves with a mass given by $m_j(t) = \{\lambda_j(\Psi(\rho(t,x_j)))\}^{1/\alpha}$, in terms of $\rho$. Such a statement is natural, given that under
\( \rho^N \), not far from \( \mathcal{B}^{N}_{\alpha} \), there are \( O(N) \) particles at \( k_{j,N} \). Differently, when \( \beta_j > \alpha \), we will observe initially a superlinear \( O(N^{\beta_j/\alpha}) \) number of particles at \( k_{j,N} \) due to the relative entropy bound assumption. This level of condensation will not evolve much for \( t > 0 \) and, as a result, leads to a macroscopic boundary condition \( \rho(t, x_j) = c_0 \) where \( c_0 \) is the parameter of the reference measure \( \mathcal{B}^{N}_{\alpha} \), reflecting a level of initial condensation at \( x_j \). The derivation of such Dirichlet boundary condition at super-slow sites (\( \beta_j > \alpha \)), to our knowledge, is novel.

Taken together, the hydrodynamic limit \( \pi_t \) on \( T \) will be in form

\[
\pi_t = \rho(t, x)dx + \sum_{j: \beta_j = \alpha} m_j(t)\delta_{x_j}(dx), \quad t > 0.
\]

At time \( t = 0 \), \( \pi_t \) reduces to \( \pi_0 \) mentioned earlier. In general, the PDE (1.1) for the bulk mass \( \rho \) will not be closed without \( m_j(t) \) being specified. In fact, the limit \( \pi_t \) is characterized as the unique weak solution to a system in terms of \( (\rho(t, x), \{m_j(t)\}_{j}) \):

\[
\partial_t \pi_t = \partial_{xx} \Phi(\rho)
\]

with boundary behaviors of \( \rho(t, x) \) at slow sites \( x_j \) given earlier; see Definition 1.1 and uniqueness Theorem 1.1. See Example 4.7 for a specific discussion when there is only one defect in the system. We mention in [17], when \( g(n) \equiv n \) (\( \alpha = 1 \)), that is for independent particles, a case of the above hydrodynamic limit may be inferred when \( \beta_j = \alpha = 1 \), with \( \Phi(u) \equiv u \).

1.1.2. Bounded rates \( g \). Our second result (Theorem 4.2) concerns the process with a rate function \( g(n) \), bounded and increasing say to level \( g_\infty = 1 \) in the limit as \( n \uparrow \infty \). Slow sites \( k_{j,N} \) that we consider will be those where the jump rate is \( \frac{1}{\lambda_j N^{\beta_j}} g(n) = \frac{1}{\lambda_j} g(n) \), when \( \beta_j = 0 \) and \( \lambda_j > 1 \), when there are \( n \) particles at \( k \). As before, with a finite number of slow sites \( k_{j,N} \), there is a family of product invariant measures \( \{\mathcal{B}^{N}_{\alpha}\} \), but now with densities \( c \) limited to \( 0 \leq \max_j \{\lambda_j N^{\beta_j} \vee 1\} \Phi(c) \leq g_\infty \). There is no non-trivial product invariant measure for \( c \) above this level. With this understanding, it is natural to focus on \( \beta_j = 0 \), as if we would slow down with \( \beta_j > 0 \), there would be no such non-trivial invariant measure, as the fugacity \( \lambda_j N^{\beta_j} \Phi(c) \) would exceed \( g_\infty \) for a finite \( N \).

Phenomenologically, the slow site effect is different here than when \( g(n) \sim n^\alpha \) in that the system and in particular the slow sites under \( \mathcal{B}^{N}_{\alpha} \) have at most \( O(N) \) particles on them. As before, between macroscopic defects \( \{x_j \sim k_{j,N}/N\} \), the hydrodynamic limit satisfies (1.1). An atom of mass \( m_j(t) \) may also form though at macroscopic site \( x_j \sim k_{j,N}/N \) where \( \beta_j = 0 \) and \( \lambda_j > 1 \).

However, unlike in the case \( g(n) \sim n^\alpha \), this atomic mass may be evanescent: We show that, at such a defect \( x_j \), the hydrodynamic density \( \rho(t, x_j) \) satisfies a bound \( \rho(t, x_j) \leq c_{j,max} \) where \( c_{j,max} := \Phi^{-1}(g_\infty/\lambda_j) \) for all \( t > 0 \). When this inequality is strict, it holds necessarily that \( m_j(t) = 0 \). In particular, the boundary condition near \( x_j \) may bounce between being periodic and Dirichlet.

For an informal description, suppose that \( \rho(t, x_j) \) is strictly less than \( c_{j,max} \) at \( t = t_0 \) and \( m_j(t_0) = 0 \). Here, the macroscopic flow will evolve as if the defect at \( x_j \) were not present, namely, we will observe a periodic boundary condition at \( x = x_j \). This will last until the first moment \( t = t_1 \) when the density \( \rho(t, x_j) \) tends toward exceeding the threshold value \( c_{j,max} \). The defect is then ‘set off’ to keep \( \rho(t, x_j) \) at the value \( c_{j,max} \), with any excess mass stored at the defect to form a delta mass. In other words, we start to observe \( m_j(t) > 0 \) as well as a Dirichlet boundary condition \( \rho(t, x_j) = c_{j,max} \) after \( t = t_1 \). Besides being a place of
storage, the slow site \( x_j \) also serves as a source of mass to maintain the mentioned Dirichlet boundary when needed as long as \( m_j(t) > 0 \). When the stored mass is used up (when \( m_j(t) \) becomes 0) and the density \( \rho(t, x_j) \) tends to drop below \( c_{j, \max} \), say at \( t = t_3 \), the boundary condition switches to the periodic one again after \( t = t_3 \). Such a ‘bouncing’ phenomenon between boundary conditions takes place at all slow sites. Finally, characterization of the limit

\[
\pi_t = \rho(t, x)dx + \sum_{j; \beta_j=0, \lambda_j>0} m_j(t)\delta_{x_j}(dx)
\]

is given through a weak formulation with boundary conditions described earlier, shown to have a unique solution; see Definition 1.2 and uniqueness Theorem 1.2.

We remark that a sufficient condition to not feel the defects \( \{x_j\} \) in the limit would be to start the process from initial (local equilibrium) measures \( \mu^N \) associated to \( \pi_0(dx) = \rho_0(x)dx \), where \( \|\rho_0\|_\infty < \Phi^{-1}(\max_j\{\lambda_j\})^{-1} \), so that by say attractiveness \( \rho(t, x) \) also satisfies this bound. On the other hand, a sufficient condition so that atom masses form is that the initial density \( \rho_0(x) > \Phi^{-1}(\max_j\{\lambda_j\})^{-1} \) on \( \mathbb{T} \): Indeed, if no atoms are formed, by the maximum principle, \( \rho(t, x) > \Phi^{-1}(\max_j\{\lambda_j\})^{-1} \) on \( \mathbb{T} \), a contradiction. We believe the results in Theorem 4.2 are the first for a general class of bounded symmetric zero-range processes with defects.

1.2. Proof ideas for Theorems 4.1 and 4.2 We now discuss ideas in the proofs. We use the general scheme of the ‘entropy’ method, discussed in [18], although there are several departures since the dynamics is not translation invariant. In particular, mixing in time estimates are used to make microscopic to macroscopic homogeneizations in the ‘bulk’. However, to capture the boundary effects and correspondences with macroscopic mass of atoms, we need to estimate the local behavior near defects, for which we give a more refined argument.

Let \( G \) be a test function on \( \mathbb{T} \) when \( g \) is bounded, and with compact support away from super-slow sites \( x_j \in \mathcal{D}_s \) when \( g \sim n^\alpha \). Consider the stochastic differential

\[
d(G, \pi^N_t) = \frac{1}{N} \sum_{k \in \mathbb{T}_N} \Delta_N G(k/N)g_k(\eta_t(k))dt + dM^N(t).
\]

Here, \( g_k = \frac{1}{\lambda_jN} \eta \) when \( k = k_jN \) and \( g_k = g \) otherwise. Also, \( M^N(t) \) is a martingale, which will vanish as \( N \uparrow \infty \). We state a ‘bulk’ replacement in Lemma 8.3. The proof that we give makes use of local replacements, as these will be useful to deduce boundary behaviors at the defects.

In particular, we may replace the local function \( g(\eta_t(k)) \) at regular sites \( k \) by \( \Phi(\eta^N_t) \) where \( \eta^N_t \) is the \( \ell \)-window average \( \frac{1}{\lambda_j N} \sum_{|y-k|\leq \ell} \eta(y) \), by use of an entropy inequality and a ‘Rayleigh’ estimate. The ‘Rayleigh’ estimate controls the difference between expectations of a local function under non-equilibrium and equilibrium measures in terms of a Dirichlet form and a term written in terms of the spectral gap of the process localized to an interval. No special bound is required–this gap needs only to be positive, or equivalently that the localized dynamics is ergodic. We remark, as a technical device, attractiveness is used to handle large densities, difficult to analyze otherwise, in the proofs of these local replacement, ‘1-block’ Lemma 8.1 and ‘2-block’ Lemma 8.2. As a consequence of these estimates, the hydrodynamic equation (1.1) may be derived in the ‘bulk’. We now discuss derivation of the boundary conditions in the two settings.
1.2.1. **Boundary derivations for** $g(n) \sim n^\alpha$. In the setting where $g(n) \sim n^\alpha$ at a defect $k_{j,N}$, when $0 < \beta_j < \alpha$, under $\mu^N$, the number of particles $\eta_t(k_{j,N}) = O(N^{\beta_j/\alpha})$ is sublinear in $N$. Hence, with respect to the empirical measure, the term

$$\langle G, N^{-1} \delta_{k_j,N} \rangle = N^{-1} \eta_t(k_{j,N}) G(k_{j,N}) \leq \|G\|_\infty N^{-1} \eta_t(k_{j,N}) = O(N^{-1+\beta_j/\alpha})$$

vanishes as $N \uparrow \infty$. In effect, we can ignore such defects in the continuum limit.

However, when $\beta_j \geq \alpha$, to deduce a non-trivial boundary condition, we consider the rate $g_{k_j,N}(\eta_t(k_{j,N})) = \frac{1}{N} \sum_{y=1}^{\ell} \eta_t(y)$ at the slow site $k_{j,N}$. The following discussion leading to (1.3) will hold actually for both $g$ of $n^\alpha$ type and $g$ bounded settings. To fix ideas, let us say $k_{j,N} = 0$, the origin. Under $\mu^N$, not far from the reversible $\mathfrak{P}_\alpha^N$ in terms of relative entropy, we show that this rate is close to the neighboring rate $g(\eta_t(1))$ say; see Lemma 9.1. By the local ‘1-block’ estimate, we show that

$$\eta_t(1) = 0$$

implies

$$\eta_t(0) = 0, \text{see Lemma 9.2.}$$

Now, by the local ‘2-block’ bound, we have $\Phi(\eta_t^{\ell,+}(0))$ is close to the macroscopic quantity $\Phi((\eta_t^{\theta N,+}(0))$ for $\theta > 0$ small. Together, we conclude the fundamental relation,

$$\Phi(\eta_t^{\theta N,+}(0)) \sim g_0(\eta_t(0)). \quad (1.3)$$

After establishment of tightness of the empirical measure trajectories, and absolute continuity estimates in Lemmas 6.1 and 7.1, we may consider a subsequent $\sigma$-finite limit $\pi$ on $\mathbb{T} \setminus \mathcal{D}$, in the form $\pi = \rho(t, x) dx + \sum_{j: \beta_j = \alpha} m_j(t)$. From (1.3), one has for this limit point that

$$\Phi(\rho(t, 0)) \sim g_0(\eta_t(0)).$$

When $\beta_j = \alpha$, we have

$$\Phi(\rho(t, 0)) \sim g_0(\eta_t(0)) \sim (N^{-1} \eta_t(0))^\alpha \lambda_j^{-1} \sim (m_j(t))^\alpha \lambda_j^{-1},$$

which is the boundary condition relating the atom strength to the local density in Theorem 4.1 see Lemma 7.3.

When $\beta_j > \alpha$, we may take the same passage to obtain

$$\Phi(\rho(t, 0)) \sim g_0(\eta_t(0)) \sim (N^{-\beta_j/\alpha} \eta_t(0))^\alpha \lambda_j^{-1}.$$
1.2.2. Boundary derivations for bounded rate \( g \). We now discuss the case of bounded rate function \( g : \|g\|_{\infty} = g_{\infty} < \infty \). There are subtleties here with respect to boundary estimates, different than in the \( n^{th} \) rate setting, although the bulk hydrodynamic limit between defects \( x_{j} \sim k_{j,N}/N \) follows the same procedure as above to derive the equation \( (1.1) \).

At a slow site \( k_{j,N} \), say again at the origin, slowed down by \( \lambda_{j}^{-1} \) (and \( \beta_{j} = 0 \)), we deduce

\[
\Phi(\rho(t,0)) \sim \frac{1}{\lambda_j} g(\eta_t(0)) \leq \frac{g_{\infty}}{\lambda_j}.
\]

Hence, the density near the slow site \( x_{j} = 0 \) is restricted,

\[
\rho(t,0) \leq \Phi^{-1}(g_{\infty}/\lambda_j).
\]

When the restriction is strict, the intuition is that the particle mass is not high enough to impact much the flow of particle numbers through the slow site 0. However, when equality in the above relation holds, an atom at the slow site 0 may form to store excess mass while maintaining the boundary condition. More precisely, we have \( m_{j}(t) \), the atomic mass accumulated by \( x_{j} = 0 \), satisfies \( m_{j}(t) = m_{j}(t)\mathbb{1}_{\rho(t,0) = \Phi^{-1}(g_{\infty}/\lambda_j)} \); this is established by showing a microscopic version of the following key relation

\[
m_{j}(t)(\Phi^{-1}(g_{\infty}/\lambda_j) - \rho(t,0)) = 0;
\]

see Lemma 7.5. These prescriptions give the boundary conditions specified in Theorem 4.2.

1.3. Paper outline. The plan of the paper is as follows: In Section 2 and 3 we introduce carefully the zero-range model with defects, their invariant measures, and the initial measures considered. In Section 4 we state results; see Section 4.7 for a discussion of the example when there is only one defect in the system. In Section 5 we give the proof outline of the main results, Theorems 4.1 and 4.2 referring to estimates in the sequel. In Section 6 we discuss tightness of the empirical measures. In Section 7 we discuss properties of limit points, including importantly their boundary behaviors near macroscopic defects. In Section 8 we show local ‘1-block’ and ‘2-block’ replacements, and as a consequence ‘bulk’ replacement. In Section 9 we discuss replacements near the boundaries of defects needed to derive the macroscopic boundary conditions. In Sections 10 and 11 we derive energy estimates and prove uniqueness theorems for the weak formulations.

2. Model description

We will consider symmetric zero-range processes on the discrete torus \( T_{N} := \mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N - 1\} \) with a finite number \( n_{0} \geq 0 \) of defects located in \( T_{N} \). We will always assume that \( N > n_{0} \) so that there is enough space in \( T_{N} \) to record the defects.

More carefully, the structure of the defects is the following: Let \( J \) be the index set \( \{1, 2, \ldots, n_{0}\} \). For each \( j \in J \), fix \( (x_{j}, \beta_{j}, \lambda_{j}) \) such that \( x_{j} \in \mathbb{T} \), \( \beta_{j} \in \mathbb{R} \) and \( \lambda_{j} \in (0, \infty) \). Here, \( \mathbb{T} \) stands for \( [0, 1) \) viewed as the unit torus. We will assume that all \( x_{j} \)'s are different, that is macroscopically separated. For each \( j \), the point \( x_{j} \) denotes the macroscopic location of a defect and \( (\beta_{j}, \lambda_{j}) \) characterizes its strength. Let \( \mathcal{D} := \{x_{j}\}_{j \in J} \) be the set of all macroscopic defect locations. For each \( j \in J \) and \( N \in \mathbb{N} \), we now define \( k_{j,N} = [x_{j}N] \) to be the integer part of \( x_{j}N \). Then, the set \( \mathcal{D}_{N} := \{k_{j,N}\}_{j \in J} \) stands for the set of microscopic locations of the defects.

Let now \( N_{0} := \{0\} \subseteq \mathbb{N} \). For each \( N \), let \( \Omega_{N} := \mathbb{N}_{0}^{T_{N}} \) be the space of all particle configurations on \( T_{N} \). With respect to \( \xi \in \Omega_{N} \), at a normal site \( k \in T_{N} \setminus \mathcal{D}_{N} \), a particle jumps to neighboring sites \( k \pm 1 \) equally likely at rate \( g(\xi(k))/\xi(k) \) where \( \xi(k) \) is the number of particles at site \( k \), and \( g : \mathbb{N}_{0} \rightarrow (0, \infty) \) is a jump rate function. At a defect site \( k = k_{j,N} \), the
jump rate is $(\lambda_j N^{\beta_j})^{-1} g(\xi(k))/\xi(k)$. In particular, the site $k_j, N$ is a slow site if $\lambda_j N^{\beta_j} > 1$ and a fast or normal site if $\lambda_j N^{\beta_j} \leq 1$ for $N$ large. Let

$$g_{k, N}(\cdot) = \begin{cases} g(\cdot) & k \in \mathbb{T}_N \setminus \mathcal{D}_N, \\ \frac{g(\cdot)}{\lambda_j N^{\beta_j}} & k = k_j, N, \; j \in J. \end{cases}$$

The zero-range process is a Markov process $\xi_t$ with the generator

$$L_N f(\xi) = \sum_{k \in \mathbb{T}_N} \left\{ g_{k, N}(\xi(k))(f(\xi^{k,k+1}) - f(\xi)) + g_{k, N}(\xi(k))(f(\xi^{k,k-1}) - f(\xi)) \right\}$$

(2.1)

where

$$\xi^{x,y}(k) = \begin{cases} \xi(x) - 1 & k = x, \\ \xi(y) + 1 & k = y, \\ \xi(k) & k \neq x, y. \end{cases}$$

(2.2)

So that the process is irreducible, we will assume that the jump rate function $g$ is such that $g(n) = 0$ exactly when $n = 0$. We will also assume the following condition.

**Condition 2.1.** The jump rate function $g(\cdot)$ satisfies

1. **Lipschitz:** there exists $g^* > 0$ such that $|g(n + 1) - g(n)| \leq g^*$ for all $n \in \mathbb{N}_0$.
2. **Power Interaction:** there exists $\alpha \in [0, 1]$ such that $g(n) \sim n^{\alpha}$, that is

$$\lim_{n \to \infty} \frac{g(n)}{n^\alpha} = 1.$$

3. **Monotonicity:** $g(n) \leq g(n + 1)$ for all $n \in \mathbb{N}_0$.

When the constant $\alpha = 0$, in the power interaction condition, we note the function $g(n)$ increases to $g_\infty := 1$ as $n \to \infty$. We will refer to this case as ‘$g$ is of bounded type’ or simply, $g$ is bounded. On the other hand, when $\alpha \in (0, 1]$, we will say ‘$g$ is of $n^\alpha$ type’.

We remark that, when $g(n) \equiv n$, the dynamics is non-interactive, a superposition of independent random walks. The Lipschitz condition, which rules out the case $\alpha > 1$, is convenient to bound $g(n)$ by $g^* n$, and in using the entropy inequality in places. Moreover, the assumption that $g$ is increasing, an ‘attractive’ dynamics assumption, allows use of the ‘basic coupling’ in our proofs; for more discussion of this point, see Section 3.2.

Also, as discussed in the introduction, we will suppose in the $g$ bounded setting that, at a defect site $k_j, N$, the parameter $\beta_j \leq 0$. Such a condition ensures existence of a non-trivial family of invariant measures; see Section 2.1.

**Condition 2.2.** In the $g$ bounded setting, we will assume $\beta_j \leq 0$ for a defect site $k = k_j, N$.

Our goal in this work is to study hydrodynamic limits of the dynamics generated by $L_N$. In particular, in the limit macroscopic flow, there will be different behaviors at the defect locations depending on the associated strength parameters. To prepare for these statements, it will be helpful to introduce a partition on the index set $J = J_s \cup J_c \cup J_b$. Here, the subscripts $s, c,$ and $b$ stand for super-critical or ‘super-slow’, critical, and sub-critical respectively.

There will be different partitions of $J$ depending on whether $g$ is of $n^\alpha$ type or bounded. When $g$ is of $n^\alpha$ type, we let

- $J_s := \{ j \in J : \beta_j > \alpha \}$,
- $J_c := \{ j \in J : \beta_j = \alpha \}$, and
- $J_b := \{ j \in J : \beta_j < \alpha \}$.

When $g$ is bounded, we take
As we first introduce the partition function:

\[ \mathcal{E}_g \]

Invariant measures.

We note, in the case \( \Phi \) may parametrize the family of distributions of the defects, are then divided into corresponding subsets. For example, we have \( \mathcal{D}_s := \{ x_i \in \mathcal{D} : j \in J_s \} \) and \( \mathcal{D}_{s,N} := \{ k_j \in \mathcal{D}_N : j \in J_s \} \). The sets \( \mathcal{D}_c, \mathcal{D}_b, \mathcal{D}_{c,N}, \) and \( \mathcal{D}_{b,N} \) are defined in the same fashion.

2.1. Invariant measures. The construction of invariant measures under \( L_N \) is based on \( \{ \mathcal{P}_\phi \} \), a family of Poisson-like distributions indexed by ‘fugacities’ \( \phi \). In order to define \( \mathcal{P}_\phi \), we first introduce the partition function:

\[ Z(\phi) := \sum_{n=0}^{\infty} \frac{\phi^n}{g(n)!} \]

Let \( r_g \) be the convergence radius of \( Z(\cdot) \). It holds that \( r_g = \lim_{n \to \infty} g(n) \). In particular, when \( g \) is of \( n^a \) type, we have \( r_g = \infty \), namely, the “FEM” condition (cf. p. 69, [18]) is satisfied. When \( g \) is bounded, \( r_g = g_\infty = 1 \). In either case, it holds that \( \lim_{\phi \uparrow r_g} Z(\phi) = \infty \).

For each \( \phi \in [0,r_g) \), define \( \mathcal{P}_\phi \) by

\[ \mathcal{P}_\phi(n) = \frac{1}{Z(\phi)} \frac{\phi^n}{g(n)!}, \quad \text{for } n \geq 0. \] (2.3)

Here, \( g(0)! := 1 \) and \( g(n)! := \prod_{k=1}^{n} g(k) \) for \( n \geq 1 \). Let \( R(\phi) = E_{\mathcal{P}_\phi}[X] \), where \( X(n) = n \), be the mean of the distribution \( \mathcal{P}_\phi \). A direct computation yields that \( R(0) = 0 \) and \( \lim_{\phi \uparrow r_g} R(\phi) = \infty \). Moreover, it holds \( R'(\phi) = \sigma^2(\phi)/\phi \) and \( \lim_{\phi \downarrow 0} R'(\phi) = 1/g(1) \) where \( \sigma^2(\phi) = \text{Var}_{\mathcal{P}_\phi}[X] \) is the variance of \( X \) under \( \mathcal{P}_\phi \).

Since \( R \) is strictly increasing, it has an inverse, denoted by \( \Phi : [0,\infty) \mapsto [0,r_g) \). We may parametrize the family of distributions \( \mathcal{P}_\phi \) by their means: For \( \rho \geq 0 \), let \( \mathcal{Q}_\rho = \mathcal{P}_{\Phi(\rho)} \), so that \( E_{\mathcal{Q}_\rho}[X] = E_{\mathcal{P}_{\Phi(\rho)}}[X] = R(\Phi(\rho)) = \rho \). A straightforward computation yields that \( E_{\mathcal{P}_\phi}[g(X)] = \phi \) for \( \phi \geq 0 \). Thus,

\[ \Phi(\rho) = E_{\mathcal{P}_{\Phi(\rho)}}[g(X)] = E_{\mathcal{Q}_\rho}[g(X)], \quad \rho \geq 0. \] (2.4)

As \( g(n) \leq g^* n \), we have that \( \Phi(\rho) \leq g^* \rho \). As \( \Phi(\cdot) = R^{-1}(\cdot) \), we have \( \Phi'(\rho) = \Phi(\rho)/\sigma^2(\Phi(\rho)) \).

Under our assumptions on \( g \), in fact, it holds that \( 0 \leq \Phi'(\rho) \leq g^* \) for all \( \rho \geq 0 \) (cf. p. 33, [18]).

In particular, \( \Phi \in C^1[0,\infty) \) is an increasing function with a uniformly bounded derivative.

We note, in the case \( g(n) \equiv n \), that \( \Phi(\rho) \equiv \rho \) and \( \mathcal{P}_\phi \) is a Poisson measure with mean \( \phi \).

We now introduce the invariant measures. For each \( N \), let

\[ q_N = \max\{1,\lambda_j N^{\beta_j} : j \in J\}. \]

For \( c \) so that \( \Phi(c) \in [0,r_g/q_N) \), denote by \( \mathcal{R}_c^N \) the product measure on \( \Omega_N \) whose marginals are given by

\[ \mathcal{R}_c^N(\xi(k) = n) = \begin{cases} \mathcal{P}_{\Phi(c)}(n) & \text{for } k \in \mathbb{T}_N \setminus \mathcal{D}_N \text{ and } n \geq 0, \\ \mathcal{P}_{\lambda_j N^{\beta_j}, \Phi(c)}(n) & \text{for } k = k_j, n \in J, \text{ and } n \geq 0. \end{cases} \] (2.5)

Notice that the condition \( \Phi(c) \in [0,r_g/q_N) \) is needed since the distributions \( \{ \mathcal{P}_\phi \} \) are defined for \( \phi \in (0,r_g) \). When \( g \) is of \( n^a \) type, as \( r_g = \infty \), we have that \( \{ \mathcal{R}_c^N \} \) are defined for all \( c \in [0,\infty) \). However, when \( g \) is bounded, we have \( r_g = g_\infty = 1 \), and thus the measures \( \{ \mathcal{R}_c^N \} \) are defined only for \( c \in [0, R(1)/q_N) \). Moreover, in the \( g \) bounded setting, we have that \( r_g/q_N \) is bounded away from \( 0 \) for all large \( N \) exactly when \( J_s = \emptyset \), that is \( \beta_j \leq 0 \), the reason for the assumption \( J_s = \emptyset \) in this case (cf. Condition 2.2).
With $\mathcal{B}^N_c$ defined, it is straightforward (cf. [1], [11]) to check the following lemma.

**Lemma 2.3.** For $c$ so that $\Phi(c) \in [0, r_q/q_N)$, $\mathcal{B}^N_c$ is invariant and reversible with respect to the generator $L_N$ in [2.1].

### 2.2. Static limit.
Before studying the hydrodynamic limits, it will be useful to understand the particle mass behavior under an invariant measure $\mathcal{B}^N_c$. For a configuration $\xi \in \Omega_N$, define the associated scaled mass empirical measure:

$$\hat{\pi}^N(dx) := \frac{1}{N} \sum_{k \in T_N} \xi(k) \delta_{k/N}(dx).$$

(2.6)

In this formulation, each particle has mass $N^{-1}$. Here and in the sequel, $\delta_z$ refers to a delta point mass at $z$.

For a test function $G \in C(\mathbb{T})$, let

$$\langle G, \pi^N \rangle := N^{-1} \sum_{k \in T_N} \xi(k) G(k/N).$$

More generally, we will use the notation $\langle G, \mu \rangle := \int G d\mu$. We now compute the limit of $\langle G, \hat{\pi}^N \rangle$ with respect to a sequence of invariant measures $\mathcal{B}^N_c$ with $c$ fixed as $N \to \infty$.

We assume first $g$ is of $n^\alpha$ type and $c > 0$. Because of the product structure of $\mathcal{B}^N_c$, $\{\xi(k)\}_{k \in T_N}$ are independent and have a common marginal $\mathcal{P}_{\Phi(c)}$ for all $k \not\in \mathcal{D}_N$. As $\mathcal{P}_{\Phi(c)}$ has expectation $c$ and finite variance, we have $N^{-1} \sum_{k \notin \mathcal{D}_N} \xi(k) G(k/N)$ converges in probability to $\int G(x) c \, dx$ as $N \to \infty$.

It remains to investigate the behavior of $N^{-1} \xi(k)$ for $k = k_{j,N} \in \mathcal{D}_N$. As $\xi(k_{j,N})$ has distribution $\mathcal{P}_{\lambda_j N^\beta_j \Phi(c)}$, by the later Lemma 2.6 for all $\beta_j > 0$,

$$E_{\mathcal{B}^N_c} [\xi(k_{j,N})] \sim (\lambda_j \Phi(c))^{1/\alpha} N^{\beta_j/\alpha}, \quad \text{and} \quad \text{Var}_{\mathcal{B}^N_c} [\xi(k_{j,N})] = o(N^{2\beta_j/\alpha}).$$

Then, according to the value of $\beta_j$, there are three different types of behaviors at $k \in \mathcal{D}_N$:

- (1) if $\beta_j < \alpha$, $N^{-1} \xi(k_{j,N}) \to 0$ in probability;
- (2) if $\beta_j = \alpha$, $N^{-1} \xi(k_{j,N}) \to (\lambda_j \Phi(c))^{1/\alpha}$ in probability;
- (3) if $\beta_j > \alpha$, $N^{-1} \xi(k_{j,N}) \to (\lambda_j \Phi(c))^{1/\alpha}$ in probability.

In other words, the defect site $k_{j,N}$ becomes macroscopically invisible when $\beta_j < \alpha$ as typically it contains $o(N)$ number of particles and each particle has mass $N^{-1}$. In the case $\beta_j = \alpha$, typically the number of particles at $k_{j,N}$ is of order $N$ and a delta mass of magnitude $(\lambda_j \Phi(c))^{1/\alpha}$ emerges. When the site is super-slow, that is $\beta_j > \alpha$, the particle number at $k_{j,N}$ is of order $N^{\beta_j/\alpha}$, corresponding to an infinite macroscopic mass. Recall, the partition $J = J_0 \cup J_s \cup J_0$ in Section 2 matches with this classification of the $\beta_j$’s.

As the macroscopical mass at $x_j \in \mathcal{D}_s$ explodes, to consider the remaining mass, we define microscopic empirical measures which exclude the super-critical defect set $\mathcal{D}_{s,N}$:

$$\pi^N(dx) := \frac{1}{N} \sum_{k \in T_N \setminus \mathcal{D}_{s,N}} \xi(k) \delta_{k/N}(dx).$$

(2.7)

Then, we may summarize the above discussion as follows.

**Proposition 2.4.** Assume $g$ is of $n^\alpha$ type. Then, for any $G \in C(\mathbb{T})$, $c \geq 0$, and $\delta > 0$:

$$\lim_{N \to \infty} \mathcal{B}^N_c \left[ \left| \langle G, \pi^N \rangle - \langle G, \pi \rangle \right| > \delta \right] = 0$$
where \( \pi(dx) = c\, dx + \sum_{j \in J} (\lambda_j \Phi(c))^{1/\alpha} \delta_{x_j}(dx) \). Moreover, for all \( j \in D_s \), we have
\[
\lim_{N \to \infty} \mathcal{R}_c^N \left[ |N^{-\beta_j/\alpha} \xi(k_j, N) - (\lambda_j \Phi(c))^{1/\alpha}| > \delta \right] = 0.
\]

Now we turn to the case when \( g \) is bounded. Recall, in this case, \( \mathcal{R}_c^N \) is defined for \( c \) such that \( \Phi(c) \in [0, 1/qN) \) where \( qN = \max\{1, \lambda_j N^{\beta_j} | j \in J\} \). Recall also, in this setting, that \( \beta_j \leq 0 \), that is \( J_s = \emptyset \). When \( J_c \neq \emptyset \), we will define \( \lambda_{\max} := \max_{j \in J_c} \{\lambda_j\} \). Otherwise, when \( J_c = \emptyset \), we will take \( \lambda_{\max} = 1 \). Hence, the domain for \( c \) so that \( \mathcal{R}_c^N \) is defined for all \( N \) is \([0, R(1/\lambda_{\max})]\).

For \( c \in [0, R(1/\lambda_{\max})] \), we also have \( N^{-1} \sum_{k \notin D_N} \xi(k)G(k/N) \) converges in probability to \( \int_0^\infty G(x)c\, dx \) as \( N \to \infty \). Given \( \xi(k) \) has finite \( \mathcal{R}_c^N \)-expectation and variance for all \( k \in D_N \), the following is easily obtained.

**Proposition 2.5.** Assume \( g \) is bounded and \( J_s = \emptyset \). Then, for \( c \in [0, R(1/\lambda_{\max})] \), for any \( G \in C(T) \) and \( \delta > 0 \), we have
\[
\lim_{N \to \infty} \mathcal{R}_c^N \left[ \left| \langle G, \pi^N \rangle - \int_T G(x)c\, dx \right| > \delta \right] = 0.
\]

We finish this section with a technical lemma used in Proposition 2.4.

**Lemma 2.6.** Assume \( g(k) \sim k^\alpha \) for some \( \alpha \in (0, 1] \). For each \( \varphi > 0 \), let \( X \) be a random variable with distribution \( P_\varphi \) (cf. (2.3)). Then, for each \( n \in \mathbb{N} \), \( E[X^n] \sim \varphi^{n/\alpha} \) as \( \varphi \to \infty \). As a result, in \( Z(\varphi) \sim \alpha \varphi^{1/\alpha} \) and \( \text{Var}[X] = o(\varphi^{2/\alpha}) \) as \( \varphi \to \infty \).

**Proof.** We first show \( E[X^n] \sim \varphi^{n/\alpha} \) for all \( n \in \mathbb{N} \). To this end, let us assume for now
\[
\sum_{k=1}^\infty \frac{k^{n\alpha} \varphi^k}{g(k)!} \sim \sum_{n=0}^\infty \frac{\varphi^k}{g(j-n)!}.
\] (2.8)

Let \( Y = X^\alpha \). Then,
\[
E[Y^n] = \frac{1}{Z(\varphi)} \sum_{k=1}^\infty \frac{k^{n\alpha} \varphi^k}{g(k)!} \sim \frac{1}{Z(\varphi)} \sum_{k=n}^\infty \frac{\varphi^k}{g(k-n)!} = \varphi^n.
\]

As \( \alpha \in (0, 1] \), we may find \( p \in (0, 1) \) and \( l \in \mathbb{N} \) such that \( \alpha^{-1} = p + (1-p)l \). Also, since \( E[X^n] = E[Y^{n\alpha}] \), by Jensen’s and Hölder’s inequalities,
\[
E[Y^{n\alpha}]^{1/\alpha} \leq E[X^n] \leq E[Y^{n\alpha}]^{1/p} E[Y^{nl}]^{1-p}.
\]

Since \( E[Y^n] \sim \varphi^n \) and \( E[Y^{nl}] \sim \varphi^{nl} \), we obtain \( E[X^n] \sim \varphi^{n/\alpha} \).

For the limit behavior of \( E[X^n] \), it remains to show the claim (2.8). As \( g(k) \sim k^\alpha \), for any \( A > 0 \), we may find \( \lambda_1 = \lambda_1(A) \) and \( \lambda_2 = \lambda_2(A) \), such that \( \lambda_1 k^\alpha \leq g(k) \leq \lambda_2 k^\alpha \) for all \( k \geq A \) and \( \lim_{A \to \infty} \lambda_1 = \lim_{A \to \infty} \lambda_2 = 1 \). Then, for all \( k \geq A + n \),
\[
\lambda_2^{-n} \leq \frac{k^{n\alpha}}{\prod_{k-n < k' \leq k} g(k')} \leq \left( \frac{A + n}{A} \right)^n \lambda_1^{-n}.
\]

Therefore,
\[
\lambda_2^{-n} \sum_{k=A+n}^\infty \frac{\varphi^k}{g(k-n)!} \leq \sum_{k=A+n}^\infty \frac{k^{n\alpha} \varphi^k}{g(k)!} \leq \left( \frac{A + n}{A} \right)^n \lambda_1^{-n} \sum_{k=A+n}^\infty \frac{\varphi^k}{g(k-n)!}
\]

Notice that, if \( \varphi \) is sent to infinity in the above display, we may replace \( \sum_{k \geq A+n} \) by either \( \sum_{k \geq 0} \) or \( \sum_{k \geq n} \). Then the claim (2.8) follows from taking \( \varphi \to \infty \) and then \( A \to \infty \).
We have shown $E[X^n] \sim \varphi^{n/\alpha}$ for all $n \in \mathbb{N}$. Then, it follows that $\text{Var}[X] = o(\varphi^{2/\alpha})$ as $E[X^2] \sim E[X]^2 \sim \varphi^{2/\alpha}$. To prove $\ln Z(\varphi) \sim \alpha \varphi^{1/\alpha}$, notice $\frac{d}{d\varphi} \ln Z(\varphi) = \varphi^{-1} E[X] \sim \varphi^{1/\alpha - 1}$ and then apply L'Hospital's rule, to finish the argument. \hfill \Box

3. Initial measures

In this section, we specify the assumptions on the initial measures $\{\mu^N\}$ we use to start our dynamics. Roughly speaking, $\{\mu^N\}$ should be associated with a macroscopic profile which gives the initial condition for the hydrodynamic limit. We will also require $\mu^N$ to possess certain relative entropy estimates and to be stochastically bounded with respect to invariant measures.

To specify these conditions, recall, for $\mu, \nu$, two probability measures on $\Omega_N$, we say that $\mu \leq \nu$, that is $\mu$ is stochastically bounded by $\nu$, if for all $f : \Omega_N \rightarrow \mathbb{R}$ coordinately increasing, we have $E_\mu(f) \leq E_\nu(f)$. Fix also $\pi$, a nonnegative measure on $\mathbb{T}$, such that

$$\pi(dx) = \rho_0(x)dx + \sum_{j \in J_c} m_{0,j} \delta_{x_j}(dx) \quad (3.1)$$

where $\rho_0(x) \in L^1(\mathbb{T})$ and $m_{j,0} \geq 0$ for $j \in J_c$. Recall also, the empirical measure $\pi^N$ defined in $(2.7)$.

Throughout this work, we will assume the following on the sequence of initial measures $\{\mu^N\}_{N \in \mathbb{N}}$ on $\Omega_N$.

**Condition 3.1. The following hold:**

1. $\{\mu^N\}_{N \in \mathbb{N}}$ has macroscopic profile $\pi$ on $\mathbb{T} \setminus \mathcal{D}_s$, i.e. for all $G(x) \in C(\mathbb{T})$ and $\delta > 0$,

$$\lim_{N \rightarrow \infty} \mu^N \left[ \left| \langle G, \pi^N \rangle - \langle G, \pi \rangle \right| > \delta \right] = 0.$$

2. There exists $c_0 > 0$ such that the relative entropy of $\mu^N$ with respect to $\mathcal{R}^N_{c_0}$ is of order $N$: Let $f_N := d\mu^N / d\mathcal{R}^N_{c_0}$. Then, $H(\mu^N | \mathcal{R}^N_{c_0}) := \int f_N \ln f_N d\mathcal{R}^N_{c_0} = O(N)$. 

3. When $g$ is of $n^\alpha$ type, there exists $c'$ such that $\mu^N$ is stochastically bounded by $\mathcal{R}^N_{c'}$ for all $N$.

3'. When $g$ is bounded, there exists $c'$ such that, when restricted to $\mathcal{T}_N \setminus \mathcal{D}_c, \mu^N$ is stochastically bounded by $\kappa^N_{c'}$, that is $E_{\mu^N}[f] \leq E_{\kappa^N_{c'}}[f]$ for coordinate increasing functions supported on $\{\eta(k) : k \in \mathbb{T} \setminus \mathcal{D}_c \}$. Here,

$$\kappa^N_{c'} = \prod_{k \in \mathcal{T}_N \setminus \mathcal{D}_{N}} \mathcal{P}_{\Phi(k)} \times \prod_{k=k_j, N \in \mathcal{D}_{N}} \mathcal{P}_{\lambda_N \beta_j \Phi(k')}.$$

We comment that item (3) is sufficiently general in the $n^\alpha$ setting to allow $\{\mu^N\}$ to be associated with profiles of the form $\pi_0(dx) = \rho_0(x)dx + \sum_{j \in J_c} m_{0,j} \delta_{x_j}(dx)$, for densities $\rho_0$ and masses $\{m_{0,j}\}$, as demonstrated in Section 3.3. In the bounded $g$ case, however, if $\{\mu^N\}$ satisfied item (3), then $O(N)$ accumulations at points $\mathcal{D}_{c,N}$ would not be allowed. In this case, the only profiles allowed would be of form $\pi_0(dx) = \rho_0(x)dx$, where $\|\Phi(\rho_0)\|_{\infty} \leq \min_{j \in J_c} \frac{1}{\lambda_j}$ (cf. Section 3.2). As mentioned in the introduction, hydrodynamic evolution from such profiles would not see any defects.

In this context, item (3') is formulated so that, with respect to $\{\mu^N\}, O(N)$ accumulations are possible on $\mathcal{D}_c, N$ as well as later point masses on $\mathcal{D}_c$ in the hydrodynamic evolution.

In fact, conditions (3) and (3') can be made to accommodate a larger class of initial measures $\{\mu^N\}$. For example, one may remove stochastic bounded assumptions on coordinates
in $D_{s,N}$. We have however chosen to state the (3) and (3') in the forms given to streamline arguments and avoid more piecemeal calculations.

Define now $\mu^N_t$ as the distribution of the process $\xi_t$, with initial measure $\mu^N$.

In the rest of this section, we make remarks on the consequences of the relative entropy bound in terms of $\mu^N$-particle numbers, discuss the use of attractiveness, and also provide a large class of examples of $\mu^N$ which satisfy Condition 3.1.

3.1. $\mu^N$-particle numbers and relative entropy. We first comment on the $\mu^N$-particle numbers in the system. Denote $H_N := H(\mu^N | \mathcal{R}^N_{c_0}) = O(N)$. Notice that the entropy does not increase in $t$, that is $\mu^N_t = H(\mu^N_t | \mathcal{R}^N_{c_0}) \leq H_N$ for $t \geq 0$ (cf. pp 340, [15]).

Assume now that $g$ is of $n^\alpha$ type. By the entropy inequality $E_{\mu^N}[f] \leq H(\mu^N | \nu) + \ln E_{\nu}[e^f]$ (cf. p.338 [15]),
\[
E_{\mu^N} \left[ \sum_{k \in \mathcal{D}_{s,N}} \xi(k) \right] \leq H(\mu^N | \mathcal{R}^N_{c_0}) + \ln E_{\mathcal{R}^N_{c_0}} \left[ e^{\sum_{k \in \mathcal{D}_{s,N}} \xi(k)} \right].
\]
As $\mathcal{R}^N_{c_0}$ is product measure, the ln term is written as $\sum_{k \in \mathcal{D}_{s,N}} \ln E_{\mathcal{R}^N_{c_0}} [e^{\xi(k)}]$ which is $O(N)$ by Lemma 2.7. Thus, the condition $H_N = O(N)$ specifies $O(N)$ particles on the sites $T_N \setminus D_{s,N}$ for times $t \geq 0$.

On the other hand, at a super-slow site $k_j,N \in D_{s,N}$, the $\mathcal{R}^N_{c_0}$-particle number is typically of order $O(N^{\beta_j/\alpha})$, as discussed in Section 2.2. Then, because $H_N = O(N)$, as consequence of the entropy inequality, we may conclude, with respect to $\mu^N$, that $N^{-\beta_j/\alpha} \xi(k_j,N)$ converges in probability to $(\lambda_j \Phi(c_0))^{1/\alpha}$; see near (7.7) for a proof.

Therefore, the net exchange of particle numbers between super-slow sites $D_{s,N}$ and the rest of the system is of order $N$ and the total particle number on $T_N \setminus D_{s,N}$ remains $O(N)$ for all times $t \geq 0$.

We now turn to case when $g$ is bounded. Since in this setting $J_s = \emptyset$, that is $D_s = \emptyset$ (Condition 2.2), by the previous entropy inequality discussion, it follows that the total number of particles in $T_N$ is of order $N$ for all times $t \geq 0$.

3.2. Basic coupling. We discuss the use of the stochastic boundedness assumption (3) and (3') in Condition 3.1. Since $g(n)$ is an increasing function in $n$, the dynamics generated by $L_N$ is ‘attractive’: if initially $\xi_0$ is distributed according to measures $\mu \leq \nu$, then we have $\mu_t \leq \nu_t$ for all $t \geq 0$ where $\mu_t$ and $\nu_t$ are distributions of $\xi_t$ (cf. [1], Chapter II in [22]).

In the case when $g$ is of $n^\alpha$ type, as $\mathcal{R}^N_{c_0}$ is invariant, the assumption $\mu^N \leq \mathcal{R}^N_{c_0}$ implies that $\mu^N_t \leq \mathcal{R}^N_{c_0}$ for all $t \geq 0$. However, when $g$ is bounded, the domain for $c$ in $\mathcal{R}^N_{c}$ is $c < \rho(1/\lambda_{\max})$ (cf. Proposition 2.5). Then, an assumption $\mu^N \leq \mathcal{R}^N_{c}$ would imply that $\pi$, the macroscopic profile associated to $\mu^N$, is $\pi(dx) = \rho_0(x)dx$ with $\|\rho_0\|_\infty < \rho(1/\lambda_{\max}) < \infty$. To accommodate more initial profiles $\pi$ (and observe more involved limit evolutions), we have assumed (3') in Condition 3.1 instead, that is, for all coordinateely increasing $f : \Omega_N \rightarrow \mathbb{R}$ depending only on $\{\xi(k)\}_{k \in \mathcal{D}_{s,N}}$, we have $E_{\mu^N}[f(\xi)] \leq E_{\mathcal{R}^N_{c}}[f(\xi)]$.

We now illustrate how we will apply attractiveness under assumption (3'). Instead of an evolution with respect to particle numbers in $\mathbb{N}_0$ and configurations in $\Omega_N^T$, we may consider a dynamics corresponding to $\Omega_0 := \mathbb{N}_0 \cup \{\infty\}$ and $\Omega_N := \overline{\Omega}_N^T$. Recall the constant $c'$ in (3'). Define $\overline{\mathcal{S}}_{k,N}(\cdot)$ by $\overline{\mathcal{S}}_{k,N}(n) = g_{k,N}(n)$ for $n \in \mathbb{N}_0$ and $\overline{\mathcal{S}}_{k,N}(\infty) = \Phi(c')$. Consider now the following generator on $\overline{\Omega}_N$
\[
\mathcal{T}_N f(\xi) = \sum_{k \in \overline{\Omega}_N} \left\{ \overline{\mathcal{S}}_{k,N}(\xi(k)) \left( f(\xi^{k+1}) - f(\xi) \right) + \overline{\mathcal{S}}_{k,N}(\xi(k)) \left( f(\xi^{k-1}) - f(\xi) \right) \right\}.
\]
Here, $\xi^{x,y}$ is defined as in (2.2) by using the convention $\infty \pm 1 = \infty$. When starting with configurations in $\Omega_N$, $\mathcal{T}_N$ coincides with $L_N$. Once a site starts with $\infty$ particles, it will serve as a ‘reservoir’ which pumps particles into its neighbors at the rate of $\Phi(\cdot')$. Let $\delta_\infty$ be the Dirac measure on the extended number $\infty$. Define $\kappa_N$ as the product measure on $\mathcal{T}_N$ that coincides with $\kappa_N$ on $k \notin \mathcal{D}_{c,N}$ and has marginal $\delta_\infty$ for $k \in \mathcal{D}_{c,N}$. It is straightforward to check that $\kappa_N$ is invariant under $\mathcal{T}_N$. Also, we have that $\mu_N \leq \kappa_N$ by (3') in Condition 3.3. As attractiveness and the basic coupling are still in effect for $\mathcal{T}_N$, we obtain that the later time $t$ distribution satisfies $\mu_N \leq \kappa_N$ for any $t \geq 0$. In particular, for a coordinate-increasing function $f(\xi)$ which does not depend on $\{\xi(k)\}_{k \in \mathcal{G}_{c,N}}$, we have $E_{\mu_N}[f] \leq E_{\kappa_N}[f] = E_{\kappa_N'}[f]$.

Finally, as a general remark, we make technical use of ‘attractiveness’ most essentially in the cutoff of large densities in the local replacements (Lemma 8.1 and Lemma 8.2), and in proving the limit mass profile has $L^2$ absolutely continuous part (Lemma 7.1). The latter property is needed in the proof of uniqueness of weak solutions (Theorem 11.1 and 11.2).

### 3.3. Local equilibrium

We now give explicit examples of initial measures that satisfy the Condition 3.3. These examples will be denoted by $\mu_{\text{le}}^N$ as they are related with the usual ‘local equilibrium’ measures in setting without defects (cf. [18]).

Let $\pi$ be as in 3.1 with $\rho_0(x) \in L^\infty(\mathbb{T})$ and $m_{0,j} \geq 0$ for each $j \in J_c$. For each $k \in \mathbb{T}_N$, define

$$\rho_{k,N} = N \int_{(k-1)/N}^{k/N} \rho_0(x)dx.$$  

We will construct $\{\mu^N\}$ separately for $g$ of $n^\alpha$ and bounded types. Consider first $g$ of $n^\alpha$ type. Fix $c_0 > 0$ and define

$$\varphi_{k,N} = \begin{cases} 
\Phi(\rho_{k,N}) & k \in \mathbb{T}_N \setminus \mathcal{D}_N \\
0 & k = k_{j,N} \in \mathcal{D}_{b,N} \\
(Nm_{0,j})^\alpha & k = k_{j,N} \in \mathcal{D}_{c,N} \\
\lambda_j N^\beta \Phi(c_0) & k = k_{j,N} \in \mathcal{D}_{s,N}. 
\end{cases}$$

For each $N \in \mathbb{N}$, let $\mu_{\text{le}}^N$ be the product measure on $\Omega_N$ with marginals given by

$$\mu_{\text{le}}^N = \mu_{\text{le}}^N(\cdot | n) = P_{\varphi_{k,N}}(n), \quad \text{for } k \in \mathbb{T}_N, \ n \geq 0.$$  

**Lemma 3.2.** Suppose $g$ is of $n^\alpha$ type. Then, $\{\mu^N = \mu_{\text{le}}^N\}$ satisfies Condition 3.3.

**Proof.** Let $c'$ be such that $\Phi(\cdot') = \max \{\Phi([\rho_0]|\infty) \cdot (m_{0,j})^\alpha \cdot \Phi(c_0)\}_{j \in J_c}$. As $P_{\varphi_{\lambda_j}} \leq P_{\varphi_{\lambda_j}}$ if $\phi_1 \leq \phi_2$ (cf. [22], pp. 32), we have that the product measure $\mu_{\text{le}}^N$ is stochastically bounded by $\mathcal{G}_{c'}$ for all $N$. As $G$ is uniformly continuous, that $\mu_{\text{le}}^N$ is associated with the given macroscopic profile $\pi$ holds straightforwardly from Chebychev’s inequality and Lemma 2.0.

It remains to check the desired entropy bound $H(\mu_{\text{le}}^N | \mathcal{F}_{c_0}) = O(N)$. Let $\varphi = \Phi(c_0)$. We compute

$$H(\mu_{\text{le}}^N | \mathcal{F}_{c_0}) = \sum_{j \in J_c} H(P_{\varphi_{x_{j,N},N}} | P_{\lambda_j N^\beta \varphi}) + \sum_{k \in \mathbb{T}_N \setminus \mathcal{D}_N} H(P_{\varphi_{k,N}} | P_{\varphi}).$$

Note that, for any $\phi_1 \geq 0$ and $\phi_2 > 0$,

$$H(P_{\varphi_1}, P_{\varphi_2}) = \ln \frac{\phi_1}{\phi_2} E_{P_{\varphi_2}} [X] + \ln \frac{Z(\phi_2)}{Z(\phi_1)}$$

where $X$ is defined as $X(n) = n$. We also adopt $0 \ln 0 = 0$ in the case $\phi_1 = 0$. By Lemma 2.0, we conclude the desired relative entropy bound $H(\mu_{\text{le}}^N | \mathcal{F}_{c_0}) = O(N)$.

\[\square\]
We now consider the $g$ bounded case. Note by assumption (Condition 2.2) that $J_\epsilon = \emptyset$ in this setting. Let $\tilde{\varphi}_{k,N} = \Phi(\omega_{k,N})$ for $k \in \mathbb{T}_N \setminus \mathcal{D}_N$ and $\tilde{\varphi}_{k,N} = 0$ for $k \in \mathcal{D}_0,N$. Also, let $\mathcal{P}_N'$ denote the Poisson distribution with mean $\lambda$.

We define $\tilde{\mu}_{le}^N$ as the product measure with marginals $\mathcal{P}_{\tilde{\varphi}_{k,N}}$ on sites $k \notin \mathcal{D}_{e,N}$ and $\mathcal{P}_{m_{0,j,N}}'$ for $k = k_{j,N} \in \mathcal{D}_{e,N}$. It is straightforward that (1) and (3') in Condition 3.1 hold with $\mu^N = \tilde{\mu}_{le}^N$. The choice of Poisson distributions at sites $\mathcal{D}_{e,N}$ allows for some explicit computation.

Fix any $c_0 \in (0, R(1/\lambda_{\text{max}}))$, we now argue that $H(\tilde{\mu}_{le}^N|\mathcal{P}_{c_0}) = O(N)$. It suffices to check that $H(\mathcal{P}_{a,N}'|\mathcal{P}_\varphi) = O(N)$ for any fixed $a \geq 0$ and $\varphi \in (0,1)$, where $g_\infty = 1$. To see this, let $f_N(n) = \mathcal{P}_{a,N}(X = n)$ and $f(n) = \mathcal{P}_\varphi(X = n)$ where $X(n) = n$. Then $H(\mathcal{P}_{a,N}'|\mathcal{P}_\varphi) = E_{\mathcal{P}_{a,N}'}[\ln f_N(X)] - E_{\mathcal{P}_{a,N}'}[\ln f(X)]$. The term $E_{\mathcal{P}_{a,N}'}[\ln f_N]$ is computed as $aN\ln(aN) - aN - E_{\mathcal{P}_{a,N}'}[\ln X]$. By Stirling’s formula, $n! \geq \sqrt{2\pi ne^{-n}n^n} \geq e^{-n}n^n$, and Jensen’s inequality, we have $E_{\mathcal{P}_{a,N}'}[\ln X!] \geq E_{\mathcal{P}_{a,N}'}[X \ln X - X] \geq aN\ln(aN) - aN$. Therefore, we have $E_{\mathcal{P}_{a,N}'}[\ln f_N]$ is $O(N)$. For the term $E_{\mathcal{P}_{a,N}'}[\ln f]$, we may write it as $aN\ln \varphi - \ln Z(\varphi) - E_{\mathcal{P}_{a,N}'}[\ln g(X)]$. Note that $g(X) \leq 1$, so that $E_{\mathcal{P}_{a,N}'}[\ln f] = O(N)$. Hence, $H(\tilde{\mu}_{le}^N|\mathcal{P}_{c_0}) = O(N)$.

We now summarize the above calculations.

**Lemma 3.3.** Suppose $g$ is bounded. Then $\mu^N = \tilde{\mu}_{le}^N$ satisfies Condition 3.1.

4. Results

Suppose that $\{\mu^N\}$ satisfies Condition 3.1. Consequently, $\mu^N$ has macroscopic profile

$$
\pi(dx) = \rho_0(x)dx + \sum_{j \in J_c} m_{0,j} \delta_{x_j}(dx)
$$

on $\mathbb{T} \setminus \mathcal{D}_s$ and we have $H(\mu^N|\mathcal{P}_{c_0}) = O(N)$ for some $c_0 \geq 0$.

On $\mathbb{T}_N$, we will observe the evolution of the zero-range process speeded up by $N^2$, in diffusive scale. Denote the process $\eta_t := \xi_{N^2t}$, generated by $N^2L_N$ (cf. (2.1)), for times $0 \leq t \leq T$. Here, $T > 0$ refers to a fixed time horizon. We will access the space-time structure of the process through the scaled mass empirical measure:

$$
\pi_t^N(dx) := \frac{1}{N} \sum_{k \in \mathbb{T}_N \setminus \mathcal{D}_{s,N}} \eta_t(k) \delta_{k/N}(dx).
$$

Throughout, we will view each $\pi_t^N$ as a member of $\mathcal{M}$, the space of finite nonnegative measures on $\mathbb{T} \setminus \mathcal{D}_s$. We will place a metric $d(\cdot, \cdot)$ on $\mathcal{M}$ which realizes the vague convergence on $\mathbb{T} \setminus \mathcal{D}_s$, (see Section 6 for a definitive choice). Here, the trajectories $\{\pi_t^N : 0 \leq t \leq T\}$ are elements of the Skorokhod space $D([0,T],\mathcal{M})$, endowed with the associated Skorokhod topology.

In the following, the process measure and associated expectation governing $\eta_t$ starting from $\mu$ will be denoted by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$. When the process starts from $\{\mu^N\}_{N \in \mathbb{N}}$, in the class satisfying Condition 3.1, we will denote by $\mathbb{P}_N := \mathbb{P}_{\mu^N}$ and $\mathbb{E}_N := \mathbb{E}_{\mu^N}$, the associated process measure and expectation.

4.1. Hydrodynamic limits. We now state our main results. Recall the initial profile $\pi$ and the constant $c_0$ from the beginning of this section.

**Theorem 4.1.** Assume $g$ is of $n^\alpha$ type. Then, for any $t > 0$, test function $G \in C(\mathbb{T})$, and $\delta > 0$, we have

$$
\lim_{N \to \infty} \mathbb{P}_N \left[ |\langle G, \pi_t^N \rangle - \langle G, \pi_t \rangle | > \delta \right] = 0.
$$

(4.2)
where $\pi_t(dx) = \rho(t,x)dx + \sum_{j \in J_c} m_j(t) \delta_{x_j}(dx)$ is the unique weak solution to
\begin{align*}
\partial_t \pi_t - \partial_{xx} \Phi(\rho(t,x)), & \quad x \in \mathbb{T}, \quad t \in (0,T), \\
\pi_{t=0} = \pi, & \quad \Phi(\rho(t,x_j)) = \Phi(c_0), \quad t \in (0,T), \quad j \in J_s, \\
m_j(t) = (\lambda_j \Phi(\rho(t,x_j)))^{1/\alpha}, & \quad t \in (0,T), \quad j \in J_c.
\end{align*}

(4.3)

Theorem 4.2. Assume $g$ is bounded. Then, for any $t > 0$, test function $G \in C(\mathbb{T})$, and $\delta > 0$, we have
\[
\lim_{N \to \infty} P_N \left[ \left| \langle G, \pi_t^N \rangle - \langle G, \pi_t \rangle \right| > \delta \right] = 0,
\]
where $\pi_t(dx) = \rho(t,x)dx + \sum_{j \in J_c} m_j(t) \delta_{x_j}(dx)$ is the unique weak solution to
\begin{align*}
\partial_t \pi_t - \partial_{xx} \Phi(\rho(t,x)), & \quad x \in \mathbb{T}, \quad t \in (0,T), \\
\pi_{t=0} = \pi, & \quad \Phi(\rho(t,x_j)) \leq 1/\lambda_j, \quad t \in (0,T), \quad j \in J_c, \\
m_j(t) = m_j(t) \mathbb{1}_{\Phi(\rho(t,x_j))=1/\lambda_j}, & \quad t \in (0,T), \quad j \in J_c.
\end{align*}

(4.4)

We now define the weak solutions to the limit PDE (4.3) and (4.5).

Definition 4.3. Let $f(t,x)$ and $g(t,x)$ be in $L^1_{loc}([0,T] \times D)$ where $D$ is a domain of $x$. We say $f$ is weakly differentiable with respect to $x \in D$ if for all $G(t,x) \in C^1_{loc}([0,T] \times D)$ that
\[
\int_0^T \int_D \partial_x G(t,x) f(t,x) dx dt = - \int_0^T \int_D G(t,x) g(t,x) dx dt.
\]
The weak derivative will be denoted by $\partial_x f(t,x)$ and $\partial_x f(t,x) := g(t,x)$.

We comment that if $f(t,x)$ is weakly differentiable with respect to $x \in D$ as defined above, then for a.e. $t \in [0,T]$, $f(t, \cdot)$ is absolutely continuous and $f(t,b) - f(t,a) = \int_a^b \partial_x f(t,x) dx$ for all connected $a,b \in D$. In particular, the evaluation of $f(t,x') = f(t,x)|_{x=x'}$ at a given value $x' \in D$ is well defined.

Definition 4.4. We say $\pi_t(dx) = \rho(t,x)dx + \sum_{j \in J_c} m_j(t) \delta_{x_j}(dx)$ is a weak solution to the system (4.3) if
\begin{enumerate}
\item $\rho(t,x)$ is in $L^2([0,T] \times \mathbb{T})$ and $\Phi(\rho(t,x))$ is weakly differentiable with respect to $x \in \mathbb{T}$ with $\partial_x \Phi(\rho(t,x)) \in L^2([0,T] \times \mathbb{T})$;
\item $\Phi(\rho(t,x_j)) = \Phi(c_0)$, for almost all $t \in (0,T)$ and all $x_j \in \mathbb{D}_s$;
\item $m_j(t) = (\lambda_j \Phi(\rho(t,x_j)))^{1/\alpha}$ for almost all $t \in (0,T)$ and $j \in J_c$;
\item for all $G(t,x) \in C^\infty_c((0,T] \times (\mathbb{T} \setminus \mathbb{D}_s))$
\[
\int_0^T \int_T \partial_x G(t,x) \pi_t(dx) dt + \int_T G(t,x) \pi(dx) + \int_0^T \int_T \partial_{xx} G(t,x) \Phi(\rho(t,x)) dx dt = 0.
\]
\end{enumerate}

(4.6)

Definition 4.5. We say $\pi_t(dx) = \rho(t,x)dx + \sum_{j \in J_c} m_j(t) \delta_{x_j}(dx)$ is a weak solution to the system (4.5) if
\begin{enumerate}
\item $\rho(t,x)$ is in $L^2([0,T] \times \mathbb{T})$ and $\Phi(\rho(t,x))$ is weakly differentiable with respect to $x \in \mathbb{T}$ with $\partial_x \Phi(\rho(t,x)) \in L^2([0,T] \times \mathbb{T})$;
\item $\Phi(\rho(t,x_j)) \leq 1/\lambda_j$, for almost all $t \in (0,T)$ and all $x_j \in \mathbb{D}_c$;
\item $m_j(t) = m_j(t) \mathbb{1}_{\Phi(\rho(t,x_j))=1/\lambda_j}$ for almost all $t \in (0,T)$ and $j \in J_c$;
\item the weak formulation (4.6) holds for all $G(t,x) \in C^\infty_c((0,T] \times \mathbb{T})$.
\end{enumerate}

Remark 4.6. In Definitions 4.4 and 4.5, we have assumed $\Phi(\rho(t,x))$ to be weakly differentiable with respect to $x \in \mathbb{T}$. Note that $\Phi(\cdot)$ is invertible and the inverse function $\Phi^{-1}(\cdot)$ is a continuous mapping from $[0,r_g]$ to $[0,\infty)$ where $r_g = \infty$ for $g$ of n$^\alpha$ type and $r_g = 1$
for $g$ bounded (cf. Section 2.1). Then $\rho(t,x) = \Phi^{-1}(\Phi(\rho(t,x)))$ is continuous in $x$ for a.e. $t \in [0,T]$ (viewed as $[0,\infty]$ valued function for $g$ bounded). In particular, the evaluation of $\Phi(\rho(t,x))$ at $x = x_j$ may be written in terms $\rho(t,x_j)$ instead.

Moreover, notice that, under the settings of Theorems 4.1 and 4.2 it holds $\|\rho\|_\infty < \infty$; see Lemma 7.7. As $\Phi^{-1}(\cdot)$ is Lipschitz on $[0,\phi]$ where $\phi = \Phi(\|\rho\|_\infty)$ (cf. Section 2.1), we have that $\rho(t,x)$ is also weakly differentiable with respect to $x \in \mathbb{T}$ and $\partial_x \rho(t,x) \in L^2([0,T] \times \mathbb{T})$.

To illustrate the relation between boundary conditions and effects on defect sites, we consider the case with only a single defect site.

**Example 4.7 (Effects of a single slow site).** As an example to illustrate how defects affect the macroscopic bulk evolution, we look at systems with a single defect site. Without loss of generality, we may assume the defect location is at 0 and the altered jump rate is $\Phi(\beta)$ of $\rho$.

Consider first $g$ of $n^\alpha$ type. By Theorem 4.1 the hydrodynamic limit $\pi_t$ is governed by the PDE (4.3). As the defect site is at $x = 0$, we have $\pi_t(dx) = \rho(t,x)dx$ when restricted to open interval $(0,1)$. Indeed, (4.3) is now reduced to a nonlinear heat equation regarding only $\rho(t,x)$ with different boundary conditions at $x = 0$ and $x = 1$ depending on the value of $\beta$. Precisely, we have the following.

1. When $\beta < \alpha$, the defect site is invisible in the limit and (4.3) becomes
   $$
   \begin{cases}
   \partial_t \rho(t,x) = \partial_x \Phi(\rho(t,x)), & x \in \mathbb{T}, \ t \in (0,T), \\
   \rho(0,x) = \rho_0(x),
   \end{cases}
   $$
   that is $\rho(t,x)$ satisfies periodic boundary conditions.

2. When $\beta = \alpha$, we have $\pi_t(dx) = \rho(t,x)dx + m(t)\delta_0(dx)$ with $m(t) = (\lambda \Phi(\rho(t,0)))^{1/\alpha}$.

   The atomic mass $m(t)$ can also be expressed in terms of the initial value $m_0$ and net change of the bulk mass as the total mass is conserved:
   $$
   m_0 + \int_0^1 \rho_0(x)dx = m(t) + \int_0^1 \rho(t,x)dx, \quad \text{for all } t > 0.
   $$

   Therefore, we have $\Phi(\rho(t,0)) = \lambda^{-1} \left[ m_0 + \int_0^1 (\rho_0(x) - \rho(t,x))dx \right]^{\alpha}$. Noticing that $x = 0$ and $x = 1$ coincide on $\mathbb{T}$, we obtain
   $$
   \begin{cases}
   \partial_t \rho(t,x) = \partial_x \Phi(\rho(t,x)), & x \in (0,1), \ t \in (0,T), \\
   \rho(0,x) = \rho_0(x), \\
   \Phi(\rho(t,0)) = \Phi(\rho(t,1)) = \lambda^{-1} \left[ m_0 + \int_0^1 (\rho_0(x) - \rho(t,x))dx \right]^{\alpha}.
   \end{cases}
   $$

   We comment that, in a system with more than one critical slow sites ($\beta = \alpha$), conservative of mass is not sufficient to determine the atomic masses individually. For a closed equation, one needs to stay with a form such as (4.3).

3. When $\beta > \alpha$, as $\mathbb{D} = \mathbb{D}_s = \{0\}$, the PDE (4.3) is
   $$
   \begin{cases}
   \partial_t \rho(t,x) = \partial_x \Phi(\rho(t,x)), & x \in (0,1), \ t \in (0,T), \\
   \rho(0,x) = \rho_0(x), \\
   \rho(t,0) = \rho(t,1) = 0.
   \end{cases}
   $$

   We now turn to case when $g$ is bounded. By Theorem 4.2 the macroscopic flow of $\pi_t$ is described by (4.3). Notice that here we have $\beta \leq 0$. When the defect is a fast site, that is $\beta < 0$ or $\beta = 0$ with $\lambda < 1$, the defect site is invisible macroscopically and the limit evolution is the usual nonlinear heat equation (4.7) with periodic boundary condition.
When the defect is a slow site, that is $\beta = 0$ and $\lambda > 1$, we can write $m(t)$, the atomic mass at the slow site, in term of ‘mass conservation’ as in part (2) above to form a closed equation of $\rho(t, x)$:

\[
\begin{align*}
\partial_t \rho(t, x) &= \partial_x \Phi(\rho(t, x)), \quad x \in \mathbb{T}, \ t \in (0, T), \\
\rho(0, x) &= \rho_0(x), \quad \rho(t, 0) = \rho(t, 1) \leq \Phi^{-1}(1/\lambda), \\
m(t) &= m(t) \mathbb{1}_{\rho(t, 0) = \Phi^{-1}(1/\lambda)}, \quad m(t) = m_0 + \int_0^t (\rho_0(x) - \rho(t, x)) \, dx.
\end{align*}
\]

Informally, during the evolution, we observe periodic boundary conditions when $m(t) = 0$ and Dirichlet boundary conditions $\rho(t, 0) = \rho(t, 1) = \Phi^{-1}(1/\lambda)$ when $m(t) > 0$.

4.2. **Remarks on general defects.** In our model, we have assumed that the jumping rate at a defect site $k, N$ is slowed down by a factor of $\lambda k N^{\beta_j}$. However, the form of $\lambda k N^{\beta_j}$ is only a choice for convenience. In fact, Theorems 4.1 and 4.2 hold for more general choices of slow down factors, as can be perused.

We may assume the jumping rate at a defect site $k, N$ is $g(\cdot)/\beta_j N$. (In terms of the current setting $\beta_j, N = \lambda_j N^{\beta_j}$. Assume that $\beta_j, N > 0$ and $\lim_{N \to \infty} \beta_j, N / g(\cdot)$ exists (might be $\infty$ though). To extend Theorems 4.1 and 4.2 to the new setting, we need to clarify the sets of $J_s, J_c, \text{ and } \lambda_j$ for $j \in J_c$. When $g$ is of $n^a$ type, $J_c = \{j \in J : \beta_j, N \sim N^a\}$ and when $g$ is bounded, $J_c = \{j \in J : \lim_{N \to \infty} \beta_j, N > 1\}$. In either case of $g$, the super-critical index set $J_s$ consists of $j$’s such that $\beta_j, N \gg g(N)$ and $J_0 = J \setminus \{J_s \cup J_c\}$. The constant $\lambda_j$ for $j \in J_c$ can be defined as $\lambda_j := \lim_{N \to \infty} \beta_j, N / g(N)$.

5. **Proof outline**

In this section, we prove Theorems 4.1 and 4.2. We begin with analyzing $\langle G, \pi t \rangle$ by computing its stochastic differential in terms of certain martingales.

5.1. **Stochastic differentials.** Let $G$ be a smooth function on $[0, T] \times \mathbb{T}$ and we write $G_t(x) := G(t, x)$. For $t \geq 0$, consider $\hat{\pi} t \otimes N$, the empirical measure associated with $\eta_t$ on $T_N$,

\[
\hat{\pi} t \otimes N(dx) := \frac{1}{N} \sum_{k \in T_N} \eta_t(k) \delta_k/N(dx). \quad (5.1)
\]

Note that if $G$ has compact support on $[0, T] \times \mathbb{T} \setminus D_s$, it holds $\langle G_t, \hat{\pi} t \otimes N \rangle = \langle G_t, \pi t \otimes N \rangle$ for all $N$ large (cf. (4.1)).

We have

\[
M_t^{N,G}(G_t, \hat{\pi} t \otimes N) = \langle G_t, \hat{\pi} t \otimes N \rangle - \langle G_0, \hat{\pi} 0 \otimes N \rangle - \int_0^t \left\{ \langle \partial_s G_s, \hat{\pi} s \otimes N \rangle + N^2 L_N \langle G_s, \hat{\pi} s \otimes N \rangle \right\} \, ds
\]

is a mean zero martingale. Denote the discrete Laplacian $\Delta_N$ by

\[
\Delta_N \left( \frac{k}{N} \right) := N^2 \left( G \left( \frac{k+1}{N} \right) + G \left( \frac{k-1}{N} \right) - 2G \left( \frac{k}{N} \right) \right).
\]

Then, for $N$ large, we compute

\[
N^2 L_N \langle G, \hat{\pi} s \otimes N \rangle = \frac{1}{N} \sum_{k \in T_N} \Delta_N G_s \left( \frac{k}{N} \right) g_{k,N}(\eta_s(k)). \quad (5.2)
\]

The quadratic variation of $M_t^{N,G}$ is given by

\[
\langle M^{N,G} \rangle_t = \int_0^t \left\{ N^2 L_N \left( \langle G_s, \hat{\pi} s \otimes N \rangle \right)^2 \right\} - 2 \langle G_s, \hat{\pi} s \otimes N \rangle N^2 L_N \langle G_s, \hat{\pi} s \otimes N \rangle \, ds
\]
which by standard calculation equals
\[
\int_0^t \sum_{k \in \mathbb{T}_N} g_{k,N}(\eta_k(k)) \left\{ \left( G_s\left( \frac{k+1}{N} \right) - G_s\left( \frac{k}{N} \right) \right)^2 + \left( G_s\left( \frac{k-1}{N} \right) - G_s\left( \frac{k}{N} \right) \right)^2 \right\} ds.
\]

This variation may be bounded as follows.

**Lemma 5.1.** For any test function \( G(x) \in C^\infty(\mathbb{T}) \), there is a constant \( C \) independent of \( N \) such that, for all \( N \) large,
\[
\sup_{0 \leq t \leq T} \mathbb{E}_N(M^{N,G}_t) \leq CN^{-1}.
\]

**Proof.** Since \( G \) is smooth, we obtain, for \( N \) large
\[
\mathbb{E}_N(M^{N,G}_t) \leq 2(\|\partial_x G\|_\infty)^2 N^{-1} \mathbb{E}_N \left[ \int_0^t \frac{1}{N} \sum_{k \in \mathbb{T}_N} g_{k,N}(\eta_k(k)) ds \right].
\]

Then, the result follows from (5.6) in the next Lemma 5.2.

It will be useful to bound local particle numbers and rates. Although there are different ways to prove parts of the following statements, say using the entropy inequality and \( \square \), then the result follows from (5.6) in the next Lemma 5.2.

**Lemma 5.2.** We have the following:

1. The expectation of total mass at all but super-critical sites is uniformly bounded:
   \[
   \sup_{N \in \mathbb{N}} \sup_{t \geq 0} \mathbb{E}_N \left[ \frac{1}{N} \sum_{k \in \mathbb{T}_N \setminus \mathcal{D}_N} \eta_k(k) \right] < \infty; \quad (5.3)
   \]

2. The expected particle number at each regular site \( k \notin \mathcal{D}_N \) is uniformly bounded:
   \[
   \sup_{N \in \mathbb{N}} \sup_{t \geq 0} \mathbb{E}_N \left[ \eta_k(k) \right] < \infty; \quad (5.4)
   \]

3. The expectation of weighted jump rate \( N^{-1}g_{k,N} \) vanishes uniformly as \( N \to \infty \):
   \[
   \lim_{N \to \infty} \sup_{t \geq 0} \mathbb{E}_N \left[ \frac{1}{N} g_{k,N}(\eta_k) \right] = 0; \quad (5.5)
   \]

4. The expectation of total weighted jump rate is uniformly bounded:
   \[
   \sup_{N \in \mathbb{N}} \sup_{t \geq 0} \mathbb{E}_N \left[ \frac{1}{N} \sum_{k \in \mathbb{T}_N} g_{k,N}(\eta_k(k)) \right] < \infty. \quad (5.6)
   \]

**Proof.** First, in both rate \( g \) settings, the total mass estimate (5.3) follows directly from the initial bound of the entropy \( H(\mu^N) = O(N) \) and the entropy inequality; see Section 3.1 for an explicit proof.

Moreover, (5.4) and (5.5) follow by straightforwardly applying attractiveness, \( \mu^N \leq A^N \) and \( \mu^N \leq \overline{A}^N \) for \( g \) of \( n^\alpha \) type and bounded type respectively (see Section 3.2). Indeed, in the \( g(n) \sim n^\alpha \) case, we have, for all \( k \notin \mathcal{D}_N \), that \( \mathbb{E}_N[\eta_k(k)] \leq E_{\overline{A}^N}[\xi(k)] = c' \). Also, as \( g \) is increasing, \( \mathbb{E}_N[g_{k,N}(\eta_k(k))] \leq E_{\overline{A}^N}[g_{k,N}(\xi(k))] = \Phi(c') \) (cf. (2.4)). When \( g \) is bounded, we have \( \mathbb{E}_N[\eta_k(k)] \leq E_{\text{A}^N}[\xi(k)] = c' \) for regular sites \( k \notin \mathcal{D}_N \). For the bound of \( \mathbb{E}_N[g_{k,N}(\eta_k(k))] \), it holds trivially that \( \mathbb{E}_N[g_{k,N}(\eta_k(k))] \leq g_\infty = 1 \) when \( k \notin \mathcal{D}_N \) and \( \mathbb{E}_N[g_{k,N}(\eta_k(k))] \leq 1/\lambda_j \) when \( k = k_{j,N} \) with \( \beta_j = 0 \). When \( k = k_{j,N} \) with \( \beta_j < 0 \), we use attractiveness again \( \mathbb{E}_N[g_{k,N}(\eta_k(k))] \leq E_{\text{A}^N}[g_{k,N}(\xi(k))] = \Phi(c'). \)
Finally, to show (5.6), we separate the sum of \( k \) into two sums consisting of the defect sites \( \mathcal{D}_N \) and the regular sites respectively: The sum over \( \mathcal{D}_N \) is bounded via (5.3) and that \( \mathcal{D}_N \) has finite cardinality \( n_0 \) independent of \( N \). Recall that \( g_{k,N}(\cdot) = g(\cdot) \) for \( k \notin \mathcal{D}_N \). Now, bounding \( g(n) \leq g^*n \), the sum over the regular sites is bounded due to (5.3).

5.2. Proof outline of Theorems 4.1 and 4.2

Let \( Q^N \) be the probability measure on the trajectory space \( D([0,T],\mathcal{M}) \) governing \( \pi^N \) when the process starts from \( \mu^N \). For both cases of \( g \) in Section 4.2, we prove the desired results (4.2) and (4.4). Let \( D \) be the density of particles. To be precise, let \( \eta \)

\[ \text{Let } D \text{ be the probability measure on the trajectory space } D([0,T],\mathcal{M}) \text{ ruling } \pi^N \text{ when the process starts from } \mu^N \text{. By Lemma 6.1, the family of measures } \{Q^N\}_{N \in \mathbb{N}} \text{ is tight with respect to the uniform topology, stronger than the Skorokhod topology. Let now } Q \text{ be any limit measure. For both cases of } g, \text{ we first show that } Q \text{ is supported on a class of weak solutions to the associated nonlinear PDE (4.3) or (4.5). Then, by uniqueness results from Section 11 we prove the desired results (4.2) and (4.4).}

Step 1. Let \( G(t,x) \) be a smooth function with compact support in \([0,T] \times (\mathcal{T} \setminus \mathcal{D}_s)\). Notice that \( \langle G_t, \pi^N \rangle = \langle G_t, \tilde{\pi}^N \rangle \) and recall the martingale \( M^{N,G}_t \) and its quadratic variation \( \langle M^{N,G} \rangle_t \) in the last section. By Lemma 6.1 we have \( \mathbb{E}_N \left[ M^{N,G}_T \right]^2 = \mathbb{E}_N \langle M^{N,G} \rangle_T \) vanishes as \( N \to \infty \). For each \( \delta > 0 \), by Chebychev’s inequality,

\[ \mathbb{P}_N \left[ \left| \langle G_t, \pi^N \rangle - \langle G_0, \pi^N_0 \rangle - \int_0^T \left( \langle \partial_t G_s, \pi^N_s \rangle + N^2 L_N \langle G_s, \pi^N_s \rangle \right) ds \right| > \delta \right] \]

\[ = \mathbb{P}_N \left[ \left| \langle M^{N,G}_T \rangle \right| > \delta \right] \leq \frac{1}{\delta^2} \mathbb{E}_N \langle M^{N,G} \rangle_T \to 0 \text{ as } N \to \infty. \]

Note that \( G_T(x) = 0 \) and recall the evaluation of \( N^2 L_N \langle G_s, \pi^N_s \rangle \) in (5.2). Then,

\[ \lim_{N \to \infty} \mathbb{P}_N \left[ \left| \langle G_0, \pi^N_0 \rangle + \int_0^T \left\{ \langle \partial_t G_s, \pi^N_s \rangle + \frac{1}{N} \sum_{k \in \mathbb{N}} \Delta N G_s(k) g_{k,N}(\eta_s(k)) \right\} ds \right| > \delta \right] = 0. \]

As \( \mathbb{E}_N \left[ \sum_{k \in \mathbb{N}} g_{k,N}(\eta_s(k)) \right] \) is uniformly bounded by (5.6), we may replace the discrete Laplacian \( \Delta N G_s(\cdot) \) in (5.7) by \( \partial_{xx} G_s(\cdot) \).

Let \( \mathcal{D}^c = \bigcup_{j \in \mathbb{Z}} (x_j - \epsilon, x_j + \epsilon) \) and \( F_{\epsilon}(s,x) = 1_{\mathcal{T} \setminus \mathcal{D}^c}(x) \partial_{ss} G(s,x) \). To prepare for the ‘bulk’ replacement in the next step, we will further replace \( \partial_{xx} G_s(\cdot) \) by its approximation \( F_{\epsilon}(s,\cdot) \). It suffices to show

\[ \lim_{\epsilon \to 0} \sup_{N \geq 0} \mathbb{E}_N \left[ N^{-1} \sum_{k \in \mathbb{N} \setminus \mathcal{D}^c} g_{k,N}(\eta_s(k)) \right] = 0. \]

As the sum of \( k \) is over a set of cardinality at most \((2\epsilon N + 1)n_0\), the desired limit follows from separating \( k \)'s into defects and regular ones, and applying (5.3) and (5.4). Therefore, we obtain

\[ \lim_{\epsilon \to 0} \lim_{N \to \infty} \mathbb{P}_N \left[ \left| \langle G_0, \pi^N_0 \rangle - \int_0^T \left\{ \langle \partial_t G_s, \pi^N_s \rangle + \frac{1}{N} \sum_{k \in \mathbb{N}} F_{\epsilon}(k/N) g_{k,N}(\eta_s(k)) \right\} ds \right| > \delta \right] = 0. \]

Step 2. We now replace the nonlinear term \( g_{k,N}(\eta_s(k)) \) by a function of the empirical density of particles. To be precise, let \( \eta^l(x) = \frac{1}{2l + 1} \sum_{|y - x| \leq l} \eta(y) \), that is the average
density of particles in the box centered at \( x \) with length \( 2l + 1 \). Therefore, using the Bulk Replacement Lemma (Lemma 8.4), we obtain from (5.8),

\[
\lim_{\varepsilon \to 0} \lim_{\theta \to 0} \limsup_{N \to \infty} \mathbb{P}_N \left[ \left( G_0, \pi_0^N \right) + \int_0^T \left\{ \langle \partial_s G_s, \pi_s^N \rangle ight\} + \frac{1}{N} \sum_{k \in \mathbb{V}_N} F_\varepsilon(s, \frac{k}{N}) \Phi \left( \eta^\varepsilon_{\theta N}(k) \right) \right] > \delta = 0. \tag{5.9}
\]

**Step 3.** For each \( \theta > 0 \), take \( \ell_\theta = (2\theta)^{-1} 1_{[-\theta, \theta]} \). The average density \( \eta^\varepsilon_{\theta N}(k) \) is written as a function of the empirical measure \( \pi^N_t \)

\[
\eta^\varepsilon_{\theta N}(k) = \frac{2\theta N}{2\theta N + 1} \langle \ell_\theta(\cdot - k/N), \pi^N \rangle.
\]

Since \( \Phi \) is Lipschitz continuous and the total number of particles has expectation of order \( N \) on the bulk (cf. Lemma 5.2), we may replace \( \eta^\varepsilon_{\theta N}(k) \) by \( \langle \ell_\theta(\cdot - k/N), \pi^N \rangle \). Hence, we get from (5.9) in terms of the induced distribution \( Q^N \) that

\[
\lim_{\varepsilon \to 0} \lim_{\theta \to 0} \limsup_{N \to \infty} Q^N \left[ \left( G_0, \pi_0^N \right) + \int_0^T \left\{ \langle \partial_s G_s, \pi_s^N \rangle ight\} + \int_T F_\varepsilon(s, x) \Phi \left( \langle \ell_\theta(\cdot - x), \pi_s^N \rangle \right) dx \right] > \delta = 0. \tag{5.10}
\]

Notice that the discrete sum on \( k \) is also replaced by the corresponding integral.

As the absolute value term in (5.10) is continuous (with respect to the uniform topology) on \( D([0, T], \mathcal{M}) \), the set of trajectories in (5.10) is open. Taking \( N \to \infty \), we obtain

\[
\lim_{\varepsilon \to 0} \sup_{\theta \to 0} \limsup_{N \to \infty} Q \left[ \left( G_0, \pi_0 \right) + \int_0^T \left\{ \langle \partial_s G_s, \pi_s \rangle ight\} + \int_T F_\varepsilon(s, x) \Phi \left( \langle \ell_\theta(\cdot - x), \pi_s \rangle \right) dx \right] > \delta = 0. \tag{5.11}
\]

**Step 4.** We show in Lemma 7.1 that \( Q \) is supported on trajectories

\[
\pi_s(dx) = \rho(s, x) dx + \sum_{j \in J_s} \mathbf{m}_j(s) \delta_{x_j}(dx).
\]

Then, for \( x \notin \mathcal{D}^c \) and \( \theta < \varepsilon \), \( \langle \ell_\theta(\cdot - x), \pi_s \rangle = (2\theta)^{-1} \int_{x-\theta}^{x+\theta} \rho(s, u) du \). Note that \( \Phi \) is Lipschitz and \( \rho \) is integrable on \([0, T] \times \mathbb{T}\). Hence, for all \( \varepsilon \) small, we have \( Q \)-almost surely

\[
\lim_{\theta \to 0} \int_T \int_T F_\varepsilon(s, x) \Phi \left( \langle \ell_\theta(\cdot - x), \pi_s \rangle \right) dx ds = \int_0^T \int_T F_\varepsilon(s, x) \Phi(\rho(s, x)) dx ds.
\]

Since almost sure convergence implies convergence in probability, we obtain from (5.11) that, for all \( \delta > 0 \)

\[
\lim_{\varepsilon \to 0} Q \left[ \left( G_0, \pi_0 \right) + \int_0^T \left\{ \langle \partial_s G_s, \pi_s \rangle + \int_T F_\varepsilon(s, x) \Phi(\rho(s, x)) dx \right\} ds \right] > \delta = 0.
\]

Taking \( \varepsilon \to 0 \) we may also replace \( F_\varepsilon \) by \( \partial_{xx} G \). As \( \delta \) is arbitrary, we have

\[
Q \left[ \left( G_0, \pi_0 \right) + \int_0^T \left\{ \langle \partial_s G_s, \pi_s \rangle + \int_T \partial_{xx} G_s(x) \Phi(\rho(s, x)) dx \right\} ds \right] = 1.
\]

By Condition 8.1 the initial condition \( \pi_0 = \pi \) holds. Thus, we conclude the limit measure \( Q \) is concentrated on trajectories \( \pi \) that satisfies the weak formulation (4.6).
Step 5. That \( \rho(s, x) \in L^2(\mathbb{R} \times \mathbb{T}) \) follows from Lemma 10.1. The weak spatial differentiability of \( \Phi(\rho(t, x)) \) and the \( L^2 \) integrability of the weak derivative are addressed in Proposition 10.1 and Remark 10.2. When \( g \) is of \( n^\alpha \) type, by Lemma 10.2 and Lemma 10.4 we obtain the boundary conditions \( m_j(t) = (A_j \Phi(\rho(t, x_j)))^{1/\alpha} \) for all \( j \in J_s \) and \( \Phi(\rho(t, x_j)) = \Phi(c_0) \) for all \( j \in J_s \). When \( g \) is bounded, by Lemma 7.5 it holds that \( \Phi(\rho(t, x_j)) \leq 1/\lambda_j \) and \( m_j(t) = m_j(t) 1_{\Phi(\rho(t, x_j)) = 1/\lambda_j} \) for all \( j \in J_s \). Therefore, \( \pi \) is a weak solution to (9.3) or (9.5) when \( g \) is of \( n^\alpha \) type or bounded respectively (cf. Definitions 4.4 and 4.5).

In Section 11 we show that there is at most one weak solution \( \pi \) to (11.3) or (11.5). We conclude then that the sequence of \( Q^N \) converges weakly to the Dirac measure on \( \pi \). Finally, as \( Q^N \) converges to \( Q \) with respect to the uniform topology, we have weak convergence of \( \pi^N_t \) for each \( 0 \leq t \leq T \). That is, for all \( G \in C_c(\mathbb{T} \setminus D_s) \), \( \langle G, \pi_t \rangle \) weakly converges to the constant \( \langle G, \pi_t \rangle \), and therefore convergence in probability as stated in (4.2) and (4.4). To finish the proof, it remains to extend the test function from \( G \in C_c(\mathbb{T} \setminus D_s) \) to \( G \in C(\mathbb{T}) \).

When \( g \) is bounded, it holds that \( C_c(\mathbb{T} \setminus D_s) = C(\mathbb{T}) \) as \( D_s = \emptyset \). When \( g \) is of \( n^\alpha \) type, notice that \( \rho(t, \cdot) \) is in \( L^1(\mathbb{T}) \) for all \( t \geq 0 \) (cf. Lemma 7.1). Then, the desired extension follows from standard approximations and (5.2) of Lemma 5.2.

6. Tightness

In this section, we address the tightness of the probability measures associated with the trajectories of empirical measures \( \pi^N \). Recall that \( D_s \) is the macroscopic locations of the super-slow sites; in particular, \( D_s = \emptyset \) when \( g \) is bounded. To unify treatment, in both rate \( g \) settings, we let \( M \) be the space of locally finite nonnegative measures on \( \mathbb{T} \setminus D_s \). Let \( C_c(\mathbb{T} \setminus D_s) \) be the space of continuous functions with compact support on \( \mathbb{T} \setminus D_s \). Let \( \{ f_k \}_{k \in \mathbb{N}} \) be a countable dense set in \( C_c(\mathbb{T} \setminus D_s) \) in the sense that for all \( f \in C_c(\mathbb{T} \setminus D_s) \) there exists a subsequence \( \{ f_{n_k} \} \) such that \( \sup_{f_{n_k}} \subset \sup_{f} \) for all \( k \) and \( f_{n_k} \) converges uniformly to \( f \). Equipped with the distance

\[
d(\mu, \nu) = \sum_{k=1}^{\infty} 2^{-k} \frac{|\int_0^1 f_k(d\mu - d\nu)|}{1 + |\int_0^1 f_k(d\mu - d\nu)|},
\]

the space \( (M, d(\cdot, \cdot)) \) is a complete separable metric space. The metric \( d(\cdot, \cdot) \) realizes the vague topology, that is, \( \lim_{n \to \infty} d(\mu_n, \mu) = 0 \) if and only if \( \lim_{n \to \infty} \int f d\mu_n = \int f d\mu \) for all \( f \in C_c(\mathbb{T} \setminus D_s) \).

Let \( Q^N \) be the probability measure on the trajectory space \( D([0, T], M) \) governing \( \pi^N \) when the process starts from \( \mu^N \). We show that \( \{ Q^N \} \) is tight with respect to the uniform topology, stronger than the Skorokhod topology on \( D([0, T], M) \).

**Lemma 6.1.** \( \{ Q^N \}_{N \in \mathbb{N}} \) is relatively compact with respect to the uniform topology. As a consequence, all limit points \( Q \) are supported on trajectories \( \{ \pi_t \}_{t \in [0, T]} \) vaguely continuous on \( \mathbb{T} \setminus D_s \), that is, for all \( G \in C_c(\mathbb{T} \setminus D_s) \), the mapping \( t \in [0, T] \mapsto \langle G, \pi_t \rangle \) is continuous.

**Proof.** To deduce that \( \{ Q^N \} \) is relatively compact with respect to uniform topology, we show the following items (cf. Theorem 15.5 in [8]).

1. For each \( t \in [0, T], \epsilon > 0 \), there exists a compact set \( K_{t, \epsilon} \subset M \) such that

\[
\sup_N Q^N \left[ \pi^N : \pi^N_t \notin K_{t, \epsilon} \right] \leq \epsilon.
\]
(2) For every $\epsilon > 0$,
\[
\lim_{r \to 0} \limsup_{N \to \infty} Q^N \left[ \pi^N_r : \sup_{|t-s| < r} d(\pi^N_t, \pi^N_s) > \epsilon \right] = 0. \quad (6.2)
\]

Step 1. We first consider \[6.1\]. Notice that, for any $A > 0$, the set $\{ \mu \in M : (1, \mu) \leq A \}$ is compact in $M$. Also, by \[5.3\], we have $E_N \left[ N^{-1} \sum_{k \notin D_{s,N}} \eta_t(k) \right] \leq C$ for some constant $C < \infty$ independent of $N$. As $Q^N \left[ \{ \pi^N_1 \} \right] \leq A^{-1} E_N \left[ N^{-1} \sum_{k \notin D_{s,N}} \eta_t(k) \right]$, the first condition \(6.1\) is checked by taking $A$ large.

Step 2. To verify the second condition \[6.2\], it is enough to show a counterpart of the condition for the distributions of $\langle G, \pi_t \rangle$ where $G$ is any smooth test function compactly supported on $T \setminus D_s$ (cf. p. 54, \[18\]). In other words, we need to show, for every $\epsilon > 0$,
\[
\lim_{r \to 0} \limsup_{N \to \infty} Q^N \left[ \pi^N_r : \sup_{|t-s| < r} \left| G, \pi^N_t - G, \pi^N_s \right| > \epsilon \right] = 0. \quad (6.3)
\]
Note that $\langle G, \pi^N_1 \rangle = \langle G, \pi^N_t \rangle + \int_0^t N^2 L_N \langle G, \pi^N_s \rangle \, ds + M_t^{N,G}$. Then, only need to consider the oscillations of $\int_0^t N^2 L_N \langle G, \pi^N_t \rangle \, ds$ and $M_t^{N,G}$ respectively.

Step 3. For the oscillations of the martingale $M_t^{N,G}$, by $|M_t^{N,G} - M_s^{N,G}| \leq |M_t^{N,G}| + |M_s^{N,G}|$, we have $E_N \left[ \sup_{0 \leq s < t} \left| M_t^{N,G} - M_s^{N,G} \right| > \epsilon / 2 \right] \leq 2 E_N \left[ \sup_{0 \leq s < t} \left| M_t^{N,G} \right| > \epsilon / 2 \right]$. Using Chebyshev and Doob’s inequality, we further bound it by
\[
\frac{8}{c^2} E_N \left[ \left( \sup_{0 \leq t \leq T} \left| M_t^{N,G} \right| \right)^2 \right] \leq \frac{32}{c^2} E_N \left[ (M_T^{N,G})^2 \right] = \frac{32}{c^2} E_N \langle M_N, G \rangle_T.
\]
By Lemma \[5.1\], $E_N \langle M_N, G \rangle_T = O(N^{-1})$. Then, we conclude
\[
\lim_{r \to 0} \limsup_{N \to \infty} \mathbb{P}_N \left[ \sup_{|t-s| < r} \left| M_t^{N,G} - M_s^{N,G} \right| > \epsilon \right] = 0.
\]

Step 4. To conclude \[6.3\], it suffices to show
\[
\lim_{r \to 0} \limsup_{N \to \infty} Q^N \left[ \sup_{|t-s| < r} \int_s^t N^2 L_N \langle G, \pi^N_t \rangle \, ds > \epsilon \right] = 0.
\]
The absolute value term is bounded above by $\|\Delta G\|_{\infty} \frac{1}{N} \sum_{k \notin D_{s,N}} g(k, \eta_t(k))$ (cf. \[5.2\]) applied to $\pi^N_t$. Then, by Markov’s inequality, it remains to show
\[
\lim_{r \to 0} \limsup_{N \to \infty} E_N \left[ \sup_{|t-s| < r} \int_s^t \frac{1}{N} \sum_{k \notin T_N \setminus D_{s,N}} g(k, \eta_t(k)) \, dt \right] = 0.
\]
By Lemma \[5.2\], $E_N \left[ \int_0^T N^{-1} g(k, \eta_t(k)) \, dt \right]$ vanishes as $N \to \infty$ for defect sites $k \in D_{s,N}$ or $D_{k,N}$. Therefore, we may restrict the summation term in the previous display over $k \in T_N \setminus D_N$. Note that for each such $k$ we have $g(k, \eta_t(k)) = g(\cdot)$. When $g$ is bounded, as $N^{-1} \sum_{k \notin D_{s,N}} g(\eta_t(k)) \leq \|g\|_{\infty}$, the lemma is proved for this case.

The rest of this proof focuses on the case when $g$ is of $n^\alpha$ type. Since $g(\cdot)$ grows at most linearly, we are left to show
\[
\lim_{r \to 0} \limsup_{N \to \infty} E_N \left[ \sup_{|t-s| < r} \int_s^t \frac{1}{N} \sum_{k \in T_N \setminus D_N} \eta_t(k) \, dt \right] = 0. \quad (6.4)
\]
Step 5. To show (6.4), we introduce a truncation \( \mathbb{1}_{\eta(k) \leq A} \) with \( A > 0 \). Notice that
\[
E_N \left[ \sup_{|s-t|<r} \left| \frac{1}{N} \sum_{k \notin \mathcal{D}_N} \eta_r(k) \mathbb{1}_{\eta_r(k) \leq A} d\tau \right| \right] \leq E_N \left[ \sup_{|s-t|<r} \left| \frac{1}{N} \sum_{k \notin \mathcal{D}_N} \eta_r(k) d\tau \right| \right] \leq rA
\]
which vanishes when taking \( N \to \infty \), \( r \to 0 \), and \( A \to \infty \) in order. It remains to show the error term with \( \mathbb{1}_{\eta(k) > A} \) also vanishes in the limit. Note that the error term is estimated by
\[
E_N \left[ \sup_{|s-t|<r} \left| \frac{1}{N} \sum_{k \notin \mathcal{D}_N} \eta_r(k) \mathbb{1}_{\eta_r(k) > A} d\tau \right| \right] \leq E_N \left[ \int_0^T \frac{1}{N} \sum_{k \notin \mathcal{D}_N} \eta_r(k) d\tau \right]
= \int_0^T E_N \left[ \frac{1}{N} \sum_{k \notin \mathcal{D}_N} \eta_r(k) \mathbb{1}_{\eta_r(k) > A} \right] d\tau.
\]
Recall the initial entropy bound \( H(\mu^N|\mathcal{A}^N_{\varphi_0}) < CN \). By entropy inequality, for any \( \tau \in [0, T] \) and \( B > 0 \), the term \( E_N \left[ N^{-1} \sum \eta_r(k) \mathbb{1}_{\eta_r(k) > A} \right] \) is further bounded above by
\[
\frac{C}{B} + \frac{1}{BN} \ln E_{\mathcal{A}^N_{\varphi_0}} \left[ \exp \left\{ B \sum_{k \notin \mathcal{D}_N} \eta(k) \mathbb{1}_{\eta(k) > A} \right\} \right].
\]
(6.5)
Since, with respect to \( \mathcal{A}^N_{\varphi_0} \), the \( \eta(k) \)'s in (6.3) are i.i.d. with common distribution \( \mathcal{P}_\varphi \) for \( \varphi = \Phi(c_0) \), we have (6.3) is equal to
\[
\frac{C}{B} + \frac{N - n_0}{BN} \ln E_{\mathcal{P}_\varphi} \left[ e^{BX_1 \mathbb{1}_{X > A}} \right] \leq \frac{C}{B} + B^{-1} \ln E_{\mathcal{P}_\varphi} \left[ e^{BX_1 \mathbb{1}_{X > A}} \right].
\]
Here, \( n_0 \) is the number of defect sites and the distribution of \( X \) is \( \mathcal{P}_\varphi \). As \( E_{\mathcal{P}_\varphi} [e^{BX}] < \infty \) for all \( B > 0 \), we have \( B^{-1} \ln E_{\mathcal{P}_\varphi} [e^{BX_1 \mathbb{1}_{X > A}}] \to 0 \) when taking \( A \to \infty \) and then \( B \to \infty \). Hence, we obtain
\[
\lim_{A \to \infty} \lim_{r \to 0} \lim_{N \to \infty} E_N \left[ \sup_{|s-t|<r} \left| \frac{1}{N} \sum_{k \notin \mathcal{D}_N} \eta_r(k) \mathbb{1}_{\eta_r(k) > A} d\tau \right| \right] = 0
\]
which completes the proof. \( \square \)

7. Properties of Limit Measures

By Lemma 6.1 the sequence \( \{Q^N\} \) is relatively compact with respect to the uniform topology. Consider any convergent subsequence of \( Q^N \) and relabel so that \( Q^N \Rightarrow Q \). We now consider absolute continuity and boundary behaviors at defect sites for trajectories under \( Q \).

7.1. Absolute continuity. We now address absolute continuity.

Lemma 7.1. We have \( Q \) is supported on trajectories whose restriction on \( \mathbb{T} \setminus \mathcal{D}_c \) is absolutely continuous and the density is in \( L^\infty \) uniformly in \( t \): there exists \( c \geq 0 \) such that
\[
Q \left[ \pi : \pi(t)(dx) = \rho(t, x) dx + \sum_{j \in J_c} m_j(t) \delta_{x_j}(dx) \right. \text{ with } \|\rho(t, \cdot)\|_\infty < c \text{ for all } 0 \leq t \leq T \right] = 1.
\]

Proof. Let \( D := \mathbb{T} \setminus \{\mathcal{D}_c \cup \mathcal{D}_c^c\} \) and \( C_c^+(D) \) be the space of nonnegative continuous functions with compact support on \( D \), equipped with topology of uniform convergence on compact
sets. Take \( \{G_n\}_{n \in \mathbb{N}} \) be a dense sequence of \( C_c^+(D) \). The lemma will follow if we have, for some \( c \geq 0 \),

\[
Q \left[ \pi : (G_n, \pi_t) \leq (G_n, c) \text{ for all } 0 \leq t \leq T \text{ and } n \in \mathbb{N} \right] = 1.
\]

Here \( \langle G, c \rangle \) is shorthand for \( c \int_0^T G(x(t))dx \).

To this end, recall \( \kappa_N^c \) defined in Condition 3.1. Let \( \nu^N \) denote \( \mathcal{A}_c^N \) when \( g \) is of \( n^a \) type and \( \kappa_N^c \) when \( g \) is bounded. By the product structure of \( \nu^N \) and Chebyshev inequality, for each \( \delta > 0 \), we have \( \nu^N[|\langle G_n, \pi^N \rangle - \langle G_n, c' \rangle| > \delta] \rightarrow 0 \) as \( N \rightarrow \infty \).

Fix \( \varepsilon > 0 \) and \( t \in [0, T] \). By attractiveness (cf. Section 3.2), \( Q^N[\langle G_n, \pi^N \rangle \leq \langle G_n, c' \rangle + \varepsilon] \) is bounded from below by \( \nu^N[\langle G_n, \pi^N \rangle \leq \langle G_n, c' \rangle + \varepsilon] \). Then, we have, for all \( t \geq 0 \) and \( \varepsilon > 0 \)

\[
\lim_{N \rightarrow \infty} Q^N \left[ \langle G_n, \pi^N_t \rangle \leq \langle G_n, c' \rangle + \varepsilon \right] = 1.
\]

As compactness of \( \{Q^N\} \) was shown in the uniform topology in Lemma 6.1, the distribution of \( \langle G_n, \pi^N \rangle \) under \( Q^N \) converges weakly to \( \langle G_n, \pi_t \rangle \) under \( Q \). Hence, for any \( \varepsilon > 0 \) and \( t \in [0, T] \), we have

\[
Q \left[ \langle G_n, \pi_t \rangle \leq \langle G_n, c' \rangle + \varepsilon \right] \geq \limsup_{N \rightarrow \infty} Q^N \left[ \langle G_n, \pi^N_t \rangle \leq \langle G_n, c' \rangle + \varepsilon \right] = 1.
\]

Since \( Q \) is supported on vaguely continuous trajectories by Lemma 6.1, we obtain for all \( \varepsilon > 0 \) that \( Q \left[ \langle G_n, \pi_t \rangle \leq \langle G_n, c' \rangle + \varepsilon \text{ for all } 0 \leq t \leq T, n \in \mathbb{N} \right] = 1 \). Then, we conclude the lemma by taking \( \varepsilon \rightarrow 0 \) and choosing \( c = c' \).

\[ \square \]

7.2. Boundary behavior. We proceed by showing the behavior of the limit density \( \rho(t, x) \) around the defects \( x_j \) in \( \mathcal{D}_c \) or \( \mathcal{D}_e \). Notice that, part of the results presented here are in terms of \( \rho(t, x) \) as \( x \rightarrow x_j^- \). The corresponding results for \( x \rightarrow x_j^+ \) follow from similar arguments. In the following, we will make use of a ‘local’ replacement lemma Lemma 9.2 proved in Section 9.

For each \( \theta \in (0, 1) \), let \( i_\theta : (0, \theta) \rightarrow [0, 1] \) be a compactly supported continuous function such that \( i_\theta(x) = 1 \) for \( x \in (\theta^2, \theta - \theta^2) \). Let \( \|i_\theta\| \) denote the \( L^1 \) norm \( \int_0^\theta i_\theta(x)dx \) and \( \|i_\theta\|^{-1} := \left(\|i_\theta\|^{-1}\right)^{-1} \). Notice that \( \|i_\theta\|^{-1} - \theta^{-1} \) is bounded from above for all \( \theta \) small since \( \|i_\theta\|^{-1} - \theta^{-1} \) is less than \( \|\theta - 2\theta^2\|^{-1} - \theta^{-1} \) which approaches 2 as \( \theta \rightarrow 0 \).

We first describe behavior near ‘super-slow’ sites in the \( n^a \) setting.

Lemma 7.2. Let \( g \) be of \( n^a \) type. Then, for any \( j \in J_s, \delta > 0 \), and \( G \in C[0, T] \), we have

\[
\lim_{\theta \rightarrow 0} Q \left[ \left| \int_0^T G(s) \left( \Phi(\rho^+, \theta(s)) - \Phi(c_0) \right) ds \right| > \delta \right] = 0 \tag{7.1}
\]

where \( \rho^+_{x_j}(s) := \|i_\theta\|^{-1}(i_\theta(-x_j), \rho) = \|i_\theta\|^{-1} \int_0^\theta i_\theta(x-x_j)\rho(s, x)dx \). Consequently,

\[
Q \left[ \Phi(\rho(t, x_j)) = \Phi(c_0) \right. \text{ for a.e. } t \in [0, T] \text{ and all } j \in J_s \left. \right] = 1. \tag{7.2}
\]

Proof. The proof of is split into steps. We show (7.1) in the first four steps and the last step derives (7.2) from (7.1).

Step 1. Fix \( j \in J_s \) and also a test function \( G \in C[0, T] \) for later use. We first show that

\[
\lim_{N \rightarrow \infty} \sup \mathbb{E}_N \left[ \left| N^{-\beta} g(\eta_s(k_j, N)) - N^{-\beta} g(\eta_s(k_j, N)) \right| \right] = 0. \tag{7.3}
\]

Note that \( g(n) \sim n^a \). For each \( \epsilon > 0 \), let \( A = A(\epsilon) \) be such that \( |n^a / g(n) - 1| < \epsilon \) for all \( n > A \). Then, the expectation term in (7.3) is bounded above by \( E_1 + E_2 \) where \( E_1 \) and \( E_2 \) are
the same expectation as in \([7.3]\) with the integrand multiplied by indicators \(\mathbb{1}_{\eta_s(k_j,N) \leq A}\) and \(\mathbb{1}_{\eta_s(k_j,N) > A}\) respectively. For each \(A\), the term \(E_1\) vanishes as \(N \to \infty\). By the definition of \(A\) and Lemma \([5.2]\), the term \(E_2\) is further bounded above by \(\epsilon \sup_{s \geq 0} E_N \left[ N^{-\beta_j} g(\eta_s(k_j,N)) \right] = O(\epsilon)\). Letting \(\epsilon \to 0\), we conclude \([7.3]\).

**Step 2.** We now argue that \(N^{-\beta_j} (\eta_h(k_j,N))^\alpha\) may be replaced by \(N^{-\beta_j} (\eta_0(k_j,N))^\alpha\). Let \(r_{N,s} = N^{-\beta_j/\alpha} (\eta_h(k_j,N) - \eta_0(k_j,N))\). Let \(\tau > 0\) be a constant such that there are no other defects within a \(\tau\)-neighborhood of \(x_j\). Take a test function \(F : \mathbb{T} \to [0,1]\) such that \(F\) has compact support in \((x_j - \tau, x_j + \tau)\) and \(F(x) = 1\) for \(|x - x_j| \leq \tau/2\). Recall \(\pi_t^N\) from \([5.2]\) and notice that

\[
|r_{N,s}| \leq N^{1-\beta_j/\alpha} \left| \langle F, \hat{\pi}_N^s \rangle - \langle F, \hat{\pi}_0^N \rangle \right| + N^{-\beta_j/\alpha} \sum_k (\eta_h(k) + \eta_0(k)) := I_1 + I_2.
\]

where the \(\sum_k\) term is summation over \(k \in \mathbb{T}_N\) such that \(|k/N - x_j| \leq \tau\) and \(k \neq k_j\).

As \(\beta_j > \alpha\), \([6.2]\) of Lemma \([5.2]\) asserts that \(\sup_{s \geq 0} E_N \left[ I_2 \right]\) vanishes as \(N \to \infty\). To show \(E_N[I_1]\) vanishes, notice that \(\langle F, \hat{\pi}_N^s \rangle - \langle F, \hat{\pi}_0^N \rangle = \int_0^s N^2 L_N \langle F, \hat{\pi}_N^t \rangle dt + M_{s,F}^N\). Using the generator computation \([5.2]\) and the jump rate bound \([5.4]\), we have

\[
\sup_{s \geq 0} E_N \left[ N^2 L_N \langle F, \hat{\pi}_N^s \rangle \right] \leq \sup_{N \geq 0} \left\| \Delta_N F(\cdot) \right\|^\infty E_N \left[ \frac{1}{N} \sum_{k \in \mathbb{T}_N} g_k, N(\eta(k)) \right] < \infty.
\]

Also, by Lemma \([5.1]\) the martingale term vanishes as its variance \(\sup_{0 \leq s \leq T} E_N \langle M_{s,F}^N \rangle_s = o(1)\). Thus, as \(\beta_j > \alpha\), we have \(E_N[I_1]\) and, therefore, \(E_N[|r_{N,s}|]\) vanishes as \(N \to \infty\) uniformly for all \(s \in [0,T]\).

As \(\alpha \in (0,1]\), by the elementary inequality \(|x| + |y|) \leq |x|^\alpha + |y|^\alpha\), we have

\[
N^{-\beta_j} (|\eta_h(k_j,N)|^\alpha - |\eta_0(k_j,N)|^\alpha) \leq |r_{N,s}|^\alpha.
\]

Therefore, we conclude

\[
\lim_{N \to \infty} \sup_{0 \leq s \leq T} E_N \left[ N^{-\beta_j} \left| (\eta_h(k_j,N))^\alpha - (\eta_0(k_j,N))^\alpha \right| \right] = 0. \quad (7.4)
\]

In particular, it follows from \([7.3]\) and \([7.4]\) that, for any \(\delta > 0\),

\[
\lim_{N \to \infty} \mathbb{P}_N \left[ \left| \int_0^T G(s) (N^{-\beta_j} g(\eta_s(k_j,N))) - N^{-\beta_j} (\eta_0(k_j,N))^\alpha ds \right| > \delta \right] = 0. \quad (7.5)
\]

**Step 3.** In this step, we show that \(N^{-\beta_j} (\eta_0(k_j,N))^\alpha\) may be replaced by \(\varphi = \lambda_\beta \Phi(c_0)\). Precisely, as \(\eta_0\) has distribution \(\mu^N\), we show that

\[
\lim_{N \to \infty} \mu^N \left[ \left| N^{-\beta_j/\alpha} \chi(k_j,N) - \varphi^{1/\alpha} \right| > \delta \right] = 0. \quad (7.6)
\]

Indeed, let \(\mathcal{U}_N := \left\{ \left| N^{-\beta_j/\alpha} \chi(k_j,N) - \varphi^{1/\alpha} \right| \geq \delta \right\}\). Fix \(\tau_0 \in (1, \beta_j/\alpha)\). By the entropy inequality,

\[
\mu^N \left[ \mathcal{U}_N \right] = E_{\mu^N} \left[ \mathbb{1}_{\mathcal{U}_N} \right] \leq N^{-\tau_0} H(\mu^N | \mathcal{R}_c^N) + N^{-\tau_0} \ln E_{\mathcal{R}_c^N} \left[ e^{N^{\tau_0} \mathcal{U}_N} \right].
\]

Since \(H(\mu^N | \mathcal{R}_c^N) = O(N)\) (cf. Condition \([5.1]\)) and \(E_{\mathcal{R}_c^N} \left[ e^{N^{\tau_0} \mathcal{U}_N} \right] \leq 1 + e^{N^{\tau_0} \mathcal{R}_c^N \left[ \mathcal{U}_N \right]}\), we have

\[
\lim_{N \to \infty} \mu^N \left[ \mathcal{U}_N \right] \leq \max \{ 0, \lim_{N \to \infty} N^{-\tau_0} \ln \left( e^{N^{\tau_0} \mathcal{R}_c^N \left[ \mathcal{U}_N \right]} \right) \}.
\]

By the following Lemma \([7.3]\) we have \(\lim_{N \to \infty} N^{-\tau_0} \ln \mathcal{R}_c^N \left[ \mathcal{U}_N \right] = -\infty\), and therefore, \([7.6]\) holds.
Step 4. Putting together (7.5) and (7.6), we have
\[
\lim_{N \to \infty} \mathbb{P}_N \left[ \int_0^T G(s) \left( N^{-\beta} g(\eta_s(k_j,N)) - \lambda_j \Phi(c_0) \right) ds > \delta \right] = 0.
\]
For each \( \theta > 0 \), let \( \eta_N^{\theta,+,+}(k) = (\theta N)^{-1} \sum_{0 < j - k \leq \theta N} \eta_s(k) \). By Lemma 9.2, we may further replace \( N^{-\beta} g(\eta_s(k_j,N)) \) by \( \lambda_j \Phi(\eta_N^{\theta,+,+}(k_j,N)) \) and obtain
\[
\lim_{N \to \infty} \sup_{\theta > 0} \mathbb{P}_N \left[ \int_0^T G(s) \left( \Phi(\eta_N^{\theta,+,+}(k_j,N)) - \Phi(c_0) \right) ds > \delta \right] = 0. \quad (7.7)
\]
Notice that \( \left| \eta_N^{\theta,+,+}(k_j,N) - \|i_\theta\|^{-1} \langle i_\theta(\cdot - x_j), \pi_s^N \rangle \right| \) is bounded above by
\[
\|i_\theta\|^{-1} \theta^{-1} \sum_{0 \leq k \leq \theta N} \eta_s(k) + (\theta N)^{-1} \sum_{(k/N - x_j) \in (0, \theta \bar{\beta}^2 \cup [\theta - \theta^2, \theta])} \eta_s(k).
\]
Using the particle numbers bound (5.1) and that \( \|i_\theta\|^{-1} - \theta^{-1} \) is bounded for all \( \theta \) small, the \( \mathbb{E}_N \) expectation of the previous display vanishes as \( N \to \infty \). Since \( \Phi(\cdot) \) is Lipschitz, we may replace \( \eta_N^{\theta,+,+}(k_j,N) \) in (7.7) by \( \|i_\theta\|^{-1} \langle i_\theta(\cdot - x_j), \pi_s^N \rangle \) to obtain
\[
\lim_{\theta \to 0} \sup_{N \to \infty} Q^N \left[ \int_0^T G(s) \left( \Phi(\|i_\theta\|^{-1} \langle i_\theta(\cdot - x_j), \pi_s^N \rangle) - \Phi(c_0) \right) ds > \delta \right] = 0.
\]
Since the absolute value term in the previous display is continuous (in the uniform topology) when viewed as a function on the trajectory space \( D([0, T], \mathcal{M}) \), we obtain
\[
\lim_{\theta \to 0} Q \left[ \int_0^T G(s) \left( \Phi(\|i_\theta\|^{-1} \langle i_\theta(\cdot - x_j), \pi_s \rangle) - \Phi(c_0) \right) ds > \delta \right] = 0.
\]
By Lemma 7.1, \( \|i_\theta\|^{-1} \langle i_\theta(\cdot - x_j), \pi_s \rangle \) is recognized as \( \rho^{+,\theta}_x(s) \); therefore (7.1) follows.

Step 5. We now conclude the proof of the lemma by showing (7.2) from (7.1). By Lemma 6.1 and Proposition 10.1 (see also Remarks 4.3 and 10.2), it holds almost surely under \( Q \) that \( \|\rho(\cdot, \cdot)\|_\infty < \infty \) and \( \rho(t, \cdot) \in C(\mathbb{T}) \) for almost all \( t \in [0, T] \). Then, we have, \( Q \)-almost surely, \( \int_0^T G(s) \Phi(\rho^{+,\theta}_x(s)) ds \) converges to \( \int_0^T G(s) \Phi(\rho(s, x_j)) ds \). As we have shown in (7.1) that \( \int_0^T G(s) \Phi(\rho^{+,\theta}_x(s)) ds \) also converges to \( \int_0^T G(s) \Phi(c_0) ds \) in probability, we obtain, for all \( G \in C([0, T], J_x) \) and \( j \in J_x \)
\[
Q \left[ \int_0^T G(s) \left( \Phi(\rho(s, x_j)) - \Phi(c_0) \right) ds = 0 \right] = 1.
\]
As the continuous function space \( C([0, T], J_x) \) is separable with respect to the uniform norm and \( J_x \) is a finite set, we further obtain
\[
Q \left[ \int_0^T G(s) \left( \Phi(\rho(s, x_j)) - \Phi(c_0) \right) ds = 0 \right] = 1 \quad \text{for all } G \in C([0, T], J_x) \text{ and } j \in J_x,
\]
which implies (7.2), finishing the proof. \( \square \)

We now show the estimate used in Step 3 of the above argument.

Lemma 7.3. Let \( g \) be of \( n^\alpha \) type. Let \( X_N \) be distributed according to \( \mathbb{P}_{N^\beta \varphi} \) for some \( \beta > \alpha \) and \( \varphi \geq 0 \). Then, for any \( r_0 < \beta / \alpha \) and \( \delta > 0 \)
\[
\lim_{N \to \infty} N^{-r_0} \ln P \left[ \left| N^{-\beta / \alpha} X_N - \varphi^{1 / \alpha} \right| \geq \delta \right] = -\infty.
\]
Lemma 7.2 shows that

\[ \lim_{N \to \infty} N^{-r_0} \ln P[X_N \geq (\phi + \delta)N^{\beta/\alpha}] = \lim_{N \to \infty} N^{-r_0} \ln P[X_N \leq (\phi - \delta)N^{\beta/\alpha}] = -\infty. \]

To this end, we notice that, for any \( x > 0 \),

\[ \ln P[X_N \geq (\phi + \delta)N^{\beta/\alpha}] \leq \ln \left( e^{-(\phi+\delta)N^{\beta/\alpha}}E[e^{xX_N}] \right) = -(\phi + \delta)N^{\beta/\alpha}x + \ln \frac{Z(e^{\alpha N^{\beta}})}{Z(N^{\beta})}. \]

By Lemma 2.6, \( \ln \frac{Z(e^{\alpha N^{\beta}})}{Z(N^{\beta})} \sim \alpha N^{\beta/\alpha}(e^{x/\alpha} - 1) \). Then, for

\[ K_{\delta, \phi, \alpha} := \max_{x > 0} \{(\phi + \delta)x - \alpha \phi(e^{x/\alpha} - 1)\}, \]

we have

\[ \limsup_{N \to \infty} N^{-\beta/\alpha} \ln P[X \geq (\phi + \delta)N^{\beta/\alpha}] \leq -K_{\delta, \phi, \alpha} < 0. \]

Since \( r_0 < \beta/\alpha \), we obtain

\[ \lim_{N \to \infty} N^{-r_0} \ln P[X_N \geq (\phi + \delta)N^{\beta/\alpha}] = -\infty. \] (7.8)

On the other hand, for all \( x > 0 \), we also have

\[ \ln P[X_N \leq (\phi - \delta)N^{\beta/\alpha}] \leq \ln \left( e^{(\phi-\delta)N^{\beta/\alpha}}E[e^{-xX_N}] \right). \]

By a similar argument employed to prove (7.8), we have

\[ \lim_{N \to \infty} N^{-r_0} \ln P[X_N \leq (\phi - \delta)N^{\beta/\alpha}] = -\infty. \]

The lemma is now proved. \( \square \)

We now consider the behavior near ‘critical’ slow sites in the \( n^\alpha \) setting used in Step 5.

Lemma 7.4. Let \( g \) be of \( n^\alpha \) type. Then, for any \( j \in J_c \), \( \delta > 0 \), and \( G \in C[0, T] \), we have

\[ \lim_{\delta \to 0} Q\left[ \left| \int_0^T G(s) \left( \lambda_j \Phi(x_{j}^\theta(s)) - (m_j(s))^\alpha \right) ds \right| > \delta \right] = 0, \] (7.9)

where \( \rho_j^\theta(s) = \| \iota_\theta \|^{-1} \iota_\theta(\cdot - x_j), \rho \). Consequently,

\[ Q[\rho_j(t) = (\lambda_j \Phi(\rho(t, x_j)))^{1/\alpha} \text{ for a.e. } t \in [0, T] \text{ and all } j \in J_c] = 1. \] (7.10)

Proof. Fix \( j \in J_c \) and \( G \in C[0, T] \). As \( g(n) \sim n^\alpha \), arguments as in Step 1 of the proof of Lemma 7.2 show that \( N^{-\alpha}g(\eta_s(k_j,N)) \) may be replaced by \( N^{-\alpha}(\eta_s(k_j,N))^\alpha \). Hence, we have

\[ \lim_{N \to \infty} P_N \left[ \left| \int_0^T G(s) \left( N^{-\alpha}g(\eta_s(k_j,N)) - (N^{-1}\eta_s(k_j,N))^\alpha \right) ds \right| > \delta \right] = 0. \] (7.11)

Let \( \{F_\theta(x)\} \) be a sequence of nonnegative smooth functions such that \( F_\theta \)'s are supported on \( (x_j - \theta, x_j + \theta) \), \( \|F_\theta\|_\infty \leq 1 \), and \( F_\theta(x) = 1 \) for \( |x - x_j| \leq \theta/2 \). Then,

\[ |\langle F_\theta, \pi_s^N \rangle - N^{-1}\eta_s(k_j,N)| \leq B_{\theta,N}(\eta_s) \]

where \( B_{\theta,N}(\eta) = N^{-1} \sum_{k \not\in D_N} \mathbb{1}_{(-\theta,\theta)(k/N-x_j)}\eta(k) \) for \( \theta \) small and \( N \) large. As \( 0 < \alpha \leq 1 \), we have

\[ E_N \left[ \left| \langle F_\theta, \pi_s^N \rangle - (N^{-1}\eta_s(k_j,N))^\alpha \right| \right] \leq E_N \left[ (B_{\theta,N})^\alpha \right] \leq E_N \left[ B_{\theta,N} \right]^\alpha, \]
which vanishes as $N \to \infty$ and $\theta \to 0$ by (7.11) in Lemma 7.2. Therefore, we may replace $N^{-1} \eta(s(k,N))$ in (7.11) to obtain
\[ \lim \limsup_{N \to \infty} \sup_{\theta \to 0} \left[ \int_0^T G(s) \left( N^{-\alpha} g(\eta(s(k,N))) - \langle F_\theta, \pi_s^{\alpha} \rangle \right) ds \right] = 0. \]

By Lemma 9.2, $N^{-\alpha} g(\eta(s(k,N)))$ may be replaced by $\lambda_j \Phi(\eta^{N,+}_s(k_j,N))$. Moreover, following Step 4 in the proof of Lemma 7.2, we obtain (7.10) to replace $\eta^{N,+}_s(k_j,N)$ with $\rho_{x_j}^{+,(\theta)}(s)$, we obtain
\[ \lim_{\theta \to 0} Q \left[ \int_0^T G(s) \left( \lambda_j \Phi(\rho_{x_j}^{+,(\theta)}(s)) - \langle F_\theta, \pi_s \rangle \right) ds \right] > \delta = 0. \]

Moreover, by Lemma 7.1, \( \int_0^T G(s) \langle F_\theta, \pi_s \rangle \alpha ds \) converges to
\[ \int_0^T G(s) \pi_s(x_j)) \alpha ds = \int_0^T G(s)(m_j(s)) \alpha ds \]
almost surely with respect to $Q$. Hence, with these observations, (7.10) in the lemma holds. Following the argument in Step 5 of Lemma 7.2, we obtain (7.10) from (7.9), finishing the proof.

We now turn to the behavior near ‘critical’ slow sites in the $g$ bounded setting.

**Lemma 7.5.** Let $g$ be bounded. Then, for any $j \in J_c$, $\delta > 0$, and $G \in C[0,T]$, we have
\[ \lim_{\theta \to 0} Q \left[ \int_0^T |G(s)| (\Phi(\rho_{x_j}^{+,(\theta)}(s)) - \phi_j) ds > \delta \right] = 0 \]
and
\[ \lim_{\theta \to 0} Q \left[ \int_0^T |G(s)|(m_j(s) \wedge 1)(\phi_j - \Phi(\rho_{x_j}^{+,(\theta)}(s))) ds > \delta \right] = 0. \]
where $\rho_{x_j}^{+,(\theta)}(s) := \|i_\theta\|^{-1}(i_\theta(-x_j), \rho)$, $\phi_j = \lambda_j^{-1}$, and $a \wedge b = \min\{a, b\}$. Consequently,
\[ Q \left[ \Phi(\rho(t,x_j)) \leq \lambda_j^{-1}, m_j(t) = m_j(t) \right] \Phi(\rho(t,x_j)) = \lambda_j^{-1}. \]
for a.e. $t \in [0,T]$ and all $j \in J_c$.

**Proof.** Fix $j \in J_c$ and $G \in C[0,T]$. We first address (7.12). As $g(\cdot) \leq 1$, it is trivial that, for all $N$
\[ \mathbb{P}_N \left[ \int_0^T |G(s)| (\lambda_j^{-1} g(\eta(s(k,N))) - \phi_j) ds > \delta \right] = 0. \]

We now use the local ‘replacement’ Lemma 9.2 to replace $\lambda_j^{-1} g(\eta_s(k,N))$ by $\Phi(\eta^{N,+}_s(k_j,N))$, to obtain
\[ \lim \limsup_{N \to \infty} \sup_{\theta \to 0} \mathbb{P}_N \left[ \int_0^T |G(s)| (\Phi(\eta^{N,+}_s(k_j,N)) - \phi_j) ds > \delta \right] = 0. \]

Then, (7.12) follows by taking $N \to \infty$, cf. Step 4 in Lemma 7.2.

To show (7.13), we take the same sequence $\{F_\theta(x)\}$ from the proof of Lemma 7.1. We now observe a key microscopic boundary relation:
\[ \lim \limsup_{\theta \to 0} \mathbb{E}_N \left[ \int_0^T |(F_\theta, \pi_s^N) \wedge 1| (\phi_j - \lambda_j^{-1} g(\eta_s(k_j,N))) |ds| \right] = 0. \]

Indeed, as $\lim_{n \to \infty} g(n) = 1$, let $A = A(\varepsilon)$ be such that $|g(n) - 1| < \varepsilon$ for all $n \geq A$. Then, in the above display, on the one hand, the absolute value is bounded above by $\varepsilon$ when $\eta_s(k_j,N) \geq A$. On the other hand, $\langle F_\theta, \pi_s^N \rangle$ is less than $N^{-1}(A + \sum_k \eta(k))$ when
expression derived from a Feynman-Kac bound. These bounds will hold in either bounded settings.

These estimates are obtained through a Rayleigh-type estimation of a variational eigenvalue generator.

Consider the generator \( \lambda_j^s \) g(\eta_s(k,j,N)) by \( \Phi(\eta_s^{\theta,N,+}(k,j,N)) \) and obtain

\[
\lim_{\theta \to 0} \sup_{N \to \infty} \mathbb{P}_N \left[ \int_0^T G(s) \left( \langle F_\theta, \pi_s^N \rangle \wedge 1 \right) \left( \phi_j - \Phi(\eta_s^{\theta,N,+}(k,j,N)) \right) ds > \delta \right] = 0.
\]

Again, by following Step 4 in Lemma 7.2 we have

\[
\lim_{\theta \to 0} Q \left[ \int_0^T G(s) \left( \langle F_\theta, \pi_s \rangle \wedge 1 \right) \left( \phi_j - \Phi(\rho_s^{\theta,N,+}(s)) \right) ds > \delta \right] = 0.
\]

Also, by Lemma 7.1, it holds that \( \lim_{\theta \to 0} \int_0^T \left( |\langle F_\theta, \pi_s \rangle - 1| - |m_j(s) - 1| \right) ds = 0 \) almost surely with respect to \( Q \). As \( \phi_j \) and \( \Phi(\cdot) \) are bounded, we conclude the proof of (7.13).

It remains to show (7.14). Following Step 5 in Lemma 7.2 we obtain from (7.12) that, \( Q \)-almost surely, \( \Phi(\rho(s, x_j)) \leq \phi_j \) for a.e. \( t \in [0, T] \) and all \( j \in J_\alpha \). To finish the proof, it suffices to show that \( m_j(s) = 0 \) whenever \( \Phi(\rho(s, x_j)) < \phi_j \). In fact, by (7.13), we have \( (m_j(t) \wedge 1)(\phi_j - \Phi(\rho(s, x_j))) = 0 \). As we have \( \Phi(\rho(s, x_j)) \leq \phi_j \), we have that \( m_j(s) \wedge 1 = 0 \) (therefore \( m_j(s) = 0 \)) when \( \Phi(\rho(s, x_j)) < \phi_j \), completing the proof.

\[\square\]

8. Local 1 and 2-block estimates of bulk sites

In this section we address the 1 and 2-block estimates for non-defect sites \( \mathbb{T}_N \setminus \mathcal{D}_N \). These estimates are obtained through a Rayleigh-type estimation of a variational eigenvalue expression derived from a Feynman-Kac bound. These bounds will hold in either \( g(n) \sim n^a \) or \( g \) bounded settings.

8.1. Local 1-block estimate. We start from recalling the concept of spectral gap which is used to prove our local 1-block estimate. For \( k \in \mathbb{T}_N \) and \( l \geq 1 \), define the set \( \Lambda_{k,l} = \{k - l, k - l + 1, \ldots, k + l\} \subset \mathbb{T}_N \). Let \( \Omega_{k,l} = \mathbb{N}_0^{\Lambda_{k,l}} \) be the state space of configurations restricted on sites \( \Lambda_{k,l} \). Define the state space of configurations with exactly \( j \) particles on the sites \( \Lambda_{k,l} \):

\[ \Omega_{k,l,j} = \{ \eta \in \Omega_{k,l} : \sum_{x \in \Lambda_{k,l}} \eta(x) = j \} \].

Consider the generator \( L_{k,l} \) on \( \Omega_{k,l} \) given by

\[ L_{k,l} f(\eta) = \sum_{x,x+1 \in \Lambda_{k,l}} \left\{ g(\eta(x)) \left[ f(\eta^{x,x+1}) - f(\eta) \right] + g(\eta(x^+)) \left[ f(\eta^{+1,x}) - f(\eta) \right] \right\} \].

Recall the generator \( L_N \) from (2.1). Notice that, for each \( k, l \) such that \( \Lambda_{k,l} \cap \mathcal{D}_N = \emptyset \), the generator \( L_{k,l} \) coincides with \( L_N \) localized on \( \Lambda_{k,l} \).

For any \( \rho > 0 \), let \( \nu_\rho \) be the product measure on \( \Omega = \mathbb{N}_0^{\mathbb{T}_N} \) with common marginal \( \mathbb{P}_\Phi(\rho) \) on each site \( k \in \mathbb{T}_N \), and let \( \nu_{k,l}^\rho \) be its restriction to \( \Omega_{k,l} \). Let \( \nu_{k,l,j}^\rho \) be \( \nu_{k,l}^\rho \) conditioned on total number of particles on \( \Lambda_{k,l} \) being \( j \). Notice that \( \nu_{k,l,j}^\rho \) does not depend on \( \rho \). It is well-known that both \( \nu_{k,l}^\rho \) and \( \nu_{k,l,j}^\rho \) are invariant measures with respect to the localized generator \( L_{k,l} \) (cf. (1)). For \( \kappa = \nu_{k,l}^\rho \) or \( \nu_{k,l,j}^\rho \), the corresponding Dirichlet form is given by

\[ E_\kappa \left[ f(-L_{k,l} f) \right] = \sum_{x,x+1 \in \Lambda_{k,l}} E_\kappa \left[ g(\eta(x)) \left( f(\eta^{x,x+1}) - f(\eta) \right)^2 \right] \] .

(8.1)
For \( j \geq 1 \), let \( b_{l,j} \) be the spectral gap of \(-L_{k,l}\) on \( \Omega_{k,l,j} \) (cf. p. 374, [13]):

\[
b_{l,j} := \inf_{f} \frac{E_{v_{k,l,j}} [f(-L_{k,l} f)]}{\operatorname{Var}_{v_{k,l,j}} (f)}
\]  

(8.2)

where the infimum is taken over all \( L^2 (\nu_{k,l,j}) \) functions \( f \) from \( \Omega_{k,l,j} \) to \( \mathbb{R} \). For all \( l, j \geq 1 \), as \( \Omega_{k,l,j} \) is a finite space and the localized process is irreducible, we have \( b_{l,j} > 0 \). As a consequence, we have the following Poincaré inequality: for all \( f \in L^2 (\nu_{k,l,j}) \)

\[
\operatorname{Var}_{v_{k,l,j}} (f) \leq C_{l,j} E_{v_{k,l,j}} [f(-L_{k,l} f)]
\]  

(8.3)

where \( C_{l,j} := b_{l,j}^{-1} < \infty \) for \( j \geq 1 \) and \( C_{l,0} = 0 \). We remark that even though, for a large class of \( g(\cdot) \)'s, sharp estimates of \( b_{l,j} \) are available in the literature, we will only need that \( b_{l,j} \) is strictly positive for all \( l, j \geq 1 \).

We now prove the local 1-block estimate for regular sites:

**Lemma 8.1 (Local 1-block estimate).** For any bounded function \( G \) on \([0, T] \times \mathbb{T} \), we have

\[
\lim_{l \to \infty} \limsup_{N \to \infty} \sup_{k, k'} \mathbb{E}_N \left[ \left| \int_0^T G(s, k'/N) \left( g(\eta_k(k')) - \Phi(\eta'_k(k)) \right) ds \right| \right] = 0
\]

where the sup is taken over all \( k \) and \( k' \) such that \( k' \in \Lambda_{k,l} \) and \( \Lambda_{k,l} \cap \mathcal{D}_N = \emptyset \).

**Proof.** We separate the argument into 5 steps.

**Step 1.** We first introduce a cutoff of large densities. Let

\[
V_{k,k',l}(s, \eta) := G(s, k'/N) \left( g(\eta(k')) - \Phi(\eta'(k)) \right)
\]

As \( g(n) \leq g^* n \) and \( \Phi(x) \leq g^* x \), we have

\[
\mathbb{E}_N \left[ \int_0^T V_{k,k',l}(s, \eta_k) \mathbb{1}_{\eta'_k(k) > A} ds \right] \leq g^* \| G \|_\infty \int_0^T \mathbb{E}_N \left[ (\eta_k(k') + \eta'_k(k)) \mathbb{1}_{\eta'_k(k) > A} \right] ds.
\]

By attractiveness (cf. Section 3.2), in both \( g(n) \sim n^\alpha \) and bounded settings (as \( \Lambda_{k,l} \cap \mathcal{D}_N = \emptyset \), the stochastic bound \( \kappa_N^\alpha \) and \( \mathcal{P}_{\alpha}^N \) agree with \( \nu_c \) on \( \Lambda_{k,l} \)), the last expectation is bounded above by

\[
\mathbb{E}_{\mathcal{P}_{\alpha}^N} \left[ (\eta(k') + \eta'_k(k)) \mathbb{1}_{\eta'_k(k) > A} \right] \leq A^{-1} \mathbb{E}_{\nu_c} \left[ (\eta(k') \eta'_k(k)) + (\eta'(k))^2 \right] \leq A^{-1} \mathbb{E}_{\nu_c} \left[ (\eta(k'))^2 + 2 (\eta'(k))^2 \right].
\]

Notice that \( (\eta'(k))^2 \leq (2l + 1)^{-1} \sum_{j \in \Lambda_{k,l}} (\eta(j))^2 \) and, under \( \nu_c \), \( \{\eta(j)\}_{j \in \Lambda_{k,l}} \) has common distribution \( \mathcal{P}_{\nu(c)}^N \). We obtain \( \mathbb{E}_N \left[ \int_0^T V_{k,k',l}(s, \eta_k) \mathbb{1}_{\eta'_k(k) > A} ds \right] \to 0 \) as \( N \to \infty \) in this order.

Therefore, to prove the lemma, it will be enough to show, for all \( A > 0 \), that

\[
\lim_{l \to \infty} \limsup_{N \to \infty} \sup_{k, k'} \mathbb{E}_N \left[ \int_0^T V_{k,k',l,A}(s, \eta_k) ds \right] = 0
\]

where \( V_{k,k',l,A}(s, \eta) := V_{k,k',l}(s, \eta) \mathbb{1}_{\eta'_k(k) \leq A} \).

**Step 2.** An \( H(\mu^N | \mathcal{P}_{\nu_c}^N) \leq C N \) for some \( C < \infty \), it follows from the entropy inequality

\[
\mathbb{E}_N \left[ \int_0^T V_{k,k',l,A}(s, \eta_k) ds \right] \leq \frac{C}{\gamma} + \frac{1}{\gamma^N} \ln \mathbb{E}_{\mathcal{P}_{\nu(c)}^N} \left[ \exp \left\{ \gamma N \int_0^T V_{k,k',l,A}(s, \eta_k) ds \right\} \right].
\]
For $c$ where

$$\lambda_{N,t}(s)$$

is the largest eigenvalue of $N^2L_N + \gamma N V_{k,l,A}(s, \eta)$. We now use the Rayleigh expansion (cf. [18], pp. 375–376, Appendix 3, Theorem 1.1)

$$\lambda_{N,t}(s) = \sup \left\{ E_{\mathscr{R}_{\alpha}^N} [V_{k,l,A} f] - \gamma^{-1} N E_{\mathscr{R}_{\alpha}^N} \left[ \sqrt{f} (-L_N \sqrt{f}) \right] \right\},$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}_{\alpha}^N$ (cf. [18], p. 377).

Let $f_{k,l} = E_{\mathscr{R}_{\alpha}^N} [f|\Omega_{k,l}]$, be the conditional expectation of $f$ given the variables on $\Lambda_{k,l}$.

Fix $\mu_{k,l}$ be such that $\mu_{k,l} = \nu_{k,l}^\alpha$. Since the Dirichlet form $E_{\mathscr{R}_{\alpha}^N} \left[ \sqrt{1} \right]$ is convex, we have

$$\left( \gamma N \right)^{-1} \lambda_{N,t} \leq \sup_{f_{k,l}} \left\{ E_{\mu_{k,l}} [V_{k,l,A} f_{k,l}] - \gamma^{-1} N E_{\mu_{k,l}} \left[ \sqrt{f} (-L_{k,l} \sqrt{f}) \right] \right\}.$$

Step 4. We now decompose $f_{k,l} d\mu_{k,l}$ with respect to sets $\Omega_{k,l,j}$ of configurations with total particle number $j$ on $\Lambda_{k,l}$:

$$E_{\mu_{k,l}} [V_{k,l,A} f_{k,l}] = \sum_{j \geq 0} c_{k,l,j}(f) \int V_{k,l,A} f_{k,l,j} d\mu_{k,l,j},$$

(8.4)

where $c_{k,l,j}(f) = \int_{\Omega_{k,l,j}} f_{k,l,j} d\mu_{k,l}$ and $f_{k,l,j} = c_{k,l,j}(f)^{-1} \mu_{k,l,j} (\Omega_{k,l,j}) f_{k,l}$. Here, $\sum_{j \geq 0} c_{k,l,j} = 1$ and $f_{k,l,j}$ is a density with respect to $\mu_{k,l,j}$.

Straightforwardly, on $\Omega_{k,l,j}$, we have

$$\frac{L_{k,l} \sqrt{f_{k,l}}}{\sqrt{f_{k,l,j}}} = \frac{L_{k,l} \sqrt{f_{k,l,j}}}{\sqrt{f_{k,l,j}}}.$$

Using (8.4), we write

$$E_{\mu_{k,l}} \left[ \sqrt{f_{k,l}} (-L_{k,l} \sqrt{f_{k,l}}) \right] = \sum_{j \geq 0} c_{k,l,j}(f) E_{\mu_{k,l,j}} \left[ \sqrt{f_{k,l,j}} (-L_{k,l} \sqrt{f_{k,l,j}}) \right].$$

Then, we get

$$\left( \gamma N \right)^{-1} \lambda_{N,t} \leq \sup_{0 \leq j \leq A(2l+1)} \sup_{f} \left\{ E_{\mu_{k,l,j}} [V_{k,l,A} f] - \gamma^{-1} N E_{\mu_{k,l,j}} \left[ \sqrt{f} (-L_{k,l} \sqrt{f}) \right] \right\},$$

where the second supremum is on all densities $f$ with respect to $\mu_{k,l,j}$.

Step 5. Let

$$\tilde{V}_{k,l,A} = V_{k,l,A} - E_{\mu_{k,l,j}} [V_{k,l,A}].$$

Let $C_{l,A,G}$ be such that $\| \tilde{V}_{k,l,A} \|_{\infty} \leq C_{l,A,G}$. Recall $C_{l,j}$ the inverse spectral gap of $L_{k,l}$ (cf. (8.3)). We now use the Rayleigh expansion (cf. [18], pp. 375–376, Appendix 3, Theorem 1.1)

$$E_{\mu_{k,l,j}} \left[ \tilde{V}_{k,l,A} f \right] - \gamma^{-1} N E_{\mu_{k,l,j}} \left[ \sqrt{f} (-L_{k,l} \sqrt{f}) \right] \leq \frac{\gamma N^{-1}}{1 - 2C_{l,A,G} C_{l,j} \gamma N^{-1} E_{\mu_{k,l,j}} \left[ \tilde{V}_{k,l,A} (-L_{k,l}) \right]^{-1} \tilde{V}_{k,l,A} \right].$$

(8.5)
The spectral gap of $L_{k,l}$ also implies that $\|L_{k,l}^{-1}\|_2$, the $L^2(\mu_{k,l,j})$ norm of the operator $L_{k,l}^{-1}$ on mean zero functions, is less than or equal to $C_{l,j}$. Now, by Cauchy-Schwarz and the estimate of $\|L_{k,l}^{-1}\|_2$, we have

$$E_{\mu_{k,l,j}} \left[ \tilde{V}_{k,k',l,A} ( - L_{k,l} )^{-1} \tilde{V}_{k,k',l,A} \right] \leq C_{l,j} E_{\mu_{k,l,j}} \left[ \tilde{V}_{k,k',l,A}^2 \right] \leq C_{l,j} C_{l,A,G}^2.$$ 

Accordingly, retrace our steps, noting (8.3), we have that $E_N \left[ \int_0^T V_{k,k',l,A}(\eta_s)ds \right]$ is less than or equal to

$$\frac{C_0}{\gamma} + \sup_{0 \leq j \leq A(2l+1)} \frac{T\gamma N^{-1}C_{l,j}C_{l,A,G}^2}{1 - 2C_{l,A,G} C_{l,j} \gamma N^{-1}} + T \sup_{0 \leq j \leq A(2l+1)} E_{\mu_{k,l,j}} \left[ V_{k,k',l,A} \right].$$

Taking $N \to \infty$, first sup term vanishes. Notice that the expression sup $E_{\mu_{k,l,j}} \left[ V_{k,k',l,A} \right]$ is independent of $N$ and vanishes as $l \to \infty$. In fact, as $\mu_{k,l,j} = \nu_{k,l,j}$ is translation-invariant

$$\left| E_{\mu_{k,l,j}} \left[ V_{k,k',l,A} \right] \right| \leq \|G\|_\infty \left| E_{\nu_{k,l,j}} [g(\eta(0))] - E_{\nu_{l,j}}[g(\eta(0))] \right|.$$

By equivalence of ensembles (cf. p.355, [18]), the right hand side of the above display vanishes as $l \to \infty$, uniformly for $\rho = j/(2l+1) \in [0,A]$. The lemma now is proved by letting $\gamma \to \infty$. \hfill \Box

### 8.2. Local 2-block estimate.

We now detail the local 2-block estimate following the outline of the local 1-block estimate. Recall the notation $\Lambda_{k,l}$ from the 1-block estimate and let $\Lambda_{k,k',l} = \Lambda_{k,l} \cup \Lambda_{k',l}$ for $|k-k'| > l$. Define the generator $L_{k,k',l}$ on $\Omega_{k,k',l} = \mathbb{R}^{\Lambda_{k,k',l}}$:

$$L_{k,k',l} f(\eta) = L_{k,l} f(\eta) + L_{k',l} f(\eta) + g(\eta(k+l))[f(\eta^{k+l,k'-l}) - f(\eta)] + g(\eta(k'-l))[f(\eta^{k'-l,k+l}) - f(\eta)].$$

When $|k-k'|$ is large, the process governed by $L_{k,k',l}$ in effect treats the blocks $\Lambda_{k,l}$ and $\Lambda_{k',l}$ as adjacent, with a connecting bond.

Let $\Theta_{k,k',l,j} := \{ \eta \in \Omega_{k,k',l} : \sum_{x \in \Lambda_{k,k',l}} \eta(x) = j \}$. As before, the localized measure $\nu_{k,k',l,j}$ defined by $\nu_{k,k',l,j}$ limited to sites in $\Lambda_{k,k',l}$, as well as $\nu_{k,k',l,j}$, the canonical measure of $\nu_{k,k',l}$ on $\Theta_{k,k',l,j}$, are both invariant and reversible with respect to $L_{k,k',l}$.

The corresponding Dirichlet form, with measure $\kappa$ given by $\mu_{k,k',l}$ or $\mu_{k,k',l,j}$, is given by

$$E_{\kappa} \left[ f(-L_{k,k',l} f) \right] = \sum_{x,x+1 \in \Lambda_{k,k',l}} E_{\kappa} \left[ g(\eta(x)) \left[ f(\eta^{x,x+1}) - f(\eta) \right]^2 \right]$$

$$+ E_{\kappa} \left[ g(\eta(k+l)) \left[ f(\eta^{k+l,k'-l}) - f(\eta) \right]^2 \right].$$

For $l,j \geq 1$, let $b_{l,j}$ be the spectral gap of $-L_{k,k',l}$ on $\Theta_{k,k',l,j}$ (cf. 8.2). As $b_{l,j}$ is strictly positive, we have the following Poincaré inequality (cf. 8.3): for all $f \in L^2(\nu_{k,k',l,j})$

$$\text{Var}_{\nu_{k,k',l,j}}(f) \leq C_{l,j} E_{\nu_{k,k',l,j}} \left[ f(-L_{k,k',l} f) \right]$$

(8.7)

where $C_{l,l,j} := b_{l,j}^{-1}$ for $j \geq 1$ and $C_{l,l,0} = 0$.

We now state and show a local 2-blocks estimate. The scheme is similar to that of the local 1-block estimate.

**Lemma 8.2 (Local 2-block estimate).** We have

$$\lim_{l \to \infty} \lim_{\theta \to 0} \lim_{N \to \infty} \sup_{k,k''} E_N \left[ \int_0^T |\Phi(\eta_s^l(k)) - \Phi(\eta_s^N(k''))| ds \right] = 0$$

(8.8)
where the sup is taken over all \( k \) and \( k' \) such that \( \Lambda_{k,l} \subset \Lambda_{k'',\theta N} \) and \( \Lambda_{k'',\theta N} \cap \Omega_N = \emptyset \).

**Proof.** We separate the argument into steps.

**Step 1.** Since \( \Phi(\cdot) \) is Lipschitz, to prove the lemma, it suffices to show \( |\eta_s^\theta(k') - \eta_s^\theta(k)| \leq 2l + 1 \) for \( l \leq k' \leq k \leq \theta N \).

By attractiveness (as explained in Step 1 of the proof of the local 1-block Lemma \ref{lem:local1block}), we may apply a cutoff of large densities. Therefore, to prove the lemma, it suffices to show \( \eta_s^\theta(k) - \eta_s^\theta(k') \) is bounded uniformly in \( k, k' \) as in \( (8.9) \).

Let \( \eta_s^\theta(k) = \eta_s^\theta(k') \) and the sup is over \( k, k' \) as in \( (8.9) \).

Let \( U_{k',l,A}(\eta) := |\eta_s^\theta(k) - \eta_s^\theta(k')| \mathbb{1}_{\{\eta_s^\theta(k,k') \leq A\}}. \) Following the proof of Lemma \ref{lem:local1block}, for fixed \( l, \theta, N, k, k' \), in order to estimate \( \mathbb{E}_N \left[ \int_0^T U_{k',l,A}(\eta_s^\theta) ds \right] \), it suffices to bound

\[
(\gamma N)^{-1} \lambda \mathbb{E}_N \left[ \int_0^T U_{k',l,A}(\eta_s^\theta) ds \right] = 0
\]

where the supremum is over all \( f \) which are densities with respect to \( \mathcal{R}_c^\theta_N \).

**Step 3.** Recall the generator \( L_{k,k',l} \) and its Dirichlet form defined in the beginning of this subsection. We now argue the following Dirichlet form inequality

\[
\mathbb{E}_{X_0} \left[ \sqrt{f(-L_{k,k',l} f)} \right] \leq \theta N \mathbb{E}_{X_0} \left[ \sqrt{f(-L_N f)} \right].
\]

The Dirichlet form with respect to the full generator \( L_N \) under \( \mathcal{R}_c^\theta_N \) is given by

\[
\mathbb{E}_{X_0} \left[ f(-L_N f) \right] = \sum_{k \in \mathcal{T}_N} \mathbb{E}_{X_0} \left[ g_{k,N}(\eta(k))(f(\eta^{k,k+1}) - f(\eta))^2 \right].
\]

First, writing out the Dirichlet form in \( \mathcal{S}_0^\theta \), in terms of the product measure \( \mathcal{R}_c^\theta_N \), we have

\[
\mathbb{E}_{X_0} \left[ f(-L_{k,k',l} f) \right] = \sum_{x,x+1 \in \Lambda_{k,k',l}} \mathbb{E}_{X_0} \left[ g(\eta(x))(f(\eta^{x,x+1}) - f(\eta))^2 \right] + \mathbb{E}_{X_0} \left[ g(\eta(k+l))(f(\eta^{k+l,k'+l-1}) - f(\eta))^2 \right].
\]
Next, by adding and subtracting at most \( \theta N \) terms, we have
\[
[f(\eta^{k+l,k'+l}) - f(\eta)]^2 \leq (k' - k - 2l) \sum_{q=0}^{k'-k-2l-1} [f(\eta^{k+l,k+l+q}) - f(\eta^{k+l,k+l+q+1})]^2.
\]

By the change of variables \( \xi = \eta^{k+l,k+l+q} \), which takes away a particle at \( k+l \) and adds one at \( k+l+q \), we have \( R_N(\eta) = \frac{g(\eta(k+l+q)+1)}{g(\eta(k+l))} R_N(\xi) \). Then
\[
E_{\mathcal{R}_N^0} \left( g(\eta(k+l)) \left[ f(\eta^{k+l+q+1}) - f(\eta^{k+l+q}) \right]^2 \right)
= \sum_{\xi} R_N(\eta) g(\eta(k+l)) \left[ f(\eta^{k+l+q+1}) - f(\eta) \right]^2
= E_{\mathcal{R}_N^0} \left( g(\eta(k+l+q)) \left[ f(\eta^{k+l+q+1}) - f(\eta) \right]^2 \right).
\]

From these observations, (8.11) follows.

**Step 4.** Let \( \mu_{k,k'} \) be the restriction of \( \mu = \mathcal{R}_N^0 \) to \( \Lambda_{k,k'} \). Clearly, \( \mu_{k,k'} = \nu_{k,k'} \).

Inputting (8.11) into (8.10), and considering the conditional expectation of \( f \) with respect to \( \Omega_{k,k'} \) as in the 1-block estimate proof, we have
\[
(\gamma N)^{-1} \lambda_{N,1} \leq \sup_{f_{k,k'}} \left\{ E_{\mu_{k,k'}}[U_{k,k',i}f_{k,k'}] - \frac{1}{\theta\gamma} E_{\mu_{k,k'}} \left( \sqrt{\tilde{f}_{k,k',i}}(-L_{k,k',i})\sqrt{\tilde{f}_{k,k',i}} \right) \right\},
\]
where the supremum is over densities \( f_{k,k'} \) with respect to \( \mu_{k,k'} \).

Again, as in the proof of the 1-block estimate, decomposing \( f_{k,k'} d\mu_{k,k'} \) along configurations with common total number \( j \), we need only to bound
\[
\sup_{0 \leq j \leq A(2l+1)} \sup_f \left\{ E_{\nu_{k,k',i,j}}[U_{k,k',i,A}f] - \frac{1}{\theta\gamma} E_{\nu_{k,k',i,j}} \left( \sqrt{\tilde{f}}(-L_{k,k',i})\sqrt{\tilde{f}} \right) \right\},
\]
where the supremum is over densities \( f \) with respect to \( \nu_{k,k',i,j} \).

**Step 5.** Consider the centered object
\[
\hat{U}_{k,k',i,A} = U_{k,k',i,A} - E_{\nu_{k,k',i,j}}[U_{k,k',i,A}].
\]
Recall \( C_{l,i,j} \), the inverse spectral gap of \( L_{k,k'} \) from (8.7) and note that \( \|\hat{U}_{k,k',i,A}\|_{\infty} \leq A \).

Using the Rayleigh expansion (cf. p. 375, [18]), we have
\[
E_{\nu_{k,k',i,j}}[\hat{U}_{k,k',i,A}f] - \left( \theta\gamma \right)^{-1} E_{\nu_{k,k',i,j}} \left[ \sqrt{\tilde{f}}(-L_{k,k',i})\sqrt{\tilde{f}} \right]
\leq \frac{\theta\gamma}{1 - 2AC_{l,i,j}} E_{\nu_{k,k',i,j}} \left[ \hat{U}_{k,k',i,A}(-L_{k,k',i})^{-1}\hat{U}_{k,k',i,A} \right]
\leq \frac{\theta\gamma C_{l,i,j}}{1 - 2AC_{l,i,j}} E_{\nu_{k,k',i,j}} \left[ \hat{U}_{k,k',i,A}^2 \right] \to 0 \text{ as } \theta \to 0.
\]

**Step 6.** Recall the definition of \( U_{k,k',i,A} \) in Step 2. To finish, we still need to estimate the centering term \( E_{\nu_{k,k',i,j}}[U_{k,k',i,A}] \). By adding and subtracting \( j/(4l+2) \), we need only bound \( E_{\nu_{k,k',i,j}}[(\eta^j(k) - j/(4l+2)]) \). By exchangeability and an equivalence of ensemble estimate (cf. p. 355 [18]), the canonical variance
\[
E_{\nu_{k,k',i,j}}[(\eta^j(k) - j/(4l+2))^2] = O(l^{-1}) E_{\nu_{k,k',i,j}}[(\eta^j(k) - j/(4l+2))^2]
+ O(1) E_{\nu_{k,k',i,j}}[(\eta(k) - j/(4l+2))((\eta(k) - j/(4l+2))]
\]
and is further bounded by $C(A)\text{Var}_{V, k', l} (\eta^j(k))$ for some constant $C(A)$ depending only on $A$. This variance is of order $O(l^{-1})$, since the single site variance $\text{Var}_{V, k', l} (\eta(k))$ is uniformly bounded for $j/(4l + 2) \leq A$. Hence, $\sup_{0 \leq j \leq A(4l + 2)} E_{V, k', l} [V_{k, k', l, A}]$ is of order $O(l^{-1/2})$, vanishing as $l \uparrow \infty$. This finishes the proof.

**Remark 8.3.** We comment that Lemmas 8.1 and 8.2 with the choice $k = k' = k''$ gives the replacement of $g(\eta_s(k))$ by $\Phi(\eta^g_N(k))$ which will be used in the ‘bulk’ replacement below. Moreover, choosing $k' = x + 1$, $k = x + l + 1$, and $k'' = x + \theta N + 1$, we may replace $g(x+1)$ by $\Phi(\eta^g_N(x+1))$. Here $\eta^g(x) = \sum_{y=x+1}^{x+2l+1} T_y$ is the average number of particles over the $2l+1$ neighboring sites to the right of $x$. Such an average will be used to treat ‘replacement’ near the boundary of a defect site in Section 9.

8.3. **Bulk Replacement Lemma.** Let $G(t, x)$ be a bounded function on $[0, T] \times \mathbb{T}$ with compact support on $[0, T] \times (\mathbb{T} \setminus \mathcal{D})$. As in Remark 8.3, Lemma 8.1 implies that

$$\lim_{l \to \infty} \sup_{N \to \infty} \sup_{s \in \mathbb{T}} \mathbb{E}_N \left[ \frac{1}{N} \sum_{k \in \mathbb{T}_N} \left| \int_0^T G(s, k/N) (g(\eta_s(k)) - \Phi(\eta^g_s(k))) \, ds \right| \right] = 0$$

and by Lemma 8.2

$$\lim_{l \to \infty} \sup_{\theta \to 0} \sup_{N \to \infty} \sup_{s \in \mathbb{T}} \mathbb{E}_N \left[ \frac{1}{N} \sum_{k \in \mathbb{T}_N} \left| \int_0^T G(s, k/N) \left( \Phi(\eta^g_s(k)) - \Phi(\eta^g_N(k)) \right) \, ds \right| \right] = 0.$$

By Markov’s inequality and triangle inequality, we obtain

**Lemma 8.4** (Bulk Replacement Lemma). For each bounded function $G(t, x)$ on $[0, T] \times \mathbb{T}$ with compact support on $[0, T] \times (\mathbb{T} \setminus \mathcal{D})$, and $\delta > 0$, we have

$$\lim_{\theta \to 0} \sup_{N \to \infty} \mathbb{P}_N \left[ \frac{1}{N} \sum_{k \in \mathbb{T}_N} \left| \int_0^T G(s, k/N) \left( g(\eta_s(k)) - \Phi(\eta^g_s(k)) \right) \, ds \right| \geq \delta \right] = 0.$$

**Remark 8.5.** We comment, in the $g(n) \sim n^\alpha$ setting (where ‘FEM’ as stated in 18 holds), the “attractiveness” assumption used in the local 1 and 2-block estimates to introduce cutoffs of large densities in a local region, may be dropped in the statement of the ‘global average’ Lemma 8.1.

9. **Replacement at the boundary**

In this section, we show a local replacement near the defect sites, used in Subsection 7.2 in two steps. In Lemma 9.1, we show that the jump rate $g_{k, N}(\eta_s(k))$ at any site $k \in \mathbb{T}_N$ is close to $g_{k+1, N}(\eta_s(k + 1))$, that of its neighbor site $k + 1$.

In later use, when $k$ is a defect site in $\mathcal{D}_N$, this neighbor site will be a non-defect or regular site for $N$ large. Then, we may apply local 1 and 2-blocks estimates from last section to obtain our local replacement Lemma 9.2 near the defect sites.

In both lemmas, the replacements will hold in the $g(n) \sim n^\alpha$ and $g$ bounded settings.

**Lemma 9.1.** Let $G(\cdot) : [0, T] \to \mathbb{R}$ be bounded. Then, we have

$$\lim_{N \to \infty} \sup_{s \in \mathbb{T}} \mathbb{E}_N \left[ \left| \int_0^T G(s) \left( g_{k, N}(\eta_s(k)) - g_{k+1, N}(\eta_s(k + 1)) \right) \, ds \right| \right] = 0.$$
Proof. Let

\[ U_{s,k}(\eta) = 2N^{-1}(G(s))^2(g_{k,N}(\eta(k)) + g_{k+1,N}(\eta(k+1))). \]

By Lemma 5.2 \[ \lim_{N \to \infty} \sup_{k \in \mathbb{T}_N} \mathbb{E}_N \left[ \int_0^T U_{s,k}(\eta_s)ds \right] = 0. \] Let

\[ V_{s,k}(\eta) = G(s)(g_{k,N}(\eta(k)) - g_{k+1,N}(\eta(k+1))). \]

Then, to prove the lemma, it suffices to show that

\[ \lim_{N \to \infty} \sup_{k \in \mathbb{T}_N} \mathbb{E}_N \left[ \int_0^T V_{s,k}(\eta_s)ds - \kappa \int_0^T U_{s,k}(\eta_s)ds \right] = 0. \tag{9.1} \]

Recall the initial entropy bound \( H(\mu^N|\mathcal{A}_c^N) \leq CN. \) Then, for \( c = c_0 \) and \( \kappa > 0, \) by the entropy inequality, the expectation in the previous display is bounded from above by

\[ \frac{C}{\kappa} + \frac{1}{\kappa N} \ln \mathbb{E}_N \left[ \exp \left\{ \kappa N \int_0^T \left| V_{s,k}(\eta_s)ds \right| - \kappa \int_0^T U_{s,k}(\eta_s)ds \right\} \right]. \]

The absolute value in the right hand side of last inequality can be dropped by using \( e^{|x|} \leq e^x + e^{-x}. \) By Feynman-Kac formula (cf. p.336, [18]), we have

\[ \frac{1}{\kappa N} \ln \mathbb{E}_N \left[ \exp \left\{ \kappa N \int_0^T (V_{s,k} - \kappa U_{s,k})ds \right\} \right] \leq \frac{1}{\kappa N} \int_0^T \lambda_{N,k}(s)ds \]

where \( \lambda_{N,k}(s) \) is the largest eigenvalue of \( N^2L_N + \kappa N(V_{s,k}(\eta) - \kappa U_{s,k}(\eta)). \) Fix \( s \in [0,T] \) and note the variational formula for \( \lambda_{N,k}: \)

\[ (\kappa N)^{-1} \lambda_{N,k} = \sup_f \left\{ \mathbb{E}_N \left[ (V_{s,k} - \kappa U_{s,k})f \right] - \kappa^{-1}N \mathbb{E}_N \left[ \sqrt{f(-L_N \sqrt{T})} \right] \right\} \]

where the supremum is over all \( f \) which are densities with respect to \( \mathcal{A}_c^N \) (cf. [18], p.377). Thus, to prove (9.1), it remains to show, for any density \( f, \)

\[ E_{\mathcal{A}_c^N}[V_{s,k}f] \leq \mathbb{E}_N \left[ \kappa U_{s,k}f \right] + \kappa^{-1}N \mathbb{E}_N \left[ \sqrt{f(-L_N \sqrt{T})} \right]. \tag{9.2} \]

By the product structure of \( \mathcal{A}_c^N, \) we have

\[ g_{k,N}(\eta(k))\mathcal{A}_c^N(\eta) = g_{k+1,N}(\eta(k+1)+1)\mathcal{A}_c^N(\eta^{k,k+1}). \]

Thus, we compute that

\[ E_{\mathcal{A}_c^N}[V_{s,k}f] = E_{\mathcal{A}_c^N} \left[ G(s) \left( g_{k,N}(\eta(k)) - g_{k+1,N}(\eta(k+1)) \right) f(\eta) \right] \]

\[ = E_{\mathcal{A}_c^N} \left[ G(s)g_{k,N}(\eta(k)) \left( f(\eta) - f(\eta^{k,k+1}) \right) \right] \]

\[ = E_{\mathcal{A}_c^N} \left[ G(s)g_{k,N}(\eta(k)) \left( \sqrt{f(\eta)} - \sqrt{f(\eta^{k,k+1})} \right) \left( \sqrt{f(\eta)} + \sqrt{f(\eta^{k,k+1})} \right) \right]. \]

By Cauchy-Schwarz, for any \( A > 0, \) the above display is estimated from above by

\[ A \mathbb{E}_N \left[ g_{k,N}(\eta(k)) \left( \sqrt{f(\eta)} - \sqrt{f(\eta^{k,k+1})} \right)^2 \right] + A^{-1} \mathbb{E}_N \left[ G(s)^2 g_{k,N}(\eta(k)) \left( \sqrt{f(\eta)} + \sqrt{f(\eta^{k,k+1})} \right)^2 \right]. \]
Notice that the first expectation in the above display is bounded by $E_{\mathcal{A}^N} \left[ \sqrt{T} (-L_N \sqrt{T}) \right]$ (cf. (8.12)). Take $A = \kappa^{-1} N$. The second summand is estimated from above by

$$
2\kappa N^{-1} E_{\mathcal{A}^N} \left[ (G(s))^2 g_{k,N}(\eta(k)) (f(\eta) + f(\eta^{k,k+1})) \right] \\
= 2\kappa N^{-1} E_{\mathcal{A}^N} \left[ (G(s))^2 (g_{k,N}(\eta(k)) + g_{k+1,N}(\eta(k+1))) f(\eta) \right] \\
= E_{\mathcal{A}^N} [\kappa U_{s,t} f].
$$

Retracing the terms, we obtain (9.2), finishing the proof. \hfill \Box

We now finish this section with a local replacement lemma at defect sites:

**Lemma 9.2** (Local replacement at defect sites). Let $G(\cdot) : [0, T] \mapsto \mathbb{R}$ be bounded. Then, for each defect site $k_{j,N} \in \mathcal{D}_N$, we have

$$
\lim_{\theta \to 0} \limsup_{N \to \infty} E_N \left[ \int_0^T G(s) \left( g_{k_{j,N},N}(\eta_s(k_{j,N})) - \Phi(\eta_s^{0,N,+}(k_{j,N})) \right) ds \right] = 0
$$

where $\eta_s^{l,+}(k) := (2l + 1)^{-1} \sum_{k+1 \leq x \leq k+2l+1} \eta(x)$. \hfill \Box

**Proof.** Lemma 9.1 shows we may replace $g_{k_{j,N},N}(\eta_s(k_{j,N}))$ by $g_{k_{j,N}+1,N}(\eta_s(k_{j,N}+1))$. To finish, notice $k_{j,N}+1 \in \mathcal{T}_N \setminus \mathcal{D}_N$ for $N$ large. Then, we may further replace $g_{k_{j,N}+1,N}(\eta_s(k_{j,N}+1))$ by $\Phi(\eta_s^{0,N,+}(k_{j,N}))$, using Lemma 8.1 and Lemma 8.2; see Remark 8.3. The proof is now complete. \hfill \Box

**10. Energy estimate**

By Lemma 7.1, we know that $Q$ is supported on paths $\pi_t(dx)$ which can be decomposed into an absolute continuous part $\rho(t,x)dx$ and atoms $\sum_{j \in J} m_j(t) \delta_{x_j}(dx)$. In this section, we prove an energy estimate for $\rho(t,x)$.

**Proposition 10.1.** $Q$ is supported on paths $\pi_t(dx)$ such that $\Phi(\rho(t,x))$ is weakly differentiable with respect to $x$ on $[0, T] \times \mathbb{T}$ and $\partial_x \Phi(\rho(t,x))$, the weakly derivative, satisfies

$$
\int_0^T \int_\mathbb{T} \frac{(\partial_x \Phi(\rho(t,x)))^2}{\Phi(\rho(t,x))} dx dt < \infty. \tag{10.1}
$$

**Remark 10.2.** As $\rho(t,x) \in L^\infty([0, T] \times \mathbb{T})$ (see Lemma 7.1) and $\Phi(\cdot)$ is Lipschitz, it follows from (10.1) that $\partial_x \Phi(\rho(t,x)) \in L^2([0, T] \times \mathbb{T})$.

**Proof of Proposition 10.1** The proof presented here is based on the one of Theorem 7.1, p. 102, [12]. However, because of the presence of the defect sites and the difference in the underlying topology, many details are different in subtle ways.

Let $\{ H_j \}_{j \in \mathbb{N}}$ be a dense sequence in $C^\infty_0([0, T] \times (\mathbb{T} \setminus \mathcal{D}))$ under the norm $\| H \|_\infty + \| \partial_x H \|_\infty$. We split the proof into steps. In the first of two steps, we show there is a constant $K_0$ such that, for all $\epsilon$ small,

$$
\max_{1 \leq j \leq m} \left\{ \int_0^T W_N(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \leq K_0 \tag{10.2}
$$
where
\[ W_N(\epsilon, H(\cdot), \eta) := \sum_{x \in \mathbb{T}_N} \frac{H(x/N)}{\epsilon N} (g_{x,N}(\eta(x)) - g_{x+\epsilon N,N}(\eta(x + \epsilon N))) \]
\[ - \frac{2}{N} \sum_{x \in \mathbb{T}_N} \frac{H^2(x/N)}{\epsilon N} \sum_{0 \leq k \leq N} g_{x+k,N}(\eta(x + k)). \]

**Step 1.** Let \( c = c_0 \). By the entropy inequality, the expectation in (10.2) is bounded from above by
\[ \frac{H(\mu^N|\mathcal{R}^N_c)}{2N} + \frac{1}{2N} \ln \mathbb{E}_{\mathcal{R}^N_c} \left[ \exp \left\{ \max_{1 \leq j \leq m} \left\{ 2N \int_0^T W_N(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \right\} \right]. \]
For convenience, we set \( c = c_0 \). Using \( H(\mu^N|\mathcal{R}^N_c) \leq CN \) and \( \epsilon^a \leq \sum \omega^a \), the limsup in \( N \) of the previous display is estimated from above by
\[ \frac{C}{2} + \max_{1 \leq j \leq m} \limsup_{N \to \infty} \frac{1}{2N} \ln \mathbb{E}_{\mathcal{R}^N_c} \left[ \exp \left\{ 2N \int_0^T W_N(\epsilon, H_j(s, \cdot), \eta_s) ds \right\} \right]. \]
By Feynman-Kac formula, for any fixed index \( j \), the limsup term in the previous display is less than or equal to
\[ \limsup_{N \to \infty} \int_0^T \sup_f \left\{ E_{\mathcal{R}^N_c} [W_N(\epsilon, H_j(s, \cdot), \eta) f(\eta)] - \frac{N}{2} E_{\mathcal{R}^N_c} \left[ \sqrt{f(-L_N \sqrt{f})} \right] \right\} ds \]
where the supremum is over all \( f \) which are densities with respect to \( \mathcal{R}^N_c \).

**Step 2.** To show (10.2), it now remains to show, for all \( H \) in \( C^1(T \setminus \mathbb{D}) \), that
\[ E_{\mathcal{R}^N_c} [W_N(\epsilon, H(\cdot), \eta) f(\eta)] - N E_{\mathcal{R}^N_c} \left[ \sqrt{f(-L_N \sqrt{f})} \right] \leq 0. \]

We first compute that \( E_{\mathcal{R}^N_c} [W_N(\epsilon, H(\cdot), \eta) f(\eta)] \) equals
\[ E_{\mathcal{R}^N_c} \left[ \sum_{x \in \mathbb{T}_N} \frac{H(x/N)}{\epsilon N} (g_{x,N}(\eta(x)) - g_{x+\epsilon N,N}(\eta(x + \epsilon N))) f(\eta) \right] \]
\[ - \frac{2}{N} E_{\mathcal{R}^N_c} \left[ \sum_{x \in \mathbb{T}_N} \frac{H^2(x/N)}{\epsilon N} \sum_{0 \leq k \leq \epsilon N} g_{x+k,N}(\eta(x + k)) f(\eta) \right]. \]
Let \( \delta_x \) be the configuration with the only particle at \( x \) and \( \eta + \delta_x \) be the configuration obtaining from adding one particle at \( x \) to \( \eta \). By the definition of \( \mathcal{R}^N_c \), we have, for each \( x \),
\[ E_{\mathcal{R}^N_c} [g_{x,N}(\eta(x)) f(\eta)] = \varphi_x E_{\mathcal{R}^N_c} [f(\eta + \delta_x)] \]
where \( \varphi = \Phi(c) \). Then, the first expectation in (10.4) is written as
\[ \sum_{x \in \mathbb{T}_N} \frac{\varphi H(x/N)}{\epsilon N} E_{\mathcal{R}^N_c} \left[ f(\eta + \delta_x) - f(\eta + \delta_{x+\epsilon N}) \right]. \]
which is rewritten as
\[ E_{\mathcal{R}^N_c} \left[ \sum_{x \in \mathbb{T}_N} \sum_{0 \leq k \leq \epsilon N - 1} \frac{\varphi H(x/N)}{\epsilon N} \left( \sqrt{f(\eta + \delta_x + k)} + \sqrt{f(\eta + \delta_{x+k+1})} \right) \right] \]
\[ \times \left( \sqrt{f(\eta + \delta_x)} - \sqrt{f(\eta + \delta_{x+k+1})} \right). \]
Using $2ab \leq a^2 + b^2$, for any $A > 0$, (10.7) is bounded from above by

$$E_{\mathcal{C}_{N}} \sum_{x \in \mathbb{T}_N} \sum_{0 \leq k \leq n-1} \frac{\varphi H^2(x/N)}{2cN^2} \left( \sqrt{f(\eta + \delta_x + k)} + \sqrt{f(\eta + \delta_x + k + 1)} \right)^2$$

$$+ E_{\mathcal{C}_{N}} \sum_{x \in \mathbb{T}_N} \sum_{0 \leq k \leq n-1} \frac{\varphi A}{2cN} \left( \sqrt{f(\eta + \delta_x + k)} - \sqrt{f(\eta + \delta_x + k + 1)} \right)^2 := I_1 + I_2. \quad (10.8)$$

The $I_2$ term in (10.8) is rewritten as $\sum_{x \in \mathbb{T}_N} \frac{\varphi A}{2cN} E_{\mathcal{C}_{N}} \left[ \left( \sqrt{f(\eta + \delta_x)} - \sqrt{f(\eta + \delta_x + 1)} \right)^2 \right]$ which, by the change of variable formula (10.5), is further recognized as the Dirichelet form $\frac{A}{2} E_{\mathcal{C}_{N}} \sqrt{f(-L_N f)}$, cf. (8.12). For the first expectation $I_1$ in (10.8), using first $(\sqrt{a} + \sqrt{b})^2 \leq 2(a + b)$ and then (10.5), it is bounded from above by

$$\sum_{x \in \mathbb{T}_N} \frac{2H^2(x/N)}{cN^2} \sum_{0 \leq k \leq n-1} E_{\mathcal{C}_{N}} \left[ g_{x+k,N}(x + k) f(\eta) \right]. \quad (10.9)$$

Notice that the summation of $k$ is ranging from $0 \leq k \leq \epsilon N$ instead of $0 \leq k \leq \epsilon N - 1$. Now, we set $A = N$. Putting together (10.4) and (10.9), we obtain (10.3) with $K_0 = C/2$.

**Step 3.** Recall that $H_j$’s have compact support in $[0, T] \times (\mathbb{T} \setminus \mathcal{Q})$. Let $\delta = (2\delta)^{-1} \mathbb{1}_{[-\delta, \delta]}$. Applying the Bulk Replacement Lemma (Lemma 8.4) to (10.2) and taking $N \to \infty$, we obtain

$$\lim_{\delta \to 0} \sup_{\mathcal{C}} E_{\mathcal{C}} \left[ \max_{1 \leq j \leq m} \left\{ \int_0^T \int_{\mathbb{T}} H_j(s, x) c^{-1}(\Phi(\delta \ast \rho_s(x)) - \Phi(\delta \ast \rho_s(x + \epsilon))) dx ds \right. \right. \left. \left. - 2 \int_0^T \int_{\mathbb{T}} H_j^2(s, x) c^{-1} \int_x^{x+\epsilon} \Phi(\delta \ast \rho_s(u)) du dx ds \right\} \right] \leq K_0,$$

Here $\delta \ast \rho_s(\cdot) = \int_{\mathbb{T}} \delta(x - \cdot) \rho(s, x) dx$ and $\rho(s, x) \in L^1([0, T] \times \mathbb{T})$ is the absolute continuous part of each limit path $\pi_s$ (cf. Lemma 7.1).

Sending $\delta \to 0$, applying a discrete integration by parts, and then taking $\epsilon \to 0$, we have

$$E_{\mathcal{C}} \left[ \max_{1 \leq j \leq m} \left\{ \int_0^T \int_{\mathbb{T}} \partial_x H_j(s, x) \Phi(\rho(s, x)) dx ds \right. \right. \left. \left. - 2 \int_0^T \int_{\mathbb{T}} H_j^2(s, x) \Phi(\rho(s, x)) dx ds \right\} \right] \leq K_0,$$

By monotone convergence, the max$_{1 \leq j \leq m}$ above can be replaced by max$_{1 \leq j < \infty}$. Furthermore, as $H_j$ is dense in $C_{c}^{0,1}([0, T] \times (\mathbb{T} \setminus \mathcal{Q}))$ with respect to the norm $\|H\|_{\infty} + \|\partial_x H\|_{\infty}$, we conclude

$$E_{\mathcal{C}} \left[ \sup_{H} \left\{ \int_0^T \int_{\mathbb{T}} \partial_x H(s, x) \Phi(\rho(s, x)) dx ds - 2 \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds \right\} \right] \leq K_0, \quad (10.10)$$

where the sup is over $H \in C_{c}^{0,1}([0, T] \times (\mathbb{T} \setminus \mathcal{Q}))$.

**Step 4.** As a result of (10.10), for $Q$-a.e path $\pi_t(dx)$, there exists $B = B(\pi_t)$ such that, for all $H \in C_{c}^{0,1}([0, T] \times (\mathbb{T} \setminus \mathcal{Q}))$,

$$\int_0^T \int_{\mathbb{T}} \partial_x H(s, x) \Phi(\rho(s, x)) dx ds - 2 \int_0^T \int_{\mathbb{T}} H^2(s, x) \Phi(\rho(s, x)) dx ds \leq B.$$
Define on $C^{0,1}_{\rho}([0, T] \times (T \setminus \mathcal{D}))$ a linear functional $l(H) := \int_0^T \int_T \partial_x H(s, x) \Phi(\rho(s, x)) dx ds$. Also define $\|H\|_{2, \rho} := \left( \int_0^T \int_T H^2(s, x) \Phi(\rho(s, x)) dx ds \right)^{1/2}$. Then we have, for all $a \in \mathbb{R}$

$$a l(H) - 2a^2 (\|H\|_{2, \rho})^2 \leq B.$$  

Maximizing the left hand side over $a \in \mathbb{R}$, we obtain $l(\cdot)$ is a bounded linear functional under the norm $\| \cdot \|_{2, \rho}$. By Riesz representation theorem, there exists $F$ such that $\|F\|_{2, \rho} < \infty$ and it holds that $l(H) = \langle H, F \rangle_{\rho} := \int_0^T \int_T H(s, x) F(s, x) \Phi(\rho(s, x)) dx ds$. Define $\partial_x \Phi(\rho(t, x)) := -F(t, x)\Phi(\rho(t, x))$. Then, we have shown that $\Phi(\rho(t, x))$ is weakly differentiable with respect to $x$ on $[0, T] \times (T \setminus \mathcal{D})$ and the weak derivative $\partial_x \Phi(\rho(t, x))$ satisfies (10.1).

To finish, we argue that the weak differentiability of $x$ may be extended from $T \setminus \mathcal{D}$ to $T$ with the same weak derivative $\partial_x \Phi(\rho(t, x))$. The weak differentiability on $[0, T] \times (T \setminus \mathcal{D})$ implies that, for almost all $t \in [0, T]$, $\Phi(\rho(t, \cdot))$ is absolutely continuous on $T \setminus \mathcal{D}$. It suffices to show that, for almost all $t \in [0, T]$, $\Phi(\rho(t, \cdot))$ is continuous on the defects $\mathcal{D}$. Notice that, as $\partial_x \Phi(\rho(t))^2_\rho \Phi(\rho)$ and $\partial_x \Phi(\rho)$ are both in $L^1([0, T] \times T)$, by Cauchy-Schwarz, we have $\partial_x \Phi(\rho)$ is in $L^1([0, T] \times T)$ as well. Then, for almost all $t \in [0, T]$, we have that the limits $\lim_{x \to x_j \pm} \Phi(\rho(t, x))$ exist and are finite for all $x_j \in \mathcal{D}$ (cf. Remark 1.6). Moreover, by Lemma 2.2, the left and right limits match. Therefore, we conclude the continuity of $\Phi(\rho(t, \cdot))$ on $\mathcal{D}$, finishing the proof. 

11. Uniqueness

In this section, we present the uniqueness of the weak solutions to equations (4.3) and (4.5). The proof is based on an energy argument (cf. [17]). Recall Definitions 4.4 and 4.5.

**Theorem 11.1.** There exists at most one weak solution to (4.3).

**Proof.** Let $\pi_t^{(1)}$ and $\pi_t^{(2)}$ be two weak solutions of (4.3) such that $\pi_t^{(i)} = \rho_i(t, x) dx + \sum_{j \in J_c} m_j^{(i)}(t) \delta_{x_j}(dx)$ for $i = 1, 2$. As $m_j^{(i)}(t) = \left[ \lambda_j \Phi(\rho_i(t, x_j)) \right]^{1/\alpha}$, to prove the theorem, it suffices to show $\rho_1 = \rho_2$ for almost all $t$.

**Step 1.** Define $\overline{\Phi}(t, x) := \Phi(\rho_1(t, x)) - \Phi(\rho_2(t, x))$. By Definition 4.4 we have that $\overline{\Phi}$ is weakly differentiable with respect to $x \in T$ and $\partial_x \Phi$ (and therefore $\overline{\Phi}$) is in $L^2([0, T] \times T)$. As $\Phi(\rho_i(t, x_j)) = \Phi(c_0)$ for each $i = 1, 2$ and $j \in J_c$, we have $\overline{\Phi}(t, x_j) = 0$ for all $j \in J_c$.

We first show that $\overline{\Phi}$ can be approximated ‘well’ by a sequence $\{\overline{\Phi}_\varepsilon\}_{\varepsilon>0}$ in the sense that (1) $\overline{\Phi}_\varepsilon$ is smooth and compactly supported on $[0, T] \times (T \setminus \mathcal{D})$; (2) $\overline{\Phi}_\varepsilon \to \overline{\Phi}$ in $L^2([0, T] \times T)$; and (3) $\overline{\Phi}_\varepsilon(t, x_j) \to \overline{\Phi}(t, x_j)$ in $L^2(0, T)$ for all $j \in J_c$. Indeed, to verify this approximation, for $\delta > 0$, let $\mathcal{D}_\delta = \bigcup_{j \in J_c} (x_j - \delta, x_j + \delta)$. We define $F_\delta$ be such that $F_\delta(t, x) = 0$ on $[0, T] \times \overline{\mathcal{D}}_\delta$ and $F_\delta(t, x) = \overline{\Phi}(t, x)$ on $[0, T] \times (T \setminus \overline{\mathcal{D}}_\delta)$. For $(t, x) \in \mathcal{D}_\delta \setminus \mathcal{D}_\delta$, define

$$F_\delta(t, x) = \begin{cases} 
\overline{\Phi}(t, 2x - x_j - 2\delta) & \text{on } [0, T] \times [x_j + \delta, x_j + 2\delta], \\
\overline{\Phi}(t, 2x - x_j + 2\delta) & \text{on } [0, T] \times [x_j - 2\delta, x_j - \delta]. 
\end{cases}$$

We note that $F_\delta$ has compact support in $[0, T] \times (T \setminus \mathcal{D})$. It is easy to check that $F_\delta$ and $\partial_x F_\delta$ approximate $\overline{\Phi}$ and $\partial_x \overline{\Phi}$ in $L^2([0, T] \times T)$ respectively. Since $F_\delta$ and $\overline{\Phi}$ match at $x = x_j$ for all $j \in J_c$, to find a desired sequence $\{F_\delta\}$, it suffices to show that, for each small $\delta$, there exists $\{F_\delta^{(x)}\}$ that approximates $F_\delta$ ‘well’.
To this end, let $\tau_\varepsilon(x)$ be a standard mollifier supported on $[-\varepsilon, \varepsilon]$. With $F_\delta(t, x)$ extended to be 0 for $t \not\in [0, T]$, we define

$$F_{\delta, \varepsilon}(t, x) := \int_\mathbb{R} \int_T^T F_\delta(t-s, x-u) \tau_\varepsilon(s) \tau_\varepsilon(u) \, du \, ds.$$

Notice that $F_{\delta, \varepsilon} \in C_c^\infty(\mathbb{R} \times (T \setminus \mathcal{D}_\delta))$ for $\varepsilon < \delta$. When restricted on $[0, T] \times \mathbb{T}$, it is standard that $F_{\delta, \varepsilon}$ and $\partial_x F_{\delta, \varepsilon}$ approximates $F_\delta$ and $\partial_x F_\delta$ respectively in $L^2([0, T] \times \mathbb{T})$ as $\varepsilon \to 0$. It remains to show that $F_{\delta, \varepsilon}(t, x_j)$ approximates $F_\delta(t, x_j)$ in $L^2[0, T]$. Write that

$$\int_0^T (F_{\delta, \varepsilon}(t, x_j) - F_\delta(t, x_j))^2 \, dt$$

$$= \int_0^T \left[ \int_\mathbb{R} \int_T^T (F_\delta(t-s, x_j-u) - F_\delta(t, x_j)) \tau_\varepsilon(s) \tau_\varepsilon(u) \, du \, ds \right]^2 \, dt.$$  

By adding and subtracting $F_\delta(t, x_j)$, the above is bounded above by $I_1 + I_2$ where

$$I_1 := 2 \int_0^T \int_\mathbb{R} \int_T^T (F_\delta(t-s, x_j-u) - F_\delta(t-s, x_j)) \tau_\varepsilon(s) \tau_\varepsilon(u) \, du \, ds \, dt,$$

$$I_2 := 2 \int_0^T \int_\mathbb{R} \int_T^T (F_\delta(t-s, x_j) - F_\delta(t, x_j)) \tau_\varepsilon(s) \, ds \, dt.$$  

As $\int_\mathbb{R} F_\delta(t-s, x_j) \tau_\varepsilon(s) \, ds$ approximates $F_\delta(t, x_j)$ in $L^2[0, T]$, the term $I_2$ vanishes as $\varepsilon \to 0$. For the term $I_1$, using $F_\delta(t-s, x_j-u) - F_\delta(t-s, x_j) = \int_{x_j}^{x_j-u} \partial_x F_\delta(t-s, x) \, dx$, we have

$$I_1 \leq 2 \int_0^T \int_\mathbb{R} \int_T^T \left( \int_{x_j}^{x_j-u} \partial_x F_\delta(t-s, x) \, dx \right)^2 \tau_\varepsilon(s) \tau_\varepsilon(u) \, du \, ds \, dt.$$

The square term above is further bounded by $u^2 \int_{x_j}^{x_j-u} (\partial_x F_\delta(t, x))^2 \, dx$. As $\tau_\varepsilon(u) = 0$ for $u \not\in [-\varepsilon, \varepsilon]$, we have that $I_1 \leq 2 \varepsilon^2 \int_0^T \int_T (\partial_x F_\delta(t, x))^2 \, dx \, dt$ which vanishes as $\varepsilon \to 0$.

**Step 2.** We now proceed to the uniqueness of weak solutions. Let $\mathbf{p}(t) := \rho_1 - \rho_2$ and $\mathbf{m}_j(t) := m_j^{(1)}(t) - m_j^{(2)}(t)$ for each $j \in J_c$. As $\pi_t^{(1)}$ and $\pi_t^{(2)}$ both satisfy (4.10), we have, for all $G(t, x) \in C_c^\infty([0, T] \times (T \setminus \mathcal{D}_\delta))$,

$$\int_0^T \int_T \partial_t G(t, x) \mathbf{p}(t, x) \, dx \, dt + \sum_{j \in J_c} \int_0^T \partial_t G(t, x) \mathbf{m}_j(t) \, dt = \int_0^T \int_T \partial_x G(t, x) \partial_x \overline{\mathbf{F}}(t, x) \, dx \, dt.$$  

(11.1)

Let $\overline{\mathbf{F}}$ be approximated ‘well’ by some $\{\overline{\mathbf{F}}_\varepsilon\}$ as in Step 1. Taking $G(t, x) = -\int_t^T \overline{\mathbf{F}}_\varepsilon(s, x) \, ds$ and then letting $\varepsilon \to 0$, we obtain

$$\int_0^T \int_T \overline{\mathbf{F}}_\varepsilon(t, x) \mathbf{p}(t, x) \, dx \, dt + \sum_{j \in J_c} \int_0^T \overline{\mathbf{F}}_\varepsilon(t, x_j) \mathbf{m}_j(t) \, dt$$

$$= -\int_0^T \int_\mathbb{R} \int_T^T \left[ \int_t^T \partial_x \overline{\mathbf{F}}_\varepsilon(s, x) \, ds \right] \partial_x \overline{\mathbf{F}}_\varepsilon(t, x) \, dx \, dt.$$  

(11.2)

The right hand side of the above is computed as $-\frac{1}{2} \int_T \left[ \int_0^T \partial_x \overline{\mathbf{F}}_\varepsilon(t, x) \, dt \right]^2 \, dx \leq 0$. However, for the left hand side, we have $\overline{\mathbf{F}}(t, x) \mathbf{p}(t, x) \geq 0$ and $\overline{\mathbf{F}}(t, x_j) \mathbf{m}_j(t) \geq 0$ for all $t$, $x$, and $j$. Then, we deduce that $\int_0^T \int_T \overline{\mathbf{F}}_\varepsilon(t, x) \mathbf{p}(t, x) \, dx \, dt = 0$ which means $\overline{\mathbf{F}}(t, x) \mathbf{p}(t, x) = 0$ a.e. and,
therefore, \( \overline{\rho}(t, x) = 0 \) a.e. Since \( \overline{\rho} \) is continuous in \( x \) for almost all \( t \) (cf. Remark 4.6), the theorem is proved. \( \square \)

**Theorem 11.2.** There exists at most one weak solution to (4.5).

**Proof.** For \( i = 1, 2 \), let \( \pi_i(t) = \rho_i(t, x)dx + \sum_{j \in J_c} \delta_{\rho(t, x_j) = c_j} \) be two weak solutions to (4.5). Notice that \( \overline{\pi}_{j}^{(i)}(t) = \overline{\pi}_{j}^{(i)}(t) 1_{\rho(t, x_j) = c_j} \) and \( \rho(t, x_j) \leq c_j \) implies \( \overline{\rho}(t, x_j) \overline{\pi}_{j}^{(i)}(t) \geq 0 \) where \( \overline{\rho} := \overline{\rho}(\rho_1) - \Phi(\rho_2) \) and \( \overline{\pi} := \overline{\pi}_{j}^{(1)} - \overline{\pi}_{j}^{(2)} \). Following the proof of Theorem 11.1 we have \( \overline{\rho}(\rho_1) - \Phi(\rho_2) = 0 \) for almost all \( t \). To conclude, it remains to show \( \overline{\pi}_{j}^{(i)}(t) = 0 \) for each \( j \in J_c \) for almost all \( t \). To this end, we fix any \( j \in J_c \) and take \( F \in C^\infty(T) \) such that \( F(x_j) = 1 \) and \( \text{supp} F \cap J_c = x_j \). Letting \( G(t, x) = \int_t^T h(s)F(x)ds \) in (11.11) for any \( h(t) \in C^\infty(0, T) \) and using \( \overline{\rho} = \overline{\rho} = 0 \), we obtain \( \int_0^T h(t)\overline{\pi}_{j}^{(i)}(t)dt = 0 \) and therefore, \( \overline{\pi}_{j}^{(i)}(t) = 0 \) for almost all \( t \), finishing the proof. \( \square \)

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