Three-dimensional noncompact $\kappa$-solutions that are Type I forward and backward

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As indicated by the third author in [19], there is a gap in the previous version of this paper by the first two authors [5]. We provide in this version an argument to fix the aforementioned gap. The main proposition, whose proof uses Perelman's techniques, is implied by Ding [9] and is covered by [19]. Our approach, however, is different from theirs. In addition, we prove a necessary and sufficient condition for a three-dimensional $\kappa$-solution to form a forward singularity. We hope that this condition is helpful in the classification of all three-dimensional $\kappa$-solutions. Up to now, the only main progress on such a classification, as conjectured by Perelman, is by Brendle [2].

1 Introduction

A complete solution to the backward Ricci flow $(M^n, g(\tau)), \tau \in (0, \infty)$, is a Type I $\kappa$-solution if $|\text{Rm}|(\tau) \leq \frac{C}{\tau}$ for some constant $C$ and each $g(\tau)$ is $\kappa$-noncollapsed below all scales. In this definition we do not assume nonnegativity of the curvatures.

As a special case of his result Proposition 0.1 in [16], Lei Ni has classified 3-dimensional closed Type I $\kappa$-solutions. In all dimensions Ni first showed that if $(M^n, g(\tau)), \tau \in (0, \infty)$, is a solution to the backward Ricci flow on a compact manifold and satisfies $|\text{Rm}|(\tau) \leq \frac{A}{\tau}$, then there exists a constant $C(n, A)$ such that

$$\text{diam}(g(\tau)) \leq \max\{\text{diam}(g(1)), C(n, A)\} \sqrt{\tau} \quad \text{for} \quad \tau \in (1, \infty).$$

(1)
In particular, by Lemma 8.3(b) in Perelman [17], there exists \( C = C(n, A) \) such that
\[
\frac{\partial}{\partial \tau} d_\tau(x_1, x_2) \leq \frac{C(n, A)}{2\sqrt{\tau}}
\]
for any \( x_1, x_2 \in \mathcal{M} \) with \( d_\tau(x_1, x_2) \geq C(n, A)\sqrt{\tau} \).
Then (1) follows from the consequence that for \( x_1, x_2 \in \mathcal{M} \) and \( \tau \geq 1 \) we have
\[
\frac{d_\tau(x_1, x_2)}{\sqrt{\tau}} < C(n, A) \text{ or } \frac{\partial}{\partial \tau} \frac{d_\tau(x_1, x_2)}{\sqrt{\tau}} \leq 0.
\]

Using this, Ni proved that if \((\mathcal{M}^n, g(\tau))\), \( \tau \in (0, \infty) \), is a closed Type I \( \kappa \)-solution with positive curvature operator (PCO), then \((\mathcal{M}, g(\tau))\) is isometric to a shrinking spherical space form. In particular, since \( g(\tau) \) has PCO and \( \mathcal{M} \) is closed, by Hamilton [11], [12] when \( n = 3, 4 \) and Böhm and Wilking [1] when \( n \geq 5 \), \( g(\tau) \) converges to a constant positive sectional curvature (CPSC) metric \( g_0 \) as \( \tau \to 0 \). By 11.2 in [17], fixing \( \rho \), there exist \( q_\rho \) such that \( \epsilon^\rho(\rho, 0)(q_\rho, i) \leq \frac{n}{2}\) and \((\mathcal{M}^n, \mathcal{i}^{-1} g(\tau), q_i)\) subconverges in the Cheeger–Gromov sense to a complete nonflat shrinking gradient Ricci soliton (GRS) \((\mathcal{M}^n_\infty, g_\infty(\tau), q_\infty)\). By (1), we have
\[
\text{diam} \left( \frac{1}{i} g(\tau) \right) \leq \max \{ \text{diam} (g(-1)), C(n, A) \} \sqrt{\tau} \quad \text{for } \tau \in \left( \frac{1}{i}, \infty \right).
\]
Thus \( \mathcal{M}_\infty \) is compact and diffeomorphic to \( \mathcal{M} \). Since \((\mathcal{M}, g_\infty(\tau))\) is irreducible with nonnegative curvature operator on a topological spherical space form, \( g_\infty(\tau) \) must be a CPSC metric. By all of the above, after rescaling, \( g(\tau) \) converges to a metric which is isometric to a constant multiple of \( g_\infty(1) \) as either \( \tau \to 0 \) or \( \tau \to \infty \). This implies that Perelman’s invariant \( \nu(g(\tau)) \) must be constant, which implies that \( g(\tau) \) is a shrinking GRS and hence a CPSC metric.

As a corollary, any 3-dimensional closed Type I \( \kappa \)-solution must be isometric to a shrinking spherical space form. The reason is as follows. By B.-L. Chen [7], \( \text{Rm} \geq 0 \). If \( \text{Rm} > 0 \), then \( g(\tau) \) is a CPSC metric by Ni’s theorem. On the other hand, if the sectional curvatures are not positive, then \( \mathcal{M}^3 \) is covered by \( S^2 \times \mathbb{R} \). Since any closed such solution is \( \kappa \)-collapsed, we are done.

Observe that, by Brendle and Schoen [3] and Brendle [2] (the latter enabling Perelman’s \( \kappa \)-solution theory to extend), Ni’s theorem holds under Brendle–Schoen positivity of curvature.

In this note we observe that the combined results of Perelman [17], Naber [15], Enders, Müller and Topping [10], and Zhang and the first author [6] yield the following special case of the assertion by Perelman (private communication to Ni) that any 3-dimensional Type I \( \kappa \)-solution with PCO must be a shrinking CPSC metric. As we mentioned in the abstract, this result is implied by the earlier work of Ding [9] and is generalized in the recent work of the third author [19], where the condition of being Type I forward in time is removed.

**Proposition 1** Suppose that \((\mathcal{M}^3, g(\tau))\), \( \tau \in (0, \infty) \), is a \( \kappa \)-solution to the backward Ricci flow with PCO forming a singularity at \( \tau = 0 \) and satisfying \( |\text{Rm}|(\tau) \leq \frac{\Delta}{4} \), then \( \mathcal{M} \) is closed and \( g(\tau) \) is a shrinking CPSC metric.

Note that we have assumed that the solution is Type I both forward and backward in time. Applications of this result to the study of shrinking gradient
2 Proof of the proposition

Before we proceed to prove Proposition 1, we prove the following lemma that asserts the existence of a singular point at the forward singular time on a 3-dimensional $\kappa$-solution. This is crucial in proving that the blow-up limit is nonflat. The existence of such a point is an issue because of the non compactness of $\mathcal{M}$; see Remark 1.1 in [10]. We actually prove that every point of $\mathcal{M}$ is a singular point.

**Lemma 2** Let $(\mathcal{M}^{3}, g(\tau))$, where $\tau \in (0, \infty)$, be a $\kappa$-solution that forms a singularity at $\tau = 0$ in the sense that $\lim_{\tau \to 0^+} \sup_{x \in \mathcal{M}} R(x, \tau) = \infty$, where $R$ denotes the scalar curvature. Then every $p \in \mathcal{M}$ is a singular point in the sense that $\lim_{\tau \to 0^+} R(p, \tau) = \infty$.

**Proof.** Since 0 is a singular time, by definition we may find a sequence $\{(x_{i}, \tau_{i})\}_{i=1}^{\infty}$, such that $\tau_{i} \searrow 0$ and $R(x_{i}, \tau_{i}) \to \infty$. Suppose $p \in \mathcal{M}$ is not a singular point. Then there exists $C < \infty$ such that $R(p, \tau_{i}) \leq C$ for every $i \in \mathbb{N}$. By Hamilton’s trace Harnack estimate [13], we have $\frac{\partial R}{\partial \tau} \leq 0$. Hence $R(p, \tau_{i}) \in [c, C]$, for all $i \in \mathbb{N}$, where we denote $c = R(p, \tau_{i}) > 0$. Define $g_{i}(\tau) = g(\tau + \tau_{i})$. Then we can use Perelman’s $\kappa$-compactness theorem [17] to extract a (not relabelled) subsequence from $\{(\mathcal{M}, g_{i}(\tau), (p, 0))_{\tau \in [0, \infty]}\}_{i=1}^{\infty}$, which converges to a $\kappa$-solution $(\mathcal{M}_{\infty}, g_{\infty}(\tau), (p_{\infty}, 0))_{\tau \in [0, \infty]}$. In particular, $(\mathcal{M}_{\infty}, g_{\infty}(0))$ has bounded curvature. Let $A < \infty$ be the curvature bound of $(\mathcal{M}_{\infty}, g_{\infty}(0))$. By the definition of pointed smooth Cheeger–Gromov convergence and by passing to a suitable subsequence, there exists a sequence of open precompact sets $\{U_{i}\}_{i=1}^{\infty}$ exhausting $(\mathcal{M}_{\infty}, g_{\infty}(0))$, where each $U_{i}$ contains $p_{\infty}$, and there exists a sequence of diffeomorphisms

$$
\psi_{i} : U_{i} \to V_{i} \subset (\mathcal{M}, g_{i}(0)),
\psi_{i}(p_{\infty}) = p_{i},
$$

with the following properties. We have $\overline{B_{g_{i}(0)}(p, i)} \subset V_{i}$ and that $\psi_{i}^{*}g_{i}(0)$ is $i^{-1}$-close to $g_{\infty}(0)$ on $U_{i}$ with respect to the $C^{3}$-topology. Notice here that we actually have Cheeger–Gromov convergence of the solutions of the backward Ricci flow on the whole time interval $[0, \infty)$, but we need only to use the convergence on the time zero slice. Let $i_{1} \in \mathbb{N}$ be large enough so that $R(x_{i_{1}}, \tau_{i_{1}}) > 100A$, where the existence of $i_{1}$ is guaranteed by the assumption that $R(x_{i_{1}}, \tau_{i_{1}}) \to \infty$. Then we select $i_{2} > i_{1}$ such that $\text{dist}_{g_{i_{1}}(0)}(p, x_{i_{1}}) = \text{dist}_{g(\tau_{i_{1}})}(p, x_{i_{1}}) < 100^{-1}i_{2}$. Since the Ricci flow with nonnegative curvature shrinks distances forward in time, it follows that $\text{dist}_{g(\tau_{i_{2}})}(p, x_{i_{1}}) < 100^{-1}i_{2}$ and hence that $x_{i_{1}} \in \overline{B_{g_{i_{2}}(0)}(p, i_{2})} \subset V_{i_{2}}$. Moreover, by Hamilton’s trace Harnack
estimate we have $R(g_i(0))(x_i) = R(x_i, \tau_i) \geq R(x_i, \tau_i) > 100A$, since $\tau_2 < \tau_1$. This yields a contradiction when $i_2$ is large enough (say $i_2 > 10000$) since $\psi_i^{-1}(x_i)$ is contained in the set $U_i$ on which $\psi_i^{-1}g_i(0)$ is $i_2^{-1}$-close to $g(0)$ with respect to the $C^{i_2}$-topology, while the curvature of $g(0)$ is bounded by $A$. ■

We now give the proof of our main result.

**Proof of Proposition 1.** By Ni’s theorem, we may suppose that $\mathcal{M}$ is noncompact, so that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^3$. By the first part of Theorem 3.1 in [15], for any $x \in \mathcal{M}$, $\tau_+ \to 0$, and $\tau_- \to \infty$, $(\mathcal{M}, (\tau_+^\pm)^{-1}g(\tau_\pm\tau), (x, 1))$ subconverges to a noncompact shrinking GRS $(\mathcal{M}^\pm, g^\pm(\tau), (x^\pm, 1))$ which does not contain any embedded $\mathbb{R}P^2$. By Theorem 1.1 in [14] and by Lemma 2 above, $(\mathcal{M}^-, g^-(\tau))$ is nonflat since every point is a singular point, whereas by Theorem 4.1 in [6] (see also the statements in its proof), $(\mathcal{M}^+, g^+(\tau))$ is nonflat, since both of these results apply to noncompact manifolds. By Lemma 1.2 in Perelman [18], $g^\pm(\tau)$ cannot have PCO. Thus the $(\mathcal{M}^\pm, g^\pm(\tau))$ are isometric to (shrinking) round cylinders $S^2 \times \mathbb{R}$. By the second part of Theorem 3.1 in [15], we conclude that the same is true for $(\mathcal{M}^3, g(\tau))$, which contradicts $g(\tau)$ having PCO. ■

**Remark 3** In [19] by the third author, it shown that there do not exist 3-dimensional noncompact PCO $\kappa$-solutions only assuming the solution is Type I backward. This confirms an assertion that Grisha Perelman made to Lei Ni.

### 3 A criterion for ancient solutions to form forward singularities

In this section we present an application of Lemma 2 which gives a necessary and sufficient condition for a 3-dimensional $\kappa$-solution to form a forward singularity.

**Corollary 4** A 3-dimensional $\kappa$-solution forms a forward singularity if and only if at some time slice $\inf_{\mathcal{M}} R > 0$.

**Proof.** Let $(\mathcal{M}^3, g(\tau))$, where $\tau \in (0, \infty)$, be a $\kappa$-solution to the backward Ricci flow that forms a singularity at $\tau = 0$. By Lemma 2 for every $p \in \mathcal{M}$, $R(p, \tau)$ increases to infinity as $\tau \searrow 0$. By integrating Perelman’s derivative estimate [17]

$$\left| \frac{\partial R}{\partial \tau} \right| \leq \eta R^2,$$

where $\eta$ depends only on $\kappa$, from 0 to $\tau$, we have

$$R(p, \tau) \geq \frac{1}{\eta \tau}.$$
for every \( p \in \mathcal{M} \) and \( \tau \in (0, \infty) \). It follows immediately that \( \inf_{p \in \mathcal{M}} R(p, \tau) > 0 \) for every \( \tau \in (0, \infty) \).

On the other hand, suppose \((\mathcal{M}^3, g(\tau))\), where \( \tau \in [0, \infty) \), is a \( \kappa \)-solution to the backward Ricci flow such that \( \inf_{p \in \mathcal{M}} R(p, T) = c > 0 \) for some \( T > 0 \). We use an idea of Perelman [18] to show that the solution cannot be extended forward to time infinity. Up to scaling the solution by a constant factor, we can find a sequence \( x_i \to \infty \), such that \( \lim_{i \to \infty} R(x_i, T) = 1 \). Applying the \( \kappa \)-compactness theorem [17], we can extract a (not relabelled) subsequence of \( \{(\mathcal{M}, g(\tau + T), (x_i, 0))\}_{i=1}^{\infty} \), converging to a \( \kappa \)-solution \((\mathcal{M}_{\infty}, g_{\infty}(\tau), (x_{\infty}, 0))\), which must be the shrinking cylinder since we have splitting at infinity; see [17]. Moreover, we have \( R_{\infty}(x_{\infty}, 0) = 1 \) and \((\mathcal{M}_{\infty}, g_{\infty}(\tau))\) has unbounded curvature as \( \tau \to -1 \). Then we can conclude that \((\mathcal{M}, g(\tau))\) becomes singular as \( \tau \to T-1 \). For suppose this is not the case. Then there exists an \( \varepsilon > 0 \) such that \( R(g(\tau)) \) is uniformly bounded for \( \tau \in [T-1-\varepsilon, \infty) \). It then follows that the limit flow \((\mathcal{M}_{\infty}, g_{\infty}(\tau))\) exists and has bounded curvature for \( \tau \in [-1-\varepsilon, \infty) \), which is a contradiction. ■

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