Covering spheres with spheres

Ilya Dumer

Abstract

Given a sphere of any radius \( r \) in an \( n \)-dimensional Euclidean space, we study the coverings of this sphere with solid spheres of radius one. Our goal is to design a covering of the lowest covering density, which defines the average number of solid spheres covering a point in a bigger sphere. For growing dimension \( n \), we design a covering that has covering density of order \( (n \ln n)^2 \) for the full Euclidean space or for a sphere of any radius \( r > 1 \). This new upper bound reduces two times the asymptotic order of \( n \ln n \) established in the classical Rogers bound.

1 Introduction

Spherical coverings. Let \( B^n_r(x) \) be a ball (solid sphere) of radius \( r \) centered at some point \( x = (x_1, \ldots, x_n) \) of an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \):

\[
B^n_r(x) \overset{\text{def}}{=} \left\{ z \in \mathbb{R}^n \mid \sum_{i=1}^n (z_i - x_i)^2 \leq r^2 \right\}.
\]

We also use a simpler notation \( B^n_r \) if a ball is centered at the origin \( x = 0 \). For any subset \( A \subseteq \mathbb{R}^n \), we say that a subset \( \text{Cov}(A, \varepsilon) \subseteq \mathbb{R}^n \) forms an \( \varepsilon \)-covering (an \( \varepsilon \)-net) of \( A \) if \( A \) is contained in the union of the balls of radius \( \varepsilon \) centered at points \( x \in \text{Cov}(A, \varepsilon) \). In this case, we use notation

\[
\text{Cov}(A, \varepsilon) : A \subseteq \bigcup_{x \in \text{Cov}(A, \varepsilon)} B^n_\varepsilon(x).
\]

By changing the scale in \( \mathbb{R}^n \), we can always consider the rescaled set \( A/\varepsilon \) and the new covering \( \text{Cov}(A/\varepsilon, 1) \) with unit balls \( B^n_1(x) \). Without loss of generality, below we consider these (unit) coverings. One of the classical problems is to obtain tight bounds on the covering size \( |\text{Cov}(B^n_r, 1)| \) for any ball \( B^n_r \) of radius \( r \) and dimension \( n \).

Another related covering problem arises for a sphere

\[
S^n_1 \overset{\text{def}}{=} \left\{ z \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} z_i^2 = r^2 \right\}.
\]

Then a unit ball \( B^{n+1}_1(x) \) intersects this sphere with a spherical cap

\[
C^n_r(\rho, y) = S^n_r \cap B^{n+1}_1(x),
\]

which has some center \( y \in S^n_r \), half-chord \( \rho \leq 1 \), and the corresponding half-angle \( \alpha = \arcsin \rho/r \). The biggest possible cap \( C^n_r(1, y) \) is obtained if the center \( x \) of the corresponding ball \( B^{n+1}_1(x) \) is centered at the distance

\[
||x|| = \sqrt{r^2 - 1}
\]

\( \star \)The author is with the University of California, Riverside, USA (e-mail: dumer@ee.ucr.edu)
from the origin. To obtain a minimal covering, we shall consider the biggest caps $C^n_r(1, y)$ assuming that all the centers $x$ satisfy (1).

**Covering density.** Given a set $A \subseteq \mathbb{R}^n$, let $|A|$ denote $n$-dimensional volume (Lebesque measure) of $A$. We then consider any unit covering $\text{Cov}(A, 1)$ and define minimum covering density

$$\vartheta(A) = \min_{\text{Cov}(A, 1)} \sum_{x \in \text{Cov}(A, 1)} \frac{|B^n_r(x) \cap A|}{|A|}.$$  

Minimal coverings have been long studied for the spheres $S^n_r$ and the balls $B^n_r$. The celebrated Coxeter-Few-Rogers lower bound [1] shows that for a sufficiently large ball $B^n_r$,

$$\vartheta(S^n_r) \geq c_0 n. \quad (2)$$

Here and below $c_i$ denote some universal constants. A similar result also holds for any sphere $S^n_r$ of radius $r \geq n$ (see Example 6.3 in [4]).

Various upper bounds on the minimum covering density are obtained for $B^n_r$ and $S^n_r$ by Rogers in the classic papers [2] and [3]. In particular, it follows from these papers that for a sufficiently large radius $r$, a ball $B^n_r$ and a sphere $S^n_r$ can be covered with density

$$\vartheta \leq \left(1 + \frac{\ln \ln n}{\ln n} + \frac{5}{\ln n} \right) n \ln n. \quad (3)$$

Despite recent improvements obtained in [4] and [5], respectively, for spheres $S^n_r$ and balls $B^n_r$ of a relatively small radius $r$, the Rogers bound (3) is still the best asymptotic bound known for sufficiently large spheres, balls, and complete spaces $\mathbb{R}^n$ of growing dimension $n$.

For a sphere $S^n_r$ of any dimension $n \geq 3$ and an arbitrary radius $r > 1$, the best universal upper bound known to date is obtained in [4] (see Corollary 1.2 and Remark 5.1):

$$\vartheta^* \leq \left(1 + \frac{2}{\ln n} \right) \left(1 + \frac{\ln \ln n}{\ln n} + \frac{\sqrt{e}}{n \ln n} \right) n \ln n. \quad (4)$$

Our main result is presented in Theorem 1 which reduces about two times the present upper bounds (3) and (4) for $n \to \infty$.

**Theorem 1** Unit balls can cover a sphere $S^n_r$ of any radius $r > 1$ and any dimension $n \geq 3$ with density

$$\vartheta(S^n_r) \leq \left(\frac{1}{2} + \frac{3 \ln \ln n}{\ln n} + \frac{3}{\ln n} \right) n \ln n. \quad (5)$$

The following corollary to Theorem 1 (see also [8]) shows that the Rogers bound can also be reduced about two times for the coverings of complete Euclidean spaces $\mathbb{R}^n$.

**Corollary 2** For $n \to \infty$, unit balls can cover the entire Euclidean space $\mathbb{R}^n$ with density

$$\vartheta(\mathbb{R}^n) \leq \left(\frac{1}{2} + o(1) \right) n \ln n. \quad (6)$$

### 2 Preliminaries: embedded coverings

In this section, we obtain an estimate on $\vartheta(S^n_r)$ that is similar to (4) but uses a different technique. This technique of embedded coverings will be substantially extended in Section 3 to improve the former bounds (3) and (4). We will also use most of our calculations performed in this section.
Consider a sphere $S^n_\rho$ of some dimension $n \geq 3$ and radius $r > 1$. We use notation $C(\rho,y)$ for a cap $C^n_r(\rho,y)$ whenever parameters $n$ and $r$ are fixed; we also use a shorter notation $C(\rho)$ when a specific center $y$ is of no importance. In this case, $\text{Cov}(\rho)$ will denote any covering of $S^n_\rho$ with spherical caps $C(\rho)$. By definition, a covering $\text{Cov}(\rho)$ has covering density

$$\varrho_\rho = \Omega_\rho |\text{Cov}(\rho)|$$

where $\Omega_\rho$ is the fraction of the surface of the sphere $S^n_\rho$ covered by a cap $C(\rho)$,

$$\Omega_\rho = \frac{|C(\rho)|}{|S^n_\rho|}$$

We begin with two preliminary lemmas, which will simplify our calculations. Let $f_1(x)$ and $f_2(x)$ be two positive differentiable functions. We say that $f_1(x)$ moderates $f_2(x)$ for $x \geq a$ if for all $x \geq a$,

$$f'_1(x) f_2(x) f_1(x) \geq f'_2(x) f_2(x) f_1(x)$$

**Lemma 3** Consider $m$ functions $f_i(x)$ such that $f_1(x)$ moderates each function $f_i(x)$, $i \geq 2$, for $x \geq a$. Then inequality $f_1(x) \geq \sum_{i=2}^{m} f_i(x)$ holds for any $x \geq a$ if it is valid for $x = a$.

**Proof.** Note that $f_i(x) = f_i(a) \exp \{s_i(x)\}$, where $s_i(x) \equiv \int_a^x \frac{f'_i(t)}{f_i(t)} \, dt$. Also, $s_i(x) \leq s_1(x)$ for all $i \geq 2$. Therefore,

$$f_1(x) \geq \sum_{i=2}^{m} f_i(a) \exp \{s_1(x)\} \geq \sum_{i=2}^{m} f_i(a) \exp \{s_i(x)\} = \sum_{i=2}^{m} f_i(x),$$

which completes the proof. \( \square \)

Let $\rho = 1 - \varepsilon$. We first estimate the volumes of the caps $C(\varepsilon,x)$ and $C(\rho,x)$ in relation to the volume $\Omega_1 = 1/2$ of a hemisphere $C(1)$. Let $k_n$ be the volume of the unit Euclidean $n$-ball $B_n$.

**Lemma 4** The caps $C(\varepsilon,x)$ and $C(\rho,x)$ in a sphere $S^n_\rho$ have volumes

$$\Omega_\varepsilon \geq \varepsilon^n/\left(3 \sqrt{1-\varepsilon^2} \sqrt{(n+1)2\pi}\right) \quad (7)$$

$$\Omega_\rho \geq \frac{1}{2} - \frac{n\varepsilon}{\pi}, \varepsilon < \pi/4n \quad (8)$$

**Proof.** Inequality (7) follows from (4), Lemma 3.1. Next, we prove (8). Let $\alpha = \arcsin \rho$. Then a cap $C(\rho,x)$ have volume

$$\Omega_\rho = nk_n \int_0^\alpha \sin^n \beta \, d\beta \geq \Omega_1 - nk_n (\pi/2 - \alpha)$$

Note that $\pi/2 - \alpha \leq \sqrt{2\varepsilon}$. Indeed,

$$\sin(\pi/2 - \alpha) = \cos \alpha = \sqrt{2\varepsilon - \varepsilon^2} \leq \sqrt{2\varepsilon (1 - \varepsilon/4)}$$

On the other hand, for any $\varepsilon < 3/4$

$$\sin(\sqrt{2\varepsilon}) \geq \sqrt{2\varepsilon (1 - \varepsilon^2/3)} \geq \sqrt{2\varepsilon (1 - \varepsilon/4)}$$
Here we used inequality $\sin x \geq x(1 - x^2/6)$ for any $x \in (0, \pi/2)$. Thus, we obtain (8) since

$$\Omega_1 \geq k_n \sqrt{n \pi/2}$$
$$\Omega_1 - \Omega_\rho \leq nk_n \sqrt{2e} \leq \Omega_1 \sqrt{4n \varepsilon/\pi}$$

An embedded algorithm. We employ the following parameters

$$\varepsilon = \frac{1}{2n \ln^2 n}, \quad \rho = 1 - \varepsilon, \quad \lambda = 1 + \frac{2 \ln \ln n}{\ln n}$$

(9)

For $n \geq 20$, these parameters simplify bounds (7) as follows

$$\Omega_\varepsilon \geq \varepsilon n/(8\sqrt{n})$$
$$\Omega_\rho \geq \theta_n = \frac{1}{2} - \frac{1}{\ln n} \sqrt{\frac{1}{2\pi}}$$

(10)

Here the first bound for $\Omega_\varepsilon$ is verified numerically.

To design a covering $\text{Cov}(1)$ of the sphere $S^n_r$, we first randomly choose $N$ points $y \in S^n_r$, where

$$\frac{\lambda n \ln n}{\Omega_\rho} - 1 < N \leq \frac{\lambda n \ln n}{\Omega_\rho} \frac{\Omega_\rho}{\Omega_\varepsilon}$$

(11)

Given the set $\{C(\rho, y)\}$ of $N$ caps, we then consider another covering

$$\text{Cov}(\varepsilon) : S^n_r \subseteq \bigcup_{u \in \text{Cov}(\varepsilon)} C(\varepsilon, u)$$

with smaller caps $C(\varepsilon, u)$. Then we take all centers $u' \in \text{Cov}(\varepsilon)$ that are left uncovered by the set $\{C(\rho, y)\}$ and form the extended set $\{x\} = \{y\} \cup \{u'\}$. This set covers the entire set $\text{Cov}(\varepsilon)$. By expanding the caps $C(\rho, x)$ to the caps $C(1, x)$, we obtain a unit covering

$$\text{Cov}(1) : S^n_r \subseteq \bigcup_{x \in \{x\}} C(1, x).$$

The following lemma yields slightly larger residual terms for density $\vartheta(S^n_r)$ than those obtained in (4); however, it will allow us to further improve estimates in Section 3.

Lemma 5 For $n \geq 20$, covering $\{x\}$ of a sphere $S^n_r$ with unit caps $C(1, x)$ has density

$$\vartheta(S^n_r) \leq \left(1 + \frac{2 \ln \ln n}{\ln n} + \frac{2}{\ln n}\right) n \ln n.$$  

(12)

Proof. Any point $u$ is covered by some cap $C(\rho, y)$ with probability $\Omega_\rho$. Let $N'$ be the expected number of centers $u'$ in $\text{Cov}(\varepsilon)$ that are left uncovered after $N$ random trials. Then the lower bound of (11) gives

$$N' = (1 - \Omega_\rho)^N \cdot |\text{Cov}(\varepsilon)| \leq 2 (1 - \Omega_\rho)^{(\lambda n \ln n)/\Omega_\rho}$$

Note that $(1 - x)^{1/x}$ declines with $x \in (0, 1)$. We then take the minimum value $\Omega_\rho = \theta_n$ of (10). This gives the upper bound

$$(1 - \theta_n)^N \leq 2 (1 - \theta_n)^{(\lambda n \ln n)/\theta_n} \leq 2 e^{nt_n},$$

$$t_n = (\ln n + 2 \ln \ln n) \frac{\ln (1 - \theta_n)}{\theta_n}$$
Finally, inequality (10) gives
\[ |\text{Cov}(\varepsilon)| = \vartheta_\varepsilon / \Omega_\varepsilon \leq (\vartheta_\varepsilon / \Omega_1) \left[ 2n \ln^2 n \right]^n 8\sqrt{n} \] (13)
and
\[ N' \leq 2e^{n_1n} |\text{Cov}(\varepsilon)| \leq 2e^{n_1d_n} \vartheta_\varepsilon / \Omega_1, \] (14)
\[ d_n = t_n + \ln^{8} n + \frac{\ln n}{\ln 2} + \ln (n \ln^2 n) \]
Note that function \( d_n \) has moderating term \( t_n \). Then straightforward calculations give \( d_n \leq 20 < -0.4 \).

Consider a covering \( \{x\} \) with caps \( C(1, x) \) that has size at most \( N + N' \). Then (11) and (8) give the covering density of \( \{x\} \):
\[ \vartheta_1 = \Omega_1 (N + N') \leq (\lambda n \ln n) / (\Omega_1 - \varepsilon) e^{-n/3} \]
\[ \leq (\lambda n \ln n) / \left( 1 - \sqrt{2/\pi} \ln^{-1} n \right) + e^{-n/3} \vartheta_\varepsilon \] (15)
For any given \( n \), bound (15) only depends on the density \( \vartheta_\varepsilon \). Next, we can change the scale in \( \mathbb{R}^{n+1} \) and replace a covering \( \text{Cov}(1) \) of a sphere \( S^n_{r/\varepsilon} \) with the covering \( \text{Cov}(\varepsilon) \) of the sphere \( S^n_r \). This rescaling shows that we can replace \( \vartheta_\varepsilon \) in (15) with any known density \( \vartheta_1 \). Thus, this iteration process yields the upper bound
\[ \vartheta_1 \leq \frac{\lambda n \ln n}{\left( 1 - \sqrt{2/\pi} \ln^{-1} n \right) (1 - e^{-n/3})}, \] (16)
which we replace with a weaker bound (12). Here we again use Lemma 3 and verify that the estimate (12) exceeds the estimate (16) for \( n = 20 \) and moderates it for larger \( n \).

3 New covering algorithm for a sphere \( S^n_r \)

Covering design. In this section, we obtain a covering of the sphere \( S^n_r \) with asymptotic density \( (n \ln n) / 2 \). The new design will use both the former covering \( \text{Cov}(\varepsilon) \) and another covering \( \text{Cov}(\mu) \) with a larger radius \( \mu \) that has asymptotic order of \( n^{-1/2} \). Namely, we use parameters
\[ \varepsilon = (2n \ln^2 n)^{-1}, \ \rho = 1 - \varepsilon \]
\[ \beta = \frac{1}{2} + \frac{3 \ln n}{\ln n}, \ \lambda = \beta + \frac{1}{2 \ln n} \]
\[ \mu = n^{-\beta/4} = 1 / \left( 4\sqrt{\pi} \ln^3 n \right) \]
\[ d = 1 - 2 \varepsilon - \mu^2 = 1 - \frac{1}{n \ln^2 n} - \frac{1}{16n \ln^6 n} \] (17)
and proceed as follows.

A. Let a sphere \( S^n_r \) be covered with two different coverings \( \text{Cov}(\mu) \) and \( \text{Cov}(\varepsilon) \):
\[ \text{Cov}(\mu) : S^n_r \subseteq \bigcup_{z \in \text{Cov}(\mu)} C(\mu, z), \]
\[ \text{Cov}(\varepsilon) : S^n_r \subseteq \bigcup_{u \in \text{Cov}(\varepsilon)} C(\varepsilon, u). \]
We assume that both coverings have the former density $\varrho_x$ of (12) or less.

B. Randomly choose $N$ points $y \in S^n_\varrho$ and consider the corresponding spherical caps $C(\rho, y)$, where

$$N = \lfloor (\lambda n \ln n) / \Omega_d \rfloor$$

(18)

C. Let $C(\mu, \bar{z})$ be any cap in $\text{Cov}(\mu)$ that contains at least one center $u \in \text{Cov}(\varepsilon)$ not covered by the $\rho$-caps. We consider all such centers $\bar{z}$ and form the joint set $\{x\} = \{y\} \cup \{\bar{z}\}$. This set covers $\text{Cov}(\varepsilon)$ with $\rho$-caps and therefore forms the required covering, by extension to the caps $C(1, x)$ :

$$\text{Cov}(1) : S^n_\rho \subseteq \bigcup_{x \in \{x\}} C(1, x).$$

We now proceed with preliminary discussion, which outlines the main steps of the proof.

**Outline of the proof.** Let us first assume that we keep the design of Section 2 but apply it to the new covering $\text{Cov}(\mu)$ instead of $\text{Cov}(\varepsilon)$. This will require taking $\rho = 1 - \mu$ to cover the centers of the caps $C(\mu, \bar{z})$ and then expanding $\rho$ to 1 to cover the whole $\mu$-caps. Contrary to our former choice of $\rho = 1 - \varepsilon$ in (9), it can be proven that this expansion causes the covering density to grow exponentially in $n$. To circumvent this problem, we keep $\rho = 1 - \varepsilon$ in (17) but change our design as follows.

1. Given any cap $C(\mu, \bar{z})$, we say that a cap $C(\rho, y)$ is $d$-close if $y$ falls within distance $t < \rho$ to $z$. In our proof, we refine the selection of the caps $C(\rho, y)$ and count only $d$-close caps, instead of the $\rho$-close caps considered in Section 2. It is easy to verify that distance $d$ of (17) is so close to $\rho$ that

$$\Omega_{d}/\Omega_{d} \rightarrow 1, \quad n \rightarrow \infty.$$ 

(19)

For this reason, counting only $d$-close caps instead of the former $\rho$-close caps will carry no overhead to the covering size (18).

2. On the other hand, we will show in Lemma 6 that the $\mu$-cap becomes almost completely covered by a cap $C(\rho, y)$ when the latter becomes $d$-close instead of $\rho$-close. Namely, only a vanishing fraction $\omega < \exp(-2 n^2)$ of a $\mu$-cap is left uncovered in this case.

3. We shall also use the fact that a typical $\mu$-cap is covered by multiple $d$-close caps. According to (18), the average number $\Omega_d N$ of these $d$-close caps has the exact order of $\lambda n \ln n$ :

$$\lambda n \ln n - \Omega_d < \Omega_d N \leq \lambda n \ln n$$

(20)

In our proof, we first define insufficiently covered $\mu$-caps. Namely, we call a cap $C(\mu, z')$ non-saturated if it has only $s$ or fewer $d$-close $\rho$-caps, where $s$ has a lower order,

$$s = \lfloor n/q \rfloor, \quad q = 6 \ln n$$

(21)

This choice of $s$ will achieve two goals.

4. We prove in Lemma 7 that non-saturated caps typically form a very small fraction among all $\mu$-caps. This fraction has the order below $\exp(-\lambda n \ln n)$. On the other hand, it is easy to see that the quantity

$$|\text{Cov}(\mu)| \leq \vartheta \left(S^n_\mu / \Omega_\mu \right)$$

exceeds $N$ by the exponential factor $\Omega_d/\Omega_\mu \sim \exp(\beta n \ln n)$ or less. Then our choice of $\beta$ and $\lambda$ in (17) gives the expected number $N' = o(N)$ of non-saturated caps. Thus, non-saturated caps typically form a vanishing fraction of not only $\mu$-caps but also of $\rho$-caps.

5. Next, we proceed with saturated $\mu$-caps and count all those centers $u'' \in \text{Cov}(\varepsilon)$ that are left uncovered by random $\rho$-caps. All caps $C(\mu, z'')$ that contain uncovered centers $u''$ are called porous. For
a given \( s \), we show in Lemma 8 that the set \( \{ u'' \} \) forms a very small portion of \( \text{Cov}(\varepsilon) \) that has the expected order \( \omega^{s+1} < \exp \left[ -n \ln^2 n / (3 \ln \ln n) \right] \). Note that the quantity
\[
|\text{Cov}(\varepsilon)| \leq \vartheta \left( S_n^r \right) / \Omega_{\varepsilon}
\]
exceeds \( N \) by the factor \( \Omega_{d} / \Omega_{\varepsilon} \) that only grows as \( \exp \left[ n \ln n \right] \). Therefore, the expected size of \( \{ u'' \} \) is \( N'' = o(N) \).

6. Finally, the centers of all non-saturated and porous caps are combined into the set \( \bar{z} = \{ z', z'' \} \). Then the set \( \{ x \} = \{ y, \bar{z} \} \) completely covers the set \( \text{Cov}(\varepsilon) \) with the caps \( C(\rho, x) \). Therefore, \( \{ x \} \) also covers \( S_n^r \) with unit caps.

**Main proofs.** To prove Theorem 1, we first observe (by numerical comparison) that the existing bound (12) is tighter for \( n \leq 100 \) than bound (5) of Theorem 1. For this reason, we shall only consider dimensions \( n \geq 100 \). The proof is based on three lemmas.

Consider two caps \( C(\mu, Z) \) and \( C(\rho, Y) \) with centers \( Y \) and \( Z \), which are \( d \)-close. These caps are represented in Fig. 1. Here the origin \( O \) is the center of \( S_n^\mu \).

**Lemma 6** For any cap \( C(\mu, Z) \), a randomly chosen \( d \)-close cap \( C(\rho, Y) \) fails to cover any given point \( x \) of \( C(\mu, Z) \) with probability \( p(x) \leq \omega \), where
\[
\omega \leq \frac{1}{4 \ln n} \exp \{ (n - 1) \nu_n / 2 \}, \tag{22}
\]
\[
\nu_n = \ln \left( 1 - \frac{4 \ln^2 n}{n} \right) < -\frac{4 \ln^2 n}{n}
\]

*Proof.* The caps \( C(\mu, Z) \) and \( C(\rho, Y) \) have bases \( PQRSA \) and \( PMRTB \), which form the balls \( B_n^\mu(A) \) and \( B_n^\rho(B) \). Below we consider the boundary \( PQRS \) of \( C(\mu, Z) \), which forms the sphere \( S_n^{\mu-1}(A) \). The bigger cap \( C(\rho, Y) \) covers this boundary, with the exception of the cap \( PQR \) centered at \( Q \). We first consider the case, when \( x \) is a boundary point and belongs to \( PQRS \). Then the probability \( p(x) \) is the fraction \( \Omega \) of the entire boundary contained in uncovered cap \( PQR \). We first estimate the half-angle \( \alpha = \angle PAQ \) formed by the cap \( PQR \).
Let \(d(H, G)\) denote the distance between any two points \(H\) and \(G\). Also, let \(\sigma(H)\) be the distance from a point \(H\) to the line \(OBY\) that connects the origin \(O\) with the center \(B\) of the bigger base \(B^\mu_\rho(B)\) and with the center \(Y\) of the cap \(C(\rho, Y)\). We use inequalities
\[
\sigma(A) \leq \sigma(Z) \leq d(Z, Y) \leq d.
\]

On the other hand, consider the base \(PNR\) of the uncovered cap \(PQR\). Here \(N\) denotes the center of this base. Then both lines \(AN\) and \(BN\) are orthogonal to this base. Also, \(d(B, P) = \rho\), and \(d(N, P) \leq d(A, P) = \mu\). Thus,
\[
d(B, N) = \sqrt{d^2(B, P) - d^2(N, P)} \geq \sqrt{\rho^2 - \mu^2} \geq \rho - \mu^2 - d = \varepsilon.
\]

Finally, the center line \(OBY\) is orthogonal to the entire base \(PMRTB\) and its line \(BN\). Then \(d(B, N) = \sigma(N)\) and
\[
d(A, N) \geq \sigma(N) - \sigma(A) \geq \rho - \mu^2 - d = \varepsilon.
\]

Now we consider the right triangle \(ANP\) and deduce that
\[
\cos \alpha = d(A, N)/d(A, P) \geq \varepsilon/\mu = (2 \ln n) / \sqrt{n} \tag{24}
\]
\[
\sin^2 \alpha \leq \ln \left( 1 - \frac{4 \ln^2 n}{n} \right) = \exp\{-\nu_\mu\}.
\]

We can now estimate the fraction \(\Omega\) of the boundary sphere \(S^\mu_n(A)\) contained in the uncovered cap \(PQR\). Here we use the bound (see [4], Corollary 3.2) immediately follows from
\[
\Omega < \frac{\sin^{-1} \alpha}{\sqrt{2\pi(n-1)}} \cos \alpha
\]

To obtain (22), we simply replace denominator with a smaller quantity \(4 \ln n\) using (25).

Finally, consider the second case, when a point \(x\) does not belong to the boundary \(PQRS\). Therefore, \(x\) is taken from a smaller cap \(C(\mu', Z) \subset C(\mu, Z)\) with the same center \(Z\) and a half-chord \(\mu' < \mu\). Similarly to the first case, we define the boundary \(S^\mu_n(A')\) of \(C(\mu', Z)\), where \(A'\) is some center on the line \(AZ\). Again, \(p(x)\) is the fraction \(\Omega'\) of this boundary left uncovered by the bigger cap \(C(\rho, Y)\). To obtain the upper bound on \(\Omega'\), we only need to replace \(\mu\) with a smaller \(\mu'\) in (25). This gives the smaller angle \(\alpha' \leq \arccos(\varepsilon/\mu')\), which is reduced to 0 if \(\varepsilon \geq \mu'\). In particular, the center \(Z\) of the cap is always covered by any \(d\)-close cap. Thus, we see that any internal layer \(S^\mu_n(A')\) of the cap \(C(\mu, Z)\) has a smaller uncovered fraction \(\Omega' \leq \Omega\). This gives the required condition \(p(x) \leq \Omega' \leq \Omega < \omega\) for any point \(x\) and proves our lemma. \(\square\)

**Remarks.** First, note that (25) can be used only for \(n \geq 75\). For \(n < 75\), inequality (25) gives \(\cos \alpha \geq 1\), which only shows that \(\alpha = 0\). In this case, a \(d\)-close cap entirely covers \(\mu\)-cap in Fig. 1. Second, note that even a marginal increase in \(d\) completely changes our setting. Namely, it can be proven that about half the base of the \(\mu\)-cap is uncovered if a \(\rho\)-cap is \((d + \varepsilon)\)-close.

Our next goal is to estimate the expected number \(N'\) of non-saturated caps \(C(\mu, z')\) left after \(N\) trials, where \(N = (\lambda n \ln n) / \Omega_d\) according to (20).

**Lemma 7** For \(n \geq 100\), the number of non-saturated caps \(C(\mu, z')\) left after \(N\) trials has expectation \(N' < e^{-n/4-1}N\).

**Proof.** Given any center \(z\), a randomly chosen center \(y\) is \(d\)-close to \(z\) with the probability \(\Omega_d\). Then the probability to obtain at most \(s\) such caps \(C(d, y)\) is
\[
P = \sum_{i=0}^{s} \binom{N}{i} \Omega_d^i (1 - \Omega_d)^{N-i}.
\]

(26)
First, we use (8) for \( \Omega_d \) and take \( d = 1 - 2\varepsilon - \mu^2 \) in (17). Then
\[
\Omega_d \geq \theta_n = \frac{1}{2} - \sqrt{\frac{1}{\pi \ln^2 n} + \frac{1}{16\pi \ln^6 n}}
\]

It is easy to verify that \( \theta_n \geq \theta_{100} > \theta \), where \( \theta \equiv 0.377 \). For brevity, let \( \Omega_d = x \). Then \( (1 - x)^{N-s} \leq (1 - x)^{N-s} \). Note also that \( N/L = L \geq \lambda_n \ln n - 1/2 \) according to (20). Then we use Taylor series \( \ln (1 - x) = -x - x^2/2 - \ldots \) and see that
\[
\ln(1 - x)^{N-s} = -L - x(L/2 - s) - x^2(L/3 - s/2) - \ldots
\]
\[
= L [\ln(1 - x)]/x - s \ln(1 - x)
\]

Every term \( x^i \) in the first line has negative coefficient \( L/(i+1) - s/i \) since \( s = n/q \). Thus, we can take \( x = \theta \), which gives the upper bound
\[
\ln(1 - x)^{N-s} \leq L [\ln(1 - \theta)]/\theta - s \ln(1 - \theta)
\]
\[
\leq -n [\tau_1 \lambda \ln n - \tau_2/ (6 \ln \ln n)] + 1
\]
with coefficients
\[
\tau_1 = [\ln(1 - \theta)]/\theta = 1.255, \tau_2 = - \ln(1 - \theta) = 0.473
\]

Now we proceed with the remaining terms in (26). Here \( \binom{N}{s} x^i < (N/\theta)^{i!} \leq (\lambda_n \ln n)^{i!} \) and
\[
\sum_{i=0}^{\infty} \binom{N}{s} x^i \leq \frac{\lambda_n \ln n)^s}{s!} \sum_{i=0}^{\infty} \frac{(\lambda q \ln n)^i}{s!} \leq 3 \frac{(\lambda_n \ln n)^s}{s!}
\]
\[
\leq \frac{(\lambda q \ln n)^s}{s/e^s} = \exp \{ nh_n \}, \quad h_n = \ln(\lambda q \ln n)/q
\]

Direct calculations give the bound \( \sum_{i=0}^{\infty} (\lambda q \ln n)^{-i} < 3 \) for any \( n \geq 100 \). Then we use Sterling formula \( s! > (2\pi s)^{1/2} (s/e)^s \). Summarizing, we have
\[
P \leq \exp \{ n|h_n - \tau_1 \lambda \ln n + \tau_2/ (6 \ln \ln n) \},
\]

Next, we calculate the size \( |\text{Cov}(\mu)| \) needed to cover \( S^n_\mu \). Comparing parameters \( \vartheta_* \) of (4) and \( \lambda \) of (17), we see that \( \vartheta_* \leq 2\lambda n \ln n - 1 \leq 2\Omega_d N \) for any \( n \geq 100 \). Thus, we can cover \( S^n_\mu \) with density \( \vartheta_* \) which gives
\[
|\text{Cov}(\mu)| \leq \vartheta_* / \Omega_* \leq 2\Omega d / \Omega_\mu
\]

Finally, we use (7), which gives \( \Omega_\mu \geq \Omega_1 \mu^n / (8\sqrt{n}) \) for \( n \geq 100 \) and
\[
|\text{Cov}(\mu)| \leq 16\sqrt{n} \mu^{-n} \leq 16\sqrt{n} \exp \{ \beta n \ln n + n \ln 4 \}
\]

Thus, the expected number of non-saturated caps is
\[
N' \leq |\text{Cov}(\mu)| P \leq 16\sqrt{n} \exp \{ n \Psi_n \}
\]
\[
\Psi_n = \ln(\lambda q \ln n)/q - (\tau_1 \lambda - \beta) \ln n + \tau_2/q + \ln 4 + 1/4
\]

The quantity \( \Psi_n \) has moderating term \( - (\tau_1 \lambda - \beta) \ln n \) and declines in \( n \). Direct calculation shows that \( \Psi_{100} < -0.36 \). Finally, we replace \( 16\sqrt{n} \exp \{ -n \Psi_n \} \) with a larger quantity \( \exp \{ -n/4 - 1 \} \).□

Consider now the saturated caps \( C(\mu, z) \) and the centers \( u \in \text{Cov}(\varepsilon) \) inside them.

**Lemma 8** For any \( n \geq 100 \), the number of centers \( u'' \in \text{Cov}(\varepsilon) \) left uncovered in all saturated caps \( C(\mu, z) \) has expectation \( N'' < e^{-2n} N \).
Proof. We first estimate the total number $|\text{Cov}(\varepsilon)|$ of centers $u$. We proceed similarly to (28). Then

$$|\text{Cov}(\varepsilon)| \leq 2 N \Omega_d / \Omega \leq 16 \sqrt{n} N (2 n \ln^2 n)^n.$$ 

Any cap $C(\mu, z)$ intersects with at least $s + 1$ randomly chosen caps $C(\rho, y)$. According to Lemma 6 any single $\rho$-cap fails to cover any given point $x \in C(\mu, z)$ with probability $\omega$ or less. Therefore any point $u'' \in C(\mu, z)$ is not covered with probability at most $\omega^{s+1} < \omega^{n_q} \leq \exp \{ n C_n \}$, where we use (22) and obtain

$$C_n = \frac{n - 1}{12 \ln n} \ln \left(1 - \frac{4 \ln^2 n}{n} \right) - \frac{\ln(4 \ln n)}{6 \ln n}.$$ 

Then

$$N'' \leq |\text{Cov}(\varepsilon)| \cdot \omega^{n_q} \leq 16 N \sqrt{n} \exp \{ n \Phi_n \}, \tag{30}$$

$$\Phi_n = C_n + \ln n + 2 \ln \ln n + \ln 2.$$ 

Note that the first term in $C_n$ has the order of $-(\ln^2 n) / (3 \ln \ln n)$ and moderates all other terms. Thus, $\Phi_n < \Phi_{100}$. Direct calculation shows that $\Phi_{100} < -2$, which proves the lemma. □

Proof of Theorem 7. Consider any cap $C(\mu, z)$ that contains at least one uncovered center $u \in \text{Cov}(\varepsilon)$. Such a cap is either non-saturated or porous and therefore $\{ z \} = \{ z' \} \cup \{ z'' \}$. Then, according to Lemmas 7 and 8 $\{ z \}$ has expected size $N' \leq N'' < e^{-n_q/4} N$. Thus, there exist $N$ randomly chosen centers $y$ that leave at most $e^{-n_q/4} N$ centers $z$. The extended set $\{ x \} = \{ y \} \cup \{ z \}$ forms a unit covering of $S^r_n$. This covering has density

$$\vartheta \leq \Omega_1 (N + N') \leq \Omega_1 N (1 + e^{-n_q/4}) \leq \lambda n \ln n (1 + e^{-n_q/4}) \Omega_1 / \Omega_d$$

Similarly to inequality (8), we can directly verify that $\Omega_1 / \Omega_d \leq 1 + 3 / (2 n)$ for $n \geq 100$. Finally, we take $\lambda$ of (17) and combine the last inequalities for $\vartheta$ and $\Omega_d / \Omega_1$ as follows

$$\frac{\vartheta}{n \ln n} \leq \left( \frac{1}{2} + \frac{3 \ln n}{\ln n} + \frac{1}{2 \ln n} \right) \left( 1 + \frac{3}{2 \ln n} \right) \vartheta (1 + e^{-n_q/4})$$

$$< \frac{1}{2} + \frac{3 \ln n}{\ln n} + \frac{3}{\ln n}.$$ 

Direct verification shows that the last inequality holds for $n = 100$. Then we can again use Lemma 8 for larger $n$. This completes the proof of Theorem 1. □

Finally, note that Theorem 1 directly leads to Corollary 2. Indeed, we can use the well known fact $\vartheta (S^{n-1}) = \lim_{r \to \infty} \vartheta (S^n_r)$ (see the proof in [6] or Theorem II.1 in [7], where a similar proof is detailed for packings of $\mathbb{R}^n$).

Further directions. We have proved that the classical Rogers bound (3) on the covering density of a sphere $S^n_r$ or the Euclidean space $\mathbb{R}^n$ can be reduced about two times for large dimensions $n$. One open problem is to reduce the gap between this bound and its lower counterpart (2), which is linear in $n$. In this regard, note that our design always holds if parameter $\mu$ has the order of $n^{-\beta}$ given some constant $\beta > 1/2$. However, it can be verified that choosing a smaller constant $\beta < 1/2$ will exponentially increase the covering size. Therefore, a completely new design is needed for any further reductions of the upper bound. Another important problem is to extend the above results to the balls $B^n_r$ of an arbitrary radius $r$. Our conjecture is that $\vartheta (B^n_r) \leq (1/2 + o(1)) n \ln n$ for any $r$ and $n \to \infty$.

Addendum. The published version of this paper [9] uses incorrect formula (8) $\Omega_\tau / \Omega_\rho \geq (\tau / \rho)^n$ instead of the correct inequality $\Omega_\tau / \Omega_\rho \geq (\arcsin \tau / \arcsin \rho)^n$ valid for any $\tau < \rho \leq 1$. Formula (8) was not used in main Lemmas 6,7 and 8 of paper [9]: however, it reduced the residual terms in the preliminary Section 2. The current version excludes formula (8) and also tightens several estimates of [9].

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