THE DEFINABLE CONTENT OF HOMOLOGICAL INVARIANTS I: Ext & lim

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Abstract. This is the first installment in a series of papers illustrating how classical invariants of homological algebra and algebraic topology may be enriched with additional descriptive set theoretic information. To effect this enrichment, we show that many of these invariants may be naturally regarded as functors to the category, introduced herein, of groups with a Polish cover. The resulting definable invariants provide far stronger means of classification.

In the present work we focus on the first derived functors of \( \text{Hom}(\_ , \_ ) \) and \( \text{lim}(\_ ) \). The resulting definable \( \text{Ext}(B, F) \) for pairs of countable abelian groups \( B, F \) and definable \( \text{lim}^1(A) \) for towers \( A \) of Polish abelian groups substantially refine their classical counterparts. We show, for example, that the definable \( \text{Ext}(\_ , \mathbb{Z}) \) is a fully faithful contravariant functor from the category of finite rank torsion-free abelian groups \( \Lambda \) with no free summands; this contrasts with the fact that there are uncountably many non-isomorphic such groups \( \Lambda \) with isomorphic classical invariants \( \text{Ext}(\Lambda, \mathbb{Z}) \). To facilitate our analysis, we introduce a general Ulam stability framework for groups with a Polish cover; within this framework we prove several rigidity results for non-Archimedean abelian groups with a Polish cover. A special case of our main result answers a question of Kanovei and Reeken regarding quotients of \( p \)-adic groups. Finally, using cocycle superrigidity methods for profinite actions of property (T) groups, we obtain a hierarchy of complexity degrees for the problem \( R(\text{Aut}(\Lambda) \times \text{Ext}(\Lambda, \mathbb{Z})) \) of classifying all group extensions of \( \Lambda \) by \( \mathbb{Z} \) up to base-free isomorphism, when \( \Lambda = \mathbb{Z}[1/p]^d \) for prime numbers \( p \) and \( d \geq 1 \).

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1. INTRODUCTION

Some of the best-known and most versatile invariants in mathematics arise as the (co)homology groups of (co)chain complexes. Such invariants are computed by first associating to each object \( X \) a chain complex

\[
C_\bullet := \cdots \overset{\partial_3}{\leftarrow} C_2 \overset{\partial_2}{\leftarrow} C_1 \overset{\partial_1}{\leftarrow} C_0 \overset{\partial_0}{\leftarrow} 0
\]

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of abelian groups encoding the relevant data about $X$. The $n^{th}$ homology group of $C_\bullet$ is then defined as the quotient
$$H_n := \mathbb{Z}_n/B_n := \ker(\partial_n)/\operatorname{ran}(\partial_{n+1}).$$

The overwhelming tendency is to regard these groups as discrete objects. This is despite the fact that the chain groups $C_n$ often carry a natural topology, and even one encoding data about $X$ not captured by their group structures alone. It is also despite the “trend” which Dieudonné recalls in the opposite direction “that was very popular until around 1950 (although later all but abandoned), namely, to consider homology groups as topological groups for suitably chosen topologies” [12, p. 67]. Dooming this approach seems mainly to have been the fact that the aforementioned natural topologies may badly fail to induce Hausdorff topologies on $H_n$. A case of some midcentury prominence, for example, was that of the reduced 0-dimensional Steenrod homology group $\tilde{H}_0(S)$ of a dyadic solenoid $S$: here, not only is the boundary group $B_0$ not closed in the natural topology on $Z_0$; it is dense therein (see [15]).

1.1. The definable content of homological invariants. What follows is the first in a series of papers in which we show how to endow various homological invariants with a finer structure than the quotient topology, as well as some of the benefits of doing so [5, 6, 44]. A main resource for this refinement is the field of invariant descriptive set theory, an area of mathematics which studies the Borel complexity of classification problems; see Section 2.3 below. The critical contexts for that field are Polish spaces, and the fundamental, initiating recognition for our work is the fact that many of the (co)homology groups from algebraic topology and homological algebra are naturally viewed as groups with a Polish cover, i.e., as quotients $G/N$ of a Polish group $G$ by a Polishable subgroup $N$. Examples include:

1. the strong or Čech homology groups $H_\alpha(K)$ of a compact metrizable space $K$;
2. the Čech or sheaf cohomology groups $H^n(L)$ of a locally compact metrizable space $L$;
3. the first derived group $\operatorname{Ext}(B, F)$ of the $\operatorname{Hom}(-, -)$-bifunctor applied to countable abelian groups $B, F$;
4. the first derived group $\operatorname{lim}^1(A)$ of the $\operatorname{lim}(\cdot)$-functor applied to towers of abelian Polish groups $A$.

In this first paper we restrict our attention to examples (3) and (4). These arise in purely algebraic settings and form the backbone of a variety of computations in algebraic topology — typically by way of the Universal Coefficient Theorem [15] and Milnor’s Exact Sequence [56], respectively — and, in particular, of many involving (1) and (2). Our results below will be applied in [5] to show that, in contrast to the classical Čech cohomology theory, definable Čech cohomology theory provides complete homotopy invariants for mapping telescopes of $n$-spheres and $n$-tori, as well as for maps to spheres from the latter.

We view groups with a Polish cover, and hence all of the aforementioned invariants, as objects in the category of groups with a Polish cover. Morphisms in this category are definable homomorphisms. A definable homomorphism between groups with a Polish cover $G/N$ and $G'/N'$ is any group homomorphism $f: G/N \to G'/N'$ which lifts to a Borel map $\tilde{f}: G \to G'$. We do not require $\tilde{f}$ to be a homomorphism from $G$ to $G'$. Definable homomorphisms may be thought of as those homomorphisms which can be described explicitly by a (potentially infinitary) formula, one making no essential appeal to the axiom of choice. For example, $\mathbb{R}/\mathbb{Q}$ admits $2^{2^{\aleph_0}}$ many endomorphisms as an abstract group, since it is a $\mathbb{Q}$-vector space of dimension $2^{\aleph_0}$. However, only continuum many of these endomorphisms are actually definable; see [37]. For $B, F$, and $A$ as above, the classical assignments $(B, F) \mapsto \operatorname{Ext}(B, F)$ and $A \mapsto \operatorname{lim}^1(A)$ factor through a bifunctor and functor, respectively, which take values in the additive category of abelian groups with a Polish cover. The resulting definable Ext invariants and definable $\operatorname{lim}^1$ invariants record much more information than their purely algebraic counterparts.

Consider, for example, the problem of classifying all finite-rank torsion-free abelian groups $\Lambda$ up to isomorphism. Notice that the invariant $\operatorname{Hom}(\Lambda, \mathbb{Z})$ is a free abelian group whose rank coincides with the rank of the largest free direct summand of $\Lambda$. By the following theorem, the discrete group $\operatorname{Hom}(\Lambda, \mathbb{Z})$ together with the definable $\operatorname{Ext}(\Lambda, \mathbb{Z})$ group completely classify all finite rank torsion-free abelian groups; see Corollary 7.6.

**Theorem 1.1.** The functor $\operatorname{Ext}(-, \mathbb{Z})$ is a fully faithful functor from the category of finite rank torsion-free abelian groups with no free summands to the category of groups with a Polish cover.

In particular, $\operatorname{Ext}(\Lambda, \mathbb{Z})$, up to definable isomorphism, together with the rank of $\operatorname{Hom}(\Lambda, \mathbb{Z})$, form a complete set of invariants for finite rank torsion-free abelian groups up to isomorphism.
Theorem 1.1 should be contrasted with the fact that, as a discrete group, Ext(Λ, Z) records comparatively little about a given finite-rank torsion-free group Λ. Indeed, there is a size-continuum family of non-isomorphic such groups Λ whose corresponding invariants Ext(Λ, Z) are isomorphic as abstract groups, as we show in Corollary 7.9 below. In the process of proving Theorem 1.1, we develop a definable version of Jensen’s theorem which relates the definable Ext-functor to the definable lim^1-functor. The definable content of the lim^1-functor is studied in Section 5, where we record a further result in the spirit of Theorem 1.1; see Corollary 5.15. We also provide an explicit description of lim^1(A) when A is a monomorphic tower of abelian groups; see Theorem 5.13. For further applications of these analyses, see the second author’s [45, 47], in which the definable Ext functor is applied to obtain new purely algebraic results; see also his recent [46], which underscores how canonical the framework introduced herein in fact is: in the latter, the category of groups with an abelian Polish cover is shown to form a minimal abelian extension, or more precisely heart, of the category of Polish abelian groups.

1.2. An Ulam stability framework. The main ingredient in the aforementioned theorems and corollaries is a new rigidity result for non-Archimedian abelian groups with a Polish cover. To state and prove this result, we introduce a general Ulam stability framework which specializes to the one appearing in [37–39] when the cover G of the relevant group G/N is a connected Polish group.

Ulam stability phenomena may be studied in any setting where one has: (1) a notion of morphisms; (2) a notion of approximate morphisms; and (3) a notion of closeness relating morphisms and approximate morphisms. Such phenomena have received considerable attention over the last 50 years, especially in the settings of groups, Boolean algebras, and C*-algebras; see [2, 16–20, 24, 26, 30, 35, 40, 65, 66]. An Ulam stability framework was introduced in [37] for studying definable homomorphisms \( \mathbb{R}/N \to \mathbb{R}/N' \), when N and N' are countable subgroups of \((\mathbb{R}, +)\). There, approximate morphisms are Baire-measurable maps \( \mathbb{R} \to \mathbb{R} \) which are lifts of homomorphisms \( \mathbb{R}/N \to \mathbb{R}/N' \); morphisms are approximate approximations of the form \( x \mapsto cx \) for some \( c \in \mathbb{R} \); and two morphisms are “close” to each other if they are lifts of the same homomorphisms \( \mathbb{R}/N \to \mathbb{R}/N' \). The main theorem in [37] is that every approximate morphism is close to a morphism. Hence every definable homomorphism \( \mathbb{R}/N \to \mathbb{R}/N' \) is of the form \( x \mapsto cx \).

The main theorem in [37] is that every approximate morphism is close to a morphism. Hence every definable homomorphism \( \mathbb{R}/N \to \mathbb{R}/N' \) is of the form \( x \mapsto cx \). The question of whether similar Ulam stability phenomena exist for quotients of the \( p \)-adic groups appears in [39, Section 8]. The next theorem answers this in the more general context of quotients of arbitrary abelian pro-countable groups. For definitions of trivial, approximately trivial, and approximately generically trivial homomorphisms, as well as for the proof of the theorem, we refer the reader to Section 4, where we develop the appropriate Ulam stability framework.

**Theorem 1.2.** Let \( f : G/N \to G'/N' \) be a definable homomorphism between groups with a Polish cover, where G is abelian and non-Archimedean and N is dense in G.

1. If \( N' \) is countable then \( f \) is trivial;
2. If \( N' \) is locally profinite then \( f \) is approximately trivial;
3. If \( N' \) is non-Archimedean in its Polish topology then \( f \) is approximately generically trivial.

Results (1) and (2) above are optimal, as illustrated by Examples 4.10 and 4.11.

1.3. Cocycle superrigidity methods and Borel reducibility complexity. Lastly, we establish complexity bounds for various classification problems within the Borel reduction hierarchy. Many classification problems in mathematics naturally parametrize as pairs \((X, E)\), where X is a Polish space and E is a Borel equivalence relation. Invariant descriptive set theory studies the ordering of the collection of all such classification problems according to their relative complexity, as measured by the Borel reduction relation \((X, E) \leq_B (Y, F)\); see Section 2.3.

One classification problem that naturally arises in homological computations pertinent, e.g., to shape theory [52], is that of classifying objects of a pro-Ho(Top) subcategory up to isomorphism. As a byproduct of the proof of Theorem 1.1 and the results from [1, 27, 70], we establish that even for very simple full subcategories of pro-Ho(Top), the complexity of the isomorphism relation \( \simeq_{pro} \) on objects can be highly non-trivial; see Corollary 6.8.

Let \( Z(\Lambda, Z)/B(\Lambda, Z) \) be the presentation of Ext(\( \Lambda, Z \)) as a group with a Polish cover. Since \( \Lambda \) is a countable abelian group, the problem \( R(\text{Ext}(\Lambda, Z)) \) of classifying all parametrized extensions \( x, y \in Z(\Lambda, Z) \) of \( \Lambda \) by \( Z \) up to base-preserving isomorphism (i.e., by whether \( x - y \in B(\Lambda, Z) \)) is hyperfinite, and therefore of comparatively low
prove Theorem 1.2, which is our main technical tool for analyzing the definable content of \( \text{Ext}(\Lambda, \mathbb{Z}) \) given by accounting for the natural action \( \text{Aut}(\Lambda) \curvearrowright \text{Ext}(\Lambda, \mathbb{Z}) \) provided by Theorem 1.1.

In particular, in the case of the groups \( \Lambda_p^d := \mathbb{Z}[1/p]^d \) for a prime number \( p \) and \( d \geq 1 \), we prove that the orbit equivalence relations \( E_p^d := \mathcal{R}(\text{Aut}(\Lambda_p^d) \curvearrowright \text{Ext}(\Lambda_p^d, \mathbb{Z})) \) array in the following hierarchy.

**Theorem 1.3.** Adopt the notations above. Fix \( d, m \geq 1 \) and primes \( p, q \).

1. \( E_q^m \) is not Borel reducible to \( E_p^d \) if \( m > d \);
2. \( E_q^m \) is not Borel reducible to \( E_p^d \) if \( p \) and \( q \) are distinct and \( m, d \geq 3 \);
3. \( E_p^d \) is hyperfinite and not smooth if \( d = 1 \);
4. \( E_p^d \) is not treeable if \( d \geq 2 \).

The proof of Theorem 1.3 relies on the description of \( E_p^d \) as the orbit equivalence relation of the affine action \( \text{GL}_d(\mathbb{Z}[1/p]) \rtimes \mathbb{Z}[1/p]^d \curvearrowright \mathbb{Q}_p^d \), where \( \mathbb{Q}_p \) is the additive group of the field of \( p \)-adic numbers, as well as Ioana’s cocycle superrigidity result for profinite actions [31] for property (T) groups, and ideas of Coskey and Thomas [10, 70].

### 1.4. The structure of the paper.

In Section 2, we give a brief review of the necessary definitions and collect some standard facts which we are going to need from the literature. In Section 3, we introduce the category of groups with a Polish cover. In Section 4, we develop a general Ulam stability framework for groups with a Polish cover and prove Theorem 1.2, which is our main technical tool for analyzing the definable content of \( \text{Ext}(\Lambda, \mathbb{Z}) \) given by accounting for the natural action \( \text{Aut}(\Lambda) \curvearrowright \text{Ext}(\Lambda, \mathbb{Z}) \) provided by Theorem 1.1. In Section 5, we analyze the definable content of \( \lim^1(-) \), a functor playing an essential bookkeeping role in many computations in algebraic topology and constructions in homological algebra. In Section 6, we apply the rigidity results from Sections 4 and 5 in the special case of profinite completions of \( \mathbb{Z}^d \). We also establish a lifting property for actions on the quotients of such completions, which will play a crucial role in Section 7. In Section 7, we analyze the definable content of \( \text{Ext}(-, -) \) and prove Theorem 1.1. We also compute the abstract group-isomorphism-type of the quotient \( \hat{\mathbb{Z}}^d/\mathbb{Z}^d \) for an arbitrary profinite completion \( \hat{\mathbb{Z}}^d \) of \( \mathbb{Z}^d \). This will show how much information may be lost by neglecting the definable data about \( \hat{\mathbb{Z}}^d/\mathbb{Z}^d \) contained in its Polish cover. In Section 8, we use cocycle superrigidity methods to study the Borel complexity of the action \( \text{Aut}(\Lambda) \curvearrowright \text{Ext}(\Lambda, \mathbb{Z}) \), and conclude with the proof of Theorem 1.3.

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### 2. Preliminaries

In this section we review the basic facts from invariant descriptive set theory and category theory that we will require below. Standard references are [23, 41] for the former and [48] for the later.

#### 2.1. Polish spaces.

A Polish space is a second countable topological space whose topology is induced by a complete metric. Let \( X \) be a Polish space. The \( \sigma \)-algebra of Borel subsets of \( X \) is the smallest \( \sigma \)-algebra of subsets of \( X \) that contains all the open subsets of \( X \). Subsets of \( X \) are Borel if they belong to its Borel \( \sigma \)-algebra. A function \( f : X \to Y \) between Polish spaces is Borel if the \( f \)-preimages of Borel sets are all Borel. The image of a Borel subset of \( X \) under such a Borel function \( f \) need not be a Borel subset of \( Y \) but it will be, by definition, analytic. If \( f \), though, is an injective Borel function then \( f \) will map Borel subsets of \( X \) to Borel subsets of \( Y \) [41, Theorem 15.1]. The subspace topology renders a subset \( A \) of a Polish space \( X \) Polish if and only if \( A \) is a \( G_\delta \) subset of \( X \), i.e., if and only if \( A \) is the intersection of a countable family of open subsets of \( X \). Hence all closed subspaces of a Polish space are Polish. A Polish space \( X \) is locally compact if every point of \( X \) has an open neighborhood with compact closure. This is equivalent (in the Polish setting) to the assertion that \( X \) can be written as an increasing union of compact subspaces.

A subset \( A \) of a topological space \( X \) is meager if it is contained in the union of countably many closed nowhere dense sets. It is nonmeager if it is not meager and comeager if its complement is meager. We say that \( A \) has the Baire property if it is contained in the smallest \( \sigma \)-algebra on \( X \) generated by the open subsets and the meager subsets.
of $X$. A function $f : X \to Y$ between topological spaces is \textit{Baire measurable} if $f^{-1}(U)$ has the Baire property for every open set $U \subseteq Y$. By the Baire Category Theorem [41, Theorem 8.4], every Polish space $X$ is a \textit{Baire space}, i.e., every nonempty open subset of such an $X$ is nonmeager. Hence the intersection of countably many dense $G_δ$ subsets of $X$ will again be dense $G_δ$. Observe lastly that a subset of $X$ is comeager if and only if it contains a dense $G_δ$ subset of $X$.

2.2. \textbf{Polish groups}. A \textit{topological group} is a group endowed with a topology rendering the group operations $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ continuous. A \textit{Polish group} is a topological group whose topology is Polish. A subgroup $H$ of a Polish group $G$ endowed with the subspace topology is a Polish group if and only if it is closed in $G$; see [23, Proposition 2.2.1]. In this case the quotient group $G/H$, endowed with the quotient topology, is also a Polish group; see [23, Theorem 2.2.10]. By \textit{the category of Polish groups} we mean the category with Polish groups as objects and continuous homomorphisms as morphisms. The following fact, known as \textit{Pettis’ Lemma}, will be used repeatedly; see [23, Theorem 2.3.2] for a proof.

\textbf{Lemma 2.1} (Pettis). If a nonmeager subset $A$ of a Polish group $G$ has the Baire property then $AA^{-1}$ contains an open neighborhood of the identity of $G$.

\textbf{Corollary 2.2}. If $ϕ : G \to H$ is a Baire-measurable group homomorphism between Polish groups, then $ϕ$ is continuous.

Hence the morphisms of the aforementioned category might equivalently be taken to be the Baire measurable or “definable” homomorphisms between Polish groups.

In the following, we will regard any countable group as a topologically discrete, and therefore locally compact, non-Archimedean, Polish group. Recall that a Polish group $G$ is \textit{locally compact} if it is a locally compact topological space and is \textit{non-Archimedean} if its identity element admits a neighborhood basis consisting of open subgroups. The main focus below will be on non-Archimedean abelian Polish groups. An abelian Polish group is non-Archimedean if and only if it is \textit{procountable}, i.e., is an inverse limit of a sequence of countable groups [49, Lemma 2]. Recall that a Polish group is \textit{profinite} if it is compact and non-Archimedean or, equivalently, is isomorphic to the inverse limit of a tower of finite groups. We say that $G$ is \textit{locally profinite} if it admits a basis of neighborhoods of the identity consisting of profinite groups. Let $G$ be a non-Archimedean abelian Polish group and let $(V_n)$ be a decreasing sequence of open subgroups, beginning with $V_0 = G$, whose intersection is the identity. It is easy to check that letting $d(g,h) = 2^{-n}$ if and only if $n$ is the largest natural number with $gh^{-1} \in V_n$ defines a metric $d \leq 1$ on $G$ compatible with its topology such that:

1. $d$ is both left and right invariant, i.e., $d(fg,fh) = d(g,h)$ and $d(hg,hf) = d(g,h)$, for all $f, g, h \in G$;
2. $d$ is an ultrametric, i.e., $d(f,h) \leq \max\{d(f,g), d(g,h)\}$, for all $f, g, h \in G$.

Moreover, as is straightforward to verify, this metric is also complete; see [23, Corollary 2.2.2].

A \textit{standard Borel structure} on a set $X$ is a collection $B$ of subsets of $X$ forming the $σ$-algebra of Borel sets with respect to some Polish topology on $X$. A \textit{standard Borel space} $(X,B)$ is a set $X$ endowed with a Borel structure $B$. If $(X,B)$ and $(X’,B’)$ are standard Borel spaces, then a \textit{Borel function from $(X,B)$ to $(X’,B’)$} is, as above, one that is measurable with respect to $B$ and $B’$. A \textit{standard Borel group} is a group endowed with a standard Borel structure with the property that all the group operations are Borel. A standard Borel group is \textit{Polishable} if there exists a Polish topology that induces its Borel structure \textit{and} renders it a Polish group. It follows from Pettis’ Lemma that such a topology is always unique. The image of any continuous group homomorphism from one Polish group to another is a Polishable standard Borel group. This fact forms part of our next lemma, which we state after recalling a few more definitions.

If $X$ is a standard Borel space and $E$ is an equivalence relation on $X$ then a \textit{Borel selector} $s$ for $E$ is a Borel function $s : X \to X$ with the property that, for every $x, y \in X$, $x E s(x)$, and $x E y$ if and only if $s(x) = s(y)$ (thus $s$ selects, in a Borel manner, a point from each equivalence class). Any subgroup $H$ of a Polish group $G$ induces a canonical equivalence relation on $G$, namely that whose classes are the cosets of $H$; this equivalence relation admits a Borel selector if and only if $H$ is a closed subgroup of $G$ [41, Theorem 12.17]. Suppose that $X$ and $Y$ are standard Borel spaces and $f : X \to Y$ is a Borel function such that $f(X)$ is a Borel subset of $Y$. Then a \textit{Borel section} for $f : X \to Y$ is a Borel function $q : f(X) \to X$ such that $f \circ q$ is the identity of $f(X)$. An important special case is when $X = Y \times Z$ for some standard Borel space $Z$ and $f$ is the first-coordinate projection map. The following lemma is a corollary of the existence of Borel selectors for closed coset equivalence relations.
Lemma 2.3. Let $A, B$ be Polish groups and let $\pi: A \to B$ be a continuous homomorphism. Then $\operatorname{ran}(\pi)$ is a Polishable Borel subgroup of $B$, and $\pi: A \to \operatorname{ran}(\pi)$ has a Borel section.

\textit{Proof.} Observe that $\pi$ induces an injective Borel map $\hat{\pi}: A/\ker(\pi) \to B$ whose image is $\operatorname{ran}(\pi)$. Hence $\operatorname{ran}(\pi)$ is a Borel subgroup of $B$. Let $\rho: A \to A$ be the Borel selector for the coset equivalence relation of $\ker(\pi)$ in $A$. Since $\rho$ is constant on cosets of $\ker(\pi)$, it induces Borel map $\hat{\rho}: A/\ker(\pi) \to A$. Thus $\hat{\rho} \circ (\hat{\pi})^{-1}$ is a Borel section of $\pi: A \to \operatorname{ran}(\pi)$. Finally, since $\ker(\pi)$ is closed in $A$ the group $A/\ker(\pi)$ is Polish in the quotient topology; see [23, Proposition 2.2.1]. Hence, $\operatorname{ran}(\pi)$ admits a Polish topology; namely, the push-forward topology of $A/\ker(\pi)$ under the injective map $\hat{\pi}$. The fact that this new topology induces the same Borel structure on $\operatorname{ran}(\pi)$ follows from the existence of the Borel section. \hfill\Box

A more general version of Lemma 2.3 is the following result, which is a particular instance of [42, Lemma 3.8].

Lemma 2.4. Let $A, B$ be Polish groups, let $A_0 \subseteq A$ and $B_0 \subseteq B$ Polishable subgroups, and let $C$ be a Borel subgroup of $A$. If $f: C \to B$ is a Borel function with $f^{-1}(B_0) = A_0 \cap C$, $f(C) + B_0 = B$, and $f(x) f(y) f(xy)^{-1} \in B_0$ for every $x, y \in C$, then $f$ has a Borel section $g$ which furthermore satisfies $g(x) g(y) g(xy)^{-1} \in A_0$ for every $x, y \in B$.

2.3. \textbf{Definable equivalence relations.} Here we review the basic notions of the Borel complexity theory of classification problems. Formally, a \textit{classification problem} is a pair $(X, E)$ where $X$ is a Polish space and $E$ is an equivalence relation on $X$ which is analytic (and, in many cases, Borel) as a subset of $X \times X$. A \textit{Borel homomorphism} (respectively, a \textit{Borel reduction}) from $(X, E)$ to $(X', E')$ is a function (respectively, an injective function) $X/E \to X'/E'$ induced by some Borel lift $X \to X'$. In the standard reading, the existence of a Borel reduction from $(X, E)$ to $(X', E')$ is tantamount to the assertion that the classification problem $(X, E)$ is at most as hard as the classification problem $(X', E')$. If such a Borel reduction exists, we say that $E$ is \textit{Borel reducible} to $E'$ and write $E \leq_B E'$. If additionally $E' \leq_B E$ then $E$ and $E'$ are \textit{bireducible}; if not, then we write $E <_B E'$. Some of the Borel reductions appearing below will reflect yet stronger relations between classification problems. Following [8, 57, 58] we say that $E$ and $E'$ are \textit{classwise Borel isomorphic} if there is a bijection $f: X/E \to X'/E'$ such that $f$ is induced by a Borel map $X \to X'$, and $f^{-1}$ is induced by a Borel map $X' \to X$. Such an $f$ is called a \textit{classwise Borel isomorphism} from $E$ to $E'$.

Example 2.5. For each $d \in \omega$, let $(\mathcal{R}(d), \simeq_{\text{iso}})$ be the problem of classifying all rank $d$ torsion-free abelian groups up to isomorphism. $\mathcal{R}(d)$ readily identifies with a closed subset of the product $\{0, 1\}^{\mathbb{Q}^d}$ and thereby admits a natural Polish structure. It is not hard then to see that $\simeq_{\text{iso}}$ is a Borel equivalence relation on $\mathcal{R}(d)$. While Baer’s analysis of rank 1 torsion-free abelian groups provides concrete invariants completely classifying $(\mathcal{R}(1), \simeq_{\text{iso}})$, no satisfactory complete invariants are known for $(\mathcal{R}(d), \simeq_{\text{iso}})$ when $d$ is greater than 1. The framework of Borel complexity theory helps to explain this fact: by the cumulative work of several authors [1, 27, 70] we now know that the problems $(\mathcal{R}(d), \simeq_{\text{iso}})$ form a strictly increasing chain in the Borel reducibility order:

$$(\mathcal{R}(1), \simeq_{\text{iso}}) <_B (\mathcal{R}(2), \simeq_{\text{iso}}) <_B (\mathcal{R}(3), \simeq_{\text{iso}}) <_B \cdots$$

Within this framework, a number of complexity classes of equivalence relations are of a sufficient importance to merit separate names. The lower part of this hierarchy is stratified by the following complexity classes:

- \textit{smooth} if it is Borel reducible to the relation $(\mathbb{R}, =_\mathbb{R})$ of equality on the real numbers (or, equivalently, to the relation of equality on any other uncountable Polish space);
- \textit{hyperfinite} if it can be written as an increasing union of equivalence relations with finite classes or, equivalently, if it is Borel reducible to the orbit equivalence relation of a Borel action of the additive group of integers $\mathbb{Z}$ [33, Proposition 1.2]; see also [13, Theorem 5.1];
- \textit{treeable} if it is countable and there exists a Borel relation $R \subseteq X \times X$ such that $(X, R)$ is an acyclic graph, and the connected components of $(X, R)$ are precisely its equivalence classes or, equivalently, it is Borel reducible to the orbit equivalence relation induced by a \textit{free} action of a free countable group [33, Section 3];

1The complete invariants originating from the work of Derry [11], Mal’cev [50], and Kurosch [43] are for all practical purposes as complicated as the original classification problem is; see [22].
• countable if each one of its equivalence classes is countable;
• essentially countable/hyperfinite/treeable, respectively, if it is Borel reducible to an equivalence relation that is countable/hyperfinite/treeable, respectively.

Among the non-smooth Borel equivalence relations, there is a least one up to Borel bireducibility, which is the relation \((\{0,1\}^\omega, E_0)\), of eventual equality of binary sequences: \(xE_0y \iff \exists n \in \omega \forall n > m \; x(n) = y(n)\). This is a hyperfinite Borel equivalence relation. Any other non-smooth hyperfinite Borel equivalence relation is Borel reducible with \(E_0\) [13, Theorem 7.1].

Many classification problems are induced by continuous actions of Polish group actions.

**Definition 2.6.** Corresponding to any Borel action \(G \acts X\) of a Polish group \(G\) on a standard Borel space \(X\) is a classification problem \((X, \mathcal{R}(G \acts X))\), where \(\mathcal{R}(G \acts X)\) is the associated orbit equivalence relation, i.e., the equivalence relation given by \((x,y) \in \mathcal{R}(G \acts X)\) if and only if \(g \cdot x = y\) for some \(g \in G\).

For example, \(E_0\) is simply the orbit equivalence relation of the action of the countable group \(\bigoplus_{n \in \omega} \mathbb{Z}/2\mathbb{Z}\) on \(\prod_{n \in \omega} \mathbb{Z}/2\mathbb{Z}\) by addition and the relation \(\simeq_{iso}\) on \(\mathcal{R}(d)\) is induced by an action of \(GL_d(\mathbb{Q})\) on \(\mathcal{R}(d)\).

**2.4. Categories.** Let \(F: C \to D\) be a functor between categories, and for every two objects \(x, y\) from \(C\) let \(F_{x,y}\) be the associated map from \(\text{Hom}(x,y)\) to \(\text{Hom}(F(x), F(y))\). The functor \(F\) is called full (respectively: faithful, or fully faithful), if \(F_{x,y}\) is surjective (respectively: injective, or bijective) for every \(x, y \in C\). It is called essentially surjective, if for every object \(d \in D\) there is an object \(x \in C\) so that \(F(x)\) and \(d\) are isomorphic in \(D\). The categories \(C\) and \(D\) are equivalent if there are functors \(F: C \to D\) and \(G: D \to C\) so that both \(G \circ F\) and \(F \circ G\) are naturally isomorphic to the corresponding identity functors. Note that \(C\) and \(D\) are equivalent if and only if there is a fully faithful and essentially surjective functor \(F: C \to D\); see [48].

Recall that an \(\textbf{Ab}\)-category is a category such that every hom-set is an abelian group, and such that composition with any given arrow defines a group homomorphism (i.e., composition distributes over addition) [75, Section 1]. A functor between \(\textbf{Ab}\)-categories is additive if it induces group homomorphisms at the level of the hom-sets. A zero object in a category is an object that is both initial and terminal. An additive category is an \(\textbf{Ab}\)-category possessing a zero object and finite products. Abelian Polish groups form an additive category, in which the sum of two continuous group homomorphisms is just their pointwise sum. Here the zero object is the zero group, and the product of a sequence \((G_n)\) of Polish group is the product group \(\prod_n G_n\) endowed with the product topology. Notice that the category of abelian Polish groups is not an abelian category. Indeed, in this category the kernel of a continuous group homomorphism \(\varphi: G \to H\) is \(\{x \in G : \varphi(x) = 0\}\), while its cokernel is the quotient of \(H\) by the closure of the image of \(\varphi\). In an abelian category, an arrow whose kernel and cokernel are both zero must be an isomorphism. This fails in the category of abelian Polish groups, as for instance, if \(i\) is the inclusion of \(\mathbb{Q}\) in \(\mathbb{R}\) then it is trivial kernel and cokernel, but it is not an isomorphism.

3. Groups with a Polish cover

The central objects in all that follows are groups with a Polish cover. These are topological groups augmented with a Polish group extension (i.e., a cover) which serves, in practice, as the formal setting for definability analyses. We describe in this section the definable homomorphisms which together with these objects comprise the category of groups with a Polish cover. We will see in later sections that the aforementioned larger groups or covers arise naturally throughout algebraic topology and homological algebra, and that systematic attention to them can substantially refine many of the classical invariants of both fields.

**Definition 3.1.** A group with Polish cover is a pair \(\mathcal{G} = (N,G)\) where \(G\) is a Polish group and \(N \subseteq G\) is a Polishable normal subgroup. We often represent a group with a Polish cover \(\mathcal{G} = (N,G)\) simply by its quotient \(G/N\).

Let \(\mathcal{G} = (N,G)\) and \(\mathcal{G}' = (N',G')\) be groups with a Polish cover and let \(f: G/N \to G'/N'\) be a group homomorphism. A function \(\hat{f}: G \to G'\) is a lift of \(f\) if \(f(xN) = \hat{f}(x)N'\), for every \(x \in G\). Notice that a lift \(\hat{f}: G \to G'\) of \(f\) is not necessarily a group homomorphism. Necessary and sufficient conditions for a function \(\varphi: G \to G'\) to be a lift of some homomorphism \(G/N \to G'/N'\) are for it to satisfy \(\varphi(N) \subseteq N'\) and \(\varphi(xy)^{-1} \varphi(x) \varphi(y) \in N'\) for all \(x,y \in G\). Indeed, if \(\varphi\) is a lift of a group homomorphism \(f\) then, since \(f\) maps the trivial element to the trivial element, we must have \(\varphi(N) \subseteq N'\). Furthermore, for \(x,y \in G\), \(f(xy) = f(x)f(y)\), whence
Example 3.4. Consider the group with a Polish cover $(\mathbb{Q}, \mathbb{R})$. When $\mathbb{R}/\mathbb{Q}$ is viewed as an abstract group then it is isomorphic to a $\mathbb{Q}$-vector space of dimension $2^{2^{\aleph_0}}$. Hence there are exactly $2^{2^{\aleph_0}}$ abstract group homomorphisms from $\mathbb{R}/\mathbb{Q}$ to $\mathbb{R}/\mathbb{Q}$. Of course, the majority of these homomorphisms are not “definable” by any concrete formula but exist, rather, as essentially abstract effects of the axiom of choice. One may also view $\mathbb{R}/\mathbb{Q}$ as a topological group with the quotient topology induced from $\mathbb{R}$. This is a decidedly uninformative topology, since its only open sets are $\emptyset$ and $\mathbb{R}$. In contrast, viewing $\mathbb{R}/\mathbb{Q}$ as a group with a Polish cover affords us access to the “definable” subsets of $\mathbb{R}/\mathbb{Q}$: these are precisely the $\mathbb{Q}$-invariant Borel subsets of $\mathbb{R}$. Note that since $\mathbb{Q}$ is dense in $\mathbb{R}$, such sets are always meager or comeager in $\mathbb{R}$; see [23, Proposition 6.1.9]. Moreover, the definable homomorphisms from $\mathbb{R}/\mathbb{Q}$ to $\mathbb{R}/\mathbb{Q}$ turn out to be exactly those which are induced by maps from $\mathbb{R}$ to $\mathbb{R}$ of the form $x \mapsto c \cdot x$ for some $c \in \mathbb{R}$; see [37].

Example 3.5. Suppose that $\mathcal{I}$ and $\mathcal{J}$ are Polishable ideals of $\mathcal{P}(\omega)$; for definitions, see [67]. The symmetric difference operation $\triangle$ then endows $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{P}(\omega)$ with Polish abelian group structures. Furthermore, $\mathcal{P}(\omega)/\mathcal{I}$ and $\mathcal{P}(\omega)/\mathcal{J}$ are then groups with Polish cover, and an injective definable homomorphism from $\mathcal{P}(\omega)/\mathcal{I}$ to $\mathcal{P}(\omega)/\mathcal{J}$ is the same as a Borel $(\mathcal{I}, \mathcal{J})$-approximate $\triangle$-homomorphism in the sense of [36, Section 3.2].

We identify any Polish group $G$ with its corresponding group with a trivial Polish cover $G = G/N$ where $N$ denotes its trivial subgroup. Since, as noted above, every definable homomorphism between Polish groups is a topological homomorphism this identification realizes the category of Polish groups as a full subcategory of the category of groups with Polish cover. To each group with Polish cover $G$ we assign the following two groups with Polish cover $G^\omega$, $G^\infty$ which measure in some sense how far $G/N$ is from being Polish:

- **The weak group** of $G$ is the Polish group $G^\omega := G/\overline{N}$, where $\overline{N}$ is the closure of $N$ inside $G$.
- **The asymptotic group** of $G$ is the group with a Polish cover $G^\infty := \overline{N}/N$.

Clearly the inclusion map $\overline{N}/N \rightarrow G/N$ is an injective topological homomorphism $G^\infty \rightarrow G$, while the quotient map $G/N \rightarrow G/\overline{N}$ is a surjective topological homomorphism $G \rightarrow G^\omega$.

Lemma 3.6. The assignments $G \mapsto G^\infty$ and $G \mapsto G^\omega$ are functors from the topological category of homomorphisms between groups with a Polish cover to itself and $G^\omega$ is always a Polish group.

Proof. By definition, a topological homomorphism $f : G/N \rightarrow G'/N'$ admits a lift to a continuous homomorphism $\hat{f} : G \rightarrow G'$ such that $\hat{f}(N) \subseteq N'$. Since $\hat{f}$ is continuous, it follows that $\hat{f}(\overline{N}) \subseteq \overline{N}'$, which in turn implies that $\hat{f}$ induces a topological homomorphism $G^\infty \rightarrow (G')^\infty$ and a continuous homomorphism $G^\omega \rightarrow (G')^\omega$. The last claim follows from [23, Proposition 2.2.10].

To each group with a Polish cover one may assign a classification problem in the sense of Section 2.3:
Definition 3.7. To each group with a Polish cover \( G = G/N \) we associate the classification problem \((G, R(G))\), where \( R(G) \) is the coset equivalence relation of \( N \) inside \( G \): for \( x, y \in G \) set \((x, y) \in R(G)\) if and only if \( Nx = Ny \).

Notice that \( R(G) \) may be viewed as the orbit equivalence relation \( R(N \rtimes G) \) of the action of the Polish group \( N \) on \( G \) by translation \((h, g) \mapsto h \cdot g\). It turns out that, at least for abelian groups, a necessary condition for the coset equivalence relation of a Borel subgroup \( N \) of some Polish group \( G \) to be Borel reducible to an orbit equivalence relation \( R(H \rtimes Y) \) of a continuous Polish group action \( H \rtimes Y \), is that \( N \) is Polishable; see [68]. In analogy with the terminology from Section 2.3, we say that a group with a Polish cover \( G = G/N \) is smooth if \( N \) is a closed subgroup of \( G \), in which case the quotient topology renders \( G/N \) a Polish group. Notice that if \( G = (N, G) \) is smooth then it is definably isomorphic to the group with the trivial Polish cover \((1_{G/N}, G/N)\). In fact, the following is true:

Lemma 3.8. If \( g \) is a definable isomorphism between the groups with a Polish cover \( G = (N, G) \) and \( G' = (N', G') \) and either \( G/N \) or \( G'/N' \) is a smooth group with a Polish cover, then so is the other.

Proof. It will suffice by symmetry to assume that \( G' \) is a smooth group with a Polish cover. We may further assume that \( G' \) is the trivial Polish cover, i.e., \( N' \) is the trivial subgroup \( 1_{G'} \) of \( G' \). By assumption, there is a group isomorphism \( g : G/N \to G' \) and a Borel function \( \hat{g} : G \to G' \) so that \( g \circ \pi = \hat{g} \), where \( \pi : G \to G/N \) is the quotient map. So \( \hat{g} \) is a homomorphism. By Corollary 2.2, \( \hat{g} \) is a continuous homomorphism and therefore \( N = \ker(\hat{g}) \) is closed.

Notice that the category of groups with a Polish cover admits an object that is both initial and terminal; namely, the identity group with a Polish cover \( G/N \), where both \( G \) and \( N \) are trivial. It is moreover closed under countable products. The product of any given countable collection \( G_n/N_n \) of groups with a Polish cover is the group with a Polish cover \( G/N \), where

\[
N = \prod_{n \in \omega} N_n \subseteq \prod_{n \in \omega} G_n = G,
\]

By an abelian group with a Polish cover we mean a group with a Polish cover \( G = (N, G) \) so that \( G/N \) is an abelian group. Note that this does not necessarily imply that \( G \) is abelian. For example, if \( A \) is a separable stable continuous-trace C*-algebra, \( \text{Aut}(A) \) is the group of automorphisms of \( A \), and \( \text{Inn}(A) \) is the Polishable subgroup of \( \text{Aut}(A) \) consisting of automorphisms of \( A \) that are inner, then \( \text{Aut}(A) \) is in general not commutative. However, the group with a Polish cover \( \text{Out}(A) := \text{Aut}(A)/\text{Inn}(A) \) is abelian, being isomorphic to the second Čech cohomology group with integer coefficients of the spectrum of \( A \) by the main theorem of [60].

Suppose that \( G/N \) and \( G'/N' \) are abelian (additively denoted) groups with a Polish cover, where \( G \) and \( G' \) are additively denoted as well (but not necessarily abelian). Suppose that \( f, g : G/N \to G'/N' \) are definable homomorphisms. Then \( f + g : G/N \to G'/N' \) is the definable homomorphism defined by \( (f + g)(x) = f(x) + g(x) \). Notice that \( f + g \) is indeed a definable homomorphism. If \( f \) and \( \hat{g} \) are Borel lifts of \( f \) and \( g \), respectively, then the function \( G \to G', x \mapsto f(x) + g(x) \) is a Borel lift of \( f + g \). This turns the Hom\((G/N, G'/N')\)-set of all definable homomorphisms from \( G/N \) to \( G'/N' \) into an abelian group. Hence, the category of abelian groups with a Polish cover is an additive category. So too is its subcategory the topological category of abelian groups with a Polish cover.

4. Ulam stability of non-Archimedean abelian groups

In this section we prove several Ulam stability results for definable homomorphisms \( f : G/N \to G'/N' \) between groups with a Polish cover, where \( N \) is a dense subgroup of a non-Archimedean abelian Polish group. Our main result is Theorem 4.3: it handles cases in which \( N' \) is countable. In Theorem 4.5 we attain similar results under either of the weaker assumptions that \( N' \) is locally profinite in its Polish topology, or that \( N' \) is non-Archimedean in its Polish topology. In all this section’s paragraphs except the next, we adopt an additive notation for group operations.

Ulam stability phenomena were first considered in [54, 73] in the context of homomorphisms between metric groups. A map \( f : G \to G \) from a metric group \((G, d)\) to itself is an \( \varepsilon \)-automorphism if \( d(f(x) \cdot f(y), f(x \cdot y)) < \varepsilon \) for all \( x, y \in G \). Ulam asked for which such metric groups \((G, d)\) and \( \varepsilon > 0 \) is there some \( \delta > 0 \) so that each \( \varepsilon \)-automorphism is \( \delta \)-close to an actual homomorphism. More generally, Ulam stability phenomena may be considered in any context where one has notions of morphisms and approximate morphisms, as well as a way to measure how far
apart a given morphism and a given approximate morphism are from each other, as described in our introduction. The series [37–39] may be regarded as an exploration of Ulam stability in the setting of groups with a Polish cover. As mentioned, for example, in [37] it is shown that if $N, N'$ are subgroups of the Polish group $(\mathbb{R}, +)$ and $N'$ is countable, then every definable homomorphism $f: \mathbb{R}/N \to \mathbb{R}/N'$ lifts to a map of the form $x \mapsto c \cdot x$. In this example, a *morphism* is any continuous group homomorphism $\psi: \mathbb{R} \to \mathbb{R}$ with $\psi(N) \subseteq N'$, an *approximate morphism* is any Borel map $\varphi: \mathbb{R} \to \mathbb{R}$ that is a lift of a (definable) group homomorphism $f: \mathbb{R}/N \to \mathbb{R}/N'$, and $\varphi, \psi: \mathbb{R} \to \mathbb{R}$ are considered to be close to each other if they are lifts of the same group homomorphism. The question of whether similar rigidity phenomena hold in the case of the $p$-adic groups is posed in [39, Section 8]. Theorem 4.3 below settles the more general case where the cover $G$ of $G/N$ is an arbitrary non-Archimedean abelian Polish group. Some implications of this theorem for the case of the $p$-adic groups are discussed separately in Section 6.

We begin by describing a framework for studying Ulam stability phenomena surrounding definable homomorphisms between arbitrary groups with a Polish cover. Let $G/N$ be a group with a Polish cover. An *essential retract* of $G/N$ is a clopen subgroup $H$ of $G$ with the property that $H$ intersects every $N$-coset in $G$. Notice that $H/(H \cap N)$ is a group with a Polish cover and the map $i^*: H/(H \cap N) \to G/N$, induced by the inclusion $i: H \to G$, is a definable isomorphism. Essential retracts satisfy the following extension property, showing that an essential retract of $G/N$ is ‘essentially’ the same group with a Polish cover as $G/N$ up to an isomorphism induced by the inclusion map.

**Lemma 4.1.** Let $H$ be an essential retract of the group with a Polish cover $G/N$. Then the definable isomorphism $(i^*)^{-1}: G/N \to H/(H \cap N)$ lifts to a Borel map $r: G \to H$ with $r \circ i = \text{id}_H$. In particular, if $\hat{f}$ is a Borel lift of a definable homomorphism $f: H/(H \cap N) \to G'/N'$ then $\hat{f}$ extends to a Borel lift of $f \circ (i^*)^{-1}: G/N \to G'/N'$.

**Proof.** The Effros (standard) Borel space $\mathcal{F}(X)$ of closed subspaces of a Polish space $X$ has the closed subsets of $X$ as points, and it is endowed with the Borel structure generated by the families $\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\}$, where $U$ varies among the open subsets of $X$ [41, Section 12.C]. Notice that, for every $g \in G$, $N \cap g^{-1}H$ is a closed subset of $N$ with respect to its Polish group topology. Furthermore, the assignment $H \mapsto \mathcal{F}(N)$, $x \mapsto N \cap x^{-1}H$ is Borel. Indeed, let $\{W_n : n \in \omega\}$ be a basis of open subsets of $N$. We use the notation for the Vaught transform in reference to the continuous right action of $N$ on $G$ by right translation [41, Section 16.B]. If $U$ is an open subset of $N$, then $N \cap x^{-1}H \cap U$ is nonempty if and only if there exists $n \in \omega$ such that $W_n \subseteq x^{-1}H$. For $n \in \omega$, we have that, since $H$ is closed in $G$, $W_n \subseteq x^{-1}H \Leftrightarrow xW_n \subseteq H \Leftrightarrow \forall^* w \in W_n, xw \in H \Leftrightarrow x \in H^{*W_n}$. This concludes the proof that the assignment $H \mapsto \mathcal{F}(N)$, $x \mapsto N \cap x^{-1}H$ is Borel. The Kuratowski–Ryll-Nardzewski theorem [41, Theorem 12.13] implies that there is a Borel function $\sigma: G \to N$ such that $\sigma(g) \in N \cap g^{-1}H$ for every $g \in G$ and $\sigma(g) = g$ for all $g \in H$. Define then $r(g) := g\sigma(g)$ for every $g \in G$. Finally, notice that $\hat{f} \circ r$ is the desired lift of $f \circ (i^*)^{-1}$. \hfill $\square$

**Definition 4.2.** Let $f: G/N \to G'/N'$ be a definable homomorphism between groups with a Polish cover. We say that $f$ is *trivial* if it admits a Borel lift $\hat{f}: G \to G'$ whose restriction $(\hat{f} \mid H): H \to G'$ is a continuous group homomorphism on some essential retract $H$ of $G/N$. We say that a group with a Polish cover $G/N$ is *Ulam stable* or *rigid*, if every definable automorphism $\varphi: G/N \to G/N$ is trivial.

Notice that when $G$ is connected, then $H = G$ for any clopen non-empty $H \subseteq G$. In consequence, in the aforementioned Ulam stability results from [37] a lift of $f: \mathbb{R}/N \to \mathbb{R}/N'$ is trivial precisely when it is of the form $x \mapsto c \cdot x$. Indeed, the framework that we develop here subsumes other notions of rigidity previously considered in the literature [37–39]. The following theorem is the main result of this section. It can be viewed as an Ulam stability result in which the *approximate morphisms* are Borel lifts $G \to G'$ of some definable homomorphism $G/N \to G'/N'$ and the *morphisms* are Borel lifts which are continuous homomorphisms after passing to an essential retract.

**Theorem 4.3.** Let $f: G/N \to G'/N'$ be a definable homomorphism between groups with a Polish cover where $G$ is a non-Archimedean abelian Polish group and $N$ is dense in $G$. If $N'$ is countable, then $f$ is trivial.

The following is an immediate consequence of Theorem 4.3.

**Corollary 4.4.** If $N$ is a countable dense subgroup of a non-Archimedean abelian Polish group $G$ then the group with a Polish cover $G/N$ is Ulam stable, namely every definable automorphism of $G/N$ has a lift that is a continuous group homomorphism on some open subgroup of $G$. 


The arguments used in the proof of Theorem 4.3 can be adjusted to prove weaker rigidity results under the weaker assumption that $N'$ is locally profinite in its Polish topology, or that $N'$ is non-Archimedean in its Polish topology. In both cases, $N'$ has a basis of neighborhoods of the identity consisting of open subgroups. Let $f : G/N \to G'/N'$ be a definable homomorphism between groups with a Polish cover. We say that $f$ is approximately trivial if it admits a Borel lift $\hat{f} : G \to G'$ with the property that for every open subgroup $V$ of $N'$ (in its Polish topology) there is an essential retract $H$ of $G/N$ so that $(\hat{f} \upharpoonright H)$ is continuous and $\hat{f}(x + y) - \hat{f}(x) - \hat{f}(y) \in V$, for all $x, y \in H$. We say that $f$ is approximately generically trivial if for every $V$ as above, the same statement holds, but only for a comeager collection of pairs $(x, y) \in H \times H$.

**Theorem 4.5.** Let $f : G/N \to G'/N'$ be a definable homomorphism between groups with a Polish cover where $G$ is a non-Archimedean abelian Polish group and $N$ is dense in $G$.

1. If $N'$ is locally profinite in its Polish topology, then $f$ is approximately trivial.
2. If $N'$ is non-Archimedean in its Polish topology, then $f$ is approximately generically trivial.

The proofs of Theorem 4.3 and Theorem 4.5 will span the rest of this section. We start by showing that whenever $G$ is a non-Archimedean abelian Polish group and $N$ is a Polishable subgroup then any definable homomorphism from $G/N$ to any group with a Polish cover has a continuous lift.

Recall that every non-Archimedean abelian group $G$ admits a compatible complete invariant metric $d$ with $d \leq 1$. For every $x \in G$ we denote by $B(x, \varepsilon)$ the corresponding ball of center $x$ and radius $\varepsilon$. A map $\gamma : G \to G$ is $1$-Lipschitz if $d(\gamma(x), \gamma(y)) \leq d(x, y)$ for all $x, y \in G$.

**Proposition 4.6.** Let $f : G/N \to G'/N'$ be a definable homomorphism between two groups with a Polish cover. If $G$ is a non-Archimedean abelian Polish Group then $f$ lift to a continuous function $\varphi : G \to G'$.

**Proof.** Let $\psi : G \to G'$ be a Borel function that is a lift of $f$. Since $\psi$ is Borel, there exists a decreasing sequence $(\mathcal{U}_i)_{i \in \omega}$ of dense open subsets of $G$ such that the restriction of $\psi$ to $U := \bigcap_{i \in \omega} \mathcal{U}_i$ is continuous [23, Section 2.3]. Fix a compatible invariant ultrametric $d$ on $G$ with $d \leq 1$. We will define $1$-Lipschitz functions $\gamma_n : G \to G$ and functions $\rho_n : G \to (0, 1]$ so that:

1. $\gamma_n(x) \in B(0, \rho_n(x))$ and $\rho_{n+1}(x) \leq \rho_n(x) \leq 2^{-n}$, for all $n \in \mathbb{N}$ and all $x \in G$;
2. $B(x + \gamma_0(x) + \cdots + \gamma_n(x), \rho_n(x)) \subseteq \mathcal{U}_n$, for all $n \in \mathbb{N}$ and all $x \in G$;
3. $B(-\gamma_0(x) - \cdots - \gamma_n(x), \rho_n(x)) \subseteq \mathcal{U}_n$, for all $n \in \mathbb{N}$ and all $x \in G$.

Assuming now that we have defined the sequence $(\gamma_n)$ so that it satisfies the above. By (1), the sequences $$(x + \gamma_0(x) + \cdots + \gamma_n(x))_{n \in \omega} \quad \text{and} \quad (-(\gamma_0(x) + \cdots + \gamma_n(x)))_{n \in \omega}$$ converge for every $x \in G$. We can therefore define the maps $f_0, f_1 : G \to G$ by setting:

$$f_0(x) := \lim_{n \to \infty} (x + \gamma_0(x) + \cdots + \gamma_n(x)) \quad \text{and} \quad f_1(x) := \lim_{n \to \infty} (-(\gamma_0(x) + \cdots + \gamma_n(x))).$$

Notice that the space of all $1$-Lipschitz maps is closed under taking pointwise limits. Since $d$ is an ultrametric, it is also closed under finite sums. It follows that the functions $f_0$ and $f_1$ are $1$-Lipschitz and hence continuous.

Fix $n \in \omega$. For $k \geq n$ we have that
$$\gamma_{n+1}(x) + \cdots + \gamma_k(x) \in B(0, \rho_n(x))$$
and hence
$$x + \gamma_0(x) + \cdots + \gamma_k(x) \in B(x + \gamma_0(x) + \cdots + \gamma_n(x), \rho_n(x))$$
and
$$-(\gamma_0(x) + \cdots + \gamma_k(x)) \in B(- (\gamma_0(x) + \cdots + \gamma_n(x)), \rho_n(x))$$
As this holds for every $k \geq n$, we have
$$f_0(x) \in B(x + \gamma_0(x) + \cdots + \gamma_n(x), \rho_n(x)) \subseteq \mathcal{U}_n$$
and
$$f_1(x) \in B(-(\gamma_0(x) + \cdots + \gamma_n(x)), \rho_n(x)) \subseteq \mathcal{U}_n.$$
\[ f_1(x) \in \cap_{n \in \omega} U_n = U. \]

Since \( \psi|_U : U \to G' \) is continuous, by (2), (3) we then have that the map \( \psi : G \to G' \) defined
\[
\varphi(x) := \psi(f_0(x)) + \psi(f_1(x)).
\]
is continuous. Since \( x = f_0(x) + f_1(x) \) for every \( x \in G \) we have that \( \varphi \) is the desired map. Indeed:
\[
\varphi(x) + N' = \psi(f_0(x)) + \psi(f_1(x)) + N' = \psi(f_0(x) + f_1(x)) + N' = \psi(x) + N'.
\]

To conclude the proof, we need to define a sequence \((\gamma_n)\) of 1-Lipschitz maps from \( G \) to itself which satisfy properties (1), (2), (3) above. Set \( \gamma_1 : G \to G \) be the map that is constantly 0, set \( \rho_1 : G \to (0, 1] \) be the map that is constantly equal to 1, and set \( A_{-1} = \{ G \} \). By recursion we define for each \( n \in \omega \) a 1-Lipschitz map \( \gamma_n : G \to G \), a function \( \rho_n : G \to (0, 1] \) and a partition \( A \) of \( G \) into \( d \)-balls, so that for all \( n \in \omega \) we have that
- \( A_n \) is a refinement of \( A_{n-1} \);
- \( \rho_n \) and \( \gamma_n \) are both constant on elements of \( A_n \); and
- for every \( x \in G \), \( \rho_n(x) \leq \min \{ 2^{-n}, \rho_{n-1}(x) \} \), \( \gamma_n(x) \in B_0(0, \rho_n(x)) \), and the entire balls \( B(x + \gamma_0(x) + \cdots + \gamma_n(x), \rho_n(x)) \) and \( B(-\gamma_0(x) + \cdots + \gamma_n(x), \rho_n(x)) \) are subsets of \( U_n \).

Assume that for some \( n \in \omega \) we have defined \( A_0, \ldots, A_{n-1}, \gamma_0, \ldots, \gamma_{n-1}, \rho_0, \ldots, \rho_{n-1} \) as above. We attain \((A_n, \gamma_n, \rho_n)\) in the form of \((A', \gamma', \rho')\) by invoking the next lemma with \((A, \gamma, \rho) := (A_{n-1}, \gamma_{n-1}, \rho_{n-1})\).

**Lemma 4.7.** Let \( G \) be a non-Archimedean abelian Polish group and let \( d \) be a compatible invariant ultrametric on \( G \) with values in \([0, 1]\). Let \( A \) be a partition of \( G \) into \( d \)-balls and assume that \( \gamma : G \to G \) and \( \rho : G \to (0, 1] \) are maps such that \( \gamma \) is 1-Lipschitz and both \( \gamma \) and \( \rho \) are constant on elements of \( A \). Let \( U \) be a dense open subset of \( G \) and fix \( r > 0 \). Then there exist:
- a partition \( A' \) of \( G \) into balls that refines \( A \),
- a function \( \gamma' : G \to G \), and
- a function \( \rho' : G \to (0, 1] \)
which together satisfy the following:
1. \( \gamma'(x) \in B_0(0, \rho(x)) \) for \( x \in G \);
2. \( \rho'(x) < \min \{ \rho(x), r \} \) for \( x \in G \);
3. \( \gamma' \) is 1-Lipschitz;
4. \( \gamma' \) and \( \rho' \) are constant on elements of \( A' \);
5. for every \( x \in G \), the balls \( B(x + \gamma(x) + \gamma'(x), \rho'(x)) \) and \( B(-\gamma(x) + \gamma'(x), \rho'(x)) \) are subsets of \( U \).

**Proof.** Without loss of generality we may assume that \( \rho(x) \) is less than or equal to the diameter of that ball \( A \in A \) which contains \( x \). Define \( C \) to be the partition of \( G \) consisting of balls of the form \( B(x, \rho(x)) \) for \( x \in G \). By our assumptions on \( \rho(x) \), the partition \( C \) refines \( A \). Notice that if \( C \) is the element \( B(x, \rho(x)) \) of the partition \( C \) then
\[
\gamma(x) + C + B(-\gamma(x), \rho(x)) = C.
\]
For each such \( C \in C \), since \( \rho \) and \( \gamma \) are constant on elements of \( A \) and hence on elements of \( C \), we may choose \( R_C < \rho(x) \) small enough that \( U \cap (\gamma(x) + C) \) contains a ball of radius \( R_C \). Define then \( A' \) to be the partition of \( G \) consisting of balls \( B(x, R_C) \) for \( x \in C \), and \( C \in C \). Since \( R_C < \rho(x) \) for any \( x \in C \) and \( C \in C \), the partition \( A' \) refines \( C \), and it therefore refines \( A \) as well.

Fix \( B = B(x, R_C) \in A' \) for \( x \in C \) with \( C \in C \), and \( A \in A \) with \( B \subseteq A \). Since \( U \) is open dense, we have that
\[
(U - (x + \gamma(x))) \cap (U + \gamma(x))
\]
is open dense. Thus there exists a \( g_B \) such that
\[
g_B \in (U - (x + \gamma(x))) \cap (U + \gamma(x)) \cap B(0, R_C),
\]
and hence
\[
x + \gamma(x) + g_B \in U,
\]
satisfy (1), (2), (4), (5). It remains to prove that \( \gamma \) is 1-Lipschitz as well. It follows that if \( x \) is abelian and non-Archimedean, then \( g_B \in B(0, R_C) \subseteq A \).

Since \( U \) is open, there also exists an \( R_B < \min\{\rho(A), r\} \) such that
\[
B(x + \gamma(x) + g_B, R_B) \subseteq U
\]
and
\[
B(-(\gamma(x) + g_B), R_B) \subseteq U.
\]
Define then
\[
\gamma'(x) := g_B
\]
and
\[
\rho'(x) := R_B
\]
for each \( x \in B \) and \( B \in A' \). In particular, \( \gamma' \) and \( \rho' \) are constant on elements of \( A' \); the terms in question in fact satisfy (1), (2), (4), (5). It remains to prove that \( \gamma' \) is 1-Lipschitz.

Suppose now that \( x \) and \( y \) are elements of \( G \). Since \( d \) is an ultrametric and \( \gamma \) is 1-Lipschitz, the map \( x \mapsto x + \gamma(x) \) is 1-Lipschitz as well. It follows that if \( x, y \in G \) and \( C, D \in C \) are such that \( x \in C \) and \( y \in D \), then
\[
d(\gamma'(x), \gamma'(y))
\]
\[
= d(x + \gamma(x) + \gamma'(x), x + \gamma(x) + \gamma'(y))
\]
\[
\leq \max\{d(x + \gamma(x) + \gamma'(x), y + \gamma(x)), d(y + \gamma(x), x + \gamma(x) + \gamma'(y))\}
\]
\[
= \max\{d(\gamma'(x), 0), d(x + \gamma(x), y + \gamma(y)), d(\gamma'(y), 0), d(y + \gamma(y), x + \gamma(x))\}
\]
\[
\leq \max\{R_C, R_D, d(x, y)\}.
\]
If \( d(x, y) < \min\{R_C, R_D\} \) then \( C = D \) and \( \gamma'(x) = \gamma'(y) \). Hence, by the above inequality \( \gamma' \) is 1-Lipschitz.

The following is the second ingredient that goes into the proof of Theorem 4.3.

**Lemma 4.8.** Let \( f : G/N \to G'/N' \) be a homomorphism between groups with a Polish cover, which lifts to a continuous map \( G \to G' \). If \( G \) is abelian and non-Archimedean, \( N \) is dense in \( G \), and \( N' \) is a countable, then \( f \) is trivial.

**Proof.** Let \( \varphi : G \to G \) be the continuous lift of \( f \). Then the function \( C : G \times G \to N' \), \( (x, y) \mapsto \varphi(x+y) - \varphi(x) - \varphi(y) \) is continuous. By the Baire Category Theorem, there exists an \( m \in N' \) such that \( A := \{(x, y) \in G \times G : C(x, y) = m\} \) has nonempty interior. As \( N \) is dense in \( G \) and \( G \) has a basis of neighborhoods of the identity consisting of clopen subgroups, there exists an \( (x_0, y_0) \in N \times N \) and clopen subgroup \( H \) of \( G \) such that \( (H \times H) + (x_0, y_0) \subseteq A \). This implies that \( \varphi(x + y + x_0 + y_0) = \varphi(x + x_0) + \varphi(y + y_0) + m \) for every \( x, y \in H \).

Notice that the function \( \varphi' : G \to G' \), \( x \mapsto \varphi(x) + m \) satisfies the same assumptions as \( \varphi' \), and furthermore that \( \varphi'(x) - \varphi(x) \in N' \) for every \( x \in G \). Notice also that for \( x, y \in G \),
\[
\varphi'(x + y + x_0 + y_0) = \varphi(x + y + x_0 + y_0) + m
\]
\[
= \varphi(x + x_0) + \varphi(y + y_0) + 2m
\]
\[
= (\varphi(x + x_0) + m) + (\varphi(y + y_0) + m)
\]
\[
= \varphi'(x + x_0) + \varphi'(y + y_0).
\]
Therefore, by replacing \( \varphi \) with \( \varphi' \) if necessary, we are free to assume that for every \( x, y \in H \),
\[
\varphi(x_0 + y_0 + x + y) = \varphi(x_0 + x) + \varphi(y_0 + y).
\]
Define now
\[
\psi(z) := \varphi(x_0 + y_0 + z) - \varphi(x_0 + y_0).
\]
Then we have that
\[
\psi(z) - \varphi(z) \in N'
\]
for all $z$ in $H$. Furthermore, for $x, y \in H$,
\[ \psi(x) + \psi(y) = (\varphi(x_0 + y_0 + x) - \varphi(x_0 + y_0)) + (\varphi(x_0 + y_0 + y) - \varphi(x_0 + y_0)) \]
\[ = \varphi(x + x_0) + \varphi(y_0) - \varphi(x_0 + y_0) + \varphi(y + y_0) + \varphi(x_0) - \varphi(x_0 + y_0) \]
\[ = \varphi(x + x_0) + \varphi(y + y_0) - \varphi(x_0 + y_0) \]
\[ = \varphi(x + y + x_0 + y_0) - \varphi(x_0 + y_0) \]
\[ = \psi(x + y). \]

This concludes the proof. \(\square\)

We may now conclude the proof of Theorem 4.3.

**Proof of Theorem 4.3.** By Proposition 4.6 $f$ admits a continuous lift. The rest follows from Lemma 4.8. \(\square\)

For the proof of Theorem 4.5, we need the following variant of Lemma 4.8. Recall that the notation $\forall^x x \in X$ stands for “there is a comeager $C \subseteq X$ so that $\forall x \in C$”.

**Lemma 4.9.** Let $\varphi: G \to G'$ be a continuous map that is a lift of a homomorphism $f : G/N \to G'/N'$ between groups with a Polish cover. If $G$ is abelian and non-Archimedean, $N$ is dense in $G$, and $N'$ is non-Archimedean in its Polish topology then for every clopen subgroup $V$ of $N'$ there exist:

- a clopen subgroup $H$ of $G$;
- an element $m$ of $N'$;
- $x_0, y_0 \in G$;

so that if $\psi : G \to G'$ is given by $\psi(z) = \varphi_0(x_0 + y_0 + z) - \varphi(x_0 + y_0) - m$, then $\forall^x x \in H \forall^y y \in H$, we have:
\[ \psi(x + y) - \psi(x) - \psi(y) \in V. \]

Moreover, if $N'$ is additionally locally profinite then the last identity holds for all $x$ and $y$ in $H$.

**Proof.** The proof is essentially identical to the proof of Lemma 4.8. We define $C : G \to N'/V$ by $C(x, y) := \psi(x + y) - \psi(x) - \psi(y) + V$. Since $N'/V$ is countable, the Baire Category Theorem implies that there is $m \in N'$ so that $A = \{(x, y) \in G \times G : C(x, y) = m + V\}$ is non-meager. Since $A$ is Borel, we may find $x_0, y_0 \in G$ and a clopen subgroup $H$ of $G$ so that $\forall^x x \in H \forall^y y \in H (x + x_0, y + y_0) \in A$. The rest of the argument is the same.

If $N'$ was additionally locally profinite then we can always arrange so that $V$ is compact in the Polish topology of $N'$. But this implies that $V$ is a closed subgroup of $G'$. By continuity of $\varphi$ this implies that $A$ is closed. Hence, we may find $x_0, y_0 \in G$ and a clopen subgroup $H$ of $G$ so that $\forall x \in H \forall y \in H (x + x_0, y + y_0) \in A$. \(\square\)

**Proof of Theorem 4.5.** We record the argument for (1). The proof of (2) is similar and is left to the reader.

By Proposition 4.6, we may fix some continuous lift $\varphi : G \to G'$ of $f$. Let $(V_k)_{k \in \omega}$ be a decreasing sequence of compact open subgroups of $N'$ that forms a basis of neighborhoods of the identity, and let $(W_k)_{k \in \omega}$ be a decreasing sequence of open subgroups of $G$ that forms a basis of neighborhoods of the identity.

Set $H_{-1} = G, \varphi_{-1} = \varphi, V_{-1} = N'$. Applying Lemma 4.9, we define by recursion on $k \in \omega$:

- an open subgroup $H_k$ of $G$ such that $H_k \subseteq W_k$;
- elements $x_k, y_k \in H_k$;
- elements $m_k \in V_k$;
- a continuous function $\varphi_k : G \to G'$,

such that, for every $k \in \omega$ one has that:

1. for every $x, y \in H_{k-1}$,
\[ \varphi_{k-1}(x + y + x_k + y_k) + V_k = \varphi_{k-1}(x + x_k) + \varphi(y + y_k) + m_k + V_k \]

2. for every $z \in G$,
\[ \varphi_k(z) = \varphi_{k-1}(x_k + y_k + z) - \varphi_{k-1}(x_k + y_k) - m_k, \]

3. for every $x, y \in H_k$,
\[ \varphi_k(x + y) - \varphi_k(x) - \varphi_k(y) \in V_k. \]
Define now, for every $k \in \omega$, 
\[ z_k := (x_0 + y_0) + \cdots + (x_k + y_k). \]

We prove by induction on $k \in \omega$ that, for every $z \in G$,
\[ \varphi_k (z) = \varphi (z_k + z) - \varphi (z_k) - m_k. \]

For $k = 0$ this holds by definition of $\varphi_0$. Indeed, (2) we have that
\[ \varphi_0 (z) = \varphi (x_0 + y_0 + z) - \varphi (x_0 + y_0) - m_0 = \varphi (z_0 + z) - \varphi (z_0) - m_0. \]

Suppose that the conclusion holds for $k - 1$. Then we have that, by the induction hypothesis,
\[ \varphi_{k-1} (x_k + y_k + z) = \varphi (z_{k-1} + x_k + y_k + z) - \varphi (z_{k-1}) - m_{k-1} \]
and
\[ \varphi_{k-1} (x_k + y_k) = \varphi (z_{k-1} + x_k + y_k) - \varphi (z_{k-1}) - m_{k-1} \]

Therefore, by definition
\[ \varphi_k (z) = \varphi_{k-1} (x_k + y_k + z) - \varphi_{k-1} (x_k + y_k) - m_k = \varphi (z_{k-1} + x_k + y_k + z) - \varphi (z_{k-1} + x_k + y_k) - m_k = \varphi (z_k + z) - \varphi (z_k) - m_k. \]

Since $H_k \subseteq W_k$ for every $k \in \omega$ and $(W_k)$ is a basis of neighborhoods of the identity, we have that the sequence $(x_k + y_k)_{k \in \omega}$ converges to 0, and hence the sequence $(z_k)_{k \in \omega}$ converges to some element $z_\infty$ of $G$.

Now, we claim that, for every $k \in \omega$, $x, y \in H_k$, and $i \geq k$,
\[ \varphi_i (x + y) - \varphi_i (x) - \varphi_i (y) \in V_k. \]

We prove this by induction on $i \geq k$. For $k = i$ this holds for (3). Suppose it holds for $i - 1 \geq k$. Then we have that, for $x, y \in H_k$,
\[ \varphi_i (x + y) = \varphi_{i-1} (x_i + y_i + x + y) - \varphi_{i-1} (x_i + y_i) - m_i, \]
\[ \varphi_i (x) = \varphi_{i-1} (x_i + y_i + x) - \varphi_{i-1} (x_k + y_k) - m_i, \]
\[ \varphi_i (y) = \varphi_{i-1} (x_i + y_i + y) - \varphi_{i-1} (x_k + y_k) - m_i. \]

Considering that $x_i, y_i \in H_{i-1} \subseteq H_k$ and $m_i \in V_{i-1} \subseteq V_k$, we have that by the inductive hypothesis
\[ \varphi_i (x + y) + V_k = \varphi_{i-1} (x_i + y_i + x + y) - \varphi_{i-1} (x_i + y_i) - m_i + V_k = \varphi_{i-1} (x_i + y_i + x) - \varphi_{i-1} (x_k + y_k) - m_i + V_k + \varphi_{i-1} (x_i + y_i + y) - \varphi_{i-1} (x_k + y_k) - m_i + V_k = \varphi_i (x) + \varphi_i (y) + V_k. \]

This concludes the proof. Therefore, setting
\[ \varphi_\infty (z) = \varphi (z + z_\infty) - \varphi (z_\infty) = \lim_{i \to \infty} \varphi (z + z_i) - \varphi (z_i) = \lim_{i \to \infty} \varphi_i (z), \]
we have that, for every $k \in \omega$ and $x, y \in H_k$,
\[ \varphi_\infty (x + y) + V_k = \varphi_\infty (x) + \varphi_\infty (y) + V_k. \]

This concludes the proof. \qed

We close this section with some examples which demonstrate that in Theorem 4.3 and in Theorem 4.5(1) the conclusions cannot be strengthened any further. It is at present unclear if the same is true for Theorem 4.5(2).

**Example 4.10.** The group $\mathbb{Z}_2$ of all dyadic integers is attained as the inverse limit of the inverse system
\[ (\mathbb{Z}/2\mathbb{Z}, +) \leftarrow (\mathbb{Z}/4\mathbb{Z}, +) \leftarrow (\mathbb{Z}/8\mathbb{Z}, +) \leftarrow (\mathbb{Z}/16\mathbb{Z}, +) \leftarrow \cdots \]
where the bonding maps $\mathbb{Z}/2^k \mathbb{Z} \to \mathbb{Z}/2^{k-1} \mathbb{Z}$ are given by $a \mapsto (a \mod 2^{k-1})$. This is clearly a profinite abelian group and hence Polish in the topology that it inherits as a subgroup of $\prod \mathbb{Z}/2^k$, the latter being endowed with the product topology. Notice that the map $k \mapsto ((k \mod 2), (k \mod 4), \ldots)$ realizes $\mathbb{Z}$ as a Polishable subgroup of $\mathbb{Z}_2$. In the group $\mathbb{Z}_2/\mathbb{Z}$ with a Polish cover every definable homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ that maps $\mathbb{Z}$ into $\mathbb{Z}$ has
the form \( y \mapsto cy \) for some \( c \in \mathbb{Z} \). Hence the Borel homomorphism \( \varphi : \mathbb{Z}/\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}, x + \mathbb{Z} \mapsto f_1(x) + \mathbb{Z} \), where \( f_1(x) \in \mathbb{Z} \) is such that \( 2f_1(x) - x \in \mathbb{Z} \), does not have a lift to a continuous homomorphism on \( \mathbb{Z}/\mathbb{Z} \). Nevertheless, \( \varphi \) is trivial, as it is induced by the continuous homomorphism \( 2\mathbb{Z} \to \mathbb{Z}, y \mapsto \frac{1}{2}y \) defined on the clopen subgroup \( 2\mathbb{Z} \) of \( \mathbb{Z} \). This shows that one cannot strengthen the conclusions of Theorem 4.3 and obtain a lift that is a continuous group homomorphism \( G \to G' \).

**Example 4.11.** Consider the groups with a Polish cover \( \mathbb{Z}_2/\mathbb{Z} \) and \( \mathbb{Z}_2/\mathbb{Z}' \), where \( \mathbb{Z}' \) and \( \mathbb{Z}_2 \) are each endowed with the product topology. Consider the Borel homomorphism \( \varphi : \mathbb{Z}_2/\mathbb{Z} \to \mathbb{Z}_2/\mathbb{Z}' \) defined by

\[
x + \mathbb{Z} \mapsto (f_0(x), f_1(x), f_2(x), \ldots) + \mathbb{Z}',
\]

where \( f_n(x) \in \mathbb{Z}_2 \) is such that \( 2^nf_n(x) - x \in \mathbb{Z} \) for \( n \in \omega \). Any clopen subgroup of \( \mathbb{Z}_2 \) is of the form \( 2^k\mathbb{Z} \) for some \( k \in \omega \). A continuous homomorphism from \( 2^k\mathbb{Z} \) to \( \mathbb{Z}_2 \) that maps \( 2^k\mathbb{Z} \) to \( \mathbb{Z}' \) is of the form \( y \mapsto (2^{-k}c_0y, 2^{-k}c_1y, \ldots) \) for some \( c_0, c_1, \ldots \in \mathbb{Z} \). Thus \( \varphi \) is not induced by a continuous homomorphism defined on a clopen subgroup of \( \mathbb{Z}_2 \).

This shows that in Theorem 4.5(1) it is not sufficient to assume that \( N' \) is non-Archimedean.

### 5. The Definable Content of the \( \lim^1 \)-Functor

Many computations and constructions in algebraic topology and homological algebra naturally give rise to groups with a Polish cover \( G/N \) which satisfy the assumptions of Theorem 4.3. Examples include the computations of Čech cohomology groups of mapping telescopes, the computations of Steenrod homology groups of solenoids, and the constructions of the invariant \( \text{Ext}(\Lambda, \mathbb{Z}) \) of all extensions of finite rank torsion-free abelian groups \( \Lambda \) by \( \mathbb{Z} \). We treat the first two of these examples in later installments of this series (see [5] for the first); the study of the “definable content” of \( \text{Ext}(\Lambda, \mathbb{Z}) \) will occupy Section 7 below. In all these examples, the computation of the pertinent group \( G/\Lambda \) involves an application of the first derived functor \( \lim^1(-) \) of the \( \lim^1 \)-functor on towers of countable abelian groups.

In this section we develop the theory of the “definable content” of the \( \lim^1 \)-functor. In particular, we show that when applied to the category of towers of abelian Polish groups it takes values in the category of groups with a Polish cover. We use Theorem 4.3 to deduce that the \( \lim^1 \)-functor is fully faithful when it is further restricted to the full subcategory of all filtrations, and we provide an explicit description of the objects in its image in terms of pro-countable completions of countable abelian groups; see Theorem 5.13 and Corollary 5.15. It follows that for any two filtrations \( A, B, \lim^1 A \) and \( \lim^1 B \) are definably isomorphic if and only if \( A \) and \( B \) are isomorphic as objects in the category of towers of countable abelian groups; see Corollary 5.17.

#### 5.1. Towers of Polish groups

A tower of Polish groups is an inverse sequence \( G = (G^{(m)}, p^{(m,m+1)}) \) of Polish groups and continuous homomorphisms \( p^{(m,m+1)} : G^{(m+1)} \to G^{(m)} \), where \( m \) ranges over \( \omega \). These morphisms determine the more general family of morphisms \( p^{(m,m')} = p^{(m,m+1)} \circ \cdots \circ p^{(m',m+1)} : G^{(m+1)} \to G^{(m')} \) for \( m < m' \leq \omega \), along with \( p^{(m,m)} = 1_{G^{(m)}} \), for each \( m \in \omega \). Let \( G = (G^{(m)}, p^{(m,m+1)}) \) and \( H = (H^{(m)}, p^{(m,m+1)}) \) be towers of Polish groups. By an inv-map from \( G \) to \( H \) we mean a sequence \( (m_k, f^{(k)})_{k \in \omega} \) in which:

- \((m_k)\) is an increasing sequence in \( \omega \), and
- for all \( k \in \omega \), \( f^{(k)} : G^{(m_k)} \to H^{(k)} \) is a continuous homomorphism so that \( p^{(k,k+1)}f^{(k+1)} = f^{(k)}p^{(m_k,m_{k+1})} \).

It follows that \( p^{(k_0,k_1)}f^{(k_1)} = f^{(k_0)}p^{(m_{k_0},m_{k_1})} \) for all \( k_0 < k_1 \). The collections of inv-maps forms a category with towers of Polish groups as objects. The identity inv-map \( \text{id}_G \) of \( G \) is \( (m_k, f^{(k)}) \), where \( m_k = k \) and \( f^{(k)} = 1_{G^{(k)}} \). The composition \( (k_i, g^{(i)}) \circ (m_k, f^{(k)}) \) of the inv-map \( (m_k, f^{(k)}) \) from \( G \) to \( H \) and the inv-map \( (k_i, g^{(i)}) \) from \( H \) to \( L \) is the inv-map \( (m_k, g^{(i)}f^{(k)}) \) from \( G \) to \( L \). So defined, the collection of all inv-maps forms a category. However, many natural functors on this category vary inv-maps as equivalent and therefore fail to be faithful. To remedy this, we define a congruence on inv-maps and pass to the related homotopy category.

Let \((m_k, f^{(k)})\) and \((m'_k, f'^{(k)})\) be inv-maps from \( G \) to \( H \). We say that \((m_k, f^{(k)})\) and \((m'_k, f'^{(k)})\) are congruent if there exists an increasing sequence \( \tilde{m}_k \) such that \( \tilde{m}_k \geq \max \{m_k, m'_k\} \) for \( k \in \omega \) and \( f^{(k)}p^{(m_k, \tilde{m}_k)} = f'^{(k)}p^{(m'_k, \tilde{m}_k)} \) for every \( k \in \omega \); [51, Section 1.1]. It is easy to check that this defines an equivalence relation among inv-maps from \( G \) to \( H \). A pro-map from \( G \) to \( H \), or simply a map from \( G \) to \( H \), is the congruence class \( [m_k, f^{(k)}] \) of some inv-map \((m_k, f^{(k)})\) from \( G \) to \( H \). The identity map of \( G \) is the congruence class of the identity inv-map of \( G \). The
composition of inv-maps \((m_k, f^{(k)})\) from \(G\) to \(H\) and \((k_l, g^{(l)})\) from \(H\) to \(L\) is given by setting:

\[
[k_l, g^{(l)}] \circ [m_k, f^{(k)}] = \left( [k_l, g^{(l)}] \circ (m_k, f^{(k)}) \right)
\]

It is easy to verify that this indeed defines a category that has towers of Polish groups as objects.

**Definition 5.1.** The maps between towers of Polish groups comprise the morphisms in the category of towers of Polish groups. Prominent full subcategories of this category are those of towers of abelian Polish groups and of towers of countable abelian Polish groups.

We have some obvious closure properties for these categories. For example, the trivial tower \(G\), in which each \(G^{(k)}\) is the trivial group, is an initial and terminal object in the category of towers of Polish groups. If \(G\) and \(H\) are towers, then let \(G \times H\) denote the tower \((G^{(k)} \times H^{(k)}))_{k \in \omega}\) with bonding maps the products of the bonding maps of \(G\) and \(H\). \(G \times H\) is the product and coproduct of \(G\) and \(H\) in the category of towers of Polish groups. Finally, the (additively denoted) category of towers of abelian Polish groups forms an additive category. To see this, if \([m_k, f^{(k)}]\) and \([m'_k, f^{(k)}]\) are maps from \(G\) to \(H\) then we may assume without loss of generality that \(m_k = m'_k\) for every \(k \in \omega\). Hence, we may define \([m_k, f^{(k)}] + [m'_k, f^{(k)}]\) to be the map \([m_k, f^{(k)} + f^{(k)}]\).

### 5.2. The \(\lim^1\)-functor.

We now restrict our discussion to the category of towers of abelian Polish groups. To each tower \(A = (A^{(m)}, p^{(m,m+1)})\) of abelian Polish groups one associates the inverse limit \(\lim A\) of \(A\) given by

\[
\lim A := \{ (x_m) \in \prod_{m \in \omega} A^{(m)} : \forall m, m' \in \omega, x_m = p^{(m,m')}(x_{m'}) \}.
\]

Clearly, \(\lim A\) is a closed, hence Polish, abelian subgroup of the Polish product group. Moreover, given any inv-map \((m_k, f^{(k)}) : A \to B\), we have a continuous homomorphism \(\lim (m_k, f^{(k)}) : \lim A \to \lim B\), defined by setting

\[
\left( \lim (m_k, f^{(k)}) \right)(x_m) = (f^{(k)}(x_{m_k}))_{k \in \omega}.
\]

This induces an assignment \([m_k, f^{(k)}] \mapsto \lim [m_k, f^{(k)}]\), since any two congruent inv-maps induce the same homomorphism \(\lim A \to \lim B\). It is straightforward to verify that the assignments \(A \mapsto \lim A\) and \([m_k, f^{(k)}] \mapsto \lim [m_k, f^{(k)}]\) define an additive functor from the category of towers of abelian Polish groups to the category of abelian Polish groups. This functor however fails to be right exact. That is, the image of a short exact sequence

\[
0 \to A \to B \to C \to 0
\]

under the \(\lim\)-functor will generally fail to give an epimorphism \(\lim B \to \lim C\); see [51, Example 11.23]. The failure of right exactness is measured by the sequence \((\lim^n)_{n \in \omega}\) of the right derived functors of \(\lim\); see [34, 51, 75]. For every tower \(A\) as above, the group \(\lim^n A\) is computed as the \(n\)-th cohomology group of a cochain complex

\[
C^n(A) := 0 \to C^0(A) \xrightarrow{\delta^1} C^1(A) \xrightarrow{\delta^2} C^2(A) \xrightarrow{\delta^3} C^3(A) \to \cdots
\]

which associated to \(A\) as in [51, Section 11.5]. It turns out that within the context of towers, these functors vanish for \(n \geq 2\); see [51, Section 11.6]. We proceed to the definition of \(\lim^1 A\).

Let \(A = (A^{(m)}, p^{(m,m+1)})\) be a tower of abelian Polish groups. We follow [51, Section 11.5] with the only exception that our coboundary operators \(\delta^0, \delta^1\) inherit the orientation from \((m_0 \leq m_1 \leq m_2)^{op} := m_2 \leq m_1 \leq m_0\) rather than from \(m_0 \leq m_1 \leq m_2\). This choice of orientation is more natural for some identifications later on. Of course, the resulting groups \(Z(A), B(A), \text{ and } \lim^1 A\) stay the same.

- \(C^0(A), C^1(A), \text{ and } C^2(A)\), are the following abelian Polish group endowed with the product topology:

  \[
  C^0(A) := \prod_{m \in \omega} A^{(m)}, \quad C^1(A) := \prod_{(m_0,m_1) \in \omega^2} A^{(m_0,m_1)}, \quad C^2(A) := \prod_{m_0 \leq m_1 \leq m_2} A^{(m_0,m_1,m_2)}.
  \]

- \(\delta^1 : C^0(A) \to C^1(A)\) and \(\delta^2 : C^1(A) \to C^2(A)\) are the continuous group homomorphisms given by

  \[
  (\delta^1(x))_{m_0,m_1} = x_{m_0} - p^{(m_0,m_1)}(x_{m_1}), \text{ for all } x = (x_m) \in C^0(A), \text{ and }
  \]

  \[
  (\delta^2(x))_{m_0,m_1,m_2} = x_{m_0,m_1} - x_{m_0,m_2} + p^{(m_0,m_1)}(x_{m_1,m_2}), \text{ for all } x = (x_{m_0,m_1}) \in C^1(A).
  \]
• \(Z(A)\) is the closed, hence Polish, subgroup \(\ker(\delta^2)\) of \(C^1(A)\). Explicitly:
\[
Z(A) := \{ (x_{m_0,m_1}) \in C^1(A) : x_{m_0,m_2} = x_{m_0,m_1} + p^{(m_0,m_1)}(x_{m_1,m_2}), \text{ for all } m_0 \leq m_1 \leq m_2 \}.
\]

• \(B(A)\) is the Polishable subgroup \((\delta)(C^0(A))\) of \(C^1(A)\); see Lemma 2.3. Explicitly:
\[
B(A) := \{ (x_{m_0,m_1}) \in C^1(A) : x_{m_0,m_1} = z_{m_0} - p^{(m_0,m_1)}(z_{m_1}), \text{ for some } (z_m) \in C^0(A), \text{ and all } m_0 \leq m_1 \}.
\]

**Definition 5.2.** Let \(A = (A^{(m)}, p^{(m,m+1)})\) be a tower of abelian Polish groups. We denote by \(\underset{\leftarrow}{\lim}^1 A\) the following group with a Polish cover:
\[
\text{lim}_{\leftarrow}^1 A := Z(A)/B(A)
\]

**Remark 5.3.** Notice that every element \(x = (x_{m_0,m_1})\) of \(Z(A)\) is completely determined by the values \(x_{m,m+1}\) for \(m \in \omega\), as each \(x_{m,m}\) necessarily equals zero and for \(m_0 < m_1\) we have:
\[
x_{m_0,m_1} = x_{m_0,m_0+1} + p^{(m_0,m_0+1)}(x_{m_0+1,m_0+2}) + \cdots + p^{(m_0,m_1-1)}(x_{m_1-1,m_1})
\]
We will sometimes tacitly apply this remark in what follows.

As in the case of the \(\text{lim}\)-functor, a map \([m_k, f^{(k)}] : A \to B\) of towers of abelian Polish groups induces a continuous homomorphism \(Z(A) \to Z(B)\) which maps \(B(A)\) into \(B(B)\). This induces a definable homomorphism \(\text{lim}_{\leftarrow}^1[m_k, f^{(k)}] : \text{lim}_{\leftarrow}^1 A \to \text{lim}_{\leftarrow}^1 B\) between groups with a Polish cover given by:
\[
\left(\text{lim}_{\leftarrow}^1[m_k, f^{(k)}]\right)(\langle x_{m,m'} \rangle_{m \leq m'}) = (f^{(k)}(x_{m,k,m'}))_{k \leq k'}.
\]
It is straightforward to verify the following statement.

**Proposition 5.4.** The assignments \(A \mapsto \text{lim}_{\leftarrow}^1 A, [m_k, f^{(k)}] \mapsto \text{lim}_{\leftarrow}^1[m_k, f^{(k)}]\), define an additive functor from the category of towers of abelian Polish groups to the category of groups with Polish cover.

The next lemma shows that \(\text{lim}_{\leftarrow}^1 A\) fails to be Polish whenever it does not vanish. Hence, Proposition 5.4 provides the right framework for studying the definable content of the \(\text{lim}_{\leftarrow}\)-functor.

**Lemma 5.5.** For every tower \(A = (A^{(m)}, p^{(m,m+1)})\) of abelian Polish groups \(B(A)\) is dense in \(Z(A)\).

**Proof.** Suppose that \(n_0 \in \omega\) and \(a \in Z(A)\) is such that \(a_{n,n+1} = 0\) for \(n \geq n_0\). Then setting
\[
b_k := a_{k,n_0} = a_{k,n} \text{ for } n \geq n_0
\]
one obtains \(b = (b_k) \in C^0(A)\) such that \(\delta^1(b) = a\). \(\square\)

In [55] it is observed that if \(\text{lim}_{\leftarrow}^1 A \neq 0\) then \(\text{lim}_{\leftarrow}^1 A\) is uncountable. One consequence of the above lemma is that, in the Polish context, the observation from [55] can be strengthened to the following; see Section 2.3 for definitions.

Observe that if \(N\) is a comeager subgroup of a Polish group \(G\), then \(N = G\). Indeed, if \(g \in G\), then \(N \cap Ng \neq \emptyset\) and hence \(g \in N^{-1}N \subseteq N\).

**Corollary 5.6.** Let \(A\) be a tower of abelian Polish groups. If \(\text{lim}_{\leftarrow}^1 A \neq 0\) then \(E_0 \leq_{B} R(C^0(A) \cap Z(A))\).

**Proof.** If \(\text{lim}_{\leftarrow}^1 A \neq 0\), then \(B(A)\) is a proper dense subgroup of \(Z(A)\). Whence, it is not \(G_\delta\). Therefore, the action \(C^0(A) \cap Z(A)\) has no \(G_\delta\) orbit. The conclusion thus follows from [23, Theorem 6.2.2] \(\square\)

We now record a useful criterion for the vanishing of \(\text{lim}_{\leftarrow}^1 A\). A tower of abelian Polish groups \(A\) is **epimorphic** when all the bonding maps \(p^{(m,m+1)} : A^{(m+1)} \to A^{(m)}\) are surjective. When \(A\) is an epimorphic tower then \(\text{lim}_{\leftarrow}^1 A\) vanishes. More generally, \(\text{lim}_{\leftarrow}^1 A\) vanishes if \(A\) satisfies the **Mittag-Leffler condition** condition, i.e., if for every \(m \in \omega\) the decreasing sequence \((p^{(m,k)}(A^{(k)}))_{k \geq m}\) of subgroups of \(A^{(m)}\) is eventually constant.

**Lemma 5.7.** Suppose that \(A\) is a tower of abelian Polish groups and consider the following assertions:

1. \(A\) satisfies the Mittag–Leffler condition;
2. \(A\) is isomorphic to an epimorphic tower;
3. \(\text{lim}_{\leftarrow}^1 A = 0\)
Then (1) and (2) are equivalent and imply (3). If each \( A^{(m)} \) in \( A \) is countable, then (1), (2), and (3) are equivalent.

**Proof.** For a proof see [52, Ch. II, Section 6.2]. \( \square \)

For an example of a tower of abelian Polish groups where (3) does not imply (1) and (2), see [55, Example 4.5].

### 5.3 Monomorphic towers and filtrations

For the remainder of this section we will restrict our attention to the category of towers of countable abelian Polish groups. A tower \( A = (A^{(m)}, p^{(m,m+1)}) \) of countable abelian Polish groups is **monomorphic** if \( p^{(m,m+1)} \) is an injection for all \( m \in \omega \). In this case, \( \lim_{\leftarrow}^1 \mathcal{A} \) admits general description as a group with a Polish cover; this is the content of Theorem 5.13 below.

**Definition 5.8.** Let \( A \) be a countable abelian group. A **filtration** \( A = (A^{(m)}) \) of \( A \) is a tower \( (A^{(m)}, p^{(m,m+1)}) \), so that \( A = A^{(0)}, p^{(m,m+1)} \) is the inclusion map \( A^{(m+1)} \subseteq A^{(m)} \), and \( \bigcap_m A^{(m)} = 0 \). The category of **filtrations** is the associated full subcategory of the category of towers of Polish groups. Its objects are all filtrations of all countable abelian groups.

Given any monomorphic tower \( A = (A^{(m)}, p^{(m,m+1)}) \), we can canonically assign to \( A \) a filtration \( A^{\text{fil}} \). First, by replacing \( A \) with an isomorphic tower, we may always assume that \( A^{(m+1)} \subseteq A^{(m)} \) for every \( m \in \omega \) and that \( p^{(m,m+1)} : A^{(m+1)} \to A^{(m)} \) is the inclusion map. Let \( A^{(\infty)} \) be the intersection of these \( A^{(m)} \) \( (m \in \omega) \). Define \( A^{\text{fil}} \) to be the tower of groups \( (A^{(m)})/A^{(\infty)}, p^{(m,m+1)}) \) where \( p^{(m,m+1)} : (A^{(m)})/A^{(\infty)} \to A^{(m)}/A^{(\infty)} \) is the inclusion map. Notice that this defines a functor \( A \mapsto A^{\text{fil}} \) from the category of monomorphic towers of countable abelian groups to the subcategory of filtrations.

**Lemma 5.9.** If \( A \) is a monomorphic tower, then \( \lim_{\leftarrow}^1 \mathcal{A} \) and \( \lim_{\leftarrow}^1 A^{\text{fil}} \) are definably isomorphic.

**Proof.** The quotient maps \( A^{(m)} \to A^{(m)}/A^{(\infty)} \) for \( m \in \omega \) induce a morphism \( \pi : A \to A^{\text{fil}} \), which induces in turn a surjective homomorphism \( \bar{\pi} : \lim_{\leftarrow}^1 \mathcal{A} \to \lim_{\leftarrow}^1 A^{\text{fil}} \) which lifts to a continuous homomorphism \( Z(\mathcal{A}) \to Z(A^{\text{fil}}) \). Hence \( \bar{\pi} \) is definable. We claim that it is also injective. Indeed, if \( a \in Z(\mathcal{A}) \) is such that \( \bar{\pi}_A(a) \in B(A^{\text{fil}}) \), then there exists \( c \in C^0(\mathcal{A}) \) such that \( a_{m,m+1} + A^{(\infty)} = c_m - c_{m+1} + A^{(\infty)} \) for every \( m \in \omega \). Hence there exist \( r_m \in A^{(\infty)} \) such that \( a_{m,m+1} + r_m = c_m - c_{m+1} \) for \( m \in \omega \). Define then

\[
d_m := c_m + r_0 + \cdots + r_{m-1} \in A^{(m)}
\]

for \( m \in \omega \). Notice that

\[
d_m - d_{m+1} = c_m + r_0 + \cdots + r_{m-1} - (c_{m+1} + r_0 + \cdots + r_m)
\]

\[
= c_m - c_{m+1} - r_m = a_m.
\]

Hence \( a \) witnesses that \( \bar{\pi} \) is a definable isomorphism. \( \square \)

**Remark 5.10.** It is easy to check that if we set \( \pi_A := \pi \) for the homomorphism \( \pi : A \to A^{\text{fil}} \) defined in the above proof, then \( A \mapsto \pi_A \) is a natural transformation between the functors \( A \mapsto \lim_{\leftarrow}^1 \mathcal{A} \) and \( A \mapsto \lim_{\leftarrow}^1 A^{\text{fil}} \).

By Lemma 5.9 we may restrict our study of monomorphic towers to the category of filtrations. In this category \( \lim_{\leftarrow}^1 \mathcal{A} \) has a concrete description to which Theorem 4.3 directly applies, as we will see.

### 5.4 Completions of topological groups and towers

**Definition 5.11.** A topological group \( G \) is **pro-countable** if it is isomorphic to the inverse limit of a tower of countable groups.

As we have noted, if \( G \) is abelian then \( G \) is pro-countable if and only if it is a non-Archimedean Polish group; see [49, Lemma 2].

**Definition 5.12.** Let \( A \) be a filtration of a countable abelian group \( A \). Then \( A \) gives rise to the pro-countable abelian group \( \hat{A} \) which is the inverse limit of the tower of countable abelian groups \( \left( \hat{A}/A^{(m)}, f^{(m,m+1)}_* \right) \), where \( f^{(m,m+1)}_* : A/A^{(m+1)} \to A/A^{(m)} \) is the epimorphism induced by the inclusion \( f^{(m,m+1)} : A^{(m+1)} \to A^{(m)} \). We say that \( \hat{A} \) is the completion of \( A \) with respect to \( A \).
Notice that the canonical homomorphism \( a \mapsto (a + A(0), a + A(1), \ldots) \) from \( A \) to \( \hat{A} \) is injective, since the sequence \( \{A^{(m)} : m \in \omega\} \) has trivial intersection. The resulting image of \( A \) is clearly dense in \( \hat{A} \). This induces an assignment \( A \mapsto \hat{A}/A \), which maps the tower \( A \) to the group with a Polish cover \( \hat{A}/A \). This assignment determines a functor from the category of filtrations to the category of groups with a Polish cover. To see this first notice that if \( \hat{A}_{m_0} \) is the completion of the subgroup \( A^{(m_0)} \subseteq A \) with respect to the filtration \( (A^{(m)} : m \geq m_0) \) then \( \hat{A}_{m_0} \) is a clopen subgroup of \( \hat{A} \). Since \( A \) is dense in \( \hat{A} \) we also have that \( \hat{A}_{m_0} + A = \hat{A} \) and therefore \( \hat{A}_{m_0} \) is an essential retract of \( \hat{A}/A \); see the discussion preceding Definition 4.2. Notice now that if \( f = (m_k, f(k)) \) is an inv-map from \( A \) to some filtration \( B \) then the homomorphism \( f(0) : A^{(m_0)} \to B \) extends to a continuous homomorphism \( \hat{f} : \hat{A}_{m_0} \to \hat{B} \) such that \( \hat{f}(A^{(m_k)}) \subseteq B \). The resulting definable homomorphism \( f : \hat{A}_{m_0}/A^{(m_0)} \to \hat{B}/B \) induces a (trivial) definable homomorphism \( f : \hat{A}/A \to \hat{B}/B \) by Lemma 4.1. It is easily verified that this defines an additive functor from the category of filtrations to the category of groups with a Polish cover.

5.5. \( \lim^1 \) and completions. In [55, Section 4] it is shown that, if \( A \) is a tower of countable abelian groups, then \( \lim^1 A \) vanishes if and only if \( \hat{A}/A \) does as well. We have the following generalization, which was already known without the definability claim; see [25, Remark 2.13].

**Theorem 5.13.** Suppose that \( \hat{A} \) is the completion of a countable group \( A \) with respect to a filtration \( A \) of \( A \). Then there is a definable isomorphism between \( \lim^1 A \) and \( \hat{A}/A \).

**Proof.** Let \( a \in Z(A) \) and notice that the sequence \( k \mapsto a_{0,k} = a_{0,1} + a_{1,2} + \cdots + a_{k-1,k} \) is Cauchy in \( \hat{A} \) since
\[
(a_{0,k} - a_{0,l}) \in A^{(k)} + A^{(k+1)} + \cdots + A^{(l-1)} = A^{(l-1)} \subseteq \hat{A}^{(l-1)}, \text{ for all } k < l.
\]
So let \( \sigma(a) = \lim_{k \to \infty} a_{0,k} \) be the limit of the above sequence in \( \hat{A} \). Notice that this defines a continuous homomorphism \( \sigma : Z(A) \to \hat{A} \). Moreover, if \( a \in B(A) \), then there exists \( b \in C^0(A) \) such that \( a_{k,k+1} = b_k - b_{k+1} \) for every \( k \in \omega \), and hence, \( \sigma(a) = b_0 \in \hat{A} \). This shows that \( \sigma \) induces a (trivial) definable homomorphism \( \lim^1 A \to \hat{A}/A \).

We now show that such a homomorphism is surjective. If \( b \in \hat{A}(0) \), then \( b \) is the limit of a Cauchy sequence \( (b_n)_{n \in \omega} \) from \( A \). After passing to a subsequence, we may assume that \( b_n - b_{n+1} \in A^{(n)} \) for every \( n \in \omega \). Define \( a \in Z(A) \) by setting \( a_{0,1} = b \) and \( a_{n,n+1} = b_n - b_{n+1} \) for \( n \geq 1 \). Observe that
\[
a_{0,n+1} = a_{0,1} + a_{1,2} + \cdots + a_{n,n+1} = b_0 + (b_1 - b_0) + (b_2 - b_1) + \cdots + (b_n - b_{n-1}) = b_n
\]
for every \( n \in \omega \). Hence
\[
\sigma(a) = \lim_{n \to \infty} a_{0,n+1} = \lim_{n \to \infty} b_n = b.
\]
We now show that the definable homomorphism \( \lim^1 A \to \hat{A}/A \) is injective. Suppose that \( a \in Z(A) \) is such that \( \sigma(a) \in \hat{A} \). We claim that \( a \in B(A) \). Let \( m \in \omega \) and notice that \( \hat{A}(m) \cap A = A^{(m)} \), where \( \hat{A}(m) \) is the clopen subgroup of \( \hat{A} \) corresponding to the completion of \( A^{(m)} \) with respect to the filtration \( \{A(l) : m \leq l\} \). Define
\[
b_m := \lim_{k \to \infty} a_{m,k} \in \hat{A}(m) \cap A = A^{(m)},
\]
and notice that \( a_{m,m+1} = b_m - b_{m+1} \), for every \( m \in \omega \). Thus \( a \in B(A) \).

**Remark 5.14.** It is easy to check that if we set \( \sigma_A := \sigma \) for the homomorphism \( \sigma(a) = \lim_{k \to \infty} a_{0,k} \) defined in the above proof, then \( A \mapsto \sigma_A \) is a natural transformation between the functors \( A \mapsto \lim^1 A \) and \( A \mapsto \hat{A}/A \) defined on the category of all filtrations of countable groups.

We may now invoke Theorem 4.3 to get the following rigidity theorem for the \( \lim^1 \) functor, restricted to the category of filtrations of countable groups. Recall that a functor between categories is fully faithful if it is bijective when restricted to hom-sets.

**Corollary 5.15.** The functor \( A \mapsto \lim^1 A \) from the category of filtrations of countable abelian groups to the category of groups with a Polish cover is fully faithful.
Proof. In view of Theorem 5.13, it suffices to show that the additive functor $A \mapsto \hat{A}/A$ from the category of filtrations of countable abelian groups to the category of groups with a Polish cover is fully faithful.

We begin by showing that such a functor is faithful. Suppose that $A$ and $B$ are filtrations of countable groups $A$ and $B$, respectively. Let $[m_k, f^{(k)}]$ be a morphism from $A$ to $B$ and let $f : \hat{A}/A \to \hat{B}/B$ be the corresponding definable homomorphism. As in the paragraph preceding Theorem 5.13, $f$ is a trivial definable homomorphism and since it is induced by the extension $f^{(0)} : \hat{A}(m_0) \to \hat{B}$, where $\hat{A}(m_0) \subseteq \hat{A}$ is the completion of $A'(m_0)$ with respect to $\{A(k) : k \geq m_0\}$. In particular, $f(x + A) = f^{(0)}(x) + A$ for $x \in \hat{A}(m_0)$.

Suppose now that $f$ is the zero homomorphism $f(\hat{A}) = 0$ from $\hat{A}/A$ to $\hat{B}/B$. Then $f^{(0)}(\hat{A}(m_0)) \subseteq B$. As $B$ is countable, by the Baire Category Theorem, there is $b_0 \in B$ such that $G(b_0) = \{x \in \hat{A}(m_0) : f^{(0)}(x) = b_0\}$ is nonmeager. By Pettis’ Lemma, $G(b_0) - G(b_0)$ contains an open neighborhood of 0. It follows that

$$\hat{A}(m_0) \subseteq G(b_0) - G(b_0) \subseteq \{x \in \hat{A}(m_0) : f^{(0)}(x) = 0\},$$

for some $n_0 \geq m_0$. It follows that $f^{(0)}|_{\hat{A}(m_0)} = 0$, and therefore $[m_k, f^{(k)}]$ is the zero morphism from $A$ to $B$.

We now show that the functor is full. Suppose as above that $A$ and $B$ are filtrations of countable groups $A$ and $B$, respectively. Let $f$ be a definable homomorphism from $\hat{A}/A$ to $\hat{B}/B$. We claim that there exists a morphism $[m_k, f^{(k)}]$ from $A$ to $B$ such that $f$ is the definable homomorphism induced by $[m_k, f^{(k)}]$. By Theorem 4.3, there exist $m_0 \in \omega$ and a continuous homomorphism $g : \hat{A}(m_0) \to \hat{B}$ such that $f(x + A) = g(x) + A$ for every $x \in \hat{A}(m_0)$. Since $g$ is continuous, there exists an increasing sequence $(m_k)$ in $\omega$ such that $g$ maps $\hat{A}(m_k)$ to $\hat{B}(k)$ for every $k \in \omega$. One can then define $f^{(k)} : \hat{A}(m_k) \to \hat{B}(k)$ to be the restriction of $g$ to $A'(m_k)$ for $k \in \omega$. Then $[m_k, f^{(k)}]$ is a morphism from $A$ to $B$ that induces the Borel homomorphism $f$. This concludes the proof that the functor is full. \qed

Remark 5.16. Notice that the conclusion of Corollary 5.15 does not hold for monomorphic towers of the form $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$, when $\lim_{m} A(m) \neq 0$. Similarly, it does not generally hold for towers that are not monomorphic, as the $\lim^1$ of any epimorphic tower vanishes.

Corollary 5.17. Let $A$ and $B$ be filtrations of the countable abelian groups $A$ and $B$. Let also $\hat{A}$ and $\hat{B}$ be the associated completions. Then, the following are equivalent:

1. $A$ and $B$ are isomorphic objects in the category of filtrations;
2. $\hat{A}/A$ and $\hat{B}/B$ are definably isomorphic;
3. $\lim^1 A$ is definably isomorphic to $\lim^1 B$.

Proof. This is an immediate corollary of Corollary 5.15 and Theorem 5.13. \qed

We should observe here that the above three equivalent conditions do not necessarily imply that the groups $\hat{A}$ and $\hat{B}$ are isomorphic; e.g. see Example 6.9.

6. Locally profinite completions of $\mathbb{Z}^d$ and $\mathbb{Q}^d$

Consider the assignment $\hat{A} \mapsto \hat{A}/A$ which maps the completion $\hat{A}$ of each abelian group $A$ with respect to some filtration $A$ to the corresponding quotient group $\hat{A}/A$. When $\hat{A}/A$ is viewed as an abstract group up to isomorphism then $\hat{A}/A$ remembers little of the group structure $\hat{A}$. For example, if $\mathbb{Z}^d$ denotes the profinite completion of $\mathbb{Z}^d$ with respect to the filtration $((m!) \cdot \mathbb{Z}^d)_{m \in \omega}$, then by [59], the group $\mathbb{Z}^d$ has the same finite quotients as $\mathbb{Z}^d$, and therefore $\mathbb{Z}^d$ and $\hat{\mathbb{Z}}^d$ are non-isomorphic when $d \neq d'$. However, $\mathbb{Z}^d/\mathbb{Z}$ and $\hat{\mathbb{Z}}^d/\mathbb{Z}$ are isomorphic as abstract groups since they both are $\mathbb{Q}$-vector spaces of the same dimension; see Theorem 7.7 below. By Corollary 5.17, the assignment $\hat{A} \mapsto \hat{A}/A$ provides a much stronger invariant for classifying completions (or filtrations) of countable abelian groups, if $\hat{A}/A$ is instead viewed as a group with a Polish cover up to definable isomorphism.

In this section we refine our analysis of the information captured by a definable isomorphism in the context of profinite completions of $\mathbb{Z}^d$. As a corollary of Theorem 6.6 below, we show that the existence of a definable isomorphism between quotients $\mathbb{Z}^d_\Lambda / \mathbb{Z}^d$ of profinite completions $\mathbb{Z}^d_\Lambda$ of $\mathbb{Z}^d$, where $\Lambda$ is a filtration of $\mathbb{Z}^d$ consisting of rank-$d$ subgroups, is equivalent to the existence of an isomorphism between certain finite rank torsion-free abelian groups which are functorially associated to these profinite completions. This theorem very concretely implies that, in classifying $\mathbb{Z}^d_\Lambda$ up to $\mathbb{Z}^d$-preserving isomorphism, $\mathbb{Z}^d_\Lambda / \mathbb{Z}^d$ considered up to definable isomorphism is a strong
invariant, particularly in comparison to these same groups \( \hat{Z}_d^d / Z^d \) considered up to abstract isomorphism; see Corollary 7.9 below.

In general, despite the fact that \( \hat{Z}_d^d / Z^d \) is Ulam stable for each filtration \( \Lambda \) by Corollary 4.4, one can often find definable automorphisms of \( \hat{Z}_d^d / Z^d \) which do not lift to topological group automorphisms of \( \hat{Z}_d^d \). In the second part of this section we show that \( \hat{Z}_d^d / Z^d \) is definably isomorphic to a group with a Polish cover \( \hat{Q}_d^d / \Lambda \), where \( \hat{Q}_d^d \) is a certain locally profinite completion of \( Q^d \) and \( \Lambda \) is a countable subgroup of \( Q^d \) canonically associated to \( \Lambda \). We then show that when \( \Lambda \) is “symmetric” enough then every definable automorphism of \( \hat{Z}_d^d / Z^d \), when transferred to \( \hat{Q}_d^d / \Lambda \), lifts to a continuous group automorphism of \( \hat{Q}_d^d \). This will play a crucial role in Section 8.

6.1. Profinite completions of \( Z^d \). Recall that the rank of an abelian group is the cardinality of a maximal linearly independent (over \( \mathbb{Z} \)) subset. We say that the group \( A \) has no free summand if for every decomposition \( A = A_0 \oplus A_1 \), if \( A_0 \) is free then \( A_0 = 0 \). By a rank \( d \) filtration we mean any filtration \( \Lambda = (\Lambda^{(m)}) \) of \( Z^d \) with the additional property that each \( \Lambda^{(m)} \) is a rank \( d \) subgroup of \( Z^d \). We denote by Fil(\( Z^* \)) the category of finite-rank filtrations, with the tower maps \( [m, f(k)] \) as morphisms. Notice that if \( A \leq Z^d \), then

\[
A \text{ is of finite index in } Z^d \iff A \text{ is rank } d \iff A \text{ is isomorphic to } Z^d.
\]

We can therefore associate to each object \( \Lambda \) from Fil(\( Z^* \)) the profinite completion \( \hat{Z}_d^\Lambda \) of \( Z^d \) given by

\[
\hat{Z}_d^\Lambda := \lim_{\longrightarrow} (Z^d / \Lambda^{(m)}).
\]

As in Section 5, notice that \( \hat{Z}_d^d \) contains a canonical copy of \( Z^d \) as a dense subgroup. Conversely, every profinite completion of \( Z^d \) is of the above form for appropriately chosen \( \Lambda \).

By a rank \( d \) cofiltration we mean any increasing sequence \( A_m = (\Lambda^{(m)}) \) of rank \( d \) subgroups of \( Q^d \) with \( A_0 = \mathbb{Z}^d \) and so that \( \Lambda := \bigcup_m A^{(m)} \) has no free summand. A cotower map \( [m, f(k)] : (\Lambda^{(m)}) \to (B^{(m)}) \) between cofiltrations, also known as a morphism of inductive sequences, is the congruence class of a sequence of group homomorphisms \( f(k) : A^{(k)} \to B^{(m_k)} \) with \( f(k+1) \mid A^{(k)} = f(k) \), where \( (m_k, f(k)) \) is congruent to \( (m'_k, f'_k) \) if for all \( k \in \omega \) we have that \( f(k) = f'_k \) as maps from \( A^{(k)} \) to \( B^{(\text{max}(m_k, m'_k))} \). Composition of cotower maps is defined in analogy with the composition of tower maps given in Section 5. We denote by coFil(\( Z^* \)) the category of finite-rank cofiltrations, with the cotower maps \( [m, f(k)] \) as morphisms.

We lastly consider the category Groups_{+}(Z^*, Q^*) whose objects are groups having no free summands satisfying \( Z^d \leq \Lambda \leq Q^d \) for some \( d \in \omega \) and whose morphisms are simply the homomorphisms between any two such groups. We are going to show that the following three categories are equivalent:

\[
\text{Fil}(Z^*), \quad \text{coFil}(Z^*), \quad \text{Groups}_{+}(Z^*, Q^*).
\]

**Lemma 6.1.** The functor \( \lim_{\longrightarrow} : \text{coFil}(Z^*) \to \text{Groups}_{+}(Z^*, Q^*) \) which maps \( [m, f(k)] : (\Lambda^{(m)}) \to (B^{(m)}) \) to the homomorphism \( \bigcup_k f(k) : \bigcup_k A^{(k)} \to \bigcup_k B^{(k)} \) is fully faithful and essentially surjective.

**Proof.** The functor \( \lim_{\longrightarrow} \) is full since for every homomorphism \( f : \bigcup_k A^{(k)} \to \bigcup_k B^{(k)} \) and every \( k \in \omega \) the group \( A^{(k)} \) is finitely-generated, and hence \( f(A^{(k)}) \) is contained in some \( B^{(m_k)} \), for large enough \( m_k \in \omega \). It is also faithful, since \( \bigcup_k f(k) \) being the constant 0 homomorphism implies the same for each \( f(k) \). We lastly show that it is also essentially surjective. For any group \( \Lambda \) in Groups_{+}(Z^*, Q^*), with \( Z^d \leq \Lambda \leq Q^d \), let \( \{l_1, l_2, \ldots \} \) be an enumeration of \( \Lambda \) and let \( A^{(m)} \) be the smallest subgroup of \( \Lambda \) which contains \( Z^d \), as well as \( l_1, \ldots, l_m \). Clearly \( (\Lambda^{(m)}) \) is a rank \( d \) cofiltration, with \( \lim_{\longrightarrow} (\Lambda^{(m)}) = \Lambda \), and therefore \( \lim_{\longrightarrow} \) is surjective on objects. \( \square \)

**Remark 6.2.** By the previous lemma, the functor \( \lim_{\longrightarrow} : \text{coFil}(Z^*) \to \text{Groups}_{+}(Z^*, Q^*) \) is an equivalence of categories. As a consequence \( \lim_{\longrightarrow} \) admits a (non-unique) inverse up to isomorphism. We define an explicit such inverse \( (\lim_{\longrightarrow})^{-1} : \text{Groups}_{+}(Z^*, Q^*) \to \text{coFil}(Z^*) \) here. Fix an enumeration of \( Q^{<\omega} := \bigcup_{d \in \omega} Q^d \). For each object \( \Lambda \) of Groups_{+}(Z^*, Q^*) we let \( (\lim_{\longrightarrow})^{-1}(\Lambda) \) be the cofiltration built as in the proof of Lemma 6.1 but with the unique choice of enumeration \( \{l_1, l_2, \ldots \} \) of \( \Lambda \) which agrees with the global enumeration of \( Q^{<\omega} \). Having chosen the assignment \( \Lambda \mapsto (\lim_{\longrightarrow})^{-1}(\Lambda) \), there is a canonical way to extend this definition to the desired functor

\[
(\lim_{\longrightarrow})^{-1} : \text{Groups}_{+}(Z^*, Q^*) \to \text{coFil}(Z^*).
\]
Next we define a contravariant functor $\text{Adj}: \text{coFil}(\mathbb{Z}^*) \to \text{Fil}(\mathbb{Z}^*)$. We will need a few definitions.

**Definition 6.3.** Consider the pairings $\langle \cdot, \cdot \rangle : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^d} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}/\mathbb{Z}$ on $\mathbb{R}^d$ defined by

$$\langle (a_1, \ldots, a_d), (b_1, \ldots, b_d) \rangle = a_1 b_1 + \cdots + a_d b_d$$

for $a, b \in \mathbb{R}^d$. For any subgroup $A$ of $\mathbb{R}^d$, the *annihilator* of $A$ is the following closed subgroup of $\mathbb{R}^d$:

$$A_\perp = \{ x \in \mathbb{R}^d : \forall y \in A \langle x, y \rangle_{\mathbb{R}^d} = 0 \}. $$

Observe that the assignment $A \mapsto A_\perp$ defines an order-reversing permutation of the set of finitely-generated subgroups of $\mathbb{Q}^d$ with fixed point $\mathbb{Z}_+^d = \mathbb{Z}^d$. Moreover, if $f : A' \to A$ is a homomorphism between two groups $A' \leq \mathbb{Q}^d$, $A \leq \mathbb{Q}^d$ and $A$ is of rank $d$, then $f$ extends uniquely to a $\mathbb{Q}$-linear map from $\mathbb{Q}^d$ to $\mathbb{Q}^d$. We can therefore identify $f$ with an element $F$ of $M_{d \times d'}(\mathbb{Q})$, i.e., with a $d \times d'$ matrix with entries in $\mathbb{Q}$. Let $F^T$ be the transpose matrix. As a linear transformation, $F^T$ maps $\mathbb{Q}^d$ to $\mathbb{Q}^d$, so that $F^T(A_\perp) \subseteq A'_\perp$. Indeed, if $x \in A_\perp$ and $y \in A'$, then

$$\langle F^T(x), y \rangle = \langle x, F(y) \rangle = 0.$$

We define $f^T : A_\perp \to A'_\perp$ to be the restriction of the linear transformation $F^T$ to $A_\perp$. Since $(F \cdot F')^T = (F')^T \cdot (F)^T$ for every two composable matrices, the assignment $f \mapsto f^T$ is a functor from the category of all homomorphisms $f : A' \to A$, where $A \leq \mathbb{Q}^d$ is of rank $d$ and $A' \leq \mathbb{Q}^d$ is of rank $d'$, to its opposite category.

**Lemma 6.4.**

1. If $A$ is a rank $d$ finitely-generated subgroup of $\mathbb{Q}^d$, then $A_\perp$ is naturally isomorphic to $\text{Hom}(A, \mathbb{Z})$.
2. If $A$ is a rank $d$ finitely-generated subgroup of $\mathbb{Q}^d$, then $(A_\perp)_\perp = A$.
3. If $(A(m))_{m \in \omega}$ is a cofiltration then $(A(m))_{m \in \omega}$, where $A(m) := (A(m))_\perp$, is a rank $d$ filtration.
4. If $(A(m))_{m \in \omega}$ is a rank $d$ filtration then $(A(m))_{m \in \omega}$, where $A(m) := (A(m))_\perp$, is a rank $d$ cofiltration.

**Proof.** For (1), notice that an element $b$ of $A_\perp$ defines an element $\varphi_b \in \text{Hom}(A, \mathbb{Z})$ given by $\varphi_b(a) = \langle b, a \rangle$. The assignment $b \mapsto \varphi_b$ defines an injective group homomorphism $\Phi_A : A_\perp \to \text{Hom}(A, \mathbb{Z})$. We now show that $\Phi_A$ is onto. Suppose that $f \in \text{Hom}(A, \mathbb{Z})$. Let $e_1, \ldots, e_d$ be the elements of the canonical basis of $\mathbb{Q}^d$, and fix a $\mathbb{Z}$-basis $v_1, \ldots, v_d$ of $A$. Fix $\psi \in \text{GL}_d(\mathbb{Q})$ such that $\psi(v_i) = e_i$ for $i \in \{1, 2, \ldots, d\}$. Define now $w = f(v_1)e_1 + \cdots + f(v_d)e_d$, and $b := \psi^T w \in \mathbb{Q}^d$. It follows that $f = \varphi_b$, since:

$$\langle b, v_i \rangle = \langle \psi^T w, v_i \rangle = \langle w, \psi(v_i) \rangle = \langle w, e_i \rangle = f(v_i).$$

For naturality one checks that if $f : B \to A$ is an isomorphism from a rank $d'$ finitely-generated subgroup $B$ of $\mathbb{Q}^d$, to a rank $d$ finitely-generated subgroup $A$ of $\mathbb{Q}^d$, then for the induced $\text{Hom}(f, \mathbb{Z}) : \text{Hom}(A, \mathbb{Z}) \to \text{Hom}(B, \mathbb{Z})$ we have $\text{Hom}(f, \mathbb{Z}) \circ \Phi_A = \Phi_B \circ f^T$. The statement (2) follows from (1), since $a \mapsto (\varphi \mapsto \varphi(a))$ induces a natural isomorphism from $A$ to $\text{Hom}(\text{Hom}(A, \mathbb{Z}))$. For (3), notice first that (1), each term $A(m)$ is finitely-generated of rank $d$ since so is $\text{Hom}(A(m), \mathbb{Z})$. We also have that $A(0) = \mathbb{Z}^d$ since $(\mathbb{Z}^d)_\perp = \mathbb{Z}^d$. Since $A \to A_\perp$ is order reversing we have that $A(0) \supseteq A(1) \supseteq \cdots$. We are left to show that $A(m)_\perp = 0$. Let $A := \bigcup_{m} A(m)$. By [29, Theorem 8.47], $A$ can be written as a direct sum $A_0 \oplus A_1$, so that $A_0$ is finitely-generated, $A_1$ has no free summand, and $\text{Hom}(A, \mathbb{Z}) = \text{Hom}(A_0, \mathbb{Z})$. Since $(A(m))_{m \in \omega}$ is a cofiltration we have that $A_0 = 0$, and therefore $\text{Hom}(A_0, \mathbb{Z}) = 0$. But $\bigcap_{m} A(m) \subseteq \bigcap_{m} A_\perp$, where $\bigcap_{m} A_\perp = 0$ by (1). A similar argument proves (4). \hfill $\Box$

**Lemma 6.5.** The assignment $[m_k, f(k)] \mapsto \text{Adj}([m_k, f(k)])$ which sends a cotower map $[m_k, f(k)]$ between cofiltrations of finite rank to the tower map $[m_k, f(k)]$, with $f(k) := f^T(k)$, is a fully faithful and essentially surjective contravariant functor from $\text{coFil}(\mathbb{Z}^*)$ to $\text{Fil}(\mathbb{Z}^*)$.

**Proof.** Let $[m_k, f(k)] : (A(m))_m \to (B(m))_m$ be a cotower map from a rank $d'$ cofiltration to a rank $d$ cofiltration. Notice that the sequence $f(k)$ is entirely determined by $f(0)$, since $\mathbb{Q}$-linear combinations of $A(0) = \mathbb{Z}^d$ span $\mathbb{Q}^d$. This shows, on the one hand, that $(m_k, f(k))$ is an inv-map from $(B(m))_m$ to $(A(m))_m$, and on the other hand, that if $[m'_k, f'_k) = [m_k, f(k)]$ then $m'_k, f'(k) = [m_k, f(k)]$. Hence $[m_k, f(k)] \mapsto \text{Adj}([m_k, f(k)])$ maps indeed elements of
coFil($\mathbb{Z}^\ast$) to elements contravariantly Fil($\mathbb{Z}^\ast$). It is clearly a functor since $(F \cdot F')^T = (F')^T \cdot (F)^T$ for every two composable matrices.

By (2), (3), and (4) of Lemma 6.4 we have that Adj is surjective on objects of Fil($\mathbb{Z}^\ast$). It is also fully faithfiful since every cotower map can be identified with a matrix $(d \times d')$-matrix which maps $\mathbb{Z}^d$ to $\mathbb{Z}^d$ and in this category of matrices the transpose $F \mapsto F^T$ is its own inverse.

The following theorem is an immediate consequence of Lemmas 6.1 and 6.5, Remark 6.2, and Corollary 5.15. Notice that the inclusion of $\text{Groups}_+(\mathbb{Z}^\ast, \mathbb{Q}^\ast)$ into the category of torsion-free finite rank abelian groups with no free summands is fully faithful and essentially surjective. In what follows we can therefore identify these categories.

**Theorem 6.6.** The following composition of functors provides a fully faithful contravariant functor $\Lambda \mapsto \hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$:

$$\Lambda \longmapsto \lim_{\leftarrow} \Lambda \longmapsto \Lambda \longmapsto \lim_{\rightarrow} \hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d,$$

from the category of finite-rank torsion-free abelian groups with no free direct summands to the category of groups with a Polish cover. The first two functors are equivalences between the categories:

$$\text{Groups}_+(\mathbb{Z}^\ast, \mathbb{Q}^\ast), \ coFil(\mathbb{Z}^\ast), \ Fil(\mathbb{Z}^\ast).$$

The following immediate corollary provides a combinatorial criterion for the existence of a definable isomorphism between the quotients $\hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$ and $\hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$. Let also $\Lambda_\Lambda, \Lambda_B$ be the countable groups $\lim_{\leftarrow}(\text{Adj}(\Lambda)), \lim_{\rightarrow}(\text{Adj}(B))$. The following are equivalent:

1. $\hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$ and $\hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$ are definably isomorphic;
2. $\Lambda$ and $B$ are isomorphic objects in the category of filtrations;
3. $\Lambda_\Lambda$ and $\Lambda_B$ are isomorphic as discrete groups.

Notice that the collection $\text{Obj}(\text{Fil}(\mathbb{Z}^d))$ of all objects $\Lambda$ in the full subcategory Fil($\mathbb{Z}^d$) of Fil($\mathbb{Z}^\ast$), which consists of all filtrations on $\mathbb{Z}^d$, inherits from $(\mathbb{Z}^d)^\mathbb{N}$ a Polish topology. We write $\Lambda \simeq_{\text{pro}} B$ to declare that $\Lambda$ and $B$ are isomorphic as objects in the category Fil($\mathbb{Z}^d$). We have the following corollary:

**Corollary 6.7.** Within the Borel reduction hierarchy, the classification problems $(\text{Obj}(\text{Fil}(\mathbb{Z}^d)), \simeq_{\text{pro}})$ satisfy:

$$(\text{Obj}(\text{Fil}(\mathbb{Z}^1)), \simeq_{\text{pro}}) <_B (\text{Obj}(\text{Fil}(\mathbb{Z}^2)), \simeq_{\text{pro}}) <_B (\text{Obj}(\text{Fil}(\mathbb{Z}^3)), \simeq_{\text{pro}}) <_B \cdots$$

**Proof.** This follows from Corollary 6.7, Remark 6.2, and the results from [1],[27],[70]; see Example 2.5. \hfill \square

### 6.2. Locally profinite completions of $\mathbb{Q}^d$

Given a definable homomorphism $f : \hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d \rightarrow \hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$ it is not necessarily true that $f$ lifts to a continuous homomorphism from $\hat{\mathbb{Z}}^d_{\Lambda} \rightarrow \hat{\mathbb{Z}}^d_{\Lambda}$. In particular, using the following example one can construct a profinite completion $\hat{\mathbb{Z}}^2_{\Lambda \times B}$ of $\mathbb{Z}^2$ so that $\hat{\mathbb{Z}}^2_{\Lambda \times B}/\mathbb{Z}^2$ admits definable automorphisms which do not lift to a topological group isomorphism of $\hat{\mathbb{Z}}^2_{\Lambda \times B}$.

**Example 6.9.** Consider the filtrations $\Lambda = (A^{(m)})_m$ and $B = (B^{(m)})_m$ of $\mathbb{Z}$, where $A^{(0)} = B^{(0)} = \mathbb{Z}$, $A^{(m)} = 3 \cdot 2^{m-1} \cdot \mathbb{Z}$, and $B^{(m)} = 2^{m-1} \cdot \mathbb{Z}$ for $m > 0$. Then the maps $f^{(k)} : A^{(k+1)} \rightarrow B^{(k)}$ and $g^{(k)} : B^{(k)} \rightarrow A^{(k)}$, with $f^{(k)}(x) = \frac{1}{3} \cdot x$ and $g^{(k)}(x) = 3 \cdot x$, provide an isomorphism between the objects $\Lambda$ and $B$ in the category of filtrations. However, $\hat{\mathbb{Z}}_{\Lambda}$ and $\hat{\mathbb{Z}}_B$ are not isomorphic since $\hat{B}$ does not have any index 3 subgroup.

We now show how to replace each group with Polish cover of the form $\hat{\mathbb{Z}}^d_{\Lambda}/\mathbb{Z}^d$ with a definably isomorphic group with Polish cover $\hat{\mathbb{Q}}^d_{\Lambda}/\Lambda$, which often enjoys stronger lifting properties. Fix $d \geq 1$ and let $\Lambda$ be a rank $d$ torsion-free abelian group with no free summand. Let $\Lambda_\Lambda = (\Lambda^{(0)} \leq \Lambda^{(1)} \leq \cdots)$ be the cofiltration associated to $\Lambda$ via the functor $(\lim_{\leftarrow})^{-1}$, and let $\Lambda = (\Lambda^{(0)} \geq \Lambda^{(1)} \geq \cdots)$ be dual filtration. We denote by $\hat{\Lambda^{(m)}}$ the profinite completion of $\Lambda^{(m)}$ with respect to the filtration:

$$(\Lambda^{(m)} \geq \Lambda^{(0)} \geq \Lambda^{(1)} \geq \cdots).$$

For each $m \in \omega$, the inclusion map $\Lambda^{(m)} \rightarrow \Lambda^{(m+1)}$ extends to a topological embedding $\hat{\Lambda^{(m)}} \rightarrow \hat{\Lambda^{(m+1)}}$. We define $\hat{\mathbb{Q}}^d_{\Lambda}$ to be the inductive limit of the sequence $(\hat{\Lambda^{(m)}})_{m \in \omega}$. Notice that $\hat{\mathbb{Q}}^d_{\Lambda}$ is a locally profinite group that contains
Let $\Lambda$ as a dense subgroup. We say that $\hat{\mathbb{Q}}_d^+\Lambda$ is the \textit{locally profinite completion of $\mathbb{Q}^d$ with respect to $\Lambda$}. Notice that $\hat{\mathbb{Z}}_\Lambda^d$ is an essential retraction of $\hat{\mathbb{Q}}_d^+\Lambda$ with $\hat{\mathbb{Z}}_\Lambda^d \cap \Lambda = \Lambda^{(0)} = \mathbb{Z}^d$. By Lemma 4.1 the inclusion map $\hat{\mathbb{Z}}_\Lambda^d \to \hat{\mathbb{Q}}_d^+\Lambda$ induces a definable isomorphism between $\hat{\mathbb{Z}}_\Lambda^d/\mathbb{Z}^d$ and $\hat{\mathbb{Q}}_d^+\Lambda/\Lambda$. The following example illustrates that the above procedure is a group-theoretic analogue of the construction of the field of fractions in the theory of rings.

**Example 6.10.** Let $p$ be a prime number and let $\mathbb{Q}_p$ be the field of all $p$-adic numbers, i.e. the field of fractions of the ring $\mathbb{Z}_p$ of all $p$-adic integers — see [61, Chapter 1] for a primer on the rings $\mathbb{Z}_p$ and $\mathbb{Q}_p$. Let also $\mathbb{Z}[1/p]$ be the subring of $\mathbb{Q}_p$ that is generated by $1/p$. If $\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{Z}[1/p]$ are viewed as Polish abelian groups with respect to their additive structure then $\mathbb{Q}_p$ is simply the locally profinite completion of $\mathbb{Q}$ with respect to $\Lambda := \mathbb{Z}[1/p]$, as above. More generally, if $\Lambda := \mathbb{Z}[1/p]^d \leq \mathbb{Q}^d$, then $\hat{\mathbb{Z}}_\Lambda^d$ is isomorphic to $\mathbb{Z}_p^d$ and $\hat{\mathbb{Q}}_d^+\Lambda$ is isomorphic to $\mathbb{Q}_p^d$.

Let $\Lambda, \Lambda' \in \text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$ be finite rank torsion-free abelian groups with no free summands. A homomorphism $g: \Lambda' \to \Lambda$ is a T-	extit{homomorphism} if $g^T$ maps $\Lambda$ to $\Lambda'$. Here, $g^T$ is the homomorphism from the divisible hull of $\Lambda$ to the divisible hull of $\Lambda'$ associate with the transpose of the matrix with rational coefficients associated with $g$ with respect to some choice of maximal independent sets in $\Lambda'$ and $\Lambda$ respectively. Let $\text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$ be the subcategory of $\text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$ which contains the same objects but whose arrows are precisely all T-homomorphisms. Notice that this new category contains strictly fewer isomorphisms than $\text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$.

**Example 6.11.**

1. Let $\Lambda = \Lambda' = \mathbb{Q}^d$ and notice that the \textit{zero homomorphism} $(x, y) \mapsto (0, 0)$ is a T-homomorphism since its transpose is also $(x, y) \mapsto (0, 0)$.
2. Let $\Lambda = \Lambda' = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/6] \leq \mathbb{Q}^2$ and let $\varphi$ be the assignment $(x, y) \mapsto (x, 2x+y)$. Notice that $\varphi: \Lambda' \to \Lambda$ is an isomorphism. However, the assignment $\varphi^T: (x, y) \mapsto (x+2y, y)$ will fail to map $(0, 1/6) \in \Lambda$ to $\Lambda'$.

Let now $g: \Lambda' \to \Lambda$ be a T-homomorphism and notice that $g^T: \Lambda \to \Lambda'$ extends to a continuous homomorphism:

$$
\hat{g}^T: \hat{\mathbb{Q}}_\Lambda^d \to \hat{\mathbb{Q}}_{\Lambda'}^d.
$$

To see this, notice that it will suffice to show that for each $m \in \omega$, there is some $n_m \in \omega$ so that $\hat{g}^T \upharpoonright \hat{\Lambda}_{(m)}$ is a continuous homomorphism from $\hat{\Lambda}_{(m)}$ to $\hat{\Lambda}'_{(n_m)}$. It is easy to see that any choice of $n_m$ with $g^T(\Lambda_{(m)}) \subseteq \Lambda'_{(n_m)}$ works. Since $\hat{g}^T: \hat{\mathbb{Q}}_\Lambda^d \to \hat{\mathbb{Q}}_{\Lambda'}^d$, is a continuous homomorphism with $\hat{g}^T(\Lambda) \subseteq \Lambda'$, it induces a definable homomorphism:

$$
\hat{g}^T: \hat{\mathbb{Q}}_\Lambda^d/\Lambda \to \hat{\mathbb{Q}}_{\Lambda'}^d/\Lambda'.
$$

It follows that the assignment $\Lambda \mapsto \hat{\mathbb{Q}}_\Lambda^d/\Lambda$, $g \mapsto \hat{g}^T$ is a contravariant functor from the category $\text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$ to the category of groups with a Polish cover. Definable homomorphisms $\hat{\mathbb{Q}}_\Lambda^d/\Lambda \to \hat{\mathbb{Q}}_{\Lambda'}^d/\Lambda'$, which are of the form $\hat{g}^T$ for some $g: \Lambda' \to \Lambda$, admit by definition a lift $\tilde{g}^T: \tilde{\mathbb{Q}}_\Lambda^d \to \tilde{\mathbb{Q}}_{\Lambda'}^d$ which has the additional property of being a continuous homomorphism. It turns out that every definable homomorphism which has this additional property is of the form $\hat{g}^T$ for some $g: \Lambda' \to \Lambda$:

**Theorem 6.12.** The assignment $\Lambda \mapsto \hat{\mathbb{Q}}_\Lambda^d/\Lambda$, $g \mapsto \hat{g}^T$ defines a fully faithful contravariant functor from the category $\text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$ to the category of groups with a Polish cover $G/N$, where morphisms $G/N \to G'/N'$ are continuous homomorphisms $G \to G'$ that map $N$ to $N'$.

**Proof.** We first show that the functor is faithful. Let $g: \Lambda' \to \Lambda$ be in $\text{Groups}_+^T(\mathbb{Z}^*, \mathbb{Q}^*)$ and assume that $\tilde{g}^T: \tilde{\mathbb{Q}}_\Lambda^d/\Lambda \to \tilde{\mathbb{Q}}_{\Lambda'}^d/\Lambda'$ is the zero homomorphism. Since $\tilde{g}^T$ is a continuous and $\Lambda'$ is countable, there exists $b \in \Lambda'$ so that the set

$$
G(b) := \{x \in \tilde{\mathbb{Q}}_\Lambda^d: \tilde{g}^T(x) = b\}
$$

is non-meager. By Pettis’ Lemma $G(b) - G(b)$ contains a clopen subgroup of $\tilde{\mathbb{Q}}_\Lambda^d$. Thus, there is $m \in \omega$ so that

$$
\text{cl}(\Lambda^{(m)}) \cap \tilde{\mathbb{Z}}_\Lambda^d = \text{cl}(\Lambda^{(m)}) \cap \hat{\Lambda}_{(0)} = G(b) - G(b).
$$

It follows that $g^T(\Lambda^{(m)}) = 0$ and since $\Lambda^{(m)}$ is a rank $d$ subgroup of $\mathbb{Z}^d$, we have that $g^T = 0$. Hence $g = 0$.

We now show that the functor is full. Let $f: \tilde{\mathbb{Q}}_\Lambda^d \to \tilde{\mathbb{Q}}_{\Lambda'}^d$ be a continuous homomorphism with $f(\Lambda) \subseteq \Lambda'$. In particular, $h := f \upharpoonright \Lambda$ is a homomorphism from $\Lambda$ to $\Lambda'$. We set $g := h^T$ and we claim that $g$ is the desired
T-homomorphism. To this end, we only need to show that \( g \) is a homomorphism from \( \Lambda' \) to \( \Lambda \). Notice that since \( f \upharpoonright \Lambda'_{0} = \hat{\mathbb{Z}}_{\Lambda}^{d} \) is continuous, there exists an increasing sequence \((m_{k})_{k \in \omega}\) so that \( h(\Lambda(\Lambda')_{0}) \subseteq (\Lambda')^{(k)} \). Hence,

\[
g(\Lambda'_{(k)}) = h^{T}((\Lambda')^{(k)}) \subseteq \Lambda^{(m_{k})} = \Lambda(\Lambda'_{0}),
\]

and therefore \( g \) is a homomorphism from \( \Lambda' \) to \( \Lambda \).

As a consequence we have the following corollary.

**Corollary 6.13.** Let \( \Lambda \) be a subgroup of \( \mathbb{Q}_{d}^{d} \) with the property that for all \( \alpha \in \text{Aut}(\Lambda) \) we have that \( \alpha^{T} \in \text{Aut}(\Lambda) \). Then every definable automorphism of \( \hat{\mathbb{Q}}^{d}_{\Lambda}/\Lambda \) lifts to a topological group automorphism of \( \hat{\mathbb{Q}}^{d}_{\Lambda} \).

### 7. The definable content of the Ext functor

Let \( F \) and \( B \) be two countable abelian groups. By an extension \( \mathcal{E} \) of \( B \) by \( F \) we mean any short exact sequence

\[
0 \longrightarrow F \longrightarrow E \longrightarrow B \longrightarrow 0
\]

of homomorphisms of abelian groups. We call \( F \) the fiber of the extension \( \mathcal{E} \) and we identify it with its image in \( E \). We also call \( B \) the base of the extension \( \mathcal{E} \). The extensions \( \mathcal{E} \) and \( \mathcal{E}' \) are isomorphic if there is a group isomorphism \( E \to E' \) which makes the following diagram commute.

![Diagram](https://via.placeholder.com/150)

We denote by \( \text{Ext}(B, F) \) the collection of all isomorphism classes of extensions of \( B \) by \( F \). The fact that \( \text{Ext}(B, F) \) admits an abelian group structure goes back to Schreier and Baer [3, 64]. The induced bifunctor \( \text{Ext} \) is the first derived functor of the bifunctor \( \text{Hom} \), and it has played a fundamental role in the development of homological algebra since the seminal work [15] of Eilenberg and MacLane, where it was shown that classical invariants from algebraic topology can be defined, and related, in terms of \( \text{Ext} \).

While \( \text{Ext}(B, F) \) carries a natural group topology, in standard treatments of \( \text{Ext}(B, F) \) this topology is ignored, and \( \text{Ext}(B, F) \) is treated as a discrete group. One of the reasons may be that this natural topology is not always Hausdorff. This fact was already understood by Eilenberg and MacLane in [15], where \( \text{Ext}(B, F) \) was termed a “generalized topological group” whenever the natural topology was not Hausdorff. In this section we show how to view \( \text{Ext} \) as a bifunctor from the category of countable abelian groups to the category of abelian groups with a Polish cover. The resulting bifunctor \textit{definable} \( \text{Ext} \) is a much stronger invariant for classifying algebraic systems. For example, for every \( d, d' \in \mathbb{N} \) we have that we have that \( \text{Ext}(\mathbb{Q}_{d}^{d}, \mathbb{Z}) \) and \( \text{Ext}(\mathbb{Q}_{d'}^{d'}, \mathbb{Z}) \) are isomorphic but not definably isomorphic. More generally, the following is the main theorem of this section, whose comparison with Corollaries 7.7, 7.8, and 7.9, illustrates how much coarser the usual \( \text{Ext}(\cdot, \mathbb{Z}) \) functor is as a group invariant.

**Theorem 7.1.** The definable \( \text{Ext}(\cdot, \mathbb{Z}) \) functor is fully faithful when restricted to the category of finite rank torsion-free abelian groups with no free summands.

Let \( F \) and \( B \) be countable abelian groups. One may describe any extension \( E \) of \( B \) by \( F \) directly using \( F \) and \( B \) alone. Indeed, if \( s \colon B \to E \) is a section of the epimorphism \( E \to B \) the multiplication table of \( E \) is entirely determined by the unique function \( c_{s} \colon B \times B \to F \) which satisfies

\[
s(x) + s(y) = s(x + y) + c_{s}(x, y).
\]

If \( t \colon B \to E \) is another section of the same epimorphism, then the function \( h \colon B \to F \) with \( h(x) = s(x) - t(x) \) is a witness to the fact that \( c_{s} \) and \( c_{t} \) represent the same extension, which is entirely defined in terms of \( F \) and \( B \). These observations give rise to the following concrete description of \( \text{Ext}(B, F) \) and related group \( \text{PExt}(B, F) \); see [15, 21, 63].
• $Z(B,F)$ is the closed subgroup of the abelian Polish group $F_B \times B$, consisting of all cocycles on $B$ with coefficients in $F$. The topology on $F_B \times B$ is the product topology and $F$ is the discrete group. By a cocycle on $B$ with coefficients in $F$ we mean any function $c : B \times B \to F$ so that for all $x, y, z \in B$ we have:
  (1) $c(x, 0) = c(0, y) = 0$;
  (2) $c(x, y) + c(x + y, z) = c(x, y + z) + c(y, z)$;
  (3) $c(x, y) = c(y, x)$, for all $x, y \in B$.

• $C(B,F)$ is the abelian Polish group $F_B$, where $F$ is endowed with the discrete topology and $F_B$ with the product topology. We have a continuous group homomorphism $\delta : C(B,F) \to Z(B,F)$, given by:
  \[
  \delta(h)(x, y) := h(x) + h(y) - h(x + y).
  \]

• $B(B,F)$ is the Polishable Borel subgroup $\delta(C(B,F))$ of $Z(B,F)$; see Lemma 2.3.Explicitly:
  \[
  B(B,F) := \{ c \in Z(B,F) : c(x, y) = h(x) + h(y) - h(x + y), \text{ for some } h \in C(B,F) \}.
  \]

A coboundary on $B$ with coefficients in $F$ is a cocycle $c : B \times B \to F$, which lies in $B(B,F)$.

• $B_\omega(B,F)$ is the closed subgroup of $Z(B,F)$ consisting of all $c \in Z(C,A)$ with $(c \upharpoonright S \times S) \in B(S,F)$, for every finite subgroup $S$ of $B$. A weak coboundary on $B$ with coefficients in $F$ is any element of $B_\omega(B,F)$.

**Definition 7.2.** Let $F$ and $B$ be countable abelian groups.

1. Let $\text{Ext}(B,F) := Z(B,F)/B(B,F)$ be the group with a Polish cover of all extensions of $B$ by $F$;
2. Let $\text{PExt}(B,F) := B_\omega(B,F)/B(B,F)$ be the group with a Polish cover of all pure extensions of $B$ by $F$;
3. Let $\text{Ext}_w(B,F) := Z(B,F)/B_\omega(B,F)$ be the Polish group of all weak extension classes of $B$ by $F$.

To justify the terminology in point (2) above, recall that a subgroup $F$ of $E$ is called pure if for all $\ell \in \mathbb{Z}$ we have that $F \cap (\ell E) = \ell F$. A well known property about extensions $0 \to F \to E \to S \to 0$ of finite abelian groups $S$ is that they are trivial (that is, they correspond to coboundaries), if and only if $F$ is a pure subgroup of $E$; see [21, Chapter V]. Since $\text{Ext}_w(B,F)$ is a Polish group, the non-trivial definable content of $\text{Ext}(B,F)$ concentrates in $\text{PExt}(B,F)$. In fact, notice that under the assumption of Theorem 7.1 that $B$ is torsion-free, we have that every extension of $B$ by $F$ is pure. That is, $\text{Ext}(B,F) = \text{PExt}(B,F)$. We can therefore concentrate on analyzing the definable content of $\text{PExt}(B,F)$.

It is not difficult to see that $\text{Ext}(-,-)$ and $\text{PExt}(-,-)$ are additive bifunctors from the category of countable abelian groups to the category of abelian Polish groups with a Polish cover, which are contravariant in the first coordinate and covariant in the second coordinate. Similarly, $\text{Ext}_w(-,-)$ is a bifunctor to the category of Polish groups which is contravariant in the first argument and covariant in the second argument. The next lemma expresses the groups with a Polish cover associated to $\text{PExt}(-,-)$ in terms of the bifunctor $\text{Hom}(-,-)$.

**Lemma 7.3.** Let $F$, $B$, and $G$ be countable abelian groups. Given any pure extension $E$ of $B$ by $F$:

\[
0 \to F \xrightarrow{f} E \xrightarrow{\pi} B \to 0,
\]

with $\text{Pext}(E,G) = 0$, the group with a Polish cover $\text{PExt}(B,G)$ is definably isomorphic to

\[
\frac{\text{Hom}(F,G)}{f^*(\text{Hom}(F,G))},
\]

where $f^* : \text{Hom}(E,G) \to \text{Hom}(F,G)$ is the image of $f : F \to E$ under the contravariant functor $\text{Hom}(-,G)$.

**Proof.** Let $c_E \in B_\omega(B,F)$ be a weak coboundary representing the pure extension $E$. Notice that $c_E$ induces a continuous group homomorphism $(c_E)^* : \text{Hom}(F,G) \to B_\omega(B,G)$ defined by

\[
(c_E)^*(\eta) = \eta \circ c_E.
\]

This induces a the group homomorphism $E^* : \text{Hom}(F,G) \to \text{PExt}(B,G)$ given by

\[
E^*(\eta) = \eta \circ c_E + B(B,F).
\]

As shown in [21, Theorem 53.7], $E^*$ defines the connecting morphism in the exact sequence

\[
0 \to \text{Hom} (B,G) \to \text{Hom} (E,G) \to \text{Hom} (F,G) \xrightarrow{E^*} \text{PExt} (B,G) \to \text{PExt} (E,G) \to \text{PExt} (F,G) \to 0.
\]

The rest follows by the assumption that $\text{PExt}(E,G) = 0$. \qed
By an inductive sequence of countable abelian groups we mean a collection \((B_n, \eta_{n+1,n})_{n<\omega}\) of homomorphisms \(\eta_{n+1,n} : B_n \to B_{n+1}\) between countable abelian groups. As in the case of towers in Section 5, we also consider the induced maps \(\eta_{m,n} : B_n \to B_m\) for all \(n \leq m\), where \(\eta_{n,n} = \text{id}_{B_n}\), and \(\eta_{m,n} := \eta_{m,m-1} \circ \cdots \circ \eta_{n+1,n}\) if \(m > n\). Notice that cotowers are special types of inductive sequences. Morphisms of inductive sequences are defined similarly to the cotower maps; see Section 6. Notice that when \(B, G\) are countable abelian groups then \(\text{Hom}(B, G)\) is a Polish abelian group. Moreover, since \(\text{Hom}(-, -)\) is contravariant in the first argument, any inductive sequence

\[ B_0 \to B_1 \to B_2 \to \cdots \]

of countable abelian groups, induces a tower of Polish abelian groups:

\[ \text{Hom}(B_0, G) \leftarrow \text{Hom}(B_1, G) \leftarrow \text{Hom}(B_2, G) \leftarrow \cdots \]

The following theorem is essentially due to C.U. Jensen; see [63, Theorem 6.1] and [34].

**Theorem 7.4** (Jensen). Let \(G\) be a countable abelian group and let \((B_n, \eta_{n+1,n})\) be an inductive sequence of countable abelian groups with \(\text{PExt}(B_n, G) = 0\) for all \(n \in \omega\). Then the groups with a Polish cover \(\varprojlim (\text{Hom}(B_n, G))\) and \(\text{PExt}(\varprojlim_n B_n, G)\) are naturally definably isomorphic.

**Proof.** Set \(B := \varprojlim_n B_n\) and let \(\oplus_n B_n\) denote the direct sum of the groups \(B_n\), indexed by \(n \in \omega\):

\[ \bigoplus_{n \in \omega} B_n. \]

For every \(n \in \omega\) and \(b \in B_n\), let \(b_n\) be the element of \(\oplus_n B_n\) with all the coordinates equal to 0 apart from the \(n\)th coordinate, which is equal to \(b\). Notice that the homomorphism \(\delta : \oplus_n B_n \to \oplus_n B_n\) which is defined on the generators \(b_n\) of \(\oplus_n B_n\) by \(\delta(b_n) = b_n - \eta_{n+1,n}(b)e_{n+1}\) is injective. Indeed, every \(b \in \oplus_n B_n\) is of the form

\[ b = b_0e_{n_0} + b_1e_{n_1} + \cdots + b_\ell e_{n_\ell}, \text{ with } n_0 < n_1 < \cdots < n_\ell. \]

But then, \(\delta(b) = 0\) implies that \(b_0 = 0\); and inductively for each \(i < \ell\), \(\delta(b_i) = 0\), \(b_0 = 0, \ldots, b_i = 0\), imply \(b_{i+1} = 0\). It is also immediate that the cokernel of \(\oplus_n B_n\) is \(B\). Hence \(\delta\) defines an extension of \(B\) by \(\oplus_n B_n\):

\[ 0 \to \oplus_n B_n \xrightarrow{\delta} \oplus_n B_n \to B \to 0. \]

This extension is also pure. Indeed, using (2) above and by inducting on \(i\) (\(0 \leq i \leq \ell\)), one shows that if \(\delta(b)\) is divisible by \(\ell \in \mathbb{Z}\) in \(\oplus_n B_n\), then each \(b_i\) is divisible by \(\ell \) in \(B_n\), which implies that \(b\) is divisible by \(\ell \) in \(\oplus_n B_n\). Moreover, by standard computations (see, [75, Proposition 3.3.4]) and since \(\text{PExt}(B_n, G) = 0\), for all \(n\), we have

\[ \text{PExt}(\oplus_n B_n, G) \cong \prod_{n \in \omega} \text{PExt}(B_n, G) = 0. \]

By Lemma 7.3 we have that \(\text{PExt}(B, G)\) is definably isomorphic to

\[ \frac{\text{Hom}(\oplus_n B_n, G)}{\delta^*(\text{Hom}(\oplus_n B_n, G))}. \]

We will show that the latter is definable isomorphic to \(\varprojlim A := Z(A)/B(A)\), where \(A\) is the tower of Polish groups

\[ A^{(n)} := \text{Hom}(B_n, G) \]

and bonding maps \(p^{(n,n+1)} := \text{Hom}(\eta_{n+1,n}, G)\). Consider the map

\[ F : Z(A) \to \text{Hom}(\oplus_n B_n, G), \]

which sends \((a_{m_0, m_1})_{m_0 \leq m_1}\) to the homomorphism \(ce_n \mapsto a_{n,n+1}(c)\). Clearly \(F\) is a continuous and surjective homomorphism with \(F(B(A)) = \delta^*(\text{Hom}(\oplus_n B_n, G))\). The fact that it is injective follows from Remark 5.3. \(\square\)

Notice that every countable abelian group \(B\) can be written as an increasing union of finitely-generated abelian groups \(B_n\). By the classification theorem of finitely-generated abelian groups we have that \(\text{PExt}(B_n, F)\) vanishes for each \(n\). Hence, the above theorem can be used to calculate \(\text{PExt}(B, F)\) for every pair of countable abelian groups \(B\) and \(F\). In the rest of this section we calculate \(\text{Ext}(\Lambda, Z)\) for each finite-rank torsion-free abelian group \(\Lambda\).

Let \(\Lambda\) be a finite-rank torsion-free abelian group. Notice that \(\Lambda\) can be written as a direct sum \(\Lambda_0 \oplus \Lambda_1\) where \(\Lambda_0\) has no free summand and \(\Lambda_1\) is finitely-generated free abelian group \(\mathbb{Z}^n\). We have that

\[ \text{Ext}(\Lambda, Z) = \text{Ext}(\Lambda_0, Z) \oplus \text{Ext}(\Lambda_1, Z) = \text{Ext}(\Lambda_0, Z). \]
Thus, without any loss of generality, we may restrict our attention to the case where $\Lambda$ has no free summand. Recall from Section 6 that we have a fully faithful contravariant functor from the category $\text{Groups}_+(\mathbb{Z}^*, \mathbb{Q}^*)$ of finite rank torsion-free abelian groups with no free summands to the category of groups with a Polish cover which assigns to each group $\Lambda$ in $\text{Groups}_+(\mathbb{Z}^*, \mathbb{Q}^*)$ the quotient $\hat{\mathbb{Z}}^d_\Lambda / \mathbb{Z}^d$ of a profinite completion $\hat{\mathbb{Z}}^d_\Lambda$ of $\mathbb{Z}^d$ by $\mathbb{Z}^d$; see Theorem 6.6. The following theorem, without the definability claim, is proved for the special case of rank 1 abelian groups in [15, Appendix B]

**Theorem 7.5.** The functors implementing $\Lambda \mapsto \hat{\mathbb{Z}}^d_\Lambda / \mathbb{Z}^d$ and $\Lambda \mapsto \text{Ext}(\Lambda, \mathbb{Z})$ from the category $\text{Groups}_+(\mathbb{Z}^*, \mathbb{Q}^*)$ to the category of groups with a Polish cover are naturally isomorphic.

**Proof.** Recall that the functor implementing $\Lambda \mapsto \hat{\mathbb{Z}}^d_\Lambda / \mathbb{Z}^d$ in Section 6 is the composition of the functors $(\lim)^{-1}$, $\text{Adj}$, and $\lim^1$. By Lemma 6.4(1) we have that the functor implementing $\Lambda \mapsto \hat{\mathbb{Z}}^d_\Lambda / \mathbb{Z}^d$ is naturally isomorphic to the functor implementing $\Lambda \mapsto \lim^1(\text{Hom}(\Lambda(n), G))$ in Theorem 7.4, where $(\Lambda(n)) = (\lim)^{-1}(\Lambda)$. But Theorem 7.4 exhibits a definable isomorphism from $\lim^1(\text{Hom}(\Lambda(n), G))$ to $P\text{Ext}(\Lambda, G)$. By a routine diagram chasing of the proof of Theorem 7.4 we see that the latter definable isomorphisms are components of a natural transformation from the functor implementing $\Lambda \mapsto \lim^1(\text{Hom}(\Lambda(n), G))$ to the functor implementing $\Lambda \mapsto P\text{Ext}(\Lambda, G)$. Finally, notice that since each $\Lambda$ is torsion-free we have that $\text{Ext}(\Lambda, \mathbb{Z}) = P\text{Ext}(\Lambda, \mathbb{Z})$.

We have the following immediate corollary.

**Corollary 7.6.** The functor $\Lambda \mapsto \text{Ext}(\Lambda, \mathbb{Z})$ from the category of finite-rank torsion-free abelian groups with no free summands to the category of groups with a Polish cover is fully faithful. In particular, two such groups $\Lambda$ and $\Lambda'$ are isomorphic, if and only if $\text{Ext}(\Lambda, \mathbb{Z})$ and $\text{Ext}(\Lambda', \mathbb{Z})$ are definably isomorphic.

The following corollaries demonstrate how forgetting the definable content of $\text{Ext}$ results in much weaker invariants. We denote by $\mathcal{P}$ the set of prime numbers. The following theorem is an adaptation of [74, Theorem 2].

**Theorem 7.7.** Let $\Lambda$ be a rank $d$ torsion-free abelian group with no free summands. For $p \in \mathcal{P}$ let $n_p \in \{0, 1, \ldots, d\}$ be such that the $p$-corank of $\Lambda$, $[\Lambda : p\Lambda]$, is $p^{n_p}$. Then $\text{Ext}(\Lambda, \mathbb{Z})$ is isomorphic as a discrete group to

$$\mathbb{Q}^{(2^{n_p})} \oplus \bigoplus_{p \in \mathcal{P}} \mathbb{Z}(p^\infty)^{n_p}.$$ 

In particular, if $\Lambda$ and $\Lambda'$ are finite rank torsion-free abelian groups with no free summands, then $\text{Ext}(\Lambda, \mathbb{Z})$ and $\text{Ext}(\Lambda', \mathbb{Z})$ are isomorphic as discrete groups if and only if $[\Lambda : p\Lambda] = [\Lambda' : p\Lambda']$ for every $p \in \mathcal{P}$.

**Corollary 7.8.** The relation $E$ for finite-rank torsion-free abelian groups defined by $(\Lambda, \Lambda') \in E$ if and only if $\text{Ext}(\Lambda, \mathbb{Z})$ and $\text{Ext}(\Lambda', \mathbb{Z})$ are isomorphic as discrete groups, is smooth.

Now adopt the following notations. For every sequence $\mathbf{m} = (m_p)_{p \in \mathcal{P}} \in \mathbb{N}^\mathcal{P}$, where $\mathbb{N}$ is the set of strictly positive integers, define $\mathbb{Z}[\frac{1}{\mathbf{m}}]$ to be the set of rational numbers of the form $a/b$ where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and for every $p \in \mathcal{P}$ and $k \in \mathbb{N}$, if $p^k$ divides $b$ then $k \leq m_p$. Write $\mathbf{m} = \ast \mathbf{n}$ if and only if $\{p \in \mathcal{P} : m_p \neq n_p\}$ is finite. From Theorem 7.5 we obtain the following.

**Corollary 7.9.** Fix $d \geq 1$. For every $\mathbf{m}, \mathbf{n} \in \mathbb{N}^\mathcal{P}$, $\text{Ext}(\mathbb{Z}[\frac{1}{\mathbf{m}}]^d, \mathbb{Z})$ and $\text{Ext}(\mathbb{Z}[\frac{1}{\mathbf{n}}]^d, \mathbb{Z})$ are isomorphic as discrete groups, and are Borel isomorphic if and only if $\mathbf{m} = \ast \mathbf{n}$. In particular, the collection

$$\left\{ \text{Ext}(\mathbb{Z}[\frac{1}{\mathbf{m}}]^d, \mathbb{Z}) : \mathbf{m} \in \mathbb{N}^\mathcal{P} \right\}$$

contains a continuum of groups with a Polish cover that are pairwise isomorphic as discrete groups but not definably isomorphic.

**Proof.** Fix $\mathbf{m} \in \mathbb{N}^\mathcal{P}$. For $\mathbf{m}, \mathbf{n} \in \mathbb{N}^\mathcal{P}$ it follows from Baer’s classification of countable rank-1 torsion-free abelian groups [22, Chapter 12, Theorem 1.1] that $\mathbb{Z}[\frac{1}{\mathbf{m}}]$ and $\mathbb{Z}[\frac{1}{\mathbf{n}}]$ are isomorphic if and only if $\mathbf{m} = \ast \mathbf{n}$. It follows from this and [22, Chapter 12, Theorem 3.5] that $\mathbb{Z}[\frac{1}{\mathbf{m}}]^d$ and $\mathbb{Z}[\frac{1}{\mathbf{n}}]^d$ are isomorphic if and only if $\mathbf{m} = \ast \mathbf{n}$. Together with Corollary 6.7, this concludes the proof of the first assertion. Observe now that for every $p \in \mathcal{P}$, the $p$-corank of $\mathbb{Z}[\frac{1}{\mathbf{m}}]^d$ is equal to $d$. Thus the second assertion is an immediate consequence of Theorem 7.7. \(\square\)
Remark 7.10. For Baer invariants $m$ that have a prime with infinite multiplicity, it is no longer true that the groups $\text{Ext}(\mathbb{Z}[\frac{1}{m}], \mathbb{Z})$ are all isomorphic as discrete groups.

8. Actions by definable automorphisms and Borel reduction complexity

Let $G$ denote the group with a Polish cover $0 \to N \to G \to G/N \to 0$. The group $\text{Aut}(G)$ of all definable automorphisms of $G$ is the automorphism group of $G$ in the category of groups with a Polish cover. Explicitly, $\text{Aut}(G)$ consists of those group automorphisms $\varphi : G/N \to G/N$ which admit a Borel map $\hat{\varphi} : G \to G$ as a lift. This defines a canonical action $\text{Aut}(G) \curvearrowright G/N$ of $\text{Aut}(G)$ on the quotient $G/N$.

Definition 8.1. By a definable action of a discrete group $\Gamma$ on a group with a Polish cover $G$ we mean a group homomorphism $\varphi : \Gamma \to \text{Aut}(G)$. The assignment $\gamma \mapsto \varphi_\gamma$ induces an action $(\varphi_\gamma)_\gamma : \Gamma \actson G/N$ on the quotient $G/N$. If $\Gamma \actson G/N$ is definable, we let $\mathcal{R}(\Gamma \actson G/N)$ be the equivalence relation on $G$ so that for $x, y \in G$ we have:

$$x, y \in \mathcal{R}(\Gamma \actson G/N) \iff \exists \gamma \in \Gamma \ (\varphi_\gamma(Nx) = Ny).$$

Notice that when both $\Gamma$ and $N$ are countable, then $\mathcal{R}(\Gamma \actson G/N)$ is a countable Borel equivalence relation.

The first of these problems is particularly natural and it can be thought of as the “base-free” analogue of the extension problem we show in Section 7. Indeed, for any countable abelian groups $B, F$ the Polish space $Z(B, F)$ parametrizes all ways of describing an extension $E$ of the base $B$ by $F$, and the coset equivalence relation induced by $B(F, B)$ is definably isomorphic.

Theorem 6.12, and $\mathcal{R}(\Gamma \actson G/N)$ represents the problem of classifying all ways of describing extensions of $B$ by $F$ up to base-free isomorphisms. That is, isomorphisms which preserve the base $B$ only up to an automorphism of $G$.

In the context of Theorem 7.5, and since $\mathbb{Z}_\Lambda^d / \mathbb{Z}^d$ and $\hat{\mathbb{Q}}^d_\Lambda$ are definably isomorphic when $\Lambda = \lim_{\Lambda}$ (Adj($\Lambda$)), the next lemma implies that the above three equivalence relations are of the exact same Borel reduction complexity when $\Lambda$ is fixed.

Definition 8.2. Two definable actions $(\varphi_\gamma)_\gamma : \Gamma \actson G/N$ and $(\varphi'_\gamma)_\gamma : \Gamma' \actson G'/N'$ are definably isomorphic if there is a group isomorphism $\alpha : \Gamma \to \Gamma'$ and a definable isomorphism $\psi : G/N \to G'/N'$ so that for all $\gamma \in \Gamma$ we have:

$$\psi \circ \varphi_\gamma = \varphi'_\alpha(\gamma) \circ \psi.$$

The following lemma implies that isomorphic definable actions generate equivalence relations of equal complexity in the Borel reduction hierarchy. The proof follows directly from the fact that, for countable Borel equivalence relations, being classwise Borel isomorphic is the same as being Borel bireducible; see [37, Theorem 2.5].

Lemma 8.3. Let $\Gamma \actson G/N$ and $\Gamma' \actson G'/N'$ be definably isomorphic actions on the groups with a Polish cover $G = (N, G, G/N)$ and $G' = (N', G', G'/N')$, where $\Gamma, \Gamma', N, N'$ are countable. Then, $\mathcal{R}(\Gamma \actson G/N)$ and $\mathcal{R}(\Gamma' \actson G'/N')$ are classwise Borel isomorphic. In particular, if $G/N$ is definably isomorphic to $G'/N'$, then $\mathcal{R}(\Gamma \actson G/N)$ and $\mathcal{R}(\Gamma' \actson G'/N')$ are classwise Borel isomorphic.

Many results in the complexity theory of Borel reductions between equivalence relations $R$ depend on expressing the pertinent equivalence relation as an orbit equivalence relation of a continuous (or measure preserving) action of a Polish group on the underlying space of $R$. In general, there is no natural continuous or (Haar) measure preserving action on $Z(\Lambda, \mathbb{Z}), \mathbb{Z}_\Lambda^d$, or $\hat{\mathbb{Q}}^d_\Lambda$ which induces the equivalence relations from (3) above. However, when $\Lambda$ has the additional property that $\alpha \in \text{Aut}(\Lambda) \iff \alpha T \in \text{Aut}(\Lambda)$ for every $\alpha \in \text{GL}_d(\mathbb{Q})$, then Corollary 6.13 implies that $\mathcal{R}(\text{Aut}(\hat{\mathbb{Q}}^d_\Lambda) \actson \hat{\mathbb{Q}}^d_\Lambda)$ is simply the orbit equivalence relation $\mathcal{R}(\text{Aut}(\Lambda) \actson \hat{\mathbb{Q}}^d_\Lambda)$.
of a continuous action of the countable group Aut(Λ) ↷ Λ on the locally profinite space \( \hat{Q}^d \). In particular, if we set \( \Lambda := \mathbb{Z}[1/p]^d \leq \mathbb{Q}^d \) to be as in Example 6.10, we have that
\[
R(\text{Aut}(\mathbb{Q}_p^d / \mathbb{Z}[1/p]^d) \bowtie \mathbb{Q}_p^d / \mathbb{Z}[1/p]^d)
\]
is classwise Borel isomorphic to the orbit equivalence relation
\[
R(\text{GL}_d(\mathbb{Z}[1/p]) / \mathbb{Z}[1/p]^d \bowtie \mathbb{Q}_p^d),
\]
associated with the affine action \( \text{GL}_d(\mathbb{Z}[1/p]) \bowtie \mathbb{Z}[1/p]^d \bowtie \mathbb{Q}_p^d \). The following is the main result of this section. The proof will rely on ideas and results from [1, 10, 28, 31, 32, 70, 71] which we review below.

**Theorem 8.4.** Fix \( m, d \geq 1 \), and prime numbers \( p, q \geq 2 \). Let \( \Gamma \) be a subgroup of \( \text{GL}_m(\mathbb{Z}[1/q]) \) containing a finite index subgroup of \( \text{SL}_m(\mathbb{Z}) \), and \( \Delta \) be a subgroup of \( \text{GL}_d(\mathbb{Z}[1/p]) \).

1. If \( m > d \), then
\[
R(\Gamma \bowtie \mathbb{Z}[1/q]^m \bowtie \mathbb{Q}_q^m) \not \leq_B R(\Delta \bowtie \mathbb{Z}[1/p]^d \bowtie \mathbb{Q}_p^d).
\]
2. If \( p, q \) are distinct, \( m \geq 3 \), and \( d \geq 2 \) then
\[
R(\Gamma \bowtie \mathbb{Z}[1/q]^m \bowtie \mathbb{Q}_q^m) \not \leq_B R(\Delta \bowtie \mathbb{Z}[1/p]^d \bowtie \mathbb{Q}_p^d).
\]
3. If \( \Delta \) is abelian then \( R(\Delta \bowtie \mathbb{Z}[1/p]^d \bowtie \mathbb{Q}_p^d) \) is hyperfinite and not smooth.
4. If \( d \geq 2 \) then \( R(\Gamma \bowtie \mathbb{Z}[1/q]^d \bowtie \mathbb{Q}_q^d) \) is not treeable.

The proof of Theorem 8.4 will be concluded at the very end of this section. We recall now some definitions regarding measure preserving dynamics which will be used throughout the rest of this section.

A standard atomless probability space \( X \) is a standard Borel space endowed with an atomless probability measure on its Borel \( \sigma \)-algebra. A probability-measure-preserving (pmp) equivalence relation on \( X \) is the orbit equivalence relation \( R(\Gamma \bowtie X) \) associated with a measure-preserving action of a countable group \( \Gamma \) on such an \( X \). We say that \( R(\Gamma \bowtie X) \) is ergodic if the action \( \Gamma \bowtie X \) is ergodic, i.e., if the only invariant sets under the action are of measure 0 or 1. If \( E \) is a pmp equivalence relation on \( X \) and \( F \) is a Borel equivalence relation on a standard Borel space \( Y \), then an almost-everywhere (a.e.) homomorphism from \( E \) to \( F \) is a Borel function \( f : X \to Y \) such that, for some conull subset \( X_0 \) of \( X \), \( f \mid X_0 \) is a Borel homomorphism from \( E \mid X_0 \) to \( F \).

**Definition 8.5.** Let \( E \) be a pmp equivalence relation on \( X \), and \( F \) be a Borel equivalence relation. Two a.e. homomorphisms \( f_0, f_1 \) from \( E \) to \( F \) are almost everywhere (a.e.) \( F \)-homotopic if there exists a conull subset \( X_0 \) of \( X \) such that, for every \( x \in X_0 \), \( f_0(x) \) and \( f_1(x) \) belong to the same \( F \)-equivalence class. The pmp equivalence relation \( E \) is \( F \)-ergodic if it is ergodic and every a.e. homomorphism from \( E \) to \( F \) is a.e. \( F \)-homotopic to a constant map.

Let \( \Gamma \bowtie X \) and \( \Delta \bowtie Y \) be two actions. If \( \varphi : \Gamma \to \Delta \) is a group homomorphism and \( f : X \to Y \) is a function so that for all \( \gamma \in \Gamma \) and \( x \in X \) we have \( f(\gamma \cdot x) = \varphi(\gamma) \cdot f(x) \), then we say that \( (\varphi, f) \) is a homomorphism of permutation groups. We similarly define the notion of an a.e. homomorphism of permutation groups, whenever \( X \) is endowed with a probability measure.

### 8.1. Hyperfiniteness and treeability

We now prove items (3) and (4) of Theorem 8.4 in a slightly more general setup. Let \( S \) be a nonempty set of primes and let \( \mathbb{Z}[1/S] \) be the subring of \( \mathbb{Q} \) generated by \( 1/p \) where \( p \) ranges in \( S \) and let \( \mathbb{Q}_S \) be the product of \( \mathbb{Z}_p \) where \( p \) ranges in \( S \). Let also
\[
\mathbb{Q}_S := \{(x_p)_{p \in S} \in \prod_{p \in S} \mathbb{Q}_p : x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \in S\}.
\]
be the restricted product of the \( p \)-adic numbers \( \mathbb{Q}_p \) with respect to the subrings \( \mathbb{Z}_p \) of the \( p \)-adic integers, where \( p \) ranges in \( S \). Since \( \mathbb{Q} \) is a subring of \( \mathbb{Q}_S \), \( \mathbb{Q}_S \) may be viewed as a \( \mathbb{Q} \)-vector space. This determines an action of \( \text{GL}_d(\mathbb{Q}) \) on \( \mathbb{Q}_S \). We also have an action of \( \mathbb{Z}[1/S]^d \) on \( \mathbb{Q}_S^d \) by translation. Together these actions induce an action of the affine group \( \text{GL}_d(\mathbb{Z}[1/S]) \bowtie \mathbb{Z}[1/S]^d \bowtie \mathbb{Q}_S^d \). The following propositions generalize (3) and (4) of Theorem 8.4

**Proposition 8.6.** Fix \( d \geq 1 \). Let \( S \) be a nonempty set of primes, and \( \Gamma \) be an abelian subgroup of \( \text{GL}_d(\mathbb{Z}[1/S]) \). Then \( R(\Gamma \bowtie \mathbb{Z}[1/S]^d \bowtie \mathbb{Q}_S^d) \) is hyperfinite and not smooth.
Proof. Notice first that $\mathbb{Q}_d^2$ is Polish, as a locally compact, metrizable $K_σ$; see [41, Theorem 5.3]. Since $Γ$ is abelian, $Γ \rtimes \mathbb{Z}[1/\mathbb{S}]$ is nilpotent. Therefore $R(Γ \rtimes \mathbb{Z}[1/\mathbb{S}]^d \rhd \mathbb{Q}_d^2)$ is hyperfinite by the main result of [62]. As the action $Γ \rtimes \mathbb{Z}[1/\mathbb{S}] \rhd \mathbb{Q}_\mathbb{S}$ has dense orbits, $R(Γ \rtimes \mathbb{Z}[1/\mathbb{S}]^d \rhd \mathbb{Q}_d^2)$ is not smooth. □

Proposition 8.7. Fix $d ≥ 2$, a nonempty set of primes $S$, and a subgroup $Γ$ of $\text{GL}_d(\mathbb{Z}[1/\mathbb{S}])$ containing a finite index subgroup of $\text{SL}_d(\mathbb{Z})$. Then $R(Γ \rtimes \mathbb{Z}[1/\mathbb{S}]^d \rhd \mathbb{Q}_d^2)$ is not treeable.

Proof. Let $Δ := Γ \cap \text{SL}_d(\mathbb{Z})$. As $R(Δ \rtimes \mathbb{Z}_d^2 \rhd \mathbb{Q}_d^2)$ is a subequivalence relation of the restriction of $R(Γ \rtimes \mathbb{Z}[1/\mathbb{S}]^d \rhd \mathbb{Q}_d^2)$ to $\mathbb{Z}_d^2$, it suffices by [33, Proposition 3.3] to show that $R(Δ \rtimes \mathbb{Z}_d^2 \rhd \mathbb{Q}_d^2)$ is not treeable. Let $μ_σ$ be the Haar measure on $\mathbb{Z}_d^2$ and $μ_p$ be the Haar measure on $\mathbb{Z}_p^2$ for all $p ∈ S$. Notice that for every nontrivial element $g$ of $Δ$ the subgroup of $\mathbb{Z}_d$ consisting of fixed points for $g$ is of infinite index in $\mathbb{Z}_d$. Hence, for all $p ∈ S$, the $μ_p$-measure preserving action $Δ \rtimes \mathbb{Z}_d \rhd \mathbb{Z}_p^2$ is free almost everywhere [31, Lemma 1.7], and therefore the $μ_σ$-measure preserving action $Δ \rtimes \mathbb{Z}_d \rhd \mathbb{Z}_d^2$ is free almost everywhere. Furthermore, $Δ \rtimes \mathbb{Z}_d$ has property (T) for $d ≥ 3$, and $\mathbb{Z}_d^2 \preceq Δ \rtimes \mathbb{Z}_d$ has the relative property (T). Hence, $Δ \rtimes \mathbb{Z}_d$ does not have the Haagerup property [9, Chapter 1]. Hence, the relation $R(Δ \rtimes \mathbb{Z}_d \rhd \mathbb{Q}_d^2)$ is not treeable [72, Proposition 6]. □

Remark 8.8. The same conclusions as in Proposition 8.7 holds for every subgroup $Γ$ of $\text{GL}_d(\mathbb{Z}[1/\mathbb{S}])$ such that $\mathbb{Z}_d^2 \preceq (Γ \cap \text{SL}_d(\mathbb{Z})) \rtimes \mathbb{Z}_d^2$ has relative property (T). When $d = 2$, this is equivalent to the assertion that $Γ \cap \text{SL}_2(\mathbb{Z})$ is not virtually cyclic by [7, Section 5, Example 2].

8.2. Comparing affine actions of different dimension. Let $Γ \rhd A$ be an action of a countable group $Γ$ on a countable abelian group $A$ be automorphisms, and let $A = (A^n)$ be a filtration on $A$, consisting of $Γ$-invariant finite-index subgroups of $A$. This induces an action $Γ \rhd A$ of $Γ$ on the associated profinite completion $\hat{A}$ of $A$ by (Haar measure-preserving) topological group automorphisms of $A$. We additionally have the translation action of $A$ on $\hat{A}$. Together these actions induce a (Haar) measure-preserving action of the semidirect product $Γ \times A$ on $\hat{A}$. Since $A$ is dense in $\hat{A}$, the action $Γ \rhd A \rhd \hat{A}$ is ergodic. Additionally, if for every nontrivial element $g$ of $Γ$ the subgroup of $A$ consisting of fixed points for $g$ is of infinite index, then the action $Γ \rtimes \mathbb{Z}_d^2 \rhd \hat{A}$ is a.e. free [31, Lemma 1.7].

The following is an immediate consequence of [31, Theorem B], where: the pair $Γ_0 \preceq Γ$ in [31] corresponds to the pair $A \preceq \hat{A}$; the profinite action $Γ \rhd X$ in [31] corresponds to the action $Γ \rtimes A \rhd \hat{A}$; and the cocycle $ω: Γ \times X \rhd A$ in [31] corresponds to the unique map $(g, x) \rhd h$ below, with values in $Δ$, so that $h \cdot f(x) = f(g \cdot x)$.

Proposition 8.9 (Ioana). Let $Γ$ be a finitely-generated group acting on a countable abelian group $A$ by automorphisms so that, for every non-trivial $g ∈ Γ$, the group of elements of $A$ fixed by $g$ has infinite index in $A$. Let also $\hat{A}$ be the profinite completion of $A$ with respect to some filtration $A = (A^n)$ on $A$, consisting of $Γ$-invariant finite-index subgroups. Let finally $f: A \rhd Y$ be an a.e. homomorphism from $R(Γ \times A \rhd \hat{A})$ to $R(Δ \rhd Y)$, where $Δ \rhd Y$ is some free Borel action of a countable group $Δ$ on a standard Borel space $X$. We have the following:

1. If the pair $A \preceq Γ \rtimes A$ has the relative property (T), then there exist an $n ∈ ω$, a group homomorphism $φ: A^n \rhd Δ$, and a Borel function $f': \hat{A}^n \rhd Y$, where $\hat{A}^n$ is the closure of $A^n$ inside $\hat{A}$, such that:
   - $f'$ is a.e. $R(Δ \rhd Y)$-homotopic to $f \rhd \hat{A}^n$, and
   - $(φ, f')$ is an a.e. homomorphism of permutation groups from $A^n \rhd \hat{A}^n$ to $Δ \rhd Y$.

2. If the group $Γ \rhd A$ has property (T), then there exist an $n ∈ ω$, a group homomorphism $φ: Γ \rtimes A^n \rhd Δ$, and a Borel function $f': \hat{A}^n \rhd Y$, where $\hat{A}^n$ is the closure of $A^n$ inside $\hat{A}$, such that:
   - $f'$ is a.e. $R(Δ \rhd Y)$-homotopic to $f \rhd \hat{A}^n$, and
   - $(φ, f')$ is an a.e. homomorphism of permutation groups from $Γ \rtimes A^n \rhd \hat{A}^n$ to $Δ \rhd Y$.

Recall that $E_0$ stands for the relation of eventual equality of binary sequences. One can regard $E_0$ as the orbit equivalence relation associated with a continuous free action of a countable group on a Polish space as follows. Set

$$B_0 := \bigoplus_{n ∈ ω} \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad B := \prod_{n ∈ ω} \mathbb{Z}/2\mathbb{Z}.$$  

Then $B_0$ is a countable subgroup of the Polish group $B$ and $E_0$ is the coset relation $R(B_0 \rhd B)$. Notice that $B_0$ is a Boolean group, i.e., for all $b ∈ B_0$ we have that $2b = 0$. 

The following is a consequence of Proposition 8.9. Notice that if $X$ is a compact abelian group, and $A \subseteq X$ is a countable dense subgroup, then the translation action $A \curvearrowright X$ is ergodic.

**Lemma 8.10.** Let $\Gamma \curvearrowright A, A = (\Lambda(n))$, and $\hat{A}$ be as in the statement of Proposition 8.9 and assume that for every $n \in \omega, A(n)$ does not have an infinite Boolean group as quotient. If the pair $A \leq \Gamma \times A$ has relative property (T), then $R(\Gamma \times A \curvearrowright \hat{A})$ is $E_0$-ergodic.

**Proof.** Let $f : \hat{A} \rightarrow B$ be an a.e. homomorphism from $R(\Gamma \times A \curvearrowright \hat{A})$ to $E_0 = R(B_0 \curvearrowright B)$. By Proposition 8.9, there exist $n \in \omega$, a group homomorphism $\phi : A(n) \rightarrow B_0$, and a Borel function $g : \hat{A}(n) \rightarrow B$ such that $g$ is a.e. $E_0$-homotopic to $f \upharpoonright \hat{A}(n)$, and $(\phi, g)$ is an a.e. homomorphism of permutation groups from $A(n) \curvearrowright \hat{A}(n)$ to $B_0 \curvearrowright B$. By assumption, $K := \text{Ker}(\hat{\phi})$ has finite index in $A(n)$. Since $(\phi, g)$ is an a.e. homomorphism of permutation groups, we have that $\phi(a) \cdot g(x) = g(a \cdot x)$ for every $a \in A(n)$ and $x \in \hat{A}(n)$. Let $\hat{K}$ be the closure of $K$ in $\hat{A}(n)$, which is an open subgroup of $\hat{A}(n)$ since $K$ is a finite-index subgroup of $\hat{A}(n)$. Thus, for a.e. $x \in \hat{K}$ and for every $a \in K$ we have that $g(a \cdot x) = g(x)$. Since $K$ is dense in $\hat{K}$, the translation $A \curvearrowright \hat{K}$ is ergodic. By the above, the function $g|_{\hat{K}} : \hat{K} \rightarrow B$ is a.e. $K$-invariant. By ergodicity, this implies that $g|_{\hat{K}}$ is constant on a conull subset $W$ of $\hat{K}$. Since $\hat{K}$ is an open subgroup of $\hat{A}(n)$, $W$ is not null in $\hat{A}(n)$. Since $g$ and $f \upharpoonright \hat{A}(n)$ are $E_0$-homotopic, this implies that there exists a $B_0$-orbit $B_0 + b \subseteq B$ such that $f(W) \subseteq B_0$. Since $f : \hat{A} \rightarrow B$ is an a.e. homomorphism from $R(\Gamma \times A \curvearrowright A)$ to $E_0 = R(B_0 \curvearrowright B)$, $f^{-1}(B_0 + b)$ is (up to null sets) a $\Gamma \times A$-invariant subset of $\hat{A}$ containing $W$, and hence not null. Since the action $\Gamma \times A \curvearrowright \hat{A}$ is ergodic, we must have that $f^{-1}(B_0 + b)$ is conull, concluding the proof. □

**Theorem 8.11.** Let $\Gamma \curvearrowright A, A = (\Lambda(n))$, and $\hat{A}$ be as in the statement of Proposition 8.9 so that moreover:

1. $A(n)$ does not have an infinite Boolean group as quotient;
2. $\Gamma$ is a subgroup of $\text{SL}_m(\mathbb{Z})$, with $m \geq 3$, so that $\Gamma \times A$ has property (T).

If $q$ is a prime number and $1 \leq d < m$, then $R(\Gamma \times A \curvearrowright \hat{A})$ is $R(\text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \curvearrowright \mathbb{Q}^d)$-ergodic.

**Remark 8.12.** Notice that if $m \geq 3$, then $\text{SL}_m(\mathbb{R}) \ltimes \mathbb{R}^m$ has property (T) [4, Corollary 1.4.16]. Since $\text{SL}_m(\mathbb{Z}) \ltimes \mathbb{Z}^m$ is a lattice in $\text{SL}_m(\mathbb{R}) \ltimes \mathbb{R}^m$, it also has property (T).

The proof of Theorem 8.11 is modeled after the proofs of [71, Theorem 1.1] and [10, Theorem 3.6]. We will use the following lemmas. The first lemma is an instance of [71, Theorem 4.4]; see also the proof of [71, Theorem 4.3].

**Lemma 8.13 (Thomas).** Suppose that $m \geq 2$ and $G$ is an algebraic $\mathbb{Q}$-group of dimension less than $m^2 - 1$. If $G$ is a finite-index subgroup of $\text{SL}_m(\mathbb{Z})$ and $\psi : \Gamma \rightarrow G(\mathbb{Q})$ is a homomorphism, then $\text{ker}(\psi)$ has finite index in $\Gamma$.

**Lemma 8.14.** Let $\Delta$ be a subgroup of $\text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$, and let $Y$ be a $\Delta$-invariant Borel subset of $\mathbb{Q}^d$, so that the action $\Delta \curvearrowright Y$ is free. Under the assumptions of Theorem 8.11, we have that every a.e. homomorphism from $R(\Gamma \times A \curvearrowright \hat{A})$ to $R(\Delta \curvearrowright Y)$ is $R(\text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \curvearrowright \mathbb{Q}^d)$-homotopic to a constant map.

**Proof.** Suppose that $f : \hat{A} \rightarrow X$ is an a.e. homomorphism from $R(\Gamma \times A \curvearrowright \hat{A})$ to $R(\Delta \curvearrowright Y)$. By Proposition 8.9, after replacing $A$ with $A_n$ for some $n \in \omega$, we can assume that there exist a group homomorphism $\phi : \Gamma \times A \rightarrow \Delta$, and a Borel function $g : \hat{A} \rightarrow Y$, such that $g$ is $R(\Delta \curvearrowright Y)$-homotopic to $f$, and $(\phi, g)$ is an a.e. homomorphism of permutation groups from $\Gamma \times A \curvearrowright \hat{A}$ to $\Delta \curvearrowright Y$. Let $\pi : \text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \rightarrow \text{GL}_d(\mathbb{Q})$ be the canonical quotient map. By Lemma 8.13, after replacing $\Gamma$ with a finite-index subgroup, we can assume without loss of generality that $\Gamma$ is entirely contained in the kernel of $(\pi \circ \phi)$. By Lemma 8.10, $R(\Gamma \times A \curvearrowright \hat{A})$ is $E_0$-ergodic. But if $\Lambda := (\pi \circ \phi)(\Gamma \times A) = (\pi \circ \phi)(A)$, then $g : \hat{A} \rightarrow Y$ is an a.e. homomorphism from $R(\Gamma \times A \curvearrowright \hat{A})$ to $R(\Lambda \times \mathbb{Q}^d \curvearrowright \mathbb{Q}^d)$. As $\Lambda \times \mathbb{Q}^d$ is nilpotent, we have that $R(\Lambda \times \mathbb{Q}^d \curvearrowright \mathbb{Q}^d)$ is Borel reducible to $E_0$ by the main result of [62]. As $R(\Gamma \times A \curvearrowright \hat{A})$ is $E_0$-ergodic, it follows that $g$, and hence $f$, are $R(\Lambda \times \mathbb{Q}^d \curvearrowright \mathbb{Q}^d)$-homotopic to a constant map. □

**Proof of Theorem 8.11.** Suppose that $f : \hat{A} \rightarrow \mathbb{Q}^d_q$ is an a.e. homomorphism from $R(\Gamma \times A \curvearrowright \hat{A})$ to $R(\text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \curvearrowright \mathbb{Q}^d_q)$. We will show that $f$ is a.e. $R(\text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \curvearrowright \mathbb{Q}^d_q)$-homotopic to a constant map. We identify $\mathbb{Q}^d_q$ with the tensor product $\mathbb{Q}_q \otimes \mathbb{Q}^d$. By a $\mathbb{Q}$-subspace of $\mathbb{Q}^d_q$ we mean any $\mathbb{Q}_q$-linear subspace of $\mathbb{Q}^d_q$ that is of the form $\mathbb{Q}_q \otimes \mathbb{Q}^d$, where $V \subseteq \mathbb{Q}^d \subseteq \mathbb{Q}^d_q$ is a $\mathbb{Q}$-linear subspace of $\mathbb{Q}^d$. An affine $\mathbb{Q}$-variety is a subset of $\mathbb{Q}^d_q$ of the form $a + V$, where $V$ is a $\mathbb{Q}$-subspace and $a \in \mathbb{Q}^d$. 


\textbf{Claim.} Suppose that $M \in M_d(\mathbb{Q})$ is a $d \times d$ matrix with rational coefficients and set

$$\ker_q(M) := \{x \in \mathbb{Q}^d : Mx = 0\} \quad \text{and} \quad \ker_q(M) := \{x \in \mathbb{Q}^d : Mx = 0\}.$$

Then $\ker_q(M) = \mathbb{Q}_p \otimes \ker_q(M)$. In particular, $\ker_q(M)$ is $\mathbb{Q}$-subspace of $\mathbb{Q}^d$.

\textbf{Proof of Claim.} By the Gaussian elimination procedure, we can assume that $M$ is in reduced row echelon form. It is then clear from the reduced row echelon form of $M$ that the dimension of $\ker_q(M)$ as a $\mathbb{Q}$-vector space is equal to the dimension of $\ker_q(M)$ as a $\mathbb{Q}_q$-vector space. Thus, $\mathbb{Q}_q \otimes \ker_q(M)$ and $\ker_q(M)$ are $\mathbb{Q}_q$-vector spaces of the same dimension. Since $\mathbb{Q}_q \otimes \ker_q(M) \subseteq \ker_q(M)$, we must have $\mathbb{Q}_q \otimes \ker_q(M) = \ker_q(M)$. \hfill $\square$

\textbf{Claim.} Suppose that $M \in M_d(\mathbb{Q})$ is a $d \times d$ matrix with rational coefficients, and $t \in \mathbb{Q}^d$. Set

$$W_{Q}\gamma = \{x \in \mathbb{Q}^d : Mx = t\} \quad \text{and} \quad W = \{x \in \mathbb{Q}^d : Mx = t\}.$$

Then $W$ is nonempty if and only if $W_{Q}\gamma$ is nonempty. In this case, $W_{Q}\gamma$ is a free $\mathbb{Q}$-variety of $\mathbb{Q}^d$.

\textbf{Proof of Claim.} Again, by the Gaussian elimination procedure, we can assume that $M$ is in reduced row echelon form. It is then clear that $W$ is nonempty if and only if $W_{Q}\gamma$ is nonempty. Suppose thus that $W_{Q}\gamma$ (or, equivalently, $W_{Q}$) is nonempty. Pick $x_0 \in W_{Q}\gamma$. Then we have that $W_{Q} = \ker_q(M) + x_0$. By the previous claim, $\ker_q(M)$ is a $\mathbb{Q}$-subspace of $\mathbb{Q}^d$. Since $x_0 \in \mathbb{Q}^d$, we have that $W_{Q}\gamma = \ker_q(M) + x_0$ is an affine $\mathbb{Q}$-variety of $\mathbb{Q}^d$. \hfill $\square$

\textbf{Claim.} Fix $(\gamma) := \{x \in \mathbb{Q}^d : \gamma \cdot x = x\}$ is an affine $\mathbb{Q}$-variety, for all $\gamma \in \mathbb{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$.

\textbf{Proof of Claim.} Notice that there exist $M \in M_d(\mathbb{Q})$ and $t \in \mathbb{Q}^d$ such that $\gamma \cdot x = Mx + t$ for $x \in \mathbb{Q}^d$. Thus,

$$\fix(\gamma) = \{x \in \mathbb{Q}^d : (I - M)x = t\}.$$

This is an affine $\mathbb{Q}$-variety by the previous claim. \hfill $\square$

Let $\mathcal{A}_\mathbb{Q}$ be the set of all affine $\mathbb{Q}$-varieties of $\mathbb{Q}^d$, ordered by inclusion. As the intersection of affine $\mathbb{Q}$-varieties is an affine $\mathbb{Q}$-variety, for every $y \in Y$ there is a (unique) smallest $\mathbb{Q}$-variety $V_y \in \mathcal{A}_\mathbb{Q}$ containing $y$. If $\gamma \in \mathbb{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$ and $W \in \mathcal{A}_\mathbb{Q}$, then $\gamma \cdot W := \{\gamma \cdot x : x \in W\}$ is also an affine $\mathbb{Q}$-variety. Thus, $V_{\gamma \cdot y} = \gamma \cdot V_y$, for every $y \in \mathbb{Q}^d$.

\textbf{Claim.} Fix $V \in \mathcal{A}_\mathbb{Q}$ and let $Y := \{y \in \mathbb{Q}^d : V_y = V\} \subseteq Y$. If $\Delta$ is the group of affine transformations of $V$ obtained as setwise stabilizers of $V$ in $\mathbb{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$. Then the action $\Delta \acts Y$ is free.

\textbf{Proof of Claim.} Suppose that $\delta(e) = e$, for some $\delta \in \Delta$ and $e \in Y$. We have that $e \in \fix(\delta) = \{x \in \mathbb{Q}^d : \gamma \cdot x = x\}$. By the previous claim, $\fix(\delta)$ is an affine $\mathbb{Q}$-variety of $\mathbb{Q}^d$. Therefore, $V = V_y \subseteq \fix(\delta)$. This shows that $\gamma \cdot y = y$ for every $y \in V$. Thus $\delta$ is the trivial element of $\Delta$. \hfill $\square$

Since $\mathcal{A}_\mathbb{Q}$ is countable, we can assume without loss of generality that there exists $V \in \mathcal{A}_\mathbb{Q}$ such that $V_f(x) = V$ for every $x \in \hat{A}$. To see this, as in the proof of [69, Lemma 5.1], pick $V \in \mathcal{A}_\mathbb{Q}$ such that $X_0 := \{x \in \hat{A} : V_f(x) = V\}$ is nonnull. By ergodicity of the action $\Gamma \times A \acts \hat{A}$ we have that

$$X_1 := \bigcup_{\alpha \in \Gamma \times A} \alpha \cdot X_0$$

has full measure. Let $c : X_1 \to X_0$ be a Borel function such that $c(x) \in (\Gamma \times A) \cdot x \cap X_0$ for every $x \in X_1$. Let also $x_0$ be a point in $X_0$. Without loss of generality replace $f$ with the Borel function $g$ defined by

$$g : x \mapsto \begin{cases} (f \circ c)(x) & x \in X_1 \\ f(x_0) & x \in A \setminus X_1. \end{cases}$$

This function satisfies $V_{g(x)} = V$ for every $x \in \hat{A}$.

So let us thus assume that $V_f(x) = V$ for every $x \in \hat{A}$. Let $(u_1, \ldots, u_d)$ be the canonical basis of $\mathbb{Q}^d$ over $\mathbb{Q}_q$. Define $Y := \{y \in V : V_y = V\}$. Let also $\Delta$ be the group of affine transformations of $V$ obtained as restrictions to $V$ of elements of the setwise stabilizer of $V$ inside $\mathbb{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$.

\textbf{Claim.} The action $\Delta \acts Y$ is free, and $f : \hat{A} \to Y$ is an a.e. homomorphism from $\mathcal{R}(\Gamma \times A \acts \hat{A})$ to $\mathcal{R}(\Delta \acts Y)$.
Proof of Claim. It follows from the previous claim that the action $\Delta \act Y$ is free. We now show that the function $f : \hat{A} \to Y$ is a Borel homomorphism from $R(\Gamma \times_A \hat{A})$ to $R(\Delta \act Y)$. Since $f$ is an a.e. homomorphism from $R(\Gamma \times_A \hat{A})$ to $R(GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \rtimes \mathbb{Q}^d)$, there is a countable subset $X_0$ of $\hat{A}$ such that, whenever $x_0, x_1 \in X_0$ and $\gamma \in \Gamma \times_A \hat{A}$ are such that $\gamma \cdot x_0 = x_1$, there exists $\delta \in GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$ such that $\delta \cdot f(x_0) = f(x_1)$. Suppose now that $x_0, x_1 \in X_0$ and $\gamma \in \Gamma \times_A \hat{A}$ are such that $\gamma \cdot x_0 = x_1$. Then there exists $\delta \in GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$ such that $\delta \cdot f(x_0) = f(x_1)$. Since $f(x_0), f(x_1) \in \mathbb{Q}^d \rtimes \mathbb{Q}^d$ we have that $V = V_x(x_1) = V_{\delta \cdot f(x_0)} = \delta \cdot V_f(x_0)$. Therefore, $\delta$ belongs to the setwise stabilizer of $V$ inside $GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$. Thus, $\delta |_V \in \Delta$ and $\delta |_V \cdot f(x_0) = f(x_1)$, This shows that $f : \hat{A} \to Y$ is an a.e. homomorphism from $R(\Gamma \times_A \hat{A})$ to $R(\Delta \act Y)$. 

The conclusion now follows from Lemma 8.14. 

8.3. Comparing affine actions associated to different primes. Let $L$ be a closed subgroup of a compact group $K$. The space $K/L$ is then endowed with a normalized Haar measure, and the $K$-action is measure-preserving. If $\Gamma$ is a countable subgroup of $K$, one may consider the induced action of $\Gamma$ on $K/L$. Such an action is ergodic if and only if $\Gamma$ is dense in $K$ by [53, Lemma 4.1.1]. We will need the following consequence of [10, Lemma 2.3.2], which is suggested in [10, Remark 2.5] as it plays a crucial role in the proof of [10, Theorem 3.6].

Lemma 8.15 (Coskey). For $i \in \{0, 1\}$, let $L_i$ be a closed subgroup of a compact group $K_i$. If $\Gamma_0$ is a countable dense subgroup of $K_0$ and $(\phi, f)$ is an a.e. homomorphism of permutation groups from $\Gamma_0 \act K_0/L_0$ to $K_1 \act K_1/L_1$, then there exist a closed subgroup $\hat{K}_1$ of $K_1$, a transitive $\hat{K}_1$-space $\hat{X}_1 \subseteq K_1/L_1$, a closed normal subgroup $H$ of $\hat{K}_1$, and a continuous surjective homomorphism $\Phi : K_0 \to \hat{K}_1/H$ such that:

1. $\Phi|_{\Gamma_0} = \pi \circ \phi$, where $\pi : \hat{K}_1 \to \hat{K}_1/H$ is the quotient map;
2. $H$ is the kernel of the action $\hat{K}_1 \act \hat{X}_1$;
3. $(\Phi, f)$ is an a.e. homomorphism of permutation groups from $K_0 \act K_0/L_0$ to $\hat{K}_1/H \act \hat{X}_1$.

Proof. Define $\hat{K}_1$ to be the closure of $\phi(\Gamma_0)$ inside $K_1$. Since $\Gamma_0$ is dense in $K_0$, the action $\Gamma_0 \act K_0/L_0$ is ergodic. Hence there exists $z \in K_1/L_1$ such that, $\{x \in K_1/L_1 : f(x) \in \hat{K}_1 \cdot z\}$ is countable. Define $\hat{X}_1 = \hat{K}_1 \cdot z$ and $H$ to be the kernel of the action $\hat{K}_1 \act \hat{X}_1$. Let $\pi : \hat{K}_1 \to \hat{K}_1/H$ be the quotient map. By [10, Lemma 2.3.2] applied to $f : K_0/L_0 \to \hat{X}_1$ and $\pi \circ \phi : \Gamma_0 \to \hat{K}_1$, we get the desired homomorphism $\Phi : K_0 \to \hat{K}_1/H$.

Recall that, if $p$ is a prime number, then a pro-$p$ group is a profinite group $G$ such that every open subgroup has index equal to a power of $p$. Notice that closed subgroups and quotients of pro-$p$ groups are pro-$p$ groups. A profinite group $G$ is a virtually pro-$p$ group if it contains a clopen pro-$p$ subgroup.

Fix $m \geq 2, k \geq 1$, and a prime $p \geq 3$, and let $SL_m(p^k \mathbb{Z}_p)$ be the kernel of the canonical surjective homomorphism $SL_m(\mathbb{Z}_p) \to SL_m(\mathbb{Z}/p^k \mathbb{Z})$. Then $SL_m(p^k \mathbb{Z}_p)$ is a clopen torsion-free pro-$p$ subgroup of $SL_m(\mathbb{Z}_p)$ when $k \geq 1$ and $p \geq 3$, or $k \geq 2$ and $p = 2$; see [14, Theorem 5.2]. In particular, $SL_m(\mathbb{Z}_p)$ is a virtually pro-$p$ group.

Lemma 8.16. Let $p, q$ be distinct prime numbers, let $m \geq 3$, and let $n, d \geq 1$. If $\Gamma$ is a subgroup of $SL_m(\mathbb{Z})$ with property (T) such that $\Gamma \times \mathbb{Z}^m$ has property (T), $\Delta$ is a subgroup of $GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d$, and $Y \subseteq Q^d$ is a Borel $\Delta$-invariant set such that the action $\Delta \act Y$ is free, then every a.e. homomorphism $f$ from $R(\Gamma \times p^n \mathbb{Z}^m \act p^n \mathbb{Z}_{p^d}^m)$ to $R(\Delta \act Y)$ is $R(GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \rtimes Q^d)$-homotopic to a constant map.

Proof. By Lemma 8.10, it suffices to consider the case when $d \geq 2$.

Suppose that $f : p^n \mathbb{Z}_{p^d}^m \to Y$ is an a.e. homomorphism from $R(\Gamma \times p^n \mathbb{Z}^m \act p^n \mathbb{Z}_{p^d}^m)$ to $R(\Delta \act Y)$. Then by Proposition 8.9, after replacing $n$ with a larger integer, we can assume that there exist a group homomorphism $\phi : \Gamma \times p^n \mathbb{Z}^m \to \Delta$, and a Borel function $g : p^n \mathbb{Z}_p^m \to Y$, such that $g$ is $R(\Delta \act Y)$-homotopic to $f$, and $(\phi, g)$ is an a.e. homomorphism of permutation groups from $\Gamma \times p^n \mathbb{Z}^m \act p^n \mathbb{Z}_{p^d}^m$ to $\Delta \act Y$. It suffices to show that $g$ is a.e. $R(GL_d(\mathbb{Q}) \ltimes \mathbb{Q}^d \rtimes Q^d)$-homotopic to a constant map.

Claim. We may assume without loss of generality that $\phi(\Gamma) \subseteq SL_d(\mathbb{Z}) \times \mathbb{Z}^d$

Proof of Claim. Let $\pi : GL_d(\mathbb{Q}) \times Q^d \to GL_d(\mathbb{Q})$ be the canonical quotient map. As $\Gamma$ has property (T), after replacing $\Gamma$ with a finite index subgroup we can assume that $(\pi \circ \phi)(\Gamma) \subseteq SL_d(\mathbb{Q})$. As $\Gamma$ is finitely-generated, there exists a finite set $F = \{q_1, \ldots, q_n\}$ of prime numbers such that $\phi(\Gamma) \subseteq SL_d(\mathbb{Z}[1/F]) \times \mathbb{Z}[1/F]^d \subseteq SL_d(\mathbb{Z}_F) \times Q^d$. 

\[ \phi(\Gamma) \subseteq SL_d(\mathbb{Z}[1/F]) \times \mathbb{Z}[1/F]^d \subseteq SL_d(\mathbb{Z}_F) \times Q^d. \]
For $1 \leq \ell \leq t$, let $\tau_\ell : \text{SL}_d(\mathbb{Q}_F) \rtimes \mathbb{Q}_F^d \to \text{SL}_d(\mathbb{Q}_q)$ be the canonical quotient map. Consider the homomorphism $\tau_\ell \circ \phi : \Gamma \to \text{SL}_d(\mathbb{Q}_q)$. We have that by [53, Theorem VIII.3.10] the Zariski closure of $(\tau_\ell \circ \phi)(\Gamma)$ inside $\text{SL}_d(\mathbb{Q}_q)$ is semisimple. Hence, $(\tau_\ell \circ \phi)(\Gamma)$ has compact closure inside $\text{SL}_d(\mathbb{Q}_q)$ by [53, VII.5.16]. Since $\text{SL}_d(\mathbb{Q}_q)$ is an open subgroup of $\text{SL}_d(\mathbb{Q}_p)$, we have that $\text{SL}_d(\mathbb{Q}_q)$ is a finite index subgroup inside $(\tau_\ell \circ \phi)(\Gamma)$. Hence, after replacing $\Gamma$ with a finite index subgroup, we can assume that $(\tau_\ell \circ \phi)(\Gamma) \subseteq \text{SL}_d(\mathbb{Q}_q)$ for every $\ell \in \{1, 2, \ldots, t\}$.

This implies that $(\pi \circ \phi)(\Gamma) \subseteq \text{SL}_n(\mathbb{Z})$ and hence $\phi(\Gamma) \subseteq \text{SL}_d(\mathbb{Z}) \rtimes \mathbb{Z}^d[1/F]$. Since $\Gamma$ is finitely-generated, we have that $\phi(\Gamma) \subseteq \text{SL}_d(\mathbb{Z}) \rtimes \frac{1}{N}\mathbb{Z}^d$ for some $N \geq 1$. Since $\text{SL}_d(\mathbb{Z}) \rtimes \frac{1}{N}\mathbb{Z}^d$ is a finite index subgroup of $\text{SL}_d(\mathbb{Z}) \rtimes \mathbb{Z}^d$, after replacing $\Gamma$ with a finite index subgroup we can assume that $\phi(\Gamma) \subseteq \text{SL}_d(\mathbb{Z}) \rtimes \mathbb{Z}^d$.

Let $G_0$ be the closure of $\phi(\Gamma)$ inside $\text{SL}_m(\mathbb{Z}_p)$, and let $G_1$ be the closure of $\phi(\Gamma)$ inside $\text{SL}_d(\mathbb{Z}_q) \rtimes \mathbb{Z}_q^d$. Since $\text{SL}_m(\mathbb{Z}_p)$ is virtually pro-$p$ and $\text{SL}_d(\mathbb{Z}_q) \rtimes \mathbb{Z}_q^d$ is virtually pro-$q$, after replacing $\Gamma$ with a finite index subgroup we can assume that $G_0$ is a pro-$p$ group and $G_1$ is a pro-$q$ group.

By Lemma 8.15 there exists a $G_1$-invariant closed subset $\tilde{Y}$ of $Y$ such that for a.e. $x \in p^n\mathbb{Z}_p^m$, $g(x) \in \tilde{Y}$, and if $H$ is the kernel of the action $G_1 \curvearrowright \tilde{Y}$ and $p : G_1 \to G_1/H$ is the projection map, then $p \circ \phi : \Gamma \to G_1/H$ extends to a continuous homomorphism $\Phi : G_0 \to G/H$ such that $(\Phi, g)$ is a homomorphism of permutation groups from $G_0 \rtimes p^n\mathbb{Z}_p^m$ to $G_1/H \rtimes \mathbb{Z}_q^d$. Since $G_0$ is a pro-$p$ group and $G_1/H$ is a pro-$q$ group, we have that $\Phi$ is trivial. Hence, $p \circ \phi$ is trivial and $\phi(\Gamma) \subseteq H$.

Therefore, if $\Lambda := (\pi \circ \phi)(\Gamma) \subseteq H$, then we have that $\Lambda$ is a $\text{SL}_m(\mathbb{Z})$-homomorphism from $\text{SL}_d(\mathbb{Z}) \rtimes p^n\mathbb{Z}_p^m$ to $\text{SL}_d(\mathbb{Z}) \rtimes \mathbb{Z}_q^d$. As $\Lambda \ltimes \mathbb{Q}_d^d$ is nilpotent we have that $\text{SL}(\Lambda \ltimes \mathbb{Q}_d^d \rtimes \mathbb{Q}_d^d)$ Borel reducible to $\text{SL}(\mathbb{Q}_d^d)$ by the main result of [62]. As $\text{SL}(\Lambda \ltimes \mathbb{Q}_d^d \rtimes \mathbb{Q}_d^d)$ is $\text{SL}(\mathbb{Q}_d^d)$-ergodic, it follows that $\Lambda$, and hence $\phi$, are $\text{SL}(\mathbb{Q}_d^d \rtimes \mathbb{Q}_d^d)$-ergodic to the constant map.

**Theorem 8.17.** Let $p, q$ be distinct prime numbers, let $m \geq 3$, and let $n, d \geq 1$. If $\Gamma$ is subgroup of $\text{SL}_m(\mathbb{Z})$ with property $(T)$, then $\text{R}(\Gamma \ltimes \mathbb{Z}^d \rtimes \mathbb{Z}_p^m)$ is $\text{R}(\text{GL}_d(\mathbb{Q}) \ltimes \mathbb{Q}_d^d \rtimes \mathbb{Q}_d^d)$-ergodic.

**Proof.** One may proceed as in the proof of Theorem 8.11, using Lemma 8.16 in the place of Lemma 8.14.

**8.4. Conclusion.** We may now conclude with the proof of Theorem 8.4.

**Proof of Theorem 8.4.** Define $E := \text{R}(\Gamma \ltimes \mathbb{Z}[1/q]^m \rtimes \mathbb{Q}_q^m)$ and $F := \text{R}(\Lambda \ltimes \mathbb{Z}[1/p]^d \rtimes \mathbb{Q}_d^d)$.

(1): When $d = 1$ and $m = 2$, then the conclusion follows from items (3) and (4). When $m \geq 3$, Theorem 8.11 in the case when $A = \mathbb{Z}[q]^m$ and $A = \mathbb{Z}[q]^m$ implies that $\text{R}(\Gamma \ltimes \text{SL}_m(\mathbb{Z})) \rtimes \mathbb{Z} \rtimes \mathbb{Z}_q^m$ is $F$-ergodic. The rest follows from the fact that the inclusion map $\mathbb{Z}^m \subseteq \mathbb{Q}_q^m$ is a Borel homomorphism from $\text{R}(\Gamma \ltimes \text{SL}_m(\mathbb{Z})) \rtimes \mathbb{Z} \rtimes \mathbb{Z}_q^m$ to $E$ which is not $E$-homotopic to the constant map.

(2): By Theorem 8.17, $\text{R}(\Gamma \ltimes \text{SL}_m(\mathbb{Z})) \ltimes \mathbb{Z} \rtimes \mathbb{Z}_q^m$ is $F$-ergodic. The rest follows as in (1) above.

(3): This is a consequence of Proposition 8.6.

(4): This is a consequence of Proposition 8.7.

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