Tail asymptotics in any direction of the stationary distribution in a two-dimensional discrete-time QBD process

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Abstract
We consider a discrete-time two-dimensional quasi-birth-and-death process (2d-QBD process for short) \( \{(X_n, J_n)\} \) on \( \mathbb{Z}_+^2 \times S_0 \), where \( X_n = (X_{1,n}, X_{2,n}) \) is the level state, \( J_n \) the phase state (background state) and \( S_0 \) a finite set, and study asymptotic properties of the stationary tail distribution. The 2d-QBD process is an extension of usual one-dimensional QBD process. By using the matrix analytic method of the queueing theory and the complex analytic method, we obtain the asymptotic decay rate of the stationary tail distribution in an arbitrary direction. This result is an extension of the corresponding result for a certain two-dimensional reflecting random walk without background processes, obtained by using the large deviation techniques and the singularity analysis methods. We also present a condition ensuring the sequence of the stationary probabilities geometrically decays without power terms, asymptotically. Asymptotic properties of the stationary tail distribution in the coordinate directions in a 2d-QBD process have already been studied in the literature. The results of this paper are also important complements to those results.

Keywords Quasi-birth-and-death process · Markov modulated reflecting random walk · Markov additive process · Asymptotic decay rate · Stationary distribution · Matrix analytic method

Mathematics Subject Classification 60J10 · 60K25
1 Introduction

We deal with a two-dimensional discrete-time quasi-birth-and-death process (2d-QBD process for short), which is an extension of ordinary one dimensional QBD process (see, for example, Latouche and Ramaswami [14]), and study asymptotic properties of the stationary tail distribution in any direction. The 2d-QBD process is also a two-dimensional skip-free Markov modulated reflecting random walk (2d-MMRRW for short), and the 2d-MMRRW is a two-dimensional skip-free reflecting random walk (2d-RRW for short) having a background process.

Let a Markov chain \( \{ Y_n \} = \{ (X_n, J_n) \} \) be a 2d-QBD process on the state space \( \mathbb{Z}_+^2 \times S_0 \), where \( X_n = (X_{1,n}, X_{2,n}) \), \( S_0 \) is a finite set with cardinality \( s_0 \), i.e., \( S_0 = \{ 1, 2, \ldots, s_0 \} \). The process \( \{ X_n \} \) is called the level process, \( \{ J_n \} \) the phase process (background process), and the transition probabilities of the level process vary according to the state of the phase process. This modulation is space homogeneous except for the boundaries of \( \mathbb{Z}_+^2 \). The level process is assumed to be skip free, i.e., for any \( n \geq 0 \), \( X_{n+1} - X_n \in \{ -1, 0, 1 \}^2 \). Stochastic models arising from various Markovian two-queue models and two-node queueing networks such as two-queue polling models and generalized two-node Jackson networks with Markovian arrival processes and phase-type service processes can be represented as two-dimensional continuous-time QBD processes, and their stationary distributions can be analyzed through the corresponding two-dimensional discrete-time QBD processes obtained by the uniformization technique; see, for example, Refs. [20, 26, 29]. In that sense, 2d-QBD processes are more versatile than 2d-RRWs, which have no background processes. This is a reason why we are interested in stochastic models with a background process. Here we emphasize that the assumption of skip-free is not so restricted since any 2d-MMRRW with bounded jumps can be represented as a 2d-MMRRW with skip-free jumps (i.e., 2d-QBD process); see Introduction of Ozawa [31].

Asymptotics of the stationary distributions in various 2d-RRWs without background processes have been investigated in the literature for several decades, especially, by Masakiyo Miyazawa and his colleagues (see a survey paper [19] of Miyazawa and references therein). Some of their results have been extended to the 2d-QBD process in Ozawa [26], Miyazawa [20] and Ozawa and Kobayashi [27]. In [26] and [20], the asymptotic decay rates of the stationary distribution in the coordinate directions and a condition ensuring the sequence of the stationary probabilities geometrically decays without power terms have been obtained (cf. results in Miyazawa [18] in the case without background processes). In [20], the tail decay rates of the marginal stationary distribution in an arbitrary direction have also been obtained. In these studies, the 2d-QBD process is represented as a one-dimensional QBD process with infinitely many phase states, where one of the coordinate axes is taken as a level, and the asymptotics of the stationary distribution are derived through the matrix geometric solution of the stationary distribution in the one-dimensional QBD process. In [20], the author has also proposed certain tractable conditions for application, which make the relation between model parameters and performance measures clear. In [27], the exact asymptotic formulae of the stationary distribution in the coordinate directions have been obtained (cf. results in Kobayashi and Miyazawa [13] in the case without background processes), where the vector generating function of the stationary probabilities along
one of the coordinate axes is given in terms of the stationary probabilities along the other coordinate axis. The exact asymptotic formulae are given by applying the complex analytic method to the vector generating functions. The formulations used in [20, 26, 27] are specialized to analysis of asymptotics of the stationary distribution in the coordinate directions, and other analytic manners were necessary for analyzing tail asymptotics in an arbitrary direction.

One of such analytic manners has been given in Ozawa [31], where the asymptotic decay rate of the occupancy measure in any direction in a multi-dimensional MMRW were obtained. Ozawa [30] is an early version of [31] and it deals with only two-dimensional models. This paper relies on [30] more than [31]. Block state processes were obtained in [30, 31], and it enables us to analyze asymptotics in an arbitrary direction. To be precise, if the asymptotics of the stationary distribution in the direction given by the vector $c = (1, 1)$ are obtained, those in an arbitrary direction can also be obtained through the block state process derived from the original 2d-QBD process. A 2d-MMRW is a discrete-time Markov chain on $\mathbb{Z}^2 \times S_0$, having no reflecting boundaries. On the other hand, a 2d-QBD process is a discrete-time Markov chain on $\mathbb{Z}_+^2 \times S_0$, having the reflecting boundaries on the coordinate axes, where $\mathbb{Z}_+$ is the set of all nonnegative integers. Hence, in order to analyze the asymptotics of the stationary distribution in an arbitrary direction in the 2d-QBD process, we need another analytic manner to evaluate the effect of the boundaries on the asymptotics. In [27], it has been done through the vector generating functions of the stationary probabilities along the coordinate axes, mentioned above; see (A.12) of this paper, which corresponds to one of those vector generating functions. The vector generating functions have been derived by using matrix generating functions one of whose variables is a matrix. This analytic manner cannot directly be applied to the case of this paper. Instead, we introduce a compensation equation (see Sect. 2.2), which enable us to obtain the vector generating function of the stationary probabilities along the direction given by the vector $c = (1, 1)$. This vector generating function is represented in terms of the stationary probabilities along both the coordinate axes. The asymptotics of the stationary distribution in the direction $c = (1, 1)$ are obtained by applying the complex analytic method to the vector generating function, and those in an arbitrary direction through the block state process derived from the original 2d-QBD process. Here we note that the compensation equation holds true not only for the 2d-QBD process but also for higher-dimensional QBD processes.

Denote by $\mathcal{J}_2$ the set of all the subsets of $\{1, 2\}$, i.e., $\mathcal{J}_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and we use it as an index set. Divide $\mathbb{Z}_+^2$ into $2^2 = 4$ exclusive subsets defined as

$$\mathbb{B}^\alpha = \{ x = (x_1, x_2) \in \mathbb{Z}_+^2; x_i > 0 \text{ for } i \in \alpha, x_i = 0 \text{ for } i \in \{1, 2\} \setminus \alpha, \alpha \in \mathcal{J}_2. $$

The class $\{\mathbb{B}^\alpha; \alpha \in \mathcal{J}_2\}$ is a partition of $\mathbb{Z}_+^2$. $\mathbb{B}^\emptyset$ is the set containing only the origin, and $\mathbb{B}^{\{1,2\}}$ is the set of all positive points in $\mathbb{Z}_+^2$. Let $P$ be the transition probability matrix of the 2d-QBD process $\{Y_n\}$ and represent it in block form as $P = (P_{x,x'}; x, x' \in \mathbb{Z}_+^2)$, where $P_{x,x'} = (p(x,j),(x',j'); j, j' \in S_0)$ and $p(x,j),(x',j'); Y_0 = (x, j) \Rightarrow P(Y_1 = (x', j') | Y_0 = (x, j))$. For $\alpha \in \mathcal{J}_2$ and $i_1, i_2 \in \{-1, 0, 1\}$, let $A_{i_1,i_2}^\alpha$ be a one-step transition probability block from a state in $\mathbb{B}^\alpha$, which is defined as
Fig. 1 Transition probability blocks

\[ [A_{i_1, i_2}^{\alpha}]_{j_1, j_2} = \mathbb{P}(Y_1 = (x + (i_1, i_2), j_2) \mid Y_0 = (x, j_1)) \] for any \( x \in \mathbb{B}^\alpha \),

where we assume the blocks corresponding to impossible transitions are zero (see Fig. 1). For example, if \( \alpha = \{1\} \), we have \( A_{i,-1}^{\alpha} = O \) for \( i \in \{-1, 0, 1\} \). Since the level process is skip free, for every \( x, x' \in \mathbb{Z}^2 \), \( P_{x,x'} \) is given by

\[
P_{x,x'} = \begin{cases} A_{x'-x}^{\alpha} & \text{if } x \in \mathbb{Z} \times \{0\} \text{ and } x' - x \in \{-1, 0, 1\} \times \{0, 1\}, \\ A_{x'-x}^{1,2} & \text{if } x \in \mathbb{Z} \times \mathbb{N} \text{ and } x' - x \in \{-1, 0, 1\}^2, \\ O, & \text{otherwise}. \end{cases} \tag{1.1}
\]

We assume the following condition throughout the paper.

**Assumption 1.1** The 2d-QBD process \( \{Y_n\} \) is irreducible and aperiodic.

Next, we define several Markov chains derived from the 2d-QBD process. For a nonempty set \( \alpha \in \mathcal{I}_2 \), let \( \{Y_\alpha^n\} = \{(X_\alpha^n, J_\alpha^n)\} \) be a process derived from the 2d-QBD process \( \{Y_n\} \) by removing the boundaries that are orthogonal to the \( x_i \)-axis for each \( i \in \alpha \). To be precise, the process \( \{Y_\alpha^n\} \) is a Markov chain on \( \mathbb{Z} \times \mathbb{Z}^+ \times S_0 \) whose transition probability matrix \( P^{[1]} = (P^{[1]}_{x,x'}; x, x' \in \mathbb{Z} \times \mathbb{Z}^+) \) is given as

\[
P^{[1]}_{x,x'} = \begin{cases} A_{x'-x}^{[1]} & \text{if } x \in \mathbb{Z} \times \{0\} \text{ and } x' - x \in \{-1, 0, 1\} \times \{0, 1\}, \\ A_{x'-x}^{1,2} & \text{if } x \in \mathbb{Z} \times \mathbb{N} \text{ and } x' - x \in \{-1, 0, 1\}^2, \\ O, & \text{otherwise}, \end{cases} \tag{1.2}
\]

where \( \mathbb{N} \) is the set of all positive integers. The process \( \{Y_\alpha^{[2]}\} \) on \( \mathbb{Z}^+ \times \mathbb{Z} \times S_0 \) and its transition probability matrix \( P^{[2]} = (P^{[2]}_{x,x'}; x, x' \in \mathbb{Z} \times \mathbb{Z}) \) are analogously defined.

The process \( \{Y_\alpha^{[1,2]}\} \) is a Markov chain on \( \mathbb{Z}^2 \times S_0 \), whose transition probability matrix \( P^{[1,2]} = (P^{[1,2]}_{x,x'}; x, x' \in \mathbb{Z}^2) \) is given as

\[
P^{[1,2]}_{x,x'} = \begin{cases} A_{x'-x}^{[1,2]} & \text{if } x' - x \in \{-1, 0, 1\}^2, \\ O, & \text{otherwise}. \end{cases} \tag{1.3}
\]

Regarding \( X_\alpha^{(1)} \) as the additive part, we see that the process \( \{Y_\alpha^{[1]}\} = \{(X_\alpha^{(1)}, (X_\alpha^{(1)}, J_\alpha^{[1]}))\} \) is a Markov additive process (MA-process for short) with the background state.
(\(X_{2,n}^{(1)}, J_{n}^{(1)}\)) (with respect to MA-processes, see, for example, Ney and Nummelin [24]). The process \(\{Y_{n}^{(2)}\} = \{(X_{2,n}^{(2)}, J_{1,n}^{(2)})\}\) is also an MA-process, where \(X_{2,n}^{(2)}\) is the additive part and \((X_{1,n}^{(2)}, J_{n}^{(2)})\) the background state, and \(\{Y_{n}^{(1,2)}\} = \{(X_{1,n}^{(1,2)}, J_{n}^{(1,2)})\}\) an MA-process, where \((X_{1,n}^{(1,2)}, X_{2,n}^{(1,2)})\) the additive part and \(J_{n}^{(1,2)}\) the background state. We call them the induced MA-processes derived from the original 2d-QBD process. Their background processes are called induced Markov chains in Fayolle et. al. [6]. Let \(\{\bar{A}_{i}^{(1)}; i \in \{-1, 0, 1\}\}\) be the Markov additive kernel (MA-kernel for short) of the induced MA-process \(\{Y_{n}^{(1)}\}\), which is the set of transition probability blocks and defined as, for \(i \in \{-1, 0, 1\}\),

\[
\bar{A}_{i}^{(1)} = \left(\bar{A}_{i,(x_{2}, x_{2}')}^{(1)}; x_{2}, x_{2}' \in \mathbb{Z}_{+}\right),
\]

\[
\bar{A}_{i,(x_{2}, x_{2}')}^{(1)} = \begin{cases} A_{i,x_{2} - x_{2}'}^{(1)} & \text{if } x_{2} = 0 \text{ and } x_{2}' - x_{2} \in \{0, 1\}, \\ A_{i,x_{2} - x_{2}'}^{(2)} & \text{if } x_{2} \geq 1 \text{ and } x_{2}' - x_{2} \in \{-1, 0, 1\}, \\ O & \text{otherwise.} \end{cases}
\]

Let \(\{\bar{A}_{i}^{(2)}; i \in \{-1, 0, 1\}\}\) be the MA-kernel of \(\{Y_{n}^{(2)}\}\), defined in the same manner. With respect to \(\{Y_{n}^{(1,2)}\}\), the MA-kernel is given by \(\{A_{i_{1},i_{2}}^{(1,2)}; i_{1}, i_{2} \in \{-1, 0, 1\}\}\). We assume the following condition throughout the paper.

**Assumption 1.2** The induced MA-processes \(\{Y_{n}^{(1)}\}, \{Y_{n}^{(2)}\}\) and \(\{Y_{n}^{(1,2)}\}\) are irreducible and aperiodic.

Let \(\bar{A}_{*}^{(1)}(z)\) and \(\bar{A}_{*}^{(2)}(z)\) be the matrix generating functions of the MA-kernels of \(\{Y_{n}^{(1)}\}\) and \(\{Y_{n}^{(2)}\}\), respectively, defined as

\[
\bar{A}_{*}^{(1)}(z) = \sum_{i \in \{-1, 0, 1\}} z^{i} \bar{A}_{i}^{(1)}, \quad \bar{A}_{*}^{(2)}(z) = \sum_{i \in \{-1, 0, 1\}} z^{i} \bar{A}_{i}^{(2)}.
\]

The matrix generating function of the MA-kernel of \(\{Y_{n}^{(1,2)}\}\) is given by \(A_{*,*}^{(1,2)}(z_{1}, z_{2})\), defined as

\[
A_{*,*}^{(1,2)}(z_{1}, z_{2}) = \sum_{i_{1}, i_{2} \in \{-1, 0, 1\}} z_{1}^{i_{1}} z_{2}^{i_{2}} A_{i_{1},i_{2}}^{(1,2)}.
\]

Note that we use generating functions instead of moment generating functions in the paper because the generating functions are more suitable for complex analysis. Let \(\Gamma^{(1)}, \Gamma^{(2)}\) and \(\Gamma^{(1,2)}\) be regions in which the convergence parameters of \(\bar{A}_{*}^{(1)}(e^{\theta_{1}})\), \(\bar{A}_{*}^{(2)}(e^{\theta_{2}})\) and \(A_{*,*}^{(1,2)}(e^{\theta_{1}}, e^{\theta_{2}})\) are greater than 1, respectively, i.e.,
\[ \Gamma^{[1]} = \{ (\theta_1, \theta_2) \in \mathbb{R}^2; \text{cp}(\bar{A}_u^{[1]}(\theta_1)) > 1 \}, \quad \Gamma^{[2]} = \{ (\theta_1, \theta_2) \in \mathbb{R}^2; \text{cp}(\bar{A}_u^{[2]}(\theta_2)) > 1 \}, \]
\[ \Gamma^{[1,2]} = \{ (\theta_1, \theta_2) \in \mathbb{R}^2; \text{cp}(\bar{A}_u^{[1,2]}(\theta_1, \theta_2)) > 1 \}, \]

where, for a nonnegative square matrix \( A \) with a finite or countable dimension, \( \text{cp}(A) \) denote the convergence parameter of \( A \), i.e., \( \text{cp}(A) = \sup\{ r \in \mathbb{R}_+; \sum_{n=0}^{\infty} r^n A^n < \infty, \text{ entry-wise} \} \). By Lemma A.1 of Ozawa [31], \( \text{cp}(\bar{A}_u^{[1]}(\theta))^{-1} \) and \( \text{cp}(\bar{A}_u^{[2]}(\theta))^{-1} \) are log-convex in \( \theta \), and the closures of \( \Gamma^{[1]} \) and \( \Gamma^{[2]} \) are convex sets; \( \text{cp}(\bar{A}_u^{[1,2]}(\theta_1, \theta_2))^{-1} \) is also log-convex in \( (\theta_1, \theta_2) \), and the closure of \( \Gamma^{[1,2]} \) is a convex set. Furthermore, by Proposition B.1 of Ozawa [31], \( \Gamma^{[1,2]} \) is bounded under Assumption 1.2. For \( i \in \{1, 2\} \), define \( \theta^*_i \) as
\[ \theta^*_i = \sup\{ \theta_i \in \mathbb{R} : (\theta_1, \theta_2) \in \Gamma^{[i]} \}. \]

We depict an example of the domains \( \Gamma^{[1,2]} \), \( \Gamma^{[1]} \) and \( \Gamma^{[2]} \) in Fig. 2.

Assuming \( \{ Y_n \} \) is positive recurrent (a condition for this will be given in Lemma 2.1 of Sect. 2), we denote by \( \nu \) the stationary distribution of \( \{ Y_n \} \), where \( \nu = (\nu_x, x \in \mathbb{Z}_+^2) \), \( \nu_x = (\nu(x,j), j \in S_0) \) and \( \nu(x,j) \) is the stationary probability that the 2d-QBD process is in the state \( (x, j) \) in steady state. Let \( c = (c_1, c_2) \in \mathbb{N}^2 \) be an arbitrary discrete direction vector and, for \( i \in \{1, 2\} \), define a real value \( \theta^*_{c,i} \) as
\[ \theta^*_{c,i} = \sup\{ \langle c, \theta \rangle : \theta \in \Gamma^{[i]} \cap \Gamma^{[1,2]} \}, \quad (1.4) \]

where \( \langle a, b \rangle \) is the inner product of vectors \( a \) ad \( b \). As depicted in the left figure of Fig. 3, the value of \( \theta^*_{c,1} \) is determined by the line with normal vector \( c \) contacting the intersection of the domains \( \Gamma^{[1]} \) and \( \Gamma^{[1,2]} \). We depict three such lines, which correspond to three different values of \( c \), in that figure. The value of \( \theta^*_{c,2} \) is determined analogously; see the right figure of Fig. 3. Our main aim is to demonstrate under certain conditions that, for any \( j \in S_0 \),
\[ \lim_{k \to \infty} \frac{1}{k} \log \nu(kc,j) = -\min\{ \theta^*_{c,1}, \theta^*_{c,2} \}, \quad (1.5) \]
i.e., the asymptotic decay rate of the stationary distribution in the direction $c$ is given by the smaller of $\theta_{c,1}^+$ and $\theta_{c,2}^+$. This result will be stated in Theorem 3.2 of Sect. 3. We also present a condition ensuring the sequence $\{\nu(kc,j)\}_{k \geq 0}$ geometrically decays without power terms. We prove them by using the matrix analytic method of the queueing theory as well as the complex analytic method; the former has been introduced by Marcel Neuts and developed in the literature; see, for example, Refs. [1, 14, 22, 23]. Our model is a kind of multidimensional reflecting process, and asymptotics in various multidimensional reflecting processes have been investigated in the literature for several decades; see Miyazawa [19] and references therein. 0-partially homogeneous ergodic Markov chains discussed in Borovkov and Mogul’skii [2] are Markov chains on the positive quadrant including 2d-RRWs as a special case. For those Markov chains, a formula corresponding to (1.5) have been obtained by using the large deviations techniques; see Theorem 3.1 of Borovkov and Mogul’skii [2] and also see Proposition 5.1 of Miyazawa [18] for the case of 2d-RRW. They have considered only models without background processes. With respect to the exact asymptotic expansion of the stationary distribution in any direction, Malyshew [16] has obtained it for a simple 2d-RRW on the positive quadrant; Also see Fayolle et al. [7]. We will compare his results with ours in Sect. 4. In Dai and Miyazawa [4] (also see Dai and Miyazawa [3]), the results parallel to ours have been obtained for a two-dimensional continuous-state Markov process, named semimartingale-reflecting Brownian motion (SRBM for short). The 2d-SRBM is also a model without background processes. For asymptotic analysis of 2d-SRBM, also see Ernst and Franceschi [5] and Franceschi and Kourkova [10]. In [5], the exact asymptotic formula of the occupancy density for two-dimensional reflected Brownian motion (2d-RBM for short) in a half-plain has been obtained, where the 2d-RBM corresponds to the induced MA-process $\{Y_{n}^{[1]}\}$ in our framework. In [10], the asymptotic expansion of the stationary distribution density in 2d-SRBM in the quarter plane has been obtained by applying the analytic method developed in [16]. With respect to models with a background process, the asymptotics of the stationary distribution in a Markov modulated fluid network with a finite number of stations have recently been studied in Miyazawa [21], where upper and lower bounds for the stationary tail decay rate in various directions were obtained by using so-called Dynkin’s formula.
The rest of the paper is organized as follows. In Sect. 2, we give a stability condition for the 2d-QBD process and define the asymptotic decay rates of the stationary distribution. In the same section, we introduce a key formula representing the stationary distribution in terms of the fundamental (potential) matrix of the induced MA-process \( \{Y_n^{(1,2)}\} \). We call it a compensation equation. Furthermore, we define block state processes derived from the original 2d-QBD process, which will be used for proving propositions in the following sections. A summary of their properties is given in Appendix A. In Sect. 3, we obtain the asymptotic decay rate of the stationary distribution in an arbitrary direction. First, we obtain it in the case where the direction vector is given by \( e = (1, 1) \). The asymptotic decay rate for an arbitrary direction vector is obtained from the results in the case of \( e = (1, 1) \), by using the block state process. In Sect. 4, we explain a geometric property of the asymptotic decay rates and give an example of two-queue model. In the two-queue model, the asymptotic decay rate corresponds to the decreasing rate of the joint queue length probability in steady state when the queue lengths of both the queues simultaneously enlarge. In the same section, we compare our results with those for a 2d-RRW on the quarter plane, obtained by Malyshev [16]. The paper concludes with remarks about the exact asymptotic formula and higher-dimensional QBD processes in Sect. 5.

Notation for vectors and matrices. For a matrix \( A \), we denote by \([A]_{i,j}\) the \((i, j)\)-entry of \( A \) and by \( A^\top \) the transpose of \( A \). If \( A = (a_{i,j}) \), \( |A| = (|a_{i,j}|) \). Similar notations are also used for vectors. For a finite square matrix \( A \), we denote by \( \text{spr}(A) \) the spectral radius of \( A \), which is the maximum modulus of eigenvalue of \( A \). If \( A \) is nonnegative, \( \text{spr}(A) \) corresponds to the Perron-Frobenius eigenvalue of \( A \) and we have \( \text{spr}(A) = \text{cp}(A)^{-1} \), where \( \text{cp}(A) \) is the convergence parameter of \( A \). \( O \) is a matrix of 0’s, \( I \) is a column vector of 1’s and \( 0 \) is a column vector of 0’s; their dimensions, which are finite or countably infinite, are determined in context. \( I \) is the identity matrix. For an \( n_1 \times n_2 \) matrix \( A = (a_{i,j}) \), \( \text{vec}(A) \) is the vector of stacked columns of \( A \), i.e., \( \text{vec}(A) = (a_{1,1}, \ldots, a_{n_1,1}, a_{1,2}, \ldots, a_{n_1,2}, \ldots, a_{1,n_2}, \ldots, a_{n_1,n_2})^\top \).

2 Preliminaries

2.1 Stability condition

Let \( a^{(1)} \), \( a^{(2)} \) and \( a^{(1,2)} \) be the mean drifts of the additive part in the induced MA-processes \( \{Y_n^{(1)}\} \), \( \{Y_n^{(2)}\} \) and \( \{Y_n^{(1,2)}\} \), respectively, i.e.,

\[
a^{(i)} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X^{(i)}_{i,k} - X^{(i)}_{i,k-1}), \quad i = 1, 2, \quad a^{(1,2)} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (X^{(1,2)}_{k} - X^{(1,2)}_{k-1}),
\]

where \( a^{(1,2)} = (a^{(1,2)}_{1}, a^{(1,2)}_{2}) \). By Corollary 3.1 of Ozawa [29], the stability condition of the 2d-QBD process \( \{Y_n\} \) is given as follows:
Lemma 2.1 (i) In the case where \( a_{1}^{[1,2]} < 0 \) and \( a_{2}^{[1,2]} < 0 \), the 2d-QBD process \( \{Y_{n}\} \) is positive recurrent if \( a^{[1]} < 0 \) and \( a^{[2]} < 0 \), and it is transient if either \( a^{[1]} > 0 \) or \( a^{[2]} > 0 \).

(ii) In the case where \( a_{1}^{[1,2]} \geq 0 \) and \( a_{2}^{[1,2]} < 0 \), \( \{Y_{n}\} \) is positive recurrent if \( a^{[1]} < 0 \), and it is transient if \( a^{[1]} > 0 \).

(iii) In the case where \( a_{1}^{[1,2]} < 0 \) and \( a_{2}^{[1,2]} \geq 0 \), \( \{Y_{n}\} \) is positive recurrent if \( a^{[2]} < 0 \), and it is transient if \( a^{[2]} > 0 \).

(iv) If one of \( a_{1}^{[1,2]} \) and \( a_{2}^{[1,2]} \) is positive and the other is non-negative, then \( \{Y_{n}\} \) is transient.

Each mean drift is represented in terms of the stationary distribution of the corresponding induced Markov chain, i.e., the background process of the corresponding induced MA-process; for their expressions, see Subsection 3.1 of Ozawa [29] and its related parts. We assume the following condition throughout the paper.

Assumption 2.1 The condition in Lemma 2.1 that ensures the 2d-QBD process \( \{Y_{n}\} \) is positive recurrent holds.

2.2 Compensation equation

Consider the induced MA-process \( \{Y_{n}^{[1,2]}\} \) on \( \mathbb{Z}^{2} \times S_{0} \). Its transition probability matrix is given by \( P^{[1,2]} = (P_{x,x'}, n, x, x' \in \mathbb{Z}^{2}) \). Denote by \( \Phi^{[1,2]} = (\Phi_{x,x'}^{[1,2]} ; x, x' \in \mathbb{Z}^{2}) \) the fundamental matrix (potential matrix) of \( P^{[1,2]} \), i.e., \( \Phi^{[1,2]} = \sum_{n=0}^{\infty} (P^{[1,2]})^{n} \). Under Assumption 2.1, since at least one element of the mean drift vector of \( \{Y_{n}^{[1,2]}\} \), \( a_{1}^{[1,2]} \) or \( a_{2}^{[1,2]} \), is negative, \( \Phi^{[1,2]} \) is entry-wise finite. Since the transition probabilities of \( \{Y_{n}^{[1,2]}\} \) are space-homogeneous with respect to the additive part, we have for every \( x, x' \in \mathbb{Z}^{2} \) and for every \( l \in \mathbb{Z}^{2} \) that

\[
\Phi_{x,x'}^{[1,2]} = \Phi_{x-l,x'-l}^{[1,2]}. \tag{2.1}
\]

Furthermore, \( \Phi^{[1,2]} \) satisfies the following property.

Proposition 2.1 \( \Phi^{[1,2]} \) is entry-wise bounded.

Proof By (2.1), it suffices to show that, for every \( j, j' \in S_{0} \),

\[
\sup_{x \in \mathbb{Z}^{2}} [\Phi_{x,0}^{[1,2]}]_{j,j'} < \infty, \tag{2.2}
\]

where we use the fact that \( S_{0} \) is finite. Let \( \tau(j') \) be the first hitting time of \( \{Y_{n}^{[1,2]}\} \) to the state \((0, 0, j')\), i.e.,

\[
\tau(j') = \inf \{n \geq 0 ; Y_{n}^{[1,2]} = (0, 0, j') \}.
\]
Since \( \tau(j') \) is a stopping time, we have by the strong Markov property of \( \{Y_{n}^{1,2}\} \) that

\[
[\Phi_{x,x'}^{1,2}]_{j,j'} = \sum_{k=0}^{\infty} \mathbb{E} \left( \sum_{n=0}^{\infty} 1(Y_{n}^{1,2} = (0, 0, j')) \left| \tau(j') = k, Y_{0}^{1,2} = (x, j) \right. \right) \mathbb{P}(\tau(j') = k | Y_{0}^{1,2} = (x, j))
\]

\[
= \sum_{n=0}^{\infty} 1(Y_{n}^{1,2} = (0, 0, j')) Y_{0}^{1,2} = (0, 0, j') \mathbb{P}(\tau(j') < \infty | Y_{0}^{1,2} = (x, j)) \leq [\Phi_{0,0}^{1,2}]_{j',j'}.
\]

(2.3)

Since \( \Phi_{0,0}^{1,2} \) is entry-wise finite, this implies inequality (2.2).

**Remark 2.1** From the proof of the proposition, we see that, for every \( (x, j), (x', j') \in \mathbb{Z}^{2} \times S_{0} \),

\[
[\Phi_{x,x'}^{1,2}]_{j,j'} \leq \max_{j'' \in S_{0}} [\Phi_{0,0}^{1,2}]_{j'',j''}.
\]

(2.4)

From \( \{Y_{n}^{1,2}\} \), we construct another Markov chain on \( \mathbb{Z}^{2} \times S_{0} \), denoted by \( \{\tilde{Y}_{n}^{1,2}\} \), by replacing the transition probabilities from the states in \( \mathbb{B}^{0} \cup \mathbb{B}^{1} \cup \mathbb{B}^{2} \) with those of the original 2d-QBD process. To be precise, the transition probability matrix of \( \{\tilde{Y}_{n}^{1,2}\} \), denoted by \( \tilde{\Phi}_{x,x'}^{1,2} = (\tilde{\Phi}_{x,x'}^{1,2}; x, x' \in \mathbb{Z}^{2}) \), is given as

\[
\tilde{\Phi}_{x,x'}^{1,2} = \begin{cases} 
\Phi_{x,x'}^{1,2}, & x \in \mathbb{B}^{\alpha} \text{ for some } \alpha \in \{1, 2, \emptyset\} \text{ and } x' - x \in \{-1, 0, 1\}^{2}, \\
\Phi_{x,x'}^{1,2}, & x \not\in \mathbb{B}^{\alpha} \text{ for any } \alpha \in \{1, 2, \emptyset\} \text{ and } x' - x \in \{-1, 0, 1\}^{2}, \\
0, & \text{otherwise}.
\end{cases}
\]

The subspace \( \mathbb{Z}_{+}^{2} \times S_{0} \) is a unique closed communication class (irreducible class) of the Markov chain \( \{\tilde{Y}_{n}^{1,2}\} \) and its stationary distribution, \( \tilde{\nu} = (\tilde{\nu}_{x}, x \in \mathbb{Z}^{2}) \), is given as

\[
\tilde{\nu}_{x} = \begin{cases} 
\nu_{x}, & x \in \mathbb{Z}_{+}^{2}, \\
0, & \text{otherwise},
\end{cases}
\]

where \( \nu = (\nu_{x}, x \in \mathbb{Z}_{+}^{2}) \) is the stationary distribution of the original 2d-QBD process. The stationary distribution \( \tilde{\nu} \) satisfies the stationary equation \( \tilde{\nu} \tilde{\Phi}_{x,x'}^{1,2} = \tilde{\nu} \). We have the following.

**Proposition 2.2** Under Assumption 2.1, \( \tilde{\nu} \Phi_{0,0}^{1,2} \) is elementwise bounded.

**Proof** Since \( \tilde{\nu} \) is a probability distribution, by Remark 2.1, we have for any \( (x, j) \in \mathbb{Z}^{2} \times S_{0} \) that

\[
[\tilde{\nu} \Phi_{x}^{1,2}]_{x,j} \leq \max_{j'' \in S_{0}} [\Phi_{0,0}^{1,2}]_{j'',j''}.
\]

(2.5)

Hence, the assertion of the proposition holds. \( \square \)
Since $\Phi^{[1,2]}$ is entry-wise finite, we have
\[(I - P^{[1,2]})\Phi^{[1,2]} = \Phi^{[1,2]}(I - P^{[1,2]}) = I.\] (2.6)

By Proposition 2.2, we obtain the following.

**Lemma 2.2**
\[
\tilde{\nu} = \tilde{\nu}(\tilde{P}^{[1,2]} - P^{[1,2]})\Phi^{[1,2]}. \tag{2.7}
\]

**Proof** By the Fubini’s theorem, we have $\tilde{\nu} \tilde{P}^{[1,2]} \Phi^{[1,2]} = \tilde{\nu} \Phi^{[1,2]} < \infty$, elementwise. Hence,
\[
\tilde{\nu}(\tilde{P}^{[1,2]} - P^{[1,2]})\Phi^{[1,2]} = \tilde{\nu}(I - P^{[1,2]})\Phi^{[1,2]} = \tilde{\nu},
\]
where $\tilde{\nu}(I - P^{[1,2]})\Phi^{[1,2]}$ corresponds to a Riesz decomposition of $\tilde{\nu}$ in the case where the harmonic function term is equivalent to zero; see Theorem 3.1 of Nummelin [25]. \[\square\]

Equation (2.7) can also be derived by the compensation method discussed in Keilson [12]. We, therefore, call it a compensation equation. Its remarkable point is that the nonzero entries of $\tilde{P}^{[1,2]} - P^{[1,2]}$ are restricted to the transition probabilities from the states in $\mathbb{B}^0 \cup \mathbb{B}^{[1]} \cup \mathbb{B}^{[2]}$. Hence, we immediately obtain, for $x \in \mathbb{Z}^2$,
\[
\tilde{\nu}_x = \sum_{i_1, i_2 \in \{-1, 0, 1\}} \nu(0, 0) (A^{[\emptyset]}_{i_1, i_2} - A^{[1,2]}_{i_1, i_2})_x \Phi^{[1,2]}_{(i_1, i_2)} + \sum_{k=1}^{\infty} \sum_{i_1, i_2 \in \{-1, 0, 1\}} \nu(k, 0) (A^{[1]}_{i_1, i_2} - A^{[1,2]}_{i_1, i_2})_x \Phi^{[1,2]}_{(k+i_1, i_2)} + \sum_{k=1}^{\infty} \sum_{i_1, i_2 \in \{-1, 0, 1\}} \nu(0, k) (A^{[2]}_{i_1, i_2} - A^{[1,2]}_{i_1, i_2})_x \Phi^{[1,2]}_{(i_1, k+i_2)}, \tag{2.8}
\]
where any $A^{[\alpha]}_{i_1, i_2}$ corresponding to impossible transitions is assumed to be zero. Equation (2.8) plays a crucial role in the following section.

### 2.3 Asymptotic decay rates

Let $\mathbf{e} = (c_1, c_2) \in \mathbb{N}^2$ be an arbitrary discrete direction vector. For $(x, j) \in \mathbb{Z}^2_+ \times S_0$, define lower and upper asymptotic decay rates $\xi_{e}(x, j)$ and $\bar{\xi}_{e}(x, j)$ as
\[
\xi_{e}(x, j) = -\limsup_{k \to \infty} \frac{1}{k} \log \nu(x+ke, j), \quad \bar{\xi}_{e}(x, j) = -\liminf_{k \to \infty} \frac{1}{k} \log \nu(x+ke, j).
\]

By the Cauchy–Hadamard theorem, the radius of convergence of the power series of the sequence $\{
u(x+ke, j)\}_{k \geq 0}$ is given by $e^{\bar{\xi}_{e}(x, j)}$. If $\xi_{e}(x, j) = \bar{\xi}_{e}(x, j)$, we denote them by $\xi_{e}(x, j)$ and call it the asymptotic decay rate. Under Assumption 1.2, the following property holds.
Proposition 2.3  For every \((x, j), (x', j') \in \mathbb{N}^2 \times S_0\), \(\xi(x, j) = \xi(x', j')\) and \(\xi_c(x, j) = \bar{\xi}(x', j')\).

Since the proof of the proposition is elementary, we give it in Appendix B. Hereafter, we denote \(\xi(x, j), \bar{\xi}(x, j)\) and \(\xi_c(x, j)\) by \(\xi_e, \bar{\xi}_e\) and \(\bar{\xi}_c\), respectively. The asymptotic decay rates in the coordinate directions, denoted by \(\xi(1,0)\) and \(\xi(0,1)\), have already been obtained in Ozawa [26].

Let \(\{\tilde{Y}_n^{1,2}\} = \{(\tilde{X}_n^{1,2}, \tilde{J}_n^{1,2})\}\) be a lossy Markov chain derived from the induced MA-process \(\{Y_n^{1,2}\}\) by restricting the state space of the additive part to \(\mathbb{N}^2\). To be precise, the process \(\{\tilde{Y}_n^{1,2}\}\) is a Markov chain on \(\mathbb{N}^2 \times S_0\) whose transition probability matrix \(\tilde{P}^{1,2}_{x,x'}(x, x' \in \mathbb{N}^2)\) is given as

\[
\tilde{P}^{1,2}_{x,x'} = \begin{cases} 
A^{1,2}_{x' - x}, & \text{if } (x \in (\mathbb{N} \setminus \{1\})^2 \text{ and } x' - x \in \{-1, 0, 1\}^2) \\
\text{otherwhise}, & \text{otherwise}, 
\end{cases}
\]

where \(\tilde{P}^{1,2}\) is strictly substochastic. Let \(\Phi^{1,2}_{x,x'} = \left(\Phi^{1,2}_{x,x'}(x, x' \in \mathbb{N}^2)\right)\) be the fundamental matrix (potential matrix) of \(\tilde{P}^{1,2}\), i.e.,

\[
\Phi^{1,2}_{x,x'} = \sum_{n=0}^{\infty} (\tilde{P}^{1,2})^n.
\]

We assume the following condition throughout the paper.

Assumption 2.2 \(\{\tilde{Y}_n^{1,2}\}\) is irreducible and aperiodic.

This condition implies that the induced MA-process \(\{Y_n^{1,2}\}\) is irreducible and aperiodic, cf. Assumption 1.2. By Theorem 5.1 of Ozawa [31], we have, for any direction vector \(e \in \mathbb{N}^2\), every \(x = (x_1, x_2) \in \mathbb{N}^2\) such that \(x_1 = 1\) or \(x_2 = 1\), every \(l \in \mathbb{Z}_+^2\) and every \(j_1, j_2 \in S_0\),

\[
\lim_{k \to \infty} \frac{1}{k} \log \left[\Phi^{1,2}_{x,ke+l}\right]_{j_1,j_2} = -\sup\{\langle c, \theta \rangle; \theta \in \Gamma^{1,2}\}. 
\]

Since the stationary distribution of the 2d-QBD process can be represented in terms of the entries of \(\Phi^{1,2}\) (see Section 6 of Ozawa [31]), this formula leads us to the following.

Lemma 2.3

\[
\bar{\xi}_c \leq \sup\{\langle c, \theta \rangle; \theta \in \Gamma^{1,2}\}. 
\]
2.4 Block state process

For \( b = (b_1, b_2) \in \mathbb{N}^2 \), we consider another 2d-QBD process derived from the original 2d-QBD process \( \{ Y_n \} = \{ (X_n, J_n) \} \) by regarding each \( b_1 \times b_2 \) block of level as a level (see, for example, Subsection 4.2 of Ozawa [31]).

For \( i \in \{ 1, 2 \} \), denote by \( bX_{i,n} \) and \( bM_{i,n} \) the quotient and remainder of \( X_{i,n} \) divided by \( b_i \), respectively, i.e.,

\[
X_{i,n} = b_i bX_{i,n} + bM_{i,n},
\]

where \( bX_{i,n} \in \mathbb{Z}_+ \) and \( bM_{i,n} \in \{ 0, 1, ..., b_i - 1 \} \). Define a process \( \{ bY_n \} \) as

\[
bY_n = (bX_n, (bM_n, bJ_n)),
\]

where \( bX_n = (bX_{1,n}, bX_{2,n}) \) is the level state and \( (bM_n, bJ_n) = (bM_{1,n}, bM_{2,n}, J_n) \) the phase state. The process \( \{ bY_n \} \) is a 2d-QBD process and its state space is given by \( \mathbb{Z}_+^2 \times (\mathbb{Z}_0 b_1 - 1 \times \mathbb{Z}_0 b_2 - 1 \times S_0) \), where \( \mathbb{Z}_0 b_1 - 1 = \{ 0, 1, ..., b_i - 1 \} \). We call \( \{ bY_n \} \) a \( b \)-block state process. The transition probability matrix of \( \{ bY_n \} \), denoted by \( bP = (bP_{x,x'}) ; x, x' \in \mathbb{Z}_+^2 \), has the same block structure as \( P \). For \( \alpha \in \mathcal{J}_2 \) and \( i_1, i_2 \in \{ -1, 0, 1 \} \), denote by \( bA_{i_1,i_2} \) the transition probability block of \( bP \) corresponding to \( A_{i_1,i_2} \) of \( P \), then \( bA_{i_1,i_2} \) can be represented by using \( A_{i_1,i_2}' \), \( \alpha' \in \mathcal{J}_2 \), \( i_1', i_2' \in \{ -1, 0, 1 \} \). We omit the explicit expressions for the transition probability blocks since we do not use them directly. Let \( b\nu = (b\nu_x ; x = (x_1, x_2) \in \mathbb{Z}_+^2) \) be the stationary distribution of \( \{ bY_n \} \), where

\[
b\nu_x = (\nu_{(b_1 x_1 + i_1, b_2 x_2 + i_2)} ; i_1 \in \mathbb{Z}_0 b_1 - 1, i_2 \in \mathbb{Z}_0 b_2 - 1)
\]

and \( \nu = (\nu_x ; x \in \mathbb{Z}_+^2) \) is the stationary distribution of the original 2d-QBD process. Denote by \( b\xi_{(1,0)} \) and \( b\xi_{(0,1)} \) the asymptotic decay rates of the sequences \( \{ b\nu_{(k,0)} \}_{k \geq 0} \) and \( \{ b\nu_{(0,k)} \}_{k \geq 0} \), respectively, i.e., for \( i_1 \in \mathbb{Z}_0 b_1 - 1, i_2 \in \mathbb{Z}_0 b_2 - 1 \) and \( j \in S_0 \),

\[
b\xi_{(1,0)} = - \lim_{k \to \infty} \frac{1}{k} \log [b\nu_{(k,0)}]_{i_1,i_2}, \quad b\xi_{(0,1)} = - \lim_{k \to \infty} \frac{1}{k} \log [b\nu_{(0,k)}]_{i_1,i_2},
\]

where \( b\xi_{(1,0)} \) and \( b\xi_{(0,1)} \) do not depend on any of \( i_1, i_2 \) and \( j \). Since \( \{ bY_n \} \) is a 2d-QBD process and inherits the nature of the original 2d-QBD process, the results obtained in Refs. [26, 27] also hold for \( \{ bY_n \} \). For example, we have \( b\xi_{(1,0)} = b_1\xi_{(1,0)} \) and \( b\xi_{(0,1)} = b_2\xi_{(0,1)} \). For later use, we summarize the properties of \( \{ bY_n \} \) in Appendix A, and here define, for \( c \in \mathbb{N}^2 \), the asymptotic decay rate in direction \( c \), \( b\xi_c \), as

\[
b\xi_c = - \lim_{k \to \infty} \frac{1}{k} \log [b\nu_{kc}]_{i_1,i_2},
\]

where \( i_1, i_2 \) and \( j \) are arbitrary.
3 Asymptotics in an arbitrary direction

Hereafter, we use the following notation: For \( r > 0 \), \( \Delta_r \) and \( \partial \Delta_r \) are the open disk and circle of center 0 and radius \( r \) on the complex plane, respectively. For \( r_1, r_2 > 0 \) such that \( r_1 < r_2 \), \( \Delta_{r_1,r_2} \) is the open annular domain defined as \( \Delta_{r_1,r_2} = \Delta_{r_2} \setminus (\Delta_{r_1} \cup \partial \Delta_{r_1}) \).

3.1 Methodology and preparation

For \( c = (c_1, c_2) \in \mathbb{N}^2 \), define the generating function of the stationary probabilities of the 2d-QBD process in direction \( c \), \( \phi^c(z) \), as

\[
\phi^c(z) = \sum_{k=0}^{\infty} z^k \nu_{ke} = \sum_{k=-\infty}^{\infty} z^k \tilde{\nu}_{ke},
\]

where \( z \) is a complex variable. Furthermore, define real values \( \theta^\text{min}_c \) and \( \theta^\text{max}_c \) as

\[
\theta^\text{min}_c = \inf \{ \langle c, \theta \rangle; \theta \in \Gamma^{[1,2]} \}, \quad \theta^\text{max}_c = \sup \{ \langle c, \theta \rangle; \theta \in \Gamma^{[1,2]} \}.
\]

For any \( x \in [0, e^{\theta^\text{max}_c}] \), if the power series \( \phi^c(z) \) is absolutely convergent at \( z = x \), then we have by the Cauchy-Hadamard theorem that \( \xi_c \geq \theta^\text{max}_c \). This and (2.11) imply \( \xi_c = \theta^\text{max}_c \). We use this procedure for obtaining the asymptotic decay rate \( \xi_c \) when \( \xi_c \) is given by \( \theta^\text{max}_c \).

On the other hand, when \( \xi_c \) is less than \( \theta^\text{max}_c \), we demonstrate that for a certain point \( x_0 \in [0, e^{\theta^\text{max}_c}] \),

\[
\lim_{x \uparrow x_0} (x_0 - x) \phi(x) = g \quad \text{for some positive vector} \ g, \quad (3.1)
\]

and that the complex function \( \phi^c(z) \) is analytic in \( \Delta_{x_0} \cup (\partial \Delta_{x_0} \setminus \{x_0\}) \). In this case, by Theorem VI.4 of Flajolet and Sedgewick [8], the exact asymptotic formula for the sequence \( \{\nu_{ke}\} \) is given by \( x_0^{-k} \) and we have \( \xi_c = \log x_0 \).

For \( k \in \mathbb{Z} \), let \( \mathbb{Z}_{\leq k} \) and \( \mathbb{Z}_{\geq k} \) be the set of integers less than or equal to \( k \) and that of integers greater than or equal to \( k \), respectively. We introduce additional assumptions.

Assumption 3.1 (i) The lossy Markov chain derived from the induced MA-process \( \{Y_n^{[1,2]}\} \) by restricting the state space to \( \mathbb{Z}_{\leq 0} \times \mathbb{Z}_{\geq 0} \times S_0 \) is irreducible and aperiodic.

(ii) The lossy Markov chain derived from \( \{Y_n^{[1,2]}\} \) by restricting the state space to \( \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\leq 0} \times S_0 \) is irreducible and aperiodic.

Remark 3.1 Let \( c = (c_1, c_2) \in \mathbb{N}^2 \) be a direction vector, and define subspaces \( S^L_c \) and \( S^R_c \) as

\[
S^L_c = \{(x_1, x_2, j) \in \mathbb{Z}^2 \times S_0 : c_1 x_2 - c_2 x_1 \geq 0 \}, \quad S^R_c = \{(x_1, x_2, j) \in \mathbb{Z}^2 \times S_0 : c_1 x_2 - c_2 x_1 \leq 0 \}.
\]
\( S_c^L \) is the upper-left space of the line \( c_1 x_2 - c_2 x_1 = 0 \), and \( S_c^R \) the lower-right space of the same line. Due to the space-homogeneity of \( \{Y_n^{[1,2]}\} \) with respect to the additive part, under part (i) of Assumption 3.1, the lossy Markov chain derived from \( \{Y_n^{[1,2]}\} \) by restricting the state space to \( S_c^L \) is irreducible and aperiodic, and under part (ii) of Assumption 3.1, that derived from \( \{Y_n^{[1,2]}\} \) by restricting the state space to \( S_c^R \) is irreducible and aperiodic. We will use this property later.

Assumption 3.1 seems rather strong and it can probably be replaced with other weaker one. We adopt the assumption since it makes discussions simple; see Remark 3.2 in the following subsection.

For \( x \in \mathbb{Z}^2 \), define the matrix generating function of the blocks of \( \Phi^{[1,2]} \) in direction \( c, \Phi_c^{x,*}(z) \), as

\[
\Phi_c^{x,*}(z) = \sum_{k=-\infty}^{\infty} z^k \Phi^{[1,2]}_{x,kc}.
\]

The matrix generating function \( \Phi_c^{x,*}(z) \) satisfies the following.

**Proposition 3.1** For every \( x \in \mathbb{Z}^2 \), \( \Phi_c^{x,*}(z) \) is absolutely convergent and entry-wise analytic in the open annular domain \( \Delta_{e_c^{\min},e_c^{\max}} \).

**Proof** For every \( x \in \mathbb{Z}^2 \), we have for any \( \theta = (\theta_1, \theta_2) \in \Gamma^{[1,2]} \) that

\[
\sum_{n=0}^{\infty} (A_{x,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2}))^n = \sum_{x' \in \mathbb{Z}^2} (x', \theta) \Phi^{[1,2]}_{0,x'}
\]

\[
\sum_{k=-\infty}^{\infty} e^{(kc-x, \theta)} \Phi^{[1,2]}_{0,kc-x} = e^{-(x, \theta)} \Phi_c^{x,*}(e^{(c, \theta)}),
\]

where we use the identity \( \Phi^{[1,2]}_{0,kc-x} = \Phi^{[1,2]}_{x,kc} \). Since the closure of \( \Gamma^{[1,2]} \) is a convex set, \( \Phi_c^{x,*}(z) \) is, therefore, absolutely convergent in \( \Delta_{e_c^{\min},e_c^{\max}} \). As a result, \( \Phi_c^{x,*}(z) \) is analytic in \( \Delta_{e_c^{\min},e_c^{\max}} \) since each entry of \( \Phi_c^{x,*}(z) \) is represented as a Laurent series of \( z \) (see, for example, Section II.1 of Markushevich [17]).

By compensation equation (2.8), \( \varphi^c(z) \) is given in terms of \( \Phi_c^{x,*}(z) \) by

\[
\varphi^c(z) = \varphi_0^c(z) + \varphi_1^c(z) + \varphi_2^c(z),
\]

where

\[
\varphi_0^c(z) = \sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(0,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(i_1,i_2),*}(z),
\]

\[
\varphi_1^c(z) = \sum_{k=1}^{\infty} \sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(k,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(k+i_1,i_2),*}(z),
\]

\[\sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(0,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(i_1,i_2),*}(z),
\]

\[\sum_{k=1}^{\infty} \sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(k,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(k+i_1,i_2),*}(z),
\]

\[\sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(0,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(i_1,i_2),*}(z),\]

\[\sum_{k=1}^{\infty} \sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(k,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(k+i_1,i_2),*}(z),\]

\[\sum_{i_1, i_2 \in \{-1, 0, 1\}} v_{(0,0)}(A_{i_1,i_2}^{[1]} - A_{i_1,i_2}^{[1,2]}) \Phi_c^{(i_1,i_2),*}(z),\]
\[ \{ \hat{Y}_n \} = \{( \hat{X}_n, \hat{J}_n \) \}

\[ \mathbf{c} = (1, 1) \]

\[ \varphi^c_2(z) = \sum_{k=1}^{\infty} \sum_{i_1, i_2 \in [-1,0,1]} \mathbf{v}_{(0,k)} \left( (A_{i_1,i_2}^{[2]} - A_{i_1,i_2}^{[1,2]}) \Phi^c_{(i_1,k+i_2),*} \right) (z). \] (3.6)

For \( i \in \{1, 2, 3\} \), let \( \xi_{c,i} \) be the supremum point of the convergence domain of \( \varphi^c_i(e^\theta) \), defined as

\[ \xi_{c,i} = \sup\{ \theta \in \mathbb{R}; \varphi^c_i(e^\theta) \text{ is absolutely convergent, elementwise} \}. \]

By Proposition 3.1, we immediately obtain the following.

**Proposition 3.2** \( \varphi^c_0(z) \) is absolutely convergent and elementwise analytic in \( \Delta_{e^{-c_{\min}}, e^{-c_{\max}}} \), and we have \( \xi_{c,0} \geq \theta^\max_c \).

We analyze \( \varphi^c_1(z) \) and \( \varphi^c_2(z) \) in the following subsection.

3.2 In the case of direction vector \( \mathbf{c} = (1, 1) \)

In overall this subsection, we assume \( \mathbf{c} = (1, 1) \).

First, focusing on \( \varphi^c_2(z) \), we construct a new skip-free MA-process from \( \{ \hat{Y}_n^{[1,2]} \} \) and apply the matrix analytic method to the MA-process. The new MA-process is denoted by \( \{ \hat{Y}_n \} = \{( \hat{X}_n, \hat{J}_n \) \)} = \{( \hat{X}_{1,n}, \hat{X}_{2,n}, (\hat{R}_n, \hat{J}_n) \)}, where \( \hat{X}_{1,n} = X_{1,n}^{[1,2]}, \hat{X}_{2,n} \) and \( \hat{R}_n \) are the quotient and remainder of \( X_{2,n}^{[1,2]} - X_{1,n}^{[1,2]} \) divided by 2, respectively, and \( \hat{J}_n = J_n^{[1,2]} \) (see Fig. 4).

The state space of \( \{ \hat{Y}_n \} \) is \( \mathbb{Z}^2 \times \{ 0, 1 \} \times S_0 \), and \( \hat{X}_n \) and \( \hat{J}_n \) are the additive part (level state) and background state (phase state), respectively. The additive part of \( \{ \hat{Y}_n \} \) is skip free, and this is a reason why we consider this new MA-process. From the definition, if \( \hat{X}_n = (x_1, x_2) \) and \( \hat{R}_n = r \) in the new MA-process, it follows that \( X_{1,n}^{[1,2]} = x_1, X_{2,n}^{[1,2]} = x_1 + 2x_2 + r \) in the original MA-process. Hence, \( \hat{Y}_n = (k, 0, 0, j) \) means \( Y_n^{[1,2]} = (k, k, j) \). Here we note that \( \{ \hat{Y}_n \} \) is slightly different from the \( (1, 2) \)-block state process derived from \( \{ Y_n^{[1,2]} \} \); see Sect. 2.4. In the latter process,
the state \((x_1, x_2, r, j)\) corresponds to the state \((x_1, 2x_2 + r, j)\) of the original MA-process. Denote by \(\hat{P} = (\hat{P}_{x',x}; x, x' \in \mathbb{Z}^2)\) the transition probability matrix of \(\{\hat{Y}_n\}\), which is given as

\[
\hat{P}_{x',x} = \begin{cases} 
\hat{A}_{x',x}^{[1,2]} & \text{if } x' - x \in \{-1, 0, 1\}^2, \\
O & \text{otherwise,}
\end{cases}
\]

where

\[
\begin{align*}
\hat{A}_{-1,1}^{[1,2]} &= \left( \begin{array}{cc} \hat{A}_{-1,1}^{[1,2]} O \\
A_{-1,0}^{-1} & A_{-1,2}^{[1,2]} \end{array} \right), \\
\hat{A}_{0,1}^{[1,2]} &= \left( \begin{array}{cc} O & O \\
A_{0,1}^{[1,2]} & O \end{array} \right), \\
\hat{A}_{1,1}^{[1,2]} &= \left( \begin{array}{cc} O & O \\
A_{1,1}^{[1,2]} & O \end{array} \right), \\
\hat{A}_{-1,0}^{[1,2]} &= \left( \begin{array}{cc} \hat{A}_{-1,0}^{[1,2]} O \\
O & \hat{A}_{-1,-1}^{[1,2]} \end{array} \right), \\
\hat{A}_{0,0}^{[1,2]} &= \left( \begin{array}{cc} O & A_{0,0}^{[1,2]} \\
\hat{A}_{0,-1}^{[1,2]} & \hat{A}_{0,0}^{[1,2]} \end{array} \right), \\
\hat{A}_{1,0}^{[1,2]} &= \left( \begin{array}{cc} O & \hat{A}_{1,0}^{[1,2]} \\
\hat{A}_{1,-1}^{[1,2]} & \hat{A}_{1,0}^{[1,2]} \end{array} \right).
\end{align*}
\]

Denote by \(\hat{\Phi} = (\hat{\Phi}_{x',x}; x, x' \in \mathbb{Z}^2)\) the fundamental matrix of \(\hat{P}\), i.e., \(\hat{\Phi} = \sum_{n=0}^{\infty} (\hat{P})^n\), and for \(x = (x_1, x_2) \in \mathbb{Z}^2\), define a matrix generating function \(\hat{\Phi}_{x,*}(z)\) as

\[
\hat{\Phi}_{x,*}(z) = \sum_{k=-\infty}^{\infty} z^k \hat{\Phi}_{x,(k,0)} = \begin{pmatrix} \Phi_{c(x_1,x_1+2x_2),*}(z) & \Phi_{c(x_1,x_1+2x_2-1),*}(z) \\
\Phi_{c(x_1,x_1+2x_2+1),*}(z) & \Phi_{c(x_1,x_1+2x_2),*}(z) \end{pmatrix}. \tag{3.7}
\]

By Proposition 3.1, for every \(x \in \mathbb{Z}^2\), \(\hat{\Phi}_{x,*}(z)\) is entry-wise analytic in the open annual domain \(\Delta_{e^{\theta_{\min}}, e^{\theta_{\max}}}^\circ\). Define blocks \(\hat{A}_{i_1,i_2}^{[2]}, i_1, i_2 \in \{-1, 0, 1\}\), as \(\hat{A}_{-1,1}^{[2]} = \hat{A}_{1,0}^{[2]} = \hat{A}_{-1,-1}^{[2]} = O\) and

\[
\begin{align*}
\hat{A}_{0,1}^{[2]} &= \begin{pmatrix} O & \hat{A}_{0,1}^{[2]} \\
\hat{A}_{0,0}^{[2]} & O \end{pmatrix}, \\
\hat{A}_{0,0}^{[2]} &= \begin{pmatrix} \hat{A}_{0,0}^{[2]} \hat{A}_{0,1}^{[2]} \\
\hat{A}_{0,0}^{[2]} & O \end{pmatrix}, \\
\hat{A}_{1,0}^{[2]} &= \begin{pmatrix} O & \hat{A}_{1,0}^{[2]} \\
\hat{A}_{1,1}^{[2]} & O \end{pmatrix}, \\
\hat{A}_{1,1}^{[2]} &= \begin{pmatrix} O & \hat{A}_{1,1}^{[2]} \\
\hat{A}_{1,0}^{[2]} & O \end{pmatrix}.
\end{align*}
\]

For \(i_1, i_2 \in \{-1, 0, 1\}\), define the following matrix generating functions:

\[
\begin{align*}
\hat{A}_{*,i_2}^{[1,2]}(z) &= \sum_{i \in \{-1,0,1\}} z^i \hat{A}_{i,i_2}^{[1,2]}, \\
\hat{A}_{i_1,*}^{[1,2]}(z) &= \sum_{i \in \{-1,0,1\}} z^i \hat{A}_{i_1,i}^{[1,2]}, \\
\hat{A}_{*,i_2}^{[2]}(z) &= \sum_{i \in \{0,1\}} z^i \hat{A}_{i,i_2}^{[2]}, \\
\hat{A}_{i_1,*}^{[2]}(z) &= \sum_{i \in \{-1,0,1\}} z^i \hat{A}_{i_1,i}^{[2]}.
\end{align*}
\]
Define a vector generating function \( \hat{\phi}_2^c(z) \) as
\[
\hat{\phi}_2^c(z) = (\hat{\phi}_{2,1}^c(z), \hat{\phi}_{2,2}^c(z)) = \sum_{l=-\infty}^{\infty} z^l \sum_{k=1}^{\infty} \sum_{i_1,i_2 \in \{-1,0,1\}} \hat{v}(0,k)(\hat{A}_{i_1,i_2}^{[2]} - \hat{A}_{i_1,i_2}^{[1,2]})\hat{\Phi}_{(i_1,k+i_2),l}(0,0),
\]
(3.8)
where, for \( x = (x_1, x_2) \in \mathbb{Z}^2 \),
\[
\hat{v}(x_1,x_1+2x_2) = (\hat{v}_{(x_1,x_1+2x_2)} \hat{v}_{(x_1,x_1+2x_2+1)})
\]
and hence, for \( k \geq 0 \), \( \hat{v}(0,k) = (\nu(0,2k) \nu(0,2k+1)) \). Since
\[
\hat{\phi}_{2,1}^c(z) = \phi_2^c(z) - \sum_{i_1,i_2 \in \{-1,0,1\}} \nu(0,1)(\hat{A}_{i_1,i_2}^{[2]} - \hat{A}_{i_1,i_2}^{[1,2]})\phi_{(i_1,1+i_2),z}(z),
\]
(3.9)
\[
\hat{\phi}_{2,2}^c(z) = \sum_{k=2}^{\infty} \sum_{i_1,i_2 \in \{-1,0,1\}} \nu(0,k)(\hat{A}_{i_1,i_2}^{[2]} - \hat{A}_{i_1,i_2}^{[1,2]})\phi_{(i_1,k+i_2-1),z}(z),
\]
(3.10)
we analyze \( \hat{\phi}_2^c(z) \) instead of \( \phi_2^c(z) \). Let \( \hat{\xi}_{c,2} \) be the supremum point of the convergence domain of \( \hat{\phi}_2^c(e^\theta) \), defined as
\[
\hat{\xi}_{c,2} = \sup(\theta \in \mathbb{R}; \hat{\phi}_2^c(e^\theta) \text{ is absolutely convergent, elementwise}).
\]
By (3.9), we have \( \xi_{c,2} \geq \hat{\xi}_{c,2} \).

Next, we obtain a tractable representation for \( \hat{\phi}_2^c(z) \). Since \( \{\hat{Y}_n\} \) is space-homogeneous with respect to the additive part, we have, for every \( x = (x_1, x_2) \in \mathbb{Z}^2 \),
\[
\hat{\Phi}_{(x_1,x_2),z}(z) = z^{x_1} \hat{\Phi}_{(0,x_2),z}(z).
\]
(3.11)
Define a stopping time \( \tau_0 \) as
\[
\tau_0 = \inf\{n \geq 1; \hat{X}_{2,n} = 0\}.
\]
This \( \tau_0 \) is the first hitting time to the subspace \( \hat{S}_0 = \{(x_1, x_2, r, j) \in \mathbb{Z}^2 \times \{0, 1\} \times \{0\}; x_2 = 0\} \), which corresponds to the subspace \( \{(x_1, x_2, j) \in \mathbb{Z}^2 \times \{0\}; x_2 = x_1 \text{ or } x_2 = x_1 + 1\} \) of the original MA-process (see Fig. 4). For \( k \geq 1, x_1, x'_1 \in \mathbb{Z} \) and \( (r, j), (r', j') \in \{0, 1\} \times \{0\} \), let \( g_{(r,j),(r',j')}^{(k)}(x_1, x'_1) \) be the probability that the MA-process \( \{\hat{Y}_n\} \) starting from \( (x_1, k, r, j) \) visits a state in \( \hat{S}_0 \) for the first time and the state is \( (x'_1, 0, r', j') \), i.e.,
\[
g_{(r,j),(r',j')}^{(k)}(x_1, x'_1) = \mathbb{P}(\hat{Y}_{\tau_0} = (x'_1, 0, r', j'), \tau_0 < \infty | \hat{Y}_0 = (x_1, k, r, j)).
\]
We denote the matrix of them by $\hat{G}^{(k)}_{x_1,x_1'}$, i.e., $\hat{G}^{(k)}_{x_1,x_1'} = (\hat{g}^{(k)}_{(r,j),(r',j')} (x_1, x_1'); (r, j), (r', j') \in \{0, 1 \times S_0\})$. When $k = 1$, we omit the superscript $(k)$ such as $\hat{G}^{(1)}_{x_1,x_1'} = \hat{G}_{x_1,x_1'}$. Since $\{\hat{Y}_n\}$ is space-homogeneous, we have $\hat{G}^{(k)}_{x_1,x_1'} = \hat{G}^{(k)}_{0,x_1'-x_1}$. By the strong Markov property, for $k \geq 1$ and $x_1 \in \mathbb{Z}$, $\hat{\Phi}_{(0,k),(x_1,0)}$ is represented in terms of $\hat{G}^{(k)}_{0,x_1'}$ as

$$\hat{\Phi}_{(0,k),(x_1,0)} = \sum_{x_1'=-\infty}^{\infty} \hat{G}^{(k)}_{0,x_1'} \hat{\Phi}_{(x',0),(x_1)}, \quad (3.12)$$

and this leads us to

$$\hat{\Phi}_{(0,k),*}(z) = \hat{G}^{(k)}_{0,*}(z) \hat{\Phi}_{(0,0),*}(z), \quad (3.13)$$

where

$$\hat{G}^{(k)}_{0,*}(z) = \sum_{x_1'=-\infty}^{\infty} z^{x_1'} \hat{G}^{(k)}_{0,x_1'}, \quad (3.14)$$

and we use (3.11). Since $\{\hat{Y}_n\}$ is skip free and space-homogeneous, we have by the strong Markov property that

$$\hat{G}^{(k)}_{0,*}(z) = \hat{G}_{0,*}(z)^k. \quad (3.15)$$

As a result, by (3.8), (3.11), (3.13) and (3.15), we obtain

$$\hat{\phi}^\epsilon_2(z) = \sum_{k=1}^{\infty} \sum_{i_2 \in \{-1,0,1\}} \hat{\nu}_{(0,k)}(A^{[2]}_{*,i_2}(z) - A^{[1,2]}_{*,i_2}(z)) \hat{G}^{(k)}_{0,*}(z)^{k+i_2} \hat{\Phi}_{(0,0),*}(z). \quad (3.16)$$

We make a preparation for analyzing $\hat{\phi}^\epsilon_2(z)$ through (3.16). Define a matrix generating function of transition probability blocks as

$$A^{[1,2]}_{*,*}(z_1, z_2) = \sum_{i_1,i_2 \in \{-1,0,1\}} z_1^{i_1}z_2^{i_2}A^{[1,2]}_{i_1,i_2}.$$ 

Define a domain $\hat{\Gamma}^{[1,2]}$ as

$$\hat{\Gamma}^{[1,2]} = \{ (\theta_1, \theta_2) \in \mathbb{R}^2; \text{spr}(A^{[1,2]}_{*,*}(e^{\theta_1}, e^{\theta_2})) < 1 \},$$

whose closure is a convex set, and define the extreme values $\hat{\theta}_1^{min}$ and $\hat{\theta}_1^{max}$ of $\hat{\Gamma}^{[1,2]}$ as

$$\hat{\theta}_1^{min} = \inf\{\theta_1 \in \mathbb{R}; (\theta_1, \theta_2) \in \hat{\Gamma}^{[1,2]}\}, \quad \hat{\theta}_1^{max} = \sup\{\theta_1 \in \mathbb{R}; (\theta_1, \theta_2) \in \hat{\Gamma}^{[1,2]}\}.$$ 

For $\theta_1 \in [\hat{\theta}_1^{min}, \hat{\theta}_1^{max}]$, let $\hat{\eta}_2^1(\theta_1)$ and $\hat{\eta}_2^2(\theta_1)$ be the two real roots to the following equation:

$$\text{spr}(A^{[1,2]}_{*,*}(e^{\theta_1}, e^{\theta_2})) = 1,$$
counting multiplicity, where \( \hat{\eta}_2^*(\theta_1) \leq \hat{\eta}_2^*(\theta_1) \) (see Fig. 5). The matrix generating function \( \hat{G}_{0,*}(z) \) corresponds to a so-called G-matrix in the matrix analytic method of the queueing theory (see, for example, Neuts [22]). For \( \theta \in [\hat{\theta}_1^{\min}, \hat{\theta}_1^{\max}] \), consider the following matrix quadratic equation:

\[
\hat{A}_{*,1}^{[1,2]}(e^{\theta}) + \hat{A}_{*,0}^{[1,2]}(e^{\theta})X + \hat{A}_{*,1}^{[1,2]}(e^{\theta})X^2 = X, \tag{3.17}
\]

then \( \hat{G}_{0,*}(e^{\theta}) \) is given by the minimum nonnegative solution to the equation, and we have by Lemma 2.5 of Ozawa [31] that

\[
\text{spr}(\hat{G}_{0,*}(e^{\theta})) = e^{\hat{\eta}_2^*(\theta)}. \tag{3.18}
\]

Let \( \alpha_1(z) \) be the maximum eigenvalue of \( \hat{G}_{0,*}(z) \) and \( \alpha_i(z) \), \( i = 2, 3, \ldots, 2s_0 \), be other eigenvalues, counting multiplicity. We have, for \( \theta \in [\hat{\theta}_1^{\min}, \hat{\theta}_1^{\max}] \), \( \alpha_1(e^{\theta}) = \text{spr}(\hat{G}_{0,*}(e^{\theta})) = e^{\hat{\eta}_2^*(\theta)} \). \( \hat{G}_{0,*}(z) \) satisfies the following properties.

**Proposition 3.3**

(i) \( \hat{G}_{0,*}(z) \) is absolutely convergent and entry-wise analytic in the open annual domain \( \Delta = e^{\hat{\theta}_1^{\min}}e^{\hat{\theta}_1^{\max}} \).

(ii) For every \( z \in \Delta = e^{\hat{\theta}_1^{\min}}e^{\hat{\theta}_1^{\max}} \), \( \text{spr}(\hat{G}_{0,*}(z)) \leq \text{spr}(\hat{G}_{0,*}(|z|)) = e^{\hat{\eta}_2^*(\log|z|)} \). Furthermore, if \( z \neq |z| \), then \( \text{spr}(\hat{G}_{0,*}(z)) < \text{spr}(\hat{G}_{0,*}(|z|)) = e^{\hat{\eta}_2^*(\log|z|)} \).

(iii) For every \( \theta \in [\hat{\theta}_1^{\min}, \hat{\theta}_1^{\max}] \), \( \alpha_1(e^{\theta}) > |\alpha_i(e^{\theta})| \) for every \( i \in \{2, 3, \ldots, 2s_0\} \).

**Proof** Since \( \hat{G}_{0,*}(z) \) is given by Laurent series (3.14) and it is absolutely convergent in the closure of \( \Delta = e^{\hat{\theta}_1^{\min}}e^{\hat{\theta}_1^{\max}} \), we immediately obtain part (i) of the proposition (see, for example, Section II.1 of Markushevich [17]). By Lemma 4.1 of Ozawa and Kobayashi [27], for every \( z \in \Delta = e^{\hat{\theta}_1^{\min}}e^{\hat{\theta}_1^{\max}} \), \( \text{spr}(\hat{G}_{0,*}(z)) \leq \text{spr}(|\hat{G}_{0,*}(z)|) \leq \text{spr}(\hat{G}_{0,*}(|z|)) \), and we obtain the first half of part (ii) of the proposition. The second half is obtained by part (i) of Lemma 4.3 of Ref. [27]. Since, under Assumption 3.1, the lossy Markov chain derived from \( \{\hat{Y}_n\} \) by restricting the state space to \( \mathbb{Z} \times \mathbb{Z}_+ \times \{0, 1\} \times S_0 \) is irreducible (see Remark 3.1), every column of \( \hat{G}_{0,*}(e^{\theta}) \) is positive or zero (see Appendix C of Ozawa [31]; a result similar to that holding for rate matrices also holds for G-matrices). Hence,
nonnegative matrix $\hat{G}_{0,+}(e^\theta)$ has just one primitive class (irreducible and aperiodic class), and this implies part (iii) of the proposition.

We get back to (3.16) and apply results in Ref. [27]. For $i \in \{1, 2\}$, recall the definition of $\theta_i^*$ and define $\theta_i^\dagger$:

$$\theta_i^* = \sup\{\theta_i \in \mathbb{R}; (\theta_1, \theta_2) \in \Gamma^{(i)}\}, \quad \theta_i^\dagger = \sup\{\theta_i; (\theta_1, \theta_2) \in \Gamma^{(3-i)} \cap \Gamma^{(1,2)}\},$$

(3.19)

then we have $\xi_{(1,0)} = \min\{\theta_1^*, \theta_1^\dagger\}$ and $\xi_{(0,1)} = \min\{\theta_2^*, \theta_2^\dagger\}$. For another equivalent definition of $\theta_i^*$ and $\theta_i^\dagger$ and for the properties of $\xi_{(1,0)}$ and $\xi_{(0,1)}$, see Appendix A.

Since $\hat{v}_{(0,k)} = (v_{(0,2k)} v_{(0,2k+1)})$ for $k \geq 0$, the radius of convergence of the power series of the sequence $\{\hat{v}_{(0,k)}\}_{k \geq 0}$ is given by $e^{2\xi_{(0,1)}}$. Taking this point into account, we define $\hat{\theta}_1^\dagger$ and $\hat{\theta}_1^{\dagger,\lessgtr}$ as

$$\hat{\theta}_1^\dagger = \max\{\theta \in [\hat{\theta}_1^{\min}, \hat{\theta}_1^{\max}]; \hat{\theta}_2^\dagger(\theta) \leq 2\theta_2^*\}, \quad \hat{\theta}_1^{\dagger,\lessgtr} = \max\{\theta \in [\hat{\theta}_1^{\min}, \hat{\theta}_1^{\max}]; \hat{\theta}_2^\dagger(\theta) \leq 2\xi_{(0,1)}\}.$$

Since $\xi_{(0,1)} = \min\{\theta_2^*, \theta_2^\dagger\}$, we have $\hat{\theta}_1^\dagger \geq \hat{\theta}_1^{\dagger,\lessgtr}$. The following is a key proposition for analyzing $\hat{\varphi}_2^\xi(z)$.

**Proposition 3.4** (i) We always have $\hat{\varphi}_{e,2} \geq \hat{\theta}_1^{\dagger,\lessgtr}$.

(ii) If $\hat{\theta}_1^\dagger < \hat{\theta}_1^{\max}$ and $\theta_2^* < \theta_2^\dagger$, then $\hat{\varphi}_2^\xi(z)$ is elementwise analytic in $\Delta_{1,e^\theta_1} \cup (\partial \Delta_{e^\theta_1} \setminus \{e^\theta_1\})$.

(iii) If $\hat{\theta}_1^\dagger < \hat{\theta}_1^{\max}$ and $\theta_2^* < \theta_2^\dagger$, then $\hat{\varphi}_{e,2} = \hat{\theta}_1^{\dagger}$ and, for some positive vector $\hat{g}_2^\xi$,

$$\lim_{\theta \uparrow \hat{\theta}_1^{\dagger}} (e^{\theta_1} - e^\theta) \hat{\varphi}_2^\xi(e^\theta) = \hat{g}_2^\xi.$$  

(3.20)

Before proving the proposition, we give one more proposition. Let $(1,2)Y_{(0,k)}, k \in \mathbb{N}$, be the stationary probability vectors in the $(1,2)$-block state process $\{(1,2)Y_n\}$ derived from the original 2d-QBD process; see Sect. 2.4 and Appendix A. Since $\{(1,2)Y_n\}$ is a 2d-QBD process, we can apply results in Ref. [27]. Define a vector generating function $\hat{\varphi}_2(z)$ as

$$\hat{\varphi}_2(z) = \sum_{k=1}^{\infty} z^k \hat{v}_{(0,k)},$$

which is identical to $(1,2)\varphi_2(z) = \sum_{k=1}^{\infty} z^k (1,2)v_{(0,k)}$ since we have $\hat{v}_{(0,k)} = (v_{(0,2k)} v_{(0,2k+1)}) = (1,2)v_{(0,k)}$ for every $k \in \mathbb{N}$. If $(1,2)\theta_2^* = 2\theta_2^* < (1,2)\theta_2^\dagger = 2\theta_2^\dagger$ (for the definitions of $(1,2)\theta_2^*$ and $(1,2)\theta_2^\dagger$, see Appendix A), then $(1,2)Y_n$ is classified into Type I ($\psi_2(\hat{z}_2^*) > 1$) or Type II in the notation of Ref. [27], where inequality $\psi_2(\hat{z}_2^*) > 1$ corresponds to $(1,2)\theta_2^* < (1,2)\theta_2^{\max}$. In our case, inequality $\theta_2^* < \theta_2^\dagger$.
implies this condition since $\theta_2^{\dagger} \leq \theta_2^{max}$ and $(1,2)\theta_2^{max} = 2\theta_2^{max}$. Therefore, if $\theta_2^* < \theta_2^{\dagger}$, we see by Corollary 5.1 of Ref. [27] that $z = e^{(1,2)\theta_2^*} = e^{2\theta_2^*}$ is a pole of $(1,2)\varphi_2(z)$, and the same property also holds for $\hat{\varphi}_2(z)$. Define $\hat{U}_2(z)$ as

$$\hat{U}_2(z) = \hat{A}_{0,*}(z) + \hat{A}_{1,*}(z)\hat{G}_{*,0}(z),$$

where $\hat{G}_{*,0}(z)$ is the G-matrix generated from the triplet $\{\hat{A}_{-1,*}(z), \hat{A}_{0,*}(z), \hat{A}_{1,*}(z)\}$ (see Subsection 4.1 of Ref. [27]) and satisfies the following matrix quadratic equation:

$$\hat{A}_{-1,*}(z) + \hat{A}_{0,*}(z)X + \hat{A}_{1,*}(z)X^2 = X.$$  \hspace{1cm} (3.21)

By the definition, $\hat{U}_2(z)$ is identical to $\hat{(1,2)}U_2(z)$ of the $(1, 2)$-block state process (for the definition of $\hat{(1,2)}U_2(z)$, see Appendix A). Let $\hat{u}_U(z)$ and $\hat{v}_U(z)$ be the left and right eigenvectors of $\hat{U}_2(z)$ with respect to the maximum eigenvalue of $\hat{U}_2(z)$, satisfying $\hat{u}_U(z)\hat{v}_U(z) = 1$. By Corollary 5.1 of Ref. [27], considering correspondence between $\hat{\varphi}_2(z)$ and $(1,2)\varphi_2(z)$, we immediately obtain the following.

**Proposition 3.5** If $\theta_2^* < \theta_2^{\dagger}$, then $\hat{\varphi}_2(\hat{\theta}_2^{\dagger}) = 2\theta_2^*$ and for some positive constant $\hat{g}_2$,

$$\lim_{\theta \to 2\theta_2^*} (e^{2\theta_2^*} - e^{\theta})\hat{\varphi}_2(e^{\theta}) = \hat{g}_2\hat{u}_U(e^{2\theta_2^*}),$$  \hspace{1cm} (3.22)

where $\hat{u}_U(e^{2\theta_2^*})$ is positive.

Note that, under Assumption 3.1, the modulus of every eigenvalue of $\hat{G}_{*,0}(e^{2\theta_2^*})$ except for the maximum one is less than $\text{spr}(\hat{G}_{*,0}(e^{2\theta_2^*}))$ (see Proposition 3.3), and it is not necessary for using Corollary 5.1 of Ref. [27] to assume all the eigenvalues of $\hat{G}_{*,0}(e^{2\theta_2^*})$ are distinct (i.e., Assumption 2.5 of Ref. [27]).

**Proof of Proposition 3.4** Temporary, define $D(z, w)$ as

$$D(z, w) = \hat{A}_{2,*,-1}(z) + \hat{A}_{2,*0}(z)w + \hat{A}_{2,*1}(z)w^2 - Iw,$$

where $w$ is a matrix or scalar. By (3.16) and (3.17), $\hat{\varphi}_2(z)$ is represented as

$$\hat{\varphi}_2(z) = a(z, \hat{G}_{*,0}(z))\hat{\varphi}(0,*,0)(z),$$

where

$$a(z, w) = \sum_{k=1}^{\infty} \hat{\varphi}_{(0,k)}D(z, \hat{G}_{*,0}(z))w^{k-1}.$$  \hspace{1cm}

Later, we will prove that $\hat{\theta}_1^{min} = \theta_1^{min}$ and $\hat{\theta}_1^{max} = \theta_1^{max}$ (see (3.32)). Hence, by Proposition 3.1 and expression (3.7), $\hat{\varphi}(0,0,*,0)(z)$ is absolutely convergent in $\Delta_{e^{\hat{\varphi}_1^{min}},e^{\hat{\varphi}_1^{max}}}$. \hspace{1cm}

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We, therefore, focus on \( a(z, \hat{G}_{0,*}(z)) \). Let \( \hat{G}_{0,*}(z) = V(z) J(z) V(z)^{-1} \) be the Jordan decomposition of \( \hat{G}_{0,*}(z) \). Since \( \hat{G}_{0,*}(z)^{k-1} = V(z) J(z)^{k-1} V(z)^{-1} \), we have

\[
a(z, \hat{G}_{0,*}(z))^\top = (V(z)^{-1})^\top \sum_{k=1}^{\infty} \left( \hat{\nu}_{(0,k)} \otimes \left( J(z)^{\top} \right)^{k-1} \right) \text{vec}(D(z, \hat{G}_{0,*}(z)) V(z)^{\top}),
\]

(3.23)

where \( \otimes \) is the Kronecker product and we use the identity \( \text{vec}(ABC) = (C^\top \otimes A) \text{vec}(B) \) for matrices \( A, B \) and \( C \) (for the identity, see Horn and Johnson [11]). Define a real value \( \theta_1' \) as

\[
\theta_1' = \arg \min_{\theta \in [\hat{\theta}_1^{\min}, \hat{\theta}_1^{\max}]} \hat{h}_2^z(\theta),
\]

(3.24)

then \( \hat{h}_2^z(\theta) \) is strictly increasing in \( (\theta_1', \hat{\theta}_1^{\max}) \). Hence, by part (ii) of Proposition 3.3, for every \( z \in \Delta_{e_1', e_1^\top} \), \( \text{spr}(\hat{G}_{0,*}(z)) \leq \text{e}^{\hat{h}_2^z(\log |z|)} \leq \text{e}^{\hat{h}_2^z(\hat{\theta}_1^{\max})} \). Since \( \text{e}^{2\hat{\theta}_1(0,1,i)} \) is the radius of convergence of the power series of the sequence \( \{\hat{\nu}_{(0,k)}\}_{k \geq 0} \), we see that, for every \( i \in \{1, 2, ..., 2s_0\} \), each entry of \( \sum_{k=1}^{\infty} |\hat{\nu}_{(0,k)}| \left( J(z)^{\top} \right)^{k-1} \) is absolutely convergent in \( z \in \Delta_{e_1', e_1^\top} \). As a result, \( \hat{\Phi}_2^z(z) \) as well as \( a(z) \) is absolutely convergent in \( \Delta_{e_1', e_1^\top} \) and we obtain \( \hat{\epsilon}_{e,2} \geq \hat{\theta}_1^{\top,0} \). This completes the proof of part (i) of the proposition.

Next, supposing \( \hat{\theta}_1^\top < \hat{\theta}_1^{\max} \), we consider the case where \( \theta_2^* < \theta_1^\top \). In this case, we have \( \hat{h}_2^z(\hat{\theta}_1^\top) = 2\hat{\xi}(0,1) = 2\hat{\theta}_2^* < 2\theta_2^{\max} \) and \( \hat{\theta}_1^\top = \hat{\theta}_1^{\max} \) since \( \hat{\xi}(0,1) = \min\{\theta_2^*, \theta_2^\top\} \) and \( \theta_2^\top \leq \theta_2^{\max} \). We prove part (ii) of the proposition in a manner similar to that used in the proof of Proposition 5.1 of Ref. [27], which is given in Ozawa and Kobayashi [28]. Let \( X = (x_{kl,i}) \) be an \( 2s_0 \times 2s_0 \) complex matrix. For \( z \in \Delta_{e_1', e_1^\top} \), if \( |w| < \text{e}^{\hat{h}_2^z(\hat{\theta}_1^\top)} \), \( a(z,w) \) is absolutely convergent, and by Lemma 3.2 of Ref. [27], we see that if \( \text{spr}(X) < \text{e}^{\hat{h}_2^z(\hat{\theta}_1^\top)} \), each element of \( a(z, X) \) is absolutely convergent. This implies that each element of \( a(z, X) \) is analytic as a complex function of \( 4s_0^2 + 1 \) variables in \( \{(z, x_{kl}: k = 1, 2, ..., 2s_0) \in \mathbb{C}^{4s_0^2+1}; e^{\hat{h}_2^{\min}} < |z| < e^{\hat{h}_2^{\max}}, \text{spr}(X) < e^{\hat{h}_2^z(\hat{\theta}_1^\top)}\} \). By parts (i) and (ii) of Proposition 3.3, for any \( z_0 \in \Delta_{e_1', e_1^\top} \cup (\partial \Delta_{e_1^\top} \setminus \{e_{1,1}^\top\}) \), \( \hat{G}_{0,*}(z) \) is entry-wise analytic at \( z = z_0 \) and \( \text{spr}(\hat{G}_{0,*}(z_0)) < \text{e}^{\hat{h}_2^z(\hat{\theta}_1^\top)} \), where \( \theta_1' \) is given by (3.24). Hence, the composite function \( a(z, \hat{G}_{0,*}(z)) \) is elementwise analytic in \( z \in \Delta_{e_1', e_1^\top} \cup (\partial \Delta_{e_1^\top} \setminus \{e_{1,1}^\top\}) \). Under Assumption 2.1, since we have \( \hat{h}_2^z(\hat{\theta}_1^\top) = 2\hat{\theta}_2^* > 0, \hat{h}_2^z(\theta_1') \leq 0 \) and \( \hat{h}_2^z(0) \leq 0 \), if \( \theta_1' > 0 \) then \( \hat{h}_2^z(\theta) \leq 0 < \hat{h}_2^z(\theta_1') \) for every \( \theta \in [0, \theta_1'] \). Hence, we can replace \( e^{\theta_1'} \) with \( e^0 = 1 \) and see that \( a(z, \hat{G}_{0,*}(z)) \) is elementwise analytic in \( z \in \Delta_{1,e_1^\top} \cup (\partial \Delta_{e_1^\top} \setminus \{e_{1,1}^\top\}) \). By Proposition 3.1 and expression (3.7), \( \hat{\Phi}_{(0,0),*}(z) \) is also entry-wise analytic in the same domain. This completes the proof of part (ii) of the proposition.

Finally, supposing \( \hat{\theta}_1^\top < \hat{\theta}_1^{\max} \), we consider the case where \( \theta_2^* < \theta_1^\top \) again. By part (iii) of Proposition 3.3, \( \text{spr}(\hat{G}_{0,*}(e_{1,1}^\top)) = e^{\hat{h}_2^z(\hat{\theta}_1^\top)} = e^{2\theta_2^*} \) is a simple eigenvalue of
\( \hat{G}_{0,*}(e^{\hat{\theta}_1^\dagger}) \), and the modulus of every eigenvalue of \( \hat{G}_{0,*}(e^{\hat{\theta}_1^\dagger}) \) except for \( e^{2\theta_2^*} \) is less than \( e^{2\theta_2^*} \). Hence, we have, for every \( i \in \{1, 2, \ldots, 2s_0\} \),

\[
\lim_{\theta \uparrow \hat{\theta}_1^\dagger} (e^{\hat{\theta}_1^\dagger} - e^\theta) \sum_{k=1}^{\infty} [\hat{\nu}(0,k)]_i (J(e^\theta)^\top)^{k-1} = \lim_{\theta \uparrow \hat{\theta}_1^\dagger} (e^{\hat{\theta}_1^\dagger} - e^\theta)[\hat{\varphi}_2(e^{\hat{\theta}_2^*(\theta)})]_i e^{-\hat{\theta}_2^*(\theta)} \text{diag}(1 0 \cdots 0),
\]

where we assume \([J(e^\theta)]_{1,1} = \alpha_1(e^\theta) = e^{\hat{\theta}_2^*(\theta)} \) without loss of generality. By (3.23), this leads us to

\[
\lim_{\theta \uparrow \hat{\theta}_1^\dagger} (e^{\hat{\theta}_1^\dagger} - e^\theta) a(e^\theta, \hat{G}_{0,*}(e^\theta)) = \lim_{\theta \uparrow \hat{\theta}_1^\dagger} (e^{\hat{\theta}_1^\dagger} - e^\theta) \hat{\varphi}_2(e^{\hat{\theta}_2^*(\theta)}) e^{-\hat{\theta}_2^*(\theta)} D(e^{\hat{\theta}_1^\dagger}, e^{\hat{\theta}_2^*(\theta)}) \hat{\nu}_G(e^{\hat{\theta}_1^\dagger}) \hat{\nu}_G(e^{\hat{\theta}_1^\dagger}),
\]

(3.25)

where \( \hat{\nu}_G(e^\theta) \) and \( \hat{\nu}_G(e^\theta) \) are the left and right eigenvectors of \( \hat{G}_{0,*}(e^\theta) \) with respect to the eigenvalue \( e^{\hat{\theta}_2^*(\theta)} \), satisfying \( \hat{\nu}_G(e^\theta) \hat{\nu}_G(e^\theta) = 1 \). By Proposition 3.5, we have

\[
\lim_{\theta \uparrow \hat{\theta}_1^\dagger} (e^{\hat{\theta}_1^\dagger} - e^\theta) \hat{\varphi}_2(e^{\hat{\theta}_2^*(\theta)}) = \lim_{\theta \uparrow \hat{\theta}_1^\dagger} \frac{e^{\hat{\theta}_1^\dagger} - e^\theta}{e^{\hat{\theta}_2^*(\theta)}(e^{\hat{\theta}_2^*(\theta)} - e^{\hat{\theta}_2^*(\theta)})} \hat{\varphi}_2(e^{\hat{\theta}_2^*(\theta)})
\]

\[
= \hat{\gamma}_2^* \hat{u}_U(e^{\hat{\theta}_2^*(\theta)^\dagger}),
\]

(3.26)

where \( \hat{\gamma}_2^* = \hat{\gamma}_2^{*} e^{\hat{\theta}_1^\dagger} / \hat{\theta}_2^*, \hat{\theta}_2^*(x) = \frac{d}{dx} \hat{\theta}_2^*(x) \) and \( \hat{\gamma}_2^* \) is a positive constant. Since \( \hat{\theta}_2^*(\theta) \) is strictly increasing in \( (\hat{\theta}_1^\dagger, \hat{\theta}_1^\dagger_{\text{max}}) \), we have \( \hat{\theta}_2^*,(\hat{\theta}_1^\dagger) > 0 \), and this implies \( \hat{\gamma}_2^* > 0 \). As a result, we obtain

\[
\lim_{\theta \uparrow \hat{\theta}_1^\dagger} (e^{\hat{\theta}_1^\dagger} - e^\theta) \hat{\varphi}_2^*(\theta) = \hat{\gamma}_2^* e^{-\hat{\theta}_2^*(\theta)^\dagger} \hat{u}_U(e^{\hat{\theta}_2^*(\theta)^\dagger}) D(e^{\hat{\theta}_1^\dagger}, e^{\hat{\theta}_2^*(\theta)^\dagger}) \hat{\nu}_G(e^{\hat{\theta}_1^\dagger}) \hat{\nu}_G(e^{\hat{\theta}_1^\dagger}) \hat{\Phi}_{0,*,}(\hat{\theta}_1^\dagger).
\]

(3.27)

In a manner similar to that used in the proof of Lemma 5.5 (part (1)) of Ref. [27], it can be seen that \( \hat{u}_U(e^{\hat{\theta}_2^*(\theta)^\dagger}) D(e^{\hat{\theta}_1^\dagger}, e^{\hat{\theta}_2^*(\theta)^\dagger}) \hat{\nu}_G(e^{\hat{\theta}_1^\dagger}) > 0 \). Since \( P^{[1,2]} \) is irreducible, \( \hat{\Phi}_{0,*,}(\hat{\theta}_1^\dagger) \) is positive, and it implies that \( \hat{u}_U(e^{\hat{\theta}_2^*(\theta)^\dagger}) \hat{\Phi}_{0,*,}(\hat{\theta}_1^\dagger) \) is also positive. This completes the proof of part (iii) of the proposition. \( \square \)

**Remark 3.2** In Ref. [27], the matrix corresponding to \( \hat{G}_{0,*}(\hat{\theta}_1^\dagger) \) is assumed to have distinct eigenvalues, but that assumption is not necessary in our case. In the proof of Proposition 3.4, the condition required for \( \hat{G}_{0,*}(e^\theta) \) is that when \( \theta = \hat{\theta}_1^\dagger \), the maximum eigenvalue \( \alpha_1(e^{\hat{\theta}_1^\dagger}) \) is simple and satisfies \( \alpha_1(e^{\hat{\theta}_1^\dagger}) > |\alpha_i(e^{\hat{\theta}_1^\dagger})| \) for every \( i \in \{2, 3, \ldots, 2s_0\} \). As a condition ensuring this point, we have adopted Assumption 3.1. Under the assumption, the same property also holds for every direction vector in \( \mathbb{N}^2 \), see the following subsection.

Proposition 3.4 is represented in terms of the parameters given based on the MA-process \( \{\hat{Y}_n\} \) such as \( \hat{\theta}_1^\dagger_{\text{max}} \) and \( \hat{\theta}_1^\dagger \). We redefined those parameters so that they are...
given based on the induced MA-process \( \{ Y_{n}^{[1,2]} \} \). Define a matrix generating function \( B(z_1, z_2) \) as

\[
B(z_1, z_2) = [A_{i,-1}^{[1,2]}(z_1)]_{0,0} z_2^{-2} + [A_{i,-1}^{[1,2]}(z_1)]_{0,1} z_2^{-1} + [A_{-1,0}^{[1,2]}(z_1)]_{0,0} z_2^2 + [A_{-1,0}^{[1,2]}(z_1)]_{0,1} z_2 + [A_{1,1}^{[1,2]}(z_1)]_{0,0} z_2^2
+ A_{1,-1}^{[1,2]} z_2^{-2} + A_{1,0}^{[1,2]} z_2^{-1} + A_{0,-1}^{[1,2]} z_2 + A_{0,-1}^{[1,2]} z_1^{-1} + A_{0,0}^{[1,2]} + A_{1,1}^{[1,2]} z_1
+ A_{1,0}^{[1,2]} z_2 + A_{0,1}^{[1,2]} z_2 + A_{-1,1}^{[1,2]} z_2
\]

where, for a block matrix \( A \), we denote by \([A]_{i,j}\) the \((i, j)\)-block of \( A \). This matrix function satisfies

\[
B(e^{\theta_1}, e^{\theta_2}) = A_{*,*}^{[1,2]}(e^{\theta_1 - \theta_2}, e^{\theta_2}). \tag{3.28}
\]

By Remark 2.4 of Ozawa [31], we have

\[
\text{spr}(A_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = \text{spr}(B(e^{\theta_1}, e^{\theta_2/2})), \tag{3.29}
\]

and this leads us to

\[
\text{spr}(A_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = \text{spr}(A_{*,*}^{[1,2]}(e^{\theta_1 - \theta_2/2}, e^{\theta_2/2})). \tag{3.30}
\]

For \( \theta \in [\theta_{c,1}^{\min}, \theta_{c,1}^{\max}] \), let \( (\eta_{c,1}^{R}(\theta), \eta_{c,2}^{R}(\theta)) \) and \( (\eta_{c,1}^{L}(\theta), \eta_{c,2}^{L}(\theta)) \) be the two real roots of the simultaneous equations:

\[
\text{spr}(A_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = 1, \quad \theta_1 + \theta_2 = \theta, \tag{3.31}
\]

counting multiplicity, where \( \eta_{c,1}^{L}(\theta) \leq \eta_{c,1}^{R}(\theta) \) and \( \eta_{c,2}^{L}(\theta) \geq \eta_{c,2}^{R}(\theta) \) (see Fig. 6). Since equation \( \text{spr}(A_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = 1 \) is equivalent to \( \text{spr}(A_{*,*}^{[1,2]}(e^{\theta_1 - \theta_2/2}, e^{\theta_2/2})) = 1 \), we have

\[
\hat{\theta}_1^{\min} = \theta_{c,1}^{\min}, \quad \hat{\theta}_1^{\max} = \theta_{c,1}^{\max}, \quad \hat{\eta}_2^{c}(\theta_1) = 2\eta_{c,2}^{R}(\theta_1). \tag{3.32}
\]
and \( \hat{\theta}^{\dagger} \) and \( \hat{\theta}^{\dagger,\xi} \) are given by

\[
\hat{\theta}^{\dagger} = \max\{\theta \in [\theta_{c}^{\min}, \theta_{c}^{\max}]; \eta_{c,2}(\theta) \leq \theta_{2}^{\ast}\}, \quad (3.33)
\]

\[
\hat{\theta}^{\dagger,\xi} = \max\{\theta \in [\theta_{c}^{\min}, \theta_{c}^{\max}]; \eta_{c,2}(\theta) \leq \xi(0,1)\}. \quad (3.34)
\]

Hereafter, we denote \( \hat{\theta}^{\dagger} \) and \( \hat{\theta}^{\dagger,\xi} \) by \( \theta_{c,2}^{\ast} \) and \( \theta_{c,2}^{\ast,\xi} \), respectively, and use (3.33) and (3.34) as their definitions. Note that, for \( \theta_{c,2}^{\ast} \) and \( \theta_{c,2}^{\ast,\xi} \), we use subscript “2” instead of “1” since they are defined by using \( \theta_{2}^{\ast} \) and \( \xi(0,1) \). \( \theta_{c,2}^{\ast} \) has already been defined in Sect. 1. Here we redefine it for the case of \( c = (1, 1) \). After, we also redefine \( \theta_{c,1}^{\dagger} \). In terms of these parameters, we rewrite Proposition 3.4 as follows.

**Corollary 3.1**

(i) We always have \( \hat{\xi}_{c,2} \geq \theta_{c,2}^{\ast} \).

(ii) If \( \theta_{c,2}^{\ast} \) and \( \theta_{2}^{\ast} \), then \( \hat{\phi}_{2}(z) \) is elementwise analytic in \( \Delta_{1,\theta_{c,2}^{\ast}} \cup (\partial \Delta_{c_{2}, \theta_{c,2}^{\ast}, \theta_{c,2}^{\ast}} \setminus \{\theta_{c,2}^{\ast}\}) \).

(iii) If \( \theta_{c,2}^{\ast} \) and \( \theta_{2}^{\ast} \), then \( \hat{\xi}_{c,2} = \theta_{c,2}^{\ast} \) and, for some positive vector \( \hat{g}_{c,2}^{\ast} \),

\[
\lim_{\theta \uparrow \theta_{c,2}^{\ast}} (e^{\theta_{c,2}^{\ast}} - e^{\theta}) \hat{\phi}_{2}(e^{\theta}) = \hat{g}_{c,2}^{\ast}. \quad (3.35)
\]

Define \( \hat{\phi}_{1}^{\ast}(z) = (\hat{\phi}_{1,1}^{\ast}(z) \hat{\phi}_{1,2}^{\ast}(z)) \) and \( \hat{\xi}_{c,1} \) analogously to \( \hat{\phi}_{2}(z) = (\hat{\phi}_{2,1}(z) \hat{\phi}_{2,2}(z)) \) and \( \hat{\xi}_{c,2} \), respectively. Then, we have \( \hat{\xi}_{c,1} \geq \hat{\xi}_{c,1} \). Define \( \theta_{c,1}^{\dagger} \) and \( \theta_{c,1}^{\dagger,\xi} \) as

\[
\theta_{c,1}^{\dagger} = \max\{\theta \in [\theta_{c}^{\min}, \theta_{c}^{\max}]; \eta_{c,1}(\theta) \leq \theta_{1}^{\ast}\}, \quad (3.36)
\]

\[
\theta_{c,1}^{\dagger,\xi} = \max\{\theta \in [\theta_{c}^{\min}, \theta_{c}^{\max}]; \eta_{c,1}(\theta) \leq \xi(1,0)\}. \quad (3.37)
\]

With respect to \( \hat{\phi}_{1}^{\ast}(e^{\theta}) \), interchanging the \( x_{1}\)-axis with the \( x_{2}\)-axis, we immediately obtain by Corollary 3.1 the following.

**Corollary 3.2**

(i) We always have \( \hat{\xi}_{c,1} \geq \theta_{c,1}^{\dagger,\xi} \).

(ii) If \( \theta_{c,1}^{\dagger} < \theta_{c}^{\max} \) and \( \theta_{1}^{\ast} < \theta_{c,1}^{\dagger} \), then \( \hat{\phi}_{1}^{\ast}(z) \) is elementwise analytic in \( \Delta_{1,\theta_{c,1}^{\dagger}} \cup (\partial \Delta_{c_{1}, \theta_{c,1}^{\dagger}} \setminus \{\theta_{c,1}^{\dagger}\}) \).

(iii) If \( \theta_{c,1}^{\dagger} < \theta_{c}^{\max} \) and \( \theta_{1}^{\ast} < \theta_{c,1}^{\dagger} \), then \( \hat{\xi}_{c,1} = \theta_{c,1}^{\dagger} \) and, for some positive vector \( \hat{g}_{1}^{\ast} \),

\[
\lim_{\theta \uparrow \theta_{c,1}^{\dagger}} (e^{\theta_{c,1}^{\dagger}} - e^{\theta}) \hat{\phi}_{1}(e^{\theta}) = \hat{g}_{1}^{\ast}. \quad (3.38)
\]

By Proposition 3.2 and Corollaries 3.1 and 3.2, we obtain a main result of this subsection as follows.
**Theorem 3.1** We have $\xi_\epsilon = \xi_{1,1} = \min(\theta_{c,1}^+, \theta_{c,2}^+)$, and if $\xi_\epsilon < \theta_{c}^{\text{max}}$, the sequence $\{v_{(k,k)}\}_{k \geq 0}$ geometrically decays with ratio $e^{-\xi_\epsilon}$ as $k$ tends to infinity, i.e., for some positive vector $g$.

$$v_{(k,k)} \sim ge^{-\xi_\epsilon k} \text{ as } k \to \infty.$$  

**Proof** Recall that $\varphi^\epsilon(z) = \varphi_0^\epsilon(z) + \varphi_1^\epsilon(z) + \varphi_2^\epsilon(z)$. This $\varphi^\epsilon(z)$ is absolutely convergent and elementwise analytic in $\Delta_{\epsilon}^\epsilon$ since $\hat{e}_\epsilon$ is the radius of the convergence of $\varphi^\epsilon(z)$.

With respect to the values of $\theta_{c,1}^+$ and $\theta_{c,2}^+$, we consider the following cases.

1. $\theta_{c,1}^+ = \theta_{c,2}^+ = \theta_{c}^{\text{max}}$. By Proposition 3.2 and Corollaries 3.1 and 3.2, we have

$$\xi_\epsilon \geq \min(\xi_{c,0}, \xi_{c,1}, \xi_{c,2}) \geq \min(\xi_{c,0}, \hat{\xi}_{c,1}, \hat{\xi}_{c,2}) \geq \min(\theta_{c,1}^+, \theta_{c,2}^+) \geq \min(\theta_{c,1}^+, \theta_{c,2}^+) = \theta_{c}^{\text{max}}.$$  

By (2.11), we have $\bar{\xi}_\epsilon \leq \theta_{c}^{\text{max}}$, and hence $\xi_\epsilon = \theta_{c}^{\text{max}} = \min(\theta_{c,1}^+, \theta_{c,2}^+)$.

2. $\theta_{c,2}^+ < \theta_{c,1}^+ \leq \theta_{c}^{\text{max}}$. By Proposition 3.2 and Corollary 3.2, $\xi_{c,0} \geq \theta_{c}^{\text{max}} > \theta_{c,2}^+$ and $\xi_{c,1} \geq \theta_{c,1}^+ > \theta_{c,2}^+$. We have $\theta_{c,1}^+ = \eta_{c,1}^L(\theta_{c,1}^+) > \eta_{c,1}^L(\theta_{c,2}^+)$ and this implies that $\theta_{c,2}^+ = \eta_{c,2}^L(\theta_{c,2}^+) < \eta_{c,1}^L(\theta_{c,2}^+) \leq \theta_{c,1}^+$ (see Fig. 7, where we assume $c = (1,1)$).

Hence, by part (iii) of Corollary 3.1, $\hat{\varphi}_2(z)$ elementwise diverges at $z = e^{\theta_{c,2}^+}$, and we have $\xi_\epsilon = \theta_{c,2}^+$. Since $\xi_\epsilon < \theta_{c}^{\text{max}} \leq \xi_{c,0}$ and $\hat{\xi}_\epsilon < \theta_{c,1}^+ \leq \hat{\xi}_{c,1}$, $\varphi_0^\epsilon(z)$ and $\varphi_1^\epsilon(z)$ as well as $\hat{\varphi}_1(z)$ are elementwise analytic on $\partial \Delta_{\epsilon}^\epsilon$. By part (ii) of Corollary 3.1, $\varphi_2^\epsilon(z)$ as well as $\hat{\varphi}_2(z)$ is elementwise analytic on $\partial \Delta_{\epsilon}^\epsilon \setminus \{\hat{e}_\epsilon\}$. Hence, $\varphi^\epsilon(z)$ is elementwise analytic in $\Delta_{\epsilon}^\epsilon \cup (\partial \Delta_{\epsilon}^\epsilon \setminus \{\hat{e}_\epsilon\})$. As a result, by part (iii) of Corollary 3.1 and Theorem VI.4 of Flajolet and Sedgewick [8], the sequence $\{v_{(k,k)}\}_{k \geq 0}$ geometrically decays with ratio $e^{-\theta_{c,2}^+}$ as $k$ tends to infinity and we obtain $\xi_\epsilon \geq \xi_{c,0} = \theta_{c,1}^+ = \min(\theta_{c,1}^+, \theta_{c,2}^+) < \theta_{c}^{\text{max}}$.

3. $\theta_{c,1}^+ < \theta_{c,2}^+ \leq \theta_{c}^{\text{max}}$. This case is symmetrical to the previous case.

4. $\theta_{c,1}^+ = \theta_{c,2}^+ < \theta_{c}^{\text{max}}$. Set $\theta = \theta_{c,1}^+ (= \theta_{c,2}^+)$. By Proposition 3.2, $\xi_{c,0} \geq \theta_{c}^{\text{max}} > \theta$. We have $\theta_{c,1}^+ = \eta_{c,1}^L(\theta) < \eta_{c,1}^R(\theta) \leq \theta_{c,1}^+$ and $\theta_{c,2}^+ = \eta_{c,2}^R(\theta) < \eta_{c,2}^L(\theta) \leq \theta_{c,1}^+$. 

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Hence, in a manner similar to that used in part (2) above, we see that the sequence \( \{v_{(k,k)}\}_{k \geq 0} \) geometrically decays with ratio \( e^{-\theta} \) as \( k \) tends to infinity and obtain
\[
\xi_c = \theta = \min\{\theta_{e,1}^\dagger, \theta_{e,2}^\dagger\} < \theta_e^{\max}.
\]

\[\square\]

### 3.3 In the case of general direction vector \( c \)

Letting \( c = (c_1, c_2) \in \mathbb{N}^2 \) be a direction vector, we obtain the asymptotic rate \( \xi_c \). For the purpose, we consider the \( c \)-block state process, introduced in Sect. 2.4, derived from the original 2d-QBD process, \( \{cY_n\} = \{(cX_{1,n}, cX_{2,n}), (cM_{1,n}, cM_{2,n}, cJ_n)\} \), whose state space is \( \mathbb{Z}_+^2 \times \mathbb{Z}_{0,c_1-1} \times \mathbb{Z}_{0,c_2-1} \times S_0 \). The \( c \)-block state process \( \{cY_n\} \) is a 2d-QBD process given by regarding each \( c_1 \times c_2 \) block of level in the original 2d-QBD process as a level, where for \( i = 1, 2 \), \( cX_{i,n} \) and \( cM_{i,n} \) are the quotient and remainder of \( X_{i,n} \) divided by \( c_i \), respectively, i.e., \( X_{i,n} = c_iX_{i,n} + cM_{i,n} \), and \( cJ_n = J_n \). Since the state \( (k, k, 0, 0, j) \) of \( \{cY_n\} \) corresponds to the state \( (c_1k, c_2k, j) \) of the original 2d-QBD process, we have for any \( j \in S_0 \) that
\[
\xi_c = \xi_{c(1,1)} = -\lim_{k \to \infty} \frac{1}{k} \log c v_{(k,k,0,0,j)},
\]
where \( \{v_{(x_1,x_2,r_1,r_2,j)}; (x_1, x_2, r_1, r_2, j) \in \mathbb{Z}_+^2 \times \mathbb{Z}_{0,c_1-1} \times \mathbb{Z}_{0,c_2-1} \times S_0\} \) is the stationary distribution of \( \{cY_n\} \). Therefore, applying the results of the previous subsection to \( \{cY_n\} \), we can obtain \( \xi_c \).

Denote by \( cA_{*,*}^{[1,2]}(z_1, z_2) \) the matrix generating function of the transition probability blocks of \( \{cY_n\} \), corresponding to \( A_{*,*}^{[1,2]}(z_1, z_2) \) of the original 2d-QBD process (see Appendix A). The simultaneous equations corresponding to (3.31) are given by
\[
\spr(cA_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = 1, \quad \theta_1 + \theta_2 = \theta, \quad (3.40)
\]
Since we have by Proposition 4.2 of Ozawa [31] that
\[
\spr(cA_{*,*}^{[1,2]}(e^{c_1\theta_1}, e^{c_2\theta_2})) = \spr(A_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})), \quad (3.41)
\]
simultaneous equations (3.40) are equivalent to
\[
\spr(A_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = 1, \quad c_1\theta_1 + c_2\theta_2 = \theta. \quad (3.42)
\]
For \( \theta \in [\theta_e^{\min}, \theta_e^{\max}] \), let \( (\eta_{e,1}^R(\theta), \eta_{e,2}^R(\theta)) \) and \( (\eta_{e,1}^L(\theta), \eta_{e,2}^L(\theta)) \) be the two real roots of simultaneous equations (3.42), counting multiplicity, where \( \eta_{e,1}^L(\theta) \leq \eta_{e,1}(\theta) \) and \( \eta_{e,2}^L(\theta) \geq \eta_{e,2}^R(\theta) \). Redefine real values \( \theta_{e,1}^\dagger \) and \( \theta_{e,2}^\dagger \) as
\[
\theta_{e,1}^\dagger = \max\{\theta \in [\theta_e^{\min}, \theta_e^{\max}]; \eta_{e,1}^L(\theta) \leq \theta_e^*,\} \quad (3.43)
\]
\[
\theta_{e,2}^\dagger = \max\{\theta \in [\theta_e^{\min}, \theta_e^{\max}]; \eta_{e,2}^R(\theta) \leq \theta_e^*,\} \quad (3.44)
\]
which are equivalent to definitions (1.4). Since the block state process \( \{cY_n\} \) is derived from the original 2d-QBD process, the former inherits all assumptions for the latter, including Assumption 3.1. Hence, by Theorem 3.1, we immediately obtain the following.

**Theorem 3.2** For any direction vector \( c \in \mathbb{N}^2 \), \( \xi_c = \min(\theta^\dagger_{c,1}, \theta^\dagger_{c,2}) \), and if \( \xi_c < \theta^\text{max}_c \), the sequence \( \{\psi_{kc}\}_{k \geq 0} \) geometrically decays with ratio \( e^{-\xi_c} \) as \( k \) tends to infinity, i.e., for some constant vector \( g \),

\[
\psi_{kc} \sim ge^{-\xi_c k} \quad \text{as} \quad k \to \infty.
\]

### 4 Geometric property and an example

#### 4.1 The value of the asymptotic decay rate \( \xi_c \)

Geometric consideration (see, for example, Miyazawa [19]) is also useful in our case. Here we reconsider Theorem 3.2 geometrically. Define two points \( Q_1 \) and \( Q_2 \) as \( Q_1 = (\theta^*_1, \bar{\eta}_1(\theta^*_1)) \) and \( Q_2 = (\bar{\eta}_1(\theta^*_2), \theta^*_2) \), respectively. For the definition of \( \theta^*_1 \) and \( \theta^*_2 \), see (3.19), and for the definition of \( \bar{\eta}_1(\theta) \) and \( \bar{\eta}_2(\theta) \), see Appendix A. Using these points, we define the following classification (see Fig. 8).

- **Type 1**: \( \theta^*_1 \geq \bar{\eta}_1(\theta^*_2) \) and \( \bar{\eta}_2(\theta^*_1) \leq \theta^*_2 \).
- **Type 2**: \( \theta^*_1 < \bar{\eta}_1(\theta^*_2) \) and \( \bar{\eta}_2(\theta^*_1) > \theta^*_2 \).
- **Type 3**: \( \theta^*_1 \geq \bar{\eta}_1(\theta^*_2) \) and \( \bar{\eta}_2(\theta^*_1) > \theta^*_2 \).
- **Type 4**: \( \theta^*_1 < \bar{\eta}_1(\theta^*_2) \) and \( \bar{\eta}_2(\theta^*_1) \leq \theta^*_2 \).

Let \( c = (c_1, c_2) \in \mathbb{N}^2 \) be an arbitrary direction vector. For \( i \in \{1, 2\} \), define \( \theta^*_i \) as \( \theta^*_i = \inf\{\theta_i \in \mathbb{R}; (\theta_1, \theta_2) \in \Gamma[i]\} \), then \( \Gamma[i] \) satisfies \( \Gamma[i] = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta^*_i < \theta_i < \theta^*_i\} \). Hence, by (1.4), we have, for \( i \in \{1, 2\} \),

\[
\theta^\dagger_{c,i} = \sup\{c_1\theta_1 + c_2\theta_2; (\theta_1, \theta_2) \in \Gamma[1,2], \theta^*_3-i < \theta_3-i < \theta^*_3-i\}.
\]

From this representation for \( \theta^\dagger_{c,i} \), we see that the asymptotic decay rate in the direction \( c \) is given depending on the geometrical relation between \( Q_1 \) and \( Q_2 \), as follows (see Figs. 7 and 8).

- **Type 1**: If \( -c_1/c_2 \leq \bar{\eta}_2(\theta_1^*) \), then \( \xi_c = c_1\theta_1^* + c_2\bar{\eta}_2(\theta_1^*) \), where \( \bar{\eta}_2(x) = (d/dx)\bar{\eta}_2(x) \); If \(-c_1/c_2 \geq \bar{\eta}_1(\theta_2^*) \), then \( \xi_c = c_1\bar{\eta}_1(\theta_2^*) + c_2\theta_2^* \), where \( \bar{\eta}_1(x) = (d/dx)\bar{\eta}_1(x) \); Otherwise (i.e., \( \bar{\eta}_2(\theta_1^*) < -c_1/c_2 < 1/\bar{\eta}_1(\theta_2^*) \)), \( \xi_c = \theta^\text{max}_c \).
- **Type 2**: If \(-c_1/c_2 \leq (\theta_2^* - \bar{\eta}_2(\theta_1^*)/(\bar{\eta}_1(\theta_2^*) - \theta_1^*) \), then \( \xi_c = c_1\theta_1^* + c_2\bar{\eta}_2(\theta_1^*) \); Otherwise (i.e., \(-c_1/c_2 > (\theta_2^* - \bar{\eta}_2(\theta_1^*)/(\bar{\eta}_1(\theta_2^*) - \theta_1^*) \)), \( \xi_c = c_1\bar{\eta}_1(\theta_2^*) + c_2\theta_2^* \).
- **Type 3**: \( \xi_c = c_1\bar{\eta}_1(\theta_2^*) + c_2\theta_2^* \).
- **Type 4**: \( \xi_c = c_1\theta_1^* + c_2\bar{\eta}_2(\theta_1^*) \).

This also holds true for the case where \( c = (1, 0) \) or \( c = (0, 1) \).

For a 2d-RW, which is a 2d-QBD process having no background processes, the exact asymptotic expansion of the stationary distribution in any direction have already
been obtained by Malyshev [16]. The same results are also written in the book of Fayolle et al. [7]. Using the formulation used in [7] and the same notation used for the 2d-QBD process in this paper, we compare those results with ours. In the case of 2d-RRW, all matrices $A_{\{1,2\}}, A_{i,j}^{(1)}, A_{i,j}^{(2)}$ become scalars, and the functions defined in [7], $Q(x, y), q(x, y)$ and $\tilde{q}(x, y)$, are given by

\begin{equation}
Q(x, y) = xy \left( A_{\{1,2\}}^{(1)}(x, y) - 1 \right) = xy \left( \text{spr}(A_{\{1,2\}}^{(1)}(x, y)) - 1 \right),
\end{equation}

\begin{equation}
q(x, y) = x \left( \sum_{i \in \{-1,0,1\}} \sum_{j \in \{0,1\}} A_{i,j}^{(1)} x^i y^j - 1 \right),
\end{equation}

\begin{equation}
\tilde{q}(x, y) = y \left( \sum_{i \in \{0,1\}} \sum_{j \in \{-1,0,1\}} A_{i,j}^{(2)} x^i y^j - 1 \right).
\end{equation}

Let $c = (c_1, c_2) \in \mathbb{N}^2$ be a direction vector, and define $\gamma$ as $\gamma = c_2/c_1$. Let $(x_3(\gamma), y_3(\gamma))$ be the real solution to the system of equations:

\begin{equation}
Q(x, y) = 0, \quad y \frac{\partial}{\partial y} Q(x, y) = \gamma x \frac{\partial}{\partial x} Q(x, y),
\end{equation}
satisfying $1 \leq x_3(\gamma) \leq e^{\theta_1^{\max}}$ and $1 \leq y_3(\gamma) \leq e^{\theta_2^{\max}}$. Letting $x = e^{\theta_1}$ and $y = e^{\theta_2}$, (4.5) becomes

$$Q(e^{\theta_1}, e^{\theta_2}) = 0, \quad \frac{d\theta_2}{d\theta_1} = -\frac{x \frac{\partial}{\partial x} Q(x, y)}{y \frac{\partial}{\partial y} Q(x, y)} = -\frac{1}{\gamma}. \quad (4.6)$$

Hence, on the $\theta_1$-$\theta_2$ plane, the point $(\log x_3(\gamma), \log y_3(\gamma))$ is the contact point of the closed curve $Q(e^{\theta_1}, e^{\theta_2}) = 0$ and a line whose normal vector is $e$ (see Fig. 9). This implies that $c_1 \log x_3(\gamma) + c_2 \log y_3(\gamma) = \theta e^{\max}$. Define the sets of direction vectors as follows:

$$C_{--} = \{(c_1, c_2) \in \mathbb{N}^2; q(x_3(\gamma), e^{\theta_2} (\log y_3(\gamma))) \leq 0, \tilde{q}(e^{\theta_1} (\log y_3(\gamma)), y_3(\gamma)) \leq 0\},$$

$$C_{++} = \{(c_1, c_2) \in \mathbb{N}^2; q(x_3(\gamma), e^{\theta_2} (\log y_3(\gamma))) > 0, \tilde{q}(e^{\theta_1} (\log y_3(\gamma)), y_3(\gamma)) \leq 0\},$$

where $\gamma = c_2/c_1$. Note that $y_3(\gamma) = e^{\tilde{\theta}_2 (\log x_3(\gamma))}$ and $x_3(\gamma) = e^{\tilde{\theta}_1 (\log y_3(\gamma))}$. Define $C_{-+}$ and $C_{+-}$ accordingly. Let $(x_0, y_0)$ be the real solution to the following system of equations except for $(1, 1)$:

$$Q(x, y) = 0, \quad q(x, y) = 0, \quad \gamma = c_2/c_1. \quad (4.7)$$

and $(x_5, y_5)$ that to the system of equations:

$$Q(x, y) = 0, \quad \tilde{q}(x, y) = 0. \quad (4.8)$$

In the case of 2d-RRW, since the background process of the induced MA-process $\{Y^{[1]}\}$ derived from the original 2d-RRW becomes just a birth-and-death process, $\theta_1^*$ is given by

$$\theta_1^* = \begin{cases} 
\log x_0 & \text{if } q(e^{\theta_1^{\max}}, e^{\theta_2 (\theta_1^{\max})}) \leq 0, \\
\theta_1^{\max} & \text{otherwise.}
\end{cases} \quad (4.9)$$

About this point, see Remark 4.1. $\theta_2^*$ is analogously given in terms of $y_5$ and $\theta_2^{\max}$.

Let the points $(x_0(\gamma), y_0(\gamma))$ and $(x_5(\gamma), y_5(\gamma))$ be the solutions to the respective systems of equations

$$\begin{cases} 
Q(x, y) = 0, \quad q(x, \xi y) = 0, & \text{for } (x_0(\gamma), y_0(\gamma)), \\
Q(x, y) = 0, \quad \tilde{q}(\eta x, y) = 0, & \text{for } (x_5(\gamma), y_5(\gamma)),
\end{cases} \quad (4.10)$$

satisfying $1 \leq x_0(\gamma) \leq e^{\theta_1^{\max}}$ and $1 \leq y_5(\gamma) \leq e^{\theta_2^{\max}}$; for the definition of $\xi$ and $\eta$, see Section 7.2.1 of [7]. These points are symmetric to $(x_0, y_0)$ and $(x_5, y_5)$, respectively, and they are given by

$$\begin{cases} 
(x_0(\gamma), y_0(\gamma)) = (x_0, e^{\tilde{\theta}_2 (\log x_0)}) = (e^{\theta_1^*}, e^{\tilde{\theta}_2 (\theta_1^*)}) & \text{if } q(e^{\theta_1^{\max}}, e^{\theta_2 (\theta_1^{\max})}) \leq 0, \\
(x_5(\gamma), y_5(\gamma)) = (e^{\tilde{\theta}_1 (\log y_5)}, y_5) = (e^{\tilde{\theta}_1 (\theta_2^*)}, e^{\theta_2^*}) & \text{if } \tilde{q}(e^{\theta_1^{\max}}, e^{\theta_2^{\max}}) \leq 0,
\end{cases} \quad (4.11)$$
Remark 4.1 In the case of 2d-RRW, \( \theta_1 = \theta_1^* \) (\( \theta_2 = \theta_2^* \)) is given as a solution to the system of equations \( Q_1(e^{\theta_1}, e^{\theta_2}) = 0 \) and \( q(e^{\theta_1}, e^{\theta_2}) = 0 \) (resp. \( Q_2(e^{\theta_1}, e^{\theta_2}) = 0 \) and \( \tilde{q}(e^{\theta_1}, e^{\theta_2}) = 0 \)), where the function \( Q(x, y) \) is represented only in terms of \( A_{i,j}^{[1,2]} \), \( q(x, y) \) only in terms of \( A_{i,j}^{[1]} \) and \( \tilde{q}(x, y) \) only in terms of \( A_{i,j}^{[2]} \). This formulation is simple and tractable. On the other hand, in the case of general 2d-QBD process, \( \theta_1 = \theta_1^* \) is given as a solution to equation \( cp(\tilde{A}_{i,j}^{[1]}(e^{\theta_1})) = 1 \), where the matrix \( \tilde{A}_{i,j}^{[1]}(x) \) is represented in terms of both \( A_{i,j}^{[1]} \) and \( A_{i,j}^{[1,2]} \). According to the results in [26], \( \theta_1 = \theta_1^* \) can also be given as a solution to equation \( spr(U_1(e^{\theta_1})) = 1 \), where \( U_1(x) = A_{i,0}^{[1]}(x) + A_{i,j}^{[1]}(x)G_1(x) \) and \( G_1(x) \) is a G-matrix; see Appendix A. \( \theta_2^* \) can also be given in the same way. As indicated by Miyazawa [20], this formulation is not tractable since the formula defining \( U_1(x) \) includes the G-matrix \( G_1(x) \). However, it seems difficult to obtain another tractable formulation for a general 2d-QBD process. With respect to this point, see Section 2.3 and Appendix 3 in [20], in which a condition making it possible to give a tractable formulation has also been proposed.

4.2 An example

We consider the same queueing model as that used in Ozawa and Kobayashi [28]. It is a single-server two-queue model in which the server visits the queues alternatively, serves one queue (queue 1) according to a 1-limited service and the other queue (queue 2) according to an exhaustive-type \( K \)-limited service (see Fig. 10). Customers arrive at queue 1 (resp. queue 2) according to a Poisson process with intensity \( \lambda_1 \) (resp. \( \lambda_2 \)). Service times are exponentially distributed with mean \( 1/\mu_1 \) in queue 1 (\( 1/\mu_2 \) in queue 2). The arrival processes and service times are mutually independent. We refer to this model as a \((1, K)\)-limited service model. In this model, the asymptotic decay rate \( \xi_c \) indicates how the joint queue length probability in steady state decreases as the queue lengths of queue 1 and queue 2 simultaneously enlarge.
Let $X_1(t)$ be the number of customers in queue 1 at time $t$, $X_2(t)$ that of customers in queue 2 and $J(t) \in S_0 = \{0, 1, ..., K\}$ the server state. When $X_1(t) = X_2(t) = 0$, $J(t)$ takes one of the states in $S_0$ at random; When $X_1(t) \geq 1$ and $X_2(t) = 0$, it also takes one of the states in $S_0$ at random; When $X_1(t) = 0$ and $X_2(t) \geq 1$, it takes the state of $0$ or $1$ at random if the server is serving the $K$-th customer in queue 2 during a visit of the server at queue 2 and takes the state of $j \in \{2, ..., K\}$ if the server is serving the $(K - j + 1)$-th customer in queue 2 during a visit of the server at queue 2. The process $\{(X_1(t), X_2(t), J(t))\}$ becomes a continuous-time 2d-QBD process on the state space $\mathbb{Z}_+^2 \times S_0$. By uniformization with parameter $\nu = \lambda + \mu_1 + \mu_2$, we obtain the corresponding discrete-time 2d-QBD process, $\{(X_{1,n}, X_{2,n}, J_n)\}$. For the description of the transition probability blocks...
such as $A_{1,2}^{(1,2)}$, see Ref. [28]. This $(1, K)$-limited service model satisfies Assumptions 1.1, 1.2, 2.2 and 3.1.

In numerical experiments, we treat two cases: a symmetric parameter case (see Fig. 11 and Table 1) and an asymmetric parameter case (see Fig. 12 and Table 2). In both the cases, the value of $K$ is set at 1, 5 or 10. In Figs. 11 and 12, the closed curves of spr$(A_{e,s}^{(1,2)}(e^{\theta_1}, e^{\theta_2})) = 1$ are drawn with points Q1 and Q2. Define points P1, P2 and R as $P_1 = (\theta_1^{max}, \eta_2(\theta_1^{max}))$, $P_2 = (\eta_1(\theta_2^{max}), \theta_2^{max})$ and $R = (\eta_c^R(\theta_c^{max}), \eta_{e,2}(\theta_c^{max}))$, respectively. For the definition of $\theta_1^{max}$ and $\theta_2^{max}$, see Appendix A. These points are also written on the figures. From the figures, we see that all the cases are classified into Type 1. If Q1 = P1 and Q2 = P2 (see Figs. 11a, b and 12b), then, for any $c = (c_1, c_2) \in \mathbb{N}^2$, $\xi_c$ is given by $\theta_c^{max}$. On the other hand, in the symmetric case of $K = 10$ (see Fig. 11c), $\xi_c$ is given by $\theta_c^{max}$ only if $-c_1/c_2 > \eta_2(\theta_1^{*}) = -9.87$; In the asymmetric case of $K = 1$ (see Fig. 12a), it is given by $\theta_c^{max}$ only if $-c_2/c_1 > \eta_1(\theta_2^{*}) = -1.73$; In that of $K = 10$ (see Fig. 12c), it is given by $\theta_c^{max}$ only if $-c_1/c_2 > \eta_2(\theta_1^{*}) = -3.88$. Tables 1 and 2 shows the normalized values of $\xi_c$, i.e., $\xi_c/\|c\|$, where $\|c\| = \sqrt{c_1^2 + c_2^2}$. From the tables, it can be seen how the values of the asymptotic decay rate vary according to the direction vector.

### Table 1

Asymptotic decay rates ($\lambda_1 = \lambda_2 = 0.3, \mu_1 = \mu_2 = 1$)

| $K$ | $\theta_1^{max}$ | $\theta_1^*$ | $\theta_2^{max}$ | $\theta_2^*$ | $\xi_1(0)$ | $\xi_2(0)/\sqrt{5}$ | $\xi_1(1)/\sqrt{2}$ | $\xi_2(1)/\sqrt{5}$ | $\xi(0,1)$ |
|-----|------------------|--------------|------------------|--------------|------------|----------------------|----------------------|----------------------|------------|
| 1   | 0.677            | ←            | 0.677            | ←            | 0.667      | 0.714                | 0.722                | 0.714                | 0.677      |
| 5   | 0.511            | ←            | 1.30             | ←            | 0.511      | 0.734                | 0.866                | 0.986                | 1.30       |
| 10  | 0.513            | 0.511        | 1.41             | ←            | 0.511      | 0.757                | 0.901                | 1.03                 | 1.41       |

### Table 2

Asymptotic decay rates ($\lambda_1 = 0.24, \lambda_2 = 0.7, \mu_1 = 1.2, \mu_2 = 1$)

| $K$ | $\theta_1^{max}$ | $\theta_1^*$ | $\theta_2^{max}$ | $\theta_2^*$ | $\xi_1(0)$ | $\xi_2(0)/\sqrt{5}$ | $\xi_1(1)/\sqrt{2}$ | $\xi_2(1)/\sqrt{5}$ | $\xi(0,1)$ |
|-----|------------------|--------------|------------------|--------------|------------|----------------------|----------------------|----------------------|------------|
| 1   | 1.29             | ←            | 0.223            | 0.110        | 1.29       | 0.98                 | 0.740                | 0.500                | 0.110      |
| 5   | 0.091            | ←            | 0.331            | ←            | 0.091      | 0.136                | 0.164                | 0.198                | 0.331      |
| 10  | 0.094            | 0.090        | 0.520            | ←            | 0.090      | 0.161                | 0.208                | 0.267                | 0.520      |

### 5 Concluding remarks

**Exact asymptotic formula in the case where $\xi_c = \theta_c^{max}$:**

In Theorem 3.2, the exact asymptotic formula was given only in the case where $\xi_c < \theta_c^{max}$. We think it can also be obtained in the case where $\xi_c = \theta_c^{max}$, in a manner similar to that used in [27, 28]. Its power term is probably given by $k^{-\frac{1}{2}}$. As mentioned in the previous section, the power term $k^{-\frac{1}{2}}$ has already been obtained for a 2d-RRW in Malyshev [16], by considering a saddle point on a Riemann surface; see Theorem 1 of [16]. In order to obtain such a result for the 2d-QBD process in our manner, several
problems still remain to be solved, and we make it a further study. Here, we briefly sketch out a possible way.

Let \( c = (1, 1) \) and consider \( \hat{\phi}_2^c(z) \) given by expression (3.16). It is known that \( \hat{\Phi}(0,0,\ast)(z) \) in (3.16) is given as

\[
\hat{\Phi}(0,0,\ast)(z) = \sum_{n=0}^{\infty} \left( A_{\ast,-1}^{[1,2]}(z) \hat{G}_{0,\ast}(z) + A_{\ast,0}^{[1,2]}(z) + A_{\ast,1}^{[1,2]}(z) \hat{G}_{0,\ast}(z) \right)^n,
\]

where \( \hat{G}_{0,\ast}(z) \) is the G-matrix in the reverse direction and given by the minimum nonnegative solution to the follow equation:

\[
A_{\ast,-1}^{[1,2]}(z)X^2 + A_{\ast,0}^{[1,2]}(z)X + A_{\ast,1}^{[1,2]}(z) = X.
\]

We have spr(\( \hat{G}_{0,\ast}(e^\theta) \)) = \( e^{-\hat{\xi}_c(\theta)} \). From the results of [27, 28], we can see that if \( \hat{\xi}_c = \theta_c^{\max} \), the point \( z = e^{\theta_c^{\max}} \) is a branch point of both \( \hat{G}_{0,\ast}(z) \) and \( \hat{G}_{0,\ast}(z) \). Hence, if we can demonstrate that \( \hat{\Phi}(0,0,\ast)(z) \) diverges at \( z = e^{\theta_c^{\max}} \), we see that the point \( z = e^{\theta_c^{\max}} \) is a pole and branch point of \( \hat{\Phi}(0,0,\ast)(z) \). As a result, if we can also demonstrate that \( \hat{\phi}_2^c(z) \) diverges at \( z = e^{\theta_c^{\max}} \), we see that the point \( z = e^{\theta_c^{\max}} \) is a pole and branch point of \( \hat{\phi}_2^c(z) \), and this leads us to the power term \( k^{-\frac{1}{2}} \).

**Higher dimensional models:**

Consider a \( k \)-dimensional QBD process (kd-QBD process for short) and denote by \( \mathcal{A}_k \) the set of all the subsets of \( \{1, 2, \ldots, k\} \). In a manner similar to that used for the 2d-QBD process, we can define \( 2^k - 1 \) induced MA-processes derived from the kd-QBD process, \( \{Y_n^\alpha\} \), \( \alpha \in \mathcal{A}_k \setminus \emptyset \), and for each \( \{Y_n^\alpha\} \), we can also define the domain \( \Gamma^\alpha \) in which the convergence parameter of the matrix moment generating function for the MA-kernel of \( \{Y_n^\alpha\} \) is greater than 1. Let \( c \) be a \( k \)-dimensional vector in \( \mathbb{Z}^k_+ \).

From the results of the paper, it can be conjectured that the asymptotic decay rate of the stationary distribution in the direction \( c \) in the kd-QBD process is determined by the domains and a hyper plane with normal vector \( c \) that contacts one of the domains or the intersection of some of the domains. Which domains contribute to determine the asymptotic decay rate in the direction \( c \) probably depends on which boundaries affect the asymptotics in the same direction. Critical paths used in the large deviation techniques are expected to give a clue to clear this point.

The large deviation techniques are often used for investigating the asymptotics of the stationary distributions in Markov processes on the positive quadrant (see, for example, Miyazawa [19] and references therein). In analysis using them, the upper and lower bounds for the asymptotic decay rates are represented in terms of the large deviation rate function, and that rate function is given by the variational problem minimizing the total variances of the critical path. Set \( \mathbb{D}^{[1]} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0, x_2 = 0\} \), \( \mathbb{D}^{[2]} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 = 0, x_2 > 0\} \) and \( \mathbb{D}^{[1,2]} = \{(x_1, x_2) \in \mathbb{R}^2; x_1 > 0, x_2 > 0\} \). For two-dimensional reflected Brownian motion on the positive quadrant, the following three kinds of path of point \( p \) moving from the origin to a positive point \( p_0 \in \mathbb{D}^{[1,2]} \) are often used as options for the critical path (see, for example, Dai and Miyazawa [4]).
• Type-0 path: $p$ directly moves from the origin to $p_0$ through $D^{(1,2)}$.
• Type-1 path: First, $p$ moves from the origin to some point on $D^{(1)}$ through $D^{(1,2)}$, and then it moves to $p_0$ through $D^{(1,2)}$.
• Type-2 path: Replace $D^{(1)}$ with $D^{(2)}$ in the definition of Type-1 path.

In our analysis, the generating function $\phi^c(z)$ was divided into three parts: $\phi^c_0(z)$, $\phi^c_1(z)$ and $\phi^c_2(z)$, through compensation equation (2.8). In some sense, $\phi^c_0(z)$ evaluates Type-0 paths and “$\xi_c = \theta_{c,1}^+ < \theta_{c}^{max}$” corresponds to the case where the critical path is of Type-0; $\phi^c_1(z)$ evaluates Type-1 paths and “$\xi_c = \theta_{c,1}^+ < \theta_{c}^{max}$” corresponds to the case where the critical path is of Type-1; $\phi^c_2(z)$ evaluates Type-2 paths and “$\xi_c = \theta_{c,2}^+ < \theta_{c}^{max}$” corresponds to the case where the critical path is of Type-2. For related topics with respect to queueing networks, see Foley and McDonald [9], where paths called jitter, bridge and cascade ones are considered. Type-0 paths above correspond to jitter ones, and Type-1 and Type-2 paths to cascade ones.

This analogy gives us some insight to investigate the asymptotics of the stationary tail distribution in a higher-dimensional QBD process. In analysis of three-dimensional reflected Brownian motion in the octant, spiral paths are added to options for the critical path (see, for example, Liang and Hasenbein [15]). In a spiral path, the point $p$ moves along several boundaries. A way to handle such spiral paths in our framework seems crucial for analyzing the asymptotics of the stationary tail distribution in a three or more than three dimensional QBD process.

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A Asymptotic properties of the block state process

For $b = (b_1, b_2) \in \mathbb{N}^2$, let $\{bY_n\} = \{(bX_n, (bM_n, bJ_n))\}$ be the $b$-block state process derived from a 2d-QBD process $\{Y_n\} = \{(X_n, J_n)\}$, introduced in Sect. 2.4. Since the $b$-block state process is also a 2d-QBD process, we obtain by the results of Refs. [26, 27, 31] the following.

Define vector generating functions $b\nu_{(*)}(z)$ and $b\nu_{(0,*)}(z)$ as

$$b\nu_{(*)}(z) = \sum_{k=1}^{\infty} z^k b\nu_{(k,0)}, \quad b\nu_{(0,*)}(z) = \sum_{k=1}^{\infty} z^k b\nu_{(0,k)}.$$ 

Define a matrix function $bA_{*,*}^{[1,2]}(z_1, z_2)$ as

$$bA_{*,*}^{[1,2]}(z_1, z_2) = \sum_{i_1, i_2 \in \{-1, 0, 1\}} z_1^{i_1} z_2^{i_2} bA_{i_1,i_2}^{[1,2]},$$

and a domain $b\Gamma^{[1,2]}$ as

$$b\Gamma^{[1,2]} = \{(\theta_1, \theta_2) \in \mathbb{R}^2, \text{spr}\left(bA_{*,*}^{[1,2]}(e^{\theta_1}, e^{\theta_2})\right) < 1\}.$$
For $\{\theta_1, \theta_2\} \subseteq \Gamma_1^{[1,2]}$, we have counting multiplicity, where $b\eta_2(\theta_1) \leq b\bar{\eta}_2(\theta_1)$ (see Fig. 13). For $\theta_2 \in [b\theta_2^{\min}, b\theta_2^{\max}]$, $b\eta_1(\theta_2)$ and $b\bar{\eta}_1(\theta_2)$ are analogously defined. Hereafter, if $b = (1, 1)$, we omit the left superscript $b$; for example, $bA_{1,*,1}^{[1,2]}(e^{\theta_1}, e^{\theta_2})$ is denoted by $A_{1,*,1}^{[1,2]}(e^{\theta_1}, e^{\theta_2})$ and $b\Gamma^{[1,2]}$ by $\Gamma^{[1,2]}(A_{1,*,1}^{[1,2]}(e^{\theta_1}, e^{\theta_2})$ and $\Gamma^{[1,2]}$ have already been defined in Sect. 1). By Proposition 4.2 of Ozawa [31], we have

$$\text{spr}(bA_{1,*,1}^{[1,2]}(e^{\theta_1}, e^{\theta_2})) = 1.$$  \hspace{1cm} (A.3)

This implies that, for example, $b\theta_1^{\max} = b_1\theta_1^{\max}$ and $b\eta_2(b_1\theta_1) = b_2\eta_2(\theta_1)$.

For $i \in \{-1, 0, 1\}$ and $i' \in \{0, 1\}$, define matrix functions $bA_{i',*,*}^{[\varnothing]}(z)$, $bA_{i,*,*}^{[1]}(z)$, $bA_{i',*,*}^{[2]}(z)$ and $bA_{i,*,*}^{[1,2]}(z)$ as

$$bA_{i',*,*}^{[\varnothing]}(z) = bA_{i',0}^{[\varnothing]} + z bA_{i',1}^{[\varnothing]}, \quad bA_{i,*,*}^{[1]}(z) = bA_{i,0}^{[1]} + z bA_{i,1}^{[1]},$$

$$bA_{i',*,*}^{[2]}(z) = z^{-1} bA_{i',-1}^{[2]} + z bA_{i',1}^{[2]}, \quad bA_{i,*,*}^{[1,2]}(z) = z^{-1} bA_{i,-1}^{[1,2]} + bA_{i,0}^{[1,2]} + z bA_{i,1}^{[1,2]}.$$  

For $i \in \{-1, 0, 1\}$ and $i' \in \{0, 1\}$, analogously define matrix functions $bA_{i',*,*}^{[\varnothing]}(z)$, $bA_{i,*,*}^{[1]}(z)$, $bA_{i,*,*}^{[2]}(z)$ and $bA_{i,*,*}^{[1,2]}(z)$. For $z_1 \in [e^{b\theta_1^{\min}}, e^{b\theta_1^{\max}}]$ and $z_2 \in [e^{b\theta_2^{\min}}, e^{b\theta_2^{\max}}]$,
let $bG_1(z_1)$ and $bG_2(z_2)$ be the minimum nonnegative solutions to quadratic matrix equations (A.4) and (A.5), respectively:

$$
bA_{*,-1}^{[1,2]}(z_1) + bA_{*,0}^{[1,2]}(z_1)X + bA_{*,1}^{[1,2]}(z_1)X^2 = X, \quad (A.4)
$$

$$
bA_{-1,*}^{[1,2]}(z_2) + bA_{0,*}^{[1,2]}(z_2)X + bA_{1,*}^{[1,2]}(z_2)X^2 = X, \quad (A.5)
$$

where $bG_1(z_1)$ and $bG_2(z_2)$ are called G-matrices in the queueing theory. By Lemma 2.5 of Ozawa and Kobayashi [31], we have

$$\text{spr}(bG_1(e^{\theta_1})) = e^{b\eta_2(\theta_1)}, \quad \text{spr}(bG_2(e^{\theta_2})) = e^{b\eta_2(\theta_2)}. \quad (A.6)$$

Define matrix functions $bU_1(z_1)$ and $bU_2(z_2)$ as

$$bU_1(z_1) = bA_{*,0}^{[1]}(z_1) + bA_{*,1}^{[1]}(z_1)bG_1(z_1), \quad bU_2(z_2) = bA_{0,*}^{[2]}(z_2) + bA_{1,*}^{[2]}(z_2)bG_2(z_2),$$

and, for $i \in \{1, 2\}$, a real value $b\theta_i^*$ as

$$b\theta_i^* = \sup\{\theta \in [b\theta_i^{\min}, b\theta_i^{\max}]; \text{spr}(bU_i(e^{\theta})) < 1\}.$$

Define real values $b\theta_1^+$ and $b\theta_2^+$ as

$$b\theta_1^+ = \max\{\theta \in [b\theta_1^{\min}, b\theta_1^{\max}]; b\eta_2(\theta) \leq b\theta_2^*, \quad b\theta_2^+ = \max\{\theta \in [b\theta_2^{\min}, b\theta_2^{\max}]; b\eta_1(\theta) \leq b\theta_1^*\}.$$

Note that if $b = (1, 1)$, then, for $i \in \{1, 2\}$, inequality $\text{spr}(bU_i(e^{\theta})) = \text{spr}(U_i(e^{\theta})) < 1$ is equivalent to $\text{cp}(\bar{A}_e^{[i]}(e^{\theta})) > 1$ (for the definition of $\bar{A}_e^{[i]}(z)$, see Sect. 1). Hence, for $i \in \{1, 2\}$, $\Gamma^{[i]}$ defined in Sect. 1 satisfies

$$\Gamma^{[i]} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \text{spr}(U_i(e^{\theta_i})) < 1\}. \quad (A.7)$$

Furthermore, for $i \in \{1, 2\}$,

$$\theta_i^* = \sup\{\theta_i; (\theta_1, \theta_2) \in \Gamma^{[i]}\}, \quad \theta_i^+ = \sup\{\theta_i; (\theta_1, \theta_2) \in \Gamma^{[3-i]} \cap \Gamma^{[1,2]}\}. \quad (A.8)$$

By Lemma 2.6 of Ozawa and Kobayashi [27], we have the following.

**Lemma A.1** The asymptotic decay rates $b\xi_{(1,0)}$ and $b\xi_{(0,1)}$ are given by

$$b\xi_{(1,0)} = \min\{b\theta_1^*, b\theta_1^+, b\xi_{(1,0)} = \min\{b\theta_2^*, b\theta_2^+, (A.9)$$

By (A.3), we have

$$b\theta_1^* = b_1\theta_1^*, \quad b\theta_1^+ = b_1\theta_1^+, \quad b\xi_{(1,0)} = b_1\xi_{(1,0)}, \quad (A.10)$$

$$b\theta_2^* = b_2\theta_2^*, \quad b\theta_2^+ = b_2\theta_2^+, \quad b\xi_{(0,1)} = b_2\xi_{(0,1)}. \quad (A.11)$$
Proof of Proposition 2.3

By Proposition 3.3 of Ozawa and Kobayashi [27], \( b_{\psi(0,\ast)}(z) \) satisfies the following equation:

\[
\begin{align*}
    b_{\psi(0,\ast)}(z) &= \sum_{k=1}^{\infty} b_{\psi(k,0)} \sum_{i \in \{-1,0,1\}} (b_{A^{[1]}_{i,\ast}}(z) - b_{A^{[1,2]}_{i,\ast}}(z)) b_{G}(G(z)^{k+i} (I - bU_{2}(z))^{-1} \\
    &+ b_{\psi(0,0)} \sum_{i \in \{0,1\}} (b_{A^{[0]}_{i,\ast}}(z) - b_{A^{[2]}_{i,\ast}}(z)) b_{G}(G(z)^{i} (I - bU_{2}(z))^{-1}.
\end{align*}
\]

(A.12)

This is a kind of compensation equation. An equation similar to (A.12) also holds for \( b_{\psi(\ast,0)}(z) \).

B Proof of Proposition 2.3

Proof of Proposition 2.3 For a sequence \( \{a_{n}\}_{n \geq 1} \), we denote by \( \tilde{a}_{k} \) the partial sum of the sequence defined as \( \tilde{a}_{k} = \sum_{n=1}^{k} a_{n} \). Let \( c = (c_{1}, c_{2}) \) be a vector of positive integers. Let \( (x, j) \) and \( (x', j') \) be arbitrary states in \( \mathbb{N}^{2} \times S_{0} \) such that \( (x, j) \neq (x', j') \). Since the induced MA-process \( \{Y^{n}_{2}\} \) is irreducible, there exist \( k_{0} \geq 1 \), \( n_{0} \geq 1 \) and sequence \( \{(l_{n}, m_{n}, j_{n}) \in \{-1,0,1\}^{2} \times S_{0}; 1 \leq n \leq n_{0}\} \) such that \( x + k_{0}c + (\tilde{l}_{k}, \tilde{j}_{k}) > (0,0) \) and \( (x + k_{0}c + (\tilde{l}_{n_{0}}, \tilde{m}_{n_{0}}), j_{n_{0}}) \neq (x' + k_{0}c, j') \) for every integer \( k \in [1, n_{0} - 1] \), \( x + k_{0}c + (\tilde{l}_{n}, \tilde{m}_{n}), j_{n} = (x' + k_{0}c, j') \) and

\[
p^{*} = [A^{[1]}_{l_{1},m_{1}}]^{j_{1}j_{1}} \prod_{n=2}^{n_{0}-1} [A^{[1,2]}_{l_{n},m_{n}}]^{j_{n}j_{n}} [A^{[1,2]}_{j_{n_{0}},m_{n_{0}}}]^{j_{n_{0}}j_{n_{0}}}, j' > 0.
\]

Such a sequence gives a path from \( Y^{1,2}_{0} = (x + k_{0}c, j) \) to \( Y^{1,2}_{n_{0}} = (x' + k_{0}c, j') \) on \( \mathbb{N}^{2} \times S_{0} \), and that path is also a path from \( Y_{0} = (x + k_{0}c, j) \) to \( Y_{n_{0}} = (x' + k_{0}c, j') \) in the original 2d-QBD process \( \{Y_{n}\} \). For \( k \geq 1 \), let \( \tau(k) \) be the first hitting time to the state \( (x + kc, j) \) in \( \{Y_{n}\} \), i.e., \( \tau(k) = \inf \{n \geq 1; Y_{n} = (x + kc, j)\} \), and denote by \( q^{(k)}_{(x'', j'')} \); \( (x'', j'') \in \mathbb{Z}^{2} \times S_{0} \) the occupation measure defined as

\[
q^{(k)}_{(x'', j'')} = \mathbb{E} \left( \sum_{n=0}^{\tau(k)-1} 1(Y_{n} = (x'', j'')) \mid Y_{0} = (x + kc, j) \right).
\]

Then, we have

\[
v_{(x' + kc, j')} = q^{(k)}_{(x' + kc, j')} v_{(x + kc, j)}.
\]

Due to the space homogeneity of \( \{Y^{1,2}_{n}\} \) with respect to the additive part, for every \( k \geq k_{0} \), there exists a path from \( Y^{1,2}_{0} = (x + kc, j) \) to \( Y^{1,2}_{n_{0}} = (x' + kc, j') \) given by the same sequence as \( \{(l_{n}, m_{n}, j_{n}) \in \{-1,0,1\}^{2} \times S_{0}; 1 \leq n \leq n_{0}\} \) mentioned
above, and it is also a path from $Y_0 = (x + kc, j)$ to $Y_{n_0} = (x' + kc, j')$ in the original 2d-QBD process. Hence, we have $q_{(x'+kc,j')}^{(k)} \geq p^*$ and obtain

$$v_{(x'+kc,j')} \geq p^* v_{(x+kc,j)},$$

(B2)

where $p^*$ does not depend on $k$. This leads us to $\xi_c(x', j') \leq \xi_c(x, j)$ and $\bar{\xi}_c(x', j') \leq \bar{\xi}_c(x, j)$. Interchanging $(x, j)$ with $(x', j')$, we analogously obtain $\xi_c(x, j) \leq \xi_c(x', j')$ and $\bar{\xi}_c(x, j) \leq \bar{\xi}_c(x', j')$. This completes the proof.□

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