Decomposition-Coordination Method for Finite Horizon Bandit Problems

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We consider a peer-to-peer microgrid where houses exchange energy, and we formulate it as a large-scale stochastic optimization problem.

How to manage such network in an (almost) optimal way?
Mix of spatial and temporal decompositions

Figure: The case of price decomposition
Increase in execution time with state dimension
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Multistage stochastic optimal control formulation

- Let $T \geq 1$ be an integer (finite) representing the horizon

- At each discrete time stage $t \in [0, T-1]$, a decision-maker (DM) makes a decision and gets a reward as follows
  - At the beginning of the time interval $[t, t+1[$, the DM selects an arm $a \in A$ (finite set)
  - At the end of the time interval $[t, t+1[,$ the arm $a$ delivers a random variable $W^a_{t+1} \in \{B, G\}$, (“bad” B, “good” G)

- The corresponding probabilities are unknown to the DM

$$\rho^a = (\rho^B, \rho^G) = \left(\mathbb{P}\{W^a_{t+1} = B\}, \mathbb{P}\{W^a_{t+1} = G\}\right) \in \Sigma$$

where $\Sigma = \{p = (p^B, p^G) \in \mathbb{R}_+^2 \mid p^B + p^G = 1\}$ is the one-dimensional simplex

- We suppose that the DM holds a prior beta distribution $\pi^a_0 = \beta(n^B, n^G)$ over the unknown $\rho^a = (\rho^B, \rho^G) \in \Sigma$
Decision model for arm selection

We consider a sequence \( U = \{U_t\}_{t \in [0, T-1]} \) of r.v on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where

\[
U_t = \{U^a_t\}_{a \in A}, \ \forall t \in [0, T-1]
\]

\[
U^a_t \in \{0, 1\}, \ \forall a \in A, \ \forall t \in [0, T-1]
\]

Values \( U^a_t \in \{0, 1\} \) represent that, at the beginning of the time interval \([t, t+1]\),

- either arm \( a \) has been selected \((U^a_t = 1)\)
- or arm \( a \) has not been selected \((U^a_t = 0)\)

Since, at each given time, one and only one arm has to be selected, we add the constraint

\[
\sum_{a \in A} U^a_t = 1, \ \forall t \in [0, T-1]
\]

This way of modeling the selection of a unique arm is not the most common in the bandit literature.
Multistage stochastic optimal problem

We formulate a maximization problem
\[ V_0(\pi_0) = V_0((n_0^{Ba})_{a \in A}, (n_0^{Ga})_{a \in A}) = \sup \left[ \int_{\Delta(S)^A} \prod_{a \in A} \pi_0^a(d\rho^a) \right] \]

subject to constraints \( \forall t \in [0, T-1] \)

\[ \sum_{a \in A} U_t^a = 1 \quad \text{(only one arm is selected)} \]

The supremum is taken over \( U = \{U_t^a\}_{a \in A, t \in [0, T-1]} \)

\[ \sigma(U_t) \subset \sigma(U_0, \{U_0^aW_1^a\}_{a \in A}, \ldots, U_{t-1}, \{U_{t-1}^aW_t^a\}_{a \in A}) \quad \text{(information)} \]
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- Multistage stochastic optimal control formulation
- Dynamic programming and arm decomposition

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Dynamic programming and arm decomposition (1/2)

By weak duality — for the the coupling constraint $\sum_{a \in A} U_t^a = 1$ with a deterministic multiplier $\mu_t$ — we obtain the upper bound

$$V_0((n_{Ba}^a)_{a \in A}, (n_{Ga}^a)_{a \in A}) \leq \inf_{\mu \in \mathbb{R}^T} \left( \sum_{a \in A} \underbrace{V_0^a[\mu](n_{Ba}^a, n_{Ga}^a)}_{\text{arm } a \text{ value function}} + \sum_{t=0}^{T-1} \mu_t \right)$$

where, for any vector $\mu = \{\mu_t\}_{t \in [0, T-1]} \in \mathbb{R}^T$ of multipliers,

$$V_T^a[\mu](n_{Ba}, n_{Ga}) = 0, \quad \forall (n_{Ba}, n_{Ga}) \in \mathbb{N} \times \mathbb{N}$$

$$V_t^a[\mu](n_{Ba}, n_{Ga}) = \max \left\{ V_{t+1}^a[\mu](n_{Ba}, n_{Ga}), -\mu_t \right\}$$

$$+ \frac{n_{Ba}}{n_{Ba} + n_{Ga}} (L_t^a(B) + V_{t+1}^a[\mu](n_{Ba} + 1, n_{Ga}))$$

$$+ \frac{n_{Ga}}{n_{Ba} + n_{Ga}} (L_t^a(G) + V_{t+1}^a[\mu](n_{Ba}, n_{Ga} + 1))$$
Dynamic programming and arm decomposition (2/2)

The global stochastic optimal control problem $V_0(\pi_0)$ is, theoretically, solvable by dynamic programming using value functions $\{V_t\}_{t \in [0, T]} : \prod_{a \in A} \Delta(\Sigma) \rightarrow \mathbb{R} \cup \{+\infty\}$.

However, computing $V_0(\pi_0)$ using Dynamic Programming faces the curse of dimensionality, as the priors are of the form $\pi_0 = \{\pi^a_0\}_{a \in A} \in \prod_{a \in A} \Delta(\Sigma)$.

The DeCo algorithm consists in replacing

$$V_{t+1}(\{\pi^a_{t+1}\}_{a \in A}) \leadsto \sum_{a \in A} V^a_{t+1}[\mu](\pi^a_{t+1})$$

for a suitable vector $\mu \in \mathbb{R}^T$ in order compute a policy by

$$\mathcal{U}_t(\pi_t) \in \arg \max_{\substack{\pi_t = \{u^a_t\}_{a \in A} \in \{0,1\}^A \\ \sum_{a \in A} u^a_t = 1}} \left(\tilde{L}_t(\pi_t, u_t) \right)$$

$$= \sum_{a \in A} V^a_{t+1}[\mu](\pi^a_{t+1}) + \int_{\Delta(\Sigma)} \left(V_{t+1}(\pi_{t+1}) k_t(\pi_{t+1} | \pi_t, u_t) \right) \, d\pi_{t+1}.$$
The **DeCo** algorithm as a nonstationary index policy

For a suitable value of $\mu$, when the state of the multi-armed bandit is given by $(n^{Ba}, n^{Ga})_{a \in A}$ at stage $t$, the **DeCo** algorithm selects an arm

\[
A_t^*[\mu](\{(n^{Ba}, n^{Ga})\}_{a \in A}) \in \operatorname{arg\,max}_{a \in A} \left[ \begin{array}{c}
\frac{n^{Ba}}{n^{Ba} + n^{Ga}} L_t^a(B) + \frac{n^{Ga}}{n^{Ba} + n^{Ga}} L_t^a(G) \\
+ \frac{n^{Ba}}{n^{Ba} + n^{Ga}} V_{t+1}^a[\mu](n^{Ba} + 1, n^{Ga}) + \\
\frac{n^{Ga}}{n^{Ba} + n^{Ga}} V_{t+1}^a[\mu](n^{Ba}, n^{Ga} + 1) \\
- V_{t+1}^a[\mu](n^{Ba}, n^{Ga}) \end{array} \right]
\]
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Numerical experiments (small number $|A|$ of arms)

The **DeCo** algorithm compared to brute force **BF** algorithm (global DP) for small number $|A|$ of arms and short horizon $T$

| arms $|A|$ | horizon $T$ | DeCo | BF |
|--------|-------------|------|----|
| 3      | 10          | 6.411| 6.409 |
| 3      | 20          | 13.458| 13.465 |
| 5      | 10          | 6.645| 6.659 |

Comparison in term of estimated total expected reward (higher is better)
Numerical experiments (small number $|A|$ of arms)

The DeCo algorithm compared to BF algorithm and others for $|A| = 2$ arms and horizon $T$ up to 100

Comparison in term of (to be defined later) Expected Bayesian Regret (lower is better)
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Numerical experiments for larger number $|A|$ of arms

$BF$ cannot be used anymore because of the curse of dimensionality. $FH$-Gittins is used as a proxy supposed to be close to the optimal solution.

| arms $|A|$ | horizon $T$ | DeCo | FH-Gittins |
|------|---------|------|-----------|
| 5    | 20      | 14.21| 14.28     |
| 5    | 40      | 29.85| 30.06     |
| 15   | 20      | 14.59| 14.67     |
| 15   | 40      | 31.54| 31.63     |

Comparison in term of estimated total expected reward (higher is better). The performance of DeCo is close to the optimal solution while keeping the computational cost reasonable.
Numerical experiments: comparison with other methods

▶ We then tested DeCo against

▶ Thomson Sampling (Ts) [10, 2]
▶ Kullback-Leibler upper-confidence bound (KL-UCB) [3]
▶ Information-Directed Sampling (IDS) [9]
▶ Finite Horizon Gittins index (FH-GITTINS) [6, 8, 7]
▶ In the case of two arms, exact DP

▶ The solutions $U = \{U_t^a\}_{a \in A, t \in [0, T-1]}$ are compared using the Expected Bayesian Regret given by

$$R(U) = \int_{\Delta(\Sigma)^A} \prod_{a \in A} \pi_0^a(\text{d}p^a) \left\{ \mathbb{E}_{\{p^a\}_{a \in A}} \left[ \sum_{t=0}^{T-1} \sum_{a \in A} (U_{t,B_A}^a - U_t^a) W_{t+1}^a \right] \right\}$$

▶ $L_t^a$ equal to 1 on $G$ and 0 on $B$
▶ $B_A$: best arm policy is, for all $a \in A$, given by

$$U_{t,B_A}^a = 1 \iff a \in \text{arg max}_{a' \in A} p_{G}^{a'}$$
▶ the prior is supposed to be the uniform distribution for all arms

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3For IDS we used the library [1]
Numerical experiments: comparison with other methods

The two arms case where brute force algorithm can be used

Comparison in term of Expected Bayesian Regret (lower is better)
Numerical experiments: comparison with other methods

Increasing the number of arms

(a) 5 arms

(b) 10 arms

(c) 20 arms

On all cases, DECO

- beats both TS and KL-U�CB with a comfortable margin
- and is comparable to IDS
Numerical experiments: comparison with other methods

- On all cases, DeCo beats both Ts and Kl-Ucb with a comfortable margin, and is comparable to IDS
- For the two arms case DeCo is very close to the optimal solution, computed by DP (we used the Julia BinaryBandit library)
- Expected Bayesian Regret is numerically obtained by Monte Carlo simulations
  - Expectation with respect to the prior: a sample of size 1000
  - Expectation with respect to the arms parameters: a sample of size 1000 or of size 100 (for large $T$)
  - Same samples for all the evaluated policies
Numerical experiments: $\text{DECO}$ provides a lower bound

- $R^L$: lower bound provided by $\text{DECO}$ (using the dual bound)

$$R(U) \geq R^L = \frac{|A|}{|A| + 1} T - \left( \sum_{a \in A} V_0^a[\mu^*](\pi_0^a) + \sum_{t=0}^{T-1} \mu_t^* \right)$$

- $R^L$, $\text{DECO}$, $T_s$ and $\text{KL-UCB}$ regret as a function of the number of arms for $T = 100$ and $T = 500$

The lower bound is of no use (lower than 0) for $|A| \leq 5$

When $|A| \uparrow$ the regrets of $\text{DECO}$ and $R^L$ become quite close, which indicates that $\text{DECO}$ is close to being optimal.
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Conclusion (pros)

- The numerical results illustrate the value of the decomposition-coordination approach (observed in other applications): DeCo is a simple algorithm and its performances are close to the optimal Bayesian solution for several configurations of arms and horizons, while keeping the computing time reasonable.
- Empirically, DeCo offers performances comparable to FH-Gittins but with a much smaller computation burden.
- DeCo can deal with time varying reward functions, and can even include a final reward.
- In particular, DeCo can be applied to nonstationary settings, whereas FH-Gittins cannot.
As of now, the approach main limitation is that the horizon $T$ is supposed to be known in advance and to be reasonably small, whereas many multi-armed bandit algorithms do not require $T$ as an input.

In addition, the usage of dynamic programming might make DECO too burdensome for some applications with long horizon $T$.

Also, since the DECO algorithm requires a Bayesian prior, the question of the impact of a wrong prior on the performance is left open.
Conclusion (perspectives)

- We could explore the possibility to adapt the multiplier $\mu$ as time goes on and we receive bandit feedback.
- Further works include:
  - A theoretical analysis of the DECO policy.
  - An extension to the discounted infinite horizon case.
  - As well as adapting the heuristic to other use cases.
References

[1] D. Baudry, Y. Russac, and A. Filiot. Information-directed sampling. https://github.com/DBaudry/, 2019.

[2] O. Chapelle and L. Li. An empirical evaluation of thompson sampling. In J. Shawe-Taylor, R. Zemel, P. Bartlett, F. Pereira, and K. Q. Weinberger, editors, Advances in Neural Information Processing Systems, volume 24. Curran Associates, Inc., 2011.

[3] A. Garivier and O. Cappé. The KL-UCB algorithm for bounded stochastic bandits and beyond. In Proceedings of the 24th annual conference on learning theory, pages 359–376. JMLR Workshop and Conference Proceedings, 2011.

[4] J. C. Gilbert and X. Jonsson. LIBOPT – An environment for testing solvers on heterogeneous collections of problems, 2007.

[5] J. Gittins and Y.-G. Wang. The Learning Component of Dynamic Allocation Indices. The Annals of Statistics, 20(3):1625 – 1636, 1992.

[6] E. Kaufmann. On Bayesian index policies for sequential resource allocation. The Annals of Statistics, 46(2):842 – 865, 2018.

[7] T. Lattimore. Regret analysis of the finite-horizon gittins index strategy for multi-armed bandits. In Conference on Learning Theory, pages 1214–1245. PMLR, 2016.

[8] J. Nino-Mora. Computing a classic index for finite-horizon bandits. INFORMS Journal on Computing, 23(2):254–267, 2011.

[9] D. Russo and B. Van Roy. Learning to optimize via posterior sampling. Mathematics of Operations Research, 39(4):1221–1243, 2014.

[10] W. R. Thompson. On the likelihood that one unknown probability exceeds another in view of the evidence of two samples. Biometrika, 25(3/4):285–294, 1933.

[11] R. Weber. On the Gittins Index for Multiarmed Bandits. The Annals of Applied Probability, 2(4):1024 – 1033, 1992.
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The **DeCo** algorithm

- **DeCo** (decomposition-coordination algorithm)
- Stands for the decentralized control policy obtained by arm decomposition
- By contrast with the (brute force) dynamic programming solution (BF), we have to solve Bellman equations for each arm,
  \[\Rightarrow\text{ dynamic programming with state of dimension 2 no matter the number of arms}\]
- The **DeCo** algorithm is made of
  - an offline computation phase
  - an online computation phase
Offline phase of the DECo algorithm

Minimization of the dual function

\[
\varphi(\mu) = \left( \sum_{a \in A} V_0^a[\mu](\pi_0^a) + \sum_{t=0}^{T-1} \mu_t \right)
\]

for a given family \( \left\{ \pi_0^a \right\}_{a \in A} = \left\{ \beta(n_0^B, n_0^G) \right\}_{a \in A} \) of beta priors.

\[
\mu_t^{(k+1)} = \mu_t^{(k)} - \rho_t \Delta_t^{(k)}
\]

Multiplier \( (\mu_t^{(k)})_{t \in [0, T-1]} \)

Arm 1, compute \( V_1^t[\mu^k] \)

\[
\Delta_t^{(k)} = \mathbb{E} \left[ \sum_{a=1}^{A} U_t^{a,(k)}(\cdot) - 1 \right], \quad \forall t \in [0, T-1]
\]

Monte Carlo estimation

Arm A compute \( V_A^t[\mu^k] \)
Offline phase of the DECo algorithm (continued)

1. Choose an initial vector $\mu^{(0)} \in \mathbb{R}^T$ of multipliers.
2. Iteration $k$, given multipliers $\mu^{(k)} \in \mathbb{R}^T$, compute the Bellman functions \( \{ V_t^a[\mu^{(k)}] \} \) \( t \in [0, T], a \in A \) and optimal controls.
   - The computation is performed in parallel, arm per arm.
   - $V_t^a[\mu^{(k)}]$ is to be evaluated only on the finite grid \( \{ (n_{0B} + n_{Ba}, n_{0G} + n_{Ga}) \mid n_{Ba} + n_{Ga} \leq t \} \).
   - If all the arms share the same prior and instantaneous reward, then all the arms share the same sequence of Bellman value functions.
Offline phase of the \texttt{DECO} algorithm (continued)

3. Once gotten \( \{ V_t^a[\mu^{(k)}] \}_{a \in A} \) at time \( t = 0 \) and iteration \( k \)
   - update the multipliers by a gradient step to obtain \( \mu^{(k+1)} \)
   - The gradient of the dual function \( \varphi \) with respect to the multipliers is obtained by computing the expectation of the dualized constraint.
   - Numerically, the expectation is obtained by Monte Carlo simulations.
   - The gradient phase can be replaced by a more sophisticated algorithm such as the conjugate gradient or the quasi-Newton method.
   - In some of our numerical experiments, we use a solver (limited memory \texttt{BFGS}) of the \texttt{MODULOPT} library from \texttt{INRIA} [4]. To obtain a global \( O(T^3) \) running time, the computing budget allocated to this iterated gradient phase does not depend on \( T \).

4. Stop the iterations (stopping criterion) or go back to 2 with multiplier \( \mu^{(k+1)} \).
The global stochastic optimal control problem is, theoretically, solvable by dynamic programming.

Using the Bellman value functions \( \{ V_t \}_{t \in [0, T]} \), an optimal policy would be given by the feedback (where \( \pi_t = \{ \pi^a_t \}_{a \in A} = \{ \beta(n^B_t, n^G_t) \}_{a \in A} \))

\[
U_t(\pi_t) \in \arg \max_{u_t=\{u^a_t\}_{a \in A} \in \{0,1\}^A} \left( \tilde{L}_t(\pi_t, u_t) + \int_{\Delta(\Sigma)} V_{t+1}(\pi_{t+1}) k_t(d\pi_{t+1} | \pi_t, u_t) \right)
\]

The DeCo algorithm consists in replacing the Bellman value function \( V_{t+1} \) by \( \sum_{a \in A} V^a_{t+1}[\mu] \), using the collection \( \{ V^a_{t+1}[\mu] \}_{a \in A} \), of Bellman value functions given by the offline phase and a suitable vector \( \mu \in \mathbb{R}^T \).
Online phase of the DECO algorithm (continued)

We obtain the following policy: when the state of the multi-armed bandit is given by \((n^Ba, n^Ga)_{a \in A}\) at time \(t\), the DECO algorithm selects an arm \(A^*_t[\mu]\left(\{(n^Ba, n^Ga)\}_{a \in A}\right)\) in

\[
\arg\max_{a \in A} \left[ -V^a_{t+1}[\mu](n^Ba, n^Ga) + \frac{n^Ba}{n^Ba + n^Ga}(L^a(B) + V^a_{t+1}[\mu](n^Ba+1, n^Ga)) + \frac{n^Ga}{n^Ba + n^Ga}(L^a(G) + V^a_{t+1}[\mu](n^Ba, n^Ga+1)) \right]
\]

This is a nonstationary index policy

The DECO policy used in numerical experiments is the policy \(A^*[\mu^*]\), where \(\mu^*\) is given by the offline phase of the DECO algorithm.
Interpretation

- The index in DECO is the sum of an exploration term and an exploitation term.

- We define the value of the information to be gained from pulling arm $a$ at time $t$ as

$$
\delta^a_t[\mu](n^{Ba}, n^{Ga}) = \frac{n^{Ba}}{n^{Ba} + n^{Ga}} V^a_{t+1}[\mu](n^{Ba} + 1, n^{Ga})
+ \frac{n^{Ga}}{n^{Ba} + n^{Ga}} V^a_{t+1}[\mu](n^{Ba}, n^{Ga} + 1) - V^a_{t+1}[\mu](n^{Ba}, n^{Ga})
$$

- Using $\delta^a_t[\mu](n^{Ba}, n^{Ga})$, we can write

$$
V^a_t[\mu](n^{Ba}, n^{Ga}) = V^a_{t+1}[\mu](n^{Ba}, n^{Ga})
+ \left(\delta^a_t[\mu](n^{Ba}, n^{Ga}) + \frac{n^{Ba}}{n^{Ba} + n^{Ga}} L^a_t(B) + \frac{n^{Ga}}{n^{Ba} + n^{Ga}} L^a_t(G) - \mu_t\right)^+
$$

- The arm is pulled in the decomposed problem only if the sum of the information gain ($\delta_t$) and the expected reward is greater than $\mu_t$. 
Interpretation (continued)

- \( \mu_t \) interpreted as an equilibrium price of a “bandit market”
- Each bandit is handled by an independent profit maximizing agent, which is required to pay the market price \( \mu_t \) to pull the arm of its bandit at time \( t \)
- This is different but connected to the fair charge metaphor proposed in [11] for the Gittins index. Here the price depends on a market made of several arms, whereas for the Gittins index, the fair charge is arm specific.
- Last, the selected arm (in online phase) is the one maximizing

\[
\delta_t^a[\mu](n^{B_a}, n^{G_a}) + \frac{n^{B_a}}{n^{B_a} + n^{G_a}} L_t^a(B) + \frac{n^{G_a}}{n^{B_a} + n^{G_a}} L_t^a(G)
\]

- Such exploration term is reminiscent of the exploration term encountered in UCB. Also, [5] refers to a learning component in the Gittins index as the difference between the index value and the immediate expected reward. More recently, the notion of information gain is also important in [9].
Computational complexity

- Solving the global maximization problem by DP is only possible for $|A|$ small and $T$ small computational cost $O((2|A|)^T)$
- FH-Gittins: time complexity $O(T^6)$
- DeCo: DP phase cost running time $O(T^3)$
  - indeed for each time $t \in [1, T]$, we need a grid of $T \times T$ for the 2 dimensional prior parameter (number of successes and failures)
- In the experiment, we fixed the number of gradient calls, so that the overall computing cost was $O(T^3)$ in time
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Probabilistic model

Let $\Sigma = \{p = (p_B, p_G) \in \mathbb{R}_+^2 \mid p_B + p_G = 1\}$ be the one-dimensional simplex.

For any $p = (p_B, p_G) \in \Sigma$, we consider on the space $\{B, G\}^T$ the probability $\mathcal{B}(p_B, p_G) = \bigotimes_{t=1}^T (p_B \delta_B + p_G \delta_G)$ — probability law of a sequence of independent (Bernoulli) random variables with values in $\{B, G\}$.

For $\{p_a\}_{a \in A} = \{(p_{Ba}, p_{Ga})\}_{a \in A} \in \prod_{a \in A} \Sigma$, we consider the probability $\bigotimes_{a \in A} \mathcal{B}(p_{Ba}, p_{Ga})$ on the product space $\prod_{a \in A} \{B, G\}^T$ — which corresponds to independence between arms in $A$.

We denote by $\mathbb{E}\{p^a\}_{a \in A}$ the corresponding mathematical expectation.

We suppose that the DM holds a prior $\pi_0^a$ over the unknown $p^a = (p_{Ba}, p_{Ga}) \in \Sigma$, for every arm $a \in A$.

In practice, we consider a beta distribution $\beta(n^B, n^G)$ on $\Sigma$, with positive integers $n^B > 0$ and $n^G > 0$ as parameters.
Probabilistic model (continued)

We consider the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) where

\[ \Omega = \prod_{a \in A} \Sigma \times \{B, G\}^T, \]

\[ \mathcal{F} = 2^\Omega, \]

\[ \mathbb{P} = \bigotimes_{a \in A} \pi_0^a\left(\text{d}(p^B_a, p^G_a)\right) \otimes \mathcal{B}(p^B_a, p^G_a). \]

Then, \(W^a = \{W^a_t\}_{t \in [1, T]}\) denotes the coordinate mappings for every arm \(a \in A\), with \(W^a_t\) a random variable having values in the set \{B, G\}.

For a given family \(\{(\bar{\rho}_B^a, \bar{\rho}_G^a)\}_{a \in A} \in \prod_{a \in A} \Sigma\) and for \(\pi_0^a = \delta(\bar{\rho}_B^a, \bar{\rho}_G^a)\), for every arm \(a \in A\), the family \(\{W^a_t\}_{a \in A, t \in [1, T]}\) consists of independent random variables, where \(W^a_t\) has (Bernouilli) probability distribution with parameter \(\bar{\rho}_G^a \in [0, 1]\), that is, \(\mathbb{P}(W^a_t = B) = 1 - \bar{\rho}_G^a\) and \(\mathbb{P}(W^a_t = G) = \bar{\rho}_G^a\). With this probabilistic model, we represent the sequential independent outcomes of \(|A|\) independent arms.
Information and admissible controls

- The DM observes the random variable
  \( Y_{t+1} = \{ U_t^a W_{t+1}^a \}_{a \in A} \), \( \forall t \in \mathbb{[}0, T - 1\mathbb{]} \)
  - When the arm \( a \) has been selected at stage \( t \) (\( U_t^a = 1 \)),
    the DM observes the outcome of the r.v. \( W_{t+1}^a \in \{ B, G \} \).
  - When the arm \( a \) has not been selected at stage \( t \) (\( U_t^a = 0 \)),
    the DM observes nothing.

- The admissible controls \( U = \{ U_t \}_{t \in \mathbb{[}0, T - 1\mathbb{]} } \) are those that satisfy
  \[
  \sigma(U_t) \subset \sigma(Y_0, U_0, Y_1, \ldots, U_{t-1}, Y_t), \ \forall t \in \mathbb{[}0, T - 1\mathbb{]} ,
  \]
  where \( \sigma(Z) \subset \mathcal{F} \) is the \( \sigma \)-field generated by the random variable \( Z \) on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).
Random rewards

- We consider given a family \( \{L^a_t\}_{a \in A, t \in [0, T-1]} \) of instantaneous reward functions \( L^a_t : \{B, G\} \rightarrow \mathbb{R} \),

- The total random reward associated with the control \( U = \{U_t\}_{t \in [0, T-1]} \) is given by

\[
\sum_{t=0}^{T-1} \sum_{a \in A} U^a_t L^a_t(W^a_{t+1})
\]

- When the arm \( a \) has been selected at stage \( t \) \( (U^a_t = 1) \), the r.v. \( W^a_{t+1} \) materializes and the DM receives the payoff \( 1 \times L^a_t(W^a_{t+1}) = U^a_t L^a_t(W^a_{t+1}) \).

- When the arm \( a \) has not been selected at stage \( t \) \( (U^a_t = 0) \), the DM receives the payoff \( 0 = U^a_t L^a_t(W^a_{t+1}) \).
Optimality criteria in the Bayesian framework

Let \( \pi_0 = \{ \pi_0^a \}_{a \in A} \in \prod_{a \in A} \Delta(\Sigma) \) be the family of initial priors.

\( \Delta(\Sigma) \): set of probability distributions on \( \Sigma \).

We formulate a maximization problem

\[
V_0(\pi_0) = \sup \int_{\Delta(\Sigma)^A} \prod_{a \in A} \pi_0^a(\cdot dp^a) \mathbb{E}\{p^a\}_{a \in A} \left[ \sum_{t=0}^{T-1} \sum_{a \in A} U_t^a L_t^a(W_{t+1}^a) \right]
\]

The supremum is taken over \( U = \{ U_t^a \}_{a \in A, t \in [0, T-1]} \) subject to constraints

\[
\sum_{a \in A} U_t^a = 1 \ , \ \forall t \in [0, T - 1]
\]

\[
\sigma(U_t) \subset \sigma(Y_0, U_0, Y_1, \ldots, U_{t-1}, Y_t) \ , \ \forall t \in [0, T - 1]
\]