Control of a HVAC unit via PDE-Constrained Optimization

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Abstract—Efficiency, comfort, and convenience are three major aspects in the design of control systems for residential Heating, Ventilation, and Air Conditioning (HVAC) units. In this paper we propose an optimization-based algorithm for HVAC control that minimizes the energy consumption while maintaining a desired temperature in a room. Our algorithm uses a Computer Fluid Dynamics model, mathematically formulated using Partial Differential Equations (PDEs), to describe the interactions between temperature, pressure, and air flow. Our model allows us to naturally formulate problems such as controlling the temperature of a small region of interest within a room, or to control the direction of the air flow at the vents at the same time we control its speed, which are hard to describe using finite-dimensional ODE models. Our results show that our algorithm produces significant energy savings without a decrease of comfort.

I. INTRODUCTION

Heating, Ventilation, and Air Conditioning (HVAC) units in buildings are complex, highly optimized, systems that control the climate of all kinds of buildings, large and small, residential and commercial. Yet, there is a wide gap between the information HVAC units need to make optimal decisions, such as the temperature and air flow in every point in a building, and the information HVAC units have access to, which is the temperature captured by a handful of thermostats (i.e., the temperature in a handful of discrete points). Also, the problem of controlling climate variables using HVAC units is seriously underactuated, with large sets of configurations inaccessible even for the most advanced commercially available systems.

In this paper we aim to experimentally investigate, using simulations, how the energy efficiency of HVAC units is improved when, first, we use a Computer Fluid Dynamic (CFD) model based on Partial Differential Equations (PDEs) to estimate the distributed values of the climate variables (temperature, pressure, and air flow), and second, we pair the results from the CFD model together with improvements in the actuation of the vents in the building (e.g., by controlling them independently, or by controlling the angle of the air flow at the vents). Our results follow from the formulation of a PDE-constrained optimal control problem that considers the CFD model and our desired goals (i.e., regulate the temperature in the building), which we numerically solve after using state-of-the-art discretization methods.

CFD models rely on the Navier-Stokes set of nonlinear PDEs, enough theoretical analysis has been done to guarantee the existence of solutions for laminar dynamics with non-turbulent Reynolds Numbers and sufficiently smooth initial conditions [1], [2]. Other authors, such as Neitzel et al. [3] and Petzold et al. [4], have considered PDE-constrained optimal control problems, yet their algorithms are not general enough to consider our CFD models.

The dynamic behavior of the air flow is complex due to turbulent dynamical responses. Nonetheless, even if the flow is turbulent inside air ducts at reasonable ventilation rates, it is laminar in larger areas [5], [6], thus it can be analyzed using simpler non-turbulent CFD models. Many results can be found in the literature using different variations of CFD models to study heating and ventilation situations in buildings, such as the papers by Bathe et al. [7], Sinha et al. [5], van Schijndel [8], Waring and Siegel [9], and the book by Awbi [10]. Well-established numerical solvers are used by researchers to find solutions of CFD models (e.g., COMSOL Multiphysics [8]). Yet, these solvers are not suitable for integration with gradient-based optimization algorithms, which require the explicit formulation of all the approximating equations and their gradients. It is for this reason that we formulated our own numerical discretization of the CFD model using the Finite Elements Method (FEM) and mixed Dirichlet and Neumann boundary conditions.

On the other hand, many results exist in the control of HVAC units using optimization-based methods. Goyal and Barooah [11] studied in detail the use of RC network circuits to model the temperature within buildings. Kelman and Borelli [12], as well as Hazyuk et al. [13], [14], used low-dimension ODE-based models to control a HVAC unit using Model Predictive Control (MPC). Aswani et al. [15] used a learning-based MPC algorithm [16] to account for unmodeled dynamics and disturbance in ODE models when controlling HVAC units. Domahidi et al. [17] and Fux et al. [18] also used a learning-based method and MPC, the first using ADABOOST to estimate uncertainties and the second using an Extended Kalman Filter. Ma et al. [19] used Stochastic MPC to handle disturbances in the control of HVAC units, also using ODE models. Our main contribution in this paper is that we use CFD models and numerical optimal control techniques to pose problems that ODE models cannot describe, such as the effect that the geometry of HVAC vents have in the energy efficiency, or how HVAC control techniques can be improved with information regarding the positions of the occupants in the building.

The paper is organized as follows: Section II contains the description of the CFD model and the formulation of the optimal control problem; Section III contains the details regarding the discretization of the PDEs forming the CFD model, and the discretization of the optimal control problem;
and Section [IV] contains the results of our simulated experiments after we incorporate small changes in the actuators of the HVAC unit, such as independent control of each vent in a room or the control of the angle of the air flow in a vent. Our results validate our hypothesis that sizable energy savings can be obtained by introducing small improvements in the actuation of HVAC units, thanks to the use of accurate CFD models to describe the dynamical and distributed behavior of the climate variables in a building.

II. PROBLEM DESCRIPTION

A commonly missing key feature in many physical climate models used to control HVAC systems is the ability to capture the real-time spatial variability of the temperature and air flow, depending on the floor plan and configuration of the building (e.g., open or closed door and windows). For this reason, we use a Computer Fluid Dynamic (CFD) model, which explicitly considers temporal and spatial variations, to describe the interactions between the temperature, air flow, and pressure. We then formulate an optimal control problem where our CFD model appears as a constraint, and whose objective function aims to minimize the energy consumption of the HVAC system while maintaining the temperature constant at a desired reference. The inclusion of spatial variation in our description of the climate variables will not only improve the accuracy of our estimations, but it will also allow us to naturally formulate richer problems, such as only focusing on a specific region in a room, as shown in Section [IV].

In the remainder of this section we introduce in detail the CFD model and the optimal control problem.

A. CFD Model

The foundation of our model is the Navier-Stokes equation, which couples temperature with free flow convection (as explained Section 8 in [10], among other references). As shown in the literature, atmospheric air can be modeled as an incompressible Newtonian fluid when the temperature is between $-20^\circ C$ and $100^\circ C$ [20], [21]. Hence, we can use the Navier-Stokes equation for incompressible laminar flows, together with the convection-diffusion temperature model for fluids.

Throughout the paper we make two major simplifications to the CFD model. First, we only consider two-dimensional air flows moving parallel to the ground. This assumption intuitively makes sense since the air flow on the top half of a room can be accurately estimated using a two-dimensional model [5], [22], mostly due to the lack of obstacles (such as furniture). Second, we assume that the air flow behaves as a laminar fluid which reaches a steady-state behavior much faster than the temperature in the building. As mentioned in Section [I], both laminar and turbulent flows are present in general in a residential building. Yet, Sun [22] found only minor differences between laminar and turbulent models in a geometry similar to ours. Hence, we only consider a stationary Navier-Stokes equation to describe the fluid behavior, and we consider a time-dependent equation to describe the temperature behavior. Both assumptions allow us to greatly simplify the computational complexity of our CFD model (measured in the number of variables and number of equality constraints of the model), which in turn allow us to compute results in near real-time calculations, as shown in Section [IV].

Let $\Omega \subset \mathbb{R}^2$ be the area of interest, assumed to be compact. We will denote the boundary of $\Omega$ by $\partial \Omega$. Let $u: \Omega \to \mathbb{R}^2$ be the stationary air flow velocity, and $p: \Omega \to \mathbb{R}$ be the stationary air pressure in $\Omega$. Also, given $T > 0$, let $T_e: \Omega \times [0, T] \to \mathbb{R}$ be the temperature in $\Omega$. Then, using the formulation found in [23], the convection-diffusion of temperature in $\Omega$ can be described by the following Partial Differential Equation (PDE):

$$\frac{\partial T_e(x, t)}{\partial t} = \frac{1}{Pr Re} \nabla_x T_e(x, t) + u(x) \cdot \nabla_x T_e(x, t) = -g_{Te}(x, t), \quad (1)$$

where $g_{Te}: \Omega \times [0, T] \to \mathbb{R}$ represents the heat sources in the room, $Pr$ is the Prandtl Number of the air (i.e., the quotient of the kinematic viscosity divided by the thermal diffusivity), $Re$ is the Reynolds Number of the air (inversely proportional to the kinematic viscosity), $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian operator, and $\nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right)^T$ is the gradient operator.

Similarly, the stationary air flow in $\Omega$ is governed by the following set of incompressible Navier-Stokes stationary PDEs:

$$- \frac{1}{Re} \Delta_x u(x) + (u(x) \cdot \nabla_x) u(x) + \frac{1}{\rho} \nabla_x p(x) = g_a(x); \quad (2)$$

$$\nabla_x \cdot u(x) = 0, \quad (3)$$

where $g_a: \Omega \to \mathbb{R}^2$ represents all the external forces applied to the air (such as fans), $\rho$ is the density of the air, $u(x) \cdot \nabla_x = u_1(x) \frac{\partial}{\partial x_1} + u_2(x) \frac{\partial}{\partial x_2}$ is the advection operator, and $\nabla_x = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ is the divergence operator. Since we do not model the vertical dimension of $\Omega$, we omit the naturally occurring buoyancy term proportional to $T_e$, which is typically included on the right-hand side of (2) (e.g., see the V-momentum equation in Section 2 of [5]). Also note that (3) follows by our assumption that the air is incompressible in the temperature range of interest.

We assume that the HVAC unit acts over $\Omega$ through air outlets, denoted $\Gamma_o$, and has a single air return inlet, denoted $\Gamma_i$, both located at the boundary of $\Omega$, i.e., $\Gamma_o, \Gamma_i \subset \partial \Omega$. We denote the rest of the boundary of $\Omega$ as $\Gamma_w$, i.e., $\Gamma_w = \partial \Omega \setminus (\Gamma_o \cup \Gamma_i)$.

We use a mix of Neumann and Dirichlet boundary conditions to model the effect of the HVAC system in the room, as explained below. Let $\hat{n}(x)$ is the inward-pointing unit vector perpendicular to the boundary at $x \in \partial \Omega$. The air flow has the following boundary conditions:

- The HVAC unit’s fan sets the air flow at $\Gamma_o$, hence $u(x) = u_o \hat{n}_o(x)$ for each $x \in \Gamma_o$, where $u_o > 0$ is the HVAC fan speed, and $\hat{n}_o(x)$ is the unit vector with angle $\theta$ w.r.t. $\hat{n}(x)$.
• The airflow at the inlet is not constrained, hence \( u(x) \) is free for each \( x \in \Gamma_i \).
• The airflow satisfies a no-transverse condition at the walls, hence \( u(x) \cdot \hat{n}(x) = 0 \) for each \( x \in \Gamma_w \).

The temperature has the following boundary conditions:
• The HVAC unit’s heater changes the temperature at \( \Gamma_o \) using a Neumann boundary condition, hence
  \[
  \frac{\partial T_i}{\partial n(x)}(x,t) = v(t) \quad \text{for each} \quad x \in \Gamma_o \quad \text{and} \quad t \in [0, T].
  \]
• The walls and inlet are not adiabatic, hence
  \[
  \frac{\partial T_i}{\partial n(x)}(x,t) = -\alpha \left( T_i(x,t) - T_A \right) \quad \text{for each} \quad x \in \Gamma_w \cup \Gamma_i,
  \]
where \( \alpha > 0 \) is the heat loss constant, and \( T_A \) is the atmospheric temperature.

We only apply a boundary condition for the pressure equation

\[
\minimizes \text{the following cost function:}
\]

constant.

necessary since (2) only determines the pressure up to a reference set by the user, and all our experiments.

add inequality box constraints for all the controlled variables, in (1)-(3) and its boundary conditions as constraints. We also problem using the cost in (4), together with the CFD model conditions. We formulate a PDE-constrained optimal control

approximations of the desired optimal control, as described

ear programming problems. After those two transformation, we obtained by, first, using the Finite Element Method (FEM) to transform the CFD model in (1)-(3) and its boundary conditions as constraints. We also add inequality box constraints for all the controlled variables, so they remain within safety limits.

As explained in Section [IV] some of our experiments introduce variations to the cost function in (4) depending on the number of available actuators. Regardless, the goal of regulating the temperature will remain the same throughout all our experiments.

III. NUMERICAL IMPLEMENTATION

Our numerical implementation of the PDE-constrained optimal control problem described in Section [II-B] is obtained by, first, using the Finite Element Method (FEM) to transform the CFD model in (1)-(3) to a set of Ordinary Differential Equations (ODEs) as described in Chapters 3 and 4 of [24], and second, using the consistent approximation technique described in Chapter 4 of [25] which transforms optimal control problems (with ODE constraints) into nonlinear programming problems. After those two transformation, we use commercially available numerical solvers to find approximations of the desired optimal control, as described in Section [IV]

A. FEM Discretization

Among the many existing discretization techniques for PDEs, FEM stands out for being compatible with complex geometries of the domain \( \Omega \). Intuitively speaking, FEM approximates PDEs by dividing the domain into polygons, and then finding a set of ODEs for each vertex, and possibly each facet, of each polygon. The resulting set of ODEs has the property that each ODE is only dependent on its neighbors.

Before we can formally describe the FEM discretization we need to introduce some extra notation. Let \( H^1(\Omega, \mathbb{R}^n) \) be the set of functions from \( \Omega \) to \( \mathbb{R}^n \) belonging to \( L^2(\Omega, \mathbb{R}^n) \), whose weak derivative is also in \( L^2(\Omega, \mathbb{R}^n) \) [26]. Note that \( H^1(\Omega, \mathbb{R}^n) \), endowed with the dot product \( \langle f, g \rangle = \int_{\Omega} f(x) \cdot g(x) \, dx \), is a Hilbert space. Similarly, we denote \( \langle f, g \rangle_S = \int_{S} f(x) \cdot g(x) \, dx \).

Let \( \{ W_k \}_{k=1}^{N_p} \) be a polygonal partition of \( \Omega \), i.e., \( \bigcup_{k=1}^{N_p} W_k = \Omega \), \( \text{int}(W_k) \cap \text{int}(W_j) = \emptyset \) for each \( k \neq j \), and each \( W_k \) is a polygon. If \( \{ x_k \}_{k=1}^{N_p} \) is the set of vertices in the polygonal partition, then we define the test functions \( \{ \xi_k \}_{k=1}^{N_p} \), \( \{ \psi_k \}_{k=1}^{N_p} \subset H^1(\Omega, \mathbb{R}) \), and \( \{ \varphi_k \}_{k=1}^{2N_p} \subset H^1(\Omega, \mathbb{R}^2) \) with the following properties for each \( k \in \{1, \ldots, N_p\} \):

\[
\begin{align*}
\xi_k, \varphi_k, & \text{ and } \psi_k \text{ are continuous;} \\
\xi_k, \varphi_k, & \text{ and } \psi_k \text{ are nonzero only in the polygons containing } x_k; \text{ and,} \\
\xi_k(x_k) = \psi_k(x_k) = 1, \quad \varphi_{2k-1}(x_k) = [\frac{1}{2}], \text{ and} \\
\varphi_{2k}(x_k) = [\frac{1}{2}].
\end{align*}
\]

Then, from (1)-(3), and using Green’s Formulas (see Appendix C.2 in [27]), we get the following Galerkin identities (as described in Chapter 3.6 of [24]):

\[
\begin{align*}
\langle \frac{\partial T_e}{\partial t} (\cdot, t), \xi_k \rangle - \frac{1}{Pr \cdot Re} \left( \langle \nabla_x T_e (\cdot, t), \nabla_x \xi_k \rangle + \langle (\nabla_x T_e (\cdot, t) - T_A, \xi_k \rangle_{\Gamma_w} \right) & + \langle u \cdot \nabla_x T_e (\cdot, t), \xi_k \rangle = \langle g_{\ell}(\cdot, \pi), \xi_k \rangle; \quad (5) \\
- \frac{1}{Re} \langle \nabla_x u, \nabla_x \varphi_k \rangle + \langle (u \cdot \nabla_x) u, \varphi_k \rangle + \langle \nabla_x p, \varphi_k \rangle = & = \langle g_{u}, \varphi_k \rangle; \quad \text{and,} \quad (6) \\
\langle \nabla_x u, \psi_k \rangle = & = 0, \quad (7)
\end{align*}
\]

for each \( k \in \{1, \ldots, N_p\} \) and almost every \( t \in [0, T_f] \), where \( \mathbb{1} : \Omega \to \mathbb{R} \) is the constant function \( \mathbb{1}(x) = 1 \) for each \( x \in \Omega \).

Now, given \( N_{T_e}, N_{\varphi}, N_{\psi} \in \mathbb{N} \), consider the linearly independent sets of basis functions \( \{ \xi_j \}_{j=1}^{N_{T_e}} \), \( \{ \psi_j \}_{j=1}^{N_{\psi}} \subset H^1(\Omega, \mathbb{R}) \), and \( \{ \varphi_j \}_{j=1}^{N_{\varphi}} \subset H^1(\Omega, \mathbb{R}^2) \). Using these basis functions we can project the variables of our CFD model into finite-dimensional subspaces, i.e.:

\[
T_e(x, t) = \sum_{j=1}^{N_{T_e}} \eta_{T_e, j}(t) \hat{\xi}_j(x), \quad u(x) = \sum_{j=1}^{N_{\psi}} \eta_{u, j} \hat{\psi}_j(x), \quad p(x) = \sum_{j=1}^{N_{\psi}} \eta_{p, j} \hat{\psi}_j(x).
\]

Applying the representations in (8) to the Galerkin identities in (7) results in a set of \( N_v \) ODEs with state variables


\{(\eta_{t,j}, \rho_{t,j})\}_{j=1}^{N_{T_t}}. \text{ Similarly, applying the representations to (7) results in a set of } 2N_e \text{ algebraic nonlinear equations with parameters } \{(u_{u,j}, \eta_{u,j})\}_{j=1}^{N_u}, \{(u_{p,j}, \eta_{p,j})\}_{j=1}^{N_p}. \text{ All these differential and algebraic equations are, in practice, parametrized by constants corresponding to the inner products between basis and test functions, as well as their gradients. We omit the technical details of the final set of equations due to space constraints, and we refer the interested reader to Chapter 3 in [24] for more information.}

\section*{B. Optimal Control Discretization}

After the FEM discretization we effectively have an optimization problem subject to \(N_v\) ODEs (due to (5)), \(2N_v\) nonlinear equality constraints (due to (6)), and \(N_v\) linear equality constraints (due to (7)), together with several extra equality constraints due to the boundary conditions of the air flow and the pressure, as described in Section II-A. The actual number of constraints due to boundary conditions depends on the number of vertices in the polygonal partition \(\{W_k\}_{k=1}^{N_p}\) over the boundary.

The consistent approximation of this type of optimal control problem is studied in Chapter 4 of [25]. We follow the procedure described there, i.e., we first normalize the problem using the technique described in Chapter 4.1.2 of the same book, and then we use the Forward-Euler discretization method to transform the ODEs into a sequence of equality constraints. Again, we omit the technical details of the final equality-constrained nonlinear programming problem due to space constraints.

\section*{IV. Experimental Results}

Our main goal in this paper is to experimentally show that small improvements in the actuators of a residential HVAC unit can produce sizable energy savings. For this reason we consider six different experimental scenarios. The first three scenarios correspond to a single room and an HVAC system with one fixed outlet, two fixed outlets, and one variable-angle outlet (e.g., using smart vents [28], [29]), respectively. The next three scenarios are similar to the first three, but now we only aim to regulate the temperature in a small \(1.44\text{[m]}^2\) region, for example in the case when we can measure the position of an occupant (e.g., using indoors beacons [30]).

All our experiments consider the same room, shown in Figure 1 under different configurations. The room dimensions are \(4\text{[m]} \times 4\text{[m]}\), the length of the inlets and outlets is \(1.2\text{[m]}\), and the smaller region \(\Omega_{t}\) has dimensions \(1.2 \times 1.2\text{[m]}\), located \(0.2\text{[m]}\) from the top wall, and \(1\text{[m]}\) from the left wall. The fluid mechanics of the air is assumed to be governed by the constants \(Re = 1\) and \(Pr = 2\). The atmospheric pressure is assumed as \(p_A = 101.3\text{[kPa]}\), and the atmospheric temperature \(T_A = 26.85\text{[°C]}\) (or 80.33\text{[°F]}). The heat loss constant is assumed as \(\alpha = \frac{1\text{[m]}}{1\text{[min]}}\), and the air’s density \(\rho = 1.13\text{[kg/m]}^3\). We set the desired temperature to \(T_e^* = 27.85\text{[°C]}\) (or 82.13\text{[°F]}). We set the time horizon as \(t_f = 60\text{[s]}\), and we assume the functions \(g_{x}, g_{u}\) are identically zero (i.e., no heat sources and not fans inside the room). The parameters in the cost function (4) are \(\lambda = \frac{3}{100}\) and \(\mu = \frac{1}{10}\).

We discretized the area \(\Omega\) using a grid of \(8 \times 8\) squares, each then divided in two similar triangles. Hence, the number of elements is \(N_p = 64\), and the number of vertices \(N_v = 128\). We used first-order Lagrange elements to define the test and basis functions \(\xi_k, \xi_k, \psi_k, \text{ and } \hat{\psi}_k\), thus \(N_{T,T} = N_p = 81\). We used second-order Lagrange elements to define the test and basis functions \(\varphi_k\) and \(\varphi_k\), thus \(N_u = 162\). More details regarding these functions can be found in Chapter 3.3.1 of [31]. The ODE discretization time step was chosen as \(\Delta t = 2\text{[s]}\).

In Section IV-A we only consider the outlet \(\Gamma_o\) with angle \(\theta = 0\), and we compute the optimal heater power \(v(t)\) and optimal fan speed \(u_o\). In Section IV-B we consider both outlets \(\Gamma_o\) and \(\Gamma_{o}^\prime\), again with angle \(\theta = 0\), thus we compute the optimal heater power for both outlets, \(v(t)\) and \(v'(t)\), and the optimal fan speeds, \(u_o\) and \(u_o'\). Finally, in Section IV-C we only consider the outlet \(\Gamma_o\), this time with variable angle \(\theta\), thus we compute the optimal \(v(t), u_o\), and \(\theta\). In each of the sections we first regulate the temperature in the whole room \(\Omega\), and then we compare those results to the regulation in the smaller target area \(\Omega_t\).

We calculate the total energy usage as the sum of the heater energy usage, calculated as \(\int_{0}^{t_f} v(t) + v'(t) \text{dt}\), and the fan energy usage, calculated as \(\int_{0}^{t_f} \int_{\Gamma_o \cup \Gamma_{o}'} \|u(x)\| p(x) \text{dxdt}\). Our results were computed using a 16-core Xeon E5-2680 computer running at 2.7[GHz], with 128[GB] of RAM. We wrote our code using Python, the FEM discretization was computed using tools from the FEniCS Project [31], and the nonlinear programming problem was numerically solved using the SNOPT library [32]. The computation time ranged between 15[min] and 40[min] for each experiment.

\subsection*{A. One Fixed Outlet}

The cost function for this experiment is shown in (4) when the target area is \(\Omega\). When the target area is \(\Omega_t\) we use a cost function similar to (4), except that the average of the temperature error is computed only over the area \(\Omega_t\), as opposed to \(\Omega\). We set the safety box constraints to \(v(t) \in \mathbb{R}\).
Table I shows that, as expected, when the target area is smaller the optimal control uses roughly 33% less energy. Moreover, when the target area is $\Omega_\ell$ the temperature reaches a steady-state within the time horizon of 60[s]. Hence, our results support our hypothesis that important energy savings can be obtained from localizing the occupants in a building provided the HVAC unit can control heater and fan separately.

Note that the temperature distribution is not perfectly constant at time $t_f$, as shown in Figure 2c due to the heat loss through the walls with constant $\alpha$. Regardless, the distribution shown in this figure (and also in Figures 3c and 4c) have reached a steady-state behavior.

### B. Two Fixed Outlets

The cost functions for this experiment are similar to those used in Section IV-A except that we added two terms penalizing $v(t)$ and $v_\ell(t)$, corresponding to the heater power and fan speed at the outlet $\Gamma_\ell$, respectively. We set the safety box constraints to $v(t), v_\ell(t) \in [0, 5][kW]$, and $u_\alpha, u_\ell \in [0.3, 2][\frac{m}{s}]$, half of those used in Section IV-A to maintain the total power of the HVAC unit constant even though we have twice as many outlets. Some results from these experiments are shown in Figure 3.

As shown in Table I this experiment shows that, for a small room, having more outlets results in greater power consumption (roughly 35% higher than in the previous experiment). This result was contrary to our intuition, since we expected that more actuators in a room would lead to a better control. Yet, we experimentally found that the larger energy usage is due to the imposed safety minimum fan speed (necessary in practical implementations so the heaters do not overheat), which implies a larger overall air speed in the room.

### C. One Variable-Angle Outlet

The cost functions and safety box constraints for this experiment are identical to those used in Section IV-A. Some results from these experiments are shown in Figure 4.

As shown in Table I controlling the angle of the air flow at the outlets results in important energy savings when compared to the experiments in Section IV-A roughly 38% when the target area is $\Omega$, and roughly 19% when the target area is $\Omega_\ell$. Moreover, when we compare the original case of a fixed outlet heating a whole room, which resembles a standard HVAC unit, to the case when the angle of the air flow can be controlled at the outlet (e.g., using a smart vent [28], [29]) and we can localize the occupants in the room (e.g., using indoors beacons [30]), then the energy usage is roughly cut in half. Most of the energy savings come from the heater running at a lower power in this situation. Note that, as also shown in Table I the energy savings are not a consequence of a decrease in comfort, since the temperature is still regulated to similar levels in all the experiments.

### V. Conclusion

As shown in Section IV, our results open the door to a large number of exiting opportunities to improve the energy efficiency of buildings. By making small improvements to existing HVAC units it is possible to dramatically increase the efficiency of HVAC units without a decrease in user's comfort. It is worth noting that our results do not require, in principle, the use of expensive variable-speed fans or variable-power heaters, since those control signals can be implemented using switched strategies [33], [34]. More
importantly, our simplifications have allowed us to obtain results in the order of tens of minutes while still capturing the distributed behavior of the climate variables, which is much closer to real-time applications than previous results [21].

As future work we will first focus on the real-time implementation of our algorithm so it can be used as part of a MPC loop. Then, we will turn our attention to the validation of our results using a small-scale implementation in a real building.

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