Shell sources as a probe of relativistic effects in neutron star models

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A perturbing shell is introduced as a device for studying the excitation of fluid motions in relativistic stellar models. We show that this approach allows a reasonably clean separation of radiation from the shell and from fluid motions in the star, and provides broad flexibility in the location and timescale of perturbations driving the fluid motions. With this model we compare the relativistic and Newtonian results for the generation of even parity gravitational waves from constant density models. Our results suggest that relativistic effects will not be important in computations of the gravitational emission except possibly in the case of excitation of the neutron star on very short time scales.

I. INTRODUCTION AND OVERVIEW

For most astrophysical objects Newton’s classical theory of gravity gives a fully satisfactory description. Only when gravitational fields become strong need one consider the possibility that general relativistic effects may play a significant role. A standard index of field strength is $GM/Rc^2$, where $M$ is an object’s mass, and $R$ is its characteristic size. This index is of order unity for black holes and for the universe itself, and much smaller than unity for almost all stars, galaxies, and other astronomical entities. One exception is neutron stars; for compact neutron stars $GM/Rc^2$ is on the order of 0.2, and exotic equations of state could lead to even larger values. Despite this, Newtonian gravity is used almost exclusively in studying neutron stars. The obvious reason is the significant increase in difficulty in giving a fully relativistic treatment of neutron star structure, and the enormous increase of difficulty in dealing with fully relativistic dynamics, i.e., with oscillations of neutron stars.

Newtonian physics has been used even in studies (see for example Refs. [1–3]) of neutron stars as sources of gravitational waves. Of course, Newtonian gravity per se has no gravitational waves, so the typical procedure is to compute gravitational wave generation as a postprocessing step. More specifically, Newtonian gravity is used to find the fluid motions inside a neutron star associated with some event (core collapse in a supernova, precession of a rotating neutron star, etc.). Those fluid motions are then used as sources in the “quadrupole formula” of general relativity, just as known charge motions would be used in the dipole formula of electromagnetism. If Newtonian theory predicts periodic oscillations, the “adiabatic approximation” can be used to give the damping of the oscillations due to gravitational wave emission: the energy of the oscillations is taken to decrease at the rate at which gravitational waves remove energy.

The sufficiency of this approximate procedure caused little worry until relativistic modes of oscillation of neutron stars were discovered [4–6] that had no counterpart in Newtonian theory. The so-called $w$ modes are qualitatively oscillations of the spacetime, like black hole quasinormal (QN) modes, rather than oscillations of the neutron star material like the $f$ (fluid) and $p$ (pressure) modes of the Newtonian description. Partly due to the existence of these $w$ modes, the general question of the sufficiency of Newtonian theory for dealing with gravitational wave processes was emphasized by Andersson and Kokkotas [7], and stirred considerable interest.

To give a clear answer to this question requires a specific and astrophysically plausible event for which gravitational wave generation can be computed in both Newtonian theory and fully relativistically. One would want, for instance, initial data for the fluid and spacetime of a neutron star formed in a supernova core collapse, but the possibility of giving such initial data is at least several years off. A more tractable model was needed. Three separate groups [8–10] studied the problem of emission of gravitational waves by a relativistic neutron star due to the close passage of a perturbing particle. This model had the advantage of definitiveness; there was no freedom in choosing (and biasing) the initial spacetime perturbations to be particularly larger or smaller than the fluid perturbations. This model allowed an investigation of whether the excitation of $w$ modes was significant, but the model had three serious shortcomings. (i) It was too restrictive. The only parameters were the two constants specifying the particle orbit (say, energy at infinity and impact parameter). There could be significant excitation only if the particle passed close to the neutron star, and this constrained the perturbation to be neither very fast nor very slow. Not only was the timescale limited, but it was coupled to the choice of location of the perturbation. (ii) It was difficult to separate the radiation due to the neutron star (the radiation of interest) from the radiation coming (in some sense) from the orbiting particle. (iii) It was not clear how to compare the fully relativistic computation from the Newtonian computation; at least there was no attempt to do this.
We introduce here a very different model for investigating the importance of relativistic effects in neutron stars, and possibly for answering other questions. We consider a perturbative spherical shell around a neutron star at some radius $R_{\text{shell}}$. In that shell we dictate the time dependence of a multipole of surface mass-energy density of the shell. There is no equation of state of the shell material constraining our choice. The equations of motion of the shell fix the surface stress once we have specified the surface energy density, so picking that single function of time fully specifies the source. In this manner we can independently choose where the perturbation of the neutron star arises and what its timescale is. We can probe the response of the star to close and far perturbations with slow motions or fast motions. The shell probe has the nice feature that it is straightforward to do the calculation in Newtonian theory, so that a comparison can be made with the relativistic result. A further advantage is that both the Newtonian and relativistic computations allow a separation of the gravitational waves from the shell and those from the star. In the Newtonian computation, this is completely straightforward. The waves from the star are those found from the quadrupole formula applied to the motions only of the stellar fluid. In the relativistic calculation the separation is only approximate, and more care is needed. From the star+shell results it is necessary to subtract the radiation from the shell itself generated in the background spacetime of the star. (The details of this procedure will be given below.)

Our main purpose in the present paper is to present the method of using a shell probe and to display its advantages. For that reason we limit the application of this method to the simplest model of a star, the homogeneous incompressible perfect fluid (HIF) model. (See [11] for details about the QN modes of this model.) For this model the only mode associated with motions of the stellar material is a single $f$ mode. An important element of our results is that we will present answers not only about excitation of $w$ modes, but about the difference in the Newtonian and relativistic predictions of excitation of the $f$ mode. In the interest of brevity we limit the analysis to even parity perturbations. The excitation of $w$ modes for odd parity should not be remarkably different from the excitation in even parity, and odd parity motions do not couple to fluid motions.

The paper is organized as follows: The shell source model is introduced in Sec. II, and the equations governing even parity perturbations due to the shell source are given in Sec. III. Some details of the computational implementation are given in Sec. IV along with a discussion of the method used for subtracting the shell contribution from the relativistic calculation of the gravitational wave due to the shell and star. Numerical results are presented and discussed in Sec. V, and conclusions are given in Sec. VI. Details of the Newtonian calculation are given in the Appendix. Throughout the paper we use geometric units $G = c = 1$, the metric signature $(- + + +)$ and the conventions of Misner, Thorne and Wheeler [12].

II. MODEL: PERTURBATION OF A STATIC STAR BY MATTER MOVING ON A SPHERICAL SHELL

We start with a static and spherically symmetric spacetime background metric

\[ ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\varphi^2] , \]  

(1)

describing both the interior and exterior of a star of a barotropic, ideal fluid of mass $M$ and radius $r = R$. The stress energy of the fluid is

\[ T_{\alpha\beta} = (\rho + p) u_\alpha u_\beta + pg_{\alpha\beta} , \]  

(2)

where $\rho$ is the mass-energy density and $p = p(\rho)$ is the pressure. The mass function $m(r)$ is defined by $e^{-\lambda(r)} = 1 - 2m(r)/r$, and the structure of the stellar interior is found by solving the hydrostatic equilibrium equations of general relativity. (See, e.g., Eq. (3) of [10].) For simplicity, we limit considerations to homogeneous incompressible fluid (HIF) stellar models whose unperturbed interior metric is given by Eqs. (5) and (6) of [10]. The exterior metric is simply the Schwarzschild metric, with $m(r)$ equal to a constant $M$, and $\nu(r) = -\lambda(r)$. A spherical thin shell of coordinate radius $r = R_{\text{shell}} > R$ surrounds the star. We treat the shell as a perturbation of the spacetime inside and outside the star and we analyze the perturbations only to first order in the parameter of the perturbation. In this order, the shell has spherical geometry described by the 3-metric,

\[ ds^2_{\text{shell}} = -\left(1 - \frac{2M}{R_{\text{shell}}} \right) dt^2 + R_{\text{shell}}^2 [d\theta^2 + \sin^2 \theta d\varphi^2] , \]  

(3)

induced by the Schwarzschild metric. The metric of the perturbed spacetime can be written as

\[ g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + h_{\alpha\beta} \]  

(4)

where the “(0)” index denotes the background solution, that of Eq. (1). The Einstein equations to first order in perturbations are...
\[ \delta G_{\alpha}^{\beta} = 8\pi[\delta T_{\alpha}^{\beta}_{\text{fluid}} + \delta T_{\alpha}^{\beta}_{\text{shell}}] . \]  

The perturbed stress energy has two contributions. One, denoted by \( \delta T_{\alpha}^{\beta}_{\text{fluid}} \), is that of the fluid star perturbations and is nonzero only inside the star. The other, denoted by \( \delta T_{\alpha}^{\beta}_{\text{shell}} \), is the stress energy of the matter in the thin shell and is nonzero only outside the star. Its form, in the coordinates of Eq. (1), is

\[ \delta T_{\text{shell}}^{\alpha\beta} = \sqrt{1 - 2M/r} S^{\alpha\beta}(r - R_{\text{shell}}) , \]  

where

\[ S^{\alpha\beta} = \lim_{\epsilon \to 0} \int^{R_{\text{shell}} + \epsilon}_{R_{\text{shell}} - \epsilon} T^{\alpha\beta} dr \sqrt{1 - 2M/r} \]  

is the surface stress energy of the shell. (See e.g., \cite{14}.)

\[ S^{ab}_{\mid \! \! b} = 0, \quad S^{r\alpha} = 0, \quad a, b = t, \theta, \phi \]  

for the components of the surface stress energy tensor. Here \( " \mid \! \! \mid \) is the covariant derivative with respect to the shell 3-metric in Eq. (3).

Due to the spherical symmetry of the shell, we can decompose \( S^{00}, S^{0\theta}, S^{ij}; i, j = \theta, \phi \) in scalar, vector and tensor spherical harmonics respectively. Restricting attention to the even parity harmonics, we write these decompositions as:

\[ S^{00} = \sum_{l} S^{3}_{00}(t) Y_{l0}(\theta) \]  

\[ S^{0\theta} = \sum_{l} S^{4}_{0\theta}(t) \frac{\partial}{\partial \theta} Y_{l0}(\theta) \]  

\[ S^{ij} = \sum_{l} [S^{5}_{ij}(t) \Phi_{l0ij}(\theta) + S^{6}_{ij}(t) \Psi_{l0ij}(\theta)] . \]  

Here \( Y_{l0} \) is the scalar spherical harmonic and \( \Phi_{l0\theta\phi}/\sin^{2}\theta = \Phi_{l0\theta\phi} = Y_{l0}, \Phi_{l0\theta\phi} = 0 \) and \( \Psi_{l0\theta\phi} = \partial^{2}/\partial \theta^{2} Y_{l0}, \Psi_{l0\theta\phi} = 0 \) are the even parity Regge-Wheeler \cite{13} tensor harmonics. Azimuthal symmetry guarantees that the azimuthal index \( m \) does not enter into any of the equations after multipole decomposition, so with no loss of generality we consider only \( m = 0 \) axially symmetric motions.

Upon substitution of Eq. (9) into Eq. (8), we get,

\[ \frac{R_{\text{shell}}^{2}}{1 - 2M/R_{\text{shell}}} \frac{dS^{3}_{00}}{dt} + l(l + 1)S^{3}_{00} = 0 \]  

\[ - \frac{R_{\text{shell}}^{2}}{1 - 2M/R_{\text{shell}}} \frac{dS^{4}_{0\theta}}{dt} - S^{4}_{0\theta} + [l(l + 1) - 1] S^{6}_{0\theta} = 0 \]  

\[ \frac{MR_{\text{shell}}}{(1 - 2M/R_{\text{shell}})^{2}} S^{3}_{0\theta} - 2S^{5}_{0\theta} + l(l + 1)S^{6}_{0\theta} = 0 . \]  

As these equations show, we only have one degree of freedom. The choice of the surface mass-energy density \( S^{00}_{10}(t) \) uniquely determines all the other components of the shell’s stress energy through Eq. (10). In this work, we make the choice

\[ S^{00}_{10}(t) = \epsilon \frac{e^{-at^{2}}}{M} , \]  

where \( \epsilon \) is the perturbation parameter. Since all perturbation equations will be proportional to \( \epsilon \) we will omit it henceforth. The use of a Gaussian time dependence for the surface density on the shell gives us a source that it localized in time and allows us a choice of timescale for the process that drives the stellar fluid motions.
III. EQUATIONS GOVERNING EVEN PARITY PERTURBATIONS

A multipole decomposition of the even parity quantities in Eq. (5) leads to a set of coupled partial differential equations in the variables \(t, r\) for the coefficients of the metric perturbation \(h_{\alpha\beta}\) and velocity of the fluid star \([14]\). We adopt the notation of Regge and Wheeler \([13]\), Thorne and Campolattaro \([14]\) and of Moncrief \([16]\). For simplicity, we make the Regge-Wheeler \([13]\) gauge choice, which for even parity means that the only nonvanishing metric perturbations, for a particular \(l\), are

\[
\begin{align*}
    h_{00} &= e^{\nu(r)} H_{0}^{0}(r, t) Y_{l0}(\theta) \
    h_{0r} &= H_{0}^{0}(r, t) Y_{l0}(\theta) \
    h_{rr} &= e^{\lambda(r)} H_{2}^{0}(r, t) Y_{l0}(\theta) \
    h_{jk} &= r^{2} K_{l0}^{0}(r, t) \Phi_{l0jk}, \quad j, k = \theta, \varphi, \\
\end{align*}
\]

where \(\Phi_{l0jk}\) is one of the even parity tensor harmonics defined above, after Eq. (9). It is useful to divide the perturbed Einstein equations (5) into those governing perturbations inside the star and those for perturbations outside the star.

A. The interior equations

Inside the star the only nonzero stress energy is due to the perturbed fluid. Its independent components, are

\[
\begin{align*}
    \delta T_{00}^{\text{fluid}} &= -\delta \rho \
    \delta T_{rr}^{\text{fluid}} &= \delta T_{\theta\theta}^{\text{fluid}} = \delta T_{\varphi\varphi}^{\text{fluid}} = \delta p \
    \delta T_{\theta}^{a} \text{ fluid} &= (\rho + p) u_{a} \delta u^{a}, \
    \delta T_{0}^{a} \text{ fluid} &= (\rho + p) \delta u^{a} 0, \quad a = r, \theta, \varphi, \\
\end{align*}
\]

where \(\delta \rho, \delta p\) are the Eulerian changes in density and pressure, \(u^{0} = e^{-\nu(r)/2}\) is the only nonzero component of the velocity of the unperturbed star and \(\delta u^{a}\) is the velocity of the perturbed fluid. It is convenient to introduce, at this point, the quantity

\[
\delta h \equiv \frac{\delta p}{\rho + p}. \tag{17}
\]

For barotropic fluids, \(\delta h\) is the Eulerian perturbation of the relativistic enthalpy and

\[
\delta \rho = \frac{(\rho + p)^2}{p \gamma} \delta h, \tag{18}
\]

where \(\gamma\) is the adiabatic index,

\[
\gamma = \frac{p + \rho}{\rho} \frac{dp/dr}{d\rho/dr}. \tag{19}
\]

We decompose the stress energy components in Eqs. (16) into spherical harmonics, and for a single multipole have

\[
\begin{align*}
    \delta h &= \delta h_{l}(r, t) Y_{l0}(\theta) \
    \delta u^{0} &= \frac{1}{2} e^{-\nu(r)/2} H_{0}^{0}(r, t) Y_{l0}(\theta) \
    \delta u^{r} &= -\frac{e^{-\nu(r) + \lambda(r)/2}}{r^{2}} \frac{\partial}{\partial t} W_{l0}(r, t) Y_{l0}(\theta) \
    \delta u^{\theta} &= \frac{e^{-\nu(r)/2}}{r^{2}} \frac{\partial}{\partial t} V_{l0}(r, t) \frac{\partial}{\partial \theta} Y_{l0}(\theta). \\
\end{align*}
\]

With a similar decomposition of the perturbed Einstein tensor into tensor harmonics, Eq. (5) leads to a set of coupled equations for \(H_{0}^{0}, H_{1}^{0}, K_{0}^{0}, H_{2}^{0}, W_{l0}, V_{l0}\) and \(\delta h_{l0}\). One of the equations \([14]\) provides the important simplification,

\[
H_{2}^{0}(r, t) = H_{0}^{0}(r, t). \tag{24}
\]
Using this result, the other equations, can be reduced to a coupled system of two equations [17] in which all fluid functions have been eliminated and the only dependent variables are the metric functions $H_0^{10}, K^{10}$. For barotropic fluids, these equations are

$$e^{[\lambda(r)-3\nu(r)]/2[\rho]}[\rho^{1/2}(r) - \lambda(r)]/2K^{10}_{r,r} + 2\left(\frac{1}{r} + \frac{1}{2}\nu', r\right)K^{10}_{r} - e^{\lambda(r)-\nu(r)}K^{10}_{tt} - (l-1)(l+2)\frac{e^{\lambda(r)}}{r^2}K^{10} - H^{10}_{0,rr} + \left(\frac{2}{r} + \frac{\lambda_r}{2} - \frac{5\nu_r}{2}\right)H^{10}_{0,r} + \left(\frac{l(l+1)}{2}\frac{e^{\lambda(r)}}{r^2} - \frac{2}{r^2}(1-r\nu', r) - e^{[\lambda(r)-3\nu(r)]/2[\rho]}[\rho^{1/2}(r) - \lambda(r)]/2\nu', r\right)_r - 8\pi e^{\lambda(r)}(\rho + p)H^{10}_{0} + e^{\lambda(r)-\nu(r)}H^{10}_{0,tt} = 0,$$ (25)

and

$$K^{10}_{r,r} + \left[\frac{3}{r} - \frac{\lambda_r}{2} + \frac{p + \rho}{p\gamma}\left(\frac{1}{r} + \frac{1}{2}\nu', r\right)\right]K^{10}_{r} - e^{\lambda(r)}(l-1)(l+2)\left(\frac{1 + \rho + \rho}{p\gamma}\right)K^{10} - e^{\lambda(r)-\nu(r)}\frac{p + \rho}{p\gamma}K^{10}_{tt} - \left(1 - \frac{p + \rho}{p\gamma}\right)H^{10}_{0,r} - \left[-\frac{\lambda_r}{r} + \frac{1}{r^2} + \frac{l(l+1)}{2r^2}\frac{e^{\lambda(r)}}{p\gamma} + \frac{p + \rho}{p\gamma}\left(\frac{1}{r^2} - \frac{l(l+1)}{2r^2}\frac{e^{\lambda(r)}}{p\gamma}\right)\nu', r p + \rho\right]H^{10}_{0} = 0.$$ (26)

For HIF models, the adiabatic index $\gamma$ is effectively infinite since $\rho$ = constant, and thus all the terms proportional to $(p + \rho)/p\gamma$ in Eq. (28) are zero.

**B. Reduction of the exterior equations to the Zerilli equation**

In the Schwarzschild spacetime outside the star, it can be shown [17], [18] that all the perturbation equations can be obtained from the two first order equations

$$l(l+1)H^{10}_{1} + 2rH^{10}_{0,t} - 2r^2K^{10}_{1,r} + \frac{6M - 2r}{1 - 2M/r}K^{10}_{1,t} = D^{10}(r,t)$$ (27)

$$\frac{2M}{r^2}H^{10}_{1} + \left(1 - \frac{2M}{r}\right)H^{10}_{1,r} - K^{10}_{r} - H^{10}_{0,t} = B^{10}(r,t)$$ (28)

together with the algebraic identity

$$F = \left[(l-1)(l+2) + \frac{6M}{r}\right]H^{10}_{0,t} - \left[(l-1)(l+2) + \frac{2M(r-3M)}{r(r-2M)}\right]K^{10}_{1,t} - \frac{2r^2}{1 - 2M/r}K^{10}_{1,tt} + 2rH^{10}_{1,tt} + M\frac{l(l+1)}{r^2}H^{10}_{1} = 0$$ (29)

and the shell’s equations of motion, Eq. (10). The source terms in Eqs. (27), (28) are

$$D^{10} = 32\pi r \frac{dS^{10}}{dt} \sqrt{1 - \frac{2M}{r}} \delta(r - R_{shell})$$ (30)

$$B^{10} = 16\pi \left(S^{10}_{t0} - \frac{dS^{10}_{t0}}{dt}\right) \sqrt{1 - \frac{2M}{r}} \delta(r - R_{shell}).$$ (31)

Equations (27), (28) and (29), in turn, can be combined into the single wave equation,

$$\frac{\partial^2 Z^{10}}{\partial t^2} - \frac{\partial^2 Z^{10}}{\partial r^2} + V_l(r)Z^{10} = S_{10}(r,t)$$ (32)

for the Zerilli [18] function,

$$Z^{10}(r,t) \equiv \frac{r(r - 2M)}{(r\lambda + 3M)(\lambda + 1)}[H^{10}_{0} - rK^{10}_{r}] + \frac{r}{\lambda + 1}K^{10}.$$ (33)

In Eq. (32), $r^*$ is the usual tortoise coordinate,
\[ r^* = r + 2M \log[r/2M - 1] + \text{constant}, \]  
(34)
the constant \( \Lambda \) is
\[ \Lambda \equiv \frac{(l-1)(l+2)}{2}, \]  
(35)
and the potential \( V_I \) is
\[ V_I(r) = \frac{2\Lambda^2(\Lambda+1)r^3 + 6\Lambda^2Mr^2 + 18\Lambda M^2r + 18M^3}{r^2(\Lambda r + 3M)^2} \left( 1 - \frac{2M}{r} \right). \]  
(36)
The source term in Eq. (32) is
\[ S_{l0}(r, t) = -\frac{C_{l0}(r,t)f(r)}{h(r)} \delta[r - R_{\text{shell}}] + \frac{\partial}{\partial r^*} \left[ \frac{C_{l0}(r,t)\delta[r - R_{\text{shell}}]}{h(r)} \right] \] 
\[ + \frac{16\pi}{R_{\text{shell}}} \left( 1 - \frac{2M}{R_{\text{shell}}} \right)^{3/2} S_{l0}^6(t) \delta (r - R_{\text{shell}}), \]  
(37)
where
\[ C_{l0}(r,t) = -\frac{16\pi R_{\text{shell}}^2}{l(l+1)\sqrt{1 - 2M/R_{\text{shell}}}} S_{l0}^6(t) \]  
(38)
\[ f(r) = \frac{6M^2 + 3\Lambda Mr + r^2\Lambda(\Lambda+1)}{r^2(3M + \Lambda r)} \]  
(39)
\[ h(r) = \frac{3M + \Lambda r}{r - 2M}. \]  
(40)
Together with appropriate boundary conditions, the set of equations (25), (26) and (32) govern the even parity perturbations of the star due to the shell.

**C. Boundary conditions**

1. **Regularity at the center of the star**

The system of equations (25-26) admits two linearly independent regular solutions. Near the center they admit the series expansion [17],
\[ R^{l0}(r,t) = k_0(t)r^l + k_2(t)r^{l+2} + O(r^{l+4}) \]  
(41a)
\[ H_0^{l0}(r,t) = k_0(t)r^l + h_2(t)r^{l+2} + O(r^{l+4}). \]  
(41b)
The \( r \)-independent “constants,” \( k_0, k_2, h_2 \) are \textit{a priori} arbitrary. When Eq. (26) is expanded about \( r = 0 \) and the above expansions are used, we find
\[ -\left[ \frac{3}{2}l + 3 + \frac{l^2}{2} \right] h_2 + \left[ \frac{l^2}{2} + \frac{11}{2}l + 9 \right] k_2 - \frac{8\pi}{3} \rho_0 [2l + l^2 - 3] k_0 = 0 \]  
(42)
where \( \rho_0, p_0, \gamma_0 \) are the values of the density and pressure at the center of the star and we used the fact that \( \gamma = \infty \) for a HIF. Equation (12) allows us to eliminate \( k_0 \), so that the only remaining unknown coefficients are \( k_2 \) and \( h_2 \). These turn out to be fixed, up to an overall scaling, by conditions at the stellar surface.

2. **Vanishing of the pressure at the stellar surface**

The pressure must vanish at the perturbed surface of the star by definition of that boundary. This amounts to the vanishing of the Lagrangian perturbation of the pressure at the stellar surface \( r = R \),
\[
\delta p(R) - \frac{e^{-\lambda(R)/2}}{R^2} W(R, t)p_r(R) = 0.
\]

When the density \( \rho \) vanishes at \( r = R \), the second term in Eq. (43) is zero and this condition is the same as the vanishing of the Eulerian perturbation of the pressure. But this is not so when \( \rho(R) \neq 0 \) as is the case for a HIF.

After differentiating Eq. (43) twice with respect to time and doing a decomposition in spherical harmonics, we can rewrite it purely in terms of the metric functions \( K^{00}, H^{00}_0 \) and its derivatives \([17]\). The resulting expression, when \( \rho(R) \neq 0 \), is

\[
\left\{ -\frac{1}{r} K_{ttt}^{00} - \frac{\nu_r}{4} \frac{l(l+1)}{r^2} e^{\nu(r)} K^{00}_r - \frac{l(l+1)(l+2)}{2} e^{\lambda(r)} K^{00}_{rr} - e^{\lambda(r)-\nu(r)} K^{00}_{ttrt} - \frac{1}{r} \frac{\nu_r}{2} \left( 1 - \frac{\nu_r}{2} \right) K^{00}_{tt} \right. \\
\left. + \frac{1}{r} H^{00}_{0,rtt} + \frac{l(l+1)}{4r^2} e^{\nu(r)} \nu_r H^{00}_{0,r} + \left[ \frac{\nu_r}{2} - \frac{1}{r^2} + \frac{l(l+1)}{2r^2} e^{\lambda(r)} \right] H^{00}_{0,t} + \left( \nu_r \right)^2 e^{\nu(r)} \frac{l(l+1)}{4r^2} H^{00}_0 \right\}_{r=R} = 0.
\]  

(44)

3. The relation between the interior and the exterior metric functions

If the density of the star is not zero at \( r = R \), the first radial derivative of \( K^{00}, H^{00}_0 \) is not continuous at \( r = R \), although the functions themselves are. Denoting by a superscript “+” the exterior metric functions and by a superscript “−” the interior ones, we have \([17]\), for a particular multipole \( l \), that

\[
K^+(R, t) = K^-(R, t)
\]

(45a)

\[
H^+(R, t) = H^-(R, t)
\]

(45b)

\[
K_{tltt}^+(R, t) = K_{tltt}^-(R, t) - \frac{1}{2R^2} \left\{ l(l+1) \left( 1 - \frac{2M}{r} \right) K_{rt}^- + 2r \left[ \frac{r - 3M}{r - 2M} K_{rt}^- + r K_{r}^- \right] - l(l+1) \left( 1 - \frac{2M}{r} \right) H_{0,r}^- - 2r H_{0,t}^- - 2M \frac{l(l+1)}{r^2} H_0^- \right\}_{r=R}.
\]

(45c)

If \( \rho(R) = 0 \) then both metric functions and their first radial derivatives would be continuous at \( r = R \).

D. Fourier transform of the perturbation equations

The simplest way to solve the above partial differential equations in \( r, t \) is to write all time dependent quantities as Fourier integrals, reducing the problem to that of ordinary differential equations in \( r \). Thus we write \( Z_{10}(r, t) \) as,

\[
Z_{10}(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{Z}_{10}(r, \omega) d\omega
\]

(46)

transforming Eq. (32) into a second order equation for \( \tilde{Z}_{10}(r, \omega) \),

\[
\frac{d^2 \tilde{Z}_{10}}{dr^2} + \left[ \omega^2 - V(r) \right] \tilde{Z}_{10} = -\tilde{S}_{10}(r, \omega)
\]

(47)

where

\[
\tilde{S}_{10}(r, \omega) = \int_{-\infty}^{\infty} S_{10}(r, t)e^{i\omega t} dt
\]

(48)

is the Fourier transform of the source term. We can proceed similarly with Eqs. (25) and (24). The resulting equations for \( \tilde{H}_{10}^0(r, \omega), \tilde{K}^{00}(r, \omega) \) can be obtained directly from Eqs. (25) and (24) by substituting \( -i\omega \) for \( \partial_t \), and by replacing functions of \( r, t \) by their Fourier transforms.

At infinity, we impose the boundary condition that the wave be purely outgoing

\[
Z_{10}(r \to \infty, u \equiv t - r^*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_{10}(\omega) e^{-i\omega u} d\omega.
\]

(49)
The even parity gravitational energy, radiated in a single multipole component, is then given \[ \frac{dE_l}{d\omega} = \frac{1}{64\pi^2} \frac{(l + 2)!}{(l - 2)!} \omega^2 |A_{l0}(\omega)|^2 . \] (50)

From the wave function \( Z_{l0}(r, t) \) we can construct the multipoles of the metric perturbations. The transverse traceless (TT) part of the metric perturbation is of particular interest, since it completely characterizes the radiation at infinity. Extracting the TT part is easiest if we realize that the Zerilli function \( Z_{l0}(r, t) \) is numerically equal to Moncrief’s \[ 14 \] gauge invariant wavefunction. Since the Moncrief invariant can be evaluated in any gauge, we choose the gauge to be asymptotically flat so that all multipole perturbations fall off faster than \( 1/r \) except the TT components. From this procedure we find that the TT perturbations are related to the Zerilli function by

\[
h^{TT}_{jk} = \frac{1}{r} \sum_{l=2}^{\infty} Z_{l0}(r, t) \sqrt{\frac{(l + 2)!}{2(l - 2)!}} T^{E2, l0}_{jk}(\theta), \quad j, k = \theta, \varphi ,
\] (51)

where

\[
T^{E2, l0}_{jk}(\theta) = \sqrt{\frac{2(l - 2)!}{(l + 2)!}} \left[ \Psi_{0jk} + \frac{l(l + 1)}{2} \Phi_{0jk} \right]
\] (52)
is the orthonormal even parity TT tensor harmonic and \( \Psi_{0jk}, \Phi_{0jk} \) are the tensor harmonics introduced in Eq. (4).

IV. COMPUTATIONAL IMPLEMENTATION

A. Solution for \( A_{l0}(\omega) \)

If the background spacetime is due to a star, a solution of Eq. (47) must be found that corresponds to outgoing waves at infinity and that matches the regular solution of Eq. (25), (26) at the unperturbed surface of the star according to the junction conditions of Eq. (45). The Green function solution is found in the usual way. (See e.g., Ref. [10].) We define \( y_{l0}^{\text{out}}(r, \omega) \) as the homogeneous solution of Eq. (47) with the asymptotic form

\[
y_{l0}^{\text{out}}(r, \omega) \rightarrow e^{\omega r^*} \quad r^* \rightarrow \infty .
\] (53)

For our second independent solution of Eq. (47) we start by finding, in the stellar interior, a solution of Eqs. (25), (26) satisfying the condition Eq. (11) at the stellar center. Our second solution \( y_{l0}^{\text{reg}}(r, \omega) \) is taken to be the homogeneous solution of Eq. (47) that joins to the interior solution through the matching conditions of Eq. (45).

We then define the Wronskian of these two homogeneous solutions, an \( r \) independent quantity, to be

\[
W_l(\omega) = y_{l0}^{\text{reg}} \frac{dy_{l0}^{\text{out}}}{dr^*} - y_{l0}^{\text{out}} \frac{dy_{l0}^{\text{reg}}}{dr^*} .
\] (54)

With the above definitions, and from the Green function solution, we obtain (see, e.g., Ref. [10]) the Fourier amplitude \( A_{l0} \) defined in Eq. (48)

\[
A_{l0}(\omega) = -\frac{1}{W_l(\omega)} \int_{r < R_{\text{shell}}} S_{l0}(r, \omega) \frac{y_{l0}^{\text{reg}}(r, \omega)}{1 - 2M/R} dr .
\] (55)

Combining Eq. (10), (11), (37) and (48) we get explicitly,

\[
A_{l0}(\omega) = -\frac{16\pi}{W_l(\omega) \sqrt{1 - 2M/R_{\text{shell}}}} \sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/(4\alpha)} \left\{ \frac{R_{\text{shell}}}{M\hbar(r)(l + 1)} \frac{dy_{l0}^{\text{reg}}}{dr} (R_{\text{shell}}, \omega) + \frac{y_{l0}^{\text{reg}}(R_{\text{shell}}, \omega)}{1 - 2M/R_{\text{shell}}} \frac{1}{(l - 1)(l + 2)} \left( \frac{2R_{\text{shell}}^2}{M\hbar(r)} + \frac{R_{\text{shell}}^2}{M\hbar(r)} \right) \right\} .
\] (56)

What is required for a solution, then, is a numerical determination of \( y_{l0}^{\text{reg}}(R_{\text{shell}}, \omega) \) and its radial derivative.
B. Numerical method to find \( y_{l}^{reg} \) for a HIF

The numerical problem of finding \( y_{l}^{reg}(R_{\text{shell}}, \omega) \) and its derivative for a HIF, can be divided in two parts: Integration of Eqs. (25) and (26) from the center of the star to \( r = R \) and integration of Eq. (32) from \( r = R \) to \( r = R_{\text{shell}} \), so that we can evaluate \( y_{l}^{reg}, dy_{l}^{reg}/dr \) at this radius.

To find \( K^{-}(R, \omega), H_{0}^{-}(R, \omega) \) and \( K_{+}^{-}(R, \omega) \), for a HIF we first find an “A solution” by starting from the center with \( k_0 = 1, k_2 = 0 \) and \( k_2 \) obtained from Eq. (22); we then integrate Eqs. (25), (26) out to \( r = R \), to find the “A solution” there. The procedure for the “B solution” at \( r = R \) is the same except we start with the central conditions \( k_0 = 0, k_2 = 1 \). The general solution for \( H_{0}^{-} \) and \( K^{-} \) can be written

\[
H_{0}^{-} = \alpha H_{0}^{A^{-}} + \beta H_{0}^{B^{-}} \quad K^{-} = \alpha K^{A^{-}} + \beta K^{B^{-}}.
\]

where \( \alpha \) and \( \beta \) are arbitrary constants. The overall scale of \( y_{l}^{reg}(r, \omega) \) is arbitrary. Since \( y_{l}^{reg}(r, \omega) \) occurs both in the integrand in Eq. (23) and in the Wronskian in the denominator, the scale cancels out so only the ratio \( \alpha/\beta \) is of importance. This ratio can be found by substituting the expression in Eq. (57) and its derivatives in the vanishing of the Lagrangian pressure condition Eq. (14).

We can next compute \( K^{+}(R, \omega), H^{+}(R, \omega) \) and \( K_{+}^{+}(R, \omega) \) using Eq. (47). The final step is to use Eq. (33) to find the starting values for integrating the Zerilli equation,

\[
y_{l}^{reg}(R, \omega) = \frac{R(R - 2M)}{(RA + 3M)(A + 1)} [H_{0}^{10+} - rK_{r}^{10+}]_{r=R} + \frac{r}{A + 1} K^{10+}(R, \omega)
\]

and to integrate the Zerilli equation Eq. (22) out to \( r = R_{\text{shell}} \) in order to find \( y_{l}^{reg}(R_{\text{shell}}, \omega) \) and \( dy_{l}^{reg}/dr(R_{\text{shell}}, \omega) \).

C. Radiation due only to the shell

The waveform \( Z_{0} \), its Fourier transform \( A_{0}(\omega) \), and the energy computed in Eq. (50) refer, of course, to radiation from the star and the shell. As might be expected the radiation that can be attributed directly to the shell is much larger than the radiation from the perturbed fluid motions in the neutron star. In order to have the clearest comparison of Newtonian and relativistic predictions of radiation from the neutron star, it is useful to remove the contribution due to the shell.

For the Newtonian computation presented in the appendix, this presents no problem. As is evident in Eq. (A18), the density perturbations due to the shell and due to the star are distinct, and it is evident in Eq. (A22) that their contribution to the quadrupole moment are distinct. This clear distinction does not exist in the relativistic calculation. The equations of Sec. 11 contain information about the shell entangled with information about the star. To get an approximate idea of what radiation can be ascribed to the shell, and not to the shell, one can compute the waveform due (in a sense) to the shell itself, and subtract this waveform from the total star+shell waveform. This subtraction, however, is somewhat subtle.

In particular, it is not useful to consider the shell in a flat spacetime background. If we consider a shell with the surface stress energy of Eqs. (14), (15) radiating in a flat background, and the same (in some sense) shell, and surface stresses, radiating in another fixed background, there will be a large difference simply due to the different radial null geodesics, the spacetime lines on which perturbations propagate. If we are to have the same shell-only radiation as that due to the shell in the shell+star problem, we must have the two signals propagate on the same spacetime background. To accomplish this, we describe the waves at radius \( r > R \) for the shell-only problem with the same Zerilli equation as we use in the shell+star problem. Equation (22) is then part of both problems for \( r > R \). In the shell+star problem the perturbations in the interior are, of course, treated with the equations of Sec. 11. For the shell-only problem, we need instead something like a Zerilli equation suitable to the fixed spacetime of the stellar interior. Such an equation is provided by the equation for the propagation of massless scalar perturbations in the neutron star background. These waves do not excite oscillations either of the spacetime or of the neutron star fluid. They are described only by an equation identical in form to Eq. (22), but with no source, and with the Zerilli potential of Eq. (34) replaced by
\[ V^{\text{scalar}}_l = e^{\nu} \left[ \frac{l(l+1)}{r^2} + \frac{\nu e - \lambda e^{-\lambda}}{2r} \right]. \] (60)

(The idea of freezing the fluid perturbations to study the \( w \) modes described in [24] is somewhat similar in spirit to our method but is different in practice.) With this mixed mathematical description we find the regular solution of the Zerilli equation and then compute, using Eq. (56), the Fourier transform, and with Eq. (49), a wave function \( Z_0 \). By subtracting these, respectively from the \( A_0 \) and \( Z_0 \) for the shell+star, we arrive at results \( A^{\text{star}}_{10} = Z^{\text{star}}_{10} \) meant to describe radiation only due to the oscillations of the neutron star fluid. From the square of \( A^{\text{star}}_{10} \), we can compute an energy spectrum and a total radiation energy attributed to the motion of the stellar fluid.

Our method of finding the radiation from the shell alone is, of course, only an approximation. Einstein’s equations couple oscillations of the fluid and oscillations of the spacetime, so that there can be no completely meaningful way of separating the two. One sign of this is that the potential Eq. (60) is not unique. The potential does not influence the radial null geodesics, and any potential with the same general behavior at \( r = 0 \) is equally good. In situations for which the details of the potential are important, our method of computing shell-only radiation is not justified. But such details are not relevant for most of our models. This can be seen in the reasonable success of our method in removing appearances of shell radiation in the star-only results presented below.

An additional important point to understand about our shell subtraction method is that in principle it should subtract the \( w \) modes. Since the \( w \) modes are due to the spacetime background, not to the fluid motion, the shell-only radiation should have \( w \) modes, and the star-only waves that result from subtraction should have only waves due to fluid motion. This will be discussed further in connection with the numerical results presented below.

V. NUMERICAL RESULTS

We focus first on a very compact HIF stellar model of radius \( R = 2.5M \). By starting with an extreme, though astrophysically implausible, model we will be able to see relativistic features that will be absent in less compact, and more plausible, models. In Fig. 1(a) we present the waveform \( Z_{20} \), the Zerilli function for quadrupole \( (l = 2) \) perturbations in the case that the Gaussian parameter in Eq. (11) is \( a = 0.1 M^{-2} \). The shell was placed at a large distance from the star, \( R_{\text{shell}} = 110M \), to have the waveform clearly display the time profile of events. The first burst, at around \( u = t - r^* = -160M \) is radiation coming directly from the stress energy of the shell. Later, at around \( u = t - r^* \approx 100M \), a burst arrives representing the ingoing radiation from the shell that has “reflected” off \( r = 0 \). At around the same time, radiation from oscillations of the fluid and central region of the spacetime arrive.

Two curves are shown in Fig. 1(a). One is the waveform for the relativistic star+shell. The second curve is that for the shell itself, computed by the method described in Sec. IV C. The two curves are nearly identical up to around \( u = 160M \), confirming that the early radiation is that due to the shell. From \( u = 160M \) to around 400M or more, there are damped oscillations of the dominant (least damped) \( w \) mode. The \( w \) modes depend on the details of wave propagation in the innermost strong field regions, and are not the same for the star+shell problem and for the somewhat ad hoc spacetime we constructed for the shell-only problem. The \( w \) mode frequency for the constant density \( R = 2.5M \) model is \( \omega = (0.24 + 0.0217)M^{-1} \), while that for the spacetime of Sec. IV C, \( \omega = (0.438 + 0.029)M^{-1} \), is more rapidly damped. We would, of course, find a different \( w \) mode frequency if in the shell-only problem we used a potential other than that in Eq. (60).

It is clear that there is no point in subtracting the shell-only radiation from the star+shell. The difference waveform would contain an artifact corresponding to the difference of the \( w \) modes, and hence would be dominated by features completely irrelevant to fluid motions of the neutron star. Figure 1(a) helps to demonstrate that subtraction is pointless when \( w \) mode radiation is of importance in the waveform. It is only the fact that \( w \) mode radiation is a minor feature for realistically compact stars that makes subtraction useful.

In addition to the difference in frequency of the \( w \) modes there is another, more important, difference between the two curves in Figure 1(a). The star+shell curve shows an oscillation with imperceptible damping at a frequency less than half that of the \( w \) mode. This is the \( f \) mode of the fluid of the neutron star. This mode is missing, as it should be, from the shell-only computation.

Both curves in Fig. 1(a) show relativistic results. A comparison between the Newtonian and relativistic star+shell results would not be of much use. The slow motion condition underlying the Newtonian approximation would be strongly violated since the radius of the shell \( R_{\text{shell}} = 110M \) is much larger than the characteristic time scale of the shell stress-energy oscillations \( (\Delta t \sim a^{-1/2} \sim \text{several } M) \) or of the modes of oscillation of the spacetime or of the stellar fluid. The Newtonian computation, based on the slow motion approximation (time scale \( \ll \) light travel time across source) would be completely inappropriate. In order to construct a more justifiable Newtonian comparison for a \( R = 2.5M \) HIF model, we consider in Fig. 1(b), a shell with radius \( R_{\text{shell}} = 5M \) and with Gaussian parameter \( a = 0.001M^{-2} \) (and hence timescale \( \sim 30M \)). The waveform in this case is not of primary interest, since the radiation
from the shell and from the stellar fluid will not be clearly distinguishable. The energy spectrum, however, gives the answer to the most important questions. It shows, for example, that there is negligible difference between the broad spectra of the Newtonian and the relativistic results. But the broad spectrum is due to the shell. Of more astrophysical interest is the \( f \) mode excitation. The relativistic computation shows a lower frequency \( f \) mode containing almost an order of magnitude more radiation energy than the \( f \) mode peak of the Newtonian computation. This conclusion, of course, applies to a model that is too compact to be astrophysically relevant.

The feature in the relativistic spectrum at \( \omega \sim 0.3 M^{-1} \) is due to the repetition of the shell radiation with a time delay of \( \Delta t \sim 10 M \). This produces a modulation of the form \( \cos^2(\omega \Delta t/2) \), and hence a dramatic decrease at \( \omega \sim \pi/\Delta t \sim 0.3 \). The location of such features is dependent on \( R_{\text{shell}} \) and, of course, is unrelated to the physics of the neutron star. It is worthwhile noting that these features are absent in the Newtonian spectrum. Since that spectrum is based on the slow motion approximation the whole star+shell source is treated as if it radiates in phase, and there can be no repetition of the shell radiation.

In Fig. 2 we show results for a HIF star of radius \( R = 5 M \), a typical radius of a neutron star. The shell is located at \( R_{\text{shell}} = 10 M \) and the Gaussian parameter for the time profile of the shell stress energy is \( a = 0.01 M^{-2} \). In Fig. 2(a) the dotted curve gives the full Zerilli quadrupole waveform of the star+shell computation, and shows why subtraction of shell radiation is useful: the waveform is completely dominated by the shell radiation, and the much smaller \( f \) mode is barely visible. The solid curve is the waveform from the shell-only computation. It is clear in Fig. 2(a) that the two waveforms are nearly identical at early times, and at late times are different in that the star+shell result has \( f \) mode oscillations, while the shell-only waveform, of course, does not. This suggests that subtraction will be very effective in isolating the radiation due to the stellar fluid, and this turns out to be true. The star-only curve (the result of subtracting the shell-only from the star+shell) is given in the inset to Fig. 2(a) and shows the \( f \) mode oscillation as a dominant feature.

Figure 2(b) shows the energy spectrum for the subtracted (i.e., star-only) case shown in the inset of Fig. 2(a). Both the relativistic and Newtonian spectra are given. The two spectra are roughly similar in general appearance, except for the strong feature in the relativistic spectrum at \( \omega \sim 0.5 M^{-1} \). Of particular interest in Fig. 2(b) is the excitation of the \( f \) mode (shown in detail in the inset). Perhaps the most important feature of our model is that we can compare the excitation given by a relativistic and a Newtonian computation.

In order to illustrate how the methods of this paper work for a very weakly relativistic model, Fig. 3 shows two spectra for a HIF star with \( R = 20 M \). In both, the Newtonian star-only spectrum is compared with the relativistic star-only (i.e., subtracted) spectrum, and for both \( R_{\text{shell}} = 22 M \). Figure 3(a) shows the case for a Gaussian parameter \( a = 0.01 M^{-2} \), while (b) shows the case \( a = 0.00001 M^{-2} \). The Newtonian and relativistic computations of \( f \) mode excitation (shown in the inset) agree quite well in Fig. 3(a), and there is general agreement in the overall shape of the spectrum but, as in Fig. 3(b), the relativistic case has structure that is missing in the Newtonian case. This is due to the fact that the shell stress energy source has a timescale \( \sim a^{-1/2} \sim 10 M \) that is less than the light travel time across the star, and the star is not radiating in phase. By comparison, in Fig. 3(b) the shell timescale is \( \sim a^{-1/2} \sim 300 M \) and the slow approximation is justified. (Note that the \( f \) mode oscillations, at \( \omega \sim 0.01 M^{-1} \) are also slow compared to the light travel time across the star.) Because of this, the Newtonian and relativistic spectra in Fig. 3(b) are in excellent overall agreement.
Newtonian for the shell-only model is \(0.1 M^{-2}\), in order to show a clear separation of the initial shell radiation and the radiation due to the stellar fluid. Waveforms are shown for the relativistic star+shell, and for the relativistic shell-only model discussed in the text. For the spectra in (b) the shell radius is small, \(R_{\text{shell}} = 5M\), and the shell time scale is slow, \(a = 0.001 M^{-2}\), so that the model is approximately a slow motion source. The spectrum of the relativistic star+shell model is compared with the Newtonian shell+star model. The agreement is good, but is dominated by the shell radiation. For the \(R = 2.5M\) model the relativistic \(f\) mode is \((0.18 + i0.000037)M^{-1}\), and the Newtonian \(f\) mode is \((0.226 + i0.00131) M^{-1}\); the least damped \(w\) mode is \((0.42 + i0.022) M^{-1}\), while the least damped \(w\) mode for the shell-only model is \((0.44 + i0.029) M^{-1}\).

FIG. 1. A waveform and spectrum for an extremely compact HIF model with \(R = 2.5M\). For the waveform in (a) the shell radius was taken to be large, \(R_{\text{shell}} = 110M\), and the time scale for the shell stress energy small, \(a = 0.1 M^{-2}\), in order to show a clear separation of the initial shell radiation and the radiation due to the stellar fluid. Waveforms are shown for the relativistic star+shell, and for the relativistic shell-only model discussed in the text. For the spectra in (b) the shell radius is \(R_{\text{shell}} = 5M\), and the shell time scale is slow, \(a = 0.001 M^{-2}\), so that the model is approximately a slow motion source. The spectrum of the relativistic star+shell model is compared with the Newtonian shell+star model. The agreement is good, but is dominated by the shell radiation. For the \(R = 2.5M\) model the relativistic \(f\) mode is \((0.18 + i0.000037)M^{-1}\), and the Newtonian \(f\) mode is \((0.226 + i0.00131) M^{-1}\); the least damped \(w\) mode is \((0.42 + i0.022) M^{-1}\), while the least damped \(w\) mode for the shell-only model is \((0.44 + i0.029) M^{-1}\).

FIG. 2. Results for a HIF model with \(R = 5M\), \(R_{\text{shell}} = 10M\), and Gaussian parameter is \(a = 0.01 M^{-2}\). In (a) the radiation waveform from the relativistic shell+star is compared to that from the relativistic shell-only to demonstrate the value of subtraction. In (b) the spectrum for the star-only (i.e., subtracted) relativistic computation is compared with the Newtonian star-only spectrum. The \(f\) mode frequency for the \(R = 5M\) model is \((0.08 + i0.000033) M^{-1}\) in relativity and \((0.08 + i0.000082) M^{-1}\) in the Newtonian computation. The least damped star+shell \(w\) mode frequency is \((0.50 + i0.31) M^{-1}\).

FIG. 3. A comparison of Newtonian and relativistic star-only computations for a weakly relativistic HIF model. In both figures the stellar radius is \(R = 20M\), and the shell radius is \(R_{\text{shell}} = 22M\). In (a) the Gaussian parameter is \(a = 0.01 M^{-2}\), so the shell is not a "slow motion" source. In (b) the Gaussian parameter is \(a = 0.00001 M^{-2}\), so that the shell is a slow motion source, and the Newtonian and relativistic results agree in all features. The \(f\) mode frequency for this model is \((0.01 + i0.32) 10^{-4} M^{-1}\) in Newtonian theory and approximately the same in relativistic computations.

VI. CONCLUSIONS

We have presented a method of probing the gravitational wave properties of neutron stars by using time varying stress energy in a spherical shell. In particular, we have shown that this method can give more useful answers about the neutron star physics than those given by studies of the scattering of gravitational waves [8] or by the response of the star to a close particle orbit [8-10]. The main motivation for considering such a probe is to compare relativistic and Newtonian computations of gravitational radiation. We have shown that the shell probe is well suited for this purpose, since both the Newtonian and relativistic computations can be carried out for the model.
Two additional features of the shell probe have been shown to be important or useful. One is the possibility of approximately distinguishing the radiation that can be ascribed to the star from the radiation due to the shell. Since the radiation from the shell is stronger than that from the stellar fluid, this separation is valuable in bringing out the physical radiation that is of primary interest.

A fundamentally important feature of the shell probe model is the ability to choose the timescale of the shell stress energy. This has allowed us to direct attention to the fact that the Newtonian approximation is not only a weak field approximation, but a “slow” approximation. That is, the quadrupole approximation used in a quasi-Newtonian gravitational wave calculation supposes that the light travel time across the source is much smaller than the period of the waves generated. We have shown that even for a weakly relativistic stellar model there is not good agreement in the details of the Newtonian and relativistic computations if the timescale for the shell stress energy is short.

The results shown in the previous section make this clear. The structure and dynamics of the $R = 20M$ model of Fig. 3(a) is well described by Newtonian physics, but there are large differences between the Newtonian and relativistic spectra when the star is excited by a short time scale perturbation. The difference between the Newtonian and relativistic results is even larger for the $R = 5M$ model of Fig. 2(b). For smaller Gaussian parameters $a$ (i.e., for driving perturbations with longer time scales) the “relativistic-only” structure seen in Figs. 2(b) and 3(a) decreases.

What becomes clear from these results is that the question of whether Newtonian physics is adequate for neutron star dynamics is inseparable from the question of the timescale of the excitation of the neutron star. If the timescale is imposed from a distance many times the neutron star radius, then the excitation will be slow and our results (based on a very limited exploration of models) suggest that Newtonian physics will suffice. On the other hand, rapid processes, due to impacts, collapse, etc., may be of short timescale only several times $GM/c^3$, the slow approximation may be violated, and Newtonian calculations may be significantly in error.

As explained in Sec. I, the motivation for the shell probe for neutron star oscillations is a sequence of two questions. In Ref. [7] Andersson and Kokkotas questioned whether Newtonian physics was adequate for neutron star physics. The second question is whether this can be adequately studied with the particle-scattering model [8–10] and its inflexible timescales. The results presented here suggest that in cases in which neutron stars are excited on a very short time scale, those particle-scattering computations are not a sufficient basis for the conclusion that relativistic effects are unimportant in neutron star models of gravitational radiation sources.

Our main purpose here has been to introduce the shell probe, and the motivation for it. We have applied it only to a single simple neutron star model. It is quite possible, of course, that for some equations of state the importance of relativistic effects might be quite different. If other equations of state are to be studied towards this end, we suggest that the method of time varying stress energy in a shell be considered as a good way of getting the clearest comparison of Newtonian and relativistic predictions.

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APPENDIX A: THE NEWTONIAN LIMIT OF A HIF

1. The newtonian perturbation equations and their solution

We consider a HIF, excited by a spherical shell whose mass is much smaller than the star’s mass. Assuming that the motion of the fluid consists of small perturbations around a spherically symmetric and static fluid ball (the equilibrium star), we can decompose all perturbative scalar quantities, namely the gravitational potential, the pressure, and the density perturbations, in scalar spherical harmonics, and can decompose the fluid’s velocity in even parity vector spherical harmonics. We write these decompositions as

\[
U = U_{eq}(r) + \sum_l \delta U_l(r, t) Y_{l0}(\theta)
\]

\[
p = p_{eq}(r) + \rho_{eq} \sum_l \delta h_l(r, t) Y_{l0}(\theta)
\]
\[ \rho = \rho_{eq} + \sum_{l} \left\{ \frac{e^{-at^2}}{M} \delta[r - R_{shell}] - \frac{W_l(r,t)}{r^2} \rho_{eq} \delta[r - R] \right\} Y_0(\theta) \] (A3)

\[ v^r = -\frac{1}{r^2} \sum_{l} \frac{\partial}{\partial r} W_l(r,t) Y_0(\theta) \] (A4)

\[ v^\theta = \frac{1}{r^2} \sum_{l} \frac{\partial}{\partial \theta} W_l(r,t) Y_0(\theta) \] (A5)

where \( U_{eq}, \rho_{eq}, \rho_{eq} = \text{const} \) are the gravitational potential, pressure and constant density of the spherical (equilibrium) star. The perturbation of the density has two contributions: one from the matter in the shell and the other from the star itself. In writing the first one, we supposed that \( \delta T_{00,\text{shell}} \approx \delta \rho_{\text{shell}} \) is dominant over all the other components of the stress energy tensor of the shell (weak field, slow motion approximations) and that \( \delta T_{00,\text{shell}} \) is given by Eqs. (4) and (11). The density due to the fluid perturbation is zero everywhere inside the star (since the star is incompressible), but not at the unperturbed surface of the star [2]. In writing the perturbation of the pressure, we used the definition Eq. (17) and the fact that \( \rho_{eq} \ll \rho_{eq} \) in the weak field limit.

We then substitute these expressions in the fluid equations for a perfect barotropic fluid, which are Poisson’s equation for the gravitational potential,

\[ \nabla^2 U = -4\pi \rho , \] (A6)

the continuity equation,

\[ \rho, t = -\nabla . (\rho \vec{v}) , \] (A7)

and Euler’s equation,

\[ \vec{v}, t + (\vec{v} . \nabla) \vec{v} = \nabla U - \frac{1}{\rho} \nabla \rho , \] (A8)

and we keep terms only to first order in the perturbations. The resulting linearized fluid equations can then be reduced to two equations: one for \( \delta U_l \) and another for \( \delta h_l \). These equations are the Newtonian limit of the relativistic even parity perturbation equations derived by Lindblom et al. [21]. In the special case of a HIF, they reduce to two decoupled second order equations,

\[ \delta U_{l,t} + \frac{2}{r} \delta U_{l,r} - \frac{l(l+1)}{r^2} \delta U_l = -4\pi \left\{ \frac{e^{-at^2}}{M} \delta[r - R_{shell}] - \frac{W_l(r,t)}{r^2} \rho_{eq} \delta[r - R] \right\} \] (A9)

and

\[ \delta h_{l,t} + \frac{2}{r} \delta h_{l,r} - \frac{l(l+1)}{r^2} \delta h_l = 0 . \] (A10)

Since the left hand sides of Eqs. (A9) and (A10) are equivalent to the multipole decomposition of the Laplacian, it is straightforward to write down the solutions that are well behaved at the center of the star and that vanish at infinity:

\[ \delta U_l(r,t) = \begin{cases} \alpha_l(t) r^l + \frac{4\pi}{2l+1} \frac{r^l}{MR_{shell}} e^{-at^2}, & r \leq R \\ \frac{\beta_l}{r^{l+1}} + \frac{4\pi}{2l+1} \frac{R_{shell}^l}{M r^{l+1}} e^{-at^2}, & R \leq r \leq R_{shell} \\ \frac{\beta_l}{r^{l+1}} + \frac{4\pi}{2l+1} \frac{R_{shell}^{l+2}}{M r^{l+1}} e^{-at^2}, & r \geq R_{shell} \end{cases} \] (A11)

and

\[ \delta h_l(r,t) = \mu_l(t) r^l, \quad r < R . \] (A12)

The three constants can be easily determined, by requiring the vanishing of the Lagrangian pressure at \( r = R \), as in Eq. (13), by requiring the continuity of \( \delta U_l \) at the surface of the star, and by integrating Eq. (A9) about \( r = R \) and by using the linearized Euler equation for the radial component of the fluid’s velocity. The result is
\[ \beta_l = R^{2l+1} \alpha_l \quad (A13a) \]
\[ \mu_l = (2l + 1) \frac{\alpha_l}{3} \quad (A13b) \]
\[ \frac{d^2 \alpha_l}{dt^2} + \omega_l^2 \alpha_l = \frac{16\pi^2}{(2l + 1)^2 M R_{\text{shell}}^{l-1}} l \rho_{\text{eq}} e^{-a t^2} \quad (A13c) \]
\[ \omega_l^2 = \frac{8\pi \rho_{\text{eq}} l(l - 1)}{3} \frac{2l + 1}{2l + 1} . \quad (A13d) \]

The frequency \( \omega_l \) is the \( f \) mode of vibration of the star.

2. Damping of the \( f \) mode oscillation

Once set into vibration at its \( f \) mode frequency by the shell perturbation, the star would oscillate forever, since there is no damping mechanism for the perfect fluid in Newtonian theory. This is clear from Eq. (A13d), which is the equation of an undamped harmonic oscillator. But in practice the star will radiate away its vibrational energy through gravitational waves, over a long period of time. The time, \( \tau_l \), for gravitational wave damping of the \( f \) mode oscillation can be computed, in the weak field, slow motion, approximation using energy conservation \cite{22, 23} to be,

\[ \tau_l = \frac{4(l - 1)^2(2l + 1)(2l - 1)!!}{3(l + 1)(l + 2) \omega_l^{2l+2} R^{2l+1}} . \quad (A14) \]

To introduce this damping we replace Eq. (A13c) by

\[ \frac{d^2 \alpha_l}{dt^2} + \omega_l^2 \alpha_l + \frac{2 \alpha_l}{\tau_l} \frac{d \alpha_l}{dt} = \frac{16\pi^2}{(2l + 1)^2 M R_{\text{shell}}^{l-1}} l \rho_{\text{eq}} e^{-a t^2} . \quad (A15) \]

(For a similar procedure see \cite{1} and \cite{3}). The retarded solution of this equation is

\[ \alpha_l(t) = \frac{16\pi^2 l \rho_{\text{eq}}}{(2l + 1)^2 M R_{\text{shell}}^{l-1} \omega_l} \int_0^\infty d\nu e^{-\nu t} t^{-a t^2} e^{-a t^2} \sin[\omega_l \nu] . \quad (A16) \]

From Eq. (A13a) and the linearized Euler equation we obtain

\[ \frac{W_l(r, t)}{R^2 \rho_{\text{eq}}} = \frac{2l + 1}{4\pi} \frac{R^l}{R_{\text{shell}}} \alpha_l . \quad (A17) \]

Combined with Eq. (A10) and Eq. (A3) for \( l = 2 \), this leads to the following expression for the quadrupole perturbation of the density

\[ \delta \rho_2(r, t) = \frac{e^{-a t^2}}{M} \delta[r - R_{\text{shell}}} + \frac{8\pi R \rho_{\text{eq}}}{3\omega_2 M R_{\text{shell}}} \int_0^\infty d\nu e^{-\nu t^2} t^{-a t^2} \sin[\omega_2 \nu] \sin[\nu t] \delta[r - R] . \quad (A18) \]

3. The energy radiated by a vibrating, semi-Newtonian HIF

The energy radiated in gravitational waves by an oscillating Newtonian star can be computed by regarding the star’s gravitational field as a small perturbation of Minkowski spacetime. For details see Ref. \cite{24}. The energy radiated in the quadrupole is

\[ \frac{dE_2}{d\omega} = \frac{1}{32\pi^2 \omega^6} |A_{20}(\omega)|^2 \quad (A19) \]

where \( A_{20} \) is the Fourier amplitude of the mass quadrupole,

\[ A_{20}(\omega) = \int_{-\infty}^{\infty} d\nu e^{i\omega \nu} I_{20}(\nu) . \quad (A20) \]
In the slow motion approximation, $I_{20}(u)$ is simply

$$I_{20}(u) = \frac{16\pi}{5\sqrt{3}} \int_0^{\infty} \delta p_2(u,r)r^4 dr .$$

(A21)

In this approximation, Eqs. (A18) and (A20) can be combined to give the Fourier amplitude

$$A_{20}(\omega) = \frac{16\pi}{5\sqrt{3}M}\left( \frac{\pi}{a} e^{-\omega^2/(4a)} \right) \left\{ R_{\text{shell}}^4 + \frac{8\pi \rho_{\text{eq}} R_{\text{shell}}^3}{5} \left( \frac{\omega_2^2 - \omega^2}{\omega_2^2 - \omega^2} + 2i\omega/\tau_2 \right)^2 \right\} ,$$

(A22)

which can then be used in Eq. (A19) to compute the Newtonian quadrupole energy spectrum. The first term in curly brackets in the Fourier amplitude Eq. (A22) is the contribution only of the shell to the total energy radiated. The second term is the contribution only of the star’s oscillations to the energy. This should be contrasted with the relativistic procedure in which no such exact identification of the shell and star contributions can be made.

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