Structured mapping problems for linearly structured matrices

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Abstract. Given an appropriate class of structured matrices \( S \), we characterize matrices \( X \) and \( B \) for which there exists a matrix \( A \in S \) such that \( AX = B \) and determine all matrices in \( S \) mapping \( X \) to \( B \). We also determine all matrices in \( S \) mapping \( X \) to \( B \) and having the smallest norm. We use these results to investigate structured backward errors of approximate eigenpairs and approximate invariant subspaces, and structured pseudospectra of structured matrices.

Keywords. Structured matrices, structured backward errors, Jordan and Lie algebras, eigenvalues, eigenvectors, invariant subspaces.

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1 Introduction

Consider a stable linear time-invariant (LTI) control system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = 0, \\
y &= Cx + Du,
\end{align*}
\]

with \( A \in \mathbb{K}^{n \times n}, \ B \in \mathbb{K}^{n \times p}, \ C \in \mathbb{K}^{p \times n} \) and \( D \in \mathbb{K}^{p \times p} \). Here \( \mathbb{K} := \mathbb{R} \) or \( \mathbb{C} \), \( u \) is the input, \( x \) is the state and \( y \) is the output. The system (1) is said to be passive if the Hamiltonian matrix

\[
\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^* \end{bmatrix} := \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^* \\ -C^*R^{-1}C & -(A - BR^{-1}C)^* \end{bmatrix}
\]

has no purely imaginary eigenvalues, where \( R := D + D^* \), see [3] [6] [2]. A matrix \( \mathcal{H} \in \mathbb{K}^{2n \times 2n} \) of the form \( \mathcal{H} = \begin{bmatrix} A & F \\ G & -A^* \end{bmatrix} \) is called Hamiltonian, where \( G^* = G \) and \( F^* = F \). Equivalently, \( \mathcal{H} \) is Hamiltonian \( \iff (\mathcal{J}\mathcal{H})^* = \mathcal{J}\mathcal{H} \), where \( \mathcal{J} := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \) and \( I \) the identity matrix of size \( n \).

For passivation problem, when purely imaginary eigenvalues occur, one tries to perturb \( \mathcal{H} \) by a Hamiltonian matrix \( \mathcal{E} \) with small norm so that the perturbed matrix \( \mathcal{H} + \mathcal{E} \) has no purely imaginary eigenvalues. If such an \( \mathcal{E} \) exists, then for some \( X \in \mathbb{K}^{2n \times p} \) and \( D \in \mathbb{K}^{p \times p} \), we have

\[
(\mathcal{H} + \mathcal{E})X = XD \implies \mathcal{E}X = B := \mathcal{H}X - XD.
\]

This leads us to the following mapping problem.

**Problem 1. (Hamiltonian mapping problem)** Given \( X, B \in \mathbb{K}^{2n \times p} \), consider

\[
\begin{align*}
\text{Ham}(X, B) & := \{ \mathcal{H} \in \mathbb{K}^{2n \times 2n} : (\mathcal{J}\mathcal{H})^* = \mathcal{J}\mathcal{H} \text{ and } \mathcal{H}X = B \}, \\
\sigma^{\text{Ham}}(X, B) & := \inf \{ \| \mathcal{H} \| : \mathcal{H} \in \text{Ham}(X, B) \}.
\end{align*}
\]

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• Characterize $X, B \in \mathbb{K}^{2n \times p}$ for which $\text{Ham}(X, B) \neq \emptyset$ and determine all matrices in $\text{Ham}(X, B)$.

• Also determine all optimal solutions $\mathcal{H}_o \in \text{Ham}(X, B)$ such that $\|\mathcal{H}_o\| = \sigma_{\text{Ham}}(X, B)$.

Motivated by PROBLEM 1, we now consider structured mapping problem for various classes of structured matrices. Let $\mathcal{S}$ denote a class of structured matrices in $\mathbb{K}^{n \times n}$. The class $\mathcal{S}$ we consider in this paper is either a Jordan or a Lie algebra associated with an appropriate scalar product on $\mathbb{K}^n$. This provides a general setting that encompasses important classes of structured matrices such as Hamiltonian, skew-Hamiltonian, symmetric, skew-symmetric, pseudo-Hermitian, persymmetric, Hermitian, Skew-Hermitian, pseudo-Hermitian, pseudo-skew-Hermitian, to name only a few, see [9]. We, therefore, consider the following problem.

PROBLEM 2. (Structured Mapping Problem) Let $\mathcal{S} \subset \mathbb{K}^{n \times n}$ be a class of structured matrices and let $X, B \in \mathbb{K}^{n \times p}$. Set

$$\mathcal{S}(X, B) := \{ A \in \mathcal{S} : AX = B \},$$

$$\sigma^\mathcal{S}(X, B) := \inf \{ \| A \| : A \in \mathcal{S}(X, B) \}. $$

• Existence: Characterize $X, B \in \mathbb{K}^{n \times p}$ for which $\mathcal{S}(X, B) \neq \emptyset$.

• Characterization: Determine all matrices in $\mathcal{S}(X, B)$. Also determine all optimal solutions $A_o \in \mathcal{S}(X, B)$ such that $\|A_o\| = \sigma^\mathcal{S}(X, B)$.

We mention that structured backward error of an approximate invariant subspace of a structured matrix also leads to a structured mapping problem. A subspace $\mathcal{X}$ is invariant under $A$ if $A\mathcal{X} \subset \mathcal{X}$.

PROBLEM 3. (Structured backward error) Let $\mathcal{S} \subset \mathbb{K}^{n \times n}$ be a class of structured matrices and $A \in \mathcal{S}$. Let $\mathcal{X}$ be a subspace of $\mathbb{K}^n$. Set

$$\omega^\mathcal{S}(A, \mathcal{X}) := \min \{ \| \Delta A \| : \Delta A \in \mathcal{S} \text{ and } (A + \Delta A)\mathcal{X} \subset \mathcal{X} \}. $$

Find all $E \in \mathcal{S}$ such $(A + E)\mathcal{X} \subset \mathcal{X}$ and $\| E \| = \omega^\mathcal{S}(A, \mathcal{X})$.

If such a matrix $E \in \mathcal{S}$ exists then $(A + E)\mathcal{X} \subset \mathcal{X} \Rightarrow EV = VR - AV =: B$ for some $R$ and a full column rank matrix $V$ whose columns form a basis of $\mathcal{X}$. This shows that structured mapping problem naturally arises when analyzing structured backward error of an approximate invariant subspace.

Solutions of structured and unstructured mapping problems for a pair of vectors $x$ and $b$ in $\mathbb{K}^n$ have been studied extensively, see [9] and the references therein. In fact, for a pair of vectors $x$ and $b$ in $\mathbb{K}^n$, a complete solution of the structured mapping problem has been provided in [9] when the class $\mathcal{S} \subset \mathbb{K}^{n \times n}$ of structured matrices is a Jordan or a Lie algebra associated with an orthosymmetric scalar product on $\mathbb{K}^n$. For a pair of matrices $X$ and $B$ in $\mathbb{K}^{n \times p}$, existence and characterization of solutions to $AX = B$ have been discussed for Hermitian solutions in [10, 13] and for skew-Hermitian and symmetric solutions in [14]. Also, for the Frobenius norm, [13] provides an optimal Hermitian solution and [14] provides optimal skew-Hermitian and symmetric solutions to $AX = B$.

The main contributions of this paper are as follows. We provide a complete solution of the structured mapping problem (PROBLEM 2) when the class $\mathcal{S} \subset \mathbb{K}^{n \times n}$ of structured matrices is a Jordan or a Lie algebra associated with an orthosymmetric scalar product on $\mathbb{K}^n$. We show that for the spectral norm there are infinitely many optimal solutions whereas for the Frobenius norm the optimal solution is unique. We determine all optimal solutions for the spectral and the Frobenius norms. We show that the results in [9] obtained for a pair of vectors follow as special cases of our general results. Finally, as an application of the structured mapping problem, we analyze structured backward errors of approximate invariant subspaces, approximate eigenpairs, and structured pseudospectra of structured matrices.
Theorem 2.1 (Davis-Kahan-Weinberger, [4]) \(\text{Let } A, B, C \text{ be given matrices. Then for any positive number } \mu \text{ satisfying (1), there exists } D \text{ such that } \| [A C] \|_2 \leq \mu. \) Indeed, those \( D \) which have this property are exactly those of the form

\[
D = -KA^HL + \mu(I - KK^H)^{1/2}Z(I - L^HL)^{1/2},
\]
where \( K^H := (\mu^2I - A^HA)^{-1/2}B^H \), \( L := (\mu^2I - AA^H)^{-1/2}C \) and \( Z \) is an arbitrary contraction, that is, \( \|Z\|_2 \leq 1 \).

We mention that when \((\mu^2I - A^HA)\) is singular, the inverses in \( K^H \) and \( L \) are replaced by their Moore-Penrose pseudo-inverses (see, [11]). An interesting fact about Theorem 2.1 is that if \( T(D) \) is symmetric or skew-symmetric or Hermitian or skew-Hermitian then the solution matrices \( D \) are, respectively, symmetric or skew-symmetric or Hermitian or skew-Hermitian [1].

### 3 Solution of structured mapping problem

For compact representations of our results, in the rest of the paper, we write \( A^* \) to denote the transpose \( A^T \) or the conjugate transpose \( A^H \). Often we write \( A^* \) with \( * \in \{T, H\} \). With this notational convention, define the map \( \mathcal{F}_*: \mathbb{K}^{n \times p} \times \mathbb{K}^{n \times p} \to \mathbb{K}^{n \times n} \) by

\[
\mathcal{F}_*(X, B) := \begin{cases} 
BX^T + (BX^T)^* - (X^T)^*(X^*B)^*X^T, & \text{if } (X^*B)^* = X^*B \\
BX^T - (BX^T)^* - (X^T)^*(X^*B)^*X^T, & \text{if } (X^*B)^* = -X^*B \\
BX^T, & \text{else}
\end{cases}
\]

where \( X^T \) is the Moore-Penrose pseudoinverse of \( X \) and \( * \in \{T, H\} \).

We write \( \mathcal{F}_* = \mathcal{F}_T \) when \( * = T \) and \( \mathcal{F}_* = \mathcal{F}_H \) when \( * = H \). Then it follows that \( \mathcal{F}_* \) has the following properties.

1. If \( X^T B \) is symmetric/skew-symmetric then \( \mathcal{F}_T(X, B) \in \text{sym/skew-sym} \).
2. If \( X^H B \) is Hermitian/skew-Hermitian then \( \mathcal{F}_H(X, B) \in \text{Herm/skew-Herm} \).
3. If \( BX^T = B \) then \( \mathcal{F}_*(X, B)X = B \).

This shows that the matrix \( \mathcal{F}_*(X, B) \) is a potential candidate for solution of the structured mapping problem for four classes of structured matrices, namely, \( \text{sym/skew-sym}, \text{Herm} \) and \( \text{skew-Herm} \). More generally, the following result provides a necessary and sufficient condition for existence of solution of the structured mapping problem.

**Theorem 3.1 (Existence)** Let \((X, B) \in \mathbb{K}^{n \times p} \times \mathbb{K}^{n \times p} \) and \( S \in \{\mathbb{J}, \mathbb{L}\} \). Then there is a matrix \( A \in S \) such that \( AX = B \) if and only if (a) \( BX^T = B \) and (b) the condition in Table 1 holds.

| M       | S = \( \mathbb{J} \) | S = \( \mathbb{L} \) |
|---------|----------------------|----------------------|
| \( M^T = M \) | \((X^TMB)^T = X^TMB\) | \((X^TMB)^T = -X^TMB\) |
| \( M^T = -M \) | \((X^TMB)^T = -X^TMB\) | \((X^TMB)^T = X^TMB\) |
| \( M^H = M \) | \((X^HMB)^H = X^HMB\) | \((X^HMB)^H = -X^HMB\) |
| \( M^H = -M \) | \((X^HMB)^H = -X^HMB\) | \((X^HMB)^H = X^HMB\) |

**Table 1:** Condition for \( S(X, B) \neq \emptyset \).

**Proof:** Suppose that there exists \( A \in S \) such that \( AX = B \). Then \( X^*MB = X^*MAX \) for \( * \in \{T, H\} \). Since \( MA \in MS \), by (6) it follows that \( X^*MB \) is symmetric/skew-symmetric (resp., Hermitian/skew-Hermitian) when \( * = T \) (resp., \( * = H \)). Hence the conditions in (b) are satisfied. Again since \( AX = B \), we have \( BX^T = AX X^T = AX = B \).

Conversely, suppose that the conditions are satisfied. Then setting \( A := M^{-1} \mathcal{F}_*(X, MB) \), it follows from (6) and the properties of \( \mathcal{F}_* \) that \( AX = B \) and \( A \in S \), where \( * \in \{T, H\} \). This completes the proof. ■
Remark 3.2 If $X$ has full column rank then $X^\dagger X = I$. Consequently, $BX^\dagger X = B$. Thus for a full column rank matrix $X$ the condition (a) in Theorem 3.1 is automatically satisfied.

We mention that for the special case when $x \in \mathbb{K}^n$ and $b \in \mathbb{K}^n$, by Theorem 3.1 we obtain the following necessary and sufficient condition provided in [9]:

| $M$ | $S = \mathbb{J}$ | $S = \mathbb{L}$ |
|-----|------------------|------------------|
| $M^T = M$ | any $x, b \in \mathbb{K}^n$ | $x^T Mb = 0$ |
| $M^T = -M$ | $x^T Mb = 0$ | any $x, b \in \mathbb{K}^n$ |
| $M^H = M$ | $x^H Mb \in \mathbb{R}$ | $x^H Mb \in i\mathbb{R}$ |
| $M^H = -M$ | $x^H Mb \in i\mathbb{R}$ | $x^H Mb \in \mathbb{R}$ |

Table 2: Necessary and sufficient condition for $S(x, b) \neq \emptyset$

Now given a pair of matrices $X$ and $B$ in $\mathbb{K}^{n \times p}$ satisfying the conditions in Theorem 3.1 the following result characterizes solution of the structured mapping problem.

**Theorem 3.3 (Characterization)** Let $(X, B) \in \mathbb{K}^{n \times p} \times \mathbb{K}^{n \times p}$ and $S \in \{\mathbb{J}, \mathbb{L}\}$. Suppose that $S(X, B) \neq \emptyset$. Then $A \in S(X, B)$ if and only if $A$ is of the form

$$A = \begin{cases} M^{-1}F_T(X, MB) + M^{-1}(I - XX^\dagger)^T Z(I - XX^\dagger), & \text{if } MS \in \{\text{sym, skew-sym}\}, \\ M^{-1}F_R(X, MB) + M^{-1}(I - XX^\dagger)Z(I - XX^\dagger), & \text{if } MS \in \{\text{Herm, skew-Herm}\}, \end{cases}$$

for some $Z$ such that $Z \in MS$.

(a) **Frobenius norm:** Define $A_o := M^{-1}F_T(X, MB)$ when $MS \in \{\text{sym, skew-sym}\}$, and $A_o := M^{-1}F_R(X, MB)$ when $MS \in \{\text{Herm, skew-Herm}\}$. Then $A_o$ is the unique matrix in $S(X, B)$ such that $A_oX = B$ and that

$$\sigma^S(X, B) = \|A_o\|_F = \begin{cases} \sqrt{2\|BX\|_F^2 - \text{Tr}(MBX^\dagger(MBX^\dagger)^HXX^\dagger)}, & \text{when } MS \in \{\text{sym, skew-sym}\}, \\ \sqrt{2\|BX\|_F^2 - \text{Tr}(MBX^\dagger(MBX^\dagger)^HXX^\dagger)}, & \text{when } MS \in \{\text{Herm, skew-Herm}\}. \end{cases}$$

(b) **Spectral norm:** Set $\sigma^\text{unstruc}(X, B) := \sigma^S(X, B)$ when $S = \mathbb{K}^{n \times n}$. Then

$$\sigma^S(X, B) = \|BX\|_2 = \sigma^\text{unstruc}(X, B).$$

Suppose that $\text{rank}(X) = r$. Consider the SVD $X = USV^H$ and partition $U$ as $U = [U_1 \ U_2]$, where $U_1 \in \mathbb{K}^{n \times r}$.

**Case-I.** If $MS \in \{\text{sym, skew-sym}\}$ then consider the matrix

$$A_o := M^{-1}F_T(X, MB) - M^{-1}(I - XX^\dagger)^TKU_1^HMBX^\dagger K^T(I - XX^\dagger) + M^{-1}f(Z),$$

where $f(Z) = \mu(U_1 - U_2^H K^H U_2)^{1/2}Z(I - U_2^H K^H U_2)^{1/2}U_2^H$, $\mu = \|BX\|_2$,

$$K = \begin{cases} MBX^\dagger U_1(\mu^2 I - U_1^H MBX^\dagger MBX^\dagger U_1)^{-1/2}, & \text{when } MS = \text{sym}, \\ MBX^\dagger U_1(\mu^2 I + U_1^H MBX^\dagger MBX^\dagger U_1)^{-1/2}, & \text{when } MS = \text{skew-sym}, \end{cases}$$

and $Z$ is an arbitrary contraction such that $Z = Z^T$ (resp. $Z = -Z^T$) when $MS = \text{sym}$ (resp. $MS = \text{skew-sym}$). Then $A_o \in S(X, B)$ and $\|A_o\|_2 = \sigma^S(X, B)$.  

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Case-II. If $MS = \text{Herm}$ then consider the matrix

$$A_o := M^{-1}F_0(X, MB) - M^{-1}(I - XX^\dagger)KU_1^HMBX^\dagger U_1K^H(I - XX^\dagger) + M^{-1}f(Z),$$

where $f(Z) = \mu U_2(I - U_2^HKKU_2)^{1/2}Z(I - U_2^HKKU_2)^{1/2}U_2^H\mu = \|BX\|_2,$

$$K = MBX^\dagger U_1(\mu^2I - U_1^HMBX^\dagger MBX^\dagger U_1)^{-1/2},$$

and $Z = Z^H$ is an arbitrary contraction. Then $A_o \in S(X, B)$ and $\|A_o\|_2 = \sigma^S(X, B).$

**Proof:** First, observe that $AX = B \iff MAX = MB.$ Consequently, we have $A \in S(X, B) \iff MA \in MS(X, MB).$ Further, since $M$ is unitary, $\sigma^S(X, B) = \sigma^{MS}(X, MB)$ for the spectral and the Frobenius norms. By (9), $MS \in \{\text{sym, skew-sym, Herm, skew-Herm}\}.$ Thus, it boils down to proving the results for symmetric, skew-symmetric, Hermitian and skew-Hermitian matrices. Indeed, if $\phi(X, B)$ is a symmetric or skew-symmetric or Hermitian or skew-Hermitian solution of $AX = B$ then $M^{-1}\phi(X, MB) \in S(X, B).$

Further, note that a skew-Hermitian solution of $AX = B$ can be obtained from a Hermitian solution of $iAX = iB$ and vice versa. Indeed, if $AX = B$ and $X^H B$ is skew-Hermitian then $iAX = iB$ and $iX^H B$ is Hermitian. Hence $\phi(X, iB)$ is a Hermitian solution of $iAX = iB$ and $-i\phi(X, iB)$ is a skew-Hermitian solution of $AX = B.$ Consequently, we only need to prove the results for symmetric, skew-symmetric and Hermitian matrices. We prove these results separately in Theorem 3.5. Hence the proof. \[ \blacksquare \]

**Remark 3.4**

(a) We mention that the solution set $S(X, B)$ as characterized in Theorem 3.5 can be written compactly as

$$S(X, B) = \begin{cases} M^{-1}F_0(X, MB) + M^{-1}(I - XX^\dagger)^TMS(I - XX^\dagger), & \text{if } MS \in \{\text{sym, skew-sym}\}, \\ M^{-1}F_0(X, MB) + M^{-1}(I - XX^\dagger)MS(I - XX^\dagger), & \text{if } MS \in \{\text{Herm, skew-Herm}\}. \end{cases}$$

Here $x + S := \{x + s : s \in S\}.$

(b) We also mention that when $X$ has full column rank, for the Frobenius norm, we have

$$\sigma^S(X, B) = \sqrt{2\|B(X^HX)^{-1/2}\|_F^2 - \|(X^TXX^T) - 1/2X^TMB(X^HX)^{-1/2}\|_F^2}$$

when $MS \in \{\text{sym, skew-sym}\},$ and

$$\sigma^S(X, B) = \sqrt{2\|B(X^HX)^{-1/2}\|_F^2 - \|(X^HX)^{-1/2}MXX^TMB(X^HX)^{-1/2}\|_F^2}$$

when $MS \in \{\text{Herm, skew-Herm}\}.$ Indeed, when $X$ has full column rank, we have $X^\dagger X = I$ and $X^\dagger = (X^HX)^{-1}X^T$ and hence the results follow. On the other hand, for the spectral norm, we have $\sigma^S(X, B) = \|B(X^HX)^{-1/2}\|_2.$

To complete the proof of Theorem 3.5 for the rest of this section, we consider $S$ to be such that $S \in \{\text{sym, skew-sym, Herm, skew-Herm}\}.$ Observe that if $A \in \mathbb{K}^{n \times n}$ is given by

$$A := \begin{bmatrix} A_{11} & \pm A_{12} \\ A_{12} & A_{22} \end{bmatrix}$$

then $\|A\|_F = \left(2\left\|\begin{bmatrix} A_{11} \\ A_{12} \end{bmatrix}\right\|_F^2 - \|A_{11}\|_F^2 + \|A_{22}\|_F^2\right)^{1/2}.$ \[ (8) \]

We repeatedly use this fact in the sequel.
Theorem 3.5 (Special solutions) Let \((X, B) \in \mathbb{K}^{n \times p} \times \mathbb{K}^{n \times p}\) and \(S \in \{\text{sym, skew-sym, Herm}\}\). Suppose that \(S(X, B) \neq \emptyset\). Then \(\mathcal{S}(X, B)\) is a unique matrix in \(\mathcal{S}(X, B)\) such that

\[
A = \begin{cases} 
F_T(X, B) + (I - XX^\dagger)^T Z(I - XX^\dagger), & \text{if } S \in \{\text{sym, skew-sym}\}, \\
F_u(X, B) + (I - XX^\dagger) Z(I - XX^\dagger), & \text{if } S = \text{Herm},
\end{cases}
\]

for some \(Z \in \mathbb{S}\).

(a) Frobenius norm: Consider \(A_o := F_T(X, B)\) when \(S \in \{\text{sym, skew-sym}\}\), and \(A_o := F_u(X, B)\) when \(S = \text{Herm}\). Then \(A_o\) is a unique matrix in \(\mathcal{S}(X, B)\) such that

\[
\sigma^S(X, B) = \|A_o\|_F = \begin{cases} 
\sqrt{2} \|BX^\dagger\|^2_F - \text{Tr}(BX^\dagger(BX^\dagger)^H(XX^\dagger)^T), & \text{when } S \in \{\text{sym, skew-sym}\}, \\
\sqrt{2} \|BX^\dagger\|^2_F - \text{Tr}(BX^\dagger(BX^\dagger)^HXX^\dagger), & \text{when } S = \text{Herm}.
\end{cases}
\]

(b) Spectral norm: We have

\[
\sigma^S(X, B) = \|BX^\dagger\|_2 = \sigma^{\text{unstruc}}(X, B).
\]

Suppose that \(\text{rank}(X) = r\). Consider the SVD \(X = U \Sigma V^H\) and partition \(U\) as \(U = [U_1 \ U_2]\), where \(U_1 \in \mathbb{K}^{n \times r}\).

Case-I. If \(S \in \{\text{sym, skew-sym}\}\) then consider the matrix

\[
A_o := F_T(X, B) - (I - XX^\dagger)^T KU_1^H BX^\dagger U_1 K^T(I - XX^\dagger) + f(Z),
\]

where \(f(Z) = \mu U_2(I - U_2^H K H^H U_2)^{1/2}Z(I - U_2^H K H^H U_2)^{1/2}U_2^H\), \(\mu = \|BX^\dagger\|_2\),

\[
K = \begin{cases} 
BX^\dagger U_1 (\mu^2 I - U_1^H BX^\dagger BX^\dagger U_1)^{-1/2}, & \text{when } S = \text{sym}, \\
BX^\dagger U_1 (\mu^2 I + U_1^H BX^\dagger BX^\dagger U_1)^{-1/2}, & \text{when } S = \text{skew-sym},
\end{cases}
\]

and \(Z\) is an arbitrary contraction such that \(Z = Z^T\) (resp., \(Z = -Z^T\)) when \(S = \text{sym}\) (resp., \(S = \text{skew-sym}\)). Then \(A_o \in \mathcal{S}(X, B)\) and \(\|A_o\|_2 = \sigma^S(X, B)\).

Case-II. If \(S = \text{Herm}\) then consider the matrix

\[
A_o := F_u(X, B) - (I - XX^\dagger) KU_1^H BX^\dagger U_1 K^T(I - XX^\dagger) + f(Z),
\]

where \(f(Z) = \mu U_2(I - U_2^H K H^H U_2)^{1/2}Z(I - U_2^H K H^H U_2)^{1/2}U_2^H\), \(\mu = \|BX^\dagger\|_2\),

\[
K = BX^\dagger U_1 (\mu^2 I - U_1^H BX^\dagger BX^\dagger U_1)^{-1/2},
\]

and \(Z = Z^H\) is an arbitrary contraction. Then \(A_o \in \mathcal{S}(X, B)\) and \(\|A_o\|_2 = \sigma^S(X, B)\).

Proof: First, suppose that \(S = \text{sym}\). By assumption there exists \(A \in \mathcal{S}\) be such that \(AX = B\). Note that \(\text{Range}(X) = \text{Range}(U_1)\). Thus representing \(A\) relative to the decomposition

\[
A : \text{Range}(X) \oplus \text{Range}(X)^\perp \rightarrow \text{Range}(X) \oplus \text{Range}(X)^\perp,
\]

we have \(A = \overline{U} \ U^H \ A \ U U^H\). Set \(\hat{A} = U^T A U = \begin{bmatrix} A_{11} & A_{12}^T \\ A_{12} & A_{22} \end{bmatrix}\). Then \(\hat{A} \in \mathbb{S}\) and \(\|A\| = \|\hat{A}\|\) for the spectral and the Frobenius norms. Set \(\Sigma_1 := \Sigma(1 : r, 1 : r)\). Let \(V = [V_1, V_2]\) be a conformal partition of \(V\) such that \(X = U_1 \Sigma_1 V_1\). Now \(AX = B \Rightarrow \overline{U} \hat{A} U^H X = B\). This gives

\[
\begin{bmatrix} A_{11} & A_{12}^T \ U_1^H \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} U_1 \ U_2^T \end{bmatrix} X = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} B \Rightarrow \begin{bmatrix} A_{11} & A_{12}^T \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 V_1^H \\ 0 \end{bmatrix} = \begin{bmatrix} U_1^T B \\ U_2^T B \end{bmatrix}
\]
and consequently $A_1 \Sigma_1 V_1^H = U_1^T B$ and $A_2 \Sigma_1 V_1^H = U_2^T B$.

Therefore, we have $A_1 = U_1^T B \Sigma_1^{-1} V_1^T$ and $A_2 = U_2^T B \Sigma_1^{-1} V_1^T$. Notice that $A_1$ is symmetric if and only if $X^T B = B^T X$ and $B X^T X = B$. Indeed $X^T B = B^T X$ gives $X_1 \Sigma_1 U_1^T B = B^T U_1 \Sigma_1 V_1^H$ and thus $B^T U_1 = \Sigma_1 U_1^T B \Sigma_1^{-1}$. Now

\[
(U_1^T B X^T U_1)^T = U_1^T (X^T B)^T U_1 = U_1^T \Sigma_1^{-1} V_1^T B^T U_1 = \Sigma_1^{-1} V_1^T \Sigma_1 U_1^T B \Sigma_1^{-1} V_1
\]

as desired. Thus we have

\[
\hat{A} = \begin{bmatrix}
U_1^T B X^T U_1 & (U_2^T B \Sigma_1^{-1})^T
\end{bmatrix}
\]

(9)

Then by \[3\] we have $\| \hat{A} \|^2 = 2 \| B X^T \|^2 - \text{Tr}(B X^T B X^T)^H (X X^T)^T + \| A_{22} \|^2$. Hence, for the Frobenius norm, setting $A_{22} = 0$ we obtain a unique matrix

\[
A = U \begin{bmatrix}
U_1^T B X^T U_1 & (U_2^T B \Sigma_1^{-1})^T
\end{bmatrix}
\]

such that $A \in \mathbb{S}(X, B)$ and $\| A \|_F = \sqrt{2 \| B X^T \|^2 - \text{Tr}(B X^T B X^T)^H (X X^T)^T} = \sigma^2(X, B)$.

Now from \[3\] we have

\[
A = U \begin{bmatrix}
U_1^T B X^T U_1 & (U_2^T B \Sigma_1^{-1})^T
\end{bmatrix}
\]

which is arbitrary. Relative to the decomposition

\[
\mu := \left\| \begin{bmatrix}
U_1^T B \Sigma_1^{-1} & U_1^T B \Sigma_1^{-1} \end{bmatrix}
\right\|_2 = \| B X^T \|_2 = \| B X^T \|_2.
\]

Then it follows that $\| \hat{A} \|_2 \geq \mu$. Now by Theorem \[2.1\] we have $\| \hat{A} \|_2 = \mu$ when

\[
A_{22} = -K \hat{A}_{11} K^T + \mu (I - K K^T)^{1/2} Z (I - K K^T)^{1/2}
\]

(10)

where $K = \mu^2 I - U_2^T B X^T U_1 (\mu^2 I - U_2^T B X^T U_1)^{-1} U_2^T B X^T U_1$ and $Z$ is an arbitrary contraction such that $Z = Z^T$ (resp., $Z = -Z^T$) when $S = \text{sym}$ (resp., $S = \text{skew-sym}$). Thus we have

\[
A = B X^T + (B X^T)^T - (X X^T)^T B X^T + \Sigma_2 A_{22} U_2^H
\]

such that $A \in \mathbb{S}, AX = B$ and $\| A \|_2 = \| B X^T \|_2 = \sigma^2(X, B)$, where $A_{22}$ is given in (10). Upon simplification we obtain the desired form of $A$.

Next suppose that $S = \text{skew-sym}$. Again representing $A$ relative to the decomposition

\[
A : \text{Range}(X) \oplus \text{Range}(X)^\perp \rightarrow \text{Range}(X) \oplus \text{Range}(X)^\perp,
\]
we have \( A = U U^H A U U^H \). Set \( \hat{A} = U^T A U = \begin{bmatrix} A_{11} & -A_{12}^T \\ A_{12} & A_{22} \end{bmatrix} \). Then \( \hat{A} \in \mathbb{S} \) and \( \|A\| = \|\hat{A}\| \) for the spectral and the Frobenius norms. By assumption \( AX = B \Rightarrow U \hat{A} U^H X = B \). This gives

\[
\begin{bmatrix} A_{11} & -A_{12}^T \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} U^H \\ U_2^H \end{bmatrix} X = U^T B = \begin{bmatrix} U^T \\ U_2^T \end{bmatrix} B \Rightarrow \begin{bmatrix} A_{11} & -A_{12}^T \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 V_1^H \\ 0 \end{bmatrix} = \begin{bmatrix} U^T \\ U_2^T \end{bmatrix} B.
\]

Consequently we have \( A_{11} \Sigma_1 V_1^H = U_1^T B \) and \( A_{12} \Sigma_1 V_1^H = U_2^T B \) and hence

\[
\hat{A} = \begin{bmatrix} U_1^T B X_1^U_1 & - (U_2^T B V_1 \Sigma_1^{-1})^T \\ U_2^T B V_1 \Sigma_1^{-1} & A_{22} \end{bmatrix}.
\] (11)

As before, setting \( A_{22} = 0 \) in (11) we obtain the desired results for the Frobenius norm.

As for the spectral norm, applying Theorem 2.1 to the matrix \( \hat{A} \) in (11) and following steps similar to those in the case when \( S = \text{sym} \), we obtain the desired results.

Finally, suppose that \( S = \text{Herm} \). Then representing \( A \) relative to the decomposition

\[ A : \text{Range}(X) \oplus \text{Range}(X)^\perp \rightarrow \text{Range}(X) \oplus \text{Range}(X)^\perp, \]

we have \( A = U U^H A U U^H \). Set \( \hat{A} = U^H A U = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \). Then \( \hat{A} \in \mathbb{S} \) and \( \|A\| = \|\hat{A}\| \) for the spectral and the Frobenius norms. Now \( AX = B \Rightarrow U \hat{A} U^H X = B \). This gives

\[
\begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} U^H \\ U_2^H \end{bmatrix} X = U^H B = \begin{bmatrix} U^H \\ U_2^H \end{bmatrix} B \Rightarrow \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 V_1^H \\ 0 \end{bmatrix} = \begin{bmatrix} U^H \\ U_2^H \end{bmatrix} B.
\]

Consequently, we have \( A_{11} \Sigma_1 V_1^H = U_1^T B \) and \( A_{12} \Sigma_1 V_1^H = U_2^T B \) and hence

\[
\hat{A} = \begin{bmatrix} U_1^H B X_1^U_1 & (U_2^H B V_1 \Sigma_1^{-1})^H \\ U_2^H B V_1 \Sigma_1^{-1} & A_{22} \end{bmatrix}.
\] (12)

Now by (8) we have \( \|\hat{A}\|_F^2 = 2\|B X_1^U_1\|_F^2 - \text{Tr}(B X_1^U_1 (B X_1^U_1)^H X X^\top) + \|A_{22}\|_F^2 \). Hence setting \( A_{22} = 0 \) in (12), we obtain a unique matrix

\[
A = U \begin{bmatrix} U_1^H B X_1^U_1 & (U_2^H B V_1 \Sigma_1^{-1})^H \\ U_2^H B V_1 \Sigma_1^{-1} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ U_2^H \end{bmatrix} = BX_1^U + (B X_1^U)^H - X X^\top B X_1^U = \mathcal{F}_0(X, B)
\]

such that \( A \in S(X, B) \) and \( \|A\|_F = \sqrt{2\|B X_1^U\|_F^2 - \text{Tr}(B X_1^U (B X_1^U)^H X X^\top)} = \sigma_0(X, B) \).

Now from (12) we have

\[
A = U \begin{bmatrix} U_1^H B X_1^U_1 & U_2^H B V_1 \Sigma_1^{-1} \\ U_2^H B V_1 \Sigma_1^{-1} & A_{22} \end{bmatrix} \begin{bmatrix} U^H \\ U_2^H \end{bmatrix} = (U_1 U_1^H) B X_1^U + (V_1 \Sigma_1^{-1} U_1^H B H U_2 U_2^H + (I - U_1 U_1^H) B X_1^U + U_2 A_{22} U_2^H
\]

\[
= X X^\top B X_1^U + (X_1^U)^H B H (I - X X^\top) + (I - X X^\top) B X_1^U + U_2 A_{22} U_2^H
\]

\[
= B X_1^U + (B X_1^U)^H - (X_1^U)^H X X^\top B X_1^U + (I - X X^\top) Z (I - X X^\top)
\]

\[
= \mathcal{F}_0(X, B) + (I - X X^\top) Z (I - X X^\top),
\]

where \( Z \in \mathbb{S} \) is arbitrary.

For the spectral norm, again consider the matrix \( \hat{A} \) given in (12) and set

\[
\mu := \left\| \begin{bmatrix} U_1^H B V_1 \Sigma_1^{-1} \\ U_2^H B V_1 \Sigma_1^{-1} \end{bmatrix} \right\|_2 = \left\| U H B V_1 \Sigma_1^{-1} \right\|_2 = \|B X_1^U\|_2.
\]
Then it follows that $\|\hat{A}\|_2 \geq \mu$. Now by Theorem 2.1 we have $\|\hat{A}\|_2 = \mu$ when
\[
A_{22} = -K_1A_{11}K_1^H + \mu(I - K_1K_1^H)^{1/2}Z(I - K_1K_1^H)^{1/2}
\]  
(13)
where $K_1 = U_1^HBX^1U_1(\mu^2I - U_1^HBX^1BX^1U_1)^{-1/2}$ and $Z = Z^H$ is an arbitrary contraction. Thus we have $A = BX^1 + (BX^1)^H - XX^1BX^1 + U_2A_{22}U_2^H = F_\delta(X, B) + U_2A_{22}U_2^H$ such that $A \in S(X, B)$ and $\|A\|_2 = \|BX^1\|_2 = \sigma^0(X, B)$, where $A_{22}$ is given in (13). Finally, upon simplification, we obtain the desired form of $A$. This completes the proof.  

4 Applications of structured mapping problem

We now consider a few applications of the structured mapping problem. As before, we consider $S \in \{J, L\}$. Given a full column rank matrix $X \in \mathbb{K}^{n \times p}$, a matrix $D \in \mathbb{K}^{p \times p}$ and a structured matrix $A \in S$, we say that $(X, D)$ is an invariant pair $A$ if $AX = XD$. Then a partially prescribed inverse eigenvalue problem can be stated as follows.

PROBLEM-I. Given a full column rank matrix $X \in \mathbb{K}^{n \times p}$ and a matrix $D \in \mathbb{K}^{p \times p}$, find a matrix $A_o \in S$, if it exists, such that $A_oX = XD$ and $\|A_o\| = \tau^S(X, D)$, where
\[
\tau^S(X, D) := \inf\{\|A\| : A \in S \text{ and } AX = XD\}.
\]

When $A_o \in S$ exists, $(X, D)$ provides a partial spectral decomposition of $A_o$ in the sense that $\Lambda(D) \subset \Lambda(A)$ and that $\text{span}(X)$ is an invariant subspace of $A$ corresponding to $\Lambda(D)$. Obviously, PROBLEM-I is a structured mapping problem for $X$ and $B := XD$. Consequently, it has a solution if and only if $X$ and $B := XD$ satisfy the conditions in Theorem 3.1 that is, if and only if $(X^*MXD)^* = X^*MXD$ or $(X^*MXD)^* = -X^*MXD$ for $* \in \{T, H\}$. An optimal solution $A_o$ is then given by Theorem 3.3 with $\tau^S(X, D) = \|A_o\| = \sigma^S(X, XD)$ for the spectral and the Frobenius norms. Note that PROBLEM-I has a solution when $X^*MX = 0$ and in such a case the subspace spanned by the columns of $X$ is called $M$-neutral - in short $X$ is $M$-neutral. Thus if $X$ is $M$-neutral then for any $D \in \mathbb{K}^{p \times p}$, $(X, D)$ is an invariant pair for some $A \in S$.

A related problem which arises when analyzing backward errors of approximate invariant pairs is as follows.

PROBLEM-II. Given a full column rank matrix $X \in \mathbb{K}^{n \times p}$, a matrix $D \in \mathbb{K}^{p \times p}$ and a structured matrix $A \in S$, find a structured matrix $E_o \in S$, if it exists, such that $E_oX = XD$ and $\|E_o - A\| = \eta^S(A, X, D)$, where $\eta^S(A, X, D) := \inf\{\|E - A\| : E \in S \text{ and } EX = XD\}$.

PROBLEM-II occurs, for example, when Krylov subspace method such as the Arnoldi method is used to compute a few eigenvalues of a (large) matrix $A$. Indeed, starting with a unit vector $v_1 \in \mathbb{C}^n$, after $p$ steps of Arnoldi method we have
\[
MV_p = V_pH_p + \alpha v_{p+1}v_{p+1}^T,
\]
where $V_p := [v_1, \ldots, v_p]$, $H_p$ is upper hessenberg and $\{v_1, \ldots, v_{p+1}\}$ is an orthonormal basis of the Krylov subspace $K(v_1, A) := \text{span}(v_1, Av_1, \ldots, A^p v_1)$. Thus, when $|\alpha|$ is small, $(V_p, H_p)$ is an approximate invariant pair of $A$ and hence the backward error of $(V_p, H_p)$ may be gainfully used in analyzing errors in the computed eigenvalues of $A$.

Writing $E = A + \Delta A$, it follows that PROBLEM-II is a structured mapping problem for $X$ and $B := XD - AX$. Indeed, if $\Delta A \in S$ is such that $(A + \Delta A)X = XD$ then $\Delta AX = XD - AX = B$. Consequently, $\Delta A \in S$ exists if and only if $X$ and $B = XD - AX$ satisfy the conditions in Theorem 3.1. An optimal solution $\Delta A_o$ is then given by Theorem 3.3 with
\[
\eta^S(A, X, D) = \|\Delta A_o\| = \sigma^S(X, XD - AX)
\]
for the spectral and the Frobenius norms. The quantity \( \eta^S(A, X, D) \) is the structured backward error of \((X, D)\) as an approximate invariant pair of \(A\). Note that the conditions in Theorem 3.1 are satisfied if and only if \(X^TMXD - XMXX \) is Hermitian or skew-Hermitian (resp., symmetric or skew-symmetric). In particular, this condition is satisfied when \(M^H = -M\), \(S = \mathbb{L}\) and \(X \in \mathbb{K}^{n \times p}\) is \(M\)-neutral. This fact plays an important role in spectral perturbation analysis of Hamiltonian matrices, see [2].

We now consider a special case when \(X = x \in \mathbb{C}^n\) and \(D = \lambda \in \mathbb{C}\) and derive structured backward error \(\eta^S(A, x, \lambda)\) of an approximate eigenpair \((\lambda, x)\) of \(A\). Set \(r := \lambda x - Ax\). Then there is a matrix \(E \in \mathbb{S}\) such that \((A + E)x = \lambda x\) if and only if \(x\) and \(r\) satisfy the condition in Table 2. Let \(\eta^S(A, x, \lambda)\) denote \(\eta^S(A, x, \lambda)\) for the spectral norm (resp., Frobenius norm). Then by Theorem 3.1 we have the following.

**Corollary 4.1.** Let \(\mathcal{S} \in \{\mathbb{J}, \mathbb{L}\}\) and \(A \in \mathcal{S}\). Suppose that \(\lambda \in \mathbb{C}\) and \(x \in \mathbb{C}^n \setminus \{0\}\) satisfy the condition in Table 2. Then we have

\[
\eta^S(A, x, \lambda) = \|r\|_2/\|x\|_2 \quad \text{and} \quad \eta^S_F(A, x, \lambda) = \sqrt{2\|r\|_2^2 - |(r, x)_M|^2}.
\]

Define \(E\) by

\[
E := \begin{cases} 
(x^TR)M^{-1}xx^T + M^{-1}xx^T(I - xx^H) + M^{-1}(I - xx^H)rxx^H, & \text{if } MA \in \text{sym} \\
M^{-1}(I - xx^H)rxx^H - M^{-1}xx^T(I - xx^H), & \text{if } MA \in \text{skew-sym} \\
(H^T)rM^{-1}xx^H + M^{-1}xx^T(I - xx^H) + M^{-1}(I - xx^H)rxx^H, & \text{if } MA \in \text{Herm}
\end{cases}
\]

Then \(E \in \mathcal{S}\) is a unique matrix such that \((A + E)x = \lambda x\) and \(\|E\|_F = \eta^S_F(A, x, \lambda)\). Further, when \(\|r\|_2 \neq |x^Hr|\), define

\[
\Delta A := \begin{cases} 
E - \frac{x^T(M^{-1}(I - xx^H))^T(1 - xx^H)}{\|r\|_2^2 - |x^Hr|^2}, & \text{if } MA \in \text{sym} \\
E, & \text{if } MA \in \text{skew-sym} \\
E - \frac{x^T(M^{-1}(I - xx^H)rxx^H)}{\|r\|_2^2 - |x^Hr|^2}, & \text{if } MA \in \text{Herm}
\end{cases}
\]

else set \(\Delta A := E\). Then \(\Delta A \in \mathcal{S}\) such that \((A + \Delta A)x = \lambda x\) and \(\|\Delta A\|_2 = \eta^S(A, x, \lambda)\).

Finally, yet another related problem which arises when dealing with backward errors of approximate invariant subspaces as well as in inverse eigenvalue problem with a specified invariant subspace is as follows.

**Problem-III.** Let \(\mathcal{X}\) be a \(p\)-dimensional subspace of \(\mathbb{K}^n\) and \(A \in \mathcal{S}\). Then find a structured matrix \(E \in \mathcal{S}\), if it exists, such that \(E\mathcal{X} \subset \mathcal{X}\) and \(\|A - E\| = \omega^S(A, \mathcal{X})\), where

\[
\omega^S(A, \mathcal{X}) := \inf\{\|E - A\| : E \in \mathcal{S} \text{ and } E\mathcal{X} \subset \mathcal{X}\}.
\]

Let \(U \in \mathbb{K}^{n \times p}\) be an isometry such that \(\text{span}(U) = \mathcal{X}\). If \(E \in \mathcal{S}\) is such that \(E\mathcal{X} \subset \mathcal{X}\) then \(EU = UD\) for some \(p\)-by-\(p\) matrix \(D\). Then setting \(\Delta A := E - A\), we have \(\Delta AU = UD - AU\) and hence by Theorem 3.1 \(\omega^S(A, \mathcal{X}) = \|AU - UD\|_2\) for the spectral norm. The choice of \(D\) that minimizes \(\|AU - UD\|\) is given by the following result.

**Proposition 4.2.** Let \(U \in \mathbb{K}^{n \times p}\) be an isometry and \(A \in \mathbb{K}^{n \times n}\). Set \(P := UU^H\). Then for the spectral and the Frobenius norms, we have

\[
\min_D \|AU - UD\| = \|AU - U(U^HAU)\| = \|(I - P)AP\|,
\]

where the minimum is taken over \(\mathbb{K}^{p \times p}\). Further, if \(A\) is Hermitian (resp., skew-Hermitian) then so is the minimizer \(D\). On the other hand, if \(A\) is symmetric (resp., skew-symmetric) then so is the minimizer \(D\) provided \(U\) is real.
Proof: Let $Z := [U, V]$ be unitary. Then for the spectral and the Frobenius norms, we have

$$\|AU - UD\| = \|Z^H (AU - UD)\| = \| \begin{bmatrix} U^H AU - D \\ V^H AU \end{bmatrix} \| \geq \|V^H AU\|$$

and the equality holds for $D = U^H AU$. Now $\|V^H AU\| = \|VV^H AUU^H U\| = \|(I - P)AP\|$ gives the desired minimum. That $D$ inherits the structure of $A$ when $A$ is Hermitian or skew-Hermitian, and under the additional assumption of $U$ being real when $A$ is symmetric or skew-symmetric, is immediate. □

Thus, for a solution of Problem III, we choose an isometry $U$ whose columns form an orthonormal basis of $X$ and set $D := U^H AU$. Then Problem-III reduces to Problem-II for the pair $(U, D)$. However, since $U$ is an isometry and $D = U^H AU$, existence criterion as well as solutions of Problem III have simpler forms. Indeed, when $MS = \text{Herm}$ (resp., $MS = \text{skew-Herm}$), we have $U^H M(I - P)AU$ is Hermitian (resp., skew-Hermitian) and hence by Theorem 3.1 Problem-III has a solution. On the other hand, when $MS = \text{sym}$ (resp., $MS = \text{skew-sym}$), we have $U^T M(I - P)AU$ is symmetric (resp., skew-symmetric) provided $U$ is real. Hence by Theorem 3.1 Problem-III has a solution. We mention once again that Problem-III has a solution for the case when $MS \in \{\text{sym, skew-sym}\}$ provided that the orthogonal projection $P$ is real. Thus, in either case, if $E \in S$ and $EX \subset X$ then by Theorem 3.3 we have

$$E = \begin{cases} A + M^{-1} \mathcal{F}_H(U, M(I - P)AU) + M^{-1}(I - P)Z(I - P), & \text{if } MS \in \{\text{Herm, skew-Herm}\} \\ A + M^{-1} \mathcal{F}_T(U, M(I - P)AU) + M^{-1}(I - P)Z(I - P), & \text{if } MS \in \{\text{sym, skew-sym}\} \end{cases}$$

where $Z \in MS$ is arbitrary. Further, we have the following result from Theorem 3.3

**Theorem 4.3** Let $X$ be a $p$-dimensional subspace of $\mathbb{K}^n$ and $A \in S$. Let $P$ be the orthogonal projection on $X$ given by $P := UU^*$. Define $E_o := A + M^{-1} \mathcal{F}_H(U, M(I - P)AU)$ when $MS \in \{\text{Herm, skew-Herm}\}$, and $E_o := A + M^{-1} \mathcal{F}_T(U, M(I - P)AU)$ when $MS \in \{\text{sym, skew-sym}\}$ and $U$ is real. Then $E_oX \subset X$. Further, $\omega^S(A, X) = \|A - E_o\|_2 = \|(I - P)AP\|_2$ for the spectral norm, and

$$\omega^S(A, X) = \|A - E_o\|_F = \sqrt{2\|(I - P)AP\|_F^2 - \|PM(I - P)AP\|_F^2}$$

for the Frobenius norm.

In particular, when $S \in \{\text{Herm, skew-Herm, sym, skew-sym}\}$ we have $E_o = PAP + (I - P)A(I - P)$ and $\omega^S(A, X) = \|A - E_o\|_2 = \|(I - P)AP\|_2$ for the spectral norm, and $\omega^S(A, X) = \|A - E_o\|_F = \sqrt{2\|(I - P)AP\|_F^2}$ for the Frobenius norm.

We mention that $E_o$ is a unique solution of Problem-III for the Frobenius. By contrast, Problem-III has infinitely many solutions for the spectral norm, which are of the form $E_o + G$ for appropriate $G \in S$ as given in Theorem 3.3. We also mention that for the special case when, for example, $S = \text{Herm}$, the results in Theorem 4.3 can be obtained easily without resorting to Theorem 3.3. Indeed, if $E \in \text{Herm}$ and $EX \subset X$ then $(I - P)EP = 0 = PE(I - P)$ which gives $E = PEP + (I - P)E(I - P)$ or in (operator) matrix notation, we have

$$E = \begin{bmatrix} PEP \\ 0 \end{bmatrix},$$

Since $A = PAP + PA(I - P) + (I - P)AP + (I - P)A(I - P)$ or in matrix notation

$$A = \begin{bmatrix} PAP \\ (I - P)AP \end{bmatrix},$$

it follows that $\|E - A\|$ is minimized for the spectral norm as well as for the Frobenius norm when $E = PAP + (I - P)A(I - P)$ and the minimum is given by $\|(I - P)AP\|_2$ for the spectral norm, and $\sqrt{2\|(I - P)AP\|_F^2}$ for the Frobenius norm.
4.1 Structured pseudospectra

Let $A \in \mathbb{S}$. Recall that $\eta^S(A, x, \lambda)$ is the structured backward error of $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ as an approximate eigenpair of $A$ with the convention $\eta^S(A, x, \lambda) = \infty$ when $x$ and $r := \lambda x - Ax$ do not satisfy the condition for structured mapping. For the spectral norm, set

$$
\eta^S(\lambda, A) := \inf \{ \eta^S(A, x, \lambda) : x \in \mathbb{C}^n \text{ and } ||x||_2 = 1 \}.
$$

Then $\eta^S(\lambda, A)$ is the structured backward error of $\lambda$ as an approximate eigenvalue of $A$. The backward error $\eta^S(\lambda, A)$ can be used to define structured pseudospectrum $\Lambda^S(\lambda)_{\varepsilon}$ of $A$:

$$
\Lambda^S(\lambda) := \bigcup_{\|\Delta A\|_2 \leq \varepsilon} \{ \Lambda(A + \Delta A) : \Delta A \in \mathbb{S} \} = \{ \lambda \in \mathbb{C} : \eta^S(\lambda, A) \leq \varepsilon \}.
$$

See [15] for more on pseudospectra and [8] [9] [12] for structured pseudospectra.

We denote $\eta^S(\lambda, A)$ by $\eta(\lambda, A)$ when $\mathbb{S} = \mathbb{C}^{n \times n}$. Then obviously $\eta(\lambda, A)$ is the unstructured backward error of $\lambda$ as an approximate eigenvalue of $A$ and we have $\eta(\lambda, A) = \sigma_{\min}(A - \lambda I)$, where $\sigma_{\min}(A)$ is the smallest singular value of $A$. In contrast, for many important structures determination of $\eta^S(\lambda, A)$ is a hard optimization problem and hence computation of $\Lambda^S(\lambda)$ is a challenging task. Nevertheless, for certain structures it turns out that $\eta^S(\lambda, A) = \eta(\lambda, A)$ holds for all $\lambda \in \mathbb{C}$, and for some other structures the equality holds only for certain $\lambda \in \mathbb{C}$. We denote $\Lambda^S(\lambda)$ by $\Lambda(\lambda)$, the unstructured pseudospectrum of $A$, when $\mathbb{S} = \mathbb{C}^{n \times n}$.

Theorem 4.4 Let $S := J$ when $M^T = M$, and $S \in \{ J, L \}$ when $M^T = -M$. Let $A \in \mathbb{S}$. Then for $\lambda \in \mathbb{C}$, there exists $\Delta A \in \mathbb{S}$ such that $\lambda \in \Lambda(A + \Delta A)$ and $\eta^S(\lambda, A) = ||\Delta A||_2 = \eta(\lambda, A)$. Consequently, we have

$$
\Lambda^S(\lambda) = \Lambda(\lambda).
$$

Proof: Consider the case $S := J$ when $M^T = M$. Then $MA \in \text{sym}$. Let $\lambda \in \mathbb{C}$. It follows that $A - \lambda I \in \mathbb{S}$, that is, $(M(A - \lambda I))^T = M(A - \lambda I)$. Since $M(A - \lambda I)$ is complex symmetric there exists a unitary matrix $U$ such that the symmetric Takagi factorization $\Sigma U^T$ holds, where $\Sigma$ is a diagonal matrix containing singular values of $M(A - \lambda I)$ ordered in descending order of magnitude. Note that $\eta(\lambda, A) = \sigma_{\min}(A - \lambda I) = \sigma_{\min}(M(A - \lambda I)) \Sigma(n, n)$. Let $u := U(:, 1 : n)$. Then we have $M(A - \lambda I)\Sigma = \eta(\lambda, A)\Sigma$. This gives $(A - \lambda I)\Sigma = \eta(\lambda, A)\Sigma$. Setting $\Delta A := -\eta(\lambda, A)M^{-1}uu^T$, we have $\Delta A \in \mathbb{S}$ and $||\Delta A||_2 = \eta(\lambda, A)$. Hence the results follow.

Next, consider the case $S \in \{ J, L \}$ when $M^T = -M$. Then $MA \in \{ \text{skew-sym} \}$. First consider $MA \in \text{skew-sym}$. For $\lambda \in \mathbb{C}$, we have $M(A - \lambda I) \in \mathbb{S}$, that is, $(M(A - \lambda I))^T = -M(A - \lambda I)$. Since $M(A - \lambda I)$ is complex skew-symmetric, we have the skew-symmetric Takagi factorization $\Sigma U^T$

$$
M(A - \lambda I) = U\text{diag}(d_1, \ldots, d_m)U^T,
$$

where $U$ is unitary, $d_j := \begin{bmatrix} 0 & s_j \\ -s_j & 0 \end{bmatrix}$, $s_j \in \mathbb{C}$ is nonzero and $|s_j|$ are singular values of $M(A - \lambda I)$. Here the blocks $d_j$ appear in descending order of magnitude of $|s_j|$. Note that $M(A - \lambda I)U = U\text{diag}(d_1, \ldots, d_m)$. Let $u := U(:, 1 : n)$. Then $M(A - \lambda I)U = u\sigma_m = u\sigma_m u^T$. This gives $(A - \lambda I)\sigma_m = \eta(\lambda, A)\sigma_m$. Hence taking $\Delta A := -M^{-1}uu^T$, we have $\lambda \in \Lambda(A + \Delta A)$, $\Delta A \in \mathbb{S}$ and $||\Delta A||_2 = \sigma_m = \sigma_{\min}(M(A - \lambda I)) = \sigma_{\min}(A - \lambda I) = \eta(\lambda, A)$. Hence $\eta^S(\lambda, A) = \eta(\lambda, A)$ and the desired result follows.

Finally, consider the case $M^T = -M$ and $MA \in \text{sym}$. Let $\lambda \in \mathbb{C}$. Note that $\sigma_{\min}(M(A - \lambda I)) = \sigma_{\min}(A - \lambda I) = \eta(\lambda, A)$. Let $u$ and $v$ be unit left and right singular vector of $M(A - \lambda I)$ corresponding to $\eta(\lambda, A)$. Then $M(A - \lambda I)v = \eta(\lambda, A)u$. This gives $Av = \lambda v$. Let $E \in \mathbb{C}^{n \times n}$ be such that $E = E^T$, $Ev = u$ and $||E|| = 1$. Such a matrix always exists (Theorem 3.3). Then setting $\Delta A := -\eta(\lambda, A)M^{-1}E$, we have $\lambda(A + \Delta A)v = \lambda v$. Hence the result follows. ■
For Lie and Jordan algebras corresponding to sesquilinear form induced by $M$, we have partial equality between structured and unstructured pseudospectra.

**Theorem 4.5** Let $S := \mathbb{J}$ when $M^H = \pm M$. Let $A \in S$. Then for $\lambda \in \mathbb{R}$, there exists $\Delta A \in S$ such that $\lambda \in \Lambda(A + \Delta A)$ and $\eta^S(\lambda, A) = \|\Delta A\|_2 = \eta(\lambda, A)$. Consequently, we have

$$\Lambda^S(\lambda) \cap \mathbb{R} = \Lambda_s(\lambda) \cap \mathbb{R}.$$  

Next, consider $S := \mathbb{L}$ when $M^H = \pm M$. Let $A \in S$. Then for $\lambda \in \mathbb{R}$, there exists $\Delta A \in S$ such that $\lambda \in \Lambda(A + \Delta A)$ and $\eta^S(\lambda, A) = \|\Delta A\| = \eta(\lambda, A)$. Consequently, we have

$$\Lambda^S(\lambda) \cap i\mathbb{R} = \Lambda_s(\lambda) \cap i\mathbb{R}.$$  

**Proof:** Consider the case $S := \mathbb{J}$ when $M^H = \pm M$. Then $MA \in \text{Herm}$ when $M = M^H$ and $MA \in \text{skew-Herm}$ and $M^H = -M$. First consider $MA \in \text{Herm}$ when $M = M^H$. Now for $\lambda \in \mathbb{R}$, $M(A - \lambda I) \in S$. Since $M(A - \lambda I)$ is Hermitian, we have the spectral decomposition $M(A - \lambda I) = U \text{diag}(\mu_1, \ldots, \mu_n) U^H$, where $U$ is unitary and $\mu_j$'s appear in descending order of their magnitudes. Note that $|\mu_j| = \sigma_{\text{min}}(M(A - \lambda I)) = \sigma_{\text{min}}(A - \lambda I) = \eta(\lambda, A)$. Now defining $\Delta A := -\mu_j M^{-1} U(:, j) U(:, n)^H$, we have $\lambda \in \Lambda(A + \Delta A)$, $\Delta A \in S$ and $\|\Delta A\|_2 = \eta(\lambda, A) = \eta^S(\lambda, A)$. Hence the result follows. The proof is similar for the case when $MA \in \text{skew-Herm}$ and $M^H = -M$.

Finally, consider the case $S := \mathbb{L}$ when $M^H = \pm M$. Then $MA \in \text{skew-Herm}$ when $M^H = M$ and $MA \in \text{Herm}$ when $M^H = -M$. First consider $MA \in \text{skew-Herm}$ when $M^H = M$. Then for $\lambda \in i\mathbb{R}$, the set of purely imaginary numbers, we have $M(A - \lambda I) \in S$, that is, $M(A - \lambda I)$ is skew Hermitian. Hence the result follows from spectral decomposition of $M(A - \lambda I)$. The proof is similar for the case when $MA \in \text{Herm}$ and $M^H = -M$.

**Conclusion.** We have provided a complete solution of the structured mapping problem (Theorem 3.3) for certain classes of structured matrices. More specifically, given a pair of matrices $X$ and $B$ in $\mathbb{K}^{n \times p}$ and a class $S \subset \mathbb{K}^{n \times n}$ of structured matrices, we have provided a complete characterization of structured solutions of the matrix equation $AX = B$ with $A \in S$ for the case when $S$ is either a Jordan algebra or a Lie algebra associated with an orthosymmetric scalar product on $\mathbb{K}^n$. We have determined all optimal solutions in $S$, that is, structured solutions which have the smallest norm. We have shown that optimal solution is unique for the Frobenius norm and that there are infinitely many optimal solutions for the spectral norm. We have shown that the results in [9] obtained for a pair of vectors follow as special cases of our general results. Finally, as an application of the structured mapping problem, we have analyzed structured backward errors of approximate invariant subspaces, approximate eigenpairs, and structured pseudospectra of structured matrices.

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