Removing the ISW-lensing bias from the local-form primordial non-Gaussianity estimation

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Abstract. The Integrated Sachs-Wolfe (ISW) effect produces a secondary temperature anisotropy of the cosmic microwave background (CMB), as CMB photons travel through time-varying potentials along the line-of-sight. The main contribution comes from redshifts \( z \lesssim 2 \), where dark energy leads to a decay of potentials. As the same photons are gravitationally lensed by these decaying potentials, there exists a high degree of correlation between the ISW effect and CMB lensing, leading to a non-zero three-point correlation (bispectrum) of the observed temperature anisotropy. This ISW-lensing bispectrum, whose shape resembles that of the so-called “local-form” primordial bispectrum parametrized by \( f_{\text{NL}} \), is known to be the largest contamination of \( f_{\text{NL}} \). In order to avoid a spurious detection of primordial non-Gaussianity, we need to remove the ISW-lensing bias. In this work, we investigate three debiasing methods: (I) subtraction of an expected, ensemble average of the ISW-lensing bispectrum; (II) subtraction of a measured ISW-lensing bispectrum; and (III) direct subtraction of an estimated ISW signal from an observed temperature map. One may use an estimation of the ISW map from external non-CMB data or that from the CMB data themselves. As the methods II and III are based on fewer assumptions about the nature of dark energy, they are preferred over the method I. While the methods I and II yield unbiased estimates of \( f_{\text{NL}} \) with comparable error bars, the method III yields a biased result when the underlying primordial \( f_{\text{NL}} \) is non-zero and the ISW map is estimated from a lensing potential reconstructed from the observed temperature map. One of the sources of the bias is a lensing reconstruction noise bias which is independent of \( f_{\text{NL}} \) and can be calculated precisely, but other \( f_{\text{NL}} \)-dependent terms are difficult to compute reliably. We thus conclude that the method II is the best, model-independent way to remove the ISW-lensing bias of \( f_{\text{NL}} \), enabling us to test the physics of inflation with smaller systematic errors.
1 Introduction

Convincing detection of the so-called “local-form” three-point correlation function (bispectrum) of primordial curvature perturbations from inflation has profound implications for our understanding of the physics of inflation, as it would rule out all single-field inflation models [1, 2], provided that an initial quantum state of the curvature perturbation is in a preferred state called the Bunch-Davies state [3, 4] and that the curvature perturbation does not evolve outside the horizon due to a non-attractor solution [5, 6].

The local-form bispectrum is defined as (e.g., [7])

$$\langle \Phi_{k_1} \Phi_{k_2} \Phi_{k_3} \rangle = (2\pi)^3 \delta(k_1 + k_2 + k_3)(2f_{NL}) \left[ P_\Phi(k_1)P_\Phi(k_2) + \text{(2 perm)} \right],$$

(1.1)

where $\Phi$ is Bardeen’s curvature perturbation in the matter era given by the trace of the space-space metric, i.e., $\sqrt{\det(g_{ij})} = a^3(t)(1 + 3\Phi)$, and $a(t)$ is the Robertson-Walker scale factor. The function $P_\Phi(k)$ is the power spectrum of $\Phi$ defined as $\langle \Phi_{k_1} \Phi_{k_2} \rangle = (2\pi)^3 \delta(k_1 + k_2)P_\Phi(k)$. The latest measurements suggest $P_\Phi(k) \propto k^{n_s-4}$ with $n_s = 0.96 \pm 0.01$ (68% CL) [8–10]. It follows from this wavenumber-dependence of $P_\Phi(k)$ that the local-form bispectrum is largest in the so-called squeezed configurations, where one of the wavenumbers is much smaller than

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1 Also see workshop summaries of “Critical Tests of Inflation Using Non-Gaussianity” in http://www.mpa-garching.mpg.de/~komatsu/meetings/ng2012/.
the other two, e.g., $k_3 \ll k_1 \approx k_2$ [11]. In the squeezed limit, $k_3 \to 0$, all single-field models give $f_{\text{NL}} = \frac{5}{3}(1 - n_s) = O(10^{-2})$ [1, 12].

The latest WMAP 9-year limit is $f_{\text{NL}} = 39.8 \pm 19.9$ (68% CL). The WMAP team then subtracts $\delta f_{\text{NL}} = 2.6$ from this measurement in order to correct for the bias due to the “ISW-lensing bispectrum,” reporting the final limit of $f_{\text{NL}} = 37.2 \pm 19.9$ (68% CL) [13].

What is the ISW-lensing bispectrum? The Integrated Sachs-Wolfe (ISW) effect is a secondary temperature anisotropy caused by time-varying gravitational potential wells between the last-scattering surface and us [14]. The (linear) ISW effect vanishes during the matter era, while it becomes important at low redshifts, $z \lesssim 2$, where dark energy leads to a decay of potential wells. The same potential wells gravitationally deflect the paths of CMB photons (see Ref. [15] for a review). Therefore, there is a correlation between the ISW effect, which is important only at low multipoles, $l \lesssim 10$, and a change in CMB anisotropy due to lensing, which is important at high multipoles, $l \gtrsim 1000$. This leads to a non-zero bispectrum of the observed temperature anisotropy [16], which is largest in the squeezed configuration, e.g., $l_3 \ll l_1 \approx l_2$ [17]. Therefore, the ISW-lensing bispectrum yields a contamination of the primordial local-form bispectrum [17–20]. We need to properly remove the ISW-lensing bias in order to avoid a spurious detection of primordial non-Gaussianity.

What is the best way to remove the ISW-lensing bias? The most straightforward way is to calculate the expected ISW-lensing bispectrum given a cosmological model, and subtract it from the measured bispectrum (“Method I”). This is what was done by the WMAP team for the nine-year analysis [13]. However, as one can only predict the ensemble average of the ISW-lensing bispectrum, this method ignores a realization-dependent term. One may then assume that the shape of the ISW-lensing bispectrum is known but the amplitude is not, and include the amplitude of the ISW-lensing bispectrum as a free parameter [21]. (i.e., one marginalizes over the amplitude of the ISW-lensing bispectrum.)

The methods we explore in this paper go beyond these simple treatments in two ways. For one, we first measure the ISW-lensing cross-power spectrum from data directly, and use this measured cross-power spectrum to compute the ISW-lensing bispectrum (“Method II”). In this way we can fully capture the ISW-lensing bias that is actually there in the sky. We show that, not only does this method yield an unbiased estimate of $f_{\text{NL}}$, but also yields a statistical uncertainty in $f_{\text{NL}}$ which is as small as Method I, and thus it is optimal.

For another, we first clean the ISW effect by removing an estimate of the ISW effect from an observed temperature map, and then measure the bispectrum (“Method III”). To the extent that the estimator of the ISW effect is accurate, this method allows us to remove the ISW-lensing coupling before measuring the bispectrum from data. However, we find that this method, as currently implemented, yields a biased result, if the underlying, primordial $f_{\text{NL}}$ is non-zero and the ISW estimation comes from a lensing potential reconstructed from the CMB data themselves, rather than from external non-CMB data.

The outline of this paper is as follows. In Section 2, we briefly describe the ISW-lensing bispectrum. In Section 3, we describe our simulations of lensed non-Gaussian CMB temperature maps with noise. In Section 4, we describe three methods for removing the ISW-lensing bias of $f_{\text{NL}}$. In Section 5, we apply these methods to simulated data and present the results. We conclude in Section 6. In Appendix A, we describe our estimator of the lensing potential. In Appendix B, we derive the estimator of $f_{\text{NL}}$ for Method II. In Appendix C, we derive the noise bias in the reduced bispectrum of an ISW-subtracted map of Method III, which arises from the reconstruction noise of lensing potential.
2 The ISW-lensing bispectrum

2.1 Bispectrum-lensing estimator and Fisher matrix

The CMB anisotropy measured over the whole-sky is conveniently decomposed in terms of spherical harmonics as \( T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}) \), where \( \hat{n} \) is a unit vector pointing toward a given direction in the sky, \( a_{lm} \) a decomposition coefficient, and \( Y_{lm}(\hat{n}) \) a spherical harmonic function. The expectation value of a 3-point correlation is given by \( \langle a_{l_1m_1} a_{l_2m_2} a_{l_3m_3} \rangle = G_{l_1l_2l_3}^{m_1m_2m_3} \sum_i \sum_{j=1}^6 b_{i,l_1l_2l_3}^{(i)} \left[ \delta \right] \), where \( \langle \ldots \rangle \) denotes the ensemble average over many realizations of universes, and \( G_{l_1l_2l_3}^{m_1m_2m_3} \) is defined by \( G_{l_1l_2l_3}^{m_1m_2m_3} = \int d^2 \hat{n} Y_{l_1m_1}(\hat{n}) Y_{l_2m_2}(\hat{n}) Y_{l_3m_3}(\hat{n}) \). Here, \( (i) \) denoted various sources of non-Gaussianity such as “local” and “ISW-lensing,” etc., \( b_{i,l_1l_2l_3}^{(i)} \) the reduced bispectrum of a particular shape, and \( f_{NL}^{(i)} \) the corresponding amplitude. See Ref. [7] for the expression of \( b_{i,l_1l_2l_3}^{\text{local}} \) and Eq. (2.6) for \( b_{i,l_1l_2l_3}^{\text{ISW-lensing}} \).

Given the CMB data, we may estimate \( f_{NL}^{(i)} \) from \( f_{NL}^{(i)} = \sum_j (F^{-1})_{ij} S_j \), where (see Ref. [21] for a review)

\[
S_i = \frac{1}{6} \sum_{lm} G_{l_1l_2l_3}^{m_1m_2m_3} b_{i,l_1l_2l_3}^{(i)} \left[ (C^{-1} a)_{l_1m_1} (C^{-1} a)_{l_2m_2} (C^{-1} a)_{l_3m_3} - 3 (C^{-1} a)_{l_1m_1,l_2m_2} (C^{-1} a)_{l_3m_3} \right],
\]

(2.1)

with \( C \) being the covariance matrix of data including CMB and noise, and \( F \) is the Fisher matrix given by

\[
F_{ij} = \frac{1}{6} \sum_{lm} \sum_{l'm'} G_{l_1l_2l_3}^{m_1m_2m_3} b_{i,l_1l_2l_3}^{(i)} (C^{-1} a)_{l_1m_1,l_2m_2} (C^{-1} a)_{l_3m_3,l_3m_3} b_{j,l_1l_2l_3}^{(j)} G_{l_1l_2l_3}^{m_1m_2m_3}.
\]

(2.2)

These expressions simplify greatly to those in Ref. [22] when the covariance matrix is diagonal and isotropic, i.e., \( C_{lm,l'm'} = C_{l} \delta_{ll'} \delta_{mm'} \). The 1σ uncertainty in \( f_{NL}^{(i)} \) is given by \( \sqrt{(F^{-1})_{ii}} \).

2.2 The bias due to the ISW-lensing bispectrum

The ISW effect is produced by the blue-shifting and red-shifting of photons as photons fall in and climb out of potential wells in their pathway, and is given in terms of a time derivative of \( \Psi - \Phi \) along the line-of-sight [14]:

\[
T_{\text{ISW}}(\hat{n}) = \int_0^{\chi^*} d\chi [\dot{\Psi} - \dot{\Phi}] (\chi \hat{n}, \eta_0 - \chi),
\]

(2.3)

where \( \Psi(\chi, \eta) \) is the Newtonian potential given by the time-time metric, \( g_{00} = -(1 + 2\Psi) \), the dot denotes a derivative with respect to the conformal time, \( \partial / \partial \eta \), \( \eta_0 \) is the present-day conformal time, and \( \chi^* \) is the comoving distance to the last scattering surface. The ISW effect is not present during matter domination (in which \( \Psi = 0 = \Phi \)), but becomes important at low redshifts, \( z \lesssim 2 \), where dark energy starts to affect the evolution of \( \Psi \) and \( \Phi \).

Traveling along the line-of-sight, CMB photons are also gravitationally lensed by the same potential as \( T(\hat{n}) \rightarrow T(\hat{n} + \nabla \psi) \) (see [15] for a review). Here, \( \psi \) is a “lensing potential,” given by

\[
\psi(\hat{n}) = - \int_0^{\chi^*} d\chi \frac{f_K(\chi_0 - \chi)}{f_K(\chi_0)} [\Psi - \Phi] (\chi \hat{n}, \eta_0 - \chi),
\]

(2.4)
where $f_K(\chi)$ is the comoving angular-diameter distance, which is equal to $\chi$ in a flat universe ($K = 0$). As most of the lensing effect also comes from $z \lesssim 2$, there is a correlation between CMB anisotropy and the lensing potential.

Let us decompose the lensing potential into spherical harmonics as $\psi_{lm} = \int d^2 n \, \psi(\hat{n}) \, Y_{lm}^*(\hat{n})$, and define the temperature-lensing cross-power spectrum as $C_{TT}^{T\psi} \equiv \langle a_{lm} \psi_{lm}^* \rangle$. While this cross-power spectrum at low multipoles, $l \lesssim 20$, is almost entirely dominated by the above late ISW-lensing correlation, there are also other small contributions (see Fig. 4 of Ref. [23]).

We use the CAMB code [24] to include these smaller effects as well. The linear theory calculation is an excellent approximation, as non-linear effects have no impact on the contamination of $f_{NL}$ [25]. In Fig. 1, we show the cross-correlation power spectrum, $C_{TT}^{T\psi}$, for the best-fit $\Lambda$CDM parameters given by the “WMAP7+BAO+H0” combination in Ref. [26].

The temperature-lensing correlation generates the following bispectrum even in the absence of primordial non-Gaussianity [16]:

$$
\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle|_{f_{NL}=0} = G_{l_1 l_2 l_3}^{m_1 m_2 m_3} b_{l_1 l_2 l_3}^{\text{ISW-lensing}},
$$

where

$$
b_{l_1 l_2 l_3}^{\text{ISW-lensing}} = l_2(l_2+1) + l_3(l_3+1) - l_1(l_1+1) \frac{C_{ll_1}^{TT} C_{ll_2}^{\psi} + (5 \text{ perm})}{2},
$$

where $C_{TT}^{l}$ is the lensed (rather than unlensed, as pointed out by [27]) power spectrum of CMB temperature anisotropy. As $C_{TT}^{l}$ falls off rapidly with multipoles (see Fig. 1), this bispectrum is largest in the squeezed configurations, e.g., $l_3 \ll l_1 \approx l_2$, just like the local-form bispectrum. This is the reason why the ISW-lensing bispectrum results in a contamination of $f_{NL}$. The expected bias in $f_{NL}^{\text{local}}$ due to the ISW-lensing coupling, $\delta f_{NL}$, can be computed.
Table 1. The expected bias and the corresponding $1\sigma$ uncertainty of $f_{\text{NL}}$ for a Planck-like experiment and a cosmic-variance-limited experiment measuring temperature anisotropy up to $l = 2500$. The uncertainty presented here does not include the statistical fluctuation from the ISW-lensing bias.

| Sensitivity                      | $1\sigma$ | Bias |
|----------------------------------|-----------|------|
| Planck                           | 5.1       | 7.8  |
| Cosmic-variance-limited ($l \leq 2500$) | 3.3       | 13.0 |

In Table 1, we show the expected bias and the corresponding $1\sigma$ uncertainty in $f_{\text{NL}}$ for Planck as well as for a cosmic-variance-limited experiment measuring temperature anisotropy up to $l = 2500$ (no polarization information is used). The ISW-lensing bias exceeds the expected $1\sigma$ uncertainty of the upcoming Planck data, and thus it must be removed. For a cosmic-variance-limited experiment, the expected bias is four times the $1\sigma$ uncertainty.

3 Simulation

To test validity of our methods for removing the ISW-lensing bias of $f_{\text{NL}}$ described in the next section, we apply our methods to simulated lensed non-Gaussian temperature maps with noise. We use 1000 simulated unlensed non-Gaussian temperature maps produced by Elsner and Wandelt [28]. The cosmological parameters of the simulations are: $\Omega_{\Lambda} = 0.728$, $\Omega_c h^2 = 0.1123$, $\Omega_b h^2 = 0.0226$, $h = 0.704$, $n_s = 0.963$, $\tau = 0.087$, and $\Delta^2(k_0) = 2.441 \times 10^{-9}$ with $k_0 = 0.002$ Mpc$^{-1}$. These simulations provide a Gaussian piece, $a_{l,m}^{\text{L}}$, and a non-Gaussian piece, $a_{l,m}^{\text{NL}}$, for $l \leq 3500$. The total anisotropy is then given by $a_{l,m} = a_{l,m}^{\text{L}} + f_{\text{NL}} a_{l,m}^{\text{NL}}$.

As the lensing potential is not available from these simulations, we need to generate the lensing potential such that it has a proper correlation with the pre-generated total CMB anisotropy, $a_{l,m}$. In order to do this, we use a “constrained Gaussian realization” method [29–31]. We have checked that the correlation between the simulated lensing potential and $a_{l,m}$ agrees with the theoretical expectation. We then use the LensPix code [32] to lens the simulated CMB maps with the lensing potential we have generated. Finally, to these lensed CMB maps we add Gaussian, white, and homogeneous noise with a given noise power spectrum. (For simplicity we do not include inhomogeneity of Planck noise.) We use the FUTURCMB code [33] to calculate the noise power spectrum corresponding to the expected sensitivity of Planck [34]. In the left panel of Fig. 2, we show the noise power spectrum together with the CMB temperature power spectrum.

Once maps are generated, we use the method of Ref. [22] to estimate $f_{\text{NL}}$ from full-sky maps (i.e., no mask is applied). The functions required for computing the local-form CMB bispectrum, $\alpha(r)$ and $\beta(r)$ [7, 22], are computed by the CAMB code [24] with 1954 points in the radial coordinates, $r$. These are chosen in accordance with CAMB’s internal $k$ sampling, which are due to the $j_l(kr)$ terms in the integrand for the $\alpha(r)$ and $\beta(r)$.
Figure 2. (Left) Power spectra of CMB and the expected Planck noise bias. (Right) Power spectra of the lensing potential and the expected Planck reconstruction noise bias (computed from Eq. (A.2)).

| $f_{NL}$ | Planck Sensitivity | Cosmic Variance Limited ($l \leq 2500$) |
|---------|--------------------|----------------------------------------|
| 0       | 7.6 ± 5.1          | 12.8 ± 3.9                              |
| 20      | 27.7 ± 5.7         | 33.1 ± 4.9                              |
| 40      | 47.7 ± 7.1         | 53.3 ± 6.7                              |

Table 2. Mean and standard deviation of $f_{NL}$ estimated from 1000 simulated lensed CMB maps with Planck-like noise (the second column) and with no noise (the third column) for the input values of $f_{NL} = 0$, 20, and 40.

In Table 2, we show the average and standard deviation of $f_{NL}$ estimated from 1000 lensed CMB plus Planck-like noise simulations (the second column) as well as from cosmic-variance-limited simulations (the third column) before removing the ISW-lensing bias, for the input values of $f_{NL} = 0$, 20, and 40. For all cases, the biases we find agree with the expectations given in Table 1. The standard deviation increases for larger values of $f_{NL}$ due to the contribution of non-Gaussian terms to the covariance matrix of the bispectrum. As $f_{NL}$ increases, the error bars become more dominated by non-Gaussian contributions to the covariance matrix. Therefore a Gaussian piece, which includes instrumental noise, becomes less important.

4 Removing the ISW-lensing bias

In this section, we describe three methods for removing the ISW-lensing bias.

4.1 Fitting out the ISW-lensing bispectrum (Method Ia and Ib)

The simplest possible method to remove the bias is to subtract the ensemble average of the ISW-lensing bias given by Eq. (2.7) from the measured $f_{NL}$ (Method Ia). While this is the simplest method, it comes with a couple of caveats. First, it assumes a perfect knowledge of the ISW-lensing bispectrum. While this assumption is not too unreasonable given the success of the minimal ΛCDM model, it might still be too restrictive given the fact that we do not know the precise nature of dark energy. Second, even if dark energy is a cosmological constant, one can only predict the ensemble average of the ISW-lensing bispectrum, whereas the bias in $f_{NL}$ is caused by a particular realization of potentials in our past light cone.
To partially mitigate both issues, one may simultaneously fit $f_{NL}$ and the overall amplitude of the ISW-lensing bispectrum [21] (Method Ib). This is equivalent to marginalizing over the amplitude of the ISW-lensing bispectrum, i.e., we assume that the shape of the ISW-lensing bispectrum is known precisely, but the amplitude is not.

### 4.2 Realization-dependent debiasing (Method II)

Can we do better? The answer is yes, if we have information on the lensing potential. How do we obtain information on the lensing potential? One way is to use deep galaxy survey data (e.g., Euclid [35, 36]) to measure the lensing potential totally independent of the CMB data, and another way is to use non-Gaussian signatures of CMB temperature anisotropy caused by gravitational lensing to reconstruct the lensing potential. (For discussion on practical applications of the lensing reconstruction technique to the Planck data, see [37, 38]). Throughout this paper, we shall use the lensing potential reconstructed from the temperature data.

Once we obtain a map of the lensing potential, we can remove the ISW-lensing bias in two ways. The first method, which we shall call “Method II,” is to estimate the ISW-lensing bispectrum:

$$\hat{v}_{ISW-lensing}^{TT} = \frac{l_2(l_2 + 1) + l_3(l_3 + 1) - l_1(l_1 + 1)}{2} C_{l_2}^{TT} \hat{c}_{l_3}^{TT} + (5 \text{ perm}), \quad (4.1)$$

where $C_{l}^{TT}$ is the theoretical temperature power spectrum and $\hat{c}_{l}^{TT} = (2l + 1)^{-1} \sum_{m=-l}^{l} a_{lm} \hat{\psi}_{lm}^T$ is the temperature-lensing cross-power spectrum estimated from the data. Here, $\hat{\psi}_{lm}$ is a lensing potential reconstructed from the CMB data. Using Eq. (A.1) and (A.8), one can show that the expectation value of $C_{l_2}^{TT} \hat{c}_{l_3}^{TT}$ is given by (see Sec. B for details):

$$\langle C_{l_2}^{TT} \hat{c}_{l_3}^{TT} \rangle = C_{l_2}^{TT} C_{l_3}^{TT} + f_{NL} C_{l_2}^{TT} \xi_l, \quad (4.2)$$

where

$$\xi_l = \frac{N_l}{2L + 1} \sum_{l' l''} \sqrt{(2l' + 1)(2l'' + 1)(2L + 1)} \left( \begin{array}{ccc} l' & l'' & L \\ 0 & 0 & 0 \end{array} \right) \left( C_{l_1}^{TT} F_{l_1}^{T} F_{l_2}^{T} + C_{l_2}^{TT} F_{l_2}^{T} F_{l_1}^{T} \right) \frac{b_{l_1}^{local}}{2C_{l_2}^{TT} C_{l_2}^{TT}}, \quad (4.3)$$

and $C_{l}^{Tot}$ is the sum of $C_{l}^{TT}$ and the noise power spectrum. Noting this result, we obtain the unbiased estimator of $f_{NL}$ as

$$\hat{f}_{NL} = S_{local} \left( \frac{1}{6} \sum_{l m l' m'} g^{l_1 l_2 l_3}_{l_1 l_2 l_3} \bar{b}_{l_1 l_2 l_3}^{local} \bar{c}_{l_1 l_2 l_3}^{TT} \bar{v}_{ISW-lensing}^{l_1 l_2 l_3} \right), \quad (4.4)$$

where

$$S_{local} = \frac{1}{6} \sum_{l m} g^{3m_1 2m_2 3m_3}_{l_1 l_2 l_3} \bar{b}_{l_1 l_2 l_3}^{local} \bar{c}_{l_1 l_2 l_3}^{TT} \left[ (C^{-1})_{l_1 l_2} (C^{-1})_{l_2 l_3} \right] - 3(C^{-1})_{l_1 l_2} (C^{-1})_{l_2 l_3} \right) \xi_{l_3}, \quad (4.5)$$
\[ \tilde{a}_{l_1l_2l_3} = b_{l_1l_2l_3} - \left( \frac{l_2(l_2 + 1) + l_3(l_3 + 1) - l_1(l_1 + 1)}{2} C_{l_2}^{TT} \xi_{l_3} + (5 \text{ perm}) \right). \]  

### 4.3 Subtracting ISW from an observed temperature map (Method III)

The second method, which we shall call “Method III,” is to estimate the ISW effect in our sky from a map of the lensing potential, and subtract it from an observed temperature map. To the extent that the estimated ISW effect is accurate, this ISW-subtracted map should yield a vanishing ISW-lensing bispectrum. This is indeed possible, as the lensing potential and ISW effect are highly correlated. The harmonic coefficients of an ISW-subtracted map, \( \hat{a}_{lm} \), are given by

\[ \hat{a}_{lm} = a_{lm} - \frac{C_{l}^{T^p}}{C_{l}^{\psi \psi}} \hat{\psi}_{lm}, \]  

where \( \hat{\psi}_{lm} \) is the lensing potential estimated from data, and \( C_{l}^{T^p} \) and \( C_{l}^{\psi \psi} \) are calculated from a given cosmological model. The power spectrum of \( \hat{a}_{lm} \) is given by

\[ C_{TT}^{l} = \frac{(C_{l}^{T^p})^2}{C_{l}^{\psi \psi}}, \]

and the cross-correlation between \( \hat{a}_{lm} \) and the lensing potential vanishes:

\[ \langle \hat{a}_{lm} \psi_{lm}^* \rangle = 0. \]

### 5 Results

The results of Method Ia and Ib are shown in Table 3 and 4, respectively. Both methods remove the bias successfully, while Method Ib (which marginalizes over the amplitude of the ISW-lensing bispectrum) yields slightly larger uncertainties in \( f_{NL} \).

In order to apply Method II and III, we need a map of the lensing potential. In this paper we estimate a map of the lensing potential from simulated CMB temperature maps using the lensing reconstruction technique of Ref. [40] with the unlensed CMB power spectrum in the filter replaced by the lensed CMB power spectrum to account for higher-order terms in the lensing potential [27]. See Appendix A for details of our estimator.

The results of Method II are shown in Table 5. We find that the uncertainties in \( f_{NL} \) from Method II are quite comparable to those of Method I. Method II is superior to Method I, as it is based on fewer assumptions about the nature of dark energy than Method I while keeping optimality of the estimator. We thus recommend Method II as the best method to remove the ISW-lensing bias.

Finally, the results of Method III are shown in Table 6: \( f_{NL} \) is estimated from ISW-subtracted maps given by Eq. (4.7). There is one subtlety in this method. As a map of the lensing potential is reconstructed from CMB data themselves and the estimated \( \hat{\psi}_{lm} \) is given by a product of two \( a_{lm} \)'s, there is a non-zero three-point correlation between \( a_{lm} \) and the reconstruction error (i.e., the difference between the true \( \psi_{lm} \) and the reconstructed one):

\[ \langle a_{l_1m_1} a_{l_2m_2} (\hat{\psi}_{l_3m_3} - \psi_{l_3m_3}) \rangle \neq 0. \]

This correlation produces the noise bias in the bispectrum measured from ISW-subtracted maps. One can calculate and subtract this noise bias. We derive the formula for the noise bias in Appendix C. We find that the noise bias in \( f_{NL} \) is \( \delta f_{NL} = -25.3 \) for the Planck noise level, and \( \delta f_{NL} = -8.1 \) for the cosmic-variance-limited case. These biases have been subtracted already in the values quoted in Table 6.

However, as the lensing reconstruction relies on non-Gaussian signatures of temperature anisotropy induced by lensing, the presence of primordial non-Gaussianity yields a small but
Table 3. Method Ia: subtraction of the predicted ISW-lensing bispectrum.

| $f_{NL}$ | Planck Sensitivity | Cosmic Variance Limited ($l \leq 2500$) |
|----------|-------------------|-----------------------------------------|
| 0        | $-0.15 \pm 5.1$   | $-0.13 \pm 3.9$                        |
| 20       | $19.9 \pm 5.7$    | $20.1 \pm 4.9$                         |
| 40       | $40.0 \pm 7.1$    | $40.3 \pm 6.7$                         |

Table 4. Method Ib: the overall amplitude of the ISW-lensing bispectrum is fitted simultaneously with $f_{NL}$.

| $f_{NL}$ | Planck Sensitivity | Cosmic Variance Limited ($l \leq 2500$) |
|----------|-------------------|-----------------------------------------|
| 0        | $-0.03 \pm 5.3$   | $0.13 \pm 4.0$                        |
| 20       | $20.0 \pm 5.9$    | $20.3 \pm 5.0$                         |
| 40       | $40.1 \pm 7.4$    | $40.5 \pm 7.0$                         |

Table 5. Method II: the ISW-lensing bispectrum is computed from the *measured* temperature-lensing cross-power spectrum.

| $f_{NL}$ | Planck Sensitivity | Cosmic Variance Limited ($l \leq 2500$) |
|----------|-------------------|-----------------------------------------|
| 0        | $-0.44 \pm 5.8$   | $-0.35 \pm 4.0$                        |
| 20       | $16.8 \pm 5.8$    | $16.0 \pm 4.1$                         |
| 40       | $30.6 \pm 6.4$    | $27.3 \pm 4.8$                         |

Table 6. Method III: $f_{NL}$ is estimated from ISW-subtracted temperature maps given by Eq. (4.7). The lensing reconstruction noise bias has been subtracted.

non-negligible impact on the reconstructed lensing potential, especially on large angular scales [41]. Specifically, the lensing estimator uses the fact that the cross-power spectrum between different multipoles, $\langle a_{lm} a_{l'm'} \rangle$, is correlated with a long-wavelength lensing potential, $\psi_{LM}$. This is similar to what the local-form bispectrum does: $\langle a_{lm} a_{l'm'} \rangle$ is correlated with a long-wavelength mode, $a_{LM}$. The lensing correlation peaks at $L \approx 50$, whereas the local form peaks at $L = 2$. As a result, the reconstructed lensing potential map at small $L$ receives contributions from $f_{NL}$-dependent terms, giving a bias in $f_{NL}$. Unlike Method II, for which we are able to calculate the $f_{NL}$-dependent terms accurately (see Appendix B), the $f_{NL}$-dependent terms in the estimator of Method III are difficult to compute reliably. We thus conclude that Method III, as currently implemented, yields a biased result, if the ISW is estimated from a lensing potential reconstructed from the observed CMB temperature data themselves, and the underlying primordial $f_{NL}$ is non-zero.

6 Conclusion

The ISW-lensing bispectrum, whose shape is similar to that of the local-form primordial bispectrum, biases the estimation of $f_{NL}$. For a Planck-like experiment and a cosmic-variance-limited experiment measuring the temperature anisotropy up to $l = 2500$, we expect the bias
on $f_{\text{NL}}$ to be 7.8 and 13, respectively. In order to avoid a spurious detection of the local-form primordial bispectrum, we must remove this ISW-lensing bias.

The method used by the WMAP team [13] assumes that we have a perfect knowledge of the ISW-lensing bispectrum (Method Ia). One can relax this assumption by marginalizing over the amplitude of the ISW-lensing bispectrum (Method Ib). While these methods remove the ISW-lensing bias in $f_{\text{NL}}$ successfully, they rely on the assumption that we understand the precise nature of dark energy.

Moreover, what produces the bias in $f_{\text{NL}}$ is the ISW-lensing correlation in our sky, rather than the ensemble average of the ISW-lensing correlation. Therefore, a better method is to use the measured ISW-lensing correlation to compute the ISW-lensing bispectrum and subtract it from the measured bispectrum (Method II). We find that this method also successfully eliminates the bias in $f_{\text{NL}}$, and yields statistical uncertainties which are as small as those of Method I. Therefore, not only is Method II model-independent, but it is also optimal.

Another method, which removes an estimate of the ISW effect directly from a map (Method III), is also promising, provided that the ISW estimation comes from external, non-CMB data, such as galaxy surveys. However, if the ISW is estimated from a lensing potential reconstructed from the CMB data themselves, then $f_{\text{NL}}$ estimated from the ISW-subtracted map is biased in two ways: (1) the lensing reconstruction noise produces a noise bias in the bispectrum of the ISW-subtracted map; and (2) the presence of primordial $f_{\text{NL}}$ biases the lensing potential reconstruction [41]. While the former effect is precisely calculable, the latter effect is difficult to estimate reliably.

Nevertheless, it may be worth pursuing Method III further, as removing the ISW from the observed temperature map has an added benefit. While the standard estimator derived by Ref. [22] is optimal when the underlying $f_{\text{NL}}$ is zero, it becomes sub-optimal when non-zero $f_{\text{NL}}$ is detected with high statistical significance. A method to make the estimator optimal even in the case of high signal-to-noise ratio detection of $f_{\text{NL}}$ relies on our knowledge of large-scale temperature anisotropy at the decoupling epoch [42, 43]; however, this information cannot be extracted precisely due to the presence of the ISW effect in a low-redshift universe. Therefore, one can achieve a smaller statistical uncertainty on $f_{\text{NL}}$, if the ISW effect can be removed from the temperature map [44]. This is precisely what we have attempted to do in this paper, but the ISW signal estimated from the lensing potential reconstructed from the temperature data was biased due to the presence of $f_{\text{NL}}$ affecting the lensing reconstruction. Whether one can mitigate this issue by using, e.g., lensing potential reconstructed only from polarization data; an improved (perhaps iterative) estimator of the lensing potential in the presence of $f_{\text{NL}}$; etc, remains to be seen.

In summary, we regard Method II as the best, model-independent way to remove the ISW-lensing bias in $f_{\text{NL}}$ from the forthcoming Planck data as well as from cosmic-variance-limited data. 2

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2After this paper was submitted, the Planck collaboration reported constraints on $f_{\text{NL}}$ [45]. They use Method Ib to remove the ISW-lensing bias.
A Quadratic estimator of lensing potential

An estimate of the lensing potential in harmonic space, $\hat{\psi}_{LM}$, may be reconstructed by the quadratic estimator as follows [27, 40]:

$$\hat{\psi}_{LM} = N_L \sum_{l_1 m_1} \sum_{l_2 m_2} (-1)^{M} \left( \frac{m_1^2 m_2^2}{l_1 l_2} - M \right) \left( C_{l_1} F_{l_1 l_1} + C_{l_2} F_{l_2 l_2} \right) \frac{a_{l_1 m_1} a_{l_2 m_2}}{2C_{l_1}^\text{tot} C_{l_2}^\text{tot}}, \quad (A.1)$$

where $C_l$ is the power spectrum of the lensed CMB (without noise), and

$$N_L \equiv \frac{2L + 1}{\sum_{l_1} \sum_{l_2} (C_{l_1} F_{l_1 l_1} + C_{l_2} F_{l_2 l_2})^2}, \quad (A.2)$$

$$F_{l_1 l_2} \equiv \frac{L(L+1) + l_2(l_2 + 1) - l_1(l_1 + 1)}{2} I_{l_1 l_2}, \quad (A.3)$$

$$I_{l_1 l_2} \equiv \sqrt{\frac{(2l_1+1)(2L+1)(2l_2+1)}{4\pi}} \left( \begin{array}{ccc} l_1 & L & l_2 \end{array} \right). \quad (A.4)$$

Here, $C_l^\text{tot}$ is the sum of $C_l$ and the noise power spectrum. One can show that the power spectrum of the reconstruction noise bias is equal to $N_L$ [40]. We can rewrite Eq. (A.1) into the form that can be computed more efficiently:

$$\hat{\psi}_{LM} = \frac{1}{2} \int d^2 \hat{n} \left[ L(L+1)A(\hat{n}) B(\hat{n}) + A(\hat{n}) B(\hat{n}) - \tilde{A}(\hat{n}) B(\hat{n}) \right] Y_{LM}^*(\hat{n}), \quad (A.5)$$

where

$$A(\hat{n}) \equiv \sum_{l m} \frac{a_{l m}}{C_l} Y_{lm}(\hat{n}), \quad B(\hat{n}) \equiv \sum_{l m} \frac{C_l a_{l m}}{C_l^\text{tot}} Y_{lm}(\hat{n}), \quad (A.6)$$

$$\tilde{A}(\hat{n}) \equiv \sum_{l m} \frac{l(l+1) a_{l m}}{C_l^\text{tot}} Y_{lm}(\hat{n}), \quad \tilde{B}(\hat{n}) \equiv \sum_{l m} \frac{l(l+1) C_l a_{l m}}{C_l^\text{tot}} Y_{lm}(\hat{n}). \quad (A.7)$$

Using the HEALPix code [46, 47], forward and backward spherical harmonic transformation necessary for the equation above can be done efficiently. Alternatively, Eq. (A.5) can be expressed in the integral form involving the gradient of spherical harmonics [37, 40].

Given the finite pixel size of the map we use for this real-space estimator, we find that accuracy of the lensing reconstruction of the lowest multipoles ($L \lesssim 10$), which relies on the highest multipoles available in the temperature map, is compromised by the finite pixel effect, even when we use the highest resolution of HEALPix ($N_{\text{side}} = 8192$). Therefore, we use the harmonic-space estimator (Eq. (A.1)) for the reconstruction of low multipoles ($L \leq 30$) and the real-space estimator (Eq. (A.5)) for the reconstruction of high multipoles ($30 < L \leq 2500$).

\[Eq. (A.5)\] shows that the lensing potential estimator is dominated by the second and third terms when $L$ is small. These terms contain $\tilde{A}(\hat{n})$ and $\tilde{B}(\hat{n})$, which have an extra factor of $l(l+1)$ (see Eq. (A.7)) and thus are more sensitive to the finite pixel effect.
Finally, difference between the true lensing potential, $\psi_{LM}$, and an estimated one, $\hat{\psi}_{LM}$, is given by

$$n_{LM} \equiv \hat{\psi}_{LM} - \psi_{LM} = N_l \sum_{l_1'} \sum_{l_2'} \sum_{m_1'} \sum_{m_2'} (-1)^M \left( \frac{l_1' m_1'}{l_2' m_2'} - L \right) \left( C_{l_1' l_1 l_2'} + C_{l_2' l_1 l_2'} \right) \frac{a_{l_1' m_1'} a_{l_2' m_2'} - (a_{l_1' m_1'} a_{l_2' m_2'})_{\text{CMB}}}{2 C_{l_1'} C_{l_2'}}.$$  \hfill (A.8)

where

$$(a_{l_1 m_1} a_{l_2 m_2})_{\text{CMB}} = C_{l_1} \delta_{l_1 l_2} \delta_{m_1 m_2} (-1)^{m_2} + \sum_{L'M'} (-1)^{M'} \left( \frac{l_1}{m_1} \frac{l_2}{m_2} - L' \right) (C_{l_1} F_{l_1 L_{l_1}} + C_{l_2} F_{l_1 L_{l_2}}) \psi_{L'M'}.$$  \hfill (A.9)

### B Estimator of the ISW-lensing bispectrum for Method II

Using the reconstructed lensing potential, $\hat{\psi}_{lm}$, and the temperature data, one can compute the temperature-lensing cross-power spectrum as $\hat{C}_{TT}^{\psi} = (2l + 1)^{-1} \sum_{m=-l}^{l} a_{lm} \hat{\psi}_{lm}^s$, hence the ISW-lensing bispectrum (c.f. Eq. (2.6)):

$$\hat{\theta}_{\text{ISW-lensing}}^{lm} = \frac{l_2(l_2 + 1) + l_3(l_3 + 1) - l_1(l_1 + 1)}{2} C_{l_1} C_{l_2} C_{l_3} + (5 \text{ perm}),$$  \hfill (B.1)

where $C_{TT}$ is the theoretical temperature power spectrum and $\hat{C}_{TT}^{\psi} = (2l + 1)^{-1} \sum_{m=-l}^{l} a_{lm} \hat{\psi}_{lm}^s$ is the temperature-lensing cross-power spectrum estimated from the data.

However, if $\hat{\psi}_{lm}$ is estimated from the temperature data themselves, $\hat{\psi}_{lm}$ contains a product of two $a_{lm}$’s, and thus the ensemble average of $\hat{C}_{TT}^{\psi}$ picks up the bispectrum of $a_{lm}$.

In the absence of primordial non-Gaussianity this is not an issue; however, the presence of primordial non-Gaussianity (such as $f_{NL}$) produces a bias in $\hat{C}_{TT}^{\psi}$. This effect needs to be taken into account when we write down an estimator of the ISW-lensing bispectrum.

Using Eq. (A.1), we find

$$\langle \hat{C}_{TT}^{\psi} \rangle = (2l + 1)^{-1} \sum_{m=-l}^{l} \langle a_{lm} \hat{\psi}_{lm}^s \rangle = \frac{N_l}{2l + 1} \sum_{l_1} \sum_{l_2} \sum_{m_1} \sum_{m_2} \left( \frac{l_1 m_1}{l_2 m_2} \right) \left( C_{l_1} F_{l_1 l_{l_1}} + C_{l_2} F_{l_1 l_{l_2}} \right) \langle a_{l_1 m_1} a_{l_2 m_2} a_{lm} \rangle 2 C_{l_1'} C_{l_2'}.$$ \hfill (B.2)

where we have used $\hat{\psi}_{lm} = (-1)^m \hat{\psi}_{l,-m}$. As discussed in Sec. 2, the expectation value of the 3-point correlation is given by

$$\langle a_{l_1' m_1'} a_{l_2' m_2'} a_{lm} \rangle = G_{l_1 l_2 l_3}^{m_1 m_2 m_3} \left( f_{NL} b_{l_1 l_2 l_3}^{\text{local}} + b_{l_1 l_2 l_3}^{\text{ISW-lensing}} \right),$$ \hfill (B.3)

where

$$G_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \sqrt{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1) \frac{l_1 l_2 l_3}{4\pi} \binom{l_1}{0} \binom{l_2}{0} \binom{l_3}{0} \binom{m_1}{m_2} \binom{m_2}{m_3}}.$$ \hfill (B.4)
As $C_l^{TT}$ decreases rapidly with $l$, $b_{l_1 l_2}^{\text{SW-lensing}}$ of a squeezed configuration ($l \ll l_1' \approx l_2'$) is given by

\[ b_{l_1 l_2}^{\text{SW-lensing}} \approx \frac{l_2'(l_2' + 1) + l(l + 1) - l_1'(l_1' + 1)}{2} C_{l_2}'^{TT} C_l^{TT} + \frac{l_1'(l_1' + 1) + l(l + 1) - l_1'(l_1' + 1)}{2} C_{l_1}'^{TT} C_l^{TT}. \]  

(B.5)

Plugging Eqs. (B.3) and (B.5) into Eq. (B.2) and then using Eqs. (A.2), (A.3), and (A.4), we find

\[
\langle \hat{C}_l^{TT} \hat{C}_l' \rangle \approx \frac{N_l}{2l + 1} \sum_{l''} \sum_{l_1' l_2'} \left( \frac{2l_1' + 1)(2l_2' + 1)(2l + 1)}{4\pi} \left( \begin{array}{c} 1' \ 0 \ 0' \\ 0 \ 0 \ 0' \end{array} \right) \left( C_{l_1} F_{l_1 l_1'} + C_{l_2} F_{l_2 l_2'} \right) \right)^2 \]

(B.6)

\[
= \sum_{l} \frac{N_l}{2l + 1} \sum_{l''} \sqrt{(2l_1' + 1)(2l_2' + 1)(2l + 1)} \left( \begin{array}{c} 1' \ 0 \ 0' \\ 0 \ 0 \ 0' \end{array} \right) \left( C_{l_1} F_{l_1 l_1'} + C_{l_2} F_{l_2 l_2'} \right) \]

\[
= C_l^{TT} + \sum_{l} \frac{N_l}{2l + 1} \sum_{l''} \sqrt{(2l_1' + 1)(2l_2' + 1)(2l + 1)} \left( \begin{array}{c} 1' \ 0 \ 0' \\ 0 \ 0 \ 0' \end{array} \right) \left( C_{l_1} F_{l_1 l_1'} + C_{l_2} F_{l_2 l_2'} \right) \]

(B.7)

where

\[
\xi_l \equiv \frac{N_l}{2l + 1} \sum_{l''} \sqrt{(2l_1' + 1)(2l_2' + 1)(2l + 1)} \left( \begin{array}{c} 1' \ 0 \ 0' \\ 0 \ 0 \ 0' \end{array} \right) \left( C_{l_1} F_{l_1 l_1'} + C_{l_2} F_{l_2 l_2'} \right) \]

and, in the second line, we have used the identity of the Wigner 3j symbol:

\[
\sum_{m_1 m_2} \left( \begin{array}{c} l_1 \ l_2 \ l_3 \\ m_1 \ m_2 \ m_3 \end{array} \right) \left( \begin{array}{c} l_1 \ l_2 \ l_3' \\ m_1 \ m_2 \ m_3' \end{array} \right) = \delta_{l_3 l_3'} \delta_{m_3 m_3'} \frac{2l_3 + 1}{2l_3 + 1}. \]  

(B.8)

Using the result above, we finally find

\[
\langle C_{l_2}^{TT} \hat{C}_{l_3}^{TT} \rangle = C_{l_2}^{TT} C_{l_3}^{TT} + f_{\text{NL}} C_{l_2}^{TT} \xi_l. \]  

(B.9)
C Reconstruction noise bias in the ISW-subtracted bispectrum for Method III

The reduced bispectrum of the “ISW-subtracted map” is

\[ \hat{b}_{\ell_1 \ell_2 \ell_3} = b_{\ell_1 \ell_2 \ell_3}^{\text{local}} + b_{\ell_1 \ell_2 \ell_3}^{\text{ISW-lensing}} - (G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3})^{-1} \left[ \langle a_{l_1 m_1} a_{l_2 m_2} \frac{C_{l_3}^{T \psi}}{C_{l_3}^{\psi \psi}} \hat{\psi}_{l_3 m_3} \rangle + (2 \text{ perm}) \right] + \mathcal{O}(\psi^2) \]

\[ = b_{\ell_1 \ell_2 \ell_3}^{\text{local}} - (G_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3})^{-1} \left[ \langle a_{l_1 m_1} a_{l_2 m_2} \frac{C_{l_3}^{T \psi}}{C_{l_3}^{\psi \psi}} \hat{\psi}_{l_3 m_3} \rangle + (2 \text{ perm}) \right] + \mathcal{O}(\psi^2). \]  

(C.1)

Here, \( \hat{\psi}_{LM} \) is the reconstructed lensing potential, which is the sum of the true lensing potential \( \psi_{LM} \) and reconstruction noise \( n_{LM} \): \( \hat{\psi}_{LM} = \psi_{LM} + n_{LM} \). Using the quadratic estimator, we may reconstruct the lensing potential, \( \hat{\psi}_{LM} \), by Eq. (A.1), where the reconstruction noise, \( n_{LM} \), is given by Eq. (A.8). In order to compute \( \langle a_{l_1 m_1} a_{l_2 m_2} n_{LM} \rangle \), we need to compute \( \langle a_{l_1 m_1} a_{l_2 m_2} \hat{a}_{l_3 m_3} \rangle \) and \( \langle a_{l_1 m_1} a_{l_2 m_2} (a_{l_3 m_3}^{\text{CMB}}) \rangle \). While both of these contain the term that is linearly proportional to the power spectrum of the lensing potential, \( C_{l}^{\psi \psi} \), these linear terms cancel out in the difference:

\[ \langle a_{l_1 m_1} a_{l_2 m_2} \hat{a}_{l_3 m_3} \rangle - \langle a_{l_1 m_1} a_{l_2 m_2} (a_{l_3 m_3}^{\text{CMB}}) \rangle = C_{l_1}^{\text{tot}} \delta_{l_1 l_3} \delta_{m_1 - m_3} (-1)^{m_1 + 1} C_{l_2}^{\text{tot}} \delta_{l_2 l_3} \delta_{m_2 - m_3} (-1)^{m_2} 
+ C_{l_1}^{\text{tot}} \delta_{l_1 l_3} \delta_{m_1 - m_3} (-1)^{m_1} C_{l_2}^{\text{tot}} \delta_{l_2 l_3} \delta_{m_2 - m_3} (-1)^{m_2} 
+ \mathcal{O}[(C_{l}^{\psi \psi})^2]. \]  

(C.2)

Therefore, we obtain

\[ \langle a_{l_1 m_1} a_{l_2 m_2} n_{LM} \rangle = N_L \left( \begin{array}{c} l_1 \\ m_1 \\ l_2 \\ m_2 \\ L \\ M \end{array} \right) (C_{l_1}^{T \psi} F_{l_2} + C_{l_2}^{T \psi} F_{l_1}) \]

\[ = G_{l_1 l_2}^{m_1 m_2 M} N_L \left[ \frac{L(L+1) + l_1(l_1+1) - l_2(l_2+1)}{2} C_{l_1} + \frac{L(L+1) + l_2(l_2+1) - l_1(l_1+1)}{2} C_{l_2} \right]. \]  

(C.3)

Here, we have used \( l_1 + l_2 + L = \text{even} \) (parity invariance), \( m_1 + m_2 + M = 0 \) (triangular condition), and \( \left( \begin{array}{c} l_1 \\ m_1 \\ l_2 \\ m_2 \\ L \\ -M \end{array} \right) = \left( \begin{array}{c} l_1 \\ l_2 \\ L \\ -m_1 \\ -m_2 \\ -M \end{array} \right) \) (parity invariance). Plugging Eq. (C.3) into Eq. (C.1), we find

\[ \hat{b}_{l_1 l_2 l_3} = b_{l_1 l_2 l_3}^{\text{local}} + b_{l_1 l_2 l_3}^{\text{noise}} + \mathcal{O}(\psi^2), \]  

(C.4)

where the reconstruction noise bias \( b_{l_1 l_2 l_3}^{\text{noise}} \) is given by:

\[ b_{l_1 l_2 l_3}^{\text{noise}} = -\frac{C_{l_1}^{T \psi}}{C_{l_3}^{\psi \psi}} N_L \left[ \frac{l_3(l_3+1) + l_1(l_1+1) - l_2(l_2+1)}{2} C_{l_1} + \frac{l_3(l_3+1) + l_2(l_2+1) - l_1(l_1+1)}{2} C_{l_2} \right] + (2 \text{ perm}). \]  

(C.5)

The noise bias in \( f_{NL} \) is \( \delta f_{NL} = -25.3 \) for the Planck noise level, and \( \delta f_{NL} = -8.1 \) for the cosmic-variance-limited case. These biases have been subtracted already in the values quoted in Table 6.
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