Spin & Statistics in Nonrelativistic Quantum Mechanics, I

Bernd Kuckert
Korteweg-de Vries Instituut voor Wiskunde
Amsterdam, The Netherlands

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Abstract

A necessary and sufficient condition for Pauli’s spin-statistics relation is given for nonrelativistic anyons, bosons, and fermions in two and three spatial dimensions.

For any point particle species in two spatial dimensions, denote by $J$ the total (i.e., spin plus orbital) angular momentum of a single particle, and let $j$ be the total angular momentum of the corresponding two-particle system with respect to its center of mass. In three spatial dimensions, write $J_z$ and $j_z$ for the $z$-components of these vector operators.

In two spatial dimensions, the spin statistics connection holds if and only if there exists a unitary operator $U$ such that $j = 2UJU^*$. In three dimensions, the analogous relation cannot hold as it stands, but restricting it to an appropriately chosen subspace of the state space yields a sufficient and necessary condition for the spin-statistics connection.

1 Introduction

The best-known derivations of Pauli’s spin-statistics connection (which will, as usual, be called “the” spin-statistics connection in what follows) have been

*Present address: II. Institut für Theoretische Physik, Luruper Chaussee 149, 22761 Hamburg, Germany
found in quantum field theory, where various proofs of increasing generality have been given over the decades. Fierz and Pauli [11, 25] treated free fields, and Lüders, Zumino, and Burgoyne [23, 17] considered finite-component general Wightman fields in 1+3 spacetime dimensions (see also [27]). Similar results were obtained in the setting of algebraic quantum field theory in 1+3 dimensions for both localizable charges (Thm. 6.4 in [8]) and topological charges [6].

Recently it has been found that for massive single-particle states of quantum field theories, the spin-statistics connection can be derived from the Unruh effect [14] (cf. also [13, 41]) or from a special form of PCT-symmetry [17], which follows from the Unruh effect by an argument given in [14]. Using an argument given in [5], one can further improve the result of [17]: the homogeneous part of the symmetry group does not need to be the universal covering of the restricted Lorentz group; it suffices to have the universal covering of the rotation group as symmetry group [18]. The strategy used in [14, 17] has also led to spin-statistics theorems for anyons and plektons in two spatial dimensions [22, 24], conformal fields [15], and quantum fields on curved spacetimes [16, 31].

Another approach to the spin-statistics connection is purely classical ([28, 29], cf. also [1]). It provides an illustration of some crucial steps in the quantum field theory proofs (cf. also the remarks made in [30]) rather than a derivation from first principles.

There have also been attempts to derive the spin-statistics connection in the setting of nonrelativistic quantum mechanics. But all arguments suggested so far turned out to be based on too restrictive assumptions, or they have been falsified by counterexamples (cf. the discussions and references in [10, 9, 32]). It has been shown in [3] that this also holds for the recent attempt by Berry and Robbins [2].

It is well known that quantum mechanics as such admits – like quantum field theory, see [26] – systems violating the spin-statistics connection: the easiest examples are spinless fermions, i.e., single-component wavefunctions that are antisymmetric under particle exchange, also counterexamples with nonzero spin are easy to find, and their second quantization is straightforward as well (see, e.g., [33]). Each derivation of the spin-statistics connection must rely on some additional assumption ruling out these counterexamples.

In this Letter, we consider nonrelativistic anyons, bosons, and fermions in two or three spatial dimensions and give a necessary and sufficient condition for the connection between such a particle’s spin $\sigma$ and its statistics phase
\( \kappa \in S^1 \), which Pauli discovered to be

\[ e^{2\pi i \kappa} = \kappa. \]  

(1)

In three spatial dimensions, it has been shown that \( \kappa \in \{\pm 1\} \), whereas in two dimensions, \( \kappa \) can be any element of \( S^1 \) \[20\] \[21\].

For a theory with nonabelian statistics, some additional structure (e.g., a Markov trace) is needed to define a statistics phase and, hence, to make the problem of finding a spin-statistics connection well posed. The issue whether and how the subsequent argument can be generalized to this case will be left open here.

In classical mechanics, the total angular momentum of two indistinguishable particles with respect to their center of mass is twice the angular momentum of each of the two with respect to the same point of reference. Does this fact have a quantum counterpart? Evidently, the observables to be compared with each other will typically live in different Hilbert spaces, so any analogous equality can, at most, be one up to a similarity transformation by a unitary operator between these two Hilbert spaces, i.e., up to unitary equivalence.

In the setting of two spatial dimensions, \( J \) will denote the total angular momentum operator of a given single particle in its one-particle space, and \( j \) will be the total angular momentum of the corresponding two-particle system with respect to its center of mass. It turns out that the spin-statistics connection holds if and only if there is a unitary operator \( U \) such that

\[ j = 2UJU^*. \]  

(2)

This strong result is possible since the rotation group \( S^1 \) is abelian, the consequence being that adding angular momenta is analogous to the classical addition.

In three spatial dimensions, the situation is more involved, since the rotation group and its universal covering are nonabelian, and therefore the addition of angular momenta and spins is well known to be more involved than in two dimensions. Denoting by \( J_z \) the \( z \)-component of the single particle total angular momentum and by \( j_z \) the \( z \)-component of the two-particle system’s total angular momentum with respect to its center of mass, one finds that the condition

\[ j_z = 2UJ_zU^* \]  

cannot hold as it stands. Nevertheless one can look for some subspace restricted to which the relation (2) is meaningful and provides a sufficient and

\[ ^3 \text{In this paper, the word “total” is, as usual, to be read as “spin plus orbital”}. \]
necessary criterion for the spin-statistics connection.

To this end, the analysis of the three-dimensional case will be confined to the Hilbert spaces $\mathcal{H}^\uparrow$ and $\mathcal{H}^{\uparrow\uparrow}$ of all one-particle and two-particle states where the $z$-components of all particle spins take their maximum values. Evidently these spaces are not invariant under most time evolutions. This does, however, not affect the argument given below, since it is purely kinematical: no Hamiltonian is specified, and the free Hamiltonian, which one may use to specify the particle mass as a further characteristic property of the particle, does commute with spin.

Within $\mathcal{H}^{\uparrow\uparrow}$, denote by $\mathcal{H}_+$ and $\mathcal{H}_-$ the eigenspaces of the $z$-parity operator $P_z : (x, y, z) \mapsto (x, y, -z)$ consisting of the functions in that are even or odd in the $z$-component, respectively. It can be shown that Condition (2) holds either when restricted to $\mathcal{H}_+$ or when restricted to $\mathcal{H}_-$. The spin-statistics connection is equivalent to the first alternative, and its violation is equivalent to the second.

After specifying some setting and notation in Sect. 2, the two-dimensional case is discussed in Sect. 3. The three-dimensional situation is discussed in Sect. 4, and some concluding remarks are made in Sect. 5.

### 2 Setting and Notation

The space of pure states of $n$ indistinguishable Bose or Fermi particles in $s$ spatial dimensions can be defined by imposing either symmetry or antisymmetry under particle exchange on a wave function in $L^2(\mathbb{R}^s)$. Alternatively, one may first reduce the classical configuration space by identifying indistinguishable configurations, and then consider all wave functions on this space. In three dimensions, it is a matter of taste which approach one wishes to use. In two dimensions, however, particles whose statistics is neither (para-) bosonic nor (para-) fermionic can occur, and these particles can only be described in the second approach. For this reason, this approach will be used in what follows.

Following Laidlaw and DeWitt [20], the configuration space of $n$ distinguishable particles in $\mathbb{R}^s$ is described by the set $Y(n, s)$ of all $n$-tuples of $s$-vectors no two of which coincide:

$$Y(n, s) := \{ y = (y_1, \ldots, y_n) \in (\mathbb{R}^s)^n : y_i \neq y_j \text{ for } i \neq j \}.$$
An action of the symmetric group $S_n$ on this space is defined by
\[ \pi y := (y_{\pi^{-1}(1)}, \ldots, y_{\pi^{-1}(n)}), \quad \pi \in S_n. \]

The orbits of $S_n$ in $Y(n, s)$ yield the configuration space $X(n, s) := Y(n, s)/S_n$ of $n$ indistinguishable particles. It is straightforward to endow $X(n, s)$ with the structure of a pathwise connected topological space for $s \geq 2$ whose fundamental group is $S_n$ for $s = 3$ and the braid group $B_n$ for $s = 2$. The fact that there are only two scalar unitary representations of $S_n$ implies the Bose-Fermi alternative for $s = 3$ [20], in two spatial dimensions, arbitrary fractional statistics can occur as well [21].

For $n = 1$, one has $X(1, s) = \mathbb{R}^s$, and a pure state of one particle whose $z$-component of spin equals its maximum possible value $\sigma$ can be described by a one-component wave function on $\mathbb{R}^s$ (all possible other spinor components vanish). As usual, the $z$-component of the orbital angular momentum is described by the self-adjoint operator $L_z$, and the $z$ component of the total angular momentum operator is $J_z = L_z + \sigma$.

For $n = 2$, center of mass coordinates can be used to describe the configuration space of two indistinguishable particles as the cartesian product of $\mathbb{R}^s$ and a relative coordinate space $C$ [21]. In two spatial dimensions, $C$ is the cone obtained (using planar polar coordinates) from the half plane
\[ \overline{H} := \{(r, \varphi) \in \mathbb{R}^2 : r \geq 0, -\pi/2 \leq \varphi \leq \pi/2\} \]
by identifying $(r, -\pi/2)$ with $(r, \pi/2)$ for each $r > 0$ and by removing the origin at $r = 0$. For $s = 3$, $C$ is obtained from the half space
\[ \overline{H} := \{(x, y, z) \in \mathbb{R}^3 : x \geq 0\} \]
by identifying $(0, y, z)$ with $(0, -y, -z)$ for all $(y, z) \in \mathbb{R}^2$ and by removing the origin.

In both two and three dimensions, $H$ will denote the interior of $\overline{H}$.

## 3 Two spatial dimensions

In two spatial dimensions, the pure-state space of two indistinguishable particles both having spin $\sigma$ is canonically isomorphic with the space $L^2(\mathbb{R}^2 \times C)$. The wave functions have one component only, since the universal covering
group of the rotation group is abelian, and since, as a consequence, its irreducible representations are one-dimensional. Corresponding to the interrelation between $\mathcal{C}$ and $H$ just discussed, any choice of the above coordinates induces an isomorphism from $L^2(\mathbb{R}^2 \times \mathcal{C})$ onto $L^2(\mathbb{R}^2 \times H)$ in a straightforward fashion. Accordingly, the state space under consideration is
\[ L^2(\mathbb{R}^2 \times H, 2d^2 \mathbf{R} \, d^2 \mathbf{r}) \cong L^2(\mathbb{R}^2, d^2 \mathbf{R}) \otimes L^2(H, 2d^2 \mathbf{r}). \]

The orbital angular momentum operator in $L^2(\mathbb{R}^2) \otimes L^2(H)$ with respect to the system’s center of mass is of the form $1 \otimes \ell$, where $\ell$ is a self-adjoint operator in $L^2(H)$. On the test functions with compact support in the interior of $H$, the operator $\ell$ coincides with the familiar differential operator $-i\partial_\varphi$. But this hermitian differential operator possesses many self-adjoint extensions $\ell$, which yield different unitaries $R := e^{\pi i \ell}$. Since by definition, the orbital angular momentum operator generates a representation of the rotation group, one has $R^2 = e^{2\pi i \ell} = 1$.

If for some $\lambda \in \mathbb{R}$, we define $j := \ell + \lambda$ as the total angular momentum operator with respect to the center of mass, then the statistics phase $\kappa \in S^1$ is related to $j$ by $\kappa = e^{\pi i j} = Re^{\pi \lambda i}$. It is to be emphasized that $\lambda$ is not assumed to equal $2\sigma$ from the outset; it will, however, be found as a result that $\lambda - 2\sigma$ is an even integer if Pauli’s spin-statistics connection holds.

Denoting the orbital angular momentum operator $-i\partial_\varphi$ in the single-particle space $L^2(\mathbb{R}^2)$ by $L$, the following theorem can be shown.

**Theorem 1** For $s = 2$, the spin-statistics connection holds if and only if there exists a unitary operator $U : L^2(\mathbb{R}^2) \to L^2(H)$ such that
\[ j = 2(ULU^* + \sigma). \]  

\(^2\)As an aside, note that the exchange of two (indistinguishable) pairs of (indistinguishable) particles yields a braid diagram with four crossings, so the statistics phase of a two-particle system is $\kappa^4$; cf. also [12]. For arbitrary $n \in \mathbb{N}$, a little braid group diagrammatics shows that a rotation of $n$ indistinguishable particles with respect to their center of mass is accompanied by a statistics phase $\kappa^{n(n-1)}$, whereas the $n$-particle system’s statistics phase is $\kappa^n$, so one can assign a statistics phase $\kappa^{n(n-1)}$ to the relative motion and a phase $\kappa^n$ to the center of mass motion. This remark is redundant if $\kappa \in \{\pm 1\}$ as in three dimensions, but since in two dimensions, $\kappa$ can take any value in $S^1$, it should be in place here.

\(^3\)To be more precise, the differential operator $-i\partial_\varphi$ is well defined and essentially self-adjoint on the dense domain of test functions; $L$ denotes its (self-adjoint) closure.
Proof. Two lemmas will be used:

**Lemma 2** For every integer \( \nu \) with \((-1)^\nu R = 1\), there exists a unitary operator \( U_\nu : L^2(\mathbb{R}^2) \rightarrow L^2(H) \) with \( 2L = U_\nu^* U_\nu + \nu \).

**Proof.** Denote the slit plane \( \mathbb{R}^+ \times (-\pi, \pi) \) by \( H^2 \), and define

\[
U_\nu \Psi(r, \phi) := e^{-\nu \phi i} \Psi(r, 2\phi), \quad \Psi \in C_0^\infty(H^2), \quad (r, \phi) \in H.
\]

By Stone’s theorem, it suffices to prove that

\[
e^{-\vartheta i \cdot 2L} \Psi = e^{-\vartheta i \cdot (U_n^* U_\nu + \nu)} \Psi
\]

for every \( \Psi \in C_0^\infty(H^2) \) and every \( \vartheta \in \mathbb{R} \). It turns out that this can be accomplished by pointwise evaluating \( e^{-\vartheta i \cdot 2L} \Psi \) and \( e^{-\vartheta i \cdot (U_n^* U_\nu + \nu)} \Psi \) almost everywhere. Namely, choose any \((r, \phi) \in H^2\) such that there exists a unique integer \( n_{\vartheta, \phi} \) with

\[-\pi < \vartheta - 2\phi + 2\pi n_{\vartheta, \phi} < \pi.\]

It is well known that \( e^{-\vartheta i \cdot 2L} \Psi(r, \phi) = \Psi(r, \phi - 2\vartheta + 2\pi n_{\vartheta, \phi}) \).

One concludes from this that

\[
e^{-\vartheta i \cdot (U_\nu U_n + \nu)} \Psi(r, \phi) = e^{-\nu \phi i} U_\nu e^{-\vartheta i \cdot \ell} U_\nu \Psi(r, \phi) = e^{\nu i(\phi/2 - \vartheta)} e^{-\nu \phi i} U_\nu \Psi(r, \phi/2)
\]

\[
= e^{\nu i(\phi/2 - \vartheta)} e^{-\vartheta i \cdot \ell} e^{-\nu i(\phi/2)} \Psi(r, \phi)
\]

\[
= e^{\nu i(\phi/2 - \vartheta)} R^{n_{\vartheta, \phi}} e^{-i(\vartheta - \pi n_{\vartheta, \phi}) \ell} e^{-\nu i(\phi/2)} \Psi(r, \phi)
\]

\[
= e^{\nu i(\phi/2 - \vartheta)} (-1)^{\nu n_{\vartheta, \phi}} e^{-\nu i(\phi/2 - \vartheta + \pi n_{\vartheta, \phi})} \Psi(r, \phi - 2\vartheta + 2\pi n_{\vartheta, \phi})
\]

\[
= (-1)^{\nu n_{\vartheta, \phi}} (-1)^{\nu n_{\vartheta, \phi}} \Psi(r, \phi - 2\vartheta + 2\pi n_{\vartheta, \phi}) = e^{-\vartheta i \cdot 2L} \Psi(r, \phi).
\]

Since this reasoning applies for almost all \((r, \phi) \in H^2\), this completes the proof.

**Lemma 3** Any two of the following three conditions imply the third one:

(i) \( e^{\pi i} = e^{2\pi i} \), i.e., Eq. (7).

(ii) \( \lambda \in 2\sigma + 2\mathbb{Z} \).

(iii) \( R = 1 \).

\[\square\]
Proof. If Condition (i) holds, then

\[ 1 = e^{i\pi \ell} \cdot e^{i(\lambda - 2\sigma)} = Re^{i(\lambda - 2\sigma)}. \]

It follows that Condition (i) implies [(ii) ⇔ (iii)]. It remains to show that [(ii) ∧ (iii)] implies (i): \( e^{i\pi \ell} e^{i(\lambda - 2\sigma)} = e^{2\pi i \sigma} \).

Finally, to prove Thm. 1, assume Condition (3). One then has

\[ \kappa = e^{i\pi j} = e^{i(2(UU^* + \sigma)} = e^{2\pi i \sigma} U e^{2\pi i L} U^* = e^{2\pi i \sigma}, \]

which is Eq. (1).

Conversely, assume Eq. (1) to hold. Then \( e^{i\pi \ell} = Re^{i\lambda} = \pm e^{\pi \lambda i} = e^{2\pi i \sigma} \), so \( \lambda - 2\sigma \in \mathbb{Z} \).

If \( \lambda - 2\sigma \) is even, then Lemma 3 implies \( R = 1 = (-1)^{\lambda - 2\sigma} \), so by Lemma 2, there exists a unitary intertwiner between \( 2L \) and \( \ell + \lambda - 2\sigma \), which is Condition (3).

If \( \lambda - 2\sigma \) is odd, then Lemma 3 implies \( R = -1 = (-1)^{\lambda - 2\sigma} \). Again, Lemma 2 implies Condition (3). \(\)

4 Three spatial dimensions

In three spatial dimensions, a relation analogous to Eq. (3) cannot hold on the whole Hilbert space. In order to see this, first note that such a condition would, in particular, have to hold for the one-particle- and two-particle states where the spins of all particles are prepared at their highest possible values \( \sigma \in \frac{1}{2} \mathbb{Z} \) (one-particle states) and \( \lambda \in \mathbb{Z} \) (two-particle states), respectively. The corresponding one-particle and two-particle spaces will be denoted by \( \mathcal{H}^\uparrow \) and \( \mathcal{H}^\uparrow \uparrow \), respectively.

\( \mathcal{H}^\uparrow \) is canonically isomorphic with \( L^2(\mathbb{R}^2) \), and \( \mathcal{H}^\uparrow \uparrow \) is canonically isomorphic with the space \( L^2(\mathbb{R}^3 \times \mathbb{C}) \), since all spinor components except one vanish. Any choice of the above coordinates induces an isomorphism from \( L^2(\mathbb{R}^3 \times \mathbb{C}) \) onto \( L^2(\mathbb{R}^3 \times H) \) in a straightforward (while coordinate-dependent) fashion, so the state space under consideration is

\[ \mathcal{H}^\uparrow \uparrow = L^2(\mathbb{R}^3 \times H, 2d^3\mathbf{r}) \]

\[ \cong L^2(\mathbb{R}^3, d^3\mathbf{r}) \otimes L^2(H, 2d^3\mathbf{r}) =: \mathcal{H}^\uparrow \uparrow_{\text{CM}} \otimes \mathcal{H}^\uparrow \uparrow_{\text{rel}}. \]
The $z$-component of the orbital angular momentum operator in $\mathcal{H}_{\text{CM}}^{\uparrow} \otimes \mathcal{H}_{\text{rel}}^{\uparrow}$ with respect to the system’s center of mass is of the form $1 \otimes \ell_z$, where $\ell_z$ is a self-adjoint operator in $\mathcal{H}_{\text{rel}}^{\uparrow} \cong L^2(H)$. When restricted to $C_0^\infty(H)$, the self-adjoint operator $\ell_z$ coincides with the familiar hermitian differential operator $-i\partial_\varphi$. Since by definition, the orbital angular momentum operator generates a representation of the group of rotations around the $z$-axis in $L^2(H)$, the operator $R_z := e^{\pi i \ell_z}$ is an involution, i.e., $R_z^2 = e^{2\pi i \ell_z} = 1$.

Now define $P_z \Psi(r, \varphi, z) = \Psi(r, \varphi, -z)$, and denote by $j_z := \ell_z + \lambda$ the $z$-component of the total angular momentum with respect to the center of mass. Since

$$\kappa = P_z e^{\pi i j_z} = e^{\pi \lambda i} P_z R_z,$$

(cf. the remark made in Sect. 2 concerning the role of $\lambda$), and since $R_z^2 = P_z^2 = \kappa^2 = 1$, one finds $R_z = \pm P_z$.

Next note that in this setting, the condition of Thm. 1 would still lead to

$$1 = U e^{2\pi i L_z U^*} = e^{\pi i (2UL_z U^*)} = e^{\pi i (j_z - \lambda)} \in S^1 R_z = S^1 P_z,$$

so it cannot hold as it stands.

Defining $\mathcal{H}_\pm := \{ \Psi \in \mathcal{H}_{\text{rel}}^{\uparrow} : P_z \Psi = \pm \Psi \}$, one can prove

**Theorem 4** (i) For $s = 3$, the spin-statistics connection (1) holds if and only if there exists a unitary operator $U : \mathcal{H}^{\uparrow} \to \mathcal{H}_{\text{rel}}^{\uparrow}$ such that

$$j_z|_{\mathcal{H}_{\pm}} = 2U(L_z + \sigma)U^*|_{\mathcal{H}_{\pm}}. \quad (4)$$

(ii) For $s = 3$, Eq. (4) does not hold if and only if there exists a unitary operator $U : \mathcal{H}^{\uparrow} \to \mathcal{H}_{\text{rel}}^{\uparrow}$ such that

$$j_z|_{\mathcal{H}_{\pm}} = 2U(L_z + \sigma)U^*|_{\mathcal{H}_{\pm}}. \quad (5)$$

**Proof.** The three-dimensional counterpart of Lemma 2 is

**Lemma 5** (i) For every integer $\nu$ with $R_z(-1)^\nu|_{\mathcal{H}_{\pm}} = 1$, there exists a unitary operator $U_\nu : L^2(\mathbb{R}^3) \to L^2(H)$ with

$$(\ell_z + \nu)|_{\mathcal{H}_{\pm}} = 2U_\nu L_z U_\nu^*|_{\mathcal{H}_{\pm}}.$$

(ii) For every integer $\nu$ with $R_z(-1)^\nu|_{\mathcal{H}_{\pm}} = 1$, there exists a unitary operator $U_\nu : L^2(\mathbb{R}^3) \to L^2(H)$ with

$$(\ell_z + \nu)|_{\mathcal{H}_{\pm}} = 2U_\nu L_z U_\nu^*|_{\mathcal{H}_{\pm}}.$$
Proof. In analogy to Lemma 2, define $U_\nu \Psi(r, \varphi, z) := e^{-\nu \varphi i} \Psi(r, 2\varphi, z)$ (using cylinder coordinates), which, as above, intertwines between the hermitian differential operators $-i\partial_\varphi + \nu$ defined on the domain $C_0^\infty(H)$ and $-2\partial_\varphi$ defined on the domain $C_0^\infty(H^2)$, respectively. Using this operator in both cases, the proofs are completely analogous to that of Lemma 2. □

The three-dimensional counterpart of Lemma 3 is

**Lemma 6** (i) Any two of the following three conditions imply the third one.

(i.i) $\kappa = P_z e^{\pi i j_z} |_{\mathcal{H}_+} = e^{\pi i j_z} |_{\mathcal{H}_+} = e^{2\pi i \sigma}$.

(ii) $\lambda \in 2\sigma + \mathbb{Z}$.

(iii) $R_z |_{\mathcal{H}_+} = P_z |_{\mathcal{H}_+} = 1$.

(ii) Any two of the following three conditions imply the third one.

(ii.i) $\kappa = P_z e^{\pi i j_z} |_{\mathcal{H}_-} = -e^{\pi i j_z} |_{\mathcal{H}_-} = -e^{2\pi i \sigma}$.

(ii.ii) $\lambda \in 2\sigma + 1 + 2\mathbb{Z}$.

(ii.iii) $R_z |_{\mathcal{H}_-} = P_z |_{\mathcal{H}_-} = -1$.

Proof. The proofs of the two statements are completely analogous to the proof of Lemma 3 and will not be spelled out here. □

Finally, to prove Thm. 4 assume Condition (3). One then has

$\kappa = P_z e^{\pi i j_z} |_{\mathcal{H}_+} = e^{\pi i 2(U L_z U^* + \sigma)} |_{\mathcal{H}_+} = e^{2\pi i \sigma} U e^{2\pi i L_z U^*} |_{\mathcal{H}_+} = e^{2\pi i \sigma}$,

which is Eq. (1).

Conversely, assume Eq. (1). Then Condition (i.i) in Lemma 6 holds. If $\lambda - 2\sigma$ is even, then Lemma 5 implies $R_z = 1 = (-1)^{\lambda - 2\sigma}$. Lemma 5 then implies that on $\mathcal{H}_+$, there exists a unitary intertwiner between $2L_z$ and $\ell + \lambda - 2\sigma$, whence Condition (4) follows.

If $\lambda - 2\sigma$ is odd, then Lemma 6 implies $R_z = -1 = (-1)^{\lambda - 2\sigma}$, and again, Lemma 5(i) implies that on $\mathcal{H}_+$, there is a unitary intertwiner between $L_z$ and $\ell_z + \lambda - 2\sigma$. This proves Statement (i).

The proof of Statement (ii) is completely analogous. □

It is instructive to see what the conditions and statements of Theorem 4 look like when applied to the example of bound states of two spinless Bose or Fermi particles interacting via some attractive central potential.
The parity of each bound state $\Psi \in \mathcal{H}_{\text{rel}}^{\uparrow \uparrow}$ is $(-1)^l$, where $l$ is the azimuthal quantum number. By the indistinguishability of the two particles, only states with either even or odd $l$ can occur, depending on whether the particles are bosons or fermions, respectively. Evidently, the latter violate the spin-statistics connection in the spinless case.

If the two particles are bosons, then $l$ must be even, and the spin-statistics connection holds, so the additivity of angular momenta must hold in $\mathcal{H}_+$ by Thm. 4. For each bound state $\Psi \in \mathcal{H}_+$, the difference $l - m$ must be even as well, because $P_z \Psi = (-1)^{l-m} \Psi$. It follows that $m$ is even and that $U^* \Psi$ is an eigenvector of $L_z$ with the integer eigenvalue $m/2$.

If on the other hand, the two particles are fermions, then only bound states with odd $l$ occur, and since the spin-statistics connection is violated, the additivity of angular momenta holds in the space $\mathcal{H}_-$ by Thm 4. Reasoning as before, one obtains that $l - m$ is odd for bound states in $\mathcal{H}_-$ and that $m$, in turn, is even. Again, $U^* \Psi$ is an eigenvector of $L_z$ with the integer eigenvalue $m/2$.

We find that the space $\mathcal{H}_+$ or $\mathcal{H}_-$ where the additivity condition holds contains precisely the bound states in $\mathcal{H}_{\text{rel}}^{\uparrow \uparrow}$ with even magnetic quantum numbers, as it should be.

### 5 Conclusion and Outlook

The fact that in classical mechanics, the total angular momentum of a system of two identical particles with respect to its center of mass is twice that of each of the two particles, does, in parts, have a quantum mechanical counterpart.

For nonrelativistic quantum mechanics in two spatial dimensions, it turns out that this condition — stated in terms of unitary equivalence of the corresponding operators $\ell$ and $2L$ — is both sufficient and necessary for the spin-statistics connection.

In three spatial dimensions, the nonabelianness of $SU(2)$ implies that the analysis has to be confined to the $z$-components of the vector operators involved. It is customary to confine the discussion to those states where all particle spins have maximal $z$-components. Within the space of these states, the analogue to the two-dimensional additivity condition holds for either the wave functions that are even in the $z$-component of their relative coordinate or for the corresponding odd functions.

It turns out that the first alternative is equivalent to the spin-statistics
relation, whereas the second alternative is equivalent to its violation.

The above results can be reformulated in a way that may be considered as more natural, since the relative-coordinate space $C$ does not need to be “cut open” there in order to obtain the half space $H$ used above. This is currently being worked out together with Jens Mund, and a corresponding joint paper will be published shortly [19].

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