Nonnegative Matrix Factorization via Rank-One Downdate

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This preliminary version of the manuscript still has an incomplete literature review and is missing the section on computational testing. It contains a complete proof of the main theorem. Please check back here after June 15, 2008 for a more complete version of this manuscript.

Abstract

Nonnegative matrix factorization (NMF) was popularized as a tool for data mining by Lee and Seung in 1999. NMF attempts to approximate a matrix with nonnegative entries by a product of two low-rank matrices, also with nonnegative entries. We propose an algorithm called rank-one downdate (R1D) for computing a NMF that is partly motivated by singular value decomposition. This algorithm computes the dominant singular values and vectors of adaptively determined submatrices of a matrix. On each iteration, R1D extracts a rank-one submatrix from the dataset according to an objective function. We establish a theoretical result that maximizing this objective function corresponds to correctly classifying articles in a nearly separable corpus. We also provide computational experiments showing the success of this method in identifying features in realistic datasets.

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1 Nonnegative Matrix Factorization

Several problems in information retrieval can be posed as low-rank matrix approximation. The seminal paper by Deerwester et al. [5] on latent semantic indexing (LSI) showed that approximating a term-document matrix describing a corpus of articles via the SVD led to powerful query and classification techniques. A drawback of LSI is that the low-rank factors in general will have both positive and negative entries, and there is no obvious statistical interpretation of the negative entries. This led Lee and Seung [13] among others to propose nonnegative matrix factorization, that is, approximation of a matrix $A \in \mathbb{R}^{m \times n}$ as a product of two factors $WH^T$, where $W \in \mathbb{R}^{m \times k}$, $H \in \mathbb{R}^{n \times k}$, both have nonnegative entries, and $k \leq \min(m, n)$. Lee and Seung showed intriguing results with a corpus of images. In a related work, Hofmann [11] showed the application of NMF to text retrieval. Nonnegative matrix factorization has its roots in work of Gregory [10], Paatero [14] and Cohen and Rothblum [4].

Since the problem is NP-hard [18], it is not surprising that no algorithm is known to solve NMF to optimality. Heuristic algorithms proposed for NMF have generally been based on incrementally improving the objective $\|A - WH^T\|$ in some norm using local moves. A particularly sophisticated example of local search is due, e.g., to Kim and Park [12]. A drawback of local search is that it is sensitive to initialization and also is sometimes difficult to establish convergence.

We propose an NMF method based on greedy rank-one downdating that we call R1D. R1D is partly motivated by Jordan’s algorithm for computing the SVD, which is described in Section 2. Unlike local search methods, greedy methods do not require an initial guess. In Section 3, we compare our algorithm to Jordan’s SVD algorithm, which is the archetypal greedy downdating procedure. Previous work on greedy downdating algorithms for NMF is the subject of Section 4. In Section 5, we present the main theoretical result of this paper, which states that in a certain model of text due to Papadimitriou et al. [15], optimizing our objective function means correctly identifying a topic in a text corpus. Similarly, optimization of the objective function corresponds to identifying a feature in a certain model of an image database, as demonstrated in Section 6. We then turn to computational experiments: in Section 8, we present results for R1D on image databases, and in Section 9, we present results on text.
2 Algorithm and Objective Function

Rank-one downdate (R1D) is based on the simple observation that the leading singular vectors of a nonnegative matrix are nonnegative. This is a consequence of the Perron-Frobenius theorem [9]. Based on this observation, it is trivial to compute rank-one NMF. This idea can be extended to approximate higher order NMF. Suppose we compute the rank-one NMF and then subtract it from the original matrix. The original matrix will no longer be nonnegative, but all negative entries can be forced to be zero or positive and the procedure can be repeated.

An improvement on this idea takes only a submatrix of the original matrix and applies the Perron-Frobenius theorem. The point is that taking the whole matrix will in some sense average the features, whereas a submatrix can pick out particular features. A second point of taking a submatrix is that a correctly chosen submatrix may be very close to having rank one, so the step of forcing the residuals to being zero will not introduce significant inaccuracy (since they will already be close to zero).

The outer loop of the R1D algorithm is as follows.

\[
\text{function } [W, H] = \text{R1D}(A, k) \nonumber \\
\text{Inputs: } A \in \mathbb{R}^{m \times n}, k > 0. \nonumber \\
\text{Outputs: } W \in \mathbb{R}^{m \times k}, H \in \mathbb{R}^{n \times k}. \nonumber \\
\phantom{\text{function } [W, H] = \text{R1D}(A, k)} \langle 1 \rangle \quad \text{for } \mu = 1, \ldots, k \nonumber \\
\phantom{\text{function } [W, H] = \text{R1D}(A, k)} \langle 2 \rangle \quad [M, N, u, v, \sigma] = \text{ApproxRankOneSubmatrix}(A); \nonumber \\
\phantom{\text{function } [W, H] = \text{R1D}(A, k)} \langle 3 \rangle \quad W(M, \mu) = u(M). \nonumber \\
\phantom{\text{function } [W, H] = \text{R1D}(A, k)} \langle 4 \rangle \quad H(N, \mu) = \sigma v(N). \nonumber \\
\phantom{\text{function } [W, H] = \text{R1D}(A, k)} \langle 5 \rangle \quad A(M, N) = 0. \nonumber \\
\phantom{\text{function } [W, H] = \text{R1D}(A, k)} \langle 6 \rangle \quad \text{end for} \nonumber 
\]

Here, \(M\) is a subset of \(\{1, \ldots, m\}\), \(N\) is a subset of \(\{1, \ldots, n\}\), \(u \in \mathbb{R}^m\), \(v \in \mathbb{R}^n\) and \(\sigma \in \mathbb{R}\), and \(u, v\) are both unit vectors. We follow Matlab subscripting conventions, so that \(u(M)\) denotes the subvector of \(u\) indexed by \(M\). In the above algorithm, \(u(\{1, \ldots, m\} - M) = 0\) and \(v(\{1, \ldots, n\} - N) = 0\). The function \text{ApproxRankOneSubmatrix} selects \(M, N, u(M), v(N), \sigma\) so that \(A(M, N)\) (i.e., the submatrix of \(A\) indexed by row set \(M\) and column set \(N\)) is approximately rank one, and in particular, is approximately equal to \(u(M)\sigma v^T(N)\).

This outer loop for NMF may be called “greedy rank-one downdating” since it greedily tries to fill the columns of \(W\) and \(H\) from left to right by
finding good rank-one submatrices of \( A \) and subtracting them from \( A \). The classical greedy rank-one downdating algorithm is Jordan’s algorithm for the SVD, described in Section 3. Related work on greedy rank-one downdating for NMF is the topic of Section 4.

The subroutine \texttt{ApproxRankOneSubmatrix}, presented later in this section, is a heuristic routine to maximize the following objective function:

\[
f(M, N, u, \sigma, v) = \|A(M, N)\|_F^2 - \gamma \|A(M, N) - u(M)\sigma v(N)^T\|_F^2.
\] (1)

Here, \( \gamma \) is a penalty parameter. The Frobenius norm of an \( m \times n \) matrix \( B \), denoted \( \|B\|_F \), is defined to be \( \sqrt{B(1,1)^2 + B(1,2)^2 + \cdots + B(m,n)^2} \). The rationale for (1) is as follows: the first term in (1) expresses the objective that \( A(M, N) \) should be large, while the second term penalizes departure of \( A(M, N) \) from being a rank-one matrix.

Since the optimal \( u, \sigma, v \) come from the SVD (once \( M, N \) are fixed), the above objective function can be rewritten just in terms of \( M \) and \( N \) as

\[
f(M, N) = \sum_{i=1}^{p} \sigma_i(A(M, N))^2 - \gamma \sum_{i=2}^{p} \sigma_i(A(M, N))^2
\]
\[
= \sigma_1(A(M, N))^2 - (\gamma - 1) \cdot (\sigma_2(A(M, N))^2 + \cdots + \sigma_p(A(M, N))^2),
\] (2)

where \( p = \min(|M|, |N|) \). The penalty parameter \( \gamma \) should be greater than 1 so that the presence of low-rank contributions is penalized rather than rewarded.

We conjecture that maximizing (1) is NP-hard (see Section 7), so we instead propose a heuristic routine for optimizing it. The procedure alternates improving \((v, N)\) and \((u, M)\). The rationale for this alternation is that for fixed \((v, N)\), the objective function (1) is separable by rows of the matrix. Similarly, for fixed \((u, M)\), the objective function is separable by columns.

Let us state and prove this as a lemma.

\textbf{Lemma 1.} Let \((v, N)\) be the optimizing choice of these variables in (1). Then the optimal \( M \) is determined as follows. Define

\[
g_i = -A(i, N)A(i, N)^T + \tilde{\gamma}(A(i, N)v(N))^2,
\] (3)

where

\[
\tilde{\gamma} = \gamma / (\gamma - 1).
\] (4)
Then \( i \in M \) if \( g_i \geq 0 \). (If exact equality \( g_i = 0 \) holds, then including \( i \) or not does not affect optimality.) Furthermore, \( \sigma u_i \) is optimally chosen to be \( A(i, N) v(N) \).

**Remark 1.** The lemma gives the formula for optimal \( \sigma u_i \) for each \( i \), i.e., the formula for the optimal \( \sigma u(M) \). To obtain a formula for optimal \( u \) and \( \sigma \) separately, we define \( \sigma := \|\sigma u(M)\| \) and \( u(M) := \sigma u(M) / \|\sigma u(M)\| \).

**Remark 2.** Assuming instead that the optimizing choice \((u, M)\) is given, there is a similar formula for determining membership in \( N \). Define

\[
f_j = -A(M, j)^T A(M, j) + \bar{\gamma} (A(M, j)^T u(M))^2,
\]

and take \( N = \{j : f_j \geq 0\} \).

**Proof.** Observe that

\[
f(M, N, u, \sigma, v) = \sum_{i=1}^{m} \chi_M(i) \left( \|A(i, N)\|^2 - \gamma \|A(i, N) - \beta_i v(N)^T\|^2 \right),
\]

where \( \beta_i = \sigma u_i \) and \( \chi_M(i) = 1 \) for \( i \in M \) and \( \chi_M(i) = 0 \) for \( i \notin M \). Observe that \( \beta_i \) occurs only in the \( i \)th term of the above summation, hence assuming \( v \) and \( N \) are optimal, each term may be optimized separately. The optimal \( \beta_i \) (that is, the minimizer of \( \|A(i, N) - \beta_i v(N)^T\| \)) is \( A(i, N) v(N) \), the solution to a simple linear least-squares minimization. Thus, we conclude that putting row \( i \) into index set \( M \) is improves the objective function if and only if \( g_i \geq 0 \), where

\[
g_i = \|A(i, N)\|^2 - \gamma \|A(i, N) - A(i, N) v(N) v(N)^T\|^2.
\]

The formula for \( g_i \) can be simplified as follows:

\[
g_i = A(i, N) A(i, N)^T - \gamma (A(i, N) - A(i, N) v(N) v(N)^T) (A(i, N) - A(i, N) v(N) v(N)^T)^T = - (\gamma - 1) A(i, N) A(i, N)^T + \gamma (A(i, N) v(N))^2.
\]

Rescaling by \( \gamma - 1 \) (which does not affect the acceptance criterion) and substituting (4), we that row \( i \) makes a positive contribution to the objective function provided \( \bar{\gamma} (A(i, N) v(N))^2 - A(i, N) A(i, N)^T > 0 \). 

The next issue is choice of starting guess for \( M, N, u, v, \sigma \). The algorithm should be initialized with a starting guess that has a positive score, else the
rules for discarding rows and columns could conceivable discard all rows or columns. More strongly, in order to improve the score of converged solution, it seems sensible to select a starting guess with a high score. For this reason, R1D uses as its starting guess a single column of $A$, and in particular, the column of $A$ with the greatest norm. (A single row may also be chosen.) It then chooses $u$ to be the normalization of this column. This column is exactly rank one, so for the correct values of $\sigma$ and $v$ the first penalty term of (1) is zero. We have derived the following algorithm for the subroutine \texttt{ApproxRankOneSubmatrix} occurring in statement (2) in R1D.

\begin{verbatim}
function $[M,N,u,v,\sigma] = \texttt{ApproxRankOneSubmatrix}(A)$;
Input: $A \in \mathbb{R}^{m \times n}$.
Outputs: $M \subset \{1, \ldots, m\}$, $N \subset \{1, \ldots, n\}$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$.
Parameter: $\bar{\gamma} > 1$

\begin{align*}
(1) & \quad \text{Select } j_0 \in \{1, \ldots, n\} \text{ to maximize } \|A(:,j_0)\|. \\
(2) & \quad M = \{1, \ldots, m\}. \\
(3) & \quad N = \{j_0\}. \\
(4) & \quad \sigma = \|A(:,j_0)\|. \\
(5) & \quad u = A(:,j_0)/\sigma. \\
\end{align*}

\begin{verbatim}
(6) Repeat
(7) \quad v = A(M,:)'u(M). \\
(8) \quad N = \{j : \bar{\gamma} v(j)^2 - \|A(M,j)\|^2 > 0\}. \\
(9) \quad v(N) = v(N)/\|v(N)\|. \quad /* Other entries of v unused */ \\\n(10) \quad \bar{u} = A(:,N)'v(N). \\
(11) \quad M = \{i : \bar{\gamma}\bar{u}(i)^2 - \|A(i,N)\|^2 > 0\}. \\
(12) \quad \sigma = \|u(M)\|. \\
(13) \quad u(M) = \bar{u}(M)/\sigma. \quad /* Other entries of u unused */ \\
\end{verbatim}

\begin{verbatim}(14) until stagnation in $M, N, u, v, \sigma$.
\end{verbatim}

The ‘Repeat’ loop is guaranteed to make progress because each iteration increases the value of the objective function. On the other hand, there does not seem to be any easy way to derive a useful prior upper bound on its number of iterations. In practice, it proceeds quite quickly, usually converging in 10–15 iterations. But to guarantee fast termination, monotonicity can be forced on $M$ and $N$ by requiring $M$ to shrink and $N$ to grow. In other words, statement (8) can be replaced by

\[ N = N \cup \{j : \bar{\gamma} v(j)^2 - \|A(M,j)\|^2 > 0\}, \]
and statement (11) by

\[ M = M - \{ i : \bar{\gamma} \bar{u}(i)^2 - \| A(i, N) \|^2 \leq 0 \}. \]

Our experiments indicate that this change does not have a major impact on the performance of R1D.

Another possible enhancement to the algorithm is as follows: we modify the objective function by adding a second penalty term

\[ -\rho |M| \cdot |N| \quad (6) \]

to (1) where \( \rho > 0 \) is a parameter. The purpose of this term is to penalize very low-norm rows or columns from being inserted into \( A(M, N) \) since they are probably noisy. For data with larger norm, the first term of (1) should dominate this penalty. Notice that this penalty term is also separable so it is easy to implement: the formula in (11) is changed to \( \bar{\gamma} \bar{u}(j)^2 - \| A(M, j) \|^2 - \bar{\rho}|M| > 0 \) while the formula in (11) becomes \( \bar{\gamma} \bar{u}(i)^2 - \| A(i, N) \|^2 - \bar{\rho}|N| > 0 \), where \( \bar{\rho} = \rho/(\gamma - 1) \). We may select \( \bar{\rho} \) so that the third term is a small fraction (say \( \bar{\eta} = 1/20 \)) of the other terms in the initial starting point. This leads to the following definition for \( \rho \):

\[ \rho = \bar{\eta}(\bar{\gamma} - 1)\sigma^2/m, \]

which may be computed immediately after (4).

### 3 Relationship to the SVD

The classical rank-one greedy downdating algorithm is Jordan’s algorithm for computing the singular value decomposition (SVD) [17]. Recall that the SVD takes as input an \( m \times n \) matrix \( A \) and returns three factors \( U, \Sigma, V \) such that \( U \in \mathbb{R}^{m \times k} \) and \( U \) has orthonormal columns (i.e., \( U^T U = I \)), \( \Sigma \in \mathbb{R}^{k \times k} \) and is diagonal with nonnegative diagonal entries, and \( V \in \mathbb{R}^{n \times k} \) also with orthonormal columns, such that \( U \Sigma V^T \) is the optimal rank-\( k \) approximation to \( A \) in either the 2-norm or Frobenius norm. (Recall that the 2-norm of an \( m \times n \) matrix \( B \), denoted \( \| B \|_2 \), is defined to be \( \sqrt{\lambda_{\text{max}}(B^T B)} \), where \( \lambda_{\text{max}} \) denotes the maximum eigenvalue.)

\[ [U, \Sigma, V] = \text{JordanSVD}(A, k); \]

Input: \( A \in \mathbb{R}^{m \times n} \) and \( k \leq \min(m, n) \).
Outputs: $U, \Sigma, V$ as above.

1. for $\mu = 1, \ldots, k$
2. Select a random nonzero $\bar{u} \in \mathbb{R}^m$.
3. $\sigma = \|\bar{u}\|_2$.
4. $u = \bar{u}/\sigma$.
5. Repeat /* power method */
6. $\bar{v} = A^T u$.
7. $v = \bar{v}/\|\bar{v}\|_2$.
8. $\bar{u} = A v$.
9. $\sigma = \|\bar{u}\|_2$.
10. $u = \bar{u}/\sigma$.
11. until stagnation in $u, \sigma, v$.
12. $A = A - u \sigma v^T$;
13. $U(:, \mu) = u$;
14. $V(:, \mu) = v$;
15. $\Sigma(\mu, \mu) = \sigma$;
16. end for

Thus, we see that R1D is quite similar to the SVD. The principal difference is that R1D tries to find a submatrix indexed by $M \times N$ at the same time that it tries to identify the optimal $u$ and $v$. Because of this similarity, the formulas for $u$ and $v$ occurring in (9) and (13) of subroutine ApproxRankOneSubmatrix, which were presented earlier as solutions to a least-squares problem, may also be regarded as steps in a power method. In fact, if $M$ and $N$ are fixed, then the inner Repeat-loop of that subroutine will indeed converge to the dominant singular triple of $A(M, N)$.

As noted earlier, use of the SVD on term-document matrices dates back to latent semantic indexing due to Deerwester et al. [5]. Its effectiveness at creating a faithful low-dimensional model of a corpus in the case of separable corpora was established by Papadimitriou et al. [15]. Although not originally proposed specifically as a clustering tool, the SVD has been observed to find good clusters in some settings [6].

The SVD, however, has a significant shortcoming as far as its use for clustering. Consider the following term-document matrix $A$, which is a sum
of a completely separable matrix $B$ and noise matrix $E$:

$$A = B + E = \begin{pmatrix} 1.01 & 1.01 & 0 & 0 \\ 1.01 & 1.01 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} -0.02 & -0.02 & 0.02 & 0.02 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It should be clear that there are two separate topics in $A$ given by the two diagonal blocks, and a reasonable NMF algorithm ought to be able to identify the two blocks. In other words, for $k = 2$, one would expect an answer close to

$$W = H = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Perhaps unexpectedly, the dominant right singular vector of $A$ is very close to being proportional to $[1; 1; 1; 1]$, i.e., the two topics are entangled in one singular vector. The reason for this behavior is that the matrix $B$ has two nearly equal singular values, so its singular vectors are highly sensitive to small perturbations (such as the matrix $E$). R1D avoids this pitfall by computing the dominant singular vector of a submatrix of the original $A$ instead of the whole matrix.

## 4 Related Work

As mentioned in the introduction, most algorithms proposed in the literature are based on forming an initial $W$ and $H$ and then improving them by local search on an objective function. The objective function usually includes a term of the form $\|A - WH^T\|$ in some norm, and may include other terms.

A few previous works follow an approach similar to ours, namely, greedy subtraction of rank-one matrices. This includes the work of Bergmann et al. [2], who identify the rank-one matrix to subtract as the fixed point of an iterative process. Asgarian and Greiner [1] find the dominant singular pair and then truncate it. Gillis [8] finds a rank-one underestimator and subtracts that. Boutsidis and Gallopoulos [3] consider the use of a greedy algorithm for initializing other algorithms and make the following interesting observation: The nonnegative part of a rank-one matrix has rank at most 2.
The main innovation herein is the idea that the search for the rank-one submatrix should itself be an optimization subproblem. This observation allows us to compare and rank one candidate submatrix to another. (Gillis also phrases his subproblem as optimization, although his optimization problem does not explicitly seek submatrices like ours.) A second innovation is our analysis showing in Section 5 that if the subproblem were solved optimally, then R1D would be able to accurately find the topics in the Papadimitriou et al. [15] model of $\epsilon$-separable corpora.

5 Behavior of this objective function on a nearly separable corpus

In this section, we establish the main theoretical result of the paper, namely, that the objective function given by (1) is able to correctly identify a topic in a nearly separable corpus. We define our text model as follows. There is a universe of terms numbered $1, \ldots, m$. There is also a set of topics numbered $1, \ldots, t$. Topic $k$, for $k = 1, \ldots, t$, is a probability distribution over the terms. Let $P(i, k)$ denote the probability of term $i$ occurring in topic $k$. Thus, $P$ is a singly stochastic matrix, i.e., it has nonnegative entries with column sums exactly 1. We assume also that there is a probability distribution over topics; say the probability of topic $k$ is $\tau_k$, for $k = 1, \ldots, t$. The text model is thus specified by $P$ and $\tau_1, \ldots, \tau_t$. We use the Zipf distribution as the model of document length. In particular, there is a number $L$ such that all documents have length less than $L$, and the probability that a document of length $l$ occurs is

$$\frac{1}{1 + 1/2 + \cdots + 1/(L - 1)}.$$  

We have checked that the Zipf model is a good fit for several common datasets. \[SHOW\ SOME\ DATA.\]

A document is generated from this text model as follows. First, topic $k$ is chosen at random according to the probability distribution $\{\tau_1, \ldots, \tau_t\}$. Then, a length $l$ is chosen at random from $\{1, \ldots, L - 1\}$ according to the Zipf distribution. Finally, the document itself is chosen at random by selecting $l$ terms independently according to the probability distribution $P(i; k)$. A corpus is a set of $n$ documents chosen independently using this text model. Its term-document matrix is the $m \times n$ matrix $A$ such that $A(i, j)$ is the frequency of term $i$ in document $j$. 

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We further assume that the text model is \( \epsilon \)-separable, meaning that each topic \( k \) is associated with a set of terms \( T_k \subset \{1, \ldots, m\} \), that \( T_1, \ldots, T_t \) are mutually disjoint, and that \( P(i,k) \leq \epsilon \) for \( i \notin T_k \), i.e., the probability that a document on topic \( k \) will use a term outside of \( T_k \) is small. Parameter \( \epsilon \) must satisfy some inequalities described below. This corpus model is quite similar to the model of Papadimitriou et al. [15]. One difference is in the document length model. Our model also relaxes several assumptions of Papadimitriou et al.

Our main theorem is that the objective function in the previous section can correctly find documents associated with a particular topic in a corpus.

**Theorem 2.** Let \((P, (\tau_1, \ldots, \tau_t))\) specify a text model, and let \( \alpha > 0 \) be chosen arbitrarily. Suppose there exists an \( \epsilon \geq 0 \) satisfying (15) below such that the text-model is \( \epsilon \)-separable with respect to \( T_1, \ldots, T_t \), the subsets of terms defining the topics. Let \( A \) be the term-document matrix of a corpus of \( n \) documents drawn from this model when the document-length parameter is \( L \). Choose \( \gamma = 4 \) in (1). Then with probability tending to 1 as \( n, L \to \infty \) (refer to Assumption A1 below), the optimizing pair \((M,N)\) of (1) satisfies the following. Let \( D_1, \ldots, D_t \) be the partitioning of the columns of \( A \) according to topics. There exists a topic \( k \in \{1, \ldots, t\} \) such that \( A(M,N) \) and \( A(T_k, D_k) \) are nearly coincident in the following sense.

\[
\sum_{(i,j) \in (M \times N) \triangle (T_k \times D_k)} A(i,j)^2 \leq \alpha \sum_{(i,j) \in M \times N} A(i,j)^2.
\]

Here, \( X \triangle Y \) denotes the set-theoretic symmetric difference \((X - Y) \cup (Y - X)\).

The organization of the proof of this theorem is as follows. We first analyze the Zipf distribution and propose some assumptions that hold with probability tending to 1 as \( n, L \to \infty \). Under these assumptions, we make some preliminary estimates of norms of submatrices of \( A \). Then we prove a sequence of lemmas as follows.

- In Lemma 3, we establish a lower bound on the optimal value of the objective function by analyzing the objective function value with the choice \( M = T_k, N = D'_k \), where \( D'_k \subset D_k \) are the ‘acceptable’ documents from \( D_k \) (defined below).
In Lemma 4, we establish an upper bound on the contribution from unacceptable entries to the optimal solution.

In Lemma 5, we deduce as a consequence of the two preceding lemmas that heavy acceptable entries must compose a significant portion of the optimal solution. Here, an entry \( A(i, j) \) is heavy if \( P(i, k) \geq \chi \), where \( \chi \) is a scalar defined below and \( k \) is the topic of document \( j \).

In Lemma 7, we show that the optimal solution cannot contain heavy acceptable entries from two different topics.

Thus, the preceding lemmas imply that heavy acceptable entries from a single topic \( k \) must dominate the optimal solution. Therefore, we show in Lemma 8 that the left and right singular vectors of the optimal \( A(M, N) \) can be estimated from \( P(M, k) \) and the vector of lengths of documents indexed by \( N \) respectively.

In Lemma 9, we give a general condition under which adding a row or column to \( M \) or \( N \) could improve the objective function value.

In Lemma 10, we show that any column in \( D'_k \) satisfies the condition given by Lemma 9 (because of the estimate of the left singular vector given by Lemma 8), and therefore \( D'_k \subset N \) if \( N \) is optimal.

In Lemma 11, we establish using analogous reasoning that the heavy terms \( H_k \) of topic \( k \) must be a subset of \( M \) if \( M \) is optimal.

Finally, the theorem can be proved because all entries of \( A(M, N) \) that are not from \( H_k \times D'_k \) are either not heavy or unacceptable but in either case, must have small norm.

We start by stating the inequality that \( \epsilon \) must satisfy in order for the theorem to hold. It should be noted that the constants that follow are quite large but are likely large overestimates. Let \( P_{\text{min}} = \min\{P(i, k) : i \in T_k, k = 1, \ldots, t\} \). Without loss of generality, \( P_{\text{min}} > 0 \) since any row \( i \in T_k \) such that \( P(i, k) = 0 \) may be removed from \( T_k \) without affecting the validity of the model. The proof requires four parameters, \( \epsilon, \theta, \phi \) and \( \chi \). The parameters depend on \( \alpha, m, t \), and \( P_{\text{min}} \). They do not depend on \( n \) and \( L \) (since the theorem requires \( n \to \infty \) and \( L \to \infty \)).

First, we define

\[
\chi = \min(\chi_1, \chi_2),
\]
where
\begin{align*}
\chi_1 &= \sqrt{\frac{3}{(32 \cdot 16^2 \cdot 256 t)/m}}, \quad (7) \\
\chi_2 &= \sqrt{\frac{3\alpha}{(32 \cdot 512 t)/m}}. \quad (8)
\end{align*}

Next, we choose
\[ \theta = \min(\theta_1, \theta_2, \theta_3), \]
where
\begin{align*}
\theta_1 &= P_{\text{min}}/2, \quad (9) \\
\theta_2 &= \sqrt{\frac{3}{(8 \cdot 18 \cdot 278 \cdot t)/m}}, \quad (10) \\
\theta_3 &= \chi \cdot \sqrt{1/(6 \cdot 278 \cdot mt)}. \quad (11)
\end{align*}

Third, take
\[ \phi = \min(\phi_1, \phi_2, \phi_3), \]
where
\begin{align*}
\phi_1 &= \frac{3}{(16 \cdot 256 \cdot 3 \cdot 2 \cdot 25^2 mt)}, \quad (12) \\
\phi_2 &= \frac{3\chi^2}{(64 \cdot 6 \cdot 278 \cdot mt)}, \quad (13) \\
\phi_3 &= \frac{\alpha}{(32 \cdot 512 mt)}. \quad (14)
\end{align*}

Last,
\[ \epsilon = \min(\sqrt{\frac{3}{(10m)}}, \chi, \theta). \quad (15) \]

We start with our assumptions of the form that \( n \) or \( L \) must be sufficiently large. The inequalities in this assumption are needed below. Let \( n_k = |D_k| \), that is, the number of documents on topic \( k \), \( k = 1, \ldots, t \). Let \( \tau_{\text{min}} = \min(\tau_1, \ldots, \tau_t) \).

**Assumption A1.** Let \( q = \log_2 L \). Assume \( n \) and \( L \) are sufficiently large so that all of the following are valid.

\begin{align*}
\exp(-2L^{1/2}\theta^2) &\leq \phi, \\
L &\geq 3q/(16\phi), \\
n &\geq q^3, \\
n\tau_{\text{min}} &\geq 20q.
\end{align*}
The first two inequalities are lower bounds on $L$, and the last two say that $L$ cannot grow much faster than $n$. Also, assume $L$ is a power of 4 so that $q$ is an even integer. (This last assumption is not necessary but simplifies notation.)

The next four assumptions are also needed for our analysis and are valid with probability tending to 1 as $n, L \to \infty$ provided Assumption A1 holds. The mean value for $n_k$ is $n\tau_k$, so let us impose the following assumption.

**Assumption A2.** For each $k$, $n_k\tau_k/2 \leq n_k \leq 2n_k\tau_k$.

By the Chernoff-Hoeffding bound and the union bound, this assumption will fail with probability at most $t \exp\left(-2n\tau_{\text{min}}\right)$. This quantity tends to 0 as $n \to \infty$.

Next, let us provide some estimates for the Zipf distribution. Let us partition $D_k$ into subsets $C_{k,1}, \ldots, C_{k,q}$ where $C_{k,\iota}$ contains documents of $D_k$ of length $[2^{\iota-1}, 2^\iota)$ and $q = \log_2 L$, an even integer by assumption. It follows from underestimating the Zipf distribution using an integral that the probability that a document lies in $C_{k,\iota}$ is at least $(n_k/q - 2)/n_k = 1/q - 2/n_k$.

We have assumed in A2 that $n_k \geq n\tau_{\text{min}}/2$ and in A1 that $n\tau_{\text{min}}/2 \geq 10q$. The choice of lengths are independent trials, and the mean size of $C_{k,\iota}$ is at least $n_k/q - 2$. All of these bounds lead to the following.

**Assumption A3.** For each $k = 1, \ldots, t$ and $\iota = 1, \ldots, q$, $|C_{k,\iota}| \geq n_k/(2q)$.

The probability of failure of this assumption is at most $qt \exp(-n_k/(8q))$ which again tends to zero since $n_k/(8q) \geq n\tau_{\text{min}}/(16q)$ by A2 and $n\tau_{\text{min}}/(16q) \geq q^2\tau_{\text{min}}/16$ by A1, and finally, $q \to \infty$. A consequence of A3 is that the number of documents that have length at least $L^{1/2}$ (i.e., those in $C_{k,q/2+1} \cup \cdots \cup C_{k,q}$) is at least $n_k/4$. We also need an upper bound on $|C_{k,\iota}|$. The mean value of this quantity is at most $n_k/q + 1$ using an integral to overestimate the Zipf distribution. Since the documents are chosen independently, we obtain the following.

**Assumption A4.** For each $k = 1, \ldots, t$ and $\iota = 1, \ldots, q$, $|C_{k,\iota}| \leq 2n_k/q$.

The probability of failure is $qt \exp(-2n_k/q^2)$ by the Chernoff-Hoeffding bound. Using arguments similar to those in the previous paragraph, this tends to 0 as $n, L \to \infty$ under A1.

Let $j \in D_k$ index a document on topic $k$ whose length we denote as $l_j$. The mean value for $A(\cdot, j)$ is $l_j P(\cdot, k)$ by the properties of the multinomial distribution. Let us now consider the probability that any $A(i, j)$ diverges significantly from the mean, e.g., say $|A(i, j) - l_j P(i, k)| \geq l_j \theta$. Again, by the
Chernoff-Hoeffding bound, this probability is at most \( \exp(-2l_j \theta^2) \), so using a union bound, the probability that any entry will diverge by \( l_j \theta \) from its mean is at most \( m \exp(-2l_j \theta^2) \). If we further assume \( l_j \geq L^{1/2} \), this quantity is at most \( m \exp(-2L^{1/2} \theta^2) \). We have assumed in A1 that \( L \) is sufficiently large so that \( m \exp(-2L^{1/2} \theta^2) \leq \phi \), where \( \phi \) is the parameter given by (12)–(14).

We say that a column \( j \in D_k \) is acceptable if its length \( l_j \) is at least \( L^{1/2} \) and if the distance of each entry from its mean is at most \( \theta l_j \). Let \( D_{\text{acc}} \) denote the subset of \( \{1, \ldots, n\} \) of acceptable documents and \( D_{\text{unacc}} \) its complement. By the assumptions so far, the number of documents with length at least \( L^{1/2} \) in topic \( k \) is at least \( n_k/4 \). Let \( D'_k \) denote \( D_k \cap D_{\text{acc}} \), i.e., the acceptable subset of \( D_k \) and let \( C'_{k,q/2+1}, \ldots, C'_{k,q} \) denote the acceptable subsets of \( C_{k,q/2+1}, \ldots, C_{k,q} \). We now impose the last assumption.

**Assumption A5.** The acceptable subset of each \( C_{k,t} \) \((t = q/2 + 1, \ldots, q)\) has size at least \( |C_{k,t}|(1 - 2\phi) \).

By the union bound, this assumption fails with probability at most

\[
\sum_{k=1}^t \sum_{\iota=q/2+1}^{q} \exp(-2|C_{k,\iota}| \phi),
\]

which, according to prior assumptions, is at most \((qt/2) \exp(-n\phi \tau_{\min}/(2q))\). Again, from A1 and A2, this probability tends to 0 for large \( n \) and \( L \).

Let us now derive some inequalities useful for the upcoming analysis. A simple inequality is

\[
\|A(:, j)\| \leq l_j \tag{16}
\]

which follows because of the inequality \( \|x\|_2 \leq \|x\|_1 \). Another simple inequality is that if \( a, b \) are both nonnegative, then

\[
(a - b)^2 \leq a^2 + b^2. \tag{17}
\]

Let \( \mathbf{l} \in \mathbb{R}^n \) denote the vector \((l_1, \ldots, l_n)\) of document lengths. We now
establish some needed norm estimates for $l$.

$$\|l(D'_k)\|^2 = \sum_{j \in D'_k} l_j^2$$

$$= \sum_{\iota = q/2+1}^{q} \sum_{j \in C'_{k,\iota}} l_j^2$$

$$\geq \sum_{\iota = q/2+1}^{q} 2^{2\iota - 2}$$

$$= \sum_{\iota = q/2+1}^{q} 2^{2\iota - 2} |C'_{k,\iota}|$$

$$\geq \sum_{\iota = q/2+1}^{q} 2^{2\iota - 2}(1 - 2\phi)|C'_{k,\iota}|$$

$$\geq \sum_{\iota = q/2+1}^{q} 2^{2\iota - 2}(1 - 2\phi) n_k/(2q)$$

$$\geq (1 - 2\phi) n_k L^2/(8q). \quad (18)$$

Here, Assumption A5 was used for the fifth line and A3 for second. A useful upper bound is:

$$\|l(D_k)\|^2 = \sum_{j \in D_k} l_j^2$$

$$= \sum_{\iota = 1}^{q} \sum_{j \in C_{k,\iota}} l_j^2$$

$$\leq \sum_{\iota = 1}^{q} \sum_{j \in C_{k,\iota}} 2^{2\iota}$$

$$= \sum_{\iota = 1}^{q} 2^{2\iota} |C'_{k,\iota}|$$

$$\leq \sum_{\iota = 1}^{q} 2^{2\iota} \cdot 2 n_k/q$$

$$\leq 8n_k L^2/(3q). \quad (19)$$
Since $||l||^2 = ||l(D_1)||^2 + \cdots + ||l(D_t)||^2$, 

$$||l||^2 \leq 8nL^2/(3q).$$ (20)

Some final estimates concern the sum of squares of lengths of unacceptable documents. A document can be unacceptable either because its length is less than $L^{1/2}$ (i.e., it lies in $C_{k,i}$ for some $k = 1, \ldots, t$ and some $i = 1, \ldots, q/2$) or else because its term frequencies deviate too much from the mean (by more than $l_j\theta$ in some position). In the former case, all document lengths are bounded by $L^{1/2}$, hence squared document lengths are bounded by $L$. For the latter case, we can apply Assumption A5. Thus, we have the following estimate on unacceptable documents:

$$||l(D_k - D'_k)||^2 = ||l(C_{k,1} \cup \cdots \cup C_{k,q/2})||^2$$
$$+ ||l(C_{k,q/2+1} \cup \cdots \cup C_{k,q} - C'_{k,q/2+1} - \cdots - C'_{k,q})||^2$$

$$\leq n_kL + \sum_{i=q/2+1}^{q} |C_{k,i} - C'_{k,i}|^2 2^{2i}$$

$$\leq n_kL + \sum_{i=q/2+1}^{q} 2\phi |C_{k,i}|^2 2^{2i}$$

$$\leq n_kL + 16\phi n_kL^2/(3q).$$

Here, Assumption A3 was used for the fourth line. We can combine these contributions from individual topics to obtain the upper bound:

$$||l(D_{\text{unacc}})||^2 \leq nL + 16\phi nL^2/(3q).$$ (21)

These estimates can be extended to sum of squares of the entries of $A$:

$$||A(:, D_k - D'_k)||^2_F = \sum_{j \in D_k - D'_k} \sum_{i=1}^{m} A(i, j)^2$$

$$\leq \sum_{j \in D_k - D'_k} l_j^2$$

$$\leq n_kL + 16\phi n_kL^2/(3q).$$

The second line follows from (16).

Thus,

$$||A(:, D_{\text{unacc}})||^2_F \leq nL + 16\phi nL^2/(3q).$$
Recalling from Assumption A1 that \( L \geq \frac{3q}{(16\phi)} \), the second term dominates in the above four inequalities, so
\[
\|\mathcal{L}(D_k - D'_k)\|^2 \leq 32\phi n_k L^2 / (3q),
\]
\[
\|A(\cdot, D_k - D'_k)\|_F^2 \leq 32\phi n_k L^2 / (3q),
\]
\[
\|\mathcal{L}(D_{\text{unacc}})\|^2 \leq 32\phi n L^2 / (3q),
\]
\[
\|A(\cdot, D_{\text{unacc}})\|_F^2 \leq 32\phi n L^2 / (3q).
\] (22)

Because of (12),
\[
\|\mathcal{L}(D_k - D'_k)\|^2 \leq n_k L^2 / (16 \cdot 256 \cdot 25^2 \cdot \theta),
\]
\[
\|A(\cdot, D_k - D'_k)\|_F^2 \leq n_k L^2 / (16 \cdot 256 \cdot 25^2 \cdot \theta),
\]
\[
\|\mathcal{L}(D_{\text{unacc}})\|^2 \leq n L^2 / (16 \cdot 256 \cdot 25^2 \cdot \theta),
\]
\[
\|A(\cdot, D_{\text{unacc}})\|_F^2 \leq n L^2 / (16 \cdot 256 \cdot 25^2 \cdot \theta).
\] (23)

With these preliminary inequalities in hand, we may now begin the first lemma in the proof of the main theorem.

**Lemma 3.** Under Assumptions A1–A5,
\[
f^{\text{opt}} \geq n L^2 / (256 \theta),
\] (28)
where \( f^{\text{opt}} \) denotes the optimal value of (1).

**Proof.** The proof follows from estimating the value of the objective function for the choices \( M = T_k \) and \( N = D'_k \). We can estimate the first term in (1) as
\[
\|A(M, N)\|_F^2 = \sum_{j \in D'_k} \sum_{i \in T_k} A(i, j)^2
\]
\[
\geq \sum_{j \in D'_k} \sum_{i \in T_k} l_j^2 (P(i, k) - \theta)^2
\]
\[
\geq \sum_{j \in D'_k} \sum_{i \in T_k} l_j^2 P(i, k)^2 / 4
\]
\[
= \|\mathcal{L}(D'_k)\|^2 \cdot \sum_{i \in T_k} P(i, k)^2 / 4
\]
\[
\geq (1 - 2\phi) n_k L^2 \|P(T_k, k)\|^2 / (32q)
\]
\[
\geq (1 - 2\phi) n_k L^2 / (64q m)
\]
\[
\geq n_k L^2 / (128q m).
\] (29)
The second line follows by the definition of ‘acceptable.’ The third follow because \( \theta \leq P(i, k)/2 \) by (9). The fifth line relies on (18), the next on the fact that \( \|P(T_k, k)\|^2 \geq 1/(2m) \) because \( \|P(:, k)\|^2 \geq 1/m \) (which follows from \( \|P(:, k)\|_1 = 1 \) and \( \|P(\{1, \ldots, m\} - T_k, k)\|^2 \leq m^2 \leq 3/(10m) \) from (15). The last line uses \( \phi \leq 1/4 \) because of (12).

Now we turn to the second term in (1). Choose \( u, v, \sigma \) so that \( u \sigma v^T = P(T_k, k)I(N)^T \), a rank-one matrix, where as above \( N = D_k' \). Since \( |A(i, j) - l_j P(i, k)| \leq l_j \theta \) when \( j \) is acceptable, we have the following bound for the second term.

\[
\gamma \|A(M, N) - u\sigma v^T\|_F^2 = \gamma \sum_{j \in D_k'} \sum_{i \in T_k} (A(i, j) - l_j P(i, k))^2 \\
\leq \gamma \sum_{j \in D_k'} \sum_{i \in T_k} l_j^2 \theta^2 \\
= \gamma \theta^2 \|1(D_k)\|^2 \cdot |T_k| \\
\leq 8\gamma \theta^2 m n_k L^2 / (3q) \\
\leq n_k L^2 / (256qm).
\]

The fourth line follows from (19), and the last follows from (10) (taking \( \gamma = 4 \)). Thus, subtracting the above right-hand side from (29) shows that \( f^{\text{opt}} \geq n_k L^2 / (16 \cdot 256 \cdot 25^2 q m) \). This inequality is valid for all \( k = 1, \ldots, t \), so we may assume it is true for the \( k \) that maximizes \( n_k \). This value of \( n_k \) is therefore at least \( n/t \). This establishes (28). \( \square \)

For a particular topic \( k \), say that a term index \( i \in T_k \) is heavy if \( P(i, k) \geq \chi \), where \( \chi \) was defined by (7)–(8) above. Let \( H_k \subseteq T_k \) be the heavy indices. We use the notation \( \text{top}(j) \) to denote the topic of document \( j \), \( j \in \{1, \ldots, n\} \).

Say that an entry \( A(i, j) \) of \( A(M, N) \) is a heavy entry if \( i \in H_k \) where \( k = \text{top}(j) \). Finally, say that an entry \( A(i, j) \) of \( A(M, N) \) is acceptable and heavy if it is heavy and \( j \) is acceptable.

**Lemma 4.** Under Assumptions A1–A5, the sum of squares of entries of \( A \) that are not heavy but are acceptable is at most \( n L^2 / (16 \cdot 256 \cdot 25^2 q m) \).
Proof. This is a straightforward estimate:

\[
\sum_{A(i, j) \text{ not heavy \& } j \text{ acceptable}} A(i, j)^2 = \sum_{k=1}^{t} \sum_{j \in D'_k} \sum_{i \in H_k} A(i, j)^2 \\
\leq \sum_{k=1}^{t} \sum_{j \in D'_k} \sum_{i \in H_k} l_j^2 (P(i, k) + \theta)^2 \\
\leq \sum_{k=1}^{t} \sum_{j \in D'_k} \sum_{i \in H_k} l_j^2 (\chi + \theta)^2 \\
= (\chi + \theta)^2 \sum_{k=1}^{t} (m - |H_k|) \sum_{j \in D'_k} l_j^2 \\
\leq m(\chi + \theta)^2 \|1\|^2 \\
\leq 8m(\chi + \theta)^2 nL^2/(3q) \\
\leq 32m\chi^2 nL^2/(3q) \\
\leq nL^2/(16 \cdot 256 \cdot 25^2 qtm). \quad (31)
\]

The second line follows by definition of ‘acceptable.’ The third follows because \(P(i, k) < \chi\) if \(i\) is not heavy in topic \(k\). The sixth line follows from (20), the seventh because \(\theta \leq \chi\) (refer to (11)) and the last from (7).

Lemma 5. Under Assumptions A1–A5, The sum of squares of acceptable and heavy entries in \(A(M, N)\), where \(M, N\) are the optimizers of (1), is at least \(nL^2/(512qt)\).

Proof. The sum of squares of entries in \(A(M, N)\) from unacceptable documents is bounded above by the sum of squares of entries in \(A\) of unacceptable documents, for which we have the estimate given by (27). The sum of squares of entries of \(A(M, N)\) which are acceptable but not heavy is bounded above by the same quantity for all of \(A\), which is given by (31). Adding these two upper bounds gives a quantity less than half of the lower bound in (28), which proves the result.

The following lemma is stated more generally than the others of this section (i.e., without Assumptions A1–A5 and without assuming \(\gamma = 4\)) because it is more broadly applicable.
Lemma 6. Let $A$ be an $m \times n$ matrix with nonnegative entries. Let $M \subset \{1, \ldots, m\}$ be the optimizing choice of $M$ for (1). Assume $\gamma > 2$. Let $j, j'$ index two columns of $A$ such that

$$\frac{A(M, j)^T A(M, j')}{\|A(M, j)\| \cdot \|A(M, j')\|} < 1 - 2/\gamma.$$  \hfill (32)

Then at least one of $j$ or $j'$ is not a member of the optimizing choice of $N$.

Remark. The lemma is also true when the roles of $M$ and $N$ are reversed since the value of the objective function (2) is unchanged under matrix transposition.

Proof. Let unit vector $u$ be the optimizing choice for (1). As noted in Lemma 1, $j$ and $j'$ are included in the optimal $N$ provided $f_j, f_{j'} > 0$, where

$$f_j = \gamma (A(M, j)^T u)^2 - (\gamma - 1) \|A(M, j)\|^2,$$

$$f_{j'} = \gamma (A(M, j')^T u)^2 - (\gamma - 1) \|A(M, j')\|^2.$$  

Here, we have simplified notation by allowing $u$ to stand for $u(M)$. We will now show that for any possible choice of $u$, either $f_j < 0$ or $f_{j'} < 0$, meaning that at least one of $j$ or $j'$ cannot be in $N$.

To proceed, let us define normalizations $r = A(M, j)/\|A(M, j)\|$ and $s = A(M, j')/\|A(M, j')\|$. With these definition, (32) is rewritten $r^T s < 1 - 2/\gamma$. Since multiplying by a positive scalar does not affect the signs of $f_j$ or $f_{j'}$, it suffices to redefine them using the normalized vectors:

$$f_j = \gamma (r^T u)^2 - \gamma + 1$$

and

$$f_{j'} = \gamma (s^T u)^2 - \gamma + 1.$$  

Thus,

$$f_j + f_{j'} = \gamma (r^T u)^2 + \gamma (s^T u)^2 - 2\gamma + 2$$  \hfill (33)

$$= \gamma \left\| \begin{pmatrix} r^T \\ s^T \end{pmatrix} u \right\|^2 - 2\gamma + 2$$

$$\leq \gamma \left\| \begin{pmatrix} r^T \\ s^T \end{pmatrix} \right\|^2 - 2\gamma + 2$$

$$= \gamma \lambda_{\text{max}} \left( \begin{pmatrix} 1 & r^T s \\ r^T s & 1 \end{pmatrix} \right) - 2\gamma + 2. \hfill (34)$$
Lemma 7. Assume In this inequality we used the notation $\lambda_{\text{max}}$ to denote the maximum eigenvalue of a symmetric matrix. We also used the identity that for any matrix $B$, $\|B\|_2 = (\lambda_{\text{max}}(BB^T))^{1/2}$. The eigenvalues of the $2 \times 2$ matrix above can be easily determined as $1 \pm r^Ts$. Thus,

$$f_j + f_j' \leq (1 + r^Ts)\gamma - 2\gamma + 2 = (r^Ts - 1)\gamma + 2.$$ 

Since $r^Ts < 1 - 2/\gamma$, the right-hand side is negative, thus showing that either $f_j$ or $f_j'$ is negative. 

We can now apply the previous lemma to the text corpus under analysis.

**Lemma 7.** Assume A1–A5 hold. Suppose that $(i, j), (i', j')$ are two acceptable heavy entries in the optimal solution $(M, N)$. Then $(i, j)$ and $(i', j')$ must be from the same topic $k$.

**Proof.** Suppose that $(i, j)$ is an acceptable heavy entry on topic $k$, and $(i', j')$ is an acceptable heavy entry topic $k'$ such that $k' \neq k$. Suppose also that both $i, i' \in M$. We will prove that either $j$ or $j'$ is not in $N$. Let $r = A(M,j)/\|A(M,j)\|$ and $s = A(M,j')/\|A(M,j')\|$. Let us split $r$ and $s$ into three subvectors: $r_1, s_1$ contain those entries indexed $M \cap T_k$; $r_2, s_2$ contain entries indexed by $M \cap T_{k'}$; and $r_3, s_3$ contain the remaining entries of $M$. Since $j \in T_k$, $(i, j)$ is heavy, and $j$ is acceptable, this means that $i \in M \cap T_k$ and $A(i,j) \geq l_j(\chi - \theta)$, so that $\|A(M \cap T_k,j)\|^2 \geq l_j^2(\chi - \theta)^2$. Since $\theta \leq \chi/2$ (refer to (11)), this quantity is at least $l_j^2\chi^2/4$. On the other hand, $\|A(M - T_k,j)\|^2 \leq ml_j^2(\epsilon + \theta)^2$ since column $j$ is acceptable and $P(i,k) \leq \epsilon$ for $i \notin T_k$. Thus, after rescaling,

$$\|r_2; r_3\| = \|r(M - T_k)\|$$

$$= \|A(M - T_k,j)\|/\|A(M,j)\|$$

$$\leq \|A(M - T_k,j)\|/\|A(M \cap T_k,j)\|$$

$$\leq 2m^{1/2}(\epsilon + \theta)/\chi.$$ 

The inequality $\chi \geq 10m^{1/2} (\epsilon + \theta)$ follows from (11) and the fact that $\epsilon \leq \theta$ from (15). Thus, $\|r_2; r_3\| \leq 1/5$. Similarly, $\|s_1; s_3\| \leq 1/5$. Hence,

$$|r^Ts| \leq |r_1^Ts_1| + |r_2^Ts_2| + |r_3^Ts_3|$$

$$\leq \|s_1\| + \|r_2\| + \|r_3\| \cdot \|s_3\|$$

$$\leq 1/5 + 1/5 + 1/25 = 0.44.$$ 

Thus, by Lemma 6, since $0.44 < 1 - 2/\gamma$ when $\gamma = 4$, either $j$ or $j'$ is not present in the optimal $N$. 

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The next lemma shows that the left and right singular vectors of the optimal solution to (1) are determined largely by the document lengths and probability distribution for topic $k$.

**Lemma 8.** Let $(M, N)$ be the index sets that optimize (1). Let $k$ be the index of the topic of the heavy entry occurring in $(M, N)$ (which is uniquely determined according to Lemma 7). Let $r$ and $s$ be the right and left singular vectors of $A(M, N)$, respectively. Assume $A_1$–$A_5$ hold. Then there exists a positive scalar $\kappa_r$ such that

$$\frac{\|\kappa_r r - r_0\|}{\|r_0\|} \leq 1/6,$$

(35)

where $r_0$ is defined by

$$r_0(j) = \begin{cases} l_j & \text{for } j \in D_k \cap N, \\ 0 & \text{else}. \end{cases}$$

(36)

Similarly, there exists a positive scalar $\kappa_s$ such that

$$\frac{\|\kappa_s s - s_0\|}{\|s_0\|} \leq 1/6,$$

(37)

where $s_0 = P(M, k)$. In addition, $r_0$ satisfies the following inequality:

$$\|r_0\| \geq nL^2/(278qt m)$$

(38)

**Proof.** Let $B$ be a matrix with the same dimensions as $A(M, N)$, indexed the same way $A$ is indexed (i.e., $B$ and $B(M, N)$ denote the same matrix) and whose $(i, j)$ entry is $B(i, j) = P(i, k)r_0(j)$. We will also use $r_0$ and $r_0(N)$ synonymously, and similarly for $r$, $s$ and $s_0$. Observe that $B$ is a rank-one matrix since $B = P(M, k)r_0^T$. The only nonzero singular value of $B$ is $\|r_0\| \cdot \|P(M, k)\|$.

Let us partition $N$ into three sets $N_1 \cup N_2 \cup N_3$ given by

$$N_1 = N \cap D_{\text{unacc}}; \quad N_2 = N \cap (D_{\text{acc}} - D_k'); \quad N_3 = N \cap D_k'.$$

(39)

From this partition,

$$\|B - A(M, N)\|_F^2 \leq \|B - A(M, N)\|_F^2$$

$$= \|B(M, N_1) - A(M, N_1)\|_F^2 + \|B(M, N_2) - A(M, N_2)\|_F^2 + \|B(M, N_3) - A(M, N_3)\|_F^2.$$
where we now obtain upper bounds on the three terms individually.

\[
\|B(M, N_1) - A(M, N_1)\|_F^2 \leq \|B(M, N_1)\|_F^2 + \|A(M, N_1)\|_F^2 = \|P(M, k)\|^2 \|I(N_1 \cap D_k)\|^2 + \sum_{j \in N_1} \sum_{i \in M} A(i, j)^2 \leq \|I(D_{\text{unacc}})\|^2 + \|A(\cdot, D_{\text{unacc}})\|_F^2 \leq 2nL^2/(256 \cdot 3 \cdot 2 \cdot 25^2 qtm).
\]

In the above derivation, we used (17) for the first line, the relationship \(N_1 \subset D_{\text{unacc}}\) and \(\|P(\cdot, k)\| \leq 1\) for the third line, and (26) and (27) for the last.

Next, observe that \(B(M, N_2) = 0\) since \(N_2 \cap D_k = \emptyset\), hence

\[
\|B(M, N_2) - A(M, N_2)\|_F^2 = \|A(M, N_2)\|_F^2.
\]

All entries of the right-hand side are acceptable and not heavy; they are acceptable by choice of \(N_2\), and they are not heavy because Lemma 7 shows that there cannot be an acceptable heavy entry from a topic other than \(k\) in the optimal solution. Thus, from (31),

\[
\|B(M, N_2) - A(M, N_2)\|_F^2 \leq nL^2/(16 \cdot 256 \cdot 25^2 qtm).
\]

Finally, for \(j \in N_3\), \(r_0(j) = l_j\) since \(N_3 \subset D_k\). Thus,

\[
\|B(M, N_3) - A(M, N_3)\|_F^2 = \sum_{j \in N_3} \sum_{i \in M} (l_j P(i, k) - A(i, j))^2 \leq \sum_{j \in N_3} \sum_{i \in M} l_j^2 \theta^2 = |M|\theta^2 \|I(N_3)\|^2 \leq m\theta^2 \|I(D_k)\|^2 \leq 8m\theta^2 nL^2/(3q) \leq nL^2/(3 \cdot 25^2 \cdot 256 qtm)
\]

Here, the definition of ‘acceptable’ was used for the second line, and (19) was used for the fifth line and (10) for the last line.

Thus, we see that \(\|B(M, N) - A(M, N)\|^2 \leq nL^2/(25^2 \cdot 256 qtm)\). On the other hand, \(\|A(M, N)\|^2 \geq nL^2/(256 qtm)\) by (28). Thus \(\|B(M, N) - A(M, N)\|/\|A(M, N)\| \leq 1/25\). This means by the triangle inequality that \(\|B(M, N) - A(M, N)\|/\|B(M, N)\| \leq 1/24\).
Now we can apply Theorem 8.6.5 of Golub and Van Loan [9] on the perturbation of singular vectors to conclude that the normalized left singular vector of $A(M, N)$ differs from $r_0/\|r_0\|$ by at most $1/6$. (Note that in applying the theorem, we use the fact that the second singular value of $B(M, N)$ is zero since $B(M, N)$ has rank one.) Similarly, the normalized right singular vector of $A(M, N)$ differs from $s_0/\|s_0\|$ by at most $1/6$.

Finally, to establish (38), we observe that

$$\|r_0\|^2 = \|B\|^2/\|P(M, k)\|^2 \geq (24/25)^2 nL^2/(256 qtm).$$

which implies (38). The first line follows because $B$ is rank-one, and the second because $\|B\| \geq (24/25)\|A(M, N)\|$ as established above, and $\|P(:, k)\| \leq 1$ since $\|P(:, k)\| = 1$.

The following lemma is used to determine when adding a row or column to the sets $M$ or $N$ will increase the objective function (1).

**Lemma 9.** Let $u$ be a nonzero vector and $d_1, d_2$ perturbations such that $\|d_1\| \leq \|u\|/6$ and $\|d_2\| \leq \|u\|/6$. Suppose $a = \kappa_1(u + d_1)$ and $b = \kappa_2(u + d_2)$, where $\kappa_1, \kappa_2$ are positive scalars. Then

$$a^T a - \gamma \|a - \beta b\|^2 > 0$$

for $\gamma = 4$ and for at least one choice of $\beta$.

**Proof.** Let us take $\beta = \kappa_1/\kappa_2$. Then

$$a^T a - \gamma \|a - \beta b\|^2 = \kappa_1^2 (u + d_1)^T (u + d_1) - \gamma \|\kappa_1 (u + d_1) - \kappa_1 (u + d_2)\|^2$$

$$= \kappa_2^2 [u^T u + 2u^T d_1 + (1 - \gamma)d_1^T d_1 - 2\gamma d_1^T d_2 - \gamma d_2^T d_2]$$

$$= \kappa_2^2 [u^T u + 2u^T d_1 - 3d_1^T d_1 - 8d_1^T d_2 - 4d_2^T d_2]$$

$$\geq \kappa_2^2 \|u\|^2 - 2\|u\| \cdot \|d_1\| - 3\|d_1\|^2 - 8\|d_1\| \cdot \|d_2\| - 4\|d_2\|^2$$

$$> 0.$$

The point of Lemma 9 is as follows. Suppose $(M, N)$ is a putative solution for maximizing (1) and $j \notin N$. Suppose the right singular vector of $A(M, N)$
is $s$, and suppose that $A(M, j)$ and $s$ are both within relative distance of $1/6$ from another vector $P(M, k)$ after rescaling, where $k$ is the topic of column $j$. Recall that once $M$ and $u$ are fixed, the objective function of (1) becomes separable by columns, i.e., it is possible to choose the $j$th entry of $v$ considering only the contribution of column $j$ to the total objective function. The previous lemma says that there is a way to choose $v_j$ so that $(M, N \cup \{j\})$ has a higher objective function value than $(M, N)$, where we take the same $M, u, \sigma$ and extend $v$ with the particular choice of $v_j$. This means that in fact $N$ is not optimal, because it should also include $j$. The lemma can also be used on rows using the analogous argument.

Now let us apply this lemma to deduce the contents of the optimal $M$ and $N$.

**Lemma 10.** Assume A1–A5 hold. In the optimal solution, $D'_k \subset N$.

**Proof.** Let us consider a column $j \in D'_k$, that is, an acceptable column for topic $k$. Observe that $A(M, j) = l_j(P(M, k) + d_2)$, where each entry of $d_2$ has absolute value at most $\theta$ by definition of ‘acceptable.’ Now we observe that $\|d_2\|_\infty \leq \theta m^{1/2} \leq \chi/6 \leq \|P(M, k)\|/6$; the first follows because $\|d_2\|_\infty \leq \theta$, the second follows from (11), and the third follows because $M$ contains at least one heavy row of $k$.

Thus, $A(M, j) = l_j(P(M, k) + d_2)$ with $d_2 \leq \|P(M, k)\|/6$ and the left singular vector $s$ of $A(M, j)$ satisfies $s = (P(M, k) + d_1)/\kappa_s$, with $\|d_1\|_2 \leq \|P(M, k)\|/6$ by (37).

Thus, by Lemma 9, column $j \in D'_k$ increases the value of the objective function since $A(M, j)$ and the left singular value of $A(M, N)$ are both scalar multiplies of perturbations of $P(M, k)$, where the relative perturbation size is at most $1/6$. This proves that all columns of $D'_k$ will lie in $N$. \hfill \Box

Recall that $H_k$ denotes the subset of $T_k$ of heavy rows (terms) associated with topic $k$.

**Lemma 11.** Assume A1–A5 hold. In the optimal solution, $H_k \subset M$.

**Proof.** Let us consider a row $i \in H_k$. Let us write $A(i, N) = P(i, k)(r_0 + d_2)$ and try to estimate $d_2$. By definition of $H_k$, $P(i, k) \geq \chi$. We can obtain an upper bound on $d_2 = A(i, N)/P(i, k) - r_0(N)$ as follows. Use the partition
of \(N\) given by (39). Then

\[
\|d_2(N_1)\|^2 \leq \|A(i, N_1)/P(i, k)^2 - r_0(N_1)\|^2 \\
\leq (1/\chi^2)\|I(N_1)\|^2 + \|I(N_1)\|^2 \\
\leq (1 + 1/\chi^2)\|I(D_{\text{unacc}})\|^2 \\
\leq 32\phi(1 + 1/\chi^2)nL^2/(3q),
\]

using (17) for the first line, (36), (16) and \(P(i, k) \geq \chi\) for the second line, \(N_1 \subset D_{\text{unacc}}\) for the third, (22) for the fourth.

Next,

\[
\|d_2(N_2)\|^2 = \|A(i, N_2)/P(i, k)^2 - r_0(N_2)\|^2 \\
= \|A(i, N_2)\|^2/P(i, k)^2 \\
= (1/P(i, k)^2) \sum_{j \in N_2} A(i, j)^2 \\
\leq (1/P(i, k)^2) \sum_{j \in N_2} l_j^2(P(i, \text{top}(j)) + \theta)^2 \\
\leq (1/\chi^2) \sum_{j \in N_2} l_j^2(\epsilon + \theta)^2 \\
\leq ((\epsilon + \theta)/\chi)^2\|l\|^2 \\
\leq 8((\epsilon + \theta)/\chi)^2nL^2/(3q).
\]

For the second line we used the fact that \(r_0(N_2) = 0\), which follows from (36) and (39). For the fourth line we used the fact that \(j\) is acceptable. For the fifth we used \(P(i, k) \geq \chi\) and \(P(i, \text{top}(j)) \leq \epsilon\) since \(i \in H_k \subset T_k\) and \(j \notin D_k\). For the last line we used (20).

Finally,

\[
\|d_2(N_3)\|^2 = \sum_{j \in N_3} (A(i, j)/P(i, k) - l_j)^2 \\
\leq \sum_{j \in N_3} l_j^2\theta^2 \\
\leq \theta^2\|I(D_k)\|^2 \\
\leq 8\theta^2n_kL^2/(3q).
\]

The second line follows because \(N_3 \subset D_{\text{acc}}\). The last line follows from (19). Thus,

\[
\|d_2\| \leq (1 + 1/\chi^2)\frac{32\phi nL^2}{3q} + \frac{8(\epsilon + \theta)^2nL^2}{\chi^2 \cdot 3q} + \frac{8\theta nL^2}{3q}.
\]

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Now apply (13) to the first term (plus the fact \((1 + 1/\chi^2) \leq 2/\chi^2\)), (11) to the second and (10) to the third term to conclude that

\[
\|d_2\| \leq \frac{nL^2}{6 \cdot 278 \cdot qmt}.
\]

Comparing to (38), \(\|d_2\| \leq \|r_0\|/6\). Thus, \(A(i, N)\) is a perturbation of \(r_0\) of relative size at most \(1/6\). By (35), the right singular vector of \(A(M, N)\) is also a perturbation of \(r_0\) of relative size at most \(1/6\). By Lemma 9, inserting \(i\) into \(M\) can only increase the objective function value. \(\Box\)

Now finally we can prove Theorem 2.

**Proof.** Consider an entry \((i, j) \in (T_k \times D_k) \triangle (M \times N)\). We take two cases: the first case is \((i, j) \in (T_k \times D_k) - (M \times N)\). In this case, since \(H_k \subset M\) and \(D_k' \subset N\) as proved in the two preceding lemmas, it must be the case that either \(j \in D_k - D_k'\) or \(i \in T_k - H_k\), i.e., either the entry is in an unacceptable column or it is a acceptable but not heavy.

The second case is \((i, j) \in (M \times N) - (T_k \times D_k)\). Thus, either \(j\) is on a topic other than \(k\) (i.e., \(j \notin D_k\)), or it is on topic \(k\) but is not a heavy entry (i.e., \(j \in D_k\) but \(i \notin T_k\), so \(i \notin H_k\)). Thus, either \(j\) is an unacceptable column, or else \(i\) is not a heavy entry, because if by Lemma 7, \(A(M, N)\) cannot contain any acceptable and heavy entries except on topic \(k\).

Thus, we see that all entries indexed by \((T_k \times D_k) \triangle (M \times N)\) are either unacceptable or not heavy. The maximum norm of unacceptable entries is given by

\[
\|A(:, D_{unacc})\|_F^2 \leq \frac{32\phi nL^2}{(3q)} \leq \frac{\alpha nL^2}{(512qtm)},
\]

where the first line comes from (23) and the second from (14).

The maximum norm of entries that are acceptable but not heavy is

\[
\sum_{A(i, j) \text{ not heavy} \& j \text{ acceptable}} A(i, j)^2 \leq \frac{32m\chi^2 nL^2}{(3q)} \leq \frac{\alpha nL^2}{(512qtm)},
\]

where the first line comes from (30) and the second from (8). Thus, adding the two previous inequalities shows that the sum of entries indexed by the symmetric difference \((T_k \times D_k) \triangle (M \times N)\) is at most \(\alpha nL^2/(256qtm)\). This is a fraction of at most \(\alpha\) times the optimal value as shown by (28). \(\Box\)
6 Behavior of the objective function on decomposable bitmap images

We consider the behavior of objective function (1) on decomposable bitmap images. A bitmap image is one in which each pixel is either white (0) or black (1). Suppose that $A$ is an $m \times n$ matrix encoding a family of images; here $m$ is the number of pixels per image and $n$ is the number of images. Since the images are assumed to be bitmaps, every entry of $A$ is either 0 or 1.

A collection of images is decomposable if there exists a partitioning of the pixel positions $\{1, \ldots, m\}$ into $t$ subsets $T_1, \ldots, T_t$, called features, such that for every $k$, every image is either black in all of $T_k$ or is white in all of $T_k$. Clearly any collection of images is decomposable into individual pixels (i.e., $T_1 = \{1\}$, $T_2 = \{2\}$, etc.), so the interesting case is when $t \ll m$. Donoho and Stodden [7] have considered a particular kind of decomposable bitmap image database.

We now consider a simple probabilistic model of generating a database of $n$ decomposable bitmap images (i.e., an $m \times n$ matrix) and prove that the objective function (1) is able to identify a feature in the database with high probability. There are many other ways to define a model for which a similar theorem could be proved.

**Theorem 12.** Let $T_1, \ldots, T_t$, the features, be a partition of $\{1, \ldots, m\}$ with $t > 1$. Let $m_{\text{min}}, m_{\text{max}}$ denote $\min_{k=1,\ldots,t} |T_k|$, $\max_{k=1,\ldots,t} |T_k|$ respectively. Let $l$ be an integer in $1, \ldots, t/2$. Assume that each of the $n$ images in the matrix $A$ is generated independently by selecting exactly $l$ features uniformly at random out of the possible $t$. Finally, assume that $\gamma > 4m/m_{\text{min}}$. Then with probability tending to 1 as $n \to \infty$, the optimizer of (1) applied to this $A$ will select $M = T_k$ for some $k$ such that $|T_k| = m_{\text{max}}$.

**Proof.** Let $(M, N)$ be the optimizing solution of (1). Observe that, for any $k$, all the rows of $A$ indexed by $T_k$ are identical by construction. Therefore, it follow from (3) that if any row from $T_k$ lies in $M$, then all of $T_k$ must be included in $M$ since all have the same $g_i$ value. Thus, $M$ is a union of some of the $T_k$’s.

We claim that it is impossible that the optimal $N$ contains two columns $j$ and $j'$ such that bitmap $j$ contains feature $T_k$ for $T_k \subset M$ while $j'$ does not contain $T_k$. The reason is that in this case, $A(M, j)$ consists of a vector with 1’s in positions indexed by $T_k$ and while $A(M, j')$ has 0’s in these positions.
Let $m_1$ be the number of ‘1’ pixels in $j$ and $m_2$ be the number of ‘1’ pixels in $j'$. Observe $m_1 \leq m$ and $m_2 \leq m$. Let $q = |T_k|$ so that $q \geq m_{\text{min}}$. Then the number of ‘1’ pixels in common between images $j$ and $j'$ is at most $\min(m_1 - q, m_2)$. Consider the left-hand side of (32):

$$\frac{A(M, j)^T A(M, j')}{\|A(M, j)\| \cdot \|A(M, j')\|} \leq \frac{\min(m_1 - q, m_2)}{\sqrt{m_1 m_2}}$$

$$\leq \begin{cases} \frac{m_1 - q}{\sqrt{m_1 m_2}}, & \text{if } m_1 - q \leq m_2, \\ \frac{m_2}{\sqrt{m_1 m_2}}, & \text{if } m_1 - q \geq m_2 \end{cases}$$

$$\leq \begin{cases} \frac{m_1 - q}{\sqrt{m_1 (m_1 - q)}}, & \text{if } m_1 - q \leq m_2, \\ \frac{m_1 - q}{\sqrt{m_1}}, & \text{if } m_1 - q \geq m_2 \end{cases}$$

$$= \sqrt{1 - q/m_1}$$

$$\leq \sqrt{1 - m_{\text{min}}/m}$$

$$\leq 1 - m_{\text{min}}/(2m). \quad (40)$$

By assumption, $\gamma > 4m/m_{\text{min}}$, so the right-hand side of (40) is less than $1 - 2/\gamma$. Thus, by Lemma 6, not both $j$ and $j'$ can be in the optimal choice of $N$.

Thus, we conclude that all columns taking part in the optimal solution must have all 1’s (or all 0’s) in positions indexed by $M$. Ignore the columns of all 0’s since their presence does not affect the objective function value. Consider now a feature $k$ such that $|T_k| = m_{\text{max}}$. Feature $k$ is expected to occur in the fraction $l/t$ of columns of $A$. For any $\varepsilon > 0$, by choosing $n$ sufficiently large, we can assume with probability arbitrarily close to 1 that this choice occurs in the fraction at least $l/t - \varepsilon$ of the columns. Therefore,

$$f(T_k, N) \geq n(l/t - \varepsilon)m_{\text{max}}, \quad (41)$$

for any $\varepsilon > 0$ and $n$ sufficiently large, where $N$ is the set of columns containing feature $k$.

Now consider any other possible choice of $M$; suppose e.g., that $M$ has $s$ of the features. By the preceding argument, the optimal choice of $N$ that could accompany this $M$ contains only columns that use all $s$ features. This union of $s$ features is expected to occur in the fraction

$$\frac{l(l - 1) \cdots (l - s + 1)}{t(t - 1) \cdots (t - s + 1)}$$
of the columns. Thus, for any $\epsilon > 0$, for $n$ sufficiently large,

$$f(M, N) \leq n \cdot \frac{ll(l-1)\cdots(l-s+1) + \epsilon}{t(t-1)\cdots(t-s+1)} \cdot sm_{max}.$$  \hspace{1cm} (42)

(The factor $sm_{max}$ is the maximum contribution to $\|A(M, N)\|_F$ from a particular column of $N$.) Now it is a simple matter to check that for any positive integers $l, t$ such that $l \leq t/2$ and $s \leq l$,

$$\frac{ll(l-1)\cdots(l-s+1)}{t(t-1)\cdots(t-s+1)} \cdot s \leq \frac{l}{t}$$

with strict inequality for $s > 1$. Thus, by comparing (41) with (42), we conclude that as $n \to \infty$, with probability tending to 1, $f(T_k, N)$ is the optimal value of the objective function.

It should be noted that the previous theorem states that the optimal $M$ includes a single feature $k$ but says nothing about the optimal $N$. Indeed, as noted in the proof, we can take the optimal $N$ to be $\{1, \ldots, n\}$. In some situations it might be desirable for the optimal $N$ to include only those images that use feature $k$. This can be achieved by including a penalty term (6) into the objective function in which $\rho$ is chosen to be a very small positive coefficient.

7 On the complexity of maximizing $f(M, N)$

In this section, we observe that the problem of globally maximizing (2) is NP-hard at least in the case that $\gamma$ is treated as an input parameter. This observation explains why R1D settles for a heuristic maximization of (2) rather than exact maximization. First, observe that the maximum biclique (MBC) problem is NP-hard as proved by Peeters [16]. We show that the MBC problem can be transformed to an instance of (2).

Let us recall the definition of the MBC problem. The input is a bipartite graph $G$. The problem is to find an $(m,n)$-complete bipartite subgraph $K$ (sometimes called a biclique) of $G$ such that $mn$ is maximized, i.e., the number of edges of $K$ is maximized.

Suppose we are given $G$, an instance of the maximum biclique problem. Let $A$ be the left-right adjacency matrix of $G$, that is, if $G = (U, V, E)$ where
$U \cup V$ is the bipartition of the node set, then $A$ has $|U|$ rows and $|V|$ columns, and $A(i, j) = 1$ if $(i, j) \in E$ for $i \in U$ and $j \in V$, else $A(i, j) = 0$.

Consider maximizing (2) for this choice of $A$. We require the following preliminary linear-algebraic lemma.

**Lemma 13.** Let $A$ be a matrix that has either of the following as a submatrix:

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } U_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{43}$$

Then $\sigma_2(A) > 0.618$.

**Proof.** First, observe that if $U$ is a submatrix of $A$, then $\|U\|_2 \leq \|A\|_2$. This follows directly from the operator definition of the matrix 2-norm.

Next, recall the following preliminary well known fact (Theorem 2.5.3 of [9]): For any matrix $A$, $\sigma_i(A) = \min\{\|A - B\|_2 : \text{rank}(B) = i - 1\}$. This fact implies the following generalization of the result in the previous paragraph: if $U$ is a submatrix of $A$, say $U = A(M, N)$, then for any $i$, $\sigma_i(U) \leq \sigma_i(A)$. The reason is that if $\bar{B} \in \mathbb{R}^{M \times N}$ is a rank-$k$ matrix, then $B \in \mathbb{R}^{m \times n}$ defined by padding with zeros is also rank $k$, and $\|A - B\|_2 \geq \|U - \bar{B}\|_2$ by the result in the previous paragraph. Now finally the lemma is proved, since $\sigma_2(U_1) = 1$ and $\sigma_2(U_2) > 0.618$. \qed

This lemma leads to the following lemma.

**Lemma 14.** Suppose all entries of $A \in \mathbb{R}^{m \times n}$ are either 0 or 1, and suppose and at least one entry is 1. Suppose $M, N$ are the optimal solution for maximizing $f(M, N)$ given by (2). Suppose also that the parameter $\gamma$ is chosen to be $2.7mn + 1$ or larger. Then the optimal choice of $M, N$ must yield a matrix $A(M, N)$ of all 1’s, possibly augmented with some rows or columns that are entirely zeros.

**Proof.** First, note that the optimal objective function value is at least 1 since we could take $M = \{i\}$ and $N = \{j\}$ where $(i, j)$ are chosen so that $A(i, j) = 1$. In this case, $f(M, N) = 1$.

Let $(M, N)$ be a pair of index sets that is a putative optimum for (2). Suppose $A(i, j) = 0$, where $(i, j) \in M \times N$. One possibility is that either row $i$ or column $j$ is entirely made of 0’s, in which case $(M, N)$ conforms to the claim in the lemma. The other case is that $A(i, j) = 0$ and yet there is an $i' \in M$ and $j' \in N$ such that $A(i', j') = 1$. In this case, $A(M, N)$
has as a submatrix (using rows \(i', i\) and columns \(j', j\)) one of the two special matrices \(U_1\) or \(U_2\) from (43). Thus, \(\sigma_2(A(M, N)) \geq 0.618\). On the other hand, \(\sigma_1(A(M, N))^2 \leq \|A(M, N)\|_F^2 \leq mn\). Therefore, \(f(M, N) \leq mn - (\gamma - 1)(0.618^2) \leq 0\) since \((\gamma - 1)(0.618) > mn\) by choice of \(\gamma\). In particular, this means \((M, N)\) cannot be optimal. \(\square\)

If \(A(M, N)\) includes a row or column entirely of zeros, then this row or column may be dropped without affecting the value of the objective function (2). Hence it follows from the lemma that without loss of generality that the optimizer \((M, N)\) of (2) indexes a matrix of all 1’s. In that case, \(\sigma_1(A(M, N)) = \sqrt{|M| \cdot |N|}\) while \(\sigma_2(A(M, N)) = \cdots = \sigma_p(A(M, N)) = 0\) (where \(p = \min(|M|, |N|)\)), and hence \(f(M, N) = |M| \cdot |N|\). Thus, the value of the objective function corresponds exactly to the number of edges in the biclique. This completes the proof that biclique is reducible in polynomial time to maximizing (2).

We note that Gillis [8] also uses the result of Peeters for a similar purpose, namely, to show that the subproblem arising in his NMF algorithm is also NP-hard.

The NP-hardness result in this section requires that \(\gamma\) be an input parameter. We conjecture that (2) is NP-hard even when \(\gamma\) is fixed (say \(\gamma = 4\) as used herein).

8 Image database test cases

9 Text database test cases

10 Conclusions

We have proposed an algorithm called R1D for nonnegative matrix factorization. It is based on greedy rank-one downdating according to an objective function, which is heuristically maximized. We have shown that the objective function is well suited for identifying topics in the \(\epsilon\)-separable text model and on a model of decomposable bitmap images. Finally, we have shown that the algorithm performs well in practice.

This work raises several interesting open questions. First, the \(\epsilon\)-separable text model seems rather too simple to describe real text, so it would be interesting to see if the results generalize to more realistic models.
One straightforward generalization is to consider power-law models for document lengths that generalize the Zipf law: suppose that the probability of length $l$ occurring is proportional to $l^{-p}$ for some $p$. Our proof of Theorem 2 generalizes to cover the case $0 \leq p < 1$ without too much difficulty. However, proving the theorem in the case $p > 1$ appears to be much more difficult (and perhaps the theorem is not true in this case). When $p > 1$, short documents dominate the corpus, and short documents are not easily analyzed using Chernoff-Hoeffding bounds.

A second question is to generalize the model of image databases for which a theorem can be established.

A third question asks whether a result like Theorem 2 will hold for the R1D algorithm using our proposed definition of the heuristic subroutine ApproxRankOneSubmatrix. When ApproxRankOneSubmatrix is applied to an $\epsilon$-separable corpus, does it successfully identify a topic? Here is an example of a difficulty. Suppose $n \to \infty$ much faster than $L$. In this case, the document $j$ with the highest norm will be the one in which $l_j$ is very close to $L$ and in which one entry $A(i, j)$ is very close to $L$ while the rest are mostly zeros. This is because the maximizer of $\|x\|_2$ subject to the constraint that $\|x\|_1 = C$ occurs when one entry of $x$ is equal to $C$ and the rest are zero. It is likely that at least one instance of such a document will occur regardless of the matrix $P(\cdot, k)$ if $n$ is sufficiently large. This document will then act as the seed for expanding $M$ and $N$, but it may not be similar to any topic.

The scenario described in the preceding paragraph can apparently be prevented by requiring $n$ and $L$ to grow proportionately, but the analysis appears to be complicated in this case. If we assume that the initial column $j$ selected by R1D is well approximated by $l_j P(\cdot, k)$ for some $k$ (i.e., the column is ‘acceptable’ in the terminology of Theorem 2), then the rest of $M$ and $N$ is likely to be the topics and terms associated with topic $k$. This is because Lemma 7 indicates that it is unlikely that a column or row associated to another topic can improve the objective function (1), whereas Lemmas 10 and 11 indicate that a document or term associated with topic $k$ will be favored by (1).

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