EXTENSION MAPS IN BEURLING ULTRAHOLOMORPHIC CLASSES

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ABSTRACT. In [9] Thilliez proved a Borel-Ritt type theorem for Beurling ultraholomorphic classes defined via weight sequences. Under the same assumptions as in Thilliez’s result, we show in the present article the existence of a continuous linear extension map (i.e. a continuous linear right inverse for the asymptotic Borel map) in these classes. This improves a previous result by Schmets and Valdivia [8].

1. Introduction

This article deals with the Borel-Ritt theorem in the setting of ultraholomorphic classes of functions defined in a sector of the Riemann surface $\Sigma$ of the logarithm. This problem has attracted much attention since the 1980’s and plays an important role in the study of formal power series solutions to various kind of equations [1]; see [3] for an overview of known results and a list of references.

Given a sequence $M = (M_p)_{p \in \mathbb{N}}$ of positive numbers and a sector $S_\gamma = \{ z \in \Sigma \mid |\text{Arg } z| < \frac{\gamma \pi}{2} \}$, $\gamma > 0$, we consider the space of ultraholomorphic functions of class $(M)$ (of Beurling type) in $S_\gamma$

$$A(M)(S_\gamma) := \{ f \in \mathcal{O}(S_\gamma) \mid \sup_{p \in \mathbb{N}} \sup_{z \in S_\gamma} \frac{n^p |f^{(p)}(z)|}{p! M_p} < \infty, \quad \forall n \in \mathbb{N} \}$$

and the associated sequence space

$$\Lambda(M)(\mathbb{N}) := \{ (c_p)_{p \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} \mid \sup_{p \in \mathbb{N}} \frac{n^p |c_p|}{p! M_p} < \infty, \quad \forall n \in \mathbb{N} \}.$$

We endow these spaces with their natural Fréchet space structure. The asymptotic Borel map is defined as

$$(1.1) \quad B : A(M)(S_\gamma) \to \Lambda(M)(\mathbb{N}), \quad f \mapsto (f^{(p)}(0))_{p \in \mathbb{N}},$$

where $f^{(p)}(0) := \lim_{z \to 0^+} f^{(p)}(z)$. Obviously, this map is well-defined and continuous.

In this context the Borel-Ritt problem consists in finding conditions relating $\gamma$ (which determines the aperture of the sector $S_\gamma$) and the sequence $M$ that ensure that the map $(1.1)$ is surjective. In [9 Corollary 3.4.1] Thilliez showed that, under some standard conditions on $M$, the map $(1.1)$ is surjective if $\gamma < \gamma(M)$, where $\gamma(M)$ denotes the

2010 Mathematics Subject Classification. Primary. 30E05, 30D60, 46M18, 46A63 .

Key words and phrases. Beurling ultraholomorphic classes; Borel-Ritt theorem; continuous linear right inverses; the linear topological invariant ($\Omega$).

A. Debrouwere was supported by FWO-Vlaanderen through the postdoctoral grant 12T0519N.
growth index of $M$; see Section 2 for the definition of $\gamma(M)$. The condition $\gamma < \gamma(M)$ seems to be near to optimal for the map (1.1) to be surjective (cf. [3]). Thilliez’s result improves a previous one by Schmets and Valdivia [8, Theorem 4.5], who proved the surjectivity of the map (1.1) under much more restricted conditions on $\gamma$ and $M$. However, in contrast to the technique of Thilliez, their proof method also provides the existence of a continuous linear extension map (i.e. a continuous linear right inverse for the map (1.1)). In this article we show that the map (1.1) has a continuous linear right inverse if one merely assumes that $\gamma < \gamma(M)$. This result seems to be new even for the Gevrey sequences.

Since, by the aforementioned result of Thilliez, we already know that the map (1.1) is surjective, we may apply results from the splitting theory of Fréchet spaces [10] to show that it has a continuous linear right inverse. By doing so and using a simple ramification argument (see Section 4), we reduce the problem to showing that a class of weighted Fréchet spaces of holomorphic functions on the right half-plane satisfies the linear topological invariant ($\Omega$) [7] (see Section 3). To this end, we use a similar technique as in [5], where it is shown that a class of weighted ($LB$)-spaces of holomorphic germs near $\mathbb{R}$ satisfies an ($\Omega$)-type condition (more precisely, that their strong duals satisfy ($DN$)).

We are much indebted to the work of Langenbruch for his construction of holomorphic cut-off functions [5, Lemma 2.3], which are essential for the present article.

2. BEURLING ULTRAHOLONOMORPHIC CLASSES AND THE ASYMPTOTIC BOREL MAP

A sequence $M = (M_p)_{p \in \mathbb{N}}$ of positive numbers is called a weight sequence if $M_0 = M_1 = 1$ and $\lim_{p \to \infty} M_p^{1/p} = \infty$. We set $m_p = M_p/M_{p-1}$ for $p \in \mathbb{Z}_+$. For $s > 0$ we define $M^s = (M^s_p)_{p \in \mathbb{N}}$. We consider the following conditions on a weight sequence $M$:

(M.1) ($\log$-convexity) $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{Z}_+$.

(M.2) ($\text{moderate growth}$) $M_{p+q} \leq CL^{p+q}M_pM_q$, $p,q \in \mathbb{N}$, for some $C, L > 0$.

(M.3) ($\text{strong non-quasianalyticity}$) $\sum_{q=p+1}^{\infty} \frac{1}{m_q} \leq \frac{Cp}{m_{p+1}}$, $p \in \mathbb{Z}_+$, for some $C > 0$.

We refer to [4] for the meaning of these conditions. Following Thilliez [9], we call a weight sequence $M$ strongly regular if it satisfies (M.1) and (M.2), and $(p!M_p)_{p \in \mathbb{N}}$ satisfies (M.3). The most important examples of strongly regular weight sequences are the Gevrey sequences $p^{\alpha}$, $\alpha > 0$.

Let $M$ be a weight sequence and let $\gamma > 0$. We say that $M$ satisfies property $(P_\gamma)$ if there exists a sequence $m' = (m'_p)_{p \in \mathbb{Z}_+}$ of positive numbers such that $(m'_p/p^\gamma)_{p \in \mathbb{Z}_+}$ is non-decreasing and there is $C > 0$ such that $C^{-1}m_p \leq m'_p \leq Cm_p$ for all $p \in \mathbb{Z}_+$. We define the growth index $\gamma(M)$ of $M$ as [9, Definition 1.3.5]

$$\gamma(M) := \sup\{\gamma > 0 \mid M \text{ satisfies } (P_\gamma)\}.$$  

If $M$ is strongly regular, then $0 < \gamma(M) < \infty$ [9, Lemma 1.3.2]. We shall need the following characterization of $\gamma(M)$ in terms of the condition (M.3):

**Lemma 2.1.** [2, Corollary 3.12(iii)] Let $M$ be a weight sequence satisfying (M.1) and let $\gamma > 0$. Then, $\gamma < \gamma(M)$ if and only if $M^{1/\gamma}$ satisfies (M.3).
The associated function of a weight sequence $M$ is defined as

$$\omega_M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p}{M_p}, \quad t \geq 0.$$  

Suppose that $M$ satisfies (M.1). Condition (M.2) implies that [4, Proposition 3.6]

$$(2.1) \quad 2 \omega_M(t) \leq \omega_M(Lt) + \log C, \quad t \geq 0,$$

for some $C, L > 0$, whereas $\gamma(M) > 0$ implies that [2, Corollary 2.14 and Corollary 4.6]

$$(2.2) \quad \omega_M(2t) \leq L \omega_M(t) + \log C, \quad t \geq 0,$$

for some $C, L > 0$.

We denote by $\Sigma$ the Riemann surface of the logarithm. Given an open subset $\Omega$ of $\Sigma$ (or $\mathbb{C}$), we define $\mathcal{O}(\Omega)$ as the space of holomorphic functions in $\Omega$. For $\gamma > 0$ we consider the sectors

$$S_\gamma := \{ z \in \Sigma \mid |\operatorname{Arg} z| < \frac{\gamma \pi}{2} \},$$

where $\operatorname{Arg}$ denotes the principal value of the argument.

Let $M$ be a weight sequence and let $\gamma > 0$. We define the space $\mathcal{A}(M)(S_\gamma)$ of ultraholomorphic functions of class $(M)$ (of Beurling type) in $S_\gamma$ as the Fréchet space consisting of all $f \in \mathcal{O}(S_\gamma)$ such that

$$\sup_{p \in \mathbb{N}} \sup_{z \in S_\gamma} \frac{n^p |f^{(p)}(z)|}{p! M_p} < \infty, \quad \forall n \in \mathbb{N}.$$  

Let $f \in \mathcal{A}(M)(S_\gamma)$. Then, the limits

$$f^{(p)}(0) := \lim_{z \rightarrow 0^+} f^{(p)}(z) \in \mathbb{C}, \quad p \in \mathbb{N},$$

exist.

We define $\Lambda_{(M)}(\mathbb{N})$ as the Fréchet space consisting of all $(c_p)_{p \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that

$$\sup_{p \in \mathbb{N}} \frac{n^p |c_p|}{p! M_p} < \infty, \quad \forall n \in \mathbb{N}.$$  

The asymptotic Borel map is defined as

$$B : \mathcal{A}(M)(S_\gamma) \rightarrow f \mapsto (f^{(p)}(0))_{p \in \mathbb{N}}.$$  

Obviously, this map is well-defined and continuous. Thilliez [9] showed the following important result about the surjectivity of the asymptotic Borel map.

**Theorem 2.2.** [9, Corollary 3.4.1] Let $M$ be a strongly regular weight sequence and let $0 < \gamma < \gamma(M)$. Then, $B : \mathcal{A}(M)(S_\gamma) \rightarrow \Lambda_{(M)}(\mathbb{N})$ is surjective.

The main result of this article asserts that, under the same assumptions as in Theorem 2.2, $B : \mathcal{A}(M)(S_\gamma) \rightarrow \Lambda_{(M)}(\mathbb{N})$ has a continuous linear right inverse:

**Theorem 2.3.** Let $M$ be a strongly regular weight sequence and let $0 < \gamma < \gamma(M)$. Then, $B : \mathcal{A}(M)(S_\gamma) \rightarrow \Lambda_{(M)}(\mathbb{N})$ has a continuous linear right inverse.

The proof of Theorem 2.3 is given in Section 4.
3. Weighted spaces of holomorphic functions on the right half-plane

A function \( \omega : [0, \infty) \to [0, \infty) \) is called a weight function if it is continuous and non-decreasing, \( \omega(0) = 0 \) and \( \lim_{t \to \infty} \omega(t) = \infty \). A weight function \( \omega \) is called non-quasianalytic if
\[
\int_{1}^{\infty} \frac{\omega(t)}{t^2} dt < \infty
\]
and strongly non-quasianalytic if there is \( C > 0 \) such that
\[
\int_{t}^{\infty} \frac{\omega(s)}{s^2} ds \leq \frac{C \omega(t)}{t} \quad t \geq 1.
\]

Remark 3.1. Let \( M \) be a weight sequence satisfying (M.1). Then, \( \omega_M \) is a weight function. Moreover, if \( M \) is strongly non-quasianalytic (i.e. \( M \) satisfies (M.3)), then so is \( \omega_M \) [4, Proposition 4.4].

Let \( \omega \) be a weight function. We denote by \( \mathcal{O}(\omega)(S_1) \) the Fréchet space consisting of all \( f \in \mathcal{O}(S_1) \) such that
\[
\|f\|_{\omega,n} := \sup_{z \in S_1} |f(z)| e^{n\omega(|z|)} < \infty, \quad \forall n \in \mathbb{N}.
\]
The space \( \mathcal{O}(\omega)(S_1) \) is non-trivial if and only if \( \omega \) is non-quasianalytic [6, Section 2.1].

A Fréchet space \( E \) with a fundamental system \( \{U_n\}_{n \in \mathbb{N}} \) of neighborhoods of zero is said to satisfy property (\( \Omega \)) if
\[
\forall n \in \mathbb{N} \exists m \in \mathbb{N} \forall k \in \mathbb{N} \exists C,L > 0 : U_m \subset e^{-r} U_n + Ce^{Lr} U_k.
\]
We refer to the book [7] for more information on the property (\( \Omega \)).

The goal of this section is to show the following result; it may be interpreted as a decomposition theorem (with respect to decay) for holomorphic functions in \( S_1 \).

Theorem 3.2. Let \( \omega \) be a strongly non-quasianalytic weight function. Then, \( \mathcal{O}(\omega)(S_1) \) satisfies (\( \Omega \)).

The proof of Theorem 3.2 is inspired by [5, Section 2]. The key point is the construction of suitable holomorphic cut-off functions in \( S_1 \):

Lemma 3.3. Let \( 0 < \lambda < 1 \). There exist \( \psi_r = \psi_r^\lambda \in \mathcal{O}(S_1) \), \( r > 0 \), satisfying the following property: There are \( B,C,L > 0 \) such that for all \( r > 0 \)
\[
\begin{align*}
(3.1) & \quad |\psi_r(z)| \leq C e^{Lr}, \quad z \in S_1, \\
(3.2) & \quad |\psi_r(z)| \leq C e^{-r|z|^\lambda}, \quad z \in S_1, |z| \geq B, \\
(3.3) & \quad |1 - \psi_r(z)| \leq C e^{-r/|z|^\lambda}, \quad z \in S_1, |z| \leq 1/B.
\end{align*}
\]

We shall prove Lemma 3.3 by first analyzing Langenbruch’s construction of holomorphic cut-off functions on horizontal strips [5, Lemma 2.3] and then using a conformal map. For \( t > 0 \) we set \( V_t = \{z \in \mathbb{C} | \|\operatorname{Im} z\| < t\} \). For \( r > 0 \) we define [3 p. 226]
\[
H_r(z) := \frac{1}{D_r} \int_{\gamma_z} \cosh \xi e^{-r \cosh \xi} d\xi, \quad z \in V_{\pi/2},
\]
where \( D_r = \int_{-\infty}^{\infty} \cosh \xi e^{-r \cosh \xi} d\xi \) and \( \gamma_z \) is a path in \( V_{\pi/2} \) from \( -\infty \) to \( z \).
**Lemma 3.4.** The functions $H_r, \ r > 0,$ belong to $\mathcal{O}(V_{\pi/2})$ and satisfy the following property: For each $t \in (0, \pi/2)$ there are $B, C, c, L > 0$ such that for all $r > 0$

(3.4) \[ |H_r(z)| \leq C e^{Lr}, \quad z \in V_t, \]

(3.5) \[ |H_r(z)| \leq C e^{-c r \cos \xi}, \quad z \in V_t, \ Re \ z \leq -B, \]

(3.6) \[ |1 - H_r(z)| \leq C e^{-c r \cos \xi}, \quad z \in V_t, \ Re \ z \geq B. \]

**Proof.** This follows from an inspection of the proof of [5, Lemma 2.3] (and replacing $V_1$ by $V_{\pi/2}$ there) but we repeat the argument here for the sake of completeness. Since

(3.7) \[ \cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y, \quad x, y \in \mathbb{R}, \]

we have that

\[ |e^{\cosh(x+iy)}| = e^{\cosh x \cos y}, \quad x, y \in \mathbb{R}, \]

and

\[ |\cosh(x + iy)|^2 = \cosh^2 x - \sin^2 y, \quad x, y \in \mathbb{R}. \]

The latter equality yields that

\[ |\cosh(x + iy)| \leq |\cosh x|, \quad x, y \in \mathbb{R}. \]

Hence, the integral defining $H_r$ is convergent on $V_{\pi/2}$. Consequently, $H_r$ is well-defined (by Cauchy’s integral theorem) and holomorphic on $V_{\pi/2}$. Note that

\[ D_r = 2 \int_0^\infty \cosh \xi e^{-r \cosh \xi} d\xi \geq 2 \int_0^\infty \sinh \xi e^{-r \cosh \xi} d\xi = \frac{2}{re^r}. \]

Fix $t \in (0, \pi/2)$. We first prove (3.4). For $z = x + iy \in V_t$ we have that

\[
|H_r(z)| = \frac{1}{D_r} \left| \int_{-\infty}^x \cosh(\xi + iy)e^{-r \cosh(\xi + iy)} d\xi \right| \leq \frac{1}{D_r} \int_{-\infty}^\infty \cosh \xi e^{-r \cosh \xi \cos t} d\xi
\]

\[
\leq \frac{2}{D_r} \int_0^1 \cosh \xi e^{-r \cosh \xi \cos t} d\xi + \frac{4}{D_r} \int_1^\infty \sinh \xi e^{-r \cosh \xi \cos t} d\xi
\]

\[
\leq 2 \left( 1 + \frac{2e^{-r \cosh 1}}{D_r r \cos t} \right) e^{r \cosh(1-\cos t)} \leq 2 \left( 1 + \frac{1}{\cos t} \right) e^{r \cosh(1-\cos t)}.
\]

This shows (3.4) for suitable $C$ and $L$. Next, we prove (3.5). For $z = x + iy \in V_t$ with $x < -1$ we have that

\[
|H_r(z)| = \frac{1}{D_r} \left| \int_{-\infty}^x \cosh(\xi + iy)e^{-r \cosh(\xi + iy)} d\xi \right| = \frac{1}{D_r} \left| \int_{|x|}^\infty \cosh(\xi - iy)e^{-r \cosh(\xi - iy)} d\xi \right|
\]

\[
\leq \frac{1}{D_r} \int_{|x|}^\infty \cosh \xi e^{-r \cosh \xi \cos t} d\xi \leq \frac{2}{D_r} \int_{|x|}^\infty \sinh \xi e^{-r \cosh \xi \cos t} d\xi
\]

\[
= \frac{2}{D_r \cos t} e^{-r \cosh(|x|) \cos t} \leq \frac{1}{\cos t} e^{-r (\cosh(|x|) \cos t - 1)}.
\]

Since

\[ \cosh(|x|) \cos t - 1 \geq \frac{e^{|x| \cos t}}{2} - 1 \geq \frac{e^{|x| \cos t}}{4}, \quad |x| \geq \log \left( \frac{4}{\cos t} \right), \]
we obtain that
\[ |H_r(z)| \leq \frac{1}{\cos t} e^{-(\cos t/4)r\text{e}^{|z|}} , \quad z = x + iy \in V_t, -x < -\log \left( \frac{4}{\cos t} \right). \]

This shows (3.5) for suitable $B$ and $C$. Finally, we prove (3.6). By Cauchy’s integral formula we have that for $z = x + iy \in V_{\pi/2}$
\[ 1 - H_r(z) = \frac{1}{D_r} \int_{-\infty}^{\infty} \cosh \xi e^{-r \cosh \xi} d\xi - \frac{1}{D_r} \int_{-\infty}^{x} \cosh(\xi + iy) e^{-r \cosh(\xi + iy)} d\xi \]
\[ = \frac{1}{D_r} \int_{x}^{\infty} \cosh(\xi + iy) e^{-r \cosh(\xi + iy)} d\xi \]
\[ = \frac{1}{D_r} \int_{-\infty}^{-x} \cosh(\xi - iy) e^{-r \cosh(\xi - iy)} d\xi = H_r(-z). \]

Hence, (3.6) follows from (3.5). \qed

Proof of Lemma 3.3 Let $c$ be the constant occurring in Lemma 3.4 with $t = \lambda \pi/2$. For $r > 0$ we set $\psi_r(z) = H_{e/(\lambda \log z)} z \in S_1$, where $\log$ denotes the principal value of the logarithm. Lemma 3.4 implies that the functions $\psi_r$, $r > 0$, satisfy all requirements. \qed

Proof of Theorem 3.2 It suffices to show that for all $n, k \in \mathbb{N}$ there are $D, D', Q > 0$ such that for each $f \in \mathcal{O}^{(\omega)}(S_1)$ with $\|f\|_{\omega, n+1} \leq 1$ there exist $g_r \in \mathcal{O}^{(\omega)}(S_1)$, $r > 0$, such that for all $r > 0$
\[ f - g_r \|_{\omega, n} \leq De^{-r}, \]
\[ \|g_r\|_{\omega, k} \leq D'e^{Qr}. \]

Since $\omega$ is strongly non-quasianalytic, \[2\] Corollary 2.13 implies that there is $0 < \lambda' < 1$ and $C', L' > 0$ such that
\[ \omega(st) \leq L's^{\lambda'}\omega(t) + \log C', \quad s \geq 1, t \geq 0. \]

Fix $\lambda' < \lambda < 1$ and consider the functions $\psi_r = \psi_r^{\lambda}$, $r > 0$, from Lemma 3.3. Throughout this proof the constants $B, C, L$ refer to those occurring in (3.1)-(3.3). We may assume that $B \geq 1$. For $r > 0$ we choose $A_r > 0$ such that $\omega(A_r/B) = (L+1)r$. We define $g_r(z) = \psi_r(z/A_r)f(z)$, $z \in S_1$, for $r > 0$. Property (3.1) yields that $g_r \in \mathcal{O}^{(\omega)}(S_1)$ for all $r > 0$. Next, we show (3.8). Property (3.3) implies that for $z \in S_1$ with $|z| \leq A_r/B$
\[ |f(z) - g_r(z)| \leq |1 - \psi_r(z/A_r)| \leq Ce^{-r(A_r/|z|)^\lambda} \leq Ce^{-B^r} \leq Ce^{-r}. \]

On the other hand, by (3.1), we have that for all $z \in S_1$ with $|z| \geq A_r/B$
\[ |f(z) - g_r(z)| \leq |1 - \psi_r(z/A_r)|e^{-(A_r/B)^{\omega(|z|)}} \leq (1 + C)e^{Lr - \omega(A_r/B)} = (1 + C)e^{-r}. \]

This shows that (3.8) holds for suitable $D$. Finally, we show (3.9). Properties (3.1) and (3.10) imply that for $z \in S_1$ with $|z| \leq BA_r$
\[ |g_r(z)| \leq |\psi_r(z/A_r)|e^{k\omega(BA_r)} \leq CC'e^{Lr + kl^2B2^{\lambda'}\omega(A_r/B)} \leq CC'e^{Lr + kl(L+1)L'2^{\lambda'}}. \]
Set $\varepsilon = \lambda - \lambda' > 0$. By (3.2) and (3.10) we have that for $z \in S_1$ with $|z| \geq BA_r$
\[
|g_r(z)|e^{K_0(|z|)} \leq C' \left| \psi_r(z/A_r) \right| e^{k L' |A_r| A_r} \left| \omega(A_r/B) \right|
\leq C C' e^{-r(|z|/A_r)^{\gamma}+k L'(L+1) B^{\lambda'}(|z|/A_r)^{\lambda'} r}
= C C' e^{k L'(L+1) B^{\lambda'}(|z|/A_r)^{\gamma} - r(|z|/A_r)^{\lambda'}}.
\]
Set $\tilde{L} = k L'(L + 1) B^{\lambda'}$. For $z \in S_1$ with $|z| \geq \tilde{L}^{1/\varepsilon} A_r$ it holds that
\[
|g_r(z)|e^{K_0(|z|)} \leq C C'',$
while for $z \in S_1$ with $BA_r \leq |z| \leq \tilde{L}^{1/\varepsilon} A_r$ we have that
\[
|g_r(z)|e^{K_0(|z|)} \leq C C' e^{(\tilde{L}/B^{\varepsilon}-1) \tilde{L}^{\lambda'/r}}.
\]
This shows that (3.9) holds for suitable $D$ and $Q$.

4. The main result

This section is devoted to the proof of Theorem 2.3. Let $\alpha = (\alpha_j)_{j \in \mathbb{N}}$ be a sequence of positive numbers such that $\alpha_n \not\to \infty$ as $n \to \infty$. We define the power series space $\Lambda_\infty(\alpha)$ of infinite type as the Fréchet space consisting of all $(c_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that
\[
\sum_{j=0}^{\infty} |c_j| e^{\alpha_j} < \infty, \quad \forall n \in \mathbb{N}.
\]
The space $\Lambda_\infty(\alpha)$ is called stable if $\sup_{j \in \mathbb{N}} \alpha_{j+1}/\alpha_j < \infty$. We will use the following version of the splitting theorem for Fréchet spaces of Vogt and Wagner in the proof of Theorem 2.3.

**Theorem 4.1.** [10, Theorem 4.1] Let $E$ be a Fréchet space and let $\Lambda_\infty(\alpha)$ be a stable power series space of infinite type. Let $T : E \to \Lambda_\infty(\alpha)$ be a surjective continuous linear mapping. If $\ker T$ satisfies $(\Omega)$, then $T$ has a continuous linear right inverse.

Next, we introduce an auxiliary space of ultraholomorphic functions (cf. [3]). Let $M$ be a weight function and let $\gamma > 0$. A function $f \in \mathcal{O}(S_\gamma)$ admits the formal power series $\sum_{p=0}^{\infty} a_p z^p$, $a_p \in \mathbb{C}$, as its uniform $(M)$-asymptotic expansion in $S_\gamma$ if for all $h > 0$ there is $C > 0$ such that
\[
|f(z) - \sum_{q=0}^{n-1} a_q z^q| \leq Ch^p M_p |z|^p, \quad z \in S_\gamma.
\]
In such a case, basic results about asymptotic expansions [11, Section 4.4] imply that for all $0 < \gamma' < \gamma$ the limits
\[
f^{(p)}(0) = \lim_{z \to 0^+} f^{(p)}(z) \in \mathbb{C}, \quad p \in \mathbb{N},
\]
and...
exist and that \( a_p = f(p)(0)/p! \) for \( p \in \mathbb{N} \). In particular, the coefficients \( a_p \) are unique. We define \( \mathfrak{A}(M)(S_\gamma) \) as the space consisting of all \( f \in \mathcal{O}(S_\gamma) \) that admit a uniform \((M)\) asymptotic expansion in \( S_\gamma \) and endow it with the Fréchet space structure generated by the norms

\[
\|f\|_{M,n} := \sup_{p \in \mathbb{N}} \sup_{z \in S_\gamma} \left| \frac{n^p}{p!} \left( f(z) - \sum_{q=0}^{p-1} \frac{f^{(q)}(0)}{q!} z^q \right) \right| < \infty, \quad n \in \mathbb{N}.
\]

Let \( 0 < \gamma' < \gamma \). Taylor’s theorem and Cauchy’s integral formula for derivatives imply that the following continuous inclusions hold

\[
(4.1) \quad \mathfrak{A}(M)(S_\gamma) \subset \mathfrak{A}(M)(S_\gamma) \subset \mathfrak{A}(M)(S_{\gamma'}).
\]

Consequently, the asymptotic Borel map \( B : \mathfrak{A}(M)(S_\gamma) \to \Lambda(M)(\mathbb{N}) \) is well-defined and continuous.

**Proof of Theorem 2.3** By the second inclusion in (4.1) it is enough to show that \( B : \mathfrak{A}(M)(S_\gamma) \to \Lambda(M)(\mathbb{N}) \) has a continuous linear right inverse for each \( 0 < \gamma < \gamma(M) \). In view of the first inclusion in (4.1), Theorem 2.2 yields that \( \mathfrak{A}(M)(S_\gamma) \to \Lambda(M)(\mathbb{N}) \) is surjective. Set \( \mathfrak{A}(M,0)(S_\gamma) = \ker (B : \mathfrak{A}(M)(S_\gamma) \to \Lambda(M)(\mathbb{N})) \). By Theorem 4.1 and the fact that \( \Lambda(M)(\mathbb{N}) \cong \Lambda_\infty(n) \) it suffices to show that \( \mathfrak{A}(M,0)(S_\gamma) \) satisfies (\( \Omega \)). Note that

\[
\mathfrak{A}(M,0)(S_\gamma) = \{ f \in \mathcal{O}(S_\gamma) \mid \|f\|_{M,n} = \sup_{z \in S_\gamma} |f(z)| e^{\gamma M(n/|z|)} < \infty, \quad \forall n \in \mathbb{N} \}.
\]

Properties (2.1) and (2.2) imply that \( f \in \mathcal{O}(S_\gamma) \) belongs to \( \mathfrak{A}(M,0)(S_\gamma) \) if and only if

\[
|f|_{M,n} := \sup_{z \in S_\gamma} |f(z)| e^{\gamma M(1/|z|)} < \infty, \quad \forall n \in \mathbb{N},
\]

and that \( \{ | \cdot |_n \}_{n \in \mathbb{N}} \) is a fundamental system of continuous norms for \( \mathfrak{A}(M,0)(S_\gamma) \). Note that \( \gamma \omega_{M/\gamma}(t) = \omega_M(t') \) for \( t \geq 0 \). Hence,

\[
\mathfrak{A}(M,0)(S_\gamma) \to \mathcal{O}(\omega_{M/\gamma})(S_1), \quad f \mapsto (z \mapsto f(1/z^\gamma))
\]

is a topological isomorphism. In view of Lemma 2.1 and Remark 3.1 \( \mathfrak{A}(M,0)(S_\gamma) \cong \mathcal{O}(\omega_{M/\gamma})(S_1) \) satisfies (\( \Omega \)) by Theorem 3.2.

\[\square\]

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