Generating weighted Hurwitz numbers

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Abstract

Multicurrent correlators associated to KP $\tau$-functions of hypergeometric type are used as generating functions for weighted Hurwitz numbers. These are expressed as formal Taylor series and used to compute generic, simple, rational and quantum weighted single Hurwitz numbers.

1 Introduction

It is well-known that KP $\tau$-functions of hypergeometric type serve as generating functions for weighted Hurwitz numbers \cite{28, 29, 30, 5, 14, 15, 20, 16}. An efficient way of computing the latter is to make use of the associated multicurrent correlators \cite{3, 4}. This method is implemented in the following, without recourse to topological recursion \cite{7, 9, 2, 4, 24} or matrix integral representations \cite{1}.

Section 2 recalls the definition of weighted Hurwitz numbers \cite{14, 15, 16, 20} for degree $N$ branched covers of the Riemann sphere corresponding to a weight generating function $G(z)$ and the KP $\tau$-functions that serve as generating functions for these. We also introduce the associated pair correlator $K^G(x, y)$ and express the multicurrent correlators $W^G_n(x_1, \ldots, x)$ in terms of these.

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In Section 3, using the multicurrent correlators as generating functions, the weighted single Hurwitz numbers are computed explicitly for ramification profile $\mu$ of the unweighted branch point of weight $\ell(\mu) =: n$ equal to 1, 2 or 3. This is first done for the generic case, with arbitrary weight generating function $G(z)$, in terms of its Taylor expansion coefficients and then for various specializations: the exponential case $G(z) = e^z$, which determines simple Hurwitz numbers, first studied by Okounkov and Pandharipande [28, 29]; polynomial and rational weight generating functions, which include, in particular, strictly and weakly monotonic Hurwitz numbers and weighted enumeration of Belyi curves [1, 13, 15, 20, 35] and, finally, for quantum Hurwitz numbers [15, 17]. The results are displayed in Appendix B for small values of $N$, $n$ and genus $g$ of the covering curve.

2 Generating functions for weighted Hurwitz numbers

2.1 Pure and weighted Hurwitz numbers

We recall the definition of pure Hurwitz numbers [10, 11, 33, 21, 22, 25] and weighted Hurwitz numbers [14, 15, 20, 16].

Definition 2.1 (Combinatorial). For a set of $k$ partitions $\{\mu^i\}_{i=1,\ldots,k}$ of $N \in \mathbb{N}^+$, the pure Hurwitz number $H(\mu^1, \ldots, \mu^k)$ is $\frac{1}{N!}$ times the number of distinct ways that the identity element $I_N \in S_N$ in the symmetric group in $N$ elements can be expressed as a product

$$I_N = h_1 \cdots h_k$$

(2.1)

of $k$ elements $\{h_i \in S_N\}_{i=1,\ldots,k}$, such that for each $i$, $h_i$ belongs to the conjugacy class $\text{cyc}(\mu^{(i)})$ whose cycle lengths are equal to the parts of $\mu^{(i)}$:

$$h_i \in \text{cyc}(\mu^{(i)}), \quad i = 1, \ldots, k.$$  

(2.2)

An equivalent definition involves the enumeration of branched coverings of the Riemann sphere.
Definition 2.2 (Geometric). For a set of partitions \( \{\mu^{(i)}\}_{i=1,\ldots,k} \) of weight \(|\mu^{(i)}| = N\), the pure Hurwitz number \( H(\mu^{(1)}, \ldots, \mu^{(k)}) \) is defined geometrically \([21, 22]\) as the number of inequivalent \(N\)-fold branched coverings \( C \to \mathbb{P}^1 \) of the Riemann sphere with \( k \) branch points whose ramification profiles are given by the partitions \( \{\mu^{(1)}, \ldots, \mu^{(k)}\} \), normalized by the inverse \( 1/|\text{aut}(C)| \) of the order of the automorphism group of the covering.

The equivalence of the two follows from the monodromy homomorphism
\[
\mathcal{M} : \pi_1(\mathbb{P}^1/\{Q^{(1)}, \ldots, Q^{(k)}\}) \to S_N
\]
from the fundamental group of the Riemann sphere punctured at the branch points into \( S_N \) obtained by lifting closed loops from the base to the covering.

The Frobenius-Schur formula determines Hurwitz numbers \( H(\mu^{(1)}, \ldots, \mu^{(k)}) \) in terms of irreducible characters of the symmetric group \( S_N \)
\[
H(\mu^{(1)}, \ldots, \mu^{(k)}) = \sum_{\lambda, \ |\lambda| = N} h^{k-2}(\lambda) \frac{\chi^{(0)}_{1}(\mu^{(i)})}{z_{\mu^{(i)}}},
\]
where \( \chi^{(0)}_{\lambda}(\mu) \) is the character of the irreducible representation corresponding to the partition \( \lambda \) of \( N \) evaluated on the conjugacy class \( \text{cyc}(\mu) \subset S_N \) with cycle structure given by the partition \( \mu \), \( h(\lambda) \) is the product of hook lengths of \( \lambda \) and
\[
z_{\mu} = \prod_{i=1}^{\mu_1} m_i(\mu)! \left| m_i(\mu) \right|
\]
is the order of the stability subgroup of any the elements of the conjugacy class \( \text{cyc}(\mu) \), where \( m_i(\mu) \) is the number of parts of \( \mu \) equal to \( i \). Computing \( H(\mu^{(1)}, \ldots, \mu^{(k)}) \) using this formula requires the character table for \( S_N \), which becomes increasingly computationally complex \([6]\) for rising \( N \) and \( k \).

To define weighted Hurwitz numbers, we introduce a weight generating function \( G(z) \), either as an infinite sum
\[
G(z) = 1 + \sum_{i=1}^{\infty} g_i z^i
\]
or an infinite product
\[
G(z) = \prod_{i=1}^{\infty} (1 + c_i z)
\]
or, in dual form

\[ \tilde{G}(z) = \prod_{i=1}^{\infty} (1 - d_i z)^{-1}, \quad (2.8) \]

which may also be developed as an infinite sum,

\[ \tilde{G}(z) = 1 + \sum_{i=1}^{\infty} \tilde{g}_i z^i, \quad (2.9) \]

either formally, or under suitable convergence conditions imposed upon the parameters \( \{c_i\}, \{d_i\}_{i \in \mathbb{N}^+} \) or the Taylor coefficients \( \{g_i\}_{i \in \mathbb{N}^+}, \{\tilde{g}_i\}_{i \in \mathbb{N}^+} \). The independent parameters determining the weighting may be viewed as any of these. Cases (2.7) and (2.8) may be viewed as generating functions for elementary \( \{e_i\}_{i \in \mathbb{N}} \) and complete symmetric functions \( \{h_i\}_{i \in \mathbb{N}} \), respectively, giving

\[ g_i = e_i(c), \quad \tilde{g}_i = h_i(d), \quad i \in \mathbb{N}^+, \quad (2.10) \]

where \( c = (c_1, c_2, \ldots), \ d = (d_1, d_2, \ldots) \).

The particular case of exponential weight generating function

\[ G(z) = \exp(z) = e^z \quad (2.11) \]

is of special interest, since it corresponds to a Dirac measure on the space of \( k \)-tuples of partitions \( \{\mu^{(1)}, \ldots, \mu^{(k)}\} \), supported uniformly on \( k \)-tuples of 2-cycles

\[ \mu^{(i)} = (2, (1)^{n-2}), \quad i = 1, \ldots, k \quad (2.12) \]

which, from the viewpoint of enumeration of branched covers of the Riemann sphere, corresponds to simple branching \[28, 30\]. A number of other special cases of particular interest will be considered in the following.

The first of these is the family of rational weight generating functions

\[ G_{c,d}(z) := \frac{\prod_{i=1}^{L}(1 + c_i z)}{\prod_{j=1}^{M}(1 - d_j z)}, \quad (2.13) \]

where the \( L + M \) complex numbers \( \{c_1, \ldots, c_L, d_1, \ldots, d_M\} \) determine the zeros and poles. The Taylor series coefficients in this case are

\[ g_i(c, d) := \sum_{j=0}^{i} e_j(c) h_{i-j}(d), \quad (2.14) \]
where \( \mathbf{c} = (c_1, \ldots, c_L) \), \( \mathbf{d} = (d_1, \ldots, d_M) \). Particular cases include: linear or quadratic polynomials \((L, M) = (1, 0)\) or \((2, 0)\), which correspond to two or three branch points (Belyi curves) \([35, 24]\) and the case \((L, M) = (0, 1)\) which, in the double weighted Hurwitz number case (which actually corresponds to a 2D Toda \(\tau\)-function), is equivalent to the Itzykson-Zuber-Harish-Chandra matrix integral \([19, 23]\) as generating function \([13]\).

Also of particular interest is the case of the quantum exponential function

\[
G(z) = H_q(z) := \prod_{i=0}^{\infty} (1 - q^i z)^{-1} =: e_q \left( z (1 - q)^{-1} \right),
\]

which is the weight generating function for (one version of) quantum Hurwitz numbers \([15, 17]\), whose Taylor coefficients are given by

\[
g_i(q) := \frac{1}{(q; q)_i}, \quad i \in \mathbb{N}^+, \tag{2.16}
\]

where \((q; q)_i\) is the \(q\)-Pochhammer symbol evaluated at \((q, q)\):

\[
(q; q)_i := (1 - q) \cdots (1 - q^i).
\]

**Definition 2.3** (Weighted Hurwitz numbers). For the case \((2.7)\), choosing a positive integer \(d\) and a fixed partition \(\mu\) of weight \(|\mu| = N\), the weighted (single) Hurwitz number \(H_G^d(\mu)\) is defined as the weighted sum over all \(k\)-tuples \((\mu^{(1)}, \ldots, \mu^{(k)})\)

\[
H_G^d(\mu) := \sum_{k=1}^{d} \sum'_{|\mu^{(i)}| = N, \sum_{i=1}^{k} \ell^*(\mu^{(i)}) = d} W_G(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu), \tag{2.18}
\]

where \(\sum'\) denotes a sum over all \(k\)-tuples of partitions \(\{\mu^{(1)}, \ldots, \mu^{(k)}\}\) of \(N\) other than the cycle type of the identity element \((1^N)\),

\[
\ell^*(\mu^{(i)}) := |\mu^{(i)}| - \ell(\mu^{(i)}) = N - \ell(\mu^{(i)}) \tag{2.19}
\]

is the colength of the partition \(\mu^{(i)}\), and the weight factor is defined to be

\[
W_G(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{1 \leq b_1 < \cdots < b_k \leq M} c_{b_1}^{\sigma(1)} \cdots c_{b_k}^{\sigma(k)} \ell^*(\mu^{(1)}) \cdots \ell^*(\mu^{(k)}).
\]
Here \( m_\lambda(c) \) is the monomial symmetric function of the parameters \( c := (c_1, c_2, \ldots) \)

\[
m_\lambda(c) = \frac{1}{|\text{aut}(\lambda)|} \sum_{\sigma \in S_k} \sum_{1 \leq b_1 < \cdots < b_k} c_{b_\sigma(1)}^{\lambda_1} \cdots c_{b_\sigma(k)}^{\lambda_k},
\]

(2.21)

indexed by the partition \( \lambda \) of weight \(|\lambda| = d\) and length \( \ell(\lambda) = k\), whose parts \( \{\lambda_i\} \) are equal to the colengths \( \{\ell^*(\mu^{(i)})\} \) (expressed in weakly decreasing order)

\[
\{\lambda_i\}_{i=1}^k \sim \{\ell^*(\mu^{(i)})\}_{i=1}^k, \quad \lambda_1 \geq \cdots \geq \lambda_k > 0
\]

(2.22)

and

\[
|\text{aut}(\lambda)| := \prod_{i \geq 1} m_i(\lambda)!
\]

(2.23)

where \( m_i(\lambda) \) is the number of parts of \( \lambda \) equal to \( i \). We similarly denote by

\[
\tilde{H}^d_G(\mu) := \sum_{k=1}^d \sum'_{|\mu^{(1)}, \ldots, \mu^{(k)}| = N, \sum_{i=1}^k \ell^*(\mu^{(i)}) = d} \mathcal{W}_G(\mu^{(1)}, \ldots, \mu^{(k)}) \tilde{H}(\mu^{(1)}, \ldots, \mu^{(k)}, \mu)
\]

(2.24)

the connected weighted Hurwitz numbers corresponding to the weight generating function \( G(z) \).

Recall that the sum

\[
d := \sum_{i=1}^k \ell^*(\mu^{(i)})
\]

(2.25)

of the colengths of the ramification profiles \( \{\mu^{(i)}\} \) of the weighted branch points determines the Euler characteristic \( \chi \) of the covering curve through the Riemann-Hurwitz formula

\[
\chi = N + \ell(\mu) - d,
\]

(2.26)

which for connected coverings \( \mathcal{C} \to \mathbf{P} \) is related, as usual, to the genus \( g \) by

\[
\chi = 2 - 2g.
\]

(2.27)
Remark 2.1. Note that we can rearrange the sum \((2.18)\) as follows:

$$\begin{align*}
H^d_G(\mu) &= \sum_{\lambda: |\lambda| = d} m_\lambda(c) \frac{|\text{aut}(\lambda)|}{\ell(\lambda)!} \sum_{\{\ell_1, \ldots, \ell_K\} \in \text{Ana}(\lambda)} \sum_{\substack{\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(K)}, \mu \in |\mu^{(i)}| = |\mu|}} H(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(K)}, \mu) \tag{2.28}
\end{align*}$$

where \(k = \ell(\lambda)\). Here \(\text{Ana}(\lambda) = (\ell_1, \ldots, \ell_k)\) denotes the distinct rearrangements (or anagrams) of the numbers \(\lambda_1, \ldots, \lambda_k\). But there are precisely \(\frac{|\ell(\lambda)|!}{|\text{aut}(\lambda)|!}\) such anagrams, so if we denote

$$F_\lambda(\mu) := \sum_{\mu^{(i)} = \ell_j \atop |\mu^{(i)}| = |\mu|} H(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}, \mu) \tag{2.29}$$

it is clear that \(F_\lambda(\mu)\) is invariant under permutations of the \(\ell_j\)'s. The counting over anagrams therefore exactly cancels the factor \(\frac{|\text{aut}(\lambda)|!}{k!}\), and \((2.18)\) is equivalent to

$$H^d_G(\mu) = \sum_{\lambda: |\lambda| = d} m_\lambda(c) F_\lambda(\mu). \tag{2.30}$$

The same is true, of course, for the connected version \(\tilde{H}^d_G(\mu)\), with \(F_\lambda(\mu)\), replaced by

$$\tilde{F}_\lambda(\mu) := \sum_{\mu^{(i)} = \ell_j \atop |\mu^{(i)}| = |\mu|} \tilde{H}(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}, \mu). \tag{2.31}$$

For the dual generating function \(\tilde{G}(z)\), the weighted Hurwitz numbers are defined as

$$H^d_G(\mu) := \sum_{k=1}^d \sum_{\{\mu^{(1)}, \ldots, \mu^{(k)}\} \in |\mu^{(i)}| = N} \sum_{\sum_{i=1}^k \ell^*(\mu^{(i)}) = d} W_G(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu), \tag{2.32}$$

where

$$W_G(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{(-1)^{\sum_{i=1}^k \ell^*(\mu^{(i)}) - k}}{k!} \sum_{\sigma \in S_k} \sum_{1 \leq b_1 \cdots b_k \leq M} d_{b_1}^{\sigma(1)} \cdots d_{b_k}^{\sigma(k)} = \frac{|\text{aut}(\lambda)|}{k!} f_\lambda(d). \tag{2.33}$$
Here $f_\lambda(d)$ is the “forgotten” symmetric function \[f_\lambda(d) = \frac{(-1)^{\ell(\lambda)}}{|\text{aut}(\lambda)|} \sum_{\sigma \in \mathfrak{S}_k} \sum_{1 \leq b_1 \leq \ldots \leq b_k} d_{b_{\sigma(1)}}^{\lambda_1} \cdots d_{b_{\sigma(k)}}^{\lambda_k},\]
indexed again by the partition $\lambda$ of weight $|\lambda| = d$ and length $\ell(\lambda) = k$, whose parts $\{\lambda_i\}$ are equal to the colengths $\{\ell^*(\mu^{(i)})\}$, as in (2.22).

For rational weight generating functions $G_{c,d}(z)$ as defined in (2.13), the weighted Hurwitz numbers are
\[H^d_{G_{c,d}(\mu)} := \sum_{k=1}^d \sum_{\mu^{(1)}, \ldots, \mu^{(k)}, |\mu^{(i)}| = N} \mathcal{W}_{G_{c,d}(\mu)}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu),\]
where the weight factor is defined to be
\[\mathcal{W}_{G_{c,d}(\mu)}(\mu^{(1)}, \ldots, \mu^{(k)}) := \frac{(-1)^{\sum_{i=1}^k \ell^*(\mu^{(i)})} - k}{k!} \sum_{\sigma, \sigma' \in \mathfrak{S}_k} \sum_{1 \leq a_1 < \ldots < a_k \leq M} c_{\sigma(1)}^{\ell^*(\mu^{(1)})} \cdots c_{\sigma(k)}^{\ell^*(\mu^{(k)})} \times d_{b_{\sigma(1)}}^{\ell^*(\mu^{(1)})} \cdots d_{b_{\sigma(k)}}^{\ell^*(\mu^{(k)})}.\]

Finally, for the case of the generating function $H_q(z)$ defined in (2.15), the quantum weighted Hurwitz number is
\[H^d_{H_q}(\mu) := \sum_{k=1}^d \sum_{\mu^{(1)}, \ldots, \mu^{(k)}, |\mu^{(i)}| = N} \mathcal{W}_{H_q}(\mu^{(1)}, \ldots, \mu^{(k)}) H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu),\]
where the weight factor is
\[\mathcal{W}_{H_q}(\mu^{(1)}, \ldots, \mu^{(k)}) := \sum_{\sigma \in \mathfrak{S}_k} \prod_{j=1}^k \frac{1}{1 - q \sum_{i=1}^j \ell^*(\mu^{(\sigma(i))})}.\]

In order to compute the weighted Hurwitz numbers for any given weight generating function $G(z)$, it is necessary not only to know the pure
Hurwitz numbers $H(\mu^{(1)}, \ldots, \mu^{(k)}, \mu)$ or $\tilde{H}(\mu^{(1)}, \ldots, \mu^{(k)}, \mu)$ entering in the sums (2.18), (2.24), but also to express the weights (2.36) or (2.33) as weighted homogeneous polynomials in the Taylor coefficients $\{g_i\}_{i \in \mathbb{N}^+}$ or $\{\tilde{g}_i\}_{i \in \mathbb{N}^+}$ of $G$ or $\tilde{G}$, respectively. This requires the transition matrices [26] relating the various bases $\{m_\lambda\}$, $\{f_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$ of the ring of symmetric functions which, in turn, require the Kostka matrices $K_{\lambda\mu}$ which are the transition matrices between the Schur function bases $\{s_\lambda\}$ and the monomial symmetric function bases $\{m_\lambda\}$. The latter is readily computable for any given weight $d$, but no known formula exists. Therefore, the computation of the weighted Hurwitz numbers $H^d_G(\mu)$ and their connected version $\tilde{H}^d_G(\mu)$ using their definitions directly is a lengthy and complex task. As will be shown in the following, however, this may all be circumvented through the use of generating functions, which neither require a knowledge of the $S_N$ group characters nor the Kostka matrices.

### 2.2 Hypergeometric $\tau$-functions as generating functions of weighted Hurwitz numbers

We recall the definition of KP $\tau$-functions of hypergeometric type [31], which serve as generating functions for weighted Hurwitz numbers [15, 20, 16]. For any weight generating function $G(z)$ of the above type, and nonzero parameter $\beta$, we define two doubly infinite sequences of numbers $\{r^{(G, \beta)}_i, \rho_i\}_{i \in \mathbb{Z}}$, labeled by integers

$$
r^{(G, \beta)}_i := \beta G(i\beta), \quad i \in \mathbb{Z}, \quad \rho_0 = 1, \quad \rho_i := \prod_{j=1}^{i} r^{(G, \beta)}_j, \quad \rho_{-i} := \prod_{j=0}^{i-1} (r^{(G, \beta)}_{-j})^{-1}, \quad i \in \mathbb{N}^+, \tag{2.39}
$$

related by

$$
r^{(G, \beta)}_i = \frac{\rho_i}{\rho_{i-1}}, \quad i \in \mathbb{N}^+. \tag{2.41}
$$

where $\beta$ is viewed as a small parameter for which $G(i\beta)$ does not vanish for any integer $i \in \mathbb{Z}$. (It is possible to extend this by requiring that, if $G(i\beta) = 0$ for some smallest positive integer $i \in \mathbb{N}^+$, then $\rho_j := 0$ for all $j \geq i$.}
For the exponential generating function \( G(z) = e^z \), we have
\[
r_i = e^{i\beta}, \quad \rho_i = \beta^i e^{\frac{1}{2}i(i+1)\beta}, \quad \rho_{-i} = \beta^{-i} e^{\frac{1}{2}i(i-1)\beta}, \quad g_i = \frac{1}{i!}.
\] (2.42)

For rational weight generating functions \( G_{c,d}(z) \) as defined in (2.13), the parameters \( \{\rho_i\}_{i \in \mathbb{Z}} \) and coefficients \( \{g_i\}_{i \in \mathbb{N}} \) become
\[
\rho_i := \beta^i \prod_{k=0}^{i} \prod_{j=1}^{L} (1 + kc_i \beta), \quad \rho_{-i} := \beta^{-i} \prod_{k=1}^{i-1} \prod_{j=1}^{M} (1 + kd_j \beta), \quad i \in \mathbb{Z},
\] (2.43)
\[
g_i = \sum_{j=0}^{i} e_j(c) h_{i-j}(d).
\] (2.44)

For the quantum exponential generating function \( H_q(z) \) defined in (2.15), the parameters \( \{\rho_i\}_{i \in \mathbb{Z}} \) and coefficients \( \{g_i\}_{i \in \mathbb{N}} \) become
\[
\rho_i := \beta^i \prod_{k=1}^{i} \prod_{j=0}^{\infty} (1 - kq^j \beta)^{-1}, \quad \rho_{-i} := \beta^{-i} \prod_{k=1}^{i-1} \prod_{j=0}^{\infty} (1 + kq^j \beta), \quad i \in \mathbb{Z},
\] (2.45)
\[
g_i = \frac{1}{(q; q)_i},
\] (2.46)
where
\[
(q; q)_i := \prod_{j=1}^{i} (1 - q^j).
\] (2.47)

For each partition \( \lambda \) of \( N \), we define the associated content product coefficient
\[
r_{\lambda}^{(G,\beta)} := \prod_{(i,j) \in \lambda} r_{i-j}^{(G,\beta)}.
\] (2.48)

The KP \( \tau \)-function of hypergeometric type associated to these parameters is defined as the Schur function series [20, 14, 15]:
\[
\tau^{(G,\beta)}(t) := \sum_{N=0}^{\infty} \sum_{|\lambda|=N} (h(\lambda))^{-1} r_{\lambda}^{(G,\beta)} s_\lambda(t),
\] (2.49)
where $h(\lambda)$ is the product of the hook lengths of the partition $\lambda$ and $t = (t_1, t_2, \ldots)$ is the infinite sequence of KP flow parameters, which may be equated to the sequence of normalized power sums $(p_1, \frac{1}{2}p_2, \ldots)$

$$t_i := \frac{1}{i} \sum_a x_a^i = \frac{1}{i} p_i \quad (2.50)$$

in a finite or infinite set of auxiliary variables $(x_1, x_2, \ldots)$.

Using the Schur character formula [12, 26]

$$s_\lambda = \sum_{\mu, |\mu| = \lambda = N} \chi_{\lambda}(\mu) \frac{p_\mu}{z_\mu}, \quad (2.51)$$

where $\chi_{\lambda}(\mu)$ is the irreducible character determined by $\lambda$ evaluated on the conjugacy class $\text{cyc}(\mu)$, $z_\mu$ is the order of the stabilizer of the elements of this conjugacy class, as given in (2.5), and

$$p_\mu := \prod_{i=1}^{\ell(\mu)} p_{\mu_i} \quad (2.52)$$

is the power sum symmetric function corresponding to partition $\mu$, we may re-express the Schur function series (2.49) as an expansion in the basis $\{p_\mu\}$ of power sum symmetric functions. As shown in [14, 15, 16, 20], this gives the following:

**Theorem 2.1.** The $\tau$-function $\tau^{(G, \beta)}(t)$ may equivalently be expressed as

$$\tau^{(G, \beta)}(t) = \sum_{\mu, |\mu| = \mu} \sum_{d=0}^\infty \beta^d H_G^d(\mu) p_\mu(t). \quad (2.53)$$

It is thus a generating function for the weighted Hurwitz numbers $H_G^d(\mu)$. This holds similarly for the dual weight generating function $\tilde{G}(z)$ and the weighted Hurwitz numbers $H_{\tilde{G}}^d(\mu)$

$$\tau^{(\tilde{G}, \beta)}(t) = \sum_{\mu, |\mu| = \nu} \sum_{d=0}^\infty \beta^d H_{\tilde{G}}^d(\mu) p_\mu(t). \quad (2.54)$$
Particular cases of weighted Hurwitz numbers for which hypergeometric \( \tau \)-functions correspond to the weight generating functions \( G(z) \) defined above include: *simple* Hurwitz numbers \([28, 30]\) (both single and double), with weight generating function \( G(z) = e^z \); *weakly monotonic* Hurwitz numbers \([13]\) (or, equivalently, *signed* Hurwitz numbers \([20, 15]\)), with weight generating function \( G(z) = \frac{1}{1-z} \); *strongly monotonic* Hurwitz numbers \([14, 15]\), with weight generating function \( G(z) = 1 + z \) (or, equivalently, weighted Hurwitz numbers for Belyi curves and *dessins d’enfants* \([1, 24, 35]\)); polynomially weighted Hurwitz numbers \([20, 5, 2, 3, 4]\); quantum Hurwitz numbers \([16, 15, 17]\) and *multispecies* Hurwitz numbers \([18]\).

2.3 The pair correlator \( K^G(x, y) \)

An essential rôle in the computation of weighted Hurwitz numbers is played by the *pair correlator* associated to the \( \tau \)-function, defined as

\[
K^G(x, y) := \frac{\tau^{G, \beta}(x) - [x] - [y])}{x - y},
\]

where \([x]\) denotes the infinite sequence

\[
[x] = (x, x^2/2, x^3/3, \ldots, x^n/n, \ldots).
\]

This has the following series expansion \([2, 4]\)

\[
K^G(x, y) = \frac{1}{x - y} + \beta^{-1} \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} (-1)^b \rho_{ab} \beta^{-a} \rho_{b-1} \frac{1}{a!b!(1 + a + b)} \left( \frac{x}{\beta} \right)^a \left( \frac{y}{\beta} \right)^b.
\]

Defining

\[
\rho_{ab}(\beta) := (-1)^b \prod_{i=-b}^{a} \frac{G(i\beta)}{a!b!(a + b + 1)},
\]

we have

\[
\rho_{ab}(\beta) = (-1)^{a+b} \rho_{ba}(-\beta)
\]

and the nonsingular part of \( K^G(x, y) \) is

\[
K^G_0(x, y) := K^G(x, y) - \frac{1}{x - y} = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \rho_{ab} x^a y^b.
\]
Taking the (formal) Taylor series expansion

$$\rho_{ab}(\beta) := \sum_{d=0}^{\infty} \rho_{ab}^d \beta^d,$$

(2.61)

the coefficients $\rho_{ab}^d(g)$ are weighted degree $d$ homogeneous polynomials in the Taylor coefficients $\{g_i\}_{i \in \mathbb{N}^+} := g$ of $G(z)$. It follows from (2.59) that they satisfy the symmetry conditions

$$\rho_{ab}^d = (-)^{a+b+d} \rho_{ba}^d.$$

(2.62)

Their values for small values of $a$, $b$, $d$ are displayed in Tables 1-3 of Appendix A.

In particular, for the case of the rational weight generating function $G_{c,d}(z)$, we have

$$K^{G_{c,d}(z)}(x, y) = \frac{1}{x-y} + \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} (-1)^b \left( \prod_{k=-b}^{a} \prod_{i=1}^{M} (1 + i c_i \beta) \right) \frac{x^a y^b}{a! b!(1 + a + b)},$$

(2.63)

and for the quantum exponential weight generating function $G_{H_q}(z)$,

$$K^{G_{H_q}(z)}(x, y) = \frac{1}{x-y} + \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} (-1)^b \left( \prod_{k=-b}^{a} \prod_{j=0}^{\infty} (1 - k q^j \beta)^{-1} \right) \frac{x^a y^b}{a! b!(1 + a + b)}.$$

(2.64)

2.4 The multicurrent correlator $W_n(x_1, \ldots, x_n)$ as generating function for weighted Hurwitz numbers

Following [2, 3, 4], define the derivations:

**Definition 2.4.** For any parameter $x$

$$\nabla(x) := \sum_{i=1}^{\infty} x^{i-1} \frac{\partial}{\partial t_i}, \quad \tilde{\nabla}(x) := \sum_{i=1}^{\infty} \frac{x^i}{i} \frac{\partial}{\partial t_i}$$

(2.65)
In terms of these, we introduce the following correlators

\[ W^G_n(x_1, \ldots, x_n) := \left. \left( \prod_{i=1}^{n} \nabla(x_i) \right) \tau^{(G,\beta)}(t) \right|_{t=0}, \tag{2.66} \]

\[ \bar{W}^G_n(x_1, \ldots, x_n) := \left. \left( \prod_{i=1}^{n} \nabla(x_i) \right) \ln \tau^{(G,\beta)}(t) \right|_{t=0}, \tag{2.67} \]

\[ F^G_n(x_1, \ldots, x_n) := \left. \left( \prod_{i=1}^{n} \bar{\nabla}(x_i) \right) \tau^{(G,\beta)}(t) \right|_{t=0}, \tag{2.68} \]

\[ \bar{F}^G_n(x_1, \ldots, x_n) := \left. \left( \prod_{i=1}^{n} \bar{\nabla}(x_i) \right) \ln \tau^{(G,\beta)}(t) \right|_{t=0}. \tag{2.69} \]

which are related by

\[ W^G_n(x_1, \ldots, x_n) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} F^G_n(x_1, \ldots, x_n), \tag{2.70} \]

\[ \bar{W}^G_n(x_1, \ldots, x_n) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} \bar{F}^G_n(x_1, \ldots, x_n). \tag{2.71} \]

As shown in [3], \( W_n(x_1, \ldots, x_n) \) has a fermionic representation as a multcurrent correlator. Moreover, we have the following result from [4], which shows that \( F_n(x_1, \ldots, x_n) \) is another type of generating function for the weighted Hurwitz numbers \( H^d_G(\nu) \), and \( \bar{F}_n(x_1, \ldots, x_n) \) a generating function for the connected Hurwitz numbers \( \bar{H}^d_G(\mu) \).

**Proposition 2.2** ([2], [4]).

\[ F^G_n(x_1, \ldots, x_n) = \sum_{d=0}^{\infty} \sum_{\ell(\nu)=n} \beta^d H^d_G(\nu) | \text{aut}(\nu)| m_\nu(x_1, \ldots, x_n), \tag{2.72} \]

\[ \bar{F}^G_n(x_1, \ldots, x_n) := \sum_{d=0}^{\infty} \sum_{\ell(\mu)=n} \beta^d \bar{H}^d_G(\mu) | \text{aut}(\mu)| m_\mu(x_1, \ldots, x_n) \tag{2.73} \]

\[ = \sum_{g=0}^{\infty} \beta^{2g-2+n} \bar{F}_{g,n}^G(x_1, \ldots, x_n), \tag{2.74} \]

where

\[ \bar{F}_{g,n}^G(x_1, \ldots, x_n) = \sum_{\ell(\mu)=n} \bar{H}^{2g-2+n+|\mu|}_G(\mu) | \text{aut}(\mu)| m_\mu(x_1, \ldots, x_n) \tag{2.75} \]

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and \( m_\mu(x_1, \ldots, x_n) \) is the monomial symmetric polynomial in the indeterminates \( (x_1, \ldots, x_n) \).

Finally, we quote the following result from [2, 4], which expresses the connected multicurrent correlators in terms of pair correlators.

**Proposition 2.3.**

\[
\tilde{W}_G^1(x) = \lim_{x' \to x} \left( K_G^G(x, x') - \frac{1}{x - x'} \right),
\]

\[ \tilde{W}_G^2(x_1, x_2) = \left( -K_G^G(x_1, x_2)K_G^G(x_2, x_1) - \frac{1}{(x_1 - x_2)^2} \right), \]

and for \( n \geq 3 \)

\[
\tilde{W}_G^n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n^{1\text{-cycle}}} \text{sgn}(\sigma) \prod_i K_G^G(x_i, x_{\sigma(i)}),
\]

where the last sum is over all permutations in \( S_n \) consisting of a single \( n \)-cycle.

The corresponding nonconnected quantities are obtained by inverting the cumulant relations:

\[
\tilde{W}_G^1(x_1) = W_1(x_1)
\]

\[
\tilde{W}_G^2(x_1, x_2) = W_2(x_1, x_2) - W_1(x_1)W_2(x_2)
\]

\[
\tilde{W}_G^3(x_1, x_2, x_3) = W_3(x_1, x_2, x_3) - W_1(x_1)W_2(x_2, x_3) - W_1(x_2)W_2(x_1, x_3) - W_1(x_3)W_2(x_1, x_2) + 2W_1(x_1)W_1(x_2)W_1(x_3)
\]

or, more generally for \( n \geq 3 \),

\[
W_n(x_1, \ldots, x_n) = \sum_{\ell \geq 1} \sum_{I_1 \cup \cdots \cup I_\ell = \{1, \ldots, n\}} \prod_{i=1}^\ell \tilde{W}_G^{|I_i|}(x_j, j \in I_i),
\]

with identical relations holding between the \( \tilde{F}_n \)'s and \( F_n \)'s.
3 Generating function computation of $\tilde{H}^d_G(\mu)$ for $\ell(\mu) = 1, 2, 3$

Using the series expansion (2.64) for the pair correlator $K^G(x, y)$, and expressing the connected multicurrent correlators $\tilde{W}_n(x_1, \ldots, x_n)$ in terms of these via eqs. (2.76), (2.77), (2.78), we can compute all weighted connected or nonconnected single Hurwitz numbers $\tilde{H}^d_G(\mu)$, $H^d_G(\mu)$ for coverings of any degree $N$, Euler characteristic $\chi$ and unweighted partition $\mu$ of $N$, as homogeneous polynomials of weighted degree $d$ in the coefficients $\{g_i\}_{i \in \mathbb{N}^+}$ of the series expansion of the weight generating function $G(z)$.

In particular, for $\ell(\mu) \leq 3$, we have:

**Proposition 3.1** (Weighted Hurwitz numbers for $\ell(\mu) = 1, 2$ or $3$). The following formulae determine the connected weighted Hurwitz numbers $\tilde{H}^d_G(\mu)$ for $\mu$ of length $\ell(\mu) \leq 3$.

$\ell(\mu_1) = 1$:

$$\tilde{H}^d_G((\mu_1)) = H^d_G((\mu_1)) = 1 + \frac{1}{\mu_1 - 1} \sum_{a=0}^{\mu_1 - 1} \rho^d_{a, \mu_1 - a - 1}. \quad (3.1)$$

$\ell(\mu_1, \mu_2) = 2$:

$$\tilde{H}^d_G((\mu_1, \mu_2)) = \tilde{H}^{(d, 2)}_{(G, 1)}((\mu_1, \mu_2)) + \tilde{H}^{(d, 2)}_{(G, 2)}((\mu_1, \mu_2)),$$  

where the linear part $\tilde{H}^{(d, 2)}_{(G, 1)}((\mu_1, \mu_2))$ is given by

$$\tilde{H}^{(d, 2)}_{(G, 1)}((\mu_1, \mu_2)) = \frac{1 + (-1)^{d + |\mu|}}{\mu_1 \mu_2 |\text{aut}(\mu_1, \mu_2)|} \sum_{b=0}^{\mu_2 - 1} \rho^d_{b, \mu_1 + \mu_2 - b - 1} \quad (3.3)$$

and the quadratic part $\tilde{H}^{(d, 2)}_{G, 2}((\mu_1, \mu_2))$ by

$$\tilde{H}^{(d, 2)}_{(G, 2)}((\mu_1, \mu_2)) = -\frac{1}{\mu_1 \mu_2 |\text{aut}(\mu_1, \mu_2)|} \sum_{a=0}^{\mu_1 - 1} \sum_{b=0}^{\mu_2 - 1} \sum_{j=0}^{d} \rho^j_{a, \mu_2 - b - 1} \rho^d_{b, \mu_1 - a - 1} \quad (3.4)$$

$\ell(\mu_1, \mu_2, \mu_3) = 3$:

$$\tilde{H}^{(d, 3)}_G((\mu_1, \mu_2, \mu_3)) = \tilde{H}^{(d, 3)}_{(G, 1)}((\mu_1, \mu_2, \mu_3)) + \tilde{H}^{(d, 3)}_{(G, 2)}((\mu_1, \mu_2, \mu_3)) + \tilde{H}^{(d, 3)}_{(G, 3)}((\mu_1, \mu_2, \mu_3)), \quad (3.5)$$
where the linear part $\tilde{H}^{d,3}_{(G,1)}((\mu_1, \mu_2))$ is given by

$$\tilde{H}^d_{(G,1)}((\mu_1, \mu_2, \mu_3)) = \frac{1 - (-1)^{d+|\mu|}}{\mu_1 \mu_2 \mu_3 |\text{aut}(\mu_1, \mu_2, \mu_3)|} \sum_{b=0}^{\mu_2-1} \left( \rho_{\mu_2-b-1, \mu_1+b} - \rho_{\mu_1+\mu_2-b-1, \mu_3+b} \right),$$

(3.6)

the quadratic part by

$$\tilde{H}^{(d,3)}_{(G,2)}((\mu_1, \mu_2, \mu_3)) = K^d_{(G,3)}(\mu_1, \mu_2; \mu_3) + K^d_{(G,3)}(\mu_1, \mu_3; \mu_2) + K^d_{(G,3)}(\mu_3, \mu_2; \mu_1),$$

(3.7)

where

$$K^d_{(G,3)}(\mu_1, \mu_2; \mu_3) := \frac{(1 - (-1)^{d+|\mu|})}{\mu_1 \mu_2 \mu_3 |\text{aut}(\mu_1, \mu_2, \mu_3)|} \sum_{a=0}^{\min(\mu_1-1, \frac{1}{2}(\mu_1+\mu_2)-1)} \sum_{c=0}^{\mu_3-1} \sum_{j=0}^{d-j} \sum_{k=0}^{d-k} \rho^j_{c, \mu_1+\mu_2-a-1} \rho^k_{a, \mu_3-c-1},$$

(3.8)

with $\text{aut}(\mu_1, \mu_2, \mu_3)$ for a composition $(\mu_1, \mu_2, \mu_3)$ (i.e., an unordered set of positive integers) defined in the same way (2.23) as for a partition, and the cubic part by

$$\tilde{H}^{(d,3)}_{(G,3)}((\mu_1, \mu_2, \mu_3)) = \frac{1 - (-1)^{d+|\mu|}}{\mu_1 \mu_2 \mu_3 |\text{aut}(\mu_1, \mu_2, \mu_3)|} \sum_{a=0}^{\mu_1-1} \sum_{b=0}^{\mu_2-1} \sum_{c=0}^{\mu_3-1} \sum_{j=0}^{d-j} \sum_{k=0}^{d-k} \rho^j_{a, \mu_2-b-1} \rho^k_{b, \mu_3-c-1} \rho^d_{c, \mu_1-a-1}.$$  

(3.9)

**Proof.** Comparing the expansion (2.73) with the series obtained from applying Proposition 2.3 using relation (2.71), gives the weighted connected Hurwitz numbers $\tilde{H}^d_G(\mu)$ for all partitions $\mu$ as polynomials in the coefficients $\rho^d_{ab}$. \hfill \Box

**Remark 3.1.** The nonconnected versions of the weighted Hurwitz numbers are obtained by applying the cumulant relations (2.79), which give

$$H^d_G(\mu_1) = \tilde{H}^d_G(\mu_1),$$

(3.10)

$$H^d_G(\mu_1, \mu_2) = \tilde{H}^d_G(\mu_1, \mu_2) + \frac{1}{|\text{aut}(\mu_1, \mu_2)|} \sum_{k=0}^{d} H^k_G(\mu_1) H^{d-k}_G(\mu_2),$$

(3.11)

$$H^d_G(\mu_1, \mu_2, \mu_3) = \tilde{H}^d_G(\mu_1, \mu_2, \mu_3) + \frac{1}{|\text{aut}(\mu_1, \mu_2, \mu_3)|} \sum_{k=0}^{d} \left( |\text{aut}(\mu_2, \mu_3)| H^k_G(\mu_1) H^{d-k}_G(\mu_2, \mu_3) \right),$$

(3.12)
\[
\begin{align*}
&+ |\text{aut}(\mu_1, \mu_3)| H_k^G(\mu_2) H_{G-d-k}^d(\mu_1, \mu_3) + |\text{aut}(\mu_1, \mu_2)| H_k^G(\mu_3) H_{G-d-k}^d(\mu_1, \mu_2) \\
&- 2 \sum_{j=0} H^j(\mu_1)) H^k(\mu_2)) H^{d-j-k}(\mu_3)). 
\end{align*}
\]

(3.12)

**Remark 3.2.** Similar formulae, consisting of degree \(n\) polynomials in the coefficients \(\rho_{ab}^d(g)\), may be obtained for all values of \(n = \ell(\mu)\), determining thereby the corresponding weighted Hurwitz numbers \(\tilde{H}_G^d(\mu)\), \(H_G^d(\mu)\) from the series expansions of \(\tilde{W}_n(x_1, \ldots, x_n)\) for all \(n\).

**Remark 3.3.** Inserting the expressions for the coefficients \(\rho_{ab}^d(g)\) as graded homogeneous polynomials in the Taylor coefficients \(g = \{g_i\}_{i \in \mathbb{N}}\) of the generating function \(G(z)\), gives \(\tilde{H}_G^d(\mu)\) as a weight \(d\) graded homogeneous polynomial in the \(g_i\)'s.

Tables 4 - 13 of Appendix B display \(H_G^d(\mu)\) and \(\tilde{H}_G^d(\mu)\) for \(n = 1, 2, 3\) and small values of \(N\) and \(d\), first for generic values of the Taylor coefficients \(\{g_i\}_{i \in \mathbb{N}}\), then for the particular cases of: exponential, rational and quantum exponential weight generating functions \(e^z\), \(G_{(c,d)}(z)\) and \(H_q(z)\), respectively.

**Appendices**

A  Table of coefficients \(\rho_{ab}^d(g)\)

Table 1: \(\rho_{ab}^1(g)\) for \(0 \leq a, b \leq 4\)

| \(a\) | \(0\) | \(1\) | \(2\) | \(3\) | \(4\) |
|-----|-----|-----|-----|-----|-----|
| \(b\) |
| 0   | 0   | \(\frac{1}{2}g_1\) | \(-\frac{1}{2}g_1\) | \(\frac{4}{3}g_1\) | \(-\frac{4}{3}g_1\) |
| 1   | \(\frac{1}{2}g_1\) | 0   | \(-\frac{1}{2}g_1\) | \(\frac{4}{3}g_1\) | \(-\frac{4}{3}g_1\) |
| 2   | \(\frac{1}{2}g_1\) | \(\frac{1}{2}g_1\) | 0   | \(\frac{4}{3}g_1\) | \(-\frac{4}{3}g_1\) |
| 3   | \(\frac{1}{2}g_1\) | \(-\frac{1}{2}g_1\) | \(\frac{4}{3}g_1\) | 0   | \(-\frac{4}{3}g_1\) |
| 4   | \(\frac{1}{2}g_1\) | \(-\frac{1}{2}g_1\) | \(\frac{4}{3}g_1\) | \(-\frac{4}{3}g_1\) | 0   |

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Table 2: $\rho_{ab}^2(g)$ for $0 \leq a, b \leq 4$. The last column $\rho_{a4}^2(g)$ is determined by applying the symmetry property (2.62) and setting $\rho_{44}^2 = -\frac{5}{864}(g_1^2 - 2g_2)$.

| $a$ | $0$ | $1$ | $2$ | $3$ |
|-----|-----|-----|-----|-----|
| $b$ |
| $0$ | $0$ | $\frac{1}{2}g_2$ | $\frac{1}{6}(2g_1^2 + 5g_2)$ | $\frac{1}{6}(11g_1^2 + 14g_2)$ |
| $1$ | $\frac{1}{2}g_2$ | $\frac{1}{3}(g_1^2 - 2g_2)$ | $\frac{1}{3}(-g_1^2 + 6g_2^2)$ | $\frac{1}{3}(-g_1^2 + 3g_2^2)$ |
| $2$ | $\frac{1}{3}(2g_1^2 + 5g_2)$ | $\frac{1}{3}(g_1^2 - 6g_2)$ | $\frac{1}{3}(-g_1^2 + 2g_2)$ | $\frac{1}{3}(5g_1^2 - 19g_2)$ |
| $3$ | $\frac{1}{3}(11g_1^2 + 14g_2)$ | $\frac{1}{3}(g_1^2 + 3g_2)$ | $\frac{1}{3}(-5g_1^2 + 19g_2)$ | $\frac{1}{3}(g_1^2 - 2g_2)$ |
| $4$ | $\frac{1}{3}(7g_1^2 + 6g_2)$ | $\frac{1}{9}(25g_1^2 + 31g_2^2)$ | $\frac{1}{9}(g_1^2 + 5g_2)$ | $\frac{1}{9}(7g_1^2 - 22g_2)$ |

Table 3: $\rho_{ab}^3(g)$ for $0 \leq a, b \leq 4$. The last column $\rho_{a4}^3(g)$ is determined by applying the symmetry property (2.62) and setting $\rho_{44}^3 = 0$.

| $a$ | $0$ | $1$ | $2$ | $3$ |
|-----|-----|-----|-----|-----|
| $b$ |
| $0$ | $0$ | $\frac{1}{2}g_3$ | $-\frac{1}{6}(6g_1g_2 - 9g_3)$ | $\frac{1}{24}(6g_1^2 + 48g_1g_2 - 96g_3)$ |
| $1$ | $\frac{1}{2}g_3$ | $0$ | $\frac{1}{4}(g_1^2 - 2g_2)$ | $\frac{1}{6}(-g_1^2 + 8g_1g_2)$ |
| $2$ | $\frac{1}{6}(6g_1g_2 + 9g_3)$ | $\frac{1}{4}(g_1^2 - 2g_2)$ | $0$ | $\frac{1}{24}(5g_1^2 + 10g_1g_2 - 9g_3)$ |
| $3$ | $\frac{1}{4}(g_1^2 + 8g_1g_2 + 6g_2^2)$ | $\frac{1}{6}(g_1^2 - 8g_1g_2 - g_2^2)$ | $\frac{1}{24}(-5g_1^2 + 10g_1g_2)$ | $0$ |
| $4$ | $\frac{1}{12}(g_1^2 + 4g_1g_2 + 2g_2^2)$ | $-\frac{1}{48}(5g_1^2 + 6g_1g_2 + 13g_2^2)$ | $\frac{1}{48}(-5g_1^2 + 22g_1g_2)$ | $\frac{1}{144}(7g_1^2 - 14g_1g_2)$ |
## B Evaluation of weighted Hurwitz numbers

### B.1 Generic case

Table 4: Nonconnected weighted Hurwitz numbers $H^d_G(\mu)$ with $\ell(\mu) = 1$ or $2$, $N = 2, 3$ or $4$

| $N$ | $\mu$ | $H^N_G(\ell(\mu)-2)$ | $H^N_G(\ell(\mu))$ |
|-----|-------|----------------------|----------------------|
| 2   | (2)   | $\frac{1}{2}g_1$     | $\frac{1}{2}g_3$     |
| 3   | (3)   | $\frac{1}{3}(g_1^2 + g_2)$ | $\frac{1}{3}(g_2^2 + 4g_1g_3 + 5g_4)$ |
| 3   | (21)  | $g_1g_2 + \frac{2}{2}g_3$ | $3g_1g_4 + 2g_2g_3 + \frac{11}{2}g_5$ |
| 4   | (4)   | $\frac{1}{4}(g_1^3 + 3g_1g_2 + g_3)$ | $\frac{7}{4}g_1^3g_3 + \frac{4}{2}g_1g_2^2 + \frac{11}{2}g_4$ |
| 4   | (31)  | $g_1^2g_2 + \frac{2}{3}g_2^2 + \frac{11}{2}g_1g_3 + \frac{8}{3}g_4$ | $g_1(6g_1g_4 + 8g_2g_3 + 24g_5) + g_2(g_2^2 + 16g_4) + 7g_3^2 + 22g_6$ |
| 4   | (22)  | $\frac{1}{2}g_1^2g_2 + \frac{1}{2}g_2^3 + g_1g_3 + \frac{1}{2}g_4$ | $g_1\left(\frac{11}{3}g_1g_4 + \frac{15}{4}g_2g_3 + 9g_5\right) + g_2\left(\frac{1}{4}g_2^2 + \frac{11}{2}g_4\right) + \frac{11}{2}g_3^2 + \frac{22}{4}g_6$ |

Table 5: Connected weighted Hurwitz numbers $\hat{H}^d_G(\mu)$ with $\ell(\mu) = 1$ or $2$, $N = 2, 3$ or $4$

| $N$ | $\mu$ | $\hat{H}^N_G(\ell(\mu)-2)$ | $\hat{H}^N_G(\ell(\mu))$ |
|-----|-------|----------------------|----------------------|
| 2   | (2)   | $\frac{1}{2}g_1$     | $\frac{1}{2}g_3$     |
| 3   | (3)   | $\frac{1}{3}(g_1^2 + g_2)$ | $\frac{1}{3}(g_2^2 + 4g_1g_3 + 5g_4)$ |
| 3   | (21)  | $g_1g_2 + g_3$      | $3g_1g_4 + 2g_2g_3 + 5g_5$ |
| 4   | (4)   | $\frac{1}{4}(g_1^3 + 3g_1g_2 + g_3)$ | $\frac{7}{4}g_1^3g_3 + \frac{4}{2}g_1g_2^2 + \frac{11}{2}g_4$ |
| 4   | (31)  | $g_1^2g_2 + g_2^2 + 2g_1g_3 + g_4$ | $g_1(6g_1g_4 + 8g_2g_3 + 20g_5) + g_2(g_2^2 + 14g_4) + 8g_3^2 + 15g_6$ |
| 4   | (22)  | $\frac{1}{2}g_1^2g_2 + \frac{1}{2}g_2^3 + g_1g_3 + \frac{1}{2}g_4$ | $g_1\left(\frac{11}{3}g_1g_4 + \frac{15}{4}g_2g_3 + 9g_5\right) + g_2\left(\frac{1}{4}g_2^2 + \frac{11}{2}g_4\right) + \frac{11}{2}g_3^2 + \frac{22}{4}g_6$ |

Further examples, with $\ell(\mu) = 3$ and $N = 4$ or $5$, are given in Tables 6 and 7.
Table 6: Nonconnected weighted Hurwitz numbers $H^d_G(\mu)$ with $\ell(\mu) = 3$, $N = 3, 4, 5$ or 6

| $N$ | $\mu$ | $d$ | $H^d_G(\mu)$ |
|-----|-------|-----|--------------|
| 3   | (1,1,1) | 2   | $\frac{1}{2}g_2$ |
| 4   | (2,1,1) | 3   | $\frac{1}{2}(g_1g_2 + g_3)$ |
| 5   | (2,2,1) | 2   | $\frac{1}{2}g_1^2$ |
| 5   | (2,2,1) | 4   | $g_1^2g_2 + \frac{1}{2}g_2^2 + \frac{1}{2}g_4$ |
| 6   | (3,2,1) | 3   | $\frac{1}{6}(g_1^2 + g_1g_2)$ |
| 6   | (3,2,1) | 5   | $\frac{1}{12}(11g_1^2g_2 + 31g_1^2g_3 + 15g_2g_3 + 2g_1(9g_2^2 + 13g_4)) + g_5$ |

Table 7: Connected weighted Hurwitz numbers $\tilde{H}^d_G(\mu)$ with $\ell(\mu) = 3$, $N = 3, 4, 5$ or 6. By the Riemann-Hurwitz formula, $\tilde{H}^d_G(1,1,1) = 0$ for $d < 4$, $\tilde{H}^d_G(2,1,1) = 0$ for $d < 5$, $\tilde{H}^d_G(2,2,1) = 0$ for $d < 6$ and $\tilde{H}^d_G(3,2,1) = 0$ for $d < 7$.

| $N$ | $\mu$ | $d$ | $\tilde{H}^d_G(\mu)$ |
|-----|-------|-----|----------------------|
| 3   | (1,1,1) | 4   | $\frac{1}{4}(g_1^2 + g_1g_3 + 2g_4)$ |
| 4   | (2,1,1) | 5   | $\frac{1}{2}(2g_1^2g_3 + 7g_2g_3 + g_1(3g_2^2 + 7g_4 + 5g_5))$ |
| 4   | (2,1,1) | 7   | $\frac{1}{4}(2g_1^2g_3 + 11g_2g_3 + 17g_3g_4 + g_1(5g_2^2 + 13g_4) + 13g_2g_5 + 3g_1^2(11g_2g_3m - 9g_5) + g_1(7g_2^2 + 22g_3^2 + 36g_2g_4 + 23g_5) + 7g_7))$ |
| 5   | (2,2,1) | 6   | $g_1^2g_2 + g_1^2g_3 + 5g_2g_4 + g_1^2(2g_2^2 + 5g_4) + g_1(9g_2g_3 + 7g_5) + 3(g_3^2 + g_6)$ |
| 6   | (2,2,2) | 7   | $\frac{1}{6}(2g_1^2g_3 + 11g_2g_3 + 17g_3g_4 + g_1(5g_2^2 + 13g_4) + 13g_2g_5 + 3g_1^2(11g_2g_3m - 9g_5) + g_1(7g_2^2 + 22g_3^2 + 36g_2g_4 + 23g_5) + 7g_7))$ |
| 6   | (3,2,1) | 7   | $\frac{1}{6}(2g_1^2g_2 + 18g_2^2g_3 + 31g_3g_4 + g_1^2(5g_2^2 + 13g_4) + 18g_2g_5 + g_1^2(35g_2g_3 + 27g_5) + g_1(10g_2^2 + 22g_3^2 + 43g_2g_4 + 23g_6) + 7g_7)$ |

B.2 Exponential case (simple Hurwitz numbers)

Choosing the exponential function $G(z) = e^z$, with Taylor coefficients

$$g_i = \frac{1}{i!}$$  \hspace{1cm} (B.1)

as weight generating function corresponds \[^{15,16}\] to a Dirac measure for the weighted Hurwitz numbers supported uniformly on simple branch points,
with ramification profiles that are all 2-cycles:

$$(\mu^{(1)}, \ldots, \mu^{(k)}) = ((2, (1)^{N-2}), \ldots, (2, (1)^{N-2})), \tag{B.2}$$

the case considered in [28, 30]. Note that for this case $d = k$ and

$$H^d_{\text{exp}}(\mu) = \frac{1}{d!} H((2, (1)^{N-2}), \ldots, (2, (1)^{N-2}), \mu). \tag{B.3}$$

Tables 8-11 give the evaluations of $\tilde{H}^d_{\text{exp}}(\mu)$ and $H^d_{\text{exp}}(\mu)$ for this case.

Table 8: Connected and nonconnected weighted Hurwitz numbers $H^d_{\text{exp}}(\mu)$ with $\ell(\mu) = 1$ or 2, $N = 2, 3$ or 4

| $N$ | $\mu$ | $\tilde{H}^N_{\text{exp}}^{\ell(\mu)-2}$ | $\tilde{H}^N_{\text{exp}}^{\ell(\mu)}$ | $H^N_{\text{exp}}^{\ell(\mu)-2}$ | $H^N_{\text{exp}}^{\ell(\mu)}$ |
|-----|-------|----------------------------------------|----------------------------------------|-------------------------------|-------------------------------|
| 2   | (2)   | $\frac{1}{2}$                           | $\frac{1}{17}$                         | $\frac{1}{7}$                | $\frac{1}{12}$                |
| 3   | (3)   | $\frac{1}{2}$                           | $\frac{1}{7}$                         | $\frac{1}{7}$                | $\frac{2}{7}$                |
| 3   | (21)  | $\frac{3}{2}$                           | $\frac{3}{7}$                         | $\frac{3}{7}$                | $\frac{21}{3}$               |
| 4   | (4)   | $\frac{1}{2}$                           | $\frac{1}{7}$                         | $\frac{1}{7}$                | $\frac{1}{7}$                |
| 4   | (31)  | $\frac{1}{2}$                           | $\frac{3}{11}$                        | $\frac{3}{7}$                | $\frac{93}{3}$               |
| 4   | (22)  | $\frac{1}{2}$                           | $\frac{3}{11}$                        | $\frac{3}{7}$                | $\frac{121}{7}$              |

Table 9: Connected and nonconnected weighted Hurwitz numbers $H^d_{\text{exp}}(\mu)$ with $\ell(\mu) = 3$, $N = 3, 4, 5$ or 6

| $N$ | $\mu$ | $d$ | $\tilde{H}^d_{\text{exp}}(\mu)$ | $H^d_{\text{exp}}(\mu)$ |
|-----|-------|-----|----------------------------------|---------------------------|
| 3   | (1, 1, 1) | 4   | $\frac{1}{6}$                   | $\frac{1}{10}$           |
| 4   | (2, 1, 1) | 5   | $\frac{1}{6}$                   | $\frac{41}{30}$          |
| 4   | (2, 1, 1) | 7   | $\frac{13}{12}$                 | $\frac{23}{63}$          |
| 5   | (2, 2, 1) | 6   | $\frac{2}{2}$                   | $\frac{221}{180}$        |
| 6   | (2, 2, 2) | 7   | $\frac{4}{3}$                   | $\frac{9855}{5760}$      |
| 6   | (3, 2, 1) | 7   | $\frac{9}{7}$                   | $\frac{2011}{160}$       |
B.3 Rational case

The Taylor coefficients \( \{g_i(c, d)\}_{i \in \mathbb{N}^+} \) for rational weight generating functions \( G = G_{c,d} \), are given in eq. (2.14) in terms of the elementary and complete symmetric polynomials \( \{e_i(c)\} \) and \( \{h_i(d)\} \) in the parameters \( c = (c_1, \ldots, c_L) \) and \( d = (d_1, \ldots, d_M) \). Substituting these in Tables [4] - [7] we obtain the specialization to rationally weighted Hurwitz numbers.

In particular, the case of weakly monotonic (or signed) single Hurwitz numbers \((L = 0, M = 1, d_1 = 1)\) is obtained by substituting the values

\[
g_i = 1, \quad \forall i \in \mathbb{N}^+ \quad (B.4)
\]

in Tables [4] - [7]. The case \( L = 2, M = 0 \), gives the weighted enumeration of Belyi curves, with three branch points, two weighted ones, with ramification profiles \((\mu^{(1)}, \mu^{(2)})\) and one unweighted one, with profile \( \mu \). It is obtained by substituting

\[
g_1 = c_1 + c_2, \quad g_2 = c_1c_2, \quad g_i = 0, \quad \forall i \geq 3 \quad (B.5)
\]

in Tables [4] - [7].

B.4 Quantum case

For the case of the quantum weight generating function \( G = H_q \), the Taylor coefficients are given, by eq. (2.16),

\[
g_i = \frac{1}{(q;q)_i}. \quad (B.6)
\]

Substituting these in Tables [10] - [13] we obtain the specialization to quantum weighted Hurwitz numbers given in Tables [10][13].
Table 10: Nonconnected quantum weighted Hurwitz numbers $H^d_{G,H_q}(\mu)$ with $\ell(\mu) = 1$ or 2, $N = 2, 3$ or 4

| $N$ | $\mu$ | $H^N+\ell(\mu)-2$ | $H^N+\ell(\mu)$ |
|-----|-------|-------------------|-----------------|
| 2   | (2)   | $\frac{1}{(q;q)_1}$ | $\frac{1}{(q;q)_1}$ |
| 3   | (3)   | $\frac{2+q}{3(q;q)_3}$ | $10+5q+6q^2+5q^3+q^4$ |
| 3   | (21)  | $\frac{5+2q+2q^2}{3(q;q)_3}$ | $21+10q+14q^2+14q^3+4q^4+4q^5$ |
| 4   | (4)   | $\frac{5+5q+5q^2+q^3}{4(q;q)_4}$ | $5(14+14q+21q^2+24q^3+25q^4+14q^5+11q^6+4q^7+q^8)$ |
| 4   | (31)  | $\frac{25+20q+27q^2+23q^3+10q^4+3q^5}{3(q;q)_3}$ | $rac{84+77q+125q^2+156q^3+198q^4+191q^5+163q^6+12q^7+9q^8+52q^9+21q^{10}+10q^{11}+q^{12}}{4(q;q)_5}$ |
| 4   | (22)  | $\frac{10+10q+13q^2+12q^3+5q^4+2q^5}{4(q;q)_4}$ | $rac{231+231q+370q^2+409q^3+589q^4+579q^5+480q^6+376q^7+281q^8+161q^9+62q^{10}+30q^{11}+2q^{12}}{8(q;q)_6}$ |

Table 11: Connected quantum weighted Hurwitz numbers, $\tilde{H}^d_{G,H_q}(\mu)$ with $\ell(\mu) = 1$ or 2, $N = 2, 3$ or 4

| $N$ | $\mu$ | $\tilde{H}^N+\ell(\mu)-2$ | $\tilde{H}^N+\ell(\mu)$ |
|-----|-------|-----------------|-----------------|
| 2   | (2)   | $\frac{1}{(q;q)_1}$ | $\frac{1}{(q;q)_1}$ |
| 3   | (3)   | $\frac{2+q}{3(q;q)_3}$ | $10+5q+6q^2+5q^3+q^4$ |
| 3   | (21)  | $\frac{2+q+q^2}{3(q;q)_3}$ | $10+5q+6q^2+5q^3+q^4+2q^5+2q^6$ |
| 4   | (4)   | $\frac{5+5q+5q^2+q^3}{4(q;q)_3}$ | $5(14+14q+21q^2+24q^3+25q^4+14q^5+11q^6+4q^7+q^8)$ |
| 4   | (31)  | $\frac{5+5q+7q^2+6q^3+3q^4+q^5}{4(q;q)_4}$ | $70+70q+115q^2+145q^3+185q^4+180q^5+156q^6+120q^7+9q^8+51q^9+21q^{10}+10q^{11}+q^{12}$ |
| 4   | (22)  | $\frac{9+9q+12q^2+11q^3+5q^4+2q^5}{4(q;q)_4}$ | $\frac{114+114q+185q^2+212q^3+292q^4+287q^5+243q^6+18q^7+180q^8+80q^9+31q^{10}+15q^{11}+q^{12}}{4(q;q)_5}$ |
Table 12: Nonconnected quantum weighted Hurwitz numbers $H^d_{G_H^q}(\mu)$ with $\ell(\mu) = 3$, $N = 4, 5$ or 6.

| $N$ | $\mu$ | $d$ | $H^d_{G_H^q}(\mu)$ |
|-----|-------|-----|----------------------|
| 3   | (1,1,1) | 2   | $\frac{2(q;q)_2}{(q;q)_2}$ |
| 4   | (2,1,1) | 3   | $\frac{5^2+q+q^2}{4(q;q)_1}$ |
| 5   | (2,2,1) | 2   | $\frac{8(q;q)_2}{(q;q)_2}$ |
| 5   | (2,2,1) | 4   | $\frac{14+16q+21q^2+26q^3+9q^4+4q^5}{4(q;q)_1}$ |
| 6   | (3,2,1) | 3   | $\frac{(2+q)(1+q+q^2)}{6(q;q)_3}$ |
| 6   | (3,2,1) | 5   | $\frac{107+172q+287q^2+360q^3+409q^4+519q^5+133q^6+51q^7+11q^8}{6(q;q)_5}$ |

Table 13: Connected quantum weighted Hurwitz numbers $\tilde{H}^d_{G_H^q}(\mu)$ with $\ell(\mu) = 3$, $N = 4, 5$ or 6.

| $N$ | $\mu$ | $d$ | $\tilde{H}^d_{G_H^q}(\mu)$ |
|-----|-------|-----|-----------------------------|
| 3   | (1,1,1) | 4   | $\frac{1+2q+3q^2+2q^3+q^4}{3(q;q)_1}$ |
| 4   | (2,1,1) | 5   | $\frac{24+24q+39q^2+44q^3+44q^4+28q^5+28q^6+8q^7+3q^8}{2(q;q)_5}$ |
| 4   | (2,1,1) | 7   | $\frac{36+54q+99q^2+141q^3+189q^4+204q^5+204q^6+179q^7}{(q;q)_7}$ |
| 5   | (2,2,1) | 6   | $\frac{216+432q+891q^2+1485q^3+2295q^4+3484q^5+4416q^6+5199q^7+5199q^8+4157q^9+3863q^{10}}{(q;q)_7}$ |
| 6   | (2,2,2) | 7   | $\frac{240+480q+1002q^2+1664q^3+2584q^4+3484q^5+4416q^6+5288q^7+5139q^8+4687q^9+4157q^{10}}{(q;q)_7}$ |
| 6   | (3,2,1) | 7   | $\frac{363q^{11}+2990q^{12}+2063q^{13}+1311q^{14}+711q^{15}+338q^{16}+128q^{17}+32q^{18}+2q^{19}}{(q;q)_7}$ |

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