HIGHER CONNECTIVITY OF GRAPH COLORING COMPLEXES

SONJA LJ. ĆUKIĆ AND DMITRY N. KOZLOV

Abstract. The main result of this paper is a proof of the following conjecture of Babson & Kozlov:

Theorem. Let $G$ be a graph of maximal valency $d$, then the complex $\text{Hom}(G, K_n)$ is at least $(n - d - 2)$-connected.

Here $\text{Hom}(-, -)$ denotes the polyhedral complex introduced by Lovász to study the topological lower bounds for chromatic numbers of graphs.

We will also prove, as a corollary to the main theorem, that the complex $\text{Hom}(C_{2r+1}, K_n)$ is $(n - 4)$-connected, for $n \geq 3$.

1. Introduction.

Given a graph $G$ and a positive integer $n$, there is a construction of a topological space, which can be thought of as a space of all $n$-colorings of $G$. This space, denoted $\text{Hom}(G, K_n)$, is a polyhedral complex, whose set of vertices coincides with the set of all allowed vertex colorings of $G$ that use at most $n$ colors.

If $n < \chi(G)$, then this space is empty, otherwise, it possesses a meaningful topology. Intuitively, the cells of $\text{Hom}(G, K_n)$ of positive dimension encode "homotopies of colorings", i.e., in some sense, continuous procedures of changing one allowed coloring into another one. The general definition is given in subsection 2.2.

$\text{Hom}(-, -)$-complexes were introduced by Lovász, [10], to study topological lower bounds for chromatic numbers of graphs. The special case $\text{Hom}(K_2, G)$ is the motivating example, since it turns out to be homotopy equivalent to the previously well-studied neighborhood complex $\mathcal{N}(G)$. Recall that neighborhood complexes are the ones which were so spectacularly used by Lovász to resolve the Kneser Conjecture, see [8, 9].

Meanwhile, the other $\text{Hom}(-, -)$-complexes are not as well-understood. The family of complexes $\text{Hom}(C_{2r+1}, G)$ constitute one instance that has been studied lately, in connection with the proof by Babson & Kozlov of the Lovász Conjecture, [1, 2, 3].

It has also been recently proven by the authors, see [6], that for two arbitrary cycles $C_m$ and $C_n$, the connected components of the complex $\text{Hom}(C_m, C_n)$ are either points, or are homotopy equivalent to circles.

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The intuition tells us that, if the maximal valency of the graph \( G \) is small and the number \( n \) is relatively large, then there should be a lot of freedom in completing the \( n \)-colorings of \( G \) locally. Expressed topologically, we may hope that the complexes \( \text{Hom}(G, K_n) \) will be highly connected.

To say that \( \text{Hom}(G, K_n) \) is \((-1)\)-connected is the same as to say that it is nonempty. It is well-known that a sufficient condition is provided by requiring that the maximal valency of \( G \) is at most \( n - 1 \) (the most primitive coloring procedure works). Next, it was shown in [2, Proposition 2.4] that, if the maximal valency of \( G \) is at most \( n - 2 \), then \( \text{Hom}(G, K_n) \) is connected (= 0-connected).

Motivated by these special cases and by some further computational evidence, Babson & Kozlov made the conjecture which initiated our present study.

**Conjecture 1.1.** [2, Conjecture 2.5].

The following inequality is valid for an arbitrary graph \( G \):

\[
\text{conn} \text{Hom}(G, K_n) \geq n - \text{maxval}(G) - 2.
\]

In other words: for an arbitrary graph \( G \), if \( \text{maxval}(G) < n - k \), then \( \text{Hom}(G, K_n) \) is \((k - 1)\)-connected.

The main purpose of this paper is to present a proof of this conjecture. First, we analyze the situation for \( n \geq \text{maxval}(G) + 3 \). In this case, we find an explicit algorithm for deforming an arbitrary loop inside \( \text{Hom}(G, K_n) \) to a point, thereby verifying the triviality of the fundamental group.

Not surprisingly, it is virtually impossible to generalize these explicit homotopies to higher dimensions. Instead, we choose to use Hurewicz theorem, and calculate the nullity of the appropriate homology groups instead. Again, we give an explicit algorithmic procedure for reducing an arbitrary cycle to zero, by means of adding to it appropriately chosen boundaries.

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2. **Basic notations and definitions.**

2.1. **Notations and terminology.**

For an arbitrary natural number \( n \) we introduce a shorthand notation: \([n]=\{1,2,\ldots,n\}\).

For any graph \( G \), we denote the set of its vertices by \( V(G) \), and the set of its edges by \( E(G) \), where \( E(G) \subseteq V(G) \times V(G) \). In this paper we will consider only undirected graphs, so \((x,y)\in E(G)\) implies that \((y,x)\in E(G)\). Also, our graphs are finite and do not contain loops.

Let \( G \) be a graph and \( S \subseteq V(G) \). Then we denote by \( G[S] \) the graph with \( V(G[S])=S \) and \( E(G[S])=(S\times S)\cap E(G) \). The graph \( G[V(G)\setminus S] \) we will denote by \( G-S \).

For a graph \( G \) and \( v\in V(G) \), let \( N(v)=\{w\in V(G)\mid (v,w)\in E(G)\} \), \( N(v) \) is the set of neighbors of \( v \). Denote the valency of \( v \) by \( d(v) \), clearly \( d(v)=|N(v)| \). Also denote the maximal valency of \( G \), \( \text{maxval}(G) \).

For a graph \( G \) and a natural number \( n \), let \( \text{vgap}(G,n) \) denote what we call the \( n \)-color-valency gap, namely, set \( \text{vgap}(G,n):=n-\text{maxval}(G)-1 \). With these notations, the equation (1.1) can be rewritten as:

\[
\text{conn} \text{Hom}(G, K_n) \geq \text{vgap}(G, n) - 1.
\]
In other words, the first possibly nontrivial homotopy group of $\text{Hom}(G, K_n)$ is indexed by the $n$-color-valency gap, $\text{vgap}(G, n)$.

Denote with $K_n$, and $C_n$ the complete graph (no loops), resp. the cycle with $n$ vertices, i.e., $V(K_n) = V(C_n) = [n]$, and $E(K_n) = \{(x, y) \mid x, y \in [n], x \neq y\}$, 
$E(C_n) = \{(x, x+n), (x+n, x) \mid x \in [n]\}$. The maximal valency of $C_n$ is 2, for $n \geq 3$, while the maximal valency of $K_n$ is $n - 1$.

For any two graphs $G$ and $H$, let $G \coprod H$ denote the disjoint union of these graphs.

2.2. $\text{Hom}$ complexes: definition, examples and basic properties.

A standard generalization of graph colorings is provided by the following definition.

**Definition 2.1.** For two graphs $G$ and $H$, a graph homomorphism from $G$ to $H$ is a map $\phi : V(G) \to V(H)$ such that if $(x, y) \in E(G)$, then $(\phi(x), \phi(y)) \in E(H)$.

We denote the set of all homomorphisms from $G$ to $H$ by $\text{Hom}_0(G, H)$.

**Definition 2.2.** [$\text{Hom}_0(G, H)$]

$\text{Hom}(G, H)$ is a polyhedral complex whose cells are indexed by all functions $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $(x, y) \in E(G)$, then for all $x \in \eta(x)$ and $y \in \eta(y)$, $(x, y) \in E(H)$.

The closure of a cell $\eta$ consists of all cells indexed by $\bar{\eta} : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ which satisfy the condition that $\bar{\eta}(v) \subseteq \eta(v)$, for all $v \in V(G)$.

**Note.** We follow $[\text{1}]$ in our notations.

The set of vertices of $\text{Hom}(G, H)$ is $\text{Hom}_0(G, H)$. We note that cells of $\text{Hom}(G, H)$ are direct products of simplices, and that the dimension of a cell $\eta$ is equal to $\sum_{v \in V(G)} |\eta(v)| - |V(G)|$.

One can describe a labeling of the cells in $\text{Hom}(G, K_n)$ rather directly: they are indexed by all $p$-tuples of nonempty subsets of $[n]$, $(A_1, \ldots, A_p)$, where $p = |V(G)|$, and such that, if $(i, j)$ is an edge in $G$, then $A_i \cap A_j = \emptyset$. Also, for a cell $\eta \in \text{Hom}(G, K_n)$, we will denote with $A^\eta_j$ its $j$-th coordinate set (and sometimes we will refer to it as $j$-th color list of $\eta$). With these notations, $\eta = (A^\eta_1, \ldots, A^\eta_p)$.

For any three graphs $G$, $H$, and $K$, the following is true:

\begin{equation}
\text{Hom}(G \coprod H, K) = \text{Hom}(G, K) \times \text{Hom}(H, K).
\end{equation}

Two examples of complexes $\text{Hom}(G, K_n)$ are shown on Figure $[\text{1}]$ Note that $\text{Hom}(K_{1,3}, K_3)$ is a 3-dimensional complex, with 3 solid cubes and 3 intervals. Further examples can be found in $[\text{2}]$.

3. The fundamental group of $\text{Hom}(G, K_n)$.

We are now ready to study the fundamental group of the graph coloring complexes.

Let $\lambda(G)$, resp. $p(G)$, denote the cardinality of a maximal independent set in $G$, resp. the number of vertices of $G$. We shall write $\lambda$ and $p$ whenever it is clear which $G$ is meant. For an arbitrary graph $G$, label the vertices with $x_1, \ldots, x_p$, such that $\{x_1, \ldots, x_\lambda\}$ is a maximal independent set.

**Lemma 3.1.** Let $G$ be an arbitrary graph, and $n \geq 2$. Assume that $\text{vgap}(G, n) \geq 2$. For an arbitrary $1 < i \leq n$, any closed edge-path in the 1-skeleton of $\text{Hom}(G, K_n)$, such that the first $\lambda$ coordinates of every vertex in this path are elements of the
set \( \{i - 1,i,\ldots,n\} \), can be deformed by subsequent homotopies so that these \( \lambda \) coordinates will be elements of the set \( \{i,i + 1,\ldots,n\} \).

Proof. Each vertex of the complex \( \text{Hom}(G,K_n) \) can be described by \( p \)-tuple \( (a_1,a_2,\ldots,a_p) \), where \( a_d \in [n] \), for \( d \in [p] \).

First of all, in any closed edge-path \( u_1,u_2,\ldots,u_s = u_1 \) we may assume that successive vertices are distinct. Otherwise, if \( u_d = u_{d+1} \) for some \( d \), we could delete \( u_d \) and obtain a homotopic path. Hence, from now on, we shall implicitly replace every path with a homotopic path where no two successive vertices are equal.

We will give an algorithm which performs the deformation, the existence of which is claimed in our lemma. Formal description of steps, together with proofs of their correctness, will be given after description of the algorithm.

Let \( u^{(\lambda+1)}_1,u^{(\lambda+1)}_2,\ldots,u^{(\lambda+1)}_m = u^{(\lambda+1)}_1 \) be the vertices of this path. Note that each \( u^{(\lambda+1)}_k \), for \( k \in [m-1] \), is described by a \( p \)-vector and \( u^{(\lambda+1)}_k \) differs from \( u^{(\lambda+1)}_{k+1} \) in exactly one component.

The input for our algorithm is the path \( u^{(\lambda+1)} \), and the algorithm consists of two parts. In the first part, we repeat both of the following steps for all \( j = \lambda + 1,\ldots,p \), in increasing order.

**Step 1:** Inserting two new vertices between all those neighboring pairs \( u^{(j)}_k \) and \( u^{(j)}_{k+1} \) from the path \( u^{(j)} \) which both have \( i \) on \( j \)-th position; result of step 1 is a path \( v^{(j)} \).

**Step 2:** Deleting those vertices from the path \( v^{(j)} \) which use color \( i \) on the graph vertex \( x_j \); result of this step is the path \( u^{(j+1)} \).

After the first phase, we obtain a path \( u^{(p+1)} \) with the property that colors used on vertices \( x_{\lambda+1},\ldots,x_p \) are elements of the set \( [n] \setminus \{i\} \).
Now, input for the next stage is the path $w^{(1)} = u^{(p+1)}$, and we repeat both steps 3 and 4 for all $j = 1, \ldots, \lambda$, again in the increasing order.

**Step 3:** Inserting two new vertices between those neighboring vertices of the path $w^{(j)}$, $u_k^{(j)}$ and $u_{k+1}^{(j)}$, which both have $i - 1$ on $j$-th position; result of step 3 is the path $q^{(j)}$.

**Step 4:** Deleting those vertices from the path $q^{(j)}$ which use color $i - 1$ on $x_j$; result of this step is the path $w^{(j+1)}$.

Output of this algorithm is a path $w^{(\lambda+1)}$, and for each vertex of this path, colors used on $x_1, \ldots, x_\lambda$ are from the set $\{i, i+1, \ldots, n\}$.

Now we give detailed description of all the steps.

**Step 1.** If $i$ occurs in $j$-th position for both $u_k^{(j)}$ and $u_{k+1}^{(j)}$, we will “separate” $u_k^{(j)}$ and $u_{k+1}^{(j)}$ by adding new vertices to the given path. We have

\[
\begin{align*}
  u_k^{(j)} &= (\ldots, x, \ldots, i, \ldots) \\
  u_{k+1}^{(j)} &= (\ldots, y, \ldots, i, \ldots)
\end{align*}
\]

where $x \neq y$. We have indicated only $i$ and components which are different in these two vertices. Let

\[ A = \{\mu\text{-th coordinates of } u_k^{(j)}\} \cup \{\mu\text{-th coordinates of } u_{k+1}^{(j)}\}, \]

where both sets are indexed with those $\mu$, for which $x_{\mu} \in N(x_j)$. Obviously, we have $|A \cup \{i\}| \leq \maxval(G) + 2$ and, since $n \geq \maxval(G) + 3$, there exists $z \in [n] \setminus (A \cup \{i\}) \neq \emptyset$, such that

\[
\begin{align*}
  U_k^{(j)} &= (\ldots, x, \ldots, z, \ldots) \\
  U_{k+1}^{(j)} &= (\ldots, y, \ldots, z, \ldots)
\end{align*}
\]
are vertices of $\text{Hom}(G, K_n)$. In another words, we have a vertex of $G$ of bounded

degree, and two colorings that differ in only one vertex, so we find an alternative

color left for the vertex $x_j$.

Deformation of the subpath $u_k^{(j)}u_{k+1}^{(j)}$ into $u_k^{(j)}U_k^{(j)}u_{k+1}^{(j)}u^{(j)}_{k+1}$ is a homotopy over
the 2-cell $(\ldots, \{x, y\}, \ldots, \{z, i\}, \ldots)$, see Figure 3.

![Figure 3.](image)

Let $v^{(j)} = v_1^{(j)}, v_2^{(j)}, \ldots, v_1^{(j)}$ be the path obtained by the first step.

**Step 2.** In this step we will remove from the path $v^{(j)}$ those vertices which have $i$
in the $j$-th position. In this case, as a result of the first step, we have the following
situation:

$$v_{k-1}^{(j)} = (\ldots, x, \ldots)$$

$$v_k^{(j)} = (\ldots, \ i, \ \ldots)$$

$$v_{k+1}^{(j)} = (\ldots, y, \ \ldots)$$

where $x, y \neq i$.

Deletion of $v_k^{(j)}$ is either a homotopy over the 2-cell $(\ldots, \{x, y, i\}, \ldots)$, or, if
$x = y$, a homotopy over the 1-cell $(\ldots, \{x, i\}, \ldots)$, see Figure 4.

![Figure 4.](image)

Let $u^{(j+1)} = u_1^{(j+1)}, u_2^{(j+1)}, \ldots, u_1^{(j+1)}$ be the path obtained after this step. It is
clear that this path does not have any vertices which have $i$ in the $j$-th position.
After the first stage of our algorithm, we obtain a path $u^{(p+1)}$. Set $w^{(1)} = u^{(p+1)}$, that is $w^{(1)} = w_1^{(1)}, w_2^{(1)}, \ldots, w_i^{(1)}$, where $w_d^{(1)} = u_d^{(p+1)}$ for all $d \in [l-1]$ ($l$ is the length of the path $u^{(p+1)}$). In this path, $i$ is not in $j$-th position of any vertex, where $j \in \{1, \ldots, p\}$.

**Step 3.** If $i-1$ occurs in $j$-th position for both $w_k^{(j)}$ and $w_{k+1}^{(j)}$ we will, similar to the first step, “separate” them by adding new vertices to the given path. We start with

$$w_k^{(j)} = (\ldots, \underline{i-1}, \ldots, x, \ldots)$$

$$w_{k+1}^{(j)} = (\ldots, i-1, \ldots, y, \ldots)$$

where $x \neq y$. Since vertices labeled $x_1, \ldots, x_3$ form an independent set, we know that $N(x_j) \subseteq \{x_{\lambda+1}, \ldots, x_p\}$. Hence, the color $i$ does not occur among the neighbors of $x_j$ in $w_k^{(j)}$, or in $w_{k+1}^{(j)}$. Therefore,

$$W_k^{(j)} = (\ldots, \underline{i}, \ldots, x, \ldots)$$

$$W_{k+1}^{(j)} = (\ldots, i, \ldots, y, \ldots)$$

are legal $n$-colorings, and hence are also vertices of $\text{Hom}(G, K_n)$. Then, the deformation of the subpath $w_k^{(j)}w_{k+1}^{(j)}$ into $w_k^{(j)}W_k^{(j)}W_{k+1}^{(j)}w_{k+1}^{(j)}$ is a homotopy over the 2-cell $(\ldots, \{i-1, i\}, \ldots, \{x, y\}, \ldots)$.

Let now $q^{(j)} = q_1^{(j)}, q_2^{(j)}, \ldots, q_{l-1}^{(j)}$ be the path obtained by the third step.

**Step 4.** Similar to the Step 2, all the vertices from the path $q^{(j)}$ which have $i-1$ in the $j$-th position are deleted:

$$q_{k-1}^{(j)} = (\ldots, \underline{x}, \ldots)$$

$$q_k^{(j)} = (\ldots, i-1, \ldots)$$

$$q_{k+1}^{(j)} = (\ldots, y, \ldots)$$

where $x, y \neq i-1$.

Deletion of $q_k^{(j)}$ is homotopy over the 2-cell $(\ldots, \{x, y, i-1\}, \ldots)$ (or, if $x = y$, over 1-cell $(\ldots, \{x, i-1\}, \ldots)$).

Let $w^{(j+1)} = w_{l+1}^{(j+1)}, w_{l+2}^{(j+1)}, \ldots, w_1^{(j+1)}$ be the obtained path. We can see that after this step $j$-th coordinate of any vertex from this path is in the set $\{i, \ldots, n\}$.

Finally, we arrive at the path $w^{(\lambda+1)}$ which has been obtained by a sequence of elementary homotopies from the original path and has the additional property that the first $\lambda$ coordinates of every vertex from this path are elements of the set $\{i, \ldots, n\}$.

\[\Box\]

**Note.** This proof was motivated by the ideas from [4].

We would like to remark, that the Steps 1 and 2 can be combined to obtain one reduction step encoding the following transformation: we find the first vertex where $i$ occurs in $j$-th position of two vertices in a row, then we glue in a square, as in Step 1, and then we clip off a triangle, as in Step 2, see Figure 5.
The outcome of this procedure will be a shortening of the undesired part (here meaning $i$ is in the $j$-th position) by 1. With this line of argument, one has two special cases to attend to. First, if the undesired vertices come only as singletons, then we clip them all off as in Step 2. Second, if all vertices are undesired, then gluing in an arbitrary square, as in Step 1, reduces this to the case which we considered first.

The Lemma 3.1 is the crucial step in the proof of the main result of this chapter.

**Theorem 3.2.** Let $G$ be any graph. If $\text{vgap}(G,n) \geq 2$, then $\text{Hom}(G,K_n)$ is simply connected.

**Proof.** We will prove that $\text{Hom}(G,K_n)$ is simply connected using induction on the maximal valency of $G$. We will use the same notations as in the previous lemma.

Suppose that the maximal valency of a graph $G$ is 0 and that $n \geq 3$. Then $G$ is disjoint union of $p$ points and, using Lemma 3.1, we conclude:

$$\text{Hom}(G,K_n) = \text{Hom}(K_1,K_n) \times \text{Hom}(K_1,K_n) \times \cdots \times \text{Hom}(K_1,K_n).$$

$\text{Hom}(K_1,K_n)$ is a simplex, so $\text{Hom}(G,K_n)$ is contractible, and hence 1-connected.

Assume now that the maximal valency of $G$ is equal to $d \geq 1$ and that $\text{vgap}(G,n) \geq 2$. Let $\beta_1, \beta_2, \ldots, \beta_1$ be any closed edge-path in 1-skeleton of $\text{Hom}(G,K_n)$. Since $\text{Hom}(G,K_n)$ is a polyhedral complex, it is clear that it is sufficient to consider only these paths.

Using Lemma 3.1 iteratively, we can homotopically transform this path to a path $\alpha_1, \alpha_2, \ldots, \alpha_1$ which has the property that the first $\lambda$ coordinates of any vertex from this path are equal to $n$, where $\{x_1, x_2, \ldots, x_\lambda\}$ is a maximal independent set in $G$. Hence, the original path is homotopical to a path lying inside the subcomplex $\text{Hom}(G - \{x_1, x_2, \ldots, x_\lambda\}, K_{n-1})$. That path is contractible to a point by induction hypothesis, since $\{x_1, \ldots, x_\lambda\}$ is a maximal independent set, and therefore the maximal valency of $G - \{x_1, x_2, \ldots, x_\lambda\}$ is strictly less than $d$. □

We will now evaluate the fundamental group of $\text{Hom}(G,K_{n+1})$ for one case when the maximal valency of a graph $G$ equals $n - 1$. 
Proposition 3.3. $\pi_1(\text{Hom}(K_n, K_{n+1}))$ is the free product of $\alpha_n$ copies of $\mathbb{Z}$, where $\alpha_n = n!\frac{n^2-n-2}{2} + 1$.

Proof. Since $\text{Hom}(K_n, K_{n+1})$ is a connected graph, we can choose a spanning tree and contract it. Then we get a bouquet of $e - v + 1$ circles, where $v$ is the number of vertices and $e$ number of edges of this graph. It is easy to see that $v = (n + 1)!$ and, since $\text{Hom}(K_n, K_{n+1})$ is $n$-regular, $e = \frac{n^2}{2}$. This gives us the claim of the proposition. \qed

4. Homology groups of $\text{Hom}(G, K_n)$ complexes and the main theorem.

In the previous section we proved that, if the maximal valency of $G$ is $d$, then $\text{Hom}(G, K_n)$ is simply connected (or 1-connected), for all $n \geq d + 3$. Here we will prove a more general statement using the notations and ideas from Lemma 3.1 and Theorem 3.2.

First, we will introduce a new notation. Let $G$ be a graph with the set of vertices $V(G) = \{x_1, \ldots, x_p\}$ and let $1 \leq j \leq p$ be an integer. Then, for all $i \in [n]$, we define a subcomplex $X_i(G, j)$ of $\text{Hom}(G, K_n)$ in the following way:

$$X_i(G, j) = \{\eta \in \text{Hom}(G, K_n) \mid \eta(x_q) \subseteq \{i, i+1, \ldots, n\}, \text{ for all } q \in [j]\}.$$ 

In other words, only colors $i, \ldots, n$ have been used in the first $j$ vertices. For example, Lemma 3.1 is exactly pushing the loops from $X_{i-1}(G, \lambda)$ into the subcomplex $X_i(G, \lambda)$. If it is clear which graph $G$ is meant, we will use the notation $X_i(j)$ instead of $X_i(G, j)$.

Let us again label the vertices of $G$ in the same way as we did in Lemma 3.1 that is so that the vertices labeled $x_1, \ldots, x_\lambda$ form a maximal independent set.

Lemma 4.1. Let $G$ be a graph with the maximal valency equal to $d \geq 1$. If $C$ is a $t$-cycle in $X_{i-1}(\lambda)$, where $1 \leq t \leq n - d - 2$ and $i \in \{2, 3, \ldots, n\}$, then there exist a $t$-cycle $C'$ in $X_i(\lambda)$ such that $C$ and $C'$ represent the same element in $H_t(\text{Hom}(G, K_n), \mathbb{Z})$.

Proof. Recall that a cell $\eta$ from $\text{Hom}(G, K_n)$ can be described by the $p$-tuple $(A_1^\eta, A_2^\eta, \ldots, A_p^\eta)$, where $A_j^\eta \subseteq [n]$ for all $j \in [p]$ ($p = |V(G)|$). Rephrasing the definition above, we get

$$X_\lambda(\lambda) = \{\eta \in \text{Hom}(G, K_n) \mid A_1^\eta, \ldots, A_\lambda^\eta \subseteq \{q, q + 1, \ldots, n\}\},$$

for $q \in [n]$. The orientation of the cells $\eta \in \text{Hom}(G, K_n)$ can be chosen so that the boundary operator $\partial$ is given by

$$\partial(\eta) = \partial(A_1^\eta, \ldots, A_p^\eta) = \sum (-1)^{c(x)}(A_1^\eta, \ldots, A_q^\eta \setminus \{x\}, \ldots, A_p^\eta),$$

where $s(x)$ denotes the index, for which $x \in A_s^\eta(x)$, $c(x) = |A_1^\eta| + \cdots + |A_{s(x)-1}^\eta| + |\{y \mid y \in A_s^\eta(x) \text{ and } y < x\}|$, and the sum is taken over all $x \in A_1^\eta \cup \cdots \cup A_p^\eta$, such that $|A_{s(x)}^\eta| \geq 2$.

Like in the proof of Lemma 3.1 we give a description of an algorithm whose output is a cycle $C'$, whose existence is claimed in this lemma. General scheme of the proof is the same as for already mentioned Lemma 3.1. Namely, we first get rid of color $i$ on vertices $x_{\lambda+1}, \ldots, x_p$, and afterwards we are removing color $i - 1$ from color lists on all vertices from the chosen maximal independent set, see Figure 2.

Input for our algorithm is the $t$-cycle $C_{\lambda+1} = C$. For each $j = \lambda + 1, \ldots, p$, in the increasing order of $j$, we repeat both of the next two inductive steps.
Step 1: This is the only step in our proof where we use the assumption about
maximal valency of \( G \). In this step we eliminate all cells \( \eta \) from the cycle
\( C_j \) such that \( A^\eta_j = \{ i \} \). Result of this step is a cycle \( C^j_2 \) with the property
that \( j \)-th color list of each cell from \( C^j_2 \) which contains \( i \) has length \( \geq 2 \).

Now we have new, iterative, step. Namely, step 2 is repeated for cycles \( C^j_2 \),
starting from \( l = 2 \), until there exists a cell \( \sigma \in C^j_2 \) such that \( i \in A^\sigma_j \).

Step 2: Transforming the cycle \( C^j_2 \) to a homologous cycle \( C^{j+1}_2 \) with the prop-erty
that, for all \( \eta \in C^{j+1}_2 \), if \( i \in A^\eta_j \), then \( |A^\eta_j| \geq l + 1 \).

Since length of each color list is certainly less or equal to \( n \), repetition of step
2 will stop after finite number of iterations, and we will get a cycle \( C_{j+1} \).

Input for the second stage of this algorithm is the cycle \( C_1 = C_{p+1} \). For each cell
\( \sigma \) of this cycle, \( i \notin A^\sigma_{p+1} \). Again, we repeat the following inductive steps
for each \( j = 1, \ldots, \lambda \), in the increasing order:

Step 3: We eliminate all cells \( \eta \) from the cycle \( C_j \) such that \( A^\eta_j = \{ i - 1 \} \).

Result of this step is a cycle \( C^j_2 \) with the property that \( j \)-th color list of
each cell from \( C^j_2 \) which contains \( i - 1 \) has length \( \geq 2 \).

Step 2 is repeated for cycles \( C^j_2 \), starting from \( l = 2 \), until there exists a cell
\( \sigma \in C^j_2 \) such that \( i - 1 \in A^\sigma_j \).

Step 4: Transforming the cycle \( C^j_2 \) to a cycle \( C^{j+1}_2 \), such that \( C^j_2 \) and \( C^{j+1}_2 \)
represent the same homology element, and such that, if \( i - 1 \in A^\eta_j \), then
\( |A^\eta_j| \geq l + 1 \), for all cells \( \eta \in C^{j+1}_2 \).

By the same argument as in the first part, repetition of step 4 will stop after
finite number of iterations, and we will get a cycle \( C_{j+1} \). In the case when
\( j = \lambda \), we label the resulting cycle with \( C' \).

After describing the algorithm, we give a detailed description of steps:

Step 1. Let \( \eta \) be a \( t \)-cell appearing in the chain \( C_j \), such that \( A^\eta_j = \{ i \} \), and let \( \eta \)
have the coefficient \( k \in \mathbb{Z} \) in \( C_j \). The dimension of \( \eta \) is equal to \( t \), and we have:

\[
t = \sum_{r=1}^{p} (|A^\eta_r| - 1) = \left( \sum_{r \mid x_r \in \mathbb{N}(x_j)} |A^\eta_j| - |\mathbb{N}(x_j)| \right) + \left( |A^\eta_j| - 1 \right) +
\]
\[
+ \left( \sum_{r \mid x_r \notin \mathbb{N}(x_j), \ r \neq j} |A^\eta_r| - (p - |\mathbb{N}(x_j)| - 1) \right) \geq \sum_{r \in [p]} |A^\eta_r| - d,
\]
since the second term in the middle sum is 0, the third term in the middle sum is
nonnegative, and \( |\mathbb{N}(x_j)| \leq d \). It follows that

\[
\sum_{r \in [p], x_r \in \mathbb{N}(x_j)} |A^\eta_r| \leq t + d \leq n - 2;
\]
as we recall that, by assumption of the lemma, \( t \leq n - d - 2 \). Hence, there exists
\( I(\eta) \in [n] \setminus \{ i \} \) such that \( \eta' = (A^\eta_0, \ldots, A^\eta_{p-1}, \{ i, I(\eta) \}, \ A^\eta_{p+1}, \ldots, A^\eta_p) \in \mathbb{H}om(G, K_n) \).

Obviously, \( \dim \eta' = t + 1 \). Set \( C'_j = C_j - (-1)^{K(\eta')} k \partial \eta' \), where \( (-1)^{K(\eta')} \) comes
from the appropriate incidence number, i.e., in our notations
\[ K(\eta^i) = |A^q_i| + \cdots + |A^q_{j-1}| + \begin{cases} 1, & \text{if } i < I(\eta); \\ 0, & \text{otherwise.} \end{cases} \]

Clearly, \( C'_j \) represents the same homology element as \( C_j \). Furthermore, the number of cells \( \sigma \) appearing in \( C'_j \), such that \( A^q_\gamma = \{i\} \), is strictly less than number of such cells appearing in \( C_j \), since we have just eliminated the appearance of the cell \( \eta \). We repeat this procedure until we get cycle \( C'_j \) in which there are no cells \( \sigma \) such that \( A^q_\gamma = \{i\} \).

**Step 2.** Suppose now that we have a cycle \( C'_j \), \( l \geq 2 \), which represents the same element in homology as \( C_j \), and which has the additional property that \(|A^q_\gamma| \geq l\), for each cell \( \eta \) appearing in \( C'_j \), such that \( i \in A^q_\gamma \). If no such cell exists, we are done with this case, and we set \( C_{j+1} := C'_j \). Since \( l \leq n \), we will always come to this case after a finite number of steps.

Assume now there exists a cell \( \eta \) such that \( i \in A^q_\gamma \), and let us construct \( C'^{l+1}_j \). Assume further that \( C'_j = \sum_{q \in Q} k_q \sigma_q \), where \( Q \subseteq \mathbb{N} \), \( k_q \in \mathbb{Z} \), \( k_q \neq 0 \), and \( \sigma_q \)'s are pairwise different \( t \)-cells from \( \mathrm{Hom}(G, K_n) \). Then we can write \( C'_j = D_l + D_{>l} + D_0 \), where

\[
D_l = \sum_{q \in Q_l} k_q \sigma_q \quad \text{where } Q_l = \{q \in Q \mid i \in A^q_\gamma \text{ and } |A^q_\gamma| = l\},
\]

\[
D_{>l} = \sum_{q \in Q_{>l}} k_q \sigma_q \quad \text{where } Q_{>l} = \{q \in Q \mid i \in A^q_\gamma \text{ and } |A^q_\gamma| > l\},
\]

\[
D_0 = \sum_{q \in Q_0} k_q \sigma_q \quad \text{where } Q_0 = \{q \in Q \mid i \notin A^q_\gamma\}.
\]

If \( D_l = 0 \), then we set \( C'^{l+1}_j = C'_j \), so assume \( D_l \neq 0 \).

Clearly, \( \partial C'^{l+1}_j = \partial C'_j \), where \( \partial \) is a subchain consisted of all cells \( \sigma \) from the boundary such that \( |A^q_\gamma| = l \), \( \gamma \) is a chain of all cells from \( \partial D_l \) such that \( j \)-th set has \( l-1 \) elements and contains \( i \); finally \( \delta \) consists of all cells where we do not have \( i \) in \( j \)-th coordinate set. Since \( 0 = \partial C'_j \) and the chain \( \partial D_{>l} + \partial D_0 \) cannot contain any cell \( \sigma \), such that \( i \in A^q_\gamma \) and \( |A^q_\gamma| = l-1 \), we conclude that \( \gamma = 0 \).

Furthermore, let us fix the sets \( A_b \), for all \( b \in [p] \setminus \{j\} \). We denote by \( \gamma_{A_1,\ldots,A_j,\ldots,A_p} \) the chain of all cells \( \xi = (A^\xi_1, \ldots, A^\xi_p) \) from \( \gamma \) such that \( A^\xi_b = A_b \), for all \( b \in [p] \setminus \{j\} \). Obviously, all these parts \( \gamma_{A_1,\ldots,A_j,\ldots,A_p} \) must also be 0.

Let us also consider the corresponding subchains
\[
D_{l:A_1,\ldots,A_j,\ldots,A_p} = \sum_{q \in Q_{l:A_1,\ldots,A_j,\ldots,A_p}} k_q \sigma_q
\]
in the chain \( D_l \), where
\[
Q_{l:A_1,\ldots,A_j,\ldots,A_p} = \{q \in Q_l \mid A^q_\gamma = A_b, \text{ for all } b \in [p] \setminus \{j\}\}.
\]

It is clear that \( \gamma_{A_1,\ldots,A_j,\ldots,A_p} \) must be the part of the boundary of \( D_{l:A_1,\ldots,A_j,\ldots,A_p} \), where the deleted element is in \( A^q_\gamma \), and is different from \( i \).

Let us construct a new cycle \( C'_j \), homologous to \( C'_j \), with the property that for all cells \( \xi \) appearing in it with \( i \in A^\xi_\gamma \) and \( A^\xi_b = A_b \), for \( b \in [p] \setminus \{j\} \), we have
\(|A^q_j| \geq l + 1\), that is with the corresponding \(D_{1;A_1,\ldots,\hat{A_j},\ldots,A_p} = 0\).

- **Case** \(l = 2\): Let \(\eta\) be a cell from \(D_{2;A_1,\ldots,\hat{A_j},\ldots,A_p}\) with coefficient \(k \neq 0\), and let \(A^q_j = \{i, x\}\), for some \(x \in [n] \setminus \{i\}\). Since \(\gamma_{A_1,\ldots,\hat{A_j},\ldots,A_p} = 0\), there must exist another cell \(\xi\) in \(D_{2;A_1,\ldots,\hat{A_j},\ldots,A_p}\), such that \(A^q_j = \{i, y\}\), for \(y \in [n] \setminus \{i, x\}\). Let us denote \((A_1,\ldots,A_{j-1}, \{i, x, y\}, \ldots, A_p)\) with \(\sigma\). It is clear that \(\sigma \in \text{Hom}(G, K_n)\).

Also, the number of cells of the corresponding \(D_{2;A_1,\ldots,\hat{A_j},\ldots,A_p}\) for the cycle \(\tilde{C}_2^q = C_2^q - (-1)^k k \partial \sigma\), where

\[
K = |A^q_j| + \cdots + |A^q_{j-1}| + \begin{cases} 
1, & \text{if } i < y < x \text{ or } x < y < i; \\
0, & \text{otherwise},
\end{cases}
\]

is reduced at least by one. Repeating this procedure, we will eventually get a chain \(\tilde{C}_2^q\) with the corresponding \(D_{2;A_1,\ldots,\hat{A_j},\ldots,A_p}\) equal to zero.

- **Case** \(l \geq 3\): Define a map \(f_{A_1,\ldots,\hat{A_j},\ldots,A_p} : X_{A_1,\ldots,\hat{A_j},\ldots,A_p} \to \Delta_i\), where \(X_{A_1,\ldots,\hat{A_j},\ldots,A_p}\) is equal to

\[
\{\eta \in \text{Hom}(G, K_n) \mid A^q_j = A_b, \text{ for } b \in [p] \setminus \{j\}, i \in A^q_j, \text{ and } |A^q_j| \geq 2\},
\]

and \(\Delta_i\) is simplex with the vertex set \([n] \setminus \{i\}\), in the following way: for a cell \(\eta \in X_{A_1,\ldots,\hat{A_j},\ldots,A_p}\), set

\[
f_{A_1,\ldots,\hat{A_j},\ldots,A_p}(\eta) = (-1)^{|\{y \in A^q_j \text{ and } y > i\}|} (A^q_j \setminus \{i\})
\]

and then extend it by linearity. Function \(f_{A_1,\ldots,\hat{A_j},\ldots,A_p}\) is clearly a bijection between cells and \(f_{A_1,\ldots,\hat{A_j},\ldots,A_p} : D_{l;A_1,\ldots,\hat{A_j},\ldots,A_p}\) is a \((l-2)\)-chain in \(\Delta_i\). Let us now prove that this chain is in fact a cycle.

For \(q \in Q_{l;A_1,\ldots,\hat{A_j},\ldots,A_p}\), let \(A^q_j = [v^q_1, \ldots, v^q_{s(q)}; i, v^q_{s(q)+1}, \ldots, v^q_{l-1}]\) where \(v^q_1 < \cdots < v^q_{s(q)} < i < v^q_{s(q)+1} < \cdots < v^q_{l-1}\). In order to avoid "ugly" formulas we will use \(Q^q_l\) instead of \(Q_{l;A_1,\ldots,\hat{A_j},\ldots,A_p}\). Then we have:

\[
\partial (f_{A_1,\ldots,\hat{A_j},\ldots,A_p} (\sum_{q \in Q^q_l} k_q \sigma_q)) = \sum_{q \in Q^q_l} k_q \partial (f_{A_1,\ldots,\hat{A_j},\ldots,A_p}(\sigma_q)) = \sum_{q \in Q^q_l} k_q \partial((-1)^{l-1-s(q)} [v^q_1, \ldots, v^q_{s(q)}, v^q_{s(q)+1}, \ldots, v^q_{l-1}]) = \sum_{q \in Q^q_l} (-1)^{l-1-s(q)} k_q \left( \sum_{b=1}^{l-1} (-1)^{b-1} [v^q_1, \ldots, v^q_b, v^q_{l-1}] \right).
\]

On the other hand we have

\[
0 = \gamma_{A_1,\ldots,\hat{A_j},\ldots,A_p} = (-1)^{\sum_{b=1}^{l-1} s(q)} \left( \sum_{q \in Q^q_l} (-1)^{b-1} (A^q_1, \ldots, A^q_j \setminus \{v^q_b\}, \ldots, A^q_p) + \sum_{b=s(q)+1}^{l-1} (-1)^b (A^q_1, \ldots, A^q_j \setminus \{v^q_b\}, \ldots, A^q_p) \right).
\]
and hence,
\[
0 = f_{A_1, \ldots, A_j, \ldots, A_p}(\tau_{A_1, \ldots, A_j, \ldots, A_p}) = \\
= (-1)^{j-1} \sum_{q \in Q^j_i} \sum_{b=1}^{l-1} \left[ k_q \left( \sum_{b=1}^{l-1} (-1)^{b-1} (-1)^{l-1-s(q)} [v^q_1, \ldots, v^q_b, \ldots, v^q_{l-1}] + \\
\sum_{b=s(q)+1}^{l-1} (-1)^b (-1)^{l-2-s(q)} [v^q_1, \ldots, v^q_b, \ldots, v^q_{l-1}] \right) \right] = \\
= (-1)^{j-1} \sum_{q \in Q^j_i} \left( (-1)^{l-1-s(q)} \sum_{b=1}^{l-1} (-1)^{b-1} [v^q_1, \ldots, v^q_b, \ldots, v^q_{l-1}] \right) = \\
= (-1)^{j-1} \sum_{q \in Q^j_i} \partial(f_{A_1, \ldots, A_j, \ldots, A_p}(\sum_{q \in Q^j_i} k_q \sigma_q)).
\]

Since a simplex is acyclic, there exists an \((l-1)\)-dimensional chain \(\tau\) in \(\Delta_i\) such that
\[
\partial \tau = f_{A_1, \ldots, A_j, \ldots, A_p}(\sum_{q \in Q^j_i} k_q \sigma_q).
\]

Let now \(\eta = f_{A_1, \ldots, A_j, \ldots, A_p}(\tau)\). Clearly, \(\eta\) is a \((t+1)\)-chain. We need to check that \(\eta \in \text{Hom}(G, K_n)\). The condition for that is \(A^{\eta}_{x_1} \cap A^{\eta}_{x_2} = \emptyset\), for \(x_{1,2}\) adjacent to \(x_{1,2}\). This is clear for \(i_1 \neq j, i_2 \neq j\), since then \(A^{\eta}_{x_1} = A^{i_2}_{x_2}\), for \(b = 1, 2\). Assume that there exists \(x_{1,2}\) adjacent to \(x_{1,2}\), such that \(A^{\eta}_{x_1} \cap A^{\eta}_{x_2} \neq \emptyset\). By construction of \(\tau\), we know that \(A^{\eta}_{x_i} \subseteq \bigcup_{q \in Q^j_i} A^\tau_q\). Since \(A^\tau_q \cap A_{x_i} = \emptyset\), for any \(q \in Q^j_i\), we arrive at a contradiction.

Let now \(\partial \eta = \beta + \gamma + \delta\), where
\[
\beta = \{\xi \text{ from } \partial \eta \text{ such that } |A^\xi_j| = |A^\eta_j| = l+1\},
\gamma = \{\xi \text{ from } \partial \eta \text{ such that } |A^\xi_j| = l \text{ and } i \in A^\xi_j\},
\delta = \{\xi \text{ from } \partial \eta \text{ such that } i \notin A^\xi_j\}.
\]

Similarly to what we have done before, one can prove that
\[
f_{A_1, \ldots, A_j, \ldots, A_p}(\gamma \tau) = \varepsilon \partial f_{A_1, \ldots, A_j, \ldots, A_p}(\eta \tau) = \varepsilon \partial \tau = \\
= f_{A_1, \ldots, A_j, \ldots, A_p}(\varepsilon D_{l; A_1, \ldots, A_j, \ldots, A_p}),
\]

where \(\varepsilon = (-1)^{j-1} \sum_{q \in Q^j_i} k_q\). Since \(f_{A_1, \ldots, A_j, \ldots, A_p}\) is a bijection, we conclude that \(\gamma \tau = \varepsilon D_{l; A_1, \ldots, A_j, \ldots, A_p}\). Finally, let \(C^\tau_j = C^\xi_j - \varepsilon \partial \eta\).

After repeating the procedure described above for all combinations of sets \(A_1, \ldots, A_{j-1}, A_{j+1}, \ldots, A_p\) appearing in the decomposition of \(D_l\) to subchains of the form \((b, a)\), we will get a chain \(C^\tau_{j+1}\) such that \(|A^\xi_j| \geq l + 1\), for all cells \(\xi\) appearing in it with nonzero coefficients, and with \(i \in A^\xi_j\). The following observation is important for our argument. If for all cells \(\xi\) from \(C_j\) we have that \(A^\xi_b \subseteq S_b\), for \(b \in [p]\) and \(S_b \subseteq [n]\), then also for all \(\xi\) from \(C_{j+1}\) and \(b \in [p]\), we have \(A^\xi_b \subseteq S_b\).
After first phase, we shall eventually obtain a chain $C_1 := C_{p+1}$ which has the following additional property: for all cells $\eta$ appearing in this chain with a nonzero coefficient, we have $A_\eta^j \subseteq \{i-1, i, \ldots, n\}$, for $j \in [\lambda]$ (the original condition of the lemma preserved), and $i \not\in A_\eta^j$, for $j \in \{\lambda + 1, \ldots, p\}$.

**Step 3.** This step is almost the same as Step 1. Namely, assume $\eta$ is a cell from $C_j$ such that coefficient of $\eta$ in $C_j$ is $k$, and $A_\eta^j = \{i-1\}$. Then $\eta' = (A_\eta^1, \ldots, A_{j-1}^q, \{i-1\}, A_{j+1}^3, \ldots, A_\eta^n) \in \text{Hom}(G, K_n)$, since $i \not\in A_\eta^2$, for all $q$ such that $x_q \in \mathbb{R}(x_j)$.

Again, $C_j' = C_j - (-1)^{|K(\eta')|}K\partial\eta'$, where $K(\eta') = |A_\eta^1| + \cdots + |A_{j-1}^q| + 1$, is a chain homologous to $C_j$ and number of cells $\sigma$ from $C_j'$ such that $A_\sigma^j = \{i-1\}$ is strictly less then number of such cells from $C_j$. We repeat this procedure until we get cycle $C_j^2$ in which there are no cells $\sigma$ such that $A_\sigma^j = \{i-1\}$.

**Step 4.** This step is very similar to the Step 2. The only difference is that we are "removing" $i-1$ (instead of $i$) from $j$-th coordinate sets of each cell from the chain $C_j$ and obtain a new chain $C_{j+1}$ such that $i-1 \not\in A_\eta^j$ for each $\eta$ from $C_{j+1}$. By the observation that we have made after Step 2, we see that, for all cells $\eta$ from $C_{j+1}$ and for all $q \in [j]$, we have $A_\eta^j \subseteq \{i, i+1, \ldots, n\}$. In the case when $j = \lambda$, we set $C' := C_{j+1}$.

Now it is easy to see that $C' \in X_1(\lambda)$ and that $C - C'$, is a boundary of some $(t+1)$-chain of $\text{Hom}(G, K_n)$, and we have proved our claim. \hfill \Box

**Example.** Now we will illustrate the proof of Lemma 4.1 on a cycle from $\text{Hom}(C_4, K_7)$. For simplicity, we will use slightly different notations for cells, for example we will write $(1, 6, 7, 234, 2)$ instead of $\{1\}, \{6, 7\}, \{2, 3, 4\}, \{2\}$. We have ordered vertices of $C_4$ as described in Lemma 3.1, see Figure 6(a).

Let $C = C_3 = -((1, 1, 7, 234, 2) + (1, 6, 234, 2) - (1, 67, 34, 2) + (1, 67, 24, 2) + (1, 7, 235, 2) - (1, 7, 234, 2) + (1, 67, 35, 2) - (1, 67, 25, 2) \in X_1(2)$. It is easy to check that $\partial C = 0$. Figure 6(b) depicts the part of the complex $\text{Hom}(C_4, K_7)$ formed by all the cells from $C$.

![Figure 6](image-url)

First we will eliminate 2 from $A_3^3$ and $A_4^2$, for all cells $\eta$ from our chain $C$.

For $j = 3$ we obviously do not have Step 1. Hence, $C = C_3 = D_{2,1,67,\Lambda_3,2} + D_{2,2,2} + D_0$, where $D_{2,1,67,\Lambda_3,2} = (1, 67, 24, 2) - (1, 67, 25, 2)$, $D_0 = (1, 67, 35, 2) - (1, 67, 34, 2)$, and $D_{2,2,2}$ is the rest of $C_3$. Let $\eta = (1, 67, 24, 2)$ and $\xi = (1, 67, 25, 2)$.
Then \( \sigma = (1, 67, 245, 2), k = +1, K = 3 \) and finally,

\[
C_3 = \tilde{C}_3 = C'_3 + \partial \sigma = (1, 6, 234, 2) - (1, 7, 234, 2) - (1, 67, 34, 2) + (1, 7, 235, 2) - (1, 6, 235, 2) + (1, 67, 35, 2) - (1, 7, 245, 2) + (1, 6, 245, 2) - (1, 67, 45, 2).
\]

Again, \( D_{3,1,7,\hat{A}_j,2} = -(1, 7, 234, 2) + (1, 7, 235, 2) - (1, 7, 245, 2) \). 

Let us now remove 1 from the first two coordinate sets of all cells from \( C \).

First we apply Step 3: Let \( \eta \) be an arbitrary graph, then \( \eta = (1, 67, 34, 2, k = -1). \) We see that, in this case, \( I(\eta) \) could be any element from the set \( \{3, 4, 5\} \), but for convenience, we will choose \( \eta' = (1, 67, 34, 23) \). Then \( C'_3 = C_3 + \partial \eta \). Repeating this for all cells such that their fourth coordinate set is \( \{2\} \), with additional remark that we always pick \( I(\eta) = 3 \), we get a chain \( C_1 = -(1, 67, 34, 3, 1) + (1, 67, 35, 3) - (1, 7, 345, 3) + (1, 6, 345, 3) - (1, 67, 45, 3) \).

Let us now remove 1 from the first two coordinate sets of all cells from \( C_1 \).

First we apply Step 3: Let \( \eta = (1, 67, 34, 3) \). Then \( \eta' = (12, 67, 34, 3) \) and \( C'_1 = C_1 - \partial \eta \). If we do the same thing for all other cells from \( C_1 \), finally we get \( C' = -(2, 67, 34, 3) + (2, 67, 35, 3) - (2, 7, 345, 3) + (2, 6, 345, 3) \in X_2(2) \).

**Theorem 4.2.** Let \( G \) be an arbitrary graph, then

\[
H_t(\text{Hom}(G, K_n), \mathbb{Z}) = 0, \text{ for all } 1 \leq t \leq \text{vgap}(G, n) - 1.
\]

**Note.** As mentioned above, it was proved in \[2\] that \( H_0(\text{Hom}(G, K_n), \mathbb{Z}) = \mathbb{Z} \) if \( \text{vgap}(G, n) \geq 1 \). Also, it is a direct corollary of Theorem \[3.2\] that \( H_1(\text{Hom}(G, K_n), \mathbb{Z}) = 0 \) if \( \text{vgap}(G, n) \geq 2 \). Hence, we can assume that \( t \geq 2 \).

**Proof.** Let \( p \) and \( \lambda \) be the same as in Lemma \[3.1\] and Lemma \[4.1\]. Also, let \( C \) be a \( t \)-cycle from \( \text{Hom}(G, K_n) \), that is dimension of all cells from \( C \) is equal to \( t \) and \( \partial C = 0 \), where the boundary operator is defined in the proof of Lemma \[1.1\].

In order to prove that \( H_t(\text{Hom}(G, K_n), \mathbb{Z}) = 0 \), we need to prove that \( C \) bounds. Again, we will use induction on the maximal valency of \( G \).

Let the maximal valency of a graph \( G \) be zero. In the Theorem \[4.2\] we proved that in this case \( \text{Hom}(G, K_n) \) is contractible and hence, \( H_t(\text{Hom}(G, K_n), \mathbb{Z}) = 0 \) for all \( 0 \leq t \leq n - 2 \) and there exists a chain \( D \) such that \( C = \partial D \).

Assume now that the maximal valency of \( G \) is \( d \geq 1 \). By Lemma \[4.1\] there exists a cycle \( C' \) which is homologous to \( C \) and which has the property that, for all cells \( \eta \) with nonzero coefficients in this cycle and for all \( q \in [\lambda] \), \( A^\eta_q = \{n\} \).

Hence, \( C' \) is isomorphic to a cycle inside \( \text{Hom}(G - \{x_1, x_2, \ldots, x_\lambda\}, K_{n-1}) \). The set \( \{x_1, x_2, \ldots, x_\lambda\} \) is a maximal independent set in \( G \), therefore, the maximal valency of \( G - \{x_1, x_2, \ldots, x_\lambda\} \) is strictly less than the maximal valency of \( G \). It follows that \( C' \) is a boundary in \( \text{Hom}(G - \{x_1, x_2, \ldots, x_\lambda\}, K_{n-1}) \) by the induction hypothesis. We conclude that there exists a \((t+1)\)-chain \( D \) in \( \text{Hom}(G, K_n) \), such that \( \partial D = C \).

\[\square\]

**Remark 4.3.** It was proved in \[4\] that:

\[
H_i(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) = 0, \text{ for } 1 \leq i \leq n - 4.
\]
This fact is a direct corollary of the previous theorem since the maximal valency of \( C_{2r+1} \) is equal to 2.

Finally, we are able to put the pieces together and prove the Conjecture 1.1.

**Proof of the Conjecture 1.1.**

The cases when \( \text{vgap}(G, n) = 0, 1 \) were observed in [2, Proposition 2.4]. The case \( \text{vgap}(G, n) = 2 \) is proven in Theorem 3.2.

Let us now deal with case \( \text{vgap}(G, n) \geq 3 \). By the Theorem 3.2, \( \text{Hom}(G, K_n) \) is 1-connected, and by the Theorem 4.2 we have,

\[
H_t(\text{Hom}(G, K_n), \mathbb{Z}) = H_t(\text{Hom}(G, K_n)) = 0,
\]

for all \( 1 \leq t \leq \text{vgap}(G, n) - 1 \). By a standard corollary to the Hurewitz theorem (see, for example, [5, 10.10 Corollary, page 479]), we have that \( \pi_t(\text{Hom}(G, K_n), *) = 0 \), for all \( t \leq \text{vgap}(G, n) - 1 \), and, hence \( \text{conn}(\text{Hom}(G, K_n)) \geq \text{vgap}(G, n) - 1 \) by definition. \( \square \)

**Remark 4.4.** We know that the result of Conjecture 1.1 is sharp for several classes of graphs, for example for odd cycles and complete graphs.

**Remark 4.5.** By the same corollary we used in the proof of Conjecture 1.1, we conclude that \( \pi_{\text{vgap}(G, n)}(\text{Hom}(G, K_n), *) \approx H_{\text{vgap}(G, n)}(\text{Hom}(G, K_n)) \).

**Corollary 4.6.** The complex \( \text{Hom}(C_r, K_n) \) is \( (n - 4) \)-connected, for arbitrary integers \( r, n \geq 3 \).

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**Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden**

E-mail address: cukic@math.kth.se

**Institute of Theoretical Computer Science, ETH-Zürich, CH-8092 Zürich, Switzerland, and Department of Mathematics, Royal Institute of Technology, Stockholm, Sweden (on leave)**

E-mail addresses: dkozlov@inf.ethz.ch, kozlov@math.kth.se