Compactons in $\mathcal{PT}$-symmetric generalized Korteweg-de Vries Equations

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In an earlier paper Cooper, Shepard, and Sodano introduced a generalized KdV equation that can exhibit the kinds of compacton solitary waves that were first seen in equations studied by Rosenau and Hyman. This paper considers the $\mathcal{PT}$-symmetric extensions of the equations examined by Cooper, Shepard, and Sodano. From the scaling properties of the $\mathcal{PT}$-symmetric equations a general theorem relating the energy, momentum, and velocity of any solitary-wave solution of the generalized KdV equation is derived, and it is shown that the velocity of the solitons is determined by their amplitude, width, and momentum.

I. INTRODUCTION

In a previous investigation Cooper, Shepard, and Sodano introduced and Khare and Cooper$^2$ studied further the first-order Lagrangian

$$L(l,p) = \int dx \left[ \frac{\varphi_x \varphi_t}{2} + \frac{(\varphi_x)^4}{l(l-1)} - \alpha(\varphi_x)^p(\varphi_{xx})^2 \right].$$

This Lagrangian is a modification of the original compacton equations of Rosenau and Hyman. (Unless otherwise specified, the range of $x$ integration is over the entire real axis $-\infty < x < \infty$.) This Lagrangian gives rise to a general class of KdV equations of the form

$$u_t + u^{l-2}u_x + \alpha[2u^p u_{xxx} + 4pu^{p-1}u_x u_{xx}] + p(p-1)u^{p-2}(u_x)^3 = 0,$$  \hspace{2cm} (2)

where the solution $u(x,t)$ to the generalized KdV equation is defined by $u(x,t) = \varphi_x(x,t)$. For $0 < p \leq 2$ and $l = p + 2$ these models admit compacton solutions whose width is independent of the amplitude. For $p > 2$ the derivatives of the solution are not finite at the boundaries of the compacton where $u \to 0$. Cooper, Khare, and Saxena$^3$ analyzed the stability of the general compacton solutions of this equation and showed that solutions are stable provided that

$$2 < l < p + 6.$$  \hspace{2cm} (3)

There has been some recent interest in complex $\mathcal{PT}$-symmetric extensions of the ordinary KdV equation. Such extensions exist in the complex plane but also lead to new PDEs that are entirely real. The first extension of the KdV equation by Bender et al$^2$ was

$$u_t - iu^4 = 0,$$  \hspace{2cm} (4)

which reduces to the usual KdV equation when $\epsilon = 1$. This equation was analyzed by Bender et al$^2$ for $\epsilon = 3$. This extension of the KdV equation is not a Hamiltonian dynamical system at arbitrary $\epsilon$. A more recent study by Fring was based on a Hamiltonian formulation. The Hamiltonian studied by Fring is related to a special case of the system of generalized KdV equations examined here.

To find extensions of the generalized KdV equation that are invariant under the joint operation of space reflection (parity) $\mathcal{P}$ and time reversal $\mathcal{T}$, we make the following definitions: spatial reflection $\mathcal{P}$ consists of making the replacement $x \to -x$. Also, because $u$ is a velocity, under $\mathcal{P}$ we replace $u$ by $-u$. The effect of the time reversal operation $\mathcal{T}$ is to change the signs of $i$, $t$, and $u$: $i \to -i$, $t \to -t$, and $u \to -u$. Therefore, the combination $iu_x$ is $\mathcal{PT}$ even, so a $\mathcal{PT}$-symmetric generalization of the Lagrangian $\mathcal{l}$ is

$$L_{\mathcal{PT}} = \int dx \left[ \frac{\varphi_x \varphi_t}{2} + \frac{(\varphi_x)^4}{l(l-1)} + \alpha(\varphi_x)^p(i\varphi_{xx})^m \right].$$  \hspace{2cm} (5)

For this Lagrangian we must find the correct $\mathcal{PT}$-symmetric contour that lies on the real axis when $m = 2$. For $\mathcal{PT}$ to be a good symmetry, branch cuts must be taken along the positive imaginary axis in the complex-$x$ plane. The Hamiltonian resulting from the above Lagrangian is

$$H = \int dx \left[ \frac{u_t}{l(l-1)} - \alpha u^p(iu_x)^m \right].$$  \hspace{2cm} (6)

When $m$ is an even integer, a convenient choice for $\alpha$ that allows for solitary-wave solutions and that gives a real equation for the generalized KdV system is

$$-\alpha(m - 1)i^m = 1.$$  \hspace{2cm} (7)

For simplicity, we choose $\alpha$ as in (4) for most of this paper. The $\mathcal{PT}$ generalization of (4) has the same canonical structure as the KdV equation. From Lagrange’s equations or from Hamilton’s equations for the generalized...
KdV equations, we obtain the equations of motion for $u(x, t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \frac{\delta H}{\delta u} \right) = \{u, H\},$$

(8)

where the Poisson bracket structure is:

$$\{u(x), u(y)\} = \partial_x \delta(x - y).$$

(9)

The resulting equation becomes

$$0 = u_t + u_x u_{xx}^{l-2} + u^{p-2} u_x^{m-3} [(m - 2)u u_{xx}^2 + 2mpu_{xx}u_x^2 + mu^2 u_{xxx} u_x + (p - 1)pu_x^3].$$

(10)

This system of equations has three obvious conservation laws: conservation of mass $M$, momentum $P$, and energy $E$, where the energy is the value of the Hamiltonian $H$ and

$$M = \int dx \, u(x, t), \quad P = \int dx \frac{1}{2} u^2(x, t).$$

(11)

The case $m = 2$ leads to the well known compacton solutions. Fring studied a Hamiltonian similar to the subclass of this $PT$-symmetric class of Hamiltonians corresponding to $l = 3$ and $p = 0$, but with slightly different coefficients for the two terms in the Hamiltonian.

This paper is organized as follows: Section II considers the scaling properties of the nonlinear wave equation (10) and discusses the energy and momentum of solitary waves, and Sec. III continues the discussion of these conserved quantities. Traveling-wave solutions are discussed in Sec. IV and the conserved quantities for these solutions are examined in Sec. V. Some special cases are described in Sec. VI. The question of stability is addressed in Secs. VII and VIII.

II. SCALING PROPERTIES

Let us examine the scaling properties of (10). We require that solutions transform into solutions under the scaling

$$x \rightarrow \lambda x, \quad t \rightarrow \lambda^n t, \quad u \rightarrow \lambda^\beta u,$$

(12)

and we find that

$$\beta - \eta = (l - 1)\beta - 1 = \beta(p + m - 1) - m - 1.$$  

(13)

Solving for $\beta$ we obtain

$$\beta = \frac{m}{p + m - l}. $$

(14)

We also find that

$$1 - \eta = \beta(l - 2). $$

(15)

Suppose that we have a traveling solitary wave of the form $f(x - ct)$. Then $c$ scales as $x/t$ or as $\lambda^{1-\eta}$. Therefore, $c$ scales as

$$c \propto \lambda^{\beta(l-2)}. $$

(16)

In terms of the velocity, $x$ scales as $\lambda$ and

$$\lambda \propto c^{i_1}, \quad i_1 = \frac{p + m - l}{m(l - 2)}. $$

(17)

From the equation for $i_1$ we see that the width of the solitary wave does not depend on the velocity when

$$l = p + m. $$

(18)

This is a generalization of the result for $m = 2$. The conserved momentum scales like $u^2 x \propto \lambda^{2\beta+1}$, so

$$P \propto c^{i_2}, \quad i_2 = \frac{3m - l + p}{m(l - 2)}. $$

(19)

Finally, the conserved energy scales as $u^l x$, so

$$E \propto \lambda^{\beta l+1} \propto c^{(m+p+m-l)(p+m-l)}. $$

(20)

Eliminating $c$ in favor of $P$ in this formula, we get

$$E \propto P^{-r}, $$

(21)

where

$$r = -\frac{lm + p + m - l}{3m - l}. $$

(22)

This reduces to our previous results for the case $m = 2$. We show below how to make these scaling laws more precise and how to determine the constants of proportionality that relate the conserved quantities.

III. ANOTHER WAY TO RELATE ENERGY AND MOMENTUM OF SOLITARY WAVES

In Ref. [4] a general theorem was derived that relates the energy, momentum, and velocity of solitary waves of the generic form $u(x,t) = AZ[\beta(x - q(t))]$ for $m = 2$. Here, we generalize this result to arbitrary $m$ and we use the results obtained by studying the scaling properties of the alternative action $\Phi$ to justify the previous derivation. We start from the action

$$\Gamma = \int dt \, L, $$

(23)

where $L$ is given in (15). We now assume that the exact solitary-wave solution has the generic form

$$\phi_x = u = AZ[\beta(x - ct)]. $$

(24)

Using this form, it is easy to calculate the value of the Hamiltonian (10) for the solitary wave

$$H = -C_1(l) \frac{A^l}{\beta(l-1)} + A^{l+m} \beta^{m-1} C_2(p,m), $$

(25)
where

\[ C_1(l) = \int dz Z^l(z), \quad C_2(p, m) = \int dz [Z'(z)]^m Z^p. \]

Since \( H \) and momentum \( P \) are conserved, we can rewrite the parameter \( A \) in terms of \( P \):

\[ P = \frac{1}{2} \int dx u^2 = \frac{A^2}{2\beta} C_5, \quad C_5 = \int dz Z^2(z). \]

Replacing \( A \) by \( P \), we rewrite the Hamiltonian \( H \) as

\[ H = -C_3(l) P^{l-2} \beta^{l-2} + C_4(p, m) P^{\frac{p+m-2}{2}} \beta^{\frac{p+m-2}{2}}, \]

where

\[ C_3(l) = \frac{C_1(l)}{l(l-1)} \sqrt{2/C_5}, \quad C_4(p, m) = \alpha C_2(p, m) (2/C_5)^{(p+m)/2}. \]

At this point, we note that the exact solutions have the property that they are the functions of the parameter \( \beta \) that minimize the Hamiltonian with respect to \( \beta \) when the momentum \( P \) is fixed. Using \( \partial H/\partial \beta = 0 \), we obtain

\[ \beta = P^{\frac{p+m-2}{2}} \left[ \frac{C_4(p, m)(p + 3m - 2)}{C_3(l)(l-2)} \right]^{\frac{l-2}{l-3m}}. \]

This leads to

\[ H = f(l, p, m) P^{-r}, \]

where \( r \) is given by (22) and

\[ f(l, p, m) = C_3(l) \frac{p + m - l}{p + 3m - 2} \times \left[ \frac{C_4(p, m)(p + 3m - 2)}{C_3(l)(l-2)} \right]^{\frac{l-2}{l-3m}}. \]

Hamilton’s equation \( \dot{q} = \partial H/\partial P \) yields the relationship

\[ -\dot{q} = c = rH/P. \]

From this analysis it is again easy to show that the momentum \( P \), amplitude \( A \), and width parameter \( \beta \) functionally depend on the velocity \( c \) (note that \( c = -\dot{q} \)):

\[ P \propto c^{\frac{p+3m-4}{3m-l}}, \quad A \propto c^{\frac{1}{l-2}}, \quad \beta \propto c^{\frac{l+p-m}{m(l-2)}}. \]

Here, the proportionality constants depend on \( C_i(l, p, m) \) \((i = 1, 2, 3, 4, 5)\) defined above, and once an exact solution is obtained, these constants can be calculated easily.

We make three observations: (i) When \( l = p + m \), the width parameter \( \beta \) is independent of the velocity \( c \) and momentum \( P \) and hence of the amplitude \( A \) of the solitary wave. (ii) The \( c \) dependence of the amplitude \( A \) depends solely on the parameter \( l \), and it is independent of the parameters \( p \) and \( m \). (iii) The stability problem when \( l = p + 2 \) was studied by Dey and Khare for the case \( m = 2 \) using the results of Karpman. In Sec. VII we extend their results to arbitrary even integer \( m \).

**IV. TRAVELING-WAVE SOLUTIONS**

We begin with the wave equation (10), which can be reexpressed as

\[ u_t + u^{l-2} u_x - \frac{p}{m-1} \left[ u^{p-1} u_x \right]^m = 0, \]

and assume that

\[ u(x, t) = f(x - ct) = f(y). \]

Then,

\[ cf' = f^{l-2} f' + \frac{1}{m-1} \left( p \left[ f^{p-1} (f')^m \right]' - m \left[ f^p (f')^{m-1} \right]'' \right), \]

and integrating once we obtain

\[ cf = \frac{f^{l-1}}{l-1} + m f^p (f')^{m-2} f'' + p (f')^m f^{p-1} + K_1, \]

where \( f' \equiv df/dy \).

For compact solutions \( K_1 = 0 \). Setting the integration constant \( K_1 \) to zero, multiplying this equation by \( f \), and integrating over \( y \), we obtain

\[ c I_2 = \frac{1}{l-1} I_1 - \frac{p + m}{m-1} J_{m,p}. \]

For noncompact solutions if \( K_1 \neq 0 \), we also have a term that includes \( I_1 \), which is the conserved mass. We multiply (39) by \( f' \) and integrate again with respect to \( y \) to get the following nonlinear differential equation for the traveling waves:

\[ \frac{c}{2} f^2 - \frac{f^l}{l(l-1)} - (f')^m f^p = K_1 f + K_2. \]

We see that \( K_2 \) must also be zero for solutions \( f \) that are compact. Now, if we set \( K_1 = 0 \) and \( K_2 = 0 \) and integrate with respect to \( y \), we obtain

\[ J_{m,p} = \frac{c}{2} I_2 - \frac{1}{l(l-1)} I_1. \]

From (50) and (52) we can solve for \( J_{m,p} \) and \( I_1 \) in terms of \( I_2 \). We obtain

\[ J_{m,p} = \frac{(l-2)(m-1)}{2[p + m + (m-1)l]} c I_2 \]

and

\[ I_1 = \frac{l(l-1)(p + 3m - 2)c}{2[p + m(l + 1) - l]} I_2. \]
Notice that when \( l \to 2, \, c = 1 \) for consistency. This is related to the fact that the equation for \( u(x,t) \) becomes a linear equation with propagation velocity \( c = 1 \). The energy of the solitary wave is given by

\[
H = \frac{1}{m-1} J_{m,p} - \frac{1}{l(l-1)} I_l
\]  
(45)

and the momentum \( P = I_2/2 \). From Eqs. (43) and (45) we deduce that the energy, momentum, and velocity of the solitary wave are related by

\[
H = Pc/r,
\]  
(46)

where \( r \) is given in (22).

### A. Weak solutions

We are interested in compacton solutions that are a combination of a compact function \( f(x) \) confined to a region (initially \(-x_0 < x < x_0 \) and zero elsewhere). At the boundaries \( \pm x_0 \) the function \( f(x) \) is assumed to be continuous but higher derivatives most likely are not. For there to be a weak solution we require that the jump in

\[
\frac{c}{2} f^2 - \frac{f^l}{l(l-1)} - (f')^m f^p - K_1 f - K_2
\]  
(47)

be zero when we cross from \( x_0 - \epsilon \) to \( x_0 + \epsilon \). Since \( f(x_0) \) is assumed to vanish, the requirement for a weak solution is

\[
\text{Disc}[\{f'(x)\}^m(x)f^p(x)]|_{x_0} = 0,
\]  
(48)

where \( \text{Disc} \) is the discontinuity across the boundary \( x_0 \). This is always satisfied if there is no infinite jump in the derivative of the function.

We are concerned mostly with the cases where the integration constants are set equal to zero. We can then rewrite (17) as

\[
\frac{c}{2} f^2 - \frac{f^l}{l(l-1)} = (f')^m.
\]  
(49)

Notice that for \( m = 2 \) we recover

\[
\frac{c}{2} f^2 - \frac{f^l}{l(l-1)} = (f')^2,
\]  
(50)

which was studied previously. The solitary wave for (50) is obtained by joining the positive and negative solutions of the square root of (49), and for \( m = 2n \) one again obtains a real solution by joining the positive and negative real parts of the solutions of the square root of (49), appropriately shifted so that the maximum is at the origin \( y = 0 \).

If we are looking for compactons, then the finiteness of the derivative when \( f \to 0 \) requires that

\[
p \leq 2, \quad p \leq l.
\]  
(51)

The case \( m = 2 \) gives both compactons and real equations. For \( m = 2, \) when

\[
l = p + 2,
\]  
(52)

the width of the compacton is independent of the velocity. In our previous discussion of scaling we found that when \( l = p + m \), the width of the compacton is indeed independent of the velocity.

### B. Compacton solutions when \( m \) is an even integer

Compacton solutions are constructed by patching a compact portion of a periodic solution that is zero at both ends to a solution that vanishes outside the compact region to give the weak solution described above. Let us look at the generalizations of the compacton equation when we go from \( m = 2 \) to \( m = 4 \). Consider the case when \( p = 1 \) and let \( l = 3 \) and \( l = 4 \). For \( p = 1, \, l = 3 \) the equation for the solitary wave is

\[
\frac{c}{2} f - \frac{1}{6} f^2 = (f')^m.
\]  
(53)

For the positive branch of the solution, we get

\[
x - ct = \int_0^t \frac{du}{\left(\frac{c}{2} u - \frac{1}{6} u^2\right)^{1/m}}.
\]  
(54)

Performing the integral, we obtain

\[
x - ct = 2^{1/2} 3^{m-1} c^{-m^2} B_{\frac{m-1}{m}} \left(\frac{m-1}{m}, \frac{m-1}{m}\right),
\]  
(55)

where \( B_{m}(x,y) \) is the incomplete beta function while \( B(x,y) \) (see below) is the complete beta function.

For \( m = 2 \) this simplifies to

\[
x - ct = 2\sqrt{3} \sin^{-1}\left(\sqrt{f/(3c)}\right),
\]  
(56)

which leads to the previous compacton result

\[
f = 3c \sin^2 \left[\frac{1}{2\sqrt{3}}(x - ct)\right].
\]  
(57)

For \( m = 4 \) we get for the positive real fourth root

\[
x - ct = 2^{1/4} 3^{3/4} \sqrt{c} \left[B_{\frac{5}{4}} \left(\frac{3}{4}, \frac{1}{4}\right)\right].
\]  
(58)

In Fig. 1 we plot \( B_{5/4} (3/4, 1/4) \) and its mirror image as a function of \( f/(3c) \). Here, \( y = (x - ct)^{2-1/3-3/4c^{-1/2}} \).

Now consider the case \( p = 1 \) and \( l = 4 \), where the solitary-wave equation becomes

\[
\frac{c}{2} f - \frac{1}{12} f^3 = (f')^m.
\]  
(59)

For the positive branch of the solution, we get

\[
x - ct = \int_0^f \frac{du}{\left(\frac{c}{2} u - \frac{1}{12} u^3\right)^{-1/m}},
\]  
(60)
or

\[ x - ct = (3/2)^{\frac{m-1}{m}} c^{\frac{m}{m+2}} B \varphi \left( \frac{m-1}{2m}, \frac{m-1}{m} \right). \]  

(61)

By adding the positive-real- and negative-real-root solutions (for \( m \) an even integer) we get the complete compacton profile. The compacton vanishes elsewhere.

For \( m = 2 \) this leads to

\[ x - ct = \left[ 3/(2c) \right]^{1/4} B \varphi \left( \frac{1}{2}, \frac{1}{2} \right). \]  

(62)

When \( m = 2 \), the equation for the compacton can in fact be directly solved in terms of elliptic function. In particular, consider the equation at \( m = 2 \)

\[ \frac{c}{2} f - \frac{1}{12} f^3 = (f')^2. \]  

(63)

Assuming a solution of the form

\[ f = A \cn^2(\beta y, k^2 = 1/2), \]  

(64)

we find that

\[ \beta = c^{1/4}(96)^{-1/4}, \quad A = \sqrt{6}c, \]  

(65)

where \( \cn(x, k) \) is the Jacobi elliptic function with modulus \( k \).

### C. Hyperelliptic compactons

Consider next the generalization of the hyperelliptic compactons discussed in Ref. [4]. For this purpose we assume that we can parametrize the solutions by

\[ f = A Z^a[\beta(x - ct)], \]  

(66)

and we demand that

\[ (Z')^m = 1 - Z^{2\tau}. \]  

(67)

This immediately leads to the relations

\[ a = \frac{m}{m + p - 2}, \quad \tau = \frac{m(l - 2)}{2(m + p - 2)}. \]  

(68)

We also find that

\[ A^m a^m \beta^m = \frac{c}{2} A^{2-p} = \frac{A^{l-p}}{l(l - 1)}. \]  

(69)

This leads to

\[ A = [cl(l - 1)/2]^{1/(l-2)} \]  

(70)

and

\[ \beta = \frac{1}{a[l(l - 1)]^{1/m}} [cl(l - 1)/2]^{(l-p-m)/(m(l-2))}. \]  

(71)

Note that this ansatz gives the correct scaling behavior of the amplitude parameter \( A \) and the width parameter \( \beta \) for the velocity \( c \).

The solution to the differential equation (67) has \( m \) branches corresponding to the various values of \( e^{2\pi n/m} \) when \( m \) is an integer and \( n = 1, 2, \ldots, m \). For even integer \( m \), the positive root can be integrated to give

\[ y = \int_0^Z dx (1 - x^{2\tau})^{-1/m} = Z \, 2F_1 \left( \frac{1}{m}, \frac{1}{2\tau}; 1 + \frac{1}{2\tau}; Z^{2\tau} \right), \]  

(72)

where \( 2F_1 \) denotes the hypergeometric function. For even \( m \) we get the full solution for the compacton by adding the positive-real-root and the negative-real-root solutions to get the complete compacton profile. The compacton vanishes elsewhere.

### V. CONSERVED QUANTITIES

For solutions satisfying (67) it is possible to determine explicitly the conserved quantities in terms of the velocity \( c \) of the wave and the parameters \( l, p, \) and \( m \) of the differential equation. We have already found that the parameters \( A \) and \( \beta \) are given by (66) and (67).

We use the generic integral

\[ \int_0^1 dz z^\alpha (1 - z^{2\tau})^\beta = \frac{\Gamma \left( \frac{\alpha + 1}{2\tau} \right) \Gamma \left( \beta + 1 \right)}{2\tau \Gamma \left( \frac{\alpha + 1}{2\tau} + \beta + 1 \right)}. \]  

(73)

For the mass we get

\[ M = \int dx AZ^a(\beta x) = \frac{A}{\beta} \int dZ (dZ/dy)^{-1} Z^a \]  

\[ = 2\frac{A}{\beta} \int_0^1 dZ Z^a (1 - Z^{2\tau})^{-1/m} \]  

\[ = A \Gamma \left( \frac{m-1}{m} \right) \frac{\Gamma \left( \frac{m+1}{2\tau} \right)}{\beta \Gamma \left( 1 - \frac{1}{m} + \frac{1}{2\tau} \right)}. \]  

(74)

Here, we have used the fact that the total area under the solitary wave is twice the area coming from the positive.
and velocity. The relevant function to invert is \( F \) or \( \tau \). Thus, we consider here the set \( p, l \) where \( a = m/(m + p - 2) \) and the energy is \( E = cP/r \).

VI. SPECIAL CASES

There are two types of special cases. The first occurs when \( \tau, l, \) and \( p \) are integers. From Eq. (68) it follows that for \( m = 4 \)

\[
l = 2 + \tau(p + 2)/2.
\]

(76)

We consider here the set \( p = \{1, 2\} \). For \( p = 1, \tau = 2n \) and \( l = 3n + 2 \). For \( p = 2, \tau = 2n + 2 \). \( n = 1, 2, \ldots \). For \( m = 6 \)

\[
l = 2 + \tau(p + 1)/3.
\]

(77)

Thus, \( \tau = 3n, \) \( l = 2 + n(p + 1) \) yields simple integer solutions for integer \( n, p = 1, 2 \).

The other interesting case arises when the width of the solitary wave is independent of the velocity. This occurs when

\[
l = p + m.
\]

(78)

We study the cases for which \( \{p, l\} = \{1, 1 + m\} \) and \( \{p, l\} = \{2, 2 + m\} \). When \( l = p + m, \tau \) is given by

\[
\tau = m/2
\]

(79)

and here we also have \( a = m/(l - 2) \).

A. Case \( m = 4 \)

For this case

\[
\tau = \frac{2l - 4}{p + 2}, \quad a = \frac{4}{p + 2}.
\]

(80)

For \( \tau = 2, l = p + 4 \) and the width is independent of velocity. The relevant function to invert is

\[
y = \int_{0}^{Z} \frac{dx}{(1 - x^4)^{1/4}} = Z_2 F_1 \left( \frac{1}{4}, \frac{3}{4}; \frac{1}{2}; Z^4 \right).
\]

(81)

For a compacton centered about the origin \( y = 0 \), the two halves of the compacton are given by

\[
y_\pm = \pm f_1(Z) \mp f_1(Z = 1),
\]

(82)

where

\[
f_1(x) = x_2 F_1 \left( \frac{1}{4}, \frac{3}{4}; \frac{5}{4}; x^4 \right)
\]

(83)

and

\[
f_1(Z = 1) = \Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{5}{4} \right) = \frac{1}{4} \pi \sqrt{2} = 1.11072\ldots
\]

(84)

The result for \( Z[y] \) is shown in Fig. 2.

For \( p = 1, l = 5 \), we get \( A = (c/2)^{1/3} \) and the solution goes as \( Z^{4/3} \). For the case \( p = 2, l = 6 \), we get \( A = (c/2)^{1/4} \) and the solution is linear in \( Z \).

When \( \tau = 3 \), there is another special case with integer values: \( p = 2, l = 8, A = 1 \). Then, the relevant function to invert is

\[
y = \int_{0}^{Z} dx (1 - x^6)^{-1/4} = Z_2 F_1 \left( \frac{1}{6}, \frac{1}{4}; \frac{1}{4}; Z^6 \right).
\]

(85)

Since

\[
x_2 F_1 \left( \frac{1}{6}, \frac{1}{4}; \frac{1}{4}; 1 \right) = \Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{5}{4} \right) / \Gamma \left( \frac{11}{12} \right).
\]

(86)

we again use both the positive and negative solutions to make up the entire function \( Z(y) \). The case \( p = 0 \) does not correspond to compacton solutions and \( p > 2 \) does not allow for a finite derivative when the amplitude of the solitary wave becomes zero.

B. Case \( m = 6 \)

For integer \( \tau = 3n \) we discuss the \( \tau = 3 \) case. Here, \( l = p + 3 \), and there are two possibilities: Let \( p = 1, l = 4 \). Then,

\[
A \propto c^{1/2}, \quad a = 6/5.
\]

(87)

For \( p = 2, l = 5, \)

\[
A \propto c^{1/3}, \quad a = 1.
\]

(88)
The form of the function $Z[y]$ is now similar to the previous case:

$$f_2(Z) = \int_0^Z dx \left(1 - x^6\right)^{-1/6} = Z_{2F1} \left(\frac{1}{6}, \frac{1}{3}; \frac{5}{6}; Z^6\right).$$

We obtain the two halves of the function $Z(y)$ by inverting

$$y_\pm = \pm f_2(Z) \mp f_2(Z = 1),$$

where

$$f_2(Z = 1) = \Gamma \left(\frac{5}{6}\right) \Gamma \left(\frac{5}{3}\right) = \frac{5}{3} = 1.0472\ldots$$

VII. ALTERNATIVE GENERATING FUNCTION AND STABILITY

Solitary waves of the form $f(y) = f(x - ct)$ can be derived by considering the following function:

$$\Phi[f(y), f'(y)] = \int dx \left( H[f, f'] + P[f, c] \right) = \int dx \varphi[f, f'] \hspace{0.2cm} \theta(f_1 = 0).$$

Notice that $\varphi$ is the negative of the Lagrangian density. That is, the original equation for the solitary wave can be written as

$$\partial_x \frac{\delta \Phi}{\delta f} = 0.$$  

The once-integrated equation (with no integration constants) is obtained from

$$\frac{\delta \Phi}{\delta f} = 0,$$

or equivalently from the Euler-Lagrange equation

$$\frac{\partial \varphi}{\partial f} = \frac{d}{dx} \left( \frac{\partial \varphi}{\partial f'} \right).$$

We have explicitly

$$\Phi = \int dy \left[ -\frac{f'}{l(l-1)} + f^p(f')^m + \frac{1}{2} c^p f^2 \right].$$

The first variation after an integration by parts can be written as

$$\delta \Phi = \int dy \left[ -\frac{f'^{-1}}{l-1} + cf - pf^{p-1}(f')^m \right] \delta f.$$

The second variation which is important for the linear stability analysis can be written as

$$\delta^2 \Phi = \int dy \delta f \ L \ \delta f,$$

where $L$ is the operator

$$L = c - f^{p-2} - p(p-1)f^{p-2}(f')^m - mp^{p-1}(f')^{m-2} f''$$

$$\left( -mp^{p-1}(f')^{m-1} - m(m-2)f^p(f')^{m-3} f'' \right) \frac{d}{dy}$$

$$-m(f')^{m-2} \frac{d^2}{dy^2}.$$  

When $m = 2$, this reduces to the result given in Dey and Khare.

One can write $\Phi$ in terms of $I_1$ and $J_{m,p}$:

$$\Phi[f] = \frac{1}{m-1} J_{m,p} - \frac{1}{l(l-1)} I_1 + \frac{c}{2} I_2.$$  

Following Derrick, we consider the scale transformation $x \rightarrow \lambda x$. Under this transformation

$$I_1[f(\lambda y)] = \frac{1}{\lambda} I_1, \hspace{0.2cm} J_{m,p}[f(\lambda y)] = \lambda^{m-1} J_{m,p},$$

so that

$$\Phi[f(\lambda y)] = \frac{1}{m-1} \lambda^{m-1} J_{m,p} - \frac{I_1}{\lambda(l-1)} + \frac{c}{2} I_2.$$  

If we assume that taking the derivative of $\Phi$ with respect to $\lambda$ and setting $\lambda = 1$ gives a solution, we get

$$\frac{d\Phi(\lambda)}{d\lambda} \bigg|_{\lambda=1} = 0 = J_{m,p} + \frac{I_1}{l(l-1)} - \frac{c}{2} I_2.$$  

This is precisely (12), the equation of motion integrated over space that we found earlier. [Derrick looked to see if the second derivative of (103) became negative which would indicate that the solitary wave was unstable.] If we calculate the second derivative, we obtain

$$\frac{d^2 \Phi(\lambda)}{d\lambda^2} = c I_2 - \frac{1}{2(l-1)} \frac{l-2}{(l-1)!}(1+m)\lambda_1^2 + \frac{1}{2(l-1)} \frac{l-2}{(l-1)!}(1+m)\lambda_1^2$$

$$-ml^2 - (m^2 - 7m - 2p + 4) l + 6m + 2p - 4 \right) \cdot (104).$$

This does not factor to give a simple criterion for stability. However, another choice leads to a simple stability criterion. Suppose we instead make the scaling

$$f(y) \rightarrow \lambda^p f(\lambda y).$$  

This again leads to the equations of motion plus a boundary term because

$$\frac{d\Phi}{d\lambda} \bigg|_{\lambda=1} = \int dy \left[ \frac{\partial \varphi}{\partial f} - \frac{d}{dx} \frac{\partial \varphi}{\partial f'} \right] \left( \rho f + x f' \right)$$

$$\left[ \frac{\partial \varphi}{\partial f'} \right]_{\rho f + x f''} \bigg|_{\text{ymax}} = 0.$$  

Assuming that the boundary term vanishes at the edges of the compacton, we recover the equation of motion

$$\Phi[\lambda^p f(\lambda y)] = \frac{1}{m-1} \lambda^{m-1} + \rho(m+p) J_{m,p}$$

$$\frac{I_1}{l(l-1)} \lambda^{p-1} + \frac{c}{2} \lambda^{2p-1} I_2.$$  

(107)
The condition for a minimum is
\[ \frac{d\Phi(\lambda)}{d\lambda} \bigg|_{\lambda=1} = 0 = \frac{(l\rho - 1)I_i}{l(l-1)} - \frac{m-1 + \rho(m+p)}{m-1}J_{m,p} - \frac{c}{2}(2\rho-1)I_2. \]  
(108)

The particular case \( \rho = 1/2 \) is special in that the conserved momentum \( P \) is invariant under this transformation; that is,
\[ P[\lambda^{1/2}f(\lambda y)] = P[f(y)]. \]  
(109)

If we choose \( \rho = 1/2 \), when we vary \( \Phi \), we are varying the Hamiltonian with the constraint that \( P \) is held fixed. This is exactly what happens in a trial variational calculation where the parameter \( \lambda \), now thought of as a variational parameter, is a constraint variable to be eliminated (43). For arbitrary \( \rho \), the second derivative does not factor into a simple form that allows one to say when it changes sign. However, for \( \rho = 1/2 \) the answer does factor and the second derivative yields
\[ \Phi''(\lambda) \bigg|_{\lambda=1} = \frac{Pc(l-2)(3m + p - l)(3m + p - 2)}{4l(-1 + m) + m + p}. \]  
(111)

We also learn from the conditions in (51) that a weak solution that is compact can exist if \( \rho \leq 2 \), \( p \leq l \). This leads to the statement that solitary waves will be unstable under this type of deformation when
\[ l > p + 3m. \]  
(112)

More general scale transformations involving two parameters, such as
\[ f(y) \rightarrow \mu^{1/2}f(\lambda y), \]  
(113)
have been discussed by Karpman and Dey and Khare.

### A. Linear Stability

In this section we extend the analysis of Karpman to our generalized KdV equation. To study linear stability we assume that we can write
\[ u(x, t) = f(y) + v(x, t), \quad |v| \ll 1, \quad (u, v) = 0, \]  
(114)

where
\[ (f, g) = \int_{-\infty}^{\infty} dx f^* g. \]  
(115)

We change variables to \( y = x - vt \) and \( T = t \), so that we parametrize an arbitrary addition to \( f(y) \) as \( v(y,T) \). The linearized equation of motion for \( v(y,t) \) is then
\[ \frac{\partial}{\partial T} v(y,T) = \frac{\partial}{\partial y} [Lv(y,T)], \]  
(116)

where \( L \) is given by (99).

The equation (57) for the solitary wave can be written as
\[ L\partial_y f(y) = 0. \]  
(117)

Thus, \( f'(y) \) is a zero eigenfunction of \( L \) corresponding to the translation invariance of the solution (that is, the Goldstone mode).

Now, if we take the derivative of the first integral of the solitary wave equation (38) with respect to the velocity \( c \), we obtain
\[ L\frac{\partial f}{\partial c} = -f. \]  
(118)

Inverting, we get
\[ \frac{\partial f}{\partial c} = -L^{-1}f. \]  
(119)

This result will be useful later. We also have
\[ (f, f') = 0 \]  
(120)
from integrating by parts. If we now consider
\[ v(y,T) = e^{-i\omega T}\psi(y) + e^{i\omega T}\psi^*(y), \]  
(121)
then \( \psi \) satisfies
\[ \omega\psi(y) = i\partial_y L\psi(y). \]  
(122)

When \( m \) is an even integer, \( L \) is a Hermitian operator. In that case there is a theorem that all the \( \omega \) are real if one of the two operators on the right side of (122) is positive definite. (The more general case will be studied elsewhere.) A sufficient condition for real eigenvalues is
\[ (\psi, L\psi) > 0, \]  
(123)

where \( \psi \) is orthogonal to \( f \) and \( f' \). This condition is exactly the same as requiring that the second variation of \( \Phi = H + Pc \) be a minimum at \( f \). If the solitary-wave solution is a minimum of \( \Phi \), then the solution is linearly stable.

Our objective is to find the extremal value of \( (\psi, L\psi) \) and to find the criterion that guarantees that it is positive. To find the extrema one solves the constrained variation condition
\[ \delta[(\psi, L\psi) - \Lambda(\psi, \psi) - C(\psi, f)] = 0, \]  
(124)
\[(L - \Lambda)\psi = Cf. \quad (125)\]

Using \((\psi, f) = 0\), we find that
\[\langle \psi, L\psi \rangle = \Lambda, \quad (\psi, \psi) = 1. \quad (126)\]

One solves this equation by expanding \(\psi\) and \(f\) in a series of eigenfunctions of the operator \(L\). Letting \(L\phi_n = \lambda_n \phi_n\) and assuming the ordering \(\lambda_n > \lambda_m\) if \(n > m\), we find that \(\lambda_1 = 0\) and \(\phi_1 = f'\). Letting
\[f = \sum_{n \neq 1} b_n \phi_n, \quad (127)\]
we find that
\[\psi = C \sum_{n \neq 1} \frac{b_n}{\lambda_n - \Lambda} \phi_n. \quad (128)\]

From \((\psi, f) = 0\) we obtain
\[r(\Lambda) = \sum_{n \neq 1} \frac{|b_n|^2}{\lambda_n - \Lambda} \phi_n = 0. \quad (129)\]

We are interested in the lowest solution \(\Lambda_{min}\) that solves this equation because this gives the minimum of \((\psi, L\psi)\) that we seek. Assuming along with Karpman et al. and de Dey and Khare that \(L\) has only one negative eigenstate, one then deduces that \((\psi, L\psi) > 0\) is satisfied if \(r(0) < 0\). However, when \(\Lambda = 0\) we get
\[L\psi_{\Lambda = 0} = Cf. \quad (130)\]
Thus, from (118) we find that
\[\psi_{\Lambda = 0} = \frac{\partial f}{\partial c}, \quad (131)\]
and
\[r(0) = -\left( \frac{\partial f}{\partial c} f \right) = -\frac{\partial P}{\partial c}. \quad (132)\]
This criterion, namely that
\[\frac{\partial P}{\partial c} > 0 \quad (133)\]
for stability, gives exactly the same sufficient result for stability as all the other criteria used.

Since for all of our solutions \(P \propto c^{(p+3m-1)/(l-2)m}\), it immediately follows that these solutions are linearly stable provided that
\[2 < l < p + 3m. \quad (134)\]

B. Lyapunov Stability

Lyapunov stability uses sharp estimates and has been used by Weinstein and Karpman et al. Here we want to show that the compacton solution is a minimum of the Hamiltonian for fixed momentum \(P\). We show this using Holder’s Inequality and follow the arguments of Weinstein, Kuznetsov, Karpman, and de Dey and Khare. We do this by first showing that the Hamiltonian for fixed momentum \(P\) is bounded below and then that the compacton solution satisfies the condition that it is a particular lower bound. We can write the Hamiltonian \(H\) in terms of \(I_l\) and \(J_{m,p}\) as
\[H[f] = \frac{1}{m-1} J_{m,p} - \frac{1}{l(l-1)} I_l, \quad (135)\]
where
\[I_2 = 2P. \quad (136)\]
Now consider that
\[I_l \leq (\max f^{l-2}) \int dy f^2. \quad (137)\]
We want to bound \(I_l\) by a function of \(J_{m,p}\) and the conserved momentum \(P\). To do this it is convenient to write
\[f^{l-2} = [f^a]^{(l-2)/a}. \quad (138)\]
Then, writing
\[f^a = \int dy df^a/df = a \int dy f^{a-k} \frac{df}{df} f^k, \quad (139)\]
we use the Holder inequality to show that
\[f^a \leq a \left( \int dy |f^{a-k} f^m| (f^m)^m \right)^{1/m} \times \left( \int dy |f^k| \right)^{1/l} . \quad (140)\]
We can dispense with the absolute values when \(m\) is an even integer. Thus by choosing
\[k = 0; \quad p = m(a - 1 - k), \quad (141)\]
we can relate the second term of (135) to \(P\) and \(J_{m,p}\).

Moreover, to relate the bound to the energy of the solitary wave (rather than having a general lower bound that depends on the choice of \(a\)), one must further choose
\[ma = p + 3m - 2, \quad (142)\]
from which we find that
\[j = \frac{m}{m - 1}. \quad (143)\]
Doing this we obtain
\[ H = \min_{J_{m,p}} \left| \frac{J_{m,p} - \alpha_{p,l,m} J_{m,p}^{(l-2)/(p+3m-2)} (2P)^{\gamma}}{m-1} \right|, \]

where
\[ \gamma = \frac{(l-2)(m-1) + p + 3m - 2}{p + 3m - 2}, \]

and
\[ \alpha_{p,l,m} = \frac{1}{l(l-1)} \left( \frac{p + 3m - 2}{m} \right)^{m(l-2)/(p+3m-2)}. \]

Minimizing with respect to \( J_{m,p} \) for fixed momentum \( P \), we obtain
\[ H_{\text{min}} = \frac{l - p - 3m}{(m-1)(l-2)} J_{m,p}. \]

For the solitary wave which obeys the generalized KdV equation we get
\[ J_{m,p} = \frac{(l-2)(m-1)}{p + m + (m-1)l} P_{\text{sol}} c. \]

Thus, the solitary wave is a minimum of the Hamiltonian and satisfies
\[ E_{\text{sol}} = E_{\text{min}} = P_{\text{sol}} c/r, \]

where \( r \) is as given by (P). Thus, as long as \( 2 < l < p + 3m \), one has stable solitary waves.

**VIII. VARIATIONAL STABILITY OF SOLUTIONS**

Suppose that we have found an exact solution of the form \( AZ \{ \beta(x - g(t)) \} \). Then one can find sufficient conditions for instability of this type of solution by seeing if the solution is a minimum rather than a maximum of the Hamiltonian as a function of \( \beta \) with the conserved momentum \( P \) held fixed.

We know that the exact solutions are stationary for fixed \( P \) under variations in \( \beta \). That is,
\[ \frac{\partial H}{\partial \beta} = 0. \]

We can write the Hamiltonian in the generic form (\( P \) fixed)
\[ H = -C_1 \beta^a + C_2 \beta^b, \]

where the constants depend on \( P \) and the parameters that define the Lagrangian and \( a \) and \( b \) also depend on the parameters in the Lagrangian.

The stationarity condition is
\[ \frac{\partial H}{\partial \beta} = 0 = \frac{1}{\beta} [C_1 a \beta^{a-1} + C_2 b \beta^{b-1}], \]

from which we infer that
\[ C_1 a \beta^{a-1} = C_2 b \beta^{b-1}. \]

The edge of stability of these solutions is given by
\[ \frac{\partial^2 H}{\partial \beta^2} = 0 = -C_1 a (a-1) \beta^{a-2} + C_2 b (b-1) \beta^{b-2}. \]

At the minimum this leads to the condition that
\[ a = (l-2)/2, \quad b = (p + 3m - 2). \]

Thus, the critical case is
\[ l = p + 3m, \]

which agrees with (134). We expect that the solutions we have found are stable as long as
\[ l < p + 3m. \]

**A. Approximate variational solutions**

To study stability it is useful to have approximate solutions that are close to the exact solutions to see if they relax to the exact solutions or become unstable. For this purpose it is useful to study the *post-Gaussian* trial functions
\[ f_V(x - ct) \equiv g(x - ct) = A \exp \left[ -|\beta(x - ct)|^{2n} \right], \]

where \( A, \beta, \) and \( n \) are continuous variational parameters chosen to minimize the action. These trial wave functions have earlier been successfully used to approximate various solitary waves in both KdV systems and NLSE applications.

The advantage of these trial functions is that the action as well as all of the conserved quantities can be explicitly evaluated using the formula
\[ \int_{0}^{\infty} dx x^n e^{-\beta|x|^{2n}} = \frac{1}{b} \frac{\Gamma(\frac{n+1}{2n})}{\beta^{\frac{n+1}{2n}}}. \]

In terms of our previous notation we have
\[ Z(z) = A \exp \left( -|z|^{2n} \right). \]

Thus, using the trial function, we get
\[ C_1 = \int dz Z^l(z) = 2 \int_{0}^{\infty} dz e^{-l|z|^{2n}} = l \frac{\Gamma \left( 1 + \frac{1}{2n} \right)}{2n}. \]
We also have
\[ C_2(p, m) = \int dz [Z'(z)]^m Z^p \]
\[ = 2 \int_0^\infty dz (2)^m e^{-m x^2 - p n x^2 (2n-1)} \]
\[ = \frac{1}{n} (-2n)^m (m + p)^{m-2n-1} \Gamma (m - \frac{m-1}{2n}) . \] (162)

These expressions depend on the variational parameter \( n \), which determines the shape of the solution; \( n = 1/2 \) gives a \textit{peakon} shape and \( n = 1 \) is the usual Gaussian. From these we can determine the quantities
\[ C_4(p, m) = K(p, m, n) \Gamma \left( 1 + \frac{1}{2n} \right)^{-\frac{m+p}{2}} \Gamma \left( m - \frac{m-1}{2n} \right) . \] (163)
where
\[ K(p, m, n) = (-n)^{m-1} \frac{\Gamma(nm + m + 2n p + p)}{\Gamma(nm - 2n - 1)} \] (164)
and
\[ C_3(l) = \frac{2^{2n+l} \Gamma(l-1-l) \Gamma \left( 1 + \frac{1}{2n} \right)^{\frac{1}{2}}}{(l-1)^{\frac{1}{2}}} . \] (165)

For the trial function we get
\[ A = \sqrt{2\beta P/C_5} \] (166)
and
\[ \beta = P \frac{C_4(p, m)(p + 3m - 2)}{C_3(l)(l - 2)} \] (167)

To determine the best trial function in this class we must also minimize the Hamiltonian with respect to the parameter \( n \). As in our discussion of conservation laws, we get
\[ H = f(l, p, m, n) P^{-r} , \] (168)
where \( r \) is given in (222) and \( f(l, p, m, n) \) is given in (322).

For solutions that are compact and cover half of the period of a positive periodic function, an alternative choice for a variational trial function is
\[ u_2(x) = A [\cos(\beta x)]^\gamma \] (169)
where \( \beta \) and \( \gamma \) are the variational parameters. For integer \( p, m, \) and \( l \) it is again possible to obtain an explicit expression for \( H[\beta, \gamma] \). One can perform the minimization with respect to \( \beta \) explicitly. Determining the global minimum in the parameter \( \gamma \) must be done numerically.

**B. Case \( p = 1, l = 3, m \)**

First, consider \( m = 2 \), where the exact solitary wave solution is
\[ f(y) = 3c \sin^2 \left( \frac{y}{2\sqrt{6}} + \frac{\pi}{2} \right) . \] (170)

For simplicity we normalize our functions by choosing \( P = 1 \). Then, this solution is
\[ f(Z) = 2^{5/4} 3^{-3/4} \pi^{-1/2} \cos \left( \frac{z}{2\sqrt{6}} \right) . \] (171)

This belongs to the class of variational solutions of the second type, and we would have obtained this exact answer from our variational minimization procedure. For the post-Gaussian trial functions, the lowest-energy variational solution having \( P = 1 \) is found to be
\[ u(x) = 0.583578 \exp \left( -0.0314705 x^{2.308} \right) . \] (172)

Fig. 3 compares the exact and variational function for \( P = 1 \). Note that apart from the region where the compacton goes to zero the agreement is excellent.

For \( m = 4 \) the value of \( n \) that minimizes the first trial function is \( n = 0.920655 \). Again, normalizing to \( P = 1 \) (which yields \( c \approx 1/3 \), we find that the best function in this class is:
\[ u(x) = 0.995936 \exp \left( -0.396108 x^{1.84131} \right) . \] (173)

If we use the second type of trial function we find that the values of \( \beta \) and \( \gamma \) that give a global minimum in the reduced space are
\[ \beta = 0.342787, \quad \gamma = 5.67846, \] (174)
which leads to \( A = 0.97067 \) for \( P = 1 \). Thus, the best trial function in the second class is given by
\[ u_2(x) = 0.97067 [\cos(0.342787x)]^{5.67846} . \] (175)

In Fig. 4 we compare the two variational approximations and note that apart from the fact that the \( u(x) \) is not compact, the agreement is quite good. Both solutions are global minima of the respective reduced Hamiltonians, which depend on two parameters.
To compare our implicit exact result with our variational approximations, we change variables to $y = \frac{x}{2^{1/4}3^{3/4}c^{1/2}}$ and redisplay the approximate solitary waves shown in Fig. 5 along with the exact result of Fig. 1.

FIG. 4: $u(x)$ and $u_2(x)$ versus $x$ for $m = 4$, $l = 3$, $p = 1$.

FIG. 5: $f(y)/3c$ versus $y$ for $m = 4$, $l = 3$, $p = 1$. Exact answer versus two variational solitary wave solutions.