Group Actions on Vector Bundles on the Projective Plane

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Abstract: We study the action of the group of automorphisms of the projective plane on the Maruyama scheme of sheaves $\mathcal{M}_{P^2}(r, c_1, c_2)$ of rank $r$ and Chern classes $c_1 = 0$ and $c_2 = n$ and obtain sufficient conditions for unstability in the sense of Mumford’s geometric invariant theory. The conditions are in terms of the splitting behaviour of sheaves when restricted to lines in the projective plane. A very strong parallel is observed with Mumford’s theory for the action of the automorphism group of the projective plane on the spaces of curves of a fixed degree.

1. Introduction

The problem we study belongs mainly to the domain of geometric invariant theory. The action of the group of automorphisms of projective space on its hypersurfaces, or more generally on the Hilbert scheme of subvarieties with a fixed Hilbert polynomial has been studied since Mumford. What if we replace the Hilbert scheme of subvarieties with a suitable moduli space of sheaves or vector bundles? The group of automorphisms of the projective space $PGL(n)$ acts naturally on sheaves by pullback. What is the nature of the semistable or unstable points in the sense of Mumford’s GIT? This question was first posed by C.Simpson [25]. We study this problem for sheaves on the projective plane.

For rank 2 vector bundles the problem is intimately related to the action of $SL(3)$ on the space of curves of degree $n$ in $P^2$. Mumford has given a complete classification of stable, semistable and unstable points for this action when the degree of the curve is less than 7. In general, for any degree, the description of the semistable and unstable points becomes exceedingly difficult. However two fairly general statements about the ‘best points’ and the ‘worst points’ remain true in all cases:

Let $C$ be a curve of degree $n$ in $P^2$.
1) If $n \geq 3$, and $C$ is nonsingular, then $C$ is stable under the $SL(3)$ action,
2) If $C$ has a point of multiplicity $> \frac{2n}{3}$ then $C$ is unstable.

The purpose of this paper is to show that there is a remarkably similar picture when we consider the action of $SL(3)$ on sheaves on the projective plane. As a suitable candidate for the space of sheaves on the projective plane, we use Maruyama’s scheme $\mathcal{M}_{P^2}(r, c_1, c_2)$, of $S$-equivalence classes of Geiseker semistable sheaves on $P^2$ with fixed rank $r$ and Chern classes $c_1$ and $c_2$. The precise definitions will follow in the subsequent section. On the open subset of $\mathcal{M}_{P^2}(r, c_1, c_2)$ consisting of Geiseker stable sheaves $S$-equivalence reduces to isomorphism of sheaves. We now state our main theorem

Theorem 1 Let $SL(3)$ act on $\mathcal{M}_{P^2}(r, 0, n)$ by pullback of sheaves. Under a suitable linearization of the action the following statements hold;
1) For \( n \geq 3 \), a generic point in \( \mathcal{M}_{P^2}(r,0,n) \) is stable.

2) For \( F \in \mathcal{M}_{P^2}(r,0,n) \), if there is a line \( l \subseteq P^2 \) such that \( F|_l = \oplus \mathcal{O}(d_i) \) with \( \Sigma d_i \geq 0, d_i > \frac{2n}{r} \), then \( F \) is unstable.

In particular if \( F \) is rank 2 with \( c_1 = 0 \) and \( c_2 = n \) such that there is a line \( l \subseteq P^2 \) with \( F|_l = \mathcal{O}(d) \oplus \mathcal{O}(-d) \) and \( d > \frac{2n}{r} \), then \( F \) is unstable under the \( SL(3) \) action on \( \mathcal{M}_{P^2}(2,0,n) \). Note the similarity with the classical result for curves. The problem for vector bundles of rank 2 is actually very closely related with the case of curves. The connection is through the so called Barth map [2], that assigns to each vector bundle of rank 2 in \( \mathcal{M}_{P^2}(2,0,n) \) a curve of degree \( n \) in \( P^2^* \), called its curve of jump lines. The map is \( SL(3) \) equivariant and a comparison theorem of Reichstein [24] allows us to compare the stability of bundles with the stability of curves. In higher rank, this facility of the Barth map is lost and a new technique becomes necessary to study stability. To do this we consider actions on suitable master spaces obtained by monadic constructions of Beilinson [4], Barth [3] and Horrocks [12]. This is the main contribution of the paper.

The paper is organised as follows. We start with recalling some basic definitions and preliminaries in the next section. Then we work out the special case for rank 2 sheaves, show how it gets related to the case of curves, and thus explain what to expect in general. In the final section we give a proof for higher rank.

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2. Definitions and Preliminaries.

Here we recall some basic definitions and preliminaries that will be used throughout. We begin with the fundamental notion of a semistable sheaf on a projective space. It is a well known fact that \( P_F(m) = \dim H^0(P^n, F(m)) \) is a polynomial in \( m \) for large enough \( m \), called the Hilbert polynomial of \( F \). Let \( r \) be the rank of \( F \). The polynomial \( p_F(m) = P_F(m)/r \) is called the reduced Hilbert polynomial of \( F \). Define \( \mu(F) = \frac{c_1(F)}{r} \).

**Definition 2** A coherent torsion free sheaf \( F \) is called semistable (stable) if for every coherent subsheaf \( F' \subseteq F \) of rank \( r' \) with \( 0 < r' < r \), we have \( p_{F'} \leq p_F \) (\( < \)), the polynomials being ordered lexicographically, where coefficients are compared starting with the highest degree term in the polynomial.

On \( P^2 \) the reduced Hilbert polynomial looks like

\[
p_F(m) = \frac{1}{2}m^2 + \left( \frac{3}{2} + \mu(F) \right)m + \chi(F).
\]

Thus \( p_{F'} \leq p_F \) means that \( \mu(F') \leq \mu(F) \) and in case of equality we require \( \chi(F')/r' \leq \chi(F)/r \).
The Riemann Roch theorem on $P^2$ reads

$$\chi(F) = r + \frac{1}{2}c_1(c_1 + 3) - c_2$$

where $r$ is the rank of $F$ and $c_1$ and $c_2$ are the first and second Chern classes respectively.

**Proposition 3** Let $F$ be a non trivial semistable sheaf on $P^2$ of rank $r$ and $c_1 = 0$. Then $H^0(F(-1)) = H^0(F) = H^2(F(-1)) = H^2(F) = 0$. Also $h^1(F(-2)) = h^1(F(-1)) = c_2$, and $h^1(F) = c_2 - r$. In particular, $c_2 \geq r$.

**Proof.** The proof is a straightforward application of definitions and the Riemann Roch theorem. We first show that $H^0(F(-1)) = 0$. Suppose not. Then there is a non-trivial map $O \to F(-1)$. If $G$ is the image of $O$, then since $F(-1)$ and $O$ are semistable we must have $p_O \leq p_G \leq p_F(-1)$. But this requires $\mu(O) \leq \mu(F(-1))$, that is $0 \leq -1$, which is false. Hence $H^0(F(-1)) = 0$. Also, by Serre duality $H^2(F(-1)) \cong H^0(F^*(-2))^* = \text{Hom}(F, O(-2))^*$. A non-trivial map $F \to O(-2)$ results in $\mu(F) \leq \mu(O(-2))$, that is $0 \leq -2$, which is false. Hence $H^2(F(-1)) = 0$. Therefore $\chi(F(-1)) = -\dim H^1(F(-1)) \leq 0$, and by Riemann Roch $\chi(F(-1)) = -c_2$. This implies that $c_2 \geq 0$ and $h^1(F(-1)) = c_2$. By exactly the same line of argument we get $h^1(F(-2)) = c_2$.

Now we show that $H^0(F) = 0$. Suppose not. Then there is a non-trivial map $O \to F$, leading to $p_O \leq p_F$. Since we have $\mu(O) = \mu(F)$ semistability requires that $\chi(O) \leq \chi(F)/r$, that is $1 \leq \frac{1 - c_2}{r} = 1 - \frac{c_2}{r}$. Now since $c_2 \geq 0$, we have $c_2 = 0$. But there are no semistable sheaves other that the trivial one with $c_1 = c_2 = 0$, and we have assumed that $F$ is non-trivial. Hence $H^0(F) = 0$. Similarly $H^2(F) = 0$ and hence by Riemann Roch $\chi(F) = r - c_2 = -h^1(F)$, so that $h^1(F) = c_2 - r$. $\blacksquare$

To give interesting and useful examples of stable bundles we will first look at bundles $F$ with $c_1 = 0$ that acquire a section after a single twist by $O(1)$, that is, $H^0(F(1)) \neq 0$. Such bundles are called Hulsbergen, after Wilfred Hulsbergen who initiated their study in his thesis [13]. If $s$ is a section of $F(1)$ then we get an exact sequence

$$0 \to O \to F(1) \to I_Z(2) \to 0.$$ 

where $Z$ is the zero scheme for $s$. We will first prove the following

**Proposition 4** The Hulsbergen bundle $F$ is Geiseker stable if and only if not all points of $Z$ lie on a line in $P^2$.

**Proof.** The proof rests on a simple observation that a rank 2 bundle on $P^2$ with $c_1 = 0$ is stable if and only if $H^0(F) = 0$. If $F$ is stable then clearly $H^0(F) = 0$ by the above proposition. Conversely, suppose that $H^0(F) = 0$. Let $G \subseteq F$ be a rank 1 subsheaf. Because $F$ is reflexive we may assume that $G$ is in fact a bundle. Hence $G = O(d)$ for some integer $d$. But this means that $F(-d)$ has a section and hence $H^0(F(-d)) \neq 0$. This implies that $d < 0$ and hence $\mu(G) < \mu(F)$ proving stability.
Now consider the Hulsbergen bundle $F$ with the above sequence. Twisting by $O(-1)$ we see that $H^0(F) = 0$ if and only if $H^0(I_Z(1)) = 0$, which is equivalent to the geometric condition that not all points of $Z$ lie on a line in $P^2$. The proof is complete.

Let $F$ be a semistable sheaf on $P^2$ with reduced Hilbert polynomial $p_F$. Then there is a filtration $F_0 = (0) \subseteq F_1 \subseteq \ldots \subseteq F_n$, such that the quotients $F_i/F_{i-1}$ are stable with reduced Hilbert polynomial $p_F$. Define $gr(F) = \oplus F_i/F_{i-1}$. It can be shown that $gr(F)$ is independent of the filtration chosen for $F$.

**Definition 5** Two semistable sheaves $F$ and $G$ are said to be $S$-equivalent if $gr(F) \simeq gr(G)$.

This is an equivalence relation on the set of semistable sheaves. On stable sheaves $S$-equivalence reduces to isomorphism of sheaves.

**Theorem 6** (Maruyama [21]) There exists a moduli space $M_{P^2}(r, c_1, c_2)$ of $S$-equivalence classes of semistable sheaves of rank $r$ and Chern classes $c_1$ and $c_2$. The moduli space is projective. If $(\chi, r, c_1) = 1$ the moduli space is smooth and fine. The dimension of $M_{P^2}(r, c_1, c_2)$ is $r^2(2\Delta - 1)+1$, where $\Delta = \frac{2rc_2-(r-1)c_1^2}{2r^2}$.

**Theorem 7** (Drezet, Le Potier [19]) The space $M_{P^2}(r, c_1, c_2)$ is irreducible.

**Theorem 8** (Drezet [8]) The Picard group $Pic M_{P^2}(r, c_1, c_2)$ is isomorphic to $\mathbb{Z}$ if $r = n$, and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ if $r \neq n$.

Next we state some basic definitions and facts about group actions.

Let $G$ be a reductive algebraic group acting linearly on a vector space $V = \mathbb{C}^{n+1}$. Then $G$ acts on the projective space $P^n = P(V)$. Let $Y \subseteq P^n$ be a closed $G$ invariant subvariety. If such an embedding of $Y$ in $P^n$ is defined by the ample line bundle $L$, then the action of $G$ is said to be linearized with respect to $L$.

**Definition 9** A point $a \in Y$ is said to be semi-stable if there is a homogeneous $G$ invariant polynomial of positive degree that does not vanish at $a$. Let $Y^{ss}$ denote the set of semi-stable points of $Y$. A point $a \in Y$ is said to be stable if it is semi-stable and if the orbit map $\varphi_a : G \to Y^{ss}$ defined by $\varphi_a(g) = g.a$ is proper.

The following fact can be verified easily [22].

**Proposition 10** Let $a' \in V \setminus \{0\}$ be any representative of $a \in Y$. Then $a$ is semi-stable if $0 \notin orb(a')$, where $orb(a')$ denotes the closure of the orbit of $a'$ in $V$. Also, $a$ is stable if $orb(a')$ is closed and the stabilizer $stab(a')$ is finite.
A non-trivial morphism of algebraic groups $\lambda : C^* \to G$ is called a 1-parameter subgroup of $G$. If $G$ acts linearly on $V$, then $C^*$ acts on $V$ by composition. By reductivity and commutativity of $C^*$ we get a decomposition of $V$ as $V = \oplus V_m$ where $\lambda(t)(v) = t^m v$, for $v \in V_m$. Let $v \in V$ be written as $v = \Sigma v_m$. Then we define

$$
\mu(v, \lambda) = -\min\{m/v_m \neq 0\}.
$$

Let $Y \subseteq P^n = P(V)$ be a closed $G$ invariant subvariety and let $y \in Y$. Let $y' \in V \setminus \{0\}$ be any representative for $y$. We put

$$
\mu(y, \lambda) = \mu(y', \lambda).
$$

It is easy to check that the definition is independent of the choice of the representative of the point $y$. The following theorems are central to all our computations.

**Theorem 11 (Hilbert-Mumford)** Let $G$ be a reductive group acting linearly on a vector space $V = C^{n+1}$. A point $x \in P(V)$ is semi-stable (stable) under the action of $G$ if and only if it is semi-stable (stable) under the action of all 1-parameter subgroups of $G$.

**Theorem 12 (Hilbert-Mumford criterion)** Let $G$ act on $V = C^{n+1}$ linearly. Let $Y \subseteq P^n$ be a closed $G$ invariant subvariety. A point $y \in Y$ is semi-stable (stable) with respect $G$ if and only if $\mu(y, \lambda) \geq 0$ for all 1-parameter subgroups $\lambda$ of $G$.

We now recall the classical result for curves.

**Proposition 13** Consider the action of $\text{SL}(3)$ on the space $P^N$ of curves of degree $n$ in $P^2$. If $C \in P^N$ has a point of multiplicity greater than $\frac{2n}{3}$ then $C$ is unstable.

**Proof.** Let $f(X, Y, Z) = 0$ be the equation for the curve $C$ in $P^2$. Without loss of generality we can assume that the point of multiplicity $> \frac{2n}{3}$ is $(0; 0; 1)$ in the projective space $P^N$. Then the equation of $C$ looks like

$$
f(X, Y, Z) = f_m(X, Y, Z) + f_{m+1}(X, Y, Z) + \ldots + f_i(X, Y, Z) \text{ where } f_i = \Sigma c^{(i)} X^a Y^b Z^c \text{ with } a + b + c = n \text{ and } a + b = i.
$$

In particular, $f_m(X, Y, Z) = \Sigma c^{(m)} X^a Y^b Z^c$ with $a + b = m$ and not all $c^{(m)} = 0$, since then the multiplicity would not be $m$. The coordinates of $C \in P^N$ are just the coefficients $c^{(j)}$ of the monomials in $X, Y$ and $Z$. Note that $a + b = m > \frac{2n}{3}$ and $a + b + c = n$ implies that $c < \frac{2n}{3}$.

Take the 1-parameter group $\lambda$ that defines the action $\lambda(t)X = tX, \lambda(t)Y = tY$ and $\lambda(t)Z = t^{-2}Z$ on the coordinates of the projective plane. Under the action of this 1-parameter group the equation $f(X, Y, Z)$ gets transformed into

$$
\lambda f(X, Y, Z) = \Sigma c^{(m)} X^{a+b-2c} Y^b Z^c + \Sigma c^{(m+1)} t^{p+q-2r} X^p Y^q Z^r + \ldots
$$

In all of the above sums $a + b - 2c > 0, p + q - 2r > 0, \ldots$ etc. That is, the power of $t$ is always $> 0$ in all the sums. This is because $a + b > \frac{2n}{3}$ and $2c < \frac{2n}{3}$. 

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Thus we see that if \( t \rightarrow 0 \) the point \( C \) in \( P^N \) tends to 0 implying that 0 is in the closure of the orbit of \( \tilde{C} \) a representative of \( C \) in \( \mathbb{C}^{N+1} \). Thus \( C \) is unstable. 

Let \( G \times H \) act on a projective variety \( X \) linearized with respect to an ample line bundle \( L \), where \( G \) and \( H \) are reductive algebraic groups. Then the group \( H \) acts on \( X \) by restriction. Let \( X^{ss}(H) \) be the set of semistable points for the action of \( H \) on \( X \). Let \( Y = X//H \) be the GIT quotient and let \( \pi \) be the quotient map \( X^{ss}(H) \rightarrow Y \). By GIT, \( Y \) comes equipped with an ample line bundle \( E \) such that \( \pi^*(E) = L^{\otimes m}|_{X^{ss}(H)} \) for some \( m > 0 \). Now \( G \) acts on the quotient space \( Y \) and the action is linearized with respect to the ample line bundle \( E \). We will need to know the unstable points for this action. Let us call this set \( Y^{un}(G) \). Let \( X^{un}(G \times H) \) be the locus of unstable points for the action of \( G \times H \) on \( X \) linearized by the line bundle \( L \). Consider the intersection \( V = X^{un}(G \times H) \cap X^{ss}(H) \subseteq X^{ss}(H) \).

**Proposition 14** \( \pi(V) = Y^{un}(G) \).

**Proof.** The proof follows from the equality 
\[
H^0(Y, E^{\otimes n}|_G) = H^0(X, L^{\otimes mn}|_{G \times H}) \text{ for all } n, \text{ and } m \text{ as described in the above paragraph.}
\]
Now let \( y \in Y \) be unstable. Let \( x \) be a preimage under \( \pi \). By definition of an unstable point, if \( s \in H^0(Y, E^{\otimes n}|_G) \) then \( s(y) = 0 \), for any \( n \). But from the above equality of groups, if \( t \in H^0(X, L^{\otimes k}|_{G \times H}) \) then \( t(x) = 0 \), for any \( k \). And this means that \( x \in X^{ss}(H) \) is \( G \times H \) unstable. The converse also follows.

**3. The Barth Map and GIT**

In this section we will show how the theory of jump lines can be used to study stability of sheaves. We begin with the classical notion of a jump line.

**Definition 15** Let \( F \) be a torsion free coherent sheaf on \( P^2 \) with \( c_1 = 0 \) and rank \( r \). A line \( l \subseteq P^2 \) is called a jump line for \( F \) if \( F|_l \neq r\mathcal{O} \).

The following proposition is a corollary of a famous theorem of Grauert and Mullich[13].

**Proposition 16** Let \( F \) be a semistable sheaf of rank 2 on \( P^2 \) with \( c_1(F) = 0 \). Then for a general line \( l \subseteq P^2 \) we have \( F|_l = \mathcal{O} \oplus \mathcal{O} \). That is, a generic line is non jumping.

The set of jump lines, in fact, has a better structure, as is proved by Barth.

**Theorem 17** (W.Barth[3]) Let \( F \) be semistable of rank 2 on \( P^2 \) with \( c_1 = 0 \). The set of jump lines of \( F \), \( J(F) \subseteq P^{2^*} \) is a curve of degree equal to \( c_2 \).

We explain the scheme structure of \( J(F) \) now. Consider the incidence manifold \( I \subseteq P^2 \times P^{2^*} \) defined by \( I = \{(x, l)/x \in l\} \). Let \( p_1 \) and \( p_2 \) be the projections \( p_1 : I \rightarrow P^2 \) and \( p_2 : I \rightarrow P^{2^*} \). Consider the sheaf \( R^1p_2_*p_1^*F(-1) \). Define \( J(F) \) to be the zero scheme of the Fitting ideal associated to the sheaf \( R^1p_2_*p_1^*F(-1) \).
Proposition 18. The Barth map, \( b \), which is the assignment \( g(H) \) defined on \( S \) is well defined on \( S^- \) equivalence classes of semistable sheaves. Hence we have a regular map on the moduli space \( b: \mathcal{M}_{P^2}(2,0,n) \to P^N \), where \( N = \frac{(n+1)(n+2)}{2} - 1 \).

Proof. Let \( F \) be semistable. Then there is a Jordan-Holder filtration \( (0) \subseteq F_1 \subseteq F \) such that both quotients \( F_1 \) and \( F/F_1 = G_1 \) have the same reduced Hilbert polynomial as \( F \). In particular, \( c_1 = 0 \) for both \( F_1 \) and \( G_1 \). We claim that \( J(F) = J(F_1) \cup J(G_1) \). In view of this let \( l \in J(F_1) \cup J(G_1) \). Then either \( H^0(F_1(-1)|_l) \neq 0 \) or \( H^0(G_1(-1)|_l) \neq 0 \) implying that \( H^0(F(-1)|_l) \neq 0 \). Conversely, if \( l \notin J(F_1) \cup J(G_1) \) then \( H^0(F_1(-1)|_l) = H^0(G_1(-1)|_l) = H^0(F(-1)|_l) = 0 \) which means that \( l \) is not a jump line for \( F \). Thus if \( f = 0 \) and \( g = 0 \) are the equations of \( J(F_1) \) and \( J(G_1) \) respectively then \( J(F) \) has equation \( fg = 0 \). Since the Jordan-Holder filtration is unique up to isomorphism of quotients, \( J(F) \) is well defined on \( S^- \) equivalence classes of sheaves. This completes the proof. \( \square \)

The Hulsbergen bundles offer examples of curves of jump lines. Let \( F \) be a Hulsbergen bundle defined by the sequence

\[
0 \to \mathcal{O} \to F(1) \to \mathcal{I}_Z(2) \to 0
\]

where \( Z \) is a zero scheme in \( P^2 \). Suppose that \( d + 1 \) point of \( Z \) lie on a line \( l \subseteq P^2 \). When restricted to this line \( l \) the sheaf \( \mathcal{I}_Z \) is simply \( \mathcal{O}(-d) \). The above defining sequence gives a surjection \( F|_l \to \mathcal{O}(-d) \to 0 \). Since \( F \) is rank 2 and \( c_1(F) = 0 \), the kernel of the map must be \( \mathcal{O}(d) \). Since \( H^i(F, \mathcal{O}(2d)) = 0 \) for \( d \geq 0 \), the restricted sequence splits and we get the decomposition \( F|_l = \mathcal{O}(d) \oplus \mathcal{O}(-d) \). Hence if \( d \geq 1 \) the line \( l \) is a jump line for \( F \).

One can in fact write down equations for the curve of jump lines. Let \( Z \) be the zero scheme of \( n + 1 \) distinct points \( \{x_1, \ldots, x_{n+1}\} \) in \( P^2 \) no three of which are collinear. Let \( L_1, \ldots, L_{n+1} \) be linear forms that are equations for the lines \( x_1^1, \ldots, x_{n+1} \) in \( P^{2n} \). Let \( f_i = \Pi_{j \neq i} L_j \). Let \( W = \text{span}\{f_1, \ldots, f_{n+1}\} \subseteq H^0(P^2, \mathcal{O}(n)) \). Hulsbergen proves that there is an isomorphism \( \text{Ext}(I_Z(2), \mathcal{O}) \simeq W = \mathbb{C}^{n+1} \) such that if \( F \in \text{Ext}(I_Z(2), \mathcal{O}) \) corresponds to \( (a_1, \ldots, a_{n+1}) \in W \) the the curve of jump lines of \( F, J(F) \), is given by the equation

\[
a_1 f_1 + \ldots + a_{n+1} f_{n+1} = 0.
\]

For a generic \( F \), the curve of jump lines is clearly nonsingular.

In order to exploit the Barth map to understand stability of sheaves we need the following comparison theorem of Reichstein [24] on a relative Hilbert-Mumford criterion.
Theorem 19 (Reichstein) Let $X$ and $Y$ be projective varieties on which a reductive group acts. Let $f : X \to Y$ be a $G$-equivariant map. Let the actions of $G$ on $X$ and $Y$ be linearized by ample line bundles $L$ and $M$ respectively. Let $L_d = f^* M^{\otimes d} \otimes L$. For large $d$ the following holds

(i) If $f(x) = y$ and $y$ is unstable, then $x$ is unstable under the linearization $L_d$,

(ii) If $f(x) = y$ and $y$ is stable, then $x$ is stable under the linearization $L_d$.

We now prove our first theorem for rank 2 sheaves.

Theorem 20 Let $F \in \mathcal{M}_{P^2}(2,0,n)$. Suppose there is a line $l \subseteq P^2$ such that $F|_l = \mathcal{O}(d) \oplus \mathcal{O}(-d)$ with $d > 2n^3$, then $F$ is unstable for the $SL(3)$ action suitably linearized. Also, a generic point in $\mathcal{M}_{P^2}(2,0,n)$ is stable.

Proof. With the previous background the proof is quite easy. Consider the Barth map $b : \mathcal{M}_{P^2}(2,0,n) \to P^N$. On $J(F)$ the line $l$ corresponds to a point of multiplicity $m$, with $m \geq d$. This can be seen in Barth’s fundamental paper \[3\]. Now since $d > 2n^3$, we have $m > 2n^3$. Hence $J(F)$ is unstable. Hence by Reichstein’s theorem $F$ is unstable under suitable linearization.

To prove that a generic point is stable we simply produce a stable point. Genericness follows from the irreducibility of the moduli space. Consider a zero scheme $Z$ of length $n + 1$ in $P^2$ such that no three points are on a line. Let $F$ be a vector bundle on $P^2$ given by the Hulsbergen sequence

$$0 \to \mathcal{O} \to F(1) \to I_Z(2) \to 0.$$ 

Then the curve of jump lines $J(F)$ is nonsingular and hence stable by Mumford’s result. But then by Reichstein’s theorem $F$ is also stable. The proof is complete. $\blacksquare$

3. Higher Rank: Group Action on $\mathcal{M}_{P^2}(r,0,n)$

When the rank $r \geq 3$ the facility of the Barth map is lost. Hulek\[14\] has tried to recover the theory of jump lines for higher rank. However, it turns out that in the event that a curve of jump lines can be assigned to a vector bundle, a lot of information is lost in the process. For one, the degree of the curve is much larger than $c_2$ and the multiplicities of points on the curve are very weakly related to the splitting of the bundle over the line. Thus in order to get sharp results we need to adopt a different approach.

In 1970 Horrocks realized a remarkable philosophy of recovering a vector bundle from its cohomology groups and certain maps between them. This is his theory of monads. Monadic representations allow us to view the moduli space as a GIT quotient of subvarieties of products of Grassmanians. In this sense it replaces the so-called Quot scheme. This structure is more convenient to study Picard groups and group actions. The GIT is reduced to calculations on products of Grassmanians. We start by stating a fundamental result of Beilinson on spectral sequences.
**Theorem 21** (Beilinson [3]) Let $E$ be a rank $r$ torsion free sheaf on $P^n$. Then there is a spectral sequence $E^p_q$, for $0 \leq q \leq n$ and $-n \leq p \leq 0$, with $E_1$ term given by $E_1^{p,q} = H^q(P^n, E(p)) \otimes \Omega^{-p}(-p)$, which converges to

$$E^i = E \text{ if } i = 0,$$

$$E^i = 0 \text{ otherwise.}$$

**Definition 22** A monad is a complex of sheaves

$$0 \to A \to B \to C \to 0$$

exact at $A$ and $C$ but not necessarily exact in the middle.

**Proposition 23** Let $E$ be a semistable sheaf of rank $r$ on $P^2$ with Chern classes $c_1$ and $c_2$. Assume that $E$ is normalized, that is, $-r < c_1 \leq 0$. Then $E$ is the middle cohomology of the monad

$$0 \to H^1(E(-2)) \otimes \mathcal{O}(-1) \to H^1(E(-1)) \otimes \Omega(1) \to H^1(E) \otimes \mathcal{O} \to 0.$$

**Proof.** Consider the $E_1$ term

$$E_1^{p,q} = H^q(P^n, E(p)) \otimes \Omega^{-p}(-p).$$

Since our base space is $P^2$, $E_1^{p,q} = 0$ if $q > 2$. Thus $q = 0, 1, 2$ are the only values of $q$ yielding non zero $E_1$ terms. Moreover, by semistability of $E$ we have $H^0(P^2, E(p)) = H^2(P^2, E(p)) = 0$ for $p = 0, 1, 2$. Thus $E_1^{0,0} = E_1^{p,2} = 0$ for $p = 0, -1$ and $-2$. The differential maps at level 1 are defined as $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$. We now have $\ker d_2^{-2,1} = E_2^{-2,1} = A$ (say), $\ker d_2^{-1,1}/imaged_2^{-2,1} = E_2^{-1,1} = B$ (say) and $\ker d_1^{-1,1} = E_1^{0,1} = C$ (say). The differentials at level 2 are defined $d_2^{p,q} : E_2^{p,q} \to E_2^{p+1,q}$ and hence are degenerate. The spectral sequence therefore degenerates at the $E_2$ term. By Beilinson’s theorem $E_2^{-2,1} = E_2^{0,1} = 0$, and $E_2^{-1,1} = E$. This simply means that we have a complex

$$0 \to E_1^{-2,1} \to E_1^{-1,1} \to E_1^{0,1} \to 0,$$

whose middle cohomology is $E$. Reading the terms of the complex we get the required monad whose middle cohomology is $E$. ■

**Corollary 24** Let $E$ be semistable of rank $r$ on $P^2$ with $c_1 = 0$ and $c_2 = r$. Then $E$ occurs as the cokernel in the exact sequence

$$0 \to \mathbb{C}^r \otimes \mathcal{O}(-1) \to \mathbb{C}^r \otimes \Omega(1) \to E \to 0.$$

**Proof.** First note that $\dim H^1(E) = -\chi(E) = c_2 - r = 0$. Adding this information in the statement of the above proposition we see that $E$ is the cokernel in the exact sequence of the form

$$0 \to H^1(E(-2)) \otimes \mathcal{O}(-1) \to H^1(E(-1)) \otimes \Omega(1) \to E \to 0.$$
By Riemann Roch theorem each of $H^1(E(-2))$ and $H^1(E(-1))$ are $r$ dimensional vector spaces. The result follows. ■

The corollary of Beilinson’s theorem helps us to construct the moduli space $\mathcal{M}(r, 0, n)$ as a GIT quotient of a certain ‘space of monads’. We describe this construction below.

Let $K, H, L$ and $V$ be fixed vector spaces of dimensions $n, n, n - r$ and 3 respectively. The projective plane under consideration will be $P^2 = P(V)$.

Consider the product of grassmannians $G(n, H \otimes V) \times G(H \otimes V^*, n - r)$, where $G(n, H \otimes V)$ is the grassmannian of subspaces of $H \otimes V$ of dimension $n$ and $G(H \otimes V^*, n - r)$ is the grassmannian of quotients of $H \otimes V^*$ of dimension $n - r$. A pair $(K, L)$ in $G(n, H \otimes V) \times G(H \otimes V^*, n - r)$ gives maps $a : K \otimes \mathcal{O}(-1) \rightarrow H \otimes \Omega(1)$, and $b : H \otimes \Omega(1) \rightarrow L \otimes \mathcal{O}$. Consider the subspace $\mathfrak{M} \subseteq G(n, H \otimes V) \times G(H \otimes V^*, n - r)$ consisting of pairs $(K, L)$ such that $b \circ a = 0$. This is the master space of monads. The group $SL(H)$ acts on $\mathfrak{M}$.

The following theorem of Le Potier is very crucial for us;

**Theorem 25 (Le Potier)** For the polarization $(rm - n, n)$ on $G(n, H \otimes V) \times G(H \otimes V^*, n - r)$ restricted to $\mathfrak{M}$, with $m$ sufficiently large, we have the isomorphism

$$\mathfrak{M} / SL(H) \simeq \mathcal{M}_{P^2}(r, 0, n).$$

The GIT quotient $\mathcal{M}_{P^2}(r, 0, n)$ comes equipped with an ample line bundle which we denote by $E_m$. Our discussion of stability is with respect to the action of $SL(3)$ linearized with respect to $E_m$.

We will need the following lemma.

**Lemma 26** Let $(K, L)$ be a pair defining a semistable sheaf $F$. Let $v \in V$ and let $v^\perp \subseteq V^*$ be the annihilating space for $v$. The injection $K \rightarrow H \otimes V$ gives a natural map $\alpha_v : K \rightarrow \text{Hom}(v^\perp, H)$ and this map is injective.

**Proof.** Suppose that there is a $v \in V$, $v \neq 0$ such that the map $\alpha_v$ is not injective. Then there is a $k \in K$, such that $k \neq 0$ and $\alpha_v(k) = 0$.

Blow up $P^2$ at the point $x = \mathbb{C}.v \in P^2$ and denote the $\sigma$ process at $x$ by $\sigma : \hat{P}^2 \rightarrow P^2$. Embed $\hat{P}^2$ in $P^2 \times P^1$ as an incidence manifold. Let $p$ and $q$ denote the projections to $P^2$ and $P^1$ respectively. Let $E \subset \hat{P}^2$ be the exceptional curve. Then we have $\sigma^* (\mathcal{O}_{P^2}(1)) = q^* (\mathcal{O}_{P^1}(1)) \otimes \mathcal{L}(E)$. Now we have the exact sequence

$$0 \rightarrow p^* F(-2) \otimes q^* (\mathcal{O}_{P^1}(-1)) \rightarrow p^* F(-1) \otimes q^* \mathcal{O}_{P^1} \rightarrow p^* F(-1) \otimes q^* \mathcal{O}_{P^1}|_{\hat{P}^2} \rightarrow 0.$$  

The direct image of this sequence under $q$ gives the following sequence on $P^1$,

$$0 \rightarrow q_* (p^* F(-2) \otimes q^* (\mathcal{O}_{P^1}(-1)))|_{\hat{P}^2} \rightarrow K \otimes \mathcal{O}_{P^1}(-1) \rightarrow H \otimes \mathcal{O}_{P^1}.$$  

A point $l \in P^1$ corresponds to a line $L \subset P^2$ such that the equation for $L$ is $z = 0$ for $z \in v^\perp \subseteq V^*$. Thus the rank one subsheaf $k \otimes q_* (p^* F(-1) \otimes q^* \mathcal{O}_{P^1})$
lies in the kernel of the right hand arrow. Hence \( q_*(p^* F(-1) \otimes q^* \mathcal{O}_{P^1}) \) contains \( \mathcal{O}_{P^1}(-1) \) as a subsheaf and we have

\[
p^* F(-1) \otimes q^* \mathcal{O}_{P^1}(1) \simeq \sigma^* F \otimes \mathcal{L}(E)^{-1} \subset \sigma^* F
\]

which implies that \( \sigma^* F \) contains a nontrivial section. But this is a contradiction since we know that by semistability \( H^0(F) = 0 \). This concludes the proof. □

In a similar vein we have the following proposition.

**Proposition 27.** If \( V^t \subseteq V^* \) is a one dimensional subspace of \( V^* \) then the image of the restricted map \( H \otimes V^t \to L \) is \( (0) \).

**Proof.** We know that \( 0 \to L^* \to H^* \otimes V \). If \( v \in V \), we will show that \( L^* \cap (H^* \otimes (v)) = (0) \). The proof is along the same lines as the previous lemma except that we need to use properties of \( F^* \) namely that it is torsion free and \( H^0(F^*(-1)) = (0) \). By Serre duality \( L^* = H^1(F)^* \simeq H^1(F^*(-3)) \), and similarly \( H^* = H^1(F(-1))^* \simeq H^1(F^*(-2)) \). Suppose that we have \( \beta \in L^* \cap (H^* \otimes (v)) \), \( \beta \neq 0 \). Let \( v^\perp = \text{ann}(v) \subseteq V^* \) be the annihilating subspace in \( V^* \).

Blow up \( P^2 \) at the point \( x = \mathbb{C}.v \in P^2 \), and denote this \( \sigma \) process at \( x \) by \( \sigma; \widehat{P^2} \to P^2 \). Embed \( \widehat{P^2} \) in \( P^2 \times P^1 \) as an incidence manifold. Let \( p \) and \( q \) denote the projections to \( P^2 \) and \( P^1 \) respectively. Let \( E \subseteq \widehat{P^2} \) be the exceptional curve. Then we have \( \sigma^* (\mathcal{O}_{P^1}(1)) = q^*(\mathcal{O}_{P^1}(1)) \otimes \mathcal{L}(E) \). Now we have the exact sequence

\[
0 \to p^* F^*(-3) \otimes q^* (\mathcal{O}_{P^1}(1)) \to p^* F^*(-2) \otimes q^* \mathcal{O}_{P^1} \to p^* F^*(-2) \otimes q^* \mathcal{O}_{P^1}|_{\widehat{P^2}} \to 0.
\]

The direct image under \( q \) gives the following sequence on \( P^1 \),

\[
0 \to q_*(p^* F^*(-2) \otimes q^* \mathcal{O}_{P^1})|_{\widehat{P^2}} \to L^* \otimes \mathcal{O}_{P^1}(-1) \to H^* \otimes \mathcal{O}_{P^1}.
\]

A point \( l \in P^1 \) corresponds to a line \( L \subseteq P^2 \) such that the equation for \( L \) is \( z \in v^\perp \subseteq V^* \).

Thus the rank one subsheaf \( k \otimes q_* (p^* F^*(-2) \otimes q^* \mathcal{O}_{P^1}) \) lies in the kernel of the rightmost arrow. Thus \( q_*(p^* F^*(-2) \otimes q^* \mathcal{O}_{P^1}) \) contains \( \mathcal{O}_{P^1}(-1) \) as a subsheaf and hence

\[
p^* F^*(-2) \otimes q^* F^*(1) \simeq \sigma^* F^*(-1) \otimes \mathcal{L}(E)^{-1} \subset \sigma^* F^*(-1).
\]

This implies that \( \sigma^* F^*(-1) \) contains a non trivial section. But this is a contradiction since we know that \( H^0(F^*(-1)) = 0 \). This concludes the proof. □

We now begin to study actions on products on Grassmannians. For convenience we will consider Grassmannians of subspaces only rather than Grassmannians of quotients. Results for the latter can be obtained simply by dualizing the actions.

**Proposition 28.** Let \( G = SL(H) \times SL(V) \) act on the space \( Y = G(n, H \otimes V) \times G(m, H \otimes V) \) linearized by the line bundles \( (p, q) \in \text{Pic}(X) \). Then \( SL(V) \) acts
on $Y$ by restriction. A point $(K, L) \in Y$ is unstable with respect to the $SL(V)$ action if and only if there is a subspace $V' \subseteq V$ such that $K' = K \cap H \otimes V'$ and $L' = L \cap H \otimes V'$ then
\[
\frac{p \dim K' + q \dim L'}{\dim V'} > \frac{p \dim K + q \dim L}{\dim V}.
\]

**Proof.** First assume that the pair $(K, L)$ satisfies
\[
\frac{p \dim K' + q \dim L'}{\dim V'} \leq \frac{p \dim K + q \dim L}{\dim V}
\]
for every $V' \subseteq V$. We will prove that $(K, L)$ is semistable. Let $\lambda$ be a 1--parameter subgroup of $SL(V)$. Choose subspaces $V_j \subseteq V$ such that $(v_j) = t^{u_j}v_j$ for $v_j \in V_j$ and $t \in \lambda$. Let $F_i = \oplus_{p=1}^{i} H \otimes V_p$. Without loss of generality we can assume that $u_1 > u_2 > ...u_s$. Then we have the flag
\[
F_1 \subseteq F_2 \subseteq ...F_s = H \otimes V.
\]
Define $K_i = K \cap F_i$ and $L_i = L \cap F_i$ for $i = 1, ...s$. Now we know that
\[
-\mu((K, L), \lambda) = -\mu(K, \lambda) - q\mu(L, \lambda).
\]
Hence
\[
-\mu((K, L), \lambda) = p\sum_{i=1}^{s} u_i (\dim K_i - \dim K_{i-1}) + q\sum_{i=1}^{s} u_i (\dim L_i - \dim L_{i-1}).
\]
The assumed inequality for $(K, L)$ gives us
\[
(u_i - u_{i})(p \dim K_{i-1} + q \dim L_{i-1}) \leq c(\dim F_{i-1})(u_i - u_{i})
\]
for $i = 2, ...s$ and
\[
u_s(p \dim K_s + q \dim L_s) \leq c \dim F_s u_s.
\]
Adding all these inequalities we get $\mu \geq 0$. By the Hilbert-Mumford criterion this means that $(K, L)$ is a semistable point.

Conversely, suppose that there is a subspace $V' \subseteq V$ such that we have the inequality
\[
\frac{p \dim K' + q \dim L'}{\dim V'} \geq \frac{p \dim K + q \dim L}{\dim V}.
\]
Let $V = V' \oplus V''$. Take the 1--parameter group $\lambda \subseteq SL(V)$ that has the action defined by $t(v') = t^{\dim V'}v'$ for $v' \in V'$ and $t(v'') = t^{-\dim V''}v''$ for $v'' \in V''$. Calculating $\mu$ we find that $\mu((K, L), \lambda) < 0$, which proves unstability.

For any subspace $V' \subseteq V$, there is an induced map $L \to L' \to 0$, where $L'$ is defined as follows: The injection $V' \to V$ gives a surjection $V^* \to V'^*$ with a kernel $W^*$. We get a map $H \otimes W^* \to L$ by restriction. If $L''$ is the image then $L'$ is defined as $L/L''$. We use this construction in the proposition below.
Corollary 29 Let $SL(H) \times SL(V)$ act on the space $X = G(k, H \otimes V) \times G(H \otimes V^*, m)$ where the action of $SL(V)$ on $V^*$ is defined by transpose. Let the action be linearized by the ample line bundle $(p, q)$ on $X$. A point $(K, L)$ is unstable for the $SL(V)$ action if and only if there is a subspace $V' \subseteq V$ such that if $K' = K \cap H \otimes V'$ and $L'$ is as defined above we have

$$\frac{p \ dimK' + q \ dimL'}{dimV'} > \frac{p \ dimK + q \ dimL}{dimV}.$$ 

Proof. By dualizing we can think of $G(H \otimes V^*, m)$ as the grassmannian of subspaces $G(m, H^* \otimes V)$. Under this identification let the pair $(K, L)$ correspond to the pair $(K, W)$. Then from the proof of the above proposition, $(K, W) \in G(m, H^* \otimes V)$ is $SL(V)$ unstable if and only if there is a subspace $V' \subseteq V$ such that if $K' = K \cap H \otimes V'$ and $W' = W \cap H^* \otimes V'$, we have

$$\frac{p \ dimK' + q \ dimW'}{dimV'} > \frac{p \ dimK + q \ dimW}{dimV}.$$ 

The vector spaces $L$ and $W$ are related by the th equality $W = L^*$. The subspace $W'$ of $W$ thus corresponds to a subspace $L'^*$ of $L^*$. Hence we get the quotient $L \rightarrow L'$. Since $L'$ and $W'$ are of the same dimension the above inequality reduces to

$$\frac{p \ dimK' + q \ dimL'}{dimV'} > \frac{p \ dimK + q \ dimL}{dimV'},$$

which is the stated criterion. The proof is complete. \[\square\]

Corollary 30 Let $F \in \mathcal{M}_{P^2}(r, 0, n)$ and let $(K, L) \in G(n, H \otimes V) \times G(H \otimes V^*, n-r)$ be a pair corresponding to $F$. If $(K, L)$ is such that there is a vector subspace $V' \subseteq V$ with

$$\dim K' > \frac{\dim V' \cdot \dim K}{\dim V},$$

where $K' = H \otimes V' \cap K$. Then $(K, L)$ is $SL(V)$ unstable. Moreover, for the above inequality to hold we need $\dim V' = 2$.

Proof. Note that for the inequality in the hypothesis to hold we must have $\dim V' = 2$. For if $\dim V' = 1$, then $\dim K' = 0$, by the lemma proved before and the inequality does not hold. We know that $(K, L)$ is unstable if and only if there is a subspace $V' \subseteq V$ such that if $K' = K \cap H \otimes V'$ and $L'$ as defined before we have

$$\frac{(rm - n) \ dimK' + n \ dimL'}{dimV'} > \frac{(rm - n) \ dimK + n \ dimL}{dimV}$$

where $m >> 0$. Simplifying the above inequality gives

$$\frac{\dim K'}{\dim V'} > \frac{\dim K}{\dim V} + \frac{1}{rm - n} \left( -n \frac{\dim L'}{\dim V'} + n \frac{\dim L}{\dim V} \right).$$
Since $m$ is very large the above inequality is equivalent to
\[ \dim K' > \frac{\dim V' \dim K}{\dim V} \pm \varepsilon \]
where $\varepsilon > 0$, and is very small. Since $\dim K'$ is an integer, the inequality in the hypothesis,
\[ \dim K' > \frac{\dim V' \dim K}{\dim V}, \]
implies the inequality for unstability. The proof is complete. \( \blacksquare \)

This will be crucial in the final steps of the proof. Roughly, the 'undemocratic' polarization leads the second factor to eventually drop out of the analysis.

The following theorem relates the geometry of $F$ to its $SL(V)$ unstability.

**Theorem 31** Let $F \in \mathcal{M}_{P^2}(r, 0, n)$. If there is a line $l \subseteq P^2$ such that $F|_l = \oplus \mathcal{O}(d_i) \oplus_{j=1,...,p} C_j$ with
\[ \Sigma d_i \geq 0 \quad d_i = 1, \ldots, p, \quad C_j \]
then $F$ is unstable under the $SL(3)$ action on $\mathcal{M}_{P^2}(r, 0, n)$ with respect to the polarization $E_m$.

**Proof.** Assume that there is a line $l \subseteq P^2$ with the given property of $F$. Let $(K, L)$ be a point in the space of monads that defines $F$. Let $l = P(V')$, where $\dim V' = 2$. We prove that $V'$ is a destabilizing subspace for the pair $(K, L)$.

We have the exact sequence
\[ 0 \rightarrow F(-2) \rightarrow F(-1) \rightarrow F(-1)|_l \rightarrow 0, \]
from which we get the cohomology sequence
\[ 0 \rightarrow H^0(F(-1)|_l) \rightarrow K \rightarrow H \rightarrow H^1(F(-1)|_l) \rightarrow 0. \]

Then $K' = K \cap H \otimes V'$ is also the image of $H^0(F(-1)|_l)$ in $K$. To check this first recall that $K$ is a subspace of $H \otimes V$. For $l \in V^*$ the map $\phi_l : K \rightarrow H$ is obtained as follows. Let $k \in K$ be written as $k = \Sigma h_i \otimes v_i$. Then $\phi_l(k) = \Sigma \phi_l(v_i) h_i$. Since $l = P(V')$, $\phi_l$ annihilates $V'$ and hence if $k \in K \cap H \otimes V'$ then $\phi_l(k) = 0$ which means that $k \in H^0(F(-1)|_l)$.

Conversely, assume that $\phi_l(k) = 0$. Think of $k$ as $k = h_1 \otimes v_1 + h_2 \otimes v_2 + h_3 \otimes v_3$ where $v_1, v_2 \in V'$ and $v_3 \notin V'$. Then $\phi_l(k) = \phi_l(v_3) h_3 = 0$, which implies that $h_3 = 0$, and hence $k \in K \cap H \otimes V'$. Thus $K'$ is the image of $H^0(F(-1)|_l)$ in $K$. A simple computation on $P^1$ yields $\dim K' = \Sigma d_i \geq 0 + p$. Since $\dim V = 3$ and $\dim K = n$, the given inequality is therefore the same as
\[ \dim K' > \frac{\dim V' \dim K}{\dim V}, \]
which is the unstability criterion for the pair \((K, L)\) for the \(SL(V)\) action on the space of monads. Hence the sheaf \(F\) is unstable for the \(SL(3)\) action by pullback, linearized with respect to the line bundle \(E_m\). The proof is complete.

Note that when \(F\) is a vector bundle there are no skyscraper sheaves in the splitting of \(F\) over a line.

Our next task is to prove that for the action of \(SL(3)\) on \(\mathcal{M}_{P^2}(r, 0, n)\) a generic point is stable. We will prepare for this direction by first giving examples of semistable sheaves of arbitrary high rank that are \(SL(3)\) unstable. The Serre criterion helps us construct bundles of rank 2 with desired splitting properties when restricted to lines in \(P^2\). Choose \(n + 1\) points \(x_1, \ldots, x_{n+1}\) in \(P^2\) such that \(d + 1\) of them lie on a line \(l\), \(d \neq n\). Let \(Z\) be the zero scheme corresponding to the \(n + 1\) points. Let \(F'\) be given as an extension

\[
0 \rightarrow \mathcal{O}_{P^2} \rightarrow F' \rightarrow I_Z(2) \rightarrow 0.
\]

Let \(F = F' \otimes \mathcal{O}(-1)\). Then \(c_1(F) = 0\) and \(c_2(F) = n\). Also, since not all the points are on a line, \(F\) is a stable bundle. Moreover, \(F|_l = \mathcal{O}(d) \oplus \mathcal{O}(-d)\). Choosing \(d > \frac{2n}{d+1}\), \(d \neq n\), we have lots of examples of stable bundles that are \(SL(3)\) unstable. The following proposition enables a construction of \(SL(3)\) unstable sheaves in higher rank.

**Proposition 32** Fix an integer \(q < n\), and a line \(l \subseteq P^2\). Then there are \(F \in \mathcal{M}_{P^2}(r, 0, n)\) such that \(F|_l = \oplus \mathcal{O}(d_i) \oplus_{j=1,...,p} C_j\) where \(C_j\) are skyscraper sheaves and

\[
\Sigma_{d_i \geq 0} d_i + p = q.
\]

**Proof.** Note first that it is enough to prove the existence of such an \(F\) for some line \(l' \subseteq P^2\). For then, we can choose an automorphism \(f\) of \(P^2\) such that \(f^{-1}(l') = l\), and then \(f^*(F)\) provides the required example. The proof will be by induction on the rank of the sheaf. The statement is true for rank 2 sheaves by the Serre construction described above. Assume now that the statement holds for rank \(r\). We need to construct rank \(r + 1\) sheaves with the desired property.

Suppose that \(F\) is semistable of rank \(r\) such that there is a line \(l \subseteq P^2\) with \(F|_l = \oplus \mathcal{O}(d_i) \oplus_{j=1,...,p} C_j\) and \(\Sigma_{d_i \geq 0} d_i + p = q\). Recall that \(H^1(F) = \text{Ext}(\mathcal{O}, F) = n - r\). Consider the extension

\[
0 \rightarrow F \rightarrow G \rightarrow \mathcal{O} \rightarrow 0.
\]

Then \(G\) is semistable and we have \(0 \rightarrow H^0(F(-1)|_l) \rightarrow H^0(G(-1)|_l) \rightarrow 0\). Let \(G|_l = \oplus \mathcal{O}(e_i) \oplus_{j=1,...,s} D_j\). Then

\[
H^0(G(-1)|_l) = \Sigma_{e_i \geq 0} e_i + s = H^0(F(-1)|_l) = \Sigma_{d_i \geq 0} d_i + p = q.
\]

Choosing \(q > \frac{2n}{r+1}\) we can construct examples of unstable sheaves of arbitrary rank.
Theorem 33 For the $SL(3)$ action on $\mathcal{M}_{P^2}(r, 0, n)$, linearized by the line bundle $E_m$, a generic point is stable.

Proof. It is enough to produce one stable point. Since the space $\mathcal{M}_{P^2}(r, 0, n)$ is irreducible genericness will follow. We know, from the Hulsbergen construction that a generic point in $\mathcal{M}_{P^2}(2, 0, n)$ is stable. The proof will be by induction on the rank of the sheaf $r \leq n$. Assume that there is a $SL(3)$ stable point $G$ in $\mathcal{M}_{P^2}(r-1, 0, n)$. Consider a non split extension $0 \rightarrow G \rightarrow F \rightarrow O \rightarrow 0$. Then $F$ is Geiseker semistable with rank $r$ and Chern classes $c_1 = 0$ and $c_2 = n$ and hence determines a point in $\mathcal{M}_{P^2}(r, 0, n)$. We will show that $F$ is stable under the $SL(3)$ action. In order to show this we need to prove that the pair $(K(F), L(F))$, where $K(F) \simeq H^1(F(-2))$ and $L(F) \simeq H^1(F)$, is $SL(H(F)) \times SL(V)$ stable in the space of monads under the action of $SL(H(F)) \times SL(V)$, where $H(F) \simeq H^1(F(-1))$. Let $T_G = SL(H(G)) \times SL(V)$ and $T_F = SL(H(F)) \times SL(V)$. Note that from the sequence defining $F$ we get $K(G) \simeq K(F) = K$(say) and $H(G) \simeq H(F) = H$(say). Thus $T_G \simeq T_F = T$ (say). For computational convenience we will think of the space of monads as a subset of product of grassmanians of subspaces rather than quotients. A pair $(K, L)$ corresponds to a pair $(K, W)$ under this identification.

Let $\lambda \subseteq T$ be a 1-parameter subgroup. Then $\lambda = \lambda_1, \lambda_2$ where $\lambda_1 \subseteq SL(H)_t$ and $\lambda_2 \subseteq SL(V)$ are 1-parameter subgroups. Choose $H_i \subseteq H$ such that $H = \oplus_i H_i$ and $\lambda_1(h_i) = t^{p_i} h_i$ for $h_i \in H_i$, and $V_j \subseteq V$ such that $\lambda_2(v_j) = t^{q_j} v_j$ for $v_j \in V_j$. Then rearrange so that

$$H \otimes V = \oplus_{i,j} E_{i,j} = \oplus_{p=1,\ldots,s} E_p$$

where $E_{i,j} = H_i \otimes V_j$ and $\lambda(e_p) = t^{u_p} e_p$ for $e_p \in E_p$ and $u_1 > u_2 > \ldots > u_s$. Note that by construction if $E_p = E_{i,j}$ then $u_p = p_i + q_j$. Let $F_i = \oplus_{p=1,\ldots,s} E_p$. Then we have a flag

$$F_1 \subseteq F_2 \subseteq \ldots \subseteq F_s = H \otimes V.$$ Define $K_i = K \cap F_i$ and $W_i = W \cap F_i$ for $i = 1, \ldots, s$.

We know that $\mu((K(G), W(G)), \lambda) > 0$, by the stability of $G$. We need to prove that $\mu((K(F), W(F)), \lambda) > 0$. Stability of $F$ will follow from the Hilbert-Mumford criterion. Now for the action linearized by $(p, q)$,

$$-\mu((K, W), \lambda) = -p\mu(K(G), \lambda) - q\mu(W(G), \lambda)$$

hence

$$-\mu((K(G), W(G)), \lambda) = p\Sigma u_i (\dim K_i - \dim K_{i-1}) + q\Sigma u_i (\dim W_i(G) - \dim W_{i-1}(G))$$

and

$$-\mu((K(F), W(F)), \lambda) = p\Sigma u_i (\dim K_i - \dim K_{i-1}) + q\Sigma u_i (\dim W_i(F) - \dim W_{i-1}(F))$$

Simplifying the equalities gives

$$\frac{-1}{p}\mu((K(G), W(G)), \lambda) = \Sigma u_i (\dim K_i - \dim K_{i-1} - \frac{q}{p}(\dim W_i(G) - \dim W_{i-1}(G))).$$
Recall that for the action on $M_{\mathcal{P}^2(r-1,0,n)}$, $p = (r-1)m - n$ and $q = n$. For large $m$, we have $p >> q$ and hence
\[
\frac{-1}{p} \mu((K(G), W(G)), \lambda) \sim \Sigma u_i(dimK_i - dimK_{i-1}) < 0,
\]
since $\frac{-1}{p} \mu((K(G), W(G)), \lambda) > 0$. When we consider the action on $M_{\mathcal{P}^2(r,0,n)}$, $p = rm - n$ and $q = n$ again for large $m$, $p >> q$ and we have $\frac{-1}{p} \mu((K(F), W(F)), \lambda) \sim \Sigma u_i(dimK_i - dimK_{i-1})$. Since the right hand side has been proved to be negative, we get $\mu((K(F), W(F)), \lambda) > 0$ thus proving stability of $F$.

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