Freezing and Thawing of Wet Soil from the Surface and Around the Underground Pipeline

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Abstract. This paper reveals a method for an approximate solution to the heat exchange problem with a phase transition in a wet finely dispersed soil in an infinite one-dimensional axisymmetric region. Some artificial techniques help to replace non-monotonic functions with a certain combination of monotonic ones. Then it becomes possible to apply comparison theorems or integral inequalities to draw the boundaries of the solution. The solution has the form of a system of functions that alternately find a majorant of the desired solution from top to bottom. The solution error is the value known and controlled at each stage of the solution. The proposed method can be used in engineering calculations, as well as a reference for estimating the accuracy of numerical and approximation techniques.

1. Introduction
The paper presents a mathematical apparatus that allows extending the results [3] to heat exchange problems in a domain with axial symmetry.

The technique for problems of freezing and thawing in a region with plane-parallel symmetry was explicitly developed and described in a review paper [3]. Being modified the method can be applied to heat exchange problems in all spatial domains for which Green's functions are known. It is obvious, for example, that one can solve a problem in any one-dimensional cylindrical region. In practical terms, this is a solution to the problem of freezing – thawing near a pipeline or an oil (gas) well.

The relevance of the work is determined by the fact that the solution of the thermal conductivity problem can be used in engineering calculations, as well as in defining the accuracy of various approximate methods.

Scientific novelty is related to a fundamentally new theoretical idea of applying the method of developing a narrowing rating system to a region with axial symmetry.

2. Formulation of the problem
Heat transfer processes with phase transitions, for example, seasonal thermophysical processes in wet soils, are usually described by the Stefan problem [1]. Accurate solutions to the Stefan problem are either obtained with restrictive assumptions [1], or cannot be applied for practical calculations. In engineering practice, various approximate methods are used [2-7, 10-22]. When a rigorous approach is demanded, numerical methods are applied.
It is possible to solve the problem of determining the front of the phase transition using the method of forming a narrowing system of estimates. If the convergence of the recurrent procedure for obtaining estimates is proved, we get a solution of a specific structure, the properties of which are similar to the properties of a convergent alternating sign series. Namely, at each step one can easily get the value of the maximum magnitude of the absolute error. The sign of error is also known, which is also important for applied problems.

The technique for problems of freezing and thawing in a region with plane-parallel symmetry was explicitly developed and described in a review paper [3]. This paper outlines a mathematical apparatus that allows extending the results [3] to heat exchange problems in a domain with axial symmetry. The problem of heat exchange with a phase transition in wet fine soil in an infinite one-dimensional axisymmetric region is written as follows [8]:

\[
\frac{\partial t}{\partial \tau} = a^2 \left( \frac{\partial^2 t}{\partial r^2} + \frac{a^2}{r} \frac{\partial t}{\partial r} - \frac{\tau}{c} \frac{\partial w(t)}{\partial \tau} \right)
\]

(1)

\[
t(\tau_0, \tau) = F(\tau), \quad t(\tau,0) = 0, \quad \tau > 0, \quad r > \tau_0
\]

\[
|t(r, \tau)| < M, \quad M = \text{const} > 0, \quad F(0) = 0
\]

(2)

where \( a^2 = \frac{\lambda}{c} \) - temperature conductivity coefficient, \( t, c, \lambda, \tau, w, \chi \) - temperature, heat capacity, heat conductivity coefficient, time, spatial coordinate, the content of unfrozen moister, latent heat of water phase transition, respectively.

For simplicity, we assume that \( F(\tau) \) is a monotone function, differentiable for \( \tau > 0 \) with bounded variation. Transition from monotone boundary conditions to non-monotonic ones is explicitly described in [3].

The first two summands on the right-hand side form the product of the coefficient by the one-dimensional Laplace operator in cylindrical coordinates, and \( \frac{\chi}{c} \frac{\partial w}{\partial \tau} \) is the source function. Here is

\[
\frac{\partial w(t)}{\partial \tau} = \frac{dw(t)}{dt} \cdot \frac{\partial t}{\partial \tau}, \quad \text{where} \quad \frac{dw}{dt} > 0 \text{ is experimentally determined function for this type of soil ("unfrozen water curve").}
\]

3. Theory (theoretical part)

It is necessary to know the Green function for the integral representation of problem (1) - (2). The Green function has a simple form only for a plane semi-infinite domain [1].

The essence of the proposed approach is that equation (1) can be formally considered as an equation for a plane semi-infinite domain \( r > \tau_0 \). In this case, on the right-hand side \( \frac{\partial^2 t}{\partial r^2} \) is the Laplace operator, and the two summands are considered as the source function.

Then the integral representation of problem (1) - (2) has the form:

\[
t(r, \tau) = n_0(r, \tau) - \frac{\chi}{c} \int \left\{ G(r, \xi, \tau - y) \frac{\partial w}{\partial \xi} d\xi \right\} dy + a \int \frac{\chi}{c} \frac{\partial t}{\partial \xi} G(r, \xi, \tau - y) \frac{1}{\xi} d\xi dy
\]

(3)
where
\[ G(r, \xi, \tau - y) = \frac{1}{2\sqrt{\pi a^2(\tau - y)}} \left( e^{\frac{-(r - \xi)^2}{4a^2(\tau - y)}} - e^{\frac{-(r + \xi)^2}{4a^2(\tau - y)}} \right) \]
is the Green's function,
\[ t_0(r, \tau) \] is solution of the equation
\[
\frac{\partial t_0}{\partial \tau} = a^{2} \frac{\partial^{2} t_0}{\partial r^{2}} \text{ with conditions of type (2)}.
\]

To get rid of the derivatives \( \frac{\partial w(t(\xi), y)}{\partial y}, \frac{\partial t(\xi, y)}{\partial \xi} \) under the integral sign, the integration formula is used by parts.

The result:
\[
t(r, \tau) = V(r, \tau) + \frac{\chi}{c} \int_{r_0}^{r} \[ \psi(r, \xi, \tau - y)w(t(\xi), y)dy - a^{2} \int_{r_0}^{r} K(r, \xi, \tau - y)t(\xi, y)dy - \]
\[
- a^{2} \int_{r_0}^{r} \int_{r_0}^{r} G(r, r_0, \tau - y) t(r_0, y)dy
\]
(4)

where

\[
\psi(r, \xi, \tau - y) = \frac{\partial G(r, \xi, \tau - y)}{\partial y}, \quad K(r, \xi, \tau - y) = -\frac{\partial}{\partial \xi} \left( \frac{G(r, \xi, \tau - y)}{\xi} \right),
\]
\[
V(r, \tau) = t_0(r, \tau) + \frac{\chi}{c} w(t(r, 0)) \left[ \text{erf} \left( \frac{r}{2\sqrt{a^2\tau}} \right) - \frac{1}{2} \text{erf} \left( \frac{r + r_0}{2\sqrt{a^2\tau}} \right) + \frac{1}{2} \text{erf} \left( \frac{r - r_0}{2\sqrt{a^2\tau}} \right) \right]
\]
\[
- \frac{\chi}{c} w(t(r, \tau))
\]

Integral equation (4) has the form suitable for obtaining a narrowing system of estimates for the solution of problem (1) - (2). Suppose we have estimates \( t_1, t_2 \) such that \( t_1 \leq t \leq t_2 \).

We define \( \psi, \psi_1, \psi_2 \) as follows:
\[
\psi_1 = \psi, \psi_2 \equiv 0 \text{ with } \psi > 0
\]
\[
\psi_1 = 0, \psi_2 = -\psi \text{ with } \psi \leq 0.
\]

It's obvious that
\[
\psi(r, \xi, \tau - y) = \psi_1(r, \xi, \tau - y) - \psi_2(r, \xi, \tau - y).
\]

Similarly, we define \( K_1, K_2 \):
\[
K_1 = K, \quad K_2 \equiv 0 \text{ with } K > 0
\]
\[
K_1 \equiv 0, \quad K_2 = -K \text{ with } K \leq 0.
\]

Then
\[
K(r, \xi, \tau - y) = K_1(r, \xi, \tau - y) - K_2(r, \xi, \tau - y).
\]
Using estimates $t_1, t_2$ for $t(r, \tau)$, we obtain refined boundaries $t_3, t_4$ from the following expressions:

$$
t_3(r, \tau) = V(r, \tau) + \sum_{\xi} \int_{0}^{\infty} \left[ \psi_1(r, \xi, \tau - y)w(t_1(\xi, y)) - \psi_2(r, \xi, \tau - y)w(t_2(\xi, y)) \right] d\xi dy - \frac{a^2}{r_0} \int_{0}^{\infty} \left[ K_1(r, \xi, \tau - y)t_1(\xi, \tau) - K_2(r, \xi, \tau - y)t_2(\xi, \tau) \right] d\xi dy - \frac{a^2}{r_0} \int_{0}^{\infty} G(r, r_0, \tau - y) t(r_0, y) dy
$$

$$
t_4(r, \tau) = V(r, \tau) + \sum_{\xi} \int_{0}^{\infty} \left[ \psi_1(r, \xi, \tau - y)w(t_2(\xi, y)) - \psi_2(r, \xi, \tau - y)w(t_1(\xi, y)) \right] d\xi dy - \frac{a^2}{r_0} \int_{0}^{\infty} \left[ K_1(r, \xi, \tau - y)t_2(\xi, \tau) - K_2(r, \xi, \tau - y)t_1(\xi, \tau) \right] d\xi dy - \frac{a^2}{r_0} \int_{0}^{\infty} G(r, r_0, \tau - y) t(r_0, y) dy
$$

(5)

(6)

It is clear that $t_1 \leq t_3 \leq t \leq t_4 \leq t_2$.

Using $t_3, t_4$, according to the formulas similar to (5) - (6), we obtain the estimates $t_5, t_6$ and then the procedure is recursively repeated.

The work [3] shows that it is not difficult to ensure that the condition of convergence of such a procedure is performed.

The proof of the convergence of the procedure, as well as the description of the economical numerical scheme used to implement it, is beyond the scope of this work.

It should be noted, however, both of them are performed according to the general methodology described in [3].

Thus, there has been obtained the scheme to solve the problem of soil freezing-thawing in a one-dimensional axisymmetric region. The scheme represents the procedure for obtaining a narrowing family of estimates of the desired function from top to bottom.

4. The results of numerical modeling
The numerical implementation of the procedure of this method for problem (5) - (6) was carried out on a computer according to the scheme specified in [3], and the following function was used as the boundary condition:

$$
F(x) = A \sin(\omega x + \phi), \text{where } A = 20^\circ C, \omega = 0.00071726, \phi = 3.141592
$$

The following thermophysical characteristics were taken:

$$
c = 500 \frac{\text{kDj}}{\text{m}^3 \cdot \text{K}}, a = 0.0028 \frac{\text{m}^2}{\text{s}}, \chi = 80 \frac{\text{kDj}}{\text{m}^3 \cdot \text{s}}, w_0 = 0.3, r_0 = 0.25 \text{ m}.
$$

The content of unfrozen soil moisture was calculated using the formulas:

$$
w = \begin{cases} 
  w_0 \cdot e^{-\frac{t^2}{t^2}}, & \text{if } t < 0 \\
  w_0, & \text{if } t \geq 0
\end{cases}
$$
The computational results are recorded in Table 1. Moreover, it should be noted that a similar solution can be provided for a one-dimensional region with central symmetry. Such a solution is especially valuable for a qualitative study of the freeze-thaw processes in areas of spherical shape [9].

| Table 1. Soil freezing temperature. |
|-------------------------------------|
| $\tau (h)$ – time, $r (m)$ – freezing radius. |
| $r$ (m) | 0.25 | 0.45 | 0.65 | 0.85 | 1.05 | 1.25 | 1.45 | 1.65 | 1.85 | 2.05 |
| 24    | -0.3 | -0.1 | -0.1 | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  | 0.0  |
| 240   | -3.4 | -1.3 | -0.7 | -0.3 | -0.3 | -0.2 | -0.1 | -0.1 | -0.1 | -0.0 |
| 480   | -6.8 | -3.3 | -1.2 | -0.4 | -0.3 | -0.2 | -0.2 | -0.1 | -0.1 | -0.1 |
| 720   | -9.9 | -4.6 | -1.8 | -0.6 | -0.3 | -0.2 | -0.2 | -0.1 | -0.1 | -0.1 |
| 960   | -12.7| -5.3 | -2.5 | -0.6 | -0.3 | -0.2 | -0.2 | -0.2 | -0.2 | -0.1 |
| 1200  | -15.2| -5.8 | -3.4 | -0.7 | -0.4 | -0.3 | -0.2 | -0.2 | -0.2 | -0.1 |
| 1440  | -17.2| -7.3 | -4.1 | -1.2 | -0.5 | -0.3 | -0.3 | -0.2 | -0.2 | -0.1 |
| 1680  | -18.7| -9.1 | -5.7 | -2.2 | -0.7 | -0.4 | -0.3 | -0.3 | -0.2 | -0.1 |
| 1920  | -19.6| -10.2| -6.9 | -3.2 | -0.9 | -0.4 | -0.3 | -0.3 | -0.2 | -0.1 |
| 2160  | -20.0| -10.4| -7.1 | -3.3 | -0.9 | -0.4 | -0.3 | -0.3 | -0.2 | -0.1 |

5. Conclusion

Thus, there has been drawn up the scheme to solve the problem of soil freezing-thawing in a one-dimensional axisymmetric region. The scheme represents the procedure for obtaining a narrowing family of estimates of the desired function from top to bottom.

In theoretical terms, the extended application of the results of work [3] to the region with axial symmetry appears to be fundamentally new.

The proposed solution can be used in engineering calculations, as well as to define the accuracy of various approximate methods.

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