Graphical representation of the partition function for a 1-D $\delta$-function Bose gas

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Abstract

One-dimensional repulsive $\delta$-function bose system is studied. By only using the Bethe ansatz equation, $n$-particle partition functions are exactly calculated. From this expression for the $n$-particle partition function, the $n$-particle cluster integral is derived. The results completely agree with those of the thermal Bethe ansatz (TBA). This directly proves the validity of the TBA. The theory of partitions and graphs is used to simplify the discussion.

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1 Introduction

We study statistical mechanics of a one-dimensional gas of Bose particles interacting through a repulsive delta function potential. The Hamiltonian for the system with \( n \) particles reads

\[
H_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i>j} \delta(x_i - x_j).
\]

Throughout the paper, we set \( \hbar = 1, 2m = 1 \) and assume the potential is repulsive, \( c > 0 \).

The eigenvalues and the eigenstates are obtained by the Bethe ansatz (BA) method \[2, 1\], and the quantum inverse scattering method (QISM) \[3, 4, 5, 6, 7\]. As a periodic boundary condition, the Bethe ansatz (BA) equation is derived. Using the BA equation, thermodynamic quantities are calculated by the thermal Bethe ansatz (TBA) \[10, 9\]. In TBA, an “interpretation” of the particle density and the state density enables us to define the entropy of \( n \)-particle system in the thermodynamic limit.

With the quantum Gelfand-Levitan equation \[3, 7\], the field operator is expressed as a series of the terms each of which is a product of creation and annihilation operators in the scattering data space. The grand partition function is written as a field operator \[5\]. Creamer, Thacker and Wilkinson \[4\] calculated the grand partition function using creation and annihilation operators of this system, but the analysis involves a delicate regularization that \( 2\pi\delta(0) \) is replaced by the volume \( L \).

To be rigorous, it is desirable to examine these results by a different approach. In this paper, we present a method to calculate the thermodynamical quantities only by use of the BA equation which is derived exactly as a periodic boundary condition from both QISM and BA method. We calculate the \( n \)-particle partition function and evaluate the \( n \)-particle cluster integral.

The paper is organized as follows. The \( n \)-particle partition function and the \( n \)-particle cluster integral are derived with a method, which we call a direct method, in section 2. This method is shown explicitly for \( n = 3 \). In section 3, we consider the \( n \)-particle case, and reformulate the results in graphical expressions. The last section is devoted to concluding remarks and discussions. To avoid complexities, the details of a mathematical proof are summarized in Appendices A.
2 Direct method

We assume a finite system size \( L \) and the periodic boundary condition. It is known that the total energy \( E \) and the wave numbers \( k_j \) of the system (1.1) are determined by the following relations,

\[
E = \sum_{j=1}^{n} k_j^2, \quad (2.1)
\]
\[
k_j L = 2\pi m_j + \sum_{j' \neq j} \Delta (k_{j'} - k_j) , \quad (2.2)
\]

where \( m_j \) are integers or half-integers, and \( \Delta (k) \) is the phase shift of two-particle scattering,

\[
m_j \in \begin{cases} 
\mathbb{Z} & \text{if } N = \text{odd} \\
\mathbb{Z} + \frac{1}{2} & \text{if } N = \text{even,}
\end{cases} \quad (2.3)
\]
\[
m_j < m_{j+1}, \quad (2.4)
\]
\[
\Delta (k) = 2 \arctan \left( \frac{k}{c} \right), \quad (2.5)
\]
\[-\pi < \Delta (k) < \pi.
\]

Eq.(2.2) is called the Bethe ansatz (BA) equation. Main objects to be calculated are the \( n \)-particle partition function \( Z_n \);

\[
Z_n = \text{Tr} e^{-\beta H_n}, \quad \beta = 1/k_B T, \quad (2.6)
\]

and the \( n \)-particle cluster integral \( b_n \);

\[
\sum_{n \geq 1} b_n z^n = \log \left( \sum_{n \geq 0} Z_n z^n \right), \quad (2.7)
\]

where \( z = e^{\beta \mu} \) with the chemical potential \( \mu \). By definition, \( Z_0 = 1 \) and we simply have

\[
b_1 = L^{-1} Z_1 = \int \frac{dk}{2\pi} e^{-\beta k^2}. \quad (2.8)
\]

We explain a direct method to evaluate the partition function for 3-particle case. To be explicit, the total energy is

\[
E = k_1^2 + k_2^2 + k_3^2, \quad (2.9)
\]
and the BA equation is

\[ k_1 L = 2\pi m_1 + \Delta (k_2 - k_1) + \Delta (k_3 - k_1) \]

\[ k_2 L = 2\pi m_2 + \Delta (k_1 - k_2) + \Delta (k_3 - k_2) \]

\[ k_3 L = 2\pi m_3 + \Delta (k_1 - k_3) + \Delta (k_2 - k_3), \]

\[ m_1 < m_2 < m_3 \in \mathbb{Z}. \] (2.10)

By use of these relations (2.9)–(2.11), we can calculate the partition function \( Z_3 \) as follows,

\[ Z_3 = \sum_{m_1 < m_2 < m_3} e^{-\beta (k_1^2 + k_2^2 + k_3^2)} \]

\[ = \frac{1}{6} \sum_{m_1, m_2, m_3} e^{-\beta (k_1^2 + k_2^2 + k_3^2)} - \frac{1}{2} \sum_{m_1, m_2 = m_3} e^{-\beta (k_1^2 + k_2^2 + k_3^2)} + \frac{1}{3} \sum_{m_1 = m_2 = m_3} e^{-\beta (k_1^2 + k_2^2 + k_3^2)} \]

\[ = \frac{1}{6} \sum_{m_1, m_2, m_3} \int dm_1dm_2dm_3 e^{-\beta (k_1^2 + k_2^2 + k_3^2)} + 2\pi i(m_1 \tilde{m}_1 + m_2 \tilde{m}_2 + m_3 \tilde{m}_3) \]

\[ - \frac{1}{2} \sum_{m_1', m_2'} \int dm_1'dm_2' e^{-\beta (2k_1^2 + k_2^2)} + 2\pi i(m_1' \tilde{m}_1' + m_2' \tilde{m}_2') \]

\[ + \frac{1}{3} \sum_{m''} \int dm'' e^{-\beta (3k^2)} + 2\pi im'' \tilde{m}'' \]

\[ = \frac{1}{6} \sum_{m_1, m_2, m_3} \int dk_1dk_2dk_3 \left| \frac{\partial m}{\partial k} \right| e^{-\beta (k_1^2 + k_2^2 + k_3^2)} + iL(k_1m_1 + k_2m_2 + k_3m_3) \]

\[ \times \left( \frac{k_1 - k_2 + ic}{k_1 - k_2 - ic} \right)^{m_2 - \tilde{m}_1} \left( \frac{k_2 - k_3 + ic}{k_2 - k_3 - ic} \right)^{m_3 - \tilde{m}_2} \left( \frac{k_3 - k_1 + ic}{k_3 - k_1 - ic} \right)^{\tilde{m}_1 - m_3} \]

\[ - \frac{1}{2} \sum_{m_1', m_2'} \int dk_1'dk_2' \left| \frac{\partial m'}{\partial k} \right| e^{-\beta (2k_1^2 + k_2^2)} + iL(k_1'm_1' + k_2'm_2') \left( \frac{k_1' - k_2 + ic}{k_1' - k_2 - ic} \right)^{\tilde{m}_2 - 2\tilde{m}_1} \]

\[ + \frac{1}{3} \sum_{m''} \int dk'' \frac{\partial m''}{\partial k''} e^{-\beta (3k^2)} + iLk'' \tilde{m}'' \], \quad (2.12)

where \( |\partial m/\partial k| \), \( |\partial m'/\partial k'| \) and \( \partial m''/\partial k'' \) are the Jacobians to be explained shortly. We have written explicitly all the steps of calculations which are common to those for general \( n \) \([11]' [12]' [13]'\). In the second equality, we use a symmetry of the BA equation with respect to the exchange \( m_i, k_i \leftrightarrow m'_i, k'_i \), in the third equality, we apply the Poisson’s summation formula, and in the last equality, we change variables of integration from \( m \) to \( k \).

The relation between \( k \) and \( m \) is defined by the BA equation. In (2.12), \( k', m', k'' \) and
\( m'' \) are related as follows,

\[
\begin{align*}
  k'_1 L &= 2\pi m'_1 + 2\Delta (k'_2 - k'_1) \\
  k'_2 L &= 2\pi m'_2 + \Delta (k'_1 - k'_2) \quad (2.13) \\
  k'' L &= 2\pi m''.
\end{align*}
\]

Thus, the Jacobians, \(|\partial m / \partial k|\), \(|\partial m'/\partial k'|\) and \(\partial m''/\partial k''\), are given by

\[
\begin{align*}
  (2\pi)^3 \left| \frac{\partial m}{\partial k} \right| &= L^3 + 2L^2K(k_1 - k_2) + 2L^2K(k_2 - k_3) + 2L^2K(k_3 - k_1) \\
  &+ 3LK(k_2 - k_1)K(k_3 - k_1) + 3LK(k_1 - k_2)K(k_3 - k_2) \\
  &+ 3LK(k_1 - k_3)K(k_2 - k_3), \quad (2.15) \\
  (2\pi)^2 \left| \frac{\partial m'}{\partial k'} \right| &= L^2 + 3LK(k_1 - k_2), \quad (2.16) \\
  \frac{\partial m''}{\partial k''} &= \frac{1}{2\pi}L, \quad (2.17)
\end{align*}
\]

where

\[
K(k) \equiv \frac{d\Delta(k)}{dk} = \frac{2c}{k^2 + c^2}. \quad (2.18)
\]

It is readily shown that all terms except \( \tilde{m}_i = 0, \tilde{m}_i' = 0 \) and \( \tilde{m}'' = 0 \) in (2.12) exponentially decay as \( L \) gets large. Although the discussion may include these decaying terms, we hereafter consider the expressions in the thermodynamic limit. Then, \( Z_3 \) becomes

\[
Z_3 = \frac{1}{6} \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \frac{dk_3}{2\pi} \left( L^3 + 6L^2K(k_1 - k_2) + 9LK(k_2 - k_1)K(k_3 - k_1) \right) e^{-\beta(k_1^2 + k_2^2 + k_3^2)} \\
- \frac{1}{2} \int \frac{dk'_1}{2\pi} \frac{dk'_2}{2\pi} \left( L^2 + 3LK(k_1 - k_2) \right) e^{-\beta(2k'_1^2 + k'_2^2)} \\
+ \frac{1}{3} \int \frac{dk''}{2\pi} Le^{-\beta(3k^2)}. \quad (2.19)
\]

It is much easier to show

\[
Z_2 = \frac{1}{2} \int \frac{dk'_1}{2\pi} \frac{dk'_2}{2\pi} \left( L^2 + 2LK(k_1 - k_2) \right) e^{-\beta(k'_1^2 + k'_2^2)} \\
- \frac{1}{2} \int \frac{dk''}{2\pi} Le^{-\beta(2k''^2)}. \quad (2.20)
\]

From (2.8), (2.19) and (2.20), the cluster integral \( b_3 \) is given by

\[
b_3 = \frac{1}{L} \left( Z_3 - Z_2Z_1 + \frac{1}{3}Z'_1 \right)
\]
Fig. 3.1: Vertices of this graph are the elements of $\Pi_3$, and if $x < y$ then $y$ is drawn “above” $x$ (i.e., with a higher vertical coordinate). It is called the Hasse diagram \cite{14} of $\Pi_3$.

$$
123 = \hat{1}_3
$$

$$
2-13
$$

$$
1-2-3 = \hat{0}_3
$$

Thanks to the effective interaction $K(k)$, $b_3$ consists of only 3 terms. In the next section, we show that $Z_n$ and $b_n$ can be calculated in the same way.

## 3 Partition function and graphs

### 3.1 Partition function

In order to present a general structure of the partition function, we need to explain some terminology in the theory of partition \cite{14}; the partially ordered set (poset, for short).

Let $\Pi(S)$ denote a set of all partitions of a finite set $S$. In what follows, $[n]$ means a set \{1, 2, $\cdots$, $n$\}, and write $\Pi_n$ for $\Pi([n])$. We define partially order in $\Pi(S)$ by refinement, that is, define $x \leq y \in \Pi(S)$ if every block of $x$ is contained in a block of $y$. For example, if $x \in \Pi_9$ has blocks 137-2-46-58-9 and $y \in \Pi_9$ has blocks 13467-2589, then $x \leq y$. Special elements in $\Pi(S)$ are $\hat{0}_S$ and $\hat{1}_S$ such that $x \geq \hat{0}_S$ and $x \leq \hat{1}_S$ for all $x \in \Pi(S)$. We write $\hat{0}_n$ and $\hat{1}_n$ for $\hat{0}_{[n]}$ and $\hat{1}_{[n]}$. Fig. 3.1 shows the partially ordered elements in $\Pi_3$, that is, 1-2-3 < 1-23, 2-13, 3-12 < 123. In addition, we define Möbius function of $\Pi(S)$ inductively as follows,

$$
\mu(x, x) = 1 \quad \text{for all} \ x \in \Pi(S),
$$

$$
\mu(x, y) = - \sum_{x < z < y} \mu(x, z) \quad \text{for all} \ x < y \text{ in } \Pi(S).
$$

\hspace{3pt} (3.1)
By use of those definitions, we can show some useful relations. For brevity, $\mathbb{N}$ and $\mathbb{C}$ stand respectively nonnegative integers and complex numbers. (1). Let $\hat{f}, \hat{g} : \Pi(S) \rightarrow \mathbb{C}$, then,

$$\hat{g}(x) = \sum_{y \geq x} \hat{f}(y), \quad \text{for all } x \in \Pi(S),$$

if and only if

$$\hat{f}(x) = \sum_{y \geq x} \mu(x, y) \hat{g}(y), \quad \text{for all } x \in \Pi(S). \quad (3.2)$$

This is called Möbius inversion formula. (2). Let $\hat{g}, \hat{J} : \Pi(S) \rightarrow \mathbb{C}$ and $f, h : \mathbb{N} \rightarrow \mathbb{C}$. If

$$\sum_{n \geq 0} h(n) \frac{u^n}{n!} = \exp \left( \sum_{n \geq 1} f(n) \frac{u^n}{n!} \right), \quad (3.3)$$

$$h(N_S) = \sum_{x \in \Pi(S)} \mu(\hat{0}_S, x) \hat{g}(x), \quad (3.4)$$

$$\hat{g}(x) = \sum_{\xi \in \Pi(x)} \prod_{y \in \xi} \hat{J}(y), \quad (3.5)$$

then

$$f(n) = \sum_{x \in \Pi_n} \mu(\hat{0}_n, x) \hat{J}(x). \quad (3.6)$$

We prove this relation in Appendix A.

With these two relations, we can derive the partition function $Z_n$ and the cluster integral $b_n$ in a compact way. First we define $f, h : \mathbb{N} \rightarrow \mathbb{C}$ and $\hat{h}, \hat{g} : \Pi(S) \rightarrow \mathbb{C}$,

$$f(n) \equiv n!Lb_n, \quad (3.7)$$

$$h(n) \equiv n!Z_n, \quad (3.8)$$

$$\hat{h}(x) \equiv \sum_{m_1 \neq \cdots \neq m'_i} e^{-\beta(k_1^2 + \cdots + k_n^2)}, \quad (3.9)$$

$$\hat{g}(x) \equiv \sum_{m_1 \cdots m'_i} e^{-\beta(k_1^2 + \cdots + k_n^2)}, \quad (3.10)$$

where

$$x \in \Pi(S), \quad \sigma_i \ni s_j \Rightarrow m'_i = m_j, \quad x = \{\sigma_1, \cdots, \sigma_l\}, \quad S = \{s_1, \cdots, s_n\}. \quad (3.11)$$

7
Recall that $m$ and $k$ are related by the BA equation. We see that Eq.(3.3) holds from the definition of cluster integral $b_n$, Eq.(2.7). From the definition (3.8)$\sim$(3.10), it is easy to show that

\[ \hat{g}(x) = \sum_{y \geq x} \hat{h}(y), \quad (3.12) \]

\[ \hat{h}(\hat{0}_n) = h(n). \quad (3.13) \]

Due to the Möbius inversion formula (3.2), (3.12) is equivalent to

\[ \hat{h}(x) = \sum_{y \geq x} \mu(x, y) \hat{g}(y). \quad (3.14) \]

Substituting $\hat{0}_n$ for $x$ in this equation, we obtain

\[ h(n) = \sum_{y \in \Pi_n} \mu(\hat{0}_n, y) \hat{g}(y). \quad (3.15) \]

This is nothing but the condition (3.4).

As we have mentioned in the previous section, we do not include exponentially decaying terms. Then, (3.10) is written as

\[ \hat{g}(x) = \int \prod_{\sigma \in x} dm_{\sigma} e^{-\beta(k_{\sigma 1}^2 + k_{\sigma 2}^2 + \cdots + k_{\sigma n}^2)} \quad (3.16) \]

\[ = \int \prod_{\sigma \in x} dk_{\sigma} \left| \frac{\partial m_{\sigma}}{\partial k_{\sigma}} \right| e^{-\beta(\sum_{\sigma \in x} N_{\sigma} k_{\sigma}^2)}. \quad (3.17) \]

The transformation matrix and the Jacobian are given as follows;

\[ \frac{\partial m_{\sigma}}{\partial k_{\sigma'}} = \frac{1}{2\pi} \times \begin{cases} L + \sum_{\sigma'' \neq \sigma} N_{\sigma''} K(k_{\sigma''} - k_{\sigma'}) & \text{if } \sigma = \sigma' \\ -N_{\sigma'} K(k_{\sigma'} - k_{\sigma'}) & \text{if } \sigma \neq \sigma' \end{cases} \quad (3.18) \]

\[ \left| \frac{\partial m_{\sigma}}{\partial k_{\sigma'}} \right| = \sum_{\xi \in \Pi(x) \gamma \xi} \frac{L}{(2\pi)^{N_\sigma}} \left( \sum_{\sigma \in y} N_{\sigma} \right) \sum_{t \in \gamma \xi} \prod_{\sigma \in y} N_{\sigma}^{N_{\sigma}(\sigma, t) - 1} \left( \prod_{b \in \Xi(t)} K(k_{\sigma_1(b)} - k_{\sigma_2(b)}) \right). \quad (3.19) \]

We explain notations in (3.18) and (3.19). The number of elements in a set $\sigma$ is denoted by $N_\sigma$. $\gamma(S)$ denotes a set of all undirected trees with a vertex set $S$. For instance, all the elements in $\gamma(\{\sigma_1, \sigma_2, \sigma_3\})$ are depicted in Fig.3.2. We call a connection of two vertices a branch. $\Xi(t)$ denotes a set of all branches contained in a tree $t$. $n(\sigma, t)$ is the number of
Fig. 3.2: A set of undirected trees, \( \mathcal{V}(\{\sigma_1, \sigma_2, \sigma_3\}) \), contains three elements branches with which the vertex \( \sigma \) are connected in the tree \( t \). \( \sigma_1(b) \) and \( \sigma_2(b) \) denote two end-vertices connected by a branch \( b \).

A remaining task is a consistency with (3.5). We define a function \( \hat{J} : \Pi(S) \to \mathbb{C} \) by
\[
\hat{J}(x) \equiv \int \prod_{\sigma \in x} dk'_\sigma \left| \frac{\partial m'}{\partial k'} \right|_c e^{-\beta \left( \sum_{\sigma \in x} N_\sigma k'^2_\sigma \right)}, \tag{3.20}
\]
where
\[
\left| \frac{\partial m'}{\partial k'} \right|_c \equiv \frac{L}{(2\pi)^N} \left( \sum_{\sigma \in x} N_\sigma \right) \sum_{t \in \mathcal{V}(x)} \left( \prod_{\sigma \in x} N^{(\sigma,t)-1}_\sigma \right) \left( \prod_{b \in \Xi(t)} K \left( k'_{\sigma_1(b)} - k'_{\sigma_2(b)} \right) \right). \tag{3.21}
\]
We can change the expression of the Jacobian \( \left| \partial m'/\partial k' \right| \) from (3.19) into a sum of forests, that is, sets of trees. On the other hand, the right hand side of (3.21) is a sum of connected forests, that is, trees. Therefore, we put a subscript \( c \) as \( \left| \partial m'/\partial k' \right|_c \). With this definition, we see that the condition (3.5) follows from (3.17).

In this way, three conditions (3.3)∼(3.5) are shown to be satisfied, which indicates that Eq.(3.6) holds. We write Eq.(3.6) explicitly,
\[
b_n = \frac{1}{n!L} \sum_{x \in \Pi_n} \mu \left( \tilde{\theta}_n, x \right) \int \prod_{\sigma \in x} dk'_\sigma \left| \frac{\partial m'}{\partial k'} \right|_c e^{-\beta \left( \sum_{\sigma \in x} N_\sigma k'^2_\sigma \right)}. \tag{3.22}
\]
In fact, (3.17) is equivalent to
\[
Z_n = \frac{1}{n!} \sum_{x \in \Pi_n} \mu \left( \tilde{\theta}_n, x \right) \int \prod_{\sigma \in x} dk'_\sigma \left| \frac{\partial m'}{\partial k'} \right|_c e^{-\beta \left( \sum_{\sigma \in x} N_\sigma k'^2_\sigma \right)}. \tag{3.23}
\]
The Jacobians in (3.22) and (3.23) are respectively (3.21) and (3.19). Explicit form of \( \mu(\tilde{\theta}_n, x) \) can be derived from (3.1),
\[
\mu \left( \tilde{\theta}_n, x \right) = \prod_{\sigma \in x} (-1)^{N_\sigma-1} (N_\sigma - 1)!. \tag{3.24}
\]
It is easily shown \cite{12,13} that the cluster integrals $b_n$ agree with those derived from the thermal Bethe ansatz (TBA). It is also possible to derive the integral equation in TBA from (3.22) \cite{13}.

We give two remarks here. First, the Jacobian $|\partial m'/\partial k'|$ is essentially the inner product of the Bethe wave functions \cite{8}. Second, the cluster integrals consist of only a finite number of terms.

### 3.2 Graph representation

We further develop a graphical representation of the cluster integral and the partition function. We draw an $l$ times rolled coil for a Boltzmann weight $e^{-\beta lk^2}$ (Fig. 3.3). We call it $l$-toron following Montroll and Ward \cite{15,16}. The tree consists of a toron or torons connected by branches. The forest consists of the trees. We denote by $\mathcal{V}_n$ a set of all the trees which satisfy the following two conditions: (1) all the vertices of the tree are made of torons, and (2) the sum of rolled number of torons composing the tree is $n$. Fig. 3.4 shows all the elements in $\mathcal{V}_3$. $\mathcal{W}_n$ is a set of all the forests which satisfy the following two conditions: (1) all the vertices of the forest are made of torons, and (2) the sum of rolled number of torons composing the forest is $n$. Simply, $\mathcal{V}_n$ is a subset of $\mathcal{W}_n$. Fig. 3.5 exhibits all the elements in $\mathcal{W}_3$.

In terms of these terminologies, we may rewrite (3.23) and (3.22) as

$$Z_n = \sum_{f \in \mathcal{W}_n} \frac{\text{Sym}(f)}{n!} \prod_{t \text{ in the forest } f} S(t), \quad (3.25)$$

Fig. 3.5: A set of the forests, $\mathcal{W}_3$. Three graphs in the first line are not trees because the vertices are not connected on partially connected by branch(es).

\[
b_n = \sum_{t \in \mathcal{V}_n} \frac{\text{Sym}(t)}{n!L} S(t).
\]

(3.26)

Here, $\text{Sym}(t)$ and $\text{Sym}(f)$ indicate symmetrical factors of graphs. In the case $t \in \mathcal{V}_n$ or $f \in \mathcal{W}_n$, $\text{Sym}(t)$ or $\text{Sym}(f)$ means the number of different ways in which a set $\{1, \cdots, n\}$ can be distributed to all the vertices of $t$ or $f$ at a time, on the condition that $l$ elements are placed in a vertex made of $l$-toron. For example

\[
\text{Sym}\left(\begin{array}{c}
\bigcirc \bigcirc \bigcirc
\end{array}\right) = 12,
\]

(3.27)

\[
\text{Sym}\left(\begin{array}{c}
\bigcirc
\end{array}\right) = 3,
\]

(3.28)

\[
\text{Sym}\left(\begin{array}{c}
\bigcirc \bigcirc
\end{array}\right) = 1.
\]

(3.29)

From (3.22) and (3.23), we can show that $S(t)$ in (3.25) and (3.26) is given as follows,

\[
S(t) \equiv L \sum_\omega N_\omega \left( \prod_\omega (-1)^{N_\omega-1} (N_\omega - 1)!N_\omega^{n(\omega,t)-1} \right)
\times \int \prod_\omega \frac{dk_\omega}{2\pi} \left( \prod_{b \in \Xi(t)} K(k_{\sigma_1(b)} - k_{\sigma_2(b)}) \right) e^{-\beta\left(\sum_\omega N_\omega k_\omega\right)},
\]

(3.30)

where $N_\omega$ denotes the rolled number of toron $\omega$, and $\sum_\omega$ (or $\prod_\omega$) denotes a sum (or a product) with respect to all the toron $\omega$ in the tree $t$. For example,

\[
S\left(\begin{array}{c}
\bigcirc
\end{array}\right) = L \times (2 + 1) \times \left((-1)^1 \times 2^0 \times (-1)^0 \times 0^0\right)
\times \int \frac{dk_{\omega_1}}{2\pi} \frac{dk_{\omega_2}}{2\pi} (K(k_{\omega_1} - k_{\omega_2})) e^{-\beta(2k_{\omega_1} + k_{\omega_2})}.
\]

(3.31)

Substitution of (3.28) and (3.31) gives the second term in (2.21).
As examples, we list a graphical representation of the cluster integrals \( b_1 \sim b_4 \),

\[
\begin{align*}
  b_1 &= \quad \circ \quad \\
  b_2 &= \quad \circ \quad \circ + \circ \quad \\
  b_3 &= \quad \circ \quad \circ + \circ \quad \circ + \circ \quad \\
  b_4 &= \quad \circ \quad \circ + \circ \quad \circ + \circ \quad \circ + \circ \quad + \circ \quad \circ + \circ \quad + \circ \quad \circ.
\end{align*}
\]

(3.32) (3.33) (3.34) (3.35)

4 Concluding remarks and discussions

In this paper, we have studied a one-dimensional \( \delta \)-function Bose gas, (1.1), and have derived directly the partition function and the cluster integral, (3.23) and (3.22), from the Bethe ansatz equation, (2.2). This should be regarded as a proof of the thermal Bethe ansatz (TBA). The derivation is simplified by use of the partially ordered set in the theory of partition.

This direct method has some advantages. First, the method is rigorous. In the quantum field theoretic method, calculation is done at the infinite volume, therefore an interpretation of \( 2\pi \delta(0) \) as the volume is unavoidable. In the TBA, it is necessary to define the \( n \)-particle entropy. This procedure is based on an interpretation of the particle and state densities. In this sense, both field theoretic method and TBA remained to be proved. On the other hand, the direct method is free from such interpretations, and all calculations are traceable step by step.

Second, the method has a wide applicability. It should be remarked that in the direct method all the dependencies on the systems come only from the BA equation. In other words, the direct method may have the generality at least as well as TBA.

We conclude that the direct method will be useful and important in calculating the thermodynamic quantities in various integrable systems.
Appendix A A proof of (3.6)

We use the same notation as in subsection 3.1. It can be shown that, on condition that $h, f : \mathbb{N} \to \mathbb{C}$, we have

$$\sum_{n \geq 0} h(n) \frac{u^n}{n!} = \exp \left( \sum_{n \geq 1} f(n) \frac{u^n}{n!} \right)$$ \hspace{1cm} (A.1)

if and only if

$$h(n) = \sum_{x \in \Pi_n} \prod_{\sigma \in x} f(N_\sigma) \hspace{1cm} (A.2)$$
$$h(0) = 1. \hspace{1cm} (A.3)$$

This is known as the cumulant expansion formula. Eq.(A.2) is equivalent to

$$\prod_{\sigma \in x} h(N_\sigma) = \sum_{y \leq x} \prod_{\sigma \in y} f(N_\sigma). \hspace{1cm} (A.4)$$

For $\hat{H}, \hat{F} : \Pi(S) \to \mathbb{C}$, the following relation holds,

$$\hat{H}(x) = \sum_{y \leq x} \hat{F}(y) \iff \hat{F}(x) = \sum_{y \leq x} \mu(x, y) \hat{H}(y). \hspace{1cm} (A.5)$$

This is the Möbius inversion formula, which is a dual form of (3.2). With this formula, Eq.(A.4) becomes

$$\prod_{\sigma \in x} f(N_\sigma) = \sum_{y \leq x} \mu(y, x) \prod_{\sigma \in y} h(N_\sigma). \hspace{1cm} (A.6)$$

Substitution of $\hat{1}_n$ for $x$ in (A.4) yields

$$f(n) = \sum_{x \leq \hat{1}_n} \mu(y, \hat{1}_n) \prod_{\sigma \in y} h(N_\sigma). \hspace{1cm} (A.7)$$

We suppose the existence of $\hat{g} : \Pi(S) \to \mathbb{C}$ which satisfies

$$h(N_S) = \sum_{x \in \Pi(S)} \mu(\hat{0}_S, x) \hat{g}(x). \hspace{1cm} (A.8)$$

Using this relation, Eq.(A.6) is written as

$$f(n) = \sum_{x \in \Pi_n} \mu(x, \hat{1}_n) \prod_{\sigma \in x} \sum_{y \in \Pi(\sigma)} \mu(\hat{0}_\sigma, y) \hat{g}(y)$$
$$= \sum_{x \in \Pi_n} \mu(\hat{0}_n, x) \sum_{\xi \in \Pi(\hat{x})} \mu(\xi, \hat{1}_x) \prod_{y \in \xi} \hat{g}(y). \hspace{1cm} (A.9)$$
In the second equality, we have used the following two relations,

\[ \sum_{x \in \Pi_n} F(N_x) \prod_{\sigma \in x \atop y \in \Pi(\sigma)} \hat{G}(y) = \sum_{x \in \Pi_n} \sum_{\xi \in \Pi(x)} F(N_\xi) \prod_{y \in \xi} \hat{G}(y), \tag{A.10} \]

for \( F : \mathbb{N} \to \mathbb{C} \), \( \hat{G} : \Pi(S) \to \mathbb{C} \), and

\[ \mu(\hat{0}_n, x) = \prod_{\sigma \in x} \mu(\hat{0}_{N_\sigma}, \hat{1}_{N_\sigma}), \tag{A.11} \]

for \( x \in \Pi_n \). And, we suppose the existence of a map \( \hat{J} : \Pi(S) \to \mathbb{C} \) which satisfies

\[ \hat{g}(x) = \sum_{\xi \in \Pi(x) \atop y \in \xi} \prod_{y \in \xi} \hat{J}(y). \tag{A.12} \]

Then, a main part of the right hand side of Eq. (A.9) becomes

\[ \sum_{\xi \in \Pi(x)} \mu(\xi, \hat{1}_x) \prod_{y \in \xi} \hat{g}(y) = \sum_{\xi \in \Pi(x)} \mu(\xi, \hat{1}_x) \prod_{y \in \xi} \sum_{\zeta \in \Pi(y)} \prod_{z \in \zeta} \hat{J}(z) \]

\[ = \sum_{\zeta \in \Pi(x)} \left( \sum_{\lambda \in \Pi(\zeta)} \mu(\lambda, \hat{1}_\zeta) \right) \prod_{z \in \zeta} \hat{J}(z) \]

\[ = \hat{J}(x). \tag{A.13} \]

The third equality is due to the following relation,

\[ \sum_{x \in \Pi(S)} \mu(x, \hat{1}_S) = \delta(1, N_S). \tag{A.14} \]

By use of the relation (A.13), (A.14) becomes

\[ f(n) = \sum_{x \in \Pi_n} \mu(\hat{0}_n, x) \hat{J}(x), \tag{A.15} \]

which is (3.6). To summarize, using (A.1), (A.8) and (A.12), we have proved (3.6).
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