EULER’S TRANSFORMATION, ZETA FUNCTIONS AND GENERALIZATIONS OF WALLIS’ FORMULA

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Abstract. In this note, we extend Euler’s transformation formula from the alternating series to more general series. Then we give new expressions for the Riemann zeta function \( \zeta(s) \) by the generalized difference operator \( \Delta_c \), which provide analytic continuation of \( \zeta(s) \) and new ways to evaluate the special values of \( \zeta(-m) \) for \( m = 0, 1, 2, \ldots \). Applying these results, we further extend Huylebrouck’s generalization of Wallis’ well-known formula for \( \pi \) in the half planes \( \text{Re}(s) > 0 \) and \( \text{Re}(s) > -1 \), respectively. They imply several interesting special cases including

\[
\frac{2\pi}{3\pi} = \frac{3^{\frac{3}{2}} \cdot 2^{\frac{3}{2}} \cdot 3^{\frac{3}{2}} \cdot 4^{\frac{3}{2}} \cdot 6^{\frac{3}{2}} \cdot 6^{\frac{3}{2}}}{2^{\frac{3}{2}} \cdot 4^{\frac{3}{2}} \cdot 5^{\frac{3}{2}} \cdot 5^{\frac{3}{2}} \cdot 4^{\frac{3}{2}} \cdot 4^{\frac{3}{2}}} \ldots ,
\]

and

\[
\left( 3 \left( \frac{2\pi e^\gamma}{A^{12}} \right)^2 \right)^\frac{1}{2} = \frac{3^{\frac{3}{2}} \cdot 3^{\frac{3}{2}} \cdot 6^{\frac{3}{2}} \cdot 6^{\frac{3}{2}} \cdot 9^{\frac{3}{2}} \cdot 9^{\frac{3}{2}}}{2^{\frac{3}{2}} \cdot 4^{\frac{3}{2}} \cdot 5^{\frac{3}{2}} \cdot 7^{\frac{3}{2}} \cdot 8^{\frac{3}{2}} \cdot 10^{\frac{3}{2}}} \ldots ,
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( A \) is the Glaisher-Kinkelin constant.

1. Introduction

The Riemann zeta function is defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

for \( \text{Re}(s) > 1 \). It can be analytically continued to a meromorphic function in the complex plane with a simple pole at \( s = 1 \). And the Dirichlet eta function is an alternating form of \( \zeta(s) \),

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},
\]

for \( \text{Re}(s) > 0 \). It can be analytically continued to the complex plane without any pole. For \( \text{Re}(s) > 0 \), (1) and (2) are connected by the following equation

\[
\eta(s) = (1 - 2^{1-s})\zeta(s).
\]
The Dirichlet eta function $\eta(s)$ appeared as a tool for Euler’s derivation of the functional equation for $\zeta(s)$. In fact, according to Weil’s history [13, p. 273–276], Euler “proved”

$$
\frac{\eta(1-s)}{\eta(s)} = \frac{-\Gamma(s)(2^s - 1)\cos(\pi s/2)}{(2^{s-1} - 1)\pi^s},
$$
then from (3) he got the functional equation of $\zeta(s)$.

In order to apply $\eta(s)$ to calculate the special values $\zeta(-m)$ for $m = 0, 1, 2, \ldots$, Euler introduced the following transformation of alternating series (see [4, volume 10, p. 222–227]). Let $A_2 = \sum_{n=1}^{\infty} (-1)^{n+1} b_n$ be an alternating series (see (19)). It can be written as

$$
A_2 = b_1 - b_2 + b_3 - b_4 + \cdots = \frac{1}{2} b_1 + \frac{1}{2} [(b_1 - b_2) - (b_2 - b_3) + \cdots]
$$

$$
= \frac{1}{2} b_1 + \frac{1}{4} (b_1 - b_2) + \frac{1}{4} [(b_1 - 2b_2 + b_3) - (b_2 - 2b_3 + b_4) + \cdots].
$$

So inductively, in general, we have

$$
\sum_{n=1}^{\infty} (-1)^{n+1} b_n = \sum_{j=0}^{k-1} \frac{\Delta^j b_1}{2^{j+1}} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\Delta^k b_n}{2^k},
$$
where the sequence of difference operators $\{\Delta^k\}_{k=1}^{\infty}$ is defined recursively by $\Delta^0 b_n = b_n$ and

$$
\Delta^k b_n = \Delta^{k-1} b_n - \Delta^{k-1} b_{n+1}
$$
for $k \geq 1$. In 1994, by using Euler’s transformation, Sondow [11] obtained a new expression for $\eta(s)$, which implies an analytic continuation of the Riemann zeta function $\zeta(s)$ to complex numbers $s \neq 1$. His main result is as follows.

**Theorem 1** (Sondow [11, p. 423, (8)]). For $k \geq 1$, the expression

$$
\eta(s) = (1 - 2^{1-s}) \zeta(s)
$$

$$
= \sum_{j=0}^{k-1} \frac{\Delta^j b_1}{2^{j+1}} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \Delta^k n^{-s},
$$

provides the analytic continuation of $\zeta(s)$ on the punctured half plane $\text{Re}(s) > 1 - k$, $s \neq 1$ where the infinite series converges absolutely and uniformly on compact sets to a holomorphic function. Moreover, except that the convergence will not be absolute in the strip $-k < \text{Re}(s) \leq 1 - k$, this remains true for $k \geq 0$ and $\text{Re}(s) > -k$, $s \neq 1$. Especially, taking $k = 1$, we have

$$
\eta(s) = (1 - 2^{1-s}) \zeta(s)
$$

$$
= \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} (n^{-s} - (n+1)^{-s})
$$
for $\text{Re}(s) > -1$ and $s \neq 1$. 

From this he successfully got a new expression of \( \zeta(-m) \) for \( m = 0, 1, 2, \ldots \),

\[
\zeta(-m) = \frac{1}{1 - 2^{m+1}} \sum_{j=0}^{m} \frac{\Delta^j 1^m}{2^{j+1}}
\]

(see [11, p. 423, Corollary]).

Euler’s transformation and Sondow’s result show a connection between Wallis’ well-known formula for \( \pi \) and zeta functions. In a book published in 1656 [12], John Wallis presented the following remarkable infinite product representation of \( \pi \)

\[
\pi = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots},
\]

which has been quoted by many calculus books. Well-known proofs include an application of the formula for integrals of powers of \( \sin x \) from the inductive method or an application of the infinite product expansion of \( \sin x \).

Taking derivatives on both sides of (9) we obtain

\[
\eta'(0) = \frac{1}{2} \log \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots} \right).
\]

From this, Yung (in 1999) and Sondow (in 2002) found a new proof of Wallis’ formula (see [10]).

In 2015, by extending Yung and Sondow’s methods to other values of \( s \), Huylebrouck [7] obtained the following result.

**Theorem 2** (Huylebrouck [7, p. 371, Theorem 1]). For appropriate values of \( s \) (and if \( \sqrt[n]{n} \) is interpreted as \( n \)),

\[
e^{2\eta'(s)} = \frac{\sqrt[2]{2} \cdot \sqrt[3]{3} \cdot \sqrt[4]{4} \cdot \sqrt[5]{5} \cdot \sqrt[6]{6} \cdots}{\sqrt[1]{1} \cdot \sqrt[3]{3} \cdot \sqrt[5]{5} \cdot \sqrt[7]{7} \cdots}.
\]

The \( s = 0 \) case of the above result recovers Wallis’ formula, the \( s = 1 \) case implies a first generalization of Wallis’ formula:

\[
2^{(2\gamma - \log 2)} = \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{4}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{5}} \cdots}{1^\frac{1}{1} \cdot 3^\frac{1}{3} \cdot 5^\frac{1}{3} \cdot 7^\frac{1}{5} \cdots},
\]

where \( \gamma = 0.5772156649 \cdots \) is the Euler-Mascheroni constant, and the \( s = 2 \) case implies the second generalization of Wallis’ formula:

\[
\left( \frac{4\pi e^\gamma}{A^{12}} \right)^2 = \frac{2^{\frac{1}{2}} \cdot 2^{\frac{1}{3}} \cdot 4^{\frac{1}{3}} \cdot 4^{\frac{1}{4}} \cdot 6^{\frac{1}{4}} \cdot 6^{\frac{1}{5}} \cdots}{1^\frac{1}{1} \cdot 3^\frac{1}{3} \cdot 5^\frac{1}{3} \cdot 7^\frac{1}{5} \cdots},
\]

where \( A = 1.2824271291 \cdots \) is the Glaisher-Kinkelin constant (see [14] for the definition).

In this note, we go to a more general case. Let \( c \geq 2 \) be an integer, for \( \text{Re}(s) > 0 \), if denote by

\[
\zeta_c(s) = (1 - c^{1-s})\zeta(s),
\]

then we have

\[
\zeta_c(s) = \sum_{n=1}^{\infty} \frac{a_{c,n}}{n^s},
\]
where

\[
\begin{aligned}
    a_{c,n} &= \begin{cases} 
        1 - c & \text{if } n \equiv 0 \pmod{c}; \\
        1 & \text{if } n \not\equiv 0 \pmod{c}
    \end{cases}
\end{aligned}
\]  

(see [8, p. 326]).

Inspiring by the representation of \(\zeta_{(c)}(s)\) (see (17)), we extend Euler’s transformation formula from the alternating series to more general series with the form \(A_c = \sum_{n=1}^{\infty} a_{c,n} b_n\), where \(a_{c,n}\) is given in (18). (See Theorem [3]). Then we generalize the above Sondow’s result to give expressions for \(\zeta_{(c)}(s)\) by the generalized difference operator \(\Delta_c\). It provides analytic continuation of the Riemann zeta function \(\zeta(s)\) and new ways to evaluate \(\zeta(-m)\) for \(m = 0, 1, 2, \ldots\) (see Theorem [5] and Corollary [1]). Based on these results, with the help of \(\zeta_{(c)}(s)\) we further extend Huylebrouck’s generalization of Wallis’ formula in the half planes \(\text{Re}(s) > 0\) and \(\text{Re}(s) > -1\), respectively (see Theorems [7] and [9]), which imply several interesting special cases including

\[
\begin{aligned}
    2\pi &= \frac{3\pi}{2} \cdot \frac{2\pi}{4} \cdot \frac{3\pi}{8} \cdot \frac{2\pi}{4} \cdot \frac{5\pi}{8} \cdot \frac{2\pi}{5} \cdot \frac{3\pi}{8} \cdot \frac{6\pi}{5} \cdot \frac{2\pi}{6} \cdots \\
    3^{\gamma - \frac{\log 3}{2}} &= \frac{3\pi}{2} \cdot \frac{3\pi}{4} \cdot \frac{6\pi}{5} \cdot \frac{9\pi}{7} \cdot \frac{9\pi}{4} \cdots 
\end{aligned}
\]

and

\[
\left(3 \left(\frac{2\pi e^\gamma}{A^{12}}\right)^2\right)^{\frac{1}{15}} = \frac{3\pi}{2} \cdot \frac{3\pi}{4} \cdot \frac{6\pi}{5} \cdot \frac{9\pi}{7} \cdot \frac{9\pi}{4} \cdots 
\]

where \(\gamma\) is the Euler-Mascheroni constant and \(A\) is the Glaisher-Kinkelin constant (see the last section).

2. Euler’s Transformation and Zeta Functions

In this section, we shall generalize Euler’s transformation from the alternating series to a more general setting. Then applying this, we give new expressions for \(\zeta_{(c)}(s)\), which provide analytic continuation of the Riemann zeta function \(\zeta(s)\) and new ways to evaluate \(\zeta(-m)\) for \(m = 0, 1, 2, \ldots\).

Let

\[
A_c = \sum_{n=1}^{\infty} a_{c,n} b_n
\]

be a complex series, where \(a_{c,n}\) is defined in (18).

For \(c = 2\), the series (19) is just the alternating series which can also be written in the following form

\[
A_2 = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots
\]

\[
= \frac{1}{2}(2b_1 - b_2) - \frac{1}{2}[(b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) - (b_5 - b_6) + \cdots]
\]

\[
= \frac{1}{2}(2b_1 - b_2) - \frac{1}{4}(2b_2 - 3b_3 + b_4) + \frac{1}{4}[(b_3 - 2b_4 + b_5) - (b_3 - 2b_4 + b_5) + \cdots].
\]
In general, for $k \geq 1$ we have

$$
\sum_{n=1}^{\infty} a_{n,2}b_n = \sum_{j=0}^{k-1} \frac{(-1)^j}{2j+1} (2\Delta_2^j b_1 - \Delta_2^j b_2) + \sum_{n=1}^{\infty} \frac{(-1)^k}{2k} a_{n,2} \Delta_2^k b_n,
$$

where $\Delta_2^0 b_n = b_n$, $\Delta_2^k b_n = \Delta_2^{k-1} b_{n+1} - \Delta_2^{k-1} b_{n+2}$ for $k \geq 1$. It needs to mention that here $\Delta_2^1 b_n = b_{n+1} - b_{n+2}$, which is slightly different from the difference operator in Euler’s transformation. In that case, we have $\Delta_2^1 b_n = b_n - b_{n+1}$ (see (19)).

For $c = 3$, the series (19) becomes to

$$
\sum_{j=0}^{k-1} \frac{(-1)^j}{3j+1} (3\Delta_3^j b_1 + 4\Delta_3^j b_2 - 4\Delta_3^j b_3) + \sum_{n=1}^{\infty} \frac{(-1)^k}{3k} a_{n,3} \Delta_3^k b_n,
$$

where $\Delta_3^0 b_n = b_n$, $\Delta_3^k b_n = \Delta_3^{k-1} b_{n+1} + \Delta_3^{k-1} b_{n+2} - 2\Delta_3^{k-1} b_{n+3}$. For $c = 4$, the series (19) becomes to

$$
\sum_{j=0}^{k-1} \frac{(-1)^j}{4j+1} (4\Delta_4^j b_1 + 5\Delta_4^j b_2 + 6\Delta_4^j b_3 - 9\Delta_4^j b_4) + \sum_{n=1}^{\infty} \frac{(-1)^k}{4k} a_{n,4} \Delta_4^k b_n,
$$

where $\Delta_4^0 b_n = b_n$, $\Delta_4^k b_n = \Delta_4^{k-1} b_{n+1} + \Delta_4^{k-1} b_{n+2} + \Delta_4^{k-1} b_{n+3} - 3\Delta_4^{k-1} b_{n+4}$. In general, for arbitrary $c$, we have the following result, in which, a generalized difference operator $\Delta_c$ is introduced.

Theorem 3. For $c \geq 2$ and $k \geq 1$, we have

$$
A_c = \sum_{j=0}^{k-1} \frac{(-1)^j}{cj+1} \left( \sum_{i=0}^{c-2} (c+i) \Delta_c^i b_{i+1} - (c-1)^2 \Delta_c^i b_c \right)
$$

$$
+ \sum_{n=1}^{\infty} \frac{(-1)^k}{c^k} a_{n,c} \Delta_c^k b_n,
$$
where the sequence of generalized difference operators \( \{\Delta^k_c\}_{k=1}^\infty \) is defined recursively by \( \Delta^0_c b_n = b_n \) and

\[
\Delta^k_c b_n = \sum_{i=1}^c a_{c,i} \Delta^{k-1}_c b_{n+i}
\]

for \( k \geq 1 \). Furthermore, if the series \( (19) \) is convergent, so does the right hand side of (23).

**Remark 4.** In some cases, although \( (19) \) is divergent, the series on the right hand side of (23) may still converge, so we can endow its sum to \( (19) \) as a generalized sum. This leads to a possible way for the analytic continuation of zeta functions (see Theorem 5).

**Proof of Theorem 3.** We prove it from the induction on \( k \). By manipulating the series \( (19) \), we formally get

\[
A_c = b_1 + \cdots + b_{c-1} - (c - 1)b_c + b_{c+1} + \cdots + b_{2c-1} - (c - 1)b_{2c} + \cdots
\]

\[
= \frac{1}{c} \left( (c+1)b_2 + \cdots + (2c-2)b_{c-1} - (c-1)^2 b_c \right)
\]

\[
= \frac{1}{c} \left( \sum_{i=0}^{c-2} (c+i) b_{i+1} - (c-1)^2 b_c \right)
\]

\[
= \frac{1}{c} \left( \sum_{i=0}^{c-2} (c+i) \Delta^{0}_c b_{i+1} - (c-1)^2 \Delta^{0}_c b_c \right) - \frac{1}{c} \sum_{n=1}^\infty a_{c,n} \Delta^{1}_c b_n,
\]

which is (23) in the case of \( k = 1 \). We assert that if the series \( \sum_{n=1}^\infty a_{c,n} b_n \) is convergent, then the right hand side of (25) is also convergent and (25) is established. Indeed, let

\[
S_{c,n} = \frac{1}{c} \left( \sum_{i=0}^{c-2} (c+i) \Delta^{0}_c b_{i+1} - (c-1)^2 \Delta^{0}_c b_c \right) - \frac{1}{c} \sum_{k=1}^n a_{c,k} \Delta^{1}_c b_k
\]

be the partial sum of the above series. By writing \( n = cm + d \) with \( m, d \in \mathbb{Z} \), \( m \geq 1 \), \( 0 \leq d < c \), we have

\[
\sum_{k=1}^{n+1} a_{c,k} b_k = S_{c,n} + \sum_{k=1}^{cm+1} a_{c,k} b_k \]

\[
= \frac{1}{c} \left( - \sum_{i=2}^{c-1} (-c - i + 1)b_{cm+i} + (c-1)^2 b_{cm+1} \right)
\]

\[
+ \sum_{k=cm+2}^{cm+d+1} a_{c,k} b_k - \frac{1}{c} \sum_{k=cm+1}^{cm+d} a_{c,k} \Delta^{1}_c b_k
\]

\[
+ \sum_{k=cm+2}^{cm+d+1} a_{c,k} b_k - \frac{1}{c} \sum_{k=cm+1}^{cm+d} a_{c,k} \sum_{i=1}^c a_{c,i} b_{k+i},
\]
which is a \( Z \)-linear combination of finite many terms \( b_k \) \((cm + 2 \leq k \leq cm + c + d)\). Thus if the series \( \sum_{n=1}^{\infty} a_{c,n} b_n \) converges, then \( b_k \to 0 \) as \( k \to \infty \) and by (26)

\[
\lim_{n \to \infty} \left( \sum_{k=1}^{n+1} a_{c,k} b_k - S_{c,n} \right) = 0,
\]

which is equivalent to

\[
A_c = \sum_{n=1}^{\infty} a_{c,n} b_n
\]

(27)

\[
= \frac{1}{c} \left( \sum_{i=0}^{c-2} (c+i) \Delta_i^0 b_{i+1} - (c-1)^2 \Delta_i^0 b_c \right) - \frac{1}{c} \sum_{n=1}^{\infty} a_{c,n} \Delta_c^1 b_n.
\]

If we assume the theorem is true for \( k \), then for \( k + 1 \) we have

\[
A_c = \sum_{j=0}^{k-1} \frac{(-1)^j}{c^{j+1}} \left( \sum_{i=0}^{c-2} (c+i) \Delta_i^j b_{i+1} - (c-1)^2 \Delta_i^j b_c \right) + \sum_{n=1}^{\infty} \frac{(-1)^k}{c^k} a_{c,n} \Delta_c^k b_n
\]

\[
= \sum_{j=0}^{k-1} \frac{(-1)^j}{c^{j+1}} \left( \sum_{i=0}^{c-2} (c+i) \Delta_i^j b_{i+1} - (c-1)^2 \Delta_i^j b_c \right) + \frac{(-1)^k}{c^k} \sum_{n=1}^{\infty} a_{c,n} \Delta_c^k b_n
\]

and the convergence is ensured from \( k \) to \( k + 1 \) by the same procedure as above. This completes our proof. \( \square \)

From Theorem 3 we get the following generalization of Sondow’s result.
Theorem 5. For \( c \geq 2 \) and \( k \geq 1 \) the product
\[
\zeta(c)(s) = (1 - c^{1-s}) \zeta(s)
\]
\[
= \sum_{j=0}^{k-1} \frac{(-1)^j}{c^{j+1}} \left( \sum_{i=0}^{c-2} (c+i) \Delta_c^j (i+1)^{-s} - (c-1)^2 \Delta_c^j c^{-s} \right)
\]
\[
+ \lim_{N \to \infty} \sum_{n=1}^{cN} \left( -\frac{1}{e} \sum_{i=1}^{c} a_{c,n} \Delta_c^k n^{-s} \right)
\]
provides the analytic continuation of \( \zeta(s) \) on the punctured half plane \( \text{Re}(s) > 1 - k, \ s \neq 1 \) where the infinite series converges absolutely and uniformly on compact sets to a holomorphic function. Moreover, except that the convergence will not be absolute in the strip \( -k < \text{Re}(s) \leq 1 - k, \ s \neq 1 \), this remains true for \( k \geq 0 \) and \( \text{Re}(s) > -k, \ s \neq 1 \). Especially, taking \( k = 1 \), we have
\[
\zeta(c)(s) = (1 - c^{1-s}) \zeta(s)
\]
\[
= \frac{1}{c} \left( \sum_{i=0}^{c-2} (c+i)(i+1)^{-s} - (c-1)^2 c^{-s} - \lim_{N \to \infty} \sum_{n=1}^{cN} a_{c,n} \sum_{i=1}^{c} a_{i,c}(n+i)^{-s} \right)
\]
for \( \text{Re}(s) > -1, \ s \neq 1 \).

Proof. By setting \( b_n = n^{-s} \) in (23), for \( \sigma = \text{Re}(s) > 1 \), we get
\[
\zeta(c)(s) = (1 - c^{1-s}) \zeta(s)
\]
\[
= \sum_{j=0}^{k-1} \frac{(-1)^j}{c^{j+1}} \left( \sum_{i=0}^{c-2} (c+i) \Delta_c^j (i+1)^{-s} - (c-1)^2 \Delta_c^j c^{-s} \right)
\]
\[
+ \sum_{n=1}^{\infty} \left( -\frac{1}{e} \sum_{i=1}^{c} a_{c,n} \Delta_c^k n^{-s} \right)
\]
where \( \Delta_c^0 n^{-s} = n^{-s} \) and by (24)
\[
\Delta_c^k n^{-s} = \sum_{i=1}^{c} a_{c,i} \Delta_c^{k-1} (n+i)^{-s}
\]
for \( k \geq 1 \).

In the following, we investigate the convergent area in \( \mathbb{C} \) for the series on the right hand side of (30). Let \( (s)_0 = 1 \) and
\[
(s)_k = s(s+1) \cdots (s+k-1)
\]
for \( k \geq 1 \). It can be checked directly for \( k \geq 1 \) that
\[
\Delta_c^k n^{-s} = (s)_k J_k,
\]
where
\[
J_k = \sum_{i_1, \ldots, i_k=1}^{c-1} \int_{i_1}^{c} \cdots \int_{i_1}^{c} (n + x_1 + \cdots + x_k)^{-s-k} dx_1 \cdots dx_k,
\]
so we have the estimation

\[(34) \quad |\Delta^k_c n^{-s}| \leq \frac{|(s)_k| \left(\frac{c(c-1)}{2}\right)^k}{n^{\sigma+k}},\]

where \(\sigma + k \geq 0\) and \(k = 0, 1, 2, \ldots\). By (31) we have

\[(35) \quad \sum_{n=c+1}^{cN} a_{c,n} \Delta^k_c n^{-s} = \sum_{m=2}^{N} (\Delta^k_c (cm - c + 1)^{-s} + \cdots + \Delta^k_c (cm - 1)^{-s} - (c - 1)\Delta^k_c (cm)^{-s})

= \sum_{m=2}^{N} \sum_{i=1}^{c} a_{c,i} \Delta^k_c (cm - c + i)^{-s}

= \sum_{m=2}^{N} \Delta^k_{c+1} (cm - c)^{-s}.

Thus from the estimation (34), we get

\[
\left| \sum_{n=c+1}^{cN} a_{c,n} \Delta^k_c n^{-s} \right| = \left| \sum_{m=2}^{N} \Delta^k_{c+1} (cm - c)^{-s} \right| \leq \sum_{m=2}^{N} \frac{|(s)_{k+1}| \left(\frac{c(c-1)}{2}\right)^{k+1}}{(cm - c)^{\sigma+k+1}},
\]

so in the half plane \(\sigma > -k\), the series

\[(36) \quad \lim_{N \to \infty} \sum_{n=1}^{cN} a_{c,n} \Delta^k_c n^{-s} = \sum_{n=1}^{c} a_{c,n} \Delta^k_c n^{-s} + \lim_{N \to \infty} \sum_{n=c+1}^{cN} a_{c,n} \Delta^k_c n^{-s}

converges uniformly on any compact set.

In addition, by the estimation (34) and \(|a_{c,n}| \leq c - 1\) for \(n \in \mathbb{N}\), we have

\[(37) \quad \left| \sum_{n=1}^{cN} a_{c,n} \Delta^k_c n^{-s} \right| \leq \sum_{n=1}^{cN} |a_{c,n} \Delta^k_c n^{-s}|

\leq (c - 1) \sum_{n=1}^{cN} \frac{|(s)_k| \left(\frac{c(c-1)}{2}\right)^k}{n^{\sigma+k}},

thus in the half plane \(\sigma > 1 - k\), the series (36) converges absolutely and uniformly on any compact set, so does the right hand side of (28). From these, we conclude that (28) provides the analytic continuation of \(\zeta(s)\) as described by the theorem. \(\square\)

The above theorem implies the following new expressions for the special values \(\zeta(-m)\), which generalize Sondow’s formula (10).
Corollary 6. For \( m = 0, 1, 2, \ldots \), we have
\[
\zeta(-m) = (1 - c^{m+1})^{-1} \zeta(c)(-m)
\]
\[
= \frac{1}{1 - c^{m+1}} \sum_{j=0}^{m} \frac{(-1)^j}{j+1} \left( \sum_{i=0}^{c-2} (c + i) \Delta_c^i (i+1)^m - (c-1)^2 \Delta_c^m \right).
\]

Proof. In this case \((-m)_j = 0\) for \( j > m \), so by (32), \( \Delta_c^j n^m = 0 \). Thus if letting \( s = -m \) in (30), then we get our result.

3. Generalizations of Wallis’ formula

With the above preparations, we can extend Yung, Sondow and Huylebrouck’s methods to \( \zeta(c)(s) \) and obtain the following two generalizations of Wallis’ formula.

Theorem 7. For any integer \( c \geq 2 \) and \( \Re(s) > 0 \), we have
\[
e^\zeta(c)(s) = \prod_{n=1}^{\infty} n \left( \frac{-1}{n} \right)^{a_{c,n}} n^s,
\]
where the coefficients \( a_{c,n} \) is defined as in (18).

Remark 8. Setting \( c = 2 \) in Theorem 7 we have \( a_{2,n} = (-1)^{n-1} \), so (39) becomes to
\[
e^{\eta}(s) = \frac{2 \frac{1}{\pi} \cdot 4 \frac{1}{\pi} \cdot 6 \frac{1}{\pi} \cdots}{3 \frac{1}{\pi} \cdot 5 \frac{1}{\pi} \cdot 7 \frac{1}{\pi} \cdots},
\]
which is equivalent to (13) above.

Proof of Theorem 7. Recall that
\[
\zeta(c)(s) = (1 - c^{1-s})\zeta(s).
\]
Since the only pole for \( \zeta(s) \) is at \( s = 1 \) and the factor \( 1 - c^{1-s} \) has a simple zero at \( s = 1 \), \( \zeta(c)(s) \) is analytic on the whole complex plane \( \mathbb{C} \). By (18), for any \( N \in \mathbb{N} \), the partial sums \( \sum_{n=1}^{N} a_{c,n} \) are all bounded, thus by Jensen-Cahen’s theorem (see, e.g., Conrad’s lecture note on Dirichlet series [3, Theorem 9]), the series \( \sum_{n=1}^{\infty} \frac{a_{c,n}}{n^s} \) is convergent and analytic on the half plane \( \Re(s) > 0 \), with its derivative there computable termwise. So
\[
\zeta(c)(s) = \sum_{n=1}^{\infty} \frac{a_{c,n}}{n^s}
\]
is an identity of analytic functions for \( \Re(s) > 0 \) and in this area we have
\[
\zeta'(c)(s) = \sum_{n=1}^{\infty} a_{c,n} (-1)^{n-1} \log n
\]
\[
= \log \prod_{n=1}^{\infty} n \left( \frac{-1}{n} \right)^{a_{c,n}} n^s.
\]
This is equivalent to
\[(41)\quad e^{G_c(s)}(s) = e^{\log \prod_{n=1}^{\infty} n^{ \frac{(-1)^{a_{c,n}}}{n^s} }} = \prod_{n=1}^{\infty} n^{ \frac{(-1)^{a_{c,n}}}{n^s} }, \]
which is our result.

\[\square\]

**Theorem 9.** For any integer \(c \geq 2\) and \(\text{Re}(s) > -1\), we have
\[(42)\quad e^{G_c(s)}(s) = \left( \prod_{i=2}^{(c-1)} i^{-\frac{s+i-1}{c+s}} \right) \cdot c^{ \frac{(c-1)^2}{c^s+1}} \lim_{N \to \infty} cN \prod_{n=1}^{cN} \prod_{i=1}^{c} (n+i)^{a_{c,n}(n+i)^c}, \]
where the coefficients \(a_{c,n}\) is defined as in (18). Note that the first product is defined to be 1 for \(c = 2\).

**Proof.** Taking derivatives on the both sides of (29), we have
\[(39)\quad \zeta'(c)(s) = -\frac{1}{c} \left( \sum_{i=2}^{(c-1)} (c+i-1)i^{-s} \log i - (c-1)^2c^{-s} \log c \right) \]
\[(43)\quad + \frac{1}{c} \lim_{N \to \infty} \sum_{n=1}^{cN} \sum_{i=1}^{c} a_{c,n}(n+i)^{-s} \log(n+i) \]
\[= \log \left( \prod_{i=2}^{(c-1)} i^{-\frac{s+i-1}{c+s}} \right) \cdot c^{ \frac{(c-1)^2}{c^s+1}} \lim_{N \to \infty} cN \prod_{n=1}^{cN} \prod_{i=1}^{c} (n+i)^{a_{c,n}(n+i)^c}, \]
so
\[(44)\quad e^{G_c(s)}(s) = \left( \prod_{i=2}^{(c-1)} i^{-\frac{s+i-1}{c+s}} \right) \cdot c^{ \frac{(c-1)^2}{c^s+1}} \lim_{N \to \infty} cN \prod_{n=1}^{cN} \prod_{i=1}^{c} (n+i)^{a_{c,n}(n+i)^c}, \]
which is the desired result. \(\square\)

### 4. Examples

In this section, we show some examples for our extensions of Wallis’ formula in the above section. First, we go to the case when \(c = 3\). In this case, by (17) we have
\[(39)\quad \zeta(3)(s) = 1 + \left( \frac{1}{2^s} - \frac{2}{3^s} + \frac{1}{4^s} \right) + \left( \frac{1}{5^s} - \frac{2}{6^s} + \frac{1}{7^s} \right) + \ldots \]
and
\[(39)\quad \zeta(3)(s) = ( (-1)2^{-s} \log 2 + 2 \cdot 3^{-s} \log 3 + (-1)4^{-s} \log 4 ) \]
\[+ ( (-1)5^{-s} \log 5 + 2 \cdot 6^{-s} \log 6 + (-1)7^{-s} \log 7 ) + \ldots , \]
\[= \log \left( \frac{3^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} \cdot 6^{\frac{1}{5}} \cdot 6^{\frac{1}{6}} \cdot 9^{\frac{1}{10}} \cdot 9^{\frac{1}{10}}}{2^{\frac{1}{2}} \cdot 4^{\frac{1}{3}} \cdot 5^{\frac{1}{5}} \cdot 7^{\frac{1}{6}} \cdot 8^{\frac{1}{7}} \cdot 10^{\frac{1}{10}} \ldots} \ldots \right) . \]
So (39) becomes to
\[(45)\quad e^{G_3(s)}(s) = \frac{3^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} \cdot 6^{\frac{1}{5}} \cdot 6^{\frac{1}{6}} \cdot 9^{\frac{1}{10}} \cdot 9^{\frac{1}{10}}}{2^{\frac{1}{2}} \cdot 4^{\frac{1}{3}} \cdot 5^{\frac{1}{5}} \cdot 7^{\frac{1}{6}} \cdot 8^{\frac{1}{7}} \cdot 10^{\frac{1}{10}} \ldots} , \]
for \( \text{Re}(s) > 0 \). Furthermore, by [2, Eq. (3)] we have the following Laurent series expansion of \( \zeta(s) \) around \( s = 1 \)

\[
\zeta(s) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} (s - 1)^n,
\]

where \( \gamma_n \) is the Stieltjes constant in 1885 (see Stieltjes’ original article [9] and Ferguson [5]) and \( \gamma = \gamma_0 \) is the Euler-Mascheroni constant. Since

\[
3^{1-s} = e^{(1-s) \log 3}
\]

\[
\zeta'(3)(1) = \log 3 \left( \gamma - \frac{\log 3}{2} \right).
\]

In fact, for any integer \( c \geq 2 \), from the same reasoning we can also get

\[
\zeta'(c)(1) = \log c \left( \gamma - \frac{\log c}{2} \right).
\]

So (45) implies another generalization of Wallis’ formula,

\[
3^{\frac{\log 3}{2}} = \frac{3^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot 6^{\frac{1}{2}} \cdot 6^{\frac{1}{2}} \cdot 9^{\frac{1}{2}} \cdot 9^{\frac{1}{2}}}{2^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 7 \cdot 8^{\frac{1}{2}} \cdot 10^{\frac{1}{2}}} \cdots.
\]

Since \( \zeta(2) = \frac{\pi^2}{6} \) and

\[
\zeta'(2) = \frac{\pi^2}{6} \left( \gamma + \log 2\pi - 12 \log A \right),
\]
where \( \gamma \) is the Euler-Mascheroni constant and \( A \) is the Glaisher-Kinkelin constant (see \[14, \text{Eq. (13)}\]), by \([16]\) we have
\[
\zeta'(3)(2) = \frac{1}{3} \zeta(2) \log 3 + \frac{2}{3} \zeta'(2)
\]
\[
= \frac{1}{3} \left( \frac{\pi^2}{6} \right) \log 3 + \frac{2}{3} \left( \frac{\pi^2}{6} (\gamma + \log 2\pi - 12 \log A) \right)
\]
\[
= \frac{\pi^2}{18} \log \left( 3 \cdot \left( \frac{2\pi e^7}{A^{12}} \right)^2 \right).
\]
So \([45]\) also implies the following generalization of Wallis' formula,
\[
\left( 3 \left( \frac{2\pi e^7}{A^{12}} \right)^2 \right)^{\frac{\pi^2}{18}} = \frac{3 \frac{\pi}{3} \cdot 3 \frac{\pi}{3} \cdot 6 \frac{\pi}{3} \cdot 6 \frac{\pi}{3} \cdot 9 \frac{\pi}{3} \cdot 9 \frac{\pi}{3}}{2 \frac{\pi}{3} \cdot 4 \frac{\pi}{3} \cdot 5 \frac{\pi}{3} \cdot 7 \frac{\pi}{3} \cdot 8 \frac{\pi}{3} \cdot 10 \frac{\pi}{3}} \ldots .
\]
Although Theorem \([7]\) can not be applied to the point \( s = 0 \), we can use Theorem \([9]\) By \([29]\) we have an identity of analytic functions on the half plane \( \text{Re}(s) > -1 \),
\[
\zeta(3)(s) = \frac{1}{3} \left( 3 \cdot 1^{-s} + 4 \cdot 2^{-s} - 4 \cdot 3^{-s} \right)
\]
\[- \frac{1}{3} \lim_{N \to \infty} \sum_{n=1}^{3N} a_{n,3}((n+1)^{-s} + (n+2)^{-s} - 2(n+3)^{-s}),
\]
so
\[
\zeta'(3)(s) = - \frac{1}{3} (4 \cdot 2^{-s} \log 2 - 4 \cdot 3^{-s} \log 3) + \frac{1}{3} \lim_{N \to \infty} \sum_{n=1}^{3N} a_{n,3}((n+1)^{-s} \log(n+1)
\]
\[- (n+2)^{-s} \log(n+2) - 2(n+3)^{-s} \log(n+3))
\]
\[
= \log \frac{3 \cdot \frac{\pi}{3} \cdot 2 \cdot \frac{\pi}{3} \cdot 3 \cdot \frac{\pi}{3} \cdot 3 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 6 \cdot \frac{\pi}{3} \cdot 6 \cdot \frac{\pi}{3}}{2 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3}} \ldots .
\]
Thus \([42]\) becomes to
\[
e^{\zeta'(3)(s)} = \frac{3 \cdot \frac{\pi}{3} \cdot 2 \cdot \frac{\pi}{3} \cdot 3 \cdot \frac{\pi}{3} \cdot 3 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 6 \cdot \frac{\pi}{3} \cdot 6 \cdot \frac{\pi}{3}}{2 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3}} \ldots .
\]
On the other hand, since \( \zeta(0) = -\frac{1}{2} \) and \( \zeta'(0) = -\frac{1}{2} \log 2\pi \) (see \([6]\) p. 1049, 9.542}), by \([16]\) we have
\[
\zeta'(3)(0) = 3\zeta(0) \log 3 + (-2)\zeta'(0)
\]
\[
= 3 \left( -\frac{1}{2} \right) \log 3 + (-2) \left( -\frac{1}{2} \log 2\pi \right)
\]
\[
= \log \frac{2\pi}{3^2},
\]
so \([53]\) implies another generalization of Wallis' formula,
\[
\frac{2\pi}{3^2} = \frac{3 \cdot \frac{\pi}{3} \cdot 2 \cdot \frac{\pi}{3} \cdot 3 \cdot \frac{\pi}{3} \cdot 3 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 6 \cdot \frac{\pi}{3} \cdot 6 \cdot \frac{\pi}{3}}{2 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 4 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3} \cdot 5 \cdot \frac{\pi}{3}} \ldots .
\]
Finally, we need to remark here that similar forms with (54), (51) and (52) are also established for $c = 4, 5, 6, \ldots$. For example, if $c = 4$, then (54) becomes to

$$(55) \frac{(2\pi)^\frac{3}{2}}{16} = \frac{4^{\frac{3}{2}}}{2^2} \cdot \frac{3}{2^2} \cdot \frac{4^{\frac{3}{2}}}{5^2} \cdot \frac{3}{5^2} \cdot \frac{4^{\frac{3}{2}}}{5^2} \cdot \frac{3}{5^2} \cdot \frac{4^{\frac{3}{2}}}{6^2} \cdot \frac{3}{6^2} \cdot \frac{4^{\frac{3}{2}}}{6^2} \cdot \frac{3}{6^2} \cdot \frac{4^{\frac{3}{2}}}{7^2} \cdot \frac{3}{7^2} \cdot \frac{4^{\frac{3}{2}}}{7^2} \cdot \frac{3}{7^2} \cdot \frac{5}{8^2} \cdot \frac{3}{8^2} \cdot \frac{8^{\frac{3}{2}}}{8^2} \cdot \frac{3}{8^2} \cdot \frac{8^{\frac{3}{2}}}{8^2} \cdots .$$

REFERENCES

[1] T.M. Apostol, Introduction to analytic number theory, Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.
[2] B.K. Choudhury, The Riemann zeta-function and its derivatives, Proc. Roy. Soc. London Ser. A 450 (1995), no. 1940, 477–499.
[3] K. Conrad, Dirichlet series, https://kconrad.math.uconn.edu/math5121s18/handouts/dirichletseries.pdf
[4] L. Euler, Opera Omnia, Series Prima, Teubner, Leipzig and Zurich, 1911–1956.
[5] R.P. Ferguson, An application of Stieltjes integration to the power series coefficients of the Riemann zeta function, Amer. Math. Monthly 70 (1963), 60–61.
[6] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series, and products, translated from the fourth Russian edition, fifth edition, translation edited and with a preface by Alan Jeffrey, Academic Press, Inc., Boston, MA, 1994.
[7] D. Huylebrouck, Generalizing Wallis’ formula, Amer. Math. Monthly 122 (2015), no. 4, 371–372.
[8] H. Knospe and L.C. Washington, Dirichlet series expansions of $p$-adic L-functions, Abh. Math. Semin. Univ. Hambg. 91 (2021), no. 2, 325–334.
[9] T.J. Stieltjes, Table des valeurs des sommes $S_k = \sum_{n=1}^{\infty} n^{-k}$, Acta Math. 10 (1887), no. 1, 299–302.
[10] J. Sondow and E.W. Weisstein, “Wallis Formula,” From MathWorld—A Wolfram Web Resource.
[11] J. Sondow, Analytic continuation of Riemann’s zeta function and values at negative integers via Euler’s transformation of series, Proc. Amer. Math. Soc. 120 (1994), no. 2, 421–424.
[12] J. Wallis, Arithmetica Infinitorum, Oxford, England, 1656.
[13] A. Weil, Number theory, An approach through history, From Hammurapi to Legendre, Birkhäuser Boston, Inc., Boston, MA, 1984.
[14] E.W. Weisstein, “Glaisher-Kinkelin Constant,” From MathWorld—A Wolfram Web Resource.

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