Mass formulas for local Galois representations (with an appendix by Daniel Gulotta)

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February 13, 2007

Abstract
Bhargava has given a formula, derived from a formula of Serre, computing a certain count of extensions of a local field, weighted by conductor and by number of automorphisms. We interpret this result as a counting formula for permutation representations of the absolute Galois group of the local field, then speculate on variants of this formula in which the role of the symmetric group is played by other groups. We prove an analogue of Bhargava’s formula for representations into a Weyl group in the $B_n$ series, which suggests a possible link with integration on $p$-adic groups. We also obtain analogous positive results in odd residual characteristic, and negative results in residual characteristic 2, for the $D_n$ series (in the appendix) and the exceptional group $G_2$.

1 Introduction

Serre [14 Théorème 2] gave the following astonishing “mass formula” counting totally ramified degree $n$ extensions of a local field $K$ with residue field $\mathbb{F}_q$: if $S_n$ is a set of representatives of the isomorphism classes of such extensions of $K$, then

$$\sum_{L \in S_n} \frac{1}{w(L)q^{c(L)-n+1}} = 1,$$

where $w(L)$ is the number of automorphisms of $L$ and $c(L)$ is the discriminant exponent of $L$ over $K$. The automorphism contribution is no surprise, as it invariably occurs in counting problems of this ilk (essentially because of Burnside’s formula); the distinguishing feature of Serre’s formula is the weighting by conductor.
In the context of deriving heuristics on the number of number fields of given degree with discriminant bounded in a certain range (consistent with the theorems of Davenport-Heilbronn \[5\] in the cubic case and Bhargava \([2, 3]\) in the quartic and quintic cases; see Belabas’s Séminaire Bourbaki notes \([1]\) for an overview), Bhargava \([4,\text{ Theorem 1}]\) has derived from Serre’s formula the following mass formula counting étale \(K\)-algebras of degree \(n\).

**Theorem 1.1** (Bhargava). Let \(K\) be a local field with residue field \(\mathbb{F}_q\), and let \(\Sigma_n\) be a set of representatives for the isomorphism classes of étale \(K\)-algebras of degree \(n\). For \(L \in \Sigma_n\), let \(w(L)\) be the number of automorphisms of \(L\) and let \(c(L)\) be the discriminant exponent of \(L\). Then one has

\[
\sum_{L \in \Sigma_n} \frac{1}{w(L)q^{c(L)}} = \sum_{k=0}^{n} \frac{P(n, n-k)}{q^k},
\]

(1.1.1)

where \(P(n, n-k)\) denotes the number of partitions of the integer \(n\) into exactly \(n-k\) parts, or equivalently the number of partitions of \(k\) into at most \(n-k\) parts. (Note: \(P(n,0) = 1\) for \(n = 0\) and 0 for \(n > 0\).)

The purpose of this paper is twofold. We first reformulate Bhargava’s formula as a counting formula for permutation representations of the absolute Galois group of a local field, and exhibit a straightforward deduction of the latter formula from Serre’s formula using standard techniques from combinatorics (notably the Exponential Formula). We then pose some questions about possible mass formulas for other types of representations, and establish affirmative and negative answers in some classes of cases. More precisely, we ask (Question \(\text{[7.1]}\)) about representations into the Weyl group of a semisimple Lie algebra, motivated by a potential link to integration on \(p\)-adic Lie groups. Bhargava’s formula answers Question \(\text{[7.1]}\) affirmatively for the \(A_n\) series; imitating the \(A_n\) proof, we are able to resolve Question \(\text{[7.1]}\) affirmatively for the \(B_n\) series. We also check the \(G_2\) case by direct calculation; here we discover a surprising negative answer to Question \(\text{[7.1]}\) for local fields of residual characteristic 2, which renders any potential link to integration on \(p\)-adic groups even more mysterious.

In the appendix by Daniel Gulotta, Question \(\text{[7.1]}\) is answered affirmatively for the \(D_n\) series in the case of odd residual characteristic. By machine calculation, it is also shown that this affirmative answer cannot in general extend to residual characteristic 2.

**Notational conventions**

In this paper, a **local field** is a complete discretely valued field (of either mixed or equal characteristics) with finite residue field. For \(K\) a local field, let \(\mathfrak{o}_K\) denote the ring of integers of \(K\), and let \(\mathfrak{m}_K\) denote the maximal ideal of \(\mathfrak{o}_K\). For \(L/K\) a finite separable extension of local fields, let \(f(L/K)\) denote the degree of the induced extension on residue fields, and let \(c(L/K)\) denote the discriminant exponent. For \(K\) any field, let \(G_K = \text{Gal}(K^{\text{sep}}/K)\) denote the absolute Galois group of \(K\).
Acknowledgments

Some of this material was presented at the workshop “Rings of low rank” in June 2006, hosted by the Lorentz Center in Leiden. Thanks to Manjul Bhargava for providing a copy of his preprint [4] and for additional helpful discussions, and to Melanie Wood for comments and corrections on a previous draft. Kedlaya was supported by NSF grant DMS-0400747, NSF CAREER grant DMS-0545904, and a Sloan Research Fellowship. Gulotta was supported by MIT’s Undergraduate Research Opportunities Program.

2 Total mass

To begin with, we define a numerical invariant attached to a local field and a finite linear group. First, let us fix notation for local fields.

Definition 2.1. Let $K$ be a local field, let $L$ be a finite Galois extension of $K$, and put $G = \text{Gal}(L/K)$. For $s \in G \setminus \{e\}$, let $i_G(s)$ be the largest integer $n$ such that $v_L(x^s - x) \geq n$ for all $x \in \mathfrak{o}_L$. Define the Artin character $a_G : G \rightarrow \mathbb{Z}$ by

$$a_G(s) = \begin{cases} -f(L/K)i_G(s) & s \neq e \\ -\sum_{t \neq e} a_G(t) & s = e. \end{cases}$$

By a theorem of Artin [15, Theorem VI.1], $a_G$ is the character of a representation of $G$. Hence for any function $\chi : G \rightarrow \mathbb{C}$ which is the character of a complex representation of $G$, the inner product

$$\frac{1}{|G|} \sum_{s \in G} a_G(s)\chi(s)$$

is a nonnegative integer, called the Artin conductor of $\chi$. For $\rho : G \rightarrow \text{GL}_n(\mathbb{C})$ a representation with character $\chi$, we write $c(\rho)$ for the conductor of $\chi$; note that $c(\rho_1 \oplus \rho_2) = c(\rho_1) + c(\rho_2)$.

Definition 2.2. Let $n$ be a positive integer and let $\Gamma$ be a finite subgroup of $\text{GL}_n(\mathbb{C})$. For $K$ a local field with residue field $\mathbb{F}_q$, define the total mass of the pair $(K, \Gamma)$, denoted $M(K, \Gamma)$, as follows. Let $S_{K,\Gamma}$ be the set of continuous homomorphisms $\rho : \text{Gal}(K^{\text{sep}}/K) \rightarrow \Gamma$. For $\rho \in S_{K,\Gamma}$, identify $\rho$ with the linear representation obtained from $\rho$ by embedding $\Gamma$ into $\text{GL}_n(\mathbb{C})$. Put

$$M(K, \Gamma) = \frac{1}{|\Gamma|} \sum_{\rho \in S_{K,\Gamma}} \frac{1}{q^{c(\rho)}},$$

assuming that the sum converges.

Remark 2.3. It can be shown that the sum in (2.2.1) is finite if $K$ has mixed characteristics, and is convergent if $K$ has equal characteristics, so the definition always makes sense.

By Burnside’s theorem, we can reformulate this definition as follows. Let $\Sigma_{K,\Gamma}$ be a set of representatives of the isomorphism classes (under conjugation within $\Gamma$) of continuous
homomorphisms $\rho : G_K \to \Gamma$. For $\rho \in \Sigma_{K,\Gamma}$, let $w(\rho)$ be the order of the centralizer in $\Gamma$ of the image of $\rho$. Then also

$$M(K,\Gamma) = \sum_{\rho \in \Sigma_{K,\Gamma}} \frac{1}{w(\rho)q^{c(\rho)}}. \quad (2.3.1)$$

**Remark 2.4.** If we view $\Gamma$ as a group equipped with a faithful (i.e., injective) linear representation, then adding a trivial summand to the representation does not change any conductors, and so does not change the total mass.

**Remark 2.5.** In some cases, it might be useful to allow $\Gamma$ to be equipped with a non-faithful linear representation, rather than to view it as a subgroup of $\text{GL}_n(\mathbb{C})$. As this does not materially enrich the situation from our point of view, we will not do so.

**Lemma 2.6.** Let $\Gamma_1 \subseteq \text{GL}_m(\mathbb{C})$ and $\Gamma_2 \subseteq \text{GL}_n(\mathbb{C})$ be finite subgroups, and view $\Gamma_1 \times \Gamma_2$ as a subgroup of $\text{GL}_{m+n}(\mathbb{C})$. Then for any local field $K$,

$$M(K,\Gamma_1 \times \Gamma_2) = M(K,\Gamma_1)M(K,\Gamma_2).$$

**Proof.** Given $\rho : G_K \to \Gamma_1 \times \Gamma_2$ continuous, let $\rho_1 : G_K \to \Gamma_1$ and $\rho_2 : G_K \to \Gamma_2$ be the results of composing $\rho$ with the projections from $\Gamma_1 \times \Gamma_2$ to its two factors. Then $c(\rho) = c(\rho_1) + c(\rho_2)$, from which the desired result follows. \qed

**Remark 2.7.** Bhargava [4, §8] suggests introducing a parameter $s$ in the exponent of $q$ in the definition of the total mass; this gives rise to local factors which one then multiplies together to give a global Dirichlet series, whose asymptotics one hopes resemble those of a Dirichlet series which actually counts certain representations of a global Galois group into $\Gamma$. As verified by Wood [19], this heuristic in fact reproduces Malle’s predicted asymptotics for counting number fields with prescribed Galois group [11, 12]. We will omit any further consideration in this direction in this paper; to do so, we omit the parameter $s$, which amounts to setting $s = 1$. (Taking $s$ to be a positive integer amounts to replacing the linear representation of $\Gamma$ by its $s$-th tensor power.)

### 3 Bhargava’s formula and permutation representations

Before proving Bhargava’s theorem, we first check that the left-hand side of (1.1.1) is equal to $M(K, S_n)$, where $S_n$ is embedded in $\text{GL}_n(\mathbb{C})$ via its standard permutation representation, by matching up corresponding terms. Using this equality, we will establish (1.1.1) in the next section.

We start with the usual equivalence of categories between étale $K$-algebras and finite $G_K$-sets.

**Lemma 3.1.** For any field $K$, there is a natural bijection between isomorphism classes of étale $K$-algebras of degree $n$ and isomorphism classes, under conjugation within $S_n$, of continuous homomorphisms $\rho : G_K \to S_n$. Under this bijection, finite separable field extensions of $K$ correspond to homomorphisms with transitive image.
Proof. A $K$-algebra $L$ of degree $n$ is étale if and only if there exists an isomorphism of $K^{\text{sep}}$-algebras $L \otimes_K K^{\text{sep}} \cong (K^{\text{sep}})^n$, that is, if $L \otimes_K K^{\text{sep}}$ contains $n$ minimal idempotents. In fact, these idempotents all lie in $L \otimes_K F$, for any Galois extension $F/K$ containing a copy of each component of $L$. Now equip $L \otimes_K K^{\text{sep}}$ with the action of $G_K$ which is trivial on the first factor and the usual action on the second factor. The action on minimal idempotents yields a continuous homomorphism $\rho : G_K \rightarrow S_n$.

Conversely, given a continuous homomorphism $\rho : G_K \rightarrow S_n$, we obtain an action of $G_K$ on $(K^{\text{sep}})^n$ via permutations. We can construct a full set of invariants under some finite separable extension of $K^{\text{sep}}$, which obviously must coincide with $K^{\text{sep}}$ itself. These invariants form a $K$-subalgebra of $(K^{\text{sep}})^n$ of $K$-dimension $n$, which by construction is étale. The functors just described yield the desired bijection.

We next verify that the bijection of Lemma 3.1 matches up the two automorphism contributions.

**Lemma 3.2.** Under the bijection of Lemma 3.1, let $L$ be an étale $K$-algebra corresponding to a continuous homomorphism $\rho : G_K \rightarrow S_n$. Then $w(L) = w(\rho)$.

**Proof.** The group $\text{Aut}(L/K)$ is isomorphic to the group of automorphisms of $L \otimes_K K^{\text{sep}}$ which are semilinear for the $K$-action on $K^{\text{sep}}$. This group in turn coincides with the $G_K$-equivariant permutations of the set of minimal idempotents of $L \otimes_K K^{\text{sep}}$, yielding the claim.

Finally, we have the equality of the discriminant and conductor contributions; for this, we need the conductor-discriminant formula.

**Lemma 3.3.** Let $H$ be an open subgroup of $G = G_K$ with fixed field $L$, and let $\rho : H \rightarrow \text{GL}_n(\mathbb{C})$ be a continuous representation, where $\text{GL}_n(\mathbb{C})$ carries the discrete topology. Then

$$c(\text{Ind}_H^G \rho) = f(L/K)c(\rho) + nc(L/K).$$

**Proof.** See [15, Proposition VI.6, Corollary 1].

**Lemma 3.4.** Under the bijection of Lemma 3.1, let $L$ be an étale $K$-algebra corresponding to a continuous homomorphism $\rho : G_K \rightarrow S_n$. Then $c(L/K) = c(\rho)$.

**Proof.** Since both functions are additive over direct sums, we may reduce to the case where $L$ is a field and $\rho$ has transitive image. In that case, let $F$ be the normal closure of $L/K$, and put $G = \text{Gal}(F/K)$ and $H = \text{Gal}(F/L)$. Then the linear representation derived from $\rho$ is isomorphic to the representation induced from the trivial representation on $H$. The desired result thus follows from Lemma 3.3 applied to the trivial one-dimensional representation.

Putting together the three lemmas, we see at once that the left side of (1.1.1) equals $M(K, S_n)$, as claimed.
4 Reduction to the mass formula

We now give a proof of Theorem 1.1 by reduction to Serre’s mass formula. This is essentially Bhargava’s proof in [4] except that we use a standard device from enumerative combinatorics, the Exponential Formula (for an exposition of which see [16, Chapter 5]), in lieu of explicitly combining fields into étale algebras as in [4, Propositions 1–3].

Proof of Theorem 1.1. We first reinterpret Serre’s formula as in the previous section. Let $I$ be the inertia subgroup of $G_K$. We say a continuous homomorphism $\rho : G_K \to S_n$ with transitive image is totally ramified if $\rho^{-1}(S_{n-1})$ and $I$ together generate $G_K$. Then Serre’s formula states that

$$\frac{1}{n!} \sum_{\rho} \frac{1}{q^{c(\rho)}} = q^{1-n},$$

where $\rho$ runs over all totally ramified homomorphisms $G_K \to S_n$ with transitive image.

Let $T_{K,S_n}$ denote the subset of $S_{K,S_n}$ (in the notation of Definition 2.2) consisting of homomorphisms with transitive image. Given $\rho \in T_{K,S_n}$, let $f = f(\rho)$ be the index of the image of $\rho^{-1}(S_{n-1})$ in $G_K/I$; this index necessarily divides $n$. Let $K_f$ be the unramified extension of $K$ of degree $f$, and put $G_f = G_{K_f}$. Then the restriction of $\rho$ to $G_f$ splits as a direct sum of $f$ totally ramified representations which are isomorphic to the conjugates of some representation $\rho_f : G_f \to S_{n/f}$ by a generator $\sigma$ of $G_K/G_f$. By Frobenius reciprocity, $\rho$ is isomorphic to the induced representation $\text{Ind}_{G_f}^{G_K} \rho_f$, and $c(\rho) = fc(\rho_f)$.

Given a choice of $f$ and $\rho_f$, one can reconstruct such a $\rho$ by choosing a partition of $\{1, \ldots, n\}$ into $f$ labeled blocks of size $n/f$, then choosing a bijection between each group and $\{1, \ldots, n/f\}$. However, each $\rho$ is produced $f/r$ times, where $r$ is the smallest positive integer such that $\rho_f$ is isomorphic to its conjugate by $\sigma^r$. In fact the same $\rho$ is produced by all $r$ of the conjugates of $\rho_f$ under $\sigma$, so we need to divide both by $f/r$ and by $r$ to account for this. In addition, there is a further overcount by a factor of $w(\rho_f)$ (the number of automorphisms of $\rho_f$). Therefore

$$\frac{1}{n!} \sum_{\rho \in T_{K,S_n}, f(\rho) = f} \frac{1}{q^{c(\rho)}} = \frac{1}{n!} \frac{1}{f} \sum_{\rho_f} \frac{1}{w(\rho_f)(q^f)^{c(\rho_f)}},$$

where the sum on $\rho_f$ runs over isomorphism classes of totally ramified representations $\rho_f : G_f \to S_{n/f}$. By Serre’s formula, we have

$$\frac{1}{n!} \sum_{\rho \in T_{K,S_n}, f(\rho) = f} \frac{1}{q^{c(\rho)}} = \frac{q^{f-n}}{f}. \quad (4.0.1)$$

Passing from (4.0.1) (after summing over $f$) to the total mass amounts to an application
of the Exponential Formula:

$$\sum_{n=0}^{\infty} M(K, S_n)x^n = \exp \left( \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{\rho \in \mathcal{T}_{K, S_n}} \frac{1}{q^{|\rho|}} \right)$$

$$= \exp \left( \sum_{n=1}^{\infty} x^n \sum_{f|n} \frac{q^{f-n}}{f} \right)$$

$$= \exp \left( \sum_{i=1}^{\infty} \sum_{f=1}^{\infty} x^{f_i} q^{f(1-i)} \frac{1}{f} \right)$$

$$= \exp \left( \sum_{i=1}^{\infty} \log(1 - x^i q^{1-i})^{-1} \right)$$

$$= \prod_{i=1}^{\infty} (1 - x^i q^{1-i})^{-1}.$$ 

Substituting $xq$ for $x$ yields

$$\sum_{n=0}^{\infty} M(K, S_n)x^n q^n = \prod_{i=1}^{\infty} (1 - x^i q)^{-1}.$$ 

The coefficient of $x^n q^{n-k}$ on the right side is visibly equal to $P(n, n-k)$. Since we checked in Section 3 that $M(K, S_n)$ equals the left side of (1.1.1), we may deduce the desired result. \Box

**Remark 4.1.** Bhargava’s proof in [4] shows that one can formulate more precise versions of the mass formula that sum over a single splitting type for étale algebras of degree $n$, the case of a single component being (4.0.1). In fact, such statements can be read off from the Exponential Formula; in particular, it will be possible to formulate such results in other cases (e.g., in Theorem 8.5), though we will not explicitly do so.

## 5 Uniformity for other groups: tame case

We now propose a context into which Theorem 1.1 can potentially be generalized.

**Definition 5.1.** Let $n$ be a positive integer and let $\Gamma$ be a finite subgroup of $\text{GL}_n(\mathbb{C})$. If $S$ is a class of local fields, we say $\Gamma$ is *uniform for $S$* if there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that for any local field $K \in S$ with residue field $\mathbb{F}_q$, we have $M(K, \Gamma) = P(q^{-1})$. (There is a natural candidate function $P(q^{-1})$, but it is not always a polynomial; see Proposition 5.3.) If $S$ is the class of all local fields, we say $\Gamma$ is *uniform for local fields*.

**Remark 5.2.** If $\Gamma_i$ is a finite subgroup of $\text{GL}_{n_i}(\mathbb{C})$ which is uniform for local fields for $i = 1, 2$, then $\Gamma_1 \times \Gamma_2 \subset \text{GL}_{n_1+n_2}(\mathbb{C})$ is uniform for local fields, by Lemma 2.6.
By Theorem 1.1, the group $S_n \subset \text{GL}_n(\mathbb{C})$ is uniform for local fields; we may then ask what other groups have this property. We obtain a candidate formula for the total mass by calculating what happens in the tamely ramified case, i.e., when the residue characteristic of $K$ is coprime to the order of $\Gamma$. The result is a quasi-polynomial in $q^{-1}$, i.e., a function which agrees with different polynomials on different residue classes; the failure of this quasi-polynomial to be a true polynomial (as in Example 6.1 below) constitutes a simple obstruction to $\Gamma$ being uniform for local fields.

**Proposition 5.3.** Let $n$ be a positive integer, and let $\Gamma$ be a finite subgroup of $\text{GL}_n(\mathbb{C})$. For $g \in \Gamma$, let $e(g)$ denote the number of eigenvalues of $g$ not equal to 1, and define the quasi-polynomial $P_\Gamma(q^{-1})$ by

$$P_\Gamma(q^{-1}) = \frac{1}{|\Gamma|} \sum_{g, h \in \Gamma : hgh^{-1} = g^q} q^{-e(g)}.$$

Then for any local field $K$ whose residue field $\mathbb{F}_q$ has characteristic prime to $|\Gamma|$, we have $M(K, \Gamma) = P_\Gamma(q^{-1})$.

**Proof.** Since $K$ has residue characteristic prime to $|G|$, any continuous homomorphism $\rho : G_K \to \Gamma$ factors through the maximal tame quotient of $K$. That quotient is topologically generated by $x, y$ subject to the relation $yxy^{-1} = x^q$; thus the homomorphisms are determined by pairs $(g, h)$ as in the proposition, and the quantity $e(g)$ is precisely the conductor of the corresponding homomorphism.

**Corollary 5.4.** With notation as in Proposition 5.3, we have

$$M(K, \Gamma) = P_\Gamma(q^{-1}) = \sum q^{-e(g)},$$

where the sum runs over a set of representatives of those conjugacy classes of $\Gamma$ which are stable under the $q$-th power map.

**Proof.** For fixed $g \in \Gamma$, the set of $h$ such that $hgh^{-1} = g^q$ is empty if $g$ is not conjugate to $g^q$, and otherwise is a left coset of the centralizer of $g$. This yields the claim by Burnside’s formula.

**Corollary 5.5.** Let $n$ be a positive integer and let $\Gamma$ be a finite subgroup of $\text{GL}_n(\mathbb{C})$. Then $\Gamma$ is uniform for local fields of residue characteristic prime to $|\Gamma|$ if and only if the character table of $\Gamma$ has rational integral entries.

**Proof.** By Corollary 5.4 plus Dirichlet’s theorem on primes in arithmetic progressions, $\Gamma$ is uniform for local fields of residue characteristic prime to $|\Gamma|$ if and only if for each $m$ coprime to $|\Gamma|$, each element of $\Gamma$ is conjugate to its $m$-th power. It is a standard result of representation theory for finite groups [13, Chapter 13, Theorem 29] that this condition is equivalent to the rationality of the entries of the character table of $\Gamma$.

**Remark 5.6.** In case $\Gamma$ arises from a permutation representation, $e(g)$ coincides with Malle’s index function. This is related to the fact that one can recover Malle’s heuristics by considering local masses; see Remark 2.7.
6 Examples and counterexamples

In this section, we mention some examples that help clarify the extent to which uniformity for local fields holds. We start with an illustration of Proposition 5.3.

Example 6.1. If $\Gamma = \mathbb{Z}/3\mathbb{Z}$ embedded into $\text{GL}_1(\mathbb{C})$ as the cube roots of unity, and $K$ is a local field of residue field $\mathbb{F}_q$ of characteristic $\neq 3$, then

$$M(K, \Gamma) = \begin{cases} 1 + 2q^{-1} & q \equiv 1 \pmod{3} \\ 1 & q \equiv 2 \pmod{3} \end{cases},$$

so $\Gamma$ cannot be uniform for local fields.

Example 6.1 illustrates that the conclusion of Corollary 5.5 imposes a strong restriction on groups which can be uniform for local fields. However, the conclusion of Corollary 5.5 does not give a sufficient condition for uniformity for all local fields; that is because it depends only on the group $\Gamma$ and not on its embedding into $\text{GL}_n(\mathbb{C})$. Here is an example to illustrate what can go wrong when one changes the embedding.

Example 6.2. Let $\Gamma$ be the group $\mathbb{Z}/2\mathbb{Z}$, viewed as a subgroup of $\text{GL}_2(\mathbb{C})$ via its regular representation. We know that $\Gamma$ is uniform for local fields by Theorem 1.1; let us check a bit of this explicitly. For any local field $K$ whose residue field $\mathbb{F}_q$ has odd residue characteristic, $M(K, \Gamma) = 1 + q^{-1}$ by Proposition 5.3.

However, for $K = \mathbb{Q}_2$, there are eight continuous homomorphisms $\rho : G_{\mathbb{Q}_2} \to \Gamma$, which all factor through $\text{Gal}(L/\mathbb{Q}_2)$ for $L = \mathbb{Q}_2(\zeta_3, i, \sqrt{2})$. We can compute the conductors of these as follows. Apply local class field theory to identify the eight homomorphisms with the homomorphisms $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2 \to \mathbb{Z}/2\mathbb{Z}$. Writing $\mathbb{Q}_2^* = \mathbb{Z} \times \mathcal{O}_{\mathbb{Q}_2}^*$, we can identify $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ with $\mathbb{Z}/2\mathbb{Z} \times \mathcal{O}_{\mathbb{Q}_2}^*/(\mathcal{O}_{\mathbb{Q}_2}^*)^2$. Now

$$\mathcal{O}_{\mathbb{Q}_2}^* = 1 + 2\mathcal{O}_{\mathbb{Q}_2}$$

and $\rho : G_{\mathbb{Q}_2} \to \Gamma$ has conductor 0, 2, 3 according to whether the kernel of the corresponding homomorphism $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2 \to \mathbb{Z}/2\mathbb{Z}$ kills $1 + 2\mathcal{O}_{\mathbb{Q}_2}$, kills $1 + 4\mathcal{O}_{\mathbb{Q}_2}$ but not $1 + 2\mathcal{O}_{\mathbb{Q}_2}$, or does not kill $1 + 4\mathcal{O}_{\mathbb{Q}_2}$. There are thus 2 representations of conductor 0, 2 of conductor 2, and 4 of conductor 3, yielding

$$M(\mathbb{Q}_2, \Gamma) = 1 + 2^{-2} + 2 \cdot 2^{-3} = 1 + 2^{-1}.$$

In fact, one can make a similar calculation for any local field of characteristic 2, as in [14, Exemple 2(b)] (which in turn follows [18, Lemma 4.3]).

Now consider the same group $\Gamma = \mathbb{Z}/2\mathbb{Z}$, but now embedded into $\text{GL}_4(\mathbb{C})$ via two copies of its regular representation. (This example will appear again in Proposition 9.3.) Then $P_\Gamma(q^{-1}) = 1 + q^{-2}$, but

$$M(\mathbb{Q}_2, \Gamma) = 1 + 2^{-4} + 2 \cdot 2^{-6} \neq 1 + 2^{-2}.$$
Hence $\Gamma$ is not uniform for local fields. In fact $\Gamma$ is not even uniform just for local fields of residual characteristic 2, as we may see by calculating the total mass over $\mathbb{F}_2((t))$. In this case, there are 2 representations of conductor 0, and for each positive integer $i$, there are $2^i$ representations of conductor $2 \cdot 2^i$, yielding

$$M(\mathbb{F}_2((t)), \Gamma) = 1 + \frac{1}{2}\sum_{i=1}^{\infty} 2^i \cdot 2^{-4i} = 1 + \frac{1}{14} = 1 + 2^{-4} + 2 \cdot 2^{-6}.$$ 

7 Weyl groups

Serre gave two proofs of his mass formula (1.0.1). One is a direct computation using $p$-adic integration on a suitable space of Eisenstein polynomials, but we find more suggestive the other proof, which is a simple application of the Weyl integration formula on a rank $n$ division algebra over $K$. Since the units of that division algebra constitute a twisted form of the group $\text{GL}_n$ over $K$, Serre’s second proof suggests that the group $S_n$ is arising in Theorem 1.1 as the Weyl group of the Lie algebra $\mathfrak{gl}_n$, and prompts the following question. (Note that the semisimple case reduces to the simple case by Lemma 2.6)

**Question 7.1.** Let $\Gamma$ be the Weyl group of a (semi)simple Lie algebra over $\mathbb{C}$, embedded in the group of linear transformations of the root space. Is $\Gamma$ uniform for local fields of all residual characteristics, or if not, for which ones?

Since $W(A_n) = S_n$ equipped with its standard representation, Theorem 1.1 asserts that the Weyl group $W(A_n)$ is uniform for local fields. In the remainder of the paper, we assemble some additional answers to Question 7.1. Namely, let $\Gamma$ be a Weyl group, and let $S$ be a set of prime numbers. Then $\Gamma$ is uniform for local fields of residual characteristics in $S$ in each of the following cases:

- $\Gamma$ is arbitrary and $S$ consists only of primes not dividing $|G|$ (Proposition 5.3);
- $\Gamma = W(A_n)$ and $S$ is arbitrary (Theorem 1.1);
- $\Gamma = W(B_n)$ and $S$ is arbitrary (Theorem 8.5);
- $\Gamma = W(D_n)$ and $2 \notin S$ (Theorem A.7; see Appendix);
- $\Gamma = W(G_2)$ and $2 \notin S$ (Proposition 9.2);

but not in the following cases:

- $\Gamma = W(D_4)$ and $S$ properly contains $\{2\}$ (Proposition A.8 see Appendix);
- $\Gamma = W(G_2)$ and $2 \in S$ (Proposition 9.3).

Keeping in mind the exceptional isomorphisms $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$, we see that the remaining cases of Question 7.1 are:
\begin{itemize}
  \item $\Gamma = W(D_n)$ for $n \geq 5$ and $2 \in S$;
  \item $\Gamma = W(E_6), W(E_7), W(E_8), W(F_4)$.
\end{itemize}

Moreover, we lack an interpretation of Question 7.1 in terms of $p$-adic integration analogous to Serre’s Weyl integration proof of his formula; the negative results suggest that any such interpretation may have to be a bit subtle.

**Remark 7.2.** One may also pose Question 7.1 for other finite Coxeter groups. However, Corollary 5.5 implies that if $\Gamma$ is uniform for local fields, then $\Gamma$ has rational character table. This is true for all Weyl groups but not typically for other finite Coxeter groups (like dihedral groups). It may be better to consider only local fields which are algebras over an appropriate cyclotomic field (over which the representations of the group are defined); we have not investigated this possibility in any detail. Something loosely analogous has been observed in the global context of counting number fields, where a counterexample to a conjecture of Malle [11], [12] has been given by Klüners [10], by distinguishing based on the presence or absence of an appropriate cyclotomic subextension.

### 8 The groups $W(B_n)$

We now treat Question 7.1 for the Weyl groups $W(B_n) = W(C_n)$, in a fashion parallel to that of Section 3. Recall that $W(B_n)$ can be identified with the wreath product of $\mathbb{Z}/2\mathbb{Z}$ by $S_n$, or the set of $n \times n$ signed permutation matrices; in particular, there is a surjection $W(B_n) \to S_n$.

**Definition 8.1.** Given a tower of fields $M/L/K$, let $w(M/L/K)$ denote the number of automorphisms of the tower over $K$, that is, preserving but not necessarily fixing $L$.

**Lemma 8.2.** For any field $K$, there is a natural bijection between isomorphism classes of towers $M/L/K$, where $L/K$ is a separable field extension of degree $n$ and $M$ is an étale $L$-algebra of degree 2, and isomorphism classes under conjugation within $W(B_n)$ of continuous homomorphisms $\rho : G_K \to W(B_n)$ which have transitive image in $S_n$.

**Proof.** Given a tower $M/L/K$, let $U$ and $V$ be the sets of minimal idempotents of $L \otimes_K K^{\text{sep}}$ and of $M \otimes_K K^{\text{sep}} = M \otimes_L (L \otimes_K K^{\text{sep}})$, respectively. Then $U$ and $V$ form $G_K$-sets of cardinality $n$ and $2n$, respectively, with $U$ transitive. Moreover, each element of $U$ splits as the sum of two elements of $V$; this defines a partition of $V$. We thus obtain a continuous action of $G_K$ on $V$ factoring through $W(B_n)$, and conversely as in the proof of Lemma 3.1.

**Lemma 8.3.** Under the bijection of Lemma 8.2, $w(M/L/K) = w(\rho)$.

**Proof.** Analogous to Lemma 3.4.

**Lemma 8.4.** Under the bijection of Lemma 8.2, $c(\rho) = f(L/K)c(M/L) + c(L/K)$.
Proof. The étale algebra $M/L$ corresponds to a permutation representation of $G_L$ of degree 2, and hence to a one-dimensional linear representation of $G_L$. The induction of that representation to $G_K$ is precisely the linear representation corresponding to $\rho$. Hence the claim follows from the conductor-discriminant formula (Lemma 3.3).

Theorem 8.5. The Weyl group $\Gamma = W(B_n)$ is uniform for local fields.

Proof. We first compute the contribution to total mass of homomorphisms with transitive image in $S_n$; let $T_{K,\Gamma}$ be the set of such homomorphisms. Switching to isomorphism classes, we can rewrite that contribution as a sum

$$
\sum_{\rho} \frac{1}{w(\rho)q^{c(\rho)}}
$$

over isomorphism classes (up to conjugation within $W(B_n)$) of continuous homomorphisms $\rho : G_K \to W(B_n)$ with transitive image in $S_n$. By Lemmas 8.2, 8.3, and 8.4, this sum in turn equals the sum

$$
\sum_{M/L/K} \frac{1}{w(M/L/K)q^{f(M/L)+c(L/K)}} = \sum_{M/L/K} \frac{1}{w(M/L/K)q^{c(M/L)}q^{c(L/K)}},
$$

where the sum runs over isomorphism classes of towers $M/L/K$ as in Lemma 8.2.

The contribution from towers with $M = L \oplus L$ is

$$
\sum_{L/K} \frac{1}{2w(L/K)q^{c(L/K)}} = \sum_{f|n} \frac{q^{f-n}}{2f}
$$

by (4.0.1). For the other towers, $M$ is a field, so we may sum separately over $M/L$ and $L/K$. Before doing so, we account for automorphisms as in the proof of Theorem 1.1. Namely, $w(M/L/K)$ is equal to twice the number $r$ of automorphisms of $L/K$ which extend to $M$; but when we count separately over $M/L$ and $L/K$, the number of times we count the same tower $M/L/K$ is equal to $w(L/K)/r$, so we need to divide by a factor of $2rw(L/K)/r = 2w(L/K)$.

We now combine the analysis of the previous paragraph, Serre’s formula in degree 2, and (4.0.1) to obtain a mass contribution of

$$
\sum_{f|n} \sum_{L:\, f(L/K)=f} \frac{1}{w(L/K)q^{c(L/K)}} \sum_{M/L} \frac{1}{2q^{c(M/L)}} = \sum_{f|n} \sum_{L:\, f(L/K)=f} \frac{1}{w(L/K)q^{c(L/K)}} \left( \frac{1}{2} + q^{-f} \right)
$$

$$
= \sum_{f|n} \frac{q^{f-n}}{f} \left( \frac{1}{2} + q^{-f} \right)
$$

$$
= \sum_{f|n} \left( \frac{q^{f-n}}{2f} + \frac{q^{-n}}{f} \right).
$$
Putting this together with (8.5.1), we conclude that the mass contribution from $T_{K,W(B_n)}$ is

$$\sum_{f|n} \frac{q^{-f-n} + q^{-n}}{f}.$$  \tag{8.5.2}

In particular, this contribution is a polynomial in $q^{-1}$, as then is the total mass by the Exponential Formula.

**Remark 8.6.** As in Theorem 1.1 we may use the Exponential Formula to compute a generating function for the total mass. We obtain

$$\sum_{n=0}^{\infty} M(K, W(B_n)) x^n = \prod_{i=1}^{\infty} (1 - x^i q^{-i})^{-1} (1 - x^i q^{1-i})^{-1}. \tag{8.6.1}$$

As computed in Theorem 1.1

$$\prod_{i=1}^{\infty} (1 - x^i q^{1-i})^{-1} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{P(n, n - k)}{q^k}.$$  

On the other hand,

$$\prod_{i=1}^{\infty} (1 - x^i q^{-i})^{-1} = \sum_{n=0}^{\infty} x^n q^{-n} P(n),$$

where $P(n)$ denotes the number of partitions of $n$ into any number of parts. Hence the total mass is

$$q^{-n} \sum_{j=0}^{n} \sum_{k=0}^{j} P(j, j - k)P(n - j)q^{j-k} = q^{-n} \sum_{j=0}^{n} \sum_{i=0}^{j} P(j, i)P(n - j)q^{i}.$$  

**Remark 8.7.** It is also possible to check uniformity for $\Gamma = W(D_n)$ in this fashion, by determining which towers in Lemma 8.2 correspond to homomorphisms to $W(B_n)$ whose images lie in $W(D_n)$; this is done in the Appendix (Theorem A.7).

**Remark 8.8.** Wood [19] has generalized Theorem 8.5 to arbitrary iterated wreath products of symmetric groups, where $W(B_n)$ is viewed as the wreath product of $S_2$ by $S_n$. However, at this level of generality, the mass must be computed using a counting function which is apparently not the Artin conductor of a linear representation; instead, it is a more general function of the images of the higher ramification groups.

### 9 The group $W(G_2)$

**Convention 9.1.** Throughout this section, for $H$ a subgroup of $\Gamma$, let $\mu(K, H)$ denote the contribution to the total mass of $(K, \Gamma)$ coming from homomorphisms $\rho : G_K \to \Gamma$ with image equal to $H$. 

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Proposition 9.2. *The Weyl group $\Gamma = W(G_2)$ is uniform for local fields of odd residual characteristic.*

*Proof.* Identify $W(G_2)$ with the dihedral group

$$\mathrm{Di}_6 = \langle s, r | s^6 = r^2 = rsr^{-1} = e \rangle$$

equipped with its natural two-dimensional representation. (We write $\mathrm{Di}_n$ to avoid confusion with the Lie algebra $D_n$.) By Proposition 5.3 it suffices to check that the total mass of $(K, \Gamma)$ equals $1 + 2q^{-1} + 3q^{-2}$ whenever $K$ has residual characteristic 3. We enumerate the subgroups of $\Gamma$ as follows:

$$C_d = \langle s^{6/d} \rangle \quad (d = 1, 2, 3, 6)$$

$$D_{d,i} = \langle rs^i, s^{6/d} \rangle \quad (d = 1, 2, 3, 6; \quad i = 0, \ldots, 6/d - 1).$$

Note that $\mu(K, D_{d,i})$ is independent of $i$. Write $\Gamma = D_{3,0} \times C_2$; given a homomorphism $\rho : G_K \to \Gamma$, let $\sigma : G_K \to D_{3,0}$ and $\tau : G_K \to \{\pm 1\}$ be the homomorphisms induced by the projections. For $H$ a subgroup of $D_{3,0}$, write $\nu(K, H)$ for the contribution to the total mass of $(K, \Gamma)$ coming from those $\rho$ for which the associated $\sigma$ has image $H$.

By applying Theorem 1.1 we compute

$$\mu(K, C_1) = \frac{1}{12}$$

$$\mu(K, C_1) + \mu(K, D_{1,0}) = \frac{1}{6} + \frac{1}{6}q^{-1}$$

$$\mu(K, C_1) + 2\mu(K, D_{1,0}) + \mu(K, C_2) + \mu(K, D_{2,0}) = \frac{1}{3} + \frac{2}{3}q^{-1} + \frac{1}{3}q^{-2}$$

$$\mu(K, C_1) + 3\mu(K, D_{1,0}) + \mu(K, C_3) + \mu(K, D_{3,0}) = \frac{1}{2} + \frac{1}{2}q^{-1} + \frac{1}{2}q^{-2}$$

and hence

$$\mu(K, D_{1,0}) = \frac{1}{12} + \frac{1}{6}q^{-1} \quad (9.2.1)$$

$$\mu(K, C_2) + \mu(K, D_{2,0}) = \frac{1}{12} + \frac{1}{3}q^{-1} + \frac{1}{3}q^{-2} \quad (9.2.2)$$

$$\mu(K, C_3) + \mu(K, D_{3,0}) = \frac{1}{6} + \frac{1}{2}q^{-2} \quad (9.2.3)$$

Remember that we have assumed that $K$ has residual characteristic 3. By direct calculation, we have

$$\nu(K, C_1) = \frac{2}{12} + \frac{2}{12}q^{-2} \quad (9.2.4)$$

$$\nu(K, D_{1,i}) = \frac{2}{12} + \frac{8}{12}q^{-1} + \frac{2}{12}q^{-2} \quad (i = 0, 1, 2). \quad (9.2.5)$$
Since \( \tau \) is at most tamely ramified, we have \( c(\rho) = c(\sigma) \) whenever the linear representation \( \sigma \) contains no tamely ramified subrepresentations. If \( \sigma \) has image \( C_3 \), then it is tamely ramified if and only if it is unramified. Hence

\[

\nu(K, C_3) = \frac{2}{12}(2 + 2q^{-2}) + 4 \left( \mu(K, C_3) - \frac{2}{12} \right) = -\frac{4}{12} + \frac{4}{12}q^{-2} + 4\mu(K, C_3).

\] (9.2.6)

If \( \sigma \) instead has image \( D_{3,0} \), then it cannot be unramified since \( D_{3,0} \) is not cyclic. It also cannot be tamely ramified: otherwise, if \( M \) were the fixed field of \( \ker(\sigma) \), then \( M \) would have a quadratic subextension \( L \) over which it would be tame of degree 3, hence unramified, but then \( M/K \) would have an unramified, hence Galois, subextension of degree 3 over \( K \), and so could not have Galois group \( D_{3,0} \). Thus we have

\[

\nu(K, D_{3,0}) = 4\mu(K, D_{3,0}).

\] (9.2.7)

Combining (9.2.3) through (9.2.7), we obtain

\[

\mu(K, \Gamma) = 1 + 2q^{-1} + 3q^{-2},

\]

verifying the desired result.

**Proposition 9.3.** The Weyl group \( \Gamma = W(G_2) \) is not uniform for local fields of residual characteristic 2.

**Proof.** Retain notation as in Proposition 9.2. In case \( K \) has residual characteristic 2, using (9.2.1) and (9.2.2) we obtain

\[

\nu(K, C_1) = \frac{1}{12} + \mu(K, C_2)
\]

(9.3.1)

\[

\nu(K, D_{1,i}) = \frac{1}{4} + \frac{2}{3}q^{-1} + \frac{1}{3}q^{-2} - \mu(K, C_2) \quad (i = 0, 1, 2).
\]

(9.3.2)

In this setting, if \( \sigma \) has image \( C_3 \), then it is at most tamely ramified. Moreover, if \( \sigma \) has image \( D_{3,0} \), then it is also at most tamely ramified, as otherwise \( D_{3,0} \) would have to have a nontrivial normal subgroup of 2-power order. In both of these cases, when \( \tau \) is unramified, we have \( c(\rho) = c(\sigma) \), and otherwise we have \( c(\rho) = c(\tau) \). Note that there are two tamely ramified homomorphisms into \( C_2 \), and there are eight nontrivial homomorphisms into \( C_3 \) and \( D_{3,0} \): if the residue field of \( K \) contains \( F_4 \), there are eight homomorphisms into \( C_3 \) and none into \( D_{3,0} \), and otherwise there are two homomorphisms into \( C_3 \) and six into \( D_{3,0} \). In either case, we have

\[

\nu(K, C_3) + \nu(K, D_{3,0}) = \frac{1}{3} + q^{-2} + 8 \left( \mu(K, C_2) - \frac{1}{12} \right).
\]

(9.3.3)

Adding up (9.3.1), (9.3.2), (9.3.3) yields

\[

M(K, W(G_2)) = \frac{1}{2} + 2q^{-1} + 2q^{-2} + 6\mu(K, C_2).
\]

We deduce that \( W(G_2) \) is uniform for local fields if and only if \( C_2 \) is; however, we have seen a failure of this in Example 6.2. \( \square \)
Remark 9.4. It may be helpful to see how the calculation of total mass in the \( G_2 \) case works over \( K = \mathbb{Q}_2 \), by going through the contributions from different Galois extensions. (All assertions below may be confirmed either by direct verification or by consulting the Database of Local Fields [9].) The trivial extension contributes a mass of 

\[
\frac{1}{12}.
\]

The quadratic extensions were enumerated in Example 6.2; each quadratic extension contributes one homomorphism with image in \( C_2 \) and six with images in the various \( D_{0,i} \), for a mass contribution of

\[
\frac{1}{12} (7 + 12 \cdot 2^{-2} + 24 \cdot 2^{-3} + 2 \cdot 2^{-4} + 4 \cdot 2^{-6}).
\]

Each \( \mathbb{Z}/3\mathbb{Z} \)-extension contributes two homomorphisms; the only such extension is unramified, for a total mass of

\[
\frac{2}{12}.
\]

Each \( \mathbb{Z}/6\mathbb{Z} \)-extension contributes two homomorphisms; there are seven of these, given by the composita of the unramified \( \mathbb{Z}/3\mathbb{Z} \)-extension with each of the seven quadratic extensions. The conductors coincide with the squares of the conductors of the quadratic extensions, yielding a mass contribution of

\[
\frac{1}{12} (2 + 4 \cdot 2^{-4} + 8 \cdot 2^{-6}).
\]

Any homomorphism with image \((\mathbb{Z}/2\mathbb{Z})^2\) can be formed from two homomorphisms with image \( \mathbb{Z}/2\mathbb{Z} \), and the conductor is the sum of the conductors of those extensions; each unordered pair gives six homomorphisms. This yields a mass contribution of

\[
\frac{1}{12} (12 \cdot 2^{-2} + 24 \cdot 2^{-3} + 6 \cdot 2^{-4} + 48 \cdot 2^{-5} + 36 \cdot 2^{-6}).
\]

Each \( S_3 \)-extension contributes six homomorphisms; the only such extension is the Galois closure of the cubic extension \( \mathbb{Q}_2[z]/(z^3 - 2) \) of discriminant exponent 2, yielding a mass contribution of

\[
\frac{1}{12} (6 \cdot 2^{-2}).
\]

Each \( \text{Di}_6 = (S_3 \times C_2) \)-extension contributes six homomorphisms; there are seven of these, given by the composita of the \( S_3 \)-extension with the seven quadratic extensions. The unramified quadratic contributes \( \frac{1}{12} (6 \cdot 2^{-2}) \); the other quadratics dominate the conductor, yielding a contribution from the \( \text{Di}_6 \)-extensions of

\[
\frac{1}{12} (6 \cdot 2^{-2} + 12 \cdot 2^{-4} + 24 \cdot 2^{-6}).
\]

Adding it up yields

\[
M(K, W(G_2)) = \frac{1}{12} (12 + 36 \cdot 2^{-2} + 48 \cdot 2^{-3} + 24 \cdot 2^{-4} + 48 \cdot 2^{-5} + 72 \cdot 2^{-6}) = \frac{83}{32}.
\]
The calculation of Proposition 9.3 predicts a total mass of
\[ \frac{1}{2} + 2 \cdot 2^{-1} + 2 \cdot 2^{-2} + 6\mu(K, C_2), \]
and from Example 6.2 (after taking out the contribution from the trivial homomorphism), we have
\[ 6\mu(K, C_2) = \frac{1}{2} + 2^{-4} + 2^{-6}. \]
So the predicted total mass is also 83/32, agreeing with the direct calculation.

10 Final remarks

One may interpret what we have been doing as counting local Galois representations which are Hodge-Tate with all weights equal to 0. It would also be natural to try to enumerate \( p \)-adic Galois representations with other Hodge-Tate weights, e.g., two-dimensional de Rham representations with Hodge-Tate weights 0 and 1. These may lead to heuristics for counting global Galois representations with particular geometric origins, e.g., those arising from the étale cohomology of elliptic curves.

On a more algebro-geometric note, it might make sense to think about counting representations of \( G_K \) into a group \( \Gamma \) as counting \( K \)-valued points of \( B\Gamma \), the classifying stack of \( \Gamma \)-torsors. This gives a natural interpretation of the automorphism contribution to total mass; it is entirely possible that the conductor contribution also has a natural, possibly Arakelov-theoretic interpretation.

A Appendix (by Daniel Gulotta)

In this appendix, we study Question 7.1 for the Weyl group \( W(D_n) \).

Let \( \sigma_n : W(B_n) \to C_2 \) be the map whose kernel is \( W(D_n) \). Then \( \rho : \text{Gal}(K_{\text{sep}}/K) \to W(B_n) \) should be counted in the total mass of \( W(D_n) \) iff \( \sigma_n \circ \rho \) is the trivial homomorphism.

**Lemma A.1.** Let \( L/K \) be a finite algebraic extension of fields. Let \( A \) and \( A' \) be bases for \( L/K \). Let \( f \) be a linear transformation that takes \( A \) to \( A' \). Then \( \Delta(A') = (\det f)^2 \Delta(A) \).

**Proof.** See [20, Proposition 12.1.2].

Since the ratio of the discriminants of any two bases of \( L/K \) is a square, we can make the following definition.

**Definition A.2.** Let \( L/K \) be a finite algebraic extension of fields. If \( \Delta \) is the discriminant of some basis of \( L/K \), then the discriminant root field of \( L/K \) is \( K(\sqrt{\Delta}) \). (The terminology is from [9].)
Lemma A.3. Let $M/L/K$ be a tower of finite separable algebraic field extensions. Let $A$ be a basis for $L/K$, and let $B$ be a basis for $M/L$. Let $C = \{ab | a \in A, b \in B\}$. If $C$ is considered as a basis for $M/K$, then $\Delta(C) = \Delta(A)^{[M:L]} \text{Norm}_{L/K} \Delta(B)$.

Proof. See [21, Theorem 39].

Lemma A.4. Let $M/L/K$ be a tower of fields corresponding to a map $\rho : \text{Gal}(K^{\text{sep}}/K) \to W(B_n)$. Assume that the characteristic of $K$ is not 2. Then the fixed field of the kernel of $\sigma_n \circ \rho$ is the discriminant root field of $M/K$.

Proof. Apply Lemma A.3 Since $[M : L] = 2$, $\Delta(C)$ is in the same class as $K^*/(K^*)^2$ as $\text{Norm}_{L/K} \Delta(B)$. For each conjugate of $\Delta(B)$, choose one of its square roots. The product of all of these is a square root of $\text{Norm}_{L/K} \Delta(B)$. Automorphisms in $W(D_n)$ will change an even number of signs and will therefore preserve the square roots of $\text{Norm}_{L/K} \Delta(B)$, while automorphisms in $W(B_n) \setminus W(D_n)$ will flip them. Hence an element of $\text{Gal}(K^{\text{sep}}/K)$ fixes the discriminant root field of $M/K$ if and only if it is in the kernel of $\sigma_n \circ \rho$.

In the following three results, assume that $q$ is odd, let $\alpha$ be a primitive $(q - 1)$-st root of unity, and let $\pi$ be a uniformizing element of $K$.

Lemma A.5. If $M/L$ is unramified and $e(L/K)$ is odd, then the discriminant root field of $M/K$ is $K(\sqrt{\alpha})$. If $M/L$ is unramified and $e(L/K)$ is even, then the discriminant root field of $M/K$ is $K$.

Proof. From Lemma A.3, one can see that the discriminant root field lies in an unramified extension of $K$, and therefore must be in $K(\sqrt{\alpha})$. The Frobenius automorphism acts on the conjugates of a primitive $(q^{2f} - 1)$-st root of unity as a $2f$-cycle. The total number of $2f$-cycles is $e$ and each one contributes an odd number of minus signs, so the Frobenius homomorphism is in $W(D_n)$ if and only if $e$ is even. If $n$ is odd, then the discriminant root field cannot be $K$, so it must be $K(\sqrt{\alpha})$. If $n$ is even, then the discriminant root field cannot be $K(\sqrt{\alpha})$ since the Frobenius automorphism does not fix this field. Therefore it must be $K$.

Lemma A.6. If $M/L$ is ramified and $f(L/K)$ is odd, then the discriminant root field of $M/K$ is $K(\sqrt{\pi})$ or $K(\sqrt{\alpha \pi})$. If $M/L$ is ramified and $f(L/K)$ is even, then the discriminant root field of $M/K$ is $K$ or $K(\sqrt{\alpha})$. In either case, the mass is divided equally between the two possibilities.

Proof. Since $M/L$ is ramified of degree two, the discriminant of $M/L$ has odd valuation in $L$. Therefore the norm of this element has odd valuation in $K$ if and only if $f$ is even.

Let $\beta$ be a primitive $(q^f - 1)$-st root of unity. Then for any uniformizing element $\pi_L$ of $L$, Lemma A.3 shows that the discriminant of $L(\sqrt{\beta \pi_L})/K$ with respect to the basis $(1, \sqrt{\beta \pi_L})$ is equal to a primitive $(q - 1)$-st root of unity times the discriminant of $L(\sqrt{\pi_L})/K$ with respect to basis $(1, \sqrt{\pi_L})$. Thus the two extensions have different discriminant root fields. They have equal amounts of mass.

Theorem A.7. The group $W(D_n)$ is uniform for local fields of odd residual characteristic.
Proof. Let $G$ be the group of continuous homomorphisms $\text{Gal}(K^{\text{sep}}/K) \to C_2$; this group is abelian. For any $\chi \in \hat{G}$, let

$$f(\chi) = \sum_n \frac{x^n}{2^n n!} \sum_{\rho} q^{-c(\rho)} \chi(\sigma_n \circ \rho)$$

(A.7.1)

$$= \exp \left[ \sum_n \frac{x^n}{2^n n!} \sum_{\rho \text{ transitive}} q^{-c(\rho)} \chi(\sigma_n \circ \rho) \right],$$

where in both cases $\rho$ runs over continuous homomorphisms from $\text{Gal}(K^{\text{sep}}/K)$ to $W(B_n)$.

Then the generating function for $M(K, W(D_n))$ is

$$\sum_{n=0}^{\infty} M(K, W(D_n)) x^n = \frac{4}{|G|} \sum_{\chi \in \hat{G}} f(\chi).$$

For $q$ odd, $G$ is isomorphic to $V_4$. Define $a$ by

$$\log(a) = \sum_{n=1}^{\infty} \sum_{M/L/K, [L:K]=n} \frac{x^n}{w(M/L/K) q^c(M/L/K)} ,$$

where the sum runs over towers where $M$ splits or is a field with discriminant root field $K$. Let $\log(b)$ sum over towers with discriminant root field $K(\sqrt{\alpha})$, and let $\log(c)$ sum over towers with discriminant root field $K(\sqrt{\pi})$; note that $\log(c)$ also equals the sum over towers with discriminant root field $K(\sqrt{\alpha\pi})$. Then

$$\sum_{n=0}^{\infty} M(K, W(D_n)) x^n = \frac{1}{4} \left( abc^2 + \frac{ab}{c^2} + 2 \frac{a}{b} \right).$$

We now imitate the proof of Theorem 8.5 using Lemmas A.5 and A.6 to sort terms. We get

$$\log(a) = \sum_{n=1}^{\infty} x^n \sum_{f|n} \frac{q^{f-n}}{2f} + \sum_{n=1}^{\infty} x^{2n} \sum_{f|n} \frac{q^{f-2n}}{2f} + \sum_{n=1}^{\infty} x^{2n} \sum_{f|n} \frac{q^{-2n}}{2f}$$

by adding the sum over $M$ split, the sum over $M/L$ unramified and $e$ even, and half the sum over $M/L$ ramified and $f$ even. We get

$$2 \log(c) = \sum_{n=1}^{\infty} x^n \sum_{f|n} \frac{q^{f-n}}{f} - \sum_{n=1}^{\infty} x^{2n} \sum_{f|n} \frac{q^{-2n}}{2f}$$

by adding the sum over $M/L$ ramified, then subtracting the sum over $M/L$ ramified and $f$. 

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even. Exponentiating, adding in (8.6.1), and solving, we get

\[
a = \prod_{n=1}^{\infty} (1 - q^{1-n}x^n)^{-1/2} (1 - q^{1-2n}x^{2n})^{-1/2} (1 - q^{-2n}x^{2n})^{-1/4} \tag{A.7.2}
\]

\[
b = \prod_{n=1}^{\infty} (1 - q^{2-2n}x^{2n-1})^{-1/2} (1 - q^{-2n}x^{2n})^{-1/4} \tag{A.7.3}
\]

\[
c = \prod_{n=1}^{\infty} (1 - q^{-2n}x^{2n})^{1/4} (1 - q^{-n}x^n)^{-1/2}. \tag{A.7.4}
\]

Therefore

\[
\sum_{n=0}^{\infty} M(K, W(D_n))x^n = \frac{1}{4} \left[ \prod_{n=1}^{\infty} (1 - q^{1-n}x^n)^{-1} \right] \left[ \prod_{n=1}^{\infty} (1 - q^{-n}x^n)^{-1} \right] \left[ \prod_{n=1}^{\infty} (1 + q^{-n}x^n)^{-1} \right] + \frac{1}{2} \prod_{n=1}^{\infty} (1 - q^{1-2n}x^{2n})^{-1}, \tag{A.7.5}
\]

proving the desired uniformity.

\[ \square \]

**Proposition A.8.** The group \( W(D_4) \) is not uniform for all local fields.

**Proof.** Put \( K = \mathbb{Q}_2 \) and

\[
\log(a) = \sum_{n=1}^{\infty} \sum_{M/L/K, [L:K]=n} \frac{x^n}{w(M/L/K)q^{-c(M/L/K)}}, \tag{A.8.1}
\]

where the sum runs over towers where \( M \) splits or the discriminant root field of \( M/K \) is \( \mathbb{Q}_2 \). Let \( \log(b), \log(c), \log(d) \) sum over towers with discriminant root field \( \mathbb{Q}_2(\sqrt{-3}), \mathbb{Q}_2(\sqrt{-1}), \mathbb{Q}_2(\sqrt{2}) \), respectively. Then

\[
\sum_{n=0}^{\infty} M(\mathbb{Q}_2, W(D_n))x^n = \frac{1}{8} \left( abc^2d^4 + \frac{abc^2}{d^4} + 2 \frac{ab}{c^2} + 4 \frac{a}{b} \right).
\]

We claim that

\[
a = \left[ \prod_{n=1}^{\infty} (1 - 2^{1-n}x^n)^{-1/2} \right] \exp \left[ \frac{45}{128}x^2 + \frac{11}{256}x^3 + \frac{691}{4096}x^4 + O(x^5) \right] \tag{A.8.2}
\]

\[
b = \exp \left[ \frac{1}{2}x + \frac{45}{128}x^2 + \frac{257}{768}x^3 + O(x^4) \right] \tag{A.8.3}
\]

\[
c = \exp \left[ \frac{1}{8}x + \frac{7}{128}x^2 + \frac{23}{768}x^3 + O(x^4) \right] \tag{A.8.4}
\]

\[
d = \exp \left[ \frac{1}{16}x + \frac{1}{64}x^2 + \frac{1}{192}x^3 + O(x^4) \right] \tag{A.8.5}
\]
This precision suffices to imply \( M(\mathbb{Q}_2, W(D_4)) = \frac{1611}{1024} \). This in turn proves the desired result: if \( W(D_4) \) were uniform for local fields, we would have to have \( M(\mathbb{Q}_2, W(D_4)) = \frac{51}{32} = \frac{1632}{1024} \) in order to agree with the coefficient of \( x^4 \) in (A.7.5) for \( q = 2 \).

It remains to explain how \( a, b, c, d \) were computed. The only calculation which is nontrivial to verify by hand is the coefficient \( \frac{691}{4096} \) in (A.8.2), i.e., the contribution to (A.8.1) from terms with \( n = 4 \) and \( M \) nonsplit. We do this by feeding data from [9] into two different programs [7]. One (called \text{lf}) is a combination of C++ and Perl scripts, reading the data from the HTML served by [9]; the other (called \text{gap-check}) uses GAP [6] within SAGE [17] to read in a raw data file available from [9], then uses SAGE to tabulate the results.

Here are some details about the verification that the sum of the terms of (A.8.1) with \( n = 4 \) and \( M \) nonsplit is \( \frac{691}{4096} \). In [9], we find a table of the 1823 isomorphism classes of fields \( M \) of degree 8 over \( K = \mathbb{Q}_2 \); each entry includes (among other information) the discriminant root field of \( M \), the order of \( \text{Aut}(M/K) \), the discriminant exponent of \( M/K \), the Galois group of the normal closure of \( M/K \) as a permutation group, and a list of the isomorphism classes of degree 4 fields which occur among the subfields of \( M \). Data about these degree 4 fields can be looked up in [9] in an analogous table.

For each \( M \) in the table with discriminant root field \( K \), we loop over the isomorphism classes of degree 4 subfields. We check the Galois groups of the two fields to identify one exceptional case (see below). In all other cases, there is a unique isomorphism class of towers \( M/L/K \) with \( M \) as chosen above and \( L \) in the chosen isomorphism class, and \( \# \text{Aut}(M/K) = \# \text{Aut}(M/L/K) \). We compute \( c(M/L/K) = c(M/K) - c(L/K) \) by the conductor-discriminant formula (Lemma 3.3), and obtain one of the desired terms of (A.8.1). This count is performed by the program \text{lf} of [7].

To determine which Galois groups yields exceptions to the above argument, we use GAP as follows (see the program \text{gap-check} in [7]). For each \( G \) in a set of representatives under conjugacy for the transitive subgroups of \( S_8 \) (a precomputed list in GAP), we loop over representatives \( H \) of subgroups of \( G \) up to conjugacy. For each \( H \) and each orbit of \( H \) of length 2, we pick an element \( t \) of the orbit, then compute the number of conjugates \( g^{-1}Hg \) of \( H \) which contain \( \text{Stab}_G(t) \), first for \( g \) running over \( G \), then over \( \text{Norm}_G(H) \), then over \( \text{Norm}_G(\text{Stab}_G(t)) \). Call these numbers \( c_1, c_2, c_3 \). If \( c_1 = c_3 \), then there is a unique isomorphism class of towers \( M/L/K \) with \( M \) the fixed field of \( G \) and \( L \) the fixed field of some conjugate of \( H \). If \( c_1 = c_2 \), then for any such tower, \( \# \text{Aut}(M/K) = \# \text{Aut}(M/L/K) \).

The only exceptional cases are found in the case where \( G = \text{Di}_4 \) and \( H \) equal to a non-normal subgroup of order 2. (Again, we write \( \text{Di} \) for dihedral groups to avoid confusion with the \( D \) series of Lie algebras.) In this case, \( c_1 = c_3 \neq c_2 \), and in fact \( \# \text{Aut}(M/K) = 8 \) and \( \# \text{Aut}(M/L/K) = 4 \). We simply replace the automorphism contribution by 4 to obtain the desired term of (A.8.1).

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