Conditional limit theorems for critical continuous-state branching processes

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Abstract We study the conditional limit theorems for critical continuous-state branching processes with branching mechanism \( \psi(\lambda) = \lambda^{1+\alpha}L(1/\lambda) \), where \( \alpha \in [0, 1] \) and \( L \) is slowly varying at \( \infty \). We prove that if \( \alpha \in (0, 1] \), there are norming constants \( Q_t \to 0 \) (as \( t \to +\infty \)) such that for every \( x > 0 \), \( P_x(Q_tX_t \in \cdot | X_t > 0) \) converges weakly to a non-degenerate limit. The converse assertion is also true provided the regularity of \( \psi \) at 0. We give a conditional limit theorem for the case \( \alpha = 0 \). The limit theorems we obtain in this paper allow infinite variance of the branching process.

Keywords continuous-state branching process, conditional laws, regular variation

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1 Introduction

A \([0, +\infty)\)-valued strong Markov process \( X = \{X_t : t \geq 0\} \) with probabilities \( \{P_x : x > 0\} \) is called a continuous-state branching process (CB process) if it has paths that are right continuous with left limits, \( P_x(X_0 = x) = 1 \) for every \( x > 0 \), and it employs the following branching property: For any \( \lambda \geq 0 \) and \( x, y > 0 \),

\[
E_{x+y}(e^{-\lambda X_t}) = E_x(e^{-\lambda X_t})E_y(e^{-\lambda X_t}). \tag{1.1}
\]

It can be characterized by the branching mechanism \( \psi \) which is also the Laplace exponent of a Lévy process with non-negative jumps. Hereafter, we use “:=” as a way of definition. Set \( \rho := \psi'(0+) \), then \( E_xX_t = xe^{-\rho t} \). We call a CB process supercritical, critical or subcritical as \( \rho < 0, \rho = 0, \) or \( \rho > 0 \).

Let \( \tau := \inf\{t \geq 0 : X_t = 0\} \) denote the extinction time of \( X_t \) and \( q(x) := P_x(\tau < +\infty) \) denote the extinction probability. When \( q(x) < 1 \) for some (and then for all) \( x > 0 \), the asymptotic behavior of \( X_t \) was studied in [3]. It was proved that there are positive constants \( \eta_t \) such that \( \eta_tX_t \) converges almost surely to a non-degenerate random variable as \( t \) goes to infinity. Note that \( q(x) \equiv 1 \) if and only if \( X \) is subcritical or critical with \( \psi \) satisfying

\[
\int_{\theta}^{+\infty} \frac{1}{\psi(\xi)} d\xi < +\infty,
\]

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for some $\theta > 0$. In this case, one can study the asymptotic behavior of $X$ by conditioning on $\{\tau > t\}$ (see [6, 7, 10, 11] and the references therein). In the subcritical case, it was proved that $P_x(X_t \in \cdot \mid \tau > t)$ converges weakly to the so-called Yaglom distribution as $t$ goes to infinity. However in the critical case, the limiting distribution of $X_t$ conditioned on non-extinction is trivial, converging to the Dirac measure at $\infty$. To evaluate the asymptotic behavior of $X_t$ more accurately, we therefore have to normalize the process appropriately.

Throughout this paper, we assume $\psi$ satisfies

$$\psi(\lambda) = \lambda^{1+\alpha}L(1/\lambda), \quad \forall \lambda \geq 0,$$

(1.2)

where $\alpha \in [0, 1]$ and $L$ is slowly varying at infinity. Our assumption on $\psi$ does not require the finiteness of $E_x X_t^2$. It is well known that a CB process can be viewed as the analogue of Galton-Watson branching process in continuous time and continuous state space. Before anything starts, let us first take a look at the asymptotic behavior of critical Galton-Watson branching processes. Let $f(s)$ denote the probability generating function of the offspring law of the critical Galton-Watson process $Z_n$. Slack [15, 16] proved that $P_1(\bar{F}(n)Z_n \leq y \mid Z_n > 0)$ converges weakly to a non-degenerate limit if and only if

$$f(s) = s + (1-s)^{1+\alpha}L\left(\frac{1}{1-s}\right),$$

(1.3)

for some $\alpha \in (0, 1]$ and $L$ slowly varying at $+\infty$. Later Nagaev et al. [8] obtained a conditional limit theorem for $f(s)$ satisfying (1.3) with $\alpha = 0$. Recently, Pakes [9] generalized the above results to continuous time Markov branching process. The proofs given in [9], based on Karamata’s theory for regular varying functions, are more concise and intuitive. But even for the discrete-state branching processes, there leaves open the question of whether or not (1.3) is implied by the more general conditional convergence of $P_1(b_nZ_n \leq y \mid Z_n > 0)$ for some positive sequence $\{b_n\}$ with $b_n$ goes to 0.

This paper is structured as follows: In Section 2, we collect some basic facts about regularly varying functions and CB processes. Section 3 is devoted to the conditional limit theorems for $\psi$ with $\alpha \in (0, 1]$. We prove that there exist positive norming constants $Q_t \to 0$ such that $P_x(Q_t X_t \in \cdot \mid \tau > t)$ converges weakly to a non-degenerate limit. An admissible norming is $Q_t = P_1(\tau > t)$. This is analogous to the result we mentioned in the above paragraph for discrete-state branching processes. Later, we prove that the converse assertion is also true provided some regularity of $\psi$ at 0. In Section 4, we give a conditional limit theorem for the case $\alpha = 0$. Its discrete state analogue is proved independently in [8, 9]. In Section 5, we provide some concrete examples and deduce their conditional limit theorems. Some of the branching mechanisms in these examples are taken from [12].

## 2 Preliminaries

In the rest of this paper, we shall use the notation $f(x) \sim g(x)$ for functions $f$ and $g$ to mean that $f(x)/g(x) \to 1$ as $x \to +\infty$ or $x \to 0$. Let $x \wedge y := \min\{x, y\}$.

By (1.1) we may always write for every $x > 0$, $\lambda \geq 0$ and every positive integer $n$,

$$E_x(e^{-\lambda X_t}) = E_{x/n}(e^{-\lambda X_t})^n.$$

(2.1)

If we define for $\lambda, t \geq 0$,

$$g(t, \lambda, x) := -\log E_x(e^{-\lambda X_t}), \quad u_t(\lambda) := g(t, \lambda, 1) = -\log E_1(e^{-\lambda X_t}),$$

then (2.1) implies that for any positive integers $m$ and $n$,

$$g\left(t, \lambda, \frac{m}{n}\right) = \frac{m}{n} u_t(\lambda).$$

(2.2)
A standard argument using (2.2) and the monotonicity of \(g(t, \lambda, x)\) in \(x\) shows that for all \(x > 0\),
\[
g(t, \lambda, x) = xu_t(\lambda).
\]
In other words, we have,
\[
E_x(e^{-\lambda X_t}) = e^{-xu_t(\lambda)}, \quad \forall x > 0.
\]  
(2.3)

It follows from Markov property and (2.3) that for all \(x > 0\) and \(t, s, \lambda \geq 0\),
\[
e^{-xu_{t+s}(\lambda)} = E_x[E(e^{-\lambda X_{t+s}} \mid X_t)] = E_x(e^{-u_s(\lambda)X_t}) = e^{-xu_s(u_s(\lambda))}.
\]
This means \(u_t(\lambda)\) obeys the semigroup property
\[
u_t+s(\lambda) = u_t(u_s(\lambda)).
\]  
(2.4)

Silverstein [14] proved that if \(u_t(\lambda)\) is the Laplace functional given by (2.3) of some CB process \(X_t\), then it is differentiable in \(t\) and satisfies the backward equation
\[
\frac{\partial}{\partial t}u_t(\lambda) = -\psi(u_t(\lambda)), \quad u_0(\lambda) = \lambda,
\]  
(2.5)

where for \(\lambda \geq 0\),
\[
\psi(\lambda) = -q - a\lambda + \frac{1}{2}a^2\lambda^2 + \int_{(0, +\infty)}(e^{-\lambda x} - 1 + \lambda x 1_{(x < 1)})\Lambda(dx),
\]  
(2.6)

\(q \geq 0, \ a \in \mathbb{R}, \ \sigma \geq 0, \text{ and } \Lambda\) is a non-negative measure on \((0, +\infty)\) satisfying \(\int_{(0, +\infty)}(1 \wedge x^2)\Lambda(dx) < +\infty\).

It is easy to deduce with the help of dominated convergence theorem that \(\psi\) satisfying (2.6) is convex and infinitely differentiable on \((0, +\infty)\). We refer to \(\psi\) as the branching mechanism of \(X_t\). \(\Lambda\) is called the Lévy measure of \(\psi\). By (2.4) and (2.5) we can also get the forward equation
\[
\frac{\partial}{\partial t}u_t(\lambda) = -\psi(\lambda)\frac{\partial}{\partial \lambda}u_t(\lambda), \quad u_0(\lambda) = \lambda.
\]  
(2.7)

A CB process \(X\) is said to be conservative if for all \(x, t > 0\), \(P_x(X_t \leq +\infty) = 1\). As is shown in [3], \(q = 0\) is a necessary condition for \(X\) to be conservative.

In the present work, we restrict our attention to the class of conservative CB processes for which the branching mechanism \(\psi\) has the property
\[
\rho := \psi'(0+) = -a - \int_{[1, +\infty)} x\Lambda(dx) = 0.
\]
In other words, we assume \(\psi\) has the representation
\[
\psi(\lambda) = b\lambda^2 + \int_{(0, +\infty)}(e^{-\lambda x} - 1 + \lambda x)\Lambda(dx) \quad \lambda \geq 0,
\]  
(2.8)

where \(\Lambda\) is a non-negative measure on \((0, +\infty)\) satisfying \(\int_{(0, +\infty)}(x \wedge x^2)\Lambda(dx) < +\infty\) and \(b \geq 0\). As is shown in [3], the CB process associated to (2.8) is conservative. It is also critical in the sense that \(E_xX_t = x\) for all \(x > 0\) and \(t \geq 0\). We impose another assumption on \(\psi\),
\[
\int_{+\infty}^{+\infty} \frac{1}{\psi(\xi)} d\xi < +\infty,
\]  
(2.9)

for some \(\theta > 0\). (2.9) is necessary and sufficient for the related critical CB process to be extinct in a finite time, i.e., \(P_x(\tau < +\infty) = 1\) for all \(x > 0\). Also note that (2.9) implies that \(b > 0\) or \(\Lambda\) is non-trivial. It is known that under (2.8) and (2.9), \(\psi\) is the Laplace exponent of a spectrally positive Lévy process with \(\psi(+\infty) = +\infty\). Moreover, \(\psi\) is a strictly increasing function on \([0, +\infty)\) and \(\psi(\lambda) = 0\) only if \(\lambda = 0\). We refer to [5, Subsection 10.2] for proofs of these statements. Hereafter, we assume that \(\psi\) satisfies (2.8) and (2.9), and that \(X\) is a CB process with branching mechanism \(\psi\).
By [3], \( X \) being conservative implies \( \int_{0+} 1/\psi(z) d\xi = +\infty \). With this and (2.9), we can define
\[
\phi(z) := \int_{z}^{+\infty} \frac{1}{\psi(\xi)} d\xi, \quad \text{for all } z > 0.
\]
Obviously, \( \phi(\theta) \) is strictly decreasing on \([0, +\infty)\) with \( \phi(0) = +\infty \) and \( \phi(+\infty) = 0 \). We use \( \varphi \) to denote the inverse function of \( \phi \). From (2.5), we have
\[
\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(\xi)} d\xi = t, \quad \forall \lambda, t > 0.
\]
Hence,
\[
u_t(\lambda) = \varphi(t + \phi(\lambda)), \quad \forall \lambda, t > 0.
\]
(2.10)
Since \( \phi(+\infty) = 0 \), we have \( u_t(+\infty) = \varphi(t) \) by (2.10), and for every \( x > 0 \) and \( t \geq 0 \),
\[
P_x(\tau > t) = P_x(X_t > 0) = 1 - \lim_{\lambda \to +\infty} e^{-xu_t(\lambda)} = 1 - e^{-x\varphi(t)}.
\]
Throughout this paper, we define
\[
\bar{F}(t) := P_1(\tau > t), \quad \forall t > 0.
\]
(2.12)
Obviously by (2.11), we have
\[
\bar{F}(t) \sim \varphi(t) \quad \text{as } t \uparrow +\infty.
\]
(2.13)
Results about regular varying functions will be used many times in the rest of this paper, so we collect some basic definitions and facts here. For an overall study of the theory of regular variation, see [1]. A positive measurable function \( L \) is said to be slowly varying at \( \infty \) if it is defined on \((0, +\infty)\) and
defined on \((0, +\infty)\) satisfies that
\[
\lim_{x \to +\infty} \frac{L(\lambda x)}{L(x)} = 1 \quad \text{for all } \lambda > 0.
\]
This convergence holds uniformly with respect to \( \lambda \) on every compact subset of \((0, +\infty)\). Let \( S \) denote the set of all slowly varying functions at \( \infty \). An elementary property of slowly varying functions is that if \( L \in S \), then for any \( \delta > 0 \), \( \lim_{x \to +\infty} x^\delta L(x) = +\infty \), and \( \lim_{x \to +\infty} x^{-\delta} L(x) = 0 \). If a positive function \( f \) defined on \((0, +\infty)\) satisfies that \( f(\lambda x)/f(x) \to \lambda^p \) as \( x \to +\infty \) (resp. \( 0 \)) for any \( \lambda > 0 \), then \( f \) is called regularly varying at \( \infty \) (resp. \( 0 \)) with index \( p \in (-\infty, +\infty) \), denoted by \( f \in \mathcal{R}_p(\infty) \) (resp. \( f \in \mathcal{R}_p(0) \)). Obviously, \( f(x) \in \mathcal{R}_p(0) \) is equivalent to \( f(1/x) \in \mathcal{R}_{-p}(\infty) \). If \( f \in \mathcal{R}_p(\infty) \) (resp. \( f \in \mathcal{R}_p(0) \)), it can be represented by \( f(x) = x^p L(x) \) (resp. \( f(x) = x^{-p} L(1/x) \)) for some \( L \in S \).

3 The case \( 0 < \alpha \leq 1 \)

The following technical lemma follows from [1, Theorems 1.5.2 and 1.5.12]. We omit its proof here.

**Lemma 3.1.**

1. If \( p \in (-\infty, +\infty) \), \( f \in \mathcal{R}_p(\infty) \) (resp. \( \mathcal{R}_p(0) \)), \( T_1(t), T_2(t) \to +\infty \) (resp. \( 0 \)) and \( T_1(t) \sim T_2(t) \) as \( t \uparrow +\infty \), then \( f(T_1(t)) \sim f(T_2(t)) \).

2. Suppose \( f \in \mathcal{R}_p(\infty) \), \( T_1(t), T_2(t) \to +\infty \) as \( t \to +\infty \), and \( f(T_1(t))/f(T_2(t)) \sim c \in (0, +\infty) \). If \( p > 0 \), then \( T_1(t)/T_2(t) \sim c^{1/p} \); otherwise if \( p < 0 \) and \( f \) has inverse function \( f^{-1} \), then \( f^{-1} \in \mathcal{R}_1/p(0) \) and \( T_1(t)/T_2(t) \sim c^{1/p} \).

**Theorem 3.2.**

If \( \psi \) satisfies (1.2) with \( 0 < \alpha \leq 1 \), then for all \( x > 0 \) and \( y \geq 0 \),
\[
\lim_{t \to +\infty} P_x(\bar{F}(t)X_t \leq y | \tau > t) = H_\alpha(y),
\]
(3.1)
where \( \bar{F}(t) \) is defined by (2.12), and \( H_\alpha(y) \) is a probability distribution function with Laplace transform given by
\[
h_\alpha(\theta) = \int_{(0, +\infty)} e^{-\theta y} dH_\alpha(y) = 1 - (1 + \theta^{-\alpha})^{-1/\alpha}.
\]
(3.2)
Moreover, \( \bar{F}(t) \) is regularly varying at \( +\infty \) with index \(-1/\alpha \), and consequently, for any \( \delta > 0 \),
\[
\lim_{t \to +\infty} t^{\frac{1}{\alpha} + \delta} \bar{F}(t) = +\infty, \quad \lim_{t \to +\infty} t^{\frac{1}{\alpha} - \delta} \bar{F}(t) = 0.
\]
Proof. For any $z > 0$, set $g(z) := \phi(1/z) = \int_0^z c^{-1} / L(\xi) \, d\xi$. Then by Karamata’s theorem (see [1, Theorem 1.5.11]), we have $g \in \mathcal{R}_\alpha(\infty)$, more specifically, $g(z) \sim \alpha^{-1} z^{-\alpha} L(1/z)^{-1}$ as $z \to +\infty$. Consequently, we get $\phi \in \mathcal{R}_{-\alpha}(0)$, $\phi(z) \sim \alpha^{-1} z^{-\alpha} L(1/z)^{-1}$ as $z \downarrow 0$, and $\varphi \in \mathcal{R}_{-1/\alpha}(\infty)$.

Since $1 - e^{-u} \sim u$ as $u \downarrow 0$, we have for all $x, \theta > 0$,

$$
\lim_{t \to +\infty} E_x(e^{-\theta F(t)} | \tau > t) = 1 - \lim_{t \to +\infty} \frac{1 - \exp\{-x\varphi(t + \theta \bar{F}(t))\}}{1 - \exp\{-x\varphi(t)\}} = 1 - \lim_{t \to +\infty} \frac{\varphi(t + \theta \bar{F}(t))}{\varphi(t)}.
$$

(3.3)

It follows from Lemma 3.1 and (2.13) that

$$
\phi(\theta \bar{F}(t)) \sim \phi(\theta \varphi(t)) \sim \theta^{-\alpha} \phi(\varphi(t)) = \theta^{-\alpha} t, \quad \text{as } t \uparrow +\infty.
$$

Hence, we have $\varphi(t + \phi(\theta \bar{F}(t))) \sim \varphi((1 + \theta^{-\alpha}) t)$. By (3.3) and the regularity of $\varphi$ at $\infty$, we get

$$
\lim_{t \to +\infty} E_x(e^{-\theta F(t)} X_t | \tau > t) = 1 - \lim_{t \to +\infty} \frac{\varphi((1 + \theta^{-\alpha}) t)}{\varphi(t)} = 1 - (1 + \theta^{-\alpha})^{-1/\alpha} = h_\alpha(\theta).
$$

(3.4)

Let $H_\alpha(y)$ be the distribution function uniquely determined by $h_\alpha(\theta)$. Theorem 3.2 follows from the continuity theory for Laplace transforms (see [2, Subsection 6.6]).

Remark 3.3. The stationary-excess operation on $H_\alpha(y)$ is defined by

$$
\bar{H}_\alpha(y) := \int_{(0,y]} H_\alpha(x) \, dx / \int_{(0,+\infty)} H_\alpha(x) \, dx,
$$

where $\bar{H}_\alpha(y) = 1 - H_\alpha(y)$. $\bar{H}_\alpha(y)$ is also a probability distribution function, and a simple calculation shows that its Laplace transform is $(1 + \theta^{-\alpha})^{-1/\alpha}$. $\bar{H}_\alpha(y)$ is often called a generalized positive Linnik law. When $\alpha = 1$, it gives the well-known standard exponential law. For more information on Linnik law, we refer readers to [9, Subsection 4] and references therein.

The remaining part of this section is devoted to the converse assertions to Theorem 3.2. Suppose $X_t$ is a critical CB process. First we note that if there exist $x > 0$ and positive constants $Q_t$ such that $P_x(Q_t X_t \in \cdot | \tau > t)$ converges weakly to a non-degenerate limit as $t$ goes to infinity, then $\liminf_{t \to +\infty} Q_t / \bar{F}(t) > 0$. This is because by Fatou’s lemma, we have

$$
0 < \liminf_{t \to +\infty} \int_0^{+\infty} P_x(Q_t X_t > y | \tau > t) \, dy = \liminf_{t \to +\infty} E_x(Q_t X_t | \tau > t) = \liminf_{t \to +\infty} Q_t / \bar{F}(t).
$$

Lemma 3.4. If $\psi$ is regularly varying at 0, then $\psi \in \mathcal{R}_{1+\alpha}(0)$ with $\alpha \in [0, 1]$.

Proof. Suppose $\psi(\lambda) = \lambda^p L(1/\lambda)$ for some $p \in (-\infty, +\infty)$ and $L \in \mathcal{S}$. Since

$$
0 = \psi'(0+) = \lim_{\lambda \downarrow 0} \frac{\psi(\lambda)}{\lambda} = \lim_{\lambda \downarrow 0} \lambda^{p-1} L(1/\lambda),
$$

we have $p \geq 1$. If $p > 2$, then

$$
\psi''(0+) = \lim_{\lambda \downarrow 0} \frac{2\psi(\lambda)}{\lambda^2} = \lim_{\lambda \downarrow 0} 2\lambda^{p-2} L(1/\lambda) = 0.
$$

(3.5)

Note that $\psi''(\lambda) = 2b + \int_{(0, +\infty)} x^2 e^{-\lambda x} \Lambda(dx)$ for some $b \geq 0$ and non-negative measure $\Lambda$ such that $\int_{(0, +\infty)} (x^2 \Lambda(dx) < +\infty$. (3.5) implies that $b = 0$ and $\Lambda(dx) \equiv 0$, which contradicts (2.9). Hence $p \leq 2$. We set $\alpha = p - 1$, thus proving the conclusion.
Theorem 3.5. If there exist \( x > 0 \) and a non-degenerate probability distribution function \( H(y) \) such that \( P_x(F(t), X_t \leq y \mid \tau > t) \) converges weakly to \( H(y) \) as \( t \) goes to infinity, then \( \psi \) satisfies (1.2) with \( \alpha \in (0, 1] \).

Proof. Let \( H(y, t) := P_x(F(t), X_t \leq y \mid \tau > t) \). Under the assumption, we have

\[
\lim_{t \to +\infty} \int_{[0, +\infty)} g(y) dH(y, t) = \int_{[0, +\infty)} g(y) dH(y),
\]

for all bounded continuous function \( g \) on \([0, +\infty)\). Suppose \( \theta > 0 \). Using (3.6) with \( g(y) = e^{-\theta y} \), we get

\[
h(\theta) := \int_{[0, +\infty)} e^{-\theta y} dH(y) = \lim_{t \to +\infty} \int_{[0, +\infty)} e^{-\theta y} dH(y, t)
= \lim_{t \to +\infty} E_x(e^{-\theta F(t)} \mid \tau > t) = 1 - \lim_{t \to +\infty} \frac{1 - \exp(-x u_t(\theta F(t)))}{1 - \exp(-x \varphi(t))}.
\]

So

\[
u_t(\theta \bar{F}(t)) \sim \bar{h}(\theta) \varphi(t) \sim \bar{h}(\theta) \bar{F}(t), \quad \text{as } t \uparrow +\infty,
\]

where \( \bar{h}(\theta) := 1 - h(\theta) \). Immediately, we have

\[
\lim_{t \to +\infty} \nu_t(\theta \bar{F}(t)) = \lim_{t \to +\infty} \bar{h}(\theta) \bar{F}(t) = 0.
\]

On the other hand, using (3.6) with \( g(y) = y e^{-\theta y} \), we obtain

\[
\bar{h}'(\theta) := \frac{\partial}{\partial \theta} \bar{h}(\theta) = \int_{[0, +\infty)} ye^{-\theta y} dH(y) = \lim_{t \to +\infty} \int_{[0, +\infty)} ye^{-\theta y} dH(y, t)
= \lim_{t \to +\infty} E_x(\bar{F}(t) X_t e^{-\theta \bar{F}(t)} \mid \tau > t) = \lim_{t \to +\infty} \frac{\bar{F}(t) E_x(X_t e^{-\theta \bar{F}(t)} \mid \tau > t)}{1 - e^{-x \varphi(t)}}.
\]

It follows from (2.5) and (2.7) that

\[
\frac{\partial}{\partial \lambda} u_t(\lambda) = \frac{\psi(u_t(\lambda))}{\psi(\lambda)}, \quad \forall \lambda > 0.
\]

Thus

\[
E_x(X_t e^{-\lambda X_t}) = -\frac{\partial}{\partial \lambda} e^{-x u_t(\lambda)} = xe^{-x u_t(\lambda)} \frac{\psi(u_t(\lambda))}{\psi(\lambda)}.
\]

Substituting (3.11) into (3.10) and using (3.9) and (2.13), we get

\[
\bar{h}'(\theta) = \lim_{t \to +\infty} \frac{\bar{F}(t)}{1 - e^{-x \varphi(t)}} e^{-x u_t(\theta \bar{F}(t))} \frac{\psi(u_t(\theta \bar{F}(t)))}{\psi(\theta \bar{F}(t))}
= \lim_{t \to +\infty} \frac{\bar{F}(t)}{\varphi(t)} \frac{\psi(u_t(\theta \bar{F}(t)))}{\psi(\theta \bar{F}(t))} \frac{\psi(\bar{h}(\theta) \bar{F}(t))}{\psi(\bar{F}(t))},
\]

The last equality follows from a standard argument using the continuity and monotonicity of \( \psi \). Let \( \lambda(\theta) := \bar{h}(\theta) / \theta \). Using integration by parts formula, we have \( \lambda(\theta) = \int_{0}^{+\infty} e^{-\theta y} H(y) dy \) where \( H(y) := 1 - H(y) \in [0, 1] \). Thus \( \lambda(\theta) \) is decreasing on \((0, +\infty)\). Since \( \bar{F}(t) \) decreases continuously to 0 as \( t \uparrow +\infty \) and \( \psi \) is monotone on \((0, +\infty)\), (3.12) implies that

\[
\lim_{s \to 0} \frac{\psi(\lambda(\theta)s)}{\psi(s)} = \xi(\lambda(\theta)), \quad \forall \theta > 0,
\]

for some function \( \xi \) such that \( \xi(\lambda(\theta)) = \bar{h}'(\theta) \). From the continuity and monotonicity of \( \lambda(\theta) \), we have for any \( \lambda \in (0, \lambda(0+)) \),

\[
\lim_{s \to 0} \frac{\psi(\lambda s)}{\psi(s)} = \xi(\lambda).
\]
Characterization theorem (see [1, Theorem 1.4.1]) says that (3.14) holds for all $\lambda > 0$, and there exists $p \in (-\infty, +\infty)$ such that $\xi(\lambda) \equiv \lambda^p$, i.e., $\psi$ is regularly varying at 0 with index $p$. Let $\alpha = p - 1$, then $\alpha \in [0, 1]$ by Lemma 3.4. If $\alpha = 0$, we have
\[
\frac{\bar{h}(\theta)}{\theta} = \lambda(\theta) = \xi(\lambda(\theta)) = \bar{h}'(\theta).
\] (3.15)

(3.15) has the solution $h(\theta) = 1 - c\theta$ for some constant $c$. This is the Laplace transform of a distribution function if and only if $c = 0$, in which case $H(y) \equiv 1$ is the distribution function of Dirac measure at 0. Therefore $\alpha > 0$.

Suppose $\mu$ is a positive measure supported on $(0, +\infty)$. We say that $\mu$ is regularly varying at $+\infty$ if $u(x) := \mu(0, x]$ is regularly varying at $+\infty$.

**Theorem 3.6.** Let $\Lambda$ be the Lévy measure of $\psi$. Suppose $x^2 \Lambda(dx)$ is regularly varying at $+\infty$. If there exist $x > 0$, positive constants $Q_t$ satisfying $\lim_{t \to +\infty} Q_t = 0$, and a non-degenerate probability distribution function $H(y)$ such that $P_x(Q_t X_1 < y \mid \tau > t)$ converges weakly to $H(y)$ as $t$ goes to infinity, then $\psi$ satisfies (1.2) with $\alpha \in (0, 1]$. Moreover, $Q_t / \bar{F}(t) \sim c$ as $t \uparrow +\infty$ for some positive constant $c$, and the Laplace transform of $H(y)$ is given by
\[
h(\theta) = \int_{(0, +\infty)} e^{-\theta y} dH(y) = 1 - (1 + e^{-\alpha \theta^{-\alpha}})^{-1/\alpha}.
\]

To prove Theorem 3.6, we need the following lemma.

**Lemma 3.7.** $\psi$ is regularly varying at 0 if and only if $x^2 \Lambda(dx)$ is regularly varying at $+\infty$.

**Proof.** First note that under (2.9), we have $\psi''(0+) = 2b + \int_{(0, +\infty)} x^2 \Lambda(dx) \in (0, +\infty]$. Let $U(z) := \int_{[0, z]} x^2 \Lambda(dx)$ and $\bar{U}(\theta) := \int_{(0, +\infty)} e^{-\theta x} dU(x)$. If $0 < \psi''(0+) < +\infty$, then $\psi \in R_2(0)$ and $\int_{(0, +\infty)} x^2 \Lambda(dx) < +\infty$. Immediately $\lim_{z \to +\infty} U(z) = \int_{(0, +\infty)} x^2 \Lambda(dx) < +\infty$, which implies that $x^2 \Lambda(dx)$ is slowly varying at $+\infty$.

Now we consider $\psi''(0+) = +\infty$, in which case $\int_{(1, +\infty)} x^2 \Lambda(dx) = +\infty$. If $\psi$ is regularly varying at 0 with index $p \in [1, 2]$, then for any $A > 0$, using L’Hospital rule, we have
\[
A^p = \lim_{\lambda \to 0^+} \frac{\psi(A\lambda)}{\psi(\lambda)} = \lim_{\lambda \to 0^+} A^2 \frac{\psi''(A\lambda)}{\psi''(\lambda)} = \lim_{\lambda \to 0^+} A^2 \frac{2b + U(A\lambda)}{2b + U(\lambda)} = \lim_{\lambda \to 0^+} A^2 \frac{\bar{U}(A\lambda)}{\bar{U}(\lambda)}.
\]

(3.16)
The last equality is because $\lim_{\theta \to 0^+} \bar{U}(\theta) = \lim_{\theta \to 0^+} \int_{(0, +\infty)} e^{-\theta x} x^2 \Lambda(dx) = +\infty$. Thus $\bar{U}$ is regularly varying at 0 with index $p - 2 \in [-1, 0]$. By Tauberian theorem (see [1, Theorem 1.7.1]), $x^2 \Lambda(dx)$ is regularly varying at $+\infty$ with index $2 - p \in [0, 1]$. The converse assertion is clear through the equalities in (3.16).

**Proof of Theorem 3.6.** The proof is similar to that of Theorem 3.5. We provide details here for the reader’s convenience. For any $y, t \geq 0$ and $\theta > 0$, let $H(y, t) := P_x(Q_t X_1 < y \mid \tau > t)$, $h(\theta) := \int_{[0, +\infty]} e^{-\theta y} dH(y, t)$, $\bar{h}(\theta) := 1 - h(\theta)$ and $\bar{h}'(\theta) := \frac{\partial}{\partial \theta} \bar{h}(\theta)$. Similarly, we can get the analogues to (3.8) and (3.12),
\[
u_t(\theta Q_t) \sim \bar{h}(\theta) \varphi(t) \sim \bar{h}(\theta) \bar{F}(t), \quad \text{as} \ t \uparrow +\infty,
\]
and
\[
\lim_{t \to +\infty} \frac{Q_t}{\bar{F}(t)} \frac{\psi(\nu_t(\theta Q_t))}{\psi(\theta Q_t)} = \bar{h}'(\theta).
\]

(3.17)

(3.18)

It follows from Lemma 3.7 that $\psi$ is regularly varying at 0. By Lemma 3.1, (3.17) and (3.18), we have
\[
\lim_{t \to +\infty} \frac{Q_t}{\bar{F}(t)} \frac{\psi(\bar{h}(\theta) \bar{F}(t))}{\psi(\theta Q_t)} = \bar{h}'(\theta).
\]

(3.19)
In view of Lemmas 3.7 and 3.4, we have \( \psi \in \mathcal{R}_{1+\alpha}(0) \) with \( \alpha \in [0,1] \). Suppose \( \psi \) satisfies (1.2) with \( \alpha \in (0,1] \). Let \( g(z) := (z\psi(1/z))^{-1}, z > 0 \). (3.19) implies that
\[
\lim_{t \to +\infty} \frac{g(1/\theta Q_t)}{g(1/h(\theta)F(t))} = \lim_{t \to +\infty} \frac{\psi(h(\theta)F(t))}{\psi(\theta Q_t)} \frac{\theta Q_t}{h(\theta)F(t)} = \frac{\theta}{h(\theta)} \bar{h}'(\theta), \quad \forall \theta > 0. \tag{3.20}
\]
Since \( g \in \mathcal{R}_\alpha(\infty) \), by Lemma 3.1, we have for every \( \theta > 0 \),
\[
\frac{\theta Q_t}{h(\theta)F(t)} \sim \left( \frac{\theta}{h(\theta)} \bar{h}'(\theta) \right)^{-1/\alpha}, \quad \text{as } t \uparrow +\infty,
\]
or equivalently,
\[
\frac{Q_t}{F(t)} \sim \left( \frac{\theta}{h(\theta)} \right)^{-1/\alpha-1} \bar{h}'(\theta)^{-1/\alpha}, \quad \text{as } t \uparrow +\infty.
\]
Since \( \theta > 0 \) is arbitrary, we conclude that \( Q_t/F(t) \sim c \) for some constant \( c \in (0, +\infty) \), and
\[
\left( \frac{\theta}{h(\theta)} \right)^{-1/\alpha-1} \bar{h}'(\theta)^{-1/\alpha} \equiv c, \quad \forall \theta > 0.
\]
In view of the initial condition \( \bar{h}(0) = 1 \), the above equation has the unique solution \( h(\theta) = 1 - (1 + c^{-\alpha}\theta^{-\alpha})^{-1/\alpha} \).

Otherwise if \( \psi(\lambda) = \lambda L(1/\lambda) \), set \( l(\lambda) := L(1/\lambda) \). By (3.19), we have
\[
\lim_{t \to +\infty} \frac{\bar{l}(h(\theta)F(t))}{\theta l(\theta Q_t)} = \bar{h}'(\theta), \quad \forall \theta > 0. \tag{3.21}
\]
Note that \( l \) is slowly varying at 0. For every \( \theta > 0 \), we have \( l(h(\theta)F(t)) \sim l(F(t)) \) and \( l(\theta Q_t) \sim l(Q_t) \) as \( t \uparrow +\infty \). It follows from this and (3.21) that
\[
\lim_{t \to +\infty} \frac{l(F(t))}{l(Q_t)} = \frac{\theta}{h(\theta)} \bar{h}'(\theta), \quad \forall \theta > 0.
\]
Thus there exists a constant \( c_1 \) independent of \( \theta \) such that
\[
\frac{\theta}{h(\theta)} \bar{h}'(\theta) \equiv c_1, \quad \forall \theta > 0.
\]
This has the solution \( h(\theta) = 1 - c_2 \theta^{c_1} \) for some constant \( c_2 \). \( h(\theta) \) is the Laplace transform of a distribution function only if \( c_2 = 0 \), in which case \( H(y) \equiv 1 \) for all \( y \geq 0 \) is the distribution function of the Dirac measure at 0. This contradicts our assumption that \( H \) is the distribution function of a non-degenerate random variable. Hence \( \alpha > 0 \). This completes the proof of Theorem 3.6.

**Remark 3.8.** Through the above proof we see that for \( \psi \) satisfying (1.2) with \( \alpha = 0 \), the limit distribution of \( P_x(Q_t X_t \in \cdot | \tau > t) \), if exists, must be the Dirac measure at 0.

### 4 The case \( \alpha = 0 \)

In this section, we assume \( \psi(\lambda) = \lambda L(1/\lambda) \), where \( L \in \mathcal{S} \). From Remark 3.8 we know that in this case, any possible positive sequence \( Q_t \) satisfying \( \lim_{t \to +\infty} Q_t = 0 \) overnormalizes \( X_t \). So we need to find an alternative way to normalize \( X_t \). [9] considers the analogous conditional limit theorem for critical Markov branching processes with the offspring generating function \( f(s) = s + (1-s)L(1/(1-s)) \), where \( L \in \mathcal{S} \). The proof in [9] can be adapted here to get convergence result for the critical CB process.

Define
\[
V(x) := \phi(1/x) = \int_{1/x}^{+\infty} \frac{1}{\psi(\xi)} d\xi = \int_{0}^{x} \frac{1}{\xi L(\xi)} d\xi, \quad \forall x \geq 0.
\]
Immediately, we have $V(0) = 0$, $V(+\infty) = +\infty$, and $V'(x) = x^{-1}L(x)^{-1}$ for all $x > 0$. $V$ is strictly increasing on $(0, +\infty)$. It follows from [1, Proposition 1.5.9a] that $V$ is slowly varying at $+\infty$, and $V(x)L(x) \to +\infty$ as $x \to +\infty$.

Let $R$ denote the inverse function of $V$. $R$ is continuous, strictly increasing on $(0, +\infty)$ with $R(+\infty) = +\infty$ and $R(0) = 0$. Moreover, $R(x) = 1/\varphi(x)$ for all $x \geq 0$, thus

$$F(t) \sim R(t)^{-1}, \quad \text{as } t \uparrow +\infty. \quad (4.1)$$

By [1, Theorem 2.4.7], $R$ belongs to the class of Karamata rapidly varying functions denoted by $KR_\infty$. We refer readers to [1, Subsection 2.4] for more information about $KR_\infty$. Since $y = V(R(y))$, we have

$$1 = V'(R(y))R'(y) = \frac{R'(y)}{R(y)L(R(y))}, \quad \forall y > 0,$$

or equivalently

$$\frac{R'(y)}{R(y)} = L(R(y)), \quad \forall y > 0.$$

Therefore, there exist constants $c, A > 0$ such that

$$R(y) = c \exp \left\{ \int_A^y L(R(z))dz \right\}, \quad \forall y \in [A, +\infty). \quad (4.2)$$

**Lemma 4.1** (See [9, Lemma 5.2]). As $t \uparrow +\infty$, $I(y, t) := \int_t^{t+y/L(R(t))} L(R(z))dz \to y$, and this convergence holds locally uniformly with respect to $y \in (-\infty, +\infty)$.

**Theorem 4.2.** If $\psi$ satisfies (1.2) with $\alpha = 0$, then

$$V(F(t)^{-1}) \sim t, \quad \text{as } t \uparrow +\infty, \quad (4.3)$$

and for all $x > 0$ and $y \geq 0$

$$\lim_{t \to +\infty} P_x(L(F(t)^{-1})V(X_t) \leq y \mid \tau > t) = 1 - e^{-y}. \quad (4.4)$$

**Proof.** Since $V \in \mathcal{S}$, it follows from (4.1) and Lemma 3.1 that $V(F(t)^{-1}) \sim V(R(t)) = t$ as $t \uparrow +\infty$. Now we only need to prove (4.4). (4.4) is obviously true for $y = 0$. Henceforth we suppose $y > 0$. By the monotonicity of $V$, we have

$$P_x(L(F(t)^{-1})V(X_t) \leq y \mid \tau > t) = P_x(X_t \leq R(y/L(F(t)^{-1})) \mid \tau > t). \quad (4.5)$$

Applying similar argument as in (3.3), we have for every $\theta > 0$,

$$\lim_{t \to +\infty} P_x \left( \exp \left\{ -\theta \frac{X_t}{R(y/L(F(t)^{-1}))} \right\} \mid \tau > t \right) = 1 - \lim_{t \to +\infty} \frac{\varphi(t + \varphi(\theta/R(y/L(F(t)^{-1}))))}{\varphi(t)} = 1 - \lim_{t \to +\infty} \frac{R(t)}{R(t + \varphi(\theta/R(y/L(F(t)^{-1}))))}. \quad (4.6)$$

By $V \in \mathcal{S}$ and (4.1), we have

$$\phi(\theta/R(y/L(F(t)^{-1})))) = V \left( \frac{1}{\theta} R(y/L(F(t)^{-1})) \right) \sim V(R(y/L(F(t)^{-1})))$$

$$= \frac{y}{L(F(t)^{-1})} \sim \frac{y}{L(R(t))}, \quad \text{as } t \uparrow +\infty.$$

In other words,

$$\phi(\theta/R(y/L(F(t)^{-1})))) = \frac{y}{L(R(t))}(1 + \delta(t)), \quad (4.7)$$
where \( \lim_{t \to \infty} \delta(t) = 0 \). By (4.7) and (4.2), we have
\[
\frac{R(t)}{R(t + \phi(\theta)/R(y/L(F(t)^{-1})))} = \frac{R(t)}{R(t + \frac{y}{L(R(t))}(1 + \delta(t)))} = \exp \left\{ - \int_{t}^{t+y(1+\delta(t))/L(R(t))} L(R(z))dz \right\} = \exp\{-I(y(1+\delta(t)), t)\},
\]
where \( I \) is defined in Lemma 4.1. By the locally uniform convergence of \( I \) in Lemma 4.1, we have \( \exp\{-I(y(1+\delta(t)), t)\} \to e^{-y} \) as \( t \uparrow +\infty \). Thus using (4.6) and (4.8), we obtain
\[
\lim_{t \to +\infty} P_{x}\left( \exp \left\{ \frac{-\theta \left( X_{t} \right)}{R(y/L(F(t)^{-1}))} \right\} \mid \tau > t \right) = 1 - e^{-y}.
\]
Note that \( 1 - e^{-y} \) is the Laplace transform of the defective law which assigns mass \( 1 - e^{-y} \) at 0 and no mass in \((0, +\infty)\). It follows from the continuity theory for Laplace transform (see [2, Subsection 6.6]) that
\[
\lim_{t \to +\infty} P_{x}(X_{t} \leq R(y/L(\bar{F}(t)^{-1})) \mid \tau > t) = 1 - e^{-y},
\]
or equivalently by (4.5)
\[
\lim_{t \to +\infty} P_{x}(L(\bar{F}(t)^{-1})V(X_{t}) \leq y \mid \tau > t) = 1 - e^{-y}.
\]

5 Examples

In this section, we give a few examples of critical branching mechanisms and deduce their conditional limit theorems. Branching mechanisms in Examples 5.1, 5.2 and 5.4 are well-known. It follows from [12, Proposition 5.2] that \( \psi(\lambda) = \lambda f(\lambda) \) is a critical branching mechanism if and only if \( f \) is a Bernstein function, and there exist \( b \geq 0 \) and a non-negative decreasing function \( g \) satisfying \( \int_{0}^{\infty} (x \wedge 1)g(x)dx < \infty \), such that \( f(\lambda) = b\lambda + \int_{0}^{\infty} (1 - e^{-x\lambda})g(x)dx \). Branching mechanisms in Examples 5.3 and 5.5 are given in this form. We refer readers to [12] for more information on the connections between branching mechanisms and Bernstein functions, and [13] for more examples of Bernstein functions.

Example 5.1. Let \( \psi(\lambda) = c\lambda^{1+\alpha} \), where \( c > 0 \) and \( \alpha \in (0, 1] \). In this case \( \phi(t) = (ca)^{-1}t^{-\alpha} \) and \( \varphi(t) = (c\alpha t)^{-1/\alpha} \). Thus we have
\[
\bar{F}(t) = 1 - \exp\{-(c\alpha t)^{-1/\alpha}\} \sim (c\alpha t)^{-1/\alpha}, \quad \text{as} \; t \uparrow +\infty.
\]
Using similar arguments as that in (3.4), we get
\[
\lim_{t \to +\infty} E_{x}(e^{-\theta t^{-1/\alpha}X_{t}} \mid \tau > t) = 1 - \lim_{t \to +\infty} \frac{\varphi(t + \phi(\theta t^{-1/\alpha}))}{\varphi(t)} = 1 - \left( (ca)^{-1}\theta^{-\alpha} \right)^{-1/\alpha}.
\]
Therefore for all \( y \geq 0 \),
\[
\lim_{t \to +\infty} P_{x}(t^{-1/\alpha}X_{t} \leq y \mid \tau > t) = H_{\alpha}(y),
\]
where \( H_{\alpha}(y) \) is uniquely determined by its Laplace transform
\[
h_{\alpha}(\theta) = \int_{0}^{+\infty} e^{-\theta y}dH_{\alpha}(y) = 1 - \left( (ca)^{-1}\theta^{-\alpha} \right)^{-1/\alpha}.
\]

Remark 5.2. This case was excluded in Pakes et al. [10,11], and was studied independently in Haas et al. [4] and Zhang [17]. More specifically, [4] discussed Example 5.1 as a special case of self-similar Markov process, while [17] viewed the corresponding CB process as the scaling limit of a certain sequence of Markov branching processes, and then exploited its limit theorems for some general conditioning events.
Example 5.3. If $\psi''(0+) = \sigma < +\infty$, then $\psi$ satisfies (1.2) with $\alpha = 1$ and $\lim_{s\to 0} L(1/s) = \sigma/2$. By Karamata’s theorem, we have $\phi(z) \sim z^{-1} L(1/z)^{-1} \sim 2/\sigma z$ as $z \downarrow 0$, and $\varphi \in \mathcal{R}_{-1}(\infty)$. Thus we have
\[
\lim_{t \to +\infty} \mathbb{E}_x(e^{-\theta X_t/t} \mid \tau > t) = 1 - \lim_{t \to +\infty} \frac{\varphi((1 + \frac{2}{\sigma} \theta^{-1})t)}{\varphi(t)} = 1 - \left(1 + \frac{2}{\sigma} \theta^{-1}\right)^{-1}.
\]
Therefore,
\[
\bar{F}(t) \sim \frac{2}{\sigma t}, \quad \text{as } t \uparrow +\infty,
\]
and for any $y \geq 0$,
\[
\lim_{t \to +\infty} \mathbb{P}_x(X_t/t > y \mid \tau > t) = e^{-\frac{2}{\sigma} y}.
\]
This conditional convergence was proved independently in [6, 7].

Example 5.4. Let $\psi(\lambda) = \lambda(\lambda^{-\alpha} + \lambda^{-\beta})^{-1}$, where $0 < \beta < \alpha \leq 1$. Note that $\psi(\lambda) = \lambda^{1+\alpha} L(1/\lambda)$ with $L(z) = (1 + z^{-\alpha+\beta})^{-1}$. By Karamata’s theorem, we have $g(z) := \phi(1/z) = \int_0^z \xi^{1-1} L(\xi) d\xi \in \mathcal{R}_\alpha(\infty)$, and
\[
g(z) \sim z^{1-1} z^\alpha L(z)^{-1} \sim z^{1-1} z^\alpha =: h(z), \quad \text{as } z \uparrow +\infty.
\]
Both $g$ and $h$ are strictly increasing on $(0, +\infty)$. Let $g^{-1}$ and $h^{-1}$ denote the inverse functions of $g$ and $h$, respectively. By (5.1), we have
\[
g(g^{-1}(z))/h(h^{-1}(z)) \sim g(g^{-1}(z))/h(h^{-1}(z)) = 1, \quad \text{as } z \uparrow +\infty.
\]
Recall that $g \in \mathcal{R}_\alpha(\infty)$. By (5.2) and Lemma 3.1, we have $g^{-1}(z) \sim h^{-1}(z) = (\alpha z)^{1/\alpha}$ as $z \uparrow +\infty$. Consequently, $\varphi(t) = 1/g^{-1}(t) \sim (\alpha t)^{-1/\alpha}$ as $t \uparrow +\infty$. Therefore, we have
\[
\bar{F}(t) \sim (\alpha t)^{-1/\alpha}, \quad \text{as } t \uparrow +\infty,
\]
and for any $y \geq 0$,
\[
\lim_{t \to +\infty} \mathbb{P}_x(t^{-1/\alpha} X_t \leq y \mid \tau > t) = H_\alpha(y),
\]
where $H_\alpha(y)$ has the Laplace transform $h_\alpha(\theta) = 1 - (1 + \alpha^{-1} \theta^{-\alpha})^{-1/\alpha}$.

Example 5.5. Let $\psi(\lambda) = \lambda^{1+\beta} + \lambda^{1+\gamma}$, $0 < \gamma < \beta \leq 1$. Then $\psi(\lambda) = \lambda^{1+\gamma} L(1/\lambda)$ with $L(z) = 1 + z^{\gamma-\beta} \in \mathcal{S}$. Using similar arguments as that in Example 5.3, we have $\bar{F}(t) \sim (\gamma t)^{-1/\gamma}$ as $t \to +\infty$, and for any $y \geq 0$,
\[
\lim_{t \to +\infty} \mathbb{P}_x(t^{-1/\gamma} X_t \leq y \mid \tau > t) = H_\gamma(y),
\]
where $H_\gamma(y)$ has the Laplace transform $h_\gamma(\theta) = 1 - (1 + \gamma^{-1} \theta^{-\gamma})^{-1/\gamma}$.

Example 5.6. Let $\psi(\lambda) = \lambda \log z (1 + \lambda^{-1})$, $\beta \in (0, 1]$. Then $\psi$ satisfies (1.2) with $\alpha = 0$ and $L(z) = \log z (1 + \lambda^{-1})$. Immediately, we have $V(z) \sim (\beta + 1)^{-1} \log z$ and $L(z) \sim \log z$ as $z \uparrow +\infty$. Inserting the asymptotic equivalents of $V$ and $L$ into Theorem 4.2, we get
\[
- \log \bar{F}(t) \sim [(\beta + 1)t]^{1/\beta+1}, \quad \text{as } t \uparrow +\infty,
\]
and for all $x > 0$ and $y \geq 0$,
\[
\lim_{t \to +\infty} \mathbb{P}_x \left( \frac{\log z X_t}{(\beta + 1) \log(\bar{F}(t)^{-1})} \leq y \mid \tau > t \right) = 1 - e^{-y}.
\]

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