Abstract

If $K$ is a commutative ring and $A$ is a $K$-algebra, for any sequence $\sigma$ of positive integers there exists an higher order analogue $dR_\sigma$ of the standard de Rham complex $dR \equiv dR_{(1,\ldots,1,\ldots)}$, which can also be defined starting from suitable (=differentially closed) subcategories of $A-\text{Mod}$. The main result of this paper is that the cohomology of $dR_\sigma$ does not depend on $\sigma$, under some smoothness assumptions on the ambient category. In [VV], a weaker result was proved by completely different methods.

Before proving the main theorem we give a rather detailed exposition of all relevant (for our present purposes) functors of differential calculus on commutative algebras. This part can be also of an independent interest.

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1 Introduction

Higher analogues of the standard de Rham complex were found by one of the authors about 20 years ago in course of his study of basic functors of differential calculus (see \[Vi1\]). More exactly, such an analogue $dR_\sigma$ can be associated with any sequence $\sigma$ of positive integers. The standard de Rham complex turns out to be associated with the simplest sequence of this kind, namely, with $\sigma = (1,1,...)$. Differentials of these complexes are natural differential operators of, generally, higher orders. Complexes $dR_\sigma$’s as well as related functors they represent play an important role inside differential calculus in the sense of \[Vi1\] and as such are worth to be better investigated. It is very plausible that they may have important applications to the formal theory of partial differential equations as well as to Secondary Calculus (see \[VI3\]), to mention the most direct ones. Of course, the first natural question concerning these complexes is: what are their cohomologies? At the end of eighties the second author jointly with Yu. N. Torkhov tested some simple cases (unpublished) in order to check a natural feeling that all complexes $dR_\sigma$ have the same cohomology. After that in 1994 the authors proved (see \[VV-esi\] and \[VV\]) an analogue of the "infinitesimal Stokes formula" for all $dR_\sigma$’s with non-decreasing $\sigma$, which allows to develop a homotopy techniques sufficient to show that these complexes are quasi-isomorphic. One year later we found a somewhat involved proof of this fact in full generality. Its essentially simplified version is the main result of this paper.

In this paper we present also for the first time a rather detailed exposition of all relevant for our purposes functors of differential calculus (sections 1-3) and as such it could be of an independent interest.

A preliminary version of this paper appeared as a Preprint \[VV-sns\] of the Scuola Normale Superiore, Pisa.

Finally we note that recently A.M.Verbovetsky sketched in \[Ve\] an alternative proof of the main theorem of this paper based on the theory of compatibility complexes.

Notations and Conventions

$K$: a commutative ring with unit;

$A$: a commutative $K$-algebra with unit;

$R - \text{Mod}$: the category of $R$-modules for a commutative ring $R$;

$\text{Diff}_A$: the category whose objects are the $A$-modules and the morphisms are differential operators (Section 1) between them;

$\text{Ens}$: the category of sets;

$[C,C]$: the category of functors $C \rightarrow C$, $C$ being a category;

$\text{Ob}(C)$: the objects of $C$, $C$ being a category; $C \in \text{Ob}(C)$ means that $C$ is an object of $C$;

$A - \text{BiMod}$: the category of $A$-bimodules, whose objects are understood as ordered couples $(P,P^+)$ of $A$-modules and whose morphisms are the usual morphisms of bimodules.

Note that $P$ and $P^+$ coincide as $K$-modules, hence as sets;

$*_A - \text{BiMod}$: the category of "biads" (see Section 1);

$K(A - \text{Mod})$ (resp. $K(K - \text{Mod})$, resp. $K(\text{Diff}_A)$): the category of complexes in $A - \text{Mod}$ (resp. $K - \text{Mod}$, resp. $\text{Diff}_A$);
are triples of objects in ∆ if it exists \( \tau \in \text{Ob}(\mathcal{D}) \) and a functorial isomorphism \( T \simeq \text{Hom}_A(\tau, \cdot) \) in \( [\mathcal{D}, \mathcal{D}] \).

\( A\text{-}\text{BiMod}_\mathcal{D} \) (resp. \( *A\text{-}\text{BiMod}_\mathcal{D} \), resp. \( \text{K}(\text{Diff}_{A, \mathcal{D}}) \)) will be the subcategory of \( A\text{-}\text{BiMod} \) whose objects are couples of objects in \( \mathcal{D} \) (resp. the subcategory of \( *A\text{-}\text{BiMod} \) whose objects are triples of objects in \( \mathcal{D} \), resp. the subcategory of \( \text{K}(\text{Diff}_A) \) whose objects are complexes of objects in \( \mathcal{D} \)).

A sequence \( T_1 \to T_2 \to T_3 \) of functors \( T_i : \mathcal{D} \to \mathcal{D} \), \( i = 1, 2, 3 \), (and functorial morphisms) \( \mathcal{D} \) being an abelian subcategory of \( A\text{-}\text{Mod} \), is said exact in \( [\mathcal{D}, \mathcal{D}] \) if it is exact in \( \mathcal{D} \) when applied to any object of \( \mathcal{D} \).

Let \( \mathbb{N}^\infty = \lim_{k \to \infty} \mathbb{N}^k \) be the set of infinite sequences of positive integers. If \( \sigma \in \mathbb{N}^n \) (or \( \sigma \in \mathbb{N}^\infty \)) then \( \sigma(r) = (\sigma_1, ..., \sigma_r) \) for \( r \leq n \) (or any \( r \in \mathbb{N}^+ \)). We denote by \( 1 \) the element \((1, ..., 1, 1, ...)\) \( \in \mathbb{N}^\infty \).

\[ \begin{align*} \delta_a : \text{Hom}_K(P, Q) &\to \text{Hom}_K(P, Q) \\ \Phi &\mapsto \{ \delta_a \Phi : p \mapsto \Phi(ap) - a\Phi(p) \} \quad p \in P \end{align*} \]

(1) \( \text{Diff}_k(P, Q) \) of differential operators of order \( \leq k \) from \( P \) to \( Q \) is an element \( \Delta \in \text{Hom}_K(P, Q) \) such that:

\[ [\delta_{a_0} \circ \delta_{a_1} \circ ... \circ \delta_{a_s}] (\Delta) = 0, \quad \forall \{a_0, a_1, ..., a_s\} \subset A. \]

We will write synthetically \( \delta_{a_0, ..., a_s} \) for \( \delta_{a_0} \circ \delta_{a_1} \circ ... \circ \delta_{a_s} \).

The set \( \text{Diff}_k(P, Q) \) of differential operators of order \( \leq k \) from \( P \) to \( Q \) is endowed naturally with two different \( A \)-module structures:

(i) \( (\text{Diff}_k(P, Q), \tau) \equiv \text{Diff}_k(P, Q) \) (left),

\[ \tau : A \times \text{Diff}_k(P, Q) \to \text{Diff}_k(P, Q) : (a, \Delta) \mapsto \tau(a, \Delta) : p \mapsto a\Delta(p) \]

(ii) \( (\text{Diff}_k(P, Q), \tau^+) \equiv \text{Diff}_k^+(P, Q) \) (right),

\[ \tau^+ : A \times \text{Diff}_k(P, Q) \to \text{Diff}_k(P, Q) : (a, \Delta) \mapsto \tau^+(a, \Delta) : p \mapsto \Delta(ap). \]

We will often write, to be concise, \( \tau(a, \Delta) \equiv a\Delta \) and \( \tau^+(a, \Delta) \equiv a^+\Delta. \) It is easy to see that \( (\text{Diff}_k(P, Q), (\tau, \tau^+)) \equiv (\text{Diff}_k(P, Q), \text{Diff}_k^+(P, Q)) \equiv \text{Diff}_k^{(+)}(P, Q) \) is an \( A \)-bimodule.

Remark 2.1 Since

\[ \delta_{a_0}(\Delta) \equiv 0 \iff \Delta(a_0 p) = a_0 \Delta(p), \quad \forall a_0 \in A, \forall p \in P \]
\textit{Diff}_0(P,Q) \text{ and } \textit{Hom}_A(P,Q) \text{ are identified as } A\text{-}(bi)\text{modules:}

$$\textit{Hom}_A(P,Q) \simeq \textit{Diff}_0(P,Q) \simeq \textit{Diff}_0^+(P,Q).$$

The obvious inclusion (of sets):

$$\textit{Diff}_k(P,Q) \hookrightarrow \textit{Diff}_l(P,Q), \ k \leq l$$

induces a monomorphism of $A$-bimodules:

$$\textit{Diff}_k^{(+)}(P,Q) \hookrightarrow \textit{Diff}_l^{(+)}(P,Q), \ k \leq l;$$

the direct limit of the system in $A\text{-BiMoD}$:

$$\text{Diff}_0^{(+)}(P,Q) \hookrightarrow \textit{Diff}_1^{(+)}(P,Q) \hookrightarrow \cdots \hookrightarrow \textit{Diff}_n^{(+)}(P,Q) \hookrightarrow \cdots$$

is denoted by $\textit{Diff}^{(+)}(P,Q) = (\textit{Diff}(P,Q), \textit{Diff}^+(P,Q))$.

With a given $A$-module $P$ the following three functors are associated

$$\text{Diff}_k: P \mapsto \textit{Diff}_k(P,Q), \quad \text{Diff}_k^+: P \mapsto \textit{Diff}_k^+(P,Q), \quad \text{Diff}_k^{(+)}: P \mapsto \textit{Diff}_k^{(+)}(P,Q).$$

Let us put $\textit{Diff}_k^{(+)}(A,Q) \equiv \textit{Diff}_k^{(+)}(Q)$. Remark 2.1 implies $\textit{Diff}_0^+ = \textit{Diff}_0 = \text{Id}_{A\text{-Mod}}$.

To simplify notations we will write $\textit{Diff}_0^{(+)}$ instead of $\textit{Diff}_0^{(+)} \circ \cdots \circ \textit{Diff}_0^{(+)}$.

\textbf{Definition 2.2} For any $s, t \geq 0$, define an $A$-module homomorphism

$$C_{s,t}(P): \textit{Diff}_s^+(\textit{Diff}_t^+P) \rightarrow \textit{Diff}^+_{s+t}P$$

by

$$C_{s,t}(P)(\Delta): a \mapsto \Delta(a)(1), \quad \Delta \in \textit{Diff}_s^+(\textit{Diff}_t^+P).$$

Then $P \mapsto C_{s,t}(P)$ defines a morphism $\textit{Diff}_s^+(\textit{Diff}_t^+P) \rightarrow \textit{Diff}^+_{s+t}$ of functors called the composition or "gluing" morphism.

Note that $D_{(k)}(Q) \equiv \{\Delta \in \textit{Diff}_kQ \mid \Delta(1) = 0\}$ is an $A$-submodule of $\textit{Diff}_kQ$ (but not of $\textit{Diff}^+_kQ$!). The functor $D_{(k)}: Q \rightarrow D_{(k)}(Q)$ allows to form the following short exact sequence:

$$0 \rightarrow D_{(k)} \xrightarrow{i_k} \textit{Diff}_k \xrightarrow{p_k} \text{Id}_{A\text{-Mod}} \rightarrow 0 \quad (1)$$

in $[A\text{-Mod}, A\text{-Mod}]$, where $i_k$ is the canonical inclusion and $p_k$ is defined by:

$$p_k(Q): \textit{Diff}_kQ \rightarrow Q: \Delta \mapsto \Delta(1), \quad \Delta \in \textit{Diff}_kQ$$

for any $A$-module $Q$. The functor monomorphism $\text{Id}_{A\text{-Mod}} \equiv \textit{Diff}_0 \hookrightarrow \textit{Diff}_k$ splits (1), so that $\textit{Diff}_k = D_{(k)} \bigoplus \text{Id}_{A\text{-Mod}}$. Note that $D_{(1)}(Q)$ is nothing but the $A$-module of all $Q$-valued $K$-linear \textit{derivations} on $A$, denoted in the literature usually by $\text{Der}_{A/K}(Q)$ (see for example [Bou X]).
Let $Q$ be an $A$-module and $P$, $P^+$ be the left and right $A$-modules corresponding to an $A$-bimodule $P^{(+)} \equiv (P, P^+)$. Let’s denote by $\text{Diff}_k^+ (Q, P^+)$ (resp. $D_{(k)}^+ (P^+)$) the $A$-module which coincides with $\text{Diff}_k (Q, P^+)$ (resp. $D_{(k)} (P^+)$) as $K$-modules and whose $A$-module structures are inherited by that of $P$.

$$(\text{mult. by } a \text{ in } \text{Diff}_k^+ (Q, P^+)) \quad (a^\bullet \Delta)(q) \triangleq a\Delta(q)$$

$$(\text{mult. by } a \text{ in } D_{(k)}^+ (P^+)) \quad (a^\bullet \delta)(q) \triangleq a\delta(q)$$

where both $a\Delta(q)$ and $a\delta(q)$ denote the multiplication by $a$ in $P$. The correspondence

$$D_{(k)}^+ : P^{(+)} \longrightarrow (D_{(k)}^+ (P^+), D_{(k)}^+ (P^+))$$

defines a functor $A - \text{BiMod} \longrightarrow A - \text{BiMod}$ in an obvious way. If $Q = A$ we write $\text{Diff}_k^+ (P^+)$ for $\text{Diff}_k^+ (A, P^+)$. Obviously, $D_{(k)}^+ (P^+)$ is an $A$-submodule of $\text{Diff}_k^+ (P^+)$. 

Let us introduce now the category of biads, $A - \text{BiMod}$, whose objects are ordered triples of $A$-modules:

$$(P, P^+; Q)$$

with $P^{(+)} \equiv (P, P^+)$ being an $A$-bimodule and $Q$ an $A$-submodule of $P$. The corresponding morphisms are those of underlying $A$-bimodules ”respecting” the selected submodules, i.e.:

$$f : (P, P^+) \longrightarrow (P, P^+) \text{ such that } f(Q) \subset \overline{Q}.$$  

Examples 2.1 (i) If $P$ is an $A$-module and $s \leq k$, then $(\text{Diff}_k P, \text{Diff}_k^+ P; D_{(s)} (P))$ is a biad for each $s \leq k$.

(ii) If $P^{(+)} \equiv (P, P^+)$ is an $A$-bimodule, then the following are biads:

$$(\text{Diff}_k^+ P^+, \text{Diff}_k^+ P^+; D_{(k)}^+ (P^+))$$

$$(\text{Diff}_k P, \text{Diff}_k^+ P; D_{(k)}^+ (P))$$

$$(\text{Diff}_k P^+, \text{Diff}_k^+ P^+; D_{(k)}^+ (P^+)).$$

We associate to a biad $(P^{(+)}; Q)$ the following $K$-modules:

$$\text{Diff}_k (Q \subset P^+) \equiv \{ \Delta \in \text{Diff}_k P^+ \mid \text{im}(\Delta) \subset Q \}$$

$$D_{(k)} (Q \subset P^+) \equiv \{ \Delta \in D_{(k)}^+ (P^+) \mid \text{im}(\Delta) \subset Q \}.$$  

The $A$-module structure of $\text{Diff}_k^+ (P^+)$ (resp., of $\text{Diff}_k^+ (P^+)$ or of $D_{(k)}^+ (P^+)$) induces an $A$-module structure on $\text{Diff}_k (Q \subset P^+)$ (resp., on $\text{Diff}_k^+ (Q \subset P^+)$, resp. on $D_{(k)} (Q \subset P^+)$), which is denoted by $\text{Diff}_k^+ (Q \subset P^+)$ (resp. $\text{Diff}_k^+ (Q \subset P^+)$ or $D_{(k)} (Q \subset P^+)$). So, the following inclusions take place in $A - \text{Mod}$

\footnote{These $A$-module structures are well defined due to the fact that $(P, P^+) \equiv P^{(+)}$ is a bimodule. Moreover one can give similar definitions with $P^+$ replaced by $P$ i.e. using the canonical involution $A - \text{BiMod} \longrightarrow A - \text{BiMod} : (P, P^+) \rightarrow (P^+, P).$}
Moreover there are three obvious forgetful functors

\[
\begin{align*}
\text{Diff}^+_k(Q \subset P^+) & \subset \text{Diff}^+_k(P^+) \\
\text{Diff}^+_k(Q \subset P^+) & \subset \text{Diff}^+_k(P^+) \\
D_{(k)}(Q \subset P^+) & \subset D^*_{(k)}(P^+).
\end{align*}
\]

We have then a biad:

\[
(\text{Diff}^+_k(Q \subset P^+), \text{Diff}^+_k(Q \subset P^+), D_{(k)}(Q \subset P^+))
\]

which is a sub-biad of \((\text{Diff}^+_k(P^+), \text{Diff}^+_k(P^+), D^*_{(k)}(P^+))\). The following result is straightforward.

**Lemma 2.2** If \(\mathcal{P}^{(+)\text{+}}\) is a sub-bimodule of \(P^{(+)\text{+}}\) and \(Q \subset \mathcal{P}\), then we have:

\[
\begin{align*}
D_{(k)}(Q \subset \mathcal{P}) &= D_{(k)}(Q \subset P^+) \\
\text{Diff}^+_k(Q \subset \mathcal{P}) &= \text{Diff}^+_k(Q \subset P^+) \\
\text{Diff}^*(Q \subset \mathcal{P}) &= \text{Diff}^*(Q \subset P^+).
\end{align*}
\]

Canonical functors

\[
A - \text{Mod} \xleftarrow{i} A - \text{BiMod} \xrightarrow{j} *A - \text{BiMod}
\]

are defined as follows

\[
\begin{align*}
i : P & \mapsto (P, P), j : (P, P^+) \mapsto (P, P^+; P), \\
h : (P, P^+; Q) & \mapsto (P, P^+) \quad \text{(forgetful)}.
\end{align*}
\]

Obviously, \(i\) is fully faithful while \(h\) is left inverse and adjoint to \(j\). So, \(j\) is fully faithful. Moreover there are three obvious forgetful functors \(p_1, p_2, p_3 : *A - \text{BiMod} \to A - \text{Mod}\) with \(p_1(P, P^+; Q) = P, p_2(P, P^+; Q) = P^+, p_3(P, P^+; Q) = Q\).

We define now some absolute functors we need (see Definition 2.3 for their ”relative” version).

**Definition 2.3** For \(k \geq 0\), \(\mathcal{P}_{(k)} : *A - \text{BiMod} \to *A - \text{BiMod}\) is the functor

\[
\mathcal{P}_{(k)} : \big( P^{(+)\text{+}}; Q \big) \mapsto \big( \text{Diff}^*_k(Q \subset P^+), \text{Diff}^+_k(Q \subset P^+); D_{(k)}(Q \subset P^+) \big)
\]

For \(\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_n^+\), define \(\mathcal{P}_\sigma \equiv \mathcal{P}_{(\sigma_1, \ldots, \sigma_n)} : *A - \text{BiMod} \to *A - \text{BiMod}\) as the composition:

\[
\mathcal{P}_{(\sigma_1, \ldots, \sigma_n)} = \mathcal{P}_{(\sigma_1)} \circ \cdots \circ \mathcal{P}_{(\sigma_n)}
\]

We put

\[
\begin{align*}
\mathcal{P}^*_{(\sigma_1, \ldots, \sigma_n)} &= p_1 \circ \mathcal{P}_{(\sigma_1, \ldots, \sigma_n)} \\
\mathcal{P}^+_p_{(\sigma_1, \ldots, \sigma_n)} &= p_2 \circ \mathcal{P}_{(\sigma_1, \ldots, \sigma_n)} \\
D^*_{(\sigma_1, \ldots, \sigma_n)} &= p_3 \circ \mathcal{P}_{(\sigma_1, \ldots, \sigma_n)}.
\end{align*}
\]

Thanks to the full faithfulness of \(i\) and \(j\) mentioned above, we can simply write \(\mathcal{P}_{(\sigma_1, \ldots, \sigma_n)}\) for both \(\mathcal{P}_{(\sigma_1, \ldots, \sigma_n)} \circ j\) and \(\mathcal{P}_{(\sigma_1, \ldots, \sigma_n)} \circ j \circ i\) by specifying the source category only in case of a possible confusion.
Remark 2.2 (Words [V1]) If \( \tau \in \mathbb{N}^k_+ \) and \( \sigma \in \mathbb{N}^n_+ \), the functor

\[
\mathcal{P}_{\sigma}(D^* \subset \mathcal{P}^+_{\sigma}) \equiv (P^*_{\tau}(D^*_{\sigma} \subset \mathcal{P}^+_{\sigma}), P^+_{\tau}(D^*_{\sigma} \subset \mathcal{P}^+_{\sigma}); D^*_{\tau}(D^*_{\sigma} \subset \mathcal{P}^+_{\sigma}))
\]

is obviously identified with \( \mathcal{P}_{(\tau_1) \circ \ldots \circ \mathcal{P}_{(\tau_k)} \circ \mathcal{P}_{\sigma}} \), i.e.

\[
\mathcal{P}_{\sigma}(D^* \subset \mathcal{P}^+_{\sigma}) = \mathcal{P}_{(\tau,\sigma)}
\]

where \((\tau,\sigma) \equiv (\tau_1,\ldots,\tau_k,\sigma_1,\ldots,\sigma_n) \in \mathbb{N}^{n+k}_+ \). This gives a ”closed” functorial meaning to ”words” composed of functors defined above.

If \( P \) is an \( A \)-module we write \( D_{(\sigma_1,\ldots,\sigma_n)}(P) \) for \( D^*_{(\sigma_1,\ldots,\sigma_n)}(j \circ i (P)) \). It is not difficult to see that

\[
D_{(\sigma_1,\ldots,\sigma_n)}(P) = D_{(\sigma_1)}(D_{(\sigma_2,\ldots,\sigma_n)}(P) \subset \text{Diff}^+_{(\sigma_2,\ldots,\sigma_n)}(P))
\]

Furthermore,

\[
D_{(\sigma_1,\ldots,\sigma_n)}(P) \hookrightarrow D^*_{(\sigma_1)}(\text{Diff}^+_{(\sigma_2,\ldots,\sigma_n)}(P)) \hookrightarrow \text{Diff}^*_{\sigma_1}(\text{Diff}^+_{(\sigma_2,\ldots,\sigma_n)}(P))
\]

are inclusions in \( A - \text{Mod} \), while the inclusion \( D_{(\sigma_1,\ldots,\sigma_n)}(P) \hookrightarrow \text{Diff}^+_{(\sigma_2,\ldots,\sigma_n)}(P) \) is a DO of order \( \leq \sigma_1 \). So, the functors \( D^*_{\sigma} \)'s can be defined also inductively:

**Definition 2.4** Let \( \sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n,\ldots) \in \mathbb{N}^\infty_+ \) and \( \sigma (n) = (\sigma_1,\ldots,\sigma_n) \). Then functors \( D_{\sigma(n)} : A - \text{Mod} \to A - \text{Mod} \) are defined by induction:

\[
D_{\sigma(1)} = D_{(\sigma_1)}, \quad D_{\sigma(n)} : P \mapsto D_{(\sigma_1)}(D_{(\sigma_2,\ldots,\sigma_n)}(P) \subset \text{Diff}^+_{(\sigma_2,\ldots,\sigma_n)}(P))
\]

If \( \sigma = (1,1,1,\ldots) \) we also write \( D_n \) for \( D_{\sigma(n)} \).

For any \( \sigma \in \mathbb{N}^\infty_+ \) and \( n \in \mathbb{N}_+ \), the sequence in \([A - \text{Mod}, A - \text{Mod}]/\):

\[
0 \to D_{\sigma(n)} \xrightarrow{I_{\sigma(n)}} D_{\sigma(n-1)} \circ \text{Diff}^{(+)}_{\sigma(n)} \xrightarrow{\pi_{\sigma(n)}} D_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_{n})}
\]

where \( I_{\sigma(n)} \) is the natural inclusion and \( \pi_{\sigma(n)} \) is the composition (see Definition 2.2)

\[
D^*_{\sigma(n-1)} \circ \text{Diff}^{(+)}_{\sigma_{n-1}} \xrightarrow{D^*_{\sigma(n-2)} \circ \text{Diff}^{(+)}_{\sigma(n-1)}} D^*_{\sigma(n-1)+\sigma_{n}}
\]

is exact.

**Remark 2.3** Let \( \sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n) \in \mathbb{N}^n_+ \). We have a canonical split exact short exact sequence in \([A - \text{Mod}, A - \text{Mod}]/\):

\[
0 \to D_{\sigma} \xrightarrow{I_{\sigma}} \mathcal{P}^*_{\sigma} \xrightarrow{\psi_{\sigma}} D_{(\sigma_2,\ldots,\sigma_n)} \to 0
\]

where \( I_{\sigma} \) is the canonical inclusion, \( \psi_{\sigma} \) is given by \( P \) is an \( A \)-module

\[
\psi_{\sigma}(P)(\Delta) = \Delta (1)
\]

and \( \rho_{\sigma} \) by

\[
\rho_{\sigma}(P)(\Delta) = \Delta - \Delta (1).
\]

Hence \( \mathcal{P}^*_{\sigma} \simeq D_{(\sigma_2,\ldots,\sigma_n)} \oplus D_{\sigma} \).
Now we define the relative (i.e. relative to an arbitrary $A$-module $P$) functors.

**Definition 2.5** If $k \geq 0$ and $P$ is an $A$-module, define

$$\mathcal{P}_k[P] : \ast A - \text{BiMod} \rightarrow A - \text{BiMod}$$

to be the functor

$$(Q, Q^+; S) \mapsto (\text{Diff}_k^+(P, S \subset Q^+), \text{Diff}_k^+(P, S \subset Q^+)).$$

If $\sigma(n) = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_+^n$, $n > 1$, define the functor

$$\mathcal{P}_{\sigma(n)}[P] : \ast A - \text{BiMod} \rightarrow (A, A) - \text{BiMod}$$

as the composition

$$\mathcal{P}_{\sigma(n)}[P] = \mathcal{P}_{\sigma(1)}[P] \circ \mathcal{P}_{\sigma(2), \ldots, \sigma(n)}.$$

As in the absolute case (Def. 2.3), we set

$$\mathcal{P}^\bullet_{\sigma(n)}[P] = p_1 \circ \mathcal{P}_{\sigma(n)}[P]$$

$$\mathcal{P}^+_{\sigma(n)}[P] = p_2 \circ \mathcal{P}_{\sigma(n)}[P]$$

and still denote by $\mathcal{P}_{\sigma(n)}[P]$ both $\mathcal{P}_{\sigma(n)}[P] \circ j$ and $\mathcal{P}_{\sigma(n)}[P] \circ j \circ i$.

By Lemma 2.2, we have $$(\mathcal{P}^\bullet_{\sigma(n)}, \mathcal{P}^+_{\sigma(n)}) \equiv (\mathcal{P}^\bullet_\sigma[A], \mathcal{P}^+_\sigma[A]).$$ Moreover, $\mathcal{P}_{\sigma(n)}[P]$ is (contravariantly) functorial in $P$ and we have

**Lemma 2.3** If $0 \rightarrow P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3 \rightarrow 0$ is exact (resp. split exact) in $A - \text{Mod}$, then

$$0 \rightarrow \mathcal{P}^\bullet_{\sigma(n)}[P_3] \xrightarrow{g^\vee} \mathcal{P}^\bullet_{\sigma(n)}[P_2] \xrightarrow{f^\vee} \mathcal{P}^\bullet_{\sigma(n)}[P_1]$$

$$0 \rightarrow \mathcal{P}^+_{\sigma(n)}[P_3] \xrightarrow{g^\vee} \mathcal{P}^+_{\sigma(n)}[P_2] \xrightarrow{f^\vee} \mathcal{P}^+_{\sigma(n)}[P_1]$$

are exact (resp.

$$0 \rightarrow \mathcal{P}^\bullet_{\sigma(n)}[P_3] \xrightarrow{g^\vee} \mathcal{P}^\bullet_{\sigma(n)}[P_2] \xrightarrow{f^\vee} \mathcal{P}^\bullet_{\sigma(n)}[P_1] \rightarrow 0$$

$$0 \rightarrow \mathcal{P}^+_{\sigma(n)}[P_3] \xrightarrow{g^\vee} \mathcal{P}^+_{\sigma(n)}[P_2] \xrightarrow{f^\vee} \mathcal{P}^+_{\sigma(n)}[P_1] \rightarrow 0$$

are exact).

**Proof.** Straightforward, by induction on $n$. ■

If $P$ is an $A$-module, there are exact sequences in $[A - \text{Mod}, A - \text{Mod}]$

$$0 \rightarrow \mathcal{P}^\bullet_{\sigma(n)}[P] \rightarrow \mathcal{P}^\bullet_{\sigma(n-1)}[P] \circ \text{Diff}_{\sigma(n)}^+[Q_{\sigma(n-1), \ldots, \sigma(n-2), \sigma(n-1)+\sigma(n)}][P]$$

(3)
where the monomorphism is the natural inclusion while \( q_{n} \) is induced by the "gluing" morphism with respect to the pair of indexes \((\sigma_{n-1}, \sigma_{n})\), i.e.:

\[
\mathcal{P}_{(\sigma_{1}, ..., \sigma_{n-1})}^{\bullet} \circ \text{Diff}_{\sigma_{n}}^{(+)} (Q) \ni \Delta \mapsto q_{\sigma_{n}}(\Delta) = \overline{\Delta} \in \mathcal{P}_{(\sigma_{1}, ..., \sigma_{n-2}, \sigma_{n-1}+\sigma_{n})}^{\bullet} [P] (Q)
\]

\[
(\cdot\cdot\cdot ((\overline{\Delta} (p)) (a_{1})) \cdot\cdot\cdot) (a_{n-2}) = ((\cdot\cdot\cdot ((\Delta (p)) (a_{1})) \cdot\cdot\cdot) (a_{n-2})) (1),
\]

where \( p \in P \) and \( a_{1},...,a_{n-2} \in A \). We have analogous exact sequences in \([A - \text{Mod}, A - \text{Mod}]\):

\[
0 \rightarrow \mathcal{P}_{\sigma_{(n)}}^{\bullet} [P] \hookrightarrow \text{Diff}_{\sigma_{1}}^{\bullet} (P, \cdot) \circ \mathcal{P}_{(\sigma_{2}, ..., \sigma_{n})}^{\bullet} [P] \rightarrow \mathcal{P}_{(\sigma_{1}+\sigma_{2}, \sigma_{3}, ..., \sigma_{n})}^{\bullet} [P]
\]

(4)

where \( g_{\sigma_{n}} : \Delta \mapsto \overline{\Delta} \) with \( \overline{\Delta} (p) = \Delta (p) (1), p \in P \) (i.e. we "glue" with respect to the first two indexes); the upper boldface dot in \( \text{Diff}_{\sigma_{1}}^{\bullet} (P, \cdot) \) denotes the \( A \)-module structure induced by \( \mathcal{P}_{(\sigma_{2}, ..., \sigma_{n})}^{\bullet} [P] \).

The following definition will allow us to be concise in the next Section:

**Definition 2.6** For any \( n > 0 \) and any \( \sigma \in \text{N}_{+}^{n} \), the functors (in \([A - \text{Mod}, A - \text{Mod}]\) \( \mathcal{P}_{\sigma}^{\bullet}, D_{\sigma}^{\bullet} \) are called the relevant absolute functors while, if \( P \) is an \( A \)-module, the functors \( \mathcal{P}_{\sigma}^{\bullet} [P] \), are called the relevant functors relative to the \( A \)-module \( P \).

**3 Absolute and relative representative objects**

In this Section we consider (strict) representative objects of the functors introduced in the previous Section. We obtain, as particular cases, the standard modules of Kähler differential forms of Algebraic Geometry and the de Rham forms of Differential Geometry. We emphasize that in our approach all these (and not only those of degree one) are obtained as representative objects of suitable functors. One of the major advantages of this approach is to allow natural generalizations.

Let \( \mathcal{D} \) be a full subcategory of \( A - \text{Mod} \). We denote by \( A - \text{BiMod}_{\mathcal{D}} \) the subcategory of \( A - \text{BiMod} \) whose objects are couples of objects of \( \mathcal{D} \) and by \( *A - \text{BiMod}_{\mathcal{D}} \) the subcategory of \(*A - \text{BiMod} \) consisting of triples whose elements are objects of \( \mathcal{D} \) (Section 1).

**Definition 3.1** A full abelian subcategory \( \mathcal{D} \) of \( A - \text{Mod} \) is said to be differentially closed if the following properties are satisfied:

(a) each functor defined in the previous Section, when restricted to \( \mathcal{D} \) (resp. \( A - \text{BiMod}_{\mathcal{D}}, \text{resp. } *A - \text{BiMod}_{\mathcal{D}} \)) has values in \( \mathcal{D} \) (resp. \( A - \text{BiMod}_{\mathcal{D}}, \text{resp. } *A - \text{BiMod}_{\mathcal{D}} \));

(b) if \( T : A - \text{Mod} \rightarrow A - \text{Mod} \) is a relevant absolute functor or a relevant relative functor, relative to an object of \( \mathcal{D} \), then \( T|_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D} \) is strictly representable in \( \mathcal{D} \);

(c) \( A \in \text{Ob}(\mathcal{D}) \);

(d) \( \mathcal{D} \) is closed under tensor product (over \( A \));

(e) \( \mathcal{D} \) is closed under taking subobjects (i.e. if \( P \subseteq Q \in A - \text{Mod} \) and \( Q \) is in \( \mathcal{D} \) then \( P \) is in \( \mathcal{D} \)).

Condition (a) is needed to have an ambient category which is "closed" with respect to functorial differential calculus; as it will be clear in the following, since among the functors of Section 1 there are also compositions of relevant "elementary" ones, we would like that
Lemma 3.1 Let \( A \) be an \( A \)-module. Then \( A \) is strictly representable by the so-called \( k \)-jet module \( J^k(P) \), if existing, could be expressed in terms of representative objects of the relevant "elementary" ones \((D_{(s)}\) and \(\text{Diff}_{t} \) in the example). Condition \((d)\) makes it possible.

\( \mathcal{D} \) being abelian and satisfying \((b)\), exactness of sequences of strictly representable functors yields exactness of the "dual" sequences of representative objects in \( \mathcal{D} \). Condition \((e)\) is related to the existence of canonical generators for some representative objects and will become clear in the sequel. Note also that \((e)\) implies that if \( f \) is a morphism in \( \mathcal{D} \), \( \text{im}(f) \) (resp. \( \ker(f) \)) is the same when considered in \( \mathcal{D} \) or in \( A - \text{Mod} \).

Let us recall some elementary facts about bimodules, mainly to fix our notations.

If \( P^{(+)} = (P, P^{+}) \) is an \( A \)-bimodule and \( a \in A \), we write \( a \) for the multiplication in \( P \) and \( a^{+} \) for the multiplication in \( P^{+} \). If \( Q \) is an \( A \)-module we denote by:

(I) \( P^{+} \otimes_{A}^{*} Q \) the \( A \)-module obtained by the abelian group \( P^{+} \otimes_{A} Q \) with multiplication by elements of \( A \) defined as

\[
a^{*}(p \otimes q) \doteq (ap) \otimes q, \quad a \in A, \ p \in P^{+}, \ q \in Q
\]

(note that \( a^{*}(p \otimes q) \neq p \otimes aq \)). Then \( P^{+} \otimes_{A}^{*} Q \doteq (P^{+} \otimes_{A}^{*} Q, P^{+} \otimes_{A} Q) \) is an \( A \)-bimodule;

(II) \( \text{Hom}_{A}^{(*)}(Q, P^{+}) \) the \( A \)-module obtained by the abelian group \( \text{Hom}_{A}(Q, P^{+}) \) with multiplication by elements of \( A \) defined as :

\[
[a^{*}f](p) \doteq a_{0} \cdot (f(p)), \quad a \in A, \ p \in P, \ f \in \text{Hom}_{A}(Q, P^{+}).
\]

Denote by \( \text{Hom}_{A}^{(*)}(Q, P^{+}) \doteq (\text{Hom}_{A}^{(*)}(Q, P^{+}), \text{Hom}_{A}(Q, P^{+})) \) the corresponding \( A \)-bimodule;

(III) \( \text{Hom}_{A}^{(+)}(P, Q) \) the \( A \)-module obtained by the abelian group \( \text{Hom}_{A}(P, Q) \) with multiplication by elements of \( A \):

\[
[a^{+}f](p) \doteq f(a^{+}p).
\]

Then

\[
\text{Hom}_{A}^{(+)}(P, Q) \doteq (\text{Hom}_{A}(P, Q), \text{Hom}_{A}^{+}(P, Q))
\]

is an \( A \)-bimodule.

In the same way we can define the \( A \)-modules \( P \otimes_{A}^{+} Q, \text{Hom}_{A}^{+}(P, Q) \) and \( \text{Hom}_{A}^{*}(P^{+}, Q) \).

Example 3.1 If \( P \) and \( Q \) are \( A \)-modules, we have an isomorphism in \( [A - \text{Mod}, A - \text{Mod}] \)

\[
\text{Diff}_{A}(\cdot, Q) \simeq \text{Hom}_{A}^{*}(\cdot, \text{Diff}_{A}^{+} Q).
\]

It is not difficult to prove the following

Lemma 3.1 Let \( R \) and \( P \) be \( A \)-modules and \( (Q, Q^{+}) \) an \( A \)-bimodule. Then we have a canonical isomorphism in \( A - \text{BiMod} \):

\[
\text{Hom}_{A}^{(*)}(R, \text{Hom}_{A}^{+(Q, P)}) \xrightarrow{\sim} \text{Hom}_{A}^{(*)}(Q^{+} \otimes_{A}^{*} R, P)
\]

Proposition 3.2 Let \( P \) be an \( A \)-module and \( k \in \mathbb{N}_{+} \). Then \( \text{Diff}_{k}(\cdot, \cdot) : A - \text{Mod} \to A - \text{Mod} \) is strictly representable by the so-called \( k \)-jet module \( J^{k}(P) \).

---

2Relevant or not.
Proof. See [KLV], p. 12. □

In other words, there exists a universal DO $j_k(P) : P \to J^k(P)$, of order $\leq k$ (often denoted simply by $j_k$), such that for each DO $\Delta : P \to Q$ of order $\leq k$, there is a unique $A$-homomorphism $f^\Delta$ and a commutative diagram

$$
P \xrightarrow{j_k(P)} J^k(P) \quad \xrightarrow{\Delta} \quad \downarrow f^\Delta \quad Q.$$

The $A$-module $J^k(P)$ is generated by $\{j_k(p) \mid p \in P\}$. Moreover, $J^k(P)$ has a bimodule structure $J^k_+(P) = (J^k(P), J^k_+(P))$ which can be described in the following, purely functorial, way.

Suppose $D \subseteq A - \text{Mod}$ is a subcategory such that $\forall P \in \text{Ob}(D)$ the functor $\text{Diff}_k(P, \cdot)$ when restricted to $D$ has values in $D$ and is strictly representable in $D$ (e.g. $D = A - \text{Mod}$ by proposition 3.1). Let $J^k_D(P)$ be the corresponding representative object i.e. $\text{Hom}_A(J^k_D(P), \cdot) \simeq \text{Diff}_k(P, \cdot)$ in $[D, D]$. If $j^D_k : P \to J^k_D(P)$ corresponds to the identity morphism of $J^k_D(P)$, we can define, for each $a \in A$, the DO

$$a^+ : P \to J^k_D(P) : p \mapsto j^D_k(ap).$$

The corresponding $A$-endomorphism of $J^k_D$ is still denoted by $a^+$ and gives the required second $A$-module structure $J^k_D_+$ on the abelian group $J^k_D$.

Using the bimodule $J^k_+$ and Proposition 3.2, we get an isomorphism in $[A - \text{Mod}, A - \text{Mod}]$

$$\text{Diff}^+_k \simeq \text{Hom}^+_A(J^k, \cdot).$$

(5)

If $P, Q$ are $A$-modules, then:

$$\text{Hom}_A(J^k_+ \otimes_A P, Q) \simeq \text{Hom}^+_A(P, \text{Hom}_A^+(J^k, Q))$$

by Lemma 3.1; therefore

$$\text{Hom}^+_A(P, \text{Hom}_A^+(J^s, Q)) \simeq \text{Hom}^+_A(P, \text{Diff}^+_s Q).$$

and, by (5) and Example 3.1, we finally get:

$$\text{Hom}^+_A(P, \text{Diff}^+_s Q) \simeq \text{Diff}_s(P, Q).$$

Since representative objects of the same functor are canonically isomorphic, we have proved:

**Lemma 3.3** There are canonical isomorphisms in $[A - \text{Mod}, A - \text{Mod}]$:

$$J^k(\cdot) \simeq J^k_+ \otimes_A (\cdot)$$

$$J^k_+(\cdot) \simeq J^k_+ \otimes_A (\cdot).$$

We are now able to prove a basic result
Lemma 3.4 (a) If $\tau \in \mathbb{N}_+^\infty$, $n > 0$, $t \geq 0$ and $D_{\tau(n)}$ is strictly representable in $A \to \text{Mod}$ by $\Lambda^{\tau(n)}$, then $D_{\tau(n)} \circ \text{Diff}_t^{(+)}$ is strictly representable in $A \to \text{Mod}$ by $J^t(\Lambda^{\tau(n)})$.

(b) If $s, t \geq 0$, then

$$P^*_s \circ \text{Diff}_t^{(+)} \equiv \text{Diff}_s \circ \text{Diff}_t^{(+)} : A \to \text{Mod}$$

is strictly representable by $J^t(J^s)$.

(c) If $s, t \geq 0$ and $P$ is an $A$-module, then

$$P^*_s[P] \circ \text{Diff}_t^{(+)} \equiv \text{Diff}_s \left( P, \text{Diff}_t^{(+)}(\cdot) \right) : A \to \text{Mod}$$

is strictly representable by $J^t(J^s(P))$.

(d) If $P^*_\sigma(n)$ is strictly representable by $\text{Hol}^{\sigma(n)}$, then

$$P^*_\sigma(n) \circ \text{Diff}_k^{(+)} : A \to \text{Mod}$$

is strictly representable by $J^k(\text{Hol}^{\sigma(n)})$.

(e) If $P$ is an $A$-module and $P^*_\sigma(n)[P]$ is strictly representable by $\text{Hol}^{\sigma(n)}[P]$, then

$$P^*_\sigma(n)[P] \circ \text{Diff}_k^{(+)} : A \to \text{Mod}$$

is strictly representable by $J^k(\text{Hol}^{\sigma(n)}[P])$.

Proof. The proofs are very similar. We prove only (b) and (d).

(b)

$$\text{Diff}_s(\text{Diff}_t^{+}P) \simeq \text{Hom}_A^*(J^s, \text{Diff}_t^{+}P) \simeq \text{Hom}_A^*(J^s, \text{Hom}_A^t(J^t, P)) ;$$

by (5), this is isomorphic to $\text{Hom}_A(J^t \otimes J^s, P)$ and, finally by Proposition 3.3, to $\text{Hom}_A(J^t(J^s), P)$.

(d)

$$P^*_\sigma(n) \left( \text{Diff}_k^{(+)}(P) \right) \simeq \text{Hom}_A^*(\text{Hol}^{\sigma(n)}, \text{Diff}_k^{+}P) \simeq \text{Hom}_A(\text{Hol}^{\sigma(n)}, \text{Hom}_A^+(J^k, P)) \simeq \text{Hom}_A( J^k \otimes \text{Hol}^{\sigma(n)}, P ) \simeq \text{Hom}_A \left( J^k \left( \text{Hol}^{\sigma(n)} \right), P \right).$$

Remark 3.1. Note that, for example, $\text{Diff}_k^* \circ \text{Diff}_t^{(+)} \circ \text{Diff}_m^+$ is representable but not strictly representable in $A \to \text{Mod}$:

$$\text{Diff}_k^* (\text{Diff}_t^{+} (\text{Diff}_m^+ P)) \simeq \text{Hom}_A(J^l(J^k), \text{Diff}_m^+ P) \simeq \text{Hom}_A(J^l(J^k), \text{Hom}_A^+(J^m, P)) \simeq \text{Hom}_A(J^m \otimes A J^l(J^k), P) \simeq \text{Hom}_A^+(J^m(J^l)^k, P).$$
We conclude this preliminaries with the following elementary result

Lemma 3.5 ("Third-representable" lemma) Let $0 \to T_1 \xrightarrow{i} T_2 \xrightarrow{\phi} T_3$ be an exact sequence in $[A - \text{Mod}, A - \text{Mod}]$, with $T_2$ and $T_3$ strictly representable by $\tau_2$ and $\tau_3$, respectively; then $T_1$ is also strictly representable by:

$$\frac{\tau_2}{\varphi^\vee(\tau_3)},$$

where $\varphi^\vee : \tau_3 \to \tau_2$ is the dual-representative of $\varphi$.

Proof. If $P$ is an $A$-module, the morphism

$$\chi_P : T_1(P) \to \text{Hom}_A(\frac{\tau_2}{\varphi^\vee(\tau_3)}, P)$$

$$q \mapsto \chi_P(q) : [t_2]_{\text{mod}, \varphi^\vee(\tau_3)} \mapsto \hat{q}(t_2)$$

where $\hat{q} = i(P)(q)$, is well defined (since $\hat{q} \circ \varphi^\vee = \varphi(P)(\hat{q}) = 0$) and is an isomorphism, natural in $P$. □

The next Theorem, collecting some of the results above, asserts that $A - \text{Mod}$ is itself differentially closed (see Definition 3.1).

Theorem 3.6 Let $P$ be an $A$-module, $\sigma \in \mathbb{N}_+^\infty$ and $k \in \mathbb{N}$. Then:

(i) $\text{Diff}_k(P, \cdot)$ is strictly representable in $A - \text{Mod}$ by the $k$-jet module $J_k(P)$;

(ii) for each $n > 0$, the functor $D_{\sigma(n)}$ is strictly representable in $A - \text{Mod}$ by the so-called higher de Rham forms’ module of type $\sigma(n)$, $A^{\sigma(n)}$;

(iii) $P^{*}_{\sigma(n)}$ and $P^{*}_{\sigma(n)}[P] : A - \text{Mod} \to A - \text{Mod}$ are strictly representable by the so-called absolute holonomy module of type $\sigma(n)$, $\text{Hol}^{\sigma(n)}$ and relative holonomy module of type $\sigma(n)$, $\text{Hol}^{\sigma(n)}[P]$.

Proof. (i) is Proposition 3.2

(ii) The strict representability of $D_{\sigma(n)}$ in $A - \text{Mod}$ may be proved by induction on $n$. The case $n = 1$ follows from the exact sequence $[\text{1}]$, the "third-representable" lemma and (i). Now, suppose we have proved strict representability of $D_{\tau(k)}$ for each $\tau \in \mathbb{N}_+^\infty$ and each $k \leq n - 1$. From the exact sequence in $[A - \text{Mod}, A - \text{Mod}]$

$$0 \to D_{\sigma(n)} \hookrightarrow D^*_{\sigma(n-1)} \circ \text{Diff}^+_n \longrightarrow D_{(\sigma(n-2), \sigma_{n-1}+\sigma_n)}; \quad (6)$$

the last morphism being

$$\Delta \mapsto \hat{\Delta}$$

$$\left(\left(\left(\hat{\Delta}(a_1)\right)(a_2)\right)\ldots\right)(a_{n-2}) (a_{n-1}) \equiv \left((\left(\left(\hat{\Delta}(a_1)\right)(a_2)\right)\ldots\right)(a_{n-2}) (a_{n-1}) \right) (1)$$

(i.e. we use the "gluing" morphism of definition 2.2 with respect to the last two indexes), Lemma 3.4 (a) and the "third-representable" lemma, we obtain strict representability for $D_{\sigma(n)}$.

(iii) the case of $P_{\sigma(n)}^{*}$ follows, as for (ii), by induction via Lemma 3.4, Lemma 3.5 and by any of the following two exact sequences in $[A - \text{Mod}, A - \text{Mod}]$.
\[ 0 \to \mathcal{P}^*_{\sigma(n)} \hookrightarrow \mathcal{P}^*_{\sigma(n-1)} \circ \text{Diff}^{(+)}_{\sigma_n} \to \mathcal{P}^*_{(\sigma(n-2),\sigma_{n-1}+\sigma_n)} \] (7)

\[ 0 \to \mathcal{P}^*_{\tau(n)} \hookrightarrow \text{Diff}^*_{\sigma_1} \circ \mathcal{P}^+_{\sigma_2}(\text{Diff}^{(+)}_{\sigma_3,...,\sigma_n}) \to \mathcal{P}^*_{(\sigma_1+\sigma_2,\sigma_3,...,\sigma_n)} \] (8)

where:
(a) the upper boldface dot in \( \text{Diff}^*_{\sigma_1} \) in (7) refers to the \( A \)-bimodule structure \( (\mathcal{P}^*_{(\sigma_2,...,\sigma_n)}, \mathcal{P}^+_{(\sigma_2,...,\sigma_n)}) \);
(b) the morphisms on the right are defined in the only natural way by using the "gluing" morphism of Definition 2.2: for (7) we "glue" with respect to the last two indexes while in (8) we "glue" with respect to the first two. The case of \( \mathcal{P}^*_{\sigma(n)}[P] \) is proved analogously, using (7) instead of (6) or (5) in place of (8). \[ \blacksquare \]

**Remark 3.2** For any \( k > 0 \), we have \( \Lambda^{(k)} \cong \frac{1}{k+1} I \) where \( I \) is the kernel of the ring multiplication \( A \otimes_K A \to A \); hence \( \Lambda^{(1)} \cong \Omega^1_{A/K} \) is just the \( A \)-module of Kähler differentials (relative to \( K \)). Moreover it is not difficult to show ([KLV] p. 17) that \( \Lambda^{(1,...,1)} \cong \Lambda^n \cong \Lambda^1 \wedge \ldots \wedge \Lambda^1 \) (\( n \) times) \( \cong \Omega^n_{A/K} \) and that for each \( k,l \in \mathbb{N}_+ \) the map \( \Delta : \Delta \to \Delta \) induces a monomorphism \([A \otimes \text{Mod}, A \otimes \text{Mod}]\):

\[ D_{1(k+l)} \hookrightarrow D_{1(k)} \circ D_{1(l)}. \]

whose dual representative \( A \)-homomorphism is just the wedge product \( \wedge : \Lambda^k(A) \otimes_A \Lambda^l(A) \to \Lambda^{k+l}(A) \).

If \( \mathcal{D} \) is a differentially closed subcategory, we will denote the strict representatives in \( \mathcal{D} \) of the relevant functors by adding \( \mathcal{D} \) as a subscript to the symbol used to denote the corresponding representative object in \( A \otimes \text{Mod} \); for example, we write \( \Lambda^{\sigma(n)}_{\mathcal{D}} \) for the representative object in \( \mathcal{D} \) of the functor \( D_{\sigma(n)} : \mathcal{D} \to \mathcal{D} \).

**Remark 3.3** As we did for \( J^1_{\mathcal{D}} = \text{Hol}^{(1)}_{\mathcal{D}} \), we can exhibit another compatible \( A \)-module structure on \( \text{Hol}^{(1,...,n)}_{\mathcal{D}} \), \( \forall n > 0 \). Let \( a \in A \) and \( \hat{a} \in \mathcal{P}^*_{\sigma(n)}(\text{Hol}^{(1,...,n)}_{\mathcal{D}}) \) correspond to the identity of \( \text{Hol}^{(1,...,n)}_{\mathcal{D}} \) under the representability isomorphism. Since \( \mathcal{P}^*_{\sigma(n)}(\text{Hol}^{(1,...,n)}_{\mathcal{D}}) \) and \( \mathcal{P}^+_{\sigma(n)}(\text{Hol}^{(1,...,n)}_{\mathcal{D}}) \) coincide as sets, we can consider \( a^+ \hat{a} \) (multiplication in \( \mathcal{P}^+_{\sigma(n)}(\text{Hol}^{(1,...,n)}_{\mathcal{D}}) \)) as an \( A \)-endomorphism of \( \text{Hol}^{(1,...,n)}_{\mathcal{D}} \). It is easy to verify that this choice defines another \( A \)-module structure on \( \text{Hol}^{(1,...,n)}_{\mathcal{D}} \), denoted by \( \text{Hol}^{(1,...,n)}_{\mathcal{D},+} \) and that \( \left( \text{Hol}^{(1,...,n)}_{\mathcal{D},+}, \text{Hol}^{(1,...,n)}_{\mathcal{D}} \right) \) is an \( A \)-bimodule.

**Lemma 3.7** If \( P \in \text{Ob}(\mathcal{D}) \) we have a canonical isomorphism \( \text{Hol}^{(1,...,n)}_{\mathcal{D}}[P] \cong \text{Hol}^{(1,...,n)}_{\mathcal{D},+} \otimes \mathcal{D}P \) in \( \mathcal{D} \).

**Proof.** Let \( Q \) be an object in \( \mathcal{D} \). By lemma 3.1, we have

\[ \text{Hom}\left( \text{Hol}^{(1,...,n)}_{\mathcal{D},+} \otimes \mathcal{D}P, Q \right) \cong \text{Hom}^{\mathcal{D}}\left( P, \text{Hom}^{\mathcal{D}}\left( \text{Hol}^{(1,...,n)}_{\mathcal{D},+}, Q \right) \right) \cong \text{Hom}^{\mathcal{D}}\left( P, \text{Hol}^{(1,...,n)}_{\mathcal{D},+} \otimes \mathcal{D}Q \right) \cong \text{Hom}^{\mathcal{D}}\left( P, \text{Hol}^{(1,...,n)}_{\mathcal{D}} \otimes \mathcal{D}Q \right) \cong \text{Hom}\left( \text{Hol}^{(1,...,n)}_{\mathcal{D}} \otimes \mathcal{D}P, \text{Hol}^{(1,...,n)}_{\mathcal{D}} \otimes \mathcal{D}Q \right) \cong \text{Hom}\left( \text{Hol}^{(1,...,n)}_{\mathcal{D}} \otimes \mathcal{D}P, Q \right) \]
\[\simeq \text{Hom}^\bullet \left( P, \mathcal{P}^\sigma_{\sigma(n)}(Q) \right) \]

and by definition of $\mathcal{P}^\sigma_{\sigma(n)}$ and example \[3.1\]

\[\text{Hom}^\bullet \left( P, \mathcal{P}^\sigma_{\sigma(n)}(Q) \right) \simeq \mathcal{P}^\bullet_{\{\sigma_1\}} \left[ P \right] \left( \mathcal{P}^{\sigma_2,\ldots,\sigma_n}(Q) \right) = \mathcal{P}^\bullet_{\{\sigma_1,\ldots,\sigma_n\}} \left[ P \right] (Q).\]

\[\blacksquare\]

**Remark 3.4** For any differentiable closed subcategory $\mathcal{D} \subseteq A - \text{Mod}$ it is still true, as in the case $\mathcal{D} = A - \text{Mod}$, that $J^k_D(P)$ is generated as an $A$-module by

\[\{ j_k(p) \mid p \in P \}.\]

In fact, let $J^k_D(P)^\sim$ denote the $A$-submodule of $J^k_D(P)$ generated by $\{ j_k(p) \mid p \in P \}$; this is still an object of $\mathcal{D}$ by Definition \[3.1\] (e). Now, the composition

\[P \xrightarrow{j_k^D} J^k_D(P) \xrightarrow{\pi} J^k_D(P)/J^k_D(P)^\sim\]

is the DO of order $\leq k$ corresponding to $\pi$ under the isomorphism

\[\text{Hom}_A \left( J^k_D(P), J^k_D(P)/J^k_D(P)^\sim \right) \simeq \text{Diff}_k \left( P, J^k_D(P)/J^k_D(P)^\sim \right).\]

But $\pi \circ j_k^D$ is zero hence $\pi = 0$ and we conclude. Note that this also shows that the canonical morphism

\[J^k(P) \to J^k_D(P)\]

is an $A$-epimorphism.

If $P \in \text{Ob}(\mathcal{D})$, the monomorphism in $[\mathcal{D}, \mathcal{D}]$:

\[\text{Diff}_s(P, \cdot) \subset \text{Diff}_t(P, \cdot), \quad t \geq s,\]

gives rise to a $\mathcal{D}$-epimorphism (also an $A$-epimorphism by Remark \[3.4\]) between representative objects:

\[\pi_{t,s}(P) : J^s_D(P) \to J^s_D(P)\]

which fits in the commutative diagram

\[\begin{array}{ccc}
P & \xrightarrow{j_s(P)} & J^s_D(P) \\\\ & \downarrow{j_t(P)} & \uparrow{\pi_{t,s}(P)} \\
J^t_D(P) & & \\
\end{array}\]

The rule $P \mapsto J^t_D(P)$ defines in the obvious way a (covariant) functor $\mathcal{D} \to \mathcal{D}$ (\[K_i\] or \[KLV\]).

The following example shows the importance of the appropriate choice of the differentially closed subcategory of $A - \text{Mod}$ in determining the ” geometrical effectiveness” and size of the representative objects of the relevant functors.
Example 3.2 Let $M$ be a smooth real manifold (which we assume Hausdorff and with a countable basis), $K = \mathbb{R}$ and $A = C^\infty(M; \mathbb{R})$. Then $(\Lambda^\sigma(n))_{\Lambda^\sigma(n)}$ is in general neither projective nor of finite type over $A$: in particular, when $\sigma = (1,\ldots,1,\ldots)$, it does not coincide with the $A$-module of differential $n$-forms on the manifold $M$. To obtain these "geometrical" objects we must choose an appropriate subcategory of $A - \text{Mod}$ : in our approach, choosing a "geometry" is equivalent to select a differentially closed subcategory $\mathcal{D}$. For finite dimensional (real) differential geometry we may choose $\mathcal{D} = A - \text{Mod}_{\text{geom}}$, the full subcategory of geometric $A$-modules, i.e. of $A$-modules $P$ such that $\bigcap_{x \in M} I_x P = (0)$, $I_x$ being the maximal ideal of smooth functions on $M$ vanishing at $x \in M$.

Note that $A - \text{Mod}_{\text{geom}} \supset A - \text{Mod}_{\text{pr, f.t.}}$, the full subcategory of projective $A$-modules of finite type, since $A$ itself is a geometric $A$-module; however, $A - \text{Mod}_{\text{pr, f.t.}}$ is not differentially closed because it is not abelian (and does not satisfy (e)). Another reason that makes us prefer working with the bigger $A - \text{Mod}_{\text{geom}}$ is its better functoriality with respect to change of algebras induced by pull backs of smooth mappings of manifolds.\footnote{If $f : M \to N$ is a smooth map and $P$ is a geometric $C^\infty(M)$-module then $P$ is still geometric when viewed as a $C^\infty(N)$-module via the pull back $f^* : C^\infty(N) \to C^\infty(M)$. Projectivity is not preserved, instead.} $A - \text{Mod}_{\text{geom}}$ is differentially closed due to the fact that the "geometrization" functor

$$(\cdot)_{\text{geom}} : A - \text{Mod} \to A - \text{Mod}_{\text{geom}}
\quad P \mapsto P_{\text{geom}} \cong \bigcap_{x \in M} I_x P$$

sends representative objects in $A - \text{Mod}_{\text{geom}}$ to representative objects in $A - \text{Mod}_{\text{geom}}$ for all the relevant functors ($[KL]$). "Geometrical" objects are obtained as representative objects; for example $\Lambda_{\text{geom}}^{(1,\ldots,1)}$, with $(1,\ldots,1) \in \mathbb{N}^k$, is isomorphic to $\Gamma \left( \bigwedge^k T^* M \right)$, the module of sections of the $k$-th exterior power of the cotangent bundle of $M$, i.e. the module of $k$-differential forms of $M$.

It is possible to encode the "smoothness" of the geometry we want to describe, completely in the choice of the differentially closed subcategory:

**Definition 3.2** A differentially closed subcategory $\mathcal{D}$ of $A - \text{Mod}$ is called smooth if $\Lambda_{\mathcal{D}}^{(1)}$ is a projective $A$-module of finite type.

**Examples 3.8** (i) If $K$ is an algebraically closed field of zero characteristic and $A$ is the coordinate ring of a regular affine $K$-variety, then $\mathcal{D} = A - \text{Mod}$ is smooth ($[Ha]$, II.8).

(ii) If $M$ is a smooth manifold and $A = C^\infty(M; \mathbb{R})$ then $A - \text{Mod}_{\text{geom}}$ is smooth while $A - \text{Mod}$ is not.

It can be proved (as in the proof of Theorem 3.6) that if $\mathcal{D}$ is smooth then all the representative objects of relevant functors are indeed projective and of finite type as $A$-modules. However, we want to stress that since representative objects may be constructed also in non-smooth cases, our approach works also in describing singular and even infinite dimensional geometrical situations. However, to resort with useful objects one has to make an adequate choice of $\mathcal{D}$.

The following proposition will be useful in the next sections:
Proposition 3.9 Let $\mathcal{D}$ be smooth. Then

(i) $\Lambda_{\mathcal{D}}^1 = (0)$ for $n >> 0$;

(ii) If $P \in \text{Ob}(\mathcal{D})$, $\text{Hol}_{\mathcal{D}}^1[P] = (0)$ for $n >> 0$.

Proof. By lemma 3.7 and remark 2.3, (ii) follows from (i). Remark 3.2 together with the fact that $\Lambda_{\mathcal{D}}^1$ is of finite type proves (i). □

4 Higher de Rham complexes

In this Section we use the functors introduced in Section 1 and their representative objects (Section 2) to build higher order analogs of the de Rham complex. Their cohomology will be studied in Section 4.

Let $\mathcal{D}$ be a differentially closed subcategory of $A-\text{Mod}$. The dual representative of the monomorphism in $[\mathcal{D}, \mathcal{D}]$:

$$D_{(\sigma(n),k)} \hookrightarrow D^*_{\sigma(n)} \circ \text{Diff}_k^{(+)}$$

$\sigma(n) \in N^0_+, k \in N_+$,

is a $\mathcal{D}$-epimorphism:

$$J^k_D(\Lambda_{\mathcal{D}}^{\sigma(n)}) \to \Lambda_{\mathcal{D}}^{(\sigma(n),k)};$$

define $d_{(\sigma(n),k)}^D$ to be the composition

$$\Lambda_{\mathcal{D}}^{\sigma(n)} \xrightarrow{j_k} J^k_D(\Lambda_{\mathcal{D}}^{\sigma(n)}) \to \Lambda_{\mathcal{D}}^{(\sigma(n),k)}. \quad (9)$$

Obviously, $d_{(\sigma(n),k)}^D$ is a DO of order $\leq k$.

Definition 4.1 If $\sigma \in N^\infty_+$, the sequence in $\text{DIFF}_A$

$$0 \to A \xrightarrow{d_{(\sigma)}^D} \Lambda_{\mathcal{D}}^{(\sigma)} \xrightarrow{d_{(\sigma_1,\sigma_2)}^D} \Lambda_{\mathcal{D}}^{(\sigma_2)} \to \ldots \xrightarrow{d_{(\sigma_k)}^D} \Lambda_{\mathcal{D}}^{(\sigma_k)} \to \ldots \quad (10)$$

is called higher de Rham sequence of type $\sigma$ of the $K$-algebra $A$ and is denoted by $dR_{\sigma}^D(A)$ or simply by $dR_{\sigma}^D$; each $d_{(\sigma(k))}^D$, $k > 0$, is called higher de Rham differential and is a DO of order $\leq \sigma_k$.

Remark 4.1 When $\sigma = 1 \in N^\infty_+$, the corresponding de Rham sequence is called ordinary. In this case we write $\Lambda_{\mathcal{D}}^k$ for $\Lambda_{\mathcal{D}}^{(1,...,1)}$, $(1,...,1) \in N^k_+$, $\forall k > 0$, so that:

$$dR_{(1)}^D \equiv dR^D: 0 \to A \xrightarrow{d} \Lambda_{\mathcal{D}}^1 \xrightarrow{d} \Lambda_{\mathcal{D}}^2 \to \ldots \xrightarrow{d} \Lambda_{\mathcal{D}}^k \to \ldots \quad (11)$$

and each differential is a DO of order $\leq 1$.

Each $d^D$ in (11) is in fact a differential according to the following:

Proposition 4.1 $\forall \sigma \in N^\infty_+$ the higher de Rham sequence $dR_{\sigma}^D$ is a complex.
Proof. Let \( n \geq 0 \) and consider the diagram defining two consecutive higher de Rham differentials:

\[
\begin{array}{c}
\Lambda^{\sigma(n)}_{\mathcal{D}} \\
\downarrow d^{\mathcal{D}}_{\sigma(n+1)} \\
\Lambda^{\sigma(n+1)}_{\mathcal{D}} \\
\downarrow j_{\sigma_{n+1}} \\
\Lambda^{\sigma(n+2)}_{\mathcal{D}} \\
\downarrow d^{\mathcal{D}}_{\sigma(n+2)} \\
\end{array}
\]

Since \( d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)} \equiv \pi_2 \circ j_{\sigma_{n+2}} \circ j_{\sigma_{n+1}} \) is a DO of order \( k + l \), there exists a unique \( A \)-homomorphism

\[
\varphi_{d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)}} : J^{\sigma_{n+2} + \sigma_{n+1}}_{\mathcal{D}}(\Lambda^{\sigma(n)}_{\mathcal{D}}) \rightarrow \Lambda^{\sigma(n+2)}_{\mathcal{D}}
\]

which makes the following diagram commutative:

\[
\begin{array}{c}
\Lambda^{\sigma(n)}_{\mathcal{D}} \\
\downarrow d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)} \\
\downarrow j_{\sigma_{n+2} + \sigma_{n+1}} \\
J^{\sigma_{n+2} + \sigma_{n+1}}_{\mathcal{D}}(\Lambda^{\sigma(n)}_{\mathcal{D}}) \\
\end{array}
\]

It is not difficult to check that \( \varphi_{d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)}} \) is just the dual representative of the composition:

\[
D_{\sigma(n+2)} \rightarrow D^{\bullet}_{\sigma(n+1)} \circ \text{Diff}^{(+)}_{\sigma_{n+2}} \rightarrow D^{\bullet}_{\sigma(n)} \circ \text{Diff}^{(+)}_{\sigma_{n+1} + \sigma_{n+2}}
\]

which is immediately checked to be zero; therefore \( \varphi_{d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)}} = 0 \) and \( d^{\mathcal{D}}_{\sigma(n+2)} \circ d^{\mathcal{D}}_{\sigma(n+1)} = 0 \) as well. \( \blacksquare \)

Remark 4.2 (i) Let \( \mathcal{D} = A - \text{Mod} \). By induction on \( n \) we can prove (using for \( n = 1 \) the explicit description of \( \Lambda^{(\sigma_1)} \) given in Section 2) that formula (1) implies that \( \Lambda^{\sigma(n)} \) is generated by the set

\[
\left\{ d_{\sigma(n)} \left( a_1 d_{\sigma(n-1)} \left( a_2 \ldots d_{\sigma(1)} (a_n) \ldots \right) \right) \mid a_1, a_2, \ldots, a_n \in A \right\}
\]

In fact, \( J^\sigma_{\mathcal{D}}(\Lambda^{\sigma(n-1)}) \) is known to be generated over \( A \) by the elements \( j_{\sigma_{n}}(\omega), \omega \in \Lambda^{\sigma(n-1)} \) and \( J^\sigma_{\mathcal{D}}(\Lambda^{\sigma(n-1)}) \rightarrow \Lambda^{\sigma(n)} \) is an epimorphism. This result still holds for \( \Lambda^{\sigma(n)}_{\mathcal{D}} \) with \( d \) replaced by \( d^{\mathcal{D}} \), \( \mathcal{D} \subseteq A - \text{Mod} \) being any differentiable closed subcategory: the proof is analogous to the argument used in Remark 3.4 (a). This also shows that the canonical morphisms

\[
\Lambda^{\sigma(n)} \rightarrow \Lambda^{\sigma(n)}_{\mathcal{D}}
\]

are \( A \)-epimorphisms.

(ii) In the "ordinary" case \( (\sigma_1, \ldots, \sigma_n) = (1, \ldots, 1) \equiv 1 \), the \( A \)-module structure of \( \text{Hol}_{\mathcal{D},+}^{1+n} \) (Remark 2.3) can be expressed via the isomorphism (Remark 2.3)

\[
\text{Hol}_{\mathcal{D}}^{(\sigma_1, \ldots, \sigma_n)} \simeq \Lambda^{(\sigma_2, \ldots, \sigma_n)}_{\mathcal{D}} \oplus \Lambda^{(\sigma_1, \ldots, \sigma_n)}_{\mathcal{D}}
\]

as \( a^+ (\rho, \omega) = (a \rho, a \omega + (d_{(1)} a) \wedge \rho) \).
(iii) If $\mathcal{D} = A - \text{Mod}$, $\text{dR}_1$ coincides with the usual algebraic de Rham complex of the $K$-algebra $A$ ([Bou III] and [Bou X]).

(iv) If $K = \mathbb{R}, M$ is a smooth manifold, $A = C^\infty (M; \mathbb{R})$ and $\mathcal{D}$ is the category of geometric $A$-modules (see Section 2) then $\text{dR}_1^2$ is the geometric de Rham complex on $M$. It turns out that any natural differential operator occurring in differential geometry can be recovered functorially using our approach: see [VV] for the case of the Lie derivative and the corresponding homotopy formula.

If $\sigma, \tau \in \mathbb{N}_+^\infty$ with $\sigma \geq \tau$ (i.e. $\sigma_i \geq \tau_i$, $\forall i$), then for each $n > 0$ we have a monomorphism $\text{D}_{\tau(n)} \hookrightarrow \text{D}_{\sigma(n)}$ in $[\mathcal{D}, \mathcal{D}]$; this induces a $\mathcal{D}$-epimorphism on representatives $\Lambda_{\mathcal{D}}^{\sigma(n)} \rightarrow \Lambda_{\mathcal{D}}^{\tau(n)}$, $\forall n > 0$. By [4.2] this is also an $A$-epimorphism. All these epimorphisms commute with higher de Rham differentials and therefore define a morphism of complexes

$$\text{dR}_{\sigma}^\mathcal{D} \rightarrow \text{dR}_{\tau}^\mathcal{D}$$ (12)

(if $\sigma \geq \tau$). So we can consider the $(A$-epimorphic) inverse system $\{\text{dR}_{\sigma}^\mathcal{D}\}_{\sigma \in \mathbb{N}_+^\infty}$ and give the following:

**Definition 4.2** The infinitely prolonged (or, simply, infinite) de Rham complex of the $K$-algebra $A$, is the complex in $\text{K}(K - \text{Mod})$

$$\text{dR}_{\infty}^\mathcal{D}(A) = \lim_{\sigma \in \mathbb{N}_+^\infty} \text{dR}_{\sigma}^\mathcal{D}(A),$$

$$\text{dR}_{\infty}^\mathcal{D}(A) : 0 \rightarrow A \xrightarrow{d_{(\infty)}} \Lambda_{\mathcal{D}}^{(\infty)} \xrightarrow{d_{(\infty, \infty)}} \Lambda_{\mathcal{D}}^{(\infty, \infty)}_2 \rightarrow \cdots \rightarrow \Lambda_{\mathcal{D}}^{(\infty, \cdots, \infty)}_n \rightarrow \cdots$$

where $\Lambda_{\mathcal{D}}^{(\infty, \cdots, \infty)}_n = \lim_{\sigma \in \mathbb{N}_+^\infty} \Lambda_{\mathcal{D}}^{\sigma(n)}$, $\forall n > 0$.

**Remark 4.3** Three descriptions of DO’s between strict representative objects.

We work in a fixed differentially closed subcategory $\mathcal{D}$ of $A - \text{Mod}$ and all representative objects will be in $\mathcal{D}$.

Let $F_1$ and $F_2$ be representative objects of differential functors $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. Suppose that $\mathcal{F}_1$ has an associated functor $\mathcal{F}_1^\bullet$ (having as domain $A - \text{BiMod}_{\mathcal{D}}$) such that $\mathcal{F}_1^\bullet (\text{Diff}_k^{(+)} )$ is strictly representable by $J^k (F_1)$: this is the case, for example, of $\mathcal{F}_1 = \text{D}_{\sigma(n)}$ or $\text{Diff}_1$. Let

$$\Delta : F_1 \rightarrow F_2$$ (14)

be a DO of order $\leq k$. Then, there exists a unique $A$-homomorphism ([K3]: jet-associated to $\Delta$)

$$f_{\Delta} : J^k (F_1) \rightarrow F_2$$ (15)

which represents $\Delta$ by duality: $\Delta = f_{\Delta} \circ j_k (F_1)$. Since $J^k (F_1)$ is the representative object of $\mathcal{F}_1^\bullet (\text{Diff}_k^{(+)} )$, $f_{\Delta}$ defines a unique morphism in $[\mathcal{D}, \mathcal{D}]$:

$$f_{\Delta} : F_2 \rightarrow \mathcal{F}_1^\bullet (\text{Diff}_k^{(+)} )$$,

called generator morphism of $\Delta$. 19
Formulas (15) and (16) give two different descriptions of a DO between representative objects. Formula (16) allows one to identify it with a functorial morphism which, as a rule, may be established in a straightforward way and can then be used to define the corresponding natural DO (14). The following examples show this procedure at work in two canonical cases; we assume for simplicity $\mathfrak{D} = A - \text{Mod}$.

(i) Higher de Rham differential $d_\sigma(n)$.
If $F_2 = D_\sigma(n), F_1 = D_\sigma(n-1), k = \sigma_n$ and we take for (16) the natural inclusion $D_\sigma(n) \hookrightarrow D_\sigma^{\ast}(D^{\ast}_{\sigma_n})$, then $d_\sigma(n) : \Lambda_\sigma^{\ast}(n-1) \to \Lambda_\sigma^{\ast}(n)$ is the corresponding DO (14).

(ii) "Absolute" jet-operator $j_k$.
In this almost tautological case, $F_1 = \text{Hom}_A(A,\cdot) \equiv \text{Diff}^0_0$ and $F_2 = \text{Diff}_k \equiv \text{Hom}_\cdot (A,\cdot) (\text{Diff}_k^{(\ast)+})$; if we take (16) to be the identity $\text{Id} : \text{Diff}_k \to \text{Hom}_\cdot (A,\cdot) (\text{Diff}_k^{(\ast)+}) \equiv \text{Diff}_k$, then (14) is just $j_k : A \to J^k$.

Rigidity of higher de Rham cohomology
In this Section we prove the main result of this paper i.e. that in the smooth case the higher-order de Rham cohomologies coincide with the ordinary (i.e. lowest order) one. Essentially this amounts to a fairly intuitive assertion: raising the order of the natural DOs involved in the $\text{dR}$-complexes does not change the cohomological information, provided the situation in which we are working is smooth.

In this Section (and in the Appendix), $A$ is a $K$-algebra of zero characteristic, containing $K$ as a subring and $\mathfrak{D}$ a differentially closed smooth subcategory of $A - \text{Mod}$. As in the previous Section, all representative objects, unless otherwise stated, will be considered in $\mathfrak{D}$.

Smoothness of $\mathfrak{D}$ implies that for any $k, l \geq 0$, the gluing morphism in $[\mathfrak{D}, \mathfrak{D}]$

$$\text{Diff}_k^{\ast} \circ \text{Diff}_l^{(\ast)+} \xrightarrow{C_{k,l}} \text{Diff}_{k+l}$$

is surjective i.e. that any DO can be expressed as a composition of lower order ones. This can be seen as follows. Let us fix $k$ and proceed by induction on $l$. The case $l = 0$ is trivial since $\text{Diff}_0 = \text{id}_\mathfrak{D}$. To prove the inductive step let us consider the commutative diagram

$$\begin{array}{ccc}
\text{Diff}_k^{\ast} \circ \text{Diff}_{l+1}^{(\ast)+} & \xrightarrow{C_{k,l+1}} & \text{Diff}_{k+l+1} \\
\uparrow & & \uparrow \\
\text{Diff}_k^{\ast} \circ \text{Diff}_{l}^{(\ast)+} & \xrightarrow{C_{k,l}} & \text{Diff}_{k+l}
\end{array}$$

(where the vertical arrows are natural inclusions) and suppose $C_{k,l}$ is epic. Passing to the corresponding diagram of representative objects completing it with kernels, we get a commutative
diagram ([KLV] p. 52) with exact columns

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
S^{k+l+1}(\Lambda^1) & S^{l+1}(\Lambda^1) \otimes J^k \\
\downarrow & \downarrow \\
J^{k+l+1} & J^{l+1}(J^k) \\
\downarrow & \downarrow \\
J^{k+l} & J^l(J^k) \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

where \( S^r \) denotes the \( r \)-th symmetric power and \( C^{s,t} \) is the dual-representative of \( C_{s,t} \). By duality it is enough to prove that \( C^{k,l+1} \) is monic. By induction hypothesis \( C^{k,l} \) is monic so we are reduced to showing that \( \rho \) is monic. It is not difficult to prove (e.g. again by induction on \( l \)) that \( \rho \) is just the composition

\[
\begin{align*}
S^{k+l+1}(\Lambda^1) & \xrightarrow{\alpha} S^{l+1}(\Lambda^1) \otimes S^k(\Lambda^1) \xrightarrow{\beta} S^{l+1}(\Lambda^1) \otimes J^k \\
\end{align*}
\]

where \( \alpha : \omega_1 \cdots \omega_{k+l+1} \mapsto \sum (\omega_{i_1} \cdots \omega_{i_{l+1}}) \otimes (\omega_{j_1} \cdots \omega_{j_k}) \) where the sum is extended to all partitions \( ((i_1, \ldots, i_{l+1}), (j_1, \ldots, j_k)) \) of \( \{1, \ldots, k+l+1\} \) of (ordered) length \( (l+1,k) \) and \( \beta = \text{id}_{S^{l+1}(\Lambda^1)} \otimes i \) with

\[
i : S^k(\Lambda^1) \hookrightarrow J^k
\]

the inclusion of the kernel of \( J^k \to J^{k-1} \) ([KLV], p. 52). \( A \) is of zero characteristic hence \( \alpha \) is well defined and monic; \( \Lambda^1 \) is projective hence \( \beta \) is monic too. Thus \( \rho \) is monic and we conclude.

As a consequence \( \forall n \geq 1 \), we have the following short exact sequence in \([\mathcal{D}, \mathcal{D}]\):

\[
0 \to D_{\sigma(n)} \hookrightarrow D_{\sigma(n)}^* \circ \text{Diff}^{(+)}_{\sigma(n)} \to D_{(\sigma_1, \ldots, \sigma_{n-2}, \sigma_{n-1}+\sigma_n)} \to 0
\]

(17)

(the new fact is that the last arrow of the sequence is epic since it is induced by the gluing morphism).

The \( n \)-th cohomology \( K \)-module of the complex

\[
dR_\sigma : \quad 0 \to A \xrightarrow{d_\sigma(1)} \Lambda^\sigma(1) \to \ldots \to \Lambda^\sigma(n) \xrightarrow{d_\sigma(n+1)} \Lambda^\sigma(n+1) \to \ldots
\]

is denoted by:

\[
H^n_{\sigma} = \frac{\ker \left( d_{\sigma(n+1)} \right)}{\text{im} \left( d_{\sigma(n)} \right)} \equiv H^n(\text{dR}_\sigma).
\]

Since \( H^n_{\sigma} \) only depends on \( \sigma(n+1) \), we will write also \( H^n_{\sigma(n+1)} \) in place of \( H^n_{\sigma} \).

Note that in the situation of Remark 4.2 (iv), \( H^n_{\sigma(n+1)} \) is the \( n \)-th de Rham cohomology \( \mathbb{R} \)-vector space of the smooth manifold \( M \).

The rest of this Section will be devoted to proving the following result:
Theorem 4.2 ("Smooth" rigidity of higher de Rham cohomologies)

If $\mathcal{D}$ is a smooth subcategory of $A-\text{Mod}$, then, for each $\tau, \sigma \in \mathbb{N}_+^\infty$ with $\tau \geq \sigma$, the canonical $\mathcal{D}$-epimorphism \([12]\):

$$dR_\tau \to dR_\sigma$$

is a quasi-isomorphism; so:

$${}^{(18)} H^\sigma_n \simeq H^n_\tau, \forall n \geq 0.$$  

Corollary 4.3

(i) If $M$ is a smooth manifold, $A = C^\infty (M; \mathbb{R})$ and $\mathcal{D} = C^\infty (M; \mathbb{R}) - \text{Mod}_{\text{geo}}$, then the higher de Rham cohomologies coincide with the standard de Rham cohomology of $M$.

(ii) If $K$ is an algebraically closed field of zero characteristic and $A$ is the coordinate ring of a regular affine variety over $K$, then the higher de Rham cohomologies coincide with the standard algebraic one.

Note that the last Corollary is false, in general, for a singular manifold or a non-regular affine variety.

The strategy of the proof of Theorem 4.2 is the following.

Keeping $n \geq 0$ fixed, we prove the thesis by reducing, step by step, each entry of $\sigma (n + 1)$ to 1, starting from $\sigma(n+1)$, i.e. we prove the chain of isomorphisms

$$H^n_{\sigma (n+1)} \simeq H^n_{\sigma(n),1} \simeq H^n_{\sigma(n-1),1,1} \simeq \cdots \simeq H^n_{\sigma_1,1,...,1} \simeq H^n_{dR}$$ \hspace{1cm} (19)

where $H^n_{dR}$ stands for $H^n_{(1,...,1)}$, $(1,...,1) \in \mathbb{N}_+^{n+1}$ (the $n$-th ordinary de Rham cohomology).

The first step in the chain (19) is obtained via the following Lemma:

Lemma 4.4 Let $n \in \mathbb{N}_+$. If $\sigma, \tau \in \mathbb{N}_+^\infty$ are such that $\sigma(n) = \tau(n)$, then:

(i) $\ker d_{\sigma(n+1)} = \ker d_{\tau(n+1)}$;

(ii) $\text{im} (d_{\sigma(n+1)}) \simeq \text{im} (d_{\tau(n+1)})$

(where $\simeq$ means $K-\text{Mod}$-isomorphism).

Proof. (ii) follows trivially from (i). Let $\sigma \in \mathbb{N}^n_+$ and $k > 1$. Consider the short exact sequence:

$$0 \to D_{(\sigma,k-1,1)} \to D^*_\sigma \circ \text{Diff}_1 \to D_{(\sigma,k)} \to 0$$

whose dual-representative:

$$0 \to \Lambda^{(\sigma,k)} \xleftarrow{i^\vee} J^1 \left( \Lambda^{(\sigma,k-1)} \right) \to \Lambda^{(\sigma,k-1,1)} \to 0$$ \hspace{1cm} (20)

is likewise exact (in $\mathcal{D}$). We embed the latter in the commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \Lambda^{(\sigma,k)} & \xleftarrow{i^\vee} & J^1 \left( \Lambda^{(\sigma,k-1)} \right) & \to & \Lambda^{(\sigma,k-1,1)} & \to & 0 \\
\downarrow d_{(\sigma,k)} & & \uparrow f_1 & & \uparrow f_1 & & \downarrow d_{(\sigma,k-1)} & & \\
\Lambda^\sigma & \xleftarrow{j^\vee} & \Lambda^{(\sigma,k-1)} & & \Lambda^{(\sigma,k-1)} & & \Lambda^{(\sigma,k-1)} & & \\
\end{array}
\]

\footnote{This Lemma has been proved, independently, also by Yu. Torkhov.}

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Now, if $\omega \in \Lambda^\sigma$ is such that $d(\sigma,k-1)(\omega) = 0$, then $j_1 (d(\sigma,k-1)(\omega)) = 0$ and, by commutativity, $(i^\vee \circ d(\sigma,k))(\omega) = 0$. But $i^\vee$ is a monomorphism, so:

$$\ker d(\sigma,k-1) \subseteq \ker d(\sigma,k), \ \forall k > 1.$$  

Since the inverse inclusion is obvious, (i) is proved.

To prove the "$k$-th step" of chain (19), it is enough to show that:

$$H_n(\sigma(k-1),\sigma_k+1,1,...,1) \cong H_n(\sigma(k-1),\sigma_k,1,...,1)$$

(21)

where we write $\rho \in N^r_\infty$.

To prove (21) we construct an auxiliary complex.

Let $P$ be an object in $D$ and $\tau \in N^\infty_\infty$. As shown at the beginning of this section, smoothness of $D$ implies that $\forall n > 0$ the "relative" sequence (3)

$$0 \rightarrow P^\bullet(\tau(n)) \rightarrow \cdots \rightarrow \delta_{\tau(n+1)} \rightarrow \cdots$$

(25)

is defined to be the DO whose description (14) of Remark 4.3 is the canonical inclusion $P_{\tau(n+1)} \rightarrow P_{\tau(n)} \circ \text{Diff}_{\tau(n+1)}$.

Furthermore, when $P$ varies in $\text{Ob}(D)$, (22) gives rise to a short exact sequence in $[D,D]$. We will refer to $\text{Hol}^{\tau(n)}[P]$ as the $\text{Hol}$-object of type $\tau(n)$ of the $A$-module $P$; we have

$$\text{Hol}^{\tau(n)}[P] \cong \frac{J_{\tau(n)}(\text{Hol}^{\tau(n-1)}[P])}{\text{HoI}(\tau(n-2),\tau_{n-1}+\tau_n)[P]}$$

(24)

as $A$-modules. This allows us to give the following:

**Proposition and Definition 4.5** Let $P \in \text{Ob}(D)$ and $\tau \in N^\infty_\infty$. We define the sequence in $\text{DIFF}_{A,D}$:

$$\text{Hol}^\tau(P) : 0 \rightarrow \text{Hol}^0[P] \rightarrow \text{Hol}^1[P] \rightarrow \cdots \rightarrow \text{Hol}^{\tau(n)}[P] \rightarrow \cdots$$

where, for each $n \geq 0$,

$$\delta_{\tau(n+1)}[P] : \text{Hol}^{\tau(n)}[P] \rightarrow \text{Hol}^{\tau(n+1)}[P]$$

(25)

is defined to be the DO whose description (14) of Remark 4.3 is the canonical inclusion $P_{\tau(n+1)} \rightarrow P_{\tau(n)} \circ \text{Diff}_{\tau(n+1)}$;

equivalently, $\delta_{\tau(n+1)}[P]$ is the composition:
where \( \psi^{\tau(n)}[P] \) is the canonical quotient projection. \( \mathbf{Hol}^\tau [P] \) is a complex in \( \mathbf{DIFF}_{A, \mathfrak{D}} \), called \( \mathbf{Hol}^\tau \)-complex of \( P \); moreover, \( \mathbf{Hol}^\tau [P] \) is natural in \( P \) and defines a functor \( \mathbf{Hol}^\tau : \mathfrak{D} \rightarrow \mathbf{K}(\mathbf{DIFF}_{A, \mathfrak{D}}) \).

**Proof.** As always, it is better to work with functors (i.e. differential operators) than with representative objects. In the notations of Remark 4.3, we have that \( \varphi^{\delta_{\tau(n)}(\tau(n))} \) coincides with the composition:

\[
\begin{align*}
\mathcal{P}^\bullet_{\tau(n+1)} [P] & \xrightarrow{\varphi^{\delta_{\tau(n+1)}}} \mathcal{P}^\bullet_{\tau(n)} [P] \circ \mathbf{Diff}^{(+)\tau_{n+1}} \\
\mathcal{P}^\bullet_{\tau(n)} [P] & \circ \mathbf{Diff}^{(+)\tau_{n+1}} \circ \mathbf{Diff}^{(+)\tau_{n+1}}[C_{\tau_{n+1}, \tau_{n+1}}] \\
\mathcal{P}^\bullet_{\tau(n-1)} [P] & \circ \mathbf{Diff}^{(+)\tau_{n+1}+\tau_{n}} \\
\mathcal{P}^\bullet_{\tau(n-1)} [P] & \circ \mathbf{Diff}^{(+)\tau_{n+1}+\tau_{n}}
\end{align*}
\]

where the first two arrows are monomorphisms and the last is the "gluing" morphism with respect to the indexes \( (\tau_{n+1}, \tau_{n+1}) \). This composition is zero. In fact, if \( Q \in \text{Ob}(\mathfrak{D}) \) and \( \Delta \in \mathcal{P}^\bullet_{\tau(n+1)} [P] (Q) \), then the image \( \Delta \) of \( \Delta \) via this composition, is defined by:

\[
(\Delta (p)) (a_1) \cdots (a_{n-1}) = ((\Delta (p)) (a_1) \cdots (a_{n-1})) (1),
\]

and is zero because \( (\Delta (p)) (a_1) \cdots (a_{n-1}) \in \mathbf{Diff}^{(+)\tau_{n+1}}(Q) \), for each \( p \in P, a_1, \ldots, a_{n-1} \in A \).

Now we show that if \( \tau \in \mathbf{N}_+^\infty \) is regular, then, for any object \( P \in \mathfrak{D}, \mathbf{Hol}^\tau [P] \) is acyclic. In order to do this, we will exhibit (functorially in \( P \)) a trivializing homotopy.

Define:

\[
\begin{align*}
\varphi_0 (P) : & \mathcal{P}^\bullet_{\tau(0)} [P] \xrightarrow{0} \mathcal{P}^\bullet_{\tau(0)} [P] = \mathbf{Hom}_A (P, \cdot) \\
\varphi_{\tau(1)} (P) : & \mathcal{P}^\bullet_{\tau(0)} [P] \xrightarrow{\varphi_{\tau(1)}} \mathcal{P}^\bullet_{\tau(0)} [P] = \mathbf{Hom}_A (P, \cdot) \hookrightarrow \mathcal{P}^\bullet_{\tau(1)} [P] \equiv \mathbf{Diff}_A (P, \cdot)
\end{align*}
\]

which are morphisms in \( [\mathfrak{D}, \mathfrak{D}] \); then define, by induction on \( n \),

\[
\varphi^{(n+1)} \left( P \right) = \tilde{\varphi}_{\tau(n+1)} (P) - \tilde{\varphi}_{\tau(n)} (P),
\]

where

\[
\tilde{\varphi}_{\tau(n+1)} (P) : \mathcal{P}^\bullet_{\tau(n)} [P] \simeq \mathcal{P}^\bullet_{\tau(n)} [P] \circ \mathbf{Diff}^{(+)\tau_{n+1}}[\mathbf{Diff}^{(+)\tau_{n+1}}]
\]

\( ^5 \)We recall that \( \mathbf{K}(\mathbf{DIFF}_{A, \mathfrak{D}}) \) denotes the category of complexes of differential operators formed by objects of \( \mathfrak{D} \).

\( ^6 \)We write shortly \( \delta_{\tau(k)} \) instead of \( \delta_{\tau(k)} [P] \), for any \( k \geq 0 \).

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so that, by formula (24),

\[ \hat{\varphi}_{\tau(n)}(P) : \mathcal{P}_{\tau(n)}^\bullet[P] \hookrightarrow \mathcal{P}_{\tau(n-1)}^\bullet[P] \circ \text{Diff}_{\tau_n}^{(+)} \xrightarrow{\varphi_{\tau(n+1)}(P)} \text{Diff}_{\tau_{n+1}}^+(P) \]

\[ \rightarrow \mathcal{P}_{\tau(n)}^\bullet[P] \circ \text{Diff}_{\tau_n}^{(+)} \hookrightarrow \mathcal{P}_{\tau(n)}^\bullet[P] \circ \text{Diff}_{\tau_{n+1}}^+(P). \]

With this definition, \( \varphi_{\tau(n+1)}(P) : \mathcal{P}_{\tau(n)}^\bullet[P] \rightarrow \mathcal{P}_{\tau(n+1)}^\bullet[P] \circ \text{Diff}_{\tau_{n+1}}^+(P) \), but it is easy to resolve the inductive definition in the following one:

\[
\{ [\varphi_{\tau(n+1)}(P)(Q)](\Delta) \} (p)(a_1)\cdots(a_n) = a_n \Delta(p)(a_1)\cdots(a_n-1) + \\
\sum_{k=1}^{n-1} (-1)^{n-k} \Delta(p)(a_1)\cdots(a_k a_{k+1})\cdots(a_n) + \\
+ (-1)^n \Delta(a_1 p)(a_2)\cdots(a_n)
\]

\((Q \in \text{Ob}(\mathfrak{D}), p \in P, a_1, \ldots, a_n \in A \text{ and } \Delta \in \mathcal{P}_{\tau(n)}^\bullet[P](Q)).\) This shows that actually

\[ \varphi_{\tau(n+1)}(P) : \mathcal{P}_{\tau(n)}^\bullet[P] \rightarrow \mathcal{P}_{\tau(n+1)}^\bullet[P]. \]

Therefore, we get a family

\[ \left\{ \varphi_{\tau(n)}(P) : \mathcal{P}_{\tau(n-1)}^\bullet[P] \rightarrow \mathcal{P}_{\tau(n)}^\bullet[P] \right\}_{n>0} \]

of morphisms in \([\mathfrak{D}, \mathfrak{D}].\) Of course, formula (28) can equally be taken as the definition of the family \(\{\varphi_{\tau(n)}(P)\}_{n>0},\) but the inductive definition can be "dualized", to representative objects, to give the following (keeping the notations of Proposition 4.3):

\[ \varphi^0(P) : \text{Hol}^0[0\rightarrow P \rightarrow \text{Hol}^0(-1)[P] \hat{=} 0 \]

\[ \varphi^1(P) : \text{Hol}^1[P] \hat{=} \mathcal{J}_{\tau_1}(P) \rightarrow \text{Hol}^0[P] \hat{=} P \text{ (natural projection)} \]

\[ \varphi^{\tau(n+1)}(P) : \text{Hol}^{\tau(n+1)}[P] \rightarrow \text{Hol}^{\tau(n)}[P] \]

where \(\varphi^{\tau(n+1)}(P)\) is the only \(\mathfrak{D}\)-morphism corresponding to the DO of order \(\leq \tau_{n+1}\)

\[ \Delta_{\tau_n} \hat{=} \text{id}_{\text{Hol}^{\tau(n)}[P]} - \delta_{\tau_n}(P) \circ \varphi^{\tau(n)}(P) : \text{Hol}^{\tau(n)}[P] \rightarrow \text{Hol}^{\tau(n)}[P]. \]

As before, but dually\(^7\), this definition gives apparently a \(\mathfrak{D}\)-morphism:

\[ \hat{\varphi}^{\tau(n+1)}(P) : \mathcal{J}_{\tau_{n+1}}\left(\text{Hol}^{\tau(n)}[P]\right) \rightarrow \text{Hol}^{\tau(n)}[P] \]

(\(\tau\) being regular) but formula (28) shows that actually

\[ \ker\left(\hat{\varphi}^{\tau(n+1)}(P)\right) \supset \text{Hol}^{\tau(n-1),\tau_n+\tau_{n+1}}[P] \]

so that, by formula (24), \(\hat{\varphi}^{\tau(n+1)}(P)\) induces, by passing to the quotient, the morphism \(\varphi^{\tau(n+1)}(P)\) we wanted.

Now we have a family of \(\mathfrak{D}\)-morphisms \(\left\{ \varphi^{\tau(n)}(P) : \text{Hol}^{\tau(n)}[P] \rightarrow \text{Hol}^{\tau(n-1)}[P] \right\}_{n \geq 0}\) dual to

\[ \left\{ \varphi_{\tau(n)}(P) : \mathcal{P}_{\tau(n)}^\bullet[P] \rightarrow \mathcal{P}_{\tau(n-1)}^\bullet[P] \right\}_{n \geq 0}. \]

\(^7\)Subfunctors of strictly representable functors correspond to quotient objects of the representatives.
Proposition 4.6 For each object \( P \) in \( \mathcal{D} \) and for each regular \( \tau \in \mathbb{N}_+^\infty \), \( \{ \varphi^{(n)}(P) \}_{n \geq 0} \) is a trivializing homotopy for \( \text{Hol}^\tau(P) \). Furthermore, \( \{ \varphi^{(n)}(P) \}_{n \geq 0} \) is natural in \( P \).

Proof. We must show that the sum \( L + R \) of the two compositions:

\[
L : P_{\tau(n)}^* [P] \hookrightarrow P_{\tau(n-1)}^* [P] \circ \text{Diff}^+(\varphi^{(n)}(P)(\text{Diff}^+_n)) \\
\rightarrow P_{\tau(n)}^* [P] \circ \text{Diff}^+_n \hookrightarrow P_{\tau(n)}^* [P] \circ \text{Diff}^+_n
\]

\[
R : P_{\tau(n)}^* [P] \xrightarrow{\varphi^{(n+1)}(P)} P_{\tau(n+1)}^* [P] \xrightarrow{\text{Diff}^+_n} P_{\tau(n)}^* [P] \circ \text{Diff}^+_n
\]

equals \( \text{id}_{P_{\tau(n)}^* [P]} \) (which is then homotopic to the zero map) or, equivalently, that the diagram:

\[
\begin{array}{c}
P_{\tau(n)}^* [P] \\
\xrightarrow{L+R} P_{\tau(n)}^* [P] \circ \text{Diff}^+_n \\
\xleftarrow{\text{id}} P_{\tau(n)}^* [P] \circ \text{Diff}^+_n \\
\end{array}
\]

is commutative. For \( Q \in \text{Ob} (\mathcal{D}) \) and \( \Delta \in P_{\tau(n)}^* [P] (Q) \), we have by (28):

\[
L (\Delta) (p) (a_1) \cdots (a_n) =
\]

\[
= \left[ a_{n-1}^+ \Delta (p) (a_1) \cdots (a_{n-2}) + \sum_{s=1}^{n-2} (-1)^{n-1-s} \Delta (p) (a_1) \cdots (a_s a_{s+1}) \cdots (a_{n-1}) + \\
+ (-1)^{n-1} \Delta (a_1 p) (a_2) \cdots (a_{n-1}) \right] (a_n) = - \sum_{s=1}^{n-1} (-1)^{n-s} \Delta (p) (a_1) \cdots (a_s a_{s+1}) \cdots (a_n) + \\
+ (-1)^{n-1} \Delta (a_1 p) (a_2) \cdots (a_n)
\]

while

\[
R (\Delta) (p) (a_1) \cdots (a_n) = a_n \Delta (p) (a_1) \cdots (a_{n-1}) + \\
\sum_{k=1}^{n-1} (-1)^{n-k} \Delta (p) (a_1) \cdots (a_k a_{k+1}) \cdots (a_n) + \\
+ (-1)^n \Delta (a_1 p) (a_2) \cdots (a_n)
\]

so that

\[
(L + R) (\Delta) (p) (a_1) \cdots (a_n) = a_n \Delta (p) (a_1) \cdots (a_{n-1})
\]

i.e. \( (L + R) (\Delta) \) coincides with the image of \( \Delta \) via the inclusion.
\[ P^*_{\tau(n)} [P] (Q) \simeq P^*_{\tau(n)} [P] \circ \text{Diff}^{(+)}_0 (Q) \hookrightarrow P^*_{\tau(n)} [P] \circ \text{Diff}^{(+)}_{\tau(n+1)} (Q). \]

We now use acyclicity of the \( \text{Hol}^1 \)-complex, \( i = (1, \ldots, 1, 1, \ldots, 1, \ldots) \in \mathbb{N}^\infty \), to prove the ”\( k \)-th step” i.e. formula (21). Let \( \sigma = (\sigma_1, \ldots, \sigma_k + 1) \in \mathbb{N}_+^k \), \( (\sigma, 1) = (\sigma_1, \ldots, \sigma_k + 1, 1, 1, \ldots, 1, \ldots) \in \mathbb{N}_+^k \) and
\[ K^{(k)}_{(\sigma, 1)} := \ker \left( dR_{(\sigma, 1)} \to dR_{(\sigma_1, 1, \ldots, \sigma_k + 1, 1, \ldots, 1)} \right). \]

For each \( (\mu)_s = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_+^s \), \( 1 \leq r \leq s, r, s \in \mathbb{N}_+ \), we put:
\[ K^{(r)}_{(\mu)_s} := \ker \left( \Lambda^{(\mu_1, \ldots, \mu_r, \ldots, \mu_s)} \to \Lambda^{(\mu_1, \ldots, \mu_r - 1, \ldots, \mu_s)} \right). \]

To prove the ”\( k \)-th step” it is enough to show acyclicity of \( K^{(k)}_{(\sigma, 1)} \). We claim that there exists a resolution of \( K^{(k)}_{(\sigma, 1)} \) of the form:
\[ \cdots \to \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + l + 1)} \right] [-k - l] \xrightarrow{\psi_l_{[-k-l]}} \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + l)} \right] [-k - l + 1] \to \cdots \]
\[ \cdots \xrightarrow{\psi_2_{[-(k+2)]}} \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 2)} \right] [-k - 1] \xrightarrow{\psi_1_{[-k-1]}} \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)} \right] [-k] \xrightarrow{\rho} K^{(k)}_{(\sigma, 1)} \to 0 \]
where if \( r \in \mathbb{N}_+ \), \( (\cdot) [r] \) denotes, as usual, the \( r \)-shift both for complexes and morphisms of complexes. We postpone in the Appendix the definition of the maps of complexes
\[ \psi_l : \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + l + 1)} \right] \xrightarrow{} \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + l)} \right] [1], \quad l \in \mathbb{N}_+, \]
\[ \rho : \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)} \right] [-k] \xrightarrow{} K^{(k)}_{(\sigma, 1)} \]
and the proof that (28) is actually a resolution.

Assuming the existence of resolution (28), the acyclicity of \( K^{(k)}_{(\sigma, 1)} \) is then an immediate consequence of acyclicity of \( \text{Hol}^1 \)-complexes together with the following elementary fact

**Lemma 4.7** Let \( C', P_i', i > 0 \), be cochain complexes in \( A - \text{Mod} \) and
\[ \cdots \to P_n' \to P_{n-1}' \to \cdots \to P_1' \to C' \to 0 \]
be a resolution of \( C' \). Suppose that \( \forall i \geq 1, P^k_i = (0) \forall k < 0 \) (so that \( C'^k = (0) \forall k < 0 \) too). If each \( P_i' \) is acyclic then so is \( C' \).

**Proof.** It follows from the hypotheses that \( C' \) is isomorphic in the derived category \( D^+ (A - \text{Mod}) \) to the total complex associated to the double complex induced by
\[ \cdots \to P_n' \to P_{n-1}' \to \cdots \to P_1' \]
which is acyclic.

For those readers who feel uncomfortable with derived categories, here is a more "step-by-step" proof. By the usual sign-trick we can associate to the given resolution a 1\textsuperscript{st}-quadrant double complex \( R = (R^p_q) \) which is mixed: homological in the vertical (i.e. with \( p \) fixed) direction and cohomological in the horizontal (i.e. with \( p \) fixed) direction. We turn it into a 2\textsuperscript{nd}-quadrant homological double complex \( \hat{R} . = \left( \hat{R}^p_q \right) \) with \( \hat{R}^p_q \equiv R^p_q \).

Consider the spectral sequence induced by the "filtration by rows" on \( \hat{R} . \) (e.g. [We] p. 142):

\[
\check{H}^0_{pq} = \hat{R}^p_q
\]

with \( \check{H}^0_{pq} \) given by the horizontal differential in \( \hat{R} . \). Then \( \{\check{H}^1_{pq}, \check{d}^1_{pq}\} \) is just:

\[
\begin{array}{c|c|c|c|c|c}
q & n+1 & n & n-1 & \cdots & 0 \\
\hline
q=n+1 & - & H^{n+1}_C & - & 0 & 0 \\
q=n & - & H^n_C & - & 0 & 0 \\
q=2 & - & H^2_C & - & 0 & 0 \\
q=1 & - & H^1_C & - & 0 & 0 \\
q=0 & - & H^0_C & - & 0 & 0 \\
\hline
p & -1 & 0 & 1 & 2 & \cdots & n
\end{array}
\]

(with differential induced by vertical differential in \( \hat{R} . \)). So \( \check{H} \) degenerates at \( \check{H}^1 \). But \( \hat{R} . \) is a 2\textsuperscript{nd}-quadrant double complex hence \( \check{H} \) converges to \( H^* \left( \text{Tot} \left( \hat{R} . \right) \right) \), \( \text{Tot} \left( \hat{R} . \right) \) being the total complex associated to \( \hat{R} . \), which is zero since by hypothesis \( \hat{R} . \) has exact columns. So \( \check{H}^1_{pq} = (0) \ \forall p, q \) and \( C^* \) is acyclic. □

**Corollary 4.8** Let \( \mathcal{D} \subseteq A - \text{Mod} \) be a differentially closed smooth subcategory. If we define the stable infinite de Rham complex to be

\[
dR_{st}^\infty \doteq \varprojlim_{k \geq 0} dR_k
\]

(where \( k \doteq (k, \ldots, k, k, \ldots, k, \ldots) \) ) then the canonical morphism \( dR_{st}^\infty \rightarrow dR_1 \) is a quasi-isomorphism.

**Proof.** We use the following facts:

(i) the index category of the inverse system which defines \( dR_{st}^\infty \) is countable;

(ii) the canonical \( \mathcal{D} \)-morphisms \( dR_k \rightarrow dR_{k'} \), \( k' \geq k \), in the inverse system are epimorphisms.

If we denote by \( \varprojlim_k \) the first right derived functor of \( \varprojlim_k \), (i) and (ii) imply, via standard spectral sequence's arguments (e.g. [Lu] Cor. 1.1, p. 535), that there is a short exact sequence

\[
0 \rightarrow \varprojlim_k H^{n-1}(dR_k) \rightarrow H^n(dR_{st}^\infty) \rightarrow \varprojlim_k H^n(dR_k) \rightarrow 0.
\]

By theorem [4.3], the term on the right is isomorphic to \( H^n_{dR} \), so we are left to prove that \( \varprojlim_k H^{n-1}(dR_k) = (0) \). But \( \varprojlim_k \)'s right exact and

\[
H^{n-1}(dR_k) \equiv H^{n-1}_k \equiv H^{n-1}_{dR} \equiv H^{n-1}_1, \ \forall n \geq 1,
\]

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by theorem 4.2, therefore it will be enough to prove the vanishing of \( \lim_{k>0}^{1} \) for the constant inverse system

\[
\cdots \to H_{dR}^{n-1} \xrightarrow{id} H_{dR}^{n-1} \xrightarrow{id} H_{dR}^{n-1} \to \cdots
\]

But this is an easy consequence of the following description (due to Eilenberg) of \( \lim_{k>0}^{1} \) for constant systems.

If we define

\[
D_0 : \prod_{k \in \mathbb{N}_+} H_{dR}^{n-1} \to \prod_{k \in \mathbb{N}_+} H_{dR}^{n-1} ; (\alpha_k)_{k \in \mathbb{N}_+} \mapsto (\alpha_{k+1} - \alpha_k)_{k \in \mathbb{N}_+} ;
\]

then

\[
\text{co ker } (D_0) = \lim_{k>0}^{1} [H_{dR}^{n-1}] .
\]

Let \((\omega_k)_{k>0} \in \prod_{k \in \mathbb{N}_+} H_{dR}^{n-1}\) and define \((\varpi_k)_{k>0}\) as \(\varpi_k = \sum_{i=1}^{k-1} \omega_i\); then

\[
D_0 ((\varpi_k)_{k>0}) = (\varpi_{k+1} - \varpi_k)_{k>0} = (\omega_k)_{k>0} .
\]

Therefore \(D_0\) is surjective and we conclude. ■

**Corollary 4.9** Let \( \mathcal{D} \subseteq A \text{-Mod} \) be a differentially closed smooth subcategory such that \( \forall \sigma \in \mathbb{N}_\infty^+ , \exists n_\sigma \in \mathbb{N}_+ : \)

\[
\Lambda^{\sigma(r)} = (0) , \forall r > n_\sigma .
\]

Then the canonical morphism (Def. 4.2)

\[
dR_\infty \to dR_\sigma
\]

is a quasi-isomorphism \( \forall \sigma \in \mathbb{N}_\infty^+ \).

**Proof.** Under our hypotheses

\[
\hat{\mathbb{N}}_+ = \bigsqcup_{k \in \mathbb{N}_+} \{ k \in \mathbb{N}_\infty^+ \mid k(n) \equiv (k, \ldots, k) \in \mathbb{N}_+^n, \forall n \in \mathbb{N}_+ \}
\]

is cofinal in the index category of the system \( \{ dR_\sigma \} \); hence \( dR_\infty \simeq \lim_{k>0} dR_k \) and the thesis follows from corollary 4.8. ■

## 5 Appendix

This appendix is devoted to defining the maps in the sequence (28) and to showing that (28) is exact.

There are two kinds of maps of complexes to be defined:

\[
\rho \equiv (\rho^n : \text{Hol}^{1-n-k}_n [K_{(\sigma_1, \ldots, \sigma_k+1)}(k)] \to K_{(\sigma_1, \ldots, \sigma_k+1, 1, \ldots, 1)}^{(k)} n \geq 0)
\]
\[
\psi_l \equiv (\psi_l^p)_{n \geq 0} : \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + l + 1)} \right] \to \text{Hol}^1 \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + l)} \right] [1].
\]

Let us first define \( \rho \). We will define a functorial morphism

\[
\Theta : D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_n \to \mathcal{P}^{\bullet}_{n-k} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)} \right]
\]

and show that the sequence

\[
0 \to D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_n \xrightarrow{\Phi} D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_n \xrightarrow{\Theta} \mathcal{P}^{\bullet}_{n-k} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)} \right]
\]

is exact so that the dual representative of \( \Theta \) will pass to the quotient defining our surjective \( \rho \).

Define \( \Theta \) to be the following composition:

\[
D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_n \cong D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_{n+1} \left( D_{n-k} \subset \text{Diff}^+_{n-k} \right) \cong
\]

\[
\cong \text{Hom}_A \left( \Lambda^{(\sigma_1, \ldots, \sigma_k + 1)}, D_{n-k} \subset \text{Diff}^+_{n-k} \right) \circ d(\sigma_1, \ldots, \sigma_k + 1, 1) \to \text{Diff}_n \left( \Lambda^{(\sigma_1, \ldots, \sigma_k + 1)}, D_{n-k} \subset \text{Diff}^+_{n-k} \right) \cong
\]

\[
\cong \mathcal{P}^{\bullet}_{n-k} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k + 1)} \right] \mathcal{P}^{\bullet}_{n-k} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k + 1)} \right] \mathcal{P}^{\bullet}_{n-k} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)} \right]
\]

where

\[
0 \to K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)} \xrightarrow{j} \Lambda^{(\sigma_1, \ldots, \sigma_k + 1)} \xrightarrow{\pi} \Lambda^{(\sigma_1, \ldots, \sigma_k)} \to 0.
\]

Then, \( \Phi \circ \Theta \) coincides with the following

\[
D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_n \cong D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_{n+1} \left( D_{n-k} \subset \text{Diff}^+_{n-k} \right) \cong
\]

\[
\cong \text{Hom}_A \left( \Lambda^{(\sigma_1, \ldots, \sigma_k)}, D_{n-k} \subset \text{Diff}^+_{n-k} \right) \circ d(\sigma_1, \ldots, \sigma_k - 1) \to \text{Diff}_n \left( \Lambda^{(\sigma_1, \ldots, \sigma_k)}, D_{n-k} \subset \text{Diff}^+_{n-k} \right) \cong
\]

\[
\cong \mathcal{P}^{\bullet}_{n-k} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k)} \right] \mathcal{P}^{\bullet}_{n-k} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k)} \right] \mathcal{P}^{\bullet}_{n-k} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k)} \right];
\]

but \( \pi \circ j = 0 \) hence \( \text{im} (\Phi) \subseteq \ker (\Theta) \). We prove the reverse inclusion.

Let \( P \) be an object in \( \mathfrak{D} \) and \( h \in \text{Hom}_A \left( \Lambda^{(\sigma_1, \ldots, \sigma_k + 1, 1)}, D_{n-k} \subset \text{Diff}^+_{n-k} \left( P \right) \right) \cong D(\sigma_1, \ldots, \sigma_k + 1, 1, \ldots, 1)_n \left( P \right) \) be such that

\[
\Theta (h) = h \circ d(\sigma_1, \ldots, \sigma_k + 1, 1) \circ j = 0;
\]

we claim that \( h \in \text{im} (\Phi) \). Now

\[
h \circ d(\sigma_1, \ldots, \sigma_k + 1, 1) \circ j = h \circ j' \circ d(\sigma_1, \ldots, \sigma_k + 1, 1) \mid K^{(k)}_{(\sigma_1, \ldots, \sigma_k + 1)}
\]
where \[ 0 \to K^{(k)}_{(\sigma_1, \ldots, \sigma_k+1, 1)} \xrightarrow{j'} \Lambda^{(k)}_{(\sigma_1, \ldots, \sigma_k+1, 1)} \xrightarrow{\pi'} \Lambda_{(\sigma_1, \ldots, \sigma_k)} \to 0. \]

But \( h \in \text{im}(\Phi) \) if \( h \circ j' = 0 \) so it is enough to show that \( \text{im} \left( d_{(\sigma_1, \ldots, \sigma_k+1, 1)} |_{K^{(k)}_{(\sigma_1, \ldots, \sigma_k+1)}} \right) \)
generates \( K^{(k)}_{(\sigma_1, \ldots, \sigma_k+1, 1)} \) over \( A \) (since both \( h \) and \( j' \) are \( A \)-homomorphisms). We know that \( \text{im} \left( j_1 : Q \to J^1_{(Q)} \right) \) generates \( J^1_{(Q)} \) over \( A \) for any object \( Q \) in \( \mathcal{D} \) (Section 2). Moreover, the \( 3 \times 3 \) lemma\(^8\) gives us an exact commutative diagram

\[
\begin{array}{ccc}
0 & \to & K^{(k)}_{(\sigma_1, \ldots, \sigma_k+2)} \\
\downarrow & & \downarrow \\
0 & \to & J^1_{(\sigma_1, \ldots, \sigma_k+2)} \\
\downarrow & & \downarrow \\
0 & \to & J^1_{(\sigma_1, \ldots, \sigma_k+1)} \\
\downarrow & & \downarrow \\
0 & \to & J^1_{(\sigma_1, \ldots, \sigma_k)} \\
\end{array}
\]

(where we used the fact that the functor \( J^k \) (\( \cdot \)) is exact if \( \mathcal{D} \) is smooth: this follows from lemma \[ \mathbb{Z} \] since \( J^k \) is projective); but \( d_{(\sigma_1, \ldots, \sigma_k+1, 1)} |_{K^{(k)}_{(\sigma_1, \ldots, \sigma_k+1)}} = t \circ j_1 \) (by definition of \( d \)), hence
\[
\text{im} \left( d_{(\sigma_1, \ldots, \sigma_k+1, 1)} |_{K^{(k)}_{(\sigma_1, \ldots, \sigma_k+1)}} \right)
\]
generates \( K^{(k)}_{(\sigma_1, \ldots, \sigma_k+1, 1)} \) over \( A \) and we have finished.

Now let’s turn ourselves to the definition of
\[
\psi_l \equiv \left( \psi_l^n : \text{Hol}^{n+1}_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \to \text{Hol}^{1+1}_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \right)_{n \geq 0}.
\]

First of all
\[
\psi_l^0 = 0 : \left( \text{Hol}^1_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \right)^0 = (0) \to \left( \text{Hol}^1_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \right)^1 = J^1_{(\sigma_1, \ldots, \sigma_k+l+1)}.
\]

For \( n > 0 \)
\[
\psi_l^n : \text{Hol}^{n+1}_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \to \text{Hol}^{1+1}_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right]
\]
will be defined as the dual representative of a functorial morphism
\[
\psi_l^l : \mathcal{P}^{1+1}_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \to \mathcal{P}^{1+1}_{(\sigma_1, \ldots, \sigma_k+l+1)} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right].
\]

From the exact sequence
\[
0 \to K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \xrightarrow{i} \Lambda_{(\sigma_1, \ldots, \sigma_k+l+1)} \xrightarrow{p} \Lambda_{(\sigma_1, \ldots, \sigma_k+l+1)} \to 0
\]
\[(\text{resp. } 0 \to K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \xrightarrow{i'} \Lambda_{(\sigma_1, \ldots, \sigma_k+l+1)} \xrightarrow{p'} \Lambda_{(\sigma_1, \ldots, \sigma_k+l+1)} \to 0) \]

\(^8\)Smoothness of \( \mathcal{D} \) enters here.
and the fact that $\Lambda^{(\sigma_1, \ldots, \sigma_k, l-1)}$ (resp. $\Lambda^{(\sigma_1, \ldots, \sigma_k, l)}$) is projective we get (Prop. 2.3) an exact sequence of functors $\mathcal{D} \rightarrow \mathcal{D}$

$$0 \rightarrow \mathcal{P}^{\bullet}_{1,n+1} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \xrightarrow{\epsilon} \mathcal{P}^{\bullet}_{1,n+1} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right] \xrightarrow{\eta} \mathcal{P}^{\bullet}_{1,n+1} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k, l)} \right] \rightarrow 0$$

(resp. $0 \rightarrow \mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right] \xrightarrow{\epsilon'} \mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \xrightarrow{\eta'} \mathcal{P}^{\bullet}_{1,n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \rightarrow 0$)

with $\epsilon = \mathcal{P}^{\bullet}_{1,n+1} [p]$ and $\eta = \mathcal{P}^{\bullet}_{1,n+1} [i]$ (resp. $\epsilon' = \mathcal{P}^{\bullet}_{1,n} [p']$ and $\eta' = \mathcal{P}^{\bullet}_{1,n} [i']$). To define $\psi^l_n$ it will be then enough to define

$$\overline{\psi}^l_n : \mathcal{P}^{\bullet}_{1,n+1} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right] \rightarrow \mathcal{P}^{\bullet}_{1,n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k, l+1)} \right]$$

and show that $\overline{\psi}^l_n \circ \epsilon = 0$. We know (Section 2) that there is an exact sequence

$$0 \rightarrow \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \xrightarrow{\alpha} J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right) \xrightarrow{\beta} \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \rightarrow 0$$

and this (Prop. 2.3) gives us the exact sequence

$$0 \rightarrow \mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \xrightarrow{\alpha} \mathcal{P}^{\bullet}_{1,n} \left[ J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right) \right] \xrightarrow{\beta} \mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \rightarrow 0$$

(with $\alpha = \mathcal{P}^{\bullet}_{1,n} [q]$ and $\beta = \mathcal{P}^{\bullet}_{1,n} [s]$). Then we take $\overline{\psi}^l_n$ to be the composition

$$\mathcal{P}^{\bullet}_{1,n+1} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right) \right] \xrightarrow{\beta} \mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right]$$

Now we show that $\overline{\psi}^l_n \circ \epsilon = 0$.

Note that using the identifications

$$\mathcal{P}^{\bullet}_{1,n} \left[ J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l)} \right) \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right) \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k, l+1)} \right]$$

and

$$\mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right) \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k, l+1)} \right]$$

(resp. the identifications

$$\mathcal{P}^{\bullet}_{1,n} \left[ \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k, l+1)} \right) \right] \simeq \mathcal{P}^{\bullet}_{1,n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k, l+1)} \right]$$

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and
\[ \text{Hom}_A^\bullet \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right) \simeq \mathcal{P}^\bullet_{1_n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right] \]

\( \beta \) (resp. \( \eta' \)) is given by
\[
\begin{align*}
\text{Hom}_A^\bullet \left( J^1 \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+1)} \right) \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right) & \xrightarrow{\psi_n^{-1} \circ \epsilon} \\
\text{Hom}_A^\bullet \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right)
\end{align*}
\]

(resp. by
\[
\begin{align*}
\text{Hom}_A^\bullet \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right) & \xrightarrow{\psi_n^{-1} \circ \epsilon} \\
\text{Hom}_A^\bullet \left( J^1 \left( K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right)
\end{align*}
\]

With a similar analysis we see that \( \epsilon \), viewed as a morphism
\[
\begin{align*}
\text{Hom}_A^\bullet \left( \text{Hol}^{(1,1)} \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l-1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right) & \xrightarrow{\epsilon} \\
\text{Hom}_A^\bullet \left( \text{Hol}^{(1,1)} \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right), D_{1_{n}} \subset \text{Diff}^+_{1_{n}} \right)
\end{align*}
\]

is given by taking the composition with \( \left[ J^1 \left( J^1 \left( p \right) \right) \right] \) where
\[
\begin{align*}
\left[ J^1 \left( J^1 \left( p \right) \right) \right] & : J^1 \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l-1)} \right) \right) \xrightarrow{\text{Hol}^{(1,1)} \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l-1)} \right) \rightarrow} \\
& \text{Hol}^{(1,1)} \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \xrightarrow{\left[ J^1 \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \right) \right]}
\end{align*}
\]

is the quotient map of \( J^1 \left( J^1 \left( p \right) \right) : J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l-1)} \right) \rightarrow J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \).

Therefore \( \psi_n^{-1} \circ \epsilon \), viewed as a morphism
\[
\begin{align*}
\text{Hom}_A^\bullet \left( \text{Hol}^{(1,1)} \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l-1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right) & \rightarrow \\
\text{Hom}_A^\bullet \left( J^1 \left( K^{(k)}_{(\sigma_1, \ldots, \sigma_k+l+1)} \right), D_{1_{n-1}} \subset \text{Diff}^+_{1_{n-1}} \right)
\end{align*}
\]

is given by
\[
f \mapsto f \circ \left[ J^1 \left( J^1 \left( p \right) \right) \right] \circ \xi_{(\sigma_1, \ldots, \sigma_k+l)} \circ J^1 \left( s \right) \circ J^1 \left( \iota' \right)
\]

where
\[
\xi_{(\sigma_1, \ldots, \sigma_k+l)} : J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \rightarrow J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right)
\]

is the natural projection. (Recall from Section 4 that \( \delta_{(1,1)} \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \) is given by the composition
\[
J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \xrightarrow{j_1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right)} J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \xrightarrow{\xi_{(\sigma_1, \ldots, \sigma_k+l)}} J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_k+l)} \right) \]
But \([J^1_1 (J^1 (p))] \circ \xi\) coincides with

\[
J^1_1 \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_{k+l})} \right) \right) \xrightarrow{\xi} J^1_1 \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_{k+l-1})} \right) \right) \xrightarrow{\xi} J^1_1 \left( J^1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_{k+l})} \right) \right) \xrightarrow{\xi}
\]

\((\xi(\sigma_1, \ldots, \sigma_{k+l-1})\) being again the natural projection), so that

\[
\overline{\psi}_{n} \circ \epsilon : f \mapsto f \circ \xi(\sigma_1, \ldots, \sigma_{k+l}) \circ J^1_1 (p) \circ J^1_1 (s) \circ J^1_1 \left( i' \right) = \overline{\psi}_{n} \circ \epsilon = J^1_1 \circ \Lambda \left( \sigma_1, \ldots, \sigma_{k+l-1} \right)\]

Again as above, the 3 \times 3 lemma gives us an exact commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l+1})} & \to & J^1_1 \left( K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l})} \right) & \to & K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l+1})} & \to & 0 \\
0 & \to & \Lambda^{(\sigma_1, \ldots, \sigma_{k+l+1})} & \xrightarrow{\delta} & J^1_1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_{k+l})} \right) & \to & \Lambda^{(\sigma_1, \ldots, \sigma_{k+l+1})} & \to & 0 \\
0 & \to & \Lambda^{(\sigma_1, \ldots, \sigma_{k+l})} & \xrightarrow{\delta} & J^1_1 \left( \Lambda^{(\sigma_1, \ldots, \sigma_{k+l-1})} \right) & \to & \Lambda^{(\sigma_1, \ldots, \sigma_{k+l-1})} & \to & 0 \\
0 & \to & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0
\end{array}
\]

which finally shows that \(J^1_1 (p) \circ s \circ i' = 0\) and hence \(\overline{\psi}^t_{n} \circ \epsilon = 0\).

Therefore\[
\overline{\psi}^t_{n} : \mathcal{P}^{*}_{1, n+1} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l})} \right] \to \mathcal{P}^{*}_{1, n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l+1})} \right]
\]
is well defined as well as its dual representative\[
\overline{\psi}^n_{l} : \text{Hol}^{1, n} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l+1})} \right] \to \text{Hol}^{1, n+1} \left[ K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+l})} \right]
\]
as we wanted.

Just as in the case of \(\rho\), an easy application of the 3 \times 3 lemma proves that \(\text{im} \left( \overline{\psi}^t_{n+1} \right) = \ker \left( \overline{\psi}^n_{l} \right) \).

It is easy to verify that \(\rho\) and \(\psi\) so defined are maps of complexes; therefore \([23]\) is a resolution of \(K^{(k)}_{(\sigma_1, \ldots, \sigma_{k+1, l})}\) as desired.

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