NONPARAMETRIC ESTIMATION FOR INTERACTING PARTICLE SYSTEMS: MCKEAN-VLASOV MODELS

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ABSTRACT. We consider a system of $N$ interacting particles, governed by transport and diffusion, that converges in a mean-field limit to the solution of a McKean-Vlasov equation. From the observation of a trajectory of the system over a fixed time horizon, we investigate nonparametric estimation of the solution of the associated nonlinear Fokker-Planck equation, together with the drift term that controls the interactions, in a large population limit $N \to \infty$. We build data-driven kernel estimators and establish oracle inequalities, following Lepski’s principle. Our results are based on a new Bernstein concentration inequality in McKean-Vlasov models for the empirical measure around its mean, possibly of independent interest. We obtain adaptive estimators over anisotropic Hölder smoothness classes built upon the solution map of the Fokker-Planck equation, and prove their optimality in a minimax sense. In the specific case of the Vlasov model, we derive an estimator of the interaction potential and establish its consistency.

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1. INTRODUCTION

1.1. Setting. We continuously observe a stochastic system of $N$ interacting particles evolving in an Euclidean ambient space $\mathbb{R}^d$, that solves

$$\begin{align*}
&\frac{dX_t^i}{dt} = b(t, X_t^i, \mu_t^N)dt + \sigma(t, X_t^i)dB_t^i, \quad 1 \leq i \leq N, \quad t \in [0, T], \\
&\mathcal{L}(X_0^1, \ldots, X_0^N) = \mu_0^\otimes N,
\end{align*}$$

where $\mu_t^N = N^{-1} \sum_{i=1}^{N} \delta_{X_t^i}$ is the empirical measure of the system. The $B^i$ are independent $\mathbb{R}^d$-valued Brownian motions, and the transport and diffusion coefficients $b$ and $\sigma$ are sufficiently regular so that $\mu_t^N \rightarrow \mu_t$ weakly as $N \rightarrow \infty$, where $\mu_t$ is a weak solution of the parabolic nonlinear equation

$$\begin{align*}
&\partial_t \mu_t + \text{div}(b(t, \cdot, \mu_t)\mu_t) = \frac{1}{2} \sum_{k,k'=1}^{d} \partial^2_{kk'}((\sigma \sigma^\top)(t, \cdot)_{kk'}\mu_t), \\
&\mu_{t=0} = \mu_0,
\end{align*}$$

see Section 2 below. In this context, we are interested in estimating nonparametrically from data (1) the solution $(t, x) \mapsto \mu_t(x)$ of (3) and the drift function $(t, x, \mu) \mapsto b(t, x, \mu) \in \mathbb{R}^d$ at the value $(t, x, \mu) = (t, x, \mu_t)$. The time horizon $T$ is fixed and asymptotics are taken as $N \rightarrow \infty$.

A particular case of interest is a homogeneous drift with a linear dependence in the measure argument. The drift term in (2) then takes the form

$$\begin{align*}
b(t, X_t^i, \mu_t^N) &= \int_{\mathbb{R}^d} \tilde{b}(X_t^i, y)\mu_t^N(dy) = N^{-1} \sum_{j=1}^{N} \tilde{b}(X_t^i, X_t^j),
\end{align*}$$

for some function $\tilde{b} : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. In the paper, when we specialise on this case, we consider $\tilde{b}$ of the form

$$\tilde{b}(x, y) = F(x - y) + G(x)$$
for some regular $F,G : \mathbb{R}^d \to \mathbb{R}^d$. In this case, we have
\[
b(t, X_i^t, \mu_t^N) = F \ast \mu_t^N(X_i^t) + G(X_i^t),
\]
where $\ast$ denotes convolution. The function $F$ plays the role of an interaction force applied to the particle system while $G$ accounts for an external force in the motion of each particle. If the forces $F$ and $G$ derive from smooth potentials $V, W : \mathbb{R}^d \to \mathbb{R}$, the interaction then takes the form $F(x) = -\nabla W(x)$ and we have a confinement $G(x) = -\nabla V(x)$, see e.g. [5, 34, 17]. However, we work on a fixed time horizon $[0, T]$ in the paper, and will not need this point of view\(^1\).

In the semi-linear representation (4), we are interested in estimating nonparametrically from data (1) the interaction force $x \mapsto F(x)$; the parameter $x \mapsto G(x)$ is considered as a nuisance.

1.2. Motivation. Stochastic systems of interacting particles and associated nonlinear Markov processes in the sense of McKean [53] date back to the 1960’s and originated from statistical physics in plasma physics. Their importance in probability theory progressively grew in the following decades, and a versatility of fundamental tools were developed in this context like e.g. coupling methods, geometric inequalities, propagation of chaos, concentration and fluctuations in abstract functional spaces, see Sznitman [63, 12], Tanaka and Hitsuda [64], Fernandez et al. [24], Méléard [54], Malrieu [50], Cattiaux et al. [17], Bolley et al. [8], among a myriad of references. However, until the early 2000’s, a modern formulation of a statistical inference program in this context was out of reach (with some notable exceptions like e.g. Kasonga [41]), at least for two reasons: first, the fine probabilistic tools required for nonparametric adaptive estimation were still in full development; second and perhaps more importantly, microscopic particles systems issued from statistical physics are not naturally observable and the motivation for statistical inference is not obvious in this context. The situation progressively evolved around the 2010’s, with the start of a kind of renaissance of McKean-Vlasov type models in several application fields that model collective and observable dynamics, ranging from mathematical biology (neurosciences, Baladron et al. [3], structured models in population dynamics, Mogilner et al. [55], Burger et al. [11]) to social sciences (opinion dynamics, Chazelle et al. [19], cooperative behaviours, Canuto et al. [13]) and finance (systemic risk, Fouque and Sun [25]). More recently, mean-field games (Cardalliaguet et al. [14], Cardalliaguet and Lehalle [15]) appear as a new frontier for statistical developments, see in particular the recent contribution of Giesecke [30]. The field has reached enough maturity for the necessity and interest of a systematic statistical inference program, starting with nonparametric estimation. This is the topic of the paper.

Parallel to understanding collective dynamics from a statistical point of view, some interest in the study of statistical models related to PDE’s has progressively emerged over the last decade. Typical examples include nonparametric Bayes and uncertainty quantification for inverse problems, see Abraham and Nickl [1], Monard et al. [56], Nickl [39, 58], and the references therein, or inference in structured models from microscopic data (Doumic et al. [22, 23], Hoffmann and Olivier [36], Boumezoued et al. [9], Ngoc et al. [35], Maïda et al. [49]). The analysis of elliptic, parabolic or transport-fragmentation equations sheds new light on the underlying nonparametric structure of companion statistical experiments and enrich the classical theory. To that extent, we provide in this paper a first step in that direction for a certain kind of nonlinear parabolic equations, in the sense of McKean [53]. Finally, our work can also be embedded in the framework of functional data analysis, where we observe $N$ diffusion processes with common dynamics, see

\[^1\] usually required to control the model for convergence to equilibrium when $T$ is large.
1.3. Results and organisation of the paper. In Section 2, we detail the notations and assumptions of the model and build kernel estimators for $\mu_t(x)$ and $b(t, x, \mu_t)$. Whereas the estimation for $\mu_t$ is standard, the estimation of the drift requires a smoothing in both time and space of the empirical measure $\pi^N(dt, dx) = N^{-1} \sum_{i=1}^N \delta_{X_i}(dx)X_i(dt)$, that estimates the intermediate function $\pi(t, x) = b(t, x, \mu_t)\mu_t(x)$. We then use a quotient estimator to recover $b$.

In Section 3, we adopt the Goldenshluger-Lepski method [31, 32, 33] to tune the bandwidths of both estimators in a data driven way and obtain oracle inequalities in Theorems 7 and 9 for both $\mu$ and $b$. We further develop a minimax theory in Section 4 when $\mu$ and $b$ belong to anisotropic Hölder spaces in time and space variable, that are built upon the solution of the parabolic nonlinear limiting equation (3) and prove the optimality and smoothness adaptivity or our estimators in Theorems 14 and 15. We finally explore in Section 5 the identification of the interaction force $F$ in the Vlasov model where the drift takes the form $b(x, \mu) = F(x)\mu + G(x)$, for some sufficiently well localised functions $F$ and $G$. We prove in Theorem 17 that one can consistently estimate $F$ (hence $G$) by means of a Fourier type estimator, inspired by blind deconvolution methods, see e.g. Johannes [37].

We develop the probabilistic tools we need to undertake our statistical estimates in Section 6. We study the fluctuations of $\mu^N_t$ around its mean $\mu_t$ in Theorem 18, with time dependent extension to the fluctuations of $\mu^N_t(dx)\rho(dt)$ around $\nu(dt, dx) = \mu(dx)\rho(dt)$ for arbitrary weight measures $\rho(dt)$. We prove a Bernstein concentration inequality that reads

$$
\text{Prob}
\left(N^{-1} \sum_{i=1}^N \int_0^T \phi(t, X_i^t)\rho(dt) - \int_{[0,T] \times \mathbb{R}^d} \phi(t, y)\mu_t(dy)\rho(dt) \geq x\right)
\leq \kappa_1 \exp\left(-\frac{\kappa_2 Nx^2}{|\phi|_{L^2(\nu)}^2 + |\phi|_{L^\infty}^2}\right), \quad \forall x \geq 0,
$$

over test functions $\phi$ and for some $\kappa_i = \kappa_i(T, \sigma, b, \mu_0) > 0$. It improves on variance estimates based on coupling or geometric inequalities that usually need $\phi$ to be 1-Lipschitz, whereas nonparametric statistical estimation requires $\phi = \phi_N$ to mimic a Dirac mass as $N \to \infty$ that can be controlled in $L^2$-norm but behaves badly in Lipschitz norm. Bernstein’s inequality for a range of deviation valid for all $x \geq 0$ is also the gateway to nonparametric adaptive estimation; it is not provided by concentration in Wasserstein distance like in Bolley et al. [8] that moreover has the drawback of adding an additional unavoidable dimensional penalty in the rates of convergence.

Our method of proof avoids coupling by relying on a Girsanov argument, following classical ideas, recently revisited for instance by Lacker [44], a key reference for our work. We classically require strong ellipticity for the diffusion coefficient and Lipschitz continuity in the space variable. As for the drift, we assume at least Lipschitz continuity

$$
|b(t, x, \mu) - b(t, x', \mu')| \leq C(|x - x'| + W_1(\mu, \mu'))
$$

in Wasserstein-1 metric $W_1$. In our approximation argument, the logarithm of the Girsanov density between the law of the data and a companion coupled system of independent particles is of order $N \int_0^T \int_{\mathbb{R}^d} |b(s, x, \mu^N_s) - b(s, x, \mu_s)|^2 \mu^N_s(dx)ds \lesssim N \sup_{0 \leq t \leq T} W_1(\mu^N_t, \mu_t)^2$ for which we need
sharp integrability properties uniformly in $N$. In order to circumvent the unavoidable dimensional effect of the approximation $W_1(\mu_N^t, \mu_t) \approx N^{-1/\max(2,d)}$, see Fournier and Guillin [26], we assume moreover $k$-linear differentiability for $b$ in the measure variable, see Assumption 4 in Section 2. This enables us to control by a sub-Gaussianity argument for $U$-statistics each term of a Taylor-like expansion of $b(t,x, \mu_N^t) - b(t,x, \mu_t)$ in the measure variable; we obtain the desired result, provided $k \geq d/2$ (with a special modification in dimension $d = 2$). In particular, in the Vlasov model, we formally have $k = \infty$ and the result is valid in all dimension $d \geq 1$.

Section 6 and 7 are devoted to the proofs and an Appendix (Section 8) contains auxiliary technical results.

2. **Model assumptions and construction of estimators**

2.1. **The system of interacting particles and its limit.** We fix an integer $d \geq 1$ and a time horizon $T > 0$. The random processes take their values in $\mathbb{R}^d$. We write $| \cdot |$ for the Euclidean distance (on $\mathbb{R}$ or $\mathbb{R}^d$) or sometimes for the modulus of a complex number, and $x^\top x = |x|^2$.

**Functions.** We consider functions that are mappings defined on products of metric spaces (typically $[0,T] \times \mathbb{R}^d \times \mathcal{P}_1$ or subsets of these) with values in $\mathbb{R}$ or $\mathbb{R}^d$. Here, $\mathcal{P}_1$ denotes the set of probability measures on $\mathbb{R}^d$ with a first moment, endowed with the Wasserstein 1-metric

$$W_1(\mu, \nu) = \inf_{m \in \Gamma(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| m(dx, dy) = \sup_{|\phi|_{Lip} \leq 1} \int_{\mathbb{R}^d} \phi d(\mu - \nu),$$

where $\Gamma(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mu$ and $\nu$. All the functions in the paper are implicitly measurable with respect to the Borel-sigma field induced by the product topology. A $\mathbb{R}^d$-valued function $f$ is written componentwise as $f = (f^k)_{1 \leq k \leq d}$ where the $f^k$ are real-valued. The product $f \otimes g$ of two $\mathbb{R}^d$-valued functions is the $\mathbb{R}^{2d}$-valued function with $\mathbb{R}^{2d}$ variables with components $(f \otimes g)^k(x, y) = f^k(x)g^k(y)$. If $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we set $|f|_\infty = \sup_{t,x} |f(t,x)|$ and $|f|_p = (\int_{[0,T] \times \mathbb{R}^d} |f(t,x)|^p dx dt)^{1/p}$ for $1 \leq p < \infty$. Depending on the context, if $f : [0,T] \to \mathbb{R}$ or $f : \mathbb{R}^d \to \mathbb{R}$ is a function of time or space only, we sometimes write $|f|_p$ for $(\int_0^T |f(t)|^p dt)^{1/p}$ or $(\int_{\mathbb{R}^d} |f(x)|^p dx)^{1/p}$ when no confusion is possible.

**Constants.** We repeatedly use positive quantities $\kappa_i, \varpi_i, C_i, i = 1, 2, \ldots$ that do not depend on $N$, that we call constants, but that actually may (continuously) depend on model parameters. In most cases, they are explicitly computable. We also use special letters like $\kappa, \delta, \tau, c_\pm, \ldots$ but they will only appear once. The generic notation $C$ is sometimes used before it is set depending on a model parameter. The notation $\varpi_i$ stands for quantities that need to be tuned in an algorithm (like an estimator).

**Assumptions.** We work under strong ellipticity and Lipschitz smoothness assumptions on the diffusion matrix $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and the drift $b : [0,T] \times \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d$, as well as strong integrability properties for the initial condition $\mu_0$.

**Assumption 1.** For some $\gamma_0 > 0, \gamma_1 \geq 1$, the initial condition $\mu_0$ satisfies

$$\int_{\mathbb{R}^d} \exp(\gamma_0|x|^2)\mu_0(dx) \leq \gamma_1.$$
Assumption 2. The diffusion matrix $\sigma$ is measurable and for some $C \geq 0$, we have
\[ |\sigma(t, x') - \sigma(t, x)| \leq C|x' - x|. \]
Moreover, $c = \sigma \sigma^\top$ is such that $\sigma_n^2 |y|^2 \leq (c(t, x)y)^\top y \leq \sigma_s^2 |y|^2$ for some $\sigma_s > 0$.

As for the regularity of the drift
\[ b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d, \]
the notion of linear differentiability, commonly used in the literature of mean-field games and McKean-Vlasov equations in order to quantify the smoothness of $\mu \mapsto b(t, x, \mu)$ as a mapping $\mathcal{P}_1 \to \mathbb{R}^d$ will be the most useful in our setting. We refer in particular to the illuminating section 2.2. in Jourdain and Tse [39] and the references therein.

Definition 3. A mapping $f : \mathcal{P}_1 \to \mathbb{R}^d$ is said to have a linear functional derivative, if there exists $\delta f : \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d$ (sometimes denoted by $\frac{\delta f}{\delta \mu}$) such that
\[ f(\mu') - f(\mu) = \int_0^1 \int_{\mathbb{R}^d} \delta f(y, (1 - \vartheta)\mu + \vartheta \mu')(\mu' - \mu) (dy) d\vartheta \]
with the following smoothness properties
\[ |\delta f(y', \mu') - \delta f(y, \mu)| \leq C(W_1(\mu', \mu) + |y' - y|), \]
\[ |\partial_y (\delta f(y, \mu')) - \delta f(y, \mu)| \leq CW_1(\mu', \mu) \]
for some $C \geq 0$.

We can iterate the process described in (6) and obtain a notion of $k$-linear functional derivative via the existence of mappings
\[ \delta^k f : (\mathbb{R}^d)^k \times \mathcal{P}_1 \to \mathbb{R}^d \text{ for } k = 1, \ldots, k \]
defined recursively by $\delta^0 f = \delta f$ and enjoying associated smoothness properties.

Assumption 4. The drift $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d$ is measurable and
\[ b_0 = \sup_{t \in [0, T]} |b(t, 0, \delta_0)| < \infty. \]
Moreover, one of the following three conditions is satisfied for some $C > 0$:

(i) (Lipschitz continuity.) We have $d = 1$ and
\[ |b(t, x', \mu') - b(t, x, \mu)| \leq C(|x' - x| + W_1(\mu', \mu)). \]

(ii) (Existence of a functional derivative of order $k$.) Let $k \geq 1$. For $(d = 1$ and $k \geq 1$ or $(d = 2$ and $k \geq 2$) or $(d \geq 3$ and $k \geq d/2$), we have (i) and the map
\[ \mu \mapsto b(t, x, \mu) \]
admits a functional derivative of order $k$ in the sense of Definition 3. Moreover, the following representation holds
\[ \delta^k b(t, x, (y_1, \ldots, y_k), \mu) = \sum_{j \in \{1, \ldots, k\}, m \geq 1} \bigotimes_{j \in j} (\delta^k b)_{j, j, m}(t, x, y_j, \mu), \]
where the sum in $m$ is finite with at most $m_k$ terms and the mappings $(\delta^k b)_{j, j, m} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}_1 \to \mathbb{R}^d$ are such that
\[ |(\delta^k b)_{j, j, m}(t, x', y', \mu) - (\delta^k b)_{j, j, m}(t, x, y, \mu)| \leq C(|x' - x| + |y' - y|). \]
(iii) (Vlasov case.) We have $d \geq 1$ and
\begin{equation}
 b(t, x, \mu) = \int_{\mathbb{R}^d} \tilde{b}(t, x, y) \mu(dy)
\end{equation}
for some measurable $\tilde{b} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that:
\begin{equation}
 |\tilde{b}(t, x', y') - \tilde{b}(t, x, y)| \leq C(|x' - x| + |y' - y|).
\end{equation}

We let $|b|_{\text{Lip}}$ denote the smallest $C \geq 0$ for which Assumption 4 (i) holds and $|\delta_{\mu} b|_{\text{Lip}}$ the smallest constant $C \geq 0$ for which Assumption 4 (ii) holds for the highest order of differentiability.

**Remark 5.** 1) The representation (7) in Assumption 4 (ii) is merely technical and enables one to obtain a control in $W_1$-distance of the remainder term in Taylor-like expansions of $b(t, x, \mu)$ in an easy way, see in particular the proof of Proposition 19, Step 2 below. It can presumably be relaxed, but will be sufficient for the level of generality intended in the paper. It accommodates in particular drifts of the form
\begin{equation}
 b(t, x, \mu) = \sum_j F_j(t, x, \int_{\mathbb{R}^d} G_j(t, x, \int_{\mathbb{R}^d} H_j(t, x, z) \mu(dz), z') \lambda_j(dz'))
\end{equation}
for smooth mappings $F_j(t, x, \cdot) : \mathbb{R}^{q_2} \to \mathbb{R}^d, G_j(t, x, \cdot) : \mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \to \mathbb{R}^{q_3}, H_j(t, x, \cdot) : \mathbb{R}^d \to \mathbb{R}^{q_4}$ and positive measures $\lambda_j$ on $\mathbb{R}^d$ in some cases and combinations of these, see Jourdain and Tse [39]. Explicit examples of mean-field models where the structure of the drift is of the form (ii) rather than (i) or (iii) are given for instance in [20, 60, 54, 38]. 2) Condition (iii) is stronger than (ii): under Assumption 4 (iii),
\begin{equation}
 \int_{\mathbb{R}^d} \tilde{b}(t, x, \cdot) d(\mu' - \mu) \leq C \sup_{|\phi|_{\text{Lip}} \leq 1} \int_{\mathbb{R}^d} \phi d(\mu' - \mu) = CW_1(\mu', \mu),
\end{equation}
thus $|b(t, x', \mu') - b(t, x, \mu)| = |\int_{\mathbb{R}^d} (\tilde{b}(t, x', \cdot) - \tilde{b}(t, x, \cdot)) d\mu' - \int_{\mathbb{R}^d} \tilde{b}(t, x, \cdot) d(\mu - \mu')| \leq C(|x' - x| + W_1(\mu', \mu))$ and Assumption 4 (i) holds true. Moreover $\delta_{\mu} b(t, x, \mu) = b(t, x, y)$ and Assumption 4 (ii) holds true as well.

We let $C = C([0, T], (\mathbb{R}^d)^N)$ denote the space of continuous functions on $(\mathbb{R}^d)^N$, equipped with the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ induced by our observation, namely the canonical mappings
\begin{equation}
 X_t(\omega) = (X^1_t(\omega), \ldots, X^N_t(\omega)) = \omega_t
\end{equation}
and modified to be right-continuous for safety. For $\mu_0 \in \mathbb{P}_1$, the probability $\mathbb{P}^N$ on $(C, \mathcal{F}_T)$ under which the canonical process $X = (X^1_t, \ldots, X^N_t)_{0 \leq t \leq 1}$ is a weak solution of (2) for the initial condition $\mu_0^\otimes N$ is uniquely defined under Assumptions 1, 2 and 4. Recommended reference (that covers our set of assumptions) is the textbook by Carmona and Delarue [16] or the lectures notes of Lacker [43]).

**2.2. Kernel estimators.** We pick two bounded and compactly supported kernel functions $H : (0, T) \to \mathbb{R}$ and $K : \mathbb{R}^d \to \mathbb{R}$ such that
\begin{equation}
 \int_0^T H(s) ds = \int_{\mathbb{R}^d} K(y) dy = 1.
\end{equation}
Let $\ell \geq 1$ be an integer. We say that the kernels $H$ or $K$ have order $\ell$ if, for $k = 0, \ldots, \ell - 1$, we have
\begin{equation}
 \int_0^T s^k H(s) ds = \int_{\mathbb{R}^d} (y^1)^k K(y) dy^1 = \ldots = \int_{\mathbb{R}^d} (y^d)^k K(y) dy^d = 1_{\{k=0\}},
\end{equation}
with $y = (y^1, \ldots, y^d)$. 
Construction of an estimator of $\mu_t(x) \in \mathbb{R}$. Let $(t_0, x_0) \in (0, T] \times \mathbb{R}^d$. For $h > 0$ we obtain a family of estimators of $\mu_{t_0}(x_0)$ by setting

$$(9) \quad \hat{\mu}_h^N(t_0, x_0) = \int_{\mathbb{R}^d} K_h(x_0 - x) \mu_{t_0}^N(dx),$$

with $K_h(x) = h^{-d}K(h^{-1}x)$.

Construction of an estimator of $b(t, x, \mu_t) \in \mathbb{R}$. Let $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$. Abusing notation slightly, define the $\mathbb{R}^d$-valued random measure

$$\pi^N(dt, dx) = N^{-1} \sum_{i=1}^N X_i(t) \delta_{X_i}(dx),$$

defined by

$$\int_{[0,T] \times \mathbb{R}^d} \phi(t, x) \pi^N(dt, dx) = N^{-1} \sum_{i=1}^N \int_0^T \phi(t, X_i^i) dX_i^i$$

for a test function $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$. Letting

$$(H \otimes K)h(t, x) = H_{h_i}(t)K_{h_2}(x) \quad \text{for} \quad h = (h_1, h_2), h_i > 0,$$

with $H_{h_i}(t) = h^{-1}H(h^{-1}t)$, we obtain a family of estimators of $\pi(t_0, x_0) = b(t_0, x_0, \mu_{t_0})\mu_{t_0}(x_0)$ by setting

$$(10) \quad \hat{\pi}_h^N(t_0, x_0) = \int_{[0,T] \times \mathbb{R}^d} (H \otimes K)h(t_0 - t, x_0 - x) \pi^N(dt, dx)$$

$$= \left( \int_{[0,T] \times \mathbb{R}^d} (H \otimes K)h(t_0 - t, x_0 - x) (\pi^N)^h(dt, dx) \right)_{1 \leq k \leq d},$$

i.e. by smoothing componentwise $\pi^N(dt, dx)$. We finally estimate $b(t_0, x_0, \mu_{t_0})$ by

$$\hat{b}_{h, h}^N(t_0, x_0) = \frac{\hat{\pi}_h^N(t_0, x_0)}{\mu_h^N(t_0, x_0)} \in \mathbb{R}^d, \quad h = (h_1, h_2),$$

with $h, h_i > 0$ and some threshold $\varepsilon > 0$ that prevents the estimator to blow-up for small values of $\hat{\mu}_h^N(t_0, x_0)$.

3. Nonparametric Oracle Estimation

Our results involve positive quantities that continuously depend on real-valued parameters of the problem, namely

$$b = (\gamma_0, \gamma_1, b_0, |b|_{\text{Lip}}, |\delta b|_{\text{Lip}}, m_b, \sigma, T, d),$$

defined in Assumptions 1, 2 and 4, together with the dimension $d \geq 1$ of the ambient space and the value of the terminal time $T > 0$. In the following, the notation $A_N \lesssim B_N$ means the existence of $C > 0$ (possibly depending on $b$ but not $N$) such that $A_N \leq CB_N$ for every $N \geq 1$. 
3.1. Oracle estimation of \( \mu_t(x) \). We fix \((t_0, x_0) \in (0, T) \times \mathbb{R}^d \) and implement a variant of the Goldenshluger-Lepski’s algorithm [31, 32, 33] for pointwise estimation. Pick a discrete set
\[
\mathcal{H}^N_1 \subset \left\{ N^{-1/d} (\log N)^{2/d}, 1 \right\},
\]
of admissible bandwidths such that \( \text{Card}(\mathcal{H}^N_1) \leq N \). The algorithm, based on Lepski’s principle, requires the family of estimators
\[
\left( \hat{\mu}^N_h(t_0, x_0), h \in \mathcal{H}^N_1 \right)
\]
defined in \( (9) \) and selects an appropriate bandwidth \( \hat{h}^N \) from data \( \mu^N_0(dx) \). Writing \( \{x\}_+ = \max(x, 0) \), define
\[
A^N_h = \max_{h' \leq h, h' \in \mathcal{H}^N_1} \left\{ \left( \hat{\mu}^N_h(t_0, x_0) - \hat{\mu}^N_{h'}(t_0, x_0) \right)^2 - (V^N_h + V^N_{h'}) \right\}_+,
\]
where
\[
V^N_h = \varpi_1 |K|^2_2 (\log N)^{1/2} N^{-1} h^{-d}, \quad \varpi_1 > 0.
\]
Let
\[
\hat{h}^N \in \arg\min_{h \in \mathcal{H}^N_1} (A^N_h + V^N_h).
\]
The data driven Goldenshluger-Lepski estimator of \( \mu_{t_0}(x_0) \) defined by
\[
\hat{\mu}^N_{\text{GL}}(t_0, x_0) = \hat{\mu}^N_{\hat{h}^N}(t_0, x_0)
\]
is specified by \( K \) and \( \varpi_1 \).

**Remark 6.** The choice of the penalty \( A^N_h \) and the threshold \( V^N_h \) in \( (11) \) and \( (12) \) are standard in the GL methodology: \( A^N_h \) is a kind of proxy for the estimation of the variance of \( \hat{\mu}^N_h(t_0, x_0) \) while \( V^N_h \) is the exact penalty needed in order to balance the size of the variance of the estimator in \( h \), of order \( |K|^2_2 N^{-1} h^{-d} \), inflated by a logarithmic term \( \log N \) and tuned with \( \varpi_1 > 0 \). This enables one to control all the stochastic deviation terms. See in particular the proof of Theorem 7. We also refer to the original sources in Goldenshluger and Lepski [31, 32, 33].

**Oracle estimate.** We need some notation. Given a kernel \( K \), the bias at scale \( h > 0 \) of \( \mu \) at point \((t_0, x_0)\) is defined as
\[
\mathcal{B}^N_h(\mu)(t_0, x_0) = \sup_{h' \leq h, h' \in \mathcal{H}^N_1} \left| \int_{\mathbb{R}^d} K_{h'}(x_0 - x) \mu_{t_0}(x) dx - \mu_{t_0}(x_0) \right|.
\]
We are ready to give the performance of our estimator of \( \mu_t(x) \), by means of an oracle inequality.

**Theorem 7.** Work under Assumptions 1, 2 and 4. Let \((t_0, x_0) \in (0, T) \times \mathbb{R}^d \). The following oracle inequality holds true:
\[
\mathbb{E}^N \left[ \left( \hat{\mu}^N_{\text{GL}}(t_0, x_0) - \mu_{t_0}(x_0) \right)^2 \right] \lesssim \min_{h \in \mathcal{H}^N_1} \left( A^N_h(\mu)(t_0, x_0)^2 + V^N_h \right),
\]
for large enough \( N \), up to a constant depending on \((t_0, x_0), |K|_{\infty} \) and \( \mathfrak{b} \), provided \( \hat{\mu}^N_{\text{GL}}(t_0, x_0) \) is calibrated with \( \varpi_1 \geq 16 \kappa_2^{-1} \kappa_3 \), where \( \kappa_2 \) is specified in Theorem 18 and \( \kappa_3 \) is a (local) upper bound of \( \mu_{t_0} \), see Lemma 23 below.
Some remarks are in order: 1) Up to an inessential logarithmic factor, our estimator achieves the optimal bias-variance tradeoff among every possible bandwidth $h \in \mathcal{H}_N^1$. 2) The requirement $N h^d \geq (\log N)^{1+\epsilon}$ for $h \in \mathcal{H}_N^1$ could be tightened to $N h^d \geq (\log N)^{1+\epsilon}$ for an arbitrary $\epsilon > 0$; this is slightly more stringent than the usual bound $N h^d \geq \log N$ in the literature [31, 32, 33], but this has no consequence for the subsequent minimax results. 3) The choice of a pointwise loss function at $(t_0, x_0)$ is inessential here: other integrated norms like $\| \cdot \|_p$ would work as well, following the general strategies of Lepski’s principle. 4) The construction of the estimator of $\mu_{h_0}(x_0)$ requires a lower bound on $\omega_1$ that has to be set prior to the data analysis. The bound we obtain are presumably too large. In practice, $\omega_1$ has to be tuned by other methods, possibly using data. Such approaches in the context of Lepski’s methods have been recently introduced by Lacour et al. [45]. This weakness is common to all nonparametric methods that depend on a data-driven bandwidth.

3.2. Oracle estimation of $b(t, x, \mu_t)$. Similarly to the estimation of $\mu_t(x)$, we pick a discrete set

$$\mathcal{H}_2^N \subset \left[ N^{-1/(d+1)} (\log N)^{2/(d+1)}, (\log N)^{-2} \right] \times \left[ N^{-1/(d+1)} (\log N)^{2/(d+1)}, 1 \right],$$

with cardinality $\text{Card} \mathcal{H}_2^N \leq N$. We assume that $\mathcal{H}_2^N$ is equipped with some ordering $\preceq$ such that for every $h, h' \in \mathcal{H}_2^N$, we have either $h \preceq h'$ or $h' \preceq h$. The construction uses $\hat{\mu}_{GL}^N(t_0, x_0)$, given in addition the family of estimators

$$(\hat{\pi}_N^N(t_0, x_0), h \in \mathcal{H}_2^N)$$

defined in (10) and constructed with the kernel $H \otimes K$. Define

$$A_h^N = \max_{h' \preceq h, h' \in \mathcal{H}_2^N} \left\{ \| \hat{\pi}_h^N(t_0, x_0) - \hat{\pi}_{h'}^N(t_0, x_0) \|^2 - (V_h^N + V_{h'}^N) \right\}^+,\tag{16}$$

where

$$V_h^N = \omega_2 |H \otimes K|^3 (\log N) N^{-1} h_1^{-1} h_2^{-d}, \quad \omega_2 > 0.\tag{17}$$

Let

$$\hat{h}_N^N \in \text{argmin}_{h \in \mathcal{H}_2^N} (A_h^N + V_h^N).$$

The data-driven Goldenshluger-Lepski estimator of $b(t_0, x_0, \mu_{h_0})$ is defined as

$$\hat{b}_{GL}^N(t_0, x_0) = \hat{\mu}_{h_N^N}(t_0, x_0) \omega_3 = \frac{\hat{\pi}_{h_N^N}(t_0, x_0)}{\mu_{h_N^N}(t_0, x_0)} \vee \omega_3.\tag{18}$$

and is specified by $H, K, \omega_1, \omega_2$ and the threshold $\omega_3 > 0$ that prevents the estimator to blow-up for small values of $\hat{\mu}_{h_N^N}(t_0, x_0)$.

**Remark 8.** The same comments as in Remark 6 apply here for the specification of $A_h^N$ in (16) and $V_h^N$ in (17), noting that in the anisotropic case, the variance of $\hat{\pi}_h^N(t_0, x_0)$ is now of order $|H \otimes K|^3 (\log N) N^{-1} h_1^{-1} h_2^{-d}$.

**Oracle estimates.** Given a kernel $H \otimes K$, the bias at scale $h$ of $\pi = b \mu$ at point $(t_0, x_0)$ is defined as

$$\mathcal{B}_h^N(\pi)(t_0, x_0) = \sup_{h' \preceq h, h' \in \mathcal{H}_2^N} \left| \int_{[0, \tau] \times \mathbb{R}^d} (H \otimes K)_{h'}(t_0 - t, x_0 - x) \pi(t, x) dx dt - \pi(t_0, x_0) \right|.\tag{19}$$

We are ready to give an oracle bound for the estimation of $b(t, x, \mu_t)$.
**Theorem 9.** Work under Assumptions 1, 2 and 4. Let \((t_0, x_0) \in (0, T) \times \mathbb{R}^d\). The following oracle inequality holds

\[
\mathbb{E}_N \left[ \left( \hat{p}^N_{GL}(t_0, x_0) - b(t_0, x_0, \mu(t_0)) \right)^2 \right] \leq \min_{h \in \mathcal{H}^N_1} \left( \mathcal{B}_h(\mu)(t_0, x_0)^2 + \mathcal{V}_h \right) + \min_{h \in \mathcal{H}^N_2} \left( \mathcal{B}_h(\pi)(t_0, x_0)^2 + \mathcal{V}_h \right),
\]

for large enough \(N\), up to a constant depending on \((t_0, x_0)\), \(|H \otimes K|\), and \(b\), provided

\[
\tilde{\omega}_2 \geq 12d\kappa_4 \max(12T\kappa_2^{-1}\kappa_5^2, 25|\text{Tr}(c)|_\infty) \quad \text{and} \quad \tilde{\omega}_3 \leq \kappa_4,
\]

where \(\kappa_2\) is defined in Theorem 18, and \(\kappa_3, \kappa_4, \kappa_5\) are (local) upper or lower bounds on \(\mu\) and \(b\) defined in Lemma 23 below.

Some remarks: 1) The same remarks as 1), 2), 3 and 4) after Theorem 7 are in order. This includes the calibration of \(\tilde{\omega}_2, \tilde{\omega}_3\) and the requirement \(Nh_1h_2^d \geq (\log N)^2\) for \(h \in \mathcal{H}^N_1\) that could be tightened to \(Nh_1h_2^d \geq (\log N)^{1+\epsilon}\) for an arbitrary \(\epsilon > 0\). However, the requirement \(h_1 \leq (\log N)^{-2}\) is a bit unusual and necessary for technical reason: it enables us to manage the delicate term \(II\) in the proof. Fortunately, this has no consequence for the subsequent minimax results, since we always look for oracle bandwidths of the form \(N^{-\epsilon}\) that are much smaller than \((\log N)^{-2}\).

2) The estimator \(\hat{p}^N_{GL}\) is a quotient estimator that estimates the ratio of \(\pi(t, x) = b(t, x, \mu(t))\mu(x)\) and \(\mu(x)\), very much in the sense of a Nadaraya-Watson (NW) type estimator in regression \([57]\).

Its performance is similar to the worst performance of the estimation of the product \(\pi\) and \(\mu\). However, the smoothness of \(\mu\) is usually no worse than the smoothness of \(b\) and we do not lose in terms of approximation results, see Section 4 and Proposition 13 below for the formulation of a minimax theory in this setting.

4. Adaptive minimax estimation

4.1. An isotropic Hölder smooth classes for McKean-Vlasov models.

**Definition 10.** Let \(x_0 \in \mathbb{R}^d\) and \(U\) a neighbourhood of \(x_0\). We say that \(f : \mathbb{R}^d \to \mathbb{R}\) belongs to \(\mathcal{H}^\alpha(x_0)\) with \(\alpha > 0\) if for every \(x, y \in U\)

\[
|D^sf(y) - D^sf(x)| \leq C|y - x|^{|\alpha|}
\]

for any \(s\) such that \(|s| \leq \lfloor \alpha \rfloor\), the largest integer strictly smaller than \(\alpha\); \(s \in \mathbb{N}^d\) is a multi-index with \(|s| = s_1 + \ldots + s_d\) and \(D^s = \partial^{s_1}_{x_1} \ldots \partial^{s_d}_{x_d}\).

The definition depends on \(x_0\) via \(U\) but this is further omitted in the notation for simplicity. We obtain a semi-norm by setting

\[
|f|_{\mathcal{H}^\alpha(x_0)} = \sup_{x \in U} |f(x)| + C(f),
\]

where \(C(f)\) is the smallest constant \(C\) for which (20) holds. The extension of Definition 10 for \(\mathbb{R}^d\)-valued functions is straightforward by considering coordinate functions. For time-varying functions defined on \((0, T)\) we have the

**Definition 11.** Let \((t_0, x_0) \in (0, T) \times \mathbb{R}^d\) and \(\alpha, \beta > 0\). The function \(f : (0, T) \times \mathbb{R}^d \to \mathbb{R}\) belongs to the anisotropic Hölder class \(\mathcal{H}^{\alpha, \beta}(t_0, x_0)\) if

\[
|f|_{\mathcal{H}^{\alpha, \beta}(t_0, x_0)} = \sum_{s, a, b} |f(\cdot, x_0)|_{\mathcal{H}^a(t_0)} + |f(t_0, \cdot)|_{\mathcal{H}^b(x_0)} < \infty.
\]

Again, the extension to \(\mathbb{R}^d\)-valued functions is straightforward: a mapping \(f = (f^k)_{1 \leq k \leq d} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d\) belongs to \(\mathcal{H}^{\alpha, \beta}(t_0, x_0)\) if \(f^k \in \mathcal{H}^{\alpha, \beta}(t_0, x_0)\) for every \(k = 1, \ldots, d\).
We model Hölder smoothness classes for the density function \( \mu_t(x) \) and the drift \( b(t, x, \mu_t) \). The McKean-Vlasov model (2) is parametrised by \((b, \sigma, \mu_0)\), or rather \((b, c, \mu_0)\), with \( c = \sigma \sigma^\top \). We denote by \( \mathcal{P} = \mathcal{P}(b) \) the class of \((b, c, \mu_0)\) satisfying Assumptions 1, 2 and 4 with model parameter \( b \). We let

\[
(b, c, \mu_0) \mapsto \mu = S(b, c, \mu_0)
\]
denote the solution (or forward) map of (3).

**Definition 12.** Let \( \alpha, \beta > 0 \). The anisotropic Hölder class \( S^{\alpha,\beta}(t_0, x_0) \) to the solution of (3) is defined by

\[
S^{\alpha,\beta}(t_0, x_0) = \left\{ (b, c, \mu_0) \in \mathcal{P}, \mu = S(b, c, \mu_0) \in \mathcal{H}^{\alpha,\beta}(t_0, x_0) \right\}.
\]

Before establishing minimax rates of convergence, we briefly investigate how rich is the class \( S^{\alpha,\beta}(t_0, x_0) \). 1) If \( b(t, x, \mu) = b(t, x) \) does not involve an interaction, then we have explicit formulas for \( \mu \) in some cases when \( c \) is regular, see e.g. Genon-Catalot and Jacod [29] and the formulas become relatively tractable in dimension \( d = 1 \), especially when \( b(t, x) = b(x), c(t, x) = c(x) \) and \( \mu_0 \) is the invariant distribution of the diffusion process \( X^b_t \) provided it exists. In that case, one can construct \( \mu \in \mathcal{H}^{\alpha,\beta}(t_0, x_0) \) with arbitrary \( \alpha, \beta > 0 \) for specific choices of \( c \) and \( b \) thanks to Feller’s classification of scalar diffusions (see e.g. Revuz and Yor [61]). 2) For a non-trivial representation of \( b(t, x, \mu) \) as in the Vlasov model, we have the following result, that shows how versatile the classes \( \mathcal{H}^{\alpha,\beta}(t_0, x_0) \) can be.

**Proposition 13.** Let \( c(t, x) = \frac{1}{2} \sigma^2 \text{Id} \) with \( \sigma > 0 \) and \( b(t, x, \mu) = b(x, \mu) = F * \mu(x) + G(x) \), with \( F \) having compact support, \( \mu_0 \in \mathcal{P}_1 \) with a continuous bounded density satisfying Assumption 1 and

\[
|G|_{3\beta} + |F|_{2\beta + \epsilon} + |\mu_0|_{2\beta + \epsilon} < \infty,
\]

for some \( \beta, \beta' > 1 \) and \( \beta'' > 0 \) (and \( \beta \) non-integer for technical reason). Here, \( \mathcal{H}^{\beta} \) denotes the global Hölder space (obtained when taking \( \mathcal{U} = \mathbb{R}^d \) in Definition 10). Then, for every \((t_0, x_0) \in (0, T) \times \mathbb{R}^d \), we have \( \mu \in \mathcal{H}^{\alpha,\beta+1}(t_0, x_0) \) and \((\cdot, \cdot, \mu) \in \mathcal{H}^{\alpha,\beta}(t_0, x_0) \) with \( \alpha = (\beta + 1)/2 \).

The proof relies on classical estimates for parabolic equations, see e.g. the textbook by Bogachev et al. [7]. See also Méleard and Jourdain [38] for analogous results. It is sketched in Appendix 8.3.

4.2. Minimax adaptive estimation of \( \mu_t(x) \). For \( \alpha, \beta, L > 0 \), we set

\[
S^{\alpha,\beta}_L(t_0, x_0) = \left\{ (b, c, \mu_0) \in \mathcal{P}, |S(b, c, \mu_0)|_{\mathcal{H}^{\alpha,\beta}(t_0, x_0)} \leq L \right\},
\]

that shall serve as a smoothness model for the unknown \( \mu \), where the semi-norm \( |\cdot|_{\mathcal{H}^{\alpha,\beta}(t_0, x_0)} \) is defined in (21).

Since the estimator \( \hat{\mu}^N_{GL}(t_0, x_0) \) is built on \( \mu_t^N \) solely and not the whole process \( \mu_t^N \) of order \( \ell \geq 1 \) as defined in (8). For every \((t_0, x_0) \in (0, T) \times \mathbb{R}^d \), we have,

\[
\sup_{(b, c, \mu_0)} \left( \mathbb{E}_{\mathbb{P}^N} \left[ (\hat{\mu}^N_{GL}(t_0, x_0) - \mu_{t_0}(x_0))^2 \right] \right)^{1/2} \leq \left( \frac{\log N}{N} \right)^{\beta \land \ell/(2\beta + d)}
\]

for large enough \( N \), up to a constant that depends on \( b, K, \beta \) and \( L \) only. Moreover

\[
\inf_{\hat{\mu}} \sup_{(b, c, \mu_0)} \mathbb{E}_{\mathbb{P}^N} \left[ |\hat{\mu} - \mu_{t_0}(x_0)| \right] \geq N^{-\beta/(2\beta + d)}
\]
for large enough $N$. The infimum in (24) is taken over all estimators constructed with $\mu_t^N$. The supremum in (23) and (24) is taken over $S^{\alpha,\beta}_L(t_0, x_0)$ for arbitrary $\alpha, \beta, L > 0$.

Some remarks: 1) We obtain the (nearly) optimal rate of convergence $N^{-\beta/(2\beta+d)}$ for estimating $\mu_t(x_0)$ as a function of the $d$-dimensional state variable $x_0$ for fixed $t_0$. The result obviously does not depend on the smoothness of $t \mapsto \mu_t(x_0)$ and is constructed from data $(X^1_{t_0}, \ldots, X^N_{t_0})$ for fixed $t_0$. 2) The extra logarithmic pay-off is unavoidable for pointwise estimation, as a result of the classical Lepski-Low phenomenon [47, 48]. 3) Although the result is stated for arbitrary $\alpha, \beta > 0$, the mapping $(t, x) \mapsto \mu_t(x)$ is locally smooth; there is no contradiction and result must be understood as bounds that are valid over smooth functions $\mu$ having prescribed $S^{\alpha,\beta}(t_0, x_0)$ semi-norms. 4) Our concentration result Theorem 18 for the fluctuation of $\mu_t^N$ around $\mu_t$ enables us to improve on the estimation result in Proposition 2.1 of Bolley et al. [8] that achieves the (non-adaptive) suboptimal rate $N^{-\beta/(2\beta+2d+2)}$. This is due to the fact that Bolley et al. [8] rely on controlling the fluctuations of $\mu_t^N$ around $\mu_t$ in Wasserstein distance, and therefore have to accomodate a dimension effect that we do not have here.

4.3. Minimax adaptive estimation of $b(t, x, \mu_t)$. For $\alpha, \beta > 0$ and $L > 0$, in analogy to the class $S^{\alpha,\beta}_L(t_0, x_0)$ defined in (22) above, we model the smoothness of the function $(t, x) \mapsto b(t, x, \mu_t)$ via the class

$$D^{\alpha,\beta}_L(t_0, x_0) = \{(b, c, \mu) \in P, \ |b(\cdot, S(b(c, \mu_0)))|_{S^{\alpha,\beta}(t_0, x_0)} \leq L\}.$$

Define the effective anisotropic smoothness $s_d(\alpha, \beta)$ by

$$\frac{1}{s_d(\alpha, \beta)} = \frac{1}{\alpha} + \frac{d}{\beta}.$$ 

We have the following adaptive upper bound and accompanying lower bound for estimating the drift $b$:

**Theorem 15.** Work under Assumptions 1, 2 and 4. Let $\hat{b}^N_{GL}(t_0, x_0)$ be constructed with kernels $H$ and $K$ of order $\ell \geq 1$. For every $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$, we have

$$\sup_{(b, c, \mu)} \left( \mathbb{E}_{\mathbb{P}^N} \left[ \left| \hat{b}^N_{GL}(t_0, x_0) - b(t_0, x_0, \mu_t) \right|^2 \right] \right)^{1/2} \lesssim \left( \frac{\log N}{N} \right)^{s_d(\alpha, \beta) / \ell_d + (2s_d(\alpha, \beta) / \ell_d + 1)},$$

for large enough $N$, with $\ell_d = \ell / d$, up to constants that depend on $b$, $H \otimes K$ and $\alpha, \beta, L$ only. Moreover,

$$\inf_{\hat{b}} \sup_{(b, c, \mu)} \mathbb{E}_{\mathbb{P}^N} \left[ \left| \hat{b} - b(t_0, x_0, \mu_t) \right| \right] \gtrsim N^{-s_d(\alpha, \beta) / (2s_d(\alpha, \beta) + 1)},$$

for large enough $N$. The infimum in (26) is taken over all estimators constructed with $(\mu_t^N)_{0 \leq t \leq T}$. The supremum in (25) and (26) is taken over $S^{\alpha,\beta}_L(t_0, x_0) \cap D^{\alpha,\beta}_L(t_0, x_0)$ for some known non-decreasing parametrisation $\alpha = \alpha(\beta) > 0$ such that $\alpha(\beta)/\beta$ is non-increasing, with $\beta > 0$, $L > 0$.

Some remarks: 1) Theorem 15 establishes that estimating $b(t, x, \mu_t)$ has the same complexity as estimating $d$ functions of $1 + d$ variables in the time and space domain. Each component $t \mapsto b^k(t, x, \mu_t)$ has smoothness $\alpha$, while $x^k \mapsto b^k(t, x^1, \ldots, x^k, \ldots, x^d, \mu_t)$ has smoothness $\beta$ for $\ell = 1, \ldots, d$, resulting in an anisotropic function $(t, x) \mapsto b^k(t, x, \mu_t)$ of $1 + d$ variables with smoothness index $(\alpha, \beta, \ldots, \beta)$. We therefore recover the usual anisotropic minimax rate of convergence, with effective smoothness $s_d(\alpha, \beta)$ obtained as the arithmetico-geometric mean of the smoothness index $(\alpha, \beta, \ldots, \beta)$. 2) For technical simplicity, we consider $\alpha$ and $\beta$ to be linked, as for instance in Proposition 13 where we have $\alpha = \alpha(\beta) = (\beta + 1)/2$. This somehow weakens our anisotropic adaptation result, but enables us to easily construct a well-behaved ordering $\leq$. 


for $3\zeta_N^2$ that behaves well with respect to the bias at scale $h$ as defined in (19). Dropping this restriction is possible in principle, as in the original paper of Goldenshluger and Lepski [32], yet at a significant additional technical cost. The remarks 2) and 3) of Theorem 14 are valid here as well.

5. Estimation of the interaction in the Vlasov model

In this section, we work under Assumptions 1 and 2 with a constant $c(t, x) = \frac{1}{2}\sigma^2\text{Id}$ for some $\sigma > 0$ and Assumption 4 (iii), i.e. in the Vlasov case

$$b(t, x, \mu) = b(x, \mu) = \int_{\mathbb{R}^d} \tilde{b}(x, y)\mu(dy),$$

for $d \geq 1$ and with a time homogeneous drift kernel

$$\tilde{b}(x, y) = F(x - y) + G(x), \quad x, y \in \mathbb{R}^d.$$ Model (2) then reads

$$dX_i^t = G(X_i^t)dt + N^{-1}\sum_{j=1}^N F(X_i^t - X_j^t)dt + \sigma dB_i^t, \quad 1 \leq i \leq N, \quad t \in [0, T],$$

$$\mathcal{L}(X_0^1, \ldots, X_N^N) = \mu_0^{\otimes N}.$$ We assume that each component $F^k \in L^1(\mathbb{R}^d)$ for every $k = 1, \ldots, d$. We are interested in identifying the interaction function $x \mapsto F(x)$ from data (1) and possibly $x \mapsto G(x)$, rather considered here as a nuisance parameter. We have

$$b(x, \mu_t) = G(x) + \int_{\mathbb{R}^d} F(x - y)\mu_t(y)dy$$

where $f \ast \mu_t(x) = (\int_{\mathbb{R}^d} f(x - y)\mu_t(y)dy)_{1 \leq k \leq d}$ denotes the convolution between $f : \mathbb{R}^d \to \mathbb{R}^d$ and $\mu_t$.

5.1. Identification of the interaction $F$. Introduce the linear form $\mathcal{L}$ acting on test functions $\varphi : [0, T] \to \mathbb{C}$ defined by

$$\mathcal{L}\varphi = \int_{[0,T]} \varphi(t)w(t)\rho(dt),$$

where $\rho$ is a probability distribution on $[0, T]$ and $w : [0, T] \to \mathbb{R}$ a bounded weight function such that $\int_{[0,T]} w(t)\rho(dt) = 0$. Note that $\mathcal{L}1 = 0$ where 1 denotes the constant function. Applying $\mathcal{L}$ on both sides of (27), we obtain

$$\mathcal{L}b(x, \mu) = F \ast \mathcal{L}\mu(x)$$

by Fubini’s theorem. For $f = (f^k)_{1 \leq k \leq d} : \mathbb{R}^d \to \mathbb{R}^d$ with $f^k \in L^1(\mathbb{R}^d)$, define a Fourier transform

$$\mathcal{F}(f)(\xi) = (\int_{\mathbb{R}^d} e^{-2\pi i \xi^\top x} f^k(x)dx)_{1 \leq k \leq d}, \quad \xi \in \mathbb{R}^d,$$

so that whenever $g \in L^1(\mathbb{R}^d)$, we have $\mathcal{F}(f \ast g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$. We infer from (29)

$$\mathcal{F}(\mathcal{L}b(\cdot, \mu)) = \mathcal{F}(F) \cdot \mathcal{F}(\mathcal{L}\mu).$$

In particular, it is a first step toward the interesting problem of testing the hypothesis $F = 0$ against a set of local alternatives that quantify how far $F$ is from being constant.
This yields the formal decomposition

\[ F(F) = \mathcal{F}(\mathcal{L}b(\cdot, \mu)) / \mathcal{F}(\mathcal{L}\mu), \]

provided the quotient is well defined.

5.2. Consistent estimation of \( F \). A first estimation strategy consists in plugging-in our estimators of \( b(t, x, \mu_t) \) and \( \mu_t \) in (30) above. A somewhat simpler estimator of \( F(\mathcal{L}\mu)(\xi) = \mathcal{L}F(\mu)(\xi) \) is given by the periodogram

\[ L\left( \int_{\mathbb{R}^d} e^{-2i\pi \xi^\top x} \mu^N(dx) \right) = \int_{\mathbb{R}^d} e^{-2i\pi \xi^\top x} \mathcal{L}\mu^N(dx) = \mathcal{F}(\mathcal{L}\mu^N)(\xi) \]

for which we need not tune a bandwidth. Following Johannes [37], we obtain an estimator of \( F \) by the formula

\[ \mathcal{F}(\hat{F}^N) = \frac{\mathcal{F}(\mathcal{L}\left( \hat{b}^N_{h,h}(t, x_0) \right)) \cdot \mathcal{F}(\mathcal{L}\mu^N)}{\mathcal{F}(\mathcal{L}\mu^N)^2} 1_{\{\mathcal{F}(\mathcal{L}\mu^N)^2 \geq \omega\}} \in \mathbb{R}^d \]

for some threshold \( \omega > 0 \) vanishing as \( N \to \infty \), with the estimator

\[ \hat{b}^N_{h,h}(t_0, x_0) = \hat{b}^N_{h,h}(t_0, x_0) \omega^{1/2} \]

of \( b(t_0, x_0, \mu_{t_0}) \), constructed in Section 2.2 for some threshold \( \omega' > 0 \) and bandwidths \( h > 0 \) and \( b > 0 \). We also set the estimator to be equal to 0 outside \( |x| \leq r \) for some \( r > 0 \). We obtain a consistency result under the following additional assumption:

Assumption 16. We have \( |\mathcal{F}(\mathcal{L}\mu)(\xi)| > 0 \) \( d\xi \)-almost everywhere.

Theorem 17. Work under the assumptions of Proposition 13 and Assumption 16. Assume moreover that \( G \) is in \( L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \) componentwise. If \( w \) has compact support in \((0, T)\), there exists a choice of \((\omega, \omega', b, h) = (\omega_N, \omega'_N, b_N, h_N) \to 0 \) and \( r = r_N \to \infty \) such that

\[ \mathbb{E}_{\mu_N} \left| \hat{F}^N_{\omega, \omega'} - F^2 \right| \to 0 \quad \text{as} \quad N \to \infty. \]

Some remarks: 1) Theorem 17 proves that we can reconstruct the interaction force \( F \) from data (1), while the function \( G \) remains a nuisance parameter. This is a first step for the construction of a statistical test based on data (1) for the presence against the absence of interaction between particles in the Vlasov model. 2) Although we obtain consistency, we do not have a rate of convergence and our result is not uniform in the model parameter. A glance at the proof of Theorem 17 shows that it is possible to cook-up a result with a rate of convergence and some uniformity in the parameter, provided we have a sharp control from below on the decay \( |\mathcal{F}(\mathcal{L}\mu)(\xi)| \) or rather \( |\mathcal{F}(\mu)(\xi)| \) as \( |\xi| \to \infty \) as well as the decay of \( \inf_{t \in [r_1, r_2], |x| \leq r} \mu_t(x) \) for given \( [r_1, r_2] \subset (0, T) \) as \( r \to \infty \). This requires the exact knowledge of the smoothness of the solution map \( \mu = \delta(b, c, \mu_0) \), and it is a delicate issue, see Proposition 13; we can anticipate ill-posedness. 3) From our estimator of \( F \), we may construct a plug-in estimator for the function \( G \) by setting

\[ \hat{G}^N(x, \omega) = -\hat{b}^N_{h,h}(t, x) \omega + \hat{F}^N_{\omega, \omega'} \hat{\mu}^N_{h,h}(t, x). \]

A consistency result can be obtained in the same way as for Theorem 17. 4) the deconvolution method we employ here requires quite a stringent localisation assumption on the external force \( G \) and the interaction force \( F \). As pointed out by a referee, it does not apply for gradient forces of the form \( G = -\nabla V \) and \( F = -\nabla W \) where \( V \) and \( W \) diverge polynomially at infinity like e.g. in [5, 34, 17] for which alternative methods yet need to be constructed.
6. Probabilistic tools: a concentration inequality

6.1. A Bernstein inequality. Let $\rho(dt)$ be a probability measure on $[0, T]$. We establish a deviation inequality for the sequence of signed measures $\nu^N(dt, dx) = \mu^N(dx) \otimes \rho(dt)$ and $\nu(dt, dx) = \mu(x)dx \otimes \rho(dt)$.

We have a Bernstein concentration inequality:

**Theorem 18.** Work under Assumptions 1, 2 and 4. Let $(\mu_t)_{0 \leq t \leq T}$ denote the unique solution of (3) with $\mu_{t=0} = \mu_0$ satisfying Assumption 1. Then there exist $\kappa_1, \kappa_2 > 0$ depending on $b$ such that

$$
\mathbb{P}^N \left( \int_{[0,T] \times \mathbb{R}^d} \phi(t, y) \left( \nu^N(dt, dy) - \nu(dt, dy) \right) \geq x \right) \leq \kappa_1 \exp \left( - \frac{\kappa_2 x^2}{\| \phi \|_{L^2(\nu)}^2 + \| \phi \|_{\infty} x} \right)
$$

for every $x \geq 0$, for every bounded $\phi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, and for any probability measure $\rho(dt)$ on $[0, T]$.

As a corollary, for a bounded $\phi : \mathbb{R}^d \to \mathbb{R}$ and any $0 \leq t_0 \leq T$, picking $\rho(dt) = \delta_{t_0}(dt)$, we obtain

$$
\mathbb{P}^N \left( \int_{\mathbb{R}^d} \phi(y) \left( \mu^N_{t_0}(dy) - \mu_0(y)dy \right) \geq x \right) \leq \kappa_1 \exp \left( - \frac{\kappa_2 x^2}{\| \phi \|_{L^2(\mu_{t_0})}^2 + \| \phi \|_{\infty} x} \right).
$$

Several remarks: 1) Up to the constants $\kappa_i$, the result is quite satisfactory and comparable to the Bernstein deviation inequality for independent data, see e.g. Massart [52]. Theorem 18 is the gateway to derive sharp nonparametric estimators, although it has an independent interest as a deviation inequality. 2) The constants $\kappa_i$ are explicitly computable, but certainly far from being optimal with the method of proof employed here. 3) Our method of proof uses a change of measure argument based on Girsanov’s theorem, in the spirit of the recent work of Lacker [44]. It can presumably be extended to path dependent coefficients in (2), but it is essential that the diffusion coefficient does not depend on $\mu^N$. 4) We have an interplay between the smoothness $k$ of the drift $b$ in its measure argument and the dimension $d$ of the ambient state space. This is explained by the fact that we need to control an exponential moment of $\sum_{i=1}^N \int_t^{t+\delta} |b(s, X^i_s, \mu^N_s) - b(s, X^i_s, \mu_s)|^2 ds$ over small intervals $[t, t+\delta]$ in order to approximate the law of the data by the law of independent particles. This approximation is roughly controlled by $NW_1(\mu^N_t, \mu_t)^2$ for which a dimensional effect drastically deteriorates the rate of convergence, see e.g. Fournier and Guillin [26]. The $k$-linear differentiability of $\mu \mapsto b(t, x, \mu)$ enables us to mitigate this effect. In particular, in the Vlasov case covered by Assumption 4 (iii), we formally have $k = \infty$, and the result is valid in any dimension $d \geq 1$.

The remainder of Section 6 is devoted to the proof of Theorem 18.

6.2. Preparation for the proof of Theorem 18. We let $\mathbb{P}^N$ denote the unique probability measure on $(\mathcal{C}, \mathcal{F}_T)$ under which the canonical process $X = (X^1, \ldots, X^N)$ solves

$$
\begin{cases}
    dX^i_t = b(t, X^i_t, \mu_t)dt + \sigma(t, X^i_t)d\mathcal{B}_t^i, & 1 \leq i \leq N, \ t \in [0, T], \\
    \mathcal{L}(X^0_0, \ldots, X^N_0) = \mu_0^\otimes N,
\end{cases}
$$

where

$$
\mathcal{B}^i_s = \int_0^s c(s, X^i_s)^{-1/2} (dX^i_s - b(s, X^i_s, \mu_s)ds), \ 1 \leq i \leq N,
$$
are independent \(d\)-dimensional \(P^N\)-Brownian motions. The existence of \(\overline{\mathcal{M}}^N\) follows from Carmona and Delarue [16] or the lectures notes of Lacker [43]. In turn, the real-valued process

\[
\overline{\mathcal{M}}^N_t = \sum_{i=1}^N \int_0^t ((c^{-1/2} b)(s, X^i_s, \mu^N_s) - (c^{-1/2} b)(s, X^i_s, \mu_s))^T d\overline{\mathcal{M}}^N_s,
\]

is a \(P^N\)-local martingale. Here, \(c^{-1/2}\) denotes any square-root of \(c^{-1} = (\sigma^\top \sigma)^{-1}\).

The following estimate is the central result of the section. It is the key ingredient that enables us to implement a change of probability argument in order to obtain our concentration estimates. Its proof is delayed until Section 6.4.

**Proposition 19.** Work under Assumptions 1, 2 and 4. For every \(\tau > 0\), there exists \(\delta_0 > 0\) depending on \(\tau\) and \(b\) such that

\[
\sup_{N \geq 1} \sup_{t \in [0, T - \delta]} \mathbb{E}^{P^N} \left[ \exp \left( \tau \left( \overline{\mathcal{M}}^N_{t+\delta} - \overline{\mathcal{M}}^N_t \right) \right) \right] \leq C_1,
\]

for every \(0 \leq \delta \leq \delta_0\) and some \(C_1\) that depends on \(b\) and \(\tau\).

Let \(\mathcal{E}^N(\overline{\mathcal{M}}^N) = \exp \left( \overline{\mathcal{M}}^N_t - \frac{1}{2} \langle \overline{\mathcal{M}}^N_t \rangle_t \right)\) denote the (martingale) exponential of \(\overline{\mathcal{M}}^N_t\) and \(\langle \overline{\mathcal{M}}^N \rangle_t\) its predictable compensator. By Novikov’s criterion – in its version developed in the classical textbook [40], Lemma 5.14, p.198 – Proposition 19 shows that the local martingale \(\mathcal{E}^N(\overline{\mathcal{M}}^N)\) is indeed a true martingale. This enables us to define a new probability measure on \((\mathcal{C}, \mathcal{F}_T)\) by setting

\[
\tilde{P}^N = \mathcal{E}_T(\overline{\mathcal{M}}^N). \quad \tilde{P}^N
\]

By Girsanov’s theorem, under \(\tilde{P}^N\), the canonical process solves (2). By uniqueness of the weak solution of (2), this proves \(\tilde{P}^N = P^N\) and shows in particular that \(P^N \ll \tilde{P}^N\) and

\[
\frac{dP^N}{d\tilde{P}^N} = \mathcal{E}_T(\overline{\mathcal{M}}^N).
\]

**6.3. Proof of Theorem 18.** Step 1: Let \(A^N \in \mathcal{F}_T\). Since \(P^N\) and \(\tilde{P}^N\) coincide on \(\mathcal{F}_0\), we have

\[
P^N(A^N) = \tilde{P}^N \left[ \mathbb{P}^N(A^N | \mathcal{F}_0) \right] = \tilde{P}^N \left[ \mathbb{P}^N(A^N | \mathcal{F}_0) \right].
\]

Next, for any subdivision \(0 = t_0 < t_1 < \ldots < t_K \leq T\) and any \(\mathcal{F}_T\)-measurable event \(A^N\), we claim

\[
\mathbb{E}^{P^N} \left[ \mathbb{P}^N(A^N | \mathcal{F}_0) \right] \leq \mathbb{E}^{P^N} \left[ \mathbb{P}^N(A^N | \mathcal{F}_{t_K}) \right]^{1/4} \prod_{j=1}^K \mathbb{E}^{P^N} \left[ \exp \left( 2 (\langle \overline{\mathcal{M}}^N \rangle_{t_j} - \langle \overline{\mathcal{M}}^N \rangle_{t_{j-1}}) \right) \right]^{1/4}. \tag{34}
\]

It follows that

\[
P^N(A^N) \leq \tilde{P}^N \left[ \mathbb{P}^N(A^N | \mathcal{F}_T) \right]^{1/4} \prod_{j=1}^K \mathbb{E}^{P^N} \left[ \exp \left( 2 (\langle \overline{\mathcal{M}}^N \rangle_{t_j} - \langle \overline{\mathcal{M}}^N \rangle_{t_{j-1}}) \right) \right]^{1/4}
\]

\[
\leq \tilde{P}^N \left[ A^N \right]^{1/4} \sup_{N \geq 1} \sup_{t \in [0, T - \delta_0]} \left( \mathbb{E}^{P^N} \left[ \exp \left( 2 (\langle \overline{\mathcal{M}}^N \rangle_{t+\delta} - \langle \overline{\mathcal{M}}^N \rangle_t) \right) \right] \right)^{K(K+1)/8}
\]

\[
\leq C_1^{K(K+1)/8} \mathbb{P}^N \left[ A^N \right]^{1/4K}
\]

by (34) and Proposition 19 with \(\tau = 2, t_j = jT/K\) and \(K\) large enough so that \(t_j - t_{j-1} \leq \delta_0\).
Step 2: Let
\[ A^N = \left\{ \int_{[0,T] \times \mathbb{R}^d} \phi(t,y) \left( \nu^N(dt,dy) - \nu(dt,dy) \right) - \nu^N(dt,dy) \right\} \geq N \]
so that \( A^N \in \mathcal{T}_T \). Recall Bernstein’s inequality: if \( Z_1, \ldots, Z_N \) are real-valued independent random variables bounded by some constant \( Q \) and such that \( \mathbb{E}[Z_i] = 0 \), we have
\[
\mathbb{P}(\sum_{i=1}^{N} Z_i \geq y) \leq \exp \left( -\frac{y^2}{2(\sum_{i=1}^{N} \mathbb{E}[Z_i^2] + Q^2)} \right) \quad \text{for every} \quad y \geq 0.
\]
Under \( \mathbb{P}^N \), the random processes \( (X^i_t)_{0 \leq t \leq T} \) are independent and identically distributed processes. Noticing that \( \int_{[0,T] \times \mathbb{R}^d} \phi(t,x) \mu(t,dy) \rho(dt) = \mathbb{E}^N \left[ \int_0^T \phi(t,X^i_t) \rho(dt) \right] \), we apply Bernstein inequality with \( Z_t = \int_0^T \phi(t,X^i_t) \rho(dt) - \mathbb{E}^N \left[ \int_0^T \phi(t,X^i_t) \rho(dt) \right] \), \( y = N \) and \( Q = 2|\phi|_{\infty} \) to infer
\[
\mathbb{P}^N (A^N) \leq \exp \left( -\frac{N x^2}{2(|\phi|_{L^2(\mu)}^2 + \frac{4}{3}|\phi|_{\infty}^2)} \right),
\]
using \( \mathbb{E}[Z_t^2] \leq \mathbb{E}^N \left[ \left( \int_0^T \phi(t,X^i_t) \rho(dt) \right)^2 \right] \leq |\phi|^2_{L^2(\mu)} \) by Jensen’s inequality. By (35), we also have
\[
\mathbb{P}^N (A^N) \leq C_1^{K(K+1)/8} \mathbb{P}^N (A^N)^{1/4K} \leq C_1^{K(K+1)/8} \exp \left( -\frac{N x^2}{2 \cdot 4^K \left( |\phi|_{L^2(\mu)}^2 + \frac{2}{3}|\phi|_{\infty}^2 \right)} \right),
\]
and we obtain Theorem 18 with \( \kappa_1 = C_1^{K(K+1)/8} \) and \( \kappa_2 = 2^{-1 \cdot 4^{-K}} \).

Step 3: It remains to prove the key estimate (34), adapted from large deviation techniques, see e.g. Gärtner [28] and the estimate (4.2) in Theorem 2.6 in Lacker [44]. We proceed by induction. First,
\[
\mathbb{E}^N \left[ \mathbb{P}^N (A^N | \mathcal{F}_{t_{i-1}}) \right] = \mathbb{E}^N \left[ \mathbb{E}^N_C \left[ \mathbb{P}^N (A^N | \mathcal{F}_{t_{i}}) | \mathcal{F}_{t_{j-1}} \right] \right]
\]
\[ = \mathbb{E}^N \left[ \mathbb{E}^N \left[ \frac{\mathcal{E}_{t_j} (M^N)}{\mathcal{E}_{t_{j-1}} (M^N)} \mathbb{P}^N (A^N | \mathcal{F}_{t_{i}}) | \mathcal{F}_{t_{j-1}} \right] \right]
\]
\[ = \mathbb{E}^N \left[ \frac{\mathcal{E}_{t_j} (M^N)}{\mathcal{E}_{t_{j-1}} (M^N)} \mathbb{P}^N (A^N | \mathcal{F}_{t_{i}}) \right],
\]
see e.g. Lemma 3.5.3 p. 193 in [40]. Next,
\[
\frac{\mathcal{E}_{t_j} (M^N)}{\mathcal{E}_{t_{j-1}} (M^N)} = \mathcal{E}_{t_j} \left( 2(M^N_{t_{j-1}} - M^N_{t_{j-1}}) \right)^{1/2} \left( \exp \left( (\overline{M}^N_{t_{j}} - \overline{M}^N_{t_{j-1}}) \right) \right)^{1/2}.
\]
As shown before, under \( \mathbb{P}^N \), the process \( \mathcal{E}_{t_j} \left( 2(M^N_{t_{j-1}} - M^N_{t_{j-1}}) \right) \) is a martingale and
\[
\mathbb{E}^N \left[ \mathcal{E}_{t_j} \left( 2(M^N_{t_{j-1}} - M^N_{t_{j-1}}) \right) \right] = 1.
\]
Using (36) and Cauchy-Schwarz’s inequality twice together with (37), we obtain
\[ E_{pN} \left[ \frac{\mathcal{E}_{t_j}(\mathcal{M}_N)}{\mathcal{E}_{t_{j-1}}(\mathcal{M}_N)} \right] \mathbb{P}^N(A^N \mid \mathcal{F}_{t_j}) \leq E_{pN} \left[ \mathbb{P}^N(A^N \mid \mathcal{F}_{t_j})^2 \exp \left( (\mathcal{M}_N)_{t_j} - (\mathcal{M}_N)_{t_{j-1}} \right)^{1/2} \right] \]
\[ \leq E_{pN} \left[ \mathbb{P}^N(A^N \mid \mathcal{F}_{t_j}) \right]^{1/4} \mathbb{E}_{pN} \left[ \exp \left( 2((\mathcal{M}_N)_{t_j} - (\mathcal{M}_N)_{t_{j-1}}) \right) \right]^{1/4}. \]

By Jensen’s inequality, we infer
\[ E_{pN} \left[ \mathbb{P}^N(A^N \mid \mathcal{F}_{t_{j-1}}) \right] \leq E_{pN} \left[ \mathbb{P}^N(A^N \mid \mathcal{F}_{t_j}) \right]^{1/4} \mathbb{E}_{pN} \left[ \exp \left( 2((\mathcal{M}_N)_{t_j} - (\mathcal{M}_N)_{t_{j-1}}) \right) \right]^{1/4}. \]

Repeating the argument over the subdivision \( 0 = t_0 < t_1 < \ldots < t_K \) proves (34). The proof of Theorem 18 is complete.

### 6.4. Proof of Proposition 19.

**Preparation.** Recall that the notation \( A_N \lesssim B_N \): it means the existence of \( C > 0 \) possibly depending on \( b \) and also \( \tau \) in this part of the paper, but not \( N \), such that \( A_N \leq C B_N \) for every \( N \geq 1 \). The following classical moment estimate will be needed.

**Lemma 20.** In the setting of Theorem 18, for every \( p \geq 1 \), we have
\[ \sup_{t \in [0, T]} E_{pN} \left[ |X|^2p \right] \leq p! C_p^2. \]

for some \( C_2 > 0 \) that depends on \( b \) only.

In particular, the \( X_t^i \) are sub-Gaussian under \( \mathbb{P}^N \) and satisfy
\[ E_{pN} \left[ e^{2C_2^2 |X_t^i|^2} \right] = 1 + \sum_{p \geq 1} \frac{2^{-p}}{p!C_p^2} E_{pN} \left[ |X_t^i|^{2p} \right] \leq 2. \]

The proof is classical (see e.g. estimates of this type in Méléard [54] or Sznitman [12]) and postponed to Appendix 8.1.

**Completion of Proof of Proposition 19.** Let \( \tau > 0 \). All we need to show is that for small enough \( \delta \), we have
\[ \sup_{t \in [0, T-\delta]} E_{pN} \left[ \exp \left( \tau (\mathcal{M}_N^N)_{t+\delta} - (\mathcal{M}_N^N)_{t} \right) \right] \lesssim 1. \]

We start with a useful estimate.

**Lemma 21.** Let \( \xi^N_t(x) = b(s, x, \mu^N_s) - b(s, x, \mu_s) \). We have
\[ \left| \xi^N_t(x^i) \right|^2 \leq \left\{ \begin{array}{ll}
\mathcal{W}_1(\mu^N_s, \mu_s)^2 & \text{under Assumption 4(i),} \\
\sum_{k=1}^{K-1} \left| \int_{[\mathbb{R}^d]^k} \delta_t^k b(s, X_s^i, y^i, \mu_s) (\mu^N_s - \mu_s)^{\otimes d}(dy^i) \right|^2 + \mathcal{W}_1(\mu^N_s, \mu_s)^2 \wedge \mathcal{W}_1(\mu^N_s, \mu_s)^{2k} & \text{under Assumption 4(ii),} \\
\int_{\mathbb{R}^d} \bar{b}(s, X_s^N, y)(\mu^N_s - \mu_s)(dy)^2 & \text{under Assumption 4(iii),}
\end{array} \right. \]

for some explicit \( C \geq 1 \) depending on \( b \).
Proof. The estimate follows from the Lipschitz continuity of \(b\) under Assumption 4(i) with \(C = \|b\|_{\text{Lip}}\) and from the definition of the Vlasov case with \(C = 1\) under Assumption 4(iii). We turn to the estimate under (ii). The \(k\)-linear differentiability of \(b\) enables us to write

\[
\xi_s^N(X_s^i) = \sum_{\ell=1}^{k-1} \frac{1}{\ell!} \int_{(\mathbb{R}^d)^\ell} \delta^\ell_{\mu}(s, X_s^i, y^\ell, \mu_s)(\mu_s^N - \mu_s)^{\otimes \ell}(dy^\ell) + \mathcal{R}_k,
\]

where \(y^\ell = (y_1, \ldots, y_\ell) \in (\mathbb{R}^d)^\ell\) and

\[
\mathcal{R}_k = \frac{1}{(k-1)!} \int_0^1 (1 - \vartheta)^{k-1} \sum_{j \subseteq \{1, \ldots, k\}, m \geq 1} \int_{(\mathbb{R}^d)^j} \bigotimes_{j \in j} \delta^k_{\mu}(s, X_s^i, y, \mu_s^N, \mu_s) \mu_s^N - \mu_s)^{\otimes k}(dy^k)d\vartheta,
\]

with \([\mu_s^N, \mu_s]_\vartheta = (1 - \vartheta)\mu_s + \vartheta \mu_s^N\), as stems from the definition of linear differentiability and the iteration \(\delta^k_{\mu} = \delta_{\mu} \circ \delta^k_{\mu} b\), see also Lemma 2.2. of Chassagneux et al. [18] where (40) is established by induction.

Thanks to representation (7), the remainder term \(\mathcal{R}_k\) equals

\[
\frac{1}{(k-1)!} \int_0^1 (1 - \vartheta)^{k-1} \sum_{j \subseteq \{1, \ldots, k\}, m \geq 1} \int_{(\mathbb{R}^d)^j} \bigotimes_{j \in j} \delta^k_{\mu}(s, X_s^i, y, \mu_s^N, \mu_s) \mu_s^N - \mu_s)^{\otimes k}(dy^k)d\vartheta.
\]

Note that the product integral vanishes for all terms in the sum in \(j\) except \(j = \{1, \ldots, k\}\) since \((\mu_s^N - \mu_s)(\mathbb{R}^d) = 0\). By definition,

\[
|\langle (\delta^k_{\mu} b)_{(1, \ldots, k), j, m}(s, X_s^i, [\mu_s^N, \mu_s]) \rangle|_{\Lip} \leq |\delta^k_{\mu} b|_{\Lip} \text{ for every } (j, m)
\]

and the sum in \(m\) has at most \(m_b\) terms by assumption. It follows that

\[
|\mathcal{R}_k| = \frac{1}{(k-1)!} \int_0^1 (1 - \vartheta)^{k-1} \sum_{m=1}^{m_b} \prod_{m=1}^{k} \delta^k_{\mu}(s, X_s^i, y, \mu_s^N, \mu_s) \mu_s^N - \mu_s)^{\otimes k}(dy^k)d\vartheta
\]

\[
\leq \frac{k!}{(k-1)!} \int_0^1 (1 - \vartheta)^{k-1} |\delta^k_{\mu} b|_{\Lip} \left( \sup_{|\varphi|_{\Lip} \leq 1} \int_{\mathbb{R}^d} \varphi d(\mu_s^N - \mu_s) \right)^k \|\mathcal{R}_k\|_{\Lip} \leq \frac{m_b |\delta^k_{\mu} b|_{\Lip} \mathcal{W}_1(\mu_s^N, \mu_s)^k}{k!}.
\]

Writing \(y^\ell = (y^{\ell-1}, y) \in (\mathbb{R}^d)^{\ell-1} \times \mathbb{R}^d\), we also have the rough bound

\[
|\int_{(\mathbb{R}^d)^\ell} \delta^\ell_{\mu}(s, X_s^i, y^\ell, \mu_s)(\mu_s^N - \mu_s)^{\otimes \ell}(dy^\ell)|
\]

\[
\leq \int_{(\mathbb{R}^d)^{\ell-1}} \left| \int_{\mathbb{R}^d} \delta^\ell_{\mu}(s, X_s^i, (y^{\ell-1}, y), \mu_s)(\mu_s^N - \mu_s)(dy) \right| (\mu_s^N + \mu_s)^{\otimes (\ell-1)}(dy^{\ell-1})
\]

\[
\leq \sup_{y^{\ell-1} \in (\mathbb{R}^d)^{\ell-1}} |\delta^\ell_{\mu}(s, X_s^i, (y^{\ell-1}, \cdot), \mu_s)|_{\Lip}
\]

\[
\times \left| \sup_{|\varphi|_{\Lip} \leq 1} \int_{\mathbb{R}^d} \varphi(y)(\mu_s^N - \mu_s)(dy) \right| \int_{(\mathbb{R}^d)^{\ell-1}} (\mu_s^N + \mu_s)^{\otimes (\ell-1)}(dy^{\ell-1})
\]

\[
= 2^{\ell-1} \sup_{y^{\ell-1} \in (\mathbb{R}^d)^{\ell-1}} |\delta^\ell_{\mu}(s, X_s^i, (y^{\ell-1}, \cdot), \mu_s)|_{\Lip} \mathcal{W}_1(\mu_s^N, \mu_s)
\]

\[
\leq 2^{\ell-1} |\delta^\ell_{\mu} b|_{\Lip} \mathcal{W}_1(\mu_s^N, \mu_s).
\]
Plugging this estimate in (40) and using the Lipschitz property for $b$, we obtain

$$ |R_k| \leq (|b|_{\text{Lip}} + \sum_{\ell=1}^{k-1} \frac{2^{\ell-1}}{\ell!} |\delta_\mu^\ell b|_{\text{Lip}}) W_1(\mu_N^N, \mu_s) $$

and we conclude

(41) $$ |R_k| = |R_k(s, X^i_s, \mu_N^N, \mu_s)| \leq C' W_1(\mu_N^N, \mu_s) \wedge W_1(\mu_N^N, \mu_s)^k, $$

with

$$ C' = \max \left( \frac{m_b}{k!} |\delta_\mu^k b|_{\text{Lip}}, |b|_{\text{Lip}} + \sum_{\ell=1}^{k-1} \frac{2^{\ell-1}}{\ell!} |\delta_\mu^\ell b|_{\text{Lip}} \right). $$

From (40) and (41) we conclude

$$ |\xi^N_s(X^i_s)|^2 \leq k \left( \sum_{\ell=1}^{k-1} \frac{1}{(\ell!)^2} \int_{(\mathbb{R}^d)^\ell} \delta_\mu^\ell b(s, X^i_s, y^\ell, \mu_s)(\mu_N^N - \mu_s)^{\otimes \ell}(dy^\ell)^2 + |R_k(s, X^i_s, \mu_N^N, \mu_s)|^2 \right) $$

$$ \leq C \left( \sum_{\ell=1}^{k-1} \int_{(\mathbb{R}^d)^\ell} \delta_\mu^\ell b(s, X^i_s, y^\ell, \mu_s)(\mu_N^N - \mu_s)^{\otimes \ell}(dy^\ell)^2 + W_1(\mu_N^N, \mu_s)^2 \wedge W_1(\mu_N^N, \mu_s)^{2k} \right), $$

where $C = k \max(1, (C')^2)$ incorporates the constant in (41). \hfill \Box

We now establish (39). By (33) and Lemma 21, we have

$$ \tau((M^N_N)_{t+\delta} - (M^N_N)_t) $$

$$ = \tau \sum_{i=1}^N \int_t^{t+\delta} (b(s, X^i_s, \mu_N^N) - b(s, X^i_s, \mu_s)) \mu(c^{-1}(b(s, X^i_s, \mu_N^N) - b(s, X^i_s, \mu_s))) ds $$

$$ \leq \tau |\text{Tr}(c^{-1})|_\infty \sum_{i=1}^N \int_t^{t+\delta} |\xi^N_s(X^i_s)|^2 ds, $$

(42) $$ \leq \kappa \left\{ \begin{array}{ll}
N \int_t^{t+\delta} W_1(\mu_N^N, \mu_s)^2 ds & \text{under 4(i),} \\
\int_t^{t+\delta} \sum_{i=1}^N \sum_{\ell=1}^{k-1} \int_{(\mathbb{R}^d)^\ell} \delta_\mu^\ell b(s, X^i_s, y^\ell, \mu_s)(\mu_N^N - \mu_s)^{\otimes \ell}(dy^\ell)^2 ds \\
+ N \int_t^{t+\delta} W_1(\mu_N^N, \mu_s)^2 \wedge W_1(\mu_N^N, \mu_s)^{2k} ds & \text{under 4(ii),} \\
\int_t^{t+\delta} \sum_{i=1}^N \int_{(\mathbb{R}^d)^d} \delta_\mu(b(s, X^i_s, y)(\mu_N^N - \mu_s)(dy)^2 ds, & \text{under 4(iii).} \\
\end{array} \right. $$

with $\kappa = \tau |\text{Tr}(c^{-1})|_\infty C$, where $C$ is the constant of Lemma 21. We now heavily rely on the sharp deviation estimate

(43) $$ \sup_{0 \leq s \leq t} \mathbb{P}_s^N(\omega_1(\mu_N^N, \mu_s) \geq x) \lesssim \varepsilon_N(x), $$

with

$$ \varepsilon_N(x) = \left\{ \begin{array}{ll}
\exp(-cN x^2) & \text{if } d = 1, \\
\exp(-cN x^{2d/(\log(2 + 1/x)^2)}) 1_{\{x \leq 1\}} + \exp(-cN x^2) 1_{\{x > 1\}} & \text{if } d = 2, \\
\exp(-cN x^2) 1_{\{x \leq 1\}} + \exp(-cN x^2) 1_{\{x > 1\}} & \text{if } d \geq 3, \\
\end{array} \right. $$

(44)
extracted from Theorem 2 of Fournier and Guillin [26]. Here $C$ depends on $C_2$ and $d$ only, thanks to (38) that guarantees that Condition (1) of Theorem 2 in [26] is satisfied, hence the uniformity in $s \in [0, T]$.

We complete the proof of (39) under Assumption 4 (i) that implies in particular $d = 1$. From (42), we infer

$$
E_{\mathbb{P}^N} \left[ \exp \left( \tau \left( \langle M^N \rangle_{t+\delta} - \langle M^N \rangle_t \right) \right) \right] 
\leq \delta^{-1} \int_t^{t+\delta} E_{\mathbb{P}^N} \left[ \exp \left( \kappa \delta k W_1(\mu^N_s, \mu_s)^2 \right) \right] ds 
\leq \sup_{s \in [0, T]} E_{\mathbb{P}^N} \left[ \exp \left( \kappa \delta N W_1(\mu^N_s, \mu_s)^2 \right) \right] 
\leq 1 + \kappa \delta \sup_{s \in [0, T]} \int_0^{\infty} \exp(\kappa \delta z) \mathbb{P}^N\left( W_1(\mu^N_s, \mu_s) \geq N^{-1/2} z^{1/2} \right) dz 
\approx \int_0^{\infty} \exp \left( (\kappa \delta - C) z \right) dz,
$$

(45)

where $C$ is the constant in (44). The integral in (45) is finite as soon as $\delta \leq \tau^{-1} |\text{Tr}(c^{-1})|_{\infty}^{-1} |b|_{\text{Lip}}^{-1}$. $C$ and (39) follows.

We next complete the proof of (39) under Assumption 4 (ii). When $d = 1$, we can rely on the previous case. Assume now that $(d = 2$ and $k \geq 2$) or $(d \geq 3$ and $k \geq d/2$). By Jensen’s inequality

$$
E_{\mathbb{P}^N} \left[ \exp \left( \tau \left( \langle M^N \rangle_{t+\delta} - \langle M^N \rangle_t \right) \right) \right] \leq I + II,
$$

with

$$
I = \frac{1}{\delta k N} \int_t^{t+\delta} \sum_{i=1}^{N} \sum_{j=1}^{k-1} E_{\mathbb{P}^N} \left[ \exp \left( \kappa \delta k N \int_{\mathbb{R}^j} \delta^i_{\mu} b(s, X^i_s, y^i_s, \mu_s)(\mu^N_s - \mu_s) \otimes (dy^i_s)^2 \right) \right] ds,
$$

$$
II = \frac{1}{\delta k} \int_t^{t+\delta} E_{\mathbb{P}^N} \left[ \exp \left( \kappa \delta k N W_1(\mu^N_s, \mu_s)^2 \wedge W_1(\mu^N_s, \mu_s)^{2k} \right) \right] ds.
$$

We first estimate the remainder term $II$: by inequality (43), we have

$$
II \leq \frac{1}{\delta k} \left[ 1 + \sup_{t \in [0, T]} \kappa \delta k \int_0^{\infty} e^{\kappa \delta k z} \mathbb{P}^N(\mathbb{W}_1(\mu^N_t, \mu_t)^2 \wedge \mathbb{W}_1(\mu^N_t, \mu_t)^{2k} \geq z) dz \right] 
\leq 1 + \sup_{t \in [0, T]} \int_0^{N} e^{\kappa \delta k z} \mathbb{P}^N(\mathbb{W}_1(\mu^N_t, \mu_t) \geq N^{-1/(2k)} z^{1/(2k)}) dz 
+ \sup_{t \in [0, T]} \int_0^{\infty} e^{\kappa \delta k z} \mathbb{P}^N(\mathbb{W}_1(\mu^N_t, \mu_t) \geq N^{-1/2} z^{1/2}) dz.
$$

We first estimate the integral over $[0, N]$:

$$
\int_0^{N} e^{\kappa \delta k z} \mathbb{P}^N(\mathbb{W}_1(\mu^N_t, \mu_t) \geq N^{-1/(2k)} z^{1/(2k)}) dz \leq \int_0^{N} \exp(\kappa \delta k z - CN^{-d/(2k)} z^{d/(2k)}) dz
= N \int_0^{1} \exp(N(\kappa \delta k z - C z^{d/(2k)})) dz \leq 1
$$
for $2 < d \leq 2k$ as soon as $\delta \leq k^{-1}\kappa^{-1} \mathcal{C}$. The case ($d = 2$ and $k \geq 2$) is slightly more technical but elementary and we omit it. For the integral over $[N, \infty)$, we proceed as under Assumption 4 (i) to obtain
\[
\int_N^\infty e^{\kappa \delta kN} \mathbb{P}^N \left( W_t(\mu^N_t, \mu_t) \geq N^{-1/2} z^{1/2} \right) dz \lesssim \int_0^\infty \exp \left( \left( \kappa \delta k - \mathcal{C} \right) z \right) dz \lesssim 1
\]
for $\delta \leq k^{-1}\kappa^{-1} \mathcal{C}$ and we conclude $II \lesssim 1$ in that case.

We next turn to the term $I$. Observe first that by exchangeability
\[
I \leq \sup_{1 \leq \ell \leq k-1, t \in [0, T]} \mathbb{E}^N_\ell \left[ \exp \left( \kappa \delta N \right) \int_{\mathbb{R}^d} \delta^\ell b(t, X^N_t, y^\ell, \mu_t)(\mu^N_t - \mu_t)1_\mathcal{C} (dy) \right]^2 \]
\[
= 1 + \sum_{p \geq 1} \frac{(\kappa \delta N)^p}{p!} \sup_{1 \leq \ell \leq k-1, t \in [0, T]} \mathbb{E}^N_\ell \left[ \int_{\mathbb{R}^d} \delta^\ell b(t, X^N_t, y^\ell, \mu_t)(\mu^N_t - \mu_t)1_\mathcal{C} (dy) \right]^{2p} \]
We then use the following estimate, reminiscent of moment bounds for $U$-statistics, however in a weaker and simpler form in our context. For an integer $\ell \geq 1$, we call $\mathcal{S}_\ell$ the class of functions $f : [0, T] \times \mathbb{R}^d \times (\mathbb{R}^d)^\ell \rightarrow \mathbb{R}$ that are Lipschitz continuous in the space variables.

**Lemma 22.** Let $p \geq 1$. For $1 \leq \ell \leq k$, $f \in \mathcal{S}_\ell$ and $N \geq k+1$, we have
\[
\mathbb{E}^N_\ell \left[ \int_{\mathbb{R}^d} f(t, X^N_t, y^\ell)(\mu^N_t - \mu_t)1_\mathcal{C} (dy) \right]^{2p} \leq \frac{p! K^p_\ell}{(N-k)^p} \| f(t, \cdot, \mu_t) \|_{\text{Lip}}^{2p}
\]
for some explicitly computable $K_\ell = K_\ell(b) > 0$.

The proof of Lemma 22 is quite elementary, yet technical, and is delayed until Appendix 8.2. The remainder of the proof of (39) is then straightforward: by Lemma 22 with $f(t, x, y^\ell) = \delta^\ell b(t, x, y^\ell, \mu_t)$, it follows that
\[
I \leq 1 + \sum_{p \geq 1} \frac{(\kappa \delta N)^p}{p!} \frac{K^p_\ell}{(N-k)^p} \sup_{t \in [0, T]} |\delta^\ell b(t, \cdot, \mu_t)|_{\text{Lip}}^{2p} \lesssim 1
\]
as soon as $\delta < \kappa^{-1}k^{-1}(k+1)^{-1}K^{-1} \sup_{t \in [0, T]} |\delta^\ell b(t, \cdot, \mu_t)|_{\text{Lip}}^{2}$. Since $II \lesssim 1$ is established as well, we obtain Proposition 19 under Assumption 4 (ii) provided Lemma 22 is proved.

We finally prove (39) under Assumption 4 (iii) when $d \geq 1$ is arbitrary. By (42) and Jensen’s inequality together with the exchangeability of $(X^i_t)_{1 \leq i \leq N}$, we have
\[
\mathbb{E}^N \left[ \exp \left( \tau \left( \frac{1}{M^N_t} - \frac{1}{N} \right) \right) \right] \leq \frac{1}{\delta} \int_{\tau}^{\tau+\delta} \mathbb{E}^N \left[ \exp \left( \kappa \delta N \right) \sum_{i=1}^N \int_{\mathbb{R}^d} \tilde{b}(s, X^i_t, y)(\mu^N_s - \mu_s)(dy) \right] ds \leq \sup_{s \in [0, T]} \mathbb{E}^N \left[ \exp \left( \kappa N \right) \int_{\mathbb{R}^d} \tilde{b}(s, X^N_s, y)(\mu^N_s - \mu_s)(dy) \right] \leq 1 + \sum_{p \geq 1} \frac{(\kappa \delta N)^p}{p!} \sup_{s \in [0, T]} \mathbb{E}^N \left[ \int_{\mathbb{R}^d} \tilde{b}(s, X^N_s, y)(\mu^N_s - \mu_s)(dy) \right]^{2p} \lesssim 1
\]
as soon as $\delta < \frac{1}{2} \kappa^{-1} K^{-1} \sup_{s \in [0, T]} [\tilde{b}(s, \cdot)]_{\text{Lip}}^{-2}$ by Lemma 22. Therefore (39) is established under Assumption 4 (iii) and this completes the proof of Proposition 19.
7. Proof of the Nonparametric Estimation Results

We will repeatedly use estimates of the form
\begin{equation}
\int_0^\infty \exp(-z^r)dz \leq 2r^{-1}\nu^{1-r}\exp(-\nu^r), \quad \nu, r > 0, \quad \nu \geq (2/r)^{1/r}.
\end{equation}
and
\begin{equation}
\int_0^\infty \exp\left(-\frac{az^p}{b + cz^p/2}\right)dz \leq C_p \max\left(\left(\frac{a}{b}\right)^{-1/p}, \left(\frac{a}{c}\right)^{-2/p}\right), \quad a, b, c, p > 0,
\end{equation}
with \(C_p = 2 \int_0^\infty \exp\left(-\frac{1}{2}(\min(\sqrt{z}, 2))^p\right)dz\), stemming from the rough bound
\begin{equation}
\exp\left(-\frac{az^p}{b + cz^p/2}\right) \leq \exp\left(-\frac{az^p}{2b}\right) + \exp\left(-\frac{az^p}{2c}\right), \quad z > 0.
\end{equation}
The estimate (48) is far from being optimal, but will be sufficient for our purpose.

7.1. Proof of Theorem 7.

Preliminaries. We first state local upper and lower estimates on \((t, x) \mapsto \mu_t(x)\).

Lemma 23. Work under Assumptions 1, 2 and 4. Let \((t_0, x_0) \in (0, T] \times \mathbb{R}^d\). Let \(r > 0\) and \([r_1, r_2] \subset (0, T)\).

(i) There exists \(\kappa_5\) depending on \((t_0, x_0), r\) and \(b\) such that
\begin{equation}
\sup_{t \in [0, T], \|x - x_0\| \leq r} |b(t, x, \mu_t)| \leq \kappa_5.
\end{equation}

(ii) There exist \(\kappa_3, \kappa_4\) depending on \(x_0, r_1, r_2, r\) and \(b\) such that
\begin{equation}
0 < \kappa_4 \leq \inf_{t \in [r_1, r_2], \|x - x_0\| \leq r} \mu_t(x) \leq \sup_{t \in [r_1, r_2], \|x - x_0\| \leq r} \mu_t(x) \leq \kappa_3.
\end{equation}

In turn, for a compactly supported kernel \(K\), this implies the existence of \(r = r(K)\) such that the estimate
\begin{equation}
|K_h(x_0 - \cdot)|_{L^2(\mu_t)}^2 = \int_{\mathbb{R}^d} h^{-2d}K(h^{-1}x)^2 \mu_t(x_0 - x)dx \leq \kappa_3(r)h^{-d}|K|_2^2
\end{equation}
holds true.

Proof. For \(x\) such that \(|x - x_0| \leq r\), we have
\begin{equation}
|b(t, x, \mu_t)| \leq |b(t, 0, \delta_0)| + |b|_{\text{Lip}}(|x| + W_1(\mu_t, \delta_0))
\end{equation}
\begin{equation}
\leq \sup_{t \in [0, T]} |b(t, 0, \delta_0)| + |b|_{\text{Lip}}(|x_0| + r + \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |y|\mu_t(dy))
\end{equation}
that defines \(\kappa_5\) thanks to Assumption 4 and Lemma 20. This establishes (49). The estimate (50)
follows from classical Gaussian tail estimates for the solution of parabolic equations. We refer for example in our context to Corollary 8.2.2 of [7]: for every compact interval \([r_1, r_2] \subset (0, T)\), there exist constants \(c_\pm > 0\) depending on \(r_1, r_2\) and \(b\) only such that
\begin{equation}
\exp\left(-c_-(1 + |x|^2)\right) \leq \mu_t(x) \leq \exp\left(c_+(1 + |x|^2)\right)
\end{equation}
for every \((t, x) \in [r_1, r_2] \times \mathbb{R}^d\). This establishes (50). Actually, if we moreover have 1/2-Hölder
smoothness in time for the diffusion coefficient, investigating further Theorem 7.3.3 and Example
8.3.10 of [7], it is possible to prove \(\sup_{x \in \mathbb{R}^d} \mu_t(x) < \infty\) uniformly in \(t \in [r_1, r_2]\), hence \(\kappa_3\) can be
taken independently of \(r\).

□
We next prove a standard bias-variance estimate for the quadratic risk of \( \hat{\mu}_h^N(t_0, x_0) \).

**Lemma 24.** In the setting of Theorem 7, if \( K \) is a bounded and compactly supported kernel and \( h \in \mathcal{H}_N \), we have

\[
E_{P^N} \left[ (\hat{\mu}_h^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right] \lesssim \mathcal{B}_h^N(\mu)(t_0, x_0)^2 + V_h^N,
\]

up to a constant that depends (continuously) on \((t_0, x_0), |K|_\infty \) and \( h \), and where \( \mathcal{B}_h^N(\mu)(t_0, x_0) \) is defined in (14) and \( V_h^N \) in (12).

**Proof.** Write \( \hat{\mu}_h^N(t_0, x_0) - \mu_{t_0}(x_0) = I + II \), with

\[
I = \int_{\mathbb{R}^d} K_h(x_0 - x) \mu_{t_0}(x) dx - \mu_{t_0}(x_0)
\]

and

\[
II = \int_{\mathbb{R}^d} K_h(x_0 - x) (\mu_h^N(dx) - \mu_{t_0}(x) dx).
\]

We have \( I^2 \leq \mathcal{B}_h^N(\mu)(t_0, x_0)^2 \) for the squared bias term. For the variance term, using successively Theorem 18 and the estimate (51) we have

\[
E_{P^N} [II]^2 = \int_0^\infty P^N(|II| \geq z^{1/2}) dz
\]

\[
\leq 2\kappa_1 \int_0^\infty \exp \left( -\frac{\kappa_2 N z}{|K_h(x_0 - \cdot)|^2_{L^2(\mu_{t_0})} + |K_h(x_0 - \cdot)|_{\infty} z^{1/2}} \right) dz
\]

\[
\leq 2\kappa_1 \int_0^\infty \exp \left( -\frac{\kappa_2 N h^d z}{\kappa_3 |K|^2 + |K|_{\infty} z^{1/2}} \right) dz
\]

\[
\lesssim (Nh^d)^{-1} (1 + (Nh^d)^{-1}) \lesssim V_h^N
\]

where we used (48) and the fact that \( \max_{h \in \mathcal{H}_N} (Nh^d)^{-1} \lesssim 1 \).

**Completion of proof of Theorem 7.** We essentially repeat the main argument of the Goldenshluger-Lepski method (see e.g. [31, 32, 33] for the pointwise risk). We nevertheless give a proof for sake of completeness. Recall that \( \hat{h}^N \) denotes the data-driven bandwidth defined in (13).

**Step 1**: For \( h \in \mathcal{H}_N \), we successively have

\[
E_{P^N} \left[ (\hat{\mu}_{GL}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right]
\]

\[
\lesssim E_{P^N} \left[ (\hat{\mu}_{GL}^N(t_0, x_0) - \mu_{t_0}(0))^2 \right] + E_{P^N} \left[ (\mu_{t_0}(x_0))^2 \right]
\]

\[
\lesssim E_{P^N} \left[ (\hat{\mu}_{h}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right] - V_h^N + V_h^N + V_h^N + V_h^N + E_{P^N} \left[ (\mu_{h}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right]
\]

\[
\lesssim E_{P^N} \left[ (\mu_{h}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right] + E_{P^N} \left[ (\mu_{h}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right]
\]

\[
\lesssim E_{P^N} \left[ (\mu_{h}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right] + E_{P^N} \left[ (\mu_{h}^N(t_0, x_0) - \mu_{t_0}(x_0))^2 \right]
\]

where we applied Lemma 24 in order to obtain the last line.
Step 2: We first estimate $A^N_h$. Write $\mu_h(t_0, x_0)$ for $\int_{\mathbb{R}^d} K_h(x_0 - x) \mu_{t_0}(x) dx$. For $h, h' \in \mathcal{N}_1$ with $h' \leq h$, since
\[
(\hat{\mu}_h^N(t_0, x_0) - \hat{\mu}_{h'}^N(t_0, x_0))^2 \leq 4(\hat{\mu}_h^N(t_0, x_0) - \mu_h(t_0, x_0))^2 + 4(\mu_h(t_0, x_0) - \mu_{t_0}(x_0))^2 + 4(\hat{\mu}_{h'}^N(t_0, x_0) - \mu_{h'}(t_0, x_0))^2,
\]
we have
\[
(\hat{\mu}_h^N(t_0, x_0) - \hat{\mu}_{h'}^N(t_0, x_0))^2 - V^N_h - V^N_{h'} \leq 8B^N_h(\mu)(t_0, x_0)^2 + 4(\hat{\mu}_h^N(t_0, x_0) - \mu_h(t_0, x_0))^2 - V^N_h
\]
\[+ 4(\hat{\mu}_{h'}^N(t_0, x_0) - \mu_{h'}(t_0, x_0))^2 - V^N_{h'} \]
using $h' \leq h$ in order to bound $(\hat{\mu}_h^N(t) - \mu_h(t_0, x_0))^2$ by the bias at scale $h$. Taking maximum over $h' \leq h$, we obtain
\[
\max_{h' \leq h} \left\{ (\hat{\mu}_h^N(t_0, x_0) - \hat{\mu}_{h'}^N(t_0, x_0))^2 - V^N_h - V^N_{h'} \right\}_+
\leq 8B^N_h(\mu)(t_0, x_0)^2 + \left\{ 4(\hat{\mu}_h^N(t_0, x_0) - \mu_h(t_0, x_0))^2 - V^N_h \right\}_+ + \max_{h' \leq h} \left\{ 4(\hat{\mu}_{h'}^N(t_0, x_0) - \mu_{h'}(t_0, x_0))^2 - V^N_{h'} \right\}_+.
\]

Step 3: We estimate the expectation of the first stochastic term in the right-hand side of (53). We refine the computation of the term $\Pi$ in the proof of Lemma 24. By Theorem 18 and using estimates of the form (47) and (48), we have
\[
\mathbb{E}_{\mathbb{P}_N}\left[ 4(\hat{\mu}_h^N(t_0, x_0) - \mu_h(t_0, x_0))^2 - V^N_h \right]_+
= \int_0^{\infty} \mathbb{P}_N(4(\hat{\mu}_h^N(t_0, x_0) - \mu_h(t_0, x_0))^2 - V^N_h \geq z) dz
= \int_0^{\infty} \mathbb{P}_N(\hat{\mu}_h^N(t_0, x_0) - \mu_h(t_0, x_0) \geq \frac{1}{2}(V^N_h + z)^{1/2}) dz
\leq 2\kappa_1 \int_0^{\infty} \exp \left( -\frac{\kappa_2 Nh^d}{K^2} \frac{1}{2} z \right) \frac{1}{K^{1/2} + |K| \frac{1}{2} z^{1/2}} dz
\lesssim \int_0^{\infty} \exp \left( -\frac{\kappa_2 Nh^d z}{8\kappa_3 |K|^2} \right) dz + \int_0^{\infty} \exp \left( -\frac{\kappa_2 Nh^d z^{1/2}}{4|K|} \right) dz
\lesssim (Nh^d)^{-1} \exp \left( -\frac{\kappa_2 Nh^d \mathbb{E}_h V^N_h}{8\kappa_3 |K|^2} \right) + (Nh^d)^{-2} Nh^d (V^N_h)^{1/2} \exp \left( -\frac{\kappa_2 Nh^d (V^N_h)^{1/2}}{4|K|} \right)
\lesssim (Nh^d)^{-1} N^{-\frac{1}{2}} + (Nh^d)^{-3/2} (\log N)^{1/2} \exp \left( -\frac{\kappa_2 |K| \sigma_1^{1/2}}{4|K|} (\log N)^{5/2} \right),
\lesssim N^{-2}
\]
as soon as $\sigma_1 \geq 16\kappa_2^{-1}\kappa_3$, thanks to $\max_{h \in \mathcal{N}_1} (Nh^d)^{-1} \lesssim 1$, and using $\min_{h \in \mathcal{N}_1} h \geq (N^{-1}(\log N)^2)^{1/d}$ to show that the second term is negligible in front of $N^{-2}$.

Step 4: For the second stochastic term, we have the rough estimate
\[
\mathbb{E}_{\mathbb{P}_N}\left[ \max_{h' \leq h} \left\{ 4(\hat{\mu}_h^N(t_0, x_0) - \mu_{h'}(t_0, x_0))^2 - V^N_{h'} \right\}_+ \right]
\]
\[ \sum_{h' \leq h} \mathbb{E}_{\mathbb{P}^N} \left[ \{ 4(\hat{\mu}_{h'}(t_0,x_0) - \mu_{h'}(t_0,x_0))^2 - V_{h'} \}^+ \right] \leq \text{Card}(\mathcal{H}_N^2) N^{-2} \leq N^{-1} \]

where we used Step 3 to bound each term \( \mathbb{E}_{\mathbb{P}^N} \left[ \{ 4(\hat{\mu}_{h'}(t_0,x_0) - \mu_{h'}(t_0,x_0))^2 - V_{h'} \}^+ \right] \) independently of \( h \) together with \( \text{Card}(\mathcal{H}_N^2) \leq N \). In conclusion, we have through Steps 2-4 that \( \mathbb{E}_{\mathbb{P}^N} [A_h^N] \leq N^{-1} + B_h^N(\mu)(t_0,x_0)^2 \). Therefore, from Step 1, we conclude

\[ \mathbb{E}_{\mathbb{P}^N} \left[ (\hat{\pi}_N^N(t_0,x_0) - \pi(t_0,x_0))^2 \right] \leq B_h^N(\mu)(t_0,x_0)^2 + V_h^N + N^{-1} \]

for any \( h \in \mathcal{H}_N^2 \). Since \( N^{-1} \leq V_h^N \) always, the proof of Theorem 7 is complete.

7.2. Proof of Theorem 9.

Preliminaries. The assumptions of Theorem 9 are in force in this section. We first study the fluctuations of the random measure \( \pi^N(dt,dx) - \pi(t,x)dt dx \), where \( \pi^N(dt,dx) = N^{-1} \sum_{i=1}^{N} \delta_{X_i^t}(dx)X^t(dt) \).

**Lemma 25.** Let \( \phi : (0,T) \times \mathbb{R}^d \to \mathbb{R} \) be bounded and compactly supported. The following decomposition holds

\[
\int_{[0,T] \times \mathbb{R}^d} \phi(t,x)(\pi^N(dt,dx) - \pi(t,x)dt dx)
\]

\[ = \int_0^T \int_{\mathbb{R}^d} \phi(t,x) (b(t,x,\mu_t)\mu^N_t(dx) - \mu_t(x)) dx + \xi^N_t(x) \mu^N_t(x) dx dt + \mathcal{M}^N_t(\phi), \]

where \( \xi^N_t(x) = b(t,x,\mu^N_t) - b(t,x,\mu_t) \) and \( \mathcal{M}^N_t(\phi) = (\mathcal{M}^N_t(\phi)^1, \ldots, \mathcal{M}^N_t(\phi)^d) \) is a \( d \)-dimensional \( \mathbb{P}^N \)-continuous martingale with predictable compensator such that

\[
\langle \mathcal{M}^N_t(\phi)^k \rangle_t \leq N^{-1} |\text{Tr}(\phi)|_\infty \int_0^t \int_{\mathbb{R}^d} \phi(s,x)^2 \mu^N_s(dx) ds.
\]

**Proof.** we have

\[
\int_{[0,T] \times \mathbb{R}^d} \phi(t,x)(\pi^N(dt,dx) - \pi(t,x)dt dx)
\]

\[ = N^{-1} \sum_{i=1}^{N} \int_0^T \phi(t,X_i^t) (\sigma(t,X_i^t)dB_t^i + b(t,X_i^t,\mu^N_t) dt) - \int_{[0,T] \times \mathbb{R}^d} \phi(t,x)b(t,x,\mu_t)\mu_t(x) dt dx
\]

\[ = \int_0^T \int_{\mathbb{R}^d} \phi(t,x) (b(t,x,\mu^N_t)\mu^N_t(dx) - b(t,x,\mu_t)\mu_t(x) dx dt + \mathcal{M}^N_t(\phi),
\]

where

\[
\mathcal{M}^N_t(\phi) = N^{-1} \sum_{i=1}^{N} \int_0^t \phi(s,X_i^s)\sigma(s,X_i^s)dB_s^i
\]

is a martingale with bracket satisfying (54). The result follows.

We next have a bias-variance estimate for the quadratic risk of \( \hat{\pi}_h(t_0,x_0) \), in the same spirit as in Lemma 24.

**Lemma 26.** Assume that \( H \otimes K \) is a bounded and compactly supported kernel on \( (0,T) \times \mathbb{R}^d \). Let \( h \in \mathcal{H}_N^2 \). Then

\[ \mathbb{E}_{\mathbb{P}^N} \left[ (\hat{\pi}_h^N(t_0,x_0) - \pi(t_0,x_0))^2 \right] \leq B_h^N(\pi)(t_0,x_0)^2 + V_h^N, \]
up to a constant that (continuously) depends on \((t_0, x_0), |H \otimes K|_\infty\) and \(b\), and where \(B_h^N(\pi)(t_0, x_0)\) is defined in (19) and \(V_h^{N}\) in (17).

Proof. Write \(\hat{\pi}_h^N(t_0, x_0) - \pi(t_0, x_0) = I + II\), with

\[
I = \int_0^T \int_{\mathbb{R}^d} (H \otimes K)_h(t_0 - t, x_0 - x) \pi(t, x) dx dt - \pi(t_0, x_0)
\]
and

\[
II = \int_0^T \int_{\mathbb{R}^d} (H \otimes K)_h(t_0 - t, x_0 - x)(\pi^N(dt, dx) - \pi(t, x) dt dx).
\]

We have \(|I|^2 \leq B_h^N(\pi)(t_0, x_0)^2\) for the squared bias term. For the variance term, applying the decomposition of Lemma 25 with test function \(\phi(t, x) = (H \otimes K)_h(t_0 - t, x_0 - x)\), we obtain

\[
|II|^2 \lesssim III + IV + V,
\]
with

\[
III = \left| \int_0^T \int_{\mathbb{R}^d} \phi(t, x) b(t, x, \mu)(\mu_t^N(dx) - \mu_t(x) dx) dt \right|^2,
\]
\[
IV = \left| \int_0^T \int_{\mathbb{R}^d} \phi(t, x) \xi_t^N(x) \mu_t^N(dx) dt \right|^2,
\]
\[
V = |M_t^N(\phi)|^2,
\]

where \(\xi_t^N(x) = b(t, x, \mu_t^N) - b(t, x, \mu_t)\). Writing \(b = (b^1, \ldots, b^d)\) in components, note first that for \(\nu(dt, dx) = \mu_t(dx)T^{-1} dt\), we have

\[
|\phi \cdot b^k(\cdot, \mu)|^2_{L^2(\nu)} \leq \kappa_5 |H \otimes K|_\infty(h_1h_2^d)^{-1}, \quad |\phi \cdot b^k(\cdot, \mu)|_\infty \leq \kappa_5 |H \otimes K|_\infty(h_1h_2^d)^{-1},
\]

by Lemma 23 and the compactness of the support of \(\phi\). By Theorem 18 applied to \(\nu^N(dt, dx) - \nu(dt, dx) = (\mu_t^N(dx) - \mu_t(dx))T^{-1} dt\), it follows that

\[
\mathbb{E}_{\nu^N}[III] \lesssim d \sum_{k=1}^d \int_0^T \int_{\mathbb{R}^d} \mathbb{P}^N(\int_{[0, T] \times \mathbb{R}^d} \phi(t, x) b^k(t, x, \mu_t)(\mu_t^N(dx) - \mu_t(x) dx)T^{-1} dt \geq z^{1/2}) dz
\]
\[
\leq 2d\kappa_1 \int_0^T \int_{\mathbb{R}^d} \mathbb{P}^N(\int_{[0, T] \times \mathbb{R}^d} \phi(t, x) b^k(t, x, \mu_t)(\mu_t^N(dx) - \mu_t(x) dx)T^{-1} dt \geq z^{1/2}) dz
\]
\[
\lesssim (Nh_1h_2^d)^{-1}(1 + (Nh_1h_2^d)^{-1}) \lesssim V_h^N,
\]

using \(\max_{h \in \mathcal{H}} (Nh_1h_2^d)^{-1} \leq 1\). We conclude

\[
(56) \quad \mathbb{E}_{\nu^N}[III] \lesssim V_h^N.
\]

We next turn to the term \(IV\). We need a deviation result for the fluctuation \(\xi_t^N(x) = b(t, x, \mu_t^N) - b(t, x, \mu_t)\) that will also be helpful later.

Lemma 27. There exist positive numbers \(\kappa_6, \kappa_7\) and \(\kappa_8\), depending on \(b\), such that for large enough \(N\)

\[
\sup_{0 \leq t \leq T} \mathbb{P}^N(\xi_t^N(X_t^N) \geq u) \leq \kappa_6 \exp \left(-\frac{\kappa_7 Nu^2}{1 + Nu^{1/2}}\right) \text{ for } u \geq \kappa_8 N^{-1/2}.
\]
Proof. Writing \( \xi^N_t(X^N_t) = (\xi^N_t(X^N_t)^1, \ldots, \xi^N_t(X^N_t)^d) \) in components, we have

\[
P^N(|\xi^N_t(X^N_t)| \geq u) \leq \sum_{\ell=1}^d P^N(|\xi^N_t(X^N_t)^\ell| \geq ud^{-1})
\]

It suffices thus to prove the result for each component \( \xi^N_t(X^N_t)^\ell \), substituting \( \kappa_6 \) and \( \kappa_8 \) by \( d\kappa_6 \) and \( d\kappa_8 \) to obtain the general case. For notational simplicity, we drop the superscript \( \ell \) and prove the result for \( \xi^N_t(X^N_t) \) instead of \( |\xi^N_t(X^N_t)| \), up to a inflation of \( \kappa_6 \) by a factor 2.

By (40) in the proof of Proposition 19, we may write

\[
\xi^N_t(X^N_t) = \zeta^N_t(X^N_t) + \mathcal{R}_k,
\]

with

\[
\zeta^N_t(X^N_t) = \sum_{\ell=1}^{k-1} \frac{1}{\ell!} \int_{(\mathbb{R}^d)^\ell} \delta^\ell b(t, X^N_t, y^\ell, \mu_t)(\mu^N_t - \mu_t) \otimes^\ell (dy^\ell),
\]

having \( k = 1 \) under Assumption 4 (i), with \( |\mathcal{R}_k| \lesssim W_1(\mu^N_t, \mu_t) \wedge W_1(\mu^N_t, \mu_t)^k \) under Assumption 4 (ii) by (41), and having \( k = 1 \) with \( \delta^1 b = \bar{b} \) and \( \mathcal{R}_k = 0 \) under Assumption 4 (iii). It is enough to prove the deviation bound for each term separately.

Let \( u \geq 0 \). We first bound the remainder term \( \mathcal{R}_k \). Applying (35) in the proof of Theorem 18 for the event \( A^N = \{ W_1(\mu^N_t, \mu_t) \wedge W_1(\mu^N_t, \mu_t)^k \geq u \} \) we obtain

\[
P^N(W_1(\mu^N_t, \mu_t) \wedge W_1(\mu^N_t, \mu_t)^k \geq u) \leq C_1^{K(K+1)/8} P^N(W_1(\mu^N_t, \mu_t) \wedge W_1(\mu^N_t, \mu_t)^k \geq u)^4 - K
\]

where the last estimate stems from the deviation inequality (43) of Fournier and Guillin [26]. Under Assumption 4 (i), with \( (d = 1 \) and \( k = 1 \) or under Assumption 4 (ii) with \( (d = 2 \) and \( k \geq 2 \) or \( (d \geq 3 \) and \( k \geq d/2 \)),

\[
\varepsilon_N(u^{1/k}) \wedge \varepsilon_N(u) \lesssim \exp(-C_N u^2) \text{ for every } u \geq 0
\]

as follows from the definition of \( \varepsilon_N(x) \) in (43). Therefore \( \mathcal{R}_k \) has the right order. As for the main term, we first note that for every \( p \geq 2 \), we have

\[
\mathbb{E}_{\Xi^N} \left[ |\xi^N_t(X^N_t)|^p \right] \leq e^{p-1} \sum_{\ell=1}^{k-1} \frac{1}{\ell!} \mathbb{E}_{\Xi^N} \left[ \int_{(\mathbb{R}^d)^\ell} \delta^\ell b(t, X^N_t, y^\ell, \mu_t)(\mu^N_t - \mu_t) \otimes^\ell (dy^\ell)^p \right]
\]

by Lemma 22 and Cauchy-Schwarz’s inequality, for large enough \( N \) and some \( C_5 \) depending on \( b \). With no loss of generality, we take \( C_5 \geq 1 \). In particular, by Cauchy-Schwarz’s inequality,

\[
|\mathbb{E}_{\Xi^N}[\xi^N_t(X^N_t)]| \leq \sqrt{2} C_5 N^{-1/2} = \kappa_8 N^{-1/2}
\]

that defines the constant \( \kappa_8 \). We next use the following version of Bernstein inequality that can be found in Lemma 8 in Birgé and Massart [6]: if \( Z \) is a real-valued random variable such that \( \mathbb{E}[|Z|^p] \leq \frac{4^p u^{2p} c^{p-2}}{v^2 + cu} \) for \( c, v > 0 \) and every \( p \geq 2 \), then

\[
P(Z - \mathbb{E}[Z] \geq u) \leq \exp \left( -\frac{u^2}{v^2 + cu} \right) \text{ for every } u \geq 0.
\]
We then apply (57) to $Z = \zeta_N^t(X_N^t)$ with $c = C_5 N^{-1/2}$ and $v = \sqrt{N} N^{-1/2} C_5$ and obtain

$$\mathbb{P}^N(\zeta_N^t(X_N^t) - \mathbb{E}_{\mathbb{P}^N}[\zeta_N^t(X_N^t)] \geq u) \leq \exp\left(-\frac{C_6 N u^2}{1 + N^{1/2} u}\right) \text{ for every } u \geq 0,$$

with $C_6 = (4 C_5^2)^{-1}$ using $C_5 \geq 1$. Finally, for $u \geq \kappa_8 N^{-1/2}$, setting $u' = u - \kappa_8 N^{-1/2}$ and applying (35) in the proof of Theorem 18 for the event $\{\zeta_N^t(X_N^t) \geq u\}$, we derive

$$\mathbb{P}^N(\zeta_N^t(X_N^t) \geq u) \lesssim N^{-1}(\|\phi\|_2^2 N^{-1/2} \|\phi\|_4^2 + N^{-1} \|\phi\|_\infty^2)$$

and the lemma follows with $\kappa_7 = 4^{-K} C_6$. \hfill \Box

We are ready to bound the term IV. Applying Cauchy-Schwarz’s inequality twice, we have

$$\mathbb{E}_{\mathbb{P}^N}[IV] \leq \mathbb{E}_{\mathbb{P}^N}\left[\left(\int_0^T \int_{\mathbb{R}^d} |\phi(t,x)|^2 \mu_t^N(dx)dt\right)^2\right]^{\frac{1}{2}} \mathbb{E}_{\mathbb{P}^N}\left[\left(\int_0^T \int_{\mathbb{R}^d} |\xi_t^N(x)|^2 \mu_t^N(dx)dt\right)^2\right]^{\frac{1}{2}}.$$

On the one hand, by exchangeability and Lemma 27, we have

$$\mathbb{E}_{\mathbb{P}^N}\left[\left(\int_0^T \int_{\mathbb{R}^d} |\xi_t^N(x)|^2 \mu_t^N(dx)dt\right)^2\right]^{\frac{1}{2}} \leq T \int_0^T \mathbb{E}_{\mathbb{P}^N}[|\xi_t^N(X_t^N)|^4] dt^{\frac{1}{2}} \lesssim \sup_{0 \leq t \leq T} \left(\int_0^\infty \mathbb{P}^N(\xi_t^N(X_t^N) \geq z^{1/4})dz\right)^{1/2} \lesssim \left(\kappa_4^2 N^{-2} + \kappa_6 \int_0^\infty \exp\left(-\frac{\kappa_7 N z^{1/2}}{1 + N^{1/2} z^{1/4}}\right)dz\right)^{1/2} \lesssim N^{-1}.$$

(58)

On the other hand

$$\mathbb{E}_{\mathbb{P}^N}\left[\left(\int_0^T \int_{\mathbb{R}^d} |\phi(t,x)|^2 \mu_t^N(dx)dt\right)^2\right]^{1/2} \lesssim \int_0^T \int_{\mathbb{R}^d} |\phi(t,x)|^2 \mu_t(x)dxdt + \mathbb{E}_{\mathbb{P}^N}\left[\left(\int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)|^2 (\mu_t^N(dx) - \mu_t(dx))dt\right)^2\right]^{1/2} \lesssim |\phi|_2^2 + \left(\int_0^\infty \mathbb{P}^N\left(\left|\int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)|^2 (\mu_t^N(dx) - \mu_t(dx))dt\right| \geq z^{1/2}\right)dz\right)^{1/2} \lesssim |\phi|_2^2 + \left(2 \kappa_1 \int_0^\infty \exp\left(-\frac{\kappa_2 N z}{\kappa_3 T^{1-1/2} \|\phi\|_4^2 + \|\phi\|_\infty^2 z^{1/2}}\right)dz\right)^{1/2} \lesssim |\phi|_2^2 + (N^{-1} |\phi|_4^4 + N^{-2} |\phi|_\infty^4)^{1/2} \lesssim |\phi|_2^2 + N^{-1/2} |\phi|_4^2 + N^{-1} |\phi|_\infty^2,$$

(59)

using Lemma 23 and the fact that $\phi$ is compactly supported to obtain the first term and Theorem 18 applied to $\nu^N(dt, dx) - \nu(dt, dx) = (\mu_t^N(dx) - \mu_t(dx)) T^{-1} dt$ together with $|\phi|_2^2 \leq \kappa_3 T^{-1} |\phi|_4^4$ to obtain the second term. Putting together (58) and (59), we conclude

$$\mathbb{E}_{\mathbb{P}^N}[IV] \lesssim N^{-1} (|\phi|_2^2 + N^{-1/2} |\phi|_4^2 + N^{-1} |\phi|_\infty^2)$$
\begin{align}
(60) \quad \Leftrightarrow (Nh_1h_2^d)^{-1}(1 + (Nh_1h_2^d)^{-1/2} + (Nh_1h_2^d)^{-1}) \lesssim V_h^N.
\end{align}

Finally, by Lemma 25 and Theorem 18 again we have
\[
\mathbb{E}_{\pi} [V] = \sum_{k=1}^{d} \mathbb{E}_{\pi} [\langle M^N (\phi)_k \rangle_T]
\leq dN^{-1} |\text{Tr}(c)|_\infty \mathbb{E}_{\pi} \left[ \int_{[0,T] \times \mathbb{R}^d} \phi(s, x)^2 \mu_s^N (dx) ds \right]
\lesssim N^{-1} |\phi|^2 + N^{-1} \mathbb{E}_{\pi} \left[ \int_{[0,T] \times \mathbb{R}^d} \phi(s, x)^2 (\mu_s^N (dx) - \mu_s (dx)) T^{-1} ds \right]
\lesssim N^{-1} |\phi|^2 + N^{-1} \int_0^\infty \mathbb{P}(\int_{[0,T] \times \mathbb{R}^d} \phi(s, x)^2 (\mu_s^N (dx) - \mu_s (x)) T^{-1} ds \geq z) dz
\leq N^{-1} |\phi|^2 + N^{-1} 2\kappa_1 \int_0^\infty \exp \left( - \frac{\kappa_2 N z^2}{\kappa_3 T^{-1} |\phi|^2 + |\phi|^2 \infty} \right) dz
\lesssim N^{-1} |\phi|^2 + N^{-3/2} |\phi|^2 + N^{-2} |\phi|^2 \infty
\end{align}

(61) \quad \lesssim (Nh_1h_2^d)^{-1}(1 + (Nh_1h_2^d)^{-1/2} + (Nh_1h_2^d)^{-1}) \lesssim V_h^N.

Putting together (56), (60) and (61) establishes \( \mathbb{E}_{\pi} [II^2] \lesssim V_h^N \) and concludes the proof of Lemma 26.

Completion of proof of Theorem 9. Let \( (h, h) \in \mathcal{H}_1^N \times \mathcal{H}_2^N \) and \( (t_0, x_0) \in (0, T) \times \mathbb{R}^d \). Remember that we set \( \pi(t, x) = b(t, x, \mu_t) \mu_t (x) \).

**Step 1:** We plan to use the decomposition
\[
\tilde{b}_{h,h}^N (t_0, x_0) - b(t_0, x_0, \mu_{t_0}) = I + II,
\]
with
\[
I = \pi(t_0, x_0) (\mu_{t_0}(x_0) - \hat{\mu}_h^N (t_0, x_0) \vee \omega_3) \bigg/ \mu_{t_0}(x_0) \hat{\mu}_h^N (t_0, x_0) \vee \omega_3
\]
and
\[
II = \left( \hat{\pi}_h^N (t_0, x_0) - \pi(t_0, x_0) \right) \mu_{t_0}(x_0) \bigg/ \mu_{t_0}(x_0) \hat{\pi}_h^N (t_0, x_0) \vee \omega_3.
\]

First, we have
\[
|I| \leq \frac{\kappa_5}{\omega_3} |\mu_{t_0}(x_0) - \hat{\mu}_h^N (t_0, x_0) \vee \omega_3| \lesssim |\mu_{t_0}(x_0) - \hat{\mu}_h^N (t_0, x_0)|
\]
as soon as \( \omega_3 \leq \kappa_4 \) by Lemma 23, for some (small) \( r > 0 \) fixed throughout. In the same way,
\[
|II| \leq \omega_3^{-1} |\hat{\pi}_h^N (t_0, x_0) - \pi(t_0, x_0)|.
\]

Picking \( h = \hat{h}^N, h = \hat{h}^N \), taking square and expectation, we have thus established
\[
\mathbb{E}_{\pi} \left[ |\tilde{b}_{h,h}^N (t_0, x_0) - b(t_0, x_0, \mu_{t_0})|^2 \right]
\lesssim \mathbb{E}_{\pi} \left[ |\hat{\pi}_h^N (t_0, x_0) - \mu_{t_0}(x_0)|^2 \right] + \mathbb{E}_{\pi} \left[ |\hat{\pi}_h^N (t_0, x_0) - \pi(t_0, x_0)|^2 \right]
\]
as soon as \( \omega_3 \leq \kappa_4 \). By Theorem 7, we already have the desired bound for the first term.
Step 2: We study the second term in the right-hand side of (62). For any $h \in 3\mathcal{K}_N$, similarly to the proof of Step 1 in Theorem 7, we have

$$
\mathbb{E}_{\mathcal{F}_N} \left[ \left( \tilde{\mathcal{N}}_h^N (t_0, x_0) - \mathcal{N}_h (t_0, x_0) \right)^2 \right] \lesssim \mathbb{E}_{\mathcal{F}_N} \left[ A_h^N \right] + V_h^N + B_h^N (\mathcal{N}) (t_0, x_0)^2
$$

thanks to Lemma 26. In order to estimate $\mathbb{E}_{\mathcal{F}_N} \left[ A_h^N \right]$, we repeat Step 2 of the proof of Theorem 7 and obtain

$$
\max_{h' \leq h} \{ \left| \tilde{\mathcal{N}}_h^N (t_0, x_0) - \tilde{\mathcal{N}}_{h'} (t_0, x_0) \right|^2 - V_h^N - V_{h'}^N \} + \\
\lesssim B_h^N (\mathcal{N}) (t_0, x_0)^2 + \left\{ 4 \left| \tilde{\mathcal{N}}_h^N (t_0, x_0) - \mathcal{N}_h (t_0, x_0) \right|^2 - V_h^N \right\} + \\
\max_{h' \leq h} \left\{ 4 \left| \tilde{\mathcal{N}}_h^N (t_0, x_0) - \mathcal{N}_h (t_0, x_0) \right|^2 - V_h^N \right\}
$$

with the notation $\pi_h (t_0, x_0) = \int_0^T \int_{\mathbb{R}^d} (H \otimes K) \mu (t_0 - t, x - x) \pi (t, x) dx dt$.

Step 3: We estimate the expectation of the first stochastic term in the right-hand side of (63). By Lemma 25, setting $\phi (t, x) = (H \otimes K) \mu_{t, x} - \mu_{t, x}$, we have

$$
\left\{ 4 \left| \tilde{\mathcal{N}}_h^N (t_0, x_0) - \mathcal{N}_h (t_0, x_0) \right|^2 - V_h^N \right\} + \leq I + II + III,
$$

with

$$
I = \left\{ \int_{[0,T] \times \mathbb{R}^d} \phi (t, x) b^k (t, x, \mu_t) (\mu_t^N (dx) - \mu_t (dx)) dt \right\}^2 - \frac{1}{3} V_h^N
$$

$$
II = \left\{ \int_{[0,T] \times \mathbb{R}^d} \phi (t, x) \xi_t^N (x) (\mu_t^N (dx) - \mu_t (dx)) dt \right\}^2 - \frac{1}{3} V_h^N
$$

$$
III = \left\{ \int_{[0,T] \times \mathbb{R}^d} \mathcal{M}_t^N (\phi) \right\}^2 - \frac{1}{3} V_h^N
$$

For the term $I$, writing $b = (b^1, \ldots, b^d)$ in components, we have

$$
I \leq 12 T^2 \sum_{k=1}^d \left\{ \int_{[0,T] \times \mathbb{R}^d} \phi (t, x) b^k (t, x, \mu_t) (\mu_t^N (dx) - \mu_t (dx)) T^{-1} dt \right\}^2 - \frac{1}{36 T^2} V_h^N
$$

Applying Theorem 18 to $(\mu_t^N (dx) - \mu_t (dx)) T^{-1} dt$ and using the estimates (55) of Lemma 26 above, we infer

$$
\mathbb{E}_{\mathcal{F}_N} \left[ \left\{ \left( \int_{[0,T] \times \mathbb{R}^d} \phi (t, x) b^k (t, x, \mu_t) (\mu_t^N (dx) - \mu_t (dx)) T^{-1} dt \right)^2 - \frac{1}{36 T^2} V_h^N \right\} \right]
$$

$$
\lesssim \int_0^\infty \mathbb{P} \left[ \left| \int_{[0,T] \times \mathbb{R}^d} \phi (t, x) b^k (t, x, \mu_t) (\mu_t^N (dx) - \mu_t (dx)) T^{-1} dt \right| \geq \left( z + \frac{1}{36 T^2} V_h^N \right)^{1/2} \right] dz
$$

$$
\lesssim 2 \kappa_1 \int_{\mathcal{V}_N^{1/(36 T^2)}} \exp \left( - \frac{\kappa_2 N h_1^2 z^2}{\kappa_3 T^{-1} |H \otimes K|^2 + \kappa_5 h_1 h_2^d |H \otimes K|_{\infty} z^{1/2}} \right) dz
$$

$$
\lesssim (N h_1 h_2^d)^{-1} N^{-\left( 72 T^{-2} \right)^{-1} \kappa_3^{-1} \kappa_5^{-1} \kappa_2^{-2} \omega_2}
$$

$$
+ (\log N)^{1/2} (N h_1 h_2^d)^{-3/2} \exp \left( - \frac{\kappa_2 \kappa_5^{-1} \omega_2}{12 T^{1/2} |H \otimes K|_{\infty} (\log N)^{1/2} (N h_1 h_2^d)^{1/2}} \right)
$$

$$
\lesssim N^{-2}
$$
as soon as \( \varpi_2 \geq 144dT\kappa_2^{-1}\kappa_3\kappa_5^2 \), thanks to \( \max_{(h_1,h_2) \in \mathcal{C}_N} (Nh_1^d)^{-1} \leq 1 \). We also use the assumption \( \min_{(h_1,h_2) \in \mathcal{C}_N} Nh_1^2 \geq (\log N)^2 \) given by (15) to show that the second term is negligible in front of \( N^{-2} \). We conclude

(64) \( E_{p,N} [ I ] \lesssim N^{-2} \).

We next consider the term \( II \). For \( \tau > 0 \), introduce the event

\[ \mathcal{B}_\tau^N = \left\{ \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)|\mu_t^N(dx)dt \leq \tau \right\}. \]

By Cauchy-Schwarz’s inequality, on \( \mathcal{B}_\tau^N \), we have

\[
\left| \int_{[0,T] \times \mathbb{R}^d} \phi(t,x)\xi_t^N(x)\mu_t^N(dx)dt \right|^2 \\
\leq \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)|\mu_t^N(dx)dt \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)||\xi_t^N(x)|^2\mu_t^N(dx)dt \\
\leq \tau \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)||\xi_t^N(x)|^2\mu_t^N(dx)dt.
\]

By Lemma 27 and exchangeability, we also have the rough bound

\[
E_{p,N} \left[ \left| \int_{[0,T] \times \mathbb{R}^d} \phi(t,x)\xi_t^N(x)\mu_t^N(dx)dt \right|^4 \right] \leq |\phi|_2^4 T^2 \sup_{0 \leq t \leq T} E_{p,N} \left[ |\xi_t^N(X_t^N)|^4 \right] \\
\lesssim (h_1^d)^{-4} N^{-2},
\]

where we estimate \( \mathbb{E}_{p,N} \left[ |\xi_t^N(X_t^N)|^4 \right] \) as in (58) above. It follows that \( II \leq IV + V \), with

\[
IV = 12\tau \left\{ \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)||\xi_t^N(x)|^2\mu_t^N(dx)dt - \frac{1}{36\tau \varpi_2} \mathbb{V}_h^N \right\} + 1_{\mathcal{B}_\tau^N} \\
\leq 12\tau |K_{h_2}| \int_0^T |H_{h_1}(t_0-t)| \left\{ \int_{\mathbb{R}^d} |\xi_t^N(x)|^2\mu_t^N(dx) - \frac{1}{36\tau |H_{h_1}|K_{h_2}|^2} \mathbb{V}_h^N \right\} dt \ 1_{\mathcal{B}_\tau^N},
\]

\[
V = 12 \int_{[0,T] \times \mathbb{R}^d} \phi(t,x)\xi_t^N(x)\mu_t^N(dx)dt \ 2 \ 1_{\mathcal{B}_\tau^N}. \]

Taking expectation and using exchangeability, we further have

\[
E_{p,N} [ IV ] \lesssim |K_{h_2}| \int_0^T |H_{h_1}(t_0-t)| E_{p,N} \left[ \left\{ \left| \xi_t^N(X_t^N) \right|^2 - \frac{1}{36\tau |H_{h_1}|K_{h_2}|^2} \mathbb{V}_h^N \right\} + 1 \right] dt \\
\lesssim |K_{h_2}| \int_0^T |H_{h_1}(t_0-t)| \int_0^\infty \mathbb{P}_{\mathcal{V}^N} \left( \left| \xi_t^N(X_t^N) \right| \geq z^{1/2} \right) dz dt \\
\lesssim |K_{h_2}| \int_0^\infty \frac{1}{1+Nt^{1/2}z^{1/2}} \exp \left( - \frac{\kappa_7 N z}{1+Nt^{1/2}z^{1/2}} \right) dz \\
\lesssim N^{-1} (h_2^d)^{-1} \exp \left( - \frac{\kappa_7 \varpi_2 |H \otimes K|^2}{72 \tau |H_{h_1}|K_{h_2}|} h_1^{-1} \log N \right) \\
+ N^{-1} (h_2^d)^{-1} h_1^{-1/2} (\log N)^{1/2} \exp \left( - \frac{\kappa_7 \varpi_2 |H \otimes K|^2}{12 \tau^{1/2} |H_{h_1}|^{1/2} K_{h_2}|^{1/2}} h_1^{-1/2} (\log N)^{1/2} \right),
\]
by Lemma 27, using in particular the fact that
\[ \frac{1}{\tau |H|_{1|Kh2|\infty}} V^N_h \gtrsim N^{-1/2} (\log N)^{3} \geq \kappa_3^3 N^{-1} \]
for large enough \( N \). Since \( \max(h_1, h_2) \in \mathcal{C}_2 \), \( h_1 \leq (\log N)^{-2} \) by assumption (15), both terms are negligible in front of \( N^{-2} \) and we conclude \( \mathbb{E}_{\mathbb{P}^N} [IV] \lesssim N^{-2} \).

We turn to the term \( V \). By Cauchy-Schwarz’s inequality and (65), we have
\[
\mathbb{E}_{\mathbb{P}^N} [V] \leq 12 \mathbb{E}_{\mathbb{P}^N} \left[ \int_{[0,T] \times \mathbb{R}^d} \phi(t,x) \xi^N_t(x) \mu^N_t(dx) dt \right]^{1/2} \mathbb{P}^N \left( \left( \mathbb{B}_r \right)^c \right)^{1/2} \lesssim N^{-1} (h_1 h_2^d)^{-2} \mathbb{P}^N \left( \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)| \mu^N_t(dx) dt > \tau \right)^{1/2}.
\]

Note that \( \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)| \mu^N_t(dx) dt \leq \kappa_3 |\phi|_1 \) hence, for the choice \( \tau \geq 2 \kappa_3 |H \otimes K|_1 \), that we make from now on and that does not depend on \( N \), we have
\[
\int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)| \mu^N_t(dx) dt \leq \frac{1}{2} \tau.
\]

By triangle inequality and a union bound argument, it follows that
\[
\mathbb{P}^N \left( \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)| \mu^N_t(dx) dt > \tau \right)^{1/2} \leq \mathbb{P}^N \left( \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)| (\mu^N_t(dx) - \mu(dx)) T^{-1} dt \right) \geq \frac{1}{2} \mathbb{P}^N \left( \int_{[0,T] \times \mathbb{R}^d} |\phi(t,x)| \mu^N_t(dx) dt \right)^{1/2} \leq (2\kappa_1)^{1/2} \exp \left( -\frac{1}{\kappa_3} \frac{1}{T^{-1}} \frac{1}{|H \otimes K|_1} \right),
\]

where we applied Theorem 18. Using \( \min(h_1, h_2) \in \mathcal{C}_2 \), \( N h_1 h_2^d \geq (\log N)^2 \) granted by (15), we obtain \( \mathbb{E}_{\mathbb{P}^N} [V] \lesssim N^{-2} \). Putting together our estimates for \( IV \) and \( V \), we conclude
\[
\mathbb{E}_{\mathbb{P}^N} [II] \lesssim N^{-2}.
\]

Finally, we consider the term \( III \). The classical following deviation bound holds for continuous martingales:
\[
\mathbb{P} \left( M^N_T (\phi) \geq u, \langle M^N (\phi) \rangle_T \leq v \right) \leq \exp \left( -\frac{u^2}{2v} \right)
\]
for every \( u, v \geq 0 \), see e.g. [61]. Let \( \kappa > 0 \) to be tuned below. The choice \( v = \kappa (V^N_h)^{1/2} (\log N)^{-1} u \) entails
\[
\mathbb{P} \left( M^N_T (\phi) \geq u \right) \leq \exp \left( -\frac{1}{2} \kappa^{-1} (V^N_h)^{-1/2} (\log N) u \right) + \mathbb{P} \left( \langle M^N (\phi) \rangle_T \geq \kappa (V^N_h)^{1/2} (\log N)^{-1} u \right).
\]

It follows that
\[
\mathbb{E}_{\mathbb{P}^N} [III] \leq 12 \mathbb{E}_{\mathbb{P}^N} \left[ \left( \langle M^N (\phi) \rangle_T \right)^2 - \frac{1}{36d} V^N_h \right] \lesssim \sum_{k=1}^{d} \int_{[0,\infty]} \mathbb{P} \left( M^N_T (\phi)^k \right) \geq (z + \frac{1}{36d} V^N_h)^{1/2} ) \right) dz \lesssim VI + VII,
\]

with
\[
VI = \int_{\frac{1}{36d} V^N_h}^{\infty} \exp \left( -\frac{1}{2} \kappa^{-1} (V^N_h)^{-1/2} (\log N) z^{1/2} \right) dz,
\]

and
\[
VII = \int_{\frac{1}{36d} V^N_h}^{\infty} \mathbb{P} \left( \langle M^N (\phi) \rangle_T \geq \kappa (V^N_h)^{1/2} (\log N)^{-1} u \right) dz.
\]
\[
VII = \sum_{k=1}^{d} \int_{\mathbb{R}^d} \mathbb{P}^N_\kappa \left( \langle M^N (\phi)^k \rangle_T \geq \kappa (V^N_h)^{1/2} (\log N)^{-1/2} z^{1/2} \right) dz.
\]

Taking for instance \( \kappa = (25 \sqrt{d})^{-1} < (24 \sqrt{d})^{-1} \), we obtain
\[
VI \lesssim V^N_h (\log N)^{-1} N^{-1/(12k^2)} \lesssim N^{-2}.
\]

In order to bound the term \( VII \), we first notice that by Lemma 25, we have
\[
\langle M^N (\phi)^k \rangle_T \leq N^{-1} |\text{Tr}(c)| \int_{[0,T] \times \mathbb{R}^d} \phi(t,x)^2 \mu^N_t (dx) dt
\]
Next, the condition
\[
N^{-1} |\text{Tr}(c)| \kappa_3(h_1^d)^{-1} |H \otimes K|^2 \geq \frac{1}{2} \kappa (V^N_h)^{1/2} (\log N)^{-1/2}
\]
is equivalent to
\[
z \leq 4 \kappa^2 |\text{Tr}(c)| \kappa_3^2 |H \otimes K|^2 (V^N_h)^{-1} (\log N)^2
\]
\[
\leq 4 \kappa^2 |\text{Tr}(c)| \kappa_3^2 \omega^2 V^N_h
\]
as soon as \( \omega^2 \geq 300 d |\text{Tr}(c)| \kappa_3 \). It follows that \( VII \) is of order
\[
\int_{\mathbb{R}^d} \mathbb{P}^N\left( \int_{[0,T] \times \mathbb{R}^d} \phi(t,x)^2 (\mu^N_t (dx) - \mu_t (dx)) dt \geq \frac{1}{2} N |\text{Tr}(c)| \kappa (V^N_h)^{1/2} (\log N)^{-1/2} z^{1/2} \right) dz
\]
\[
\lesssim \int_{\mathbb{R}^d} \exp\left( - \frac{\kappa_4^2 |\text{Tr}(c)| \kappa_3^2 |H \otimes K|^2 (V^N_h)^{-1} (\log N)^2 - T^{-2/2}}{288 \kappa_4^2 |\text{Tr}(c)| \kappa_3^2 |H \otimes K|^2 (V^N_h)^{-1} (\log N)^{-1/2} T^{-1/2}} \right) du
\]
\[
\lesssim (N h_1^d)^{-2} (\log N) \exp\left( - \frac{\kappa_4^2 |\text{Tr}(c)| \kappa_3^2 |H \otimes K|^2 (V^N_h)^{-1} (\log N)^2 - T^{-2/2}}{24 d^{1/2} T |H \otimes K|^2 (\log N)^{-1} T^{-1/2}} \right)
\]
\[
\lesssim N^{-2}
\]
where we applied again Theorem 18 and used \( \min_{(h_1, h_2) \in \mathcal{E}^N_2} N h_1^d \geq (\log N)^2 \) granted by (15). We infer \( VII \lesssim N^{-2} \) and conclude
\[
(67) \quad \mathbb{E}_N [III] \lesssim N^{-2}
\]
and putting together (64), (66) and (67), we have established
\[
\mathbb{E}_N \left[ \left\{ 4 |\hat{\pi}^N_h (t_0, x_0) - \pi_t (t_0, x_0) |^2 - V^N_h \right\} \right] \lesssim N^{-2}.
\]

**Step 4:** The control of the second term in the right-hand side of (63) is done in the same way as in Step 4 of the proof of Theorem 7 and only inflates the previous bound by a factor or order \( \text{Card}(\mathcal{E}^N_2) \lesssim N \). In turn \( \mathbb{E}_N [A^N_h] \lesssim N^{-1} + B^N_h (\pi) (t_0, x_0)^2 \) and we have established by Step 2 that for any \( h \in \mathcal{E}^N_2 \),
\[
\mathbb{E}_N \left[ \left\{ \hat{\pi}^N_h (t_0, x_0) - \pi_t (t_0, x_0) \right\}^2 \right] \lesssim B^N_h (\pi) (t_0, x_0)^2 + V^N_h + N^{-1}
\]
holds true. Putting together Step 1 and Theorem 7 and using \( N^{-1} \lesssim V_N^N \) completes the proof of Theorem 9.

7.3. Proof of Theorem 14.

Proof of the lower bound (24).

**Step 1.** Pick an infinitely many times differentiable function \( V_1 : \mathbb{R}^d \to \mathbb{R} \) such that

(i) \( \nabla V_1 \) is Lipschitz continuous,
(ii) \( \limsup_{|x| \to \infty} -\nabla V_1(x)^T (x/|x|^2) < 0 \),
(iii) \( V_1 = 0 \) in a neighbourhood of \( x_0 \).

Let \( C_{V_1} = \int_{\mathbb{R}^d} \exp(-2V_1(x))dx \) and define

\[
\nu_1(x) = C_{V_1}^{-1} \exp\left(-2V_1(x)\right), \quad x \in \mathbb{R}^d.
\]

From the classical theory of multidimensional diffusion processes (see e.g. the classical textbook of Stroock and Varadhan [62]), properties (i) and (ii) imply that \( \nu_1 \) is the unique invariant measure of the diffusion process \( d\xi_t = -\nabla V_1(\xi_t)dt + dW_t \) for some Brownian motion \( W \) on \( \mathbb{R}^d \). In turn \( \nu_1(t, x) = \nu_1(x) \) as a function defined on \( [0, T] \times \mathbb{R}^d \) satisfies

\[
\nu_1 = S(b_1, \text{Id}, \nu_1) \quad \text{with} \quad b_1(t, x, \mu) = -\nabla V_1(x),
\]

assuming moreover that \( V_1 \) is such that \( \nu_1 \) satisfies Assumption 1, a choice which is obviously possible. Since \( \nu_1 \) is constant in a neighbourhood of \( (t_0, x_0) \) we may (and will) assume that \( (b_1, \text{Id}, \nu_1) \in S_{L/2}^{\alpha, \beta}(t_0, x_0) \). Next, we set

\[
V_2^N(x) = V_1(x) + C V_1 N^{-1/2} \tau_N^{d/2} \psi(\tau_N(x - x_0)), \quad 0 < \tau_N \to \infty,
\]

for some \( 0 < \varpi \leq 1 \), where \( \psi : \mathbb{R}^d \to \mathbb{R} \) is infinitely many times differentiable, compactly supported and satisfies

\[
\psi(0) = 1, \quad |\psi|_{\infty} \leq 1, \quad \int_{\mathbb{R}^d} \psi(x)dx = 0, \quad |\psi|_2 = 1,
\]

hence \( \nabla V_2^N \) satisfies (i) and (ii) and (iii). It defines in turn a solution

\[
\nu_2^N = S(b_2^N, \text{Id}, \nu_2^N) \quad \text{with} \quad b_2^N(t, x, \mu) = -\nabla V_2^N(x),
\]

having

\[
\nu_2^N(x) = C_{V_2^N}^{-1} \exp\left(-2V_2^N(x)\right) \quad \text{with} \quad C_{V_2^N} = \int_{\mathbb{R}^d} \exp(-2V_2^N(x))dx
\]

and \( \nu_2^N(t, x) = \nu_2^N(x) \) is understood as a function defined on \( [0, T] \times \mathbb{R}^d \).

**Step 2:** We claim that for every \( \beta > 0 \), setting \( \tau_N = N^{1/(2\beta + d)} \) and taking \( \varpi \) sufficiently small, we have \( (b_2^N, \text{Id}, \nu_2^N) \in S_{L/2}^{\alpha, \beta}(t_0, x_0) \) for large enough \( N \). Indeed, in a neighbourhood of \( x_0 \), we have \( V_1 = 0 \) hence for \( x \) in such a neighbourhood, we have

\[
\nu_2^N(x) = C_{V_2^N}^{-1} \exp\left(-2\varpi C_{V_1} N^{-1/2} \tau_N^{d/2} \psi(\tau_N(x - x_0))\right).
\]

On the one hand, \( \varpi N^{-1/2} \tau_N^{d/2} \|\psi(\tau_N(x - x_0))\|_{\mathcal{H}^\beta(x_0)} \lesssim \varpi \|\psi\|_{\mathcal{H}^\beta(x_0)} \) which can be taken arbitrarily small. On the other hand, we also have \( C_{V_2^N} \to C_{V_1} \) as \( N \to \infty \), see in particular (68) below, and the claim follows.
Step 3. For \((b, c, \mu_0) \in \mathcal{P}\), we write \(\mathbb{P}_{b,c,\mu_0}^N\) for \(\mathbb{P}_N\) to emphasise the model parameter \((b, c, \mu_0)\). For data extracted from \(\mu_0^N\) solely, we restrict the model to

\[
\mathbb{P}_{b,c,\mu_0}^N(t_0) = (X_{t_0}^1, \ldots, X_{t_0}^N) \circ \mathbb{P}_{b,c,\mu_0}^N,
\]

the law of \((X_{t_0}^1, \ldots, X_{t_0}^N)\) under \(\mathbb{P}_{b,c,\mu_0}^N\). Note that for a drift \(b(t, x, \mu) = b(t, x)\) independent of an interaction measure term \(\mu\), we have

\[
\mathbb{P}_{b,c,\mu_0}^N = \mathbb{P}_{b,c,\mu_0}^N \quad \text{hence} \quad \mathbb{P}_{b,c,\mu_0}^N(t_0) = \mu_0^N.
\]

By Pinsker’s inequality, it follows that

\[
\|\mathbb{P}_{b_1,1d,\nu_1}^N(t_0) - \mathbb{P}_{b_2,1d,\nu_2}^N(t_0)\|_{TV}^2 = \|\nu_1^N - (\nu_2^N) \circ (\sigma_1)\|_{TV}^2 \\
\leq \frac{N}{2} \int_{\mathbb{R}^d} \nu_1(x) \log \frac{\nu_1(x)}{\nu_2(x)} dx \\
= N \int_{\mathbb{R}^d} \nu_1(x) (V_2^N(x) - V_1(x)) dx + \frac{N}{2} \log \frac{C_{V_2}}{C_{V_1}}
\]

for large enough \(N\), where \(\|\cdot\|_{TV}\) denotes the total variation distance, using successively

\[
V_2^N(x) - V_1(x) = \varpi C_{V_1} N^{-1/2} d / 2 \psi(\tau_N(x - x_0)) = \varpi \nu_1(x)^{-1} N^{-1/2} d / 2 \psi(\tau_N(x - x_0)),
\]

since \(\nu_1(x)^{-1} = C_{V_1}\) in a neighbourhood of \(x_0\) and the fact that \(\psi(\tau_N(x - x_0)) = 0\) outside this neighbourhood, for large enough \(N\), thanks to the fact that \(\text{Supp}(\psi)\) is compact, together with the cancellation property of \(\psi\) in a neighbourhood of \(x_0\). It follows that

\[
0 \leq \vartheta^N(x) = \exp \left(2\varpi N^{-1/2} d / 2 \psi(\tau_N(x - x_0))\right) \sup_{x \in \text{Supp}(\psi)} \nu_1(x)^{-1} \leq 2
\]

for large enough \(N\). It follows that

\[
\left|\frac{C_{V_2}}{C_{V_1}} - 1\right| \leq 4\varpi^2 N^{-1} \sup_{x \in \text{Supp}(\psi)} \nu_1(x)^{-1} \lesssim N^{-1}.
\]

The inequality \(\log(1 + x) \leq x\) for \(x \geq -1\) enables us to conclude

\[
\|\mathbb{P}_{b_1,1d,\nu_1}^N(t_0) - \mathbb{P}_{b_2,1d,\nu_2}^N(t_0)\|_{TV}^2 \leq 2\varpi^2 \sup_{x \in \text{Supp}(\psi)} \nu_1(x)^{-1} \leq \frac{1}{2}
\]

for large enough \(N\) by taking \(\varpi > 0\) sufficiently small.
Step 4. We conclude by a classical two-point lower bound argument using Le Cam’s lemma: if \( P_i \), \( i = 1, 2 \) are two probability measures defined on the same probability space and \( \Psi(P_i) \in \mathbb{R} \) is a functional of \( P_i \), we have
\[
\inf_{\Psi} \max_{i=1,2} \mathbb{E}_{P_i} [ |\Psi - \Psi(P_i)| ] \geq \frac{1}{2} |\Psi(P_1) - \Psi(P_2)| (1 - \|P_1 - P_2\|_{TV}),
\]
where the infimum is taken over all estimators of \( \Psi(P_i) \), see e.g. [46] among many other references. We let
\[
\Psi(P^N_{b_1,Id,x_1}(t_0)) = \nu_1(t_0, x_0), \quad \Psi(P^N_{b_2,Id,x_2}(t_0)) = \nu_2(t_0, x_0),
\]
so that
\[
|\Psi(P^N_{b_1,Id,x_1}(t_0)) - \Psi(P^N_{b_2,Id,x_2}(t_0))| \geq \nu_1(x_0) \left( \frac{C_{V_2}}{C_{V_1}} - \exp \left( 2\omega \nu_1(x)^{-1} N^{-1/2} r_N^{d/2} \psi(0) \right) \right)
\geq N^{-1/2} r_N^{d/2} = N^{-\beta/(2\beta + d)}
\]
in the same way as before, using the properties of \( \psi \) and (68). We conclude by applying Le Cam’s lemma together with (69). The proof of the lower bound (24) is complete.

Proof of the upper bound (23). The argument is classical (see e.g. [31, 32, 33]). Pick
\[
\mathcal{H}_1^N = \{ e^{-k}, 1 \leq k \leq d^{-1} \log N - 2d^{-1} \log \log N \}.
\]
We have Card \( \mathcal{H}_1^N \leq N \) holds as well (and is actually much smaller). Moreover, for every \( h \in \mathcal{H}_1^N \):
\[
B_h^N(\mu)(t_0, x_0)^2 \leq h^{2\beta + \ell} \quad \text{and} \quad \mathcal{V}_h^N \leq N^{-\beta} \mathcal{N}^{-d}(\log N).
\]
Applying Theorem 7, we obtain
\[
\mathbb{E}_{P^N} \left[ \left( \mu_{G_i}^N(t_0, x_0) - \mu_{t_0}(x_0) \right)^2 \right] \leq \min_{h \in \mathcal{H}_1^N} \left( h^{2\beta + \ell} + N^{-1} h^{-d} (\log N) \right)
\leq \left( \frac{\log \mathcal{N}}{N} \right)^{2\beta + \ell/(2\beta + \ell + d)},
\]
for large enough \( N \), since for every \( \beta \in (0, \ell] \), we have \( e^{-(k+1)^N} \leq (N/\log N)^{-1/(2\beta + d)} \leq e^{-k^N} \) with \( k^N = \lceil \frac{\log N}{2\beta + \ell + d} \rceil \). This proves (23) and completes the proof of Theorem 14.

7.4. Proof of Theorem 15.

Proof of the lower bound (26). We apply the same strategy as for Theorem 14, establishing a two-point inequality and applying Le Cam’s lemma for two drift functions that have no interaction. We write \( P_{b,c,\mu_0}^N \) for the law of \( (X_i^t, \ldots, X_i^T)_{0 \leq t \leq T} \) parametrised by \( (b, c, \mu_0) \). We start with the following simple consequence of Girsanov’s theorem:

Lemma 28. For \( i = 1, 2 \), let \( b_i(t, x, \mu) = b_i(t, x) \) be two drift functions that satisfy Assumption 4 with no interaction. Set \( \Delta(t, x) = b_2(t, x) - b_1(t, x) \). We have
\[
\|P_{b_2,Id,\mu}^N - P_{b_1,Id,\mu_0}^N\|_{TV}^2 \leq \frac{N}{T} \int_0^T |\Delta(t, \cdot)|^2 d\mathbb{P}_{\mu_0}(dt),
\]
where \( \mu = \delta(b_1, Id, \mu_0) \) is a solution of (3) with parameter \( (b_1, Id, \mu_0) \), up to an explicitly computable constant that only depends on \( \mu_0 \) and \( b_1 \).
Proof. With the notation of Section 6.2, by Girsanov’s theorem,
\[
\frac{d\mathbb{P}^N_{b_1,\text{Id},\mu_0}}{d\mathbb{P}^N_{b_1,\text{Id},\mu_0}} = \mathbb{E}_T(M^N(\Delta)), \quad \text{where} \quad M^N_i(\Delta) = \sum_{i=1}^N \int_0^t \Delta(s, X^i_s) dB^i_s
\]
is a \(\mathbb{P}^N_{b_1,\text{Id},\mu_0}\)-martingale, with \(B^i_t = \int_0^t (dX^i_s - b_1(s, X^i_s)ds)\). Moreover, since there is no interaction term in the drift \(b_1\), we have
\[
\|\mathbb{P}^N_{b_1,\text{Id},\mu_0} - \mathbb{P}^N_{b_2,\text{Id},\mu_0}\|_{TV} \lesssim \vartheta^2 \leq \frac{1}{2}
\]
by Lemma 28, for a suitable choice of \(\vartheta > 0\).

By Pinsker’s inequality, for \(\mathbb{P}^N_{b_1,\text{Id},\mu_0} - \mathbb{P}^N_{b_2,\text{Id},\mu_0}\), we obtain
\[
\|\mathbb{P}^N_{b_1,\text{Id},\mu_0} - \mathbb{P}^N_{b_2,\text{Id},\mu_0}\|_{TV} \lesssim \vartheta^2 \leq \frac{1}{2}
\]
By Lemma 28, for a suitable choice of \(\vartheta > 0\).

Step 1. Pick now a function \(b_1 : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) satisfying Assumption 4 and any initial condition \(\mu_0\) such that \((b_1, \text{Id}, \mu_0) \in S^{\alpha,\beta}(t_0, x_0) \cap D_{L/2}^{\alpha,\beta}(t_0, x_0)\).

Let \(\psi = (\psi^1, \ldots, \psi^d) : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d\) be infinitely many times differentiable, compactly supported and such that for every \(1 \leq k \leq d\), we have
\[
\psi^k(0) = 1, \quad |\psi^k|_\infty \leq 1, \quad \int_{[0, T] \times \mathbb{R}^d} \psi(t, x) dtdx = 0, \quad |\psi^k|_2 = 1.
\]

For \(N \geq 1\) and some \(0 < \varpi \leq 1\), we define
\[
b_N^N(t, x) = b_1(t, x) + \varpi N^{-1/2} \tau_N^{1/2}(\tau_N)^{1/2} \psi(\tau_N(t - t_0), \tau_N(x - x_0)),
\]
where \(\tau_N\) and \(\bar{\tau}_N\) are defined via
\[
\tau_N^\alpha = (\bar{\tau}_N)^\alpha = N^{\alpha d(\alpha,\beta)/(2\alpha d(\alpha,\beta) + 1)}.
\]

By accommodating \(b_1\) and \(\varpi > 0\), we may (and will) assume that \((b_N^N, \text{Id}, \mu_0) \in D_{L/2}^{\alpha,\beta}(t_0, x_0)\) and \(b_N^N \in \mathbb{P}\) for every \(N \geq 1\). Setting \(\Delta_N^N(t, x) = b_N^N(t, x) - b_1(t, x)\) and noting that
\[
\int_0^T |\Delta_N^N(t, \cdot)|^2_{L^2(\mu_1)} dt \lesssim \varpi^2 N^{-1},
\]
thanks to the compactness of the support of \(\psi\) and Lemma 23, we obtain
\[
\|\mathbb{P}^N_{b_1,\text{Id},\mu_0} - \mathbb{P}^N_{b_N^N,\text{Id},\mu_0}\|_{TV} \lesssim \varpi^2 \leq \frac{1}{2}
\]
by Lemma 28, for a suitable choice of \(\varpi > 0\).
Step 2. We conclude in the same way as in the proof of the lower bound of Theorem 14: by Le Cam’s lemma (70), with

$$\Psi(\mathbb{F}_{b_1, \text{Id}, \mu_0}^N) = b_1(t_0, x_0) \quad \text{and} \quad \Psi(\mathbb{F}_{b_2, \text{Id}, \mu_0}^N) = b_2^N(t_0, x_0),$$

we have

$$|\Psi(\mathbb{F}_{b_1, \text{Id}, \mu_0}^N) - \Psi(\mathbb{F}_{b_2, \text{Id}, \mu_0}^N)| \gtrsim N^{-1/2} r_{1/2} N^{-1/2} = N^{-s_d(\alpha, \beta) / (2s_d(\alpha, \beta) + 1)}$$

and the conclusion follows. The proof of the lower bound (26) is complete.

**Proof of the upper bound (25).** Let

$$\delta^N(k) = \frac{g^{-1}(k/\log N)}{\alpha(g^{-1}(k/\log N))} k,$$

where $g^{-1}$ is the inverse of the function $g(\beta) = \frac{1}{\beta} \frac{s_d(\alpha, \beta)}{2s_d(\alpha, \beta) + 1}$, which is non-increasing for $\beta > 0$ thanks to the assumption that $\beta \mapsto \alpha(\beta)$ is non-decreasing and $\beta \mapsto \beta(\beta)$ is non-increasing. Pick

$$\mathcal{H}_2^N = \left\{(e^{-k_1}, e^{-k_2}), k_1 = \delta^N(k_2), 1 \leq k_2 \leq (d + 1)^{-1}(\log N - 2 \log \log N), \right.$$

$$\left. 2 \log \log N \leq \delta^N(k_2) \leq (d + 1)^{-1}(\log N - 2 \log \log N) \right\}.$$

The condition (15) for the grid $\mathcal{H}_2^N$ is satisfied and $\text{Card} \mathcal{H}_2^N \lesssim N$ holds as well (and is actually much smaller). Now, set $\beta_{k_2} = g^{-1}(k_2 / \log N)$. We define an ordering $\leq$ on $\mathcal{H}_2^N$ that has the right behaviour with respect to the bias of $\pi$ at scale $h$. We say that $h = (h_1, h_2) \leq h' = (h'_1, h'_2)$ if

$$h_1(k_2) + h_2(k_2) \leq h'_1(k'_2) + h'_2(k'_2),$$

where we write $h = (h_1, h_2) = (h_1(k_2), h_2(k_2)) = (\alpha(\delta^N(k_2)), \beta(\delta^N(k_2)))$ and likewise for $h'$. We have that $h \leq h'$ is equivalent to $k_2 \geq k'_2$ since

$$h_1(k_2) + h_2(k_2) = 2N^{-s_d(\alpha, \beta) / (2s_d(\alpha, \beta) + 1)}$$

and the fact that $\beta \mapsto s_d(\alpha, \beta)$ is non-decreasing, following from the assumption that $\beta \mapsto \alpha(\beta)$ is non-decreasing. Hence $h \leq h'$ or $h' \leq h$. Moreover,

$$(b, c, \mu_0) \in \mathcal{D}_L^{\alpha, \beta}(t_0, x_0) \cap \mathcal{D}_L^{\alpha, \beta}(t_0, x_0) \implies \pi \in \mathcal{D}_L^{\alpha, \beta}(t_0, x_0).$$

Therefore, for every $h = (h_1, h_2) \in \mathcal{H}_2^N$,

$$Z_h(\pi)(t_0, x_0)^2 \lesssim h_1^{2\alpha} + h_2^{2\beta}$$

and $V_h \lesssim h_1^{2\alpha} h_2^{-d} (\log N)$ thanks to the definition of the ordering $\leq$ in (71). It follows that for every $s_d(\alpha, \beta) / (0, \ell / d)$, we have

$$e^{-(k_1^N + 1)} \leq N^{-s_d(\alpha, \beta) / (2s_d(\alpha, \beta) + 1)} \leq e^{-k_1^N}$$

with $k_1^N = \left\lfloor \frac{\alpha^{-1} s_d(\alpha, \beta)}{2s_d(\alpha, \beta) + 1} \right\rfloor \log N / \alpha$. And

$$e^{-(k_2^N + 1)} \leq N^{-s_d(\alpha, \beta) / (2s_d(\alpha, \beta) + 1)} \leq e^{-k_2^N}$$

with $k_2^N = \left\lfloor \frac{\beta^{-1} s_d(\alpha, \beta)}{2s_d(\alpha, \beta) + 1} \right\rfloor \log N$. Applying Theorem 9, we obtain

$$\mathbb{E}_{\mathbb{P}_X} \left[ \left| \hat{\pi}_{GL}^N(t_0, x_0) - \pi_{t_0}(x_0) \right|^2 \right] \lesssim \min_{(h_1, h_2) \in \mathcal{H}_1^N} (h_1^{2\alpha} + h_2^{2\beta} + N^{-1} h_1^{-1} h_2^{-d} (\log N) + N^{-1} \left( \frac{\log N}{N} \right)^{2s_d(\alpha, \beta) / (2s_d(\alpha, \beta) + 1)}).$$
This proves (23) and completes the proof of Theorem 14.

7.5. Proof of Theorem 17.

Preliminary results.

Lemma 29. Work under Assumptions 1, 2, and 4. We have

$$\sup_{\xi \in \mathbb{R}^d} \mathbb{E}_P \left[ |\mathcal{F} \left( \mathcal{L}(\mu^N - \mu) \right)(\xi) |^2 \right] \lesssim N^{-1}.$$ 

Proof. Writing $\varphi_1(x) = \cos(2\pi \xi^T x)$ and $\varphi_2(x) = \sin(2\pi \xi^T x)$, we have by Jensen’s inequality and Theorem 18

$$\mathbb{E}_P \left[ \int_{\mathbb{R}^d} e^{-2i\pi \xi^T x} \mathcal{L}(\mu^N - \mu)(dx) \right]^2 = \mathbb{E}_P \left[ \int_{[0,T]} \left( \int_{\mathbb{R}^d} e^{-2i\pi \xi^T x} (\mu^N - \mu)(dx) w(t) \right)^2 \rho(dt) \right]$$

$$\leq \int_{[0,T]} \mathbb{E}_P \left[ \int_{\mathbb{R}^d} e^{-2i\pi \xi^T x} (\mu^N - \mu)(dx) \right]^2 w(t)^2 \rho(dt)$$

$$\leq \sup_{t \in [0,T]} \max_{k=1,2} \mathbb{E}_P \left( \int_{\mathbb{R}^d} |\varphi_k| d(\mu^N - \mu) \right) \geq \{ \int_{\mathbb{R}^d} |\varphi_k| \} \leq N^{-1}$$

using $|\varphi_k| \lesssim 1$ that stems from $|\varphi_k| \leq 1$. Since this bound is uniform in $\xi \in \mathbb{R}^d$, the result follows.

Completion of proof of Theorem 17. We have

$$|\tilde{F}_{\alpha,\alpha'} - F|^2 \lesssim |\mathcal{F}(\tilde{F}_{\alpha,\alpha'}) - \mathcal{F}(F)|^2 = I + II,$$

with

$$I = \left( \frac{\mathcal{F}(\mathcal{L}(\hat{b}^N_{h,h}))}{|\mathcal{F}(\mathcal{L}(\mu^N))|^2} - \mathcal{F}(F) \right) \mathbb{1}_{\{ |\mathcal{F}(\mathcal{L}(\nu^N))| \geq \alpha \}}^2,$$

$$II = \mathbb{E}_P \left[ \mathcal{F}(F) \mathbb{1}_{\{ |\mathcal{F}(\mathcal{L}(\nu^N))|^2 \leq \alpha \}}^2 \right].$$

On $\{ |\mathcal{F}(\mathcal{L}(\nu^N))|^2 \geq \alpha \}$ and using the fact that $|\mathcal{F}(\mathcal{L}(\nu^N))| > 0$ almost everywhere, we write

$$\frac{\mathcal{F}(\mathcal{L}(\hat{b}^N_{h,h}))}{|\mathcal{F}(\mathcal{L}(\mu^N))|^2} - \mathcal{F}(F)$$

$$= \left( \frac{\mathcal{F}(\mathcal{L}(\hat{b}^N_{h,h}))}{|\mathcal{F}(\mathcal{L}(\mu^N))|^2} - \mathcal{F}(F) \right) \mathbb{1}_{\{ |\mathcal{F}(\mathcal{L}(\nu^N))| \geq \alpha \}}^2,$$

$$= \mathbb{E}_P \left[ \mathcal{F}(F) \mathbb{1}_{\{ |\mathcal{F}(\mathcal{L}(\nu^N))|^2 \leq \alpha \}}^2 \right].$$

It follows that for $r > 0$, we have

$$\mathbb{E}_P \left[ I \right] \lesssim III + IV,$$

with

$$III = \alpha^{-2} \mathbb{E}_P \left[ \mathcal{F}(\mathcal{L}(\hat{b}^N_{h,h}))^2 \right],$$

$$IV = \alpha^{-2} \mathbb{E}_P \left[ \mathcal{F}(F) \right]^2.$$
By Parseval’s identity, and the boundedness of $\mathcal{L}$, we have

$$III \lesssim \omega^{-2} \sup_{t \in \text{Supp}(w), |x| \leq r} \mathbb{E}_{\mathbb{P}^N} \left[ |\hat{b}^N_{h,h}(t,x) - b(x,\mu_t)|^2 \right] + \omega^{-2} \sup_{t \in \text{Supp}(w)} \int_{|x| \geq r} |b(x,\mu_t)|^2 \, dx.$$ 

By Lemma 24 and Lemma 26, we have

$$\sup_{t \in \text{Supp}(w), |x| \leq r} \mathbb{E}_{\mathbb{P}^N} \left[ |\hat{b}^N_{h,h}(t,x) - b(x,\mu_t)|^2 \right] \lesssim \omega'(r)^{-2} (B^N_h(\mu)(t,x) + B^N_h(\pi)(t,x) + V^N_h + V^N_h).$$

Moreover, by Proposition 13 the smoothness of $F$, $G$ and $\mu_0$ entails some Hölder smoothness on $\mu$ and $b$ (hence on $\pi$) that implies in turn the estimate

$$\sup_{t \in \text{Supp}(w), |x| \leq r} B^N_h(\mu)(t,x) + B^N_h(\pi)(t,x) \lesssim C(r)(h^\gamma + h^\gamma + h^{\gamma'} + h^{\gamma''}),$$

for some $\gamma, \gamma', \gamma'' > 0$ and $C(r)$ a locally bounded function in $r$. We infer

$$\omega^{-2} \sup_{t \in \text{Supp}(w), |x| \leq r} \mathbb{E}_{\mathbb{P}^N} \left[ |\hat{b}^N_{h,h}(t,x) - b(x,\mu_t)|^2 \right] \lesssim \omega^{-2} \omega'(r)^{-2} C(r) u_N,$$

with $u_N \to 0$, for a choice $(h, h) = (h_N, h_N) \to 0$ as $N \to \infty$. Since $b(x,\mu_t) = G(x) + F * \mu_t(x)$, the second term in $III$ can be bounded as follows:

$$\int_{|x| \geq r} |b(x,\mu_t)|^2 \, dx \lesssim \int_{|x| \geq r} |G(x)|^2 \, dx + \int_{\mathbb{R}^d} \left( \int_{|x| \geq r} |F(x-y)|^2 \, dx \right) \mu_t(dy)$$

by Fubini’s theorem, and both terms converge to 0 at some rate $\widetilde{C}(r) \to 0$ as $r \to \infty$ by dominated convergence since $F$ and $G$ are in $L^2(\mathbb{R}^d)$. We conclude

$$III \lesssim \omega^{-2} (\omega'(r)^{-2} C(r) u_N + \widetilde{C}(r)).$$

By Parseval’s identity, the boundedness of $\mathcal{F}(\mathcal{L}\mu^N)$ and Lemma 29, we also have

$$IV \lesssim \omega^{-2} \sup_{\xi \in \mathbb{R}^d} \mathbb{E}_{\mathbb{P}^N} \left[ |\mathcal{F}(\mathcal{L}(\mu^N - \mu))(\xi)|^2 \right] |F|^2 \lesssim \omega^{-2} N^{-1}.$$ 

We finally turn to the term $II$. We have

$$\mathbb{E}_{\mathbb{P}^N} \left[ II \right] \lesssim \mathbb{E}_{\mathbb{P}^N} \left[ \left| \mathcal{F}(F) \mathbf{1}_{(|\mathcal{F}(\mathcal{L}(\mu^N - \mu))(\xi)|^2 \geq \omega\)} \right|^2 \right] + \mathbb{E}_{\mathbb{P}^N} \left[ \left| \mathcal{F}(F) \mathbf{1}_{(|\mathcal{F}(\mathcal{L}(\mu))(\xi)|^2 \leq \omega\)} \right|^2 \right] \lesssim \omega^{-2} \mathbb{E}_{\mathbb{P}^N} \left[ \left| \mathcal{F}(\mathcal{L}(\mu^N - \mu))(\xi)|^2 \right| |F|^2 + H(\omega) \right] \lesssim \omega^{-2} N^{-1} + H(\omega),$$

where $H(\omega) \to 0$ as $\omega \to 0$ by dominated convergence thanks to the property $|\mathcal{F}(\mathcal{L}(\mu))(\xi)| > 0$ almost everywhere of Assumption 16. We conclude

$$\mathbb{E}_{\mathbb{P}^N} \left[ |\hat{F}^N_{\omega,\omega'} - F|^2 \right] \lesssim \omega^{-2} (\omega'(r)^{-2} C(r) u_N + \widetilde{C}(r)) + N^{-1} + H(\omega).$$

Let $r_N \to \infty$ slowly enough so that $\omega'(r_N)^{-2} C(r_N) u_N \to 0$. This yields $v_N = C(r_N) u_N + \widetilde{C}(r_N) + N^{-1} \to 0$. Pick now $\omega_N \to 0$ slowly enough so that $\omega_N^{-2} v_N \to 0$. The proof of Theorem 17 follows.
8. Appendix

Characterisation of sub-Gaussian random variables. We recall a classical definition of a sub-Gaussian random variable. Recommended reference is [10].

**Definition 30.** A real-valued random variable $Z$ such that $\mathbb{E}[Z] = 0$ is sub-Gaussian if one of the following conditions is satisfied, each statement implying the next:

(i) Laplace transform condition

$$\mathbb{E}\left[ \exp(zZ) \right] \leq \exp\left( \frac{1}{2} \lambda^2 z^2 \right) \text{ for every } z \in \mathbb{R}.$$ 

(ii) Moment condition

$$\mathbb{E}[Z^{2p}] \leq p!(4\lambda^2)^p \text{ for every integer } p \geq 1.$$ 

(iii) Orlicz condition

$$\mathbb{E}\left[ \exp\left( \frac{\lambda^2}{4} Z^2 \right) \right] \leq 2.$$ 

(iv) Laplace transform condition (bis)

$$\mathbb{E}\left[ \exp(zZ) \right] \leq \exp\left( \frac{\lambda^2}{4} z^2 \right) \text{ for every } z \in \mathbb{R}.$$ 

We will also use the following additive property of sub-Gaussian random variables: if the random variables $Z_i$ are independent and $\lambda_i^2$ sub-Gaussian, then $\rho(Z_1 + Z_2)$ is $|\rho|^2(\lambda_1^2 + \lambda_2^2)$ sub-Gaussian for every $\rho \in \mathbb{R}$.

8.1. Proof of Lemma 20. By Assumption 4, the estimate

$$|b(t, x, \mu_t)| \leq b_0 + |b|_{\text{Lip}}(|x| + \mathbb{E}_{\mathbb{P}^N}[|X^i_t|])$$

holds for every $(t, x) \in [0, T] \times \mathbb{R}^d$, where $b_0 = \sup_{t \in [0, T]} |b(t, 0, \delta_0)|$. Remember that

$$\mathcal{B}_i = \int_0^t c(s, X^i_s)^{-1/2} (dX^i_s - b(s, X^i_s, \mu_s) ds), \quad 1 \leq i \leq N,$

are independent $d$-dimensional $\mathbb{P}^N$-Brownian motions. By Minkowski’s and Jensen’s inequality, we have

$$|X^i_t| \leq |X^i_0| + \int_0^t |b(t, X^i_s, \mu_s)| ds + |\int_0^t \sigma(s, X^i_s) d\mathcal{B}_s^i|$$

$$\leq |X^i_0| + b_0 t + |b|_{\text{Lip}} \int_0^t (|X^i_s| + \mathbb{E}_{\mathbb{P}^N}[|X^i_s|]) ds + |\int_0^t \sigma(s, X^i_s) d\mathcal{B}_s^i|$$

$$\leq |X^i_0| + b_0 t + |b|_{\text{Lip}} \int_0^t (|X^i_s| + \mathbb{E}_{\mathbb{P}^N}[|X^i_s|]) ds + \zeta^i_T,$$

where $\zeta^i_T = \sup_{0 \leq s \leq T} \int_0^t \sigma(s, X^i_s) d\mathcal{B}_s^i$. Integrating w.r.t. $\mathbb{P}^N$, we also have

$$\mathbb{E}_{\mathbb{P}^N}[|X^i_t|] \leq \mathbb{E}_{\mathbb{P}^N}[|X^i_0|] + b_0 T + 2|b|_{\text{Lip}} \int_0^t \mathbb{E}_{\mathbb{P}^N}[|X^i_s|] ds + \mathbb{E}_{\mathbb{P}^N}[\zeta^i_T].$$

We infer by Grönwall’s lemma

$$\mathbb{E}_{\mathbb{P}^N}[|X^i_t|] \leq (\mathbb{E}_{\mathbb{P}^N}[|X^i_0|] + b_0 T + \mathbb{E}_{\mathbb{P}^N}[\zeta^i_T]) e^{2|b|_{\text{Lip}} t}$$

and plugging this estimate in (72) we infer

$$|X^i_t| \leq |X^i_0| + b_0 T + |b|_{\text{Lip}} \int_0^t |X^i_s| ds + (\mathbb{E}_{\mathbb{P}^N}[|X^i_0|] + b_0 T + \mathbb{E}_{\mathbb{P}^N}[\zeta^i_T]) e^{2|b|_{\text{Lip}} t} + \zeta^i_T.$$
Applying Grönwall’s lemma again, we derive
\[ |X^i_t| \leq (|X^i_0| + b_0 T + E_{\mathbb{P}^N} [X^i_t] + 2b_0 T + E_{\mathbb{P}^N} [\zeta_T^i]) e^{2|b| \ln p T} + \zeta^i_T e^{|b| \ln p T}. \]

Taking the exponent \(2p\) and expectation w.r.t. \(\mathbb{P}^N\), we further obtain
\[ E_{\mathbb{P}^N} [X^i_t]^{2p} \leq 5^{2p-1} (2E_{\mathbb{P}^N} [X^i_0]^{2p} + (2b_0 T)^p + 2E_{\mathbb{P}^N} [(\zeta_T^i)^{2p}]) e^{3|b| \ln p T p} \]
\[ \leq C_6^p (E_{\mathbb{P}^N} [X^i_0]^{2p} + (b_0 T)^p + E_{\mathbb{P}^N} [(\zeta_T^i)^{2p}]) \]
with \(C_6 = 50 e^{3|b| \ln p T}.\) By Assumption 1, the initial condition \(|X^i_0|\) satisfies
\[ E_{\mathbb{P}^N} [\exp(\gamma_0 |X^i_0|)] = 1 + \sum_{p \geq 1} \frac{\gamma_0^p}{p!} E_{\mathbb{P}^N} [X^i_0]^{2p} \leq \gamma_1 \]
hence for every \(p \geq 1\), we obtain
\[ E_{\mathbb{P}^N} [X^i_0]^{2p} \leq p! \left(\frac{\gamma_1}{\gamma_0}\right)^p \]
since \(\gamma_1 \geq 1.\) By Burkholder-Davis-Gundy’s inequality with constant \((C^*)^{p/2}p^{p/2}\) for some numerical constant \(C^*\), see e.g. Barlow and Yor [4], we also have
\[ E_{\mathbb{P}^N} [(\zeta_T^i)^{2p}] \leq \left(\frac{2p}{2p-1}\right)^{2p} E_{\mathbb{P}^N} \left[ \int_0^T \sigma(t, X^i_t) dB_t \right]^{2p} \]
\[ \leq \left(\frac{2p}{2p-1}\right)^{2p} (2C^*)^{p} p^{p} E_{\mathbb{P}^N} \left[ \left( \int_0^T \text{Tr}(c(t, X^i_t)) dt \right)^p \right] \]
\[ \leq p^p (8C^* |T| \text{Tr}(c) \| c \|\infty)^p \leq p^p (8C^* e \|c\| \text{Tr}(c) \| \text{Tr}(c) \|\infty)^p. \]

Putting these estimates together, we conclude
\[ E_{\mathbb{P}^N} [X^i_t]^{2p} \leq p! C_6^p \left(\frac{\gamma_1}{\gamma_0} + T(b_0 + 8C^* e |\text{Tr}(c)\|\infty)\right)^p \]
and Lemma 20 is established with \(C_2 = C_6 \left(\frac{\gamma_1}{\gamma_0} + T(b_0 + 8C^* e |\text{Tr}(c)\|\infty)\right). \)

8.2. Proof of Lemma 22. Fix \(J_k = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}.\) For \(g : [0, T] \times (\mathbb{R}^d)^k \times (\mathbb{R}^d)^\ell \rightarrow \mathbb{R}^d,\) we define
\[ g_{\ell_k}(t, y^\ell) = g(t, X^{i_1}_t, \ldots, X^{i_k}_t, y^\ell). \]

For technical convenience, we establish a slightly stronger, replacing \(V^N_{2p} (f(t, \cdot))\) in (46) by
\[ V^N_{2p, \ell}(g_{\ell_k-t_k+1}\cdot) = E_{\mathbb{P}^N} \left[ \left( \int_{(\mathbb{R}^d)\ell} g(t, X^{i_1}_t, X^{i_2}_t, \ldots, X^{i_{k-\ell+1}_t}, y^\ell) (\mu^N - \mu_t) \otimes^\ell (dy^\ell) \right)^{2p} \right] \]
for every \(J_k = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}\) with cardinality \(k - \ell + 1\) and every function \(g : [0, T] \times (\mathbb{R}^d)^{k-\ell+1} \times (\mathbb{R}^d)^\ell \rightarrow \mathbb{R}^d,\) Lipschitz continuous in the space variables, that defines in turn a class \(\mathcal{G}_{k, \ell+1}.\) In particular \(V^N_{2p, \ell}(f(t, \cdot))\) and \(V^N_{2p, \ell}(g_{\ell_k-t_k+1}\cdot)\) agree for \(\ell = k\) in which case the class \(\mathcal{G}_{1, k}\) coincide with \(\mathcal{G}_k\) and we obtain Lemma 22. We prove the result by induction.
Step 1: The case $\ell = 1$. For $g \in G_{k,1}$, $x_k \in (\mathbb{R}^d)^k$ and $I \subset \{1, \ldots, N\}$, we derive

$$
\Lambda_I^k(g, x^k) = \int_{\mathbb{R}^d} g(t, x^k, y)(\mu_t^k - \mu_t)(dy),
$$

where we write $\mu_t^j(dx) = |\delta|^j - 1 \sum_{\delta \in \mathcal{J}} \delta x_t^I(dx)$ for the empirical measure in restriction to $\mathcal{J}$. Observe that $\Lambda_I^k(g, x^k)$ is a sum of independent and centred random variables under $\mathbb{P}^N$. We write

$$
\Lambda_{\{1, \ldots, N\}}^k(g, X_{t}^{i_1}, \ldots, X_{t}^{i_k}) = N^{-1} \sum_{i \in \mathcal{J}_k} \left( g(t, X_{t}^{i_1}, \ldots, X_{t}^{i_k}, X_t) - \int_{\mathbb{R}^d} g(t, X_{t}^{i_1}, \ldots, X_{t}^{i_k}, y)\mu_t(dy) \right)
$$

$$
+ \frac{N-k}{N} \Lambda^j_t(g, X_{t}^{i_1}, \ldots, X_{t}^{i_k}),
$$

since $|\mathcal{J}_k| = k$. We obtain the decomposition

$$
\mathcal{V}_{2p,1}^N(g_k(t, \cdot)) = \mathbb{E}_{\mathbb{P}^N} \left[ |\Lambda_{\{1, \ldots, N\}}^k(g, X_{t}^{i_1}, \ldots, X_{t}^{i_k})|^{2p} \right] \leq 2^{2p-1}(I + II),
$$

with

$$
I = \frac{k^{2p-1}}{N^{2p}} \sum_{i \in \mathcal{J}_k} \mathbb{E}_{\mathbb{P}^N} \left[ \left| g(t, X_{t}^{i_1}, \ldots, X_{t}^{i_k}, X_t) - \int_{\mathbb{R}^d} g(t, X_{t}^{i_1}, \ldots, X_{t}^{i_k}, y)\mu_t(dy) \right|^{2p} \right],
$$

$$
II = \left( \frac{N-k}{N} \right)^{2p} \mathbb{E}_{\mathbb{P}^N} \left[ |\Lambda^j_t(g, X_{t}^{i_1}, \ldots, X_{t}^{i_k})|^{2p} \right].
$$

The term $I$ is controlled by the smoothness of $g$:

$$
I \leq k^{2p-1} \frac{\mathcal{L}^p(g(t, \cdot))}{N^{2p}} \sum_{i \in \mathcal{J}_k} \mathbb{E}_{\mathbb{P}^N} \left[ \left( \int_{\mathbb{R}^d} |X_t - y|^{2p} \mu_t(dy) \right) \right] \leq N^{-2p}p(24C_2)^p \|g(t, \cdot)\|^{2p}_{\mathcal{L}^p},
$$

where the last estimate stems from Lemma 20. For the term $II$, writing $g = (g^1, \ldots, g^d)$ where the functions $g^j$ are real-valued, we further have

$$
II \leq \left( \frac{N-k}{N} \right)^{2p} d^{2p-1} \sum_{j=1}^d \mathbb{E}_{\mathbb{P}^N} \left[ \Lambda^j_t(g^j, X_{t}^{i_1}, \ldots, X_{t}^{i_k}) \right]^{2p}.
$$

Moreover, for every $x \in \mathbb{R}^d$, the term

$$
\Lambda^j_t(g^j, x^k) = \frac{1}{N-k} \sum_{i \in \mathcal{J}_k} \left( g^j(t, x^k, X_t) - \mathbb{E}_{\mathbb{P}^N} \left[ g^j(t, x^k, X_t) \right] \right)
$$

is the sum of independent centred random variables that are independent of $(X_{t}^{i_1}, \ldots, X_{t}^{i_k})$ and

$$
g^j(t, x^k, X_t) - \mathbb{E}_{\mathbb{P}^N} \left[ g^j(t, x^k, X_t) \right]
$$

is $\lambda^2$ sub-Gaussian with $\lambda^2 = 24C_2\|g^j(t, \cdot)\|^{2p}_{\mathcal{L}^p}$ via the same estimate as for $I$ and the fact that (ii) implies (iv) in Definition 30. Thanks to the additivity property of independent sub-Gaussian random variables, we further infer that $\Lambda^j_t(g^j, x^k)$ is $\tilde{\lambda}^2$ sub-Gaussian with

$$
\tilde{\lambda}^2 = \frac{1}{N-k} \lambda^2 = \frac{1}{N-k} 24C_2\|g^j(t, \cdot)\|^{2p}_{\mathcal{L}^p}.
$$

Conditioning on $(X_{t}^{i_1}, \ldots, X_{t}^{i_k})$, we derive

$$
\mathbb{E}_{\mathbb{P}^N} \left[ \Lambda^j_t(g^j, X_{t}^{i_1}, \ldots, X_{t}^{i_k}) \right]^{2p} \leq \frac{pl(96C_2)^p}{(N-k)^p} \|g^j(t, \cdot)\|^{2p}_{\mathcal{L}^p},
$$
by (ii) of Definition 30. Plugging this estimate in (73), we obtain
\[
II \leq \frac{p!(96C_2d^2)^p}{(N-k)^p}|g(t, \cdot)|_{Lip}^{2p}
\]
and putting together our estimates for I and II, we conclude
\[
\mathcal{V}_{2p,1}^N(g(t, \cdot)) \leq \frac{p!K_p}{(N-k)^p}|g(t, \cdot)|_{Lip}^{2p}
\]
with \(K_1 = 16(k^2 + 24d^2)C_2\). This establishes Lemma 22 for \(g\) in the case \(\ell = 1\).

**Step 2:** We assume that (46) holds for \(\mathcal{V}_{2p,\ell}^N(g_{k-\ell+1}(t, \cdot))\), for every \(J_{k-\ell+1} \subset \{1, \ldots, N\}\) with cardinality \(k - \ell + 1\) and every \(g \in \mathcal{G}_{k-\ell+1,\ell}\) with \(\ell < k\). Let \(g \in \mathcal{G}_{k-\ell+1,\ell}\) and \(J_{k-\ell} \subset \{1, \ldots, N\}\). We have:
\[
\mathcal{V}_{2p,\ell+1}^N(g_{k-\ell}(t, \cdot)) = \mathbb{E}_{\mathbb{P}^N}\left[\int_{(\mathbb{R}^d)^{\ell+1}} g_{k-\ell}(t, y^{\ell+1})(\mu_1 - \mu_k)^{\otimes(\ell+1)}(dy^{\ell+1})|^{2p}\right]
\]
\[
\leq 2^{2p-1}(III + IV),
\]
with
\[
III = N^{-1} \sum_{i=1}^N \mathbb{E}_{\mathbb{P}^N}\left[\left|\int_{(\mathbb{R}^d)^{\ell}} g(t, x_1, x_2, \ldots, x_{k-\ell}, y^{\ell})(\mu_t - \mu_{y^{\ell}})^{\otimes\ell}(dy^{\ell})\right|^{2p}\right],
\]
\[
IV = \int_{\mathbb{R}^d} \mathbb{E}_{\mathbb{P}^N}\left[\left|\int_{(\mathbb{R}^d)^{\ell}} g(t, x_1, x_2, \ldots, x_{k-\ell}, y^{\ell})(\mu_t - \mu_{y^{\ell}})^{\otimes\ell}(dy^{\ell})\right|^{2p}\right] \mu_t(dy).
\]
Let \(i_0 \in J_{k-\ell}^c\) and put \(J_{k-\ell+1} = J_{k-\ell} \cup \{i_0\}\). The term IV can be rewritten as
\[
IV = \int_{\mathbb{R}^d} \mathcal{V}_{2p,\ell}^N(g_{k-\ell+1}(t, \cdot))(y) \mu_t(dy),
\]
where, for fixed \(y \in \mathbb{R}^d\), the function \(g(t, x_1, x_2, \ldots, x_{k-\ell}, y^{\ell})(y) = g(t, x_1, x_2, \ldots, x_{k-\ell}, y, y^{\ell})\) with the artificial variable \(x_{i_0}\) belongs to \(\mathcal{G}_{k-\ell+1,\ell}\). By the induction hypothesis and noting that \(\sup_{y \in \mathbb{R}^d} |g(t, \cdot, y)|_{Lip} \leq |g(t, \cdot)|_{Lip}\), we infer
\[
IV \leq \frac{p!K_p}{(N-k)^p}|g(t, \cdot)|_{Lip}^{2p}.
\]
We split the sum in III over indices in \(J_{k-\ell}\) and \(J_{k-\ell}^c\). If \(i \in J_{k-\ell}\), in the same way as for IV, we write
\[
g(t, x_1, x_2, \ldots, x_{k-\ell}, y^{\ell}) = g''(t, y^{\ell})
\]
with \(J_{k-\ell+1} = J_{k-\ell} \cup \{i\}\) for some arbitrary \(i_0 \in J_{k-\ell}\) and with \(g''(t, x_1, x_2, \ldots, x_{k-\ell}, x_{i_0}, y^{\ell}) = g(t, x_1, x_2, \ldots, x_{i-1}, x_i, y^{\ell})\), where \(i\) coincides with one of the \(i_j \in J_{k-\ell}\). Also, \(g''\) belongs to \(\mathcal{G}_{k-\ell+1,\ell}\). If \(i \in J_{k-\ell}^c\), we write
\[
g(t, x_1, x_2, \ldots, x_{k-\ell}, y^{\ell}) = g''(t, y^{\ell})
\]
with \(g''(t, x_1, x_2, \ldots, x_{i-1}, x_i, y^{\ell}) = g(t, x_1, x_2, \ldots, x_{i-1}, x_i, y^{\ell})\) and \(g''\) belongs to \(\mathcal{G}_{k-\ell+1,\ell}\) as well. We infer
\[
III \leq (k-\ell)N^{-1} \mathcal{V}_{2p,\ell}^N(g''(t, \cdot)) + N^{-1} \sum_{i \in J_{k-\ell}^c} \mathcal{V}_{2p,\ell}^N(g''(t, \cdot)) \leq \frac{p!K_p}{(N-k)^p}|g(t, \cdot)|_{Lip}^{2p}.
\]
by the induction hypothesis and noting again that $|g''(t, \cdot)|_{\text{Lip}}$ and $|g'''(t, \cdot)|_{\text{Lip}}$ are controlled by $|g(t, \cdot)|_{\text{Lip}}$. We conclude
\[
V_{2p,\ell+1}(g_{a_{k-1}}(t, \cdot)) \leq 2^{2p} \frac{p!K_{\ell+1}^p}{(N-k)^p} |g(t, \cdot)|_{\text{Lip}}^{2p} = \frac{p!K_{\ell+1}^p}{(N-k)^p} |g(t, \cdot)|_{\text{Lip}}^{2p}
\]
with $K_{\ell+1} = 4K_{\ell}$. The proof of Lemma 22 is complete.

8.3. (Sketch of) proof of Proposition 13. Step 1: Thanks to Chapters 6 and 9 of [7], since $F, G$ are bounded and $\mu_0$ satisfies Assumption 1, it can be shown that (3) admits a unique probability solution $\mu$ in the sense of [7], absolutely continuous w.r.t. the Lebesgue measure, that we still denote $\mu(t, x) = \mu_t(x)$. Moreover $\mu \in \mathcal{H}^{\delta/2, \delta}_{\text{loc}} = \cap_{(t_0, x_0) \in (0, T) \times \mathbb{R}^d} \mathcal{H}^{\delta/2, \delta}(t_0, x_0)$ for every $0 < \delta < 1$. The main arguments of these properties rely on the existence of a suitable Lyapunov function associated to (3), following the terminology of [51] and [7] (for instance $x \mapsto 1 + |x|^2$) together with Sobolev embeddings.

Step 2: Define
\[
\tilde{a}_k(t, x) = G^k(x) + \int_{\mathbb{R}^d} F^k(x - y) \mu_t(y) dy, \quad k = 1, \ldots, d,
\]
and
\[
\tilde{a}(t, x) = \text{div}(G(x) + \int_{\mathbb{R}^d} F(x - y) \mu_t(y) dy),
\]
which are well defined since $\beta, \beta' > 1$. Consider next the Cauchy problem associated to (3) in its strong form:
\[
\begin{cases}
\partial_t \tilde{\mu}_t = \frac{1}{2} \sigma^2 \Delta \tilde{\mu}_t - \sum_{k=1}^d \tilde{a}_k(t, \cdot) \partial_k \tilde{\mu}_t - \tilde{a}(t, \cdot) \tilde{\mu}_t \\
\tilde{\mu}_t = 0 = \mu_0.
\end{cases}
\]

Taking $\delta = \beta - |\beta|$ we obtain $\tilde{a}, \tilde{a} \in C^1_{\text{loc}}(\beta - |\beta|)/2, \beta - |\beta|$ by Step 1.

Step 3: Using $\inf \tilde{a} > -\infty$ and the existence of a Lyapunov function associated to the problem, by Theorem 2.3 of [2], there exists a unique solution $\bar{\mu}$ of (74). Moreover, $\bar{\mu}$ is continuous on $(0, T) \times \mathbb{R}^d$ and
\[
\bar{\mu} \in C^1_{\text{loc}}(\beta - |\beta|)/2, 2 + \beta - |\beta|) \cap L^\infty((0, T) \times \mathbb{R}^d).
\]
It is also the unique solution defined in Theorem 12 of Chapter 1 of [27], therefore the unique integrable solution of the problem (3). By uniqueness, $\mu = \bar{\mu}$.

Step 4: If $|\beta| = 1$, we obtain $\mu \in \mathcal{H}^{(1+\beta)/2, 1+\beta}(t_0, x_0)$ for every $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$. Otherwise, we can iterate the process thanks to results of Section 8.12 in [42]; successively:

- Since $\partial_{x_k} \tilde{a}_k$ and $\partial_{x_k} \tilde{a}$ are in $C^1_{\text{loc}}(\beta - |\beta|)/2, \beta - |\beta|)$, we have
  \[
  \partial_{x_k} \mu \in C^1_{\text{loc}}(\beta - |\beta|)/2, 2 + \beta - |\beta|).
  \]
- Since $\partial_t \tilde{a}_k$ and $\partial_t \tilde{a}$ are now in $C^1_{\text{loc}}(\beta - |\beta|)/2, \beta - |\beta|)$, we have
  \[
  \partial_t \mu \in C^1_{\text{loc}}(\beta - |\beta|)/2, 2 + \beta - |\beta|).
  \]

Therefore, if $|\beta| = 2$, we obtain $\mu \in \mathcal{H}^{(1+\beta)/2, 1+\beta}(t_0, x_0)$ for every $(t_0, x_0) \in (0, T) \times \mathbb{R}^d$. Otherwise, we can iterate again the process and so on. The result follows.
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