The Energy-Momentum Tensor in Field Theory I *

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This is the first of three papers on the short-distance properties of the energy-momentum tensor in field theory. We study the energy-momentum tensor for renormalized field theory in curved space. We postulate an exact Ward identity of the energy-momentum tensor. By studying the consistency of the Ward identity with the renormalization group and diffeomorphisms, we determine the short-distance singularities in the product of the energy-momentum tensor and an arbitrary composite field in terms of a connection for the space of composite fields over theory space. We discuss examples from the four-dimensional $\phi^4$ theory. In the forthcoming two papers we plan to discuss the torsion and curvature of the connection.

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1. Introduction

The study of the energy-momentum tensor in field theory has a long history. The short-coming of the naïve analysis of the classical lagrangian in constructing the energy-momentum tensor, like the canonical energy-momentum tensor and Belinfante tensor, was noticed long ago. (See ref. [1] and references therein for a convenient summary of the earlier works.) The purpose of the present paper and its sequels is to give a fresh look at this old subject of the energy-momentum tensor and its short-distance singularities.

Our motivation for the present study comes partially from string field theory. One of the most important issues in perturbative string field theory is to formulate the theory using an arbitrary two dimensional non-linear sigma model as a background. (Here, the non-linear sigma field takes values on an arbitrary space-time manifold, and the model has an infinite number of parameters including the space-time metric and dilaton field.) This is necessary for any discussion of background independence to make sense at all. Our goal is, then, to express the dependence of the theory on the parameters and the world-sheet metric in a way suitable for string field theory.

The dependence of renormalized field theories on their parameters has been studied recently in the framework of the variational formula [2]. The notion of a field $\mathcal{O}_i$ conjugate to a parameter $g^i$ has been introduced, and the spatial integral over the conjugate field has been given a precise meaning. In doing so, we have introduced a connection $c_i$ for the linear space of composite fields. The short-distance singularities in the product of a conjugate field and an arbitrary composite field are expressed in terms of the scale dimension of the composite field and the connection $c_i$. This formulation of field theory in coordinate space, especially its application to conformal field theory [3], has turned out to be the necessary tool in proving the background independence of string field theory based upon a conformal field theory with continuous parameters [4].

We will study the energy-momentum tensor using the techniques similar to the above variational formula. It is possible to consider the energy-momentum tensor strictly in the realm of field theory in flat space. But, as was pointed out for the first time in ref. [5], the energy-momentum tensor is best defined in curved space, or, equivalently, it can be defined unambiguously if we couple the theory to external gravity. Hence, we study general renormalized field theories in curved space, i.e., a Riemann manifold with metric $h_{\mu\nu}$. We will define the energy-momentum tensor as the field that is conjugate to the metric $h_{\mu\nu}$.

We are interested in the general properties of the energy-momentum tensor. We would like to isolate those features of the energy-momentum tensor that specify it uniquely. We
would also like to study the short-distance singularities of the product of the energy-momentum tensor and an arbitrary composite field. We wish, in the course of this work, to take up the old subject which was initiated in refs. [5], [6], and [7]. The reader will recognize a fuller use of the renormalization group (RG) in our study.

In sect. 2 we will postulate an exact Ward identity for the energy-momentum tensor. In writing the Ward identity, the short-distance singularities are subtracted carefully to obtain a finite result. We will show that the finite counterterms in the Ward identity can be interpreted as the matrix elements of a connection for the linear space of composite fields over theory space. In sects. 3, 4, and 5 we check the consistency of the Ward identity postulated in sect. 2. In sect. 3 we will demand that the Ward identity is consistent with the RG. Consequently we will find that the energy-momentum tensor has no anomalous dimension and that the short-distance singularities in the product of the energy-momentum tensor and an arbitrary composite field are determined by the connection introduced in sect. 2. In sect. 4 we will see that the consistency of the Ward identity with the variational formula that determines how the theory depends on the parameters gives a derivation of the famous trace anomaly. We will obtain an expression of the trace of the energy-momentum tensor as a sum over the fields conjugate to the parameters. In sect. 5 we will require consistency with diffeomorphisms. We will write down the euclidean analogue of the commutator between the energy-momentum tensor and an arbitrary composite field in terms of the connection of sect. 2. In sect. 6 we will study further the short-distance singularities of the energy-momentum tensor, found in sects. 3 and 5. In sect. 7 we will discuss the characteristic properties of the energy-momentum tensor which specify it unambiguously. In sect. 8 we will discuss $\phi^4$ theory in four dimensional flat space to elucidate the general discussions of the preceding sections. We give concluding remarks in sect. 9.

2. The exact Ward identity of the energy-momentum tensor

We consider a renormalized field theory with renormalized parameters $g^i (i = 1, ..., N)$ on a $D$-dimensional manifold with a positive definite metric $h_{\mu\nu}$. We regard the parameters $g^i$ and the metric $h_{\mu\nu}$ as local coordinates of the theory space.

We introduce the renormalization group (RG) transformation on the theory space as

$$\frac{d}{dt} g^i = \beta^i (g) \quad (2.1a)$$

$$\frac{d}{dt} h_{\mu\nu} = -2h_{\mu\nu} \quad (2.1b)$$
Note that eq. (2.1b) implies that the physical distance between two points \( r^\mu \) and \( r^\mu + \delta r^\mu \) goes as \( e^{-t} \) under the RG transformation.

Let \( \{ \Phi_a \} \) be a basis of composite fields. We only take covariant local fields for simplicity. We can define a new basis by

\[
\Phi'_a = (N(h, g))_a^b \Phi_b ,
\]

(2.2)

where \( N(h, g) \) is an invertible matrix which can depend on both the parameters \( g^i \) and the metric \( h_{\mu\nu} \), curvature, and its covariant derivatives. The composite fields satisfy the RG equations

\[
\frac{d}{dt} \Phi_a = (\gamma(h, g))_a^b \Phi_b ,
\]

(2.3)

where \( \gamma \) is the matrix of the full scale dimension, in the sense that the correlation function of \( n \) arbitrary composite fields satisfies the following RG equation:

\[
\frac{d}{dt} \langle \Phi_{a_1}(P_1)...\Phi_{a_n}(P_n) \rangle_{h,g} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \langle \Phi_{a_1}(P_1)...\Phi_{a_n}(P_n) \rangle_{(1-\Delta t)h,g+\Delta t\beta} - \langle \Phi_{a_1}(P_1)...\Phi_{a_n}(P_n) \rangle_{h,g} \right] 
\]

\[
= \sum_{k=1}^{n} [\gamma(h(P_k), g)]_{a_k}^b \langle \Phi_{a_1}(P_1)...\Phi_{b}(P_k)...\Phi_{a_n}(P_n) \rangle_{h,g} .
\]

(2.4)

We note that the convention of the RG equations adopted here differs somewhat from the convention adopted on flat space for which the metric is

\[
h_{\mu\nu}(r) = \delta_{\mu\nu} .
\]

(2.5)

To keep the flat metric invariant under the RG, we must compensate eq. (2.1b) by a coordinate transformation

\[
r^\mu \to (1 - \Delta t)r^\mu ,
\]

(2.6)

where \( \Delta t \) is an infinitesimal change of the scale parameter \( t \). If the field \( \Phi_a \) is a tensor of rank \((m,n)\), it transforms as

\[
(\Phi_a)^{\nu_1...\nu_m}_{\mu_1...\mu_n} \to (1 + (m - n)\Delta t)(\Phi_a)^{\nu_1...\nu_m}_{\mu_1...\mu_n}
\]

(2.7)

under (2.6). Hence, in the flat space the RG transformation is given by eqs. (2.1a) for the parameters and

\[
\frac{d}{dt} \Phi_a = (m - n) \Phi_a + (\gamma(\delta, g))_a^b \Phi_b .
\]

(2.8)
for the composite fields.

We define the energy-momentum tensor $\Theta^{\mu\nu}$ as the composite field that generates infinitesimal changes of the metric tensor $h_{\mu\nu}$. More specifically we assume the existence of $\Theta^{\mu\nu}$ that satisfies the following exact Ward identity:

$$
\langle \Phi_{a_1}(P_1)\cdots\Phi_{a_n}(P_n) \rangle_{h,g} - \langle \Phi_{a_1}(P_1)\cdots\Phi_{a_n}(P_n) \rangle_{h+\delta h,g} = \lim_{\epsilon \to 0} \left[ \int_{\rho(r,P_k) \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \right. \\
\times \left. \left\langle \left( \Theta^{\mu\nu}(r) - \langle \Theta^{\mu\nu}(r) \rangle_{h,g} \right) \Phi_{a_1}(P_1)\cdots\Phi_{a_n}(P_n) \right\rangle_{h,g} + \sum_{k=1}^{n} \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \cdots \nabla_{\mu_m} \frac{1}{2} \delta h_{\mu\nu}(P_k) \\
\times \left\{ (\mathcal{K}^{\mu\nu,\mu_1\cdots\mu_m}(h(P_k),g))_{ak} - \int_{\epsilon}^{1} d\rho (\mathcal{C}^{\mu\nu,\mu_1\cdots\mu_m}(\rho; h(P_k),g))_{ak} \right\} \right. \\
\times \left. \langle \Phi_{a_1}(P_1)\cdots\Phi_{b}(P_k)\cdots\Phi_{a_n}(P_n) \rangle_{h,g} \right],
$$

(2.9)

where $\delta h_{\mu\nu}$ is an infinitesimal symmetric tensor. The Ward identity specifies only the symmetric part of $\Theta^{\mu\nu}$, and we can define $\Theta^{\mu\nu}$ to be symmetric. The symbol $\rho(r,P)$ denotes the geodesic distance between the points $r$ and $P$. We must exclude infinitesimal balls $\rho(r,P_k) \leq \epsilon$ from the domain of integration since the product $\Theta^{\mu\nu}(r)\Phi_{a_k}(P_k)$ contains short-distance singularities.

Let us elaborate on the coefficients $\mathcal{C}^{\mu\nu,\mu_1\cdots\mu_m}$ in the exact Ward identity (2.9). In a neighborhood of a point $P$ we can decompose the volume element $\sqrt{h(r)}d^D r$ to the product of the volume elements for the radial and angular parts:

$$
\sqrt{h(r)}d^D r = d\rho \ d^{D-1}\Omega_{\rho}(r,P). \quad (2.10)
$$

Then, given an arbitrary symmetric tensor $t_{\mu\nu}(r)$, we can expand the product $\Theta^{\mu\nu}(r)\Phi_{a}(P)$ as

$$
\int d^{D-1}\Omega_{\rho}(r,P) \ t_{\mu\nu}(r)\Theta^{\mu\nu}(r)\Phi_{a}(P) = \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \cdots \nabla_{\mu_m} t_{\mu\nu}(P) \cdot (\mathcal{C}^{\mu\nu,\mu_1\cdots\mu_m}(\rho; h(P),g))_{a} \Phi_{b}(P) + o \left( \frac{1}{\rho(r,P)} \right), \quad (2.11)
$$

where we only keep the part which cannot be integrated over $\rho$ to the origin. Because of this, the sum over the integer $m$ is a finite sum. The coefficients $(\mathcal{C}^{\mu\nu,\mu_1\cdots\mu_m})_{a}$ is a
tensor at the point $P$ which can depend on the geodesic distance $\rho(r, P)$, metric $h_{\mu\nu}(P)$, curvature $R_{\mu\nu\alpha\beta}(P)$, its covariant derivatives at $P$, and parameters $g^i$. The coefficient $C^{\mu\nu,\mu_1\ldots\mu_m}$ is symmetric with respect to $\mu_1, \ldots, \mu_m$.

Coming back to the exact Ward identity (2.9), we can take the limit $\epsilon \to 0$ thanks to the subtraction of the short-distance singularities (2.11). To compensate the arbitrariness of the subtraction, we need to introduce finite counterterms $K^{\mu\nu,\mu_1\ldots\mu_m}$.

It is easy to see that the finite counterterms behave as the matrix elements of a connection for the linear space of composite fields over theory space. Under the change of basis (2.2), we find that the operator product expansion (OPE) coefficients transform covariantly as

$$ (\delta h \cdot C)(\rho; h(P), g) \to N(h(P), g) \cdot (\delta h \cdot C)(\rho; h(P), g) \cdot N^{-1}(h(P), g) , $$

but the finite counterterms transform as

$$ (\delta h \cdot K)(h(P), g) \to N(h(P), g) \cdot (\delta h \cdot K)(h(P), g) \cdot N^{-1}(h(P), g) $$

$$ + (N(h(P), g) - N(h + \delta h(P), g)) N^{-1}(h(P), g) , $$

(2.13)

where we have introduced the short-hand notation

$$ (\delta h \cdot C)(h(P), g) \equiv \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \ldots \nabla_{\mu_m} \frac{1}{2} \delta h_{\mu\nu}(P) \cdot C^{\mu\nu,\mu_1\ldots\mu_m}(h(P), g) , $$

$$ (\delta h \cdot K)(h(P), g) \equiv \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \ldots \nabla_{\mu_m} \frac{1}{2} \delta h_{\mu\nu}(P) \cdot K^{\mu\nu,\mu_1\ldots\mu_m}(h(P), g) . $$

(2.14)

From the transformation property (2.13), we see that $K^{\mu\nu,\mu_1\ldots\mu_m}$ is a connection for the linear space of composite fields over theory space.

Here we should recall that the theory space has the metric $h_{\mu\nu}$ and parameters $g^i$ as local coordinates. Therefore, strictly speaking, $K^{\mu\nu,\mu_1\ldots\mu_m}$ gives the connection in the direction of the metric deformation on the theory space. The elements of the connection in the direction of the parameters $g^i$ have been introduced as $c_i$ in ref. [2]. Though only field theory in flat space is discussed in this reference, the generalization to curved space is straightforward.

For the reader’s convenience, let us summarize the results of ref. [2] relevant to this paper. The connection $c_i$ has been introduced as finite counterterms in the variational
formula that expresses how the correlation functions change under infinitesimal changes of the parameters $g^i$:

$$-rac{\partial}{\partial g^i} \langle \Phi_{a_1}(P_1)\ldots\Phi_{a_n}(P_n) \rangle_{h,g}$$

$$= \int_{\rho(r,P_k) \geq \epsilon} d^D r \sqrt{h} \left\langle \left( \mathcal{O}_i(r) - \langle \mathcal{O}_i(r) \rangle_{h,g} \right) \Phi_{a_1}(P_1)\ldots\Phi_{a_n}(P_n) \right\rangle_{h,g}$$

$$+ \sum_{k=1}^n \left[ (c_i)_a^b (h(P_k), g) \right.$$

$$- \int_{\epsilon}^1 \rho \left( (c_i)_a^b (\rho; h(P_k), g) \right)$$

$$\times \langle \Phi_{a_1}(P_1)\ldots\Phi_b(P_k)\ldots\Phi_{a_n}(P_n) \rangle_{h,g} \bigg] \ .$$

(2.15)

where $\mathcal{O}_i$ is the composite field conjugate to the parameter $g^i$, and the OPE coefficients $\mathcal{C}_i$ are defined by

$$\int d^{D-1} \Omega \rho \mathcal{O}_i(r) \Phi_a(P) = (c_i)_a^b (\rho; h(P), g) \Phi_b(P) + o \left( \frac{1}{\rho} \right) .$$

(2.16)

Note that the connection $c_i(h(P), g)$ in general depends not only on the parameters but also on the metric, curvature, and its derivatives at point $P$. Under the change of basis (2.2), the connection $c_i$ transforms as

$$c_i(h(P), g) \rightarrow N(h(P), g) \left( c_i(h(P), g) + \frac{\partial}{\partial g^i} \right) N^{-1}(h(P), g) .$$

(2.17)

In general the conjugate field $\mathcal{O}_i$ is ambiguous up to a total derivative $\nabla_\mu J^\mu_i$, but we assume that we can remove the ambiguity by demanding the absence of mixing with total derivatives under the RG. Then, the conjugate field $\mathcal{O}_i$ satisfies the RG equation

$$\frac{d}{dt} \mathcal{O}_i = D\mathcal{O}_i - \frac{\partial \beta^j}{\partial g^i} \mathcal{O}_j .$$

(2.18)

Finally we note that the consistency of the variational formula (2.15) with the RG gives the OPE coefficients $\mathcal{C}_i$ as

$$\mathcal{C}_i(1; h, g) = \partial_i \gamma + \frac{d}{dt} c_i - [\gamma, c_i] + \partial_i \beta^j \cdot c_j .$$

(2.19)

The purpose of the following three sections is to check consistency of the exact Ward identity (2.9).
3. Consistency with the RG

We demand that the exact Ward identity (2.9) be consistent with the RG equations (2.1). In ref. [2], it is shown that the consistency of the variational formula (2.15) with the RG gives rise to the expression (2.19) of the OPE coefficients $C_i$. We proceed analogously here.

We can compute

$$\Delta \equiv \langle \Phi_a(P) \rangle e^{-2\Delta t(h+\delta h),g+\Delta t\beta} - \langle \Phi_a(P) \rangle_{h,g}$$

(3.1)

to first order in $\Delta t$ and $\delta h$ in two different ways. The results must agree. The two methods are shown schematically in Fig. 1.

![Fig. 1](two different ways of evaluating the difference $\Delta$)

In the first method we calculate

$$\Delta = \left( \langle \Phi_a(P) \rangle_{(h+\delta h)e^{-2\Delta t(h+\delta h),g+\Delta t\beta}} - \langle \Phi_a(P) \rangle_{h+\delta h,g} \right)$$

(3.2)

We apply the RG eq. (2.4) to the first bracket and obtain

$$\Delta = \Delta t \gamma_b^a(h+\delta h,g) \langle \Phi_b(P) \rangle_{h+\delta h,g}$$

(3.3)

By applying the exact Ward identity (2.9), we find

$$\Delta = \Delta t \gamma_b^a(h,g) \langle \Phi_b(P) \rangle_{h,g} + \left( \langle \Phi_a(P) \rangle_{h+\delta h,g} - \langle \Phi_a(P) \rangle_{h,g} \right)$$
\[+ \Delta t \left( \delta_h \gamma_a^b(h, g) \langle \Phi_b(P) \rangle_{h,g} \right) \tag{3.4}\]
\[+ \Delta t \left( \gamma_a^b(h, g) \left[ - \int_{\rho(r,P) \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu \nu}(r) \langle \Theta^\mu^\nu(r) \Phi_b(P) \rangle_{h,g}^c \right. \right.
\[+ \left. \left( \int_{\epsilon}^1 d \rho \ \delta h \cdot \mathcal{C}(\rho; h, g) - \delta h \cdot \mathcal{K}(h, g) \right) \right]_b \langle \Phi_c(P) \rangle_{h,g} \right],\]

where \(c\) denotes the connected part, and
\[
\delta_h \gamma(h, g) = \gamma(h + \delta h, g) - \gamma(h, g). \tag{3.5}\]

In the second method we calculate
\[
\Delta = \left( \langle \Phi_a(P) \rangle_{(h+\delta h) e^{-2\Delta t}, g + \Delta t \beta} - \langle \Phi_a(P) \rangle_{h e^{-2\Delta t}, g + \Delta t \beta} \right)
\[+ \left( \langle \Phi_a(P) \rangle_{h e^{-2\Delta t}, g + \Delta t \beta} - \langle \Phi_a(P) \rangle_{h,g} \right). \tag{3.6}\]

Using the Ward identity (2.9) and the RG eq. (2.4), we obtain
\[
\Delta = - \int_{\rho(r,P,h e^{-2\Delta t}) \geq \epsilon e^{-\Delta t}} d^D r \sqrt{h} \ e^{-D \Delta t}
\[\times \frac{1}{2} \delta h_{\mu \nu}(r) e^{-2\Delta t} \langle \Theta^\mu^\nu(r) \Phi_a(P) \rangle_{h e^{-2\Delta t}, g + \Delta t \beta}^c \]
\[+ \left[ \int_{\epsilon e^{-\Delta t}}^1 d \rho \ e^{-2\Delta t} \delta h \cdot \mathcal{C}(\rho; h e^{-2\Delta t}, g + \Delta t \beta) \right. \]
\[\left. - e^{-2\Delta t} \delta h \cdot \mathcal{K}(h e^{-2\Delta t}, g + \Delta t \beta) \right]_b \langle \Phi_b(P) \rangle_{h e^{-2\Delta t}, g + \Delta t \beta}^a \]
\[\left. + \Delta t \gamma_a^b(h, g) \langle \Phi_b(P) \rangle_{h,g} \right]. \tag{3.7}\]

Applying the RG eq. (2.4), we find
\[
\Delta = \Delta t \left( \gamma_a^b(h, g) \langle \Phi_b(P) \rangle_{h,g} \right)\]
\[+ \left( \langle \Phi_a(P) \rangle_{h+\delta h, g} - \langle \Phi_a(P) \rangle_{h,g} \right) \]
\[+ \Delta t \left[ - \gamma_a^b(h, g) \int_{\rho(r,P) \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu \nu}(r) \langle \Theta^\mu^\nu(r) \Phi_b(P) \rangle_{h,g}^c \right. \]
\[\left. - \int_{\rho(r,P) \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu \nu}(r) \right. \]
\[\times \left( \left. \frac{d}{dt} \Theta^\mu^\nu(r) - (D + 2) \Theta^\mu^\nu(r) \right) \Phi_a(P) \right)_{h,g}^c \tag{3.8}\]
\[ + (\delta h \cdot C(1; h, g))_a^b \langle \Phi_b(P) \rangle_{h,g} \]
\[ + \int_\varepsilon^1 d\rho \left\{ \delta h \cdot \left( \frac{d}{dt} C(\rho; h, g) + C(\rho; h, g)\gamma(h, g) \right) \right\}_a^b \langle \Phi_b(P) \rangle_{h,g} \]
\[ - \left\{ \delta h \cdot \left( \frac{d}{dt} K(h, g) + K(h, g)\gamma(h, g) \right) \right\}_a^b \langle \Phi_b(P) \rangle_{h,g} \],

where
\[ \frac{d}{dt} C(\rho; h, g) \equiv \frac{1}{\Delta t} \left[ e^{-3\Delta t} C(\rho e^{-\Delta t}; h, g + \Delta t\beta) - C(\rho; h, g) \right] \]
\[ \frac{d}{dt} K(h, g) \equiv \frac{1}{\Delta t} \left[ e^{-2\Delta t} K(\rho e^{-\Delta t}; h, g + \Delta t\beta) - K(\rho; h, g) \right]. \]  

(3.9)

Now, eq. \( (3.8) \) must agree with eq. \( (3.4) \). This gives
\[ \left[ \delta_h \gamma + \delta h \cdot \left\{ \gamma(h, g) \left( \int_\varepsilon^1 d\rho \left( C(\rho; h, g) - K(\rho; h, g) \right) \right) \right\}_a \right] \langle \Phi_b(P) \rangle_{h,g} \]
\[ = - \int_{\rho(r, P) \geq \varepsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \]
\[ \times \left\langle \left( \frac{d}{dt} \Theta^{\mu\nu}(r) - (D + 2)\Theta^{\mu\nu}(r) \right) \Phi_a(P) \right\rangle_{h,g} \]
\[ + \left[ \delta h \cdot \left\{ C(1; h, g) + \int_\varepsilon^1 d\rho \left( \frac{d}{dt} C(\rho; h, g) + C(\rho; h, g)\gamma(h, g) \right) \right\}_a \right] \langle \Phi_b(P) \rangle_{h,g} \].

(3.10)

Since the left-hand side is local, the integrand of the integral on the right-hand side must be a total derivative. For this to be valid for an arbitrary \( \delta h_{\mu\nu} \), we must find
\[ \frac{d}{dt} \Theta^{\mu\nu} = (D + 2)\Theta^{\mu\nu}. \]  

(3.11)

This implies that the energy-momentum tensor has no anomaly under the renormalization group. The RG equations \( (2.4) \) and \( (3.11) \) imply that
\[ \frac{d}{dt} C(\rho; h, g) = [\gamma(h, g), C(\rho; h, g)] , \]  

(3.12)

where the square bracket denotes a commutator. Substituting eqs. \( (3.11) \) and \( (3.12) \) into eq. \( (3.10) \), we obtain
\[ \delta h \cdot C(1; h, g) = \delta h \gamma(h, g) + \delta h \cdot \left( \frac{d}{dt} K(h, g) - [\gamma(h, g), K(h, g)] \right) . \]  

(3.13)

This gives the OPE coefficients \( C \) in terms of the scale dimensions \( \gamma \) and connection \( K \) in the same way that eq. \( (2.13) \) gives the OPE coefficients \( C_i \) in terms of the scale dimensions \( \gamma \) and connection \( c_i \). The relation \( (3.13) \) is a main result of this paper.
4. Consistency with the variational formula

In this section we demand consistency between the exact Ward identity (2.9) and the variational formula (2.15). We recall the RG equation (2.4) for $n = 1$:

$$\langle \Phi_a(P) \rangle_{h,g} (1 - 2\Delta t) + \Delta t \beta_{\gamma^b} \langle \Phi_a(P) \rangle_{h,g} = \Delta t \gamma_{a}^{b} \langle h(P), g \rangle \langle \Phi_b(P) \rangle_{h,g}.$$  (4.1)

We can evaluate the left-hand side by using the exact Ward identity (2.9) and the variational formula (2.15). Keeping only terms of first order in $\Delta t$, we obtain

$$\langle \Phi_a(P) \rangle_{h,g} (1 - 2\Delta t) + \Delta t \beta_{\gamma^b} - \langle \Phi_a(P) \rangle_{h,g} = (\langle \Phi_a(P) \rangle_{h,g} (1 - 2\Delta t) - \langle \Phi_a(P) \rangle_{h,g}) + (\langle \Phi_a(P) \rangle_{h,g} + \Delta t \beta_{\gamma^b} - \langle \Phi_a(P) \rangle_{h,g}).$$  (4.2)

We use (2.9) for the first bracket and (2.15) for the second to obtain

$$\langle \Phi_a(P) \rangle_{(1-2\Delta t)h,g + \Delta t\beta} - \langle \Phi_a(P) \rangle_{h,g} = \left[ \int \rho(r,P) \geq \epsilon \, dD_{\rho} \langle (\Theta(r) - \beta^i O_i(r)) \Phi_a(P) \rangle_{h,g}^c ight. \right. \left. \left. + \int \epsilon \, d\rho \left( -h_{\mu\nu}(P)C^{\mu\nu} + \beta^i C_i \right)_{a}^{b} \left( \rho; h(P), g \right) \langle \Phi_b(P) \rangle_{h,g} ight] + \left( h_{\mu\nu}(P)C^{\mu\nu} - \beta^i c_i \right)_{a}^{b} \langle h(P), g \rangle \langle \Phi_b(P) \rangle_{h,g}.$$  (4.3)

where we denote the trace of the energy-momentum tensor by

$$\Theta \equiv h_{\mu\nu} \Theta^{\mu\nu}.$$  (4.4)

For eqs. (4.1) and (4.3) to agree, the difference $\Theta - \beta^i O_i$ must be a total derivative

$$\Theta - \beta^i O_i = \nabla_{\mu} J^{\mu}$$  (4.5)

so that the integral in (4.3) reduces to local terms at $P$. From (3.11) and (2.18), however, both the trace $\Theta$ and $\beta^i O_i$ have canonical dimension $D$ under the RG, and we must find

$$\frac{d}{dt} \nabla_{\mu} J^{\mu} = D \nabla_{\mu} J^{\mu}.$$  (4.6)

This implies that the current $J^{\mu}$ has no anomalous dimension except that it may mix with a conserved current $j^{\mu}$:

$$\frac{d}{dt} J^{\mu} = DJ^{\mu} + j^{\mu}.$$  (4.7)
We assume that such a non-conserved current $J^\mu$ does not exist. Therefore, we conclude that

$$\Theta = \beta^i \mathcal{O}_i .$$

This is the well-known trace anomaly. It means that the sum of the conjugate fields $\beta^i \mathcal{O}_i$ generates global scale transformations.

Finally, the consistency between eqs. (4.1) and (4.3) gives a constraint on the trace of the connection:

$$h_{\mu\nu}(P)\mathcal{K}^{\mu\nu}(h(P), g) = 2h(P) \cdot \mathcal{K}(h(P), g)$$

$$= \Psi(h(P), g) \equiv \gamma(h(P), g) + \beta^i c_i(h(P), g) .$$

This turns out to be very useful for practical applications as we will see in sect. 8.

5. Consistency with diffeomorphisms

The most interesting consistency check is that with diffeomorphisms. We can express an arbitrary infinitesimal diffeomorphism by

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + (\mathcal{L}_u h)_{\mu\nu} ,$$

where $\mathcal{L}_u$ denotes the Lie derivative along an infinitesimal vector field $u^\mu$:

$$\mathcal{L}_u h_{\mu\nu} \equiv \nabla_\mu u_\nu + \nabla_\nu u_\mu .$$

Under the diffeomorphism we must find

$$\langle \Phi_a(P) \rangle_{h + \mathcal{L}_u h, g} = \langle \Phi_a(P) \rangle_{h, g} + \langle \mathcal{L}_u \Phi_a(P) \rangle_{h, g} .$$

Recall that if $\Phi$ is a tensor of rank $(m,n)$, its Lie derivative is defined by

$$\mathcal{L}_u \Phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} \equiv u^\mu \partial_\mu \Phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} + \sum_{k=1}^m \partial_{\mu_k} u^\mu \Phi_{\mu_1 \ldots \mu_{k-1} \mu_{k+1} \ldots \mu_m}^{\nu_1 \ldots \nu_n} - \sum_{k=1}^n \partial_{\nu_k} u^{\nu_k} \Phi_{\mu_1 \ldots \mu_{k-1} \mu_{k+1} \ldots \mu_m}^{\nu_1 \ldots \nu_{k-1} \nu_{k+1} \ldots \nu_n} .$$

We can also calculate the change of the expectation value under the diffeomorphism using the Ward identity (2.9). We find

$$\langle \Phi_a(P) \rangle_{h + \mathcal{L}_u h, g} - \langle \Phi_a(P) \rangle_{h, g}$$

$$= - \int_{\rho(r, P) \geq \epsilon} d^Dr \sqrt{h} \nabla_\mu u_\nu(r) \langle \Theta^{\mu\nu}(r) \Phi_a(P) \rangle_{h, g}^c$$

$$+ \left[ \mathcal{L}_u h \cdot \left( \int_{\epsilon}^1 d\rho \mathcal{C}(\rho; h, g) - \mathcal{K}(h, g) \right) \right]_a^b \langle \Phi_b(P) \rangle_{h, g} .$$

11
Integration by parts gives

\[
\langle \Phi_a(P) \rangle_{h+\mathcal{L}_u h,g} - \langle \Phi_a(P) \rangle_{h,g} = \int_{\rho(r,P) \geq \epsilon} d^D r \sqrt{h} \ u_\nu(r) \langle \nabla_\mu \Theta^{\mu\nu}(r) \Phi_a(P) \rangle_{h,g}^c \\
+ \int_{\rho(r,P) = \epsilon} d^{D-1} \Omega \ N_\mu(r) u_\nu(r) \langle \Theta^{\mu\nu}(r) \Phi_a(P) \rangle_{h,g}^c \\
+ \left[ \mathcal{L}_u h \cdot \left( \int_{\epsilon}^1 d\rho \ C(\rho; h, g) - K(h, g) \right) \right]_a^b \langle \Phi_b(P) \rangle_{h,g} ,
\]

(5.6)

where $N^\mu(r)$ is the outward normal vector of unit length at a point $r$ on the sphere $\rho(r, P) = \epsilon$. Eq. (5.6) must agree with eq. (5.3). Hence, the integrand of the volume integral on the right-hand side of (5.6) must be a total derivative. Since the vector field $u^\mu$ is arbitrary, this condition implies that the energy-momentum tensor satisfies the conservation law:

\[
\nabla_\mu \Theta^{\mu\nu} = 0 .
\]

(5.7)

This is a well known result.

To proceed further we need some preparation on local riemannian geometry. We go back to eq. (2.11), the definition of the OPE coefficients $C$. The tensor field $t_{\mu\nu}(r)$ is regular at $P$, and we can Taylor-expand it as

\[
t_{\mu\nu}(r) = \sum_{m=0}^\infty \frac{1}{m!} v^{\mu_1}(r) \ldots v^{\mu_m}(r) \cdot \nabla_{\mu_1} \ldots \nabla_{\mu_m} t_{\alpha\beta}(P) \cdot V^\alpha_\mu(P, r) V^\beta_\nu(P, r) ,
\]

(5.8)

where $v^{\mu}(r)$ is a tangent vector at $P$ such that its image under the exponential map at $P$ is the point $r$, i.e.,

\[
\text{Exp}_v(P) = r ,
\]

(5.9)

and $V^\mu_\alpha(P, r)$ is the operator that parallel transports vectors at $r$ to vectors at $P$ along the geodesic between the two points. The covariant derivatives are symmetrized automatically in (5.8). Using the Taylor expansion (5.8), we can write

\[
\int d^{D-1} \Omega_{\rho}(r, P) \ t_{\mu\nu}(r) \Theta^{\mu\nu}(r) \Phi_a(P) = \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \ldots \nabla_{\mu_m} t_{\alpha\beta}(P) \\
\times \int d^{D-1} \Omega_{\rho}(r, P) \ v^{\mu_1}(r) \ldots v^{\mu_m}(r) V^\alpha_\mu(P, r) V^\beta_\nu(P, r) \Theta^{\mu\nu}(r) \Phi_a(P) .
\]

(5.10)
Comparing this with the definition (2.11) of the OPE coefficients, we obtain a relation
\[
\int d^{D-1}\Omega_\rho(r,P) \, v^{\mu_1}(r) \ldots v^{\mu_m}(r)V^\alpha_\mu(P,r)V^\beta_\nu(P,r)\Theta^{\mu\nu}(r)\Phi_a(P)
= (C^{\alpha\beta, \mu_1 \ldots \mu_m}(\rho; h(P), g))_a^b \Phi_b(P) + o\left(\frac{1}{\rho(r,P)}\right)
\]  
that we will use later. The symmetry with respect to \(\mu_1, \ldots, \mu_m\) is manifest in eq. (5.11).

Now we are ready to examine eq. (5.6) further. By noting that the outward unit normal vector \(N_\mu(r)\) is related to \(v_\mu(r)\) by
\[
N_\mu(r) = \frac{1}{\rho(r,P)} \, v_\alpha(r) V^\alpha_\mu(P,r),
\]  
and using the Taylor expansion
\[
u_\nu(r) = \sum_{m=0}^\infty \frac{1}{m!} v^{\mu_1}(r) \ldots v^{\mu_m}(r) \cdot \nabla_\mu_1 \ldots \nabla_{\mu_m} u_\alpha(P) \cdot V^\alpha_\mu(P,r),
\]  
we can write the second surface integral of eq. (5.6) as
\[
\int_{\rho(r,P)=\epsilon} d^{D-1}\Omega_\epsilon \, \rho \, N_\mu(r) u_\nu(r) \Theta^{\mu\nu}(r)\Phi_a(P)
= \sum_{m=0}^\infty \frac{1}{m!} \nabla_\mu_1 \ldots \nabla_{\mu_m} u_\beta(P) \cdot \left[\tilde{C}^{\alpha\beta, \mu_1 \ldots \mu_m}(\epsilon; h(P), g)\right]_a^b \Phi_b(P) + o(1),
\]  
where we define the coefficients \(\tilde{C}\) by
\[
\frac{1}{\epsilon} \int_{\rho(r,P)=\epsilon} d^{D-1}\Omega_\epsilon \, \rho \, v_\alpha(r) v^{\mu_1}(r) \ldots v^{\mu_m}(r) V^\alpha_\mu(P,r) V^\beta_\nu(P,r) \Theta^{\mu\nu}(r)\Phi_a(P)
= \left[\tilde{C}^{\alpha\beta, \mu_1 \ldots \mu_m}(\epsilon; h(P), g)\right]_a^b \Phi_b(P) + o(1),
\]  
in which we ignore terms that vanish in the limit \(\epsilon \to 0\).

The conservation law (5.7) implies that
\[
\int_{\rho_1}^{\rho_2} d\rho \int d^{D-1}\Omega_\rho \, \nabla_\mu u_\nu(r) \Theta^{\mu\nu}(r)\Phi_a(P)
= \left(\int d^{D-1}\Omega_{\rho_2} - \int d^{D-1}\Omega_{\rho_1}\right) N_\mu(r) u_\nu(r) \Theta^{\mu\nu}(r)\Phi_a(P).
\]  
Hence, by differentiating this with respect to \(\rho_2\) (and replacing \(\rho_2\) by \(\rho\)), we obtain
\[
\frac{\partial}{\partial \rho} \int d^{D-1}\Omega_\rho \, N_\mu(r) u_\nu(r) \Theta^{\mu\nu}(r)\Phi_a(P)
= \int d^{D-1}\Omega_\rho \, \nabla_\mu u_\nu(r) \Theta^{\mu\nu}(r)\Phi_a(P).
\]  

13
We extract the part of eq. (5.17) that cannot be integrated over $\rho$ up to 0, and we obtain,

from (5.14) and (2.11),

$$\frac{\partial}{\partial \rho} \left( u \cdot \tilde{C}(\rho; h, g) \right)_a^b \Phi_b(P) = (\mathcal{L}_u h \cdot \mathcal{C}(\rho; h, g))_a^b \Phi_b(P), \quad (5.18)$$

where

$$\left( u \cdot \tilde{C}(\rho; h, g) \right)_a^b \equiv \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \cdots \nabla_{\mu_m} u_\beta(P) \cdot \left( \tilde{C}^\beta_{\alpha} \cdot \mu_1 \cdots \mu_m (\rho; h, g) \right)_a^b. \quad (5.19)$$

Eq. (5.18) implies that the $\rho$ dependence of the coefficients $\tilde{C}$ is determined by the coefficients $\mathcal{C}$ which are themselves determined by eq. (3.13) in terms of the connection $\mathcal{K}$.

We can now rewrite eq. (5.6) using eqs. (5.14) and (5.18) as

$$\langle \Phi_a(P) \rangle_{h+\mathcal{L}_u h\cdot g} - \langle \Phi_a(P) \rangle_{h\cdot g} = \left[ u \cdot \tilde{C}(1; h, g) - \mathcal{L}_u h \cdot \mathcal{K}(h, g) \right]_a^b \langle \Phi_b(P) \rangle_{h\cdot g}. \quad (5.20)$$

Therefore, the consistency between the diffeomorphism (5.3) and eq. (5.20), which is a result of the Ward identity (2.9), gives the second main result of this paper:

$$\left( u \cdot \tilde{C}(1; h, g) \right)_a^b \Phi_b(P) = \mathcal{L}_u \Phi_a(P) + (\mathcal{L}_u h \cdot \mathcal{K}(h, g))_a^b \Phi_b(P). \quad (5.21)$$

This can be regarded as the initial condition for the differential equation (5.18), which we will solve in the next section.

6. Further discussion of the two main results

Eqs. (3.13) and (5.21) constitute two main results of this paper. They express the short-distance singularities $\mathcal{C}$, $\tilde{C}$ in terms of the connection $\mathcal{K}$ which was introduced as finite counterterms in the exact Ward identity (2.9).

First we determine the $\rho$ dependence of the coefficient $\mathcal{C}$, which can be obtained by solving the RG equation (3.12) using (3.13) as the initial condition. The solution is

$$\delta h \cdot \mathcal{C}(\rho; h, g) = \frac{\partial}{\partial \rho} (\delta h \cdot S(\rho; h, g)), \quad (6.1)$$
where we define
\[
\delta h \cdot S(\rho; h, g) \equiv \left[ G(\rho; h, g) \cdot \left\{ \frac{\delta h}{\rho^2} \cdot \mathcal{K} \left( \frac{h}{\rho^2}, g(\ln \rho) \right) \right\} + G(\rho; h, g) - G(\rho; h + \delta h, g) \right] \cdot G^{-1}(\rho; h, g) .
\] (6.2)

Here the matrix \( G \) is defined by
\[
\frac{d}{dt} G(\rho; h, g) \equiv \frac{1}{\Delta t} \left[ G(e^{-\Delta t \rho}; e^{-2\Delta t h, g} + \Delta t \beta) - G(\rho; h, g) \right] = \gamma(h, g) G(\rho; h, g) ,
\] (6.3)
and the running parameter is defined by
\[
\frac{\partial}{\partial t} g^i(t) = \beta^i(g(t)) , \quad g^i(0) = g^i .
\] (6.4)

Since the coefficients \( \tilde{C}(\rho) \) are given as derivatives with respect to \( \rho \), we can rewrite the Ward identity (2.9) as
\[
\langle \Phi_{a_1}(P_1) ... \Phi_{a_n}(P_n) \rangle_{h,g} - \langle \Phi_{a_1}(P_1) ... \Phi_{a_n}(P_n) \rangle_{h+\delta h,g} = \lim_{\epsilon \to 0} \left[ \int_{\rho(r,P_k) \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \times \left( \langle \Theta^{\mu\nu}(r) - \langle \Theta^{\mu\nu}(r) \rangle_{h,g} \rangle \Phi_{a_1}(P_1) ... \Phi_{a_n}(P_n) \right)_{h,g} + \sum_{k=1}^{n} [\delta h(P_k) \cdot S(\epsilon; h(P_k), g)]_{a_k} b \langle \Phi_{a_1}(P_1) ... \Phi_{b}(P_k) ... \Phi_{a_n}(P_n) \rangle_{h,g} \right] .
\] (6.5)

Now, we determine the \( \rho \) dependence of the coefficients \( \tilde{C} \) by solving the differential equation (5.18) using (5.21) as the initial condition. The solution is
\[
\left( u \cdot \tilde{C}(\rho; h, g) \right) \Phi = \mathcal{L}_u \Phi + (\mathcal{L}_u h \cdot S(\rho; h, g)) \Phi .
\] (6.6)

Going back to the original definition (5.14) of the coefficients \( \tilde{C} \), we find that eq. (6.6) gives
\[
\int_{\rho(r,P) = \epsilon} d^{D-1} \Omega \epsilon N_{\mu}(r) u_{\nu}(r) \Theta^{\mu\nu}(r) \Phi_a(P) = \mathcal{L}_u \Phi_a(P) + [\mathcal{L}_u h(P) \cdot S(\epsilon; h(P), g)] b \Phi_b(P) + o(1) .
\] (6.7)

This surface integral is nothing but the euclidean version of the regularized commutator between the energy-momentum tensor and the field \( \Phi_a \). The first term, which gives the change of the field \( \Phi_a \) under an infinitesimal diffeomorphism, is the canonical contribution to the commutator, and the second term is the Schwinger term. Hence, the connection \( \mathcal{K} \) determines the anomaly in the commutator.

To summarize, we have rewritten the main results (3.13) and (5.21) by introducing the \( \rho \) dependence. Our final results are given by eqs. (6.1) and (6.6).
7. Uniqueness of the energy-momentum tensor

We have introduced the energy-momentum tensor $\Theta^{\mu\nu}$ as a field that generates the changes of the correlation functions under the corresponding change of the metric, as given by the Ward identity (2.9) (or (6.5)). We wonder what properties characterize the energy-momentum tensor uniquely.

Let us recapitulate the properties of the energy-momentum tensor that we have obtained from consistency of the Ward identity (2.9). First, it is symmetric:

$$\Theta^{\mu\nu} = \Theta^{\nu\mu},$$  \hspace{1cm} (7.1)

second, it is conserved (see (5.7)):

$$\nabla_\mu \Theta^{\mu\nu} = 0,$$  \hspace{1cm} (7.2)

third, it satisfies the canonical RG equation (see (3.11))

$$\frac{d}{dt} \Theta^{\mu\nu} = (D + 2) \Theta^{\mu\nu},$$  \hspace{1cm} (7.3)

and fourth, its trace is given by (see (4.8))

$$\Theta = \beta^i \mathcal{O}_i.$$  \hspace{1cm} (7.4)

The last condition is independent of the choice of the parameters $g^i$, since under an arbitrary coordinate change $g^i \rightarrow g'^i$ we find

$$\mathcal{O}_i \rightarrow \mathcal{O}'_i = \frac{\partial g^j}{\partial g'^i} \mathcal{O}_j, \quad \beta^i \rightarrow \beta'^i = \frac{\partial g'^i}{\partial g^j} \beta^j.$$  \hspace{1cm} (7.5)

Thus, if there is any ambiguity in $\Theta^{\mu\nu}$, there must exist a field $\delta \Theta^{\mu\nu}$ which satisfies

$$\delta \Theta^{\mu\nu} = \delta \Theta^{\nu\mu}, \quad \nabla_\mu \delta \Theta^{\mu\nu} = 0, \quad \frac{d}{dt} \delta \Theta^{\mu\nu} = (D + 2) \delta \Theta^{\mu\nu}, \quad h_{\mu\nu} \delta \Theta^{\mu\nu} = 0.$$  \hspace{1cm} (7.6)

We assume that such $\delta \Theta^{\mu\nu}$ does not exist. Then, the four conditions (7.1), (7.2), (7.3), and (7.4) characterize the energy-momentum tensor uniquely.
8. Examples from $\phi^4$ theory

In this section we examine the energy-momentum tensor for the $\phi^4$ theory in four dimensions. For simplicity we will restrict ourselves to the flat space $(2.5)$. The energy-momentum tensor for this theory has been discussed by Brown using dimensional regularization [8]. See ref. [9] for discussions of the conjugate fields in the $\phi^4$ theory. We will do two things in this section: first we will enumerate three conditions that specify the energy-momentum tensor $\Theta^{\mu\nu}$ uniquely, and second we will determine the singularities in the product of $\Theta^{\mu\nu}$ and the elementary field $\phi$.

We recall that in flat space the theory is specified by three parameters: $\lambda$, $m^2$, and $g_1$. These satisfy the RG equations:

$$\frac{d}{dt} \lambda = \beta_\lambda(\lambda),$$  \hspace{1cm} (8.1)

$$\frac{d}{dt} m^2 = (2 + \beta_m(\lambda))m^2,$$  \hspace{1cm} (8.2)

$$\frac{d}{dt} g_1 = 4g_1 + \frac{1}{2} m^4\beta_1(\lambda).$$  \hspace{1cm} (8.3)

The parameter $g_1$ is the cosmological constant.

Let us now impose the first three conditions $(7.1)$, $(7.2)$, and $(7.3)$. We get

$$\Theta^{\mu\nu} = \Theta^{\nu\mu},$$  \hspace{1cm} (8.4)

$$\partial_\mu \Theta^{\mu\nu} = 0,$$  \hspace{1cm} (8.5)

$$\frac{d}{dt} \Theta^{\mu\nu} = 4\Theta^{\mu\nu}.$$  \hspace{1cm} (8.6)

We note that eq. (8.4) for the flat space follows from eq. (7.3) for the curved space because of eq. $(2.5)$. ($m = 0$, $n = 2$)

We wish to show that the three conditions $(8.2)$, $(8.3)$, and $(8.4)$ specify $\Theta^{\mu\nu}$ without ambiguity. In ref. [8] only eqs. (8.2) and (8.3) were considered. But these two conditions leave a well-known ambiguity in $\Theta^{\mu\nu}$. Namely, if $\Theta^{\mu\nu}$ satisfies eqs. (8.2) and (8.3), then

$$\Theta^{\mu\nu} \equiv \Theta^{\mu\nu} + f(\lambda) \left( \partial^\mu \partial^\nu - \delta^\mu_\alpha \delta^\nu_\alpha \partial^\alpha \right) O_m,$$  \hspace{1cm} (8.7)

where $O_m$ is the field conjugate to $m^2$, also satisfies the two conditions. But the conjugate field $O_m$, which is the same as the renormalized $\phi^2/2$, satisfies

$$\frac{d}{dt} O_m = (2 - \beta_m(\lambda))O_m - m^2\beta_1.$$  \hspace{1cm} (8.8)
Hence, if $\Theta^{\mu\nu}$ satisfies eq. (8.4) then $\Theta'^{\mu\nu}$ defined by (8.5) does not satisfy it for any regular function $f(\lambda)$. Therefore, the three conditions (8.2), (8.3), and (8.4) specify the energy-momentum tensor uniquely. Here we did not use the trace condition (7.4) to prove uniqueness, but alternatively we can prove uniqueness by showing the absence of a conserved symmetric traceless tensor which transforms canonically under the RG. In appendix we will give an explicit form of the energy-momentum tensor using dimensional regularization.

In ref. [5] it was argued that the energy-momentum tensor can be specified uniquely if we demand that it be coupled to external gravity. Imposing this condition is equivalent to defining the energy-momentum tensor through the Ward identity (2.9). The energy-momentum tensor $\Theta^{\mu\nu}$ that satisfies the three conditions (8.2), (8.3), and (8.4) are obtained from the energy-momentum tensor in curved space by taking the limit of the flat metric.

We now discuss the short-distance singularities in the product $\Theta^{\mu\nu}(r)\phi(P)$. The only relevant matrix element of the connection is $(K^{\mu\nu}(\delta, \lambda))^{\phi}_{\phi}$. By covariance we must find it in the form:

\[(K^{\mu\nu}(\delta, \lambda))^{\phi}_{\phi} = \delta^{\mu\nu} K(\lambda) . \quad (8.7)\]

The trace condition (8.9) implies that

\[K(\lambda) = \frac{1}{4} \Psi_{\phi}^{\phi}(\lambda) , \quad (8.8)\]

where

\[\Psi_{\phi}^{\phi}(\lambda) \equiv 1 + \gamma_{\phi}(\lambda) + \beta_{\lambda}(\lambda) (c_{\lambda})^{\phi}_{\phi}(\lambda) . \quad (8.9)\]

Here $1 + \gamma_{\phi}(\lambda)$ is the full scale dimension of $\phi$, and $c_{\lambda}$ is the connection in the $\lambda$ direction. Under the redefinition of the field $\phi$ by

\[\phi \rightarrow N(\lambda)\phi , \quad (8.10)\]

the anomalous dimension $\gamma_{\phi}$ and the connection $(c_{\lambda})^{\phi}_{\phi}$ transform as

\[\gamma_{\phi} \rightarrow \gamma_{\phi} + \beta_{\lambda}\partial_{\lambda} \ln N\]

\[(c_{\lambda})^{\phi}_{\phi} \rightarrow (c_{\lambda})^{\phi}_{\phi} - \partial_{\lambda} \ln N . \quad (8.11)\]

---

1 On curved space we can introduce a new dimensionless parameter $\eta$, whose conjugate field is $R O_m$, where $R$ is the Ricci curvature. We must take $\eta = 0$ before we take the flat metric limit.
Hence, $\Psi^\phi$ is invariant under the redefinition (8.10).

We apply eqs. (5.1) and (5.3) to the product $\Theta^{\mu\nu}(r)\phi(P)$, and obtain

$$
(C^{\mu\nu})^\phi(\rho; \lambda) = \frac{1}{\rho} \delta^{\mu\nu} \beta_\lambda(\lambda(\ln \rho)) K'(\lambda(\ln \rho))
$$

(8.12)

$$
\left(\tilde{C}^{\alpha\nu; \alpha}_{\beta} \partial_\mu^\phi\right)(\rho; \lambda) = \delta^{\mu\nu}
$$

(8.13a)

$$
\left(\tilde{C}^{\alpha\nu; \mu}_{\beta} \right)^\phi(\rho; \lambda) = \delta^{\mu\nu} K(\lambda(\ln \rho))
$$

(8.13b)

If we adopt a particular scheme in which

$$
(c_\lambda)^\phi(\lambda) = 0,
$$

(8.14)

then

$$
K(\lambda) = \frac{1}{4} \Psi^\phi(\lambda) = \frac{1}{4} (1 + \gamma_\phi(\lambda)),
$$

(8.15)

and the OPE coefficients are completely determined by the full scale dimension $1 + \gamma_\phi$ as in eqs. (8.12) and (8.13). This result, to order $\lambda^2$, was obtained a long time ago in refs. [6] and [7].

9. Concluding remarks

In this paper we have studied the energy-momentum tensor in field theory on curved space. We have introduced the energy-momentum tensor through an exact Ward identity (2.9) (or (5.3)). We have found that the singular part of the OPE of the energy-momentum tensor and an arbitrary composite field is determined in terms of a connection $K$ as in eqs. (6.1) and (6.6).

Our Ward identity (2.9) is a generalization of the Ward identity for two dimensional conformal field theory given in refs. [10] and [11]. The OPE of the energy-momentum tensor and an arbitrary composite field is completely determined by the conformal symmetry, and the connection $K$ is calculable. In fact, the OPE of two energy-momentum tensors (i.e., the central charge) and the normalization of three-point functions give enough data to construct all correlation functions [12]. This feature will not generalize to field theories in higher space dimensions, either massless or massive.

Another important result in this paper is the absence of anomalies in the RG equation of the energy-momentum tensor, eq. (3.11). This result is not new. For example, it has played a crucial role in the work of Curci and Paffuti [13] in which the canonical RG
equation of the energy-momentum tensor was used to derive a particular convention of the beta functions for the two dimensional non-linear sigma model in their discussion of string field equations.

Our discussion of the short-distance singularities in sect. 6 is not complete unless we compute the connection $\mathcal{K}$; our main results (9.1), (9.6) give relations between the short-distance singularities and the counterterms in the exact Ward identity (2.9), but $\mathcal{K}$, being independent of the beta functions and anomalous dimensions, needs to be computed separately. To find the connection $\mathcal{K}$, it helps to know the constraints on its matrix elements. We found one constraint on the trace of the connection, eq. (4.9), in sect. 4. There are additional algebraic constraints as can be seen as follows. First, eqs. (5.11) and (5.15) imply that the following conditions must be satisfied upon contraction of indices:

$$C_{\mu_1 \cdots \mu_m}(\rho; h, g) = \frac{1}{\rho^2} h_{\mu_{m+1} \mu_{m+2}} C_{\mu_1 \cdots \mu_{m+2}}(\rho; h, g) + o \left( \frac{1}{\rho^2} \right)$$

$$\tilde{C}_{\mu_1 \cdots \mu_m}(\rho; h, g) = \frac{1}{\rho^2} h_{\mu_{m+1} \mu_{m+2}} \tilde{C}_{\mu_1 \cdots \mu_{m+2}}(\rho; h, g) + o \left( \frac{1}{\rho^2} \right) .$$

(9.1)

Second, we note that eq. (6.6) determines $\tilde{C}$ with its indices partially contracted. There must exist uncontracted coefficients $\tilde{C}$ that satisfy

$$\tilde{C}_{\mu_1 \cdots \mu_m}(\rho; h, g) = h_{\mu_{m+1} \mu_{m+2}} \tilde{C}_{\mu_1 \cdots \mu_{m+2}}(\rho; h, g) .$$

(9.2)

Finally, the definitions (5.11) and (5.13) imply that $\tilde{C}$ must be related to $C$ by

$$\tilde{C}_{\mu_1 \cdots \mu_m}(\rho; h, g) = \frac{1}{\rho} C_{\mu_1 \cdots \mu_m}(\rho; h, g) + o \left( \frac{1}{\rho^2} \right) .$$

(9.3)

The connection $\mathcal{K}$ is constrained by the above three algebraic conditions. So far we have not found any simple way of rewriting these constraints as constraints directly on $\mathcal{K}$.

In part II and III of the present paper we plan to discuss the torsion $\tau$ and curvature $\Omega$ of the connection $\mathcal{K}$, respectively:

$$\tau(\delta h_1, \delta h_2; h, g) \equiv (\delta h_1 \cdot \mathcal{K}) (\delta h_2)_{\alpha \beta} \Theta^{\alpha \beta} - (\delta h_2 \cdot \mathcal{K}) (\delta h_1)_{\alpha \beta} \Theta^{\alpha \beta} ,$$

$$\Omega(\delta h_1, \delta h_2; h, g) \equiv \delta h_2 \cdot (\mathcal{K}(h + \delta h_1, g) - \mathcal{K}(h, g))$$

$$- \delta h_1 \cdot (\mathcal{K}(h + \delta h_2, g) - \mathcal{K}(h, g)) + [\delta h_1 \cdot \mathcal{K}(h, g), \delta h_2 \cdot \mathcal{K}(h, g)] .$$

(9.4)

(9.5)

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Appendix A. Construction of the energy-momentum tensor in $\phi^4$ theory using dimensional regularization

In this appendix we construct the energy-momentum tensor for $\phi^4$ theory using dimensional regularization. For the most part we follow ref. [8].

The theory is defined perturbatively in $D = 4 - \epsilon$ dimensional euclidean space by the lagrangian

$$
\mathcal{L} = \frac{1}{2} \partial^\mu \phi_0 \partial_\mu \phi_0 + m_0^2 \frac{\phi_0^2}{2} + \lambda_0 \frac{\phi_0^4}{4!} + g_0 ,
$$

(A.1)

where the bare parameters are given in terms of the renormalized parameters $\lambda$, $m^2$, and $g_1$ as

$$
\lambda_0 = Z_\lambda(\epsilon; \lambda) \lambda , \quad m_0^2 = Z_m(\epsilon; \lambda)m^2 , \quad g_0 = g_1 + z_0(\epsilon; \lambda)\frac{m^4}{2} .
$$

(A.2)

We adopt the MS scheme: $Z_\lambda - 1$, $Z_m - 1$, and $z_0(\epsilon; \lambda)$ all contain only the pole part with respect to $\epsilon$. We fix the usual arbitrary scale $\mu^2$ at 1 for simplicity. The renormalization constants are related to the beta functions (see eqs. (8.1)) as

$$
\epsilon \lambda + \beta_\lambda(\lambda) = \frac{\epsilon \lambda Z_\lambda}{\partial_\lambda (\lambda Z_\lambda)}
$$

$$
\beta_m(\lambda) = - (\epsilon \lambda + \beta_\lambda) \partial_\lambda \ln Z_m
$$

$$
\beta_1(\lambda) = - (\epsilon \lambda + \beta_\lambda) \partial_\lambda z_0 - (\epsilon + 2 \beta_m) z_0 .
$$

(A.3)

$\beta_\lambda$, $\beta_m$, and $\beta_1$ are of order $\lambda^2$, $\lambda$, and 1, respectively.

The renormalized composite fields are given as follows:

$$
\left[ \frac{\phi_0^4}{4!} \right] \equiv \partial_\lambda (\lambda Z_\lambda) \cdot \phi_0^4 \cdot 4! + m_0^2 \partial_\lambda Z_m \cdot \frac{\phi_0^2}{2} + m^4 \frac{1}{2} \partial_\lambda z_0 + z_R(\epsilon; \lambda) \partial_\lambda^2 \phi_0^2 ,
$$

(A.4)

$$
\left[ \frac{\phi_0^2}{2} \right] \equiv Z_m \frac{\phi_0^2}{2} + z_0 m^2 .
$$

(A.5)

Here $z_R(\epsilon; \lambda)$ contains only the pole part, and it is of order $\lambda^2$ due to the Feynman diagram shown in Fig. 2.

Fig. 2  the lowest order diagram that mixes $\phi^4$ with $\Delta \phi^2$

Fig 3.  the lowest order contribution to z $\mathcal{T}$
We can find the RG equations satisfied by these composite fields in the usual way as
\[
\frac{d}{dt} \left[ \frac{\phi^4}{4!} \right] = (4 - \beta_m^\prime) \left[ \frac{\phi^4}{4!} \right] - \beta_m m^2 \left[ \frac{\phi^2}{2} \right] - \beta_1 m^4 \left[ \frac{\phi^2}{2} \right] + u_R(\lambda) \partial^2 \left[ \frac{\phi^2}{2} \right], 
\]
(A.6)
\[
\frac{d}{dt} \left[ \frac{\phi^2}{2} \right] = (2 - \beta_m) \left[ \frac{\phi^2}{2} \right] - \beta_1 m^2,
\]
(A.7)
where \( u_R \) is defined by
\[
u_R(\lambda) \equiv \frac{\partial \lambda ((\epsilon \lambda + \beta_\lambda)Z_R)}{Z_m},
\]
(A.8)
and it is of order \( \lambda^2 \).

The general formula (2.18) implies that the fields conjugate to \( \lambda, m^2 \) must satisfy the RG equations
\[
\frac{d}{dt} \mathcal{O}_\lambda = (4 - \beta_m^\prime) \mathcal{O}_\lambda - m^2 \beta_m^\prime \mathcal{O}_m - \frac{m^4}{2} \beta_1,
\]
(A.9)
\[
\frac{d}{dt} \mathcal{O}_m = (2 - \beta_m) \mathcal{O}_m - m^2 \beta_1.
\]
(A.10)

Eqs. (A.7) and (A.10) imply
\[
\mathcal{O}_m = \left[ \frac{\phi^2}{2} \right].
\]
(A.11)

On the other hand, eqs. (A.6) and (A.9) imply that the renormalized \( \phi^4/4! \) differs from \( \mathcal{O}_\lambda \) by a total derivative:
\[
\mathcal{O}_\lambda = \left[ \frac{\phi^4}{4!} \right] + f(\lambda) \partial^2 \mathcal{O}_m,
\]
(A.12)
where \( f(\lambda) \) satisfies
\[
\partial_\lambda (\beta_\lambda f) - \beta_m f + u_R = 0.
\]
(A.13)

Eq. (A.13) has a unique solution which is regular at \( \lambda = 0 \). The general formula (7.4) gives the trace of the energy-momentum tensor as
\[
\Theta = \beta_\lambda \mathcal{O}_\lambda + (2 + \beta_m)m^2 \mathcal{O}_m + 4g_1 + \frac{m^4}{2} \beta_1.
\]
(A.14)

In ref. [4] it was shown that we can construct two independent traceless symmetric tensors of dimension four:
\[
T^{\mu\nu} \equiv \phi_0 \left( \partial^\mu \partial^\nu - \frac{1}{D} \delta^{\mu\nu} \partial^2 \right) \phi_0 + z_T(\epsilon; \lambda) \left( \partial^\mu \partial^\nu - \frac{1}{D} \delta^{\mu\nu} \partial^2 \right) \frac{\phi_0^2}{2},
\]
(A.15)
\[
t^{\mu\nu} \equiv \left( \partial^\mu \partial^\nu - \frac{1}{4} \delta^{\mu\nu} \partial^2 \right) \mathcal{O}_m,
\]
(A.16)
where \( z_T \) includes only the pole part, and it is of order \( \lambda \). See Fig. 3. The traceless tensors \( T^{\mu \nu}, t^{\mu \nu} \) satisfy the RG equations:

\[
\frac{d}{dt} T^{\mu \nu} = 4 T^{\mu \nu} + \eta(\lambda) t^{\mu \nu} \quad (A.17)
\]

\[
\frac{d}{dt} t^{\mu \nu} = (4 - \beta_m) t^{\mu \nu} , \quad (A.18)
\]

where

\[
\eta(\lambda) \equiv \frac{(\epsilon \lambda + \beta \lambda) \partial_\lambda z_T}{Z_m} \quad (A.19)
\]

is of order \( \lambda \).

The traceless part of the energy-momentum tensor \( \Theta^{\mu \nu} - \frac{1}{4} \delta^{\mu \nu} \Theta \) must be a linear combination of \( T^{\mu \nu} \) and \( t^{\mu \nu} \):

\[
\Theta^{\mu \nu} - \frac{1}{4} \delta^{\mu \nu} \Theta = a(\lambda) T^{\mu \nu} + b(\lambda) t^{\mu \nu} . \quad (A.20)
\]

We can determine the coefficients \( a, b \) by demanding the conservation law

\[
\partial_\mu \left( a(\lambda) T^{\mu \nu} + b(\lambda) t^{\mu \nu} \right) = -\frac{1}{4} \partial^\nu \Theta
\]

\[
= -\frac{1}{4} \left( \beta_\lambda \left[ \frac{\phi^4}{4!} \right] + (2 + \beta_m) m^2 \mathcal{O}_m + \beta_\lambda f \partial^2 \mathcal{O}_m \right) . \quad (A.21)
\]

We find, from (A.15), that

\[
\partial_\mu T^{\mu \nu} = -\frac{1}{4} \partial^\nu \left( \beta_\lambda \left[ \frac{\phi^4}{4!} \right] + (2 + \beta_m) m^2 \mathcal{O}_m \right) + \frac{1}{4} \chi \partial^\nu \partial^2 \mathcal{O}_m , \quad (A.22)
\]

where

\[
\chi(\lambda) \equiv \frac{1}{Z_m} \left( (\epsilon \lambda + \beta \lambda) z_R + 2 + \frac{\epsilon}{2} (Z_m - 1) + z_T (3 - \epsilon) \right) . \quad (A.23)
\]

We also find trivially

\[
\partial_\mu t^{\mu \nu} = \frac{3}{4} \partial^\nu \partial^2 \mathcal{O}_m . \quad (A.24)
\]

Hence, the condition (A.21) determines

\[
a = 1 , \quad b = -\frac{1}{3} \left( \beta_\lambda f + \chi \right) . \quad (A.25)
\]

We can actually show that \( \chi \) and \( u_R \) are determined by \( \eta \) as follows. By differentiating eq. (A.23) with respect to \( \lambda \), we obtain

\[
(\epsilon \lambda + \beta \lambda) \partial_\lambda \chi - \beta_m \chi = (\epsilon \lambda + \beta \lambda) u_R - \frac{\epsilon}{2} \beta_m + (3 - \epsilon) \eta , \quad (A.26)
\]

23
where we used eqs. (A.3), (A.8), and (A.19). Eq. (A.26) has two kinds of terms: those zeroth order in $\epsilon$ and those first order in $\epsilon$. By taking the zeroth order terms, we obtain

$$ (\beta \lambda \partial_\lambda - \beta_m) \chi = \beta \lambda u_R + 3 \eta , \quad (A.27) $$

and by taking the first order terms, we obtain

$$ \lambda \partial_\lambda \chi = \lambda u_R - \frac{1}{2} \beta_m - \eta . \quad (A.28) $$

We find, from these two equations,

$$ \chi = \frac{3}{2} - \left( 3 + \frac{\beta \lambda}{\lambda} \right) \sigma , \quad (A.29) $$

and

$$ u_R = -3 \partial_\lambda \sigma - \partial_\lambda \left( \frac{\beta \lambda}{\lambda} \sigma \right) + \frac{\beta m}{\lambda} \sigma , \quad (A.30) $$

where we define

$$ \sigma \equiv \frac{1}{2} + \frac{\eta}{\beta m} . \quad (A.31) $$

Eqs. (A.29) and (A.30) determine $\chi$ and $u_R$ in terms of $\eta$ and $\beta_m$. Since eq. (A.23) implies

$$ \chi(0) = 2 , \quad (A.32) $$

we obtain, from eq. (A.29),

$$ \sigma(0) = -\frac{1}{6} . \quad (A.33) $$

Similarly, we can obtain $\sigma'(0)$, $\sigma''(0)$ from eq. (A.30) by recalling that $u_R$ is of order $\lambda^2$.

Finally, we verify that the traceless tensor (A.20) has no anomaly under the RG:

$$ \frac{d}{d \tau} (T^{\mu \nu} + b(\lambda) t^{\mu \nu}) = 4 (T^{\mu \nu} + b(\lambda) t^{\mu \nu}) . \quad (A.34) $$

This would imply

$$ (\beta \lambda \partial_\lambda - \beta_m) b + \eta = 0 . \quad (A.35) $$

Using (A.25), this condition is equivalent to

$$ (\beta \lambda \partial_\lambda - \beta_m) (\beta \lambda f + \chi) - 3 \eta = 0 , \quad (A.36) $$

which is indeed satisfied thanks to eqs. (A.13) and (A.27).

To summarize, we have found the traceless part of the energy-momentum tensor as eq. (A.20), where $a$, $b$ are given by eqs. (A.23).
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