The prospect of distinguishing astrophysical objects with LISA

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The famous no-hair theorem dictates that the multipole moments of Kerr black holes depend only on their mass and angular momentum. Thus, the measurement of multipole moments of astrophysical objects through gravitational-wave observations provides a novel way to test the theorem and distinguish black holes from other astrophysical objects. This paper studies the gravitational wave radiation from an extreme mass ratio inspiral system consisting of a supermassive Kerr black hole and a spinning stellar-mass compact object. The quadrupolar moment induced by the rotation of the stellar-mass object carries information about its internal structure. We show that the Laser Interferometer Space Antenna (LISA) can probe the imprints of quadrupolar deformation for certain astrophysical objects and thus provide a way to identify them.

I. INTRODUCTION

The detection of gravitational waves [1, 2] paved the way for a new era in observational astrophysics, which allows us to probe physics in the strong gravity regime for the very first time [3–6]. The ground-based gravitational-wave detectors successfully observed the merger of stellar-mass black holes and neutron stars. Unfortunately, these detectors are only sensitive to frequencies above $\sim 10$ Hz due to the presence of seismic noise (future third-generation detectors like the Einstein Telescope (ET) hope to evade the seismic noise by going underground and can probe signals in $\sim 0.1 - 10$ Hz frequency band [7]). The future space-based gravitational wave detectors like the Laser Interferometer Space Antenna (LISA) can probe the imprints of quadrupolar deformation for certain astrophysical objects and cosmological sources [8–16].

One primary source for LISA observations is extreme mass ratio inspiral (EMRI), a binary system where a stellar-mass object ($m_s \sim 1 - 100M_\odot$) hovers around a supermassive compact object ($M \sim 10^4 - 10^7M_\odot$) [9–11, 14, 17]. The mass ratio of the system is $q \equiv m_s/M \sim 10^{-7} - 10^{-4}$. The stellar-mass object (hereafter, the secondary) completes $\sim 10^4 - 10^5$ orbits cycles around the supermassive central object (hereafter, the primary) within the LISA frequency band before plunging [17, 18]. The gravitational waveform from the system carries very accurate information about the parameters of the binary system [10, 19] and the geometry surrounding the primary object [20–28]. Recent studies have shown that LISA can measure the mass and spin of the primary with significantly high accuracy as compared to current ground-based detectors and X-ray measurements [10, 19].

Furthermore, the EMRI system is an ideal testbed to analyze the nature of the supermassive object [20–24, 26]. The uniqueness and no-hair theorems in the context of general relativity assert that the astrophysical objects beyond a certain mass limit are Kerr black holes, and their geometry and multipole moments depend only on their mass and angular momentum. However, recently, black hole alternative models like gravastars [31, 32], boson stars [33, 34], and fuzzballs [35, 36] have gained much attention [37–47]. These objects are collectively known as exotic compact objects (ECOs)[37, 43]. They are slightly larger than the black holes with the same mass and angular momentum and have finite reflectivity. Moreover, the multipolar structure of some of these objects is drastically different from that of the Kerr black holes [46–55]. Since LISA can measure the quadrupolar moment of the primary with great precision (independent of its mass and angular momentum), the emitted gravitational radiations from the system can testify for the “Kerr-ness” of the primary [20, 23, 54]. Other notable EMRI based tests to identify the nature of primary includes the measurement of the change in tidal heating [56–58] and energy flux [59, 60] due to the presence of finite reflectivity and the measurement of tidal Love numbers [61].

Relatively less attention has been given to finding the nature of the secondary object. This is because the effect of secondary’s spin and higher-order multipole moments is expected to get suppressed by the system’s tiny mass ratio. In recent times, several authors have considered the effect of secondary’s spin on orbital dynamics and gravitational wave production [62–70]. In particular, Piovano et al. studied the adiabatic evolution of spinning secondary in circular and equatorial orbit [69, 70]. Their study shows that the gravitational wave dephasing due to secondary’s spin could be large enough for the detection. Moreover, LISA can detect string theory-inspired exotic compact object models called the superspinars that can breach the Kerr bound. Moreover, tidal effects are considered in Ref. [71]. Interestingly, Chen et al. considered the effect of quadrupolar deformation in an intermediate-mass ratio inspiral (IMRI) system [72]. In this paper, we consider the secondary as a spinning object that hovers
around a supermassive Kerr black hole in circular, equatorial orbit. Moreover, the rotation induces quadrupolar deformation in the secondary. Several authors have recently emphasized the importance of considering second-order effects like quadrupolar deformation for the correct modeling of EMRI waveform \[73, 74\]. The argument follows from the fact that over the long inspiral period \((T_i \sim M/q)\) of an EMRI system, the second-order force terms \(q^2 f_{(2)}^\alpha\) have a considerable effect on orbital dynamics \(\delta z^\alpha \sim q^2 f_{(2)}^\alpha T_i^2 \sim q^0\). Thus, one can not neglect the contribution of these terms. Since the quadrupolar moment carries information about the object’s internal structure, it can help us identify the nature of the object. In this paper, we calculate the corrections in the gravitational wave phase due to the effect and show that those corrections can be large enough for LISA to detect and thus can distinguish between black holes and other astrophysical objects.

The paper is organized as follows: In Section II, we briefly describe the equation of motion of a deformed spinning object in curved spacetime. In Section III, we describe the orbital motion of the object in Kerr spacetime. Section IV gives a brief review on the Teukolsky formalism and gravitational wave emission from the EMRI system. In Section V, we present our main results. Finally, Section VI contains our conclusion.

Notation and Convention: Throughout the paper, we adopt positive signature convention \((-\,+,\,+,\,+)\). Greek indices are used to denote four-dimensional tensors, whereas lower case roman indices are used to denote three-dimensional tensors. The fundamental constants are assumed as \(h = c = G = 1\).

II. DYNAMICS OF EXTENDED OBJECTS

A. Equation of motion

The dynamics of the stellar mass object, immersed in the gravitational field of the supermassive black hole, can be adequately described by the multi-polar approximation method \[75–77\]. It asserts that a set of multipole moments can narrate the effect of the internal structure of the secondary on its motion along a reference worldline \(z^\mu\). Since the secondary object’s size is much smaller than the curvature radius of the primary object, only a finite number of terms are required to describe the motion. Here, we consider terms up to quadrupolar order, which describes the secondary object as an extended spinning object subjected to quadrupolar deformation. Under this approximation, the energy-momentum tensor of the object can be written as follows \[77, 78\],

\[
T^{\alpha\beta} = \int d\tau \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \left( p^{(\alpha)}_{(\beta)} - \frac{1}{3} f^{\delta(\alpha} R_{\beta)\gamma} e_{\gamma} \right) \\
- \int d\tau \nabla_\gamma \left( S^{\gamma(\alpha} e_{\beta)} \right) \delta^4(x - z(\tau)) \frac{1}{\sqrt{-g}} \\
- \frac{2}{3} \int d\tau \nabla_\gamma \nabla_\delta \left( J^{\delta(\alpha\beta)} e_{\gamma(\alpha} \delta_{\beta)} \right) \frac{1}{\sqrt{-g}} + \mathcal{O}(\epsilon^3)
\]

where, \(\nu^\mu = dz^\mu/d\tau\) is the tangent to the object’s worldline, \(p^\mu\) is the momentum of the object, \(S^{\mu\nu}\) is spin tensor, and \(J^{\alpha\beta}\) is the quadrupole tensor. Here, we choose the proper time \(\tau\) as the affine parameter so that the following normalization condition is satisfied \(\nu^\mu \nu_\mu = -1\). Note that the quadrupole tensor exhibits all the algebraic symmetries of the Riemann tensor \(R_{\alpha\beta\gamma\delta}\). Following Ref. \[79\], we introduce a small parameter \(\epsilon\) to keep track of the multipole moment order.

The equation of motion of the object is given by the Mathisson-Papapetrou-Dixon (MPD) equation, which can be written as follows \[75, 77–80\]

\[
\frac{Dp^\mu}{d\tau} = -\frac{1}{2} S^{\nu\rho} e^\mu R_{\nu\rho} - \frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla_\mu R_{\alpha\beta\gamma\delta} + \mathcal{O}(\epsilon^3)
\]

\[
\frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu} \nu^{\nu]} - 3 R_{\alpha\beta\gamma\delta} [\nu^{[\gamma\alpha\beta}} J^{\nu]} + \mathcal{O}(\epsilon^3)
\]

Here, \(D/d\tau \equiv \nu^\mu \nabla_\mu\). However, the system of equations consisting of the MPD equation along with the tangent equation \(\nu^\mu = dz^\mu/d\tau\) is under-determined as the number of variables \((z^\mu, \nu^\mu, p^\mu, S^{\mu\nu})\) exceeds the number of equations. Thus, we need to impose some supplementary conditions. Here, we choose the Tulczyjew spin supplementary condition \[78–80\]

\[
p^\mu S^{\mu\nu} = 0.
\]

The above condition fixes the centre of mass of the object. Moreover, it gives a relation between the 4-velocity \(\nu^\mu\) and the momentum \(p^\mu\) which can be written as follows \[78\]

\[
\nu^\mu = \hat{\nu}^\mu + \frac{2R_{\gamma\alpha\beta} S^{\alpha\beta}}{4\mu^2 + R_{\mu\nu\alpha\beta} S^{\alpha\beta} S^{\mu\nu}} \delta^\gamma,
\]

where,

\[
\hat{\nu}^\mu = u^\mu + \frac{4}{3\mu^2} R^{[\mu}_{\alpha\beta} \gamma \alpha\beta J^{\nu] \gamma\alpha\beta}
\]

and \(p^\mu = \mu u^\mu\). The parameter \(\mu\) represents the dynamic mass of the object, which can be defined as follows \(\mu^2 = (-p^\mu p_\mu)\). For convenience, we also introduce the monopole rest mass \(m_0\) of the object, which can be defined in the following way, \(m_0 = -p^\mu u_\mu\).

The quadrupole tensor \(J^{\mu\nu\alpha\beta}\) contains the information about the deformation due to spin and tidal forces. In
this paper, we mainly focus on the distortion caused by spin effects. Thus, we choose the following form of the quadrupole tensor [78, 79, 81]

\[ J^{\alpha \beta \gamma \delta} = -\frac{3}{\mu^2} \rho^{[\alpha} Q^{\beta \gamma \delta]} [\gamma p^\delta] \]  

(6)

where, \( Q^{\alpha \beta} = C_Q S_{\mu} S^{\beta \mu}/\mu \) is the mass quadrupole tensor. Here, \( C_Q \) is the spin-induced quadrupole moment (SIQM) parameter. For rotating Kerr black holes, the \( C_Q = 1 \) in accordance to black hole no-hair theorem [82]. For rotating neutron stars, the value of \( C_Q \) varies between \( \sim 2 - 20 \) depending on the equation of state [48–50]. Interestingly, for certain exotic compact objects, like boson stars, the parameter can be quite large, ranging from \( \sim 10 - 150 \) [46, 52, 53, 83, 84]. For gravastar, the parameter can be negative [46, 47, 85].

### B. Conserved quantities

If the spacetime admits a Killing vector \( \xi_\mu \), then the following quantity [78, 79, 81]

\[ C_\xi = p^\mu \xi_\mu - \frac{1}{2} S^{\mu \nu} \nabla_\nu \xi_\mu \]  

(7)

is conserved along the trajectory of the object. Since we are interested in stationary, axisymmetric spacetime, the associated conserved quantities are the energy \( E \) and angular momentum \( J_z \). The conservation of the spin length \( S^2 = S_{\mu \nu} S^{\mu \nu}/2 \) depends on the Tulczyjew spin supplementary condition, which can be seen from the following expression.

\[
\begin{align*}
S^{\mu} \frac{dS}{d\tau} &= \frac{1}{2} S^{\mu \nu} \frac{DS^{\mu \nu}}{d\tau} \\
&= S_{\mu \nu} (p^\mu v^\nu - \frac{2}{3} R^{\alpha \beta \gamma \delta} J^{\gamma \alpha \beta \delta}) \\
&= 0 .
\end{align*}
\]  

(8)

Here, we obtain the above result by substituting Eq. (3) in the second line and then using the Tulczyjew spin supplementary condition. The dynamical mass term \( \mu \) is not conserved. However, we can define the mass term \( m_s \) given by the following expression [78, 81]

\[ m_s = \mu - \frac{\mu}{6m_0} R_{\alpha \beta \gamma \delta} J^{\alpha \beta \gamma \delta} \]  

(9)

which is approximately conserved along the trajectory. To construct the conserved mass term, we consider the following quantity \( \frac{Dp_\alpha}{d\tau} p_\nu \frac{DS^{\mu \nu}}{d\tau} \). Inserting Eq. (2) in the above expression and retaining terms up to \( O(\epsilon^3) \), we can show that \( dm_s/d\tau = O(\epsilon^3) \).

### III. Orbital Motion of the Extended Object in Kerr Background

We start with Kerr black holes in Boyer-Lindquist coordinate \((t, r, \theta, \phi)\) whose line element can be expressed as follows [86]

\[
ds^2 = -\frac{\Delta}{\Sigma} \left[ dt - a \sin^2 \theta \, d\phi \right]^2 + \frac{\Sigma}{\Delta} \left[ dr^2 + a^2 \sin^2 \theta \, d\phi^2 + d\theta^2 \right] + \sin^2 \theta \left[ adt - (r^2 + a^2) \, d\phi \right]^2
\]  

(10)

where, \( \Delta = (r^2 + a^2) - 2Mr \) and \( \Sigma = r^2 + a^2 \cos^2 \theta \). The solutions of \( \Delta = 0 \) give the position of the horizons as \( r_\pm = M \pm \sqrt{M^2 - a^2} \), where the upper (lower) sign corresponds to the event (Cauchy) horizon. The spacetime is stationary and axisymmetric; thus admits two Killing vectors \( \xi_\mu^\alpha = (\partial_\alpha) \mu \) and \( \xi_\phi^\mu = (\partial_\phi) \mu \). The Killing vector \( \eta^\mu = \xi_\mu^\alpha + \Omega_\alpha \xi_\mu^\phi \) is the generator of the event horizon. Here, \( \Omega_+ = a/2Mr_+ \) is the angular frequency of the black hole. For our convenience, we introduce orthonormal tetrad frame to describe the orbital motion [79]

\[
e_{\mu}^{(0)} = \frac{\sqrt{\Delta}}{\sqrt{\Sigma}} (1, 0, 0, -a \sin^2 \theta), \quad e_{\mu}^{(1)} = \frac{\sqrt{\Delta}}{\sqrt{\Sigma}} (0, 1, 0, 0),
\]

\[
e_{\mu}^{(2)} = (0, 0, \sqrt{\Sigma}, 0), \quad e_{\mu}^{(3)} = \frac{\sin \theta}{\sqrt{\Sigma}} (-a, 0, 0, r^2 + a^2) .
\]  

(11)

We define the spin vector through the following relation [69]

\[
S^{(a)} = -\frac{1}{2} \varepsilon^{(a)(b)(c)(d)} u_{(b)} S_{(c)(d)}
\]

\[
S^{(a)(b)} = \varepsilon^{(a)(b)(c)(d)} u_{(c)} S_{(d)}
\]  

(12)

where, \( e^{(a)(b)(c)(d)} \) is the Levi-Civita tensor.

### A. Equations of motion on the equatorial plane

In the following, we consider the secondary object is orbiting around the supermassive black hole in an equatorial plane (\( \theta = \pi/2 \)). Moreover, we choose the spin vector of the secondary \( \Omega_\sigma = (0, 0, -S, 0) \). The negative sign implies that the secondary is moving in a spin aligned configuration [79]. Using Eq. (3) and Eq. (12), we find that \( p^{(2)} = 0 \), \( S^{(2)}(a) = 0 \), \( S^{(0)(1)} = -S u^{(3)} \), \( S^{(0)(3)} = S u^{(1)} \), \( S^{(1)(3)} = S u^{(0)} \). It is useful to define dimensionless variables,

\[
\hat{r} = \frac{r}{M}, \quad \hat{a} = \frac{a}{M}, \quad \hat{E} = \frac{E}{m_s},
\]

\[
\hat{J}_z = \frac{J_z}{M m_s}, \quad \sigma = \frac{S}{M m_s} = q \chi .
\]  

(13)
where, \( q = m_s/M \) is the mass ratio. Using Eq. (7), we can write the energy and angular momentum of the object as follows

\[
\begin{align*}
\hat{E} &= \sqrt{\Delta} \hat{w}^{(0)} + \frac{\hat{a} \hat{a}^2 + \sigma}{\hat{r}^2} \hat{w}^{(3)} \\
\hat{J}_z &= \sqrt{\Delta} \left( \hat{a} + \sigma \right) \hat{w}^{(0)} + \frac{(\hat{a} \hat{r} + 1) \sigma + \hat{a}^2 \hat{r}^2 + \hat{r}^3}{\hat{r}^2} \hat{w}^{(3)}
\end{align*}
\]

where, we introduce a parameter \( w^\mu = p^\mu/m_s \) for convenience. We invert the above expression to write \( w^{(0)} \) and \( w^{(3)} \) in terms of \( \hat{E} \) and \( \hat{J}_z \), which is given as follows

\[
\begin{align*}
w^{(0)} &= \left( \hat{r}^2 + \hat{a}^2 \right) \hat{E} - \hat{a} \hat{J}_z \left( 1 + \frac{\hat{a}^2}{\hat{r}^2} \right) + \sigma \left( \hat{a} \hat{E} (\hat{r} + 1) - \hat{J}_z \right) \\
w^{(3)} &= \hat{J}_z - \hat{a} \hat{E} \left( 1 + \frac{\hat{a}^2}{\hat{r}^2} \right) - \hat{E} \sigma \hat{r}.
\end{align*}
\]

Replacing the above expression in Eq. (9), we find conserved mass as follows [78]

\[
m_s = \mu \left[ 1 + \frac{\sigma^2 C_Q}{2 \hat{r}^3} \left( 1 + 3 \left( \hat{J}_z - \hat{a} \hat{E} \right)^2 \right) \right] + O(\epsilon^3).
\]

We can obtain the expression for \( w^{(1)} \) from the following relation \( (w^{(0)})^2 - (w^{(1)})^2 = \mu^2/m_s^2 \). The relationship between normalized momenta \( u^{(a)} \) and the 4-velocity \( v^{(a)} \) turns out to be

\[
\begin{align*}
v^{(0)} &= \left( 1 + 3 \left( 1 - 8 C_Q \right) \sigma^2 (u^{(3)})^2 \right) u^{(0)} + O(\epsilon^3) \\
v^{(1)} &= \left( 1 + \frac{\sigma^2 C_Q}{\hat{r}^3} \left( 1 + (u^{(3)})^2 \right) u^{(1)} + O(\epsilon^3) \\
v^{(3)} &= \left( 1 + \frac{3 \left( 1 - 8 C_Q \right) \sigma^2 (1 + (u^{(3)})^2) \hat{r}^3 \right) u^{(3)} + O(\epsilon^3),
\end{align*}
\]

where, \( u^{(a)} \) follows the relation \( u^{(a)} = m_s w^{(a)}/\mu \). The component of 4-velocity in Boyer-Lindquist coordinate can be obtained with the following \( v^\mu = e^\mu_{(a)} v^{(a)} \) which gives the equation of motion as follows [78, 79]

\[
\begin{align*}
\Sigma_s \Lambda_s \left( \frac{\mu}{m_s} \right) \left( \frac{d\hat{t}}{d\hat{r}} \right) &= \hat{a} \left( \hat{J}_z - (\hat{a} + \sigma) \hat{E} \right) Q_s + \left( \frac{\hat{r}^2 + \hat{a}^2}{\Delta} \right) P_s, \\
\left( \frac{d\hat{r}}{d\hat{r}} \right)^2 &= V_s(\hat{r}) \equiv \frac{\hat{r}^4}{\Sigma_s} \left[ \alpha \hat{E}^2 - 2 \beta \hat{J}_z \hat{E} + \gamma \frac{\hat{J}_z^2}{\hat{r}^2} - \frac{\Delta \hat{r}^2}{m_s^2} \right], \\
\Sigma_s \Lambda_s \left( \frac{\mu}{m_s} \right) \left( \frac{d\hat{x}}{d\hat{r}} \right) &= \left( \hat{J}_z - (\hat{a} + \sigma) \hat{E} \right) Q_s + \frac{\hat{a}}{\Delta} P_s,
\end{align*}
\]

where,

\[
\begin{align*}
\Sigma_s &= \hat{r}^2 \left( 1 - \frac{\sigma^2}{\hat{r}^3} \right), \quad \Lambda_s = 1 - 3\left( 1 - 8 C_Q \right) (\hat{J}_z - \hat{E} \hat{a})^2 \sigma^2, \\
Q_s &= 1 - 3\left( 1 - 8 C_Q \right) \sigma^2, \\
P_s &= \hat{E} \left( \hat{r}^2 + \hat{a}^2 \right) + \frac{\hat{a} \sigma}{\hat{r}^2} (\hat{r} + 1) - \hat{J}_z (1 + \frac{\hat{a}}{\hat{r}}), \\
\alpha &= \left( 1 + \frac{\hat{a}^2}{\hat{r}^2} + \frac{\hat{a} \sigma}{\hat{r}^2} \left( 1 + \frac{1}{\hat{r}} \right) \right)^2 - \frac{\Delta}{\hat{r}^2} \left( \frac{\hat{a}}{\hat{r}} + \frac{\hat{r}}{\hat{a}} \right)^2, \\
\beta &= \left( 1 + \frac{\hat{a}^2}{\hat{r}^2} + \frac{\hat{a} \sigma}{\hat{r}^2} \left( 1 + \frac{1}{\hat{r}} \right) \right) \left( \frac{\hat{a}}{\hat{r}} + \frac{\hat{r}}{\hat{a}} \right) - \Delta \left( \frac{\hat{a}}{\hat{r}} + \frac{\hat{r}}{\hat{a}} \right) \left( \frac{\hat{a}}{\hat{r}} + \frac{\hat{r}}{\hat{a}} \right), \\
\gamma &= \left( \frac{\hat{a}}{\hat{r}} + \frac{\hat{r}}{\hat{a}} \right)^2 - \frac{\Delta}{\hat{r}^2}, \quad \delta = \frac{\Delta \Sigma^2}{\hat{r}^6}.
\end{align*}
\]

### B. Circular orbit, ISCO and Orbital frequency

In this paper, we focus on circular orbits. For an object moving in a circular orbit, the radial velocity and radial acceleration of the object vanishes simultaneously; leading to the condition \( V_s = 0 \) and \( dV_s/d\hat{r} = 0 \). The stability of such orbits against radial perturbation is dictated by the condition \( d^2 V_s/d\hat{r}^2 < 0 \). It is more convenient to use an effective potential term \( V_{\text{eff}} \) for the calculation which can be written as follow [78]

\[
V_{\text{eff}}(\hat{r}) = \left[ \alpha \hat{E}^2 - 2 \beta \frac{\hat{J}_z}{\hat{r}} \hat{E} + \gamma \frac{\hat{J}_z^2}{\hat{r}^2} - \delta \frac{\mu^2}{m_s^2} \right],
\]

where, the \( \alpha, \beta, \gamma, \delta \) is given in Eq. (19). Moreover, we adopt the variables \( y = 1/\hat{r} \) and \( z = \hat{J}_z - \hat{a} \hat{E} \) in place of \( \hat{r} \) and \( \hat{J}_z \). The condition for circular orbit then transformed as \( V_{\text{eff}} = 0 \) and \( dV_{\text{eff}}/dy = 0 \). Noting that the parameter \( \sigma \ll 1 \), we can expand the above mentioned equations into series of \( \sigma \). Here, we seek solution of the equations in the following form

\[
\hat{E} = \hat{E}_0 + \sigma \hat{E}_1 + \sigma^2 \hat{E}_2, \quad \hat{x} = \hat{x}_0 + \sigma \hat{x}_1 + \sigma^2 \hat{x}_2,
\]

where, \( \{ \hat{E}_0, \hat{x}_0 \} \) corresponds to the value of \( \{ \hat{E}, \hat{x} \} \) for a spinless object, whereas \( \{ \hat{E}_1, \hat{x}_1 \} \) and \( \{ \hat{E}_2, \hat{x}_2 \} \) represent the linear and quadratic corrections due to spin respectively. For stable circular orbit, \( \{ \hat{E}_0, \hat{x}_0 \} \) attains the value [87]

\[
\hat{E}_0 = \frac{1 - 2 y \hat{y}}{\sqrt{1 - 3 y^2}} \hat{y} \hat{x}_0 = \frac{1 \mp \hat{a} \sqrt{y}}{\sqrt{1 - 3 y^2}} \hat{y},
\]

Here, the upper sign represents a retrograde (counter-rotating) orbit whereas lower sign corresponds to a prograde (co-rotating) orbit. The equations for \( \{ \hat{E}_i, \hat{x}_i \} \)
(i = 1, 2) is presented in the Appendix A. We solve these equations numerically and replace them in Eq. (21) along with Eq. (22) to obtain the value of \{\dot{E}, \dot{x}\} as a function of y. By replacing \( y = 1/\dot{r} \) and \( \dot{J}_z = \dot{x} + \dot{a} \dot{E} \), we obtain the value of energy and angular momentum of the object hovering in a circular orbit with radius \( \dot{r} \).

Determination of the parameters of the innermost stable circular orbit (ISCO) needs an additional condition \( d^2V_{\text{eff}}/dy^2 = 0 \) (or \( d^2V_{\text{eff}}/dy^2 = 0 \)). Series expansion of this condition into the series of \( \sigma \) is presented in Appendix A. Similar to Eq. (21), we seek solution in the following form

\[
y = y_0 + \sigma y_1 + \sigma^2 y_2.
\]

Solving the equations \( V_{\text{eff}} = dV_{\text{eff}}/dy = d^2V_{\text{eff}}/dy^2 = 0 \) simultaneously, we obtain the parameters \{\dot{E}, \dot{x}, \dot{y}\} in the form given by Eq. (21) and Eq. (23), which in turn gives us the energy, angular momentum and position of the ISCO \{\dot{E}^{\text{isco}}, \dot{J}_z^{\text{isco}}, \dot{r}^{\text{isco}}\} (see Appendix A for more details).

The angular frequency of the circular orbits are given by

\[
\dot{\Omega} \equiv \frac{d\phi/d\tau}{d\dot{r}/d\tau} = \frac{\Delta \left( \dot{J}_z - (\dot{a} + \dot{\sigma}) \dot{E} \right) Q_s + \dot{a}P_s}{\Delta \dot{a} \left( \dot{J}_z - (\dot{a} + \dot{\sigma}) \dot{E} \right) Q_s + (i\dot{a} + \dot{a}^2) P_s},
\]

where in the second step, we use Eq. (18). Replacing Eq. (21) in Eq. (24) and expanding the expression as

\[
\dot{\Omega}(\dot{r}) = \dot{\Omega}_0(\dot{r}) + \sigma \dot{\Omega}_1(\dot{r}) + \sigma^2 \dot{\Omega}_2(\dot{r}),
\]

we obtain the angular frequency of the circular orbit. Here, \( \dot{\Omega}_0 \) corresponds the angular frequency of a non-spinning object whereas \( \dot{\Omega}_1 \) and \( \dot{\Omega}_2 \) represents linear and quadrupolar correction due to spin respectively.

### IV. GRAVITATIONAL WAVE FLUXES

In this section, we describe the gravitational wave radiation from the EMRI system. As a result of the system’s tiny mass ratio, we can study the evolution of the secondary object through perturbation methods. The system loses energy and angular momentum due to gravitational radiation, the back-reaction of which (self-force) shrinks the binary separation, and the system goes through an inspiral phase. Here, we study the gravitational back-reaction effects within the framework of adiabatic approximation [17, 88]. The motivation behind this formalism is that the orbital time scale \( T_o \) \((\sim M)\) is much shorter than the dissipative \( T_i \) \((\sim M^2/m_s \sim M/q \gg T_o)\); thus allowing us to treat the orbital dynamics as geodesics over a short time scale. Furthermore, the rate of change of the orbit’s energy and angular momentum is dictated by the time-averaged, dissipative part of the self-force [88] i.e.,

\[
\left( \frac{d\dot{E}}{dt} \right)^{\text{orbit}} = -\left( \frac{d\dot{E}}{dt} \right)^{\text{GW}}, \quad \left( \frac{d\dot{J}_z}{dt} \right)^{\text{orbit}} = -\left( \frac{d\dot{J}_z}{dt} \right)^{\text{GW}},
\]

where \( \langle \cdot \rangle \) denotes the averaging over a time period that is much larger than \( T_o \) but smaller than \( T_i \). We calculate the back-reaction effect on the orbit by solving the Teukolsky equation. This gives the adiabatic evaluation of the object from orbit to orbit. Note that the adiabatic approximation breaks down as the object crosses the ISCO and the object transits onto a geodesic plunge orbit [17]. In our study, we focus only on the adiabatic part of the motion.

#### A. Teukolsky equation

In order to study the gravitational wave radiation from the EMRI system, we adopt Teukolsky formalism. As discussed earlier, we treat the secondary as a perturbation to the Kerr background. The information about the gravitational degrees of freedom are encoded in the Weyl scalar \( \Psi_4 = (h_+ - ih_\times)/2 \), where overdot sign implies derivative with respect to \( t \). The spacetime symmetries allow us to decompose \( \Psi_4 \) as [69]

\[
\Psi_4 = \rho^4 \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} d\omega \ R_{\ell m \omega}(\dot{r}) - 2 S_{\ell m \omega}^{\dot{\Omega}}(\theta)e^{i(m\phi - \omega t)}
\]

in Fourier space, where \( \rho = |\dot{r} - i\dot{a}\cos \theta|^{-1} \) and \( -2 S_{\ell m \omega}^{\dot{\Omega}}(\theta) \) is the spin weighted spheroidal harmonics with weight \( -2 \) which satisfies the angular Teukolsky equation

\[
\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \dot{a}^2 \dot{\omega}^2 \sin^2 \theta - \left( \frac{m - 2 \cos \theta}{\sin \theta} \right)^2 \right. \\
+ 4 \dot{\omega} \cos \theta - 2 + 2iam\dot{\omega} + \lambda_{\ell m \dot{\omega}} \left. \right] - 2 S_{\ell m \omega}^{\dot{\Omega}}(\theta) = 0.
\]

where, \( \lambda_{\ell m \dot{\omega}} = E_{\ell m \dot{\omega}} - 2m\dot{\omega} + \dot{a}^2 \dot{\omega}^2 - 2 \). Hereafter, we denote \( -2 S_{\ell m \omega}^{\dot{\Omega}}(\theta) \) by \( S_{\ell m \omega}^{\dot{\Omega}}(\theta) \) the sake of brevity. The eigenfunction of the angular Teukolsky equation \( S_{\ell m \omega}^{\dot{\Omega}}(\theta) \) satisfies the following normalization condition.

\[
\int \sin \theta d\theta d\phi |S_{\ell m \omega}^{\dot{\Omega}}(\theta)e^{i\ell m \phi}|^2 = 1.
\]

We use Mathematica based Black Hole Perturbation Toolkit package [89] to calculate spin-weighted spheroidal harmonics and \( \lambda_{\ell m \dot{\omega}} \). The radial function \( R_{\ell m \dot{\omega}} \) satisfies the following equation

\[
\Delta^2 \frac{d}{d\ell} \left( \frac{1}{\Delta} \frac{dR_{\ell m \dot{\omega}}}{d\ell} \right) - V(\dot{r})R_{\ell m \dot{\omega}} = \mathcal{J}_{\ell m \dot{\omega}}
\]
where
\[ V(\hat{r}) = \frac{K^2 + 4i(\hat{r} - 1)K}{\Delta} + 8i\hat{\omega}\hat{r} + \lambda_{\ell m\omega} \]  
(31)

and \( \mathcal{J}_{\ell m\omega} \) is the source term given by Eq. (B28). The details of the calculation for the source term is presented in Appendix B. Here, we employ the Green function method to obtain \( R_{\ell m\omega}(\hat{r}) \). In terms of the linearly independent solutions of homogeneous radial Teukolsky equation \( R_{\ell m\omega}^{\text{in}}(\hat{r}) \) and \( R_{\ell m\omega}^{\text{up}}(\hat{r}) \) following purely incoming boundary conditions at the horizon and purely outgoing boundary condition at the infinity respectively, the solution of Eq. (30) can be written as
\[
R_{\ell m\omega}(\hat{r}) = \left\{ \begin{array}{ll}
\frac{1}{W} \int_{\hat{r}_+}^{\hat{r}} d\hat{r} \frac{R_{\ell m\omega}^{\text{in}}(\hat{r})}{\Delta_2} \\
+ \frac{R_{\ell m\omega}^{\text{up}}(\hat{r})}{\Delta_2} \end{array} \right. \mathcal{J}_{\ell m\omega}
\]  
(32)

where \( W = \frac{R_{\ell m\omega}^{\text{in}} \partial_r R_{\ell m\omega}^{\text{up}} - R_{\ell m\omega}^{\text{in}} \partial_r R_{\ell m\omega}^{\text{up}}}{\Delta} \) is the constant Wronskian. The asymptotic behavior of the radial function \( R_{\ell m\omega} \) is given as follows
\[
R_{\ell m\omega} = \left\{ \begin{array}{ll}
Z^{\infty}_{\ell m\omega} e^{i\omega t}, & \hat{r} \to \infty \\
Z^{H\infty}_{\ell m\omega} e^{-i(\omega - m\hat{\Omega})t}, & \hat{r} \to \hat{r}_+
\end{array} \right. 
\]  
(33)

where
\[
\hat{r}_+ = \hat{r} + \frac{2\hat{r}_+}{\hat{\mu} - \hat{r}_-} \ln \left[ \frac{\hat{r}_+ - \hat{r}_-}{2} \right] - \frac{2\hat{r}_+}{\hat{\mu} - \hat{r}_-} \ln \left[ \frac{\hat{r}_- - \hat{r}_{\infty}}{2} \right] 
\]  
(34)

is the tortoise coordinate and \( \hat{\Omega}_+ \) is the dimensionless angular frequency of the black hole. The amplitudes \( Z^{H\infty}_{\ell m\omega} \) are given by the following relation.
\[
Z^{H\infty}_{\ell m\omega} = c^{H\infty}_{\ell m\omega} \int_{\hat{r}_+}^{\infty} d\hat{r} \frac{R_{\ell m\omega}^{\text{in}}(\hat{r})}{\Delta_2} \mathcal{J}_{\ell m\omega}, 
\]  
(35)

where \( c^{H\infty}_{\ell m\omega} \) are constants; (see Eq. (B2)). For the energy-momentum tensor presented in Eq. (1), the amplitudes takes the following form (see Eq. (B36))
\[
Z^{H\infty}_{\ell m\omega} = c^{H\infty}_{\ell m\omega} \int_{-\infty}^{\infty} dt e^{i(\omega t - m\hat{\Omega} t)} h^{H\infty}_{\ell m\omega}[r(t), \theta(t)], 
\]  
(36)

where
\[
h^{H\infty}_{\ell m\omega}[r(t), \theta(t)] = \left[ A_0 - (A_1 + B_0) \frac{d}{dt} + (A_2 + B_1 + C_0) \frac{d^2}{dt^2} \right. \\
- (B_2 + C_1) \frac{d^3}{dt^3} + C_2 \frac{d^4}{dt^4} \left. \right|_{r(t),\theta(t)}. 
\]  
(37)

We have presented the details to calculate these quantities as well as the explicit form of the coefficients \( (A_i, B_i, C_i) \) \((i = 0, 1, 2)\) in the Appendix B.

Using Eq. (27) and Eq. (35), we find that the gravitational wave signal \((h = h_+ - ih_\times)\) at asymptotic infinity takes the following form
\[
h = -\frac{2}{r} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} Z^{H\infty}_{\ell m\omega}(\hat{r}) S^{\hat{\mu}\hat{\omega} 0}_{\ell m\omega}(\vartheta) e^{i(m\varphi - \hat{\omega} t_+)} 
\]  
(38)

where \( \vartheta \) is the angle between the line of sight of an observer sitting at infinity and the axis of symmetry of the central object (i.e., the \( z \) axis) and \( \varphi = \phi(t = 0) \). In what follows, we turn our attention to equatorial, circular orbits. This hugely simplifies the calculation (see Eq. (B38) in Appendix B). Moreover, for circular orbits, we have \( \phi(t) = \hat{\Omega} t \) which simplifies the expression for the amplitude in Eq. (36) as \( Z_{\ell m\omega}^{H\infty} = A_{\ell m\omega}^{H\infty} \delta(\omega - m\hat{\Omega}) \) at some specific radius \( r_0 \), where \( A_{\ell m\omega}^{H\infty} = 2\pi \omega \mathcal{H}_{\ell m\omega}^{H\infty}[r_0, \pi/2] \).

The gravitational waveform then reduces to
\[
h = -\frac{2}{r} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \frac{A_{\ell m\omega}^{H\infty}}{(m\hat{\Omega})^2} S^{\hat{\mu}\hat{\omega} 0}_{\ell m\omega}(\vartheta) e^{i(m\varphi - \hat{\Omega} t_+)}, 
\]  
(39)

We obtain energy and angular momentum flux by replacing Eq. (39) in \( dE/d\hat{A}t \hat{\omega}^{\infty} \equiv \langle (\hat{h}_+)^2 + (\hat{h}_\times)^2 \rangle/16\pi \) and taking integration over the solid angle which can be written as follows
\[
\left( \frac{dE}{dt} \right)^{\infty}_{GW} = \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} \frac{|A_{\ell m\omega}^{H\infty}|^2}{2\pi(m\hat{\Omega})^2}, 
\]  
(40)

\[
\left( \frac{dJ}{dt} \right)^{\infty}_{GW} = \frac{1}{\hat{\Omega}} \left( \frac{dE}{dt} \right)^{H}_{GW}. 
\]  
Similarly, we can write energy and angular momentum flux at the horizon as [69]
\[
\left( \frac{dE}{dt} \right)^{H}_{GW} = \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} \alpha_{\ell m} |A_{\ell m\omega}^{H\infty}|^2, 
\]  
(41)

where \( \alpha_{\ell m} = \left( 256(2\hat{r}_+)^5 \kappa (\kappa^2 + 4\varepsilon^2)(\kappa^2 + 16\varepsilon^2)(m\hat{\Omega})^3 \right) / |C_{\ell m}|^2 \)

\[
|C_{\ell m}|^2 = \left( (\lambda_{\ell m\hat{\Omega}} + 2)^2 + 4\hat{\omega}(m\hat{\Omega}) - 4\hat{\omega}^2(m\hat{\Omega})^2 \right) \times \left[ \lambda_{\ell m\hat{\Omega}}^2 + 36\hat{a}(m\hat{\Omega}) - 36\hat{a}^2(m\hat{\Omega})^2 \right] + (2\lambda_{\ell m\hat{\Omega}} + 3) \left[ 96\hat{a}(m\hat{\Omega})^2 - 48\hat{a}(m\hat{\Omega}) \right] + 144(m\hat{\Omega})^2(1 - \hat{a}^2). 
\]
The back-reaction effect is dictated by the following expression of the orbital radius and phase as a result of the gravitational wave interaction with the primary object. In the left panel, the energy flux of a non-spinning secondary object $F^{(0)}$ is presented. The middle and right panel shows the linear and quadratic correction coefficients in energy flux due to spin effects respectively. Here, we take $q = 10^{-4}$, $CQ = 10$.

\[ F = \frac{1}{q} \left[ \left( \frac{dE}{dt} \right)_{GW}^H + \left( \frac{dE}{dt} \right)_{GW}^\infty \right] = \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} F_{\ell m}. \] (42)

where, $F_{\ell m} = (|A_{\ell m}^\infty|^2 + \alpha_\ell m |A_{\ell m}^H|^2)/2\pi q(m\Omega)^2$. The energy and angular momentum of the orbit evolves adiabatically as a result of gravitational back-reaction effect over timescales $\sim T_i$ (see Eq. (26)). Here, we assume that the mass, spin, and internal structure of the secondary object remains unaltered during the evolution. The evolution of the orbital radius and phase as a result of the back-reaction effect is dictated by the following expression [69]

\[ \frac{d\hat{r}}{dt} = -q F(\hat{r}) \left( \frac{dE}{dt} \right)^{-1}, \] (43)

\[ \frac{d\phi}{dt} = \Omega(\hat{r}(\hat{t})), \] (44)

where, the expression of $\hat{E}$ and $\hat{\Omega}$ as a function of $\hat{r}$ is given in Eq. (21) and Eq. (25) respectively. The solution of Eq. (44) gives the expression for instantaneous orbital phase, which is related to the dominant mode gravitational-wave phase by $\Phi_{GW}(\ell) = 2\phi(\ell)$. As discussed earlier, the adiabatic approximation breaks down as the object crosses the ISCO radius. Since, we focus on the adiabatic evolution of the orbit, we consider the evolution in the domain $\hat{r} \in (\hat{r}_{ini}, \hat{r}_{isco})$.

V. NUMERICAL METHOD AND RESULTS

In this section, we briefly describe the numerical methods implemented to calculate the energy flux $F$. One of the main tasks to do so is to find the solutions of homogeneous Teukolsky equation, $^\text{m}_n^{\text{in}}$ and $^\text{pp}_n^{\text{in}}$. Here, we have considered two different methods to calculate these functions: (i) Mano-Suzuki-Takasugi (MST) method [90–92] as implemented in Mathematica package Black hole Perturbation Toolkit [89], (ii) Sasaki-Nakamura (SN) method as described in [69, 93]. This is due to the fact that the MST method, albeit faster, fails to deliver results with significant numerical precision for large values of $\ell$ (see [69] for further discussion). We consider the SN method to calculate the energy flux in this scenario. In our calculation, we set the numerical precision to 22 significant digits. Furthermore, we have calculated the eigenvalue $\lambda_{\ell m \omega}$ and the eigenfunction $S_{\ell m \omega}(\theta)$ of the angular Teukolsky equation Eq. (28) using Black hole Perturbation Toolkit package. With $^\text{m}_n^{\text{in}}$ and $^\text{pp}_n^{\text{in}}$ in our hand, we can calculate the flux at the infinity and the horizon using Eq. (40) and Eq. (41). However, we need to truncate the infinite sum in those equations. Here, we set $\ell_{\text{max}} = 22$ since the contribution of terms beyond $\ell > \ell_{\text{max}}$ is significantly small for all practical purposes. For each value of $\ell$, $m$ varies from 1 to $\ell$ starting with $m = \ell$. However, we neglect the contributions of the terms for which the fractional truncation
error $\Delta F \equiv |F_{t-1} - (F_{t} - F_{t-1})| < 10^{-6}$. This gives us the energy flux $F$ as a function of $\hat{r}$, $\hat{a}$, $q$, $\chi$, and $C_Q$. Since we are interested in adiabatic evaluation of the orbit, we calculate the flux $F(\hat{r})$ in the range $\hat{r} \in (\hat{r}_{\text{ins}}, \hat{r}_{\text{isco}})$ for fixed values of system’s intrinsic parameters $\hat{a}$, $q$, $\chi$ and $C_Q$. Following [69], we choose the starting point of the inspiral $\hat{r}_{\text{isco}}$ such that all the spinning objects have the same orbital frequency as a non-spinning ($\chi = 0$) secondary object at $\hat{r} = 10$. We can obtain the effect of spin by fitting the flux $F(\hat{r}, \sigma)$ with the following polynomial

$$F(\hat{r}, \sigma) = F^{(0)}(\hat{r}) + \sigma F^{(1)}(\hat{r}) + \sigma^2 F^{(2)}(\hat{r}) + O(\sigma^3),$$  \hspace{1cm} (45)$$

where $F^{(0)}(\hat{r})$ is the flux of an non-spinning secondary whereas $F^{(1)}(\hat{r})$ and $F^{(2)}(\hat{r})$ describes the linear and quadratic corrections to flux due to spin effect. In Fig. 1, we show the $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$ for stable and prograde orbits as a function of orbital radius $\hat{r}$ for different values of $\hat{a}$. Here, we take $q = 10^{-4}$, $C_Q = 10$.

We can calculate the adiabatic evaluation of the orbit by integrating Eq. (43). The integration starts at $\hat{r}_{\text{ini}}$ which marks the beginning of the inspiral phase. The integration stops when the object reaches $\hat{r}_{\text{end}} = \hat{r}_{\text{isco}} + \varepsilon$. Here, we choose $\varepsilon = 10^{-6}$. We obtain the instantaneous orbital phase $\phi(\hat{t})$ by replacing the solution of Eq. (43) in Eq. (44) and solving it using the finite difference method. The dependence of gravitational wave phase $\Phi_{GW}(\hat{t}) = 2\phi(\hat{t})$ on secondary’s spin can be understood by fitting the numeric result with a quadratic polynomial of $\chi$ in the following form [17, 69]

$$\Phi_{GW}(\hat{t}) = \Phi^{(0)}(\hat{t}) + \chi \Phi^{(1)}(\hat{t}) + q\chi^2 \delta \Phi^{(2)}(\hat{t}) + O(\chi^3),$$ \hspace{1cm} (46)$$

where $\Phi^{(0)}(\hat{t})$ denotes the phase of a non-spinning secondary object whereas $\Phi^{(1)}(\hat{t})$ and $\delta \Phi^{(2)}(\hat{t})$ represents shift in phase due to secondary’s spin respectively. In Fig. 2, we present the linear correction in phase as a function of time for different values of primary spin $\hat{a}$ and secondary’s SIQM parameter $C_Q$ for prograde orbits. As can be seen from left panel of the figure, the time of adiabatic evolution up to the ISCO increases with the increase of $\hat{a}$. Furthermore, the right panel of the figure shows $\Phi^{(1)}$ does not depend on the $C_Q$. This confirms that the body’s internal structure does not have an effect on gravitational phase up to linear order in the spin.

To see the behaviour of quadratic corrections in $\Phi_{GW}(\hat{t})$, we start with the following ansatz that $\delta \Phi^{(2)}(\hat{t})$ is proportional to $C_Q$, i.e.,

$$\delta \Phi^{(2)}(\hat{t}) = C_Q \delta \Phi^{(2)}(\hat{t})$$ \hspace{1cm} (47)$$

In the left and middle panel of Fig. 3, we show $\Phi^{(2)}$ as a function of $t$ for different values of $C_Q$. The figure suggest that $\Phi^{(2)}$ is independent of $C_Q$ and thus confirming that $\delta \Phi^{(2)}(\hat{t})$ has a linear dependence on $C_Q$. The right panel of Fig. 3 shows the dependence of $\Phi^{(2)}(t)$ on $\hat{a}$. As evident from the plot, the quadratic correction to accumulated phase at the end of inspiral period $\Phi^{(2)}(t_{\text{end}})$ increases with the increase of $\hat{a}$.

### A. Fisher-information analysis and measurability of the effects of quadrupolar deformation

In this section, we discuss whether the effects of quadrupolar deformation are strong enough for the detection. Our analysis is based on the following rule of thumb: the systematic errors due to the inaccuracy in modeling should be less than the statistical errors due to detector noise [94, 95]. Given two model waveforms $h_1(t)$ and $h_2(t)$ with parameters $\hat{\vartheta} = (\vartheta^1, \vartheta^2, ..., \vartheta^D)$, we can define the *overlap* between them as follows [96]

$$F(h_1, h_2) = \frac{(h_1|h_2)}{\sqrt{(h_1|h_1)(h_2|h_2)}}.$$ \hspace{1cm} (48)
Here, \((h_1|h_2)\) represents the noise-weighted inner product \([97]\)
\[
(h_1|h_2) = 4\Re \int_0^\infty \frac{\tilde{h}_1(f)\tilde{h}_2(f)}{S_n(f)} df ,
\]
where, \(\tilde{h}(f)\) is the Fourier transform of \(h(t)\) and \(S_n(f)\) is the one-sided noise power spectrum density of the detector. The overlap is defined in such way that perfect agreement between the two waveforms corresponds to \(F = 1\). The mismatch between the two waveforms is denoted by \(M = 1 - F\). The see how the difference between the two waveforms \(\delta h\) affects the mismatch, we write \(h_1\) as \(h_2 = h_1 + \delta h\). Ref. \([96]\) showed that \(h_1\) and \(h_2\) would be indistinguishable through the measurement of any gravitational wave detector if the following condition is satisfied.
\[
(\delta h|\delta h) < 2\rho^2 M_{\text{max}},
\]
where, \(\rho = \sqrt{(h_1|h_1)}\) is the optimal signal-to-noise ratio (SNR). Here, \(M_{\text{max}}\) represents the maximum value of mismatch consistent with our target detection efficiency. If we write \(h_1\) as \(h_1 = A\exp(i\Phi)\), where \(A\) and \(\Phi\) denotes the amplitude and phase of the waveform, the difference between the waveforms can be written as \(\delta h = h_1\exp(i\delta\Phi) - h_1 \approx i\delta\Phi h_1\). Here, we considered terms upto the first order and ignored the deviation in the amplitude. Replacing the expression of \(\delta h\) in Eq. (50), we obtain \([96]\)
\[
\Delta \Phi \equiv \sqrt{(\tilde{h}_1\delta\Phi|\tilde{h}_1\delta\Phi)} < \sqrt{2M_{\text{max}}},
\]
where, \(\tilde{h}_1 = h_1/\rho\), such that \((\tilde{h}_1|\tilde{h}_1) = 1\).

To obtain the value of \(M_{\text{max}}\), we employ Fisher-information analysis \([98]\). Let us consider that \(h(t)\) denotes the true waveform. Moreover, \(\tilde{\vartheta}^i\) is the “true” value of the physical parameters and \(\tilde{\vartheta}^i + \Delta \vartheta^i\) is the best fit parameter values in presence of noise. Here, \(i\) runs from 1 to \(D\), where \(D\) is total number of parameters (not to be confused with spatial dimensions in previous sections). Assuming close proximity to the true values, we expand any generic model waveform \(h_m\) as a Taylor series \([98]\)
\[
h_m = h + h_{i} \Delta \vartheta^i + \frac{1}{2} h_{ij} \Delta \vartheta^i \Delta \vartheta^j + \mathcal{O}(\Delta \vartheta^3) .
\]
For large values of SNR, \(\Delta \vartheta^i\) have multivariate Gaussian probability distribution.
\[
p(\Delta \vartheta) = \mathcal{N} \exp \left( -\frac{1}{2} \frac{\Gamma_{ij}\Delta \vartheta^i \Delta \vartheta^j}{\Sigma_{ij}} \right) ,
\]
where, \(\Gamma_{ij} = (h_{i}|h_{j})\) is the Fisher-information matrix and \(\mathcal{N} = \det(\Gamma_{ij}/2\pi)\) is the normalization constant. The covariance matrix comes from the expectation value of \(\Delta \vartheta^i \Delta \vartheta^j\) i.e., \(\Sigma_{ij} = \Gamma_{ij}^{-1} = \langle \Delta \vartheta^i \Delta \vartheta^j \rangle\) for large values of SNR \([98]\). Substituting Eq. (52) in Eq. (48) and treating \(\Delta \vartheta^i\) as small, we obtain the mismatch due to statistical errors as \([95]\)
\[
M \equiv 1 - F = \frac{1}{2} g_{ij} \Delta \vartheta^i \Delta \vartheta^j ,
\]
where,
\[
g_{ij} = \frac{(h_{i}|h_{j})}{(h|h)} - \frac{(h|h)(h|h)}{(h|h)^2} ,
\]
By using $\Sigma^{ij} = \langle \Delta \vartheta^i \Delta \vartheta^j \rangle$, we obtain the exception of the value of the mismatch $\langle M \rangle$,
\begin{equation}
\langle M \rangle = \frac{D - 1}{2\rho^2},
\end{equation}
where, the factor $D$ appears as the result of $\Sigma^{ij} \Gamma_{ij} = \delta_i^i = D$ and $-1$ comes from the second term of Eq. (55) by removing the dependence on the amplitude of the waveform. This reduces dimension by a factor 1. Eq. (56) represents the effect of statistical error due to detector noise. Here, the rule of thumb is that the systematic error due to mismodeling should be less than the statistical error [94, 95]. Thus, replacing Eq. (56) in Eq. (51), we find that the two waveforms would be indistinguishable through the measurement of the detector if [100]
\begin{equation}
\Delta \Phi < \sqrt{\frac{D - 1}{\rho}}.
\end{equation}

In this work, we set the SNR detection threshold for LISA observation to 20 following Ref. [10, 100]. Considering $\rho$ varies as $1/d$, where $d$ is the distance of the source from the earth and the number of sources per unit distance varies as $d^2$, we obtain the average SNR for detection as $\sim 30$. Replacing the value of the average SNR in Eq. (57), we find that the waveforms would be indistinguishable if $\Delta \Phi < 0.1$ rad [100]. Thus, the effect of quadrupolar deformation would be significant for LISA observations if the following condition is satisfied
\begin{equation}
\Delta \Phi = qC_Q\chi^2\Phi^{(2)}(t_{\text{end}}) > 0.1 \text{ rad}.
\end{equation}
where, $\Phi^{(2)}(t_{\text{end}})$ is the quadrupolar correction to accumulated phase at the end inspiral period. We use Eq. (58) to check whether LISA can distinguish black holes from neutron stars and exotic objects like boson stars or gravastars.

In Fig. 4, we present our main result. Here, we show the contour plot of $\Delta \Phi$ in the $(\chi, C_Q)$ plane for $\hat{a} = 0.3$ (left panel), $\hat{a} = 0.6$ (middle panel) and $\hat{a} = 0.9$ (right panel). Here, we fix the mass ratio as $q = 10^{-4}$. As discussed earlier, the value of the SIQM parameter $C_Q$ for Kerr black holes is 1, whereas it can take values $\sim 2 - 20$ for neutron stars and $\sim 10 - 150$ for boson stars. It can also take large negative values for gravastar [46]. Thus, we vary SIQM parameter in the range $C_Q \in (0, 150)$. For a Kerr black hole, the spin of the secondary is restricted by the Kerr bound $\chi \leq 1$. The mass-shading limit restricts the spin of the neutron stars—for instance, $\chi \approx 0.3$ for the fastest pulsar [101]. However, compact astrophysical objects like white dwarfs and brown dwarfs can exceed the Kerr bound. The observational study suggests that $\chi \approx 10$ for the fastest white-dwarf [102]. Moreover, string theory predicts the existence of exotic compact objects, called superspinars that can breach the Kerr bound [103]. Thus, we set the parameter range for the secondary spin as $\chi \in (0, 10)$. In Fig. 4, the green contour line represents the threshold $\Delta \Phi = 0.1$ rad for detecting the quadrupolar deformation. The plot shows that the parameter space that allows distinction between black holes and exotic compact objects like bo-
son stars and superspinars is quite significant. The size of the parameters space which allows this distinction increases with the increase of primary spin \( \dot{a} \). However, it is highly unlikely to distinguish between a black hole and a neutron star with EMRI observations. Moreover, for a smaller mass ratio \( q \sim 10^{-5} \), we can only distinguish black holes from very fast spinning exotic compact objects (\( \chi > 2 \)) for larger values of \( \dot{a} \).

VI. CONCLUSION AND DISCUSSION

Detection of gravitational waves by LIGO-VIRGO detectors taught us a valuable lesson: accurately modeling the coalescence process is as vital as extracting accurate data to maximize the science return from the observation. With the prospect that LISA will observe hundreds of EMRI events each year, realistic modeling of the binary system is of utmost importance. In this paper, we have considered a system where a spinning stellar-mass object orbiting around a super-massive Kerr black hole in the equatorial plane and studied the system’s orbital dynamics and the emitted gravitational radiation. Moreover, we considered the effect of spin-induced quadrupolar deformation of the secondary on gravitational wave production. The effect of quadrupolar deformation is often ignored from the expectation that the information about the effect get suppressed by the tiny mass of the system. In this paper, we have shown that the impact of quadrupolar deformation can be pretty significant; thus, ignoring the contribution of such effects can create considerable estimation biases.

Moreover, our analysis shows that the gravitational signals from the EMRI system can distinguish different astrophysical objects. We show that the quadrupolar deformation adds a correction term \( \Delta \Phi = qC_\chi \Phi^{(2)}(t_{\text{end}}) \) to total accumulated phase, where \( \Phi^{(2)}(t_{\text{end}}) \) is numerical parameter which depends only on the dimensionless spin \( \dot{a} \) of the central black hole. The no-hair theorem sets the value of the SIQM parameter to unity \((C_Q = 1)\) for a Kerr black hole. However, for other astrophysical objects, the parameter’s value depends on their internal structure, ranging between 2 – 20 for neutron stars and 10 – 150 for boson stars and can even take negative values for gravastar \([46, 47, 85]\). Although the spin parameter of the black holes and neutron stars is restricted by Kerr bound and mass-shading limit, respectively, it can take large values (\( \chi > 1 \)) for string theory inspired superspinars and white dwarfs. Using Fisher-information analysis, we have shown that the effect of quadrupolar deformation would be significant for LISA observation for \( \Delta \Phi > 0.1 \) rad. In Fig. 4, we have shown that the condition is satisfied for a large parameter space (in \( \chi - C_Q \) plane). Moreover, the parameter space increases with the increase of \( \dot{a} \). This allows us to distinguish black holes from a large variety of astrophysical objects, including boson stars, superspinars, and gravastars. However, it is unlikely to distinguish between a black hole from neutron star from EMRI observations. Recently, Chen et al. \([72]\) have shown that the perturbation analysis as presented in this paper may remain valid for an intermediate-mass ratio inspiral (IMRI) system, a binary system with \( q \approx 10^{-4} - 10^{-2} \). It is possible to distinguish black holes from neutron stars in such a scenario. Another interesting aspect of our study is that gravitational waves from white-dwarf EMRI system can put tighter constraints on white dwarf equation-of-state.

A possible extension of this work is to study the effect of tidally-induced quadrupolar deformation due to gravito-electric and gravito-magnetic tidal forces on gravitational wave production \([78]\). Similar to spin-induced quadrupolar deformation, tidal deformation contains information about the equation-of-state of the object and thus can potentially distinguish different astrophysical objects \([46, 47, 85]\). Other possible extensions include the relaxation of this paper’s assumptions like equatorial circular orbit, aligned spin. Furthermore, an exciting extension of the work includes the contribution of the second-order self-force based on the two-time scale method \([74]\). Several authors recently studied such effects for EMRI systems consisting of a massive Schwarzschild black hole and a point particle \([104–106]\). It is assertive to consider the effect of second-order self-force in the presence of spinning binaries, which we left for the future.

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Appendix A: Circular orbit, ISCO and angular frequency

The condition for circular orbit is given by \( V_{\text{eff}} = 0 \) and \( dV_{\text{eff}}/dy = 0 \). We expand the equations up to quadratic order of spin \( \sigma \). We seek solutions in the form given by Eq. (21). The equations for a non-spinning object are
Linear order corrections in spin is dictated by

\[ -2a \tilde{E}_0 \tilde{x}_0 y^2 - y^2 (a^2 + \tilde{x}_0^2) + \tilde{E}_0^2 + 2\tilde{x}_0^2 y^3 + 2y - 1 = 0 , \]

\[-2 \left( 2a \tilde{E}_0 \tilde{x}_0 y + \tilde{a}^2 y + \tilde{x}_0^2 (y - 3y^2) - 1 \right) = 0 . \]

The solutions of these equations give the value \{\tilde{E}_0, \tilde{x}_0\}. Linear order corrections \{\tilde{E}_1, \tilde{x}_1\} can be found by solving the following equations

\begin{equation}
2 \left( a \tilde{E}_0^2 y^2 + \tilde{x}_0 y^2 \left( -\tilde{a} \tilde{E}_0 + \tilde{a} \tilde{x}_0 y^3 + 2\tilde{x}_1 - \tilde{x}_0 \right) \right) + \tilde{E}_0 \left( y^2 \left( -\tilde{a} \tilde{x}_0 y - \tilde{x}_0 + \tilde{E}_1 \right) \right) = 0 ,
\end{equation}

\begin{equation}
\tilde{a} \left( 2 \tilde{E}_0 \tilde{x}_0 + 5\tilde{x}_0 y^3 - 2\tilde{E}_1 \tilde{x}_0 \right) - 2 \tilde{E}_0 \tilde{x}_1 \tilde{x}_0 \left( -9 \tilde{E}_0 y + 2 \tilde{E}_0 y + 6 \tilde{x}_1 y - 2 \tilde{x}_1 \right) = 0 .
\end{equation}

The equation for quadratic corrections \{\tilde{E}_2, \tilde{x}_2\} is given by

\begin{equation}
y^2 (a^2 y^3 (C_Q (3\tilde{x}_0^2 y^2 + 1) + 2) - 2a \tilde{x}_0 \left( \tilde{E}_2 - 2\tilde{x}_1 y^3 \right) + \tilde{x}_0^2 (5\tilde{x}_0 y^3 - 2\tilde{x}_2) + 2\tilde{x}_1 \left( \tilde{E}_1 - 5\tilde{x}_0^2 y^3 \right) + \tilde{E}_0 \left( 4\tilde{E}_1 - 5\tilde{x}_0 y^3 - 2\tilde{x}_2 - 2\tilde{E}_2 \tilde{x}_0 \right) + yC_Q \left( 3\tilde{x}_0^2 y^2 (5 - 12y) - 8y + 3 \right) + 2 \left( \tilde{E}_0 \tilde{x}_1 (2 - 9y) + \tilde{E}_1 \tilde{x}_1 (2 - 9y) + \tilde{E}_0^2 (3y - 1) \right) + 3y \left( \tilde{x}_2^2 y^3 + \tilde{x}_1^2 \right) + 2 \tilde{E}_0 \tilde{x}_2 (3y - 1) - \tilde{x}_1^2 + y(3 - 8y) \right) = 0 .
\end{equation}

The ISCO is requires additional condition, \(dV_{\text{eff}}/dy^2 = 0\). For an non-spinning object (zero-th order in spin), this condition translates as

\[ -2a \tilde{E}_0 \tilde{x}_0 - \tilde{a}^2 - \tilde{x}_0^2 (6y_0 - 1) = 0 \quad (A4) \]

Linear order corrections in spin is dictated by

\begin{equation}
2 \left( \tilde{a} \left( \tilde{E}_0 \left( 10\tilde{x}_0 y_1 - \tilde{E}_1 \right) - \tilde{E}_0 \tilde{x}_1 + \tilde{E}_0^2 \right) + \tilde{x}_0 \left( \tilde{E}_0 (1 - 9y_0) + 3\tilde{x}_0 y_1 + \tilde{x}_1 (6y_0 - 1) \right) \right) = 0 \quad (A5)
\end{equation}

Quadratic correction follows the equation given below

\begin{equation}
10y_0^3 \left( a^2 (C_Q + 2) - 2a \tilde{x}_0 \left( \tilde{E}_0 - 2\tilde{x}_1 \right) + 3\tilde{x}_0^2 C_Q \right) + 6\tilde{a}^2 \tilde{x}_0^2 y_0 C_Q - 12y_0^3 \left( -5\tilde{a}^2 y_1 + C_Q + 2 \right) + 2\tilde{E}_0 \left( 2a \tilde{E}_1 - a \tilde{x}_2 - 9\tilde{x}_0 y_1 + \tilde{x}_1 \right) - 2a \tilde{E}_2 \tilde{x}_0 - 2a \tilde{E}_1 \tilde{x}_1 + 3y_0 \left( C_Q + 2 \right) \left( -3 \tilde{E}_0 \tilde{x}_1 + \tilde{x}_1 \left( 2\tilde{x}_2 - 3\tilde{E}_1 \right) + \tilde{E}_0^2 + \tilde{x}_1^2 + 1 \right) + 15\tilde{x}_0^4 y_0 (1 - 6C_Q) + 2 \tilde{E}_1 \tilde{x}_0 \left( \tilde{E}_0^2 + \tilde{x}_1^2 + 1 \right) + 6\tilde{x}_0 y_2 + 12\tilde{x}_0 \tilde{x}_1 y_1 - \tilde{x}_1^2 - 2a \tilde{E}_2 \tilde{x}_0 = 0
\end{equation}

Expanding the parameter \(y\) in Eq. (A1), Eq. (A2), Eq. (A3) as Eq. (23) and solving them together with the conditions Eq. (A4), Eq. (A5) and Eq. (A6), we obtain the values of \{\tilde{E}_i, \tilde{x}_i, y_i\} \(i = 0, 1, 2\). Replacing these values in Eq. (21) and Eq. (23), we obtain the values of \{\tilde{E}, \tilde{x}, \tilde{y}\}.

**Appendix B: The Teukolsky source term**

In the section Section IV we have calculated the flux due to the gravitational wave. We now provide some more details in this appendix. Eq. (35) gives the amplitude at the horizon and at infinity

\[ Z^{H, \infty} = c^{H, \infty} \int_\text{\textcopyright} \Delta^2 \Delta^2 , \]

As discussed in the main text, \(J_\text{\textcopyright}^{H, \infty}\) is the source term for the radial Teukolsky equation Eq. (30). \(R_\text{\textcopyright}^{H, \infty}\) are the solution of homogeneous Teukolsky equation with the following boundary condition

\begin{equation}
R_\text{\textcopyright}^{\text{H, \infty}} \sim \begin{cases} B_{\text{\textcopyright}^{\text{H, \infty}}} \frac{\tilde{x}_0^3 e^{i\omega \tilde{x}_0}}{\Delta^2} & \tilde{x}_0 \rightarrow \infty, \tilde{x}_0 \rightarrow \tilde{x}_0^+ & \tilde{x}_0 \rightarrow \tilde{x}_0^+ \end{cases}
\end{equation}

\begin{equation}
B_{\text{\textcopyright}^{\text{H, \infty}}} \sim \begin{cases} D_{\text{\textcopyright}^{\text{H, \infty}}} \frac{\tilde{x}_0^3 e^{i\omega \tilde{x}_0}}{\Delta^2} & \tilde{x}_0 \rightarrow \infty, \tilde{x}_0 \rightarrow \tilde{x}_0^+ & \tilde{x}_0 \rightarrow \tilde{x}_0^+ \end{cases}
\end{equation}

where, \(\kappa = (\omega - m\tilde{\omega})\). The constant terms \(C_{\text{\textcopyright}^{\text{H, \infty}}}^{H, \infty}\) are given by

\begin{equation}
C_{\text{\textcopyright}^{\text{H, \infty}}}^{H, \infty} = \frac{1}{2i\omega B_{\text{\textcopyright}^{\text{H, \infty}}}}, \quad c_{\text{\textcopyright}^{\text{H, \infty}}} = \frac{B_{\text{\textcopyright}^{\text{H, \infty}}}^{\text{H, \infty}}}{2i\omega B_{\text{\textcopyright}^{\text{H, \infty}}} D_{\text{\textcopyright}^{\text{H, \infty}}}^{\text{D, \textcopyright}}} \end{equation}

where, following [69], we fix the value of the \(B_{\text{\textcopyright}^{\text{H, \infty}}}^{\text{H, \infty}}\) and \(D_{\text{\textcopyright}^{\text{H, \infty}}}^{\text{D, \textcopyright}}\) as

\begin{equation}
B_{\text{\textcopyright}^{\text{H, \infty}}}^{\text{H, \infty}} = \frac{1}{d_{\text{\textcopyright}}}, \quad D_{\text{\textcopyright}^{\text{H, \infty}}}^{\text{D, \textcopyright}} = -4\omega^2 \end{equation}
with

\[ c_0 = -12i\dot{\omega} + \lambda_{\ell m}\omega (\lambda_{\ell m} + 2) - 12\hat{a}\dot{\omega}(\hat{a}\dot{\omega} - m) \]

\[ d_{\ell m} = 2\sqrt{2}\hat{r}^2 + (2 - 6i\omega - 4\dot{\omega})^2 + 3i\dot{a}m - 4 \]

\[ + 4\hat{a}\dot{\omega}m + 6i\dot{\omega}\hat{r}^2 - \hat{a}^2m^2 - 3i\dot{a}m + 2 \]

(B4)

and \( B_{\ell m}^n \) satisfies the following relation with the constant Wronskian \( W = 2i\omega B_{\ell m}^n D_{\ell m}^n \).

In Teukolsky formalism, the source term is given by the following relation \([69]\)

\[ J_{\ell m\omega} = \int d\dot{\theta} d\phi \Delta e^{i(\dot{\omega} - m) \phi} \left( J_{NN} + J_{MN} + J_{MM} \right). \]

(B5)

where,

\[ J_{NN} = -\frac{2\sin(\theta)}{\Delta^2 \rho^2} \left[ \left( \mathcal{L}_1^2 - 2i\hat{a}\rho \sin(\theta) \right) \mathcal{L}_2 S_{\ell m}^{\hat{\omega}} \right] T_{NN}, \]

\[ J_{MN} = \partial_\tau \left\{ T_{\bar{M}N} \left[ \frac{4\sin(\theta)}{\sqrt{2}\rho^3 \Delta} \left( \mathcal{L}_1^2 S_{\ell m}^{\hat{\omega}} + i\hat{a} \sin(\theta) (\hat{\rho} - \rho) S_{\ell m}^{\hat{\omega}} \right) \right] \right\} \]

\[ + T_{\bar{M}N} \mathcal{L}_1 \left[ \frac{4\sin(\theta)}{\sqrt{2}\rho^3 \Delta} \left( \left( \frac{1}{\rho} + \hat{r} \right) S_{\ell m}^{\hat{\omega}} + \hat{a} \sin(\theta) \frac{K}{\Delta} (\hat{\rho} - \rho) S_{\ell m}^{\hat{\omega}} \right) \right] \]

(B6)

\[ J_{\bar{M}M} = \{ \partial_\tau \left( T_{\bar{M}M} \left[ - \frac{\hat{r}}{\rho^3} \right] \right) + \partial_\tau \left( T_{\bar{M}M} \left[ - \left( \frac{\hat{r}}{\rho^3} + i \frac{\hat{a}}{\rho^3} \frac{K}{\Delta} \right) \right] \right) \}

\[ T_{\bar{M}M} \left[ \frac{\hat{r}}{\rho^3} \left( \frac{\partial}{\partial \tau} \left( \frac{i}{\rho} \right) - \left( \frac{\hat{r}}{\rho} + i \frac{\hat{a}}{\rho} \frac{K}{\Delta} + \frac{K}{\Delta^2} \right) \right) \right] \sin(\theta) S_{\ell m}^{\hat{\omega}}, \]

The energy-momentum tensor for deformed spinning object is given in Eq. (1)

\[ T^{\alpha\beta} = \int d\tau \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \left( p^{(\alpha)}v^{(\beta)} + \frac{1}{3} j^{(\beta)(\alpha) R^{(\beta)}} e^{(\delta)} \right) \]

\[ - \int d\tau \nabla_\gamma \left( S^{(\gamma)(\alpha) \beta}) \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \right) \]

\[ - \frac{2}{3} \int d\tau \nabla_\gamma \nabla_\delta \left( j^{(\alpha)(\beta) \gamma} \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \right). \]

In Tetrad frame,

\[ T^{(a)}_{(b)} = \int d\tau \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \left( p^{(a)}v^{(b)} - \frac{1}{3} j^{(c)(d)(e)(f) R^{(b)}} e^{(c)} \right) \]

\[ - \int d\tau \left( \delta^4(x - z(\tau)) \right) \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \]

\[ - \frac{2}{3} \int d\tau \nabla_\gamma \left( S^{(\gamma)(a) \beta}) \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \right) \]

\[ - \frac{2}{3} \int d\tau \nabla_\gamma \nabla_\delta \left( j^{(\alpha)(\beta) \gamma} \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \right). \]

(B10)
Then after simplifying we get,

\[ T^{(a)(b)} = \int dt \left( \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \mathcal{P}^{(a)(b)} + \partial_\gamma \left[ \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \mathcal{Q}^{(a)(b)\gamma} \right] \right), \]

(B11)

where,

\[ \mathcal{P}^{(a)(b)} = \frac{1}{\ell} \left[ v^{(a)} v^{(b)} - \frac{1}{3} \mathcal{J}^{de(a} R^{b)de} + \omega^{(a)} \omega^{(b)} \mathcal{S}^{(a)(b)} \right] - \omega^{(a)(b)} \mathcal{S}^{(a)(b)}, \]

\[ \mathcal{Q}^{(a)(b)\gamma} = \frac{-1}{\ell} \mathcal{S}^{(a)(b)\gamma}, \quad T^{(a)(b)\gamma} = \frac{-2}{3 \ell} T^{(a)(b)\gamma}, \quad \hat{t} = \frac{dt}{d\tau}. \]

Here we have used the following identities,

\[ e^{(a)} e^{(a)} \nabla \omega^{(a)} = -\omega^{(a)(a)}, \quad e^{(a)} e^{(m)} \nabla e^{(m)} = \omega^{(a)(m)}, \]

\[ e^{(a)} \nabla e^{(m)} = \omega^{(a)(m)}, \]

\[ e^{(a)} e^{(d)} \nabla e^{(d)} = -\epsilon^{(a)} \nabla \omega^{(d)(a)} + \omega^{(a)(m)} \omega^{(d)(m)} - \omega^{(a)(d)}, \quad \hat{t} = \frac{dt}{d\tau}. \]

Then finally we get from Eq. (B11) after performing the integral over \( \tau \),

\[ T^{(a)(b)} = \frac{\delta^3}{\sqrt{-g}} \left( \mathcal{P}^{(a)(b)} - \mathcal{Q}^{(a)(b)\gamma} \hat{t} + T^{(a)(b)\gamma} \right) \]

\[ -\frac{2}{\sqrt{-g}} \hat{t} \left( T^{(a)(b)\gamma} \right) \delta^{(3)} + \frac{1}{\sqrt{-g}} \hat{t} \left( \mathcal{Q}^{(a)(b)\gamma} \delta^{(3)} \right) \]

\[ + \partial_\gamma \partial_\gamma \left( T^{(a)(b)\gamma} \right) \delta^{(3)} \]

(B16)

We have used the last equality to obtain Eq. (B16). Hence \( T^{(a)(b)} \) mentioned in the Eq. (B16) should be viewed as an operator acting on functions.

Next we have to calculate the quantities in Eq. (B8). In order to do so, we consider an arbitrary smooth function of the form \( h(x) = h(r, \theta) e^{i(\omega \tau - m \phi)} \). Then,

\[ T_{\tilde{N}N} h(x) = N_{\tilde{N}N} \left[ \delta^{(3)} \left( \mathcal{P}_{\tilde{N}N} - i \tilde{\omega} \mathcal{Q}_{\tilde{N}N} - \tilde{\omega}^2 \mathcal{I}_{\tilde{N}N} \right) \right] h(x) \]

\[ + N_{\tilde{N}N} \partial_\gamma \left[ \left( \mathcal{Q}_{\tilde{N}N} - 2 i \tilde{\omega} \mathcal{I}_{\tilde{N}N} \right) \delta^{(3)} \right] h(x), \]

(B19)

where, \( N_{\tilde{N}N} = \frac{\Sigma}{\sqrt{-g}} \). Note that the stress tensor in our \( \frac{(b)(b)}{\ell} \) should be viewed as an operator. Finally we can write it in the following compact form,

\[ D^{\Omega\tilde{N}N} \left[ N_{\tilde{N}N} h(x) \right] = \left[ \mathcal{P}_{\tilde{N}N} - i \tilde{\omega} \mathcal{Q}_{\tilde{N}N} - \tilde{\omega}^2 \mathcal{I}_{\tilde{N}N} \right] h(x) \]

\[ + m^2 \mathcal{I}_{\tilde{N}N} \delta^{(3)} \right] \partial_\theta \left( N_{\tilde{N}N} h(x) \right) \]

\[ + \mathcal{I}_{\tilde{N}N} \delta^{(3)} \partial_\theta \left( N_{\tilde{N}N} h(x) \right), \]

(B20)

Similarly,

\[ T_{\tilde{M}N} = \delta^{(3)} D^{\Omega\tilde{M}N} \left[ N_{\tilde{M}N} h(x) \right] + D^{\Omega\tilde{M}N} \left[ N_{\tilde{M}N} h(x) \right], \]

\[ T_{\tilde{M}N} = \delta^{(3)} D^{\Omega\tilde{M}N} \left[ N_{\tilde{M}N} h(x) \right] + D^{\Omega\tilde{M}N} \left[ N_{\tilde{M}N} h(x) \right], \]

(B23)

with, \( N_{\tilde{M}MM} = \frac{\Sigma}{\sqrt{-g}} \) and \( N_{\tilde{M}MM} = \frac{\Sigma}{\sqrt{-g}} \). Here, \( D^{\Omega\tilde{M}N}, D^{\Omega\tilde{M}N}, D^{\Omega\tilde{M}N}, D^{\Omega\tilde{M}N} \) satisfy similar equations as Eq. (B21) and Eq. (B22) with the appropriate indices. Also we have used the following: \( \partial_\gamma \partial_\gamma \mathcal{I}_{\tilde{N}N} = \partial_\gamma \partial_\gamma \mathcal{I}_{\tilde{N}N} \).
With $T_{NN}$, $T_{MN}$ and $\mathcal{N}_{MM}$ in our hand, we can calculated the quantities in Eq. (B6) which turns out to be

$$J_{NN} = \delta^{(3)} D_{NN}^{(0)} f^{(0)}_{NN} + D_{NN}^{(0)} f^{(0)}_{NN} \tag{B24}$$

$$J_{MN} = \partial_{r} \{ \delta^{(3)} D_{MN}^{(1)} f^{(1)}_{MN} + D_{MN}^{(1)} f^{(1)}_{MN} \} + \delta^{(3)} D_{MN}^{(2)} f^{(2)}_{MN} + D_{MN}^{(2)} f^{(2)}_{MN} \tag{B25}$$

$$J_{MM} = - \partial_{r} \{ \delta^{(3)} D_{KK}^{(1)} f^{(1)}_{MM} + D_{KK}^{(1)} f^{(1)}_{MM} \} + \delta^{(3)} D_{KK}^{(2)} f^{(2)}_{MM} + D_{KK}^{(2)} f^{(2)}_{MM} \tag{B26}$$

where,

$$f^{(0)}_{NN} = - \frac{2 \tilde{\rho}}{\Delta \rho} \left[ \left( \mathcal{L}_{1} - 2 i \tilde{\rho} \sin(\theta) \right) \mathcal{L}_{2} \tilde{S}_{lm}^{\text{in}} \right],$$

$$f^{(0)}_{MN} = - \frac{4 \tilde{\rho}}{\sqrt{2} \rho \Delta} \left\{ \left( i \frac{K}{\Delta} + \rho + \tilde{\rho} \right) \mathcal{L}_{1} \tilde{S}_{lm}^{\text{in}} \right. - \left. i \tilde{\rho} \sin(\theta) \left[ i \frac{K}{\Delta} - 2 \rho + \tilde{\rho} \frac{K^{2}}{\Delta^{2}} \right] S_{lm}^{\text{in}}, \right.$$  

$$f^{(1)}_{MN} = - \frac{4 \tilde{\rho}}{\sqrt{2} \rho \Delta} \left( \tilde{\rho} \frac{d}{d\tilde{r}} \left( i \frac{K}{\Delta} - 2 \rho + \tilde{\rho} \frac{K^{2}}{\Delta^{2}} \right) \right. S_{lm}^{\text{in}}, \right.$$  

$$f^{(2)}_{MM} = - \left( \tilde{\rho} + \frac{i \tilde{\rho}}{\tilde{\rho}} \frac{i K}{\Delta} \right) S_{lm}^{\text{in}}, \quad f^{(2)}_{MM} = - \frac{\tilde{\rho}}{\rho} S_{lm}^{\text{in}}. \tag{B27}$$

Replacing Eq. (B24), Eq. (B25) and Eq. (B26) in Eq. (B5) and doing the integration over $\theta$ and $\phi$ we get the expression for source term,

$$J_{lm\bar{\omega}} = \int d\bar{\nu} e^{i\bar{\nu} \bar{m} \phi} \left[ \delta(\bar{r} - \bar{r}(\bar{t})) J_{lm}^{(0)} D_{DD}^{(1)} \partial_{\bar{r}} \left( J_{\bar{r}}^{(1)} \delta(\bar{r} - \bar{r}(\bar{t})) \right) \right. + \left. \partial_{\bar{r}} \left( J_{\bar{r}}^{(1)} \delta(\bar{r} - \bar{r}(\bar{t})) \right) \right] R_{lm\bar{\omega}}^{in, \text{up}}(\bar{r}) \tag{B28}$$

where,

$$J_{\bar{r}}^{(0)} = D_{NN}^{(0)} f^{(0)}_{NN} + D_{KK}^{(0)} f^{(0)}_{MM} + D_{MN}^{(0)} f^{(0)}_{MN},$$

$$J_{\bar{r}}^{(1)} = D_{NN}^{(1)} f^{(1)}_{NN} + D_{KK}^{(1)} f^{(1)}_{MM},$$

$$J_{\bar{r}}^{(2)} = D_{MN}^{(2)} f^{(2)}_{MN}.$$

With the source term for Teukolsky equation in our hand, we can calculate the amplitude given in Eq. (35), which turns out to be

$$Z_{lm\bar{\omega}}^{H, \infty} = C_{lm\bar{\omega}}^{H, \infty} \int_{-\infty}^{\infty} d\bar{r} \int_{-\infty}^{\infty} d\bar{\nu} e^{i\bar{\nu} \bar{m} \phi} \left[ \delta(\bar{r} - \bar{r}(\bar{t})) J_{\bar{r}}^{(0)} D_{DD}^{(1)} \partial_{\bar{r}} \left( J_{\bar{r}}^{(1)} \delta(\bar{r} - \bar{r}(\bar{t})) \right) \right. + \left. \partial_{\bar{r}} \left( J_{\bar{r}}^{(1)} \delta(\bar{r} - \bar{r}(\bar{t})) \right) \right] R_{lm\bar{\omega}}^{in, \text{up}}(\bar{r}) \tag{B30}$$

Note that the integrand has to be evaluated at $\theta = \theta(\bar{t}), \phi = \phi(\bar{t}).$
\[ Z_{1m}^{H, \infty} = C_{1m}^{H, \infty} \int_{-\infty}^{\infty} dt e^{i(\omega t - m \phi)} \]

\[ \left\{ \left[ O_{MN} f_{NN}^{(0)} + O_{MN} J_{MN}^{(0)} + O_{\hat{M} \hat{M}} J_{\hat{M} \hat{M}}^{(0)} \right] R_{1m}^{\text{in, up}} (\hat{r}) - \left\{ \left( O_{MN} J_{MN}^{(1)} + O_{\hat{M} \hat{M}} J_{\hat{M} \hat{M}}^{(1)} \right) \frac{\partial R_{1m}^{\text{in, up}} (\hat{r})}{\partial \hat{r}} + \left( O_{\hat{M} \hat{M}} J_{\hat{M} \hat{M}}^{(2)} \right) \frac{\partial^2 R_{1m}^{\text{in, up}} (\hat{r})}{\partial \hat{r}^2} \right\} \right] \]

\[ \theta \]

In this paper we have set \( \theta = \frac{\pi}{2} \). This further simplifies certain term. On the equatorial plane whenever one of the indices set to \( \theta \), the corresponding components of the tensor will be zero, i.e.

\[ Q^\theta \hat{\imath} \hat{\imath} = Q^\theta \hat{\imath} \hat{\jmath} = Q^\theta \hat{\gamma} \hat{\gamma} = 0, \]

\[ T^\alpha \hat{\imath} \hat{\imath}^\theta = T^\alpha \hat{\imath} \hat{\jmath}^\theta = T^\alpha \hat{\gamma} \hat{\gamma}^\theta = 0, \]

\( \alpha = \{ \hat{\imath}, \hat{\imath}, \theta, \phi \} \).
Hence,

\[ K_{NN} = T_{\hat{N}\hat{N}}^{\hat{N}}, \quad J_{NN} = I_{NN} + T_{\hat{N}\hat{N}}^{\hat{N}} \partial_t, \]
\[ I_{NN} = -\left( Q_{\hat{N}\hat{N}}^{\hat{N}} - 2i \omega T_{\hat{N}\hat{N}}^{\hat{N}} + 2im T_{\hat{N}\hat{N}}^{\hat{N}} \partial_t \right) + T_{\hat{N}\hat{N}}^{\hat{N}} \partial_t. \]  

Then expressions in Eq. (B37) get simplified.

\[ O_{NN} = P_{\hat{N}\hat{N}} - i \omega Q_{\hat{N}\hat{N}}^{\hat{N}} - \omega^2 T_{\hat{N}\hat{N}}^{\hat{N}} + i m \left( Q_{\hat{N}\hat{N}}^{\hat{N}} - 2i \omega T_{\hat{N}\hat{N}}^{\hat{N}} \right) - m^2 T_{\hat{N}\hat{N}}^{\hat{N}} + I_{NN} \partial_t. \]

(\text{B40})

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