THE SCATTERING MATRIX APPROACH
FOR THE QUANTUM BLACK HOLE

an overview

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Abstract

If one assumes the validity of conventional quantum field theory in the vicinity of the horizon of a black hole, one does not find a quantum mechanical description of the entire black hole that even remotely resembles that of conventional forms of matter; in contrast with matter made out of ordinary particles one finds that, even if embedded in a finite volume, a black hole would be predicted to have a strictly continuous spectrum.

Dissatisfied with such a result, which indeed hinges on assumptions concerning the horizon that may well be wrong, various investigators have now tried to formulate alternative approaches to the problem of “quantizing” the black hole. We here review the approach based on the assumption of quantum mechanical purity and unitarity as a starting point, as has been advocated by the present author for some time, concentrating on the physics of the states that should live on a black hole horizon. The approach is shown to be powerful in not only producing promising models for the quantum black hole, but also new insights concerning the dynamics of physical degrees of freedom in ordinary flat space-time.
1. INTRODUCTION

More and more physicists working on attempts to reconcile the theory of general relativity with the postulates of quantum mechanics are becoming aware of the obstinate problems arising when gravitational collapse might occur in microscopic systems. In classical general relativity there is nothing wrong with the “black hole solution”, whose properties can be precisely calculated using ordinary laws of physics, and which may arise naturally in astronomical objects with masses several times that of our Sun. Indeed it is not hard to indicate the classical initial conditions that inevitably will lead to a black hole, as we will briefly summarise in Sections 2 and 3. In particular one observes that the emergence of a horizon has nothing to do with small scale physics, and therefore one generally expects that black hole properties, at scales large compared to the Planck length, are indeed represented accurately by these classical calculations\textsuperscript{1, 2}.

In the classical case, the emergence of a singularity at the origin of space-time is in no way problematic. The region where it occurs is physically unobservable and so, for practical purposes, in particular when one wants to predict “experimental observations”, this singularity is of no importance. Throughout this paper it will receive little attention (see however section 17).

One would have expected that also the quantum mechanical properties of a black hole should follow naturally by applying large scale physics. Only the space-time region at the side of the observer, the “physical side” of the horizon, should be relevant. Indeed one can calculate accurately the quantummechanical effects near a large black hole, as seen by an outside observer, by first studying what an infalling observer would experience, and then performing the appropriate general coordinate transformation. As is to be expected from quantum mechanical calculations, one finds “probabilities”: chances that particles of certain types, with certain momenta, energies or other quantum numbers, emerge at certain places. It is when one wants to interpret these outcomes in terms of some Schrödinger equation for the black holes as a whole, that the first genuine problems emerge\textsuperscript{3, 4}.

Does a black hole, enclosed in a finite volume of space, possess a discrete or a continuous spectrum? We will argue in Section 7 that one would expect the spectrum to be discrete, but more careful analysis of Hawking’s result suggests a continuum to be inevitable, even if interactions are taken into account\textsuperscript{5, 6}. This situation is fundamentally different from what we have in any ordinary form of matter. It would imply, among other things, that a black hole cannot disappear at the end of its evaporation process; a remnant should then stay behind (see for instance Ref \textsuperscript{7}). Still equipped with a continuous spectrum, such remnants would defy any description in terms of conventional physical laws. No spin-statistics theorem would apply to them, and no thermodynamics can be worked
Whether remnants are acceptable or not in a logically coherent theory can now be disputed. Many physicists argue that there is as yet no conflict. Black holes will always behave in an exceptional way. Others however maintain that in the presence of remnants that can be abundantly created virtually, the construction of consistent theories that fully predict the physical effects at the Planck length will be impossible, a situation that is unprecedented in any other branch of physics. More to the point is the observation that there are many reasons to question the validity of the arguments that led to these results; more likely quantum gravity will require drastically different techniques, and the conclusion that black holes terminate their lives as remnants appears to be highly premature.

As will be argued in much more detail in this review article, on the one hand we have observations on large black holes, made by distant observers; these can probably be deduced entirely by applying conventional physics (though even here there could be surprises). On the other hand, however, questions concerning their spectrum will require new ultra short distance physics for their answers.

What makes this problem so important is that its resolution, whichever it will be, will strongly affect our views on all other laws of physics at the Planck scale as well. A theory with virtual black hole remnants will look quite different from a world where conventional quantum mechanics holds unabatedly at the tiniest distance scales. But also if conventional quantum mechanics, free from decoherence effects from black holes, can be rescued, the world at the Planck scale will be different from conventional physical theories as we will further discuss (Section 16). Yet, it is this latter avenue that this author strongly prefers for further investigation. It leads to rich physics, connected in several curious ways to string theories. It may thus provide for an improved interpretation of string theory. In short, our assumption is that Schrödinger equations can be used universally for all dynamics in the universe, including the Planck scale. This assumption can obviously not be proved from any more basic first principle; it is a first principle itself. Principles of this sort are indeed badly needed for the construction of a logically coherent theory for Planck scale physics. It can of course also be wrong. We have nothing further to say about that.

The technique that is applied is simply assuming that black holes form a discrete spectrum just as any other physical system; the density of states is then approximately prescribed by the entropy law. Combining this assumption to known laws of physics, numerous properties of the $S$-matrix elements involving black holes can actually be derived. The procedure is completely straightforward and free from ambiguities, at least in principle. Its elaboration is the subject of this article.

Relatively little attention will be paid to the generalizations of the Schwarzschild solu-
tion: the Reissner-Nordstrom and Kerr-Newman solutions$^{11,12}$. These are the solutions obtained when electric charge and angular momentum are added to the hole. Conceptually, generalization of everything we say to these cases should be straightforward. Recent literature pays a considerable amount of attention to the extremal solutions$^{13}$, which are the ones obtained when electric charge, angular momentum, or a combination of both, take the maximally allowed values. As models these cases are quite interesting, but in the present paper they are given less priority. Our motivation for this is that these limiting cases cannot be representative for the general black hole; it could well be that these solutions will be unattainable in “realistic” experiments at the Planck scale. In view of this, they might show rather pathological behavior without upsetting the logic of the underlying theory (more about this at the end of Section 2). Conclusions based on a study of these solutions alone might therefore be misleading.

For the same reason we will not pay much attention to dilaton black holes. Dilaton fields might be important at the Planck scale, but appear not to exist at macroscopic scales; macroscopic black holes should be logically consistent without dilatons. There are several recent developments which are not yet in a stage that they can be thoroughly reviewed here. One is the attempts to count black hole states in the context of string theory$^{14}$. It is claimed that the results strongly support our views, since the numbers obtained agree with the value of the Bekenstein entropy$^{15}$. Their interpretation in terms of modes living on the horizon is still obscure however, and we have no idea how to look against general coordinate transformations for these states, which is the problem addressed in our paper.

An interesting suggestion was made by Carlip$^{16}$, who claims that states can be counted for a black hole in 2+1 dimensions. His result sounds remarkable since the theory he considers is a topological one. If the characteristics of the states counted by him can be further identified, this would be a valuable piece of information for our theories. The philosophy adhered to in our paper is to start our analysis from the other end: we insist on first asking the question what the physical effects are that we wish to describe. Only after these have been identified, one may decide which mathematical tricks should be used to speed up and simplify calculations.

The paper is organised as follows. The introductory parts, Sections 1–10, contain only rather elementary material, but are needed as a background for the remainder (most of it can be skipped by a more informed reader). The coordinate frames used are exhibited in Section 2. Then, in Section 3, a picture is sketched of a black hole formed by matter. This picture should make it clear that conventional physical laws can never be such that black holes could be avoided – they are legitimate solutions of classical, (i.e., non-quantummechanical) physics. For some special conventions in our notation, see Section 4.
The standard Hawking-Unruh effect, and its relation to Bogolyubov transformations of creation and annihilation operators, are explained in Sections 5 and 6. In Section 7, the standard argument is given explaining why we expect only a finite density of physically allowed states near a horizon, by deriving the entropy of a black hole. From the “brick wall model”, in Section 8, we learn that ordinary quantum fields already provide the required number of states if we limit these to exist only outside a “brick wall”, at distance $h$ from the horizon.

All this has given us a detailed picture of the statistical behavior of particles near a black hole, but the laws of physics at a microscopic scale that should be held responsible for these effects are still obscure. In Section 9 we explain that, in order to zoom in to these laws, we need to understand the gravitational back reaction of the Hawking radiation process. The first step is the Aichelburg-Sexl metric (Section 10), which is then exploited in Section 11, where we make a first attempt to set up a description of the Hilbert space of black hole horizon states. Non-gravitational forces must be accommodated for as well (Sections 12 and 13), but the results fail to account for the finiteness of density of states on a given area of the horizon. What is needed for this is the incorporation of the sideways gravitational forces. Their source is the transverse momentum operator (Section 14), and attempts are made to incorporate these effects in the horizon operator algebra (Section 15).

The most urgent motivation for this work is its possible relevance to modelling physics at the Planck scale in ordinary flat space-time, as is further elaborated on in Section 16. In Section 17 we briefly speculate on how to proceed from here.

2. THE SCHWARZSCHILD METRIC

The space-time metric of a stationary, non-rotating and electrically neutral black hole is the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\Omega^2,$$  \hspace{1cm} (2.1)

where

$$d\Omega^2 \equiv d\theta^2 + \sin^2 \theta \, d\phi^2.$$  \hspace{1cm} (2.2)

Furthermore, $M$ stands for $Gm_{\text{BH}}$, where $G$ is Newton’s constant and $m_{\text{BH}}$ is the black hole mass. Often we will employ the Kruskal coordinates\textsuperscript{12}, which we will write as $(x, y, \theta, \phi)$, with

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = xy;$$

$$e^{r/2M} = x/y.$$  \hspace{1cm} (2.3)

5
In terms of these coordinates we have (see Fig. 1)

\[
\begin{align*}
\frac{dx}{x} + \frac{dy}{y} &= \frac{dr}{2M\left(1 - 2M/r\right)}; \\
\frac{dx}{x} - \frac{dy}{y} &= \frac{dt}{2M}; \\
ds^2 &= \frac{32M^3}{r}e^{-r/2M} \, dx \, dy + r^2 \, d\Omega^2.
\end{align*}
\]  

(2.4)

The apparent singularity at the horizon, \( r = 2M \), has disappeared. The only true singularities are at the curves \( xy = -1 \), where \( r = 0 \). The region \( \{x > 0, \, y > 0\} \) is the “outside region”, the only region from which distant observers can obtain any information. The line \( y = 0 \), where \( r = 2M \), is the “future horizon”; the line \( x = 0 \) where also \( r = 2M \), is the “past horizon”.

The local light cones are oriented everywhere as indicated.

In the region \( r \approx 2M \) one can write the metric as

\[
ds^2 \approx \frac{16M^2}{\sqrt{e}} \, dx \, dy + 4M^2 \, d\Omega^2
\]  

(2.5)

and with the coordinate substitution

\[
\begin{align*}
\frac{4M}{\sqrt{e}} \, x &= Z + T, \\
\frac{4M}{\sqrt{e}} \, y &= Z - T, \\
2M\theta &= \frac{1}{2}\pi + X, \\
2M\phi &= Y,
\end{align*}
\]  

(2.6)

close to the origin, one finds that in terms of these coordinates space-time is approximately flat:

\[
ds^2 \approx -dT^2 + dZ^2 + dX^2 + dY^2.
\]  

(2.7)
The transformation
\[ Z = \rho \cosh \tau, \quad T = \rho \sinh \tau, \]
(2.8)
brings us back to the Schwarzschild coordinates (close to the horizon), apart from normalization factors:
\[ \frac{t}{2M} = 2\tau, \quad 8M(r - 2M) = \rho^2. \]
(2.9)
The description of a flat space-time (2.7) in terms of the coordinates (2.8) is called “Rindler space”\(^{17}\). We see that close to the horizon, the Schwarzschild coordinates \( r \) and \( t \) behave as Rindler space coordinates.

The derivation of more general black hole solutions carrying electric charge and/or angular momentum can be found in the literature.\(^{11, 12}\) One finds
\[
ds^2 = -\frac{\Delta}{Y}(dt + a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{Y}(adt + (r^2 + a^2)d\phi)^2 + \frac{Y}{\Delta} dr^2 + Y d\theta^2, \]
(2.10)
with
\[ \Delta = r^2 - 2Mr + Q^2 + a^2, \quad Y = r^2 + a^2 \cos^2 \theta, \]
(2.11)
where \( J = Ma \) is the angular momentum, and \( Q \) is the electric charge in conveniently chosen units. The two horizons, \( r_+ \) and \( r_- \), are the solutions of the equation \( \Delta(r) = 0 \).

*Extreme black holes* are represented by the case
\[ Q^2 + a^2 = M^2, \]
(2.12)
in which case \( r_+ - r_- \downarrow 0 \). They (together with versions including dilaton fields) are presently being studied in string theory scenarios. Two aspects of the extreme black holes should be kept in mind:

One: both horizons are separated from the observer by an infinite distance, since \( r_+ = r_- \) implies that \( \int_{r_+}^{r_-} dr \sqrt{Y/\Delta} \) diverges logarithmically.

Two: the two horizons may appear to coincide in the extreme limit, but in reality they do not. The (timelike) distance between \( r_+ \) and \( r_- \) is in the extreme limit (2.12):
\[
\lim_{\varepsilon \to 0} \int_{r_-}^{r_+} dr \sqrt{\frac{M^2 + a^2 \cos^2 \theta}{\varepsilon^2 - (r - M)^2}} = \pi \sqrt{M^2 + a^2 \cos^2 \theta},
\]
(2.13)
which remains macroscopic at large \( M \). Thus, for an observer who stays close to the horizon there is no extreme limit; for him, the horizon at \( r = r_+ \) continues to look like an ordinary horizon, which, even in the extreme limit, stays far away from the horizon at \( r = r_- \). This is why the author believes that macroscopic extreme black holes are
physically ill-defined limits; it would be a mistake to treat an extreme horizon as just one horizon. Most likely, this limit can only be understood if one first manages to control the regular case, not vice versa. The string theory results reported in ref 14 are very encouraging, but, as stated earlier, they are not yet in a stage that they can be reviewed here.

3. BLACK HOLE FORMED BY MATTER\textsuperscript{18, 19}

So far the stationary solution was described. To see that black holes can actually be formed by ordinary matter we have to study time-dependent solutions. A class of such solutions can easily be found. Take the case that we have spherical symmetry, and matter that only consists of particles moving radially inwards and outwards with the speed of light, without interacting (here, the outward moving particles are included for later considerations; in this section they will be ignored from Eq. (3.9) onwards). The metric is taken to be

\[ ds^2 = 2A(x,y)dx dy + r^2(x,y)d\Omega^2, \]

where \( A \) and \( r \) are functions of \( x \) and \( y \) yet to be determined. Matter only contributes to the components \( T_{xx} \) and \( T_{yy} \) of the energy-momentum tensor, whereas all other components, including the trace \( T_{\mu}^{\mu} \), are zero:

\[ T_{xy} = T_{\theta\theta} = 0, \]  \( (3.2) \)

implying for the Ricci tensor \( R_{\mu\nu} \):

\[ R_{xy} = R_{\theta\theta} = 0. \]  \( (3.3) \)

If we introduce a new function \( M = M(x,y) \) and write

\[ A = \frac{2r r_x r_y}{r - 2M}, \]  \( (3.4) \)

where \( r_x \) stands for \( \left. \frac{\partial r}{\partial x} \right|_y \), etc., then the equations (3.3) become remarkably simple:

\[ r_{xy} = \frac{2M r_x r_y}{r(r - 2M)}, \quad M_{xy} = \frac{-2M_x M_y}{r - 2M}. \]  \( (3.5) \)

They both can be integrated once to give

\[ 2M_x r_x = g(x)\left(1 - \frac{2M}{r}\right), \quad 2M_y r_y = h(y)\left(1 - \frac{2M}{r}\right), \]  \( (3.6) \)
where $g(x)$ and $h(y)$ are arbitrary functions. Since

$$8\pi GT_{xx} = \frac{2g(x)}{r^2} \quad \text{and} \quad 8\pi GT_{yy} = \frac{2h(y)}{r^2},$$

we must demand that $g(x) \geq 0$ and $h(y) \geq 0$. These functions are determined by fixing the flux of ingoing and outgoing particles at infinity. At $r \to \infty$ we write

$$r \to \frac{1}{\sqrt{2}}(x + y); \quad t = \frac{1}{\sqrt{2}}(x - y),$$

$$ds^2 \to dr^2 - dt^2 + r^2 d\Omega^2.$$  

The ingoing matter depends only on the $x$ coordinate and the outgoing matter on the $y$ coordinate (see Fig. 2a).

![Figure 2a](image)

![Figure 2b](image)

2. a) Spherically symmetric configuration of matter radially moving inward and outward with the speed of light. b) Spherically symmetric black hole formed by radially inmoving lightlike matter.

Now take the case $h(y) = 0$, which corresponds to the case that there is no outgoing matter. Since we do not want $r_y$ to vanish, Eq. (3.6) implies

$$M(x, y) = M(x),$$

and since at infinity $r_x \to \frac{1}{\sqrt{2}}$ we get

$$g(x) = \sqrt{2} \frac{dM(x)}{dx}.$$  

(3.10)

Note that at $t \to -\infty$ we have

$$T_{tt} = T_{rr} = T_{rt} = T_{tr} = \frac{1}{2} T_{xx}, \quad T_{xy} = T_{yy} = 0,$$

$$E = 4\pi \int r^2 T_{tt} dr \bigg|_t = \int \frac{1}{2G} g(x) dr \bigg|_t \to \int \frac{1}{\sqrt{2}} \frac{\partial M}{\partial x} dr \bigg|_t$$

$$= \int dx \frac{dM}{G dx} = \frac{M}{G}.$$  

(3.11)
We see from this solution that matter imploding in a spherically symmetric way, and with the speed of light, without any further interactions, always collapses into a black hole. The horizon appears as soon as \( r = 2M \), at which point the local situation is still completely regular, so that there is no classical mechanism to be found that could block this collapse from happening. The clearest situation is the case where we have a thin spherically symmetric shell of imploding matter:

\[
M(x) = M_0 \Theta(x - x_0) , \quad g(x) = \sqrt{2} M_0 \delta(x - x_0) ,
\]

where we choose \( x_0 > 0 \). At \( x < x_0 \) we have flat space-time:

\[
r = \frac{1}{\sqrt{2}} (x + y) , \quad A = 1 \quad (x < x_0).
\]

At \( x > x_0 \) we define a new \( y \) variable, \( \tilde{y} \), by

\[
r(x_0, y) = \frac{1}{\sqrt{2}} (x_0 + y) ; \quad \tilde{y} = \frac{1}{x_0} \left( \frac{r(x_0, y)}{2M} - 1 \right) e^{r(x_0, y)/2M} .
\]

Then at \( x > x_0 \) we recover

\[
x \tilde{y} = \left( \frac{r(x, \tilde{y})}{2M} - 1 \right) e^{r(x, \tilde{y})/2M} ; \quad M(x, \tilde{y}) = M_0.
\]

Thus we have flat space-time inside the shell and the Schwarzschild solution outside. See Fig. 2b. The horizon is the sheet \( \tilde{y} = 0 \), corresponding to \( r(x_0, y) = 2M \) or \( y = y_1 = 2\sqrt{2} M - x_0 \). The horizon opens up for the first time at the point \( S \) defined by \( r(x, y_1) = 0 \) or

\[
x_S = -2\sqrt{2} M + x_0 , \quad y_S = 2\sqrt{2} M - x_0 ,
\]

but this situation becomes more complicated as soon as deviations from spherical symmetry are introduced.

The freely falling thin shell of matter can also be treated when this matter moves slower than the speed of light. Again one gets a flat space-time inside, and the Schwarzschild solution outside. The horizon starts at a point \( S \) well inside the flat region, and the shell moves through this horizon in a region where everything is still completely regular. Thus one concludes that if the equation of state describes non-interacting matter and the initial condition is spherically symmetric an implosion into a black hole will often be inevitable\(^\dagger\).

\[^\dagger\text{The arguments that black hole formation is also inevitable in more generic circumstances without spherical symmetry are more delicate and will not further be discussed here.}\]

In principle, a sufficiently large and dilute dust cloud can obey these conditions. The
importance of such considerations is that this way one can convince oneself that black holes must exist as part of the spectrum of physically realizable states. The question “what is the smallest possible black hole?” must be asked and answered in any viable theory of quantum gravity.‡ It is quite plausible that at the Planck scale no clear distinction will be possible between black holes and ordinary particles, but if this is the case we must ask how the “unification” between black holes and ordinary matter can be realized in mathematical terms.

4. CONVENTIONS AND NOTATION

In the equations of the previous two sections we have

\[ M \equiv Gm_{\text{BH}}, \]

where \( m_{\text{BH}} \) is the mass of the black hole and \( G \) the gravitational constant. Later we will not choose units such that \( G \) is normalized to one, because this would leave factors \( 16\pi \) in the Lagrangian. In analogy with the notation \( \hbar = h/2\pi \), one could propose the notation

\[ G = 8\pi G. \]

In any case, we will pick units such that

\[ G = 1/8\pi, \quad (4.1) \]

in view of equations (10.13) and (10.23).

In previous publications, various choices have been made for factors \( \sqrt{2} \) in the light-cone coordinates. Omitting such factors does no harm in general considerations of particular details, but leads to ambiguities in the definitions when later comparisons are made. For this reason we here put these factors in, so that as a consequence some numerical factors ar not quite the same as in previous work. We are not yet in a situation that comparison with the phenomenology of the real world can be made so that as yet factors of 2 are of no direct importance, but we will try to be precise.

The coordinates

\[ (t, r, \theta, \phi) \leftrightarrow (x, y, \theta, \phi) \leftrightarrow (T, X, Y, Z) \]

‡ There is evidence that even in classical physics, black holes that are arbitrarily small can be made, provided the initial data are sufficiently carefully chosen, see ref 20.
are as in Sect. 2. Here \((T, X, Y, Z)\) are coordinates in terms of which a small region near the horizon has an approximately Minkowskian metric \((2.7)\). In addition, we will use

\[
x^+ = \frac{1}{\sqrt{2}}(Z + T) = \frac{\rho}{\sqrt{2}}e^\tau; \quad x^- = \frac{1}{\sqrt{2}}(Z - T) = \frac{\rho}{\sqrt{2}}e^{-\tau},
\]

\[\bar{x} = \begin{pmatrix} X \\ Y \end{pmatrix},\]

and

\[
k^+ = \frac{1}{\sqrt{2}}(k^3 + k^0); \quad k^- = \frac{1}{\sqrt{2}}(k^3 - k^0);
\]

\[
\tilde{k} = \begin{pmatrix} k^1 \\ k^2 \end{pmatrix}, \quad k^0 = +\sqrt{\tilde{k}^2 + m^2 + k_3^2},
\]

so that

\[
(k \cdot X) = k^+x^- + k^-x^+ + \tilde{k} \cdot \bar{x}, \quad \text{and} \quad ds^2 = 2dx^+dx^- + d\bar{x}^2,
\]

and the mass shell condition reads

\[
k^- = - (\tilde{k}^2 + m^2)/2k^+.
\]

The physical region of a classical, permanent black hole is defined by

\[
\rho > 0 \quad \text{or} \quad r > 2M \quad \text{or} \quad x > 0; \quad y > 0 \quad \text{or} \quad Z > |T| \quad \text{or} \quad x^+ > 0; \quad x^- > 0.
\]

In Sections 11–14, heavy use is made of operators that we call the “momentum density”, of various sorts. These are naturally related to the energy-momentum tensor in light cone coordinates, but they are not quite the same thing. The relations are as follows:

\[
P\text{in}_{\bar{x}}(\bar{x}) = \int T_{++}(\bar{x}, x^+, x^-)dx^+|_{x^-=0}; \quad (4.7)
\]

\[
P\text{out}_{\bar{x}}(\bar{x}) = - \int T_{--}(\bar{x}, x^+, x^-)dx^-|_{x^+=0}; \quad (4.8)
\]

\[
\tilde{P}^a\text{in}_{\bar{x}}(\bar{x}) = \int T_{a+}(\bar{x}, x^+, 0)dx^+; \quad (4.9)
\]

\[
\tilde{P}^a\text{out}_{\bar{x}}(\bar{x}) = - \int T_{a-}(\bar{x}, 0, x^-)dx^-.
\]

Here, the transverse index \(a\) takes the values 1 or 2. The minus sign has its origin in our definition of the lightcone minus components.
5. THE RINDLER SPACE TRANSFORMATION\textsuperscript{6, 21}

Consider the Minkowski coordinate frame \{T, X, Y, Z\}, or \{T, X\} for short, and a scalar field \(\Phi(T, X)\). Let this field simply obey a Klein-Gordon equation,

\[
(\partial^2 - m^2)\Phi = 0. \tag{5.1}
\]

The quantum theory is written in the Heisenberg representation, which means that the states \(|\psi\rangle\) are space-time independent, but the fields are operators depending both on space and on time. Usually, a complete set of solutions of (5.1) is written in terms of the Fourier modes with respect to the Minkowski space coordinates, and one gets

\[
\Phi(X, T) = \int \frac{d^3k}{\sqrt{2k^0(k)(2\pi)^3}} \left( a(k)e^{ik\cdot X - ik^0T} + a^+(k)e^{-ik\cdot X + ik^0T} \right), \tag{5.2}
\]

\[
\dot{\Phi}(X, T) = \int -\frac{ik^0d^3k}{\sqrt{2k^0(k)(2\pi)^3}} \left( a(k)e^{ik\cdot X - ik^0T} - a^+(k)e^{-ik\cdot X + ik^0T} \right). \tag{5.3}
\]

Here \(k^0(k) = \sqrt{k^2 + m^2}\), and the transformation from \(a\) and \(a^+\) to \(\Phi\) and \(\dot{\Phi}\) has been designed such that the following commutation rules are maintained:

\[
[\Phi(X, T), \Phi(X', T)] = 0, \quad [\Phi(X, T), \dot{\Phi}(X', T)] = i\delta^3(X - X'), \tag{5.4}
\]

\[
[a(k), a(k')] = 0, \quad [a(k), a^+(k')] = \delta^3(k - k'). \tag{5.5}
\]

Not only do these commutation rules ensure that \(a^+\) and \(a\) act as creation and annihilation operators, but also the time dependence in (5.2) and (5.3) implies that the objects created and annihilated carry an amount of energy equal to \(k^0\).

The operator \(H_M\) that generates boosts in the time coordinate \(T\),

\[
\frac{\partial \Phi}{\partial T} = -i[\Phi, H_M], \tag{5.6}
\]

is the Minkowski-Hamiltonian

\[
H_M = \int \mathcal{H}(X)d^3X = \int d^3k k^0(k)a^+(k)a(k), \tag{5.7}
\]

We need first the transition to light-cone coordinates (4.2), and we define

\[
a(k)\sqrt{k^0} = a_1(\tilde{k}, k^+\sqrt{k^+}). \tag{5.8}
\]

Since

\[
\frac{\partial k^+}{\partial k^3} \bigg|_k = \frac{1}{\sqrt{2}} \left( 1 + \frac{k^3}{\sqrt{k^2 + k_3^2 + m^2}} \right) = \frac{k^+}{k^0}, \tag{5.9}
\]

13
the new operators $a_1$, $a_1^\dagger$ are normalized by

$$[a_1(k, k^+), a_1^\dagger(k', k'^+)] = \delta^2(k - k')\delta(k^+ - k'^+) .$$

Thus one can write

$$\Phi(T, X) = A(x^+, x^-, \bar{x}) + A^\dagger(x^+, x^-, \bar{x}) ,$$

with

$$A(x^+, x^-, \bar{x}) = \int d^2k \int_0^\infty \frac{dk^+}{\sqrt{2k^+(2\pi)^3}} a_1(\tilde{k}, k^+) e^{i(k\cdot\tilde{x} + k^+x^- + k^-x^+)} ,$$

(5.12)

In terms of the Rindler space coordinates $\{\tau, \varrho, \bar{x}\}$ of Eq. (2.8) the Klein Gordon equation (5.1) reads

$$[(\varrho \partial_\varrho)^2 - \partial^2 - \varrho^2(\partial^2 - m^2)] \Phi = 0 .$$

(5.13)

Solutions periodic in $\tau$ are†:

$$\Phi_\omega = K(\omega, \frac{1}{2}\mu \varrho \tau, \frac{1}{2}\mu \varrho \tau) e^{i\tilde{k} \cdot \bar{x}} = K(\omega, \frac{1}{2}\mu \varrho, \frac{1}{2}\mu \varrho) e^{ik \cdot \bar{x} - i\omega \tau} ,$$

(5.14)

where $\mu^2 = \tilde{k}^2 + m^2$ and

$$K(\omega, \alpha, \beta) = \int_0^\infty \frac{ds}{s} s^{i\omega} e^{-is\alpha + i\beta/s} .$$

(5.15)

This function $K$ obeys

$$K(\omega, \sigma \alpha, \beta/\sigma) = \sigma^{-i\omega} K(\omega, \alpha, \beta) ,$$

(5.16)

and can be expressed in terms of familiar Bessel and Hankel functions. Eq. (5.15) is readily obtained by taking in 4-dimensional space-time one of the plane wave solutions: $k^+ = -k^- = \mu/\sqrt{2}$, which are rewritten in terms of the coordinates $\varrho, \tau$ of Eq. (4.2), and then Fourier transformed with respect to $\tau$. It is not difficult to verify directly (using partial integration in $s$) that the partial differential equation (5.13) is obeyed.

We will now normalize the Fourier components of the fields $\Phi$ with respect to the $\tau$ coordinate as follows:

$$A(x^+, x^-, \bar{x}) = \int_{-\infty}^{\infty} d\omega \int \frac{d^2\tilde{k}}{\sqrt{2(2\pi)^3}} K(\omega, \frac{1}{2}\mu \varrho, \frac{1}{2}\mu \varrho) e^{ik \cdot \bar{x} - i\omega \tau} a_2(\tilde{k}, \omega) ,$$

(5.17)

† The notations used here differ slightly from our notation in Ref28.
since then the new operators $a_2$ are identified as

$$a_2(\tilde{k}, \omega) = (2\pi)^{-1/2} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} a_1(\tilde{k}, k^+) e^{i\omega \ln \left(\frac{k^+}{\mu}\right)},$$

(5.18)
a Fourier transformation whose inverse is:

$$a_1(\tilde{k}, k^+) \sqrt{k^+} = (2\pi)^{-1/2} \int_{-\infty}^\infty d\omega a_2(\tilde{k}, \omega) e^{-i\omega \ln \left(\frac{k^+}{\mu}\right)}.$$  

(5.19)

Substituting (5.19) into (5.12) gives us back (5.17) and (5.15). With the normalization factors chosen in (5.17)--(5.19), the operators $a_2$ and $a_2^\dagger$ again obey

$$[a_2(\tilde{k}, \omega), a_2^\dagger(\tilde{k}', \omega')] = \delta^2(\tilde{k} - \tilde{k}') \delta(\omega - \omega'),$$

(5.20)

and therefore one might think that they could serve as annihilation and creation operators in Rindler space. But then one has to realize that in the integrals (5.17), $\omega$ can take negative values. The corresponding operators $a(\omega)$ would annihilate a negative amount of Rindler energy. To cure this situation we first need to know some simple properties of the function $K$:

First of all one has:

$$K^*(\omega, \alpha, \beta) = K(-\omega, -\alpha, -\beta).$$

(5.21)

Let now $\alpha > 0$ and $\beta > 0$. In the definition Eq. (5.15) the integrand is bounded in the region $\text{Im}(s) \leq 0$. Therefore one can rotate the integration contour as follows:

$$s \rightarrow s e^{-i\phi}, \quad 0 \leq \phi \leq \pi.$$  

(5.22)

Taking the case $\phi = \pi$ we find

$$K(-\omega, \alpha, \beta) = \int_0^\infty \frac{ds}{s} s^\omega e^{-\pi \omega} e^{is\alpha - i\beta/s} = e^{-\pi \omega} K^*(\omega, \alpha, \beta) \quad \text{if} \quad \alpha > 0, \ \beta > 0,$$

(5.23)

and similarly one has:

$$K(-\omega, \alpha, \beta) = e^{i\pi \omega} K^*(\omega, \alpha, \beta) \quad \text{if} \quad \alpha < 0, \ \beta < 0.$$  

(5.24)

This allows us to rearrange the integral over $\omega$ in Eq. (5.17) to obtain for $\Phi = A + A^\dagger$ in the region $\rho > 0$: 

15
\[ \Phi = \int_0^\infty d\omega \ e^{-i\omega \tau} \int \frac{d\vec{k} \ e^{i\vec{k} \cdot \vec{x}}}{\sqrt{2(2\pi)^3}} K(\omega, \frac{1}{2} \mu_0, \frac{1}{2} \mu_0) \left( a_2(\vec{k}, \omega) + e^{-\pi \omega} a_2^\dagger(-\vec{k}, -\omega) \right) + \text{h.c.} \] (5.25)

In the opposite quadrant of Rindler space, where \( \varrho < 0 \), we have:

\[ \Phi = \int_0^\infty d\omega \ e^{-i\omega \tau} \int \frac{d\vec{k} \ e^{i\vec{k} \cdot \vec{x}}}{\sqrt{2(2\pi)^3}} K(\omega, \frac{1}{2} \mu_0, \frac{1}{2} \mu_0) \left( a_2(\vec{k}, \omega) + e^{+\pi \omega} a_2^\dagger(-\vec{k}, -\omega) \right) + \text{h.c.} \] (5.26)

At this point it is opportune to define the new creation and annihilation operators \( a_I, a_{II}, a_I^\dagger, a_{II}^\dagger \), applying the following Bogolyubov transformation, when \( \omega > 0 \):

\[
\begin{pmatrix}
    a_I(\vec{k}, \omega) \\
    a_{II}(\vec{k}, \omega) \\
    a_I^\dagger(-\vec{k}, \omega) \\
    a_{II}^\dagger(-\vec{k}, \omega)
\end{pmatrix} = \frac{1}{\sqrt{1 - e^{-2\pi \omega}}} \begin{pmatrix}
    1 & 0 & 0 & e^{-\pi \omega} \\
    0 & 1 & e^{-\pi \omega} & 0 \\
    0 & e^{-\pi \omega} & 1 & 0 \\
    e^{-\pi \omega} & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    a_2(\vec{k}, \omega) \\
    a_2^\dagger(\vec{k}, -\omega) \\
    a_2^\dagger(-\vec{k}, \omega) \\
    a_2^\dagger(-\vec{k}, -\omega)
\end{pmatrix} .
\] (5.27)

This way, at \( \varrho > 0 \) the field \( \Phi \) only depends on \( a_I \) and \( a_I^\dagger \), and at \( \varrho < 0 \) only on \( a_{II} \) and \( a_{II}^\dagger \), whereas the normalization has been chosen again such that

\[
[a_I(\vec{k}, \omega), a_I^\dagger(\vec{k'}, \omega')] = [a_{II}(\vec{k}, \omega), a_{II}^\dagger(\vec{k'}, \omega')] = \delta(\omega - \omega') \delta^2(\vec{k}, \vec{k'}) ;
\]

\[
[a_I, a_{II}] = [a_I, a_{II}^\dagger] = 0 .
\] (5.28)

This Bogolyubov transformation does not directly affect the Rindler Hamiltonian, the latter being the generator of a boost in the \( \tau \) direction:

\[
\frac{\partial \Phi}{\partial \tau} = -i[\Phi, H_R] ,
\]

\[
-\imath \omega a_2(\vec{k}, \omega) = -i[a_2(\vec{k}, \omega), H_R] ,
\] (5.29)

where

\[
H_R = \int_{-\infty}^\infty d\omega \ \omega a_2^\dagger(\vec{k}, \omega) a_2(\vec{k}, \omega) = \int_0^\infty d\omega \ \omega \left( a_I^\dagger(\vec{k}, \omega) a_I(\vec{k}, \omega) - a_{II}^\dagger(\vec{k}, \omega) a_{II}(\vec{k}, \omega) \right) = H_R^I - H_R^{II} .
\] (5.30)

Thus we observe that Hilbert space is separable into two factor spaces: \( \mathcal{H} = \mathcal{H}_I \otimes \mathcal{H}_{II} \). The space \( \mathcal{H}_I \) is described by the Hamiltonian \( H_R^I \) and \( \mathcal{H}_{II} \) is described by the Hamiltonian \( -H_R^{II} \). One may also verify that, apart from a possible additive contribution of the vacuum, if

\[
H_M = \int d^3X \mathcal{H}_M \left( \Phi(0, \mathbf{X}), \tilde{\Phi}(0, \mathbf{X}), \dot{\Phi}(0, \mathbf{X}) \right) ,
\] (5.31)
then
\[ H^I_R = \int_{\varrho > 0} d^3 \mathbf{X} \varrho \mathcal{H}_M ; \quad H^{II}_R = \int_{\varrho < 0} d^3 \mathbf{X} |\varrho| \mathcal{H}_M . \] (5.32)

Consequently, all observables made of fields in quadrant \( \text{II} \) commute with \( H^I_R \) and \textit{vice versa.}

The Rindler- or Boulware vacuum state \(|0,0\rangle\) is defined by
\[ a_I|0,0\rangle = a_{II}|0,0\rangle = 0 \] (5.33)
This is not the same as the vacuum experienced by a freely falling observer, who is said to experience the Minkowski- or Hawking vacuum \(|\Omega\rangle\), which obeys
\[ A(T, \mathbf{X})|\Omega\rangle = a_2(\tilde{\mathbf{k}}, \omega)|\Omega\rangle = 0 . \] (5.34)

It is not difficult to express this state in terms of the basis generated by \( a_I \) and \( a_{II} \):
\[ a_I(\tilde{\mathbf{k}}, \omega)|\Omega\rangle = e^{-\pi \omega} a_{II}^\dagger (-\tilde{\mathbf{k}}, \omega)|\Omega\rangle , \]
\[ a_{II}(\tilde{\mathbf{k}}, \omega)|\Omega\rangle = e^{-\pi \omega} a_I^\dagger (-\tilde{\mathbf{k}}, \omega)|\Omega\rangle , \] (5.35)
so that
\[ |\Omega\rangle = \prod_{\tilde{\mathbf{k}}, \omega} \sqrt{1 - e^{-2\pi \omega}} \sum_{n=0}^{\infty} |n\rangle_I |n\rangle_{II} e^{-\pi n \omega} , \] (5.36)
where the square root is added for normalization.

Notice that
\[ H_R|\Omega\rangle = (H^I_R - H^{II}_R)|\Omega\rangle = 0 , \] (5.37)
which confirms that \(|\Omega\rangle\) is Lorentz invariant; remember that \( H_R \) is the generator of Lorentz boosts.

If one does not have the means to observe any features at \( \varrho < 0 \) this implies that one only has at one’s disposal operators \( \mathcal{O} \) composed of the fields in region \( \text{I} \), that is, the operators \( a_I \) and \( a_I^\dagger \). These act only in the factor space \( \mathcal{H}_I \) but proportional to the identity operator in \( \mathcal{H}_{II} \):
\[ \mathcal{O}(|\psi\rangle_I|\psi'\rangle_{II}) = |\psi'\rangle_{II} \left( \mathcal{O}|\psi\rangle_I \right) . \] (5.38)

Let us limit ourselves momentarily to a single point \((\tilde{\mathbf{k}}, \omega)\). There the expectation value for such an operator in the state \(|\Omega\rangle\) is
\[ \langle \Omega|\mathcal{O}|\Omega\rangle = (1 - e^{-2\pi \omega}) \sum_{n_1, n_2} |n_1\rangle_I |n_1\rangle_I \langle n_2|_{II} \langle n_2|_{II} e^{-\pi \omega(n_1 + n_2)} \]
\[ = \sum_{n \geq 0} \langle n|\mathcal{O}|n\rangle_I e^{-2\pi n \omega} (1 - e^{-2\pi \omega}) , \]
\[ = \text{Tr}(\mathcal{O} g_{\Omega}) , \] (5.39)
where $\varrho_{\Omega}$ is the density matrix $Ce^{-\beta H}$ corresponding to a thermal state at the temperature $T = 1/2\pi$. Note that in Rindler space time, energy and temperature are dimensionless. If we scale with the appropriate factor $4M$ as in Eq. (2.9) we find the Hawking temperature,

$$T_H = 1/8\pi M = 1/8\pi Gm_{BH}. \quad (5.40)$$

It should be emphasized here that the above is a rather straightforward calculation of a result that can also be obtained in a number of other, different ways. Several of these alternative derivations are more elegant and direct, for instance one may observe that the factors $e^{\pm i\pi \omega}$ arise from a rotation over $\pi$ in Euclidean space, and that the thermal nature of the result is linked to the fact that the Euclidean extension of the Schwarzschild solution is periodic in $it$ with period $8\pi M$. The derivation given in this section was chosen because of the directness of the physical arguments, which are independent of the Euclidean properties of the solutions.

6. HAWKING RADIATION

S. Hawking\(^3\) was the first to arrive at an apparently inevitable conclusion, drawn by starting from the results sketched in Sect. 5: that a black hole will emit black body radiation, precisely corresponding to the temperature (5.40). One immediately concludes that there cannot be a lower bound to the black hole mass much higher than the Planck mass, since, once it is placed in a high enough vacuum, the black hole will loose energy:

$$\frac{dM}{dt} = C(T)T^4 R^2 \approx - \frac{C'}{M^2}; \quad M(t) \approx C''(t_0 - t)^{1/3}. \quad (6.1)$$

Here, $C$, $C'$, etc., are all constants of order one in natural units. Furthermore, if $S$-matrix theory would apply to them, the black hole instability would shift the pole $s_j$ generated by black hole of type $j$, in the $S$-matrix, into the complex plane:

$$s_j \approx m_{j,BH}^2 - iC_j M_{Pl}^2, \quad (6.2)$$

where $C_j$ is of order one in natural units.

Numerous authors investigated this chain of arguments and found that they generally agree. Indeed, suspicions that there could be deviations from this result, involving non-thermal fluctuations, or even a different temperature\(^8\) are most often not taken very seriously. Caution however is called for. We must underline that here we are dealing with a purely theoretical prediction which, whether we like it or not, is based on assumptions that cannot all be verified directly, plausible as they may seem. Black holes emitting
quantum radiation have never been observed experimentally, and indeed it is conceivable that either Quantum Mechanics or General Relativity, or both, might break down precisely at the horizon, regardless how large the horizon is. Such fears are not totally unfounded, as we will argue later in this paper, but it is very illuminating first to derive what the consequences of the conventional theories would be. Let us therefore begin with rederiving Hawking's result as precisely as we can, stating what assumptions were made.

Consider the black hole just formed out of imploding matter. The model pictured in Fig. 2b is quite suitable for our considerations, and it will not be difficult to convince oneself that the details of the imploding matter are totally irrelevant for the arguments. As in the previous section, we consider quantized fields in this space-time in a Heisenberg picture (which means that the states $\psi$ are time-independent and that the operators referring to the various observers are defined as a function of their location $x$ in this space-time). Take an operator-valued field $\Phi(x)$ interacting no other way than gravitationally (later we can relax also this condition). If we are interested in observations of particles leaving the black hole at late time $t$, this means that we are interested in modes that propagate along lines such as the line $L$ in Fig. 2b. These modes are rapidly oscillating as functions of the coordinate $y$ but practically constant as functions of $x$. Now consider these modes as they cross a Cauchy surface near the point labeled $S$ in Fig. 2b. At this point we assume that the energetic modes that we are interested in are not populated by any particle, or, the associated annihilation operators $a(k)$ are assumed to yield zero when acting on the vacuum (for short: “annihilate the vacuum”). Because the field $\Phi(x)$ obeys field equations, we can also consider some later Cauchy surface at time $t$, near the letter $L$ in Fig. 2b. We can now calculate how the operators $a(k)$ act as experienced by an observer there. To do this, we may safely assume that also the observer moving in along the curve $A$ in Fig. 2b sees no super-energetic particles along the outgoing curve $L$. Thus we can ignore any of the effects of the imploding material that created the black hole itself, and concentrate on the metric of an eternal black hole, Fig. 1, where the same curves $A$ and $L$ may be considered (see Fig. 1).

Since the wave fronts that we wish to study are tightly compressed, it is permitted to concentrate on a tiny region of the horizon, where the Minkowski coordinates $T, X, Y, Z$ are appropriate. The Hamiltonian $H_M$ that serves as the generator of boosts in the time variable $T$ can be written as

$$H_M = \int \mathcal{H}(X) d^3X.$$ (6.3)

Now for the late observer watching the radiation going out along curve $L$ the relevant coordinates are $\varrho$ for space and $\tau$ for time, where $\varrho$ and $\tau$ are defined as in Eq. (4.2). Since all our considerations are now in the upper part of space-time we can use the exact
Schwarzschild metric (2.1) or its Kruskal notation (2.4). A boost in the new time coordinate \( \tau \) is generated by the \textit{Rindler} Hamiltonian \( H_R \), defined as in Eq. (5.32). This, apart from the factor \( 4M \), is the time coordinate used by the late observer. What the results of the previous section imply, is that if the observer near the point \( S \) observes few if any ultra-energetic particles, we can say that the state we are dealing with is the Hawking vacuum \(|\Omega\rangle\). The field operators \( \Phi \) in the matter-free region outside the horizon still act exactly the same way on this state as they do in the Rindler space treatment of Sect. 5. If we now perform the general coordinate transformation to the Rindler variables, and from there to the familiar Schwarzschild coordinates \( t, r, \theta, \phi \), we find that the appropriate field variables there consist of the creation and annihilation operators \( a_I \) and \( a_I^\dagger \). The state \(|\Omega\rangle\) becomes the density matrix state \( \varrho_{\Omega} \) derived from Eq. (5.39).

This argument is highly independent of the way the black hole was formed. In case the collapse took place in a less symmetric way, or at various steps and intervals, one still finds that an observer falling in the black hole should observe the Hawking vacuum state there, and this necessarily leads to the density matrix \( \varrho_{\Omega} \). In particular, one could assume that the collapsing matter was in a \textit{pure} quantum state, and even in that case, the outgoing radiation appears to be mixed according to the matrix \( \varrho_{\Omega} \). The question to be asked is how literally this result is to be taken. One could conclude

\begin{enumerate}
  \item that black holes must be fundamentally different from other objects in nature. They do not obey a single Schrödinger equation (which after all would allow pure states to evolve only into pure states), but in stead obey probabilistic equations of motion that are not purely quantum mechanical\textsuperscript{4, 22}.
\end{enumerate}

According to this view, a more basic theory at the Planck scale would show no quantum mechanical features of the familiar kind. Scenarios where this \textit{does} happen have been proposed, but the present author is skeptical towards such proposals. The reason is not that they would be logically impossible (although difficulties with energy conservation were claimed\textsuperscript{23}), but rather that unitarity violation would be practically impossible to build in a comprehensive theory of Planckian evolution such a way that Quantum Mechanics at the atomic scale would result naturally. Existing scenarios show a complete lack of determinism (even in the sense of a Schrödinger equation) at the Planck scale and hence do not allow us even in principle to make any sound physical prediction for quantities such as the fine structure constant. Instead, perhaps,

\begin{enumerate}[resume]
  \item black holes do obey a Schrödinger equation, but this equation requires knowledge of all inaccessible observables behind the horizon, so that a black hole forms an infinitely degenerate state. In this case the black hole can never decay completely\textsuperscript{7}, but it decays into stable, infinitely degenerate, final states with masses of the order of the
Planck mass, called *remnants*. Alternatively, however, one may suspect that

The density matrix derivation depended on certain hidden assumptions of a statistical nature, such that the answer may be correct in a statistical sense, but more precise treatments may yield a purely quantum mechanical description of a black hole that nevertheless has only a finite degeneracy. This is the scattering matrix assumption, which we will further investigate from section 11 onwards.

Thus, one expects the system as a whole to react just as any other physical system does: when it absorbs infalling material, or even just infalling radiation, it should react some way or other, and enter into a state that is orthogonal to what it would have evolved into if the infalling material had been in a different mode or totally absent. This is just the experimentally observed fact that all known evolution laws in small-scale physics can be written in terms of a *unitary* evolution matrix. It is hard to understand how the world at the scale of ordinary atomic and elementary particle physics could behave quantum mechanically and evolve in a unitary way, if quantum mechanics were not at the basis of the laws of dynamics at the smallest distance scales.

In appears that the derivation of the density matrix $\varrho_\Omega$ in Eq. (5.39) cannot be exactly right, since it implies that infalling material of whatever variety should not affect the outgoing radiation at all (linearized quantum field theory was used). This would violate unitarity.

There are (at least) three different reasons to expect that the density matrix $\varrho_\Omega$ has to be replaced by a pure state. First, as already explained, the result of the computations that provided the density matrix $\varrho_\Omega$ is difficult to accept at all levels of rigor.

Secondly, there is the back reaction: of course outgoing radiation *does* depend on ingoing material. After all, the black hole mass and angular momentum, and hence also the surrounding space-time metric, are affected by whatever went in, and therefore also the location of the horizon, the temperature and other observable features. However, taking into account back reaction, is *not* sufficient to resolve the unitarity problem. Even if we keep track of *everything* that went in, the outgoing radiation nevertheless appears to be quantum mechanically mixed. Our point, however, is that the Rindler space picture may be more fundamentally flawed: the density matrix that was derived requires states in the original Minkowski space that feature particles with tremendous energies (with respect to the Kruskal coordinates), both ingoing and outgoing. These particles should interact extremely strongly gravitationally; it would be incorrect to ignore this super strong interaction, as we will see in section 10.

Thirdly, we claim that the derivation depends on assumptions that need not be true.
In Eq. (5.39) it was tacitly assumed that the expectation value of an operator $O$ should be calculated by averaging over all unseen modes $|n_2\rangle$. In statistical physics, of course, this is the standard procedure, so much so that its applicability is nearly never questioned. But is it the right way to treat information that disappeared across a gravitational horizon? If we take the laws of General Relativity literally the answer is yes. In an accelerated frame, we also cannot see what went through the acceleration horizon, and there the rule would definitely be to average over all possible unseen states. Yet it could be that General Relativity in this respect has to be modified near a horizon. One proposal was made by this author\textsuperscript{8} in 1984: it could be that the double Hilbert space $\mathcal{H}_I \otimes \mathcal{H}_{II}$, mentioned following Eq. (5.30), itself has to be regarded as a space of density matrix elements. This might look like a more economical and perhaps more natural rule whenever a gravitational field produces a horizon. The resulting theory differs from the standard one by a factor two in the temperature $T_H$.

A theory such as General Relativity could be interpreted as essentially consisting of the following ingredients: different physical situations are compared (for instance a region of space-time with a gravitational field and an other region without gravitational field). Then a transformation law is formulated enabling us to calculate what happens in one system (the one with the gravitational field), if we know how to calculate what happens in the other region (the one without gravitational field). In the conventional theory, the transformation involved here is nothing but a coordinate transformation. Quantum mechanical Hilbert space is assumed not to be affected. Now we claim that in the presence of a horizon the transformation rule could be more complicated. For instance, a state $|\psi\rangle$ in flat space could be transformed into a density matrix $\rho$ near the horizon (this then would give a factor two difference in $T_H$).

Unfortunately, this particular proposal does not resolve the unitarity problem, and implementing it in a more comprehensive theory for black holes turned out to be even harder than for the conventional view. We were unable however to rule it out completely, and all we can do at present is to take this heretic theory as a warning sign that not all may seem to be as one might expect. In principle, the transformation law could be even more intricate than the simple theories just mentioned. In this paper, we will leave this question open. At some places, we speculate on Hawking’s original formalism, which is by far the most likely to be correct. It could be, however, that the transformation from Minkowski space to Rindler space is not a one-to-one mapping in terms of states in Hilbert space, because of the divergence due to infinitely energetic particles.
The fact that the radiation emitted, as described by Eq. (5.40), is thermal, opens up the possibility to approach this phenomenon from a thermodynamical point of view. Taking $m_{BH}$ to be the energy and $T = T_H$ the temperature, one readily derives the entropy $S$:

$$TdS = dm_{BH}; \quad dS = 8\pi Gm_{BH}dm_{BH}; \quad S = 4\pi Gm_{BH}^2 + C,$$  \hspace{1cm} (7.1)

where $C$ is an unknown integration constant, to be referred to as the "entropy normalization constant". *

For the general Kerr-Newman solution (see sect. 2 and 11, 12), where $J = Ma$ is the angular momentum, $Q$ the electric charge in convenient units, $r_+$ the radius of the horizon, $\Phi$ the electric potential on the horizon, and $\Omega$ the angular velocity at the horizon, these expressions generalize into:

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}; \quad \phi = \frac{Qr_+}{r_+^2 + a^2}; \quad \Omega = \frac{a}{r_+^2 + a^2};$$  \hspace{1cm} (7.2a)

$$T_H = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}; \quad G \frac{dT}{dS} = dM - \Omega dJ - \phi dQ; \quad G S = \pi(r_+^2 + a^2) + GC. \quad (7.3a, b)$$

It is indeed not difficult to check that Eq. (7.3b) solves the differential equation (7.3a). The expressions (7.2) for the Kerr-Newman solution will not further be derived here.

However, deriving the value of the entropy normalization constant $C$ from conventional physical arguments is a great challenge, and it has not yet been accomplished. In fact, not only is $C$ still unknown, but we also have no idea where in the range between, say, $10^{-40}$ and $10^{+40}$, it might be; indeed, one may even dispute whether it is finite and well-defined at all. One could also suspect that $C$ depends on properties of the black hole (other than $Q$ or $a$) that are determined by its history ($Q$ and $a$ dependence are excluded by Eq. (7.3a)), but we consider such speculations as unlikely.

It is important to note that the expression obtained for the entropy $S$, apart from the integration constant, is always equal to $\frac{1}{4}A/G$, where $A$ is the area of the horizon, a finding that will be very much at the center of our discussions. For dilaton black holes, which will not be discussed here, the situation may be more complicated.

* There may well be subdominant $m_{BH}$-dependent terms needed in Eq. (7.1), which would imply that $C$ is not truly constant.
We would like to compare the black hole with an inflatable balloon filled with gas molecules. Through a small opening, molecules may enter or leave. When it is large, and contains many molecules, the statistical, thermodynamic treatment is appropriate. But when it is small, with only one or two molecules left, it should be more accurate to solve the Schrödinger equation. Formally, a treatment in terms of pure quantum states should always be possible. The black hole should only differ from the toy balloon by the fact that its quantum states seem to be labeled by spinlike variables, separated by approximately the Planck length, on the horizon.

Let us return to the pure Schwarzschild black hole. Connecting the entropy to the density of quantum mechanical states\(^{15}\), must be done with considerable care, since there will be two kinds of divergences: at the horizon and at spacelike infinity. In fact, one may very well question the mere existence of such quantum levels. This, however, is the key assumption of this paper: not only is the quantum mechanics of black holes meaningful, it can also be derived, and the constant \(C\) in Eqs.(7.1) and (7.3b) is finite and of order one (apart from subdominant terms). In order to enable us to judge the relation between the entropy just derived, and the density of quantum states, we now present a direct argument concerning the density of states, an argument that will also show that any infinities at the horizon must be absorbed in \(C\), but the “infrared” infinities arising from spacelike infinity should be excluded; the latter represent the radiation field far from the black hole.

The spectral density of a black hole can be derived from its Hawking temperature by applying time reversal invariance\(^9\). We have to our disposal both the emission rate (the Hawking radiation intensity), and the capture probability, or the effective cross section of the black hole for infalling matter.

The cross section \(\sigma\) is approximately:

\[
\sigma = 2\pi r_+^2 = 8\pi M^2, \tag{7.4}
\]

and slightly more for objects moving in slowly. The emission probability \(W dt\) for a given particle type, in a given quantum state, in a large volume \(V = L^3\) is:

\[
W dt = \frac{\sigma(k)v}{V} e^{-E/T} dt, \tag{7.5}
\]

where \(k\) is the wave number characterizing the quantum state, \(v\) is the particle velocity, and \(E\) is its momentum.

Now we assume that the process is also governed by a Schrödinger equation. This means that there exist quantum mechanical transition amplitudes,

\[
T_{in} = \langle M + GE | M\rangle_{BH} | M\rangle_{BH},
\]

and

\[
T_{out} = \langle M |_{out} (E) | M + GE\rangle_{BH}, \tag{7.6}
\]
where the states $|M\rangle_{BH}$ represent black hole states with mass $M/G$, and the other states are energy eigenstates of particles in the volume $V$. In terms of these amplitudes, using the so-called Fermi Golden Rule, the cross section and the emission probabilities can be written as

\[
\sigma = |T_{\text{in}}|^2 \frac{\varrho(M + GE)}{v}, \tag{7.7}
\]
\[
W = |T_{\text{out}}|^2 \frac{\varrho(M)}{V}. \tag{7.8}
\]

where $\varrho(M)$ stands for the level density of a black hole with mass $M$. The factor $v^{-1}$ in Eq. (7.7) is a kinematical factor, and the factor $V^{-1}$ in $W$ arises from the normalization of the wave function.

Now, time reversal invariance relates $T_{\text{in}}$ to $T_{\text{out}}$. To be precise, all one needs is $PCT$ invariance, since the parity transformation $P$ and charge conjugation $C$ have no effect on our calculation of $\sigma$. Dividing the expressions (7.7) and (7.8), and using (7.5), one finds:

\[
\frac{\varrho(M + GE)}{\varrho(M)} = e^{E/T} = e^{8\pi ME}. \tag{7.9}
\]

This is easy to integrate:

\[
\varrho(M) = e^{4\pi M^2/G + C} = e^S. \tag{7.10}
\]

Eq. (7.10) may be rewritten as

\[
\varrho(M) = 2^{A/A_0}, \tag{7.11}
\]

where $A$ is the horizon area and $A_0$ is a fundamental unit of area,

\[
A_0 = 4G \ln 2. \tag{7.12}
\]

This suggests a spin-like degree of freedom on all surface elements of size $A_0$.

Extension to the more general Kerr-Newman solutions is straightforward; one has to take into account that the Hawking emission intensity $e^{-E/T_H}$ is modified by chemical potential terms when the particles considered carry an electrical charge $q$ or angular momentum $j$ (in conveniently chosen units):

\[
W \propto e^{(-E + \Omega j + \phi Q)/T}. \tag{7.13}
\]

One again finds Eq. (7.3a), that is to be integrated to obtain (7.3b), where $S$ is the logarithm of the level density.

25
As stated earlier, the importance of this derivation is the fact that the expressions used as starting points are the actual Hawking emission rate and the actual black hole absorption cross section. This implies that, if in more detailed considerations divergences are found near the horizon, these divergences should not be used as arguments to adjust the relation between entropy and level density by large renormalization factors. Furthermore, the Golden Rule argument can be used only to deal with one emitted particle at the time. Hence, we should not take the outside volume $V$ so large that the dominant emission mode contains very many particles. Therefore, any divergences found when the outside volume is taken to infinity should be subtracted.

8. THE BRICK WALL MODEL

Attempts to compute the entropy normalization constant $C$ from field theory near black hole horizons are doomed to failure *(unless we would be content with the value infinity). This statement we derive from the following argument. A quantum field theory may easily exhibit global additive conservation laws such as baryon number conservation (or, more realistically, conservation of $B-L$, where $L$ is lepton number, but for simplicity, we will refer to the conserved quantity as baryon number). Such conservation laws are connected to global $U(1)$ symmetries via Noether’s theorem, in which case there is no long range gauge field coupled to this conserved current. Any quantum mechanical amplitudes derived within the framework of such a theory should display the same global symmetries. If $C$ is finite, however, our amplitudes $T$ of Eq. (7.6) cannot possibly share the symmetry. Since the cross section $\sigma$, Eq. (7.4), for baryon capture is fixed, it should be possible for the black hole to capture unlimited amounts of baryons, whereas the number of baryons and antibaryons emitted in the Hawking process should always average out to be equal, since there will be no chemical potential term (as in Eq. (7.13)) favoring one above the other. Thus, on the one hand, one can increase the baryon number of a black hole indefinitely, without increasing its mass. If, on the other hand, one has only a finite number of quantum states below a given mass value, one has a conflict unless baryon number is ill-defined for these states, which means that the symmetry is broken. Such a symmetry breaking can only come from physical mechanisms that were not part of the given quantum field theory.

* One important exception is the result obtained by Carlip, who derived the black hole entropy in 2+1 dimensions, in a theory with a cosmological constant. There are two reasons why this is an exceptional case: one is that in the limit where the cosmological constant is small his black hole is either very large or very heavy. His universe can only contain one black hole at most; second: there are no other particles in his model, the degrees of freedom are purely topological and gravitational. Nevertheless, the result is of great interest.
We would now like to see explicitly what goes wrong if one attempts to do the calculation anyway. The answer is that quantum field theory does not give to the black hole a discrete spectrum, but gives it a continuous spectrum of states, \( i.e., C = \infty \). Since we clearly have a discrete spectrum of particles in the GeV range, this result would imply that the continuum should begin somewhere, presumably at a mass value comparable to the Planck mass. This “lightest black hole” would not be able to decay, since this would require a transition from a continuous spectrum into a discrete spectrum, which in general is impossible quantummechanically. In short, one would predict the existence of so-called black hole remnants.

Remnants, which would form an infinite set of different species, all at about the same mass value, would as such disobey conventional rules of quantum statistics, and thermodynamics would not apply to them. We will not be able to prove that this is unacceptable, but, in our opinion, it is unlikely that such behavior can be accommodated in an airtight quantum theory. More to the point, however, is the remark that Quantum Field Theory in a black hole background is not more than an approximation method, in which gravitational back reaction was neglected. In a four-dimensional theory, this approximation may well be not good enough to conclude anything at all about the spectrum.

In this Section, we now present a model in which only low energy quantum fluctuations of the fields are taken into account. We apply quantum field theory op to some point \( r_1 \) close to the horizon: \( r_1 = r_+ + h, \ h > 0 \). For simplicity we only consider scalar fields \( \Phi_i(r, \theta, \phi, t) \), whose only interaction is the gravitational one with the metric; generalization towards spinor, vector or even perturbative gravitational field excitations will be straightforward. To simplify things, we just represent all those by giving the fields \( \Phi_i \) a multiplicity \( N \), so \( i = 1 \ldots, N \). At \( r = r_1 \) we impose a boundary condition:

\[
\Phi_i(r, \theta, \phi, t) = 0 \quad \text{if} \quad r \leq r_1. \quad (8.1)
\]

The quanta of the fields will be given a temperature \( T = T_H \). The question one may ask is: which value should one assign to the cutoff parameter \( h \), such that the entropy of this system precisely takes the value \( (7.1) \), so that the density of quantum states corresponds to \( (7.10) \)? We will need an infrared cutoff in the form of a box with radius \( L \):

\[
\Phi_i(r, \theta, \phi, t) = 0 \quad \text{if} \quad r \geq L. \quad (8.2)
\]

To determine the thermodynamic properties of this system, we first compute the energy levels \( E(n, \ell, \ell_3) \) of the bosons \( \Phi_i \). The Lagrange density \( \mathcal{L} \) in the metric (2.1) is

\[
\mathcal{L}(x,t) = \left(1 - \frac{2M}{r}\right)^{-1} \partial_t \Phi_i^2 - \left(1 - \frac{2M}{r}\right) \partial_r \Phi_i^2 - r^{-2} \partial_\Omega \Phi_i^2. \quad (8.3)
\]
The total Lagrangian is \( \int_{r_1}^{L} dr \int d\Omega r^2 \mathcal{L}(r, \Omega, t) \), so that the field equation for modes with angular momentum quantum numbers \( \ell \), \( \ell_3 \) and energy \( E \) reads:

\[
\left( 1 - \frac{2M}{r} \right)^{-1} E^2 \Phi + \frac{1}{r^2} \partial_r r (r - 2M) \partial_r \Phi - \left( \frac{\ell(\ell + 1)}{r^2} + m^2 \right) \Phi = 0. \tag{8.4}
\]

To enable us here to apply a WKB approximation, we first must smoothen the singularity of the second term in (8.4) at \( r = 2M \), since this singularity is too close to the cutoff point \( r - r_1 \). Therefore, temporarily write

\[
r - 2M = e^\sigma, \tag{8.5}
\]

so that

\[
\left[ rE^2 + \frac{1}{r^2} \partial_\sigma r \partial_\sigma - e^\sigma \left( \frac{\ell(\ell + 1)}{r^2} + m^2 \right) \right] \Phi = 0 \quad (r = 2M + e^\sigma). \tag{8.6}
\]

In terms of the \( \sigma \) coordinate we see that the oscillatory behavior of \( \Phi \) is well approximated by

\[
e^{\pm i} \int \kappa(\sigma) d\sigma = e^{\pm i} \int k(r) dr, \tag{8.7}
\]

where \( \kappa(\sigma) \) is determined by

\[
rE^2 - \frac{1}{r} \kappa(\sigma)^2 - e^\sigma \left( \frac{\ell(\ell + 1)}{r^2} + m^2 \right) = 0, \tag{8.8}
\]

so that

\[
k(r)^2 = e^{-2\sigma} \kappa(\sigma)^2 = \left( 1 - \frac{2M}{r} \right)^{-2} E^2 - \left( 1 - \frac{2M}{r} \right)^{-1} \left( \frac{\ell(\ell + 1)}{r^2} + m^2 \right), \tag{8.9}
\]

as long as the right hand side of this equation is positive; let us take \( k(r) = 0 \) as soon as the right hand side is negative.

The energy spectrum of Eq. (8.6) is now given by

\[
\pi n = \int_{r_1}^{L} dr \, k(r, \ell, E), \tag{8.10}
\]

where the quantum numbers \( n > 0 \), \( \ell \) and \( \ell_3 = -\ell, \ldots, \ell \) are integers. The total number \( \nu \) of wave solutions with energy not exceeding \( E \) is then given by

\[
\pi \nu = \int (2\ell + 1) d\ell \pi n \overset{\text{def}}{=} g(E)
\]

\[
= \int_{r_1}^{L} dr \left( 1 - \frac{2M}{r} \right)^{-1} \int (2\ell + 1) d\ell \sqrt{E^2 - \left( 1 - \frac{2M}{r} \right) \left( m^2 + \frac{\ell(\ell + 1)}{r^2} \right)}, \tag{8.11}
\]

28
where the $\ell$-integration goes over those values of $\ell$ for which the argument of the square root is positive.

To determine the thermodynamic properties of the system, we proceed with second quantization. Every energy level determined above may be occupied by any non-negative number of quanta. The free energy $F$ at an inverse temperature $\beta$ is given by

$$e^{-\beta F} = \sum e^{-\beta E} = \prod_{n, \ell, \ell_3} \frac{1}{1 - e^{-\beta E(n, \ell, \ell_3)}}.$$  \hspace{1cm} (8.12)

From this one computes

$$\beta F = N \sum_\nu \log \left( 1 - e^{-\beta E} \right);$$  \hspace{1cm} (8.13)

$$\pi \beta F = N \int d g(E) \log \left( 1 - e^{-\beta F} \right) = -N \int_0^\infty d E \frac{\beta g(E)}{e^{\beta E} - 1}$$

$$= -\beta N \int_0^\infty d E \int_{r_1}^L d r \left( 1 - \frac{2M}{r} \right)^{-1} \int (2\ell + 1) d \ell$$

$$\times \left( e^{\beta E} - 1 \right)^{-1} \sqrt{E^2 - \left( 1 - \frac{2M}{r} \right) \left( m^2 + \frac{\ell(\ell + 1)}{r^2} \right)} \quad (r_1 = 2M + h).$$  \hspace{1cm} (8.14)

Again the integral is taken only over those values for which the square root exists. $N$ is the multiplicity of the fields $\Phi_i$. In the approximation

$$m^2 \ll 2M/\beta^2 h, \quad L \gg 2M,$$  \hspace{1cm} (8.15)

the main contributions to this integral are found to be

$$F \approx -\frac{2\pi^3 N}{45h} \left( \frac{2M}{\beta} \right)^4 - \frac{2}{9\pi} L^3 N \int_m^\infty \frac{d E (E^2 - m^2)^{3/2}}{e^{\beta E} - 1}.$$  \hspace{1cm} (8.16)

The second part is the usual contribution from the vacuum surrounding the black hole at great distances, and as argued before, should be discarded. The first part is an intrinsic contribution from the horizon, and it is seen to diverge linearly as $h \downarrow 0$.

The contribution of the horizon to the total energy $U$ and the entropy $S$ are

$$U = \frac{\partial}{\partial \beta} (\beta F) = \frac{2\pi^3}{15h} \left( \frac{2M}{\beta} \right)^4 N,$$  \hspace{1cm} (8.17)

$$S = \beta (U - F) = \frac{8\pi^3}{45h} 2M \left( \frac{2M}{\beta} \right)^3 N.$$  \hspace{1cm} (8.18)

When this is adjusted to the Hawking value, Eq. (7.1), with $\beta = 1/T_H = 8\pi M$, we find that the cutoff parameter $h$ must be chosen to be

$$h = \frac{NG}{720\pi M}.$$  \hspace{1cm} (8.19)
The total energy $U$ of the thermally excited particles is given by

$$GU = \frac{3}{8}M,$$  \hspace{1cm} (8.20)

independently of $N$. Alternatively, one could have tuned the energy $U$ to be equal to $m_{\text{BH}}$, which would yield the same order of magnitude for $h$, but adjusting the physical degrees of freedom, \textit{i.e.}, the entropy $S$, appears to us more sensible. Clearly, it makes little sense to allow $h \to 0$, since then both the entropy and the energy would diverge.

We refer to the cutoff near the horizon as a “brick wall”. The physical distance between the brick wall and the horizon is

$$\int_{r=2M}^{r=2M+h} ds = \int_{2M}^{2M+h} \frac{dr}{\sqrt{1-2M/r}} = 2\sqrt{2Mh} = \sqrt{\frac{NG}{90\pi}},$$  \hspace{1cm} (8.21)

which is independent of the mass $m_{\text{BH}}$ of the black hole. The brick wall should be a property of any horizon of arbitrary size. If $N$ is not too large, the brick wall thickness is of the order of the Planck length.

The brick wall model, with the values of $\beta$ and $h$ fixed according to Eqs. (5.40) and (7.1), actually reproduces the thermodynamic properties of a black hole quite nicely, and could have served as a realistic model for a black hole that fully obeys Schrödinger’s equation and preserves quantum coherence, except for the fact that it also preserves all symmetries of the underlying quantum field theory; it could generate chemical potentials for the various globally conserved quantum numbers. Thus, not only the temperature must be constrained to keep the Hawking value, but also the chemical potentials are constrained to be zero. In principle, this is easy to realize, simply by introducing symmetry breaking effects in the brick wall boundary condition, but probably one would then be pushing this model too far; anyway, its most important deficiency is that we completely gave up invariance under general coordinate transformations near the horizon.

The most important lesson to be learned from the brick wall model is that Hawking radiation can indeed be seen to be compatible with quantum mechanical purity, if only one could introduce a cutoff at the Planck scale.

9. QUANTUM COHERENCE

The field equation (8.6) for non-interacting scalar fields in the Schwarzschild metric shows an important fact: \textit{in terms of the coordinates $\sigma$ and $t$ the solutions form plane waves as $\sigma \to -\infty$}, or, the region near the horizon is not compact. If the brick wall were absent, the entropy of the quanta in this region would be unbounded. But it is
equally evident that, with or without brick wall, it would be incorrect to ignore the self-interactions. At $\sigma \to -\infty$, the plane waves enter an infinitely inflated region of space-time. The effective Newton constant $G$ becomes infinitely strong here, and therefore any decomposition of Hilbert space in terms of mutually non-interacting field quanta will be hopelessly inadequate in this region. Indeed, one possible physical interpretation of the brick wall model could be that the excessively strong gravitational self interactions among the field quanta effectively form a barrier at $r < 2M + h$, crudely represented here as a brick wall.

In the derivation of the Hawking spectrum, these gravitational self-interactions were ignored. Now this could be partly justified. The general coordinate transformation which yielded this result has been performed in the region where the outgoing particles were already fairly soft, so that indeed ignoring the gravitational self-interactions was justified, if our only aim had been to compute the intensity of this radiation. What is not justified, is ignoring the gravitational self-interactions when we compute the reaction of the quantum state of the outgoing matter when new particles are sent in.

Consider the Kruskal coordinates described in Section 2. Let an ingoing particle be at the position $r \approx 2.56 \, M$ at time $t_0 = 0$. From Eqs (2.3) we read off that at this point $xy \approx 1$ and $x/y = 1$, and the particle follows the line $x \approx 1$. Let an outgoing particle arrive at $r \approx 2.56 \, M$, $xy = 1$ at time $t = t_1$. At that time, $x/y = e^{t_1/2M}$, and this particle has moved along the line $y = e^{-t_1/4M}$. The two particles have met each other at time $t = \frac{1}{2} t_1$, at the point $xy = e^{-t_1/4M}$, or

$$r = 2M + 2Me^{-t_1/4M-1}; \quad t = \frac{1}{2} t_1. \quad (9.1)$$

Apparently, if $t_1 \gg 4M$, this meeting point is far inside the trans-Planckian region, and it may not be generally correct to regard the outgoing particle as a Hawking particle that was not affected by the ingoing object.

Whether or not this is correct depends on how the outgoing particle is represented. The Rindler coordinate transformation yields a state for the outgoing particle that is a thermodynamic mixture of many different quantum states. It is this mixture which appears hardly to be affected by ingoing material. Our central assumption, however, is that this thermodynamically mixed state is merely a macroscopic description, comparable to the way one describes a vessel containing a gas consisting of a large number of molecules; but the same vessel also allows for a microscopic description, in terms of one quantum state only. Similarly, we expect a pure state description of the outgoing Hawking particles to be possible as well, but it is these individual states that will be strongly affected by the ingoing material.
Interactions between in- and outgoing material of course arise due to gauge theoretic forces as described in the Standard Model of the elementary particles, but by far the most important force here will be the gravitational one, whose strength increases without bound in the trans-Planckian region as $\sigma \to -\infty$. The main ingredient of this gravitational interaction may be conveniently described as a shift in either the $x$- or the $y$ coordinate. In Fig. 3, the shift is illustrated. Here, the $x$, $y$ coordinates are chosen such that the outgoing particle is far on its way out, so the exit time $t_1$ has been tuned to be close to zero. The ingoing particle has entered at time $t_0 = -\Lambda$, a large negative number. Consequently, the ingoing particle has been boosted to such high energies that it produces a non negligible gravitational field. The effect this field has on nearby geodesics is in first approximation a shift, as indicated, with

$$\delta y(\theta, \phi) \approx f(\theta, \phi, \theta', \phi') p_{\text{in}}(\theta', \phi'),$$

where $(\theta', \phi')$ is the angular point where the ingoing particle crosses the horizon, and $(\theta, \phi)$ is the intersection of the geodesic with the horizon. Here, $f$ is a Green function, to be further described in the next Section. $p_{\text{in}}$ is the momentum in terms of the locally flat coordinates $(X, Y, Z, T)$ of Eq. (2.6).

This effect opens the possibility to describe the response of a black hole to infalling matter in terms of a unitary scattering matrix. We refer to this theory as the $S$-matrix Ansatz. According to this theory the outgoing particles do not form a mixture of different quantum states, but just one state, which only appears to behave like a mixed state as soon as we neglect to hand in all existing information about the ingoing particles. Averaging over the various possible modes for the ingoing particles will give us back the mixed state of the outgoing objects, fully in agreement with the calculations of Section 5.
Naturally, an objection often heard is that if ingoing particles are considered that are either very energetic, or merely went in sufficiently long ago, the shift $\delta y$ will be so big that a particle originally on its way out (towards region $I$ in Fig. 3), will be sent back into the black hole, in region $III$. One could argue that we may start choosing from many different initial states (where the outgoing particles were different species, for instance), which would then all turn into the same state after the shift.

The way to avoid this objection is to assume that the dimensionality of the Hilbert space of all possible out-states is not that great at all. We could begin with an “ideal” black hole, for instance a perfectly spherical symmetric one, or else, more realistically, with one particular choice of initial state. This initial black hole produces Hawking particles \textit{in one quantum state only}. All other black hole states are then obtained from this one by adding or subtracting ingoing particles. Their effect on the outgoing state is that this state is deformed by shifts $\delta y$. We must insist that there can be no way to send particles into the hole, or even some arbitrary form of information, without effecting the outgoing particles by a shift somewhere along the horizon, or by some other detectable change.

As will be shown in subsequent chapters, one can construct a unitary scattering matrix along these lines, but a somewhat curious consequence of this scenario can already be seen here: if the gravitational shift were taken to be the \textit{only} possible interaction among particles, then there could not exist two states that are mutually independent (orthogonal) and yet identical to each other at all $y \geq \delta y$, where $\delta y$ can be arbitrarily large – if there were, then one could obliterate this information by sending an extra particle inwards, and arrive at a contradiction with unitarity. This situation is certainly quite different from any of the conventional descriptions of quantum field theory in terms of a Fock space. The existence of non-gravitational interactions, and more complicated gravitational ones as well, will give rise to complications in this simple constraint on the states (of outgoing particles), but since these other interactions may often be taken to be weak, we expect that indeed in the real world there are constraints of this remarkable nature.

Another objection often heard is the question of the quantum mechanical copying machine. Imagine some form of information being sent into the black hole. An observer in region $III$ (see Fig. 3) could pick up this information. But if, as our theory asserts, also an observer in region $I$, at very late times, can read off this information, since the ingoing and outgoing states are connected by a unitary $S$-matrix, then information concerning the quantum phases of wave packets can be extracted at two places, as if a photo-copying machine had reproduced it. According to quantum mechanics, this leads to conflicts as soon as interference experiments are considered. A quantum photo-copying machine cannot exist. One could add to this that the two observers under consideration seem to be spacelike
separated, hence the operators they have to their disposal appear to commute.

The answer\textsuperscript{25} to this “objection” is most clearly formulated in terms of Heisenberg operators (space-time dependent operators acting on states that do not depend on time). An observer can only move towards region \textit{III} if he does not encounter extremely energetic particles along the $x$-axis while he goes there. This implies that the state of outgoing particles must be chosen to be exactly the one corresponding to the given past history of the black hole. Now suppose that a late Hawking observer switches on his operator in order to measure something. If this operator has any effect at all on the states he is considering, he replaces the state of outgoing Hawking particles by another state. This other state differs from the previous one by having a given set of (or superposition of) particles now emerging from the black hole.

The first observer who needed to enter into region \textit{III} to do his observation, will be impeded in this act by the energetic particles now present on the $x$-axis. These particles are so energetic (in terms of the appropriate coordinates) that region \textit{III} is essentially blocked out for him. If he does try to enter anyway, he will find a world quite different from what he would have seen if the Hawking observer had not used his operator. All of this merely implies that the operators of observers in region \textit{III} and those of Hawking observers do not commute at all. Their commutators diverge exponentially, as the time difference increases.

The fact that operators that appear to commute in flat space-time, now cease to commute in terms of the Rindler variables implies that the Rindler coordinate transformation must be more complicated than usually considered. States naturally present in Rindler space (such as the ones containing arbitrary choices of Hawking particles or arbitrary choices of early ingoing particles), cannot be accepted in flat space because of the extreme gravitational curvature they would tend to produce. Conversely, in flat space, one would expect mutually orthogonal and independent modes in regions \textit{I} and \textit{III} which cannot be chosen independently as far as the Rindler observer is concerned. We return to this issue in Section 16.

The conclusion of this section is that assuming the survival of quantum mechanical purity in a black hole has far-reaching consequences for the nature of Hilbert space, even in flat space-time, and that the way the states transform in a transformation towards Rindler space must then be far more complex than the linearised procedures usually considered. This is actually what our theory has in common with string theory. String theories now also appear to favor the idea that black holes may form pure quantum mechanical states\textsuperscript{14}. 

34
10. **THE AICHELBURG-SEXL METRIC NEAR A BLACK HOLE**

The gravitational effect of an infalling particle in the Schwarzschild metric can be understood when we transform to a locally flat space-time, Eqs. (2.6). Consider the coordinate frames of Section 2. As Schwarzschild time $t$, or equivalently, Rindler time $\tau$, evolves, the infalling particle is Lorentz boosted, as we see in Eq. (2.8). In terms of the flat coordinates, therefore, the energy of the particles increases exponentially, and thus it quickly reaches values where gravitational effects can no longer be ignored. These effects are easy to calculate in the approximation that the source particle moves with the speed of light $c^{26,27}$.

First consider the case that the surrounding space-time is completely flat. In the rest frame we can approximate the metric as

$$ds^2 = dx^2 + \frac{2\mu}{r}dt^2 + \frac{2\mu}{r}dr^2, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

where $\mu = Gm$, and $m$ is the mass of the source particle. This we rewrite as

$$ds^2 = dx^2 + \frac{2\mu}{r}(u \cdot dx)^2 + \frac{2\mu}{r}dr^2, \quad r = \sqrt{x^2 + (u \cdot x)^2},$$

where

$$u = (0,0,0,i) ; \quad u^2 = -1.$$  \hspace{1cm} (10.2)

In these expressions, we have neglected all effects that are of higher order in the particle mass $\mu$, since we choose $\mu$ to be small. The particle’s Schwarzschild radius $r_+$ is very small, and the Lorentz boost to be considered next will only further reduce the particle’s dimensions.

The advantage of the notation chosen in Eq. (10.2) is of course that now the Lorentz boost is straightforward. In the boosted frame we can take

$$m u^\mu \Rightarrow (0,0,p,ip) = p^\mu , \quad Gp = \frac{\mu u}{\sqrt{1 - v^2/c^2}} \gg \mu .$$  \hspace{1cm} (10.4)

In the limit $\mu \Rightarrow 0, \ p$ fixed, one has $r \Rightarrow |x \cdot u|$. It will turn out to be useful to compare the metric then obtained with the flat space-time metric in two coordinate frames $y_\nu^{(\pm)}$, defined as

$$y_\nu^{(\pm)} = x_\nu \pm 2\mu u^\mu \log r .$$  \hspace{1cm} (10.5)

We have:

$$dy_\nu^{(\pm)} = dx_\nu \pm \frac{4\mu}{r}(u \cdot dx) dr - 4\mu^2 \frac{dr^2}{r^2} ;$$  \hspace{1cm} (10.6)
\[
\text{ds}^2 - \text{dy}^2 = \frac{2\mu}{r} \left[ r \mp (u \cdot x) \right]^2 + 4\mu^2 (d \log r)^2. \tag{10.7±}
\]

Now consider the limit (10.4). We keep \( p \) fixed but let \( \mu \) tend to zero. We now claim that when \( (p \cdot x) > 0 \), the metric \( \text{ds}^2 \) approaches the flat metric \( \text{dy}^2 \), whereas when \( (p \cdot x) < 0 \), we have \( \text{ds}^2 \Rightarrow \text{dy}^2 \), and at the plane defined by \( (p \cdot x) = 0 \) these two flat space-times are glued together according to

\[
y_\mu^{(+)} = y_\mu^{(-)} + 4\mu u^\mu \log |\tilde{x}|, \tag{10.8}
\]

where \( \tilde{x} = (X, Y, 0, 0) \), the transverse part of the coordinates \( y^\mu \).

Verifying the flatness of space-time away from the plane \( (p \cdot x) = 0 \), is easy, but to ascertain the connection formula (10.8), is a bit more delicate (if in Eq. (10.5), \( r \) were replaced by \(|(u \cdot x)|\), we would also obtain flat regions of space-time, but not the connection formula). The complication is that, though \( u^\mu \to \infty \), the inner product \( (u \cdot x) \) can be small. Our argument is as follows. First we note that the range of \( \log r \) in (10.7±) will diverge not worse than logarithmically in the limit, so when \( \mu^2 \to 0 \) we can ignore this last term. Next, given a number \( \lambda \), we divide space-time in three regions:

\[
A) \quad (u \cdot x) > \lambda; \\
B) \quad (u \cdot x) < -\lambda. \\
\text{and} \quad C) \quad |(u \cdot x)| \leq \lambda; \tag{10.9}
\]

In region (A), we use

\[
r - (u \cdot x) = \frac{x^2}{r + (u \cdot x)}, \tag{10.10}
\]

which is therefore bounded by \( x^2/\lambda \). Thus, the first term in Eq. (10.7+) is bounded by \( \mu/\lambda^2 \) times a coordinate dependent function (note that \( r \geq |\tilde{x}| \) ). Similarly, in region (B), Eq. (10.7–) will tend to zero as \( \mu/\lambda^2 \). In the region (C), we have that \( r \) and \( (u \cdot x) \) are both bounded by terms that are finite or proportional to \( \lambda \). So, in (C), both (10.7±) are bounded by functions of the form \( \mu \) or \( \mu \lambda^2 \). Choosing \( \lambda \) such that, as \( \mu \to 0 \), both \( \mu \lambda^2 \to 0 \) and \( \mu/\lambda^2 \to 0 \), allows us to conclude that

\[
\begin{align*}
\text{ds}^2 &\to \text{dy}^2_{(+)}, \quad \text{if} \quad (p \cdot x) \gtrsim 0, \quad \tag{10.11A} \\
\text{ds}^2 &\to \text{dy}^2_{(-)}, \quad \text{if} \quad (p \cdot x) \lesssim 0, \quad \tag{10.11B} \\
y^{(+)} &= y^{(-)} + 4\mu u^\mu \log r \quad \text{in the region} \quad (p \cdot x) \approx 0, \quad \tag{10.11C}
\end{align*}
\]

which is equivalent to Eq. (10.8). This defines the Aichelburg-Sexl metric\(^{26}\).

The effect a fast moving particle has on the surrounding space-time, is visualized in Fig. 4. Actually, the picture should be a bit more complicated since the clocks will also
Figure 4. Snapshot of the gravitational shock wave caused by a highly relativistic particle. If we have a rectangular grid and synchronized clocks before the particle passed by (region A), then, behind the particle (region B), the grid will be deformed, and the clocks desynchronized. The shift is proportional to the logarithm of the transverse distance.

begin to move after the shock wave passed. The artist stopped them by giving them a little kick to keep them in a fixed position afterwards.

In terms of the light cone coordinates of Eq. (4.2) in section 4, we have the connection formula
\[
\begin{align*}
    x^+_{(+)} - x^+_{(-)} &= 4Gp^+ \log |\tilde{x}| = 4\sqrt{2}Gp \log |\tilde{x}|; \\
    x^-_{(+)} - x^-_{(-)} &= 0.
\end{align*}
\]

Here, \(\tilde{x}\) is the transverse distance from the source particle, which is moving (highly relativistically) along the line \(x^- = \tilde{x} = 0\). The r.h.s. of Eq. (10.12) is a Green function, \(-\sqrt{2}f(\tilde{x})p\), satisfying the equation
\[
\tilde{\partial}^2 f(\tilde{x}) = -8\pi G \delta^2(\tilde{x}) ,
\]
where the sign is chosen such that \(f\) is large for small values of \(\tilde{x}\).

Next, we generalize this result for a particle moving into a black hole. For the Schwarzschild observer, its energy is taken to be so small that its gravitational field appears to be negligible. However, in Kruskal space, Section 2, the energy is seen to grow exponentially as Schwarzschild time \(t\) progresses. Let us therefore choose the Krusal coordinate frame such that the particle came in at large negative \(t\). This means (Eq. (2.3)), that \(x \approx 0\), or, the particle moves in along the past horizon. In view of the result derived above one can guess in which way the ingoing particle will deform the metric: we cut Kruskal space in halves across the \(x\)-axis, and glue the pieces together again after a shift, defined by
where \((\pm)\) now refers to the regions \(x \geq 0\). This corresponds to a metric with a delta-distributed Riemann curvature on the plane \(x = 0\). The function \(F\) is yet to be determined. Since the metric is

\[
ds^2 = 2A(x, y(\pm))dy(\pm) + r^2(x, y(\pm))d\Omega^2 \quad \text{in the regions} \quad x \geq 0,
\]

with \(A(x, y) = (16M^3/r)e^{-r(x,y)/2M}\), and \(r(x, y)\) as in Eq. (2.3), one may be tempted to write

\[
y \overset{\text{def}}{=} y(+)\Theta(x) + y(-)\Theta(-x);
\]

\[
ds^2 = 2A(x, y)dx\left(dy - \delta(x)F(y, \theta, \phi)dx\right) + r^2(x, y)d\Omega^2,
\]

but some caution is not out of place: if \(F\) depends on \(y\) this expression is not unambiguous. We will see that, fortunately, \(F\) does not depend on \(y\), but it is always better to refer to Eqs. (10.14) and (10.15) whenever the delta function in (10.16) causes difficulties.

The metric described by Eq. (10.16), and the effect it has on one of the geodesics, are sketched in Fig. 3. The shift \(\delta y\) in the \(y\)-coordinate is our function \(F\).

Let us reserve indices from the beginning of the alphabet \((a, b, \ldots)\) for indicating the transverse directions \(\theta\) and \(\phi\). The Ricci curvature has been computed in Ref 27. Making use of the fact that, on the horizon \(x = 0\), or \(r = 2M\), the functions \(A\) and \(r^2\) do not depend on the coordinate \(y\), one obtains

\[
R_{yy} = 0, \quad R_{ya} = 0, \quad R_{ab} = -h_{ab}\left(\frac{r^2}{A}\right)^{yy}F\delta = 0, \quad (10.17a)
\]

\[
R_{xy} = \left(\frac{A_{,y}^2}{A^2} - \frac{A_{,yy}}{A} - \frac{2r_{,y}A_{,y}}{rA}\right)F\delta = 0, \quad (10.17b)
\]

\[
R_{xa} = -\frac{A_{,y}}{A}F_{,a}\delta + \mathcal{O}[A_{,y}, A_{,yy}]F_{,a}F\theta\delta = 0, \quad (10.17c)
\]

\[
R_{xx} = \frac{A}{r^2}\tilde{\partial}^2F\delta - \frac{2r_{,yy}}{r}F\delta + \mathcal{O}(A_{,y}, r_{,y})(\delta, \delta_{,x}). \quad (10.17d)
\]

Here, \(h_{ab}\) stands for the spherical metric in \((\theta, \phi)\) space, \(\delta\) stands for \(\delta(x)\), and \{\}_y stands for differentiation with respect to \(y\). Furthermore, \(\tilde{\partial}^2\) stands for the angular Laplacian in the \(\theta, \phi\) variables. The last terms in Eqs. (10.17d) and (10.17e) would be ambiguous if \(A\) or \(r\) depended on \(y\). Our construction works because we take the cut to be along the past horizon, where \(x = 0\), so that the \(y\) dependence of \(A\) and \(r\) disappears.
The vacuum Einstein equation would require all $R_{\mu \nu}$ coefficients to vanish. Only (10.17e) remains, and putting this equal to zero yields

$$\tilde{\partial}^2 F - \frac{2r r_{xy}}{A} F = 0 \quad \text{(vacuum)}. \quad (10.18)$$

Here, the second term can easily be seen to originate from the Riemann curvature of the $S(2)$ sphere of the $\theta, \phi$ coordinates.

Substituting the known functions $A$ and $r$ we obtain:

$$r_{xy} = (4M^2/r)e^{-r/2M} = A/4M; \quad 2rr_{xy} = A; \quad (10.19)$$

$$\tilde{\partial}^2 F - F = 0 \quad \text{(vacuum)}. \quad (10.20)$$

In terms of the scaled coordinates $T, X, Y, Z$ of Section 4, the l.h.s. of this equation becomes

$$4M^2 \tilde{\partial}^2 F - F, \quad (10.21)$$

and now we are in a position to insert a source particle. Let it have momentum $p = p^-/\sqrt{2}$ in terms of these coordinates (see Eq. (4.3), then combining (10.21) with Eqs. (10.12) and (10.13), we find in the scaled coordinates (having the particle enter, for once, at angular coordinates $\theta = \frac{1}{2} \pi, \phi = 0$):

$$-\tilde{\partial}^2 F + F/4M^2 = 8\pi G \frac{\sqrt{e}}{4M} (p^-) \delta^2(x) = \frac{2\pi G \sqrt{2e}}{M^2} p \delta^2(\tilde{x}). \quad (10.22)$$

If we take $p$ to be the momentum with respect to the Kruskal coordinates $x$ and $y$, then this equation rescales into:

$$-\tilde{\partial}^2 F + F = 8\pi G (p_x) \delta^2(\Omega), \quad (10.23)$$

where $p_x$ (which on the mass shell is negative) is the momentum canonically conjugated to the Kruskal coordinate $x$, and $\Omega$ is the angular distance from the ingoing particle.

The solution to the partial differential equation

$$-\tilde{\partial}_\Omega^2 F(\Omega, \Omega') + F(\Omega, \Omega') = \delta^2(\Omega - \Omega'), \quad (10.24)$$

for the angular Green function $F$ can be derived by expanding $F$ in Legendre polynomials. Taking $\theta$ to be the angle between $\Omega$ and $\Omega'$, one finds

$$F(\theta) = (2\pi)^{-1} \sum_{\ell} \frac{\ell + \frac{1}{2}}{\ell(\ell + 1) + 1} P_\ell(\cos \theta). \quad (10.25)$$
An integral expression for $F(\theta)$ can be found by using the generating function for the Legendre polynomials,

$$\sum_{\ell} P_{\ell}(x)e^{-s\ell} = (1 - 2xe^{-s} + e^{-2s})^{-1/2},$$

(10.26)

together with

$$\int_0^\infty e^{-s(\ell+\frac{1}{2})} \cos(\alpha s) ds = \frac{\ell + \frac{1}{2}}{\ell(\ell + 1) + \alpha^2 + \frac{1}{4}},$$

(10.27)

so that one can rewrite Eq. (10.25) as

$$F(\theta) = \frac{1}{2\pi \sqrt{2}} \int_0^\infty \frac{\cos(\frac{1}{2} \sqrt{3} s)}{\sqrt{\cosh s - \cos \theta}} ds.$$

(10.28)

By rotating the integration contour in the complex plane one can obtain an integral of a function with a constant sign:

$$F(\theta) = \frac{1}{2\pi \sqrt{2}} \int_0^{\pi - \theta} \frac{d\omega \cosh(\frac{1}{2} \omega \sqrt{3})}{\sqrt{\cos \theta + \cos \omega}}.$$

(10.29)

From this expression it is easy to deduce that $F > 0$ for all $\theta$, and that

$$F(\pi) = \frac{1}{2 \cosh(\frac{1}{2} \pi \sqrt{3})}.$$

(10.30)

One may also verify that Eqs (10.28) and (10.29) have the proper limiting behavior at small $\theta$:

$$F(\theta) \to (1/2\pi) \log(1/\theta) \quad \text{as} \quad \theta \to 0.$$

(10.31)

11. THE S-MATRIX

As stated in the Introduction, the postulate that scattering of particles against a black hole can be described by a quantum mechanical scattering matrix is an assumption that cannot be proved from the principles of quantum field theory and general relativity alone. Indeed, it may well be at variance with these theories, if the latter would be extrapolated to beyond the Planck scale. The $S$-matrix Ansatz applied here may be seen as a new physical principle, perhaps comparable to Max Planck’s new postulate in his 1900 paper, that energies are quantized. The $S$-matrix Ansatz reads as follows:

All physical interaction processes that begin and end with free, stable particles moving far apart in an asymptotically flat space-time, therefore also all those that
involve the creation and subsequent evaporation of a black hole, can be described by one scattering matrix $S$ relating the asymptotic outgoing states $|\text{out}\rangle$ to the ingoing states $|\text{in}\rangle$.

In essence, the Ansatz will be used in the following way $^{10, 28}$: consider one state $|\text{in}_0\rangle$ and one state $|\text{out}_0\rangle$, with a possible black hole in their connecting history. We assume some value for the transition amplitude $\langle \text{out}_0 | \text{in}_0 \rangle = N$. This means that we replaced the out-state produced by the Hartle-Hawking vacuum, which actually was a quantum mechanical mixture of states, by one arbitrary choice $|\text{out}_0\rangle$. Then, using all the physical laws that we know and trust, we compute neighboring $S$-matrix elements, $\langle \text{out}_0 + \delta_{\text{out}} | \text{in}_0 + \delta_{\text{in}} \rangle$.

If there were no interactions, the effects from $\delta_{\text{in}}$ onto the out-states would not have been discernable. All amplitudes would have to be equal, and the scattering matrix thus obtained could never be unitary. Since in the calculations of Section 5, interactions between the $\Phi$ particles were ignored, those calculations were not good enough to give us our $S$-matrix. In this section, we will take only one type of interaction into account: the gravitational shift computed in the previous section. Thus, we only consider particles moving in and out in the longitudinal direction, with hyper-relativistic speeds when they are near the horizon. Far away from the horizon, as soon as $r - 2M = \mathcal{O}(2M)$, they will be allowed to go slower, indeed, out-moving particles may turn around to fall back in again. What has to be done in order to accommodate for such possibilities, is to define the $S$-matrix to consist of three ingredients:

$$ S = S_{\text{out}} S_{\text{hor}} S_{\text{in}},$$ (11.1)

where $S_{\text{in}}$ relates the asymptotic in-states to wave packets moving inwards very near the horizon, $S_{\text{out}}$ connects wave packets moving outwards very near the horizon to the asymptotic out-states, and $S_{\text{hor}}$ is the really important part telling us how particles moving inwards very near the horizon affect the outgoing particles very near the horizon. $^*$ $S_{\text{in}}$ and $S_{\text{out}}$ follow unambiguously from known laws of low-energy physics, and require little discussion.

The splitting (11.1) allows us to consider the limit $M \to \infty$ without appreciable complications. It is in this limit that the most important paradoxes arise; indeed, it is the limit that has not yet been addressed in terms of string theories. $^\dagger$ In this limit, the region

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$^*$ This phrasing is more precise than the ones in previous publications $^{28}$, but it is only partly an answer to recently uttered criticisms $^{29}$. The splitting of the wave functions into ingoing and outgoing ones at finite distance from the horizon will continue to be problematic.

$^\dagger$ In string theories $^{14}$, only near-extreme black holes can be handled, in which the horizon has a very tiny area or is degenerate. What concerns us most in this paper, is the non-degenerate horizon.
near the horizon can be described as a Rindler space. The angles θ and φ are replaced by flat transverse coordinates, and after rescaling of the momentum \( p \), the term \( F/4M^2 \) in Eq. (10.22) disappears. We recover the shift (10.12), determined by the Green function \( f \) of Eq. (10.13).

To begin our construction of the \( S \)-matrix, let us take \( \delta \) to be one extra particle going in with momentum \( \delta p_{\text{in}} \), at the transverse position \( \tilde{x}' \). Since we use the conventions of Section 4, the value of \( \delta p_{\text{in}} \) is negative. The outgoing particles, at points \( \tilde{x} \) near to the point \( \tilde{x}' \), are shifted inwards, so that \( \delta x_{\text{out}} \) is negative, and

\[
x_{\text{out}}^- \rightarrow x_{\text{out}}^- + \delta x_{\text{out}}^-(\tilde{x}) , \quad \delta x_{\text{out}}^-(\tilde{x}) = f(\tilde{x} - \tilde{x}')\delta p_{\text{in}}^- ,
\]

where \( f \) obeys Eq. (10.13), or, if from now on \( 8\pi G = 1 \),

\[
\tilde{\partial}^2 f(\tilde{x}) = -\delta^2(\tilde{x}) .
\]

We now temporarily suppress the superscripts \( \{\pm\} \), since the subscripts \( \text{in} \) and \( \text{out} \) suffice, and later we want to reintroduce \( \{\pm\} \) with different sign conventions. Any outgoing particle has a wave packet \( \psi \), oscillating as \( e^{ip_{\text{out}}x_{\text{out}}} \). With the shift \( \delta x_{\text{out}} \), this wave turns into

\[
e^{ip_{\text{out}}x_{\text{out}} - ip_{\text{out}}\delta x_{\text{out}}} = e^{-i \int d^2\tilde{x}\left[\delta x_{\text{out}}(\tilde{x})\hat{P}_{\text{out}}(\tilde{x})\right]} \psi ,
\]

where \( \hat{P}_{\text{out}}(\tilde{x}) \) is the operator\(^\dagger\) that generates a shift at transverse position \( \tilde{x} \). It is also the total momentum density of the outgoing particles at transverse position \( \tilde{x} \).

Now combining this with Eq. (11.2), we see that

\[
\psi \Rightarrow \psi' = e^{-i \int d^2\tilde{x}\left[\delta p_{\text{in}}f(\tilde{x} - \tilde{x}')\hat{P}_{\text{out}}(\tilde{x})\right]} \psi .
\]

Repeating this many times, adding (or removing) different incoming particles in the in-state, with momenta adding all up to \( P_{\text{in}}(\tilde{x}') \) at the transverse position \( \tilde{x}' \), we see that the total effect is:

\[
\psi' = e^{-i \int d^2\tilde{x}d^2\tilde{x}'\left[P_{\text{in}}(\tilde{x}')f(\tilde{x} - \tilde{x}')\hat{P}_{\text{out}}(\tilde{x})\right]} \psi .
\]

Notice the complete symmetry between in- and outgoing particles. \( P_{\text{in}}(\tilde{x}') \) refers to all momenta of particles going in during a certain epoch where we have control over the incoming particles. \( \hat{P}_{\text{out}}(\tilde{x}) \) refers to all particles seen going out during a similar epoch of observations. Before or after these two epochs we do not have the opportunity to observe

\(^\dagger\) Its relation to the energy momentum tensor is explained in Section 4.
or control. The states there are kept fixed as much as is possible. Of course, both $P_{\text{in}}$ and $\hat{P}_{\text{out}}$ are operators; from now on we omit the hat ($\hat{\cdot}$).

Noting that, according to the result of Section 7, the total number of quantum states should be finite, we have reasons to believe that, by adding or subtracting a sufficient number of particles, we can generate all in-states from $|\text{in}_0\rangle$, and for the out-states it is even more natural to have $P_{\text{out}}(\tilde{x})$ refer to all outgoing particles. It is suggested to describe the in- and out states exclusively by giving the functions $P_{\text{in}}(\tilde{x})$ and $P_{\text{out}}(\tilde{x})$.

One then obtains
\[
\langle\{P_{\text{out}}(\tilde{x})\}|\{P_{\text{in}}(\tilde{x})\}\rangle = \mathcal{N} \exp\left[-i \int d^2\tilde{x} d^2\tilde{x}' P_{\text{in}}(\tilde{x}')f(\tilde{x} - \tilde{x}')P_{\text{out}}(\tilde{x})\right], \tag{11.7}
\]
where $\mathcal{N}$ is a common normalization factor. The magnitude of this factor is fixed by requiring $S$ to be unitary; its phase cannot be determined, but in most cases it will be a freely adjustable parameter anyway, since our amplitude tends to violate global conservation laws.

This scattering matrix is indeed unitary, if one imposes the inner product
\[
\langle\{P_{\text{in}}(\tilde{x})\}|\{P_{\text{in}}'(\tilde{x})\}\rangle = \mathcal{N}' \prod_{\tilde{x}} \delta(P_{\text{in}}(\tilde{x}) - P_{\text{in}}'(\tilde{x})), \tag{11.8}
\]
for the in-states, with again some normalization parameter $\mathcal{N}'$, and we impose a similar inner product rule for the out-states.

We should hasten to add, that the $S$-matrix (11.7) cannot be the ultimate result of our theory, since the states $|\{P_{\text{in}}(\tilde{x})\}\rangle$ with the inner product (11.8) form a continuum of states, and this is not the result we want. What this really means is that we still expect some cut-off mechanism when $|\tilde{x} - \tilde{x}'|$ approaches the Planck length. Indeed, if $|\tilde{x} - \tilde{x}'|$ approaches the Planck length, our present result is invalid, since then the transverse components of the momenta also produce shifts, and those have not been taken into account. If, however, we limit ourselves to a “course grained” description, specifying only features that are large compared to the Planck length, and if it could indeed be accepted that restricting oneself to the gravitational interaction forces only (and of those only the longitudinal ones), is reasonable, then (11.7) seems to be a reasonable approximation to the $S$-matrix that we are looking for. In view of this, let us first further analyze what this $S$-matrix implies.

Consider the Hilbert space of in-states $|\{P_{\text{in}}(\tilde{x})\}\rangle$ with inner product (11.8), and define an operator $U_{\text{in}}(\tilde{x})$ that is canonically conjugated to $P_{\text{in}}(\tilde{x})$:
\[
[P_{\text{in}}(\tilde{x}), U_{\text{in}}(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}'), \tag{11.9}
\]
\[
[P_{\text{in}}(\tilde{x}), P_{\text{in}}(\tilde{x}')] = [U_{\text{in}}(\tilde{x}), U_{\text{in}}(\tilde{x}')] = 0 \tag{11.10}
\]
(We regard all these operators as acting on in-states). The eigenstates of \( U_{\text{in}}(\bar{x}) \) are the functional Fourier transforms of the eigenstates \( \{|P_{\text{in}}(\bar{x})\rangle\} \) of the operators \( P_{\text{in}} \):

\[
|\{U_{\text{in}}(\bar{x})\rangle\} = \mathcal{N}'' \int \mathcal{D} P_{\text{in}} e^{-i \int d\bar{x} P_{\text{in}}(\bar{x}) U_{\text{in}}(\bar{x})} |\{P_{\text{in}}(\bar{x})\rangle\},
\]

where \( \mathcal{N}'' \) is again a normalization factor.

Writing this as

\[
\langle \{U_{\text{in}}(\bar{x})\} | \{P_{\text{in}}(\bar{x})\} \rangle = \mathcal{N}'' e^{i \int d\bar{x} \frac{1}{2} P_{\text{in}}(\bar{x}) U_{\text{in}}(\bar{x})},
\]

we find that the states \( \{|U_{\text{in}}(\bar{x})\rangle\} \) can be expressed in terms of the states \( \{|P_{\text{out}}(\bar{x})\rangle\} \), by using Eq. (11.7). We find:

\[
U_{\text{in}}(\bar{x}') = - \int d\bar{x} f(\bar{x} - \bar{x}') P_{\text{out}}(\bar{x}),
\]

and similarly:

\[
U_{\text{out}}(\bar{x}') = \int d\bar{x} f(\bar{x} - \bar{x}') P_{\text{in}}(\bar{x}),
\]

where \( U_{\text{out}} \) is the operator canonically conjugated to \( P_{\text{out}} \), since in addition to Eqs. (11.9) and (11.10) we have for the out-states:

\[
[P_{\text{out}}(\bar{x}), U_{\text{out}}(\bar{x}')] = -i \delta^2(\bar{x} - \bar{x}'),
\]

\[
[P_{\text{out}}(\bar{x}), P_{\text{out}}(\bar{x}')] = [U_{\text{out}}(\bar{x}), U_{\text{out}}(\bar{x}')] = 0
\]

By virtue of the fact that Eqs (11.13) and (11.14) relate operators on in-states to operators on out-states, we say that these generate the \( S \)-matrix. Rewriting the equations as

\[
\tilde{\partial}^2 U_{\text{in}}(\bar{x}) = P_{\text{out}}(\bar{x}), \quad \tilde{\partial}^2 U_{\text{out}}(\bar{x}) = -P_{\text{in}}(\bar{x}),
\]

underlines the local nature of these equations with respect to the transverse coordinates \( \bar{x} \). Also:

\[
\langle \{U_{\text{out}}(\bar{x})\} | \{U_{\text{in}}(\bar{x})\} \rangle = \mathcal{N}''' \exp \left[ -i \int d^2 \bar{x} \tilde{\partial} U_{\text{out}}(\bar{x}) \cdot \tilde{\partial} U_{\text{in}}(\bar{x}) \right].
\]

Because of its local nature, this equation may be suspected to be more elementary than Eq. (11.7), which was derived earlier. Combining (11.18) with (11.12) and the analogous inner product between the \( U_{\text{out}} \) and \( P_{\text{out}} \) eigenstates, we rewrite Eq. (11.7) as

\[
\langle \{P_{\text{out}}(\bar{x})\} | \{P_{\text{in}}(\bar{x})\} \rangle = \mathcal{N} \int \mathcal{D} U_{\text{in}}(\bar{x}) \int \mathcal{D} U_{\text{out}}(\bar{x}) \exp \left[ i \int d^2 \bar{x} \left\{ -\tilde{\partial} U_{\text{out}}(\bar{x}) \cdot \tilde{\partial} U_{\text{in}}(\bar{x}) + P_{\text{in}}(\bar{x}) U_{\text{in}}(\bar{x}) - P_{\text{out}}(\bar{x}) U_{\text{out}}(\bar{x}) \right\} \right],
\]

44
where $\mathcal{N}$ is again a different but universal normalization factor (Henceforth, we write such factors simply as $\mathcal{N}$.)

Imagine now that both the in- and the out-state can be completely composed of a finite number, $N = N_{\text{in}} + N_{\text{out}}$, of particles. Let us denote the momenta of the ingoing particles as $p^{-i}_{\text{in}}, \ i = 1, \ldots, N_{\text{in}}$, entering at transverse coordinates $\tilde{x}^{i}$, and those of the outgoing particles, at transverse coordinates $\tilde{x}^{j}$, as $-p^{+j}_{\text{out}}, \ j = N_{\text{in}} + 1, \ldots, N$. The reason for the minus sign here, is that now the total momentum going into the horizon can be seen as the sum of all 4-vectors $p^{\mu}$ of the in- and outgoing particles, as it is usually done in field theory. The operators $U_{\text{in}}$ are put in a Lorentz vector $x^{\mu}$ without sign changes:

$$x^{+} = U_{\text{in}}, \quad x^{-} = U_{\text{out}}.$$  \hfill (11.20)

Substituting

$$P_{\text{in}}(\tilde{x}) = \sum_{i} p^{-i}_{\text{in}} \delta^{2}(\tilde{x} - \tilde{x}^{i}), \quad P_{\text{out}}(\tilde{x}) = -\sum_{j = N_{\text{in}} + 1}^{N} p^{+j}_{\text{out}} \delta^{2}(\tilde{x} - \tilde{x}^{j}),$$  \hfill (11.21)

one obtains

$$\langle \text{out}| \text{in} \rangle = \mathcal{N} \int Dx^{+}(\tilde{x}) \int Dx^{-}(\tilde{x}) \exp \left[ i \int d^{2}\tilde{x} \left\{ -\frac{1}{2} \partial x^{\mu}(\tilde{x}) \partial x^{\mu}(\tilde{x}) \right\} + i \sum_{i=1}^{N} p^{\mu,i} x^{\mu}(\tilde{x}^{i}) \right].$$  \hfill (11.22)

Here, the transverse components of $x^{\mu}$ are not functionally integrated over; they are the transverse coordinates. The factor $\frac{1}{2}$ compensates for double counting. The contribution of the transverse components of $x^{\mu}$ to the integrand must be subtracted, which corresponds to a renormalization of $\mathcal{N}$.

It is, however, more realistic to put the external particles in wave functions that are eigenstates of momenta only. Therefore, we must convolute this expression by transverse wave functions $e^{i\tilde{p}^{i} \cdot \tilde{x}^{i}}$, where the transverse components of the momenta, $\tilde{p}^{i}$, must be kept small compared to the Planck energy (Otherwise, it would have been illegal to ignore the transverse gravitational shifts.) We then obtain

$$\langle \text{out}| \text{in} \rangle = \mathcal{N} \left( \prod_{i} \int d^{2}\tilde{x}^{i} \right) \int Dx^{+}(\tilde{x}) \int Dx^{-}(\tilde{x}) \exp \left[ i \int d^{2}\tilde{x} \left\{ -\frac{1}{2} \partial x^{\mu}(\tilde{x}) \cdot \partial x^{\mu}(\tilde{x}) \right\} + i \sum_{i=1}^{N} p^{\mu,i} x^{\mu}(\tilde{x}^{i}) \right],$$  \hfill (11.23)

where now the effects of the wave functions are included in the contributions of the external momenta $p^{i,\mu}$ to the ‘vertex insertions’. 

45
It is here that the striking resemblance to string amplitudes should be pointed out. We have the string integrand (for closed strings), as well as the integration over moduli space, which here is formed by the points $\tilde{x}^i$ where the particles cross the horizon. The fact that the action is linearized is understandable, since all transverse dimensions have been kept large compared to the longitudinal ones. What is more surprising is the value of the string constant: it is equal to $i$, in units where $8\pi G = 1$.

The way in which here the black hole horizon is identified with a string worldsheet is sketched in Fig. 5. At $t \to -\infty$ we have ingoing closed “strings”. Arriving at the horizon these strings exchange a string, whose world sheet wraps around the horizon exactly once. The edges of the holes left behind are the outgoing closed strings.

![Figure 5](image.png)

5. The horizon as a string world sheet. Three snapshots of a collision event with a black hole intermediate state.

At this point, let us once again focus on the nature of the Hilbert space of in- and outgoing particles. Suppose that, for simplicity, we discretize the transverse coordinates $\tilde{x}$. The functional integrals then become finite-dimensional. What distinguishes this space from the usual Fock space is now, that at every point $\tilde{x}$ exactly one “particle” is allowed. The only way to mimic the usual Fock space is to assume that every elementary point particle must be given a different value for its transverse coordinate $\tilde{x}$. This constraint may be considered to be negligible, if the $\tilde{x}$ are sufficiently fine-grained, but it is somewhat puzzling how to maintain this constraint in an infinite-volume limit. Apparently, unlike ordinary Fock space, a state with two or more particles at the same transverse position $\tilde{x}$, with momenta $p^{\mu,1}$, $\ldots$, $p^{\mu,k}$, is indistinguishable from the state with just a single particle there, whose momentum is $\sum_{i=1}^k p^{\mu,i}$. This may seem to be odd, but it should be noted that this situation is identical to what one has in string theory, where the integrand for a many-particle amplitude is identical to the amplitude for fewer particles, when two
or more of the vertex insertions happen to coincide in the string world sheet.

The operators $x^{\mu}(\tilde{x})$ may be regarded as an “average position” operator for all particles ever entering or leaving the horizon at that point. This may (partly) explain why this information never “disappears” behind the horizon: there are always sufficient numbers of particles to be seen outside. This is in fact guaranteed by our brick wall model: the number of particles at distance greater than $h$ from the horizon are sufficient to represent all “information” concerning the state of the black hole.

There is a special interpretation for the commutation rule

$$[U_{\text{out}}(\tilde{x}), U_{\text{in}}(\tilde{x}')] = \int d^2\tilde{x}'' f(\tilde{x} - \tilde{x}'') [P_{\text{in}}(\tilde{x}''), U_{\text{in}}(\tilde{x}')] = -if(\tilde{x} - \tilde{x}'). \quad (11.24)$$

We could decide to interpret $-U_{\text{out}}(\tilde{x})$ as indicating the position of the horizon with respect to the particles seen to emerge from the black hole, and similarly, $-U_{\text{in}}(\tilde{x})$ as the time reverse of this: the position of the past horizon with respect to the ingoing particles. Eq. (11.24) implies an uncertainty relation for these two quantities. For ordinary black holes, $U_{\text{out}}(\tilde{x})$ is usually precisely defined, as it is determined by the momentum distribution of the ingoing particles that actually formed the black hole. $U_{\text{in}}(\tilde{x})$ is the horizon of the time-reversed, or “white hole”. In our picture, the white hole is the object formed by the Hawking particles if we follow these backwards in time. This is usually spread quantum mechanically over a large range of values. In our view, white holes are nothing but quantum super positions of black holes. They relate to black holes just like the momentum and the position of a quantum particle are related to each other.

For future use, we will define the operator $P^{\mu}(\tilde{x})$ by

$$P^{-}(\tilde{x}) = P_{\text{in}}(\tilde{x}) ; \quad P^{+}(\tilde{x}) = -P_{\text{out}}(\tilde{x}), \quad (11.25)$$

where the minus sign for the outgoing particles is in accordance with Feynman diagram conventions in field theory. The transverse components are still kept small at this stage. Eqs. (11.17) become

$$\tilde{\partial}^2 x^{\mu}(\tilde{x}) = -P^{\mu}(\tilde{x}). \quad (11.26)$$

In addition to Eq. (11.24) we have

$$[P^{-}(\tilde{x}), P^{+}(\tilde{x}')] = -i\tilde{\partial}^2 \delta^2 (\tilde{x} - \tilde{x}'). \quad (11.27)$$
12. ELECTROMAGNETISM

It is clear that the $S$-matrix derived in the previous section cannot be very precise: only the gravitational force was taken into account, so that it can only be relevant at length scales not too far away from the Planck scale, whereas beyond the Planck scale the approximation used again ceases to be valid (the transverse components of the gravitational force were ignored.) One might try to replace Eq. (11.23) by the complete Nambu string action, but the danger then is that this might be an incorrect way to reproduce Lorentz invariance near the horizon. Attempts of this nature are postponed to section 15.

In this section we advocate a more cautious and rigorous route towards improvement, which is the incorporation of other known forces among the ingoing and outgoing particles. One could include everything known from the Standard Model, and use more general quantum field theories in order to conjecture the generally possible expressions for the $S$-matrix up to details approaching the Planck length. The philosophy then to be applied is identical to the one used in the previous sections.

The simplest field theoretical interaction that can be handled is the electromagnetic one\(^{28}\). Consider the effects of electric charge on the ingoing and outgoing particles. The ingoing particles are not only characterized by their momentum distribution $P_{\text{in}}(\tilde{x})$, but also by the charge distribution $\varrho_{\text{in}}(\tilde{x})$. Computing the electromagnetic field of a particle entering the black hole along the past horizon, $x = 0$, goes in a way that is similar to the computation of its gravitational field, Section 10. The field of a static particle with charge $Q$ at the origin of a coordinate frame,

$$A_\mu(x) = \frac{Q}{4\pi} \frac{u_\mu}{r} = \frac{(Q/4\pi)u_\mu}{\sqrt{x^2 + (u \cdot x)^2}} , \quad (12.1)$$

is Lorentz boosted. One then finds that, in the limit $u_\pm \to \infty$, this approaches a pure gauge, except at the space-time points $x$ satisfying $(u \cdot x) \approx 0$, where the field is singular. To be precise, we again consider the three regions $(A)$, $(B)$, and $(C)$ of Section 10, Eq. (10.9). Let

$$\Lambda(x) = (Q/2\pi) \log r = (Q/4\pi) \log[x^2 + (u \cdot x)^2] , \quad (12.2)$$

be a gauge function. We have

$$\partial_\mu \Lambda = \frac{Q}{2\pi} \frac{(x_\mu + u_\mu(u \cdot x))}{x^2 + (u \cdot x)^2} . \quad (12.3)$$

In region $(A)$, the vector field $A_\mu$ approaches $\frac{1}{2} \partial_\mu \Lambda$, and in region $(B)$, it approaches $-\frac{1}{2} \partial_\mu \Lambda$. As in Section 10, the most delicate question is, whether in region $(C)$, $A_\mu \pm \frac{1}{2} \partial_\mu \Lambda$
approaches zero in the sense of distributions. By carefully comparing Eq. (12.1) with Eq. (12.3), one can convince oneself of this being the case.

We conclude from this calculation, that a charged particle moving with the speed of light produces an electromagnetic field that takes the form of a distribution in the transverse plane through the particle. The regions (A) and (C) are connected to each other by a gauge rotation \( \Lambda = (Q/2\pi) \log |\tilde{x}| \), where \( \tilde{x} \) is the transverse distance from the particle. Again, we can rewrite this as a Green function,

\[
\Lambda(\tilde{x}) = -f_1(\tilde{x})Q, \quad \tilde{\partial}^2 f_1(\tilde{x}) = -\delta^2(\tilde{x}).
\]  

On the horizon of the black hole, we must use for electromagnetism the Green function \( f_1(\theta, \phi) \), obeying

\[
\tilde{\partial}^2 f_1(\Omega, \Omega') = -\delta^2(\Omega - \Omega') + 1/4\pi,
\]  

which differs from the function \( F \) used for the gravitational shifts, by the absence of the second term in Eq. (10.24). (It is not difficult to verify Eq. (12.5) directly from the Maxwell equations in the Schwarzschild metric; in any case, gauge invariance would forbid any non-derivative terms in Eq. (12.5).) The constant \( 1/4\pi \) is our temporary resolution to a problem: the integral of the left hand side over all of the angles \( \Omega \) vanishes, hence the right hand side should vanish also. Actually, if the total charge \( \int d^2\Omega \rho(\Omega) \) of all ingoing particles would not vanish, the result would be a charged Reissner-Nordstrom black hole, which carries a residual field. we ignore this field (restricting ourselves to the case when the total charge is insignificant), and hence we ignore the constant.

In case of a charge distribution \( \rho(\tilde{x}) \) we find the gauge rotation

\[
\Lambda(\tilde{x}) = - \int d^2\tilde{x}' f_1(\tilde{x} - \tilde{x}') \rho(\tilde{x}').
\]  

Just as in the beginning of Section 11, we investigate the effect that a small change \( \delta \rho(\tilde{x}) \) in the charge distribution of the ingoing particles has on the outgoing particles. The wave function \( \psi(x^-, \tilde{x}) \) of an outgoing particle with charge \( Q_{out} \) is now not only gravitationally shifted, but also gauge rotated. The gauge rotation is \( e^{iQ_{out}\Lambda(\tilde{x})} \). In total, if \( \rho_{out}(\tilde{x}) \) is the charge distribution of the outgoing particles, the gauge rotation caused by the ingoing ones is

\[
e^{-i \int d^2\tilde{x} \int d^2\tilde{x}' f_1(\tilde{x} - \tilde{x}') \rho_{in}(\tilde{x}) \rho_{out}(\tilde{x}')}.
\]  

The sign in this exponent can be verified by comparing with ordinary Coulomb scattering amplitudes.
We introduce the canonically conjugated operators $\phi_{\text{in}}$ as follows:

\[
\theta_{\text{in}}(\tilde{x})d^2\tilde{x} = -i\partial/\partial \phi_{\text{in}}(\tilde{x}) ;
\]

\[
[\phi_{\text{in}}(\tilde{x}), \phi_{\text{in}}(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}') ;
\]

\[
\langle\{\phi(\tilde{x})\}|\{\theta(\tilde{x})\}\rangle = N e^i \int d^2\tilde{x} \theta(\tilde{x})\phi(\tilde{x}) (\text{for the in- and for the out-operators}).
\]

Comparing this with Eq. (12.7), we find

\[
\phi_{\text{in}}(\tilde{x}) = -\int d^2\tilde{x}' f_1(\tilde{x} - \tilde{x}') \theta_{\text{out}}(\tilde{x}') .
\]

Consequently,

\[
\theta_{\text{in}}(\tilde{x}) = \int d^2\tilde{x}' f_1(\tilde{x} - \tilde{x}') \phi_{\text{in}}(\tilde{x}') ;
\]

\[
\phi_{\text{out}}(\tilde{x}) = -i\delta^2(\tilde{x} - \tilde{x}') .
\]

As in Eq. (12.5), there is a problem with the overall constant; $\phi$ is well-defined up to an overall constant. Changing this overall constant would be nothing but a global gauge transformation, and is hence unphysical. Another important observation is to be made in connection with the last line of Eq. (12.8). Since electric charge is quantized in multiples of the electric charge unit $e$, the charge density $\theta_{\text{out}}(\tilde{x})$ can only vary by additions of the form $N e \delta^2(\tilde{x} - \tilde{x}_1)$. This implies that the amplitude (12.8) cannot change if we add a multiple of $2\pi/e$ to the field $\phi(\tilde{x})$ (fractions of this quantum, though also unphysical, do affect the overall amplitude). We deduce that the model here is actually a sigma model, where $\phi$ is periodic, or rather, the physically relevant field is $e^{2\pi i \phi/e}$, instead of $\phi$.

Defining

\[
\theta(\tilde{x}) = \theta_{\text{in}}(\tilde{x}) - \theta_{\text{out}}(\tilde{x}) ;
\]

\[
\phi(\tilde{x}) = \phi_{\text{in}}(\tilde{x}) + \phi_{\text{out}}(\tilde{x}) ;
\]

we have

\[
[\phi(\tilde{x}), \theta(\tilde{x}')] = 0 ;
\]

\[
\phi(\tilde{x}) = \int d^2\tilde{x}' f(\tilde{x} - \tilde{x}') \theta(\tilde{x}') .
\]

The gauge amplitude (12.7) can be rewritten as

\[
e^{\frac{1}{2}i} \int d^2\tilde{x} d^2\tilde{x}' f_1(\tilde{x} - \tilde{x}') \theta(\tilde{x})\theta(\tilde{x}') ,
\]

which differs from Eq. (12.7) only by two factors that renormalize the overall phase of the in-state, and the overall phase of the out-state, but are independent of the way in which these states are put together.
We may rewrite this phase as a functional integral:

\[ N \int \mathcal{D}\phi(\tilde{x}) e^{i \int d^2 \tilde{x} \left(-\frac{1}{2} \tilde{\partial} \phi(\tilde{x}) \cdot \tilde{\partial} \phi(\tilde{x}) + \phi(\tilde{x}) \varrho(\tilde{x}) \right)} \],

(12.15)

of which the classical equation

\[ \tilde{\partial}^2 \phi = -\varrho, \]

(12.16)

corresponds to Eq. (12.13).

The expressions of this section must now all be combined with those of the previous one. Hilbert space is the product of the two spaces, being spanned by the states

\[ |\{P_{in}(\tilde{x})\}, \{\varrho_{in}(\tilde{x})\}\rangle \]

(12.16)

All operators of Section 11, referring to momentum and position, commute with the operators of the present section, referring to electric charge, and the total amplitude is given by the direct product of the functional integrals (11.19) (or (11.23)) and (12.16).

Three remarks are of order:

1. Apparently, adding the electromagnetic force makes Hilbert space much larger than it already was. Actually, a cut-off is needed in any case. The electromagnetic force gives us no clue as to how this cut-off should be introduced.

2. In every respect, the electromagnetic charge density resembles a fifth component of the momentum. The conjugate parameter, \( \phi \), is a periodic coordinate. Thus, in our approach, electromagnetism automatically emerges in the form of a Kaluza-Klein theory.

3. It seems obvious that we can generalize this approach to handle other gauge forces.

13. OTHER FORCES

In the Standard Model of elementary particles, we have non-Abelian gauge forces, fermions and at least one scalar field. A complete implementation of all these variables in the black hole \( S \)-matrix appears to be difficult at present, but we will briefly discuss all of them.

One could introduce the non-Abelian forces directly by adding further compactified dimensions to space-time. This is probably the procedure that works best, but it is not quite in agreement with the strategy proposed all along in this paper. One should first prove that the physically observed charges, such as weak isospin and \( SU(3) \) color, actually necessitate the introduction of such degrees of freedom. This, however, is somewhat
problematic. If we add a single gauge-colored charge to the ingoing state, the latter will become gauge non-invariant, and therefore unphysical. Remember that we had a problem of this sort even in the Abelian case, but there it could be controlled. Secondly, non-Abelian charges are not additive, in contrast to the Abelian ones, so that a semiclassical calculation of their effects on outgoing particles, as we did this in the Abelian case, is not possible. Probably the only thing one can do in order to estimate the non-Abelian effects on the $S$-matrix, is first to compute the amplitudes with one-photon exchange between ingoing and outgoing particles. The generalization of the single periodic field $\phi$ for the electromagnetic case is then a sigma field $\Omega(\tilde{x})$, with elements in the space of gauge group transformations. Thus, the non-Abelian gauge theory generates a sigma model on the two-sphere.

From now on, we refer to the ordinary field theoretical interactions among elementary particles as “the Standard Model”. What we mean by this is that we are free to assume any kind of theory that may possibly be relevant at some energy scale between a GeV and the Planck scale, and it is assumed to contain gauge vector fields, scalar fields and spinors.

Suppose we want to repeat the procedure of Section 12 for particles interacting with a scalar Yukawa field $\phi(x)$. The mass $\mu$ of this scalar field may or may not be zero. Using the same notation as in Eq. (12.1), an ingoing particle will generate a scalar field of the form

$$\phi(x) = \frac{g}{4\pi r} e^{-\mu r} = \frac{g}{4\pi} \frac{e^{-\mu \sqrt{x^2 + (u \cdot x)^2}}}{\sqrt{x^2 + (u \cdot x)^2}},$$  \hspace{1cm} (13.1)$$

As Schwarzschild time proceeds, the particle’s velocity vector $u^\mu$ will be boosted more and more towards the past horizon. Unlike Eq. (12.1), however, there is no factor $u^\mu$ in the numerator, and while the support of this field becomes more and more flattened against the past horizon, its intensity does not increase. Outgoing particles going through this pancake shaped field will be less and less influenced by it. As we are mainly interested in the interactions between ingoing and outgoing particles very near the horizon, where their relative center of mass energy approaches the Planck energy, the effects of scalar field exchanges will be extremely small. In Section 12, we were able to produce a meaningful amplitude in the limit $u^\mu \to \infty$; in the case of scalar field exchange, this limit does not produce any non-trivial contribution to the amplitudes.

Yet there is a way in which scalar fields in the Standard Model may play a role (of course, in a complete theory, without simplifying approximations, they must be relevant). Assume that one or more of the gauge forces in our “Standard Model” are condensed into a Higgs mode. This effect may be entirely due to scalar fields, and yet its consequences for the black hole $S$-matrix are non-trivial and can be computed. Take the $U(1)$ case. The procedure is exactly as described in the Section 12, except that the Coulomb field is
assumed to have a finite range. For collisions at a very high center-of-mass energy and a relatively low impact parameter, the only force felt is the Maxwell field with a Yukawa suppression factor. Instead of Eq. (12.1) we have

$$A_\mu(x) = \frac{Q}{4\pi} \frac{u_\mu}{r} e^{-m_A r}, \quad r = \sqrt{x^2 + (u \cdot x)^2},$$

where $m_A$ is the photon mass acquired through the Higgs mechanism. The gauge rotation function $\Lambda(x)$ of Eq. (12.2) now takes the form

$$\Lambda(x) = -\frac{Q}{2\pi} \int_0^\infty d\sigma e^{-m_A \sigma}. \quad (13.3)$$

With

$$\partial_\mu \Lambda = \frac{Q}{2\pi} \frac{x_\mu + u_\mu (u \cdot x)}{r^2} e^{-m_A r}, \quad (13.4)$$

we see, as in Section 12, that in regions (A) and (B) we have $A_\mu \Rightarrow +\frac{1}{2} \partial_\mu \Lambda$ and $-\frac{1}{2} \partial_\mu \Lambda$, respectively. In region (C), both $A_\mu$ and $\partial_\mu \Lambda$ approach to zero in the distributional sense.

Thus, as in Section 12, outgoing particles are phase shifted, but now the phase shift function $\Lambda(\tilde{x})$ is given by Eq. (13.3), with $r = |\tilde{x}|$. Can we set up an algebra as in Section 12? Not quite. The function $\Lambda(\tilde{x})$ does not obey a field equation as useful as Eq. (12.5). The equation it does satisfy is:

$$\left(\tilde{\partial}^2 + m_A^2 \frac{\partial}{\partial r}\right) \Lambda = \frac{Q}{2\pi} \delta^2(\tilde{x}), \quad (13.5)$$

where the second term involves the gradient in the radial direction. This equation does not allow a linear superposition in the case of many different sources at different locations in the transverse plane. We do have, for large $r$,

$$\Lambda(r) \Rightarrow -\frac{e^{-m_A r}}{m_A r} \left(1 - \frac{1}{m_A r} + \frac{2!}{(m_A r)^2} - \cdots\right); \quad (13.6a)$$

$$\frac{\partial}{\partial r} \Lambda = \frac{e^{-m_A r}}{r}, \quad (13.6b)$$

and from this it follows that, at sufficiently large $m_A r$, our field $\Lambda$ tends to obey the Klein-Gordon equation:

$$\left(\tilde{\partial}^2 - m_A^2 \right) \Lambda(\tilde{x}) \Rightarrow \frac{Q}{2\pi} \delta^2(\tilde{x}). \quad (13.7)$$

* This treatment is more precise than the original one in ref 28
From this equation it is clear what the main \textit{infrared} effect is of a Higgs mechanism in the electromagnetic contribution to our amplitudes: the Green function \( f(\tilde{x} - \tilde{x}') \) obtains a mass term in its defining equation. In Eq. (12.14), the integrand in the exponent receives a mass term of the form

\[-\frac{1}{2} m_A^2 \phi^2(\tilde{x}).\]  

(13.8)

Here, \( m_A \) is the gauge photon mass. What is remarkable about this modification term is that it appears to break the gauge symmetry \( \phi \to \phi + \lambda \). We find that, if in the Standard Model a \textit{local} symmetry is \textit{spontaneously} broken by Higgs mechanism, then in the two-dimensional system described in Eq. (12.14), the corresponding \textit{global} symmetry is broken \textit{explicitly}.

Clearly, in this particular case, a scalar field can have an effect on the black hole back reaction amplitude, even if the outgoing particles emerge at arbitrarily late Schwarzschild time. However, the only visible effect is a modification of a field equation. This, we believe, is the only way a fundamental scalar field in the Standard model can betray its presence in the black hole scattering amplitude: it modifies other field equations. The fact that a spontaneous symmetry breaking effect now turns into an explicit symmetry breaking effect, can be interpreted as follows. The scalar field under consideration, may have some accidental value at the point where past horizon and future horizon meet. In terms of the 4-dimensional theory, a non-vanishing value of the scalar field there, may imply the spontaneous breakdown of a symmetry. But for the observer of a black hole, the scalar field is permanently present at that point. Its value will not change as the black hole ages. So, it appears to break the symmetry explicitly.

The next subject to be studied is the question how the horizon amplitudes are affected if the 4-dimensional theory includes a confinement mechanism, such as the property that keeps the quarks together in a hadron. Confinement is often understood as a condensation of magnetic charges in the theory. A non-Abelian gauge theory is first decomposed into a system of photons representing the Cartan subalgebra of the gauge group, treating the remaining photons together with the quarks as ordinary charged particles, and the inevitable gauge singularities one then gets: the magnetic monopoles. If one then assumes that these magnetic monopoles condense, one has a satisfactory description of a quark confinement mechanism\textsuperscript{37}. Before discussing this any further, we must understand how the black hole amplitudes for states with \textit{magnetic} charges can be understood.

Consider an operator \( \sigma(\tilde{x}) \) that creates a magnetic monopole in the in-state at the transverse position \( \tilde{x} \). It is not difficult to describe the electromagnetic field produced by a monopole with a speed close to that of light. An electric charge produces a Maxwell...
### Table 1

| STANDARD MODEL IN 3+1 DIMENSIONS | INDUCED 2 DIM. OPERATOR FIELD ON HORIZON |
|----------------------------------|------------------------------------------|
| **Spin 2:** $g_{\mu\nu}(x,t)$  | String variables (spin 1): $x^{\mu}(\Omega)$ |
| local gauge generator: $u^{\mu}(x,t)$ | Scalar variable (spin 0): $\phi(\Omega)$ mod $2\pi/e$ |
| **Spin 1:** $A_\mu(x,t)$       | Coupling constant                          |
| local gauge generator: $\Lambda(x,t)$ mod $2\pi/e$ | Vortex                                   |
| **Spin 0:** $\phi(x,t)$        | Magnetic monopole                          |
| **Higgs mechanism:**           | Explicit symmetry breaking: $\phi(\Omega)$ obtains same mass $m_A$. |
| “spontaneous” mass $m_A$ for vector field | Confinement in vector field $A_\mu$ |
| **Confinement in vector field $A_\mu$** | Scalar fields $W(\Omega)$, corresponding to non-linear sigma model. |
| **Non-Abelian gauge theory**   | Unknown                                   |
| **Spin $\frac{1}{2}$:** fermions | Unknown                                   |
| **Spin $\frac{3}{2}$:** gravitino | Fermion, Spin $\frac{1}{2}$ |
| local gauge generator spin $\frac{1}{2}$ | |

Field,

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \quad \text{with} \quad A_+ = \partial_+\left(\frac{1}{2}\Lambda(\tilde{x})\operatorname{sgn}(x^+)\right),$$

or,

$$F^\text{el}_{+a} = \delta(x^+)\partial_a\Lambda(\tilde{x}) ,$$

where the index $a$ is in the transverse direction, and $\Lambda(\tilde{x})$ is given by Eq. (12.2). The other components of $F_{\mu\nu}$ vanish. The field of a magnetic charge is the dual of this:

$$F^\text{magn}_{+a} = g_{+-}\varepsilon^{+ab}F^\text{el}_{+b} = -\varepsilon_{ab}\delta(x^+)\partial_b\Lambda(\tilde{x}) = \delta(x^+)\partial_a\Lambda^D(\tilde{x}) ,$$

where $\Lambda^D$ ($D$ stands for “dual”) is a gauge function that is multiply connected if we rotate around the origin: $L \rightarrow \Lambda \pm 2\pi/e$. Thus, the operator $\sigma(\tilde{x})$ produces a “frustration” in the field $\phi(\tilde{x})$, a locally stable vortex. Such operators are known as “disorder operators”.

We are now in a position to treat the confinement mechanism. The disorder operators $\sigma(\tilde{x})$ that we introduced fail to commute with the $\phi(\tilde{x}')$ fields, as follows:

$$e^{(2\pi i/e)\phi(\tilde{x})}\sigma(\tilde{x}') = \sigma(\tilde{x}')e^{(2\pi i/e)\phi(\tilde{x})}e^{i\theta(\tilde{x} - \tilde{x}')} ,$$

where $\theta(\tilde{y})$ is the angle of the 2-vector $\tilde{y}$.  

55
The disordered phase of a field theory would be characterized by a spontaneous breakdown of the symmetry
\[ \sigma \rightarrow \sigma e^{i\lambda}, \] (13.12)
but if confinement is to be described as the dual analogue of the Higgs mechanism, we must conclude that the disorder field obtains a mass term, thus becoming a short range field, and the disorder symmetry (13.12) must be broken \textit{explicitly}. After such an explicit symmetry breaking it has become impossible to transform back to the original fields \( \phi(x) \).

Not much has been done yet to understand the consequences of the presence of fermions in the Standard Model. The transformation properties of a fermion field under Lorentz boosts is intermediate between those of vectors and those of scalars. Therefore, we expect the effect of a spinorial field produced by ingoing particles also to scale away as Schwarzschild time proceeds.

In supergravity theories, fundamental fields occur with spin equal to \( \frac{3}{2} \). Such fields will be enhanced, and therefore play a more prominent role on the horizon. As is the case for gravity and for electromagnetism, the generator of a local supersymmetry transformation will correspond to an operator on the horizon. Thus, supergravity theories are expected to produce a spin \( \frac{1}{2} \) operator field on the horizon. The conclusions of this Section are summarized in Table 1.

14. THE TRANSVERSE GRAVITATIONAL FORCE\textsuperscript{28}

As was shown in the previous section, adding the effects from known (and unknown) quantum field theoretic forces in our calculations for the scattering matrix, rather adds more states to our Hilbert space than reducing it. Clearly, we would like to know which of the physical laws should be taken into account if we want to understand the area – entropy relation. A promising candidate is the transverse component of the gravitational shift effect; so-far we have only taken its longitudinal part into account.

Unfortunately, as we will show, including the transverse gravitational force is difficult. We here only give an indication as to how one could proceed along these lines, so as to further improve our theory.

Let us recapitulate our algebra. From Section 11:
\[
\begin{align*}
[P_{\text{in}}(\tilde{x}), P_{\text{in}}(\tilde{y})] &= 0 = [P_{\text{out}}(\tilde{x}), P_{\text{out}}(\tilde{y})]; \\
[U_{\text{in}}(\tilde{x}), U_{\text{in}}(\tilde{y})] &= 0 = [U_{\text{out}}(\tilde{x}), U_{\text{out}}(\tilde{y})]; \\
[P_{\text{in}}(\tilde{x}), U_{\text{in}}(\tilde{y})] &= -i\delta^2(\tilde{x} - \tilde{y}) = [P_{\text{out}}(\tilde{x}), U_{\text{out}}(\tilde{y})]; \\
P_{\text{out}}(\tilde{x}) &= \tilde{\partial}^2 U_{\text{in}}(\tilde{x}); \quad P_{\text{in}}(\tilde{x}) = -\tilde{\partial}^2 U_{\text{out}}(\tilde{x});
\end{align*}
\] (14.1 – 14.4)
\[ [U_{\text{in}}(\tilde{x}), U_{\text{out}}(\tilde{y})] = i f(\tilde{x} - \tilde{y}) \]
\[ [P_{\text{in}}(\tilde{x}), P_{\text{out}}(\tilde{y})] = -i \tilde{\partial}^2 \delta^2(\tilde{x} - \tilde{y}), \quad (14.5) \]

and from Section 12:
\[ [\varrho_{\text{in}}(\tilde{x}), \varrho_{\text{in}}(\tilde{y})] = 0 = [\varrho_{\text{out}}(\tilde{x}), \varrho_{\text{out}}(\tilde{y})]; \quad (14.6) \]
\[ [\phi_{\text{in}}(\tilde{x}), \phi_{\text{in}}(\tilde{y})] = 0 = [\phi_{\text{out}}(\tilde{x}), \phi_{\text{out}}(\tilde{y})]; \quad (14.7) \]
\[ [\varrho_{\text{in}}(\tilde{x}), \phi_{\text{in}}(\tilde{y})] = -i \delta^2(\tilde{x} - \tilde{y}) = [\varrho_{\text{out}}(\tilde{x}), \phi_{\text{out}}(\tilde{y})]; \quad (14.8) \]
\[ \varrho_{\text{out}}(\tilde{x}) = \tilde{\partial}^2 \phi_{\text{in}}(\tilde{x}); \quad \varrho_{\text{in}}(\tilde{x}) = -\tilde{\partial}^2 \phi_{\text{out}}(\tilde{x}); \quad (14.9) \]
\[ [\phi_{\text{in}}(\tilde{x}), \phi_{\text{out}}(\tilde{y})] = if(\tilde{x} - \tilde{y}) \quad [\varrho_{\text{in}}(\tilde{x}), \varrho_{\text{out}}(\tilde{y})] = -i \tilde{\partial}^2 \delta^2(\tilde{x} - \tilde{y}) \quad (14.10) \]

(since we neglect the angular curvature of the horizon, we may take \( f_1 \) to be equal to \( f \).)

These equations clearly illustrate that charge density may be seen as a fifth momentum component, and \( \phi \) as a fifth coordinate. Now, it is not difficult to see that what is missing here is the transverse components, \( \tilde{\hat{P}}(\tilde{x}) \), of the momentum densities. These should obey similar commutation rules. Indeed, an ingoing particle with large transverse momentum does produce a shift among the outgoing particles in the transverse direction.

However, transverse momentum is not an independent variable; it is related to the \( \tilde{x} \)-dependence of all other fields. Let \( f(\tilde{x}) \) be any operator valued function of the transverse coordinates. The total transverse momentum operator, \( \tilde{P}^\text{tot} \), obeys the commutation rule,
\[ [\tilde{P}^\text{tot}, f(\tilde{x})] = -i \tilde{\partial} f(\tilde{x}). \quad (14.11) \]

It is the integral of a transverse momentum density, \( \tilde{\hat{P}}(\tilde{x}) \). We distinguish the transverse momentum density of the in-states from that of the out-states, so
\[ [\tilde{P}_{\text{in}}(\tilde{x}), f_{\text{in}}(\tilde{y})] = -i \delta^2(\tilde{x} - \tilde{y}) \tilde{\partial}_{\text{in}} f_{\text{in}}(\tilde{x}), \quad (14.12) \]

and similarly for the out-states.

For an operator \( \varrho(\tilde{x}) \) that is a density, which means that under displacements it transforms with a Jacobian, such that its integral is conserved, we must demand
\[ [\tilde{P}_{\text{in}}(\tilde{x}), \varrho_{\text{in}}(\tilde{y})] = i \varrho_{\text{in}}(\tilde{x}) \tilde{\partial} \delta^2(\tilde{x} - \tilde{y}). \quad (14.13) \]

Constructing an operator \( \tilde{P}_{\text{in}}(\tilde{x}) \) with the properties (14.12) and (14.13) is straightforward; the answer is an elementary exercise in quantum field theory:
\[ \tilde{P}_{\text{in}}(\tilde{x}) = P_{\text{in}}(\tilde{x}) \tilde{\partial} U_{\text{in}}(\tilde{x}) + \varrho_{\text{in}}(\tilde{x}) \tilde{\partial} \phi_{\text{in}}(\tilde{x}) + \cdots, \quad (14.14) \]
where the dots stand for contributions of any other kinds of local operators fields that may exist in our Hilbert space. With this definition, the commutation rules (14.12) and (14.13) follow directly from Eqs. (14.1–10). We also have

\[
[\tilde{P}^a_{\text{in}}(\tilde{x}), \tilde{P}^b_{\text{in}}(\tilde{y})] = i\tilde{P}^a_{\text{in}}(\tilde{y})\partial_\beta \delta^2(\tilde{x} - \tilde{y}) + i\tilde{P}^b_{\text{in}}(\tilde{x})\partial_\alpha \delta^2(\tilde{x} - \tilde{y}). \tag{14.15}
\]

The relation between these operators and the energy-momentum density is explained in Section 4. Latin indices \(a, b, \ldots\) are in the transverse direction. The tilde (\(\tilde{}\)) on \(P^a\) is there only to distinguish this operator from the longitudinal ones. Deriving Eq. (14.15) from (14.14) is facilitated by observing the following important properties of the Dirac delta distribution:

\[
\delta^2(\tilde{x} - \tilde{y})\{f(\tilde{x}) - f(\tilde{y})\} = 0; \tag{14.16}
\]

\[
\partial_\alpha \delta^2(\tilde{x} - \tilde{y}) \{f(\tilde{x}) - f(\tilde{y})\} + \delta^2(\tilde{x} - \tilde{y})\{\partial_\alpha f(\tilde{x})\} = 0. \tag{14.17}
\]

Eq. (14.17) is, of course, obtained by differentiating (14.16) with respect to \(\tilde{x}\). The out-states can be subjected to operators \(\tilde{P}^a_{\text{out}}\) that obey an algebra similar to (14.15).

Consider now, as before, our deductive procedure for the production of scattering matrix amplitudes. This time, we bring about a small change \(\delta \tilde{P}_{\text{in}}(\tilde{x})\) in the transverse momentum distribution of the in-state; one of the ingoing particles, for instance, was replaced by a particle with a small change in its transverse momentum components. The gravitational shift, caused by this particle, is now in a slightly different direction. From Eq. (11.2), we derive that the shift in the transverse direction is

\[
\delta \tilde{x}_{\text{out}} = \int d^2 \tilde{x}' f(\tilde{x} - \tilde{x}') \delta \tilde{P}_{\text{in}}(\tilde{x}'), \tag{14.18}
\]

which rotates the phase of the outgoing state as follows:

\[
|\psi\rangle_{\text{out}} \rightarrow e^{-i \int d^2 \tilde{x} d^2 \tilde{x}' f(\tilde{x} - \tilde{x}') \delta \tilde{P}_{\text{in}}(\tilde{x}')} \cdot \tilde{P}_{\text{out}}(\tilde{x}) |\psi\rangle_{\text{out}}. \tag{14.19}
\]

This is entirely analogous to Eq. (12.7), and therefore one expects commutation rules of the form

\[
[\tilde{P}^a_{\text{in}}(\tilde{x}), \tilde{P}^b_{\text{out}}(\tilde{x}')] \not= -i\delta^{ab} \partial^2 \delta^2(\tilde{x} - \tilde{x}'). \tag{14.20}
\]

Unfortunately, Eq. (14.20) cannot be correct. For one thing, it violates the Jacobi identities when combined with Eq. (14.15). In any case, when Eq. (14.14) is substituted, much more complicated expressions for the commutator are obtained. There are several reasons for this apparent contradiction to arise: Eqs. (14.4), (14.5), (14.9) and (14.10)
should be violated due to the transverse shift, and furthermore, the non-commutativity of $\tilde{P}(\tilde{x})$ and $P(\tilde{x}')$ should be taken into account.

Presently, it is beyond our capabilities to set up a completely consistent algebra for the in- and out-states on the horizon. Perhaps, when a better understanding of string theory is achieved, we might expect that string theory could provide for a consistent prescription here. Conversely, since the first principles from which we arrived at this point are conceptually very straightforward, one might expect that these could form a better guideline for further improvement, whereas an interpretation in terms of strings, as in Section 11, may come afterwards.

15. ALGEBRAS WITH A TRANSVERSE CUT-OFF

Qualitatively, it can be understood why the transverse gravitational force will produce a cut-off at the Planck scale on the horizon. Imagine a particle going in at transverse position $\tilde{x}$, with transverse momentum $\tilde{p} \equiv \delta \tilde{P}_{\text{in}}(\tilde{x})$ whose absolute value is much larger than the Planck energy. The transverse shift brought about by this particle is non-negligible, and the out-state is rotated by the phase factor (14.19). Since the shift is large, already the small values of $\tilde{P}_{\text{out}}(\tilde{x})$ contribute. Thus, in terms of the out-states, the substitution $\tilde{P}_{\text{in}} \rightarrow \tilde{P}_{\text{in}} + \delta \tilde{P}_{\text{in}}$ does not yield particles with very large values of $\tilde{P}_{\text{out}}$, and can therefore be seen as an operation that keeps the state in the subspace of Hilbert space spanned by the low momentum particles. It is conceivable, therefore, that states with high transverse momentum can be formed by linear superposition of states with low transverse momenta.

It is clear that, when a Planckian resolution is required in the transverse coordinates, the models of sections 11 – 14 are inadequate. As long as our understanding of the situation at the Planck length is limited, we take our resort to some models. Surely, these models will show deficiencies as well, but they illustrate how discrete physics may arise naturally, and they may show how to understand the black hole entropy in terms of the density of states in Hilbert space.

Our first model is obtained by introducing discreteness in the transverse coordinates by hand, merely by substituting the continuum of the points $\tilde{x}$ by a lattice. The lattice is chosen to be a random one, since this allows us to accommodate for curvature on the horizon. In short: we have points $\{A, B, \ldots\}$, connected by links, see Fig. 6.

The “lengths” of the links are not specified; even if one would start by postulating them to be of infinitesimal length only, one will later find the generic separation between neighboring points to be of the order of the Planck length, as will be demonstrated. The
The absence of the factors $\sqrt{2}$ from the lightcone coordinate definitions (4.2), (4.3) of Section 4, in the third components of Eqs. (15.1) and (15.2), where one would have expected them, is a peculiarity of this model. The commutation rules are:

\[
\begin{align*}
[x_{i,\text{in}}^A, x_{j,\text{in}}^B] &= 0, \\
[p_{i,\text{in}}^A, p_{j,\text{in}}^B] &= 0, \\
[x_{i,\text{in}}^A, p_{j,\text{in}}^B] &= i\delta^{ij}\delta_{AB},
\end{align*}
\] (15.3)

for all $A, B$, and where $i, j$ take the values 1, 2 and 3. The commutation rules for the out-operators are identical.

In order to generate relations between the in- and the out-operators resembling Eqs. (14.4), (14.5) and (14.20), we need a discrete version of the transverse Laplacian on our lattice. Define

\[
C_{AB} = \begin{cases} 
0 & \text{if } A \text{ and } B \text{ are not directly connected}, \\
1 & \text{if } A \text{ and } B \text{ are linked neighbors}.
\end{cases}
\]

\[
C_{AA} = -N(A),
\] (15.4)

* This is a weak point in the model; since we expect no truly conserved quantities on the black hole, we should expect a dynamical evolution of the link topology. One may consider further refinements of this model that take such an evolution into account.
where \( N(A) \) is the number of neighbors linked to the point \( A \). An operator that approaches the Laplacian in the continuum limit is then given by:

\[
\lambda \sum_B C_{AB} F_B \Rightarrow \left( \tilde{\partial}^2 F \right)_A ,
\]

(15.5)

where the constant \( \lambda \) depends on the average number of neighbors that the points \( A \) have, for instance:

\[
\lambda = \begin{cases} 
1 & \text{for a square lattice;} \\
1/\sqrt{3} & \text{for a triangular lattice.}
\end{cases}
\]

(15.6)

Therefore, we may write

\[
p_{A}^{i,\text{out}} = \lambda C_{AB} x_{B}^{i,\text{in}} ;
\]

\[
p_{A}^{i,\text{in}} = -\lambda C_{AB} x_{B}^{i,\text{out}} ,
\]

(15.7)

where summation over \( B \) is implied, and

\[
[p_{A}^{i,\text{in}} , p_{B}^{j,\text{out}} ] = -i\lambda \delta^{ij} C_{AB} ;
\]

\[
[x_{A}^{i,\text{in}} , x_{B}^{j,\text{out}} ] = -i \lambda^{-1} (C^{-1})_{AB} .
\]

(15.8)

The inverse \( C^{-1} \) of the matrix \( C \) is actually ill-defined because \( C \) has a vanishing eigenvalue (corresponding to the constant function), but this can be cured as usual with an infrared cut-off; for finite black holes the horizon curvature term comes to the rescue.

In the continuum limit, these equations approach to (14.1 – 10), as well as (14.20). Now, since \( x_{A}^{\text{in}} - x_{B}^{\text{in}} \) and \( x_{A}^{\text{out}} - x_{B}^{\text{out}} \) for two linked points \( A \) and \( B \) do not commute, they fluctuate in accordance with an uncertainty relation, such that the average link length fluctuates around the Planck length. Thus, we automatically obtain a finite transverse cut-off, as was promised at the beginning of this section.

Next, consider a black hole for which both \( |x^{\text{in}}| \) and \( |x^{\text{out}}| \) are bounded, by writing

\[
\sum_{A} |x_{A}^{\text{out}}|^2 < (2Gm_{BH})^2 .
\]

(15.10)

Upon diagonalizing \( C_{AB} \), we find at each eigenmode a harmonic oscillator with a bound on its Hamiltonian. Consequently, this system has a bound on its total number of states. The bound one finds this way, however, is not yet stringent enough to yield the Bekenstein-Hawking entropy formula.

A different procedure was introduced in Ref 30. As our starting point we again use Eqs. (14.1) – (14.5), but assume these to be valid only when the functions \( U_{\text{in}}(\tilde{x}) \) and \( U_{\text{out}}(\tilde{x}) \) are slowly varying. For later convenience, we rename the transverse coordinates on the horizon as \((\sigma^1, \sigma^2)\), and now define a 2-surface \( x^\mu(\tilde{\sigma}) \) embedded in 4-space:

\[
x^+ = U_{\text{in}}, \quad x^- = U_{\text{out}}, \quad \tilde{x} = \tilde{\sigma} ,
\]

(15.11)
thus, in contradistinction with the previous procedure, we now keep the the transverse coordinates for the in- and out-states identical.

The orientation of the surface is given by the tensor

$$W_{\mu\nu}(\tilde{\sigma}) = -W_{\nu\mu} = \varepsilon^{ab} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b}.$$  \hspace{1cm} (15.12)

We have

$$\frac{\partial \tilde{x}^a}{\partial \sigma^b} = \delta_b^a.$$  \hspace{1cm} (15.13)

Now first consider the case that $x^\pm$ are slowly varying. This implies

$$W^{12} = 1; \quad W^{1\pm} = \frac{\partial x^\pm}{\partial \sigma^2};$$

$$W^{2\pm} = -\frac{\partial x^\pm}{\partial \sigma^1}; \quad W^{+\pm} = O(\partial_\sigma x^\pm)^2.$$  \hspace{1cm} (15.14)

Commutation rules follow from Eqs. (14.2) and (14.5):

$$[W^{1+}(\tilde{\sigma}), W^{2-}(\tilde{\sigma}')] = [W^{2+}(\tilde{\sigma}), W^{1-}(\tilde{\sigma}')] = i\partial_1 \partial_2 f(\tilde{\sigma} - \tilde{\sigma}');$$

$$[W^{1+}(\tilde{\sigma}), W^{1-}(\tilde{\sigma}')] = -i\frac{\partial^2}{\partial \sigma^2 \partial^2} f(\tilde{\sigma} - \tilde{\sigma}');$$

$$[W^{2+}(\tilde{\sigma}), W^{2-}(\tilde{\sigma}')] = -i\frac{\partial^2}{\partial \sigma^1 \partial^2} f(\tilde{\sigma} - \tilde{\sigma}').$$  \hspace{1cm} (15.15)

As a special case, we have

$$[W^{\mu+}(\tilde{\sigma}), W^{\mu-}(\tilde{\sigma}')] = -i\delta^2 f(\tilde{\sigma} - \tilde{\sigma}') = i \delta^2(\tilde{\sigma} - \tilde{\sigma}'),$$  \hspace{1cm} (15.16)

where the index $\mu$ is summed over. It is this equation that we can reformulate in a manifestly Lorentz covariant form. One then may hope that not only the longitudinal, but also the shifts in all other directions will have been accommodated for. Since according to Eq. (15.14), $W^{12}$ is the dominating component of the tensor $W^{\mu\nu}$, one may rewrite the right hand side of Eq. (15.16) as

$$\varepsilon^{+12} W_{12}(\tilde{\sigma}) \delta^2(\tilde{\sigma} - \tilde{\sigma}') \approx \frac{1}{2} \varepsilon^{+\mu\nu} W^{\mu\nu}(\tilde{\sigma}) \delta^2(\tilde{\sigma} - \tilde{\sigma}'), \quad \text{with} \quad \varepsilon^{+12} = i \varepsilon^{3412} = i.$$  \hspace{1cm} (15.17)

The covariant generalization is then:

$$[W^{\mu\alpha}(\tilde{\sigma}), W^{\mu\beta}(\tilde{\sigma}')] = \frac{1}{2} \delta^2(\tilde{\sigma} - \tilde{\sigma}') \varepsilon^{\alpha\beta\mu\nu} W^{\mu\nu}(\tilde{\sigma}).$$  \hspace{1cm} (15.18)

This equation, as well as (15.16), is invariant under all continuous reparametrizations of the $\tilde{\sigma}$ coordinates (note that $W^{\mu\nu}$, as defined by Eq. (15.12), transforms as a density.)
It is tempting to assume Eq. (15.18) to have a wider range of validity than the non-covariant Eqs. (14.1) – (14.10). After all, Lorentz invariance guarantees that Eq. (15.18) continues to hold when the derivatives of $x^\pm(\tilde{x})$ are arbitrarily large. Unfortunately, the equations (15.16) do not form a closed algebra, since at the left hand side the index $\mu$ is still summed over. One can, however, limit oneself to the self-dual part:

$$K^{\mu\nu} = i(W^{\mu\nu} + \frac{1}{2} \varepsilon^{\mu\nu\kappa\lambda} W^{\kappa\lambda}), \quad (15.19)$$

which has only three independent components:

$$K_1 = i(W^{23} + W^{14}); \quad K_2 = i(W^{31} + W^{24}); \quad K_3 = i(W^{12} + W^{34}), \quad (15.20)$$

and, indeed, their algebra closes. From Eq. (15.19) we derive:

$$[K_a(\tilde{\sigma}), K_b(\tilde{\sigma}')] = i\varepsilon_{abc} K_c(\tilde{\sigma})\delta^2(\tilde{\sigma} - \tilde{\sigma}'). \quad (15.21)$$

The operators $K_a(\tilde{\sigma})$ are distributions. In order to construct representations of the algebra (15.21), we introduce test functions $f(\tilde{\sigma}), g(\tilde{\sigma})$, and write

$$L_a(f) \overset{\text{def}}{=} \int K_a(\tilde{\sigma}) f(\tilde{\sigma}) d^2\tilde{\sigma}, \quad (15.22)$$

$$[L_a(f), L_b(g)] = i\varepsilon_{abc} L_c(fg). \quad (15.23)$$

Restricting oneself to test functions $f$ with $f^2 = f$, which only take the values 0 or 1, we find that the operators $L_a(f)$ obey the commutation rules of the angular momenta:

$$[L_a(f), L_b(f)] = i\varepsilon_{abc} L_c(f). \quad (15.24)$$

Notice that, since these operators $L_a$ are obtained by integrating $K_a$ over the region(s) where $f = 1$, and because the definition of $K_a$ can be traced back to Eq. (15.12), one can rewrite $L_a(f)$ as a contour integral:

$$L_1^{(f)} = i \oint_{\delta f} (x^2 dx^3 + x^1 dx^4), \quad \text{etc.}, \quad (15.25)$$

where $\delta f$ stands for the boundary of the support of $f$. 

![Diagram](image-url)
Suppose now that we have a set of test functions $f$ which are equal to 1 on domains $A$ or $B$, etc., and zero elsewhere. The domains form a lattice (of our choice) on the horizon, see Fig. 7. In each domain we have a set of three operators $L_a$ that commute as angular momentum operators. The states could be formed out of the $|\ell, m\rangle$ eigenstates of $L^2$ and $L_3$. If we combine domains to form some larger domain, the corresponding angular momentum operators must be added to form the new $L$ operators, by the use of Clebsh-Gordan coefficients. Actually, if any of the $\ell$ values is larger than the minimal value $\frac{1}{2}$ (or perhaps, in some cases, 1), one can imagine splitting the corresponding domain into smaller ones with each the $\ell$ value $\frac{1}{2}$. Thus, one may end up with a lattice where on each site one has $m = \pm \frac{1}{2}$. It would not make much sense to maintain domains which have $L = 0$, because the vanishing of the integrals (15.25) would imply that these regions have no spacial extent.

At first sight, this looks like a complete resolution of our problems. If each domain could be attributed an area equal to $4G \ln 2$ (see Eq. (7.12)), we exactly reproduce Eq. (7.11) for the level density. Unfortunately, life is not so simple. In Eq. (15.20), $W^{14} = iW^{40}$ are antihermitean operators, but $W^{12}$, $W^{23}$ and $W^{31}$ are hermitean. Therefore, the hermitean conjugates of $K_a$, and those of $L_a$, are the antiself-dual parts of $W^{\mu\nu}$. The $L_a$ operators are not hermitean, and therefore the $\ell$ and $m$ quantum numbers need not be subjected to the usual constraints of being half-integer, nor to obey the usual inequalities $|m| \leq \ell$.

16. CONSEQUENCES FOR THEORIES OF SPACE-TIME

In a completely satisfactory theory, the consequences of coordinate reparametrization invariance (or whatever replaces that), should have been exploited to the very end. It means that a black hole horizon could be transformed to exist anywhere in space-time. Flat space-time should locally be indistinguishable from an infinite size black hole. This provides us with a dilemma. If the transformation from flat space-time to Rindler space is one-to-one in terms of the elements of Hilbert space, and if indeed the horizon only has one Boolean degree of freedom (such as a spin being $\pm \frac{1}{2}$), at every surface element of size $A_0 = 4G \ln 2$, then also the Hilbert space of ordinary particles in flat space-time should carry only one Boolean degree of freedom for each surface element $A_0$. This is a big contrast with theories such as quantum field theories on lattices, in which the spacelike lattice is three-dimensional, and hence should show at least one Boolean degree of freedom for each fundamental volume element.

Information content may be a crucial concept in theories for Planck scale physics. The fact that information in a given volume $V$ must be limited to a boundary surface, can
also be deduced as follows. Let us ask what is the maximal amount of information that can be stored in any volume $V = R^3$, given only one constraint: the total energy in use should be such that, if gravitational collapse takes place, the resulting black hole should not be larger than $V$, otherwise the information cannot be retrieved.

First, let us fill $V$ with non-interacting particles. the number of different quantum states can easily be derived using thermodynamics. Since

$$E = C_1 VT^4 \lesssim R, \quad \text{and} \quad S = C_2 VT^3,$$

where $C_i$ are constants of order one in natural units, we find

$$T \lesssim C_3 R^{-\frac{1}{2}}; \quad S \lesssim C_4 R^{\frac{3}{2}} \approx C_4 \sqrt{V},$$

(16.1)

The number of states is then approximately $\exp C_4 \sqrt{V}$.

Next, suppose that, due to gravitational attraction, several black holes were formed, having energies $E_i$. Then

$$E = \sum_i E_i \lesssim R; \quad S = \sum_i \pi R_i^2 = 4\pi \sum_i E_i^2.$$

(16.2)

This is bounded by

$$S \lesssim 4\pi E^2 \approx \pi R^2,$$

(16.3)

which equals $\frac{1}{4}$ times the surrounding surface. The bound is indeed saturated if just one black hole is formed that just barely fits inside $V$. More information inside $V$ is impossible.

The information content of 3-space according to the arguments just sketched may be compared to that of a holographic registration of a three dimensional object; the photographic plate has a resolution not better than the Planck length, showing only one pixel that can be either black or white, at every unit $A_0 = 4G \ln 2$ of its surface. This rendering of our three dimensional world can be blurry at best, but because the Planck length is extremely small, nothing of this phenomenon is noticed in ordinary physics.

Attempts to construct quantum field theoretical systems with their information content limited to any given flat (infinite) surface, were made, but not with much success.

Let us return to the argument at the end of Section 9 concerning the notion of causality. It has often been raised as a point of criticism against our scattering matrix Ansatz. See Fig. 8. An observer $A$ passes through an horizon, while also an onserver $B$ detects Hawking radiation. If this were flat space-time, these two observers were considered to be spacelike separated, and therefore their measurement operators commute.
Hilbert space can be factored into a space of states whose properties can be detected by $A$, and another space of states whose properties can be detected by $B$, and possible further factors that can be seen neither by $A$ nor by $B$. If, however, this space were considered to be the horizon of a black hole, the states seen by $A$ are related to the states seen by $B$ through an $S$-matrix, and hence no longer independent. For the black hole physicist, there is no contradiction. Any measurement made by $B$, implies the introduction of states obtained from the Hartle-Hawking vacuum by acting on it with operators that create or remove particles seen by $B$, which for $A$ would be outrageously energetic. These particles would cause gravitational shifts that seriously affect the ingoing objects, including the fragile detectors used by $A$. Thus, these observations cannot be independent. What is new here, even for any possible flat space-time observer, is that trans-Planckian particles are involved (with this term we mean particles whose energies are far beyond the Planck value).

Apparently, new phenomena strongly affect the conventional form of quantum mechanical Hilbert space when trans-Planckian particles enter the scene. With trans-Planckian particles around, spacelike separated operators may no longer commute with each other.

17. OUTLOOK

Even though the philosophy, adhered to in this paper, is completely straightforward, and should not present fundamental conceptual problems, it nevertheless turned out to be extremely difficult to implement it completely. We have not yet been able to fully exploit the presence of fermions in the Standard Model, which should probably lead to anticommuting operators on the horizon. The effects of transverse gravitational shifts were also hard to implement, since these shifts do not commute with the longitudinal ones.
(because of their $\tilde{x}$-dependence). We have not mentioned a third difficulty: the mass shell conditions for the in- and outgoing particles. We took these to be essentially massless, but most particles (such as the electrically charged ones) have a lower bound for their masses. Transverse momenta and masses, however, cause outgoing particles to fall back in again. The difficulty connected to this is the fact that, close to the horizon, ingoing and outgoing states will be difficult to distinguish. Presumably, the splitting of $S$ according to $S = S_{\text{out}} S_{\text{hor}} S_{\text{in}}$ (Eq. 11.1), must be further refined.

The resemblance to string theory in our final results may suggest that one should readdress the black hole using string theory. Some caution however is called for. It is well-known that string theory requires a 10 or 26 dimensional target space, if tachyons and other unphysical features are to be avoided, but such arguments do not directly apply to our present approach: unitarity and causality look very different, as is manifest from the observation that our string constant is purely imaginary. Secondly, by considering the “information content” of the states in our Hilbert space, we infer that a cut-off at the Planck scale is required that turns our world into a discrete one at that scale. This is quite unlike the starting points of string theory. Convergence of the various approaches may well be envisioned, but it is conceivable that two-dimensional conformal quantum field theory is no more (or less) relevant here than it is in certain statistical models such as the Ising model. We should keep in mind that QCD is also a theory that shows stringlike behavior, but clearly lives in four space-time dimensions, so that apparently the formal unitarity arguments are not applicable here.

the observation of black hole–white hole complementarity (Section 11) suggests an interesting relationship between the black hole horizon and the white hole singularity, and vice versa. After all, a white hole singularity would develop as soon as one allows Hawking particles to produce a gravitational field, as one would be tempted to do when contemplating time reversal invariance. Indeed, the point $S$ in Fig. 2b, is not truly a point, but gets the extended shape of a caustic when ingoing matter is deprived of its spherical symmetry. The operator $U_{\text{out}}(\tilde{x})$ could be regarded as the one describing this caustic. When ingoing matter is allowed to enter during sufficiently large time intervals, this caustic becomes a true fractal. At the same time, this fractal may be relevant for the description of the singularity in the time-reversed black hole. A duality relationship between the black hole singularity and the horizon has been proposed in the framework of string theory.

Numerous other attempts have been made to “tame” the quantum black hole. A possible description of “quantum hair” can be imagined by introducing discrete but local symmetries, such as what happens when a Higgs field with large isospin is introduced in an
SO(3) Yang-Mills theory. Introducing such discrete local symmetries appears to be a large step, not corroborated by any observations within the frame of the Standard Model, and in our scheme unnecessary.

Several authors suggest that the black hole horizon area is quantized into multiples of a Planck-sized unit, perhaps \(4G \ln 2\). Now, since the black hole mass carries an imaginary part (see Section 6), the area is actually not very precisely defined, so that a quantization rule of this kind may have relatively little physical impact; by itself, it does not help our understanding very much. However, when combined with loop quantization of gravity, as advocated by Ashtekar, Rovelli, Smolin, and others, perhaps new avenues may be opened. It is obvious that more new ideas are needed.

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