Convergent perturbation theory for lattice models with fermions

V.K. Sazonov\textsuperscript{a,b}

\textsuperscript{a}Institute of Physics, Department of Theoretical Physics, University of Graz, Universitätsplatz 5, Graz, A-8010, Austria
\textsuperscript{b}Department of Theoretical Physics, St. Petersburg State University, Uljanovskaja 1, St. Petersburg, Petrodvorez, 198504, Russia
vasily.sazonov@uni-graz.at, vasily.sazonov@gmail.com

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Abstract

The standard perturbation theory in QFT and lattice models leads to asymptotic expansions. However, an appropriate regularization of the path or lattice integrals allows one to construct convergent series with an infinite radius of the convergence. In the earlier studies this approach was applied to the purely bosonic systems. Here, using bosonization, we develop the convergent perturbation theory for a toy lattice model with interacting fermionic and bosonic fields.

1 Introduction

One of the main tools for studying quantum field theories far away from the small coupling limit is the lattice Monte Carlo. However, lattice simulations are not always applicable. For instance, in case of the complex action they fail because of the sign problem \cite{1} and it becomes necessary to develop alternative methods. Here we do only a first step and do not consider the models with the sign problem, but develop a new approach for lattice computations based on the convergent perturbative expansion.

Application of the standard perturbation theory to QFTs and lattice models is restricted to the region of small values of coupling constants (expansion parameters). This is caused by the asymptotic character of appearing series \cite{2}. Nevertheless, it is possible to regularize initial integrals and to get the convergent expansions. One of the possible regularizations is based on the direct cut-off of the large fluctuations of the fields \cite{3,4,5,6,7}. This method works well for one dimensional integrals, but in multi-dimensional case it leads to the complicated analytically unsolvable integrals. Alternative regularization scheme was developed in \cite{8,9,10,11,12,13}. Its main advantage is that all loop integrals which appear in computations are exactly the same as in the standard perturbation theory. In this paper we generalize the latter method of constructing the convergent perturbation theory and apply it to a lattice model with interacting fermionic and bosonic fields, a toy model of lattice QED.

The standard perturbative expansions are asymptotic because of the incorrect interchange of orders of the integration and summation. Conditions under which the integration and summation are interchangeable, are given by the following version of the Fubini’s theorem, see \cite{18}.

\textbf{Theorem 1} Let \((a, b)\) is a finite or infinite interval, \(u_n(t)\) is a sequence of continuous complex functions defined on \((a, b)\) and

- \(\sum_{n=0}^{\infty} u_n(t)\) converges uniformly on every bounded interval in \((a, b)\),
- at least one of the following quantities defined with Lebesgue integrals is finite

\[
\int_{a}^{b} \left( \sum_{n=0}^{\infty} |u_n(t)| \right) dt, \quad \sum_{n=0}^{\infty} \int_{a}^{b} |u_n(t)| dt.
\]

\footnote{For the constructions of the convergent expansions without regularization see \cite{14,15,16,17}.}
Then
\[ \int_a^b \left( \sum_{n=0}^\infty u_n(t) \right) dt = \sum_{n=0}^\infty \int_a^b u_n(t)dt. \]

In Ref. [8] the convergent perturbation theory satisfying the theorem [4] was constructed for the multi-dimensional integrals in \( \mathbb{R}^N \) space. The generalization to the path integrals with the trace-class operators in the Gaussian measure was obtained in [9, 10]. In both cases the interaction part of integrals was restricted to the continuous polynomials \( P(x) \geq 0 \) with the even degree \( 2m \). In Section 2 we present the construction [8] extended to the integrals of the type

\[ I(g) = \int_{\mathbb{R}^N} e^{-\|x\|^2-gP(x)}dx \]

(1)

with the interaction represented by the bounded from below continuous polynomial \( P(x) \geq -M, M \geq 0 \) with the even degree \( 2m \). In Section 3 using the bosonization of the fermion determinant [19, 20, 21, 22, 23, 24, 25] we generalize convergent perturbation theory to the models containing fermions. We demonstrate our method on the example of toy lattice model which may be considered as a simplest approximation to the lattice QED in the Lorentz gauge.

**2 N-dimensional integrals**

We consider the integral of the form

\[ I(g) = \int_{\mathbb{R}^N} e^{-x_i K_{ij} x_j - P(x)}dx, \]

(2)

where \( K_{ij} \) is the matrix with strictly positive eigenvalues and \( P(x) \) is a bounded from below polynomial of even degree \( 2m \). The main goal of this section is to build up its power expansion satisfying the theorem [1].

The polynomial \( P(x) \) is bounded from below, but can be negative. We define its minimal value as \( P_{\text{min}} \equiv -M \) and then add and subtract \( M \) in the argument of the exponential function in (2)

\[ I(g) = e^M \int_{\mathbb{R}^N} e^{-x_i K_{ij} x_j - |P(x)| + M}dx. \]

(3)

Now the polynomial \( \tilde{P}(x) = P(x) + M \geq 0 \) and to construct converging perturbation theory we apply the program developed in [8]. Let us introduce the Fourier transform of \( \exp(-r^{2m}) \) function

\[ \varphi_m(\rho) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\rho r} e^{-r^{2m}} dr. \]

(4)

Employing the definition and positivity of \( \tilde{P}(x) \) we have

\[ e^{-\tilde{P}(x)} = \int_{-\infty}^{+\infty} \varphi_m(\rho) e^{i\rho \frac{P}{M}(x)} d\rho. \]

(5)

The equation (3) transforms to

\[ I(g) = e^M \int_{\mathbb{R}^N} e^{-x_i K_{ij} x_j} \left[ \int_{-\infty}^{+\infty} \varphi_m(\rho) e^{i\rho \frac{P}{M}(x)} d\rho \right] dx. \]

(6)

At large \( |\rho| \) the function \( \varphi_m(\rho) \) obeys

\[ |\varphi_m(\rho)| \leq C \exp\left(-|\rho|^{1+\frac{1}{2m}}\right), \]

(7)

see [11]. Then one can estimate the integrand in (6) as

\[ \left| e^{-x_i K_{ij} x_j} \varphi_m(\rho) e^{i\rho \frac{P}{M}(x)} \right| \leq C \exp\left(-|\rho|^{1+\frac{1}{2m}} - x_i K_{ij} x_j\right). \]

(8)
The integral $I(g)$ is absolutely convergent in the Lebesgue sense and according to the Fubini’s theorem it is possible to interchange the order of integrations

$$I(g) = e^M \int_{-\infty}^{+\infty} \varphi_m(\rho) \left( \int_{\mathbb{R}^n} e^{-x_i K_{ij} x_j} e^{i \rho \tilde{P} \tilde{\pi} n(x)} dx \right) d\rho. \tag{9}$$

The integral $I(g)$ may be represented as a limit of the proper integral over $\rho$

$$I(g) = \lim_{R \to \infty} J(g, R), \tag{10}$$

where

$$J(g, R) = e^M \int_{-R}^{R} \varphi_m(\rho) \left( \int_{\mathbb{R}^n} e^{-x_i K_{ij} x_j} e^{i \rho \tilde{P} \tilde{\pi} n(x)} dx \right) d\rho. \tag{11}$$

Expanding the function $e^{i \rho \tilde{P} \tilde{\pi} n(x)}$ we get

$$J(g, R) = e^M \int_{-R}^{R} \varphi_m(\rho) \left( \int_{\mathbb{R}^n} e^{-x_i K_{ij} x_j} \left( \sum_{n=0}^{\infty} \frac{i^n \rho^n \tilde{P} \tilde{\pi} n(x)}{n!} \right) dx \right) d\rho. \tag{12}$$

In this case the conditions of the theorem [1] are satisfied and

$$J(g, R) = e^M \sum_{n=0}^{\infty} \frac{A_n(m, R)}{n!} \int_{\mathbb{R}^n} \tilde{P} \tilde{\pi} n(x) e^{-x_i K_{ij} x_j} dx, \tag{13}$$

where

$$A_n(m, R) = l^n \int_{-R}^{R} \varphi_m(\rho) \rho^n d\rho = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left( \frac{d^n}{dr^n} e^{-r^2 m} \right) \sin R \frac{r}{m} dr. \tag{15}$$

The properties of the coefficients $A_n(m, R)$ were carefully studied in [8].

To proceed with the calculation of the integral over $x$ from (13), we rewrite it as

$$\int_{\mathbb{R}^n} \tilde{P}^l(x) \tilde{P} \tilde{\pi} n(x) e^{-x_i K_{ij} x_j} dx, \tag{16}$$

where $l \in \mathbb{N}$ and $k = 1, \ldots, (2m - 1)$. Let $\lambda$ be a half of the smallest eigenvalue of $K$, then we add and subtract $\lambda$ times identity matrix in the argument of the exponential function in (10)

$$\int_{\mathbb{R}^n} \tilde{P}^l(x) \left( \tilde{P}(x) e^{-\frac{\lambda x_i \delta_{ij} x_j}{2}} \right) \tilde{\pi} n(x) e^{-x_i (K_{ij} - \lambda \delta_{ij}) x_j} dx. \tag{17}$$

The function $y = \tilde{P}(x) e^{-\frac{\lambda x_i \delta_{ij} x_j}{2}}$ is positive and bounded for any $x$. Then there is such $a > 0$, that $0 \leq y < a$. According to the Weierstrass theorem the function $y^{k/(2m)}$ can be approximated by a finite degree polynomial with an arbitrary precision. For each $\delta > 0$ there exists a polynomial

$$a_0 + a_1 y + \ldots + a_k y^k,$$

that for all $y \in [0, a]$

$$|y^{k/(2m)} - a_0 - a_1 y - \ldots - a_k y^k| < \delta. \tag{18}$$

After the substitution of $y^{k/(2m)}$ by its polynomial approximation the integral over $x$ reduces to a sum of the Gaussian integral moments which can be easily taken. This finishes the construction of convergent perturbation theory for the integral (2).

Finally, we remark that instead of the function $\varphi_m(\rho)$ one may use any function of the type

$$\varphi(\rho) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i \rho \sigma} e^{-r^k} dr, \quad \text{with even } k \geq 2m. \tag{18}$$
The corresponding regularized integral then looks as

\[
J(g, R) = e^M \int_{-R}^{R} \tilde{\varphi}(\rho) \left[ \int_{\mathbb{R}^N} e^{-x_i K_{ij} x_j} e^{i \rho \tilde{B}^i(x)} dx \right] d\rho \tag{19}
\]

and still satisfies the theorem \[\text{II}\] since

\[
J(g, R) = e^M \int_{-R}^{R} \tilde{\varphi}(\rho) \left[ \int_{\mathbb{R}^N} e^{-x_i K_{ij} x_j} e^{i \rho \tilde{B}^i(x)} dx \right] d\rho \tag{20}
\]

is finite, because \( e^{-x_i K_{ij} x_j} \) decays at infinity sufficiently faster then \( e^{i \rho \tilde{B}^i(x)} \) grows.

### 3 Fermions

The method presented in Section 2 was developed only for bosonic integrals. Employing the bosonization of the fermion determinant \[\text{[19, 20, 21, 22, 23, 24, 25]}\] we extend the convergent perturbation theory to the models with fermions. The procedure which is proposed remains similar for any theory where the fermion contribution can be reduced to the strictly positive determinant. Here we focus on the toy model with interacting bosons and fermions which we derive as a simplest lattice approximation to QED in the Lorentz gauge. The Euclidean action of the continuum QED with two degenerate flavors in the Lorentz gauge is

\[
S_{QED} = \int dx \left[ \frac{1}{4} (F_{\mu \nu})^2 + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 - \sum_{f=1,2} \bar{\psi}_f (i \gamma_\mu D_\mu + m) \psi_f \right].
\]  

Integration over fermions leads to the determinant

\[
det(D + m^2) = det(\gamma_5 (D + m) \gamma_5 (D + m)) = det(-D^2 + m^2) \equiv det(B^2 + m^2), \tag{22}
\]

where \( B \equiv \gamma_5 D \) and \( D = (\gamma_\mu \partial_\mu + i e \gamma_5 A_\mu) \). To obtain the toy model of lattice QED we apply a naive discretization to the gauge part of the action (21) and to the operator \( B \). For this we define \( x \) variable on the finite 4-dimensional lattice with spacing \( a \) and volume \( V \) and substitute all derivatives by the finite difference approximations. Then, the gauge action is given by

\[
S_{gauge} = a^4 \sum_x L_{gauge} = a^4 \sum_x \left\{ \frac{1}{4} \left( \frac{A_\nu(x + a \hat{\mu}) - A_\nu(x)}{a} \right)^2 - \frac{1}{2\alpha} \left( \sum_\mu \frac{A_\mu(x + a \hat{\mu}) - A_\mu(x)}{a} \right)^2 \right\}.
\]  

The action of the operator \( B \) on some spinor \( \psi \) transforms to

\[
\hat{B}(x) \psi(x) = \gamma_5 \left[ \gamma_\mu \frac{\psi(x + a \hat{\mu}) - \psi(x)}{a} + i e A_\mu(x) \psi(x) \right]. \tag{24}
\]

The equations (23) and (24) define our model of lattice QED.

Following [24] we represent the determinant \( \det(\hat{B}^2 + m^2) \) as a result of the integration over the five dimensional bosonic fields \( \phi_n(x) \)

\[
det(\hat{B}^2 + m^2) = \lim_{L \to \infty, b \to 0} \prod_x \left[ d\phi_n^*(x) [d\phi_n(x)] [d\xi^*(x)] [d\xi(x)] \right] \exp \left\{ -a^4 b \sum_{n=0}^{N-1} \sum_x L_{\text{matter}} \right\}, \tag{25}
\]

\[
L_{\text{matter}} = \left( \frac{\phi_n(x) - \phi_{n+1}(x)}{b} - i \hat{B}(x) \phi^*_n(x) \right) \left( \frac{\phi_{n+1}(x) - \phi_n(x)}{b} + i \hat{B}(x) \phi_n(x) \right) + \sqrt{L} \left[ \xi^*(x) (m + i \hat{B}(x) \phi_n(x) + \text{h.c.}) + \frac{1}{2m} \xi^*(x) [\xi(x)] \right], \tag{26}
\]
Here $L$ - is the length of the additional dimension, $N$ is a number of sites and $b$ is a lattice spacing in this dimension,

$$L = N b , \quad 0 \leq n < N , \quad b\|\bar{B}\| \ll 1 .$$  \hfill (27)

The fields $\phi_n(x)$ are five dimensional, but have the same spinorial structure as $(3 + 1)$-dimensional fermionic fields $\tilde{\psi}(x), \psi(x)$. We impose free boundary conditions in the additional dimension

$$\phi_n = 0 , \quad n < 0 , \quad n \geq N .$$  \hfill (28)

The fields $\xi(x)$ are $(3 + 1)$-dimensional and bosonic. Substituting the expression for $\bar{B}(x)$ we obtain

$$L_{\text{matter}} = \left( \frac{\phi^{n+1}(x) - \phi_n(x)}{b} - i \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} + e \gamma_5 \gamma_\mu A_\mu(x) \phi_n(x) \right)$$

$$\cdot \left( \frac{\phi^{n+1}(x) - \phi_n(x)}{b} + i \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} - e \gamma_5 \gamma_\mu A_\mu(x) \phi_n(x) \right)$$

$$+ \sqrt{L} \left[ m \xi^*(x) \phi_n(x) + i \xi^*(x) \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} + \text{h.c.} \right]$$

$$+ \frac{1}{2m} \xi^*(x) \xi(x).$$  \hfill (29)

The Gaussian part of the Lagrangian is

$$L_0 = \left( \frac{\phi^{n+1}(x) - \phi_n(x)}{b} - i \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} \right)$$

$$\cdot \left( \frac{\phi^{n+1}(x) - \phi_n(x)}{b} + i \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} - e \gamma_5 \gamma_\mu A_\mu(x) \phi_n(x) \right)$$

$$+ \sqrt{L} \left[ m \xi^*(x) \phi_n(x) + i \xi^*(x) \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} + \text{h.c.} \right]$$

$$+ \frac{1}{2m} \xi^*(x) \xi(x).$$  \hfill (30)

The interaction part of $L_{\text{matter}}$

$$L_{\text{int}} = \left( \frac{\phi^{n+1}(x) - \phi_n(x)}{b} - i \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} \right)$$

$$\cdot \left( \frac{\phi^{n+1}(x) - \phi_n(x)}{b} + i \gamma_5 \gamma_\mu \frac{\phi_n(x + a \hat{\mu}) - \phi_n(x)}{a} - e \gamma_5 \gamma_\mu A_\mu(x) \phi_n(x) \right)$$

$$+ e \gamma_5 \gamma_\mu A_\mu(x) \phi_n(x)$$

$$+ \sqrt{L} \left[ - e \xi^*(x) \gamma_5 \gamma_\mu A_\mu(x) \phi_n(x) + \text{h.c.} \right]$$  \hfill (31)

is not bounded from below. This is caused by the separation of the Gaussian and interaction parts of the Lagrangian. To apply the method from the previous section we represent $L_{\text{int}}$ as a limit of the bounded function. There are infinitely many ways to do this, here we use the most trivial scheme

$$L_{\text{int}}(x, n) = \lim_{\epsilon \to 0} L_{\text{int}}(x, n, \epsilon) ,$$

where

$$L_{\text{int}}(x, n, \epsilon) = L_{\text{int}}(x, n) + \epsilon \left[ (A_\mu(x) A_\mu(x))^2 + (\phi_n^*(x) \phi_n(x))^2 \right.$$  

$$\left. + (\phi^{n+1}(x) \phi_{n+1}(x))^2 + (\xi^*(x) \xi(x))^2 \right]$$  \hfill (32)

is bounded from below.
The fields \( \phi, \xi \) are complex, however, one can interpret them in terms of their real components \( \phi = c + id, \xi = f + ih \). Then the partition function of the model (23), (24) may be written in the form of the equation (33)

\[
Z = \lim_{L \to \infty, b \to 0} \lim_{\epsilon \to 0} \prod_{x,n} \int [dA_x][d\bar{\phi}_{x,n}][d\phi_{x,n}][d\bar{\xi}_x][d\xi_x] \cdot \exp \left\{ -a^4 b \sum_{n=0}^{N-1} \sum_x \frac{1}{L} \mathcal{L}_{\text{gauge}} + \mathcal{L}_0 + \tilde{\mathcal{L}}_{\text{int}}(x,n,\epsilon) \right\}.
\]

Note that the limit \( \epsilon \to 0 \) is not a priori interchangeable with other limits and should be taken first.

Let \( M \) be an absolute value of the minimum of \( \tilde{\mathcal{L}}_{\text{int}}(x,n,\epsilon) \), we define \( \Lambda_{\text{int}}(x,n,\epsilon) \equiv \tilde{\mathcal{L}}_{\text{int}}(x,n,\epsilon) + M \geq 0 \) and regularize (33) as in the previous section

\[
Z(R) = e^{\tilde{V} M} \lim_{L \to \infty, b \to 0} \lim_{\epsilon \to 0} \prod_{x,n} \int_{-R}^{R} \varphi(\rho) \left[ \int [dA_x][d\bar{\phi}_{x,n}][d\phi_{x,n}][d\bar{\xi}_x][d\xi_x] \cdot \exp \left\{ -a^4 b \sum_{n=0}^{N-1} \sum_x \frac{1}{L} \mathcal{L}_{\text{gauge}} + \mathcal{L}_0 \right\} \cdot \exp \left\{ i\rho \left( a^4 b \sum_{n=0}^{N-1} \sum_x \Lambda_{\text{int}}(x,n) \right) \right\} \right] d\rho,
\]

where \( \tilde{V} = VL \) is the volume of the five dimensional lattice. The Taylor expansion of the second exponential function gives convergent perturbation theory for toy model of lattice QED.

## 4 Conclusions

The series of the standard perturbation theory for path and lattice integrals are asymptotic. This happens because of the illegal interchange of the summation and integration. However, it is possible to regularize integrals in such a way, that all conditions required for the interchanging of the summation and integration will be satisfied. This gives an approach to QFT and lattice computations valid at any arbitrary values of the coupling constants.

In this work we presented the method for constructing the convergent perturbation theory for integrals with interactions bounded from below. Employing the bosonization of the fermion determinant we extended this method to the model of lattice QED. Recently, a bosonization of the complex actions was proposed [26]. Together with the convergent perturbation theory this opens new way to avoid sign problem.

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