On Walker 4-manifolds with pseudo bi-Hermitian structures

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Abstract: \((M_{2n}, g^w, D)\) is a 4-dimensional Walker manifold and this triple is also a pseudo-Riemannian manifold \((M_{2n}, g^w)\) of signature \((+ + --)\) (or neutral), which is admitted a field of null 2-plane. In this paper, we consider bi-Hermitian structures \((\varphi_1, \varphi_2)\) on 4-dimensional Walker manifolds. We discuss when these structures are integrable and when the bi-Kähler forms are symplectic.

Key words: Almost complex structures, symplectic structures, almost Hermitian and Kähler structures, pseudobi-Hermitian structures, Walker manifold.

1. Introduction

Let \(M_{2n}\) be a manifold with a neutral metric which is a pseudo-Riemannian metric \(g\) of signature \((n, n)\). Let \(\mathfrak{A}_q^p (M_{2n})\) be the set of all tensor fields of type \((p, q)\) on \(M_{2n}\). Manifolds, tensor fields, and connections are assumed to be differentiable and of class \(C^\infty\).

The pair \((M_{2n}, \varphi)\) is called an almost complex manifold if the condition \(\varphi^2 = -I\) is hold, where \(I\) is a field of identity endomorphisms and \(\varphi\) is an affinor field \(\varphi \in \mathfrak{A}_1^1 (M_{2n})\). The affinor field \(\varphi\) is integrable if and only if there exists a torsion-free affine connection \(\nabla\) with respect to which the structure tensor \(\varphi\) is covariantly constant, i.e., \(\nabla \varphi = 0\). Moreover, if the Nijenhuis tensor of such an affinor field \(\varphi\) defined by

\[
N_{\varphi}(X, Y) = [\varphi X, \varphi Y] - \varphi [\varphi X, Y] - \varphi [X, \varphi Y] + [X, Y]
\]

is equivalent to the vanish, then the almost complex structure \(\varphi\) is called integrable. In this case, the almost complex manifold \((M_{2n}, \varphi)\) is called a complex manifold.

Let \(M_{2n}\) be a 4-dimensional complex manifold and \(\varphi_i, \) for \(i = 1, 2,\) be two independent compatible integrable almost complex structures. Here \(\varphi_1(x) \neq \varphi_2(x)\) for a point \(x\) in \(M_{2n}\). Also, \(g\) metric is a Hermitian metric with respect to both complex structures \(\varphi_1\) and \(\varphi_2\), i.e.,

\[
g(\varphi_1 X, \varphi_1 Y) = g(X, Y) \quad \text{and} \quad g(\varphi_2 X, \varphi_2 Y) = g(X, Y).
\]

In this case, the quartet \((M_{2n}, g, \varphi_1, \varphi_2)\) is called bi-Hermitian manifold. If \(\varphi_1(x) \neq \varphi_2(x)\) everywhere on \(M_{2n}\), a bi-Hermitian structure \((g, \varphi_1, \varphi_2)\) is called strongly bi-Hermitian. The real function \(p\) is defined by

\[
p = -\frac{1}{4} \text{trace} (\varphi_1 \circ \varphi_2)
\]

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or equivalently
\[ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -2pI, \]
where \( p \) is the angle function of a bi-Hermitian structure and where \( I \) is the field of identity endomorphisms [1,13].

An almost Hermitian structure on a manifold \( M_{2n} \) consists of a nondegenerate 2-form \( w \), an almost complex structure \( \varphi \) and a metric \( g \) satisfying the compatibility condition \( w(X,Y) = g(\varphi X, Y) \). If the 2-form \( w \) is closed, i.e., \( dw = 0 \), a triple \( (g, \varphi, w) \) is called an almost Kähler structure. Also, the triple \( (g, \varphi, w) \) is called Kähler structure if the almost complex structure \( \varphi \) is integrable [4].

Let \( (M_{2n}, g, \varphi_1, \varphi_2) \) be a bi-Hermitian manifold. For such a structure we define 2-forms \( w_i \) setting \( w_i(X,Y) = g(\varphi_i X, Y) \), \( i = 1,2 \). If the 2-forms \( w_i \) are closed (\( dw_i = 0 \)), the bi-Hermitian structure is called bi-Kähler. Such bi-Hermitian structures have been studied by many authors (see, e.g. [1-3, 13]).

2. Walker metrics
Let \( M_{2n} \) be a 4-dimensional manifold and \( g^w \) be a neutral metric (or \( g^w \) is of signature \( ++-- \)). \( g^w \) is called Walker metric if there exists a 2-dimensional null distribution \( D \) on \( M_{2n} \), which is parallel with respect to \( g^w \). Such metrics are studied by Walker [15] and canonical form of the metric \( g^w \) is given by
\[ g^w = (g^w_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \] (2.1)
where \( a, b, \) and \( c \) are some functions depending on the coordinates \( (x^1, x^2, x^3, x^4) \). Note that the parallel null 2-plane \( D \) is spanned locally by \( \{\partial_1, \partial_2\} \), where \( \partial_i = \frac{\partial}{\partial x^i} (i = 1, 2, 3, 4) \). Such Walker manifolds are intensively investigated (see, e.g. [4-12,14,15]).

3. Almost bi-Hermitian structures on a neutral 4-manifold
In this section, we consider 4-dimensional pseudo-Riemannian manifolds of neutral signature. For the next step, it is appropriate to state a neutral metric \( g \) and the almost complex structure \( \varphi \) in terms of an orthonormal frame \( \{e_i\}, (i = 1, 2, 3, 4) \) of vectors and its dual frame \( \{e^j\}, (j = 1, 2, 3, 4) \) of 1-forms. The metric \( g \) is given by
\[ g = (g(e_i, e_j)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \] (3.1)
Let \( (M_{2n}, g, \varphi_1, \varphi_2) \) be a bi-Hermitian manifold. From identity \( \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -2pI \), two almost complex structures \( \varphi_1 \) and \( \varphi_2 \) can be written as:
\[ \varphi_1 = (\varphi_1^i_j) = \begin{pmatrix} 0 & 3 & -2 & 2 \\ -3 & 0 & 2 & 2 \\ -2 & 2 & 0 & 3 \\ 2 & 2 & -3 & 0 \end{pmatrix}, \] (3.2)
\[
\varphi_2 = (\varphi_{2j}^i) = \begin{pmatrix}
0 & 3 & 2 & 2 \\
-3 & 0 & 2 & -2 \\
2 & 2 & 0 & 3 \\
2 & -2 & -3 & 0
\end{pmatrix}.
\] (3.3)

According to \(g\), \(\varphi_1\), and \(\varphi_2\), we have two kinds of Kähler forms on 4-manifolds which are given by
\[
w_1(X,Y) = g(\varphi_1 X, Y), \quad w_2(X,Y) = g(\varphi_2 X, Y). \quad (3.4)
\]

Equation (3.4) is equivalent to in matrix notations in the following equation
\[
w_1 = \varphi_1^T g, \quad w_2 = \varphi_2^T g, \quad (3.5)
\]
where matrix \(\varphi^T\) is the transpose matrix of matrix \(\varphi\). From (3.1)–(3.3) and (3.5), we can write
\[
w_1 = (w_{1ij}) = \begin{pmatrix}
0 & -3 & 2 & -2 \\
3 & 0 & -2 & -2 \\
-2 & 2 & 0 & 3 \\
2 & 2 & -3 & 0
\end{pmatrix}, \quad (3.6)
\]
\[
w_2 = (w_{2ij}) = \begin{pmatrix}
0 & -3 & -2 & -2 \\
3 & 0 & -2 & 2 \\
2 & 2 & 0 & 3 \\
2 & -2 & -3 & 0
\end{pmatrix} \quad (3.7)
\]

These Kähler forms in terms of the local orthonormal basis \(\{e^j\} (j = 1, 2, 3, 4)\) of 1-forms are written as:
\[
w_1 = \sum_{i<j} w_{1ij} \ e^i \wedge e^j = -3e^1 \wedge e^2 + 2e^1 \wedge e^3 - 2e^1 \wedge e^4
\]
\[-2e^2 \wedge e^3 - 2e^2 \wedge e^4 + 3e^3 \wedge e^4, \quad (3.8)
\]
\[
w_2 = \sum_{i<j} w_{2ij} \ e^i \wedge e^j = -3e^1 \wedge e^2 - 2e^1 \wedge e^3 - 2e^1 \wedge e^4
\]
\[-2e^2 \wedge e^3 + 2e^2 \wedge e^4 + 3e^3 \wedge e^4. \quad (3.9)
\]

4. Almost bi-Hermitian structures and bi-Kähler forms on Walker 4-manifolds
Let \((M_{2n}, g^w)\) be a Walker-4 manifold which is given in (2.1), where \(g^w\) is Walker metric and let \(\{e_i\} \) and \(\{\partial_i\}, (i = 1, 2, 3, 4)\) be two orthonormal frames. Also, matrix \(A = (A^i_j)\) of the change of coordinates satisfies:
\[
g = A^T g^w A, \quad (4.1)
\]
where \(A^T\) is the transpose matrix of \(A\).
Substituting (2.1) and (3.1) in (4.1), one of the matrices which we apply in the present analysis, we obtain as:

\[
A = (A_j^i) = \begin{pmatrix}
0 & -\left(\frac{1-a^2}{2}\right) & 0 & \frac{1+a}{c}
\frac{1+b}{c} & 0 & -\left(\frac{1+b}{2}\right) & 0
1 & 0 & 1 & 0
\end{pmatrix}.
\] (4.2)

Also, matrix \(A = (A_j^i)\) of the change of coordinates satisfies:

\[
\varphi = A^{-1}\varphi'A,
\] (4.3)

where \(A^{-1}\) is the inverse matrix of \(A\) and it is given by:

\[
A^{-1} = \begin{pmatrix}
0 & 1 & -\frac{c}{2} & \frac{1+b}{2}
-1 & 0 & -\left(\frac{1+a}{2}\right) & 0
0 & -1 & -\frac{c}{2} & \frac{1+b}{2}
1 & 0 & -\left(\frac{1-a}{2}\right) & 0
\end{pmatrix}.
\] (4.4)

Substituting (3.2), (4.2), and (4.4) in (4.3), the almost complex structure in (3.2) is obtained as the following:

\[
\varphi_1' = \begin{pmatrix}
-2 & 5 & 5c - 2a & \frac{1}{2}(5b - a)
-1 & 2 & \frac{1}{2}(5b - a) & 2b - c
0 & 0 & 2 & 1
0 & 0 & -5 & -2
\end{pmatrix}.
\] (4.5)

Similarly, substituting (3.3), (4.2), and (4.4) in (4.3), the almost complex structure in (3.3) is obtained as the following:

\[
\varphi_2' = \begin{pmatrix}
2 & 5 & 5c + 2a & \frac{1}{2}(5b - a)
-1 & -2 & \frac{1}{2}(5b - a) & -2b - c
0 & 0 & -2 & 1
0 & 0 & -5 & 2
\end{pmatrix}.
\] (4.6)

A matrix \(A = (A_j^i)\) of the change of coordinates for the tensor fields of type \((0, 2)\) satisfies:

\[
w = A^T w'A,
\] (4.7)

where \(A^T\) is the transpose matrix of \(A\).

Substituting (3.6) and (4.2) in (4.7), the bi-Kähler form in (3.6) is obtained as:

\[
w_1' = \begin{pmatrix}
0 & 0 & -2 & -1
0 & 0 & 5 & 2
2 & -5 & 0 & \frac{1}{2} (-a - 5b + 4c)
1 & -2 & -\frac{1}{2} (-a - 5b + 4c) & 0
\end{pmatrix}.
\] (4.8)

The bi-Kähler form in (4.8) is written in terms of the coordinate basis as follows:

\[
w_1' = \sum_{i<j} w_1'_{ij} dx^i \wedge dx^j = -2 dx^1 \wedge dx^3 - dx^1 \wedge dx^4 + 5 dx^2 \wedge dx^3 +
\]
\[
2dx^2 \wedge dx^4 + \frac{1}{2}(-a - 5b + 4c)dx^3 \wedge dx^4. \tag{4.9}
\]

Similarly, substituting (3.7) and (4.2) in (4.7), we obtain the bi-Kähler form in (3.7) as:

\[
w_2' = \left( w_2'_{ij} \right) = \begin{pmatrix}
0 & 0 & 2 & -1 \\
0 & 0 & 5 & -2 \\
-2 & -5 & 0 & -\frac{1}{2}(a + 5b + 4c) \\
1 & 2 & \frac{1}{2}(a + 5b + 4c) & 0
\end{pmatrix}. \tag{4.10}
\]

Also, in terms of the coordinate basis, the bi-Kähler form in (4.10) is written as follows:

\[
w_2' = \sum_{i<j} w_2'_{ij} dx^i \wedge dx^j = 2dx^1 \wedge dx^3 - dx^1 \wedge dx^4 + 5dx^2 \wedge dx^3 -
2dx^2 \wedge dx^4 - \frac{1}{2}(a + 5b + 4c)dx^3 \wedge dx^4. \tag{4.11}
\]

5. Integrability of \(\varphi_1\) and \(\varphi_2\) (bi-Hermitian structures)

The almost complex structure \(\varphi\) is integrable if and only if

\[
\left( N_{\varphi'} \right)^i_{jk} = \varphi'_{jm} \partial_m \varphi'^i_k - \varphi'_{km} \partial_m \varphi'^i_j - \varphi'_{mj} \partial_j \varphi'^m_k + \varphi'^i_m \partial_k \varphi'^m_j = 0. \tag{5.1}
\]

From (4.5) and (5.1), the Nijenhuis tensor of \(\varphi_1\) in (4.5) has nonzero components as follows:

\[
N^x_{xz} = -N^x_{zx} = 2a_y - 5c_y - \frac{25}{2}b_x + \frac{5}{2}a_x, \\
N^x_{xt} = -N^x_{tx} = -\frac{5}{2}b_y + \frac{1}{2}a_y - 10b_x + 5c_x, \\
N^y_{xz} = -N^y_{zx} = -10b_x + 5c_x - \frac{5}{2}b_y + \frac{1}{2}a_y, \\
N^y_{xt} = -N^y_{tx} = -8b_x + 4c_x - 2b_y + c_y + \frac{5}{2}b_x - \frac{1}{2}a_x, \\
N^x_{yz} = -N^x_{zy} = 25c_x - 10a_x + 20c_y - 8a_y - \frac{25}{2}b_y + \frac{5}{2}a_x, \\
N^y_{yz} = -N^y_{zy} = \frac{25}{2}b_x - \frac{5}{2}a_x + 5c_y - 2a_y, \\
N^x_{yt} = -N^x_{ty} = \frac{25}{2}b_x - \frac{5}{2}a_x + 5c_y - 2a_y,
\]
\[ N^y_{yt} = -N^y_{ty} = \frac{5}{2} b_y - \frac{1}{2} a_y + 10 b_x - 5 c_x, \]

\[ N^x_{zt} = -N^x_{tz} = \left( \frac{25 c}{2} - 5 a \right) b_x + \left( -\frac{5 c}{2} + 5 b \right) a_x + \left( -\frac{25 b}{2} + 5 a \right) c_x + \left( \frac{21b - a}{4} - 2c \right) a_y \]

\[ + \left( -\frac{25 b + 5a}{4} \right) b_y + (-10b + 5c) c_y, \]

\[ N^y_{zt} = -N^y_{tz} = \left( \frac{-11a - 25b}{4} + 10c \right) b_x + \left( \frac{5b - a}{4} \right) a_x + \left( \frac{5c - a}{4} \right) b_y + \left( b - \frac{c}{2} \right) a_y \]

\[ + \left( -\frac{5b + a}{2} \right) c_y + (-5c + 2a) c_x. \]

From these equations, we have:

**Theorem 5.1** The almost complex structure \( \varphi'_1 \) is integrable if and only if the following PDEs hold:

\[ 2b_x - c_x = 0, \quad 2b_y - c_y = 0, \]

\[ 5b_x - a_x = 0, \quad 5b_y - a_y = 0. \]  

(5.2)

From (4.6) and (5.1), the Nijenhuis tensor of \( \varphi'_2 \) in (4.6) has nonzero components as follows:

\[ N^x_{xz} = -N^x_{zx} = -2a_y - 5c_y - \frac{25}{2} b_x + \frac{5}{2} a_x, \]

\[ N^x_{xt} = -N^x_{tx} = -\frac{5}{2} b_y + \frac{1}{2} a_y + 10 b_x + 5 c_x, \]

\[ N^y_{xz} = -N^y_{zx} = 10b_x + 5c_x - \frac{5}{2} b_y + \frac{1}{2} a_y, \]

\[ N^y_{xt} = -N^y_{tx} = -8b_x - 4c_x + 2b_y + c_y + \frac{5}{2} b_x - \frac{1}{2} a_x, \]

\[ N^x_{yz} = -N^x_{zy} = 25c_x + 10a_x - 20c_y - 8a_y - \frac{25}{2} b_y + \frac{5}{2} a_y, \]

\[ N^y_{yz} = -N^y_{zy} = \frac{25}{2} b_x - \frac{5}{2} a_x + 5 c_y + 2a_y, \]

\[ N^x_{yt} = -N^x_{ty} = \frac{25}{2} b_x - \frac{5}{2} a_x + 5 c_y + 2a_y, \]
\[ N_{yt} = -N_{ty} = \frac{5}{2} b_y - \frac{1}{2} a_y - 10 b_x - 5 c_x, \]
\[ N_{zt} = -N_{tz} = (\frac{25c}{2} + 5a) b_x + (\frac{-5c}{2} - 5b) a_x + (\frac{-25b}{2} + \frac{5a}{2}) c_x + \]
\[ (\frac{11b + a}{4} + 2c) a_y + (\frac{25b - 5a}{4}) b_y + (10b + 5c) c_y, \]
\[ N_{yt} = -N_{ty} = (\frac{-11a - 25b}{4} - 10c) b_x + (\frac{5b - a}{4}) a_x + (\frac{5c}{2} + a) b_y + \]
\[ (\frac{-c}{2} - b) a_y + (\frac{-5b + a}{2}) c_y + (-5c - 2a) c_x. \]

From these equations, we have:

**Theorem 5.2** The almost complex structure \( \varphi_2' \) is integrable if and only if the following PDEs hold:
\[ 20b_x + 10c_x - 5b_y + a_y = 0, \]
\[ 25b_x - 5a_x + 10c_y + 4a_y = 0. \] (5.3)

From (5.2) and (5.3), we can write the following integrability conditions for almost bi-Hermitian–Walker structures.

**Theorem 5.3** The triple \((g^w, \varphi_1', \varphi_2')\) is bi-Hermitian–Walker structure if and only if the following PDEs hold:
\[ a_x = a_y = b_x = b_y = c_x = c_y = 0. \] (5.4)

### 6. Symplectic structures

In this section, we focus our attention on bi-Kähler forms \((w_1', w_2')\) which are symplectics, i.e,
\[ dw_i' = 0 \quad (i = 1, 2). \] (6.1)

From (4.9), external differential of \( w_1' \) is written as:
\[ dw_1' = -\frac{1}{2} (a_1 + 5b_1 - 4c_1) dx^1 \wedge dx^3 \wedge dx^4 - \frac{1}{2} (a_2 + 5b_2 - 4c_2) dx^2 \wedge dx^3 \wedge dx^4. \]

Therefore, we have:

**Theorem 6.1** The Kähler form in (4.9) is a symplectic form \((dw_1' = 0)\) if the following PDEs hold:
\[ a_1 + 5b_1 - 4c_1 = 0, \]
\[ a_2 + 5b_2 - 4c_2 = 0. \] (6.2)
From (4.11), external differential of $w_2'$ is written as:

$$dw_2' = -\frac{1}{2}(a_1 + 5b_1 + 4c_1)dx^1 \wedge dx^3 \wedge dx^4 - \frac{1}{2}(a_2 + 5b_2 + 4c_2)dx^2 \wedge dx^3 \wedge dx^4.$$ 

Therefore, we have:

**Theorem 6.2** The Kähler form in (4.11) is a symplectic form ($dw_2' = 0$) if the following PDEs hold:

$$a_1 + 5b_1 + 4c_1 = 0,$$

$$a_2 + 5b_2 + 4c_2 = 0.$$  \hspace{1cm} (6.3)

From Theorem 6.1 and Theorem 6.2, we can write the following theorem:

**Theorem 6.3** The quinary ($g^w, \phi_1', \phi_2', w_1', w_2'$) is bi-Kähler–Walker if and only if the following PDEs hold:

$$a_1 + 5b_1 = 0, c_1 = 0,$$

$$a_2 + 5b_2 = 0, c_2 = 0.$$  \hspace{1cm} (6.4)

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