Algorithms for enumerating and counting 
D2CS of some graphs

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Abstract

We define a D2CS of a graph $G$ to be a set $S \subseteq V(G)$ with $diam(G[S]) \leq 2$. A D2CS arises in connection with conditional coloring and radio-$k$-coloring of graphs. We study the problem of counting and enumerating D2CS of a graph. We first prove the following propositions:

(1) Let $f(k, h)$ be the number of D2CS of a complete $k$-ary tree of height $h$. Then
$$f(k, h) = \frac{1}{k^2}(f(k+1, 1) - 4)(k^{h-1} - 1) + f(k, 1)$$
and $f(k, 1) = 2^k + k + 1$.

(2) A Fibonacci tree, a variant of a binary tree is defined recursively as follows: (a) Fibonacci tree of order 0 and 1 is a single node. (b) Fibonacci tree of order $n$ ($n \geq 2$) is constructed by attaching tree of order $n - 2$ as the leftmost child of the tree of order $n - 1$. Let $g(n)$ denote the number of D2CS in a Fibonacci tree of order $n$. Then
$$g(n) = 3 \cdot 2^{n-2} - (F_{n-1} + F_{n+1}) + 2.$$

(3) A binary Fibonacci tree of order $n$ ($n > 1$) is a variant of a binary tree whose left subtree is of order $n - 1$ and right subtree of order $n - 2$. An order 0 Fibonacci tree has a single node, and an order 1 tree is $P_2$. Let $h(n)$ denote the number of D2CS in a binary Fibonacci tree of order $n$. Then
$$h(n) = 2F_n + 3F_{n+2} - 9.$$

(4) A binomial tree $B_k$ of order $k$ ($k \geq 0$) is an ordered tree defined recursively as: (i) $B_0$ is a one-vertex graph. (ii) $B_k$ consists of two copies of $B_{k-1}$ such that the root of one is the leftmost child of the root of the other. Let $b(k)$ denote the number of D2CS in a binomial tree $B_k$. Then $b(k) = k2^k + 2$.

(5) Let $G$ be a split graph with $K \subseteq V(G)$, $|K| = \omega(G) = k$, for all $v \in K$ and $r > 1$, $d(v) = k + r - 1$ and for all $v' \in V(G) \setminus K$, $d(v') = 1$. Then the number of D2CS in $G$ is $k2^{k-1}(2^r - 1) + 2^k + kr$.

We then show: all D2CS in a given graph $G$ with $n$ vertices can be enumerated in time $O(n^3/log^2n)$ for each D2CS. We finally show: all maximal D2CS in a strongly chordal graph on $n$ vertices can be enumerated and counted in time $O(n)$.

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1 Introduction

Let $G = (V(G), E(G))$ be a simple, connected, undirected graph. For a vertex $v \in V(G)$, the (open) neighborhood of $v$ in $G$ is $N(v) = \{u \in V(G) : (u, v) \in E(G)\}$, the closed neighborhood of $v$ is defined as $N[v] = N(v) \cup \{v\}$ and the degree of $v$ is $d(v) = |N(v)|$. The subgraph induced by a set $A \subseteq V(G)$ is denoted by $G[A]$. Let $\omega(G)$ (or simply $\omega$) denote the clique number of a graph $G$. The square of a graph $G$, denoted by $G^2$, has the same vertex set as $G$, and the edge set $E(G^2) = \{(u, v) : d(u, v) \leq 2 \text{ in } G\}$. The distance $d(u, v)$ between two vertices $u$ and $v$ is the minimum length of a path between $u$ and $v$. The diameter of a graph $G$ is $\text{diam}(G) = \max \{d(u, v) : u, v \in V(G)\}$. For a vertex $v \in V(G)$ we define by $N^k(v) = \{u : u \in N(v), u > k\}$. We define a distance-2-clique-set (D2CS) of a graph $G$ as a subset $S$ of $V(G)$ such that every two distinct vertices in $S$ are at a distance at most two in $G[S]$ i.e., $\text{diam}(G[S]) \leq 2$. For example $K_{1,n}$ has $2^n + n + 1$ D2CS. A D2CS is maximal if it is not properly contained in any other D2CS. A maximum D2CS is one which has the largest size among all D2CS. For undefined terms and notations see standard texts in graph theory such as [2, 5].

Recently counting and enumeration of certain specified sets in a graph have been widely investigated e.g., in data mining. In this paper we deal with the problem of counting and enumeration of D2CS of general and some restricted class of graphs. Unlike a clique and an independent set every subset of a D2CS need not be a D2CS. In general, the problem of finding all D2CS is difficult. A graph can have an exponential number of D2CS. For example the complete graph $K_n$ on $n$ vertices has $2^n$, a $k$-tree on $n$ vertices has $2^n(1 - 2^{(k+1)}) + n - k$, the ladder graph $L_n \cong P_n \Box P_2$ has $10n - 6$ and the graph $\overline{K_n}$ of $n$ vertices has just $n + 1$ D2CS. The number of D2CS in any other graph with $n$ vertices lies between $O(n)$ and $O(2^n)$.

2 D2CS of some structured graphs

We first give our results for the cases when the graph $G$ is a: complete $k$-ary tree, Fibonacci tree, binary Fibonacci tree and binomial tree. For two integers $k, h > 0$, let $f(k, h)$ be the number of D2CS of a complete $k$-ary tree of height $h$. Then we have

$$f(k, h) = f(k, h - 1) + k^{h-1}(2^{k+1} + k - 2), \quad h > 1, \text{ and}$$

$$f(k, 1) = 2^k + k + 1.$$ 

By solving the above recurrence we get

$$f(k, h) = \frac{k}{k - 1}(f(k + 1, 1) - 4)(k^{h-1} - 1) + f(k, 1). \quad (1)$$

**Proposition 1.** Let $f'(k, h)$ be the number of D2CS of a rooted tree $T$ with $\Delta(T) = k$ and height $h$. Then $2^k + k + 3h - 5 \leq f'(k, h) \leq (2^k + k - 3)(2 + l) + 4$, where $l = \frac{k}{k - 2}(k(k - 1)^{h-2} - 2)(k - 1)$.

**Proof.** It can be easily checked that $f'(k, h) \geq 2^k + k + 3h - 5$. By $f'_{max}(k, h)$ we denote the maximum possible number of D2CS of a tree $T$ with $\Delta(T) = k$ and height $h$. Considering the
root at level 0, a tree $T$ with $\Delta(T) = k$ and height $h$ has maximum number of D2CS only if it has $k(k - 1)^{i-1}$ vertices at level $i$, for all $1 \leq i \leq h$. Then it is easy to see that

$$f'_{\text{max}}(k, h) = f(k - 1, h) + f(k - 1, h - 1) + 2^k - 2.$$  

By virtue of Eq.(1) we obtain $f'_{\text{max}} = (2^k + k - 3)(2 + l) + 4$, where $l = \frac{k-1}{k-2}(k(k - 1)^{h-2} - 2)$. □

### 2.1 Fibonacci Trees and Binomial Trees

**Definition 1.** A Fibonacci tree, a variant of a binary tree is defined recursively as follows: (a) Fibonacci tree of order 0 and 1 is a single node. (b) Fibonacci tree of order $n$ ($n \geq 2$) is constructed by attaching tree of order $n - 2$ as the leftmost child of the tree of order $n - 1$. Let $g(n)$ denote the number of D2CS in a Fibonacci tree of order $n$. Then,

$$g(n) = g(n - 1) + g(n - 2) + 3 \cdot 2^{n-4} - 2, \quad n \geq 4,$$

with the initial conditions $g(2) = 2$ and $g(3) = 4$.

The (ordinary) generating function $G(z)$ for the sequence $g(n)$ is given by

$$G(z) = \frac{5z^3 - z^2 - 4z + 2}{(2z^2 - 3z + 1)(1 - z - z^2)}.$$

It then follows that

$$g(n) = 3 \cdot 2^{n-2} - (F_{n-1} + F_{n+1}) + 2 = 3 \cdot 2^{n-2} - L_n + 2,$$

where $L_n$ is the $n^{th}$ Lucas number.

**Definition 2.** A binary Fibonacci tree of order $n$ ($n > 1$) is a variant of a binary tree whose left subtree is of order $n - 1$ and right subtree of order $n - 2$. An order 0 Fibonacci tree has a single node, and an order 1 tree is $P_2$. Let $h(n)$ denote the number of D2CS in a binary Fibonacci tree of order $n$. Then

$$h(n) = h(n - 1) + h(n - 2) + 9, \quad n \geq 5$$

with the initial conditions $h(3) = 10$ and $h(4) = 21$.

The (ordinary) generating function $G(z)$ for the sequence $h(n)$ is given by

$$G(z) = \frac{10 + z + z^2}{(1 - z)(1 - z - z^2)}.$$

It then follows that

$$h(n) = 2F_n + 3F_{n+2} - 9 = 2L_{n+1} + F_{n+2} - 9.$$

**Definition 3.** A binomial tree $B_k$ of order $k$ ($k \geq 0$) is an ordered tree defined recursively as follows

(i) $B_0$ is a one-vertex graph.
(ii) $B_k$ consists of two copies of $B_{k-1}$ such that the root of one is the leftmost child of the root of the other.

Let $b(k)$ denote the number of D2CS in a binomial tree $B_k$. Then we have

$$b(k) = 2b(k-1) + 2^k - 2, \quad k \geq 1,$$

with the initial condition $b(0) = 2$.

By solving the above recurrence, we obtain $b(k) = k2^k + 2$. Thus, $b(k)$ grows exponentially in the order of binomial tree.

**Definition 4.** A graph is a split if there is a partition of its vertex set into a clique and an independent set.

**Proposition 2.** Let $G$ be a split graph with $K \subseteq V(G)$, $|K| = \omega(G) = k$, for all $v \in K, d(v) = k + r - 1$ and for all $v' \in V(G) \setminus K, d(v') = 1$. Then the number of D2CS of $G$ is $k2^{k-1}(2^r - 1) + 2^k + kr$.

**Proof.** From the given conditions it is clear that if $S$ is a D2CS of $G$ then there exists a vertex $v \in K$ such that and $S \subseteq N[v]$. We know that the number of D2CS of $G$ of size 0, 1 and 2 are respectively 1, $|V(G)| = k(r + 1)$ and $E(G) = \binom{k}{2} + kr$. Now we count the number of D2CS of size at least 3. Let $S$ be a D2CS of $G$ with $|S| \geq 3$, then $S$ fits into one of the following three cases:

Case (i) $: |S \setminus K| = 0$. Clearly the number of D2CS of this form is $2^k - \binom{k}{2} - k - 1$.

Case (ii) $: |S \setminus K| = 1$. The number of D2CS of this form is $(2^{k-1} - 1)kr$.

Case (iii) $: |S \setminus K| \geq 2$. The number of D2CS of this form is $k2^{k-1}(2^r - r - 1)$.

Therefore the number of D2CS of cardinality greater than two is $2^k - \binom{k}{2} - k - 1 + (2^{k-1} - 1)kr + k2^{k-1}(2^r - r - 1)$. So the total number of D2CS of $G$ is $1 + k(r + 1) + \binom{k}{2} + kr + 2^k - \binom{k}{2} - k - 1 + (2^{k-1} - 1)kr + k2^{k-1}(2^r - r - 1)$, which gives the result. □

3 **Algorithm for Counting and Enumerating the D2CS of a Graph**

In this section we describe an algorithm for counting and enumerating the D2CS of a graph. The basic idea is obtaining $G^2$ and generating all the cliques in $G^2$. Then all those cliques of $G^2$ which are not D2CS of $G$ are eliminated.

**Fact 1.** Every D2CS in $G$ is a clique in $G^2$.

**Proof.** Proof follows from the definitions of D2CS and $G^2$. □

Algorithm **EnumAllD2CS** enumerates and counts D2CS of a graph $G$ with $|V(G)| = n$ and $|E(G)| = m$. The algorithm outputs the number of D2CS in $G$.

**Algorithm** EnumAllD2CS($G$)

1. Enumerate all cliques in $G^2$; let $T_S = \{S : S$ is a clique in $G^2 \ & |S| \geq 3\}$
2. Eliminate those elements of $T_S$, which are not D2CS of $G$; let $T'_S = \{S : S$ is a D2CS in $G$ and $|S| \geq 3\}$
3. Return $|T'_S| + n + m + 1$. 4
Correctness and complexity: From Fact 1 it is clear that *EnumAllD2CS* doesn’t miss any D2CS of $G$. Step 2 of the algorithm ensures that no wrong D2CS is generated. Hence the correctness of *EnumAllD2CS*.

The complexity of *EnumAllD2CS* corresponds to the question of determining the number of D2CS of $G$. Clearly step 1 of the algorithm takes $O(f(n)+g(n))$ time, where $f(n)$ is the complexity of boolean matrix multiplication and $g(n)$ is the complexity of generating all the cliques in a graph on $n$ vertices and step 2 takes $O(n^3/\log^2 n)$ time for each element of $T$. Summarizing, we have:

**Theorem 1.** Let $G$ be a graph. All D2CS in $G$ can be enumerated in $O(n^3/\log^2 n)$ time for each.

**Remark.** It is easy to see that the complexity of the above algorithm is no more than $O(f(n)+g(n)n^3/\log^2 n)$, where $f(n)$ and $g(n)$ at present stands at $O(n^{2.376})$ and $O(n\log n+2^2)$ respectively.

### 4 Linear-Time Algorithm for Enumerating Maximal D2CS in a Strongly Chordal Graph

Given a graph $G$, a vertex $v \in V(G)$ is called *simplicial* in $G$ if $N[v]$ induces a clique. Fulkerson and Gross [4] showed that a graph $G$ is *chordal* if and only if it is possible to order the vertices \(\{v_1, \ldots, v_n\}\) of $V(G)$ in such a way that for each $i \in \{1, \ldots, n\}$, the vertex $v_i$ is simplicial vertex of $G_i = G[\{v_i, \ldots, v_n\}]$. Such an ordering is called a *perfect elimination ordering* (also referred as the p.e.o). Let $N_i[v]$ denote the closed neighborhood of $v$ in $G_i$. The ordering of the vertices $v_1, \ldots, v_n$ is called a *strong elimination ordering* (also referred as the s.e.o), if it is a p.e.o and for each $i < j < k$, if $v_j, v_k \in N_i[v_i]$ then $N_i[v_j] \subseteq N_i[v_k]$. Throughout this section we assume that the vertices are numbered in s.e.o. order.

**Definition 5.** [3] A graph $G$ is *strongly chordal* if and only if it admits a s.e.o.

**Lemma 1.** Let $G$ be a strongly chordal graph. Every maximal D2CS in $G$ is of the form $N[v]$, where $v \in V(G)$.

**Proof.** Let $A$ be a maximal D2CS of $G$. We assume that $w$ and $z$ represent respectively the lowest and the highest numbered vertices of $A$. Now, we have three cases:

Case (i) : $A = N[z]$. The lemma holds.

Case (ii) : $A \subset N[z]$. It is clear that $\text{diam}(N[z]) \leq 2$ and so $N[z]$ is also a D2CS. Hence our assumption that $A$ is a maximal D2CS is not correct.

Case (iii) : $A \supset N[z]$. There exist two vertices $u$ and $v$ such that $u \in A \setminus N[z], v \in N[z]$ and $d(u,v) > 2$ in the graph $G[A]$; a contradiction that $A$ is a D2CS.

Hence the lemma.

The following result follows directly from the above lemma.
Proposition 3. Let $G$ be a strongly chordal with $|V(G)| = n$. Let $X$ be the maximum possible number of D2CS in $G$. Then $X \leq n$. The equality holds iff $G$ has no edges.

Since the converse of the lemma 2 does not hold, the following algorithm enumerates all maximal D2CS in a strongly chordal graph in linear time.

Algorithm EnumMaxD2CSSChordal($G$)

Input: A strongly chordal graph $G$ with vertices 1, \ldots, $n$ labeled in s.e.o order.

Output: All maximal D2CS of $G$.

1. for $i \leftarrow 1$ to $n$
2. \hspace{1em} $P(i) \leftarrow S(i) \leftarrow NIL$
3. \hspace{1em} $u \leftarrow$ largest numbered vertex of $N(1)$
4. \hspace{1em} print $N[u]$.
5. for $i \leftarrow 2$ to $n$
6. \hspace{1em} $u' \leftarrow \max\{v : v \in N(i), v < i\}$
7. \hspace{1em} $P(i) \leftarrow \max\{v : v \in N(u'), v > u'\}$
8. \hspace{1em} $S(i) \leftarrow \max\{v : v \in N(i), v > i\}$
9. \hspace{1em} if $P(i) = NIL$ or $|N[S(i)] \setminus N[P(i)]| > 0$ then
10. \hspace{2em} print $N[S(i)]$.

Clearly the algorithm generates all maximal D2CS of $G$ and runs in linear time. Hence we have:

Proposition 4. All maximal D2CS in a strongly chordal graph on $n$ vertices can be enumerated and counted in $O(n)$ time.

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