Sylow 3-subgroups of solvable cut groups

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Abstract
A group $G$ is said to be cut if, for every $g \in G$, each generator of $\langle g \rangle$ is conjugated to either $g$ or $g^{-1}$. It is conjectured that a Sylow 3-subgroup $P$ of a cut group $G$ is cut. We prove that this is true if either $|G|$ is odd or $P$ is of nilpotency class at most 2.

Keywords—cut groups, semi-rational groups, Sylow subgroups, character theory, $B_\pi$-characters

1 Introduction
A finite group $G$ is said to be rational if every irreducible character is rational valued or, equivalently, if every element $g \in G$ is conjugated to every generator of the cyclic group $\langle g \rangle$. For long it had been conjectured that, if $G$ is rational, then a Sylow 2-subgroup $P$ of $G$ must also be rational. This conjecture was proved to be false in general in [11], even for solvable groups; however, in the same paper the conjecture is proved to be true for solvable groups assuming that $P$ is of nilpotency class at most 2.

A concept related to rational groups is the one of cut groups. Cut groups arise form the study of group rings. In fact, a finite group $G$ is said to be a cut group if $\mathbb{Z}G$ only contains trivial central units (see [1] Definition 1.1 for details). There exists, however, also some purely group theoretical conditions for a group to be cut. By [1] Proposition 2.2, $G$ is a cut group if and only if, for every element $g \in G$, each generator of the cyclic group $\langle g \rangle$ is conjugated either to $g$ or to $g^{-1}$. Equivalently, a group is cut if, for each $\chi \in \text{Irr}(G)$, $Q(\chi) = \mathbb{Q}(\sqrt{-d})$ for some non-negative integer $d$, where $Q(\chi)$ is the field of values of $\chi$. In particular, for every $\chi \in \text{Irr}(G)$, $|Q(\chi) : \mathbb{Q}| \leq 2$.

Notice that, in the literature, cut groups are also called inverse semi-rational groups (see, for instance, [3]).
It is conjectured that, if $G$ is a cut group, then a Sylow 3-subgroup $P$ of $G$ must also be cut. In this paper, we answer positively in the solvable case, under the further assumption that $P$ is of nilpotency class at most 2.

**Theorem A.** Let $G$ be a 3-solvable cut group and let $P \in \text{Syl}_3(G)$ of nilpotency class at most 2. Then $P$ is cut.

Theorem A will be proved as a consequence of a more general result involving $B_p$-characters. These characters, first introduced by Isaacs in [6], have always values in the $p^a$-cyclotomic extension $Q_{p^a}$ for $p^a = |G|_p$. We will see that this allows us to relax our hypothesis on the field of values and still obtaining meaningful results. In particular, we prove the following theorem.

**Theorem B.** Let $G$ be a $p$-solvable group and let $P \in \text{Syl}_p(G)$ of nilpotency class at most 2. Let $Q_p$ be the $p$-cyclotomic extension of the field of rational numbers. Then, every $B_p$-character has values in $Q_p$ if and only if every irreducible character of $P$ has values in $Q_p$.

Since $Q_3 = Q(\sqrt{-3})$, Theorem A will follow as a corollary.

The problem of whether a Sylow 3-subgroup of a cut group is cut has already been studied for several classes of groups. In particular, in [2, Theorem 6.6] it is proved that the property holds when the group is supersolvable, Frobenius or simple. It is also proved that a Sylow 3-subgroup of a cut group $G$ is cut when $|G|$ is odd and $O_3(G)$ is abelian. We will prove that the property holds also without assumptions on $O_3(G)$.

**Theorem C.** Let $G$ be a cut group of odd order and let $P$ be a Sylow 3-subgroup of $G$. Then, $P$ is cut. Moreover, also $O_3(G)$ is cut.

### 2 Proof of Theorem C

We first prove Theorem C since it is easier and it does not require to recall the theory of $B_p$-characters.

The reader shall keep in mind that a character $\varphi \in \text{Irr}(P)$ of a $p$-group $P$ has values in $Q_{p^a}$, i.e., the $p^a$-cyclotomic extension of $Q$, for some $a \in \mathbb{N}$ such that $p^a = |P|$. Therefore, we will need to prove that $\varphi$ has values in $Q_3 = Q(\sqrt{-3})$, since it is the only subfield of $Q_{p^a}$ to be an extension of $Q$ of degree 2, thus, the only one which can be generated by the square root of a negative integer.

**Proof of Theorem C.** Let $G$ be a cut group of odd order; it follows from [3, Remark 13] and [3, Theorem 3] that $G$ is either a 3-group, a Frobenius group of order $3 \cdot 7^a$ or a group of order $7 \cdot 3^b$. In the first case there is nothing to prove while, in the second case, the thesis follows from [2, Theorem 6.6, (2)] and from the fact either $O_3(G) = P$ or $O_3(G) = 1$. 

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Therefore, we only have to consider the case when $|G| = 7 \cdot 3^h$. In this situation, let $O = O_3(G)$ and let $H \in \text{Hall}_2(G)$; it follows from [3] Theorem 3 that $OH$ is a Frobenius group with Frobenius kernel $O$, $OT \in \text{Syl}_3(G)$ for some group $T$ of order 3 and $G/O$ is the nonabelian group of order 21 (notice that $G/O$ is a Frobenius group, too, and for this reason groups like $G$ are sometimes called 2-Frobenius or double Frobenius groups).

Since $O$ is a 3-group, $Z(O) > 1$ and, since $O$ is normal in $G$, then $Z(O) \triangleleft G$. It follows that there exists $M \leq Z(O)$ minimal normal subgroup of $G$. In particular, $M$ is elementary abelian.

Now, let $\varphi \in \text{Irr}(P)$. If $M \leq \ker \varphi$, then $\varphi$ is a character of $P/M \in \text{Syl}_3(G/M)$ and it has values in $\mathbb{Q}_3$ by induction. Thus, we may assume that $M \not\leq \ker \varphi$.

In order to prove that $\varphi$ has values in $\mathbb{Q}_3$, we need to prove that it is fixed by every element of $\text{Gal}(\mathbb{Q}_{3^b} \mid \mathbb{Q}_3)$. Thus, let $\sigma \in \text{Gal}(\mathbb{Q}_{3^b} \mid \mathbb{Q}_3)$ and let $1_M \neq \lambda \in \text{Irr}(M)$ be a constituent of $\varphi_M$. Notice that $\lambda$ is linear of order 3 because $M$ is elementary abelian; thus, $\lambda$ has values in $\mathbb{Q}_3$ and is fixed by $\sigma$.

Since $OH$ is Frobenius, no elements of $H$ fix any nontrivial element of $M \leq O$. Thus, also $H \cap I_G(\lambda) = 1$. Moreover, $O \leq I_G(\lambda)$ because $M$ is central in $O$. Thus, either $I_G(\lambda) = O$ or $I_G(\lambda) = P$.

If $I_G(\lambda) = P$, then $\varphi^P = \chi \in \text{Irr}(G)$ by Clifford theorem. Moreover, $\chi$ is fixed by $\sigma$, since $|\mathbb{Q}(\chi) : \mathbb{Q}| = 2$ by hypothesis and $o(\sigma)$ is odd. Since both $\chi$ and $\lambda$ are fixed by $\sigma$, it follows from the uniqueness in Clifford theory that also $\varphi$ is fixed.

Suppose then that $I_G(\lambda) = O$, let $\theta \in \text{Irr}(O)$ be an irreducible constituent of $\varphi_O$ lying over $\lambda$ and notice that $\theta^G = \chi \in \text{Irr}(G)$ and $\theta^P = \varphi$. By the same argument as the previous paragraph, we have that $\theta$ is fixed by $\sigma$ and, since $\sigma$ commutes with the conjugation by elements of $G$, we have that

$$\varphi^\sigma = (\theta^\sigma)^P = \theta^P = \varphi.$$

It only remains to prove that $O$ is cut. Suppose that there exists $\theta \in \text{Irr}(O)$ and $\sigma \in \text{Gal}(\mathbb{Q}_{3^b} \mid \mathbb{Q}_3)$ such that $\theta^\sigma \neq \theta$ and let $\varphi \in \text{Irr}(P)$ lying over $\theta$. Let $M \leq O$ be a minimal normal subgroup of $G$ central in $O$, as above, and let $\lambda$ be an irreducible constituent of $\theta_M$. We can assume that $\lambda \neq 1_M$, since otherwise $\theta$ is an irreducible character of $O/M = O_3(G/M)$ and the thesis follows by induction. Thus, we have that no nontrivial element of $H$ fixes $\lambda$, because $OH$ is Frobenius and $\lambda$ is linear; since $\theta_M = \theta(1)\lambda$, the same is true for $\theta$. Thus, $H \cap I_G(\theta) = 1$.

Moreover, let $T = \langle t \rangle$ for some $t \in T$ of order 3, so that $P = O(t)$. Since $\sigma$ fixes $\varphi$ and not $\theta$, we have that $\varphi_O \neq \theta$ and it follows that $\varphi_O = \theta + \theta^t + \theta^{-t}$ and, thus, $I_G(\theta) = O$. However, also $\varphi_O = \theta + \theta^\sigma + \theta^{-\sigma}$, since $\theta^\sigma, \theta^{-\sigma}$ both lie under $\varphi$. Thus, either $\theta^\sigma = \theta^t$ or $\theta^\sigma = \theta^{-t}$ and, without loss of generality, we may assume $\theta^\sigma = \theta^t$.  

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Let $1 \neq h \in H$ and let $\varphi_1 = (\theta^h)^P$. Since $\theta^h$ is not $\sigma$-invariant, $\varphi_1$ is irreducible and, for the same arguments as above, it follows that either $(\theta^h)^\sigma = (\theta^h)^t$ or $(\theta^h)^\sigma = (\theta^h)^t^{-1}$. However, we also have that $(\theta^h)^\sigma = \theta^h$. Since $HT$ is a complement for $O = I_G(\theta)$ in $G$, $HT \cap I_G(\theta) = 1$ and we have that either $th = ht$ or $th = ht^{-1}$. In the first case we have that $t$ and $h$ commute, while in the second case we have that $t^h = t^{-1}$ and, thus, $H = \langle h \rangle$ normalizes $T$. Since $HT$ is a Frobenius group, both results are absurd; thus, it follows that $\theta$ is fixed by $\sigma$.

3 Review of the $\pi$-theory

In this section, we will briefly summarize the main properties of the $B_p$-characters in $p$-solvable groups. We do it in the more general frame of the theory of characters of $\pi$-separable groups. Here we see only some basic facts, an interested reader can consult [6], or the first part of [8], for a complete exposition of the theory.

Let $\pi$ be a set of primes and denote as $\pi'$ its complementary set. A finite group is said to be a $\pi$-group if its order is a $\pi$-number, which means that all its prime divisors lie in $\pi$.

A finite group $G$ is said to be $\pi$-separable if every quotient in a composition series of the group is either a $\pi$-group or a $\pi'$-group. It is said to be $\pi$-solvable if every quotient is either a $\pi'$-group or a group of prime order.

The reader will notice that, if $\pi$ consists of the sole prime $p$, then the two concepts coincide and in this case we simply talk about $p$-solvable groups.

If $G$ is a finite group, a character $\chi \in \text{Irr}(G)$ is said to be $\pi$-special if its degree and its order are $\pi$-numbers and, for any $M \trianglelefteq G$ and any irreducible constituent $\phi$ of $\chi_M$, the order of $\phi$ is a $\pi$-number.

Sometimes, we will write $X_\pi(G)$ to describe the subset of $\text{Irr}(G)$ of all the $\pi$-special characters. Notice that, if $G$ is a $\pi'$-groups and $\phi \in X_\pi(G)$, since both $\chi(1)$ and $o(\chi)$ have to be $\pi$-numbers, it follows that $\chi(1) = o(\chi) = 1$ and, therefore, $X_\pi(G) = \{1_G\}$.

The behaviour of $\pi$-special characters when induced from, or restricted to, normal subgroups is well described by the following propositions.

**Proposition 3.1** ([4, Proposition 4.1]). Let $G$ be a finite group and $\chi \in X_\pi(G)$. If $M$ is a subnormal subgroup of $G$, then every irreducible constituent of $\chi_M$ is a $\pi$-special character.

**Proposition 3.2** ([4, Proposition 4.5]). Let $G$ be a finite group and let $N \triangleleft G$ such that $G/N$ is a $\pi'$-group. If $\psi \in X_\pi(N)$, then every irreducible constituent of $\psi^G$ is a $\pi$-special character.

**Proposition 3.3** ([4, Proposition 4.3]). Let $G$ be a finite group and let $N < G$ such that $G/N$ is a $\pi'$-group. If $\psi \in X_\pi(N)$ is $G$-invariant, then it extends to $G$ and there exists a unique extension which is a $\pi$-special character.
character. If $\psi$ is not $G$-invariant, none of the irreducible constituents of $\psi^G$ is a $\pi$-special character.

There is no need for the group to be $\pi$-separable in order to define $\pi$-special characters. However, if $G$ is a $\pi$-separable group, we have a further useful property.

**Theorem 3.4** ([4, Theorem 7.2]). Let $G$ be a $\pi$-separable group and let $\alpha, \beta \in \Irr(G)$, with $\alpha$ $\pi$-special character and $\beta$ $\pi'$-special character. Then, $\alpha \beta$ is an irreducible character of $G$ and this factorization is unique.

An irreducible character which can be written as a product of a $\pi$-special and a $\pi'$-special character is said to be a $\pi$-factorable character.

We now consider character pairs $(H, \theta)$, where $H$ is a subgroup of some fixed group $G$ and $\theta$ is an irreducible character of $H$. We say that $(H, \theta) \leq (K, \varphi)$ if $H \leq K$ and $\theta$ is an irreducible constituent of $\varphi_H$. This defines a partial order on the set of character pairs.

**Definition 3.5** ([6, Definition 3.1]). Let $G$ be $\pi$-separable. A $\pi$-factorable subnormal pair of $G$ is a character pair $(S, \theta)$, where $S \trianglelefteq G$ and $\theta$ is a $\pi$-factorable character. We write $\mathfrak{F}_\pi(G)$ to denote the set of $\pi$-factorable subnormal pairs.

**Theorem 3.6** ([6, Theorem 3.2]). Let $G$ be $\pi$-separable and let $\chi \in \Irr(G)$ then there exists a $\pi$-factorable subnormal pair $(S, \theta)$ of $G$ such that it is maximal in $\mathfrak{F}_\pi(G)$ and $(S, \theta) \leq (G, \chi)$. Moreover, if $(R, \eta)$ is another such pair, then $R = S^g$ and $\eta = \theta^g$ for some $g \in G$.

**Theorem 3.7** ([6, Theorem 4.4 and Lemma 4.5]). Let $G$ be $\pi$-separable and let $(S, \mu)$ be a maximal $\pi$-factorable subnormal pair. Let $T = I_G(S, \theta)$, where $I_G(S, \theta) = I_{NG(S)}(\theta)$. Then the induction defines a bijection between $\Irr(T \mid \mu)$ and $\Irr(G \mid \mu)$. Moreover, if $S < G$, then also $T < G$.

If $G$ is $\pi$-separable and $\chi \in \Irr(G)$, by Theorem 3.6 we have that there exists $(S, \mu) \in \mathfrak{F}_\pi(G)$ maximal such that $(S, \mu) \leq (G, \chi)$. If $T = I_G(S, \mu)$, by Theorem 3.7 there exists $\xi \in \Irr(T \mid \mu)$ such that $\xi^G = \chi$. This process associates, to the pair $(G, \chi)$ a specific pair $(T, \mu)$, determined uniquely up to conjugacy in $G$, which is called a standard inducing pair for $(G, \chi)$.

If $\chi$ is already $\pi$-factorable, then $(S, \mu) = (G, \chi)$ and, therefore, also $T = G$. Otherwise, $S < G$ and, by Theorem 3.7 also $T < G$. In this case, we can repeat the process and find a standard inducing pair for $(T, \xi)$. If we continue this way until we reach a $\pi$-factorable pair, which will happen eventually, since the group is finite, we have

$$(G, \chi) = (T_0, \xi_0) > (T_1, \xi_1) > \ldots > (T_k, \xi_k),$$
where \((T_i, \xi_i)\) is a standard inducing pair for \((T_{i-1}, \xi_{i-1})\) and \(\xi_k\) is \(\pi\)-factorable. At each stage, the pair \((T_i, \xi_i)\) is determined up to conjugacy in \(T_{i-1}\); in particular, the terminal pair \((W, \mu) = (T_k, \mu_k)\) is determined up to conjugacy in \(G\) and it is said to be a nucleus for \(\chi\).

We can now give the definition of \(B_{\pi}\)-characters.

**Definition 3.8** ([6, Definition 5.1]). Let \(\chi \in \text{Irr}(G)\), where \(G\) is a \(\pi\)-separable group, and let \((W, \mu) \in \text{nc}(\chi)\), which is unique up to conjugation for elements of \(G\). If \(\mu\) is a \(\pi\)-special character, we say that \(\chi\) is a \(B_{\pi}\)-character. We denote as \(B_{\pi}(G)\) the set of \(B_{\pi}\)-characters of the group \(G\).

It is useful to study the behaviour of the \(B_{\pi}\)-characters in relation with normal subgroups. As expected, this behaviour will be similar to the one of \(\pi\)-special characters.

**Theorem 3.9.** Let \(G\) be \(\pi\)-separable and let \(M \triangleleft G\). If \(\chi \in B_{\pi}(G)\), then every irreducible constituent of \(\chi_M\) belongs to \(B_{\pi}(M)\).

On the other hand, if \(\psi \in B_{\pi}(M)\), then there exist some characters in \(B_{\pi}(G)\) lying over \(\psi\). In particular, if \(G/M\) is a \(\pi\)-group, then every character in \(\text{Irr}(G/\psi)\) belongs to \(B_{\pi}(G)\) while, if \(G/M\) is a \(\pi'\)-group, then there exists a unique character in \(\text{Irr}(G/\psi)\) which belongs to \(B_{\pi}(G)\).

**Proof.** It is a direct consequence of [6, Theorem 6.2] and [6, Theorem 7.1].

The main property of \(B_{\pi}\)-characters, however, concerns their restriction to Hall \(\pi\)-subgroups.

**Theorem 3.10** ([6, Theorem 8.1]). Let \(\chi \in B_{\pi}(G)\), with \(G\) \(\pi\)-separable, and let \(H \in \text{Hall}_{\pi}(G)\). Then the following hold.

a) For each \(\alpha \in \text{Irr}(H)\), \(\alpha(1) \geq |\alpha, \chi_H|\chi(1)_{\pi}\).

b) There exists at least one irreducible constituent \(\alpha\) of \(\chi_H\) such that \(\alpha(1) = \chi(1)_{\pi}\).

c) If \(\alpha\) is as in b), then \([\chi_H, \alpha] = 1\), and \([\psi_H, \alpha] = 0\) for any \(\psi \in B_{\pi}\), \(\psi \neq \chi\).

**Corollary 3.11** ([6, Corollary 8.2]). Let \(G\) be \(\pi\)-separable and let \(H\) be a Hall \(\pi\)-subgroup of \(G\). Then, restriction defines an injection from the set of \(\pi\)-special characters of \(G\) into \(\text{Irr}(H)\).

Characters like the ones in Theorem 3.10, point b), play an important role in the theory of characters of \(\pi\)-separable groups. We refer to them as Fong characters.

An other consequence of Theorem 3.10 is that we have informations about the field of values of the characters in \(B_{\pi}(G)\). If \(n\) is a natural number, we write \(Q_n\) to refer to the \(n\)-cyclotomic extension of \(Q\), i.e., the extension
of the field of rational numbers obtained by adjoining a primitive \( n \)-root of unity \( \zeta_n \) to \( \mathbb{Q} \). If \( \pi \) is a set of primes, \( \mathbb{Q}_\pi \) denotes the extension of the field of rational numbers obtained by adjoining all complex \( n \)-th roots of unity of \( \mathbb{Q} \), for all \( \pi \)-numbers \( n \).

**Corollary 3.12** ([6, Corollary 12.1]). If \( \chi \in B_\pi(G) \), then it has values in \( \mathbb{Q}_\pi \), i.e., for every \( x \in G \), \( \chi(x) \in \mathbb{Q}_\pi \).

We conclude this section by citing one final result which links the field of values of \( B_\pi \)-characters of a \( p \)-solvable group \( G \) with the field of values of the characters of a Sylow \( p \)-subgroup of \( G \). It is easier to prove it as a consequence of [5, Corollary 3.3] but it can be seen also as a consequence of [10, Corollary 3.8], and it is generalized in [12] for nonsolvable groups.

**Theorem 3.13.** Let \( G \) be a \( p \)-solvable group and suppose every character in the set \( B_p(G) \cap \text{Irr}'_p(G) \) has values in \( \mathbb{Q}_p \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \), then \( P/P' \) has exponent \( p \).

**Proof.** Let \( \lambda \in \text{Lin}(P) \), then, by [9, Corollary 6.1] it is a Fong character associated with some \( \chi \in B_p(G) \cap \text{Irr}'_p(G) \) and, by [5, Corollary 3.3], it follows that \( o(\lambda) = p \). Since this holds for every \( \lambda \in \text{Lin}(P) \), then \( \exp(P/P') = p \).

### 4 Proofs of Theorem A and of Theorem B

We first prove a preliminary result which helps us to identify \( B_\pi \)-characters when the Hall \( \pi \)-subgroup is normal.

**Proposition 4.1.** Let \( G \) be a finite group with a normal Hall \( \pi \)-subgroup \( H \), then each irreducible character of \( H \) is a Fong character in \( G \). Moreover, if \( \varphi \in \text{Irr}(H) \), \( I = I_G(\varphi) \) and \( \eta \in \text{Irr}(I \mid \varphi) \), then \( \eta^G = \chi \in \text{Irr}(G) \) is a \( B_\pi \)-character if and only if \( \eta \) is a \( \pi \)-special character.

**Proof.** Let \( \varphi \in \text{Irr}(H) \), then \( (H, \varphi) \in \mathcal{F}_\pi(G) \) since \( H \) is normal in \( G \). Let \( \chi \in \text{Irr}(G) \) lying over \( \varphi \) and let \( (M, \gamma) \in \mathcal{F}_\pi(G) \) maximal such that \( (H, \varphi) \leq (M, \gamma) \leq (G, \chi) \). Then, \( \gamma = \alpha \beta \), with \( \alpha \) \( \pi \)-special, and thus \( \alpha_H = \varphi \) by Corollary 3.11.

Let \( I = I_G(\varphi) \) and let \( J = I_G(M, \gamma) \); then, \( J \leq I \), since \( \alpha_H = \varphi \) and the factorization \( \gamma = \alpha \beta \) is unique, thus, if \( g \in G \) fixes \( \gamma \), it also fixes \( \alpha \) and \( \varphi \). By iterating the Isaacs’ algorithm, we obtain \( (W, \mu) \in \text{mc}(\chi) \) such that \( (H, \varphi) \leq (W, \mu) \leq (J, \gamma) \leq (I, \eta) \), where \( \eta \in \text{Irr}(I \mid \varphi) \) such that \( \eta^G = \chi \), and \( \mu^I = \eta \) by the uniqueness in Clifford theorem.

Now, if \( \eta \) is \( \pi \)-special, then \( \eta_H \) is irreducible by Corollary 3.11 and, thus, \( \eta_W = \mu \). Since \( \mu^I = \eta \), it follows that \( (W, \mu) = (I, \eta) \), \( \mu \) is \( \pi \)-special and, therefore, \( \chi \in B_\pi(G) \).
On the other hand, if \( \chi \in B_{\pi}(G) \), then \([\eta_{H}, \varphi] \leq [\chi_{H}, \varphi] = 1\), thus, \(\eta_{H} = \varphi\), since \(\varphi\) is invariant in \(I\), and \(\eta_{W} = \mu\). It follows that \((W, \mu) = (I, \eta)\) and \(\eta\) is \(\pi\)-special, because so is \(\mu\).

Finally, by Proposition 3.3 \(X_{\pi}(I \mid \varphi)\) is nonempty, thus, \(\varphi\) is a Fong character in \(G\).

It is clear that the hypothesis in Theorem \([11]\) are preserved for group quotients. We see that, under some hypothesis, they are preserved also for normal subgroups.

**Lemma 4.2.** Let \(G\) be a \(p\)-solvable group and suppose that every \(\chi \in B_{p}(G)\) has values in \(Q_{p}\). Let \(K \triangleleft G\) such that \(p \nmid |G : K|\), then also every \(\psi \in B_{p}(K)\) has values in \(Q_{p}\).

**Proof.** Let \(\psi \in B_{p}(K)\), then by Corollary 3.12 it has values in \(Q_{|G|_{p}}\). Thus, it is enough to prove that every element of \(\text{Gal}(Q_{|G|_{p}} \mid Q_{p})\) fixes \(\psi\).

Let \(\sigma \in \text{Gal}(Q_{|G|_{p}} \mid Q_{p})\) and suppose \(\psi^{\sigma} \neq \psi\). By Theorem 3.9, there exists \(\chi \in B_{p}(G)\) lying over \(\psi\) and, by hypothesis, \(\chi\) is fixed by \(\sigma\). Thus, \(\sigma\) permutes the irreducible constituents of \(\chi_{K}\). Let \(C\) be the orbit of \(\psi\) under this permutation; since \(\sigma\) commutes with the conjugation for elements of \(G\), the orbit under \(\sigma\) of any conjugate \(\psi^{g}\) of \(\psi\) is \(C^{g}\). It follows that all the orbits of the action have cardinality \(|C|\) and, thus, \(|C| \mid |G : K|\). However, if \(\psi^{\sigma} \neq \psi\), then \(|C| > 1\) and, since \(o(\sigma)\) is a power of \(p\), then also \(|C|\) is a power of \(p\) and \(p \mid |G : K|\), in contradiction with the hypothesis.

We now prove our version of \([11]\) Theorem 3.1, for any prime \(p\). Our theorem is actually identical to a portion of \([11]\) Theorem 3.1, however since this was stated only for the prime 2, we cannot just cite it.

**Theorem 4.3.** Let \(G\) be a \(p\)-solvable group and let \(P \in \text{Syl}_{p}(G)\) of nilpotency class at most 2. Suppose that \(P/P'\) is elementary abelian and assume \(O_{p'}(G) = 1\) and \(O_{p'}'(G) = G\). Let \(F = F(G), H \in \text{Hall}_{p'}(G)\) and \(B = [F, H]\).

The following then hold.

(a) \(B \triangleleft G\) and \(C_{B}(H) = 1\);

(b) \(B\) is an elementary abelian \(p\)-group and it is central in \(F\);

(c) \(HB \triangleleft G\);

(d) \(G = N_{G}(H)B\) and \(N_{G}(H) \cap B = 1\).

**Proof.** Since \(O_{p'}(G) = 1\), we have that \(F = F_{p}(G) = O_{p}(G)\) and, since \(G\) is \(p\)-solvable, we also have that \(C_{G}(F) \leq F\). Since \(P\) has nilpotency class at most 2, it follows that \(P' \leq Z(P) \leq C_{G}(F) \leq F\). Thus, \(P/F\) is abelian and \(G/F\) has \(p\)-length at most 1 by \([7]\) Theorem 3.22. Since
however $O_{p'}(G) = G$, it follows that $G/F$ has a normal $p$-complement and, thus, $HF < G$.

Let $Z = Z(F)$, then $P' \leq Z(P) \leq Z$ and $P$ acts trivially on $P/Z$, thus it also acts trivially on $F/Z$. It follows that $p \nmid |G : C_G(F/Z)|$ and, since $C_G(F/Z) < G$, it follows that $C_G(F/Z) = G$ and, thus, $B = [F, H] \leq Z$ and it is abelian.

Now, $[F, H] = [F, H, H]$ for \cite{Fitting lemma 4.34}, thus, $B = [F, H, H] = [B, H] \leq [Z, H] \leq [F, H] = B$ and it follows that $B = [Z, H]$. Moreover, by Proposition 3.3, by Proposition 4.1, $\tau$ is a Fong character associated with $\phi$ and it is abelian.

We prove the other direction working by induction on $|\lambda|$. We are now ready to prove Theorem B.

We prove the other direction working by induction on $|\lambda|$. We can clearly assume $O_{p'}(G) = 1$, since $PO_{p'}(G)/O_{P'}(G)$ is a Sylow $p$-subgroup of $G/O_{p'}(G)$ and it is isomorphic to $P$. Moreover, by Lemma \cite{4.2} we can assume also that $O_{p'}(G) = G$ and, by Theorem \cite{4.13} we have that $P/P'$ is elementary abelian. Thus, Theorem \cite{4.13} applies and we can use its notation.

Let $\varphi \in \text{Irr}(P)$ and assume $B \nsubseteq \ker(\varphi)$, since otherwise $\varphi$ has values in $\mathbb{Q}_p$ by induction on the group order. Let $\lambda$ be an irreducible constituent of $\varphi_B$ and let $T = I_G(\lambda)$, we have that $I_T(\lambda) = T \cap P$ and $I_K(\lambda) = T \cap K$, for $K = HB < G$. Thus, let $\gamma \in \text{Irr}(T \cap P | \lambda)$ such that $\gamma^p = \varphi$ and let $\tau \in \text{Irr}(T \cap K)$ be the unique $p$-special extension of $\lambda$ to $T \cap K$, which exists by Proposition \cite{4.33}. By Proposition \cite{4.11} $\tau^K = \psi \in B_p(K)$. \hfill \qed
Now, $\psi$ is an irreducible constituent of $\lambda^K$, which is a constituent of $\varphi_B^K = \varphi^G_K$. It follows that $[\varphi^G, \psi^G] = [\varphi^G_K, \psi] \neq 0$ and there exists $\chi \in \text{Irr}(G)$ lying over both $\psi$ and $\varphi$. Since $\psi \in B_p(K)$ and $G/K$ is a $p$-group, $\chi \in B_p(G)$ by Theorem 3.9 and it has values in $\mathbb{Q}_p$.

Since $B$ is complemented in $G$ and $\lambda$ is linear, then $\lambda$ extends to $T$. Let $\mu \in \text{Irr}(T)$ be an extension of $\lambda$ of $p$-power order and notice that $\mu_{T \cap K} = \tau$ (by uniqueness, since $\mu_{T \cap K}$ is $p$-special and lies over $\lambda$). Let $\rho \in \text{Irr}(T | \lambda)$ such that $\rho^G = \chi$, then by Gallagher $\rho = \mu \xi$ for some $\xi \in \text{Irr}(T/B)$.

Now, $0 \neq [\chi_{T \cap K}, \tau] = [\chi_T, \tau^T]$, thus there exists some character $\varepsilon \in \text{Irr}(T)$ lying under $\chi$ and over $\tau$. Therefore, $\varepsilon$ lies over $\lambda$, too, and it follows that $\varepsilon = \rho$ by the uniqueness in Clifford theorem. Thus, $\rho$ lies over $\tau$ and, with the same argument, we also have that $\rho$ lies over $\gamma$.

Since $T \cap K \triangleleft T$ and $\mu_{T \cap K} = \tau$, it follows that $\rho_{T \cap K} = \xi(1)\tau$ and, in particular, $\xi \in \text{Irr}(T/T \cap K)$. Since $T/T \cap K = T/T \cap HB \cong T \cap P/B$, we have that $\xi_{T \cap P}$ is irreducible. As a consequence, $\rho_{T \cap P} = (\mu \xi)_{T \cap P}$ is irreducible, too, and $\rho_{T \cap P} = \gamma$.

Since both $\chi$ and $\lambda$ have values in $\mathbb{Q}_p$, by uniqueness in the Clifford theory $\rho$ is fixed by every element of $\text{Gal}(\mathbb{Q}(G^\lambda) | \mathbb{Q}_p)$ and, thus, it has values in $\mathbb{Q}_p$. It follows that the same is true for $\gamma = \rho_{T \cap P}$ and, finally, also for $\varphi = \gamma^P$. \hfill \Box

The proof of Theorem A then follows easily.

**Proof of Theorem A.** If $G$ is cut, then every character in $B_3(G)$ have values in $\mathbb{Q}_3 = \mathbb{Q}(\sqrt{-3})$, since it is the only subfield of $\mathbb{Q}(G_{13})$ to be an extension of $\mathbb{Q}$ of degree 2, thus the only one which can be generated by the square root of a negative integer. Then, by Theorem B it follows that every character in $P$ has values in $\mathbb{Q}_3$ and the thesis follows. \hfill \Box

Moreover, Theorem B also provides a partially different proof of (a weaker version of) the main result in [11].

**Corollary 4.4.** Let $G$ be a solvable rational group and let $P \in Syl_2(G)$ of nilpotency class at most 2. Then $P$ is rational.
Proof. By hypothesis, every character in $B_2(G)$ has values in $Q_2 = Q$. Thus, by Theorem 3 it follows that also every character in $\text{Irr}(P)$ has values in $Q$ and, therefore, $P$ is rational. \qed

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