INTERTWINING OPERATOR ASSOCIATED TO SYMMETRIC GROUPS AND SUMMABILITY ON THE UNIT SPHERE

YUAN XU

Abstract. An integral representation of the intertwining operator for the Dunkl operators associated with symmetric groups is derived for the class of functions of a single component. The expression provides a closed form formula for the reproducing kernels of $h$-harmonics associated with symmetric groups when one of the components is a coordinate vector. The latter allows us to establish a sharp result for the Cesàro summability of $h$-harmonic series on the unit sphere.

1. Introduction

Associate to a reflection group $G$, the Dunkl operators are a family of commuting first order differential-difference operators that act on smooth functions on $\mathbb{R}^d$ [6]. In the case that $G = S_d$, the symmetric group of $d$ elements, these operators are defined by

\begin{equation}
D_i f(x) = \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j=1 \atop j \neq i}^{d} \frac{f(x) - f(x(i, j))}{x_i - x_j}, \quad 1 \leq i \leq d,
\end{equation}

where $\kappa$ is a non-negative real number and $(i, j)$ denotes the transposition of exchanging $i$th and $j$th elements. The Dunkl operators commute in the sense that $D_i D_j = D_j D_i$ for $1 \leq i, j \leq d$. A linear operator, denote by $V_\kappa$, is called an intertwining operator if it satisfies the relations [7]

\begin{equation}
D_i V_\kappa = V_\kappa \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq d.
\end{equation}

This operator is uniquely determined if it also satisfies $V_\kappa 1 = 1$ and $V_\kappa : \mathcal{P}_n^d \rightarrow \mathcal{P}_n^d$, where $\mathcal{P}_n^d$ is the space of homogeneous polynomial of degree $n$ in $d$ variables.

The commuting property of the Dunkl operators leads to the definition of an analogue of the Laplace operator, $\Delta_\kappa = \sum_{i=1}^{d} D_i^2$. While the Laplace operator is essential for analysis in $L^2(\mathbb{R}^d)$, the operator $\Delta_\kappa$ plays its role in $L^2(h_\kappa; \mathbb{R}^d)$, where $h_\kappa$ is a function invariant under the reflection group $G$. In the case of $G = S_d$, the weight function $h_\kappa$ is defined by

\begin{equation}
h_\kappa(x) = \prod_{1 \leq i < j \leq d} |x_i - x_j|^\kappa, \quad x \in \mathbb{R}^d, \quad \kappa \geq 0.
\end{equation}

In particular, a homogeneous polynomial $Y$ in $d$ variables is called an $h$-harmonic if $\Delta_\kappa Y = 0$. The restriction of $h$-harmonics on the unit sphere $\mathbb{S}^{d-1}$, called spherical...
h-harmonics, are orthogonal. More precisely, let $\mathcal{H}_n^d(h^2)$ be the space of h-harmonic polynomials of degree exactly $n$. Then h-harmonics of different degrees are orthogonal: for $Y_n \in \mathcal{H}_n^d(h^2)$ and $Y_m \in \mathcal{H}_m^d(h^2)$,
$$
\int_{S^{d-1}} Y_n(x)Y_m(x)h^2(x)\,d\sigma(x) = 0, \quad n \neq m,
$$
where $d\sigma$ is the surface measure. The theory of spherical h-harmonics resembles that of ordinary spherical harmonics. In particular,
$$
\dim(n, d) := \dim \mathcal{H}_n^d(h^2) = \left( \frac{n + d - 1}{n} \right) - \left( \frac{n + d - 3}{n - 2} \right).
$$
The reproducing kernel $P_n(h^2;\cdot,\cdot)$ of the space $\mathcal{H}_n^d(h^2)$ enjoys an addition formula given in terms of the intertwining operator $V_\kappa$. Let $\{Y_{n,\ell} : 1 \leq \ell \leq \dim(n, d)\}$ be an orthonormal basis of $\mathcal{H}_n^d(h^2)$. Then the kernel $P_n(h^2; x, y)$ satisfies
$$
P_n(h^2; x, y) = \sum_{\ell=1}^{\dim(n, d)} Y_{n,\ell}(x)Y_{n,\ell}(y).
$$
The addition formula of the kernel is given by [14]
$$
P_n(h^2; x, y) = V_\kappa \left( Z^\lambda_n(\cdot, y) \right) (x), \quad x, y \in S^{d-1},
$$
where $Z^\lambda_n$ is given in terms of the classical Gegenbauer polynomial $C^\lambda_n$ by
$$
Z^\lambda_n(t) = \frac{n + \lambda}{\lambda}C^\lambda_n(t), \quad -1 \leq t \leq 1,
$$
and $\lambda_\kappa$ is a constant that is given by, when $G = S_d$,
$$
\lambda_\kappa := \left( \frac{d}{2} \right)^\kappa + \frac{d - 2}{2}.
$$
When $\kappa = 0$, $V_\kappa = id$ and the identity [15] coincides with the addition formula of ordinary spherical harmonics.

The reproducing kernel $P_n(h^2)$ is the kernel of the orthogonal projection operator $\text{proj}_n : L^2(h^2, S^{d-1}) \rightarrow \mathcal{H}_n^d(h^2)$ and it plays a central role in the study of Fourier orthogonal series in spherical h-harmonics, which we shall call spherical h-harmonic series from now on. For intrinsic properties that rely on the underlying reflection group of such series, we need a closed formula for the kernel, which calls for an explicit integral representation of $V_\kappa$: It is known [11] that there exists a nonnegative probability measure $d\mu_x$ such that $V_\kappa f(x) = \int_{S_d} f(y) d\mu_x(x)$. What we need, however, is a far more explicit representation. At the moment, such a representation is known for $G = Z_d^d$ with $h_\kappa(x) = \prod_{i=1}^d |x_i|^\kappa$, which allows us to carry out hard analysis and establish several fundamental results on the spherical h-harmonic series; see [5]. For the symmetric group $S_3$, a version of the integral representation was obtained in [5], which however is not adequate for hard analysis of the spherical h-harmonic series. It should be mentioned that an integral representation of the generalized spherical functions associate to $S_d$ was given recently in [12], which is closely related to the intertwining operator.

Our main result of the present paper is to provide an explicit integral representation for $V_\kappa f$ associate to $S_d$ when the function $f$ depends only on one component of its variables. The integral is over a regular simplex in $d - 1$ variables and is motivated by our recent work [15], where such an integral is used for a representation of $V_\kappa$ for the dihedral group. As an application, we obtain a closed formula for
the reproducing kernel \( P_n(h_2^2; x, e_j) \), where \( e_j \) is the \( j \)-th coordinate vector, which allows us to study the \( h \)-harmonic series at \( e_j \). By taking an integral average over \( S^{d-1} \), the Cesàro \((C, \delta)\) means of \( h \)-harmonic series are known \cite{14} to converge if \( \delta > \lambda_\kappa \) in \( L^1(h_\kappa^2, S^{d-1}) \) or in \( C(S^{d-1}) \), but the result is not sharp since taking average over sphere removes the action of the group inadvertently. Using the new integral representation of \( V_\kappa \), we shall show that the convergence holds if \( \delta > \lambda_\kappa \) is replaced by \( \delta > \lambda_\kappa - (d-1)\kappa \).

The paper is organized as follows. The new integral representation will be stated and proved in the next section, where several of its consequences will also be discussed. The spherical \( h \)-harmonic series is considered in Section 3, where the convergence of the \((C, \delta)\) means at coordinate vectors is established, assuming a critical estimate over an integral of the Jacobi polynomial. The latter estimate is technical and will be carried out in Section 4.

2. INTERTWINING OPERATOR ASSOCIATED TO SYMMETRIC GROUPS

Let \( V_\kappa \) be the intertwining operator associated to the symmetric group \( S_d \). Our main result in this section is the following integral representation of \( V_\kappa \). Let \( T^{d-1} \) denote the simplex

\[
T^{d-1} := \{ u \in \mathbb{R}^{d-1} : t_1 \geq 0, \ldots, t_{d-1} \geq 0, t_1 + \cdots + t_{d-1} \leq 1 \}.
\]

Written in homogeneous coordinates of \( \mathbb{R}^d \), it is equivalent to the simplex

\[
T^d = \{ (t_0, \ldots, t_{d-1}) \in \mathbb{R}^d : t_1 \geq 0, t_0 + t_1 + \cdots + t_{d-1} = 1 \}.
\]

**Theorem 2.1.** Let \( f : \mathbb{R} \to \mathbb{R} \). For \( 1 \leq \ell \leq d \), define \( F(x_1, \ldots, x_d) = f(x_\ell) \). Let

\[
V_\kappa F(x) = c_\kappa \int_{T^d} f(x_1t_0 + x_2t_1 + \cdots + x_dt_{d-1})t_{\ell-1}(t_0 \cdots t_{d-1})^{\kappa-1} dt,
\]

where the constant \( c_\kappa \) is given by

\[
c_\kappa = c_{\kappa,d} = \Gamma(d\kappa + 1)/(\kappa \Gamma(\kappa)^d).
\]

Then the operator \( V_\kappa \) satisfies

\[
D_i V_\kappa F(x) = V_\kappa(\partial_i F)(x), \quad 1 \leq i \leq d.
\]

**Proof.** The constant \( c_\kappa \) is chosen so that \( V_\kappa 1 = 1 \). By the symmetry in the formula of \( \cite{21} \), it is sufficient to consider \( \ell = 1 \). Let \( F(x) = f(x_1) \). Exchanging the variables \( t_0 \) and \( t_{j-1} \) in the integral, we see that

\[
V_\kappa F(x(1j)) = c_\kappa \int_{T^d} f(x_1t_0 + \cdots + x_dt_{d-1})t_{j-1}(t_0 \cdots t_{d-1})^{\kappa-1} dt,
\]

which leads immediately to, setting \( t_0 = 1 - t_1 - \cdots - t_{d-1} \),

\[
V_\kappa F(x) - V_\kappa F(x(1j)) = c_\kappa \int_{T^{d-1}} f(x_1t_0 + \cdots + x_dt_{d-1})t_0(t_0 - t_{j-1})(t_0 \cdots t_{d-1})^{\kappa-1} dt.
\]

Since \( \kappa(t_0 - t_j)^{\kappa-1} = \frac{\partial}{\partial t_j}(t_0 t_j)^\kappa \) for \( j \geq 1 \), integration by parts gives

\[
\kappa \frac{V_\kappa F(x) - V_\kappa F(x(1j))}{x_1 - x_j} = c_\kappa \int_{T^{d-1}} f'(x_1t_0 + \cdots + x_dt_{d-1})t_0 t_j(t_0 \cdots t_{d-1})^{\kappa-1} dt.
\]

Moreover, taking derivative shows

\[
\frac{\partial}{\partial x_1} V_\kappa F(x) = c_\kappa \int_{T^{d-1}} f'(x_1t_0 + \cdots + x_dt_{d-1}) t_0^2(t_0 \cdots t_{d-1})^{\kappa-1} dt.
\]
Hence, adding the terms together, we obtain
\[
D_i V_\kappa f(x) = c_\kappa \int_{T^{d-1}} f'(x_1 t_0 + \cdots + x_d t_{d-1}) (t_0^\kappa + t_0 (t_1 + \cdots t_{d-1})) \\
\times (t_0 \cdots t_{d-1})^{\kappa-1} dt = V_\kappa (\partial_i F)(x)
\]
upon using \( t_0 + t_1 + \cdots + t_{d-1} = 1 \). Furthermore, since \( t_0 = 1 - t_1 - \cdots - t_{d-1} \) is symmetric in \( t_1, \ldots, t_{d-1} \), it is easy to see that \( V_\kappa f(x) - V_\kappa f(x(i,j)) = 0 \) for \( 2 \leq i, j \leq d \). Moreover, for \( i \geq 2 \),
\[
\frac{\partial}{\partial x_i} V_\kappa f(x) = c_\kappa \int_{T^{d-1}} f'(x_1 t_0 + \cdots + x_d t_{d-1}) t_0 t_{i-1} (t_0 \cdots t_{d-1})^{\kappa-1} dt.
\]
Hence, it follows that, for \( i = 2, 3, \ldots, d \),
\[
D_i V_\kappa f(x) = \frac{\partial}{\partial x_i} V_\kappa f(x) + \kappa V_\kappa f(x) - V_\kappa f(x(i,1))
\]
\[
= c_\kappa \int_{T^{d-1}} f'(x_1 t_0 + \cdots + x_d t_{d-1}) (t_0 t_{i-1} - t_0 t_{i-1}) (t_0 \cdots t_{d-1})^{\kappa-1} dt
\]
\[
= 0 = V_\kappa \partial_i F(x).
\]
Putting these together, we have complete the proof. \( \square \)

The integral over the simplex \( T^d \) is also used in [15] for an integral representation of the intertwining operator associated to the dihedral group of \( d \)-regular polygon in \( \mathbb{R}^2 \), and the integral representation is also given for functions that depend only on one variable.

Although (2.1) is suggestive, we do not have an integral expression for a generic function \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) for \( d > 2 \). In the case \( d = 2 \), (1.3) becomes
\[
h_\kappa(x_1, x_2) = |x_1 - x_2|^{\kappa},
\]
for which we can deduce a complete integral representation for \( V_\kappa \). This formula, stated below, can also be deduced from the formula for the weight function \( h_{\lambda, \mu}(x) = |x_1|^\lambda |x_2|^\mu \), associated to the dihedral group \( I_2 \), by a rotation of \( 90^\circ \) and setting \( \lambda = \kappa \) and \( \mu = 0 \). For the record, we give a proof that verifies it directly from the definition.

**Theorem 2.2.** For the group \( S_2 \) and in homogeneous coordinates \( t_0 + t_1 = 1 \),
\[
V_\kappa f(x_1, x_2) = c_\kappa \int_{T^2} f(x_1 t_0 + x_2 t_1, x_1 t_0 + x_2 t_0) t_0^{\kappa-1} t_1^{\kappa-1} dt.
\]

**Proof.** We verify the righthand side of (2.2) satisfies the definition of \( V_\kappa \). First,
\[
V_\kappa f(x_1, x_2) - V f(x_2, x_1) = c_\kappa \int_{T^2} f(x_1 t_0 + x_2 t_1, x_1 t_0 + x_2 t_0) (t_0 - t_1) t_0^{\kappa-1} t_1^{\kappa-1} dt.
\]
Since \( \kappa (t_0 - t_1) t_0^{\kappa-1} t_1^{\kappa-1} = \frac{\partial}{\partial t_1} (t_0 t_1)^\kappa \), integration by parts gives
\[
\frac{\kappa}{x_1 - x_2} \frac{V f(x_1, x_2) - V f(x_2, x_1)}{x_1 - x_2} = c_\kappa \int_{T^2} \partial_1 f(x_1 t_0 + x_2 t_1, x_1 t_0 + x_2 t_0) t_0^{\kappa-1} t_1^{\kappa-1} dt
\]
\[
- c_\kappa \int_{T^2} \partial_2 f(x_1 t_0 + x_2 t_1, x_1 t_0 + x_2 t_0) t_0^{\kappa-1} t_1^{\kappa-1} dt.
\]
Taking derivative gives

\[ \frac{\partial}{\partial x_1} Vf(x_1, x_2) = c_\kappa \int_{T^2} \partial_1 f(x_1 t_0 + x_2 t_1, x_1 t_1 + x_2 t_0) t_0^{\kappa+1} t_1^{\kappa-1} dt \]

\[ + c_\kappa \int_{T^2} \partial_2 f(x_1 t_0 + x_2 t_1, x_1 t_1 + x_2 t_0) t_0^n t_1^{\kappa-1} dt. \]

Hence, adding the two terms together, we obtain

\[ D_1 Vf(x_1, x_2) = c_\kappa \int_{T^2} \partial_1 f(x_1 t_0 + x_2 t_1, x_1 t_1 + x_2 t_0) t_0^{\kappa+1} t_1^{\kappa-1} dt = V(\partial_1 f)(x), \]

where we have used \( t_0^n t_1^{\kappa+1} = t_0^{\kappa+1} t_1^{\kappa-1} \), which follows from \( t_0 = t_1 = 1 \). The same consideration works for \( D_2 V = V\partial_2 \). Notice that the denominator of the difference operator for \( D_2 \) is \( x_2 - x_1 \) instead of \( x_1 - x_2 \).

Let \( \mathcal{H}_n^d(h_\kappa^2) \) be the space of spherical \( h \)-harmonics of degree \( n \). We denote by \( e_1 := (1, 0, \ldots, 0), \ldots, e_d := (0, \ldots, 0, 1) \) the standard coordinate vectors of \( \mathbb{R}^d \).

**Proposition 2.3.** For \( 1 \leq \ell \leq d \), the reproducing kernel \( P_n(h_\kappa^2; \cdot, \cdot) \) of \( \mathcal{H}_n^d(h_\kappa^2) \) satisfies

\[ P_n(h_\kappa^2; x, e_\ell) = c_\kappa \int_{T^d} Z_n^{\lambda_\kappa}(x_1 t_0 + \cdots + x_d t_{d-1}) t_0 (t_0 \cdots t_{d-1})^{\kappa-1} dt. \]

This follows immediately from the addition formula (1.5) and the integral representation (2.1). The identity (2.3) plays an essential role in our study in the next section.

Another important extension from \( L^2(\mathbb{R}^d) \) to \( L^2(\mathbb{R}^d, h_\kappa^2) \) is an analogue of the Fourier transform in the Dunkl setting. For the symmetric group \( S_d \), this transform is defined by [7]

\[ \mathcal{F}_\kappa f(x) = c'_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(x, iy) h_\kappa^2(y) dy, \quad c'_\kappa = \frac{\Gamma\left(\frac{d}{2}\right)}{(2\pi)^{\frac{d}{2}} 2^\kappa \Gamma\left(\frac{d}{2}\kappa + \frac{d}{2}\right)} c_\kappa, \]

where the exponential function \( E_\kappa \) is defined by

\[ E_\kappa(x, y) := V_\kappa \left( e^{(x, s)} \right)(y), \quad x, y \in \mathbb{C}^d. \]

It is known that \( E_\kappa(x, y) := E_\kappa(y, x) \). Furthermore, the generalized Bessel function \( K_\kappa \) is defined by

\[ K_\kappa(x, y) = \frac{1}{d!} \sum_{\sigma \in S_d} E_\kappa(x, y\sigma). \]

For the symmetric group \( S_d \), the formula in Theorem 2.1 gives the following:

**Corollary 2.4.** For \( t = (t_0, \ldots, t_{d-1}) \in T^d \) and \( 1 \leq \ell \leq d \),

\[ E_\kappa(e_\ell, y) = c_\kappa \int_{T^d} e^{(y, t_\ell)} t_\ell (t_0 \cdots t_{d-1})^{\kappa-1} dt. \]

Furthermore, the generalized Bessel function satisfies \( K_\kappa,e(1, y) = K_\kappa,e(1, y) \) and

\[ K_\kappa,e(e_1, y) = \frac{1}{d} \sum_{j=1}^d E_\kappa(e_1, y(j)) = \frac{c_\kappa}{d} \int_{T^d} e^{(y, t_\ell)} (t_0 \cdots t_{d-1})^{\kappa-1} dt. \]
Proof. The symmetric group $S_d$ can be decomposed as the left cosets of $S_{d-1}$ given by $S_{d-1}(1, j)$, $1 \leq j \leq d$, which proves the first identity in $K_{\kappa, d}(e_1, y)$. From the expression of $E_\kappa(e_t, y)$, it is easy to see that $E_\kappa(e_1, y(1, j)) = E_\kappa(e_j, y)$, so that the second expression for $K_{\kappa, d}$ follows readily from $t_0 + \cdots + t_{d-1} = 1$. □

For $d = 2$, we can use Proposition 2.2 to write $K_{\kappa, 2}(x, iy)$ in terms of the classical Bessel function $J_\nu$, which satisfies the integral formula

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} e^{iz(1 - t^2)^{-\frac{1}{4}}} dt.$$  

Theorem 2.5. For $x, y \in \mathbb{R}^2$,

$$(2.4) \quad K_{\kappa, 2}(x, iy) = \sqrt{\pi} \Gamma(\frac{\kappa}{2} + \frac{1}{2}) e^{\frac{(x_1 + y_2)(y_1 + y_2)}{2}} \times \left( \frac{2}{(x_1 - x_2)(y_1 - y_2)} \right)^{\kappa - \frac{1}{2}} J_{\kappa - \frac{1}{2}} \left( \frac{2}{2} \frac{(x_1 - x_2)(y_1 - y_2)}{2} \right).$$

Furthermore, for $d > 2$, the generalized Bessel function satisfies

$$(2.5) \quad K_{\kappa, d}(e_1, iy) = \frac{c_{\kappa, d}}{c_{\kappa, d-1}} \int_{0}^{1} e^{y y_4} K_{\kappa, d-1}(e_1, i(1 - r)y') r^{\kappa - 1} (1 - r)^{(d-1)\kappa - 1} dr,$$

where $y = (y', y_d) \in \mathbb{R}^d$ and $e_1 = (1, 0, \ldots, 0)$ in either $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$.

Proof. Using (2.2) with $t_0 = 1 - t$ and $t_1 = t$, and then changing variable $t \mapsto (1 + s)/2$, we obtain

$$K_{\kappa, 2}(x, iy) = \frac{\Gamma(2\kappa + 1)}{2\Gamma(\kappa + 1)} \int_{0}^{1} e^{i(x, y) - (x_1 - x_2)(y_1 - y_2)t} (1 - t)^{\kappa - 1} t^{\kappa - 1} dt$$

$$= e^{i(x_1 + y_2)(y_1 + y_2)} \frac{\Gamma(\kappa + \frac{1}{2})}{\Gamma(\kappa)} \int_{-1}^{1} e^{-\frac{(x_1 - y_2)(y_1 - y_2)}{2}} (1 - s^2)^{\kappa - 1} ds,$$

where the constant has been simplified using the formula for $\Gamma(2a)$. Writing the last integral in terms of $J_{\kappa - \frac{1}{2}}$ proves (2.4).

For $d > 2$, we denote by $T^d_\rho$ the simplex $\{t \in \mathbb{R}^d : t_0 + t_1 + \cdots + t_{d-1} = \rho\}$. Then

$$\int_{T^d} g(u) du = \int_{0}^{1} \int_{T^{d-1}_{t_{d-1}}} g(t_0, \ldots, t_{d-2}, t_{d-1}) dt$$

$$= \int_{0}^{1} (1 - r)^{d-2} \int_{T^{d-1}_{r}} g((1 - r)s, r) ds dr,$$

where we set $t_{d-1} = r$ and $t_i = (1 - r)s_i$ for $i = 1, 2, \ldots, d - 2$, which also implies that $t_0 = (1 - r)s_0$. Setting $g(t) = e^{t(0y_1 + \cdots + t_{d-1}y_d)}$ in the above identity, the recursive formula (2.5) follows readily. □

For $x, y$ in the domain $\{x \in \mathbb{R}^d : x_1 + x_2 + \cdots + x_d = 0\}$, a fairly involved recursive formula for the generalized Bessel functions associated to the symmetric group $S_d$, or root system $A_{d-1}$, is given in [11]. The domain, however, does not contain coordinate vectors $e_i$. In the case of $d = 2$, it agrees with (2.4) with $x_2 = -x_1$ and $y_2 = -y_1$, apart from an extra constant $\sqrt{\pi}$.

Finally, let us mention a property of the intertwining operator $V_\kappa$ that we shall need in the next section. We denote by $a_\kappa$ the normalization constant of $h^2_\kappa$ defined...
by \( a_\kappa \int_{S^{d-1}} h^2_\kappa(x) d\sigma(x) = 1 \). For the symmetric group \( S_d \), we have [9] p. 216 and Thm 10.6.17]

\[
a_\kappa = \frac{2^{(d\kappa)} \Gamma((d\kappa + \frac{d}{2}) \prod_{j=2}^d \frac{\Gamma(\kappa + 1)}{\Gamma(j\kappa + 1)}},
\]

where \( \omega_d = 2\pi^{\frac{d}{2}} / \Gamma(\frac{d}{2}) \) is the surface area of \( S^{d-1} \).

**Lemma 2.6.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that both integrals below are defined. Then for \( x \in \mathbb{R}^d \),

\[
(2.6) \quad a_\kappa \int_{S^{d-1}} V_\kappa[f(\langle x, \cdot \rangle)](y) h^2_\kappa(y) d\sigma(y) = b_\lambda \int_{-1}^1 f(||x||t) (1 - t^2)^{\lambda - \frac{d}{2}} dt,
\]

where \( b_\lambda \) is the constant so that \( b_\lambda \int_{-1}^1 (1 - t^2)^{\lambda - \frac{d}{2}} dt = 1 \).

A more general identity holds for \( V_\kappa f \) for generic function \( f : \mathbb{R}^d \to \mathbb{R} \), where the righthand side is replaced by the integral of \( f(x) \) with respect to \( (1 - ||x||^2)^{\lambda - 1} \) over the unit ball of \( \mathbb{R}^d \) [13]. The identity shows that taking the average over the sphere removes the action of the reflection group.

### 3. Spherical h-harmonic series

In the first subsection we outline the background and what is known for the spherical \( h \)-harmonic series in the setting of a generic reflection group. The new result for the symmetric group is given in the second subsection.

#### 3.1. Spherical h-harmonic series

Let \( G \) be a given reflection group. Let \( h_\kappa \) be the \( G \)-invariant function with respect to which that spherical \( h \)-harmonics are orthogonal. When \( G \) is the symmetric group \( S_d \), the function \( h_\kappa \) is given in (1.3). Another case of interests for our discussion is \( G = \mathbb{Z}_2^d \), the group of sign changes, that has

\[
(3.1) \quad h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \quad \kappa_i \geq 0, \quad 1 \leq i \leq d.
\]

Unless specified otherwise, the discussion in this subsection holds for spherical \( h \)-harmonics associated with a generic reflection group \( G \); see, for example, [9].

For \( n \geq 0 \), let \( \{Y_{n,\ell} : 1 \leq \ell \leq \dim(n, d)\} \) be an orthonormal basis of \( \mathcal{H}_n(h^2_\kappa) \), normalized with respect to the inner product

\[
\langle f, g \rangle_\kappa = a_\kappa \int_{S^{d-1}} f(x)g(x) h^2_\kappa(x) d\sigma(x).
\]

For \( f \in L^2(S^{d-1}, h^2_\kappa) \), the spherical \( h \)-harmonic series of \( f \) is defined by

\[
L^2(h^2_\kappa) = \bigoplus_{n=0}^\infty \mathcal{H}_n(h^2_\kappa) : \quad f = \sum_{n=0}^\infty \sum_{\ell=1}^{\dim(n, d)} \langle f, Y_{n,\ell} \rangle_\kappa Y_{n,\ell}.
\]

The projection operator \( \text{proj}^n_\kappa : L^2(h^2_\kappa) \to \mathcal{H}_n(h^2_\kappa) \) and the \( n \)-th partial sum operator \( S_n(h^2_\kappa; f) \) are defined by

\[
\text{proj}^n_\kappa f = \sum_{\ell=1}^{\dim(n, d)} \langle f, Y_{n,\ell} \rangle_\kappa Y_{n,\ell} \quad \text{and} \quad S_n(h^2_\kappa; f) = \sum_{m=0}^n \text{proj}^m_\kappa f.
\]
By the definition of the reproducing kernel $P_n(h^2_\kappa)$ in [14], it follows readily that
\begin{equation}
\text{proj}_n^\kappa f(x) = a_\kappa \int_{S^{d-1}} f(y) P_n(h^2_\kappa; x, y) h^2_\kappa(y) d\sigma(y)
= a_\kappa \int_{S^{d-1}} f(y) V_\kappa \left[Z_\kappa^\lambda(\langle x, \cdot \rangle)\right] (y) h^2_\kappa(y) d\sigma(y),
\end{equation}
where the second identity follows from the addition formula (1.5). By the definition of $Z_\kappa^\lambda$ in (1.6), the partial sum operator is related to that of Fourier series in the Gegenbauer polynomials.

The $n$-th partial sum operator $S_n(h^2_\kappa; f)$ converges to $f$ in the $L^2(h^2_\kappa)$ norm by the classical Hilbert space theory. For $p \neq 2$, we consider the convergence of the Cesàro means, which often serve as a test stone of our knowledge on summability methods. For $\delta > 1$, the classical Hilbert space theory. For $p < \infty$, we use the definition in the next section. Then we can derive from (1.6) that
\begin{equation}
S_n^\delta(h^2_\kappa; f) := \frac{1}{(n+\delta)_n} \sum_{k=0}^n \binom{n-k+\delta}{n-k} \text{proj}_n^\kappa f
= a_\kappa \int_{S^{d-1}} f(y) K_n^\delta(h^2_\kappa; x, y) h^2_\kappa(y) dy,
\end{equation}
where $K_n^\delta(h^2_\kappa; \cdot, \cdot)$ is the $(C, \delta)$ means of $Z_\kappa^\lambda(\langle \cdot, \cdot \rangle)$ and the second identity follows from (3.2). Let $l_n^\delta(w; s, t)$ be the Gegenbauer weight function. Denote by $l_n^\delta(w; s, t)$ the kernel of the $(C, \delta)$ means of the Fourier–Gegenbauer series on $[-1, 1]$; see the definition in the next section. Then we can derive from (1.6) that
\begin{equation}
K_n^\delta(h^2_\kappa; x, y) = V_\kappa \left[k_n^\delta(w_{\lambda_\kappa}; \langle x, \cdot \rangle, 1)\right] (y).
\end{equation}
Since the $(C, \delta)$ means are linear integral operators, we know that $S_n^\delta(h^2_\kappa; f)$ converges in $L^1(h^2_\kappa; S^{d-1})$ or in $C(S^{d-1})$ if and only if
\begin{equation}
\sup_{x \in S^{d-1}} \int_{S^{d-1}} |K_n^\delta(h^2_\kappa; x, y)| h^2_\kappa(y) d\sigma(y) < \infty.
\end{equation}
For $1 \leq p \leq \infty$, let $\| \cdot \|_{p, \kappa}$ denote the $L^p(h^2_\kappa; S^{d-1})$ norm for $1 \leq p < \infty$ and the uniform norm of $C(S^{d-1})$ for $p = \infty$. A sufficient condition for the convergence of the $(C, \delta)$ means of spherical $h$-harmonics was given in [14].

**Theorem 3.1.** Let $f \in L^p(h^2_\kappa; S^{d-1})$, $1 \leq p < \infty$ or $f \in C(S^{d-1})$. Then the $(C, \delta)$ means $S_n^\delta(h^2_\kappa; f)$ converges to $f$ in $\| \cdot \|_{p, \kappa}$ norm if $\delta > \lambda_\kappa$.

The proof follows from Lemma 2.3 and (3.4), which reduces the boundedness in (3.3) to the boundedness of $\int_{-1}^1 [k_n^\delta(w_{\lambda_\kappa}; \langle x, \cdot \rangle, 1)] w_{\lambda_\kappa}(t) dt$, and the latter holds if and only if $\delta > \lambda_\kappa$ by the classical result of Szegő [13] Theorem 9.1.3. The case $1 < p < \infty$ follows from the Riesz interpolation. The theorem holds for spherical $h$-harmonics series associated with all reflection groups. The use of (2.6), however, removes the action of reflection group altogether and, as a consequence and not surprisingly, we pay the price that the condition $\delta > \lambda_\kappa$ is not sharp in general. This is first illustrated in the case when $G = Z_2^d$.

For the weight function $h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}$ given in (3.1), the intertwining operator $V_\kappa$ has an integral representation
\begin{equation}
V_\kappa f(x) = c_\kappa \int_{[-1, 1]^d} f(x_1 t_1, \ldots, x_d t_d) \prod_{i=1}^d (1 + t_i) (1 - t_i) \kappa_i^{-1} dt.
\end{equation}
This leads to, by Theorem 3.1, a closed formula for the kernel $K_\kappa^\delta(h_\kappa^\delta; \cdot, \cdot)$, which makes it possible to obtain a sharp estimate of the kernel that can be used to determine the critical index of the $(C, \delta)$ means. Indeed, while Theorem 3.1 establishes the convergence for $\delta > \lambda$ in this setting, it was proved in [10] that the Cesàro means $S_n^\delta(h_\kappa^\delta; f)$ converges to $f$ in $\| \cdot \|_{\kappa,p}$ norm for $p = 1$ or $\infty$ if and only if $\delta > \lambda - \min_{1 \leq i \leq d} \kappa_i$ for $G = \mathbb{Z}_2^d$.

We shall show in the next subsection that our new integral representation of $V_\kappa$ for the symmetric group will allow us to establish a similar result for the symmetric group, albeit only for convergence of $S_n^\delta(h_\kappa^\delta; f)$ at coordinate vectors.

3.2 Spherical $h$-harmonics series associated to symmetric group. In this subsection, $G$ is the symmetric group $S_d$ and $h_\kappa$ is given by (1.3). Recall that $\lambda_\kappa = (\frac{d}{2}) \kappa + \frac{d-2}{2}$ by (1.7). Our main result is the following theorem on the Cesàro $(C, \delta)$ means of spherical $h$-harmonic series.

**Theorem 3.2.** Let $h_\kappa$ be defined as in (1.3) and $f \in C(S^{d-1})$. Then $S_n^\delta(h_\kappa^\delta; f)$ converges to $f$ at the coordinate vectors $e_\ell$, $1 \leq \ell \leq d$, if

$$\delta > \lambda_\kappa - (d-1)\kappa = \left(\frac{d-1}{2}\right) \kappa + \frac{d-2}{2}.$$  

The proof requires a sharp estimate of the kernel $K_\kappa^\delta(h_\kappa^\delta; x, e_\ell)$, which comes down to estimate an integral of Jacobi polynomials over the simplex. We start by recalling the Jacobi polynomials and the definition of the kernel $k_\kappa^\delta$.

The Jacobi polynomials $P_n^{(\alpha, \beta)}$ are orthogonal with respect to the weight function $w_{\alpha, \beta}(t) = (1-t)^\alpha (1+t)^\beta$ on $[-1, 1]$. For $g \in L^1(w_{\alpha, \beta}; [-1, 1])$, let $s_n^{(\delta)}(w_{\alpha, \beta}; g)$ denote the Cesàro $(C, \delta)$ means of the Fourier-Jacobi series. Then

$$s_n^{(\delta)}(w_{\alpha, \beta}; g) = c_{\alpha, \beta} \int_{-1}^{1} g(s)k_\kappa^\delta(w_{\alpha, \beta}; \cdot, s)w_{\alpha, \beta}(s)ds,$$

where the kernel $k_\kappa^\delta(w_{\alpha, \beta}; \cdot, \cdot)$ is given by

$$k_\kappa^\delta(w_{\alpha, \beta}; s, t) = \frac{1}{(\kappa + \delta)} \sum_{k=0}^{\infty} \left(\begin{array}{c} n - \kappa + \delta \\ n - k \end{array}\right) P_k^{(\alpha, \beta)}(s)P_k^{(\alpha, \beta)}(t),$$

in which $h_k^{(\alpha, \beta)}$ is the $L^2(w_{\alpha, \beta}, [-1, 1])$ norm of $P_k^{(\alpha, \beta)}$. The Gegenbauer polynomials are related to the Jacobi polynomials by

$$C_n^\lambda(t) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(t)$$

and they are orthogonal with respect to $w_\lambda(t) = w_{\lambda - \frac{1}{2}, \lambda - \frac{1}{2}}(t)$ on $[-1, 1]$. In particular, $k_\kappa^\delta(w_{\lambda}; \cdot, \cdot) = k_\kappa^\delta(w_{\lambda - \frac{1}{2}, \lambda - \frac{1}{2}}; \cdot, \cdot)$.

Throughout the rest of this paper, we denote by $c$ a generic constant that may depend on fixed parameters such as $\kappa$ and $d$, and its value may change from line to line. Furthermore, we write $A \sim B$ if $A \leq cB$ and $B \leq cA$.

Our starting point is the following result in [13] p. 261, (9.4.13)], which shows that the main term of $k_\kappa^\delta(w_{\alpha, \beta}; \cdot, 1)$ is a Jacobi polynomial.
Lemma 3.3. For any $\alpha, \beta > -1$ such that $\alpha + \beta + 3 > 0$,

\[
(3.7) \quad k_n^\delta(w_{\alpha, \beta}, t, 1) = \sum_{j=0}^{J} b_j(\alpha, \beta, \delta, n) P_n^{(\alpha + \delta + j + 1, \beta)}(t) + G_n^\delta(t),
\]

where $J$ is a fixed integer and

\[
G_n^\delta(t) = \sum_{j=J+1}^{\infty} d_j(\alpha, \beta, \delta, n) k_n^{\delta+j}(w_{\alpha, \beta}, 1, t);
\]

moreover, the coefficients satisfy the inequalities,

\[
|b_j(\alpha, \beta, \delta, n)| \leq cn^{\alpha + 1 - \delta - j} \text{ and } |d_j(\alpha, \beta, \delta, n)| \leq cj^{-\alpha - \beta - \delta - 4}.
\]

When $\delta$ is large, the kernel $k_n^\delta(w_{\alpha, \beta})$ is non-negative and satisfies an estimate given in the lemma below, which was first used in [2] and [3].

Lemma 3.4. Let $\alpha, \beta \geq -1/2$. If $\delta \geq \alpha + \beta + 2$, then

\[
0 \leq k_n^\delta(w_{\alpha, \beta}, t, 1) \leq cn^{-1}(1 - t + n^{-2})^{-(\alpha + 3/2)}.
\]

We shall use Lemma 3.3 to write $K_n^\delta(h_n^2)$ as two terms. For the second term, we use the above lemma to estimate the $G_n^\delta$ term, which is relatively easy to handle. The main effort in estimating the first term lies in the proof of the following theorem.

Theorem 3.5. Let $\kappa > 0$ and let $\varphi$ be a $C^{\infty}$ function on $\mathcal{T}^d$. If $\alpha \geq \beta$ and $\alpha \geq (d - 1)\kappa - \frac{1}{4}$, then

\[
(3.8) \quad \left| \int_{\mathcal{T}^d} P_n^{(\alpha, \beta)}(x_1 t_0 + x_2 t_1 + \cdots + x_d t_d - 1) \varphi(t_0 t_1 \cdots t_d - 1) k_n^{\delta-1} dt \right| \leq cn^{-(d-1)\kappa - \frac{1}{4}} \sum_{i=1}^{d} \frac{\prod_{j=1, j \neq i}^{d} |x_j - x_i|^{-\kappa}}{(\sqrt{1 - |x_i|} + n^{-1})^{\alpha + \frac{1}{4} - (d-1)\kappa}}.
\]

The proof of this theorem is technical and will be given in the next section. In the rest of this subsection we use this theorem to provide a proof of Theorem 3.2 which relies on the following proposition.

Proposition 3.6. Let $\kappa > 0$. Then

\[
(3.9) \quad |K_n(h_n^2; x, e_\ell)| \leq cn^{\lambda_{\kappa} - (d-1)\kappa - \delta} \sum_{i=1}^{d} \frac{\prod_{j=1, j \neq i}^{d} |x_j - x_i|^{-\kappa}}{(\sqrt{1 - |x_i|} + n^{-1})^{\lambda_{\kappa} - (d-1)\kappa + 1}} + cn^{-1}V_{\kappa}\left(1 - \langle \cdot, e_\ell \rangle + n^{-2}\right)^{-(\lambda_{\kappa} + 1)}(x).
\]

Proof. By (3.4) and the integral representation of $V_{\kappa}$ in (2.1), we obtain

\[
K_n^\delta(h_n^2; x, e_\ell) := c_\delta \int_{\mathcal{T}^d} k_n^\delta(w_{\lambda_{\kappa}}, t_0 x_1 + \cdots + t_d x_d - 1) t_{d-1} (t_0 \cdots t_{d-1})^{\kappa - 1} dt.
\]

We replace the kernel $k_n^\delta(w_{\lambda_{\kappa}}) = k_n^\delta(w_{\lambda_{\kappa} - \frac{1}{2}, \lambda_{\kappa} - \frac{1}{2}})$ by the expansion in Lemma 3.3. Let $J = \lfloor 2\lambda_{\kappa} + 1 \rfloor$. Then

\[
K_n^\delta(h_n^2; x, e_\ell) = \sum_{j=0}^{J} b_j(\lambda_{\kappa} - \frac{1}{2}, \lambda_{\kappa} - \frac{1}{2}, \delta, n) \Omega_j(x) + \Omega(x),
\]
where

\[ \Omega_j(x) = c_n \int_{T^d} P_n^{\lambda_\kappa+\delta+j+\frac{\delta}{2},\lambda_\kappa-\frac{\delta}{2}}(t_0 x_1 + \cdots + t_{d-1} x_d) t_{\ell-1} (t_0 \cdots t_{d-1})^{\kappa-1} dt \]

and

\[ \Omega(x) = c_n \int_{T^d} G_n^\delta(t_0 x_1 + \cdots + t_{d-1} x_d) t_{\ell-1} (t_0 \cdots t_{d-1})^{\kappa-1} dt. \]

For \( \Omega_j \), we apply the estimate (3.8) with \( \varphi(t) = t_{\ell-1} \), \( \alpha = \lambda_\kappa + \delta + j + \frac{\delta}{2} \) and \( \beta = \lambda_\kappa - \frac{\delta}{2} \). Together with the estimate of \( b_j(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2}, \delta, n) \), we obtain that \( |b_j(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2}, 0 \delta, n) \Omega_j(x)| \) is bounded by the first term in the righthand side of (3.9), hence, so is the sum of these terms over \( 0 \leq j \leq J \). Our choice of \( J \) allows us to use the estimate in Lemma 3.3 to obtain

\[ |G_n^\delta(t)| \leq cn^{-1}(1-t+n^{-2})^{\lambda_\kappa+1}, \]

where we have used the fact that \( \sum_{j=1}^{\infty} |d_j(\lambda_\kappa - \frac{1}{2}, \lambda_\kappa - \frac{1}{2}, \delta, n)| \) is bounded. Consequently, by (3.11), it follows that \( |\Omega(x)| \) is bounded by the second term in the righthand side of (3.9). This completes the proof.

**Proof of Theorem 3.2** By (3.8), we need to prove that

\[ I_n := \int_{S^{d-1}} |K_n^\delta(h_n^{\kappa}; x, e_\ell)| h_n^\kappa(x) d\sigma(x) < \infty \]

if \( \delta > \lambda_\kappa - (d-1)\kappa \). By (3.9), \( I_n \) is bounded by

\[ I_n \leq \sum_{i=1}^d I_{n,i} + cn^{-1} \int_{S^{d-1}} V_n \left[ (1 - \langle \cdot, e_\ell \rangle + n^{-2})^{-(\lambda_\kappa+1)} \right] \langle x, h_n^\kappa(x) \rangle d\sigma(x), \]

where \( I_{n,i} \) is defined by

\[ I_{n,i} = cn^{\lambda_\kappa-(d-1)\kappa-\delta} \int_{S^{d-1}} \prod_{j=1, j \neq i}^d \frac{|x_j - x_i|^{-\kappa}}{(1 - |x_i| + n^{-1})^{\lambda_\kappa-(d-1)\kappa+\delta+1}} h_n^\kappa(x) d\sigma(x). \]

For the second term in the righthand side of (3.10), we use (2.6) to bound it by

\[ cn^{-1} \int_{-1}^1 \frac{(1-t^2)^{\lambda_\kappa-\frac{\delta}{2}}}{(1-t+n^{-2})^{\lambda_\kappa+1}} dt = cn^{-1} \int_0^\pi \frac{(\sin \theta)^{2\lambda_\kappa}}{(1-\cos \theta + n^{-2})^{\lambda_\kappa+1}} d\theta \sim n^{-1} \int_0^{\pi/2} \frac{\theta^{2\lambda_\kappa}}{(\theta^2 + n^{-2})^{\lambda_\kappa+1}} d\theta = \int_0^{\pi/2} \frac{s^{2\lambda_\kappa}}{(1+s^2)^{\lambda_\kappa+1}} ds < \infty. \]

Next we estimate \( I_{n,i} \) for \( 1 \leq i \leq d \). If \( j \neq i \) and \( k \neq i \), then, for \( x \in S^{d-1} \),

\[ |x_j - x_k|^2 \leq 2(x_j^2 + x_k^2) \leq 2 \sum_{j \neq i} x_j^2 = 2(1 - x_i^2) \leq 4(1 - |x_i|), \]

so that \( |x_j - x_k| \leq 2\sqrt{1 - |x_i|} \). Hence, by \( h_n^\kappa(x) = \prod_{1 \leq j < k \leq d} |x_j - x_k|^{2\kappa} \), we obtain

\[ I_{n,i} \leq cn^{\lambda_\kappa-(d-1)\kappa-\delta} \int_{S^{d-1}} \prod_{j \neq i, k \neq i} \frac{|x_j - x_k|^{2\kappa}}{(1 - |x_i| + n^{-1})^{\lambda_\kappa-(d-1)\kappa+\delta+1}} d\sigma(x) \]

\[ \leq cn^{\lambda_\kappa-(d-1)\kappa-\delta} \int_{S^{d-1}} \frac{1}{(1 - |x_i| + n^{-1})^{\lambda_\kappa-(d-1)\kappa+\delta+1+2\left(\frac{d-1}{2}\right)\kappa}} d\sigma(x). \]
Since \( \lambda_\kappa - (d - 1)\kappa = \left(\frac{d - 1}{2}\right)\kappa + \frac{d - 2}{2} \), the last integral can be rewritten to give
\[
I_{n,i} \leq cn^{-d-1} \int_{S^{d-1}} \frac{1}{(1 + n\sqrt{1 - |x|^2})^\frac{d}{2} - \frac{d - 1}{2} + \frac{\kappa}{2}} d\sigma(x)
\]
\[
\sim cn^{-d-1} \int_{\pi/2}^{\pi/2} \frac{(\sin \theta)^{d - 2}}{(1 + n\theta)^{\frac{d}{2} - \frac{d - 1}{2} + \kappa + \frac{\kappa}{2}}} d\theta,
\]
where we have used the spherical coordinates by choosing \( x_i = \cos \theta \). Using \( \sin \theta \sim \theta \) and changing variable \( t = n\theta \), it is easy to see that the last integral is bounded if and only if \( \delta > \left(\frac{d - 1}{2}\right)\kappa + \frac{d - 2}{2} \). This completes the proof. \( \square \)

We conjecture that the condition (3.6) in Theorem 3.2 is sharp; that is, \( S^\delta_n (h^2_\kappa; \epsilon) \) does not converge if \( \delta = \lambda_\kappa - (d - 1)\kappa \). More precisely, we expect the inequality
\[
\int_{S^{d-1}} \int_{T^d} P_{n}^{(\alpha, \beta)}(x_1 t_0 + x_2 t_1 + \cdots + x_d t_{d-1}) t_0 t_1 \cdots t_{d-1})^{k-1} dt ||h^2_\kappa(x)||d\sigma(x)
\]
\[
\geq cn^{-(d-1)\kappa - \frac{1}{2} \log n}
\]
to hold for \( \alpha = \lambda_\kappa + \frac{1}{2}, \beta = \lambda_\kappa + \frac{1}{2} \) and \( \delta = \lambda_\kappa - (d - 1)\kappa \), which would prove the sharpness of the condition by Lemma 3.3. Furthermore, taking the cue from the classical Fourier-Jacobi series and spherical \( h \)-harmonic series with \( G = \mathbb{Z}_d^2 \), we expect that the \((C, \delta)\) means \( S^\delta_n (h^2_\kappa; f) \), with \( h_\kappa \) in (1.3), converge for the same \( \delta \) that ensures the convergence at the coordinates vectors. In other words, we conjecture that the condition \( \delta > \lambda_\kappa - (d - 1)\kappa \) is the necessary and sufficient condition for the convergence of \( S^\delta_n (h^2_\kappa; f) \) in \( \| \cdot \|_{p,\kappa} \) for \( p = 1 \) and \( \infty \).

4. Proof of Theorem 3.5

The proof is based on a lemma established in [4]. First we need a definition.

**Definition 4.1.** Let \( n, v \in \mathbb{N}_0 \). A function \( f : [-1, 1] \to \mathbb{R} \) is said to be in class \( S^\mu_n (\mu) \), if there exist functions \( F_j, j = 0, 1, \cdots, v \) on \([-1, 1] \) such that \( F_j^{(j)}(x) = f(x), x \in [-1, 1], 0 \leq j \leq v, \) and
\[
|F_j(x)| \leq cn^{-2j} \left(1 + n\sqrt{1 - |x|^2}\right)^{-n - \frac{1}{2} + j}, \quad x \in [-1, 1], \quad j = 0, 1, \cdots, v.
\]

This definition is motivated by the following two properties of the Jacobi polynomials. The first one is the well-known identity
\[
P_n^{(\alpha+1, \beta+1)}(y) = \frac{2}{n + \alpha + \beta + 2} \frac{d}{dy} P_n^{(\alpha, \beta)}(y)
\]
and the second one is the pointwise estimate of the Jacobi polynomials in the lemma below (13 (7.32.5) and (4.1.3))).

**Lemma 4.2.** For an arbitrary real number \( \alpha \) and \( t \in [0, 1] \),
\[
|F_n^{(\alpha, \beta)}(t)| \leq cn^{-1/2}(1 - t + n^{-2})^{-(\alpha + 1/2)/2}.
\]
The estimate on \([-1, 0]\) follows from the fact that \( F_n^{(\alpha, \beta)}(t) = (-1)^n F_n^{(\beta, \alpha)}(-t) \).

In particular, these two properties show that \( n^-\alpha F_n^{(\alpha, \beta)} \in S^\mu_n (\alpha) \) for all \( v \in \mathbb{N}_0 \). We can now state our main lemma.
Lemma 4.4. Let $\kappa > 0$. For a fixed $b$ with $0 < b < 1$, let $\xi \in C^\infty([0, b])$. Let $f \in T_n^\kappa(\mu)$ with $v \geq |\mu| + 2\kappa + \frac{3}{2}$. Assume $|x| \leq 1$ and $|x + at| \leq 1$ for $t \in [0, 1]$. Then,

\[
\left| \int_0^1 f(x + at) t^{\kappa-1} \xi(t) \, dt \right| \leq c n^{-2\kappa} |a|^{-\kappa} \left( 1 + n \sqrt{1 - |x|} \right)^{\kappa - \frac{1}{2} + \kappa}.
\]

Proof. Changing variable $t \mapsto 1 - t$, the integral becomes

\[
\int_0^1 f(x + at) t^{\kappa-1} \xi(t) \, dt = \int_0^1 f(x + a - at)(1 - t)^{\kappa-1} \xi(1 - t) \, dt.
\]

We can then apply Lemma 3.4 of [4] on the integral in the righthand side.

The statement of [9] Lemma 3.4] is slightly more complicated, with an assumption that $|x| \leq 1 - |a|$, but a close look at the proof shows that it suffices to assume that $|x + at| \leq 1$ for $t \in [0, 1]$. This lemma is used to prove the following estimate:

Lemma 4.4. Let $\kappa > 0$ and let $\xi$ be a $C^\infty(\mathcal{T}^d)$ function such that its support set is $\{ t \in \mathcal{T}^d : t_0 \geq (2d)^{-1} \}$. Then, for $\alpha \geq \beta$, $\alpha \geq (d - 1)\kappa - \frac{1}{2}$,

\[
\left| \int_{\mathcal{T}^d} P_n^{(\alpha, \beta)}(x_1t_0 + x_2t_1 + \cdots + x_dt_{d-1})^\kappa \xi(t)(t_0 \cdots t_{d-1})^k \, dt \right| \leq c n^{-2(d-1)\kappa} \prod_{j=2}^d |x_j - x_1|^{-\kappa} \left( \sqrt{1 - |x_1|} + n^{-1} \right)^{\alpha + \frac{1}{2} - (d-1)\kappa}.
\]

Proof. With $t_0 = 1 - t_1 - \cdots - t_{d-1}$ and $\eta(t_1, \ldots, t_{d-1}) = t_0^{k-1} \xi(t)$. Then $\eta$ is a $C^\infty(\mathcal{T}^{d-1})$ function and the integral over $\mathcal{T}^d$ can be written as

\[
I(x) := \int_{\mathcal{T}^d} P_n^{(\alpha, \beta)}(x_1t_0 + x_2t_1 + \cdots + x_dt_{d-1})^\kappa \xi(t)(t_0 \cdots t_{d-1})^k \, dt
\]

\[
= \int_0^1 \cdots \int_0^1 \int_0^{1-t_1} \cdots \int_0^{1-t_1-\cdots-t_{d-1}} P_n^{(\alpha, \beta)}(x_1 + (x_2 - x_1)t_1 + \cdots + (x_d - x_1)t_{d-1})^\kappa \xi(t_0 \cdots t_{d-1})^k \, dt_1 \cdots dt_{d-1}.
\]

Changing variable $t \mapsto u$ in the integral with $t_1 = u_1$, $t_2 = (1 - u_1)u_2$, $\ldots$, $t_{d-1} = (1 - u_1) \cdots (1 - u_{d-2})u_{d-1}$, and write $\xi(t) = \xi(t(u))$. Then $t \in T^{d-1}$ becomes $u \in [0, 1]^{d-1}$. It is easy to verify that $t_0(u) = (1 - u_1) \cdots (1 - u_{d-1})$, so that $\xi(t(u))$ is zero only if $(1 - u_1) \cdots (1 - u_{d-1}) \leq 1/(2d)$. In particular, if $1 - u_j \leq 1/(2d)$, or $u_j \geq 1 - 1/(2d)$, for some $j$, then $\xi(t(u)) = 0$. Let $\chi$ be a $C^\infty$ function such that $\chi(u) = 1$ if $u \leq 1 - 1/(2d)$ and $\chi(u) = 0$ if $u \geq 1 - 1/(3d)$. Then $\chi(u_1) \cdots \chi(u_{d-1})$ is equal to 1 over the support set of $\xi(t(u))$. Hence, we can write

\[
I(x) = \int_{[0,1]^{d-1}} P_n^{(\alpha, \beta)} \chi \left( x_1 + \sum_{j=1}^{d-1} (x_{j+1} - x_1) \prod_{i=2}^{j} (1 - u_{i-1})u_j \right) \xi(t(u))
\]

\[
\times \prod_{j=1}^{d-1} \chi(u_j) u_j^{\kappa-1} \prod_{i=2}^{j} (1 - u_{i-1})^\kappa du_{d-1} \cdots du_1.
\]
where we adopt the convention $\prod_{i=2}^{1} (1 - u_{i-1}) = 1$. For $2 \leq m \leq d - 1$, we define

$$f_m(y) = \int_{[0,1]^m} P_n^{(\alpha, \beta)} \left( y + \sum_{j=d-m}^{d-1} (x_{j+1} - x_1) \prod_{i=2}^{j} (1 - u_{i-1}) u_j \right) \xi(t(u))$$

$$\times \prod_{j=d-m}^{d-1} \chi(u_j) u_j^{\kappa-1} \prod_{i=2}^{j} (1 - u_{i-1})^\kappa du_{d-1} \cdots du_{d-m}$$

for all $y$ such that the argument of $P_n^{(\alpha, \beta)}$ is in $[-1, 1]$. It is evident that $I(x) = f_{d-1}(x_1)$. Furthermore, let $v_m := |[\alpha - m\kappa] + 2\kappa| + 3$ for $0 \leq m \leq d - 1$. For $\ell = 0, 1, \ldots, v_m$, we define

$$F_{m, \ell}(y) = B_{n, \ell} \int_{[0,1]^m} P_n^{(\alpha - \ell, \beta - \ell)} \left( y + \sum_{j=d-m}^{d-1} (x_{j+1} - x_1) \prod_{i=2}^{j} (1 - u_{i-1}) u_j \right) \xi(t(u))$$

$$\times \prod_{j=d-m}^{d-1} \chi(u_j) u_j^{\kappa-1} \prod_{i=2}^{j} (1 - u_{i-1})^\kappa du_{d-1} \cdots du_{d-m},$$

where $B_{n, \ell} = 2^\ell / \prod_{j=1}^{\ell} (n + \alpha + \beta + 1 - j)$. By (4.4), it follows readily that $F_{m, \ell}^{(\ell)} = f_m$. We now prove that $f_m$ satisfies the estimate

$$|f_m(y)| \leq cn^{-2m \kappa} \prod_{j=d-m+1}^{d} |x_j - x_1|^{-\kappa} \left( 1 + n \sqrt{1 - |y|} \right)^{-\alpha - \frac{3}{2} + m \kappa}$$

for $m = 1, 2, \ldots, d - 1$ and moreover,

$$n^{-\alpha + 2m \kappa} \prod_{j=d-m+1}^{d} |x_j - x_1|^{\kappa} f_m \in \mathcal{S}_{\alpha}^{m\kappa}(\alpha - m\kappa).$$

The proof is by induction. For $m = 1$, write $a_{d-1} = \prod_{i=2}^{d-1} (1 - u_{i-1})$. Since $\chi$ is supported on $[0, b]$, where $b = 1 - (3d)^{-1}$, and $n^{-\alpha} P_n^{(\alpha, \beta)}(x) \in \mathcal{S}_{\alpha}^{m\kappa}(\alpha)$ for $v_0 = |[\alpha] + 2\kappa| + 3$, we can apply Lemma 4.3 to obtain

$$|f_1(y)| = a_{d-1}^{\kappa} \left| \int_{0}^{1} P_n^{(\alpha, \beta)} \left( y + a_{d-1} (x_d - x_1) u_{d-1} \right) \xi(t(u)) \chi(u_{d-1}) u_{d-1}^{\kappa-1} du_{d-1} \right|$$

$$\leq c n^{-2m \kappa} |x_d - x_1|^{-\kappa} \left( 1 + n \sqrt{1 - |y|} \right)^{-\alpha - \frac{3}{2} + \kappa},$$

which establishes (4.6) for $m = 1$. Similarly, since $B_{n, \ell} \sim n^{-\ell}$, the similar estimate can be carried out for $F_{1, \ell}$ and we obtain, for $\ell = 1, \ldots, v_1$, 

$$n^{-\alpha} |F_{1, \ell}(y)| \leq c |x_d - x_1|^{-\kappa} n^{-2\kappa - 2\ell} \left( 1 + n \sqrt{1 - |y|} \right)^{-\alpha - \frac{3}{2} + \kappa + \ell},$$

which shows that (4.7) holds for $m = 1$. Assume now that (4.6) and (4.7) have been established for $f_{m-1}$. We now consider $f_m$. From the definition of $f_m$, it is easy to see that

$$f_m(y) = a_{d-m}^{\kappa} \int_{0}^{1} f_{m-1} \left( y + (x_{d-m+1} - x_1) a_{d-m} u_{d-m} \right) \xi(t(u))$$

$$\times \chi(u_{d-m}) u_{d-m}^{\kappa-1} du_{d-m},$$
where $a_{d-m} = \prod_{i=2}^{d-m} (1-u_{i-1})$ for $m = 2, 3, \ldots, d-1$, and similar iterative relations hold if we replace $f_m$ and $f_{m-1}$ by $F_{m,t}$ and $F_{m-1,t}$ for $1 \leq t \leq v_m$. Using the induction hypothesis, we can then apply Lemma 4.3 to obtain the estimate (1.6) for $f_m$, which can be carried out exactly as in the estimate of $f_1$, and similarly for $F_m$ to establish (1.7) for $f_m$. This completes the induction.

Finally, it is easy to see that the desired estimate (1.5) is equivalent to the estimate for $f_{d-1}(x_1)$ in (1.6). This completes the proof.

The support set of $\xi$ in the theorem means that we are considering the simplex with one vertex chopped off. The proposition below shows that this can be done one at a time.

**Proposition 4.5.** There exist $C^\infty$ functions $\xi_0, \xi_1, \ldots, \xi_{d-1}$ on $T^d$ such that

$$\xi_0(t) + \cdots + \xi_{d-1}(t) = 1, \quad \xi_0(t) \geq 0, \ldots, \xi_{d-1}(t) \geq 0, \quad t \in T^d,$$

and the support set of $\xi_j$ is a subset of $\{t \in T^d : t_j \geq (2d)^{-1}\}$.

**Proof.** For $d \geq 2$, let $0 < a < b < 1$ be defined by

$$a = \frac{1}{2d} \quad \text{and} \quad b = \frac{1-a}{d-1} = \frac{1}{2d} + \frac{1}{2(d-1)} < 1.$$

Let $\xi$ be a $C^\infty$ function on the real line such that $\xi(t) = 0$ if $0 \leq t \leq a$, and $\xi(t) = 1$ if $b < t \leq 1$. In particular, $\xi$ is supported on $(a, 1]$ and $1 - \xi$ is supported on $[0, b)$. For $t \in T^d$, we write $t = (t_0, t_1, \ldots, t_{d-1})$ in homogeneous coordinates, or $t_0 = 1 - t_1 - \cdots - t_{d-1}$. We define $C^\infty$ functions $\xi_0, \xi_1, \ldots, \xi_{d-1}$ by

$$\xi_0(t) = \xi(t_0),$$

$$\xi_1(t) = (1 - \xi(t_0))\xi(t_1),$$

$$\cdots,$$

$$\xi_{d-2}(t) = (1 - \xi(t_0)) \cdots (1 - \xi(t_{d-3}))\xi(t_{d-2}),$$

$$\xi_{d-1}(t) = (1 - \xi(t_0)) \cdots (1 - \xi(t_{d-3}))(1 - \xi(t_{d-2})).$$

Then it is evident that $\xi_0(t) + \cdots + \xi_{d-1}(t) = 1$. Furthermore, it is easy to see that, for $0 \leq j \leq d-2$, the support set of $\xi_j$ is $\{t \in T^d : t_0 \leq b, \ldots, t_j \leq b, \text{ and } t_j > a\}$, which is evidently a subset of $\{t \in T^d : t_j > a\}$. Moreover, the support set of $\xi_{d-1}$ is $\{t \in T^d : t_0 \leq b, \ldots, t_{d-2} \leq b\}$. Each element of this last set satisfies the inequality

$$t_{d-1} = t_0 - \cdots - t_{j-2} \geq 1 - (d-1)b = 1 - (1-a) = a$$

by the definition of $b$, so that the subset of $\xi_{d-1}$ is a subset of $\{t \in T^d : t_{d-1} > a\}$. This completes the proof.

**Proof of Theorem 4.5.** Using the partition of unity in Proposition 4.5, we can write the integral as a sum of

$$\int_{T^n} P_n^{(\alpha, \beta)} (x_1 t_0 + x_2 t_1 + \cdots + x_d t_{d-1}) \xi_i(t) \varpi(t)(t_0 t_1 \cdots t_{d-1})^{k-1} dt$$

for $i = 0, 1, \ldots, d-1$. Hence, we only need to estimate the above integral for each $i$. For $i = 0$, this is precisely the estimate carried out in Lemma 4.3 with $\xi(t) = \xi_0(t)\varpi(t)$. By the symmetry of $T_n$ and the integral, for each $i \neq 0$, we can exchange $t_i$ and $t_0$, so that the same estimate applies. This completes the proof.
References

[1] B. Amri, Note on Bessel functions of type $A_{N-1}$. Integral Transf. Spec. Funct, 25 (2014), 448–461.
[2] A. Bonami and J-L. Clerc, Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques, Trans. Amer. Math. Soc. 183 (1973), 223–263.
[3] L. Colzani, M.H. Taibleson and G. Weiss, Maximal estimates for Cesàro and Riesz means on spheres, Indiana Univ. Math. J. 33 (1984), 873–889.
[4] F. Dai and Y. Xu, Cesàro means of orthogonal expansions in several variables, Const. Approx. 29 (2009), 129–155.
[5] F. Dai and Y. Xu, Approximation theory and harmonic analysis on spheres and balls, Springer Monographs in Mathematics, Springer, 2013.
[6] C. F. Dunkl, Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc., 311 (1989), 167–183.
[7] C. F. Dunkl, Integral kernels with reflection group invariance, Can. J. Math. 43 (1991), 1213–1227.
[8] C. F. Dunkl, Intertwining operators associated to the group $S^3$. Trans. Amer. Math. Soc. 347 (1995), 3347–3374.
[9] C. F. Dunkl and Y. Xu, Orthogonal Polynomials of Several Variables, 2nd ed., Encyclopedia of Mathematics and its Applications 155, Cambridge University Press, Cambridge, 2014.
[10] Zh.-K, Li and Y. Xu, Summability of orthogonal expansions of several variables, J. Approx. Theory, 122 (2003), 267–333.
[11] M. Rösler, Positivity of Dunkl’s intertwining operator. Duke Math. J. 98 (1999), 445–464.
[12] P. Sawyer, A Laplace-type representation of the generalized spherical functions associated to the root systems of type $A$. Mediterr. J. Math. 14 (2017), 147.
[13] G. Szegő, Orthogonal polynomials, 4th edition. Amer. Math. Soc., Providence, RI, 1975.
[14] Y. Xu, Integration of the intertwining operator for $h$-harmonic polynomials associated to reflection groups, Proc. Amer. Math. Soc. 125 (1997), 2963–2973.
[15] Y. Xu, Intertwining operators associated to dihedral groups. Constr. Approx., to appear. https://doi.org/10.1007/s00365-019-09487-w

Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222.
E-mail address: yuan@uoregon.edu