Multidimensional rearrangement and Lorentz spaces

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Abstract

We define a multidimensional rearrangement, which is related to classical inequalities for functions that are monotone in each variable. We prove the main measure theoretical results of the new theory and characterize the functional properties of the associated weighted Lorentz spaces.

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1 Introduction

Recently, some authors (see [Ba], [BPSo], and [BPSt]) have considered multi-dimensional analogs of classical inequalities for monotone functions: Hardy’s inequality, Chebyshev’s inequality, embeddings for weighted Lorentz spaces, etc. (see, e.g., [AM], [Sa], [St], [CS]). We recall that the main interest in studying these results on monotone functions comes from the fact that the spaces, where the estimates hold, are rearrangement invariant function spaces (see [BS]), and hence the functions that show up in the inequalities are the nonincreasing rearrangements of general measurable functions (which are essentially all monotone functions on \( \mathbb{R}_+ \)). This observation is fundamental to understand our main purpose: we want to find the natural definition for a multidimensional rearrangement in such a way that what we get is a general decreasing function on \( \mathbb{R}^n_+ := \mathbb{R}^n \times \cdots \times \mathbb{R}_+ \). Our approach is very geometrical: we look for a measure preserving transformation taking (all) sets in \( \mathbb{R}^n \) to (all) decreasing sets in \( \mathbb{R}^n_+ \), and such that it is monotone, and leaves fixed the sets that are already decreasing (see Definition 2.2). Once we know how to rearrange sets, we can define the multidimensional rearrangement of a function by using the “Layer-cake formula”, which recovers a function by means of its level sets (see Definition 2.3).

This new definition opens up the possibility of studying whether the properties of the classical rearrangement hold true in the multidimensional setting (see Corollary 2.12 for an example which shows that the resonant property fails). In Section 3 we develop the main ideas of the new rearrangement from a measure theoretical point of view (Propositions 2.4, 2.7, 2.8, and Theorem 2.11), establish the relationship with the classical rearrangement and show that it agrees with the so called multivariate rearrangement (Corollary 2.9 and Theorem 2.13). In Section 3 we introduce the weighted Lorentz spaces associated to the multidimensional rearrangement, we find their relationship with the Lebesgue and the rearrangement invariant spaces (Theorem 3.1 and Propositions 3.2, 3.3), prove the different embeddings in the whole range of indices (Proposition 3.4), and characterize functional properties like quasinormability (Theorem 3.5) and the weights which give rise to a norm (Theorems 3.6 and 3.7).

Most of the notations we are going to use are standard as, for example, defined in [BS]: \( \lambda_f \) is the distribution function of \( f \), the nonincreasing rearrangement of \( f \) is denoted \( f^* \), \( h \downarrow \) means that \( h \) is decreasing, etc. A weight \( w \) is a locally integrable positive function (either on \( \mathbb{R}^n \) or \( \mathbb{R}^n_+ \), depending on the context), and if \( E \) is a set, \( w(E) = \int_E w \). As usual, \( |E| \) denotes the Lebesgue measure of \( E \). Two positive quantities \( A \) and \( B \), are said to be
equivalent \((A \approx B)\) if there exists a constant \(C > 1\) (independent of the essential parameters defining \(A\) and \(B\), and not the same at different occasions) such that \(C^{-1}A \leq B \leq CA\). Also, all sets that we are going to consider are always Lebesgue measurable sets.

2 Two-dimensional decreasing rearrangement

For simplicity, we are going to reduce our definitions to the two-dimensional case, although there are natural extensions to higher dimensions too. Our approach is to give a geometric definition of the rearrangement of a measurable set (so that we get a general decreasing set in \(\mathbb{R}^2_+\)), and extend it to also rearrange functions, by looking at the level sets and the use of the Layer-cake formula ([LL]). We will show in Theorem 2.13 that this definition agrees with the so called multivariate rearrangement (see [Bl]).

Definition 2.1 We say that a set \(D \subset \mathbb{R}^2_+\) is decreasing (and write \(D \in \Delta_d\)) if the function \(\chi_D\) is decreasing in each variable.

Definition 2.2 Let \(E\) be a subset of \(\mathbb{R}^2\) and \(\varphi_E(x) = |\{y \in \mathbb{R} : (x, y) \in E\}|, x \in \mathbb{R}\). Let the function \(\varphi^*_E\), defined by

\[\varphi^*_E(s) = \inf\{\lambda : |\{x \in \mathbb{R} : \varphi_E(x) > \lambda\}| \leq s\}, \quad (s \geq 0)\]

be the usual decreasing rearrangement of \(\varphi_E\) (see [BS]). Then, the two-dimensional decreasing rearrangement of the set \(E\) is the set

\[E^* = \{(s, t) \in \mathbb{R}^2_+ : 0 < t < \varphi^*_E(s)\}\].

Definition 2.3 (Layer-cake formula [LL]). The two-dimensional decreasing rearrangement \(f^*_2\) of a function \(f\) on \(\mathbb{R}^2\) is given by

\[f^*_2(x) = \int_0^\infty \chi_{\{|f| > t\}^*}(x)dt, \quad x \in \mathbb{R}^2_+\].

We give now some elementary properties for this new rearrangement definition.

Proposition 2.4 Let \(E\) and \(F\) be two subsets of \(\mathbb{R}^2\). Then,

a) \(|E| = |E^*|\), and \(E^* \subset F^*\), if \(E \subset F\).

b) \(E = E^*\), if and only if \(E\) is a decreasing set of \(\mathbb{R}^2_+\).

c) \(f^*_2 = \chi_{F^*}\), if and only if \(f = \chi_E\), and \(E^* = F^*\). In particular, \((\chi_E)^*_2 = \chi_{E^*}\).

\(d)\) If \(E \cap F = \emptyset\) then \(|(E \cup F)^* \setminus E^*| = |F|\).
Proof. a) We have

$$|E| = \int_{-\infty}^{\infty} \varphi_E(x)dx = \int_{0}^{\infty} \varphi_E^*(x)dx = |E^*|.$$ 

The second part is trivial since $\varphi_E \leq \varphi_F$.

b) If $E$ is a decreasing set, then there exists $r > 0$ such that

$$E = \{(x, y) \in \mathbb{R}^2 : 0 < x < r, 0 < y < \varphi_E(x)\}.$$ 

Since $\varphi_E$ is decreasing, then $E = E^*$. The converse implication is trivial.

c) It yields that

$$(\chi_E)^2(x) = \int_{0}^{\infty} \chi_{\{|f| > s\}}^*(x)dt = \int_{0}^{1} \chi_{E^*}(x)dt = \chi_{E^*}(x).$$ 

Conversely, suppose $f_2^* = \chi_{F^*}$:

- If $x \notin F^*$, then $x \notin \{f > t\}^*$ and hence $\{f > t\}^* \subset F^*$, for all $t > 0$.
- If $x \in F^*$, then $x \in \{f > t\}^*$, $0 < t < 1$, and $x \notin \{f > t\}^*$, $1 < t$.

Therefore, $\{f > t\}^* = F^*$, if $0 < t < 1$, and $\{f > t\}^* = \emptyset$, if $1 < t$. Thus, $t < f(x) \leq 1$, if $f(x) \neq 0$, for every $0 < t < 1$, and hence there exists a set $E$ such that $f = \chi_E$ and $E^* = F^*$.

Property d) follows easily from a). □

The following results gives more information on the level sets of $f$ and $f_2^*$.

Lemma 2.5 If $f$ is a measurable function on $\mathbb{R}^2$ and $t > 0$, then

$$\{f_2^* > t\} \subseteq \{|f| > t\}^* \subseteq \{f_2^* \geq t\}.$$ 

Proof. By definition,

$$f_2^*(x) > t \iff \int_{0}^{\infty} \chi_{\{|f| > s\}}^*(x)ds > t, \ (x = (x_1, x_2)).$$ 

But,

$$\chi_{\{|f| > s\}}^*(x) = \begin{cases} 1 & \text{if } \varphi_1^*(x_1) > x_2 \\ 0 & \text{if } \varphi_1^*(x_1) \leq x_2, \end{cases}$$
where \( \varphi_s(a) = |\{b : |f(a,b)| > s\}| \). Thus \( f_2^*(x) > t \iff |\{s : \varphi_s^*(x_1) > x_2\}| > t \). Observe that if \( s < s' \), then \( \varphi_{s'}^*(x_1) \leq \varphi_s^*(x_1) \), and hence \( \{s : \varphi_s^*(x_1) > x_2\} \) is an interval of the form \((0, s)\) or \((0, s']\). Hence,

\[
|(0, s)| > t \implies s > t \implies \varphi_s^*(x_1) > x_2 \\
\implies (x_1, x_2) = x \in |\{|f| > t\}|^*.
\]

Conversely, if \( x \in |\{|f| > t\}|^* \), then \( \varphi_s^*(x_1) > x_2, x = (x_1, x_2) \), and hence 
\[
|\{s : \varphi_s^*(x_1) > x_2\}| \geq t,
\] which implies \( f_2^*(x) \geq t \). \( \square \)

**Lemma 2.6** Let \( f \) and \( g \) be two measurable functions on \( \mathbb{R}^2 \) and \( t > 0 \). Then

\[
\chi_{\{|f+g| > t\}}(x+y) \leq \chi_{\{|f| > t/2\}}(x) + \chi_{\{|g| > t/2\}}(y),
\]

\( x = (x_1, x_2), \ y = (y_1, y_2) \in \mathbb{R}^2 \).

**Proof.** Let

\[
\varphi_{f,t}(a) := |\{b \in \mathbb{R} : |f(a,b)| > t\}| \\
\varphi_{g,t}(a) := |\{b \in \mathbb{R} : |g(a,b)| > t\}| \\
\varphi_{f+g,t}(a) := |\{b \in \mathbb{R} : |f+g(a,b)| > t\}|.
\]

We know that

\[
\varphi_{f+g,t}(a) \leq \varphi_{f,t/2}(a) + \varphi_{g,t/2}(a).
\]

Also, if \( x \notin |\{|f| > t/2\}|^* \), then \( \varphi_{f,t/2}^*(x_1) < x_2 \) and similarly, if \( y \notin |\{|g| > t/2\}|^* \), then \( \varphi_{g,t/2}^*(y_1) < y_2 \). Therefore

\[
\varphi_{f+g,t}^*(x_1+y_1) \leq (\varphi_{f,t/2} + \varphi_{g,t/2})^*(x_1+y_1) \\
\leq \varphi_{f,t/2}^*(x_1) + \varphi_{g,t/2}^*(y_1) \\
< x_2 + y_2,
\]

which means exactly that \( x + y \notin |\{|f+g| > t\}|^* \). This completes the proof of the lemma. \( \square \)

**Proposition 2.7** Suppose \( f, g, \) and \( f_n, \ (n = 1, 2, \ldots) \) are measurable functions on \( \mathbb{R}^2 \) and let \( c \in \mathbb{C} \). Then the two-dimensional decreasing rearrangement \( f_2^* \) is a nonnegative function on \( \mathbb{R}_+^2 \), decreasing in each variable. Furthermore,

a) \( |g| \leq |f| \) a.e. \( \implies g_2^* \leq f_2^* \);

b) \( (cf)_2^* = |c|f_2^* \).
c) if \( f \) is decreasing in each variable, then \( f^*_2 = f \);

d) \((f + g)^*_2(x + y) \leq 2(f^*_2(x) + g^*_2(y))\);

e) \( |f| \leq \liminf_{n \to \infty} |f_n| \implies f^*_2 \leq \liminf_{n \to \infty} (f_n)^*_2\), and, in particular, if \( |f_n| \uparrow |f| \implies (f_n)^*_2 \uparrow f^*_2\);

f) \((f^*_2(x))^p = (f^p(x))^*_2, (0 < p < \infty)\);

g) if \( f \) is a symmetric function (i.e. \( f(x_1, x_2) = f(x_2, x_1) \)), then \( f^*_2 \) is symmetric.

**Proof.** That \( f^*_2 \) is nonnegative and decreasing follows from Definition 2.3 and the fact that the characteristic function of a decreasing set is a decreasing function.

a) By Definition 2.2, it follows that

\[
\{|g| > t\} \subset \{|f| > t\} \implies \{|g| > t\}^* \subset \{|f| > t\}^*.
\]

Thus \( \chi_{\{|g| > t\}^*} \leq \chi_{\{|f| > t\}^*} \) and \( g^*_2 \leq f^*_2 \).

b) Trivial.

c) If \( f \) is a decreasing function in each variable, then the level set \( \{|f| > t\} \) is a decreasing set (see also [BPSQ]) and c.f. Proposition 2.4

\[
\{|f| > t\}^* = \{|f| > t\}.
\]

We get the desired equality by using Definition 2.3.

d) By Lemma 2.6 and b) of this proposition we have

\[
(f + g)^*_2(x + y) = \int_0^\infty \chi_{\{|f+g| > t\}^*}(x + y)dt
\]

\[
\leq \int_0^\infty \chi_{\{|f| > t/2\}^*}(x)dt + \int_0^\infty \chi_{\{|g| > t/2\}^*}(y)dt
\]

\[
= 2(f^*_2(x) + g^*_2(y)).
\]

e) Let

\[
E^t := \{(x, y) : |f(x, y)| > t\}
\]

and

\[
E^t_n := \{(x, y) : |f_n(x, y)| > t\}.
\]

Set \( f_n^t(y) := f(x, y) \) and

\[
\varphi_{f,t}(x) = |\{y : |f(x, y)| > t\}| = \chi_{f_n}(t),
\]
where $\lambda_{f_x}$ is the usual distribution function (see [BS]). Then

$$|f| \leq \liminf_{n \to \infty} |f_n| \quad \Rightarrow \quad |f_x| \leq \liminf_{n \to \infty} |f_{x,n}| \text{ a.e.}$$

$$\Rightarrow \quad \lambda_{f_x} \leq \liminf_{n \to \infty} \lambda_{f_{x,n}}$$

$$\Rightarrow \quad \varphi_{f,t} \leq \liminf_{n \to \infty} \varphi_{f_{n,t}}, \text{ a.e., } \forall t > 0$$

$$\Rightarrow \quad \varphi_{f,t}^* \leq \liminf_{n \to \infty} \varphi_{f_{n,t}}^*, \text{ a.e., } \forall t > 0$$

$$\Rightarrow \quad \chi_{(E^*)^*} \leq \liminf_{n \to \infty} \chi_{(E^*_n)^*}$$

$$\Rightarrow \quad f_2^* \leq \liminf_{n \to \infty} (f_n)^*_2.$$  

The second part is an immediate consequence of the first.

f) We have

$$\left(f_2^*(x)\right)^p = \left(\int_0^\infty \chi_{\{|f| > t\}^*}(x)dt\right)^p$$

$$= \int_0^\infty \chi_{\{|f| > t\}^*}(x)dt$$

$$= p \int_0^\infty \chi_{\{|f| > t\}^*}(x)t^{p-1}dt.$$  

In view of Lemma 2.5 we have

$$\chi_{\{|f| > t\}^*} \geq \chi_{\{f_2^* \geq t\}}$$

and, hence,

$$\left(f_2^*(x)\right)^p \geq p \int_0^\infty t^{p-1}\chi_{\{f_2^* \geq t\}}(x)dt$$

$$= p \int_0^{f_2^*(x)} t^{p-1}dt = \left(f_2^*(x)\right)^p.$$  

On the other hand, if we take $0 < r < 1$, then by Lemma 2.5 we have

$$\chi_{\{|f| > rt\}^*} \leq \chi_{\{f_2^* \geq rt\}},$$

and, hence,

$$\left(f_2^*(x)ight)^p \leq p \int_0^\infty t^{p-1}\chi_{\{f_2^* \geq rt\}}(x)dt$$

$$= p \int_0^{f_2^*(x)/r} t^{p-1}dt = \left(\frac{f_2^*(x)}{r}\right)^p.$$  

Since this is true for all $0 < r < 1$, we get

$$f_2^*(x) \geq \left(f_2^*(x)\right)^p \geq f_2^*(x).$$
g) This is just an observation which follows immediately by using the definition of $f_2^*$.

The following proposition will be very useful for proving our main results, since it will allow us to consider the special and easier case of simple functions.

**Proposition 2.8** If $f$ is a measurable function on $\mathbb{R}^2$, then there exists a sequence $(s_n)_n$ of simple measurable functions such that:

a) $0 \leq (s_1)_2^* \leq \ldots \leq (s_n)_2^* \leq f_2^*$,

b) $(s_n)_2^* \rightarrow f_2^*$ as $n \rightarrow \infty$ a.e.

**Proof.** The existence of the sequence is standard, and the rest is just a consequence of Proposition 2.7 a) and e), and the following remark: If $s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}$, with $a_1 > a_2 > \ldots > a_n > 0$ and $E_j \cap E_i = \emptyset$, $i \neq j$, then

$$s_2^*(x) = \sum_{j=1}^{n} a_j \chi_{F_j^* \setminus F_{j-1}^*}(x),$$

where $F_j = \bigcup_{k=1}^{j} E_k$, and $F_0 = \emptyset$. Observe that from Proposition 2.4 we have that $|F_j^* \setminus F_{j-1}^*| = |E_j|$. □

As a corollary, we can obtain several properties relating our two-dimensional rearrangement and the classical one. In particular, we see that the new rearrangement is finer and gives more information than the other.

**Corollary 2.9** Let $f$ and $g$ be two measurable functions in $\mathbb{R}^2$.

a) If $f_2^* = g_2^*$, then $f^* = g^*$, and the converse is not true in general.

b) $(f_2^*)^* = f^*$.

**Proof.** To prove a) we observe that if $f_2^* = g_2^*$, then

$$\int_{0}^{\infty} \chi_{\{f > t\}}(x) \, dt = \int_{0}^{\infty} \chi_{\{g > t\}}(x) \, dt,$$

and hence $\{f > t\}^* = \{g > t\}^*$. Using now Proposition 2.4 a), we get that $|\{f > t\}| = |\{g > t\}|$ which shows that $f^* = g^*$. To see that the converse does not hold, consider the decreasing sets $A = (0,1) \times (0,2)$, $B = (0,2) \times (0,1)$ and the functions $f = \chi_A$ and $g = \chi_B$. Then, $f^* = g^* = \chi_{(0,2)}$ but $f_2^* = f \neq g = g_2^*$.

The proof of b) follows immediately by checking what happens for simple functions and using Proposition 2.8. We observe that from b) we can also give an alternative proof of a) □
We consider next integral inequalities, for the two-dimensional rearrangement, related to the Hardy–Littlewood inequality (see [BS]). Again we observe that what we obtain is a better estimate. We begin with an elementary but useful lemma.

**Lemma 2.10** Let $g$ be a nonnegative simple function on $\mathbb{R}^2$ and let $E$ be an arbitrary set of $\mathbb{R}^2$. Then

$$\int_E g(x) \, dx \leq \int_{E^*} g_2^*(x) \, dx.$$ 

**Proof.** Let

$$g(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $a_1 > a_2 > \cdots > a_n > 0$, $a_{n+1} = 0$, and $E_j \subset \mathbb{R}^2$ are of finite measure such that $E_j \cap E_i = \emptyset$, $i \neq j$. Another representation of $g$ is

$$g(x) = \sum_{j=1}^{n} b_j \chi_{F_j}(x),$$

where $b_j > 0$, $b_j = a_j - a_{j+1}$, and $F_j = \bigcup_{i=1}^{j} E_i$. Then,

$$g_2^*(x) = \int_{0}^{\infty} \chi_{\{g > t\}^*}(x) dt = \int_{a_1}^{a_2} \chi_{E_1^*}(x) dt + \int_{a_2}^{a_3} \chi_{(E_1 \cup E_2)^*}(x) dt + \cdots + \int_{a_n}^{a_{n+1}} \chi_{(E_1 \cup \cdots \cup E_n)^*}(x) dt = \chi_{E_1^*}(x)(a_1 - a_2) + \chi_{(E_1 \cup E_2)^*}(x)(a_2 - a_3) + \cdots + \chi_{(E_1 \cup \cdots \cup E_n)^*}(x)a_n = \sum_{j=1}^{n} b_j \chi_{F_j^*}(x).$$

Thus, since $(F_j \cap E)^* \subset F_j^* \cap E^*$, we have that

$$\int_E g(x) \, dx = \sum_{j=1}^{n} b_j \int_{E} \chi_{F_j}(x) \, dx = \sum_{j=1}^{n} b_j |F_j \cap E| = \sum_{j=1}^{n} b_j |(F_j \cap E)^*| = \sum_{j=1}^{n} b_j \int_{(F_j \cap E)^*} \, dx \leq \sum_{j=1}^{n} b_j \int_{F_j^* \cap E^*} \, dx = \int_{E^*} g_2^*(x) \, dx. \quad \square$$
Theorem 2.11 If $f$ and $g$ are measurable functions on $\mathbb{R}^2$, then
\[
\int_{\mathbb{R}^2} |f(x)g(x)| \, dx \leq \int_{\mathbb{R}^2} f^*_2(x)g^*_2(x) \, dx \leq \int_0^\infty f^*(t)g^*(t) \, dt.
\]

Proof. It is enough to prove the statement for $f$ and $g$ nonnegative. By Proposition 2.7 e) and in view of the monotone convergence theorem there is no loss of generality in assuming $f$ and $g$ to be simple functions. Let
\[
f(x) = \sum_{j=1}^n a_j \chi_{E_j}(x),
\]
where $E_1 \subset E_2 \subset \ldots \subset E_n \subset \mathbb{R}^2$, are of finite measure, and $a_j > 0$. Then, by Lemma 2.10, we have that
\[
\int_{\mathbb{R}^2} f(x)g(x) \, dx = \sum_{j=1}^n a_j \int_{E_j} g(x) \, dx \leq \sum_{j=1}^n a_j \int_{E_j^*} g^*_2(x) \, dx
\]
\[
= \int_{\mathbb{R}^2} \sum_{j=1}^n a_j \chi_{E_j^*}(x)g^*_2(x) \, dx = \int_{\mathbb{R}^2} f^*_2(x)g^*_2(x) \, dx.
\]
The second inequality follows from Corollary 2.9 b). □

Corollary 2.12 If $f$ is a nonnegative measurable function on $\mathbb{R}^2$, and $D$ is a decreasing set, then
\[
\sup_{E^* = D} \int_E f(x) \, dx \leq \int_D f^*_2(x) \, dx \leq \int_0^{|D|} f^*(t) \, dt,
\]
and both inequalities can hold strictly for some $f$ and $D$.

Proof. That the inequalities hold is a consequence of Theorem 2.11 applied with $g = \chi_E$. To show that the first inequality can be strict, consider the sets $A = (3, 4) \times (0, 1)$, $B = (4, 6) \times (0, 2)$, $D = (0, 1) \times (0, 2)$, and the function $f(x) = 2\chi_A(x) + \chi_B(x)$. Then, it is easy to see that for every set $E$ such that $E_2^* = D$, we have
\[
\int_E f(x) \, dx \leq 2 < 3 = \int_D f^*_2(x) \, dx.
\]
For the second inequality, consider $D_\varepsilon = (0, \varepsilon) \times (0, 1/\varepsilon)$ and $f$ as before. Then,
\[
\int_0^{|D_\varepsilon|} f^*(t) \, dt = 2, \quad \text{for every } \varepsilon > 0,
\]
but
\[
\lim_{\varepsilon \to 0} \int_{D_{\varepsilon}} f_2^*(x) \, dx = 0. \quad \square
\]

As we have mentioned in the introduction, our definition of the two-dimensional rearrangement is based on a geometric approach: we first look at the rearrangement of the level sets of the function, and then we recover the rearrangement of the function by summing up all these level sets (Layer-cake formula). In the next theorem, we are going to prove a direct way of calculating the two-dimensional rearrangement as an iterative procedure with respect to the usual rearrangement in each variable (see [Bl] for some related work).

In order to clarify the notation used in the proof, given a function \( f(x, y) \) defined on \( \mathbb{R}^2 \), we write \( R_t(x) = (f_x)^*(y) \), where \( f_x(y) = f(x, y) \) and \( t > 0 \) (i.e., \( R_t \) is the usual rearrangement of the function \( f_x \), with respect to the variable \( y \)). Similarly, we set \( \tilde{f}(s, t) = (R_t)^*(s) \), \( s, t > 0 \). It is very easy to show that, in general, we do not get the same function if we first rearrange with respect to \( x \) and then with respect to \( y \).

**Theorem 2.13** If \( f \) is a measurable function on \( \mathbb{R}^2 \), then
\[
f_2^*(s, t) = \tilde{f}(s, t), \quad \forall \ s, t > 0.
\]

**Proof.** Using Proposition 2.8, it suffices to consider \( f \) to be a simple function. Hence, let \( f(x, y) = \sum_{j=1}^n a_j \chi_{E_j}(x, y) \), with \( a_1 > a_2 > \cdots > a_n \), \( E_j \cap E_k = \emptyset, j \neq k \). Set \( F_k = \bigcup_{j=1}^k E_j, \ F_0 = \emptyset \), so that
\[
f_2^*(s, t) = \sum_{j=1}^n a_j \chi_{F_j \setminus F_{j-1}}(s, t).
\]

Recall that \( \varphi_E(x) = |\{y : (x, y) \in E\}| \) and \( E^* = \{(s, t) : 0 < t < \varphi_E^*(s)\} \).

Hence,
\[
\chi_{E^*}(s, t) = \chi_{(0,\varphi_E^*(s))}(t) = \chi_{(0,\lambda_{\varphi_E}(t))}(s).
\]

Thus,
\[
\chi_{F_j \setminus F_{j-1}}(s, t) = \chi_{F_j^*}(s, t) - \chi_{F_{j-1}^*}(s, t) = \chi_{[\lambda_{\varphi_{F_{j-1}}(t)}, \lambda_{\varphi_F}(t))}(s),
\]

which gives
\[
f_2^*(s, t) = \sum_{j=1}^n a_j \chi_{[\lambda_{\varphi_{F_{j-1}}(t)}, \lambda_{\varphi_F}(t))}(s). \quad (2)
\]
On the other hand, since
\[ f_x(y) = \sum_{j=1}^{n} a_j \chi_{E_j(x)}(y), \]
where \( E(x) = \{ y : (x, y) \in E \} \), we have that
\[
R_t(x) = (f_x)^*/y(t) = \sum_{j=1}^{n} a_j \chi_{[F_{j-1}(x), F_j(x)]}(t) \\
= \sum_{j=1}^{n} a_j \chi_{[\varphi_{F_{j-1}}(x), \varphi_{F_j}(x)]}(t) = \sum_{j=1}^{n} a_j \chi_{H_j(t)}(x),
\]
where \( H_j(t) = \{ y : \varphi_{F_{j-1}}(y) \leq t < \varphi_{F_j}(y) \} \). Therefore,
\[
\tilde{f}(s, t) = (R_t)^*/x(s) = \sum_{j=1}^{n} a_j \chi_{[G_{j-1}(t), G_j(t)]}(s), \tag{3}
\]
where \( G_j(t) = \bigcup_{k=1}^{j} H_k(t), \ G_0(t) = \emptyset \). Thus looking at (4) and (3) it suffices to proving that
\[ |G_j(t)| = \lambda_{\varphi_{F_j}}(t). \]
But, in fact
\[
|G_j(t)| = \sum_{k=1}^{j} |H_k(t)| = \sum_{k=1}^{j} |\{ y : \varphi_{F_{k-1}}(y) \leq t < \varphi_{F_k}(y) \}| \\
= |\{ y : t < \varphi_{F_j}(y) \}| = \lambda_{\varphi_{F_j}}(t),
\]
and the proof is complete. \( \square \)

**Corollary 2.14** If \( g \) and \( h \) are two measurable functions on \( \mathbb{R} \), and \( f(x, y) = g(x)h(y) \), then \( f^*_2(s, t) = g^*(s)h^*(t) \).

Another application of Theorem 2.13 is that the inequality proved in Theorem 2.7 (d) can be improved to obtain the classical subadditivity condition: \( (f + g)_2^*(x + y) \leq f^*_2(x) + g^*_2(y) \) (we leave the details to the interested reader).
3 A new multidimensional Lorentz space

In this section we prove some properties of a new type of space, defined using the two-dimensional decreasing rearrangement. Recall the definition of the classical Lorentz space: If \( v \) is a weight in \( \mathbb{R}^+ \) and \( 0 < p < \infty \),

\[
\Lambda^p(v) = \left\{ f : \mathbb{R}^n \to \mathbb{C} : \| f \|_{\Lambda^p(v)} := \left( \int_0^\infty (f^*(t))^p v(t) \, dt \right)^{1/p} < \infty \right\}.
\]

We now say that a measurable function \( f \) on \( \mathbb{R}^2 \) belongs to the (multidimensional) Lorentz space \( \Lambda^p_2(w) \), provided \( \| f \|_{\Lambda^p_2(w)} \), defined by

\[
\| f \|_{\Lambda^p_2(w)} := \left( \int_{\mathbb{R}^2_+} (f^*_2(x))^p w(x) \, dx \right)^{1/p},
\]

is finite. Here \( w \) is a nonnegative, locally integrable function on \( \mathbb{R}^2_+ \), not identically 0.

The next result gives an alternative description of the \( L^p_{\mathbb{R}^2} \) norm in terms of the two-dimensional decreasing rearrangement, i.e., the spaces defined above generalize naturally the Lebesgue spaces.

**Theorem 3.1** If \( 0 < p < \infty \), then \( \Lambda^p_2(1) = L^p_{\mathbb{R}^2} \).

**Proof.** By Fubini’s theorem and Proposition 2.7 f) we have

\[
\int_{\mathbb{R}^2} |f(x)|^p \, dx = \int_0^\infty \int_{|f|^p > t} dx \, dt = \int_0^\infty \int_{|f|^p > t} \chi_{|f|^p > t} \, dx \, dt
\]

\[
= \int_{\mathbb{R}^2_+} \int_0^\infty \chi_{|f|^p > t} \, dx \, dt = \int_{\mathbb{R}^2_+} \left( f^p \right)^*_2(x) \, dx
\]

\[
= \int_{\mathbb{R}^2_+} (f^*_2(x))^p \, dx. \quad \square
\]

We are interested in studying functional properties of the spaces \( \Lambda^p_2(w) \) and their relationship with the classical rearrangement invariant spaces (see [BS]). The following results show that these two kinds of spaces only agree in very particular cases:

**Proposition 3.2** If \( \| \cdot \|_{\Lambda^p_2(w)} \) is a rearrangement invariant norm, then \( w \) is constant, and hence \( \Lambda^p_2(w) = L^p_{\mathbb{R}^2} \).

**Proof.** Fix \((x, y) \in \mathbb{R}^2_+, 0 < \varepsilon < \min(x, y)\), and define \( R = (0, x) \times (0, y) \), \( P_\varepsilon = (x-\varepsilon, x) \times (y-\varepsilon, y) \), \( Q_\varepsilon = (x, x+\varepsilon) \times (0, \varepsilon) \), and \( A_\varepsilon = (R \setminus P_\varepsilon) \cup Q_\varepsilon \). Then \( |R| = |A_\varepsilon| \), and hence \( \| \chi_R \|_{\Lambda^p_2(w)} = \| \chi_{A_\varepsilon} \|_{\Lambda^p_2(w)} \), which gives

\[
\int_{P_\varepsilon} w(x) \, dx = \int_{Q_\varepsilon} w(x) \, dx.
\]
Now, letting \( \varepsilon \to 0 \), using the Lebesgue differentiation theorem, and a symmetric argument changing \( x \) and \( y \), we obtain that \( w \) is constant. \( \square \)

In a similar way, one can prove the following:

**Proposition 3.3** There exists a weight \( v \) in \( \mathbb{R}^2 \) such that \( \Lambda_p^p(w) = L_p^{\mathbb{R}^2}(v) \) if and only if \( \Lambda_2^p(w) = L_p^{\mathbb{R}^2} \).

It is very easy to see that embedding results for the spaces \( \Lambda_2^p(w) \) are equivalent to embeddings for the cone of decreasing functions on \( L_p^{\mathbb{R}^2} \), which have been completely characterized in all cases (see [BPS] and [BPS]). The result reads as follows:

**Proposition 3.4** Let \( 0 < p_1, p_2 < \infty \) and \( w_1, w_2 \) be two weights in \( \mathbb{R}_+^2 \).

a) If \( p_1 \leq p_2 \), then \( \Lambda_2^{p_1}(w_1) \subset \Lambda_2^{p_2}(w_2) \), if and only if,

\[
\sup_{D \in \Delta_d} \frac{w_2(D)^{1/p_2}}{w_1(D)^{1/p_1}} < \infty.
\]

b) If \( p_1 > p_2 \), then \( \Lambda_2^{p_1}(w_1) \subset \Lambda_2^{p_2}(w_2) \), if and only if,

\[
\sup_{0 \leq h \downarrow} \int_0^\infty w_1(D_{h,t})^{-r/p_1} d(-w_2(D_{h,t})^{r/p_2}) < \infty,
\]

where \( D_{h,t} = \{ x \in \mathbb{R}_+^2 : h(x) > t \} \), and \( 1/r = 1/p_2 - 1/p_1 \).

The characterization of the quasinormability, in the case of the classical Lorentz spaces, was proved in [US] to be equivalent to a doubling condition on the weight (the \( \Delta_2 \)-condition). We show that a similar result holds for the two-dimensional rearrangement.

First we note that the spaces \( \Lambda_2^p(w) \), \( 0 < p < \infty \), have the following (quasi)norm properties:

\[
\| cf \|_{\Lambda_2^p(w)} = |c| \| f \|_{\Lambda_2^p(w)},
\]

(See Proposition 2.7 b)), and if \( w \) is strictly positive (which we assume in the sequel)

\[
\| f \|_{\Lambda_2^p(w)} = 0 \iff f = 0 \text{ a.e.}
\]

Thus, in order to investigate if \( \| \cdot \|_{\Lambda_2^p(w)} \) is a norm (quasi-norm) we only have to check that the triangle (quasi-triangle) inequality holds.
Theorem 3.5 Let $0 < p < \infty$. Then, $\| \cdot \|_{\Lambda_p^2(w)}$ is a quasinorm if and only if there exists a constant $C > 0$ such that

$$\int_D w(2x) dx \leq C \int_D w(x) dx,$$

for all decreasing sets $D \subset \mathbb{R}_+^2$. Moreover, with this quasinorm, $\Lambda_p^2(w)$ becomes a complete quasinormed space.

Proof. For sufficiency we use Proposition 2.7 d), Theorem 2.2 d) in [BPSo], with $p = q$, and we get:

$$\| f + g \|_{\Lambda_p^2(w)}^p = \int_{\mathbb{R}_+^2} ((f + g)^* w(x))^p w(x) dx$$

$$\leq C \int_{\mathbb{R}_+^2} (f_2^* (x/2) + g_2^* (x/2))^p w(x) dx$$

$$\leq C \left( \int_{\mathbb{R}_+^2} (f_2^* (x))^p w(x) dx + \int_{\mathbb{R}_+^2} (g_2^* (x))^p w(x) dx \right)$$

$$\leq C \left( \int_{\mathbb{R}_+^2} (f_2^* (x))^p w(2x) dx + \int_{\mathbb{R}_+^2} (g_2^* (x))^p w(2x) dx \right)$$

$$\leq C \left( \int_{\mathbb{R}_+^2} (f_2^* (x))^p w(x) dx + \int_{\mathbb{R}_+^2} (g_2^* (x))^p w(x) dx \right)$$

$$= C \| f \|_{\Lambda_p^2(w)}^p + \| g \|_{\Lambda_p^2(w)}^p,$$

and it follows that $\| f + g \|_{\Lambda_p^2(w)} \leq C (\| f \|_{\Lambda_p^2(w)} + \| g \|_{\Lambda_p^2(w)})$.

Conversely, let $D$ and $D_1$ be two sets of $\mathbb{R}^2$ with $D \cap D_1 = \emptyset$ and $D^* = D_1^*$, and such that if $D^*$ has the representation

$$D^* = \{ (x_1, x_2) : 0 < x_1 < r, 0 < x_2 < \phi(x_1); r > 0 \},$$

(with $\phi \downarrow$), then

$$(D \cup D_1)^* = \{ (x_1, x_2) : 0 < x_1 < 2r, 0 < x_2 < \phi(x_1/2); r > 0 \},$$

(this is easily done by taking $D_1$ to be a translation of the form $D_1 = D + (N, 0)$, where $N > 0$ is big enough). If $\| \cdot \|_{\Lambda_p^2(w)}$ is a quasinorm, then

$$\| f + g \|_{\Lambda_p^2(w)}^p \leq C (\| f \|_{\Lambda_p^2(w)}^p + \| g \|_{\Lambda_p^2(w)}^p),$$

and if we take $f = \chi_D$ and $g = \chi_{D_1}$, then we get
\[ \int_{(D \cup D_1)^*} w(x) \, dx \leq C \int_{D^*} w(x) \, dx. \tag{8} \]

We denote by \( E := (D \cup D_1)^* \), and by
\[ E_1 := \{ (x_1, x_2) : 0 < x_1 < 2r, \phi(2x_1) < x_2 < 2\phi(x_1/2); \, r > 0 \}. \]

Obviously, \( E_1 \cup E = 2D^* \). Since \( E_1^* = E = E^* \) we can apply (8) with \( D = E, D_1 = E_1 \) and get
\[ \int_{2D^*} w(x) \, dx = \int_{(E \cup E_1)^*} w(x) \, dx \leq C \int_{E^*} w(x) \, dx \]
\[ = C \int_{(D \cup D_1)^*} w(x) \, dx \leq C \int_{D^*} w(x) \, dx, \]

which is obviously equivalent to condition (7). Thus, in view of (5) and (6), the first statement is proved.

To prove that \( \Lambda^p_w(w) \) is complete we have to show that if \( (f_k)_k \subset \Lambda^p_w(w) \) is a Cauchy sequence, then there exists a function \( f \in \Lambda^p_w(w) \) such that
\[ \| f_j - f \|_{\Lambda^p_w(w)} \to 0 \text{ as } j \to \infty. \]

Since \( \| \cdot \|_{\Lambda^p_w(w)} \) is quasinorm and \( (f_k)_k \) is Cauchy, there exists a constant \( C > 0 \) such that
\[ \| f_j \|_{\Lambda^p_w(w)} \leq C < \infty, \quad \forall j \in \mathbb{N}. \]

Also since \( (f_j - f_k)_k^* \) is decreasing in each variable, for a fixed \( x \in \mathbb{R}^2 \), if we set \( Q_x = \{ y \in \mathbb{R}^2_+ : 0 < y_k \leq x_k, \ k = 1, 2 \} \), then
\[ (f_j - f_k)_2^p(x) \int_{Q_x} w(y) \, dy \leq \int_{\mathbb{R}^2_+} (f_j - f_k)_2^p(y) w(y) \, dy. \]

Therefore
\[ (f_j - f_k)_2^* \to 0, \text{ a.e.} \]

This implies
\[ \lambda_{(f_j - f_k)_2^*} \to 0, \text{ a.e.} \]

and hence
\[ \lambda_{(f_j - f_k)} \to 0, \text{ a.e.,} \]
i.e., \( (f_k)_k \) is Cauchy in measure. Hence there is a subsequence \( (f_{k_j})_j \) which converges pointwise, say to a function \( f \) which is measurable. By Proposition 2.7 e) and by Fatou’s lemma we have that \( f \in \Lambda^p_w(w) \). Moreover,
\[ \lim_{j \to \infty} | f_{k_j}(x) - f_i(x) | = | f(x) - f_i(x) |, \ x \in \mathbb{R}^2. \]
Using Fatou’s lemma again and the fact that \((f_k)_k\) is a Cauchy sequence, we finally get
\[\|f - f_i\|_{L^p_w} \leq C \left( \|f - f_{k_j}\|_{L^p_w} + \|f_i - f_{k_j}\|_{L^p_w} \right) \to 0, \text{ as } i, j \to \infty. \]

Finally, we are now going to prove the main result of this section, namely, the characterization of the weights \(w\) for which \(\| \cdot \|_{L^p_w}\) is a norm. We begin by showing the following necessary condition on the index \(p\):

**Theorem 3.6** Let \(0 < p < \infty\). If \(\Lambda^p_w\) is a Banach space, then \(p \geq 1\).

**Proof.** Since \(\Lambda^p_w\) is a Banach space, there exists \(\| \cdot \|\), a norm on \(\Lambda^p_w\), such that
\[\| f \|_{\Lambda^p_w} \approx \| f \|.

Hence
\[\left\| \sum_{k=1}^N f_k \right\|_{\Lambda^p_w} \leq C \sum_{k=1}^N \| f_k \| \leq C \sum_{k=1}^N \| f_k \|_{\Lambda^p_w},
\]
for all \(N \in \mathbb{N}\). Suppose \(0 < p < 1\) and take a decreasing sequence of domains
\[A_{k+1} \subset A_k \subset \ldots \subset \mathbb{R}^2,
\]
such that \(\int_{A_k^*} w(x) \, dx = 2^{-kp}\). If \(f_k = 2^k \chi_{A_k}\), then \(\| f_k \|_{\Lambda^p_w} = 1\).

But for a fixed \(N\), we have that
\[\frac{1}{N} \left\| \sum_{k=1}^N f_k \right\|_{\Lambda^p_w} \leq \tilde{C} < \infty.
\]

On the other hand, since \(\left( \sum_{k=1}^N 2^k \chi_{A_k} \right)^*_2 = \sum_{k=1}^N 2^k \chi_{A_k}^*\) (by (12)), and \(A_{k+1}^* \subset \ldots \subset \mathbb{R}^2\), we have (taking \(A_{N+1} = \emptyset\))
\[\frac{1}{N} \left\| \sum_{k=1}^N f_k \right\|_{\Lambda^p_w} = \frac{1}{N} \left\| \sum_{k=1}^N 2^k \chi_{A_k} \right\|_{\Lambda^p_w} = \frac{1}{N} \left( \int_{\mathbb{R}^2} \left( \sum_{k=1}^N 2^k \chi_{A_k^*} \right) \, (x) w(x) \, dx \right)^{1/p}
\]
\[= \frac{1}{N} \left( \int_{\mathbb{R}^2} \left( \sum_{k=1}^N \sum_{j=1}^{2^k} \chi_{A_k^* \setminus A_{k+1}^*} \right) \, (x) w(x) \, dx \right)^{1/p}
\]
\[= \frac{1}{N} \left( \int_{\mathbb{R}^2} \left( \sum_{k=1}^N \sum_{j=1}^{2^k} \chi_{A_k^* \setminus A_{k+1}^*} \right) \, (x) w(x) \, dx \right)^{1/p}
\]
\[= \frac{1}{N} \left( \int_{\mathbb{R}^2} \left( \sum_{k=1}^N \sum_{j=1}^{2^k} \chi_{A_k^* \setminus A_{k+1}^*} \right) \, (x) w(x) \, dx \right)^{1/p}
\]
\[= \frac{1}{N} \left( \int_{\mathbb{R}^2} \left( \sum_{k=1}^N \sum_{j=1}^{2^k} \chi_{A_k^* \setminus A_{k+1}^*} \right) \, (x) w(x) \, dx \right)^{1/p}
\]
\[= \frac{1}{N} \left( \int_{\mathbb{R}^2} \left( \sum_{k=1}^N \sum_{j=1}^{2^k} \chi_{A_k^* \setminus A_{k+1}^*} \right) \, (x) w(x) \, dx \right)^{1/p}
\]
\[
\frac{1}{N} \left( \sum_{k=1}^{N} \left( \sum_{j=1}^{k} 2^j \right)^p \left( \int_{A_k^*} w(x) \, dx - \int_{A_{k+1}^*} (x)w(x) \, dx \right) \right)^{1/p} \\
\geq \frac{C}{N} \left( \sum_{k=1}^{N} \left( 1 - 2^{-k} \right)^p \right)^{1/p} \\
\geq \frac{C}{N} \left( \sum_{k=1}^{N} 2^{-p} \right)^{1/p} = C \frac{N^{1/p}}{N} \to \infty, \text{ as } N \to \infty,
\]

which is a contradiction. Hence \( p \geq 1. \)  \( \square \)

**Theorem 3.7** Let \( 1 \leq p < \infty \) and \( w \) be a weight in \( \mathbb{R}^2_+ \). Then, the following conditions are equivalent:

a) \( \| \cdot \|_{\Lambda^p_2(w)} \) is a norm.

b) For every \( A, B \subset \mathbb{R}^2 \), \( w((A \cap B)^*) + w((A \cup B)^*) \leq w(A^*) + w(B^*) \).

c) There exists a decreasing weight \( v \) on \( \mathbb{R}^+ \) such that \( w(s, t) = v(t), s, t > 0 \).

**Proof.** If \( \| \cdot \|_{\Lambda^p_2(w)} \) is a norm, take \( A, B \subset \mathbb{R}^2 \), \( \delta > 0 \) and define the functions

\[
\begin{align*}
  f(x) & = \begin{cases} 
  1 + \delta, & \text{if } x \in A \\
  1, & \text{if } x \in (A \cup B) \setminus A \\
  0, & \text{otherwise},
\end{cases} \\
  g(x) & = \begin{cases} 
  1 + \delta, & \text{if } x \in B \\
  1, & \text{if } x \in (A \cup B) \setminus B \\
  0, & \text{otherwise}.
\end{cases}
\end{align*}
\]

Then,

\[
\begin{align*}
  f_2^*(x) & = (1 + \delta)\chi_{A^*}(x) + \chi_{(A \cup B)^* \setminus A^*}(x), \\
  g_2^*(x) & = (1 + \delta)\chi_{B^*}(x) + \chi_{(A \cup B)^* \setminus B^*}(x), \\
  (f + g)_2^*(x) & = (2 + 2\delta)\chi_{(A \cap B)^*}(x) + (2 + \delta)\chi_{(A \cup B)^* \setminus (A \cap B)^*}(x),
\end{align*}
\]

and, hence, the triangle inequality and the fact that \( 1/p \leq 1 \) imply

\[
\|f + g\|_{\Lambda^p_2(w)} = \left( (2 + 2\delta)^p w((A \cap B)^*) + (2 + \delta)^p w((A \cup B)^* \setminus (A \cap B)^*) \right)^{1/p} \\
\leq \|f\|_{\Lambda^p_2(w)} + \|g\|_{\Lambda^p_2(w)} \\
= \left( (1 + \delta)^p w(A^*) + w((A \cup B)^* \setminus A^*) \right)^{1/p}
\]
Collecting terms, dividing both sides by \(2^{p-1}(1 + \delta)^p - 1\) and letting \(\delta \to 0\), we finally obtain

\[
w((A \cap B)^*) + w((A \cup B)^*) \leq w(A^*) + w(B^*),
\]

which is b). Thus a) implies b).

Assume now that b) holds. Fix \(s, t > 0\), and consider, for \(\varepsilon > 0\) small, the sets

\[
A = (0, \varepsilon) \times (0, t) \cup (\varepsilon, s) \times (0, t - \varepsilon),
B = (0, \varepsilon) \times (0, t - \varepsilon) \cup (\varepsilon, s) \times (0, t).
\]

Then,

\[
A^* = A, \\
B^* = (0, s - \varepsilon) \times (0, t) \cup (s - \varepsilon, s) \times (0, t - \varepsilon), \\
(A \cap B)^* = (0, s) \times (0, t - \varepsilon), \\
(A \cup B)^* = (0, s) \times (0, t).
\]

Hence using b) we obtain that

\[
w((s - \varepsilon, s) \times (t - \varepsilon, t)) = w((A \cup B)^*) - w(B^*) \leq w(A^*) - w((A \cap B)^*) \\
= w((0, \varepsilon) \times (t - \varepsilon, t)).
\]

Thus, dividing both sides by \(\varepsilon^2\) and letting \(\varepsilon \to 0\) we obtain that \(w(s, t) \leq w(0, t)\).

Similarly, taking now

\[
A = (0, s) \times (0, t), \\
B = (0, \varepsilon) \times (\varepsilon, t + \varepsilon) \cup (\varepsilon, s - \varepsilon) \times (0, t) \cup (s - \varepsilon, s) \times (0, t - \varepsilon),
\]

we obtain that

\[
A^* = A, \\
B^* = (0, s - \varepsilon) \times (0, t) \cup (s - \varepsilon, s) \times (0, t - \varepsilon), \\
(A \cap B)^* = (0, s - 2\varepsilon) \times (0, t) \cup (s - 2\varepsilon, s) \times (0, t - \varepsilon), \\
(A \cup B)^* = (0, \varepsilon) \times (0, t + \varepsilon) \cup (\varepsilon, s) \times (0, t).
\]
Therefore by using b) we obtain that
\[
  w((0, \varepsilon) \times (t, t + \varepsilon)) = w((A \cup B)^*) - w(A^*) \leq w(B^*) - w((A \cap B)^*)
\]
\[
  = w((s - 2\varepsilon, s - \varepsilon) \times (t - \varepsilon, t)).
\]

Hence, dividing both sides by $\varepsilon^2$ and letting $\varepsilon \to 0$, we obtain that $w(0, t) \leq w(s, t)$ and, thus,
\[
  w(s, t) = w(0, t) = v(t).
\]

To finish we will prove that $v(b) = w(0, b) \leq w(0, a) = v(a)$ if $0 < a \leq b$:
for $\varepsilon > 0$ small, take now
\[
  A = (0, \varepsilon) \times (0, a),
\]
\[
  B = (0, \varepsilon) \times (0, b).
\]

Then,
\[
  A^* = A,
\]
\[
  B^* = (0, \varepsilon) \times (0, b - \varepsilon),
\]
\[
  (A \cap B)^* = (0, \varepsilon) \times (0, a - \varepsilon),
\]
\[
  (A \cup B)^* = (0, \varepsilon) \times (0, b).
\]

Hence using b) we obtain that
\[
  w((0, \varepsilon) \times (b - \varepsilon, b)) = w((A \cup B)^*) - w(B^*) \leq w(A^*) - w((A \cap B)^*)
\]
\[
  = w((0, \varepsilon) \times (a - \varepsilon, a)).
\]

Thus, dividing both sides by $\varepsilon^2$ and letting $\varepsilon \to 0$ we obtain that $w(0, b) \leq w(0, a)$.

Finally, we are now going to prove that c) implies a). By Theorem 2.13
we know that $f^*_x(s, t) = (f^*_x(t))^{*x}(s)$. Thus, using the fact that $\| \cdot \|_{A^p(v)}$ is a norm, if $v$ is decreasing (see [4]), and Minkowski’s inequality, we obtain
\[
  \|f + g\|_{A^p_v(w)} = \left( \int_{\mathbb{R}^+} (f + g)^*_v(s, t)^p w(s, t) \, ds dt \right)^{1/p}
\]
\[
  = \left( \int_0^\infty \left( \int_0^\infty \left[ (f_x + g_x)^*_v(t) \right]^p \, ds \right)^p \, v(t) \, dt \right)^{1/p}
\]
\[
  = \left( \int_0^\infty \left( \int_{\mathbb{R}} \left[ (f_x + g_x)^*_v(t) \right]^p \, dx \right)^p \, v(t) \, dt \right)^{1/p}
\]
\[
  = \left( \int_{\mathbb{R}} \left( \int_0^\infty \left[ (f_x + g_x)^*_v(t) \right]^p v(t) \, dt \right)^p \, dx \right)^{1/p}
\]

20
\[ \leq \left( \int_{\mathbb{R}} \left[ \left( \int_{0}^{\infty} \left[ (f_x)^{\ast y}(t) \right]^p v(t) \, dt \right) \right]_{\mathbb{R}}^{1/p} + \left( \int_{0}^{\infty} \left[ (g_x)^{\ast y}(t) \right]^p v(t) \, dt \right) \right]_{\mathbb{R}}^{1/p} \right) \]
\[ \leq \left( \int_{\mathbb{R}} \left( \int_{0}^{\infty} \left[ (f_x)^{\ast y}(t) \right]^p v(t) \, dt \right) \right)_{\mathbb{R}}^{1/p} + \left( \int_{\mathbb{R}} \left( \int_{0}^{\infty} \left[ (g_x)^{\ast y}(t) \right]^p v(t) \, dt \right) \right)_{\mathbb{R}}^{1/p} \]
\[ = \left( \int_{\mathbb{R}^{2+}} (f_2^*(s, t))^p w(s, t) \, ds \, dt \right)_{\mathbb{R}^{2+}}^{1/p} + \left( \int_{\mathbb{R}^{2+}} (g_2^*(s, t))^p w(s, t) \, ds \, dt \right)_{\mathbb{R}^{2+}}^{1/p} \]
\[ = \| f \|_{\Lambda^p_2(w)} + \| g \|_{\Lambda^p_2(w)}. \]

Thus, in view of (3) and (4), the proof is complete. \[\Box\]

**Remark 3.8** Observe that the equivalences proved in Theorem 3.7 in particular say that \( \Lambda^p_2(w) = L^p(\Lambda^p(v, dy), dx) \), which is a mixed norm space.

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