Injective Chromatic Number of Outerplanar Graphs

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Abstract. An injective coloring of a graph is a vertex coloring where two vertices with common neighbor receive distinct colors. The minimum integer $k$ such that $G$ has a $k$-injective coloring is called injective chromatic number of $G$ and denoted by $\chi_i(G)$. In this paper, the injective chromatic number of outerplanar graphs with maximum degree $\Delta$ and girth $g$ is studied. It is shown that every outerplanar graph $G$ has $\chi_i(G) \leq \Delta + 2$, and this bound is tight. Then, it is proved that for an outerplanar graph $G$ with $\Delta = 3$, $\chi_i(G) \leq \Delta + 1$ and the bound is tight for outerplanar graphs of girth 3 and 4. Finally, it is proved that, the injective chromatic number of 2-connected outerplanar graphs with $\Delta = 3$, $g \geq 6$ and $\Delta \geq 4$, $g \geq 4$ is equal to $\Delta$.

1. Introduction

All graphs we have considered here are finite, connected and simple. A plane graph is a planar drawing of a planar graph in the Euclidean plane. The vertex set, edge set, face set, minimum degree and maximum degree of a plane graph $G$, are denoted by $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree $k$ is called a $k$-vertex. For vertex $v \in V(G)$, $N_G(v)$ is the set of neighbors of $v$ in $G$. The girth of a graph $G$, $g(G)$, is the length of a shortest cycle in $G$. If there is no confusion, we delete $G$ in the notations. A face $f \in F(G)$ is denoted by its boundary walk $f = [v_1v_2...v_k]$, where $v_1, v_2, ..., v_k$ are its vertices in the clockwise order. Also, the vertices $v_1$ and $v_k$ as end vertices of $f$ are denoted by $v_{L_f}$ and $v_{R_f}$, respectively. An outerplanar graph is a graph with a planar drawing for which all vertices belong to the outer face of the drawing. It is known that a graph $G$ is an outerplanar graph if and only if $G$ has no subdivision of complete graph $K_4$ and complete bipartite graph $K_{2,3}$. A path $P : v_1, v_2, ..., v_k$ is called a simple path in $G$ if $v_2, ..., v_{k-1}$ are all 2-vertices in $G$. The length of a path is the number of its edges. We say that a face $f = [v_1v_2...v_k]$ is an end face of an outerplane graph $G$, if $P : v_1, v_2, ..., v_k$ is a simple path in $G$. An end block in graph $G$ is a maximal 2-connected subgraph of $G$ that contains a unique cut vertex of $G$.

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A proper $k$-coloring of a graph $G$ is a mapping from $V(G)$ to the set of colors \{1, 2, \ldots, k\} such that any two adjacent vertices have different colors. The chromatic number, $\chi(G)$, is the minimum integer $k$ that $G$ has a proper $k$-coloring. A coloring $c$ of $G$ is called an injective coloring if for every two vertices $u$ and $v$ which have common neighbor, $c(u) \neq c(v)$. That means, the restriction of $c$ to the neighborhood of any vertex is an injective function. The injective chromatic number, $\chi_i(G)$, is the least integer $k$ such that $G$ has an injective $k$-coloring. Note that an injective coloring is not necessarily a proper coloring. In fact, $\chi_i(G) \leq \chi(G^2)$, where $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $u$ and $v$ have a common neighbor in $G$. The square of graph $G$, denoted by $G^2$, is a graph with vertex set $V(G)$, where two vertices are adjacent in $G^2$ if and only if they are at distance at most two in $G$. Since $G^2$ is a subgraph of $G^2$, obviously, $\chi_i(G) \leq \chi(G^2)$. The concept of injective coloring is introduced by Hahn et al. in 2002 [7]. It is clear that for every graph $G$, $\chi_i(G) \geq \Delta$. In general, in [7] Hahn et al. proved that $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$. In [13], Wegner raised the following conjecture for the chromatic number of the square of planar graphs.

**Conjecture 1.1.** [13] If $G$ is a planar graph with maximum degree $\Delta$, then

- For $\Delta = 3$, $\chi(G^2) \leq \Delta + 2$.
- For $4 \leq \Delta \leq 7$, $\chi(G^2) \leq \Delta + 5$.
- For $\Delta \geq 8$, $\chi(G^2) \leq \lfloor 3\Delta/2 \rfloor + 1$.

Since $\chi_i(G) \leq \chi(G^2)$, Lužar and Škrekovski in [10] proposed the following conjecture for the injective chromatic number of planar graphs.

**Conjecture 1.2.** [10] If $G$ is a planar graph with maximum degree $\Delta$, then

- For $\Delta = 3$, $\chi_i(G) \leq \Delta + 2$.
- For $4 \leq \Delta \leq 7$, $\chi_i(G) \leq \Delta + 5$.
- For $\Delta \geq 8$, $\chi_i(G) \leq \lfloor 3\Delta/2 \rfloor + 1$.

The injective coloring of planar graphs with respect to its girth and maximum degree is studied in [1–6, 9, 11]. In [8], Lih and Wang proved upper bound $\Delta + 2$ for the chromatic number of square of outerplanar graphs.

**Theorem 1.3.** [8] If $G$ is an outerplanar graph, then $\chi(G^2) \leq \Delta + 2$.

Since $\chi_i(G) \leq \chi(G^2)$, Conjecture 1.2 is true for outerplanar graphs.

**Corollary 1.4.** If $G$ is an outerplanar graph, then $\chi_i(G) \leq \Delta + 2$. 
In Figure 1.1, an outerplanar graph with $\Delta = 4$, $g = 3$ and $\chi_i(G) = \Delta + 2 = 6$ is shown. Therefore, the given bound in Corollary 1.4 is tight.

Figure 1.1: An outerplanar graph with $\Delta = 4$, $g = 3$ and $\chi_i = 6$.

In this paper, we study the injective chromatic number of outerplanar graphs. The main results of Section 2 are as follows. If $G$ is an outerplanar graph with maximum degree $\Delta$ and girth $g$, then

- (Theorem 2.1) For $\Delta = 3$, $\chi_i(G) \leq \Delta + 1 = 4$.

- (Theorem 2.2) For $\Delta = 3$ and $g \geq 5$, with no face of degree $k$, $k \equiv 2 \pmod{4}$, $\chi_i(G) = \Delta$.

- (Theorem 2.4) For $\Delta = 3$ and $g \geq 6$, $\chi_i(G) = \Delta$.

- (Theorems 2.5 and 2.8) For $\Delta \geq 4$ and $g \geq 4$, $\chi_i(G) = \Delta$.

2. Main results

First, we prove a tight bound for the injective chromatic number of outerplanar graphs with $\Delta = 3$. Note that if $\Delta = 2$, then $G$ is an union of paths and cycles, which obviously $\chi_i(G) \leq 3 = \Delta + 1$. Moreover, if $G$ is an arbitrary path or is a cycle of length $k$, where $k \equiv 0 \pmod{4}$, then $\chi_i(G) = 2$. Otherwise, $\chi_i(G) = 3$ [7].

**Theorem 2.1.** If $G$ is an outerplanar graph with $\Delta = 3$, then $G$ has a 4-injective coloring such that in every simple path of length three, at most three colors appear. Moreover, the bound is tight.

**Proof.** We prove the theorem by the induction on $|V(G)|$. In Figure 2.1, all outerplanar graphs with $\Delta = 3$ of order 4 and 5 with an injective coloring with desired property are shown. Obviously, in the left side graph, $\chi_i(G) = 4$. Hence, bound $\Delta + 1$ is tight.

Now suppose that $G$ is an outerplane graph with $\Delta = 3$ and the statement is true for all outerplanar graphs with $\Delta = 3$ of order less than $|V(G)|$. The following two cases can be caused.
If an end block of $G$ is an edge, say $uv$, where $\deg(u) = 1$, then we consider the maximal simple path $P : (v_1 = u), (v_2 = v), v_3, \ldots, v_k$ in $G$. Since $P$ is a maximal simple path and $\Delta(G) = 3$, we have $\deg(v_k) = 3$. Suppose that $N(v_k) = \{ w_1, w_2, w_{k-1} \}$ and $c$ is a 4-injective coloring of $G \setminus \{ v_1, v_2, \ldots, v_{k-1} \}$ with colors $\{ \alpha, \beta, \gamma, \lambda \}$ such that every simple path of length three has at most three colors. Note that $w_1$ and $w_2$ have a common neighbor $v_k$ therefore, $c(w_1) \neq c(w_2)$. In this case, we assign to the ordered vertices $v_{k-1}, v_{k-2}, \ldots, v_2, v_1$ of path $P$ the ordered string ($ssttsstt\ldots$), where $s \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_k), c(w_1), c(w_2)\}$ and $t = c(v_k)$.

If the minimum degree of every end block of $G$ is at least two in $G$, then we consider an end face $f = [v_i v_{i+1} \ldots v_j]$ in an end block $B$ of $G$ in clockwise order, where $v_1$ is the vertex cut of $G$ belongs to $B$. Note that, since $\Delta(G) = 3$, if $G$ is a block, then $G$ has an end face $f = [v_i v_{i+1} \ldots v_j]$. Let $H$ be the induced subgraph of $G$ on 2-vertices of $f$. If $\Delta(G \setminus H) = 2$, then we color the ordered vertices $v_j, v_{j+1}, \ldots, v_{i-1}, v_i$ of $G \setminus H$ by ordered string $(\alpha \beta \gamma \lambda \alpha \beta \gamma \lambda \ldots)$. If $|V(G \setminus H)| \equiv 2 \pmod{4}$, then change the color of $v_{i-1}$ and $v_i$ to $\beta$ and $\alpha$, respectively. If $\Delta(G \setminus H) = 3$, then by the induction hypothesis $G \setminus H$ has a 4-injective coloring $c$ with colors $\{ \alpha, \beta, \gamma, \lambda \}$, such that every simple path of length three has at most three colors. Hence, in $G \setminus H$ at most three colors are used for vertices $v_{i-1}, v_i, v_j, v_{j+1}$. Now we extend $c$ to an injective coloring of $G$ with the desired property.

If $c(v_i) = c(v_j)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string ($ssttsstt\ldots$), where $s \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$ and $t \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_i) = c(v_j), c(v_{j+1}), s\}$.

If $c(v_i) \neq c(v_j)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string ($ssttsstt\ldots$), where $s \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$. If $j - i - 1 \equiv 1, 2 \pmod{4}$, then $t \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_j), s\}$. If $j - i - 1 \equiv 0, 3 \pmod{4}$, then $t \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_{i-1}), c(v_j), s\}$. In the case $j - i - 1 \equiv 0 \pmod{4}$, if $t = c(v_j)$, then change the color of $v_{j-2}$ to $t' \in \{ \alpha, \beta, \gamma, \lambda \} \setminus \{c(v_j) = t, s\}$. Note that, since by the induction hypothesis $|\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}| \leq 3$, in each cases the colors $s$ and $t$ exist. It can be easily seen that the given coloring is a 4-injective coloring for $G$ such that every simple path of length three in $G$ has at most three colors as well.

Graph $G$ in Figure 2.2 is an outerplanar graph of girth 4 with maximum degree three.
and injective chromatic number 4. Since each pair of set \{u, v, w\} have a common neighbor, in every injective coloring of G, they must have three different colors. In the similar way, we need three different colors for the vertices \{x, y, z\}. Without loss of generality, color the vertices u, v, w with color \alpha, \beta and \gamma, respectively. Now by devoting any permutation of these colors to vertices x, y and z, it can be checked that in each case we need a new color for the other vertices. Therefore, bound \Delta + 1 in Theorem 2.1 is tight for outerplanar graphs with \Delta = 3, g = 4 and \chi_i = 3 (see also Figure 2.1).

![Figure 2.2: An outerplanar graph with \Delta = 3, g = 4 and \chi_i = 4.](image)

In the next theorems, we improve bound \Delta + 1 to \Delta for outerplanar graphs with \Delta = 3 of girth greater than 4.

**Theorem 2.2.** If G is a 2-connected outerplanar graph with \Delta = 3, g \geq 5 and no face of degree k, where k \equiv 2 \pmod{4}, then G has a 3-injective coloring such that in every simple path of length three, exactly three colors appear.

**Proof.** We prove it by the induction on |V(G)|. In Figure 2.3 the 2-connected outerplanar graphs with \Delta = 3 and g \geq 5 of order at most 10 with an injective coloring with desired property are shown.

![Figure 2.3: Outerplanar graphs with \Delta = 3 and g \geq 5 of order 8 and 10.](image)

Now suppose that G is a 2-connected outerplane graph with \Delta = 3, g \geq 5 and no face of degree k, where k \equiv 2 \pmod{4} and the statement is true for all such 2-connected outerplanar graphs of order less than |V(G)|.
Let $f = [v_i v_{i+1} \ldots v_j]$ be an end face of $G$ in clockwise order and $H$ be the induced subgraph of $G$ on 2-vertices of $f$. If $\Delta(G \setminus H) = 3$, then by the induction hypothesis $G \setminus H$ has a 3-injective coloring $c$ with colors $\{\alpha, \beta, \gamma\}$, such that every simple path of length three has exactly three colors.

If $\Delta(G \setminus H) = 2$, then we color the vertices of $G \setminus H$ as follows. If $G \setminus H = C_t$, where $t > 5$ and $t \equiv 0, 1 \pmod{3}$, then color the ordered vertices $v_{i-1}, v_i, v_j, v_{j+1}, \ldots, v_{i-2}$ with the ordered string $(\alpha\beta\gamma\alpha\beta\gamma\ldots)$. If $t > 5$ and $t \equiv 2 \pmod{3}$, then color the ordered vertices $v_{i-1}, v_i, v_j, v_{j+1}, \ldots, v_{i-5}$ with the ordered string $(\alpha\beta\gamma\alpha\beta\gamma\ldots)$. Then color the vertices $v_{i-4}, v_{i-3}$ and $v_{i-2}$ with colors $\beta, \gamma$ and $\alpha$, respectively. One can check that every simple path of length three in $G \setminus H$ has exactly three colors. If $G \setminus H = C_5$, then since $|V(G)| > 10$, $f = [v_i v_{i+1} \ldots v_j]$ is a cycle of length at least 8. In this case, we consider the end face $f' = [v_j v_{j+1} \ldots v_i]$ and follow the above proof when $H$ is induced subgraph of $G$ on 2-vertices of $f'$. In the following, we extend injective coloring $c$ of $G \setminus H$ to an injective coloring of $G$ with the desired property.

If $c(v_i) = c(v_j)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \ldots)$, where $s_1 = c(v_{j+1})$. Since $G$ has no face of degree $k$ where $k \equiv 2 \pmod{4}$, we have following cases. If $j - i - 1 \equiv 1 \pmod{4}$, then let $s_2 = c(v_{i-1})$, $s_3 = s_4 = c(v_i) = c(v_j)$ and change the color of vertices $v_{i-2}$ and $v_{i-1}$ to $c(v_{j+1})$ and $c(v_{i-1})$, respectively. If $j - i - 1 \equiv 2 \pmod{4}$, then let $s_2 = s_1 = c(v_{j+1})$, $s_3 = c(v_{i-1})$ and $s_4 = c(v_i) = c(v_j)$ and change the color of $v_{i-1}$ to $c(v_{i-1})$. If $j - i - 1 \equiv 3 \pmod{4}$, then let $s_2 = s_1 = c(v_{j+1})$, $s_3 = c(v_{i-1})$ and $s_4 = c(v_i) = c(v_j)$.

If $c(v_i) \neq c(v_j)$ and $c(v_{i-1}) = c(v_{j+1})$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \ldots)$, where $s_1 = c(v_i)$. If $j - i - 1 \equiv 1, 2 \pmod{4}$, then $s_2 = c(v_j)$ and $s_3 = s_4 = c(v_{i-1}) = c(v_{j+1})$. In the case $j - i - 1 \equiv 1 \pmod{4}$, we change the color of vertices $v_{j-2}$ and $v_{j-1}$ to $c(v_j)$ and $c(v_i)$, respectively. If $j - i - 1 \equiv 3 \pmod{4}$, then let $s_2 = c(v_{i-1}) = c(v_{j+1})$ and $s_3 = s_4 = c(v_j)$.

If $c(v_i) \neq c(v_j)$ and $c(v_{i-1}) = c(v_i)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \ldots)$, where $s_1 = c(v_{j+1})$. If $j - i - 1 \equiv 1 \pmod{4}$, then let $s_2 = c(v_j)$, $s_3 = c(v_i) = c(v_{i-1})$, $s_4 = s_1$ and change the color of vertex $v_{i-1}$ to $c(v_j)$. If $j - i - 1 \equiv 2 \pmod{4}$, then let $s_2 = c(v_j)$, $s_3 = s_4 = c(v_{i-1}) = c(v_i)$. If $j - i - 1 \equiv 3 \pmod{4}$, then we assign to the ordered vertices $v_{j-1}, v_{j-2}, \ldots, v_{i+1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \ldots)$, where $s_1 = c(v_j)$, $s_2 = c(v_{j+1})$, $s_3 = s_4 = c(v_i) = c(v_{i-1})$ and change the colors of $v_{i+1}$ to $c(v_{j+1})$.

If $c(v_i) \neq c(v_j)$ and $c(v_j) = c(v_{j+1})$, then we assign to the ordered vertices $v_{j-1}, v_{j-2}, \ldots, v_{i+1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \ldots)$, where $s_1 = c(v_{i-1})$. If $j - i - 1 \equiv 1 \pmod{4}$, then $s_2 = c(v_i)$, $s_3 = c(v_{j+1}) = c(v_j)$, $s_4 = s_1$ and change the color of $v_{i+1}$ to $c(v_i)$. If $j - i - 1 \equiv 2 \pmod{4}$, then $s_2 = c(v_i)$ and $s_3 = c(v_{j+1}) = c(v_j)$, $s_4 = s_1$ and change the color of $v_{i+1}$ to $c(v_i)$. If $j - i - 1 \equiv 3 \pmod{4}$, then $s_2 = c(v_i)$ and $s_3 = s_4 = c(v_{j+1})$. If $j - i - 1 \equiv 3 \pmod{4}$, then $s_2 = c(v_i)$ and $s_3 = c(v_{j+1}) = c(v_j)$, $s_4 = s_1$ and change the color of $v_{i+1}$ to $c(v_i)$.
(mod 4), then we assign to the ordered vertices \(v_{i+1}, v_{i+2}, \ldots, v_{j-1}\) the ordered string \((s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \ldots)\), where \(s_1 = c(v_i)\), \(s_2 = c(v_{i-1})\) and \(s_3 = s_4 = c(v_j) = c(v_{j+1})\) and change the color of \(v_{j-1}\) to \(c(v_{i-1})\). It can be seen that the given coloring is a 3-injective coloring for \(G\) such that every simple path of length three in \(G\) has exactly three colors. \(\square\)

In Theorem 2.4, we improve bound \(\Delta + 1\) in Theorem 2.1 to \(\Delta\) for outerplanar graph with \(\Delta = 3\) and \(g \geq 6\). First, we need the following theorem.

**Theorem 2.3.** [12] Let \(G\) be a connected graph and \(L\) be a list-assignment to the vertices, where \(|L(v)| \geq \deg(v)\) for each \(v \in V(G)\). If

1. \(|L(v)| > \deg(v)\) for some vertex \(v\), or
2. \(G\) contains a block which is neither a complete graph nor an induced odd cycle,

then \(G\) admits a proper coloring such that the color assign to each vertex \(v\) is in \(L(v)\).

**Theorem 2.4.** If \(G\) is an outerplanar graph with \(\Delta = 3\) and \(g \geq 6\), then \(\chi_i(G) = \Delta\).

**Proof.** Since \(\chi_i(G) \geq \Delta\), it is enough to show that \(\chi_i(G) \leq \Delta\). Let \(G\) be a minimal counterexample for this statement. That means \(G\) is an outerplane graph with \(\Delta = 3\), \(g \geq 6\) and \(\chi_i(G) \geq \Delta + 1\), such that every proper subgraph of \(G\) has a \(\Delta\)-injective coloring. Obviously \(\delta(G) \geq 2\). Now consider an end face \(f = [v_i v_{i+1} \ldots v_j]\) in an end block \(B\) of \(G\) in clockwise order, where \(v_i\) is the vertex cut of \(G\) belonging to \(B\). Since \(\Delta = 3\) and \(g \geq 6\), the degree of face \(f\) is at least 6 and the degree of \(v_i\) and \(v_j\) are three. Let \(H\) be the induced subgraph of \(G\) on 2-vertices of \(f\). If \(\Delta(G \setminus H) = 3\), then by the minimality of \(G\), we have \(\chi_i(G \setminus H) \leq \Delta(G \setminus H) \leq \Delta(G)\). Also, if \(G \setminus H\) is a cycle, then \(\chi_i(G \setminus H) \leq 3 = \Delta\).

Now, we extend the \(\Delta\)-injective coloring of \(G \setminus H\) to a \(\Delta\)-injective coloring of \(G\), which contradicts our assumption. Each of the vertices \(v_i\) and \(v_j\) has at most \(\Delta - 1 = 2\) neighbors except \(v_{i+1}\) and \(v_{j-1}\), respectively. Hence, for each of vertices \(v_{i+1}\) and \(v_{j-1}\) there is at least one available color. Also, among the colored vertices in \(G \setminus H\), the only forbiden colors for vertices \(v_{i+2}\) and \(v_{j-2}\) are colors of the vertices \(v_i\) and \(v_j\), respectively. The other vertices have three available colors. Now consider induced subgraph of \(G^{(2)}\) on the vertices of \(H\), denoted by \(G^{(2)}[H]\), and list of available colors for each vertex of \(H\). The components of \(G^{(2)}[H]\) are some paths satisfying the assumption of Theorem 2.3. Thus, we have a proper \(\Delta\)-coloring for \(G^{(2)}[H]\) using the available colors which is a \(\Delta\)-injective coloring of \(H\) as desired. \(\square\)

Now, we are ready to determine the injective chromatic number of 2-connected outerplanar graphs with maximum degree and girth greater than three. We prove this fact by two different methods for the cases \(\Delta = 4\) and \(\Delta \geq 5\).
Theorem 2.5. If $G$ is a 2-connected outerplanar graph with $\Delta = 4$ and $g \geq 4$, then $G$ has a 4-injective coloring $c$ such that for every adjacent vertices $v$ and $u$ of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$.

Proof. We prove it by the induction on $|V(G)|$. In Figure 2.4 the 2-connected outerplanar graphs with $\Delta = 4$ and $g \geq 4$ of order 8 and 9 with an injective coloring of desired property are shown.

![Graphs with Δ = 4 and g ≥ 4 of order 8 and 9.](image)

Figure 2.4: 2-connected outerplanar graphs with $\Delta = 4$ and $g \geq 4$ of order 8 and 9.

Now suppose that $G$ is a 2-connected outerplane graph with $\Delta = 4$, $g \geq 4$ and the statement is true for all 2-connected outerplanar graphs with $\Delta = 4$ and $g \geq 4$ of order less than $|V(G)|$.

Let $f = [v_i v_{i+1} \ldots v_j]$ be an end face of $G$ in clockwise order. If $\deg(v_i) = \deg(v_j) = 3$, then consider induced subgraph $H$ on 2-vertices of face $f$. Thus, $G \setminus H$ is a 2-connected outerplane graph with $\Delta(G \setminus H) = 4$ and $g(G \setminus H) \geq 4$. Hence, by the induction hypothesis, $G \setminus H$ has a 4-injective coloring such that for every adjacent vertices $v$ and $u$ of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$. If there are exactly four colors in $\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$, then consider graph $G^{(2)}[H]$ and list of available colors for each vertex of $H$. Graph $G^{(2)}[H]$ satisfy the assumption of Theorem 2.3 Thus, we have a $\Delta$-coloring for $G^{(2)}[H]$ which is a $\Delta$-injective coloring of $H$. If there are at most three colors in $\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$, then color $v_{i+1}$ with one of its colors not in $\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$ and color $v_{j-1}$ with one of its available colors such that $c(v_{i+1}) \neq c(v_{j-1})$. Then color the other vertices of $H$ with one of their available colors similar to above. It can be easily seen that for every adjacent vertices $v$ and $u$ of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$.

Now suppose that each face of $G$ has an end vertex of degree 4. We have two following cases.

Case 1. There is an end face $f$ with one end vertex of degree 4 and the other one of degree less than 4.

In this case, suppose that $G$ has an end face $f = [v_i v_{i+1} \ldots v_j]$, where $\deg(v_i) = 4$ and $\deg(v_j) = 3$. Consider induced subgraph $H$ on 2-vertices of face $f$. If $\Delta(G \setminus H) = 4$, then...
then by the induction hypothesis, $G \setminus H$ has a 4-injective coloring such that for every adjacent vertices $v$ and $u$ of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, \(\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}\). Now we extend the 4-injective coloring of $G \setminus H$ to $G$. If $\deg(v_{j+1}) = 3$, then suppose that $v_s$ is the other neighbor of $v_{j+1}$ except $v_j$ and $v_{j+2}$. If there are exactly three colors in \(\{c(v_i), c(v_j), c(v_{j+1}), c(v_{j+2}), c(v_s)\}\), then color vertex $v_{j-1}$ with one of its colors not in \(\{c(v_i), c(v_j), c(v_{j+1}), c(v_{j+2}), c(v_s)\}\) and color the other vertices of $H$ with one of their available colors as explained in above. If $|\{c(v_i), c(v_j), c(v_{j+1}), c(v_{j+2}), c(v_s)\}| = 4$ or $\deg(v_{j+1}) \neq 3$, then by Theorem 2.3, we obtain a 4-injective coloring of $G$. Note that, since $g(G) \geq 4$, in this case there is no two adjacent vertices of degree three.

**Case 2.** For each end face $f$, its two end vertices are of degree 4.

In this case, consider the induced subgraph $H$ on 2-vertices of $f = [v_i, v_{i+1}, \ldots, v_j]$, where $\deg(v_i) = \deg(v_j) = 4$. Since $\deg(v_i) = 4$, $G \setminus H$ has an end face $f'$ with two ends of degree 4. Hence, $\Delta(G \setminus H) = 4$ and by the induction hypothesis, $G \setminus H$ has a 4-injective coloring such that for every adjacent vertices $v$ and $u$ of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, \(\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}\). Now by Theorem 2.3, color the vertices of $H$ with their available colors such that obtained coloring is a 4-injective coloring of $G$. Obviously, for every adjacent vertices $v$ and $u$ of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, \(\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}\).

Now we consider 2-connected outerplanar graphs with $\Delta = 5$ and $g \geq 4$. First, we need to prove the following theorem on the structure of 2-connected outerplanar graphs.

**Theorem 2.6.** If $G$ is a 2-connected outerplanar graph, then $G$ has an end face $f = [v_i, v_{i+1}, \ldots, v_j]$, where either $\deg(v_i) < 5$ or $\deg(v_j) < 5$.

**Proof.** First replace every simple path in boundary of each end face of $G$ with a path of length two and name this graph $G'$. Graph $G'$ is also a 2-connected outerplane graph that each end face of $G'$ is of degree three (for example see Figure 2.5). If $G'$ is a cycle, then we are done. Now, let $\Delta(G') \geq 3$ and $C : v_1v_2 \ldots v_n$ be a Hamilton cycle of $G'$ in clockwise order. Also, let $f = [v_iv_{i+1}v_{i+2}]$ be an end face of $G'$. If $\deg(v_{i+2})$ is at least 5, then we present an algorithm that find an end face of $G'$ such that the degree of at least one of its end vertices is less than 5. Since by assumption $\deg(v_{i+2}) \geq 5$, $v_{i+2}$ has at least two other
neighbors except $v_i$, $v_{i+1}$ and $v_{i+3}$, named $v_i'$ and $v_{j'}$ such that the number of vertices between $v_{i+3}$ and $v_{i'}$ in clockwise order is less than the number of vertices between $v_{i+3}$ and $v_{j'}$ in clockwise order.

Figure 2.5: Two graphs $G$ and $G'$.

Algorithm 2.7.

1. $k = 0$.

2. $f_0 = [v_i v_{i+1} v_{i+2}]$.

3. If $f_k = [v_t v_{t+1} v_{t+2}]$ is an end face of $G'$, then do steps 4 to 7, respectively.

4. Suppose that $v_{L_{f_k}} = v_t$ and $v_{R_{f_k}} = v_{t+2}$. Let $v_{i'_{k}}$ and $v_{j'_{k}}$ be another neighbors of $v_{t+2}$ except $v_t$, $v_{t+1}$ and $v_{t+3}$ such that the number of vertices between $v_{R_{f_k}}$ and $v_{i'_{k}}$ in clockwise order is less than the number of vertices between $v_{R_{f_k}}$ and $v_{j'_{k}}$ in clockwise order.

5. If $\deg(v_{L_{f_k}}) \leq 4$ or there is no $v_{i'_{k}}$ or $v_{j'_{k}}$, then stop the algorithm and give the face $f_k$ as output of the algorithm.

6. $k = k + 1$.

7. $f_k = [v_{R_{f_{k-1}}} v_{R_{f_{k-1}}} + 1 \ldots v_{k-1}]$ and go to step three.

8. If $f_k$ is not an end face of $G'$, then there exists an end face $f$ in $f_k$. Do steps 9 and 10, respectively.

9. $k = k + 1$.

10. $f_k = f$ and go to step three.

Note that, the neighbors of all vertices $v_{R_{f_k}}$ are between $v_{i+2}$ and $v_{j'}$ in clockwise order; otherwise there is a subdivision of $K_4$ on $G'$ and it is a contradiction with the assumption that $G'$ is an outerplanar graph. Therefore, the algorithm terminates. Moreover, if $f_k = [v_k v_{k+1} v_{k+2}]$ is the output of the algorithm, then by line 5 of the algorithm, the degree of
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$v_{k+2}$ is less than 5. Finally, by returning the contracted paths to $G'$; we have an end face of $G$ that one of its ends is of degree less than 5.

\[ \]

**Theorem 2.8.** If $G$ is a 2-connected outerplanar graph with $\Delta \geq 5$ and $g \geq 4$, then $\chi_i(G) = \Delta$.

**Proof.** Since $\chi_i(G) \geq \Delta(G)$, it is enough to show that $\chi_i(G) \leq \Delta(G)$. We prove it by the induction on $|V(G)|$. In Figure 2.6, the 2-connected outerplanar graphs with $\Delta \geq 5$ and $g \geq 4$ of order 10 and 11 with a $\Delta$-injective coloring are shown.

![Figure 2.6: 2-connected outerplanar graphs with $\Delta \geq 5$ and $g \geq 4$ of order 10 and 11.](image)

Now suppose that $G$ is a 2-connected outerplane graph with $\Delta \geq 5$, $g \geq 4$ and the statement is true for all 2-connected outerplanar graphs with $\Delta \geq 5$ and $g \geq 4$ of order less that $|V(G)|$. By Theorem 2.6, $G$ has an end face $f$ of degree at least 4 such that at least one of its end vertices is of degree at most 4. Now consider the induced subgraph $H$ on 2-vertices of end face $f$. If $\Delta(G \setminus H) \geq 5$, then by induction hypothesis, $\chi_i(G \setminus H) = \Delta(G \setminus H) \leq \Delta(G)$. If $\Delta(G \setminus H) = 4$, then by Theorem 2.5, $G \setminus H$ has a 4-injective coloring. Now consider the end face $f = [v_i v_{i+1} \ldots v_j]$ and suppose that $\deg(v_i) \leq \Delta$ and $\deg(v_j) \leq 4$. Since $\Delta \geq 5$, the vertices $v_{i+1}$ and $v_{j-1}$ have at least one and two available colors, respectively. The other vertices of $H$ has at least three available colors. Now consider the graph $G^{(2)}[H]$ and list of available colors for each vertex of $H$. It can be easily seen that $G^{(2)}[H]$ is union of paths and isolated vertices, which satisfy the assumption of Theorem 2.3. Hence, $G^{(2)}[H]$ can be colored by at most $\Delta$ colors and the obtained coloring is a $\Delta$-injective coloring of $H$.

**Remark 2.9.** Applying the same idea and by laboriously proof, the results of Theorems 2.2, 2.5 and 2.8 can be generalized for the outerplanar graphs containing some cut vertices.

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