Quantum Baxter-Belavin R-matrices and multidimensional Lax pairs for Painlevé VI

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Abstract

The quantum elliptic $R$-matrices of Baxter-Belavin type satisfy the associative Yang-Baxter equation in $\text{Mat}(\mathbb{N}, \mathbb{C})^\otimes 3$. The latter can be considered as noncommutative analogue of the Fay identity for the scalar Kronecker function. In this paper we extend the list of $R$-matrix valued analogues of elliptic function identities. In particular, we propose counterparts of the Fay identities in $\text{Mat}(\mathbb{N}, \mathbb{C})^\otimes 2$. As an application we construct $R$-matrix valued $2\mathbb{N}^2 \times 2\mathbb{N}^2$ Lax pairs for the Painlevé VI equation (in elliptic form) with four free constants using $\mathbb{Z}_N \times \mathbb{Z}_N$ elliptic $R$-matrix. More precisely, the four free constants case appears for an odd $N$ while even $N$’s correspond to a single constant.
1 Introduction and summary

In this paper we continue the study of identities for quantum (and classical) $R$-matrices, which are similar to the elliptic functions identities for scalar elliptic functions [13, 8]. More concretely, we prove the Fay identities in $\text{Mat}(N, \mathbb{C})^\otimes 2$. It allows us to construct multidimensional Lax pairs for the Painlevé VI equation with the $R$-matrices as matrix elements.

We start with the list of properties and identities for elliptic functions, and then give their $R$-matrix version. Most of the properties are known from [2, 4], [14], [3, 15], [13] and [8].

Consider the following functions:

$$\phi(z, u) = \frac{\vartheta'(0)\vartheta(z + u)}{\vartheta(z)\vartheta(u)},$$

$$E_1(z) = \frac{\vartheta'(z)}{\vartheta(z)}, \quad E_2(z) = -\partial_z E_1(z) = \wp(z) - \frac{1}{3} \frac{\vartheta''''(0)}{\vartheta''(0)},$$

where $\vartheta(z)$ is the odd Riemann theta-function

$$\vartheta(z) = \vartheta(z|\tau) = \sum_{k \in \mathbb{Z}} \exp \left( \pi i \tau (k + \frac{1}{2})^2 + 2\pi i (z + \frac{1}{2})(k + \frac{1}{2}) \right)$$

and $\wp(z)$ is the Weierstrass $\wp$-function.

Following [16] the function (1.1) is referred to as the Kronecker function, and (1.2) are called the (first and the second) Eisenstein functions.

The Kronecker function can be considered as a section of the Poincaré bundle $\mathcal{P}$ over $\Sigma_{\tau} \times \Sigma'_{\tau}$. Here $\Sigma_{\tau}$ is the elliptic curve

$$\Sigma_{\tau} = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z}), \quad \Im \tau > 0,$$

$\Sigma'_{\tau}$ is its Jacobian ($\Sigma'_{\tau} \sim \Sigma_{\tau}$). The Poincaré bundle $\mathcal{P}$ is a line bundle over $\Sigma_{\tau} \times \Sigma'_{\tau}$.
specialized by (1.6), (1.7), (1.10) and (1.11).

The properties of theta-function (1.3) (including Riemann identities, see [11]) provides the following set of properties and relations for the functions (1.1)-(1.2):

- **Arguments symmetry:**
  \[ \phi(z, u) = \phi(u, z), \quad z \in \Sigma_\tau, \ u \in \Sigma'_\tau, \]  
  (1.6)

- **Local expansion:**
  \[ \phi(z, u) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \phi(u)) + O(z^2), \]  
  (1.7)

- **Residues:**
  \[ \text{Res}_{z=0} \phi(z, u) = \text{Res}_{u=0} \phi(z, u) = \text{Res}_{z=0} E_1(z) = 1, \]  
  (1.8)

- **Parity:**
  \[ \phi(-z, -u) = -\phi(z, u), \quad E_1(-z) = -E_1(z), \quad E_2(-z) = E_2(z), \]  
  (1.9)

- **(Quasi)periodicity properties:**
  \[ \phi(z + 1, u) = \phi(z, u), \quad E_1(z + 1) = E_1(z), \quad E_2(z + 1) = E_2(z), \]  
  \[ \phi(z + \tau, u) = e^{-2\pi i u} \phi(z, u), \quad E_1(z + \tau) = E_1(z) - 2\pi i, \quad E_2(z + \tau) = E_2(z), \]  
  (1.10) (1.11)

- **Heat equation:**
  \[ 2\pi i \partial_\tau \phi(z, u) = \partial_z \partial_u \phi(z, u), \]  
  (1.12)

- **Derivatives:**
  \[ \partial_u \phi(z, u) = \phi(z, u)(E_1(z + u) - E_1(u)), \]  
  \[ \partial_z \phi(z, u) = \phi(z, u)(E_1(z + u) - E_1(z)), \]  
  (1.13) (1.14)

- **Fay (trisecant) identity [6]:**
  \[ \phi(x, u)\phi(y, w) = \phi(x - y, u)\phi(y, u + w) + \phi(y - x, w)\phi(x, u + w), \]  
  (1.15)
Degenerated Fay identities:
\[ \phi(x, z)\phi(x, w) = \phi(x, z + w)(E_1(x) + E_1(z) + E_1(w) - E_1(x + z + w)), \]  
(1.16)

or
\[ \phi(x, z)\phi(y, z) = \phi(x + y, z)(E_1(x) + E_1(y) + E_1(z) - E_1(x + y + z)), \]  
(1.17)

\[ \phi(x, z)\phi(x, -z) = E_2(x) - E_2(z) = \wp(x) - \wp(z). \]  
(1.18)

Geometric interpretation: The Kronecker function \( \phi(z, u) \) is a section of the Poincaré bundle \( \mathcal{P} \). It is a line bundle over \( \Sigma_r \times \Sigma_r \), defined by the conditions (1.6), (1.7), (1.10), (1.11).

Green function: The Kronecker function is the Green function for the operator \( \bar{\partial} \) in the space of one forms \( \mathcal{A}^{(1,0)}(\Sigma_r) \) with the boundary conditions (1.10) and (1.11):
\[ \bar{\partial}\phi(z, u) = \delta^2(z, \bar{z}). \]  
(1.19)

Quantum \( R \)-matrices. Consider \( \mathbb{Z}_N \times \mathbb{Z}_N \) (Baxter-Belavin’s) elliptic \( R \)-matrix [2, 4] in the fundamental representation (see also [13]). It is defined via the finite-dimensional representation of the Heisenberg group:
\[ Q, \Lambda \in \text{Mat}(N, \mathbb{C}) : \quad Q_{kl} = \delta_{kl} \exp \left( \frac{2\pi i}{N} k \right), \quad \Lambda_{kl} = \delta_{k-l+1=0 \mod N}, \quad k, l = 1, ..., N, \]  
(1.20)

\[ \exp \left( 2\pi i \frac{\gamma_1 \gamma_2}{N} \right) Q^{\gamma_1} \Lambda^{\gamma_2} = \Lambda^{\gamma_2} Q^{\gamma_1}, \quad \gamma_1, \gamma_2 \in \mathbb{Z}. \]  
(1.21)

Introduce the sin-algebra basis in \( \text{Mat}(N, \mathbb{C}) \):
\[ T_{\gamma} := T_{\gamma_1 \gamma_2} = \exp \left( \frac{2\pi i}{N} \frac{\gamma_1 \gamma_2}{N} \right) Q^{\gamma_1} \Lambda^{\gamma_2}, \quad \gamma_1, \gamma_2 = 0, ..., N - 1. \]  
(1.22)

The same definition is used for any \( \gamma \in \mathbb{Z}^{\times 2} \). Then
\[ T_{\alpha}T_{\beta} = \kappa_{\alpha, \beta} T_{\alpha + \beta}, \quad \kappa_{\alpha, \beta} = \exp \left( \frac{\pi i}{N} (\beta_1 \alpha_2 - \beta_2 \alpha_1) \right), \]  
(1.23)

where \( \alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2) \). The \( R \)-matrix is defined as
\[ R_{12}^{h}(u) = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} \varphi_{\alpha}(u, \omega_\alpha + \bar{h}) T_{\alpha} \otimes T_{-\alpha} \in \text{Mat}(N, \mathbb{C})^{\otimes 2}, \]  
(1.24)

where
\[ \varphi_{\alpha}(u, \omega_\alpha + \bar{h}) = \exp(2\pi i u \partial_\tau \omega_\alpha) \phi(u, \omega_\alpha + \bar{h}), \quad \omega_\alpha = \frac{\alpha_1 + \alpha_2 \tau}{N}. \]  
(1.25)

The \( \mathbb{Z}_N \times \mathbb{Z}_N \) symmetry means that for \( g = Q, \Lambda \)
\[ (g \otimes g) R_{12}^{h}(u)(g^{-1} \otimes g^{-1}) = R_{12}^{h}(u). \]  
(1.26)

\[ \text{Here } \partial_\tau \omega_\alpha = \alpha_2 / N. \]
For $N = 1$ the $R$-matrix (1.24) is the scalar Kronecker function $\phi(h, u)$ (1.1). Notice that (1.24) is normalized in such a way that the unitarity condition acquires the form:

$$R^h_{12}(u)P^h_{21}(-u) = N^2 \phi(Nh, u) \phi(Nh, -u) 1 \otimes 1 = N^2 (\varphi(Nh) - \varphi(u)) 1 \otimes 1 .$$

(1.27)

The latter can be considered as an analogue of (1.18). Here $R^h_{21}(z) = P_{12} R_{12}(z) P_{12}$, where

$$P_{12} = \frac{1}{N} \sum_{\alpha} T_{\alpha} \otimes T_{-\alpha} = \sum_{i,j=1}^N E_{ij} \otimes E_{ji} , \quad (E_{ij})_{kl} = \delta_{ik} \delta_{jl}$$

(1.28)

is the permutation operator. We also use notation $R^h_{ab}(z)$ which differs from (1.27) by $T_{\alpha} \otimes T_{-\alpha} = 1 \otimes \ldots 1 \otimes T_{\alpha} \otimes 1 \ldots \otimes 1$ instead of $T_{\alpha} \otimes T_{-\alpha}$ (i.e. $T_{\alpha}$ and $T_{-\alpha}$ are in the $a$-th and $b$-th components). The number of components in the tensor product is an integer $\tilde{N}$. It means that $R^h_{ab}$ is considered as an element of $\text{Mat}(N, \mathbb{C})^{\otimes \tilde{N}}$, i.e. $N^{\tilde{N}} \times N^{\tilde{N}}$ matrix.

The properties and identities (1.8)-(1.17) have the following analogues for $R$-matrices:

- **Arguments symmetry:**
  $$R^h_{12}(z) = R^h_{12}(Nh)P_{12} ,$$

  (1.29)

- **Local expansion** in $h$ is the classical limit:
  $$R^h_{12}(z) = h^{-1} 1 \otimes 1 + r_{12}(z) + h m_{12}(z) + O(h^2) ,$$

  (1.30)

where $r_{12}(z)$ is the classical (Belavin-Drinfeld [4]) $r$-matrix:

$$r_{12}(z) = E_1(z) 1 \otimes 1 + \sum_{\alpha \neq 0} \varphi_\alpha(z) T_{\alpha} \otimes T_{-\alpha}$$

(1.31)

and

$$m_{12}(z) = \frac{E_1^2(z) - \varphi(z)}{2} 1 \otimes 1 + \sum_{\alpha \neq 0} \exp(2\pi iz \partial_u \omega_\alpha) \partial_u \varphi(z, u) \big|_{u = \omega_\alpha} T_{\alpha} \otimes T_{-\alpha} .$$

(1.32)

Similarly to (1.7) we have:

$$r_{12}^2(z) - 2m_{12}(z) = 1 \otimes 1 N^2 \varphi(z) ,$$

(1.33)

i.e. the quantum $R$-matrix is a matrix analogue of the Kronecker function (1.1) while the classical one is the analogue of the first Eisenstein function (1.2).

Expansion with respect to $z$ (near $z = 0$) is as follows:

$$R^h_{12}(z) = \frac{NP_{12}}{z} + R^{h,(0)}_{12} + O(z) ,$$

(1.34)

where

$$R^{h,(0)}_{12} = \sum_{\alpha} T_{\alpha} \otimes T_{-\alpha} (E_1(h + \omega_\alpha) + 2\pi i \partial_u \omega_\alpha) .$$

(1.35)

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$^2$ $R^{h,(0)}_{12}$ appears as a part of the inverse inertia tensor for relativistic tops [9].
\( R^{h}_{12}(z) = -R^{-h}_{21}(-z) \), \( r_{12}(z) = -r_{21}(-z) \), \( m_{12}(z) = m_{21}(-z) \). \hspace{1cm} (1.37)

The \( R \)-matrix analogue of \( E_{2}(u) = E_{2}(-u) \) appears as \( F_{12}^{0}(u) = -\partial_{u}r_{12}(u) \) (It is natural because \( r_{12}(u) \) is the analogue of \( E_{1}(u) \)). The classical \( r \)-matrix is odd. Hence \( F_{12}^{0}(u) \) is even matrix function. The same answer follows from the local expansions (1.7), (1.30): \( E_{2}(u) = -\partial_{u}\phi(z, u) \mid_{z=0} \), then \(-\partial_{u}R^{\pm}_{12}(u) \mid_{z=0} = -\partial_{u}r_{12}(u) \).

\textbf{(Quasi)periodicity properties:}

\[ R^{h}_{12}(z + N\omega_{\gamma}) = \exp(-2\pi \i \hbar \partial_{z} \omega_{\gamma}) (T_{\gamma}^{-1} \otimes 1) R^{h}_{12}(z)(T_{\gamma} \otimes 1), \]
\[ R^{h+\omega_{\gamma}}_{12}(z) = \exp(-2\pi \i \omega_{\gamma}) (T_{\gamma}^{-1} \otimes 1) R^{h}_{12}(z)(1 \otimes T_{\gamma}). \]

In particular,
\[ R^{h}_{12}(z + 1) = (Q^{-1} \otimes 1) R^{h}_{12}(z)(Q \otimes 1), \]
\[ R^{h}_{12}(z + \tau) = \exp(-2\pi \hbar \partial_{z} \tau) (\Lambda^{-1} \otimes 1) R^{h}_{12}(z)(\Lambda \otimes 1), \]
\[ R^{h+\tau}_{12}(z) = R^{h}_{12}(z), \]
\[ R^{h+1}_{12}(z) = R^{h}_{12}(z), \]
\[ r_{12}(z + 1) = (Q^{-1} \otimes 1) r_{12}(z)(Q \otimes 1), \]
\[ r_{12}(z + \tau) = (\Lambda^{-1} \otimes 1) r_{12}(z)(\Lambda \otimes 1) - 2\pi \i 1 \otimes 1. \]

Let us also rewrite (1.39) as follows:
\[ R^{h+1/N}_{ab}(z_{a} - z_{b}) = Q^{-1}_{a} R^{h}_{ab}(z_{a} - z_{b}) Q_{b}, \]
\[ R^{h+\tau/N}_{ab}(z_{a} - z_{b}) = \exp(-2\pi \i \frac{z_{a} - z_{b}}{N}) \Lambda^{-1}_{ab} R^{h}_{ab}(z_{a} - z_{b}) \Lambda_{b}. \]

Recall now the \( R \)-matrix valued Lax matrix for \( g_{\tilde{N}} \) Calogero-Moser model \([8]\):
\[ \mathcal{L}(h) = \sum_{a,b=1}^{\tilde{N}} \tilde{E}_{ab} \otimes \mathcal{L}_{ab}(h), \quad \mathcal{L}_{ab}(h) = \delta_{ab} p_{a} 1_{a} \otimes 1_{b} + \nu (1 - \delta_{ab}) R^{h}_{ab}(z_{a} - z_{b}). \]
(1.45)

where \( \tilde{E}_{ab} \) is the standard basis of \( g_{\tilde{N}} \): \( \tilde{E}_{ab} \) \( cd = \delta_{ac} \delta_{bd}, a, b, c, d = 1...\tilde{N} \). Then it follows from (1.43)-(1.44) that

\[ \mathcal{L}(h + 1/N) = Q^{-1} \mathcal{L}(h) Q, \]
(1.46)

\[ \mathcal{L}(h + \tau/N) = \exp(-Z/N) \Lambda^{-1} \mathcal{L}(h) \Lambda \exp(Z/N), \]

where
\[ Q = \bigoplus_{a=1}^{\tilde{N}} Q_{a}, \quad \Lambda = \bigoplus_{a=1}^{\tilde{N}} \Lambda_{a}, \quad Z = \bigoplus_{a=1}^{\tilde{N}} z_{a} 1_{a} \]

are block diagonal matrices. The number of blocks is \( \tilde{N} \times \tilde{N} \), the size of a block is \( N\tilde{N} \times N\tilde{N} \).
• Heat equation:
\[2\pi i \partial_z R^{h}_{12}(z) = \partial_z \partial_h R^{h}_{12}(z).\]  
(1.48)

• Derivative:\[3\]:
\[\partial_h R^{h}_{12}(z) = \frac{1}{2} \left( r_{12}(z + Nh) R^{h}_{12}(z) + R^{h}_{12}(z) r_{12}(z - Nh) \right) \]
\[+ \frac{N}{2} \left( E_1(z + Nh) - E_1(z - Nh) - 2E_1(Nh) \right) R^{h}_{12}(z),\]  
(1.49)
\[\partial_z R^{h}_{12}(z) = \frac{1}{2N} \left( r_{12}(z + Nh) R^{h}_{12}(z) - R^{h}_{12}(z) r_{12}(z - Nh) \right) \]
\[+ \frac{1}{2} \left( E_1(z + Nh) + E_1(z - Nh) - 2E_1(z) \right) R^{h}_{12}(z).\]  
(1.50)

• The Fay identity in Mat(N, \mathbb{C})^{\otimes 3} [1, 13, 8]:
\[R^{h}_{ab} R^{h'}_{bc} = R^{h'}_{ac} R^{h-h'}_{ab} + R^{h-h'}_{bc} R^{h}_{ac}, \quad R^{h}_{ab} = R^{h}_{ab}(z_a - z_b).\]  
(1.51)
Both parts of the identity are elements of Mat(N, \mathbb{C})^{\otimes 3}. It was used in [8] for constructing higher-dimensional Lax pairs for Calogero-Moser models. Here we will prove another analogue of (1.15) – in Mat(N, \mathbb{C})^{\otimes 2}.

• The Fay identity in Mat(N, \mathbb{C})^{\otimes 2}:
\[R^{h}_{12}(z) R^{h'}_{21}(-w) = \]
\[N \phi(N h', \frac{z-w}{N}) + h' - h) R^{h-h'}_{12}(z + Nh') - N \phi(N h, \frac{z-w}{N}) + h' - h) R^{h-h'}_{12}(w + Nh) \]
\[+ N \phi(-w, \frac{z-w}{N}) + h' - h) R^{h-h'}_{12}(w + Nh) - N \phi(-z, \frac{z-w}{N}) + h' - h) R^{h-h'}_{12}(z + Nh').\]  
The scalar analogue of this identity is obtained as follows: apply (1.15) (with \(x = h, \ y = h'\)) to \(\phi(h, z)\phi(h', -w)\), and then apply (1.15) once again to the obtained r.h.s. Then we get the scalar analogue of r.h.s. of (1.52).

• Degenerated Fay identities in Mat(N, \mathbb{C})^{\otimes 3} [1, 13, 8]:
\[R^{h}_{ab} R^{h}_{bc} = R^{h}_{ac} r_{ab} + r_{bc} R^{h}_{ac} - \partial_h R^{h}_{ac}, \]
(1.53)
\[R^{h}_{ab}(z) R^{h'}_{bc}(-z) = R^{h'}_{ac} R^{h-h'}_{ab}(z) + R^{h-h'}_{bc}(-z) R^{h}_{ac} + N P_{bc}^{h-h}(z) P_{ac}, \]
(1.54)
where \(P^{h}_{ab}(u) = \partial_u R^{h}_{ab}(u)\) and \(R^{h}_{ab}(0)\) is from (1.31)-(1.35).

\(\text{The identities for derivatives of } R\text{-matrix with respect to the Planck constant and spectral parameter were found in [3] and [15] respectively. Authors of [3] [15] used different normalization of the } R\text{-matrix.}\)
The Painlevé VI equation in the elliptic form \[12\] is follows from the nondegenerated one (1.51) and local expansions (1.30), (1.34), (1.49), (1.50) were obtained in \[3, 15\]. Degenerated Fay identities (1.53), (1.54) in Mat(\(\mathbb{C}\)) follows from the unitarity condition (1.27) in the classical limit (1.30). Identities for derivatives "arguments symmetry" property (1.29) and the scalar Fay identities (1.15)-(1.17).

Degenerated Fay identities in Mat(\(\mathbb{C}\))

\[
R_{12}^h(z)R_{21}^h(-w) = N\phi(z, w, Nh) (r_{12}(z + Nh) - r_{12}(w + Nh))
\]

\[
+ N\phi(z, w, Nh) R_{12}^h (z + Nh) - N\phi(z, w, N) R_{12}^h (w + Nh)
\]

\[
+ N^2 1 \otimes 1 \phi(z, w, Nh) (E_1(Nh) - E_1(Nh + z - w))
\]

and

\[
R_{12}^h(z)R_{21}^{h'}(-z) = N\phi(h' - h, -z) (r_{12}(z + Nh) - r_{12}(z + Nh'))
\]

\[
- N\phi(h' - h, Nh) R_{12}^{h-h'} (z + Nh) + N\phi(h' - h, Nh') R_{12}^{h-h'} (z + Nh')
\]

\[
+ N^2 1 \otimes 1 \phi(h' - h, -z) (E_1(z) - E_1(z + h - h'))
\]

- Geometric interpretation. Due to the quasi-periodicities (1.38)-(1.41) the R-matrix have the following geometrical interpretation. Let \(V_1 (V_2)\) be a rank \(N\) and degree one vector bundle over elliptic curve \(\Sigma^{(1)}\) with coordinate \(z_1 (\Sigma^{(2)}_r\) with coordinate \(z_2)\). Consider the bundle \(V_1 \otimes V_2\) over \(\Sigma^{(1)}_r \times \Sigma^{(2)}_r\). Let \(Aut_{\text{PGL}(N)}(V_1 \otimes V_2)\) be the automorphism group of the bundle (the gauge group). The sections \(\Gamma(Aut_{\text{PGL}(N)}(V_1 \otimes V_2))\) depends only on the anti-diagonal \(\tilde{\Sigma}_r\) of \(\Sigma^{(1)}_r \times \Sigma^{(2)}_r\) with the coordinate \(z = z_1 - z_2\). Let \(\tilde{\Sigma}_r'\) be the dual curve, \(h\) is the coordinate on \(\tilde{\Sigma}_r\) and \(P\) is the Poincaré bundle \(P\) over \(\tilde{\Sigma}_r\) \(\oplus \tilde{\Sigma}_r'\). Then the R-matrix \(1.24\) is a section

\[
R_{12}^h(z) \in \Gamma ((\text{Aut}_{\text{PGL}(N)}(V_1 \otimes V_2)) \otimes P)
\]

- Green function. Similarly to \(1.19\) the R-matrix can be considered as the Green function of \(\bar{\partial}\)-operator:

\[
\bar{\partial}R_{12}^h(z) = NP_{12}\delta^2(z, \bar{z})
\]

Properties (1.30)-(1.48) simply follows from their scalar counterparts except (1.33) which follows from the unitarity condition (1.27) in the classical limit (1.30). Identities for derivatives (1.49), (1.50) were obtained in \[3, 15\]. Degenerated Fay identities (1.53), (1.54) in Mat(\(N, \mathbb{C}\))\(\otimes^3\) follows from the nondegenerated one (1.51) and local expansions (1.30), (1.34).

Our main interest (in this paper) is the Fay identity in Mat(\(N, \mathbb{C}\))\(\otimes^2\) (1.52) and its degenerations (1.55), (1.56). We prove them below. The computational trick is based on the "arguments symmetry" property (1.29) and the scalar Fay identities (1.15)-(1.17).

Painlevé VI. As an application of the obtained formulae we construct higher-dimensional Lax pairs for the Painlevé VI equation. Denote the half-periods of the elliptic curve \(\Sigma_r\) as

\[
\{\Omega_a, a = 0, 1, 2, 3\} = \{0, \frac{1}{2}, \frac{1 + \tau}{2}, \frac{\tau}{2}\}
\]

The Painlevé VI equation in the elliptic form \[12\] is

\[
\frac{d^2 u}{d\tau^2} = -\sum_{a=0}^{3} \nu_a^2 \psi'(u + \Omega_a)
\]
Let $N$ be an odd (positive) integer. Consider the following pair of block-matrices:

$$L(h) = \frac{1}{2} \frac{du}{d\tau} \begin{pmatrix} 1 \otimes 1 & 0 \\ 0 & -1 \otimes 1 \end{pmatrix} + \sum_{a=0}^{3} \frac{\nu_a}{N\sqrt{-2}} \begin{pmatrix} 0 & \mathcal{R}_{12}^{h,a}(u) \\ \mathcal{R}_{21}^{h,a}(-u) & 0 \end{pmatrix}$$

(1.60)

$$M(h) = \sum_{a=0}^{3} \frac{\nu_a}{N\sqrt{-2}} \begin{pmatrix} 0 & \mathcal{F}_{12}^{h,a}(u) \\ \mathcal{F}_{21}^{h,a}(u) & 0 \end{pmatrix}$$

(1.61)

where

$$\mathcal{R}_{12}^{h,a}(u) = \exp(2\pi i N h \partial_\tau \Omega_a) R_{12}^{h}(u + N \Omega_a),$$

(1.62)

$$\mathcal{R}_{21}^{h,a}(-u) = \exp(-2\pi i N h \partial_\tau \Omega_a) R_{21}^{h}(-u - N \Omega_a),$$

and

$$\mathcal{F}_{12}^{h,a}(u) = \exp(2\pi i N h \partial_\tau \Omega_a) F_{12}^{h}(u + N \Omega_a),$$

(1.63)

$$\mathcal{F}_{21}^{h,a}(-u) = \exp(-2\pi i N h \partial_\tau \Omega_a) F_{21}^{h}(-u - N \Omega_a)$$

with

$$F_{ab}^{h}(u) = \partial_u R_{ab}^{h}(u).$$

(1.64)

The matrices $L(h), M(h) \in \text{Mat}(2, \mathbb{C}) \otimes \text{Mat}(N, \mathbb{C})^{\otimes 2}$. Their size equals $2N^2 \times 2N^2$. The Painlevé VI equation (1.59) is equivalent to the monodromy preserving equation

$$\frac{d}{d\tau} L(h) - \left( \frac{1}{2\pi i} \right) \frac{d}{dh} M(h) = [L(h), M(h)],$$

(1.65)

where the Planck constant $\hbar$ plays the role of the spectral parameter (see [8]).

For $N = 1$ the answer (1.60), (1.61) reproduces the elliptic $2 \times 2$ Lax pair proposed in [17].

The Lax pair (1.60), (1.61) works for even $N$’s as well. But the Painlevé equation in this case has only one free constant:

$$\frac{d^2 u}{d\tau^2} = -\nu^2 \psi'(u), \quad \nu^2 = \sum_{a=0}^{3} \nu_a^2.$$  

(1.66)

2 \quad **Kronecker double series and Baxter-Belavin $R$-matrix**

Following idea suggested in [13] we derive here the Baxter-Belavin $R$-matrix as generalization of the Kronecker series.

$R$-matrix in Jacobi variables. Represent the elliptic curve $\Sigma_\tau$ (1.4) in the Jacobi form

$$C_q = \mathbb{C}/q^\mathbb{Z}, \quad q = e(\tau) = \exp 2\pi i \tau.$$  

Consider the product $C_q \times C_q$ with the coordinates $s = e(u), t = e(z)$. Instead of the Kronecker function $\phi(z, u)$ we consider the distribution $g(s, t)$ on the space of the Laurent polynomials $\mathbb{C}[[s^{-1}, t^{-1}, s, t]]$. For $|q| < |t| < 1$ it can be represented as the series

$$g(s, t| q) = \sum_{n \in \mathbb{Z}} \frac{t^n}{q^n s - 1}.$$  

(2.1)

\[\footnote{The coefficient $1/\sqrt{-2}$ gives the normalization of the constants as in (1.59).}\]
If simultaneously \(|q| < |s| < 1\) then
\[
g(s, t| q) = -g^+(s, t| q) + g^-(s, t| q), \quad g^+(s, t| q) = \sum_{i,n \geq 0} s^i q^i t^n, \quad g^-(s, t| q) = \sum_{i,n < 0} s^i q^i t^n \tag{2.2}
\]
or
\[
g(s, t| q) = 1 - \frac{1}{1-t} - \frac{1}{1-s} + g^-(s, t) - \sum_{i,n > 0} s^i q^i t^n. \tag{2.3}
\]
In the domain \(|q| < |t| < 1\) and \(|q| < |s| < 1\) we have
\[
g(s, t| q) = \frac{1}{2\pi i} \ln u, \quad t = \frac{1}{2\pi i} \ln z = \phi(z, u). \tag{2.4}
\]
The distribution \(g(s, t| q)\) has the properties analogous to (1.6)-(1.9). In particular,
\[
g(s, t| q) = g(t, s| q). \tag{2.5}
\]
It follows from (2.2) that
\[
g(s^{-1}, t^{-1}| q) = -g(s, t| q) + \delta(t) + \delta(s) - 2, \tag{2.6}
\]
where \(\delta(s)\) is the distribution on the space of the Laurent polynomials
\[\mathbb{C}[t, t^{-1}] = \{\psi(t) = \sum t^i \alpha_i\},\]
defined by the functional \(\langle \delta, \psi \rangle = Res|_{t=0} \psi(t)\) and represented by the formal series
\[
\delta(t) = \sum_{n \in \mathbb{Z}} t^n. \tag{2.7}
\]
The analog of the quasiperiodic property (1.11) is the following. The distribution \(g(s, t)\) is a solution of the difference equation on \(t\) (the Green function) variable
\[
sg(s, t| q) - g(s, t| q) = \delta(t) - 1. \tag{2.8}
\]
It defines the continuation of \(g(s, t| q)\) from the annulus \(|q| < |t| < 1\) to \(\mathbb{C}^\ast\). Due to (2.5) the similar equation can be written with respect to the \(s\) variable.

Let \(\eta = e(h)\). The \(R\)-matrix (1.24) takes the following form in variables \((s, t, \eta)\):
\[
R^h_{12}(s) = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} s^{\alpha_2/N} g(s, \omega_{\alpha} + h) T_{\alpha} \otimes T_{-\alpha} = \sum_{\alpha \in \mathbb{Z}_N \times \mathbb{Z}_N} s^{\alpha_2/N} \left( \sum_{\alpha, \beta} e(n \alpha_1/N) q^{(m+\alpha_2/N)\eta} s^m \right) T_{\alpha} \otimes T_{-\alpha}. \tag{2.9}
\]
It plays the role of the Green function for the difference operator
\[
\eta (\Lambda \otimes 1) R^h_{12}(s q) (\Lambda^{-1} \otimes 1) - R^h_{12}(s) = (\delta(s) - 1) P_{12}. \tag{2.10}
\]

**Kronecker double series [16]**

The distribution \(g(s, t| q)\) (and \(\phi(z, u)\)) can be represented as a Kronecker double series. Consider the lattice in \(\mathbb{C}\)
\[
W = \{\gamma = m + n\tau, \ m, n \in \mathbb{Z}\}.
\]
Represent the argument $u$ of $\phi(z, u)$ as $u = u_1 + u_2 \tau$ ($u_1, u_2$ are real), and let

$$\chi_u(\gamma) = e(-mu_2 + nu_1)$$

be a character of the lattice $W$ ($\chi_u(\gamma) : W \rightarrow S^1$), parameterized by $u \in \Sigma\tau$. The Kronecker double series is defined as:

$$S(z, u|\tau) = \sum_{\gamma \in W} \frac{\chi_u(\gamma)}{z + \gamma}.$$  \hspace{1cm} (2.11)

From the definition we find that

$$S(z + 1, u|\tau) = e(u_2)S(z, u|\tau),$$

$$S(z + \tau, u|\tau) = e(-u_1)S(z, u|\tau).$$  \hspace{1cm} (2.12)

It was proved in \cite{16} that $S(z, u|\tau)$ is related to the Kronecker function as

$$S(z, u|\tau) = e(u_2z)\phi(z, u),$$  \hspace{1cm} (2.13)

or in the Jacobi coordinates

$$S(t, s|q) = t^{u_2}g(s, t|q).$$  \hspace{1cm} (2.14)

Let us now pass to the $R$-matrix and describe it in terms of the Kronecker double series $S(z, u|\tau)$ (2.11).

Define the lattice $W$ by the two generators $(\alpha_1/N + h_1, (\alpha_2/N + h_2)\tau)$, where $h = h_1 + h_2\tau$, $h_{1,2} \in \mathbb{R}$. The corresponding character of $W$ is

$$\chi_{(m,n)}(\alpha, h) = e\left(-m(\alpha_2/N + h_2) + n(\alpha_1/N + h_1)\right).$$  \hspace{1cm} (2.15)

Then the $R$-matrix (1.24) is defined in terms of the Kronecker double series (2.11) as

$$R^h_{12}(z) = e(-h_2z) \sum_{(m,n) \in \mathbb{Z}\otimes\mathbb{Z}} \frac{\chi_{(m,n)}(\alpha, h) T_\alpha \otimes T_{-\alpha}}{z + m + n\tau}. \hspace{1cm} (2.16)$$

The quasi-periodicities (1.40), (1.41) now become evident. It follows from (2.13) that the singular behavior $z, h \rightarrow 0$ of this representation is in agreement with (1.36).

We pass from $R^h_{12}(z)$ to the modified matrix

$$\tilde{R}^h_{12}(z) = e(h_2z)R^h_{12}(z).$$

It satisfies the Yang-Baxter equation and has the quasi-periodicities

$$\tilde{R}^h_{12}(z + 1) = e(h_2)(Q^{-1} \otimes 1)\tilde{R}^h_{12}(z)(Q \otimes 1),$$

$$\tilde{R}^h_{12}(z + \tau) = e(h_1) (\Lambda^{-1} \otimes 1)\tilde{R}^h_{12}(z)(\Lambda \otimes 1),$$

(compare with (1.40)). In contrast with (1.41) $\tilde{R}$ is not holomorphic in $h$ and is double-periodic.
Remark 1 The representation (2.16) means that the elliptic $\tilde{R}$-matrix is represented as the averaging of the Yang matrix $z^{-1}P_{12}$ along the lattice $W$ twisted by the character (2.15).

From (1.30) we also find the representation for the classical $r$-matrix:

$$r_{12}(z) = E_1(z) 1 \otimes 1 + \sum_{m,n \in (Z \oplus Z) \setminus (0,0)} \frac{\sum_{\alpha \in Z_N \times Z_N} \chi_{(m,n)}(\alpha, 0) T_\alpha \otimes T_{-\alpha}}{z + m + n\tau} \frac{1}{z + m + n\tau}$$

and

$$m_{12}(z) = \frac{E_2^2(z) - \varphi(z)}{2} 1 \otimes 1 + \sum_{m,n \in (Z \oplus Z) \setminus (0,0)} \frac{\sum_{\alpha \in Z_N \times Z_N} (z + m + n\tau) \chi_{(m,n)}(\alpha, 0) T_\alpha \otimes T_{-\alpha}}{(z + m + n\tau)(\bar{\tau} - \tau)}.$$

3 Derivation of identities

Proposition 3.1 The $R$-matrix (1.24) satisfies the arguments symmetry property (1.29).

Proof: Using definitions (1.28) and (1.23) we have

$$R_{12}^N(N\hbar)P_{12} = \frac{1}{N} \sum_{\alpha,\beta} T_\alpha T_\beta \otimes T_{-\alpha} T_{-\beta} \varphi_\alpha(N\hbar, \omega_\alpha + \frac{z}{N})$$

$$= \frac{1}{N} \sum_{\alpha,\beta} \kappa_{\alpha,\beta}^2 T_{\alpha + \beta} \otimes T_{-\alpha - \beta} \varphi_\alpha(N\hbar, \omega_\alpha + \frac{z}{N}).$$

(3.1)

Since $\kappa_{\alpha,\beta} = \kappa_{\alpha,\alpha + \beta}$, the property (1.29) is equivalent to the following set of $N^2$ identities:

$$\frac{1}{N} \sum_\alpha \kappa_{\alpha,\gamma}^2 \varphi_\alpha(N\hbar, \omega_\alpha + \frac{z}{N}) = \varphi_\gamma(z, \omega_\gamma + \hbar), \ \forall \gamma \in \mathbb{Z}^\times 2$$

(3.2)

or

$$\frac{1}{N} \sum_\alpha \kappa_{\alpha,\gamma}^2 \varphi_\alpha(z, \omega_\alpha + \hbar) = \varphi_\gamma(N\hbar, \omega_\gamma + \frac{z}{N}), \ \forall \gamma \in \mathbb{Z}^\times 2.$$

(3.3)

The latter is verified by comparing residues. To do it we also need the relation for the sums of $N$-th roots of 1 (it also follows from $P_{12}^2 = 1$):

$$\sum_\alpha \kappa_{\alpha,\gamma}^2 = N^2 \delta_{\gamma,0}.$$

(3.4)

Let us calculate the residue of both parts of (3.2) at $\hbar = -\omega_\mu$. The answer for the r.h.s. is obviously $\delta_{\mu,\gamma} \exp(2\pi i \partial_\tau \omega_\gamma z)$ due to (1.8). For the l.h.s. we have:

$$\text{Res}_{\hbar = -\omega_\mu} \frac{1}{N} \sum_\alpha \kappa_{\alpha,\gamma}^2 \varphi_\alpha(N\hbar, \omega_\alpha + \frac{z}{N}) = \text{Res}_{\hbar = 0} \frac{1}{N} \sum_\alpha \kappa_{\alpha,\gamma}^2 \varphi_\alpha(N\hbar - N\omega_\mu, \omega_\alpha + \frac{z}{N})$$

$$= \frac{1}{N} \sum_\alpha \kappa_{\alpha,\gamma}^2 \exp(2\pi i \partial_\tau \omega_\mu z) \varphi_\alpha(N\hbar, \omega_\alpha + \frac{z}{N}).$$

(3.5)

$$\frac{1}{N} \sum_\alpha \kappa_{\alpha,\gamma}^2 \exp(2\pi i \partial_\tau \omega_\mu z) \frac{1}{N} \delta_{\mu,\gamma} \exp(2\pi i \partial_\tau \omega_\mu z). \blacksquare$$
Proposition 3.2 The R-matrix \((1.24)\) satisfies the Fay identity \((1.52)\) in Mat\((N, \mathbb{C})^{\otimes 2}\).

Proof: Consider

\[
R_{12}^h(z)R_{21}^{h'}(-w) = -\sum_{\alpha, \beta} \kappa_{\alpha, \beta}^2 T_{\alpha+\beta} \otimes T_{-\alpha-\beta} \varphi_{\alpha}(z, \omega_{\alpha} + h) \varphi_{\beta}(w, \omega_{\beta} - h') = \tag{3.6}
\]

Here we already used \(R_{21}^{h'}(-w) = -R_{12}^{-h'}(w)\). Apply the Fay identity \((1.15)\), then \((3.3)\), and then \((1.15)\) again:

\[
= -\sum_{\alpha, \beta} \kappa_{\alpha, \beta}^2 T_{\alpha+\beta} \otimes T_{-\alpha-\beta} \varphi_{\alpha}(z - w, \omega_{\alpha} + h) \varphi_{\alpha+\beta}(w, \omega_{\alpha+\beta} + h - h')
\]

\[
- \sum_{\alpha, \beta} \kappa_{\alpha, \beta}^2 T_{\alpha+\beta} \otimes T_{-\alpha-\beta} \varphi_{\beta}(w - z, \omega_{\beta} - h') \varphi_{\alpha+\beta}(z, \omega_{\alpha+\beta} + h - h')
\]

\[
= -N \sum_{\gamma} T_{\gamma} \otimes T_{-\gamma} \varphi_{\gamma}(Nh, \omega_{\gamma} + \frac{z - w}{N}) \varphi_{\gamma}(w, \omega_{\gamma} + h - h') + N \sum_{\gamma} T_{\gamma} \otimes T_{-\gamma} \varphi_{\gamma}(Nh', \omega_{\gamma} + \frac{z - w}{N}) \varphi_{\gamma}(z, \omega_{\gamma} + h - h')
\]

\[
= N \sum_{\gamma} T_{\gamma} \otimes T_{-\gamma} \left(-\phi(Nh, \frac{z - w}{N} + h' - h) \varphi_{\gamma}(w + Nh, \omega_{\gamma} + \frac{z - w}{N}) + \phi(Nh', \frac{z - w}{N} + h' - h) \varphi_{\gamma}(z + Nh', \omega_{\gamma} + \frac{z - w}{N}) + \phi(z, h - h' - \frac{z - w}{N}) \varphi_{\gamma}(z + Nh', \omega_{\gamma} + \frac{z - w}{N})\right). \tag{3.9}
\]

Proposition 3.3 The R-matrices \((1.24)\) and \((1.31)\) satisfies the degenerated Fay identities \((1.52)\), \((1.56)\) in Mat\((N, \mathbb{C})^{\otimes 2}\).

Proof: We begin with \((1.55)\). Consider

\[
R_{12}^h(z)R_{21}^h(-w) = -\sum_{\alpha, \beta} \kappa_{\alpha, \beta}^2 T_{\alpha+\beta} \otimes T_{-\alpha-\beta} \varphi_{\alpha}(z, \omega_{\alpha} + h) \varphi_{\beta}(w, \omega_{\beta} - h). \tag{3.10}
\]

Subdivide it into two parts: \(\sum_{\alpha, \beta} = \sum_{\alpha \neq -\beta} + \sum_{\alpha = -\beta}\). The first part is transformed as in the previous Proposition (via \((1.15)\), then \((3.3)\), and then \((1.15)\) again)

\[
\sum_{\alpha \neq -\beta} = -\sum_{\alpha \neq -\beta} \kappa_{\alpha, \beta}^2 T_{\alpha+\beta} \otimes T_{-\alpha-\beta} \varphi_{\alpha}(z - w, \omega_{\alpha} + h) \varphi_{\alpha+\beta}(w, \omega_{\alpha+\beta})
\]

\[
- \sum_{\alpha \neq -\beta} \kappa_{\alpha, \beta}^2 T_{\alpha+\beta} \otimes T_{-\alpha-\beta} \varphi_{\beta}(w - z, \omega_{\beta} - h) \varphi_{\alpha+\beta}(z, \omega_{\alpha+\beta}) \tag{3.11}
\]
\begin{align}
= ... &= -N \phi \left( \frac{z - w}{N}, Nh \right) \sum_{\gamma \neq 0} T_{\gamma} \otimes T_{-\gamma} \varphi_{\gamma}(w + Nh, \omega_{\gamma}) \\
&
-N \phi \left( \frac{w - z}{N}, w \right) \sum_{\gamma \neq 0} T_{\gamma} \otimes T_{-\gamma} \varphi_{\gamma}(w + Nh, \omega_{\gamma} + \frac{z - w}{N}) \\
&
+N \phi \left( \frac{z - w}{N}, Nh \right) \sum_{\gamma \neq 0} T_{\gamma} \otimes T_{-\gamma} \varphi_{\gamma}(z + Nh, \omega_{\gamma}) \\
&
+N \phi \left( \frac{w - z}{N}, z \right) \sum_{\gamma \neq 0} T_{\gamma} \otimes T_{-\gamma} \varphi_{\gamma}(z + Nh, \omega_{\gamma} + \frac{z - w}{N})
\end{align}

(3.12)

By adding (and subtracting) scalar terms \((1 \otimes 1)\) to each line one obtains the first and the second lines of (1.55). The input to the scalar part should be summed up together with

\[\sum_{\alpha = -\beta} = 1 \otimes 1 \sum_{\alpha} \varphi_{\alpha}(z, \omega_{\alpha} + \hbar) \varphi_{\alpha}(-w, \omega_{\alpha} + \hbar)\]

\[= 1 \otimes 1 \sum_{\alpha} \varphi_{\alpha}(z - w, \omega_{\alpha} + \hbar) \left( E_{1}(z) - E_{1}(w) + E_{1}(\hbar + \omega_{\alpha}) - E_{1}(z - w + \hbar + \omega_{\alpha}) \right)\]

(3.13)

The latter expression is transformed via (3.3) for \(\gamma = 0\)

\[\sum_{\alpha} \varphi_{\alpha}(z - w, \omega_{\alpha} + \hbar) = N \phi(Nh, \frac{z - w}{N})\]

and its derivative (1.13), (1.14) with respect to \(\hbar\):

\[\sum_{\alpha} \varphi_{\alpha}(z - w, \omega_{\alpha} + \hbar) \left( E_{1}(z - w + \hbar + \omega_{\alpha}) - E_{1}(\hbar + \omega_{\alpha}) \right)\]

\[= N^{2} \phi(Nh, \frac{z - w}{N}) \left( E_{1}(Nh + \frac{z - w}{N}) - E_{1}(Nh) \right)\]

This finishes the proof of (1.55). The identity (1.56) can be derived similarly. Equivalently, (1.56) follows from (1.55) by using the properties (1.29) and (1.37).

\section{Higher-dimensional elliptic Lax pairs for Painlevé VI}

Different types of matrix-valued Lax pairs for Painlevé equations are known (see e.g. [7, 5, 10]). In this section we construct \(R\)-matrix valued generalization of the elliptic \(2 \times 2\) Lax pair suggested in [17].

\textbf{Proposition 4.1} The Painlevé VI equation in the elliptic form (1.59) is equivalent to the monodromy preserving equation (1.65) with the Lax pair (1.60)- (1.64) and the elliptic \(R\)-matrix (1.24) for odd \(N\).

\textit{Proof} is similar to the one given in [17] for the scalar \((N = 1)\) case. First, notice that \(\frac{d}{d\tau} L(h) = \frac{d}{d\tau} \partial_{\alpha} L(h) + \partial_{\tau} L(h),\) where the last term is the derivative by explicit dependence on \(\tau\). It is canceled out by \(\frac{1}{2\pi i} \frac{d}{dh} M(h)\) due to the heat equation (1.48) \(2\pi i \partial_{\tau} R_{bc}^{h,a}(u) = \partial_{h} F_{bc}^{h,a}(u)\).
The main statement which we need to verify is that for \( a \neq b \)
\[
[L^a, M^b] + [L^b, M^a] = 0 , \tag{4.2}
\]
i.e. the input to \([L(h), M(h)]\) comes only from \([L^a, M^a]\). Indeed, it follows from the unitarity condition (1.27) that
\[
\mathcal{R}_{12}^{h,a}(u)\mathcal{R}_{21}^{h,a}(-u) = R_{12}^h(u + N\Omega_a)R_{21}^h(-u - N\Omega_a) = N^2(\varphi(Nh) - \varphi(u + N\Omega_a)) . \tag{4.3}
\]
Differentiating (4.3) with respect to \( u \) we get
\[
\mathcal{F}_{12}^{h,a}(u)\mathcal{R}_{21}^{h,a}(-u) - \mathcal{R}_{12}^{h,a}(u)\mathcal{F}_{21}^{h,a}(-u) = -N^2 \varphi'(u + N\Omega_a) . \tag{4.4}
\]
This identity provides the equation of motion. Notice that in order to have all four constants \( N \) should be odd since \( \varphi'(u + N\Omega_a) = \varphi'(u + \Omega_a) \) in this case. If \( N \) is even then \( \varphi'(u + N\Omega_a) = \varphi'(u) \), and we have only one constant as in (1.66).

To prove (4.2) let us recall that in the scalar case this followed from
\[
\varphi_a(h, u + \Omega_a) f_b(h, -u - \Omega_b) - f_b(h, u + \Omega_b)\varphi_a(h, -u - \Omega_a) = 0 , \tag{4.5}
\]
where
\[
f_a(z, u + \Omega_a) = \exp(2\pi i \partial \Omega_a h)\partial_w \phi(h, w) |_{w = u + \Omega_a}
\]
is the scalar analogue of \( \mathcal{F}_{12}^{h,a}(u) \). The identity (4.5) appears from (1.10) and (1.10)-(1.11) as follows:
\[
\varphi_a(h, u + \Omega_a)\varphi_b(h, -u - \Omega_b) + \varphi_b(h, u + \Omega_b)\varphi_a(h, -u - \Omega_a) = \varphi_{a+b}(h, \Omega_a + \Omega_b)(2E_1(h) - E_1(h + \Omega_a - \Omega_b) - E_1(h + \Omega_b - \Omega_a)) . \tag{4.6}
\]
The r.h.s. of (4.6) is independent of \( u \). The derivative of (4.6) with respect to \( u \) gives (4.5).

Similarly to (4.6) it follows from the degenerated Fay identity (1.55) that
\[
\mathcal{R}_{12}^{h,a}(u)\mathcal{R}_{21}^{h,b}(-u) + \mathcal{R}_{12}^{h,b}(u)\mathcal{R}_{21}^{h,a}(-u) = N^2 1 \otimes 1 \varphi_{a+b}(N\hbar, \Omega_a + \Omega_b)(2E_1(N\hbar) - E_1(N\hbar + \Omega_a - \Omega_b) - E_1(N\hbar + \Omega_b - \Omega_a)) . \tag{4.7}
\]
It can be verified directly using (1.10)-(1.11) which can be re-written as
\[
\phi(z, w + \Omega_a) = \exp(-2\pi i z \partial \Omega_a)\phi(z, w - \Omega_a) .
\]

The r.h.s. of (4.7) is scalar and independent of \( u \). The derivative of (4.7) with respect to \( u \) gives
\[
\mathcal{F}_{12}^{h,a}(u)\mathcal{R}_{21}^{h,b}(-u) - \mathcal{R}_{12}^{h,a}(u)\mathcal{F}_{21}^{h,b}(-u) + \mathcal{F}_{12}^{h,b}(u)\mathcal{R}_{21}^{h,a}(-u) - \mathcal{R}_{12}^{h,b}(u)\mathcal{F}_{21}^{h,a}(-u) = 0 . \tag{4.8}
\]
This identity underlies (4.2). \( \blacksquare \)
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