Convergence rates for empirical barycenters in metric spaces: curvature, convexity and extendible geodesics

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Abstract

This paper provides rates of convergence for empirical barycenters of a Borel probability measure on a metric space under general conditions. Our results are given in the form of sharp oracle inequalities. Our main assumption connects ideas from metric geometry to the theory of empirical processes and is studied in two meaningful scenarios. The first one is a geometrical constraint on the underlying space referred to as $(k, \alpha)$-convexity, compatible with a positive upper curvature bound in the sense of Alexandrov. The second scenario considers the case of a nonnegatively curved space on which geodesics, emanating from a barycenter, can be extended. While not restricted to this setting, our results are discussed in the context of Wasserstein spaces.

1 Introduction

Given a metric space $(S, d)$, define $\mathcal{P}_2(S)$ as the set of Borel probability measures on $S$ such that

$$\int_S d(x_0, x)^2 dP(x) < +\infty,$$

for some (hence all) $x_0 \in S$. A barycenter of $P \in \mathcal{P}_2(S)$ (also called a Fréchet mean) is any element $b^* \in S$ such that

$$b^* \in \arg \min_{b \in S} \int_S d(x, b)^2 dP(x). \quad (1.1)$$

When it exists, a barycenter stands as a natural analog of the mean of a (square integrable) probability measure on $\mathbb{R}^d$. Extending the notion of mean value to the case of probability measures on spaces $S$ with no Euclidean (or Hilbert) structure has a number of applications ranging from Geometry and Optimal Transport to Statistics and Data Science (Villani, 2003, 2008; Ambrosio et al., 2008; Santambrogio, 2015; Cuturi and Peyré, 2018) and the context of abstract metric spaces provides a unifying framework encompassing many non-standard settings. Barycenters in metric spaces have been widely studied and are well understood in several scenarios including the case where $S$ has nonpositive curvature (see, for instance, Sturm, 2003) or more generally upper bounded curvature (Yokota, 2016, 2017) (Here and throughout, curvature of a metric spaces is understood in the sense of Alexandrov. Precise definitions and necessary background are reported in Appendix B). Significant contributions have also focused on the more delicate context of spaces $S$ with lower bounded curvature (Yokota, 2012; Ohta, 2012). Focus on this last context arises, in particular, from the increasing interest for questions related to optimal transport, where spaces of central importance are the Wasserstein...
spaces $(S, d) = (\mathcal{P}_2(E), W_2)$, for some auxiliary metric space $E$, where $W_2$ denotes the 2-Wasserstein metric. Indeed, provided $E$ has nonnegative curvature (which includes the important setting where $E$ is a Hilbert space), the space $(\mathcal{P}_2(E), W_2)$ is known to have nonnegative curvature (see Section 7.3 in Ambrosio et al., 2008, for the case where $E$ is a separable Hilbert space and Appendix A in Lott and Villani, 2009, for the case where $E$ is a compact Riemannian manifold).

The performance of estimators of barycenters in metric spaces is less fully understood. Two remarkable trends are however modifying rapidly the landscape. On the one hand, recent algorithmic breakthroughs in computational optimal transport (see, for instance, Cuturi, 2013; Benamou et al., 2015; Genevay et al., 2016, and the references therein) are opening the way for the efficient computation of barycenters in the context where $S = \mathcal{P}_2(E)$. From a more theoretical perspective, many contributions have improved our understanding of these algorithmic successes and the behavior of some statistical procedures in general (we refer to Bhattacharya and Patrangenaru, 2003; Kendall and Le, 2011; Le Gouic and Loubes, 2017; Altschuler et al., 2017; Dvurechensky et al., 2018; Bigot et al., 2018, for works in this direction).

Our contribution is in the spirit of this second trend, providing nonasymptotic rates of convergence for empirical barycenters defined as follows. Consider random variables $X_1, \ldots, X_n$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (the expectation under $\mathbb{P}$ will be denoted $\mathbb{E}$ in the sequel), independent and with common distribution $\mathbb{P}$. Let $\mathbb{P}_n$ be the associated empirical distribution defined by

$$P_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}.$$  \hfill (1.2)

Then, for some $B \subset S$, we define as an empirical barycenter of $\mathbb{P}$ any point $b_n \in \text{arg min}_{b \in B} \int_S d(x, b)^2 \, d\mathbb{P}_n(x) = \text{arg min}_{b \in B} \frac{1}{n} \sum_{i=1}^{n} d(X_i, b)^2$. \hfill (1.3)

Throughout the paper, both $B$ and the support of $\mathbb{P}$ are assumed to be of bounded diameter. The need for this restriction appears as a limitation of our proof techniques. We believe that this requirement can be removed via a truncation approach at the price of additional technicality. By avoiding these technicalities, we wish to isolate more clearly the main geometrical insights and hopefully account for some practical considerations as the minimization step over $S$ may be computationally unrealistic.

In Section 2, we provide general convergence results which rely, in particular, on the following assumption.

**Assump. (A1):** Let $\bar{b} \in B$ be such that

$$\bar{b} \in \text{arg min}_{b \in B} \int_S d(x, b)^2 \, d\mathbb{P}(x).$$

Then, there exists constants $K > 0$ and $\beta \in (0, 1]$ such that, for all $b \in B$,

$$d(b, \bar{b})^2 \leq K \left( \int_S (d(b, x)^2 - d(\bar{b}, x)^2) \, d\mathbb{P}(x) \right)^{\beta}.$$

Assumption (A1) is interesting in several respects. The reader familiar with the theory of empirical processes, will identify in the proof of Theorem 1 (see Remark 24) that this condition is instrumental in obtaining fast rates of convergence as it implies a so called Bernstein-type condition on the class of functions indexing our empirical process. In particular, this assumption may be understood in our context as an analog of the Mammen-Tsybakov low-noise assumption (Mammen and Tsybakov, 1999).
used in binary classification (see also section 5.2 in the survey by Boucheron et al., 2005).

In Section 3 we show that Assumption (A1) is strongly connected to geometrical properties of the underlying set $B$. As an illustration, we mention the following result establishing a strong connection between so called variance inequalities, in the flavour of Assumption (A1), and the curvature properties of the underlying space.

**Theorem** (Theorem 4.9 in Sturm, 2003). Given a complete metric space $(S, d)$, the following statements are equivalent.

(i) $S$ has globally nonpositive curvature;

(ii) (Variance inequality) For any $P \in \mathcal{P}_2(S)$, there exists a barycenter $b^* \in S$ of $P$ such that, for all $b \in S$,

$$d(b, b^*)^2 \leq \int_S (d(b, x)^2 - d(b^*, x)^2) \, dP(x).$$

The implication (i) $\Rightarrow$ (ii) provides a sufficient condition for (A1) to hold: If $(S, d)$ has nonpositive curvature, then (A1) holds at least for $B = S$, $K = 1$ and $\beta = 1$. Up to our knowledge, however, the interplay between the curvature properties of $B$, the convexity properties of the metric $d$ and variance inequalities in the style of (A1) remains partially understood for arbitrary $B \subset S$, $K > 0$ and $\beta \in (0, 1]$. In the spirit of Ohta (2007), Section 3 provides a contribution in that direction by identifying conditions, compatible in particular with a positive upper curvature bound, in which (A1) holds for other values of parameters $K$ and $\beta$.

Section 4 provides alternative conditions under which Assumption (A1) is satisfied. Contrary to the results of Section 3, results in Section 4 are compatible with spaces of nonnegative curvature (possibly unbounded from above) which is of strong interest for applications in the context of Wasserstein spaces. We show for instance, in Theorem 18, that a variance inequality in the style of (A1) holds provided all shortest paths, emanating from a barycenter of $P$, can be extended to a shortest path (extended so to remain a shortest path between endpoints) by the same factor $\lambda > 0$. Theorem 18, is further investigated in the context of Wasserstein spaces and we show in Theorem 19 that the extendibility of shortest paths between two measures in $S = \mathcal{P}_2(E)$ can be interpreted in terms of the strong convexity of Kantorovitch potentials.

Finally, proofs are reported in Section 5. Appendix A reports some bounds in expectation derived from our main results in deviation and Appendix B gathers some background from metric geometry used in Sections 3 and 4.

## 2 Convergence results

Let $(S, d)$ be a metric space. Let $P \in \mathcal{P}_2(S)$ and let $b^* \in S$ be a barycenter of $P$ as defined in (1.1). For a fixed subset $B \subset S$, define $b_n$ as in (1.3) and consider

$$\tilde{b} \in \arg \min_{b \in B} \int_S d(x, b)^2 \, dP(x),$$

where the dependence of $b_n$ and $\tilde{b}$ on $B$ is omitted in the notation. Throughout the paper, we consider the following assumptions.

**Assump. (A1):** There exists constants $K > 0$ and $\beta \in (0, 1]$ such that, for all $b \in B$,

$$d(b, \tilde{b})^2 \leq K \left( \int_S (d(b, x)^2 - d(\tilde{b}, x)^2) \, dP(x) \right)^\beta.$$
2 Convergence results

Assump. (A2): There exists constants $C > 0$ and $D > 0$ such that, for all $0 < \varepsilon \leq r \leq \rho := \text{diam}(B)$,

$$\log N(B \cap B(\bar{b}, r), d, \varepsilon) \leq \left(\frac{Cr}{\varepsilon}\right)^D.$$

For $A \subset S$ and $\varepsilon > 0$, notation $N(A, d, \varepsilon)$ stands for the $\varepsilon$-covering number of $A$ in $S$ defined as follows. First, define an $\varepsilon$-net for $A$ as a finite subset \( \{x_1, \ldots, x_m\} \subset S \) such that

$$A \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon),$$

where $B(x, \varepsilon) := \{y \in S : d(x, y) \leq \varepsilon\}$. Then, $N(A, d, \varepsilon)$ stands for the smallest integer $m$ such that there exists an $\varepsilon$-net $\{x_1, \ldots, x_m\}$ for $A$ in $S$. Next is the main result of this section.

**Theorem 1.** Suppose that the support of $P$, and the set $B$, have bounded diameter. Suppose that Assumptions (A1) and (A2) hold. Then for all $n \geq 1$ and all $t > 0$,

$$\int_S (d(b_n, x)^2 - d(\bar{b}, x)^2) \, dP(x) \leq A \cdot \max \left\{ \theta_n, \left(\frac{t}{n}\right)^{\frac{\beta}{2}}, \frac{t}{n} \right\},$$

with probability at least $1 - 2e^{-t}$, where

$$\theta_n = \begin{cases} 
    n^{-\frac{1}{2}} & \text{if } D < 2, \\
    (\log n)^{\frac{1}{2}} n^{-\frac{1}{2}} & \text{if } D = 2, \\
    n^{-\frac{1}{2} D} & \text{if } D > 2,
\end{cases}$$

and $A$ is an explicit constant given in the proof depending only on $K, \beta, C, D, \rho$ and $\max\{d(b, x) : b \in B, x \in \text{supp}(P)\}$.

As will be discussed in the next section, many relevant situations imply that Assumption (A1) holds with the exponent $\beta$ equal to 1. For readability, we include the following direct specialization of Theorem 1 to this case.

**Corollary 2.** Suppose that the support of $P$, and the set $B$, have bounded diameter. Suppose that Assumption (A1), with $\beta = 1$, as well as Assumption (A2) hold. Then for all $n \geq 1$ and all $t > 0$,

$$\int_S (d(b_n, x)^2 - d(\bar{b}, x)^2) \, dP(x) \leq A \cdot \max \left\{ \theta_n, \frac{t}{n} \right\},$$

with probability at least $1 - 2e^{-t}$, where

$$\theta_n = \begin{cases} 
    n^{-1} & \text{if } D < 2, \\
    (\log n)^{2} n^{-1} & \text{if } D = 2, \\
    n^{-\frac{1}{2} D} & \text{if } D > 2,
\end{cases}$$

where $A$ is as in Theorem 1.

**Remark 3.** Under Assumption (A1), and since $b_n \in B$ by construction, it follows immediately that

$$d(b_n, \bar{b})^2 \leq K \left( \int_S (d(b_n, x)^2 - d(\bar{b}, x)^2) \, dP(x) \right)^\beta,$$

so that Theorem 1, Corollary 2 and Theorem 4 (see below) provide also rates of convergence of $d(b_n, \bar{b})$ towards 0.
Note that more detailed results, namely Statements 25, 26 and 27, are provided in the proofs section. The bound of Theorem 1 may be translated into so called oracle inequalities involving the approximation properties of the chosen set $B$ relative to $b^*$. Indeed, Theorem 1, Remark 3 and the definition of $\bar{b}$, imply that inequalities

$$
\int_S (d(b_n, x)^2 - d(b^*, x)^2) \, dP(x) = \int_S (d(\bar{b}, x)^2 - d(b^*, x)^2) \, dP(x) + \int_S (d(b_n, x)^2 - d(\bar{b}, x)^2) \, dP(x)
$$

$$\leq \inf_{b \in B} \int_S (d(b, x)^2 - d(b^*, x)^2) \, dP(x) + \int_S (d(b_n, x)^2 - d(\bar{b}, x)^2) \, dP(x)
$$

and

$$
d(b_n, b^*) \leq d(\bar{b}, b^*) + d(b_n, \bar{b}) \leq d(\bar{b}, b^*) + K 2^{\frac{q}{2}} \max \left\{ \theta_n, \left( \frac{t}{n} \right)^{\frac{1}{\theta_n}}, \frac{t}{n} \right\} \theta
$$

both hold with probability at least $1 - 2e^{-t}$. The approximation terms on the right-hand side of these inequalities, namely

$$
\inf_{b \in B} \int_S (d(b, x)^2 - d(b^*, x)^2) \, dP(x)
$$

account for the misspecification of set $B$. In particular, both terms vanish if $b^* \in B$ since one may take $\bar{b} = b^*$ in this case by definition.

Assumption (A2) is imposing a rather mild complexity restriction on the neighbourhoods of $\bar{b}$ in $B$. This condition allows for a wide range of situations as discussed at the end of this section. Before discussing examples, we provide a result based on a even weaker assumption which, contrary to (A2), controls only the complexity of the set $B$ as a whole.

**Assump. (A3):** There exists constants $C > 0$ and $D > 0$ such that, for all $0 < \varepsilon \leq \rho := \text{diam}(B)$,

$$\log N(B, d, \varepsilon) \leq \left( \frac{C \rho}{\varepsilon} \right)^D.$$

Note indeed that Assumption (A2) implies Assumption (A3) but that the converse does not hold in general. The next result is in the spirit of Theorem 1.

**Theorem 4.** Suppose that the support of $P$, and the set $B$, have bounded diameter. Suppose that Assumptions (A1) and (A3) hold. Then for all $n \geq 1$ and all $t > 0$,

$$
\int_S (d(b_n, x)^2 - d(\bar{b}, x)^2) \, dP(x) \leq A^* \cdot \max \left\{ \theta_n, \left( \frac{t}{n} \right)^{\frac{1}{\theta_n}}, \frac{t}{n} \right\},
$$

with probability at least $1 - 2e^{-t}$, where

$$
\vartheta_n = \begin{cases} 
\frac{n^{-\frac{2}{\beta}}} {\log n} & \text{if } D < 2, \\
\frac{(\log n)/\sqrt{n}} {n} & \text{if } D = 2, \\
\frac{n^{-1/D}} {n} & \text{if } D > 2,
\end{cases}
$$

and $A^*$ is an explicit constant available in the proof depending only on $K, \beta, C, D, \rho$ and $\max\{d(b, x) : b \in B, x \in \text{supp}(P)\}.$
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Next, we provide a few examples in which Assumptions (A2) or (A3) are satisfied.

Example 5 (Finite dimensional sets). Suppose \((S, d)\) is metric space equipped with a Borel measure \(\mu\). Fix \(m > 0\) and denote \(\alpha_m : S \times (0, +\infty) \rightarrow [0, +\infty)\) the function defined, for any \(x \in S\) and any \(r > 0\), by

\[
\alpha_m(x, r) = \frac{\mu(B(x, r))}{r^m}.
\]  

(2.1)

Let \(B \subset S\) be bounded, denote \(\rho = \text{diam}(B)\), and suppose there exists constants \(0 < \alpha_- \leq \alpha_+ < +\infty\) such that, for all \(x \in B^\rho = \bigcup_{z \in B} B(z, \rho)\) and all \(r \in (0, 3\rho/2]\),

\[
\alpha_- \leq \alpha_m(x, r) \leq \alpha_+.
\]  

(2.2)

Then, for any \(x \in B\) and any \(0 < \varepsilon \leq r \leq \rho\),

\[
\frac{\alpha_-}{\alpha_+} \left( \frac{r}{\varepsilon} \right)^m \leq N(B(x, r), d, \varepsilon) \leq \frac{\alpha_+}{\alpha_-} \left( \frac{3r}{\varepsilon} \right)^m.
\]  

(2.3)

(See the proof in Section 5.) Therefore, given any \(x \in B\) and any \(0 < \varepsilon \leq r \leq \rho\), the set \(B \cap B(x, r)\) is obviously included in \(B(x, r)\) and we have

\[
\log N(B \cap B(x, r), d, \varepsilon) \leq \log \left( \frac{\alpha_+}{\alpha_-} \right) + m \log \left( \frac{3r}{\varepsilon} \right),
\]

so that Assumption (A2) is satisfied for all \(D > 0\) with a constant \(C\) depending on \(D\). Note that condition (2.2) is satisfied in particular if \(S\) is an \(m\)-dimensional smooth Riemannian manifold equipped with the geodesic distance \(d\) and the volume measure \(\mu = \text{vol}_m\) (\(m\)-dimensional Hausdorff measure).

Example 6 (Wasserstein spaces). Suppose \(S = \mathcal{P}_2(\mathbb{R}^d)\) is equipped with the Wasserstein metric \(W_2\) and \(B = \mathcal{P}_2(B(x_0, \rho))\) for some fixed \(x_0 \in \mathbb{R}^d\) and some fixed radius \(\rho > 0\). Combining the result of Appendix A in Bolley et al. (2007) with a classical bound on the covering number of Euclidean balls (see Example 5), it follows that for all \(0 < \varepsilon \leq \rho\),

\[
\log N(B, W_2, \varepsilon) \leq 2 \left( \frac{6\rho}{\varepsilon} \right)^d \log \left( \frac{8\rho}{\varepsilon} \right).
\]

In particular, for any \(D > d\), there exists \(C > 0\) depending on \(D\) such that, for all \(0 < \varepsilon \leq \rho\),

\[
\log N(B, W_2, \varepsilon) \leq \left( \frac{C\rho}{\varepsilon} \right)^D,
\]

so that Assumption (A3) is satisfied in this scenario. To have the stronger Assumption (A2) satisfied in this case requires that, for all \(0 < \varepsilon \leq r \leq \rho\),

\[
\log N(B \cap B(b, r), W_2, \varepsilon) \leq \left( \frac{C\rho}{\varepsilon} \right)^D,
\]

where it is understood here that \(B(b, r)\) is a ball in \(\mathcal{P}_2(\mathbb{R}^d)\) of radius \(r\) around \(b \in B\). The previous discussion therefore guarantees that a sufficient condition for this property to hold is that there exists \(C' > 0\) and points \(\{x_r : 0 < r \leq \rho\}\) in \(\mathbb{R}^d\) such that, for all \(0 < r \leq \rho\),

\[
B \cap B(b, r) \subset \mathcal{P}_2(B(x_r, C'r)).
\]

We leave the identification of bounded subsets \(B \subset \mathcal{P}_2(\mathbb{R}^d)\) for which this last (sufficient) condition holds as an interesting question for future work.
3 Convexity, curvature and variance inequality

This section investigates scenarios in which Assumption (A1) holds, connecting it to notions of convexity and curvature. Note that the results considered next rely on ideas from metric geometry outlined in Appendix B for convenience. We start with a few definitions.

Definition 7 (Convex set). Suppose that $(S, d)$ is a geodesic space. A subset $B \subset S$ is said to be convex if for any two points $x, y \in B$, and any (constant speed) shortest path $\gamma_{xy} : [0, 1] \to S$ joining $x$ to $y$, we have $\gamma_{xy}([0, 1]) \subset B$.

Definition 8 ($(k, \alpha)$-convex function). Suppose that $(S, d)$ is a geodesic space and $B \subset S$ is convex in the sense of Definition 7. Then, for $k \geq 0$ and $\alpha \geq 0$, a function $f : B \to \mathbb{R}$ is said $(k, \alpha)$-convex if for any two points $x, y \in B$, and any (constant speed) shortest path $\gamma_{xy} : [0, 1] \to S$ joining $x$ to $y$, the function

$$t \in [0, 1] \mapsto f(\gamma_{xy}(t)) - \frac{k}{2}t^{2}d(x, y)^{\alpha},$$

is convex on $[0, 1]$.

Note that, equivalently, the function $f : B \to \mathbb{R}$ is said $(k, \alpha)$-convex if for any two points $x, y \in B$, and any (constant speed) shortest path $\gamma_{xy} : [0, 1] \to S$ joining $x$ to $y$, we have

$$\forall t \in [0, 1], \quad f(\gamma_{xy}(t)) \leq (1 - t)f(x) + tf(y) - \frac{k}{2}t(1 - t)d(x, y)^{\alpha}.$$

Definition 9 ($(k, \alpha)$-convex set). Suppose that $(S, d)$ is a geodesic space. For $k \geq 0$, $\alpha \geq 0$ and $B \subset S$, we say that set $B$ is a $(k, \alpha)$-convex set if it is convex in the sense of Definition 7 and if, for all $p \in B$, the function $d(p, \cdot)^{2} : B \to [0, +\infty)$ is $(k, \alpha)$-convex in the sense of Definition 8.

Definition 9 is a slight generalisation of the definition of a $k$-convex set introduced by Ohta (2007). A $k$-convex set $B$, in the sense of Ohta (2007), is a $(k, 2)$-convex set in the sense of Definition 9. Hence, we simply give additional freedom through the introduction of parameter $\alpha \geq 0$. Note that, if $\text{diam}(B) \leq 1$ and $\alpha \geq 2$, every $(k, 2)$-convex set $B$ is also $(k, \alpha)$-convex. Hence $\alpha \geq 2$ introduces more flexibility for small sets compared to Ohta’s definition. On the contrary if $\text{diam}(B) > 1$ and $\alpha > 2$, $(k, \alpha)$-convexity is actually imposing a more restrictive condition on geodesics connecting points far apart so that there is a priori no possible comparison between $(k, 2)$-convex sets and $(k, \alpha)$-convex sets of diameter strictly larger than 1. Finally, note that when $\alpha < 2$, the concavity of $x \geq 0 \mapsto x^\alpha$ makes the notion of $(k, \alpha)$-convexity less relevant. The next result connects the notion of $(k, \alpha)$-convexity to Assumption (A1).

Theorem 10. If $B$ is $(k, \alpha)$-convex in the sense of Definition 9 for some $k > 0$ and some $\alpha \geq 2$, then Assumption (A1) holds with $K = 2/k$ and $\beta = 2/\alpha$.

As clearly pointed out by Ohta (2007) in the context of a $(k, 2)$-convex set, the $(k, \alpha)$-convexity of a set $B$ is connected to its curvature properties. For instance, observe that (see Appendix B) a geodesic space $(S, d)$ is said to have curvature bounded from above by 0 in the large or globally (in this case $S$ is also called a CAT(0) space or a global NPC space) if $S$ is a $(2, 2)$-convex set in the sense of Definition 9. To insist on the practical interest of the previous result, we therefore mention without proof the following.

Corollary 11. Assumption (A1), with $K = 1$ and $\beta = 1$, is satisfied for any convex subset of a CAT(0) (or global NPC) space such as a Banach space, the hyperbolic plane (or Poincaré disc), a riemannian manifold with sectional curvature upper bounded by 0 or any (metric) tree.

The next result, in the spirit of Proposition 3.1 in Ohta (2007), exhibits further the link between curvature and the notion of $(k, \alpha)$-convexity.
Theorem 12. Suppose that $B$ is convex, complete, with $d := \text{diam}(B) < \pi/2$, and with curvature upper bounded by 1 in the large (i.e. $B$ is a CAT(1) space). Then for any $\alpha \geq 2$, $B$ is $(k(\alpha), \alpha)$-convex for

$$k(\alpha) = \frac{8d}{\alpha^2} \tan \left( \frac{\pi}{2} - d \right).$$

Hence, for all $\alpha \geq 2$, Assumption (A1) is in this case satisfied for $K = 2/k(\alpha)$ and $\beta = 2/\alpha$.

We end the section with a result in the spirit of Theorem 4.9 in Sturm (2003) connecting $(k,2)$-convexity to a series of comparable statements.

Theorem 13. Let $(S, d)$ be a geodesic space and fix $k > 0$. Then we have $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$, where

(a) For any $P \in \mathcal{P}_2(S)$ having a barycentre $b^*$, and any $b \in S$,

$$d(b, b^*)^2 \leq \int_S d(x, b)^2 \, dP(x) - \frac{k}{2} \int_S d(x, b^*)^2 \, dP(x). \quad (3.1)$$

(b) $S$ is $(k, 2)$-convex.

(c) For any $P \in \mathcal{P}_2(S)$, any barycentre $b^*$ of $P$ and any $b \in S$,

$$\frac{k}{2} d(b, b^*)^2 \leq \int_S (d(x, b)^2 - d(x, b^*)^2) \, dP(x). \quad (3.2)$$

(d) For all $x, y, b \in S$, any constant speed shortest path $\gamma_{xy} : [0, 1] \to S$ from $x$ to $y$ and all $t \in [0, 1]$,

$$\frac{k}{2} d(b, \gamma_{xy}(t))^2 \leq (1 - t)d(b, x)^2 + td(b, y)^2 - t(1 - t)d(x, y)^2.$$

In addition, if $k \leq 2$ and if $d(b, b^*)^2 \leq \int_S d(x, b^*)^2 \, dP(x)$, then $(3.2) \Rightarrow (3.1)$.

This result expresses in particular that the $(k, 2)$-convexity of the space $S$ implies a variance inequality, with constant $k/2$, in the spirit of (1.4). For $k \leq 2$, the last line of the previous theorem also shows that if a variance inequality holds, then $S$ is "locally" $(k, 2)$-convex, in the sense that a small enough neighborhood of a barycenter will be $(k, 2)$-convex. It is not clear to us yet if this "local" restriction is necessary or not.

4 Extendible geodesics

This section provides a different perspective in the context where the ambient space $S$ has nonnegative curvature and builds upon the geometric properties of the tangent cone $T_b S$ of $S$ at a barycentre $b^*$ of $P \in \mathcal{P}_2(S)$. The reader unfamiliar with the notion of a tangent cone is referred to Appendix B which in particular reports essential definitions and basic properties of this delicate object.

Definition 14. Let $(S, d)$ be a geodesic space. For $x \in S$, denote $C(x)$ the set of points $y \in S$ such that there is strictly more than one constant speed shortest path connecting $x$ to $y$.

Note that $C(x) = \emptyset$ if $S$ is a global NPC space (i.e. has curvature upper bounded by 0 in the large). However, for spaces of nonnegative curvature, $C(x)$ may be nonempty. A notion, related to that of a barycentre, is that of an exponential barycentre (Émery and Mokobodzki, 1991) defined as follows.

Definition 15 (Exponential barycentre). Let $S$ be a global NNC space and $P \in \mathcal{P}_2(S)$. A point $b \in S$ is said to be an exponential barycentre of $P$ if

$$\int_S \int_S \langle \log_b(x), \log_b(y) \rangle_b \, dP(x) dP(y) = 0.$$
For the definition of \( \log_b : S \backslash C(x) \to T_bS \) and \( \langle \cdot, \cdot \rangle_b \), we refer the reader to Appendix B. The definition of an exponential barycentre mimics that of the Pettis-integral of a Hilbert-valued function and stands as one of the many possible ways to define the analog of the mean value of an element of \( \mathcal{P}_2(S) \). A useful result in our setting, Theorem 45 in Yokota (2012), states that if \( b \) is an exponential barycentre of \( P \in \mathcal{P}_2(S) \), and \( S \) is a global NNC space, then the linear hull of the subset \( \text{supp}(\log_{b,2} P) \) (the support of the image of \( P \) by \( \log_b \)), included in \( T_bS \), is isometric to a Hilbert space. This geometrical characterisation of the tangent cone at an exponential barycentre allows to conduct computations in a flat space, much easier to handle. As it turns out, any barycentre, in the sense considered so far, is also an exponential barycentre.

**Theorem 16.** Suppose that \((S, d)\) is a global NNC space and let \( P \in \mathcal{P}_2(S) \). Then if \( b^* \) is a barycentre of \( P \), it is an exponential barycentre of \( P \).

The next result exploits this last fact to provide a variance inequality in global NNC spaces.

**Theorem 17.** Suppose \((S, d)\) is a global NNC space. Let \( P \in \mathcal{P}_2(S) \). Let \( b^* \in S \) be a barycentre of \( P \). For all \( x, b \in \text{supp}(P) \setminus C(b^*) \), define

\[
  k_0(x, b, b^*) := 1 - \frac{||\log(x) - \log(b)||^2 - d(x, b)^2}{d(b, b^*)^2},
\]

where we have denoted \( \log := \log_{b^*} \) and \( \|\cdot\| := \|\cdot\|_{b^*} \). Then, we obtain

\[
  d(b, b^*)^2 \int_S k_0(x, b, b^*) \, dP(x) \leq \int_S (d(b, x)^2 - d(b^*, x)^2) \, dP(x).
\]

The previous inequality indicates that, provided

\[
  k_0 := \inf_{b \in \text{supp}(P)} \int_S k_0(x, b, b^*) \, dP(x) > 0,
\]

then Assumption (A1) holds with \( K = k_0^{-1}, \beta = 1 \) and \( B = S \). It is worth noticing that in the context where \( S \) is a global NPC space, the quantity \( k_0(x, b, b^*) \) can also be defined in a similar fashion and satisfies \( k_0(x, b, b^*) \geq 1 \). In particular, we recover in this case Proposition 4.4 in Sturm (2003). The next result provides a sufficient condition for \( k_0 > 0 \) to hold in a global NNC space.

**Theorem 18.** Let \((S, d)\) be a global NNC space. Let \( P \in \mathcal{P}_2(S) \) and let \( b^* \in S \) be a barycentre of \( P \). Fix \( \lambda > 0 \). Suppose that, for \( P \)-almost every \( x \) in \( S \), the following two properties holds.

- Any constant speed geodesic \( \gamma_x : [0, 1] \to S \) joining \( b^* \) to \( x \) can be extended to a function \( \gamma_x^+ : [0, 1 + \lambda] \to S \) that remains a shortest path between its endpoints.
- Let \( X \) be an \( S \)-valued random variable with distribution \( P \) and let \( P_\lambda \) be the distribution of \( \gamma_x^+(1 + \lambda) \) where \( \gamma_x^+ \) is as above. Then \( b^* \) is the unique barycentre of \( P_\lambda \).

Then, for all \( b \in S \),

\[
  d(b, b^*)^2 \leq \frac{1 + \lambda}{\lambda} \int_S (d(b_n, x)^2 - d(b^*, x)^2) \, dP(x),
\]

and Assumption (A1) holds for \( B = S \) with \( K = (1 + \lambda)/\lambda \) and \( \beta = 1 \).

Below, we investigate further the previous result in the context where \( S = \mathcal{P}_2(H) \) is the Wasserstein space over a Hilbert space \( H \). Consider a (separable) Hilbert space \( H \) with scalar product \( \langle \cdot, \cdot \rangle \) and associated norm \( \|\cdot\| \). For \( \alpha \geq 0 \), recall that a map \( \varphi : H \to \mathbb{R} \) is called \( \alpha \)-strongly convex if, for all \( x, y \in H \) and all \( t \in (0, 1) \),

\[
  \varphi((1 - t)x + ty) \leq (1 - t)\varphi(x) + t\varphi(y) - \frac{\alpha}{2}t(1 - t)\|x - y\|^2.
\]
A convex function corresponds to a 0-strongly convex function. Finally, if \( \varphi : H \to \mathbb{R} \) is convex, its subdifferential \( \partial \varphi \subset H^2 \) is defined by
\[
(x, y) \in \partial \varphi \iff \forall z \in H, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle.
\]

**Theorem 19.** Let \( S = \mathcal{P}_2(H) \) be the Wasserstein space over a separable Hilbert space \( H \). Let \( \mu \) and \( \nu \) be two elements of \( S \) and let \( \gamma : [0, 1] \to S \) be a constant speed shortest path joining \( \mu \) to \( \nu \) in \( S \). Then, \( \gamma \) can be extended by a factor \( 1 + \lambda \) (in the sense of the first point in Theorem 18) if and only if the support of the optimal transport plan \( \pi \) of \( (\mu, \nu) \) lies in the subdifferential \( \partial \varphi \) of a \( \frac{1}{1 + \lambda} \)-strongly convex function. Finally, if \( \beta \) is the gradient of the convex function
\[
\varphi(x) = \frac{1}{2} \langle x - m_\mu, A(x - m_\mu) \rangle + \langle x, m_\nu \rangle,
\]
which is \( \lambda_{\text{min}}(A) \)-strongly convex. It easily follows from this observation that the support of the optimal transport plan \( (\text{id} \otimes T) \pi \mu \) is supported on the subdifferential \( \partial \varphi = \{ (x, \nabla \varphi(x)) : x \in \mathbb{R}^n \} \) of the strongly convex \( \varphi \).

## 5 Proofs

### 5.1 Proof of Theorem 1

The proof of Theorem 1 is based on three auxiliary results. The first lemma below provides an upper bound on the largest fixed point of a random nonnegative function. The proof follows from a combination of arguments presented in Theorem 4.1, Corollary 4.1 and Theorem 4.3 in Koltchinskii (2011).

**Lemma 21** (Koltchinskii, 2011). Let \( \{ \varphi(\delta) : \delta \geq 0 \} \) be non-negative random variables (indexed by all deterministic \( \delta \geq 0 \)) such that, almost surely, \( \varphi(\delta) \leq \varphi(\delta') \) if \( \delta \leq \delta' \). Let \( \{ \beta(\delta, t) : \delta \geq 0, t \geq 0 \} \), be (deterministic) real numbers such that \( \beta(\delta, t) \leq \beta(\delta, t') \), as soon as \( t \leq t' \), and such that
\[
\mathbb{P}(\varphi(\delta) \geq \beta(\delta, t)) \leq e^{-t}.
\]

Finally, let \( \delta \) be a nonnegative random variable, a priori upper bounded by a constant \( \delta > 0 \), and such that, almost surely,
\[
\delta \leq \varphi(\delta).
\]

Then defining, for all \( t \geq 0 \),
\[
\beta(t) := \inf \left\{ \alpha > 0 : \sup_{\delta \geq \alpha} \frac{\beta(\delta, \alpha \delta)}{\delta} \leq 1 \right\},
\]
we obtain, for all \( t \geq 0 \),
\[
\mathbb{P}(\delta(\beta(t)) \leq 2e^{-t}.
\]
The second lemma is due to Bousquet (2002) and improves upon the work of Talagrand (1996) by providing explicit constants. Given a metric space \((S, d)\) and a family \(\mathcal{F}\) of functions \(f : S \to \mathbb{R}\), denote
\[
|P - P_n|_{\mathcal{F}} := \sup_{f \in \mathcal{F}} (P - P_n)f \quad \text{and} \quad \sigma^2_{\mathcal{F}} := \sup_{f \in \mathcal{F}} P(f - P f)^2.
\]

**Lemma 22** (Bousquet, 2002). Suppose that all functions in \(\mathcal{F}\) are \([a, b]\)-valued, for some \(a < b\). Then, for all \(n \geq 1\) and all \(t > 0\),
\[
|P - P_n|_{\mathcal{F}} \leq \mathbb{E}|P - P_n|_{\mathcal{F}} + \sqrt{\frac{2t}{n} \left( \sigma^2_{\mathcal{F}} + 2(b - a) \mathbb{E}|P - P_n|_{\mathcal{F}} \right)} + \frac{(b - a)t}{3n},
\]
with probability larger than \(1 - e^{-t}\).

For simplicity, we avoid discussions of measurability issues of the suprema involved in the previous result in the general setting. As detailed below, this issue is addressed in our case based on Assumption (A2). For background on empirical processes, including the proof of Lemma 22, we refer the reader to Giné and Nickl (2015). Finally, the third result we need is the following generalized version of Dudley’s entropy bound (see, for instance, Theorem 5.31 in van Handel, 2016).

**Lemma 23.** Let \((X_t)_{t \in E}\) be a real valued process indexed by a (pseudo) metric space \((E, d)\). Suppose that the three following conditions hold.

- (Separability) There exists a countable subset \(E' \subset E\) such that, for any \(t \in E\),
  \[
  X_t = \lim_{t' \to t, t' \in E'} X_{t'}, \quad \text{a.s.}
  \]

- (Subgaussian process) For all \(s, t \in E\), \(X_s - X_t\) is subgaussian in the sense that
  \[
  \forall \theta \in \mathbb{R}, \quad \log \mathbb{E} \exp(\theta \{X_s - X_t\}) \leq \frac{\theta^2 d(s, t)^2}{2}.
  \]

- (Lipschitz property) There exists a random variable \(C\) such that, for all \(s, t \in E\),
  \[
  |X_s - X_t| \leqCd(s, t), \quad \text{a.s.}
  \]

Then, for any \(T \subset E\) and any \(\varepsilon \geq 0\), we have
\[
\mathbb{E} \sup_{t \in T} X_t \leq 2\varepsilon \mathbb{E}|C| + 12 \int_{\varepsilon}^{+\infty} \sqrt{\log N(T, d, u)} \, du.
\]

We are now in position for proving Theorem 1.

**Proof of Theorem 1.** For any \(\delta \geq 0\), denote
\[
B(\delta) := \{b \in B : P(d(., b)^2 - d(., \bar{b})^2) \leq \delta\},
\]
and
\[
\varphi_n(\delta) := \sup\{(P - P_n)(d(., b)^2 - d(., \bar{b})^2) : b \in B(\delta)\}.
\]
Let us point out that, as a consequence of Assumption (A2), the set \(B\) is separable. Hence, the quantity \(\varphi_n(\delta)\) is measurable, as well as all suprema involved in the rest of the proof. Abbreviate,
\[
\delta_n := P(d(., b_n)^2 - d(., \bar{b})^2),
\]
for simplicity. By definition of \(b_n\),
\[
P_n(d(., b_n)^2 - d(., \bar{b})^2) \leq 0.
\]
Since, $$\delta_n \leq (P - P_n)(d(\cdot, b_n)^2 - d(\cdot, \bar{b})^2) \leq \varphi_n(\delta_n), \quad (5.1)$$

As a result, in order to provide an upper bound on $$\delta_n$$ holding with high probability, it is enough to provide an upper bound on $$\varphi_n(\delta)$$, for fixed $$\delta \geq 0$$, and apply Lemma 21. Denoting

$$\sigma^2(\delta) := \sup\{P(d(\cdot, b)^2 - d(\cdot, \bar{b})^2) : b \in B(\delta)\},$$

$$R := \max\{d(b, x) : b \in B, x \in \text{supp}(P)\}$$ and $$\rho := \text{diam}(B)$$, we get, for all $$b \in B$$ and all $$x \in \text{supp}(P),$$

$$|d(x, b)^2 - d(x, \bar{b})^2| \leq \rho := 2\rho R,$$

and it follows from Lemma 22 that inequality

$$\varphi_n(\delta) \leq E\varphi_n(\delta) + \sqrt{\frac{2t}{n}}(\sigma^2(\delta) + 4\rho E\varphi_n(\delta)) + \frac{2\rho t}{3n},$$

holds with probability at least $$1 - e^{-t}$$. Using that $$\sqrt{u + v} \leq \sqrt{u} + \sqrt{v}$$ and that $$2\sqrt{uv} \leq u + v$$, we further deduce that

$$\varphi_n(\delta) \leq 2E\varphi_n(\delta) + \sigma(\delta)\sqrt{\frac{2t}{n}} + \frac{8\rho t}{3n}, \quad (5.2)$$

with probability at least $$1 - e^{-t}$$. Let us now focus on upper bounding $$\sigma(\delta)$$. For all $$b \in B$$ and all $$x \in \text{supp}(P),$$

$$(d(x, b)^2 - d(x, \bar{b})^2)^2 = (d(x, b) + d(x, \bar{b}))^2(d(x, b) - d(x, \bar{b}))^2$$

$$\leq (d(x, b) + d(x, \bar{b}))^2d(b, \bar{b})^2$$

$$\leq 4\max\{d(u, y)^2 : u \in B, y \in \text{supp}(P)\}d(b, \bar{b})^2,$$

and therefore, using Assumption (A1),

$$P(d(\cdot, b)^2 - d(\cdot, \bar{b})^2) \leq 4R^2d(b, \bar{b})^2$$

$$\leq 4R^2K(P(d(\cdot, b)^2 - d(\cdot, \bar{b})^2)^{\beta}, \quad (5.3)$$

where, as above, we have denoted $$R = \max\{d(b, x) : b \in B, x \in \text{supp}(P)\}.$$ 

**Remark 24.** We briefly interrupt the proof for an important comment. Inequality (5.3) exhibits a property, sometimes referred to as the Bernstein property, of the functional class $$\{d(\cdot, b)^2 - d(\cdot, \bar{b})^2 : b \in B\}$$ indexd by our empirical process, allowing to control its second moment by a power of its first moment. The importance of such a condition for obtaining fast convergence rates for empirical risk minimizers has been emphasized in one way or another by many authors including Mammen and Tsybakov (1999); Massart (2000); Mendelson (2002); Blanchard et al. (2003); Bartlett et al. (2005, 2006); Kolchinskii (2006) and Bartlett and Mendelson (2006) among others. Bernstein’s condition is known to be sometimes connected to the convexity properties of the loss function in statistical learning theory. It is of interest to point out that, in the context of our study, this condition receives a strong geometrical interpretation as described in Sections 3 and 4.

It follows from the definition of $$\sigma(\delta)$$ and equation (5.3) that

$$\sigma(\delta) \leq 2R\sqrt{K}\delta^\beta. \quad (5.4)$$
Next, we provide an upper bound for $\mathbb{E}\varphi_n(\delta)$. Let $\sigma_1, \ldots, \sigma_n$ be a sequence of i.i.d. random signs, i.e. such that $\mathbb{P}(\sigma_i = -1) = \mathbb{P}(\sigma_i = 1) = 1/2$, independent from the $X_i$’s. The symmetrization principle (see, e.g., Lemma 7.4 in van Handel, 2016) indicates that

$$\mathbb{E}\varphi_n(\delta) \leq 2\mathbb{E} \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} \sigma_i (d(X_i, b) - d(X_\bar{b}, \bar{b})) : b \in B(\delta) \right\}$$

$$= 2\mathbb{E} \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} \sigma_i d(X_i, b) : b \in B(\delta) \right\}.$$

By the contraction principle (see Theorem 4.12 in Ledoux and Talagrand, 1991), the fact that $d(X_i, b) \leq R$ almost surely and the fact that $x \mapsto x^2$ is $2R$-Lipschitz on $[0, R]$, we further deduce that

$$\mathbb{E}\varphi_n(\delta) \leq 4R \mathbb{E} \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} \sigma_i d(X_i, b) : b \in B(\delta) \right\}.$$

Now observe that, conditionally on the $X_i$’s, the process

$$\xi_b := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_i d(X_i, b), \quad b \in B$$

satisfies the separability condition of Lemma 23 due to the separability of $B$. In addition, this process is subgaussian since

$$\forall b, b' \in B, \forall \theta \in \mathbb{R}, \quad \log \mathbb{E}[\exp(\theta(\xi_b - \xi_{b'}))]|X_1, \ldots, X_n] \leq \frac{\theta^2 d(b, b')^2}{2},$$

and satisfies the Lipschitz condition $|\xi_b - \xi_{b'}| \leq \sqrt{nd(b, b')}$. Hence, applying Lemma 23 we obtain

$$\mathbb{E}\varphi_n(\delta) \leq 4R \inf_{\varepsilon \geq 0} \left\{ 2\varepsilon + 12 \sqrt{n} \int_{\varepsilon}^{+\infty} \sqrt{\log N(B(\delta), d, u)} \, du \right\}.$$

Using once again Assumption (A1) yields $B(\delta) \subset B(\bar{b}, \sqrt{K\delta^3})$ so that

$$\mathbb{E}\varphi_n(\delta) \leq 4R \inf_{\varepsilon \geq 0} \left\{ 2\varepsilon + 12 \sqrt{n} \int_{\varepsilon}^{+\infty} \sqrt{\log N(B \cap B(\bar{b}, \sqrt{K\delta^3}), d, u)} \, du \right\}$$

$$= 4R \inf_{\varepsilon \geq 0} \left\{ 2\varepsilon + 12 \sqrt{n} \int_{\varepsilon}^{\sqrt{K\delta^3}} \sqrt{\log N(B \cap B(\bar{b}, \sqrt{K\delta^3}), d, u)} \, du \right\}. \quad (5.5)$$

Now consider the case $D \in (0, 2)$ where parameter $D$ appears in Assumption (A2). In this case, the integrand on the right-hand side of (5.5) is integrable at 0. Hence, taking $\varepsilon = 0$ in (5.5) and using Assumption (A2), it follows that

$$\mathbb{E}\varphi_n(\delta) \leq 48R \int_{0}^{\sqrt{K\delta^3}} \left( \frac{C\sqrt{K\delta^3}}{u} \right)^{D} \frac{d}{du} \int_{0}^{\sqrt{K\delta^3}} \left( \frac{C\sqrt{K\delta^3}}{u} \right)^{D} \frac{d}{du}$$

$$= 96RC^{D} K^{\frac{D}{2}} \frac{\sqrt{\delta^3}}{2 - D} \frac{\sqrt{\delta^3}}{n}. \quad (5.6)$$

Combining (5.2), (5.4) and (5.6) implies therefore that inequality

$$\varphi_n(\delta) \leq \beta_n(\delta, t) := c_1 \sqrt{\frac{\delta^3}{n}} + c_2 \sqrt{\frac{t\delta}{n}} + c_3 t,$$
holds with probability at least $1 - e^{-t}$ with $c_1 := 192RC^2\frac{\bar{D}K^\frac{1}{2}}{2 - D}$, $c_2 = 2R\sqrt{2K}$ and $c_3 = 16\rho R/3$. Using the fact that $\delta_n \leq \varphi_n(\delta_n)$ and Lemma 21, it follows that

$$\delta_n \leq \beta_n(t) := \inf \left\{ \tau > 0 : \sup_{\delta \geq \tau} \delta^{-1} \beta_n(\delta, \frac{\delta}{\tau}) \leq 1 \right\},$$

with probability larger that $1 - 2e^{-t}$. It therefore remains to provide an upper bound for $\beta_n(t)$. Straightforward computations, using the fact that $\beta \in (0, 1]$, indicate that, for any $\tau > 0$,

$$\sup_{\delta \geq \tau} \delta^{-1} \beta_n(\delta, \frac{\delta}{\tau}) = \frac{c_1}{\sqrt{n}} \tau^2 - 1 + c_2 \sqrt{\frac{t}{n}} \tau^2 - 1 + \frac{c_3 t}{\tau n}.$$ 

Observing finally that for any nonincreasing functions $h_j : [0, +\infty) \to [0, +\infty)$, we have

$$\inf \{ \tau > 0 : h_1(\tau) + \cdots + h_m(\tau) \leq 1 \} \leq \max_{1 \leq j \leq m} \inf \{ \tau > 0 : h_j(\tau) \leq 1/m \},$$

it follows that

$$\beta_n(t) \leq \max \left\{ A_1 \left( \frac{1}{n} \right)^{\frac{1}{2 - \beta}}, A_2 \left( \frac{t}{n} \right)^{\frac{1}{2 - \beta}}, A_3 \left( \frac{t}{n} \right) \right\},$$

with

$$A_1 = (3c_1)^{\frac{2}{2 - \beta}}, \quad A_2 = (3c_2)^{\frac{2}{2 - \beta}} \quad \text{and} \quad A_3 = 3c_3.$$

As a consequence, we obtain the following statement.

**Statement 25** (Case $D \in (0, 2)$). If $D \in (0, 2)$, then for all $n \geq 1$ and all $t > 0$,

$$\int_S (d(x, b_n)^2 - d(x, \bar{b})^2) dP(x) \leq \max \left\{ A_1 \left( \frac{1}{n} \right)^{\frac{1}{2 - \beta}}, A_2 \left( \frac{t}{n} \right)^{\frac{1}{2 - \beta}}, A_3 \left( \frac{t}{n} \right) \right\},$$

with probability at least $1 - 2e^{-t}$, with

$$A_1 := \left( \frac{576RC^2\frac{\bar{D}K^\frac{1}{2}}{2 - D}}{2 - D} \right)^{\frac{2}{2 - \beta}}, \quad A_2 := (6R\sqrt{2K})^{\frac{2}{2 - \beta}}, \quad A_3 := 16\rho R,$$

where $\rho = \text{diam}(B)$ and $R := \max\{d(b, x) : b \in B, x \in \text{supp}(P)\}$.

Now suppose $D = 2$. Combining Assumption (A2) with inequality (5.5) yields in this case

$$\mathbb{E} \varphi_n(\delta) \leq 8R\varepsilon + 48RC \sqrt{\frac{\bar{D}K^\beta}{n}} \log \left( \frac{\sqrt{\bar{D}K^\beta}}{\varepsilon} \right),$$

for all $\varepsilon > 0$. Taking $\varepsilon = \sqrt{\bar{D}K^\beta/n}$ provides the upper bound

$$\mathbb{E} \varphi_n(\delta) \leq 8R \left( \sqrt{\frac{\bar{D}K^\beta}{n}} + 24RC(\log n) \sqrt{\frac{\bar{D}K^\beta}{n}} \right) = 8R\sqrt{\bar{D}K(1 + 3C\log n)} \sqrt{\frac{\delta^\beta}{n}}.$$

Then, applying the exact same steps as in the case $D \in (0, 2)$ with $c_1' = 16R\sqrt{\bar{D}K(1 + 3C\log n)}$ instead of $c_1$, we obtain the following result.

**Statement 26** (Case $D = 2$). If $D = 2$, then for all $n \geq e$ and all $t > 0$,

$$\int_S (d(x, b_n)^2 - d(x, \bar{b})^2) dP(x) \leq \max \left\{ A_1'(\log n)^{\frac{2}{2 - \beta}} n^{\frac{1}{2 - \beta}}, A_2 \left( \frac{t}{n} \right)^{\frac{1}{2 - \beta}}, A_3 \left( \frac{t}{n} \right) \right\},$$

with probability at least $1 - 2e^{-t}$, with

$$A_1' := (48R\sqrt{\bar{D}K} \max\{1, 3C\})^{\frac{2}{2 - \beta}}, \quad A_2 := (6R\sqrt{2\bar{D}K})^{\frac{2}{2 - \beta}}, \quad A_3 := 16\rho R,$$

where $\rho = \text{diam}(B)$ and $R := \max\{d(b, x) : b \in B, x \in \text{supp}(P)\}$. 

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5.2 Proof of Theorem 4

Suppose finally $D > 2$. Combining Assumption (A2) with inequality (5.5) yields in this case

$$\mathbb{E}\varphi_n(\delta) \leq 8R\varepsilon + \frac{96RC^3K^2D^{3/2}D\varepsilon^{D/2}}{D - 2} \sqrt{\frac{1}{\varepsilon^{D/2}}} - \frac{1}{\varepsilon^{D/2}} - \frac{1}{(K\delta^3)^{D/2}}$$

for all $\varepsilon > 0$. Taking $\varepsilon$ so that the two terms on the right-hand side coincide, i.e.

$$\varepsilon = C\sqrt{K} \left( \frac{12}{D - 2} \right)^{\beta} \delta^{\beta/n} n^{-\frac{1}{\beta}}$$

we obtain

$$\mathbb{E}\varphi_n(\delta) \leq 16RC\sqrt{K} \left( \frac{12}{D - 2} \right)^{\beta} \delta^{\beta/n} n^{-\frac{1}{\beta}}$$

and therefore

$$\varphi_n(\delta) \leq \beta_n(\delta, t) := c''_1\delta^{\beta/n} n^{-\frac{1}{\beta}} + c_2\sqrt{\frac{12\delta}{n}} + \frac{c_3t}{n} \text{ where } c''_1 := 32RC\sqrt{K} \left( \frac{12}{D - 2} \right)^{\beta},$$

with probability at least $1 - e^{-t}$ where $c_2$ and $c_3$ are as in the previous cases. The proof now ends using the same strategy as above and we obtain the following.

**Statement 27** (Case $D > 2$). If $D > 2$, we obtain for all $n \geq 1$ and all $t > 0$,

$$\int_S (d(x, b_n)^2 - d(x, \bar{b})^2) dP(x) \leq \max \left\{ A_1''n^{-\frac{1}{2(\beta-1)}}, A_2 \left( \frac{t}{n} \right)^{\frac{1}{2(\beta-1)}}, A_3 \left( \frac{t}{n} \right) \right\},$$

with probability at least $1 - 2e^{-t}$, with

$$A_1'' := (48CR\sqrt{K})^{2/\beta} \left( \frac{12}{D - 2} \right)^{\frac{1}{2(\beta-1)}}, \quad A_2 := (6R\sqrt{2K})^{2/\beta}, \quad A_3 := 16\rho R,$$

where $\rho = \text{diam}(B)$ and $R := \max\{d(b, x) : b \in B, x \in \text{supp}(P)\}$.

The proof is complete. \qed

5.2 Proof of Theorem 4

The proof of this result is almost identical to that of Theorem 1 and we therefore only point out the slight modification. As the control on the complexity of set $B$ is only here of global nature, the inequality (5.5), i.e.

$$\mathbb{E}\varphi_n(\delta) \leq 4R\inf_{\varepsilon \geq 0} \left\{ 2\varepsilon + \frac{12}{\sqrt{n}} \int_\varepsilon^{\sqrt{K}\delta^3} \sqrt{\log N(B \cap B(b, \sqrt{K}\delta^3), d, u)} du \right\},$$

which is still valid, cannot be exploited as such. However, replacing it by the crude upper bound

$$\mathbb{E}\varphi_n(\delta) \leq 4R\inf_{\varepsilon \geq 0} \left\{ 2\varepsilon + \frac{12}{\sqrt{n}} \int_\varepsilon^{\sqrt{K}\delta^3} \sqrt{\log N(B, d, u)} du \right\},$$

and using Assumption (A3), yields the desired result by following the same lines as in the proof of Theorem 1. Precisely, we obtain the following results, in the same spirit as Statements 25, 26 and 27.
Statement 28. Under the assumptions of the theorem, for all \( n \geq 1 \), for all \( t > 0 \) and with probability at least \( 1 - 2e^{-t} \), we have

- If \( D < 2 \),
  \[
  \int_S (d(x, b_n)^2 - d(x, \bar{b})^2) \, dP(x) \leq \max \left\{ \mathcal{A}_1 n^{-\frac{2}{4-D}}, \mathcal{A}_2 \left( \frac{t}{n} \right)^{\frac{1}{2-D}}, \mathcal{A}_3 \left( \frac{t}{n} \right) \right\},
  \]
  with
  \[
  \mathcal{A}_1 = \left( \frac{576RC}{2-D} \right)^{\frac{2-D}{4-D}}, \quad \mathcal{A}_2 := (6R\sqrt{2K})^{\frac{2}{2-D}}, \quad \mathcal{A}_3 := 16\rho R;
  \]
- If \( D = 2 \),
  \[
  \int_S (d(x, b_n)^2 - d(x, \bar{b})^2) \, dP(x) \leq \max \left\{ \mathcal{B}_1 \frac{\log n}{\sqrt{n}}, \mathcal{B}_2 n^{-\frac{1}{2}}, \mathcal{B}_3 \left( \frac{t}{n} \right)^{\frac{1}{2}}, \mathcal{B}_4 \left( \frac{t}{n} \right) \right\},
  \]
  with
  \[
  \mathcal{B}_1 = 196RC, \quad \mathcal{B}_2 = (64R\sqrt{K})^{\frac{2}{2}}, \quad \mathcal{B}_3 := (8R\sqrt{2K})^{\frac{2}{2}}, \quad \mathcal{B}_4 := \frac{64}{3}\rho R;
  \]
- If \( D > 2 \),
  \[
  \int_S (d(x, b_n)^2 - d(x, \bar{b})^2) \, dP(x) \leq \max \left\{ \mathcal{C}_1 n^\frac{D}{D-2}, \mathcal{C}_2 \left( \frac{t}{n} \right)^{\frac{1}{2}}, \mathcal{C}_3 \left( \frac{t}{n} \right) \right\},
  \]
  with
  \[
  \mathcal{C}_1 = C\rho \left( \frac{12}{D-2} \right)^{\frac{D}{2}}, \quad \mathcal{C}_2 := (6R\sqrt{2K})^{\frac{2}{2}}, \quad \mathcal{C}_3 := 16\rho R;
  \]

where \( \rho = \text{diam}(B) \) and \( R := \max\{d(b, x) : b \in B, x \in \text{supp}(P)\} \).

5.3 Proof for Example 5

Let us prove inequalities (2.3). Let \( x \in B \) and \( 0 < \varepsilon \leq r \leq \rho \) be fixed. Let \( x_1, \ldots, x_N \) be a collection of minimal size \( N = N(B(x, r), d, \varepsilon) \) such that \( B(x, r) \subset \bigcup_{i=1}^N B(x_i, \varepsilon) \). Without loss of generality, observe that we can assume that each \( x_i \) belongs to \( B^\rho \) since otherwise \( B(x, r) \cap B(x_i, \varepsilon) = \emptyset \). Then, by subadditivity of measure \( \mu \), it follows that

\[
\alpha_m(x, r) r^m = \mu(B(x, r)) \leq \sum_{i=1}^N \mu(B(x_i, \varepsilon)) = \varepsilon^m \sum_{i=1}^N \alpha(x_i, \varepsilon).
\]

By definition of \( \alpha_- \) and \( \alpha_+ \), we deduce that \( \alpha_- r^m \leq \alpha_+ N \varepsilon^m \) which proves the first inequality. To prove the second inequality, define the \( \varepsilon \)-packing number \( M(B(x, r), d, \varepsilon) \) of \( B(x, r) \) as the maximal number \( M \) of points \( x_1, \ldots, x_M \in B(x, r) \) such that \( d(x_i, x_j) > \varepsilon \) for all \( i \neq j \). A collection of such points is called an \( \varepsilon \)-packing of \( B(x, r) \). It is a classical fact that the covering and packing numbers satisfy the duality property

\[
N(B(x, r), d, \varepsilon) \leq M(B(x, r), d, \varepsilon) \leq N(B(x, r), d, \varepsilon/2).
\]

In particular, to upper bound \( N(B(x, r), d, \varepsilon) \), it suffices to upper bound \( M(B(x, r), d, \varepsilon) \). Hence, let \( x_1, \ldots, x_M \) be a maximal \( \varepsilon \)-packing of \( B(x, r) \) of size \( M = M(B(x, r), d, \varepsilon) \). Notice that the balls
5.4 Proof of Theorem 10

Let \( p, x, y \in B \) be arbitrary and let \( \bar{p}, \bar{x}, \bar{y} \in M^2_1 \) be a comparison triangle in \( M^2_1 \), the Euclidean sphere of radius 1 in \( \mathbb{R}^3 \). Since \( B \) has curvature upper bounded by 1 in the large, it follows by definition of curvature and completeness of \( B \) that this property is equivalent to the fact that for any constant speed shortest path \( \gamma = \gamma_{\bar{b}b} : [0, 1] \to S \) joining \( \bar{b} \) to \( b \) and any \( t \in [0, 1] \),

\[
d(x, \gamma(t))^2 \leq (1 - t)d(x, \bar{b})^2 + td(x, b)^2 - \frac{k}{2}t(1 - t)d(b, \bar{b})^\alpha.
\]

After rearranging the terms, we obtain, for all \( t \in (0, 1) \),

\[
\frac{k}{2}d(b, \bar{b})^\alpha \leq \frac{d(x, b)^2 - d(x, \bar{b})^2}{1 - t} + \frac{d(x, b)^2 - d(x, \gamma(t))^2}{t(1 - t)}.
\]

Integrating this inequality with respect to \( P(dx) \), and using the definition of \( \bar{b} \), it follows that

\[
\frac{k}{2}d(b, \bar{b})^\alpha \leq \frac{P(d(., b)^2 - d(., \bar{b})^2)}{1 - t},
\]

for all \( t \in (0, 1) \), where we have used the fact that, by definition of \((k, \alpha)\)-convexity we have \( \gamma(t) \in B \) for all \( t \). The result then follows by letting \( t \) tend to 0.

5.5 Proof of Theorem 12

Let \( p, x, y \in B \) be arbitrary and let \( \bar{p}, \bar{x}, \bar{y} \in M^2_1 \) be a comparison triangle in \( M^2_1 \), the Euclidean sphere of radius 1 in \( \mathbb{R}^3 \). Since \( B \) has curvature upper bounded by 1 in the large, it follows by definition of curvature and completeness of \( B \) that this property is equivalent to the fact that for any constant speed shortest path \( \gamma \) between \( x \) and \( y \) in \( B \) and any constant speed shortest path \( \bar{\gamma} \) between \( \bar{x} \) and \( \bar{y} \) in \( M^2_1 \), we have

\[
d(p, \gamma(1/2)) \leq d_1(p, \bar{\gamma}(1/2)),
\]

where \( d_1 \) denotes the angular metric on \( M^2_1 \). Hence, to show that \( B \) is \((k, \alpha)\)-convex, it is enough to show that for any pairwise distinct \( \bar{p}, \bar{x}, \bar{y} \in M^2_1 \) we have

\[
kd_1(\bar{x}, \bar{y})^{\alpha - 2} \leq \frac{8}{d_1(\bar{x}, \bar{y})^2} \left( \frac{1}{2}d_1(\bar{p}, \bar{x})^2 + \frac{1}{2}d_1(\bar{p}, \bar{y})^2 - d_1(\bar{p}, \bar{\gamma}(1/2))^2 \right).
\]
Since \( \alpha \geq 2 \), it is enough for this property to hold that, for all pairwise distinct \( \bar{p}, \bar{x}, \bar{y} \in M^2_1 \), we have

\[
kd^{\alpha - 2} \leq \frac{8}{d_1(\bar{x}, \bar{y})^2} \left( \frac{1}{2}d_1(\bar{p}, \bar{x})^2 + \frac{1}{2}d_1(\bar{p}, \bar{y})^2 - d_1(\bar{p}, \gamma(t))^2 \right).
\]

As proven by Ohta (2007), the right-hand side is upper bounded by

\[
8d \tan \left( \frac{\pi}{2} - d \right),
\]

which concludes the proof.

### 5.6 Proof of Theorem 13

(a) \( \Rightarrow \) (b): For any \( x, y \in S \) let \( \gamma_{xy} : [0,1] \to S \) be a constant speed shortest path joining \( x \) to \( y \). For any \( t \in [0,1] \), applying property (a) with \( P = (1-t)\delta_x + t\delta_y \), and noticing that \( b^* = \gamma_{xy}(t) \) is a barycentre of \( P \), yields property (b).

(b) \( \Rightarrow \) (c): This implication follows from the proof of Theorem 10 with \( B = S \) and \( \alpha = 2 \).

(c) \( \Rightarrow \) (d): As for the first implication, the result follows from applying property (c) to the measure \( P = (1-t)\delta_x + t\delta_y \) for all \( x, y \in S \) and all \( t \in [0,1] \).

The last statement follows from the fact that \( u + av \leq au + v \) whenever \( 0 \leq a \leq 1 \) and \( u \leq v \). Indeed, applying this inequality with \( a = k/2 \), \( u = d^2(b, b^*) \) and \( v = Pd^2(\cdot, b^*) \) yields the desired result.

### 5.7 Proof of Theorem 16

In the proof, the barycenter \( b^* \) is fixed and we denote \( \log = \log_{b^*} \), \( \| \cdot \| = \| \cdot \|_{b^*} \) and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{b^*} \) for brevity. Since \( S \) is a global NNC space, the first statement of Theorem 45 in Yokota (2012) implies that

\[
\int_S \int_S \langle \log(x), \log(y) \rangle \, dP(x) dP(y) \geq 0.
\]

Moreover, Proposition 39 also ensures that \( d(x, y) \leq \| \log(x) - \log(y) \| \) for all \( x, y \in S \) with equality if \( x = b^* \) or \( y = b^* \). In particular, \( b^* \) minimizes also

\[
y \mapsto \int_S \| \log(x) - \log(y) \|^2 \, dP(x).
\]

Thus, using the properties of \( \| \cdot \| \), discussed in Appendix B, we get for all \( y \in S \) and \( t > 0 \),

\[
\int_S \| \log(x) \|^2 dP(x) = \int_S \| \log(x) - \log(b^*) \|^2 dP(x)
\leq \int_S \| \log(x) - t \cdot \log(y) \|^2 dP(x)
\]

\[
= \int_S (\|x\|^2 + \|t \cdot \log(y)\|^2 - 2\langle \log(x), t \cdot \log(y) \rangle) dP(x),
\]

so that

\[
2 \int_S \langle \log(x), \log(y) \rangle \, dP(x) \leq t \| \log(y) \|^2.
\]

Letting \( t \to 0^+ \), we get

\[
\int_S \langle \log(x), \log(y) \rangle dP(x) \leq 0.
\]
Then, integrating with respect to \( y \), we obtain

\[
\int_S \int_S \langle \log(x), \log(y) \rangle \, dP(x) \, dP(y) \leq 0.
\]

Combining this observation with the first inequality of the proof yields the desired result.

### 5.8 Proof of Theorem 17

In the proof, the barycenter \( b^* \) is fixed and we denote \( \log = \log_{b^*}, \| \cdot \| = \| \cdot \|_{b^*} \) and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{b^*} \) for brevity. Also, we implicitly identify a point \( x \in S \) and its image \( \log(x) \) in \( T_{b^*} S \) for readability. In particular, notation \( \| x - y \| \) should be understood as \( \| \log(x) - \log(y) \| \). According to Theorem 16, the barycenter \( b^* \) of \( P \) is also an exponential barycentre of \( P \). Hence, as recalled in Appendix B, Theorem 45 in Yokota (2012) implies that \( \text{supp}(\log_{b^*} P) \subset T_{b^*} S \) can be isometrically embedded in a Hilbert space. In particular, for any \( b \in S \), if we denote \( b, t \in [0, 1] \) a constant speed geodesic joining \( b^* \) to \( b \), it follows from the previous observation and the geometry of a Hilbert space, that for \( x \in S \) and any \( t \in [0, 1] \),

\[
\| x - b_t \|^2 = (1 - t)\| x - b^* \|^2 + t\| x - b \|^2 - t(1 - t)\| b - b^* \|^2.
\]

Now, since \( S \) has nonnegative curvature, recall the fact that, for all \( x, y \in S \), we have \( d(x, y) \leq \| x - y \| \) with equality if \( x = b^* \) or \( y = b^* \). Hence, it follows that, for all \( x \in S \) and all \( t \in (0, 1) \),

\[
t(1-t)d(b, b^*)^2
\]

\[
= t(1-t)\| b - b^* \|^2
\]

\[
= (1-t)\| x - b^* \|^2 + t\| x - b \|^2 - \| x - b_t \|^2
\]

\[
= (1-t)d(x, b^*)^2 + t\| x - b \|^2 - \| x - b_t \|^2
\]

\[
= t(\| x - b \|^2 - d(x, b^*^2)) + (d(x, b^*)^2 - \| x - b_t \|^2)
\]

\[
= t(d(x, b)^2 - d(x, b^*)^2) + (d(x, b^*)^2 - \| x - b_t \|^2) + t(\| x - b \|^2 - d(x, b)^2)
\]

\[
= t(d(x, b)^2 - d(x, b^*)^2) + (d(x, b^*)^2 - \| x - b_t \|^2) + t(1 - k_0(x, b, b^*)) d(b, b^*)^2.
\]

Then by dividing by \( t > 0 \), we obtain

\[
(k_0(x, b, b^*) - t) d(b, b^*)^2 = (d(x, b)^2 - d(x, b^*)^2) + \frac{1}{t}(d(x, b^*)^2 - \| x - b_t \|^2)
\]

\[
\leq (d(x, b)^2 - d(x, b^*)^2) + \frac{1}{t}(d(x, b^*)^2 - d(x, b_t)^2).
\]

Integrating with respect to \( P(dx) \), we obtain

\[
(Pk_0(\cdot, b, b^*) - t) d^2(b, b^*) \leq P(d(\cdot, b)^2 - d(\cdot, b^*)^2) + \frac{P(d(\cdot, b^*)^2 - d(\cdot, b_1)^2)}{t},
\]

where \( Pf(\cdot) \) denotes \( \int f(x) \, dP(x) \). Since \( b^* \) is a barycentre of \( P \), \( P(d(\cdot, b^*)^2 - d(\cdot, b_t)^2) \leq 0 \) and the result follows by letting \( t \to 0 \).

### 5.9 Proof of Theorem 18

We proceed to the same identifications as in the proof of the previous Theorem. First remark that since \( b^* \) is barycentre of \( P \), it is also an exponential barycentre of \( P \) by Theorem 16. Since \( \text{supp}(P) \) is flat on \( T_{b^*} S \), \( b^* \) is also an exponential barycentre of \( P_\lambda \) and, since \( P_\lambda \) has a unique barycentre by assumption, it is also the barycentre of \( P_\lambda \).
Now let $x, b \in \text{supp}(P)$ and denote $x^\lambda := \gamma^+_{x}(1+\lambda)$. Remark that on $T_xS$ (through the identification by $\log_x$) we have $x^\lambda = x + \lambda(x - b^*)$. Now take $t = 1/(1+\lambda)$. Then, on $T_xS$,

$$x = tx^\lambda + (1-t)b^*.$$  

(5.7)

As $S$ is a global NNC space, $T_xS$ itself is a global NNC space by Proposition 39. Hence, we deduce that

$$\|x - b\|_x^2 \geq (1-t)\|b - b^*\|_x^2 + t\|x^\lambda - b\|_x^2 - t(1-t)\|x^\lambda - b^*\|_x^2$$

$$= \frac{\lambda}{1+\lambda}\|b - b^*\|_x^2 + \frac{1}{1+\lambda}\|x^\lambda - b\|_x^2 - \frac{\lambda}{(1+\lambda)^2}\|x^\lambda - b^*\|_x^2,$$

which, using the fact $\|x^\lambda - b^*\|_x^2 = (1+\lambda)^2\|x - b^*\|_x^2$, we can rewrite as

$$\frac{\lambda}{1+\lambda}\|b - b^*\|_x^2 \leq d^2(x, b) - \frac{1}{1+\lambda}\|x^\lambda - b\|_x^2 + \lambda d^2(x, b^*)$$

$$= d^2(x, b) - d^2(x, b^*) - \frac{1}{1+\lambda}\|x^\lambda - b\|_x^2 + (1+\lambda)d^2(x, b^*).$$

Now, letting

$$c(\lambda) = 1 + \lambda - \left(\frac{\int \|x^\lambda - b\|_x^2 dP(x)}{\int d(x, b^*)^2 dP(x)}\right)^{1/2},$$

it follows that

$$\int_S \left(\frac{1}{1+\lambda}\|x^\lambda - b\|_x^2 + (1+\lambda)d^2(x, b^*)\right) dP(x) = \left(2c(\lambda) - \frac{c(\lambda)^2}{1+\lambda}\right) \int_S d^2(x, b^*) dP(x).$$

Hence, since $\|b - b^*\|_x \geq d(b, b^*)$ and $\|b - x\|_x = d(b, x)$, we deduce from the previous computations that

$$\frac{\lambda}{1+\lambda}d(b, b^*)^2 \leq \int_S (d(x, b)^2 - d(x, b^*)^2) dP(x) + 2c(\lambda) \int_S d(x, b^*)^2 dP(x).$$

Finally, we observe that since $b^*$ is the unique barycentre of $P_{\lambda}$, we have

$$c(\lambda) \leq 1 + \lambda - \left(\frac{\int d^2(x^\lambda, b^*) dP(x)}{\int d(x, b^*)^2 dP(x)}\right)^{1/2} \leq 0,$$

which concludes the proof.

### 5.10 Proof of Theorem 19

We start by a technical lemma. Below $H$ is a Hilbert space with scalar product $\langle.,.\rangle$ and associated norm $\|\|$. 

**Lemma 29.** Let $\varphi : H \to \mathbb{R}$ be a convex function and $\partial \varphi \subset H^2$ its subdifferential defined by

$$(x, y) \in \partial \varphi \iff \forall z \in H, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle.$$ 

Then, for all $c > 0$,

$$(x, y) \in \partial \varphi \iff \forall z \in H, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle - c\|x - z\|^2.$$
Proof of the Lemma. One implication is obvious. For the second implication, suppose \((x, y) \in H^2\) is such that
\[
\forall z \in H, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle - c\|z - x\|^2. \tag{5.8}
\]
Then, on the one hand, we get by convexity of \(\varphi\) that, for all \(z \in H\) and all \(t \in (0, 1),\)
\[
\varphi(z) - \varphi(x) \geq \frac{1}{t}(\varphi(tz + (1-t)x) - \varphi(x)). \tag{5.9}
\]
On the other hand, applying (5.8), we obtain for all \(z \in H\) and all \(t \in (0, 1),\)
\[
\varphi(tz + (1-t)x) - \varphi(x) \geq t\langle y, z - x \rangle - ct\|z - x\|^2. \tag{5.10}
\]
Hence, combining (5.9) and (5.10), we deduce that for all \(z \in H\) and all \(t \in (0, 1),\)
\[
\varphi(z) - \varphi(x) \geq \langle y, z - x \rangle - ct\|z - x\|^2.
\]
Letting \(t \to 0\) proves the other implication.

We are now in position to prove Theorem 19. Fix \(\mu, \nu \in S = \mathcal{P}_2(H)\) and denote \(\gamma : [0, 1] \to S\) a constant speed shortest path between \(\mu\) and \(\nu\). By the Knott-Smith optimality criterion (see Theorem 2.12 in Villani, 2003), \(\pi\) is an optimal transport plan of \((\mu, \nu)\) if and only if its support lies in the graph of the subdifferential of a convex function \(\varphi\), i.e.
\[
(x, y) \in \text{supp } \pi \Rightarrow \forall z \in H, \quad \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle. \tag{5.11}
\]
Suppose first that, for some \(\lambda > 0\), \(\gamma : [0, 1] \to S\) can be extended to a function \(\gamma^+ : [0, 1+\lambda] \to S\) that remains a shortest path between its endpoints \(\mu = \gamma^+(0) = \gamma(0)\) and \(\nu^\lambda := \gamma^+(1+\lambda)\). Then, by Theorem 7.2.2 in Ambrosio et al. (2008), there exists an optimal transport plan \(\pi^\lambda\) of \((\mu, \nu^\lambda)\) such that \(\{\pi\} = \Gamma_o(\mu, \nu)\) is that the law of
\[
(X, Y^\lambda) := \left(X, \frac{\lambda}{1+\lambda}X + \frac{1}{1+\lambda}Y^\lambda \right)
\]
where \((X, Y^\lambda) \sim \pi^\lambda\). In particular \(Y^\lambda = (1+\lambda)Y - \lambda X\). Therefore, there exist convex functions \(\varphi, \varphi^\lambda\) (\(\varphi^\lambda\) is unique up to an additive constant by Lemma 7.2.1 of Ambrosio et al. (2008)) such that denoting \(y^\lambda = (1+\lambda)y - \lambda x\)
\[
(x, y) \in \text{supp } \pi \Leftrightarrow (x, y^\lambda) \in \text{supp } \pi^\lambda
\]
\[
\Rightarrow \forall z \in H, \varphi^\lambda(z) \geq \varphi^\lambda(x) + \langle y^\lambda, z - x \rangle
\]
\[
\Leftrightarrow \forall z \in H, \varphi^\lambda(z) \geq \varphi^\lambda(x) + (1+\lambda)\langle y, z - x \rangle - \lambda\langle x, z - x \rangle
\]
\[
\Leftrightarrow \forall z \in H, \varphi^\lambda(z) + \frac{\lambda}{2}\|z\|^2 \geq \varphi^\lambda(x) + \frac{1}{2}\|x\|^2 + (1+\lambda)\langle y, z - x \rangle + \lambda\frac{1}{2}\|x - z\|^2
\]
\[
\Rightarrow \forall z \in H, \frac{\varphi^\lambda(z)}{1+\lambda} + \frac{\lambda\|z\|^2}{2(1+\lambda)} \geq \frac{\varphi^\lambda(x)}{1+\lambda} + \frac{\lambda\|x\|^2}{2(1+\lambda)} + \langle y, z - x \rangle \tag{5.12}
\]
Since \(\varphi\) is unique up to an additive constant \(c\), (5.11) and (5.12) imply
\[
\frac{\varphi^\lambda}{1+\lambda} + c = \varphi - \frac{\lambda}{2(1+\lambda)} \|x\|^2,
\]
which is a convex function, i.e. \(\varphi\) is \(\frac{1}{1+\lambda}\)-convex.

Conversely, suppose that \(\varphi\) is \(\frac{1}{1+\lambda}\)-convex and supp\(\pi\) for \(\pi \in \Gamma_o(\mu, \nu)\) lies in the subdifferential of \(\varphi\). Denote \((X, Y) \sim \pi\) and set \(Y^\lambda = X + \lambda(Y - X) \sim \nu^\lambda\) and \((X, Y^\lambda) \sim \pi^\lambda\). Then \(\pi^\lambda\) is an optimal transport plan between \(\mu\) and \(\nu^\lambda\) if and only if there exists a convex function \(\varphi^\lambda\) such that the supp\(\pi^\lambda\)
lies in $\partial \varphi^\lambda$. In that case, $W^2_2(\mu, \nu^\lambda) = E\|Y^\lambda - X\|^2 = (1 + \lambda)^2 W^2_2(\mu, \nu)$ so that by Lemma 7.2.1 of Ambrosio et al. (2008), $\nu$ is in the shortest path joining $\mu$ to $\nu^\lambda$. It thus just remains to prove that there exists a convex function $\varphi^\lambda$ such that $\text{supp } \pi^\lambda \subseteq \partial \varphi^\lambda$.

Set 

$$
\varphi^\lambda = (1 + \lambda) \varphi - \frac{1}{2} \lambda \|\cdot\|^2.
$$

$\varphi^\lambda$ is convex since $\varphi$ is $\frac{\lambda}{1 + \lambda}$-convex. Then, denoting again $y^\lambda = (1 + \lambda)y - \lambda x$,

$$(x, y^\lambda) \in \text{supp } \pi^\lambda \Leftrightarrow (x, y) \in \text{supp } \pi
\Leftrightarrow \forall z \in H, \varphi(z) \geq \varphi(x) + \langle y, z - x \rangle
\Leftrightarrow \forall z \in H, \frac{\varphi^\lambda(z)}{1 + \lambda} + \frac{\lambda \|z\|^2}{2(1 + \lambda)} \geq \frac{\varphi^\lambda(x)}{1 + \lambda} + \frac{\lambda \|x\|^2}{2(1 + \lambda)} + \langle y, z - x \rangle
\Leftrightarrow \forall z \in H, \varphi^\lambda(z) \geq \varphi^\lambda(x) + \langle y^\lambda, z - x \rangle - \lambda \frac{1}{2} \|x - z\|^2
$$

(5.13)

Since $\varphi^\lambda$ is convex, by Lemma 29, (5.13) is equivalent to

$$
\forall z \in H, \varphi^\lambda(z) \geq \varphi^\lambda(x) + \langle y^\lambda, z - x \rangle
$$

### A Bound in expectation

This section provides, for completeness, a bound in expectation derived from Theorem 1. A similar extension of Theorem 4 is left to the reader.

**Theorem 30.** Suppose that the support of $P$ and the set $B$ have finite diameter. Then, under Assumptions (A1) and (A2), and for all $n \geq 1$,

$$
E \int_S (d(b_n, x) - d(\bar{b}, x))^2 \, dP(x) \leq \begin{cases} 
(A_1 + \alpha A_2)n^{-\frac{1}{\beta - 1}} + \frac{\alpha A_3}{n} & \text{if } D < 2, \\
A'_1 (\log n)^2 n^{-\frac{1}{\beta - 1}} + \alpha A_2 n^{-\frac{1}{\beta - 1}} + \frac{\alpha A_3}{n} & \text{if } D = 2, \\
A''_1 n^{-\frac{1}{\beta - 1}} + \alpha A_2 n^{-\frac{1}{\beta - 1}} + \frac{\alpha A_3}{n} & \text{if } D > 2,
\end{cases}
$$

where $A_1, A'_1, A''_1, A_2, A_3$ are as in Statements 25, 26 and 27 and where finally $\alpha = 2$ if $\beta = 1$ and $\alpha = 1 + 4/\varepsilon(2 - \beta)$ if $\beta \in (0, 1)$.

**Proof.** We only outline the proof for $D < 1$ as the other cases are exactly the same. As in the proof of Theorem 1, let

$$
\delta_n := \int_S (d(b_n, x) - d(\bar{b}, x))^2 \, dP(x).
$$

Abbreviate

$$
a_1 := A_1 \left(\frac{1}{n}\right)^{\frac{1}{\beta - 1}}, a_2 := A_2 \left(\frac{1}{n}\right)^{\frac{1}{\beta - 1}}, \quad \text{and} \quad a_3 := \frac{A_3}{n},
$$

for simplicity, where $A_1, A_2$ and $A_3$ are the constants involved in Statement 25. Then, it follows that

$$
E\delta_n = \int_0^{+\infty} P(\delta_n > t) \, dt 
\leq a_1 + \int_{a_1}^{+\infty} P(\delta_n > a_1 + t) \, dt
= a_1 + (a_2 + a_3) \int_0^{+\infty} P(\delta_n > a_1 + (a_2 + a_3)t) \, dt.
$$
If $\beta = 1$, we deduce from Statement 25 that
\[
E\delta_n \leq a_1 + (a_2 + a_3) \int_0^{+\infty} 2e^{-t} \, dt = a_1 + 2(a_2 + a_3),
\]
which proves the statement. Now if $\beta \in (0, 1)$, and using the lines above, we write
\[
E\delta_n \leq a_1 + (a_2 + a_3) \int_0^{+\infty} P(\delta_n > a_1 + (a_2 + a_3)t) \, dt
\]
\[
\leq a_1 + (a_2 + a_3) + \frac{2(a_2 + a_3)}{2 - \beta} \int_1^{+\infty} t^{2/\beta - 1} P(\delta_n > a_1 + (a_2 + a_3)t^{2/\beta}) \, dt
\]
\[
\leq a_1 + (a_2 + a_3) + \frac{2(a_2 + a_3)}{2 - \beta} \int_1^{+\infty} P(\delta_n > a_1 + a_2t^{2/\beta} + a_3t) \, dt
\]
\[
\leq a_1 + (a_2 + a_3) + \frac{2(a_2 + a_3)}{2 - \beta} \int_1^{+\infty} 2e^{-t} \, dt
\]
\[
\leq a_1 + (a_2 + a_3) + \frac{4(a_2 + a_3)}{e(2 - \beta)},
\]
where the last inequality follows again from Statement 25. This completes the proof.

\[\square\]

**B Background in metric geometry**

This appendix gathers material used throughout the paper. General references, for topics discussed below can be found for instance in Burago et al. (2001) or Alexander et al. (2017).

**B.1 Geodesic spaces**

Let $(S, d)$ be a metric space. We call path in $S$ a continuous map $\gamma : I \to S$ defined on a nonempty interval $I \subset \mathbb{R}$. Define the length $L(\gamma) \in [0, +\infty]$ of a path $\gamma : I \to S$ by
\[
L(\gamma) := \sup_{n \geq 1} \sup_{t_1 < \cdots < t_n \in I} \sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i+1})).
\]
A path $\gamma : [a, b] \to S$ is said to connect $x$ to $y$ if $\gamma(a) = x$ and $\gamma(b) = y$. The space $S$ is called a length space if, for all $x, y \in S$,
\[
d(x, y) = \inf_{\gamma} L(\gamma), \tag{B.1}
\]
where the infimum is taken over all paths $\gamma$ that connect $x$ to $y$. A length space is said to be geodesic if, for all $x, y \in S$, the infimum on the right hand side of (B.1) is attained. A path, realizing this infimum is called a shortest path (or geodesic) between $x$ and $y$ and needs not be unique. In a geodesic space, a constant speed shortest path between $x$ and $y$ is a path $\gamma : [0, 1] \to S$ such that $\gamma(0) = x$, $\gamma(1) = y$ and such that, for all $0 \leq s \leq t \leq 1$,
\[
L(\gamma_{[s,t]}) = (t - s)d(x, y),
\]
where $\gamma_{[s,t]}$ denotes the restriction of $\gamma$ to $[s, t]$. A unit speed shortest path between $x$ and $y$ is a path $\gamma : [0, T] \to S$ such that $\gamma(0) = x$, $\gamma(T) = y$ and such that, for all $0 \leq s \leq t \leq 1$,
\[
L(\gamma_{[s,t]}) = (t - s).
\]
In this situation, note in particular that $T = d(x, y)$. The following characterization of geodesic spaces is helpful.
Proposition 31. Let \((S, d)\) be a metric space.

1. If \(S\) is a geodesic space, then any two points \(x, y \in S\) admit a midpoint, i.e. a point \(z \in S\) such that
\[
d(x, z) = d(y, z) = \frac{1}{2} d(x, y).
\]

2. Conversely, if \(S\) is complete and if any two points in \(S\) admit a midpoint, then \(S\) is a geodesic space.

B.2 Curvature

Geodesic spaces of interest are those with bounded curvature in the sense of Alexandrov. In this setting, curvature bounds are defined by comparison arguments involving model surfaces defined as follows. For \(\kappa \in \mathbb{R}\), we'll denote \((M_\kappa^2, d_\kappa)\) the 2-dimensional model surface (also referred to as the \(\kappa\)-plane) defined by

- The Hyperbolic plane (modeled as the Poincaré disc for instance), with metric multiplied by \(1/\sqrt{-\kappa}\), if \(\kappa < 0\),
- The Euclidean plane equipped with its Euclidean metric, if \(\kappa = 0\),
- The Euclidean sphere in \(\mathbb{R}^3\) of radius \(1/\sqrt{\kappa}\) equipped with the angular metric, if \(\kappa > 0\).

In addition to these model surfaces, the definition of curvature bounds given below requires to define the notion of comparison triangles. For any triangle in \(S\), i.e. any three points set \(\{p, x, y\}\) in \(S\), a comparison triangle in \(M_\kappa^2\) is defined as a set \(\tilde{p}, \tilde{x}, \tilde{y}\) in \(M_\kappa^2\) with the property that 
\[
d_\kappa(\tilde{p}, \tilde{x}) = d(p, x), d_\kappa(\tilde{p}, \tilde{y}) = d(p, y) \text{ and } d_\kappa(\tilde{x}, \tilde{y}) = d(x, y).
\]

Definition 32. Let \(\kappa \in \mathbb{R}\) and \((S, d)\) be a geodesic space.

1. The space \((S, d)\) is said to have curvature bounded below by \(\kappa\) if any point \(z \in S\) has a neighborhood \(U_z\) for which the following holds: for any points \(p, x, y \in U_z\), any comparison triangle \(\tilde{p}, \tilde{x}, \tilde{y}\) in \(M_\kappa^2\), any constant speed shortest path \(\gamma\) joining \(x\) to \(y\) in \(S\) (whose image is contained in \(U_z\)) and any constant speed shortest path \(\tilde{\gamma}\) joining \(\tilde{x}\) to \(\tilde{y}\) in \(M_\kappa^2\), we have
\[
\forall t \in [0, 1], \quad d(p, \gamma(t)) \geq d_\kappa(\tilde{p}, \tilde{\gamma}(t)). \tag{B.2}
\]

2. The space \((S, d)\) is said to have curvature bounded above by \(\kappa\) if the above definition holds with opposite inequality in \(\text{(B.2)}\).

3. The space \((S, d)\) is said to be nonnegatively curved (NNC) if it has curvature bounded below by 0.

4. The space \((S, d)\) is said to be nonpositively curved (NPC) if it has curvature bounded above by 0.

The previous definition has a natural geometric interpretation: in a space \(S\) with curvature lower (resp. upper) bounded by \(\kappa\), a (small enough) triangle (meaning here three points joined by shortest paths) looks fatter (resp. thinner) than a corresponding comparison triangle in \(M_\kappa^2\). In the context of NNC and NPC spaces, the above definition may be given an alternative form of practical interest.

Proposition 33. A geodesic space \((S, d)\) is NNC if, and only if, any point \(z \in S\) has a neighborhood \(U_z\) for which the following holds: for any points \(p, x, y \in U_z\) and any constant speed shortest path \(\gamma\) joining \(x\) to \(y\), whose image is contained in \(U_z\), we have
\[
\forall t \in [0, 1], \quad d(p, \gamma(t))^2 \geq (1 - t)d(p, x)^2 + td(p, y)^2 - t(1 - t)d(x, y)^2.
\]

A space \((S, d)\) is NPC if, and only if, the same statement holds with opposite inequality.
The proof follows immediately from Definition 32 by exploiting the geometry of the flat Euclidean plane. For \( \kappa \neq 0 \), an equivalent formulation of Definition 32, given only in terms of the ambient metric \( d \), is given in the next subsection using the notion of angle. It is worth noting that the previous definitions are of local nature as they require inequalities to be valid in the neighborhood of a point. Some spaces have the additional property that the comparison inequalities of Definition 32 are valid without restriction on the points \( p, x \) and \( y \). Such spaces are said to have curvature bounded from below (or above) by \( \kappa \) in the large (or globally). Note that \( (M^2_n, d_n) \) has diameter \( \pi/\sqrt{\kappa} \) when \( \kappa > 0 \). Hence, for \( \kappa > 0 \), one can find a comparison triangle for \( \{p, x, y\} \) provided the perimeter \( \text{peri}\{p, x, y\} := d(p, x) + d(p, y) + d(x, y) \leq 2\pi/\sqrt{\kappa} \). The next definition accounts for this observation.

**Definition 34.** Let \((S, d)\) be a geodesic space.

- If \( \kappa > 0 \), then \((S, d)\) is said to have curvature bounded from above (or below) by \( \kappa \) in the large, if the comparison inequality of Definition 32 (from above or below) holds for all points \( p, x, y \in S \) such that \( \text{peri}\{p, x, y\} \leq 2\pi/\sqrt{\kappa} \).

- If \( \kappa \leq 0 \), then \((S, d)\) is said to have curvature bounded from above (or below) by \( \kappa \) in the large, if the comparison inequality of Definition 32 (from above or below) holds for all points \( p, x, y \in S \).

A geodesic space with curvature upper bounded by some \( \kappa \geq 0 \), in the large, is sometimes referred to as a CAT(\( \kappa) \) space in the literature in reference to the contributions of E. Cartan, A.D. Alexandrov and V.A. Topogonov. A CAT(0) space is also referred to as a global NPC space. Similarly, a space with curvature lower bounded by 0 in the large will be called a global NNC space. Note that a CAT(\( \kappa) \) space is also a CAT(\( \kappa') \) space for all \( \kappa' > \kappa \). Riemannian manifolds with sectional curvature at most \( \kappa \) and injectivity radius not less than \( \pi/\sqrt{\kappa} \) are typical examples of CAT(\( \kappa) \) spaces. An important fact is that in a CAT(\( \kappa) \) space, two points \( x, y \) with \( d(x, y) < \pi/\sqrt{\kappa} \) have a unique shortest path connecting them (up to reparametrization). The following results are known as globalization theorems.

**Theorem 35 (Globalization).** Let \((S, d)\) be a geodesic space.

- Suppose \((S, d)\) is complete, simply connected and has curvature bounded from above by some \( \kappa \leq 0 \). Then \((S, d)\) has curvature bounded from above by \( \kappa \) in the large (i.e. is a CAT(\( \kappa) \) space).

- Suppose \((S, d)\) is complete and has curvature bounded from below by some \( \kappa \in \mathbb{R} \). Then \((S, d)\) has curvature bounded from below by \( \kappa \) in the large.

### B.3 Angles, spaces of directions and tangent cones

Metric spaces considered so far have a priori no differentiable structure. In this context, an analog of a tangent space is provided by the notion of a tangent cone. This section shortly reviews this notion. We first need the notion of angles and spaces of directions.

For \( \kappa \in \mathbb{R} \) and a triangle \( \{p, x, y\} \) in \( S \), such that \( \text{peri}\{p, x, y\} < 2\pi/\sqrt{\kappa} \) if \( \kappa > 0 \), we define the comparison angle \( \angle_\kappa(x, p, y) \in [0, \pi] \) at \( p \) by

\[
\cos \angle_\kappa(x, p, y) := \begin{cases} \frac{d(p, x)^2 + d(p, y)^2 - d(x, y)^2}{2d(p, x)d(p, y)} & \text{if } \kappa = 0, \\ \frac{C_\kappa(d(x, y) - C_\kappa(d(p, x))C_\kappa(d(p, y))}{\kappa S_\kappa(d(p, x))S_\kappa(d(p, y))} & \text{if } \kappa \neq 0, \end{cases}
\]

where \( C_\kappa := S'_\kappa \) and

\[
S_\kappa(r) := \begin{cases} \sin(r\sqrt{\kappa})/\sqrt{\kappa} & \text{if } \kappa > 0, \\ \sinh(r\sqrt{-\kappa})/\sqrt{-\kappa} & \text{if } \kappa < 0. \end{cases}
\]

The notion of angle allows to define bounded curvature in several equivalent manners.
\textbf{Proposition 36.} For $\kappa \in \mathbb{R}$, a geodesic space $(S, d)$ has curvature lower (resp. upper) bounded by $\kappa$ in the sense of Definition 32 if one of the two following equivalent statements holds.

- (Quadruple comparison) Any point in $S$ has a neighbourhood $U$ in which the following holds: for any four distinct points set $\{p, x, y, z\} \subset U$, we have
  \[ \angle_\kappa(x, p, y) + \angle_\kappa(x, p, z) + \angle_\kappa(y, p, z) \leq (\text{resp.} \geq) 2\pi. \]

- (Angle monotonicity) Any point in $S$ has a neighbourhood $U$ in which the following holds: for any any three distinct points set $\{p, x, y\} \subset U$ and any constant speed shortest path $\gamma$ from $p$ to $y$, we have
  \[ 0 \leq s \leq t \leq 1 \Rightarrow \angle_\kappa(x, p, \gamma(s)) \geq (\text{resp.} \leq) \angle_\kappa(x, p, \gamma(t)). \]

Now for a point $p$ in $(S, d)$ geodesic, and two constant speed shortest paths $\alpha : [0, 1] \rightarrow S$ and $\beta : [0, 1] \rightarrow S$ such that $\alpha(0) = \beta(0) = p$, the angle $\angle(\alpha, \beta)$ between $\alpha$ and $\beta$ is defined by
\[
\angle(\alpha, \beta) := \lim_{s, t \rightarrow 0} \angle_\kappa(\alpha(s), p, \beta(t)),
\]
when it exists. If $S$ is geodesic with upper or lower bounded curvature, the angle monotonicity guarantees that $\angle(\alpha, \beta)$ always exists (and does not depend on $\kappa$). Given a third constant speed shortest path $\gamma : [0, 1] \rightarrow S$ such that $\gamma(0) = p$, we have the triangular inequality
\[
\angle(\alpha, \beta) \leq \angle(\alpha, \gamma) + \angle(\gamma, \beta). \tag{B.3}
\]

Now for fixed $p \in S$, let $\Sigma'_p$ be the set of all constant speed shortest paths emanating from $p$, two of them being identified if the angle between them is 0. On the set $\Sigma'_p$, the angle $\angle$ defines a metric by (B.3) and we call space of directions at $p$ the metric completion $(\Sigma'_p, \angle)$ of $(\Sigma'_p, \angle)$ (with a slight abuse of notation, the metric in $\Sigma'_p$ is still denoted $\angle$). Following Petrunin (2007), we denote $\Sigma^p \subset \Sigma_p$ the set of all directions (or classes) of constant speed shortest paths from $p$ to $x$ and, for all $x \in S$, we select $\Sigma^p \subset \Sigma_p$ one such direction.

\textbf{Definition 37} (Tangent cone). Let $(S, d)$ be a geodesic space, with upper or lower bounded curvature in the sense of Definition 32. Let $p \in S$. The tangent cone $T_pS$ at $p$ is the Euclidean cone over the space of directions $(\Sigma_p, \angle)$. In other words, $T_pS$ is the metric space:

- Whose underlying set consists in equivalent classes in $\Sigma_p \times [0, +\infty)$ for the equivalence relation $\sim$ defined by
  \[ (\alpha, s) \sim (\beta, t) \Leftrightarrow (s = 0 \text{ and } t = 0) \text{ or } (s = t \text{ and } \alpha = \beta). \]

  With a slight abuse of notation, we'll consider that a point in $T_pS$ is either the tip of the cone $\alpha_p$, i.e. the class $\Sigma_p \times \{0\}$, or a couple $(\alpha, s) \in \Sigma_p \times (0, +\infty)$ (hence identified to the class $\{(x, s)\}$).

- Whose metric $d_p$ is defined (without ambiguity, while using the abuse of notation mentioned above) by
  \[ d_p((\alpha, s), (\beta, t)) := \sqrt{s^2 + t^2 - 2st \cos \angle(\alpha, \beta)}. \]

  For $u = (\alpha, s)$ and $v = (\beta, t) \in T_pS$, we often denote $\|u - v\|_p := d_p(u, v)$,
  \[ \|u\|_p := d_p(\alpha_p, u) = s \quad \text{and} \quad \langle u, v \rangle_p := st \cos \angle(\alpha, \beta) = (\|u\|_p^2 + \|v\|_p^2 - \|u - v\|_p^2)/2. \]

On $T_pS$, a Hilbert-like structure can be described as follows. For a point $u = (\alpha, t)$ and $\lambda \geq 0$, we define $\lambda \cdot x := (\alpha, \lambda t)$. Then, the sum of points $u, v \in T_pS$ is defined as the mid-point of $2 \cdot u$ and $2 \cdot v$. It may be checked, using the previous definitions, that for any $u, v \in T_pS$ and any $\lambda \geq 0$, we get
  \[ \|\lambda \cdot u\|_p = \lambda \|u\|_p \quad \text{and} \quad \langle \lambda \cdot u, v \rangle_p = \langle u, \lambda \cdot v \rangle_p = \lambda \langle u, v \rangle_p. \]

Next, we define the notion of logarithmic and exponential maps that play an important role in the sequel.
**Definition 38** (Logarithmic and exponential maps). Let \((S, d)\) be a geodesic space with curvature bounded from above or below in the sense of Definition 32 and fix \(p \in S\). Then we denote

\[
\log_p : x \in S \mapsto (\gamma_{px}, d(p, x)) \in T_p S.
\]

Note that the definition of \(\log_p\) hence depends on the choice of the collection \(\{\gamma_{px}\}_{x \in S}\). However, letting \(C(p)\) be the set of points \(x \in S\) such that there exists strictly more than one shortest path joining \(p\) to \(x\), then for all \(x \in S \setminus C(p)\), we have

\[
\log_p(x) = (\gamma_{px}, d(p, x)),
\]

where \(\gamma_{px}\) denotes the unique constant speed shortest path from \(p\) to \(x\). In particular, for any constant speed \(\gamma : [0, 1] \to S\) emanating from \(p\) and any \(t \in [0, 1]\), \(\log_p(\gamma(t)) = (\gamma, t)\). Also, for a neighbourhood \(U \subset T_p S\) of \(o_p\), we denote by \(\exp_p\) any function \(U \to S\) such that

\[
\log_p \circ \exp_p = \text{Id}_U
\]

In the previous definition, note that \(C(p)\) is empty for spaces of curvature bounded from above. We end by mentioning a connection used in the paper between the curvature properties of \(S\) and that of the tangent cone.

**Proposition 39.** Let \((S, d)\) be a geodesic space and \(p \in S\) be fixed. If \(S\) is has curvature bounded from below in the large by some \(\kappa \in \mathbb{R}\), then the tangent cone \(T_p S\) is a global NNC space. In addition, for any \(x, y \in S\),

\[
d(x, y) \leq \|\log_p(x) - \log_p(y)\|_p,
\]

with equality if \(x = p\) or \(y = p\).

The first statement of Proposition 39 follows from Proposition 28 in Yokota (2012). The second statement follows by the monotonicity property of angles in spaces with lower curvature bound. We use several times in the proofs the following result (Theorem 45 in Yokota, 2012).

**Theorem 40.** Let \(b^*\) be an exponential barycentre of \(P \in \mathbb{P}_2(S)\). Denote \(l_P\) the image of the support of \(P\) through \(\log_{b^*}\). Then \(l_P \subset T_{b^*} S\) can be isometrically embedded in a Hilbert space and the same holds with its convex hull.

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