Abstract

We give a survey on projective ring lines and some of their substructures which in turn are more general than a projective line over a ring.

Keywords: Projective line over a ring, distant graph, connected component, elementary linear group, subspace of a chain geometry, Jordan system, projective line over a strong Jordan system

1 Distant graph and connected components

The projective line $\mathbb{P}(R)$ over any ring $R$ (associative with $1 \neq 0$) can be defined in terms of the free left $R$-module $R^2$ as follows [11], [24]: It is the orbit of a starter point $R(1, 0)$ under the action of the general linear group $GL_2(R)$ on $R^2$. A basic notion on $\mathbb{P}(R)$ is its distant relation: Two points are called distant (in symbols: $\triangle$) if they can be represented by the elements of a two-element basis of $R^2$. The distant graph $(\mathbb{P}(R), \triangle)$ has as vertices the points of $\mathbb{P}(R)$ and as edges the pairs of distant points. The distant graph is connected precisely when $GL_2(R)$ is generated by the elementary linear group $E_2(R)$, i.e., the subgroup of $GL_2(R)$ which is generated by elementary transvections, together with the set of all invertible diagonal matrices [7]. The orbit of $R(1, 0)$ under $E_2(R)$ is a connected component of the distant graph. It admits a parametrisation in terms of infinitely many formulas [7], [8]. The situation is less intricate for a ring $R$ of stable rank 2 (see

---

*Email: havlicek@geometrie.tuwien.ac.at*
As it gives rise to a connected distant graph with diameter $≤ 2$, the above-mentioned parametrisation turns into Bartolone’s parametrisation \[^{11}\] of $\mathbb{P}(R)$, namely

$$\mathbb{P}(R) = \{ R(t_2t_1 - 1, t_2) \mid t_1, t_2 \in R \} \quad (R \text{ of stable rank } 2).$$

Refer to the seminal paper of P. M. Cohn \[^{16}\] for the algebraic background, and to the work of A. Blunck \[^{5}, \,^{6}\] for orbits of the point $R(1, \, 0)$ under other subgroups of $\text{GL}_2(R)$.

## 2 Chain Geometries, subspaces and Jordan Systems

Let $R$ be an algebra over a commutative field $K$; by identifying $K$ with $K \cdot 1_R$ the projective line $\mathbb{P}(K)$ is embedded in $\mathbb{P}(R)$. For $R \neq K$ the projective line $\mathbb{P}(R)$ can be considered as the point set of the chain geometry $\Sigma(K, R)$; the $\text{GL}_2(R)$ orbit of $\mathbb{P}(K)$ is the set of chains \[^{11}, \,^{24}\]. The geometries of Möbius, Minkowski and Laguerre are well known examples of chain geometries \[^{2}\]. A crucial property is that any three mutually distinct points are on a unique chain. The chain geometry $\Sigma(K, R)$ may be viewed as a refinement of the distant graph, since two points of $\mathbb{P}(R)$ are distant if, and only if, they are on a common chain. There are cases though, when the word “refinement” is inappropriate in its strict sense: Let $R = \text{End}_F(V)$ be the endomorphism ring of a vector space $V$ over a (not necessarily commutative) field $F$ and let $K$ denote the centre of $F$. Then the $K$-chains of $\mathbb{P}(R)$ can be defined solely in terms of the distant graph $(\mathbb{P}(R), \triangle)$ \[^{10}\].

Each chain geometry $\Sigma(K, R)$ is a chain space; see \[^{11}\], where also the precise definition of subspaces of a chain space is given. The algebraic description of subspaces of $\Sigma(K, R)$ is due to A. Herzer \[^{23}\] and H.-J. Kroll \[^{29}, \,^{30}, \,^{31}\]. It is based on the following notions: A Jordan system is a $K$-subspace of $R$ satisfying two extra conditions: (i) $1 \in J$; (ii) If $b \in J$ has an inverse in $R$ then $b^{-1} \in J$. (See \[^{33}\] for relations with Jordan algebras and Jordan pairs and compare with \[^{18}, \,^{34}\].) A Jordan system $J$ is called strong if it satisfies a (somewhat technical) condition which guarantees the existence of “many” invertible elements in $J$. Strong Jordan systems are closed under triple multiplication, i. e., $xyx \in J$ for all $x, y \in J$. The projective line $\mathbb{P}(J)$ over a strong Jordan system $J \subset R$ is defined by restricting the parameters $t_1, \, t_2$ to $J$ in Bartolone’s parameterization. We wish to emphasize that in general a point of $\mathbb{P}(J)$ cannot be written as $R(a, b)$ with $a, b \in J$, unless $J$ is even a subalgebra of $R$. The main theorem about subspaces is as follows: If $R$ is a strong algebra then any connected subspace of $\Sigma(K, R)$ is projectively equivalent to a projective line over a strong Jordan system of $R$.

Projective lines over strong Jordan systems admit many applications: For example, one may use them to describe subsets of Grassmannians which are closed.
under reguli [23] or chain spaces on quadrics [4]. See also [3], [25], [26], [27], and the numerous examples given in [11].

Finally, let us mention one of the many questions that remain: Is it possible to replace the strongness condition for Jordan systems by closedness under triple multiplication without affecting the known results? A partial affirmative answer was given in [9] for case when $R$ is the ring of $n \times n$ matrices over a field $F$ with an involution $\sigma$ and $J$ is the (not necessarily strong) Jordan system of $\sigma$-Hermitian matrices. The proof is based upon the verification that the projective line over this $J$ is, up to some notational differences, nothing but the point set of a dual polar space [14] or, in the terminology of [38], the point set of a projective space of $\sigma$-Hermitian matrices.

A wealth of further references can be found in [2], [11], [19], [24], [28], [35], [37], and [38]. Refer to [12], [13], [17], [20], [21], [22], and [32] for deviating definitions of projective lines which we cannot present here.

References

[1] C. Bartolone. Jordan homomorphisms, chain geometries and the fundamental theorem. Abh. Math. Sem. Univ. Hamburg, 59:93–99, 1989.

[2] W. Benz. Vorlesungen über Geometrie der Algebren. Springer, Berlin, 1973.

[3] A. Blunck. Chain spaces over Jordan systems. Abh. Math. Sem. Univ. Hamburg, 64:33–49, 1994.

[4] A. Blunck. Chain spaces via Clifford algebras. Monatsh. Math., 123:98–107, 1997.

[5] A. Blunck. Geometries for Certain Linear Groups over Rings — Construction and Coordinatization. Habilitationsschrift, Technische Universität Darmstadt, 1997.

[6] A. Blunck. Projective groups over rings. J. Algebra, 249:266–290, 2002.

[7] A. Blunck and H. Havlicek. The connected components of the projective line over a ring. Adv. Geom., 1:107–117, 2001.

[8] A. Blunck and H. Havlicek. Jordan homomorphisms and harmonic mappings. Monatsh. Math., 139:111–127, 2003.

[9] A. Blunck and H. Havlicek. Projective lines over Jordan systems and geometry of Hermitian matrices. Linear Algebra Appl., 433:672–680, 2010.
[10] A. Blunck and H. Havlicek. Geometric structures on finite- and infinite-dimensional Grassmannians. *Beitr. Algebra Geom.*, to appear.

[11] A. Blunck and A. Herzer. *Kettengeometrien – Eine Einführung*. Shaker Verlag, Aachen, 2005.

[12] U. Brehm. Algebraic representation of mappings between submodule lattices. *J. Math. Sci. (N. Y.)*, 153(4):454–480, 2008.

[13] U. Brehm, M. Greferath, and S. E. Schmidt. Projective geometry on modular lattices. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 1115–1142. Elsevier, Amsterdam, 1995.

[14] P. J. Cameron. Dual polar spaces. *Geom. Dedicata*, 12(1):75–85, 1982.

[15] H. Chen. *Rings Related to Stable Range Conditions*, volume 11 of *Series in Algebra*. World Scientific, Singapore, 2011.

[16] P. M. Cohn. On the structure of the $GL_2$ of a ring. *Inst. Hautes Etudes Sci. Publ. Math.*, 30:365–413, 1966.

[17] C.-A. Faure. Morphisms of projective spaces over rings. *Adv. Geom.*, 4(1):19–31, 2004.

[18] D. Goldstein, R. M. Guralnick, L. Small, and E. Zelmanov. Inversion invariant additive subgroups of division rings. *Pacific J. Math.*, 227(2):287–294, 2006.

[19] H. Havlicek. From pentacyclic coordinates to chain geometries, and back. *Mitt. Math. Ges. Hamburg*, 26:75–94, 2007.

[20] H. Havlicek, A. Matraš, and M. Pankov. Geometry of free cyclic submodules over ternions. *Abh. Math. Semin. Univ. Hambg.*, 81(2):237–249, 2011.

[21] H. Havlicek, J. Kosiorek, and B. Odehnal. A point model for the free cyclic submodules over ternions. *Results Math.*, to appear.

[22] H. Havlicek and M. Saniga. Vectors, cyclic submodules, and projective spaces linked with ternions. *J. Geom.*, 92(1-2):79–90, 2009.

[23] A. Herzer. On sets of subspaces closed under reguli. *Geom. Dedicata*, 41:89–99, 1992.

[24] A. Herzer. Chain geometries. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 781–842. Elsevier, Amsterdam, 1995.
[25] A. Herzer. Konstruktion von Jordansystemen. *Mitt. Math. Ges. Hamburg*, 27:203–210, 2008.

[26] A. Herzer. Die kleine projektive Gruppe zu einem Jordansystem. *Mitt. Math. Ges. Hamburg*, 29:157–168, 2010.

[27] A. Herzer. Korrektur und Ergänzung zum Artikel *Die kleine projektive Gruppe zu einem Jordansystem* in *Mitt. Math. Ges. Hamburg* 29, Armin Herzer. *Mitt. Math. Ges. Hamburg*, 30:15–17, 2011.

[28] L.-P. Huang. *Geometry of Matrices over Ring*. Science Press, Beijing, 2006.

[29] H.-J. Kroll. Unterräume von Kettengeometrien und Kettengeometrien mit Quadrikenmodell. *Results Math.*, 19:327–334, 1991.

[30] H.-J. Kroll. Unterräume von Kettengeometrien. In N. K. Stephanidis, editor, *Proceedings of the 3rd Congress of Geometry (Thessaloniki, 1991)*, pages 245–247, Thessaloniki, 1992. Aristotle Univ.

[31] H.-J. Kroll. Zur Darstellung der Unterräume von Kettengeometrien. *Geom. Dedicata*, 43:59–64, 1992.

[32] A. Lashkhi. Harmonic maps over rings. *Georgian Math. J.*, 4:41–64, 1997.

[33] O. Loos. *Jordan Pairs*, volume 460 of *Lecture Notes in Mathematics*. Springer, Berlin, 1975.

[34] S. Mattarei. Inverse-closed additive subgroups of fields. *Israel J. Math.*, 159:343–347, 2007.

[35] M. Pankov. *Grassmannians of Classical Buildings*, volume 2 of *Algebra and Discrete Mathematics*. World Scientific, Singapore, 2010.

[36] F. D. Veldkamp. Projective ring planes and their homomorphisms. In R. Kaya, P. Plaumann, and K. Strambach, editors, *Rings and Geometry*, pages 289–350. D. Reidel, Dordrecht, 1985.

[37] F. D. Veldkamp. Geometry over rings. In F. Buekenhout, editor, *Handbook of Incidence Geometry*, pages 1033–1084. Elsevier, Amsterdam, 1995.

[38] Z.-X. Wan. *Geometry of Matrices*. World Scientific, Singapore, 1996.