Relativistic antifragility

Sauro Succi *1,2,3

1Center for Life Nano Science @Sapienza, Istituto Italiano di Tecnologia - 295 Viale Regina Elena, I-00161 Rome, Italy
2Istituto per le Applicazioni del Calcolo CNR, Via dei Taurini 19, 00185 Rome, Italy
3Institute for Applied Computational Science, Harvard John A. Paulson School of Engineering and Applied Sciences - Cambridge, MA 02138, USA

Friday 17th January, 2020

Abstract

It is shown that the barbell distribution of a gas of relativistic molecules above its critical temperature, can be interpreted as an antifragile response to the relativistic constraint of subluminal propagation.

1 Introduction

In the recent years, the notion of anti-fragility, as introduced by Nassim Taleb [1], has gained a boost of popularity across most walks of science and society. Although to a physicist the term "anti-fragility" sounds a bit like a clever renaming of an old wine in a new bottle, the value of extending and applying the notion beyond the context of the natural sciences, finance first, up to a philosophy of life, should not be underestimated.

In this short note, we wish to point out that relativistic kinetic theory, namely the statistical theory of particles which obey Einstein’s (special) relativity, shows distinct signatures of anti-fragility, most notably the emergence of extreme (bell-bar) statistics in lieu of standard ‘Mediocristian’ gaussian distribution, in response to increasing thermal load. The reason for such transition from Mediocristan to Extremistan, to borrow from Taleb’s terminology is adamant: material particles cannot move faster than light.

2 Maxwell-Boltzmann statistics

It is well known that a gas of non-relativistic molecules at equilibrium obeys the Maxwell-Boltzmann (MB) distribution (one spatial dimension for simplicity) [2]:

\[ f_{MB}(v) = \frac{n}{(2\pi)^{1/2}v_T} e^{-\frac{(u-v)^2}{2v_T^2}} \]  

In the above \( n \) is the gas number density (number of molecules per unit volume), \( u \) is the macroscopic gas velocity and \( v_T = \sqrt{\frac{k_B T}{m}} \) is the thermal speed, \( T \) being the temperature and \( m \) the mass of the particle. According to kinetic theory, the macroscopic gas velocity \( u \) coincides with the mean molecular velocity, namely:

\[ u = \langle v \rangle = \frac{1}{n} \int_{-\infty}^{+\infty} f(v)vdv \]

Note that the integral in velocity space runs from minus infinity to plus infinity since classical mechanics sets no restrictions on the magnitude of the particle velocity. Likewise, \( mv_T^2/2 = k_B T/2 \) is the kinetic energy contained in the molecular fluctuations.

\[ k_B T = \langle m(v-u)^2 \rangle = \frac{1}{n} \int f(mv^2)dv \]

*Electronic address: s.succi@iac.cnr.it; Corresponding author
In other words, the kinetic energy of the gas splits into the sum of a macroscopic (mechanical) and a microscopic (thermal) components

$$E_K = \frac{mu^2}{2} + \frac{mv_T^2}{2}$$

In classical kinetic theory both terms are potentially unlimited.

A few comments are in order.

First, the mean velocity $u$ is also the most probable one, in the sense that the MB distribution attains its peak value precisely at $v = u$. This is typical conformistic-behaviour, most molecules “go with the flow”, they move at the same speed as the average.

Second, such conformistic behaviour is fairly intolerant of outliers, namely particles which move much faster or slower than the average are exponentially suppressed. For instance, molecules moving at three thermal speed faster than average (FTA) are suppressed by more than 1:1000, and at five FTA their number goes down to about one in a million! Extreme behaviour is suppressed in Mediocristan. Hence, ‘socially’ speaking, the MB distribution speaks for a comfortable and stable world in which most individuals behave like the average and those who don’t are exponentially suppressed. For good or for worst, this is the most stable statistics a gas of classical molecules can achieve, as sharply pinpointed by Boltzmann’s H-theorem, showing that any different distribution is bound to converge to $f_{MB}$ in order to maximize its entropy, basically a microscopoc underpinning of the second principle of Thermodynamics.

Third, the MB distribution encodes the non-relativistic principle of Galilean invariance, i.e. the statistics does not depend on the absolute molecular velocity $v$, but on its speed relative to the average, $v - u$, also known as peculiar velocity, the one characterizing microscopic fluctuations.

3 Conservation constraints

The MB distribution maximizes entropy under the constraint of mass, momentum and energy conservation, i.e

$$\int_{-\infty}^{+\infty} f_{MB}dv = n$$  \hspace{1cm} (2)

$$\int_{-\infty}^{+\infty} f_{MB}vdv = nu$$  \hspace{1cm} (3)

$$\int_{-\infty}^{+\infty} f_{MB}v^2dv = nu^2 + nv_T^2.$$  \hspace{1cm} (4)

In Taleb’s parlance, constraints are the ’stressors’, sources of stress, since they constrain the freedom of the system.

A few numbers won’t hurt. For air in standard conditions, 300 Kelvin degrees, the thermal speed is close to 300 meters per second (basically the sound speed). If the gas is macroscopically at rest, $u = 0$, the probability of finding a molecule moving faster than one thermal speed, in either direction, is a sizeable 16 percent. At two thermal speeds, the number goes down to about 2 percent, and at above three thermal speeds, we are left with just one in thousands. That means that in a sample of thousands molecules, on average, only one moves faster than $3 \times 300 = 900$ m/s. By iterating the game, numbers get rapidly ridiculously small, at five thermal speeds, (1500 m/s) only one in a billion is left. This speaks clearly for outlier suppression.

Now suppose that we set the gas in motion, with a substantial macroscopic speed, say $u = 30$ m/s (100 Km/h, a pretty strong wind). How does the ‘molecular society’ adjust to such stressor? The answer is fairly straightforward: by simply shifting the MB distribution towards positive values so that the new peak locates precisely at $v = u$. This breaks the left/right symmetry, the probability of finding a particle moving right at a given speed $+v$ is higher than the probability of moving left with the same but opposite speed $-v$. The probability of finding a molecule moving at 300 m/s is 20/100, slightly larger than 16/100 and the probability of moving at $-300$ m/s is 12/100, slightly smaller than 16/100, but the distribution keeps the same bell-shaped form, symmetric around $u = 30$ m/s. The gas of molecules adjusts to the constraint without compromising any of the three basic features described above, Mediocristan still rules.

Now suppose you keep the gas at rest, but increase its temperature instead, say 600 Kelvin instead of 300. How does the gas adjust to this thermal stressor?

Again the policy is adamant; by simply ’broadening’ its distribution, so that the average kinetic energy in the fluctuations is doubled, while the mean velocity is left unchanged. This means that now there are sixteen out of hundred molecules which travel at 600 m/s instead of 300 m/s. This is how the system manages to increase its random energy content, but again, none of three distinctive features above are broken. In particular, while the high velocity region gets more populated, the most probable molecules remain those that move at mean speed, in this case zero. The outliers are less suppressed, but they don’t take over the conformists.
At this point, it is worth noting that the conformists contribute a little precious nothing to the temperature constraint, since they carry no energy at all! But they do contribute to the constraint that the gas should not move on average, so they still have a role in this business. This argument is flawed, but in classical mechanics the flaw remains silent, as we shall see shortly.

In classical kinetic theory you can play the game ad libitum, heat to the point of making the sound speed equal to the speed of light, and the most probable molecules will still be the ones at rest, although by an hardly appreciable extent, since the MB distribution becomes utterly flat. The outliers are no longer such, granted, but the "conformist" are still there.

Since the outliers can afford unlimited speed, the system can absorb virtually any amount of thermal energy, by simply increasing their population at expense of the conformists. Yet, the thermal constraint can always be matched without destroying the conformist in the process, even though they contribute nothing to the energy budget!

The availability of unlimited speed provides the tolerance towards the zero-speed population of molecules.

4 Relativistic kinetic theory

Next we move to relativistic molecules.

The relativistic analogue of the MB distribution, known as Maxwell-Juettner (MJ) distribution, reads as follows [3, 4]:

$$f_{MJ}(v) = nA(z)\gamma_v^3e^{-z\gamma_v\gamma_u(1-uv)}$$

(5)

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$ and $\gamma_u = (1 - u^2/c^2)^{-1/2}$ are the molecular and gas Lorentz factors, $z = mce^2/kT$ is the rest energy in thermal units and $A(z)$ a normalization prefactor [4]. The MJ distribution follows directly from the expression of the relativistic energy, $E^2 = m^2c^4 + p^2c^2$, which in turn encodes Lorentz rather than Galilean invariance. This means that the MJ depends no longer on the peculiar speed $v - u$, but on the (scalar in multi-dimensions) product $vu$.

In addition, it obeys the boundary condition $f(v \to c) \to 0$, in compliance with the relativistic constraint $v \leq c$.

Crucial to the MJ statistics is the term $\gamma_v^3$, which stems from the metric transformation from the momentum distribution $f(p)$ to the velocity distribution $f(v)$. Such factor exhibits a cubic divergence in the limit $v/c \to 1$, thereby promoting depletion of the conformist population in favour of the outliers. This divergence, in turn, results from the fact that while relativistic velocities are bound, the corresponding momenta are not, which
Figure 2: The response of a relativistic gas at rest and $z = 1$ and $z = 20$, and to a net flow $u/c = 0.25$ still at $z = 1$.

This divergence is tamed by the exponential term $e^{-\gamma v}$ which enforces the boundary condition $f(v = c) = 0$. The MJ distribution reflects the basic competition between these two terms, whose outcome is strongly sensitive to temperature via the parameter $z$.

In the non-relativistic limits $v/c \ll 1$ and $1/z \to 0$, the MJ reduces to the MB distribution, hence it reacts to flow and thermal constraints in a similar way.

In the genuinely relativistic limit $\gamma v, \gamma u \gg 1$, however, the response is completely different. To appreciate the point, let’s consider again the case at rest, $u = 0$. It can be readily checked that upon increasing the temperature, i.e. $z \to 0$, the MJ undergoes a depletion of the conformist region in favor of the emergence of a barbell distribution, with two distinct peaks away from the origin. More precisely, the transition from a MB-like unimodal to the bimodal barbell distribution occurs slightly above a critical temperature $kT_{crit} \sim mc^2$, namely below $z \sim 1$. This stands in sharp contrast with the reaction of the MB distribution, which broadens indefinitely in response to the constraint of increasing temperature, without ever developing any double-humped structure.

Why such a different strategy? Simply because the relativistic molecules cannot afford infinite-speed, the fast runners must nevertheless stop at $v = \pm c$. Hence, the only chance for the system to meet an increasing thermal constraint is to enhance the fastest-running populations as much as possible. However, since the outliers can no longer afford infinite speed, a point comes where the conformists are no longer sustainable, and above the critical temperature they begin to be suppressed.

But, didn’t we say that the conformists are needed to comply with the net flow constraint, i.e. zero net motion for the case in point?

That is true, but now the system "realizes" that the same constraint can be fulfilled without the conformists, by simply keeping an exact balance between the left and right movers. In other words, the relativistic barrier $v \leq c$ exposes the 'uselessness' of the conformists and the system gets rid of them: that’s Extremistan in full action!

But how about the case of a net macroscopic motion, say $u > 0$?

Detailed inspection of the MJ distribution [5], shows that this is accomodated by simply developing a positive bias on the right-moving hump and a negative one on the left-moving one.

This mechanism, known as "skweness", reflects Lorentz invariance, as opposed to the rigid shift of the non-relativistic case, which reflects Galilean invariance instead.

Quantitative analysis permits to compute the exact location of the humps and their width as a function of the macroscopic parameters $u/c$ and $z$, but does not add anything substantial to the essence of the story. And the essence is that, above a critical temperature, the relativistic constraint $v \leq c$ makes the conformist simply unsustainable.

The emergence of the barbell distribution then appears as quintessential antifragility, i.e an innovative
survival strategy which was left silent in the non-relativistic case, just due to the availability of unlimited-speed runners.

5 "Social" relativity

The "biological" interpretation of the MJ distribution is adamant: a survival strategy against thermal constraints in a finite-resource (velocity) environment. The social one, as usual, is a bit less straightforward. The collapse of the conformist in relativistic gas is strongly conducive to the economic nosedive of the middle class in the modern "the winner takes it all" global economy.

With a big pinch of imagination, one might even posit that this is due to the mind-boggling acceleration of financial transactions, a sort of analogue of the relativistic limit \( v \to c \). In a world where data and algorithms can evaporate a lifetime savings at keystroke speed, quick-witted 'influencers' and fake news, win hands down over 'slow-paced' engineers, spelling doom for the real economy against the virtual one.

The second consideration is for the conformists, which relativity reveals in their true colors, basically as a disposable population. In human society, some individuals don't move because they don't want to, but some others don't because they really can't. The antifragile policy is fair for the former but not for the latter.

Acknowledgments

This research leading has received funding from the European Research Council under the European Union’s Horizon 2020 Framework Programme (No. FP/2014-2020)/ERC Grant Agreement No. 739964 (COPMAT).

References

[1] N. Taleb, Antifragile: Things That Gain From Disorder, Random House, NY (2012)
[2] L. Boltzmann, Lectures on Gas Theory, University of California Press, Berkeley, (1964)
[3] F. Juettner, Annalen der Physik. 339 (5): 856-882 (1911),
[4] C. Cercignani and G.M. Kremer, The Relativistic Boltzmann Equation: Theory and Applications, Birkhauser, Berlin (2002).
[5] M. Mendoza, N. Arajuno, H. Herrmann and S. Succi, Sci. Rep, 2, 611, (2012)