Digital implementation of continuous-time observers for nonlinear networked control systems

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ABSTRACT
We propose a digital implementation of continuous-time observers to networked control systems. We introduce a sufficiently small constant time step to construct the Euler model of continuous-time observers parameterized by this time step. Then, we propose a real-time iteration of this Euler model to implement continuous-time observers digitally and we give the sufficient conditions that the proposed digital implementation can estimate the state at sampling times semiglobally and practically. Numerical examples are also given to illustrate the proposed digital implementation of continuous-time observers.

1. Introduction

Recently, the use of wired or wireless communication networks, in which sensor data and control commands are communicated, becomes popular, because of several merits such as cost of implementation of sensors, actuators, and controllers, and ease of their implementation and maintenance. Such control systems are called networked control systems. For these systems, we must consider restrictions induced by communication networks such as communication delays, packet dropouts, time-varying sampling intervals, and the algorithm called a protocol that assigns a communication node to connect a network. Under these restrictions, it is required to guarantee the stability and the performance of networked control systems (for details, see [1, 2] and references therein).

In [3, 4], hybrid dynamical model representations of protocols and networked control systems have been introduced and the classes of uniformly globally exponentially stable (UGES) protocols and uniformly globally asymptotically stable (UGAS) protocols have been given. Then, the nonlinear hybrid control theory has been used to give the sufficient conditions that continuous-time controllers that input-to-output (or state) stabilize continuous-time systems, also input-to-output (or state) stabilize networked control systems. The size of the maximal allowable transmission interval (MATI) has been also given. Following the approach in [3, 4], the analysis and design of observers and tracking controllers and the size of MATI have been also discussed for networked control systems [5–9]. But the practical digital implementation of continuous-time observers and controllers and its influence to the size of MATI have not been discussed. Following the nonlinear sampled-data control theory based on the approximate discrete-time models [10, 11], the analysis and design of discrete-time controllers have been also proposed for networked control systems [12–14]. Then, we need nominal sampling intervals to construct approximate models that are used for design purposes. But it is difficult to design discrete-time controllers that have a good control performance due to the difference between practical time-varying sampling intervals and nominal sampling intervals. Moreover, the size of MATI usually become very small.

In this paper, we propose a digital implementation of continuous-time observers to networked control systems when continuous-time observers are designed. If a sampling interval is constant and known, it is possible to implement continuous-time observers by using the approximate discrete-time models parameterized by a constant sampling interval [15, 16]. For networked control systems, sampling intervals are usually time-varying and unknown. Then, we introduce a sufficiently small constant time step $\tau > 0$ and implement continuous-time observers by the Euler model parameterized by $\tau > 0$. We use a real-time iteration of the Euler model to estimate the state at sampling times of networked control systems.

This paper is organized as follows. In Section 2, we first introduce a discrete-time representation of communication protocols such as Round-Robin (RR) protocols and Try-Once-Discard (TOD) protocols. Then, we propose a digital implementation of
continuous-time observers to networked control systems. In Section 3, we show that the proposed digital implementation can semiglobally and practically estimate the state at sampling times of networked control systems with UGAS protocols. In Section 4, we give numerical examples to illustrate the proposed digital implementation of continuous-time observers.

Notation: Let $|\cdot|$ be the norm of vectors and matrices and $B_r = \{x \in \mathbb{R}^r \ | |x| \leq r\}$. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $K$ ($\in K$) if it is continuous, zero at zero, and strictly increasing. It is of class $K_{\infty}$ if it is of class $K$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $KL$, if for any fixed $s \geq 0$ $\beta(s, \cdot)$ is decreasing to zero as its argument tends to infinity [17].

2. Problem Formulation

Consider

$$\dot{x}_c = f(x_c(t)), \quad y_c(t) = h(x_c(t)), \quad (1)$$

where $x_c \in \mathbb{R}^n$ is the state, $y_c \in \mathbb{R}^p$ is the measurement output, $f$ and $h$ are locally Lipschitz, $f(0) = 0$, and $h(0) = 0$. Let $s_k \geq 0$ for any $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ be the sampling times (or sampling points) that satisfy $0 = s_0 < s_1 < \ldots < s_k < s_{k+1} < \ldots$ and $y(k) = y_c(s_k) = h(x_c(s_k)).$ Then sampling intervals $T_k = s_k - s_{k-1}, k \in \mathbb{N}_0$ are bounded, time-variant, and unknown. The difference equations corresponding to the exact (discretized) model and the Euler model of the system (1) are given by

$$x(k+1) = F_{T_{k}}^c(x(k)) = x(k) + \int_{s_k}^{s_{k+1}} f(x_c(s)) \, ds, \quad (2)$$

$$y(k) = h(x(k)), \quad x(k+1) = F_{T_{k}}^c(x(k)) = x(k) + T_k f(x(k)), \quad (3)$$

respectively. Here, note that $x(k) = x_c(s_k)$ and $y(k) = y_c(s_k)$ for the exact model (2) and the Euler model (3) cannot be used for design purposes.

Let $\tau > 0$ be a sufficiently small constant time step. Then, we define $M_k \in \mathbb{N}_0$ for any $k \in \mathbb{N}_0$ by

$$M_0 = s_0 = 0, \quad (M_{k-1} - 1) \tau < s_k \leq M_k \tau \quad \forall k \geq 1 \quad (4)$$

and we set $N_k = M_{k+1} - M_k$ and $T_k = N_k \tau$ for any $k \in \mathbb{N}_0$.

**Remark 2.1:** Equation (4) implies $\tilde{T}_k - \tau \leq T_k \leq \tilde{T}_k + \tau.$ If there exists $T^* > 0$ that satisfies $T_k \in (0, T^*)$ for any $k \in \mathbb{N}_0$ and we set $\tilde{T}^* = T^* + \tau$, then we have $\tilde{T}_k \in (0, \tilde{T}^*)$.

Assume that a continuous-time observer

$$\dot{\hat{z}}_c = \phi(z_c(t), y_c(t)) \quad (5)$$

that estimates the state $x_c(t)$ is designed for the system (1) where $z_c \in \mathbb{R}^n$, $\phi$ is locally Lipschitz, and $\phi(0, 0) = 0$. Then consider the emulation of the continuous-time observer (5)

$$\dot{z}_c = \phi(z_c(t), y(k)) \quad \forall t \in [s_k, s_{k+1}), \forall k \in \mathbb{N}_0. \quad (6)$$

Since $T_k$ is unknown, it is impossible to estimate the state $x(k)$ of the exact model (2) by the approximate model of (6) depending on $T_k$. Let $m \in \mathbb{N}_0$,

$$z_t(0) = z_c(0), \quad y_t(m) = y(k), \quad M_k \leq m < M_{k+1} \quad (7)$$

and consider the difference equation

$$z_t(m+1) = \phi_t(z_t(m), y_t(m)), \quad (8)$$

where $\phi_t(z, y) = z + \tau \phi(z, y)$ is the Euler model of $\phi(z, y)$ parameterized by $\tau > 0$. Then, we use the Euler model (8) iteratively to calculate $z_t(m)$ for any $m \in \mathbb{N}_0$ in real time and we estimate the state $x(k)$ of the exact model (2) by

$$z(k) = z_t(M_k). \quad (9)$$

If we set $\phi^1_t(z, y) = \phi_t(z, y)$ and $\phi^{i+1}_t(z, y) = \phi_t(\phi^i_t(z, y), y)$ for any $i \geq 1$, then the discrete-time model of $z(k)$ is given by

$$z(k+1) = \Phi^N_{\tilde{T}_k}(z(k), y(k)), \quad z(0) = z_c(0), \quad (10)$$

where $\Phi^N_{\tilde{T}_k}(z, y) = \phi^N_{\tilde{T}_k}(z, y)$.

Next, we consider restrictions induced by communication networks. In this case, $y(k)$ is no longer available for observers and the output $\tilde{y}(k)$ of communication networks that is generated by $y(k)$ is used for observers. Typical network protocols such as RR and TOD protocols are modelled by the following form:

$$\tilde{y}(k+1) = y(k+1) + g(k+1, e(k), y(k), y(k+1)) \quad (11)$$

for any $k \in \mathbb{N}_0$ when $\tilde{y}(0)$ is suitably given. Here, $e = \tilde{y} - y$ is an error induced by networks. Since $y(k) = h(x(k))$, $y(k+1) = h(x(k+1)) = h(F_{\tilde{T}_k}^c(x(k)))$, (11) can be rewritten by

$$e(k+1) = G(k+1, e(k), x(k)), \quad (12)$$

where $G(k+1, e, x) = g(k+1, e, h(x), h(F_{\tilde{T}_k}^c(x)))$. The system (12) is also called a protocol.

**Example 2.1:** Let $y \in \mathbb{R}^p$ be partitioned into $l$ nodes and $y = [y_1^T \cdots y_l^T]^T$ where $y_i \in \mathbb{R}^{p_i}$. RR protocol connects the node $i$ to a network at time $s_{i+k}$ for any $i \in \{1, \ldots, l\}$ and $k \in \mathbb{N}_0$. When an initial output $\tilde{y}(0)$ of a network is suitably given, we...
have $\tilde{y}_i(i + ml) = y_i(i + ml)$, $\tilde{y}_j(i + ml) = \tilde{y}_j(i - 1 + ml)$, $j \neq i$ for any $i \in [1, \ldots, l]$ and $m \in \mathbb{N}_0$. Then, we have $\tilde{y}_i(k + 1) = \Delta(k + 1)y_i(k + 1) + [I - \Delta(k + 1)]\tilde{y}(k)$ and (11) with $g(k + 1, e(k), y(k), y(k + 1)) = [I - \Delta(k + 1)]\psi(y(k) - y(k + 1) + e(k))$ where $\Delta(k) = \text{diag}(\Delta_1(k), \ldots, \Delta_l(k))$, $\Delta_i(k) = \delta_i(k)I_{p_i}$, and

$$\delta_i(k) = \begin{cases} 1, & k = i + ml, \\ 0, & k \neq i + ml. \end{cases}$$

TOD protocol connects the node $i$, which has the maximal weighted error at time $s_i$, to a network at time $s_{i+1}$ and we have $\tilde{y}_i(k + 1) = y_i(k + 1)$, $\tilde{y}_j(k + 1) = \tilde{y}_j(k)$, $j \neq i$ for any $k \in \mathbb{N}_0$. When $\tilde{y}(0)$ is suitably given, we obtain $\tilde{y}(k + 1) = \psi(e(k))\psi(y(k + 1) + [I - \psi(e(k))]\tilde{y}(k)$ for any $k \geq 0$ and (11) with $g(k + 1, e(k), y(k), y(k + 1)) = [I - \psi(e(k))]\psi(y(k) - y(k + 1) + e(k))$ where $\psi(e) = \text{diag}(\psi_1(e)I_{p_1}, \ldots, \psi_l(e)I_{p_l})$ and

$$\psi_j(e) = \begin{cases} 1, & j = \min \left(\frac{\arg\max|e_j|}{e_j} \right), \\ 0, & \text{otherwise.} \end{cases}$$

When we consider restrictions of communication networks, we replace $y$ in (7)–(10) by $\tilde{y}$ and we use the real-time iteration of (8) with $z_\tau(0) = z_\tau(0)$, $v_\tau(m) = \tilde{y}(k)$, $M_\tau \leq m < M_{\tau+1}$ to calculate $z_\tau(m)$. Then, we discretize the emulation of the continuous-time observer

$$\dot{z}_\tau = \phi(z_\tau(t), \tilde{y}(k)) \quad \forall t \in [s_\tau, s_{\tau+1}), \forall k \in \mathbb{N}_0$$

by $z(k) = z_\tau(M_\tau)$ or equivalently

$$z(k + 1) = \Phi_{\tilde{y}, \tau}^{\tau}(z(k), \tilde{y}(k)), \quad z(0) = z_\tau(0). \quad (14)$$

We also consider the difference equations corresponding to the exact model and the Euler model of (13) given, respectively, by

$$z(k + 1) = \Phi_{\tilde{y}, \tau}^{\tau}(z(k), \tilde{y}(k)) = z(k) + \int_{M_{\tau+1}}^{M_{\tau}} \phi(z_\tau(s), \tilde{y}(k)) \, ds,$$

$$z(k + 1) = \Phi_{\tilde{y}, \tau}^{\tau}(z(k), \tilde{y}(k)) = z(k) + \tilde{T}_\tau \phi(z(k), \tilde{y}(k)).$$

**Remark 2.2:** (1) As a practical and digital implementation of the system (13), we use the real-time iteration of (8) with $z_\tau(0) = z_\tau(0)$ and $v_\tau(m) = \tilde{y}(k)$, $M_\tau \leq m < M_{\tau+1}$. We also use the equivalent discrete time model (14) to prove the convergence of $z(k)$ to the state $x(k)$ of the exact model (2).

(2) Let $\tau$ be one calculation (execution) time of (8), then $z(k)$ obtained by $M_\tau$ iterations of (8) is given at time $M_\tau \tau$. The output $\tilde{y}(k)$ is available for the calculation of (8) at time $M_\tau \tau$ (or $\tilde{y}(k - 1)$ is updated to $\tilde{y}(k)$ in calculation at time $M_\tau \tau$). By the update from $\tilde{y}(k - 1)$ to $\tilde{y}(k)$, a computer can recognize $z(k)$ at time $M_\tau \tau$.

In this paper, we give the sufficient conditions that the real-time iteration of (8) with $y(k)$ replaced by $\tilde{y}(k)$, or equivalently the approximate model (14) parameterized by $\tau > 0$, can estimate $x(k)$ of the exact model (2).

### 3. A main result

Let $\Omega \subset \mathbb{R}^n$ and $\Omega_p \subset \mathbb{R}^p$ be compact sets containing the origin. For the systems (1), (5), and (8), we introduce the following assumptions:

A1: There exist $L_f$, $L_h > 0$ satisfying $|f(x_1) - f(x_2)| \leq L_f|x_1 - x_2|$ and $|h(x_1) - h(x_2)| \leq L_h|x_1 - x_2|$ for any $x_1, x_2 \in \Omega$

A2: There exists $L_\phi > 0$ satisfying $|\phi(z_1, y_1) - \phi(z_2, y_2)| \leq L_\phi|z_1 - z_2| + |y_1 - y_2|$ for any $(z_1, y_1)$, $(z_2, y_2) \in \Omega \times \Omega_p$

A3: There exists a continuously differentiable function $V(x, z)$ such that

$$\alpha_1(|x|) \leq V(x, z) < \alpha_2(|x|), \quad (15)$$

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial z} \phi(z, h(x)) \leq -\alpha_3(|x|), \quad (16)$$

for any $x, z \in \mathbb{R}^n$ where $\xi = z - x$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$.

**Remark 3.1:** (1) By A2 and $\phi_\tau(z, y) = z + \tau \phi(z, y)$, we have $|\phi_\tau(z_1, y_1) - \phi_\tau(z_2, y_2)| \leq (1 + \tau L_\phi)|z_1 - z_2| + \tau L_\phi|y_1 - y_2|$ for any $(z_1, y_1)$, $(z_2, y_2) \in \Omega \times \Omega_p$

A2 implies $|\phi_\tau(z, y) - \phi_\tau(z, \tilde{y})| \leq (\tau + 1 + \tau L_\phi)^i - 1)|y - \tilde{y}|$ for any $i \geq 1$ and we obtain

$$|\Phi_{\tilde{y}, \tau}^{\tau}(z, y) - \Phi_{\tilde{y}, \tau}^{\tau}(z, \tilde{y})| \leq \bar{a}(\tau)|y - \tilde{y}|,$$

where $\bar{a}(\tau) = (1 + \tau L_\phi)^N - 1 \in \mathcal{K}_\infty$

A3 implies that $z(\tau(t))$ of continuous-time observer (5) converges to $x(\tau(t))$ of the system (1) globally and asymptotically.

By A1 and A2, there exist $\gamma_\Phi \in \mathcal{K}$, $\tilde{T} > 0$, and $\tau^* \in (0, \tilde{T}]$ that depends on a fixed $\tilde{T} \in (0, \tilde{T}^*)$ such that

$$|\Phi_{\tilde{y}, \tau}^{\tau}(z, y) - \Phi_{\tilde{y}, \tau}^{\tau}(z, \tilde{y})| \leq \tilde{T} \gamma_\Phi(\tau) \quad (17)$$

for any $(z, y) \in \Omega \times \Omega_p$, $\tilde{T} \in (0, \tilde{T}^*)$, and $\tau \in (0, \tau^*)$ [10].

By the one-step consistency between the Euler model and the exact model, there exist $\gamma_\Phi \in \mathcal{K}$ and
For any given $\alpha_{\pi}$, $\alpha_{\sigma} \in K_{\infty}$ and $\pi \in [0, 1)$ such that for all $k \in N_0$ and $e \in \mathbb{R}^p$
\begin{align}
\alpha_{\pi}(|e|) & \leq W(k, e) \leq \alpha_{\sigma}(|e|), \\
W(k + 1, \tilde{G}(k + 1, e)) & \leq \pi W(k, e),
\end{align}
where $\tilde{G}(k + 1, e) = G(k + 1, e, 0)$. In particular, if $\alpha_{\pi}(s) = a_\delta$, $i = 1, 2$ with $0 < a_1 \leq a_2$, then protocol (12) is UGAS with Lyapunov function $W$.

A5: Let $x(k) \in \Omega$ for any $k \in N_0$ and $|\hat{y}(0)| \leq \bar{D}$ for some $\bar{D} \in (0, \infty)$. Then there exists a compact set $\Omega_e \subset \mathbb{R}^p$ containing the origin such that $e(k) \in \Omega_e$ for any $k \in N_0$.

A6: For any given $\nu > 0$, there exist $L_G$, $T^* > 0$ such that
\begin{align}
|G(k + 1, e, x) - \tilde{G}(k + 1, e)| & \leq T_k L_G(|x| + \nu),
\end{align}
for any $x \in \Omega$ and $T_k \in (0, T^*)$.

Remark 3.2: (1) RR and TOD protocols are UGAS with Lyapunov function $W$. See Propositions 4 and 5 in [3].

(2) Let $x(k) \in \Omega$ for any $k \in N_0$, $|\hat{y}(0)| \leq \bar{D}$ for some $\bar{D} \in (0, \infty)$, $\Delta_x \geq \sup_{x \in \Omega} |h(x)|$, $\Delta_p \geq \max_{x \in \Omega} |\bar{h}(x)|$, and $\Omega_e = B_{\bar{D} + (l + 1)\Delta_p}$. Then $|\hat{y}(k)| \leq \Delta_p$ for any $k \in N_0$. For RR and TOD protocols, we have $|\hat{y}(k)| \leq \Delta_p + \bar{D}$ by the construction of $\hat{y}(k)$. Since $|e(k)| \leq |\hat{y}(k)| + |\bar{y}(k)| \leq \bar{D} + (l + 1)\Delta_p$, $e(k) \in \Omega_e$ for any $k \in N_0$ and hence RR and TOD protocols satisfy A5.

(3) For RR and TOD protocols, we have $|I - \Delta(k + 1)|, |I - \Psi(e(k)))| \leq 1$, and $|G(k + 1, e, x) - \tilde{G}(k + 1, e)| \leq |h(x)| - h(F^e_{\tilde{I}_k}(x))$. Then by the one-step consistency between the Euler model and the exact model and the locally Lipschitzness of $f$ and $h$, we obtain
\begin{align}
|G(k + 1, e, x) - \tilde{G}(k + 1, e)| & \leq L_h|x - F^e_{\tilde{I}_k}(x)| + F^e_{\tilde{I}_k}(x) - F^e_{\tilde{I}_k}(x)| \\
& \leq T_k L_h L_f |x| + y\varphi(T_k)/L_f).
\end{align}

Since $W(k, e)$ is continuous in $e$ and uniformly in $k$, for any $\delta_1 > 0$, A6 implies the existence of $T^* > 0$ such that $|W(k + 1, G(k + 1, e, x)) - W(k + 1, \tilde{G}(k + 1, e))| < \delta_1$ for any $x \in \Omega$, and $T_k \in (0, T^*)$. Then, we have
\begin{align}
W(k, e(k)) & \leq \pi^k W(0, e(0)) + \sum_{j=0}^{k-1} \pi^{k-j-1}\delta_1 \\
& \leq \pi^k W(0, e(0)) + \delta_2,
\end{align}
where $\delta_2 = \delta_1/(1 - \pi)$. Hence by (20) and (22), we have
\begin{align}
|e(k)| & \leq \alpha_{\pi}^{-1}(|\exp(-\alpha_{\sigma}k)e(0)|) + \delta_2 \\
& \leq \beta_v(|e(0)|, k) + \delta,
\end{align}
where $\beta_v(s, k) = \alpha_{\pi}^{-1}(2\exp(-\alpha_{\sigma}k)e(0)) \in KL$, $\alpha_{\pi} = -\ln\pi > 0$, and $\delta = \alpha_{\pi}^{-1}(2\delta_2)$.

Similar to [3], we assume that there exists an arbitrarily small $\epsilon > 0$ satisfying $T_k > \epsilon$ for any $k \in N_0$ in the remaining of this paper. Then, we have the following result for the system:
\begin{align}
x(k + 1) &= F^e_{\tilde{I}_k}(x(k)), \\
\hat{y}(k + 1) &= y(k + 1) + g(k + 1, e(k), y(k), y(k + 1)), \\
z(k + 1) &= \Phi^d_{\tilde{I}_k}(z(\hat{y}(k))).
\end{align}

Theorem 3.1: Let $\tau > 0$ be chosen arbitrarily, $|\hat{y}(0)| \leq \bar{D}$ for some $\bar{D} \in (0, \infty)$, and assume A1–A6. Then
(1) For any given $\delta > 0$, there exists $T^* > 0$ such that the error $e(k)$ induced by networks satisfies (23).

(2) The observer (14) (or equivalently, the real-time iteration of (8) with $y(k)$ replaced by $\hat{y}(k)$) is semiglobal in $T_k$ and practical in $T_k$ and $\tau$ for the exact model (2), i.e. for given positive real numbers $0 < d_1 < D$ and a compact set $\Omega \subset \mathbb{R}^p$, there exist $\beta \in KL$, $T^* > 0$, and $\tau^* > 0$ that depends on $\epsilon > 0$ and
\[ d_2 \in (0, D - d_1) \text{ such that } |z(0) - x(0)| \leq D \text{ and } x(k) \in \Omega \text{ for any } k \in N_0 \text{ imply } \\
|z(k) - x(k)| \leq \beta(|z(0) - x(0)|, s_k) + d_1 + d_2 \]

for any \( T_k \in (\epsilon, T^*) \) and \( \tau \in (0, \tau^*) \).

**Remark 3.3:** Under practical situations, \( \epsilon > 0 \) and \( T^* > \epsilon \) cannot be chosen arbitrarily due to performance limitations of hardware such as A/D converters and D/A converters and networks. Then there usually remains an offset \( d_1 > 0 \) that depends on \( T^* \) and \( \epsilon \). But \( \tau^* > 0 \) depends only on the performance of digital computers that are improved very fast and hence we can always choose sufficiently small \( \tau^* \in (0, \epsilon) \) to make \( d_2 \) small.

### 3.1. Proof of Theorem 3.1

Let \((\Delta_x, \Delta_z, \Delta_e)\) be positive real numbers, \( R_x = \Delta_x + 1 \), and \( R_z = \Delta_z + 1 \). Let \( l_f, l_h, l_\phi > 0 \) be Lipschitz constants on \(|x| \leq R_x, |z| \leq R_z, \) and \(|y| \leq \max_{|x| \leq R_x} |h(x)| + \Delta_o \), and let \( b > 0 \) satisfy

\[
\max \left\{ \frac{\partial V}{\partial x}(x, z), \frac{\partial V}{\partial z}(x, z), |f(x)|, |\phi(h(x) + e)| \right\} \leq b
\]

for any \(|x| \leq R_x, |z| \leq R_z, \) and \(|e| \leq \Delta_e \). Without loss of generality, we assume \( b \geq 1 \). Let \((x, z, e) \in B_{\Delta_x} \times B_{\Delta_z} \times B_{\Delta_e} \text{ and } \hat{y} = h(x) + e \). Then there exist \( T^*_0 > 0 \) and \( \gamma_T \in K \) satisfying (18) for any \( T \in (0, T^*_0) \) and there exist \( T^*_0 \) and \( \gamma_T \in K \) satisfying (19) for any \( T \in (0, T^*_0) \). Moreover, there exist \( T^*_{02} > 0 \) and \( \tau^* > 0 \) satisfying (17) for any \( \tau \in (0, \tau^*_0) \). We set

\[
T^*_1 = \min \left\{ T^*_0, T^*_0, T^*_{02}, \frac{1}{2b}, \gamma_T^{-1} \left( \frac{1}{2b} \right), \gamma_f^{-1} \left( \frac{1}{2b} \right) \right\},
\]

\[
\tau^*_1 = \min \left\{ \tau^*_0, \gamma_T^{-1} \left( \frac{1}{2b} \right) \right\}.
\]

Let \( \tilde{\tau}, \tilde{T} \in (\epsilon, T^*_0) \) and \( \tau \in (0, \tau^*_0) \). For any \((x, z, e) \in B_{\Delta_x} \times B_{\Delta_z} \times B_{\Delta_e} \) the solutions \( x_t(t) \) and \( z_t(t) \) of the initial value problems \( \dot{x}_t = f(x_t(t), x_0) = x, \) and \( \dot{z}_t = \phi(z_t(t), \hat{y}) = z \) satisfy \( |x_t(t)| \leq \Delta_x + \frac{1}{2} \) for all \( t \in [0, \tilde{T}] \) and \( |z_t(t)| \leq \Delta_z + \frac{1}{2} \) for all \( t \in [0, \tilde{T}] \), respectively. Then, we have \( |f_{\tilde{T}}(x)| \leq \Delta_x + \frac{1}{2} < R_x \) and \( |\phi_{\tilde{T}}(z, \hat{y})| \leq \Delta_z + \frac{1}{2} < R_z \). Remarks 3.1, (4), (5), \( b \geq 1, \) and the definitions of \( T^*_1 \) and \( \tau^*_1 \) imply \( |\Phi_{\tilde{T}}(z, \hat{y})| \leq |\Phi_{\tilde{T}}(z)| + |\Phi_{\tilde{T}}(\hat{y})| \leq \Delta_x + \frac{1}{2} + \Delta_z + \frac{1}{2} \), \( \gamma_{\Phi}(\tau^*_0) < R_x \) and Remarks 3.1, (5) with the same discussion also imply \( |F_{\Phi}(x)| < R_x \) and \( |\Phi_{\tilde{T}}(z, \hat{y})| < R_z \).

Now, we assume that all conditions in Theorem 3.1 are satisfied and we introduce the following lemmas. Proofs are given in the Appendix.

**Lemma 3.2:** For given positive real numbers \((\Delta_x, \Delta_z, \Delta_e, v_1)\), there exist \( T^* \in (\epsilon, T^*_1) \) and \( \tau^* \in (0, \tau^*_1) \) that depends on \( \epsilon > 0 \) and arbitrary \( v_2 > 0 \) such that

\[
\frac{\Delta V_k}{T_k} \leq -\alpha_3(|\xi(k)|) + v_1 + v_2
\]

for any \((x(k), z(k)) \in B_{\Delta_x} \times B_{\Delta_z}, T_k \in (\epsilon, T^*), \) and \( \tau \in (0, \tau^*) \) where \( \Delta V_k = V(P^k_{T_k}(x(k)), \Phi_{\tilde{T}_k, \tau}^{\epsilon}(z(k), h(x(k)))) - V(x(k), z(k)) \).

**Lemma 3.3:** For given positive real numbers \((\Delta_x, \Delta_z, \Delta_e, v_1)\), there exist \( T^* \in (\epsilon, T^*_1) \) and \( \tau^* \in (0, \tau^*_1) \) that depends on \( \epsilon > 0 \) and arbitrary \( v_2 > 0 \) such that

\[
\frac{\Delta V_k}{T_k} \leq -\alpha_3(|\xi(k)|) + v_1 + v_2
\]

for any \((x(k), z(k), e(k)) \in B_{\Delta_x} \times B_{\Delta_z} \times B_{\Delta_e}, T_k \in (\epsilon, T^*), \) and \( \tau \in (0, \tau^*) \) where \( \Delta V_k = V(P^k_{T_k}(x(k)), \Phi_{\tilde{T}_k, \tau}^{\epsilon}(z(k), h(x(k))) + e(k)) - V(x(k), z(k)) \).

**Proof of Theorem 3.1:** Proof of (1) is obvious from Remarks 3.2, (4). Let \( \Omega \subset \mathbb{R}^b \) be a compact set containing the origin, \( D > d_1 > 0, \) and \( \tilde{D} \in (0, \infty) \). Let \( |z(0) - x(0)| \leq D, |\dot{y}(0)| \leq \tilde{D}, \) and \( x(k) \in \Omega \) for any \( k \in \mathbb{N}_0 \). For simplicity of notation, let \( x = x(k), z = z(k), e = e(k), \xi = \xi(k), \rho x = x(k + 1), \) and \( \rho z = z(k + 1) \).

Let \( 0 < r < R \) be given arbitrarily. Then, we first show that if \( r \leq V(x, z) \leq R, \) then there exist \( T^*_2 \in (\epsilon, T^*_1) \) and \( T^*_2 \in (0, \tau^*_1) \) that depends on \( \epsilon, v_2 > 0 \) such that

\[
\frac{\Delta V_k}{T_k} \leq -\frac{1}{2} y_3(|\xi|)
\]

for any \( T_k \in (\epsilon, T^*_2) \) and \( \tau \in (0, \tau^*_2) \). Let \( \Delta_x \) and \( \Delta_z \) be such that \( \Delta_x \geq \sup_{x \in \overline{\Omega}} |x| \text{ and } \Delta_z \geq \sup_{x \in \overline{\Omega}} |x| + 1 \text{ for any } k \in \mathbb{N}_0 \). By A5 and Remark 3.2, (2), there exist a compact set \( \Omega_0 \subset \mathbb{R}^b \) containing the origin such that \( e(k) \in \Omega_0 \) for any \( k \in \mathbb{N}_0 \) and we have \(|e(k)| \leq \sup_{x \in \overline{\Omega}} |e| =: \Delta_e \text{ for any } k \in \mathbb{N}_0 \). Let \( v_1, v_2 > 0 \) be such that \( v_1 + v_2 \leq \alpha_3(\alpha_2^{-1}(r))/2 \). Then by Lemma 3.3, there exist \( T^*_2 \in (\epsilon, T^*_1) \) and \( T^*_2 \in (0, \tau^*_1) \) such that

\[
\frac{\Delta V_k}{T_k} \leq -\alpha_3(|\xi|) + v_1 + v_2
\]

for any \( T \in (\epsilon, T^*_2) \) and \( \tau \in (0, \tau^*_2) \). Moreover, \( V(x, z) \geq r \) and (15) imply \(|\xi| \geq \alpha_1^{-1}(V(x, z)) \geq \alpha_1^{-1}(r) \) and we have \( v_1 + v_2 \leq \alpha_3(\alpha_2^{-1}(r))/2 < \alpha_3(|\xi|)/2 \). Hence by (29), we obtain (28).
If \( V(x,z) \leq r \), then \( V(\rho x, \rho z) \leq r + T_k(v_1 + v_2) \). Let \( T^*_k \in (\epsilon, T^*_k) \) be such that \( r + T^*_k(v_1 + v_2) \leq R \) and \( T_k \in (\epsilon, T^*_k) \) for any \( k \in \mathbb{N}_0 \).

Let
\[
\eta_T(t) = V(x,z) + \frac{t - s_k}{T_k} [V(\rho x, \rho z) - V(x,z)]
\]
for any \( t \in [s_k, s_{k+1}) \) and \( k \in \mathbb{N}_0 \). Note that \( \eta_T(t) \) is continuous, nonnegative, piecewise linear function of \( t \), and its time derivative is given by
\[
\dot{\eta}_T(t) = \frac{\Delta V_k}{T_k}
\]
for any \( t \in [s_k, s_{k+1}) \) and almost all \( t \). Then \( V(x,z) \geq r \) implies (28) and we have \( \dot{\eta}_T(t) \leq 0 \), \( V(\rho x, \rho z) \leq V(x,z) \), and \( \eta_T(t) \leq V(x,z) = \eta_T(s_k) \) for any \( t \in [s_k, s_{k+1}) \). By (15), we have \( |\xi(k)| \geq \alpha_2^{-1}(V(x,z)) \geq \alpha_2^{-1}(\eta_T(t)) \) for any \( t \in [s_k, s_{k+1}) \) and we obtain
\[
\dot{\eta}_T(t) \leq -\frac{1}{2} \alpha_3((\xi(k))^2) \leq -\frac{1}{2} \alpha_3 \circ \alpha_2^{-1}(\eta_T(t))
\]
for any \( t \in [s_k, s_{k+1}) \) and \( k \in \mathbb{N}_0 \). Hence if \( V(x(k), z(k)) \geq r \) for any \( k \in \mathbb{N}_0 \), then there exists \( \beta_1 \in KL \) that depends on \( \alpha_3 \circ \alpha_2^{-1} \) such that \( \eta_T(t) \leq \beta_1(\eta_T(0), t) \) for any \( t \geq 0 \). Since \( V(x,z) = \eta_T(s_k) \) and \( \eta_T(0) = V(x(0), z(0)) \), we have \( V(x,z) \leq \beta_1(V(x(0), z(0)), s_k) \).

Hence \( V(x(0), z(0)) \leq R \) implies
\[
V(x(k), z(k)) \leq \max\{\beta_1(V(x(0), z(0)), s_k), r + T^*_k(v_1 + v_2)\}.
\]
By (15) and (30), we have
\[
|\xi(k)| \leq \max\{\alpha_1^{-1}(\beta_1(\alpha_2(|\xi(0)|), s_k)), \alpha_1^{-1} \times (r + T^*_k(v_1 + v_2))\}
\leq \beta(|\xi(0)|, s_k) + \alpha_1^{-1}(2(r + T^*_k v_1))
\leq \alpha_1^{-1}(T^*_k v_2),
\]
where \( \beta(s,t) = \alpha_1^{-1}(\beta_1(\alpha_2(s,t))) \in KL \). For given \( D > d_1 > 0 \), let \( R = \alpha_2(D) \), \( r = \alpha_1(d_1)/4 \), and let \( T^* \in (\epsilon, T^*_k) \) be such that \( v_1 T^* \leq \alpha_1(d_1)/4 \). Then, we have \( |\xi(0)| \leq \alpha_2(R) \) and \( \alpha_1^{-1}(2(r + T^*_k v_1)) \leq \alpha_1^{-1}(\alpha_2(d_1)) = d_1 \). Moreover, for given \( d_2 \in (0, D - d_1) \), there exists \( \tau^* \in (0, T^*_k) \) that depends on \( \epsilon, d_2 > 0 \) such that \( 2T^*_k \leq \alpha_1(d_2) \) for any \( \tau \in (0, \tau^*) \). Since \( \alpha_1^{-1}(2T^*_k v_2) \leq d_2 \), we have the assertion.

4. Numerical examples

Example 4.1: Consider
\[
\hat{x}_c = f(x_c(t)), \quad y_c(t) = h(x_c(t)) = Cx_c(t),
\]
where \( x_c = [x_{c1} x_{c2} x_{c3} x_{c4}]^T \), \( f(x) = Ax + g(x), C = [10000], \)
\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]
\[
g(x) = \begin{bmatrix} 0 \\ 0 \\ -a_1 x_1 + a_1 x_2 - 1.25 x_3 \\ a_2 x_1 - a_2 x_2 - 3.3 \sin x_3 \end{bmatrix},
\]
a_1 = 48.6, and \( a_2 = 19.5 \). The system (31) comes from the flexible joint robotic arm model without control input [5]. Then a continuous-time high gain observer
\[
\hat{z}_c = \phi(z_c(t), y_c(t)) = Az_c(t) + g(z_c(t)) + \theta \Delta^{-1} S^{-1} \times C^T [y_c(t) - Cz_c(t)]
\]
achieves the convergence to \( x_c(t) \) exponentially for sufficiently large \( \theta > 1 \) where \( \Delta = \text{diag}[I_{2 \times 2}, \theta^{-1} I_{2 \times 2}] \) and \( S > 0 \), which is the solution of the algebraic Lyapunov equation \( SA + A^T S - C^T C = -S \), is given by
\[
S = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}.
\]
Moreover, faster convergence of \( z_c(t) \) to \( x_c(t) \) is obtained for larger \( \theta > 1 \).

In the following, we set \( \theta = 40 \) as in [5] and we replace \( y_c(t) \) by the sampled observation \( y(k) \) that is divided into two nodes, i.e. \( y(k) = [y_1(k) y_2(k)]^T \) where \( y_i(k) = x_{ci}(s_k) \), \( i = 1, 2 \). Then consider the networked system (31) with the RR protocol whose sampling intervals \( T_k = s_{k+1} - s_k \in [0.002, 0.01] \) are time-varying and unknown. Here, we use \( T^* = 0.01 \) given in [5]. It guarantees that the hybrid dynamical model representation of the observer (32) and the RR protocol achieves the convergence to \( x_c(t) \) exponentially. But it cannot be implemented in digital computers. We introduce a sufficiently small time step \( \tau > 0 \) and we use the real-time iteration of (7)–(9) with \( y(k) \) replaced by \( \hat{y}(k) \) to implement the emulation of continuous-time observer
\[
\hat{z}_c = \phi(z_c(t), \hat{y}(k)) \quad \forall \ t \in [s_k, s_{k+1}),
\]
digitally.

Let the initial conditions of the system (31), the observer (33), and the output from the network be \( x_c(0) = [1 2 - 2 1]^T \), \( z(0) = 0_{4 \times 1} \), and \( \hat{y}(0) = 0_{2 \times 1} \), respectively. Moreover, let \( \tau = 0.001 \) [5]. Figure 1 shows a simulation result of the time response of the state \( x_{c3}(t) \) of the system (31), the state \( z_{c3}(t) \) of the continuous-time observer (32), and \( z_3(k) \) that is obtained by the real-time iteration of (7)–(9) with \( y(k) \) replaced by \( \hat{y}(k) \) where the lines from the top express \( z_3(k), z_{c3}(t), \) and \( x_{c3}(t) \), respectively. Figure 2 shows a simulation result of the time response of \( |x_c(s_k) - \hat{z}(k)| \) for \( t \in [2, 10] \). By Figure 1, we see that the proposed digital implementation of (32) gives a good estimate of \( x_c(s_k) \) and recovers the performance of continuous-time observer (32) for \( T_k \in [0.002, 0.01] \) and \( \tau = 0.001 \).
Example 4.2: For a continuous-time system

\[ \dot{x}_c = f(x_c(t)) = Ax_c(t) + g(x_c(t)), \quad y_c(t) = Cx_c(t), \quad (34) \]

a continuous-time observer

\[ \dot{z}_c = \phi(z_c(t), y_c(t)) = Az_c(t) + g(y_c(t)) \]
\[ + K[y_c(t) - Cx_c(t)] \quad (35) \]

achieves the convergence of \( z_c(t) \) to \( x_c(t) \) exponentially where

\[ A = \begin{bmatrix} -1.1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad g(x) = \begin{bmatrix} [x_1] \\ 0 \\ 0 \end{bmatrix}, \]
\[ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} -0.1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}. \]

Then \( V(x,z) = \xi^T X \xi \) satisfies A3 with \( \alpha_1(s) = 0.3 \xi^2 \), \( \alpha_2(s) = 1.32 \xi^2 \), and \( \alpha_3(s) = s^2 \) where \( X > 0 \) is the solution of the algebraic Lyapunov equation \((A - KC)^T X + X(A - KC) = -I\).

We now estimate \( T^* > 0 \) that satisfies (24). By Remark 3.3, we make \( d_2 > 0 \) and \( \nu_2 > 0 \) in Lemmas 3.2 and 3.3 sufficiently small for sufficiently small \( \tau > 0 \). From the proofs of Theorem 3.1 and Lemma 3.2, let \( R = \alpha_2(D), \ r = \alpha_1(d_1)/4, \) and \( \nu_1 < \alpha_3^{-1}(\alpha_2(r))/2 \) for given \( D > d_1 > 0 \). Then \( T^* > 0 \) is the maximum of \( T \) that satisfies \( \Omega_1(x,z,T) \leq \nu_1, \ r + T \nu_1 \leq R, \) and \( \nu_1 T \leq r \) where \( \Omega_1(x,z,T) \) is given in the proof of Lemma 3.2. Note that \( T^* \leq 1/2 \) by (25) and \( \nu_1 T \leq r \) implies \( r + T \nu_1 \leq R \) from \( 2r \leq R \). By [18], we have \( \gamma_f(T) = T(L_f^2/2) \exp(L_f T) \) where \( L_f \) is a Lipschitz constant of \( f \). For (34), \( L_f \simeq 3.28 \).

Let \( d_1 = 1 \). Then we have \( r = 7.5 \times 10^{-2}, \nu_1 = 4.31 \times 10^{-2}, \) and we have \( \nu_1 T \leq r \) for any \( T \in (0, 1/2] \). Since \( \gamma_f(T) \leq \Omega_1(x,z,T) \leq \nu_1, \gamma_f(0.01) = 5.56 \times 10^{-2}, \) and \( \gamma_f(0.005) = 2.73 \times 10^{-2}, \) we consider \( T = 0.005 \) as a candidate of \( T^* > 0 \). But \( T^* = 0.005 \) is conservative. In fact, Figure 3 shows the simulation result of time responses of \( z_3(k), z_{C3}(t), \) and \( x_{C3}(t) \) from the top and Figure 4 shows the time response of \( |x_c(s_k) - z(k)| \) for \( t \in [5, 20] \). As we see Figures 3 and 4, we obtain the convergence of \( z(k) \) to \( x_c(s_k) \), the performance recovery of the continuous-time observer (35), and much smaller offset \( d_1 + d_2 \) than 1 for \( (T, \tau) = (0.01, 0.001) \).
Figure 3. Time response of $x_{c3}(t)$, $x_{c3}(s_k)$, and $z_3(k)$.

Figure 4. Time response of $|x_c(s_k) - z(k)|$.

Remark 4.1: Example 4.2 implies that $T^* > 0$, which is obtained based on the proofs of Theorem 3.1 and Lemma 3.2, becomes conservative. On the other hand, the hybrid dynamical model representation of continuous-time observers and protocols considered in [5, 7, 8] give MATIs that guarantee exponential convergence of the estimated state to that of continuous-time systems. Although it cannot be implemented in digital computers, it is still possible to use MATIs as a candidate of $T^* > 0$. In Example 4.1, we used the MATI given in [5] as $T^* > 0$.

5. Conclusion

In this paper, a digital implementation of continuous-time observers have been considered for networked control systems. First, we have proposed a real-time iteration of the Euler model parameterized by a sufficiently small constant time step to implement continuous-time observers digitally. Then, we have given sufficient conditions that the proposed digital implementation can estimate the state at sampling times semiglobally and practically. Numerical examples have also been given to illustrate the proposed digital implementation of continuous-time observers.

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Notes on Contributor

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Appendix. Proof of Lemmas 3.2 and 3.3

For simplicity of notation, let \( T = T_k, \bar{T} = \bar{T}_k, x = x(k), z = z(k), \xi = \xi(k), e = e(k), f = f(x), \) and \( F_1^E = F_1^E(x), i = e, E. \)

Proof of Lemma 3.2: For \( \theta \in (0, 1), \) let \( x_1^* = \theta F_1^E + (1 - \theta) F_2^E, x_2^* = \theta F_2^E + (1 - \theta) x = x + \theta T \phi f, z_1^* = \theta \Phi_{1, \tau}^E, z_2^* = \theta \Phi_{2, \tau}^E + (1 - \theta) \Phi_{2, \tau}^E, \) and \( z_3^* = \theta \Phi_{1, \tau}^E + (1 - \theta) z = z + \theta T \phi f \) where \( \Phi_{1, \tau}^E = \Phi_{1, \tau}^E(z(h(x)), \phi), \phi_{1, \tau}^E = \phi_{1, \tau}^E(z(h(x)), i = e, E, \) and \( \phi = \phi(z(h(x))). \) Then, we have \(|x_1^*| \leq R_k \) and \(|x_2^*| \leq R_k, i = 1, 2, j = 1, 2, 3. \)

Note that

\[
\Delta V_k = I(1) + I(2) + I(3),
\]

\[
I(1) = \left[ \frac{\partial V}{\partial x} (x, z) + \frac{\partial V}{\partial z} (x, z) \phi \right],
\]

\[
I(2) = \left[ V(F_1^E, \Phi_{1, \tau}^E) - V(F_2^E, \Phi_{2, \tau}^E) \right] \Phi_{1, \tau}^E
\]

\[
I(3) = \left[ V(F_1^E, \Phi_{1, \tau}^E) - V(F_2^E, \Phi_{2, \tau}^E) \right] \frac{\partial V}{\partial x} (x, z) \phi
\]

By A3, we have \( I(1) \leq -T \alpha_0(x) |\xi| \) and by the mean value theorem \([17], \) Remark 3.1, (4), (5), and \( b \geq 1, \) we obtain

\[
I(2) = V(F_1^E, \Phi_{1, \tau}^E) - V(F_2^E, \Phi_{2, \tau}^E)
\]

\[
- V(F_1^E, \Phi_{2, \tau}^E) - V(F_2^E, \Phi_{2, \tau}^E)
\]

\[
\leq \left[ \frac{\partial V}{\partial x} (x_1^*, \Phi_{1, \tau}^E) \right] |F_1^E - F_2^E|\]

\[
+ \left[ \frac{\partial V}{\partial z} (F_1^E, z_1^*) \right] |\Phi_{1, \tau}^E - \Phi_{2, \tau}^E|\]

\[
+ \left[ \frac{\partial V}{\partial z} (F_2^E, z_2^*) \right] |\Phi_{2, \tau}^E - \Phi_{2, \tau}^E|
\]

\[
\leq b|\tau T g(T) + \tau T g(T) + \tau T g(T)|
\]

\[
\leq b|T| g(T) + \tau T g(T) + \tau T g(T)
\]

Similarly, by the mean value theorem and Remark 2.1, we have

\[
I(3) = \left[ \frac{\partial V}{\partial x} (x_1^*, \Phi_{1, \tau}^E) - \frac{\partial V}{\partial x} (x, z) \right] f
\]

\[
+ \left[ \frac{\partial V}{\partial z} (x, z_1^*) \phi - \frac{\partial V}{\partial z} (x, z) \phi \right]
\]

\[
= \left[ \frac{\partial V}{\partial x} (x, z + \theta T \phi f) - \frac{\partial V}{\partial x} (x, z) \phi \right]
\]

\[
+ \left[ \frac{\partial V}{\partial z} (x, z) + \theta T \phi f \right]
\]

\[
+ \left[ \frac{\partial V}{\partial z} (x, z_2^*) - \frac{\partial V}{\partial z} (x, z + \theta T \phi f) \phi
\]

\[
\leq T b \Delta_1(x, z, T)
\]

\[
+ T b \Delta_2(x, z, T, r) + b^2 r,
\]
where
\[ \Lambda_1(x, z, T) = \frac{\partial V}{\partial x} (x^*_1, \Phi^E_1) - \frac{\partial V}{\partial x} (x, z) ] + \frac{\partial V}{\partial z} (x, z + \theta T \phi) - \frac{\partial V}{\partial z} (x, z) \right) \]
\[ \Lambda_2(x, z, T, \tau) = \frac{\partial V}{\partial x} (x^*_1, \Phi^E_1) - \frac{\partial V}{\partial x} (x^*_1, \Phi^E_1) + \frac{\partial V}{\partial z} (x, z^*_1) - \frac{\partial V}{\partial z} (x, z + \theta T \phi) \right) . \]

Hence we obtain
\[ \frac{\Delta \tilde{V}_k}{T} \leq -\alpha_3(|\xi|) + \Omega_1(x, z, T) + \Omega_2(x, z, T, \tau), \]
where
\[ \Omega_1(x, z, T) = b [\gamma_\phi(T) + \gamma_\Phi(T) + \Lambda_1(x, z, T)], \]
\[ \Omega_2(x, z, T, \tau) = b \left( |\gamma_\phi(T + \tau) - \gamma_\phi(T)| + \Lambda_2(x, z, T, \tau) + \frac{T}{T} \gamma_\phi(\tau) + \frac{T}{T} [b + \gamma_\phi(T)] \right) . \]

Let \( v_1 > 0 \) be arbitrary. By the continuity of \( \frac{\partial V}{\partial x} \) and \( \frac{\partial V}{\partial z} \), \( \gamma_\phi, \gamma_\Phi \in \mathcal{K} \), and the definitions of \( x^*_1 \) and \( z^*_1 \), there exists \( T^* \leq T^*_1 \) such that \( \Omega_1(x, z, T) \leq v_1 \) for any \( T \in (\epsilon, T^*) \). Let \( T \in (\epsilon, T^*) \) and \( v_2 > 0 \) be given arbitrarily. Then, there exists \( \tau^* \leq T^*_1 \) such that \( \Omega_2(x, z, T, \tau) \leq v_2 \) for any \( T \in (0, \tau^*) \), since \( \gamma_\phi \in \mathcal{K} \), \( |z^*_1 - (z + \theta T \phi)| \leq b \tau \), \( |\Phi^E_1 - \Phi^E_1| \leq b \tau \), \( \tau / T \leq \tau / \epsilon \), and \( T / T \leq 1 + (\tau / \epsilon) \). Hence we have (26) for any \( T \in (\epsilon, T^*) \) and \( \tau \in (0, \tau^*). \]

**Proof of Lemma 3.3:** Note that \( \Delta V_k = \Delta \tilde{V}_k + I(4) \) where \( I(4) = V(P_{T_1}^e(x), \Phi^E_{T_1} (z, h(x) + e)) - V(P_{T_2}^e(x), \Phi^E_{T_2} (z, h(x))) \). By Lemma 3.2, there exist \( T^* > 0 \) and \( \tau^*_1 > 0 \) such that
\[ \frac{\Delta \tilde{V}_k}{T} \leq -\alpha_3(|\xi|) + v_1 + v_2 \]
for any \( T \in (\epsilon, T^*) \) and \( \tau \in (0, \tau^*_1) \). Let \( T \in (\epsilon, T^*), \tau \in (0, \tau^*_1), \theta \in (0, 1) \), and \( z^*_1 = \theta \Phi^E_{T_1} (z, h(x) + e) + (1 - \theta) \Phi^E_{T_1} (z, h(x)) \). Then, we have \( |z^*_1| < R_1 \) and we apply the mean value theorem and Remark 3.1, (3) to \( I(4) \) to obtain
\[ I(4) \leq \left| \frac{\partial V}{\partial z} (P_{T_1}^e, z^*_1) \right| |\Phi^E_{T_1} (z, h(x) + e) - \Phi^E_{T_1} (z, h(x))| \leq b |\phi_{T_1} (z, h(x) + e) - \phi_{T_1} (z, h(x))| \leq b \bar{a}(\tau)|e| \leq b \Delta_s \bar{a}(\tau). \]

Hence we have
\[ \frac{\Delta V_k}{T} \leq -\alpha_3(|\xi|) + v_1 + v_2 + \frac{b \Delta_s \bar{a}(\tau)}{T}. \]

Since \( \bar{a} \in \mathcal{K}_\infty \), there exists \( \tau^* \leq \tau^*_1 \) such that
\[ \tilde{v}_2 + \frac{b \Delta_s \bar{a}(\tau)}{T} \leq \tilde{v}_2 + \frac{b \Delta_s \bar{a}(\tau)}{\epsilon} \leq v_2 \]
for any \( \tau \in (0, \tau^*). \) Hence there exist \( T^* > 0 \) and \( \tau^* > 0 \) that depends on \( \epsilon > 0 \) and \( v_2 > 0 \) satisfying (27) for any \( T \in (\epsilon, T^*) \) and \( \tau \in (0, \tau^*). \)