1 Introduction

The Bianchi-Bäcklund transforms, which were introduced by Bianchi [2] for surfaces of positive constant Gauss curvature (equivalently, solutions of sinh-Gordon equation), are an extension of the Bäcklund transforms [7] for surfaces of negative constant Gauss curvature (equivalently, solutions of sine-Gordon equation). In contrast to the negative case, a Bianchi-Bäcklund transform of a real surface is in general complex. In order to obtain a new real surface with positive constant Gauss curvature (CGC $K > 0$), one has to apply two successive Bianchi-Bäcklund transforms, where the second transform has to be matched to the first in a particular way.

In [11], A. Mahler showed in a constructive way that the classical Bianchi-Bäcklund procedure for obtaining a new real CGC $K > 0$ surface $\tilde{f}$ out of an old one $f$ amounts to dressing the extended framing associated to the Gauss map of $f$, which is an harmonic map, by a certain dressing matrix. In this paper we shall give an alternative approach to Mahler’s work. Following the philosophy developed by Terng and Uhlenbeck [15], we start with certain basic elements, the simple factors, for which the dressing action can be computed explicitly. We show that each single Bianchi-Bäcklund transform corresponds to the dressing action of a certain simple factor. A nice geometrical parameterization of these simple factors is available and we shall see how to relate it with the classical parameterization of Bianchi-Bäcklund transforms. As a consequence, we recover the result announced by Mahler. Moreover:

Sophus Lie proved that every Bäcklund transformation is a combination of transformations of Lie and Bianchi. Lie transformations correspond to the invariance of sine-Gordon equation under Lorentz transforms and Bianchi transformations are Bäcklund transformations for which the tangent spaces at corresponding points on the original surface and the new surface are orthogonal. On the other hand, Bonnet observed that sinh-Gordon
equation admits an invariance similar to that of Lie. In this paper, we shall show how this invariance of sinh-Gordon equation can be combined with those Bianchi-Bäcklund transformations satisfying the orthogonality condition of Bianchi in order to produce all Bianchi-Bäcklund transformations.

Finally, we shall mention that this correspondence between Bianchi-Bäcklund transforms and dressing actions of simple type has already been established within another setting [3, 9, 10]. In fact, each CGC $K > 0$ surface corresponds to a pair of parallel constant mean curvature (CMC) surfaces. Hence, by applying the two-step Bianchi-Bäcklund procedure, we can transform a CMC surface in a new CMC surface. Since CMC surfaces are isothermic, they also allow Darboux transformations via sphere congruences. Burstall [3] showed that Darboux transformations for isothermic surfaces correspond to certain loop group actions of the simple type. On the other hand, Darboux transforms of CMC surfaces are equivalent to Bianchi-Bäcklund transforms of CMC surfaces [9, 10]. Hence, the two-step Bianchi-Bäcklund procedure for CGC $K > 0$ surfaces corresponds to loop group actions of the simple type. In our approach, we study CGC $K > 0$ surfaces via their Gauss maps, which are harmonic maps into the unit sphere $S^2$. Since there is no well established theory of CMC surfaces unifying the harmonic map and isothermic surface theories, we do not know how to relate our dressing action to that of F. Burstall [3] (the underlying symmetry groups seem quite different).

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2 CMC Surfaces, CGC Surfaces, and Harmonic Maps

We start by reviewing some well known facts about CMC and CGC surfaces, and their correspondence with harmonic maps into $S^2$. We refer the reader to [8] for details.

Consider on $\mathbb{C}$ the standard conformal structure and let $(\cdot, \cdot)$ be the standard inner product of $\mathbb{R}^3$. A smooth map $\varphi : \mathbb{C} \to S^2$ is harmonic if, and only if, the one form $\varphi \times *d\varphi$ is closed. Hence, if $\varphi : \mathbb{C} \to S^2$ is harmonic, there exists $F : \mathbb{C} \to \mathbb{R}^3$ such that $dF = \varphi \times *d\varphi$. Moreover, $F$ is an immersion if, and only if, $\varphi$ is an immersion. In this case: $F$ is a CGC $K = 1$ surface; the conformal structure on $\mathbb{C}$ is given by the second fundamental form $\Pi_F$ of $F$; and $F$ has an umbilic point at $p \in \mathbb{C}$ if, and only if, $\varphi$ is conformal at $p$. Conversely, given a CGC $K = 1$ immersion $F : \mathbb{C} \to \mathbb{R}^3$, $\Pi_F$ is definite and the corresponding Gauss map $\varphi : (\mathbb{C}, \Pi_F) \to S^2$ is harmonic.

Any harmonic map $\varphi : \mathbb{C} \to S^2$ which is everywhere non-conformal is the Gauss map
of two conformal CMC $H = \frac{1}{2}$ immersions. These immersions have no umbilic points and are given by $f^\pm = F \pm \varphi : \mathbb{C} \to \mathbb{R}^3$, where $F$ (not necessarily an immersion) is such that $dF = \varphi \times * d\varphi$. Conversely, if $f : \mathbb{C} \to \mathbb{R}^3$ is a conformal CMC $H = \frac{1}{2}$ immersion without umbilic points, the corresponding Gauss map $\varphi : \mathbb{C} \to S^2$ is such that $dF = \varphi \times * d\varphi$, with $F = f + \varphi$, which means that $\varphi$ is harmonic and everywhere non-conformal.

When a conformal CMC $H = \frac{1}{2}$ immersion has no umbilic points, it is well known that we can always choose a conformal coordinate $z = x + iy$ with respect to which the second fundamental form of $f$ is diagonal, that is, $z$ is a conformal curvature line coordinate on $\mathbb{C}$. More precisely, we have

$$I_f = e^{2\omega}(dx^2 + dy^2), \quad \Pi_f = e^{\omega}(\sinh \omega dx^2 + \cosh \omega dy^2),$$

where $\omega : \mathbb{C} \to \mathbb{R}$ is a solution to the Gauss (sinh-Gordon) equation

$$\triangle \omega + \sinh \omega \cosh \omega = 0 . \quad (1)$$

So, away from the points where $\omega$ vanishes, $F = f + \varphi$ is a CGC $K = 1$ immersion and we have

$$I_F = \cosh^2 \omega dx^2 + \sinh^2 \omega dy^2, \quad \Pi_F = - \sinh \omega \cosh (dx^2 + dy^2) . \quad (2)$$

Bonnet observed that if $\omega$ is a solution of (1) and $\sigma \in \mathbb{C}$, then $\omega \circ R_\sigma$ is also a solution, where

$$R_\sigma(x, y) = (\cos \sigma x - \sin \sigma y, \sin \sigma x + \cos \sigma y) \quad (\text{cf. [2]}).$$

Hence, to every CGC $K = 1$ surface $F$, there is an associated $S^1$-family of CGC $K = 1$ surfaces. We can enlarge this family: if $\omega$ is a solution of sinh-Gordon equation, then so is $\omega^\lambda(z, \bar{z}) = \omega(\lambda z, \lambda^{-1} \bar{z})$, for each $\lambda \in \mathbb{C}^*$; hence, there is an associated smooth $\mathbb{C}^*$-family of CGC $K = 1$ surfaces $F^\lambda$. We denote $F^\lambda = S_\sigma(F)$, with $\lambda = e^{-i\sigma}$. In general, for $\lambda \notin S^1$, observe that $\omega^\lambda$ is a complex solution and, consequently, $F^\lambda$ is a complex surface.

**Pseudospherical surfaces.** Denote by $(\mathbb{R}^2, \epsilon)$ the Minkowski space, where $\epsilon := dudv$ is the Minkowski metric in characteristic coordinates: $u = x + y$ and $v = x - y$. Define the star operator on one-forms by $(Adu + Bdv) = Adu - Bdv$. Again, a smooth map $\varphi : (\mathbb{R}^2, \epsilon) \to S^2$ is (Lorentz) harmonic if, and only if, the one form $\varphi \times * d\varphi$ is closed (cf. [12]). Hence, if $\varphi : (\mathbb{R}^2, \epsilon) \to S^2$ is harmonic, there exists $F : \mathbb{R}^2 \to \mathbb{R}^3$ such that $dF = \varphi \times * d\varphi$. Again, $F$ is an immersion if, and only if, $\varphi$ is an immersion. In this case: $F$ is a CGC $K = -1$ surface; the Minkowski metric $\epsilon$ and the second fundamental
form $\Pi_F$ are conformally equivalent; if $\varphi$ is weakly regular, that is, if $d\varphi$ never vanishes on the characteristic directions, the corresponding surface admits a global parametrization in asymptotic coordinates $\xi, \eta$ such that the two fundamental forms become

$$I_F = d\xi^2 + 2\cos \omega d\xi d\eta + d\eta^2, \quad \Pi_F = 2\sin \omega d\xi d\eta,$$

where $\omega$ satisfies the sine-Gordon equation $\omega_{\xi\eta} = \sin \omega$ (cf. [12]). Lie observed that this equation is invariant under Lorentz transformations: if $\omega$ is a solution, then so is $\omega(\lambda \xi, \lambda^{-1} \eta)$, with $\lambda$ belonging to the multiplicative group $\mathbb{R}^*$ of nonzero real numbers. Hence, to every CGC $K = -1$ surface $F$, we can associate a smooth $\mathbb{R}^*$-family of CGC $K = -1$ surfaces $F^\lambda$. $F^\lambda$ is called a Lie transform of $F$ and we denote $F^\lambda = L_\lambda(F)$.

### 3 Bianchi-Bäcklund Transforms

The Bianchi-Bäcklund transforms, which were introduced by Bianchi [2] for positive CGC surfaces, are an extension of the Bäcklund transformation [7] for surfaces of negative constant Gauss curvature. We shall now review briefly this theory, following [13].

Let $F : \mathbb{C} \to \mathbb{R}^3$ be a CGC $K = 1$ surface and $z = x + iy$ a curvature line coordinate on $\mathbb{C}$, with fundamental forms $I_F$ and $\Pi_F$ given by (2), where $\omega$ is a solution to the sinh-Gordon equation (1). In particular, $F$ has no umbilic points and the coordinate $z$ is conformal for the parallel CMC surface but not for $F$. Let $\varphi$ be the corresponding Gauss map. Define the orthonormal frame $[e_1, e_2, e_3]$ by:

$$e_1 = \frac{1}{\cosh \omega} F_x, \quad e_2 = \frac{1}{\sinh \omega} F_y, \quad e_3 = e_1 \times e_2.$$

Let $\tilde{F} : \mathbb{C} \to (\mathbb{R}^3)^\mathbb{C} \cong \mathbb{C}^3$ be a complex surface with Gauss map $\tilde{\varphi}$.

**Definition 1.** [13] We say that $\tilde{F}$ is a Bianchi-Bäcklund transform of the CGC $K = 1$ surface $F$ if it satisfies the following properties: $z$ is a curvature line coordinate with respect to $F$ and $\tilde{F}$; $(\tilde{F} - F, \varphi) = (\tilde{F} - F, \tilde{\varphi}) = 0$; $\tilde{F} - F$ has constant length; the normals have a constant angle with each other.

So, suppose that $\tilde{F}$ is a Bianchi-Bäcklund transform of $F$. Let $\phi : \mathbb{C} \to \mathbb{C}$ be the angle formed by the tangent line $\tilde{F} - F$ and $e_1$. Then, $\tilde{F} = F + \mu (\cos \phi e_1 + \sin \phi e_2)$ for some $\mu \in \mathbb{C} \setminus \{0\}$. We also have

$$(\varphi, \tilde{\varphi}) = \cos \sigma \quad \text{and} \quad \varphi \times \tilde{\varphi} = \sin \sigma(\cos \phi e_1 + \sin \phi e_2),$$

for some constant angle $\sigma$. 
Theorem 1. \[ \tilde{F} \] is a new CGC \( K = 1 \) surface and
\[ \tilde{F} = F + \frac{1}{\sinh \beta} (\cosh \theta e_1 + i \sinh \theta e_2), \] (4)

where \( \beta \in \mathbb{C} \setminus \{i\pi, n \in \mathbb{Z}\} \) is a constant, \( \mu = \frac{1}{\sinh \beta} \), \( \cot \sigma = -i \cosh \beta \), and \( \theta : \mathbb{C} \to \mathbb{C} \), with \( \theta = -i \phi \), is a solution of
\[ \begin{align*}
\theta_x + i \omega_y &= \sinh \beta \sinh \theta \cosh \omega + \cosh \beta \cosh \theta \sinh \omega \\
i \theta_y + \omega_x &= -\sinh \beta \cosh \theta \sinh \omega - \cosh \beta \sinh \theta \cosh \omega
\end{align*} \] (5)

The first and second fundamental forms of \( \tilde{F} \) are given by
\[ I_{\tilde{F}} = \cosh^2 \theta dx^2 + \sinh^2 \theta dy^2, \quad \Pi_{\tilde{F}} = -\sinh \theta \cosh \theta (dx^2 + dy^2). \]

Hence, the classical Bianchi-Bäcklund transformations are determined by two complex numbers: the spectral parameter \( \beta \in \mathbb{C} \setminus \{i\pi, n \in \mathbb{Z}\} \) and an initial angle \( \theta_0 \in \mathbb{C} \).

From the analytical point of view, the Bianchi-Bäcklund transformations may be interpreted as a procedure of obtaining new solutions of the sinh-Gordon equation from an old one. In fact, if \( \omega : \mathbb{R}^2 \to \mathbb{C} \) is a solution of the sinh-Gordon equation, then any solution \( \theta \) of the Bianchi-Bäcklund PDEs (5) will also solve the sinh-Gordon equation.

We denote by \( BB_\beta(F) \) the family of Bianchi-Bäcklund transforms of \( F \) with spectral parameter \( \beta \) and by \( BB_{\beta, \theta_0}(F) \) the Bianchi-Bäcklund transform of \( F \) determined by \( (\beta, \theta_0) \).

Lemma 1. Suppose that \( \beta_1, \beta_2 \in \mathbb{C} \setminus \{i\pi, n \in \mathbb{Z}\} \) satisfy \( \sinh \beta_1 = -\sinh \beta_2 \) and \( \cosh \beta_1 = -\cosh \beta_2 \). Then \( BB_{\beta_1}(F) = BB_{\beta_2}(F) \).

Proof. Let \( \theta_1 \) be a solution of (5) for \( \beta_1 \). Then \( \theta_2 = i\pi + \theta_1 \) is a solution of (5) for \( \beta_2 \). Taking account formula (4), we conclude that the pairs \( (\beta_1, \theta_1) \) and \( (\beta_2, \theta_2) \) produce the same surface, and we are done. \( \square \)

In contrast to the negative CGC case, the solution \( \theta \) of the Bianchi-Bäcklund PDEs is in general complex and so \( \tilde{F} \) will be complex. To obtain a new real solution of sinh-Gordon equation we must perform two iterations of Bianchi-Bäcklund transformations:

Theorem 2. \[ I \] Start with the real CGC \( K = 1 \) surface \( F \). If the reality condition \( \beta^* = i\pi - \beta \) holds (hence \( \theta^* = -\theta \)), then \( BB_{\beta^*, \theta_0^*}(BB_{\beta, \theta_0}(F)) \) is real.

Finally we recall the Bianchi–Bäcklund Permutability theorem:

Theorem 3. \[ I \] Let \( F \) be a CGC \( K = 1 \) surface and \( \beta, \beta^* \in \mathbb{C} \setminus \{i\pi, n \in \mathbb{Z}\} \). Then
\[ BB_{\beta^*}(BB_{\beta}(F)) = BB_{\beta}(BB_{\beta^*}(F)). \]
Bäcklund transforms. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a $K = -1$ pseudospherical surface. A surface $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^3$ is a Bäcklund transform of $F$ if it satisfies the following properties [7]: The coordinates $u, v$ correspond to parametrization along asymptotic lines with respect to $F$ and $\tilde{F}$; $(\tilde{F} - F, \varphi) = (\tilde{F} - F, \tilde{\varphi}) = 0$; $\tilde{F} - F$ has constant length; the normals have a constant angle $\sigma$ with each other. We denote $\tilde{F} = B_{\beta}(F)$, with $\beta = \tan \sigma/2$. In the particular case $\beta = 1$ (the tangent planes at corresponding points on $F$ and $\tilde{F}$ are orthogonal), $\tilde{F}$ is called a Bianchi transform of $F$. Lie observed that every Bäcklund transformation is a combination of transformations of Lie and Bianchi (cf. [7]). More precisely, $B_{\beta} = L_{\beta}^{-1} \circ B_1 \circ L_{\beta}$.

4 Harmonic Maps and Loop Groups

In this section we review the reformulation of harmonicity equations, for maps from a Riemann surface into a compact symmetric space, in terms of loops of flat connections, referring the reader to [4, 16] for details.

Let $G$ be a compact (connected) semisimple matrix Lie group, with identity $e$ and Lie algebra $\mathfrak{g}$. Equip $G$ with a bi-invariant metric. Let $G^C$ be the complexification of $G$, with Lie algebra $\mathfrak{g}^C$ (thus $\mathfrak{g}^C = \mathfrak{g} \otimes \mathbb{C}$). Consider a symmetric space $N = G/K$ with automorphism $\tau$ and associated symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Let $\phi : \mathbb{C} \to N$ be a smooth map and take a lift $\psi : \mathbb{C} \to G$ of $\phi$, that is, we have $\phi = \pi \circ \psi$ where $\pi : G \to G/K$ is the coset projection. Corresponding to the symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ there is a decomposition of $\alpha = \psi^{-1} d\psi$, $\alpha = \alpha_\mathfrak{k} + \alpha_\mathfrak{m}$. Let $\alpha_\mathfrak{m} = \alpha'_\mathfrak{m} + \alpha''_\mathfrak{m}$ be the type decomposition of $\alpha_\mathfrak{m}$ into $(1,0)$-form and $(0,1)$-form of $\mathbb{C}$. Consider the loop of 1-forms $\alpha_\lambda = \lambda^{-1} \alpha'_\mathfrak{m} + \alpha_\mathfrak{k} + \lambda \alpha''_\mathfrak{m}$. We may view $\alpha_\lambda$ as a $\Lambda_\tau \mathfrak{g}$-valued 1-form, where

$$\Lambda_\tau \mathfrak{g} = \{ \xi : S^1 \to \mathfrak{g} \text{ (smooth)} | \tau(\xi(\lambda)) = \xi(-\lambda) \text{ for all } \lambda \in S^1 \}.$$  \hspace{1cm} (6)

It is well known that $\phi$ is harmonic if, and only if, $d + \alpha_\lambda$ is a loop of flat connections on the trivial bundle $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^n$. Hence, if $\phi$ is harmonic, we can define a smooth map $\Phi : \mathbb{C} \to \Lambda_\tau G$, where $\Lambda_\tau G$ is the infinite-dimensional Lie group corresponding to the loop Lie algebra [6],

$$\Lambda_\tau G = \{ \gamma : S^1 \to G \text{ (smooth)} | \tau(\gamma(\lambda)) = \gamma(-\lambda) \text{ for all } \lambda \in S^1 \},$$

such that $\Phi^{-1} d\Phi = \alpha_\lambda$. The smooth map $\Phi$ is called an extended framing (associated to $\phi$) and gives rise to a smooth $S^1$-family of harmonic maps $\phi^\lambda = \pi \circ \Phi_\lambda$ (here we are using the notation $\Phi_\lambda(z) = \Phi(z)(\lambda)$), with $\phi^1 = \phi$. 

Inspired by the classical theory of Bianchi-Bäcklund transforms, where we have to deal with complex surfaces, we generalize the notion of extended framing as follows: a smooth map \( \Phi : \mathbb{C} \to \Lambda_r \mathbb{C} \) is called a complex extended framing if \( \Phi \) satisfies
\[
\Phi^{-1}d\Phi = (\lambda^{-1}A + B)dz + (C + \lambda D)d\bar{z},
\]
where \( B, C : \mathbb{C} \to \mathfrak{t}^\mathbb{C} \) and \( A, D : \mathbb{C} \to \mathfrak{m}^\mathbb{C} \). Of course, an extended framing is a complex extended framing satisfying the reality condition \( \Phi(\lambda) = \overline{\Phi(\lambda)} \).

In order to recover our CGC \( K = 1 \) surface and the corresponding Gauss map from an extended framing, we proceed as follows:

As \( \text{SO}(3, \mathbb{R}) \)-modules, we can identify \( (\mathbb{R}^3, \times) \cong \mathfrak{so}(3, \mathbb{R}) \) via \( \mathbb{R}^3 \ni u \mapsto \xi_u \in \mathfrak{so}(3, \mathbb{R}) \), where \( \xi_u(v) = u \times v \). The inner product on \( \mathfrak{so}(3, \mathbb{R}) \) inherited from \( \mathbb{R}^3 \) is \( (\xi, \eta) = -\frac{1}{2} \text{Tr} \xi \eta \).

Let \( e_1, e_2, e_3 \) be the canonical orthonormal basis of \( \mathbb{R}^3 \) and \( K \subset \text{SO}(3, \mathbb{R}) \) the stabilizer of \( e_1 \in S^2 \). Consider the automorphism \( \tau \) of \( \text{SO}(3, \mathbb{R}) \) given by conjugation by
\[
Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]
Then \( K \) is the identity component of the fixed set \( K^\tau \) of \( \tau \). The corresponding symmetric decomposition \( \mathfrak{so}(3, \mathbb{C}) = \mathfrak{t}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C} \) is given by:
\[
\mathfrak{t}^\mathbb{C} = \{ x\xi_{e_1} : x \in \mathbb{C} \}, \quad \mathfrak{m}^\mathbb{C} = \{ y\xi_{e_2} + z\xi_{e_3} : y, z \in \mathbb{C} \}.
\]

Given a complex extended framing \( \Phi : \mathbb{C} \to \Lambda_r \text{SO}(3, \mathbb{C}) \), consider the smooth map \( \varphi^\lambda : \mathbb{C} \to \mathfrak{so}(3, \mathbb{C}) \) defined by \( \varphi^\lambda = \Phi_\lambda \cdot \varphi_0 \), where \( \varphi_0 = \xi_{e_1} \). Then one can easily prove that the smooth map
\[
F^\lambda = -i\lambda \frac{\partial \Phi}{\partial \lambda} \Phi^{-1}_\lambda : \mathbb{C} \to \mathfrak{so}(3, \mathbb{C})
\]
satisfies \( dF^\lambda = [\varphi^\lambda, *d\varphi^\lambda] \). In particular, \( F^\lambda \) is a (real or complex) CGC \( K = 1 \) immersion if \( \varphi^\lambda \) is an immersion. Moreover, \( F^\lambda = S_\sigma(F) \), with \( \lambda = e^{-i\sigma} \).

Reformulation of Lorentz harmonicity in terms of loops of flat connections. Let \( \varphi : (\mathbb{R}^2, \epsilon) \to S^2 \) be a smooth map and take a lift \( \psi : \mathbb{R}^2 \to \text{SO}(3, \mathbb{R}) \) of \( \varphi \), that is, we have \( \phi = \pi \circ \psi \) where \( \pi : \text{SO}(3, \mathbb{R}) \to S^2 \cong \text{SO}(3, \mathbb{R})/\text{SO}(2, \mathbb{R}) \) is the coset projection. Corresponding to the symmetric decomposition (\ref{SO3decomp}) of \( \mathfrak{so}(3, \mathbb{R}) \), \( \mathfrak{so}(3, \mathbb{R}) = \mathfrak{t} \oplus \mathfrak{m} \), there is a decomposition of \( \alpha = \psi^{-1}d\psi \), \( \alpha = \alpha_\mathfrak{t} + \alpha_\mathfrak{m} \). Write \( \alpha_\mathfrak{m} = A'_1 du + A''_1 dv \) and \( \alpha_\mathfrak{t} = A'_0 du + A''_0 dv \). Consider the loop of 1-forms
\[
\alpha^\lambda = \lambda^{-1}A'_1 du + A'_0 du + A''_0 dv + \lambda A''_1 dv,
\]
where \( u, v \) are characteristic coordinates, with \( \lambda \) belonging to the multiplicative group \( \mathbb{R}^* \) of nonzero real numbers. We may view \( \alpha_\lambda \) as a \( \Lambda^\epsilon \mathfrak{so}(3, \mathbb{R}) \)-valued 1-form, where

\[
\Lambda^\epsilon \mathfrak{so}(3, \mathbb{R}) = \left\{ \xi : \mathbb{C}^* \to \mathfrak{so}(3, \mathbb{C}) \text{ (smooth)} \mid \tau(\xi(\lambda)) = \xi(-\lambda), \, \overline{\xi(\lambda)} = \xi(\overline{\lambda}) \right\}.
\]

Observe that the reality condition \( \overline{\xi(\lambda)} = \xi(\overline{\lambda}) \) forces any element \( \xi \) of \( \Lambda^\epsilon \mathfrak{so}(3, \mathbb{R}) \) to assume values in \( \mathfrak{so}(3, \mathbb{R}) \) for all \( \lambda \in \mathbb{R}^* \). The smooth map \( \varphi \) is (Lorentz) harmonic if, and only if, \( d + \alpha_\lambda \) is a flat connection for all \( \lambda \in \mathbb{R}^* \). Hence, if \( \varphi \) is harmonic, we can define a smooth map \( \Phi : \mathbb{R}^2 \to \Lambda^\epsilon \mathfrak{so}(3, \mathbb{R}) \), where \( \Lambda^\epsilon \mathfrak{so}(3, \mathbb{R}) \) is the infinite-dimensional Lie group corresponding to \( \Lambda^\epsilon \mathfrak{so}(3, \mathbb{R}) \), such that \( \Phi^{-1}d\Phi = \alpha_\lambda \). Again, the smooth map \( \Phi \) is called an extended framing, \( \varphi \) is recovered from \( \Phi \) via \( \varphi = \pi \circ \Phi_1 \), and the corresponding \( \mathbb{R}^* \)-family of pseudospherical surfaces is given by

\[
F^\lambda = -\lambda \frac{\partial \Phi}{\partial \lambda} \Phi_1^{-1} : \mathbb{R}^2 \to \mathbb{R}^3.
\]

## 5 Dressing Action

Harmonicity equations for maps from a Riemann surface into a compact symmetric space admit an infinite dimensional group of symmetries:

Let \( G \) be a compact semisimple Lie group and \( \tau : G \to G \) an involution with fixed set \( K \). Fix an Iwasawa decomposition of \( K^\mathbb{C} : K^\mathbb{C} = KB \), where \( B \) is a solvable subgroup of \( K^\mathbb{C} \). Fix \( 0 < \varepsilon < 1 \). Let \( C_\varepsilon \) and \( C_{1/\varepsilon} \) denote the circles of radius \( \varepsilon \) and \( 1/\varepsilon \) centered at \( 0 \in \mathbb{C} \); define

\[
I_\varepsilon = \{ \lambda \in \mathbb{P}^1 \mid |\lambda| < \varepsilon \}, \quad I_{1/\varepsilon} = \{ \lambda \in \mathbb{P}^1 \mid |\lambda| > 1/\varepsilon \}, \quad E_\varepsilon = \{ \lambda \in \mathbb{P}^1 \mid \varepsilon < |\lambda| < 1/\varepsilon \};
\]

put \( I^\varepsilon = I_\varepsilon \cup I_{1/\varepsilon} \) and \( C^\varepsilon = C_\varepsilon \cup C_{1/\varepsilon} \) so that \( \mathbb{P}^1 = I^\varepsilon \cup C^\varepsilon \cup E^\varepsilon \). Consider the infinite-dimensional twisted Lie groups

\[
\Lambda^\varepsilon G^\mathbb{C} = \{ \gamma : C^\varepsilon \to G^\mathbb{C} \text{ (smooth)} \mid \tau \gamma(\lambda) = \gamma(-\lambda) \}\]
\[
\Lambda^\varepsilon_E G^\mathbb{C} = \{ \gamma \in \Lambda^\varepsilon G^\mathbb{C} \mid \gamma \text{ extends holomorphically to } \gamma : E^\varepsilon \to G^\mathbb{C} \}\]
\[
\Lambda^\varepsilon_{I, B} G^\mathbb{C} = \{ \gamma \in \Lambda^\varepsilon G^\mathbb{C} \mid \gamma \text{ extends holomorphically to } \gamma : I^\varepsilon \to G^\mathbb{C} \text{ and } \gamma(0) \in B \}.\]

The basis of our action is the following decomposition:

**Theorem 4.** [12] The multiplication map

\[
\mu : \Lambda^\varepsilon_E G^\mathbb{C} \times \Lambda^\varepsilon_{I, B} G^\mathbb{C} \to \Lambda^\varepsilon G^\mathbb{C}, \quad (g_E, g_I) \mapsto g_E g_I
\]

is a diffeomorphism onto an open dense subset \( U \) of a component of \( \Lambda^\varepsilon G^\mathbb{C} \).
Remark 1. Denote by $\Lambda^\varepsilon G$ the subgroup of $\Lambda^\varepsilon G^C$ formed by the loops $\gamma$ satisfying the reality condition $\gamma(\lambda) = \overline{\gamma(1/\lambda)}$. In this case, the multiplication map $\mu$ gives a diffeomorphism of $\Lambda^\varepsilon_E G \times \Lambda^\varepsilon_{I,B} G$ onto $\Lambda^\varepsilon G$. The dressing action on harmonic maps studied by Burstall and Pedit in [5] is based on this decomposition of $\Lambda^\varepsilon G$.

For $g_I \in \Lambda^\varepsilon_{I,B} G^C$, let $U^\varepsilon_{g_I}$ be the open neighborhood of the identity 1 in $\Lambda^\varepsilon_E G^C$ defined by: $g_E \in U^\varepsilon_{g_I}$ if, and only if, there are unique $\hat{g}_E \in \Lambda^\varepsilon_E G^C$ and $g_I \in \Lambda^\varepsilon_{I,B} G^C$ such that $g_I g_E = \hat{g}_E g_I$ on $C^\varepsilon$. Write $g_I \# \varepsilon g_E$ for $\hat{g}_E$. Thus $g_I \# \varepsilon g_E = g_I g_E \hat{g}_I^{-1}$. One can prove easily the following:

**Lemma 2.** a) $U^\varepsilon_1 = \Lambda^\varepsilon_E G$ and $1 \# \varepsilon g_E = g_E$ for all $g_E \in \Lambda^\varepsilon_E G$. b) For all $g_I \in \Lambda^\varepsilon_{I,B} G$, $g_I \# \varepsilon 1 = 1$. c) Let $g_I, g_2 \in \Lambda^\varepsilon_I G$, $g_E \in U^\varepsilon_{g_I}$ and suppose $g_I \# \varepsilon g_E \in U^\varepsilon_{g_2}$ so that $g_2 \# \varepsilon (g_I \# \varepsilon g_E)$ is defined. Then, $g_E \in U^\varepsilon_{g_2 g_I}$ and $(g_2 g_I) \# \varepsilon g_E = g_2 \# \varepsilon (g_I \# \varepsilon g_E)$.

Hence we conclude that $g_I \# \varepsilon g_E$ defines a local action of $\Lambda^\varepsilon_I G$ on $\Lambda^\varepsilon_E G^C$.

For $0 < \varepsilon < \varepsilon' < 1$ we have injections $\Lambda^\varepsilon_{I,B} G^C \subset \Lambda^\varepsilon'_{I,B} G^C$ and $\Lambda^\varepsilon_E G^C \subset \Lambda^\varepsilon'_E G^C$. Similarly, for $0 < \varepsilon < 1$, we have $\Lambda_{\text{hol}} G^C \subset \Lambda^\varepsilon_E G^C$, where

$$
\Lambda_{\text{hol}} G^C = \bigcap_{0 < \varepsilon < 1} \Lambda^\varepsilon_E G^C.
$$

It is easy to see that

$$
\Lambda_{\text{hol}} G^C = \{ \gamma : \mathbb{C}^* \rightarrow G^C \mid \gamma \text{ is holomorphic and } \tau \gamma(\lambda) = \overline{\gamma(-\lambda)} \}.
$$

The dressing actions are compatible with these inclusions:

**Theorem 5.** For $0 < \varepsilon < \varepsilon' < 1$, $\gamma \in \Lambda^\varepsilon_{I,B} G^C \subset \Lambda^\varepsilon'_{I,B} G^C$, and $g \in \Lambda^\varepsilon_E G^C \subset \Lambda^\varepsilon'_E G^C$, we have: $g \in U^\varepsilon_{\gamma}$ if and only if $g \in U^\varepsilon'_{\gamma}$: $\gamma \# \varepsilon g = \gamma \# \varepsilon' g \in \Lambda^\varepsilon_E G^C$.

**Proof.** We argue as in [5], Proposition 2.3. Suppose that $g \in U^\varepsilon_{\gamma}$. Then, on $C^\varepsilon'$ we can write

$$
\gamma \# \varepsilon' g = \gamma g(\gamma g)_{1/\varepsilon'}^{-1}.
$$

Since $\gamma \# \varepsilon' g$ has a holomorphic extension to $E^\varepsilon'$ while $\gamma g(\gamma g)_{1/\varepsilon'}^{-1}$ has an holomorphic extension to $I^\varepsilon' \cap E^\varepsilon$, it follows from a theorem of Painlevé that $\gamma \# \varepsilon' g$ has a holomorphic extension to $E^\varepsilon' \cap C^\varepsilon' \cup (I^\varepsilon' \cap E^\varepsilon) = E^\varepsilon$. Thus, $(\gamma g)_{1/\varepsilon'} \in \Lambda^\varepsilon_{I,B} G^C$ while $\gamma \# \varepsilon' g \in \Lambda^\varepsilon_E G^C$, which means that $g \in U^\varepsilon_{\gamma}$ and, from the uniqueness of the factorization of $\Lambda^\varepsilon G^C$, $\gamma \# \varepsilon g = \gamma \# \varepsilon' g$.

Conversely, suppose that $g \in U^\varepsilon_{\gamma}$. Then, on $C^\varepsilon$ we can write

$$
(\gamma g)_{1/\varepsilon} = (\gamma g)_{1/\varepsilon'}^{-1} \gamma g.
$$
Since \((\gamma g)_{I^c}\) has a holomorphic extension to \(I^c\) while \((\gamma g)_{E^c}^{-1}\) has an holomorphic extension to \(I^c \cap E^c\), it follows from a theorem of Painlevé that \((\gamma g)_{I^c}\) has a holomorphic extension to \(I^c \cap E^c\). Thus, \((\gamma g)_{I^c} \in \Lambda^c_{I^c,B^c,G^c}\) while \(\gamma_{E^c}^{-1} g \in \Lambda^c_{E^c,G^c}\); which means that \(g \in U^c_{I^c}\) and, from the uniqueness of the factorization of \(\Lambda^c_{G^c}\), \(\gamma_{E^c}^{-1} g = \gamma_{E^c}^{-1} g\).

Corollary 1. The (local) action of each \(\Lambda^c_{I^c,B^c,G^c}\) preserves \(\Lambda^c_{hol,G^c}\) and, for \(0 < \epsilon < \epsilon' < 1\), \(\gamma \in \Lambda^c_{I^c,B^c,G^c}\) and \(g \in U^c_{\gamma} \cap \Lambda^c_{hol,G^c}\), we have \(\gamma_{E^c}^{-1} g = \gamma_{E^c}^{-1} g\).

Henceforth, we write \(\gamma g\) for this (local) action on \(\Lambda^c_{hol,G^C}\).

Now, let \(\Phi : \mathbb{C} \to \Lambda^c_{hol,G^c}\) be a smooth map and \(g_I \in \Lambda^c_{I^c,B^c,G^c}\). Define the (smooth) map \(g_I \# \Phi : \Phi^{-1}(U^c_{\gamma}) \subset \mathbb{C} \to \Lambda^c_{hol,G^c}\) by

\[
(g_I \# \Phi)(p) = g_I(\Phi(p)).
\]

Observe that we can see a complex extended framing \(\Phi\) as a map into \(\Lambda^c_{hol,G^c}\), because \(\Phi^{-1}d\Phi\) is holomorphic in \(\lambda\) on \(\mathbb{C}^*\).

The relevance of the local action \(\#\) is contained in the following theorem:

Theorem 6. If \(\Phi : \mathbb{C} \to \Lambda^c_{hol,G^c}\) is a complex extended framing then so is \(g_I \# \Phi\).

Proof. To see that \(g_I \# \Phi\) is a complex extended framing, write \(g_I \Phi = ab\), where \(a = g_I \# \Phi\) and \(b : \Phi^{-1}(U^c_{\gamma}) \subset \mathbb{C} \to \Lambda^c_{I^c,B^c,G^c}\). Then

\[
a^{-1} da = Ad_b(\Phi^{-1}d\Phi - b^{-1} db),
\]

so that

\[
\lambda a^{-1} da = Ad_b(\lambda \Phi^{-1}d\Phi - \lambda b^{-1} db)
\]

and

\[
\lambda^{-1} a^{-1} da = Ad_b(\lambda^{-1} \Phi^{-1}d\Phi - \lambda^{-1} b^{-1} db).
\]

Now, all the ingredients on the right side of (11) are holomorphic in \(\lambda\) on a neighborhood of 0 so that \(\lambda a^{-1} da\) is also; similarly, all the ingredients on the right side of (12) are holomorphic in \(\lambda\) on a neighborhood of \(\infty\) so that \(\lambda^{-1} a^{-1} da\) is also. Whence, \(a\) is a complex extended framing.

From equation (10) we see that the \((1,0)\)-part of \((a^{-1} da)_{mc}\) lies on an adjoint orbit of the \((1,0)\)-part of \((\Phi^{-1}d\Phi)_{mc}\). Then:

Proposition 1. \(\varphi^\lambda = a_\lambda \cdot \varphi_0\) is conformal if, and only if, \(\varphi^\lambda = \Phi^\lambda \cdot \varphi_0\) is conformal.
6 Simple factors

Given $\gamma \in \Lambda_{I,B}^C G^C$ and $g \in \Lambda_{\text{hol}} G^C$, a basic problem is to compute $\gamma \# g$. This is a Riemann-Hilbert problem and, in general, explicit solutions are not available. However, as the philosophy underlying the work of Terng and Uhlenbeck [15] suggests, there should be certain elements of $\gamma \in \Lambda_{I,B} G^C$, the simple factors, for which one can compute explicitly $\gamma \# g$ by algebra alone. In this section we construct the simple factors that are relevant to our geometric problem.

Let $L$ be an 1-dimensional isotropic subspace of $(\mathbb{R}^3)^C \cong \mathbb{C}^3$: $(L, L) = 0$. Let $Q \in \text{SO}(3, \mathbb{R})$ be defined by (7), suppose that $QL \neq L$ and consider the decomposition

$$\mathbb{C}^3 = L \oplus QL \oplus L_0,$$

where $L_0 = (L \oplus QL)\perp$. Denote by $\pi_L$, $\pi_{QL}$ and $\pi_{L_0}$ the corresponding projections. For each $\alpha \in \mathbb{C} \setminus \{0\}$ set

$$p_{\alpha,L}(\lambda) = \frac{\alpha - \lambda}{\alpha + \lambda} \pi_L + \pi_{L_0} + \frac{\alpha + \lambda}{\alpha - \lambda} \pi_{QL}. \quad (13)$$

Thus, $p_{\alpha,L} : \mathbb{P}^1 \setminus \{\pm \alpha\} \to \text{SO}(3, \mathbb{C})$ and $p_{\alpha,L}(0) = 1$. Moreover, each $p_{\alpha,L}$ is twisted, that is, $\tau p_{\alpha,L}(\lambda) = p_{\alpha,L}(-\lambda)$. Then $p_{\alpha,L} \in \Lambda_{I,B}^C \text{SO}(3, \mathbb{C})$ for some $\varepsilon < 1$. The key to computing the dressing action of $p_{\alpha,L}$ is the following proposition:

**Proposition 2.** [3] Let $\Phi$ be a germ at $\alpha$ of a holomorphic map into $\text{SO}(3, \mathbb{C})$ such that $\tau \Phi(\lambda) = \Phi(-\lambda)$. Suppose further that $Q \Phi^{-1}(\alpha)L \neq \Phi^{-1}(\alpha)L$. Then $p_{\alpha,L} \Phi^{-1}(\alpha)L \in \Lambda_{I,B}^C \text{SO}(3, \mathbb{C})$ and

$$p_{\alpha,L} \Phi p_{\alpha,L}^{-1}(\alpha)L$$

is holomorphic and invertible at $\alpha$.

**Corollary 2.** Let $\langle \varphi_0 \rangle$ be the subspace of $\mathbb{R}^3$ generated by $\varphi_0$ and denote by $\langle \varphi_0^\perp \rangle$ its real orthogonal complement in $\mathbb{R}^3$. Then, given $g \in \Lambda_{\text{hol}} \text{SO}(3, \mathbb{C})$, $g \in U_{\varphi_0,L}^C$ if, and only if, $g^{-1}(\alpha)L$ is not contained in $\langle \varphi_0^\perp \rangle^C$. For $g \in U_{\varphi_0,L}^C$, we have

$$p_{\alpha,L} \# g = p_{\alpha,L} g p_{\alpha,L}^{-1}(\alpha)L. \quad (14)$$

**Proof.** The eigenspaces of $Q$ are $\langle \varphi_0 \rangle$ and $\langle \varphi_0^\perp \rangle$. Hence, since $g^{-1}(\alpha)L$ is isotropic and no real subspace is isotropic, $Q g^{-1}(\alpha)L \neq g^{-1}(\alpha)L$ if, and only if, $g^{-1}(\alpha)L$ is not contained in $\langle \varphi_0^\perp \rangle^C$.

The first part of Proposition 2 assures us that $p_{\alpha,g^{-1}(\alpha)L} \in \Lambda_{I,B}^C \text{SO}(3, \mathbb{C})$. So we just need to prove that $p_{\alpha,L} \# g$ given by (14) is an element of $\Lambda_{\text{hol}} \text{SO}(3, \mathbb{C})$. Clearly $p_{\alpha,L} \# g$ is
twisted, since it is a product of maps with this property. The holomorphicity at \( \alpha \) follows directly from Proposition 2 and then we get holomorphicity at \( \pm \alpha \) from the twisting condition.

\[ \square \]

**Remark 2.** In [6], the authors gave the following general definition of simple factors: Let \( G \) be a compact Lie group and \( \rho : G^\mathbb{C} \to \text{GL}(V) \) a representation. A semisimple element \( H \in g^\mathbb{C} \) is said to be \( \rho \)-integral if \( \rho(H) \in \text{End}(V) \) has only integer values. For any \( \alpha \in \mathbb{C} \setminus \mathbb{R} \) and \( \rho \)-integral element \( H \in i g \), the loop

\[ p_{\alpha,H}(\lambda) = \exp \left( \ln \left( \frac{\lambda - \alpha}{\lambda + \alpha} \right) H \right) \]  \hspace{1cm} (15)

is a simple factor, with \( \lambda \) belonging to \( \mathbb{R}^* \), the multiplicative group of nonzero real numbers. The condition \( H \in i g \) ensures that \( p_{\alpha,H} \) satisfies the reality condition \( p_{\alpha,H}(\lambda) = \overline{p_{\alpha,H}(\overline{\lambda})} \). Hence, on the real axis \( p_{\alpha,H} \) takes values in \( G \). Moreover, if \( N = G/K \) is a symmetric space with automorphism \( \tau \) and associated symmetric decomposition \( g = k \oplus m \), then the conditions \( H \in i m \) and \( \alpha = ir \), with \( r \in \mathbb{R} \setminus \{0\} \), ensure that \( p_{\alpha,H} \) is also twisted: \( \tau p_{\alpha,H}(\lambda) = p_{\alpha,H}(-\lambda) \). In our case, we are not imposing any reality condition to the simple factors \([13]\), which can also be written as:

\[ p_{\alpha,H}(\lambda) = \exp \left( \ln \left( \frac{\alpha - \lambda}{\alpha + \lambda} \right) H \right), \]

where \( H \in m^\mathbb{C} \) is \( \rho \)-integral with respect to the standard representation of \( \text{SO}(3, \mathbb{C}) \).

**Remark 3.** Let \( L \) be an 1-dimensional isotropic subspace of \( (\mathbb{R}^3)^\mathbb{C} \cong \mathbb{C}^3 \). Suppose that \( QL \neq L \). The cross product multiplication table for the decomposition \( \mathbb{C}^3 = L \oplus QL \oplus L_0 \), where \( L_0 = (L \oplus QL)^\perp \), is given by

|   | L   | L_0  | QL  |
|---|-----|------|-----|
| L | 0   | L    | L_0 |
| L_0 | L  | 0    | QL  |
| QL | L_0 | QL   | 0   |

7 Bianchi-Bäcklund transforms via dressing actions

In this section we prove that the actions of these simple factors amount to Bianchi-Bäcklund transformations.

Start with an everywhere non-conformal harmonic map \( \varphi : \mathbb{C} \to S^2 \). Let \( \Phi : \mathbb{C} \to \Lambda_{\text{hol}} \text{SO}(3, \mathbb{C}) \) be an extended framing associated to \( \varphi \). By applying formula \([9]\) to \( \Phi \), we get a map \( F^\lambda : \mathbb{C} \to \mathbb{R}^3 \), for each \( \lambda \in \mathbb{C}^* \), such that \( dF^\lambda = [\varphi^\lambda, *d\varphi^\lambda] \), that is, a CGC
$K = 1$ surface (which is real when $\lambda \in S^1$) without umbilics, with normal $\varphi^\lambda = \Phi_\lambda \cdot \varphi_0$. Assume that $\Phi_\lambda(z_0) = 1$ for all $\lambda \in \mathbb{C}^*$.

Choose $\alpha \in \mathbb{C} \setminus \{0, \pm \lambda\}$. Consider the action of a simple factor $p_{\alpha, L}$ on $\Phi$:

$$
\Phi = p_{\alpha, L} \Phi = p_{\alpha, L} \Phi p_{\alpha, L}^{-1} : \Phi^{-1}(U_{p_{\alpha, L}}) \to \Lambda_{\text{hol}} \text{SO}(3, \mathbb{C}),
$$
where $\tilde{L} = \Phi(\alpha)^{-1} L$. Set $h = p_{\alpha, L}$ and $\tilde{h} = p_{\alpha, L}$. Since

$$
-i \lambda \frac{\partial \Phi}{\partial h} \Phi^{-1} = -i \lambda h_\lambda \left( \frac{\partial \Phi}{\partial \lambda} \Phi^{-1} - \Phi_\lambda \tilde{h}_\lambda^{-1} \frac{\partial \tilde{h}}{\partial \lambda} \Phi^{-1} + h_\lambda \frac{\partial h}{\partial \lambda} \right) h^{-1},
$$
our new CGC $K = 1$ surface equals

$$
\tilde{F}^\lambda = F^\lambda + i \lambda \Phi_\lambda \tilde{h}_\lambda^{-1} \frac{\partial \tilde{h}}{\partial \lambda} \Phi^{-1},
$$
up to (complex) Euclidean motions – translation by $h^{-1}_\lambda \frac{\partial h}{\partial \lambda}$ followed by a rotation (conjugation by $h_\lambda$). The corresponding normal is $\tilde{\varphi}^\lambda = \Phi_\lambda \tilde{h}_\lambda^{-1} \cdot \varphi_0$. By Proposition $\Box$, $\tilde{\varphi}^\lambda$ is also everywhere non-conformal, whence $\tilde{F}^\lambda$ has no umbilic points.

**Theorem 7.** $\tilde{F}^\lambda \in \text{BB}_\beta(F^\lambda)$, with $\beta = \ln (\alpha / \lambda)$. Moreover, any Bianchi-Bäcklund transform of $F$ is of the form (17) for some simple factor $p_{\alpha, L}$.

**Proof.** 1) Since $\varphi^\lambda$ and $\tilde{\varphi}^\lambda$ are harmonic with respect to the same conformal structure $z = x + iy$ on $\mathbb{C}$, we have $\Pi_{F^\lambda} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \Pi_{\tilde{F}^\lambda} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = 0$, that is, $z$ is a curvature line coordinate with respect to $F^\lambda$ and $\tilde{F}^\lambda$.

2) Since

$$
\tilde{h}_\lambda^{-1} \frac{\partial \tilde{h}}{\partial \lambda} = A_\lambda(\alpha) (\pi_{QL} - \pi_L),
$$
where $A_\lambda(\alpha) = \frac{2a}{(\alpha - \lambda)(\alpha + \lambda)}$, we have:

$$
(\tilde{F}^\lambda - F^\lambda, \tilde{F}^\lambda - F^\lambda) = \frac{\lambda^2}{2} \text{Tr} \left( A_\lambda(\alpha) \Phi_\lambda (\pi_{QL} - \pi_L) \Phi_\lambda^{-1} \right)^2 = \lambda^2 A^2_\lambda(\alpha),
$$
that is, $\tilde{F}^\lambda - F^\lambda$ has constant length.

3) Since $[Q, \varphi_0] = 0$ and $\pi_{QL} = Q \pi_L Q^{-1}$, one can easily check that

$$
(\tilde{F}^\lambda - F^\lambda, \varphi^\lambda) = (\tilde{F}^\lambda - F^\lambda, \tilde{\varphi}^\lambda) = 0.
$$

4) Recall that $\varphi_0 = \xi_{e_1}$, where $e_1, e_2, e_3$ is the canonical basis of $\mathbb{R}^3$ and $\xi_{e_1}(v) = e_1 \times v$. Since $\tilde{L}$ is isotropic and $QL \neq \tilde{L}$, $\tilde{L}$ is generated by a vector $\tilde{v}$ of the form $\tilde{v} = \frac{1}{2} e_1 + \tilde{a} e_2 + \tilde{b} e_3$, with $\tilde{a}^2 + \tilde{b}^2 = -\frac{1}{4}$. Note that $e_1 = \tilde{v} + Q \tilde{v}$.

Now:

$$
(\varphi^\lambda, \tilde{\varphi}^\lambda) = -\frac{1}{2} \text{Tr} \left( \Phi_\lambda \varphi_0 \Phi_\lambda^{-1} \Phi_\lambda \tilde{h}_\lambda^{-1} \varphi_0 \tilde{h}_\lambda \Phi^{-1} \right) = -\frac{1}{2} \text{Tr} \left( \varphi_0 \tilde{h}_\lambda^{-1} \varphi_0 \tilde{h}_\lambda \right).$$


Set $a_\lambda(\alpha) = \frac{\alpha + \lambda}{\alpha - \lambda}$. Fix $X \in L_0$, $Y \in \bar{L}$ and $Z \in Q\bar{L}$. By using the cross product multiplication table with respect to the decomposition $\mathbb{C}^3 = \bar{L} \oplus L_0 \oplus Q\bar{L}$ (see Remark 3) and the triple product expansion $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$, one can straightforwardly compute $\rho(X)$, $\rho(Y)$ and $\rho(Z)$. For example:

$$\rho(X) = \varphi_0 h_\lambda^{-1} \varphi_0 h_\lambda(X) = \varphi_0 h_\lambda^{-1} \left( \vec{v} \times X + Q\vec{v} \times X \right) = \varphi_0 \left( a_\lambda(\alpha)\vec{v} \times X + a_\lambda(\alpha)^{-1}Q\vec{v} \times X \right)$$

$$= a_\lambda(\alpha)Q\vec{v} \times (\vec{v} \times X) + a_\lambda(\alpha)^{-1}\vec{v} \times (Q\vec{v} \times X) = -\frac{a_\lambda(\alpha) + a_\lambda(\alpha)^{-1}}{2}X.$$

Similarly, we have:

$$\rho(Y) = -\frac{a_\lambda(\alpha)^{-1}}{2}Y + a_\lambda(\alpha)^{-1}Q\vec{v} \times (Q\vec{v} \times Y) \in Q\bar{L}$$

and

$$\rho(Z) = -\frac{a_\lambda(\alpha)}{2}Z + a_\lambda(\alpha)\vec{v} \times (\vec{v} \times Z) \in \bar{L}.$$

Then,

$$\cos \sigma = (\varphi^\lambda, \tilde{\varphi}^\lambda) = -\frac{1}{2} \text{Tr} \rho = \frac{a_\lambda(\alpha) + a_\lambda(\alpha)^{-1}}{2},$$

and we conclude that $(\varphi^\lambda, \tilde{\varphi}^\lambda)$ is constant.

So, $\tilde{F}^\lambda$ is a Bianchi-Bäcklund transformation of $F^\lambda$. Next we shall find the corresponding $\beta$ parameter.

By (11) and (17), we get

$$\frac{1}{\sinh \beta} = \pm \lambda A_\lambda(\alpha).$$

Taking account (3) and (4), one can compute $\sin \sigma$:

$$\sin \sigma = (\varphi^\lambda \times \tilde{\varphi}^\lambda, \sinh \beta(F^\lambda - \tilde{F}^\lambda)) = \pm i(\varphi^\lambda \times \tilde{\varphi}^\lambda, \Phi_\lambda(\pi_{Q\bar{L}} - \pi_{\bar{L}})\Phi_\lambda^{-1}) = \pm i\frac{a_\lambda(\alpha) - a_\lambda(\alpha)^{-1}}{2}.$$

Hence,

$$-i \cosh \beta = \cot \sigma = \mp i\frac{a_\lambda(\alpha) + a_\lambda(\alpha)^{-1}}{a_\lambda(\alpha) - a_\lambda(\alpha)^{-1}}.$$

By Lemma 11 we can take

$$\frac{1}{\sinh \beta} = \lambda A_\lambda(\alpha) = \frac{a_\lambda(\alpha) - a_\lambda(\alpha)^{-1}}{2} \quad \text{and} \quad \cosh \beta = \frac{a_\lambda(\alpha) + a_\lambda(\alpha)^{-1}}{a_\lambda(\alpha) - a_\lambda(\alpha)^{-1}};$$

hence $\beta = \ln(\alpha/\lambda)$.

It remains to prove the converse, that is, any Bianchi-Bäcklund transformation of $F^\lambda$ amounts to the dressing action of some simple factor $p_{\alpha,L}$ (up to conjugation by a constant complex matrix):
Evaluating $\tilde{F}^\lambda - F^\lambda$ at $z_0$ gives

$$\tilde{F}^\lambda(z_0) - F^\lambda(z_0) = i\lambda A_\lambda(\pi_{QL} - \pi_L).$$

For some fixed spectral parameter $\beta \in \mathbb{C} \setminus \{in\pi, n \in \mathbb{Z}\}$, we choose $\alpha = e^{\beta \lambda}$. On the other hand, if $L$ is generated by $v = \frac{1}{2}e_1 + ae_2 + be_3$, for some $a, b \in \mathbb{C}$ such that $a^2 + b^2 = -\frac{1}{4}$, we have

$$\pi_{QL} - \pi_L = \begin{pmatrix} 0 & 2a & 2b \\ -2a & 0 & 0 \\ -2b & 0 & 0 \end{pmatrix},$$

and so any non-isotropic direction in $<\varphi^0_\beta>^C$ is generated by a vector of the form $\pi_{QL} - \pi_L$.

We therefore deduce from the uniqueness of solutions to the Bianchi-Bäcklund PDEs that each Bianchi-Bäcklund transformation of $F^\lambda$ amounts to the dressing action of some simple factor $p_{\alpha,L}$ (up to conjugation by a constant complex matrix). \hfill \Box

In the pseudospherical case, Lie observed that every Bäcklund transformation is a combination of transformations of Lie and Bianchi. In the spherical case, the tangent planes at corresponding points on $F$ and $\tilde{F} = BB_{i\pi/2}(F)$ are orthogonal, and we have:

**Corollary 3.** $BB_\beta = S_{-1}^{-1}_{\beta+i\pi/2} \circ BB_{i\pi/2} \circ S_{\beta+i\pi/2}$.

**Bäcklund transforms and dressing actions.** In [16] (see also [14]), Uhlenbeck made the observation that Bäcklund transforms for pseudospherical surfaces (CGC $K < 0$ surfaces) amount to dressings of the simplest type. She used SU(2) as symmetry group. If we want to use SO(3, $\mathbb{R}$) as symmetry group, we shall proceed as follows:

The basis of the action is the same decomposition of Theorem[4] restricted to the loops satisfying the reality condition $\bar{\xi}(\lambda) = \xi(\bar{\lambda})$, and the simple factors are those of the form (15) with $H \in i\mathfrak{m}$ and $\alpha \in i\mathbb{R}$. More explicitly, the simple factors we need in this case are of the form

$$p_{\alpha,L}(\lambda) = \frac{\alpha - \lambda}{\alpha + \lambda} \pi_L + \frac{\alpha + \lambda}{\alpha - \lambda} \pi_{QL},$$

with $QL = L$.

**Remark 4.** The simple factors (18) generate the group of all rational maps $\gamma : \mathbb{P}^1 \to SO(3, \mathbb{C})$, holomorphic at 0 and $\infty$, satisfying the reality condition $\overline{\gamma(\lambda)} = \gamma(\bar{\lambda})$ and the twisted condition $\tau \gamma(\lambda) = \gamma(-\lambda)$ (cf. [6]).
7.1 Bianchi-Bäcklund Permutability theorem

Taking account the results we have obtained above, the Bianchi-Bäcklund Permutability theorem is a direct consequence of the following:

**Theorem 8.** Consider simple factors \( p_{\alpha_1, L_1} \) and \( p_{\alpha_2, L_2} \) with \( \alpha_1^2 \neq \alpha_2^2 \). Set \( L'_1 = p_{\alpha_2, L_2}(\alpha_1)L_1 \) and \( L'_2 = p_{\alpha_1, L_1}(\alpha_2)L_2 \). Assume that \( QL'_i \neq L'_i \), \( i = 1, 2 \). Then

\[
p_{\alpha_1, L_1}p_{\alpha_2, L_2} = p_{\alpha_2, L_2}p_{\alpha_1, L_1}.
\]  

(19)

This theorem is a simple adaptation of Proposition 4.15 of [3] to our setting and its proof can be carried out similarly, without any reality assumption.

7.2 Getting a real solution from an old one

If we want to obtain a new real CGC \( K = 1 \) surface from an old one, we have to perform two dressing actions with simple factors, as the classical theory suggests.

For each pair \((\alpha, L)\), we introduce the holomorphic map \( q_{\alpha, L} : \mathbb{P}^1 \setminus \{0, \pm \alpha\} \to \text{SO}(3, \mathbb{C}) \) defined by

\[
q_{\alpha, L}(\lambda) = p_{\alpha, L}(\infty)p_{\alpha, L}(\lambda) = \frac{\lambda - \alpha}{\lambda + \alpha} \pi_L + \pi_{L_0} + \frac{\lambda + \alpha}{\lambda - \alpha} \pi_{QL},
\]

where, as before, \( L \) is an isotropic line in \((\mathbb{R}^3)^C \cong \mathbb{C}^3\) such that \( QL \neq L \). Consider the automorphism \( R : \text{ASO}(3, \mathbb{C}) \to \text{ASO}(3, \mathbb{C}) \) defined by \( R(\gamma)(\lambda) = \overline{\gamma(1/\lambda)} \). Clearly, \( R \) is an involution. One can easily check that

\[
R(p_{\alpha, L}(\lambda)) = q_{\frac{1}{R(\lambda)}, L}(\lambda).
\]  

(20)

**Theorem 9.** Set \( L_1 = L, \ L_2 = \overline{L}, \ \alpha_1 = \alpha, \ \alpha_2 = \frac{1}{\alpha_1}, \ L'_2 = p_{\alpha_2, L_2}(\alpha_2)L_2 \) and \( L'_1 = p_{\alpha_2, L_2}(\alpha_1)L_1 \). There exists \( k \in K^C \) such that \( kp_{\alpha_2, L_2}p_{\alpha_1, L_1} \in \Lambda_{i,B}^C \text{SO}(3, \mathbb{R}) \) for some \( 0 < \varepsilon < 1 \).

**Proof.** Set \( u = p_{\alpha_2, L_2}p_{\alpha_1, L_1} \). We want to find \( k \in K^C \) satisfying \( R(ku) = ku \); and this is the same to \( k^{-1}k = uR(u)^{-1} \). Now, it follows from (19) and (20) that:

\[
uR(u)^{-1} = p_{\alpha_2, L_2}p_{\alpha_1, L_1}q_{\alpha_2, L_2}^{-1}q_{\alpha_1, L_2}^{-1} = p_{\alpha_1, L_1}p_{\alpha_2, L_2}q_{\alpha_2, L_2}^{-1}q_{\alpha_1, L_2}^{-1} = p_{\alpha_2, L_2}(\infty)q_{\alpha_1, L_2}^{-1}.
\]  

(21)

But

\[
\overline{L_2} = p_{\alpha_2, L_2}(\alpha_2)L_2 = q_{\alpha_2, L_2}(\alpha_1)L_1 = p_{\alpha_2, L_2}(\infty)L'_1.
\]

Hence

\[
q_{\alpha_1, L_1}\overline{L_2} = p_{\alpha_2, L_2}(\infty)q_{\alpha_1, L_1}p_{\alpha_2, L_2}(\infty).
\]  

(22)
From (21) and (22), we obtain
\[ u_{R}(u)^{-1} = p_{\alpha_{1}, L_{1}}(\infty)p_{\alpha_{2}, L_{2}}(\infty). \]

Then, \( P = u_{R}(u)^{-1} \) does not depend on \( \lambda \) and, since all the simple factors are twisted, we have \( P \in K^{\tau, C} \). Finally, observe that
\[ P \overline{P} = P R(P) = u_{R}(u)^{-1} R(u_{R}(u)^{-1}) = 1. \]

The existence of such \( k \in K^{\tau, C} \) follows now from Lemma 3.

**Lemma 3.** If \( P \in K^{\tau, C} \) satisfies \( P \overline{P} = 1 \), then \( P = k^{-1} \overline{k} \) for some \( k \in K^{\tau, C} \).

**Proof.** The complex group \( \mathbb{C}^{*} \) double covers \( K^{\tau, C} \) via
\[ z \in \mathbb{C}^{*} \mapsto \rho(z) = \text{Ad} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \in \text{Ad}_{\text{SL}(2, \mathbb{C})} \cong \text{SO}(3, \mathbb{C}). \]

Observe that \( \rho(\overline{z}) = \overline{\rho(z)}^{-1} \). Given \( P \in K^{\tau, C} \) such that \( P \overline{P} = 1 \), one can find \( z_{0} \in \mathbb{C}^{*} \) such that \( \rho(z_{0}) = P \) and \( z_{0} \overline{z_{0}}^{-1} = 1 \); in particular, we can fix \( z_{0} \in \mathbb{R}^{+} \). This means that there exists \( u_{0} \in \mathbb{C}^{*} \) such that \( z_{0} = u_{0} \overline{u_{0}} \). Hence
\[ P = \rho(z_{0}) = \rho(u_{0}) \rho(\overline{u_{0}}) = \rho(u_{0}) \rho(u_{0})^{-1}. \]

Set \( k = \rho(u_{0})^{-1}. \) Then \( P = k^{-1} \overline{k}. \)

Let \( \Phi : \mathbb{C} \rightarrow \Lambda_{\text{hol}} \text{SO}(3, \mathbb{R}) \) be an extended framing associated to an everywhere non-conformal harmonic map \( \varphi : \mathbb{C} \rightarrow S^{2} \). Applying formula (9) to \( \Phi \), and evaluating at \( \lambda = 1 \), we get a real CGC \( K = 1 \) surface \( F \). With the notations of Theorem 9, \( \tilde{\Phi} \) is a new (real) extended framing. Therefore, formula (9) applied to \( \tilde{\Phi} \) gives a new real CGC \( K = 1 \) surface \( F^{*} \), evaluating again at \( \lambda = 1 \). Up to an Euclidean motion, this surface is obtained out of \( F \) by applying two successive Bianchi-Bäcklund transformations: if \( \beta_{1} \in \mathbb{C} \setminus \{ n \pi, n \in \mathbb{Z} \} \) is such that \( \beta_{1} = \ln \alpha_{1} \), then, taking account Lemma 1 and Theorem 7, \( F^{*} \) belongs to \( \text{BB}_{\ln \alpha_{1} - \beta_{1}}(\text{BB}_{\beta_{1}}(F)) \), up to an Euclidean motion, which agrees with Theorem 2.

**Remark 5.** The multiplication map \( \mu \), restricted to loops \( \gamma \) satisfying the reality condition \( \gamma(\lambda) = \overline{\gamma(1/\lambda)} \), gives a diffeomorphism of \( \Lambda^{\varepsilon}_{E} \text{SO}(3, \mathbb{R}) \times \Lambda^{\varepsilon}_{I, B} \text{SO}(3, \mathbb{R}) \) onto \( \Lambda^{\varepsilon} \text{SO}(3, \mathbb{R}) \). Hence, the extended framing \( \tilde{\Phi} = (kp_{\alpha_{2}, L_{2}^{*}p_{\alpha_{1}, L_{1}}}) \# \Phi \) is well-defined everywhere.
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