An imaginary PBW basis for quantum affine algebras of type 1.

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Abstract. Let \( \hat{\mathfrak{g}} \) be an affine Lie algebra of type 1. We give a PBW basis for the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) with respect to the triangular decomposition of \( \hat{\mathfrak{g}} \) associated with the imaginary positive root system.

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1. Introduction

Although affine Lie algebras are infinite dimensional analogs of finite dimensional semisimple Lie algebras, they have special features that do not have analogs in the finite dimensional theory. One such feature is the existence of closed partitions of the root system into sets of positive and negative roots which are not equivalent under the action of the Weyl group to the standard partitions of the root system. Such partitions are called nonstandard partitions. The classification of closed subsets of the root system for affine Kac-Moody algebras was obtained by Jakobsen and Kac [JK85, JK89], and independently by Futorny [Fut90, Fut92]. In particular, it is shown that for affine Lie algebras there are only a finite number of Weyl-equivalency classes of these nonstandard partitions. Corresponding to each non-standard partition we have non-standard Borel subalgebras from which one may induce other non-standard Verma-type modules and these typically contain both finite and infinite dimensional weight spaces. For example, for the affine Lie algebra \( \mathfrak{sl}(2) \), the only non-standard modules of Verma-type are the imaginary Verma modules [Fut94]. In this paper we focus on the imaginary Verma modules for affine Lie algebras of type 1.

Let \( \hat{\mathfrak{g}} \) be an affine Lie algebra of type 1 and \( U_q(\hat{\mathfrak{g}}) \) denote the associated quantum affine algebra introduced independently by Drinfeld [Dri85] and Jimbo [Jim85]. One of the problems in dealing with nonstandard partitions of root systems is that the associated triangular decomposition of \( \hat{\mathfrak{g}} \) can not be lifted to a triangular decomposition of \( U_q(\hat{\mathfrak{g}}) \). In [CFKM97], the imaginary Verma module for the affine Lie algebra \( \mathfrak{sl}(2) \) is \( q \)-deformed in such a way that the weight multiplicities, both

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finite and infinite-dimensional, are preserved. This construction is generalized to
the imaginary Verma modules for any affine Lie algebra \( \hat{\mathfrak{g}} \) of type 1 in [FGM98].
Furthermore, in [CFKM97], the authors used a special technique involving the
Diamond Lemma to construct a PBW type basis for the quantum imaginary Verma
module for \( U_q(\mathfrak{sl}(2)) \) which does not generalize to other affine Lie algebras. In
this paper we use a different approach and construct a PBW type basis for the
quantum imaginary Verma module for any quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) of type 1
(see Theorem 3.4.7 and Theorem 4.1.1).

A reader might want to compare our results with those appearing in the work
of Beck, Chari and Pressley [BCP99]. In this cited paper their Lemma 1.5 should
be compared to our definition of \( X_{\Delta^+} \) given below in section 3. The difference of our
decomposition with that given in the work of Beck, Chari and Pressley has more
to do with the breakdown of different quantized Borel subalgebras. In [BCP99],
the authors have a decomposition of the quantized Borel subalgebra coming from
the standard positive root system as

\[
U^+ \cong U^+(\langle \rangle) \otimes U^+(0) \otimes U^+(\langle \rangle)
\]

where \( U^+(\langle \rangle) \) (resp. \( U^+(\langle \rangle) \)), is the subalgebra generated by root vectors
having a root from the set \( \{ \alpha + k\delta \mid k \geq 0, \alpha \in \Delta_{0,+} \} \) (resp. \( \{ -\alpha + k\delta \mid k > 0, \alpha \in \Delta_{0,+} \} \)) and \( U^+(0) \) is the subalgebra generated by certain root vectors for the set
of imaginary positive roots \( \{ k\delta \mid k > 0 \} \). Here \( \Delta_0 \) denotes the set of roots of \( \mathfrak{g} \)
with chosen set of positive/negative roots \( \Delta_{0,\pm} \). Let \( U_q^+(S) \) generated by the root
vectors coming from the “natural” or synonymously imaginary partition of positive
roots \( S = \{ \alpha + k\delta \mid k \in \mathbb{Z}, \alpha \in \Delta_{0,+} \} \cup \{ k\delta \mid k \geq 0 \} \) for the natural Borel subalgebra.
This subalgebra gives rise to an imaginary Verma module where one induces up
using the natural Borel. If one chooses another partition instead of the natural or
standard positive root system one obtains other Verma type modules. In this paper
we focus on quantized imaginary Verma modules which are in a sense the simplest
quantized Verma type modules. We don’t decompose \( U^+ \), rather we decompose the
positive part of the quantized enveloping algebra \( U_q^+(S) \). Our main result given
below in Theorem 3.4.7 is essentially

\[
U^+(S) \cong U^+(\langle \rangle) \otimes \Omega(U^+(\langle \rangle)) \otimes U^+(0) \otimes U^0
\]

where the algebras on the right are what is defined above as in [BCP99] and \( \Omega \)
is an anti-automorphism. The braid group action on the root vectors of the quantized
enveloping algebra plays a fundamental role in the proof of the our result. We use
them to transform results appearing in Beck (see [Bec94a] and [Bec94b]), and
Damiani’s work (see [Dam98] and [Dam00]) to our setting.

In [BCP99] the authors give an algebraic characterization of the affine canonical
basis corresponding to the standard set of positive roots generalizing results
of Lusztig [Lus90] and Kashiwara [Kas91]. In future work we will construct an
analog of what we call the Kashiwara algebra \( \mathcal{K}_q \) for the imaginary Verma module
\( M_q(\lambda) \) for the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) by introducing certain Kashiwara-type
operators. Then we will prove that a certain quotient \( \mathcal{N}_q^{-} \) of \( U_q(\hat{\mathfrak{g}}) \) is a simple
\( \mathcal{K}_q \)-module. This has already been done in the setting of \( U_q(\mathfrak{sl}(2)) \) (see [CFM10]).
Our eventual aim is to provide an algebraic characterization of a “reduced” canonical
basis for the quantized imaginary reduced Verma module constructed from the
imaginary Borel defined by the set of positive roots \( S \).
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2. The affine Lie algebra \( \hat{g} \).

We begin by recalling some basic facts and constructions for the affine Kac-Moody algebra \( \hat{g} \) and its imaginary Verma modules. See [Kac90] for Kac-Moody algebra terminology and standard notations.

2.1. Let \( I = \{0, \ldots, N\}, I_0 = \{1, 2, \ldots, N\} \), and \( A = (a_{ij})_{0 \leq i, j \leq N} \) be a generalized affine Cartan matrix of type 1 for an untwisted affine Kac-Moody algebra \( \hat{g} \). Let \( D = (d_0, \ldots, d_N) \) be a diagonal matrix with relatively prime integer entries such that the matrix \( DA \) is symmetric. Then \( \hat{g} \) has the loop space realization

\[
\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d,
\]

where \( g \) is the finite dimensional simple Lie algebra over \( \mathbb{C} \) with Cartan matrix \((a_{ij})_{1 \leq i, j \leq N} \), \( c \) is central in \( \hat{g} \), \( d \) is the degree derivation, so that \([d, x \otimes t^n] = nx \otimes t^n\) for any \( x \in g \) and \( n \in \mathbb{Z} \), and \([x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + \delta_{n+m,0}n(x)y \) for all \( x, y \in g \), \( n, m \in \mathbb{Z} \).

An alternative Chevalley-Serre presentation of \( \hat{g} \) is given by defining it as the Lie algebra with generators \( e_i, f_i, h_i \ (i \in I) \) and \( d \) subject to the relations

\[
\begin{align*}
(2.1) & \quad [h_i, h_j] = 0, \quad [d, h_i] = 0, \\
(2.2) & \quad [h_i, e_j] = a_{ij}e_j, \quad [d, e_j] = \delta_{0,j}e_j, \\
(2.3) & \quad [h_i, f_j] = -a_{ij}f_j, \quad [d, f_j] = -\delta_{0,j}f_j, \\
(2.4) & \quad [e_i, f_j] = \delta_{ij}h_i, \\
(2.5) & \quad (ad e_i)^{1-a_{ij}}(e_j) = 0, \quad (ad f_j)^{1-a_{ij}}(f_j) = 0, \quad i \neq j.
\end{align*}
\]

We set \( \mathfrak{h} \) to be the span of \( \{h_0, \ldots, h_N, d\} \).

Let \( \Delta_0 \) be the set of roots of \( g \) with chosen set of positive/negative roots \( \Delta_{0, \pm} \). Let \( Q_0 \) be the free abelian group with basis \( \alpha_i \), \( 1 \leq i \leq N \) which is the root lattice of \( g \). Let \( Q_0 = \sum \mathbb{Z}h_i \) be the coroot lattice of \( g \). The co-weight lattice is defined to be \( P_0 = \text{Hom}(Q_0, \mathbb{Z}) \) with basis \( \omega_i \) defined by \( \langle \omega_i, \alpha_j \rangle = \delta_{i,j} \). The simple reflections \( s_i : P_0 \rightarrow P_0 \) are defined by \( s_i(x) = x - \langle \alpha_i, x \rangle h_i \). The \( s_i \) also act on \( Q_0 \) by \( s_i(y) = y - \langle y, h_i \rangle \alpha_i \). The Weyl group of \( g \) is defined as the subgroup \( W_0 \) of \( \text{Aut} P_0 \) generated by \( s_1, \ldots, s_N \). The affine Weyl group is defined as \( W = W_0 \ltimes Q_0 \). Let \( \theta = \sum_{i=1}^N a_i\alpha_i \) be the highest positive root with \( a_i \) labels of the extended Dynkin diagram and set \( s_\theta = (s_\theta, -\theta) \) where

\[
s_\theta(\lambda) = \lambda - \langle \lambda, \theta \rangle \theta
\]

for all \( \lambda \in \mathfrak{h}^* \). Note if \( \alpha = \sum_i k_i\alpha_i \), then

\[
\tilde{\alpha} = \sum_i \frac{\langle \alpha_i, \alpha \rangle}{\langle \alpha, \alpha \rangle} k_i\alpha_i
\]

(see [Kac90] formula (5.1.1) and Ch. 2 for the notation \( \langle \cdot, \cdot \rangle \)). Note also \( h_i = \tilde{\alpha}_i \).

Then \( W \) is generated by \( s_0, \ldots, s_N \). Let \( W = W_0 \ltimes P_0 \cong T \ltimes W \) be the generalized affine Weyl group where \( T \) is the group of Dynkin diagram automorphisms. The length of element \( \tilde{w} = \tau w \in \tilde{W} \) with \( \tau \in T \) and \( w \in W \) is defined by \( l(\tilde{w}) = l(w) \).
Let $\Delta$ be the root system of $\hat{g}$ with positive/negative set of roots $\Delta_{\pm}$ and simple roots $\Pi = \{\alpha_0, \ldots, \alpha_N\}$. Define $\delta = \alpha_0 + \theta$. Extend the root lattice $Q_0$ of $g$ to the affine root lattice $Q := Q_0 \oplus \mathbb{Z}\delta$, and extend the form $(\cdot, \cdot)$ to $Q$ by setting $(q\delta, \delta) = 0$ for all $q \in Q_0$ and $(\delta, \delta) = 0$. The generalized affine Weyl group $\hat{W}$ acts on $Q$ as an affine transformation group. In particular if $z \in \hat{P}_0$ and $1 \leq i \leq N$, then $z(\alpha_i) = \alpha_i - (z, \alpha_i)\delta$. Let $Q_+ = \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i \oplus \mathbb{Z}_{\geq 0}\delta$.

The root system $\Delta$ of $\hat{g}$ is given by

$$\Delta = \{\alpha + n\delta \mid \alpha \in \Delta_0, n \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z}, k \neq 0\}.$$ 

The roots of the form $\alpha + n\delta$, $\alpha \in \Delta, n \in \mathbb{Z}$ are called real roots, and those of the form $k\delta$, $k \in \mathbb{Z}, k \neq 0$ are called imaginary roots. We let $\Delta^r$ and $\Delta^i$ denote the sets of real and imaginary roots, respectively. The set of positive real roots of $\hat{g}$ is $\Delta^r_+ = \Delta_{0,+} \cup \{\alpha + n\delta \mid \alpha \in \Delta_0, n > 0\}$ and the set of positive imaginary roots is $\Delta^i_+ = \{k\delta \mid k > 0\}$. The set of positive roots of $\hat{g}$ is $\Delta_+ = \Delta^r_+ \cup \Delta^i_+$. Similarly, on the negative side, we have $\Delta_− = \Delta^r_− \cup \Delta^i_−$, where $\Delta^r_− = \Delta_{0,-} \cup \{\alpha + n\delta \mid \alpha \in \Delta_0, n < 0\}$ and $\Delta^i_− = \{k\delta \mid k < 0\}$. The weight lattice $P$ of $\hat{g}$ is $P = \{\lambda \in \hat{h}^* \mid \lambda(h_i) \in \mathbb{Z}, i \in I, \lambda(d) \in \mathbb{Z}\}$. Let $B$ denote the associated braid group with generators $T_0, T_1, \ldots, T_N$.

2.2. Consider the partition $\Delta = \mathbb{Z} \cup \mathbb{Z}$ of the root system of $\hat{g}$ where $S = \{\alpha + n\delta \mid \alpha \in \Delta_0, n \in \mathbb{Z}\} \cup \{k\delta \mid k > 0\}$. This is a non-standard partition of the root system $\Delta$ in the sense that $S$ is not Weyl equivalent to the set $\Delta_+$ of positive roots. There are other non-standard partitions of the root system $\Delta$, but we leave the study of Verma type modules arising from these other partitions to future work. The reason we stick to the case of the above $S$ for imaginary Verma modules is that they are perhaps the least technically complicated to work with when considering all non-conjugate non-standard partitions of $\Delta$.

3. The quantum affine algebra $U_q(\hat{g})$

3.1. The quantum affine algebra $U_q(\hat{g})$ is the $\mathbb{C}(q^{1/2})$-algebra with 1 generated by $E_i, F_i, K_\alpha, \gamma_i^{\pm 1/2}, D_i, 0 \leq i \leq N, \alpha \in Q,$ and defining relations:

$$DD^{-1} = D^{-1}D = \gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1,$$

$$K_\alpha K_\beta = K_{\alpha + \beta}, K_0 = 1,$$

$$[\gamma_i^{\pm 1/2}, U_q(\hat{g})] = [D, K_i^{\pm 1}] = [K_i, K_j] = 0,$$

$$\gamma_i^{\pm 1/2} = K_i^\pm 1,$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$K_\alpha E_i K_\alpha^{-1} = q^{(\alpha, \alpha)} E_i, K_\alpha F_i K_\alpha^{-1} = q^{-(\alpha, \alpha)} F_i,$$

$$DE_i D^{-1} = q^{\delta_i} E_i, DF_i D^{-1} = q^{-\delta_i} F_i,$$

$$\sum_{a=0}^{1-\delta_{ij}} (-1)^a F_i^{1-(a, -s)} E_j E_i^{(s)} = 0 = \sum_{a=0}^{1-\delta_{ij}} (-1)^a F_i^{(1-a, -s)} F_j E_i^{(s)}, \quad i \neq j.$$
where
\[ q_i := q^{a_i}, \quad [n]_i = \frac{q^n_i - q^{-n}_i}{q_i - q^{-1}_i}, \quad [n]_i! := \prod_{k=1}^{n} [k]_i \]

and \( K_i = K_{\alpha_i}, \quad E_i^{(s)} = E_i/\lbrack s \rbrack_i! \) and \( F_i^{(s)} = F_i/\lbrack s \rbrack_i! \) (see [Bec94a] and [Lus88]).

The quantum affine algebra \( U_q(\hat{g}) \) is a Hopf algebra with a comultiplication given by

\[
\begin{align*}
\Delta(K^\pm_1) &= K^\pm_1 \otimes K^\pm_1, \\
\Delta(D^\pm_1) &= D^\pm_1 \otimes D^\pm_1, \\
\Delta(\gamma^{\pm 1/2}) &= \gamma^{\pm 1/2} \otimes \gamma^{\pm 1/2}, \\
\Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\
\Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i,
\end{align*}
\]

and an antipode given by

\[
\begin{align*}
s(E_i) &= -E_i K_i^{-1}, \\
s(F_i) &= -K_i F_i, \\
s(K_i) &= K_i^{-1}, \\
s(D) &= D^{-1}, \\
s(\gamma^{1/2}) &= \gamma^{-1/2}.
\end{align*}
\]

Let \( \Phi : U_q(\hat{g}) \to U_q(\hat{g}) \) be the \( \mathbb{C} \)-algebra automorphism defined by

\[
\begin{align*}
\Phi(E_i) &= F_i, \quad \Phi(F_i) = E_i, \quad \Phi(K_\alpha) = K_\alpha, \\
\Phi(D) &= D, \quad \Phi(\gamma^{\pm 1/2}) = \gamma^{\pm 1/2}, \quad \Phi(q^{\pm 1/2}) = q^{\mp 1/2},
\end{align*}
\]

and let \( \Omega : U_q(\hat{g}) \to U_q(\hat{g}) \) be the \( \mathbb{C} \)-algebra anti-automorphism defined by

\[
\begin{align*}
\Omega(E_i) &= F_i, \quad \Omega(F_i) = E_i, \quad \Omega(K_\alpha) = K_{-\alpha}, \\
\Omega(D) &= D^{-1}, \quad \Omega(\gamma^{\pm 1/2}) = \gamma^{\mp 1/2}, \quad \Omega(q^{\pm 1/2}) = q^{\mp 1/2},
\end{align*}
\]

(see [Bec94a, Section 1]).

3.2. There is an alternative realization for \( U_q(\hat{g}) \), due to Drinfeld [Dri85], which we shall also need. We will use the formulation due to J. Beck [Bec94a]. Let \( U_q(\hat{g}) \) be the associative algebra with 1 over \( \mathbb{C}(q^{1/2}) \)- generated by

\[
x_i^{\pm 1}, \quad h_is, \quad K_i^{\pm 1}, \quad \gamma^{\pm 1/2}, \quad D^{\pm 1} \quad 1 \leq i \leq N, \quad r, s \in \mathbb{Z}, \quad s \neq 0,
\]
with defining relations:

\[ DD^{-1} = D^{-1}D = K_iK_i^{-1} = K_i^{-1}K_i = \gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1, \]

\[ \gamma^{1/2}U_q(\mathfrak{g}) = [D, K_i^{\pm1}] = [K_i, K_j] = [K_i, h_{ij}] = 0, \]

\[ Dh_{ir}D^{-1} = q^r h_{ir}, \quad Dx_{ir}^\pm D^{-1} = q^r x_{ir}^\pm, \]

\[ K_i x_{ir}^\pm K_i^{-1} = q_i^{(\alpha_i|\alpha_j)} x_{ir}^\pm, \]

\[ [h_{ik}, h_{jl}] = \delta_{k,-l} \frac{1}{k} [ka_{ij}]_i \frac{\gamma^k - \gamma^{-k}}{q_j - q_j}, \]

\[ [h_{ik}, x_{jl}^\pm] = \pm \frac{1}{k} [ka_{ij}]_i \gamma^{\pm |kj|/2} x_{jl}^\pm, \]

\[ x_{i,k+1}^\pm x_{j,l}^\pm - q^{\pm (\alpha_i|\alpha_j)} x_{j,l+1}^\pm x_{i,k+1}^\pm = q^{\pm (\alpha_i|\alpha_j)} x_{i,k}^\pm x_{j,l+1}^\pm - x_{j,l+1}^\pm x_{i,k}^\pm, \]

\[ [x_{ik}^\pm, x_{jl}^\pm] = \delta_{ij} \frac{1}{q_i - q_i} \left( \gamma^{\pm 1} \psi_{i,k+1} - \gamma^{\pm 2} \phi_{i,k+1} \right), \]

where \( \sum_{k=0}^{\infty} \psi_{ik} z^k = K_i \exp \left( (q_i - q_i^{-1}) \sum_{l>0} h_{il} z^l \right), \) and

\[ \sum_{k=0}^{\infty} \phi_{i-k} z^{-k} = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{l>0} h_{i,-l} z^{-l} \right), \]

For \( i \neq j, \) \( n := 1 - a_{ij} \)

\[ \text{Sym}_{k_1, k_2, \ldots, k_n} \sum_{r=0}^{n} (-1)^r \left[ \sum_{p=0}^{\infty} \phi_{ip} u^{-p} \right] x_{ik_1}^\pm \cdots x_{ik_r}^\pm x_{i,k_{r+1}}^\pm \cdots x_{i,k_n}^\pm = 0. \]

Note that Beck’s paper [Bec94a] on page 565 has a typo in it where he has \( \phi_{i-k} z^k \) instead of \( \phi_{i-k} z^{-k}. \)

In the above last relation Sym means symmetrization with respect to the indices \( k_1, \ldots, k_n. \) Also in Drinfeld’s notation one has \( e^{\hbar c/2} = \gamma \) and \( e^{\hbar/2} = q. \)

The algebras given above and in \( \S 3.1 \) are isomorphic [Dri85]. If one uses the formal sums

\[ \phi_i(u) = \sum_{p \in \mathbb{Z}} \phi_{ip} u^{-p}, \quad \psi_i(u) = \sum_{p \in \mathbb{Z}} \psi_{ip} u^{-p}, \quad x_i^\pm(u) = \sum_{p \in \mathbb{Z}} x_{ip}^\pm u^{-p} \]

Drinfeld’s relations (3.11)-(3.14) can be written as

\[ \phi_i(u), \phi_j(v) = 0 = [\psi_i(u), \psi_j(v)] \]

\[ \phi_i(u) \psi_j(v) \phi_i(u)^{-1} \psi_j(v)^{-1} = g_{ij}(uv^{-1} \gamma^{-1})/g_{ij}(uv^{-1} \gamma) \]

\[ \phi_i(u)x_j^\pm(v) \phi_i(u)^{-1} = g_{ij}(uv^{-1} \gamma^{1/2} \pm 1) x_j^\pm(v) \]

\[ \psi_i(u)x_j^\pm(v) \psi_i(u)^{-1} = g_{ij}(uv^{-1} \gamma^{1/2} \mp 1) x_j^\pm(v) \]

\[ (u - q^{\pm (\alpha_i|\alpha_j)} v) x_j^\pm(u) x_j^\pm(v) = (q^{\pm (\alpha_i|\alpha_j)} u - v) x_j^\pm(v) x_j^\pm(u) \]

\[ [x_i^\pm(u), x_j^\pm(v)] = \delta_{ij} (q_i - q_i^{-1})^{-1} \left( \delta(u/v \gamma) \psi_i(v \gamma^{1/2}) - \delta(u \gamma/v) \phi_i(u \gamma^{1/2}) \right) \]

where \( g_{ij}(t) = g_{ij,q}(t) \) is the Taylor series at \( t = 0 \) of the function \( (q^{(\alpha_i|\alpha_j)} t - 1)/(t - q^{(\alpha_i|\alpha_j)}) \) and \( \delta(z) = \sum_{k \in \mathbb{Z}} z^k \) is the formal Dirac delta function.
3.3. Let $U_q^+ = U_q^+ (\widehat{g})$ (resp. $U_q^- = U_q^- (\widehat{g})$) be the subalgebra of $U_q (\widehat{g})$ generated by $E_i$ (resp. $F_i$), $i \in I$, and let $U_q^0 = U_q^0 (\widehat{g})$ denote the subalgebra generated by $K_i^{\pm 1}$ ($i \in I$) and $D^{\pm 1}$.

Beck in [Bec94a] and [Bec94b] has given a total ordering of the root system $\Delta$ and a PBW-like basis for $U_q (\widehat{g})$. Below we follow the construction developed by Damiani [Dam98], Gavarini [Gav99] and [BK96] and let $E_\beta$ denote the root vectors for each $\beta \in \Delta_+$ counting with multiplicity for the imaginary roots. One defines $F_\beta = E_{-\beta} := \Omega(E_\beta)$ for $\beta \in \Delta_+$ (refer to (3.6)).

For any affine Lie algebra $\widehat{g}$, there exists a map $\pi : Z \rightarrow I$ such that, if we define

$$
\beta_k = \begin{cases} 
\pi(k); & \text{if } k < 0, \\
\pi(k) & \text{if } k = 0, \\
\pi(k+1) & \text{if } k = 1,
\end{cases}
$$

then the map $\pi' : Z \rightarrow \Delta^\text{im}$ given by $\pi'(k) = \beta_k$ is a bijection. Note that the map $\pi$, and hence the total ordering, is not unique. We fix $\pi$ so that $\{\beta_k | k \leq 0\} = \{\alpha + n\delta | \alpha \in \Delta_{0,+}, n \geq 0\}$ and $\{\beta_k | k \geq 1\} = \{-\alpha + n\delta | \alpha \in \Delta_{0,+}, n > 0\}$. One also defines the set of imaginary roots with multiplicity as

$$
\Delta_+ (\text{im}) := \Delta_+^{\text{im}} \times I_0,
$$

where we recall $I_0 = \{1, ..., N\}$.

It will be convenient for us to invert Beck’s original ordering of the positive roots (see [BK96, §1.4.1]). Let

$$
(3.24) \beta_0 > \beta_{-1} > \beta_{-2} > \cdots > \delta > 2\delta > \cdots > \beta_2 > \beta_1.
$$

([Gav99, §2.1] for this ordering). We define $-\alpha < -\beta$ iff $\alpha > \beta$ for all positive roots $\alpha, \beta$, so we obtain a corresponding ordering on $\Delta_-$.

The following elementary observation on the ordering will play a crucial role later. Write $A < B$ for two sets $A$ and $B$ if $x < y$ for all $x \in A$ and $y \in B$. Then Beck’s total ordering of the positive roots can be divided into three sets:

$$
\{\alpha + n\delta | \alpha \in \Delta_{0,+}, n \geq 0\} > \{k\delta | k > 0\} > \{-\alpha + n\delta | \alpha \in \Delta_{0,+}, k > 0\}.
$$

Similarly, for the negative roots, we have,

$$
\{-\alpha - n\delta | \alpha \in \Delta_{0,+}, n \geq 0\} < \{-k\delta | k > 0\} < \{-\alpha - k\delta | \alpha \in \Delta_{0,+}, k > 0\}.
$$

The action of the braid group generators $T_i$ on the generators of the quantum group $U_q(\widehat{g})$ is given by the following.

$$
T_i(E_i) = -F_iK_i, \quad T_i(F_i) = -K_i^{-1}E_i,
$$

$$
T_i(E_j) = \sum_{r=0}^{\infty} (-1)^r a_{ij} q_r^{E_i(a_{ij} - r)} E_j E_i^{(r)}, \quad \text{if } i \neq j,
$$

$$
T_i(F_j) = \sum_{r=0}^{\infty} (-1)^r a_{ij} q_r^{F_j(r)} F_i F_j^{(r)}, \quad \text{if } i \neq j,
$$

$$
T_i(K_j) = K_j K_i^{-a_{ij}}, \quad T_i(K_j^{-1}) = K_i^{-1} K_j^{a_{ij}},
$$

$$
T_i(D) = DK_i^{-\delta_{i,0}}, \quad T_i(D^{-1}) = D^{-1} K_i^{\delta_{i,0}}.
$$
For each $\beta_k \in \Delta_r^e$, define the root vector $E_{\beta_k}$ in $U_q(\hat{g})$ by

$$E_{\beta_k} = \begin{cases} 
T_{\pi(0)}^{-1} T_{\pi(-1)}^{-1} \cdots T_{\pi(k+1)}^{-1}(E_{\pi(k)}) & \text{for all } k < 0, \\
E_{\pi(0)} & k = 0, \\
E_{\pi(1)} & k = 1, \\
T_{\pi(1)} T_{\pi(2)} \cdots T_{\pi(k-1)}(E_{\pi(k)}) & \text{for all } k > 1.
\end{cases} \quad (3.25)$$

The following result is due to Iwahori, Matsumoto and Tits (see [Bec94a], Section 2).

**Proposition 3.3.1.** Suppose $w \in \hat{W}$ and $w = \tau s_{i_1} \cdots s_{i_r}$ is a reduced decomposition in terms of simple reflections and $\tau \in T$. Then $T_w := \tau T_{i_1} \cdots T_{i_n}$ does not depend on the reduced decomposition of $w$ chosen, but rather only on $w$.

Orient the Dynkin diagram of $g$ by defining a map $o : V \to \{\pm 1\}$ so that for adjacent vertices $i$ and $j$ one has $o(i) = -o(j)$. Beck defines $\hat{T}_{\omega_i} = o(i)T_{\omega_i}$ and obtains ([Bec94a, Section 4]) for $i \in I_0$ and $k \in \mathbb{Z}$,

$$x_{ik}^- := \hat{T}_{\omega_i}^{-k}(F_i), \quad x_{ik}^+ := \hat{T}_{\omega_i}^{-k}(E_i).$$

Fix $i \in I_0$ and $k \geq 0$. The proposition above in the particular case of the reduced decomposition of $\omega_i = \tau s_{i_1} \cdots s_{i_r} \in \hat{P}_0 \subset \hat{W}$ where $\tau$ is a diagram automorphism and the $s_{i}$ are simple reflections, gives

$$x_{ik}^+ = \hat{T}_{\omega_i}^{-k}(E_i) = o(i)^k(\tau T_{i_1} \cdots T_{i_r})^{-k}(E_i) = o(i)^k T_{i_1} \cdots T_{i_m} \tau^{-k}(E_i),$$

for some $j_t \in I$.

Fixing still $i \in I_0$ and $k \geq 0$, choose now $w_{\alpha_i + k\delta} \in \hat{W}$, and $j \in I$, such that $w_{\alpha_i + k\delta}(\alpha_j) = \alpha_i + k\delta$. Writing $w_{\alpha_i + k\delta} = s_{i_1} \cdots s_{i_p}$ as a reduced decomposition of simple reflections, Beck defines

$$E_{\alpha_i + k\delta} := T_{w_{\alpha_i + k\delta}}(E_j) = T_{i_1} \cdots T_{i_p}(E_j),$$

which according to Lusztig is independent of the choice of $w_{\alpha_i + k\delta}$, its reduced decomposition and $j \in I$. In particular we can choose $j = \tau^{-k}(i)$ and $w = s_{j_1} \cdots s_{j_m}$, so that $s_{j_1} \cdots s_{j_m}(\alpha_j) = s_{j_1} \cdots s_{j_m}(\alpha_{\tau^{-k}(i)}) = \alpha_i + k\delta$. Then

$$E_{\alpha_i + k\delta} = T_{j_1} \cdots T_{j_m}(E_{\tau^{-k}(i)}) = o(i)^k x_{ik}^+. \quad (3.26)$$

Now one defines

$$F_{\alpha_i + k\delta} = \Omega(E_{\alpha_i + k\delta}) = o(i)^k \Omega(x_{ik}^+) = o(i)^k \Omega(\hat{T}_{\omega_i}^{-k}(E_i)) \quad (3.27)$$

$$= o(i)^k \hat{T}_{\omega_i}^{-k}(\Omega(E_i)) = o(i)^k \hat{T}_{\omega_i}^{-k}(F_i) = o(i)^k x_{i,-k},$$

as $T_{\tau} \Omega = \Omega T_{\tau}$ and $T_{\tau} \Omega = \Omega T_{\tau}$.

If $k < 0$ and $i \in I_0$, then $-\alpha_i - k\delta \notin \Delta_r^e$, so that $-\alpha_i - k\delta = \beta_l = s_{\pi(1)} \cdots s_{\pi(l-1)}(\alpha_{\pi(l)})$ for $l > 1$ and $-\alpha_i - k\delta = \beta_l = \alpha_{\pi(1)}$ if $l = 1$. Then for $l > 1$,

$$E_{-\alpha_i - k\delta} = E_{\beta_l} = T_{\omega_i}^{-k} T_{i}^{-1}(E_i) = -T_{\omega_i}^{-k}(K_i^{-1} F_i) \quad (3.28)$$

$$= -o(i)^k T_{\omega_i}^{-k}(K_i^{-1}) x_{i,-k}^- = -o(i)^k K_i^{-1} x_{i,-k}^-.$$
as \( \omega_i(-\alpha_i) = -\alpha_i + \delta \) (see \S2.1) so that \( \omega_i^{-k}s_i(\alpha_i) = \omega_i^{-k}(-\alpha_i) = -\alpha_i - k\delta \) and \( T_{\omega_i}(K_i^{-1}) = K_{-\alpha_i+\delta} \). Now

\[
(3.29) \quad F_{-\alpha_i-k\delta} = \Omega(E_{-\alpha_i-k\delta}) = -o(i)^{k}\Omega(K_i^{-1}\gamma^{-k}x_{i,-k}^+ - \gamma^{-k}x_{i,k}^-) = -o(i)^{k}K_i\gamma^kx_{i,k}^+.
\]

He also defines for \( k > 0 \)

\[
\psi_{ik} = (q_i-q_i^{-1})\gamma^{k/2}[E_i, \hat{T}_\omega(F_i)] = (q_i-q_i^{-1})\gamma^{k/2}[E_i, x_{i,k}^-],
\]

\[
\phi_{i,-k} = (q_i-q_i^{-1})\gamma^{-k/2}[F_i, \hat{T}_\omega(E_i)] = (q_i-q_i^{-1})\gamma^{-k/2}[F_i, x_{i,-k}^+],
\]

\( \psi_{i,0} := K_i, \phi_{i,0} := K_i^{-1}, \) and for any \( \tau \in T \),

\[
(3.30) \quad T_\tau(E_i) := E_{\tau(i)}, \quad T_\tau(F_i) := F_{\tau(i)}, \quad T_\tau(K_i) := K_{\tau(i)}.
\]

One writes \( \tau \) for \( T_\tau \). Note also that \( \tau s_i\tau^{-1} = s_{\tau(i)} \) for all \( 0 \leq i \leq n \).

Each real root space is 1-dimensional, but each imaginary root space is \( N \)-dimensional. Hence, for each positive imaginary root \( k\delta \) \( (k > 0) \) we define \( N \) imaginary root vectors, \( E_{k\delta}^{(i)}(i \in I_0) \) by

\[
(3.31) \quad \exp \left( (q_i-q_i^{-1})\sum_{k=1}^{\infty} E_{k\delta}^{(i)} z^k \right) = 1 + (q_i-q_i^{-1})\sum_{k=1}^{\infty} K_{-1}[E_i, x_{i,k}^-]z^k = 1 + \sum_{k=1}^{\infty} K_{-1}\psi_{ik} \left( \gamma^{-1/2}z \right)^k = \exp \left( (q_i-q_i^{-1})\sum_{l>0} h_{il}\gamma^{-1/2}z^l \right).
\]

So \( E_{k\delta}^{(i)} = h_{ik}\gamma^{-k/2} \) for all \( k > 0 \). For \( k < 0 \) we also define \( E_{k\delta}^{(i)} := \Omega(E_{-k\delta}^{(i)}) = h_{ik}\gamma^{k/2} \). Our definition of \( E_{k\delta}^{(i)} \) is the same as \cite{Dam98}, Definition 7. In particular

\[
K_{-1}[E_i, x_{i,k}^-] = -o(i)^{k}\gamma^{-k}K_{-1}[E_i, K_iE_{-\alpha_i+k\delta}]
= -o(i)^{k}\gamma^{-k}K_{-1}(E_iK_iE_{-\alpha_i+k\delta} - K_iE_{-\alpha_i+k\delta}E_i)
= -o(i)^{k}\gamma^{-k}(q_i^{-2}E_iE_{-\alpha_i+k\delta} - E_{-\alpha_i+k\delta}E_i).
\]

Using these sets and a symbol \( \ast \in \{ \pm\infty, \pm \infty, \text{im} \} \), one defines \( U_q^+(\ast) \) as the subalgebra of \( U_q \) generated by \( \{ E_\alpha | \alpha \in \Delta_+(\ast) \} \) and

\[
(3.32) \quad U_q^{\geq0}(\ast) := U_q^+(\ast)U_q^0, \quad U_q^-(\ast) := \Omega(U_q^+(\ast)), \quad U_q^{\leq0}(\ast) := \Omega(U_{q}^{\geq0}(\ast)).
\]

Recall that the \( R \)-"matrices" are defined having values in \( U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}}) \) (see \cite{Lus93} for the definition of \( U_q(\hat{\mathfrak{g}}) \hat{\otimes} U_q(\hat{\mathfrak{g}}) \) and \cite{Bec94a}, Section 5) for \( 1 \leq i \leq N \)
by

\[
R_i = \sum_{n \geq 0} (-1)^n q_i^{-\frac{n(n-1)}{2}} (q_i - q_i^{-1})^n [n]_i ! T_i (F_i^{(n)}) \otimes T_i (E_i^{(n)}),
\]

\[
= \sum_{n \geq 0} (q_i^{-1} - q_i)^n q_i^{-\frac{n(n-1)}{2}} [n]_i ! E_i^n K_i^{-n} \otimes F_i^n K_i^n,
\]

(3.34) \[\overline{R}_i = T_i^{-1} \otimes T_i^{-1} \circ \tilde{\phi}_i^{-1} = \sum_{n \geq 0} \frac{n(n-1)}{2} (q_i - q_i^{-1})^n [n]_i ! F_i^{(n)} \otimes E_i^{(n)}.
\]

These operators have inverses

\[
R_i^{-1} = \sum_{n \geq 0} (q_i - q_i^{-1})^n q_i^{\frac{n(n+1)}{2}} [n]_i ! E_i^n K_i^{-n} \otimes F_i^n K_i^n
\]

\[
\overline{R}_i^{-1} = \sum_{n \geq 0} q_i^{-\frac{n(n-1)}{2}} (q_i^{-1} - q_i)^n [n]_i ! F_i^n \otimes E_i^n
\]

Suppose \( w \in \tilde{W} \) and \( \tau s_i \cdots s_k \) is a reduced presentation for \( w \) such that \( \tau \) is defined as in (3.30). Beck defines the following “\( R \)-matrices”:

(3.35) \[ R_w = \tau (S_i, S_{i+1} \cdots S_{i-1} (R_{i-1}) \cdots (R_2 R_1)) (\overline{R}_1 \cdots \overline{R}_{i-1} \overline{R}_i),
\]

(3.36) \[ \overline{R}_w = \tau (S_i^{-1}, S_{i-1}^{-1} \cdots S_{i+1}^{-1} (R_{i+1}^{-1}) \cdots (R_2^{-1} R_1^{-1})) (\overline{R}_1 \cdots \overline{R}_{i-1} \overline{R}_i).
\]

Using the root partition \( S = \{ \alpha + k \delta \mid \alpha \in \Delta_{0,+}, k \in \mathbb{Z} \} \cup \{ l \delta \mid l \in \mathbb{Z}_{>0} \} \) from Section 2.3, we define:

\( U_r^+(S) \) to be the subalgebra of \( U_q(\hat{g}) \) generated by \( x_{i,k}^+ \) \((1 \leq i \leq N, k \in \mathbb{Z}) \) and \( h_{i,l} \) \((1 \leq i \leq N, l > 0)\);

\( U_r^-(S) \) to be the subalgebra of \( U_q(\hat{g}) \) generated by \( x_{i,k}^- \) \((1 \leq i \leq N, k \in \mathbb{Z}) \) and \( h_{i,-l} \) \((1 \leq i \leq N, l > 0)\), and

\( U_r^0(S) \) to be the subalgebra of \( U_q(\hat{g}) \) generated by \( K_i^{\pm 1} \) \((1 \leq i \leq N)\), \( \gamma^{\pm 1/2} \), and \( D^{\pm 1} \). Thus \( U_r^0(S) = U_r^0(\hat{g}) \).

3.4. Let \( \omega \) denote the standard \( \mathbb{C} (q^{1/2}) \)-linear antiautomorphism of \( U_q(\hat{g}) \), and set \( E_{-\alpha} = \omega (E_{\alpha}) \) for all \( \alpha \in \Delta_+ \). Then \( U_q \) has a basis of elements of the form \( E_{-HE_{+}} \), where \( E_{\pm} \) are ordered monomials in the \( E_{\alpha}, \alpha \in \Delta_{\pm} \), and \( H \) is a monomial in \( K_i^{\pm 1}, \gamma^{\pm 1/2}, \) and \( D^{\pm 1} \) (which all commute).

Furthermore, this basis is, in Beck’s terminology, convex, meaning that, if \( \alpha, \beta \in \Delta_+ \) and \( \beta > \alpha \), then

(3.37) \[ E_\beta E_\alpha - q^{(\alpha|\beta)} E_\alpha E_\beta = \sum_{\alpha < \gamma_1 < \cdots < \gamma_r < \beta} c_{\gamma} E_{\gamma_1}^a \cdots E_{\gamma_r}^a \]

for some integers \( a_1, \ldots, a_r \) and scalars \( c_{\gamma} \in \mathbb{C} [q, q^{-1}] \), \( \gamma = (\gamma_1, \ldots, \gamma_r) \) [BK96, Proposition 1.7c], [LS99], and similarly for the negative roots. The above is called the Levendorski and Soibelman convexity formula.
Set $\Lambda = \mathbb{C}[q^{1/2}, q^{-1/2}, q_i, i \in I, n > 1]$. We first begin with a slightly different $\Lambda$-form than in [FGM98]. Namely we define this algebra $U_\Lambda = U_\Lambda(\hat{\gamma})$ to be the $\Lambda$-subalgebra of $U_q(\hat{\gamma})$ with 1 generated by the elements

$$\gamma_{1,rs}^{\pm 1}, h_{is}, K_{i}^{\pm 1}, \gamma_1^{\pm 1/2}, D^{\pm 1}, \left[ K_i : s \right]_n, \left[ D : s \right]_n, \left[ \gamma ; s \right]_1, \left[ \gamma \psi_i ; k,l \right]$$

for $1 \leq i \leq N, r, s \in \mathbb{Z}, s \neq 0$ where following [Lus88], for each $i \in I, s \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we define the Lusztig elements in $U_q(\hat{\gamma})$:

$$\left[ \gamma ; s \right]_i = \frac{\gamma^s - \gamma^{-s}}{q_i - q_i'},$$

$$\left[ \gamma \psi_i ; k,l \right] = \frac{\gamma^{q_{i+k,l}} e_{i+k,l} - \gamma^{q_{i+k,l}} e_{i+k,l}^{-1}}{q_i - q_i'},$$

$$\left[ K_i : s \right]_n = \prod_{r=1}^{n} K_{i, i+r}^{s-r+1} - K_{i, i-r}^{-1} q_i^r - q_i'^{-r},$$

and

$$\left[ D : s \right]_n = \prod_{r=1}^{n} D q_0^{s-r+1} - D^{-1} q_0^{-r},$$

where $q_0 = q^{d_0}$. This $\Lambda$-form can be shown to be the same as that in [FGM98] with the exception that we have added the generators $\gamma_1^{\pm 1/2}, \left[ \gamma ; s \right]_i$ and $\left[ \gamma \psi_i ; k,l \right]$. Let $U^+_\Lambda$ (resp. $U^-_\Lambda$) denote the subalgebra of $U_\Lambda$ generated by the $x_{ik}^+, h_{ij}$, where $k \in \mathbb{Z}, l \in \mathbb{Z}_{>0}, 1 \leq i \leq N$ (resp. $x_{ik}^-, h_{ij}$, where $k \in \mathbb{Z}, l \in -\mathbb{Z}_{>0}\{0\}, 1 \leq i \leq N$), $i \in I$, and let $U^0_\Lambda$ denote the subalgebra of $U_\Lambda$ generated by the elements $\gamma_1^{\pm 1/2}, K_i^{\pm 1}, \left[ K_i : s \right]_n, D^{\pm 1}, \left[ D : s \right]_n, \left[ \gamma ; s \right]_1, \left[ \gamma \psi_i ; k,l \right]_i$. Note if $k + l > 0$

$$\left[ \gamma \psi_i ; k,l \right] \in U^+_\Lambda \quad \text{(resp.} \left[ \gamma \psi_i ; k,l \right] \in U^-_\Lambda).$$

**Lemma 3.4.1.** [Bec94a, Prop. 3.10, Lemma 3.15] Set $a = q^2_1 \gamma_1^{1/2}$, $b = q^2_1 \gamma_1^{-1/2}$, $c = -q^{a+1}_1 \gamma_1^{1/2}, d = -q^{b+1}_1 \gamma_1^{-1/2}$ for $i \neq j, r > 0, m \in \mathbb{Z}$. Then

$$[\psi_ir, x_{im}^+] = -\gamma^{1/2}[2]_i \sum_{s=1}^{r-1} q^{k-1}_{i-s} (q^s_i - q^{-s}_i) \psi_{i, r-k} x_{i, m+k}^+ + a^{r-1} x_{i, m+r}^+$$

$$[\psi_ir, x_{im}^-] = \gamma^{1/2}[2]_i \sum_{s=1}^{r-1} b^{k-1}(q^s_i - q^{-s}_i) x_{i, m+k}^+ \psi_{i, r-k}^+ + b^{r-1} x_{i, m+r}^-$$

$$[\psi_jr, x_{jm}^+] = \gamma^{1/2}[a_{ij}] \sum_{s=1}^{r-1} c^{k-1} (q^s_i - q^{-s}_i) \psi_{i, r-k} x_{j, m+k}^+ + a^{r-1} x_{j, m+r}^+$$

$$[\psi_jr, x_{jm}^-] = -\gamma^{-1/2}[a_{ij}] \sum_{s=1}^{r-1} d^{k-1}(q^s_i - q^{-s}_i) x_{j, m+k}^+ \psi_{i, r-k} + d^{r-1} x_{j, m+r}^-.$$

The anti-automorphism $\Omega$ sends $\psi_{ik}$ to $\phi_{ik}$ and since $\Omega T_i = T_i \Omega$, $\Omega$ sends $x_{im}^+$ to $x_{im}^+$ (see [Bec94a, Section 1 and 3.11] and [Lus93]). Thus one gets a similar set of commutator formulas for $\phi_{ir}$, but since we don’t need them explicitly we will not write them down.
The next result follows from direct calculations using Lusztig’s elements.

**Proposition 3.4.2.** The following commutation relations hold between the generators of $U_\k$. For $k \in \mathbb{Z}$, $l \in \mathbb{Z}_{>0}$, $1 \leq i, j \leq N$,

\[
x_{ik}^+ \begin{bmatrix} K_j \; s \\ n \end{bmatrix} = \begin{bmatrix} K_j \; s - a_{ji} \\ n \end{bmatrix} x_{ik}^+,
\]

\[
x_{ik}^+ \begin{bmatrix} D \; s \\ n \end{bmatrix} = \begin{bmatrix} D \; s - k \\ n \end{bmatrix} x_{ik}^+,
\]

\[
\begin{bmatrix} K_j \; s \\ n \end{bmatrix} x_{ik}^- = x_{ik}^- \begin{bmatrix} K_j \; s - a_{ji} \\ n \end{bmatrix},
\]

\[
\begin{bmatrix} D \; s \\ n \end{bmatrix} x_{ik}^- = x_{ik}^- \begin{bmatrix} D \; s - k \\ n \end{bmatrix},
\]

\[
\begin{bmatrix} D \; s \\ n \end{bmatrix} h_{ik} = h_{ik} \begin{bmatrix} D \; s + k \\ n \end{bmatrix},
\]

\[
[\gamma^{\pm 1/2}, U_\k] = [D, K_i^\pm 1] = [K_i, K_j] = [K_i, h_{jk}] = 0,
\]

\[
Dh_{ir} = q^r h_{ir} D, \quad Dx_{ir}^+ = q^s x_{ir}^+ D,
\]

\[
K_i x_{jr}^+ = q_i^{\pm (\alpha_i | \alpha_j)} x_{jr}^+ K_i,
\]

\[
[h_{ik}, h_{jl}] = \delta_{k,l} \left( \frac{1}{k} [ka_{ij}] \right)^{\gamma \gamma} \begin{bmatrix} \gamma / k \\ 1 \end{bmatrix},
\]

\[
[h_{ik}, x_{jl}^+] = \pm \frac{1}{k} [ka_{ij}] \gamma^{\pm (k/2)} x_{jl}^+, k, l,
\]

\[
[x_{ik}^+, x_{jl}^-] = \delta_{ij} \left( \gamma \gamma ; k, l \right)^{1 \gamma}. \]

**Corollary 3.4.3.** The algebra $U_\k$ inherits the standard triangular decomposition of $U_q(\mathfrak{g})$. In particular, any element $u$ of $U_\k$ can be written as an $\k$-linear combination of monomials of the form $u^{-} u^{+}$ where $u^{-} \in U_\k^-$ and $u^{+} \in U_\k^+$.

**Proof.** We first consider an element of the form $wx_{jl}^+$ where $w$ is a monomial in $U_\k^0$ or $U_\k^+$. We need to move the $w$ past $x_{jl}^+$. If $w = x_{il}^+$, then

\[
x_{ik}^+ x_{jl}^+ = x_{jl}^+ x_{ik}^+ + \delta_{ij} \left( \gamma \gamma ; k, l \right)^{1 \gamma},
\]

which now has summands that are in $U_\k^- U_\k^0 U_\k^+$. Similarly if $w$ is any of the following elements

\[
h_{is}, s > 0, \quad K_i^\pm 1, \quad \gamma^{\pm 1/2}, \quad D^{\pm 1}, \quad [K_i ; s] ; \quad [D ; s] ; \quad \gamma ; s
\]

then the Proposition above shows that $wx_{jl}^+ \in U_\k^- U_\k^0 U_\k^+$. If $w = \left( \gamma \gamma ; k, l \right)^{1 \gamma}$, then Lemma 3.4.1 tells us (using induction on $k$), that $wx_{jl}^+ \in U_\k^- U_\k^0 U_\k^+$.

If we consider elements of the form $wz$ where $z = h_{ik}, k < 0$, then if $w$ is any of the elements

\[
x_{ij}^+, h_{js}, s > 0, \quad K_i^\pm 1, \quad \gamma^{\pm 1/2}, \quad D^{\pm 1}, \quad [K_i ; s] ; \quad [D ; s] ; \quad \gamma ; s
\]

then the Proposition above shows that $wh_{ik} \in U_\k^- U_\k^0 U_\k^+$. 


If we consider elements of the form \( wz \) where
\[
z = \left[ \gamma \psi_i ; k, l \right] = \frac{-\gamma \phi_{i,k+l}}{q_i - q_{i-1}}
\]
k + l < 0, and if \( w \) is any of the elements
\[
x_{jl}^+, h_{js}, s > 0, \quad K_i^{\pm 1}, \quad \gamma^{\pm 1/2}, D, N_n, \quad \left[ K_i ; s \right]_n, \quad \left[ D ; s \right]_n, \quad \left[ \gamma ; s \right]_n, \quad \left[ \gamma \psi_i ; k, l \right]_n
\]
then Lemma 3.4.1 and Proposition 3.4.2 above shows that \( w \left[ \gamma \psi_i ; k, l \right] \in U^- U^+_{\pm} U^+_{\pm}. \)

Let \( \text{Aut}(\Gamma) \) be the set of automorphisms of the affine Dynkin diagram \( \Gamma \). Recall
\[
I_0 = \{1, \ldots, N\}, \quad \text{and let } \pi : Z \ni r \mapsto \pi_r \in I, N_1, \ldots, N_n \in \mathbb{Z}_{\geq 0}, \quad \tau_1, \ldots, \tau_n \in \text{Aut}(\Gamma)
\]
be such that:

1). \( N_i = \sum_{j=1}^l l(\omega_j) \forall i \in I_0 \) (where \( \langle \omega_i, \alpha_j \rangle = \delta_{ij} \) for all \( i, j \in I_0 \));

2). \( s_{\pi_1} \cdots s_{\pi_N}, \tau_i = \sum_{j=1}^l \omega_j \forall i \in I_0 \);

3). \( \tau_{r+N_n} = \tau_n(\pi_r) \forall r \in \mathbb{Z} \);

(These conditions imply that for all \( r < r' \in Z \) \( s_{\pi_r} s_{\pi_{r+1}} \cdots s_{\pi_{r'-1}} s_{\pi_r} \) is a reduced expression, see [IM65] and [Kac90]).

Then \( \pi \) induces a map
\[
Z \ni r \mapsto w_r \in W \text{ defined by } w_r = \begin{cases} 
s_{\pi_0} \cdots s_{\pi_{r+1}} & \text{if } r < 0, \\
1 & \text{if } r = 0, 1, \\
s_{\pi_1} \cdots s_{\pi_{r-1}} & \text{if } r > 1,
\end{cases}
\]
a bijection
\[
Z \ni r \mapsto \beta_r = w_r(\alpha_{\pi_r}) \in \Phi^+_+, \n\]
and of course a bijection \( \{\pm\} \times Z \leftrightarrow \Phi^+_+ \).

For all \( \alpha = \beta_r \in \Phi^+_+ \) as in (3.25) the root vectors \( E_{\alpha} \) can be written as:
\[
E_{\beta_r} = \begin{cases} 
T_{w_r}^{-1}(E_{\pi_r}) & \text{if } r < 0, \\
E_{\pi(0)} & \text{if } r = 0, \\
E_{\pi(1)} & \text{if } r = 1, \\
T_{w_r}(E_{\pi_r}) & \text{if } r > 1
\end{cases}
\]
and we define
\[
F_{\alpha} = \Omega(E_{\alpha}).
\]

For \( r \in \mathbb{Z} \), we define
\[
\beta_r^\pm = \begin{cases}
\pm \beta_r & \text{if } r \leq 0, \\
\mp \beta_r & \text{if } r > 0
\end{cases}
\]
then of course
\[
\{\beta_r^+ \mid r \in \mathbb{Z}\} = \{m\delta + \alpha \in \Phi \mid m \in \mathbb{Z}, \alpha \in \Delta_{0,+}\},
\]
\[
\{\beta_r^- \mid r \in \mathbb{Z}\} = \{m\delta - \alpha \in \Phi \mid m \in \mathbb{Z}, \alpha \in \Delta_{0,+}\}.
\]
The root vectors do depend on $\pi$ (for example if $a_{ij} = a_{ji} = -1$ we have $T_i(E_j) \neq T_j(E_i)$). What is independent of $\pi$ are the root vectors relative to the roots $m\delta + \alpha$:

$$E_{m\delta + \alpha} = T_{\omega_i}^{-m}(E_i), \quad E_{m\delta - \alpha} = T_{\omega_i}^{-m}T_{\omega_i}^{-1}(E_i).$$

**Lemma 3.4.4.** Let $\omega = \omega_1 + \cdots + \omega_n$ and let $m \in \mathbb{Z}_{\geq 0}, \alpha \in \Delta_{0,+}, h \in \mathbb{Z}$. Then $m\delta + \alpha \in \Delta_+$ implies

$$T_h^m(E_{m\delta + \alpha}) = \begin{cases} E_{(m-h(\omega,\alpha))\delta + \alpha} & \text{if } m-h(\omega,\alpha) \geq 0 \\ -F_{(h(\omega,\alpha)-m)\delta - \alpha} & \text{if } m-h(\omega,\alpha) < 0; \end{cases}$$

and $m\delta - \alpha \in \Delta_+$ implies

$$T_h^m(E_{m\delta - \alpha}) = \begin{cases} E_{(m+h(\omega,\alpha))\delta - \alpha} & \text{if } m+h(\omega,\alpha) > 0 \\ -K_{-1}^{(m+h(\omega,\alpha))\delta + \alpha} & \text{if } m+h(\omega,\alpha) \leq 0. \end{cases}$$

**Proof.** The function $\pi$ is defined to have the following property:

- If $m\delta + \alpha \in \Delta_+$, then there exists $r \leq 0$ such that $m\delta + \alpha = \beta_r = s_{\pi_0}s_{\pi_1} \cdots s_{\pi_{r-1}}(\alpha_{\pi_r})$ (if $r = 0$, then $\beta_0 = \alpha_{\pi_0}$);

- If $m\delta - \alpha \in \Delta_+$, then there exists $r \geq 1$ such that $m\delta - \alpha = \beta_r = s_{\pi_1} \cdots s_{\pi_{r-1}}(\alpha_{\pi_r})$ (if $r = 1$, then $\beta_1 = \alpha_{\pi_1}$).

Recall $\omega(m\delta \pm \alpha) = m\delta \pm (\alpha \cdot \langle \omega, \alpha \rangle \delta)$. By the definition of $\pi$ we have $\omega = s_{\pi_1} \cdots s_{\pi_N} \tau_n = \tau_n s_{\pi_1} \cdots s_{\pi_N} = \tau_n s_{\pi_1-N_n} \cdots s_{\pi_0}$. It follows that

$$\omega w_r = \begin{cases} w_{r+N_n} \tau_n & \text{if } r \geq 1 \text{ or } r \leq -N_n \\ w_{r+N_n} s_{\pi_{r+N_n}} \tau_n = w_{r+N_n} \tau_n s_{\pi_r} & \text{if } -N_n < r \leq 0, \end{cases}$$

and

$$\omega(\beta_r) = \begin{cases} \beta_{r+N_n} & \text{if } r \geq 1 \text{ or } r \leq -N_n, \\ -\beta_{r+N_n} & \text{if } -N_n < r \leq 0, \end{cases}$$

This means $\omega(\beta_r^\pm) = \beta_{r+N_n}^\pm$. Moreover

- If $r \geq 1$, then $T_\omega T_{w_r} = T_{w_{r+N_n}} \tau_n$.
  - If $r \leq 0$, then

$$T_\omega T_{w_r}^{-1} = \begin{cases} T_{\pi_{r+N_n}^{-1}}^{-1} \tau_n & \text{if } r \leq -N_n, \\ T_{w_{r+N_n}} T_{\pi_{r+N_n}} \tau_n & \text{if } -N_n < r \leq 0. \end{cases}$$

Thus

$$T_\omega(E_{\beta_r}) = \begin{cases} E_{\beta_{r+N_n}} & \text{if } r \geq 1 \text{ or } r \leq -N_n, \\ T_{w_{r+N_n}} T_{\pi_{r+N_n}} (E_{\pi_{r+N_n}}) = -F_{\beta_{r+N_n}} K_{\beta_{r+N_n}} & \text{if } -N_n < r \leq 0. \end{cases}$$
Now assuming the result above for $h > 0$,

$$T_{m\delta + \alpha}^{h+1} = \begin{cases} E_{m-(h+1)(\omega, \alpha)}\delta + \alpha & \text{if } m - h(\omega, \alpha) \geq 0, \\ -T_{m+1}(F_{h(\omega, \alpha)-m})\delta - \alpha & \text{if } m - h(\omega, \alpha) < 0; \end{cases}$$

$$= \begin{cases} E_{m-(h+1)(\omega, \alpha)}\delta + \alpha & \text{if } m - h(\omega, \alpha) \geq 0, \\ -\Omega T_{m}(E_{h(\omega, \alpha)-m})\delta - \alpha & \text{if } m - h(\omega, \alpha) < 0; \end{cases}$$

$$= \begin{cases} E_{m-(h+1)(\omega, \alpha)}\delta + \alpha & \text{if } m - h(\omega, \alpha) \geq 0, \\ -F_{h(\omega, \alpha)-m}\delta - \alpha & \text{if } m - h(\omega, \alpha) < 0; \end{cases}$$

$$= \begin{cases} E_{m-(h+1)(\omega, \alpha)}\delta + \alpha & \text{if } m - h(\omega, \alpha) \geq 0, \\ -F_{h(\omega, \alpha)+m}\delta - \alpha & \text{if } m - h(\omega, \alpha) < 0. \end{cases}$$

Let $m \in \mathbb{Z}$, $\alpha \in \Delta_{0,+}$ be such that $m \delta + \alpha \in \Delta$; consider the following modified root vectors:

$$X_{m\delta + \alpha} = \begin{cases} E_{m\delta + \alpha} & \text{if } m \geq 0, \\ -F_{-m\delta - \alpha}K_{-m\delta - \alpha} & \text{if } m < 0, \end{cases}$$

$$X_{m\delta - \alpha} = \begin{cases} -K_{m\delta - \alpha}^{-1}E_{m\delta - \alpha} & \text{if } m > 0, \\ F_{-m\delta + \alpha} & \text{if } m \leq 0, \end{cases}$$

$$(\Omega(X_{m\delta + \alpha}) = X_{-m\delta + \alpha}).$$

Equivalently

$$X_{\beta^+_r} = \begin{cases} E_{\beta^+_r} & \text{if } r \leq 0, \\ -F_{\beta^+_r}K_{\beta^+_r} & \text{if } r \geq 1, \end{cases} \quad X_{\beta^-_r} = \begin{cases} F_{\beta^-_r} & \text{if } r \leq 0, \\ -K_{\beta^-_r}^{-1}E_{\beta^-_r} & \text{if } r \geq 1. \end{cases}$$

It follows from the proof of Lemma 3.4.4 that:

$$\forall r \in \mathbb{Z} \ T_{\omega}(X_{\beta^+_r}) = X_{\beta^+_r+N_{\alpha}}.$$

**Corollary 3.4.5.** If $X$ is a finite subset of $\{X_{\beta} | \beta \in \Delta^+ \}$, then there exists $h \in \mathbb{Z}$ such that $T_{\omega}^{h}(X) \subseteq \{X_{\beta^+_r} | r \leq 0 \}$.

**Remark 3.4.6.** The imaginary root vectors are fixed points for the action of the $T_{\omega}$, and in particular of $T_{\omega}$.

**Theorem 3.4.7.** Given $m : \mathbb{Z} \ni r \mapsto m_r \in \mathbb{Z}_{\geq 0}$ such that $|\{r \in \mathbb{Z} | m_r \neq 0\}| < \infty$ define

$$X^{-}(m) = \prod_{r \in \mathbb{Z}} X^{-}_{\beta^+_r}, \quad X^{+}(m) = \prod_{r \in \mathbb{Z}} X^{+}_{\beta^-_r},$$

where one chooses a fixed ordering for the products.

Given $l : \Delta_{\pm}(im) \to \mathbb{Z}_{\geq 0}$ such that $|\{(r, \delta, i) \in \Delta_{\pm}(im) | l(r, \delta, i) \neq 0\}| < \infty$ define

$$E^{im}(l) = \prod_{(r, \delta, i) \in \Delta_{\pm}(im)} E^{l(r, \delta, i)}, \quad F^{im}(l) = \Omega(E^{im}(l)).$$
where \( E_{(r,s,i)} = E_{r,s}^{(i)} \). Then the set

\[
\{ X^{-}(m)F_{i}^{r}\kappa D^r\gamma^{a/2}E^{im}(l')X^{+}(m') \}, \quad r, s \in \mathbb{Z}, \quad \alpha \in \mathbb{Q}_0
\]

is a basis of \( U_q(\mathfrak{g}) \).

Proof. Since we are dealing with a finite set, up to applying a suitable power of \( T_\omega \) we can suppose \( m_r = m_r' = 0 \) \( \forall r \geq 1 \) (notice that \( Q \ni \gamma \mapsto \omega(\gamma) = \gamma - (\omega|\gamma)\delta \in Q \) is a bijection).

Then the elements \( X^{-}(m)F_{i}^{r}\kappa E^{im}(l')X^{+}(m') \) are elements of the “classical” PBW-basis, and the claim is obvious.

The fact that the products \((3.42)\) span \( U_q(\mathfrak{g}) \) is similar to the proof of [FGM98, Proposition 1, Theorem 1] (see Theorem 4.0.8 below). The Levendorskii-Soibelman formula \((3.37)\) implies that

\[
\{ X^{-}(m)F_{i}^{r}\kappa \}, \quad \{ K_\gamma E^{im}(l')X^{+}(m') \}
\]

span two subalgebras of \( U_q \). Indeed consider the left set. Then after applying the anti-automorphism \( \Omega \) to the Levendorskii-Soibelman formula we get

\[
F_{r,s}^{(i)}X_{\beta^{-}} - X_{\beta^{-}}F_{r,s}^{(i)} = \begin{cases} \sum_{(r,s,i) < \nu_1 < \ldots < \nu_s} \bar{c}_s F_{\nu_1} \cdots F_{\nu_s} & \text{if } s \leq 0, \\ K_{\beta}^{-1}(K_{\beta}^{-1}E_{\beta} + E_{\beta}F_{r,s}^{(i)}) & \text{if } s \geq 1, \end{cases}
\]

where \( \bar{c}_s \in \mathbb{Q}(q^{1/2}) \).

From [Dam00], Theorem 5.3.2 (4), one has

\[
[ E_{r,s}^{(i)}, F_{k\delta}^{\alpha} ] = \begin{cases} -x_{ijr} E_{(r-k)\delta + \alpha} K_{\alpha} & \text{if } r \geq k, \\ x_{ijr} F_{(k-r)\delta - \alpha} K_{\alpha}^{-1} & \text{if } r < k, \end{cases}
\]

for some \( x_{ijr} \in \mathbb{Q}(q^{1/2}) \). Applying \( \Omega \) to the above gives us

\[
[ F_{r,s}^{(i)}, E_{k\delta}^{\alpha} ] = \begin{cases} x_{ijr} K_{\alpha}^{-1} F_{(r-k)\delta + \alpha} & \text{if } r \geq k, \\ -x_{ijr} K_{\alpha}^{-1} E_{(k-r)\delta - \alpha} & \text{if } r < k, \end{cases}
\]

and both of these are in the span of \( \{ X^{-}(m)F_{i}^{r}\kappa \} \). Hence the left hand side \( F_{r,s}^{(i)}X_{\beta^{-}} - X_{\beta^{-}}F_{r,s}^{(i)} \) above is in the span of \( \{ X^{-}(m)F_{i}^{r}\kappa \} \).

Let \( U_q' \) be the span of

\[
\{ X^{-}(m)F_{i}^{r}\kappa E_{i}^{l'}E^{im}(l')X^{+}(m') \}
\]

then the previous paragraph implies that \( U_q' \) is stable under left product by the \( X^{-}(m)F_{i}^{r}\kappa \)'s and under right product by the \( K_\gamma E^{im}(l')X^{+}(m') \)'s.

The classical PBW-basis is a subset of

\[
\{ X^{-}(m)F_{i}^{r}\kappa F(p)K_{i}E(p')E^{im}(l')X^{+}(m') \}
\]

where

\[
F(p) = \prod_{r \in \mathbb{Z}^+} F_{\beta}^{p_r}, \quad E(p) = \prod_{r \in \mathbb{Z}^+} E_{\beta}^{p_r}
\]
with $p : \mathbb{Z}_+ \to \mathbb{Z}_{>0}$, $\# \{r \in \mathbb{Z}_+ | p_r \neq 0 \} < \infty$. This implies that in order to prove that $U_q^p = U_q$ it is enough to prove that $F(p)E(p') \in U_q'$. This is done by induction on the “height” $h$ of $F(p)E(p')$:

$$h = \sum_{r \in \mathbb{Z}_+} (p_r + p'_r) h(\beta_r), \text{ where } h(\sum_{i \in I} m_i \alpha_i) = \sum_{i \in I} m_i,$$

the cases $p \equiv 0$ or $p' \equiv 0$ being obvious. Recall from the definition of $U_q(\hat{g})$, §3.1, that $F(p)E(\beta_k) = E(\beta_k)F(p) + \sum fke$ with $k$ a monomial in the $K_i$’s ($i \in I$), $f \in U_q^-$ and $e \in U_q^+$ homogeneous, and the “height” of $fe$ less than that of $F(p)E(\beta_k)$. The scholium follows from the fact that the “height” of $F(p)E(p'')$ and that of $feE(p''')$ are strictly less than $h$, where $p''_r = p'_r - \delta_{rk}$. Hence the theorem follows. □

### 4. Imaginary Verma Modules

The algebra $\hat{g}$ has a triangular decomposition $\hat{g} = \hat{g}^- \oplus \hat{h} \oplus \hat{g}^+$, where $\hat{g}^+ = \oplus_{\alpha \in S} \hat{g}_\alpha$ and $S$ is defined in §2.2. Let $U(\hat{g}_S)$ (resp. $U(\hat{g}^-)$) denote the universal enveloping algebra of $\hat{g}_S$ (resp. $\hat{g}^-$).

Let $\lambda \in P$, where $P$ is the weight lattice of $\hat{g}$. A weight (with respect to $\hat{h}$) $U(\hat{g})$-module $V$ is called an $S$-highest weight module with highest weight $\lambda$ if there is some nonzero vector $v \in V$ such that

(i). $u^+ \cdot v = 0$ for all $u^+ \in \hat{g}_S$;
(ii). $V = U(\hat{g}) \cdot v$.

Let $\lambda \in P$. We make $\mathbb{C}$ into a 1-dimensional $U(\hat{g}_S \oplus \hat{h})$-module by picking a generating vector $v$ and setting $(x + h) \cdot v = \lambda(h)v$, for all $x \in \hat{g}_S, h \in \hat{h}$. The induced module

$$M(\lambda) = U(\hat{g}) \otimes_{U(\hat{g}_S \oplus \hat{h})} \mathbb{C}v = U(\hat{g}^-) \otimes \mathbb{C}v$$

is called the imaginary Verma module with $S$-highest weight $\lambda$. Imaginary Verma modules are in many ways similar to ordinary Verma modules except they contain both finite and infinite-dimensional weight spaces. They were studied in [Fut94], from which we summarize.

**Proposition 4.0.8 ([Fut94], Proposition 1, Theorem 1).** Let $\lambda \in P$, and let $M(\lambda)$ be the imaginary Verma module of $S$-highest weight $\lambda$. Then $M(\lambda)$ has the following properties.

(i). The module $M(\lambda)$ is a free $U(\hat{g}_S)$-module of rank 1 generated by the $S$-highest weight vector $1 \otimes 1$ of weight $\lambda$.
(ii). $M(\lambda)$ has a unique maximal submodule.
(iii). Let $V$ be a $U(\hat{g})$-module generated by some $S$-highest weight vector $v$ of weight $\lambda$. Then there exists a unique surjective homomorphism $\phi : M(\lambda) \rightarrow V$ such that $\phi(1 \otimes 1) = v$.
(iv). $\dim M(\lambda) = 1$. For any $\mu = \lambda - k\delta$, $k$ a positive integer, $0 < \dim M(\lambda)_\mu < \infty$. If $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and $\dim M(\lambda)_\mu \neq 0$, then $\dim M(\lambda)_\mu = \infty$.
(v). Let $\lambda, \mu \in \hat{h}^\vee$. Any non-zero element of $\text{Hom}_{U(\hat{g})}(M(\lambda), M(\mu))$ is injective.
(vi). The module $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$. 
4.1. The Subalgebras $U_q(-S)$ and $U_q^+(S)$ of $U_q(\hat{g})$. Let $U_q(\pm S)$ be the subalgebra of $U_q = U_q(\hat{g})$ generated by $\{X_{\beta^+} \mid r \in \mathbb{Z}\} \cup \{E_{\pm k\delta}^{(i)} \mid 1 \leq i \leq N, k > 0\}$, and let $B_q^r$ denote the subalgebra of $U_q(\hat{g})$ generated by $U_q(S) \cup U_q^0(\hat{g})$ (the superscript $r$ is used to remind us that it is generated in part by root vectors). Let $U_q^+(S)$ be the subalgebra of $U_q(\hat{g})$ generated by $\{x_{ik}^{\pm} \mid 1 \leq i \leq N, k \in \mathbb{Z}\} \cup \{h_{kl} \mid 1 \leq i \leq N, l \in \pm \mathbb{Z}_{\geq 0}\}$, and let $B_q^d$ denote the subalgebra of $U_q(\hat{g})$ generated by $U_q^+(S) \cup U_q^0(\hat{g})$. (The superscript $d$, is used to remind us that the respective subalgebras are generated in part by Drinfeld generators).

Let $\lambda \in P$. A $U_q(\hat{g})$ weight module $V_q^\lambda$ is called an $S$-highest weight module with highest weight $\lambda$ if there is a non-zero vector $v \in V_q^\lambda$ such that:

(i). $u^+ \cdot v = 0$ for all $u^+ \in U_q(S) \setminus \mathbb{C}(q^{1/2})$;

(ii). $V_q^\lambda = U_q(\hat{g}) \cdot v$.

Note that, in the absence of a general quantum PBW theorem for non-standard partitions, we cannot immediately claim that an $S$-highest weight module $V_q^\lambda$ is generated by $U_q(-S)$. This is in contrast to the classical case.

Let $\mathbb{C}(q^{1/2}) \cdot v$ be a 1-dimensional vector space over $\mathbb{C}(q^{1/2})$. Let $\lambda \in P$, and set $X_{\beta^+} \cdot v = 0$, for all $r \in \mathbb{Z}$ and $E_{k\delta}^{(i)} \cdot v = 0$ for $k < 0$ and $1 \leq i \leq N$, $K_i^{\pm \lambda(h_i)} v$ ($i \in I$) and $D^{\pm \lambda} \cdot v = q^{\pm \lambda(h_i)} v$. Define $M_q^\lambda(\lambda) = U_q(\hat{g}) \otimes_{B_q^d} \mathbb{C}(q^{1/2}) v$. Then $M_q^\lambda(\lambda)$ is an $S$-highest weight $U_q$-module called the quantum imaginary Verma module with highest weight $\lambda$. If we let $L_q^\lambda$ be the left ideal in $U_q$ generated by $X_{\beta^+}$ for all $r \in \mathbb{Z}$ and $E_{k\delta}^{(i)}$ for $k < 0$ and $1 \leq i \leq N$, $K_i^{\pm \lambda(h_i)}$ ($i \in I$) and $D^{\pm \lambda} - q^{\pm \lambda(d)}$, then $U_q/L_q^\lambda \cong M_q^\lambda(\lambda)$ which is induced by $1 \mapsto v$.

We obtain the following refinement of [FGM98, Theorem 3.4]:

**Theorem 4.1.1.** As a vector space, $M_q^\lambda(\lambda)$ has a basis consisting of the ordered monomials

$$
\{X_-(m)F^m(l)v\}.
$$

**Proof.** This module is free over this set of vectors by Theorem 3.4.7. □

**Corollary 4.1.2.** $M_q^\lambda(\lambda)$ is free as a module over $U_q(-S)$.

Recall the notation from §3.3. Let $M_q^d(\lambda) = U_q/L_q^d$ where $L_q^d$ is the left ideal generated by the Drinfeld generators $x_{ik}^+, h_i$, $i \in I_0$, $k \in \mathbb{Z}$, $l > 0$, together with $K_i^{\pm 1} - q^{\pm \lambda(h_i)}, \gamma^{\pm 1/2} - q^{\pm \lambda(c)}/2$ and $D^{\pm 1} - q^{\pm \lambda(d)}$. Let $B_q^d$ be the subalgebra of $U_q$ generated by $U_q(S)$ and $U_q^0(\hat{g})$ and let $\mathbb{C}(q^{1/2}) \lambda$ be the one dimensional $B_q^d$-module where $x_{ik}^+ 1 = 0$, $h_i 1 = 0$, $K_i^{\pm 1} = q^{\pm \lambda(h_i)} 1$, $i \in I_0$, $k \in \mathbb{Z}$, $l > 0$, $\gamma^{\pm 1/2} = q^{\pm \lambda(c)/2} 1$ and $D^{\pm 1} = q^{\pm \lambda(d)} 1$. Note that $B_q^d \subseteq B_q^r$ as $E_{\alpha_i + k\delta} = o(i)^k x_{ik}^+$ for $k \geq 0$, $F_{-\alpha_i} \cdot k\delta = -o(i)^k K_i \gamma^{k/2} x_{ik}^+$ for $k < 0$, and $E_{k\delta}^{(i)} = \gamma^{-k/2} h_{ik}$ (see (3.26) and (3.29)).

By universal mapping properties of quotients and the tensor products one has $M_q^d(\lambda) \cong U_q \otimes_{B_q^d} \mathbb{C}(q^{1/2}) \lambda$.

Since $L_q^d \subseteq L_q^r$, there is a surjective $U_q$-module homomorphism $\pi : M_q^d(\lambda) \to M_q^r(\lambda)$.

**Corollary 4.1.3.** $M_q^d(\lambda)$ is isomorphic to $M_q^r(\lambda)$ as $U_q$-modules.
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Proof. Due to the previous Corollary there exists a $U_q(-S)$-module homomorphism $\psi: M_q^d(\lambda) \to M_q^d(\lambda)$ such that $\psi \circ \pi$ is the identity on $M_q^d(\lambda)$. The image of $\psi$ is a $U_q(-S)$-submodule of $M_q^d(\lambda)$. But $x_{ik} \in U_q(-S)U_q^0$, for $k \in \mathbb{Z}$, and $h_{il} \in U_q(-S)U_q^0$ for $l \in -\mathbb{Z}_{>0}$ by (3.27), (3.28) and (3.31). Moreover $U_q(-S)$ is generated by these elements and thus $M_q^d(\lambda) \supseteq U_q(-S)v = U_q(-S)U_q^0v \supseteq U_q(-S)v = M_q^d(\lambda)$ by Corollary 3.4.3.

□

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