A Refined Enumeration of $p$-ary Labeled Trees

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Abstract. Let $T_n^{(p)}$ be the set of $p$-ary labeled trees on $\{1, 2, \ldots, n\}$. A maximal decreasing subtree of an $p$-ary labeled tree is defined by the maximal $p$-ary subtree from the root with all edges being decreasing. In this paper, we study a new refinement $T_n^{(p)}_{n,k}$ of $T_n^{(p)}$, which is the set of $p$-ary labeled trees whose maximal decreasing subtree has $k$ vertices.

1. Introduction

Let $p$ be a fixed integer greater than 1. A $p$-ary tree $T$ is a tree such that:

(1) Either $T$ is empty or has a distinguished vertex $r$ which is called the root of $T$, and
(2) $T - r$ consists of a weak ordered partition $(T_1, \ldots, T_p)$ of $p$-ary trees.

A 2-ary (resp. 3-ary) tree is called binary (resp. ternary) tree. Figure 1 exhibits all the ternary trees with 3 vertices. A full $p$-ary tree is a $p$-ary tree, where each vertex has either 0 or $p$ children. It is well known (see [Sta99, 6.2.2 Proposition]) that the number of full $p$-ary trees with $n$ internal vertices is given by the $n$th order-$p$ Fuss-Catalan number [GKP89, p.361] $C_n^{(p)} = \frac{1}{pn+1} \binom{pn+1}{n}$. Clearly a full $p$-ary tree $T$ with $m$ internal vertices corresponds to a $p$-ary tree with $m$ vertices by deleting all the leaves in $T$, so the number of $p$-ary trees with $n$ vertices is also $C_n^{(p)}$.

![Figure 1. All 12 ternary trees with 3 vertices](image)

An $p$-ary labeled tree is a $p$-ary tree whose vertices are labeled by distinct positive integers. In most cases, a $p$-ary labeled tree with $n$ vertices is identified with an $p$-ary tree on the

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Figure 2. The maximal decreasing subtree of the ternary labeled tree $T$

vertex set $[n] := \{1, 2, \ldots, n\}$. Let $T_n^{(p)}$ be the set of $p$-ary labeled trees on $[n]$. Clearly the cardinality of $T_n^{(p)}$ is given by

$$|T_n^{(p)}| = n! C_n^{(p)} = (pn)_{(n-1)},$$

where $m(k) := m(m-1) \cdots (m-k+1)$ is a falling factorial.

For a given $p$-ary labeled tree $T$, a maximal decreasing subtree of $T$ is defined by the maximal $p$-ary subtree from the root with all edges being decreasing, denoted by $MD(T)$. Figure 2 illustrates the maximal decreasing subtree of a given ternary tree $T$. Let $T_{n,k}^{(p)}$ be the set of $p$-ary labeled trees on $[n]$ with its maximal decreasing subtree having $k$ vertices.

In this paper we present a formula for $|T_{n,k}^{(p)}|$, which makes a refined enumeration of $T_n^{(p)}$, or a generalization of equation (1). Note that a similar refinement for rooted labeled trees and ordered labeled trees were done before (see [SS12a, SS12b]), but the $p$-ary case is much more complicated and has quite different features.

2. Main results

From now on we will consider labeled trees only. So we will omit the word “labeled”. Recall that $T_{n,k}^{(p)}$ is the set of $p$-ary trees on $[n]$ with its maximal decreasing ordered subtree having $k$ vertices. Let $\mathcal{Y}_{n,k}^{(p)}$ be the set of $p$-ary trees $T$ on $[n]$, where $T$ is given by attaching additional $(n-k)$ increasing leaves to a decreasing tree with $k$ vertices. Let $\mathcal{F}_{n,k}^{(p)}$ be the set of (non-ordered) forests on $[n]$ consisting of $k$ $p$-ary trees, where the $k$ roots are not ordered. In Figure 3 the first two forests are the same, but the third one is a different forest in $\mathcal{F}_{4,2}^{(2)}$.

Figure 3. Forests in $\mathcal{F}_{4,2}^{(3)}$
Define the numbers
\[
t(n, k) = \left| \mathcal{T}_{n,k}^{(p)} \right|,
\]
\[
y(n, k) = \left| \mathcal{Y}_{n,k}^{(p)} \right|,
\]
\[
f(n, k) = \left| \mathcal{F}_{n,k}^{(p)} \right|.
\]

We will show that a \( p \)-ary tree can be “decomposed” into a \( p \)-ary tree in \( \bigcup_{n,k} \mathcal{Y}_{n,k}^{(p)} \) and a forest in \( \bigcup_{n,k} \mathcal{F}_{n,k}^{(p)} \). Thus it is important to count the numbers \( y(n, k) \) and \( f(n, k) \).

**Lemma 1.** For \( n \geq 2 \), the number \( y(n, k) \) satisfies the recursion:
\[
y(n, k) = \sum_{m=0}^{p} \binom{n-1}{m} \binom{p}{m} m! ((k-1)p - n + m + 2) y(n-m-1, k-1) \quad \text{for } 1 \leq k < n
\]
with the following boundary conditions:
\[
y(n, n) = \prod_{j=0}^{n-1} (1 + (p-1)j) \quad \text{for } n \geq 1
\]
\[
y(n, k) = 0 \quad \text{for } k < \max\left(\frac{n-1}{p}, 1\right).
\]

**Proof.** Consider a tree \( Y \) in \( \mathcal{Y}_{n,k}^{(p)} \). The tree \( Y \) with \( n \) vertices consists of its maximal decreasing tree with \( k \) vertices and the number of increasing leaves is \( n-k \). Note that the vertex 1 is always contained in \( \text{MD}(Y) \).

If the vertex 1 is a leaf of \( Y \), consider the tree \( Y' \) by deleting the leaf 1 from \( Y \). The number of vertices in \( Y' \) and \( \text{MD}(Y') \) are \( n-1 \) and \( k-1 \), respectively. So the number of possible trees \( Y' \) is \( y(n-1, k-1) \). Since we cannot attach the vertex 1 to \( n-k \) increasing leaves of \( Y' \), there are \( (k-1)p - (n-2) \) ways of recovering \( Y \). Thus the number of \( Y \) with the leaf 1 is
\[
((k-1)p - n+2) \cdot y(n-1, k-1).
\]

If the vertex 1 is not a leaf of \( Y \), then the vertex 1 has increasing leaves \( \ell_1, \ldots, \ell_m \), where \( 1 \leq m \leq p \). Consider the tree \( Y'' \) obtained by deleting \( \ell_1, \ldots, \ell_m \) from \( Y \). Clearly 1 is a leaf of \( Y'' \) and the number of vertices in \( Y'' \) and \( \text{MD}(Y'') \) are \( n-m \) and \( k \), respectively. Thus by [3], the number of possible trees \( Y'' \) is \( ((k-1)p - (n-m)+2) \cdot y(n-m-1, k-1) \). To recover \( Y \) is to relabel all the vertices except 1 of \( Y'' \) with the label set \( \{2, 3, \ldots, n\} \setminus \{\ell_1, \ldots, \ell_m\} \) and to attach the leaves \( \ell_1, \ldots, \ell_m \) to the vertex 1 of \( Y'' \). Clearly \( \ell_1, \ldots, \ell_m \) is a subset of \( \{2, 3, \ldots, n\} \). It is obvious that a way of attaching \( \ell_1, \ldots, \ell_m \) to vertex 1 can be regarded as an injection from \( \ell_1, \ldots, \ell_m \) to \( [p] \). Thus the number of \( Y \) without the leaf 1 is
\[
\binom{n-1}{m} \binom{p}{m} m! ((k-1)p - (n-m) + 2) \cdot y(n-m-1, k-1).
\]
Since \( m \) may be the number from 1 to \( p \) and substituting \( m = 0 \) in (6) yields (5), we have the recursion (2).

Since \( \mathcal{Y}_{n,n}^{(p)} \) is the set of decreasing \( p \)-ary trees on \([n]\), the equation (3) holds (see [BFS92]). If the inequality \( pk - (k - 1) < n - k \) holds, \( \mathcal{Y}_{n,k}^{(p)} \) should be empty. For \( n \geq 1 \) and \( k = 0 \), \( \mathcal{Y}_{n,k}^{(p)} \) is also empty. Thus the equation (4) also holds.

The table for \( y(n,k) \) with \( p = 2 \) is shown in Table 1.

Now we calculate \( f(n,k) \) which is the number of forests on \([n]\) consisting of \( k \) \( p \)-ary trees, where the \( k \) components are not ordered. Here we use the convention that the empty product is 1.

**Lemma 2.** For \( 0 \leq k \leq n \), we have

\[
f(n,k) = \binom{n}{k} pk \prod_{i=1}^{n-k-1} (pn - i) \quad \text{if } n > k,
\]

else \( f(n,n) = 1 \).

**Proof.** Consider a forest \( F \) in \( \mathcal{F}_{n,k}^{(p)} \). The forest \( F \) consists of (non-ordered) \( p \)-ary trees \( T_1, \ldots, T_k \) with roots \( r_1, r_2, \ldots, r_k \), where \( r_1 < r_2 < \cdots < r_k \). The number of ways for choosing roots \( r_1, r_2, \cdots, r_k \) from \([n]\) is equal to \( \binom{n}{k} \). From the reverse Prüfer algorithm (RP Algorithm) in [SS07], the number of ways for adding \( n - k \) vertices successively to \( k \) roots \( r_1, r_2, \cdots, r_k \) is equal to

\[
\frac{p(n)(pn - 1)(pn - 2) \cdots (pn - n + k + 1)}{(n-k-1)!}
\]

for \( 0 < k < n \), thus the equation (7) holds. For \( 0 = k < n \), \( \mathcal{F}_{n,0}^{(p)} \) is empty, so \( f(n,0) = 0 \) included in (7). For \( 0 \leq k = n \), \( \mathcal{F}_{n,n}^{(p)} \) is the set of forests with no edges, so \( f(n,n) = 1 \). 

Since the number \( y(n,k) \) is determined by the recurrence relation (2) in Lemma 1, we can count the number \( t(n,k) \) with the following theorem.

Table 1. \( y(n,k) \) with \( p = 2 \)
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Theorem 3. For \( n \geq 1 \), we have

\[
t(n, k) = \sum_{m=k}^{n} \binom{n}{m} \frac{m-k}{n-k} (pn-pk)_{(n-m)} y(m, k) \quad \text{if } 1 \leq k < n,
\]

else \( t(n, n) = \prod_{j=0}^{n-1} (pj - j + 1) \), where \( a(l) := a(a-1) \cdots (a-\ell+1) \) is a falling factorial.

Proof. Given a \( p \)-ary tree \( T \) in \( \mathcal{T}_{n,k}^{(p)} \), let \( Y \) be the subtree of \( T \) consisting of \( \text{MD}(T) \) and its increasing leaves. If \( Y \) has \( m \) vertices, then \( Y \) is a subtree of \( T \) with \( (m-k) \) increasing leaves. Also, the induced subgraph \( Z \) of \( T \) generated by the \( (n-k) \) vertices not belonging to \( \text{MD}(T) \) is a (non-ordered) forest consisting of \( (m-k) \) \( p \)-ary trees whose roots are increasing leaves of \( Y \). Figure 4 illustrates the subgraph \( Y \) and \( Z \) of a given ternary tree \( T \).

Now let us count the number of \( p \)-ary trees \( T \in \mathcal{T}_{n,k}^{(p)} \) with \( |V(Y)| = m \) where \( V(Y) \) is the set of vertices in \( Y \). First of all, the number of ways for selecting a set \( V(Y) \subset [n] \) is equal to \( \binom{n}{m} \). By attaching \( (m-k) \) increasing leaves to a decreasing \( p \)-ary tree with \( k \) vertices, we can make a \( p \)-ary trees on \( V(Y) \). So there are exactly \( y(m, k) \) ways for making such a \( p \)-ary subtree on \( V(Y) \). Since all the roots of \( Z \) are determined by \( Y \), by the definition of \( \mathcal{F}_{n,k}^{(p)} \) and Lemma 2, the number of ways for constructing the other parts on \( V(T) \setminus V(\text{MD}(T)) \) is equal to

\[
f(n-k, m-k) / \binom{n-k}{m-k} = \frac{m-k}{n-k} (pn-pk)_{(n-m)}.
\]

Since the range of \( m \) is \( k \leq m \leq n \), the equation (8) holds.

Finally, \( \mathcal{T}_{n,k}^{(p)} \) is the set of decreasing \( p \)-ary trees on \( [n] \), so

\[
t(n, n) = y(n, n) = \prod_{j=0}^{n-1} (pj - j + 1)
\]
The sequence \( t(n, k) \) with \( p = 2 \) is listed in Table 2. Note that each row sum is equal to \( n!C_n^{(p)} \) with \( p = 2 \).

**Remark.** Due to Lemma 1 and Theorem 3 we can calculate \( t(n, k) \) for all \( n, k \). In particular we express \( t(n, k) \) as a linear combination of \( y(k, k), y(k+1, k), \ldots, y(n, k) \). However a closed form, a recurrence relation, or a (double) generating function of \( t(n, k) \) have not been found yet.

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**References**

[BFS92] François Bergeron, Philippe Flajolet, and Bruno Salvy. Varieties of increasing trees. In *CAAP ’92 (Rennes, 1992)*, volume 581 of *Lecture Notes in Comput. Sci.*, pages 24–48. Springer, Berlin, 1992.

[GKP89] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1989. A foundation for computer science.

[SS07] Seunghyun Seo and Heesung Shin. A generalized enumeration of labeled trees and reverse Prüfer algorithm. *J. Combin. Theory Ser. A*, 114(7):1357–1361, 2007.

[SS12a] Seunghyun Seo and Heesung Shin. On the enumeration of rooted trees with fixed size of maximal decreasing trees. *Discrete Math.*, 312(2):419–426, 2012.

[SS12b] Seunghyun Seo and Heesung Shin. A refinement for ordered labeled trees. *Korean J. Math.*, 20(2):255–261, 2012.
[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

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