Bottleneck Routing Games with Low Price of Anarchy

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Abstract. We study bottleneck routing games where the social cost is determined by the worst congestion on any edge in the network. In the literature, bottleneck games assume player utility costs determined by the worst congested edge in their paths. However, the Nash equilibria of such games are inefficient since the price of anarchy can be very high and proportional to the size of the network. In order to obtain smaller price of anarchy we introduce exponential bottleneck games where the utility costs of the players are exponential functions of their congestions. We find that exponential bottleneck games are very efficient and give a poly-log bound on the price of anarchy: $O(\log L \cdot \log |E|)$, where $L$ is the largest path length in the players’ strategy sets and $E$ is the set of edges in the graph. By adjusting the exponential utility costs with a logarithm we obtain games whose player costs are almost identical to those in regular bottleneck games, and at the same time have the good price of anarchy of exponential games.

1 Introduction

Motivated by the selfish behavior of entities in communication networks, we study routing games in general networks where each packet’s path is controlled independently by a selfish player. We consider non-cooperative games with $N$ players, where each player has a pure strategy profile from which it selfishly selects a single path from a source node to a destination node such that the selected path minimizes the player’s utility cost function (such games are also known as atomic or unsplittable-flow games). We focus on bottleneck games where the objective for the social outcome is to minimize $C$, the maximum congestion on any edge in the network. Typically, the congestion on an edge is a non-decreasing function on the number of paths that use the edge; here, we consider the congestion to be simply the number of paths that use the edge.

Bottleneck congestion games have been studied in the literature [1–4] where each player’s utility cost is the worst congestion on its path edges. In particular, player $i$ has utility cost function $C_i = \max_{e \in p_i} C_e$ where $p_i$ is the path of the player and $C_e$ denotes the congestion of edge $e$. In [1] the authors observe that bottleneck games are important in networks for various practical reasons. In wireless networks the maximum congested edge is related to the lifetime of the network since the nodes adjacent to high congestion edges transmit large number
of packets which results to higher energy utilization. Thus, minimizing the maximum edge congestion immediately translates to longer network lifetime. High congestion edges also result to congestion hot-spots in the network which may slow down the performance of the whole network. Hot spots may also increase the vulnerability of the network to malicious attacks which aim to increase the congestion of edges in the hope to bring down the network or degrade its performance. Thus, minimizing the maximum congested edge results to hot-spot avoidance and also to more secure networks.

Bottleneck games are also important from a theoretical point of view since the maximum edge congestion is immediately related to the optimal packet scheduling. In a seminal result, Leighton et al. [5] showed that there exist packet scheduling algorithms that can deliver the packets along their chosen paths in time very close to $C + D$, where $D$ is the maximum chosen path length. When $C \gg D$, the congestion becomes the dominant factor in the packet scheduling performance. Thus, smaller $C$ immediately implies faster packet delivery time.

A natural problem that arises concerns the effect of the players’ selfishness on the welfare of the whole network measured with the social cost $C$. We examine the consequence of the selfish behavior in pure Nash equilibria which are stable states of the game in which no player can unilaterally improve her situation. We quantify the effect of selfishness with the price of anarchy (PoA) [6, 7], which expresses how much larger is the worst social cost in a Nash equilibrium compared to the social cost in the optimal coordinated solution. The price of anarchy provides a measure for estimating how closely do Nash equilibria of bottleneck routing games approximate the optimal $C^*$ of the respective coordinated routing optimization problem.

Ideally, the price of anarchy should be small. However, the current literature results have only provided weak bounds for bottleneck games. In [1] it is shown that if the edge-congestion function is bounded by some polynomial with degree $d$ (with respect to the packets that use the edge) then $\text{PoA} = O(|E|^d)$, where $E$ is the set of edges in the graph. In [2] the authors consider the case $d = 1$ and they show that $\text{PoA} = O(L + \log |V|)$, where $L$ is the maximum path length in the players strategies and $V$ is the set of nodes. This bound is asymptotically tight since there are game instances with $\text{PoA} = \Omega(L)$. Note that $L \leq |E|$, and further $L$ may be significantly smaller than $|E|$. However, $L$ can still be proportional to the size of the graph, and thus the price of anarchy can be large.

1.1 Contributions

The lower bound in [2] suggests that in order to obtain better price of anarchy in bottleneck games (where the social cost is the bottleneck edge $C$), we need to consider alternative player utility cost functions. Towards this goal, we introduce exponential bottleneck games whose social cost is the bottleneck $C$ and the player cost functions are exponential expressions of the player congestions along their paths. These games can be easily converted to “almost” regular bottleneck games that preserve the good price of anarchy of the exponential games.
In the exponential bottleneck games the player utilities are exponential functions on the congestion of the edges along the chosen paths. In particular, the player utility cost function for player $i$ is: $\tilde{C}_i = \sum_{e \in p_i} 2^{C_e}$. Note that the new utility cost is a sum of exponential terms on the congestion of the edges in the path (instead of the max that we described earlier). Using the new utility cost functions we show that exponential games have always Nash equilibria which can be obtained by best response dynamics. Furthermore, for the bottleneck social cost $C$ we prove that the price of anarchy is poly-log:

$$\text{PoA} = O(\log L \cdot \log |E|),$$

where $L$ is the maximum path length in the players strategy set and $E$ is the set of edges in the graph. This price of anarchy bound is a significant improvement over the price of anarchy from the regular utility cost functions described earlier.

It can be shown that exponential games can be easily converted to equivalent games with player cost $C'_i$ which are closely related to the bottleneck cost $C_i$. In particular, we can obtain equivalent games which have similar stabilization properties while exactly preserving the price of anarchy by taking the monotonic cost function $C'_i = \log(\tilde{C}_i)$. It holds that $C_i \leq C'_i \leq C_i + \log n$, where $n$ is the number of nodes in the graph. Thus, in the resulting game with utility cost $C'_i$, the player cost functions are very close to $C_i$, and also the price of anarchy is the same as in Equation 1.

Exponential games are interesting variations of bottleneck games not only because they can provide good price of anarchy but also because they represent real-life problems. It has been shown that in wireless networks the power used by individual nodes to transmit messages along an edge with guaranteed rate is exponentially proportional to the flow of the edge. Thus, exponential game equilibria represent also power game equilibria in wireless networks, where small price of anarchy translates to small power utilization by the nodes. Exponential cost functions on edge congestion have been used before in a different context for online routing optimization problems [8][Chapter 13]. However, here we use the exponential functions for the first time in the context of routing games. Our technical proofs are based on the novel idea of proving the existence and exploring properties of expansion chains of players and edges in Nash equilibria. This technique differs significantly from the potential function analysis used in other literature.

1.2 Related Work

Koutsoupias and Papadimitriou [6] introduced the notion of price of anarchy in the specific parallel link networks model in which they provide the bound $\text{PoA} = 3/2$. Roughgarden and Tardos [9] provided the first result for splittable flows in general networks in which they showed that $\text{PoA} \leq 4/3$ for a player cost which reflects to the sum of congestions of the edges of a path. Pure equilibria with atomic flow have been studied in [2, 10–12] (our work fits into this category), and with splittable flow in [13, 14, 9, 15]. Mixed equilibria with atomic flow have been studied in [16–18, 6, 19, 20, 7], and with splittable flow in [21, 22].
Most of the work in the literature uses a cost metric measured as the sum of congestions of all the edges of the player’s path \([10, 14, 9, 15, 12]\). Our work differs from these approaches since we adopt the exponential metric for player cost. The vast majority of the work on routing games has been performed for parallel link networks, with only a few exceptions on general network topologies \([2, 10, 21, 13]\), which we consider here.

Our work is close to \([2]\), where the authors consider the player cost \(C_i\) and social cost \(C\). They prove that the price of stability is 1. They show that the price of anarchy is bounded by \(O(L + \log n)\), where \(L\) is the maximum allowed path length. They also prove that \(\kappa \leq P_oA \leq c(\kappa^2 + \log^2 n)\), where \(\kappa\) is the size of the largest edge-simple cycle in the graph and \(c\) is a constant. That work was extended in \([3, 4]\) to the \(C + D\) routing problem. Bottleneck congestion games have also been studied in \([1]\), where the authors consider the maximum congestion metric in general networks with splittable and atomic flow (but without considering path lengths). They prove the existence and non-uniqueness of equilibria in both the splittable and atomic flow models. They show that finding the best Nash equilibrium that minimizes the social cost is a NP-hard problem. Further, they show that the price of anarchy may be unbounded for specific edge congestion functions.

2 Definitions

2.1 Path Routings

Consider an arbitrary graph \(G = (V, E)\) with nodes \(V\) and edges \(E\). Let \(\Pi = \{\pi_1, \ldots, \pi_N\}\) be a set of packets such that each \(\pi_i\) has a source \(u_i\) and destination \(v_i\). A routing \(p = [p_1, p_2, \cdots, p_N]\) is a collection of paths, where \(p_i\) is a path for packet \(\pi_i\) from \(u_i\) to \(v_i\). We will denote by \(E(p_i)\) the set of edges in path \(p_i\).

Consider a particular routing \(p\). The edge-congestion of an edge \(e\), denoted \(C_e\), is the number of paths in \(p\) that use edge \(e\). For any set of edges \(A \subseteq E\), we will denote by \(C_A = \max_{e \in A} C_e\). For any path \(q\), the path-congestion is \(C_q = C_{E(q)}\).

For any path \(p_i \in p\), we will also use the notation \(C_i = C_{p_i}\). The network congestion is \(C = C_E\), which is the maximum edge-congestion over all edges in \(E\).

We continue with definitions of exponential functions on congestion. Consider a routing \(p\). For any edge \(e\), we will denote \(\tilde{C}_e = 2^{C_e}\). For any set of edges \(A \subseteq E\), we will denote \(C_A = \sum_{e \in A} C_e\). For any path \(q\), we will denote \(\tilde{C}_q = \tilde{C}_{E(q)}\). For any path \(p_i \in p\) we will denote \(\tilde{C}_i = \tilde{C}_{p_i}\). We denote the length (number of edges) of any path \(q\) as \(|q|\). Whenever necessary we will append \((p)\) in the above definitions to signify the dependence on routing \(p\). For example, we will write \(C(p)\) instead of \(C\).

2.2 Routing Games

A routing game in graph \(G\) is a tuple \(\mathcal{R} = (G, \mathcal{N}, \mathcal{P})\), where \(\mathcal{N}\) is the set of \(N\) players such that each corresponds to one of the packets \(\pi_i\) with source \(u_i\) and
Lemma 1. If in routing \( p \) a player \( \pi_i \) performs a greedy move, then the resulting routing \( p' \) has \( C_E(p') > C_E(p) \).
Since the result of the potential function cannot be smaller than zero, Lemma 1 implies that best response dynamics converge to Nash-routings. Thus, we have:

**Theorem 1.** Every exponential game instance $\mathcal{R} = (G, N, \mathcal{P})$ has a Nash-routing.

We would like to note that there exist instances of exponential games that have multiple Nash-routings. Next we bound the price of anarchy with respect to the worst Nash-routing.

### 4 Price of Anarchy

We bound the price of anarchy in exponential bottleneck games. Consider an exponential bottleneck routing game $\mathcal{R} = (G, N, \mathcal{P})$. Let $p = [p_1, \ldots, p_N]$ be an arbitrary Nash-routing with social cost $C$; from Theorem 1 we know that $p$ exists. Let $p^* = [p^*_1, \ldots, p^*_N]$ represent the routing with coordinated optimal social cost $C^*$. We will obtain an upper bound on the price of anarchy $\text{PoA} \leq C/C^*$. In the analysis, we will use several parameters as defined in the table below. The proof relies on the notion of self-sufficient set of players.

| Param. | Definition |
|--------|------------|
| $p, p^*$ | Nash routing and optimal routing, respectively |
| $C, C^*$ | congestion in $p$ and $p^*$, respectively |
| $L, L^*$ | maximum allowed path length in players' strategy set $\mathcal{P}$; longest path length in optimal routing $p^*$ (notice that $L^* \leq L$) |
| $\hat{C}, 2\hat{C}$ | upper bound on congestion, upper bound on player costs in $p$ |
| $l^*, l_1^*$ | $l^* = \log L^*$, $l_1^* = \log(L^* - 1)$ (logarithms are base 2) |

**Definition 1 (Self-sufficient player set).** Consider an arbitrary set of players $S$ in Nash-routing $p$. For each player $\pi_i \in S$ let $q_i$ be the routing where all players in $S$ have the same paths as in $p$ except for player $\pi_i$ whose path is now $p_i^*$ (there are no other paths in $q_i$ other than for players in $S$). We label the set of players $S$ as self-sufficient in $p$ if for each $\pi_i \in S$ it holds $pc_i(q_i) \geq pc_i(p)$. Namely, in routing $p$ player $\pi_i$ does not switch to optimal path $p_i^*$ only because of congestion caused by players in $S$.

Trivially, in a Nash-routing, the set of all players is self-sufficient. If $S$ is not self-sufficient, then $\forall i \in S, p_i^*$ are called *expansion edges* because additional players $S'$ must be present on them to guarantee the Nash-routing. We define the notion of support sets:

**Definition 2 (Support player set).** If in Nash-routing $p$ a set of players $S$ is not self-sufficient, then there is a (support) set of players $S' \neq \emptyset$, where $S \cap S' = \emptyset$, such that for each $\pi_i \in S$ it holds $pc_i(q_i) \geq pc_i(p)$, where $q_i$ is the routing where all players in $S \cup S'$ have the same paths as in $p$ except for player $\pi_i$ whose path is now $p_i^*$ (there are no other paths in $q_i$). Namely, in routing $p$ player $\pi_i$ does not switch to optimal path $p_i^*$ only because of congestion caused by players in $S \cup S'$. 
Note that there could be multiple support sets for a non self-sufficient set \( S \) in Nash-routing \( p \). If a set is not self-sufficient then there is a support set \( S' \). The set \( S \cup S' \) may not be self-sufficient either, which implies the existence of a support set \( S'' \). If \( S \cup S' \cup S'' \) is not self-sufficient then there is some new support set for it. The process repeats until we find a self-sufficient set. Every time we find a new support set, the previous set grows and we call this expansion.

### 4.1 Outline of Proof

Let \( \tilde{C} = \lceil \max_i \log_2 \tilde{C}_i \rceil \). Let ‘game cost’ \( 2^\tilde{C} \) denote the maximum cost of a player in Nash-Routing \( p \) rounded to the nearest power of 2. Note that there can be many possible Nash routings for a given game cost \( \tilde{C} \). Furthermore the bottleneck congestion (social cost) in Nash routing \( p \) is \( \tilde{C} \geq C \geq \tilde{C} - l^* \), by definition of the exponential player cost function. We will use \( \tilde{C} / C^* \) to find the upper bound on the PoA.

Let \( X \) be a (large) set of self-sufficient players. Define an expansion chain rooted at \( X_1 \) as an ordering \( X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_k \) of players in \( X \) by decreasing cost levels (where a level is a range of costs as defined below) and satisfying the following properties: 1) \( X = \bigcup_{i=1}^k X_i \) where each \( X_i \) is a set of players at the same cost level; 2) \( X_i \cap X_j = \emptyset \); 3) No prefix group of players \( \bigcup_{j=1}^t X_i \) is self-sufficient for \( 1 \leq j < k \) and at least one player from their support sets is in some \( X_t \), \( t > j \). (Note that individual \( X_i \)’s might be self-sufficient but the union, starting from \( X_1 \) is not.) For example, note that \( X_1 \)’s support set might consist of players from different levels in the chain; 4) Every player in \( X_i, i > 1 \) is in the support set of some other \( X_j, j \neq i \).

We label this as an expansion chain \( EC \) because starting with \( X_1 \), the support set of the \( \bigcup_{i} X_i \)’s is increasing by adding players of lower cost. However as we keep expanding, eventually we will arrive at a set of players at some lowest cost level \( k \), who make the entire chain seen so far self-sufficient.

For a given game cost \( 2^\tilde{C} \), we are interested in finding the minimum sized graph (number of edges) that supports both a socially optimal routing and Nash equilibrium routing with characteristics \( C^* \) and \( \tilde{C} \). Since every Nash routing will have an associated expansion chain, the equivalent goal is to find the minimum sized expansion chain. Note that expansion chains bound the number of edges in the graph. Each player in \( EC \) performs an essential function in the Nash-routing by property 4 and thus the size of \( EC \) (number of players) in some sense relates to the size of the graph \( G \). More specifically, each player in \( EC \) is in the support set for some other players and occupies the expansion edges for these players.

In our proof, we obtain a relationship between players on \( EC \) and the number of expansion edges they occupy in \( G \). We will show that any Nash-routing \( p \) with \( \tilde{C} = \Omega(\log L^*) \) is guaranteed to have a minimum sized \( EC \) that is also very large. Each \( X_i \) in this minimum \( EC \) must have a support set of players with costs close to it. The number of stages in this \( EC \) grows with \( l^* \), however the support set and expansion edges for each subsequent \( X_i \) grows exponentially. By finding the minimum sized \( EC \) for a given \( \tilde{C} \), we then find the smallest graph \( G \)
(with an exponentially large number of edges) with the given Price of Anarchy. Equivalently, for a graph of given size, we can then compute the upper bound on the PoA for any Nash routing.

We define our cost stages (cost levels) for expansion chains and player types in the following manner: let $S^{(i)}$ denote the set of players in stage $i$, $1 \leq i \leq \hat{C}$ with player costs in range $[2^{\hat{C}-i+2}, 2^{\hat{C}-i+1}]$. In stage $i$, let $A^{(i)}$ denote the set of all players occupying exactly one edge of congestion $\hat{C}-i+1$, let $B^{(i)}$ denote the set of all players whose maximum edge congestion $C'$ satisfies $\hat{C}-i \geq C' > \hat{C}-i-l^*-1$ and finally let $D^{(i)} = S^{(i)} - A^{(i)} - B^{(i)}$. $1$ is the highest stage and has at least one player of type A, B or D by definition of $\hat{C}$. Lower stages could be empty of players.

4.2 Price of Anarchy Bound for $C^* = 1$

For ease of exposition, assume $C^* = 1$, i.e every player in the coordinated socially optimal network (we will use the term network or game interchangeably with the term routing) has a unique optimal path to its destination of length at most $L^*$. The general $C^*$ case is proved later.

To find minimum sized expansion chains, we first need to determine if expansion chains of size $> 1$ exist for a given value of $\hat{C}$. Related to this, we also need to know how large is the set of these players. We first prove a sufficient condition on $\hat{C}$ for expansion chains to exist. Subsequently, we will derive the specific minimum sized expansion chain and its size.

**Lemma 2.** Any non-empty player set $X^{(i)} \subseteq \{A^{(i)} \cup B^{(i)} \cup D^{(i)}\}$ is not self-sufficient, where $1 \leq i \leq \hat{C}-l^*_1-11$.

(Please see appendix for the proofs). This leads to

**Theorem 2.** Any subset of players $S \subseteq \{S^{(1)} \cup S^{(2)} \cup \ldots \cup S^{(k)}\}$ are not self-sufficient, where $k = \hat{C}-l^*_1-11$. Equivalently there cannot exist any expansion chains consisting only of players from the first $k$ stages.

Since we are guaranteed the existence of at least one player from $S^{(1)}$ and there exists a Nash-routing, there must be a self-sufficient set of players including this player. By the above theorem, the expansion chain for this set and rooted at stage 1 cannot terminate before stage $k$. Identifying this particular minimum size expansion chain allows us to count the minimum number of edges in $G$ and hence an upper bound on the PoA.

We now want to find the minimum number of edges required to support the Nash routing with game cost $2^\hat{C}$. This corresponds to finding the smallest expansion chain rooted at stage 1. By our definition, an expansion chain consists of new players occupying the optimal path edges of players on the previous levels. It would seem that chains should consist of type B players since they occupy multiple edges and thus fewer players are required. However as the lemma below shows it is players of type A that minimize the expansion edges.
Consider an arbitrary player $\pi$ of type $B$ in $p$ occupying edges $E = \{e_1, e_2, \ldots, e_k\}$ of non-increasing congestion $c_1 \geq c_2 \geq \ldots c_k$ that are optimal edges (expansion edges) of other players, where we assume maximum congestion $c_1 \geq 2$. We want to answer the following question: Is there an alternate equilibrium/game containing player(s) with the same total equilibrium cost as $\pi$, but requiring fewer edges to support this equilibrium cost. Note that when comparing these two games, the actual routing paths (i.e., source-destinations) do not have to be the same. All we need to show is the existence of an alternate game (even with different source-destination pairs for the players) that has the same equilibrium cost.

In particular, consider an alternate game $p'$ in which $\pi$ is replaced by a set $P = \{\pi_1, \pi_2, \ldots, \pi_k\}$ of type $A$ players occupying single edges of congestion $c_1, c_2, \ldots, c_k$, where $\pi$ and the set $P$ are also in equilibrium in their respective games. The equilibrium cost of $\pi$ and set $P$ is the same ($\sum_{j=1}^{k} 2^{c_j}$) as they are occupying edges of the same congestion. Since both $\pi$ in game $p$ and the set of players $P$ are in equilibrium and occupying expansion edges of other players in their respective games, $C^* = 1$ implies they must have their own expansion edges in their respective games. Suppose we can show that the number of expansion edges required by the $k$ players in $P$ is at most those required by the single player of type $B$. Since $\pi$ is an arbitrary type $B$ player, this argument applied recursively implies that all expansion edges in the game $p$ should be occupied by type $A$ players to minimize the total number of expansion edges. Thus we will have shown that any equilibrium with cost $C$ can be supported with fewer total players if they are of type $A$ than if they are of type $B$. Let $\pi^*$ and $P^*$ denote the expansion edges of $\pi$ and the set $P$ respectively.

**Lemma 3.** $|P^*| \leq |\pi^*|$ for arbitrary players $\pi$ and set $P$ with the same equilibrium cost.

As a consequence of lemma 3, we have

**Lemma 4.** For $\hat{C} > l^* + 11$, the expansion chain rooted in stage 1 and occupying the minimum number of edges consists only of players of type $A$ (other than the root).

Next we derive the size of the smallest network required to support an equilibrium congestion of $\hat{C}$. Without loss of generality, we assume there exists at least one type $A$ player in stage 1, i.e., a single edge of congestion $\hat{C}$ and derive the minimum chain rooted at $A^{(1)}$. From lemma 4, there exists an expansion chain rooted at $A^{(1)}$ with only type $A$ players. Among all such expansion chains, the one with the minimum number of players (equivalently edges, since each type $A$ player occupies a single edge) is defined below.

**Theorem 3.** $EC_{\text{min}}$, the expansion chain with minimum number of edges that supports a self-sufficient equilibrium rooted at $A^{(1)}$ is defined by $EC_{\text{min}} : A^{(1)} \rightarrow A^{(l^*+2)} \rightarrow A^{(2l^*+3)} \rightarrow A^{(3l^*+4)} \rightarrow \ldots \rightarrow A^{(\hat{C}-1)}$. Every player in $EC_{\text{min}}$ has an optimal path whose length is the maximum allowed $L^*$. The depth of chain $EC_{\text{min}}$ is $O(\hat{C}/l^*)$. 
Theorem 3 is also the minimum sized chain when the root players are from $B^{(1)}$ or $D^{(1)}$ although the number of edges required in the supporting graph is slightly different as we see later. In these cases, all stages (other than the root) in the minimum expansion chain consist of type $A$ players by lemma 4 and the proof of Theorem 3 is immediately applicable in choosing the specific indices of the expansion stages required to support the equilibrium. As shown later, however, the PoA is maximized when the chain is rooted at $A^{(1)}$.

Theorem 4. When $C^* = 1$, the upper bound $\kappa$ on the Price of Anarchy PoA of Nash-routing $p$ is given by the minimum of 1) $\kappa = O(\log L^*)$ or 2) $\kappa(\log(\kappa L^*)) \leq \log L^* \cdot \log |E|$.

Can we get a larger upper bound on the PoA if the expansion chain is rooted at $B^{(1)}$ or $D^{(1)}$ instead of $A^{(1)}$? To examine this, let $C - q$ be the largest congestion in $p$, $q > 0$. We need $2^q$ such edges in order to satisfy the maximum player cost of $2C$. All these edges can be used as expansion edges for other players. From the analysis in Theorem 4, we note that expansion between stages occurs at a factorial rate. Thus using these $2^q$ edges as high up in the chain as possible (thereby reducing the need for new expansion edges) will minimize the expansion rate. The best choice for $q$ then is $l^*$. In this case, we have a single player $\pi_m$ in equilibrium in $p$, occupying $L^*$ edges of congestion $C - l^*$. These $L^*$ edges are also the optimal edges of $\pi_m$, i.e., its equilibrium and optimal paths are identical. Hence the first stage of expansion in this chain is for the $L^*(C - l^* - 1)$ players on the $L^*$ edges of $\pi_m$. From this point on the minimum sized chain for this graph is identical to the minimum sized chain $EC_{min}$ defined above. The total number of edges in this chain can be computed in a manner similar to above. While the number of edges is smaller than $EC_{min}$, it can be shown that the PoA is also smaller $C - l^*$. Hence the upper bound on the PoA is obtained using an expansion chain rooted at $A^{(1)}$.

4.3 Price of Anarchy Bound for $C^* > 1$

So far we have assumed the optimal bottleneck congestion $C^* = 1$ in our derivations. We now show that increasing $C^*$ decreases the PoA and hence the previous derivation is the upper bound. We first evaluate the impact of $C^* = M > 1$ on expansion chains. Having $C^* > 1$ implies that more players can share expansion edges and thus the rate of expansion as well as the depth of an expansion chain (if it exists) should decrease. We first show that expansion chains exist even for arbitrary $C^* = M$.

Lemma 5. Any subset of players $S \subseteq \{S^{(1)} \cup S^{(2)} \cup \ldots \cup S^{(k)}\}$ are not self-sufficient, where $k : C - k > 8M + l_1^* + 2$. Equivalently there cannot exist any expansion chains consisting only of players from the first $k$ stages.

Similarly Lemmas 3 and 4 can be suitably modified and the minimum sized chain in this case has the same structure as defined in Theorem 3. Analogous to
the $C^* = 1$ case, the maximum $\text{PoA}$ occurs when $EC_{\text{min}}$ is rooted at $A^{(1)}$. We calculate this $\text{PoA}$ with $C^* = M$, below.

**Theorem 5.** When $C^* = M$, the upper bound $\kappa$ on the Price of Anarchy $\text{PoA}$ of game $p$ is given by the minimum of 1) $\kappa = O(\frac{\log L^*}{M})$ or 2) $\kappa(\log(L^*\kappa)) \leq \frac{L^* \log |E|}{M}$

5 Conclusions

We show by carefully selecting appropriate player cost functions that the price of anarchy of bottleneck routing games is poly-log with respect to the size of the game parameters: $O(\log L \cdot \log |E|)$. A natural question that arises is what is the impact of polynomial cost functions to the price of anarchy. Polynomial cost functions with low degree give high price of anarchy. Consider the game instance in the figure where the player cost is $pc_i = \sum_{e \in p_i} C_e$ which is a linear function on the congestion of the edges on the player’s path.

In this game there $k$ players $\pi_1, \ldots, \pi_k$ where all the players have source $u$ and destination $v$ which are connected by edge $e$. The graph consists of $k-1$ edge-disjoint paths from $u$ to $v$ each of length $k$. There is a Nash equilibrium, depicted in the left figure where every player chooses to use a path of length 1 on edge $e$. This is an equilibrium because the cost of each player is $k$, while the cost of every alternative path is also $k$. Since the congestion of edge $e$ is $k$ the social cost is $k$. The optimal solution for the same routing problem is depicted in the right of the figure. where every player uses an edge-disjoint path and thus the maximum congestion on any edge is 1. Therefore, the price of anarchy is at least $k$. Since we can choose $k = \Theta(\sqrt{n})$, where $n$ is the number of nodes in the graph, the price of anarchy is $\Omega(\sqrt{n})$.

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