ALMOST NONNEGATIVE CURVATURE ON SOME FAKE $\mathbb{R}P^6$’S AND $\mathbb{R}P^{14}$’S

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Abstract. We apply the lifting theorem of Searle and the second author to put metrics of almost nonnegative curvature on the fake $\mathbb{R}P^6$’s of Hirsch and Milnor and on the analogous fake $\mathbb{R}P^{14}$’s.

One of the great unsolved problems of Riemannian geometry is to determine the structure of collapse with a lower curvature bound. An apparently simpler, but still intractable problem, is to determine which closed manifolds collapse to a point with a lower curvature bound. Such manifolds are called almost nonnegatively curved. Here we construct almost nonnegative curvature on some fake $\mathbb{R}P^6$’s and $\mathbb{R}P^{14}$’s.

Theorem A. The Hirsch-Milnor fake $\mathbb{R}P^6$’s and the analogous fake $\mathbb{R}P^{14}$’s admit Riemannian metrics that simultaneously have almost nonnegative sectional curvature and positive Ricci curvature.

Remark. By considering cohomogeneity one actions on Brieskorn varieties, Schwachh"{o}fer and Tuschmann observed in [15] that in each odd dimension of the form, $4k + 1$, there are at least $4^k$ oriented diffeomorphism types of homotopy $\mathbb{R}P^{4k+1}$’s that admit metrics that simultaneously have positive Ricci curvature and almost nonnegative sectional curvature.

The Hirsch-Milnor fake $\mathbb{R}P^6$’s are quotients of free involutions on the images of embeddings $\iota$ of the standard 6–sphere, $S^6$, into some of the Milnor exotic 7–spheres, $\Sigma^7_k$ ([12], [14]). Our proof begins with the observation that the $SO(3)$–actions that Davis constructed on the $\Sigma^7_k$’s in [5] leave these Hirsch-Milnor $S^6$’s invariant and commute with the Hirsch-Milnor free involution. Next we compare the Hirsch-Milnor/Davis ($SO(3) \times \mathbb{Z}_2$)–action on $\iota(S^6) \subset \Sigma^7_k$ with a very similar linear action of ($SO(3) \times \mathbb{Z}_2$) on $S^6 \subset \mathbb{R}^7$ and apply the following lifting result of Searle and the second author.

Theorem B. (See Proposition 8.1 and Theorems B and C in [17]) Let $(M_e, G)$ and $(M_s, G)$ be smooth, compact, $n$–dimensional $G$–manifolds with $G$ a compact Lie group. Suppose that the orbit spaces $M_e/G$ and $M_s/G$ are equivalent, and $M_s/G$ has almost nonnegative curvature. Then $M_e$ admits a $G$–invariant family of metrics that has almost nonnegative sectional curvature. Moreover, if the principal orbits of $(M_e, G)$ have finite fundamental group and the quotient of the principal orbits of $M_s$ has Ricci curvature $\geq 1$, then every metric in the almost nonnegatively curved family on $M_e$ can be chosen to also have positive Ricci curvature.

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We emphasize that to apply Theorem [3] $M_s/G$ need not be a Riemannian manifold, but since $M_s$ is compact, $M_s/G$ is an Alexandrov space with curvature bounded from below. The meaning of almost nonnegative curvature for Alexandrov spaces is as follows.

**Definition.** We say that a sequence of Alexandrov spaces $\{(X, \text{dist}_\alpha)\}_\alpha$ is almost nonnegatively curved if and only if there is a $D > 0$ so that

$$\sec (X, g_\alpha) \geq -\frac{1}{\alpha} \text{ and } \text{Diam} (X, g_\alpha) \leq D,$$

or equivalently, after a rescaling, $X$ collapses to a point with a uniform lower curvature bound.

The following is the precise notion of equivalence of orbit spaces required by the hypotheses of Theorem [3]

**Definition.** Suppose $G$ acts on $M_e$ and on $M_s$. We say that the orbit spaces $M_e/G$ and $M_s/G$ are equivalent if and only if there is a strata-preserving homeomorphism $\Phi : M_e/G \rightarrow M_s/G$ whose restriction to each stratum is a diffeomorphism with the following property:

Let $\pi_s : M_s \rightarrow M_s/G$ and $\pi_e : M_e \rightarrow M_e/G$ be the quotient maps. If $S \subset M_e$ is a stratum, then for any $x_e \in S$ and any $x_s \in \pi_s^{-1} (\Phi (\pi_e (x_e)))$, the action of $G_{x_e}$ on $\nu (S)_{x_e}$ is linearly equivalent to the action of $G_{x_s}$ on $\nu (S)_{x_s}$. Here $G_x$ is the isotropy subgroup at $x$ and $\nu (S)_x$ is the normal space to $S$ at $x$.

To construct the metrics on the fake $\mathbb{RP}^6$s of Theorem [1] we apply Theorem [3] with $G = (SO(3) \times \mathbb{Z}_2)$. $M_e$ will be the Hirsch-Milnor embedded image of $\mathbb{S}^6$ in $\Sigma_k^7$, and $M_s$ will be $\mathbb{S}^6$ with the following $(SO(3) \times \mathbb{Z}_2)$–action: View $\mathbb{S}^6$ as the unit sphere in $\mathbb{H} \oplus \text{Im} \mathbb{H}$, where $\mathbb{H}$ stands for the quaternions, and let $SO(3) \times \mathbb{Z}_2$ act on $\mathbb{S}^6 \subset \mathbb{H} \oplus \text{Im} \mathbb{H}$ via

$$SO(3) \times \mathbb{Z}_2 \times \mathbb{S}^6 \rightarrow \mathbb{S}^6$$

$$(g, \pm, (a, c)) \mapsto \pm (g(a), g(c)).$$

(0.0.1)

Here the $SO(3)$–action on the $\mathbb{H}$–factor is the direct sum of the standard action of $SO(3)$ on $\text{Im} \mathbb{H}$ with the trivial action on $\text{Re} (\mathbb{H})$.

Since quotient maps of isometric group actions preserve lower curvature bounds, $\mathbb{S}^6/(SO(3) \times \mathbb{Z}_2)$ has curvature $\geq 1$ ([14]). Thus to construct the metrics on the fake $\mathbb{RP}^6$s of Theorem [1] it suffices to combine Theorem [3] with the following result.

**Lemma C.** The orbit space of the Hirsch-Milnor/Davis action of $SO(3) \times \mathbb{Z}_2$ on $\iota (\mathbb{S}^6) \subset \Sigma_k^7$ is equivalent to the orbit space of the linear action (0.0.1) on $\mathbb{S}^6$.

Our metrics on fake $\mathbb{RP}^{14}$s are octonionic analogs of our metrics on fake $\mathbb{RP}^6$s. The analogy begins with Shimada’s observation that Milnor’s proof of the total spaces of certain $\mathbb{S}^3$–bundles over $\mathbb{S}^4$ being exotic spheres also applies to certain $\mathbb{S}^7$–bundles over $\mathbb{S}^8$ ([18]). Davis’s construction of the $SO(3)$–actions on $\Sigma_k^7$s is based on the fact that $SO(3)$ is the group of automorphisms of $\mathbb{H}$. Exploiting the fact that $G_2$ is the group automorphisms of the octonions, $\mathcal{O}$, Davis constructs analogous $G_2$ actions on Shimada’s exotic $\Sigma_k^{15}$s. By applying a result of Brumfiel ([3]), we will see that the Hirsch and Milnor construction of fake $\mathbb{RP}^6$s as quotients of $\iota (\mathbb{S}^6) \subset \Sigma_k^7$ also works to construct fake $\mathbb{RP}^{14}$s as quotients of $\iota (\mathbb{S}^{14}) \subset \Sigma_k^{15}$. Thus to construct the fake $\mathbb{RP}^{14}$s of Theorem [1] it suffices to show the following.
Lemma D. The orbit space of the Hirsch-Milnor/Davis action of $G_2 \times \mathbb{Z}_2$ on $\iota(S^{14}) \subset \Sigma^k_3$ is equivalent to the orbit space of the following linear action of $G_2 \times \mathbb{Z}_2$ on $S^{14} \subset \mathcal{O} \oplus \text{Im} \mathcal{O}$,

$$G_2 \times \mathbb{Z}_2 \times S^{14} \rightarrow S^{14} \quad (g, \pm, (a, c)) \mapsto \pm (g(a), g(c)). \quad (0.0.2)$$

In Section 1, we review the construction of the Hirsch-Milnor and Davis actions and explain why the Hirsch-Milnor construction works in the Octonionic case. In Section 2, we prove Lemmas C and D and hence Theorem A, and in Section 3, we make some concluding remarks. We refer the reader to page 185 of [2] for a description of how $G_2$ acts as automorphisms of the Octonions.

Remark. Explicit formulas for exotic involutions on $S^6$ and $S^{14}$ are given in (II), where it is shown, on pages 13–17, that the corresponding fake $\mathbb{R}P^6$ is diffeomorphic to the Hirsch–Milnor $\mathbb{R}P^6$ that corresponds to $\Sigma^3_3$.

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1. How to Construct Exotic Real Projective Spaces

In this section, we review Milnor spheres, the Hirsch-Milnor construction, and the Davis actions. We then explain how the Hirsch-Milnor argument gives fake $\mathbb{R}P^{14}$s.

To construct the Milnor spheres, we write $\Lambda$ for $\mathbb{H}$ or $\mathcal{O}$ and $b$ for the real dimension of $\Lambda$. To get a $S^{b-1}$–bundle over $S^b$ with structure group $SO(b)$, $(E_{h,j}, p_{h,j})$, we glue two copies of $\Lambda \times S^{b-1}$ together via

$$\Phi_{h,j} : \Lambda \setminus \{0\} \times S^{b-1} \rightarrow \Lambda \setminus \{0\} \times S^{b-1} \quad \Phi_{h,j} : (u, q) \mapsto \left( \frac{u}{|u|^2}, \left( \frac{u}{|u|} \right)^h q \left( \frac{u}{|u|} \right)^j \right). \quad (1.0.3)$$

To define the projection $p_{h,j} : E_{h,j} \rightarrow S^b$, we think of $S^b$ as obtained by gluing together two copies of $\Lambda$ along $\Lambda \setminus \{0\}$ via $u \mapsto \frac{u}{|u|^2}$. $p_{h,j}$ is then defined to be the projection to either copy of $\Lambda$.

When $h + j = \pm 1$, the smooth function

$$f : (u, q) \mapsto \frac{\text{Re}(q)}{\sqrt{1 + |u|^2}} = \frac{\text{Re}(vr^{-1})}{\sqrt{1 + |v|^2}}$$

is regular except at $(u, q) = (0, \pm 1)$. Hence, $E_{h,j}$ is homeomorphic to $S^{2b-1}$ if $h + j = \pm 1$, and a Mayer-Vietoris argument shows that $E_{h,j}$ is not homeomorphic to $S^{2b-1}$ if $h + j \neq \pm 1$. Since $f(0, \pm 1) = \pm 1$, it also follows that $f^{-1}(0)$ is diffeomorphic to $S^{2b-2}$.

From now on we assume that

$$h + j = 1, \quad (1.0.4)$$
and we set
\[ k = h - j. \] (1.0.5)
So
\[ k = 2h - 1. \]
For simplicity, we will write \( \Sigma_k^{2b-1} \) for \( E_{h,j} \) and \( \Phi_k \) for \( \Phi_{h,j} \), and set
\[ S_k^{2b-2} \equiv f^{-1}(0). \]

The Hirsch-Milnor construction ([12]) begins with the observation that the involution
\[ T : \Lambda \times S^{b-1} \longrightarrow \Lambda \times S^{b-1} \]

\[ T : (u, q) \mapsto (u, -q) \]
induces a well-defined free involution of \( \Sigma_k^{2b-1} \). Moreover, \( T \) leaves \( S_k^{2b-2} \) invariant. Lemma 3 of [12] says that the quotient of any fixed point free involution on \( S^n \) is homotopy equivalent to \( \mathbb{R}P^n \). In particular, all of our spaces
\[ P_k^{2b-2} \equiv S_k^{2b-2}/T \]
are homotopy equivalent to \( \mathbb{R}P^{2b-2} \). Hirsch and Milnor then show that when \( b = 4 \), \( P_k^6 \) is not diffeomorphic to \( \mathbb{R}P^6 \), provided \( \Sigma_k^7 \) is an odd element of \( \Theta_7 \), the group of oriented diffeomorphism classes of differential structures on \( S^7 \). According to pages 102 and 103 of [6], there are 16 oriented diffeomorphism classes among the \( \Sigma_k^8 \), and among these, 8 are odd elements of \( \Theta_7 \).

To understand how this works octonionically, we let \( \Theta_{15} \) be the group of oriented diffeomorphism classes of differential structures on \( S^{15} \), and we let \( bP_{16} \) be the set of the elements of \( \Theta_{15} \) that bound parallelizable manifolds. According to [13], \( bP_{16} \) is a cyclic subgroup of \( \Theta_{15} \) of order 8, 128 and index 2, and according to Theorem 1.3 in [3], \( \Theta_{15} \) is not cyclic. Thus
\[ \Theta_{15} \cong bP_{16} \oplus \mathbb{Z}_2 \cong \mathbb{Z}_{8,128} \oplus \mathbb{Z}_2. \]

According to Wall ([19]), a homotopy sphere bounds a parallelizable manifold if and only if it bounds a 7–connected manifold. In particular, each of the \( \Sigma_k^{15} \) is in \( bP_{16} \).

According to pages 101—107 of [6], \( \Sigma_k^{15} \) represents an odd element of \( bP_{16} \) if and only if \( \frac{h(h-1)}{2} \) is odd, that is, \( h \) is congruent to 2 or 3 mod 4.

The Hirsch-Milnor argument, combined with the fact that \( \Theta_{15} \cong bP_{16} \oplus \mathbb{Z}_2 \), implies \( P_k^{14} \) is not diffeomorphic to \( \mathbb{R}P^{14} \), if \( \Sigma_k^{15} \) is an odd element of \( bP_{16} \).

We let
\[ G^\Lambda \equiv \begin{cases} SO(3) & \text{when } \Lambda = \mathbb{H} \\ G_2 & \text{when } \Lambda = \mathbb{O}. \end{cases} \]

Davis observed that since \( G^\Lambda \) is the automorphism group of \( \Lambda \), the diagonal action
\[ G^\Lambda \times \Lambda \times S^{b-1} \longrightarrow \Lambda \times S^{b-1} \]
\[ g(u, v) = (g(u), g(v)) \] (1.0.6)
induces a well-defined \( G^\Lambda \)–action on \( \Sigma_k^{2b-1} \).

Next we observe that the Davis action leaves \( S_k^{2b-2} = f^{-1}(0) \) invariant and commutes with \( T \), giving us the \( SO(3) \times \mathbb{Z}_2 \) actions of Lemma [C] and the \( G_2 \times \mathbb{Z}_2 \) actions of Lemma [D].
2. Identifying the Orbit Spaces

In this section, we prove Lemmas \[ \text{C and D} \] simultaneously and hence Theorem \[ \text{A} \]. In Lemma \[ \text{2.1} \] (below), we identify the quotient map for the standard \[ G^A \]-action of \[ S^{2b-2} \]. In Lemma \[ \text{2.2} \] (below), we identify the quotient map for the Davis action on \[ S_k^{2b-2} \]. Then in Key Lemma \[ \text{2.3} \], we show that the two \[ G^A \] quotients are the same. It is then a simple matter to identify the two \[ G^A \times \mathbb{Z}_2 \] quotient spaces with each other.

**Lemma 2.1.** Let \[ S^{2b-2} \] be the unit sphere in \( \Lambda \oplus \text{Im} (\Lambda) \), and let \( \langle \cdot, \cdot \rangle \) be the real dot product. The map

\[
Q_s : S^{2b-2} \to Q_s(S^{2b-2}) \subset \mathbb{R}^3
\]

\[
\begin{pmatrix} a \\ c \end{pmatrix} \mapsto (|a|, \text{Re } a, \langle \text{Im } a, \text{Im } c \rangle)
\]

has the following properties.

1. The fibers of \( Q_s \) coincide with the orbits of the \( G^A \) action

\[
G^A \times S^{2b-2} \to S^{2b-2}
\]

\[
(g, (a, c)) \mapsto (g(a), g(c)).
\]

2. The image of \( Q_s \) is \( Q_s(S^{2b-2}) = \)

\[
\left\{ (x, y, z) \mid x \in [0, 1] \ y \in [-x, x], \ z \in \left[ -\sqrt{(x^2 - y^2)(1-x^2)}, \sqrt{(x^2 - y^2)(1-x^2)} \right] \right\}.
\]

3. The principal orbits are mapped to the interior of \( Q_s(S^{2b-2}) \). The fixed points are mapped to \((1,1,0),(1,-1,0)\), and the other orbits are mapped to \( \partial Q_s(S^{2b-2}) \setminus \{(1,1,0),(1,-1,0)\} \).

**Proof.** Part 2 follows from the observations that

\[
|a| \in [0, 1],
\]

\[
\text{Re } a \in [-|a|, |a|],
\]

\[
\langle \text{Im } a, \text{Im } c \rangle \in [-|\text{Im } a| |\text{Im } c|, |\text{Im } a| |\text{Im } c|],
\]

and

\[
|\text{Im } a| |\text{Im } c| \in \left[ 0, \sqrt{(|a|^2 - \text{Re } a^2)(1-|a|^2)} \right].
\]

Since the three quantities \(|a|, \text{Re } a, \langle \text{Im } a, \text{Im } c \rangle\) are invariant under \( G^A \), each orbit of \( G^A \) is contained in a fiber of \( Q_s \).

Conversely, if \((a_1, c_1)\) and \((a_2, c_2)\) satisfy \( Q_s(a_1, c_1) = Q_s(a_2, c_2) \), then

\[
|a_1| = |a_2|
\]

\[
\text{Re } (a_1) = \text{Re } (a_2), \text{ and}
\]

\[
\langle \text{Im } a_1, \text{Im } c_1 \rangle = \langle \text{Im } a_2, \text{Im } c_2 \rangle.
\]

Together with \( \text{Re } (c_i) = 0 \) and \(|a_i|^2 + |c_i|^2 = 1\), this gives

\[
|\text{Im } (a_1)| = |\text{Im } (a_2)|
\]

\[
|\text{Im } (c_1)| = |\text{Im } (c_2)|.
\]
Since we also have \( \langle \text{Im} \ a_1, \text{Im} \ c_1 \rangle = \langle \text{Im} \ a_2, \text{Im} \ c_2 \rangle \), it follows that an element of \( G^\Lambda \) carries \( (a_1, c_1) \) to \( (a_2, c_2) \). This completes the proof of Part 1.

To prove Part 3, we first note that the orbit of \( (a, c) \) is not principal if and only if
\[
|\langle \text{Im} \ a, \text{Im} \ c \rangle| = |\text{Im} \ (a)| |\text{Im} \ (c)|,
\]
and this is equivalent to \( Q_s \ (a, c) \in \partial Q_s \ (a, c) \). So the principal orbits are mapped onto the interior of \( Q_s \ \left(S^{2b-2}\right) \).

On the other hand, the fixed points are \((\pm 1, 0)\) and \( Q_s \ (\pm 1, 0) = (1, \pm 1, 0) \) as claimed.

Before proceeding, recall that we view
\[
\Sigma^{2b-1}_k = \left( \Lambda \times S^{b-1} \right) \cup_{\Phi_k} \left( \Lambda \times S^{b-1} \right),
\]
where \( \Phi_k \) is determined by Equations \( (1.0.3), (1.0.4), \) and \( (1.0.5) \). Combining this with the definition of \( S^{2b-2}_k \), we have that
\[
S^{2b-2}_k = U_1 \cup_{\Phi_k} U_2,
\]
where
\[
U_1 = \{(u, q) \in \Lambda \times S^{b-1} \mid \text{Re} \ (q) = 0\} \quad \text{and} \quad
U_2 = \{(v, r) \in \Lambda \times S^{b-1} \mid \text{Re} \ (vr^{-1}) = \text{Re} \ (\bar{v} \bar{r}) = 0\}.
\]

The quotient map of the \( G^\Lambda \)-action on \( S^{2b-2}_k \) has the following description.

**Lemma 2.2.** Let \( \phi : \mathbb{R}^n \longrightarrow \mathbb{R} \) be given by, \( \phi(v) = \frac{1}{\sqrt{1+|v|^2}}. \)

The map
\[
Q_k : S^{2b-2}_k \longrightarrow Q_k \ (S^{2b-2}_k) \subset \mathbb{R}^3
\]
\[
Q_k|_{U_1} (u, q) = \phi(u) \ (|u|, \ \text{Re} \ (vq), \ \phi(u) \ (\text{Im} \ (vq)))
\]
\[
Q_k|_{U_2} (v, r) = \phi(v) \ (|r|, \ \text{Re} \ (\bar{r}), \ \phi(v) \ (\text{Im} \ (r), \ \text{Im} \ (\bar{r})))
\]

is well-defined and has fibers that coincide with the orbits of \( G^\Lambda \).

**Proof.** To see that \( Q_k \) is well-defined, we will show
\[
Q_k|_{U_1 \setminus \{0 \times S^{b-1}\}} = Q_k|_{U_2 \setminus \{0 \times S^{b-1}\}} \circ \Phi_k|_{U_1 \setminus \{0 \times S^{b-1}\}}. \quad (2.2.1)
\]
Since
\[
\Phi_k(u, q) = \left(\frac{u}{|u|^2}, \left(\frac{u}{|u|}\right)^h q \left(\frac{u}{|u|}\right)^{-(h-1)}\right),
\]
where \( k = 2h - 1 \), the left hand side of Equation \((2.2.1)\) is
\[
Q_k|_{U_2 \setminus \{0 \times S^{b-1}\}} \circ \Phi_k|_{U_1 \setminus \{0 \times S^{b-1}\}} (u, q) = Q_k\left(\frac{u}{|u|^2}, \frac{u^hq^{-(h-1)}}{|u|}\right)
\]
\[
= \phi\left(\frac{u}{|u|^2}\right) \left(\left|\frac{u^hq^{-(h-1)}}{|u|}\right|, \ \text{Re} \ \frac{u^hq^{-(h-1)}}{|u|}, \ \phi\left(\frac{u}{|u|^2}\right) \left(\text{Im} \ \frac{u^hq^{-(h-1)}}{|u|}, \ \text{Im} \ \frac{\bar{u}^h q^{-(h-1)}}{|u|}\right)\right). \quad (2.2.2)
\]
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To see that this is equal to $Q_k|_{U_1 \setminus \{0 \times S^{b-1}\}} (u, q)$, we will simplify each coordinate separately. Before doing so we point out that

$$
\frac{1}{|u|} \phi \left( \frac{u}{|u|^2} \right) = \frac{1}{|u|} \frac{1}{\sqrt{1 + \frac{1}{|u|^2}}} = \frac{1}{\sqrt{|u|^2 + 1}} = \phi(u).
$$

(2.2.3)

So the first coordinate of the right hand side of Equation (2.2.2) is

$$
\phi \left( \frac{u}{|u|^2} \right) \frac{|u^h q u^{-(h-1)}|}{|u|} = \phi \left( \frac{u}{|u|^2} \right)
$$

and the second coordinate of the right hand side of Equation (2.2.2) is

$$
\phi \left( \frac{u}{|u|^2} \right) \text{Re} \frac{u^h q u^{-(h-1)}}{|u|} = \phi \left( \frac{u}{|u|^2} \right) \text{Re} \left( \frac{uq}{|u|} \right)
$$

$$
= \frac{1}{|u|} \phi \left( \frac{u}{|u|^2} \right) \text{Re} (uq)
$$

$$
= \phi(u) \text{Re} (uq), \text{ by Equation (2.2.3)}.
$$

Finally, we have that the third coordinate of the right hand side of Equation (2.2.2) is

$$
\phi \left( \frac{u}{|u|^2} \right)^2 \langle \text{Im} \frac{u^h q u^{-(h-1)}}{|u|}, \text{Im} \frac{\bar{u}^h q u^{-(h-1)}}{|u|^2} \rangle
$$

$$
= \phi \left( \frac{u}{|u|^2} \right)^2 \langle \text{Im} \frac{u^h q u^{-(h-1)}}{|u|}, \text{Im} \frac{u^{h-1} q u^{-(h-1)}}{|u|} \rangle
$$

$$
= \phi \left( \frac{u}{|u|^2} \right)^2 \frac{1}{|u|^2} \langle \text{Im} u^{h-1} u q u^{-(h-1)}, \text{Im} u^{h-1} (q) u^{-(h-1)} \rangle
$$

$$
= \phi(u)^2 \langle \text{Im} u q, \text{Im} q \rangle, \text{ by Equation (2.2.3)}.
$$

Combining the previous three displays with Equation (2.2.2) and the definition of $Q_k|_{U_1}$, we see that $Q_k : S^{2b-2}_k \rightarrow Q_k \left( S^{2b-2}_k \right) \subseteq \mathbb{R}^3$ is well-defined.

To see that $Q_k|_{U_1}$ is constant on each orbit of $G^\Lambda$, we use the fact that $G^\Lambda$ acts by isometries and commutes with conjugation to get

$$
\text{Re} g(u) g(q) = \langle g(u), g(q) \rangle
$$

$$
= \langle g(u), \bar{g}(q) \rangle
$$

$$
= \langle u, \bar{q} \rangle
$$

$$
= \text{Re} (u q).
$$
We also have
\[
\langle \text{Im} (g(u)) g(q), \text{Im} g(q) \rangle = \langle \text{Re} (g(u)) \text{Im} g(q) + \text{Re} (g(q)) \text{Im} g(u) + \text{Im} g(u) \text{Im} g(q), \text{Im} g(q) \rangle
\]
\[
= \langle \text{Re} (u) \text{Im} g(q) + \text{Re} (q) \text{Im} g(u), \text{Im} g(q) \rangle
\]
\[
= \langle g(\text{Re} (u) \text{Im} (q) + \text{Re} (q) \text{Im} (u)), g(\text{Im} (q)) \rangle
\]
\[
= \langle \text{Re} (u) \text{Im} (q) + \text{Re} (q) \text{Im} (u) + \text{Im} uq \text{Im} q, \text{Im} (q) \rangle
\]
\[
= \langle \text{Im} (uq), \text{Im} q \rangle.
\]
Since \(|g(u)| = |u|\) and \(\phi (gu) = \phi (u)\), it follows that
\[
Q_k|_{V_1} \begin{pmatrix} g(u) \\ g(q) \end{pmatrix} = Q_k|_{V_1} \begin{pmatrix} u \\ q \end{pmatrix}.
\]
Combining this with
\[
Q_k|_{V_2} g \begin{pmatrix} 0 \\ r \end{pmatrix} = (1, \text{Re} (r), 0)
\]
\[
= Q_k|_{V_2} \begin{pmatrix} 0 \\ r \end{pmatrix},
\]
it follows that \(Q_k\) is constant on each orbit of \(G^\Lambda\).

On the other hand, if
\[
Q_k|_{V_1} (u_1, q_1) = Q_k|_{V_1} (u_2, q_2),
\]
then
\[
\phi (u_1) |u_1| = \phi (u_2) |u_2|, \quad (2.2.5)
\]
\[
\phi (u_1)^2 \langle \text{Im} (u_1 q_1), q_1 \rangle = \phi (u_2)^2 \langle \text{Im} (u_2 q_2), q_2 \rangle, \quad \text{and} \quad (2.2.6)
\]
\[
\phi (u_1) \text{Re} u_1 q_1 = \phi (u_2) \text{Re} u_2 q_2. \quad (2.2.7)
\]
Equation (2.2.5) implies that \(|u_1| = |u_2|\) and \(\phi (u_1) = \phi (u_2)\). So
\[
\text{Re} (u_1) = \text{Re} (u_1) \langle q_1, q_1 \rangle
\]
\[
= \langle (\text{Re} (u_1) + \text{Im} (u_1)) q_1, q_1 \rangle, \quad \text{since} \ \text{Re} (q_1) = 0
\]
\[
= \langle u_1 q_1, q_1 \rangle
\]
\[
= \langle \text{Im} (u_1 q_1), q_1 \rangle, \quad \text{since} \ \text{Re} (q_1) = 0
\]
\[
= \langle \text{Im} (u_2 q_2), q_2 \rangle, \quad \text{by Equation (2.2.6) and the fact that} \ \phi (u_1) = \phi (u_2)
\]
\[
= \text{Re} (u_2)
\]
and
\[ \langle \text{Im} (u_1), q_1 \rangle = -\langle u_1, \bar{q}_1 \rangle, \text{ since } \text{Re} (q_1) = 0 \]
\[ = -\text{Re} u_1 q_1 \]
\[ = -\text{Re} u_2 q_2, \text{ by Equation 2.2.7 and the fact that } \phi (u_1) = \phi (u_2) \]
\[ = -\langle u_2, \bar{q}_2 \rangle \]
\[ = \langle \text{Im} (u_2), q_2 \rangle. \]
Together with \(|u_1| = |u_2|\) and the fact that \(q_1\) and \(q_2\) are imaginary, the previous two displays imply that \(\begin{pmatrix} u_1 \\ q_1 \end{pmatrix}\) and \(\begin{pmatrix} u_2 \\ q_2 \end{pmatrix}\) are in the same orbit.

Finally suppose that \(Q_k|_{U_2} (0, r_1) = Q_k|_{U_2} (0, r_2)\).

Then
\[ (1, \text{Re} (r_1), 0) = (1, \text{Re} (r_2), 0). \]
Since we also have that \(|r_1| = |r_2| = 1\), it follows that \((0, r_1)\) and \((0, r_2)\) are in the same \(G^\Lambda\)-orbit.

\[ \square \]

**Key Lemma 2.3.** Let \(Q_s\) be as in Lemma 2.1.

1. There is a well-defined surjective map
\[ \tilde{Q}_k : S^{2b-2}_k \to S^{2b-2}/G^\Lambda \]
whose fibers coincide with the orbits of the \(G^\Lambda\) action on \(S^{2b-2}_k\).

2. The orbit types of \(p \in S^{2b-2}_k\) and \(Q_s^{-1} \left( \tilde{Q}_k (p) \right)\) coincide.

3. For \(p \in S^{2b-2}_k\) and any \(q \in Q_s^{-1} \left( \tilde{Q}_k (p) \right)\) the isotropy representation of \(G^\Lambda_p\) and \(G^\Lambda_q\) are equivalent.

In particular, \(S^{2b-2}/G^\Lambda\) and \(S^{2b-2}_k/G^\Lambda\) are equivalent orbit spaces.

**Proof.** Motivated by [7, 20], we let \(h_1, h_2 : \Lambda \times S^{b-2} \to S^{2b-2}\) be given by
\[ h_1 (u, q) = \begin{pmatrix} uq \\ q \end{pmatrix} \phi (u) \text{ and } \]
\[ h_2 (v, r) = \begin{pmatrix} r \\ \bar{v}r \end{pmatrix} \phi (v). \]

We claim that \(Q_s\) and \(Q_k\) are related by
\[ Q_k = \begin{cases} 
Q_s \circ h_1 & \text{on } U_1 \\
Q_s \circ h_2 & \text{on } U_2.
\end{cases} \quad (2.3.1) \]

Indeed,
\[ Q_s \circ h_1 (u, q) = Q_s \left( \begin{pmatrix} uq \\ q \end{pmatrix} \phi (u) \right) \]
\[ = \phi (u) (|u|, \text{Re} uq, \phi (u) (\text{Im} uq, \text{Im} q)) \]
\[ = Q_k (u, q) \quad (2.3.2) \]
and
\[
Q_s \circ h_2 (v, r) = Q_s \left( \frac{r}{\bar{v}r} \right) \phi(v) \\
= \phi(v) (|r|, \text{Re}(r), \phi(v) (\text{Im} r, \text{Im} \bar{v}r)) \\
= Q_k (v, r),
\]
proving Equation (2.3.1).

Since \( h_1 (\Lambda \times S^{b-2}) \cup h_2 (\Lambda \times S^{b-2}) = S^{2b-2} \), Equation (2.3.1) implies that \( Q_k (S^{2b-2}) = Q_s (S^{2b-2}) \); so setting \( \tilde{Q}_k = Q_k \) gives a well-defined surjective map
\[
\tilde{Q}_k : S^{2b-2} \rightarrow S^{2b-2} / G^\Lambda,
\]
and Part 1 is proven. Parts 2 and 3 follow from the observation that \( h_1 \) and \( h_2 \) are \( G^\Lambda \)–equivariant embeddings. \( \square \)

Since the antipodal map \( A : S^{2b-2} \rightarrow S^{2b-2} \) and the involution \( T : S^{2b-2} \rightarrow S^{2b-2} \) from page 4, commute with the \( G^\Lambda \)–actions (0.0.1), (0.0.2) and (1.0.6), they induce well-defined \( \mathbb{Z}_2 \)–actions on our orbit space \( Q_s (S^{2b-2}) = Q_e (S^{2b-2}) = \{ (x, y, z) \mid x \in [0, 1], y \in [-x, x], z \in \left[ -\sqrt{(x^2 - y^2) (1 - x^2)}, \sqrt{(x^2 - y^2) (1 - x^2)} \right] \} \).

A simple calculation shows that the two \( \mathbb{Z}_2 \)–actions on \( Q_s (S^{2b-2}) \) coincide and are given by
\[
(x, y, z) \mapsto (x, -y, z).
\]

Since quotient maps of isometric group actions preserve lower curvature bounds, \( S^{2b-2} / (SO(3) \times \mathbb{Z}_2) \) has curvature \( \geq 1 \) (4). Therefore, Theorem A follows from Theorem B and Key Lemma 2.3.

3. Some Closing Remarks

In the same paper, Hirsch and Milnor also constructed exotic \( \mathbb{R}P^5 \)s, \( P^5_k \)s. The Davis action also descends to the \( P^5_k \)s where they commute with an \( SO(2) \)–action. The combined \( SO(2) \times SO(3) \)–action on the \( P^5_k \)s is by cohomogeneity one. Dearricott and Grove–Ziller observed that since these cohomogeneity one actions have codimension 2 singular orbits, Theorem E of [9] implies that they admit invariant metrics of nonnegative curvature.

Octonionically, the Hirsch-Milnor construction yields closed 13–manifolds, \( P^{13}_k \), that are homotopy equivalent to \( \mathbb{R}P^{13} \). Their proof that the \( P^5_k \)s are not diffeomorphic to \( \mathbb{R}P^5 \) breaks down, since in contrast to dimension 6, there is an exotic 14–sphere; however, Chenxu He has informed us that some of the \( P^{13}_k \)s are in fact exotic ([11]).

The Davis construction yields a cohomogeneity one action of \( SO(2) \times G_2 \) on the \( P^{13}_k \)s, only now one of the singular orbits has codimension 6. So we cannot apply Theorem E of [9]. Moreover, there are cohomogeneity one manifolds that do not admit invariant metrics with nonnegative curvature ([8] [10]). On the other hand, by the main theorem of [16], every cohomogeneity one manifold admits an invariant metric with almost nonnegative curvature.
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