Loop Calculus Helps to Improve Belief Propagation and Linear Programming Decodings of Low-Density-Parity-Check Codes

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Abstract—We illustrate the utility of the recently developed loop calculus [1], [2] for improving the Belief Propagation (BP) algorithm. If the algorithm that minimizes the Bethe free energy fails we modify the free energy by accounting for a critical loop in a graphical representation of the code. The log-likelihood specific critical loop is found by means of the loop calculus. The general method is tested using an example of the Linear Programming (LP) decoding, that can be viewed as a special limit of the BP decoding. Considering the (155, 64, 20) code that performs over Additive-White-Gaussian channel we show that the loop calculus improves the LP decoding and corrects all previously found dangerous configurations of log-likelihoods related to pseudo-codewords with low effective distance, thus reducing the code’s error-floor.

Belief Propagation (BP) constitutes an efficient approximation, as well as an algorithm, that applies to many inference problems in statistical physics [3], [4], [5], information theory [6], [7], [8], [9], and computer science [10]. All these problems can be stated in terms of computation of marginal probabilities on a factor graph. If the underlying graph structure contains no loops, i.e. it is a tree, the BP is exact, being only an approximation in the case of a general graph. The BP approximation can be restated in terms of a variational principle [11], [12], [13], where the BP equations describe a minimum of the so-called Bethe free energy, and the standard BP algorithm [6], [7] means solving the BP equations iteratively.

In coding theory BP plays a special role as the decoding of choice for Low Density Parity Check (LDPC) Codes introduced by R. Gallager [6]. These codes, described in terms of sparse (Tanner) graphs, are among the best performing codes known to date. Actually, these codes perform so well exactly due to the high-quality performance of the computationally efficient BP decoding scheme [6], [8], [9]. In the water-fall domain, i.e. at low and moderate Signal-to-Noise Ratios (SNR), the Frame-Error-Rate (FER) or Bit-Error-Rate (BER) of an LDPC code decoded using BP comes close to the optimal yet inefficient Maximum-Likelihood and Maximum-a-Posteriori decodings. However, in the low noise regime BP decoding clearly fails to approximate ML in practical (finite size) LDPC codes, thus causing the error-floor [14], [15], [16]. It is now well understood that the BP decoding failure in the error-floor domain is due to existence of pseudo-codewords [17], [18], [19], [20], [21], [22] and related instants [23], [24], defined as dangerous noise configurations causing the failures. Removing the obstacles and thus improving the BP decoding while keeping a reasonable computational complexity is on a great demand for high-performance applications, e.g. optical communications and data storage, where error-floor is a serious handicap.

Some general purpose BP-improvement strategies were already discussed in the literature [13], [25], [26]. Survey Propagation (SP) has been suggested in the context of combinatorial optimization [25]. SP, based on the so-called replica approach that originates from spin glass theory [27], applies successfully to highly degenerate problems where the standard BP would be trapped in a local unrepresentative minimum. Generalized Belief Propagation (GBP) of [13] constitutes another generalization of BP. It extends the cluster variational approximation of statistical physics [11], [12] to problems in information and computer sciences. GBP has been shown to perform well for the problems with many short loops, like Inter-Symbol Interference on a regular two-dimensional lattice [28], where transition to a coarse-grained cluster offers an improvement over the standard BP. The only yet serious drawback of the method is the expense of an overhead that scales exponentially with the cluster size. An efficient alternative to GBP, that is claimed successful in describing some random graph and random spin models on lattices, has been recently discussed in [26]. The method is based on closing the system of cavity equations and suggests a way to account for correlations (many loops) on all scales.

In spite of their potential in dealing with highly degenerate problems where the bare BP approach fails, the set of the aforementioned methods/algorithms (maybe except for a possible extension of [26]) do not seem appropriate in dealing with the error-floor problem. Indeed, one expects that any actual code (as opposed to ensemble of codes) has a discrete set of well-defined and relatively simple troublemakers (pseudo-codewords) localized on a subgraph. Therefore, for each (rare) failure of the standard BP one needs to identify and correct for a relatively long correlated configuration that is localized however on a small portion of the total number of bits.

This paper suggests an efficient approach that generalizes the BP algorithm and which is capable of reducing the undesirable error floor effect. Our method is based on the recently developed analytical tool, called loop calculus [1], [2], that represents the partition function (and, therefore,
the marginal probabilities) in terms of a finite series where each term is associated with a generalized loop on the graph and the zero order contribution corresponds to the bare BP approximation. We conjecture that for an instanton noise configuration that causes a BP failure there is always a relatively simple loop correction to the bare BP (in the terminology of the loop calculus) that provides an equal or comparable contribution to the partition function. We further suggest an improved decoding scheme based on finding this critical loop correction (in the case of the bare BP failure) followed by correcting the error. This is achieved by a proper modification of the Bethe free energy and the BP equations. These ideas are verified using an example of the Tanner \((155, 64, 20)\) code \([29]\) performing over the Additive-White-Gaussian-Noise (AWGN) channel and decoded with the Linear Programming (LP) decoding \([30]\). We build this test on analysis of the set of pseudo-codewords recently found for this code by the LP-based pseudo-codeword search algorithm \([31]\). We introduce the LP-erasure decoding which is equivalent to the standard LP decoding with full or partial erasure of information at the bits along the critical loop. We demonstrate that the LP-erasure algorithm corrects errors associated with all previously found pseudo-codewords of the Tanner code \((\sim 200\) of them) completely closing the error-floor gap between the lowest LP-instanton, with the effective distance \(\approx 16.4037\), and the Hamming distance 20 of the code.

The manuscript is organized as follows. An extended introductory Section \(\text{I}\) consists of four Subsections. Subsection \(\text{I-A}\) introduces the notations and briefly overviews decoding of a binary linear code. Subsection \(\text{I-B}\) describes the loop calculus of \([1], \ [2]\) also complementing it by some new variational interpretation that was not discussed in the original papers. Subsection \(\text{I-C}\) describes the calculation of a-posteriori log-likelihoods (magnetizations) within the loop calculus. Subsection \(\text{I-D}\) establishes a connection of the Bethe free energy approach to the loop calculus and LP decoding. Section \(\text{II}\) unveils an underlying loop structure, e.g. emergence of certain critical loops, for the family of instantons that appear in the \((155, 64, 20)\) code performing over the AWGN channel and decoded by LP \([31]\). The effective free energy approach, suggesting modification of the BP gauges to account for the critical loop, is introduced in Section \(\text{III}\). An improved LP decoding, called LP-erasure, is presented in Section \(\text{IV}\). Numerical test of the LP-erasure algorithm using an example of the \((155, 64, 20)\) code is discussed in Subsection \(\text{IV-A}\) where we also demonstrate that all previously found bare LP instantons (most damaging noise configurations) are actually corrected by the LP-erasure procedure thus completely reducing the error-floor observed for the standard LP decoding. The final Section \(\text{V}\) contains conclusions and discussions.

I. INTRODUCTION

A. Decoding in terms of Statistical Inference

A message word that consists of \(K\) bits is encoded to an \(N\)-bit long codeword, \(N > K\). In the binary linear case the code can be conveniently represented by \(M \geq N - K\) constraints, usually referred to as parity checks or simply checks. Formally, \(\pi = (\pi_1, \ldots, \pi_N)\) with \(\pi_i = \pm 1\), is one of the \(2^K\) codewords if and only if \(\prod_{i \in \alpha} \pi_i = 1\) for all checks \(\alpha = 1, \ldots, M,\) where \(i \in \alpha\) if the bit \(i\) contributes the check \(\alpha\). The relation between bits and checks (we use \(i \in \alpha\) and \(\alpha \ni i\) interchangeably) is often described in terms of an \(M \times N\) parity-check matrix \(H\) that consists of ones and zeros: \(H_{\alpha i} = 1\) if \(i \in \alpha\) and \(H_{\alpha i} = 0\) otherwise.

A bipartite graph representation of \(H\), with bits marked as circles, checks marked as squares and edges corresponding to the corresponding nonzero elements of \(H\), is usually called the Tanner graph associated with the code. For an LDPC code \(H\) is sparse, i.e. most of the entries are zeros. Transmitted through a noisy channel, a codeword gets corrupted due to the channel noise, so that the channel output at the receiver is \(x \neq \pi\). Even though an information about the original codeword is lost at the receiver, one still possesses full probabilistic information about the channel, i.e. the conditional probability \(P(x|\sigma)\) for a codeword \(\sigma\) to be a pre-image for the output word \(x\) is known. In the case of independent noise samples the full conditional probability can be decomposed into a product, \(P(x|\sigma) = \prod_i p(x_i|\sigma_i)\). The channel output at a bit can be conveniently characterized by the so-called log-likelihood \(h_i = \log(p(x_i|1)/p(x_i|0))\) measured in the units of the Signal-to-Noise-Ratio (SNR), normally defined as \(2s^2\). For a common model of the Additive White Gaussian Noise (AWGN) channel, \(p(x|\sigma) = \exp(-s^2(x - \sigma)^2/2)/\sqrt{2\pi s^2}\), \(h_i = x_i\). The decoding goal is to infer the original message from the received output \(x\).

ML decoding (that generally requires an exponentially large number \(2^K\) of steps) corresponds to finding \(\sigma\) maximizing the following weight (probability distribution) function

\[
W(\sigma) = Z^{-1} \prod_{\alpha} \delta \left( \prod_{i \in \alpha} \sigma_i + 1 \right) \exp \left( \sum_i \sigma_i h_i \right),
\]

where the normalization factor \(Z\) that enforce the \(\sum_{\sigma} W(\sigma) = 1\) condition is called the partition function in the statistical physics literature. Maximum-A-Posteriori (MAP) decoding boils down to finding a-posteriori log-likelihood (magnetization) at a bit defined according to

\[
\mathbf{m} = \sum_{\sigma} \sigma W(\sigma),
\]

followed by taking an absolute value of the result bit-wise.

Even though bits and checks play an essentially different role in the LDPC decode, it actually turns out to be formally convenient to consider them on equal footing thus putting the relevant inference problem in a more general context of graphical models \([32], \ [33], \ [34]\) where binary variables are shifted from bits/vertexes to edges of the corresponding Tanner graph.

A general vertex model, also called normal factor graph model \([33]\), is determined by the weight (probability distribution) function \(W\), which along with the partition function
\( Z \) can be represented in the following general form
\[
W(\sigma) = Z^{-1} \prod_{a \in X} f_a(\sigma_a), \quad Z = \sum_{\sigma} \prod_{a \in \tilde{X}} f_a(\sigma_a),
\]
where \( X \) defines a graph consisting of vertices and edges; \( a \) denotes a node (vertex) in the model; an elementary spin resides at the edge connecting two neighboring vertices, \( \sigma_{ab} = \pm 1, \) for \( b \in a \) and \( a \in b; \) \( \sigma \) stands for the vector built of all \( \sigma_{ab} \) with \( b \in a; \sigma \) is a particular configuration of spins on all the edges. With this notation one assumes that \( \sigma_{ab} = \sigma_{ba}. \)

The problem of LDPC decoding [1] represents a particular case of the general vertex model with bits and checks combined in one family of vertexes \( \{ a \} = \{ i \} \cup \{ \alpha \}, \) the Tanner graph \( X \) and the factor functions defined according to
\[
f_i(\sigma_i) = \begin{cases} \exp(h_i \sigma_i), & \sigma_{ia} = \sigma_{ib} = \sigma_i \forall \alpha, \beta \ni i \\ 0, & \text{otherwise;} \end{cases} \quad f_a(\alpha) = \begin{cases} 1, & \prod_{i \in a} \sigma_i = 1 \\ 0, & \prod_{i \in a} \sigma_i = -1 \end{cases}.
\]

Our strategy will be to keep the general vertex model notations whenever possible. The transition in the general formulas to our focus case of the LDPC decoding will always be simple and straightforward according to Eqs. (4-5).

**B. Loop Calculus [1, 2]**

Consider a general vertex model (3) and relax the condition \( \sigma_{ab} = \sigma_{ba} \), thus treating \( \sigma_{ab} \) and \( \sigma_{ba} \) as independent binary variables. We represent the partition function in the form:
\[
Z = \sum_{\sigma} \prod_{a} f_a(\sigma_a) \prod_{bc} \frac{1 + \sigma_{bc} \sigma_{cb}}{2}.
\]
Note that for this representations the vectors \( \sigma_a \) become independent variables. Also in the product over edges, \( (bc), \) we assume that each edge contributes only once. We further introduce a parameter vector \( \eta \) with the set of independent components \( \eta_{ab}. \)

Making use of the algebraic relation
\[
\frac{\cosh(\eta + \chi)}{(\cosh \eta + \sinh \eta)(\cosh \chi + \pi \sinh \chi)} = 1 + \frac{1}{\pi} \cot \chi + \frac{1}{\pi} \cot \eta.
\]
where \( \cot \chi = \frac{\sinh \chi}{\cosh \chi}; \) we arrive at the following representation for the partition function that is ready for a subsequent loop decomposition
\[
Z = \left( \prod_{bc} 2 \cosh(\eta_{bc} + \eta_{cb}) \right)^{-1} \sum_{\sigma} \prod_{a} P_a(\sigma_a) \prod_{bc} V_{bc},
\]
where
\[
P_a(\sigma_a) = f_a(\sigma_a) \prod_{b \in a} \left( \cosh \eta_{ba} + \sigma_{ab} \sinh \eta_{ab} \right); \quad V_{bc}(\sigma_{bc}, \sigma_{cb}) = 1 + \left( \tanh(\eta_{bc} + \eta_{cb}) - \sigma_{bc} \right) \cosh^2(\eta_{bc} + \eta_{cb}).
\]

The desired decomposition is obtained by expanding the \( V \)-terms followed by a local computation. The parameters (gauges) \( \eta \) are chosen using the criterion that subgraphs with at least one loose end do not contribute to the decomposition. This can be achieved if the parameters satisfy the following system of equations:
\[
\sum_{\sigma_a} \left( \tanh(\eta_{ab}^{bp}) + \eta_{ba}^{bp} - \sigma_{ab} \right) P_a(\sigma_a) = 0.
\]

It is also straightforward to check that the gauge-fixing condition Eq. (11) corresponds to an extremum of \( Z_0, \)
\[
\frac{\delta Z_0}{\delta \eta_{ab}} = 0,
\]
where
\[
Z_0 = \left( \prod_{bc} 2 \cosh(\eta_{bc} + \eta_{cb}) \right)^{-1} \sum_{\sigma} \prod_{a} P_a(\sigma_a) \bigg|_{\eta_{ab}}.
\]

A typical sum, entering a diagram contribution for a generalized loop \( C, \) expressed in terms of the corresponding irreducible correlation functions of the spin variables computed within BP:
\[
\mu_{ab}^{bp} = \sum_{\sigma_a} \eta_{ab}^{bp} \left( \sigma_a \right) \prod_{b \in a, c} \left( \sigma_{ab} - m_{ab}^{bp} \right),
\]
where \( m_{ab}^{bp} \) is the magnetization (a-posteriori loglikelihood) at the edge \( (ab), \) calculated within BP
\[
m_{ab}^{bp} = \sum_{\sigma_a} \eta_{ab}^{bp} (\sigma_a) \sigma_{ab}.
\]

Making use of Eqs. (8-13), one derives the following final expression for the partition function
\[
Z = Z_0 \left( 1 + \sum_{C} r(C) \right),
\]
where
\[
r(C) = \frac{\prod_{a \in C} \mu_a^{bp}}{\prod_{(ab) \in C} \left( 1 - (m_{ab}^{bp})^2 \right)^2},
\]
and \( Z_0 \) is taken at \( \eta_{ab}^{bp} \) and the summation in Eq. (17) runs over all allowed \( C \) (marked) paths in the graph associated with the model; \( (ab) \) marks the edge on the graph that connects nodes \( a \) and \( b. \) A marked path is allowed to branch at any node/vertex, however it cannot terminate at a node.
We refer to such a structure as a loop (it is actually some kind of a generalized loop since branching is allowed; we use the shorter name for convenience). An example is given in Fig. 1.

In the LDPC case the loop series expressions assumes the following form

\[
Z_{\text{LDPC}} = Z_b \left( 1 + \sum_C r(C) \right),
\]

(18)

\[
r(C) = \prod_{i,\alpha \in C} \mu_{\alpha}^{(bp)} \mu_i^{(bp)}, \quad q_i = \sum_{\alpha \in C} 1,
\]

\[
\mu_i^{(bp)} = \frac{(1 - m_i^{(bp)})q_i - 1 + (1 + m_i^{(bp)})q_i - 1}{2(1 - (m_i^{(bp)})^2)q_i - 1},
\]

\[
\mu_{\alpha}^{(bp)} = \sum_{\sigma_\alpha} b_i^{(bp)}(\sigma_\alpha) \prod_{i \in C} (\sigma_\alpha - m_i^{(bp)}), \quad m_i^{(bp)} = \sum_{\sigma_i} b_i^{(bp)}(\sigma_i)\sigma_i,
\]

where \( b_i^{(bp)}(\sigma_i) \) and \( b_{\alpha}^{(bp)}(\sigma_\alpha) \) are the beliefs defined on bits and checks respectively according to

\[
b_i^{(bp)}(\sigma_i) \propto \exp \left( \frac{\sigma_i}{q_i} - 1 \left( \sum_{\gamma \ni i} \eta_\gamma^{(bp)} - h_i \right) \right),
\]

(19)

\[
b_{\alpha}^{(bp)}(\sigma_\alpha) \propto \delta \left( \prod_{i \in \alpha} \sigma_i, +1 \right) \exp \left( \sum_{i \in \alpha} \eta_i^{(bp)} \right),
\]

(20)

\[
\delta m_{\alpha}^{(bp)} = h_i + \sum_{\beta \ni i} \tanh^{-1} \left( \prod_{j \in \beta} \tanh \delta m_{\beta}^{(bp)} \right).
\]

(21)

Eq. (21) represents a traditional form of the BP equations for LDPC codes [6], [7].

See [2] for more details of the derivations sketched in this Subsection.

C. Calculating a-posteriori log-likelihoods

Formally a-posteriori log-likelihood (magnetization) is defined by

\[
m_{ab} \equiv \sum_{\sigma} \sigma_{ab} W(\sigma).
\]

(22)

Full magnetization at an edge of a general graphical model can be recalculated using Eqs. (17) with the gauge fields fixed according to the BP rules (11). There are two complementary ways to perform these calculations. One can incorporate infinitesimal test-field, \( h_{ab} \), into the factor functions by \( f_a \rightarrow f_a \exp(\sigma_{ab} h_{ab}) \), and use the generalized partition function to generate the corresponding magnetizations by simply differentiating \( \ln Z \) (with \( Z \) taken in the loop-series representation) with respect to the test-field followed by taking the \( h_{ab} \rightarrow 0 \) limit. Alternatively and straightforwardly, one can calculate the magnetization according to the definition (22) with the probability measure taken according to Eq. (8) in the following BP-gauge form

\[
W^{(bp)}(\sigma) = \left( \prod_a P_a \prod_{bc} V_{bc} \right)^{-1} \left( \eta_{ab}^{(bp)} \right).
\]

(23)

Applying the same “marked path” rule to calculate the magnetization as we used in the previous Subsection to derive the loop-series expression for the partition function, we arrive at

\[
m_{ab} = \frac{m_{ab}^{(bp)} + \sum_C r(C)}{1 + \sum_C r(C)},
\]

(24)

\[
\delta m_{ab}^{(bp)} = \left( \prod_{(ab) \in C} \mu_{ab}^{(bp)} \prod_{(ab) \in C} \mu_{c}^{(bp)} \right)^{-1} \left( 1 - (m_{ab}^{(bp)})^2 \right),
\]

(25)

\[
\mu_{ab}^{(bp)} = \frac{\sum_{\sigma_{ab}} b_{ab}^{(bp)}(\sigma_{ab}) \prod_{\sigma_{ac}, \sigma_{bc}} (\sigma_{ac} - m_{ac}^{(bp)})}{1 - (m_{ab}^{(bp)})^2},
\]

(26)

where \( C_{a \rightarrow b} \) consists of an extended family of loops with the connectivity degree at node \( a \) being one or higher while the connectivity degree at node \( b \), as well as at any other node that belongs to \( C_{a \rightarrow b} \) and different from \( a \), being two or higher.

Note that for any generalized loop \( C \) there may be many extended loops \( C_{ab} \) propagating correlations imposed by the local loop contribution all over the graph.

Replacing the hole families of the generalized loops \( C \) and the extended generalized loops \( C_{a \rightarrow b} \) in Eq. (25) by some relevant subfamilies constitutes an approximation which provides an improvement over the bare BP approximation.

D. Bethe-Free energy and Linear Programming Decoding

The variational approximation for a general vertex model reads as follows [13]. The Bethe free energy

\[
F = \sum_a \sum_{\sigma_a} b_a \ln \left( \frac{b_a}{\sigma_a} \right) - \sum_{(ab)} \sum_{\sigma_{ab}} b_{ab} \ln b_{ab},
\]

(27)
is a functional of the beliefs \( b_a(\sigma_a), b_{ac}(\sigma_{ac}) \), defined on the vertices and edges of the graph respectively. BP equations can be introduced as equations for a conditional extremum of the Bethe-free energy. The realizability conditions (constraints) are

\[
\forall c \in a, b \in c, c \neq a, b \quad 0 \leq b_a(\sigma_a), b_{ab}(\sigma_{ac}) \leq 1, \quad \forall a \in c \quad \sum_{\sigma_a} b_a(\sigma_a) = \sum_{\sigma_{ac}} b_{ac}(\sigma_{ac}) = 1,
\]

where we assume \( \sigma_{ac} = \sigma_{ca} \). The second term on the rhs of Eq. (27) is the entropy term that corrects for “double counting” of the link contribution: any link appears twice in the entropy part of the first term on the rhs of Eq. (27).

Optimal configurations of the beliefs minimize the Bethe free energy subject to the constraints. Introducing the constraints as Lagrange multipliers in the effective Lagrangian and looking for the extremum with respect to all possible beliefs leads to

\[
b_{a}(\sigma_a) \propto f_a(\sigma_a) \prod_{b \in a} \exp(\eta_{ab}(bp) \sigma_{ab}), n_{ab}(\sigma_{ab}) \propto \exp(\eta_{ab}(bp) + \eta_{ba}(bp) \sigma_{ab}),
\]

where \( \propto \) indicates that one should use the normalization conditions to guarantee that the beliefs sum up to one. It is straightforward to check that the system of Eqs. (31,32) is fully equivalent to the BP equations discussed above.

If the aforementioned optimization procedure is performed with the compatibility conditions excluded, yet all other constraints accounted for, one still obtains Eqs. (31,32) for the beliefs with \( \eta_{bp}(\sigma_{ab}) \) replaced by yet unconditioned \( \eta \). Expressing the beliefs in terms of the \( \eta \)-fields according to the relaxed version of Eqs. (31,32) we arrive at the following expression for the Bethe free energy in terms of the \( \eta \) variables:

\[
F = F_0 + \sum_{ab} \left( \eta_{ab} m_{ab}^{(1)} + \eta_{ba} m_{ba}^{(1)} - (\eta_{ab} + \eta_{ba}) m_{ab}^{(1)} \right),
\]

\[
m_{ab}^{(1)} = \frac{\sum_{\sigma_a} f_a(\sigma_a) \prod_{c \in a} \exp(\eta_{ac} \sigma_{ac})}{\sum_{\sigma_a} f_a(\sigma_a) \prod_{c \in a} \exp(\eta_{ac} \sigma_{ac})},
\]

\[
m_{ab}^{(1)} = \tanh(\eta_{ab} + \eta_{ba}),
\]

where \( F_0(\eta) = -\ln Z_0(\eta) \) and \( m_{ab}^{(1)} \) and \( m_{ab}^{(1)} \) are two expressions for a-posteriori log-likelihoods (magnetizations) at the edge \((ab)\) which are equal to each other if the compatibility conditions are accounted for. For an arbitrary choice of \( \eta \) the magnetizations are different. However, for the special choice of \( \eta \) that corresponds to \( \eta_{bp} \) the two generally different magnetizations become equal. More formally: the Belief Propagation equations can be expressed as

\[
m_{a-b}^{(1)} = m_{a-b}^{(1)} = m_{a-b}^{(1)} = m_{a-b}^{(1)} = m_{a-b}^{(1)} = m_{a-b}^{(1)}, \quad \eta_{ab} \text{ and } \eta_{ba} \text{ turn into } \eta_{bp} \text{ and } \eta_{bp},
\]

satisfied \( \forall ab \), where \( \eta \) and \( \eta \) turn into \( \eta_{bp} \) and \( \eta_{bp} \), respectively.

There is also a relation between the Bethe free energy minimization approach and the LP decoding. Consider the Bethe free energy Eq. (27) in the \( f \to \infty \) limit of zero noise or, equivalently, large SNR. In this limit the self-energy contribution to the free energy dominates the entropy terms and the latter can be safely neglected. However, both the self energy and the constraints are linear in beliefs. Therefore this asymptotic optimization problem can be solved efficiently by means of the standard linear programming approach.

II. LOOP CALCULUS ANALYSIS OF THE LP-INSTANTONS FOR THE \( (155, 64, 20) \) CODE

We consider a family of \( (\sim 200) \) instantons, i.e. noise configurations (log-likelihoods) corresponding to the pseudo-codewords with the effective distance smaller then the Hamming distance of the code. This set of instantons and related pseudo-codewords was found in [31] for the Tanner \((155, 64, 20)\) code performing over the Additive-White-Gaussian-Noise (AWGN) channel and decoded using LP decoding.

In short the method/algorithm of [31], called pseudo-codeword search algorithm, proposes an efficient way of describing LP decoding polytope and the pseudo-codeword spectra of the code. It approximates a pseudo-codeword and the corresponding noise configuration on the error-surface surrounding the zero codeword correspondent to the shortest effective distance of the code. The algorithm starts from choosing a random initial noise configuration. The configuration is modified through a discrete number of steps. Each step consists of two sub-steps. First, one applies LP decoder to the initial noise-configuration deriving a pseudo-codeword. Second, one finds the noise configuration equidistant from both the pseudo codeword and the zero codeword. The resulting noise configuration is used as an entry for the next step. The algorithm, tested on the Tanner \((155, 64, 20)\) code and Margulis \( p = 7 \) and \( p = 11 \) codes \((672 \text{ and } 2640 \text{ bits long respectively}) \) shows very fast convergence.

Discussing the instantons, found for the \((155, 64, 20)\) code with the pseudo-codeword-search algorithm, one after another we aim to associate for each instanton the corresponding critical loop, \( \Gamma \) that generates a contribution to the loop series, comparable to the bare BP contribution.

We restrict our search for the critical loop contribution to the class of single-connected loops, i.e. loops that consist of checks and bits with each check connected to only two bits of the loop. According to Eqs. (36), such contribution to the loop series is the product of all the triads, along the loop,

\[
r(\Gamma) = \prod_{\alpha \in \Gamma} \tilde{\mu}_\alpha^{(bp)} = \prod_{\alpha \in \Gamma} \frac{\mu_\alpha^{(bp)}}{\sqrt{(1 - (m_i^{(bp)})^2)(1 - (m_j^{(bp)})^2)}},
\]

satisfied \( \forall ab \), where \( \eta \) and \( \eta \) turn into \( \eta_{bp} \) and \( \eta_{bp} \), respectively.
where for any check $\alpha$ that belongs to $\Gamma$, $i, j$ is the only pair of $\alpha$ bit neighbors that also belongs to $\Gamma$. By construction, $|\tilde{\mu}_{\alpha}^{(bp)}| \leq 1$. We immediately find that for the critical loop contribution to be exactly equal to unity (where unity corresponds to the bare BP term), the critical loop should consist of triads with all $\tilde{\mu}_{\alpha}^{(bp)}$ equal to unity by absolute value. Even if degeneracy is not exact one still anticipates the contributions from all the triads along the critical loop to be reasonably large, as an emergence of a single triad with small contributions from all the triads along the critical loop to be value. Even if degeneracy is not exact one still anticipates the first dozen of instantons has loop contribution and the BP/LP contribution are equal to scheme generates code with the lowest effective distances this triad-based efficient way to find a single connected critical loop, with the bare BP term. This consideration suggests that an $\mu^{(\beta)} \sim (\eta^{\beta}) = 1$ series (18).

Applied to the set of instantons of the Tanner (155, 64, 20) code with the lowest effective distances this triad-based scheme generates $r(\Gamma)$ that is exactly unity by the absolute value. This is the special degenerate case when the critical loop contribution and the BP/LP contribution are equal to each other by absolute value. Thus only the sixth of the first dozen of instantons has $r(\Gamma) \approx 0.82$ while all others yield $r(\Gamma) = 1$. To extend the triad-based search scheme to the instantons with larger effective distance one needs to decrease the threshold. This always results in emergence of at least one single connected loop with $r(\Gamma) \sim 1$. Note, that it may be advantageous, even thought not necessary, to include in this triad-based search for the critical loop some additional criteria. In particular, we found it useful to also require that the absolute values of the a-posteriori log-likelihoods of the bits involved in the critical loop be larger than certain threshold.

Fig. 2 shows a representative set of instantons analyzed using the described loop calculus tool. Critical loops are shown in the corresponding subplots. The resulting critical loops are typically 4 – 5 bits long (with the girth of the code that counts both bits and checks being 8). However some configurations, like instanton #192 shown in the right lower corner of Fig. 2, corresponds to a highly degenerate situation. The instanton #192 shows three distinct single connected loops giving all $r(\Gamma) = 1$ contribution to the loop series (13).

Obviously, when an instanton produces a critical loop $\Gamma$ with $r(\Gamma) = 1$ one can guarantee that this is the largest non-bare (non-BP) contribution to the loop series. However in all other cases when $r(\Gamma) < 1$, no guarantee can be given and we may not exclude a possibility that some other general loop of a more complex structure can provide a contribution to the loop series with higher $r(\Gamma)$. However, we will show in Section IV that knowledge of the $r(\Gamma) = O(1)$ critical contribution found along the way explained above is sufficient for successful decoding of these dangerous configurations.

One final remark of this Section concerns the value of magnetization calculated for degenerate critical loops, i.e. loops with $r(\Gamma) = 1$. Calculating a-posteriori log-likelihood (magnetization) at a bit which belongs to the critical loop, one finds that the first term in the numerator of Eq. (24) is completely compensated by the only relevant of the other terms, correspondent to $\Gamma_{a \rightarrow b}$ replaced just by the critical loop $\Gamma$. Therefore, if only these two contributions are accounted for the magnetization at the bit is exactly zero. This suggests that one of the effects related to a critical loop is an effective shift of log-likelihoods at the bits of the critical loop in the direction opposite to the magnetization measured at the bit by bare BP. We have also found that the bare BP a-posteriori-log-likelihoods along the critical loop are always aligned bit-wise with the corresponding log-likelihoods. This set of observations will actually be explored in Section V to construct a simple modification of the LP decoding algorithm.

III. Effective Free Energy Approach

Accounting for a single loop effect, when it is comparable to a bare (BP) contribution, can be improved through an effective free energy approach explained in this Section. This approach is akin to degenerate Hartree-Fock variational approach used in quantum mechanics (quantum chemistry) [37] in the case of a phase space degeneracy.

It was shown in the previous Section that finding the BP gauge is equivalent to optimizing (finding an extremum of) the functional $F_0(\eta) = - \ln Z_0$, where $Z_0$ is given by Eq. (13). Therefore, the $\eta$-gauge is fixed according to the first term in the series Eq. (8). Further terms in $Z$, coming from other higher-order vertex “corrections”, were calculated above with the BP gauge fixed. This resulted in Eqs. (17)(18).

As we see from an example presented in the previous Section, some number (or just one) of these corrections may be either comparable (by absolute value) or just equal to the bare $Z_0$ contribution. Any of this special $O(Z_0)$ contributions is associated with some critical generalized loop. This situation, when one or more of these critical loops emerge, is a potentially dangerous one, possibly leading (as we observed) to the bare BP failure.

In this troublesome situation a plausible solution would be to modify the BP gauge conditions Eq. (12) by

$$
\frac{\delta \exp (-F)}{\delta \eta_{ab}} \bigg|_{\eta_{\text{eff}}} = 0, \quad F = - \ln \left( Z_0 + \sum_\Gamma Z_\Gamma \right), \quad (38)
$$

where $Z_\Gamma$ is the component of the full expression for $Z$ that corresponds to a critical loop, $\Gamma$. In general Eq. (38) will be different from the standard BP equation, and we can anticipates that the corrections due to the critical loops may cure the bare BP failure in decoding.

From Eqs. (38) we derive the following set of modified
BP equations

\[ m_{a \rightarrow b}^{(s)} - m_{ab}^{(s)} = \sum_{\Gamma} \prod_{d \in \Gamma}^{(1 - (m_{a d}^{(s)})^2)} \delta m_{a \rightarrow b; \Gamma}, (39) \]

\[ \delta m_{a \rightarrow b}^{\Gamma} = \begin{cases} \frac{1 - (m_{a b}^{(s)})^2}{\mu_{a b}} \left( \prod_{c \in \Gamma} (\sigma_{bc} - m_{bc}^{(s)}) \right), (ab) \in \Gamma; \\ \frac{1 - (m_{a b}^{(s)})^2}{\mu_{a b}} \left( \prod_{c \in \Gamma, c \neq b} (\sigma_{ac} - m_{ac}^{(s)}) \right), a \in \Gamma, b \notin \Gamma; \\ \sum_{\sigma_a} P_a(\sigma_a), \mu_{a \Gamma} = \left( \prod_{b \in \Gamma; a} (\sigma_{ab} - m_{ab}^{(s)}) \right), (41) \end{cases} \]

where Eqs. 38 defined \( m_{ab}^{(s)}(\eta) \) and \( m_{ab}^{(s)}(\eta) \) and Eqs. 39 should all be taken at \( \eta \rightarrow \eta_{eff} \) as the system of equations actually defines \( \eta_{eff} \). Recast in terms of the beliefs Eqs. 39 forms a system of polynomial equations for beliefs, that become linear only if the right hand sides of Eqs. 39 turn to zero.

To find a-posteriori log-likelihoods within the degenerate approach one calculates

\[ m_{ab; eff}^{(s)} = m_{a \rightarrow b}^{(s)} + \sum_{\Gamma} \frac{\langle \sigma_a \prod_{\mu_{a'b'}} (\sigma_{a'\mu_{a'b'}} - m_{a'\mu_{a'b'}}^{(s)}) \rangle}{\prod_{(a'b')} \prod_{\mu_{a'b'}} (1 - (m_{a'\mu_{a'b'}}^{(s)})^2)} \left( \prod_{d \in \Gamma}^{(1 - (m_{a d}^{(s)})^2)} \right), (42) \]

where \( \eta \) is substituted by \( \eta_{eff} \) solving Eqs. 39.

Note a couple of special cases. First of all, if the graph consists of a set of disconnected single loops, the modified BP equations 39 are simply reduced to the standard BP, with the rhs of Eq. 39 replaced by zero. One consequence of this degeneracy is that if one chooses \( F \) based on all the single-connected loops contained in the degenerate model, the variational result would just be exact. Second, if one considers contributions to \( F \) based on some number of single-connected critical loops, \( \Gamma \), the only terms on the rhs of Eqs. 39 that will not result in exact zero for the bare BP solution will be associated with \( a \rightarrow b \) where \( a \in \Gamma \) while \( b \notin \Gamma \).

The modified free energy approach, described by equations 39 for renormalized gauges \( \eta_{eff} \), promises decoding benefit in the degenerate or close to the degenerate cases when compared with the corresponding direct truncation of the loop series given by Eqs. 17 or Eqs. 18. The approximation is also convenient as it keeps the same level of complexity as the BP equations. This is in contrary to the direct approach which requires summation according to Eq. 25 over many extended diagrams for getting renormalized values of log-likelihoods at the bits that do not belong to the critical loops.

Note that pretty much like in the case of bare BP, to define an algorithm associated with Eqs. 39 one needs to introduce an iterative algorithm based on it, and there is obviously some freedom associated with the choice of discretization. (See [38] for discussion of different discretization/iteration schemes in the context of the bare BP equation.)

We expect that accounting for just one critical loop \( \Gamma \) that corresponds to the largest value of \( r_{\Gamma} \) (calculated within the bare BP) will be already sufficient for a substantial improvement of BP in the special cases when the bare BP fails. Summarizing, we arrive at the following Loop-corrected BP algorithm

- 1. Run bare BP algorithm. Terminate if BP succeeds (i.e. a valid code word is found in terms of marginal probabilities).
- 2. If BP fails find the most relevant loop \( \Gamma \) that corresponds to the maximal (by absolute value) amplitude \( r_{\Gamma} \) in Eq. 17. A simple method for the critical loop search introduced in the previous Section may be tried first.
- 3. Solve the modified-BP equations 39 for the given \( \Gamma \). Terminate if the improved-BP succeeds.
- 4. Return to Step 2 with an improved \( \Gamma \)-loop selection. An additional loop found through an improved critical loop procedure can simply be added to the sum on the rhs of Eqs. 39.

In this manuscript we do not report any results of numerical simulations where the Loop-corrected BP algorithm would be tested directly using a sample code. We postpone this important exercise for future analysis. Instead, we use the result described in this Section as a motivation for an even simpler heuristic approach detailed in the next Section.

IV. LP-ERASURE DECODING

As it was already discussed in the literature [35], [31] and commented on in Section 10 LP decoding can be viewed as a certain (large SNR) limit of BP decoding. It is not obvious, however, that the BP-improved procedure outlined in the previous Section can be rigorously transformed into a correction to LP keeping a linear structure.

Our approach to this question will be heuristic, i.e. we simply conjecture a plausible modification of the LP scheme based on the algorithm formulated above for improved BP and then test this idea using an example of the (155, 64, 20) code.

On our way to proposing an improved LP decoding we first note that the effective free energy approach keeps the same number of degrees of freedom as the original bare BP. Therefore, if we conjecture that a modification of LP decoding should keep its linear structure and thus the number of constraints, the only actual degree of freedom left is in the log-likelihoods, i.e. in their possible modifications deduced from loop calculus. We further observe that the modifications of the BP equations discussed in the previous Section are actually well localized. Specifically, the rhs of 39 is non-zero for the \( \eta \) variables associated with the vertices that belong to the critical loop \( \Gamma \) or are immediately adjusted to it. Given that LP decoding is a special limit of BP decoding one deduces from these observations that the log-likelihoods should be renormalized just at the bits lying on the critical loop. Furthermore, taking into account the observation reported in the last paragraph of Section 11 we argue that renormalization of log-likelihoods on the bits of the critical loop should be directed against the bare log-likelihoods.
All this suggests the following LP-version of the loop-enhanced algorithm:

**LP-erasure algorithm**

1. Run LP algorithm. Terminate if LP succeeds (i.e. a valid code word is found).
2. If LP fails, find the most relevant loop $\Gamma$ that corresponds to the maximal amplitude $r(\Gamma)$ in the LP-version of Eq. (17).
3. Modify the log-likelihoods (factor-functions) along the loop $\Gamma$ introducing a shift towards zero, i.e. introduce a complete or partial erasure of the log-likelihoods at the bits. Run LP with modified log-likelihoods. Terminate if the modified LP succeeds.
4. Return to Step 2 with an improved selection principle for the critical loop.

Let us also mention that our loop-calculus based conjecture that the full or partial erasure of certain log-likelihoods is beneficial for LP decoding is akin to the statement made in [39] in regard to the positive effect of thresholding in LP decoding.

A. $(155, 64, 20)$ test of the LP-erasure algorithm

Here we describe our numerical test of the LP-erasure algorithm. The test is based on the analysis (described in Section II) of instantons, i.e. most probable out of a variety of dangerous noise configurations that lead to LP-decoding failure. The $\sim 200$ instantons found in [31] for the $155, 64, 20$ code have the effective weight lower than the Hamming distance of the code, thus leading to the undesirable error-floor. We apply the triad method of Section II to analyze all the low effective weight instantons of the $(155, 64, 20)$ code. In spite of the fact that the method did not guarantee that the critical loop was actually the one with the largest (for given configuration of the noise) $r(\Gamma)$ we still choose to try it for the next, third, step of the LP-erasure procedure: for the special marked bits we lowered the original log-likelihood uniformly multiplying all the log-likelihoods at the marked bits of the critical loop by a positive number $\epsilon$ that is smaller than one.

The results of the test are remarkable. We found out that all instantons are actually corrected already with the roughest, i.e. $\epsilon = 0$, modification corresponding to the full erasure of the information (log-likelihoods) along the critical loop. We verified that the noise configurations that are re-scaled instantons (of the same structure but with effective distance larger then one of the original instanton but smaller then the Hamming distance of the code) are also corrected by the LP-erasure algorithm successfully.

Note that the instantons shown in Fig. 4 are counted using the all “+1” configuration as the original codeword primarily for the purpose of the demonstration transparency. We did verify that the LP-erasure algorithm is invariant with respect to a change in the original codeword, i.e. that the LP-erasure algorithm corrects instantons of the bare LP and their derived configurations if counted using any other codeword as a reference point. In all our tests (with the $\sim 200$ instantons) whenever LP-erasure decoded to a codeword, the codeword was actually the right one.

The LP-erasure algorithm also shows an impressive robustness. The algorithm often forgives an inaccurate definition of the critical loop. For example, if one uses a very low threshold in identifying the bits that can possibly enter the critical loop, the resulting loop can actually be large and contain up to 20 bits. By lowering or completely erasing log-likelihoods at all these bits we often get the correct result with subsequent LP-decoding. However, in the rare cases when this loose way to define a critical loop leads to a failure one just needs to tighten the threshold and possibly use some additional thresholding (e.g. in the value of the a-posteriori-loglikelihood that belong to the loop and also in the erasure coefficient $\epsilon$) till a codeword emerges.

V. CONCLUSIONS AND DISCUSSIONS

In this manuscript we presented a proof-of-concept test demonstrating the utility of the loop calculus approach of [1], [2] to improve inference algorithms of the BP class in general, and of BP decoding of LDPC codes in particular. The key observation, that enabled the reported improvement in the decoding scheme, was emergence of a well-defined and relatively simple loop correction for each BP-dangerous configuration of log-likelihoods. Identification of the critical loop and further log-likelihood specific modification of the BP/LP algorithm has been suggested as a cure for bare BP/LP failure. LP-erasure that is the simplest algorithm based on the critical loop identification, was successfully tested using the $(155, 64, 20)$ code operated on the AWGN channel. LP-erasure was able to correct all bare-LP-dangerous noise configurations related to the previously found pseudo-codewords with the effective distance lower than the Hamming distance of the code.

This demonstration is clearly the first step in the highlighted direction where the next steps are thought of as follows. We plan to improve and continue testing the simple LP-erasure algorithm. The major improvement required is an automatization of the critical loop identification scheme. Further tests imply (a) direct comparison of LP-erasure with bare LP algorithms with the help of Monte Carlo simulations of BER/FER, (b) working with other (longer) codes, (c) working with other (e.g. correlated) channels. We will also be implementing all of the above using a more sophisticated and also better justified Loop-corrected BP scheme outlined in Section III.

These studies will certainly benefit from using some other recent developments in the field of BP/LP decoding, such as [40], [41] on reducing complexity of LP-decoding, [38] on accelerating the bare BP convergence, and [42] which suggests an alternative method of LP-decoding improvement.
Fig. 2. The panels represent the results of LP decoding and critical loop identification for 6 representative instantons found for the Tanner (155, 64, 20) code performing over AWGN channel. All instantons has an effective distance smaller than the Hamming distance, 20, of the code and the exact (ML) decoding would correctly decode them to the correct codeword (all +1 for the test shown in the Figure). The main Figures show dependence of the a-posteriori log-likelihood on the bit label/position. Bits lying on the critical loops are marked with red (filled circles). The critical loops are also shown schematically in the subplots. Values shown in the subplots next to the checks (squares) that connect pairs of bits on the critical loop are related to the of the corresponding triad contributions $\tilde{\mu}$ defined by Eq. (37). Loop-erasure decoding, with erasures applied along the critical loop, corrects all the dangerous (instanton) errors.
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