Anti-$\mathcal{PT}$-symmetric Qubit: 
Decoherence and Entanglement Entropy

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Abstract
We investigate a two-level spin system based anti-parity-time (anti-$\mathcal{PT}$)-symmetric qubit and study its decoherence as well as entanglement entropy properties. We compare our findings with that of the corresponding $\mathcal{PT}$-symmetric and Hermitian qubits. First we consider the time-dependent Dyson map to find the exact analytical dynamics for a general non-Hermitian qubit system coupled with a bath, then we specialize it to the case of the anti-$\mathcal{PT}$-symmetric qubit. We find that the decoherence function for the anti-$\mathcal{PT}$-symmetric qubit decays slower than the $\mathcal{PT}$-symmetric and Hermitian qubits. For the entanglement entropy we find that for the anti-$\mathcal{PT}$-symmetric qubit it grows more slowly compared to the $\mathcal{PT}$-symmetric and Hermitian qubits. Similarly, we find that the corresponding variance and area of Fisher information is much higher compared to the $\mathcal{PT}$-symmetric and Hermitian qubits. These results demonstrate that anti-$\mathcal{PT}$-symmetric qubits may be better suited for quantum computing and quantum information processing applications than conventional Hermitian or even $\mathcal{PT}$-symmetric qubits.

1 Introduction
One of the roadblocks for achieving viable quantum computing platforms is decoherence intrinsic to quantum systems [1]. This is particularly important for quantum information processing and storage [2]. Previously it was shown that parity-time or $\mathcal{PT}$-symmetric qubits are better than Hermitian qubits from this perspective when the $\mathcal{PT}$-symmetric quantum system is coupled to a Hermitian environment very weakly [3]. Here we explore whether another non-Hermitian realization, i.e. an anti-$\mathcal{PT}$ qubit, can further improve decoherence properties. We find that the answer is in the affirmative. A very weak coupling to the bath or environment ascertains that the system and the bath do not exchange any heat [4], which leads to what is known as pure decoherence or dephasing [5]. The choice of a Hermitian bath is only for the sake of simplicity.

Before exploring properties, we need to find the dynamics of our qubit systems. The computation of the qubit’s reduced density matrix, through the expectation value of the time evolved operator, usually is performed from a Hermitian diagonal qubit, which can be easily obtained for Hermitian and $\mathcal{PT}$-symmetric qubits [3, 6]. However, for the anti-$\mathcal{PT}$ qubit, we need to first provide a method to obtain the exact analytical expression for the evolved density matrix of a general non-Hermitian qubit system. This is accomplished through utilizing the time-dependent Dyson map to find a corresponding
Hermitian system and as a result, obtain a reduced density matrix for a time-dependent Hermitian system \[7, 8\]. The decoherence is naturally revealed in the dynamics and we can subsequently also investigate further properties such as entanglement entropy (von Neumann as well as Rényi) and Fisher information.

After the introduction of \(\mathcal{PT}\)-symmetry \[9\] and subsequently two decades of intensive research \[10\], the notion of anti-\(\mathcal{PT}\) symmetry was introduced by Ge and Türeci \[11\] in optics by an appropriate spatial arrangement of the effective optical potential. The \(\mathcal{PT}\) operator commutes with the Hamiltonian \([H, \mathcal{PT}] = 0\) whereas the anti-\(\mathcal{PT}\) operator anticommutes with the Hamiltonian \{\(H, \mathcal{PT}\)\} = 0. The anti-\(\mathcal{PT}\) symmetry has been realized in spatially coupled atom beams \[12\], electrical circuit resonators \[13\], optical waveguides with imaginary couplings \[14\] and optical four-wave mixing in cold atoms \[15\]. In addition, constant refraction optical systems \[16\] and several experiments in atomic \[17, 18\] and optical \[19, 20, 21\] systems have realized the anti-\(\mathcal{PT}\) symmetry. There are many other applications involving waveguide arrays \[21\], diffusive systems \[22\], phase transitions \[23\], spin chains \[24\], information flow \[25\] and non-Markovian aspects \[26\].

Specifically, we investigate an anti-\(\mathcal{PT}\)-symmetric quantum system very weakly coupled to a bath (or environment). This leads to a critical slowing down of decoherence that is better than its \(\mathcal{PT}\)-symmetric counterpart. Possibly, an anti-\(\mathcal{PT}\) qubit can be experimentally realized in recently demonstrated optical and microcavity settings \[27\]. A quantum circuit \[28\] and information flow \[29\] using a two-level system have also been recently discussed.

The outline of the paper is as follows. We first introduce the two-level Hermitian, \(\mathcal{PT}\)-symmetric and anti-\(\mathcal{PT}\)-symmetric qubits. For the former two qubits, we explicitly write down their reduced density matrices. In Section 2 we show a general formalism for treating non-Hermitian qubits by way of the time-dependent Dyson map for the density matrix and apply it to the anti-\(\mathcal{PT}\)-symmetric qubit. Section 3 is devoted to studying decoherence of the anti-\(\mathcal{PT}\) qubit. In Section 4 we calculate the von Neumann and Rényi entanglement entropy for the anti-\(\mathcal{PT}\) qubit. In Section 5 we invoke the use of the Kullback-Leibler divergence in order to compute the quantum Fisher information \[30, 31\]. Section 6 deals with spin vector representation for the three types of qubits. Finally, Section 7 contains a summary of our main conclusions.

### 1.1 The Model

In this paper, we will be considering the spin qubit system coupled to a bath of bosonic systems described by the following Hamiltonian \[32\]

\[
H = H_S \otimes I_B + I_S \otimes H_B + H_S \otimes V_B,
\]

where \(H_S\) denotes the system, \(H_B\) the bath

\[
H_B = \sum_k \omega_k a_k^\dagger a_k,
\]

and \(V_B\) the interaction term between the system and bath

\[
V_B = \sum_k \left( g_k a_k^\dagger + g_k^* a_k \right).
\]
Here $a_k^\dagger$ and $a_k$ denote bosonic creation and annihilation operators, respectively, $\omega_k$ are the eigenmodes of the bath and $g_k$ are the coupling constants. One of the main purposes of this paper is to compare the various features and properties between a Hermitian, $\mathcal{PT}$-symmetric and non-Hermitian qubit system, with particular focus on the anti-$\mathcal{PT}$-symmetric qubit.

### 1.2 Hermitian Qubit

To begin, let us review the dynamics for the Hermitian qubit as

$$H_S^h = \begin{pmatrix} \alpha + \theta & \xi + i\delta \\ -\xi - i\delta & -\alpha + \theta \end{pmatrix}, \quad (1.4)$$

with $\alpha, \xi, \delta, \theta \in \mathbb{R}$ to satisfy the Hermiticity condition $H_S^h = (H_S^h)^\dagger$. For convenience, we take a similarity transformation $T = \begin{pmatrix} \omega_0 - \alpha - \xi - i\delta \\ \omega_0 + \alpha & \xi + i\delta \end{pmatrix}$, where $\omega_0 = \sqrt{\alpha^2 + \delta^2 + \xi^2}$, to obtain a diagonalized Hamiltonian $H_S^{dh} = TH_S^h T^{-1}$. Subsequently, the evolved reduced density matrix for the system can be expressed in terms of the reduced density matrix for the diagonalized system with normalization as

$$\rho_S(t) = \frac{T^{-1} \rho_S^{dh}(t) (T^{-1})^\dagger}{\text{Tr}_S [T^{-1} \rho_S^{dh}(0) (T^{-1})^\dagger]} \quad (1.6)$$

and the diagonalized reduced density matrix being

$$\rho_S^{dh}(t) = \begin{pmatrix} \rho_{11}^{dh} & 0 \\ 0 & \rho_{22}^{dh} \end{pmatrix} + \begin{pmatrix} 0 & \rho_{12}^{dh} e^{2i\omega_0 t} e^{-i\omega_0 \Omega(t)} \\ \rho_{21}^{dh} e^{-2i\omega_0 t} e^{i\omega_0 \Omega(t)} & 0 \end{pmatrix} e^{-\omega_0^2 \gamma(t)}, \quad (1.7)$$

where

$$\Omega(t) = 4\theta \int_0^\infty dw J(w) \frac{wt - \sin(wt)}{w^2}, \quad (1.8)$$

$$\gamma(t) = 4 \int_0^\infty dw J(w) \frac{1 - \cos(wt)}{w^2} \coth \left( \frac{\beta w}{2} \right), \quad (1.9)$$

$$J(w) = \sum_k |g_k|^2 \delta(w - w_k), \quad (1.10)$$

are respectively the function influencing phase evolution, decoherence function and spectral density of the bath.

### 1.3 $\mathcal{PT}$-symmetric Qubit

For the $\mathcal{PT}$-symmetric case, which has been studied in [3], let us take the Hamiltonian as

$$H_S^{\mathcal{PT}} = \begin{pmatrix} \alpha + i\theta & \xi + i\delta \\ -\xi - i\delta & -\alpha - i\theta \end{pmatrix}, \quad (1.11)$$
where $\alpha, \xi, \delta, \theta \in \mathbb{R}$, $\delta^2 + \xi^2 \geq \theta^2$ and the parity and time operators to be

$$\mathcal{P} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{T} : i \rightarrow -i,$$  \quad (1.12)$$

which has the property $[\mathcal{P} \mathcal{T}, H^T_S] = 0$. Taking the same similarity transformation (1.5) as for the Hermitian system, but where

$$\omega_0 = \sqrt{\gamma^2 + \delta^2 - \theta^2},$$  \quad (1.13)$$

will lead to a diagonalized Hermitian Hamiltonian $h^T_S = T H^T_S T^{-1}$ and same evolved reduced density matrix (1.7), with the difference that $\Omega(t)$ is now $-\Omega(t)$.

1.4 Anti-$\mathcal{PT}$-symmetric Qubit

We now introduce an anti-$\mathcal{PT}$-symmetric quantum system of the general form

$$H_S = \begin{pmatrix} \alpha + i\theta & \xi + i\delta \\ -\xi + i\delta & -\alpha + i\theta \end{pmatrix},$$  \quad (1.14)$$

where $\alpha, \xi, \delta, \theta \in \mathbb{R}$ and can check that the anticommutation relation $\{\mathcal{P} \mathcal{T}, H_S\} = 0$ is satisfied taking the same parity and time operators (1.12). The total system $H$, is diagonalizable by again taking the similarity transformation (1.5) with

$$\omega_0 = \sqrt{\alpha^2 - \xi^2 - \delta^2}.$$  \quad (1.15)$$

The resulting diagonalized Hamiltonian will be given by

$$H^D = THT^{-1},$$

$$= (-\omega_0 \sigma_z + i\theta \mathbb{I}_S) \otimes \mathbb{I}_B + \mathbb{I}_S \otimes H_B + (-\omega_0 \sigma_z + i\theta \mathbb{I}_S) \otimes V_B.$$  \quad (1.16)$$

This is equivalent to the eigenbasis representation

$$THT^{-1} = \sum_n E_n |n\rangle \langle n|$$  \quad (1.17)$$

with $E_n \in \mathbb{C}$, which can be rewritten in a complete biorthonormal basis for $H$ \cite{34, 35, 36, 37}

$$H = T^{-1} \sum_n E_N |n\rangle \langle n| T,$$

$$= \sum_n E_N |\psi_n^R\rangle \langle \psi_n^L|,$$  \quad (1.18)$$

taking the set of eigenvectors $|\psi_n^R\rangle = T^{-1} |n\rangle$ and $|\psi_n^L\rangle = |n\rangle T$, satisfying the defining equations $\langle \psi_n^L| \psi_m^R\rangle = \delta_{nm}$ and $\sum_n |\psi_n^R\rangle \langle \psi_n^L| = \mathbb{I}$.

The pair of eigenvalues of the anti-$\mathcal{PT}$-symmetric system (1.14) is given by $E_\pm = i\theta \pm \omega_0$ with the difference of the two eigenvalues being $\Delta = 2\omega_0$. This represents the energy gap of our two-level system and its reality depends on the parameters $\alpha, \delta$ and $\xi$. When $\delta^2 + \xi^2 = \alpha^2$, the energy gap will be zero to give degenerate eigenvalues, but for this paper, we will be interested in the case when the energy gap is real, i.e. to study the parametric domain $\alpha^2 \geq \delta^2 + \xi^2$. 

4
For a general non-Hermitian Hamiltonian, $H$, it has been suggested in \[38, 39, 40\] that the Hamiltonian can also be viewed as a decomposition of real and imaginary parts $H = H_R + iH_I$ to give a complex extension of the Liouville-von Neumann equation

$$\rho_t = -i [H_R, \rho] + \{H_I, \rho\} - 2\rho Tr (\rho H_I),$$

(1.19)

which can be solved by the form

$$\rho(t) = \frac{U(t) \rho(0) U^\dagger(t)}{Tr [U(t) \rho(0) U^\dagger(t)]},$$

(1.20)

given the evolution operator $U(t) = e^{-iHt}$, then tracing out the bath degrees of freedom results in the reduced density matrix for the system. However, for non-Hermitian spin-boson models, calculations become quite involved. In what follows, we will present a scheme that makes computing the reduced density matrix of these systems more feasible.

## 2 Time-dependent Dyson Map for Density Matrix of a Non-Hermitian System

The key is to show that one can reformulate the non-Hermitian Hamiltonian problem in terms of a Hermitian one utilizing a time-dependent Dyson map \[41, 42, 43, 44, 45, 46, 47, 48, 49\]. Recently, this has been investigated for a $\mathcal{PT}$-symmetric bosonic system coupled to a bath of $N$ bosonic systems \[7\] and $\mathcal{PT}$-symmetric Jaynes-Cummings Hamiltonian \[8\]. Here, we shall present this method for a general non-Hermitian qubit system and in particular we will focus on the anti-$\mathcal{PT}$-symmetric case.

First, let us recall that for a Hermitian system with density matrix $\rho_{Dh}$, the Liouville-von Neumann equation is given by

$$i\dot{\rho}_{Dh} = [h^D, \rho_{Dh}],$$

(2.1)

In the non-Hermitian case, taking the diagonalized example $H^D$, we can write the Schrödinger equation and its conjugate transpose as

$$i\frac{\partial}{\partial t} |\psi_i\rangle = H^D |\psi_i\rangle,$$

(2.2)

$$-i\frac{\partial}{\partial t} \langle \psi_i | = \langle \psi_i | (H^D)^\dagger,$$

(2.3)

where, with $\rho^D = \sum_i P_i |\psi_i\rangle \langle \psi_i |$ being the density matrix, the corresponding non-Hermitian Liouville-von Neumann equation can be derived as

$$i\dot{\rho}_t^D = i \sum_i P_i \left( \frac{\partial}{\partial t} |\psi_i\rangle \langle \psi_i | + |\psi_i\rangle \frac{\partial}{\partial t} \langle \psi_i | \right),$$

(2.4)

$$= H^D \rho^D - \rho^D (H^D)^\dagger.$$

(2.5)
Let $H^D$ and $h^D$ be related by the Dyson relation

$$H^D = \eta^{-1}h^D \eta - i\eta^{-1}\eta_t,$$  \hfill (2.6)

where $\eta$ relates the states $|\phi_i\rangle$, $|\psi_i\rangle$ of the Hamiltonians $h^D$, $H^D$ respectively as $|\phi_i\rangle = \eta|\psi_i\rangle$. Substituting the Dyson relation into (2.5) and comparing with (2.1) gives the relation of density matrices between the Hermitian and non-Hermitian Hamiltonians as

$$\rho^{Dh} = \eta\rho^D \eta^\dagger.$$  \hfill (2.7)

Supposing $\rho^{Dh} = \sum_i P_i|\phi_i\rangle\langle\phi_i|$, we can check using the relation above that $\rho^D = \sum_i P_i|\psi_i\rangle\langle\psi_i|$, which shows the mapping of density matrices is able to preserve the set of probabilities $P_i$.

On the other hand, taking the Hermitian Liouville-von Neumann equation (2.1) and substituting again the Dyson relation (2.6), it can be shown

$$i\dot{\rho}^D = [H^D, \rho^D]$$  \hfill (2.8)

under the relation $\tilde{\rho}^D = \eta^{-1}\rho^{Dh}\eta = \rho^D M$, with $M = \eta^\dagger\eta$ being the metric such that for the quasi-Hermitian Hamiltonian

$$H^Q = H^D + i\eta^{-1}\eta_t,$$  \hfill (2.9)

it satisfies $\langle H^Q\psi_i|M\psi_i \rangle = \langle \psi_i|M H^Q\psi_i \rangle = \langle \phi_i|h^D\phi_i \rangle$ and $(H^Q)^\dagger M = MH^Q$. Looking at the anti-$\mathcal{PT}$-symmetric qubit of (1.16) which we now denote by $H^D_S$, we will see that the corresponding quasi-Hermitian qubit $H^Q_S$, is a two-level system with energies $E^Q_\pm = \pm\omega_0$, with same energy gap as $H^D_S$. It follows that $H^Q_S$ is interpreted as the physical operator that plays the role of energy for $H^D_S$ \cite{55, 56}, and $\tilde{\rho}^D$ is a Hermitian density matrix operator in the Hilbert space under the metric $M(t)$. We now find the corresponding Hermitian system from a time-dependent Dyson map.

Let us recall the diagonalized non-Hermitian Hamiltonian (1.16), if we take the Ansatz $\eta = e^{\epsilon(t)} e^{\phi(t)V_B}$ in the Dyson relation, $h^D$ becomes

$$h^D = -\omega_0\sigma_z + i\theta + H_B - \varphi(t)\widetilde{V}_B - \varphi^2(t)\Omega_k - \omega_0\sigma_z V_B + i\theta V_B + i\dot{\epsilon}(t) + i\dot{\varphi}(t) V_B,$$  \hfill (2.10)

where $\widetilde{V}_B = \sum_k \omega_k \left( g_k a_k^\dagger - g_k^* a_k \right)$ and $\Omega_k = \sum_k \omega_k |g_k|^2$. For $h^D$ to be Hermitian $(h^D)^\dagger = h^D$, the constraining equations are $\dot{\epsilon}(t) = -\theta$ and $\dot{\varphi}(t) = -\theta$, so we can take $\eta = e^{-\theta t} e^{-\theta V_B}$, then the corresponding Hermitian Hamiltonian is

$$h^D = -\omega_0\sigma_z + H_B - \omega_0\sigma_z V_B + \theta t\widetilde{V}_B - \theta^2 t^2\Omega_k.$$  \hfill (2.11)

Note we can also represent $h^D$ in terms of the quasi-Hermitian Hamiltonian $H^Q$ \cite{50, 51, 52}, as

$$h^D = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_G^{(n)} (H^Q),$$  \hfill (2.12)
with \( G = \theta (1 + V_B t) \) and denoting

\[
C_G^{(n)}(\mathcal{O}) = \underbrace{[G, [G, \cdots [G, \mathcal{O}]]}_{n}
\]

as the \( n \)-fold commutation for operators \( G \) and \( \mathcal{O} \), then in terms of the non-Hermitian Hamiltonian \( H^D \), \( h^D \) can be expressed as

\[
h^D = -i\theta (1 + V_B) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C_G^{(n)}(H^D).
\]

The remaining step now, is to find the density matrix of (2.11), which we will look at in the next section. Consequently, we also obtain the decoherence for our qubit system.

### 3 Decoherence of an Anti-\( \mathcal{PT} \)-symmetric Qubit

We shall consider the dynamics where the qubit is initially uncorrelated with a bath in thermal equilibrium i.e. the Gibbs state \( \Omega_B = e^{-\beta H_B} / Z \), where \( Z = Tr_B e^{-\beta H_B} \) is the partition function \cite{53}, so the initial density matrix of the total system becomes

\[
\rho^{Dh}(0) = \rho_{S}^{Dh}(0) \otimes \Omega_B,
\]

taking a general form for the qubit

\[
\rho_{S}^{Dh}(0) = \begin{pmatrix}
\rho_{11}^{Dh} & \rho_{12}^{Dh} \\
\rho_{21}^{Dh} & \rho_{22}^{Dh}
\end{pmatrix}
\]

(3.2)

with \( \sigma_\pm = \sigma_x \pm i\sigma_y \). The reduced system’s density matrix at time \( t \) is given by

\[
\rho_{S}^{Dh}(t) = \begin{pmatrix}
\frac{1}{2} (1 + \langle \sigma_z(t) \rangle) & \frac{1}{2} \langle \sigma_-(t) \rangle \\
\frac{1}{2} \langle \sigma_+(t) \rangle & \frac{1}{2} (1 - \langle \sigma_z(t) \rangle)
\end{pmatrix}
\]

(3.4)

and the expectation value of a general time evolved operator \( \mathcal{O} \) with Hermitian Hamiltonian \( h^D \) can be expressed as \( \langle \mathcal{O}(t) \rangle = Tr [\mathcal{O}(t) \rho (0)] \) where \( \mathcal{O}(t) = e^{i \int h^D dt} \mathcal{O}(0) e^{-i \int h^D dt} \), hence the time-dependent qubit operators are given by

\[
\sigma_z(t) = e^{i \int h^D(t) dt} \sigma_z e^{-i \int h^D(t) dt},
\]

(3.5)

\[
\sigma_\pm(t) = e^{i \int h^D(t) dt} \sigma_\pm e^{-i \int h^D(t) dt}.
\]

To proceed with computing the decoherence function, we want to find the expressions for the time-dependent qubit operators \( \sigma_z, \sigma_\pm \). To begin, consider the expression for time-dependent bath operator

\[
a_k(t) = e^{i \int h^D(t) dt} a_k e^{-i \int h^D(t) dt},
\]

(3.7)
by using their equations of motion respectively, which are solved by

$$ \frac{d}{dt} a_k (t) = -i \omega_k \left[ a_k (t) - \omega_0 g_k \frac{g_k}{\omega_k} \sigma_z + \theta g_k t \right], \quad (3.8) $$

and the time-dependent bath operator expression is given by

$$ a_k (t) = -\theta g_k t + e^{-i \omega_k t} \left[ a_k - A_k (t) \sigma_z + B_k (t) \right], \quad (3.9) $$

where

$$ A_k (t) = \omega_0 \frac{g_k}{\omega_k} \left( 1 - e^{i \omega_k t} \right) \quad \text{and} \quad B_k (t) = i \theta \frac{g_k}{\omega_k} \left( 1 - e^{i \omega_k t} \right). \quad (3.10) $$

Utilizing this expression, we can find the expressions for the time-dependent qubit operators similarly, by using their equations of motion respectively, which are solved by

$$ \sigma_z (t) = \sigma_z, \quad (3.11) $$

$$ \sigma_{\pm} (t) = e^{\mp 2i \omega_0 t} e^{\int_0^t \mp 2i \omega_0 \sum_k (g_k a_k^* (\tau) + g_k^* a_k (\tau)) d\tau} \sigma_{\pm}, \quad (3.12) $$

$$ = \sigma_{\pm} e^{\mp 2i \omega_0 t} e^{\mp i \theta \omega_0 \sum_k (g_k^2 + |g_k|^2) \int \frac{d\tau}{\omega_k} (1 - \cos (\omega_k \tau))} e^ {\mp \int \frac{d\tau}{\omega_k} \sum_k [2 A_k (t) a_k^* - 2 A_k^* (t) a_k]}, \quad (3.13) $$

where $e^{[\cdots]}$ is the time-ordered exponent

$$ e^{2i \omega_0 \int_0^t [1 + \sum G_k (\tau)] d\tau} = e^{2i \omega_0 \int_0^t [1 + \sum G_k (\tau)] d\tau} e^{-\omega_0 \int_0^t d\tau \int \frac{d\tau}{\omega_k} \sum_k [2 A_k (t) a_k^* - 2 A_k^* (t) a_k]} \quad (3.14) $$

$$ \langle \sigma_+ (t) \rangle = \langle \sigma_+ \rangle, \quad \langle \sigma_- (t) \rangle = \langle \sigma_- \rangle e^{\mp 2i \omega_0 t} e^{\pm i \omega_0 \int_0^t [2 A_k (t) - 2 A_k^* (t)] d\tau} \quad (3.15) $$

with $G_k (\tau)$ a time-dependent operator and $[[G_k (\tau_1), G_k (\tau_2)], G_k (\tau_3)]=0$. Now, we can express the reduced system’s density matrix at time $t$ (3.14), by computing the expectation values of $\sigma_{\pm} (t)$ and $\sigma_z (t)$

$$ \langle \sigma_z (t) \rangle = \langle \sigma_z \rangle, \quad \langle \sigma_\pm (t) \rangle = \langle \sigma_\pm \rangle e^{\mp 2i \omega_0 t} e^{\pm i \omega_0 \int_0^t [2 A_k (t) - 2 A_k^* (t)] d\tau} \quad (3.16) $$

in the continuum limit of bath modes. Here

$$ \tilde{J} (w) = \sum_k |g_k|^2 \left[ \cos (2 \theta_k) + i \sin (2 \theta_k) \right] \delta (w - w_k), \quad (3.18) $$

is the spectral density characterizing the environment with in general, a complex coupling constant $g_k = |g_k| e^{i \theta_k}$. Let us take as an example, $J (w) = J_0 w^{1+\mu} e^{-\frac{w}{\omega_c}}$ and $\tilde{J} (w) = J (w)$ with $g_k \in \mathbb{R}$. The resulting density matrix of the reduced system at time $t$ is given by
\[ \rho_{S}^{Dh}(t) = \left( \begin{array}{cc} \rho_{11}^{Dh} & 0 \\ 0 & \rho_{22}^{Dh} \end{array} \right) + \left( \begin{array}{cc} 0 & \rho_{12}^{Dh} e^{2i\omega_0 t} e^{i\omega_0 [\Omega_2(t) - \Omega_1(t)]} \\ \rho_{21}^{Dh} e^{-2i\omega_0 t} e^{-i\omega_0 [\Omega_2(t) - \Omega_1(t)]} & 0 \end{array} \right) e^{-\omega_0^2 \gamma(t)} \] (3.19)

and the decoherence function reads

\[ D(t) = e^{-\omega_0^2 \gamma(t)}, \] (3.20)

which quantifies the loss of quantum information to the environment. As this function evolves with time, we are able to see that the anti-\(\mathcal{PT}\)-symmetric (1.14) qubit decays more gradually compared to a Hermitian (1.4) or \(\mathcal{PT}\)-symmetric (1.11) qubit, as shown in Figure 1.

![Figure 1](image-url)

Figure 1: Comparison of the evolution of decoherence function with anti-\(\mathcal{PT}\)-symmetric (APT), \(\mathcal{PT}\)-symmetric (PT) and Hermitian (H) systems, with parameters \(J_0 = w_c = 1\), \(\beta = 0.5\), \(\delta = 0.56\), \(\mu = -0.5\), \(\xi = 0.81\), \(\theta = 0.86\) and \(\alpha = 1\).

There are four real parameters in our qubits, \(\alpha\), \(\delta\), \(\theta\) and \(\xi\). For the Hermitian case, \(\alpha\), \(\delta\) and \(\xi\) will contribute to increasing the decoherence. For a \(\mathcal{PT}\)-symmetric qubit, the parameters \(\delta\), \(\xi\) bring an increase and \(\theta\), a decrease of decoherence. But for our example of an anti-\(\mathcal{PT}\)-symmetric qubit, we only have one parameter, \(\alpha\) participating in increasing; the other two, \(\delta\) and \(\xi\), help to reduce decoherence.

## 4 Entanglement Entropy

Let us now study the entanglement entropy, which is a common measure of entanglement and quantum information. In particular, for our system, this will be a measure of entanglement between our qubit and environment. To do this, let us rewrite the standard representation of the reduced system’s density matrix at time \(t\) \[54, 55\]

\[ \rho_{S}^{Dh}(t) = \frac{1}{2} [1 + \vec{v}(t) \vec{\sigma}] \] (4.1)

following \[56\] in exponential form

\[ \rho_{S}^{Dh}(t) = \frac{1}{2} \sqrt{1 - v(t)^2} e^{\vec{v}(t) \cdot \vec{\sigma}}, \] (4.2)
where \( v(t) = |\vec{v}(t)| \) and \( \vec{u}(t) = \frac{\vec{v}(t)}{2v(t)} \ln \left[ \frac{1+v(t)}{1-v(t)} \right] \). The (von Neumann) entropy can consequently be calculated as

\[
S(t) = -Tr_S \left[ \rho_S^{Dh}(t) \ln \rho_S^{Dh}(t) \right],
\]

\[
= \ln 2 - \frac{1}{2} [1 + v(t)] \ln [1 + v(t)] - \frac{1}{2} [1 - v(t)] \ln [1 - v(t)].
\]

Now taking the initial state of the reduced system (3.3) as a pure state, we have \( Tr \left[ \left( \rho_S^{Dh} \right)^2 (0) \right] = 1 \), then if we choose equal populations of ground and excited state initially, i.e. \( \rho_{11}^{Dh} = \rho_{22}^{Dh} \),

\[
v(t) = e^{-\omega_0^2 \gamma(t)}.
\]

The corresponding entropy becomes

\[
S(t) = \ln 2 - \frac{1}{2} \left[ 1 + e^{-\omega_0^2 \gamma(t)} \right] \ln \left[ 1 + e^{-\omega_0^2 \gamma(t)} \right] - \frac{1}{2} \left[ 1 - e^{-\omega_0^2 \gamma(t)} \right] \ln \left[ 1 - e^{-\omega_0^2 \gamma(t)} \right].
\]

At \( t = 0 \), \( S(0) = 0 \) and as \( t \to \infty \), \( S(\infty) = \ln 2 \), as shown in Figure 2. What is also evident is that the entropy with anti-\( \mathcal{PT} \)-symmetric qubit increases more gradually compared with Hermitian or \( \mathcal{PT} \)-symmetric qubits. This result shows the anti-\( \mathcal{PT} \)-symmetric qubit entangles with the environment at a slower rate compared to Hermitian or \( \mathcal{PT} \)-symmetric qubits, which is equivalent to the ability to preserve quantum information for a longer time.

![Figure 2: Comparison of the time evolution of von Neumann entropy with anti-\( \mathcal{PT} \)-symmetric (APT), \( \mathcal{PT} \)-symmetric (PT) and Hermitian (H) systems, with parameters \( J_0 = \omega_c = 1 \), \( \beta = 0.5 \), \( \delta = 0.56 \), \( \mu = -0.5 \), \( \xi = 0.81 \), \( \theta = 0.86 \) and \( \alpha = 1 \).](image)

An extension of the von Neumann entropy is the quantum Rényi entanglement entropy [57, 58] through an introduction of a parameter \( r \). One interesting connection is the Rényi entanglement entropy’s close relation with free energy [59]. In our case, the parameter \( r \) gives a nonlinear dependence of entropy on the reduced density matrix and is given as

\[
S_r(r,t) = \frac{\ln Tr \left\{ [\rho_S^{Dh}(t)]^r \right\}}{1-r}
\]

\[
= \frac{r \ln 2}{1-r} + \frac{1}{1-r} \ln \left\{ [1 + e^{-\omega_0^2 \gamma(t)}]^r + [1 - e^{-\omega_0^2 \gamma(t)}]^r \right\}.
\]
Taking the ratio of Rényi entanglement entropy with von Neumann entropy at a fixed time for the three types of qubits with respect to varying the parameter \( r \) on the left of Figure 3, we can clearly see that in the limit of \( r \to 1 \), this reduces back to the von Neumann entropy. On the right of Figure 3, we depict the time evolution of Rényi entanglement entropy for the anti-\( \mathcal{PT} \) qubit under varying parameter \( r \).

![Figure 3: Ratio of Rényi entanglement entropy and von Neumann entropy at time 1.25 for anti-\( \mathcal{PT} \)-symmetric (APT), \( \mathcal{PT} \)-symmetric (PT) and Hermitian (H) systems (left) and evolution of Rényi entanglement entropy for the anti-\( \mathcal{PT} \)-symmetric case (right), under the parameter choices \( J_0 = w_c = 1, \beta = 0.5, \delta = 0.2, \mu = -0.5, \xi = 0.12, \theta = 0.05 \) and \( \alpha = 0.3 \).]

To summarize, through analysis of Rényi entanglement entropy and von Neumann entropy, results show the anti-\( \mathcal{PT} \)-symmetric qubit is a better choice to use in quantum computing than the Hermitian or \( \mathcal{PT} \)-symmetric qubit. We note that this calculation complements the analysis of decoherence function.

5 Fisher Information

Quantum Fisher information is an important quantity in quantum metrology [60]. It quantifies the precision that can be achieved in estimating a parameter for a given quantum state. Thus, it can be regarded as a measure of reliability of a quantum system. A higher value of Fisher information equates to higher precision of estimating a parameter. One can compute from the relative entropy (i.e. the Kullback-Leibler divergence [30, 31])

\[
\begin{align*}
D_{KL} (K, t) &= \text{Tr} \left[ \rho_S^{\text{Ph}} (\tilde{K}, t) \ln \rho_S^{\text{Ph}} (\tilde{K}, t) \right] - \text{Tr} \left[ \rho_S^{\text{Ph}} (K, t) \ln \rho_S^{\text{Ph}} (K, t) \right], \\
&= \frac{1}{2} \left\{ \ln \frac{1 - v^2 (\tilde{K}, t)}{1 - v^2 (K, t)} + v (\tilde{K}, t) \ln \frac{1 + v (\tilde{K}, t)}{1 - v (K, t)} \right\},
\end{align*}
\]

the Fisher entropy with respect to (the inverse) temperature parameter, \( K = \beta \) as
\[ S_f(\beta, t) = \frac{\partial^2}{\partial \beta^2} D_{KL}(\tilde{\beta}, t) \bigg|_{\beta=\beta} \]
\[ = \frac{\omega_0^4}{2} \left\{ \coth \left[ \omega_0^2 \gamma(\beta, t) \right] - 1 \right\} \left[ \frac{\partial}{\partial \beta} \gamma(\beta, t) \right]^2, \]

From the left of Figure 4 we see that although the maximum Fisher information for the three types of qubit are roughly equal, the anti-\(PT\)-symmetric case is still slightly higher, as shown in Table 1. However, the variance and area of Fisher information are visibly much larger for the anti-\(PT\)-symmetric qubit. This may be interpreted that we need a larger interval of time in estimating \(\beta\) accurately.

![Figure 4](image-url)

Figure 4: Comparison of the Fisher entropy with respect to inverse temperature \(\beta\) (left) and with respect to the combined qubit parameter \(\omega_0\) (right) for anti-\(PT\)-symmetric (APT), \(PT\)-symmetric (PT) and Hermitian (H) Hamiltonians, with parameters \(J_0 = w_c = 1, \mu = -0.5, \beta = \delta = 0.5, \xi = 0.8, \theta = 0.6\) and \(\alpha = 1\).

Similarly, the Fisher entropy depending on what we shall call the ‘combined qubit parameter’ \(\omega_0\) can be computed to be

\[ S_f(\omega_0, t) = \frac{\partial^2}{\partial \omega_0^2} D_{KL}(\tilde{\omega}_0, t) \bigg|_{\tilde{\omega}_0=\omega_0} \]
\[ = 2\omega_0^2 \left\{ \coth \left[ \omega_0^2 \gamma(\omega_0, t) \right] - 1 \right\} \gamma^2(\omega_0, t). \]

The right of Figure 4 clearly shows that both the maximum and area of Fisher information with respect to \(\omega_0\) for the anti-\(PT\)-symmetric qubit are much greater compared with the Hermitian and \(PT\)-symmetric cases. In particular, this represents that a higher accuracy can be obtained in measuring the parameter \(\omega_0\).

In Table 1 we compute numerically the maxima, when they occur and total areas of Fisher information.
Table 1: Comparison of the Fisher information data with respect to inverse temperature $\beta$ and the combined qubit parameter $\omega_0$ for anti-$\mathcal{PT}$-symmetric, $\mathcal{PT}$-symmetric and Hermitian Hamiltonians, with parameters $J_0 = w_c = 1, \mu = -0.5, \beta = \delta = 0.5, \xi = 0.8, \theta = 0.6$ and $\alpha = 1$.

From the Table, we can clearly see that the Fisher information maximum and total area for the anti-$\mathcal{PT}$-symmetric qubit are much larger compared with the other two qubits. In addition, the maxima times do not seem to be much affected by the choice of parameters.

### 6 Spin Vector Representation

Considering that our investigation is of a two-level system, the dynamics is naturally best revealed with the spin vector representation. Let us start by taking the reduced density matrices of our systems with $g_K \in \mathbb{R}$,

$$\rho_S^{\text{ph}}(t) = \frac{1}{2} \left( \begin{array}{cc} 1 + S_z & S_x(t) - iS_y(t) \\ S_x(t) + iS_y(t) & 1 - S_z \end{array} \right).$$

we can obtain the spin vector representation with decoherence as

$$\vec{S}(t) = \left( S_x(t), S_y(t), S_z(t) \right) = \left( \sin \theta_0 \cos \left[ \phi(t) \right] D(t), \sin \theta_0 \sin \left[ \phi(t) \right] D(t), \cos \theta_0, \right),$$

where the phase is given by

$$\phi(t) = \phi_0 - 2\omega_0 t + \omega_0 \tilde{\Omega}(t),$$

$\omega_0$ is the combined qubit parameter

$$\omega_0 = \begin{cases} \sqrt{\alpha^2 + \delta^2 + \xi^2} & \text{(Hermitian)} \\ \sqrt{\gamma^2 + \delta^2 - \theta^2} & \text{($\mathcal{PT}$-symmetric)} \\ \sqrt{\alpha^2 - \delta^2 - \xi^2} & \text{(anti-$\mathcal{PT}$-symmetric)} \end{cases},$$

and $\tilde{\Omega}(t)$ is a time-dependent function in the phase

$$\tilde{\Omega}(t) = \begin{cases} \Omega(t) & \text{(Hermitian)} \\ -\Omega(t) & \text{($\mathcal{PT}$-symmetric)} \\ \Omega_2(t) - \Omega_1(t) & \text{(anti-$\mathcal{PT}$-symmetric)} \end{cases}.$$
the Hermitian, $\mathcal{PT}$-symmetric and anti-$\mathcal{PT}$-symmetric qubit systems. We take $\phi_0 = 0$ without losing generality and draw the dynamics at different initial angles $\theta_0$, with respect to the $S_z$ axis.

\[
\begin{align*}
\theta_0 &= \pi/8 \\
\theta_0 &= 2\pi/8 \\
\theta_0 &= 7\pi/16
\end{align*}
\]

Figure 5: Spin vector representation for the evolution of the Hermitian (top left), $\mathcal{PT}$-symmetric (top right) and anti-$\mathcal{PT}$-symmetric (bottom) qubits at different initial $\theta_0$, with parameters $J_0 = w_c = 1$, $\mu = -0.5$, $\beta = 0.5$, $\delta = 0.217$, $\xi = 0.45$, $\theta = 0.488$ and $\alpha = 0.5$.

The evolution starts furthest away from the center of the Bloch sphere, then spirals inwards. Initially, all three cases start evolving clockwise, but the Hermitian and anti-$\mathcal{PT}$-symmetric cases change to anticlockwise evolution after a short time. This is visibly noticeable for the anti-$\mathcal{PT}$-symmetric case, but not so easy to see for the Hermitian case, because the speed at which the trajectory reaches the center is much greater than the time it takes to change direction. However, we can take the derivative of phase (6.3) with respect to time and obtain the spin angular velocity, which we plot for the three types of qubits after normalization, on the left of Figure 6. From the figure, we can see that taking the change of evolution direction clockwise as positive, so when the lines go into the negative region,
the trajectories are evolving anticlockwise. What is also noticeable is that the angular speed for the anti-\(\mathcal{PT}\)-symmetric case seems to increase at a faster rate compared with the other two qubit cases.

Another feature to note from the spin representation is the distance of evolution trajectory from the \(z\)-axis

\[
d(t) = \sqrt{S_x^2(t) + S_y^2(t)}
\]

\[
= \sin \theta_0 e^{-\omega_{00}^2 \gamma(t)},
\]

depicted by the center plot of Figure 6. The anti-\(\mathcal{PT}\)-symmetric distance decreases at a substantially slower rate compared to the other two qubit systems. This plays a large role in the spin linear velocity, which is given by

\[
V_L(t) = d(t) \frac{\partial}{\partial t} \phi(t).
\]

Drawn against time for the three qubits in the right panel of Figure 6 we notice a similar property of a more gradual decay, but for the linear velocity in the anti-\(\mathcal{PT}\)-symmetric case.

Figure 6: Spin angular velocity (left), distance from the \(z\)-axis (center) and spin linear velocity (right) for the Hermitian (H), \(\mathcal{PT}\)-symmetric (PT) and anti-\(\mathcal{PT}\)-symmetric (APT) qubits with parameters \(J_0 = w_c = 1, \mu = -0.5, \beta = 0.5, \delta = 0.38, \xi = 0.8, \theta = 0.6\) and \(\alpha = 0.9\) and \(\theta_0 = \frac{3\pi}{8}\).

7 Conclusions

We have studied the decoherence and entanglement (via von Neumann, Rényi and Fisher entropy) properties of an anti-\(\mathcal{PT}\) qubit comprising a two-level spin system. To this end, we utilized the time-dependent Dyson map to transform our problem of computing the reduced density matrix from a non-Hermitian to a more feasible Hermitian one. We found superior decoherence properties as compared to the \(\mathcal{PT}\)-symmetric and Hermitian qubits. We also found a slower growth for the entanglement entropy and much higher Fisher information for the anti-\(\mathcal{PT}\) qubit. These findings suggest the utility of anti-\(\mathcal{PT}\) qubits for quantum information processing and storage. It would be desirable to have a possible experimental realization (e.g. in optical waveguides [14] and microcavity systems [27]) of the anti-PT-symmetric qubit to observe the predicted superior properties.
Acknowledgments

We thank Bartłomiej Gardas and Aurélia Chenu for insightful discussions. This work was supported by the U.S. Department of Energy.

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