Nambu-Jona-Lasinio model correlation functions from QCD

Marco Frasca¹, Anish Ghoshal², Stefan Groote³

¹Via Erasmo Gattamelata, 3, 00176 Rome (Italy)
²INFN, Rome, Italy and Warsaw University, Poland
³University of Tartu, Estonia

Abstract

We treat quantum chromodynamics (QCD) using a set of Dyson-Schwinger equations derived, in differential form, with the Bender-Milton-Savage technique. In this way, we are able to derive the low energy limit that assumes the form of a non-local Nambu-Jona-Lasinio model with all the parameters properly fixed by the QCD Lagrangian and the determination of the mass gap of the gluon sector.

Keywords:

1. Introduction

Adding quarks to the Yang-Mills Lagrangian makes the theory not exactly treatable. Notwithstanding such a difficulty, full QCD can be handled with Dyson-Schwinger equations even if some approximations are needed to get the low-energy limit. Such approximations entail both the strong coupling limit and 't Hooft limit \( N \to \infty, N g^2 = constant \gg 1 \) [4, 5]. In this way, we will recover, in the low-energy limit, the equations for the correlation functions of a non-local NJL model.

Our idea is to provide a method to derive the full hierarchy of Dyson-Schwinger equations also for QCD, retaining their full differential form. This is possible provided we use a technique devised by Bender, Milton and Savage [3]. Some approximations are needed to get the low-energy limit. Such approximations entail both the strong coupling limit and 't Hooft limit \( N \to \infty, N g^2 = constant \gg 1 \) [4, 5]. In this way, we will recover, in the low-energy limit, the equations for the correlation functions of a non-local Nambu-Jona-Lasinio model.

This is an interesting result in view of the fact that it permits to recover from an error present in Ref. [6] granting anyway the conclusions. This has been recently applied to the \( g - 2 \) problem [7] to evaluate the hadronic vacuum polarization for the \( \pi\pi \) contribution.

2. Bender-Milton-Savage technique

This technique can be better explained referring to a scalar field. We consider the following partition function

\[
Z[j] = \int [D\phi] e^{S(\phi) + i \int d^4x j(x) \phi(x)}.
\]

To derive 1P-function, one has

\[
\frac{\delta S}{\delta \phi(x)} = j(x),
\]
assuming
\[ \langle \ldots \rangle = \frac{\int [D\phi] \ldots e^{iS(\phi) + \int d^4x j(x) \phi(x)}}{\int [D\phi] e^{iS(\phi) + \int d^4x j(x) \phi(x)}} \] (3)

By setting \( j = 0 \) one obtains the equation for the 1P-function. Next, we derive this equation again with respect to \( j \) to get the equation for the 2P-function. We are taking for the nP-functions the following definition
\[ \langle \phi(x_1)\phi(x_2)\ldots\phi(x_n) \rangle = \frac{\delta^n \ln[Z(j)]}{\delta j(x_1)\delta j(x_2)\ldots\delta j(x_n)} \] (4)

This implies
\[ \frac{\delta G_k(\ldots)}{\delta j(x)} = G_{k+1}(\ldots,x). \] (5)

Such a procedure can be iterated to whatever order giving, in principle, all the hierarchy of the Dyson-Schwinger equations in PDE form. Going to higher orders could imply complicated computations but this approach shows itself to be very useful when some known solutions are given for 1P- and 2P-functions as in our case.

3. 1P and 2P functions for QCD

We work in the Landau gauge as can make some computations simpler as seen in \[1\].

The Bender-Milton-Savage method yields for the 1P-functions
\[ \delta^2G^a_{1P}(x) + gf^{abc}(\delta^2G^b_{2P}(0) + \delta^2G^c_{1P}(x) - \partial_i G^b_{2P}(0)) - \partial_i G^b_{1P}(x)G^c_{1P}(x)) \]
\[ + g f^{abc}\delta^i G^c_{2P}(0) + g f^{abc}\partial_i(G^b_{1P}(x)G^c_{1P}(x)) \]
\[ + g f^{abc}g^{cdef}G^d_{2P}(0,0)G^e_{2P}(0)G^f_{1P}(x) \]
\[ + G^{ab}_{2P}(0)G^a_{1P}(x) + G^{ab}_{2P}(0)G^a_{1P}(x) + G^{ab}_{1P}(x)G^{cde}_{1P}(x) \]
\[ = g \sum_{q,i,j} \gamma_i T^a S^{ij}_{q}(0) + g \sum_{q,i} \tilde{q}^i(x)\gamma_i T^{\mu}_{q}(x) \] (6)

and for the quarks
\[ (i\partial - m_q)q^i_{1P}(x) + gT\cdot G^1_{1P}(x)q^i_{1P}(x) + gT\cdot W^0_{q}(x,x) = 0 \] (7)

Here and in the following Greek indexes (\( \mu, \nu, \ldots \)) are for the space-time and Latin index (\( a, b, \ldots \)) for the gauge group. A usual for the Dyson-Schwinger set, the equations for the lower order nP-functions depend on the higher order correlation functions. We will see how to treat this aspect in the following.

At this stage, we apply the re-mapping idea to such equations as done in \[1\]. So, we assume
\[ G^{a}_{1P}(x) \rightarrow \eta^{a}_{\mu,\nu}\phi(x) \] (8)

being \( \phi(x) \) a scalar field. Let us introduce the \( \eta \)-symbols as follows
\[ \eta^{\mu}_{\nu} = N^2 - 1, \]
\[ \eta^{\mu}_{\nu} \eta^{\lambda}_{\rho} = \delta_{\lambda,\rho}, \]
\[ \eta^{\mu}_{\nu}q^i (x) = \left( \delta_{\mu,\rho} - \delta_{\mu,\nu} \right)/2. \] (9)

All this permits to get the reduced equations
\[ \delta^2 \phi(x) + 2N g^2 \Delta(0)\phi(x) + N g^2 \phi^{3}(x) \]
\[ = \frac{1}{N^2 - 1} \left[ g \sum_{q,i,j} \eta^{\mu}_{\nu}\gamma_i T^\mu S^{ij}_{q}(0) \right. \]
\[ + g \sum_{q,i} \tilde{q}^i(x)\eta^{\mu}_{\nu}\gamma_i T^{\mu}_{q}(x) \]
\[ (i\partial - m_q)q^i_{1P}(x) + gT\cdot \eta^{\mu}_{\nu}\phi(x)q^i_{1P}(x) = 0. \] (10)

We do the same for the 2P-functions. In the Landau gauge, the gluon 2P-function takes the form
\[ G^{ab}_{2P}(x-y) = \left( \eta^{\mu}_{\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\partial^2} \right) \Delta_{q}(x-y) \] (11)

being \( \eta_{\mu\nu} \) the Minkowski metric, and \( \Delta_{q}(x-y) \) is the propagator of the \( \phi \) given the map between the scalar and the Yang-Mills fields. Finally, we can write
\[ \delta^2 \Delta_{q}(x-y) + 2N g^2 \Delta_{q}(0)\Delta_{q}(x-y) + 3N g^2 \phi^{2}(x)\Delta_{q}(x-y) \]
\[ = g \sum_{q,i,j} \tilde{Q}^{ij}_{q}(x-y)\gamma_i T^{\mu} S^{ij}_{q}(0) \]
\[ + g \sum_{q,i} \tilde{q}^i(x)\gamma_i T^{\mu}_{q}(x) \]
\[ (i\partial - m_q)\tilde{S}^{ij}_{q}(x-y) \]
\[ + g T\cdot \phi(x)S^{ij}_{q}(x-y) = \delta_{ij}\delta^2(x-y) \]
\[ \delta^2 W^{a}_{q}(x-y) + 2N g^2 \Delta_{q}(0)W^{a}_{q}(x-y) + 3N g^2 \phi^{2}(x)W^{a}_{q}(x-y) \]
\[ = g \sum_{q,i} \tilde{q}^i(x)\gamma_i T^{\mu} S^{ij}_{q}(x-y) \]
\[ (i\partial - m_q)\tilde{Q}^{ij}_{q}(x-y) + g T\cdot \phi(x)\tilde{Q}^{ij}_{q}(x-y) \]
\[ + g T^{\mu}\gamma_i \Delta_{q}(x-y)q^i_{1P}(x) = 0. \] (12)
4. 't Hooft limit

't Hooft limit means to solve the theory assuming \[4, 5\]

\[ N \to \infty, \quad N g^2 = \text{constant}, \quad N g^2 \gg 1. \quad (13) \]

In our case, the gauge group is SU(N) being \( N \) is the number of colors. To evaluate our equations in such a limit, we need a perturbation series for a very large coupling. We proposed such a technique in Ref.\[8\]. We do a rescaling, \( x \to \sqrt{N g^2} \), and write the equation for the gluon field as follows

\[ \partial^2 \phi(x') + 2 \Delta_0(0) \phi(x') + 3 \phi^3(x') = \frac{1}{\sqrt{N g^2 \sqrt{N(N^2 - 1)}}} \left[ \sum_{q,i} \eta \cdot \gamma \cdot TS_{q}^{(0)}(0) + \sum_{q,i} \tilde{q}_i(x') \eta \cdot \gamma \cdot T q_i(x') \right]. \quad (14) \]

In the 't Hooft limit, we get at the leading order for the 1P-functions

\[ \partial^2 \phi_0(x) + 2 N g^2 \Delta_0(0) \phi_0(x) + 3 N g^2 \phi_0^3(x) = 0, \]

\[ (i \partial - m_q') \tilde{q}_i(x) + g T \cdot \gamma \phi(x) q_i(x) = 0. \quad (15) \]

At the leading order the only effect is seen on masses. For the quark field, this will be clearer below. We can solve the equation for the gluon field taking

\[ \phi_0(x) = \sqrt{\frac{2 \mu^4}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}}} \times \]

\[ \eta (p \cdot x + \gamma, \gamma), \quad (16) \]

being \( \eta \) a Jacobi elliptical function, \( \mu \) and \( \chi \) arbitrary integration constants and \( m^2 = 2 N g^2 \Delta_0(0) \). We have

\[ \kappa = \frac{-m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}}{-m^2 - \sqrt{m^4 + 2 N g^2 \mu^4}}. \]

This is true provided that the following dispersion relation holds

\[ p^2 = m^2 + \frac{N g^2 \mu^4}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}}. \]

For the equations of the 2P-functions one has

\[ \partial^2 \Delta_0(x, y) + 2 N g^2 \Delta_0(0) \Delta(x - y) + 3 N g^2 \phi_0^3(x) \Delta_0(x - y) \]

\[ = g \sum_{q,i} \tilde{Q}_i(x, y) \gamma T q_i(x) \]

\[ + g \sum_{q,i} \tilde{q}_i(x) \gamma T q_i(x, y) + \delta^3(x - y) \]

\[ \partial^2 W_0^{\mu}(x, y) + 2 N g^2 \Delta_0(0) W_0^{\mu}(x, y) + 3 N g^2 \phi_0^3(x) W_0^{\mu}(x, y) \]

\[ = g \sum_{q,i} \tilde{q}_i(x) \gamma T \tilde{q}_i(x, y) \]

\[ \times T \cdot \gamma \phi(x) q_i(x, y) = 0 . \quad (19) \]

To solve these equations, let us consider

\[ \partial^2 \Delta_0(x, y) + [m^2 + 3 N g^2 \phi_0^2(x)] \Delta_0(x - y) = \]

\[ \delta^3(x - y). \]

(20)

In momenta space, the solution of this equation is given by \[11, 13\]

\[ \Delta_0(p) = M \tilde{Z}(\mu, m, N g^2) \frac{2 \mu^4}{K^3(\kappa)} \times \sum_{n=0}^{\infty} \left( \frac{(-1)^n \cdot \eta 2^{(n+1)} \kappa^{(n+1)}}{1 - e^{-(2n+1) \mu \kappa}} \right) \times \]

\[ (2n + 1)^2 \frac{1}{p^2 - m^2 + i \epsilon} \quad (21) \]

being

\[ M = \sqrt{m^2 + \frac{N g^2 \mu^4}{m^2 + \sqrt{m^4 + 2 N g^2 \mu^4}}}, \]

and \( \tilde{Z}(\mu, m, N g^2) \) a given constant. The spectrum is given by \( m_q \) and a proper gap equation \[2\].

5. QCD in the low energy limit

Using the technique devised in \[8\], the next-to-leading order term is given by

\[ \phi_1(x) = g \frac{1}{N^2 - 1} \int d^4 x' \Delta_0(x - x') \left[ \sum_{q,i} \eta \cdot \gamma \cdot T S_{q}^{(0)}(0) + \sum_{q,i} \tilde{q}_i(x') \eta \cdot \gamma \cdot T q_i(x') \right]. \quad (23) \]

Given equation for the quark 1P-function

\[ (i \partial - m_q) q_i(x) + g T \cdot \gamma \phi(x) q_i(x) = 0, \]

one has

\[ \partial^2 \tilde{q}_i(x) + g T \cdot \gamma \phi(x) q_i(x) \times \]

\[ g^2 \frac{1}{N^2 - 1} \int d^4 x' \Delta_0(x - x') \sum_{q,i} \tilde{q}_i(x') T \cdot \gamma q_i(x') = 0. \quad (25) \]
't Hooft limit implies that the $\phi_0$ term can be neglected with respect to the second one and we get

$$
(i\partial - m_q)\bar{q}_i'(x) + g^2\frac{1}{N^2-1} \int d^4 x' \Delta_0(x-x') \sum_{q,k} \bar{q}_k' T \cdot \bar{q}_k q_i'(x') \times T \cdot \bar{q}_j q_i'(x) = 0. \quad \text{(26)}
$$

For the quark propagator one has instead

$$
(i\partial - m_q)S_q^{ij}(x-y) + g T \cdot \bar{q}_i(x) S_q^{ij}(x-y) = \delta_{ij} \delta^4(x-y) \quad \text{(27)}
$$

therefore

$$
(i\partial - m_q)S_q^{ij}(x-y) + g T \cdot \bar{q}_i(x) S_q^{ij}(x-y) + g^2 \frac{1}{N^2-1} \int d^4 x' \Delta_0(x-x') \sum_{q,k} \bar{q}_k' T \cdot \bar{q}_k q_i'(x') \times T \cdot \bar{q}_j q_i'(x) = \delta_{ij} \delta^4(x-y). \quad \text{(28)}
$$

Again, by the 't Hooft limit we can neglect the $\phi_0$ term with respect to the second one and we get

$$
(i\partial - m_q)S_q^{ij}(x-y) + g T \cdot \bar{q}_i(x) S_q^{ij}(x-y) + g^2 \frac{1}{N^2-1} \int d^4 x' \Delta_0(x-x') \sum_{q,k} \bar{q}_k' T \cdot \bar{q}_k q_i'(x') \times T \cdot \bar{q}_j q_i'(x) = \delta_{ij} \delta^4(x-y). \quad \text{(29)}
$$

We can recognize here the equations for the 1P- and 2P-functions of a non-local Nambu-Jona-Lasinio model. These are not generally treatable. They should be solved straightforwardly being already quantum averaged. In order to obtain a gap equation, we need to recover the Nambu-Jona-Lasinio-model from which they can be obtained doing some kind of backtracking. Only in this way a gap equation is derived. Indeed, such a model has the Lagrangian

$$
L_{N\bar{L}} = \sum_{i,q} \left[ \bar{q}_i(x) (i\partial - m_q) q_i(x) + \frac{g^2}{N^2-1} \int d^4 x' \Delta_0(x-x') \sum_{k,q} \bar{q}_k'(x') T \cdot \bar{q}_k q_i'(x') T \cdot \bar{q}_j q_i'(x) \right]. \quad \text{(30)}
$$

From this, one can get a quark gap equation that is identical to the one given in [6] as proven in [7].

6. Conclusions

We derived the set of Dyson-Schwinger equations, till to 2P-functions, for QCD with the Bender-Milton-Savage technique. We treated them in the 't Hooft limit. The low-energy limit is a nonlocal-Nambu-Jona-Lasinio model. This was shown obtaining the corresponding 1P- and 2P-equations for its correlation functions. The Lagrangian of the model is also given.

Work is ongoing to get the gap equation and to analyze its properties in view of recent g-2 Fermilab measurement [7].

Acknowledgements

The research was supported in part by the European Regional Development Fund under Grant No. TK133.

References

[1] M. Frasca, Eur. Phys. J. Plus 132, no.1, 38 (2017) [erratum: Eur. Phys. J. Plus 132, no.5, 242 (2017)] doi:10.1140/epjpj2017-11321-4 [arXiv:1509.05292 [math-ph]].

[2] M. Frasca, Nucl. Part. Phys. Proc. 294-296, 124-128 (2018) doi:10.1016/j.nuclphysbps.2018.02.005 [arXiv:1708.06184 [hep-ph]].

[3] C. M. Bender, K. A. Milton and V. Savage, Phys. Rev. D 62, 085001 (2000) doi:10.1103/PhysRevD.62.085001 [arXiv:hep-th/9907045 [hep-th]].

[4] G. ’t Hooft, Nucl. Phys. B 72, 461 (1974) doi:10.1016/0550-3213(74)90154-0.

[5] G. ’t Hooft, Nucl. Phys. B 75, 461-470 (1974) doi:10.1016/0550-3213(74)90088-1.

[6] M. Frasca, Eur. Phys. J. C 80, no.8, 707 (2020) doi:10.1140/epjc/s10052-020-8261-7 [arXiv:1901.08124 [hep-ph]].

[7] M. Frasca, A. Ghoshal and S. Groote, [arXiv:2109.05041 [hep-ph]].

[8] M. Frasca, Eur. Phys. J. C 74, 2929 (2014) doi:10.1140/epjc/s10052-014-2929-9 [arXiv:1306.6530 [hep-ph]].