PIEZOELECTRIC BEAMS WITH MAGNETIC EFFECT AND LOCALIZED DAMPING

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Abstract. In this work we are considering a one-dimensional dissipative system of piezoelectric beams with magnetic effect and localized damping. We prove that the system is exponential stable using a damping mechanism acting only on one component and on a small part of the beam.

1. Introduction. Discovered and carried out by the brothers Pierre and Jacques Curie in France in 1880, the piezoelectric effect is presented in crystals. The Curie brothers, however, did not foresee the reverse piezoelectric effect. The inverse effect was mathematically deduced from fundamental principles of thermodynamics by Gabriel Lippmann in 1881. The Curies immediately confirmed the existence of the inverse effect, which quantitatively evidenced the complete electro-mechanical reversibility for deformations in piezoelectric crystals. In the following decades, piezoelectricity remained a laboratory curiosity. More work was done to explore and define the crystal structures that had the property of generating an electric current. This culminated in the year 1910 with the publication of the book by Woldemar Voigt Lehrbuch der Kristallphysik (mit Ausschluss der Kristalloptik), which describes 20 classes of natural crystals capable of generating current when subjected to mechanical pressure, and rigorously defined the piezoelectric constants using analysis tensor.

From what was said above, piezoelectric material presents two reverse behaviors: when subjected to electric potential it undergoes an elastic displacement and conversely when subjected to an internal elastic displacement it produces an electric...
potential. This makes piezoelectric structures, also called “smart structures”, perfect candidates as sensors or actuators and explains why they are widely used for stabilization purposes [2, 3]. An important point to be highlighted here is that the actuators, in addition to transform mechanical energy into electric one, it also turns a small portion of which into magnetic energy [11]. These effects being relatively small, models that often describe, for example, piezoelectric beams, ignore such magnetic effects because the magnetic energy has a relatively small effect on the overall dynamics. However, this magnetic contribution may limit the system performance in the following sense: “the model with the magnetic effects is proved to be not exactly observable/exponentially stabilizable in the energy space for almost all choices of material parameters. Moreover, even strong stability is not achievable for many values of the material parameters (see [13] for details”). Also Özer and Horner studied the uniform boundary observability of Finite Difference approximations of non-compactly coupled piezoelectric beam equations (see [14] for details).

In recent years, researches on piezoelectric beams systems have received much attention in the specialized literature, especially considering the presence of the magnetic effect. Among the main results in the literature for the model of one-dimensional evolution equations we highlight the one that was introduced by Morris and Özer [11], where the authors used a variational approach to construct a coupled model of piezoelectric beams with magnetic effect. So the model introduced by Morris and Özer in [11] was

\[
\begin{align*}
\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} &= 0, & \text{in } (0, L) \times (0, \infty), \\
\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} &= 0, & \text{in } (0, L) \times (0, \infty),
\end{align*}
\]

(1)

with boundary conditions

\[
\begin{align*}
v(0, t) = p(0, t) = \alpha v_x(L, t) - \gamma\beta p_x(L, t) &= 0, & \forall t \geq 0, \\
p_x(L, t) - \gamma v_x(L, t) &= \frac{-V(t)}{h}, & \forall t \geq 0,
\end{align*}
\]

(2)

and initial conditions

\[
v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad p(x, 0) = p_0(x), \quad p_t(x, 0) = p_1(x) \quad \text{in } (0, L),
\]

(3)

where the functions \(v\) and \(p\) are used to denote the longitudinal displacements of the centerline of the beam and the total load of the electric displacement along the transverse direction at each point \(x\) respectively. Here \(\rho, \alpha, \gamma, \beta, \mu, L, h\) and \(V\) denote mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermittivity, length, thickness and the prescribed voltage on electrodes of beam respectively. In addition, the relationship is considered

\[
\alpha = \alpha_1 + \gamma^2 \beta.
\]

(4)

Remark 1. It is important to note that since

\[
p(x, t) = \int_0^x D(\xi, t) d\xi,
\]

(5)

where \(D(x, t)\) represents the electric displacement in the direction \(z\), then \(p(0, t) = 0\) and still \(p(L, t) = \int_0^L D(\xi, t) d\xi\) may not be zero, because the boundary condition \(p(L, t) = 0\) does not represent the fixation of the beam on both sides. In fact, the fixation is due to the boundary condition \(v(0, t) = v(L, t) = 0\), where \(v\) is the transverse displacement of the beam.
Remark 2. Due to the relation (4), the boundary condition (2) is equivalent to
\[
\begin{align*}
    v(0,t) = v_x(L,t) = 0, & \quad \forall t \geq 0, \\
p(0,t) = p_x(L,t) = 0, & \quad \forall t \geq 0.
\end{align*}
\]

It is important to emphasize that in [12] the authors included fully dynamic magnetic effects in a model for a piezoelectric beam with voltage-driven electrodes. The wave behavior of the electromagnetic fields also had been included in the dynamics. They obtained two coupled partial differential equations, one for the stretching motion and one for the magnetic effect. So, taking \( V(t) = p_t(L,t) \), Morris and Özer proved that the partial differential equation model (1)-(3) is well-posed and the magnetic effects have a strong effect on the stabilizability of the control system. That is, for almost all system parameters the piezoelectric beam can be strongly stabilized, but is not exponentially stabilizable in the energy space.

It is well known that magnetic energy of the piezoelectric beam is relatively small, and it does not change the overall dynamics. On the other hand, a single piezoelectric beam model without the magnetic effects is known to be exactly observable and exponentially stabilizable in the energy space (see [10]). In [13] Özer showed that the uncontrolled system (1)-(3) with \( V(t) = p_t(L,t) \) is exactly observable in a space larger than the energy space, and in addition, he proves an estimate of a polynomial decay.

Since piezoelectric materials have the valuable property of converting mechanical energy to electro-magnetic energy, Ramos et. al. [18] considered the following Piezoelectric beam with magnetic field
\[
\begin{align*}
    \rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + \delta v_t &= 0, & \quad \text{in} & \quad (0,L) \times (0,\infty), \\
    \mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} &= 0, & \quad \text{in} & \quad (0,L) \times (0,\infty), \\
    v(0,t) &= v_x(L,t) - \gamma \beta p_x(L,t) = 0, & \quad \forall t \geq 0, \\
p(0,t) &= p_x(L,t) - \gamma v_x(L,t) = 0, & \quad \forall t \geq 0, \\
v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x), \quad p(x,0) = p_0(x), & \quad \text{in} & \quad (0,L), \\
p_t(x,0) &= p_1(x), & \quad \text{in} & \quad (0,L),
\end{align*}
\]
where \( \rho, \alpha, \gamma, \mu, \beta \) and \( \delta \) are positive constants. They proved that the dissipation produced by damping \( \delta v_t \) acting in the transverse displacement of the beam is strong enough to stabilize exponentially the system (7) for whatever the physical parameters of the model and reproduced a numerical counterpart in a totally discrete domain, which preserve the important decay property of the numerical energy. In the same way, in [17] Ramos, A.J.A., Freitas, M.M., Almeida, D.S. et al. proved the system’s exponential stability independent of any relation between the coefficients using terms of feedback at the boundary and consequently proved their equivalence with the exact observability at the boundary. In both papers the piezoelectric beam is used as a sensor. This means to say that when a force is applied to a piezoelectric material, a electro-magnetic energy charge is generated. This can be measured as a voltage proportional to the pressure.

On the other hand, in [19] A. Soufyane et.al. considered the following piezoelectric beams with magnetic effect and nonlinear damping and nonlinear delay terms.
Under appropriate assumptions on the weight of the delay, the authors established an energy decay rate, using a perturbed energy method and some properties of a convex functions.

Localized frictional damping was studied by several authors in one or more space dimension, (see e.g. [4, 6, 8, 9, 20, 21]). The main result of the above articles is that localized frictional damping produces exponential decay in time of the solution. A more general result occurs in one-dimensional space where the solution always decays exponentially to zero for any localized frictional damping active over an open subset of the domain. This result is no longer valid for materials configured over bounded domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ where the position of the frictional effect is important. See for example [1], where necessary and sufficient conditions are given to get stabilization of the wave equation with localized frictional damping. That is, to get the exponential stability, the damping mechanism must be present in a sufficient large neighborhood of the boundary, see also [6].

We emphasize that due to the physical phenomenon here considered, only one mechanical damping acting on the displacement is sufficient to reach the exponential decay the energy associated of the system (7) (see [18]). On the other hand, controlling a system through a damping mechanism acting on a small part of the beam is interesting from an application point of view. So, the main objective of this work is to investigate the exponential decay of total energy associated with the dissipative system of piezoelectric beams with magnetic effect and localized damping given by

$$
\begin{align*}
\rho v_{tt} - \alpha v_{xx} + \gamma \beta p_{xx} + a(x)v_t &= 0, \quad \text{in} \quad (0, L) \times (0, \infty), \\
\mu p_{tt} - \beta p_{xx} + \gamma \beta v_{xx} &= 0, \quad \text{in} \quad (0, L) \times (0, \infty), \\
v(0, t) &= \nu_0(x), \quad v_t(0, t) = v_1(x), \\
p(0, t) &= p_0(x), \\
v_t(x, t - \tau) &= f_0(x, t - \tau), \quad x \in (0, 1), \quad 0 < t < \tau.
\end{align*}
$$

where $a \in C^1(0, L)$ is a positive function and satisfy

$$
\exists \omega > 0, \quad \text{such that} \quad a(x) \geq \omega > 0 \quad \text{in} \quad (L_1, L_2) \subset (0, L).
$$

It is important to emphasize again that the novel contribution of the manuscript is to obtain the stabilization with fully localized damping (in the transverse displacement) acting in an arbitrary small region of the interval $(0, L)$. As per the result to our best knowledge the system here studied has not been considered in the literature.

**Remark 3.** To achieve the benefits from piezoelectric materials, a lot of investment in technology is needed. In addition, it is need to study more efficient mathematical models which do not neglect important physical phenomena present in modeling. A recent example of this is the magnetic effects in mathematical models of a single piezoelectric beam, which until recently were overlooked because these are very small compared to mechanical effects. In this way, as a application of the problem...
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here studied, the Center for Intelligent Material Systems and Structures (CIMSS) developed a structural health monitoring technique relies on the high frequency impedance characteristics of the structure to qualitatively detect incipient damage. An important feature of this technique is the limited PZT actuator-sensor sensing area, which has been attributed to energy dissipation through material damping and conservative joints. In [5] a study of the structural damping effect at high frequency on the localized sensing region is presented. A theoretical model of the energy dissipation, using a wave propagation approach and the correspondence principle is derived to obtain the specific damping capacity.

The paper is organized as follows. In Section 2, we study existence and uniqueness of solutions of the system (8) using the semigroup techniques. In Section 3, we prove that the semigroup \( S(t) = e^{At} \) associated with the system (8) is exponentially stable. In Section 4, we give conclusion and open question.

2. Semigroup setting. We begin with the issue of well posedness of solutions corresponding to the system (8). To do this, let us consider the Hilbert space

\( \mathcal{H} = H^1_v(0, L) \times H^1_v(0, L) \times L^2(0, L) \times L^2(0, L), \)

where

\[ H^1_v(0, L) = \{ w \in H^1(0, L) \mid w(0) = 0 \}. \]

with inner product given by

\[ \langle U, V \rangle_{\mathcal{H}} = \int_0^L (\mu u_3 v_3 + \mu u_4 v_4) \, dx + \int_0^L \left( \gamma \frac{\partial u_1}{\partial x} + \beta (\gamma u_1 - u_2) (\gamma v_1 - v_2) \right) \, dx \]

\[ = \int_0^L (\mu u_3 v_3 + \mu u_4 v_4) \, dx + \left( \begin{array}{c} \alpha \\ -\gamma \beta \end{array} \right) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right), \]

\[ = \int_0^L (\mu u_3 v_3 + \mu u_4 v_4) \, dx + \left( \begin{array}{c} \alpha \\ -\gamma \beta \end{array} \right) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right), \]

where \( U = (u_1, u_2, u_3, u_4)' \), \( V = (v_1, v_2, v_3, v_4)' \), \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) denote the \( L^2(0, L) \) norm and inner product. Indeed, (12) is an inner product since the matrix

\[ \begin{pmatrix} \alpha & -\gamma \beta \\ -\gamma \beta & \beta \end{pmatrix} \]

is positive definite.

If we denote \( \Psi = \{ v, p, v_t, p_t \} \) and \( \Psi_0 = \{ v_0, p_0, v_1, p_1 \} \) then the system (8) can be rewritten as follows

\[ \begin{cases} \frac{d\Psi}{dt} = A\Psi, & \text{for } t > 0, \\
\Psi(0) = \Psi_0, \end{cases} \]

where the operator \( A \) is given by

\[ A\{ v, p, \varphi, \psi \} = \{ \varphi, \psi, \frac{\alpha}{\rho} v_{xx} - \frac{\gamma}{\rho} p_{xx} - \frac{a(x)}{\rho} \varphi, \frac{\beta}{\mu} p_{xx} - \frac{\gamma \beta}{\mu} v_{xx} \}, \]

for \( (v, p, \varphi, \psi)' \in D(A) \), where

\[ D(A) = \{ (v, p, \varphi, \psi)' \in \mathcal{H} : v, p \in H^2(0, L); \varphi, \psi \in H^1_v(0, L); v_v(L, t) = p_v(L, t) = 0 \}. \]

Proposition 2.1 \( A \) generates a \( C_0 \)-semigroup \( e^{At} \) of contractions in \( \mathcal{H} \), and \( 0 \in \rho(A) \).

Proof. First, we prove that \( A \) is a maximal dissipative operator on the energy space \( \mathcal{H} \), and the conclusion will follow by Lummer-Phillips theorem (see [15]).
• ∀U = (v, p, ϕ, ψ)′ ∈ D(A), it is easy to see that

\[ \text{Re} \langle AU, U \rangle_H = - \int_0^L a(x) |\varphi|^2 \ dx, \] (16)

which implies that A is a dissipative linear operator on H.

• Let \( F = (f_1, f_2, f_3, f_4) \in H \).

Find \( z \in D(A) \), such that

\[ Az = F, \] (17)

which implies that

\[ \begin{align*}
\varphi &= f_1, \quad \psi = f_2, \\
v_{xx} &= \frac{1}{\alpha_1} \left( \rho f_3 + a(x)f_1 + \gamma \mu f_4 \right), \\
p_{xx} &= \frac{1}{\alpha_1} \left( \gamma \left( \rho f_3 + a(x)f_1 \right) + \frac{\alpha}{\beta} \mu f_4 \right), \\
v(0, t) &= v_x(L, t) = p(0, t) = p_x(L, t) = 0.
\end{align*} \] (18)

Using the same arguments in (see [11]), we have the Greens function corresponding to the operator \( -\frac{d^2}{dx^2} \) with the boundary conditions \( \begin{align*}
\varphi &= f_1, \quad \psi = f_2, \\
v &= \frac{1}{\alpha_1} \int_0^L K(x, r) \left( \rho f_3 + a(x)f_1 + \gamma \mu f_4 \right)(r) \ dr, \\
p &= \frac{1}{\alpha_1} \int_0^L K(x, r) \left( \gamma \left( \rho f_3 + a(x)f_1 \right) + \frac{\alpha}{\beta} \mu f_4 \right)(r) \ dr.
\end{align*} \) (19)

Using \( F \in H \), i.e. \( (f_1, f_2) \in H^1_0(0, L) \) and \( (f_3, f_4) \in L^2(0, L) \), implies that \( (\varphi, \psi) \in H^1_0(0, L) \) and \((v, p) \in H^1_0(0, L) \cap H^1_0(0, L) \) with \( v_x(L, t) = p_x(L, t) = 0 \) then \( U \in D(A) \) is uniquely defined and \( 0 \in \rho(A) \). Using Lumer-Phillips theorem ([15]), we conclude that A generates a \( C_0 \)-semigroup \( e^{At} \) of contractions in \( H \).

3. Exponential decay. In this section, we prove the uniform decay of the solutions for problem (7) with boundary conditions (2). Our method of proof is based on ([16]) and ([7]).

Along this paper we will use \( \| . \| \) to denote the norm in space \( L^2(0, L) \).

**Theorem 3.1.** The \( C_0 \)-semigroup \( e^{tA} \) is exponentially stable, i.e. there exist constant \( M \) and \( \delta > 0 \) independent of \( U_0 \) such that

\[ \| e^{tA}U_0 \|_H \leq Me^{-\delta t} \| U_0 \|_H, \quad t > 0. \] (20)

**Proof.** Following the results in ([16]) and ([7]), the following two conditions

\[ i \mathbb{R} \subset \rho(A), \] (21)

\[ \sup_{\delta \in \mathbb{R}} \left\| (i\delta I - A)^{-1} \right\| < +\infty, \] (22)

are necessary and sufficient for the exponential stability.
• **First**, Note that \( 0 \in \rho(\mathcal{A}) \), by the result of Theorem 3.1, then \( \mathcal{A}^{-1} \) is bounded and it is a bijection between \( \mathcal{H} \) and the domain \( \mathcal{D} (\mathcal{A}) \). Since \( \mathcal{D} (\mathcal{A}) \) has compact embedding into \( \mathcal{H} \) it follows that \( \mathcal{A}^{-1} \) is a compact operator, which implies that the spectrum of \( \mathcal{A} \) is discrete. To check the condition (21), we proceed as follow: let \( b \in \mathbb{R} \setminus \{0\} \) let \( U = (v, p, \varphi, \psi)' \in D(\mathcal{A}) \), with
\[
\mathcal{A} U = ibU.
\] (23)
Taking the inner product with \( U \) in \( \mathcal{H} \) and taking its real part, we deduce that
\[
\int_0^L a(x) |\varphi|^2 \, dx = 0.
\] (24)
Now, (23) is equivalent to
\[
\begin{aligned}
\varphi &= ibv, \\
\psi &= ibp, \\
v_{xx} - \gamma \beta p_{xx} &= ibp\varphi + a(x)\varphi, \\
p_{xx} - \gamma \beta v_{xx} &= ib\mu\psi,
\end{aligned}
\] (25)
then by using (25)\textsubscript{1} and (25)\textsubscript{2} in (25)\textsubscript{3} and (25)\textsubscript{4}, we get
\[
\begin{aligned}
\varphi &= ibv, \\
\psi &= ibp, \\
v_{xx} &= \frac{1}{\alpha_1} \left( -b^2 \rho v + iba(x) v - b^2 \gamma \mu p \right), \\
p_{xx} &= \frac{1}{\alpha_1} \left( -b^2 \gamma \rho v + ib\gamma a(x) v - \frac{b^2 \alpha \mu}{\beta} p \right),
\end{aligned}
\] (26)
from (24) and (25)\textsubscript{1}, we have
\[
\left\| \sqrt{a(x)} v \right\|^2 = 0,
\] (27)
using (26) and (27), we deduce that
\[
\begin{aligned}
v_{xx} &= \frac{1}{\alpha_1} \left( -\rho b^2 v - \gamma \mu b^2 p \right), \\
p_{xx} &= \frac{1}{\alpha_1} \left( -\gamma \rho b^2 v - \frac{\alpha \mu b^2}{\beta} p \right),
\end{aligned}
\] (28)
with the boundary conditions
\[
\begin{aligned}
v (0) &= p (0) = v_x (L) = p_x (L), \\
v (x) &= 0, \; \forall x \in (L_1, L_2).
\end{aligned}
\] (29)
Let \( Z = (v, v_x, p, p_x)' \), then we rewrite system (28) in the first order form
\[
\frac{d}{dx} Z = \mathcal{D} Z = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-\frac{\rho b^2}{\alpha_1} & 0 & -\frac{\gamma \mu b^2}{\alpha_1} & 0 \\
0 & 0 & 0 & 1 \\
-\frac{\gamma \rho b^2}{\alpha_1} & 0 & -\frac{\alpha \mu b^2}{\alpha_1} & 0
\end{pmatrix} Z,
\] (30)
this implies that the solution to (30) is as follow:
\[
Z = e^{\mathcal{D} x} K,
\] (31)
where \( K = (k_1, k_2, k_3, k_4)' \) is a vector with arbitrary coefficients. Using the same arguments as in ([11]), we obtain
\[ U = 0, \]
then
\[ i \mathbb{R} \subset \rho (A). \]

**Second.** Assume that the condition (22) is false. Then there is a real sequence \((\delta_n)_{n \in \mathbb{N}}\) and a sequence \((U_n)_{n \in \mathbb{N}} \in D(A), \) such that
\[ \|U_n\| = 1, \] (32)
\[ |\delta_n| \rightarrow +\infty, \text{ as } n \rightarrow \infty \] (33)
\[ \lim_{n \rightarrow +\infty} \|(i\delta_n I - A) U_n\|_H = 0, \] (34)
i.e., we have the following convergence (as \( n \rightarrow \infty \)):
\[
\begin{cases}
  i\delta_n v_n - \varphi_n \rightarrow 0 \text{ in } H^1_0(0, L), \\
i\delta_n p_n - \psi_n \rightarrow 0 \text{ in } H^1_0(0, L), \\
i\delta_n \varphi_n - \frac{\alpha}{\rho} v_{n,xx} + \frac{\gamma}{\rho} p_{n,xx} + \frac{a(x)}{\rho} \varphi_n \rightarrow 0 \text{ in } L^2(0, L), \\
i\delta_n \psi_n - \frac{\beta}{\mu} p_{n,xx} + \frac{\gamma}{\mu} v_{n,xx} \rightarrow 0 \text{ in } L^2(0, L).
\end{cases}
\]
(35)

In the following, we will check the condition (22) by finding a contradiction with (32). Our proof is divided into several steps.

**Step 1.** Taking the inner product of \((i\delta_n I - A) U_n\) with \(U_n\) in \(H\), we get:
\[ \text{Re} \langle (i\delta_n I - A) U_n, U_n \rangle_H = -\int_0^L a(x) |\varphi_n|^2 \, dx. \] (36)
Using (34), we deduce that
\[ \int_0^L a(x) |\varphi_n|^2 \, dx \rightarrow 0, \text{ as } n \rightarrow \infty, \] (37)
it follows from (35)_1 and (37) that:
\[ \|\sqrt{a(x)} \delta_n v_n\|^2 \rightarrow 0, \text{ as } n \rightarrow \infty. \] (38)

**Step 2.** Using (32), (33), (35)_1, (35)_2 and the triangular inequality, we get
\[
\begin{align*}
  \|v_n\| & \leq \frac{1}{i\delta_n} \|i\delta_n v_n - \varphi_n\| + \frac{1}{i\delta_n} \|\varphi_n\| \rightarrow 0, \\
  \|p_n\| & \leq \frac{1}{i\delta_n} \|i\delta_n p_n - \psi_n\| + \frac{1}{i\delta_n} \|\psi_n\| \rightarrow 0,
\end{align*}
\] (39)

it follows from (32), (35)_1, (35)_2 and the triangular inequality, we get
\[
\begin{align*}
  \|\delta_n v_n\| & \leq \|i\delta_n v_n - \varphi_n\| + \|\varphi_n\|, \\
  \|\delta_n p_n\| & \leq \|i\delta_n p_n - \psi_n\| + \|\psi_n\|,
\end{align*}
\]

then, we deduce that
\[ (\|\delta_n v_n\|)_{n \in \mathbb{N}} \text{ and } (\|\delta_n p_n\|)_{n \in \mathbb{N}} \text{ are uniformly bounded.} \] (40)
Step 3. Using (35) and (35)2 then (35)3 and (35)4 becomes (as $n \to \infty$):

\[
\begin{align*}
v_{n,x} + \frac{\rho \Delta^2}{\alpha} v_{n} - \frac{i}{\alpha} a(x) v_{n} + \frac{\gamma \mu \delta_n}{\alpha} (\psi_n - i \delta_n v_n) \\
- \frac{1}{\alpha} \frac{i}{\alpha} a(x) (\psi_n - i \delta_n v_n) + \frac{\gamma}{\alpha} \mu \delta_n (\psi_n - i \delta_n v_n)
\end{align*}
\to 0 \text{ in } L^2(0,L),
\]

\[\tag{41}\]

\[
\begin{align*}
p_{n,x} + \frac{\gamma \rho}{\alpha^2} p_{n} - \frac{1}{\Delta x^2} \frac{1}{\alpha^2} \frac{1}{\alpha^2} a(x) p_{n} + \frac{\alpha \mu}{\alpha^2} \frac{1}{\alpha^2} (\psi_n - i \delta_n p_n) - \frac{1}{\alpha} i \mu \delta_n (\psi_n - i \delta_n p_n)
\end{align*}
\to 0 \text{ in } L^2(0,L).
\]

Multiplying (41)1 and (41)2 by $\frac{1}{\delta_n}$, we obtain (as $n \to \infty$):

\[
\begin{align*}
\frac{v_{n,x}}{\delta_n} + \frac{\rho \delta_n}{\alpha} v_{n} - \frac{i}{\alpha} a(x) v_{n} + \frac{\gamma \mu \delta_n}{\alpha^2} (\psi_n - i \delta_n v_n) - \frac{1}{\delta_n} \frac{i}{\alpha} a(x) (\psi_n - i \delta_n v_n) \\
- \frac{\gamma}{\alpha^2} \mu \delta_n (\psi_n - i \delta_n v_n)
\end{align*}
\to 0 \text{ in } L^2(0,L),
\]

\[
\begin{align*}
\frac{p_{n,x}}{\delta_n} + \frac{\gamma \rho \delta_n}{\alpha^2} p_{n} - \frac{\gamma}{\alpha^2} i \mu (\psi_n - i \delta_n v_n) - \frac{\alpha \mu \delta_n}{\alpha^2} (\psi_n - i \delta_n v_n) + \frac{1}{\alpha} \frac{i}{\alpha} a(x) (\psi_n - i \delta_n v_n) \\
- \frac{\gamma}{\alpha^2} \mu \delta_n (\psi_n - i \delta_n v_n)
\end{align*}
\to 0 \text{ in } L^2(0,L),
\]

using (32), (33), (35)1, (35)2 and triangular inequality, then we deduce

\[
\left\| \frac{v_{n,x}}{\delta_n} \right\|_{L^2(0,L)} \text{ and } \left\| \frac{p_{n,x}}{\delta_n} \right\|_{L^2(0,L)} \text{ are uniformly bounded. (42)}
\]

Using (6), taking the inner product of (41)1 with $v_{n,x}$ and (41)2 with $p_{n,x}$ in $L^2(0,L)$, we obtain (as $n \to \infty$):

\[
\begin{align*}
- \left\| v_{n,x} \right\|^2 + \frac{\rho \delta_n}{\alpha^2} \left\| \delta_n v_n \right\|^2 - \frac{1}{\alpha} \left\langle i \delta_n v_n, a(x) v_n \right\rangle + \frac{\gamma \mu \delta_n}{\alpha^2} \left\langle (\psi_n - i \delta_n p_n), i \delta_n v_n \right\rangle \\
+ \frac{\rho}{\alpha} \left\langle (\psi_n - i \delta_n v_n), i \delta_n v_n \right\rangle + \frac{\mu \delta_n}{\alpha^2} \left\langle (\psi_n - i \delta_n p_n), i \delta_n v_n \right\rangle + \frac{\gamma \mu \delta_n}{\alpha^2} \left\langle (\psi_n - i \delta_n p_n), i \delta_n v_n \right\rangle \\
- \left\| p_{n,x} \right\|^2 + \frac{\gamma \rho}{\alpha^2} \left\| \delta_n v_n, i \delta_n p_n \right\|^2 + \frac{\alpha \mu}{\alpha^2} \left\langle (\psi_n - i \delta_n p_n), i \delta_n p_n \right\rangle + \frac{\rho \alpha}{\alpha \beta} \left\| i \alpha (x) (\psi_n - i \delta_n v_n), \delta_n p_n \right\|^2
\end{align*}
\to 0,
\]

again using (32), (35)1, (35)2 and (40), then we conclude

\[
\left\| v_{n,x} \right\|_{L^2(0,L)} \text{ and } \left\| p_{n,x} \right\|_{L^2(0,L)} \text{ are uniformly bounded. (44)}
\]

Step 4. Let $q(x) \in C^2(0,L)$ be a real function, to be fixed later. Using (44), taking the inner product of (41)1 with $q(x) \frac{v_{n,x}}{\delta_n}$ and (41)2 with $\frac{\mu}{\rho} q(x) \frac{p_{n,x}}{\delta_n}$ in $L^2(0,L)$, we obtain (as $n \to \infty$):

\[
\begin{align*}
\left\langle q_{n,x}, q v_{n,x} \right\rangle + \frac{\rho \delta_n^2}{\alpha} \left\langle v_{n,x}, q v_{n} \right\rangle - \frac{\mu \delta_n}{\alpha} \left\langle q v_{n,x}, q v_{n} \right\rangle \\
+ \frac{\gamma \mu}{\alpha} \left\langle p_{n,x}, q v_{n} \right\rangle - \left\langle \frac{\mu \delta_n}{\alpha} \left\langle q v_{n}, q v_{n} \right\rangle, q v_{n,x} \right\rangle \\
- \frac{\gamma \mu}{\alpha} \left\langle \psi_n - i \delta_n v_n, q v_{n,x} \right\rangle - \frac{\rho}{\alpha} \left\langle \psi_n - i \delta_n v_n, q v_{n,x} \right\rangle
\end{align*}
\to 0,
\]

\[
\begin{align*}
\left\langle \frac{\mu}{\rho} q_{n,x}, q p_{n,x} \right\rangle + \frac{\rho \delta_n^2}{\alpha} \left\langle p_{n,x}, q p_{n} \right\rangle - \frac{\mu \delta_n}{\alpha} \left\langle q p_{n,x}, q p_{n} \right\rangle \\
+ \frac{\alpha \mu}{\alpha} \left\langle p_{n,x}, q p_{n} \right\rangle - \frac{\gamma \mu}{\alpha} \left\langle q p_{n}, q p_{n} \right\rangle \\
- \frac{\gamma \mu}{\alpha} \left\langle \psi_n - i \delta_n p_n, q p_{n,x} \right\rangle - \frac{\rho}{\alpha} \left\langle \psi_n - i \delta_n p_n, q p_{n,x} \right\rangle
\end{align*}
\to 0,
\]

\[\tag{45}\]
using (6), we have

\[ \text{Re} \langle v_{n,x}, q(x) v_n \rangle = -\frac{1}{2} q(0) |v_{n,x}(0)|^2 - \frac{1}{2} \text{Re} \langle v_{n,x}, q_x(x) v_{n,x} \rangle , \]
\[ \text{Re} \langle p_{n,x}, q(x) p_{n,x} \rangle = -\frac{1}{2} q(0) |p_{n,x}(0)|^2 - \frac{1}{2} \text{Re} \langle p_{n,x}, q_x(x) p_{n,x} \rangle , \]  
\tag{46}

and

\[ \text{Re} \langle v_n, q(x) v_n \rangle = \frac{1}{2} q(L) |v_n(L)|^2 - \frac{1}{2} \langle v_n, q_x(x) v_n \rangle , \]
\[ \text{Re} \langle p_n, q(x) p_n \rangle = \frac{1}{2} q(L) |p_n(L)|^2 - \frac{1}{2} \text{Re} \langle p_n, q_x(x) p_n \rangle , \]  
\tag{47}

also, we have:

\[ \gamma \mu \delta_n^2 \langle p_n, q(x) v_n \rangle = \gamma \mu \delta_n^2 q(L) \langle p_n(L), v_n(L) \rangle - \gamma \mu \delta_n^2 \langle p_n, q_x(x) v_n \rangle \]
\[ -\mu \gamma \delta_n^2 \langle q(x)p_{n,x}, v_n \rangle , \]  
\tag{48}

\[ \frac{\rho}{\alpha_1} \langle i \delta_n (\varphi_n - i \delta_n v_n), q(x) v_n \rangle = \]
\[ \left( \frac{\rho}{\alpha_1} q(L) \langle i \delta_n \varphi_n(L), v_n(L) \rangle + \frac{\rho}{\alpha_1} \delta_n^2 q(L) |v_n(L)|^2 \right) \]
\[ + \frac{\rho}{\alpha_1} \langle (\varphi_n - i \delta_n v_n), i q_x(x) \delta_n v_n \rangle + \langle (\varphi_n - i \delta_n v_n)_x, i q(x) \delta_n v_n \rangle \]  
\tag{49}

\[ \frac{\gamma \mu}{\alpha_1} \langle i \delta_n (\psi_n - i \delta_n p_n), q(x) v_n \rangle = \]
\[ \left( \frac{\gamma \mu}{\alpha_1} q(L) \langle i \delta_n \psi_n(L), v_n(L) \rangle + \frac{\gamma \mu}{\alpha_1} \delta_n^2 q(L) \langle p_n(L), v_n(L) \rangle \right) \]
\[ + \frac{\gamma \mu}{\alpha_1} \langle (\psi_n - i \delta_n p_n), i q_x(x) \delta_n v_n \rangle + \frac{\gamma \mu}{\alpha_1} \langle (\psi_n - i \delta_n p_n)_x, i q(x) \delta_n v_n \rangle \]  
\tag{50}

\[ \frac{\gamma \mu}{\alpha_1} \langle i \delta_n (\varphi_n - i \delta_n v_n), q(x) p_{n,x} \rangle = \]
\[ \left( \frac{\gamma \mu}{\alpha_1} q(L) \langle i \delta_n \varphi_n(L), p_n(L) \rangle + \frac{\gamma \mu}{\alpha_1} \delta_n^2 q(L) \langle v_n(L), p_n(L) \rangle \right) \]
\[ + \frac{\gamma \mu}{\alpha_1} \langle (\varphi_n - i \delta_n v_n), q_x(x) i \delta_n p_n \rangle + \frac{\gamma \mu}{\alpha_1} \langle (\varphi_n - i \delta_n v_n)_x, q(x) i \delta_n p_n \rangle \]  
\tag{51}

and,

\[ \frac{\alpha \mu^2}{\beta \rho \alpha_1} \langle i \delta_n (\psi_n - i \delta_n p_n), q(x) p_{n,x} \rangle = \]
\[ \left( \frac{\alpha \mu^2}{\beta \rho \alpha_1} \langle i \delta_n \psi_n(L), q(L) p_n(L) \rangle + \frac{\alpha \mu^2}{\beta \rho \alpha_1} q(L) \delta_n^2 |p_n(L)|^2 \right) \]
\[ + \frac{\alpha \mu^2}{\beta \rho \alpha_1} \langle (\psi_n - i \delta_n p_n), q_x(x) i \delta_n p_n \rangle + \frac{\alpha \mu^2}{\beta \rho \alpha_1} \langle (\psi_n - i \delta_n p_n)_x, q(x) i \delta_n p_n \rangle \]  
\tag{52}

Using (6), (46), (47), (48), (49), (50), (51), (52), and (4) in the result of adding (45)\textsubscript{1} and (45)\textsubscript{2}, we have (as \( n \to \infty \)): 
(53)

\begin{align}
&\left( -\frac{1}{2}\alpha_1^2 \delta_n^2 q(L) |v_n(L)|^2 - \frac{1}{2}\alpha_1^2 \delta_n^2 |p_n(L)|^2 - \frac{1}{2}q(0) |v_{n,x}(0)|^2 \\
&- \frac{1}{2}q(0) |p_{n,x}(0)|^2 - \left( \frac{\rho}{\alpha_1} + \frac{\gamma \mu}{\alpha_1} \right) q(L) \Re \langle i \delta_n \varphi_n(L), v_n(L) \rangle \\
&- \frac{\gamma \mu}{\alpha_1} \Re \langle i \delta_n \psi_n(L), v_n(L) \rangle - \frac{1}{2} \Re \langle v_{n,x}, q_x(x) v_{n,x} \rangle \\
&- \frac{\gamma \mu}{\alpha_1} \Re \langle \varphi_n - i \delta_n v_n, q(x) i \delta_n p_n \rangle \rightarrow 0, \\
&\end{align}

\begin{align}
&\left( -\frac{1}{2}\alpha_1^2 \delta_n^2 q(L) |v_n(L)|^2 - \frac{1}{2}\alpha_1^2 \delta_n^2 |p_n(L)|^2 - \frac{1}{2}q(0) |v_{n,x}(0)|^2 \\
&- \frac{1}{2}q(0) |p_{n,x}(0)|^2 - \gamma \mu q(L) \Re \langle i \delta_n \psi_n(L), v_n(L) \rangle \\
&- \left( \frac{\rho}{\alpha_1} + \frac{\gamma \mu}{\alpha_1} \right) q(L) \Re \langle i \delta_n \varphi_n(L), v_n(L) \rangle \\
&- \frac{\gamma \mu}{\alpha_1} \Re \langle i \delta_n \psi_n(L), p_n(L) \rangle - \frac{1}{2} \Re \langle v_{n,x}, q_x(x) v_{n,x} \rangle \\
&- \frac{1}{2} \Re \langle p_{n,x}, q_x(x) p_{n,x} \rangle \rightarrow 0, \\
&\end{align}

taking in account (35), (37), (38), (39), (40) and (44), then (53) becomes (as \( n \rightarrow \infty \)):

\begin{align}
&\left( -\frac{1}{2}\alpha_1^2 \delta_n^2 q(L) |v_n(L)|^2 - \frac{1}{2}\alpha_1^2 \delta_n^2 |p_n(L)|^2 - \frac{1}{2}q(0) |v_{n,x}(0)|^2 \\
&- \frac{1}{2}q(0) |p_{n,x}(0)|^2 - \gamma \mu q(L) \Re \langle i \delta_n \psi_n(L), v_n(L) \rangle \\
&- \left( \frac{\rho}{\alpha_1} + \frac{\gamma \mu}{\alpha_1} \right) q(L) \Re \langle i \delta_n \varphi_n(L), v_n(L) \rangle \\
&- \frac{\gamma \mu}{\alpha_1} \Re \langle i \delta_n \psi_n(L), p_n(L) \rangle - \frac{1}{2} \Re \langle v_{n,x}, q_x(x) v_{n,x} \rangle \\
&- \frac{1}{2} \Re \langle p_{n,x}, q_x(x) p_{n,x} \rangle \rightarrow 0, \\
&\end{align}

at this stage, we choose \( q(x) = x \), then (54) will be written (as \( n \rightarrow \infty \)):

\begin{align}
&\left( -\frac{1}{2}\alpha_1^2 \delta_n^2 |v_n(L)|^2 - \frac{1}{2}\alpha_1^2 \delta_n^2 |p_n(L)|^2 \\
&- \left( \frac{\rho}{\alpha_1} + \frac{\gamma \mu}{\alpha_1} \right) \Re \langle i \delta_n \varphi_n(L), v_n(L) \rangle - \frac{\gamma \mu}{\alpha_1} \Re \langle i \delta_n \psi_n(L), v_n(L) \rangle \\
&- \frac{\gamma \mu}{\alpha_1} \Re \langle i \delta_n \psi_n(L), p_n(L) \rangle - \frac{1}{2} \Re \langle v_{n,x}, q_x(x) v_{n,x} \rangle \\
&- \frac{1}{2} \Re \langle p_{n,x}, q_x(x) p_{n,x} \rangle \rightarrow 0. \\
&\end{align}

(55)
Step 5. Taking the inner product of (41) with \(a(x)\) \(p_n\) and (41) with \(a(x)\) \(p_n\) in \(L^2(0, L)\), we obtain (as \(n \to \infty\)):

\[
\begin{align*}
- \langle v_{n, x}, a_x(x) p_n \rangle - \langle v_{n, x}, a(x) p_n \rangle + \frac{\rho}{\alpha_1} \langle a(x) \delta_n v_n, \delta_n p_n \rangle \\
- \langle \frac{i \delta_n}{\alpha_1} (a(x)v_n, a(x) p_n) + \frac{\gamma}{\alpha_1} \| a(x) \delta_n v_n \|^2 - \frac{1}{\alpha_1} (ip (\varphi_n - i \delta_n v_n), a(x) \delta_n p_n) \rangle \\
- \frac{\gamma}{\alpha_1} \langle (\varphi_n - i \delta_n v_n), a^2(x) p_n \rangle - \frac{\gamma}{\alpha_1} \langle i \mu (\psi_n - i \delta_n p_n), a(x) \delta_n v_n \rangle \\
\to 0,
\end{align*}
\]

substituting (56) by (56), we obtain (as \(n \to \infty\)):

\[
\begin{align*}
\left( \frac{\gamma \mu}{\alpha_1} \right) \| a(x) \delta_n p_n \|^2 - \langle v_{n, x}, a_x(x) p_n \rangle + \left( \frac{\rho}{\alpha_1} - \frac{\gamma \mu}{\beta \alpha_1} \right) \langle a(x) \delta_n v_n, \delta_n p_n \rangle \\
- \langle \frac{i \delta_n}{\alpha_1} (a(x)v_n, a(x) p_n) - \frac{1}{\alpha_1} (ip (\varphi_n - i \delta_n v_n), a(x) \delta_n p_n) \rangle \\
- \frac{\gamma}{\alpha_1} \langle (\varphi_n - i \delta_n v_n), a^2(x) p_n \rangle - \frac{\gamma}{\alpha_1} \langle i \mu (\psi_n - i \delta_n p_n), a(x) \delta_n v_n \rangle \\
+ \langle p_{n, x}, a_x(x) v_n \rangle - \frac{\gamma}{\alpha_1} \langle i \mu (\psi_n - i \delta_n p_n), a(x) \delta_n v_n \rangle \\
+ \frac{\gamma}{\alpha_1} \langle i \mu (\psi_n - i \delta_n p_n), a(x) \delta_n v_n \rangle \\
\to 0,
\end{align*}
\]

with (38), (39), (40) and (44), we obtain (as \(n \to \infty\)):

\[
\begin{align*}
|\text{Re} \langle v_{n, x}, a_x(x) p_n \rangle| & \leq C \| v_{n, x} \| \| p_n \| \to 0, \\
|\text{Re} \langle a(x) \delta_n v_n, \delta_n p_n \rangle| & \leq C \| a(x) \delta_n v_n \| \| \delta_n p_n \| \to 0, \\
|\text{Re} \langle i a(x) \delta_n v_n, a(x) p_n \rangle| & \leq C \| i a(x) \delta_n v_n \| \| a(x) p_n \| \to 0, \\
|\text{Re} \langle i (\varphi_n - i \delta_n v_n), a(x) \delta_n p_n \rangle| & \leq C \| i (\varphi_n - i \delta_n v_n) \| \| a(x) \delta_n p_n \| \to 0, \\
|\langle (\varphi_n - i \delta_n v_n), a^2(x) p_n \rangle| & \leq C \| (\varphi_n - i \delta_n v_n) \| \| a(x) \delta_n p_n \| \to 0, \\
\text{Re} \langle p_{n, x}, a_x(x) v_n \rangle & \leq C \| p_{n, x} \| \| v_n \| \to 0, \\
|\text{Re} \langle i \delta_n v_n, a(x) \delta_n v_n \rangle| & \leq C \| a(x) \delta_n v_n \| \| \delta_n v_n \| \to 0, \\
|\text{Re} \langle i \delta_n a(x) v_n, a(x) v_n \rangle| & \leq C \| \delta_n a(x) v_n \| \| a(x) v_n \| \to 0, \\
|\langle i (\varphi_n - i \delta_n v_n), a(x) \delta_n v_n \rangle| & \leq C \| i (\varphi_n - i \delta_n v_n) \| \| a(x) \delta_n v_n \| \to 0, \\
|\langle (\varphi_n - i \delta_n v_n), a^2(x) v_n \rangle| & \leq C \| (\varphi_n - i \delta_n v_n) \| \| a^2(x) v_n \| \to 0, \\
|\text{Re} \langle i \mu (\psi_n - i \delta_n p_n), a(x) \delta n v_n \rangle| & \leq C \| i \mu (\psi_n - i \delta_n p_n) \| \| a(x) \delta_n v_n \| \to 0.
\end{align*}
\]

Using (57) and (58), we deduce that (as \(n \to \infty\)):

\[
\| a(x) \delta_n p_n \|^2 \to 0.
\]
Step 6. Performing an integration by parts and using the boundary conditions (6), we get
\[
\langle v_{n,x}, q_x(x) v_{n,x} \rangle = -\left\langle \delta_n v_n, q_x(x) \frac{v_{n,xx}}{\delta_n} \right\rangle - \left\langle v_n, q_{xx}(x) v_{n,x} \right\rangle, \\
\langle p_{n,x}, q_x(x) p_{n,x} \rangle = -\left\langle \delta_n p_n, q_x(x) \frac{p_{n,xx}}{\delta_n} \right\rangle - \left\langle p_n, q_{xx}(x) p_{n,x} \right\rangle. \tag{60}
\]

Here, we choose \( q(x) = \int_0^x a(s) \, ds \) and using (60), then (54) will be written as \( n \to \infty \):
\[
\begin{aligned}
&\left( -\frac{1}{2} \frac{\rho}{\alpha_1} \delta_n^2 q(L) \left| v_n(L) \right|^2 - \frac{1}{2} \frac{\alpha_2}{\beta \rho \alpha_1} q(L) \delta_n^2 \left| p_n(L) \right|^2 \\
&- \left( \frac{\rho}{\alpha_1} + \frac{\gamma \mu}{\alpha_1} \right) q(L) \text{Re} \left\langle i \delta_n \varphi_n(L), v_n(L) \right\rangle \\
&\quad - \frac{\gamma \mu}{\alpha_1} q(L) \text{Re} \left\langle i \delta_n \psi_n(L), v_n(L) \right\rangle - \frac{1}{2} \frac{\mu^2}{\rho} \left\| \sqrt{a(x)} \delta_n p_n \right\|^2 \\
&\quad - \frac{\alpha \mu^2}{\beta \rho \alpha_1} q(L) \text{Re} \left\langle i \delta_n \psi_n(L), p_n(L) \right\rangle + \frac{1}{2} \left\langle \sqrt{a(x)} \delta_n p_n, \sqrt{a(x)} \frac{p_{n,xx}}{\delta_n} \right\rangle \right) \to 0,
\end{aligned}
\tag{61}
\]

with (38), (39), (42), (44) and (59), we have (as \( n \to \infty \)):
\[
\begin{aligned}
&\text{Re} \left\langle \sqrt{a(x)} \delta_n v_n, \sqrt{a(x)} \frac{v_{n,xx}}{\delta_n} \right\rangle \leq C \left\| \sqrt{a(x)} \delta_n v_n \right\| \left\| \frac{v_{n,xx}}{\delta_n} \right\| \to 0, \\
&\text{Re} \left\langle \sqrt{a(x)} \delta_n p_n, \sqrt{a(x)} \frac{p_{n,xx}}{\delta_n} \right\rangle \leq C \left\| \sqrt{a(x)} \delta_n p_n \right\| \left\| \frac{p_{n,xx}}{\delta_n} \right\| \to 0, \\
&\text{Re} \left\langle v_n, a_x(x) v_{n,x} \right\rangle \leq C \left\| v_n \right\| \left\| v_{n,x} \right\| \to 0, \\
&\text{Re} \left\langle p_n, a_x(x) p_{n,x} \right\rangle \leq C \left\| p_n \right\| \left\| p_{n,x} \right\| \to 0,
\end{aligned}
\tag{62}
\]

using (61) and (62), we deduce (as \( n \to \infty \)):
\[
q(L) \left( -\frac{1}{2} \frac{\rho}{\alpha_1} \delta_n^2 \left| v_n(L) \right|^2 - \frac{1}{2} \frac{\alpha_2}{\beta \rho \alpha_1} \delta_n^2 \left| p_n(L) \right|^2 \\
- \left( \frac{\rho}{\alpha_1} + \frac{\gamma \mu}{\alpha_1} \right) \text{Re} \left\langle i \delta_n \varphi_n(L), v_n(L) \right\rangle - \frac{\gamma \mu}{\alpha_1} \text{Re} \left\langle i \delta_n \psi_n(L), v_n(L) \right\rangle \\
- \frac{\alpha \mu^2}{\beta \rho \alpha_1} \text{Re} \left\langle i \delta_n \psi_n(L), p_n(L) \right\rangle - \frac{\gamma \mu}{\alpha_1} \delta_n^2 \text{Re} \left\langle v_n(L), p_n(L) \right\rangle \right) \to 0.
\tag{63}
\]

Step 7. Using (63), then from (55) we get (as \( n \to \infty \)):
\[
\left\| v_{n,x} \right\| \to 0, \quad \left\| v_{n,x} \right\| \to 0, \\
\left\| \delta_n p_n \right\| \to 0, \text{ and } \left\| \rho \delta_n v_n + \gamma \mu \delta_n p_n \right\| \to 0, \tag{64}
\]

applying triangular inequality, we have (as \( n \to \infty \)):
\[
\left\| \delta_n v_n \right\| \to 0,
\tag{65}
\]
with (35)_1, (35)_2, (64), (65) and the triangular inequality, we have (as \( n \to \infty \)):
\[
\| \varphi_n \| \to 0, \quad \| \psi_n \| \to 0.
\]
Finally from (64)_1,(64)_2 and (66) we deduce that (as \( n \to \infty \)):
\[
\| U_n \| \to 0.
\]
Therefore we get the contradiction with (32). Which completes the proof of Theorem 3.1.

4. Conclusion and open question. In this paper we established the uniform decay for the one dimensional piezoelectric beams with magnetic effect under a localized damping placed in the displacement equation, with a minimal support for the damping. Our method of proof is mainly based on semigroup theory and Prüss results and a multipliers techniques. It will be interesting to check the decay rates for the piezoelectric beams with magnetic effect with a localized nonlinear damping with/or without delay term.

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