Gauging $\mathcal{N}=4$ Supersymmetric Mechanics

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Abstract

We argue that off-shell dualities between $d=1$ supermultiplets with different sets of physical bosonic components and the same number of fermionic ones are related to gauging some symmetries in the actions of the supermultiplets with maximal sets of physical bosons. Our gauging procedure uses off-shell superfields and so is manifestly supersymmetric. We focus on $\mathcal{N}=4$ supersymmetric mechanics and show that various actions of the multiplet $(3,4,1)$ amount to some gauge choices in the gauged superfield actions of the linear or nonlinear $(4,4,0)$ multiplets. In particular, the conformally invariant $(3,4,1)$ superpotential is generated by the Fayet-Iliopoulos term of the gauge superfield. We find a new nonlinear variant of the multiplet $(4,4,0)$, such that its simplest superfield action produces the most general 4-dim hyper-Kähler metric with one triholomorphic isometry as the bosonic target metric. We also elaborate on some other instructive examples of $\mathcal{N}=4$ superfield gaugings, including a non-abelian gauging which relates the free linear $(4,4,0)$ multiplet to a self-interacting $(1,4,3)$ multiplet.

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1 Introduction

In recent years, the models of supersymmetric quantum mechanics, particularly those with extended $\mathcal{N}=4$ and $\mathcal{N}=8$, $d=1$ supersymmetries, received considerable attention (see e.g. [1] - [3] and refs. therein). The interest in these theories is mainly motivated by the hope that their thorough study could essentially clarify the geometric and quantum structure of some higher-dimensional “parent” theories which yield the supersymmetric mechanics models upon the appropriate reduction to one dimension. Another source of interest in such models lies in the fact that they can describe superextensions of some notable quantum systems like integrable Calogero-Moser system, quantum Hall effect, etc.

The $d=1$ supersymmetric models are of interest also because of some specific features of $d=1$ supersymmetry which have no direct analogs in the higher-dimensional cases. One of such peculiarities is that the $d=1$ analogs of some on-shell higher-dimensional supermultiplets are off-shell supermultiplets. For instance, the complex form of $\mathcal{N}=2$, $d=4$ hypermultiplet requires an infinite number of auxiliary fields for its off-shell description, which is naturally achieved within $\mathcal{N}=2$ harmonic superspace [4, 5]. On the other hand, there exists an off-shell multiplet of $\mathcal{N}=4$, $d=1$ supersymmetry which collects just 4 physical bosonic fields and 4 physical fermionic fields and no auxiliary fields at all [6]-[10]. This $(4, 4, 0)$ multiplet cannot be obtained by dimensional reduction from any off-shell $d>1$ multiplet. Its $\mathcal{N}=8$ analog is the off-shell $(8, 8, 0)$ multiplet [11] which also cannot be obtained by dimensional reduction from higher dimensions.

Another interesting feature of one-dimensional supersymmetries is the possibility to obtain new off-shell multiplets by replacing the time derivative of some physical bosonic field by a new independent auxiliary field with the same transformation law. This phenomenon was revealed in [6, 11] and it was called there “1D automorphic duality” (see also [12]). This duality manifests itself already in the simpler cases of $\mathcal{N}=1$ and $\mathcal{N}=2$, $d=1$ supersymmetries. For instance, in the $\mathcal{N}=2$ case one can define a real general superfield $\Phi(t, \theta, \bar{\theta})$ with the off-shell content $(1, 2, 1)$ and a complex chiral superfield $\varphi(t + i\theta\bar{\theta}, \theta)$ with the off-shell content $(2, 2, 0)$. The real or imaginary parts of the latter, say

$$\Phi_+(t, \theta, \bar{\theta}) = \varphi(t + i\theta\bar{\theta}, \theta) + \varphi(t - i\theta\bar{\theta}, \bar{\theta}),$$ (1.1)

transforms just as $\Phi(t, \theta, \bar{\theta})$, in which the auxiliary field appearing as a coefficient before the highest-degree $\theta$-monomial, i.e. $\theta\bar{\theta}A(t)$, is substituted by

$$A(t) = i\partial_t(\phi - \bar{\phi}),$$ (1.2)

where $\phi = \varphi|_{\theta=\bar{\theta}=0}$. Thus, if $\varphi$, $\bar{\varphi}$ enter some action only through the combination (1.1), one can treat the time derivative (1.2) as an auxiliary field and so dualize the action into an action of the $(1, 2, 1)$ multiplet. In refs. [6, 11, 12] it was shown that this phenomenon is generic for $d=1$ supersymmetries. For instance, in this way all off-shell $\mathcal{N}=4$ multiplets with 4 physical fermions, i.e. $(3, 4, 1)$, $(2, 4, 2)$, $(1, 4, 3)$ and $(0, 4, 4)$, can be obtained from the “master” (or “root” [13, 14]) multiplet $(4, 4, 0)$. An $\mathcal{N}=8$ analog of the latter is the multiplet $(8, 8, 0)$. Alternatively, one could start from the opposite end and substitute an auxiliary field by the time derivative of some new bosonic field of physical dimension (as in (1.1), (1.2)) thus enlarging the sets of physical fields. The direct and inverse procedures will be referred to as the “reduction” and “oxidation”, respectively$^1$.

$^1$Though these terms, while applied to $d=1$ supersymmetry, are also used in a different sense (see e.g.
In [6, 7, 11, 12, 13] (see also [3]) these $d=1$ dualities were considered at the linearized level. It was observed in [9, 16, 14] that there also exist nonlinear versions of these relations. Namely, the multiplet $(3, 4, 1)$ can be constructed as a bilinear of two multiplets $(4, 4, 0)$, so that the auxiliary field of the former is not expressed as a simple time derivative. Once again, this phenomenon can be seen already in the $\mathcal{N}=2$ case: one can alternatively construct a composite $\tilde{\Phi}$ as

$$\tilde{\Phi}(t, \theta, \bar{\theta}) = \varphi(t + i\theta\bar{\theta}, \theta)\varphi(t - i\theta\bar{\theta}, \bar{\theta}) \Rightarrow \tilde{A} = -i(\phi\partial_t\bar{\phi} - \bar{\phi}\partial_t\phi).$$

(1.3)

These dualities between different off-shell $d=1$ multiplets are very useful since they allow one to establish connections between the relevant invariant actions. For instance, the superconformally invariant actions of the $\mathcal{N}=4$ multiplet $(4, 4, 0)$ were found in [9] by substituting, into the known superconformal actions of the multiplet $(3, 4, 1)$ [16], the expression of the latter multiplet in terms of the former one. It was also observed in [8, 9] that the general actions of the reduced multiplets are always obtained from some subsets of the full variety of the actions of multiplets one started with, and these subsets are characterized by certain isometries commuting with supersymmetry. In the above two $\mathcal{N}=2$ examples the composite multiplets $\Phi_+$ and $\tilde{\Phi}$ are invariant under the translational and rotational abelian transformations of the chiral superfields, viz.

(a) $\delta \varphi = i\lambda$ and (b) $\delta \varphi = i\lambda' \varphi$;

(1.4)

where $\lambda$ and $\lambda'$ are some real group parameters. Thus the actions of the composite superfields $\Phi_+$ and $\tilde{\Phi}$ form subclasses of the general $\varphi$ action, such that they are invariant under isometries (1.4a) or (1.4b).

In the previous studies, the reduction procedure from multiplets with lesser number of auxiliary fields to those with the enlarged number and/or the inverse oxidation procedure were mostly considered at the level of components, performing the relevant substitutions of the auxiliary fields “by hand”. In the case of nonlinear substitutions proceeding this way is to some extent an “art”. The basic aim of the present paper is to put this procedure on a systematic basis, staying at all steps within the manifestly supersymmetric framework of the superfield approach. Using the formalism of harmonic $\mathcal{N}=4$ superspace as most appropriate for $\mathcal{N}=4, d=1$ supersymmetry [9], we demonstrate that the path leading from the master $(4, 4, 0)$ multiplet to its reduced counterparts in most of cases amounts to gauging some isometries of the off-shell $(4, 4, 0)$ actions by a non-propagating “topological” gauge $\mathcal{N}=4$ superfield and choosing some manifestly supersymmetric gauge in the resulting gauge-covariantized actions. We consider both the translational (shift) and rotational isometries. One of the notable outputs of our analysis is as follows. We construct a nonlinear version of the $(4, 4, 0)$ multiplet yielding in the bosonic sector of the corresponding superfield action the general Gibbons-Hawkings (GH) ansatz for hyper-Kähler 4-dimensional metrics with one triholomorphic isometry [17] and accomplish the superfield gauging of this isometry. As a result we obtain the general class of superfield actions of the multiplet $(3, 4, 1)$ which admit a formulation in the analytic harmonic $\mathcal{N}=4, d=1$ superspace [9]. Another novel point is the interpretation of the conformally invariant superpotential of the $(3, 4, 1)$ multiplet as the Fayet-Iliopoulos term of the gauge $\mathcal{N}=4$ superfield while gauging a rotational $U(1)$ isometry of some linear $(4, 4, 0)$ multiplet. We also show that the actions of the so called “nonlinear” $(3, 4, 1)$ multiplet [9, 10] can be reproduced by gauging an abelian target

[3, 15], we hope that their usage in the given context will not be misleading. Originally, the term “oxidation” was employed to denote the procedure inverse to the space-time dimensional reduction. It seems natural to make use of the same nomenclature also as regards the target bosonic manifolds.
space scale isometry in the appropriate \((4,4,0)\) actions. Our procedure is not limited solely to the \((4,4,0)\) actions. We present the HSS formulation of the multiplet \((1,4,3)\) and demonstrate on the simple example of the free action of this multiplet that the gauging of a shifting isometry of the latter gives rise to a superfield action of the fermionic multiplet \((0,4,4)\). At last, we give an example of non-abelian gauging. We gauge the triholomorphic isometry \(SU(2)_{PG}\) of the free \(q^{+a}\) action (realized as rotations of the doublet index \(a\)) and, as a result, recover a non-trivial action of a self-interacting \((1,4,3)\) multiplet.

In Sect. 2 we consider two simple \(\mathcal{N}=1\) and \(\mathcal{N}=2\) toy examples of our gauging procedure. We reproduce some actions of the multiplets \((0,1,1)\) and \((1,2,1)\) as special gauges of the gauge-covariantized free actions of the multiplets \((1,1,0)\) and \((2,2,0)\). The basic notions of \(\mathcal{N}=4,d=1\) harmonic superspace are collected in Sect. 3. In Sect. 4 we gauge shifting and rotational isometries of the superfield actions of the multiplet \((4,4,0)\) defined by a linear harmonic constraint. A generalization of these considerations to the case of the nonlinear \((4,4,0)\) multiplet producing the general GH ansatz in the bosonic sector of the relevant action is the subject of Sect. 5. Some further examples of the superfield gauging procedure are presented in Sect. 6.

## 2 \(\mathcal{N}=1\) and \(\mathcal{N}=2\) examples

The basic principles of our construction can be explained already on the simple \(\mathcal{N}=1,d=1\) example. Let the coordinate set \((t,\theta)\) parametrize \(\mathcal{N}=1,d=1\) superspace and \(\Phi(t,\theta) = \phi(t) + \theta \chi(t)\) be a scalar \(\mathcal{N}=1\) superfield comprising the \(\mathcal{N}=1\) supermultiplet \((1,1,0)\). The invariant free action of \(\Phi\) is \(^2\)

\[
S_{\mathcal{N}=1} = -i \int dt d\theta \partial_t \Phi D \Phi = \int dt \left[ (\partial_t \phi)^2 + i \chi \partial_t \chi \right],
\]

(2.1)

where

\[
D = \frac{\partial}{\partial \theta} + i \theta \frac{\partial}{\partial t}, \quad D^2 = i \partial_t, \quad \int d\theta = 1.
\]

(2.2)

The action (2.1) is invariant under constant shifts

\[
\Phi' = \Phi + \lambda.
\]

(2.3)

Let us now gauge this shifting symmetry by replacing \(\lambda \rightarrow \Lambda(t,\theta)\) in (2.3). To gauge-covariantize the action (2.1), we are led to introduce the fermionic “gauge superfield” \(\Psi(t,\theta) = \psi(t) + i \theta A(t)\) transforming as

\[
\Psi' = \Psi + D\Lambda,
\]

(2.4)

and to substitute the “flat” derivatives in (2.1) by the gauge-covariant ones

\[
S_{\mathcal{N}=1}^{\text{gauge}} = -i \int dt d\theta \nabla_t \Phi D \Phi,
\]

(2.5)

with

\[
\nabla_t \Phi = \partial_t \Phi + i D \Psi, \quad D \Phi = D \Phi - \Psi.
\]

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\(^2\)Throughout the paper we do not care about the dimension of the superfields involved, having in mind that the correct dimension of the superfield actions can be ensured by inserting appropriate normalization constants in front of them. Hereafter, without loss of generality, we set such constants equal to unity.
Taking into account that $\Lambda(t, \theta)$ is an arbitrary superfunction, one can choose the “unitary gauge” in (2.5)

$$\Phi = 0,$$

in which

$$S_{N=1}^{gauge} = - \int dt d\theta D\Psi \Psi = \int dt \left( i\psi \partial_t \psi + A^2 \right).$$

This is just the free action of the $\mathcal{N}=1$ multiplet $(0,1,1)$, with $A(t)$ being the auxiliary bosonic field. Thus we observe the phenomenon of transmutation of the physical bosonic field $\phi(t)$ of the $\mathcal{N}=1$ multiplet $(1,1,0)$ into an auxiliary bosonic field $A(t)$ of another off-shell $\mathcal{N}=1$ multiplet, the $(0,1,1)$ one. This comes about as a result of gauging a shift isometry of the action (2.1) of the former multiplet. In other words, the $(0,1,1)$ action (2.8) is a particular gauge of the covariantized $(1,1,0)$ action (2.5). This is analogous to the Higgs effect: the gauge superfield $\Psi$ “eats” the Goldstone superfield $\Phi$ (“unitary gauge”) and as a result becomes the standard (ungauged) $\mathcal{N}=1, d=1$ superfield comprising the irreducible $(0,1,1)$ $\mathcal{N}=1$ multiplet $\psi(t), A(t)$.

One can come to the same final result by choosing a Wess-Zumino gauge, in which $\Psi(t, \theta)$ takes the form

$$\Psi_{WZ}(t, \theta) = i\theta A(t), \quad \delta A(t) = \partial_t \lambda(t), \quad \lambda(t) = \Lambda(t, \theta)|_{\theta=0}.$$ 

In this gauge (2.5) becomes

$$S_{N=1}^{gauge} = \int dt \left[ (\partial_t \phi - A)^2 + i\chi \partial_t \chi \right]$$

and the residual gauge freedom acts as an arbitrary shift of $\phi(t), \phi'(t) = \phi(t) + \lambda(t)$. Fixing this freedom by the gauge condition $\phi(t) = 0$, we once again come to the $(0,1,1)$ free action.

On this simplest example we observe that the phenomenon of interchangeability between the physical and auxiliary degrees of freedom in $d=1$ supermultiplets can be given a clear interpretation in terms of gauging appropriate isometries of the relevant superfield actions. After gauging, some physical (Goldstone) bosons become pure gauge and can be eliminated, while the relevant $d=1$ “gauge fields” acquire status of auxiliary fields. This treatment can be extended to higher $\mathcal{N}, d=1$ supersymmetries. In the next Sections we shall discuss how it works in models of $\mathcal{N}=4$ mechanics.

But before turning to this, let us dwell on some unusual features of the $\mathcal{N}=1$ gauge multiplet above, shared by its $\mathcal{N}=2$ and $\mathcal{N}=4$ counterparts.

Looking at the “WZ” gauge (2.9) we see that only bosonic “gauge” field $A(t)$ remains in it, without any fermionic partner, although we started from off-shell $\mathcal{N}=1$ supersymmetry. However, no contradiction arises since, due to the residual gauge invariance, one ends up with $(0+0)$ off-shell degrees of freedom (locally). It is amusing that $\mathcal{N}=1$ supersymmetry plus compensating gauge transformation needed to preserve the WZ gauge (2.9) yield $\delta_{SUSY} A(t) = 0$! This is still compatible with the standard closure of $d=1$ supersymmetry on $d=1$ translations. Indeed, the latter act on $A(t)$ as $\delta_a A(t) = a \partial_t A(t) = \partial_t (a A)$, i.e. this group variation is a particular case of the residual gauge freedom. So one can say that, modulo the residual gauge transformations, the trivial supersymmetry transformations of $A(t)$ still have one-dimensional translations as their closure, just similarly to what one observes in WZ gauges in higher dimensions. Though locally $\Psi(t, \theta)$ does not bring any new physical degrees of freedom, globally the
field $A(t)$ surviving in the WZ gauge can differ from a pure gauge $\sim \partial_t b(t)$, and this property leads to surprising consequences when gauging isometries of the $d=1$ supersymmetry actions. This special feature suggests the name “topological” for such $d=1$ gauge multiplets.

In the $\mathcal{N}=2$ case an analog of the gauge superfield $\Psi(t,\theta)$ is the real superfield $\mathcal{V}(t,\theta,\bar{\theta})$ with the transformation law

$$\mathcal{V}'(t,\theta,\bar{\theta}) = \mathcal{V}(t,\theta,\bar{\theta}) + \frac{i}{2} \left[ \Lambda(t_L,\theta) - \bar{\Lambda}(t_R,\bar{\theta}) \right], \quad t_L = t + i\theta\bar{\theta}, \quad t_R = \overline{(t_L)}$$

(2.11)

Though this transformation law mimics that of $\mathcal{N}=1,4D$ gauge superfield, in the WZ gauge only one bosonic field survives, as in (2.9):

$$\mathcal{V}_{WZ} = \theta\bar{\theta} A(t).$$

(2.12)

So this gauge multiplet is also “topological”. By making use of it, one can study various gaugings of $\mathcal{N}=2$ supersymmetric mechanics models and establish, in this way, the relations between off-shell $\mathcal{N}=2$ multiplets ($\mathbf{2},\mathbf{2},\mathbf{0}$), ($\mathbf{1},\mathbf{2},\mathbf{1}$) and ($\mathbf{0},\mathbf{2},\mathbf{2}$). As an example, let us consider the gauging of the $U(1)$ phase invariance of the free action of the ($\mathbf{2},\mathbf{2},\mathbf{0}$) multiplet.

As was said already, this multiplet is described by the chiral $\mathcal{N}=2$ superfield $\varphi(t_L,\theta) = \phi(t_L) + \theta\psi(t_L)$, with the bilinear action

$$S_{\mathcal{N}=2}^{free} = -\int dt d^2\theta \left[ D\varphi(t_L,\theta)\bar{D}\bar{\varphi}(t_R,\bar{\theta}) + 4c \varphi(t_L,\theta)\bar{\varphi}(t_R,\bar{\theta}) \right],$$

(2.13)

where

$$D = \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial \theta} - i\theta \frac{\partial}{\partial t},$$

where $c$ is a coupling constant, with dimension $t^{-1}$. In (2.13) the first term yields the standard free action of the involved fields, $\sim \partial_t \phi \partial_t \bar{\phi} + \ldots$, while the second piece is an $\mathcal{N}=2$ superextension of the WZ-type Lagrangian $\sim i \left( \partial_t \phi \partial_t \bar{\phi} - \partial_t \bar{\phi} \right)$ (the specific normalization of the WZ term was chosen for further convenience). One can also add some potential terms as integrals over chiral and anti-chiral $\mathcal{N}=2$ superspaces, but we do not consider them here.

The action (2.13) possesses the evident $U(1)$ invariance $\varphi' = e^{-i\lambda} \varphi$, $\bar{\varphi}' = e^{i\lambda} \bar{\varphi}$ with a constant parameter $\lambda$. Under the local version of these transformations

$$\varphi' = e^{-i\Lambda} \varphi, \quad \bar{\varphi}' = e^{i\bar{\Lambda}} \bar{\varphi}, \quad \Lambda = \Lambda(t_L,\theta), \quad \bar{\Lambda} = \bar{\Lambda}(t_R,\bar{\theta})$$

(2.14)

the action ceases to be invariant and should be covariantized with the help of the “topological” gauge superfield $\mathcal{V}$ with the transformation law (2.11):

$$S_{\mathcal{N}=2}^{gauge} = -\int dt d^2\theta \left[ D\varphi(t_L,\theta)\bar{D}\bar{\varphi}(t_R,\bar{\theta}) e^{2\mathcal{V}} + 4c \varphi(t_L,\theta)\bar{\varphi}(t_R,\bar{\theta}) e^{2\mathcal{V}} + 2\xi \mathcal{V} \right].$$

(2.15)

Here

$$D = D + 2D\mathcal{V}, \quad \bar{D} = \bar{D} + 2\bar{D}\mathcal{V},$$

(2.16)

and we also added a Fayet-Iliopoulos (FI) term for $\mathcal{V}$, with coupling constant $\xi$ of the same dimension as the constant $c$. Using the gauge freedom (2.14), one can choose the manifestly supersymmetric “unitary” gauge

$$\varphi = 1.$$
Now the gauge freedom (2.14) has been fully “compensated” and $V$ becomes a general real $\mathcal{N}=2, d=1$ superfield with the off-shell content $\mathbf{(1, 2, 1)}$. The action (2.15) in this particular gauge becomes the specific action of the latter multiplet:

$$S^W_{\mathcal{N}=2} = -\int dt d^2\theta \left(DW DW + c W^2 + 2\xi \ln W\right),$$  \hspace{1cm} (2.18)$$

where we redefined $W = 2e^V$. Thus we started from the bilinear action of the multiplet $\mathbf{(2, 2, 0)}$, gauged its rotational $U(1)$ symmetry and came to the action of the multiplet $\mathbf{(1, 2, 1)}$ with a non-trivial superpotential as a result of the special gauge-fixing in the covariantized $\mathbf{(2, 2, 0)}$ action.

The superpotential in (2.18) is generated by the WZ and FI terms in the gauge-covariantized action, and it is interesting to see what kind of scalar component potential they produce. Expanding $W$ as

$$W(t, \theta, \bar{\theta}) = \rho(t) + \theta \chi(t) - \bar{\theta} \bar{\chi}(t) + \theta \bar{\theta} \omega(t),$$  \hspace{1cm} (2.19)$$

and neglecting fermions, we find

$$S^W_{\mathcal{N}=2}^{\text{bos}} = \int dt \left[(\partial_t \rho)^2 + \omega^2 - 2c \rho \omega - 2\xi \omega \rho^{-1}\right],$$  \hspace{1cm} (2.20)$$

which, after eliminating the auxiliary field $\omega(t)$, is reduced to the simple expression

$$S^W_{\mathcal{N}=2}^{\text{bos}} = \int dt \left[(\partial_t \rho)^2 - c^2 \rho^2 - \frac{\xi^2}{\rho^2} - 2c\xi\right].$$  \hspace{1cm} (2.21)$$

Thus, gauging the free action of the $\mathbf{(2, 2, 0)}$ multiplet, we finally arrived at the action of the multiplet $\mathbf{(1, 2, 1)}$ with a non-trivial scalar potential. It describes a two-particle Calogero-Moser model modulo a trivial center of mass motion and a constant shift of the energy.

Note that the action (2.13) possesses also a shift isometry $\varphi' = \varphi + \omega$, $\omega$ being a complex constant parameter. One can alternatively gauge this isometry, and in the corresponding “unitary” gauge $\varphi = 0$ also recover a superfield action of $\mathbf{(1, 2, 1)}$ multiplet. It looks like (2.18), but with the term $\sim W$ instead of $\ln W$ (this linear term can in fact be removed by a shift of $W$).

In the $\mathcal{N}=1$ and $\mathcal{N}=2$ examples considered, the inverse “oxidation” procedure is rather straightforward, though non-unique in the $\mathcal{N}=2$ case. While in the $\mathcal{N}=1$ action (2.8) one should just choose $\Psi = D\Phi$, in the actions of the $\mathbf{(1, 2, 1)}$ superfield $W$ one can substitute $W$ either as in (1.1), or as in (1.3), or even as $W = \varphi^n \varphi^n$, $n$ being some real number (for self-consistency, one should assume that $\varphi$ has a constant background part). For instance, in order to reproduce the original action (2.13), one should make in (2.18) the substitution $W \sim \varphi^{\frac{1}{n}} \varphi^{\frac{1}{n}}$. Such a non-uniqueness of the oxidation procedure as compared to the reduction one (which is just the gauging of some fixed isometry) is a general phenomenon manifesting itself also in models with higher $\mathcal{N}$.

So much for the toy examples and let us turn to the $\mathcal{N}=4$ case. In fact, various $\mathcal{N}=1$ and $\mathcal{N}=2$ gauged models can be reproduced by passing to the $\mathcal{N}=1$ or $\mathcal{N}=2$ superfield formulations of the $\mathcal{N}=4$ models considered in the next Sections.

3 $\mathcal{N}=4$, $d=1$ harmonic superspace

3.1 Definitions

We start by recalling basics of $\mathcal{N}=4$, $d=1$ harmonic superspace (HSS) [9].
The ordinary $\mathcal{N}=4, d=1$ superspace is defined as

$$ (t, \theta_i, \bar{\theta}^i), \quad \bar{\theta}^i = \overline{(\theta_i)}, $$

where $t$ is the time coordinate and the Grassmann-odd coordinates $\theta_i, \bar{\theta}^i$ form doublets of the automorphism group $SU(2)_A$. The $\mathcal{N}=4$ supertranslations act as

$$ \delta \theta_i = \epsilon_i, \quad \delta \bar{\theta}^i = \bar{\epsilon}^i, \quad \delta t = i \left( \bar{\theta}^i \epsilon_i - \bar{\epsilon}^i \theta_i \right). \quad (3.2) $$

The corresponding covariant derivatives are defined as

$$ D^i = \frac{\partial}{\partial \theta^i} + i \bar{\theta}^i \partial_t, \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} + i \epsilon_i \partial_t, \quad \{ D^i, \bar{D}_j \} = 2i \delta^i_j \partial_t, \quad \{ D^i, D^j \} = \{ \bar{D}_i, \bar{D}_j \} = 0. \quad (3.3) $$

$\mathcal{N}=4, d=1$ HSS is defined as an extension of (3.1) by the harmonics $u_i^\pm \in SU(2)_A/U(1)$. The basic relations the harmonics satisfy are $u_i^- = (u_i^\pm)^*, \ u_i^+ u_i^- = 1$. The latter constraint implies the important completeness relation

$$ u_i^+ u_k^- - u_k^+ u_i^- = \varepsilon_{ik}. \quad (3.4) $$

The coordinates of $\mathcal{N}=4, d=1$ HSS in the analytic basis are

$$ (t_A = t - i(\theta^+ \bar{\theta}^- + \bar{\theta}^+ \theta^-), \quad \theta^\pm = \theta^i u_i^\pm, \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm, \ u_k^\pm) \quad (3.5) $$

The analytic subspace of HSS is defined as the coordinate subset

$$ (t_A, \theta^+, \bar{\theta}^+, u_i^\pm) \equiv (\zeta, u). \quad (3.6) $$

It is closed under the $\mathcal{N}=4$ supersymmetry (3.2).

In the “central” basis $(t, \theta_i, \bar{\theta}^k, u^{\pm i})$ we define the harmonic derivatives and the harmonic projections of spinor derivatives as

$$ D^{\pm \pm} = \partial^{\pm \pm} = u_i^\pm \frac{\partial}{\partial u_i^\pm}, \quad D^\pm = u_i^\pm D^i, \quad \bar{D}^\pm = u_i^\pm \bar{D}_i. \quad (3.7) $$

Then in the analytic basis, the same spinor and harmonic derivatives read

$$ D^+ = \frac{\partial}{\partial \theta^+}, \quad \bar{D}^- = \frac{\partial}{\partial \bar{\theta}^-}, \quad D^- = -\frac{\partial}{\partial \theta^-} + 2i \bar{\theta}^- \partial_{t_A}, \quad \bar{D}^+ = \frac{\partial}{\partial \bar{\theta}^+} + 2i \theta^+ \partial_{t_A}, $$

$$ D^{++} = \partial^{++} - 2i \theta^+ \bar{\theta}^- \partial_{t_A} + \theta^+ \frac{\partial}{\partial \theta^-} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^-}, $$

$$ D^{--} = \partial^{--} - 2i \theta^- \bar{\theta}^+ \partial_{t_A} + \theta^- \frac{\partial}{\partial \theta^+} + \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^+}. \quad (3.8) $$

The precise form of the algebra satisfied by these derivatives can be found using their explicit expressions. The basic relations to be used in what follows are

$$ [D^{\pm \pm}, D^\pm] = D^{\pm \pm}, \quad [D^{\pm \pm}, D^\pm] = D^{\pm \pm}, \quad [D^0, D^\pm] = -\{ D^-, D^+ \} = 2i \partial_{t_A}, \quad (3.9) $$

$$ D^0 = u_i^+ \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^-} + \theta^+ \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \frac{\partial}{\partial \bar{\theta}^+} - \theta^- \frac{\partial}{\partial \theta^-} - \bar{\theta}^- \frac{\partial}{\partial \bar{\theta}^-}. \quad (3.10) $$

The second automorphism $SU(2)$ of the $\mathcal{N}=4, d=1$ Poincaré supersymmetry is hidden in this notation; it combines $\theta_i$ and $\bar{\theta}_i$ into a doublet.
The derivatives $D^+, \bar{D}^+$ are short in the analytic basis, whence it follows that there exist analytic $\mathcal{N}=4$ superfields $\Phi^{(q)}(\zeta, u)$

$$D^+ \Phi^{(q)} = \bar{D}^+ \Phi^{(q)} = 0 \Rightarrow \Phi^{(q)} = \Phi^{(q)}(\zeta, u), \quad (3.11)$$

where $q$ is the external harmonic $U(1)$ charge. This Grassmann harmonic analyticity is preserved by the harmonic derivative $\bar{D}^{++}$: when acting on $\Phi^{(q)}(\zeta, u)$, this derivative yields an analytic $\mathcal{N}=4, d=1$ superfield of the charge $(q + 2)$.

Finally, the measures of integration over the full HSS and its analytic subspace are given, respectively, by

$$\begin{align*}
dudtd^4\theta &= dudt_A(D^- \bar{D}^-)(D^+ \bar{D}^+) = \mu_A^{(-2)}(D^+ \bar{D}^+), \\
\mu_A^{(-2)} &= dud\zeta^{(-2)} = dudt_A d\theta^+ d\bar{\theta}^+ = dudt_A(D^- \bar{D}^-). \quad (3.12)
\end{align*}$$

### 3.2 $\mathcal{N}=4, d=1$ multiplets in HSS

Off-shell multiplets of $\mathcal{N}=4, d=1$ supersymmetry with four fermions admit a concise description in the HSS framework. The HSS formulations of the multiplets (4, 4, 0) and (3, 4, 1) are simpler, since these multiplets are described by analytic superfields.

The multiplet (4, 4, 0) is described by a doublet analytic superfield $q^{+a}(\zeta, u)$ of charge 1 satisfying the non-dynamical harmonic constraint

$$D^{++} q^{+a} = 0 \Rightarrow q^{+a}(\zeta, u) = f^{ia}(t)u_i^+ + \theta^+ \chi^a(t) + \bar{\theta}^+ \bar{\chi}^{a}(t) + 2i\theta^+ \bar{\theta}^+ \partial_t f^{ia}(t)u_i^-. \quad (3.13)$$

The (3, 4, 1) multiplet is described by a charge 2 analytic superfield $W^{++}(\zeta, u)$ satisfying the harmonic constraint

$$D^{++} W^{++} = 0 \Rightarrow W^{++}(\zeta, u) = w^{(ik)}(t)u_i^+ u_k^+ + \theta^+ \psi^i(t)u_i^+ + \bar{\theta}^+ \bar{\psi}^i(t)u_i^+ + i\theta^+ \bar{\theta}^+[F(t) + 2\partial_t w^{(ik)}(t)u_i^+ u_k^+]. \quad (3.14)$$

The Grassmann analyticity conditions together with the harmonic constraints (3.13) and (3.14) imply that in the central basis

$$\begin{align*}
q^{+a} &= q^{ia}(t, \theta, \bar{\theta})u_i^+, \quad D^{(i,k,a)} = \bar{D}^{(i,k,a)} = 0, \\
W^{++} &= W^{(ik)}(t, \theta, \bar{\theta})u_i^+ u_k^+, \quad D^{(i)}W^{kl} = \bar{D}^{(i)}W^{kl} = 0. \quad (3.15) \quad (3.16)
\end{align*}$$

By analogy with the $\mathcal{N}=2, d=4$ HSS [4], one can also introduce the $\mathcal{N}=4, d=1$ “gauge multiplet”. It is represented by a charge 2 unconstrained analytic superfield $V^{++}(\zeta, u)$ the gauge transformation of which reads in the abelian case

$$\delta V^{++} = D^{++}\Lambda, \quad (3.17)$$

with $\Lambda(\zeta, u)$ being a charge zero unconstrained analytic superfield parameter. Using this gauge freedom, one can choose the Wess-Zumino gauge, in which the gauge superfield becomes

$$\begin{align*}
V^{++}(\zeta, u) &= 2i(\theta^+ \bar{\theta}^+)A(t), \quad \delta A(t) = -\partial_t \Lambda_0(t), \quad \Lambda_0 = \Lambda(\zeta, u)|_{\theta=0}. \quad (3.18)
\end{align*}$$

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4For brevity, in what follows we frequently omit the index “A” of $t_A$. 

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8
We observe here the same phenomenon as in the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases of Sect. 2: the “gauge” $\mathcal{N}=4, d=1$ multiplet locally carries $(0+0)$ degrees of freedom and so it is “topological”. Globally the field $A(t)$ can differ from a pure gauge, and this feature allows for its treatment as an auxiliary field in the “unitary” gauges.

As in the $\mathcal{N}=2, d=4$ HSS [4], $V^{++}$ gauge-covariantizes the analyticity-preserving harmonic derivative $D^{++}$. Assume that the analytic superfield $\Phi^{(q)}$ is transformed under some abelian gauge isometry as

$$\delta_{A} \Phi^{(q)} = \Lambda \mathcal{I} \Phi^{(q)},$$

(3.19)

where $\mathcal{I}$ is the corresponding generator. Then the harmonic derivative $D^{++}$ is covariantized as

$$D^{++} \Phi^{(q)} \implies \mathcal{D}^{++} \Phi^{(q)} = (D^{++} - V^{++} \mathcal{I}) \Phi^{(q)}.$$  

(3.20)

One can also define the second, non-analytic harmonic connection $V^{--}$

$$\mathcal{D}^{--} = D^{--} - V^{--} \mathcal{I}, \quad \delta V^{--} = D^{--} \Lambda.$$  

(3.21)

From the requirement of preserving the algebra of harmonic derivatives (3.9),

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = D^{0}, \quad [D^{0}, \mathcal{D}^{\pm\pm}] = \pm 2 \mathcal{D}^{\pm\pm},$$

(3.22)

the well-known harmonic zero-curvature equation follows

$$D^{++}V^{--} - D^{--}V^{++} = 0,$$

(3.23)

which specifies $V^{--}$ in terms of $V^{++}$. One can also define the covariant spinor derivatives

$$\mathcal{D}^{--} = [\mathcal{D}^{--}, \mathcal{D}^{++}] = D^{--} + (D^{+}V^{--}) \mathcal{I}, \quad \bar{\mathcal{D}}^{--} = [\mathcal{D}^{--}, \bar{\mathcal{D}}^{++}] = \bar{D}^{--} + (\bar{D}^{++}V^{--}) \mathcal{I},$$

(3.24)

as well as the covariant time derivative

$$\{D^{+}, \bar{D}^{-}\} = 2i \mathcal{D}_{t}, \quad \mathcal{D}_{t} = \partial_{t} - \frac{i}{2}(D^{+}\bar{D}^{+}V^{--}) \mathcal{I}.$$  

(3.25)

The important specific feature of the $d=1$ HSS is that the vector gauge connection

$$V \equiv D^{+}\bar{D}^{+}V^{--}, \quad \delta V = -2i \partial_{t} \Lambda,$$

(3.26)

is an analytic superfield, $D^{+}V = \bar{D}^{+}V = 0$, so $\mathcal{D}_{t}$ preserves the analyticity.

In the WZ gauge (3.18) spinor gauge connections are vanishing, while

$$V \implies 2i A(t).$$  

(3.27)

We will exploit these relations in Sect. 3.

### 3.3 General $q^{+a}$ and $W^{++}$ actions

We may write a general off-shell action for the $(4, 4, 0)$ multiplet as

$$S_{q} = \int dudt d^{4} \theta \mathcal{L}(q^{+a}, q^{-b}, u^{\pm}), \quad q^{-a} \equiv D^{--}q^{+a}.$$  

(3.28)
After solving the constraint (3.13) in the central basis of HSS, the superfield $q^{±a}$ may be written in this basis as $q^{±a} = u^{±i}q^{i a}(t, \theta, \bar{\theta})$, $D^{(i}q^{k)a} = \bar{D}^{(i}q^{k)a} = 0$. We then use the notation

$$L(q^{ia}) = \int du \mathcal{L}(q^{+a}, q^{-b}, u^{±}), \quad S_q = \int dt d\theta L(q^{ia}). \quad (3.29)$$

The free action is given by

$$S_q^{\text{free}} = -\frac{1}{4} \int dtd\theta (q^{+a}q^{−a}) = i \int d\zeta (-2) (q^{+a}\partial_{\zeta}q^{−a}). \quad (3.30)$$

The action (3.28) produces a sigma-model type action in components, with two time derivatives on the bosonic fields and one derivative on the fermions. One can also construct an invariant which in components yields a Wess-Zumino type action, with one time derivative on the bosonic fields (plus Yukawa-type fermionic terms). It is given by the following general integral over the analytic subspace

$$S_{WZ}^{q} = \int dud\zeta (-2) \mathcal{L}^{+2}(q^{+a}, u^{±}). \quad (3.31)$$

A general sigma-model type action for the (3, 4, 1) multiplet can be written as

$$S_{W} = \int dtd\theta \mathcal{L}(W^{++}, W^{−−}, W^{+-}, u^{±}), \quad (3.32)$$

$$W^{−−} = \frac{1}{2} (D^{−−})^2 W^{++}, \quad W^{+-} = \frac{1}{2} D^{−−}W^{++}. \quad (3.33)$$

The free action is given by

$$S_{W}^{\text{free}} = -\frac{1}{4} \int dtd\theta W^{++} (D^{−−})^2 W^{++} = i \int d\zeta (-2) W^{++}\Pi_{−−}^{2}W^{++}, \quad (3.34)$$

where the second-order differential operator

$$\Pi_{−−}^{2} = D^{−−}\partial_{t} + \frac{i}{2} D^{−}D^{−} = \partial^{−−}\partial_{t_{A}} - \frac{i}{2} \frac{\partial}{\partial \theta^{+}} \frac{\partial}{\partial \bar{\theta}^{+}} \quad (3.35)$$

preserves the analyticity:

$$[D^{+}, \Pi_{−−}^{2}] = [\bar{D}^{+}, \Pi_{−−}^{2}] = 0, \quad [D^{++}, \Pi_{−−}^{2}] = (D_{0} - 1)\partial_{t_{A}} - \frac{i}{2} \left( \bar{D}^{−}D^{+} - D^{−}\bar{D}^{+} \right). \quad (3.36)$$

One can also define an analog of (3.31):

$$S_{W}^{WZ} = \int dud\zeta (-2) \mathcal{L}^{+2}(W^{++}, u^{±}). \quad (3.37)$$

In the component notation, it yields both the coupling to an external gauge field background and the scalar potential (plus appropriate fermionic terms).

4 Gauging isometries of the $q^{+}$ actions

In the following, we shall consider two different types of isometries admitting a realization on $q^{+a}$: shifts and rotations.
4.1 Shift isometries

Let us start with an example of shift isometry. It will be convenient for us to equivalently represent \( q^a \) by its harmonic projections:

\[
q^a \quad \iff \quad L^{++} = q^a u^+_a, \quad \omega = q^+_a u_a^-.
\]

(4.1)

The constraints (3.13) are rewritten in this notation as

\[
\begin{aligned}
(a) \quad D^{++} L^{++} &= 0, \\
(b) \quad D^{++} \omega - L^{++} &= 0.
\end{aligned}
\]

(4.2)

Infinitesimal transformations of the isometry are

\[
\delta q^a = \lambda u^a \quad \iff \quad \delta L^{++} = 0, \quad \delta \omega = \lambda,
\]

(4.3)

where \( \lambda \) is a constant parameter. The free action of the \( q^a \) superfields (3.30) is manifestly invariant under this transformation, because the integrand transforms to a total derivative

\[
\delta S^\text{free}_q = \frac{i}{2} \int dud\zeta (-2) \partial_t (\lambda u^a q^a_+) = 0.
\]

(4.4)

Let us now gauge this isometry

\[
\lambda \Rightarrow \Lambda(\zeta, u), \quad \delta S^\text{free}_q = i \int dud\zeta (-2) \partial_t \Lambda (q^a u^a_+) \neq 0.
\]

(4.5)

To gauge-covariantize the action, we apply the relations (3.20) - (3.25). In the case under consideration:

\[
\mathcal{I} q^a = u^a.
\]

(4.6)

Two steps are needed to make the theory gauge invariant.

1. Covariantization of the constraint on \( q^a \):

\[
D^{++} q^a = 0 \Rightarrow D^{++} q^a = D^{++} q^a - V^{++} u^a = 0.
\]

(4.7)

The equivalent form of this constraint in terms of the superfields \( L^{++} \) and \( \omega \), eq. (4.1), can be easily obtained by projecting on \( u^\pm \):

\[
\begin{aligned}
(a) \quad D^{++} L^{++} &= 0, \\
(b) \quad D^{++} \omega - V^{++} &= L^{++}.
\end{aligned}
\]

(4.8)

2. Covariantization of the action \( S^\text{free}_q \).

The gauge-invariant action is constructed using the vector gauge connection (3.26)

\[
S^\text{free}_q (\text{cov}) = \frac{i}{2} \int dud\zeta (-2) \left\{ q^a \partial_t q^a_+ - i V(q^a q^a_+) + \xi V^{++} \right\}.
\]

(4.9)

Here, we have taken into account that we are free to add a FI term proportional to the arbitrary constant \( \xi \). It is worth noting that (4.9) is not just a replacement of \( \partial_t \) in (3.30) by the covariant derivative (3.25). This is related to the fact that (3.30) is invariant under the rigid isometry (4.3) up to a total derivative in the integrand.
The action (4.9) may be written as a general superspace integral

\[ S_{q,(cov)}^{\text{free}} = -\frac{1}{4} \int dudt d^4\theta \left\{ q^{+a}D^{-}q_{a}^{+} - 2V^{-}(q^{+a}u_{a}^{+}) - 2i\xi\theta \bar{\theta}V^{++} \right\}. \]  

(4.10)

Using eq. (4.8a), it is convenient to replace the second term in (4.10) by

\[ -2V^{-}(q^{+a}u_{a}^{+}) \to -(D^-)^2V^{++}(q^{+a}u_{a}^{+}). \]  

(4.11)

The equivalence of these two expressions may be shown by substituting, in the second expression, \( D^{-}V^{++} = D^{++}V^{-} \), pulling \( D^{++} \) to the left using \( [D^{++}, D^{-}] = D^{0} \) and \( D^{0}V^{-} = -2V^{-} \) and, finally, integrating by parts with respect to \( D^{++} \), taking into account the constraint \( D^{++}(q^{+a}u_{a}^{+}) = 0 \).

Then, coming back to the analytic superspace notation, we get

\[ S_{q,(cov)}^{\text{free}} = i \int dud\zeta^{(-2)} \left\{ q^{+a}\partial_{t}q_{a}^{+} - 2(q^{+a}u_{a}^{+})\Pi_{2}^{-}V^{++} + \xi V^{++} \right\}, \]  

(4.12)

where the analyticity-preserving operator \( \Pi_{2}^{-} \) was defined in (3.35).

Instead of the WZ gauge, we can choose the unitary-type gauge

\[ \omega = 0, \]  

(4.13)

which may be reached since \( \delta\omega = \Lambda \). Then, identifying in this gauge \( W^{++} \equiv L^{++} = q^{+a}u_{a}^{+} \), the basic constraints (4.8) amount to

\[ \begin{align*}
(a) & \quad D^{++}W^{++} = 0, \\
(b) & \quad V^{++} + W^{++} = 0 \Rightarrow V^{++} = -W^{++}.
\end{align*} \]  

(4.14)

In the gauge (4.13), one has

\[ q^{+a}\partial_{t}q_{a}^{+} = \partial_{t}\omega L^{++} - \omega \partial_{t}L^{++} = 0, \]  

(4.15)

and the action (4.12) becomes

\[ S_{q,(cov)}^{\text{free}} = i \int dud\xi \left\{ W^{++}\Pi_{2}^{-}W^{++} - \frac{1}{2}\xi W^{++} \right\}. \]  

(4.16)

Up to the FI term, we recover the free action (3.34) for the \( W^{++} \) superfield.

Let us comment on the general sigma-model type action (3.28). A simple analysis shows that the gauge invariant subclass of such actions corresponds to the choice

\[ S_{q,(cov)} = \int dudt d\theta \mathcal{L} \left( L^{++}, D^{-}L^{++}, (D^{-})^{2}L^{++}, u \right). \]  

(4.17)

In particular, the free gauge invariant Lagrangian in the central basis (the first two terms in (4.10)) can be written as (modulo purely analytic terms vanishing under the full superspace integral)

\[ L^{++}(D^{-})^{2}L^{++}. \]  

(4.18)

Any possible gauge invariant dimensionless quantity which can be constructed from the gauge-variant projection \( \omega \) is reduced to one of the functional arguments in \( \mathcal{L} \) in (4.17), e.g.,

\[ D^{-}\omega - V^{-} = \frac{1}{2}(D^{-})^{2}L^{++}, \quad \text{etc.} \]  

(4.19)
Thus in the unitary gauge (4.13) we recover the general $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ action (3.32).

Finally, the gauge-invariant subclass of the WZ terms (3.31) is

$$S_{q(cov)}^{WZ} = \int dud\zeta (-2) L^{+2}(q^{+a}u^+_a, u^\pm). \quad (4.20)$$

Note that the simplest choice

$$L^{+2}_0 = q^{+a}u^+_a = L^{++} \quad (4.21)$$

gives a vanishing contribution in the ungauged case: indeed, representing $u^+_a = D^{++}u^+_a$, integrating by parts and taking account of the original constraint (3.13) yield zero. At the same time, performing the same manipulations in the gauged case, we obtain, due to the modified constraint (4.7) (or (4.8)), that

$$L^{+2}_0 \Rightarrow -V^{++} \neq 0$$

i.e. (4.21) is reduced to the FI term of $V^{++}$.

To summarize, passing to the gauge $\omega = 0$ in which $V^{++} = -L^{++}$ and identifying $W^{++} = L^{++}$ like in the case of the gauged free $q$-action, we found that all general gauge-invariant $q$-actions (4.17), (4.20) became their $W$-counterparts defined in (3.32), (3.37). Thus the general action of the $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet (the sum of the sigma-model and WZ pieces) is equivalent to the subclass of the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet actions enjoying symmetry with respect to the gauged shift isometry $q^{+a} \Rightarrow q^{+a} + \Lambda u^{+a}$. Of course, one could arrive at the same conclusion in the WZ gauge (3.18) for the harmonic connection $V^{++}$. One can gauge away one physical bosonic field from the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ using the residual gauge freedom with the parameter $\lambda_0(t)$, after which the “former” gauge field $A(t)$ becomes the auxiliary field enlarging the rest of $(\mathbf{3} + \mathbf{4})$ fields to the off-shell $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ multiplet. The use of the unitary gauge $\omega = 0$ is advantageous in that it preserves the standard realization of $\mathcal{N}=4$ supersymmetry at all steps.

### 4.2 Rotational isometries

We consider the following infinitesimal gauge transformations

$$\delta q^+_a = \Lambda c^b_a q^+_b, \quad c^{ab} = c^{ba}, \quad \bar{c}^{ab} = c_{ab}, \quad c^{ab}c_{ab} \equiv c^2 = 2. \quad (4.22)$$

Thus in this case

$$\mathcal{I}q^+_a = c^b_a q^+_b. \quad (4.23)$$

The constraint (3.13) may be covariantized as

$$D^{++}q^{+a} + V^{++}c^a_b q^{+b} = 0. \quad (4.24)$$

The simplest gauge invariant action is obtained by covariantizing the free action (3.30) and it reads

$$S_{q(cov)}^{free} = \frac{i}{2} \int dud\zeta^{-2}\{q^{+a}\partial_\zeta q^+_a + \frac{i}{2} Vq^{+a}c_{ab}q^{+b} + \xi V^{++}\} \quad (4.25)$$

(where we also added an independent FI term). It turns out convenient to parametrize the superfields $q^{+a}$ as in eqs. (6.65), (6.66) of [5]

$$q^+_a = (c_{ad} + i\epsilon_{ad})(u^+d - ig^{+d}u^-d)e^{-i\frac{\zeta}{2}} + (c_{ad} - i\epsilon_{ad})(u^+d + ig^{+d}u^-d)e^{+i\frac{\zeta}{2}}, \quad (4.26)$$
The constraint (4.24) can be easily rewritten in these variables. To simplify things, note that the transformation (4.22) amounts to \( \delta \omega = 2 \Lambda \), so we can again choose the supersymmetry-invariant gauge \( \omega = 0 \). In this gauge

\[
q^+ = 2(c^a du^d + g^{++} u^{-a}) = 2(c^{++} + g^{++}) u^{-a} - 2c^{-} u^{+a},
\]
\[
(q^+ u^+) = -2(c^{++} + g^{++}), \quad (q^+ u^-) = -2c^{--}.
\]

(4.27)

Here \( c^{\pm \pm} = c^a u^a u^\pm \), \( c^{+-} = c^a u^a u^- \) and

\[
c^{++} c^{--} - (c^{-})^2 = \frac{1}{2} c^2 = 1,
\]

(4.28)

where the completeness relation (3.4) is used. We may use the expressions (4.27) in eq. (4.24) and project it on \( u^+_a \) and \( u^-_a \) to obtain the constraints

\[
D^{++} g^{++} + V^{++} g^{++} c^{--} = 0,
\]

(4.29)

\[
V^{++}(1 + c^{--} g^{++}) - g^{++} = 0 \Rightarrow V^{++} = \frac{g^{++}}{1 + g^{++} c^{--}}.
\]

(4.30)

Substituting (4.30) into (4.29), we obtain the nonlinear harmonic constraint

\[
D^{++} g^{++} = -\frac{(g^{++})^2 c^{--}}{1 + c^{--} g^{++}}.
\]

(4.31)

After going to the parametrization

\[
g^{++} = \frac{l^{++}}{1 + \sqrt{1 + c^{--} l^{++}}} \leftrightarrow l^{++} = (2 + c^{--} g^{++}) g^{++},
\]

(4.32)

the harmonic constraint (4.31) simplifies to

\[
D^{++} l^{++} = 0.
\]

(4.33)

The gauge superfield \( V^{++} \) in (4.30) may now be expressed as

\[
V^{++} = \frac{l^{++}}{(1 + \sqrt{1 + c^{--} l^{++}})\sqrt{1 + c^{--} l^{++}}}.
\]

(4.34)

From the last of equations (4.27) and from (4.32), we also find

\[
q^{+a} c_{ab} q^{+b} \equiv J^{++} = 4(c^{++} + l^{++}),
\]

(4.35)

and, as a consequence of (4.33), we have

\[
D^{++} J^{++} = 0.
\]

(4.36)

It is worth pointing out that eqs. (4.36) and (4.33) are the direct corollaries of (4.24) and so hold irrespective of any particular gauge fixing.
In the gauge $\omega = 0$ that we are considering, the action (4.25) also simplifies. One can show that the first term in the integrand in (4.25), i.e. $q^+ a \partial_t q^+$, vanishes up to a full time derivative. The second term takes a form similar to that obtained in the case of a shift isometry

$$\frac{i}{4} \int \!dud\zeta (-2) \{iV q^+ a c_b q^+ b\} = \frac{i}{2} \int \!dud\zeta (-2) \{V^{++}\Pi_2^- J^{++}\}$$

$$= 2i \int \!dud\zeta (-2) \frac{l^{++}}{(1 + \sqrt{1 + c_-} l^{++}) \sqrt{1 + c_-} l^{++}} \Pi_2^- (c^{++} + l^{++}) . \quad (4.37)$$

We now have a non-trivial sigma model action for the $(3, 4, 1)$ multiplet described by the superfield $l^{++}$. Recall that we started from a free $q^+$ action with the rotational isometry, we gauged this isometry and arrived in a unitary-type gauge $\omega = 0$ to a non-trivial $\sigma$-model action for the $(3, 4, 1)$ multiplet. The kinetic term (4.37) may be written in the full superspace as

$$\frac{1}{4} \int \!dudt d^4 \theta \frac{1}{(\sqrt{1 + l^{++} c^-})^3} (D^- l^{++} D^- l^{++}) . \quad (4.38)$$

Finally, the gauge-fixed form of the FI term in the action (4.25) is

$$\frac{i}{2} \xi \int \!dud\zeta (-2) \frac{l^{++}}{(1 + \sqrt{1 + c_-} l^{++}) \sqrt{1 + c_-} l^{++}} . \quad (4.39)$$

This is just the conformally-invariant potential term involving a coupling to a Dirac monopole background [16, 9]. One can see how simple the derivation of this term is in the approach based on gauging a rotational isometry as compared to the original derivation in [9]. There, this Lagrangian was restored step by step from the requirement of superconformal invariance. It would be interesting to elaborate on the superconformal properties of the sigma model action (4.38) and compare it with the conformally-invariant actions of $q^+$ deduced in [9] in the framework of ordinary $\mathcal{N}=4, d=1$ superspace.

It is worthwhile to note that we could from the very beginning add to the action (4.25) the gauge invariant action

$$\sim \int \!dud\zeta (-2) L^{++ 2} (q^+ a c_b q^+ b, u^\pm) , \quad (4.40)$$

which in components yields a $U(1)$ invariant coupling of $f^a(t)$ to some background gauge potential [9]. In the gauge $\omega = 0$ this term, together with the FI term of $V^{++}$, produce the most general superpotential for the $(3, 4, 1)$ multiplet described by $l^{++}$ (or $J^{++} = 4(l^{++} + c^{++})$). One can also start from the most general sigma model $q^+$ action invariant under the rotational isometry considered. The corresponding gauge invariant Lagrangian is a function of three possible invariants

$$J^{++} = q^+ a c_b q^+ b , \quad J^{+-} = q^+ a c_b q^- b , \quad J^{-+} = q^- a c_b q^- b \quad (q^{-a} \equiv D^- q^a) . \quad (4.41)$$

In the gauge $\omega = 0$ the full gauge-covariantized $q^+$ action (a sum of the sigma-model and WZ-like terms) is again reduced to the most general action of the $(3, 4, 1)$ multiplet, as in the case of a shift isometry. Note that the field redefinition (4.26) relating the rotational and shifting realizations of the same isometry (on the superfields $q^+$ and $(l^{++}, \omega)$, respectively) is highly nonlinear, so the gauge-invariant actions which look very simple in the shift case can take a rather complicated form in the rotational case and vice versa. For instance, the Dirac
monopole superpotential (4.39) is just the FI term of $V^{++}$, while in the shift case it is a sum of the relevant FI term and a properly chosen function $L^{+2}(q^{+a}u_{a}^{+}, u^{\pm})$.

It is worth pointing out that the final $(3, 4, 1)$ actions do not “remember” from which gauged $q^{+a}$ action they originated. Therefore the inverse procedure of “oxidation” of these actions to the $q^{+}$ actions is highly non-unique. For instance, one can substitute $W^{++}$ in the general $W$ actions either by $q^{+} \cdot u^{+}$, or $J^{++} = q^{+a}c_{a}bq^{+b}$, with $q^{+a}$ in both cases satisfying the linear constraint (3.13). In this way one recovers general $q^{+}$ actions possessing either shift or rotational isometries [9] (actually, these subclasses are related to each other via the redefinition (4.26)). Moreover, the same $W^{++}$ actions can be “oxidized” to the actions of another type of $(4, 4, 0)$ multiplet, the nonlinear one.

5 Nonlinear $(4, 4, 0)$ multiplet and its gauging

The standard $\mathcal{N}=4$ $(4, 4, 0)$ multiplet is defined by the linear constraints (3.13) or (3.15). Its general sigma model action always yields a conformally flat metric in the bosonic sector [9]. On the other hand, as was noticed for the first time in [18], one can deform the off-shell constraint (3.13) by some nonlinear terms, so as to gain a hyper-Kähler metric in the bosonic sector of the appropriate superfield $q^{+a}$ action (non-trivial hyper-Kähler metrics are not conformally flat).

This new possibility was demonstrated in [18] for the simple example of the Taub-NUT metric (see also [19] where the Eguchi-Hanson example was constructed via the $d=1$ version of the HSS quotient construction [20]). Here we generalize this deformation of the $q^{+a}$ constraints in such a way that, in the bosonic sector of the superfield action, a general Gibbons-Hawking (GH) ansatz [17] for 4-dimensional hyper-Kähler metrics with one triholomorphic isometry is reproduced (with Taub-NUT and Eguchi-Hanson as particular cases). After this we gauge this isometry with the help of the non-propagating “topological” $V^{++}$ and as output obtain the most general subclass of the $(3, 4, 1)$ multiplet actions admitting a closed formulation in the analytic harmonic $\mathcal{N}=4, d=1$ superspace. The corresponding 3-dimensional bosonic metric is conformally flat and satisfies the three-dimensional Laplace equation. It is just a quotient of the general GH metric by the action of a $U(1)$ isometry.

5.1 Hyper-Kähler $\mathcal{N}=4$ mechanics from a nonlinear deformation of the multiplet $(4, 4, 0)$

Our starting point will be the quadratic $q^{+a}$ action

$$S = i \frac{1}{2} \int dud\zeta (-2) (q^{+a}\partial_{a}q_{+}).$$

(5.1)

As we saw above, the action (5.1) gives rise to the free dynamics for the $(4, 4, 0)$ multiplet, if the latter is defined by the linear constraints (3.13). But the same action produces a non-trivial sigma model if the superfield $q^{+a}$ is required to satisfy the nonlinear constraint

$$D^{++} q^{+a} + u^{+a} L^{+2}(q^{+} \cdot u^{+}, u^{\pm}) = 0, \quad q^{+} \cdot u^{+} \equiv q^{+a}u_{a}^{+},$$

(5.2)

where $L^{+2}$ is an arbitrary charge +2 function of its arguments. These constraints have been chosen in such a way that they are invariant under the shift isometry (4.3):

$$q^{+a} \longrightarrow q^{+a} + \lambda u^{+a}.$$  

(5.3)
In the case of generic $\mathcal{L}^{+2}$, no other symmetry (besides $\mathcal{N}=4$ supersymmetry) is respected by the theory. In particular, the $\mathcal{N}=4$ superalgebra automorphism $SU(2)_A$ symmetry acting on the doublet indices of the harmonic variables $u^{\pm i}$ is fully broken due to the explicit harmonics in $\mathcal{L}^{+2}$.

Like in the examples considered in Sect. 3, we may represent $q^a$ by its projections on $u^a$ and $u^{-a}$ and rewrite (5.2) as the constraints for these projections

\[(a) \quad D^{++}(q^+ \cdot u^+) = 0, \quad (b) \quad D^{++}(q^+ \cdot u^-) - (q^+ \cdot u^+) + \mathcal{L}^{+2}(q^+ \cdot u^+, u^\pm) = 0. \quad (5.4)\]

The action (5.1) can be rewritten in terms of these projections (modulo a total $t$-derivative) as

\[S = i \int dud\zeta^{(-2)}(q^+ \cdot u^+)\partial_t(q^+ \cdot u^-). \quad (5.5)\]

In what follows we will be interested in the bosonic field contents of (5.4) and (5.5). The bosonic field content of the superfield $q^a$ is given by

\[q^a = f^+a(t, u) + (\theta^+ \bar{\theta}^+)A^{-a}(t, u) + \text{fermionic terms}. \quad (5.6)\]

The nonlinear constraint (5.2), like (3.13), leaves four bosonic and four fermionic fields as the irreducible field content of $q^+a$ off shell.\footnote{The property that (5.2) are off-shell constraints is the crucial difference with the case of the $\mathcal{N}=2, d=4$ hypermultiplet in $\mathcal{N}=2, d=4$ HSS where analogous relations are the equations of motion [21, 5].} One could explicitly solve (5.2) and find the full expressions for all the components in the expansion (5.6) in terms of this irreducible ($4 + 4$) set. Remarkably, in order to calculate the bosonic part of (5.5) we do not need to know the full solution of the component constraints.

After performing the Grassmann integration, the bosonic fields defined in (5.6) appear in the bosonic part of the action (5.1) (or (5.5)) as

\[S_{\text{bos}} = \frac{i}{2} \int dtdu (f^+a(t, u)\partial_tA^{-a}_a(t, u) + A^{-a}(t, u)\partial_tf^+_a(t, u)) = i \int dtdu A^{-a}(t, u)\partial_tf^+_a(t, u), \quad (5.7)\]

where we integrated by parts with respect to $\partial_t$. Using the completeness of the harmonic set $u^{\pm i}$, eq. (3.4), we may write the fields $f^+a$ and $A^{-a}$ as

\[f^+a(t, u) = -\varphi^{+2}(t, u)u^{-a} + v(t, u)u^a, \quad A^{-a}(t, u) = \varphi^{-2}(t, u)u^a + \varphi(t, u)u^{-a}. \quad (5.8)\]

Being rewritten in terms of the harmonic projections $\varphi^{\pm 2}$, $\varphi$ and $v$, the action (5.7) becomes

\[S = i \int dtdu \mathcal{F}(t, u), \quad \mathcal{F}(t, u) = \varphi^{+2}\partial_t\varphi^{-2} + v\partial_t\varphi. \quad (5.9)\]

When all fermions are omitted, the bosonic part of the constraint (5.2) amounts to the following set of equations for $\varphi^{\pm 2}$, $\varphi$ and $v$

1) $\partial^{++}\varphi^{+2} = 0 \Rightarrow \varphi^{+2}(t, u) = x^{(+)}(t)u^+_i u^+_j \equiv x^{++}$, \quad (5.10)
2) $\partial^{++}v - x^{++} + \mathcal{L}^{+2}(x^{++}, u) = 0$, \quad (5.11)
3) $\partial^{++}\varphi + 2i\partial_t x^{++} = 0 \Rightarrow \varphi(t, u) = \varphi_0(t) - 2i\partial_t x^{+-}(t, u), \partial^{++}\varphi_0 = 0$, \quad (5.12)
4) $\partial^{++}\varphi^{-2} + \varphi - 2i\partial_t v - \varphi \frac{\partial\mathcal{L}^{+2}}{\partial x^{++}} = 0$. \quad (5.13)
Then, taking the harmonic integral of both sides of (5.13), it is easy to get

\[ \frac{\partial \mathcal{L}^+}{\partial x^+} \equiv \mathcal{L}, \quad \mathcal{L} \equiv \int du \mathcal{L}, \quad A^{(ik)} = \int du u^+(i \mu^k) \mathcal{L}. \] (5.14)

We denote by \( \varphi_0 \) and \( v_0 \) those parts of \( \varphi \) and \( v \) which are independent of harmonic variables. Then, taking the harmonic integral of both sides of (5.13), it is easy to get

\[ \varphi_0(1 - L) - 2i \partial_t v_0 + 2i A^{(ik)} \partial_t x_{(ik)} = 0 \implies \varphi_0 = \frac{2i}{1 - L} [\partial_t v_0 - \partial_t x_{(ik)} A^{(ik)}]. \] (5.15)

Further, using the solution (5.12) for \( \varphi \), we find that

\[ \int dt du (v \partial_t \varphi) = \int dt du [-\partial_t v_0 \varphi_0 - i(1 - \mathcal{L})(\partial_t x^+ + \partial_t x^-)]. \] (5.16)

In this calculation we firstly used the solution for \( \varphi(t, u) \), then integrated by parts with respect to \( \partial^+ \) and \( \partial_t \) and finally made use of eq. (5.11). In the same spirit, we can work out the second term \( \varphi^+ \partial_t \varphi^- = x^+ \partial_t x^- \) appearing in the bosonic action (5.9)

\[ \int dt du (x^+ \partial_t \varphi^-) = \int dt du [-i(\partial_t x^- \partial_t x^+) + \varphi_0 \partial_t x^{(ik)} A_{(ik)} + 2i(\partial_t x^+ \partial_t x^-)(1 - \mathcal{L})]. \] (5.17)

Here we have used the time derivative of eq. (5.11)

\[ \partial^+ \partial_t v = \frac{\partial}{\partial t}[x^+ - \mathcal{L}^+ (x^+, u)] = \partial_t x^+(1 - \mathcal{L}). \] (5.18)

Summing up (5.16) and (5.17), we obtain for the action (5.9)

\[ S = i \int dt du [-\partial_t v_0 \varphi_0 - 2i(\partial_t x^+ \partial_t x^- - \partial_t x^+ \partial_t x^-)(1 - \mathcal{L}) + \varphi_0 \partial_t x^{(ik)} A_{(ik)}] \]

\[ = \frac{i}{2} \int dt [\partial_t v_0 \varphi_0 - i \partial_t x^{(ik)} \partial_t x_{(ik)} (1 - \mathcal{L}) + \varphi_0 \partial_t x^{(ik)} A_{(ik)}]. \] (5.19)

To obtain the last expression, we made use of the relation

\[ \partial_t x^+ \partial_t x^- - \partial_t x^+ \partial_t x^- = \frac{1}{2} \partial_t x^{ik} \partial_t x_{ik}, \] (5.20)

which follows from the completeness condition (3.4). Substituting the expression in eq. (5.15) for \( \varphi_0 \) into (5.19), we find

\[ S = \frac{1}{2} \int dt \left\{ \frac{2}{1 - L} [\partial_t v_0 - (\partial_t x^{(ik)} A_{(ik)})]^2 + (1 - L) \partial_t x^{(ik)} \partial_t x_{(ik)} \right\}. \] (5.21)

Introducing the notations

\[ \tilde{\mathcal{A}} = -i(\tilde{\tau})^{(ik)} A^{(ik)}, \quad \tilde{x} = \frac{i}{\sqrt{2}} (\tilde{\tau})^{(ik)} x^{(ik)}, \quad \tilde{v}_0 = \sqrt{2} v_0, \] (5.22)

where \( \tau^a, a = 1, 2, 3 \), are Pauli matrices, we cast (5.21) into the form

\[ S = \frac{1}{2} \int dt \left\{ \frac{1}{1 - L} (\partial_t \tilde{v}_0 + \partial_t \tilde{x} \cdot \tilde{\mathcal{A}})^2 + (1 - L) \partial_t \tilde{x} \cdot \partial_t \tilde{x} \right\}. \] (5.23)
which is just the \( d=1 \) pullback of the general Gibbons-Hawking ansatz for four-dimensional hyper-Kähler metrics with one triholomorphic \( U(1) \) isometry. The isometry is realized as a constant shift of \( ˜v_0 \). It is easy to check that, by definition,

\[
\Delta(1 - L) = \partial_{\bar{v}_0}(1 - L) = 0, \quad \bar{\nabla} \wedge \bar{A} = \bar{\nabla}(1 - L),
\]

(5.24)

which are general defining equations of the multicenter GH ansatz with the potential \( V \equiv 1 - L \).

For the case of Taub-NUT in the parametrization considered here \( L^{+2} \) is given by [22]

\[
L^{+2} = 2\lambda \frac{(x^{++} - c^{++})}{(1 + \sqrt{1 + (x^{++} - c^{++})c^{--}})\sqrt{1 + (x^{++} - c^{++})c^{--}}},
\]

(5.25)

Then

\[
L = \int du \frac{\partial L^{+2}}{\partial x^{++}} = \lambda \int du \frac{1}{(\sqrt{1 + (x^{++} - c^{++})c^{--}})^3} = \lambda \frac{\sqrt{c^2}}{\sqrt{x^{ik}x_{ik}}} = \sqrt{2}\lambda \frac{1}{\sqrt{x^{ik}x_{ik}}},
\]

(5.26)

from which we obtain the typical Taub-Nut potential

\[
1 - L = 1 - \sqrt{2}\lambda \frac{1}{|x|}.
\]

(5.27)

It is interesting to compare this result with another form of the Taub-NUT metric, corresponding to the realization of the triholomorphic \( U(1) \) isometry of the latter not as a shift (like in the above parametrization) but as a rotation. The corresponding \( \mathcal{N}=4 \) superfield action is still given by

\[
S_{TN} = \frac{i}{2} \int dud\bar{\zeta}^{(-2)} \hat{q}^+ a \partial \hat{q}^+_a,
\]

(5.28)

while the constraint reads \(^6\)

\[
D^{++} \hat{q}^a + f \mathcal{J}^{++} c^a_b \hat{q}^b = 0, \quad \mathcal{J}^{++} \equiv \hat{q}^a c_{ab} \hat{q}^b.
\]

(5.29)

where \( f \) is a coupling constant and \( c_{ab} = c_{ba}, c^2 = 2 \). The action and constraint are invariant under the rigid \( U(1) \) transformation embedded into the \( SU(2)_{PG} \) symmetry realized on the doublet indices \( a, b \):

\[
\delta q^+_a = \lambda c^a_b q^+_b.
\]

(5.30)

Note the “conservation law” which follows from (5.29)

\[
D^{++} \mathcal{J}^{++} = 0.
\]

(5.31)

The transformation law (5.30) coincides with the rigid version of the rotational isometry transformation (4.22) of the case of linear \((4, 4, 0)\) multiplet, and (5.31) with (4.36).

Now we wish to show that, using an appropriate change of variables, this particular system can be brought into the generic form of the \( \mathcal{N}=4, d=1 \) multicenter sigma model as given above. This change of variables was introduced in [4] in the context of \( \mathcal{N}=2, d=4 \) hypermultiplets with an infinite number of auxiliary fields and we already used it in Sect. 4.2. It is given by eq.\(^6\)This constraint is equivalent to that of [18] where a particular form of the constant triplet \( c^{(ab)} \) was used.
Like in Sect. 4.2, we pass to the new analytic superfield \( l^{++} \) which is related to \( g^{++} \) in (4.26) as
\[
g^{++} = \frac{l^{++}}{1 + \sqrt{1 + c^{-}l^{++}}}. \tag{5.32}
\]
It is easy to check that \( J^{++} = 4(c^{++} + l^{++}) \). \( \tag{5.33} \)
Then, as a corollary of (5.31), it follows that
\[
D^{++}l^{++} = 0. \tag{5.34}
\]
The rotational isometry (5.30), in terms of the superfields \( \omega, l^{++} \), is realized as a shift
\[
\delta l^{++} = 0, \quad \delta \omega = 2\lambda. \tag{5.35}
\]
A direct calculation shows that, modulo total derivatives,
\[
\hat{q}^{+a} \partial_t \hat{q}_a^+ = -2l^{++} \partial_t \omega. \tag{5.36}
\]
Substituting (4.26) and (5.32) into the constraint (5.29) while taking into account its consequence (5.34), one finds that (5.29) also implies the following condition on the superfield \( \omega \):
\[
D^{++} \omega = 8f(c^{++} + l^{++}) - 2 \frac{l^{++}}{(1 + \sqrt{1 + c^{-}l^{++}}) \sqrt{1 + c^{-}l^{++}}}. \tag{5.37}
\]
Thus the original constraint (5.29), being equivalently rewritten in terms of the superfields \( \omega \) and \( l^{++} \), amounts to two constraints (5.34) and (5.37).
Now, combining \( \omega \) and \( L^{++} \equiv l^{++} + c^{++} \) into a new \( q^{+a} \) as
\[
q^{+a} = u^{+a} \frac{1}{2} \omega + u^{-a} L^{++}, \quad (q^{+} \cdot u^{+}) \equiv x^{++} = -L^{++}, \quad (q^{+} \cdot u^{-}) = \frac{1}{2} \omega, \tag{5.38}
\]
we observe that the isometry (5.30) is realized on this \( q^{+a} \) just by shifts (5.3), so we can apply the whole reasoning which led from (5.1) to (5.23). Comparing (5.36), (5.37) with (5.5) and (5.4), we find
\[
\hat{q}^{+a} \partial_t \hat{q}_a^+ = 2q^{+a} \partial_t q_a^+ = -2L^{++} \partial_t \omega + \text{total } t\text{-derivative} \tag{5.39}
\]
and
\[
L^{++}_{TN} = (1 + 4f)x^{++} - \frac{(x^{++} + c^{++})}{[1 + \sqrt{1 - c^{-}(x^{++} + c^{++})}] \sqrt{1 - c^{-}(x^{++} + c^{++})}}. \tag{5.40}
\]
The latter yields the potential
\[
V_{TN} = 1 - L_{TN} = -4f + \frac{1}{\sqrt{2 |\vec{x}|}}, \tag{5.41}
\]
which basically coincides with (5.27). Taking into account the relation (5.39), we conclude that the action (5.28) with the constraint (5.29) gives rise, up to some rescaling, to the same Taub-NUT distance as (5.1) with the constraint (5.2) for the particular choice (5.25) of the nonlinear function \( L^{+2}(q^{+} \cdot u^{+}, u) \) in (5.2).
5.2 Gauging

We now wish to gauge the shift isometry of the action (5.1) and constraint (5.2). The action becomes

\[ S = \frac{i}{2} \int d\zeta (-2) q^a \partial_t q^a \rightarrow S_g = \frac{i}{2} \int d\zeta (-2) \left\{ q^a \partial_t q^a - iV(q^+ \cdot u^+) + \xi V^+ \right\} \]

\( \Leftrightarrow S_g = \frac{i}{2} \int d\zeta (-2) \left\{ q^a \partial_t q^a - 2V^+ \Pi_2^{-}(q^+ u^+) + \xi V^+ \right\}. \)  (5.42)

The last term in eq. (5.42), as in other cases, is a FI term. The covariantized constraint reads

\[ D^{++} q^a - V^+ u^a + u^a \L^{++2}(q^+ \cdot u^+, u) = 0. \]  (5.43)

Projecting these equations on \( u^+_a \) and \( u^-_a \), we obtain

\[ D^{++}(q^+ \cdot u^+) = 0, \quad D^{++}(q^+ \cdot u^-) - (q^+ \cdot u^+) - V^+ + \L^{++2} = 0. \]  (5.44)

In the gauge \( q^+ \cdot u^- = 0 \) these relations yield

\[ V^+ = \L^{++2}(W^+, u) - W^+, \quad W^+ \equiv (q^+ \cdot u^+), \quad D^{++} W^+ = 0. \]  (5.45)

The first term in the gauged action, \( q^a \partial_t q^a \), vanishes in the gauge chosen. The only superfield which remains is \( W^+ \) satisfying the constraint \( D^{++} W^+ = 0 \). The resulting action reads

\[ S_g = \frac{i}{2} \int d\zeta (-2) \left\{ 2[W^+ - \L^{++2}(W^+, u)] \Pi_2^{-} W^+ + \xi (\L^{++2} - W^+) \right\}. \]  (5.46)

The first piece in (5.46) is the most general sigma-model action for the \((3, 4, 1)\) multiplet which admits a representation in \( N=4, d=1 \) harmonic superspace [9]. It is worth noting that the whole nonlinearity of this action originates from the function \( \L^{++2} \) which initially enters the nonlinear \( q^a \) constraint (5.2). On the other hand, the property that the same function specifies both the sigma model and WZ parts is to some extent accidental. One could, from the very beginning, add to the action (5.1) an independent WZ term invariant under (5.3), i.e. \( \L^{++2}(q^+ \cdot u^+, u) \).

This would have no impact on the structure of the sigma-model part, but would modify the WZ term in the gauge-fixed action (5.46) by adding \( \L^{++2}(W^+, u) \) and so would make the WZ term fully independent of the sigma-model one.

It is worth saying a few words about the oxidation procedure in the present case. One could forget about the origin of the sigma-model part in the action (5.46) and consider it just as a convenient parametrization of a general analytic superspace sigma-model action of the multiplet \((3, 4, 1)\), with \( \L^{++2}(W^+, u) \) fully characterizing such an action [9]. At the component level, such actions are distinguished in that the target metric satisfies the Laplace equation with respect to three target bosonic coordinates. Like in the previous cases, one can oxidize this \( W^+ \) action to the relevant subclass of actions of the linear \((4, 4, 0)\) multiplet, with one shift or rotational isometry, and the conformally flat target metric satisfying the four-dimensional analog of Laplace equation. On the other hand, being aware of the existence of nonlinear \((4, 4, 0)\) multiplet with the constraint (5.2), one can alternatively oxidize (5.46) just to this new multiplet by identifying \( W^+ \) with the relevant \( (q^+ \cdot u^+) \) and the nonlinear function in (5.2) with the function \( \L^{++2}(W^+, u) \) from (5.46). After substituting (5.4b) into (5.46), integrating by parts with respect to \( D^{++} \) and using the last relation in (3.36), one arrives just at the action
(5.5) which was the starting point in finding the HK \(d=1\) sigma model with the general GH ansatz in Sect. 5.1. A component version of this superfield oxidation procedure was described in a recent paper [23].

Finally, we would like to note that the transition to the \((3, 4, 1)\) sigma model can perhaps be even more clearly understood in the WZ gauge (3.18) for \(V^{++}\). In this gauge, the whole effect of covariantization of the bosonic action (5.23) amounts to the change \(\partial_t \tilde{v}_0 \Rightarrow \partial_t \tilde{v}_0 + \sqrt{2} A\). The residual gauge freedom acts as an arbitrary time-dependent shift of the field \(\tilde{v}_0\), so we can further fix it by the condition \(\tilde{v}_0 = 0\). In this gauge, the first term in the covariantized bosonic action (5.23) basically becomes just the square of the auxiliary field. In the absence of FI term (i.e. for \(\xi = 0\), the auxiliary field fully decouples and the remaining bosonic distance is just the second term in (5.23)

\[
\sim (1 - L) \partial_t \vec{x} \cdot \partial_t \vec{x}.
\]

It is conformally flat, with the conformal factor \((1 - L)\) being a general solution of the 3-dimensional Laplace equation. If \(\xi \neq 0\), the elimination of the auxiliary field induces a non-trivial scalar potential and the coupling to an abelian background gauge field

\[
-\frac{\xi^2}{4} (1 - L) + \frac{\xi}{\sqrt{2}} \partial_t \vec{x} \cdot \vec{A}.
\]

6 Some other \(\mathcal{N}=4\) gauged models

Here we show that one physical bosonic field of the \((4, 4, 0)\) multiplet can be traded for an auxiliary field in such a way that one ends up with a nonlinear \((3, 4, 1)\) multiplet introduced in [9, 10]. The corresponding isometry to be gauged is some non-compact scaling invariance. Also, we show how to describe the off-shell multiplet \((1, 4, 3)\) in \(\mathcal{N}=4, d=1\) HSS and, by gauging the appropriate shift symmetry of the free action of this multiplet, to reproduce the action of the fermionic multiplet \((0, 4, 4)\) [9]. Finally, we give an example of non-abelian \(SU(2)\) gauging which directly yields a \((1, 4, 3)\) multiplet, starting from the \((4, 4, 0)\) one.

6.1 Nonlinear \((3,4,1)\) multiplet from gauging \((4,4,0)\)

Besides the shift and rotational isometries, one can implement on the superfield \(q^{+a}\) also a rigid rescaling symmetry\(^7\)

\[
\delta q^{+a} = \lambda q^{+a}.
\]

The linear constraint (3.13) is obviously covariant under these rescalings. One can also define an invariant sigma-model type action with the Lagrangian which can in general depend on harmonics and all possible scale invariant ratios of the harmonic projections of \(q^{+a}\) and \(q^{-a}\), namely,

\[
(q^+ \cdot u^+)/ (q^+ \cdot u^-), \ (q^- \cdot u^-)/ (q^+ \cdot u^-), \ (q^- \cdot u^+)/ (q^+ \cdot u^-),
\]

where, in order to avoid singularities, we are led to assume that the quantities \(q^+ \cdot u^- = q^{+a}u_a^-\) and \(q^- \cdot u^+ = q^{-a}u_a^+\) start with constant parts \(^8\) (in principle, one can also divide by non-singular

\(^7\)It does not affect the superspace coordinates and so has no relation to the ordinary dilatations and super-conformal group. It can be interpreted as dilatations in the target space.

\(^8\)These objects are in a sense similar to the hypermultiplet conformal compensators in the HSS formulation of the most general off-shell version of \(\mathcal{N}=2, d=4\) Einstein supergravity [4].
linear combinations of \((q^+ \cdot u^-)\) and \((q^- \cdot u^+)\). The simplest such action is the scale-invariant analog of the free action (3.30). It admits a concise representation as an integral over the analytic HSS

\[
S^\text{scale}_q = -\frac{1}{4} \int dudtd^4\theta \left( \frac{q^- \cdot u^+}{q^+ \cdot u^-} \right) = i \frac{1}{2} \int dud\zeta (-2) \left( \frac{\partial q^+ \cdot u^+}{q^+ \cdot u^-} \right).
\]

(6.3)

It is straightforward to check that in the bosonic sector this action yields the unique scale and \(SU(2) \times SU(2)\) invariant action \(^9\)

\[
S^\text{scale}_{q(\text{bos})} = \frac{1}{2} \int dt \, G(f) \partial_t f^{ia} \partial_i \zeta, \quad G(f) = \int du \frac{1}{(f^{ia}u_i^+u_a^-)^2} = \frac{2}{(f^{ia}f_{ia})}.
\]

(6.4)

The metric function \(G(f)\) satisfies the four-dimensional harmonicity equation, like in any sigma-model \(q^+a\) action admitting a formulation in the analytic \(d=1\) HSS [9]. One can also define the scale-invariant subclass of the general analytic \(q^+\) WZ terms (3.31)

\[
S^\text{WZ(scale)}_q = \int dud\zeta (-2) \mathcal{L}^{+2} \left( \frac{q^+ \cdot u^+}{q^+ \cdot u^-}, u^\pm \right).
\]

(6.5)

Let us now gauge the isometry (6.1) as we did in Sect. 3, i.e. by replacing the rigid parameter \(\lambda\) with an unconstrained analytic function \(\Lambda(\zeta, u)\). The gauge-covariant generalization of (3.13) reads

\[
(D^{++} - V^{++}) q^+a = 0,
\]

(6.6)

or, in terms of the harmonic projections,

\[
(a) \quad (D^{++} - V^{++})(q^+ \cdot u^-) - (q^+ \cdot u^+) = 0, \quad (b) \quad (D^{++} - V^{++})(q^+ \cdot u^+) = 0.
\]

(6.7)

The action (6.3) can also be easily covariantized as

\[
S^\text{scale}_{q(\text{cov})} = \frac{1}{4} \int dudtd^4\theta \left[ \left( \frac{D^- q^+ \cdot u^+}{q^+ \cdot u^-} \right) - V^{--} \left( \frac{q^+ \cdot u^+}{q^+ \cdot u^-} \right) \right] = \frac{i}{2} \int dud\zeta (-2) \left[ \left( \frac{\partial q^+ \cdot u^+}{q^+ \cdot u^-} \right) - \frac{i}{2} V \left( \frac{q^+ \cdot u^+}{q^+ \cdot u^-} \right) \right],
\]

(6.8)

where the analytic vector gauge connection \(V(\zeta, u)\) was defined in (3.26). The term (6.5) is invariant under both rigid and gauge transformations, so it does not require any covariantization.

To reveal what the gauged model describes one can choose the WZ gauge (3.18). However, like in the previous cases, we first use a gauge which preserves manifest supersymmetry. The projection \(\omega \equiv q^+a u_a^-\) is transformed as \(\delta \omega = \Lambda \omega\). Thus, taking into account that \(\omega\) is assumed to start with a constant, one can impose the gauge \(\omega = 1\). Denoting \(N^{++} = q^+a u_a^+\), in this gauge the constraints (6.7) become

\[
(a) \quad V^{++} = -N^{++}, \quad (b) \quad D^{++}N^{++} - V^{++}N^{++} = 0 \Rightarrow D^{++}N^{++} + N^{++}N^{++} = 0.
\]

(6.9)

\(^9\)To calculate \(G(f)\), one should represent \(f^{ia} = -i \zeta^{ia} f + f^{(ia)}\) whence \(G(f) = \frac{1}{2} \int du \frac{1}{(\frac{1}{f^{(ia)}}/f)u_i^+u_a^-}^2\). The latter harmonic integral can be directly computed, e.g. by expanding the integrand in powers of \((f^{(ia)}/f)u_i^+u_a^-\) and doing the integrals in each term.
The only remaining superfield is the analytic superfield $N^{++}$, and the constraint (6.9) is just the defining constraint of the nonlinear $(3,4,1)$ multiplet $[9,10]$. Thus the gauging of the scale isometry of the $(4,4,0)$ multiplet leaves us with the nonlinear $(3,4,1)$ multiplet, in contradistinction to the gauging of the shift and rotational isometries which leads to the linear $(3,4,1)$ multiplet.

In the gauge $\omega = 1$ the first term in the second line of (6.8) becomes a total derivative, so (6.8) is reduced to

$$S_{\text{scale}}^{(\text{cov})} = \frac{1}{4} \int dud\zeta^{(-2)} V N^{++}. \quad (6.10)$$

It describes $N=4$ superextension of some $d=1$ nonlinear sigma model with 3-dimensional target manifold. The Lagrangian in (6.10) is a function of $N^{++}$ since $V = D^{+} \bar{D}^{+} V^{--}$ and $D^{++} V^{++} + D^{--} N^{++} = 0$ (recall eq. (3.23)). This action can be explicitly expressed in terms of $N^{++}$ in the central basis as

$$S_{\text{scale}}^{(\text{cov})} \sim \int dt d\theta d\bar{\theta} dudv N^{++}(u) \frac{1}{(u^{+} \cdot v^{+})^{2}} N^{++}(v), \quad (6.11)$$

where both $N^{++}$ are given at the same superspace “point” $(t, \theta, \bar{\theta})$ but at the different harmonic sets $u^{\pm}_{i}$ and $v^{\pm}_{i}$. The harmonic Green function $1/(u^{+} \cdot v^{+})^{2}$ is defined in [5].

To learn which $d=1$ sigma model underlies the action (6.10), it is simpler to make use of the WZ gauge in (6.7) and (6.8). For the bosonic action, the effect of gauging the scale invariance amounts to changing the time derivatives in (6.4) to the covariant one

$$S \sim \int dt \frac{1}{(f^{ia} f_{ia})} \nabla_{t} f^{ia} \nabla_{t} f_{ia}, \quad \nabla_{t} f^{ia} = (\partial_{t} + A) f^{ia}. \quad (6.12)$$

Fixing the gauge with respect to the residual gauge freedom $\delta f^{ia} = \lambda(t) f^{ia}$ in such a way that

$$f^{ia} f_{ia} = 1, \quad (6.13)$$

and eliminating the field $A(t)$ by its algebraic equation of motion ($A(t)$ fully decouples in the gauge (6.13)), we can reduce (6.12) to

$$S \sim \int dt \partial_{t} f^{ia} \partial_{t} f_{ia}, \quad f^{ia} f_{ia} = 1. \quad (6.14)$$

It is just the action of $d=1$ sigma model for the principal chiral field, i.e. for the coset $S^{3} \sim SO(4)/SO(3)$. Thus we started from the model the bosonic sector of which is described by the scale-invariant sigma-model action (6.4) corresponding to a four-dimensional target with a linearly realized $SO(4)$ symmetry. Then we gauged the target space scale invariance in this model and, as a result, gained the sigma model with the homogeneous three-dimensional target $\sim S^{3}$ with half of $SO(4)$ nonlinearly realized. One could naïvely expect that the gauging of the dilatations leads to the projective space $\mathbb{R}P^{3} = S^{3}/Z_{2}$ which is the space of straight lines in $\mathbb{R}^{4}$. However, the gauge fixing leaves untouched the “large” gauge transformations $f^{ia} \rightarrow -f^{ia}$ and thus $S^{3}$ emerges$^{10}$. One can obtain more general sigma models, with broken $SO(4)$, by gauging some other scale-invariant $q^{+a}$ actions having no simple description in the analytic HSS. Also,

$^{10}$ $N=4$ superextension of the same bosonic model was constructed in [10], using the description of the nonlinear $(3,4,1)$ multiplet in the ordinary $N=4, d=1$ superspace. Presumably, the superfield action of [10] is related to (6.11) via a field redefinition.
it is of clear interest to seek a scale-invariant generalization of the nonlinear \((4, 4, 0)\) constraint (5.2) and to study the relevant gauged models.

The superpotential term (6.5) in the gauge \(\omega = 1\) becomes

\[
S^W_N = \int dud\zeta \zeta^{-2} \mathcal{L}^2 \left( N^{++}, u^\pm \right). \tag{6.15}
\]

The addition of the FI term \(\sim \xi V^{++} = -\xi N^{++}\) in the present case is not crucial, since (6.5) can include, prior to any gauging, the non-vanishing term \(\sim (q^a u^+_a)/(q^a u^-_a)\) which coincides with the FI term in the gauge \(\omega = 1\).

### 6.2 From \((1,4,3)\) to \((0,4,4)\)

The off-shell multiplet \((1,4,3)\) exists in two forms which differ in the \(SU(2)\) assignment of the auxiliary fields: they can be triplets with respect to one or another automorphism \(SU(2)\) groups forming the full \(SO(4)\) automorphisms of \(\mathcal{N}=4, d=1\) Poincaré supersymmetry [24, 25].

One of these multiplets admits a simple description in the full \(\mathcal{N}=4, d=1\) HSS. It is represented by the real superfield \(\Omega(x, \theta, \bar{\theta}, u)\) satisfying the constraints

\[
\begin{align*}
(a) & \quad D^+ \bar{D}^+ \Omega = 0, \quad (b) \quad D^{++} \Omega = 0.
\end{align*}
\tag{6.16}
\]

In the analytic basis, eq. (6.16a) implies that \(\Omega\) is linear in the non-analytic coordinates,

\[
\Omega = \Sigma(\zeta, u) + i \left[ \theta^- \Psi^+(\zeta, u) + \bar{\theta}^- \bar{\Psi}^+(\zeta, u) \right],
\tag{6.17}
\]

while (6.16b) amounts to the following harmonic constraints on the analytic superfunctions in (6.17)

\[
\begin{align*}
(a) & \quad D^{++} \Psi^+ = D^{++} \bar{\Psi}^+ = 0, \quad (b) \quad D^{++} \Sigma + i \left( \theta^+ \Psi^+ + \bar{\theta}^+ \bar{\Psi}^+ \right) = 0.
\end{align*}
\tag{6.18}
\]

The general solution of (6.18) is

\[
\begin{align*}
\Psi^+ &= \psi^i u^+_i + \theta^+ s + \bar{\theta}^+ r + 2i\theta^+ \bar{\theta}^+ \partial_i \psi^ju^-_i, \\
\bar{\Psi}^+ &= -\bar{\psi}^i u^+_i - \theta^+ \bar{r} + \bar{\theta}^+ \bar{s} - 2i\theta^+ \bar{\theta}^+ \partial_i \bar{\psi}^ju^-_i, \\
\Sigma &= \sigma - i\theta^+ \psi^ju^-_i + i\bar{\theta}^+ \bar{\psi}^ju^-_i,
\end{align*}
\tag{6.19}
\]

where

\[
\text{Re } r = \partial_t \sigma.
\tag{6.21}
\]

The independent fields \(\sigma(t), \psi^i(t), s(t), \text{Im } r(t)\) constitute an off-shell \((1,4,3)\) multiplet.

The most general sigma-model type action of this multiplet is given by the following integral over the full harmonic superspace

\[
S_\Omega = \int dudtd^4 \theta \mathcal{L}(\Omega, u^\pm).
\tag{6.22}
\]

For our purposes we consider only its free part

\[
S^\text{free}_\Omega = \int dudt d^4 \Omega^2 = \int dud\zeta \zeta^{-2} \Psi^+ \bar{\Psi}^+ = \int dt \left[ (\partial_t \sigma)^2 - 2i\psi^i \partial_t \bar{\psi}^i + ss + (\text{Im } r)^2 \right].
\tag{6.23}
\]

\[
= \int dt \left[ (\partial_t \sigma)^2 - 2i\psi^i \partial_t \bar{\psi}^i + ss + (\text{Im } r)^2 \right].
\tag{6.24}
\]

25
It is invariant under the constant shifts \( \Omega \rightarrow \Omega + \lambda \), since the integral of \( \Omega \) over the full superspace vanishes as a consequence of (6.16a). Both constraints (6.16) are also manifestly invariant. Now let us gauge this isometry by replacing, as usual, \( \lambda \rightarrow \Lambda(\zeta, u) \). The superfield action (6.23) (prior to passing to the components) and the constraint (6.16a) remain unchanged, while the harmonic constraint (6.16b) needs the obvious covariantization

\[
(6.16b) \Rightarrow D^{++}\Omega - V^{++} = 0 .
\] (6.25)

Clearly, one can choose the supersymmetric gauge

\[
\Sigma = 0 ,
\] (6.26)

in which the covariantized harmonic constraint (6.25) is reduced to

\[
(a) \ D^{++}\Psi^+ = D^{++}\bar{\Psi}^+ = 0 , \quad (b) \ V^{++} = i \left( \theta^+\Psi^+ + \bar{\theta}^+\bar{\Psi}^+ \right) .
\] (6.27)

The solution of (6.27a) is just (6.19) without the additional condition (6.21). So we end up with the fermionic analog of the \( q^+ \) hypermultiplet, the off-shell \((0, 4, 4)\) multiplet formed by the fields \( \psi^i(t), s(t), r(t) \). The action (6.23) in components becomes

\[
S_{\Psi}^{\text{free}} = \int dud\zeta(-2) \Psi^+\bar{\Psi}^+ = \int dt \left( s\bar{s} + r\bar{r} - 2i\psi^i\partial_t\bar{\psi}^i \right) .
\] (6.28)

One can also add the FI term \( \sim \xi V^{++} \) which in the gauge considered here yields a “potential” component term

\[
S_{FI} = \frac{i}{2} \xi \int dud\zeta(-2) V^{++} = -\frac{\xi}{2} \int dud\zeta(-2) \left( \theta^+\Psi^+ + \bar{\theta}^+\bar{\Psi}^+ \right) = -\frac{\xi}{2} \int dt(r + \bar{r}) .
\] (6.29)

Less trivial examples can be obtained by considering a few independent \((1, 4, 3)\) multiplets and gauging some abelian isometries of the relevant sigma-model type superfield actions. One of such multiplets can always be traded for the \((0, 4, 4)\) multiplet which will non-trivially interact with the remaining \((1, 4, 3)\) multiplets. For instance, one can introduce two superfields \( \Omega_1 \) and \( \Omega_2 \), start with the scale invariant superfield Lagrangian \( \mathcal{L}(\Omega_1/\Omega_2, u) \) and gauge the scale invariance. Another possibility is to consider a complex \( \Omega \) undergoing \( U(1) \) transformations \( \delta \Omega = i\lambda \Omega \), start with the \( U(1) \) invariant Lagrangian \( \mathcal{L}(\Omega\bar{\Omega}, u) \) and gauge this \( U(1) \) isometry. We plan to consider these and some other models elsewhere.

### 6.3 An example of non-abelian gauging

As our final example we consider non-abelian gauging of the \((4, 4, 0)\) multiplet action. The covariantized action in a fixed gauge yields an action of the \((1, 4, 3)\) multiplet. Like in the previous example, we shall limit our consideration to the gauging of the free action, this time the free action of the \( q^{+a} \) multiplet (3.30), leaving the consideration of more complicated gauged systems for future study.

The action (3.30) exhibits a manifest invariance under the global \( SU(2) \) transformations which act on the doublet index \( a \) of \( q^{+a} \) and so commute with \( \mathcal{N}=4 \) supersymmetry\(^{11} \).

\[
\delta q^{+a} = \lambda^a_b q^{+b}, \quad \lambda^{0}_a = 0 .
\] (6.30)

\(^{11}\)By analogy with the \( \mathcal{N}=2, d=4 \) case such \( SU(2) \) symmetry can be called “Pauli-Gürsey” \( SU(2) \).
Let us gauge this symmetry by changing $\lambda^a_b \rightarrow \Lambda^a_b(\zeta, u)$. The constraint (3.13) is covariantized to
\[ D^{++} q^{+a} - V^{++a}_b q^{+b} = 0 , \] (6.31)
where the traceless analytic gauge connection $V^{++a}_b$ is transformed as
\[ \delta V^{++a}_b = D^{++} \Lambda^a_b + \Lambda^a_c V^{++c}_b - V^{++c}_b \Lambda^c_b . \] (6.32)

Using this freedom, one can pass to the WZ gauge as in the abelian case (3.18)
\[ V^{++a}_b = 2 i \theta^+ \bar{\theta}^+ A^a_b(t) , \quad \delta_r A^a_b = - \partial_t A^{(0)}_b + \Lambda^{(0)}_c A^c_b - A^{c} A^{(0)}_c . \] (6.33)

The action (3.30) is covariantized by changing
\[ \partial_t q^{+a} \Rightarrow \nabla_t q^{+a} = \partial_t q^{+a} - \frac{i}{2} V^a q^{+b} , \] (6.34)
where
\[ V^a = D^+ \bar{D} + V^{--a}_b , \quad D^{++} V^{--a}_b - D^{--} V^{++a}_b - V^{++a} V^{--c}_b + V^{--c} V^{++}_b = 0 . \] (6.35)

In the present case we do not know a manifestly supersymmetric gauge, so we prefer to deal with the WZ gauge (6.33) in which
\[ V^{--a}_b = 2 i \theta^- \bar{\theta}^- A^a_b(t) , \quad V^a_b = D^+ \bar{D} + V^{--a}_b = 2 i A^a_b(t) . \] (6.36)

In this gauge, the solution of the covariantized constraint (6.31) is obtained from the solution (3.13) just by the replacement
\[ \partial_t f^{ia} \Rightarrow \nabla_t f^{ia} = \partial_t f^{ia} + A^a_b f^{ib} . \] (6.37)

After substituting this covariantized solution for $q^{+a}$ into the covariantization of the action (3.30) and performing there the Grassmann and harmonic integration, we arrive at the following component action
\[ S_{na}^{bos} = \int dt \left( \nabla_t f^{ia} \nabla_t f_{ia} - i \chi^a \nabla_t \bar{\chi}^a \right) . \] (6.38)

Splitting $f^{ia}$ as
\[ f^{ia} = \varepsilon^{ia} \frac{1}{\sqrt{2}} f + f^{(ia)} , \] (6.39)
and assuming that $f$ has a non-vanishing constant vacuum part, $f = < f > + \ldots , < f > \neq 0$, one observes that the symmetric part in (6.39) can be fully gauged away by the residual $SU(2)$ gauge freedom
\[ f^{ia} \Rightarrow \varepsilon^{ia} \frac{1}{\sqrt{2}} f . \] (6.40)

In this gauge, the action (6.38) becomes
\[ S_{na}^{bos} = \int dt \left[ (\partial_t f)^2 - i \chi^a \partial_t \bar{\chi}^a + f^2 \frac{1}{2} A^{(ab)} A_{(ab)} - i \chi^{(a} \bar{\chi}^{b)} A^{(ab)} \right] \] (6.41)
where the former gauge field $A^{(ab)}$ becomes a triplet of auxiliary fields. So, gauging the “Pauli-Gürsey” $SU(2)$ symmetry of the free action of $(4, 4, 0)$ multiplet and choosing the appropriate
gauge in the resulting covariantized action, we arrived at the action (6.41) which describes an interacting system of 1 physical bosonic field $f(t)$, the fermionic doublet $\chi^a(t)$ and the triplet of auxiliary fields $A^{(ab)}(t)$, that is just the field content of off-shell $\mathcal{N}=4, d=1$ multiplet $(1, 4, 3)$.

The $\mathcal{N}=4$ supersymmetry of the action (6.41) is guaranteed, since we started from the manifestly supersymmetric action and just fixed some gauges in it. The modified supersymmetry transformations can be easily found, but we will not dwell on this. We only note that after eliminating the auxiliary field from (6.41), the latter takes the following on-shell form

$$S_{na}^{bos} = \int dt \left[ (\partial_t f)^2 - i\chi^a \partial_t \bar{\chi}_a + \frac{3}{8f^2} (\chi^a \chi_a)(\bar{\chi}_a \bar{\chi}_a) \right].$$

(6.42)

It remains to reveal whether we obtained in this way the standard linear $(1, 4, 3)$ supermultiplet [24, 25], or some new nonlinear version.

More general models of self-interacting $(1, 4, 3)$ multiplet could be obtained via the $SU(2)_{PG}$ gauging of more general sigma-model actions of $q^{+a}$. It is impossible to construct any non-trivial $SU(2)_{PG}$ invariant in the analytic HSS, besides the free action (3.30). In particular, any WZ term $L^{+2}(q^+, u)$ necessarily breaks $SU(2)_{PG}$. On the other hand, in the full superspace one can easily construct non-analytic $SU(2)_{PG}$ invariant $q^{+a}q_a^{-}$ and consider general $SU(2)_{PG}$ invariant sigma model Lagrangians as functions of this invariant and explicit harmonics. We plan to consider the $SU(2)_{PG}$ gaugings of such more general $\mathcal{N}=4, d=1$ actions elsewhere.

7 Conclusions

In this article we have shown, on a few instructive examples, that $d=1$ dualities between supermultiplets with the same number of physical fermions but varying numbers of physical and auxiliary bosonic fields amount to gauging of the appropriate isometries of the relevant superfield actions by “topological” gauge multiplets. We mainly concentrated on the case of $\mathcal{N}=4$ supersymmetric mechanics and considered various reductions of the actions of the $(4, 4, 0)$ multiplet to those of the $(3, 4, 1)$ multiplet. Both linear and nonlinear versions of these multiplets were regarded. As a by-product, we constructed a new nonlinear $(4, 4, 0)$ multiplet within a manifestly supersymmetric superfield formulation. The target metric in the bosonic sector of the simplest invariant action of this multiplet is the general 4-dimensional HK metric with one triholomorphic isometry.

Although our consideration here was limited to $\mathcal{N} \leq 4$, the essence of the method is expected to be the same for any $\mathcal{N}$, and it goes in two steps. In a first step, one uses the gauge freedom to get rid of the whole of the topological gauge supermultiplet but the gauge field. In this Wess-Zumino gauge, the gauge field does not transform at all under supersymmetry. Further, what remains of the gauge freedom is used to get rid of one physical bosonic field in a unitary-type gauge. With this additional gauge-fixing, the gauge field becomes an auxiliary field of the new (reduced) multiplet. In practice, every time when this is possible it proves more advantageous to use a manifestly supersymmetric gauge (if it exists) at each step of the procedure. The outcome naturally remains the same, but a clear merit of this alternative gauge choice is the opportunity to stay within the superfield approach. While the direct (“reduction”) procedure is always fully specified by the choice of isometry to be gauged and the form of the gauged action, the inverse one (“oxidation”) is not unique: the same reduced superfield action can be
promoted to different particular actions of the “oxidized” multiplet, depending on the choice of the substitution which expresses the reduced multiplet in terms of the “oxidized” one.

One of the possible further areas where our new approach could be efficiently employed is $\mathcal{N}=8$ supersymmetric mechanics. There exists a plethora of various $\mathcal{N}=8$ multiplets [26], as well as of associated mechanics models (see e.g. [3] and refs. therein), and our procedure could probably be very helpful in establishing interrelations between these multiplets and models. There still remain some open problems in the $\mathcal{N}=4$ case. For instance, it would be of obvious interest to understand how to reproduce in our approach the reductions of the $(4,4,0)$ multiplet to the off-shell multiplets $(2,4,2)$. These reductions, both for the linear [27, 24] and nonlinear [10] versions of the multiplet $(2,4,2)$, were described in [14] at the component level. As one of the conceivable ways of translating them into the superfield language, it is natural to simultaneously gauge two mutually commuting isometries, e.g. the rotational and scale isometries (5.30) and (6.1), realized on the same superfield $q^{+a}$. Also, only the simplest case of a nonabelian gauging which relates the multiplets $(4,4,0)$ and $(1,4,3)$ has been studied in this article. There are other possibilities which still require to be investigated.

Finally, we would like to point out that a necessary condition of applicability of our superfield gauging procedure is the commutativity of the isometries to be gauged with the global supersymmetry (triholomorphicity). For isometries which do not meet this criterion (e.g. those belonging to R-symmetry groups) a gauging seems to be possible only within an extended framework of non-dynamical superfield $d=1$ supergravities [28, 29].

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