ENTIRE SOLUTIONS AND TRAVELING WAVE SOLUTIONS OF THE ALLEN-CAHN-NAGUMO EQUATION

HIROKAZU NINOMIYA
School of Interdisciplinary Mathematical Sciences, Meiji University
4-21-1 Nakano, Nakano-ku, Tokyo 164-8525, Japan

Abstract. In this paper, the propagation phenomena in the Allen-Cahn-Nagumo equation are considered. Especially, the relation between traveling wave solutions and entire solutions is discussed. Indeed, several types of one-dimensional entire solutions are constructed by composing one-dimensional traveling wave solutions. Combining planar traveling wave solutions provides several types of multi-dimensional traveling wave solutions. The relation between multi-dimensional traveling wave solutions and entire solutions suggests the existence of new traveling wave solutions and new entire solutions.

1. Introduction. Propagation phenomena have attracted research attention in many fields. Propagation while maintaining a certain shape is observed not only in propagation phenomena, such as sound and vibration, but also in phenomena accompanying energy dissipation, such as diffusion. A solution that constantly maintains a certain profile and moves with a constant speed is called a traveling wave solution. A nonlinear mechanism and a suitable choice of profile are required for preserving a constant shape. Diverse complicated patterns and dynamics are often observed owing to the collision or synthesis of multiple waves. To understand these propagation phenomena, we need to determine a universal profile or an intrinsic shape. We may understand more complicated propagation phenomena by using universal profiles and intrinsic shapes. Toward this end, we consider a reaction-diffusion equation that consists of a diffusion term and a nonlinear term:

\[ u_t = \Delta u + f(u), \]  

where \( x \in \mathbb{R}^N \), \( u_t := \partial u / \partial t \), and \( \Delta u := \sum_{j=1}^{N} \partial^2 u / \partial x_j^2 \). The traveling wave solution of this equation has been studied by many researchers and is well known. The property of the traveling wave solution is roughly classified into two cases according to the behavior of the solution of the ordinary differential equation \( u_t = f(u) \):

- Monostable case: \( u_t = f(u) \) has a unique stable equilibrium;
- Bistable case: \( u_t = f(u) \) has two stable equilibria.

2010 Mathematics Subject Classification. Primary: 35C07, 35K57; Secondary: 35B40, 35C08.
Key words and phrases. Traveling wave, reaction-diffusion equations.

The author was partially supported by JSPS KAKENHI Grant Numbers JP26287024, JP15K04963, JP16K13778, and JP16KT0022.

* Corresponding author: H. Ninomiya.
The former case was first studied by Fisher [6] and Kolmogorov, Petrovsky, and Piskunov [15]. A typical example of the nonlinear function in this case is $f(u) = u(1 - u)$. Equation (1) is often called the \textit{Fisher-KPP equation}. The latter case was studied by Kanel [14], Hadeler and Rothe [9], etc. A typical example of the nonlinear function in this case is $f(u) = u(1 - u)(u - a)$, where $0 < a < 1$. Equation (1) is often called the \textit{Allen-Cahn equation} or the Nagumo equation. In this paper, we call it the \textit{Allen-Cahn-Nagumo equation}. The difference between the two above-mentioned cases will be explained in the next section. We focus on the latter case in this paper. Hereafter, we always assume that $f$ is a smooth function with only three zeros $0, a, 1$ satisfying

$$f'(0) < 0, \quad f'(1) < 0, \quad f'(a) > 0, \quad \int_{0}^{1} f(s)ds > 0$$

unless otherwise noted.

To study the universal profile of the solutions of (1), let us consider the solution defined for all time. This solution is called an \textit{entire solution}. However, because the reaction-diffusion equation is a partial differential equation of parabolic type, it is generally ill posed to solve the problem for $t < 0$. Therefore, the set of initial functions is restricted and essential profiles will be obtained. At this point, it is difficult to determine the set of all entire solutions of (1). Here, we will create more complicated entire solutions by composing simple entire solutions as a piece. In this paper, we review each piece as well as how to glue or combine these pieces mathematically. For a linear equation, the linear sum of solutions also satisfies the equation. For the monostable case, such as the Fisher-KPP equation, the linearized equation plays an important role in the propagation, and the linear sum of solutions becomes a subsolution. Thus, the linear sum is available as a composition of solutions for the monostable case. However, the same is not true for the bistable case, which hence requires new methods.

The remainder of this paper is organized as follows. In Section 2, we review the results of the traveling wave solutions of (1). Although the Allen-Cahn-Nagumo equation is of bistable type, it also includes the traveling wave solutions of monostable type. In Section 3, we discuss one-dimensional entire solutions. The notion of the entire solution originating from multiple fronts is introduced. Specifically, we will construct more complicated entire solutions by gluing several types of traveling wave solutions. Then, it is shown that there exist entire solutions originating from $k$ fronts if $2 \leq k \leq 4$, whereas no entire solution originates from $k$ fronts if $k \geq 5$.

In Section 4, multi-dimensional traveling wave solutions are discussed. By combining planar traveling wave solutions in the multi-dimensional space, we will construct the traveling wave solutions. First, we explain how to imagine the multi-dimensional traveling wave solutions by combining them. Thus, V-shaped traveling wave solutions and pyramidal traveling wave solutions are explained. Then, conical traveling wave solutions and convex-conical traveling wave solutions are explained.

In Section 5, the heuristic understanding between $N$-dimensional traveling wave solutions and $(N - 1)$-dimensional entire solutions will be explained. Based on these observations, new types of traveling wave solutions and entire solutions will be expected. Actually, we can show the existence of a traveling wave solution connecting the unstable stationary solution to $a$, which is called a zipping wave solution. In addition, we explain a multi-dimensional entire solution whose level sets are approximately equidistant from any convex body as $t$ tends to $-\infty$. 
2. **One-dimensional traveling wave solutions.** First, consider one-dimensional traveling wave solutions of the Allen-Cahn-Nagumo equation. By the phase plane analysis, we see that there exist three types of traveling wave solutions:

\[-c_0 \psi'_0 = \psi''_0 + f(\psi_0), \quad \psi'_0 > 0, \quad \lim_{z \to -\infty} \psi_0(z) = 0, \quad \lim_{z \to \infty} \psi_0(z) = 1, \quad (3)\]

\[-c_1 \psi'_1 = \psi''_1 + f(\psi_1), \quad \psi'_1 > 0, \quad \lim_{z \to -\infty} \psi_1(z) = 0, \quad \lim_{z \to \infty} \psi_1(z) = a, \quad (4)\]

\[-c_2 \psi'_2 = \psi''_2 + f(\psi_2), \quad \psi'_2 > 0, \quad \lim_{z \to -\infty} \psi_2(z) = a, \quad \lim_{z \to \infty} \psi_2(z) = 1, \quad (5)\]

where \(c_j (j = 0, 1, 2)\) are the speeds of the traveling wave solutions and \(c_0 \leq 0, \ c_1 > 0, \ c_2 < 0\). The pair \((c_j, \psi_j) (j = 0, 1, 2)\) is called a traveling wave solution of (1), or \(\psi_j\) is called a traveling wave solution of (1) with speed \(c_j\). It is also often called a front. We note that \(c_0\) and \(\psi_0\) are uniquely determined (up to the shift for \(\psi_0\)). Actually, when the nonlinearity is \(f(u) = u(1-u)(u-a)\), we can exactly calculate \(c_0\) and \(\psi_0\) as follows:

\[
\psi_0(x) = \frac{1}{1 + e^{-x/\sqrt{2}}}, \quad c_0 = \sqrt{2} \left( \frac{1}{2} - a \right). \quad (6)
\]

In particular, we remark that \(c_0 = 0\) when \(a = 1/2\).

Because a traveling wave solution \(\psi_1\) connects a stable state 0 and an unstable state \(a\), there exists a minimal speed \(c^*_1\) such that if \(c_1 \geq c^*_1\), there exists a traveling wave solution \(\psi_1\) satisfying (4). For \(\psi_2\), there is a minimal (negative) speed \(c^*_2\) such that \(\psi_2\) exists for any \(c_2 \leq c^*_2 < 0\). Then, there exist positive constants \(\rho_1, \rho_2,\) and \(\gamma\) satisfying

\[
\begin{align*}
\inf_{x \leq 0} \frac{\psi'_0(x)}{\psi_0(x)} & \geq \rho_1, & \inf_{x \geq 0} \left| \frac{\psi'_0(x)}{\psi_0(x) - 1} \right| & \geq \rho_1, & 0 \leq \psi'_0(x) & \leq \rho_2 e^{-\gamma |x|} \\
\inf_{x \leq 0} \frac{\psi'_1(x)}{\psi_1(x)} & \geq \rho_1, & \inf_{x \geq 0} \left| \frac{\psi'_1(x)}{\psi_1(x) - a} \right| & \geq \rho_1, & 0 \leq \psi'_1(x) & \leq \rho_2 e^{-\gamma |x|} \\
\inf_{x \leq 0} \left| \frac{\psi'_2(x)}{\psi_2(x) - a} \right| & \geq \rho_1, & \inf_{x \geq 0} \left| \frac{\psi'_2(x)}{\psi_2(x) - 1} \right| & \geq \rho_1, & 0 \leq \psi'_2(x) & \leq \rho_2 e^{-\gamma |x|}.
\end{align*}
\]

(7)

The above-mentioned traveling wave solutions are monotone. In general, non-monotone traveling wave solutions may exist. Actually, it has been shown in [9] that non-monotone traveling wave solutions connecting 0 and \(a\) exist if \(0 < a < 1/3\). Moreover, if \(|c| < 2\sqrt{f'(a)}\), there is a traveling wave solution that converges to \(a\) oscillatingly.

3. **One-dimensional entire solutions.** We denote a set of entire solutions by \(\mathcal{E}\). Recall that an entire solution refers to a solution for all time. In this section, we always assume that \(N = 1\).

First, we adduce stationary solutions as the members of \(\mathcal{E}\). There are three homogeneous stationary solutions 0, \(a\), and 1. By (2), there is a standing wave solution \(u^*\) such that

\[
u^* > 0, \quad \lim_{|x| \to \infty} u^*(x) = 0,
\]

which belongs to \(\mathcal{E}\). There are many spatially periodic solutions around \(a\), which also belong to \(\mathcal{E}\).
The connecting orbits between equilibria also belong to the set of entire solutions. The homogeneous connecting orbits $W_{\pm}(t)$ satisfy

$$W_{\pm}(t) = f(W_{\pm}(t)), \quad \lim_{t \to -\infty} W_{\pm}(t) = a, \quad \lim_{t \to \infty} W_{+}(t) = 1, \quad \lim_{t \to \infty} W_{-}(t) = 0,$$

which belong to $\mathcal{E}$. As inhomogeneous connecting orbits in $\mathcal{E}$, there exist solutions $w_{\pm}(x,t)$ that connect $u^*$ to 0 or 1. Specifically,

$$w_{\pm,t} = \Delta w_{\pm} + f(w_{\pm}), \quad \lim_{t \to -\infty} w_{\pm}(x,t) = u^*(x), \quad \lim_{t \to \infty} w_{+}(x,t) = 1, \quad \lim_{t \to \infty} w_{-}(x,t) = 0.$$

Further details can be found in the paper by Fukao, Morita and Ninomiya [7].

Next, we see that the traveling wave solutions $\psi_0, \psi_1, \text{and} \psi_2$ in Section 2 also belong to $\mathcal{E}$. To consider more complicated entire solutions, we construct them by combining these members of $\mathcal{E}$.

**Theorem 3.1** ([29]). There is an entire solution $\Phi_1$ such that

$$\lim_{t \to -\infty} \left\{ \sup_{x \leq 0} |\Phi_1(x,t) - \psi_0(-x - c_0t)| + \sup_{x \geq 0} |\Phi_1(x,t) - \psi_0(x - c_0t)| \right\} = 0.$$

Moreover,

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\Phi_1(x,t) - 1| = 0.$$

Set

$$\mathcal{F}[u] := u_t - u_{xx} - f(u). \quad (8)$$

Yagisita [29] first proved this theorem by using the theory of the invariant manifold. Fukao et al [7] introduced a method for constructing supersolutions and subsolutions. Then, this method was improved and extended to the monostable case and more general settings by Guo and Morita [8], Chen and Guo [2], and Morita and Ninomiya [16]. Let us explain how to construct the supersolutions and subsolutions for this simple case.

**Proof.** Set

$$\begin{align*}
Q_1(y,z) &:= 1 - (1-y)(1-z), \\
U(x,t) &:= Q_1(\psi_0(-x + p(t)), \psi_0(x + p(t))),
\end{align*} \quad (9)$$

where $p(t)$, which will be specified later, converges to $-\infty$ as $t$ tends to $-\infty$. By the choice of $Q_1$, we have

$$\begin{align*}
Q_1(1,z) &= Q_1(y,1) = 1, \quad Q_1(0,z) = z, \quad Q_1(y,0) = y, \\
Q_{1,y}(y,z) &= 1 - z, \quad Q_{1,z}(y,z) = 1 - y, \\
Q_{1,yy}(y,z) &= Q_{1,zz}(y,z) = 0, \quad Q_{1,yz}(y,z) = -1.
\end{align*}$$

By plugging $U$ into (8), we get

$$\begin{align*}
\mathcal{F}[U] &= Q_{1,y} \psi'_0(-x + p)p' + Q_{1,z} \psi'_0(x + p)p' \\
&\quad + (Q_{1,y} \psi'_0(-x + p) - Q_{1,z} \psi'_0(x + p))x - f(U),
\end{align*}$$

where $\psi_0'(\cdot)$ denotes the derivative of $\psi_0(\cdot)$.
where we use \( Q_{1,y} = Q_{1,y}(\psi_0(-x + p), \psi_0(x + p)) \) and so on for simplicity. Using (3) implies that

\[
\mathcal{F}[U] = Q_{1,y} \psi'_0(-x + p)(p' + c_0) + Q_{1,z} \psi'_0(x + p)(p' + c_0) \\
+ (-Q_{1,yy} \psi'_0(-x + p)^2 + 2Q_{1,yz} \psi'_0(-x + p) \psi'_0(x + p) - Q_{1,zz} \psi'_0(x + p)^2) \\
+ Q_{1,y} f(\psi_0(-x + p)) + Q_{1,z} f(\psi_0(x + p)) \\
- f(Q_1(\psi_0(-x + p(t)), \psi_0(x + p(t)))) \\
= A(x, p)(p' + c_0) - B(x, p) - G(\psi_0(-x + p), \psi_0(x + p)),
\]

where

\[
A(x, p) := Q_{1,y}(\psi_0(-x + p), \psi_0(x + p)) \psi'_0(-x + p) \\
+ Q_{1,z}(\psi_0(-x + p), \psi_0(x + p)) \psi'_0(x + p) \\
= (1 - \psi_0(-x + p)) \psi'_0(-x + p) + (1 - \psi_0(-x + p)) \psi'_0(x + p),
\]

\[
B(x, p) := -2Q_{1,yz} \psi'_0(-x + p) \psi'_0(x + p) = 2 \psi'_0(-x + p) \psi'_0(x + p),
\]

\[
G(y, z) := f(Q_1(y, z)) - Q_{1,y}(y, z) f(y) - Q_{1,z}(y, z) f(z).
\]

When \( f(u) = u(1 - u)(u - a) \), it follows that

\[
G(y, z) = y(1 - y)z(1 - z)(2 + a - 2y - 2z + yz).
\]

(10)

For a more general nonlinear function, we do not have the above-mentioned equality, but we have a similar property (see [16, Section 2]). The monotonicity of \( \psi_0 \) induces

\[
A(x, p) \geq (1 - \psi_0(0)) \psi'_0(x + p) > 0 \quad \text{for } x \geq 0 \text{ and } p \leq 0.
\]

(11)

Similarly, we also have

\[
A(x, p) \geq (1 - \psi_0(0)) \psi'_0(-x + p) > 0 \quad \text{for } x \leq 0 \text{ and } p \leq 0.
\]

(12)

By (11), (12), and the definition of \( B \), we have

\[
\frac{|B(x, p)|}{A(x, p)} \leq \begin{cases} 
\frac{2}{1 - \psi_0(0)} \psi'_0(-x + p) \leq \frac{2\rho_2}{1 - \psi_0(0)} e^{\gamma p} & \text{for } x \geq 0 \text{ and } p \leq 0, \\
\frac{2}{1 - \psi_0(0)} \psi'_0(x + p) \leq \frac{2\rho_2}{1 - \psi_0(0)} e^{\gamma p} & \text{for } x \leq 0 \text{ and } p \leq 0.
\end{cases}
\]

Using (7), (10), (11), and (12), we obtain

\[
\frac{|G(\psi_0(-x + p), \psi_0(x + p))|}{A(x, p)} \leq 
\begin{cases} 
\frac{3 + a}{(1 - \psi_0(0))\rho_1} \psi'_0(-x + p) \leq \frac{(3 + a)\rho_2}{(1 - \psi_0(0))\rho_1} e^{\gamma p} & \text{for } x \geq 0 \text{ and } p \leq 0, \\
\frac{3 + a}{(1 - \psi_0(0))\rho_1} \psi'_0(x + p) \leq \frac{(3 + a)\rho_2}{(1 - \psi_0(0))\rho_1} e^{\gamma p} & \text{for } x \leq 0 \text{ and } p \leq 0.
\end{cases}
\]

Gathering these inequalities yields

\[
\mathcal{F}[U] \geq A(x, p)(p' + c_0 - C_1 e^{\gamma p}),
\]

where

\[
C_1 := \frac{2\rho_2}{1 - \psi_0(0)} + \frac{(3 + a)\rho_2}{(1 - \psi_0(0))\rho_1}.
\]

By taking \( p = p_+ \) satisfying

\[
p'_+ = -c_0 + C_1 e^{\gamma p_+}
\]
for $t \leq 0$, $U_+(x,t) := U(x,p_+(t))$ becomes a supersolution of (1). Note that because $c_0 \leq 0$, $p_+(t)$ becomes

$$p_+(t) = -c_0 t - \frac{1}{\gamma} \log \left\{ e^{-\gamma p_+(0)} + \frac{C_1 (e^{-\gamma c_0 t} - 1)}{c_0} \right\}$$

as $t \to -\infty$. If we take $p = p_-$ satisfying

$$p_-' = -c_0 - C_1 e^{\gamma p_-}$$

for $t \leq 0$, then $U_-(x,t) := U(x,p_-(t))$ satisfies

$$\mathcal{F}[U_-] = A(x,p_-)(p_-' + c_0) - B(x,p_-) - G(x - p_-, \psi_0(x + p_-))$$

which means that $U_-$ is a subsolution of (1). Moreover, if

$$e^{-\gamma p_+(0)} + \frac{C_1}{c_0} = e^{-\gamma p_-(0)} - \frac{C_1}{c_0}$$

then there is a positive constant $C_2$ such that

$$0 \leq p_+(t) - p_-(t) \leq C_2 e^{-\gamma c_0 t}$$

for any $t \leq 0$. This implies that

$$0 \leq U_+(x,t) - U_-(x,t) \leq C_3 e^{-\gamma c_0 t}$$

for $x \in \mathbb{R}$ and $t \leq 0$ for some positive constant $C_3$. There is a unique entire solution $\Phi_1$ of (1) satisfying

$$U_-(x,t) \leq \Phi_1(x,t) \leq U_+(x,t)$$

for $x \in \mathbb{R}$ and $t \leq 0$ (see also [22, 7, 16, 27]). Thus, the assertion of this theorem can be shown easily. \hfill \Box

See Figure 1 (a) for the profile of solutions in this theorem. In the proof, we used a specific nonlinearity. Since we have an equality similar to (10) for a more general nonlinearity, this method is applicable to the bistable case as well as to the monostable case (see [8, 2]).

This theorem gives a universal profile of solutions when two opposing traveling wave solutions annihilate each other. To extend this result, we consider entire solutions $\Phi$ originating from $k$ fronts $\{(c_j, \phi_j), j = 1, 2, \cdots, k\}$ ($k \geq 2$) satisfying the condition

$$c_1 > c_2 > \cdots > c_k$$

such that

$$\lim_{t \to -\infty} \max_{1 \leq j \leq k} \sup_{w_{j-1}(t) < x < w_j(t)} \left| \Phi(x,t) - \phi_j(x - c_j t + \eta_j) \right| = 0$$

for some constants $\eta_1, \cdots, \eta_k$, where $\phi_j$ is a traveling wave solution with speed $c_j$, $w_j(t) := -(c_j + c_{j+1})t/2$, $w_0(t) := -\infty$, and $w_k(t) := \infty$. Theorem 3.1 gives us an entire solution originating from two fronts.

Morita and Ninomiya [16] provided other entire solutions originating from two fronts.
Figure 1. Profiles of supersolutions for one-dimensional entire solutions when \(-t\) is large, where \(f(u) = u(1-u)(u-1/12)\).

Theorem 3.2 ([16]). There are an entire solution \(\Phi_2\) and a constant \(x_0\) satisfying

\[
\lim_{t \to -\infty} \left\{ \sup_{x \leq (c_1+c_2)t/2} |\Phi_2(x,t) - \psi_1(x-c_1t)| + \sup_{x \geq (c_1+c_2)t/2} |\Phi_2(x,t) - \psi_2(x-c_2t)| \right\} = 0,
\]

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\Phi_2(x,t) - \psi_0(x-c_0t + x_0)| = 0.
\]

Moreover, if \(c_1^* \leq c_1 < -c_0\), then there is an entire solution \(\Phi_3\) satisfying

\[
\lim_{t \to -\infty} \left\{ \sup_{x \leq (c_0-c_1)t/2} |\Phi_3(x,t) - \psi_1(-x-c_1t)| + \sup_{x \geq (c_0-c_1)t/2} |\Phi_3(x,t) - \psi_0(x-c_0t)| \right\} = 0,
\]

\[
\lim_{t \to \infty} \sup_{x \in [-\xi, \infty)} |\Phi_3(x,t) - 1| = 0, \quad \lim_{t \to \infty} \inf_{x \in \mathbb{R}} \Phi_3(x,t) = a
\]

for any \(\xi > 0\).

Although we can add the spatial shifts in these conditions, we ignore them in this paper for simplicity. The profiles of the solutions \(\Phi_2\) and \(\Phi_3\) when \(-t\) is large are shown in Figure 1 (b) and (c), respectively. When \(f(u) = u(1-u)(u-1/12)\), the condition \(c_1^* < -c_0\) can be confirmed.

The proof of this theorem is similar to that of Theorem 3.1, but it becomes more technical (see [16] for the details). Instead of (9), the functions

\[
Q_2(y,z) := \frac{(1-a)yz}{y(z-a) + a(1-z)},
\]

\[
Q_3(y,z) := \frac{a(y+z) - (1+a)yz}{a - yz}
\]

were used to construct the supersolutions and subsolutions for \(\Phi_2\) and \(\Phi_3\), respectively.

The following question arises:

Are there any entire solutions originating from three or more fronts?

Chen, Guo, Ninomiya, and Yao [4] answered this question affirmatively through the following theorem.
Theorem 3.3 ([4]). If $c_1^* \leq c_1 < -c_0$ and $c_4^* \leq \hat{c}_1 < -c_0$, then there exist two entire solutions $\Phi_4$ and $\Phi_5$ of (1) originating from three fronts such that

$$
\lim_{t \to -\infty} \left\{ \sup_{x \leq (c_1-c_0)t/2} |\Phi_4(x,t) - \psi_0(-x-c_0t)| + \sup_{(c_1-c_0)t/2 \leq x \leq (c_1+c_2)t/2} |\Phi_4(x,t) - \psi_1(x-c_1t)| + \sup_{x \geq (c_1+c_2)t/2} |\Phi_4(x,t) - \psi_2(x-c_2t)| \right\} = 0,
$$

$$
\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} |\Phi_4(x,t) - 1| = 0,
$$

$$
\lim_{t \to -\infty} \left\{ \sup_{x \leq (\hat{c}_1-c_0)t/2} |\Phi_5(x,t) - \psi_0(-x-c_0t)| + \sup_{(c_1-c_0)t/2 \leq x \leq (\hat{c}_1-\hat{c}_2)t/2} |\Phi_5(x,t) - \psi_1(x-\hat{c}_1t)| + \sup_{x \geq (\hat{c}_1-\hat{c}_2)t/2} |\Phi_5(x,t) - \hat{\psi}_1(-x-\hat{c}_1t)| \right\} = 0,
$$

$$
\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} |\Phi_5(x,t) - \psi_0(-x+c_0t + x_0)| = 0,
$$

where $(\hat{c}_1, \hat{\psi}_1)$ is a traveling wave solution with speed $\hat{c}_1 \geq c_4^*$, which satisfies (4), and $x_0$ is a constant.

To show the existence of $\Phi_4$ and $\Phi_5$, the auxiliary functions

$$
Q_4(y,z,w) := z + (1-z) \frac{(1-y)z(w-a) + y(a-z)(1-w)}{(1-y)z(1-a) + (a-z)(1-w)},
$$

$$
Q_5(y,z,w) := z + \frac{(1-y)z(a-w)(-z) + y(a-z)w(1-z)}{(1-y)za + (a-z)w},
$$

respectively, were used in [4].

Theorem 3.4 ([4]). If $c_4^* \leq c_1 < -c_0$, then there exists a symmetric (with respect to $x = 0$) entire solution $\Phi_6$ of (1) such that

$$
\Phi_6(-x,t) = \Phi_6(x,t),
$$

$$
\lim_{t \to -\infty} \left\{ \sup_{|x| \leq (c_0-c_1)t/2} |\Phi_6(x,t) - \psi_1(|x| - c_1t)| + \sup_{|x| \geq (c_0-c_1)t/2} |\Phi_6(x,t) - \psi_0(|x| - c_0t)| \right\} = 0,
$$

$$
\lim_{t \to -\infty} \sup_{x \in \mathbb{R}} |\Phi_6(x,t) - 1| = 0.
$$

Figure 2. Profiles of supersolutions for one-dimensional entire solutions when $-t$ is large, where $f(u) = u(1-u)(u-1/12)$. 

![Figure 2](image-url)
This solution $\Phi_6$ is an entire solution originating from four fronts. Because it is difficult to construct the auxiliary function $Q$ for $\Phi_6$, a supersolution and a subsolution were constructed by using those of $\Phi_4$ and $\Phi_5$ in [4].

**Theorem 3.5** ([4]). **Under the condition (13), there is no entire solution originating from $k$ fronts if $k \geq 5$.**

This can be shown by introducing the notion of “terminated sequence.” Roughly speaking, if there is an entire solution originating from $k$ fronts with $k \geq 5$, then it violates (13), i.e., at least two fronts will collide with each other as $t$ tends to $-\infty$. See [4] for the details.

**Remark 3.6.** For the monostable case, Hamel and Nadirashivili [12, 13] studied entire solutions of the Fisher-KPP equation.

4. **Multi-dimensional traveling wave solutions.** In this section, we briefly review studies on multi-dimensional traveling wave solutions. Without loss of generality, we assume that a traveling wave solution propagates in a given direction $x_N$. We use the new notation $x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}$. With the moving coordinate $z = x_N - ct$, our problem is reduced to finding a solution $(c, V)$ of the following nonlinear elliptic equation:

$$L[V] = 0,$$

where

$$L[V] := -c V_z - \Delta V - f(V).$$

As the condition for the asymptotic behavior of the solution at infinity, we may assume that

$$\lim_{z \to -\infty} V(x', z) = 1, \quad \lim_{z \to \infty} V(x', z) = 0.$$  (15)

Obviously, the one-dimensional traveling wave solution $V(x', z) = \psi_0(-z)$ satisfies (14) and (15) with $c = -c_0 > 0$. This is often called a planar traveling wave solution. For changing the propagating direction, $\psi_0(x \cdot n - c_0 t)$ also becomes a planar traveling wave solution for any $n \in S^{N-1}$. This traveling wave solution can be regarded as the propagation of the stable state 1 from the direction $n$. Note that this planar traveling wave solution moves in the direction of $-n$ because $c_0 < 0$.

We can easily extend this situation to propagation from two different directions $n_1$ and $n_2$, where $n_i \in S^{N-1}$ $(i = 1, 2)$. Specifically, we consider the two planar traveling wave solutions $\psi_0(x \cdot n_i - c_0 t) (i = 1, 2)$ propagating from the right bottom and the left bottom to the direction $n_i$, respectively. This planar traveling wave solution can be regarded as the traveling wave solution with speed $c$ in the direction $c$ when

$$c \cdot n = c_0.$$  (16)

By an appropriate rotation, we can assume that $c$ is equal to $(0, \ldots, 0, c)$, where $c = |c|$ is the speed of the traveling wave solution. Then,

$$n_N = \frac{c_0}{|c|},$$  (17)

where $n_N$ is the $N$-th component of $n$.

Here, we focus on the two-dimensional case and set the space variable $x = (x, y) \in \mathbb{R}^2$. By (17), we have only two choices of $n_1$:

$$n_1 = \pm \sqrt{c^2 - c_0^2}.$$
Then, the function
\[ V_1^-(x, z) := \max \left\{ \psi_0 \left( \frac{\sqrt{c^2 - c_0^2}}{c} x + \frac{c_0}{c} z \right), \psi_0 \left( -\frac{\sqrt{c^2 - c_0^2}}{c} x + \frac{c_0}{c} z \right) \right\} \]
can be defined, and it becomes a subsolution that estimates a solution from below. Because both the planar traveling wave solutions are moving in the \( y \) direction with speed \( c \), this subsolution \( V_1^- \) is also propagating with speed \( c \) in the \( y \) direction. If the angle between the two planar traveling wave solutions is set as \( 2\alpha \), as shown in Figure 3, then
\[ c = \left| \frac{c_0}{\sin \alpha} \right|. \]

Then, we may expect the emergence of a V-shaped traveling wave solution. Actually, we have the following theorem:

**Theorem 4.1** ([10, 11, 19, 20]). Assume that \( N = 2 \). Then, for arbitrarily given \( c > c_0 \), there is a solution \( V_1(x, z; c) \) of (14) with (15) satisfying
\[ \lim_{R \to \infty} \sup \left| V_1(x, z) - V_1^-(x, z) \right| = 0. \]

We emphasize that the V-shaped traveling wave solution is faster than the one-dimensional traveling wave solution and a V-shaped traveling wave solution with an arbitrary speed greater than \( c_0 \) is obtained by varying the opening angle, while the traveling wave solution of (1) in one-dimensional space is unique up to the shift.

To show the existence of the V-shaped traveling wave solution, we only need to find a supersolution. Ninomiya and Taniguchi [19, 20] proved Theorem 4.1 for the two-dimensional case and studied the global asymptotic stability, while Hamel, Monneau, and Roquejoffre [10, 11] treated traveling wave solutions in two-dimensional space and a traveling wave solution with rotational symmetry around the \( x_N \)-axis for general space dimensions, which will be explained in Theorem 4.3.

If \( N \geq 3 \), then we have many choices of \( n \) satisfying (16). Take \( n_1, \cdots, n_M \in \mathbb{S}^{N-1} \) satisfying (16). As seen above, (17) holds for any \( n_j \) \((j = 1, \cdots, M)\). Then,
\[ V_2^-(x) := \psi_0 \left( \max_{1 \leq j \leq M} (n_j \cdot x) \right) \]
becomes a subsolution of (14).

**Theorem 4.2** ([23, 24]). Assume that \( N \geq 3 \). Then, for arbitrarily given \( c > c_0 \), there is a solution \( V_2(x', z; c) \) of (14) with (15) satisfying
\[ \lim_{R \to \infty} \sup_{|x'|^2 + z^2 \geq R^2} \left| V_2(x', z) - V_2^-(x', z) \right| = 0. \]
This traveling wave solution is called a *pyramidal* traveling wave solution. The other extension of Theorem 4.1 to multi-dimensional space is the traveling wave solution with rotational symmetry in $x'$. Actually, it has been shown by Hamel et al [10, 11].

**Theorem 4.3 ([10, 11]).** Assume that $N \geq 3$. Then, for arbitrarily given $c > c_0$, there is a solution $V_3(x', z; c)$ of (14) with (15) satisfying

$$V_3(x', z) = \tilde{V}_3(|x'|, z),$$

$$0 < V_3(x', z) < 1,$$

$$\lim_{|x'| \to \infty} \frac{x'}{|x'|} \cdot \nabla \gamma(x') = \frac{\sqrt{c^2 - c_0^2}}{c_0},$$

where the level set of $V_3 \{ (x', z) \mid V_3(x', z) = \lambda \}$ is represented by the graph of $\gamma$, i.e., $\{ z = \gamma(x'; \lambda) \}$ for $\lambda \in (0, 1)$.

This traveling wave solution is often called a *conical* traveling wave solution. In [25, 26], it has been shown by increasing the number of lateral faces that a pyramidal traveling wave solution converges to a conical traveling wave solution.

Taniguchi [25, 26] extended these studies to more general settings and showed the existence of traveling wave solutions whose level sets are prescribed by any strictly convex set. More precisely, take any strictly convex set $K_0$ and its boundary $\Gamma_0 = \partial K_0$, all principal curvatures of which are positive. By an appropriate shift, we can assume that there is a $C^2$ function $g$ satisfying

$$\Gamma_0 = \{ g(\omega) \omega \in \mathbb{R}^{N-1} \mid \omega \in S^{N-2} \}.$$

**Theorem 4.4 ([25, 26]).** Assume that $N \geq 3$. Then, for any $c > c_0$ and any function $g$ as described above, there is a solution $V_4$ of (14) with (15) satisfying

$$\lim_{R \to \infty} \sup_{|x'|^2 + z^2 \geq R^2} |V_4(x', z) - \min_{\omega \in S^{N-2}} \tilde{V}_3(|x' - g(\omega)\omega|, z)| = 0.$$

In this paper, we call this traveling wave solution a *convex-conical* traveling wave solution.

**Remark 4.5.** For the case where

$$\int_0^1 f(s) ds = 0,$$

the planar traveling wave solution does not move because the attractivity of the equilibria $u = 0$ and $u = 1$ of (1) is balanced. Therefore, it seems that the previous argument is not applicable to this case. In this case, the existence of multi-dimensional traveling wave solutions is also known [3]. See also [5] for non-convex traveling wave solutions.

### 5. Relation between multi-dimensional traveling wave solutions and entire solutions.

In this section, we consider the relation between traveling wave solutions and entire solutions. Before explaining the mathematical results, let us consider the above-mentioned relation heuristically. Assume that we stay at some point in two-dimensional space and observe a V-shaped traveling wave solution $V_1$ with very high speed along the $y$-direction. As $t \to -\infty$, we only observe the state $u = 0$. As $t$ increases, two fronts approach from both sides of large $|x|$. Then, the two fronts collide at some time. As $t \to \infty$, we only see the state $u = 1$. This suggests the existence of the one-dimensional entire solution $\Phi_1$ in Theorem 3.1.
Conversely, by arraying the snapshots of the \((N - 1)\)-dimensional entire solution appropriately in the \(x_N\) direction, we can expect the \(N\)-dimensional traveling wave solution. Indeed, by the existence of two entire solutions \(w_{\pm}\) connecting \(u^*\) to 0 or 1, as seen in Section 3, we can expect traveling wave solutions that look roughly like \(w_{\pm}(x', -x_N)\), i.e., a finger shape and a broom shape. These solutions will be discussed in Section 5.1.

Next, let us consider the entire solution \(\Phi_j\) in Theorem 3.2. It suggests the existence of a traveling wave solution with three states 0, \(a\), and 1, which connects \(a\) to the traveling wave solution \(\psi_0\). When \(f(u) = u(1 - u)(u - 1/2)\), the speed of the traveling wave solution is zero, and it becomes a stationary solution. The candidate multi-dimensional traveling wave solution will be shown in Theorem 5.2 of Section 5.2.

The existence of convex-conical traveling wave solutions explained in Section 4 suggests that there is an entire solution whose level sets are nearly equidistant from the convex set. See Theorem 5.5 for the details.

5.1. Traveling fingers. In this subsection, we give a traveling wave solution connecting a standing wave solution \(u^*\) and the constant stable state 0 or 1. As mentioned above, it is suggested by the entire solution \(w_{\pm}\).

**Theorem 5.1 ([17]).** Let \(\mu_+\) be a positive principal eigenvalue of

\[-\Delta' \psi + f'(u^*) \psi = \mu \psi, \quad x \in \mathbb{R}^{N-1},\]

where \(\Delta' := \sum_{j=1}^{N-1} \partial^2/\partial x_j^2\). Then, for any arbitrarily given \(c \geq c_\ast\), there exist solutions \(V_{0, \pm}\) of (14) with

\[
\lim_{x_N \to -\infty} V_{0, +}(x', x_N) = 1, \quad \lim_{x_N \to -\infty} V_{0, -}(x', x_N) = 0, \quad \lim_{x_N \to +\infty} V_{0, \pm}(x', x_N) = u^*(x'),
\]

where

\[
c_\ast := 2 \sqrt{\max \left\{ -\min_{0 \leq s \leq 1} f'(s), \mu_+ \right\}}.
\]

Because these traveling wave solutions connect the unstable standing wave solution to the constant stable state, the speeds of the traveling wave solutions are not uniquely determined, as with the Fisher-KPP equation. Actually, in the proof of this theorem, the traveling wave solutions of the Fisher-KPP equation were used. Because the profile of \(V_{0, -}\) is like a finger, it can be called a traveling finger. See [17] for the details.

5.2. Zipping wave solutions. In this subsection, we assume that \(f(u) := u(1 - u)(u - 1/2)\). We construct the traveling wave solution related to the entire solution \(\Phi_j\). There are positive minimal speeds \(c_1^*\) and \(c_2^*\) such that \(c_1 \geq c_1^*\) and \(c_2 \leq c_2^* = -c_1^*\), where \(c_1\) (resp. \(c_2\)) is the speed of the traveling wave solution \(\psi_1\) (resp. \(\psi_2\)) in Section 2. By the symmetry, we can take \(\psi_j\) and \(c_j\) \((j = 1, 2)\) satisfying

\[
\psi_1(\xi) = 1 - \psi_2(-\xi), \quad c_1 = -c_2 \geq c_1^*.
\]

Using these traveling wave solutions, we have planar solutions

\[
\psi_j(\mathbf{n}_j \cdot \mathbf{x}),
\]

where \(j = 1, 2\) and \(\mathbf{n}_j\) is any unit vector. Set

\[
\mathbf{n}_1 = (n_1, n_2), \quad \mathbf{n}_2 = (n_1, -n_2), \quad n_1^2 + n_2^2 = 1, \quad n_1 > 0, \quad n_2 > 0.
\]
More precisely, \( \psi_1(n_1 \cdot x - c_1 t) = \psi_1(n_1 x + n_2 y - c_1 t) \), \( \psi_2(n_2 \cdot x - c_2 t) = \psi_2(n_1 x - n_2 y + c_1 t) \)

are traveling wave solutions of (1) with speed \( c := c_1/n_2 \) in the \( y \) direction.

**Theorem 5.2.** Assume that \( f(u) = u(1 - u)(u - 1/2) \). Then, there is a traveling
wave solution \( Y(x, y - ct) \) of (1) satisfying

\[
\lim_{s \to \infty} Y(s \cos \theta, s \sin \theta) = \begin{cases} 
0 & (\pi - \theta_0 < \theta < \frac{3\pi}{2}), \\
\frac{1}{2} & (\theta_0 < \theta < \pi - \theta_0), \\
1 & (-\frac{\pi}{2} < \theta < \theta_0),
\end{cases}
\]

where \( \theta_0 \in (0, \pi/2) \) and

\[
c \geq \frac{3}{2\sqrt{2} \cos \theta_0}.
\]

Moreover, \( Y(0, y) = 1/2 \),

\[
\lim_{y \to -\infty} Y(x, y) = \psi_0(x),
\]

and \( Y \) is decreasing (resp. increasing) in \( y \) if \( x > 0 \) (resp. \( x < 0 \)).

The profile of the traveling wave solution is shown in Figure 4. We call this new

The profile of the traveling wave solution is shown in Figure 4. We call this new type of traveling wave solution a *zipping wave solution*, because two stable states 0 and 1 squeeze the unstable state 1/2 like a zipper. To show the existence of the zipping wave solution, we need to construct the supersolution and the subsolution. Actually, we can use the planar traveling wave solution \( \psi_2(n_2 \cdot x + c_1 t) \) as a supersolution in \( \mathbb{R}^2 \). Thus, we only need to construct the subsolution.

First, we recall that \( Q(v, w) := Q_2(v, w) = \frac{vw}{2vw - v - w + 1} \),

as stated in Section 3 with \( a = 1/2 \). This function is used to construct the one-dimensional entire solution \( \Phi_1 \) of the Allen-Cahn-Nagumo equation [16]. It can be
easily checked that

\[ Q_v(v, w) = \frac{w(1 - w)}{(2vw - v - w + 1)^2}, \quad Q_w(v, w) = \frac{v(1 - v)}{(2vw - v - w + 1)^2}, \quad (21) \]

\[ Q_{vv} = \frac{2(1 - w)w(1 - 2w)}{(2vw - v - w + 1)^3}, \quad Q_{vw} = \frac{1 - v - w}{(2vw - v - w + 1)^3}, \quad (22) \]

\[ Q_{ww} = \frac{2(1 - v)v(1 - 2v)}{(2vw - v - w + 1)^3}. \quad (23) \]

Using \( Q \), we define

\[ U^-(x, y) := Q(\psi_1(n_1x + n_2y), \psi_2(n_1x - n_2y)) \quad (24) \]

and we show that \( U^- \) is a subsolution. For simplicity, we abbreviate

\[ \psi_1 = \psi_1(n_1x + n_2y), \quad \psi_2 = \psi_2(n_1x - n_2y), \quad Q = Q(\psi_1, \psi_2), \]

\[ \psi_{1,x} = \frac{\partial}{\partial x} \psi_1(n_1x + n_2y) = n_1 \psi_1', \quad \psi_{1,y} = \frac{\partial}{\partial y} \psi_1(n_1x + n_2y) = n_2 \psi_1', \]

\[ \psi_{2,x} = \frac{\partial}{\partial x} \psi_2(n_1x - n_2y) = n_1 \psi_2', \quad \psi_{1,y} = \frac{\partial}{\partial y} \psi_2(n_1x - n_2y) = -n_2 \psi_2', \]

\[ Q_v = \frac{\partial Q}{\partial v}(\psi_1, \psi_2), \quad Q_w = \frac{\partial Q}{\partial w}(\psi_1, \psi_2), \quad Q_{vv} = \frac{\partial^2 Q}{\partial v^2}(\psi_1, \psi_2), \quad \text{etc.} \]

It follows from (21)—(23) that

\[ Q_v > 0, \quad Q_w > 0, \quad Q_{vv} < 0, \quad Q_{ww} > 0. \]

Using (4) and (5), we obtain

\[ \mathcal{L}[U^-] = -cQ_vn_2\psi_1' + cQ_wn_2\psi_2' - f(Q) \]

\[ - \left\{ Q_{vv}\psi_1'^2 + 2Q_{vw}\psi_1'\psi_2'(n_1^2 - n_2^2) + Q_{ww}\psi_1'^2 + Q_v\psi_1'' + Q_w\psi_2'' \right\} \]

\[ = -Q_{vv}\psi_1'^2 - 2Q_{vw}\psi_1'\psi_2'(n_1^2 - n_2^2) - Q_{ww}\psi_1'^2 + Q_vf(\psi_1) + Q_wf(\psi_2) \]

\[ - f(Q). \]

We estimate the right-hand side in the right half space \( x > 0 \). First, we begin with \( Q_vf(\psi_1) + Q_wf(\psi_2) - f(Q) \).

**Lemma 5.3.** Let \( \psi_1, \psi_2 \) and \( Q \) be defined as above. Then,

\[ \psi_1 + \psi_2 > 1 \]

for \( x > 0 \). Moreover,

\[ Q_v(\psi_1, \psi_2)f(\psi_1) + Q_w(\psi_1, \psi_2)f(\psi_2) - f(Q(\psi_1, \psi_2)) \]

\[ = -\frac{\psi_1(1 - \psi_1)\psi_2(1 - \psi_2)(1 - 2\psi_1)(2\psi_2 - 1)(\psi_1 + \psi_2 - 1)}{2(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^3} < 0 \]

for \( x > 0 \).

**Proof.** From the monotonicity of \( \psi_1 \) and (18), it follows that

\[ \psi_1 + \psi_2 - 1 = \psi_1(n_1x + n_2y) + \psi_2(n_1x - n_2y) - 1 \]

\[ = \psi_1(n_1x + n_2y) - \psi_1(-n_1x + n_2y) \]

\[ = 2n_1x \int_0^1 \psi_1'(\theta n_1x - (1 - \theta)n_1x + n_2y)d\theta > 0 \]

for \( x > 0 \).
Using $0 < \psi_1 < 1/2 < \psi_2 < 1$, we have

$$
\begin{align*}
2\psi_1\psi_2 + 1 - \psi_1 - \psi_2 &= -(1 - 2\psi_1)\psi_2 + 1 - \psi_1 < 1 - \psi_1, \\
2\psi_1\psi_2 + 1 - \psi_1 - \psi_2 &= -(2\psi_2 - 1)(1 - \psi_1) + \psi_2 < \psi_2, \\
2\psi_1\psi_2 + 1 - \psi_1 - \psi_2 &= (2\psi_2 - 1)\psi_1 + 1 - \psi_2 > 1 - \psi_2, \\
2\psi_1\psi_2 + 1 - \psi_1 - \psi_2 &= (1 - \psi_2)(1 - 2\psi_1) + \psi_1 > \psi_1.
\end{align*}
$$

Note that

$$0 < \frac{\psi_1(1 - \psi_2)}{(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^2} < 1.$$

The definitions of $Q$ and $f$ imply that

$$
Q_v(\psi_1, \psi_2)f(\psi_1) + Q_w(\psi_1, \psi_2)f(\psi_2) - f(Q(\psi_1, \psi_2))
= \frac{\psi_2(1 - \psi_2)\psi_1(1 - \psi_1)(\psi_1 - 1/2)}{(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^2}
+ \frac{\psi_1(1 - \psi_1)\psi_2(1 - \psi_2)(\psi_2 - 1/2)}{(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^2}
- Q(1 - Q) \left( Q - \frac{1}{2} \right).
$$

Since

$$Q(1 - Q) \left( Q - \frac{1}{2} \right) = \frac{\psi_1\psi_2(1 - \psi_1)(1 - \psi_2)(\psi_1 + \psi_2 - 1)}{2(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^3},$$

we have

$$
Q_v(\psi_1, \psi_2)f(\psi_1) + Q_w(\psi_1, \psi_2)f(\psi_2) - f(Q(\psi_1, \psi_2))
= \frac{\psi_2(1 - \psi_2)\psi_1(1 - \psi_1)(\psi_1 - 1/2)}{(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^2}
+ \frac{\psi_1(1 - \psi_1)\psi_2(1 - \psi_2)(\psi_2 - 1/2)}{(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^2}
- \frac{\psi_1\psi_2(1 - \psi_1)(1 - \psi_2)(\psi_1 + \psi_2 - 1)}{2(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^3}
\leq \frac{\psi_1(1 - \psi_1)\psi_2(1 - \psi_2)(\psi_1 - 1)(2\psi_2 - 1)(\psi_1 + \psi_2 - 1) - \psi_1 + \psi_2 - 1}{2(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^3} < 0,
$$

which follows from the first statement of this lemma.

\textbf{Proposition 5.4.} If

$$
\begin{align*}
\psi_1(1 - \psi_1)(1 - 2\psi_1)\psi_2(1 - \psi_2)(2\psi_2 - 1)(\psi_1 + \psi_2 - 1) \\
\geq 4\psi_2(1 - \psi_2)(2\psi_2 - 1)|\psi'_1|^2 + 4(\psi_1 + \psi_2 - 1)|\psi'_2|^2
- 4\psi_1(1 - \psi_1)(1 - 2\psi_1)|\psi'_2|^2,
\end{align*}
$$

then $U^-(x, y) = Q(\psi_1(n_1 x + n_2 y), \psi_2(n_1 x - n_2 y))$ becomes a subsolution of (14).

\textbf{Proof.} Using Lemma 5.3 yields

$$
- Q_{vv}|\psi'_1|^2 - 2Q_{vw}\psi'_1\psi'_2(n_1^2 - n_2^2) - Q_{ww}|\psi'_2|^2 + Q_v f(\psi_1) + Q_w f(\psi_2) - f(Q)
= \frac{-2(1 - \psi_2)\psi_2(1 - 2\psi_2)|\psi'_1|^2 - 2(1 - \psi_1 - \psi_2)|\psi'_1\psi'_2(n_1^2 - n_2^2)}{(2\psi_1\psi_2 - \psi_1 - \psi_2 + 1)^3}
$$

(25)
\[
\frac{2(1 - \psi_1)\psi_1(1 - 2\psi_1)|\psi'_2|^2}{(2\psi_1\psi_2 - \psi_1 - \psi_2 + 1)^3} + \frac{\psi_2(1 - \psi_2)\psi_1(1 - \psi_1)(2\psi_2 - 1)(1 - 2\psi_1)(\psi_1 + \psi_2 - 1)}{2(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^3} \\
= \frac{4(1 - \psi_2)\psi_2(2\psi_2 - 1)|\psi'_2|^2 + 4(\psi_1 + \psi_2 - 1)\psi'_1 \psi'_2 (n_1^2 - n_2^2)}{2(2\psi_1\psi_2 - \psi_1 - \psi_2 + 1)^3} - \frac{4(1 - \psi_1)\psi_1(1 - 2\psi_1)|\psi'_2|^2}{2(2\psi_1\psi_2 - \psi_1 - \psi_2 + 1)^3} - \frac{\psi_1(1 - \psi_1)\psi_2(1 - \psi_2)(2\psi_2 - 1)(\psi_1 + \psi_2 - 1)}{2(2\psi_1\psi_2 + 1 - \psi_1 - \psi_2)^3}.
\]

It follows from (25), \(n_1^2 - n_2^2 \leq 1\) and \((\psi_1 + \psi_2 - 1)\psi'_1 \psi'_2 \geq 0\) in \(x \geq 0\) that \(\mathcal{L}[U^-] \leq 0\). \(\square\)

Next, we give the proof of Theorem 5.2

**Proof of Theorem 5.2.** When \(f(u) = u(1 - u)(u - 1/2)\), we have the exact traveling wave solutions of (1):

\[
\psi_1 := \frac{1}{2 + 2e^{-z/(2\sqrt{2})}}, \quad \psi_2 = \frac{2 + e^{-z/(2\sqrt{2})}}{2 + 2e^{-z/(2\sqrt{2})}},
\]

with \(c_1 = -c_2 = 3/(2\sqrt{2})\). Moreover, we have

\[
\psi'_1 = \frac{\psi_1(1 - 2\psi_1)}{2\sqrt{2}}, \quad \psi'_2 = \frac{(1 - \psi_2)(2\psi_2 - 1)}{2\sqrt{2}}.
\]

(26)

It follows from (26) that

\[
4\psi_2(1 - \psi_2)(2\psi_2 - 1)|\psi'_1|^2 - 4\psi_1(1 - \psi_1)(1 - 2\psi_1)|\psi'_2|^2 = \frac{1}{2}\psi_2(1 - \psi_2)(2\psi_2 - 1)|\psi'_1|^2(1 - 2\psi_1)^2 - \frac{1}{2}\psi_1(1 - \psi_1)(1 - 2\psi_1)(1 - \psi_2)^2(2\psi_2 - 1)^2
\]

\[
= \psi_1(1 - 2\psi_1)(1 - \psi_2)(2\psi_2 - 1)(\psi_1 + \psi_2 - 1)(-2\psi_1\psi_2 + 2\psi_2 - 1),
\]

\[
4(\psi_1 + \psi_2 - 1)\psi'_1 \psi'_2
\]

\[
= \frac{1}{2}(\psi_1 + \psi_2 - 1)\psi_1(1 - 2\psi_1)(1 - \psi_2)(2\psi_2 - 1).
\]

Therefore, we have

\[
4\psi_2(1 - \psi_2)(2\psi_2 - 1)|\psi'_1|^2 + 4(\psi_1 + \psi_2 - 1)\psi'_1 \psi'_2 - 4\psi_1(1 - \psi_1)(1 - 2\psi_1)|\psi'_2|^2
\]

\[
= (\psi_1 + \psi_2 - 1)\psi_1(1 - 2\psi_1)(1 - \psi_2)(2\psi_2 - 1)\psi_2(1 - \psi_1),
\]

which implies that

\[
\psi_1(1 - \psi_1)(1 - 2\psi_1)\psi_2(1 - \psi_2)(2\psi_2 - 1)(\psi_1 + \psi_2 - 1)
\]

\[
= 4\psi_2(1 - \psi_2)(2\psi_2 - 1)|\psi'_1|^2 + 4(\psi_1 + \psi_2 - 1)\psi'_1 \psi'_2 - 4\psi_1(1 - \psi_1)(1 - 2\psi_1)|\psi'_2|^2.
\]

By Proposition 5.4, \(U^-\) is a subsolution. We note that \(U^-(0, y) = 1/2\) for any \(y \in \mathbb{R}\). Set \(U^+(x, y) := \psi_2(n_1 x - n_2 y)\). Then, it is a supersolution of \(\mathcal{L}[U] = 0\). Moreover, we have

\[
U^- = Q(\psi_1(n_1 x + n_2 y), \psi_2(n_1 x - n_2 y))
\]

\[
\leq Q(1/2, \psi_2(n_1 x - n_2 y)) = \psi_2(n_1 x - n_2 y).
\]
Consider the solution $U$ of
\[ U_t + \mathcal{L}[U] = 0, \quad U(x, y, 0) = U^-(x, y), \quad U(0, y, t) = \frac{1}{2} \]
for $x > 0$, $y \in \mathbb{R}$, $t > 0$. We see that
\[ U^-(x, y) \leq U(x, y, t) \leq U^+(x, y) \]
for $x > 0$, $y \in \mathbb{R}$, $t > 0$. Taking the limit as $t \to \infty$, we get
\[ U^\infty(x, y) := \lim_{t \to \infty} U(x, y, t), \]
which satisfies
\[ U^-(x, y) \leq U^\infty(x, y) \leq U^+(x, y). \]
The standard parabolic regularity theorem implies that $U^\infty$ is a classical solution and
\[ \mathcal{L}[U^\infty] = 0. \]
Thus, we have obtained the traveling wave solution $U^\infty$ of (1) with speed $c$ in $x > 0$. Moreover, we have
\[ U^\infty(0, y) = \lim_{t \to \infty} U(0, y, t) = \frac{1}{2}. \]
Hence, we can confirm (19) for $x \geq 0$.

Next, to check (19) for $x < 0$, we show that
\[ u(x, y, t) = 1 - u(-x, y, t). \]
(27)

Indeed, using
\[ 1 - Q(v, w) = 1 - \frac{vw}{2vw - v - w + 1} = \frac{(1 - v)(1 - w)}{2vw - v - w + 1} = Q(1 - w, 1 - v), \]
we obtain
\[ 1 - U^-(-x, y) = 1 - Q(\psi_1(-n_1x + n_2y), \psi_2(-n_1x - n_2y)) \]
\[ = Q(1 - \psi_2(-n_1x - n_2y), 1 - \psi_1(-n_1x + n_2y)) \]
\[ = Q(\psi_1(n_1x + n_2y), \psi_2(n_1x - n_2y)) = U^-(x, y). \]
The symmetry property (27) is preserved for any $t > 0$ owing to the symmetry of the nonlinear function of $f$. Therefore, $U$ also satisfies (27). We see that (19) also holds for $x < 0$. Thus, the proof is complete.

5.3. Multi-dimensional entire solutions with convex contours. In this subsection, we construct an entire solution whose level sets are close to the equidistant hypersurface from any strictly convex body as $t \to -\infty$ in $\mathbb{R}^{N-1}$. The construction of this entire solution is closely related to the work of Yagisita [28] and Taniguchi [26].

First, we take any strictly convex body $K_0$ and a function $g$ as in Section 4. Set
\[ h(x') := \max_{\omega \in S^{N-2}} (x' \cdot \nu(x') - g(\omega)\omega \cdot \nu(x')). \]
(28)
This function $h$ is equal to the signed distance from $K_0$ (see Figure 5). We can easily extend the function $h$ to a $C^2$ function $H$ by modifying it inside $K_0$.

**Theorem 5.5** ([18]). Assume that $N \geq 3$ and take a strictly convex set $K_0$ such that all principle curvatures of its smooth boundary $\Gamma_0$ are positive. Then, there exists a unique entire solution $\Phi_7$ of (1) for $x' \in \mathbb{R}^{N-1}$, $t \in \mathbb{R}$ satisfying
\[ \lim \inf_{t \to -\infty} \sup_{p \geq 0} \sup_{x' \in \mathbb{R}^{N-1}} \left| \Phi_7(x', t) - \psi_0(H(x') - p) \right| = 0. \]
Figure 5. Example of $\Gamma_0$ and the equidistant curves from $K_0$ in $\mathbb{R}^2$.

This theorem implies that the difference between the inner and outer radius does not converge to 0 and that there are many non-radially symmetric entire solutions even in the bistable case. See also [1] and [21]. For the monostable case, the linear term plays an important role in determining entire solutions. See [13] for the details of the monostable case.

REFERENCES

[1] H. Berestycki and F. Hamel, Generalized transition waves and their properties, Comm. Pure Appl. Math., 65 (2012), 592–648.
[2] X. F. Chen and J.-S. Guo, Existence and uniqueness of entire solutions for a reaction-diffusion equation, J. Differential Equations, 212 (2005), 62–84.
[3] X. F. Chen, J. S. Guo, F. Hamel, H. Ninomiya and J. M. Roquejoffre, Traveling waves with paraboloid like interfaces for balanced bistable dynamics, Annales de l’Institut Henri Poincaré C: Non Linear Analysis, 24 (2007), 369–393.
[4] Y. Y. Chen, J. S. Guo, H. Ninomiya and C. H. Yao, Entire solutions originating from monotone fronts to the Allen-Cahn equation, Physica D, 378/379 (2018), 1–19.
[5] M. del Pino, M. Kowalczyk and J. Wei, Traveling waves with multiple and nonconvex fronts for a bistable semilinear parabolic equation, Comm. Pure Appl. Math., 66 (2013), 481–547.
[6] R. A. Fisher, The wave of advance of advantageous genes, Ann. Eugenics, 7 (1937), 355–369.
[7] Y. Fukao, Y. Morita and H. Ninomiya, Some entire solutions of the Allen-Cahn equation, Taiwanese J. Math., 8 (2004), 15–32.
[8] J. S. Guo and Y. Morita, Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations, Discrete Contin. Dyn. System, 12 (2005), 193–212.
[9] K. P. Hadeler and F. Rothe, Travelling fronts in nonlinear diffusion equations, J. Math. Biol., 2 (1975), 251–263.
[10] F. Hamel, R. Monneau and J.-M. Roquejoffre, Existence and qualitative properties of multi-dimensional conical bistable fronts, Disc. Cont. Dyn. Systems, 13 (2005), 1069–1096.
[11] F. Hamel, R. Monneau and J.-M. Roquejoffre, Asymptotic properties and classification of bistable fronts with Lipschitz level sets, Disc. Cont. Dyn. Systems, 14 (2006), 75–92.
[12] F. Hamel and N. Nadirashvili, Entire solutions of the KPP equation, Comm. Pure Appl. Math., 52 (1999), 1255–1276.
[13] F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$, Arch. Ration. Mech. Anal., 157 (2001), 91–163.
[14] Y. I. Kanel, Some problems involving burning-theory equations, Soviet Math. Dokl., 2 (1961), 48–51.
[15] A. Kolmogorov, I. Petrovsky and N. Piskunov, Etude de l’equation de la diffusion avec croissance de la quantite de mateire et son application a un probleme biologique, Byul. Moskovskogo Gos. Univ. Ser. Internat. Sec. A, 1 (1937), 1–26.
[16] Y. Morita and H. Ninomiya, Entire solutions with merging fronts to reaction-diffusion equations, *J. Dynam. Differential Equations*, 18 (2006), 841–861.

[17] Y. Morita and H. Ninomiya, Monostable-type traveling waves of bistable reaction-diffusion equations in the multi-dimensional space, *Bull. Inst. Math. Acad. Sin.(NS)* 3 (2008), 567–584.

[18] H. Ninomiya, Multi-dimensional entire solutions of the Allen-Cahn-Nagumo equation, in preparation.

[19] H. Ninomiya and M. Taniguchi, Existence and global stability of traveling curved fronts in the Allen-Cahn equations, *J. Differential Equations*, 213 (2005), 204–233.

[20] H. Ninomiya and M. Taniguchi, Global stability of traveling curved fronts in the Allen-Cahn equations, *Disc. Cont. Dyn. Systems*, 15 (2006), 819–832.

[21] P. Poláčik, Symmetry properties of positive solutions of parabolic equations on $\mathbb{R}^N$: II. entire solutions, *Communications in Partial Differential Equations* 31 (2006), 1615–1638.

[22] D. H. Sattinger, On the stability of waves of nonlinear parabolic systems, *Adv. Math.*, 22 (1976), 312–355.

[23] M. Taniguchi, Traveling fronts of pyramidal shapes in the Allen-Cahn equations, *SIAM J. Math. Anal.*, 39 (2007), 319–344.

[24] M. Taniguchi, The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen–Cahn equations, *J. Differential Equations*, 246 (2009), 2103–2130.

[25] M. Taniguchi, Multi-dimensional traveling fronts in bistable reaction-diffusion, *Discrete Contin. Dyn. System*, 32 (2012), 1011–1046.

[26] M. Taniguchi, An $(N - 1)$-dimensional convex compact set gives an $N$-dimensional traveling front in the Allen–Cahn equation, *SIAM J. Math. Anal.*, 47 (2015), 455–476.

[27] X. Wang, On the Cauchy problem for reaction-diffusion equations, *Trans. Amer. Math. Soc.*, 337 (1993), 549–590.

[28] H. Yagisita, Nearly spherically symmetric expanding fronts in a bistable reaction-diffusion equation, *J. Dynam. Differential Equations*, 13 (2001), 323–353.

[29] H. Yagisita, Backward global solutions characterizing annihilation dynamics of travelling fronts, *Publ. Res. Inst. Math. Sci.*, 39 (2003), 117–164.

Received for publication February 2018.

*E-mail address*: hirokazu.ninomiya@gmail.com