Abstract. We discuss an algebro-geometric description of Witten’s phases of $N=2$ theories and propose a definition of their elliptic genus provided some conditions on singularities of the phases are met. For Landau-Ginzburg phase one recovers elliptic genus of LG models proposed in physics literature in early 90s. For certain transitions between phases we derive invariance of elliptic genus from an equivariant form of McKay correspondence for elliptic genus. As special cases one obtains Landau-Ginzburg/Calabi-Yau correspondence for elliptic genus of weighted homogeneous potentials as well as certain hybrid/CY correspondences.

1. Introduction

Elliptic genera appeared in mathematics and physics literature in the middle 80s as invariants associated with manifolds (cf. [31]). Around the same time it was realized that elliptic genus can be associated with superconformal field theories which depend on rather different type of data e.g.minimal models, Landau-Ginzburg models etc. (cf. [38], [27]). In [39], Witten proposed geometric procedure based on use of variations of symplectic quotients for specific actions of Lie groups, which relates Landau Ginzburg models to the sigma-models and hence to the manifolds. In fact Witten’s construction lead not just to Landau-Ginzburg or sigma models but to a host of others, which he called phases of $N=2$ theories, and which, besides just mentioned types, include hybrid models, gauged Landau Ginzburg models etc. The purpose of this note is to associate elliptic genus to (a generalization of) Witten’s phases of $N=2$ theories.

The key constraint for such definition it seems should be invariance of elliptic genus in transitions between $N=2$ supersymmetric phases. The cases when a relation between Landau-Ginzburg and sigma-models are involved called Landau-Ginzburg/Calabi-Yau correspondence. Study of such correspondence in the various contexts was the subject of enormous number of beautiful works over last 20 years of which we shall mention only a few. In the context of A-and B-models LG/CY correspondence was considered by Chiodo, Iritani and Ruan (cf. [14] and further references there) in particular yields relation between Gromov-Witten and FJRW’s theories. In the context of homological mirror symmetry, the LG/CY correspondence was obtained by Orlov (cf. [34]) based on Kontsevich definition of categories associated with singularities and was recently substantially extended by Ballard, Favero and Katzarkov (cf. [4]) (cf. also [24]). See a review and discussion of some of these and numerous other recent developments in [25], [26]. In the case of elliptic genus LG/CY correspondence for homogeneous polynomials was obtained by Gorbounov and Malikov (in fact this work deals with the vertex algebras associated

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with LG/CY data, cf. [22], also [23]). Extension of LG/CY correspondence for vertex algebras of (0,2)-toric models was considered in [7].

Our generalization of Witten phases are Geometric Invariant Theory (GIT) quotients of the total spaces of a $G$-equivariant line bundle ($G$ is a reductive group) on a quasi-projective manifold (cf. Def. 3.3). Fiberwise $\mathbb{C}^*$-action on the total space of such bundles, can be used to define the action on the GIT quotient (cf. Prop. 4.4). We work under assumptions that singularities of GIT quotients are not too bad with precise conditions discussed in section 3. For each compact component of the fixed point set one has a contribution into equivariant elliptic genus discussed in section 4. This contribution is a holomorphic function on $\mathbb{C} \times \mathbb{C} \times \mathcal{H}$ where $\mathcal{H}$ is the upper half plane. It depends on the equivariant Chern classes of the neighborhood of this component in the GIT quotient (cf. section 4). The elliptic genus of our phase is defined as the function $E(z, \tau)$ which is the restriction of the contribution of the fixed point set of the $\mathbb{C}^*$-action into the equivariant elliptic genus of the phase to $\Delta \times \mathcal{H} \subset \mathbb{C} \times \mathbb{C} \times \mathcal{H}$ where $\Delta$ is the diagonal of $\mathbb{C} \times \mathbb{C}$. We show that such restriction yields, under certain Calabi Yau type conditions, a function on $\mathbb{C} \times \mathcal{H}$ which is Jacobi form of expected weight and index (cf. Section 4.3). In the case when the total space of the line bundle is $\mathbb{C} \times \mathbb{C}^n$ with the action of $\mathbb{C}^*$ given by $\lambda(p, x_1, ..., x_n) = (\lambda^{-D}p, \lambda^{w_1}z_1, ..., \lambda^{w_n}z_n)$ there are two GIT quotients specified by a choice of linearization of this action (cf. example 2.7). One of the GIT quotients is the quotient of $\mathbb{C}^n$ by the action of the cyclic group and our elliptic genus $E(z, \tau)$ is just the elliptic genus of Landau Ginzburg model appearing in physics literature (cf. [1], [6], [27]). Moreover, one readily identified the elliptic genus of the second GIT quotient with the orbifold elliptic genus of hypersurface in weighted projective space. The identity between elliptic genera corresponding to different linearization represents a wall crossing phenomenon and can be derived in many cases from the equivariant version of McKay correspondence (cf. [37]) for elliptic genus obtained in [11]. In particular, the equality of elliptic genera corresponding to hybrid and other models lead to interesting new identities between Jacobi forms and several explicit examples are given in the section 5. Paper is concluded with description of possible further developments.

The point of view on Witten phases described in this paper is close to the one taken in [4] but the difference is that in the later work variation of GIT quotient occurs in the base of equivariant bundle while we consider variation GIT quotients of the total space of line $G$-bundle. The piece of the data of LG model consisting of the section of $L$ in [4] in this note is replaced by the action of the group $G$ on the total space of $L$. The potential is a $G$-equivariant section of $L$, i.e. in the case of trivial bundle on $\mathbb{C}^n$ (classical LG model) is essentially a weighted homogeneous polynomial (cf. Example 2.7). It does not play direct role in our definition of the elliptic genus of a phase.

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1 the work [4] uses differently the term “gauged Landau Ginzburg model”. We use it (cf. definition 2.9 (iii)) essentially in the way it was used in Witten paper [39].
2. GIT quotients of total spaces of line bundles.

2.1. Linearizations of actions of reductive groups and corresponding GIT quotients. Let $X$ be a smooth quasi-projective variety acted upon by a reductive group $G$. A linearization $\kappa$ of a line bundle $L$ (cf. \[33\], \[18\]) is a fiberwise linear action of $G$ on the total space $X$ of the line bundle $L$ commuting with projection of $L$ on $X$. Denote by $Pic^G(X)$ the group of linearized $G$-bundles. One has the exact sequence: (cf. \[29\], \[18\]):

$$0 \to H^1(G, \mathcal{O}(X)^*) \to Pic^G(X) \to Pic(X)^G \subset Pic(X)$$

where $Pic(X)^G$ is the group of $G$-invariant line bundles.

If $X$ is an affine space or is projective and $G$ is connected then

$$H^1(G, \mathcal{O}(X)^*) = \text{Char}(G)$$

With the data $X, G, L, \kappa, (L \in Pic^G(X))$ as above, one associates subsets of $X$, the semi-stable, stable and unstable loci, $X^s_\kappa, X^s_s, X^{ss}_\kappa$ respectively, as follows (cf. \[33\]):

$$(3) \quad X^s_\kappa = \{x \in X | \exists m > 0 \text{ and } s \in \Gamma(X, L^\otimes m)^G, s(x) \neq 0, \text{ such that } X_s \text{ is affine}\}$$

$$(4) \quad X^s_s = \{x \in X | \text{ as in } 3, G_x \text{ is finite and } \overline{tx} = Gx \text{ in } X_s\}$$

$$(5) \quad X^{ss}_\kappa = X \setminus X^s_\kappa$$

(Here $X_s = \{y \in X | s(y) \neq 0\}$, $G_x$ is the stabilizer of $x$ and $\overline{tx}$ is the closure of $G$-orbit of $x$).

**Definition 2.1.** GIT quotient $X/\!/^\kappa G$ of $X$ corresponding to linearization $\kappa$ is the categorical $G$-quotient of $X^s_\kappa$ i.e. a morphism $p : X^s_\kappa \to X^s_\kappa/\!/G$ such that for any $G$-compatible morphism $X^s_\kappa \to Z$ there exist unique factorization $X^s_\kappa \to X^s_\kappa/\!/G \to Z$ (cf. \[15\]).

If $X^s_\kappa = \text{Spec}A$ then $X^s_\kappa/\!/G = \text{Spec}A^G$ and for quasi-projective manifold $X$, the categorical quotient $X^s_\kappa/\!/G$ can be obtained via gluing quotients of affine subsets (cf. \[18\], \[33\]).

Dependence of GIT quotient on linearization is as follows (cf. \[36\], \[17\]). Let $NS^G(X)$ be the $G$-linearized Neron-Severi group i.e. the quotient of the group $Pic^G(G)$ of $G$-linearizations by algebraic equivalence. Firstly, the subsets $X^s_\kappa, X^s_s, X^{ss}_\kappa$ of $X$ depend only on the class of linearization $\kappa$ in $NS^G$ (cf. (2.1) in \[36\]). It is convenient to view linearizations as elements in $NS^G \otimes \mathbb{Q}$ (referred to as (classes) of fractional linearizations). An ample linearization $L$ is called $G$-effective if $L^n$ has a $G$-invariant section for some $n > 0$. Linearizations $\kappa$ which are $G$-effective and such that $X^s_\kappa \neq \emptyset$ generate a cone $E^G_\kappa$ in the preimage of the ample cone $A_Q \subset NS(X) \otimes \mathbb{Q}$ for the map $\nu : NS^G(X) \to NS(X) \otimes \mathbb{Q}$.

Secondly, one has the following key result:

**Theorem 2.2.** (cf. \[36\], \[17\])

1. The cone $E^G_\kappa$ is polyhedral i.e. is an intersection in $NS^G_\kappa$ of $\nu^{-1}(A_Q)$ with a polyhedron. It is a finite disjoint union of cones $C_i$ (called chambers if $dim C_i$ is maximal and called cells in general) with the following property: $X^{ss}_\kappa$ is independent of a linearization $\kappa$ as long as $\kappa$ belongs to a fixed cell.
(2) If \( t \in [-\epsilon, +\epsilon] \subset E_G^G \), where \([-\epsilon, +\epsilon]\) is viewed as a linear variation of linearizations such that \( t = 0 \) correspond to a point belonging to a wall (i.e. a codimension one cell), and \( X^{ss}(t) \) is the semi-stable locus for the linearization corresponding to \( t \), then there is inclusion \( X^{ss}(\epsilon) \subset X^{ss}(0) \) inducing a projective morphism \( X^{ss}(\epsilon) \to X^{ss}(0) \) which is birational if \( X^{ss}(0) \neq \emptyset \).

2.2. Phases. The main object of this note is a class GIT quotients appearing in the following context.

Definition 2.3. Let \( L \) be the total space of a linearized line bundle \( L \) over a smooth quasi-projective manifold \( X \) with an action of a reductive connected group \( G \). A phase corresponding to quadruple \((X, G, L, \kappa)\) where \( \kappa \) is a linearization of \( G \)-action on \( L \) is the GIT quotient of the total space of linearized bundle \( L \) on \( X \) i.e. \( L^{ss}/G \).

(1) A phase is called Landau Ginzburg, if it is biholomorphic to \( \mathbb{C}^N/H \) where \( H \) a finite subgroup of a torus in \( GL_N(\mathbb{C}) \).

(2) A phase is called hybrid if it is biholomorphic to an orbifold bundle over a projective orbifold with the fiber as in (1).

(3) A phase \( L^{ss}/G \) is called “gauged Landau Ginzburg model” if it is biholomorphic to GIT quotient \( X//H \) for a subgroup \( H \times A \) of \( G \) and \( A \) (resp. \( H \)) is a finite abelian (resp reductive) group.

(4) A phase is called Calabi Yau if it is biholomorphic to the total space \( L \) of a line bundle \( L \) over a smooth projective manifold \( X \) such that \( c_1(T(L)|_{X}) = 0 \) (here \( T(L) \) is the tangent bundle of the space \( L \)).

The following is useful for explicite descriptions of phases in the case when the bundle \( L \) is trivial.

Proposition 2.4. Let \( G \) be a reductive group acting on a smooth quasi-projective variety \( X \) and let \( \kappa, \psi \in \text{Char}G \) be two linearizations (cf.[11]) of trivial line bundle on \( X \). Then the action of \( G \) on \( L = X \times \mathbb{C} \) has the form

\[
g(p, x) = (\psi(g)p, gx)
\]

for \( \psi \in \text{Char}(G) \). As in [10], let \( (X \times \mathbb{C})_{ss}^{\kappa}, (X \times \mathbb{C})_{s}^{\kappa}, (X \times \mathbb{C})_{u}^{\kappa} \) denote semi-stable, stable and unstable loci for \( G \)-action and the linearization \( \kappa \) of the trivial bundle on \( X \times \mathbb{C} \).

(1) If \( 0 \times X_{s}^{\kappa} \subset (X \times \mathbb{C})_{s}^{\kappa} \) then \( (X \times \mathbb{C})_{s}^{\kappa}/G \) is biholomorphic to a rank one orbibundle over the quotient \( X_{s}^{\kappa}/G \).

(2) Suppose that \( 0 \times X \subseteq (X \times \mathbb{C})_{u}^{\kappa} \). Let \( H = \text{Ker}^\psi \), \( H_0 \) its connected component and \( H/H_0 \), a finite group of connected components of \( H \). If \( (0 \times X) \subset (X \times \mathbb{C})_{u}^{\kappa} \) then \( (X \times \mathbb{C})/H \) is biholomorphic of a quotient of \( X//H_0 \) by action of \( H/H_0 \).

Proof. First note, that the fibers of maps of geometric quotients can be described as follows:

Lemma 2.5. Let \( f : Y \to Z \) be a \( G \) equivariant holomorphic map of quasi-projective varieties for which both geometric quotients \( Y/G, Z/G \) exist. For \( z \in Z \) let \( G_z \) denotes the stabilizer of \( z \). Then the fiber at the orbit of \( z \in Z \) of induced map \( f_G : Y/G \to Z/G \) of the spaces of \( G \)-orbits is biholomorphic to \( f^{-1}(z)/\text{Stab}(z) \).
Projection $\pi : X \times \mathbb{C} \to X$ is $G$-equivariant and it follows from \cite{3} that $\pi$ induces the map $(X \times \mathbb{C})^s \to X^s$. Hence Prop \ref{2.4} \cite{1} is a consequence of lemma \ref{2.5}.

In the case Prop \ref{2.4} we have:

\begin{equation}
(X \times \mathbb{C})^s / G = (X \times \mathbb{C}^*)^s / G = X^s / H = (X / H_0) / (H / H_0)
\end{equation}

\[\square\]

2.3. Examples of Phases. (cf. \cite{39}).

Example 2.6. LG model and $\sigma$-model cf. \cite{39}. Let $G = \mathbb{C}^*$ acts on $\mathbb{C} \times \mathbb{C}^n$ with coordinates $(p, x_1, ..., x_n)$ via:

\begin{equation}
\lambda(p, x_1, ..., x_n) = (\lambda^{-p}, \lambda x_1, ..., \lambda x_n)
\end{equation}

(i.e. we consider the case of Prop. \ref{2.3} when $X = \mathbb{C}^n, G = \mathbb{C}^*$ and $\psi(\lambda) = \lambda^{-n}, \lambda \in \mathbb{C}^*$). Let $\mathcal{H}_p \subset \mathbb{C} \times \mathbb{C}^n$ denotes the hyperplane $p = 0$. For linearization $\kappa(\lambda) = \lambda^k, k < 0$, one has $\mathcal{H}_p = (\mathbb{C} \times \mathbb{C}^n)\lambda^k$, the subgroup $H$ of $G = \mathbb{C}^*$ from \ref{2.4} part (2) is $\text{Ker}\psi = \mu_n$ where $\mu_n$ is the group of roots of unity of degree $n$. Therefore $\mathbb{C}^n \times \mathbb{C} / (\mathbb{C}^* \times W^\lambda) = \mathbb{C}^n / H = \mathbb{C}^n / \mu_n$.

If $k > 0$, then $\mathcal{H}_p \setminus \{(0, ..., 0)\} \subset (\mathbb{C} \times \mathbb{C}^n)\lambda^{-k}$ (indeed, in this case we have $\mathbb{C} \times \mathbb{C}^n = \{(p, x_1, ..., x_n) x_1 = ... = x_n = 0\}$ and from \ref{2.4} part (1), $(\mathbb{C} \times \mathbb{C}^n) / (\mathbb{C}^* \times W^\lambda)$ is the total space of the line bundle $O(-n)p_{\mathbb{P}^{n-1}}$ (cf. Corollary \ref{a} below for relation of invariants of this GIT to invariants of hypersurfaces in $\mathbb{P}^{n-1}$ which explains the term $\sigma$-model phase).

Example 2.7. LG models and weighted projective spaces. Consider action of $G = \mathbb{C}^*$ on $\mathbb{C} \times \mathbb{C}^n$ via

\begin{equation}
\lambda \cdot (p, x_1, ..., x_n) = (\lambda^{-p}, \lambda x_1, ..., \lambda x_n).
\end{equation}

(i.e. $X = \mathbb{C}^n, \psi(\lambda) = \lambda^{-D}$ in notations of Prop. \ref{2.3}). There are two GIT quotients of $\mathbb{C}^{n+1}$ with respect to the action \cite{9} corresponding to $\mathbb{C}^*$-linearizations $\kappa(\lambda) = \lambda^k$ with $k < 0$ and $k > 0$. The former is isomorphic to $\mathbb{C}^n / \text{Ker}\psi = \mathbb{C}^n / \mu_D$ where generator $J_W$ of $\mu_D$ acts as

\begin{equation}
J_W(..., x_i, ...) = (...e^{2\pi i q_i} x_i, ...).
\end{equation}

(i.e. via exponential grading operator; here $q_i = \frac{m_i}{D}$). The later linearization yields the line bundle over the weighted projective space $\mathbb{P}(w_1, ..., w_n)$ with $c_1 = - Dh$, where $h$ is the positive generator of $H^2(\mathbb{P}(w_1, ..., w_n), \mathbb{Z})$.

Note that $\mathbb{C}^n / \mu_D$ admits as a compactification the weighted projective space $\mathbb{P}(D, w_1, ..., w_n)$. The latter is the quotient of $\mathbb{P}(1, w_1, ..., w_n)$ by the action of $\mu_D$.

Example 2.8. Hybrid model (cf. \cite{39} sect. 5.2) Product of projective spaces. Let $G = (\mathbb{C}^*)^2$ acts on $\mathbb{C}^{n+1}$ via

\begin{equation}
(\lambda_1, \lambda_2)(p, x_1, ..., x_n, y_1, ..., y_m) = (\psi(\lambda_1, \lambda_2)p, \lambda_1 x_1, ..., \lambda_1 x_n, \lambda_2 y_1, ..., \lambda_2 y_m)
\end{equation}

where $\psi(\lambda_1, \lambda_2) = \lambda_1^{-n} \lambda_2^{-m}$. Elements of $Pic^G(\mathbb{C}^{n+1}) = \text{Char}(\mathbb{C}^2)$ have the form $\kappa(\lambda_1, \lambda_2) = \lambda_1^{r_1} \lambda_2^{r_2}$. Different choice of stability conditions lead to

(1) $(H_1)$ For the cone $r_1 < 0, r_1 m - r_2 n < 0$ (which correspond to the case when unstable locus is the union of $p = 0$ and $y_1 = ... = y_m = 0$) one obtains the total space of the orbifold $\mathbb{C}^n / \mu_n$ bundle over $\mathbb{P}^{m-1}$,
(2) \((H_2)\) For the cone \(r_2 < 0, r_1 m - r_2 n > 0\) (where the unstable locus is the union of \(p = 0\) and \(x_1 = \ldots = x_n = 0\)), the total space of orbifold \(\mathbb{C}^m/\mu_m\) bundle over \(\mathbb{P}^{n-1}\).

(3) \((H_3)\) For the cone \(r_1 > 0, r_2 > 0\) one obtains the total space of line bundle over \(\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}\).

These GIT quotients are quotients of smooth quasi-projective varieties by finite abelian groups as follows:

- \(H_1\) (resp. \(H_2\)) is quotient of the total space of \(V_1 = (\oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-m))^n\) (resp. \(V_2 = (\oplus \mathcal{O}_{\mathbb{P}^{m-1}}(-n))^m\)) by the action of \(\mu_n \subset (\mathbb{C}^*)^n = \text{Aut}(V_1)\) (resp \(\mu_m\)). Both support the diagonal action of \(T = \mathbb{C}^* \subset \text{Aut}(V_i)\).

**Example 2.9. Gauged LG models:**

(1) **Linearization on the wall.** Consider the case \(m = n = 2\) in Example 2.8 (i) and linearizations satisfying \(r_1 = r_2\). The unstable locus is \(p = 0\) and hence it follows from (2.4) that \(\mathbb{C}^2 \times \mathbb{C}^4//\mathbb{C}^* = \mathbb{C}^4//H\) where \(H = \mathbb{C}^* \times \mu_2\). The action of connected component of identity \(H_0 \subset H\) on \(\mathbb{C}^4\) is given by \(\lambda(x_1, x_2, y_1, y_2) = (\lambda x_1, \lambda x_2, \lambda^{-1}y_1, \lambda^{-1}y_2)\) and the action of \(\mu_2\) is \((\lambda x_1, \lambda x_2, y_1, y_2), \lambda \in \mu_2\). The GIT quotient by \(H_0\) can be identified with the cone in \(\mathbb{C}^4\) given by \(T_1 T_4 = T_2 T_3\) where \(T_1 = x_1 y_1, T_2 = x_1 y_2, T_3 = x_2 y_1, T_4 = x_2 y_2\) (cf. [18], Ex.8.6). The action of \(\mu_2\) on the cone is the diagonal action \(\lambda(T_1, \ldots, T_4) = (\lambda T_1, \ldots, \lambda T_4)\) and this phase is biholomorphic to \(\mu_2\)-quotient of the cone. Here we have gauged LG model corresponding to \(\mu_2\) with the gauge group \(H_0 = \mathbb{C}^*\).

(2) **Toric varieties.** Let \(X\) be a projective toric variety corresponding to a simplicial fan. It has a GIT quotient presentation \(\mathbb{C}^N//\mathbb{C}^k\) where \(\rho : \mathbb{C}^*^k \to (\mathbb{C}*)^N\) is a homomorphism into the torus acting diagonally on \(\mathbb{C}^N\) (cf. [15, 16]). For a fixed \(\psi \in \text{Char}(\mathbb{C}^*)^k\) one has several phases of \(\mathbb{C}^*\)-action of \(\mathbb{C} \times \mathbb{C}^N\) given by \(g(p, x) = (\psi(g)p, \rho(g)x), g \in \mathbb{C}^*, p \in \mathbb{C}, x \in \mathbb{C}^N\), one of which is a line bundle over \(X\) (for \(\kappa \in \text{Char}\mathbb{C}^k\) such that \(([p, x]| p = 0, x \in \mathbb{C}^N]\subset (\mathbb{C} \times \mathbb{C}^N)_\kappa^*\) and the rest are toric varieties which are GIT quotients of \(\mathbb{C}^N\) by the action of an algebraic group with identity component being the torus \((\mathbb{C}^*)^{k-1}\) i.e. an example of gauge LG model in the sense of (3) Def. 2.3 (in some cases, as in example 2.8 this is also a hybrid model).

(3) Let \(\text{Mat}_{n,m}\) denote the vector space of \(n \times m\) matrices (with entries in \(\mathbb{C}\)). Consider the action of \(\text{GL}_n\) on \((\text{Mat}_{n,m}) \times \mathbb{C} (n < m)\) via multiplication on the first factor and via the character \(\psi(A) = \text{det}(A)^{-k}, k > 0\) on the second. In this case, \(\text{Pic}^{\text{GL}_n} = \mathbb{Z}\) and one has two GIT quotients one of which is the line bundle over the Grassmanian \(\text{Gr}(n, m)\) and another is the quotient of the affine cone of \(\text{Gr}(n, m)\) by the cyclic group (i.e. this phase is the gauged (with gauge group \(\text{SL}_n\) \(\mu_m\)-LG model). For \(k = m\) the first GIT quotient has trivial first Chern class (recall that \(c_1(\text{Gr}(n, m) = m\sigma\) where \(\sigma\) is a positive generator of \(H^2(\text{Gr}(n, m), \mathbb{Z})\) (cf. [3], Sect. 16.2)

3. \(\mathbb{C}^*\)-action on GIT quotients

Following Proposition gives a sufficient condition for a phase \(L//\kappa G\) (cf. def. 2.3) to support the \(\mathbb{C}^*\)-action induced by \(\mathbb{C}^*\) action on \(L\).
Proposition 3.1. Let $X, G, L, L, \kappa$ as in Def. \[2.3\] and such that $\kappa$ belongs to the interior of a GIT chamber i.e. the interior of a cone of codimension zero in $E^G_l$ described in Theorem \[2.2\]. Then the $C^*$-action on the fibers of projection $\pi : L \rightarrow X$ induces the $C^*$-action on $L/\kappa G$.

Proof. We claim that $x \in L^u_\kappa$ iff $\pi(x) \in X^u_{\kappa}$. Indeed, let $x \in L$, $K$ be the line bundle on $X$ underlying the linearization $\kappa$ and let us assume that $x$ is $\kappa$-unstable. Let $s \in \Gamma(X, K^m)^G$, $m \in \mathbb{N}$. Then $\pi^*(s) \in \Gamma(L, \pi^*(K)^m)^G$ for the linearization of $\pi^*(K)$ induced by $\kappa$ and hence $\pi^*s(x) = 0$, i.e. $s(x) = 0$.

Vice versa, let $x \in L$ and $\pi(x) \in X$ be unstable. Consider $s \in \Gamma(L, (\pi^*(K)^m)^G)$. From Leray spectral sequence: $H^p(X, R^q\pi_* (\pi^*(K)^m)) \Rightarrow H^{p+q} (L, \pi^*(K)^m)$, and vanishing of $R^q\pi_* (\pi^*(K)^m)$ for $q > 0$, it follows that

$$H^0(L, \pi^*(K)^m) = H^0(X, \pi_* (\pi^*(K)^m)) = H^0(X, K^m \otimes \pi_* (O_L)).$$

Using the decomposition of $\pi_* (O_L)$ into eigensheaves of $C^*$-action on $L$ and identifying these eigensheaves with the sheaves $L^n, n \in \mathbb{Z}$ we obtain the isomorphism:

$$\Gamma(L, (\pi^*(K)^m)^G) = \oplus_n \Gamma(X, K^m \otimes L^n)$$

Now, since $\kappa$ is in the interior of a GIT chamber, for $0 < \epsilon << 1$ and fixed $n$ the linearization $\kappa + n\epsilon \psi$, where $\psi$ is the linearization corresponding to the $G$-bundle $L$, belongs to the same GIT chamber as $\kappa$. This implies that for each $n$ and $m >> 0$ the linearization $m\kappa + n\psi = m(\kappa + \frac{n}{m} \psi)$ belongs to the interior of the same chamber as $\kappa$ and hence invariant section of $K^m \otimes L^n$ must vanish at $x$. Applying this to the finite collection on integers $n$ for which the components of decomposition (12) of $s$ are non trivial we see that any section in $H^0(X, \pi_* (\pi^*(K)^m)) = H^0(L, \pi^*(K)^m)$ vanishes at $x$ for large $m$ i.e. $x$ is unstable.

This implies that $L^{ss}$ is $C^*$-invariant and hence restriction induces the $C^*$-action on semi-stable locus. Since $L$ is a $G$-bundle, this $C^*$-action commutes with $G$-action the gluing construction of $L/\kappa G$ yields that considered $C^*$ action descends to this GIT quotient. \qed

Singularities of the open subset $X^*/G \subset X^{ss}/G$ are quotient singularities (corresponding to orbits of points with finite non-trivial stabilizers) while singularities in the complement $X^{ss} \setminus X^*$ can be more complicated (cf. \[2.8\] for a description of their resolution).

It follows from \[2.11\] that a quotient $X/G$ by the action of a torus can be represented as a global quotient i.e. there exist a smooth quasi-projective variety $\hat{X}$ and finite group $\Gamma$ such that $X/G = \hat{X}/\Gamma$. The next proposition discusses liftings of $C^*$-actions in the general context of global quotients by finite groups.

Proposition 3.2. Let $\kappa_1$ and $\kappa_2$ be two linearizations of $G$-action on $L$ as in Def. \[2.3\] and let $X_i = (L)/\kappa_i G$ for $i = 1, 2$ be corresponding GIT quotients such that there is a birational morphism $\psi : X_1 \rightarrow X_2$. Assume that

1. there exist a finite group $\Gamma$ acting on a smooth manifold $\hat{X}_2$ and morphism $\hat{\psi}$ of $\hat{X}_1 = \hat{X}_2 \times X_1$ $X_1$ making the diagram commutative:

$$\begin{array}{ccc}
\hat{X}_1 & \xrightarrow{\hat{\psi}} & \hat{X}_2 \\
\pi_1 \downarrow & & \pi_2 \downarrow \\
\hat{X}_1/\Gamma = X_1 & \xrightarrow{\psi} & \hat{X}_2/\Gamma = X_2
\end{array}$$


(2) the action of $T = \mathbb{C}^*$ on total space $L_2$ of bundle $L_2$ descends to the $\mathbb{C}^*$-action on $X_2$ (e.g. if conditions of Prop. 3.1 are met) so that it preserves the branching locus of $\pi_2$.

Then this action on $X_2$ lifts to the action of a finite cover $\tilde{T}$ of $T$ on $\tilde{X}_2, X_1, \tilde{X}_1$ so that the diagram (13) is $\tilde{T}$-equivariant.

Proof. Conditions for the local lifting of automorphisms of quotients by a finite group on ramification divisor of the quotient from [32] are satisfied for the GIT quotients $X_1, X_2$. Such a lift is unique up to possible ambiguity due to analytic continuation which produces a well defined element in the covering group so that the action of the covering group $\tilde{T}$ is well defined. □

4. Elliptic genus

4.1. Orbifold Equivariant elliptic genus of pairs. Let $\mathcal{G}$ and $\Gamma$ be respectively an algebraic group and a finite group both acting on quasi-projective manifolds $X$ via biholomorphic automorphisms so that two actions commute. The following describes expansion of holomorphic Euler characteristic of a sheaf which is both $\mathcal{G}$ and $\Gamma$ equivariant in terms of the characters of $\mathcal{G}$. Note that our choice of generators in the lattice of characters of $\mathcal{G}$ inside $\text{Char} \mathcal{G} \otimes \mathbb{Q}$ depends on $\Gamma$ and made so that the lattice of characters of the quotient of $\mathcal{G}$ acting on $X/\Gamma$ effectively will be primitive.

For the rest of the paper we will be interested in the case $\mathcal{G} = \mathbb{C}^*$ so we shall denote the algebraic group of the automorphisms as $T$.

Definition 4.1. Let $X$ be a smooth quasi-projective variety, $T$ and $\Gamma$ be a connected algebraic and finite abelian groups respectively both acting holomorphically on $X$ so that both actions commute. Let $\Gamma_0 = T \cap \Gamma$ be normal in $T$ and $T_0 = T/\Gamma_0$. Let $\mathcal{F}$ be a sheaf on $X$ which is both $T$ and $\Gamma$-equivariant. Assume that the actions on $\mathcal{F}$ commute so each $\Gamma$-eigensheaf $\mathcal{F}_\lambda$ of $\mathcal{F}$ supports the action of $T$. Equivariant Euler characteristic of $\mathcal{F}_\lambda$ is

\[
\chi^T(\mathcal{F}_\lambda) = \sum (-1)^i \dim H^i(X, \mathcal{F}_\lambda)_m \cdot e^m = \int_X ch_T(\mathcal{F}_\lambda) Td_T(X)
\]

where $m \in \text{Char}(T_0) \otimes \mathbb{Q} = \text{Char}(T) \otimes \mathbb{Q}$ and $e^m$ is corresponding to a character $m \in \text{Char}(T_0)$ element of group ring of $\text{Char}(T_0)$, $ch_T, Td_T$ are the $T$-equivariant Chern and Todd classes respectively (cf. [20] for a discussion of equivariant Riemann-Roch).

Elements of $\text{Char}T$ are linear combinations with $\mathbb{Q}$-coefficients of elements in $\text{Char}T_0$ and the series (14) has fractional exponents.

Though $T_0$ does not act on $X$ (but does so on $X/\Gamma$) the reason to express $\chi^T(\mathcal{F})$ in terms of characters of $T_0$ is that we will be interested in sheaves $\mathcal{F}$ used to describe the invariants of $X/\Gamma$. In particular we will consider equivariant version of elliptic class introduced in [11]. We refer to this paper for details of the definitions including the one for $T$-normal pair and for numerous applications. See [37] for discussion of equivariant case.

Definition 4.2. Let $(X, E)$ be a Kawamata log-terminal $\Gamma$-normal pair (in particular, $X$ is smooth and $E$ has simple normal crossings) with $E = - \sum_k \delta_k E_k$. Assume that $T$ and $\Gamma$ are as in definition [3.1] and that the action of $\Gamma$ on the set

$2$ action of $\tilde{T}$ on $X_1, X_2$ is not effective with $\text{Ker} \tilde{T} \rightarrow T$ acting trivially on $X_1, X_2$.
of components $E_k$ of $E$ is trivial. For a character $\lambda$ of the subgroup $(g, h)$ of $G$ generated by a pair of commuting elements $g, h \in G$, denote by $V_\lambda$ the $\lambda$-eigenbundle of the restriction of $TX|_{X^{g,h}}$ on the set $X^{g,h}$ of fixed points of $g$ and $h$. Let $0 \leq \lambda(g) < 1$ be the logarithm of the value of character on $g$ (i.e. the value of character at $g$ is $e^{2\pi i \lambda(g)}$). Let $x_\lambda(t) \in H^*_T (X^{g,h})$ be equivariant Chern roots of $V_\lambda$, $e_k(t) \in H^*_T (X)$ be the equivariant Chern classes of $T$-bundles $O(E_k)$ and let $0 \leq \epsilon_k < 1$ denotes the logarithm of the character of $(g, h) \subset G$ acting on the bundle $O(E_k)$ if $X^{g,h} \subset E_k$ and zero otherwise. Finally let $i_{X^{g,h}*} : H^*_T (X^{g,h}) \to H^*_T (X)$ be the Gysin and $\Psi : X \to X/\Gamma$ be the quotient map.

The orbifold elliptic class of the triple $(X, E, \Gamma)$ is an element $\Psi_* (\mathcal{EL}_T^T (X, E, \Gamma; u, z, \tau)) \in H^*_T (X/\Gamma)$ where

$$\mathcal{EL}_T^T (X, E, \Gamma; u, z, \tau) := \frac{1}{|\Gamma|} \sum_{g, h, gh = hg} \sum_{X^{g,h}} (i_{X^{g,h}*} \prod_{\lambda(g) = \lambda(h) = 0} x_\lambda(t)) \times \prod_\lambda \frac{\theta(x_\lambda(t) + \lambda(g) - \tau \lambda(h) - z)}{\theta(2x_\lambda(t) + \lambda(g) - \tau \lambda(h))} \times \prod_k \frac{\theta(x_\lambda(t)/2 \pi i) + \epsilon_k(g) \tau - (\delta_k + 1)z}{\theta(x_\lambda(t)/2 \pi i + \epsilon_k(g) \tau - \epsilon_k(h) \tau - z)} \frac{\theta(-z)}{\theta(-(\delta_k + 1)z)} e^{2\pi i \epsilon_k(h)z}. \tag{15}$$

($T_0$ is as in Def. [11])

Localization theorem represents equivariant euler characteristic as a sum over fixed components. For example, for a torus $T$ acting on a manifold $X$ one has (cf. [20]):

$$\chi^T (X, \mathcal{F}) = \sum_{P \in X^T} \chi^T (X, P, i_P^* (\mathcal{F})) \tag{16}$$

where summation is over connected components $P$ of the fixed point set $X^G$ of $G$ and $i^*$ is the pull back map of equivariant cohomology.

In the case of [20], in the term of the localization form (16) corresponding to a component $P$ of the fixed point set of the $G$-action, the class $e_k(t)$ get replaced by the class of equivariant line bundle associated with the divisor $E_k$ restricted to $P$.

**Definition 4.3.** The equivariant elliptic genus $\mathcal{EL}^T_{orb} (X, E, \Gamma; u, z, \tau) (u \in \text{Lie}(T)^* \otimes \mathbb{C})$ is the value of class $\mathcal{EL}^T_{orb} (X, E, \Gamma; u, z, \tau)$ on the fundamental class of $X$. If $P$ is a fixed irreducible component of $T$-action then contribution of $P$ is the value the localization of elliptic class $\mathcal{EL}^T_{orb} (X, E, \Gamma; u, z, \tau)$ on the fundamental class of $P$.

4.2. Elliptic genus of phases. Now we apply the set up of the last section to the case of GIT quotients. This is done with assumptions of existence of the structure of global orbifold on the GIT quotient. The definition will be stated only in the case when $E$ in Def[12] is empty.

**Definition 4.4.** (Elliptic genus of a phase) Let $X, G, L, L, \kappa$ be as in Def. [22]. Assume that this data satisfies the conditions of both Prop. [7.2] and [7.2]. In particular $L/\kappa G$ is endowed with the $T = \mathbb{C}^*$-action, there exist a smooth manifold $L/\kappa G$ acted upon by a finite abelian group $\Gamma$ and by $\mathbb{C}^*$ so that these actions commute and such that $L/\kappa G/\Gamma = L/\kappa G$. Let $P \subset L/\kappa G$ be an irreducible
component fixed by the $\mathbb{C}^*$-action on $L/\kappa G$. Consider the equivariant elliptic class $\mathcal{E}LL^\mathbb{C}^*_\text{orb}(L/\kappa G, u, z, \tau)$ where $u$ is the infinitesimal character of $\mathbb{C}^*$ acting faithfully on orbifold $L/\kappa G$. Then the elliptic genus of the phase $(X, G, L, L, \kappa)$ relative to the component $P$, denoted as $\text{Ell}_L(L/\kappa G, \Gamma, P, u, z, \tau)$ on the diagonal $u = z$ of $\mathbb{C} \times \mathbb{C} \times \mathcal{H}$:

\begin{equation}
\text{Ell}(L/\kappa G, P, z, \tau) = \mathcal{E}LL^\mathbb{C}^*_\text{orb}(L/\kappa G, \Gamma, P, z, z, \tau)
\end{equation}

More generally, the same definition will be used in the cases when $L/\kappa G$ has Kawamata log-terminal singularities and when $\text{Ell}(L/\kappa G, \Gamma)$ is well defined as the orbifold elliptic genus of a pair via a resolution of singularities and taking into account the divisor determined by the discrepancies of the resolution (cf. [11]).

We shall give examples in the next section but first we discuss modularity properties of our elliptic genus. For a discussion of modular properties of elliptic genus of manifolds see [9].

4.3. Modularity.

**Theorem 4.5.** Let $L$ be the total space of a $G$-equivariant line bundle over a quasi-projective manifold $X$ and $\mathcal{X} = L/\kappa G$ be the phase corresponding to a linearization $\kappa$ of $G$-action on $L$. Assume that $\mathcal{X}$ admits a presentation as a global quotient $\mathcal{X} = \tilde{\mathcal{X}}/\Gamma$ as in Prop. [3,2] and, in addition, that the orbifoldization group $\Gamma$ is subgroup of the lift of $\mathbb{C}^*$-action on $\tilde{\mathcal{X}}$ as also described in [3,2].

Let $P \subset \tilde{\mathcal{X}}$ be a compact component of the fixed point set for this $\mathbb{C}^*$-action, $T_{\tilde{\mathcal{X}}}|_P$ be the tangent bundle to $\tilde{\mathcal{X}}$ restricted on $P$ and $c_{1}^e(T_{\tilde{\mathcal{X}}}|_P) \in H^2_{\mathbb{C}^*}(P)$ be its equivariant first Chern class. If

(a) $c_{1}(T_{\tilde{\mathcal{X}}}|_P) = 0$,

(b) the Calabi Yau condition [19] described below is met,

then the restriction of the $\mathbb{C}^*$-equivariant elliptic genus $\mathcal{E}LL^\mathbb{C}^*_\text{orb}(P, \tilde{\mathcal{X}}, \Gamma, u, z)$ on $u = z$ is a Jacobi form of weight zero and index $d_2 = \dim_{\mathbb{Q}} \mathcal{X} - 1$.

**Proof.** Let $n$ be the order of group $\Gamma$, which is cyclic since we assume that $\Gamma \subset \mathbb{C}^*$. We shall identify it with the group of roots of unity $\mu_n$. Let $T_{\tilde{\mathcal{X}}}|_P = \oplus V_i$ be a split into a sum of rank one eigen-bundles of the above $\mathbb{C}^*$-action (some possibly equivariantly isomorphic). Let $x_i \in H^2(P, \mathbb{Z})$ be the first Chern class of $V_i$ and $q_i \in \mathbb{Q}$ is such that $q_i u$ be the infinitesimal character of $\mathbb{C}^*$-action corresponding to the bundle $V_i$ ($u$ is the infinitesimal character of the group $\mathbb{C}^*/\Gamma$ acting on $\tilde{\mathcal{X}}/\Gamma$). Note that $n q_i \in \mathbb{Z}$ since ord($\Gamma$) = $n$. For $\chi_i \in \text{Char}\mu_n$ corresponding to the action of $\Gamma = \mu_n$ on sumand $V_i$ we have: $\chi_i(1) = \exp 2\pi i q_i$. Vanishing of equivariant first Chern class implies that:

\begin{equation}
\sum_i x_i = 0,
\end{equation}
We shall assume further the following Calabi Yau condition:

\begin{equation}
\sum_{i=1}^{i=n} q_i = 1
\end{equation}

This implies that

\begin{equation}
\Gamma \text{ acts trivially on } det T_{\chi} | _{\mu}.
\end{equation}

since we assume that \( \Gamma \subset \mathbb{C}^* \). As in [10] (and Def. 4.2 above), we replace \( \chi \in \text{Char}_{\mu_n} \) by logarithm \( \lambda : \mu_n \to \mathbb{Q} \) such that \( \chi(g) = \exp(2\pi i \lambda(g)), 0 \leq \lambda(g) < n \). With these notations, one has the following expression for the contribution into the orbifold elliptic genus of \( X \)

\begin{equation}
Ell(L/\times G, P) = Ell_{orb}^{\chi}(X, P, \mu_n) = \frac{1}{n} \prod_{g,h \in \mu_n, \lambda(g) = \lambda(h) = 0} x_{\lambda} \prod_{i} \Phi(g, h, \lambda_i, x, z, \tau)[\mathcal{X}^{g,h}]
\end{equation}

The factor \( \Phi(g, h, \lambda, x, z, \tau) \) defined by:

\begin{equation}
\Phi(g, h, \lambda, x, z, \tau) = \frac{\theta(x/2\pi i + (q\lambda - 1)z + \lambda(g) - \lambda(h)\tau)}{\theta(x/2\pi i + q\lambda z + \lambda(g) - \lambda(h)\tau)} e^{2\pi i \lambda(h)z}
\end{equation}

where \( q_\lambda \in \mathbb{Q} \) is such that \( q_\lambda u \) is the infinitesimal character of the \( \mathbb{C}^* \)-action on the eigen-bundle corresponding to \( \lambda \) and \( x \in H^2(P, \mathbb{Z}) \). This is specialization to \( u = z \) of the equivariant version of the corresponding expression in [10]. For \( a, b \in \mathbb{Z}/n\mathbb{Z}, q \in 1/n \mathbb{Z} \) we define

\begin{equation}
\Psi(a, b, q, x, z, \tau) = \frac{\theta(x/2\pi i + (q - 1)z + qa - qbr)}{\theta(x/2\pi i + qz + qa - qbr)} e^{2\pi iz\Phi}
\end{equation}

so that for \( g, h \in \mathbb{Z}/n\mathbb{Z}, g = a \text{ mod } n, h = b \text{ mod } n \) and \( q = q_\lambda \) one has: \( \Phi(g, h, \lambda, x, z, \tau) = \Psi(a, b, q, x, z, \tau) \). We have the following identities:

\begin{equation}
\Psi(a - 1, b, x, q, z + 1, \tau) = -\Psi(a, b, x, z, \tau)
\end{equation}

\begin{equation}
\Psi(a, b + 1, x, q, z + \tau, \tau) = e^{x - 2\pi i (1 - 2q)z + 2\pi i q - (1 - 2q)\pi i \tau} \Psi(a, b, x, z, \tau)
\end{equation}

\begin{equation}
\Psi(a - b, b, x, q, z, \tau + 1) = \Psi(a, b, x, z, \tau)
\end{equation}

\begin{equation}
\Psi(a, b, x, q, \frac{z}{r}, \frac{1}{r}, \tau) = e^{-x + \frac{\pi i (1 - 2q)z^2}{r} - 2\pi izq} \Psi(b, -a, x, z, \tau)
\end{equation}

Identity (23) uses \( \theta(z + 1, \tau) = -\theta(z, \tau) \) (cf. [12]). The identity (25) is clear and (26) follows as the corresponding identity in the proof of the theorem 4.3 in [10]. Finally

\begin{equation}
\Psi(a, b + 1, x, z + \tau, \tau) = \frac{\theta(x/2\pi i + (q - 1)(z + \tau) + qa - q(b + 1)\tau)}{\theta(x/2\pi i + q(z + \tau) + qa - q(b + 1)\tau)} e^{2\pi i q(b + 1)(z + \tau)}
\end{equation}

\begin{equation}
\Psi(a, b, x, q, \frac{z}{r}, \frac{1}{r}, \tau) = e^{-x + \frac{\pi i (1 - 2q)z^2}{r} - 2\pi izq} \Psi(b, -a, x, z, \tau)
\end{equation}

Using \( \theta(z - \tau) = -\theta(z)e^{2\pi iz - \pi i \tau} \) we obtain (24). Now Jacobi property of (20) follows since \( \sum_{i} (1 - 2qi) = \dim X - 2 \).
4. LG elliptic genus. Another kind of elliptic genus, which is associated with weighted homogeneous polynomials, was proposed in physics literature (cf. [27], [6] and references therein).

**Definition 4.6. (LG elliptic genus)** (cf. [6]). Let \( GW \) be abelian group and \( R \) a representation of \( G \) in \( \mathbb{C}^n \) which preserves a weighted homogeneous polynomial given in coordinates of a basis \( e_i, i = 1, \ldots, n \) as \( W(x_1, \ldots, x_n) \). Let \( w_i, D \in \mathbb{N}, i = 1, \ldots, n \) (weights and degree of \( W \)) be integers such that \( W(\ldots, x_i, \ldots) = x_i^D W(\ldots, x_i, \ldots) \) and \( q_i = \frac{w_i}{D} \in \mathbb{Q} \). Assume that in basis \( e_i \) the group \( G \) acts diagonally and via \( R(g) \cdot e_i = R_i(g)e_i = \exp(2\pi i \theta_i(g))e_i, \theta_i(g) \in \mathbb{Q} \). Finally, let \( H \) be a subgroup of \( G \). Then the elliptic genus \( Z[R, H] \) of the the data \( (G, R, W, H) \) is given by

\[
Z[R, H] = \frac{1}{|H|} \sum_{h_a, h_b \in H} \prod_{i=1}^n Z[R_i](h_a, h_b)
\]

where

\[
Z[R_i](h_a, h_b) = e^{-2\pi i \theta_i(h_a)} \frac{\theta_i((1 - q_i)z + \theta_i(h_b) - \tau \theta_i(h_a), \tau)}{\theta_i(z + \theta_i(h_b) + \tau \theta_i(h_a), \tau)}
\]

The following follows by direct calculations and description of spectrum of weighted homogeneous singularities obtained in [35] (cf. also [19]; recall that the exponential grading operator \( J_W \) is given by (13)).

**Theorem 4.7.** Specialization of LG elliptic genus for \( \tau = i\infty, t = \exp(2\pi iz) \) coincides with the orbifoldization of generating function \( \sum_{\alpha \in \mathbb{Q}} \dim H^\alpha \exp(2\pi i \alpha) t^\alpha \) where \( \alpha \) runs through the spectrum of isolated singularity \( W = 0 \) and \( H^\alpha \) is the eigenspace of the \( \Gamma \)\( F^{[\alpha]} H \) graded vector space of the Hodge filtration of Milnor fiber of \( W \). Orbifoldization group \( H \) is the cyclic group generated by the exponential grading operator \( J_W \) (cf. [19]).

Now for Landau Ginzburg phase (cf. Example 2.6) the only fixed point of the \( \mathbb{C}^* \)-action on \( \mathbb{C}^n / \langle J_W \rangle \) is the origin. The infinitesimal characters of the lift of this action on \( \mathbb{C}^n \) are \( \frac{w_i}{D} \) (where \( u \) is the infinitesimal character of the \( \mathbb{C}^* \) action on \( \mathbb{C}^n / \langle J_W \rangle \)). Hence we obtain:

**Proposition 4.8.** The elliptic genus of Landau Ginzburg phase (cf. Def 2.3 (2)) coincides with the LG elliptic genus (4.6).

5. Main theorem and explicit forms of LG/CY correspondence

5.1. Equivariant elliptic genus in birational morphisms. Recall the following equivariant version of McKay correspondence for elliptic genus (theorem 5.3 in [11] and theorem 10 in [37]). We give a weaker form which assumes existence of crepant resolution since this is sufficient in all examples we consider below. We refer to [11] and [37] for versions which involves elliptic genus of pairs and includes corrections corresponding to discrepancies of a resolution map.

**Theorem 5.1.** (local equivariant McKay correspondence) Let \( X \) be a smooth quasi-projective variety, \( \Gamma \) and \( G \) are respectively finite and reductive groups acting on \( X \) so that the actions commute. Let \( T \) be the maximal torus of \( G \). Assume that

1. there exist a crepant \( T \)-equivariant resolution \( \pi : \tilde{X} \to X/\Gamma \) i.e. \( K_{\tilde{X}} = \pi^*(K_{X/\Gamma}) \).

2. fixed point sets of \( T \) action on \( X \) and \( \tilde{X} \) are compact.
Then
\[ ELL_{\text{orb}}^T(X, \Gamma, P) = \sum_{P_i} ELL^T(\tilde{X}, P_i) \]
where the sum is taken over all fixed components \( P_i \) of \( T \)-action mapped to \( P \).

**Remark 5.2.** Equivariant elliptic genera appearing in (29) are functions on \( C \times (\text{Lie}(T)^* \otimes \mathbb{C}) \times \mathcal{H} \) where the first and third coordinates correspond to the variables of theta function \( \theta(z, \tau) \) and \( \text{Lie}(T)^* \) is the dual of the Lie algebra of the maximal torus i.e. the space of infinitesimal characters of representation of \( T \). Recall again that as was done in Def. 4.4 and Theorem 4.5, we normalize variables in \( \text{Lie}(T)^* \otimes \mathbb{C} \) by choosing basis given by the characters of the quotient group of \( T \) which acts faithfully on \( X/\Gamma \) i.e. the group \( T/T \cap \Gamma \).

5.2. Main theorem.

**Theorem 5.3.** Let \( L/\kappa_1 G = X_1 = \tilde{X}_1/\Gamma, L/\kappa_2 G = X_2 = \tilde{X}_2/\Gamma, \tilde{X}_2, \Gamma \) are as in Prop. 3.2. Assume that \( \psi : X_1 \rightarrow X_2 \) is a \( K \)-equivalence i.e. \( \psi^*(K_{X_2}) = K_{X_1} \). Then
\[ \sum_{P_i} \text{Ell}(L/\kappa_1, P_i) = \text{Ell}(L/\kappa_2, P) \]
where \( P_i \) is collection of fixed point sets which \( \psi \) takes into \( P \).

**Proof.** McKay correspondence (cf. theorem 5.1) asserts that the orbifold elliptic genus, in particular yielding the elliptic genus of the phase cf. Def. 4.4, coincides with the singular elliptic genus of the quotient. The elliptic genus of the quotient, as the elliptic genus of a singular variety is given in terms of resolution of its singularities. Our assumption that phases \( L/\kappa_1 G \) and \( L/\kappa_2 G \) are related by \( K \)-equivalence imply that both expressions in terms of resolution are the same: cf. Prop. 3.7 in [10]. Hence the assertion follows.

The reset of this section considers explicit forms of the identity (30) for the examples of phases discussed by Witten in [39].

5.3. Projective space. Consider the GIT quotient \( \mathbb{C}^n \times \mathbb{C}/\mathbb{C}^* \) as in example 2.6 (1), i.e. for the action \( \lambda(p, x_1, ..., x_n) = (\lambda^{-n}p, \lambda x_1, ..., \lambda x_n) \). Assume first that \( k < 0 \) for linearization \( \kappa(\lambda) = \lambda^k \). In this case GIT quotient is the orbifold \( X_1 = \mathbb{C}^n/\mu_n \) (cf. 2.6). The \( \mathbb{C}^* \)-action from Prop. 3.4 is the action of 1-dimensional torus \( \mathbb{C}^*/\mu_n \) induced on \( X_1 \) by the diagonal action of \( \mathbb{C}^* \) on \( \mathbb{C}^n \). The contribution of the origin i.e. the only fixed point of this action (for the action of \( \mathbb{C}^* \) expressed in terms of generator \( u \in \text{Char}\mathbb{C}^*/\mu_n \)) is given by
\[ (\frac{\theta(u/n - z)}{\theta(\frac{z}{n})})^n. \]

Orbifoldized elliptic genus (31) has form (cf. (4.2)):
\[ ELL_{\text{orb}}^*(\mathbb{C}^n, P, \mu_n) = \sum_{a, b} (e^{-2\pi i \frac{ab}{n}} \theta((\frac{u}{n} - z + \frac{a-b}{n}, \tau)) \theta(\frac{u}{n} + \frac{a-b}{n}, \tau))^n \]
For \( u = z \) we obtain the LG-genus (27) where \( q_i = \frac{1}{n} \):
\[ = \frac{1}{n} \sum_{0 \leq a, b < n} \theta((\frac{1-n}{n})z + \frac{a-b}{n}) e^{2\pi i \frac{ab}{n}} n \]
Another GIT quotient corresponds to linearizations with \( k > 0 \) in which case unstable locus is the line \( x_1 = \ldots = x_N = 0 \) (cf. \[2.6(1)\]). It is biholomorphic to the total space \( X_2 \) of the bundle \( O_{\mathbb{P}^{n-1}}(-N) \). If \( \tilde{X}_2 \) is the blow up of \( \mathbb{C}^n \) at the origin, one has \( \tilde{X}_2/\mu_n = X_2 \). Let \( P \) be the exceptional \( \mathbb{P}^{n-1} \). Then the equivariant Chern class satisfies: \( c(TP) = (1 + x)^n, c(N_P) = -ux + 2\pi i u \). Therefore the contribution of the exceptional set \( P \), which is also the fixed point set of \( \mathbb{C}^* \) action, in the equivariant elliptic genus of resolution is given by:

\[
\mathcal{E}LL^* (X_2, P) = \frac{x\theta(x/2\pi i - z)}{\theta(2\pi i)} n \frac{\theta((u - \frac{ux}{2\pi i}) + u - z)}{\theta(-\frac{ux}{2\pi i} + u, \tau)}
\]

**Corollary 5.4.** *(LG-CY correspondence, cf. \[22\])* LG elliptic genus of singularity \( x_1^n + \ldots + x_n^n \) coincides with the elliptic genus of smooth CY hypersurface in \( \mathbb{P}^{n-1} \).

**Proof.** It follows from equivariant McKay correspondence (theorem 5.1) that

\[
\text{Ell}_\text{orb}^*(\mathbb{C}^n, \mu_n, \mathcal{O}) = \text{Ell}^*(\mathbb{C}^n/\mu_n, P)
\]

Hence LG genus \((\ref{ll})\), i.e. the left hand side of \((\ref{ll})\) for \( u = z \) is the elliptic genus \((\ref{ll})\) restricted to \( u = z \) i.e. can be written as

\[
\frac{x\theta(x/2\pi i - z)}{\theta(2\pi i)} n \frac{\theta((u - \frac{ux}{2\pi i}) + u - z)}{\theta(-\frac{ux}{2\pi i} + u, \tau)} [\mathbb{P}^{n-1}]
\]

Since the total Chern class of smooth hypersurface \( V_n \) of degree \( n \) in \( \mathbb{P}^{n-1} \) is \( c(V_n) = \frac{(1 + x)^n}{(1 + nx)} (x \in H^2(\mathbb{P}^{n-1}, \mathbb{Z}) \) is the positive generator) the expression \((\ref{ll})\) coincides with the elliptic genus of \( V_n \). \(\square\)

### 5.4. Weighted \( \mathbb{C}^* \)-actions.

Now we consider the extension of the previous case, to the action with unequal weights considered in example \[2.7\].

**Theorem 5.5.** Let \( T = \mathbb{C}^* \) acts on \( \mathbb{C}^{n+1} \) via \[8\] and let

\[
D = \sum w_i, q_i = \frac{w_i}{D}
\]

i.e. the Calabi Yau condition \((\ref{cy})\) is satisfied.

1. The \( \mathbb{C}^* \)-action discussed in Prop. \[32\] on \( \tilde{X}_1 = \mathbb{C}^n \) which is the lift of the \( \mathbb{C}^* \)-action on \( X_1 = \mathbb{C}^n/\mu_D = \mathbb{C}^n \times \mathbb{C}/T \) (where the latter induced from the action of \( \mathbb{C}^* \) on \( \mathbb{C}^n \) given by \( s(x, z_1, \ldots, z_n) = (s \cdot x, z_1, \ldots, z_n) \) has the form: \( s(x_1, \ldots, x_n) = (s^{w_1} x_1, \ldots, s^{w_n} x_n) \). The action on the second quotient \( X_2 \), i.e. the line bundle over \( \mathbb{P}(w_1, \ldots, w_n) \) is the fiberwise action of \( \mathbb{C}^* \). The contraction of the zero section of the line bundle \( X_2 \) is biholomorphic to \( X_1 \).

2. The restriction of local contribution of the origin \( \mathcal{O} \) of \( X_1 \) into equivariant orbifold elliptic genus \( \text{Ell}^*(\mathbb{C}^n, < J_W >, \mathcal{O}) \) obtained by restriction on subset of variables given by \( u = z \) coincides with the LG elliptic genus corresponding to the data \( (G_W, R, W, < J_W >) \) (cf. def. \[4.6\]) where \( W \) is a weighted homogeneous polynomial with weights \( w_i \) and degree \( D \), \( G_W \) generated by \( (\ldots, \exp(2\pi i w_i / D), \ldots) \) and \( < J_W > \) is cyclic subgroup of \( G_W \). generated by exponential grading operator \[(\ref{i})\].
(3) The local contribution for $C^*$-action on the total space of the line bundle over $\mathbb{P}^{n-1}(w_1,\ldots,w_n)$ in Example 2.8 is given by

$$E L L_{\text{orb}}(\mathbb{P}^{n-1}, \mu_{w_1} \times \cdots \times \mu_{w_n}) \cdot \frac{\theta(Dx + u - z)}{\theta(Dx + z, \tau)} \left[\mathbb{P}^{n-1}(w_1,\ldots,w_n)\right]$$

where $E L L_{\text{orb}}(\mathbb{P}^{n-1}, \mu_{w_1} \times \cdots \times \mu_{w_n})$ is the total orbifold elliptic class of $\mathbb{P}^{n-1}(w_1,\ldots,w_n)$ considered as the orbifold quotient of $\mathbb{P}^{n-1}$.

(4) (LG-CY correspondence) If the hypersurface $V_D$ of degree $D$ in $\mathbb{P}^{n-1}(w_1,\ldots,w_n)$ is quasi-smooth and with assumption (37) the elliptic genus (4.6) coincides with the orbifold elliptic genus of hypersurface $V_D$.

Proof. Part (1) follows from definitions. Next consider the local contribution described in (2). The map of groups of characters $j^* : \text{Char}^* / \mu_D \to \text{Char}^*$ satisfies: $j^*(u) = Dv$. To determine the contribution of the origin $\emptyset$, which is the only fixed component of the $C^*$-action, in $E L L_{\text{orb}}(\mathbb{C}^n / \mu)$ note that the infinitesimal characters of action of $C^*$ are $u w_i$. Hence the local contribution of $\emptyset$ in terms of generator $u$ of $\text{Char}^* / \mu_D$ is given as follows (where $\lambda_i(g)$ as in (1.2)):

$$\frac{1}{D} \sum_{g,h \in \mu_D} \prod_{i=1}^{\infty} \frac{\theta(u w_i - \lambda_i(g) - \lambda_i(h) \tau - z)}{\theta(u w_i - \lambda_i(g) - \lambda_i(h) \tau)} e^{2 \pi i \lambda_i(h) z}$$

Specializing this to the case $u = z$ yields the expression

$$\frac{1}{D} \sum_{0 \leq a, b < D} \prod_{i=1}^{\infty} \frac{\theta(q_i - 1) z + a - b \tau}{\theta(z q_i + a - b \tau)} e^{2 \pi i \tau}$$

identical to (27).

In the case (3) the argument is similar to the used to derive (34) but we use partial resolution in which the the only component of the exceptional set is the orbifold (i.e. $\mathbb{P}^{n-1}(w_1,\ldots,w_n)$). The formula for the contribution follows from presentation of this orbifold as global quotient which results in replacing the total elliptic class by the orbifold elliptic class.

The last statement follows from results of [11,37] due to relation between the orbifold elliptic class and the elliptic class of resolutions. \qed

5.5. Hybrid models. Consider now elliptic genus for the phase in example 2.8

Theorem 5.6. The local contributions of the fixed point set of $\mathcal{V}_1$ which is the zero section of $\mathcal{V}_1$ for the above action of $T$ (i.e. the elliptic genus of phase $H_1$) is given by:

$$\frac{1}{n} \sum_{0 \leq a, b < n} \theta(-m \frac{a}{2 \pi i} + \left(\frac{m}{n} - 1\right) z + a - b \tau) e^{\frac{2 \pi i \tau}{n}} \theta(-m \frac{a}{2 \pi i} + \frac{m}{n} z + a - b \tau) e^{\frac{2 \pi i \tau}{n}} n^{m} [\mathbb{P}^{n-1}]$$

---

3. (cf.) is the Jacobi form of weight zero and index $n - 2$ due to equality $\sum q_i = 1$ cf. Theorem 4.4.

4. in terminology of [3], $\mu_D$ is the group of phase symmetries.
(where \( x = c_1(0_{\mathbb{P}^{m-1}}(1)) \)). Similarly the elliptic genus of phase \( H_2 \) is given by

\[
\frac{1}{m} \sum_{0 \leq a, b < m} \frac{\theta(-\frac{nx}{2\pi m} + \frac{1}{m} u - z + \frac{a-b\tau}{m}, \tau)}{\theta(-\frac{nx}{2\pi m} + \frac{1}{m} z + \frac{a-b\tau}{m}, \tau)} e^{\frac{2\pi ib\theta}{\theta(xz)}} \]  

Both these contributions are equal to the contribution of the phase \( H_3 \) ("hybrid-CY correspondence"). All 3 contributions (up to a factor) coincide with the elliptic genus of CY hypersurface in \( \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \) (of bi-degree \((n, m)\)).

**Proof.** The GIT quotient in the case \((H_1)\) is the global quotient by the group \( \Gamma = \mu_n \) given by \( \mathcal{V}_1/\mu_n = [\mathcal{O}_{\mathbb{P}^{m-1}}(-m)]/\mu_n \) (where \([F]\) denotes the total space of the bundle \( F \)). \( \mathbb{C}^*\)-action is the diagonal action on the fibers of \( \mathcal{V}_1 \) i.e. the fixed point set is \( \mathbb{P}^{m-1} \). The equivariant Chern class of the restriction of the tangent bundle to \( \mathcal{V}_1 \) on \( \mathbb{P}^{m-1} \) which is the zero section is \( c(\mathcal{V}_1, F) = (1 + x)^m \).

The eigenbundles of action of \( \Gamma \) are the same as for \( \mathbb{C}^* \) action so that \( \lambda(g) \) for \( g = \exp(\frac{2\pi i}{n}) \) is \( \frac{a}{n} \). Hence the formula \((4.2)\) for equivariant elliptic genus reduces to

\[
\frac{1}{n} \sum_{0 \leq a, b < n} \frac{\theta(-mn \frac{1}{2\pi m} + \frac{1}{m} u - z + \frac{a-b\tau}{n}, \tau)}{\theta(-mn \frac{1}{2\pi m} + \frac{1}{m} z + \frac{a-b\tau}{n}, \tau)} e^{\frac{2\pi ib\theta}{\theta(xz)}} \]  

and \((40)\) follows. The case of \((H_2)\) phase is identical. The last assertion on equality of \((42)\) and the elliptic genus of CY hypersurface in \( \mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \) follows since blow up of \( \mathcal{V}_1/\mu_n \) is a K-equivalence.

This example can be easily extended to the case of weighted actions of \((\mathbb{C}^*)^2\) on \( \mathbb{C}^n \times \mathbb{C}^m \times \mathbb{C} \) given by:

\[
(\lambda_1, \lambda_2)(p, x_1, \ldots, x_n, y_1, \ldots, y_m) = (\lambda_1^{w_1} p, \lambda_2^{w_1} x_1, \ldots, \lambda_1^{w_m} x_n, \lambda_2^{w_1} y_1, \ldots, \lambda_2^{w_m} y_m) 
\]

One has three phases, one of which is a line bundle over product of weighted projective spaces \( \mathbb{P}(w_1^1, \ldots, w_1^n) \times \mathbb{P}(w_2^1, \ldots, w_2^m) \) and others are orbi-bundles over each of the factors in this product. The first two phases are the global quotients. One of the orbi-bundles is the quotient of \( V = (\mathcal{O}_{\mathbb{P}^{m-1}}(-D_1))^n \) by the orbitifold group \( \Gamma = \mu_{D_2} \times (\mu_{w_1^1} \times \cdots \times \mu_{w_1^n}) \) with \( \mu_{D_1} \) acting by multiplying \( i \)-th summand by \( \exp(\frac{2\pi i w_i^1}{D_1}) \) and the group \( \mu_{w_1^1} \times \cdots \times \mu_{w_1^n} \) acts coordinate-wise on \( \mathbb{P}^{m-1} \) so that the quotient is \( \mathbb{P}(w_1^1, \ldots, w_1^n) \). The elliptic genus of this phase is given by the formula which is a modification of expression \((40)\) as follows. The group \( \mu_n \) is replaced by the group generated by exponential grading operator and factor \( \frac{\theta(xz)}{\theta(xz)} \) by the cohomology class which is the class of the orbifold \( ELL(\mathbb{P}^{m-1}; \mu_{w_2^1} \cdots \mu_{w_2^m}) \) (cf. \((43)\)).

### 5.6. Gauged Landau Ginzburg models.

Consider now the elliptic genera in the cases discussed in Example \((2.9)\). In Example \((2.9)\) (1), the GIT quotient is the global quotient by group \( \Gamma = \mu_2 \) acting on a singular space which is the quadratic cone in \( \mathbb{C}^4 \). It can be resolve either via a small resolution or with exceptional set \( \mathbb{P}^1 \times \mathbb{P}^1 \). If \( X \) is either of these resolutions, then \( ELL_{\text{orb}}(X, \Gamma, z, z, \tau) \) yields the elliptic genus of this gauged LG model i.e. we obtain several expression for the elliptic genus of such phase.
Similarly in the case of Example 2.9 (3) the elliptic genus of this GIT quotient is the equivariant orbifold elliptic (specialized to $u = z$) of the resolution of the affine cone over the Grassmanian with the orbifold group $\mu_n$.

6. Concluding remarks.

**Remark 6.1. Elliptic genus of phases when CY condition fails.** The assumptions of the theorem 5.3 can be weakened. Firstly, the condition that $\psi$ is a K-equivalence can be eliminated by using equivariant elliptic genus of pairs described in Definition 4.2 and relating elliptic genus of the phase $L//\kappa_2$ to elliptic genus of a pair $(L//\kappa_1, E)$.

Secondly, assumption of Prop. 6.2 that $X_1, X_2$ can be replace by requiring that $X_1, X_2$ are Kawamata log-terminal.

Finally one can extend Prop. 6.2 to Kawamata log-terminal pairs and show 5.3 replacing phases $L//\kappa_i$ by pairs $(L//\kappa_i, E_i)$ where $E_i$ are $C^*$ invariant divisors and the latter pairs are klt.

**Remark 6.2. Mirror symmetry for hybrid and gauged LG models.** Existence of LG/CY correspondence suggests that mirror symmetry between Calabi Yau manifolds for which LG/CY correspondence defined, should correspond to mirror symmetry between LG models. Such construction was proposed in [5] and was studied in detail more recently in [30], [13], [8]. Correspondence described here between other classes of phases suggests that there should be a mirror correspondence between certain hybrid models and more general phases discussed above. It would be interesting to have such mirror correspondence between hybrid models explicitly extending Begrland-Hubsch mirror symmetry for weighted homogeneous polynomials.

**Remark 6.3. Limit $q \to 0$.** Such specialization yields Hodge theoretical data for sigma-models (i.e the Hirzebruch’s $\chi_y$ genuses, cf. [10]) and for Landau-Ginzburg models where it involve information about the spectrum of weighted homogeneous singularity (cf. [4], and section 5.4 above). It would be interesting to have Hodge theoretical interpretation of such limit for hybrid and gauged LG models as well.

**Remark 6.4. Possible generalizations** In recent paper [7] the authors considered generalizations of LG-CY correspondence to (0,2)-models on the level of vertex algebras (case of ordinary LG/CY on the level of vertex algebras in the homogeneous case was considered in [22]). It would be interesting to find “vertex algebras of phases” extending results of these works.

**Remark 6.5.** As was mentioned (cf. remark 5.2), equivariant orbifold elliptic genus is a holomorphic function on $C \times \text{Lie}(T)^* \otimes C \times \mathcal{H}$ while the elliptic genus of a phase, which in Calabi Yau case is a Jacobi form on $C \times \mathcal{H}$, is obtained by restricting this function on the line $u_i = z, i = 1, ..., \dim T$ in $C \times \text{Lie}(T)^* \otimes C$ (though we were concerned only with the case $\dim T = 1$ one can extend construction of this paper to the case $\dim T > 1$ as well). It would be interesting to describe modular properties of these functions on $C \times \text{Lie}(T)^* \otimes C \times \mathcal{H}$ and characterize mentioned restriction in modular terms.
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