On the long-time asymptotics of the Camassa-Holm equation in solitonic regions

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Abstract

We study the long time asymptotic behavior for the Cauchy problem of the Camassa-Holm (CH) equation with the appropriate Sobolev initial data. in the solitonic regions. Our main technical tool is the generalization of Deift-Zhou steepest descent method. Through introducing a new scale \((y, t)\) and constructing the RH problem, we derive different long time asymptotic expansion of the solution \(u(y, t)\) in different space-time solitonic regions of \(\xi = y/t\). We divide the half-plane \(\{(y, t) : -\infty < y < \infty, t > 0\}\) into two asymptotic regions: 1. phase point absent region: \(\xi \in (-\infty, -1/4) \cup (2, +\infty)\), corresponding asymptotic approximations can be characterized with an \(N(\Lambda)\)-solitons with diverse residual error order \(O(t^{-1+2\rho})\); 2. phase points region: \(\xi \in (0, 2)\) and \(\xi \in (-1/4, 0)\). The corresponding asymptotic approximations can be characterized with an \(N(\Lambda)\)-soliton and an interaction term between soliton solutions and the dispersion term with diverse residual error order \(O(t^{-3/4})\). Our results also confirm the soliton resolution conjecture and asymptotically stability of the \(N\)-soliton solutions for the CH equation.

Keywords: Camassa-Holm equation, Riemann-Hilbert problem, \(\mathcal{D}\) steepest descent method, long time asymptotics, asymptotic stability, soliton resolution.

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1 Introduction

In this paper, we study the long time asymptotic behavior for the Cauchy problem of the Camassa-Holm (CH) equation in the solitonic regions

\[ u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \]  
\[ u(x, 0) = u_0(x), \]  

where \(\omega\) is a positive constant. The CH equation (1.1) derived as a model for unidirectional propagation of small amplitude shallow water waves by Camassa and Holm [1], but already appeared earlier in a list by Fuchssteiner and Fokas [2]. It also arises in the study of the propagation of axially symmetric waves in hyperelastic materials.
rods \[3, 4\] and its high-frequency limit models nematic liquid crystals \[5, 6\]. It has attracted considerable interest and been studied extensively due to their rich mathematical structure and remarkable properties. For example, the CH equation (1.1) is an infinite-dimensional Hamiltonian system that is completely integrable \[7, 8\]. The CH equation admits non-smooth peakon-type solitary or periodic traveling wave solutions and it was shown that these solutions are orbitally stable \[9, 10\]. It has been shown that the solitary waves of CH equation are smooth if \(\omega > 0\) \[11–14\] and peaked weak solutions if \(\omega = 0\) \[1, 15–17\] . The stability of peakons and orbital stability of solitary wave solution for the CH equation were further shown by Constantin \[18, 19\]. The global well-posedness for the Cauchy problem of the CH equation (1.1) was studied \[20–22\]. The algebro-geometric quasiperiodic solutions were constructed by using algebro-geometric method \[23, 24\]. With the aid of reciprocal transformation, Darboux transformation and multi-soliton solutions for the CH equation were given \[25\]. The inverse spectral transform for the conservative Camassa-Holm equation was studied \[26–28\].

The inverse scattering transform (IST) procedure, as one of the most powerful tools to investigate solitons of nonlinear integrable models, was first discovered by Gardner, Green, Kruskal and Miura \[29\]. The CH equation admits a Lax pair which allows using the IST method to study its initial value problem with \(\omega > 0\) \[8, 30–32\]. The related Riemann-Hilbert (RH) problem are further studied by Boutet de Monvel \[33, 34\].

In general, the initial value problems of integrable systems can be solved by using inverse scattering transform or RH method only in the case of reflectionless potentials. So it is a natural idea to study the asymptotic behavior of solutions to integrable systems. The nonlinear steepest descent method \[35\] developed by Deift and Zhou a very effective tool to rigorously obtain the long-time asymptotics behavior of systems \[36–45\]. In recent years, Boutet de Monvel et al first developed a RH and Deift and Zhou steepest descent method to study long-time asymptotics for CH equation on the line and half-line with Schwartz initial data \[46, 47\]. They further found Painleve-type asymptotics for Camassa-Holm equation \[48\]. Minakov obtained long-time for the Camassa-Holm equation with step-like initial data \[49\].

In our present paper, we study the long time asymptotic behavior for the initial
value problem of the CH equation (1.1)-(1.2). It is known that if \( m(x,0) + \omega > 0 \) for all \( x \), where \( m := u - u_{xx} \), then \( m(x,t) \) exists for all \( t > 0 \); moreover, \( m(x,t) + \omega > 0 \), which justifies the equivalent form of the CH equation
\[
(\sqrt{m + \omega})_t = -(u \sqrt{m + \omega})_x,
\] (1.3)

Different from the above results [46–49], in our paper, we remove the restrictions on the Schwartz initial data by allowing presence of discrete spectrum and also allowing the weighted Sobolev initial data \( u_0(x) \in H^{k,2}(\mathbb{R}) \), where
\[
H^{k,s}(\mathbb{R}) = \left\{ f(x) \in L^2(\mathbb{R}) \mid (1 + |x|^s)\partial^j f(x) \in L^2(\mathbb{R}), \text{ for } j = 1, \ldots, k \right\}.
\]

We present different leading order asymptotic approximation for the CH equation (1.1) in different space-time solitonic regions (see Figure 1)
\[
u(x,t) = u^r(x,t; \tilde{D}) + \mathcal{O}(t^{-1+2\rho}), \quad \text{for } \xi \in (-\infty, -1/4) \cup (2, +\infty), \quad (1.4)
\]
and
\[
u(x,t) = u^r(x,t; \tilde{D}) + f_{11}t^{1/2} + \mathcal{O}(t^{-3/4}), \quad \text{for } \xi \in (-1/4, 2). \quad (1.5)
\]

Our main technical tool is the representation of the Cauchy problem (1.1)-(1.2) with an associated matrix RH problem and the consequent asymptotic analysis of this RH problem via the \( \partial \) generalization of Deift-Zhou steepest descent method. This method was a \( \partial \) generalization to the Deift-Zhou steepest descent method proposed by McLaughlin and Miller to analyze asymptotic of orthogonal polynomials with non-analytical weights [51, 52]. In recent years this method has been successfully used to investigate long-time asymptotics, soliton resolution and asymptotic stability of N-soliton solutions to integrable systems in a weighted Sobolev space [53–59].

Our paper is arranged as follows. In section 2, we recall some main results on the construction process of RH problem [46, 47], which will be used to analyze long-time asymptotics of the CH equation in our paper. In section 3, we present long-time asymptotics for the CH equation in the regions in which the jump contours admit no phase points. In this case, the main contribution to the asymptotic expansion comes from discrete spectrum and a \( \partial \)-equation. In section 4, we present long-time asymptotics for the CH equation in the regions in which there are two and four phase points.
\[ \xi = -1/4 \]

\[ u^r + f_{11} t^{-1/2} + O(t^{-3/4}) \]

\[ u^r + O(t^{-1+2\rho}) \]

\[ \xi = 2 \]

\[ \eta = 2 \]

\[ 0 \]

\[ y \]

Figure 1: Asymptotic approximations of the CH equation in different space-time solitonic regions, where without stationary phase points in yellow region; four and eight stationary phase points in green and blue regions respectively.

on the jump contours. Our results also confirm the soliton resolution conjecture and asymptotically stability of the N-soliton solutions for the CH equation.

2 Spectral analysis and the RH problem

2.1 Spectral analysis on the Lax pair

Without loss of generality, we fix \( \omega = 1 \) in the CH equation (1.1) since the result for the case \( \omega = 1 \) works for any \( \omega > 0 \) if we replace \( m \) by \( m/\omega \) and \( \lambda \) by \( \lambda \omega \). The CH equation (1.1) is completely integrable and admits the Lax pair as follow:

\[ \Psi_x = X \Psi, \quad \Psi_t = T \Psi, \quad \tag{2.1} \]

where

\[ X = \begin{pmatrix} 0 & 1 \\ \frac{1}{4} + \lambda(m+1) & 0 \end{pmatrix}, \]

\[ T = \frac{1}{2} u_x \sigma_3 + \begin{pmatrix} 0 & \frac{1}{2} u_x \sigma_3 \\ \frac{1}{2} \sigma_3 - u & 0 \end{pmatrix}, \]

The \( X \) and \( T \) in above Lax pair (2.1) are traceless matrixes, which implies that \( \det \Psi(x,t) \) is independent of \( x \) and \( t \) according to the Abel theorem.
Usually we use the $x$-part of Lax pair to analyze the initial value problem, while the $t$-part is used to determine the time evolution of the scattering data by inverse scattering transform method. Here, different from NLS and derivative NLS equations, the Lax pair (2.1) for the CH equation has singularities at $\lambda = 0, \lambda = \infty$, so the asymptotic behavior of their eigenfunctions should be controlled. The asymptotic behaviors of Lax pair (2.1) can not be directly obtained as $\lambda \to \infty$. This difficulty is solved by introducing an appropriate transformation due to Boutet de Monvel and Shepelsky. Following this idea, we need to use different transformations to analyze these singularities $\lambda = 0$ and $\lambda = \infty$ respectively, and give a new scale to construct RH problem.

**Case I: $\lambda = \infty, \ m + 1 > 0$**

Let $u(x,t)$ be a solution of (1.1) with $\omega = 1$ such that $u(x,t) \to 0$ as $|x| \to \infty$ for all $t$. Introducing

$$k^2 := -\lambda - \frac{1}{4}, \ \ Y = \begin{pmatrix} 1 & 1 \\ -ik & ik \end{pmatrix}, \ \ \tilde{\Phi} := Y^{-1} \begin{pmatrix} (m + 1)^{\frac{1}{4}} & 0 \\ 0 & (m + 1)^{-\frac{1}{4}} \end{pmatrix} \Psi$$

we change the Lax pair (2.1) into

$$\tilde{\Phi}_x = -ik\sqrt{m + 1}\sigma_3\tilde{\Phi} + U\tilde{\Phi}, \quad (2.2)$$
$$\tilde{\Phi}_t = -ik\left(\frac{1}{2\lambda} - u\sqrt{m + 1}\right)\sigma_3\tilde{\Phi} + V\tilde{\Phi}, \quad (2.3)$$

where

$$U(x,t,k) = \frac{1}{4m + 1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{8ik} \frac{m}{\sqrt{m + 1}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

$$V(x,t,k) = -uU + \frac{ik}{4\lambda} \sqrt{m + 1} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \left(\frac{u}{4ik} + \frac{ik}{4\lambda}\right) \frac{1}{\sqrt{m + 1}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{ik}{2\lambda} \sigma_3$$

Eqs. (2.2), (2.3) suggest us to define a scalar function $p(x,t,k)$ such that $p_x = \sqrt{m + 1}$ and $p_t = \frac{1}{2\lambda} - u\sqrt{m + 1}$; due to (1.3), this is possible, so we define $p$ by

$$p(x,t,k) = x - \int_x^\infty \left(\sqrt{m(\xi,t) + 1} - 1\right)d\xi + \frac{t}{2\lambda(k)}, \quad (2.4)$$

By making a transformation

$$\Phi(x,t,k) := \tilde{\Phi}e^{ikp(x,t,k)\sigma_3}, \quad (2.5)$$
(2.2)-(2.3) becomes

\[
\begin{align*}
\Phi_x &= -ikp_x [\sigma_3, \Phi] + U\Phi, \\
\Phi_t &= -ikp_t [\sigma_3, \Phi] + V\Phi,
\end{align*}
\]

In addition, the transformation above also implies that

\[\Phi(x, t, k) \sim I, \quad x \to \pm \infty.\]

The Lax pair (2.6)-(2.7) can be written into a total differential form

\[
d \left( e^{ikp\sigma_3} \Phi \right) = e^{ikp\sigma_3} [(Udx + Vdt) \Phi],
\]

which leads to two Volterra type integrals

\[
\Phi_\pm (x, t, k) = I + \int_{\pm \infty}^x e^{-ik(p(x) - p(y))\sigma_3} (U\Phi_\pm)(y, t, k) dy.
\]

Denote

\[
\Phi_\pm (x, t, k) = \begin{pmatrix} \Phi_\pm^{(1)} & \Phi_\pm^{(2)} \end{pmatrix},
\]

where \( \Phi_\pm^{(1)} \) and \( \Phi_\pm^{(2)} \) are the first and second columns of \( \Phi_\pm (x, t, k) \), respectively. Then from the Volterra integral equation (2.9), we can show that \( \Phi_\pm^{(1)} \) and \( \Phi_\pm^{(2)} \) are analytical in \( \mathbb{C}^+ \); \( \Phi_+^{(1)} \) and \( \Phi_+^{(2)} \) are analytical in \( \mathbb{C}^- \) (see yellow and white domains in Figure 2). Moreover, as \( k \to \pm \infty \), \( (\Phi_\pm^{(1)}, \Phi_\pm^{(2)}) \to I \) and \( (\Phi_+^{(1)}, \Phi_+^{(2)}) \to I \).

Next, we show the reduction conditions of the Jost functions:

**Proposition 1.** The Jost functions \( \Phi_\pm (k) \) admit two kinds of reduction conditions

\[
\Phi(x, k) = \Phi(x, -k).
\]

and

\[
\Phi^{(1)}(x, -k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Phi^{(2)}(x, k) = \Phi^{(1)}(x, k),
\]

Since \( \Phi_\pm \) are two fundamental matrix solutions of the Lax pair (2.2), there exists a linear relation between \( \Phi_+ \) and \( \Phi_- \), namely

\[
\tilde{\Phi}_+(x, t, k) = \tilde{\Phi}_-(x, t, k) S(k),
\]

where

\[
S(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]
Combing with the transformation applied to \( \tilde{\Phi} \), the equation (2.12) is changed to

\[
\Phi_+ (x, t, k) = \Phi_- (x, t, k) e^{-ikp(x,t,k)\hat{\sigma}_3} S(k), \quad k \in \mathbb{R}, k \neq 0
\] (2.13)

where \( S(k) \) is called scattering matrix

\[
S(k) = \begin{pmatrix}
\frac{a(k)}{b(k)} & b(k) \\
\frac{b(k)}{a(k)} & a(k)
\end{pmatrix} = \begin{pmatrix}
a(-k) & b(k) \\
b(-k) & a(k)
\end{pmatrix}
\] (2.14)

We define the reflection coefficients by

\[
r(z) = \frac{b(z)}{a(z)},
\] (2.15)

which admits symmetry reductions

\[
r(k) = r(-k).
\]

**Proposition 2.** If the initial data \( u_0(x) \in H^{4,2}(\mathbb{R}) \), then corresponding reflection coefficient \( r(z) \in H^{1,1}(\mathbb{R}) \).

From (2.13), \( a(k) \) and \( b(k) \) can be expressed by \( \Phi_\pm (k) \) as

\[
a(k) = \Phi_\pm^{11} \phi_\mp^{22} - \Phi_\pm^{12} \phi_\mp^{21}, \quad b(k) = (\Phi_\pm^{12} \phi_\mp^{22} - \Phi_\pm^{11} \phi_\mp^{21}) e^{2ikp}.
\] (2.16)

So \( a(k) \) is analytic on \( \mathbb{C}^+ \). In addition, \( \Phi_\pm (k) \) admit the asymptotics

\[
\Phi_\pm (k) = I + \frac{D_1}{k} + o(k^{-2}), \quad k \to \infty,
\] (2.17)

From (2.16) and (2.17), we obtain the asymptotic of \( a(k) \)

\[
a(k) = 1 + O(k^{-1}), \quad k \to \infty.
\] (2.18)

It has been proved in [...] that the zeros of \( a(k) \) in the upper half-plane lie on the imaginary axis and all zeros of \( a(k) \) in the upper half-plane are simple zeros. Suppose that \( a(k) \) has \( N \) simple zeros \( k_1, ..., k_N \) on \( \{ k \mid k \in \mathbb{C}^+ \} \). The symmetries (2.14) imply that

\[
a(k_n) = 0 \iff a(-k_n) = 0, \quad n = 1, ..., N,
\]
Therefore, the discrete spectrum is

\[ Z = \{ k_n, -k_n \}_{n=1}^{N}, \]

with \( k_n \in \mathbb{C}^+ \) and \( -k_n \in \mathbb{C}^- \). And the distribution of \( Z \) on the \( k \)-plane is shown in Figure 2.

![Diagram of analytical domains and distribution of the discrete spectrum](image)

**Figure 2:** Analytical domains and distribution of the discrete spectrum \( Z \). \( \Phi^{(1)}, \Phi^{(2)}, a(z) \) are analytical in \( \mathbb{C}^+ \) (yellow domain); \( \Phi^{(1)}_+, \Phi^{(2)}_- \) are analytical in \( \mathbb{C}^- \) (white domain);

**Case II:** \( \lambda = 0, \quad m + 1 > 0 \)

Define a new transformation

\[
\Phi^0(k) = Y^{-1} \Psi e^{i(kx + \frac{k}{2\lambda} t)\sigma_3},
\]

then

\[
\Phi^0_\pm(k) \sim I, \quad x \to \pm \infty,
\]

and the Lax pair (2.1) change to

\[
(\Phi^0)_x = -ik[\sigma_3, \Phi^0] + U^0 \Phi^0, \quad (2.21)
\]

\[
(\Phi^0)_t = \frac{-ik}{2\lambda} [\sigma_3, \Phi^0] + V^0 \Phi^0, \quad (2.22)
\]
where
\[
\begin{align*}
U^0 &= -\frac{\lambda m}{2ik}\sigma_3 + \frac{\lambda m}{2ik}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
V^0 &= u(ik\sigma_3 - \frac{\lambda m}{2ik}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}) + \frac{m}{4ik} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \\
&\quad + \frac{1}{2}u_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{u_{xx}}{4ik} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.
\end{align*}
\]

Consider asymptotic expansion as \( k \to \frac{i}{2} \),
\[
\Phi^0 = I + \Phi^0_1(k - \frac{i}{2}) + O\left((k - \frac{i}{2})^2\right),
\]
where
\[
\Phi^0_1 = \begin{pmatrix} 0 & i(u - u_x) \\ i(u + u_x) & 0 \end{pmatrix}.
\]
Since \( \Phi(k) \) and \( \Phi^0(k) \) are derived from the same Lax pair (2.1), they are linearly dependent. The relations (2.5) and (2.20) lead to
\[
\Phi_{\pm}(k) := F(x,t)\Phi^0_{\pm}(k)e^{ik\int_{\pm\infty}^{x}\sqrt{m+1-1}\xi\sigma_3},
\]
where
\[
F(x,t) = \frac{1}{2} \left( \begin{array}{cc}
(m+1)^{\frac{1}{4}} + (m+1)^{-\frac{1}{4}} & (m+1)^{\frac{1}{4}} - (m+1)^{-\frac{1}{4}} \\
(m+1)^{\frac{1}{4}} - (m+1)^{-\frac{1}{4}} & (m+1)^{\frac{1}{4}} + (m+1)^{-\frac{1}{4}}
\end{array} \right).
\]

2.2 A RH formalism with Cauchy value problem

The residue conditions are
\[
\begin{align*}
\text{Res}_{k=k_n} \left[ \frac{\Phi^0_{\pm}(k)}{a(k)} \right] &= c_n e^{-2ik_n p(k_n)} \Phi^1_{\pm}(k_n), \\
\text{Res}_{k=-k_n} \left[ \frac{\Phi^1_{\pm}(k)}{a(-k)} \right] &= c_n e^{-2ik_n p(-k_n)} \Phi^0_{\pm}(-k_n),
\end{align*}
\]
where \( c_n = b(k_n)/a'(k_n), \) \( n = 1, \cdots, N \) norming constants, and the collection \( \sigma_d = \{k_n, -k_n, c_n\}_{n=1}^N \) is called the scattering data. The solution of CH equation (1.1) is difficult to reconstruct, since \( p(x,t,k) \) is still unknown. Boutet de Monvel and
Shepelsky proposed an idea to change the spatial variable [46, 47]. Following this idea, we introduce a new scale

\[ y(x, t) = x - \int_x^{+\infty} (\sqrt{m + 1} - 1) \, d\xi. \]  

(2.28)

The price to pay for this is that the solution of the initial problem can be given only implicitly, or perimetrically: it will be given in terms of functions in the new scale, whereas the original scale will also be given in terms of functions in the new scale.

By the definition of the new scale \( y(x, t) \), we define

\[
M(k) = M(k; y, t) \triangleq \begin{cases} 
\left( \Phi^{(1)}_{-}(k; y, t), \frac{\Phi^{(2)}_{+}(k; y, t)}{a(k)} \right), & \text{as } k \in \mathbb{C}^+, \\
\left( \frac{\Phi^{(1)}_{-}(k; y, t)}{a(-k)}, \Phi^{(2)}(-k; y, t) \right), & \text{as } k \in \mathbb{C}^-,
\end{cases}
\]  

(2.29)

Denote the phase function

\[
\theta(k) = k \left[ \frac{y}{t} + \frac{1}{2\lambda(k)} \right],
\]  

(2.30)

then we can get the RH problem for the new variable \((y, t)\).

**RHP 1.** Find a matrix-valued function \( M(k) \triangleq M(k; y, t) \) which satisfies:

- **Analyticity:** \( M(k) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \) and has single poles;
- **Symmetry:** \( M(k) = M(-k) \);
- **Jump condition:** \( M(k) \) has continuous boundary values \( M_{\pm}(k) \) on \( \mathbb{R} \) and

\[
M_{\pm}(k) = M_{\mp}(k)V(k), \quad k \in \mathbb{R},
\]  

(2.31)

where

\[
V(k) = \begin{pmatrix} 1 & r(k)e^{-2i\theta} \\
r(k)e^{2i\theta} & 1 - |r(k)|^2 \end{pmatrix};
\]  

(2.32)

- **Asymptotic behaviors:**

\[
M(k) = I + \mathcal{O}(k^{-1}), \quad k \to \infty,
\]  

(2.33)

\[
M(k) = F \left[ I + \Phi^0(k - \frac{i}{2}) \right] e^{\sigma_3} + \mathcal{O} \left( (k - \frac{i}{2})^2 \right);
\]  

(2.34)
Residue conditions: \( M(k) \) has simple poles at each point in \( \mathbb{Z} \) with:

\[
\text{Res}_{k=k_n} M(k) = \lim_{k \to k_n} M(k) \begin{pmatrix} 0 & c_n e^{-2it\theta(k_n)} \\ 0 & 0 \end{pmatrix},
\]

(2.35)

\[
\text{Res}_{k=-k_n} M(k) = \lim_{k \to -k_n} M(k) \begin{pmatrix} 0 & c_n e^{2it\theta(-k_n)} \\ 0 & 0 \end{pmatrix}.
\]

(2.36)

To reconstruct the potential \( u(x,t) \), we can define two functions:

\[
\mu_1(y,t,k) := \left( \Phi^1_{11}(y,t,k) + \Phi^{21}_1(y,t,k) \right) \\
\mu_2(y,t,k) := \Phi^1_{22}(y,t,k) + \Phi^{22}_2(y,t,k)
\]

Therefore, we get the following formula

\[
u(y,t) = \frac{1}{2i} \lim_{k \to \frac{1}{2}} \left( \frac{\mu_1(y,t;k)\mu_2(y,t;k)}{\mu_1(y,t;\frac{1}{2})\mu_2(y,t;\frac{1}{2})} - 1 \right) \frac{1}{k - \frac{1}{2}}.
\]

(2.37)

\[
x(y,t) = y + \ln \left( \frac{\mu_1(y,t;\frac{1}{2})}{\mu_2(y,t;\frac{1}{2})} \right)
\]

(2.38)

Combining (2.29) and (2.37), the solution \( u(x,t) \) of the initial value problem of the CH equation can be represented by the solution of RHP2:

\[
u(y,t) = \frac{1}{2i} \lim_{k \to \frac{1}{2}} \left( \frac{(M_{11}(k) + M_{21}(k))(M_{12}(k) + M_{22}(k))}{(M_{11}(\frac{1}{2}) + M_{21}(\frac{1}{2}))(M_{12}(\frac{1}{2}) + M_{22}(\frac{1}{2}))} - 1 \right) \frac{1}{k - \frac{1}{2}}.
\]

(2.39)

2.3 Deformation of the RH problem

We note that the jump matrix and residue conditions of RHP 1 contain the exponential function \( e^{\pm 2it\theta} \), which is an oscillatory term in long-time asymptotics. Therefore we need to control the real part of \( 2it\theta \) and decompose the jump matrix by decay region.

\[
\text{Re}(2it\theta) = -2t\text{Im}\theta
\]

\[
= -2t\text{Im}k \left[ \xi - \frac{2(\text{Re}^2k - \text{Im}^2k) + (\frac{1}{2} - 4\text{Re}^2k)}{2(\text{Re}^2k - \text{Im}^2k) + (\frac{1}{2})^2 + (4\text{Re}k\text{Im}k)^2} \right],
\]

(2.40)

where \( \xi = \frac{y}{t} \). The signature of \( \text{Im}\theta \) are shown in Figure 3, which suggest to divide half-plane \(-\infty < y < \infty, \ t > 0\) in four space-time regions:
Figure 3: The classification of sign \( \text{Im}\theta \). In the green regions, \( \text{Im}\theta > 0 \), which implies that \( |e^{2it\theta}| \to 0 \) as \( t \to \infty \). While in the white regions, \( \text{Im}\theta < 0 \), which implies \( |e^{-2it\theta}| \to 0 \) as \( t \to \infty \). The green curves \( \text{Im}\theta = 0 \) are critical lines between decay and growth regions.
Figure 4: The half-plane $-\infty < y < \infty$, $t > 0$ is divided into four regions: The phase function $\theta(k)$ has no phase point on the real axis for the yellow regions; the phase function $\theta(k)$ has two phase points on the real axis for green region and four phase points on the real axis for the blue region.

Case I: $\xi < -1/4$ in Figure 3 (a), Case II: $-1/4 < \xi < 0$ in Figures 3 (b),
Case III: $0 \leq \xi < 2$ in Figure 3 (c), Case IV: $\xi > 2$ in Figure 3 (d).

For the cases I and IV, there is not any stationary phase point on the real axis, while for the cases II and III, there exist four and two stationary phase points on the real axis, denoted as $\xi_1 > ... > \xi_4$ and $\xi_1 > \xi_2$ respectively (see Figure 4 and Figure 5).

Note that the jump matrix in RHP1 also has an item of $e^{2it\theta}$, so we have to introduce a classical decomposition to make it decay in the target area.

\[
V(z) = \begin{pmatrix} 1 & 0 \\ -r e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & re^{-2it\theta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 - |r|^2 & 1 \end{pmatrix} (1 - |r|^2)^{-\sigma_3} \begin{pmatrix} 1 & 0 \\ -r e^{2it\theta} & 1 \end{pmatrix}.
\]

(2.41) 

(2.42)

We will utilize these factorizations to deform the jump contours so that the oscillating factor $e^{\pm 2it\theta}$ are decaying in corresponding region respectively.
3 Long-time asymptotic without phase point

3.1 Deformation of the RH problem

3.1.1 Conjugation

One will observe that the long time asymptotic of RHP1 is influenced by the growth and decay of the oscillatory term. In this section, our aim is to introduce a transform of $M(k;x,t) \to M^{(1)}(k;x,t)$ so that $M^{(1)}(k;x,t)$ is well behaved as $|t| \to \infty$ along the characteristic line.

For $\xi < -\frac{1}{4}$ and $\xi > 2$, we choose the decomposition (2.41),(2.42) respectively. Through simple calculation, one can get the trace formula.

$$a(k) = \prod_{n=1}^{N} \frac{k - k_n}{k + k_n} \exp \left[ -i \int_{-\infty}^{+\infty} \frac{v(s)}{s - k} ds \right], \quad (3.1)$$
\[ a(-k) = \prod_{n=1}^{N} \frac{k + k_n}{k - k_n} \exp \left[ i \int_{-\infty}^{+\infty} \frac{v(s)}{s - k} \, ds \right]. \quad (3.2) \]

From (2.16), (2.23) and (2.24):

\[ a(i^2) = e^{-\frac{1}{2} \int_{-\infty}^{+\infty} \sqrt{m+1} - 1} d\xi, \quad (3.3) \]

Introduce the notation that will be used later. Define

\[ \Omega_1 = \{ k | \text{Re} k = 0, \text{Im} k > 0 \}, \]
\[ \Omega^-_1 = \{ k | k \in \Omega_1, \text{Im} \theta(k) < -\delta \}, \Omega^+_1 = \{ k | k \in \Omega_1, \text{Im} \theta(k) > \delta \}, \]

and

\[ \Delta_1 = \Delta^+_1 \cup \Delta^-_1, \quad \Delta_1 = \{ n \in \{ 1, \cdots, N \} || \text{Im} \theta(k_n) | < \delta \} \]
\[ \Delta^+_1 = \{ n \in \{ 1, \cdots, N \} | k_n \in \Omega^+_1 \}, \Delta^-_1 = \{ n \in \{ 1, \cdots, N \} | k_n \in \Omega^-_1 \}. \]

The first step in our analysis is to introduce a transformation which renormalizes the Riemann-Hilbert problem such that it is well conditioned for \( t \to \infty \) with \( \xi \) fixed. In order to arrive at a problem which is well normalized, Let’s introduce some notation. Denote

\[ L(\xi) = \begin{cases} \emptyset, & \text{as } \xi < -1/4, \\ \mathbb{R}, & \text{as } \xi > 2. \end{cases} \quad (3.4) \]

Here, \( \emptyset \) means the integral interval is zero, i.e. for any function \( f(x) \), we make a definition of engagement

\[ \int_{\emptyset} f(x) \, dx = 0. \]

Define the function

\[ T(k) = \prod_{n \in \Delta^+_1} \frac{k + k_n}{k - k_n} \delta(k, \xi). \quad (3.5) \]

where

\[ \delta(k, \xi) = \exp\{-i \int_{L(\xi)} \frac{v_1(s)}{s - k} \, ds\}; \quad v_1(k) = \frac{1}{2\pi} \log(1 - |r|^2). \]
Proposition 3. The function defined by (3.5) has following properties:

(a) $T$ is meromorphic in $\mathbb{C}$, and for each $n \in \Delta^1_+$, $T(k)$ has simple poles at $k_n$ and simple zeros at $-k_n$; $T(k)$ is analytic and nonzero elsewhere.

(b) $\overline{T(k)} = \frac{1}{T(k)}$;

(c) $\lim_{k \to \infty} T(k) = 1$; $T(k)$ is continuous at $k = 0$, and $T(0) = 1$;

(d) As $k$ approaching the real axis from above and below, $T$ has boundary values $T_\pm$, which satisfy:

\[ T_+ (k) = (1 - |r(k)|^2) T_- (k), \quad \xi > 2; \quad (3.6) \]

\[ T_+ (k) = T_- (k), \quad \xi < -\frac{1}{4}. \quad (3.7) \]

(e) For $k = \frac{i}{2}$

\[ T(k) = \prod_{n \in \Delta^1_+} \left( \frac{\frac{i}{2} + k_n}{\frac{i}{2} - k_n} \right)^{\delta(i/2, \xi)}. \]

(f) As $k \to \frac{i}{2}$, $T(k)$ has asymptotic expansion as

\[ T(k) = T\left(\frac{i}{2}\right) J_0 + \left[ T\left(\frac{i}{2}\right) J_0 J_1 + \sum_{j,s=1,j \neq s}^{\mathcal{N}(\Delta^1_+)} \left( \frac{-2k_{n_j} (k_{n_j} + \frac{i}{2})}{(\frac{i}{2} - k_{n_j})(\frac{i}{2} - k_{n_s})^2} \right) J_0 \right] \left( k - \frac{i}{2} \right) + \mathcal{O}(\left( k - \frac{i}{2} \right)^2), \quad (3.8) \]

with

\[ J_0 = \exp\left\{ -\frac{i}{2 \pi} \int_{L(\xi)} \frac{\ln(1 - |r(s)|^2)}{s - \frac{i}{2}} ds \right\}; \quad (3.9) \]

\[ J_1 = -\frac{i}{2 \pi} \int_{L(\xi)} \frac{\ln(1 - |r(s)|^2)}{(s - \frac{i}{2})^2} ds; \quad (3.10) \]

Proof. Properties (a)-(e) can be obtain by simple calculation from the definition of $T(k)$ in (3.5). To obtain property (f), it is sufficient to do a Laurent expansion of $T(k)$ at $\frac{i}{2}$.

We define a new unknown function $M^{(1)}(k)$ using the function we defined above

\[ M^{(1)}(k) = M(k) T^{\sigma_3}(k), \]

then $M^{(1)}$ solves the following RH problem:
RHP 2. Find a matrix-valued function $M^{(1)}(k)$ which satisfies:

- **Analyticity:** $M^{(1)}(k)$ is analytic in $\mathbb{C} \setminus (\mathbb{R} \cup \mathbb{Z})$ and has single poles;

- **Jump condition:** $M^{(1)}(k)$ has continuous boundary values $M^{(1)}_{\pm}(k)$ on $\mathbb{R}$ and

$$M^{(1)}_{+}(k) = M^{(1)}_{-}(k)V^{(1)}(k, \xi), \quad k \in \mathbb{R},$$

where

$$V^{(1)}(k, \xi) = \begin{cases} 
\begin{pmatrix} 1 & T^{-2}r e^{-2it\theta} \\
-T^2 T e^{2it\theta} & 1 
\end{pmatrix}, & \text{as } \xi < -\frac{1}{4}, k \in \mathbb{R}; \\
\begin{pmatrix} 1 & r(k)e^{-2it\theta}T^{-2}(k) \\
0 & \frac{1}{1-|r(k)|^2} 
\end{pmatrix}, & \text{as } \xi > 2, k \in \mathbb{R}; 
\end{cases}$$

- **Asymptotic behaviors:**

$$M^{(1)}(k) = I + \mathcal{O}(k^{-1}), \quad k \to \infty,$$  

$$M^{(1)}(k) = Fe^{c_3T^3(i)} + \left[F e^{c_3I_3} + F \Phi_1 e^{c_3T^3(i)}(k - \frac{i}{2})\right] + \mathcal{O}\left((k - \frac{i}{2})^2\right), \quad k \to \frac{i}{2}.$$  

- **Residue conditions:** $M^{(1)}(k)$ has simple poles at each point in $\mathbb{Z}$ with:

$$\text{Res}_{k=k_n} M^{(1)}(k) = \begin{cases} 
\lim_{k \to k_n} M^{(1)}(k) \begin{pmatrix} c_n^{-1} e^{-2it\theta(k_n)}(\frac{1}{T}(k_n))^{-2} & 0 \\
0 & 0 
\end{pmatrix}, & \text{as } n \in \Delta^+_1; \\
\lim_{k \to k_n} M^{(1)}(k) \begin{pmatrix} 0 & c_n T^{-2} e^{-2it\theta(k_n)} \\
0 & 0 
\end{pmatrix}, & \text{as } n \in \Delta^-_1 \cup \Lambda; 
\end{cases}$$

$$\text{Res}_{k=-k_n} M^{(1)}(k) = \begin{cases} 
\lim_{k \to -k_n} M^{(1)}(k) \begin{pmatrix} 0 & c_n^{-1} e^{-2it\theta(-k_n)}[T'(-k_n)]^{-2} \\
0 & 0 
\end{pmatrix}, & \text{as } n \in \Delta^+_1; \\
\lim_{k \to -k_n} M^{(1)}(k) \begin{pmatrix} 0 & 0 \\
0 & c_n T^2 e^{2it\theta(-k_n)} 
\end{pmatrix}, & \text{as } n \in \Delta^-_1 \cup \Lambda; 
\end{cases}$$

### 3.1.2 A mixed $\bar{\partial}$-RH problem

The next step in our analysis is to introduce factorizations of the jump matrix whose factors admit continuous-but not necessarily analytic-extensions off the real axis.
The price we pay for this non-analytic transformation is that the new unknown has nonzero $\bar{\partial}$ derivatives inside the regions in which the extensions are introduced and satisfies a hybrid $\bar{\partial}$ Riemann-Hilbert problem.

Define the contour as follow:

$$\Upsilon_{2n+1} = \{ k \in \mathbb{C} | n\pi \leq \arg k \leq n\pi + \varphi \},$$

$$\Upsilon_{2n+2} = \{ k \in \mathbb{C} | (n+1)\pi - \varphi \leq \arg k \leq (n+1)\pi \},$$

where $n = 0, 1$. And

$$\Sigma_k = e^{(k-1)i\pi/2+i\varphi}R_+, \quad k = 1, 3,$$

$$\Sigma_k = e^{ki\pi/2-i\varphi}R_+, \quad k = 2, 4,$$

which is the boundary of $\Upsilon_k$ respectively. Take $\varphi$ be a sufficiently small fixed angle achieving following conditions:

(i): each $\Upsilon_i$ doesn’t intersect $\{ k \in \mathbb{C} | \text{Im} \theta(k) = 0 \}$,

(ii): $\sqrt{2(\cos 2\varphi + 1)} > -1 + \sqrt{1 - \frac{7}{4} \xi}$ for $\xi < -\frac{1}{4};$

(iii): $1 - \sqrt{1 - \frac{2}{7} \xi} < \sqrt{2(\cos 2\varphi + 1)} < 1 + \sqrt{1 - \frac{2}{7} \xi}$ for $\xi > 2.$

A more intuitive expression of the above process can be seen in the figure 6.

Figure 6: Without stationary phase point corresponds the case $\xi < -1/4.$ The blue regions $\Upsilon_1(\xi)$ and $\Upsilon_2(\xi)$ have same decay/grow properties; The same for yellow regions $\Upsilon_3(\xi)$ and $\Upsilon_4(\xi).$
Define a new unknown function

\[ R^{(2)}(k, \xi) = \begin{cases} 
\begin{pmatrix} 1 & R_j(k, \xi)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \xi < -\frac{1}{4}, k \in \mathcal{Y}_j, j = 1, 2; \\
\begin{pmatrix} 1 & 0 \\ R_j(k, \xi)e^{2it\theta} & 1 \end{pmatrix}, & \xi < -\frac{1}{4}, k \in \mathcal{Y}_j, j = 3, 4; \\
\begin{pmatrix} 1 & R_j(k, \xi)e^{2it\theta} \\ 0 & 1 \end{pmatrix}, & \xi > 2, k \in \mathcal{Y}_j, j = 1, 2; \\
\begin{pmatrix} 1 & R_j(k, \xi)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & \xi > 2, k \in \mathcal{Y}_j, j = 3, 4; \\
I, & \text{elsewhere};
\end{cases} \]  

(3.15)

where the $R_j$ are given in following Proposition.

**Proposition 4.** $R_j: \bar{\mathcal{Y}}_j \to C, j = 1, 2, 3, 4$ have boundary values as follow:

For $\xi \in (-\infty, -1/4)$,

\[ R_1(k, \xi) = \begin{cases} 
-rt^{-2} & k \in \mathbb{R}^+, k \in \Sigma_1; \\
0 & \text{elsewhere},
\end{cases} \quad R_2(k, \xi) = \begin{cases} 
0 & k \in \Sigma_2, k \in \mathbb{R}^+; \\
-rt^{-2} & \text{elsewhere}.
\end{cases} \]

\[ R_3(k, \xi) = \begin{cases} 
s^{-t} & k \in \mathbb{R}^-, k \in \Sigma_3; \\
0 & \text{elsewhere},
\end{cases} \quad R_4(k, \xi) = \begin{cases} 
0 & k \in \Sigma_4, k \in \mathbb{R}^+; \\
s^{-t} & \text{elsewhere}.
\end{cases} \]

$R_j$ have following properties: for $j = 1, 2, 3, 4$,

\[ |\bar{\partial}R_j(k)| \lesssim |r'(|k|)| + |k|^{-1/2}, \text{ for all } k \in \mathcal{Y}_j, j=1,2 \]

(3.16)

\[ |\bar{\partial}R_j(k)| \lesssim |r'(|k|)| + |k|^{-1/2}, \text{ for all } k \in \mathcal{Y}_j, j=3,4 \]

(3.17)

moreover,

\[ |\bar{\partial}R_j(k)| \lesssim |r'(|k|)| + |k|^{-1}, \text{ for all } k \in \mathcal{Y}_j, j=1,2 \]

(3.18)

\[ |\bar{\partial}R_j(k)| \lesssim |r'(|k|)| + |k|^{-1}, \text{ for all } k \in \mathcal{Y}_j, j=3,4 \]

(3.19)

\[ \bar{\partial}R_j(k) = 0, \quad \text{if } k \in \text{elsewhere}. \]
For $\xi \in (2, +\infty)$,

\[
R_1(k, \xi) = \begin{cases}
\frac{\bar{r}_e^{2i\theta}}{1 - |r(k)|^2} T_+^2 & k \in \mathbb{R}^+, \\
0 & k \in \Sigma_1,
\end{cases} \quad R_2(k, \xi) = \begin{cases}
0 & k \in \Sigma_2, \\
\frac{\bar{r}_e^{2i\theta}}{1 - |r(k)|^2} T_+^2 & k \in \mathbb{R}^-,
\end{cases}
\]

\[
R_3(k, \xi) = \begin{cases}
\frac{r_e^{-2i\theta}}{1 - |r(k)|^2} T_-^2 & k \in \mathbb{R}^-, \\
0 & k \in \Sigma_3,
\end{cases} \quad R_4(k, \xi) = \begin{cases}
0 & k \in \Sigma_4, \\
\frac{r_e^{-2i\theta}}{1 - |r(k)|^2} T_-^2 & k \in \mathbb{R}^+,
\end{cases}
\]

In this case, $R_j$ have the similar properties. Just replace $r, \bar{r}$ with $\frac{\bar{r}_e^{2i\theta}}{1 - |r(k)|^2}$, $\frac{r_e^{-2i\theta}}{1 - |r(k)|^2}$ respectively in (3.16)-(3.19).

**Proof.** The proof is similar with [58].

Make the matrix transformation:

\[
M^{(2)}(k) = M^{(1)}(k) R^{(2)}(k),
\]

then $M^{(2)}$ solves the mixed $\bar{\partial}$-RH problem:

**RHP 3.** Find a matrix valued function $M^{(2)}(k)$ with following properties:

- **Analyticity:** $M^{(2)}(k)$ is continuous in $\mathbb{C} \setminus \mathbb{Z}$;
- **Jump condition:** $M^{(2)}(k)$ has continuous boundary values $M^{(2)}_\pm(k)$ on $\Sigma^{(2)}$ and

\[
M^{(2)}_+(k) = M^{(2)}_-(k), \quad z \in \Sigma^{(2)}, \tag{3.20}
\]

where $\Sigma^{(2)} = \bigcup \Sigma_j, j = 1, 2, 3, 4$.

- **Asymptotic behaviors:**

\[
M^{(2)}(k) = I + O(k^{-1}), \quad k \to \infty, \tag{3.21}
\]

\[
M^{(2)}(k) = Fe^{c\sigma_3 T \sigma_3} (i) + [Fe^{c\sigma_3 I_0^{\sigma_3}} + F \Phi_1 e^{c\sigma_3 T \sigma_3} (i)](k - \frac{i}{2}) + O \left((k - \frac{i}{2})^2 \right), \quad k \to \frac{i}{2}. \tag{3.22}
\]

- **$\bar{\partial}$-Derivative:** For $k \in \mathbb{C}$ we have

\[
\bar{\partial} M^{(2)}(k) = M^{(2)}(k) \bar{\partial} R^{(2)}(k), \tag{3.23}
\]

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where

\[
\tilde{\mathcal{R}}^{(2)}(k, \xi) = \begin{cases} 
(0 \hspace{1cm} 0 & e^{-2it\theta} \\
0 & 0 
\end{cases}, \quad \xi < -\frac{1}{4}, k \in \Upsilon_j, j = 1, 2; \\
\begin{cases} 
0 \hspace{1cm} 0 & e^{2it\theta} \\
0 & 0 
\end{cases}, \quad \xi < -\frac{1}{4}, k \in \Upsilon_j, j = 3, 4; \\
\begin{cases} 
0 \hspace{1cm} 0 & e^{-2it\theta} \\
0 & 0 
\end{cases}, \quad \xi > \frac{2}{4}, k \in \Upsilon_j, j = 1, 2; \\
\begin{cases} 
0 \hspace{1cm} 0 & e^{2it\theta} \\
0 & 0 
\end{cases}, \quad \xi > \frac{2}{4}, k \in \Upsilon_j, j = 3, 4; \\
0, \quad \text{elsewhere;}
\end{cases}
\]

- Residue conditions: \(M^{(2)}(k)\) has the same residue conditions with the RHP2.

In order to solve the RHP3, we decompose it into a pure RH problem \(M^{(R)}(k)\) with \(\tilde{\mathcal{R}}^{(2)} \equiv 0\) and a pure \(\bar{\partial}\) problem \(M^{(3)}(k)\) with \(\tilde{\mathcal{R}}^{(2)} \neq 0\), that is,

\[
M^{(2)}(k) = M^{(3)}(k)M^{(R)}(k). \tag{3.25}
\]

Since the fact that Case I has no stationary point as well as the jump matrix in \(\Sigma^{(2)}\) equals to I, the result of \(M^{(R)}\) is only contributed by its poles and asymptotic behaviors.

The matrix function \(M^{(R)}\) satisfies RHP5.

**RHP 4.** Find a matrix valued function \(M^{(R)}(k)\) with following properties:

- Analyticity: \(M^{(R)}(k)\) is meromorphic in \(\mathbb{C}\);
- Jump condition:

\[
M^{(R)}_+(k) = M^{(R)}_-(k); 
\]

- Asymptotic behaviors:

\[
M^{(R)}(k) = I + O(k^{-1}), \quad k \to \infty; \tag{3.26}
\]

\[
M^{(R)}(k) = Fe^{c_3T_{\sigma_3}}(i) + [Fe^{c_3I_{\sigma_3}} + F\Phi_1e^{c_3T_{\sigma_3}}(i)](k - \frac{i}{2}) + O\left((k - \frac{i}{2})^2\right), \quad k \to \frac{i}{2}. \tag{3.27}
\]

- Residue conditions: \(M^{(R)}(k)\) has the same residue conditions with the RHP2.
3.2 Analysis on a pure RH problem

3.2.1 Classify the poles

In this section, our aim is to convert the poles away from \( \{ k \in \mathbb{C} | \text{Im } \theta(k) = 0 \} \) into jumps. Take the transform

\[
M_R(k) = M^{(R)}(k)G(k),
\]

where

\[
G(k) = \begin{cases} 
\frac{1}{-c_n^{-1}T(k)e^{2it\theta(k_n)}[(\frac{1}{T})'(k_n)]^{-1}} & \text{as } k \in \{ k : |k - k_n| = \rho \}, n \in \Delta_1^+; \\
\frac{1}{c_n T^{-2}e^{-2it\theta(k_n)}(k - k_n)^{-1}} & \text{as } k \in \{ k : |k - k_n| = \rho \}, n \in \Delta_1^-; \\
\frac{1}{-c_n^{-1}T^{-1}e^{-2it\theta(-k_n)}[T'(-k_n)]^{-1}} & \text{as } k \in \{ k : |k + k_n| = \rho \}, n \in \Delta_1^+; \\
\frac{1}{c_n T^2e^{2it\theta(-k_n)}} & \text{as } k \in \{ k : |k + k_n| = \rho \}, n \in \Delta_1^-; \\
I, & \text{elsewhere}.
\end{cases}
\]

then \( M_R \) satisfies the following RH problem:

**RHP 5.** Find a matrix valued function \( M_R(k) \) with following properties:

- **Analyticity:** \( M^{(R)}(k) \) is meromorphic in \( \mathbb{C} \);
- **Jump condition:** \( M_{+,R}(k) = M_{-,R}(k)V_R(k) \), \( k \in \Sigma^{(R)} \);
- **\( \bar{\partial} \)-Derivative:** \( \bar{\partial}R^{(2)}(k) = 0 \);
- **Asymptotic behaviors:**
  \[
  M^{(R)}(k) = I + \mathcal{O}(k^{-1}), \quad k \to \infty;
  \]
- **Residue conditions:**

\[
\text{Res}_{k=k_n} M_R(k) = \lim_{k \to k_n} M_R(k) \begin{pmatrix} 0 & c_n T^{-2}e^{-2it\theta(k_n)} \\ 0 & 0 \end{pmatrix}, \quad \text{as } n \in \bigwedge; \quad (3.30)
\]
\[
\text{Res}_{k=-k_n} M_R(k) = \lim_{k \to -k_n} M_R(k) = \begin{pmatrix}
0 & 0 \\
c_n T^2 e^{2it\theta(-k_n)} & 0
\end{pmatrix}, \quad \text{as } n \in \bigwedge;
\] 

(3.31)

\hspace{1cm}

Figure 7: Jump path of \( M_R \). The blue circle in the figure is the jump path.

In order to do the estimation error later, here are some simple parametric estimates for \( V_R \).

**Lemma 1.** The jump matrix \( V_R(k) \) defined by (3.29) satisfies

\[
\| V_R(k) - I \|_{L^\infty} = O(e^{-2\rho_0 t}),
\]

(3.32)

where

\[
\rho_0 = \min_{n \in \Delta_1 \setminus \Lambda_1} \{|\text{Im}\theta_n| > \delta_0\}.
\]

(3.33)

**Proof.** Take \( k \in \{k : |k - k_n| = \rho, n \in \Delta_1^+\} \) as an example.

\[
\| V_R(k) - I \|_{L^\infty} = |c_n^{-1} T(k)e^{2it\theta(k_n)}(\frac{1}{T})'(k_n)|^{-1} \\
\leq e^{-\text{Re}(2it\theta_n)} \leq e^{2t\text{Im}(\theta_n)} \leq e^{-2\rho_0 t}.
\]

The proofs for the rest of the cases are similar. \qed

**Corollary 1.** For \( 1 \leq p \leq +\infty \), the jump matrix \( V_R(k) \) satisfies

\[
\| V_R(k) - I \|_{L^p} \leq C_p e^{-2\rho_0 t},
\]

(3.34)

for some constant \( C_p \geq 0 \) depending on \( p \).
3.2.2 Soliton solutions

In this part, we prove the existence and uniqueness of solution of the above RHP 5. Meanwhile, we show the solution can be approximated by a soliton solution $M^\Lambda(k)$.

**Proposition 5.** For a given scattering data $D = \{r(k), \{k_n, c_n\}_{n \in \mathbb{N}}\}$, the solution $M_R(k)$ of the above RH problem exists uniquely and is equivalent to the RHP1 corresponding to the reflection-free $N$ soliton solution of the corrected scattering data $\tilde{D} = \{\tilde{r} = 0, \{k_n, \tilde{c}_n\}_{n \in \mathbb{N}}\}$ by a display transformation, where the correlation function $\tilde{c}_n = c_n \delta(k_n)$.

**Proof.** $M_R(k)$ contains a number of jump lines consisting of small circumferences and poles in $\Lambda$. Take a transformation

$$
\tilde{M}(k) = M_R(k)G^{-1}(k) \left( \prod_{n \in \Delta^+} \frac{k + k_n}{k - k_n} \right)^{-\sigma_3},
$$

(3.35)

Then the transformation has the following effect:

a. $\tilde{M}(k)$ asymptotic to I when $k \to \infty$;

b. $\tilde{M}(k)$ has no jump on the small circle circumferences;

c. When $n \not\in \Lambda$, the transformation converts the jump into the poles; For $n \in \Lambda$,

$$
\text{Res}_{k=k_n} \tilde{M}(k) = \text{Res}_{k=k_n} M_R(k)G^{-1}(k) \left( \prod_{n \in \Delta^+} \frac{k + k_n}{k - k_n} \right)^{-\sigma_3}
$$

$$
= \lim_{k \to k_n} \tilde{M}(k) \left( \prod_{n \in \Delta^+} \frac{k + k_n}{k - k_n} \right)^{-\sigma_3} \begin{pmatrix}
0 & c_n T^{-2} e^{-2it\theta(k_n)} \\
0 & 0
\end{pmatrix} G^{-1}(k) \left( \prod_{n \in \Delta^+} \frac{k + k_n}{k - k_n} \right)^{-\sigma_3}
$$

$$
= \lim_{k \to k_n} \tilde{M}(k) \begin{pmatrix}
0 & c_n \delta(k_n) e^{-2it\theta(k_n)} \\
0 & 0
\end{pmatrix},
$$

(3.36)

thus, $\tilde{c}_n = c_n \delta(k_n)$;

So far, we have proved that $\tilde{M}(k)$ is the solution of RHP 1 under the corrected scattering data $\tilde{D} = \{\tilde{r} = 0, \{k_n, c_n \delta(k_n)\}_{n \in \mathbb{N}}\}$. And the solution of this kind of RHP is existentially unique, so $M_R(k)$ exists uniquely.
Now we try to match $M_R(k)$ to the model RHP $M^\Lambda(k)$ and show in the next subsection that the error between the two is a small parametric RHP.

Suppose $M^\Lambda(k)$ is the solution of the RHP corresponding to $M_R(k)$ after removing the jump, that is, $M^\Lambda(k)$ satisfies the RHP below:

**RHP 6.** Find a matrix valued function $M^\Lambda(k)$ with following properties:

- **Analyticity:** $M^{(R)}(k)$ is meromorphic in $\mathbb{C}$;
- **Jump condition:**
  \[ M^\Lambda_+(k) = M^\Lambda_-(k); \]
- **$\partial$-Derivative:** $\partial R^{(2)}(k) = 0$;
- **Asymptotic behaviors:**
  \[ M^\Lambda(k) = I + O(k^{-1}), \quad k \to \infty; \]
- **Residue conditions:** $M^\Lambda(k)$ has the same residue conditions with the RHP5.

**Proposition 6.** For a given corrected scattering data $\tilde{\mathcal{D}}_\Lambda = \{ \tilde{r} = 0, \{ \tilde{c}_n, \tilde{c}_n \}, n \in \Lambda \}$, the solution of RHP7 exists uniquely and can be shown to be constructed:

**I:** if $\Lambda = \emptyset$, then
\[ M^\Lambda(k) = I; \quad (3.37) \]

**II:** if $\Lambda \neq \emptyset$ with $\Lambda = \{ k_n \}_{n=1}^\Xi$, where the symbol $\Xi$ denotes the number of poles in $\Lambda$, then
\[ M^\Lambda(k) = I + \sum_{n=1}^\Xi \left( \frac{\gamma_n}{k_n + k_n} \frac{\beta_n}{k_n - k_n} \right), \quad (3.38) \]

where $\alpha_n, \beta_n, \gamma_n, \zeta_n$ are determined by linearly dependant equations:

\[ c_n^{-1}T^2(k_n)e^{2it\theta(k_n)}\beta_n = 1 + \sum_{h=1}^\Xi \frac{\gamma_h}{k_n + k_h}, \quad (3.39) \]

\[ c_n^{-1}T^2(k_n)e^{2it\theta(k_n)}\alpha_n = \sum_{h=1}^\Xi \frac{\zeta_h}{k_n + k_h}, \quad (3.40) \]

\[ c_n^{-1}T^2(k_n)e^{2it\theta(k_n)}\gamma_n = \sum_{h=1}^\Xi \frac{-\beta_h}{k_n + k_h}, \quad (3.41) \]

\[ c_n^{-1}T^2(k_n)e^{2it\theta(k_n)}\zeta_n = 1 + \sum_{h=1}^\Xi \frac{-\alpha_h}{k_n + k_h}. \quad (3.42) \]
Proof. The uniqueness of solution follows from the Liouville’s theorem. The result of I can be simple obtain.

As for II, the residue and asymptotic conditions ensure that $M^A(k)$ can have an expansion of the form as mentioned above. And in order to obtain $\alpha_n, \beta_n, \gamma_n, \zeta_n$, we substitute (3.38) into (3.30) and obtain four linearly dependant equations set above.

For convenience, denote the asymptotic expansion of $M^A(k)$ as $k \to \frac{i}{2}$:

$$M^A(k) = M^A\left(\frac{i}{2}\right) + M_1^A(k - \frac{i}{2}) + O((k - \frac{i}{2})^2).$$

(3.43)

3.2.3 Error estimate between $M_R$ and $M^A$

We define $M^{err}$ as the error between $M_R$ and $M^A$. More specifically,

$$M_R(k) = M^{err}(k)M^A(k).$$

(3.44)

From the definition, we can obtain that $M^{err}(k)$ is a solution of the following small parametric RHP:

RHP 7. Find a matrix-valued function $M^{err}(k)$ with following identities:

- **Analyticity:** $M^{err}(k)$ is analytical in $\mathbb{C} \setminus \Sigma^{(R)}$;
- **Asymptotic behaviors:**
  $$M^{err}(k) \sim I + O(k^{-1}), \quad |k| \to \infty;$$

  (3.45)
  - **Jump condition:** $M^{err}(k)$ has continuous boundary values $M^{err}_{\pm}(k)$ on $\Sigma^{(R)}$ satisfying
    $$M^{err}_{\pm}(k) = M^{err}(k)V^{err}(k),$$
    where the jump matrix $V^{err}(k)$ is given by
    $$V^{err}(k) = M^A(k)V_R(k)M^A(k)^{-1}.$$  

(3.46)

Lemma 2. The jump matrix $V^{err}(k)$ satisfies

$$\| V^{err}(k) - I \|_{L^\infty} = O(e^{-2\rho_0 t}),$$

(3.47)
Proof. Take $k \in \{ k : |k - k_n| = \rho, n \in \Delta^+ \}$ as an example.

\[ |V_R(k) - I| = |M^A(k)(V_R - I)M^A(k)^{-1}| \leq c|V_R - I| \leq e^{-2\rho t}. \]

The proofs for the rest of the cases are similar. \( \square \)

Corollary 2. For $1 \leq p \leq +\infty$, the jump matrix $V^{err}(k)$ satisfies

\[ \| V^{err}(k) - I \|_{L^p} \leq C_p e^{-2\rho t}, \quad (3.48) \]

for some constant $C_p \geq 0$ depending on $p$.

Constructing the solution of the RHP 7 according to Beal-Coifman’s theorem. Consider the mundane decomposition of the jump matrix $V^{err}(k)$

\[ V^{err} = (b_-)^{-1}b_+, \quad b_- = I, \quad b_+ = V^{err}, \]

thus

\[ (\omega_e)_- = I - b_-, \quad (\omega_e)_+ = b_+ - I, \quad \omega_e = (\omega_e)_+ + (\omega_e)_- \]

\[ C_{\omega e} f = C_-(f(\omega_e)_+) + C_+(f(\omega_e)_-) = C_-(f(V^{err} - I)), \quad (3.49) \]

where $C_-$ is the Cauchy projection operator, defined as follows

\[ C_-(f) = \lim_{k' \to k \in \Sigma^R} \frac{1}{2\pi i} \int_{\Sigma^R} \frac{f(s)}{s - k'} ds, \quad (3.50) \]

and $\|C_-\|_{L^2}$ is bounded. Then the solution of the above RH problem can be expressed as

\[ M^{err}(k) = I + \frac{1}{2\pi i} \int_{\Sigma^R} \frac{\mu_e(s)(V^{err}(s) - I)}{s - k} ds, \quad (3.51) \]

where $\mu_e(k) \in L^2(\Sigma^R)$ satisfies

\[ (1 - C_{\omega e})\mu_e(k) = I. \quad (3.52) \]

Using (3.47) and (3.49), we can see that

\[ \|C_{\omega e}\|_{L^2(\Sigma^R)} \leq \|C_-\|_{L^2(\Sigma^R)}\|V^{err}(k) - I\|_{L^\infty(\Sigma^R)} \leq ce^{-2\rho^2 t}, \quad (3.53) \]
Therefore, the pre-solution operator $(1 - C_{\omega e})^{-1}$ exists such that $\mu_e$ and the solution $M_{\text{err}}(k)$ of the RH problem exist uniquely.

In order to reconstruct the solution $u(x, t)$, we need some estimates and asymptotic behaviors of $M_{\text{err}}(k)$.

**Proposition 7.** For $M_{\text{err}}(k)$ defined in (3.44), it stratifies

$$\left| M_{\text{err}}(k) - I \right| \leq ce^{-2\rho_0 t}$$

where

1. When $k = \frac{i}{2}$,

$$M_{\text{err}}\left(\frac{i}{2}\right) = I + \frac{1}{2\pi i} \int_{\Sigma(R)}^{\mu_e(s)(V_{\text{err}}(s) - I)}{ds},$$

2. As $k \to \frac{i}{2}$, the Laurent expansion of $M_{\text{err}}(k)$ is

$$M_{\text{err}}(k) = M_{\text{err}}\left(\frac{i}{2}\right) + M_1^{\text{err}}(k - \frac{i}{2}) + \mathcal{O}((k - \frac{i}{2})^2),$$

where

$$M_1^{\text{err}} = \frac{1}{2\pi i} \int_{\Sigma(R)}^{\mu_e(s)(V_{\text{err}}(s) - I)}{(s - \frac{i}{2})^2} ds.$$  

In addition, $M_{\text{err}}\left(\frac{i}{2}\right)$ and $M_1^{\text{err}}$ satisfy following long time asymptotic behavior condition:

$$\left| M_{\text{err}}\left(\frac{i}{2}\right) - I \right| \lesssim \mathcal{O}(e^{-2\rho_0 t}), \quad M_1^{\text{err}} \lesssim \mathcal{O}(e^{-2\rho_0 t}).$$

**Proof.** By (3.51),

$$M_{\text{err}}(k) - I = \frac{1}{2\pi i} \int_{\Sigma(R)}^{V_{\text{err}}(s) - I}{ds} + \frac{1}{2\pi i} \int_{\Sigma(R)}^{(\mu_e(s) - I)(V_{\text{err}}(s) - I)}{ds},$$

Further estimates can be given

$$\left| M_{\text{err}}(k) - I \right| \leq \|V_{\text{err}}(s) - I\|_{L^2(\Sigma(2R))} \left\| \frac{1}{s - k} \right\|_{L^2(\Sigma(R))} +$$

$$\|V_{\text{err}}(s) - I\|_{L^\infty(\Sigma(R))} \|\mu_e(s) - I\|_{L^2(\Sigma(R))} \left\| \frac{1}{s - k} \right\|_{L^2(\Sigma(R))} \leq ce^{-2\rho_0 t}.$$  

(3.56)-(3.57) can be obtained by doing Taylor expansion on $\frac{1}{s-k}$. At last, since $\frac{1}{(s-\frac{i}{2})^2}$ is bounded in $\Sigma(R)$, the proof of (3.58) can be given as follow:

$$|M_1^{\text{err}}| \lesssim \|V_{\text{err}} - I\|_{L^2}\|\mu_e(s) - I\|_{L^2} + \|V_{\text{err}} - I\|_{L^1} \lesssim \mathcal{O}(e^{-2\rho_0 t}).$$
3.3 Analysis on a pure $\bar{\partial}$ problem

In this subsection, we mainly consider the pure $\bar{\partial}$-problem which is defined by (3.25). One can get the RHP of $M^{(3)}$ by simple calculation.

**RHP 8. (Pure $\bar{\partial}$-problem)** Find $M^{(3)}(k)$ with following identities:

- **Analyticity:** $M^{(3)}(k)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C}$.
- **Asymptotic behavior:**
  \[ M^{(3)}(k) \sim I + O(k^{-1}), \quad k \to \infty; \quad (3.59) \]
- **$\bar{\partial}$-Derivative:** We have
  \[ \bar{\partial}M^{(3)}(k) = M^{(3)}(k)W^{(3)}(k), \quad k \in \mathbb{C}, \]
  where
  \[ W^{(3)}(k) = M^{(R)}(k)\bar{\partial}R^{(2)}(k)M^{(R)}(k)^{-1}. \quad (3.60) \]

The solution of the pure $\bar{\partial}$ problem $M^{(3)}(k)$ can be given by the following integral equation

\[ M^{(3)}(k) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}W^{(3)}}{s - k} dA(s), \quad (3.61) \]

where $dA(s)$ is the Lebesgue measure in the real plane. The equation (3.61) can also be expressed as an operator equation

\[ (I - J)M^{(3)}(k) = I \iff M^{(3)}(k) = I + JM^{(3)}(k), \quad (3.62) \]

where $J$ is the Cauchy operator

\[ Jf(k) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W^{(3)}(s)}{s - k} dA(s) = \frac{1}{\pi k} * f(k)W^{(3)}(k). \quad (3.63) \]

Next we prove that the above operator $J$ is small parametric when $t$ is sufficiently large.
Lemma 3. For $\xi \in (-\infty, -\frac{1}{4})$, the imaginary part of phase function (2.30) $\text{Im}\theta(k)$ has following estimation:

$$\text{Im} \theta(k) \leq |k||\sin \omega| \left( \xi + \frac{1}{1 + \cos 2\varphi + \sqrt{2(1 + \cos 2\varphi)}} \right), \quad \text{as} \ k \in \mathcal{Y}_1, \mathcal{Y}_2, \quad (3.64)$$

$$\text{Im} \theta(k) \geq -|k||\sin \omega| \left( \xi + \frac{1}{1 + \cos 2\varphi + \sqrt{2(1 + \cos 2\varphi)}} \right), \quad \text{as} \ k \in \mathcal{Y}_3, \mathcal{Y}_4. \quad (3.65)$$

For $\xi \in (2, +\infty)$, the estimation is shown as follow:

$$\text{Im} \theta(k) \geq |k||\sin \omega| \left( \xi + \frac{1}{1 + \cos 2\varphi - \sqrt{2(1 + \cos 2\varphi)}} \right), \quad \text{as} \ k \in \mathcal{Y}_1, \mathcal{Y}_2, \quad (3.66)$$

$$\text{Im} \theta(k) \leq -|k||\sin \omega| \left( \xi + \frac{1}{1 + \cos 2\varphi - \sqrt{2(1 + \cos 2\varphi)}} \right), \quad \text{as} \ k \in \mathcal{Y}_3, \mathcal{Y}_4. \quad (3.67)$$

Proof. We just take $k \in \mathcal{Y}_1$ as an example, and the other regions are similarly.

Since $\theta(k) = k\xi - \frac{k}{2k^2 + \frac{1}{4}}$, let $k = |k|e^{i\omega} = u + iv$, then the imaginary part of $\theta(k)$ can be expressed as

$$\text{Im} \theta(k) = k\sin \omega \left( \xi + \frac{2|k|^2 - \frac{1}{2}}{(2|k|^2 \cos 2\omega + \frac{1}{2})^2 + 4|k|^2 \sin^2 2\omega} \right),$$

Denote

$$h(x; a) = \frac{x}{(x + \frac{1}{2})^2 + a(x + \frac{1}{2}) + \frac{1}{4}},$$

where $x = 2|k|^2 - \frac{1}{2}$, $a = \cos 2\omega$.

Then

$$\text{Im} \theta = v(\xi + h(x; a)).$$

with $x \in (-\frac{1}{2}, +\infty)$, $a \in [\cos 2\varphi, 1]$.

By doing some simple simplification of $h(x, a)$, the image of the function $\frac{1}{h(x, a)}$ can be drawn, so the value domain of $\frac{1}{h(x, a)}$ is

$$(-\infty, 1 + a - \sqrt{2(1 + a)}) \cup (1 + a + \sqrt{2(1 + a)}, +\infty).$$

The conclusion of the lemma can be deduced from the above equation.
Corollary 3. There exist constants $c(\xi) < 0$, $\tilde{c}(\xi) > 0$ relative to $\xi$ that the imaginary part of phase function (2.30) $\text{Im} \, \theta(k)$ have following evaluation for $k = |k|e^{i\omega} = u + iv$:

When $\xi \in (-\infty, -1/4)$,

\begin{align*}
\text{Im} \, \theta(k) &\leq c(\xi)v, \quad \text{as } k \in \Upsilon_1, \Upsilon_2, \quad (3.68) \\
\text{Im} \, \theta(k) &\geq -\tilde{c}(\xi)v, \quad \text{as } k \in \Upsilon_3, \Upsilon_4. \quad (3.69)
\end{align*}

When $\xi \in (2, +\infty)$,

\begin{align*}
\text{Im} \, \theta(k) &\geq \tilde{c}(\xi)v, \quad \text{as } k \in \Upsilon_1, \Upsilon_2, \quad (3.70) \\
\text{Im} \, \theta(k) &\leq -\tilde{c}(\xi)v, \quad \text{as } k \in \Upsilon_3, \Upsilon_4. \quad (3.71)
\end{align*}

Proposition 8. Cauchy integral operator $J : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C})$. When $\xi \in (-\infty, -1/4) \cup (2, +\infty)$, for sufficiently large $t$, the operator $J$ is a small parametrization. And

\[ ||J||_{L^\infty \rightarrow L^\infty} \leq ct^{-\frac{1}{2}}. \quad (3.72) \]

Therefore $(1 - J)^{-1}$ exists, and thus the operator equation (3.62) has a solution.

Proof. Here, we just take the case $\xi \in (-\infty, -1/4)$ as an example. In order to get the esitmate (3.72), It is only necessary to prove that for $f \in L^\infty(\mathbb{C})$,

\[ ||Jf||_{L^\infty(\mathbb{C})} \leq ct^{-1/2}||f||_{L^\infty(\mathbb{C})}. \]

Here, we just prove the case $k \in \Upsilon_1$. By (3.63),

\[ |Jf(k)| \leq c||f||_{L^\infty} \int_{\Upsilon_1} \frac{|W^{(3)}(s)|}{|s - k|} dA(s), \quad (3.73) \]

where

\[ |W^{(3)}(s)| \leq |M^{(R)}|(|\bar{\partial}R^{(2)}|)(M^{(R)})^{-1} \leq |M^{(R)}||\bar{\partial}R^{(2)}||\sigma_2(M^{(R)})^T \sigma_2| \]

\[ \leq |M^{(R)}|^2|\bar{\partial}R^{(2)}| \leq c|\bar{\partial}R_1(k)||e^{-2it\theta}|, \quad (3.74) \]

thus

\[ \int_{\Upsilon_1} \frac{|W^{(3)}(s)|}{|s - k|} dA(s) \leq \int_{\Upsilon_1} \frac{|\bar{\partial}R_1|e^{2it\theta}}{|s - k|} dA(s) \quad (3.75) \]
Referring into (3.16) in proposition 4, the integral \( \int_{\mathbf{T}_1} \frac{\partial R_1(s)e^{2t\Im\theta}}{|s-k|} dA(s) \) can be divided to two parts:

\[
\int_{\mathbf{T}_1} \frac{\partial R_1(s)e^{2t\Im\theta}}{|s-k|} dA(s) \lesssim I_1 + I_2,
\]  
(3.76)

with

\[
I_1 = \int_{\mathbf{T}_1} \frac{|r'(|s|)|e^{2t\Im\theta}}{|s-k|} dA(s), \quad I_2 = \int_{\mathbf{T}_1} \frac{|s|^{-1/2}e^{2t\Im\theta}}{|s-k|} dA(s).
\]  
(3.77)

Write \( s = u + iv, \ k = k_R + ik_I, \) by (3.69),

\[
\Im \theta(s) \leq c(\xi)^2 t v \leq c' tv.
\]  
(3.78)

Therefore,

\[
I_1 \approx \int_0^{\infty} e^{ctv} dv \int_v^{\infty} \frac{|r'(|s|)|}{|s-k|} du \leq \int_0^{\infty} e^{ctv} ||r'(|s|)||_{L^2(v, \infty)} \left\| \frac{1}{s-z} \right\|_{L^2(v, \infty)} dv.
\]  
(3.79)

and

\[
\left\| \frac{1}{s-k} \right\|_{L^2(v, \infty)}^2 = \int_v^{\infty} \frac{1}{|s-k|^2} du \leq \int_{-\infty}^{\infty} \frac{1}{(u-k_R)^2 + (v-k_I)^2} du = \frac{1}{|v-k_I|} \int_{-\infty}^{\infty} \frac{1}{1+y^2} dy = \frac{\pi}{|v-k_I|},
\]  
(3.80)

where \( y = \frac{u-k_R}{v-k_I}. \)

\[
||r'(|s|)||_{L^2(v, \infty)}^2 = \int_v^{\infty} |r'(|s|)|^2 du \leq \frac{1}{\sqrt{2v}} \sup_{\tau} |r'(|s|)|^2 \int_0^{\infty} \frac{\sqrt{u^2 + v^2}}{u} d\tau \leq \sqrt{2} \int_{\sqrt{2v}}^{\infty} |r'(|s|)|^2 d\tau \leq ||r'(s)||_{L^2(\mathbb{R})}^2.
\]  
(3.81)

Using (3.80)-(3.81), we can directly calculate

\[
I_1 \leq c ||r'||_{L^2(\mathbb{R})} \int_0^{\infty} e^{ctv} |v-k_I|^{-1/2} dv = c ||r'||_{L^2(\mathbb{R})} \left[ \int_0^{k_I} \frac{e^{ctv}}{\sqrt{k_I-v}} dv + \int_{k_I}^{\infty} \frac{e^{ctv}}{\sqrt{v-k_I}} dv \right] \lesssim t^{-\frac{1}{2}}.
\]  
(3.82)
To estimate $I_2$, consider the following $L^p$-estimate ($p > 2$).

$$
\|s\|_{L^p(v, \infty)}^{-1/2} = \left( \int_v^\infty \frac{1}{|u|^{p/2}} \, du \right)^{1/p} = \left( \int_v^\infty \frac{1}{(u^2 + v^2)^{p/4}} \, du \right)^{1/p} \leq cv^{1/p-1/2}.
$$

Similar to the $L^p$ estimate above, it can be shown that

$$
\|s-k\|_{L^q(v, \infty)}^{-1} \leq c|v-k_i|^{1/q-1}, \quad 1/p + 1/q = 1.
$$

Using the two estimates above,

$$
I_2 \leq \int_0^\infty e^{ct} dv \int_v^\infty \frac{s^{-1/2}}{|s-k|} \, du
$$

$$
\leq \int_0^\infty e^{ct} \|s\|_{L^p(v, \infty)}^{-1/2} \|s-k\|_{L^q(v, \infty)}^{-1} \, dv.
$$

$$
\leq c \left[ \int_0^{k_f} e^{ct} v^{1/p-1/2} |v-k_i|^{1/q-1} \, dv + \int_{k_f}^\infty e^{ct} v^{1/p-1/2} |v-k_i|^{1/q-1} \, dv \right].
$$

Calculating the two integrals by permutation yields,

$$
I_2 \leq ct^{-1/2}.
$$

By (3.73)-(3.76), the result of property 8 is proved.

Proposition 9. There exist a small positive constant $\rho < 1/4$, such that the solution $M(3)(k)$ of $\bar{\partial}$-problem admits the following estimation

$$
\| M(3)(\frac{i}{2}) - I \| = \| \frac{1}{\pi} \int_C \frac{M(3)(s)W(3)(s)}{s-\frac{i}{2}} \, dA(s) \| \leq t^{-1+2\rho}.
$$

As $k \to \frac{i}{2}$, $M(3)(k)$ has asymptotic expansion

$$
M(3)(k) = M(3)(\frac{i}{2}) + M_1(3)(x,t)(k-\frac{i}{2}) + O((k-\frac{i}{2})^2),
$$

where $M_1(3)(x,t)$ is a $k$-independent coefficient with

$$
M_1(3)(x,t) = \frac{1}{\pi} \int_C \frac{M(3)(s)W(3)(s)}{(s-\frac{i}{2})^2} \, dA(s),
$$

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and \( M_1^{(3)}(x,t) \) satisfies
\[
|M_1^{(3)}(x,t)| \lesssim t^{-1+2\rho}.
\] (3.89)

**Proof.** First we estimate (3.86). The proof proceeds along the same steps as the proof of above Proposition. (3.62) and (3.72) implies that for large \( t \), \( \| M^{(3)} \|_{\infty} \lesssim 1 \).

And here we only estimate the integral on sector \( \Upsilon_1 \) as \( \xi < -1/4 \). Let \( s = u + vi \).

We also divide \( M^{(3)}(\frac{1}{2}) - I \) to two parts, but this time we use another estimation (3.18):
\[
\frac{1}{\pi} \int \int \frac{|r'(|s|)|e^{2\text{Im}\theta}}{|\frac{i}{2} - s|}dA(s) \lesssim I_3 + I_4,
\] (3.90)

with
\[
I_3 = \int \int_{\Upsilon_1} \frac{|s|^{-1}e^{2\text{Im}\theta}}{|\frac{i}{2} - s|}dA(s), \quad I_4 = \int \int_{\Upsilon_1} \frac{|s|^2e^{2\text{Im}\theta}}{|\frac{i}{2} - s|}dA(s).
\] (3.91)

For \( s \in \Upsilon_1, |\frac{i}{2} - s| \leq c \). So
\[
I_3 \leq \int_0^{\infty} \int_0^{\infty} \frac{|s|^{-1}e^{ctv}}{|\frac{i}{2} - s|}dudv \lesssim \int_0^{\infty} \| r' \|_{L^1(\mathbb{R}^+)}e^{ctv}dv \lesssim \int_0^{\infty} e^{ctv}dv \lesssim t^{-1}.
\]

As for \( I_4 \), we partition it to two parts:
\[
I_4 \leq \int_0^{\frac{1}{4}} \int_0^{\infty} \frac{|s|^2e^{ctv}}{|\frac{i}{2} - s|}dudv + \int_0^{\infty} \int_0^{\frac{1}{4}} \frac{|s|^2e^{ctv}}{|\frac{i}{2} - s|}dudv := I_{41} + I_{42}.
\] (3.92)

For \( 0 < v < \frac{1}{4}, |s - \frac{i}{2}|^2 = u^2 + (v - \frac{1}{2})^2 > u^2 + v^2 = |s|^2 \), while as \( v > \frac{1}{4}, |s - \frac{i}{2}|^2 < |s|^2 \).

Then the first integral has:
\[
I_{41} \leq \int_0^{\frac{1}{4}} \int_0^{\infty} (u^2 + v^2)^{-\frac{1}{2}-\rho}(u^2 + (v - \frac{1}{2})^2)^{-\frac{1}{2}+\rho}ue^{ctv}dv \leq \int_0^{\frac{1}{4}} \left[ \int_0^{\infty} \left( 1 + \left( \frac{u}{v} \right)^2 \right)^{-\frac{1}{2}-\rho}u^{-2\rho}d\frac{u}{v} \right] (v^2 + (v - \frac{1}{2})^2)^{-\frac{1}{2}+\rho}e^{ctv}dv \lesssim \int_0^{\frac{1}{4}} v^{-2\rho}(\frac{1}{8})^{-\frac{1}{2}+\rho}e^{ctv}dv \lesssim t^{-1+2\rho}.
\] (3.93)
The second integral can be bounded in similar way:

\[
I_{42} \leq \int_{\frac{1}{2}}^{+\infty} e^{c'tv} \|s^{-1}\|_{L^p} \left\| \frac{i}{2} - s \right\|_{L^q}^{-1} dv
\]
\[
\leq \int_{\frac{1}{2}}^{+\infty} (v^2)^{-\frac{1}{2}+\rho} |v - \frac{1}{2}|^{-2\rho} e^{c'tv} dv
\]
\[
\lesssim \int_{\frac{1}{2}}^{+\infty} (v^2)^{-\frac{1}{2}} + \int_{\frac{1}{2}}^{+\infty} (v - \frac{1}{2})^{-2\rho} e^{c'tv} dv
\]
\[
\lesssim e^{\frac{c'}{t}} \int_{\frac{1}{2}}^{1/2} (1 - v)^{-2\rho} dv + e^{\frac{c'}{t}} \int_{\frac{1}{2}}^{+\infty} (v - \frac{1}{2})^{-2\rho} e^{c't(v - \frac{1}{2})} d(v - \frac{1}{2}) \lesssim e^{\frac{c'}{t}}.
\]

This estimation is strong enough to obtain the result (3.86). And (3.89) is obtained by deflating \(|s - \frac{i}{2}|^2\) to a constant for \(s \in \Upsilon_1\).

\[\square\]

### 3.4 Long time asymptotic behaviors

Here we begin to construct the long time asymptotics of the CH equation (1.1). According to the series of RHP transformations we have done before, we have

\[
M(k) = M^{(3)}(k)M^{err}(k)M^{A}(k)G^{-1}(k)R^{(2)}(k)^{-1}T(k)^{-\sigma_3}.
\]

(3.94)

To reconstruct \(u(x, t)\) by using (2.39), in above equation, we take \(k \rightarrow \frac{i}{2}\) out of \(\Upsilon\). Further taking the Laurent expansion of each element into (3.94), we can obtain that

\[
M(k) = M^{(3)}(\frac{i}{2}) + M^{(3)}(k - \frac{i}{2}) M^{err}(k)M^{A}(k)G^{-1}(k)
\]
\[
\left( T(\frac{i}{2})J_0 + \sum_{j,s=1}^{N(\Delta^+_1)} \left( \frac{-2k_n(s + \frac{i}{2})}{(\frac{i}{2} - k_n)(\frac{i}{2} - k_n)} \right) J_0 \right) (k - \frac{i}{2})^{-\sigma_3} + O((k - \frac{i}{2})^2).
\]

(3.95)

Its long time asymptotics can be given by

\[
M(k) = M^{A}(k) \left( T(\frac{i}{2})J_0 + \sum_{j,s=1}^{N(\Delta^+_1)} \left( \frac{-2k_n(s + \frac{i}{2})}{(\frac{i}{2} - k_n)(\frac{i}{2} - k_n)} \right) J_0 \right) (k - \frac{i}{2})^{-\sigma_3}
\]
\[
+ O((k - \frac{i}{2})^2) + O(t^{-1+2\rho}),
\]

(3.96)
\[ M(i_{\frac{1}{2}}) = M^\Lambda(i_{\frac{1}{2}}) \left( T(i_{\frac{1}{2}}) J_0 \right)^{-\sigma_3} + \mathcal{O}(t^{-1+2\rho}). \] (3.97)

Substituting above estimates into (2.39) and (2.38) leads to

\[ u(x,t) = u(y(x,t),t) = \frac{1}{2i} \lim_{k \to i_{\frac{1}{2}}} \frac{1}{k - i_{\frac{1}{2}}} \left( \frac{M_{11}(k) + M_{21}(k) (M_{12}(k) + M_{22}(k))}{(M_{11}(i_{\frac{1}{2}}) + M_{21}(i_{\frac{1}{2}}) (M_{12}(i_{\frac{1}{2}}) + M_{22}(i_{\frac{1}{2}})))} - 1 \right) \]
\[ = \frac{1}{2i} \lim_{k \to i_{\frac{1}{2}}} \frac{1}{k - i_{\frac{1}{2}}} \left( \frac{([M^\Lambda]_{11}(k) + [M^\Lambda]_{21}(k)) ([M^\Lambda]_{12}(k) + [M^\Lambda]_{22}(k))}{([M^\Lambda]_{11}(i_{\frac{1}{2}}) + [M^\Lambda]_{21}(i_{\frac{1}{2}})) ([M^\Lambda]_{12}(i_{\frac{1}{2}}) + [M^\Lambda]_{22}(i_{\frac{1}{2}}))} - 1 \right) \]
\[ + \mathcal{O}(t^{-1+2\rho}) \]
\[ = u^r(x,t; \tilde{D}) + \mathcal{O}(t^{-1+2\rho}), \] (3.98)

and

\[ x(y,t) = y + \ln \left( \frac{M_{11}(i_{\frac{1}{2}}) + M_{21}(i_{\frac{1}{2}})}{M_{12}(i_{\frac{1}{2}}) + M_{22}(i_{\frac{1}{2}}) a^2(i_{\frac{1}{2}})} \right) \]
\[ = y + \int_{-\infty}^{+\infty} (\sqrt{m + 1} - 1) d\xi + \ln \left( \frac{M_{11}(i_{\frac{1}{2}}) + M_{21}(i_{\frac{1}{2}})}{M_{12}(i_{\frac{1}{2}}) + M_{22}(i_{\frac{1}{2}})} \right) + \mathcal{O}(t^{-1+2\rho}). \] (3.99)

4 Long-time asymptotic with phase points

In a similar way to the case without phase points in Section 3, we can obtain Long-time asymptotic in the regions with phase points as follows

\[ u(x,t) = u^r(x,t; \tilde{D}) + f_{11} t^{-1/2} + \mathcal{O}(t^{-3/4}), \quad \text{for} \ \xi \in (-1/4, 2). \] (4.1)

The detail analysis will be given in near future.

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