On Freud-Sobolev type orthogonal polynomials

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Abstract

In this contribution we deal with sequences of monic polynomials orthogonal with respect to the Freud Sobolev-type inner product

\[ \langle p, q \rangle_s = \int_{\mathbb{R}} p(x)q(x)e^{-x^4} \, dx + M_0 p(0)q(0) + M_1 p'(0)q'(0), \]

where \( p, q \) are polynomials, \( M_0 \) and \( M_1 \) are nonnegative real numbers. Connection formulas between these polynomials and Freud polynomials are deduced and, as a consequence, a five term recurrence relation for such polynomials is obtained. The location of their zeros as well as their asymptotic behavior is studied. Finally, an electrostatic interpretation of them in terms of a logarithmic interaction in the presence of an external field is given.

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1 Introduction

Let us consider the so called Freud type inner products

\[ \langle p, q \rangle = \int_{\mathbb{R}} p(x)q(x)d\mu(x), \quad p, q \in \mathbb{P}, \] (1)

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1 INTRODUCTION

where $d\mu(x) = \omega(x)dx = e^{-V(x)}dx$ is a positive, nontrivial Borel measure supported on the whole real line $\mathbb{R}$, and $\mathbb{P}$ is the linear space of polynomials with real coefficients. Analytic properties of such sequences of polynomials are very well known for certain values of the external potential $V(x)$.

Let us introduce the following inner product in $\mathbb{P}$

$$\langle p, q \rangle = \int_{\mathbb{R}} p(x)q(x)e^{-x^4}dx, \quad p, q \in \mathbb{P},$$

i.e., $V(x) = x^4$ in (1).

Let $\{F_n(x)\}_{n \geq 0}$ be the corresponding sequence of monic orthogonal polynomials (MOPS, in short), which constitute a family of semi–classical orthogonal polynomials (see [21], [25]), because $V(x)$ is differentiable in $\mathbb{R}$ (the support of $\mu$), and the linear functional $u$ associated with $\omega(x) = e^{-V(x)}$, i.e.

$$\langle u, p(x) \rangle = \int_{\mathbb{R}} p(x)\omega(x)dx,$$

satisfies the distributional (or Pearson) equation (see [27])

$$[\sigma(x)\omega(x)]' = \tau(x)\omega(x),$$

where $\sigma(x) = 1$ and $\tau(x) = -4x^3$. Notice that, in terms of the weight function, the above relation means that

$$\frac{\omega'(x)}{\omega(x)} = \frac{\tau(x) - \sigma'(x)}{\sigma(x)} = -V'(x).$$

In this contribution, we consider the diagonal Freud Sobolev-type inner product

$$\langle p, q \rangle_s = \langle p, q \rangle + p^T(0)Mq(0),$$

where

$$p(0) = [p(0), p'(0), \ldots, p^{(s)}(0)]^T$$

is a column vector of dimension $s + 1$, the column vector $q(0)$ is defined in an analogous way, and $M$ is the diagonal and positive definite $(s + 1) \times (s + 1)$ matrix

$$M = \text{diag}[M_0, M_1, \ldots, M_s], \quad M_k \in \mathbb{R}_+, k = 0, 1, \ldots, s.$$ 

Thus (3) reads

$$\langle p, q \rangle_s = \langle p, q \rangle + \sum_{k=0}^{s} M_k p^{(k)}(0)q^{(k)}(0).$$

(4)

We will denote by $\{Q_n(x)\}_{n \geq 0}$ the MOPS with respect to the above inner product. This is the so called diagonal case for Sobolev-type inner products, see [1]. If there are no derivatives involved therein (i.e., $s = 0$), the polynomials orthogonal with respect to [1] are known as Krall–type orthogonal polynomials, and they are orthogonal with respect to a standard inner product, because the operator of multiplication by $x$ is symmetric with respect to such an inner product, i.e. $\langle xp, q \rangle_{s=0} = \langle p, xq \rangle_{s=0}$, for every $p, q \in \mathbb{P}$. On the other hand, when $s > 0$ (3) becomes non–standard, and the corresponding polynomials are called Sobolev–type orthogonal polynomials. In this work we consider the Sobolev case, so we will refer $Q_n(x)$ as Freud–Sobolev type orthogonal polynomials.
We will also use a notation relative to the norm of the polynomials. If for any \( n \)-th degree polynomial of a sequence of orthogonal polynomials we have \( \langle f_n, f_n \rangle = ||f_n||^2 = 1 \), then the sequence is said to be orthonormal. In order to have uniqueness, we will always choose the leading coefficient of any orthonormal polynomial to be positive for every \( n \).

**Proposition 1** Let \( \{f_n(x)\}_{n \geq 0} \) denote the sequence of polynomials orthonormal with respect to (2). That is, 

\[
f_n(x) = \gamma_n F_n(x) = \gamma_n x^n + \text{lower degree terms},
\]

where

\[
\gamma_n = (||F_n||^2)^{-1/2} > 0,
\]

and 

\[
||F_n||^2 = \int_{\mathbb{R}} [F_n(x)]^2 e^{-x^4} \, dx.
\]

The following structural properties hold.

1. **Three term recurrence relation (see [22]).** Since \( \omega(x) \) is an even weight function, the family \( \{f_n(x)\}_{n \geq 0} \) is symmetric. For every \( n \in \mathbb{N} \),

\[
x f_{n-1}(x) = a_n f_n(x) + a_{n-1} f_{n-2}(x), \quad n \geq 1,
\]

with \( f_{-1} := 0, \ f_0(x) = (\int_{\mathbb{R}} \omega(x) \, dx)^{-1/2}, \ f_1(x) = a_1^{-1} x \). Also, \( a_n = \frac{\gamma_{n-1}}{\gamma_n} \), \( n \geq 1, \ a_0 = 0 \), and

\[
a_2^2 = \frac{\int_{\mathbb{R}} x^2 \omega(x) \, dx}{\int_{\mathbb{R}} \omega(x) \, dx} = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}
\]

We also have (see [12])

\[
x F_n(x) = F_{n+1}(x) + a_n^2 F_{n-1}(x), \quad n \geq 1,
\]

for the monic normalization.

2. **Ratio of the leading coefficients (see [16], [23])**

\[
a_n^2 = \left(\frac{n}{12}\right)^2 \left[1 + \frac{1}{24n^2} + O(n^{-4})\right]
\]

3. **String equation (see [27, (2.12)]).** An important feature of these polynomials is that the recurrence coefficients \( a_n \) in the above three term recurrence relation, satisfy the following nonlinear difference equation

\[
4a_n^2 (a_{n+1}^2 + a_n^2 + a_{n-1}^2) = n, \quad n \geq 1.
\]

This is known in the literature as the string equation or Freud equation (see [12], [14, (3.2.20)], among others)

4. ([22, Th. 4]). The polynomials \( f_n(x) \) defined by (6) constitute a generalized Appell sequence. More precisely,

\[
[f_n]'(x) = \frac{n}{a_n} f_{n-1}(x) + 4a_n a_{n-1} a_{n-2} f_{n-3}(x), \quad n = 1, 2, \ldots
\]
5. ([22, Th. 5]) The polynomials $f_n(x)$ satisfy
\[ [f_n]^3(x) = -4x a_n^2 f_n(x) + 4a_n \phi_n(x) f_{n-1}(x), \] (9)

and
\[ [f_n]^\nu(x) = [16a_n^4 x^2 - 4a_n^2 - 16a_n^2 \phi_n(x) \phi_{n-1}(x)] f_n(x) \]
\[ + [8a_n x + 16a_n x^3 \phi_n(x)] f_{n-1}(x), \] (10)

where (see [22, eq. (14)])
\[ \phi_n(x) = a_{n+1}^2 + a_n^2 + x^2. \]

6. Strong inner asymptotics (see [22, Th. 1], and [23, eq. (8)]). Let \{f_n\}_{n \geq 0} be the orthonormal polynomials with respect to the weight function for $\omega(x) = e^{-x^2}$. Then,
\[ f_n(x) = A e^{x^2/2} n^{-1/8} \]
\[ \sin \left\{ \frac{64}{27} n^{3/4} x + 12^{-1/4} n^{1/4} x^3 - \frac{n-1}{2} \right\} + o(n^{-1/8}), \]

where $A = \sqrt{12}/\sqrt{\pi}$, uniformly for $x$ in every compact subset $\Delta \subset \mathbb{R}$.

We are interested in the asymptotic properties of derivatives of the Freud polynomials which will be useful in the sequel. From [22, eqs. (16)-(17)], and [10], the following Lemma follows

**Lemma 1** (see [22, Th. 6]) There exists a constant $A = \sqrt{12}/\sqrt{\pi}$ such that the following estimates hold

\[ f_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} n^{-1/8} (A + o(1)), & \text{if } n \text{ is even} \end{cases}, \]

\[ [f_n]^{(0)}(0) = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} \frac{\sqrt{3}}{\sqrt{27}} n^{5/8} (A + o(1)), & \text{if } n \text{ is odd} \end{cases}, \]

\[ [f_n]^{(\nu)}(0) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{(n+1)/2} \frac{8}{3 \sqrt{3}} n^{11/8} (A + o(1)), & \text{if } n \text{ is even} \end{cases}, \]

\[ [f_n]^{(\nu)}(0) = \begin{cases} 0, & \text{if } n \text{ is even} \\ (-1)^{(n-1)/2} \frac{16 \sqrt{3}}{9 \sqrt{4}} n^{17/8} (A + o(1)), & \text{if } n \text{ is odd} \end{cases}. \]

The kernel polynomials associated with the polynomial sequence \{F_n\}_{n \geq 0} will play a key role in order to prove some of the results of the manuscript. In the remaining of this section, we analyze them in detail. The $n$-th degree reproducing kernel associated with \{F_n\}_{n \geq 0} is
\[ K_n(x, y) = \sum_{k=0}^{n} \frac{F_k(x) F_k(y)}{||F_k||^2}. \]

For $x \neq y$, the Christoffel-Darboux formula reads
\[ K_n(x, y) = \frac{1}{||F_n||^2} \frac{F_{n+1}(x) F_n(y) - F_{n+1}(y) F_n(x)}{x - y}, \] (11)
and its confluent expression becomes

\[ K_n(x, x) = \sum_{k=0}^{n} \frac{[F_k(x)]^2}{\|F_k\|^2} = \frac{[F_{n+1}](x)F_n(x) - [F_n](x)F_{n+1}(x)}{\|F_n\|^2}. \]  

\[ (12) \]

We introduce the following standard notation for the partial derivatives of the \( n \)-th degree kernel \( K_n(x, y) \)

\[ \frac{\partial^{j+k} K_n(x, y)}{\partial^j x \partial^k y} =: K_n^{(j,k)}(x, y), \quad 0 \leq j, k \leq n. \]  

\[ (13) \]

Thus,

\[ K_{n-1}^{(0,1)}(x, 0) = K_{n-1}^{(1,0)}(0, x) = \frac{1}{\|F_{n-1}\|^2} \left[ \frac{F_n(x)F_{n-1}(0) - F_{n-1}(x)F_n(0)}{x^2} + \frac{F_n(x)[F_{n-1}]'(0) - F_{n-1}(x)[F_n]'(0)}{x} \right], \]

and, considering the coefficient of \( x \) in the above expression, we have

\[ K_{n-1}^{(1,1)}(0, 0) = \frac{1}{\|F_{n-1}\|^2} \left[ \frac{[F_n]'(0)F_{n-1}(0) - [F_{n-1}]'(0)F_n(0)}{6} + \frac{[F_n]''(0)[F_{n-1}]'(0) - [F_{n-1}]''(0)[F_n]'(0)}{2} \right]. \]

\[ (14) \]

From \( (12) \)

\[ K_{n-1}(0, 0) = \frac{[F_n]'(0)F_{n-1}(0) - [F_{n-1}]'(0)F_n(0)}{\|F_{n-1}\|^2}, \]

and taking limit in \( (14) \) when \( x \to 0 \), we get

\[ K_{n-1}^{(0,1)}(0, 0) = K_{n-1}^{(1,0)}(0, 0) = \frac{1}{\|F_{n-1}\|^2} \frac{[F_n]''(0)F_{n-1}(0) - [F_{n-1}]''(0)F_n(0)}{2}. \]

\[ (15) \]

Taking a suitable index shifting in the last three expressions, we conclude

\[ K_{2n-1}(0, 0) = \frac{-[F_{2n-1}](0)F_{2n}(0)}{\|F_{2n-1}\|^2}, \]

\[ K_{2n-1}^{(0,1)}(0, 0) = K_{2n-1}^{(1,0)}(0, 0) = 0, \]

\[ K_{2n-1}^{(1,1)}(0, 0) = \frac{1}{\|F_{2n-1}\|^2} \left[ \frac{[F_{2n}]''(0)[F_{2n-1}]'(0)}{2} - \frac{[F_{2n-1}]''(0)F_{2n}(0)}{6} \right], \]

as well as

\[ K_{2n}(0, 0) = \frac{[F_{2n}](0)F_{2n}(0)}{\|F_{2n}\|^2}, \]

\[ K_{2n}^{(0,1)}(0, 0) = K_{2n}^{(1,0)}(0, 0) = 0, \]

\[ K_{2n}^{(1,1)}(0, 0) = \frac{1}{\|F_{2n}\|^2} \left[ \frac{[F_{2n+1}]''(0)F_{2n}(0)}{6} - \frac{[F_{2n}]''(0)[F_{2n+1}]'(0)}{2} \right]. \]
Another interesting property of the Freud kernels arises from the symmetry of \( \{ F_n(x) \}_{n \geq 0} \). From (12) and (13) we have

\[
K_{2n+1}(x, 0) = \sum_{i=0}^{n-1} \frac{F_{2i}(0)}{\| F_{2i} \|^2} F_{2i}(x) = K_{2n}(x, 0),
\]
\[
K_{2n}^{(0,1)}(x, 0) = K_{2n-1}^{(0,1)}(x, 0),
\]
\[
K_{2n}^{(1,1)}(x, 0) = \frac{[F_{2n}]'(x)[F_{2n}]'(0)}{\| F_{2n} \|^2} + K_{2n-1}^{(1,1)}(x, 0) = K_{2n-1}^{(1,1)}(x, 0),
\]

This fact will be useful throughout the paper. We also need explicit asymptotic expressions for the reproducing kernel and their derivatives. They are introduced in the following lemma.

**Lemma 2** For every \( n = 0, 1, \ldots \), we have

\[
K_n(0, 0) = O(n^{3/4}),
\]
\[
K_n^{(0,1)}(0, 0) = K_n^{(1,0)}(0, 0) = 0,
\]
\[
K_n^{(1,1)}(0, 0) = O(n^{9/4}).
\]

**Proof.** Writing \( K_n(0, 0), K_n^{(0,1)}(0, 0) \) and \( K_n^{(1,1)}(0, 0) \) in terms of orthonormal polynomials \( f_n \), the Lemma follows.  

The structure of the manuscript is as follows.

In Section 2 we will obtain connection formulas between monic Freud-Sobolev type and monic Freud orthogonal polynomials. We also prove that Freud-Sobolev orthogonal polynomials satisfy a five term recurrence relation and we will deduce the asymptotic behavior of the coefficients involved therein. In Section 3 we study some analytic properties of zeros of Freud-Sobolev type orthogonal polynomials, in particular interlacing and asymptotic behavior. Section 4 is focused on the second order linear differential equation that such polynomials satisfy. As a direct consequence, the electrostatic interpretation of such polynomials in terms of a logarithmic potential interaction and an external potential is presented.

## 2 Connection formulas

Let us consider the aforementioned Sobolev-type inner product (4). In the sequel, we will denote by \( \{ Q_n(x) \}_{n \geq 0} \) the corresponding sequence of monic orthogonal polynomials and by

\[
|| Q_n ||_s^2 = (Q_n, x^n)_s
\]

the norm of the \( n \)-th degree polynomial. The connection formula between \( \{ Q_n(x) \}_{n \geq 0} \) and \( \{ P_n(x) \}_{n \geq 0} \) is stated in the following lemma.

**Lemma 3** [4] For \( n \geq 1 \), we have

\[
Q_n(x) = F_n(x) - \sum_{k=0}^{s} M_k [Q_{n}]^{(k)}(0) K_{n-1}^{(0,k)}(x, 0),
\] (16)
where, for \(0 \leq k \leq s\),

\[
[Q_n]^{(k)}(0) = (\det D)^{-1} \begin{pmatrix}
1 + M_0 K_{n-1}^{(0,0)}(0,0) & \cdots & F_n(0) & \cdots & M_s K_{n-1}^{(0,s)}(0,0) \\
M_0 K_{n-1}^{(1,0)}(0,0) & \cdots & [F_n]'(0) & \cdots & M_s K_{n-1}^{(1,s)}(0,0) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
M_0 K_{n-1}^{(s,0)}(0,0) & \cdots & [F_n]^{(s)}(0) & \cdots & 1 + M_s K_{n-1}^{(s,s)}(0,0)
\end{pmatrix},
\]

with

\[
D = \begin{pmatrix}
1 + M_0 K_{n-1}^{(0,0)}(0,0) & M_1 K_{n-1}^{(0,1)}(0,0) & \cdots & M_s K_{n-1}^{(0,s)}(0,0) \\
M_0 K_{n-1}^{(1,0)}(0,0) & 1 + M_1 K_{n-1}^{(1,1)}(0,0) & \cdots & M_s K_{n-1}^{(1,s)}(0,0) \\
\vdots & \vdots & \ddots & \vdots \\
M_0 K_{n-1}^{(s,0)}(0,0) & M_1 K_{n-1}^{(s,1)}(0,0) & \cdots & 1 + M_s K_{n-1}^{(s,s)}(0,0)
\end{pmatrix}.
\]

Moreover, an easy computation shows that

\[
K_{n-1}^{(0,k)}(x,0) = \frac{1}{||F_{n-1}||^2} \left( \sum_{\eta=0}^{k} \frac{k!}{\eta!} [F_n(x)]^{(\eta)}(0) - [F_{n-1}(x)]^{(\eta)}(0) \right),
\]

and, as a consequence, we can write (16) as

\[
x^{s+1}Q_n(x) = A_s(n; x)F_n(x) + B_s(n; x)F_{n-1}(x),
\]

where

\[
A_s(n; x) = \sum_{k=0}^{s} \left( x^{s+1} - \frac{k!}{\eta!} \frac{M_k [Q_n]^{(k)}(0)}{||F_{n-1}||^2} [F_n^{(\eta)}(0) x^{s-k+\eta}] \right),
\]

\[
B_s(n; x) = \sum_{k=0}^{s} \left( \frac{k!}{\eta!} \frac{M_k [Q_n]^{(k)}(0)}{||F_{n-1}||^2} [F_n^{(\eta)}(0) x^{s-k+\eta}] \right),
\]

are polynomials of degree \(s + 1\) and \(s\), respectively.

**2.1 Connection formulas for monic polynomials**

In what follows, we restrict ourselves to study the case of only one mass point with derivative in the inner product (2), i.e., \(s = 1\), \(M_0 \geq 0\), and \(M_1 \geq 0\),

\[
\langle p, q \rangle = \langle p, q \rangle + M_0 p(0) q(0) + M_1 p'(0) q'(0).
\]

In such a case, the connection formula (17) becomes

\[
x^2 Q_n(x) = A_1(n; x) F_n(x) + B_1(n; x) F_{n-1}(x),
\]

where \(A_1(n; x) = x^2 + A_{10}(n)\), and \(B_1(n; x) = B_{11}(n) x\) with

\[
A_{10}(n) = -\frac{M_1 [Q_n]'(0) F_{n-1}(0)}{||F_{n-1}||^2}, \quad B_{11}(n) = \frac{M_0 Q_n(0) F_n(0) + M_1 [Q_n](0) [F_n]'(0)}{||F_{n-1}||^2}.
\]
To obtain $Q_n(0)$ and $[Q_n]'(0)$ in the above expression, we evaluate (19) at $x = 0$ and solve the corresponding linear system. Indeed,

$$Q_n(0) = \begin{vmatrix}
F_n(0) & M_1 K_{n-1}^{(0,1)}(0,0) \\
F'_n(0) & 1 + M_1 K_{n-1}^{(1,1)}(0,0)
\end{vmatrix},
1 + M_0 K_{n-1}(0,0) & M_1 K_{n-1}^{(0,1)}(0,0) \\
M_0 K_{n-1}^{(1,0)}(0,0) & 1 + M_1 K_{n-1}^{(1,1)}(0,0)
\end{vmatrix},
$$

$$[Q_n]'(0) = \begin{vmatrix}
1 + M_0 K_{n-1}(0,0) & F_n(0) \\
M_0 K_{n-1}^{(1,0)}(0,0) & [F_n]'(0)
\end{vmatrix},
1 + M_0 K_{n-1}(0,0) & M_1 K_{n-1}^{(0,1)}(0,0) \\
M_0 K_{n-1}^{(1,0)}(0,0) & 1 + M_1 K_{n-1}^{(1,1)}(0,0)
\end{vmatrix}.$$
If $k = n$, by comparing the leading coefficients, we obtain $c_{n,n} = \zeta_n / \gamma_n$. When $k < n$, by orthogonality we have $\langle q_n(x), f_k(x) \rangle_1 = 0$, so that
\[
c_{k,n} = -M_0 q_n(0) f_k(0) - M_1 [q_n]'(0) [f_k]'(0).
\]

Hence,
\[
\int_{\mathbb{R}} [q_n(x)]^2 e^{-x^4} \, dx = \left( \frac{\zeta_n}{\gamma_n} \right)^2 + \sum_{k=0}^{n-1} [c_{k,n}]^2 [f_k(x)]^2
\]
\[
= \left( \frac{\zeta_n}{\gamma_n} \right)^2 + M_0^2 q_n^2(0) \sum_{k=0}^{n-1} f_k^2(0) + M_1^2 ([q_n]'(0))^2 \sum_{k=0}^{n-1} ([f_k]'(0))^2
\]
\[
+ 2M_0 M_1 q_n(0) [q_n]'(0) \sum_{k=0}^{n-1} f_k(0) [f_k]'(0).
\]

On the other hand, by the orthonormality of $q_n(x)$ with respect to $[18]$,
\[
\langle q_n, q_n \rangle_1 = 1 = \int_{\mathbb{R}} [q_n(x)]^2 e^{-x^4} \, dx + M_0 q_n^2(0) + M_1 ([q_n]'(0))^2,
\]
so that
\[
\int_{\mathbb{R}} [q_n(x)]^2 e^{-x^4} \, dx = 1 - M_0 q_n^2(0) - M_1 ([q_n]'(0))^2.
\]

Therefore,
\[
1 - M_0 q_n^2(0) - M_1 ([q_n]'(0))^2 = \left( \frac{\zeta_n}{\gamma_n} \right)^2 + M_0^2 q_n^2(0) \sum_{k=0}^{n-1} f_k^2(0)
\]
\[
+ M_1^2 ([q_n]'(0))^2 \sum_{k=0}^{n-1} ([f_k]'(0))^2 + 2M_0 M_1 q_n(0) [q_n]'(0) \sum_{k=0}^{n-1} f_k(0) [f_k]'(0).
\]

Taking into account that $\sum_{k=0}^{n-1} f_k(0) [f_k]'(0) = K_{n-1}^{(0,1)}(0,0) = 0$, we rewrite the above expression as
\[
1 - M_0 q_n^2(0) - M_1 ([q_n]'(0))^2 = \left( \frac{\zeta_n}{\gamma_n} \right)^2 + M_0^2 q_n^2(0) K_{n-1}(0,0) + M_1^2 ([q_n]'(0))^2 K_{n-1}^{(1,1)}(0,0),
\]
and, as a consequence,
\[
1 = \left( \frac{\zeta_n}{\gamma_n} \right)^2 + M_0 q_n^2(0) \left[ 1 + M_0 K_{n-1}(0,0) \right] + M_1 ([q_n]'(0))^2 \left[ 1 + M_1 K_{n-1}^{(1,1)}(0,0) \right].
\]

Next, using the orthonormal versions of $[20]$ and $[21]$, respectively,
\[
q_n(0) = \frac{\zeta_n}{\gamma_n} \frac{f_n(0)}{1 + M_0 K_{n-1}(0,0)}, \quad [q_n]'(0) = \frac{\zeta_n}{\gamma_n} \frac{[f_n]'(0)}{1 + M_1 K_{n-1}^{(1,1)}(0,0)},
\]
we obtain
\[
1 = \left( \frac{\zeta_n}{\gamma_n} \right)^2 \left[ 1 + M_0 \frac{(f_n(0))^2}{[1 + M_0 K_{n-1}(0,0)]} \right] + M_1 \frac{([f_n]'(0))^2}{[1 + M_1 K_{n-1}^{(1,1)}(0,0)]}.
\]
Since
\[
K_n(0,0) - K_{n-1}(0,0) = (f_n(0))^2
\]
\[
K_n^{(1,1)}(0,0) - K_{n-1}^{(1,1)}(0,0) = ([f_n]''(0))^2
\]
we get
\[
1 = \left( \frac{\zeta_n}{\gamma_n} \right)^2 \left[ 1 + \frac{M_0 K_n(0,0) - M_0 K_{n-1}(0,0)}{1 + M_0 K_{n-1}(0,0)} \right] + \frac{M_1 K_n^{(1,1)}(0,0) - M_1 K_{n-1}^{(1,1)}(0,0)}{1 + M_1 K_{n-1}^{(1,1)}(0,0)},
\]
\[
1 = \left( \frac{\zeta_n}{\gamma_n} \right)^2 \left[ 1 + \frac{M_0 K_n(0,0)}{1 + M_0 K_{n-1}(0,0)} + \frac{M_1 K_n^{(1,1)}(0,0)}{1 + M_1 K_{n-1}^{(1,1)}(0,0)} - 1 \right]
\]
which is (23).

Now, we obtain connection formulas that relate both families of monic orthogonal polynomials.

**Proposition 3** The Freud-Sobolev type orthogonal polynomials satisfy
\[
x^2 Q_n(x) = \left[ x^2 - \frac{r_n \kappa_n^{[1]}}{4 \phi_n(0)} \right] F_n(x) + a_n^2 \left( \kappa_n^{[0]} + \kappa_n^{[1]} \right) x F_{n-1}(x), \quad n \geq 1,
\]
where
\[
\kappa_n^{[0]} = \frac{1 + M_0 K_n(0,0)}{1 + M_0 K_{n-1}(0,0)} - 1, \quad \kappa_n^{[1]} = \frac{1 + M_1 K_n^{(1,1)}(0,0)}{1 + M_1 K_{n-1}^{(1,1)}(0,0)} - 1, \quad r_n = \frac{1 - (-1)^n}{2}.
\]
Moreover, for the even and odd degrees, respectively, we have
\[
Q_{2n}(x) = F_{2n}(x) - M_0 \frac{F_{2n}(0)}{1 + M_0 K_{2n-2}(0,0)} K_{2n-2}(x,0), \quad n \geq 1,
\]
\[
Q_{2n+1}(x) = F_{2n+1}(x) - M_1 \frac{[F_{2n+1}]'(0)}{1 + M_1 K_{2n}^{(1,1)}(0,0)} K_{2n-1}^{(0,1)}(x,0), \quad n \geq 1.
\]
In other words, \( Q_{2n} \) \( (Q_{2n+1}) \) is an even (odd) polynomial.

**Proof.** Setting \( s = 1 \) in (16) we get
\[
Q_n(x) = F_n(x) - M_0 Q_n(0) K_{n-1}(x,0) - M_1 [Q_n]'(0) K_{n-1}^{(0,1)}(x,0).
\]
From (14) we have
\[
K_{n-1}^{(0,1)}(x,0) = \left( \frac{F_{n-1}(0) + x[F_{n-1}]'(0)}{x^2 ||F_{n-1}||^2} \right) F_n(x) - \left( \frac{F_n(0) + x[F_n]'(0)}{x^2 ||F_{n-1}||^2} \right) F_{n-1}(x),
\]
and taking into account (11), (20), (21), and the symmetry of the Freud polynomials, we get
\[ Q_n(x) = \left[ 1 - \frac{F_{n-1}(0)}{[F_n]'(0)} x^2 \| F_{n-1} \|^2 \right] \left( 1 + M_1 K_n^{(1,1)}(0,0) \right) - 1 \] \[ F_n(x) \]
\[ + \frac{a_n^2}{x} \left[ \left( 1 + M_0 K_n(0,0) \right) - 1 \right] + \left( 1 + M_1 K_n^{(1,1)}(0,0) \right) - 1 \] \[ F_{n-1}(x) \].

Denoting
\[ \kappa_n^{[0]} = \frac{1 + M_0 K_n(0,0)}{1 + M_0 K_{n-1}(0,0)} - 1, \quad \kappa_n^{[1]} = \frac{1 + M_1 K_n^{(1,1)}(0,0)}{1 + M_1 K_{n-1}^{(1,1)}(0,0)} - 1, \quad r_n = \frac{1 - (-1)^n}{2}, \]

and noticing that from (24) we have
\[ \frac{F_{n-1}(0)}{[F_n]'(0)} = 1/4a_n^2 \phi_n(0), \]
we obtain
\[ Q_n(x) = \left[ 1 - \frac{r_n}{4x^2 \phi_n(0)} \kappa_n^{[0]} \right] F_n(x) + \frac{a_n^2}{x} \left( \kappa_n^{[0]} + \kappa_n^{[1]} \right) F_{n-1}(x), \quad (27) \]

which is (24). On the other hand, shifting the index \( n \to 2n \), and taking into account (22) we obtain (25). For the odd case, (26) follows similarly by using (22).

**Remark 1** Notice that, from the symmetry of the Freud polynomials, we have \( \kappa_n^{[0]} = 0 \) and \( \kappa_n^{[1]} = 0 \) for \( n \geq 1 \). As a consequence, (22) becomes
\[
\begin{align*}
xF_2(x) &= xF_2(x) + \kappa_2 \phi_2 F_{n-1}(x), \quad n \geq 1, \\
x^2F_{2n+1}(x) &= x^2 - \frac{r_n \kappa_{2n+1}}{4 \phi_{2n+1}(0)} F_{2n+1}(x) + \kappa_{2n+1} x F_{2n}(x), \quad n \geq 1.
\end{align*}
\]

The following is a straightforward extension of connection formulas (25) and (24) for orthonormal polynomials.

**Corollary 1** Let \( q_0 = \zeta_n x^n + \cdots \) and \( f_n = \gamma_n x^n + \cdots \), then for \( n \geq 1 \),
\[
\begin{align*}
q_{2n}(x) &= \frac{\zeta_2}{\gamma_2} f_{2n}(x) - M_0 q_{2n}(0) K_{2n-2}(x,0), \\
q_{2n+1}(x) &= \frac{\zeta_2}{\gamma_2} f_{2n+1}(x) - M_1 [q_{2n+1}]'(0) K_{2n-1}^{(1,1)}(x,0).
\end{align*}
\]

with
\[
q_{2n}(0) = \frac{\zeta_2}{\gamma_2} \left[ \frac{f_{2n}(0)}{1 + M_0 K_{2n-2}(0,0)} \right], \quad [q_{2n+1}]'(0) = \frac{\zeta_2}{\gamma_2} \left[ \frac{[f_{2n+1}]'(0)}{1 + M_1 K_{2n-1}^{(1,1)}(0,0)} \right].
\]

**Remark 2** Notice that, by defining \( Q_{2n}(x) := P_n(x^2) \) and \( Q_{2n+1}(x) := xR_n(x^2) \), for \( n \geq 0 \) and introducing the change of variable \( x = \sqrt{y} \), we obtain the following orthogonality relations
\[
0 = \langle Q_{2n}, Q_{2m} \rangle = \int_0^\infty P_n(y) P_m(y) y^{-1/2} e^{-y^2} dy + M_0 P_n(0) P_m(0), \quad n \neq m
\]
\[
0 = \langle Q_{2n+1}, Q_{2m+1} \rangle = \int_0^\infty R_n(y) R_m(y) y^{1/2} e^{-y^2} dy + M_1 R_n(0) R_m(0), \quad n \neq m,
\]

i.e. \( \{ P_n(x) \}_{n \geq 0} \) and \( \{ R_n(x) \}_{n \geq 0} \) are MOPS with respect to standard inner products associated with the measures \( d\sigma(x) = x^{-1/2} e^{-x^2} dx + M_0 \delta(x) \) and \( xd\sigma(x) + M_1 \delta(x) \), respectively, supported on the positive real semiaxis.
2 CONNECTION FORMULAS

2.2 The five-term recurrence relation

This section deals with the five-term recurrence relation that the sequence of discrete Freud–Sobolev orthogonal polynomials \( \{Q_n(x)\}_{n \geq 0} \) satisfies. We will use the remarkable fact, which is a straightforward consequence of (18), that the multiplication operator by \( x^2 \) is a symmetric operator with respect to such a discrete Sobolev inner product. Indeed, for polynomials \( h(x), g(x) \in \mathbb{P} \)

\[
\langle x^2h(x), g(x) \rangle_1 = \langle h(x), x^2g(x) \rangle_1.
\]

Notice that

\[
\langle x^2h(x), g(x) \rangle_1 = \langle h(x), g(x) \rangle_{[2]}.
\]

An equivalent formulation of (28) is

\[
\langle x^2h(x), g(x) \rangle_1 = \langle x^2h(x), g(x) \rangle.
\]

We need a preliminary result.

**Lemma 4** For every \( n \geq 1 \), the five term connection formula

\[
x^2Q_n(x) = F_{n+2}(x) + [a_{n+1}^2 + a_n^2 + A_{10}(n) + B_{11}(n)] F_n(x)
+ a_{n-1}^2 [a_n^2 + B_{11}(n)] F_{n-2}(x)
\]

holds.

**Proof.** The result follows easily from (19) after successive applications of (7).

We are ready to find the five-term recurrence relation satisfied by \( \{Q_n(x)\}_{n \geq 0} \). Let us consider the Fourier expansion of \( x^2Q_n(x) \) in terms of \( \{Q_n(x)\}_{n \geq 0} \)

\[
x^2Q_n(x) = \sum_{k=0}^{n+2} \lambda_{n,k} Q_k(x),
\]

where

\[
\lambda_{n,k} = \frac{\langle x^2Q_n(x), Q_k(x) \rangle_1}{||Q_k||_1^2}, \quad k = 0, \ldots, n+2.
\]

(29)

Thus, \( \lambda_{n,k} = 0 \) for \( k = 0, \ldots, n-3 \), and \( \lambda_{n,n+2} = 1 \). To obtain \( \lambda_{n,n+1} \), we use (19) and get

\[
\lambda_{n,n+1} = \frac{1}{||Q_{n+1}||_1^2} \langle A_1(n;x) F_n^2(x), Q_{n+1}(x) \rangle_1 + \frac{1}{||Q_{n+1}||_1^2} \langle B_1(n;x) F_{n-1}(x), Q_{n+1}(x) \rangle_1
= \frac{1}{||Q_{n+1}||_1^2} \langle x^2F_n(x), Q_{n+1}(x) \rangle_1 = \frac{1}{||Q_{n+1}||_1^2} \langle F_n(x), x^2Q_{n+1}(x) \rangle = 0,
\]

by using Lemma 1. In order to compute \( \lambda_{n,n} \), using (19) we get

\[
\lambda_{n,n} = \frac{\langle x^2F_n(x), Q_n(x) \rangle_1}{||Q_n||_1^2} + A_{10}(n) + B_{11}(n).
\]

But, according to Lemma 3, the first term is

\[
\frac{\langle x^2F_n(x), Q_n(x) \rangle_1}{||Q_n||_1^2} = [a_{n+1}^2 + a_n^2 + A_{10}(n) + B_{11}(n)] \frac{||F_n||_1^2}{||Q_n||_1^2}.
\]
so that
\[ \lambda_{n,n} = \left[a_{n+1}^2 + a_n^2 + A_{10}(n) + B_{11}(n)\right] \frac{||F_n||^2}{||Q_n||^2_1} + A_{10}(n) + B_{11}(n). \]
A similar analysis yields \( \lambda_{n,n-1} = 0 \) and
\[
\lambda_{n,n-2} = \frac{\langle x^2 Q_n(x), Q_{n-2}(x) \rangle}{||Q_{n-2}||^2_1} = \frac{\langle a_{n-1}^2 [a_n^2 + B_{11}(n)] F_{n-2}(x), Q_{n-2}(x) \rangle}{||Q_{n-2}||^2_1} = a_{n-1}^2 [a_n^2 + B_{11}(n)] \frac{||F_{n-2}||^2}{||Q_{n-2}||^2_1}.
\]
Thus, as a conclusion:

**Theorem 1 (Five-term recurrence relation)** For every \( n \geq 1 \), the monic Freud-Sobolev type polynomials \( \{Q_n(x)\}_{n \geq 0} \), orthogonal with respect to \( [\mathcal{L}] \), satisfy the following five-term recurrence relation
\[
x^2 Q_n(x) = Q_{n+2}(x) + \lambda_{n,n} Q_n(x) + \lambda_{n,n-2} Q_{n-2}(x), \quad n \geq 1,
\]
with initial conditions \( Q_{-1}(x) = 0, Q_0(x) = 1, Q_1(x) = x, \) and \( Q_2(x) = x^2 - \lambda_{0,0} \), where
\[
\lambda_{n,n} = \left[a_{n+1}^2 + a_n^2 + A_{10}(n) + B_{11}(n)\right] \frac{||F_n||^2}{||Q_n||^2_1} + A_{10}(n) + B_{11}(n), \quad n \geq 0,
\]
\[
\lambda_{n,n-2} = a_{n-1}^2 [a_n^2 + B_{11}(n)] \frac{||F_{n-2}||^2}{||Q_{n-2}||^2_1}, \quad n \geq 2.
\]

We now proceed to analyze the asymptotic behavior of the coefficients. First, we need the following lemma.

**Lemma 5** We have
\[
\lim_{n \to \infty} \frac{||Q_n||}{||F_n||} = 1 + O(n^{-1}). \tag{31}
\]

**Proof.** Let us consider first the even case. From [13] and its analogue for \( Q_n \), as well as [23], we have
\[
\frac{||Q_{2n}||^2}{||F_{2n}||^2} = \frac{1 + M_0 K_{2n}(0,0)}{1 + M_0 K_{2n-2}(0,0)} = \frac{1 + M_0 (K_{2n-2}(0,0) + f_{2n}^2(0))}{1 + M_0 K_{2n-2}(0,0)}.
\]
Taking into account Lemmas [13] and [23] the result follows. The odd case is similar. \( \blacksquare \)

Notice that from successive applications of the three term recurrence relation [13], we get
\[
x^2 F_n(x) = F_{n+2}(x) + [a_{n+1}^2 + a_n^2] F_n(x) + a_{n-1}^2 a_n^2 F_{n-2}(x).
\]
We will show that, when \( n \to \infty \), the five term recurrence relation [30] behaves exactly as the previous equation.

**Proposition 4** We have
\[
\lim_{n \to \infty} \frac{\lambda_{n,n}}{a_{n+1}^2 + a_n^2} = 1 + O(n^{-2}), \quad \text{and} \quad \lim_{n \to \infty} \frac{\lambda_{n,n-2}}{a_{n-1}^2 a_n^2} = 1 + O(n^{-3/2}).
\]
3 THE ZEROS

Proof. In view of (30) and (31), we need estimates for \( \lim_{n \to \infty} A_{10}(n) \) and \( \lim_{n \to \infty} B_{11}(n) \). It is easy to show that \( A_{10}(2n) = 0 \), and for the odd case we have

\[
A_{10}(2n+1) = -\frac{M_1(Q_{2n+1})'(0)F_{2n}(0)}{||F_{2n}||^2} = -\frac{M_1 \left( \frac{(F_{2n+1})'(0)}{1+M_1K_{2n+1}^{-1}(0,0)} \right) F_{2n}(0)}{||F_{2n}||^2} = -\frac{M_1K_{2n}(0,0)}{1+M_1K_{2n+1}^{-1}(0,0)},
\]

where the second equality follows from (22) and the third equality from the confluent expression (55). As a consequence, using Lemma 2 we get \( A_{10}(2n+1) = O(n^{-3/2}) \). On the other side, for the even case,

\[
B_{11}(2n) = \frac{M_0Q_{2n}(0)F_{2n}(0)}{||F_{2n-1}||^2} = \frac{M_0F_{2n}(0)}{||F_{2n-1}||^2(1+M_0K_{2n-2}(0,0))} = \frac{M_0||F_{2n}||^2f_{2n}(0)}{||F_{2n-1}||^2(1+M_0K_{2n-2}(0,0))},
\]

where we have used (22) for the second equality and the fact that \( F_{2n}(0) = ||F_{2n}||^2(K_{2n}(0,0) - K_{2n-1}(0,0)) = ||F_{2n}||^2f_{2n}(0) \) for the third equality. Since \( ||F_{2n}||^2/||F_{2n-1}||^2 = a_{2n}^2 \), and by using (32) and Lemmas 1 and 2 we obtain \( B_{11}(2n) = O(n^{-1/2}) \). Finally, for the odd case, in a similar way we have

\[
B_{11}(2n+1) = \frac{M_1([F_{2n+1}]'(0))^2}{||F_{2n}||^2(1+M_1K_{2n}(0,0))^2} = \frac{M_1||F_{2n+1}||^2(f_{2n+1}'(0))^2}{||F_{2n}||^2(1+M_1K_{2n}(0,0))^2},
\]

and again from (32), and Lemmas 1 and 2 we get \( B_{11}(2n+1) = O(n^{-1/2}) \). As a consequence, we have

\[
\lim_{n \to \infty} \frac{\lambda_{n,n}}{a_{n+1}^2 + a_n^2} = \lim_{n \to \infty} \left[ \frac{a_{n+1}^2 + a_n^2 + A_{10}(n) + B_{11}(n)}{a_{n+1}^2 + a_n^2} \right] \frac{||F_n||^2}{||Q_{n+1}||^2} + A_{10}(n) + B_{11}(n),
\]

and

\[
\lim_{n \to \infty} \frac{\lambda_{n,n-2}}{a_{n-1}^2 + a_n^2} = \frac{a_{n-1}^2 + B_{11}(n)}{a_{n-1}^2 + a_n^2} \frac{||F_{n-2}||^2}{||Q_{n-2}||^2}. = 1 + O(n^{-3/2}).
\]

3 The Zeros

In this Section we analyze some properties of the zeros of the polynomials \( \{Q_n(x)\}_{n \geq 0} \).

3.1 Interlacing rupture

From (25) and (26), it is clear that the zeros of even \( Q_{2n}(x) \) and odd \( Q_{2n+1}(x) \) Freud-Sobolev type polynomials act in an independent way. From those expressions, we observe that the variation of \( M_0 \) (respectively \( M_1 \)) exclusively influences the position of the zeros of \( Q_{2n}(x) \) (respectively \( Q_{2n+1}(x) \)) without affecting the zeros of \( Q_{2n+1}(x) \) (respectively \( Q_{2n}(x) \)). This interesting phenomena leads to the destruction of the zero interlacing for two consecutive polynomials of the sequence \( \{Q_n(x)\} \) for certain values of \( M_0 \) and \( M_1 \). Notice that the zeros of \( Q_n(x), n \geq 1 \), are real and simple (see [29], Proposition 3.2). In the next two tables we provide numerical evidence that supports this fact. In the sequel, let \( \{\eta_{n,k}\}_{k=0}^n \equiv \eta_{n,1} < \eta_{n,2} < \ldots < \eta_{n,n} \) be
the zeros of $Q_n(x)$ and the zeros $\{x_{n,k}\}_{k=0}^{n}$ be the zeros of $F_n(x)$ arranged in an increasing order. Next we show the position of the second zero of the Freud-Sobolev-type polynomial of degree $n = 4$ (namely $Q_4(x)$) and the second and third zeros of $Q_5(x)$ for some choices of the masses $M_0$ and $M_1$. For $M_0 = M_1 = 0$ we obviously recover the corresponding zeros of the Freud polynomials. The first table shows the position of the aforementioned zeros for $M_0 = 0$ and several values for $M_1$. The cases when between the second and third (resp. third and fourth) zeros of $Q_5(x)$ there are no zeros of $Q_4(x)$, i.e. the zero interlacing for the sequence $\{Q_n(x)\}_{n \geq 0}$ fails, are shown in bold.

| $M_0 = 0.0$ | $\eta_{5.2}$ | $\eta_{4.2}$ | $\eta_{5.3}$ | $\eta_{4.3}$ | $\eta_{5.4}$ |
|-----------|-------------|-------------|-------------|-------------|-------------|
| $M_1 = 0.0$ | -0.655248   | -0.39615    | 0.0         | 0.39615     | 0.655248    |
| $M_1 = 0.2$ | -0.458455   | -0.39615    | 0.0         | 0.39615     | 0.458455    |
| $M_1 = 0.4$ | -0.371898   | -0.39615    | 0.0         | 0.39615     | 0.371898    |
| $M_1 = 1.0$ | -0.261023   | -0.39615    | 0.0         | 0.39615     | 0.261023    |

Table 1: Zeros of $Q_5(x)$ and $Q_4(x)$ for fixed $M_0 = 0.0$ and some values of $M_1$.

| $M_0 = 1.0$ | $\eta_{5.2}$ | $\eta_{4.2}$ | $\eta_{5.3}$ | $\eta_{4.3}$ | $\eta_{5.4}$ |
|-----------|-------------|-------------|-------------|-------------|-------------|
| $M_1 = 0.0$ | -0.655248   | -0.284325   | 0.0         | 0.284325    | 0.655248    |
| $M_1 = 0.4$ | -0.371898   | -0.284325   | 0.0         | 0.284325    | 0.371898    |
| $M_1 = 0.9$ | -0.272822   | -0.284325   | 0.0         | 0.284325    | 0.272822    |
| $M_1 = 2.0$ | -0.192081   | -0.284325   | 0.0         | 0.284325    | 0.192081    |

Table 2: Zeros of $Q_5(x)$ and $Q_4(x)$ for fixed $M_0 = 1.0$ and some values of $M_1$.

Observe that, as expected, the variation of $M_1$ only affects the position of $\eta_{5.2}$ and $\eta_{5.4}$ and the variation of $M_0$ only affects the position of $\eta_{4.2}$ and $\eta_{4.4}$. This numerical example is also reflected in Figure 1.

### 3.2 Asymptotic behavior

We are interested in the dynamics of the zeros of the Freud-Sobolev type when $M_0$ and $M_1$ tend, respectively, to infinity. To that end, let us introduce the following limit polynomials

$$G_{2n}(x) = \lim_{M_0 \to \infty} Q_{2n}(x) = F_{2n}(x) - \frac{F_{2n}(0)}{K_{2n-2}(0,0)} K_{2n-2}(x,0),$$

$$J_{2n+1}(x) = \lim_{M_1 \to \infty} Q_{2n+1}(x) = F_{2n+1}(x) - \frac{[F_{2n+1}](0)}{K_{2n-1}(0,0)} K_{2n-1}(x,0).$$

Similar polynomials have been previously studied in [20], when the discrete mass points are located outside the support of the perturbed measure. Here, we find a slightly different situation because the support of the measure is the whole real line and the discrete masses $M_0$ and $M_1$ are both located at $x = 0 \in \mathbb{R}$. As stated before, $M_0$ only affects the even degree polynomials, and
Figure 1: The figure shows, for a fixed value $M_0 = 1$, the evolution of the second zero of the Freud-Sobolev type polynomial $Q_5(x)$ for three different values of the mass $M_1$. The curve in gray color represents the Freud-Sobolev type $Q_4(x)$, which is not affected by the variation of $M_1$. The zero of $Q_5(x; M_1 = 0) = F_5(x)$ (continuous black graph) occurs at $\eta_{5,2}(M_1 = 0) = -0.655248$. For $Q_5(x; M_1 = 0.2)$ (dashed line) we have $\eta_{5,2}(M_1 = 0.2) = -0.371898$ and for $Q_5(x; M_1 = 2)$ (dotted line) occurs at $\eta_{5,2}(M_1 = 2) = -0.19208$. Notice that for $M_0 = 1$ and $M_1 = 2$ there is no zero of the polynomial $Q_4(x)$ between the second ($\eta_{5,2}(M_1 = 2) = -0.19208$) and third ($\eta_{5,3}(M_1 = 2) = 0$) roots of $Q_5(x; M_1 = 2)$, so the interlacing of the complete Freud-Sobolev type orthogonal polynomial sequence $\{Q_n(x)\}_{n \geq 0}$ has been broken.

the dynamics for the zeros of $\{Q_{2n}(x)\}_{n \geq 0}$ has been already obtained in [3]. Next, we extend those results for the odd sequence $\{Q_{2n+1}(x)\}_{n \geq 0}$.

Our goal is to obtain results concerning the monotonicity and speed of convergence of the zeros of $Q_{2n+1}(x)$. For this purpose we need the following lemma concerning the behavior and the asymptotics of the zeros of linear combinations of two polynomials with interlacing zeros, whose proof we omit (see [3, Lemma 1] or [9, Lemma 3]).

**Lemma 6** Let $f_n(x) = a(x - x_1) \cdots (x - x_n)$ and $j_n(x) = b(x - y_1) \cdots (x - y_n)$ be polynomials with real and simple zeros, where $a$ and $b$ are positive real constants.

If

$$y_1 < x_1 < \cdots < y_n < x_n,$$

then, for any real constant $c > 0$, the polynomial

$$q_n(x) = f_n(x) + cj_n(x)$$

has $n$ real zeros $\eta_1 < \cdots < \eta_n$ which interlace with the zeros of $f_n(x)$ and $j_n(x)$ as follows

$$y_1 < \eta_1 < x_1 < \cdots < y_n < \eta_n < x_n.$$

Moreover, each $\eta_k = \eta_k(c)$ is a decreasing function of $c$ and, for each $k = 1, \ldots, n$,

$$\lim_{c \to \infty} \eta_k = y_k \quad \text{and} \quad \lim_{c \to \infty} c(\eta_k - y_k) = \frac{-f_n(y_k)}{j_n'(y_k)}.$$
Before stating the main result of this Section, we will prove some auxiliary results concerning the interlacing properties of \( \{F_{2n+1}\}_{n \geq 0} \), \( \{K_{2n-1}^{(0,1)}(x,0)\}_{n \geq 0} \), and \( \{J_{2n+1}\}_{n \geq 0} \).

**Lemma 7** The zeros of \( \{K_{2n-1}^{(0,1)}(x,0)\}_{n \geq 0} \), are real and simple. Moreover, for every \( n \geq 1 \), the non vanishing zeros of \( K_{2n-1}^{(0,1)}(x,0) \) and \( K_{2n-1}^{(0,1)}(x,0) \) interlace.

**Proof.** First, since \( K_{2n-1}^{(0,1)}(x,0) \) is an odd polynomial, we can write \( K_{2n+1}^{(0,1)}(x,0) = x s_n(x^2) \), where \( s_n \) is a polynomial of degree \( n \). We will prove that \( \{s_n(y)\}_{n \geq 0} \), with \( y = x^2 \), is an orthogonal polynomial sequence with respect to the measure \( d\sigma(y) = y^{3/2} e^{-y^2} dy \), which is positive in the positive real line. Indeed, for \( n \neq m \), we have

\[
\int_0^\infty s_n(y)s_m(y)d\sigma(y) = \int_\infty^\infty K_{2n+1}^{(0,1)}(x,0) K_{2m+1}^{(0,1)}(x,0)x^3 e^{-x^4} (2xdx) = 2 \int_{-\infty}^\infty K_{2n+1}^{(0,1)}(x,0) K_{2m+1}^{(0,1)}(x,0)x^2 e^{-x^4} dx = 0,
\]

by using the reproducing property of \( K_{2n-1}^{(0,1)}(x,0) \). On the other hand, for \( n = m \), and taking into account (14) and the symmetry of the Freud polynomials, we get

\[
\int_0^\infty s_n^2(y)d\sigma(y) = \int_{-\infty}^\infty K_{2n+1}^{(0,1)}(x,0) K_{2n+1}^{(0,1)}(x,0)x^2 e^{-x^4} dx = \int_{-\infty}^\infty K_{2n+1}^{(0,1)}(x,0) x F_{2n+1}(x)[F_{2n+1}]'(0) - F_{2n+1}(x) F_{2n+2}(0) e^{-x^4} dx = \frac{1}{\|F_{2n+1}\|^2} \left( ([F_{2n-1}]'(0))^2 \|F_{2n+2}\|^2 - [F_{2n+1}]'(0)F_{2n+2}(0) \right) > 0,
\]

since \( [F_{2n+1}]'(0)F_{2n+2}(0) < 0 \). As a consequence, the zeros of \( s_n(x) \) are real, simple, and they are located in the positive real semiaxis. Moreover, the zeros of \( s_n(x) \) and \( s_{n-1}(x) \) interlace. Now, because of the symmetry, all polynomials of the sequence \( \{K_{2n-1}^{(0,1)}(x,0)\}_{n \geq 0} \) have a zero at the origin, and the remaining zeros are located symmetrically at both sides of the origin. Furthermore, if we denote by \( s_n,k \) the \( k \)th zero of \( s_n(x) \), then it is clear from the definition that \( \pm \sqrt{s_n,k} \) are zeros of \( K_{2n-1}^{(0,1)}(x,0) \). As a consequence, the (non vanishing) zeros of \( K_{2n+1}^{(0,1)}(x,0) \) and \( K_{2n-1}^{(0,1)}(x,0) \) interlace.

The next Lemma shows that the non vanishing zeros of \( F_{2n+1} \) and \( K_{2n-1}^{(0,1)}(x,0) \) also interlace.

**Lemma 8** Let \( \{x_{2n+1,k}\}_{k=1}^{2n+1} \) and \( \{z_{2n-1,k}\}_{k=1}^{2n-1} \) be the set of zeros of \( F_{2n+1} \) and \( K_{2n-1}^{(0,1)}(x,0) \), respectively, arranged in increasing order. Then, we have

\[
x_{2n+1,k} < z_{2n-1,k} < x_{2n+1,k+1}, \quad 1 \leq k \leq n - 1,
\]

\[
x_{2n+1,k+1} < z_{2n-1,k} < x_{2n+1,k+2}, \quad n + 1 \leq k \leq 2n - 1.
\]

**Proof.** Due to the symmetry of both polynomials, it suffices to prove the interlacing for the positive zeros. Since \( x_{2n+1,n+1} = z_{2n-1,n} = 0 \), we consider the case when \( n + 1 \leq k \leq 2n - 1 \). From (14) and the symmetry of the Freud polynomials, we have

\[
x^2 K_{2n-1}^{(0,1)}(x,0) = \frac{1}{\|F_{2n-1}\|^2} (xF_{2n}(x)[F_{2n-1}]'(0) - F_{2n-1}(x) F_{2n}(0)) = x F_{2n}(x)[F_{2n-1}]'(0) - \left( \frac{xF_{2n}(x) - F_{2n+1}(x)}{a_{2n}^2} \right) F_{2n}(0),
\]
where we have used \((\ref{eq1})\) on the second equality. As a consequence, evaluating the previous equation in \(x_{2n+1,k+1}\) and \(x_{2n+1,k+2}\) we obtain, respectively,

\[
x_{2n+1,k+1}^2 K_{2n-1}^{(0,1)}(x_{2n+1,k+1}, 0) = x_{2n+1,k+1} F_{2n}(x_{2n+1,k+1}) \left( [F_{2n-1}'](0) - \frac{F_{2n}(0)}{a_{2n}^2} \right),
\]

\[
x_{2n+1,k+2}^2 K_{2n-1}^{(0,1)}(x_{2n+1,k+2}, 0) = x_{2n+1,k+2} F_{2n}(x_{2n+1,k+2}) \left( [F_{2n-1}'](0) - \frac{F_{2n}(0)}{a_{2n}^2} \right).
\]

Since \(x_{2n+1,k+1}\) and \(x_{2n+1,k+2}\) are positive and the zeros of the Freud polynomials interlace, \(F_{2n}(x_{2n+1,k+1})\) and \(F_{2n}(x_{2n+1,k+2})\) have distinct sign. As a consequence, \(K_{2n-1}^{(0,1)}(x_{2n+1,k+1}, 0)\) and \(K_{2n-1}^{(0,1)}(x_{2n+1,k+2}, 0)\) differ in sign, which means that \(K_{2n-1}^{(0,1)}(x, 0)\) has a zero between the zeros \(x_{2n+1,k+1}\) and \(x_{2n+1,k+2}\). ■

\textbf{Remark 3} Notice that \(F_{2n+1}^{(0,1)}(x, 0)\) differs in two degrees. This causes that the zeros interlacing between them is not complete. Indeed, \(K_{2n-1}^{(0,1)}(x, 0)\) has not zeros in the interval \([x_{2n+1,n}, x_{2n+1,n+2}]\), i.e. between the origin and the first zeros of \(F_{2n+1}(x)\) at both sides.

We will need some results concerning the interlacing properties of the zeros of \(F_{2n+1}(x)\), \(J_{2n+1}(x)\) and \(Q_{2n+1}(x)\). By symmetry, for the zeros of \(F_{2n+1}(x)\), we have \(x_{2n+1,n+1} = 0\) and \(x_{2n+1,k} = -x_{2n+1,2n+2-k}\) for \(1 \leq k \leq n\). As a consequence, it suffices to analyze the behavior of the positive zeros. In order to simplify the notation, we denote \(x_k := x_{2n+1,n+1+k}\) for \(1 \leq k \leq n\), i.e. \(\{x_k\}_{k=1}^n\) are the \(n\) positive zeros of \(F_{2n+1}\) arranged in increasing order. A similar notation will be used for the zeros of \(Q_{2n+1}\) and \(F_{2n+1}\). The following result is a straightforward corollary of Lemma 8.

\textbf{Corollary 2} Let us denote by \(\{y_k\}_{k=1}^n\) the set of positive zeros of \(J_{2n+1}(x)\) arranged in increasing order. Then, for \(1 \leq k \leq n-1\), we have

\[
x_k < y_{k+1} < x_{k+1},
\]

i.e., positive zeros of \(J_{2n+1}(x)\) and \(F_{2n+1}(x)\) interlace.

\textbf{Proof.} Taking into account the symmetry and the fact that \([J_{2n+1}'](0) = 0\), we deduce that \(J_{2n+1}(x)\) has a zero of multiplicity 3 at the origin. This is, \(y_1 = 0\). The result follows by evaluating \((\ref{eq2})\) at two consecutive zeros \(x_k\) and \(x_{k+1}\) of \(F_{2n+1}\), for \(1 \leq k \leq n-1\), and noticing that by Lemma 8 \(J_{2n+1}(x_k)\) and \(J_{2n+1}(x_{k+1})\) have distinct sign. ■

\textbf{Remark 4} Observe that due to the triple zero at the origin, \(J_{2n+1}(x)\) does not have a zero in the interval \((0, x_1)\), i.e. between the origin and the first positive zero of \(F_{2n+1}(x)\). Since \(F_{2n+1}(x)\) only have \(n-1\) positive zeros, we have \(y_1 = 0\).

Now, we are ready to enunciate the main result of this Section.

\textbf{Theorem 2} On the positive real line, the following interlacing property holds

\[
0 = y_1 < y_1 < x_1 < y_2 < x_2 \cdots < y_k < x_k < \cdots < y_n < x_n.
\]

Moreover, each \(\eta_k := \eta_k(M_1)\) is a decreasing function of \(M_1\) and, for each \(k = 1, \ldots, n\),

\[
\lim_{M_1 \to \infty} \eta_k(M_1) = y_k,
\]
as well as
\[
\lim_{M_1 \to \infty} M_1[\eta_k(M_1) - y_k] = \frac{-F_{2n+1}(y_k)}{K_{2n-1}^{(1,1)}(0,0)[J_{2n+1}(y_k)]}.
\]  
(35)

**Proof.** Notice that the polynomials \(\{Q_{2n+1}(x)\}_{n \geq 0}\) with \(Q_{2n+1}(x) = \rho_{2n+1}Q_{2n+1}(x)\), can be represented as
\[
Q_{2n+1}(x) = F_{2n+1}(x) + M_1K_{2n-1}^{(1,1)}(0,0)J_{2n+1}(x),
\]
where
\[
\rho_{2n+1} = 1 + M_1K_{2n-1}^{(1,1)}(0,0).
\]
Thus, the interlacing follows at once from (33), (9), and Lemma 6. On the other hand, we can write
\[
xq_n(x^2) = x\hat{f}_n(x^2) + M_1K_{2n-1}^{(1,1)}(0,0)x\hat{j}_n(x^2),
\]
with
\[
\hat{f}_n = (x - x_1^2) \cdots (x - x_n^2),
\]
\[
\hat{q}_n = (x - y_1^2) \cdots (x - y_n^2),
\]
\[
\hat{j}_n = (x - \eta_1^2) \cdots (x - \eta_n^2),
\]
and by the previous results, their zeros are real, simple and interlace, so they satisfy the conditions on Lemma 6 and therefore
\[
\lim_{M_1 \to \infty} \eta_k^2 = y_k^2,
\]
and
\[
\lim_{M_1 \to \infty} = M_1K_{2n-1}^{(1,1)}(0,0)[y_k^2 - \eta_k^2] = \frac{\hat{f}_n(y_k^2)}{\hat{j}_n(y_k^2)} = \frac{2y_kF_{2n+1}(y_k)}{[J_{2n+1}]'(y_k)},
\]
and since \(\eta_k^2 - y_k^2 = (\eta_k + y_k)(\eta_k - y_k)\) and \(\lim_{M_1 \to \infty} \eta_k = y_k\), the result follows. \(\blacksquare\)

**Remark 5** Because of the symmetry, the limits (34) and (35) also hold for the negative zeros. The only difference is that those zeros are increasing functions of \(M_1\).

## 4 Holonomic equation and electrostatic interpretation

In this section, we deduce a second order differential equation satisfied by \(\{Q_n(x)\}_{n \geq 0}\) and, as an application, an electrostatic interpretation of its zeros is presented. We will use the connection formula between \(Q_n\) and \(F_n\), which for convenience will take the form (27). We will also use the structure formula (9) (for the monic normalization) and the three term recurrence relation (7). Let us rewrite these formulas as
\[
Q_n(x) = A_n(x)F_n(x) + B_n(x)F_{n-1}(x),
\]
(36)
\[
F_n(x) = a_n(x)F_n(x) + b_n(x)F_{n-1}(x),
\]
(37)
\[
F_{n+1}(x) = \beta_n(x)F_n(x) + \gamma_n(x)F_{n-1}(x),
\]
(38)
where the coefficients above are given according to (27), (9) and (7), respectively. Before stating our main result, we need the following Lemmas.
Figure 2: It illustrates the variation of the zeros of an odd degree Freud-Sobolev type polynomials when \( M_1 \) varies as described in Theorem \( \text{[2]} \). The graphs of \( Q_7(x) \) for three different values of \( M_1 \) are plotted. The black continuous, dashed, and dotted lines correspond to \( M_1 = 0.03, \ M_1 = 0.05, \) and \( M_1 = 0.09, \) respectively. We also include the graphs of \( F_7(x) \) (medium gray color) and \( J_7(x) \) (light gray color), showing that the zeros of \( Q_7(x) \) are increasing functions of \( M_1 \) in the negative real semiaxis, traveling from the negative zeros of \( F_7(x) \) to the corresponding zeros of \( J_7(x) \) as \( M_1 \) increases. Likewise, the positive zeros of \( Q_7(x) \) are decreasing functions of \( M_1 \), traveling from the positive zero of \( F_7(x) \) to the corresponding zero of \( J_7(x) \) according with Theorem \( \text{[2]} \). Observe that in this picture, the value of \( M_0 \) is irrelevant.

**Lemma 9** The monic sequences \( \{Q_n(x)\}_{n \geq 0} \) and \( \{F_n(x)\}_{n \geq 0} \) satisfy

\[
[Q_n(x)]' = C_1(x; n)F_n(x) + D_1(x; n)F_{n-1}(x) \tag{39}
\]

where

\[
C_1(x; n) = A_n'(x) + A_n(x)a(z; n) + B_n(x)\frac{b_{n-1}(x)}{\gamma_{n-1}(x)}, \tag{40}
\]

\[
D_1(x; n) = B_n'(x) + A_n(x)b_n(x) + B_n(x)\left(a_{n-1}(x) - \frac{\beta_{n-1}(x)}{\gamma_{n-1}(x)}\right).
\]

**Proof.** Notice that, combining (38) and (39) we have

\[
[F_{n-1}(x)]' = \frac{b_{n-1}(x)}{\gamma_{n-1}(x)}F_n(x) + \left(a_{n-1}(x) - \frac{\beta_{n-1}(x)}{\gamma_{n-1}(x)}\right)F_{n-1}(x).
\]

The result follows by substituting the last equation and (37) into the derivative with respect to \( x \) of (36). \( \blacksquare \)
Lemma 10 The sequences of monic polynomials \( \{Q_n(x)\}_{n \geq 0} \) and \( \{F_n(x)\}_{n \geq 0} \) are also related by

\[
Q_{n-1}(x) = A_2(x; n)F_n(x) + B_2(x; n)F_{n-1}(x),
\]

\[
[Q_{n-1}(x)]' = C_2(x; n)F_n(x) + D_2(x; n)F_{n-1}(x),
\]

where

\[
A_2(x; n) = \frac{B_{n-1}(x)}{\gamma_{n-1}(x)}, \quad B_2(x; n) = A_{n-1}(x) - B_{n-1}(x)\frac{\beta_{n-1}(x)}{\gamma_{n-1}(x)},
\]

\[
C_2(x; n) = \frac{D_1(x; n - 1)}{\gamma_{n-1}(x)}, \quad D_2(x; n) = C_1(x; n - 1) - D_1(x; n - 1)\frac{\beta_{n-1}(x)}{\gamma_{n-1}(x)}.
\]

The coefficients \( C_1(x; n - 1) \) and \( D_1(x; n - 1) \) are given in (40).

**Proof.** The expressions follow from (38) and (39), respectively, after a shift in the degree, and using (38) to express both of them in terms of \( F_n \) and \( F_{n-1} \). \( \blacksquare \)

Lemma 11 The following “inverse connection” formulas hold.

\[
F_n(x) = \frac{B_2(x; n)\Lambda(x; n)}{\Lambda(x; n)} Q_n(z) - \frac{B_2(x; n)}{\Lambda(x; n)} Q_{n-1}(x),
\]

\[
F_{n-1}(z) = -\frac{A_2(x; n)\Lambda(x; n)}{\Lambda(x; n)} Q_n(x) + \frac{A_2(x; n)}{\Lambda(x; n)} Q_{n-1}(x),
\]

where

\[
\Lambda(x; n) = A_n(x)B_2(x; n) - A_2(x; n)B_n(x).
\]

**Proof.** The result follows by solving the linear system defined by (36) and (41). \( \blacksquare \)

Now, we replace (43) and (44) in (39) and (42), respectively, to obtain the ladder equations

\[
[Q_n(z)]' = \left[ \frac{C_1(x; n)B_2(x; n)}{\Lambda(x; n)} - \frac{D_1(x; n)A_2(x; n)}{\Lambda(x; n)} \right] Q_n(x)
\]

\[
+ \left[ \frac{A_n(x)D_1(x; n)}{\Lambda(x; n)} - \frac{C_1(x; n)B_n(x)}{\Lambda(x; n)} \right] Q_{n-1}(x),
\]

\[
[Q_{n-1}(x)]' = \left[ \frac{C_2(x; n)B_2(x; n)}{\Lambda(x; n)} - \frac{A_2(x; n)D_2(x; n)}{\Lambda(x; n)} \right] Q_n(x)
\]

\[
+ \left[ \frac{A_n(x)D_2(x; n)}{\Lambda(x; n)} - \frac{C_2(x; n)B_n(x)}{\Lambda(x; n)} \right] Q_{n-1}(x),
\]

which can be written in the more compact way

\[
(\Xi(x; n, 2)I - D_x)Q_n(x) = \Xi(x; n, 1)Q_{n-1}(x),
\]

\[
(\Theta(x; n, 1)I + D_x)Q_{n-1}(x) = \Theta(x; n, 2)Q_n(x),
\]

where \( I \) and \( D_x \) are the identity and \( x \)-derivative operators, respectively, by defining the determinants

\[
\Xi(x; n, k) = \frac{1}{\Lambda(x; n)} \begin{vmatrix}
C_1(x; n) & A_k(x; n) \\
D_1(x; n) & B_k(x; n)
\end{vmatrix},
\]

\[
\Theta(x; n, k) = \frac{1}{\Lambda(x; n)} \begin{vmatrix}
C_2(x; n) & A_k(x; n) \\
D_2(x; n) & B_k(x; n)
\end{vmatrix},
\]

for \( k = 1, 2 \), where \( A_1(x; n) := A_n(x) \). As a consequence, we have the following result.
4  HOLONOMIC EQUATION AND ELECTROSTATIC INTERPRETATION

**Theorem 3** Let $b_n$ and $b^\dagger_n$ be the differential operators

$$
\begin{align*}
   b_n &= \Xi(x; n, 1)I - D_x, \\
   b^\dagger_n &= \Theta(x; n, 2)I + D_x.
\end{align*}
$$

Then,

$$
\begin{align*}
   b_n[Q_n(x)] &= \Xi(x; n, 1)Q_{n-1}(x), \\
   b^\dagger_n[Q_{n-1}(x)] &= \Theta(x; n, 2)Q_n(x),
\end{align*}
$$

where $\Xi(x; n, k)$ and $\Theta(x; n, k)$ are given in (47) and (48), respectively.

Finally, we state the main result of this section.

**Theorem 4** The Sobolev-Freud type polynomials $\{Q_n(x)\}_{n \geq 0}$ satisfy the second order linear differential equation

$$
[Q_n(x)]'' + R(x; n)[Q_n(x)]' + S(x; n)Q_n(x) = 0,
$$

(49)

where

$$
\begin{align*}
   R(x; n) &= \Theta(x; n, 1) - \Xi(x; n, 2) - \frac{[\Xi(x; n, 1)]'}{\Xi(x; n, 1)}, \\
   S(x; n) &= \Xi(x; n, 2) \left( \frac{[\Xi(x; n, 1)]'}{\Xi(x; n, 1)} - \Theta(x; n, 1) \right) - [\Xi(x; n, 2)]'.
\end{align*}
$$

**Proof.** The result follows in a straightforward way from the ladder operators provided in Theorem 3. The usual technique (see, for example [14, Th. 3.2.3]) consists in applying the raising operator to both sides of the equation satisfied by the lowering operator, i.e.

$$
\begin{align*}
   b^\dagger_n \left[ \frac{1}{\Xi(x; n, 1)} b_n[Q_n(x)] \right] &= b^\dagger_n[Q_{n-1}(x)] = \Theta(x; n, 2)Q_n(x),
\end{align*}
$$

and then using the definition $b^\dagger_n$ to compute the left hand side. After some computations, (49) follows.

We point out that we have obtained a second order linear differential equation for the complete sequence $\{Q_n(x)\}_{n \geq 0}$. However, as we have mentioned in the previous sections, the even and odd degree polynomials behave differently. Indeed, they have another connection formula, and the previous results hold in either case just by taking the coefficients of the connection formula (36) accordingly. Using Mathematica®, the expression for $R(x; n)$ was obtained according to Theorem 3. In the sequel, we provide the expressions for the odd case ($\kappa_{2n}^{[0]} = 0, \kappa_{2n}^{[1]} = 0, r_{2n+1} = 1, r_{2n} = 0$), together with an electrostatic interpretation of the zeros of $\{Q_n(x)\}_{n \geq 0}$. The even case was analyzed in [3]. We found

$$
R(x; 2n + 1) = \frac{2}{x} - 4x^3 - \frac{u'(x; 2n + 1)}{u(x; 2n + 1)},
$$

where $u(x; 2n + 1)$ is the biquartic polynomial

$$
u(x; 2n + 1) = u_4(n) x^4 + u_2(n) x^2 + u_0(n)$$

(50)
with
\[ u_4(n) = 16\phi_{2n+1}^2(0)[1 + \kappa_{2n+1}^2], \]
\[ u_2(n) = 4\phi_{2n+1}^2(0)\left[4\phi_{2n+1}^2(0) + \kappa_{2n+1}^2 + \kappa_{2n+1}^2 + (4a_{2n+1}^2 \phi_{2n+1}(0) - 1)\right], \]
\[ u_0(n) = \kappa_{2n+1}^2 - 12\phi_{2n+1}^2(0) + \kappa_{2n+1}^2 \left[1 + 8a_{2n+1}^2 \phi_{2n+1}(0) \left[-1 + 2\phi_{2n}(0)\phi_{2n+1}(0)\right]\right]. \]

Now, the evaluation of (19) at the zeros \(\{y_{2n+1,k}\}_{k=1}^{2n+1}\) of \(Q_{2n+1}\) yields
\[
\frac{[Q_{2n+1}']^2(y_{2n+1,k})}{[Q_{2n+1}]^2(y_{2n+1,k})} = -R(y_{2n+1,k}; 2n + 1) = -\frac{2}{y_{2n+1,k}} + 4(y_{2n+1,k})^3 + \frac{u'(y_{2n+1,k}; 2n + 1)}{u(y_{2n+1,k}; 2n + 1)}. \]

The above equation represents the electrostatic equilibrium condition for the zeros \(\{y_{2n+1,k}\}_{k=1}^{2n+1}\) of \(Q_{2n+1}\) and can be rewritten as (see [14])
\[
\sum_{j=1, j \neq k}^{2n+1} \frac{1}{y_{2n+1,j} - y_{2n+1,k}} + \frac{u'(y_{2n+1,k}; 2n + 1)}{2u(y_{2n+1,k}; 2n + 1)} - \frac{1}{y_{2n+1,k}} + 2(y_{2n+1,k})^3 = 0. \]

Therefore, the zeros of \(Q_{2n+1}\) are critical points of the total energy. Thus, the electrostatic interpretation of the distribution of zeros means that we have an equilibrium position under the action of the external potential
\[ V_{ext}(x, 2n + 1) = \frac{1}{2} \ln u(x; 2n + 1) - \frac{1}{2} \ln x^2 e^{-x^2}, \]
where the first term represents a short range potential which corresponds to unit charges located at the four zeros of \(u(x; 2n + 1)\), and the second term is a long range potential associated with a Christoffel perturbation of the Freud weight function.

If \(z_+(n)\) and \(z_-(n)\) are the solutions of the associated quadratic equation
\[ u_4(n)z^2 + u_2(n)z + u_0(n) = 0, \]
then the zeros of (50) are
\[ x_1(n) = +\sqrt{z_+(n)}, \quad x_2(n) = -\sqrt{z_-(n)}, \quad x_3(n) = +\sqrt{z_-(n)}, \quad x_4(n) = -\sqrt{z_+(n)}. \]

Table 3 shows the zeros of \(u(x; 2n + 1)\) for some fixed values of \(M_1\) and several values of \(n\). With just a little more effort, we can describe the asymptotic behavior with \(n\) of these four roots.

From Lemma 2 and 3, after some tedious but straightforward computations, the asymptotic behavior of the three coefficients is
\[ u_4(n) = \frac{32}{3}n\left(1 + \frac{15}{8n} + \frac{155}{128n^2} + O(n^{-3})\right), \]
\[ u_2(n) = 8\sqrt{6}n^{1/2}\left(1 + \frac{4n}{9} + \frac{287}{432n} + O(n^{-2})\right), \]
\[ u_0(n) = -\frac{9}{2}\left(1 + \frac{9}{8n} + O(n^{-2})\right). \]

Then, the asymptotic behavior of the aforementioned \(z_+\) and \(z_-\) is
\[ z_+(n) = \frac{27}{64}\sqrt{\frac{3}{2}n^{3/2}} - \frac{243}{512}\sqrt{\frac{3}{2}n^{5/2}} + O(n^{-7/2}), \]
\[ z_-(n) = -\sqrt{\frac{3}{2}n^{1/2}} - \frac{1}{4}\sqrt{\frac{3}{2}n^{5/2}} + O(n^{-7/2}). \]
The above shows that, as $n$ goes to infinity, $z_{\pm}(n)$ remains positive and $z_{-}(n)$ negative, so $u(x; 2n + 1)$ will always have two symmetric real zeros $x_1, x_2 = \pm \sqrt{z_{\pm}(n)}$, and two extra simple conjugate pure imaginary zeros $x_3, x_4 = \pm \sqrt{z_{-}(n)}$.  

Table 3: Zeros of $u(x; 2n + 1)$ for several values of $M_1$ and odd values of $n$.  

| $n$ | $M = 0.1$ | $M = 1$ | $M = 10$ |
|-----|-----------|----------|----------|
|     | $\pm \sqrt{z_{1}}$ | $\pm \sqrt{z_{2}}$ | $\pm \sqrt{z_{1}}$ | $\pm \sqrt{z_{2}}$ |
| 1   | $\pm 0.369164 \pm 0.878731 i$ | $\pm 0.745497 \pm 0.914759 i$ | $\pm 0.905303 \pm 0.928589 i$ |
| 3   | $\pm 0.397067 \pm 1.059517 i$ | $\pm 0.387740 \pm 1.089036 i$ | $\pm 0.159258 \pm 1.106825 i$ |
| 5   | $\pm 0.329766 \pm 1.181451 i$ | $\pm 0.197206 \pm 1.197172 i$ | $\pm 0.068685 \pm 1.201241 i$ |
| 7   | $\pm 0.251172 \pm 1.272375 i$ | $\pm 0.116257 \pm 1.279623 i$ | $\pm 0.038576 \pm 1.280856 i$ |
| 9   | $\pm 0.189032 \pm 1.345977 i$ | $\pm 0.076318 \pm 1.349456 i$ | $\pm 0.024825 \pm 1.349937 i$ |
| 11  | $\pm 0.144418 \pm 1.408813 i$ | $\pm 0.053943 \pm 1.410616 i$ | $\pm 0.017374 \pm 1.410839 i$ |
| 13  | $\pm 0.112816 \pm 1.464184 i$ | $\pm 0.040222 \pm 1.465192 i$ | $\pm 0.012969 \pm 1.465308 i$ |
| 15  | $\pm 0.089745 \pm 1.513969 i$ | $\pm 0.029902 \pm 1.514571 i$ | $\pm 0.004691 \pm 1.514637 i$ |
| 17  | $\pm 0.073204 \pm 1.559345 i$ | $\pm 0.024134 \pm 1.559723 i$ | $\pm 0.005169 \pm 1.559764 i$ |
| 19  | $\pm 0.060787 \pm 1.601140 i$ | $\pm 0.019950 \pm 1.601389 i$ | $\pm 0.005144 \pm 1.601416 i$ |

Figure 3: Zeros of $u(x; 2n + 1)$ for $M_1 = 1$ and odd values of $n$, from 1 to 19.
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