Second Order Langevin Equation
and Stochastic Quantisation for Gravity

Laurent Baulieu† and Siye Wu†† *

† LPTHE, Sorbonne Université, CNRS
   4 Place Jussieu, 75005 Paris, France
†† Department of Mathematics, National Tsing Hua University
   30013 Hsinchu, Taiwan

July 2018

Abstract

Euclidean quantum gravity might be defined by a stochastic quantisation that is governed by a higher order Langevin equation rather than by a first order stochastic equation, giving a transitory phase where the Minkowski time cannot be defined, so the parameter that orders the evolution of quantum gravity phenomena is the stochastic time. This may enlarge the definition of causality in the period of primordial cosmology. Here we examine the meaning of a second order Langevin equation in zero dimensions to define precisely what is second order stochastic quantisation in a soluble case.
1 Introduction

In has been proposed in [1] that quantum gravity might obey the laws of a stochastic quantisation governed by a second order rather than by a first order stochastic equation or, perhaps equivalently, by a possibly higher order Fokker-Planck equation and its supersymmetric representation.

This modification of the quantisation changes the smooth exponentially damped stochastic time relaxation phenomenon toward the quantum physics equilibrium that often occurs when the equation is purely first order. Using a second order Langevin equation rather than a first order one changes the dynamics of the stochastic time evolution: physically relevant fluctuations of the vacuum may occur at finite stochastic time in correlation with quantum effects consisting of an abundance of creations and annihilations of the quantised states of the systems [1]. Standard first order stochastic quantisation is actually a dead end in the case of gravity because it has no equilibrium distribution and its evolution is ill-defined. With higher order stochastic quantisation, new phenomena may occur. Heuristically, the second order form of the Langevin equation implies the creation of pairs of microscopic black holes, with a possible phase transition that may explain the exit from the of primordial cosmology phase toward the phase where the universe is so diluted that it can be described by the standard quantum gauge field theories in a classical gravitational background. Namely, the phase transition is marked by the exit of the inflation. In the proposal of using a second order Langevin equation, the drift force for the metric is the sum of the Einstein tensor and the energy momentum tensor of the matter, plus a possible contribution proportional to the cosmological constant, as it would be in the tentative standard first order Langevin equation. It is a logical possibility that the Wick rotation of one of the Euclidean coordinates in the correlators that stochastic quantisation computes for finite values of the stochastic time is impossible during the phase of primordial cosmology. In the absence of a Minkowski time, the only possible parameter that truly describes the evolution of processes is the stochastic time, giving a broader sense of causality (whose usual description is recovered after the exit of inflation when the Minkowski time can emerge and can be used as a phenomenological ordering parameter).

In the scenario of [1], the interpretation of the Euclidean correlators at a finite stochastic time is precisely what defines quantum gravity until, one has a sudden relaxation to classical gravity by a phase transition, identified as the exit from inflation.

It is an interesting fact that at a finite stochastic time, the correlation functions depend on their initial data, so the primordial physics is dependent on the new scales introduced by an initial Euclidean geometry. The meaning of the phase transition is that the standard relaxation toward the standard path integral formulation can be obtained. Then, the dependence on the initial data in the earlier phase is washed away in any given experiment using standard clocks. In this perspective, the Minkowski time can only emerge, and in fact emerges, after the exit from the inflation, as the consequence of the phase transition from quantum to classical gravity. This is indeed the physical description of [1], using a strong enough fluctuation for the system relaxes toward standard quantisation for long wave-length processes involving of all interactions but quantum gravity.

Independent of the goal of getting a possible definition of quantum gravity, it was suggested in [1] that, by using the idea of stochastic time with a second order time evolution, a \(d\)-dimensional Euclidean QFT can be defined as the limit of a \((d+1)\)-dimensional QFT with a double Fock space structure. The latter contains the usual Fock space for the particle degrees of freedom and another Fock space for the accessible levels of the vacuum, which is a good theoretical framework for the quantum gravity scenario in [1]. A higher order Langevin equation could also reveal some properties of simpler quantum mechanical models, such as the one with a conformal potential \(\sim 1/x^2\), a system which never reaches an equilibrium but has interesting properties at a finite stochastic time.

This \((d+1)\)-dimensional QFT is Euclidean in all coordinates but the stochastic time, and it involves
effectively a physical UV cutoff $1/\Delta T$, as a parameter that occurs naturally from dimensionality considerations in a second order Langevin equation.

Here we examine the precise meaning of a second order Langevin equation, in zero dimensions, to help understand concretely the mechanism that was proposed for defining gravity in its quantum phase.

2 Standard quantisation and first order stochastic equations

Let us consider a scalar field $\Phi(X)$ in $d$-dimensional Euclidean space with coordinates $X^\mu$. Stochastic quantisation involves an additional “stochastic time” variable $\tau$, in the sense that one promotes the field $\Phi(X)$ depending on $X$ to $\Phi(X, \tau)$ depending on both $X$ and a stochastic time $\tau$, and one defines the $\tau$-dependent correlations functions $\langle \langle \Phi(X_1, \tau_1) \cdots \Phi(X_n, \tau_n) \rangle \rangle$, with

$$\lim_{\tau \to \infty} \langle \langle \Phi(X_1, \tau_1) \cdots \Phi(X_n, \tau_n) \rangle \rangle \bigg|_{\tau_1 = \cdots = \tau_n = \tau} \equiv \langle \Phi(X_1) \cdots \Phi(X_n) \rangle. \quad (1)$$

If the limit exists, it must equate the Euclidean correlator of the $d$-dimensional Euclidean QFT obtained by the standard formalism, either canonical quantisation or the path integral.

In zero dimensions, $\Phi(X)$ reduces to a point $x$ and $\Phi(X, \tau)$ is simply a real function $x(\tau)$. So, in zero dimensions, the concept of stochastic quantisation can be handled with standard knowledge in analysis and distribution theory. The extension to QFT is a limit where the number of points becomes infinite, but the general features of the zero dimensional case must remain the same.

The proposal of [2] was that the stochastic time dependence is through a (first order) Langevin equation, where the drift force is the Euclidean equation of motion. Given a local action $S[\Phi]$ with the relevant properties, the smoothness of quantum processes, if any, comes from the property that these correlation functions are the mean values computed with a Fokker-Planck evolution kernel $P_{FP}(\Phi(X, \tau))$, which satisfies the following parabolic equation

$$\frac{\partial}{\partial \tau} P_{FP}[\Phi, \tau] = \int dX \frac{\delta}{\delta \Phi(X)} \left( \frac{\delta S}{\delta \Phi(X)} + \frac{\delta}{\delta \Phi(X)} \right) P_{FP}(\Phi(X, \tau)), \quad (2)$$

and

$$\langle \langle \Phi(X_1, \tau) \cdots \Phi(X_n, \tau) \rangle \rangle \equiv \int [d\Phi]_X \Phi(X_1) \cdots \Phi(X_n) P_{FP}(\Phi(X, \tau)). \quad (3)$$

One can choose any given initial condition $P_0(\Phi(X))$ at $\tau = \tau_0$ for $P_{FP}$, and one can prove that the limit, if it exists, is independent of $P_0(\Phi, \tau_0)$, with an exponential damping in powers of $\exp(-\tau)$ when $\tau \to \infty$.

The connection with Euclidean Feynman standard path integral is by proving that the equilibrium distribution of the Fokker-Planck distribution is

$$\lim_{\tau \to \infty} P_{FP}[\Phi, \tau] = \exp(-S[\Phi]). \quad (4)$$

All relevant QFTs for elementary particles that are not coupled to quantum gravity can be defined by such a Fokker-Planck equation. Eventually, one gets a probabilistic interpretation of all their Euclidean correlation functions. Moreover, if the Minkowski time can be defined by an analytic continuation, either at a finite stochastic time or simply when $\tau \to \infty$, that is, a consistent Wick rotation of one of the coordinates $X^\mu$, all scattering and disintegration effects that can be computed in Minkowski quantum field theory also result from the relaxation of stochastic time dependent process. Gauge invariance can be elegantly handled in the stochastic framework using the framework of TQFT [3]. Of course some actions are such that the limit (4) cannot be reached or makes no sense, as it is the case.
of gravity, because its Euclidean action is not definite positive. Here we suggest that this is not in contradiction with the possibility that its correlators are well defined at finite values of the stochastic time, with some oscillatory dependence, whose physical interpretation is in fact relevant.

A first order Langevin equation often implies a Fokker-Planck equation. The Langevin framework is more general in the sense that it allows one to define correlators at different values of the \( \tau \). It leads one to a supersymmetric formulation in \( d + 1 \) dimensions, whose Hamiltonian interpretation includes the Fokker-Planck formulation [5] for many well-behaved theories.

In the Langevin formulation, the stochastic time dependence of the correlation functions of fields is determined from the equation*

\[
\frac{\partial \Phi(X, \tau)}{\partial \tau} = \frac{\delta S[\Phi]}{\delta \Phi(X, \tau)} + \sqrt{\hbar} \eta(X, \tau),
\]

where \( \eta(X, \tau) \) is a noise whose correlation functions define the quantisation, together with some initial condition of \( \Phi \) at \( \tau = \tau_0 \), say \( \Phi(X, \tau_0) = \Phi_0(X) \).

The Langevin equation is very much like a Brownian motion equation where \( \frac{\delta S[\Phi]}{\delta \Phi(X, \tau)} \) is a drift force. In the simplest formulation the noise is Gaussian, which means

\[
\langle \langle \eta(X, \tau)\eta(X', \tau') \rangle \rangle = 2\delta(\tau - \tau')\delta(X - X').
\]

Rotational invariance is generally not ensured in \( d + 1 \) dimensions (it can be enforced e.g. for certain Chern-Simons actions [3]). The stochastic time dimension is the square of the dimension of space coordinates if the free part of the action is second order in the space derivatives.

The standard prescription to get rid of the \( \tau \) dependence and define the Euclidean correlation functions of the \( d \)-dimensional smooth quantum field theory is by computing the correlation functions of the \( (d + 1) \)-dimensional theory at equal \( \tau \) and then taking the limit \( \tau \to \infty \), assuming for example that \( \Phi = 0 \) at \( \tau = 0 \). That is, we have

\[
\langle \Phi(X_1) \cdots \Phi(X_n) \rangle_{\text{Euclidean}} \equiv \lim_{\tau \to \infty} \langle \langle \Phi(X_1, \tau) \cdots \Phi(X_n, \tau) \rangle \rangle.
\]

One can often prove the convergence of the stochastic process toward the same limit for arbitrary initial conditions at \( \tau = \tau_0 \), whatever the value of \( \tau_0 \) is. Basically, one has to invert the Langevin equation and express \( \Phi_\tau \) in function of the noise \( \eta_\tau \), and then one averages functions \( f(\Phi_\tau) \) assuming a Gaussian noise. One can then show that the Langevin equation formulation gives the same result as the Fokker-Planck formulation for \( \lim_{\tau \to \infty} \langle \langle f(\Phi_\tau) \rangle \rangle \).

In the Fokker-Planck formalism, it is rather easy to prove that the damping of the Langevin/Fokker-Planck process toward the usual path integral formula amounts to the possibility of normalising the Euclidean path integral “vacuum functional”

\[
\int [d\Phi]_X \exp \left( -\frac{1}{2\hbar}S[\Phi] \right) < \infty.
\]

The necessity of this condition is a general result of statistical physics that can be proven in many ways.

This condition is fulfilled for all renormalisable theories with a meaningful vacuum, such as 4d Yang-Mills theory, 3d Chern-Simons action, and many quantum mechanical models with a discrete spectrum and a normalisable vacuum. But it is not fulfilled in the case of 4d gravity, since in this case the field is \( g_{\mu\nu} \) and \( S[\Phi] \) amounts to the Euclidean Hilbert-Einstein action that is not positive definite,

*There is the possibility of inserting a kernel in factor of \( \frac{\delta S[\Phi]}{\delta \Phi(X, \tau)} \) to improve the convergence, if any, of the stochastic process, without changing the conclusion of the foregoing discussion.
so that (8) cannot be defined, which is consistent with the fact that time cannot be globally well defined in quantum gravity. Nor is it fulfilled in some apparently much simpler cases, as for instance the one of conformal mechanics with a potential is $1/x^2 = -\partial_x \log(|x|)$, in which case $S(x) = \frac{1}{4} \log(|x|)$, and $\int_0^\infty dx \exp(-S(x))$ is divergent. There are also cases of delta-function potentials in 2- and 3-dimensional quantum mechanics. For such theories, the standard recipes of quantisation are ineffective, except if one uses a regularisation that breaks their symmetry and destroys their foundation. For gravity one cannot even think of such a regularisation. The whole process seems to make sense only if the limiting standard QFT is well defined, but, in the case of gravity we cannot exclude the possibility that the Euclidean correlation functions can be well defined at finite values of the stochastic time. Whatever they are, they may describe some complicated Euclidean physics with no possibility of defining a Minkowski time, but there is the possibility of a phase transition at a finite value of $\tau$, such that the gravity becomes classical. Then, by definition the stochastic process becomes smooth since the standard stochastic quantisation is compatible with classical quantum field theory, including classical gravity [1]. The physical difficulties brought by the Wheeler-DeWitt equation become in fact irrelevant.

When the convergence at $\tau \to \infty$ is ensured, one can often make a Wick rotation, that is an analytic continuation of all correlators in one of the Euclidean coordinates $X^\mu$, when $\tau \to \infty$. Then all necessary Minkowskian correlators can be used to compute perturbatively $S$-matrix elements, giving a particle interpretation to the theory with well-defined scattering amplitudes and clocks.

The question of the possibility of a Wick rotation at finite stochastic time has not been extensively studied in the literature. A recent paper has studied the question of establishing Hilbert-space positivity as a precise mathematical result at finite time [4].

The equivalence proofs between standard path integral quantisation and stochastic quantisation with a first order Langevin/Fokker-Planck amount to show that there is a smooth damped relaxation of the stochastic process toward the Euclidean correlation functions defined by the path integral with “Boltzmann weight” $\exp(-S)$. It can only be done if the equilibrium process is itself well defined.

Quite generally, when the effect of the noise is less stringent, the solution is that the fields concentrate around the classical solutions, until they reach the classical limit. In fact, we call $\tau$-dependent classical states solutions to the Langevin equations when the noise is neglected all along the $\tau$ evolution. This operation can be heuristically understood by taking the limit $\hbar \to 0$ in the Langevin equation for certain fields. In fact, stochastic coherent states can be also defined as solutions that depend on the noise and are as near as possible to stochastic classical fields when one computes correlators in a natural generalisation of the Schrödinger picture.

We now come to the generalisation of a first order stochastic evolution into a second order one.

3 Stochastic quantisation with second order Langevin equations

We now generalise the first order equation (5) into

$$a^2 \frac{\partial^2 \Phi(X,\tau)}{\partial^2 \tau} + 2b \frac{\partial \Phi(X,\tau)}{\partial \tau} = \frac{\delta S[\Phi]}{\delta \Phi(X,\tau)} + \sqrt{\hbar} \eta(X,\tau).$$

In other words, we will analyse the new physical features of the dynamics of stochastic quantisation after $\partial_\tau \equiv \partial/\partial \tau$ is replaced by $a^2 \partial^2 + 2b \partial_\tau$. Here $b$ is a dimensionless positive real number, $a$ is a positive real number with the same dimension as $\sqrt{\tau}$ and can be perhaps used as a physical ultra-violet cutoff, $a \sim 1/\Delta T$, for certain theories. For $a \to 0$, the standard stochastic quantisation occurs, and it is a consistent formulation provided we have a renormalisable QFT with an appropriate potential. If $b = 0$, we have oscillatory solutions that are unlikely to becomes stationary for $\tau \to \infty$. So we
must always have a friction term, proportionally to $b$. A deeper understanding of the dissipation that occurs for $b \neq 0$ is of great interest.

For $a \neq 0$, as discussed in [1], the standard first order equation is effectively recovered when the action of the operator $a^2 \partial^2_\tau$ is much smaller than the action of $2b \partial_\tau$, at least perturbatively, if one has a renormalisable theory for $S$. The combination of the effect of the noise and of the acceleration term proportional to $a^2$ can lead the system to nontrivial relaxations toward a possible equilibrium, like oscillating ones that cannot be predicted by a genuine first order equation. This justifies heuristically the intuition that $1/a$ is a physical UV cutoff, by comparing the time scale of stochastic time oscillations with those time scales that occur in standard quantisation when $\tau \to \infty$.

In the example of [1], we considered a 6-dimensional scalar (one real and one complex field) renormalisable theory with interacting terms $g \Phi^3 + e \Phi \bar{\Psi}$. (This was to avoid the more confusing case of fields with genuine self-interactions.) In this theory the real field $\Phi$ can be be treated as a background coherent state $\Phi_{cl}$ and $\Psi$ is a complex quantum field that couples with the background $\Phi_{cl}$ by its a current $e \bar{\Psi} \Psi$, similar to a background electromagnetic field coupling to an electron current, or a classical background gravitational field coupling to a microscopic quantum black hole pair.

Eq. (9) is translation invariant, and, despite of the friction term proportional to $b$, the conservation of the stochastic time energy must be somehow ensured: something must balance the energy variations that occur for the $\tau$-oscillations in function of the vacuum $\Phi_{cl}$, computed by solving the Langevin equation for $\eta_\phi = 0$. The scenario is that it must be the quantum excitations for the field $\Psi$ during this $\tau$-dependent process, the later being pair creations of the Schwinger type. In fact, the pair creations or absorptions can be seen as a back reaction for the oscillation in stochastic time of the vacuum. The whole process is driven by the effect of the noise. The stronger $\Phi_{cl}$ is, the stronger is the counter effects of pair annihilation and creation for the field $\Psi$. In principle, this can be perturbatively computed by solving the stochastic equation including the effect of the noise $\eta_\Psi$ in the background field $\Phi_{cl}$.

This phase of strong vacuum oscillations in the stochastic time with creations and absorptions of pairs can persist until one relaxes an equilibrium, if any, by the damping of the periodic growth and contractions of the vacuum energy $\Phi_{cl}$.

It was argued in [1] that this early dynamics might even keep away the field excitations from dangerous part of the potential for a while. There is a time where the damping is very slow as compared to the rapidity of the oscillations. The illustration of this heuristic description is schematised in Figure 1.

The idea of [1] is that in the case of gravity, this equilibrium can only be the classical gravity theory (coupled ordinary standard particle QFT’s), due to a given fluctuation that suddenly increase the volume of the space, with a brutal decrease of the cosmological constant, so that there is no more room for quantum gravity effects, and Minkowski time can be then defined. In this way one avoids the usual paradox that the Minkowski time cannot be defined when quantum gravity prevails. The system has two phases, one without Minkowski time and other one that has an “emerging” Minkowski time, but both phases rely on the same microscopic theory. This may enlarge the scope of causality.

In fact, after the space metric $g_{\mu \nu}(X)$ is promoted to $g_{\mu \nu}(X, \tau)$, modulo the introduction of additional components $g_{\mu \tau}(X, \tau)$ and $g_{\tau \tau}(X, \tau)$, we obtain a covariant derivative $\nabla_\tau = \nabla / \partial_\tau$, and the gravity stochastic equation can be covariantly written using the method of equivariant cohomology (for a 2d example and the YM$_4$ case, see the last reference in [3]). That is,

$$a^2 \nabla^2_\tau g_{\mu \nu}(X, \tau) + b \nabla_\tau g_{\mu \nu}(X, \tau) = R_{\mu \nu}(X, \tau) - \frac{1}{2} g_{\mu \nu}(X, \tau) R + \kappa g_{\mu \nu}(X, \tau) - 8 \pi G T_{\mu \nu} + h \eta_{\mu \nu}(X, \tau), \quad (10)$$

where $\eta_{\mu \nu}$ is the noise for the Euclidean $g_{\mu \nu}$.

Eq. (10) is a complicated non-linear differential equation. If one solves it and eliminates the noise, it
produces correlation functions at a finite stochastic time $\tau$ that depend on the initial data $g_{\mu\nu}(X,\tau = \tau_0)$. Therefore the dimensionful parameters that may occur in the solutions of the stochastically quantised quantum gravity are not only $a$, $b$ and $\hbar$, $\kappa$ and the Newton constant $G$, but also the initial condition that describes the initial Euclidean geometry of the universe at $\tau = \tau_0$. In this sense, the value of the stochastic time measured in units of $a$ when the Universe exits from inflation depends on the initial data. After the exit from inflation, according to our scenario, gravity becomes by definition purely classical, and one has a standard classical evolution from a given geometry, which was reached by a random fluctuation and must fit the data of today’s observations.

Here we shall justify some of these speculations by a careful analysis of a zero dimensional case, where the fields $\Phi$ and $\Psi$ are replaced by two real numbers $x$ and $y$, as an extreme dimensional reduction, to explain the methodology and the way to handle the initial data.

## 4 The zero dimensional case

Consider the zero dimensional case of two real variables $x$ and $y$ with an action

\[
S(x, y) = \frac{1}{2} M^2 x^2 + \frac{1}{2} m^2 y^2 + \frac{\lambda}{2} xy^2.
\]  

(11)

The stochastic quantisation of $S(x, y)$ introduces a bulk time $\tau$ and promotes $x, y$ to $x(\tau), y(\tau)$, and one builds eventually a quantum mechanical path integral with a supersymmetric Lagrangian $\mathcal{L}_{\text{stoc}}(x, y, \psi_x, \psi_y)$, to be determined later on, such that for any observable $f(x, y)$, one has

\[
\int dx dy f(x, y) \exp \left( -\frac{1}{\hbar} S(x, y) \right)
\]

\[
= \lim_{T \rightarrow \infty} \int [dx]_{\tau} [dy]_{\tau} [d\psi_x]_{\tau} [d\psi_y]_{\tau} [d\bar{\psi}_x]_{\tau} [d\bar{\psi}_y]_{\tau} f(x(\tau), y(\tau)) \exp \left( -\frac{1}{\hbar} \int_0^T d\tau \mathcal{L}_{\text{stoc}}(x, y, \psi_x, \bar{\psi}_x, \psi_y, \bar{\psi}_y) \right).
\]

(12)
Second order stochastic quantisation means in fact that, in correspondence with such a supersymmetric representation, there is an underlying Langevin equation

\[
\left( a^2 \frac{d^2}{d\tau^2} + 2b \frac{d}{d\tau} \right) x(\tau) + M^2 x(\tau) + \frac{\lambda}{2} y^2(\tau) = \sqrt{\hbar} \eta_x(\tau),
\]

\[
\left( a^2 \frac{d^2}{d\tau^2} + 2b \frac{d}{d\tau} \right) y(\tau) + m^2 y(\tau) + \lambda x(\tau) y(\tau) = \sqrt{\hbar} \eta_y(\tau)
\]

that defines the \( \tau \) evolution. Here \( \eta_x \) and \( \eta_y \) are taken as Gaussian noises, i.e.,

\[
\langle \eta_x(\tau) \rangle = \langle \eta_y(\tau) \rangle = 0, \quad \langle \eta_x(\tau) \eta_x(\tau') \rangle = \langle \eta_y(\tau) \eta_y(\tau') \rangle = 2b \delta(\tau - \tau').
\]

(14)

All other Gaussian correlators for mean values of higher order products of the \( \eta \)'s follow by averaging with the Gaussian distribution \( \int [d\eta]_\tau \exp(-\frac{1}{2\hbar} \int d\tau \eta^2(\tau)) \).

The case \( a = 0 \) and \( b = \frac{1}{2} \) is for the standard first order stochastic quantisation of [2].

In what follows, both \( a \neq 0 \) and \( b \neq 0 \). For a more general potential \( S = \frac{1}{2} M^2 y^2 + \frac{1}{2} m^2 x^2 + \lambda V(x, y) \), we must replace \( \frac{\lambda}{2} y^2 \) and \( \lambda xy \) in Eq. (13) by \( \lambda V_x \) and \( \lambda V_y \), respectively. Of course, not all choices of the "pre-potential" \( S \) produce proper convergence of solutions at infinite \( \tau \). For \( a \neq 0 \), we need to complete each Langevin equation with two boundary conditions, instead of one condition in the first order case. Here we assume that at two values of the time, \( \tau = \tau_1 \) and \( \tau_2 \), the coordinates \( x \) and \( y \) take the values \( x(\tau_i) = x_i \) and \( y(\tau_i) = y_i \) for \( i = 1, 2 \). For regular theories, the infinite time limit is expected to be independent of the choice of such conditions.

Our aim is to consider that one of the field, here \( x(\tau) \), is a "coherent state", which is, in the Schrödinger sense, a state that minimises maximally some quantum fluctuations. So \( x(\tau) \) is a state that is as close as possible to a solution where one neglects everywhere \( \eta_x \). This situation has been advocated to in [1], to define the primordial cosmology.

Moreover, since we understand intuitively that at finite stochastic time \( y(\tau) \) undergoes quantum effects around the "strong" coherent state \( x(\tau) \), we introduce an arbitrary given fixed position \( x_{cl} \) for \( x \), which can be interpreted as the classical equilibrium value of \( \langle x(\tau) \rangle \) when \( \tau \to \infty \). We thus reduce the above coupled Langevin equations, to the case where effectively \( \eta_x = 0 \), that is,

\[
a^2 \ddot{x} + 2b \dot{x} + M^2 (x - x_{cl}) + \frac{\lambda}{2} y^2 = 0,
\]

\[
a^2 \ddot{y} + 2b \dot{y} + m^2 y + \lambda xy = \sqrt{\hbar} \eta_y.
\]

(15)

Solving perturbatively Eqs. (15) as a Taylor expansion on \( \lambda \) is of course possible. Using Green's function techniques gives a hint of the physics at finite \( \tau \). In fact, some aspects of the loop expansion that will occur for \( a \neq 0 \) are better viewed by the redefinition of fields and coupling constant,

\[
\hbar \lambda \to \lambda, \quad \frac{x}{\sqrt{\hbar}} \to x, \quad \frac{y}{\sqrt{\hbar}} \to y.
\]

(16)

After such redefinitions, the Planck constant \( \hbar \) disappears from the Langevin equations, which become

\[
D^M_\tau (x - x_{cl}) + \frac{\lambda}{2} y^2 = 0, \quad D^n_\tau y + \lambda xy = \eta_y,
\]

(17)

where \( D^M_\tau \equiv a^2 \partial^2_\tau + 2b \partial_\tau + M^2, \quad D^n_\tau \equiv a^2 \partial^2_\tau + 2b \partial_\tau + m^2 \), and the perturbative loop expansion of the correlators can be expanded in powers of the rescaled coupling constant \( \lambda \hbar \to \lambda \).

### 4.1 Perturbative expansion and finite \( \tau \) QFT behaviour

The perturbative expansion in \( \lambda \) for the \( \tau \) evolution of \( x \) and \( y \) can be represented by Feynman diagrams with insertions of the noises analogous to those in [2], except that its propagators \( G_M \) and \( G_m \) have
a double pole structure instead of being of parabolic type in [2]. There is a forward propagation of
types with positive and negative energy in the $\tau$ evolution, and one has insertions of the field $x(\tau)$ in
addition to the insertions of $n_x$ on the propagators of the $y(\tau)$. The Feynman diagrams that one
can draw to describe perturbation theory involve closed loops.

In a Fourier transformation over $\tau$, using the conjugate variable $E$, we must define particles of
type $y$ and antiparticles of type $\bar{y}$, with creation and annihilation operators acting on a Fock space,
basically because we have solutions of positive and negative energy in a symmetric way.

Because $a \neq 0$, the $\tau$ dependence implies a relativistic quantum field theory framework rather
than a non-relativistic framework as in the case $a = 0$. The closed loops can be interpreted as forward
stochastic time propagations of particles of type $y$ and antiparticles of type $\bar{y}$.

The closed loops are finite integrals over a one dimensional momentum space, with neither infra-
red nor ultra-violet divergences, since $m \neq 0$ and $M \neq 0$. They occur in the perturbative expansion of
$\langle x^p(\tau)y^q(\tau) \rangle$ and can be interpreted as creations of a virtual pairs created by the vertex $\lambda xy\bar{y}$
of particle and antiparticle $y$ and $\bar{y}$ at a given value of the stochastic time, each one propagating
forwardly in $\tau$, until they annihilate at a further stochastic time, with possible interactions with the
“classical field” $x(\tau)$.

This suggests that a double Fock space must be constructed. It is made of all possible states that
can occur for the $\tau$ evolution, one for the field $x$, one for all possible vacua that can occur and the
other one for the ordinary quanta of the field $y$. This description will become clearer by studying
the Fokker-Planck Lagrangian and Hamiltonian associated to the second order Langevin equation:
$L_{\text{stoc}}$ contains a higher order derivatives in the stochastic time, and implies a doubled phase space in
(stochastic time) Hamiltonian formalism.

Here we use the approximation that the field $x(\tau)$ is a coherent state, made up of its elementary
quanta, in a way that minimises the dispersion relations.

The elementary quantum processes that build the perturbation theory are the possible decay, 
annihilation and diffusion reactions

$$x_{\text{cl}} \rightarrow y + \bar{y}, \quad y + \bar{y} \rightarrow x_{\text{cl}}, \quad y + x_{\text{cl}} \rightarrow y + x_{\text{cl}}, \quad \bar{y} + x_{\text{cl}} \rightarrow \bar{y} + x_{\text{cl}}$$

(18)

whose strength is proportionally to $\lambda$. The $\tau$ translation symmetry of the Langevin equation implies
that at each vertex there is a conservation of the $\tau$ energy, so that the $\tau$ evolution of the field $x_{\text{cl}}$ can
be accompanied by real decay and real annihilations of pairs of $y$ and $\bar{y}$ quanta, namely by a Schwinger
type mechanism.

Because we have a friction term proportional to $b$, the phenomena that occur during the $\tau$ evolution
will disappear in the limit $\tau \rightarrow \infty$, if it exists. This is the case in our example.

Once $x(t)$ and $y(t)$ have been diagrammatically expressed at a given order of perturbation, one
can compute at the same order of perturbation theory $x^p(\tau)y^q(\tau)$, and then averaging, one can obtain
$\langle x^p(\tau)y^q(\tau) \rangle$ by using the fact that $\eta$ has a Gaussian distribution.

Thus, there is a perturbation expansion involving interactions and propagators, with closed loops,
which determines for every value of $\tau$ the expectation values $\langle x^p(\tau)y^q(\tau) \rangle$ as a formal series in $\lambda$, and
one can compute it at any given finite order in $\lambda$. The final result is expressed as

$$\lim_{T \rightarrow \infty} \langle x^p(\tau)y^q(\tau) \rangle$$

$$= \int dx dy \, x^p y^q \delta(x - x_{\text{cl}} + o(\lambda)) \exp \left( -\frac{1}{2} M^2 (x - x_{\text{cl}})^2 + \frac{1}{2} m^2 y^2 + \frac{1}{2} \lambda xy^2 \right) \delta(1 + o(\lambda^2))$$

$$= Z^p_{x_{\text{cl}}} Z^q y^q \int dy \, y^q \exp \left( -\frac{1}{2} Z^2_{m^2} m^2 y^2 + \frac{1}{2} Z \lambda x_{\text{cl}} y^2 \right).$$

(19)
The $Z$ factors are finite renormalisation factors with a Taylor expansion, $Z = 1 + \lambda Z^1 + \lambda^2 Z^2 + \cdots$, where all the finite coefficients $Z^n$ can be in principle computed in the perturbation theory in $\lambda$.

The last formula tells us that Langevin equations gives us a very complicated way to define a standard Gaussian integral in zero dimensions, with a well defined theory that computes the corrections in $\lambda$ at any given finite order of perturbation theory! The way the limit is reached is an exponential damping, with a regime of very fast oscillations, as sketched in the figure (3).

We will check this explicitly for the one point and two point functions $\langle x(\tau) \rangle$ and $\langle y(\tau)y(\tau') \rangle$.

### 4.2 0-loop and 1-loop computation of two point functions

The Green’s function $G^M(\tau)$ of the operator $D^m_\tau = a^2 \partial^2_\tau + 2b \partial_\tau + M^2$ satisfying $D^M_\tau G^M = \delta(\tau)$ can be computed by a Laplace or Fourier transform. Suppose $aM > b > 0$. It is

$$G^M(\tau) = \theta(\tau) \frac{\exp(-E^M_+ \tau) - \exp(-E^M_- \tau)}{a^2 (E^M_+ - E^M_-)} = \frac{i \theta(\tau)}{2a^2 M^2 - b^2} (\exp(-E^M_+ \tau) - \exp(-E^M_- \tau)), \quad (20)$$

where

$$E^M_{\pm} = \frac{1}{a^2} (b \pm i \sqrt{a^2 M^2 - b^2}) \quad (21)$$

satisfy $a^2 E^2 + 2bE + M^2 = (E + E^M_+)(E + E^M_-)$.

When $a \neq 0$, the free propagator has still an exponential damping factor, with a characteristic time that is proportional to $b^{-1}$ (when $\tau$ is counted in units of $a^2$), but there is a new phenomenon, which are $\tau$-oscillations that can be of a very high frequency (in units of $a$) if the mass $M$ is large enough.

This is the situation that was suggested generically in [1]. In the case of QFTs, one should replace $M^2$ by $M^2 + \vec{k}^2$, where $\vec{k}$ stands for the momentum of the particle. Care must be given to the possible UV divergencies when $\vec{k}^2$ becomes very large.

If $am > b$, we have $G^m(\tau)$ and $E^m_{\pm}$ similarly for the operator $D^m_\tau = a^2 \partial^2_\tau + 2b \partial_\tau + m^2$.

We consider the coupled Langevin equations (17), with only one noise for $y(\tau)$. We solve Eq. (17) perturbatively. Suppose

$$x(\tau) = x_0(\tau) + \lambda x_1(\tau) + o(\lambda^2), \quad y(\tau) = y_0(\tau) + \lambda y_1(\tau) + o(\lambda^2). \quad (22)$$

satisfy (17). Then the 0th order terms in $\lambda$ satisfy

$$D^M_\tau (x_0 - x_{cl}) = 0, \quad D^m_\tau y_0 = \eta \quad (23)$$

whereas the first order terms in $\lambda$ satisfy

$$D^M_\tau x_1 + \frac{1}{2} y_0^2 = 0, \quad D^m_\tau y_1 + x_0 y_0 = 0. \quad (24)$$

The solution to the first (homogeneous) equation for $x_0(\tau)$ in (23) is

$$x_0(\tau) = x_{cl} + c^M_+ \exp(-E^M_+ \tau) + c^M_- \exp(-E^M_- \tau) = x_{cl} + o(\exp(-\tau)), \quad (25)$$

where $c^M_{\pm}$ are constants that are determined by the chosen values of $x_0$ at some $\tau_1$ and $\tau_2$, and $o(\exp(-\tau))$ stands for any term that is dominated by $\exp(-\epsilon \tau)$ for some $\epsilon > 0$ (including the oscillations) as $\tau \to \infty$. Thus

$$\langle x_0(\tau) \rangle = x_{cl} + o(\exp(-\tau)). \quad (26)$$

Indeed, when the stochastic process converges, the limit at infinite stochastic time of correlators of the fields are independent of the chosen values of the fields at $\tau_1$ and $\tau_2$ as well. Since $E^M_{\pm}$ have a
positive real part, the damping in the dependence of boundary conditions is exponentially fast in \( \tau \), times some oscillations.

On the other hand, the equation in (23) for \( y_0(\tau) \) is inhomogeneous and we have

\[
y_0(\tau) = (G^m * \eta)(\tau) + c^m_+ \exp(-E^m_+ \tau) + c^m_- \exp(-E^m_- \tau) = (G^m * \eta)(\tau) + o(e^{-\tau})
\]  

(27)

for some constants \( c^m_\pm \), where * stands for the convolution. Similarly, \( c^m_\pm \) are related to the boundary values of \( y_0(\tau) \) at some \( \tau_1 \) and \( \tau_2 \), and their choice does not affect the correlators at infinite stochastic time. If \( 0 < \tau_1 \leq \tau_2 \), we have

\[
\langle \langle y_0(\tau_1) y_0(\tau_2) \rangle \rangle = \int_0^{\tau_1} d\tau_1' \int_0^{\tau_2} d\tau_2' G^m(\tau_1 - \tau_1')G^m(\tau_2 - \tau_2') \langle \langle \eta(\tau_1') \eta(\tau_2') \rangle \rangle + o(e^{-\tau_1})
\]

\[
= 2b \int_0^{\tau_1} d\tau' G^m(\tau_1 - \tau')G^m(\tau_2 - \tau') + o(e^{-\tau_1})
\]

\[
= \frac{b}{a^4(E^m_+ + E^m_-)(E^m_- - E^m_+)} \left( \frac{\exp(-E^m_+(\tau_2 - \tau_1))}{E^m_+} - \frac{\exp(-E^m_-(\tau_2 - \tau_1))}{E^m_-} \right) + o(e^{-\tau_1}).
\]  

(28)

In particular, taking \( \tau_2 = \tau_1 = \tau \), we have

\[
\langle \langle y_0(\tau)^2 \rangle \rangle = \frac{b}{a^4(E^m_+ + E^m_-)E^m_- E^m_+} + o(e^{-\tau}) = \frac{1}{2m^2} + o(e^{-\tau}).
\]  

(29)

For the next order, we solve \( x_1(\tau) \) from Eq. (24) and obtain

\[
x_1(\tau) = -\frac{1}{2} \int_0^\tau d\tau' G^M(\tau - \tau') y_0(\tau')^2 + o(e^{-\tau}).
\]  

(30)

Taking the expectation value, we have

\[
\langle \langle x_1(\tau) \rangle \rangle = -\frac{1}{4m^2} \int_0^\tau d\tau' G^M(\tau - \tau') + o(e^{-\tau})
\]

\[
= -\frac{1}{4m^2} a^2(E^M - E^M_+) \left( \frac{1}{E^M_+} - \frac{1}{E^M_-} \right) + o(e^{-\tau})
\]

\[
= -\frac{1}{4m^2 M^2} + o(e^{-\tau}).
\]  

(31)

Similarly, solving \( y_1(\tau) \) from Eq. (24), we obtain

\[
y_1(\tau) = -\int_0^\tau d\tau' G^m(\tau - \tau') x_0(\tau') y_0(\tau') + o(e^{-\tau})
\]

\[
= -x_1 \int_0^\tau d\tau' G^m(\tau - \tau') y_0(\tau') + o(e^{-\tau}).
\]  

(32)

Therefore, taking the expectation value, we have

\[
\langle \langle y_0(\tau) y_1(\tau) \rangle \rangle
\]

\[
= -x_1 \int_0^\tau d\tau' G^m(\tau - \tau') \langle \langle y_0(\tau) y_0(\tau') \rangle \rangle
\]

\[
= \frac{b x_1}{a^4(E^m_+ + E^m_-)(E^m_- - E^m_+)} \int_0^\tau d\tau' G^m(\tau - \tau') \left( \frac{\exp(-E^m_+ \tau')}{E^m_+} - \frac{\exp(-E^m_- \tau')}{E^m_-} \right) + o(e^{-\tau})
\]

\[
= -\frac{2ab(E^m_+ + E^m_-)(E^m_- E^m_+)^2}{4m^4} + o(e^{-\tau})
\]

\[
= -\frac{x_1}{4m^4} + o(e^{-\tau}).
\]  

(33)
Combining the above 0th order and the 1st order contributions to the correlators, we have

$$\lim_{\tau \to \infty} \langle x(\tau) \rangle = \lim_{\tau_0 \to \infty} \langle x_0(\tau) \rangle + \lambda \lim_{\tau \to \infty} \langle x_1(\tau) \rangle + o(\lambda^2)$$

$$= x_{cl} - \frac{\lambda}{4m^2 M^2} + o(\lambda^2)$$

from Eqs. (26) and (31), and

$$\lim_{\tau \to \infty} \langle y(\tau)^2 \rangle = \lim_{\tau_0 \to \infty} \langle y_0(\tau)^2 \rangle + 2 \lambda \lim_{\tau \to \infty} \langle y_0(\tau) y_1(\tau) \rangle + o(\lambda^2)$$

$$= \frac{1}{2m^2} + 2\lambda \cdot \left( - \frac{x_{cl}}{4m^2} \right) + o(\lambda^2)$$

$$= \frac{1}{2(m^2 + x_{cl}^2)} + o(\lambda^2)$$

from Eqs. (29) and (33). In the limit, the $a$ and $b$ dependance is washed away, as well as that on the initial conditions.

The shifts $x_{cl} \to x'_{cl} = x_{cl} - \lambda/4bm^2 M^2$ and $m^2 \to m' = m^2 + \lambda x_{cl}$ in Eqs. (34) and (35) have a simple explanation. In Eq. (17), substituting the expectation value $\langle \eta^2 \rangle$ in (29) for $y^2$ in the equation for $x$, we have

$$D^M_T (x - x_{cl}) + \frac{\lambda}{2} \langle \eta^2 \rangle = D^M_T (x - x'_{cl})$$

while substituting the expectation value $\langle x_0 \rangle$ in (26) for $x$ in the equation for $y$, we have

$$D^m_T y + \lambda x \langle y_0 \rangle = D^{m'}_T y.$$  

Thus we have the same form of Eq. (17) but with the shifted constant parameters $x'_{cl}$ and $m'$. This is consistent with Eq. (19) with the finite renormalisation constants

$$Z_x = 1 - \frac{\lambda}{4bm^2 M^2 x_{cl}} + o(\lambda^2), \quad Z_m = 1 + \frac{\lambda x_{cl}}{2m^2} + o(\lambda^2), \quad Z_{\lambda} = 1 + o(\lambda). \quad (38)$$

### 4.3 Higher loops and diagrammatic expansions

Recall the system (17) of stochastic equations. Iteratively, we have

$$x = x_{cl} - \frac{\lambda}{2} G^M \ast y^2 = x_{cl} - \frac{\lambda}{2} G^M \ast \left( G^m \ast \eta - \lambda G^m \ast (xy) \right)^2$$

$$= x_{cl} - \frac{\lambda}{2} G^M \ast (G^m \ast \eta)^2 + \lambda^2 x_{cl} G^M \ast ((G^m \ast \eta)(G^M \ast G^m \ast \eta)) + o(\lambda^3)$$

and

$$y = G^m \ast \eta - \lambda G^m \ast (xy) = G^m \ast \eta - \lambda G^m \ast \left( (x_{cl} - \frac{\lambda}{2} G^M \ast y^2)(G^m \ast \eta - \lambda G^m \ast (xy)) \right)$$

$$= G^m \ast \eta - \lambda x_{cl} G^m \ast G^m \ast \eta + \lambda^2 x_{cl} G^m \ast G^M \ast \eta + \lambda^2 \frac{G^m}{2} \ast \left( (G^M \ast (G^m \ast \eta)^2)(G^m \ast \eta)) + o(\lambda^3).$$

Diagrammatically, the above expansions of $x$ and $y$ can be represented by

$$x(\eta) = \bigcirc + \bigcirc + \bigcirc + \cdots$$

and

$$y(\eta) = \bigcirc + \bigcirc + \bigcirc + \cdots$$
\[ y(\eta) = \lambda^0 + \lambda^1 + \lambda^2 + \cdots. \]

Here a circle means an insertion of the constant term \( x_{\text{cl}} \), a cross means attaching to the external noise \( \eta \), a wiggly line means the propagator with the Green's function \( G^M \) of \( x \), and a solid line means the propagator with the Green's function \( G^m \) of \( y \). To each trivalent vertex with one wiggly line and two solid lines is assigned the coupling constant \( -\lambda \) and to each circle on a solid line is assigned the factor \( -\lambda x_{\text{cl}} \). As usual, each diagram is divided by the order of its automorphism group. Note that the quantity represented by each diagram, though \( \tau \)-dependent, is determined by solely the equations in (17) and does not rely on the initial/boundary conditions of a particular solution.

We can express \( \langle\langle x(\tau) \rangle\rangle \) and \( \langle\langle y(\tau_1)y(\tau_2) \rangle\rangle \) by diagrams. Since the same method of iteration is used both here and in Section 4.2, the results to the order \( \lambda^1 \) must agree. To the order \( \lambda^2 \), we have

\[
\langle\langle x(\tau) \rangle\rangle = \lambda^0 + \lambda^1 + \lambda^2 + \cdots
\]

and

\[
\langle\langle y(\tau_1)y(\tau_2) \rangle\rangle = \lambda^0 + \lambda^1 + \lambda^2 + \cdots
\]

where a cross on a solid line means now contraction of two \( \eta \)'s in the diagramatic expansions of \( x(\eta) \) and \( y(\eta) \) above using (14), giving rise to a factor of 2\( b \). Quantitatively, these diagrams are

\( (a) = x_{\text{cl}}, \)

\( (b) = -b\lambda \int_0^{\tau_0} d\tau' G^M(\tau - \tau') \int_0^{\tau'} d\tau'' G^m(\tau' - \tau'')^2; \)

\( (c) = 2b\lambda^2 x_{\text{cl}} \int_0^{\tau_0} d\tau' G^M(\tau - \tau') \int_0^{\tau'} d\tau'' G^m(\tau' - \tau'') \int_0^{\tau'} d\tau''' G^m(\tau'' - \tau''') G^m(\tau''' - \tau''') \)

\[ \tau_0 \]

\[ \tau_0 \]

\[ \tau_0 \]

\[ \tau_0 \]
for $\langle x(\tau) \rangle$. The diagrams (a) and (b) agree with (34) when $\tau \to +\infty$. For $\langle y(\tau_1) y(\tau_2) \rangle$, we have

$$(d) = 2b \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') G^m(\tau_2 - \tau'),$$

$$(e) = -2b \lambda x \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau'}^{\tau_2} d\tau_2' G^m(\tau_2 - \tau') G^m(\tau_2' - \tau'),$$

$$(f) = -2b \lambda x \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \int_{\tau'}^{\tau_1} d\tau_1' G^m(\tau_1 - \tau') G^m(\tau_1' - \tau'),$$

$$(g) = 2b \lambda x^2 \int_0^{\tau_1} d\tau_1' G^m(\tau_1 - \tau_1') \int_{\tau_1'}^{\tau_2} d\tau_2' G^m(\tau_2 - \tau_2') \int_0^{\min(\tau_1', \tau_2')} d\tau' G^m(\tau_1' - \tau') G^m(\tau_2' - \tau'),$$

$$(h) = 2b \lambda x^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau_1}^{\tau_1'} d\tau_1' G^m(\tau_1 - \tau_1') G^m(\tau_1' - \tau_1''),$$

$$i = 2b \lambda x^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \int_{\tau_1}^{\tau_1''} d\tau_1'' G^m(\tau_1 - \tau_1'') G^m(\tau_1' - \tau_1''),$$

$$(j) = 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau_1}^{\tau_2} d\tau_1 G^m(\tau_2 - \tau') G^m(\tau_2 - \tau'') G^m(\tau_1 - \tau_1'),$$

$$(k) = 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau_1}^{\tau_2} d\tau_1 G^m(\tau_2 - \tau') G^m(\tau_2 - \tau'') G^m(\tau_1 - \tau_1'),$$

$$(l) = 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_1 - \tau') \int_{\tau_1}^{\tau_2} d\tau_1 G^m(\tau_2 - \tau') G^m(\tau_2 - \tau'') G^m(\tau_1 - \tau_1'),$$

$$(m) = 4b^2 \lambda^2 \int_0^{\min(\tau_1, \tau_2)} d\tau' G^m(\tau_2 - \tau') \int_{\tau_2}^{\tau_1} d\tau_1 G^m(\tau_1 - \tau_1') G^m(\tau_1' - \tau_1''),$$

and diagrams (d), (e), (f) agree with (35) when $\tau_1 = \tau_2 = \tau \to +\infty$.

5 Supersymmetric representation and a possible Fokker-Planck Hamiltonian

The correlation functions stemming from a Langevin equation can be represented by a supersymmetric path integral, with the generating functional

$$Z[J_x, J_y] = \int [dx]_\tau [dy]_\tau [d\psi_x]_\tau [d\psi_y]_\tau [d\overline{\psi}_x]_\tau [d\overline{\psi}_y]_\tau [d\eta_x]_\tau [d\eta_y]_\tau \exp \left[ -\frac{1}{\hbar} \int_0^\tau d\tau \left( \mathcal{L}_{\text{susy}}(x, y, \psi_x, \psi_y, \overline{\psi}_x, \overline{\psi}_y, \eta_x, \eta_y) + x J_x + y J_y \right) \right] \tag{39}$$

using the standard determinant manipulation [5]. For simplicity, we will write $q$ for $(x, y)$, $\eta$ for $(\eta_x, \eta_y)$, $\psi$ for $(\psi_x, \psi_y)$, and $\overline{\psi}$ for $(\overline{\psi}_x, \overline{\psi}_y)$. The action is topological in the sense that $\mathcal{L}_{\text{susy}} = s_{\text{top}} (\cdots)$, where the supersymmetry transformations

$$s_{\text{top}} q = \psi, \quad s_{\text{top}} \psi = 0, \quad s_{\text{top}} \overline{\psi} = \eta \tag{40}$$
satisfy $s_{\text{top}}^2 = 0$. In our case with a second order time evolution, one has

$$L_{\text{susy}} = s_{\text{top}} \left( \overline{\psi} \left( a^2 \dot{q} + 2b \dot{q} + \frac{\partial S}{\partial q} - \frac{1}{2} \eta \right) \right)$$
$$= -\frac{1}{2} \eta^2 + \eta \left( a^2 \dot{q} + 2b \dot{q} + \frac{\partial S}{\partial q} \right) - \overline{\psi} \left( a^2 \dot{\psi} + 2b \dot{\psi} + \frac{\partial^2 S}{\partial q^2} \psi \right)$$
$$\sim \frac{1}{2} \left( a^2 \dot{q} + 2b \dot{q} + \frac{\partial S}{\partial q} \right)^2 - \overline{\psi} \left( a^2 \dot{\psi} + 2b \dot{\psi} + \frac{\partial^2 S}{\partial q^2} \psi \right),$$

(41)

This Lagrangian $L_{\text{susy}}$ was mentioned in Eq. (12) and can be called the Fokker-Planck supersymmetric Lagrangian.

When one eliminates the fermions in the path integral by Berezin integration and computes correlation functions, one recovers the same results as from solving the Langevin equations and computing mean values of functions of $q$ by elimination of the noise $\eta$.

It is actually relevant to express the Hamiltonian associated to the supersymmetric Lagrangian (41). Let us recall that, when $a = 0$, $b = \frac{1}{2}$, $\Psi$ and $\overline{\Psi}$ are self-conjugate. In this case, the supersymmetric Hamiltonian can be represented as a $2 \times 2$ matrix whose diagonal terms are both advanced and retarded Fokker-Planck Hamiltonians for the first order stochastic time evolution, as they can be computed in statistical mechanics. The property of its spectrum determines the convergence of the stochastic process [5]. In this sense, the supersymmetry is truly a topological symmetry that determines the Fokker-Planck process from first principles. The word “topological” is justified by the fact that many interesting theories express in the simplest way their physics only in the slice $\tau = \infty$ [3].

For $a \neq 0$, we have a more general supersymmetric Lagrangian that has higher order derivative terms, symbolically $L = L(q, \dot{q}, \ddot{q})$. This doesn’t change our point of view that its Hamiltonian, once suitably defined, will generate the stochastic time evolution, and define the required generalisation of the Fokker-Planck evolution kernel.

To see how it works in an elementary way, we can use the simple $\eta$-dependance of the Lagrangian. We can write the bosonic part of the Lagrangian $L_{\text{susy}}$ as

$$L_{\text{susy}}^B = -a^2 \dot{q} + b(\eta \dot{q} - q \dot{\eta}) - \frac{1}{2} \eta^2 + \eta \frac{\partial S}{\partial q},$$

(42)

up to a total derivative. Then the canonical momenta of $q$ and $\eta$ are

$$p_q \equiv \frac{\partial L_{\text{susy}}}{\partial \dot{q}} = -a^2 \dot{q} + b \eta, \quad p_\eta \equiv \frac{\partial L_{\text{susy}}}{\partial \dot{\eta}} = -a^2 \dot{\eta} - b q,$$

(43)

respectively. Taking a Legendre transform, the bosonic part of the Hamiltonian is

$$H_{\text{susy}}^B = p_q \dot{q} + p_\eta \dot{\eta} - L_{\text{susy}}^B$$
$$= -\frac{1}{a^2} (p_q - b \eta)(p_\eta + b q) + \frac{1}{2} \eta^2 - \eta \frac{\partial S}{\partial q}.$$  

(44)

It shows that we must consider a doubling of the phase space, $(q, p_q) \rightarrow (q, p_q, \eta, p_\eta)$, and that at finite values of $\tau$, the limit $a \rightarrow 0$ is not smooth. The fermionic part contains the supersymmetric partners and has the same structure. This extension of the phase space is because the Langevin equation is of a higher order.\(^1\)

If we examine the Hamiltonian (44) from a perturbative point of view, its zeroth order is given by a quadratic approximation of $S$ as a function of $x$. If we restrict to this approximation, one can diagonalise $H_{\text{susy}}^B$ in the space $(x, p_x) \rightarrow (x, p_x, \eta, p_\eta)$ and get a double harmonic oscillator with a double harmonic spectrum, one for the particles and one for the noise through the quantisation of $\eta$. In our approximation we can neglect $\eta_x$, this extra quantisation is reported on the oscillations at

\(^1\)An analogous phenomenon arises when one performs the BRST quantisation of supergravity, where the fermionic Lagrange multiplier for the gauge-fixing of the Rarita-Schwinger field becomes a propagating fermionic field.
finite time of the vacuum around $x_{cl}$, till it becomes stationary. These features survive perturbatively for a large class of potentials, and one may hope that they can be true non-perturbatively.

In a different approach, and basically with the same conclusion, we can analyse the Lagrangian $\mathcal{L}_{\text{susy}}$ directly after the elimination of $\eta$, which gives the last line in (41). The Lagrangian $\mathcal{L}_{\text{susy}}$ has higher order derivatives and can be studied by applying the standard Lagrangian/Hamiltonian formalism of Ostrogradsky (see for example [6] and references therein). Given a Lagrangian depending on $q, \dot{q}, \ddot{q}$, where the dot means $\equiv \frac{d}{dt}$, and an action $S = \int d\tau L(q, \dot{q}, \ddot{q})$, one finds that the extrema of $S$ occur when the Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial q} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \ddot{q}} = 0,$$

is satisfied. In fact, a general variation of $L$ for arbitrary variations $\delta q$ and $\delta \dot{q}$ is

$$\delta L = \left( \frac{\partial L}{\partial q} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{d\tau^2} \frac{\partial L}{\partial \ddot{q}} \right) \delta q + \frac{d}{d\tau} \left( p_0 \delta q_0 + p_1 \delta q_1 \right),$$

where

$$q_0 \equiv q, \quad q_1 \equiv \dot{q}, \quad p_0 \equiv \frac{\partial L}{\partial \dot{q}} - \frac{d}{d\tau} \frac{\partial L}{\partial q}, \quad p_1 \equiv \frac{\partial L}{\partial q}.$$  

This shows that the phase space is parametrised by the conjugate coordinates $(q_0, q_1, p_0, p_1)$. The canonical phase space is doubled, as can be simply understood because one needs twice as many initial conditions.

This doubling of the phase space has non-trivial consequences. In particular, when $a \neq 0$, $\dot{q}$ is not identified as the momentum of $q$ in the Lagrangian (41), so $q$ and $\dot{q}$ can be measured simultaneously in the quantum mechanics defined by replacing the (anti)coordinates by operators and the Poisson brackets by (anti)commutators.

In fact, the uncertainty relation holds only between $q_0$ and $p_0$ (for $a \neq 0$), and between $q_1$ and $p_1$ (see the canonical commutation relations below).

Eq. (46) shows that the conserved Hamiltonian associated to $L(q, \dot{q}, \ddot{q})$, which expresses the $\tau$-evolution, is still given by a Legendre transformation

$$H \equiv p_0 \dot{q}_0 + p_1 \dot{q}_1 - L(q, \dot{q}, \ddot{q}),$$

where $\dot{q}, \ddot{q}$ must be expressed as functions of $q_0, p_0, q_1, p_1$ using (47). One finds that the phase space equations of motion are

$$\dot{p}_0 = -\frac{\partial H}{\partial q_0}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \ddot{q}_0 = \frac{\partial H}{\partial p_1}, \quad \ddot{q}_1 = \frac{\partial H}{\partial p_0}.$$ 

They are first order equations, just as in the standard case, but a doubling. Quantisation in the operator formalism is then defined by regarding all coordinates and momenta as operators subject to the canonical relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0.$$  

and for any operator $\hat{A}$, one has $\hat{A} = [\hat{H}, \hat{A}]$. This more systematic construction explains in a different way the doubling of the Fock space that we directly derived, before the elimination of the noises. The positivity of the spectrum is not ensured, as it was obvious seen in (44), and its physical interpretation is different than of standard quantum mechanics in its Euclidean form. As explained in this paper, the later can be only obtained in limit $\tau \to \infty$, if the limit exists.

---

1The same remark applies to the models presented in [3].

2The Ostrogradsky formalism is valid for higher order Lagrangians $L(q, \dot{q}, \ddot{q}, \cdots, q^{(r)})$, giving an enlarged phase space of dimension $2r$. 

---

16
The construction of the phase space can be easily repeated for the entire supersymmetric $L_{\text{susy}}$ in Eq. (41), giving a graded symplectic structure and a supersymmetric Hamiltonian, where both the $\Psi$ and $\bar{\Psi}$ have their own independent momenta for $a \neq 0$ and there is also a doubling for the fermionic part of the phase space as compared to the case $a = 0$.

All this suggests that a well-defined Fokker-Planck Hamiltonian, which defines the stochastic time evolution by first order Hamiltonian equations, can be associated to the second order Langevin equation. This explains in a better way the expression (44), where the noise was kept in a rather artificial way. The supersymmetric representation (41) of the second order Langevin equation is expected to play a key role, for the understanding of the $\tau$ evolution as it does in the case of a first order Langevin equation.

6 Conclusion

There is interesting physics for the correlators at finite stochastic time when stochastic quantisation is second order, even in the (existing) puzzling cases such as gravity when the limit $\tau \to \infty$ is not defined. Because of the oscillations, instabilities can occur on the way, so that the phase of the system can change. The standard and known physics with its limitations, that is, the necessity that gravity be classical, can only occur in the limit $\tau \to \infty$ that can be reached after the exit from inflation.

Acknowledgment. LB wishes to thank Giorgio Parisi for very encouraging discussions and the NCTS in Hsinchu for its generous and warm hospitality. SW is supported in part by grant No. 106-2115-M-007-005-MY2 from MOST (Taiwan).

References

[1] L. Baulieu, Early universes with effective discrete time (II), lecture at the Cargèse Summer Institute, June (2016), arXiv:1611.03347 [hep-th]; Higher order stochastic equation and primordial cosmology, LPTHE internal report, April (2017).

[2] G. Parisi, and Y.-S. Wu, Perturbation theory without gauge fixing, Sci. Sinica 24 (1981) 483–496.

[3] L. Baulieu and B. Grossman, A topological interpretation of stochastic quantization, Phys. Lett. 212B (1988) 351–356; L. Baulieu, Stochastic and topological gauge theories, Phys. Lett. 232 (1989) 479–485; Extended supersymmetry for path integral representations of Langevin type equations, Prog. Theor. Phys. Suppl. 111 (1993) 151–162; L. Baulieu and D. Zwanziger, QCD$_4$ from a five-dimensional point of view, Nucl. Phys. B581 (2000) 604–640, arXiv:hep-th/9909006; L. Baulieu, A. Grassi and D. Zwanziger, Gauge and topological symmetries in the bulk quantization of gauge theories, Nucl. Phys. B597 (2001) 583–614, arXiv:hep-th/0006036.

[4] A. Jaffe, Stochastic quantization, reflection positivity, and quantum fields, J. Stat. Phys. 161 (2015) 1–15, arXiv:1411.2964 [math.PR].

[5] E. Gozzi, Ground-state wave-function “representation”, Phys. Lett. 129B (1983) 432–436.

[6] H.J. Schmidt, Stability and Hamiltonian formulation of higher derivative theories, Phys. Rev. D49 (1994) 6354–6366; Erratum, Phys. Rev. D54 (1996) 7906, arxiv:gr-qc/9404038; J.Z. Simon, Higher derivative Lagrangians, non-locality, problems, and solutions, Phys. Rev. D41 (1990) 3720–3733.