The analytic continuation of the high–energy parton–parton scattering amplitude with an IR cutoff

Enrico Meggiolaro*

Dipartimento di Fisica,
Università di Pisa,
Via Buonarroti 2,
I–56127 Pisa, Italy.

Abstract

The high–energy parton–parton scattering amplitude can be described, in the c.m.s., by the expectation value of two infinite Wilson lines, running along the classical trajectories of the two colliding particles. The above description suffers from IR divergences (typical of 3 + 1 dimensional gauge theories), which can be regularized by considering finite Wilson lines, extending in proper time from $-T$ to $T$ (and eventually letting $T \to +\infty$). Generalizing the results of a previous paper, we give here the general proof that the expectation value of two IR–regularized Wilson lines, forming a certain hyperbolic angle in Minkowski space–time, and the expectation value of two IR–regularized Euclidean Wilson lines, forming a certain angle in Euclidean four–space, are connected by an analytic continuation in the angular variables and in the IR cutoff $T$. This result can be used to evaluate the IR–regularized high–energy scattering amplitude directly in the Euclidean theory.

*E–mail: enrico.meggiolaro@df.unipi.it
1. Introduction

The parton–parton scattering amplitude, at high squared energies $s$ in the center of mass and small squared transferred momentum $t$ (that is $s \to \infty$ and $|t| \ll s$, let us say $|t| \leq 1 \text{ GeV}^2$), can be described by the expectation value of two infinite Wilson lines, running along the classical trajectories of the two colliding particles [1, 2, 3, 4].

Let us consider, for example, the case of the quark–quark scattering amplitude. If one defines the scattering amplitude $T_{fi} = \langle f | \hat{T} | i \rangle$, between the initial state $|i\rangle$ and the final state $|f\rangle$, as follows ($\hat{S}$ being the scattering operator)

$$\langle f | (\hat{S} - 1) | i \rangle = i (2\pi)^4 \delta^{(4)} (P_{fin} - P_{in}) \langle f | \hat{T} | i \rangle ,$$

(1.1)

where $P_{in}$ is the initial total four–momentum and $P_{fin}$ is the final total four–momentum, then, in the center–of–mass reference system (c.m.s.), taking for example the initial trajectories of the two quarks along the $x^1$–axis, the high–energy scattering amplitude $T_{fi}$ has the following form [explicitly indicating the color indices $(i,j,\ldots)$ and the spin indices $(\alpha,\beta,\ldots)$ of the quarks] [1, 2, 3, 4]

$$T_{fi} = \langle \psi_i(\tau_1) \psi_f(\tau_2) | \hat{T} | \psi_i(\tau_1) \psi_f(\tau_2) \rangle \sim -\frac{i}{Z_W^2} \cdot \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot 2s \int d^2 \mathbf{z} \cdot e^{i \mathbf{q} \cdot \mathbf{z}} \langle [W_{p_1}(z_t) - 1]_{ij} [W_{p_2}(0) - 1]_{kl} \rangle ,$$

(1.2)

where $q = (0, 0, \vec{q}_{\perp})$, with $t = q^2 = -\vec{q}_{\perp}^2$, is the transferred four–momentum and $z_t = (0, 0, \mathbf{z}_t)$, with $\mathbf{z}_t = (z^2, z^3)$, is the distance between the two trajectories in the transverse plane [the coordinates $(x^0, x^1)$ are often called longitudinal coordinates]. The expectation value $\langle f(A) \rangle$ is the average of $f(A)$ in the sense of the functional integration over the gluon field $A^\mu$ (including also the determinant of the fermion matrix, i.e., $\det[i\gamma^\mu D_\mu - m_0]$, where $D^\mu = \partial^\mu + igA^\mu$ is the covariant derivative and $m_0$ is the bare quark mass). The two infinite Wilson lines $W_{p_1}(z_t)$ and $W_{p_2}(0)$ in Eq. (1.2) are defined as

$$W_{p_1}(z_t) = \mathcal{T} \exp \left[ -ig \int_{-\infty}^{+\infty} A_\mu(z_t + p_1 \tilde{\tau}) p_1^\mu d\tilde{\tau} \right] ;$$

$$W_{p_2}(0) = \mathcal{T} \exp \left[ -ig \int_{-\infty}^{+\infty} A_\mu(p_2 \tilde{\tau}) p_2^\mu d\tilde{\tau} \right] ,$$

(1.3)

where $\mathcal{T}$ stands for “time ordering” and $A_\mu = A_\mu^a T^a$; the four–vectors $p_1 \simeq (E, E, 0, 0)$ and $p_2 \simeq (E, -E, 0, 0)$ are the initial four–momenta of the two quarks [$s = (p_1 + p_2)^2 = 4E^2$]. The space–time configuration of these two Wilson lines is shown in Fig. 1.
Finally, $Z_W$ in Eq. (1.2) is the residue at the pole (i.e., for $p^2 \to m^2$, $m$ being the quark pole mass) of the unrenormalized quark propagator, which can be written in the eikonal approximation as \cite{1, 4}

$$Z_W \simeq \frac{1}{N_c} \langle \text{Tr}[W_{p_1}(z_t)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{p_1}(0)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W_{p_2}(0)] \rangle,$$

where $N_c$ is the number of colours.

In a perfectly analogous way, one can also derive the high–energy scattering amplitude for an elastic process involving two partons, which can be quarks, antiquarks or gluons \cite{2, 4}. For an antiquark, one simply has to substitute the Wilson line $W_p(b)$ with its complex conjugate $W_p^*(b)$: this is due to the fact that the scattering amplitude of an antiquark in the external gluon field $A_\mu$ is equal to the scattering amplitude of a quark in the charge–conjugated (C–transformed) gluon field $A'_\mu = -A_\mu^t = -A_\mu^*$. In other words, going from quarks to antiquarks corresponds just to the change from the fundamental representation $T_a$ of $SU(N_c)$ to the complex conjugate representation $T'_a = -T_a^*$. In the same way, going from quarks to gluons corresponds just to the change from the fundamental representation $T_a$ of $SU(N_c)$ to the adjoint representation $T_{a(\text{adj})}$. So, if the parton is a gluon, one must substitute $W_p(b)$, the Wilson string in the fundamental representation, with $V_k(b)$, the Wilson string in the adjoint representation [and the renormalization constant $Z_W$ with $Z_V = \langle \text{Tr}[V_k(0)] \rangle/(N_c^2 - 1)$].

In what follows, to be definite, we shall consider the case of the quark–quark scattering
and we shall deal with the quantity

\[ g_{M(ij,kl)}(s; t) \equiv \frac{1}{Z_W} \int d^2\mathbf{z}_\perp e^{i\mathbf{q}_\perp \cdot \mathbf{z}_\perp} \langle [W_{p_1}(z_t) - 1]_{ij} [W_{p_2}(0) - 1]_{kl} \rangle , \quad (1.5) \]

in terms of which the scattering amplitude can be written as

\[ T_{fi} = \langle \psi_{i\alpha}(p'_1) \psi_{k\gamma}(p'_2) | \hat{T} | \psi_{j\beta}(p_1) \psi_{l\delta}(p_2) \rangle \sim -i \cdot 2s \cdot \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot g_{M(ij,kl)}(s; t) . \quad (1.6) \]

At first sight, it could appear that the above expression (1.5) of the quantity \( g_M \) is essentially independent on the center–of–mass energy of the two quarks and that the \( s \) dependence of the scattering amplitude is all contained in the kinematical factor \( 2s \) in front of the integral in Eq. (1.2). This is clearly in contradiction with the well–known fact that amplitudes in QCD have a very non–trivial \( s \) dependence, whose origin lies in the infrared (IR) divergences typical of 3 + 1 dimensional gauge theories. In more standard perturbative approaches to high–energy QCD, based on the direct computation of Feynman diagrams in the high–energy limit, these IR divergences are taken care of by restricting the rapidities of the intermediate gluons to lie in between those of the two fast quarks (see, e.g., [5, 6]). The classical trajectories of two quarks with a non–zero mass \( m \) and a center–of–mass energy squared \( s = 4E^2 \) are related by a finite Lorentz boost with rapidity parameter \( \log(s/m^2) \), so that the size of the rapidity space for each intermediate gluon grows as \( \log s \) and each Feynman diagram acquires an overall factor proportional to some power of \( \log s \), depending on the number of intermediate gluon propagators.

In the case of the quantity (1.5), as was first pointed out by Verlinde and Verlinde in [7], the IR singularity is originated by the fact that the trajectories of the Wilson lines were taken to be lightlike and therefore have an infinite distance in rapidity space. One can regularize this infrared problem by giving the Wilson lines a small timelike component, such that they coincide with the classical trajectories for quarks with a non–zero mass \( m \) (this is equivalent to consider two Wilson lines forming a certain \textit{finite} hyperbolic angle \( \chi \) in Minkowski space–time; of course, \( \chi \to \infty \) when \( s \to \infty \)), and, in addition, by letting them end after some finite proper time \( \pm T \) (and eventually letting \( T \to \infty \)). Such a regularization of the IR singularities gives rise to an \( s \) dependence of the quantity \( g_M \) defined in (1.5) and, therefore, to a non–trivial \( s \) dependence of the amplitude (1.2), as obtained by ordinary perturbation theory [5, 6] and as confirmed by the experiments on hadron–hadron scattering processes. We refer the reader to Refs. [7] and [8, 9] for a detailed discussion about this point.
The direct evaluation of the expectation value (1.5) is a highly non-trivial matter and it is also strictly connected with the renormalization properties of Wilson–line operators [10, 11]. A non-perturbative approach for the calculation of (1.5) has been proposed and developed in Refs. [12, 13], in the framework of the so-called “stochastic vacuum model”. In two previous papers [8, 9] we proposed a new approach, which consists in analytically continuing the scattering amplitude from the Minkowskian to the Euclidean world, so opening the possibility of studying the scattering amplitude non perturbatively by well-known and well-established techniques available in the Euclidean theory (e.g., by means of the formulation of the theory on the lattice). This approach has been recently adopted in Refs. [14, 15], in order to study the high–energy scattering in strongly coupled gauge theories using the AdS/CFT correspondence, and also in Ref. [16], in order to investigate instanton–induced effects in QCD high–energy scattering.

More explicitly, in Refs. [8, 9] we have given arguments showing that the expectation value of two infinite Wilson lines, forming a certain hyperbolic angle $\chi$ in Minkowski space–time, and the expectation value of two infinite Euclidean Wilson lines, forming a certain angle $\theta$ in Euclidean four–space, are connected by an analytic continuation in the angular variables. This relation of analytic continuation was proven in Ref. [8] for an Abelian gauge theory (QED) in the so–called quenched approximation and for a non–Abelian gauge theory (QCD) up to the fourth order in the renormalized coupling constant in perturbation theory; a general proof was finally given in Ref. [9]. The relation of analytic continuation between the amplitudes $g_M(\chi; t)$ and $g_E(\theta; t)$, in the Minkowski and the Euclidean world, was derived in Refs. [8, 9] using infinite Wilson lines, i.e., directly in the limit $T \to \infty$ and assuming that the amplitudes were independent on $T$. In other words, the results derived in Refs. [8, 9] apply to the cutoff–independent part of the amplitudes.

On the contrary, in this paper we shall consider IR–regularized amplitudes at any $T$ (including also possible divergent pieces when $T \to \infty$). Generalizing the results of Ref. [9], in Sect. 2 of this paper we shall give the general proof that the expectation value of two IR–regularized Wilson lines, forming a certain hyperbolic angle in Minkowski space–time, and the expectation value of two IR–regularized Euclidean Wilson lines, forming a certain angle in Euclidean four–space, are connected by an analytic continuation in the angular variables and in the IR cutoff $T$. This result can be used to evaluate the IR–regularized high–energy scattering amplitude directly in the Euclidean theory. The conclusions and an outlook are given in Sect. 3.
2. From Minkowskian to Euclidean theory

Let us consider the following quantity, defined in Minkowski space–time:

\[ g_M(p_1, p_2; T; t) = \frac{M(p_1, p_2; T; t)}{Z_M(p_1; T)Z_M(p_2; T)} , \]

\[ M(p_1, p_2; T; t) = \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \langle [\tilde{W}^{(T)}_{p_1}(z_t) - 1]_{ij} [\tilde{W}^{(T)}_{p_2}(0) - 1]_{kl} \rangle , \]  

(2.1)

where \( z_t = (0, 0, \vec{z}_\perp) \) and \( q = (0, 0, \vec{q}_\perp) \), so that \( t = -\vec{q}_\perp^2 = q^2 \). The Minkowskian four–momenta \( p_1 \) and \( p_2 \) are arbitrary four–vectors lying in the longitudinal plane \((x^0, x^1)\) [so that \( \vec{p}_1 \parallel = \vec{p}_2 \parallel = \vec{0}_\perp \)] and define the trajectories of the two IR–regularized Wilson lines \( W^{(T)}_{p_1} \) and \( W^{(T)}_{p_2} \):

\[ W^{(T)}_{p_1}(z_t) \equiv T \exp \left[ -ig \int_{-T}^{+T} A_\mu(z_t + \frac{p_1}{m}) \frac{p_1^\mu}{m} d\tau \right] ; \]

\[ W^{(T)}_{p_2}(0) \equiv T \exp \left[ -ig \int_{-T}^{+T} A_\mu(\frac{p_2}{m}) \frac{p_2^\mu}{m} d\tau \right] . \]  

(2.2)

\( A_\mu = A^a_\mu T^a \) and \( m \) is the quark pole mass. \( T \) is our IR cutoff.

\( Z_M(p; T) \) in Eq. (2.1) is defined as (\( N_c \) being the number of colours)

\[ Z_M(p; T) \equiv \frac{1}{N_c} \langle \text{Tr}[W^{(T)}_p(z_t)] \rangle = \frac{1}{N_c} \langle \text{Tr}[W^{(T)}_p(0)] \rangle . \]  

(2.3)

(The last equality comes from the space–time translation invariance.) This is a sort of Wilson–line’s renormalization constant: as shown in Ref. \([4]\), \( Z_M(p; T \to \infty) \) is the residue at the pole (i.e., for \( p^2 \to m^2 \)) of the unrenormalized quark propagator, in the eikonal approximation.

In an analogous way, we can consider the following quantity, defined in Euclidean four–space:

\[ g_E(p_{1E}, p_{2E}; T; t) = \frac{E(p_{1E}, p_{2E}; T; t)}{Z_E(p_{1E}; T)Z_E(p_{2E}; T)} , \]

\[ E(p_{1E}, p_{2E}; T; t) = \int d^2 \vec{z}_\perp e^{i \vec{q}_\perp \cdot \vec{z}_\perp} \langle [\tilde{W}^{(T)}_{p_{1E}}(z_{tE}) - 1]_{ij} [\tilde{W}^{(T)}_{p_{2E}}(0) - 1]_{kl} \rangle_E , \]  

(2.4)

where \( z_{tE} = (0, \vec{z}_\perp, 0) \) and \( q_E = (0, \vec{q}_\perp, 0) \), so that \( t = -\vec{q}_\perp^2 = -q_E^2 \). The expectation value \( \langle \ldots \rangle_E \) must be intended now as a functional integration with respect to the gauge
variable \( A^{(E)}_\mu = A^{(E)\mu} T_\mu \) in the Euclidean theory. The Euclidean four–momenta \( p_{1E} \) and \( p_{2E} \) are arbitrary four–vectors lying in the plane \( (x_1, x_4) \) so that \( \vec{p}_{1E} \parallel \vec{b}_\perp \) and define the trajectories of the two IR–regularized Euclidean Wilson lines \( \tilde{W}^{(E)}_{p_{1E}} \) and \( \tilde{W}^{(E)}_{p_{2E}} \):

\[
\tilde{W}^{(E)}_{p_{1E}}(z_{tE}) \equiv \mathcal{T} \exp \left[ -ig \int_{-T}^{+T} A^{(E)}(z_{tE} + \frac{p_{1E}}{m} t) \frac{p_{1E}}{m} d\tau \right] ;
\]

\[
\tilde{W}^{(E)}_{p_{2E}}(0) \equiv \mathcal{T} \exp \left[ -ig \int_{-T}^{+T} A^{(E)}(z_{tE} + \frac{p_{2E}}{m} t) \frac{p_{2E}}{m} d\tau \right].
\]  

(2.5)

\( Z_E(p; T) \) in Eq. (2.4) is defined analogously to \( Z_M(p; T) \) in Eq. (2.3):

\[
Z_E(p; T) \equiv \frac{1}{N_c} \langle \text{Tr} [\tilde{W}^{(E)}_{p_{1E}}(z_{tE})] \rangle_E = \frac{1}{N_c} \langle \text{Tr} [\tilde{W}^{(E)}_{p_{2E}}(0)] \rangle_E .
\]  

(2.6)

(The last equality comes from the translation invariance in Euclidean four–space.)

We can now use the definition of the time–ordered exponential in Eq. (2.2) to explicitly write the Wilson lines \( W^{(E)}_{p_{1E}} \) and \( W^{(E)}_{p_{2E}} \) as power series in the exponents \( g \cdot A \). Therefore, the quantity \( M(p_1, p_2; T; t) \) is defined to be the series \( M = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} M_{(n,r)} \), where \( M_{(n,r)} \) is the contribution from the piece with \( (g \cdot A)^n \) in the expansion of \( W^{(E)}_{p_{1E}} \) and from the piece with \( (g \cdot A)^r \) in the expansion of \( W^{(E)}_{p_{2E}} \); it is given by:

\[
M_{(n,r)}(p_1, p_2; T; t) = (-ig)^{(n+r)}(T^{a_1} \ldots T^{a_n})_{ij}(T^{b_r} \ldots T^{b_1})_{kl} \int d^2 \vec{z} e^{i \vec{q} \cdot \vec{z}} \cdot \ldots \cdot \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) \theta(\omega_r - \omega_{r-1}) \ldots \theta(\omega_2 - \omega_1) \times \langle A^{a_n}_{\mu_n}(z_t + \frac{p_1}{m} \tau_n) \ldots A^{a_1}_{\mu_1}(z_t + \frac{p_1}{m} \tau_1) A^{b_r}_{\nu_r}(\frac{p_2}{m} \omega_r) \ldots A^{b_1}_{\nu_1}(\frac{p_2}{m} \omega_1) \rangle .
\]  

(2.7)

Analogously, the Euclidean quantity \( E(p_{1E}, p_{2E}; T; t) \) is defined to be the series \( E = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} E_{(n,r)} \), where \( E_{(n,r)} \) is the contribution from the pieces with \( (g \cdot A^{(E)})^n \) and \( (g \cdot A^{(E)})^r \) in the expansions of the Euclidean Wilson lines \( \tilde{W}^{(E)}_{p_{1E}} \) and \( \tilde{W}^{(E)}_{p_{2E}} \) respectively; it is given by:

\[
E_{(n,r)}(p_{1E}, p_{2E}; T; t) = (-ig)^{(n+r)}(T^{a_n} \ldots T^{a_1})_{ij}(T^{b_r} \ldots T^{b_1})_{kl} \int d^2 \vec{z} e^{i \vec{q} \cdot \vec{z}} \cdot \ldots \cdot \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) \theta(\omega_r - \omega_{r-1}) \ldots \theta(\omega_2 - \omega_1) \langle A^{a_n}_{(E)\mu_n}(z_{tE} + \frac{p_{1E}}{m} \tau_n) \ldots A^{a_1}_{(E)\mu_1}(z_{tE} + \frac{p_{1E}}{m} \tau_1) A^{b_r}_{(E)\nu_r}(\frac{p_{2E}}{m} \omega_r) \ldots A^{b_1}_{(E)\nu_1}(\frac{p_{2E}}{m} \omega_1) \rangle .
\]  

(2.8)
It is known that, making use of the correspondence
\[ A_0(x) \rightarrow iA_4^{(E)}(x_E) \quad , \quad A_k(x) \rightarrow A_k^{(E)}(x_E) \]
with: \( x^0 \rightarrow -ix_E \quad , \quad \vec{x} \rightarrow \vec{x}_E \),
\[ (2.9) \]
between the Minkowski and the Euclidean world, the following relationship is derived between the gluonic Green functions in the two theories:
\[
\tilde{B}_{(1)}^{\mu_1} \ldots \tilde{B}_{(N)}^{\mu_N} \langle A_{\mu_1}^{a_1}(\vec{x}(1)) \ldots A_{\mu_N}^{a_N}(\vec{x}(N)) \rangle = B_{(1)E}^{\mu_1} \ldots B_{(N)E}^{\mu_N} \langle A_{(E)\mu_1}^{a_1}(x_{1E}) \ldots A_{(E)\mu_N}^{a_N}(x_{NE}) \rangle E ,
\]
\[ (2.10) \]
where \( x_{(k)E} = (\vec{x}_{(k)E}, x_{(k)E}) \) are Euclidean four–vectors and \( B_{(k)E} = (\vec{B}_{(k)E}, B_{(k)E}E) \) are any Euclidean four–vectors, while \( \vec{x}_{(k)} \) and \( B_{(k)} \) are Minkowski four–vectors defined as
\[
\vec{x}_{(k)} = (\vec{x}_0, \vec{x}_{(k)}) = (-ix_{(k)E1}, \vec{x}_{(k)E}) ,
\]
\[ \tilde{B}_{(k)} = (\vec{B}_{(k)}^0, \vec{B}_{(k)}) = (-iB_{(k)E1}, \vec{B}_{(k)E}) . \]
\[ (2.11) \]
In our specific case, we can use Eq. \[ (2.10) \] to state that
\[ \tilde{p}_1^\mu = \frac{\tilde{p}_1^{\mu_1}}{m} \ldots \tilde{p}_2^\nu = \frac{\tilde{p}_2^{\nu_1}}{m} \ldots \tilde{p}_{2E}^\nu = \frac{\tilde{p}_{2E}^{\nu_1}}{m} \langle A_{\mu_1}^{a_1}(z_1 + \frac{\tilde{p}_1^{\mu_1}}{m} \tau_1) \ldots A_{\mu_2}^{a_2}(z_2 + \frac{\tilde{p}_1^{\mu_2}}{m} \tau_2) \ldots A_{\nu_1}^{b_1}(\frac{\tilde{p}_{2E}^{\nu_1}}{m} \omega_1) \rangle 
\]
\[ = \frac{\tilde{p}_1^{\mu_1}}{m} \ldots \frac{\tilde{p}_2^{\nu_1}}{m} \ldots \frac{\tilde{p}_{2E}^{\nu_1}}{m} \langle A_{(E)\mu_1}^{a_1}(z_{1E} + \frac{\tilde{p}_{1E}^{\mu_1}}{m} \tau_1) \ldots A_{(E)\mu_2}^{a_2}(z_{2E} + \frac{\tilde{p}_{1E}^{\mu_2}}{m} \tau_2) \ldots A_{(E)\nu_1}^{b_1}(\frac{\tilde{p}_{2E}^{\nu_1}}{m} \omega_1) \rangle , 
\]
\[ (2.12) \]
where \( p_{kE} = (\tilde{p}_{kE}, \tilde{p}_{kE}E) \), for \( k = 1, 2 \), are the two Euclidean four–vectors introduced above and \( \tilde{p}_k \) are the two corresponding Minkowskian four–vectors, obtained according to Eq. \[ (2.11) \]:
\[
\tilde{p}_k = (\tilde{p}_k^0, \tilde{p}_k) = (-ip_{kE1}, \tilde{p}_{kE}) , \quad \text{for } k = 1, 2 .
\]
\[ (2.13) \]
By virtue of the definitions \[ (2.7) \] and \[ (2.8) \] for \( M_{(n,r)} \) and \( E_{(n,r)} \) respectively, Eq. \[ (2.12) \] implies that:
\[
E_{(n,r)}(p_{1E}, p_{2E}; T; t) = M_{(n,r)}(\tilde{p}_1, \tilde{p}_2; T; t) .
\]
\[ (2.14) \]
This relation is valid for every couple of integer numbers \( (n, r) \), so that, more generally, \( \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} E_{(n,r)}(p_{1E}, p_{2E}; T; t) = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} M_{(n,r)}(\tilde{p}_1, \tilde{p}_2; T; t) \); and therefore, by definition:
\[
E(p_{1E}, p_{2E}; T; t) = M(\tilde{p}_1, \tilde{p}_2; T; t) .
\]
\[ (2.15) \]
Moreover, one has that, changing the integration variable in the exponent of the Wilson line from the real proper time $\tau$ to the imaginary proper time $\tau' \equiv -i\tau$:

\[
W^{(T)}_{\tilde{p}_1}(z_t) = \mathcal{T} \exp \left[ -ig \int_{-T}^{+T} A_\mu(z_t + \frac{\tilde{p}_1}{m} \tau) \frac{\tilde{p}_1^\mu}{m} d\tau \right]
\]

\[
= \mathcal{T} \exp \left[ -ig \int_{iT}^{-iT} A_\mu(z_t + \frac{\tilde{p}_1}{m} \tau') \frac{\tilde{p}_1^\mu}{m} d\tau' \right]
\]

\[
= W^{(-iT)}_{\tilde{p}_1}(z_t)
\]

(2.16)

and, similarly:

\[
W^{(T)}_{\tilde{p}_2}(0) = W^{(-iT)}_{\tilde{p}_2}(0)
\]

(2.17)

where the Minkowskian four–vectors $\bar{p}_k$ are defined as:

\[
\bar{p}_1 = i\tilde{p}_1 = (p_{kE1}, i\vec{p}_{kE})
\]

for $k = 1, 2$.

(2.18)

The following prescription for the $\mathcal{T}$–ordered product of a bosonic field $B$ in the imaginary domain is used:

\[
\mathcal{T} B(\tau_1') B(\tau_2') = B(\tau_1') B(\tau_2')
\]

if $i\tau_1' > i\tau_2'$;

\[
\mathcal{T} B(\tau_1') B(\tau_2') = B(\tau_2') B(\tau_1')
\]

if $i\tau_1' < i\tau_2'$.

(2.19)

In other words, $\theta(\tau' = -i\tau) \equiv \theta(\tau)$, for every real $\tau$: this prescription is used in order to keep the $\mathcal{T}$–ordering unchanged when going from Minkowskian to Euclidean theory, $(x^0, \vec{x}) \rightarrow (-ix_{E1}, \vec{x}_{E})$. From the definition of $M(p_1, p_2; T; t)$ given in Eq. (2.1), one gets that:

\[
M(\bar{p}_1, \bar{p}_2; T; t) = M(\bar{p}_1, \bar{p}_2; -iT; t).
\]

(2.20)

And therefore, from the Eq. (2.19) derived above:

\[
E(p_{1E}, p_{2E}; T; t) = M(\bar{p}_1, \bar{p}_2; -iT; t).
\]

(2.21)

We also observe that, rescaling the four–momentum $p$ in the Wilson line by a positive constant $\alpha$:

\[
W^{(T)}_{\alpha p}(z_t) = \mathcal{T} \exp \left[ -ig \int_{-T}^{+T} A_\mu(z_t + \frac{\alpha p}{m} \tau) \frac{\alpha p^\mu}{m} d\tau \right]
\]

\[
= \mathcal{T} \exp \left[ -ig \int_{-\alpha T}^{+\alpha T} A_\mu(z_t + \frac{p}{m} \tau') \frac{p^\mu}{m} d\tau' \right]
\]

\[
= W^{(\alpha T)}_{p}(z_t), \quad \forall \alpha > 0.
\]

(2.22)
And similarly, for the Euclidean Wilson lines:

\[ \tilde{W}_{\alpha_\text{op}}^{(T)}(z_{\text{E}}) = \tilde{W}_{\text{p}_\text{E}}^{(\alpha T)}(z_{\text{E}}), \quad \forall \alpha > 0. \] (2.23)

Of course \( M \), considered as a general function of \( p_1, p_2 \) [and \( q = (0,0,\vec{q}) \)], can only depend on the scalar quantities constructed with the vectors \( p_1, p_2 \) and \( q = (0,0,\vec{q}) \): the only possibilities are \( q^2 = -\vec{q}_\perp^2 = t, p_1 \cdot p_2, p_1^2 \) and \( p_2^2 \), since \( p_1 \cdot q = p_2 \cdot q = 0 \). Therefore, from the result (2.22) found above, \( M \) is forced to have the following form:

\[ M(p_1, p_2; T; t) = f_M(u_1 T^2, u_2 T^2, (u_1 \cdot u_2) T^2; t), \] (2.24)

where \( u_1 \equiv p_1/m \) and \( u_2 \equiv p_2/m \). [One can easily derive this by first introducing two different IR cutoffs, \( T_1 \) and \( T_2 \), for the two Wilson lines \( W_{p_1} \) and \( W_{p_2} \), then by using the result (2.22) found above, so deriving the relation \( M(p_1, p_2; T_1; T_2; t) = f_M(u_1 T_1^2, u_2 T_2^2, (u_1 \cdot u_2) T_1 T_2; t) \), and finally by putting \( T_1 = T_2 \equiv T \).]

For analogous reasons, \( E \) must be of the form:

\[ E(p_{1E}, p_{2E}; T; t) = f_E(u_{1E} T^2, u_{2E} T^2, (u_{1E} \cdot u_{2E}) T^2; t), \] (2.25)

where \( u_{1E} \equiv p_{1E}/m \) and \( u_{2E} \equiv p_{2E}/m \). [A short remark about the notation: we have denoted everywhere the scalar product by a “.,” both in the Minkowski and the Euclidean world. Of course, when \( A \) and \( B \) are Minkowski four–vectors, then \( A \cdot B = A^\mu B_\mu = A^0 B^0 - \vec{A} \cdot \vec{B} \); while, if \( A_E \) and \( B_E \) are Euclidean four–vectors, then \( A_E \cdot B_E = A_{E\mu} B_{E\mu} = \vec{A}_E \cdot \vec{B}_E + A_{E4} B_{E4}. \) Therefore, the relations (2.15) and (2.21) found above can be reformulated as follows [observing that \( \vec{p}^2 = -\vec{p}_E^2 = -p_E^2 \) and \( \vec{p}_1 \cdot \vec{p}_2 = -\vec{p}_1 \cdot \vec{p}_2 = -p_{1E} \cdot p_{2E} \):]

\[ f_E(u_{1E} T^2, u_{2E} T^2, (u_{1E} \cdot u_{2E}) T^2; t) = f_M(-u_{1E}^2 T^2, -u_{2E}^2 T^2, -(u_{1E} \cdot u_{2E}) T^2; t). \] (2.26)

Since we finally want to obtain the expression (1.3) of the scattering amplitude in the c.m.s. of the two quarks, taking their initial trajectories along the \( x_1 \)–axis, we choose \( p_1 \) and \( p_2 \) to be the four–momenta of the two particles with mass \( m \), moving with speed \( \beta \) and \( -\beta \) along the \( x_1 \)–direction, i.e.,

\[ p_1 = E(1, \beta, 0, 0), \]
\[ p_2 = E(1, -\beta, 0, 0), \] (2.27)
where \( E = m/\sqrt{1-\beta^2} \) (in units with \( c = 1 \)) is the energy of each particle (so that: \( s = 4E^2 \)). We now introduce the hyperbolic angle \( \psi \) [in the plane \((x^0, x^1)\)] of the trajectory of \( W_{p_1}^{(T)} \): it is given by \( \beta = \tanh \psi \). We can give the explicit form of the Minkowski four–vectors \( u_1 = p_1/m \) and \( u_2 = p_2/m \) in terms of the hyperbolic angle \( \psi \):

\[
\begin{align*}
u_1 &= \frac{p_1}{m} = (\cosh \psi, \sinh \psi, 0, 0) , \\
u_2 &= \frac{p_2}{m} = (\cosh \psi, -\sinh \psi, 0, 0) .
\end{align*}
\]

(2.28)

Clearly, \( u_1^2 = u_2^2 = 1 \) and

\[
u_1 \cdot \nu_2 = \cosh(2\psi) = \cosh \chi ,
\]

(2.29)

where \( \chi = 2\psi \) is the hyperbolic angle [in the plane \((x^0, x^1)\)] between the two trajectories of \( W_{p_1}^{(T)} \) and \( W_{p_2}^{(T)} \).

Analogously, in the Euclidean theory we choose a reference frame in which the spatial vectors \( \vec{p}_{1E} \) and \( \vec{p}_{2E} = -\vec{p}_{1E} \) are along the \( x^1 \)–direction and, moreover, \( p_{1E}^2 = p_{2E}^2 = m^2 \); that is,

\[
\begin{align*}
p_{1E} &= m(\sin \phi, 0, 0, \cos \phi) ; \\
p_{2E} &= m(-\sin \phi, 0, 0, \cos \phi) .
\end{align*}
\]

(2.30)

where \( \phi \) is the angle formed by each trajectory with the \( x_4 \)–axis. The value of \( \phi \) is between \( 0 \) and \( \pi/2 \), so that the angle \( \theta = 2\phi \) between the two Euclidean trajectories \( \vec{W}_{p_{1E}}^{(T)} \) and \( \vec{W}_{p_{2E}}^{(T)} \) lies in the range \([0, \pi]\): it is always possible to make such a choice by virtue of the \( O(4) \) symmetry of the Euclidean theory. The two four–momenta \( u_{1E} \) and \( u_{2E} \) are, therefore:

\[
\begin{align*}
u_{1E} &= \frac{p_{1E}}{m} = (\sin \phi, 0, 0, \cos \phi) ; \\
u_{2E} &= \frac{p_{2E}}{m} = (-\sin \phi, 0, 0, \cos \phi) ,
\end{align*}
\]

(2.31)

Clearly, \( u_{1E}^2 = u_{2E}^2 = 1 \) and

\[
u_{1E} \cdot \nu_{2E} = \cos \theta .
\]

(2.32)

With this choice, one has that:

\[
\begin{align*}
\vec{p}_1 &= m(\cos \frac{\theta}{2}, i \sin \frac{\theta}{2}, 0, 0) = m(\cosh \frac{i\theta}{2}, \sinh \frac{i\theta}{2}, 0, 0) ; \\
\vec{p}_2 &= m(\cos \frac{\theta}{2}, -i \sin \frac{\theta}{2}, 0, 0) = m(\cosh \frac{i\theta}{2}, -\sinh \frac{i\theta}{2}, 0, 0) .
\end{align*}
\]

(2.33)
A comparison with the expressions (2.28) for the Minkowski four–vectors \( u_1 = p_1/m \) and \( u_2 = p_2/m \) reveals that \( \bar{p}_1 \) and \( \bar{p}_2 \) are obtained from \( p_1 \) and \( p_2 \) after the following analytic continuation in the angular variables is made:

\[
\chi \to i\theta . \tag{2.34}
\]

(We remind that \( \phi = \theta/2 \) and \( \psi = \chi/2 \).) Therefore, if we denote with \( M(\chi; T; t) \) the value of \( M(p_1, p_2; T; t) \) for \( p_1 \) and \( p_2 \) given by Eq. (2.27) and we also denote with \( E(\theta; T; t) \) the value of \( E(p_{1E}, p_{2E}; T; t) \) for \( p_{1E} \) and \( p_{2E} \) given by Eq. (2.30), we find, using the result (2.21) derived above:

\[
E(\theta; T; t) = M(\chi \to i\theta; T \to -iT; t) . \tag{2.35}
\]

This is, of course, in agreement with the relation (2.26) found above, observing that \( M(\chi ; T; t) = f_M(T^2, T^2, T^2 \cosh \chi; t) \) and \( E(\theta ; T; t) = f_E(T^2, T^2, T^2 \cos \theta; t) \).

Let us consider, now, the Wilson–line’s renormalization constant \( Z_M(p; T) \):

\[
Z_M(p; T) \equiv \frac{1}{N_c} \langle \text{Tr}[W_p^{(T)}(0)] \rangle . \tag{2.36}
\]

Again, we can use the definition of the time–ordered exponential in Eq. (2.2) to expand the Wilson line \( W_p^{(T)}(0) \) in powers of the exponent \( g \cdot A \). The quantity \( Z_M(p; T) \) is thus defined to be the series \( Z_M(p; T) = \sum_{n=1}^{\infty} Z_M^{(n)}(p; T) \), where \( Z_M^{(n)}(p; T) \) is the contribution from the piece with \( (g \cdot A)^n \) in the expansion of \( W_p^{(T)}(0) \); it is given by:

\[
Z_M^{(n)}(p; T) = \left( \frac{-ig}{N_c} \right)^n \langle \text{Tr}(T^{a_n} \ldots T^{a_1}) \int_{-T}^{+T} d\tau_1 \frac{p_{\mu_1}}{m} \ldots \int_{-T}^{+T} d\tau_n \frac{p_{\mu_n}}{m} \times \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) \langle A_{\mu_n}^{a_n}(\frac{p}{m}) \tau_n \rangle \ldots A_{\mu_1}^{a_1}(\frac{p}{m}) \tau_1 \rangle \rangle . \tag{2.37}
\]

In the Euclidean theory we have, analogously:

\[
Z_E(p_E; T) \equiv \frac{1}{N_c} \langle \text{Tr}[\bar{W}_E^{(T)}(0)] \rangle_E , \tag{2.38}
\]

and \( Z_E(p_E; T) = \sum_{n=1}^{\infty} Z_E^{(n)}(p_E; T) \), with

\[
Z_E^{(n)}(p_E; T) = \left( \frac{-ig}{N_c} \right)^n \langle \text{Tr}(T^{a_n} \ldots T^{a_1}) \int_{-T}^{+T} d\tau_1 \frac{p_{\mu_1}^{E}}{m} \ldots \int_{-T}^{+T} d\tau_n \frac{p_{\mu_n}^{E}}{m} \times \theta(\tau_n - \tau_{n-1}) \ldots \theta(\tau_2 - \tau_1) \langle A_{\mu_n}^{a_n}(\frac{p_E}{m}) \tau_n \rangle \ldots A_{\mu_1}^{a_1}(\frac{p_E}{m}) \tau_1 \rangle \rangle_E . \tag{2.39}
\]
Using Eq. (2.10), we can derive the following relation:

\[
\frac{\tilde{p}^\mu_1}{m} \ldots \frac{\tilde{p}^\mu_n}{m} \langle A_{\mu_n}^a(\frac{\tilde{p}}{m}) \tau_n \rangle \ldots \langle A_{\mu_1}^{a_1}(\frac{\tilde{p}}{m}) \tau_1 \rangle = \frac{p_E^\mu_1}{m} \ldots \frac{p_E^\mu_n}{m} \langle A_{(E)\mu_n}^a(\frac{p_E}{m}) \tau_n \rangle \ldots \langle A_{(E)\mu_1}^{a_1}(\frac{p_E}{m}) \tau_1 \rangle E ,
\]

(2.40)

where, as usual, \( p_E = (p_E^0, p, p_E^4) \) and \( \tilde{p} = (\tilde{p}^0, \tilde{p}) = (-ip_E^4, \tilde{p}) \). From this relation we obtain

\[
Z_E^{(n)}(p_E; T) = Z_M^{(n)}(\tilde{p}; T) .
\]

(2.41)

This relation is valid for every integer number \( n \), so that we also have, more generally,

\[
\sum_{n=1}^{\infty} Z_E^{(n)}(p_E; T) = \sum_{n=1}^{\infty} Z_M^{(n)}(\tilde{p}; T) ;
\]

and therefore, by definition:

\[
Z_E(p_E; T) = Z_M(\tilde{p}; -iT) .
\]

(2.42)

Moreover, from Eq. (2.16) one derives that \( Z_M(\tilde{p}; T) = Z_M(\tilde{p}; -iT) \), where, as usual, \( \tilde{p} = i\tilde{p} \). And therefore, from Eq. (2.42):

\[
Z_E(p_E; T) = Z_M(\tilde{p}; -iT) .
\]

(2.43)

From the definition (2.3), \( Z_M(p; T) \), considered as a function of a general four–vector \( p \), is a scalar function constructed with the only four–vector \( u = p/m \). In addition, by virtue of the property (2.22) of the Wilson lines, one has that \( Z_M(\alpha p; T) = Z_M(p; \alpha T) \) for every positive \( \alpha \). Therefore, \( Z_M(p; T) \) is forced to have the form

\[
Z_M(p; T) = H_M(u^2 T^2) ,
\]

(2.44)

where \( u = p/m \). In a perfectly analogous way, for the Euclidean case we have that:

\[
Z_E(p_E; T) = H_E(u_E^2 T^2) ,
\]

(2.45)

where \( u_E = p_E/m \). Therefore, the relations (2.42) and (2.43) found above can be reformulated as follows [observing that \( \tilde{p}^2 = -\tilde{p}^2 = -p_E^2 \)]:

\[
H_E(u_E^2 T^2) = H_M(-u_E^2 T^2) .
\]

(2.46)

13
Therefore, if we denote with $Z_W(T)$ the value of $Z_M(p_1; T)$ or $Z_M(p_2; T)$, for $p_1$ and $p_2$ given by Eq. (2.27), and we also denote with $Z_{WE}(T)$ the value of $Z_E(p_{1E}; T)$ or $Z_E(p_{2E}; T)$, for $p_{1E}$ and $p_{2E}$ given by Eq. (2.30), i.e.,

$$
Z_W(T) \equiv Z_M(p_1; T) = Z_M(p_2; T) = H_M(T^2) ;
Z_{WE}(T) \equiv Z_E(p_{1E}; T) = Z_E(p_{2E}; T) = H_E(T^2) ,
$$

(2.47)

we find, using the result (2.43) derived above:

$$
Z_{WE}(T) = Z_W(-iT) .
$$

(2.48)

Combining this identity with Eq. (2.33), we find that the Minkowskian and the Euclidean amplitudes, defined by Eqs. (2.1), (2.4) and (2.47), with $p_1$ and $p_2$ given by Eq. (2.27) and $p_1E$ and $p_2E$ given by Eq. (2.30), i.e.,

$$
g_M(\chi; T; t) \equiv \frac{M(\chi; T; t)}{[Z_W(T)]^2} , \quad g_E(\theta; T; t) \equiv \frac{E(\theta; T; t)}{[Z_{WE}(T)]^2} ,
$$

are connected by the following relation:

$$
g_E(\theta; T; t) = g_M(\chi \rightarrow i\theta; T \rightarrow -iT; t) ;
or : g_M(\chi; T; t) = g_E(\theta \rightarrow -i\chi; T \rightarrow iT; t) .
$$

(2.50)

We have derived the relation (2.50) of analytic continuation for a non-Abelian gauge theory with gauge group $SU(N_c)$. It is clear, from the derivation given above, that the same result is valid also for an Abelian gauge theory (QED).

Moreover, even if the result (2.50) has been explicitly derived for the case of the quark–quark scattering, it is immediately generalized to the more general case of the parton–parton scattering, where each parton can be a quark, an antiquark or a gluon. In fact, as explained in the Introduction, one simply has to use a proper Wilson line for each parton: $W_p(b)$, the Wilson string in the fundamental representation $T^a$, for a quark; $W_p^*(b)$, the Wilson string in the complex conjugate representation $T^*_a = -T^*_a$, for an antiquark; and $V_k(b)$, the Wilson string in the adjoint representation $T^a_{(adj)}$, for a gluon. The proof leading to Eq. (2.50) is then repeated step by step, after properly modifying the definitions (2.2) of the Wilson lines. [If the parton is a gluon, one must substitute the quark mass $m$ appearing in all previous formulae with an arbitrarily small mass $\mu \rightarrow 0$. The IR cutoff appears in all expressions in the form of the ratio $T/\mu$ for a gluon and $T/m$ for a quark/antiquark.]
3. Conclusions and outlook

In this paper we have completely generalized the results of Ref. [9], where we derived a relation of analytic continuation between the amplitudes $g_M(\chi; T)$ and $g_E(\theta; t)$, in the Minkowski and the Euclidean world, using infinite Wilson lines, i.e., directly in the limit $T \to \infty$ and assuming that the amplitudes were independent on $T$. In other words, we can claim that the results of Ref. [9] apply to the cutoff–independent part of the amplitudes, while, in this paper, we have derived the relation (2.50) of analytic continuation between IR–regularized amplitudes at any $T$.

The result (2.50) found in this paper can be used to evaluate the IR–regularized high–energy parton–parton scattering amplitude directly in the Euclidean theory. In fact, the IR–regularized high–energy scattering amplitude is given (e.g., for the case of the quark–quark scattering) by

$$T_{fi} = \langle \psi_i(p'_1) \psi_k(p'_2) | T | \psi_j(p_1) \psi_l(p_2) \rangle \sim -i \cdot 2s \cdot \delta_{\alpha\beta} \delta_{\gamma\delta} \cdot g_M(\chi \to \infty; T \to \infty; t) ,$$

(3.1)

where the quantity $g_M(\chi; T; t)$, defined by Eq. (2.1), is essentially a correlation function of two IR–regularized Wilson lines forming a certain hyperbolic angle $\chi$ in Minkowski space–time. For deriving the dependence on $s$ one exploits the fact that the hyperbolic angle $\chi$ is a function of $s$. In fact, from $s = 4E^2$, $E = m/\sqrt{1-\beta^2}$, and $\beta = \tanh(\chi/2)$ [see Eqs. (2.27), (2.28) and (2.29)], one immediately finds that:

$$s = 2m^2(\cosh \chi + 1) .$$

(3.2)

Therefore, in the high–energy limit $s \to \infty$ (or $\chi \to \infty$, i.e., $\beta \to 1$), the hyperbolic angle $\chi$ is essentially equal to the logarithm of $s/m^2$ (for a non–zero quark mass $m$):

$$\chi \sim s \to \infty \log \left( \frac{s}{m^2} \right) .$$

(3.3)

The quantity $g_M(\chi; T; t)$ is linked to the corresponding Euclidean quantity $g_E(\theta; T; t)$, defined by Eq. (2.4), by the analytic continuation (2.50) in the angular variables and in the IR cutoff $T$. Therefore, one can start by evaluating $g_E(\theta; T; t)$, which is essentially a correlation function of two IR–regularized Wilson lines forming a certain angle $\theta$ in
Euclidean four–space, then by continuing this quantity into Minkowski space–time by rotating the Euclidean angular variable clockwise, $\theta \rightarrow -i\chi$, and the IR cutoff (Euclidean proper time) anticlockwise, $T \rightarrow iT$: in such a way one reconstructs the Minkowskian quantity $g_M(\chi; T; t)$. As was pointed out in [15], one should note that a priori there is an ambiguity in making such an analytical continuation, depending on the precise choice of the path. This phenomenon does not appear when the Euclidean correlation function $g_E(\theta; T; t)$ has only simple poles in the complex $\theta$–plane, but in some cases the analyticity structure can contain branch cuts in the complex plane, which must be taken into account: we refer the reader to Ref. [15] for a full discussion about this point.

We want to conclude by making a remark about the problem of the IR divergences which appear in the high–energy scattering amplitudes.

A well–known feature of the parton–parton scattering amplitude is its IR divergence, which, as we have already said in the Introduction, is typical of $3 + 1$ dimensional gauge theories and which, in our formulation, manifests itself in the IR singularity of the correlation function of two Wilson lines when $T \rightarrow \infty$. In many cases these IR divergences can be factorized out.

As suggested in Ref. [15], an alternative way to eliminate this cutoff dependence is to consider an IR–finite physical quantity, like the scattering amplitude of two colourless states in gauge theories, e.g., two $q\bar{q}$ meson states. It was shown in Ref. [4] that the high–energy meson–meson scattering amplitude can be approximately reconstructed by first evaluating, in the eikonal approximation, the scattering amplitude of two $q\bar{q}$ pairs, of given transverse sizes $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$ respectively, and then folding this amplitude with two proper wave functions $\omega_1(\vec{R}_{1\perp})$ and $\omega_2(\vec{R}_{2\perp})$ describing the two interacting mesons. It turns out that the high–energy scattering amplitude of two $q\bar{q}$ pairs of transverse sizes $\vec{R}_{1\perp}$ and $\vec{R}_{2\perp}$, and impact–parameter distance $z_{\perp}$, is governed by the correlation function of two Wilson loops $\mathcal{W}_1$ and $\mathcal{W}_2$, which follow the classical straight lines for quark (antiquark) trajectories $[2, 12]$:

$$\mathcal{W}_1 \rightarrow X^\mu_{\pm 1}(\tau) = z^\mu_1 + \frac{p^\mu_1}{m} \tau \pm \frac{R^\mu_1}{2} ;$$

$$\mathcal{W}_2 \rightarrow X^\mu_{\pm 2}(\tau) = \frac{p^\mu_2}{m} \tau \pm \frac{R^\mu_2}{2} , \quad (3.4)$$

[where $R_{1\perp} = (0, 0, \vec{R}_{1\perp})$ and $R_{2\perp} = (0, 0, \vec{R}_{2\perp})$] and close at proper times $\tau = \pm T$.

The same analytical continuation (2.50), that we have derived for the case of Wilson lines, is, of course, expected to apply also to the Wilson–loop correlators: the proof can
be repeated going essentially through the same steps as in the previous section, after adapting the definitions \(2.2\) from the case of Wilson lines to the case of Wilson loops. However, in this case the cutoff dependence on \(T\) is expected to be removed together with the related IR divergence which was present for the case of Wilson lines.

In our opinion, the high–energy scattering problem could be directly investigated on the lattice using this Wilson–loop formulation. A further advantage of the Wilson–loop formulation, which makes it suitable to be studied on the lattice, is that, contrary to the Wilson–line formulation, it is manifestly gauge–invariant. (In the case of the parton–parton scattering amplitude, gauge invariance can be restored, at least for the diffractive, i.e., no-colour-exchange, part proportional to \(\langle \text{Tr}[W_{p_1}(z_t) - 1]\text{Tr}[W_{p_2}(0) - 1]\rangle\), by requiring that the gauge transformations at both ends of the Wilson lines are the same \[1, 7\].) A considerable progress is expected along this line in the near future.

Acknowledgements

I would like to thank R. Peschanski for useful discussions (during the “Sixth Workshop on Non–Perturbative Quantum Chromodynamics”, Paris, France, June 5th–9th, 2001), which have inspired this work.
References

[1] O. Nachtmann, Ann. Phys. 209 (1991) 436.

[2] O. Nachtmann, in *Perturbative and Nonperturbative aspects of Quantum Field Theory*, edited by H. Latal and W. Schweiger (Springer–Verlag, Berlin, Heidelberg, 1997).

[3] E. Meggiolaro, Phys. Rev. D 53 (1996) 3835.

[4] E. Meggiolaro, Nucl. Phys. B 602 (2001) 261.

[5] H. Cheng and T.T. Wu, *Expanding Protons: Scattering at High Energies* (MIT Press, Cambridge, Massachusetts, 1987).

[6] L.N. Lipatov, in *Review in Perturbative QCD*, edited by A.H. Mueller (World Scientific, Singapore, 1989), and references therein.

[7] H. Verlinde and E. Verlinde, Princeton University, report No. PUP T–1319 (revised 1993); [hep–th/9302104](http://arxiv.org/abs/hep-th/9302104).

[8] E. Meggiolaro, Z. Phys. C 76 (1997) 523.

[9] E. Meggiolaro, Eur. Phys. J. C 4 (1998) 101.

[10] I.Ya. Aref’eva, Phys. Lett. 93B (1980) 347.

[11] G.P. Korchemsky, Phys. Lett. B 325 (1994) 459; I.A. Korchemskaya and G.P. Korchemsky, Nucl. Phys. B 437 (1995) 127.

[12] H.G. Dosch, E. Ferreira and A. Krämer, Phys. Rev. D 50 (1994) 1992.

[13] E.R. Berger and O. Nachtmann, Eur. Phys. J. C 7 (1999) 459.

[14] R.A. Janik and R. Peschanski, Nucl. Phys. B 565 (2000) 193.

[15] R.A. Janik and R. Peschanski, Nucl. Phys. B 586 (2000) 163.

[16] E. Shuryak and I. Zahed, Phys. Rev. D 62 (2000) 085014.