A QUANTITATIVE HOPF-TYPE MAXIMUM PRINCIPLE FOR
SUBSOLUTIONS OF ELLIPTIC PDES

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Dedicated to Gisèle Ruiz Goldstein

Abstract. Suppose that \( u(x) \) is a positive subsolution to an elliptic equation in a bounded domain \( D \), with the \( C^2 \) smooth boundary \( \partial D \). We prove a quantitative version of the Hopf maximum principle that can be formulated as follows: there exists a constant \( \gamma > 0 \) such that \( \partial_n u(\tilde{x}) \) – the outward normal derivative at the maximum point \( \tilde{x} \) ∈ \( \partial D \) (necessary located at \( \partial D \), by the strong maximum principle) – satisfies \( \partial_n u(\tilde{x}) > \gamma u(\tilde{x}) \), provided the coefficient \( c(x) \) by the zero order term satisfies \( \sup_{x \in D} c(x) = -c_* < 0 \). The constant \( \gamma \) depends only on the geometry of \( D \), uniform ellipticity bound, \( L^\infty \) bounds on the coefficients, and \( c_* \). The key tool used is the Feynman–Kac representation of a subsolution to the elliptic equation.

1. Introduction. Suppose that \( D \) is a bounded domain, i.e. a connected and open set, with a \( C^2 \) smooth boundary \( \partial D \), so that the outward unit vector \( n(x) \) is defined for any \( x \in \partial D \). Assume furthermore that we are given a uniformly elliptic differential operator with bounded coefficients

\[
L[u](x) := \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j}(x) \partial^2_{x_i,x_j} u(x) + \sum_{i=1}^{n} b_i(x) \partial_{x_i} u(x) + c(x) u(x), \quad x \in D
\]

for \( u \in C^2(D) \). The classical theorem of Hopf, see e.g. Theorem 2.3.8, p. 67 of [4], concerning strict positivity of the outward normal derivative of a subsolution of an elliptic equation at the maximal point at the boundary, can be stated as follows.

Theorem 1.1. Suppose that \( D \) and operator \( L \) are as stated above. Assume furthermore that:

i) \( c(x) \leq 0, \quad x \in D \),

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Then, \( \partial_n u(\tilde{x}) \), if exists, is strictly positive (i.e. \( \partial_n u(\tilde{x}) > 0 \)), unless \( u(x) \equiv u(\tilde{x}) \).

This result can be used e.g. to prove the strong maximum principle of Hopf, see Theorem 3.2 of [2]. For a good review of existing results concerning the maximum principle for second order elliptic operators we refer a reader to the already mentioned monographs [2, 4] and paper [1].

In the present note we show that \( \partial_n u(\tilde{x}) > \gamma u(\tilde{x}) \), at a maximal point \( \tilde{x} \in \partial D \), for subsolutions satisfying \( u(\tilde{x}) \geq 0 \) and not identically equal to 0, see Theorem 2.2 below. The constant \( \gamma \) turns out to depend only on the geometry of \( D \), \( L^\infty \)-bounds on the coefficients, ellipticity constant and sup \( x \in D \) \( c(x) \), which is assumed to be strictly negative. Our main tool is the Feynmann–Kac formula to represent the subsolution of an elliptic equation, see Lemma 3.3 below. This result seems to be a natural extension of Hopf’s theorem but we could not find it formulated in such a form in the existing literature.

2. The statement of the main result. Suppose that \( D \subset \mathbb{R}^n \) is a bounded domain (i.e. a bounded, open and connected set). We start by recalling the following definition.

Definition 2.1. The boundary \( \partial D \) of \( D \) is said to be \( C^k \)-smooth if for any \( x \in \partial D \) there exists an open ball \( K \) and a \( C^k \) mapping \( \phi : K \to \mathbb{R} \) such that

D1) \( \phi(K \cap D) \subset \mathbb{R}_+ \equiv (0, +\infty) \),

D2) \( x \in \partial D \cap K \) iff \( \phi(x) = 0 \),

D3) \( \nabla \phi(x) \neq 0, x \in \partial D \cap K \).

Suppose that \( a := \{ a_{i,j} \} : D \to \mathbb{R}^2 \), \( i,j = 1, \ldots, n \), \( b := \{ b_i \} : D \to \mathbb{R}^n \), \( i = 1, \ldots, n \) and \( c : D \to \mathbb{R} \) are Borel measurable, and such that

A1) \( a_{i,j}(x) = a_{j,i}(x) \), \( x \in D \), \( i,j = 1, \ldots, n \) and there exist \( \lambda,M > 0 \) such that

\[
\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq \lambda \sum_{i=1}^n \xi_i^2, \quad (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \quad x \in D \tag{2}
\]

(\( \lambda \) is called the uniform ellipticity bound) and

\[
\sum_{i,j=1}^n |a_{i,j}(x)| + \sum_{i=1}^n |b_i(x)| + |c(x)| \leq M, \quad x \in D, \tag{3}
\]

A2) there exist extensions of \( a \) and \( b \), which with slight abuse of notation we denote by the same symbols, to the entire \( \mathbb{R}^n \) such that the corresponding martingale problem is well posed in the sense of Section 6.0 of [5],

A3) there exists \( c_* > 0 \) such that

\[
c(x) \leq -c_*, \quad x \in D. \tag{4}
\]

Remark 1. Let us comment on condition A2). For the aforementioned extensions of \( a \) and \( b \), let \( L_0 \) denote the operator defined by

\[
L_0[u](x) := \frac{1}{2} \sum_{i,j=1}^n a_{i,j}(x) \partial_{x_i,x_j}^2 u(x) + \sum_{i=1}^n b_i(x) \partial_{x_i} u(x), \quad u \in C_0^2(\mathbb{R}^n), x \in \mathbb{R}^n. \tag{5}
\]
Consider the space $C([0, +\infty); \mathbb{R}^n)$ of continuous mappings $\omega : [0, +\infty) \to \mathbb{R}^n$, equipped with the standard topology of uniform convergence over compact intervals. Let $\mathcal{F}$ be its Borel $\sigma$-algebra and let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $\sigma$-algebras corresponding to the process $X := (X(t))_{t \geq 0}$, where $X(t; \omega) := \omega(t)$. We say that the family of Borel probability measures $\mu_x, x \in \mathbb{R}^n$ is a solution to the martingale problem corresponding to $L_0$ iff for any $u \in C_0^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ the process

$$u(X(t)) - u(x) - \int_0^t L_0[u](X(s)) \, ds, \quad t \geq 0$$

is a zero mean, $(\mathcal{F}_t)_{t \geq 0}$ - martingale with respect to measure $\mu_x$ and

$$\mu_x[\omega : X(0; \omega) = x] = 1.$$ (7)

The martingale problem is said to be well posed if it admits a unique solution. According to Theorem 7.2.1 of [5] it is the case when $a$ is bounded, continuous and strictly positive definite, and $b$ is bounded and measurable.

Having an elliptic operator $L$, defined as in (1), denote, by $\mathcal{E}(D, L)$ the class of functions $u : D \to \mathbb{R}$ that satisfy the following conditions:

U1) $u \in C^2(D)$ and extends continuously to $\partial D$,

U2) $u$ satisfies inequality $L[u](x) \geq 0, x \in D$, we say then that $u$ is a subsolution of the equation $L[u] = 0$,

U3) if $\hat{x} \in \partial D$ is such that $u(x) \leq u(\hat{x})$ for all $x \in D$, then $u(\hat{x}) \geq 0$ and $u$ has the outward normal derivative $\partial_n u(\hat{x})$ at the point.

Finally, for any $u \in \mathcal{E}(D, L)$ we let

$$\mathcal{M}(u) := \{ \hat{x} \in \partial D : u(\hat{x}) = \max_{x \in D} u(x) \}.$$ (8)

It is well known, see e.g. [2, Corollary 3.2, p. 33], that if $u$ satisfies U1)-U2), then

$$\max_{x \in D} u(x) = \max_{x \in \partial D} u(x).$$

The main objective of this paper is the following result, which is a version of Hopf’s maximum principle [4, Section 2.3] with a quantitative lower bound.

**Theorem 2.2.** If $D$ is of class $C^2$, then there exists a constant $\gamma > 0$ depending only on $c_*, \lambda, M$ and $D$ such that for any $u \in \mathcal{E}(D, L)$ such that $u(x) \neq 0$ we have

$$\partial_n u(\hat{x}) > \gamma u(\hat{x}), \quad \hat{x} \in \mathcal{M}(u).$$ (8)

**Remark 2.** More specifically, the constant $\gamma$ depends on the domain $D$ only through $r$ and $\delta$ appearing in Lemma 3.1 below. See Remark 3, further down, for more details.

3. **Proof of Theorem 2.2.** First, we introduce some notations. Given $r > 0$ and $x \in \mathbb{R}^n$ we denote by

$$K(x, r) := \{ x' \in \mathbb{R}^n : |x - x'| < r \}$$

the open ball of radius $r > 0$ centered at $x$. Throughout the paper $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ or in $\mathbb{R}$. Also, we let

$$\partial K(x, r) := \{ x' \in \mathbb{R}^n : |x - x'| = r \},$$

and

$$\bar{K}(x, r) := \{ x' \in \mathbb{R}^n : |x - x'| \leq r \}.$$ (9)

For the proof of Theorem 2.2 we will also need the following lemma.
Lemma 3.1. Suppose that $D$ is a domain with $C^2$-smooth boundary $\partial D$. Then (see Figure 1 for the two-dimensional case), there exist $\delta, r > 0$ such that for any $\tilde{x} \in \partial D$ there is a ball $K(y, r)$ satisfying $K(y, r) \subset D$ and $\tilde{K}(\tilde{x}, y, r) \cap \partial D = \{\tilde{x}\}$. In addition, for any $x \in \partial K(\tilde{x}, r/2) \cap K(y, r)$ we have $K(x, \delta) \subset D$.

Proof. It is an exercise in calculus to show that, because of the assumed regularity of the boundary, for each $\tilde{x} \in \partial D$ there are $r_0 = r_0(\tilde{x})$ and $y^0 = y^0(\tilde{x})$ such that $K(y^0, r_0) \subset D$ and $\tilde{K}(\tilde{x}, y^0, r_0) \cap \partial D = \{\tilde{x}\}$. Moreover, it is clear that $r_0 > 0$ may be chosen to depend continuously on $\tilde{x} \in \partial D$. Since $\partial D$ is compact, a minimal value $r > 0$ of $r_0$ is attained at a certain point.

Then, for all $\tilde{x}$ there is a $y^0(\tilde{x})$ such that $K(y^0, r) \subset D$ and $\tilde{K}(\tilde{x}, y^0, r) \cap \partial D = \{\tilde{x}\}$: we simply take $y^0$ to be the point lying on the radius connecting $\tilde{x}$ and $y^0$ at the distance $r$ from $\tilde{x}$; then $K(y^0, r) \subset K(y^0, r_0)$ whereas $\tilde{K}(\tilde{x}, y^0, r_0) \cap \partial D = \{\tilde{x}\}$.

Taking $r < r$ and choosing $y$ as the point lying on the radius connecting $\tilde{x}$ and $y^0$ at the distance $r$ from $\tilde{x}$ we see that $K(y^0, r) \subset K(y^0, r)$ and $\tilde{K}(\tilde{x}, y^0, r) \cap \partial D = \{\tilde{x}\}$. Since $\partial K(\tilde{x}, r/2) \cap K(y, r)$ is a compact set contained in $K(y^0, r)$, its distance from the complement of $K(y^0, r)$ (which contains the complement of $D$) is larger than zero. This distance, on the other hand, depends continuously on $\tilde{x}$, and as above we conclude that a minimal nonzero distance may be chosen. Then, any $\delta$ smaller than this distance will do the job.

Fix an $\tilde{x} \in M(u)$, and let $r$ and $y$ be the radius and the point in $D$ corresponding to $\tilde{x}$, as discussed in Lemma 3.1. To simplify notation in what follows we write

$$K_1 = K(y, r) \quad \text{and} \quad K_2 := K(\tilde{x}, r/2).$$

Figure 1. The solid curve $\partial D$ separates $D$ (below) from its complement $D^c$ (above). The set $\partial K(x, r/2) \cap K(y, r)$ forms an arc on which the centers of the small dotted circles, representing $\partial K(z, \delta)$, lie.

Our next lemma gathers properties of an auxiliary function $\psi$, defined below, on these balls.

Lemma 3.2. Let

$$\psi(x) := e^{-\alpha|x-y|^2} - e^{-\alpha r^2}, \quad x \in \mathbb{R}^n,$$

where $\alpha > 0$ is a parameter. Then

(a) $\psi > 0$ on $K_1$, $\psi = 0$ on $\partial K_1$, and $\psi < 0$ outside of $K_1$, and
(b) $\psi \leq e^{-\alpha \frac{z^2}{2}}$, on $\tilde{K}_1 \cap K_2$. 

(c) on $K_1 \cap K_2$ we have
\[ L[\psi] \geq \left\{ \frac{\alpha^2}{2} \lambda r^2 - \alpha M - 2\alpha r M \right\} e^{-\alpha r^2/4} - M e^{-\alpha r^2/4} > 0, \]
provided that $\alpha > 0$ is sufficiently large.

Proof. The proof of the lemma follows a classical argument, that can be found e.g. in the proof of Theorem 2.3.7 of [4]. We invoke it here for a reader convenience. Point (a) is clear. Also, for $x \in K_1 \cap K_2$, the distance between $x$ and $y$ is no smaller than $r/2$ (since $|y - x| = r$ and $|x - \tilde{x}| \leq r/2$), implying (b).

As for (c), we have
\[ \nabla \psi(x) = -2\alpha(x - y)e^{-\alpha|x-y|^2} \]
and
\[ \nabla^2 \psi(x) = e^{-\alpha|x-y|^2} \left( 4\alpha^2(x - y) \otimes (x - y) - 2\alpha I_n \right), \]
where $I_n$ is the $n \times n$ unit matrix. Hence
\[
L[\psi](x) = \left\{ 2\alpha^2 \sum_{i,j=1}^{n} a_{i,j}(x)(x_i - y_i)(x_j - y_j) - \alpha \sum_{i=1}^{n} a_{i,i}(x) \\
- 2\alpha \sum_{i=1}^{n} (x_i - y_i)b_i(x) \right\} e^{-\alpha|x-y|^2} + c(x)\psi(x), \quad x \in \mathbb{R}^n,
\]
where $x_i, y_i$'s are coordinates of $x$ and $y$, respectively. Using assumptions (2)–(3), we see that
\[ L[\psi](x) \geq \left\{ 2\alpha^2|x - y|^2 - \alpha M - 2\alpha r M \right\} e^{-\alpha|x-y|^2} + c(x)\psi(x), \quad x \in K_1; \]
indeed,
\[ \sum_{i=1}^{n} |x_i - y_i| |b_i(x)| \leq \max_{i=1,...,n} |x_i - y_i| M \leq r M. \]

Moreover, by (3) and point (b),
\[ |c(x)\psi(x)| \leq Me^{-\alpha x_2^2}, \quad x \in K_1 \cap K_2. \]
Combining these estimates with the fact, established in the proof of (b), that for $x \in K_1 \cap K_2$, the distance between $x$ and $y$ is no smaller than $r/2$, we complete our reasoning. \hfill \square

The following result is a key ingredient to the proof of Theorem 2.2.

Lemma 3.3. There exists a $q < 1$ such that
\[ u(x) \leq qu(\tilde{x}), \quad x \in \partial K_2 \cap K_1, \quad \tilde{x} \in \mathcal{M}(u). \]

Proof. For $x \in D$, let $\mu_x$ be the path measure on $C([0, +\infty); \mathbb{R}^d)$ determined by the solution of the martingale problem corresponding to $L_0$ (cf (5)). Denote by $E_{\mu_x}$ the respective expectation.

Step 1. Without loss of generality, we assume that $\delta \leq 1$. In the first step, we will find an upper bound for $\mu_x [\tau_{x,\delta} \leq T]$, where $\tau_{x,\delta}$ is the exit time of the process $X$ from the ball $K(x, \delta)$ centered at $x \in \partial K_2 \cap K_1$. Note that $K(x, \delta) \subset D$ for $x \in \partial K_2 \cap K_1$. 

Namely, we claim that
\[ \mu_x [\tau_{x,\delta} \leq T] < \frac{1}{8}, \]  
(10)
where \( T = T(\delta) := \frac{\delta^2}{10(M+1)} \) and \( M \) is as in (3). To see this, consider the vector valued, zero \( \mu_x \)-mean \((\mathcal{F}_t)_{t \geq 0}\)-martingale \( \mathfrak{M}(t) = (\mathfrak{M}_1(t), \ldots, \mathfrak{M}_n(t)) \)
\[ \mathfrak{M}(t) := X(t \wedge \tau_{x,\delta}) - x - \int_0^{t \wedge \tau_{x,\delta}} b(X(s)) \, ds, \]
where, to recall, \( b(x) = (b_1(x), \ldots, b_n(x)) \). This martingale is bounded, and
\[ [\mathfrak{M}_i(t)]^2 - \int_0^{t \wedge \tau_{x,\delta}} a_{i,i}(X(s)) \, ds, \quad t \geq 0, \]
is also a zero \( \mu_x \)-mean, \((\mathcal{F}_t)_{t \geq 0}\)-martingale for all \( i = 1, \ldots, n \). In particular, by (2),
\[ E_{\mu_x} [\mathfrak{M}(t)]^2 = E_{\mu_x} [\sum_{i=1}^n [\mathfrak{M}_i(t)]^2] \leq Mt, \quad t \geq 0. \]  
(11)
Similarly, by (2) and the elementary inequality
\[ \sqrt{a_1^2 + \cdots + a_n^2} \leq a_1 + \cdots + a_n, \quad a_i \geq 0, \]
we see that \[ \left| \int_0^{t \wedge \tau_{x,\delta}} b(X(s)) \, ds \right| \leq Mt. \] Therefore, for any \( \omega \in [\tau_{x,\delta} \leq T] \),
\[ \sup_{0 \leq t \leq T} \| \mathfrak{M}_i \| \geq \delta - MT = \delta - \frac{\delta^2 M}{10(M+1)} \geq \delta - \frac{\delta}{10} \geq \frac{9\delta}{10}, \]
(recall that we assumed \( \delta \leq 1 \)). It follows, by virtue of Doob’s inequality and (11) that we can write
\[ \mu_x [\tau_{x,\delta} \leq T] \leq \mu_x \left[ \sup_{0 \leq t \leq T} |\mathfrak{M}_i|^2 \geq \left( \frac{9\delta}{10} \right)^2 \right] \leq \left( \frac{10}{9\delta} \right)^2 E_{\mu_x} |\mathfrak{M}_T|^2 \leq \frac{100MT}{81\delta^2} < \frac{10}{81} < \frac{1}{8}, \]
completing the proof of (10).

**Step 2.** Step 1 implies the following upper bound for \( E_{\mu_x} [e^{-c_x\tau_{x,\delta}}] \):
\[ E_{\mu_x} [e^{-c_x\tau_{x,\delta}}] \leq \mu_x \left[ \tau_{x,\delta} \leq T \right] + e^{-c_xT} \mu_x \left[ \tau_{x,\delta} > T \right] \]
(12)
\[ = \mu_x \left[ \tau_{x,\delta} \geq T \right] \left( 1 - e^{-c_xT} \right) + e^{-c_xT} \]
\[ < \frac{1}{8} \left( 1 - e^{-c_xT} \right) + e^{-c_xT} = \frac{1}{8} + \frac{7}{8} e^{-c_xT} =: q < 1, \quad x \in D_\delta, \]
where \( D_\delta := \{ x \in D : \bar{K}(x, \delta) \subset D \} \).

**Step 3.** Using the Itô formula, see [5, Theorem 4.4.1, p. 105], we check that
\[ \mathfrak{M}(t) := u(X(t \wedge \tau_{x,\delta})) \exp \left\{ \int_0^{t \wedge \tau_{x,\delta}} c(X(s)) \, ds \right\} - u(x) \]
\[ - \int_0^{t \wedge \tau_{x,\delta}} Lu(X(s)) \exp \left\{ \int_0^s c(X(\sigma)) \, d\sigma \right\} \, ds, \quad t \geq 0 \]
is a zero mean, $\mu_x$-martingale; in particular, $E_{\mu_x} M(t) = 0, t \geq 0$. Since the diffusion process $X$ is non-degenerate, we have, see Proposition 5.7.2, p. 364 of [3],

$$E_{\mu_x} \tau_{x,\delta} < +\infty.$$  \hspace{1cm} (13)

Furthermore, due to non-positivity of $c(x)$ we can estimate

$$|M(t)| \leq 2 \sup_{z \in K(x,\delta)} |u(z)| + \tau_{x,\delta} \sup_{z \in K(x,\delta)} Lu(z), \quad t \geq 0.$$  

On the other hand, as another consequence of (13),

$$\lim_{t \to +\infty} \mu_x[\tau_{x,\delta} \geq t] = 0$$

and this in turn implies that $\lim_{t \to +\infty} M(t)$ exists almost surely and equals

$$M(\infty) := u(X(\tau_{x,\delta})) \exp \left\{ \int_0^{\tau_{x,\delta}} c(X(s)) \, ds \right\} - u(x)$$

$$- \int_0^{\tau_{x,\delta}} Lu(X(s)) \exp \left\{ \int_0^s c(X(\sigma)) \, d\sigma \right\} \, ds.$$  

Thus, by the Lebesgue dominated convergence theorem,

$$0 = E_{\mu_x} \left[ \lim_{t \to +\infty} M(t) \right] = E_{\mu_x} M(\infty).$$

It follows that

$$u(x) = E_{\mu_x} \left[ u(X(\tau_{x,\delta})) \exp \left\{ \int_0^{\tau_{x,\delta}} c(X(s)) \, ds \right\} - \int_0^{\tau_{x,\delta}} Lu(X(s)) \exp \left\{ \int_0^s c(X(\sigma)) \, d\sigma \right\} \right].$$

Since $u$ satisfies $L[u] \geq 0$ (see point U2) in the definition of $\mathcal{E}(D, L)$ and $c$ is bounded from above as in (4), we conclude from the maximality of $u(\tilde{x})$ that

$$u(x) \leq E_{\mu_x} \left[ u(X(\tau_{x,\delta})) \exp \left\{ \int_0^{\tau_{x,\delta}} c(X(s)) \, ds \right\} \right]$$

$$\leq u(\tilde{x}) E_{\mu_x} \left[ \exp \left\{ \int_0^{\tau_{x,\delta}} c(X(s)) \, ds \right\} \right]$$

$$\leq u(\tilde{x}) E_{\mu_x} \left[ e^{-c_{x,\delta} \tau_{x,\delta}} \right] \leq qu(\tilde{x}),$$

with the last inequality following from (12) and non-negativity of $u(\tilde{x})$. This completes the proof.

**Proof of Theorem 2.2.** Fix $\tilde{x} \in \mathcal{M}(u)$. If $u(\tilde{x}) = 0$ the result follows directly from Theorem 1.1 above. Assume therefore that $u(\tilde{x}) > 0$. Let $K_1, K_2$ be defined by (9), and for some $\varepsilon > 0$ to be chosen later, consider the function

$$w(x) := u(x) + \varepsilon \psi(x),$$

with domain equal to the domain of $u$, where $\psi$ was defined in Lemma 3.2. We have

$$w(x) = u(x) \leq u(\tilde{x}), \quad x \in \partial K_1.$$

Moreover, by Lemma 3.2 (b) and Lemma 3.3, for $x \in \partial K_2 \cap K_1$,

$$w(x) = u(x) + \varepsilon \psi(x) \leq qu(\tilde{x}) + \varepsilon e^{-\alpha r^2/4} \leq u(\tilde{x}),$$

provided that

$$0 < \varepsilon < u(\tilde{x})(1 - q)e^{\alpha r^2/4}.$$
Also, Lemma 3.2 (c) implies that $L[w](x) > 0$ for $x \in K_1 \cap K_2$. Therefore, by the maximum principle for subsolutions of the elliptic equation, see the aforementioned [2, Corollary 3.2, p. 33], applied to $K_1 \cap K_2$

$$w(\tilde{x}) \geq w(x), \quad x \in \tilde{K}_1 \cap \tilde{K}_2.$$ 

Hence $\tilde{x}$ is a maximal point of $w(\cdot)$ in $K_1 \cap K_2$. Thanks to assumption U3), there exists a directional derivative of $w$ at $\tilde{x}$ in the outward normal direction. Since $w(\tilde{x})$ is maximal we have

$$0 \leq \partial_n w(\tilde{x}) = \partial_n u(\tilde{x}) + \varepsilon \partial_n \psi(\tilde{x}).$$

On the other hand,

$$\partial_n \psi(\tilde{x}) = \nabla \psi(\tilde{x}) \cdot n = -2\alpha re^{-\alpha r^2}.$$ 

Therefore

$$\partial_n u(\tilde{x}) \geq 2\alpha ru(\tilde{x})(1 - q)e^{-3\alpha r^2 / 4}. $$

Since this is (8) with

$$\gamma := \alpha r(1 - q)e^{-3\alpha r^2 / 4},$$

we are done. \hfill \Box

**Remark 3.** The constant $\gamma$, as defined above (in (14)), depends on $\alpha$ and $r$, as in Lemmas 3.1 and 3.2, and on $q$ of Lemma 3.3. In turn, $\alpha$, as can be seen from the proof of Lemma 3.2, depends on the uniform ellipticity constant $\lambda$ and the $L^\infty$ bound $M$ of assumptions (2) and (3) (see Lemma 3.2 (c)). Finally, $q$ defined in (12) depends on $c_* = \sup_{x \in \partial D} c(x)$ and $T = T(\delta)$ – introduced in (10); both $r$ and $\delta$ were discussed in Lemma 3.1. Hence,

$$q = q(r, \delta, \lambda, M, c_*)$$

where, roughly, $r$ and $\delta$ characterize ‘geometry’ (i.e. depend on the domain $D$) whereas $\lambda, M$ and $c_*$ pertain to the diffusion operator $L$.

The entire argument presented in this paper works also in the case where – instead of assuming that the boundary of $D$ is of class $C^2$ – one can guarantee existence of the $r$ and $\delta$ along with existence of some kind of counterpart of the normal derivative at the extremal point, e.g. in the normal direction to $\partial K_1$ at $\tilde{x}$.

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