Isolated Horizon, Killing Horizon and Event Horizon

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Abstract

We consider space-times which in addition to admitting an isolated horizon also admit Killing horizons with or without an event horizon. We show that an isolated horizon is a Killing horizon provided either (1) it admits a stationary neighbourhood or (2) it admits a neighbourhood with two independent, commuting Killing vectors. A Killing horizon is always an isolated horizon. For the case when an event horizon is definable, all conceivable relative locations of isolated horizon and event horizons are possible. Corresponding conditions are given.

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I. INTRODUCTION

Isolated horizons (IH) are defined quasi-locally and without any assumptions about isometries (e.g., stationarity) of the space-time \([1,2]\). One also has completely quasi-local definitions of mass, angular momentum etc such that the zeroth and the first law(s) of black hole mechanics hold \([3,4]\). Even quantum entropy computations are available for the non-rotating isolated horizons \([5]\). These developments are non-trivial because firstly they are quasi-local and secondly they extend the scope of black hole thermodynamics to space-times which need not be stationary. This class of space-times is parameterized by infinitely many parameters \([6]\). Primarily it is the relaxation of condition of stationarity that is the source of technical non-triviality, asymptotic structure playing virtually no role.

IH therefore are candidates for replacing Killing horizons (KH) (for stationary black holes) as well as event horizons (EH) for purposes of black hole phenomena. Since all three horizons are distinct one could have space-times wherein all are present. One could then ask if and how these are ‘related’ to each other. How they are located relative to each other? Does existence of IH imply existence of the others when potentially possible? We address these questions in stages.

We will consider space-times with an IH which could also admit Killing horizons. Thus these space-times must have at least one isometry. We will show that this alone is not sufficient to for an isolated horizon to be a Killing horizon, further conditions are needed. However, every KH will be shown to be an isolated horizon. Note that the question asked is different from asking for the existence of any Killing vector at all \([6]\). For this case asymptotic structure is irrelevant.

Next we will consider space-time which could have event horizon. We will restrict to
asymptotically flat (and strongly asymptotically predictable) space-times. We will be able to find conditions under which an IH will imply existence of event horizon. For this case stationarity is not essential. The result is not immediately obvious since IH definition does not require foliation by marginally trapped surfaces.

The paper is organized as follows: In section II we recall the definitions of IH, KH and EH and set the notation. In section III we discuss the IH-KH relation while in section IV we discuss the IH-EH connection. The last section contains summary and discussion.

The notation and conventions used are those of Chandrasekhar [8] with the metric signature (+ - - -). The notation used for isolated horizons is that of [2].

II. PRELIMINARIES

All space-times under consideration are solutions of Einstein-matter equations with matter satisfying the dominant energy condition.

An isolated horizon \( \Delta \), is defined as a null hypersurface with topology \( R \times S^2 \), vanishing expansion of its null normal (non-expanding horizon), and the Newman-Penrose spin coefficients \( \mu, \lambda \) being constant along the null generators. We also follow the procedure given in reference [2] of setting up the null tetrads on \( \Delta \) and in an infinitesimal neigh-

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1 There is an inconsistent use of metric signature in Chandrasekhar’s book eg equations 287, 293 in the first chapter. In the definitions of the Ricci scalars, equation 300, all the Ricci scalars and \( \Lambda \) should be replaced by minus these scalars. Thus \( \Phi_{00} = + R_{11}/2 \). In reference [3] this requires \( \mathcal{E} \) to be replaced by \(-\mathcal{E}\) in equations 4 and 12. None of these affect the conclusions of reference [3]. The 18 equations of [8] are also unaffected.
bourhood $U_{\Delta}$. Explicitly, the following equations hold (modulo residual boost, constant scaling and local rotation freedom for the choice of tetrads on $\Delta$):

In $U_{\Delta}$ : $\gamma = \nu = \tau = \mu - \bar{\mu} = \pi - \alpha - \bar{\beta} = \kappa = 0$.

On $\Delta$ : $\rho = \sigma = D\pi = 0, \kappa \equiv \epsilon + \bar{\epsilon}$ (a non-zero constant)

On $\Delta$ : $\Psi_0 = \Psi_1 = \Phi_{00} = \Phi_{01} = 0$. (energy conditions, Raychoudhuri equation)

On $\Delta$ : $\bar{D}\lambda = \bar{D}\mu = 0$ (Isolated horizon conditions).

The underlined derivatives are the rotation covariant derivatives (`compacted’ derivatives) $\bar{\partial}$.

Note that a non-expanding horizon admits several equivalence classes of null normals, $[\ell]$ while isolated horizon conditions select a unique equivalence class. The choice of tetrads, conditions on spin coefficients etc are available for every equivalence class on a non-expanding horizon.

A Killing horizon is defined as a null hypersurface, again denoted by $\Delta$, such that a Killing vector is normal to $\Delta$. This clearly needs existence of a Killing vector at least in a neighbourhood of $\Delta$. Note that Killing vector need not exist everywhere in the space-time. It is also not required, a priori, to be time-like. As such the topology of $\Delta$ is not stipulated. For the questions explored here however we will consider only those Killing horizons which are topologically $R \times S^2$. This will be important in the following.

Definition of an event horizon needs a notion of ‘infinity’ and its ‘structure’ to be specified. We will take this to be asymptotically flat. We will also take the space-time in this context to be strongly asymptotically predictable in order to admit possibility of the usual black holes (not necessarily stationary). If the causal past of the future null infinity does not equal to the space-time, one says that a black hole region exists. Its boundary is then defined to be the event horizon. In the asymptotically
flat and stationary context, the topology of event horizon is $R \times S^2$. In the following, we will restrict to asymptotically flat context and assume that event horizons under consideration are all smooth and with topology $R \times S^2$.

III. ISOLATED HORIZON AND KILLING HORIZON

A killing horizon is automatically a non-expanding horizon since the induced metric on the leaves is preserved by the diffeomorphism generated by the Killing vector and hence so are the areas of the leaves. Also, an isolated horizon is a non-expanding horizon with further conditions. Consider therefore a solution admitting a non-expanding horizon, $\Delta$ with topology $R \times S^2$ and an infinitesimal neighbourhood $U_\Delta$. We also assume that $\Delta$ is also “isolated” in the sense that in $U_\Delta$ there is no other non-expanding horizon. This will ensure that when a Killing vector exist in $U_\Delta$, it will be necessarily tangential to $\Delta$. All the machinery of null tetrads etc described in the previous section is available. We are interested in finding out if a Killing horizon is an isolated horizon and conversely. To this end assume further that the solution also admits at least one Killing vector, at least in $U_\Delta$.

We want to see that if the Killing vector is normal to $\Delta$, does there exist an equivalence class of null normals such that $\Delta$ becomes an isolated horizon. For the converse we first note that an isolated horizon does not imply existence of an isometry even in a neighbourhood $U_\Delta$. However we are given that an isometry exists and the question is whether there exists a Killing vector normal to the IH.

To this end let us write a Killing vector in the form (in $U_\Delta$):

$$\xi^\mu \equiv A \ell^\mu + B n^\mu + C m^\mu + \bar{C} \bar{m}^\mu$$

(1)

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2I thank Abhay Ashtekar for alerting me to the need for this assumption.
The Killing equations, valid in $U_\Delta$, can be written in terms of the $A, B, C, \bar{C}$ as:

\begin{align*}
D' A &= 0 ; \\
D' B &= - DA - \bar{\kappa} A - \bar{\pi} C - \pi \bar{C} ; \\
D' C &= \bar{\delta} A + \pi A + \mu C + \lambda \bar{C} ; \\
D B &= \bar{\kappa} B ; \\
D C &= - (\epsilon - \bar{\epsilon}) C + \bar{\delta} B - 2\pi B - \rho C - \bar{\sigma} \bar{C} ; \\
\bar{\delta} C &= - (\alpha - \bar{\beta}) C + \bar{\sigma} A - \lambda B \\
\delta C &= (\bar{\alpha} - \beta) C - \bar{\delta} \bar{C} - (\alpha - \bar{\beta}) \bar{C} + (\rho + \bar{\rho}) A - 2\mu B
\end{align*}

Remarks:

(1) These equations can be thought of ‘evolution’ equations for $A, B, C, \bar{C}$ to go off-$\Delta$ with their values being specified on $\Delta$. Thus for any consistent choice of $A, B, C$ on $\Delta$, we can obtain Killing vector in $U_\Delta$. On the horizon, the Killing vector must be tangential to the horizon and therefore, $B = 0$ on $\Delta$. The special case where $B = C = 0, A \neq 0$ on $\Delta$ corresponds to the non-expanding $\Delta$ being a Killing horizon.

(2) On $\Delta$, $C$ satisfies, $D C = 0 = \bar{\delta} C + \bar{\delta} \bar{C}$. Derivatives of $A$ along $\Delta$ and one combination of derivatives of $C$ along leaves is not explicitly specified.

The coefficient functions however must satisfy the commutator identities \[8,2\]. These applied to $C$ and $B$ imply, on $\Delta$ :

\begin{align*}
\bar{\delta}^2 A &= - 2\pi \bar{\delta} A + \lambda (DA + 2\bar{\delta} C) - (C \bar{\delta} + \bar{C} \bar{\delta}) \lambda - A D \lambda \\
\bar{\delta}^2 \bar{A} &= - (\pi \bar{\delta} + \bar{\pi} \bar{\delta}) A + \mu DA - (C \bar{\delta} + \bar{C} \bar{\delta}) \mu - A D \mu \\
\bar{\delta} \bar{D} A &= - \bar{\kappa} \bar{\delta} A - (C \bar{\delta} + \bar{C} \bar{\delta}) \pi + \pi \bar{\delta} C \\
D^2 A &= - \bar{\kappa} D A
\end{align*}
We have deliberately not used the isolated horizon conditions.

Now if $\Delta$ is a Killing horizon for the above Killing vector, then $A \neq 0$ and $C = 0$ on $\Delta$. It remains to check if the isolated horizon conditions hold.

For the special case of $B = C = 0$ on $\Delta$, the equations simplify. The equations (11) and (12) imply that $\mathcal{D}A + \bar{\kappa}A = Q$ where $Q$ is a constant. It is easy to see that $Q$ is the acceleration of the Killing vector (on $\Delta$). The off-$\Delta$ derivative of the norm of the Killing vector, evaluated on the horizon is $-2AQ$ while the norm itself is of course zero.

A Killing vector can not vanish on a hypersurface without vanishing everywhere and thus $A \neq 0$ on $\Delta$. Now by making a scaling transformation $\ell \rightarrow A^{-1}\ell$, we go to another equivalence class of null normals in which the Killing vector is just the new null normal, i.e. effectively, $A = 1$. This immediately gives $Q = \bar{\kappa}$. Furthermore, equations (9) and (10) imply the isolated horizon conditions! For the case of non-zero surface gravity, the off-$\Delta$ derivative of the norm of the Killing vector is then non-zero and hence the Killing vector is time-like at least on ‘one side’ of $\Delta$.

Thus we have shown that if a non-expanding horizon is a Killing horizon, then it admits an equivalence class $[\ell]$ with respect to which it is an isolated horizon.

To consider the converse we now take $\Delta$ as an isolated horizon.

Now we have two possibilities. Either $C = 0$ on $\Delta$ or it is non-zero. In the former case the Killing vector which is given to exist, is normal to the horizon and hence $\Delta$ is a Killing horizon. As seen above this means that the Killing vector is time-like on ‘one side’ of $\Delta$.

\footnote{I thank Abhay Ashtekar for drawing my attention to this fact.}
side’ of $\Delta$ at least when the surface gravity is non-zero. Conversely, if the Killing vector is time-like at least on ‘one side’ of $\Delta$ (i.e. $AB > C\bar{C}$), then on $\Delta$, it must be normal to $\Delta$ making it a Killing horizon.

In the latter case the norm of the Killing vector on $\Delta$ is negative. In this case, the $D'$ of the norm of the Killing vector being non-zero on $\Delta$ and norm itself being negative on $\Delta$ implies that the Killing vector is space-like in $U_\Delta$. This is for instance realized by a Killing vector of ‘axisymmetry’. Clearly, mere existence of a Killing vector in a neighbourhood of an isolated horizon is not enough for the IH to be a Killing horizon. Norm being positive on ‘one side’ is needed in addition. This could be ensured by existence of two independent commuting Killing vectors in a neighbourhood. This is seen as follows.

Suppose that $\xi_1, \xi_2$ are two Killing vectors in $U_\Delta$ which commute. This implies that there are two independent, mutually commuting isometries of $\Delta$ (the horizon is a hypersurface, so the independence of Killing vectors in the neighbourhood continues to hold on the horizon). By repeating the arguments of reference [2] one can choose a unique foliation such that one of the Killing vector, $\xi_1$, is tangential to the leaves ($A_1 = 0$). The commutativity of the Killing vectors and the spherical topology of leaves imply that the second Killing vector can not be tangential to the leaves i.e. $A_2 \neq 0$. Now however we can construct a linear combination, $\xi_3$, of these Killing vectors which has $A_3 \neq 0, C_3 = 0$. Commutativity of the Killing vectors is used for this statement. This Killing vector is normal to $\Delta$ implying that $\Delta$ is a Killing horizon. The linear combination is the precise analogue of the combination appearing in the Kerr solution. The commutativity of the Killing vectors and the spherical topology of the leaves is essential for this argument.

Thus we conclude that for an isolated horizon to be a Killing horizon there must exist a neighbourhood together with either one Killing vector which is time-like at least on
‘one side’ of $\Delta$ or two commuting Killing vectors. In the former case the Killing vector is trivially normal to $\Delta$ while in the latter there exist another Killing vector which is normal to $\Delta$. The two cases of course correspond to the usual static and the stationary-axisymmetric examples. Completeness of Killing vectors (group of isometries) is not essential for this argument.

It would be nicer to have a condition on isolated horizon spin coefficients directly which will guarantee either of the above conditions. One could of course try to define a Killing vector by taking $A = \text{constant}, \ B = C = 0$ on $\Delta$. The argument of Lewandowski [6] shows that for the special case of non-rotating, spherically symmetric, vacuum isolated horizon such a definition is in conflict with non-zero $\Psi_4$. However for general isolated horizons it is not clear what ‘obstructions’ could be there. Except for the special case mentioned above or the case where the leaves have at least one isometry of their intrinsic metric, the foliations are not unique and $\Psi_3, \Psi_4$ are not invariant under residual boosts.

IV. ISOLATED HORIZON IN POTENTIALLY BLACK HOLE SPACE-TIMES

To admit the possibility of a (usual) black hole we will consider space-times which are strongly asymptotically predictable and are solutions of Einstein-matter equations with matter stress tensor satisfying the dominant energy condition. In addition, we assume that such a space-time also has an isolated horizon. Note that there is however no condition on $\mu$ being positive/negative on $\Delta$ nor is there any assumption made about the symmetry class of the horizon.

We could of course have a special case where $\Delta$ admits a foliation such that $\mu < 0$ on a leaf and hence on $\Delta$. In this case the horizon is foliated by marginally trapped
surfaces and hence belong to the black hole region, \( M - J^{-}(J^+) \), of the space-time i.e. \( S^2 \cap J^-(J^+) = \emptyset \). In particular, an event horizon (EH) must exist.

Now the IH and EH may or may not intersect. If they do not intersect, then the IH is irrelevant for any observer outside the event horizon. If however they do intersect then they must do so on a space-like two dimensional surface, in fact \( S^2 \) or IH must be a subset of EH. In the former case null generators of the EH will be either along \( \ell \) or along \( n \) elements of the tetrad on IH. But the expansion of the generators of EH is non-negative (area theorem) while \( \mu \) is negative. Hence the generators of EH and IH must be proportional at points of intersections. The isolated horizon being non-expanding now implies that the expansion of the generators of the EH must also be zero and it can not become positive in the future. Thus the area of the instantaneous black hole corresponding to the coinciding portion of the event horizon must remain constant until possible future merger with other instantaneous black holes. Thus, in the special case of \( \mu < 0 \), we see that (a) event horizon must exist and (b) either IH is irrelevant or it coincides with an instantaneous black hole (a connected component of intersection of EH with a Cauchy surface). Its area can change only with possible future mergers with other black holes. Thus such an isolated horizon refers to the settled stage of an instantaneous black hole.

However a general isolated horizon does not require (or imply) existence of a foliation such that \( \mu < 0 \) on a leaf. The same proof which shows that marginally trapped surfaces are invisible from future null infinity can be used to explore sub-cases of the general case. The argument is as follows.

Let \( T \) be a leaf in a foliation of the isolated horizon. It is a compact, orientable, space-like submanifold of the space-time. Hence the boundary of its causal future is generated by null geodesics starting orthogonally from \( T \) and having no conjugate points (theorem
Consider now \( X \equiv T \cap J^-(\mathcal{J}^+) \). If \( X \) is empty, then IH again implies existence of EH and is contained in the black hole region. Once again IH and EH may not intersect making the IH irrelevant or IH may coincide with a portion of the EH or they may intersect non-trivially in a space-like \( S^2 \). If however \( X \) is non-empty, then boundary of causal future of \( T \) contains a whole spherical cross-section of \( \mathcal{J}^+ \). In the vicinity of future null infinity, the expansion of any null geodesic congruence orthogonal to a cross-section of the null infinity is strictly positive. Hence on a spherical cross section on the boundary, the generators must have positive expansion. Since these generators can not have conjugate points, we must have their expansions on \( T \) itself to be strictly positive. These generators thus must comprise of the \( n \) congruence and hence \( \mu > 0 \) must hold on \( T \). The condition of isolation then implies that \( \mu \) is positive on \( \Delta \).

The Gauss-Bonnet integral \([2]\) however is:

\[
\tilde{\kappa} \int_{\Sigma_2} \mu = -2\pi + \int_{\Sigma_2} (\mathcal{E} + \pi \tilde{\pi}) \quad (13)
\]

From this we see that if the surface gravity is positive then we must have the integral on the right hand side to be larger than \( 2\pi \). If the IH is non-rotating then we can choose a foliation such that \( \pi = 0 \) and if in addition it is a vacuum solution then \( \mu \) can not be strictly positive on \( \Delta \). But this means that \( X \) must be empty and hence an EH must exist. Thus a non-rotating isolated horizon with positive surface gravity in a space-time which is vacuum on the horizon, necessarily implies existence of an event horizon. In particular this implies that in *Minkowski space-time, there can not be a non-rotating isolated horizon with topology \( R \times S^2 \).* However, for a rotating isolated horizon and/or space-time which is non-vacuum on the IH, it is possible that the isolated horizon is visible from future null infinity *provided the Gauss-Bonnet integral so permits*. Note that in this case an event horizon could well exist but is not implied by the existence of an IH.
Since now we could have an IH outside of an EH for the rotating and/or non-vacuum case the IH itself is accessible from infinity unlike an EH.

V. SUMMARY AND DISCUSSION

In this work we have explored the implication of existence of an isolated horizon (topologically $R \times S^2$) for the existence of an event horizon when such is applicable and for Killing horizons when such are admissible.

For the relationship between a Killing horizon and an isolated horizon we have shown that a Killing horizon is always an isolated horizon but for the converse we need either a Killing vector which is time like on one side of the IH or we need two commuting Killing vectors. Mere isometry of space-time together with an isolated horizon does not imply existence of a Killing horizon, one needs either ‘staticity’ or additional ‘axisymmetry’.

For the relationship between IH and EH, we have shown that for the non-rotating isolated horizon in a solution (vacuum near $\Delta$) an event horizon must necessarily exist. Furthermore it corresponds (when relevant) to the settled stage of a black hole (no further area increase unless merger with other instantaneous black holes occurs). For the case of a rotating isolated horizon with or without non-trivial $\mathcal{E}$, we also have the possibility of no event horizon or at least accessibility of the IH from future null infinity.

Consider the case where we have several isolated horizons all coinciding with portions of the event horizon. Then the area theorem for event horizon implies that the isolated horizons which are to the future of a given isolated horizon will have areas larger than or equal to the area of the given IH. In this sense a second law of black hole mechanics can be seen to generalize to a family of isolated horizons.
The initial restriction to stationary black holes was justified on the grounds of uniqueness theorems and that a stationary black hole, if disturbed, will return quickly to another stationary black hole. For isolated horizons, no such results are available. Since an isolated horizon need not imply an event horizon or even when an EH is present, an IH could be outside of it means that the settling down of a disturbed IH is a non-trivial issue and so is a physical process interpretation of the first law.

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