Non-equilibrium stochastic dynamics of continuous systems and Bogoliubov generating functionals

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Abstract
Combinatorial harmonic analysis techniques are used to develop new functional analysis methods based on Bogoliubov functionals. Concrete applications of the methods are presented, namely the study of a non-equilibrium stochastic dynamics of continuous systems.
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1 Introduction

The combinatorial harmonic analysis on configuration spaces introduced and developed in [KK02], [KK05], [Kun99], [KKO04] is a natural tool for the study of equilibrium states of continuous systems in terms of the corresponding Bogoliubov or generating functionals. Originally, this class of functionals was introduced by N. N. Bogoliubov in [Bog46] to define correlation functions for statistical mechanics systems. In the context of classical statistical mechanics, this class of functionals, as a basic concept, was analyzed by G. I. Nazin. We refer to [Naz85] for historical remarks and references therein.

Apart from this specific application, and many others, the Bogoliubov functionals are, by themselves, a subject of interest in infinite dimensional analysis. This is partially due to the fact that to any probability measure $\mu$ defined on the space $\Gamma$ of locally finite configurations one may associate a Bogoliubov functional

$$L_{\mu}(\theta) := \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \, d\mu(\gamma),$$

allowing the study of $\mu$ through the functional $L_{\mu}$. Technically, this means that through the Bogoliubov functionals one may reduce measure theory problems to functional analysis ones, yielding a new method in measure theory as well as new applications in functional analysis.

From this standpoint, new perspectives were announced in [KKO04] in the setting of combinatorial harmonic analysis on configuration spaces. The purpose of this work is to carry out these technical improvements.

Of course the domain of a Bogoliubov functional $L_{\mu}$ depends on the underlying probability measure $\mu$. Conversely, the domain of a Bogoliubov functional $L_{\mu}$ carries special properties over to the probability measure $\mu$. In this work we mainly analyze the class of entire Bogoliubov functionals on a $L^1$-space (Section 3), which is a natural environment to widen the scope of this work towards Gibbs measures (or equilibrium states). This restriction allows, in particular, to recover the notion of correlation function.
As a side remark, let us mention that in the same setting further progresses, under analytical assumptions, are achieved in [Kun05] on a space of continuous functions.

The close relation between probability measures and Bogoliubov functionals is best illustrated by a “dictionary” (cf. G. I. Nazin), relating measure concepts and problems to functional analysis ones. In this “dictionary”, the translation of the Dobrushin-Lanford-Ruelle equation, defining Gibbs measures, leads to a functional equation, called the Bogoliubov (equilibrium) equation (Section 4). As a result, through analytical techniques one may derive a uniqueness result for Gibbs measures corresponding to positive potentials in the high temperature-low activity regime (Theorem 23). Although this result does not improve the known uniqueness results for Gibbs measures (see e.g. [DSI75], [PZ99], [Rue63]), its proof is technically new and presents an alternative approach to the uniqueness problem.

This work concludes with a concrete application of the Bogoliubov functionals to the study of a non-equilibrium diffusion dynamics of a continuous system (Section 5). For particles in suspension in a liquid, each particle interacts with the molecules of the fluid and the remaining particles in the suspension. At the microscopic level, the time evolution of the whole system is described by Hamiltonian dynamics. In the mesoscopic approximation, the system is described as the result of random perturbations of the particles with dynamics heuristically given by a system of stochastic differential equations

\[
\begin{align*}
 dx_k(t) &= -\frac{\beta}{2} \sum_{1 \leq i \neq k} \nabla V(x_k(t) - x_i(t)) dt + dW_k(t), \quad t \geq 0 \\
 x_k(0) &= x_k, \quad k \in \mathbb{N}
\end{align*}
\]

for a given starting configuration \( \gamma = \{x_k : k \in \mathbb{N}\} \). Here \( W_k, k \in \mathbb{N}, \) is a family of independent standard Brownian motions describing the random perturbations and \( V : \mathbb{R}^d \backslash \{0\} \to \mathbb{R} \) is the interaction potential between the particles. The problem of existence of a stochastic dynamics corresponding to the system has been well analyzed for the equilibrium stochastic dynamics case (see e.g. [AKR98], [Osa96], [Yos96]). For non-equilibrium dynamics, the existence problem is essentially open and at the moment all we have is the construction of non-equilibrium processes done in [Fri87], in the case of smooth potentials with finite range and \( d \leq 4 \), or the existence of time evolution for correlation functions described by a correlation diffusion hierarchy.
(see [KRR04] and the references therein). Our goal now is the study of the non-equilibrium case in terms of Bogoliubov functionals. The procedure that is used turns out to be an effective method for the study of other equilibrium and non-equilibrium problems for continuous systems. Further examples of applications, e.g., equations for birth-and-death and hopping type dynamics on configuration spaces in terms of Bogoliubov functionals, are now being studied and will be reported in forthcoming publications.

2 Harmonic analysis on configuration spaces

Let \( X \) be a geodesically complete connected oriented non-compact Riemannian \( C^\infty \)-manifold and \( \Gamma := \Gamma_X \) the configuration space over \( X \):

\[
\Gamma := \{ \gamma \subset X : |\gamma \cap K| < \infty \text{ for every compact } K \subset X \}.
\]

Here \(| \cdot |\) denotes the cardinality of a set. As usual we identify each \( \gamma \in \Gamma \) with the non-negative Radon measure \( \sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(X) \), where \( \varepsilon_x \) is the Dirac measure with mass at \( x \), \( \sum_{x \in \emptyset} \varepsilon_x \) is, by definition, the zero measure, and \( \mathcal{M}(X) \) denotes the space of all non-negative Radon measures on the Borel \( \sigma \)-algebra \( B(X) \). This procedure allows to endow \( \Gamma \) with the topology induced by the vague topology on \( \mathcal{M}(X) \). We denote the Borel \( \sigma \)-algebra on \( \Gamma \) by \( B(\Gamma) \).

Another description of the measurable space \((\Gamma, B(\Gamma))\) is also possible. For each \( Y \in B(X) \), let \( \Gamma_Y \) be the space of all configurations contained in \( Y \), \( \Gamma_Y := \{ \gamma \in \Gamma : |\gamma \cap (X \setminus Y)| = 0 \} \), and let \( \Gamma_Y^{(n)} \) be the subset of all \( n \)-point configurations, \( \Gamma_Y^{(n)} := \{ \gamma \in \Gamma_Y : |\gamma| = n \} \), \( n \in \mathbb{N}, \Gamma_Y^{(0)} := \{ \emptyset \} \). For \( n \in \mathbb{N} \), there is a natural surjective mapping of \( \widetilde{Y}^n \) onto \( \Gamma_Y^{(n)} \) defined by

\[
\text{sym}_Y^n : \widetilde{Y}^n \to \Gamma_Y^{(n)} \quad (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}.
\]

This leads to a bijection between the space \( \Gamma_Y^{(n)} \) and the symmetrization \( \widetilde{Y}^n / S_n \) of \( \widetilde{Y}^n \) under the permutation group \( S_n \) over \( \{1, \ldots, n\} \), and then to a metrizable topology on \( \Gamma_Y^{(n)} \). We denote the corresponding Borel \( \sigma \)-algebra on
Γ_Y^{(n)} by B(Γ_Y^{(n)}). For Λ ∈ B(X) with compact closure (Λ ∈ B_c(X) for short), one clearly has Γ_Λ = \bigcup_{n=0}^{\infty} Γ_Λ^{(n)}. In this case we endow Γ_Λ with the topology of the disjoint union of topological spaces and with the corresponding Borel σ-algebra B(Γ_Λ) defined by the disjoint union of the σ-algebras B(Γ_Λ^{(n)}), n ∈ N_0, i.e.,

\[ B(Γ_Λ) = σ\left(\{γ ∈ Γ_Λ : |γ ∩ Λ'| = n\}\right), \quad Λ' ∈ B_c(X), n ∈ N_0. \]

The measurable space (Γ, B(Γ)) is the projective limit of the measurable spaces (Γ_Λ, B(Γ_Λ)), Λ ∈ B_c(X), with respect to the projections

\[ p_Λ : Γ → Γ_Λ \]
\[ γ ↦ γ_Λ := γ ∩ Λ. \]  

Apart from the spaces described above we also consider the space of finite configurations

\[ Γ_0 := \bigsqcup_{n=0}^{\infty} Γ_X^{(n)} \]
endowed with the topology of disjoint union of topological spaces and with the corresponding Borel σ-algebra denoted by B(Γ_0).

To define the K-transform, among the functions defined on Γ_0 we distinguish the space B_{exp,ls}(Γ_0) of all complex-valued exponentially bounded B(Γ_0)-measurable functions G with local support, i.e., G|Γ_0\Γ_Λ ≡ 0 for some Λ ∈ B_c(X) and there are C_1, C_2 > 0 such that |G(η)| ≤ C_1e^{C_2|η|} for all η ∈ Γ_0. The K-transform of any G ∈ B_{exp,ls}(Γ_0) is the mapping KG : Γ → C defined at each γ ∈ Γ by

\[ (KG)(γ) := \sum_{η⊂γ \atop |η|<\infty} G(η). \]  

Note that for every G ∈ B_{exp,ls}(Γ_0) the sum in (4) has only a finite number of summands different from zero, and thus KG is a well-defined measurable cylinder function on Γ with domain of cylindricity Λ. Moreover, |(KG)(γ)| ≤ C_1e^{(C_2+1)|γ_Λ|}.

Throughout this work the so-called coherent states e_λ(f) of B(X)-measurable functions f, defined by

\[ e_λ(f, η) := \prod_{x ∈ η} f(x), \quad η ∈ Γ_0 \setminus \{\emptyset\}, \quad e_λ(f, \emptyset) := 1, \]
will play an essential role. This is partially due to the fact that the $K$-transform of this class of functions coincides with the integrand functions of the Bogoliubov functionals (Section 3). More precisely, for every bounded $\mathcal{B}(X)$-measurable function $f$ with bounded support ($f \in B_{bs}(X)$ for short), one has $e_\lambda(f) \in B_{exp,ls}(\Gamma_0)$, and
\[
(Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x)), \quad \gamma \in \Gamma.
\]

Besides the $K$-transform, we also consider the dual operator $K^*$. Let $\mathcal{M}^1_{fexp}(\Gamma)$ denote the set of all probability measures $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ with finite local exponential moments, i.e.,
\[
\int_{\Gamma} e^{\alpha|\gamma\Lambda|} d\mu(\gamma) < \infty \quad \text{for all } \Lambda \in \mathcal{B}_c(X) \text{ and all } \alpha > 0.
\]

By the definition of a dual operator, given a $\mu \in \mathcal{M}^1_{fexp}(\Gamma)$, $K^*\mu =: \rho_\mu$ is a measure defined on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ by
\[
\int_{\Gamma_0} G(\eta) d\rho_\mu(\eta) = \int_{\Gamma} (KG)(\gamma) d\mu(\gamma), \quad (5)
\]
for all $G \in B_{exp,ls}(\Gamma_0)$. The measure $\rho_\mu$ is the correlation measure corresponding to $\mu$. This definition shows, in particular, that $B_{exp,ls}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu)^1$. Moreover, on the dense set $B_{exp,ls}(\Gamma_0)$ in $L^1(\Gamma_0, \rho_\mu)$ the inequality $\|KG\|_{L^1(\mu)} \leq \|G\|_{L^1(\rho_\mu)}$ holds, allowing an extension of the $K$-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (4) still holds, now $\mu$-a.e. This means, in particular,
\[
(Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x)), \quad \mu\text{-a.a. } \gamma \in \Gamma, \quad (6)
\]
for all $\mathcal{B}(X)$-measurable functions $f$ such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$.

**Remark 1** All the notions described above as well as their relations are graphically summarized in the figure below. Having in mind the concrete application in Section 5 below, let us mention the natural meaning of this figure in the context of an infinite particle system. The state of such a system

\[\text{6}^1\text{Throughout this work all } L^p\text{-spaces, } p \geq 1, \text{ consist of complex-valued functions.}\]
is described by a probability measure $\mu$ on $\Gamma$ and the functions $F$ on $\Gamma$ are considered as observables of the system and they represent physical quantities which can be measured. The expected values of the measured observables correspond to the expectation values $\int_\Gamma F(\gamma) \, d\mu(\gamma)$. In this interpretation we call the functions $G$ on $\Gamma_0$ quasi-observables, because they are not observables themselves, but can be used to construct observables via the $K$-transform. In this way one obtains all observables which are additive in the particles, e.g., number of particles, energy.

\[
\begin{align*}
<F, \mu> &= \int_\Gamma F(\gamma) d\mu(\gamma) \\
<G, \rho_\mu> &= \int_{\Gamma_0} G(\eta) \, d\rho_\mu(\eta)
\end{align*}
\]

On the underlying measurable space $(X, \mathcal{B}(X))$ let us consider a non-atomic Radon measure $\sigma$, i.e., $\sigma(\{x\}) = 0$ for any $x \in X$. The Poisson measure $\pi_\sigma$ with intensity $\sigma$ is the probability measure defined on $(\Gamma, \mathcal{B}(\Gamma))$ by

\[
\int_\Gamma \exp \left( \sum_{x \in \gamma} \varphi(x) \right) d\pi_\sigma(\gamma) = \exp \left( \int_X \left( e^{\varphi(x)} - 1 \right) \, d\sigma(x) \right), \quad \varphi \in \mathcal{D},
\]

where $\mathcal{D} := C_0^\infty(X)$ denotes the Schwartz space of all infinitely differentiable real-valued functions on $X$ with compact support. The correlation measure corresponding to the Poisson measure $\pi_\sigma$ is the so-called Lebesgue-Poisson measure $\lambda_\sigma$ (with intensity $\sigma$)

\[
\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)},
\]
where each $\sigma^{(n)}$, $n \in \mathbb{N}$, is the symmetrization of the product measure $\sigma^\otimes n$, i.e., the image measure on $\Gamma_X^{(n)}$ of the measure $\sigma^\otimes n$ under the mapping $\text{sym}_X^n$ defined in (2). For $n = 0$ we set $\sigma^{(0)}(\{\emptyset\}) := 1$. The following Lebesgue-Poisson measure properties underline the importance of coherent states. First, $e_{\lambda}(f) \in L^p(\Gamma_0, \lambda_\sigma)$ whenever $f \in L^p(\mathcal{X}, \sigma)$ for some $p \geq 1$, and, moreover, $\int_{\Gamma_0} |e_{\lambda}(f, \eta)|^p \, d\lambda_\sigma(\eta) = \exp \left( \int_{\mathcal{X}} |f(x)|^p \, d\sigma(x) \right)$. Second, given a dense subspace $\mathcal{L} \subset L^2(\mathcal{X}, \sigma)$, the set $\{e_{\lambda}(f) : f \in \mathcal{L}\}$ is total in $L^2(\Gamma_0, \lambda_\sigma)$.

3 Bogoliubov functionals

For the case $\mathcal{X} = \mathbb{R}^d$, $d \in \mathbb{N}$, we refer to [Naz85] and his own references therein.

**Definition 2** Let $\mu$ be a probability measure on $(\Gamma, \mathcal{B}(\Gamma))$. The Bogoliubov functional $L_{\mu}$ corresponding to $\mu$ is a functional defined at each $\mathcal{B}^c(\mathcal{X})$-measurable function $\theta$ by

$$L_{\mu}(\theta) := \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \, d\mu(\gamma),$$

provided the right-hand side exists for $|\theta|$. We note that if $L_{\mu}(|\theta|) < \infty$, then the product $\prod_{x \in \gamma}(1 + \theta(x))$ is $\mu$-a.e. absolutely convergent. For the definition and properties of infinite products see [Kno64].

It is clear that the domain of a Bogoliubov functional $L_{\mu}$ depends on the measure $\mu$ fixed on $(\Gamma, \mathcal{B}(\Gamma))$. Conversely, the domain of a Bogoliubov functional reflects special properties over the underlying measure on $(\Gamma, \mathcal{B}(\Gamma))$. For instance, probability measures $\mu$ for which the Bogoliubov functional is well-defined on multiples of indicator functions $1_\Lambda$, $\Lambda \in \mathcal{B}_c(\mathcal{X})$, necessarily have finite local exponential moments, i.e., $\mu \in \mathcal{M}_{\text{exp}}(\Gamma)$. In fact, for all $\alpha > 0$ and all $\Lambda \in \mathcal{B}_c(\mathcal{X})$ we find

$$\int_{\Gamma} e^{\alpha|\gamma|} \, d\mu(\gamma) = \int_{\Gamma} \prod_{x \in \gamma} e^{\alpha 1_\Lambda(x)} \, d\mu(\gamma) = L_{\mu}((e^\alpha - 1) 1_\Lambda) < \infty.$$ 

In the sequel, for each probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ and each $\Lambda \in \mathcal{B}_c(\mathcal{X})$, we denote by $\mu^\Lambda := \mu \circ (p_\Lambda)^{-1}$ the image measure on $\Gamma_\Lambda$ of the
measure \( \mu \) under the projection \( p_\Lambda \) defined in (3), i.e., \( \mu^\Lambda \) is the projection of \( \mu \) onto \( \Gamma_\Lambda \). Given a \( \Lambda \in \mathcal{B}_c(X) \), the definition of a Bogoliubov functional \( L_\mu \) on the space of all functions \( \theta \) with support contained in \( \Lambda \) reduces to the Bogoliubov functional \( L_{\mu^\Lambda} \):

\[
L_\mu(\theta) = \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) d\mu(\gamma) = \int_{\Gamma} \prod_{x \in \gamma_{\Lambda}} (1 + \theta(x)) d\mu(\gamma) = L_{\mu^\Lambda}(\theta).
\]

Furthermore, one may straightforwardly express the \( \mu \)-measure of a large class of sets by the Bogoliubov functional \( L_{\mu^\Lambda} \). In fact, given \( z_1, \ldots, z_n \in \mathbb{C} \) and a collection of mutually disjoint sets \( \Lambda_1, \ldots, \Lambda_n \in \mathcal{B}_c(X) \), \( \Delta := \bigcup_{i=1}^n \Lambda_i \), \( n \in \mathbb{N} \), the above computation has shown that

\[
L_{\mu} \left( \sum_{i=1}^n z_i \mathbb{1}_{\Lambda_i} - \mathbb{1}_\Delta \right) = \int_{\Gamma} \prod_{x \in \gamma_{\Delta}} \left( \sum_{i=1}^n z_i \mathbb{1}_{\Lambda_i}(x) \right) d\mu(\gamma).
\]

Since \( \Gamma_{\Delta} \) may be written as the disjoint union

\[
\Gamma_{\Delta} = \bigcup_{k_1, \ldots, k_n = 0}^\infty \{ \gamma \in \Gamma_{\Delta} : |\gamma_{\Lambda_i}| = k_i, i = 1, \ldots, n \},
\]

the latter integral is then equal to

\[
\sum_{k_1, \ldots, k_n = 0}^\infty z_1^{k_1} \cdots z_n^{k_n} \mu \left( \{ \gamma \in \Gamma : |\gamma_{\Lambda_i}| = k_i, i = 1, \ldots, n \} \right).
\]

Heuristically, this means that

\[
\mu \left( \{ \gamma \in \Gamma : |\gamma_{\Lambda_i}| = k_i, i = 1, \ldots, n \} \right) = \frac{1}{k_1! \cdots k_n!} \left. \frac{\partial^{k_1+\cdots+k_n}}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} L_{\mu} \left( \sum_{i=1}^n z_i \mathbb{1}_{\Lambda_i} - \mathbb{1}_{\bigcup_{i=1}^n \Lambda_i} \right) \right|_{z_1=\ldots=z_n=0}.
\]

According to the definition of the \( \sigma \)-algebra \( \mathcal{B}(\Gamma) \), the collection of sets appearing in the left-hand side of the informal equality (7) already characterizes the measure \( \mu \).

Of course, in order to apply the above procedure we must assume that the Bogoliubov functional \( L_{\mu} \) is well-defined and differentiable on the class of linear combinations of indicator functions which appears in (7). As the linear
space spanned by indicator functions or the spaces of measurable functions are both difficult to handle, throughout this work we will consider Bogoliubov functionals on a $L^1(X, \sigma) =: L^1(\sigma)$ space, for some Radon measure $\sigma$ defined on the space $(X, \mathcal{B}(X))$. Furthermore, we will assume that the Bogoliubov functionals are entire. We observe that from the viewpoint of particle systems these restrictions are natural. Actually, even stronger properties should be expected.

In the sequel, we fix on $(X, \mathcal{B}(X))$ a non-atomic Radon measure $\sigma$ which we assume to be non-degenerate, i.e., $\sigma(O) > 0$ for all non-empty open sets $O \subset X$, and, in addition, $\sigma(X) = \infty$. We recall that a functional $A : L^1(\sigma) \to \mathbb{C}$ is entire on $L^1(\sigma)$ whenever $A$ is locally bounded, and for all $\theta_0, \theta \in L^1(\sigma)$ the mapping $\mathbb{C} \ni z \mapsto A(\theta_0 + z\theta) \in \mathbb{C}$ is entire. Thus, at each $\theta_0 \in L^1(\sigma)$, every entire functional $A$ on $L^1(\sigma)$ has a representation in terms of its Taylor expansion,

$$A(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n A(\theta_0; \theta, \ldots, \theta), \quad z \in \mathbb{C}, \theta \in L^1(\sigma),$$

see e.g. [Bar85], [Dim81]. The next theorem states a characterization result for the differentials $d^n A(\theta_0; \cdot)$ of an entire functional $A$ on $L^1(\sigma)$.

**Theorem 3** Let $A$ be an entire functional on $L^1(\sigma)$. Then each differential $d^n A(\theta_0; \cdot), n \in \mathbb{N}, \theta_0 \in L^1(\sigma)$ is defined by a (symmetric) kernel in $L^\infty(X^n, \sigma^\otimes n)$ denoted by $\frac{\delta^n A(\theta_0)}{\delta \theta_0(x_1) \ldots \delta \theta_0(x_n)}$ and called the variational derivative of $n$-th order of $A$ at the point $\theta_0$. In other words,

$$d^n A(\theta_0; \theta_1, \ldots, \theta_n) := \frac{\partial^n}{\partial z_1 \ldots \partial z_n} A\left(\theta_0 + \sum_{i=1}^{n} z_i \theta_i\right) \bigg|_{z_1 = \ldots = z_n = 0}$$

$$=: \int_{X^n} \frac{\delta^n A(\theta_0)}{\delta \theta_0(x_1) \ldots \delta \theta_0(x_n)} \prod_{i=1}^{n} \theta_i(x_i) d\sigma^\otimes n(x_1, \ldots, x_n)$$

for all $\theta_1, \ldots, \theta_n \in L^1(\sigma)$. Moreover, for all $r > 0$

$$\left\| \frac{\delta^n A(\theta_0)}{\delta \theta_0(x_1) \ldots \delta \theta_0(x_n)} \right\|_{L^\infty(X^n, \sigma^n)} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1(\sigma)} \leq r} |A(\theta_0 + \theta')| \quad (8)$$

for all $\theta_0 \in L^1(\sigma)$. In other words,
Remark 4 According to Theorem 3, the Taylor expansion of an entire functional $A$ at a point $\theta_0 \in L^1(\sigma)$ may be written in the form

$$A(\theta_0 + \theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \frac{\delta^n A(\theta_0)}{\delta \theta_0(x_1)...\delta \theta_0(x_n)} \prod_{i=1}^{n} \theta(x_i) d\sigma^\otimes n(x_1, ..., x_n),$$

for all $\theta \in L^1(\sigma)$. Using the notation

$$(D^{[n]} A)(\theta_0; \eta) := \frac{\delta^n A(\theta_0)}{\delta \theta_0(x_1)...\delta \theta_0(x_n)}$$

for $\eta = \{x_1, ..., x_n\} \in \Gamma^{(n)}_X, n \in \mathbb{N},$

this means

$$A(\theta_0 + \theta) = \int_{\Gamma_0} e_{\lambda}(\theta, \eta) \left( D^{[n]} A \right)(\theta_0; \eta) d\lambda_{\sigma}(\eta).$$

Concerning the estimate (8), we note that the entire property of $A$ does not insure that for every $r > 0$ the supremum on the right-hand side is always finite. This will hold if, in addition, the entire functional $A$ is of bounded type, that is,

$$\forall r > 0, \sup_{\|\theta\|_{L^1(\sigma)} \leq r} |A(\theta_0 + \theta)| < \infty, \forall \theta_0 \in L^1(\sigma).$$

For simplicity, throughout this work we will assume this assumption.

The proof of the first part of this result is of a technical nature outside of the present context. However, it contains a few steps which we will need to prove the second part. Because of this, we just present a sketch of the proof conveniently adapted to our aims and complemented with suitable references for a detailed proof.

Proof. According to the Cauchy formula for analytic functions on Banach spaces, each differential $d^n A(\theta_0; \cdot)$ of an entire functional $A$ on $L^1(\sigma)$ is a bounded symmetric $n$-linear functional on $L^1(\sigma)$. In particular, for $n = 1$, the first order differential $dA(\theta_0; \cdot)$ is a bounded linear functional on $L^1(\sigma)$, insuring that it can be represented by a kernel in $L^\infty(X, \sigma)$, the so-called first variational derivative $\frac{\delta A(\theta_0)}{\delta \theta_0(x)}$. Furthermore, the (usual) operator norm of the bounded linear functional $dA(\theta_0; \cdot)$ is equal to $\left\| \frac{\delta A(\theta_0)}{\delta \theta_0(\cdot)} \right\|_{L^\infty(X, \sigma)}$. 

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For higher orders, the proof of existence of the corresponding variational
derivatives is a straightforward consequence of the isometries between the
Banach spaces

\[ B_n(L^1(X, \sigma)) \simeq (L^1(X^\otimes n))' \simeq L^\infty(X^n, \sigma^\otimes n), \quad (9) \]

\( B_n(L^1(X, \sigma)) \) being the space of all bounded \( n \)-linear functionals on \( L^1(X, \sigma) \).
For the proof see e.g. [DU77], [Sch71], [Tre67]. These isometries prove, on
the one hand, the existence of the variational derivatives

\[ \frac{\delta^n A(\theta_0)}{\delta \theta_0(x_1) \ldots \delta \theta_0(x_n)} \in L^\infty(X^n, \sigma^\otimes n) \]

as kernels for \( d^n A(\theta_0; \cdot) \), and, on the other hand, that the op-
operator norm of \( d^n A(\theta_0; \cdot) \in B_n(L^1(X, \sigma)) \) is given by

\[ \| \frac{\delta^n A(\theta_0)}{\delta \theta_0(\cdot) \ldots \delta \theta_0(\cdot)} \|_{L^\infty(X^n, \sigma^\otimes n)}. \]

This shows the first part of the theorem. To prove the second one, we observe
that by the Cauchy formula, for any \( \theta \in L^1(\sigma) \) one has

\[ \frac{1}{n!} d^n A(\theta_0; \theta, \ldots, \theta) = \frac{1}{2\pi i} \int_{|z|=r} \frac{A(\theta_0 + z\theta)}{z^{n+1}} dz \]

for any \( r > 0 \) and any \( n \in \mathbb{N} \). Therefore

\[ |d^n A(\theta_0; \theta, \ldots, \theta)| \leq n! \sup_{\|\theta\|_{L^1(\sigma)} \leq r} |A(\theta_0 + \theta')| \left( \frac{\|\theta\|_{L^1(\sigma)}}{r} \right)^n, \]

and an application of the polarization identity extends this inequality to
\( \theta_1, \ldots, \theta_n \in L^1(\sigma) \):

\[ |d^n A(\theta_0; \theta_1, \ldots, \theta_n)| \leq n! \left( \frac{e}{r} \right)^n \sup_{\|\theta'\|_{L^1(\sigma)} \leq r} |A(\theta_0 + \theta')| \prod_{i=1}^{n} \|\theta_i\|_{L^1(\sigma)}, \]

see e.g. [Din81, Theorem 1.7].

**Remark 5** Observe that the first isometry in (9) is specific of \( L^1 \) spaces. The analogous result does not hold neither for other \( L^p \)-spaces, nor Banach
spaces of continuous functions, or Sobolev spaces.

Theorem 3 stated for Bogoliubov functionals yields the next result. In
particular, it gives a rigorous sense to the discussion at the beginning of this
section.
Corollary 6 Let $L_\mu$ be a Bogoliubov functional corresponding to some probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$. If $L_\mu$ is entire of bounded type on $L^1(\sigma)$, then the measure $\mu$ is locally absolutely continuous with respect to the Poisson measure $\pi_\sigma$, i.e., for all $\Lambda \in \mathcal{B}_c(X)$ the measure $\mu^\Lambda = \mu \circ (p_\Lambda)^{-1}$ is absolutely continuous with respect to $\pi_\sigma^\Lambda = \pi_\sigma \circ (p_\Lambda)^{-1}$. Moreover, for all $\Lambda \in \mathcal{B}_c(X)$ one has

$$\frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\gamma) = e^{\sigma(\Lambda)} \left(D^{[\gamma]} L_\mu\right) (-\mathbb{1}_\Lambda; \gamma) \quad \text{for } \pi_\sigma^\Lambda \text{-a.a. } \gamma \in \Gamma_\Lambda,$$

and for each $r > 0$ there exists a constant $C \geq 0$ such that

$$\left|\frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\gamma)\right| \leq e^{\sigma(\Lambda)} C|\gamma|! \left(\frac{e}{r}\right)^{|\gamma|} \quad \text{for } \pi_\sigma^\Lambda \text{-a.a. } \gamma \in \Gamma_\Lambda^{(n)}.$$

Proof. In Theorem replace $A$ by the functional $L_\mu$ and $\theta_0$ by an indicator function $-\mathbb{1}_\Lambda$ for some $\Lambda \in \mathcal{B}_c(X)$. Thus, for all functions $\theta \in L^1(\sigma)$ with support contained in $\Lambda$, we find

$$L_\mu(\theta) = L_\mu(-\mathbb{1}_\Lambda + (\theta + \mathbb{1}_\Lambda)) = \int_{\Gamma_\Lambda} \prod_{x \in \eta} (1 + \theta(x)) \left(D^{[\eta]} L_\mu\right) (-\mathbb{1}_\Lambda; \eta)d\lambda_\sigma(\eta).$$

On the other hand, according to the considerations done at the beginning of this section, we also have

$$L_\mu(\theta) = \int_{\Gamma_\Lambda} \prod_{x \in \gamma} (1 + \theta(x)) d\mu^\Lambda(\gamma).$$

Therefore

$$\int_{\Gamma_\Lambda} \prod_{x \in \gamma} (1 + \theta(x)) d\mu^\Lambda(\gamma) = \int_{\Gamma_\Lambda} \prod_{x \in \eta} (1 + \theta(x)) \left(D^{[\eta]} L_\mu\right) (-\mathbb{1}_\Lambda; \eta)d\lambda_\sigma(\eta)$$

for all functions $\theta \in L^1(\sigma)$ with support contained in $\Lambda$. The proof follows by a monotone class argument. ■

Since $\mu \in \mathcal{M}_{\text{exp}}^1(\Gamma)$ whenever the corresponding Bogoliubov functional is well-defined on the whole space $L^1(\sigma)$, one can associate the correlation
measure $\rho_\mu = K^*\mu$ to a such measure. Equalities (6) and (5) then yield a description of the functional $L_\mu$ in terms of the measure $\rho_\mu$:

$$L_\mu(\theta) = \int_{\Gamma} (K e_\lambda(\theta)) (\gamma) d\mu(\gamma) = \int_{\Gamma_0} e_\lambda(\theta, \eta) d\rho_\mu(\eta).$$  \hspace{1cm} (10)$$

Within this formalism Theorem 3 states as follows.

**Proposition 7** Let $L_\mu$ be an entire Bogoliubov functional of bounded type on $L^1(\sigma)$. Then the measure $\rho_\mu$ is absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda_\sigma$ and the Radon-Nykodim derivative $k_\mu := \frac{d\rho_\mu}{d\lambda_\sigma}$ is given by

$$k_\mu(\eta) = (D^{[n]}L_\mu) (0; \eta) \text{ for } \lambda_\sigma-a.a. \eta \in \Gamma_0.$$  

Furthermore, for each $r > 0$ there is a constant $C \geq 0$ such that

$$\left| (D^{[n]}L_\mu) (0; \eta) \right| \leq C \left| \eta \right|! \left( \frac{e}{r} \right)^{\left| \eta \right|} \text{ for } \lambda_\sigma-a.a. \eta \in \Gamma_0.$$  

In the sequel we call $k_\mu$ the correlation function corresponding to $\mu$.

**Proof.** A straightforward application of Theorem 3 yields

$$L_\mu(\theta) = \int_{\Gamma_0} e_\lambda(\theta, \eta) (D^{[n]}L_\mu) (0; \eta) d\lambda_\sigma(\eta), \hspace{0.5cm} \theta \in L^1(\sigma),$$

and

$$\left| (D^{[n]}L_\mu) (0; \eta) \right| \leq C \left| \eta \right|! \left( \frac{e}{r} \right)^{\left| \eta \right|}, \hspace{0.5cm} \lambda_\sigma-a.a. \eta \in \Gamma_0,$$

for some $C \geq 0$ depending on $r$. Expression (10) then allows to identify $k_\mu(\eta)$ with $(D^{[n]}L_\mu) (0; \eta)$.

**Remark 8** Proposition 7 shows that the correlation functions $k^{(n)}_\mu := k_\mu|_{\Gamma_0}^{(n)}$ are the Taylor coefficients of the Bogoliubov functional $L_\mu$. In other words, $L_\mu$ is the generating functional for the correlation functions $k^{(n)}_\mu$. This was also the reason why N. N. Bogoliubov introduced these functionals. Furthermore, Bogoliubov functionals are also related to the general infinite dimensional analysis on configuration spaces, cf. e.g. [KKO02]. Namely, through the unitary isomorphism $S_\lambda$ defined in [KKO02] between the space $L^2(\Gamma_0, \lambda_\sigma)$ and the Bargmann-Segal space one has $L_\mu = S_\lambda(k_\mu)$. 

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Proposition 9. For any Bogoliubov functional $L_\mu$ entire of bounded type on $L^1(\sigma)$ the following relations between variational derivatives hold:

$$(D^{[\eta]} L_\mu)(\theta; \eta) = \int_{\Gamma_0} k_\mu(\eta \cup \xi) e_\lambda(\theta, \xi) d\lambda_\sigma(\xi) \quad \text{for } \lambda_\sigma - \text{a.a. } \eta \in \Gamma_0$$ (11)

and, more generally,

$$(D^{[\eta]} L_\mu)(\theta_1 + \theta_2; \eta) = \int_{\Gamma_0} (D^{[\eta \cup \xi]} L_\mu)(\theta_1; \eta \cup \xi) e_\lambda(\theta_2, \xi) d\lambda_\sigma(\xi) \quad \text{for } \lambda_\sigma - \text{a.a. } \eta \in \Gamma_0,$$

for $\theta, \theta_1, \theta_2 \in L^1(\sigma)$.

To prove this result as well as other forthcoming ones the next lemma shows to be useful.

Lemma 10. ([FF97], [KKO02], [Rue69]) The following equality holds

$$\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\xi, \eta) d\lambda_\sigma(\eta) d\lambda_\sigma(\xi) = \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi) d\lambda_\sigma(\eta)$$

for all positive measurable functions $G : \Gamma_0 \to \mathbb{R}$ and $H : \Gamma_0 \times \Gamma_0 \to \mathbb{R}$.

Proof. According to Theorem 3 for all $\theta_1, \theta_2, \theta \in L^1(\sigma)$ one has

$$L_\mu(\theta_1 + \theta_2 + \theta) = \int_{\Gamma_0} (D^{[\eta]} L_\mu)(\theta_1 + \theta_2; \eta) e_\lambda(\theta, \eta) d\lambda_\sigma(\eta)$$

as well as

$$L_\mu(\theta_1 + \theta_2 + \theta) = \int_{\Gamma_0} (D^{[\eta]} L_\mu)(\theta_1; \eta) e_\lambda(\theta_2 + \theta, \eta) d\lambda_\sigma(\eta).$$

The bounds obtained in Theorem 3 allows to apply Lemma 10 to the latter equality yielding

$$\int_{\Gamma_0} \int_{\Gamma_0} (D^{[\eta \cup \xi]} L_\mu)(\theta_1; \eta \cup \xi) e_\lambda(\theta_2, \xi) d\lambda_\sigma(\xi) e_\lambda(\theta, \eta) d\lambda_\sigma(\eta).$$

The second stated equality follows by a monotone class argument. By Proposition 7 one sees that (11) is a special case of the derived result for $\theta_1 = 0$ and $\theta_2 = \theta$. 

A particular application of Proposition 9 yields the next two formulas well-known in statistical mechanics, see e.g. [Rue70], and in the theory of point processes, see e.g. [DVJ88].
Corollary 11 Under the conditions of Proposition 9 for all \( \Lambda \in \mathcal{B}_c(X) \) we have
\[
k_\mu(\eta) = \int_{\Gamma_\Lambda} \frac{d\mu^\Lambda}{d\pi^\Lambda_{\sigma}} (\eta \cup \gamma) d\pi^\Lambda_{\sigma}(\gamma) \text{ for } \lambda_{\sigma}-a.a. \ \eta \in \Gamma_\Lambda,
\]
and
\[
\frac{d\mu^\Lambda}{d\pi^\Lambda_{\sigma}}(\gamma) = e^{\sigma(\Lambda)} \int_{\Gamma_\Lambda} (-1)^{|\eta|} k_\mu(\gamma \cup \eta) d\lambda_{\sigma}(\eta) \text{ for } \pi^\Lambda_{\sigma}-a.a. \ \gamma \in \Gamma_\Lambda.
\]

Proof. Fixing a \( \Lambda \in \mathcal{B}_c(X) \), in Proposition 9 replace both functions \( \theta \) and \( \theta_1 \) by the function \(-1_\Lambda\) and \( \theta_2 \) by \( 1_\Lambda\). The expressions for the densities given in Corollary 6 and Proposition 7 complete the proof.

Remark 12 Corollary 11 may be stated under more general conditions. Given a probability measure \( \mu \) on \((\Gamma, \mathcal{B}(\Gamma))\) such that
\[
\int_{\Gamma} |\gamma|^{n} d\mu(\gamma), \int_{\Gamma_\Lambda} 2^{|\eta|} d\rho(\eta) < \infty \text{ for all } \Lambda \in \mathcal{B}_c(X) \text{ and all } n \in \mathbb{N},
\]
onlyout{\( \mu \) is locally absolutely continuous with respect to the Poisson measure \( \pi_{\sigma} \) if and only if the correlation measure \( \rho \) is absolutely continuous with respect to the Lebesgue-Poisson measure \( \lambda_{\sigma} \). Under these conditions, equalities (12) and (13) hold (see e.g. [KK02]).

Corollary 13 Let \( L_\mu \) be an entire Bogoliubov functional of bounded type on \( L^1(\sigma) \). For any \( \mathcal{B}(\Gamma_0) \)-measurable function \( G : \Gamma_0 \to \mathbb{R} \) such that there is a \( f \in L^1(\sigma) \) with \(|G| \leq e_\lambda(f)\), one has
\[
\int_{\Gamma_0} G(\eta) \left(D^{\eta^0}L_\mu\right)(\theta;\eta) d\lambda_{\sigma}(\eta) = \int_{\Gamma_0} \sum_{\xi \subset \eta} G(\xi) e_\lambda(\theta, \eta\setminus\xi) d\rho_{\mu}(\eta),
\]
for all \( \theta \in L^1(\sigma) \).

According to Proposition 8, the correlation function \( k_\mu \) of an entire Bogoliubov functional on \( L^1(\sigma) \) fulfills the so-called generalized Ruelle bound, that is, for any \( 0 \leq \varepsilon \leq 1 \) and any \( r > 0 \) there is some constant \( C \geq 0 \) depending on \( r \) such that
\[
k_\mu(\eta) \leq C (|\eta|!)^{1-\varepsilon} \left(\frac{e}{r}\right)^{|\eta|}, \text{ for } \lambda_{\sigma}-a.a. \ \eta \in \Gamma_0.
\]
In our case, \( \varepsilon \) is zero. We note that if (14) holds for \( \varepsilon = 1 \) and for at least one \( r > 0 \), then condition (14) is the classical Ruelle bound. For a general \( 0 < \varepsilon \leq 1 \) one may state the following result.
Proposition 14 \([\text{KK05}]\) If there are a function \(0 \leq C \in L^1_{\text{loc}}(X, \sigma)\) and a \(0 < \varepsilon \leq 1\) such that
\[
k_\mu(\eta) \leq (|\eta|!)^{1-\varepsilon} e_\lambda(C, \eta), \quad \lambda_\sigma - \text{a.a. } \eta \in \Gamma_0,
\]
then there are constants \(c_1 = c_1(\varepsilon), c_2 = c_2(\varepsilon) > 0\) such that
\[
|L_\mu(\varphi)| \leq c_1 \exp\left(\frac{\|\varphi\|^{1/\varepsilon}_{L_1(C\sigma)}}{L_1(C\sigma)}\right), \quad \varphi \in D.
\]
Furthermore, \(L_\mu\) is an entire functional of bounded type on \(L^1(C\sigma)\).

The definition of a Bogoliubov functional clearly shows that for any probability measure \(\mu \in \mathcal{M}^{1}_{\exp}(\Gamma)\) \(L_\mu\) is a normalized functional, that is, \(L_\mu(0) = 1\). If, in addition, \(L_\mu\) is an entire functional on \(L^1(\sigma)\), then, according to Corollary \(\text{C}\) for all \(\Lambda \in \mathcal{B}_c(X)\) we have
\[
(D|\eta|L_\mu)(-\mathbb{1}_\Lambda; \eta) = e^{-\sigma(\Lambda)\frac{d\mu^\Lambda}{d\pi^\Lambda}(\gamma)} \geq 0, \quad \lambda_\sigma - \text{a.a. } \gamma \in \Gamma_\Lambda.
\]
These conditions are also sufficient to insure that a generic entire functional on \(L^1(\sigma)\) is a Bogoliubov functional corresponding to some measure in \(\mathcal{M}^{1}_{\exp}(\Gamma)\).

Proposition 15 Let \(L\) be a normalized entire functional of bounded type on \(L^1(\sigma)\) such that for all \(\Lambda \in \mathcal{B}_c(X)\)
\[
(D|\eta|L)(-\mathbb{1}_\Lambda; \eta) \geq 0, \quad \lambda_\sigma - \text{a.a. } \eta \in \Gamma_\Lambda.
\]
Then there is a unique probability measure \(\mu \in \mathcal{M}^{1}_{\exp}(\Gamma)\) such that for all \(\theta \in L^1(\sigma)\)
\[
L(\theta) = \int_{\Gamma} \prod_{x \in \Gamma}(1 + \theta(x))d\mu(\gamma). \tag{15}
\]

Proof. For any \(\Lambda \in \mathcal{B}_c(X)\) let us define the function
\[
G_\Lambda(\eta) := (D|\eta|L)(-\mathbb{1}_\Lambda; \eta) \geq 0, \quad \eta \in \Gamma_\Lambda.
\]
For all \(\Lambda \in \mathcal{B}_c(X)\) we have
\[
\int_{\Gamma_\Lambda} G_\Lambda(\eta)d\lambda_\sigma(\eta) = \int_{\Gamma_0} e_\lambda(\mathbb{1}_\Lambda, \eta)(D|\eta|L)(-\mathbb{1}_\Lambda; \eta)d\lambda_\sigma(\eta)
\]
\[
= L(\mathbb{1}_\Lambda - \mathbb{1}_\Lambda) = L(0) = 1,
\]
allowing to define a family of probability measures $\mu^{\Lambda}$ on $(\Gamma, B(\Gamma))$ by

$$\mu^{\Lambda}(A) := \int_{\Gamma} \mathbb{1}_A(\eta) G^{\Lambda}(\eta) d\lambda_\sigma(\eta), \quad A \in B(\Gamma).$$

Similarly, one verifies that the family $(\mu^{\Lambda})_{\Lambda \in B_c(X)}$ is consistent. Therefore, by the version of the Kolmogorov theorem for the projective limit space $(\Gamma, B(\Gamma))$ [Par67, Chapter V, Theorem 5.1], there is a unique probability measure $\mu$ on $\Gamma$ such that the measures $\mu^{\Lambda}$ are the projections of $\mu$. From the definition of $G^{\Lambda}$ follows the relation (15) between $L$ and $\mu$ for every $\theta$ supported in $\Lambda$. The $L^1$-continuity of $L$ and monotone convergence arguments extend this relation to all non-negative functions $\theta \in L^1(\sigma)$. The general relation follows from dominated convergence results.

\section{4 Bogoliubov equations}

Particularly interesting is the characterization of Gibbs measures through the Bogoliubov functionals.

Given a pair potential $\phi : X \times X \to \mathbb{R} \cup \{+\infty\}$, that is, a symmetric measurable function, let $E : \Gamma \to \mathbb{R} \cup \{+\infty\}$ be the energy functional and $W : \Gamma_0 \times \Gamma \to \mathbb{R} \cup \{+\infty\}$ be the interaction energy defined for all $\eta \in \Gamma_0$ and all $\gamma \in \Gamma$ by

$$E(\emptyset) := E(\{-x\}) := 0$$

and

$$W(\emptyset, \gamma) := \begin{cases} \sum_{x \in \eta, y \in \gamma} \phi(x, y), & \text{if } \sum_{x \in \eta, y \in \gamma} |\phi(x, y)| < \infty, \\ +\infty, & \text{otherwise} \end{cases},$$

respectively. We set $W(\emptyset, \gamma) := 0$. A grand canonical Gibbs measure (Gibbs measure for short) corresponding to a pair potential $\phi$, the intensity measure $\sigma$, and an inverse temperature $\beta > 0$, is usually defined through the Dobrushin-Lanford-Ruelle equation. For convenience, we present here an equivalent definition through the Georgii-Nguyen-Zessin equation ((GNZ)-equation) [NZ79, Theorem 2], see also [KK03, Theorem 3.12], [Kun99, Appendix A.1]). More precisely, a probability measure $\mu$ on $(\Gamma, B(\Gamma))$ is called
a Gibbs measure if it fulfills the integral equation

\[ \int_{\Gamma} \sum_{x \in \gamma} H(x, \gamma \{ x \}) \, d\mu(\gamma) = \int_{\Gamma} \int_{X} H(x, \gamma) e^{-\beta W(x, \gamma)} \, d\sigma(x) \, d\mu(\gamma) \]  

(16)

for all positive measurable functions \( H : X \times \Gamma \to \mathbb{R} \). In particular, for \( \phi \equiv 0 \), [16] reduces to the Mecke identity, which yields an equivalent definition of the Poisson measure \( \pi_\sigma \) [Mec67, Theorem 3.1].

Correlation measures corresponding to Gibbs measures are always absolutely continuous with respect to the Lebesgue-Poisson measure \( \lambda_\sigma \). In view of this fact and Remark [12] the framework used throughout this section is restricted to measures \( \mu \in M^1_{\text{exp}}(\Gamma) \) that are locally absolutely continuous with respect to the Poisson measure \( \pi_\sigma \). Furthermore, we shall assume that the corresponding correlation functions \( k_\mu \) fulfill the so-called Ruelle type bound inequality, that is, there are \( a > 0 \) and \( a < \varepsilon \leq 1 \) such that

\[ k_\mu(\eta) \leq (|\eta|!)^{1-\varepsilon} e^{\lambda(a, \eta)} = (|\eta|!)^{1-\varepsilon} a^{|\eta|}, \quad \lambda_\sigma - \text{a.a.} \eta \in \Gamma_0. \]

According to Proposition [13] this assumption implies that

1. There are \( c_1, c_2 > 0 \) such that

\[ |L_\mu(\theta)| \leq c_1 \exp \left( c_2 \| \theta \|_{L^1(\sigma)}^{1/\varepsilon} \right) \quad \text{for all } \theta \in L^1(\sigma). \]

As a consequence of Proposition [14] the Bogoliubov functional \( L_\mu \) is entire of bounded type on \( L^1(\sigma) \).

To proceed towards the equivalent description of Gibbs measures through Bogoliubov functionals, we consider potentials \( \phi \) fulfilling the following semi-boundedness and integrability conditions:

2. \( \exists B \geq 0 : \phi(x, y) \geq -2B \quad \text{for all } x, y \in X \)

3. \( C(\beta) := \text{ess sup}_{x \in X} \int_{X} |e^{-\beta \phi(x, y)} - 1| \, d\sigma(y) < \infty \)

**Proposition 16** Given a \( \mu \in M^1_{\text{exp}}(\Gamma) \) and a pair potential \( \phi \), assume that Assumptions 1–3 are fulfilled. Then \( \mu \) is a Gibbs measure corresponding to the potential \( \phi \), the intensity measure \( \sigma \), and the inverse temperature \( \beta \) if and
only if the Bogoliubov functional $L_\mu$ corresponding to $\mu$ solves the so-called Bogoliubov (equilibrium) equation,

$$\frac{\delta L(\theta)}{\delta \theta(x)} = L((1 + \theta)(e^{-\beta \phi(x, \cdot)} - 1) + \theta), \quad \sigma - a.e.,$$

for all $\theta \in L^1(\sigma)$.

Proof. The analyticity of $L_\mu$ on $L^1(\sigma)$ implies

$$dL_\mu(\theta; f) = \int_{\Gamma} \left| \frac{d}{dz} \prod_{x \in \gamma}(1 + \theta(x) + zf(x)) \right|_{z=0} d\mu(\gamma)$$

$$= \int_{\Gamma} \sum_{x \in \gamma} f(x) \prod_{y \in \gamma \setminus \{x\}} (1 + \theta(y)) d\mu(\gamma), \quad \theta, f \in L^1(\sigma). \quad (17)$$

Thus, for a Gibbs measure $\mu$, the (GNZ)-equation yields for the right-hand side of (17)

$$\int_{\mathcal{X}} f(x) \int_{\Gamma} \prod_{y \in \gamma} (1 + \theta(y))e^{-\beta W(\{x\}, \gamma)} d\mu(\gamma)d\sigma(x). \quad (18)$$

We claim that

$$e^{-\beta W(\{x\}, \gamma)} = \prod_{y \in \gamma} \left( 1 + \left( e^{-\beta \phi(x,y)} - 1 \right) \right), \quad \beta W(\{x\}, \gamma)$$

which proof we postpone to the end. Hence (18) is given by

$$\int_{\mathcal{X}} f(x) \int_{\Gamma} \prod_{y \in \gamma} (1 + \theta(y)) \prod_{y \in \gamma} \left( 1 + \left( e^{-\beta \phi(x,y)} - 1 \right) \right) d\mu(\gamma)d\sigma(x)$$

$$= \int_{\mathcal{X}} f(x) \int_{\Gamma} \prod_{y \in \gamma} ((1 + \theta(y)) \left( e^{-\beta \phi(x,y)} - 1 \right) + 1 + \theta(y)) d\mu(\gamma)d\sigma(x).$$

In this way we show that for all $f \in L^1(\sigma)$

$$dL_\mu(\theta; f) = \int_{\mathcal{X}} f(x) L_\mu \left( (1 + \theta)(e^{-\beta \phi(x, \cdot)} - 1) + \theta \right) d\sigma(x),$$

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provided \((1 + \theta)(e^{-\beta \phi(x, \cdot)} - 1) + \theta \in L^1(X, \sigma)\). Assumption 1 then implies that 
\(L_\mu ((1 + \theta)(e^{-\beta \phi(x, \cdot)} - 1) + \theta) \in L^\infty(X, \sigma)\) which completes the first part of the proof. Conversely, the same arguments as before yield,

\[
\int_X \sum_{x \in \gamma} f(x) \prod_{y \in \gamma \setminus \{x\}} (1 + \theta(y)) d\mu(\gamma) = dL_\mu(\theta; f)
\]

\[
= \int_X f(x) L_\mu ((1 + \theta)(e^{-\beta \phi(x, \cdot)} - 1) + \theta) \ d\sigma(x)
\]

\[
= \int_X f(x) \int_\Gamma \prod_{y \in \gamma} (1 + \theta(y)) e^{-\beta W(\{x\}, \gamma)} d\mu(\gamma) \ d\sigma(x),
\]

showing that the measure \(\mu\) fulfills the (GNZ)-equation for the class of functions \(H\) of the form

\[
H(x, \gamma) = f(x) \prod_{y \in \gamma \setminus \{x\}} (1 + \theta(y)), \ \theta, f \in L^1(\sigma).
\]

The result follows by a monotone class argument.

To conclude this proof amounts to check the technical problems left open. Due to Assumptions 2 and 3 one has

\[
\|\theta e^{-\beta \phi(x, \cdot)} + e^{-\beta \phi(x, \cdot)} - 1\|_{L^1(\sigma)} \leq e^{2\beta B} \|\theta\|_{L^1(\sigma)} + C(\beta),
\]

showing \((1 + \theta)(e^{-\beta \phi(x, \cdot)} - 1) + \theta \in L^1(X, \sigma)\).

The infinite product \(\prod_{y \in \gamma} (1 + |e^{-\beta \phi(x, y)} - 1|)\) converges for \(\sigma \otimes \mu\)-a.a. \((x, \gamma)\), because Assumption 3 implies that \(\sigma\)-a.e. \(\|e^{-\beta \phi(x, \cdot)} - 1\|_{L^1(\sigma)} < \infty\) and

\[
\int_\Gamma \prod_{y \in \gamma} (1 + |e^{-\beta \phi(x, y)} - 1|) \ d\mu(\gamma) < \infty.
\]

The absolute convergence of the infinite product in (19) implies the convergence of \(\sum_{y \in \gamma} |e^{-\beta \phi(x, y)} - 1|\). Hence, either the series \(\sum_{y \in \gamma} \phi(x, y)\) converges or there is a \(y \in \gamma\) such that \(\phi(x, y) = +\infty\). In the latter case the infinite product in (19) as well as \(e^{-\beta W(\{x\}, \gamma)}\) are both zero. For the first case we obtain

\[
\prod_{y \in \gamma} (1 + (e^{-\beta \phi(x, y)} - 1)) = \exp \left(-\beta \sum_{y \in \gamma} \phi(x, y)\right) = e^{-\beta W(\{x\}, \gamma)}.
\]
For higher order derivatives the corresponding Bogoliubov equations are defined as follows.

**Corollary 17** Given a \( \mu \in \mathcal{M}^1_{\text{exp}}(\Gamma) \) and a pair potential \( \phi \), assume that Assumptions 1–3 are fulfilled. If \( \mu \) is a Gibbs measure corresponding to the potential \( \phi \), the intensity measure \( \sigma \), and the inverse temperature \( \beta \), then for all \( \theta \in L^1(\sigma) \) the following relation holds:

\[
(D^n L_\mu)(\theta; \eta) = e^{-\beta E(\eta)} L_\mu((1 + \theta) (e^{-\beta W(\eta,\cdot)} - 1) + \theta), \quad \sigma^{(n)} - a.a. \eta \in \Gamma_X^{(n)}.
\]

**Proof.** It follows from successive applications of Proposition 16 and the chain rule to the function \( L^1(\sigma) \ni \theta \mapsto (1 + \theta) (e^{-\beta \phi(x,\cdot)} - 1) + \theta \in L^1(\sigma) \).

**Proposition 18** For any pair potential \( \phi \) and any measure \( \mu \in \mathcal{M}^1_{\text{exp}}(\Gamma) \) under Assumptions 1–3, the following equations are equivalent:

(i) For all \( \theta \in L^1(\sigma) \),

\[
\frac{\delta L_\mu(\theta)}{\delta \theta(x)} = L_\mu((1 + \theta) (e^{-\beta \phi(x)} - 1) + \theta) \quad \text{for } \sigma - a.a. \ x \in X.
\]

(ii) For every \( \theta, f \in L^1(\sigma) \),

\[
L_\mu(\theta + f) - L_\mu(\theta) = \int_X f(x) \int_0^1 L_\mu((1 + \theta + tf) (e^{-\beta \phi(x)} - 1) + \theta + tf) \, dt \, d\sigma(x).
\]

Furthermore, the previous equations imply that

(iii) For all \( \theta, f \in L^1(\sigma) \),

\[
L_\mu(\theta + f) = \int_{\Gamma_0} e_\lambda(f, \eta) e^{-\beta E(\eta)} L_\mu((1 + \theta) (e^{-\beta W(\eta,\cdot)} - 1) + \theta) \, d\lambda(\eta).
\]

**Remark 19** Assumptions 1–3 are not sufficient to insure the existence of the integral on the right-hand side of the equation stated in (iii).

**Proof.** (i) \( \Rightarrow \) (ii): Since \( L_\mu \) is entire on \( L^1(\sigma) \), one has

\[
L_\mu(\theta + f) - L_\mu(\theta) = \int_0^1 \frac{d}{dt} L_\mu(\theta + tf) \, dt.
\]
and, according to (i),
\[
\frac{d}{dt} L_\mu (\theta + tf) = d L_\mu (\theta + tf, f)
\]
\[
= \int_X f(x) L_\mu ((1 + \theta + tf) (e^{-\beta \phi(x, \cdot)} - 1) + \theta + tf) \, d\sigma(x).
\]

(ii) \(\Rightarrow\) (i): Assuming (ii), for any \(\theta, f \in L^1(\sigma)\) one finds
\[
\frac{d}{dz} L_\mu (\theta + zf) \bigg|_{z=0} = \lim_{z \to 0} \frac{L_\mu (\theta + zf) - L_\mu (\theta)}{z}
\]
\[
= \lim_{z \to 0} \int_X f(x) \int_0^1 L_\mu ((1 + \theta + tzf) (e^{-\beta \phi(x, \cdot)} - 1) + \theta + tzf) \, dt \, d\sigma(x).
\]
Assumptions 1–3 allow to apply the Lebesgue dominated convergence theorem and thus, interchanging the limit with the integrals and using the continuity of \(L_\mu\) on \(L^1(\sigma)\), to obtain
\[
\int_X f(x) L_\mu ((1 + \theta) (e^{-\beta \phi(x, \cdot)} - 1) + \theta) \, d\sigma(x).
\]

(i) \(\Rightarrow\) (iii): The analyticity of \(L_\mu\) straightforwardly leads (Remark 4) to
\[
L_\mu (\theta + f) = \int_{\Gamma_0} e_\lambda(f, \eta) \left( D[\eta] L_\mu \right) (\theta; \eta) \, d\lambda_\sigma(\eta)
\]
\[
= \int_{\Gamma_0} e_\lambda(f, \eta) e^{-\beta E(\eta)} L_\mu ((1 + \theta) (e^{-\beta W(\eta, \cdot)} - 1) + \theta) \, d\lambda_\sigma(\eta),
\]
where the second equality is a consequence of Corollary 17.

Proposition 18 leads to a uniqueness result for Gibbs measures corresponding to positive potentials. As a first step towards this purpose, we must introduce additional spaces of functionals. More precisely, for each \(\alpha > 0\), let \(\text{Ent}_\alpha(L^1(\sigma))\) be the space of all entire functionals \(L\) on \(L^1(\sigma)\) such that
\[
\|L\|_\alpha := \sup_{\theta \in L^1(\sigma)} \left( |L(\theta)| e^{-\alpha \|\theta\|_{L^1(\sigma)}} \right) < \infty.
\]
It is clear that \(\|\cdot\|_\alpha\) defines a norm on \(\text{Ent}_\alpha(L^1(\sigma))\).

**Proposition 20** With respect to the norm \(\|\cdot\|_\alpha\), \(\text{Ent}_\alpha(L^1(\sigma))\) has the structure of a Banach space.
Proof. Fixing an $\alpha > 0$, let $(L_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\text{Ent}_\alpha(L^1(\sigma))$, i.e., $(L_n e^{-\alpha \|\cdot\|_{L^1(\sigma)}})_{n \in \mathbb{N}}$ is a Cauchy sequence in the Banach space consisting of all complex-valued bounded functions defined on $L^1(\sigma)$ endowed with the supremum norm. By completeness, there is a complex-valued bounded function $\bar{L}$ such that
\[
\lim_{n \to \infty} \sup_{\theta \in L^1(\sigma)} \left( |L_n(\theta)e^{-\alpha \|\theta\|_{L^1(\sigma)}} - \bar{L}(\theta)| \right) = 0. \tag{20}
\]
It remains to show that the functional $L(\theta) := \bar{L}(\theta)e^{\alpha \|\theta\|_{L^1(\sigma)}}$, $\theta \in L^1(\sigma)$, is entire on $L^1(\sigma)$. This follows from the Vitali theorem (see e.g. [HP57]), since by (20) the sequence $(L_n)_{n \in \mathbb{N}}$ converges pointwisely to $L$ and, by the inequality
\[
|L_n(\theta)| \leq \sup_{\theta \in L^1(\sigma)} \left( |L_n(\theta)| e^{-\alpha \|\theta\|_{L^1(\sigma)}} \right) e^{\alpha \|\theta\|_{L^1(\sigma)}} = \|L_n\| e^{\alpha \|\theta\|_{L^1(\sigma)}} \quad n \in \mathbb{N},
\]
the sequence $(L_n)_{n \in \mathbb{N}}$ is locally uniformly bounded in $L^1(\sigma)$. $\blacksquare$

For pair potentials $\phi$ semi-bounded from below fulfilling Assumption 3, Proposition 18 has shown that any functional $L$ in $\text{Ent}_\alpha(L^1(\sigma))$ solving the initial value problem
\[
\begin{cases}
\frac{\delta L(\theta)}{\delta \theta(x)} = L \left( (1 + \theta) \left( e^{-\beta \phi(x,\cdot)} - 1 \right) + \theta \right), & \theta \in L^1(\sigma) \\
L(0) = 1
\end{cases}
\]
is a solution of the equation
\[
L(\theta) - 1 = \int_X \theta(x) \int_0^1 L \left( (1 + t\theta) \left( e^{-\beta \phi(x,\cdot)} - 1 \right) + t\theta \right) dt d\sigma(x), \quad \theta \in L^1(\sigma).
\]
In the sequel we denote by $J$ the linear mapping defined on each space $\text{Ent}_\alpha(L^1(\sigma))$, $\alpha > 0$, by
\[
(JL)(\theta) := \int_X \theta(x) \int_0^1 L \left( (1 + t\theta) \left( e^{-\beta \phi(x,\cdot)} - 1 \right) + t\theta \right) dt d\sigma(x),
\]
for $L \in \text{Ent}_\alpha(L^1(\sigma)), \theta \in L^1(\sigma)$. 24
Proposition 21 Let $\phi$ be a positive pair potential fulfilling Assumption 3. Then, for any $\alpha > 0$, the mapping $J$ defines a bounded linear operator on $\text{Ent}_\alpha(L^1(\sigma))$. Moreover, for all $L \in \text{Ent}_\alpha(L^1(\sigma))$,

$$\|JL\|_\alpha \leq \frac{e^{\alpha C(\beta)}}{\alpha} \|L\|_\alpha.$$  

Proof. Let $\alpha > 0$ be given. For all $\theta \in L^1(\sigma)$ one has

$$|(JL)(\theta)| \leq \|L\|_\alpha \int_X |\theta(x)| \int_0^1 e^{\alpha \|1+t\theta\|_{L^1(\sigma)}(e^{-\beta \phi(x,y)}-1)+t\theta\|_{L^1(\sigma)}} dt d\sigma(x)$$

and, according to the stated assumptions on $\phi$,

$$\|1+t\theta\|_{L^1(\sigma)}(e^{-\beta \phi(x,y)}-1)+t\theta\|_{L^1(\sigma)} \leq t \int_X |\theta(y)| e^{-\beta \phi(x,y)} d\sigma(y) + \int_X |e^{-\beta \phi(x,y)} - 1| d\sigma(y)$$

$$\leq t \|\theta\|_{L^1(\sigma)} + C(\beta).$$

Therefore

$$|(JL)(\theta)| \leq \|L\|_\alpha \|\theta\|_{L^1(\sigma)} e^{\alpha C(\beta)} \int_0^1 e^{\alpha \|\theta\|_{L^1(\sigma)}} dt$$

$$= \|L\|_\alpha \frac{e^{\alpha C(\beta)}}{\alpha} \left( e^{\alpha \|\theta\|_{L^1(\sigma)}} - 1 \right) < \|L\|_\alpha \frac{e^{\alpha C(\beta)}}{\alpha} e^{\alpha \|\theta\|_{L^1(\sigma)}} - 1,$$

showing the required estimate of the norms. \[\blacksquare\]

Corollary 22 Let $\beta > 0$ be given. Then, under the assumptions of Proposition 21 on each space $\text{Ent}_\alpha(L^1(\sigma))$ with

$$\frac{e^{\alpha C(\beta)}}{\alpha} < 1$$

exists a unique solution of the equation

$$L - JL = 1.$$  \hspace{1cm} (21)

In particular, for all $\beta > 0$ such that $C(\beta) < e^{-1}$, there is a unique solution of equation (21) for a suitable choice of $\alpha$ (e.g., $\alpha = (C(\beta))^{-1}$).
Proof. According to Proposition 21 one has
\[ \| J L \|_\alpha \leq \frac{e^{\alpha C(\beta)}}{\alpha} \| L \|_\alpha < \| L \|_\alpha, \quad L \in \text{Ent}_\alpha(L^1(\sigma)). \]
That is, the operator \( J \) is a contraction on \( \text{Ent}_\alpha(L^1(\sigma)) \). Thus, by the contraction mapping principle, there is a unique solution of equation (21), namely, \((1 - J)^{-1}1\), with \((1 - J)^{-1}\) defined by the von Neumann series \( \sum_{n=0}^{\infty} J^n \). The last assertion follows by minimizing the expression \( \alpha^{-1} e^{\alpha C(\beta)} \) in the parameter \( \alpha \). \( \square \)

In this way we have proved the following uniqueness result.

Theorem 23 Let \( \phi \) be a positive pair potential fulfilling the integrability condition
\[ C(\beta) = \text{ess sup}_{x \in X} \int_X |e^{-\beta \phi(x,y)} - 1| \, d\sigma(y) < \infty. \]
For each \( \beta > 0 \) such that \( C(\beta) < e^{-1} \) there is at most one Gibbs measure fulfilling Ruelle bound and corresponding to the potential \( \phi \), the intensity measure \( \sigma \), and the inverse temperature \( \beta \).

5 Stochastic dynamic equations

To deal with the differential structures used below to study a diffusion dynamics of a continuous system, this section begins by recalling a few concepts of the intrinsic geometry on configuration spaces ([AKR98a], [KK02], [Kun99]).

5.1 Differential geometry on configuration spaces

Apart from the topological structure, the bijection defined in Section 2 between the spaces \( \Gamma^{(n)}_X \) and \( \tilde{X}^n/S_n \) also induces a differentiable structure on \( \Gamma^{(n)}_X \) (see 2). More precisely, given \( n \) charts \((h_1, U_1), ..., (h_n, U_n)\) of \( X \), where \( U_1, ..., U_n \) are mutually disjoint open sets in \( X \), one constructs a chart \( h_1 \times \ldots \times h_n \) of \( \Gamma^{(n)}_X \) defined on the open set \( U_1 \times \ldots \times U_n \) in \( \Gamma^{(n)}_X \),
\[ U_1 \times \ldots \times U_n := \{ \eta = \{x_1, ..., x_n\} \in \Gamma^{(n)}_X : \exists u \in S_n \text{ s.t. } x_{i(k)} \in U_k, k = 1, ..., n \}, \]
by
\[
(h_1 \ldots h_n)(x_1, \ldots, x_n) := (h_1(x_{i(1)}), \ldots, h_n(x_{i(n)})) \in h_1(U_1) \times \ldots \times h_n(U_n).
\]
Each set, \(\Gamma_n^X\), endowed with this geometry has the structure of an \(n \cdot \dim(X)\)-dimensional \(C^\infty\)-manifold. In this way we have also defined a differentiable structure on \(\Gamma_0\). For any vector field \(v\) on \(X\) we have
\[
(\nabla^\Gamma_0 v G)(\eta) = \sum_{x \in \eta} \langle (\nabla^\Gamma_0 G)(\eta, x), v(x) \rangle_{T_x X},
\]
yielding, in particular,
\[
(\nabla^\Gamma_0 e_{\lambda}(\theta))(\eta, x) = \nabla^X \theta(x)e_{\lambda}(\theta, \eta \\backslash \{x\}), \quad \eta \in \Gamma_0, x \in \eta,
\]
where \(\nabla := \nabla^X\) being the gradient on \(X\). For the Laplace-Beltrami operator \(\Delta^\Gamma_0\) on \(\Gamma_0\), which is defined by the direct sum of the Laplace-Beltrami operators \(\Delta^\Gamma_0^X\) on \(\Gamma_0^X\), we find
\[
(\Delta^\Gamma_0 e_{\lambda}(\theta))(\eta) = \sum_{x \in \eta} \Delta^X \theta(x)e_{\lambda}(\theta, \eta \backslash \{x\}),
\]
where \(\Delta := \Delta^X\) denotes the Laplace-Beltrami operator on \(X\).

In the sequel we use the classical notation \(C^k(\Gamma_0), k \in \mathbb{N} \cup \{\infty\}\), for the space of all real-valued \(C^k\)-functions on \(\Gamma_0\), and \(C^k_0(\Gamma_0)\) for the space of all functions \(G\) in \(C^k(\Gamma_0)\) with bounded support such that for some \(\varepsilon > 0\) one has \(G(\eta) = 0\) for all \(\eta\) containing a pair \(x, y, x \neq y\), such that \(|x - y| \leq \varepsilon\).

Through the \(K\)-transform one may introduce a differential structure on \(\Gamma\) \([KK02]\), which coincides with the one introduced in \([AKR98a]\) by ”lifting” the geometrical structure on the underlying manifold \(X\). For each \(G \in C^1_0(\Gamma_0)\),
\[
(\nabla^\Gamma(KG))(\gamma, x) := \sum_{\eta \subset \gamma : |\eta| < \infty} (\nabla^\Gamma_0 G)(\eta, x), \quad \gamma \in \Gamma, x \in \gamma,
\]
and \(\Delta^\Gamma := K \Delta^\Gamma_0 K^{-1}\) on \(\mathcal{F}\mathcal{P}(C^2_0,\Gamma)\), the set of all twice differentiable cylinder polynomials \(F\) with the property that there exists a \(\varepsilon > 0\) such that \(F(\gamma) = 0\) on all \(\gamma\) which contains a pair of points in the domain of cylindricity with distance smaller than \(\varepsilon\). Equivalently, all such functions \(F\) are of the form \(F = KG, G \in C^2_0(\Gamma_0)\).
5.2 Non-equilibrium stochastic dynamics equations

The purpose of this subsection is to investigate the problem heuristically formulated in (1). Let us first fix the framework. On the space $X = \mathbb{R}^d$, $d \in \mathbb{N}$, let us consider the intensity measure $d\sigma(x) = zdm(x)$, $m$ being the Lebesgue measure on $\mathbb{R}^d$ and $z > 0$ (activity), and a measurable function $V : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ (potential) such that $V(-x) = V(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. Accordingly, we may define a translation invariant pair potential $\phi$ on $\mathbb{R}^d$ by $\phi(x, y) := V(y - x)$. Concerning $V$, we must at least assume the standard Ruelle conditions of superstability, integrability (i.e., Assumption 3), and lower regularity ([Rue70]), which are sufficient to insure the existence of corresponding Gibbs measures, cf. e.g. [Rue70, Section 5]. In particular, this includes the class of potentials $V$ which are bounded from below and integrable at infinity, and having a small enough negative part.

The problem under consideration is the construction of a solution to the system of stochastic differential equations heuristically given by

$$
\begin{cases}
\frac{dx_k(t)}{dt} = -\frac{\beta}{2} \sum_{1 \leq i \neq k} \nabla V(x_k(t) - x_i(t))dt + dW_k(t), \quad t \geq 0 \\
x_k(0) = x_k, \quad k \in \mathbb{N}
\end{cases}
$$

where $W_k$, $k \in \mathbb{N}$, is a family of independent Brownian motions. Note that due to the symmetry in the labels, any solution $(x_k)_{k \in \mathbb{N}}$ of (24) can be interpreted (modulo collapse) as a stochastic process with paths in configuration space, that is, $\gamma(t) := \{x_k(t) : k \in \mathbb{N}\}$. Informally, the generator of this dynamics is given by

$$
(HF)(\gamma) := -\frac{1}{2} \langle \Delta^{\Gamma} F(\gamma) \rangle + \frac{\beta}{2} \sum_{x \in \gamma} \sum_{y \in \gamma \setminus \{x\}} \langle \nabla_x V(x - y), \nabla^\Gamma F(\gamma, x) \rangle,
$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^d$ and $\beta$ the inverse temperature. Note that in contrast to (24), the generator $H$ is well-defined, for example, on $\mathcal{F}\mathcal{P}(C_0^2, \Gamma)$.

In the equilibrium dynamics case, the authors in [AKR98b] have constructed a solution for a wide class of potentials $V$. More precisely, for a Gibbs measure $\mu_{\text{inv}}$ corresponding to $V$, the same as used in definition (24), it has been shown that $H$ is a positive symmetric operator on the space
$L^2(\Gamma, \mu_{\text{inv}})$ associated to the Dirichlet form

$$(HF, F)_{L^2(\mu_{\text{inv}})} = \frac{1}{2} \int_\Gamma \sum_{x \in \gamma} |\nabla^\Gamma F(\gamma, x)|^2 d\mu_{\text{inv}}(\gamma).$$

This allows the use of standard Dirichlet form techniques to construct a diffusion process corresponding to $H$ having $\mu_{\text{inv}}$ as an invariant (and, moreover, reversible) measure and starting on $\mu_{\text{inv}}$-a.a. initial points. This yields, in particular, the corresponding semigroup $T_t := e^{-Ht}$, $t \geq 0$, on $L^2(\Gamma, \mu_{\text{inv}})$, a solution of the Cauchy problem

$$\begin{cases} 
\frac{d}{dt} F_t = -HF_t, & t \geq 0 \\
F_0 
\end{cases}.$$

For further references see also [AKR98b].

An essentially more difficult and interesting question is the non-equilibrium dynamics case. This means, the construction of the dynamics without reference to any invariant measure. In this case, the above scheme does not apply, and the only general result was obtained by [Fri87] for a restrictive class of potentials and $d \leq 4$.

In the sequel we describe a new scheme for the construction of the dynamics, based on the diagram in Remark 1 (Section 2). For this purpose we shall fix a probability measure $\mu$ on $\Gamma$ as an initial distribution. In contrast to the previous situation, we now assume that the measure $\mu$ is neither an invariant measure nor a perturbation of an invariant one.

The starting point for the approach is the description of the operator $H$ in terms of quasi-observables. In fact, as $H$ is well-defined, for instance, on $\mathcal{FP}(C_0^0, \Gamma)$, its image under the $K$-transform yields on the space of quasi-observables the operator $\hat{H} := K^{-1}HK$ acting on functions $G \in C_0^2(\Gamma_0)$ by

$$\left(\hat{H}G\right)(\eta) = -\frac{1}{2} \left(\Delta_{\Gamma_0} G\right)(\eta)$$

$$+ \frac{\beta}{2} \sum_{x \in \eta} \sum_{y \in \eta \setminus \{x\}} \left\{ \langle \nabla_x V(x - y), (\nabla_{\Gamma_0} G)(\eta, x) \rangle 
+ \langle \nabla_x V(x - y), (\nabla_{\Gamma_0} G)(\eta \setminus \{y\}, x) \rangle \right\}.$$  (25)
The time evolution equation is then given by the corresponding Cauchy problem

\[
\begin{aligned}
\frac{\partial}{\partial t} G_t(\eta) &= -\hat{H} G_t(\eta), \quad t \geq 0, \eta \in \Gamma_0 \\
G_0 &\in C^\infty_0(\Gamma_0) 
\end{aligned}
\]

having the advantage of being recursively solvable, because the time derivative of each \( G_t|_{\Gamma^{(n)}_{\xi_0}} \) depends only on \( G_t|_{\Gamma^{(n)}_{\xi_0}} \) and \( G_t|_{\Gamma^{(n-1)}_{\xi_0}} \). Hence, for quasi-observables, the evolution can be always constructed. However, the difficulty is to show that this solution is regular enough to allow a reconstruction of the dynamics on the level of functions on \( \Gamma \).

The previous procedure based on the diagram in Remark \[ \Box \] allows to proceed further. Actually, we may also describe the dynamics in terms of correlation functions through the dual operator \( \hat{H}^* \) of \( \hat{H} \) in the sense

\[
\int_{\Gamma_0} (\hat{H} G)(\eta) k(\eta) d\lambda_m(\eta) = \int_{\Gamma_0} G(\eta)(\hat{H}^* k)(\eta) d\lambda_m(\eta).
\]

As an aside, let us mention that in the Hamiltonian dynamics case this approach corresponds to the well-known BBGKY-hierarchy, see e.g. [Bog46]. In our case, this leads to

\[
\begin{aligned}
\frac{\partial}{\partial t} k^{(n)}_t &= -\left( \hat{H}^* k^{(n)}_t \right) \\
k^{(n)}_0, \quad n \in \mathbb{N}_0
\end{aligned}
\]

where \( k^{(n)}_0, \ n \in \mathbb{N}_0 \), are the correlation functions corresponding to the initial distribution \( \mu \). This system of equations also has hierarchical structure in which the time derivative of each \( k^{(n)}_t \) depends on \( k^{(n+1)}_t \). Namely, written
out explicitly,
\[
\frac{\partial}{\partial t} k_t^{(n)}(x_1, \ldots, x_n) = \frac{1}{2} \sum_{k=1}^{n} \triangle x_k k_t^{(n)}(x_1, \ldots, x_n)
+ \frac{\beta}{2} \sum_{k,j=1}^{n} \frac{k \neq j}{\triangle V(x_k - x_j) k_t^{(n)}(x_1, \ldots, x_n)}
+ \frac{\beta}{2} \sum_{k,j=1}^{n} \left\langle \nabla x_k V(x_k - x_j), \nabla x_k k_t^{(n)}(x_1, \ldots, x_n) \right\rangle
+ \frac{\beta}{2} \sum_{k=1}^{n} \int_{\mathbb{R}^d} \left\langle \nabla x_k V(x_k - y), \nabla x_k k_t^{(n+1)}(x_1, \ldots, x_n, y) \right\rangle dy
+ \frac{\beta}{2} \sum_{k=1}^{n} \int_{\mathbb{R}^d} \triangle V(x_k - y) k_t^{(n+1)}(x_1, \ldots, x_n, y) dy.
\]

In theoretical physics this system of equations is known as the Bogoliubov-Streltsova diffusion hierarchy (see [Str59]). We observe that the operator \( \hat{H}^* \) can be rigorously defined, for example, on correlation functions \( k \) fulfilling the bound
\[
|\Delta k(\eta)| + |\nabla k(\eta)| + k(\eta) \leq C|\eta| e^{-\alpha E(\eta)}, \quad C, \alpha \geq 0, \lambda_m - \text{a.a.} \eta \in \Gamma_0. \tag{27}
\]

Completing our way through the diagram (Remark [Π]), one can construct a dynamics on states.

The previous construction implies, in particular, that the dynamics can be also expressed in terms of Bogoliubov functionals
\[
L_t(\theta) := \int_{\Gamma_0} e_{\lambda}(\theta, \eta) \rho_t(d\eta), \quad t \geq 0.
\]

This leads to the following result.

**Theorem 24** Under the above conditions one has
\[
\frac{\partial}{\partial t} L_t(\theta) = \frac{1}{2} \int_{\mathbb{R}^d} \triangle \theta(x) \frac{\delta L_t(\theta)}{\delta \theta(x)} dx
- \frac{\beta}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \nabla x V(x - y), \nabla \theta(x)(\theta(y) + 1) \rangle \delta^2 L_t(\theta) dx dy
- \nabla \theta(y)(\theta(x) + 1) \frac{\delta^2 L_t(\theta)}{\delta \theta(x) \delta \theta(y)} dx dy. \tag{28}
\]
for all $\theta \in C^2_0(\mathbb{R}^d):=\text{the space of all } C^2\text{-functions on } \mathbb{R}^d \text{ with compact support.}

**Remark 25** According to Theorem 24, the correlation functions $k_t$ corresponding to a solution $L_t$ for the diffusion hierarchical equation (28) fulfill the generalized Ruelle bound [KKK04].

**Proof.** According to (26),

$$
\frac{\partial}{\partial t} L_t(\theta) = \int_{\Gamma_0} e_\lambda(\theta, \eta) \left( \frac{d}{dt} \rho_t \right) (d\eta)
$$

$$
= -\int_{\Gamma_0} e_\lambda(\theta, \eta) \left( \mathcal{H}^* \rho_t \right) (d\eta)
$$

$$
= -\int_{\Gamma_0} \left( \mathcal{H}e_\lambda(\theta) \right)(\eta) \rho_t (d\eta)
$$

with

$$
\left( \mathcal{H}e_\lambda(\theta) \right)(\eta) = -\frac{1}{2} \left( \Delta^{\Gamma_0} e_\lambda(\theta) \right)(\eta)
$$

$$
+ \frac{\beta}{2} \sum_{x \in \eta} \sum_{y \in \eta \setminus \{x\}} \left\{ \langle \nabla_x V(x-y), \nabla^{\Gamma_0} e_\lambda(\theta) \rangle(\eta,x) \right. 
$$

$$
+ \left. \langle \nabla_x V(x-y), \nabla^{\Gamma_0} e_\lambda(\theta) \rangle(\eta \setminus \{y\}, x) \right\},
$$
Therefore, equalities (22) and (23) yield
\[
\frac{\partial}{\partial t} L_t(\theta) = \frac{1}{2} \int_{\Gamma_0} \sum_{x \in \eta} \Delta \theta(x) e_{\lambda}(\theta, \eta \backslash \{x\}) \rho_t(d\eta) \\
- \frac{\beta}{2} \int_{\Gamma_0} \sum_{x \in \eta} \sum_{y \in \eta \backslash \{x\}} \langle \nabla_x V(x - y), \nabla \theta(x) \rangle e_{\lambda}(\theta, \eta \backslash \{x\}) \rho_t(d\eta) \\
- \frac{\beta}{2} \int_{\Gamma_0} \sum_{x \in \eta} \sum_{y \in \eta \backslash \{x\}} \langle \nabla_x V(x - y), \nabla \theta(x) \rangle e_{\lambda}(\theta, \eta \backslash \{x, y\}) \rho_t(d\eta)
\]
\[
= \frac{1}{2} \int_{\Gamma_0} \sum_{x \in \eta} \Delta \theta(x) e_{\lambda}(\theta, \eta \backslash \{x\}) \rho_t(d\eta) \\
- \frac{\beta}{2} \int_{\Gamma_0} \sum_{\{x, y\} \subset \eta} \langle \nabla_x V(x - y), \nabla \theta(x) \rangle e_{\lambda}(\theta, \eta \backslash \{x, y\}) \rho_t(d\eta)
\]
and the proof follows by Corollary 13.

As a straightforward consequence, one may easily derive the time evolution equation of the Laplace transform corresponding to the measures $\mu_t$,
\[
L_t(\varphi) := \int_{\Gamma} \exp \langle \gamma, \varphi \rangle \mu_t(d\gamma) = L_t(e^{\varphi} - 1).
\]
In hydrodynamics this equation is related to the Hopf equation.

**Corollary 26** Under the conditions of Theorem 24 for all $\varphi \in C_0^2(\mathbb{R}^d)$ we have
\[
\frac{\partial}{\partial t} L_t(\varphi) = \frac{1}{2} \int_{\mathbb{R}^d} \left( \Delta \varphi(x) + |\nabla \varphi(x)|^2 \right) \frac{\delta L_t(\varphi)}{\delta \varphi(x)} dx \\
- \frac{\beta}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \nabla_x V(x - y), \nabla \varphi(x) - \nabla \varphi(y) \rangle \\
\frac{\delta^2 L_t(\varphi)}{\delta \varphi(x) \delta \varphi(y)} dxdy.
\]
Proof. In Theorem 24 consider the case \( \theta = e^\varphi - 1 \) with \( \varphi \in C_0^2(\mathbb{R}^d) \). This gives

\[
\frac{\partial}{\partial t} L_t(\varphi) = \frac{\partial}{\partial t} L_t(\theta) = \frac{1}{2} \int_{\mathbb{R}^d} \left( \Delta \varphi(x) + |\nabla \varphi(x)|^2 \right) e^{\varphi(x)} \frac{\delta L_t(\theta)}{\delta \theta(x)} dx
\]

\[
- \frac{\beta}{4} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \nabla_x V(x-y), \nabla \varphi(x) - \nabla \varphi(y) \rangle 
\]

\[
\cdot e^{\varphi(x)+\varphi(y)} \frac{\delta^2 L_t(\theta)}{\delta \theta(x) \delta \theta(y)} dxdy,
\]

and the proof follows because

\[
\frac{\delta L_t(\varphi)}{\delta \varphi(x)} = \frac{\delta L_t(\theta)}{\delta \theta(x)} \frac{\delta (e^\varphi - 1)(\varphi)}{\delta \varphi(x)} = \frac{\delta L_t(\theta)}{\delta \theta(x)} e^{\varphi(x)}, \quad m-a.a. \ x
\]

and

\[
\frac{\delta^2 L_t(\varphi)}{\delta \varphi(x) \delta \varphi(y)} = \frac{\delta}{\delta \varphi(x)} \left( \frac{\delta L_t(\varphi)}{\delta \varphi(y)} \right) = e^{\varphi(y)} \frac{\delta}{\delta \varphi(x)} \left( \frac{\delta L_t(\theta)}{\delta \theta(y)} \right)
\]

\[
= e^{\varphi(y)+\varphi(x)} \frac{\delta^2 L_t(\theta)}{\delta \theta(x) \delta \theta(y)}, \quad m-a.a. \ x, y.
\]

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References

[AKR98a] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154(2):444–500, 1998.

[AKR98b] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces: The Gibbsian case. *J. Funct. Anal.*, 157:242–291, 1998.

[Bar85] J. A. Barroso. *Introduction to Holomorphy*, volume 106 of *Mathematics Studies*. North-Holland Publ. Co., Amsterdam, 1985.

[Bog46] N. N. Bogoliubov. *Problems of a Dynamical Theory in Statistical Physics*. Gostekhisdat, Moscow, 1946. (in Russian). English translation in J. de Boer and G. E. Uhlenbeck (editors), *Studies in Statistical Mechanics*, volume 1, pages 1–118. North-Holland, Amsterdam, 1962.

[Din81] S. Dineen. *Complex Analysis in Locally Convex Spaces*, volume 57 of *Mathematics Studies*. North-Holland Publ. Co., Amsterdam, 1981.

[DSI75] M. Duneau, B. Souillard, and D. Iagolnitzer. Decay of correlations for infinite-range interactions. *J. Math. Phys.*, 16(8):1662–1666, 1975.

[DU77] J. Diestel and J. J. Uhl. *Vector Measures*, volume 15 of *Mathematical Surveys*. Amer. Math. Soc., Providence, Rhode Island, 1977.

[DVJ88] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Springer Verlag, NewYork, Berlin, and Heidelberg, 1988.

[FF91] K.-H. Fichtner and W. Freudenberg. Characterization of states of infinite Boson systems I. On the construction of states of Boson systems. *Comm. Math. Phys.*, 137:315–357, 1991.

[Fri87] J. Fritz. Gradient dynamics of infinite point systems. *Ann. Probab.*, 15(2):478–514, 1987.
[HP57] E. Hille and R. S. Phillips. *Functional Analysis and Semi-groups*, volume 31 of *Amer. Math. Soc. Colloq. Publ*. American Mathematical Society, 1957.

[KK02] Yu. G. Kondratiev and T. Kuna. Harmonic analysis on configuration space I. General theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(2):201–233, 2002.

[KK03] Yu. G. Kondratiev and T. Kuna. Correlation functionals for Gibbs measures and Ruelle bounds. *Methods Funct. Anal. Topology*, 9(1):9–58, 2003.

[KK05] Yu. G. Kondratiev and T. Kuna. Harmonic analysis on configuration space II. Bogoliubov functional and equilibrium states. In preparation, 2005.

[KKK04] Yu. G. Kondratiev, T. Kuna, and O. Kutoviy. On relations between a priori bounds for measures on configuration spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 7(2):195–213, 2004.

[KKO02] Yu. G. Kondratiev, T. Kuna, and M. J. Oliveira. Analytic aspects of Poissonian white noise analysis. *Methods Funct. Anal. Topology*, 8(4):15–48, 2002.

[KKO04] Yu. G. Kondratiev, T. Kuna, and M. J. Oliveira. On the relations between Poissonian white noise analysis and harmonic analysis on configuration spaces. *J. Funct. Anal.*, 213(1):1–30, 2004.

[Kno64] K. Knopp. *Theorie und Anwendung der Unendlichen Reihen*. Springer Verlag, Berlin, Heidelberg, and New York, 5th edition, 1964.

[KRR04] Yu. G. Kondratiev, A. L. Rebenko, and M. Röckner. On diffusion dynamics for continuous systems with singular superstable interaction. *J. Math. Phys.*, 45(5):1826–1848, 2004.

[Kun99] T. Kuna. *Studies in Configuration Space Analysis and Applications*. PhD thesis, Bonner Mathematische Schriften Nr. 324, University of Bonn, 1999.
[Kun05] T. Kuna. Bochner’s theorem for Bogoliubov functionals. In preparation, 2005.

[Mec67] J. Mecke. Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen. Z. Wahrsch. verw. Gebiete, 9:36–58, 1967.

[Naz85] G. I. Nazin. Method of the generating functional. J. Sov. Math., 31:2859–2886, 1985.

[NZ79] X. X. Nguyen and H. Zessin. Integral and differential characterizations of the Gibbs process. Math. Nachr., 88:105–115, 1979.

[Osa96] H. Osada. Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions. Comm. Math. Phys., 176(1):117–131, 1996.

[Par67] K. R. Parthasarathy. Probability Measures on Metric Spaces. Probability and Mathematical Statistics. Academic Press, New York and London, 1967.

[PZ99] E. Pechersky and Yu. Zhukov. Uniqueness of Gibbs state for nonideal gas in $\mathbb{R}^d$: The case of pair potentials. J. Statist. Phys., 97:145–172, 1999.

[Rue63] D. Ruelle. Correlation functions of classical gases. Ann. Phys., 25:109–120, 1963.

[Rue69] D. Ruelle. Statistical Mechanics. Rigorous Results. Benjamin, New York and Amsterdam, 1969.

[Rue70] D. Ruelle. Superstable interactions in classical statistical mechanics. Comm. Math. Phys., 18:127–159, 1970.

[Sch71] H. H. Schaefer. Topological Vector Spaces. Springer Verlag, Berlin, Heidelberg, and New York, 1971.

[Str59] E. A. Streltsova. Non-stationary processes in electrolyte theory. Ukrainian Math. J., 11:83–92, 1959.

[Tre67] F. Treves. Topological Vector Spaces, Distributions and Kernels. Academic Press, New York and London, 1967.
[Yos96] M. W. Yoshida. Construction of infinite-dimensional interacting diffusion processes through Dirichlet forms. *Probab. Theory Relat. Fields*, 106:265–297, 1996.