The Correlations of Greenberger-Horne-Zeilinger States Described by Hilbert-Schmidt Decomposition

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Using simple quantum analysis we describe the correlations of Greenberger-Horne-Zeilinger (GHZ) states by the use of Hilbert-Schmidt (HS) representation. Our conclusion is that while these states disprove local-realism they do not prove any nonlocality property.

Key words: Quantum locality, Hilbert-Schmidt decomposition, GHZ states
1. INTRODUCTION

Greenberger-Horne-Zeilinger (GHZ) states have attracted much attention in recent literature [1-11] as experiments which are done with these states disprove local-realism, without the use of Bell’s inequalities. In many articles it was claimed that GHZ states violate locality. We would like to quote here some examples: In Ref. 3 “GHZ states have been fascinating quantum systems to reveal the nonlocality of the quantum world” (our emphasis). In Ref. 7 (in the abstract): “A scheme is proposed for generating maximally entangled GHZ atomic states for testing quantum nonlocality” (our emphasis). In Ref. 8 (in the abstract): “enable various novel tests of quantum nonlocality” (our emphasis). In Ref. 10, in Caption to Figure 1: “GHZ tests of quantum nonlocality”. In Ref. 11 (in the abstract): “We propose an experimentally feasible scheme to demonstrate quantum nonlocality” (our emphasis). In Ref. 12 (in the abstract) “it is possible to demonstrate nonlocality for two particles without using inequalities”. These expressions and many more which can be found in the usual literature lead to the impression that the quantum world is nonlocal. In the present Letter we would like to analyze the correlations obtained for GHZ states and show that “locality” is not violated by these correlations. We adopt here the definition given in Ref. 12: “The assumption of locality is that the choice of measurement on one side cannot influence the outcome of any measurement on the other side.” It is obvious that once observer “a” does anything to his part of the
system, measurement on the “a” system will give a new result, and the correlations between a and the rest of the world will change. This does not contradict locality, and is obviously true also classically. However, locality implies that no matter what is done in the “a” system, it should not affect the measurements on b and c and also not affect the correlations between b and c. By following this definition we show in the following pure quantum mechanical (QM) analysis that experiments with GHZ states do not violate locality. Although we treat here a specific system, by following the present approach a similar QM analysis can be done also for other entangled systems.

2. ANALYSIS

Greenberger, Horne, Shimony and Zeilinger [1] have suggested a gedanken three-particle interferometer which has been described [1] as follows: The source emits a triple of particles 1, 2, and 3, in six beams, with the state given by

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[ |a\rangle_1 |b\rangle_2 |c\rangle_3 + |a'\rangle_1 |b'\rangle_2 |c'\rangle_3 \right]. \quad (1)$$

The three particles 1, 2, and 3 emerge either through a, b, and c apertures or through a’, b’, and c’, respectively. A phase shift $\phi_1$ is imparted to beam a’ of particle 1, and beams a and a’ are brought together on a beam-splitter before illuminating detectors d and d’. Likewise for particles 2 and 3, with their respective apertures, phase shifts and detectors. The evolution of the kets $|a\rangle_1$ and $|a'\rangle_1$ is given by

$$|a\rangle_1 \rightarrow \frac{1}{\sqrt{2}} \left[ |d\rangle_1 + i|d'\rangle_1 \right] \quad (2 - a)$$
and

\[ |a'\rangle_1 \rightarrow \left( \frac{1}{\sqrt{2}} \right) e^{i\phi_1} \left[ |d'\rangle_1 + i|d\rangle_1 \right], \quad (2 - b) \]

where the ket \(|d\rangle_1\) denotes particle 1 directed toward detector \(d_1\), etc. The particle 2 beams and the particle 3 beams are subjected to similar treatment and hence undergo similar evolutions. A state with eight terms develops from which we obtain amplitudes and hence probabilities of detection of the three particles by the triple detectors \((d, e, f)\), the triple of detectors \((d', e, f)\), etc. (An analysis of possible experiments which can be done on this system which refute local-realism is described in Ref. 1).

The quantum state given by Eq. (1) can be considered formally as a three spin-\(\frac{1}{2}\) system (denoted by \(a, b,\) and \(c\)). \(|a\rangle\) and \(|a'\rangle\) may be represented by the two levels \(\left( \begin{array}{c} 1 \\ 0 \end{array} \right)\) and \(\left( \begin{array}{c} 0 \\ 1 \end{array} \right)\) of the first spin-\(\frac{1}{2}\) system, \(|b\rangle\) and \(|b'\rangle\) are the two levels of the second spin-\(\frac{1}{2}\) system, etc.

In this representation the density matrix \(|\psi\rangle\langle\psi|\) corresponding to the state \(|\psi\rangle\) of Eq. (1) is given by:

\[
\rho = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}.
\]

(3)

While this form of the density matrix seems quite simple, the locality of GHZ states is demonstrated in a better way by using the HS representation [13,14] of this density matrix.
For an entangled state of three two-level particles (denoted by \(a, b, c\)), the HS decomposition becomes [14]:

\[
8\rho = (I)_a \otimes (I)_b \otimes (I)_c + (\vec{r} \cdot \vec{\sigma})_a \otimes (I)_b \otimes (I)_c + (I)_a \otimes (\vec{s} \cdot \vec{\sigma})_b \otimes (I)_c \\
+ (I)_a \otimes (I)_b \otimes (\vec{p} \cdot \vec{\sigma})_c + \sum_{mn} t_{mn} (I)_a \otimes (\sigma_m)_b \otimes (\sigma_n)_c \\
+ \sum_{k\ell} o_{k\ell} (\sigma_k)_a \otimes (I)_b \otimes (\sigma_\ell)_c + \sum_{ij} p_{ij} (\sigma_i)_a \otimes (\sigma_j)_b \otimes (I)_c \\
+ \sum_{\alpha,\beta,\gamma} R_{\alpha\beta\gamma} (\sigma_\alpha)_a \otimes (\sigma_\beta)_b \otimes (\sigma_\gamma)_c .
\]

Here \(I\) stands for the unit operator, \(\vec{r}, \vec{s}\), and \(\vec{p}\) belong to \(\mathbb{R}^3\), \(\sigma_n (n = 1, 2, 3)\) are the standard Pauli matrices. The coefficients \(t_{mn}, o_{k\ell}\), and \(p_{ij}\) form real \(3 \times 3\) matrices.

The coefficients \(R_{\alpha\beta\gamma}\) form a real \(3 \times 3 \times 3\) tensor, related to the density matrix \(\rho\) by

\[
R_{\alpha\beta\gamma} = \left(\frac{1}{8}\right) Tr \left[\rho (\sigma_\gamma)_c \otimes (\sigma_\beta)_b \otimes (\sigma_\alpha)_a\right].
\]

The coefficients \(t_{mn}\) are related to the density matrix \(\rho\) by

\[
t_{mn} = \left(\frac{1}{8}\right) Tr \left[\rho (\sigma_n)_c \otimes (\sigma_m)_b \otimes (I)_a\right],
\]

and similar relations hold between other coefficients and the density matrix. In deriving such relations we use the simple relation \(Tr[\sigma_i \sigma_j] = 2\delta_{ij}\). We find that the general entangled state of three two-level systems is described by 63 parameters: 9 for \(\vec{r}, \vec{s}\) and \(\vec{p}\), 27 for \(t_{mn}, o_{k\ell}\), and \(p_{ij}\) and 27 for \(R_{\alpha\beta\gamma}\). The parameters \(\vec{r}, \vec{s}\) and \(\vec{p}\) can be obtained from measurements on one arm of the measurement device, \(t_{mn}, o_{k\ell}\), and \(p_{ij}\) can be obtained from measurements on the corresponding two arms of the measurement device [e.g., by using Eq. (6)] and \(R_{\alpha\beta\gamma}\) can be obtained from measurements on the three arms of the measuring device, [e.g., by using Eq. (5)]. Local realism has been
refuted by applying different sets of measurements in the different arms of the measurement device [1]. Although the representation (4) assumes axes of measurements \( x, y, z \) of a certain basis \( F \) corresponding to \( \sigma_1, \sigma_2, \sigma_3 \), changes of axes of measurement can be obtained by rotation from the basis \( F \) to another \( F' \) [13-14]:

\[
(\vec{\sigma}^{F'})_a = O_1(\vec{\sigma}^{F})_a; \quad (\vec{\sigma}^{F'})_b = O_2(\vec{\sigma}^{F})_b; \quad (\vec{\sigma}^{F'})_c = O_3(\vec{\sigma}^{F})_c.
\] (7)

The 63 parameters defining the density matrix can be obtained by measurements in the \( x, y \) and \( z \) directions. If the measurements are done along different axes (e.g., \( x', y', z' \)) one should transform these parameters accordingly. The essential point here is that the rotation of axes in system \( a \) can be done independently of the rotation of axes in system \( b \) or \( c \) so that in addition to the 63 parameters which have been fixed by local interaction in the past, each observer can rotate individually his axis of measurement. The correlations are, of course, changed but only due to local operations.

The HS representation for the density matrix (3) corresponding to the wavefunction \( |\psi\rangle \) of Eq. (1) has been already evaluated in our previous article [14]. The parameters for this state are given by

\[
R_{122} = R_{212} = R_{221} = -1; \quad t_{33} = o_{33} = p_{33} = R_{111} = 1 ,
\] (8)

and all other parameters are equal to zero.

Experimentally the states \( |d\rangle_1 \) and \( |d'\rangle_1 \) of Eqs. (2) are obtained from the states \( |a\rangle_1 \) and \( |a'\rangle_1 \) by the use of 50-50 beam-splitter transformation. This
transformation can be described by the following unitary transformation:

\[
\begin{pmatrix}
  d \\
d'
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
  1 & -ie^{-i\phi_1} \\
  -i & e^{-i\phi_1}
\end{pmatrix} \begin{pmatrix}
  a \\
a'
\end{pmatrix} = U_1 \begin{pmatrix}
  a \\
a'
\end{pmatrix}
\]

(9)

The operation of the beam-splitter on side \(a\) leads to a change in the HS representation of Eq. 4 by changing \((\sigma_i)_a\) \((i = 1, 2, 3)\) everywhere in this equation, into \(U_1(\sigma_i)_aU_1^\dagger\), where \(U_1\) is given by Eq. (9). Here again the beam-splitter transformation on side \(a\) does not affect other sides of the system as given by the HS representation. The HS decomposition shows this very simply. The correlations between system \(a\) and systems \(b\) and \(c\) are changed by changing \((\sigma_k)_a\), \((\sigma_i)_a\) and \((\sigma_\alpha)_a\) in the 6’th, 7’th and 8’th terms of Eq. (4), respectively. The correlations between \(b\) and \(c\) given by the 5’th term of Eq. (4) and the coefficients \(t_{mn}\) are unaffected by the beam-splitter or any other interaction employed by “\(a\)”. In a similar way, one can see the local effects of the beam-splitters in other sides of the GHZ system. The correlations are, of course, changed due to the beam-splitter transformation but only due to local operations. When these occur in subsystem “\(a\)”, measurements of “\(a\)” and correlations with “\(a\)” are affected, of course. However, all other correlations are not affected, which is the essence of locality.

In conclusion, locality, as defined in the present letter, is not violated by QM. EPR correlations are fixed by the local interaction which occurred in the past. Local-realism is, however, refuted.
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