Predictive Information Rate in Discrete-time Gaussian Processes

Samer A. Abdallah and Mark D. Plumbley
Queen Mary University of London

We derive expressions for the predictive information rate (PIR) for the class of autoregressive Gaussian processes AR(N), both in terms of the prediction coefficients and in terms of the power spectral density. The latter result suggests a duality between the PIR and the multi-information rate for processes with mutually inverse power spectra (i.e. with poles and zeros of the transfer function exchanged). We investigate the behaviour of the PIR in relation to the multi-information rate for some simple examples, which suggest, somewhat counter-intuitively, that the PIR is maximised for very ‘smooth’ AR processes whose power spectra have multiple poles at zero frequency. We also obtain results for moving average Gaussian processes which are consistent with the duality conjectured earlier. One consequence of this is that the PIR is unbounded for MA(N) processes.

I. INTRODUCTION

The predictive information rate (PIR) of a bi-infinite sequence of random variables \(\ldots, X_{-1}, X_0, X_1, \ldots\), was defined by Abdallah and Plumbley \cite{Abdallah2014} in the context of information dynamics, which is concerned with the application of information theoretic methods \cite{Samo2011, Samo2012} to the process of sequentially observing a random sequence while maintaining a probabilistic description of the expected future evolution of the sequence. An observer in this situation can maintain an estimate of its uncertainty about future observations (by computing various entropies) and can also estimate the information in each observation about the as-yet unobserved future given the all the observations so far; this is the instantaneous predictive information or IPI. For stationary processes, the ensemble average of the IPI is the PIR. It is a measure of temporal structure that characterises the process as a whole, rather than on a moment-by-moment basis or for particular realisations of the process, in the same way that the entropy rate characterises its overall randomness. Abdallah and Plumbley \cite{Abdallah2014} examined several process information measures and their interrelationships. Following the conventions established there, we let \(\bar{X}_t = (\ldots, X_{t-2}, X_{t-1})\) denote the variables before time \(t\), and \(\hat{X}_t = (X_{t+1}, X_{t+2}, \ldots)\) denote those after \(t\). For a process with a shift-invariant probability measure \(\mu\), the predictive information rate \(b_\mu\) is defined as a conditional mutual information

\[
b_\mu = I(X_t; \bar{X}_t|\hat{X}_t) = H(\bar{X}_t|\hat{X}_t) - H(\bar{X}_t|\bar{X}_t, \hat{X}_t). \tag{1}
\]

Equation (1) says that the PIR is the average reduction in uncertainty about the future on learning \(X_t\), given the past. In similar terms, three other process information measures can be defined: the entropy rate \(h_\mu\), the multi-information rate \(\rho_\mu\), and the erasure or residual entropy rate \(r_\mu\), as follows:

\[
h_\mu = H(\bar{X}_t|\bar{X}_t), \tag{2}
\]

\[
\rho_\mu = I(X_t; \bar{X}_t) = H(X_t) - H(X_t|\bar{X}_t), \tag{3}
\]

\[
r_\mu = H(X_t|\hat{X}_t, \bar{X}_t). \tag{4}
\]

These measures are illustrated in an information diagram, or I-diagram \cite{Samo2011}, in fig. 1 which shows how they partition the marginal entropy \(H(X_t)\), the uncertainty about a single observation in isolation; this partitioning is discussed in depth by James et al. \cite{Samo2012}.

II. GAUSS-MARKOV PROCESSES

A Gauss-Markov, or autoregressive Gaussian process of order \(N\) is a real-valued random process \((X_t)_{t \in \mathbb{Z}}\) on the domain of integers such that

\[
X_t = U_t + \sum_{k=1}^{N} \psi_k X_{t-k}, \tag{5}
\]

where the innovations \(U_t\) form a sequence of independent and identically distributed Gaussian random variables with zero mean and variance \(\sigma^2\), and the \(\psi_k\) are the autoregressive or prediction coefficients. Thus, a realisation of the random process \(X\) is the result of applying an order-\(N\) infinite impulse response (IIR) filter to a realisation of the innovation sequence formed by the \(U_t\). The class of such processes is known as AR(\(N\)). If the autoregressive coefficients \(\psi_k\) are such that the filter is stable, the process will be stationary and thus may have well defined entropy and predictive information rates. We will assume that this is the case.

![I-diagram representing several information measures for stationary random processes. Each circle or oval represents a random variable or sequence of random variables relative to time \(t = 0\). The circles represent the 'present'. The total area is \(H(X_0) = \rho_\mu + b_\mu + r_\mu\), where \(\rho_\mu\) is the multi-information rate, \(r_\mu\) is the residual entropy rate, and \(b_\mu\) is the predictive information rate. The entropy rate is \(h_\mu = r_\mu + b_\mu\).](image-url)
A. Entropy rate

From the defining equation (5), we can immediately see that $H(X_t|X_t) = H(X_t|X_{t-N}, \ldots, X_{t-1}) = H(U_t)$, which depends only on $\sigma^2$, so

$$h_\mu = \frac{1}{2} \log 2\pi e \sigma^2.$$  

(6)

It is known that the entropy rate of a stationary Gaussian process can also be expressed in terms of its power spectral density (PSD) function $S: \mathbb{R} \rightarrow \mathbb{R}$, which is defined as the discrete-time Fourier transform of the autocorrelation sequence $\gamma_k = E(X_t X_{t-k})$:

$$S(\omega) = \sum_{k=-\infty}^{\infty} \gamma_k e^{-i\omega k}, \quad \gamma_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega)e^{i\omega k} \, d\omega.$$  

(7)

Using methods of teoplitz matrix analysis [17], the entropy rate can be shown, with suitable restrictions on the autocorrelation sequence, to be

$$h_\mu = \frac{1}{2} \left( \log 2\pi e + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S(\omega) \, d\omega \right),$$  

(8)

which is also known as the Kolmogorov-Sinai entropy for this process. Incidentally, this means that the variance of the innovations $\sigma^2$ can be expressed as

$$\sigma^2 = \exp \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S(\omega) \, d\omega \right).$$  

(9)

B. Multi-information rate

The multi-information rate of a stationary Gaussian process was found by Dubnov [8] to be expressible as

$$\rho_\mu = \frac{1}{2} \left( \log 2\pi e + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S(\omega) \, d\omega \right).$$  

(10)

We can see how this was obtained by noting that the marginal variance $E X_t^2 = \gamma_0$ can be computed from the spectral density function simply by setting $k = 0$ in the inverse Fourier transform [7], yielding

$$H(X_t) = \frac{1}{2} \left( \log 2\pi e + \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) \, d\omega \right).$$  

(11)

Since $\rho_\mu = H(X_t) - h_\mu$ and $h_\mu$ is given by the Kolmogorov-Sinai entropy of the Gaussian process [8], Dubnov’s expression (10) follows directly.

C. Predictive information rate

To derive an expression for the predictive information rate $b_\mu$, we first note that the model is $N$th-order Markov, since the trailing segment of the process $(X_{N+1}, X_{N+2}, \ldots)$ is conditionally independent of the leading segment $(\ldots, X_1, X_0)$ given intervening segment $(X_1, \ldots, X_N)$. Writing $X_{p:j}$ for the finite segment $(X_p, X_{p+1}, \ldots, X_j)$, it can be shown that

$$I(X_0; X_j|X_0) = I(X_0; X_{1:N}|X_{N-1}).$$  

(12)

Thus, to find the PIR, we need only consider the the $2N + 1$ consecutive variables around $X_0$, namely $X_{-N:N}$. Expanding the conditional mutual information in terms of entropies, we obtain

$$b_\mu = H(X_{1:N}|X_{-N-1}) - H(X_{1:N}|X_{-N}) - Nh_\mu.$$  

(13)

Since the segment $X_{-N}$ contains more than $N$ elements, the second term is just $N$ times the entropy rate, so

$$b_\mu = H(X_{1:N}|X_{-N-1}) - N h_\mu.$$  

(14)

To evaluate $H(X_{1:N}|X_{-N-1})$, we note that, for continuous random variables $Y$ and $Z$,

$$H(Y|Z) = \int H(Y|Z=z)p(z) \, dz,$$  

(15)

where $p(z)$ is $Z$’s probability density at $z$; that is, we find the entropy of $Y$ given particular values of $Z$, and then average over the possible values of $Z$. If we find that $H(Y|Z=z)$ is the same value independent of $z$, then $H(Y|Z)$ is trivially that value. This is indeed what we will find when we apply this approach here, and so we will examine the case where the variables $(X_{-N}, \ldots, X_{-1})$ have been observed with the values $(x_{-N}, \ldots, x_{-1})$ respectively. Under these conditions, we can, in effect, forget that $X_{-N-1}$ are random variables and investigate the joint distribution of $X_0:N$ implicitly conditioned on the observation $X_{-N-1}=x_{-N-1}$. Referring back to (5), we may rewrite the recursive relation between random variables $X_j$ for $1 \leq j \leq N$ given observations $x_{-N-1}$ as

$$X_j = U_j + \left( \sum_{i=1}^{j-1} \psi_i X_{j-i} \right) + \psi_j x_0 + \left( \sum_{i=j+1}^{N} \psi_i x_{j-i} \right),$$  

(16)

with a special case for $X_0$:

$$X_0 = U_0 + \left( \sum_{i=1}^{N} \psi_i x_{-i} \right).$$  

(17)

With an eye to the final sums in both the above equations, let us also define $m_j$ for $0 \leq j \leq N$ as

$$m_j = \sum_{i=j+1}^{N} \psi_i x_{j-i}.$$  

(18)
Consider now the following transformation of variables:

\[ Y_j = X_j - \left( \sum_{i=1}^{j-1} \psi_i X_{j-i} \right) - m_j. \]  

(19)

Putting \( X \equiv (X_1, \ldots, X_N)^T \), \( Y \equiv (Y_1, \ldots, Y_N)^T \) and \( m \equiv (m_1, \ldots, m_N)^T \), this can be written as a vector equation,

\[ Y = AX - m, \]  

(20)

where the matrix \( A \) is lower triangular with ones along the main diagonal:

\[ A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -\psi_1 & 1 & 0 & \ldots & 0 \\ -\psi_2 & -\psi_1 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\psi_{N-1} & -\psi_{N-2} & -\psi_{N-3} & \ldots & 1 \end{pmatrix} \]  

(21)

Equation (20) implies that \( H(Y) = H(X) + \log |A| \), but since \( A \) has the above form, \(|A| = 1\) and so \( H(X) = H(Y) \). Substituting (16) and (18) into (14) to obtain

\[ Y_j = U_j + \psi_j X_0. \]  

(22)

Expanding \( X_0 \) using (17) and writing in vector form,

\[ Y = U - U_0 a - m_0 a, \]  

(23)

where \( U \) is a spherical Gaussian random vector and \( a \) is a constant vector with components \( a_i = -\psi_i \). The determinant of the covariance matrix of \( Y \) can be found by exploiting the spherical symmetry of \( U \) and rotating into a frame of reference in which \( a \) is aligned with first coordinate axis and therefore has the components \( (\|a\|, 0, \ldots) \). In this frame of reference, the covariance matrix of \( Y \) is

\[ \sigma^2 \begin{pmatrix} 1 + \|a\|^2 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}, \]

and therefore its determinant is just \( \sigma^{2N}(1 + \|a\|^2) \). This gives us the entropy of \( Y \) and hence of \( X \):

\[ H(X) = H(Y) = \frac{1}{2} \log(2\pi e \sigma^2)^N (1 + \|a\|^2). \]  

(24)

By construction, \( H(X) = H(X_{1:N}|X_{-N-1} = x_{-N-1}) \); that is, the entropy of \( X_{1:N} \) conditioned on the particular observed values \( x_{-N-1} \), but since (24) is independent of those values, we may conclude that \( H(X_{1:N}|X_{-N-1}) = H(X) \) and substitute (24) and (6) into (14) to obtain

\[ b_\mu = H(X_{1:N}|X_{-N-1}) - Nh_\mu = \frac{1}{2} \log(2\pi e \sigma^2)^N (1 + \|a\|^2) - \frac{1}{2} N \log 2\pi e \sigma^2. \]  

(25)

Simplified and expressed in terms of the original filter coefficients \( \psi_k \),

\[ b_\mu = \frac{1}{2} \log \left( 1 + \sum_{k=1}^{\infty} \psi_k^2 \right). \]  

(26)

Let us now consider the relationship between the PIR and power spectral density. For an autoregressive process, the PSD can be computed directly from the prediction coefficients via the filter transfer function, which is the \( z \)-transform of the filter impulse response. If we set \( a_0 = 1 \) and \( a_k = -\psi_k \) for \( 1 \leq k \leq N \), and temporarily reuse the symbol \( H(\cdot) \) to denote the transfer function, then,

\[ H(z) = \frac{1}{1 - \psi_1 z^{-1} - \cdots - \psi_N z^{-N}} = \frac{1}{\sum_{k=0}^{N} a_k z^{-k}}, \]  

(27)

and \( S(\omega) = \sigma^2 |H(e^{i\omega})|^2 \). Since all the coefficients are assumed real, this gives

\[ S(\omega) = \frac{\sigma^2}{\sum_{k=0}^{N} a_k e^{ik\omega} \sum_{j=0}^{N} a_j e^{-ik\omega}} = \frac{\sigma^2}{\sum_{k=0}^{N} \sum_{j=0}^{N} a_k a_j e^{i(k-j)}}. \]

Now, consider the integral of \( \sigma^2 / S(\omega) \) over one cycle of \( \omega \) from \(-\pi\) to \( \pi\):

\[ \int_{-\pi}^{\pi} \frac{\sigma^2}{S(\omega)} \, d\omega = \int_{-\pi}^{\pi} \sum_{k=0}^{N} \sum_{j=0}^{N} a_k a_j e^{i(k-j)} \, d\omega \]

\[ = \sum_{k=0}^{N} \sum_{j=0}^{N} a_k a_j \int_{-\pi}^{\pi} e^{i(k-j)} \, d\omega \]

\[ = \sum_{k=0}^{N} \sum_{j=0}^{N} a_k a_j 2\pi \delta_{jk} = 2\pi \sum_{k=0}^{N} a_k^2. \]

Referring back to (26), this shows that, for \( \text{AR}(N) \) processes at least, the predictive information rate can be expressed in terms of the power spectrum \( S(\omega) \) as

\[ b_\mu = \frac{1}{2} \log \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma^2}{S(\omega)} \, d\omega \right). \]  

(28)

Substituting in (9) for \( \sigma^2 \), we get

\[ b_\mu = \frac{1}{2} \log \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S(\omega) \, d\omega + \log \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega \right). \]

As a final step, this can be written entirely in terms of the inverse power spectrum, in an expression which is an exact parallel of (10):

\[ b_\mu = \frac{1}{2} \left( \log \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega - \log \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega \right), \]  

(29)
exposing an intriguing duality between the multi-information and predictive information rates on the one hand, and Gaussian processes whose power spectra are mutually inverse on the other. A similar duality was noted by Abdallah and Plumbley [4] in relation to the multi-information and the bounding information in finite sets of discrete-valued random variables. Although derived for finite-order autoregressive process, we conjecture that [29] may be valid for any Gaussian process for which the required integrals exist, and return to this topic later, in our analysis of moving-average processes.

D. Residual or erasure entropy rate

Since the erasure entropy rate is \( r_\mu = b_\mu - b_\mu \), we can write \( r_\mu \) in terms of the power spectrum as follows:

\[
r_\mu = \frac{1}{2} \left( \log 2\pi e - \log \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega \right).
\]

(30)

This concurs with the results of Verdú and Weissman [9], which are presented there without proof. We outline a skeleton proof later in §IV C.

III. AUTOREGRESSIVE EXAMPLES

Here we compute \( b_\mu \) and \( \rho_\mu \) for some simple cases to get feel for their range of variation. In all cases, the processes are normalised to unit variance, so that \( \gamma_0 = E X_t^2 = 1 \) for all \( t \) and the marginal entropy \( H(X_t) = \frac{1}{2} \log 2\pi e \) and therefore \( \rho_\mu + b_\mu = H(X_0) \) is constant.

A. AR(1) processes

The simplest case we can consider is that of the AR(1) processes, which, given the variance constraint, form a one-dimensional family parameterised by the prediction coefficient \( \psi_1 \). The generative equation is

\[
X_t = U_t + \psi_1 X_{t-1}.
\]

(31)

The process will be stationary only if the corresponding IIR filter is stable, which requires that \( |\psi_1| < 1 \). Multiplying [31] by \( X_{t-k} \) and taking expectations yields

\[
E X_t X_{t-k} = E U_t X_{t-k} + \psi_1 E X_{t-1} X_{t-k},
\]

from which we obtain the Yule-Walker equations relating the autocorrelation sequence \( \gamma_k = E X_t X_{t-k} \) and the prediction coefficient \( \psi_1 \):

\[
\gamma_0 = \sigma^2 + \psi_1 \psi_2,
\]

(32)

\[
\gamma_1 = \gamma_0 \psi_1.
\]

The variance constraint means that \( \gamma_0 = 1 \) and so we find that \( \sigma^2 = 1 - \psi_1^2 \). Since \( \rho_\mu = H(X_0) - b_\mu \) and \( H(X_0) \) is fixed at \( \frac{1}{2} \log 2\pi e \), we obtain the following results:

\[
\rho_\mu = -\frac{1}{2} \log(1 - \psi_1^2),
\]

(33)

\[
b_\mu = \frac{1}{2} \log(1 + \psi_1^2).
\]

(34)

Given the stability constraints on \( \psi_1 \), both quantities are minimised when \( \psi_1 = 0 \), which corresponds to \( X \) being a unit-variance white noise sequence. Both \( \rho_\mu \) and \( b_\mu \) increase as \( \psi_1 \to \pm 1 \): the multi-information rate diverges while the PIR tends to \( \frac{1}{2} \log 2 \) or half a bit. As \( \psi_1 \to 1 \), the process becomes Brownian noise, whose sequence of first differences are white noise. The marginal variance constraint means that the innovation variance \( \sigma^2 \) simultaneously tends to zero. However, since \( b_\mu \) is invariant to rescaling of the processes, the PIR of any (discrete-time) Brownian noise can be taken to be 0.5 bits per sample. If \( \psi_1 \to -1 \), the process no longer looks like Brownian noise, but can be obtained from one by reversing the sign of every other sample, that is, by applying the map \( X_t \to (-1)^t X_t \).

B. AR(2) processes

Second-order processes can be tackled in much the same way. In this case, the generative equation is

\[
X_t = U_t + \psi_1 X_{t-1} + \psi_2 X_{t-2},
\]

(35)

and the Yule-Walker equations are

\[
\gamma_0 = \sigma^2 + \psi_1 \psi_2 + \psi_2^2,
\]

(36)

\[
\gamma_1 = \gamma_0 \psi_1 + \gamma_1 \psi_2,
\]

\[
\gamma_2 = \gamma_1 \psi_1 + \gamma_0 \psi_2.
\]

A little algebra eventually yields

\[
\gamma_0 = \frac{\sigma^2 (1 - \psi_2^2)}{1 - \psi_1^2 - \psi_1 \psi_2 - \psi_2^2 + \psi_2^2}.
\]

(37)

and therefore

\[
\rho_\mu = \frac{1}{2} \log \frac{1 - \psi_2^2}{1 - (\psi_1^2 + \psi_2^2)(1 + \psi_2^2) + \psi_2^2},
\]

(38)

\[
b_\mu = \frac{1}{2} \log(1 + \psi_1^2 + \psi_2^2).
\]

(39)

Fig. [3] illustrates how \( \rho_\mu \) and \( b_\mu \) vary with \( \psi_1 \) and \( \psi_2 \). The PIR is maximised when \( \psi_1 = \pm 2 \) and \( \psi_2 = -1 \), which
The autocorrelation sequence $\gamma_k$ can be computed from $\sigma^2$ and the prediction coefficients using what is essentially a generalisation of the methods used in §III.A and §III.B (as implemented in the MATLAB function `rlevinson`). From $\gamma_0/\sigma^2$ we compute $\rho_\mu$. The points cover a region qualitatively similar to that shown in fig. 2(b), but with different upper and lower asymptotes. Our initial investigations suggest that the upper limit of $b_\mu$ is approached if all the poles approach 1 or $-1$, at which point the prediction coefficients are the binomial coefficients and are easily computed. For example, at $N = 8$, we obtain $b_\mu < \frac{1}{2} \log 12870$. As with AR(1) and AR(2), the resulting processes are such that the $N$th differences of the sequence are white noise, but since the variance of the innovations tends to zero, the processes themselves appear increasingly smooth and are dominated by low frequencies.

IV. MOVING-AVERAGE PROCESSES

A moving-average Gaussian process of order $N$ is a real-valued random process $(X_t)_{t \in \mathbb{Z}}$ such that

$$X_t = \sum_{k=0}^{N} b_k U_{t-k},$$

where the $U_t$ form a sequence of independent Gaussian random variables with zero mean and variance $\sigma^2$. Thus, a realisation of the process $X$ is the result of applying an order-$N$ finite impulse response (FIR) filter with coefficients $b_k$ to a realisation of the sequence formed by the $U_t$. The class of such processes is known as MA($N$). Without loss of generality, we may assume $b_1 = 1$, since any overall scaling of the process can be absorbed into $\sigma^2$. We may also assume that none of the roots of the filter transfer function polynomial $B(z) = \sum_{k=0}^{N} b_k z^{-k}$ are outside the unit disk in the complex plane, by the following argument: assuming $b_0 = 1$, $B(z)$ can be expressed in terms of its $N$ roots $\beta_1, \ldots, \beta_N$ as

$$B(z) = \frac{1}{z^N} \prod_{k=1}^{N} (z - \beta_k).$$

The spectral density at angular frequency $\omega$ is therefore

$$S(\omega) = \sigma^2 |B(e^{i\omega})|^2 = \sigma^2 \prod_{k=1}^{N} |e^{i\omega} - \beta_k|^2.$$  

The Gaussian process is uniquely determined by giving either its autocorrelation sequence or its spectral density function. If we move any of the roots $\beta_k$ without changing the value of $S(\omega)$ for any $\omega \in \mathbb{R}$, the FIR filter may be different but the process itself will remain unchanged. Suppose one of the roots is at $\zeta$ and $|\zeta| > 1$. Its contribution to the PSD is a factor of

$$|e^{i\omega} - \zeta|^2 = |\zeta e^{i\omega} (1/\zeta - e^{-i\omega})|^2 = |\zeta|^2 |e^{i\omega} - \zeta|^2,$$

corresponds to a transfer function $H(z)$ (as in equation 27) with two poles in the same place at $z = \pm 1$. Similar to the AR(1) case, as $(\psi_1, \psi_2)$ approaches $(2, -1)$, $\rho_\mu$ diverges and the process becomes non-stationary, but instead of becoming Brownian noise, it becomes the cumulative sum of a Brownian noise: a quick check will verify that the sequence of second differences is white noise.

C. AR($N$) processes with random poles

In our final example, we take a look at higher-order autoregressive processes. In order to ensure that we consider only stable processes, we generate them by randomly sampling their poles inside the unit circle in the complex plane. From the poles we compute the prediction coefficients by expanding the factorised form of the transfer function, which, for $N$ poles at $\zeta_1, \ldots, \zeta_N$, is

$$H(z) = \frac{z^N}{\prod_{k=1}^{N} (z - \zeta_k)}.$$
where \( \bar{\zeta} = 1/\zeta^* \) is the reciprocal of the complex conjugate of \( \zeta \) and hence inside the unit disk. Thus, the root \( \zeta \) can be replaced with \( \bar{\zeta} \) and the only effect on the PSD is the introduction of the constant factor \( |\zeta|^2 \), which can be absorbed into \( \sigma^2 \). In this way, all the roots of \( B(z) \) that are outside the unit disk can be moved inside without changing the statistical structure of the process. Noting that \([41]\) can be written as

\[
U_t = X_t - \sum_{k=1}^{N} b_k U_{t-k},
\]

we see that the sequence \((\ldots, U_{t-1}, U_t)\) can be computed from the sequence \((\ldots, X_{t-1}, X_t)\) via a stable IIR filter with the transfer function \(1/B(z)\). These properties will be useful when we try to determine the process information measures of the MA(\(N\)) process.

The first thing to note about this model is that it is does not have the Markov conditional independence structure of the AR(\(N\)) model. Consider the graphical model of an MA(\(1\)) process depicted in fig. [5]. Even though \(X_2\) and \(X_4\) are marginally independent (since their parent node sets are disjoint and independent), they become conditionally dependent given \(X_3\). This lack of any finite-order Markov structure means that the measures \(h_\mu\), \(\rho_\mu\) and \(b_\mu\) cannot be computed from the joint distribution of any finite segment of the sequence, say \(X_{-\ell:t}\), as we did in [47], but can be obtained by using spectral methods to analyse the covariance structure in the limit \(\ell \to \infty\). From [41], we obtain the autocorrelation sequence

\[
\gamma_m = \text{E} X_t X_{t-m} = \sum_{k=0}^{N} b_k U_{t-k} \sum_{j=0}^{N} b_j U_{t-m-j} = \sum_{k=0}^{N} \sum_{j=0}^{N} b_k b_j \sigma^2 \delta_{k,m+j} = \sigma^2 \sum_{k=m}^{N} b_k b_{k-m},
\]

which is non-zero for at most \(2N+1\) values of \(m\), from \(-N\) to \(N\). Hence, the covariance matrix \(R = \text{E} XX^T\) of the multivariate Gaussian \(X \equiv (X_{-\ell}, \ldots, X_t)\), when \(\ell > N\), will be a banded toeplitz matrix. For example, for an MA(\(1\)) process it will be

\[
\begin{pmatrix}
\gamma_0 & \gamma_1 & \cdots & 0 \\
\gamma_1 & \gamma_0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \gamma_1 & \cdots & \gamma_0
\end{pmatrix},
\]

\[\text{(46)}\]

A. Entropy rate

In the case of MA processes and with our assumption that roots of the transfer function are not outside the unit disk, the Kolmogorov-Sinai entropy \([5]\) can be evaluated exactly by substituting in \([43]\) and using Jensen’s formula, which gives \(\int_{-\pi}^{\pi} \log |e^{i\omega} - \zeta| \, d\omega = 0\) if \(|\zeta| \leq 1:\)

\[
\int_{-\pi}^{\pi} \log \sigma(\omega) \, d\omega = \int_{-\pi}^{\pi} \log \sigma^2 \prod_{k=1}^{N} |e^{i\omega} - \beta_k|^2 \, d\omega = \log \sigma^2 + 2 \sum_{k=1}^{N} \int_{-\pi}^{\pi} \log |e^{i\omega} - \beta_k| \, d\omega.
\]

and hence

\[
h_\mu = \frac{1}{2} \log 2\pi e \sigma^2.
\]

(47)

This is consistent with our earlier observation that the innovations up to and including time \(t\) can be computed from the observations up to time \(t\) by IIR filtering the observations: in this case, the conditional variance of the next observation is just the variance of \(b_0 U_{t+1}\), which is \(\sigma^2\).

B. Multi-information rate

From [41] and [45], the marginal variance is \(E X_t^2 = \gamma_0 = \sigma^2 \sum_{k=0}^{N} b_k^2\); so, with \(b_0 = 1\), and \(\rho_\mu = H(X_t) - h_\mu\), the multi-information rate is

\[
\rho_\mu = \frac{1}{2} \log \left(1 + \sum_{k=1}^{N} b_k^2 \right),
\]

(48)

which is in agreement with Ihara’s result \([10] \text{§2.2}\). Note that this is dual to the result obtained for the predictive information rate in AR(\(N\)) processes \([26]\), in that the FIR filter coefficients \(b_k\) have taken the place of the IIR filter coefficients \(\psi_k\) or \(a_k\).

C. Predictive information rate

The PIR can be obtained from the erasure entropy rate \(r_\mu\) using the relation \(b_\mu = h_\mu - r_\mu\). Verdú and Weissman \([9]\) state without proof that the erasure entropy rate of a Gaussian process with power spectral density \(S(\omega)\) is

\[
r_\mu = \frac{1}{2} \log 2\pi e - \frac{1}{2} \log \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega \right),
\]

(49)
which, in combination with (8), yields
\[ b_\mu = \frac{1}{2} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S(\omega) \, d\omega + \log \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega \right), \]
which agrees with the expression we obtained earlier for AR processes. A skeleton of a proof of (49) can be obtained by considering the limit of \( H(X_\ell|X_{\ell-\ell},...,X_\ell) \) as \( \ell \to \infty \). Let \( X_\ell \) be the random vector \((X_{\ell-\ell},...,X_\ell)\) with covariance matrix \( R_\ell \) constructed from the autocorrelation sequence as shown previously in (45) and (46). If \( K^\ell = R_\ell^{-1} \) is the corresponding precision matrix, the probability density function \( p_\ell : \mathbb{R}^{2\ell+1} \to \mathbb{R} \) for \( X_\ell \) is the multivariate Gaussian:
\[ p_\ell(x) \propto \exp \left( -\frac{1}{2} x^T K^\ell x \right). \]

If we index the elements of \( x \) and \( K^\ell \) starting with \(-\ell \) and running through 0 to \( \ell \), then it can easily be shown by examining the functional dependence of \( p_\ell(x) \) on \( x_0 \) that the conditional density of the central variable \( X_0 \) given all the values of the others is Gaussian with variance \( 1/K_{00}^\ell \), and hence the conditional entropy is \( H(X_0|X_{\ell-\ell},...,X_\ell) = -\frac{1}{2} \log 2\pi K_{00}^\ell \). Now, since \( R_\ell \) is real and symmetric, it will have \( 2\ell + 1 \) orthogonal eigenvectors with real eigenvalues, and \( K^\ell \) can be represented in terms of these as
\[ K^\ell_{jk} = \sum_{n=0}^{\ell} r_n^{-1} V_{jn} V_{kn}^*, \]
where \( V_{jn} \) is the \( j \)th component of the \( n \)th eigenvector with eigenvalue \( r_n \). If \( R_\ell \) had been circulant as well as Toeplitz, its eigenvectors would have been complex exponentials of the form \( V_{jn} = e^{-2\pi i j n/\ell} / \sqrt{2\ell + 1} \), in which case, substitution into (52) would yield \( \sum_n r_n^{-1}/(2\ell + 1) \) for all the diagonal elements. Instead, the standard approach is to construct two infinite sequences of matrices with are asymptotically equivalent. The first sequence consists of covariance matrices \( R_\ell \) as \( \ell \) increases, i.e., \( R_1, R_2, \ldots \). The second is a sequence of circulant approximations of the \( R_\ell \). As \( \ell \to \infty \), the sequences converge to each other (in the weak norm sense) and many properties of the \( R_\ell \) converge to those of their circulant approximations. This does not prove that all diagonal elements of the inverse \( R_\ell^{-1} \) converge in this way, and indeed, we would not expect them to to for the extremal elements such as \( K_{00}^\ell \) as this would be inconsistent with the result for the entropy rate. However, numerical results suggest that for ‘central’ elements \( K_{jk}^\ell \) such that both \( j + \ell \) and \( \ell - j \) tend to infinity as \( \ell \) tends to infinity, we can assume that the values do converge as expected. In particular, for the middle element, we suppose that
\[ \lim_{\ell \to \infty} K_{00}^\ell = \frac{1}{2\ell + 1} \sum_{n=\ell}^{\ell} r_n^{-1}. \]

This remains to be proved, but if we accept it, then by Szegö’s theorem [7], which applies to such functions of the eigenvalues of a Toeplitz matrix, this converges to an integral expressed in terms of the spectral density function:
\[ \lim_{\ell \to \infty} \frac{1}{2\ell + 1} \sum_{n=\ell}^{\ell} r_n^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega, \]
so we obtain the expected expression for the erasure entropy rate
\[ r_\mu = \frac{1}{2} \log 2\pi e - \frac{1}{2} \log \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{S(\omega)} \, d\omega \right). \]

V. MOVING-AVERAGE EXAMPLES

The simplest non-trivial moving-average process that we can consider is the MA(1) process
\[ X_\ell = U_\ell + b_1 U_{\ell-1}, \]
where the sole parameter \( b_1 \) satisfies \( |b_1| < 1 \), according to the assumptions described at the beginning of §IV. Using (48), we find that \( p_\mu = \frac{1}{2} \log (1 + b_1^2) \), which is dual to the result (57) obtained for the PIR of the AR(1) process obtained by inverting the spectrum of this MA(1) process. The transfer function of the two-tap FIR filter from \( U \) to \( X \) is \( H(z) = 1 + b_1 z^{-1} \). If we define \( H(z) = 1/|H(z)| = 1/(1 + b_1 z^{-1}) \), we can see that \( H(z) \) is the transfer function of the 1st order IIR filter associated with an AR(1) process, where \( b_1 \) plays the role of the prediction coefficient. Clearly, the spectrum of this process, call it \( X \), will be in the inverse of the original process, and we can use the results of §II along with the duality relationship we observed relating the multi-information and predictive information rates, to compute the multi-information and predictive information rates of the moving-average process \( X \). Referring back to §II, we obtain
\[ \rho_\mu = \frac{1}{2} \log (1 + b_1^2), \]
\[ b_\mu = -\frac{1}{2} \log (1 - b_1^2). \]

Rather than repeat the process of illustrating these equations, we refer the reader back to fig. 3, where the same except for swapping the \( \rho_\mu \) and \( b_\mu \) axis labels and replacing \( \psi \) with \( b_1 \). Indeed, the same reasoning can be applied to higher-order moving-average processes, so we can reuse figure 4 for moving-average process by swapping \( \rho_\mu \) and \( b_\mu \), and replacing the prediction coefficients \( \psi_\ell \) with the moving-average coefficients \( b_\mu \).

One implication of these results is that, even in the MA(1) process, the PIR approaches infinity as \( b_1 \) approaches \( \pm 1 \). In higher-order processes, the PIR diverges as the zeros of the transfer function approach the unit circle in the complex plane. In particular, the dual of the AR(N) process identified in §III is the MA(N) process with all zeros at 1 or \(-1\), and maximises \( \rho_\mu \) as \( b_\mu \) diverges. With all zeros
at \(-1\), the coefficients of the corresponding FIR filter are the binomial coefficients, and so as the order \(N\) tends to infinity, the filter approximates a smoothing filter with a Gaussian impulse response.

VI. DISCUSSION AND CONCLUSIONS

We have found a closed-form expression for the predictive information rate in autoregressive Gaussian processes of arbitrary finite order, which is a simple function of the predictive coefficients. It can also be expressed as function of the power spectral density of the process in a form which we conjecture may apply to arbitrary Gaussian processes and not just autoregressive ones. The functional form also suggests a duality between the PIR and multi-information rate, since the PIR of a process with power spectrum \(S(\omega)\) equals the multi-information rate of a process with the inverse power spectrum \(1/S(\omega)\).

The fact that the stationary AR(1) and AR(2) processes maximising the PIR turn out to be, in the limit, Brownian motion and its (discrete time) integral is intriguing and perhaps counter-intuitive: in order to preserve finite variance, both process have vanishingly small innovations, with \(\sigma^2\) tending to zero as the limit is approached, and therefore ‘look smooth’. Indeed, as the order \(N\) is increased, the results of \cite{4} suggest that this pattern continues, with the PIR-maximising processes being increasingly ‘smooth’ and having power spectra more and more strongly peaked at \(\omega = 0\). The PIR, originally proposed \cite{1} as a potential measure of complexity or ‘interest- ingness’ (for which purpose it seems a plausible candidate, at least for discrete valued processes), is telling us that these very ‘smooth’ Gaussian processes are somehow the most ‘interesting’.

The difficulty is presented even more starkly in the case of moving-average processes, where the PIR is unbounded, and we are forced to conclude that a single observation can yield infinite information about the unobserved future. Once again, we find that very ‘smooth’ looking processes can have arbitrarily high predictive information rates.

The reason for this, we suggest, lies in the assumption that variables in a real-valued random sequence can be observed with infinite precision. Under these conditions, the tiny innovations observed in the unit-variance almost-Brownian noise of AR(1) when \(\psi_1\) approaches 1 are just as measurable as the macroscopic innovations in the non-Brownian case and are significant and informative in a predictive sense, because every innovation is preserved into the infinite future in the form of an additive shift to all subsequent values in the sequence. In addition, as soon as we have infinite precision measurements, we open the door to the possibility of infinite information; hence the divergence of \(\rho_b\) and \(b_0\) in these limiting cases. This rather un-physical situation can be remedied if we recogni- se that, in physically realisable systems, the variables can only be observed with finite precision, either by explicitly modelling a quantisation error or by introducing some ‘observation noise’, for example, by allowing infinite precision observations only of \(Z_t = X_t + N_t\), where the \(N_t\) are independent and Gaussian with some variance \(\sigma^2\). In this case, each observation can only yield a finite amount of information about \(X_t\), and it will no longer be possible to use infinitesimal variations to carry information about the future because they will be swamped by the observation noise. Recognising that what we are talking about here is essentially a hidden Markov model, we aim to establish these ideas on a more rigorous footing in future work.

ACKNOWLEDGMENTS

This research was supported by EPSRC grant EP/H01294X/1: ‘Information and neural dynamics in the perception of musical structure’.

[1] Samer A. Abdallah and Mark D. Plumbley, “Information dynamics: Patterns of expectation and surprise in the perception of music,” Connection Science 21, 89–117 (2009).
[2] Claude E. Shannon, “A mathematical theory of commu- nication,” The Bell System Technical Journal 27, 379–423,623–656 (1948).
[3] Thomas M. Cover and Joy A. Thomas, Elements of Information Theory (John Wiley and Sons, New York, 1991).
[4] Samer A. Abdallah and Mark D. Plumbley, “A measure of statistical complexity based on predictive information with application to finite spin systems,” Physics Letters A 376, 275 – 281 (2012).
[5] R.W. Yeung, “A new outlook on Shannon’s information measures,” Information Theory, IEEE Transactions on 37, 466–474 (1991).
[6] Ryan G. James, Christopher J. Ellison, and James P. Crutchfield, “Anatomy of a bit: Information in a time series observation,” Chaos 21, 037109 (2011).
[7] R.M. Gray, Toeplitz and circulant matrices: A review (Now Pub, 2006).
[8] Shlomo Dubnov, “Spectral anticipations,” Computer Mu- sic Journal 30, 63–83 (2006).
[9] S. Verdú and T Weissman, “Erasure entropy,” in IEEE International Symposium on Information Theory (ISIT 2006) (2006) pp. 98–102.
[10] S. Ihara, Information theory for continuous systems Vol. 2 (World Scientific Pub Co Inc, 1993).