ARC INDEX OF PRETZEL KNOTS OF TYPE \((-p, q, r)\)

HWA JEONG LEE AND GYO TAEK JIN

Abstract. We computed the arc index for some of the pretzel knots $K = P(-p, q, r)$ with $p, q, r \geq 2$, $r \geq q$ and at most one of $p, q, r$ is even. If $q = 2$, then the arc index $\alpha(K)$ equals the minimal crossing number $c(K)$. If $p \geq 3$ and $q = 3$, then $\alpha(K) = c(K) - 1$. If $p \geq 5$ and $q = 4$, then $\alpha(K) = c(K) - 2$.

1. Arc presentation

Let $D$ be a diagram of a knot or a link $L$. Suppose that there is a simple closed curve $C$ meeting $D$ in $k$ distinct points which divide $D$ into $k$ arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following properties:

1. Each $\alpha_i$ has no self-crossing.
2. If $\alpha_i$ crosses over $\alpha_j$ at a crossing, then $i > j$ and it crosses over $\alpha_j$ at any other crossings with $\alpha_j$.
3. For each $i$, there exists an embedded disk $d_i$ such that $\partial d_i = C$ and $\alpha_i \subset d_i$.
4. $d_i \cap d_j = C$, for distinct $i$ and $j$.

Then the pair $(D, C)$ is called an arc presentation of $L$ with $k$ arcs, and $C$ is called the axis of the arc presentation. Figure 1 shows an arc presentation of the trefoil knot. The thick round curve is the axis. It is known that every knot or link has an arc presentation \cite{3,4}. For a given knot or link $L$, the minimal number of arcs in all arc presentations of $L$ is called the arc index of $L$, denoted by $\alpha(L)$.

By removing a point $P$ from $C$ away from $L$, we may identify $C \setminus P$ with the $z$-axis and each $d_i \setminus P$ with a vertical half plane along the $z$-axis. This shows that an arc presentation is equivalent to an open-book presentation.

Given a link $L$, let $c(L)$ denote the minimal crossing number of $L$.

Theorem 1 (Jin-Park). A prime link $L$ is nonalternating if and only if $\alpha(L) \leq c(L)$.

2000 Mathematics Subject Classification. Primary 57M27; Secondary 57M25.

Key words and phrases. knot, pretzel knot, arc presentation, arc index, Kauffman polynomial.

The first author was supported in part by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2010-0024630).

The second author was supported in part by the National Research Foundation of Korea Grant funded by the Korean Government (NRF-2011-0027989).

Figure 1. An arc presentation of the right-handed trefoil knot

25 Jan 2013
Figure 2. An open-book presentation of the right-handed trefoil knot

2. KAUFFMAN POLYNOMIAL

The Kauffman polynomial $F_L(a, z)$ of an oriented knot or link $L$ is defined by

$$F_L(a, z) = a^{-w(D)} \Lambda_D(a, z)$$

where $D$ is a diagram of $L$, $w(D)$ the writhe of $D$ and $\Lambda_D(a, z)$ the polynomial determined by the rules K1, K2 and K3.

(K1) $\Lambda_O(a, z) = 1$ where $O$ is the trivial knot diagram.

(K2) For any four diagrams $D_+, D_-, D_0$ and $D_\infty$ which are identical outside a small disk in which they differ as shown below,

we have the relation

$$\Lambda_{D_+}(a, z) + \Lambda_{D_-}(a, z) = z(\Lambda_{D_0}(a, z) + \Lambda_{D_\infty}(a, z)).$$

(K3) For any three diagrams $D_+, D$ and $D_-$ which are identical outside a small disk in which they differ as shown below,

we have the relation

$$a \Lambda_{D_+}(a, z) = \Lambda_D(a, z) = a^{-1} \Lambda_{D_-}(a, z).$$

For a connected sum and a split union of two diagrams, $\Lambda$ satisfies the following properties:

(K4) If $D$ is a connected sum of $D_1$ and $D_2$, then

$$\Lambda_D(a, z) = \Lambda_{D_1}(a, z) \Lambda_{D_2}(a, z).$$

(K5) If $D$ is the split union of $D_1$ and $D_2$, then

$$\Lambda_D(a, z) = (z^{-1}a - 1 + z^{-1}a^{-1}) \Lambda_{D_1}(a, z) \Lambda_{D_2}(a, z).$$

The Laurent degree in the variable $a$ of the Kauffman polynomial $F_L(a, z)$ is denoted by $\text{spread}_a(F_L)$ and defined by the formula

$$\text{spread}_a(F_L) = \max\text{-deg}_a(F_L) - \min\text{-deg}_a(F_L)$$

Notice that $\text{spread}_a(F_L) = \text{spread}_a(\Lambda_D)$ for any diagram $D$ of $L$. The following theorem gives an important lower bound for the arc index.

**Theorem 2** (Morton-Beltrami). Let $L$ be a link. Then

$$\alpha(L) \geq \text{spread}_a(F_L) + 2.$$
If $L$ is nonsplit and alternating, then the equality holds so that $\alpha(L) = c(L) + 2$. This is shown by Bae and Park [1] using arc presentations in the form of wheel diagrams.

3. Pretzel knots

Given a sequence of integers $p_1, p_2, \ldots, p_n$, we connect two disjoint disks by $n$ bands with $p_i$ half twists, $i = 1, 2, \ldots, n$, so that the boundary of the resulting surface is a link as shown in Figure 3. This link is called the *pretzel link* of type $(p_1, p_2, \ldots, p_n)$ and denoted by $P(p_1, p_2, \ldots, p_n)$.

![Pretzel Link Diagram](image)

**Figure 3.** Pretzel links $P(p_1, p_2, \ldots, p_n)$ and $P(-p, q, r)$

In the case $n = 3$, the pretzel links satisfy the following properties:

**Proposition 1.** Let $p, q,$ and $r$ be nonzero integers.

1. The link type of $P(p, q, r)$ is independent of the order of $p, q, r$.
2. $P(p, q, r)$ is a knot if and only if at most one of $p, q, r$ is an even number.

In this work, we compute the arc index for the pretzel knots $K = P(-p, q, r)$ with $p, q, r \geq 2$. By Proposition 1, we may assume that $r \geq q$. By Theorem 3 we know that $P(-p, q, r)$ is a minimal crossing diagram of $K$, i.e., $c(K) = p + q + r$.

**Theorem 3** (Lickorish-Thistlethwaite). If a link $L$ admits a reduced Montesinos diagram having $n$ crossings, then $L$ cannot be projected with fewer than $n$ crossings.

This work was motivated by Theorem 4 which is a special case of Theorem 4.

**Theorem 4** (Beltrami-Cromwell). If $K = P(-p, q, r)$ is a knot with $p, q, r \geq 2$, then

$$\alpha(K) \leq c(K) = p + q + r.$$  

By computing spread$_a(F_K)$ and finding arc presentations of $K = P(-p, q, r)$ with the minimum number of arcs for various values of $p, q$ and $r$, we obtained sharper results.

4. Main results

**Theorem 5.** If $K = P(-2, q, r)$ is a knot with $3 \leq q \leq r$, then

$$\alpha(K) \leq c(K) - 1.$$  

**Theorem 6.** If $K = P(-p, 2, r)$ is a knot with $p \geq 3$, $r \geq 3$, then

$$\alpha(K) = c(K).$$  

**Theorem 7.** If $K = P(-p, 3, r)$ is a knot with $p \geq 3$, $r \geq 3$, then

$$\alpha(K) = c(K) - 1.$$  

**Theorem 8.** If $K = P(-p, 4, r)$ is a knot with $p \geq 5$, $r \geq 5$, then

$$\alpha(K) = c(K) - 2.$$  

**Theorem 9.** If $K = P(-3, 4, r)$ is a knot with $r \geq 5$, then

$$c(K) - 3 \leq \alpha(K) \leq c(K) - 2.$$
5. ARC PRESENTATIONS OF $P(-p, q, r)$

**Proposition 2.** If $K = P(-2, q, r)$ is a knot with $3 \leq q \leq r$, then $K$ has an arc presentation with $q + r + 1$ arcs.

![Figure 4. An arc presentation of $P(-2, q, r)$](image)

**Proof.** Figure 4 shows a pretzel diagram of $P(-2, q, r)$ and its arc presentation with $q + r + 1$ arcs. The thick curve is the axis of the arc presentation which cuts the knot at 1 place in the leftmost box, $q - 1$ places in the second, 2 places in the third, and $r - 1$ places in the fourth. The $q + r + 1$ arcs of the knot satisfies the four properties of an arc presentation. □

**Proposition 3.** If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $2 \leq q \leq 3 \leq r$, then $K$ has an arc presentation with $p + r + 2$ arcs.

![Figure 5. Arc presentations of $P(-p, q, r)$ for $q = 2, 3$](image)

**Proof.** For each of $q = 2, 3$, Figure 5 shows a pretzel diagram of $P(-p, q, r)$ and its arc presentation with $p + r + 2$ arcs. The thick curve is the axis of the arc presentation which cuts the knot at $p - 1$ places in the leftmost box, 4 places in the second, and $r - 1$ places in the third. The $p + r + 2$ arcs of the knot satisfies the four properties of an arc presentation. □

**Proposition 4.** If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $4 \leq q \leq r$, then $K$ has an arc presentation with $p + q + r - 2$ arcs.

**Proof.** In Figure 6 the diagram (a) shows a pretzel diagram of $P(-p, q, r)$ with $p \geq 3$ and $4 \leq q \leq r$. The diagram (b) is obtained from (a) by two applications of the Reidemeister move of type 3. The diagram (c) shows an arc presentation with $p + q + r - 1$ arcs. The diagram (d) is obtained from (c) by isotoping the arc labeled...
Figure 6. Arc presentations of \( P(-p,q,r) \) with \( p \geq 3 \) and \( 4 \leq q \leq r \)

\( x \) over the axis so that there are only \( p + q + r - 2 \) arcs. Each of the seven boxes, from left to right, contains \( 1, p - 3, 2, 3, q - 3, 1, \) and \( r - 3 \) arcs, respectively. \( \square \)

6. The Kauffman polynomial of the pretzel knots \( P(-p,q,r) \)

For any link diagram \( D \), the polynomial \( \Lambda_D \) is of the form

\[
\Lambda_D(a,z) = \sum_{i=m}^{n} f_i(z)a^i
\]

where \( m, n \) are integers with \( m \leq n \), and \( f_i(z) \)'s are polynomials in \( z \) with integer coefficients such that \( f_m(z) \neq 0 \) and \( f_n(z) \neq 0 \). To simplify our computation of spread, \( \Lambda_D \) we use the notations

\[
f_m(z) = \langle k_m z^{h_m} \rangle
\]
\[
f_n(z) = \langle k_n z^{h_n} \rangle
\]
\[
\sum_{i=m}^{n} f_i(z)a^i = \left[ \langle k_n z^{h_n} \rangle a^n, \langle k_m z^{h_m} \rangle a^m \right], \quad (m < n)
\]

where \( k_m z^{h_m} \) and \( k_n z^{h_n} \) are the highest degree terms in \( f_m(z) \) and \( f_n(z) \), respectively. For example, we write

\[
z(z^2-1)a^{-1} + z^2a^{-2} - 2za^{-3} = \left[ \langle z^3 \rangle a^{-1}, \langle -2z \rangle a^{-3} \right].
\]

We also use the notation \( \Lambda_{(p_1,p_2,\ldots,p_n)} \) for \( \Lambda_D \) when \( D = P(p_1,p_2,\ldots,p_n) \).
Lemma 1. Let $m, n$ be nonnegative integers. Then

\[
\Lambda_{(-m,n)} = \begin{cases} 
[z]a^{k-1}, (z^{k-1})a^{-1} & \text{if } k = m - n > 1 \\
1 & \text{if } k = m - n = 1 \\
z^{-1}a - 1 + z^{-1}a^{-1} & \text{if } k = m - n = 0 \\
a & \text{if } k = n - m = 1 \\
(z^{k-1})a, (z)a^{-(k-1)} & \text{if } k = n - m > 1
\end{cases}
\]

Proof. Since $\Lambda_D(a, z)$ is invariant under regular isotopy of diagrams, we have

\[
\Lambda_{(-m,n)} = \begin{cases} 
\Lambda_{(-k,0)} & \text{if } k = m - n \geq 1 \\
\Lambda_{(0,0)} & \text{if } k = m - n = 0 \\
\Lambda_{(0,k)} & \text{if } k = n - m \geq 1
\end{cases}
\]

Using the skein relations K1, K2, K3, and inductions on $k$, we can prove the lemma.

Lemma 2.

\[
\begin{align*}
\Lambda_{(-p,0,r)} &= \begin{cases} 
1 & \text{if } p = r = 1 \\
[z]a^p, (z^{p-1})a^0 & \text{if } p > 1, r = 1 \\
(z^r)a^p, (z^p)a^{-r} & \text{if } p > 1, r > 1
\end{cases} \\
\Lambda_{(-p,1,r)} &= \begin{cases} 
a^2 & \text{if } p = 0, r = 1 \\
(z^{r-1})a^2, (z)a^{-(r-2)} & \text{if } p = 0, r > 1 \\
a^{r-1} & \text{if } p = 1, r \geq 1 \\
(z^{r+1})a^p, (z^{p-1})a^{-r} & \text{if } p > 1, r \geq 1
\end{cases}
\end{align*}
\]

Proof. Since $P(-p, 0, r) = P(-p, 0)\# P(0, r)$, we have $\Lambda_{(-p,0,r)} = \Lambda_{(-p,0)}\Lambda_{(0,r)}$. Therefore the first formula follows from Lemma 1.

Now we consider the second formula. The first three cases follow from K1, K2, K3, and Lemma 1. The last is derived inductively by the recurrence formula

\[
\Lambda_{(-p,1,r)} = -\Lambda_{(-p+2,1,r)} + z\{\Lambda_{(-p+1,1,r)} + a^{p-1}\Lambda_{(0,1+r)}\}.
\]

Let $p = 2$. Then

\[
\begin{align*}
\Lambda_{(-p,1,r)} &= -\Lambda_{(0,1,r)} + z\{\Lambda_{(-1,1,r)} + a\Lambda_{(0,1+r)}\} \\
&= -\begin{cases} 
a^2 & \text{if } r = 1 \\
(z^{r-1})a^2, (z)a^{-(r-2)} & \text{if } r \geq 2
\end{cases} + za^{-r} + za \{z^ra, (z)a^{-r}\}
\end{align*}
\]

\[
= (z^{r+1})a^p, (z^{p-1})a^{-r}
\]
For $p > 2$, we have
\[
\Lambda_{(-p,1,r)} = -\Lambda_{(-p+2,1,r)} + z\{\Lambda_{(-p+1,1,r)} + a^{p-1}\Lambda_{(0,1+r)}\}
\]
\[
= -\left\{\begin{array}{ll}
a^{-r} & \text{(if } p = 3) \\
(z^{r+1})a^{p-2}, (z^{p-3})a^{-r} & \text{(if } p \geq 4) \\
\end{array}\right.
\]
\[+ z \left[(z^{r+1})a^{p-1}, (z^{p-2})a^{-r}\right] + za^{p-1}[(z^r)a, (z)a^{-r}]
\]
\[= \left[(z^{r+1})a^{p}, (z^{p-1})a^{-r}\right]
\]
This completes the proof.

**Proposition 5.** \(\text{spread}_a(\Lambda_{(-p,2,r)}(a,z)) = p + r \) for \(p \geq 3\), and \(r \geq 3\).

**Proof.** For \(p \geq 3\), we show that
\[
\Lambda_{(-p,2,r)} = \left\{\begin{array}{ll}
\langle z^3 \rangle a^p, (z^{p-1})a^{-2} & \text{(if } r = 1) \\
\langle z^4 \rangle a^p, (z)a^{-2} & \text{(if } r = 2) \\
\langle z^r + 2 \rangle a^p, (z^{p-2})a^{-r} & \text{(if } r \geq 3) \\
\end{array}\right.
\]
where \((z)a^{-2}\) indicates that the lowest \(a\)-degree of \(\Lambda_{(-p,2,2)}\) is not smaller than \(-2\).

Using K1, K2, K3 and Lemmas 1 and 2 we obtain
\[
\Lambda_{(-p,2,1)} = -\Lambda_{(-p,0,1)} + z\Lambda_{(-p,1,1)} + za^{-1}\Lambda_{(-p,1)}
\]
\[= -\left[(z)^p, (z^{p-1})a^0\right] + z \left[(z^4)a^p, (z^{p-1})a^{-1}\right] + za^{-1} \left[(z)a^{p-2}, (z^{p-2})a^{-1}\right]
\]
\[= \left[(z^3)a^p, (z^{p-1})a^{-2}\right],
\]
\[
\Lambda_{(-p,2,2)} = -\Lambda_{(-p,0,2)} + z\Lambda_{(-p,1,2)} + za^{-1}\Lambda_{(-p,2)}
\]
\[= -\left[(z^2)a^p, (z^p)a^{-2}\right] + z \left[(z^3)a^p, (z^{p-1})a^{-2}\right]
\]
\[+ \left[(z^2)a^{p-4}, (z^{p-2})a^{-2}\right] \text{ (if } p \geq 4) 
\]
\[= \left[(z^4)a^p, (z)a^{-2}\right],
\]
which prove the first two cases of \(1\). Now we show the third case of \(1\) by an induction on \(r\). For \(r = 3\), we have
\[
\Lambda_{(-p,2,3)} = -\Lambda_{(-p,2,1)} + z\Lambda_{(-p,2,2)} + za^{-2}\Lambda_{(-p,2)}
\]
\[= -\left[(z^3)a^p, (z^{p-1})a^{-2}\right] + z \left[(z^4)a^p, (z)a^{-2}\right]
\]
\[+ \left[(z^2)a^{p-3}, (z^{p-2})a^{-3}\right] \text{ (if } p \geq 4) 
\]
\[= \left[(z^5)a^p, (z^{p-2})a^{-3}\right],
\]
and for \(r \geq 4\), inductively, we have
\[
\Lambda_{(-p,2,r)} = -\Lambda_{(-p,2,r-2)} + z\Lambda_{(-p,2,r-1)} + za^{-(r-1)}\Lambda_{(-p,2)}
\]
\[= -\left[(z^r)a^p, (z)a^{-(r-2)}\right] + z \left[(z^{r+1})a^p, (z^{p-2})a^{-(r-1)}\right]
\]
\[+ \left[(z^2)a^{p-r-2}, (z^{p-2})a^{-r}\right] \text{ (if } p \geq 4) 
\]
\[= \left[(z^{r+2})a^p, (z^{p-2})a^{-r}\right].
\]
This completes the proof.

**Proposition 6.** \(\text{spread}_a(\Lambda_{(-p,3,r)}(a,z)) = p + r \) for \(p \geq 3\) and \(r \geq 3\).
Proof. We show that

$$\Lambda_{(-p,3,r)} = \begin{cases} \left[ (z^6)a_p, (2z^{p-3})a^{-3} \right] & (\text{if } r = 3) \\ \left[ (z^{r+3})a_p, (z^{p-3})a^{-r} \right] & (\text{if } r \geq 4) \end{cases}$$

Using K1, K2, K3 and the results above, we obtain

$$\Lambda_{(-p,3,3)} = -\Lambda_{(-p,3,1)} + z\Lambda_{(-p,3,2)} + za^{-2}\Lambda_{(-p,3)}$$

$$= -\left\{ -\Lambda_{(-p,1,1)} + z\Lambda_{(-p,2,1)} + za^{-2}\Lambda_{(-p,1)} \right\} + z\Lambda_{(-p,1,2)} + za^{-2}\Lambda_{(-p,2,2)} + za^{-2}\Lambda_{(-p,3)}$$

$$= \Lambda_{(-p,1,1)} - 2z\Lambda_{(-p,2,1)} + z^2\Lambda_{(-p,2,2)} + za^{-2}\Lambda_{(-p,1)} + za^{-2}\Lambda_{(-p,2)} + za^{-2}\Lambda_{(-p,3)}$$

$$= \left[ (z^2)a^p, (z^{p-1})a^{-1} \right] - 2z \left[ (z^3)a^p, (z^{p-1})a^{-2} \right] + z^2 \left[ (z^4)a^p, (a)a^{-2} \right] + za^{-2}\Lambda_{(-p,3)}$$

$$= \left[ (z^6)a^p, (a)a^{-2} \right] + 2za^{-2} \begin{cases} (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p = 3) \\ (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p = 4) \\ (z)a^{p-4}, (z^{p-4})a^{-1} & (\text{if } p \geq 5) \end{cases}$$

$$= \left[ (z^7)a^p, (z^{p-3})a^{-4} \right],$$

and, for $r \geq 5$, inductively, we have

$$\Lambda_{(-p,3,r)} = -\Lambda_{(-p,3,r-2)} + z\Lambda_{(-p,3,r-1)} + za^{-(r-1)}\Lambda_{(-p,3)}$$

$$= -\left[ (z^{r+1})a^p, (a)a^{-(r-2)} \right] + z \left[ (z^{r+2})a^p, (z^{p-3})a^{-(r-1)} \right] + za^{-(r-1)} \begin{cases} (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p = 3) \\ (z^{-1}a - 1 + z^{-1}a^{-1}) & (\text{if } p = 4) \\ (z)a^{p-4}, (z^{p-4})a^{-1} & (\text{if } p \geq 5) \end{cases}$$

$$= \left[ (z^{r+3})a^p, (z^{p-3})a^{-r} \right]$$

This completes the proof. □

**Proposition 7.** $\text{spread}_u(\Lambda_{(-3,4,r)}(a,z)) = r + 2$ for $r \geq 5$.

Proof. Using K1, K2, K3 and Lemmas 4 and 2 we obtain

$$\Lambda_{(-3,4,r)} = -\Lambda_{(-3,4,r-2)} + z\Lambda_{(-3,4,r-1)} + za^{-(r-1)}\Lambda_{(-3,4)}$$

$$= (z^2 - 1)\Lambda_{(-3,0,r-2)} - (z^3 - 2z)\Lambda_{(-3,1,r-2)}$$

$$- \{ za^{-3} + z^2a^{-2} + (z^3 - z)a^{-1} \} \Lambda_{(-3,2,r-2)}$$

$$- (z^3 - z)\Lambda_{(-3,0,r-1)} + (z^4 - 2z^2)\Lambda_{(-3,1,r-1)}$$

$$+ \{ z^2a^{-3} + z^3a^{-2} + (z^4 - z^2)a^{-1} \} \Lambda_{(-3,3,r-2)} + za^{-(r-2)}$$
\[ (z^2 - 1) \left( (z^{r-2})a^3, (z^3)a^{-(r-2)} \right) - (z^3 - 2z) \left( (z^{-1})a^3, (z^2)a^{-(r-2)} \right) - \left\{ \begin{array}{l} (z^{-1}a - 1 + z^{-1}a^{-1}) \\
(a, (z^{-1})a^{-(r-1)}) \end{array} \right\} \begin{array}{l} (if \ r = 5) \\
(if \ r = 6) \end{array} \\
- (z^{-3} - z) \left( (z^{-1})a^3, (z^3)a^{-(r-1)} \right) + (z^4 - 2z^2) \left( (z^r)a^3, (z^2)a^{-(r-1)} \right) + \left\{ \begin{array}{l} a \\
((z^{-5})a, (z)a^{-(r-5)}) \end{array} \right\} \begin{array}{l} (if \ r = 5) \\
(if \ r \geq 6) \end{array} \]

This completes the proof. \( \square \)

**Proposition 8.** spread,\( (\Lambda_{(-p,4,r)}(a, z)) = p + r \) for \( p \geq 5 \) and \( r \geq 5 \).

**Proof.**

\[
\Lambda_{(-p,4,r)} = -\Lambda_{(-p,4,r-2)} + z\Lambda_{(-p,4,r-1)} + za^{-(r-1)}\Lambda_{(-p,4)}
\]

\[
-\left\{ \begin{array}{l} -\Lambda_{(-p,2,r-2)} + z\Lambda_{(-p,2,r-3)} + za^{-3}\Lambda_{(-p,r-2)} \\
+ z\Lambda_{(-p,2,r-1)} + za^{-3}\Lambda_{(-p,r-1)} + za^{-(r-3)}\Lambda_{(-p,r)} \end{array} \right\} \begin{array}{l} (if \ p \geq r) \\
(p = r - 1) \end{array} \\
- z^a^{-3} \left\{ \begin{array}{l} z^{-1}a - 1 + z^{-1}a^{-1} \\
(a, (z^{-1})a^{-(r-1)}) \end{array} \right\} \begin{array}{l} (if \ p = r - 2) \\
(p = r - 3) \end{array} \\
+ \left\{ \begin{array}{l} (z)ap^{r-1}, (z^r)a^{-(r-2)} \\
(z^{p-r})a^{-1} \end{array} \right\} \begin{array}{l} (if \ p = 5) \\
(p = 6) \end{array} \\
= \left\{ \begin{array}{l} (z^{r+4})a^p, (z)a^{-r} \\
((z^{r+4})a^p, (z^{-(p-4)})a^{-r}) \end{array} \right\} \begin{array}{l} (if \ p = 5) \\
(p \geq 6) \end{array},
\]

where \( c = 2 \) if \( r = 5 \) and \( c = 1 \) if \( r \geq 6 \). This completes the proof. \( \square \)

**7. Proofs of main results and comments**

Theorem 3 is proved by Proposition 2. Table 1 shows that the upper bound ‘\( c(K) - 1 \)’ for the arc index in Theorem 3 is best possible but not so sharp. It also shows that the lower bound ‘\( \text{spread}_a(F_K) + 2 \)’ in Theorem 3 is best possible but not so sharp either.

\( ^1\)The Dowker-Thistlethwaite name. See [10].
Table 1. Examples of Theorem 5

| Pretzel knot $K$ | DT Name | spread$_s(F_K) + 2$ | arc index | $c(K) - 1$ |
|-----------------|---------|---------------------|-----------|-----------|
| $P(-2, 3, 3)$   | 8n3     | 6                   | 7         | 7         |
| $P(-2, 3, 5)$   | 10n21   | 6                   | 8         | 9         |
| $P(-2, 3, 7)$   | 12n242  | 9                   | 9         | 11        |
| $P(-2, 5, 5)$   | 12n725  | 10                  | 10        | 11        |

The proof of Theorem 6 is a combination of Propositions 3 and 5. The proof of Theorem 7 is a combination of Propositions 3 and 6. The proof of Theorem 8 is a combination of Propositions 4 and 8. The proof of Theorem 9 is a combination of Propositions 4 and 7. Table 2 shows that the upper bound $c(K) - 2$ and the lower bound $c(K) - 3$ for the arc index in Theorem 9 are both best possible.

Table 2. Examples of Theorem 9

| Pretzel knot $K$ | DT Name | $c(K) - 3$ | arc index | $c(K) - 2$ |
|-----------------|---------|-----------|-----------|-----------|
| $P(-3, 4, 5)$   | 12n475  | 9         | 10        | 10        |
| $P(-3, 4, 7)$   | 14n12205| 11        | 11        | 12        |

References

[1] Y. Bae and C. Y. Park, An upper bound of arc index of links, Math. Proc. Camb. Phil. Soc. 120 (2000) 491–500.
[2] E. Beltrami and P. R. Cromwell, Minimal arc-presentations of some nonalternating knots,Topology and its Applications. 81 (1997) 137–145.
[3] H. Brunn, Über verknotete Kurven, Mathematiker-Kongresses Zurich 1897, Leipzig (1898) 256–259.
[4] P. R. Cromwell, Embedding knots and links in an open book I: Basic properties, Topology Appl. 64 (1995) 37–58.
[5] P. R. Cromwell and Ian J. Nutt, Embedding knots and links in an open book II. Bounds on arc index, Math. Proc. Camb. Phil. Soc. 119 (1996), 309–319.
[6] G. T. Jin and W. K. Park, Prime knots with arc index up to 11 and an upper bound of arc index for nonalternating knots, J. Knot Theory Ramifications. 19(12) (2010) 1655–1672.
[7] W. B. R. Lickorish and M. B. Thistlethwaite, Some links with non-trivial polynomials and their crossing-numbers, Comment. Math. Helvetici 63 (1988) 527–539.
[8] H. R. Morton and E. Beltrami, Arc index and the Kauffman polynomial, Math. Proc. Camb. Phil. Soc. 123 (1998), 41–48.
[9] Knotscape, http://www.math.utk.edu/~morwen/knotscape.html
[10] Table of Knot Invariants, http://www.indiana.edu/~knotinfo/