One-dimensional Schrödinger equation with non-analytic potential \( V(x) = -g^2 \exp (-|x|) \) and its exact Bessel-function solvability

Ryu Sasaki\(^1,3\) and Miloslav Znojil\(^2\)

\(^1\)Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
\(^2\)Nuclear Physics Institute ASCR, Hlavní 130, 250 68 Rež, Czech Republic

E-mail: ryu@yukawa.kyoto-u.ac.jp and znojil@ujf.cas.cz

Received 24 May 2016, revised 7 September 2016
Accepted for publication 12 September 2016
Published 10 October 2016

Abstract

Exact solvability (ES) of one-dimensional quantum potentials \( V(x) \) is a vague concept. We propose that beyond its most conventional range the ES status should also be attributed to many less common interaction models for which the wave functions remain piecewise proportional to special functions. The claim is supported by constructive analysis of a toy model \( V(x) = -g^2 \exp (-|x|) \). The detailed description of the related bound-state and scattering solutions of the Schrödinger equation is provided in terms of Bessel functions which are properly matched in the origin.

Keywords: non-analytic potentials, bound states, reflection and transmission, exactly solvable, orthogonality theorems, associated Hamiltonians, supersymmetry

1. Introduction

In the current literature on quantum mechanics an almost disproportionate amount of attention is being attracted by the so called exactly solvable (ES) models of dynamics. After a more detailed inspection the concept gets split into several subcategories. The main divide may be spotted between the ES property of the piecewise constant ‘point’ interactions (cf., e.g. \([1]\)) and the ES property of a family of polynomially solvable analytic potentials (cf., e.g. one of their lists in \([2]\) and the recent developments of the \textit{exceptional} and the multi-indexed \textit{orthogonal polynomials} in \([3\text{–}6]\)).

The resulting point-or-analytic classification of the ES models is not too satisfactory, for several reasons. In what follows we intend to emphasize that one of these reasons is that the respective construction techniques (i.e. the matching of wave functions in the case of point
interactions and the closed-form constructions of wave functions in the case of analytic interactions) are independent and could be combined. Thus, in place of the point-or-analytic divide one encounters a non-empty third domain of the possible coexistence of the two approaches.

In the language of mathematics the new, intermediate ES domain will inherit some weaknesses of the former, elementary-function ES domain which admits non-polynomial wave functions and in which the related determination of spectra is partially numerical. Only the latter, analytic-solvability domain remains, strictly speaking, non-numerical and characterized not only by the polynomiality of the wave functions but also by the presence of various inherent symmetries in combination with a comparatively low degree of flexibility—pars pro toto let us mention the frequent ‘shape invariance’ property and/or the supersymmetry-representing relations between the different analytic ES potentials [2]. In contrast, the essence of the former, point-interaction approach to the one-dimensional Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E \psi(x), \quad x \in (-\infty, \infty)$$

may be seen in a suitable ad hoc split of the real line of coordinates $x$ into subintervals,

$$(-\infty, \infty) = (-\infty, a_1) \bigcup (a_1, a_2) \bigcup \ldots \bigcup (a_K, \infty).$$

Inside these subintervals the potential function $V(x)$ is chosen to be constant (cf., e.g. the square-well model Nr. 26 on p 52 in [7] with $K = 4$ or several other models of this type in [8]) or at least sufficiently elementary—cf., e.g. the $K = 2$ Coulomb + square-well model Nr. 28 on p 64 in [8] or the most recent symmetrized Morse-potential short-range-interaction one-dimensional model of [9] with $K = 1$ and $a_1 = 0$.

In practice, the piecewise-nonconstant potentials $V(x)$ are rarely treated as ES (cf., e.g. [10] or the model Nr. 28 in [8] once more). Even in the above-mentioned methodical studies [9, 10] it has been revealed that from the purely numerical point of view the traditional ‘shooting and bracketing’ algorithm seems more efficient than the direct use of the available (viz. confluent hypergeometric function) piecewise analytic and asymptotically correct representations of the individual bound-state wave functions $\psi_n(x) \in L^2(\mathbb{R})$.

Recently, a fairly persuasive counterargument against the resulting scepticism has been provided by Ishkhanyan [11]. In a way which we now perceive as disproving the discouraging experience of [9, 10] he took into consideration a new $K = 1$ model with $V(x) \sim 1/\sqrt{|x|}$ (and, in our present language, with $a_1 = 0$), he constructed its asymptotically correct wave functions $\psi_n(x) \in L^2(\mathbb{R})$ (again, in a closed form derived from the confluent hypergeometric functions) and, finally, he succeeded in showing that the matching condition (in the origin) appears numerically extremely friendly.

Ishkhanyan’s conclusions may be perceived as a gem of a new paradigm, recommending an immediate extension of the standard ES domain of the use of matching techniques which were, up to now, virtually exclusively applied just to the piecewise constant potentials. In parallel, Ishkhanyan’s experience may also be perceived as a decisive encouragement of transition to the infinite-series versions of special functions. Indeed, a transition to non-truncated, non-polynomial confluent hypergeometric wave functions is also one of the key characteristics of his ‘screened-Coulomb’ model with $V(x) \sim 1/\sqrt{|x|}$. See also the most recent [12] in this context.

In a certain sense these innovative observations may be perceived as weakening the well known exclusivity of the confluent hypergeometric functions in elementary quantum mechanics (well sampled by their applications in supersymmetric quantum mechanics [2]). Indeed, the essence of their popularity may be traced back to their ability of termination
to Laguerre polynomials). Naturally, there exist several other special functions without such a degeneracy option. In the new paradigmatic framework, their possible use in solving equation (1) should certainly be thoroughly reconsidered at present, therefore.

We shall recall here the Bessel functions as one of the most natural illustrative examples. A serendipitous additional merit of these functions is that they might provide the piecewise analytic general solutions of equation (1) in the case of various piecewise exponential potentials such that \( V(x) = a_j \exp \beta_j x \) for \( x \in (a_j, a_{j+1}) \). In what follows we shall simplify the problem (choosing \( K=1 \) and \( a_1 = 0 \)) and we shall bring a number of arguments supporting the effective ES status of the model.

2. Bound states

2.1. Problem setting

Our one-dimensional Schrödinger equation (1) will be considered with potential

\[
V(x) = -g^2 \exp(-|x|), \quad x \in (-\infty, \infty), \quad g > 0.
\]

The model may admit, first of all, the discrete and non-degenerate and finitely many bound-state energies \( E_m = -\kappa_m^2 \) at \( m = 0, 1, \ldots, M \). See the fourth paragraph in appendix in this connection. The corresponding eigenfunctions must be normalizable, \( \psi_m(x) \in L^2(\mathbb{R}) \). Since the potential is parity invariant, \( V(-x) = V(x) \), the eigenfunctions are also parity invariant,

\[
\psi_m(-x) = (-1)^m \psi_m(x).
\]

According to the conventional oscillation theorems [13] the subscript \( m \) counts the nodes of the eigenfunctions. Moreover, we may only consider the (say, positive) half-axis of \( x > 0 \), with

\[
\text{even parity: } \psi_{2m}''(0) = 0, \quad \text{odd parity: } \psi_{2m+1}''(0) = 0,
\]

i.e. with the eigenfunctions constrained by the parity-dependent boundary condition in the origin.

2.2. Eigenfunctions

Let us introduce an auxiliary function \( \rho(x) \),

\[
\rho(x) \overset{\text{def}}{=} 2g e^{-|x|/2}, \quad \frac{d\rho(x)}{dx} = \text{sign}(-x) \frac{\rho(x)}{2},
\]

which maps \([0, \infty)\) and \((-\infty, 0]\) to \((0, 2g]\). With this function, the Schrödinger equation for the discrete spectrum \( E = -\kappa^2 \) is rewritten as the equation for Bessel functions,

\[
\psi(x) = \phi(\rho(x)) \Rightarrow \frac{d^2\phi(\rho)}{d\rho^2} + \frac{1}{\rho} \frac{d\phi(\rho)}{d\rho} + \left(1 - \frac{4\kappa^2}{\rho^2}\right) \phi(\rho) = 0.
\]

The general solutions are obtained as linear combinations of two types of Bessel functions. In the MAPLE-inspired notation [14] we may use either the negative-sign ansatz,

\[
\psi(x) = A J(-2\kappa, \rho(x)) + B Y(-2\kappa, \rho(x)), \quad A, B \in \mathbb{R},
\]

or its positive-sign alternative,

\[
\psi(x) = A J(2\kappa, \rho(x)) + B Y(2\kappa, \rho(x)), \quad A, B \in \mathbb{R}.
\]
These solutions are to be constrained by the appropriate asymptotic boundary conditions and by the matching of their logarithmic derivatives in the origin.

### 2.3. Eigenvalues

Let us discuss the positive order solutions (8) in some detail. First let us note that, for $\kappa \in \mathbb{R}, \rho$, function $J(2\kappa, \rho)$ is non-singular at $\rho = 0$ or $x = \infty$:

$$J(2\kappa, \rho) \simeq \left(\frac{\rho}{2}\right)^{2\kappa} \times O(1), \quad \rho \simeq 0,$$

$$J(2\kappa, 2e^{-x/2}) \simeq g^{2\kappa}e^{-\kappa x} \times O(1), \quad x \to +\infty,$$  

whereas functions $J(-2\kappa, \rho)$ and $Y(\pm 2\kappa, \rho)$ are singular at $\rho = 0$ or $x = \infty$. Thus, wave functions satisfying the matching conditions (4) in the origin can be easily found. Since $x = 0$ corresponds to a regular point $\rho = 2g$ of the Bessel functions we have, for the even wave functions,

$$\psi^{(e)}(x) = A^{(e)}(\kappa, g)J(2\kappa, \rho(x)) + B^{(e)}(\kappa, g)Y(2\kappa, \rho(x)), \quad (10)$$

$$A^{(e)}(\kappa, g) \overset{\text{def}}{=} -Y'(2\kappa, 2g) = -\left( Y(2\kappa - 1, 2g) - \frac{\kappa}{g}Y(2\kappa, 2g) \right), \quad (11)$$

$$B^{(e)}(\kappa, g) \overset{\text{def}}{=} J'(2\kappa, 2g) = J(2\kappa - 1, 2g) - \frac{\kappa}{g}J(2\kappa, 2g). \quad (12)$$

In deriving the second expressions for $A^{(e)}$ and $B^{(e)}$, the following two relations for the general Bessel functions $Z(\nu, \rho)$ were employed,

$$\frac{dZ(\nu, \rho)}{d\rho} = \frac{1}{2}(Z(\nu - 1, \rho) - Z(\nu + 1, \rho)),$$

$$Z(\nu - 1, \rho) + Z(\nu + 1, \rho) = \frac{2\nu}{\rho}Z(\nu, \rho). \quad (13)$$

For the odd wave functions the result is equally straightforward,

$$\psi^{(o)}(x) = A^{(o)}(\kappa, g)J(2\kappa, \rho(x)) + B^{(o)}(\kappa, g)Y(2\kappa, \rho(x)), \quad (14)$$

$$A^{(o)}(\kappa, g) \overset{\text{def}}{=} -Y(2\kappa, 2g), \quad (15)$$

$$B^{(o)}(\kappa, g) \overset{\text{def}}{=} J(2\kappa, 2g). \quad (16)$$

Thus, one obtains the eigenfunctions whenever the coefficient of the singular term vanishes,

$$\psi_{2n}(x) = J(2\kappa_{2n}, \rho(x)) = \begin{cases} J(2\kappa_{2n}, 2ge^{-x/2}) & x \geq 0, \\ J(2\kappa_{2n}, 2ge^{x/2}) & x \leq 0, \end{cases} \quad E_{2n} = -\kappa_{2n}^2, \quad n = 0, 1, \ldots, \quad (17)$$

$$\psi_{2n+1}(x) = \text{sign}(x)J(2\kappa_{2n+1}, \rho(x)) = \begin{cases} J(2\kappa_{2n+1}, 2ge^{-x/2}) & x \geq 0, \\ -J(2\kappa_{2n+1}, 2ge^{x/2}) & x \leq 0, \end{cases} \quad (18)$$

$$E_{2n+1} = -\kappa_{2n+1}^2, \quad n = 0, 1, \ldots \quad (19)$$
and whenever our solutions are matched, properly, in the origin,
even: \( J'(2\kappa_{2n}, 2g) = 0 = J(2\kappa_{2n} - 1, 2g) - \frac{\kappa_{2n}}{g} J(2\kappa_{2n}, 2g), \quad n = 0, 1, \ldots \), (20)
odd: \( J(2\kappa_{2n+1}, 2g) = 0, \quad n = 0, 1, \ldots \) (21)
The latter relations may be perceived as implicit definitions of the eigenvalues,
\[-g^2 < E_0 < E_1 < E_2 < \cdots < E_M < 0 \iff g > \kappa_0 > \kappa_1 > \kappa_2 > \cdots > \kappa_M > 0.\] (22)

Finally, from the estimates (9) of the behavior of our Bessel functions \( J(2\kappa, \rho) \) near \( \rho = 0 \)
we may deduce that our eigenfunctions have the correct physical asymptotics,
\[
\psi_{2n}(x) \simeq \text{const.} \begin{cases} 
  e^{-\sqrt{-E_{2n}}x} & x \to +\infty, \\
  e^{\sqrt{-E_{2n}}x} & x \to -\infty.
\end{cases}
\] (23)
\[
\psi_{2n+1}(x) \simeq \text{const.} \begin{cases} 
  e^{-\sqrt{-E_{2n+1}}x} & x \to +\infty, \\
  -e^{\sqrt{-E_{2n+1}}x} & x \to -\infty.
\end{cases}
\] (24)

In this manner we have established that our ‘non-analytic exponential well potential’ (2) is exactly solvable. Moreover, the compact form of the solutions enables us to verify easily the mutual orthogonality between the even and odd eigenfunctions. The evaluation of the normalization coefficients from relations
\[
\int_0^\infty J(2\kappa_{2n}, 2ge^{-x/2})J(2\kappa_{2m}, 2ge^{-x/2})dx \propto \delta_{n,m},
\] (25)
\[
\int_0^\infty J(2\kappa_{2n+1}, 2ge^{-x/2})J(2\kappa_{2m+1}, 2ge^{-x/2})dx \propto \delta_{n,m},
\] (26)
is left to the reader.

### 3. Scattering problem

In the light of the results of the preceding section one may ask whether our model is also exactly solvable at the positive energies \( E = k^2 \), i.e. in the continuous-spectrum dynamical regime of scattering. In the one-dimensional version of this regime the experimentally testable information is carried by the reflection and transmission amplitudes (cf figure 1).
In order to determine the reflection amplitude \( r(k) \) and the transmission amplitude \( t(k) \), let us solve Schrödinger equation \( \hat{H} \psi_k(x) = k^2 \psi_k(x) \) with positive \( k \in \mathbb{R}_{>0} \) and with the following boundary conditions:

\[
\psi_k(x) \approx \begin{cases} 
  e^{ikx} & x \to +\infty \\
  A(k)e^{ikx} + B(k)e^{-ikx} & x \to -\infty 
\end{cases}
\]  

(27)

\[
r(k) = \frac{B(k)}{A(k)}, \quad t(k) = \frac{1}{A(k)}.
\]  

(28)

For \( \rho \to 0 \), the asymptotics of the relevant Bessel functions are

\[
\rho \to 0, \quad J(2ik, \rho) \simeq \left( \frac{\rho}{2} \right)^{2ik}, \quad J(-2ik, \rho) \simeq \left( \frac{\rho}{2} \right)^{-2ik}.
\]  

(29)

This leads to the following asymptotics in variable \( x \),

\[
x \to +\infty, \quad J(2ik, \rho(x)) \simeq g^{2ik}e^{-ikx}, \quad J(-2ik, \rho(x)) \simeq g^{-2ik}e^{ikx},
\]  

(30)

\[
x \to -\infty, \quad J(2ik, \rho(x)) \simeq g^{2ik}e^{ikx}, \quad J(-2ik, \rho(x)) \simeq g^{-2ik}e^{-ikx}.
\]  

(31)

According to (27) we have to connect

\[
x \leq 0: \quad A(k)g^{2ik}J(2ik, \rho(x)) + B(k)g^{2ik}J(-2ik, \rho(x)) \]  

(32)

\[
x \geq 0: \quad g^{2ik}J(-2ik, \rho(x)).
\]  

(33)

at \( x = 0 \). That is, these functions have to be equal at \( x = 0 \), together with the first derivatives. We obtain

\[
A(k)g^{-2ik}J(2ik, 2g) + B(k)g^{2ik}J(-2ik, 2g) = g^{2ik}J(-2ik, 2g),
\]  

(34)

\[
A(k)g^{-2ik}J'(2ik, 2g) + B(k)g^{2ik}J'(-2ik, 2g) = -g^{2ik}J'(-2ik, 2g).
\]  

(35)

This set of two linear equations can be readily solved,

\[
\begin{pmatrix} A(k) \\ B(k) \end{pmatrix} = \frac{1}{W} \begin{pmatrix} g^{2ik}J'(2ik, 2g) & -g^{2ik}J(-2ik, 2g) \\ g^{-2ik}J'(2ik, 2g) & -g^{-2ik}J(-2ik, 2g) \end{pmatrix} \begin{pmatrix} 2g^{4ik}J'(-2ik, 2g)J(-2ik, 2g) \\ -g^{2ik}J'(-2ik, 2g) \end{pmatrix}
\]  

(36)

\[
= \frac{1}{W} \begin{pmatrix} g^{4ik}J'(-2ik, 2g)J(-2ik, 2g) + J'(-2ik, 2g)J(2ik, 2g) \\ -g^{2ik}J'(-2ik, 2g) \end{pmatrix}
\]  

(37)

In this formula we abbreviated

\[
W \equiv J(2ik, 2g)J'(2ik, 2g) - J'(2ik, 2g)J(-2ik, 2g) = i\sinh(2\pi k)/(\pi g) = i|W|
\]  

(38)

and used one of the Lommel’s relations (section 3.12 equation (2) of [15])

\[
J(\nu, \rho)J'(-\nu, \rho) - J'(\nu, \rho)J(-\nu, \rho) = -2\sin(\nu\pi)/(\pi \rho).
\]  

(39)

As long as the second argument \( 2g \) of our Bessel functions is always the same we may omit it and shorten our formulae, \( J(\pm 2ik, 2g) \to J(\pm 2ik) \). This yields the amplitudes in compact form,
\[ r(k) = \frac{B(k)}{A(k)} = -\frac{(J'(2ik)J(-2ik) + J(2ik)J'(-2ik))}{2J'(-2ik)J(-2ik)} \times g^{-4ik}, \]  
(40)

\[ t(k) = \frac{1}{A(k)} = \frac{W}{2J'(-2ik)J(-2ik)} \times g^{-4ik}. \]  
(41)

For \( k > 0 \) and \( g > 0 \), it is straightforward to verify that these scattering amplitudes satisfy the unitarity relations

\[ r(k)t(k)^* + r(k)^*t(k) = 0, \quad |r(k)|^2 + |t(k)|^2 = 1, \]  
(42)

in which the star \(^*\) marks complex conjugation.

The proof proceeds as follows. Firstly, since \( J'(2ik)^* = J'(-2ik) \) and \( J'(2ik)^* = J'(-2ik) \) we may conclude that \( J'(2ik)J(-2ik) + J(2ik)J'(-2ik) \) is real whereas \( W \) in (38) is purely imaginary. The orthogonality \( r(k)t(k)^* + r(k)^*t(k) = 0 \) also holds. Thus, we obtain

\[ |r(k)|^2 + |t(k)|^2 = \frac{1}{4J'(-2ik)J(-2ik)J'(2ik)J(2ik)} \times \left( |J'(2ik)J(-2ik) + J(2ik)J'(-2ik)|^2 + |W|^2 \right). \]  
(43)

In other words we have to show that

\[ (J'(2ik)J(-2ik) + J(2ik)J'(-2ik))^2 + |W|^2 = 4J'(2ik)J(2ik)J'(-2ik)J(-2ik). \]  
(44)

Fortunately, this is rather simple due to (38),

\[ \text{lhs} - \text{rhs} = (J'(2ik)J(-2ik) - J(2ik)J'(-2ik))^2 + |W|^2 = W^2 + |W|^2 = 0. \]  
(45)

In conclusion let us notice that in our model the zeros of \( A(k) \) of equation (37) (continued to the positive imaginary axis, \( k \to ik, \kappa > 0 \)) correspond, as they should, to the discrete eigenvalues of the system. The demonstration is again easy: from formula

\[ k \to ik \quad J'(2ik, 2g)J(-2ik, 2g) \to J'(2\kappa, 2g)J(2\kappa, 2g), \]  
(46)

we see that the zeros of the first factor \( J'(2\kappa, 2g) \) correspond to the even eigenstates (20) while those of the second factor \( J(2\kappa, 2g) \) correspond to the odd eigenstates (21).

4. Associated Hamiltonians

4.1. The Crum’s sequence

According to Crum [16], to a one-dimensional Hamiltonian \( \mathcal{H} = \mathcal{H}[0] \) with the eigensystems \( \{E_n, \psi_n(x)\}, n = 0, 1, \ldots \), a sequence of iso-spectral Hamiltonian systems \( \mathcal{H}[L] L = 1, 2, \ldots \), is associated:

\[ \mathcal{H}[L]_n^L \psi_n^L(x) = E_n^L \psi_n^L(x), \quad n = L, L + 1, \ldots, \]  
(47)

\[ \mathcal{H}[L] \equiv \mathcal{H}[0] - 2\partial_x^2 \log |W| \left[ \psi_0, \psi_1, \ldots, \psi_{L-1} \right](x), \]  
(48)

\[ \psi_n^L(x) \equiv \frac{W \left[ \psi_0, \psi_1, \ldots, \psi_{L-1} \right] \psi_n(x)}{W \left[ \psi_0, \psi_1, \ldots, \psi_{L-1} \right]} \]  
(49)
in which the Wronskian of \( n \)-functions \( \{f_1, \ldots, f_n\} \) is defined by formula

\[
W[f_1, \ldots, f_n](x) \equiv \det \left( \frac{d^{i-1}f_k}{dx^{i-1}} \right)_{1 \leq i, k \leq n}.
\] (50)

This result is obtained from a multiple application of the Darboux transformations [17]. One only has to adopt the lowest eigenfunctions as the seed solutions \( \{\psi_0, \psi_1, \ldots\} \). The following properties of the Wronskians are instrumental:

\[
W[\psi_0, \psi_1, \ldots, \psi_n](x) = W[J(2\kappa_0, \rho), J(2\kappa_n, \rho)](\rho),
\]

\[
W[\psi_0, \psi_1, \psi_n](x) = \left( \frac{\rho}{2} \right)^{2+n} \cdot W[J(2\kappa_0, \rho), J(2\kappa_1, \rho), J(2\kappa_n, \rho)](\rho),
\]

\[
W[\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n](x) = \left( \frac{\rho}{2} \right)^{L(L+1)/2} \cdot W[J(2\kappa_0, \rho), J(2\kappa_1, \rho), \ldots, J(2\kappa_{n-1}, \rho), J(2\kappa_n, \rho)](\rho).
\] (54)

By using the properties of the Wronskians (51)–(52), we can reduce the Wronskians of \( \{\psi_n(x)\} \) of equation (49) to the Wronskians of the Bessel functions of the first kind, \( J(2\kappa_n, \rho) \). For example, we obtain:

\[
W[\psi_0, \psi_1](x) = (\text{sign}(-x))^{1+n} \cdot W[J(2\kappa_0, \rho), J(2\kappa_1, \rho)](\rho),
\]

\[
W[\psi_0, \psi_1, \psi_n](x) = (\text{sign}(-x))^{2+n} \left( \frac{\rho}{2} \right)^3 
\cdot W[J(2\kappa_0, \rho), J(2\kappa_1, \rho), J(2\kappa_n, \rho)](\rho),
\]

\[
W[\psi_0, \psi_1, \ldots, \psi_{n-1}, \psi_n](x) = (\text{sign}(-x))^{L+1} \left( \frac{\rho}{2} \right)^{L(L+1)/2} \cdot W[J(2\kappa_0, \rho), J(2\kappa_1, \rho), \ldots, J(2\kappa_{n-1}, \rho), J(2\kappa_n, \rho)](\rho).
\]

The next-to-elementary structure of these relations indicates that the further systematic study of the \( L \)th associated Hamiltonians remains unexpectedly friendly and transparent. In particular, we believe that the constructive solution of the related scattering problem is so straightforward that it can be left to the reader as an exercise.

4.2. The breakdown of the shape invariance

Let us consider the associated Hamiltonian systems \( \mathcal{H}^{[L]} \) with \( L = 1, 2, \ldots \) corresponding to the ‘non-analytic exponential well’ (2). Obviously they are all exactly solvable. It is easy to see that the systems are parity invariant:

\[
V^{[L]}(x) \equiv V(x) - 2\alpha^2 \log |W[\psi_0, \psi_1, \ldots, \psi_{L-1}](x)|, \quad V^{[L]}(-x) = V^{[L]}(x),
\]

\[
\psi^{[L]}_n(-x) = (-1)^{L+n} \psi^{[L]}_n(x).
\] (56)

By construction all the eigenfunctions \( \{\psi_n(x)\} \) and their first derivatives \( \{\psi'_n(x)\} \) are continuous. By using the Schrödinger equation, the second derivatives in the Wronskians (49) are replaced by

\[
\psi''_n(x) \rightarrow (V(x) + \kappa_n^2) \psi_n(x),
\]

in which \( V(x) \psi_n(x) \) is canceled by the multi-linearity of the determinant. This applies to all the even order derivatives and the Wronskian (49) contains the eigenfunctions and their first derivatives only. Thus the eigenfunctions \( \{\psi^{[L]}_n(x)\} \) of the associated systems and their first derivatives are continuous.
Because of the parity, the orthogonality relations among the even and odd eigenfunctions are trivial and those even–even and odd–odd
\[ \delta_{n,m} \propto \langle \psi^{[L]}_n | \psi^{[L]}_m \rangle = \int_{-\infty}^{\infty} \psi^{[L]}_n(x) \psi^{[L]}_m(x) dx \] (58)
can be rewritten as those on the positive x-axis
\[ \delta_{n,m} \propto \int_0^{\infty} \psi^{[L]}_{2n}(x) \psi^{[L]}_{2m}(x) dx, \] (59)
\[ \delta_{n,m} \propto \int_0^{\infty} \psi^{[L]}_{2n+1}(x) \psi^{[L]}_{2m+1}(x) dx. \] (60)

It is straightforward to evaluate
\[ -2 \partial_x^2 \log \psi_0(x) = \frac{\rho(x)^2}{2} - 2\kappa_0^2 + \frac{\rho(x)^2}{2} \times \left( \frac{J'(2\kappa_0, \rho(x))}{J(2\kappa_0, \rho(x))} \right)^2, \] (61)
which gives the potential of the first associated Hamiltonian \( \mathcal{H}^{[1]} \)
\[ V^{[1]}(x) \overset{\text{def}}{=} V(x) - 2\partial_x^2 \log \psi_0(x) = \frac{\rho(x)^2}{4} - 2\kappa_0^2 + \frac{\rho(x)^2}{2} \times \left( \frac{J'(2\kappa_0, \rho(x))}{J(2\kappa_0, \rho(x))} \right)^2. \] (62)
If the potential of \( V^{[1]}(x) \) (62) has the form (up to an additive constant),
\[ -f(g)^2 e^{-|x|}, \]
with a certain function \( f(g) \) of the parameter \( g \), the system is shape invariant, which is a sufficient condition for exact solvability [18]. This simply cannot happen, since \( \frac{J'(2\kappa_0, \rho(x))}{J(2\kappa_0, \rho(x))} \) can never be a constant. By construction, the potential of \( \mathcal{H}^{[1]} \) is a non-analytic function of \( x \) due to the \( \rho(x) = 2g e^{-|x|^2/2} \) dependence. This gives a negative answer to the question of whether the solvability is due to the shape invariance.

5. Discussion

One can consult the textbook by Watson [15] to confirm that the zeros of Bessel functions \( J(\nu, x) \) (i.e. in the notation of the book, of \( J_\nu(x) \)) as functions of subscript \( \nu \) for fixed \( x \) were not discussed in the book. A few comments on this problem were relocated to a mathematical appendix. In it, theorems 1 and 2 could be interesting, per se, as they could be very special cases of much more general theorems for complex \( x \).

Another feature of our present results which is worth emphasizing is that our one-dimensional model strongly resembles the spherical version \( V(r) = -g^2 e^{-r/\alpha} \) of the same potential in three or more dimensions. In the s-wave sector the related radial Schrödinger equation is known to reduce to the Bessel equation. From our present point of view this recipe picks up, exclusively, just the ‘odd’ solutions of our present paper. Moreover, the well known fact is that there is a minimal value of the interaction strength \( g \) which yields at least one discrete eigenstate in the s-wave setting. Indeed, one has the constraint \( g^2 a^2 \geq j_{01}/4 \), in which \( j_{01} \) is the minimal positive zero of \( J_0(x) \). \( j_{01} \approx 2.405 \). After transition to the full real line of coordinates and to the even solutions, the surprising news is that, in a way proved in the appendix, at least one discrete even-parity eigenstate always exists for any nonvanishing coupling \( g \).
We may summarize that our present model is exactly solvable not only in the bound-state dynamical regime but also in the continuous spectrum sector. We succeeded in obtaining the reflection and transmission amplitudes and in demonstrating, constructively, that they satisfy the expected unitarity relations. These constructions become more interesting when we analytically continue and reveal that in spite of the violation of the analyticity of the potential \(2\) itself (in \(x\)), the poles of the scattering amplitudes (in the complex planes of energies or momenta) correspond to the discrete spectrum.

Last but not least, in sharp contrast to the s-wave predecessors of our present constructions the one-dimensional form of the present problem also enabled us to recall Crum’s results and to point out that the existence of the sequence of the associated Hamiltonian systems should be perceived as a highly nontrivial result with, perhaps, multiple future consequences. *Pars pro toto* we emphasized here the non-shape invariance of this hierarchy of models. In this connection, let us remind the reader of another class of non-shape invariant and exactly solvable potentials, that is, the reflectionless potentials of \([19, 20]\).

**Acknowledgments**

The MZ’s participation in the project was supported by the Institutional Research Plan RVO61389005 and by GAČR Grant Nr. 16-22945S.

**Note added.** After completing the paper, we became aware of a remark in \([21]\)

Investigations about the zeros \(\nu_n\) of \(J_{\nu}(z)\) regarded as a function of \(\nu\), with fixed \(z\) have been carried out by Coulomb (1936) \([22]\). They show that for positive real values of \(z\), the \(\nu_n\) are real and simple and asymptotically near to negative integers.

Coulomb’s results were derived based on an indefinite integration formula of Bessel functions in Watson’s textbook \([15]\) (formula (13) on page 135, section 5.11)

\[
\int_0^z C_{\mu}(kz)C_{\nu}(kz)\, dz = \frac{kz [C_{\mu+1}(kz)C_{\nu}(kz) - C_{\mu}(kz)C_{\nu+1}(kz)]}{\mu^2 - \nu^2} + \frac{C_{\mu}(kz)C_{\nu}(kz)}{\mu + \nu},
\]

in which \(C_{\mu}\) and \(C_{\nu}\) are two arbitrary cylinder functions. It is easy to derive theorem 1 for the odd part (66) from the above formula by putting \(k = 1\), \(C_{\mu}(z) \to J_{\mu}(z)\), \(C_{\nu}(z) \to J_{\nu}(z)\) and integrating on \([0, z]\). The even part of theorem 1 (67) can also be obtained in the same manner by noting, in terms of (13), \(J_{\mu}’(z) = 0 \Rightarrow J_{\mu-1}(z) = J_{\mu+1}(z) = \frac{z}{\mu^2}J_{\mu}(z)\). As is clear from the above derivation, theorem 1 is valid for complex \(z\), \(\text{Re}(\mu) > 0\), \(\text{Re}(\nu) > 0\). However, for complex \(z\), the oscillation theorem is not obvious and the validity of the interlacing relations (64), (65) is unclear.

Let us emphasize, however, that in our case theorem 1 is not just definite integral formulas, but the orthogonality relation representing the exact solvability of potential (2), which in turn leads to higher orthogonality relations, theorem 2 via Crum’s sequence. It is an interesting challenge to see if theorem 2 is also valid for complex \(z\).

We found another paper \([23]\) discussing the \(\nu\) zeros of \(J_{-\nu}(x)\).
Appendix. Orthogonality theorems

The orthogonality theorems for the eigenfunctions of the original ‘non-analytic exponential well’ (25)–(26) and for the associated systems (59)–(60) can be stated as simple theorems for Bessel functions.

In this appendix we shall temporarily return to the textbook, subscripted denotation $J_n(x)$ of the present Bessel functions $J_n(x)$. Next, having fixed a positive number $x$, we shall consider the Bessel functions of the first kind $J_n(x)$ and $J'_n(x) \equiv \frac{d}{dx}J_n(x)$ as functions of the order $\nu$. This leads to the following corollary: for fixed $x$, there are finitely many $(M + 1)$ positive zeros of our interest $\{\nu_n\} = \{\lambda_0, \lambda_1, \ldots, \lambda_M\}$,

$$J'_n(x) : x > \lambda_0 > \lambda_1 > \cdots > \lambda_M > 0, \quad J'_n(x) = 0,$$
$$J_n(x) : x > \mu_0 > \mu_1 > \cdots > \mu_M > 0, \quad J_n(x) = 0.$$  

The oscillation theorem for the potential (2) tells us that these nodal zeros are interlaced, i.e.

$$x > \lambda_0 > \mu_0 > \lambda_1 > \mu_1 > \cdots > \mu_{M-1} > \lambda_M > 0,$$  

or

$$x > \lambda_0 > \mu_0 > \lambda_1 > \mu_1 > \cdots > \mu_{N-1} > \lambda_N > \mu_N > 0.$$  

This means that the number of the positive zeros of $J_n(x)$ is equal or smaller than its partner from $J'_n(x)$, i.e. $M = N$ or $M = N - 1$.

In particular, one notices a slightly counterintuitive fact that a nodeless even-parity ground state with $2 \kappa_0 = \nu$ always exists, irrespectively of the size of the strength $g = z/2$ of the attraction. An independent proof of this observation is elementary: the corresponding wave function $J_{\nu}(z)$ may be approximated by its first two terms,

$$J_{\nu}(z) \approx \left(\frac{z}{2}\right)^\nu/\Gamma(\nu + 1) - \left(\frac{z}{2}\right)^{\nu+2}/\Gamma(\nu + 2) + \cdots$$

so that the matching condition of equation (20) yields

$$z^2 = \frac{4\nu(\nu + 1)}{\nu + 2} \approx 2\nu + \ldots.$$  

Next, by the change of variables $x \rightarrow \rho$, the orthogonality relations between all of the eigenfunctions (25)–(26) of the original system can be stated as follows.

Theorem 1

$$\text{even : } \int_0^x J_{\lambda_j}(\rho)J_{\lambda_k}(\rho) \frac{d\rho}{\rho} = 0, \quad j = k,$$
$$\text{odd : } \int_0^x J_{\mu_j}(\rho)J_{\mu_k}(\rho) \frac{d\rho}{\rho} = 0, \quad j = k.$$  

Let us denote these two types of zeros by one consecutive sequence $\{\nu_j\}$:

$$\nu_0 \equiv \lambda_0, \nu_1 \equiv \mu_0, \nu_2 \equiv \lambda_1, \nu_3 \equiv \mu_1, \ldots.$$  

Likewise, the orthogonality relations of the eigenfunctions (59)–(60) of the $L$th associated Hamiltonian system can be stated as
Theorem 2

\[
\begin{align*}
\text{even : } & \int_0^\infty \frac{W[J_{\nu_1}(\mu), \ldots, J_{\nu_m}(\mu)]W[J_{\sigma_1}(\mu), \ldots, J_{\sigma_s}(\mu)]}{(W[J_{\nu_1}(\mu), \ldots, J_{\nu_m}(\mu)]^2)\rho^{2s-1}} d\rho = 0, \quad n = m, \\
\text{odd : } & \int_0^\infty \frac{W[J_{\nu_1}(\mu), \ldots, J_{\nu_m}(\mu)]W[J_{\sigma_1}(\mu), \ldots, J_{\sigma_s}(\mu)]}{(W[J_{\nu_1}(\mu), \ldots, J_{\nu_m}(\mu)]^2)\rho^{2s-1}} d\rho = 0, \quad n = m.
\end{align*}
\]

(68)

(69)

Theorem 1 is the special case \((L = 0)\) of theorem 2.

References

[1] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 Solvable Models in Quantum Mechanics (New York: Springer)
[2] Cooper F, Khare A and Sakhate M U 1995 Supersymmetry and quantum mechanics Phys. Rep. 251 267–388
[3] Gómez-Ullate D, Kamran N and Milson R 2010 An extension of Bochner’s problem: exceptional invariant subspaces J. Approx. Theory 162 987–1006
Gómez-Ullate D, Kamran N and Milson R 2009 An extended class of orthogonal polynomials defined by a Sturm–Liouville problem J. Math. Anal. Appl. 359 352–67
[4] Quesne C 2008 Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry J. Phys. A: Math. Theor. 41 392001
[5] Odake S and Sasaki R 2009 Infinitely many shape invariant potentials and new orthogonal polynomials Phys. Lett. B 679 414–7
[6] Odake S and Sasaki R 2011 Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials Phys. Lett. B 702 164–70
[7] Flügge S 1971 Practical Quantum Mechanics I (Berlin: Springer)
[8] Constantinescu F and Magyari E 1971 Problems in Quantum Mechanics (Oxford: Pergamon) ch II
[9] Znojil M 2016 Morse potential, symmetric morse potential and bracketed bound-state energies Mod. Phys. Lett. A 31 1650088
[10] Znojil M Non-analytic exponential well \(V(x) = -g^2 \exp(-|x|)\) and an innovated, analytic shooting method arXiv:1605.06780 [quant-ph]
[11] Ishkhanyan A M 2015 Exact solution of the Schrödinger equation for the inverse square root potential \(V(x) = \sqrt{x}/x\) Eur. Phys. Lett. 112 10006
[12] Znojil M Symmetrized exponential oscillator Mod. Phys. Lett. A to appear arXiv:1609.00166 [quant-ph]
[13] Hille E 1976 Ordinary Differential equations in the Complex Domain (New York: Wiley)
[14] Maple (math software) (https://maplesoft.com/products/maple/)
[15] Watson G N 1922 A Treatise on the Theory of Bessel Functions (Cambridge: Cambridge University Press)
[16] Crum M M 1955 Associated Sturm–Liouville systems Quart. J. Math. Oxford Ser. 6 121–7
[17] Darboux G 1882 Sur une proposition relative aux équations linéaires C. R. Acad. Paris 94 1456–9
[18] Gendenshtein L E 1983 Derivation of exact spectra of the Schrödinger equation by means of supersymmetry JETP Lett. 38 356–9
[19] Kay I and Moses H M 1956 Reflectionless transmission through dielectrics and scattering potentials J. Appl. Phys. 27 1503–8
[20] Sasaki R 2014 Exactly solvable potentials with finitely many discrete eigenvalues of arbitrary choice J. Math. Phys. 55 062101 14pp
[21] Erdélyi A et al 1953 Bessel Functions Higher Transcendental Functions vol 2 (New York: McGraw-Hill) ch 7, p 60
[22] Coulomb M J 1936 Sur les zéros de fonctions de Bessel considérées comme fonctions de l’ordre Bull. Sci. Math. 60 297–302
[23] Conde S and Kalla S L 1979 The \(\nu\)-Zeros of \(J_\nu(x)\) Math. Computation 33 423–6