Nonradial solutions of nonlinear scalar field equations

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Abstract
We prove new results concerning the nonlinear scalar field equation

\[-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \quad N \geq 3,
\]
\[u \in H^1(\mathbb{R}^N) \]

with a nonlinearity \(g\) satisfying the general assumptions due to Berestycki and Lions. In particular, we find at least one nonradial solution for any \(N \geq 4\) minimizing the energy functional on the Pohozaev constraint in a subspace of \(H^1(\mathbb{R}^N)\) consisting of nonradial functions. If in addition \(N \neq 5\), then there are infinitely many nonradial solutions. These solutions are sign-changing. The results give a positive answer to a question posed by Berestycki and Lions in [5, 6]. Moreover, we build a critical point theory on a topological manifold, which enables us to solve the above equation as well as to treat new elliptic problems.

Keywords: nonlinear scalar field equations, critical point theory, nonradial solutions, concentration compactness, profile decomposition, Pohozaev manifold

Mathematics Subject Classification numbers: 35J20, 58E05.

1. Introduction

We investigate the nonlinear scalar field equation

\[-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, \quad N \geq 3,
\]
\[u \in H^1(\mathbb{R}^N) \quad (1.1) \]
under the following general assumptions introduced by Berestycki and Lions in their fundamental papers [5, 6]:

(g0) \( g : \mathbb{R} \to \mathbb{R} \) is continuous and odd,

(g1) \( -\infty < \lim \inf_{s \to 0} g(s)/s \leq \lim \sup_{s \to 0} g(s)/s = -m < 0 \),

(g2) \( \lim \sup_{s \to \infty} g(s)/s^{2^* - 1} = 0 \), where \( 2^* = \frac{2N}{N-2} \),

(g3) There exists \( \xi_0 > 0 \) such that \( G(\xi_0) > 0 \), where

\[
G(s) = \int_0^s g(t) \, dt \quad \text{for} \quad s \in \mathbb{R}.
\]

The problem appears e.g. in nonlinear optics or in the study of Bose–Einstein condensates [8, 11] and (1.1) describes the propagation of solitons which are special nontrivial solitary wave solutions \( \Phi(x, t) = u(x)e^{-i\omega t} \) of the time-dependent Schrödinger equation. The general conditions (g0)–(g3) can model a wide range of nonlinear phenomena, e.g. the Kerr effect, the dual power law nonlinearity or the saturation effect arising in nonlinear optics. The parameter \( m \) is usually interpreted as a mass [5, 6].

Recall that the existence of a least energy solution \( u \in H^1(\mathbb{R}^N) \), which is positive, spherically symmetric (radial) and decreasing in \( r = |x| \) is established in [5] and the existence of infinitely many radial solutions but not necessarily positive are provided in [6]. Moreover Jeanjean and Tanaka [16] showed that \( J(u) = \inf_{M} J \), where \( M \) stands for the Pohozaev manifold defined as follows

\[
\mathcal{M} := \left\{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 2^* \int_{\mathbb{R}^N} G(u) \, dx \right\},
\]

(1.2)

and \( J : H^1(\mathbb{R}^N) \to \mathbb{R} \) is the energy functional given by

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx.
\]

(1.3)

Recall that \( \mathcal{M} \) contains all nontrivial critical points [5].

Firstly, the aim of this paper is to answer to the problem [6][section 10.8] concerning the existence and multiplicity of nonradial solutions to (1.1) for dimensions \( N \geq 4 \) under the almost optimal assumptions (g0)–(g3). The problem in \( N = 3 \) remains open. Secondly, we present a new variational approach based on a critical point theory built on the Pohozaev manifold. Since \( g \) is only continuous, this manifold need not be of class \( \mathcal{C}^{1,1} \) and it seems to be difficult to use a standard critical point theory directly on this constraint. Without the radial symmetry one has to deal with the issues of lack of compactness and we present a concentration-compactness approach in the spirit of Lions [18, 19] together with profile decompositions in the spirit of Gérard [12] and Nawa [25] adapted to a general nonlinearity satisfying (g0)–(g3); see theorem 1.4. Using these techniques we provide a new proof of the following result.

**Theorem 1.1** ([5, 16]). There is a solution \( u \in \mathcal{M} \) to (1.1) such that \( J(u) = \inf_{M} J > 0 \).

Moreover we find nonradial solutions to (1.1) provided that \( N \geq 4 \). Indeed, let us fix \( \tau \in \mathcal{O}(N) \) such that \( \tau(x_1, x_2, x_3) = (x_2, x_1, x_3) \) for \( x_1, x_2 \in \mathbb{R}^M \) and \( x_3 \in \mathbb{R}^{N-2M} \), where \( x = (x_1, x_2, x_3) \in \mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M} \) and \( 2 \leq M \leq N/2 \). We define

\[
X_{\tau} := \left\{ u \in H^1(\mathbb{R}^N) : u(x) = -u(\tau x) \quad \text{for all} \quad x \in \mathbb{R}^N \right\}.
\]

(1.4)
Clearly, if \( u \in X_r \) is radial, i.e. \( u(x) = u(r x) \) for any \( r \in \mathcal{O}(N) \), then \( u = 0 \). Hence \( X_r \) does not contain nontrivial radial functions. Then \( \mathcal{O}_1 := \mathcal{O}(M) \times \mathcal{O}(M) \times \text{id} \subset \mathcal{O}(N) \) acts isometrically on \( H^1(\mathbb{R}^N) \) and let \( H^1_{\mathcal{O}_1}(\mathbb{R}^N) \) denote the subspace of invariant functions with respect to \( \mathcal{O}_1 \).

Our first main result reads as follows.

**Theorem 1.2.** Fix \( N \geq 4 \) and \( 2 \leq M \leq N/2 \). Then there is a solution \( u \in \mathcal{M} \cap X_r \cap H^1_{\mathcal{O}_1}(\mathbb{R}^N) \) to (1.1) such that

\[
J(u) = \inf_{\mathcal{M} \cap X_r \cap H^1_{\mathcal{O}_1}(\mathbb{R}^N)} J \geq 2 \inf_{\mathcal{M}} J .
\]  

If in addition \( N = 5 \), then we may assume that \( N - 2M \neq 1 \) and let us consider \( \mathcal{O}_2 := \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(N - 2M) \subset \mathcal{O}(N) \) acting isometrically on \( H^1(\mathbb{R}^N) \) with the subspace of invariant function denoted by \( H^1_{\mathcal{O}_2}(\mathbb{R}^N) \).

**Theorem 1.3.** Fix \( N \geq 4 \), \( 2 \leq M \leq N/2 \) such that \( N - 2M \neq 1 \). Then the following statements hold.

(a) There is a solution \( u \in \mathcal{M} \cap X_r \cap H^1_{\mathcal{O}_2}(\mathbb{R}^N) \) to (1.1) such that

\[
J(u) = \inf_{\mathcal{M} \cap X_r \cap H^1_{\mathcal{O}_2}(\mathbb{R}^N)} J \geq \inf_{\mathcal{M} \cap X_r \cap H^1_{\mathcal{O}_2}(\mathbb{R}^N)} J .
\]

(b) There is an infinite sequence of distinct solutions \((u_n) \subset \mathcal{M} \cap X_r \cap H^1_{\mathcal{O}_2}(\mathbb{R}^N) \) to (1.1).

Note that the associated energy functional \( J \) is given by (1.3) is of class \( C^1 \) and has the mountain pass geometry, i.e. \( J \) is positive and bounded away from 0 on a sphere centred at the origin and with a sufficiently small radius \( r > 0 \) and there is \( \nu \) such that \( J(\nu) < 0 \) and \( \|\nu\| > r \), see [16] for details. Our problem is modelled in \( \mathbb{R}^N \), so that we have to deal with the lack of compactness of Palais–Smale sequences, i.e. sequences \((u_n)\) such that \( J(u_n) \to c > 0 \) and \( J'(u_n) \to 0 \) as \( n \to \infty \). In the classical approach [5, 6] the compactness properties can be obtained by considering only radial functions \( H^1_{\mathcal{O}_1}(\mathbb{R}^N) \) in the spirit of Strauss [29] due to \( \mathcal{O}(N) \)-invariance of \( J \). In a nonradial case, however, for instance in \( H^1(\mathbb{R}^N) \), \( X_r \cap H^1_{\mathcal{O}_1}(\mathbb{R}^N) \) or in \( X_r \cap H^1_{\mathcal{O}_2}(\mathbb{R}^N) \), the crucial radial lemma [5][Lemma A.II] is no longer available and an application of the compactness lemma of Strauss [5][Lemma A.I] is impossible. As usual, one needs to analyse the lack of compactness of Palais–Smale sequences by means of a concentration-compactness argument of Lions [19]. The main difficulty concerning the concentration-compactness analysis is that, in general, \( g(s) \) has not subcritical growth of order \( \chi^{-1} \) for large \( s \) with \( 2 < \chi < 2^* \) and \( g \) does not satisfy an Ambrosetti–Rabinowitz-type condition [1], or any monotonicity assumption, which guarantee the boundedness of Palais–Smale sequences [1, 32]. In the present paper we show how to deal with the difficulties with lack of compactness having the general nonlinearity \( g \) satisfying \((g0)\)--\((g3)\) and our argument requires a deep analysis of profiles of bounded sequences in \( H^1(\mathbb{R}^N) \); see theorem 1.4 below.

Besides the lack of compactness difficulties, it is not clear how to treat (1.1) by means of the standard variational methods. Although \( J \) has the classical mountain pass geometry [16], we do not know whether Palais–Smale sequences of \( J \) are bounded. To overcome this difficulty in the radial case in [5, 6], the authors considered the following constrained problems: the minimization of \( u \mapsto \int_{\mathbb{R}^N} |\nabla u|^2 \) on

\[
\left\{ u \in H^1_{\mathcal{O}_1}(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) \ dx = 1 \right\}
\]
and a critical point theory of the functional \( u \mapsto \int_{\mathbb{R}^N} G(u) \, dx \) on
\[
\left\{ u \in H^{1,1}_C(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 1 \right\}.
\]
Both approaches require the compactness properties and the scaling invariance of the equation (1.1) with application of Lagrange multipliers. Another method in the radial case in [15] is based on the mountain pass theorem for an extended functional in the spirit of Jeanjean [14]. Let us mention that a direct minimization method on the Pohozaev manifold in \( H^{1,1}_C(\mathbb{R}^N) \) is due to Shatah [28], who studied a nonlinear Klein–Gordon equation with a general nonlinearity. Again, the radial symmetry and the Strauss lemma played an important role in these works.

In this paper we provide a new constrained approach which allows to deal with noncompact problems and can be described in an abstract and transparent way for future applications; see section 2 for details. Let us briefly sketch our approach. Recall that if \( u \in H^1(\mathbb{R}^N) \) is a critical point of \( J \), then \( u \in W^{1,q}_0(\mathbb{R}^N) \) for any \( q < \infty \) and \( u \) satisfies the Pohozaev identity, i.e. \( M(u) = 0 \), where
\[
M(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2^* \int_{\mathbb{R}^N} G(u) \, dx.
\]
Observe that \( M : H^1(\mathbb{R}^N) \to \mathbb{R} \) is of class \( C^1 \) and but, in general, \( M' \) is not locally Lipschitz and \( \mathcal{M} = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} : M(u) = 0 \} \) need not be of class \( C^{1,1} \). Hence it seems to be impossible to use any critical point theory based on the deformation lemma involving a Cauchy problem directly on \( \mathcal{M} \), e.g. as in [33, section 5].

Our crucial observation is that \( \mathcal{M} \) is a topological manifold and there is a homeomorphism \( m_U : \mathcal{U} \to \mathcal{M} \) such that
\[
\mathcal{U} := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 \, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} G(u) \, dx > 0 \right\} \tag{1.7}
\]
is a manifold of class \( C^{1,1} \). Moreover \( J \circ m_U : \mathcal{U} \to \mathbb{R} \) is still of class \( C^1 \) and \( u \in \mathcal{U} \) is a critical point of \( J \circ m_U \) if and only if \( m_U(u) \) is a critical point of the unconstrained functional \( J \). The main difficulty is the fact that it is not clear whether a Palais–Smale sequence \( (u_n) \subset \mathcal{U} \) of \( J \circ m_U \) can be mapped into a Palais–Smale sequence \( m_U(u_n) \subset \mathcal{M} \) of the unconstrained functional \( J \). Moreover, we do not know if a nontrivial weak limit point of \( (m_U(u_n)) \) is a critical point of \( J \) and stays in \( \mathcal{M} \).

In order to overcome these obstacles we introduce a new variant of the Palais–Smale condition at level \( \beta \in \mathbb{R} \) denoted by \( (M)^{\beta}_{\mathcal{U}}(i) \) (see section 2), which roughly says that, for every Palais–Smale sequence \( (u_n) \subset \mathcal{U} \) at level \( \beta \), \( (m_U(u_n)) \) contains a subsequence converging weakly towards a point \( u \in H^1(\mathbb{R}^N) \) up to the \( \mathbb{R}^N \)-translations, which can be projected on a critical point \( m_U(u) \subset \mathcal{M} \). Moreover we may choose a proper sequence of \( \mathbb{R}^N \)-translations such that
\[
J(m_U(u)) = J(\tilde{x}_n) \leq \beta = \lim_{n \to \infty} J(m_U(u_n)).
\]
The selection of the proper translation plays a crucial role and requires the following profile decompositions of bounded sequences in \( H^1(\mathbb{R}^N) \) in the spirit of [10, 12, 13].

**Theorem 1.4.** Suppose that \( (u_n) \subset H^1(\mathbb{R}^N) \) is bounded. Then there are sequences \( (\tilde{u}_i)_{n=0}^{\infty} \subset H^1(\mathbb{R}^N), (\gamma^+_n)_{n=0}^{\infty} \subset \mathbb{R}^N \) for any \( n \geq 1 \), such that \( \gamma^+_n = 0, |\gamma^+_n - \gamma^+_i| \to \infty \) as \( n \to \infty \) for \( i \neq j \), and
passing to a subsequence, the following conditions hold for any $i \geq 0$:

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \sum_{j=0}^i \int_{\mathbb{R}^N} |\nabla \tilde{u}_j|^2 \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v'_n|^2 \, dx,
$$

(1.8)

where $v'_n := u_n - \sum_{j=0}^i \tilde{u}_j$ and

$$
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n) \, dx = \sum_{j=0}^i \int_{\mathbb{R}^N} \Psi(\tilde{u}_j) \, dx + \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(v'_n) \, dx
$$

(1.9)

for any function $\Psi : \mathbb{R} \to [0, \infty)$ of class $C^1$ such that $\Psi'(s) \leq C(|s| + |s|^{2^* - 1})$ for any $s \in \mathbb{R}$ and some constant $C > 0$. Moreover, if in addition $\Psi$ satisfies

$$
\lim_{s \to 0} \frac{\Psi(s)}{|s|^2} = \lim_{|s| \to \infty} \frac{\Psi(s)}{|s|^{2^*}} = 0,
$$

(1.10)

then

$$
\lim_{i \to \infty} \left( \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(v'_n) \, dx \right) = 0.
$$

(1.11)

In particular, taking $\Psi(s) = |s|^p$ with $p = 2$ and with $2 < p < 2^*$ we obtain [13][proposition 2.1]. Our argument relies only on new variants of Lions lemma [33][lemma 1.21]; see section 3 with lemma 3.1 and variants of theorem 1.4 in $H^1_0(\mathbb{R}^N)$ as well as in $H^1_0(\mathbb{R}^N)$.

Having a minimizing sequence of $J \circ m_{\ell}$, we find a proper translation such that a weak limit point can be projected on a critical point of $J$ in $\mathcal{M}$ and we prove theorem 1.1. The same procedure works in the subspace $X_\ell \cap H^1_0(\mathbb{R}^N) \subset H^1(\mathbb{R}^N)$, however we have to ensure that we choose a proper translation along $\mathbb{R}^{N-2M}$-variable and we get theorem 1.2.

In order to get multiplicity of critical points, we show that $J \circ m_{\ell}$ satisfies the Palais–Smale condition in $\mathcal{U} \cap X_\ell \cap H^1_0(\mathbb{R}^N)$ and in view of the critical point theorem 2.2 of section 2, $J$ has infinitely many critical points and we prove theorem 1.3.

Note that the existence and the multiplicity results concerning similar problems to (1.1) in the noncompact case present in the literature require strong growth conditions imposed on the nonlinear term, e.g. $g$ has to be of subcritical growth, i.e. $|g(s)| \leq c(|s| + |s|^{p-1})$ for some constant $c > 0$ and $2 < p < 2^*$; and, in addition, must satisfy an Ambrosetti–Rabinowitz-type condition [1, 9], or a monotonicity-type assumption [32]; see also references therein. If a nonlinear equation like (1.1) exhibits radial symmetry, then the problem of existence of nonradial solutions is particularly challenging and there are only few results in this direction. The first paper [3] due to Bartsch and Willem dealt with semilinear elliptic problems in dimension $N = 4$ and $N \geq 6$ under subcritical growth conditions and an Ambrosetti–Rabinowitz-type condition. In fact, from [3] we borrowed an idea of the decomposition of $\mathbb{R}^N$ into $\mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M}$ and $\mathcal{O}_2$-action on $H^1(\mathbb{R}^N)$ given in theorem 1.2. Further analysis of decompositions of $\mathbb{R}^N$ in this spirit has been recently studied in [21] and in the references therein. Next, Lorca and Ubilla [20] solved the similar problem in dimension $N = 5$ by considering $\mathcal{O}_1$-action on $H^1(\mathbb{R}^N)$, and recently Musso, Pacard and Wei [24] obtained nonradial solutions in any dimension $N \geq 2$; see also [2]. In these works, again, strong assumptions needed to be imposed on nonlinear terms, for
instance a nondegeneracy condition in [2, 24], which allows to apply a Liapunov-Schmidt-type reduction argument.

The paper is organized as follows. In section 2 we build a critical point theory on a general topological manifold \( M \) in the setting of abstract assumptions (A1)–(A3). Having our variants of Palais–Smale condition \((M)_\beta(i)-(ii)\), in theorem 2.2 we prove the existence of minimizers on \( M \) and the multiplicity result. The general theorem can be useful in the study of strongly indefinite problems as well, like [22, 32], where the classical linking approach due to Benci and Rabinowitz [4] does not apply and the classical Palais–Smale condition is not satisfied; see remark 2.3. Moreover, in a subsequent work [23] these techniques will be used to obtain nonradial solutions in the zero mass case problem (1.1), which has been studied in the radial case so far in [5, 7]. In section 3 we prove three variants of Lions lemma in \( H^1([\mathbb{R}^N]), H^1_{C_0}(\mathbb{R}^N) \) and in \( H^1_{C_0}(\mathbb{R}^N) \). These allow us to prove the profile decomposition theorem 1.4 and its variant corollary 3.5 in order to analyse Palais–Smale sequences in proposition 4.4 and in corollary 4.5. We complete proofs of theorems 1.1–1.3 in section 4.

Finally, we would like to point out that similarly as in [5] one can assume \((g_0), (g_1), (g_3)\) and that \( -\infty \leq \limsup_{s \to \pm \infty} g(s)/s^{2^*-1} \leq 0 \) instead of \((g_2)\). Then, in view of theorems 1.2 and 1.3 we obtain nonradial solutions as well. Indeed, we modify \( g \) in the following way. If \( g(s) \geq 0 \) for all \( s \geq \xi_0 \), then \( \tilde{g} = g \). Otherwise we set \( \xi_1 := \inf \{ \xi \geq \xi_0 : g(\xi) = 0 \} \),

\[
\tilde{g}(s) = \begin{cases} 
g(s) & \text{if } 0 \leq s \leq \xi_1, \\
0 & \text{if } s > \xi_1,
\end{cases}
\]

and \( \tilde{g}(s) = -\tilde{g}(-s) \) for \( s < 0 \). Hence \( \tilde{g} \) satisfies \((g_0)-(g_3)\), and by the strong maximum principle if \( u \in H^2(\mathbb{R}^N) \) solves \(-\Delta u = \tilde{g}(u)\), then \( |u(s)| \leq \xi_1 \) and \( u \) is a solution to (1.1).

2. Critical point theory on a topological manifold

In this section we introduce a critical point theory on a topological submanifold \( M \) of a Banach space endowed with a certain isometric group action. As we shall see we introduce variants of the Palais–Smale condition \((M)_\beta(i)\) and \((M)_\beta(ii)\) which allow to treat variational problems with the lack of compactness. The results of this section will be applied to (1.1) in section 4.

Let \( G \) be an isometric group action on a reflexive Banach space \( X \) with norm \( \| \cdot \| \) and \( J : X \to \mathbb{R} \) is a \( C^1 \)-functional. Assume that

(A1) \( J \) is \( G \)-invariant, i.e. if \( u \in X \) and \( g \in G \) then \( J(gu) = J(u) \). If \( gu = -u \), then \( u = 0 \).

Moreover if \( g_v \in G, u \in X \) and \( g_vu \to v \), then \( v = gu \) for some \( g \in G \) or \( v = 0 \).

Let \( M \subset X \setminus \{ 0 \} \) be a closed and nonempty subset of \( X \) such that

(A2) \( M \) is \( G \)-invariant and \( \inf_M J > 0 = J(0) \).

Since, in general, \( M \) has not the \( C^{1,1} \)-structure, it is difficult to build a critical point theory directly on \( M \) and deal with deformation techniques involving a Cauchy problem, e.g. as in [33, section 5]. Therefore we introduce a manifold

\[
S = \{ u \in Y : \psi(u) = 1 \}
\]

in a closed \( G \)-invariant subspace \( Y \subset X \), where \( \psi \in C^{1,1}(Y, \mathbb{R}) \) is \( G \)-invariant and such that \( \psi(u) \neq 0 \) for \( u \in S \). Clearly, from the implicit function theorem, \( S \) is a \( G \)-invariant manifold of class \( C^{1,1} \) and of codimension 1 in \( Y \) with the following tangent space at \( u \in S \)

\[
T_uS = \{ v \in Y : \psi'(u)(v) = 0 \}.
\]
Moreover we assume:

(A3) There are a $G$-invariant open neighbourhood $\mathcal{P} \subset X \setminus \{0\}$ of $\mathcal{M}$ and $G$-equivariant map $m_P : \mathcal{P} \to \mathcal{M}$ such that $m_P(u) = u$ for $u \in \mathcal{M}$ and the restriction $m_{l|U} : \mathcal{U} \to \mathcal{M}$ for $\mathcal{U} := \mathcal{S} \cap \mathcal{P}$ is a homeomorphism. Moreover $J \circ m_{l|U} = J \circ m_{l}$ is of class $C^1$ and $(J \circ m_{l|U})(u_n) \to \infty$ as $u_n \to u \in \partial \mathcal{U}$, $u_n \in \mathcal{U}$, where the boundary of $\mathcal{U}$ is taken in $\mathcal{S}$.

The above sets are described in the following diagram:

\[
\begin{array}{c}
X \supset X \setminus \{0\} \supset \mathcal{P} \\
\cup \\
Y \supset \mathcal{S} \supset \mathcal{U} = \mathcal{S} \cap \mathcal{P} \\
\end{array}
\]

As usual, we say that $(u_n) \subset \mathcal{U}$ is a $(\text{PS})_\beta$-sequence of $J \circ m_{l|U} : \mathcal{U} \to \mathbb{R}$ provided that

\[
(J \circ m_{l|U})(u_n) \to \beta \quad \text{and} \quad (J \circ m_{l|U})(u_n) \to 0.
\]

Let $K$ be the set of all critical points of $J \circ m_{l|U}$, i.e.

\[K := \{u \in \mathcal{U} : (J \circ m_{l|U})(u) = 0 \quad \text{for any } v \in T_u \mathcal{S}\}.
\]

For $u \in X$, $G^*u$ denotes the orbit of $u$

\[G^*u := \{gu : g \in G\}.
\]

$G^*u$ is called a critical orbit if $u \in K$. Clearly $G^*u \subset K$ if $u \in K$. We introduce the following variants of the Palais–Smale condition at level $\beta \in \mathbb{R}$.

$(M)_\beta$ (i) For every $(\text{PS})_\beta$-sequence $(u_n) \subset \mathcal{U}$ of $J \circ m_{l|U}$, there are a sequence $(g_n) \subset G$ and $u \in \mathcal{P}$ such that $g_n m_{l|U}(u_n) \to u$ along a subsequence, $J'(m_{l|U})(u) = 0$ and $J(m_{l|U})(u) \leq \beta$.

(ii) If the number of distinct critical orbits is finite, then there is $m_\beta > 0$ such that for every $(u_n) \subset \mathcal{U}$ such that $(J \circ m_{l|U})(u_n) \to 0$ as $n \to \infty$, $(J \circ m_{l|U})(u_n) \leq \beta$ and $\|u_n - u_{n+1}\| < m_\beta$ for $n \geq 1$, there holds $\liminf_{n \to \infty} \|u_n - u_{n+1}\| = 0$.

Note that $(M)_\beta$(i) implies that if $(J \circ m_{l|U})(u) = 0$, then $J'(m_{l|U})(u) = 0$ for $u \in \mathcal{U}$. Indeed, taking a sequence $u_n = u$, observe that $g_n m_{l|U}(u_n) \to \tilde{u}$ along a subsequence, $\tilde{u} \in \mathcal{P} \subset X \setminus \{0\}$ and by (A1), $\tilde{u} = g m_{l|U}(u)$ for some $g \in G$. Then $\tilde{u} \in \mathcal{M}$ and by (A3), $m_{l|U}(\tilde{u}) = m_{l}(g m_{l|U}(u)) = g m_{l|U}(u)$ is a critical point of $J \circ m_{l|U}$, hence by (A1), we conclude $J'(m_{l|U})(u) = 0$. Therefore critical points of $J \circ m_{l|U}$ are mapped by $m_{l|U}$ into nontrivial critical points of the unconstrained functional $J$. Observe that, however, $(m_{l}(u_n))$ need not to be a Palais–Smale sequence of the unconstrained functional $J$ if $(u_n) \subset \mathcal{U}$ is a $(\text{PS})_\beta$-sequence of $J \circ m_{l|U}$.

In what follows, for $A \subset X$ and $r > 0$, $B(A, r) := \{u \in X : \|u - v\| < r \text{ for some } v \in A\}$. Moreover, let $\Phi := J \circ m_{l|U} : \mathcal{U} \to \mathbb{R}$ and for any $a < b$ let us denote

\[
\Phi^a := \{u \in \mathcal{U} : \alpha \leq \Phi(u) \leq \beta\},
\]

\[
\Phi^b := \{u \in \mathcal{U} : \Phi(u) \leq \beta\}.
\]

Lemma 2.1. Suppose that (A1)–(A3), $(M)_\beta$ hold for some $\beta \in \mathbb{R}$ the number of distinct critical orbits is finite. If $(u_n) \subset \mathcal{U}$ is a $(\text{PS})_\alpha$-sequence for some $\alpha < \beta$ and

\[u_n \in B(K \cap \Phi^\beta, m_\beta),
\]

then passing to a subsequence $g_n u_n \to u$ for some $u \in K$ and $g_n \in G$. 

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Proof. Let \((u_n) \subset U\) be a \((\text{PS})_\beta\)-sequence such that (2.1) holds. Passing to a subsequence \(\Phi(u_n) \leq \beta\). Then we put \(w_{2n-1} := u_n\) and take any \(w_{2n} \in K\) such that
\[
\|w_{2n} - u_n\| < m_\beta
\]
and \(\Phi(w_{2n}) \leq \beta\) for any \(n \geq 1\). Take \(\bar{K} \subset K\) such that each orbit \(G * u\) has a unique representative in \(\bar{K}\) for \(u \in K\), so that \(\bar{K} \cap (G * u)\) is a singleton. Since \(\bar{K}\) is finite, passing to a subsequence we may assume that \(g_{2n}w_{2n} = u\) for some \(g_{2n} \in G\) and \(u \in \bar{K}\). Take \(g_{2n-1} = g_{2m}\) for \(n \geq 1\) and observe that by (A1), \(\Phi^j\) is \(G\)-equivariant, hence \(\Phi^j(g_{2n}u_n) \to 0\), \(\Phi(g_{2n}u_n) \leq \beta\) and
\[
\|g_{2n}u_n - g_{2n+1}w_{n+1}\| < m_\beta
\]
for \(n \geq 1\). Then, in view of \((M)_\beta\) we obtain
\[
\liminf_{n \to \infty} \|g_{2n}u_n - u\| = \liminf_{n \to \infty} \|g_{2n-1}w_{2n-1} - g_{2n}w_{2n}\| = 0,
\]
and \(g_{2n}u_n \to u \in K\) passing to a subsequence. \(\square\)

Hence, roughly speaking, lemma 2.1 says that if \((M)_\beta\) holds, then provided that \(K\) has a finite number of distinct orbits, a Palais–Smale sequence of \(\Phi = J \circ m_\iota\) which is close enough to \(K\) will contain a convergent subsequence up to the \(G\)-action.

In order to deal with the multiplicity of critical points by means of the Krasnosel'skii genus [31], we consider the following assumption:

(S) \(J\) is even, \(m_P\) is odd, \(U, M\) are symmetric, i.e. \(U = -U, M = -M\) and for any \(k \geq 1\), there exists a continuous and odd map \(\tau : S^{k-1} \to P\), where \(S^{k-1}\) is the unit sphere in \(\mathbb{R}^k\).

As we shall see later, the latter condition in (a) guarantees that the Krasnosel’skii genus of \(\Phi^j\) is sufficiently large for large \(\beta\).

Now our main result of this section reads as follows.

Theorem 2.2. Suppose that \(J : X \to \mathbb{R}\) is of class \(C^1\) and satisfies (A1)–(A3).

(a) If \((M)_\beta\) holds for \(\beta = \inf_{M} J\), then \(J\) has a critical point \(u \in M\) such that
\[
J(u) = \inf_{M} J.
\]

(b) Assume that \((M)_\beta\) hold for every \(\beta \geq \inf_{M} J\) and (S) is satisfied. Then \(J\) has infinitely many \(G\)-distinct critical points in \(M\), i.e. there is a sequence of critical points \((u_n) \subset M\) such that \((G * u_n) \cap \{G * u_m\} = \emptyset\) for \(n \neq m\).

(c) Assume that \(G = \{id\}\), (S) is satisfied and for every \((\text{PS})_\beta\)-sequence \((u_n) \subset U\) of \(J \circ m_\iota\) with \(\beta \geq \inf_{M} J\), there is \(u \in U\) such that \(J(m_\iota(u)) = 0\) and \(u_n \to u\) along a subsequence. Then \(J\) has infinitely many critical points in \(M\).

Proof. Proof of (a). Similarly as in [33, lemma 5.14] we find an odd and locally Lipschitz pseudo-gradient vector field \(v : U \setminus K \to Y\) such that \(v(u) \in T_uS\) and
\[
\|v(u)\| < 2\|\Phi'(u)\|, \quad (2.2)
\]
\[
\Phi'(u)v(u) > \|\Phi'(u)\|^2 \quad (2.3)
\]
for any \(u \in U \setminus K\). The obtained pseudo-gradient vector field allows to prove a variant of deformation lemma [33, lemma 5.15] in \(U\) and arguing as in [33, theorem 8.5], we find a minimizing sequence \((u_n) \subset U\) such that
\[
\Phi(u_n) \to c := \inf_{U \setminus K} \|J \circ m_\iota\| = \inf_M J
\]
for
and \( \Phi'(u_n) \to 0 \) as \( n \to \infty \). In view of \((M)_{\beta}(i)\) we find a nontrivial critical point \( m_P(u) \in \mathcal{M} \) of \( J \) such that passing to a subsequence \( g_n u_n \to u \) for some \( g_n \in G \). Since
\[
c \geq J(m_P(u)) \geq \inf_{\mathcal{M}} J,
\]
we get \( J(m_P(u)) = c \), which completes proof of (a).

Proof of (b) and (c) is based on the fact that the Lusternik–Schnirelman values
\[
\beta_k := \inf \{ \beta \in \mathbb{R} : \gamma(\Phi^\beta) \geq k \}. \tag{2.4}
\]
are increasing critical values of \( \Phi \) for \( k \geq 1 \), where \( \gamma \) stands for the Krasnoselskii genus for closed and symmetric subsets of \( X \), see [31, Chapter II.5] for the definition and properties of the genus. Observe that (a) implies that for any \( k \geq 1 \) there is an odd map \( \tau : S^{k-1} \to \mathcal{P} \), hence by [31, Proposition II.5.4 and Proposition II.5.2] we get \( \gamma \left( m^{-1} (m_P(\tau(S^{k-1}))) \right) \geq \gamma \left( \tau(S^{k-1}) \right) \geq \gamma \left( S^{k-1} \right) = k \). Therefore we find \( \beta > 0 \) such that
\[
\gamma(\Phi^{\beta}) \geq \gamma \left( m^{-1} (m_P(\tau(S^{k-1}))) \right) \geq k,
\]
hence \( \beta_k < \infty \). Now, if the classical Palais–Smale condition is satisfied as in (c), then one can argue as in [27, Theorem 8.10] and show that \( (\beta_k) \) is a sequence of increasing critical values and we conclude (c).

Proof of (b). Observe that we find the unique flow \( \eta : \mathcal{G} \to U \setminus \mathcal{K} \) for the pseudo-gradient vector field obtained in (a) such that
\[
\begin{align*}
\partial_t \eta(t,u) &= -v(\eta(t,u)) \\
\eta(0,u) &= u
\end{align*}
\]
where \( \mathcal{G} := \{(t,u) \in (0,\infty) \times (U \setminus \mathcal{K}) : \ t < T(u) \} \) and \( T(u) \) is the maximal time of the existence of \( \eta(t,u) \). Suppose that the number of distinct critical orbits is finite. We will show that in fact \( (\beta_k) \) is a sequence of increasing critical values and this contradiction will complete the proof. Take \( \beta \geq c \) and let
\[
\mathcal{K}^\beta := \{ u \in \mathcal{K} : \Phi(u) = \beta \}.
\]

**Claim 1.** There is \( \varepsilon_0 > 0 \) such that
\[
\mathcal{K} \cap \Phi_{\beta-\varepsilon_0} = \mathcal{K}^\beta. \tag{2.5}
\]
Indeed, observe that each orbit \( G+u \) for \( u \in \mathcal{K} \) corresponds to a single value of \( \Phi \) and since the number of distinct critical orbits is finite, \( \mathcal{K}^\beta \) is empty except for finitely many values \( \beta \).

**Claim 2.** For every \( \delta \in (0, m_{\beta+\varepsilon_0}) \) there is \( \varepsilon \in (0, \varepsilon_0) \) such that
\[
\lim_{t \to T(u)} \Phi(\eta(t,u)) < \beta - \varepsilon \quad \text{for} \quad u \in \Phi_{\beta-\varepsilon_0} \setminus B(\mathcal{K}^\beta, \delta), \tag{2.6}
\]
which means that if \( u \) is away from \( \mathcal{K}^\beta \) at a level not greater than \( \beta + \varepsilon \), then the flow \( \eta \) brings \( u \) below the level \( \beta \). This will be used to define the entrance time map below; see the conclusion. Take \( u \in \Phi_{\beta-\varepsilon_0} \setminus \mathcal{K}^\beta \) and observe that by (A2), \( \Phi(\eta(t,u)) = J(m_P(\eta(t,u))) \) is bounded from below by \( c \), and by (2.3) it is decreasing in \( t \in [0, T(u)] \). Hence \( \lim_{t \to T(u)} \Phi(\eta(t,u)) \) exists. Suppose that (2.6) does not hold, i.e. there is \( \delta \in (0, m_{\beta+\varepsilon_0}) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \)
\[
A_\varepsilon := \left\{ u \in \Phi_{\beta-\varepsilon_0} \setminus B(\mathcal{K}^\beta, \delta) : \lim_{t \to T(u)} \Phi(\eta(t,u)) \geq \beta - \varepsilon \right\} \neq \emptyset.
\]
We will now show that for any $u \in A_{r_0}$
\[
\lim_{t \to T(u) \forall v \in K^\beta} \inf_{v(t, u) - v} = \lim_{t \to T(u)} \text{dist} \{ \eta(t, u), K^\beta \} = 0. \tag{2.7}
\]
Let us start to show that for $u \in A_{r_0}$, $\lim_{t \to T(u)} \eta(t, u)$ exists. Suppose that, on the contrary, there is $0 < \eta_0 < m_{\beta + r_0}$ and there is an increasing sequence $(t_n) \subset [0, T(u))$ such that $t_n \to T(u)$ and
\[
\eta_0 < \| \eta(t_{n+1}, u) - \eta(t_n, u) \| < m_{\beta + r_0},
\]
for $n \geq 1$. Note that by (2.2) and (2.3)
\[
\eta_0 < \| \eta(t_{n+1}, u) - \eta(t_n, u) \| \leq \int_{t_n}^{t_{n+1}} \| v(\eta(s, u)) \| \, ds \leq 2 \int_{t_n}^{t_{n+1}} \| \Phi'(\eta(s, u)) \| \, ds
\]
\[
\leq 2 \sqrt{t_{n+1} - t_n} \left( \int_{t_n}^{t_{n+1}} \Phi'(\eta(s, u)) (v(\eta(s, u))) \, ds \right)^{1/2}
\]
\[
= 2 \sqrt{t_{n+1} - t_n} (\Phi(\eta(t_n, u)) - \Phi(\eta(t_{n+1}, u)))^{1/2}
\]
\[
\leq 2 \sqrt{t_{n+1} - t_n} (\beta + \varepsilon)^{1/2}.
\]
Hence $|t_{n+1} - t_n| \geq \frac{\eta_0^2}{16(\beta + \varepsilon)}$ and $T(u) = \infty$. Again, by (2.3)
\[
\int_{t_n}^{t_{n+1}} \| \Phi'(\eta(s, u)) \|^2 \, ds \leq \left( \Phi(\eta(t_n, u)) - \Phi(\eta(t_{n+1}, u)) \right) \to 0 \tag{2.9}
\]
as $n \to \infty$. Then, for every $n$, we find $t'_n \in [t_n, t_{n+1}]$ such that $\Phi'(\eta(t'_n, u)) \to 0$ and by (2.8) we may assume that $\eta_0 < \| \eta(t'_n, u) - \eta(t'_{n+1}, u) \| < m_{\beta + r_0}$ for $n \geq 1$. Therefore we get a contradiction with (M) $\beta + r_0$ (ii). Hence $u_0 = \lim_{t \to T(u)} \eta(t, u)$ exists and since $J(\eta(t, u)) \leq J(u)$ is bounded as $t \to T(u)$, by (A3) we get $u_0 \notin \partial J$. From the definition of $T(u)$, we infer that $u_0 \in K$. Moreover, by (2.5)
\[
u_0 \in K \cap \Phi_{\beta + r_0}^\beta = K^\beta,
\]
which completes the proof of (2.7).

Now observe that in view of (2.7), for $u \in A_{r_0}$ we may define
\[
\tau_0(u) := \inf \big\{ t \in [0, T(u)) : \eta(s, u) \in B(K^\beta, m_{\beta + r_0}) \quad \text{for all } s > t \big\}
\]
\[
\tau(u) := \inf \big\{ t \in [\tau_0(u), T(u)) : \eta(t, u) \in B(K^\beta, (\delta / 2)) \big\}
\]
Note that $0 \leq \tau_0(u) < \tau(u) < T(u)$ and we show that
\[
\inf_{u \in A_{r_0}} \tau(u) - \tau_0(u) \geq \frac{\delta^2}{16(\beta + \varepsilon_0)}. \tag{2.10}
\]
Indeed, if $u \in A_{r_0}$, then by (2.2) and (2.3) we have
\[
\delta \leq \| \eta(\tau_0(u), u) - \eta(\tau(u), u) \| \leq \int_{\tau_0(u)}^{\tau(u)} \| v(\eta(s, u)) \| \, ds \leq 2 \int_{\tau_0(u)}^{\tau(u)} \| \Phi'(\eta(s, u)) \| \, ds
\]
\[
\leq 2 \int_{t_0(u)}^{t(u)} \left( \Phi'(\eta(s, u))(v(\eta(s, u))) \right)^{1/2} ds \\
\leq 2 \sqrt{t(u) - t_0(u)} \left( \int_{t_0(u)}^{t(u)} \Phi'(\eta(s, u))(v(\eta(s, u))) ds \right)^{1/2} \\
= 2 \sqrt{t(u) - t_0(u)} (\Phi(\eta(t_0(u)u) - \Phi(\eta(t(u), u)))^{1/2} \\
\leq 2 \sqrt{t(u) - t_0(u)} (\beta + \varepsilon) \quad \text{and we get (2.10).}
\]

Note that \( A_{\varepsilon_0}/2 \subset A_{\varepsilon_0} \) and let

\[
\rho := \inf_{u \in A_{\varepsilon_0}/2} \int_{t_0(u)}^{t(u)} \| \Phi'(\eta(s, u)) \|^2 ds.
\]

If \( \rho = 0 \) then by (2.10) we find \( u_n \in A_{\varepsilon_0}/2 \) and \( t_n \in (t_0(u_n), t(u_n)) \) such that

\[
\Phi'(\eta(t_n, u_n)) \to 0 \quad \text{as } n \to \infty.
\]

Since \( t_n > t_0(u_n) \) we have \( \eta(t_n, u_n) \in B(\mathcal{K}^\beta, \delta_0) \) and passing to a subsequence

\[
\Phi'(\eta(t_n, u_n)) \to \alpha \leq \beta + \varepsilon_0/2 < \beta + \varepsilon.
\]

In view of lemma 2.1, passing to a subsequence, we obtain

\[
g_n \eta(t_n, u_n) \to u
\]

for some \( u \in \mathcal{K} \) and \( g_n \in G \). By (2.5) we get \( u \in \mathcal{K}^\beta \). On the other hand, since \( t_n < t(u_n) \) we obtain

\[
g_n \eta(t_n, u_n) \notin B(\mathcal{K}^\beta, \delta/2),
\]

which is a contradiction. Therefore \( \rho > 0 \) and we take

\[
\varepsilon < \min \left\{ \frac{1}{2} \varepsilon_0, \frac{1}{4} \rho \right\}.
\]

Let \( u \in A_\varepsilon \subset A_{\varepsilon_0}/2 \) and since

\[
\Phi'(\eta(t(u), u)) - \Phi'(\eta(t_0(u), u)) = - \int_{t_0(u)}^{t(u)} \Phi'(\eta(s, u))(v(\eta(s, u))) ds \\
\leq - \frac{1}{2} \int_{t_0(u)}^{t(u)} \| \Phi'(\eta(s, u)) \|^2 ds,
\]

we obtain

\[
\beta - \varepsilon \leq \lim_{t \to t(u)} \Phi'(\eta(t, u)) \leq \Phi'(\eta(t_0(u), u)) \\
\leq \beta + \varepsilon - \frac{1}{2} \int_{t_0(u)}^{t(u)} \| \Phi'(\eta(s, u)) \|^2 ds \leq \beta + \varepsilon - \frac{1}{2} \rho \\
< \beta - \varepsilon.
\]
which gives again a contradiction. Thus we have finally proved that (2.6) holds.

**Conclusion.** Now take any \( \delta < m_{\beta_{k+1}} \) such that

\[
\gamma(\text{cl} \mathcal{B}(K^\beta, \delta)) = \gamma(K^\beta).
\]

Let us define the entrance time map \( e : \Phi_{\beta_{k+1}}^{\beta+\varepsilon} \setminus B(K^\beta, \delta) \to [0, \infty) \) such that

\[
e(u) := \inf \{ t \in [0, T(u)) : \Phi(t, u) \leq \beta - \varepsilon \}.
\]

It is standard to show that \( e \) is continuous and even. Moreover we may define a continuous and odd map \( h : \Phi_{\beta_{k+1}}^{\beta+\varepsilon} \setminus B(K^\beta, \delta) \to \Phi^{\beta-\varepsilon} \) such that

\[
h(u) = \begin{cases} 
\eta(e(u), u) & \text{for } u \in \Phi_{\beta-\varepsilon} \setminus B(K^\beta, \delta), \\
u & \text{for } u \in \Phi_{\beta-\varepsilon}.
\end{cases}
\]

Let us take \( \beta = \beta_k \) defined by (2.4) for some \( k \geq 1 \). Then by [31, proposition II.5.4 (2')] and (4'))

\[
\gamma(\Phi_{\beta+\varepsilon} \setminus B(K^\beta, \delta)) \leq \gamma(\Phi^{\beta-\varepsilon}) \leq k - 1
\]

and by [31, proposition II.5.4 (3')]

\[
k \leq \gamma(\Phi_{\beta+\varepsilon}) \leq \gamma(\text{cl} \mathcal{B}(K^\beta, \delta)) + \gamma(\Phi_{\beta+\varepsilon} \setminus B(K^\beta, \delta)) \leq \gamma(K^\beta) + k - 1.
\]

Thus \( K^\beta \neq \emptyset \), and since it has finite number of orbits and (A1) holds, we easy show that there is a continuous and odd map from \( K^\beta \) with values in \( \{-1, 1\} \). Thus \( \gamma(K^\beta) = 1 \). Note that if \( \beta_k = \beta_{k+1} \) for some \( k \geq 1 \), then by (2.11) we get \( \gamma(K^{\beta_k}) - k = \gamma(\Phi^{\beta_k+\varepsilon}) - k \geq 2 \), which is a contradiction. Hence we get an infinite sequence \( \beta_1 < \beta_2 < \ldots \) of critical values, which contradicts that \( K \) consists of a finite number of distinct orbits. This completes the proof of (b).

\( \square \)

**Remark 2.3.** (a) In this paper we consider the problem (1.1) having the mountain pass geometry, hence we assume that \( Y = X = H^1(\mathbb{R}^N) \), \( \mathcal{M} \) is a Pochozay manifold in \( H^1(\mathbb{R}^N) \), and \( U \) is given by (1.7) and we show that (M)_{3(ii)} holds with translations \( G = \mathbb{R}^N \); see lemma 4.7 below. Therefore we will apply theorem 2.2(a) in proof of theorem 1.1. In order to prove theorem 1.2 we will also consider the setting of this section for \( Y = X = X_r \cap H^1_{\Omega_1}(\mathbb{R}^N) \) with \( G = \{0\} \times \{0\} \times \mathbb{R}^{N-2m} \); see lemma 4.8 below. Moreover, we apply the multiplicity result contained in theorem 2.2(c) and we show that \( J \circ m_\ell \) satisfies the Palais–Smale condition in \( U \cap X_r \cap H^1_{\Omega_1}(\mathbb{R}^N) \); see lemma 4.9.

(b) In this work, however, we do not apply (M)_{3(ii)} and theorem 2.2(b) and we would like to mention other possible applications. Note that for an indefinite functional \( J \), i.e. when \( 0 \) is not a local minimum, one needs to find a proper subspace \( Y \subset X \) such that (A3) holds and the Palais–Smale condition may not be satisfied. For instance, as in [22, 32] one can consider a generalized Nehari manifold

\[
\mathcal{M} = \{ u \in X \setminus Y' : J'(u)(u) = 0 \quad \text{and} \quad J'(u)|_{Y'} = 0 \},
\]

where \( Y' \) is an orthogonal complement of \( Y \) in \( X \). The approaches considered in these works fit into the abstract setting of this section. For instance \( X = H^1(\mathbb{R}^N) \), \( Y \) and \( Y' \) are subspaces, where the Schrödinger operator \( -\Delta + V \) is positive and negative definite respectively, \( 0 \) lies in a spectral gap of this operator and we can consider the \( \mathbb{Z}^N \)-periodic problem \( -\Delta u + V(x)u = \)

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$g(x, u)$ as in [32]. The discreteness of Palais–Smale sequences obtained in [32, lemma 2.14] implies $(M)_{\gamma}$ with $G = \mathbb{Z}^N$ and we can reprove the results of [32]. Similarly one can recover the variational approach in [22]. Hence, theorem 2.2 may be applied to Pohozhev as well as Nehari-type topological constraints.

3. Concentration compactness and profile decompositions

The following lemma is known if $\Psi(s) = |s|^p$ with $2 < p < 2^*$ and is due to Lions [33, lemma 1.21], [19]. We will use it to prove the profile decomposition theorem 1.4 with a general function $\Psi$.

**Lemma 3.1.** Suppose that $(u_n) \subset H^1(\mathbb{R}^N)$ is bounded and for some $r > 0$

$$\limsup_{n \to \infty} \int_{B(y, r)} |u_n|^2 \, dx = 0. \tag{3.1}$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_n) \, dx \to 0 \quad \text{as } n \to \infty$$

for any continuous function $\Psi : \mathbb{R} \to [0, \infty)$ such that (1.10) holds.

**Proof.** Take any $\varepsilon > 0$ and $2 < p < 2^*$ and suppose that $\Psi$ satisfies (1.10). Then we find $0 < \delta < M$ and $c_\varepsilon > 0$ such that

$$\Psi(s) \leq \varepsilon |s|^2 \quad \text{if } |s| \in [0, \delta],$$

$$\Psi(s) \leq \varepsilon |s|^{2^*} \quad \text{if } |s| > M,$$

$$\Psi(s) \leq c_\varepsilon |s|^p \quad \text{if } |s| \in (\delta, M].$$

Hence, in view of Lions lemma [33][lemma 1.21] we get

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n) \, dx \leq \varepsilon \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^2 + |u_n|^{2^*} \, dx.$$  

Since $(u_n)$ is bounded in $L^2(\mathbb{R}^N)$ and in $L^{2^*}(\mathbb{R}^N)$, we conclude by letting $\varepsilon \to 0$. \qed

Let us consider $x = (x^1, x^2, x^3) \in \mathbb{R}^N = \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{N-2M}$ with $2 \leq M \leq N/2$ such that $x^1, x^2 \in \mathbb{R}^M$ and $x^3 \in \mathbb{R}^{N-2M}$. Let $O_1 := O(M) \times O(M) \times \text{id} \subset O(N)$ and now we consider invariant functions with respect to $O_1$.

**Corollary 3.2.** Suppose that $(u_n) \subset H^1_{O_1}(\mathbb{R}^N)$ is bounded and for all $r > 0$

$$\limsup_{n \to \infty} \int_{B(y, r)} |u_n|^2 \, dx = 0. \tag{3.2}$$

Then

$$\int_{\mathbb{R}^N} \Psi(u_n) \, dx \to 0 \quad \text{as } n \to \infty$$

for any continuous function $\Psi : \mathbb{R} \to [0, \infty)$ such that (1.10) holds.
Proof. Suppose that
\[
\int_{B(0,1)} |u_n|^2 \, dx \geq c > 0
\] (3.3)
for some sequence \((y_n) \subset \mathbb{R}^N\) and a constant \(c\). Observe that in the family \(\{B(hy_n, 1)\}_{h \in \mathcal{O}_1}\) we find an increasing number of disjoint balls provided that \(|(y_n^1, y_n^2)| \to \infty\). Since \((u_n)\) is bounded in \(L^2(\mathbb{R}^N)\) and invariant with respect to \(\mathcal{O}_1\), by \((3.3)\) \(|(y_n^1, y_n^2)|\) must be bounded. Then for sufficiently large \(r \geq r_0\) one obtains
\[
\int_{B(0,0,y_n^1)} |u_n|^2 \, dx \geq \int_{B(0,1)} |u_n|^2 \, dx \geq c > 0,
\]
and we get a contradiction with \((3.2)\). Therefore \((3.1)\) is satisfied with \(r = 1\) and by lemma 3.1 we conclude.\(\Box\)

Remark 3.3. Instead of \(\mathcal{O}_1\) in corollary 3.2 one can consider any subgroup \(G = \mathcal{O}' \times \text{id} \subset \mathcal{O}(N)\) such that \(\mathcal{O}' \subset \mathcal{O}(M)\) and \(\mathbb{R}^M\) is compatible with \(\mathcal{O}'\) for some \(0 \leq M \leq N\) (in the sense of [33, definition 1.23], see [18]), i.e. if \(\lim_{y \to \infty} \in \mathbb{R}^M m(y, r) = \infty\) for some \(r > 0\), where
\[
m(y, r) := \sup \{ n \in \mathbb{N} : \text{there exist } g_1, \ldots, g_n \in \mathcal{O}' \text{ such that } B(g_i, y, r) \cap B(g_j, y, r) = \emptyset \text{ for } i \neq j\}.
\]
Now let us assume in addition that \(N - 2M \neq 1\) and
\[
\mathcal{O}_2 := \mathcal{O}(M) \times \mathcal{O}(M) \times \mathcal{O}(N - 2M) \subset \mathcal{O}(N).
\]
In view of [18], \(H^1_{\mathcal{O}_2}(\mathbb{R}^N)\) embeds compactly into \(L^p(\mathbb{R}^N)\) for \(2 < p < 2^*\). In order to deal with the general nonlinearity we need the following result.

Corollary 3.4. Suppose that \((u_n) \subset H^1_{\mathcal{O}_2}(\mathbb{R}^N)\) is bounded and \(u_n \to 0\) in \(L^2_{\text{loc}}(\mathbb{R}^N)\). Then
\[
\int_{\mathbb{R}^N} \Psi(u_n) \, dx \to 0 \quad \text{as } n \to \infty
\]
for any continuous function \(\Psi : \mathbb{R} \to [0, \infty)\) such that \((1.10)\) holds.

Proof. Observe that for all \(r > 0\)
\[
\lim_{n \to \infty} \int_{B(0,r)} |u_n|^2 \, dx = 0
\]
and similarly as in proof of corollary 3.2 we complete the proof.\(\Box\)

Proof of theorem 1.4. Let \((u_n) \subset H^1(\mathbb{R}^N)\) be a bounded sequence and \(\Psi\) as in theorem 1.4. We claim that, passing to a subsequence, there is \(K \in \mathbb{N} \cup \{\infty\}\) and there is a sequence \((\tilde{u}_i)_{i=0}^K \subset H^1(\mathbb{R}^N)\), for \(0 \leq i < K + 1\) there are sequences \((y_n^i) \subset H^1(\mathbb{R}^N)\), \((y_n^i) \subset \mathbb{R}^N\) and positive numbers \((c_i)_{i=0}^K, (r_i)_{i=0}^K\) such that \(y_n^0 = 0, r_0 = 0\) and for any \(0 \leq i < K + 1\) one has
\[
u_n((\cdot + y_n^i)) \to \tilde{u}_i \quad \text{in } H^1(\mathbb{R}^N)\]
and \(u_n((\cdot + y_n^i)\chi_{B(0,a)}) \to \tilde{u}_i \) in \(L^2(\mathbb{R}^N)\) as \(n \to \infty\), \(\quad (3.4)\)
\[
\tilde{u}_i \neq 0 \quad \text{if } i \geq 1,
\] \(\quad (3.5)\)

\footnote{If \(K = \infty\) then \(K + 1 = \infty\) as well.}
\[ |v^i_n - y^j_n| \geq n - r_i - r_j \quad \text{for } 0 \leq j \neq i < K + 1 \text{ and sufficiently large } n, \quad (3.6) \]

\[ v_n^{-1} := u_n \quad \text{and} \quad v_n^i := v_n^{-1} - \tilde{u}_i(-y^j_n) \text{ for } n \geq 1 \quad (3.7) \]

\[ \int_{B(y^n,r)} |v_n^{i-1}|^2 \, dx \geq c_1 \geq \frac{1}{2} \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |v_n^{i-1}|^2 \, dx \quad \text{for sufficiently large } n, \quad (3.8) \]

\[ r_i \geq \max\{i, r_{i-1}\}, \quad \text{if } i \geq 1, \quad \text{and } c_2 = \frac{3}{4} \lim_{r \to \infty} \sup_{n \to \infty} \int_{B(y,r)} |v_n^{i-1}|^2 \, dx > 0. \]

Moreover (1.8) is satisfied. Since \((u_n)\) is bounded, passing to a subsequence we may assume that \(\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx\) exists and

\[ u_n \rightharpoonup \tilde{u}_0 \quad \text{in } H^1(\mathbb{R}^N), \]

\[ u_n\chi_{B(0,\rho)} \to \tilde{u}_0 \quad \text{in } L^2(\mathbb{R}^N). \]

The latter convergence follows from the fact that for any \(n, H^1(B(0,n))\) is compactly embedded into \(L^2(B(0,n))\) and we find sufficiently large \(k_n\) such that \((u_n - \tilde{u}_0)\chi_{B(0,\rho)} < \rho/2\), where \(\chi_{B(0,\rho)}\) is the characteristic function of \(B(0,\rho)\) and \(|\cdot|_p\) denotes the usual \(L^p\)-norm for \(p > 1\). The subsequence \((u_{n_k})\) is then relabelled by \((u_n)\). Take \(v_n^0 := u_n - \tilde{u}_0\) and if

\[ \limsup_{n \to \infty, y \in \mathbb{R}^N} \int_{B(y,r)} |v_n^0|^2 \, dx = 0 \]

for every \(r \geq 1\), then we finish the proof of our claim with \(K = 0\). Otherwise we get

\[ \infty > \sup_{n \geq 1} \int_{\mathbb{R}^N} |v_n^0|^2 \, dx \geq c_1 := \frac{3}{4} \lim_{r \to \infty} \sup_{n \to \infty} \int_{B(y,r)} |v_n^0|^2 \, dx > 0, \]

and there is \(r_1 \geq 1\) and, passing to a subsequence, we find \((y^1_n) \subset \mathbb{R}^N\) such that

\[ \int_{B(y_n^{r_1})} |v_n^0|^2 \, dx \geq c_1 \geq \frac{1}{2} \sup_{y \in \mathbb{R}^N} \int_{B(y,r_1)} |v_n^0|^2 \, dx. \quad (3.9) \]

Note that \((y^1_n)\) is unbounded and we may assume that \(|y^1_n| \geq n - r_1\). Since \((u_n(-y^1_n))\) is bounded in \(H^1(\mathbb{R}^N)\), up to a subsequence, we find \(\tilde{u}_1 \in H^1(\mathbb{R}^N)\) such that

\[ u_n(-y^1_n) \rightharpoonup \tilde{u}_1 \quad \text{in } H^1(\mathbb{R}^N). \]

In view of (3.9), we get \(\tilde{u}_1 \neq 0\), and again we may assume that \(u_n(-y^1_n)\chi_{B(0,\rho)} \to \tilde{u}_1\) in \(L^2(\mathbb{R}^N)\). Since

\[ \lim_{n \to \infty} \left( \int_{\mathbb{R}^N} |\nabla (u_n - \tilde{u}_0)(-y^1_n)|^2 \, dx - \int_{\mathbb{R}^N} |\nabla v_n^0(-y^1_n)|^2 \, dx \right) = \int_{\mathbb{R}^N} |\nabla \tilde{u}_1|^2 \, dx, \]

where \(v_n^0 := v_n^0 - \tilde{u}_1(-y^1_n) = u_n - \tilde{u}_0 - \tilde{u}_1(-y^1_n)_n\), then

\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N} |\nabla \tilde{u}_0|^2 \, dx + \int_{\mathbb{R}^N} |\nabla \tilde{u}_1|^2 \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n^0|^2 \, dx \]

If

\[ \limsup_{n \to \infty, y \in \mathbb{R}^N} \int_{B(y,r)} |v_n^0|^2 \, dx = 0 \]

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for every $r \geq \max \{2, r_1\}$, then we finish the proof of our claim with $K = 1$. Otherwise, $c_2 := \frac{1}{2} \lim_{r \to \infty} \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,r)} |v^1_n|^2 \, dx > 0$, there is $r_2 \geq \max \{2, r_1\}$ and, passing to a subsequence, we find $(v^2_n) \subset \mathbb{R}^N$ such that

$$\int_{B(0,\varepsilon_0)} |v^1_n|^2 \, dx \geq c_2 \geq \frac{1}{2} \sup_{y \in \mathbb{R}^N} \int_{B(y,r_2)} |v^1_n|^2 \, dx$$

(3.10) and $|v^1_n| \geq n - r_2$. Moreover $|y^2_n - y^1_n| \geq n - r_2 - r_1$. Otherwise $B(y^2_n, r_2) \subset B(y^1_n, n)$ and the convergence $u_n(\cdot + y^1_n) \chi_{B(0,n)} \to u_1$ in $L^2(\mathbb{R}^N)$ contradicts (3.10). Then, passing to a subsequence, we find $\tilde{u}_2 \neq 0$ such that

$$v^1_n(\cdot + y^2_n), \ u_n(\cdot + y^2_n) \to \tilde{u}_2 \quad \text{in } H^1(\mathbb{R}^N) \text{ and } u_n(\cdot + y^2_n) \chi_{B(0,n)} \to \tilde{u}_2 \text{ in } L^2(\mathbb{R}^N).$$

Again, if

$$\limsup_{n \to \infty} \int_{B(0,r)} |v^1_n|^2 \, dx = 0,$$

for every $r \geq \max \{3, r_2\}$, where $v^1_n := v^1_n - \tilde{u}_2(\cdot - y^2_n)$, then we finish proof with $K = 2$. Continuing the above procedure, for each $i \geq 1$ we find a subsequence of $(u_n)$, still denoted by $(u_n)$, such that (3.4)–(3.8) and (1.8) are satisfied. Similarly as above, if there is $i \geq 0$ such that

$$\limsup_{n \to \infty} \int_{B(0,r)} |v^i_n|^2 \, dx = 0$$

(3.11) for every $r \geq \max \{n, r_{i-1}\}$, then $K := i$ and we finish proof of the claim. Otherwise, $K = \infty$, by a standard diagonal method and passing to a subsequence, we show that (3.4)–(3.8) and (1.8) are satisfied for every $i \geq 0$.

Now we show that (1.9) holds. Observe that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (\Psi(u_n) - \Psi(v^0_n)) \, dx = \int_{\mathbb{R}^N} \Psi(\tilde{u}_0) \, dx.$$

Indeed, by Vitali’s convergence theorem

$$\int_{\mathbb{R}^N} (\Psi(u_n) - \Psi(v^0_n)) \, dx = \int_{\mathbb{R}^N} \int_0^1 - \frac{d}{ds} \Psi(u_n - s\tilde{u}_0) \, ds \, dx$$

$$= \int_{\mathbb{R}^N} \int_0^1 \Psi'(u_n - s\tilde{u}_0) \tilde{u}_0 \, ds \, dx$$

$$\to \int_0^1 \int_{\mathbb{R}^N} \Psi'(\tilde{u}_0 - s\tilde{u}_0) \tilde{u}_0 \, dx \, ds$$

$$= \int_{\mathbb{R}^N} \int_0^1 - \frac{d}{ds} \Psi(\tilde{u}_0 - s\tilde{u}_0) \, ds \, dx$$

$$= \int_{\mathbb{R}^N} \Psi(\tilde{u}_0) \, dx$$

as $n \to \infty$. Then

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} (\Psi(u_n) \, dx = \int_{\mathbb{R}^N} \Psi(\tilde{u}_0) \, dx + \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(v^0_n) \, dx$$

(3.12)
and (1.9) holds for \( i = 0 \). Similarly as above we show that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \left( \Psi((u_n - \tilde{u}_0)(\cdot + y_n^i)) - \Psi(v_n^i(\cdot + y_n^i)) \right) \, dx = \int_{\mathbb{R}^N} \Psi(\tilde{u}_i) \, dx.
\]

In view of (3.12) we obtain

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n) \, dx = \int_{\mathbb{R}^N} \Psi(\tilde{u}_0) \, dx + \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n - \tilde{u}_0) \, dx
\]

\[
= \int_{\mathbb{R}^N} \Psi(\tilde{u}_0) \, dx + \int_{\mathbb{R}^N} \Psi(\tilde{u}_i) \, dx + \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(v_n^i) \, dx.
\]

Continuing the above procedure we prove that (1.9) holds for every \( i \geq 0 \). Now observe that, if there is \( i \geq 0 \) such that (3.11) holds for every \( r \geq \max\{i, r_i\} \), then \( K = i \). If, in addition, (1.10) holds, then in view of lemma 3.1 we obtain that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \Psi(v_n^i) \, dx = 0
\]

and we finish the proof by setting \( \tilde{u}_i = 0 \) for \( j > i \). Otherwise we have \( K = \infty \). It remains to prove (1.11) in this case. Note that by (3.8) we have

\[
c_k + 1 \leq \frac{1}{2} \int_{B(a_k, r_k + 1)} |v_n^i|^2 \, dx
\]

\[
= 2 \int_{B(y_n^i, r_k + 1)} |v_n^i|^2 \, dx + 2 \int_{B(y_n^i, r_k + 1)} \left| \sum_{j=0}^{k} \tilde{u}_j(\cdot - y_n^i) \right|^2 \, dx
\]

\[
\leq 2 \sup_{y \in \mathbb{R}^N} \int_{B(y, r_k + 1)} |v_n^i|^2 \, dx + 2(k + 1) \sum_{j=0}^{k} \int_{B(y_n^j, r_k + 1)} |\tilde{u}_j|^2 \, dx
\]

for any \( 0 \leq i < k \). Taking into account (3.6) and letting \( n \to \infty \) we get \( c_{k+1} \leq \frac{4}{3} c_k + 1 \). Take \( k \geq 1 \) and sufficiently large \( n > 4 r_k \) such that (3.8) and (3.6) are satisfied. Then we obtain

\[
\frac{3}{32} \sup_{y \in \mathbb{R}^N} \int_{B(y, r_k + 1)} |v_n^i|^2 \, dx
\]

\[
\leq \frac{3}{16} c_{k+1} \leq \frac{1}{2} \sum_{i=0}^{k-1} c_{i+1} \leq \frac{1}{2} \sum_{i=0}^{k-1} \int_{B(y_n^i, r_k + 1)} |v_n^i|^2 \, dx
\]

\[
\leq \frac{1}{k} \sum_{i=0}^{k-1} \int_{B(y_n^i, r_k + 1)} \left( |u_n|^2 + \left| \sum_{j=0}^{i} \tilde{u}_j(\cdot - y_n^i) \right|^2 \right) \, dx
\]

\[
= \frac{1}{k} \int_{B(y_n^i, r_k + 1)} |u_n|^2 \, dx + \frac{1}{k} \int_{\mathbb{R}^N} \left| \sum_{j=0}^{i} \tilde{u}_j(\cdot - y_n^j) \right|^2 \, dx
\]

\[
\leq \frac{1}{k} |u_n|^2 + \frac{1}{k} \sum_{i=0}^{k-1} \left| \sum_{j=0}^{i} \tilde{u}_j(\cdot - y_n^j) \right|^2 \, dx
\]
Observe that by (3.6) and since \( n > 4r_k \) we have

\[
B(y_{n+1}^i, r_k) \subset \mathbb{R}^N \backslash B(0, n - 3r_k)
\]

for \( 0 \leq j < i < k \) and

\[
\left| \sum_{i=0}^{k-1} \sum_{j=0}^{i} u_j (\cdot - y_{n+1}^i) \chi_{B(y_{n+1}^i, r_k)} \right|_2 \leq \frac{32}{3k} \limsup_{n \to \infty} |u_n|_2^2,
\]

as \( n \to \infty \). Hence

\[
\limsup_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(y, r_k)} |v_n|_2^2 \, dx \right) \leq \frac{32}{3k} \limsup_{n \to \infty} |u_n|_2^2,
\]

(3.13)

and suppose that (1.11) does not hold, that is

\[
\limsup_{i \to \infty} \left( \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(v_{n}^i) \, dx \right) > \delta
\]

(3.14)

for some \( \delta > 0 \). Then we find increasing sequences \((i_k), (n_k) \subset \mathbb{N}\) such that

\[
\int_{\mathbb{R}^N} \Psi(v_{n_k}^{i_k}) \, dx > \delta
\]

and

\[
\sup_{y \in \mathbb{R}^N} \int_{B(y, r_k)} |v_{n_k}^{i_k}|_2^2 \, dx \leq \limsup_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(y, r_k)} |v_{n}^{i_k}|_2^2 \, dx \right) + \frac{1}{u_k}.
\]

Since (3.13) holds, we get

\[
\lim_{k \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B(y, r_k)} |v_{n_k}^{i_k}|_2^2 \, dx \right) = 0,
\]

and in view of lemma 3.1 we obtain that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^N} \Psi(v_{n_k}^{i_k}) \, dx = 0,
\]

which is a contradiction. Hence (1.11) is satisfied. \( \square \)

Now we observe that in theorem 1.4 we may find translations \((y_n^i)_{i=0}^\infty \subset \{0\} \times \{0\} \times \mathbb{R}^{N-2M}\) provided that \((u_n) \subset H_{\mathcal{O}_1}^1(\mathbb{R}^N)\) and \(2 \leq m < N/2\).

**Corollary 3.5.** Suppose that \((u_n) \subset H_{\mathcal{O}_1}^1(\mathbb{R}^N)\) is bounded and \(2 \leq M < N/2\). Then there are sequences \((\tilde{y}_n^i)_{i=0}^\infty \subset H_{\mathcal{O}_1}^1(\mathbb{R}^N), (y_n^i)_{i=0}^\infty \subset \{0\} \times \{0\} \times \mathbb{R}^{N-2M}\) for any \(n \geq 1\), such that the statements of theorem 1.4 are satisfied.
Proof. A careful inspection of proof of theorem 1.4 leads to the following claim: there is $K \in \mathbb{N} \cup \{\infty\}$ and there is a sequence $(y_i)_{i=0}^K \subset H^1(\mathbb{R}^N)$, for $0 \leq i < K + 1$ there are sequences $(v_i) \subset H^1(\mathbb{R}^N)$, $(y_i) \subset \{0\} \times \{0\} \times \mathbb{R}^{N-2M}$ and positive numbers $(c_i)_{i=0}^K$, $(r_i)_{i=0}^K$ such that $y_i = 0$, $r_0 = 0$ and, up to a subsequence, for any $n$ and $0 \leq i < K + 1$ one has (3.4)–(3.7),

$$\int_{B(x,n)} |v_n^{-1}|^2 \, dx \geq c_i \geq \frac{1}{2} \sup_{y \in \mathbb{R}^{N-2M}} \int_{B(0,0,y)} |v_n^{-1}|^2 \, dx \geq \frac{1}{4} \sup_{r > 0, y \in \mathbb{R}^{N-2M}} \int_{B(0,0,y)} |v_n^{-1}|^2 \, dx > 0, r_i \geq \max \{i, r_{i-1}\} \text{ for } i > 1,$$

and (1.8) and (1.9) are satisfied. In order to prove (1.11) we use corollary 3.2 instead of lemma 3.1. □

Observe that if $m = N/2$, then we consider $O_2$-invariant sequences. In general we assume that $N - 2M \neq 1$ and we have the following result.

Corollary 3.6. Suppose that $(u_n) \subset H^1(\mathbb{R}^N)$ is bounded. Then passing to a subsequence we find $\tilde{u}_0 \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup \tilde{u}_0 \ \text{in} \ H^1(\mathbb{R}^N) \ \text{as} \ n \to \infty,$$

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N} |\nabla \tilde{u}_0|^2 \, dx + \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla (u_n - \tilde{u}_0)|^2 \, dx,$$

$$\limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n) \, dx = \int_{\mathbb{R}^N} \Psi(\tilde{u}_0) \, dx + \limsup_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n - \tilde{u}_0) \, dx$$

for any function $\Psi : \mathbb{R} \to [0, \infty)$ of class $C^1$ such that $\Psi'(s) \leq C(|s| + |s|^{2^*-1})$ for any $s \in \mathbb{R}$ and some constant $C > 0$. Moreover, if $\Psi$ satisfies (1.10), then

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \Psi(u_n - \tilde{u}_0) \, dx = 0$$

and if $s \mapsto |\Psi(s)|$ satisfies (1.10), then

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \Psi'(u_n)u_n \, dx = \int_{\mathbb{R}^N} \Psi'(\tilde{u}_0)\tilde{u}_0 \, dx. \quad (3.15)$$

Proof. Similarly as in proof of theorem 1.4 we show that passing to a subsequence

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx$$

exists and (3.12) holds. Due to the compact embedding of $H^1(\mathbb{R}^N)$ into $L^2_{\text{loc}}(\mathbb{R}^N)$ we may assume that $u_n - \tilde{u}_0 \to 0$ in $L^2_{\text{loc}}(\mathbb{R}^N)$. Then we apply corollary 3.4 instead of lemma 3.1. In order to prove (3.15) observe that

$$\int_{\mathbb{R}^N} |\Psi'(u_n)u_n - \Psi'(\tilde{u}_0)\tilde{u}_0| \, dx \leq \int_{\mathbb{R}^N} |\Psi'(u_n)||u_n - \tilde{u}_0| \, dx + \int_{\mathbb{R}^N} |\Psi'(u_n) - \Psi'(\tilde{u}_0)||\tilde{u}_0| \, dx.$$

Take any $\varepsilon > 0$, $2 < p < 2^*$ and we find $0 < \delta < M$ and $c_\varepsilon > 0$ such that

$$|\Psi'(s)| \leq \varepsilon(|s| + |s|^{2^*-1}) \quad \text{if } |s| \in [0, \delta) \text{ or } |s| > M,$$
\[ |\Psi'(s)| \leq c_s s^{2/(1-p)} \] if \(|s| \in (\delta, M]\).

Then, passing to a subsequence, \(u_n \to \tilde{u}_0\) in \(L^p(\mathbb{R}^N)\) and we infer that \(\int_{\mathbb{R}^N} |\Psi'(u_n)| |u_n - \tilde{u}_0| \, dx \to 0\) as \(n \to \infty\).

\[ \square \]

4. Proofs of theorems 1.1, 1.2 and 1.3

Let us consider the standard norm of \(u\) in \(H^1(\mathbb{R}^N)\) given by
\[ \|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \, dx. \]

In view of [5][theorem A.VI], \(J : H^1(\mathbb{R}^N) \to \mathbb{R}\) given by (1.3), i.e. \(J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx\), is of class \(C^1\). In the next subsections we build the variational setting according to section 2.

4.1. Critical point theory setting

Let \(X = Y = H^1(\mathbb{R}^N)\) and let \(M, \psi : H^1(\mathbb{R}^N) \to \mathbb{R}\) be given by
\[ M(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2^* \int_{\mathbb{R}^N} G(u) \, dx, \quad \text{and} \]
\[ \psi(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \quad \text{for} \, u \in H^1(\mathbb{R}^N). \]

**Proposition 4.1.** Let us denote
\[ \mathcal{M} := \{ u \in H^1(\mathbb{R}^N) : M(u) = 0 \}, \]
\[ \mathcal{S} := \{ u \in H^1(\mathbb{R}^N) : \psi(u) = 1 \}, \]
\[ \mathcal{P} := \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) \, dx > 0 \}, \]
\[ \mathcal{U} := \mathcal{S} \cap \mathcal{P}. \]

Then the following holds.

(i) There is a continuous map \(m_P : \mathcal{P} \to \mathcal{M}\) such that \(m_P(u)(x) = u(rx)\) for \(x \in \mathbb{R}^N\) with
\[ r = r(u) = \left( \frac{2^* \int_{\mathbb{R}^N} G(u) \, dx}{\psi(u)} \right)^{1/2} > 0. \] (4.1)

(ii) \(m_\mathcal{U} := m_P|_\mathcal{U} : \mathcal{U} \to \mathcal{M}\) is a homeomorphism with the inverse \(m_\mathcal{M}^{-1}(u) = u(\psi(u)^{1/2} \cdot)\).

\(J \circ m_P : \mathcal{P} \to \mathbb{R}\) is of class \(C^1\) with
\[ (J \circ m_P)'(u)(v) = J'(m_P(u))(v(r(u))) \]
\[ = r(u)^{2-N} \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle \, dx - r(u)^{-N} \int_{\mathbb{R}^N} g(u)v \, dx \]
for \(u \in \mathcal{P}\) and \(v \in H^1(\mathbb{R}^N)\).
(iii) $J$ is coercive on $\mathcal{M}$, i.e. for $(u_m) \subset \mathcal{M}$, $J(u_m) \to \infty$ as $\|u_m\| \to \infty$, and

$$c := \inf_{\mathcal{M}} J > 0. \quad (4.2)$$

(iv) If $u_n \to u$, $u_n \in \mathcal{U}$ and $u \in \partial \mathcal{U} = \{ u \in \mathcal{S} : \int_{\mathbb{R}^N} G(u) \, dx = 0 \}$, where the boundary of $\mathcal{U}$ is taken in $\mathcal{S}$, then $(J \circ m_U)(u) \to \infty$ as $n \to \infty$.

Proof. (i) If $u \in \mathcal{P}$ then

$$M(u(r \cdot)) = r^{-N} \left( r^2 \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2^+ \int_{\mathbb{R}^N} G(u) \, dx \right) = 0$$

for $r = r(u)$ given by (4.1). Let $m_{\mathcal{P}} : \mathcal{P} \to \mathcal{M}$ be a map such that

$$m_{\mathcal{P}}(u) := u(r(u) \cdot).$$

Let $u_n \to u_0$, $u_n \in \mathcal{P}$ for $n \geq 0$. Observe that $r(u_n) \to r(u_0)$ and

$$\psi(m_{\mathcal{P}}(u_n) - m_{\mathcal{P}}(u_0))
= \int_{\mathbb{R}^N} |\nabla (u_n(r(u_n) \cdot) - u_0(r(u_0) \cdot))|^2 \, dx
\leq 2(r(u_n)^2 - r(u_0)^2) \psi(u_n - u_0) + 2 \int_{\mathbb{R}^N} |\nabla (u_n(r(u_n) \cdot) - u_0(r(u_0) \cdot))|^2 \, dx
\leq 2(r(u_n)^2 - r(u_0)^2) \psi(u_n - u_0) + \left( r(u_0)^2 - r(u_0)^2 \right) \psi(u_0)
+ 4r(u_0)^2 \int_{\mathbb{R}^N} \left( \nabla u_0 - \nabla u_0 \left( \frac{r(u_n)}{r(u_0)} \right) \right) \cdot \nabla u_0 \, dx
\to 0$$

passing to a subsequence. Similarly we show that $m_{\mathcal{P}}(u_n) \to m_{\mathcal{P}}(u_0)$ in $L^2(\mathbb{R}^N)$, hence $m_{\mathcal{P}}$ is continuous.

(ii) Observe that

$$m_{\mathcal{P}}^{-1}(u) := \{ v \in \mathcal{P} : m_{\mathcal{P}}(v) = u \} = \{ u_\lambda : u_\lambda = u(\cdot \lambda), \quad \lambda > 0 \}.$$ 

Then $1 = \psi(u_\lambda) = \lambda^{2-N} \psi(u)$ if and only if $\lambda = \psi(u)^{\frac{1}{2-N}}$. Therefore $m^{-1}(u) = u(\psi(u)^{\frac{1}{2-N}} \cdot) \in \mathcal{U}$. Similarly as in (i) we show the continuity of $m^{-1} : \mathcal{M} \to \mathcal{U}$. Moreover for $u \in \mathcal{P}$ and $v \in \mathcal{X}$ one obtains

$$(J \circ m_{\mathcal{P}})'(u)(v)
= \lim_{t \to 0} \frac{J(m_{\mathcal{P}}(u + tv)) - J(m_{\mathcal{P}}(u))}{t}
= \lim_{t \to 0} \frac{r(u + tv)^2 - r(u)^2}{2t} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + r(u + tv)^{2-N} \int_{\mathbb{R}^N} (\nabla (2u + tv) \cdot \nabla v) \, dx
- \lim_{t \to 0} \frac{r(u + tv)^{2-N} - r(u)^{2-N}}{t} \int_{\mathbb{R}^N} G(u) \, dx + r(u + tv)^{-N} \int_{\mathbb{R}^N} G(u + tv) - G(u) \, dx.$$
\[
\begin{align*}
\frac{m}{2} + N \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + r(u)^2 \int_{\mathbb{R}^N} (\nabla u, \nabla v) \, dx \\
- \left( -N \right) r(u)^{-N-1} r'(u)(v) \int_{\mathbb{R}^N} G(u) \, dx + r(u)^{-N} \int_{\mathbb{R}^N} g(u) v \, dx \\
= \frac{2 - N}{2} r(u)^{-N-1} r'(u)(v) \left( r(u)^2 \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2 \int_{\mathbb{R}^N} G(u) \, dx \right) \\
+ r(u)^{-N} \left( r(u)^2 \int_{\mathbb{R}^N} (\nabla u, \nabla v) \, dx - \int_{\mathbb{R}^N} g(u) v \, dx \right) \\
= \frac{2 - N}{2} r(u)^{-N-1} r'(u)(v) M(m(u)) + J'(m(u))(v(r(u))) \\
= J'(m(u))(v(r(u))).
\end{align*}
\]

(iii) Let us introduce the following auxiliary functions \( g_1(s) = \max\{g(s) + ms, 0\} \) and \( g_2(s) = g_1(s) - g(s) \) for \( s \geq 0 \) and \( g_1(s) = -g_1(-s) \) for \( s < 0 \). Then \( g_1(s), g_2(s) \geq 0 \) for \( s \geq 0 \),

\[
\lim_{s \to 0^+} g_1(s)/s = \lim_{s \to +\infty} g_1(s)/s^{2-1} = 0 \tag{4.3}
\]

\[
g_2(s) \geq ms \quad \text{for } s \geq 0, \tag{4.4}
\]

and let

\[
G_i(s) = \int_0^s g_i(t) \, dt \quad \text{for } i = 1, 2.
\]

The condition (4.3) will be important e.g. to apply theorem 1.4 with \( \Psi(s) = G_i(s) \), whereas (4.4) is used below to estimate the \( L^2 \)-norm. Namely, suppose that for some \( (u_n) \subset U \)

\[
J(m(u_n)) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla m_i(u_n)|^2 \, dx
\]

is bounded. Then we obtain that \( m_i(u_n) \) is bounded in \( L^2(\mathbb{R}^N) \) and by (4.3), \( \int_{\mathbb{R}^N} G_i(m(u_n)) \, dx \) is bounded as well. By (4.4) and since \( m_i(u_n) \in M \), we infer that \( m_i(u_n) \) is bounded in \( H^1(\mathbb{R}^N) \). Thus \( J \) is coercive on \( M \). Observe that for some constants \( 0 < C_1 < C_2 \) one has

\[
|m_i(u_n)|^2 + |m_i(u_n)|^2 \leq C_1 \int_{\mathbb{R}^N} (|\nabla m_i(u_n)|^2 + 2^*_2 G_2(m_i(u_n))) \, dx \\
= C_1 2^* \int_{\mathbb{R}^N} G_i(m_i(u_n)) \, dx \\
\leq |m_i(u_n)|^2 + C_2 |m_i(u_n)|^2
\]

and we conclude that \( |m_i(u_n)|^2 \geq C_2^{-1/(2^*-2)} > 0 \). Hence \( c = \inf_M J > 0 \).

(iv) Note that if \( u_n \to u \in \partial U \) and \( u_0 \in U \), then \( r(u_n) \to 0 \) and

\[
\|m_i(u_n)\|^2 = r(u_n)^{2-N} + r(u_n)^{-N} |u_n|^2 \to \infty
\]

as \( n \to \infty \). Hence by the coercivity, \( J(m_i(u_n)) \to \infty \) as \( n \to \infty \).
Now observe that we may consider the group of translations \( G = \mathbb{R}^N \) acting on \( X = H^1(\mathbb{R}^N) \), i.e.

\[
(yu)(x) = u(x + y)
\]

for \( y \in \mathbb{R}^N, u \in X, x \in \mathbb{R}^N \), and in view of proposition 4.1 conditions (A1)–(A3) are satisfied.

In the similar way we may consider the following subgroup of translations \( G = \{0\} \times \mathbb{R}^{N-2M} \) acting on \( X = X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N) \) and conditions (A1)–(A3) are satisfied provided that instead of \( \mathcal{M}, \mathcal{S}, Y = X = H^1(\mathbb{R}^N) \), \( U, m_U \) and \( m \), we consider \( \mathcal{M} \cap X, \mathcal{S} \cap X, Y = X = X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N), U \cap X, m_{|\mathcal{S} \cap X} : \mathcal{P} \cap X \to \mathcal{M} \cap X \) and \( m_{|\mathcal{S} \cap X} : U \cap X \to \mathcal{M} \cap X \) respectively.

Finally, in case of \( X = X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N) \) we consider the trivial group \( G = \{(0,0)\} \) acting on \( X \).

**Remark 4.2.** We show how to easily construct functions in \( X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N) \). Let \( u \in H^1_0(B(0,R)) \cap L^\infty(B(0,R)) \) be \( \mathcal{O}(N) \)-invariant (radial) function, \( R > 1 \) and take any odd and smooth function \( \varphi : \mathbb{R} \to [0,1] \) such that \( \varphi(x) = 1 \) for \( x \geq 1 \) and \( \varphi(x) = -1 \) for \( x \leq -1 \). Note that, defining

\[
\hat{u}(x_1, x_2, x_3) := u \left( \sqrt{|x_1|^2 + |x_2|^2 + |x_3|^2} \right) \varphi(|x_1| - |x_2|) \quad \text{for } x_1, x_2 \in \mathbb{R}^M, x_3 \in \mathbb{R}^{N-2M},
\]

we get \( \tilde{u} \in X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N) \). Take \( A := \text{ess sup}|u| \) and \( B := \max_{s \in [0,A]}|G(s)| \). Let us denote \( r = |x| \) and \( r_i = |x_i| \) for \( i = 1, 2, 3 \). Observe that

\[
\int_{\mathbb{R}^N} G(\tilde{u}) \, dx = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} G(\tilde{u}) r_1^{-1} r_2^{-1} r_3^{-1} \, dr_1 \, dr_2 \, dr_3
\]

\[
= \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{2\pi} \int_{0}^{\infty} G(\tilde{u}) r_1^{-1} r_2^{-1} r_3^{-1} \, dr_1 \, dr_2 \, dr_3
\]

\[
\geq \int_{\mathbb{R}^N} G(u) \, dx - c_1 B \left( \sum_{j=N-M}^{N-1} R^j \right)
\]

for some constant \( c_1 > 0 \) dependent only on \( N \). In [5][page 325], for any \( R > 0 \) one can find a radial function \( u \in H^1_0(B(0,R)) \cap L^\infty(B(0,R)) \) such that \( \int_{\mathbb{R}^N} G(u) \, dx \geq c_2 R^N - c_3 R^{N-1} \) for some constants \( c_2, c_3 > 0 \). Therefore we get \( \int_{\mathbb{R}^N} G(\tilde{u}) \, dx > 0 \) for sufficiently large \( R \), hence \( \mathcal{P} \cap X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N) \neq \emptyset \) and \( \mathcal{M} \cap X_r \cap H^1_{\mathcal{O}}(\mathbb{R}^N) \neq \emptyset \).
4.2. $\theta$-analysis of Palais–Smale sequences

Below we explain the role of $\theta$ in the analysis of Palais–Smale sequences of $J \circ m_{\|}$.

**Lemma 4.3.** Suppose that $(u_n) \subset U$ is a (PS)$_\theta$-sequence of $J \circ m_{\|}$ such that

$$m_{\|}(u_n) \cdot + y_n \rightharpoonup \tilde{u} \neq 0 \quad \text{in } H^1(\mathbb{R}^N)$$

for some sequence $(y_n) \subset \mathbb{R}^N$ and $\tilde{u} \in H^1(\mathbb{R}^N)$. Then $\tilde{u}$ solves

$$-\theta \Delta u = g(u), \quad \text{where } \theta := \psi(\tilde{u})^{-1} \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx,$$  \hspace{1cm} (4.5)

and passing to a subsequence

$$\theta = \lim_{n \to \infty} \psi(m_{\|}(u_n))^{-1} \int_{\mathbb{R}^N} g(m_{\|}(u_n)) m_{\|}(u_n) \, dx.$$  \hspace{1cm} (4.6)

Moreover $\theta \neq 0$ and

$$\theta = 2^* \psi(\tilde{u})^{-1} \int_{\mathbb{R}^N} G(\tilde{u}) \, dx.$$  \hspace{1cm} (4.7)

If $\theta > 0$, then $m_{\|}(\tilde{u}) \in M$ is a critical point of $J$. If $\theta \geq 1$, then $J(m_{\|}(\tilde{u})) \leq \beta$.

**Proof.** For $v \in X$ we set $v_n(x) = v(r(u_n)^{-1}x - y_n)$ and observe that passing to a subsequence $m_{\|}(u_n)(x + y_n) \rightharpoonup \tilde{u}(x)$ for a.e. $x \in \mathbb{R}^N$ and by Vitali’s convergence theorem

$$(J \circ m_{\|})(u_n)(v_n) = \int_{\mathbb{R}^N} \langle \nabla m_{\|}(u_n) \cdot + y_n, \nabla v \rangle \, dx - \int_{\mathbb{R}^N} g(m_{\|}(u_n) \cdot + y_n)v \, dx$$

$$\rightarrow \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) v \, dx.$$  \hspace{1cm} (4.8)

We find the following decomposition

$$v_n = \left( \int_{\mathbb{R}^N} \langle \nabla u, \nabla v_n \rangle \, dx \right) u_n + \tilde{v}_n$$

with $\tilde{v}_n \in T_{u_n} \mathcal{S}$. In view of proposition 4.1(iii) we get that $r(u_n)$ is bounded from above, bounded away from 0 and passing to a subsequence $r(u_n) \rightarrow r_0 > 0$. Note that $(v_n)$ is bounded, hence $(\tilde{v}_n)$ is bounded and $(J \circ m_{\|})(u_n)(\tilde{v}_n) \rightarrow 0$. Moreover

$$\int_{\mathbb{R}^N} \langle \nabla u, \nabla v_n \rangle \, dx = r(u_n)^{N-2} \int_{\mathbb{R}^N} \langle \nabla m_{\|}(u_n), \nabla v_n(r(u_n)) \rangle \, dx$$

$$= r(u_n)^{N-2} \int_{\mathbb{R}^N} \langle \nabla m_{\|}(u_n)(\cdot + y_n), \nabla v \rangle \, dx$$

$$\rightarrow r_0^{N-2} \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx = 0$$

provided that $\int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx = 0$. Hence

$$(J \circ m_{\|})(u_n)(v_n) = \left( \int_{\mathbb{R}^N} \langle \nabla u, \nabla v_n \rangle \, dx \right) (J \circ m_{\|})(u_n) + (J \circ m_{\|})(u_n)(\tilde{v}_n) \rightarrow 0$$

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and by (4.8) we obtain
\[
\int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) v \, dx = 0
\]
for any \( v \) such that \( \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx = 0 \). We define \( \xi : H^1(\mathbb{R}^N) \to \mathbb{R} \) by the following formula
\[
\xi(v) = \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) v \, dx
\]
\[
- \left( \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx \right) \psi(\tilde{u})^{-1} \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx.
\]
Observe that any \( v \in H^1(\mathbb{R}^N) \) has the following decomposition
\[
v = \left( \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx \right) \tilde{u} + \tilde{v}
\]
such that \( \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla \tilde{v} \rangle \, dx = 0 \). Note that \( \xi(\tilde{u}) = 0 \) and
\[
\xi(v) = \left( \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx \right) \xi(\tilde{u}) + \xi(\tilde{v})
\]
\[
= \xi(\tilde{v}) = 0
\]
for any \( v \in H^1(\mathbb{R}^N) \). Then
\[
0 = \xi(v) = \int_{\mathbb{R}^N} \left( 1 - \left( \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx \right) \psi(\tilde{u})^{-1} \right) \langle \nabla \tilde{u}, \nabla v \rangle \, dx
\]
\[
- \int_{\mathbb{R}^N} g(\tilde{u}) v \, dx \tag{4.9}
\]
and \( \tilde{u} \) is a weak solution to the problem (4.5) with
\[
\theta = 1 - \left( \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx \right) \psi(\tilde{u})^{-1} = \psi(\tilde{u})^{-1} \int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx.
\]
Now we show (4.6). Let us define a map \( \eta : \mathcal{P} \to (H^1(\mathbb{R}^N))^* \) by the following formula
\[
\eta(u)(v) = (J \circ m_P)'(u)(v) - (J \circ m_P)'(u) \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle \, dx
\]
for \( u \in \mathcal{P} \) and \( v \in H^1(\mathbb{R}^N) \). Observe that any \( v \in H^1(\mathbb{R}^N) \) has the unique decomposition
\[
v = \left( \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v \rangle \, dx \right) u_n + \tilde{v}_n
\]
such that \( \tilde{v}_n \in T_{u_n} S \). Note that
\[
\eta(u_n)(v) = \left( \int_{\mathbb{R}^N} \langle \nabla u_n, \nabla v \rangle \, dx \right) \eta(u_n)(u_n) + \eta(u_n)(\tilde{v}_n)
\]
\[
= \eta(u_n)(\tilde{v}_n) = (J \circ m_{u_n})'(u_n)(\tilde{v}_n).
\]
Since \((u_n)\) is a \((PS)_c\)-sequence of \(J \circ m_d\), we obtain \(\eta(u_n) \to 0\) in \((H^1(\mathbb{R}^N))^\ast\). On the other hand, in view of proposition 4.1(ii)

\[
\eta(u_n)(v(r(u_n)^{-1}x - y_n)) = \int_{\mathbb{R}^N} \left(1 - r(u_n)N^{-2} (J \circ m_P)'(u_n)(u_n) \right) \\
\times \langle \nabla m_d(u_n)(\cdot + y_n), \nabla v \rangle \, dx \\
- \int_{\mathbb{R}^N} g(m_d(u_n)(\cdot + y_n))v \, dx \\
= \int_{\mathbb{R}^N} \theta_n \langle \nabla m_d(u_n)(\cdot + y_n), \nabla v \rangle \, dx \\
- \int_{\mathbb{R}^N} g(m_d(u_n)(\cdot + y_n))v \, dx,
\]

where

\[
\theta_n = r(u_n)^{N-2} \int_{\mathbb{R}^N} g(m_d(u_n))m_d(u_n) \, dx = \psi(m_d(u_n))^{-1} \int_{\mathbb{R}^N} g(m_d(u_n))m_d(u_n) \, dx.
\]

Passing to a subsequence \(\theta_n \to \bar{\theta}\) and

\[
0 = \lim_{n \to \infty} \eta(u_n)(v(r(u_n)^{-1}x - y_n)) = \int_{\mathbb{R}^N} \bar{\theta} \langle \nabla \tilde{u}, \nabla v \rangle \, dx - \int_{\mathbb{R}^N} g(\tilde{u})v \, dx
\]

for any \(v \in H^1(\mathbb{R}^N)\). Taking into account (4.9) we obtain that \(\bar{\theta} = \theta\) and (4.6) is satisfied. Now we show that \(\bar{\theta} \neq 0\). Suppose that \(\bar{\theta} = 0\), hence \(g(\tilde{u}(x)) = 0\) for a.e. \(x \in \mathbb{R}^N\). Take \(\Sigma := \{x \in \mathbb{R}^N : g(\tilde{u}(x)) = 0\}\) and clearly \(\Sigma = \{x \in \mathbb{R}^N : g(\tilde{u}(x)) = 0\}\). Suppose that \(\varepsilon := \text{essinf}_{x \in \Omega} |\tilde{u}(x)| > 0\). Since \(\tilde{u} \in L^2(\mathbb{R}^N)\setminus\{0\}\), we infer that \(\Omega\) has finite positive measure and note that

\[
\int_{\mathbb{R}^N} |\tilde{u}(x + h) - \tilde{u}(x)|^2 \, dx \geq \varepsilon \int_{\mathbb{R}^N} |\chi_\Omega(x + h) - \chi_\Omega(x)|^2 \, dx
\]

holds for any \(h \in \mathbb{R}^N\), where \(\chi_\Omega\) is the characteristic function of \(\Omega\). In view of [34][Theorem 2.1.6] we infer that \(\chi_\Omega \in H^1(\mathbb{R}^N)\) and we get the contradiction with the assumption \(\varepsilon > 0\). Therefore, \(\varepsilon = \text{essinf}_{x \in \Omega} |\tilde{u}(x)| = 0\), and we find a sequence \((x_n) \subset \mathbb{R}^N\) such that \(\tilde{u}(x_n) \to 0\), \(\tilde{u}(x_n) \neq 0\) and \(g(\tilde{u}(x_n)) = 0\). Thus, by (4.3) and (4.4) we obtain the next contradiction

\[
0 = \lim_{n \to \infty} \frac{g(\tilde{u}(x_n))}{|\tilde{u}(x_n)|} \leq -m < 0.
\]

Therefore \(\bar{\theta} \neq 0\) and by the elliptic regularity we infer that \(\tilde{u} \in W^{2,q}_{\text{loc}}(\mathbb{R}^N)\) for any \(q < \infty\). In view of the Pohozaev identity

\[
\bar{\theta} \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = 2^* \int_{\mathbb{R}^N} G(\tilde{u}) \, dx,
\]

hence (4.7) holds. Now suppose that \(\bar{\theta} > 0\). Then

\[
r(\tilde{u}) = \left(2^* \int_{\mathbb{R}^N} G(\tilde{u}) \, dx \frac{1}{\psi(\tilde{u})} \right)^{1/2} = \left(\int_{\mathbb{R}^N} g(\tilde{u}) \tilde{u} \, dx \frac{1}{\psi(\tilde{u})} \right)^{1/2} = \bar{\theta}^{1/2}.
\]
Observe that for \( v \in X \) and \( \nu \equiv v(r(\tilde{u})^{-1}) \) one has
\[
J'(m_P(\tilde{u}))(v) = r(\tilde{u})^{-N} \int_{\mathbb{R}^N} \langle \nabla \tilde{u}, \nabla v \rangle \, dx - r(\tilde{u})^{-N} \int_{\mathbb{R}^N} g(\tilde{u}) v \, dx
\]
\[
= r(\tilde{u})^{-N} \left( \int_{\mathbb{R}^N} \langle \nabla \theta \tilde{u}, \nabla v \rangle \, dx - \int_{\mathbb{R}^N} g(\tilde{u}) v \, dx \right) = 0,
\]
which finally shows that \( m_P(\tilde{u}) \) is a critical point of \( J \). If \( \theta \geq 1 \), then
\[
\beta = \lim_{n \to \infty} J(m_P(u_n)) \geq \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx \geq r(\tilde{u})^{-N} \times \left( \frac{1}{2} - \frac{1}{2^*} \right) \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 \, dx = J(m_P(\tilde{u})).
\]

The main difficulty in the analysis of Palais–Smale sequences of \( J \circ m_U \) is to find proper translations \( (y_n) \subset \mathbb{R}^N \) such that \( \theta > 0 \) in lemma \( 4.3 \). In order to check \( (M)_i \) condition one needs to ensure that even \( \theta \geq 1 \). This can be performed with the help of the following result providing decompositions for Palais–Smale sequences of \( J \circ m_U \), which is based on the profile decomposition theorem 1.4. Observe that in the usual variational approach e.g. due to Struwe [30] or Coti Zelati and Rabinowitz [9], such decompositions of Palais–Smale sequences are finite. In our case, however, a finite procedure cannot be performed in general, since we do not know whether a weak limit point of a Palais–Smale sequence of \( J \circ m_U \) is a critical point. Therefore we need to employ the profile decompositions from theorem 1.4 and corollary 3.5.

**Proposition 4.4.** Let \( (u_n) \subset U \) be a Palais–Smale sequence of \( J \circ m_U \) at level \( \beta = c \). Then there is \( K \in \mathbb{N} \cup \{ \infty \} \) and there are sequences \( (\tilde{u})_i \subset H^1(\mathbb{R}^N) \), \( (\theta_i)_i \subset \mathbb{R} \), for any \( n \geq 1 \), \( (y_n^i)_{i=0}^K \subset \mathbb{R}^N \) is such that \( y_n^0 = 0 \), \( |y_n^i| \to \infty \) as \( n \to \infty \) for \( i \neq j \), and passing to a subsequence, the following conditions hold:
\[
m_P(u_n)(\cdot + y_n^i) \to \tilde{u}_i \quad \text{in } H^1(\mathbb{R}^N) \quad \text{as } n \to \infty \quad \text{for } 0 \leq i < K + 1,
\]
\[
\tilde{u}_i \text{ solves } (4.5) \quad \text{with } \theta_i \quad \text{for } 0 \leq i < K + 1, \tilde{u}_i \neq 0 \quad \text{and } (4.7) \text{ holds for } 1 \leq i < K + 1,
\]
if \( \tilde{u}_0 \neq 0 \), then \( \theta_0 \neq 0 \) and satisfies (4.7),
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} G_1(m_P(u_n)) \, dx = \sum_{i=0}^K \int_{\mathbb{R}^N} G_1(\tilde{u}_i) \, dx,
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} G_2(m_P(u_n)) \, dx \geq \sum_{i=0}^K \int_{\mathbb{R}^N} G_2(\tilde{u}_i) \, dx
\]
\[
\lim_{n \to \infty} \psi(m_P(u_n)) \geq \sum_{i=0}^K \psi(\tilde{u}_i).
\]

**Proof.** Since \( J \) is coercive on \( \mathcal{M} \), we know that \( m_P(u_n) \) is bounded and passing to a subsequence we may assume that \( \lim_{n \to \infty} \int_{\mathbb{R}^N} G_1(m_P(u_n)) \, dx \), \( \lim_{n \to \infty} \int_{\mathbb{R}^N} G_2(m_P(u_n)) \, dx \) exist. In view of theorem 1.4 we obtain sequences \( (\tilde{u})_{i=0}^K \subset H^1(\mathbb{R}^N) \) and \( (y_n^i)_{i=0}^K \subset \mathbb{R}^N \) for \( n \geq 1 \), such that (1.8) and (1.9) are satisfied. If \( (\tilde{u})_{i=1}^K \) contains exactly \( K \) nontrivial functions, then we may assume that \( \tilde{u}_i \neq 0 \) for \( i = 1, \ldots, K \). Otherwise we set \( K = \infty \). In view of lemma \( 4.3 \), \( \tilde{u}_i \) solves...
Corollary 4.4. Suppose that \( X = H_{0}^{1}(\mathbb{R}^{N}) \cap X_{e} \), and \( 2 \leq M < N/2 \). Let \((u_{n}) \subseteq \mathcal{U} \cap X \) be a Palais–Smale sequence of \( J_{X} \circ m_{\mathcal{U}}|_{U^{X}} \) at level \( \beta = \inf_{J_{X} \circ m_{\mathcal{U}}|_{U^{X}}} J \). Then there is \( K \in \mathbb{N} \cup \{\infty\} \) and there are sequences \((\tilde{u}_{i})_{i=0}^{K} \subseteq X, (\theta_{i})_{i=0}^{K} \subseteq \mathbb{R}, (\gamma_{i})_{i=0}^{K} \subseteq \{0\} \times \{0\} \times \mathbb{R}^{N-2M} \) for any \( n \geq 1 \) such that \( y_{n}^{i} = 0, |y_{n}^{i} - y_{n}^{j}| \to \infty \) as \( n \to \infty \) for \( i \neq j \), and passing to a subsequence, (4.10)–(4.14) are satisfied.

**Proof.** We argue as in proof of proposition 4.4, but instead of theorem 1.4 we use corollary 4.3. Arguing as in lemma 4.3 we obtain that

\[
\theta_{i} \int_{\mathbb{R}^{N}} \langle \nabla \tilde{u}_{i}, \nabla v \rangle \, dx = \int_{\mathbb{R}^{N}} g(\tilde{u}_{i}) v \, dx \quad \text{for every} \quad v \in X,
\]

and \( \tilde{u}_{i} \neq 0 \) for \( 1 \leq i < K + 1 \). By the Palais principle of symmetric criticality [26], \( \tilde{u}_{i} \) solves (4.5) with \( \theta_{i} \), and \( \beta_{i} \) in place of \( \beta \). In lemma 4.3 we show that \( \beta_{i} \neq 0 \), and by the Pohozaev identity (4.7) holds for \( \tilde{u}_{i} \) and \( \beta_{i} \) for \( 1 \leq i < K + 1 \). If \( \tilde{u}_{0} = 0 \), then \( \beta_{0} = 0 \), otherwise \( \tilde{u}_{0} \) is given by (4.5) and that the Pohozaev identity holds always (4.7).

**Remark 4.6.** An important consequence of proposition 4.4 and corollary 4.5 is the existence of a sequence of translations \((y_{i}^{0})\) such that \( \theta_{i} \geq 1 \) for some \( i \geq 0 \). Indeed, in view of (4.11) we get

\[
\theta_{i} \psi(\tilde{u}_{i}) = 2^{*} \left( \int_{\mathbb{R}^{N}} G_{1}(\tilde{u}_{i}) \, dx - \int_{\mathbb{R}^{N}} G_{2}(\tilde{u}_{i}) \, dx \right)
\]

for \( 0 \leq i < K + 1 \). Then by (4.12)–(4.14) we obtain

\[
\sum_{i=0}^{K} \theta_{i} \psi(\tilde{u}_{i}) = 2^{*} \left( \sum_{i=0}^{K} \int_{\mathbb{R}^{N}} G_{1}(\tilde{u}_{i}) \, dx - \sum_{i=0}^{K} \int_{\mathbb{R}^{N}} G_{2}(\tilde{u}_{i}) \, dx \right)
\]

\[
\geq 2^{*} \left( \lim_{n \to \infty} \int_{\mathbb{R}^{N}} G_{1}(m_{\mathcal{U}}(u_{n})) \, dx - \lim_{n \to \infty} \int_{\mathbb{R}^{N}} G_{2}(m_{\mathcal{U}}(u_{n})) \, dx \right)
\]

\[
= \lim_{n \to \infty} \psi(m_{\mathcal{U}}(u_{n})) \geq \sum_{i=0}^{K} \psi(\tilde{u}_{i})
\]

Therefore there is \( \theta_{i} \geq 1 \) for some \( 0 \leq i < K + 1 \).

### 4.3. Proof of theorems 1.1, 1.2 and 1.3

**Lemma 4.7.** \( J \circ m_{\mathcal{U}} \) satisfies \((M_{\beta})\) (i) for \( \beta = c \).

**Proof.** Let \((u_{n}) \subseteq \mathcal{U} \) be a \((PS)_{c}\)-sequence of \( J \circ m_{\mathcal{U}} \). Since \( J \) is coercive on \( M, (m_{\mathcal{U}}(u_{n})) \) is bounded and in view of proposition 4.4 and remark 4.6 we find a sequence \((y_{n}) \subseteq \mathbb{R}^{N} \) such that \( m_{\mathcal{U}}(u_{n})(x + y_{n}) \to \tilde{u} \in H_{0}^{1}(\mathbb{R}^{N}) \) for some \( \tilde{u} \neq 0 \) and \( \theta \geq 1 \) given by (4.7). Observe that \( \tilde{u} \in \mathcal{P} \) and by lemma 4.3 we conclude.

Now, let us consider \( O_{1} \)-invariant functions.

### Lemma 4.8.** Suppose that \( X = H_{0}^{1}(\mathbb{R}^{N}) \cap X_{e} \) and \( 2 \leq M < N/2 \). Then \( J_{X} \circ m_{\mathcal{U}}|_{U^{X}} \) satisfies \((M_{\beta})\) (i) for \( \beta = \inf_{J_{X} \circ m_{\mathcal{U}}|_{U^{X}}} J \).
**Proof.** Let \((u_n) \subset U \cap X\) be a \((PS)_\beta\)-sequence of \(J|_X \circ m_\beta|_{U \cap X}\). Similarly as in proof of lemma 4.7, in view of corollary 4.5 and and remark 4.6 we find a sequence \((y_n) \subset \{0\} \times \{0\} \times \mathbb{R}^N - \mathbb{M}\) such that \(m_\beta(u_n) y_n \to \bar{u}\) in \(X\) for some \(\bar{u} \neq 0\) and \(\theta \geq 1\) given by (4.7). Observe that \(\bar{u} \in \mathcal{P} \cap X\) and as in lemma 4.3, \(J(m_\beta(\bar{u})) \leq \beta\).

More can be said for \(O_2\)-invariant functions.

**Lemma 4.9.** Suppose that \(X = H^1_{\alpha}\mathbb{R}^N \cap X\). If \((u_n) \subset U \cap X\) is a \((PS)_\beta\)-sequence of \((J|_X \circ m_\beta)(u_n) \to 0\) and \((J|_X \circ m_\beta)(u_n) \to \beta\). Since \(J\) is coercive on \(\mathcal{M}\), \((m_\beta(u_n))\) is bounded and in view of corollary 3.6 we find a sequence \((\bar{u})\) and passing to subsequence we obtain

\[
\int_{\mathbb{R}^N} G_1(m_\beta(u_n)) \, dx \to \int_{\mathbb{R}^N} G_1(\bar{u}) \, dx
\]

as \(n \to \infty\). If \(\bar{u} = 0\), then by (4.4)

\[
\min \left\{ 1, \frac{m}{2} \right\} \|m_\beta(u_n)\|^2 \leq \int_{\mathbb{R}^N} |\nabla m_\beta(u_n)|^2 \, dx + 2\int_{\mathbb{R}^N} G_2(m_\beta(u_n)) \, dx
\]

\[
= 2\int_{\mathbb{R}^N} G_1(m_\beta(u_n)) \, dx \to 0,
\]

which contradicts the fact that \(\inf_{\mathcal{M} \cap X} J > 0\). Therefore \(\bar{u} \neq 0\). Now, observe that applying corollary 3.6 with (3.16) for \(\Psi(s) = G_1(s)\) and passing to subsequence we obtain

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} g_1(m_\beta(u_n)) m_\beta(u_n) \, dx = \int_{\mathbb{R}^N} g_1(\bar{u}) \bar{u} \, dx,
\]

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} g_2(m_\beta(u_n)) m_\beta(u_n) \, dx \geq \int_{\mathbb{R}^N} g_2(\bar{u}) \bar{u} \, dx.
\]

Similarly as in lemma 4.3 we infer that (4.6) and (4.7) hold and

\[
\psi(\bar{u}) \leq \lim_{n \to \infty} \psi(m_\beta(u_n)) = \lim_{n \to \infty} \int_{\mathbb{R}^N} g_1(m_\beta(u_n)) m_\beta(u_n) \, dx
\]

\[
- \lim_{n \to \infty} \int_{\mathbb{R}^N} g_2(m_\beta(u_n)) m_\beta(u_n) \, dx
\]

\[
\leq \int_{\mathbb{R}^N} g_1(\bar{u}) \bar{u} \, dx - \int_{\mathbb{R}^N} g_2(\bar{u}) \bar{u} \, dx = \theta \psi(\bar{u}).
\]

Hence \(\theta \geq 1\) and \(m_\beta(\bar{u}) \in \mathcal{M} \cap X\) is a critical point of \(J|_X \circ m_\beta|_{U \cap X}\). In view of (4.6) we get

\[
\theta = \lim_{n \to \infty} \psi(m_\beta(u_n))^{-1} \int_{\mathbb{R}^N} g(m_\beta(u_n)) m_\beta(u_n) \, dx \leq \psi(\bar{u})^{-1} \int_{\mathbb{R}^N} g(\bar{u}) \bar{u} \, dx = \theta,
\]

hence

\[
\lim_{n \to \infty} \psi(m_\beta(u_n)) = \psi(\bar{u})
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} g_2(m(u_n))m(u_n) \, dx = \int_{\mathbb{R}^N} g_2(\tilde{u}) \tilde{u} \, dx.
\]

Note that \( g_2(s) = ms + g_3(s) \), where \( g_3(s) := \max \{0, -g(s) - ms\} \geq 0 \) for \( s \geq 0 \) and \( g_3(s) := -\max \{0, -g(s) - ms\} \) for \( s \leq 0 \). Then \( g_3(s) \) for \( s \in \mathbb{R} \) and we easily infer that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |m(u_n)|^2 \, dx = \int_{\mathbb{R}^N} |\tilde{u}|^2 \, dx.
\]
Therefore \( m(u_n) \to \tilde{u} \) in \( H^1(\mathbb{R}^N) \), \( \tilde{u} \in \mathcal{M} \cap X \) and \( u_n \to u_0 := m^{-1}(\tilde{u}) \) in \( \mathcal{U} \cap X \). Since \( \theta = 1 \), \( J'_\theta(m(u_0)) = J'_\theta(\tilde{u}) = 0 \).

**Proof of Theorem 1.1.** Since \( J \) satisfies (A1)–(A3) and \((M_\beta)\) (i), proof follows from theorem 2.2(a). □

**Proof of Theorem 1.2.** If \( X = H^1_{\Omega_1}(\mathbb{R}^N) \cap X \), and \( 2 \leq m < N/2 \), then \( J|_X \circ m|_{|X|} \) satisfies (A1)–(A3) and \((M_\beta)\) (i). Then, in view of theorem 2.2(a) there is a critical point \( u \in \mathcal{M} \cap X \) of \( J|_X \) such that
\[
J(u) = \inf_{\mathcal{M} \cap X} J.
\]
In view of the Palais principle of symmetric criticality [26], \( u \) solves (1.1). Let
\[
\Omega_1 := \{ x \in \mathbb{R}^N : |x_1| > |x_2| \},
\]
\[
\Omega_2 := \{ x \in \mathbb{R}^N : |x_1| < |x_2| \}.
\]
Since \( u \in X_\tau \cap H^1_{\Omega_1}(\mathbb{R}^N) \), we get \( \chi_{\Omega_1}u \in H^1_0(\Omega_1) \subset H^1(\mathbb{R}^N) \) and \( \chi_{\Omega_2}u \in H^1_0(\Omega_2) \subset H^1(\mathbb{R}^N) \). Moreover \( \chi_{\Omega_1}u \in \mathcal{M} \) and
\[
J(u) = J(\chi_{\Omega_1}u) + J(\chi_{\Omega_2}u) = 2J(\chi_{\Omega_1}u) \geq 2 \inf_{\mathcal{M}} J,
\]
which completes the proof of (1.5). The remaining case \( 2 \leq m = N/2 \) is contained in theorem 1.3. □

**Proof of Theorem 1.3.** If \( X = H^1_{\Omega_1}(\mathbb{R}^N) \cap X \), then \( J|_X \circ m|_{|X|} \) satisfies (A1)–(A3), and note that lemma 4.9 holds. In view of [6][theorem 10], for any \( k \geq 1 \) we find an odd continuous map \( \tau : S^{k-1} \to H^1_0(B(0, R)) \cap L^\infty(B(0, R)) \) such that \( \tau(\sigma) \) is a radial function and \( \tau(\sigma) \neq 0 \) for all \( \sigma \in S^{k-1} \). Moreover
\[
\int_{B(0, R)} G(\tau(\sigma)) \, dx \geq c_2 R^N - c_3 R^{N-1}
\]
for any \( \sigma \in S^{k-1} \) and some constants \( c_2, c_3 > 0 \). As in remark 4.2 we define a map \( \tilde{\tau} : S^{k-1} \to H^1_0(B(0, R)) \cap L^\infty(B(0, R)) \) such that \( \tilde{\tau}(\sigma)(x_1, x_2, x_3) = \tau(\sigma)(x_1, x_2, x_3) \phi(|x_1| - |x_2|) \). Observe that \( \tilde{\tau}(\sigma) \in X \) and \( \int_{B(0, R)} G(\tilde{\tau}(\sigma)) \, dx > 0 \) for \( \sigma \in S^{k-1} \) and sufficiently large \( R \). Therefore (a) is satisfied and proof follows from theorem 2.2(c) and from the Palais principle of symmetric criticality [26]. □
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