Abstract

Gauge invariant conserved conformal currents built from massless fields of all spins in 4d Minkowski space-time and $AdS_4$ are described within the unfolded dynamics approach. The current cohomology associated with non-zero conserved charges is found. The resulting list of charges is shown to match the space of parameters of the conformal higher-spin symmetry algebra in four dimensions.
## 1 Introduction

Gauge-invariant conserved currents of different spins in 4d Minkowski space were constructed in [1] in terms of generalized higher-spin (HS) Weyl curvatures introduced originally in [2]. The latter describe on-shell nontrivial gauge-invariant combinations of derivatives of fields which generalize the spin-one Maxwell tensor and linearized spin-two Weyl tensor. Later conserved HS currents were also considered in [3, 4] while nontrivial conserved currents leading to non-zero charges were identified in [5].

In [6] it was shown that global conformal HS symmetries of 4d massless fields of all spins are described by the Weyl algebra $A_4$ of eight oscillators. Algebras of symmetries of equations of motion of irreducible free fields and supermultiplets were also found in [6] while extensions to higher dimensions were elaborated in [7, 8, 9, 10].

A convenient way to analyze conserved quantities is by using the language of differential forms in the framework of so-called unfolded approach [11] in which all fields and field equations are formulated in terms of differential forms (for a review see e.g. [12]). Closed forms describing the gauge-invariant conservation laws in 4d Minkowski space-time in the unfolded approach were found in [4]. In this paper we extend these results to the case of $AdS_4$ and analyze the current cohomology characterizing nontrivial conserved charges. Namely, in [13] it was shown that the space of closed three-forms which can give rise to conserved charges is far larger than the space of HS conserved charges that can be associated with symmetries of massless fields. Hence, it was conjectured in [13] that most of the found closed three-forms are exact. In this paper, we show that this is indeed true and that the current cohomology matches the anticipated higher-spin global symmetries. We focus on the gauge-invariant currents built in terms of generalized Weyl tensors. Note that non-gauge-invariant conserved currents built in terms of HS connections are also available [14] in 4d Minkowski space but not yet in $AdS_4$.

The extension of the construction of gauge-invariant conserved charges to the $AdS_4$ background performed in this paper may have various applications allowing in particular to compute HS charges carried by HS black-hole solutions [15, 16] in the HS theory which requires the $AdS_4$ background.
The rest of the paper is organized as follows. In Section 2 the description of higher-rank fields in the unfolded approach is briefly recalled. In Section 3 we recall the unfolded form of free 4d HS equations in $AdS_4$ proposed in [17] and their flat limit. In Section 4 conserved HS currents in the $AdS_4$ space-time are constructed in terms of covariantly-constant oscillators and De Rham cohomology of gauge-invariant conserved conformal currents built from massless fields of all spins is found. It is shown that resulting nontrivial charges match the space of parameters of the HS symmetry algebra. In Section 5 conserved currents of 4d Minkowski space are reconstructed from the flat limit of those in $AdS_4$.

## 2 Higher-rank fields

Conformal massless fields of all spins in four dimensions can be described [6] by a rank–one zero-form $C(Y|x)$ where $Y^A$ are auxiliary spinor variables ($A, B = 1, ..., 4$, are Majorana spinor indices). It is convenient to interpret $C(Y|x)$ as a vector

$$|C(Y|X)\rangle \tag{2.1}$$

in the Fock space $F$ of the algebra of oscillators $Y^A$ and $Z_A$ that satisfy commutation relations

$$[Y^A, Y^B] = 0, \quad [Z_A, Z_B] = 0, \quad [Z_A, Y^B] = \delta_A^B. \quad (2.2)$$

The Fock vacua $|0\rangle$ and $\langle 0|$ are defined to obey

$$Z_A|0\rangle = 0, \quad \langle 0|Y^A = 0. \quad (2.3)$$

In these terms the rank–one equation of [3] takes the form

$$D|C(Y|X)\rangle := (d + W(Y, Z|X))|C(Y|X)\rangle = 0, \quad (2.4)$$

where $W(Y, Z|X)$ satisfies the flatness condition

$$D^2 = 0 : \quad dW + \frac{1}{2}[W, \wedge W] = 0. \quad (2.5)$$

It is convenient to choose $W(Y, Z|X)$ belonging to the $\mathfrak{sp}(8)$ realized by bilinears of $Y^A$ and $Z_A$

$$W(Y, Z|X) = f_{AB}(X) Y^A Y^B + h^{AB}(X) Z_A Z_B + \frac{1}{2} \omega_A^B(X) \{Y^A, Z_B\}, \quad (2.6)$$

where $h^{AB}, f_{AB}$ and $\omega_A^B$ are components of the one-form connection. The $\mathfrak{sp}(8)$ flatness conditions are

$$R^{AB} := d h^{AB} - \omega^A_C \wedge h^{CB} = 0, \quad R_{AB} := d f_{AB} + \omega^A_C \wedge f_{CB} = 0, \quad (2.7)$$

$$R_A^B := d \omega^B_A + \omega^C_A \wedge \omega^B_C - f_{AC} \wedge h^{CB} = 0.$$
From the oscillator realization it is obvious that the massless field equations (2.4) are invariant under the global symmetry associated with the full Weyl algebra of the oscillators $Y$ and $Z$. Indeed, suppose that $|C(Y|X)\rangle$ solves the rank–one equation. Then any $|\tilde{C}(Y|X)\rangle$ of the form

$$|\tilde{C}(Y|X)\rangle = \eta(Y,Z|X)|C(Y|X)\rangle$$

with $\eta(Y,Z|X)$ satisfying

$$D\eta(Y,Z|X) := d\eta(Y,Z|X) + [W(Y,Z|X),\eta(Y,Z|X)] = 0$$

also solves the rank–one equation. Since the equations (2.9) are consistent with $D^2 = 0$ by virtue of the flatness condition (2.5), their general solution is reconstructed uniquely in terms of $\eta(Y,Z|X_0)$ at any given point $X_0$ (denoted by 0 in the sequel) hence being characterized by an arbitrary function of $Y$ and $Z$.

Because $W(Y,Z|X)$ is bilinear in the oscillators $Y$ and $Z$, the equation (2.9) is homogeneous in the oscillators. In particular, one can solve Eq. (2.9) for $\eta(Y,Z|X)$ linear in $Y$ and $Z$. Clearly, there are eight independent solutions of this type which we denote

$$A_A(Y,Z|X), \quad B_B(Y,Z|X),$$

where $A, B = 1, \ldots, 4$ label independent solutions normalized so that

$$A_A(Y,Z|X_0) = Y^A, \quad B_A(Y,Z|X_0) = Z_A.$$  \hspace{1cm} (2.10)

This normalization guarantees that $A_A(X)$ and $B_B(X)$ obey canonical commutation relations at any $X$

$$[A^A_A, A^B_B] = 0, \quad [B^A_A, B^B_B] = 0, \quad [B^A_A, A^B_B] = \delta^B_B.$$  \hspace{1cm} (2.11)

Indeed, since the oscillators $A_A$ and $B_B$ are covariantly constant with respect to $D$, their commutator is also covariantly constant. Since the commutator of linear combinations of the oscillators $Y^A$ and $Z_B$ is independent of the oscillators, $D$ acts on the commutator as $d$.

In terms of $A_A$ and $B_B$, general solution of the equation (2.9) is

$$\eta(Y,Z|X) = \eta(A,B), \quad \eta(A,B) \equiv \eta(A(Y,Z|X),B(Y,Z|X)).$$  \hspace{1cm} (2.12)

Thus, in agreement with the results of [6], global conformal HS symmetries form the Lie algebra associated with the Weyl algebra $A_4$ of four pairs of oscillators. It contains the $\mathfrak{sp}(8)$ subalgebra of bilinears of oscillators.

In fact, the generating elements of the symmetry parameters $A_A$ and $B_B$ can be interpreted as supergenerators

$$Q^A_A = A^A_A, \quad Q^B_B = B^B_B,$$  \hspace{1cm} (2.13)

which, together with their anticommutators

$$T^A_B = Q^A_A Q^B_B, \quad T^{AB} = Q^A_A Q^B_B, \quad T^A_C = \frac{1}{2} \{Q^A_A, Q^B_B\},$$  \hspace{1cm} (2.14)

form $\mathfrak{osp}(1,8)$. 


The rank-\( r \) equations for \( r \) species of oscillators \( Y^A, Z_A \rightarrow Y^A_i, Z^i_A \) \((i, j = 1 \ldots r)\) can be considered analogously with the Fock space realization of the rank-\( r \) field \(|C(Y_i|X)\)

\[
D|C(Y|X)\rangle := (d + W^{(r)}(Y, Z|X))|C(Y|X)\rangle = 0, \quad W^{(r)}(Y, Z|X) = \sum_{i=1}^{r} W(Y_i, Z^i).
\]

(2.16)

Obviously, these equations are invariant under the global symmetry with parameters \( \eta(A, B) \) valued in the Weyl algebra \( A_{4r} \) generated by

\[
A^i_A = A^i_A(Y_i, Z^i|X), \quad B^i_A = B^i_A(Y_i, Z^i|X).
\]

(2.17)

Let \( \mathfrak{hs}(n; \mathbb{C}) \) be the complex Lie superalgebra resulting from \( A_n \) via the \( \mathbb{Z}_2 \) graded commutator \([f, g] \) where homogeneous elements \( f(Z, Y) \) are associated with even and odd elements, \( f(Z, Y) = (-1)^{\nu} f(-Z, -Y) \). Then the (complexified) rank-\( r \) equations are invariant under \( \mathfrak{hs}(4r; \mathbb{C}) \). The rank-one HS algebra \( \mathfrak{hs}(4; \mathbb{C}) \) belongs to \( \mathfrak{hs}(4r; \mathbb{C}) \). Among a number of inequivalent embeddings of \( \mathfrak{hs}(4; \mathbb{C}) \) into \( \mathfrak{hs}(4r; \mathbb{C}) \), the principal embedding where the element \( f(Y, Z) \in \mathfrak{hs}(4; \mathbb{C}) \) is represented by \( \sum_{i=1}^{r} f(Y_i, Z_i) \) is most important.

In the 4\( d \) Minkowski setup it is convenient to use two-component spinor notation. In these terms,

\[
X^{AB} = (x^{\alpha\beta'}, x^{\alpha\beta}, \bar{x}^\alpha\beta'), \quad Y^A = (y^\alpha, \bar{y}^\beta), \quad Z_A = (z_\alpha, \bar{z}_\beta),
\]

(2.18)

where \( y^\alpha, z_\alpha, \bar{y}^\beta, \bar{z}_\beta \) are conjugated two-component spinor oscillators with nonzero commutation relations

\[
[y^\alpha, z_\beta] = \delta_\alpha^\beta, \quad [\bar{y}^\beta, \bar{z}_\alpha] = \delta_\beta^\alpha.
\]

(2.19)

The 4\( d \) conformal algebra \( \mathfrak{su}(2, 2) \sim \mathfrak{so}(4, 2) \) has connections \( h^{\alpha\beta'}, \omega_\alpha^\beta, \bar{\omega}_\alpha^{\beta'} \), \( b \) and \( f_{\alpha\alpha'} \). Extending \( \mathfrak{su}(2, 2) \) to \( \mathfrak{u}(2, 2) \) by adding a central helicity generator with the gauge connection \( \bar{b} \), the \( \mathfrak{u}(2, 2) \) flatness conditions read as

\[
R^{\alpha\beta'} := dh^{\alpha\beta'} - \omega_\gamma^\alpha \wedge \eta^{\beta'} - \bar{\omega}_{\gamma'}^\beta \wedge h^{\alpha\gamma'} = 0,
\]

(2.20)

\[
R_{\alpha\beta'} := df_{\alpha\beta'} + \omega_\gamma^\alpha \wedge f_{\gamma\beta'} + \bar{\omega}_{\gamma'}^\beta \wedge f_{\alpha\gamma'} = 0,
\]

(2.21)

\[
R_\alpha^\beta := dh_\alpha^\beta - \omega_\gamma^\alpha \wedge \eta^\beta - f_{\gamma\alpha'} \wedge h^{\gamma\beta'} = 0,
\]

(2.22)

\[
\bar{R}_{\alpha'}^\beta' := df_{\alpha'}^\beta' - \bar{\omega}_{\gamma'}^\beta \wedge \bar{\omega}_{\gamma'}^\alpha \wedge f_{\gamma\alpha'} \wedge h^{\gamma\beta'} = 0.
\]

(2.23)

Here the traceless parts \( \omega^{\alpha\beta}_L \) and \( \bar{\omega}^{\alpha\beta'}_L \) of \( \omega_\alpha^\beta \) and \( \bar{\omega}_\alpha^{\beta'} \) describe the Lorentz connection, while the traces are associated with the gauge fields of dilatation \( b \) and helicity generator \( \bar{b} \)

\[
b = \frac{1}{2}\left(\omega_\alpha^\alpha + \bar{\omega}_{\alpha'}^{\alpha'}\right), \quad \bar{b} = \frac{1}{2}\left(\omega_\alpha^\alpha - \bar{\omega}_{\alpha'}^{\alpha'}\right).
\]

(2.24)

Reduction of the unfolded equations \([2.4]\) to the \( \mathfrak{u}(2, 2) \subset \mathfrak{sp}(8) \) invariant setup for the massless field \( C(y, \bar{y}|x) \) is

\[
D^\mu_u C(y, \bar{y}|x) = 0,
\]

(2.25)
where
\[ D_{tw}^u = D_{tw}^c \frac{1}{2} \bar{b}(y^\alpha z_\alpha - \bar{y}^{\alpha'} \bar{z}_{\alpha'}) , \]
and \( D_{tw}^c \) is the conformal covariant derivative
\[ D_{tw}^c = d + \omega^L_{\alpha \beta} y^\alpha z_\beta + \bar{\omega}^L_{\alpha' \beta'} \bar{y}^{\alpha'} \bar{z}_{\beta'} + f_{\alpha \alpha'} y^\alpha \bar{y}^{\alpha'} + h^{\alpha \alpha'} z_\alpha \bar{z}_{\alpha'} + \frac{1}{2} b(2 + y^\alpha z_\alpha + \bar{y}^{\alpha'} \bar{z}_{\alpha'}) . \]

The \( AdS_4 \) description with the background fields valued in \( \mathfrak{sp}(4) \subset \mathfrak{su}(2, 2) \) results from the Ansatz
\[ h^{\alpha \alpha'} = -\lambda e^{\alpha \alpha'}, \quad f_{\alpha \alpha'} = \lambda e^{\alpha \alpha'}, \quad b = \tilde{b} = 0 \]
and
\[ D_{tw}^{\text{ads}} = D^L + \lambda e^{\alpha \alpha'} y_\alpha \bar{y}_{\alpha'} - \lambda e^{\alpha \alpha'} z_\alpha \bar{z}_{\alpha'}, \quad D^L = d + (\omega^L_{\alpha \beta} y_\alpha z_\beta + \bar{\omega}^L_{\alpha' \beta'} \bar{y}^{\alpha'} \bar{z}_{\beta'}). \]

Here indices are raised and lowered by the two-component symplectic forms \( \varepsilon_{\alpha \beta} \) and \( \varepsilon_{\alpha' \beta'} \)
\[ A_\beta = A^\alpha \varepsilon_{\alpha \beta}, \quad A_{\beta'} = A^{\alpha'} \varepsilon_{\alpha' \beta'}, \quad A^\alpha = A_\beta \varepsilon^{\alpha \beta}, \quad A^{\alpha'} = A_{\beta'} \varepsilon^{\alpha' \beta'}. \]

The rank-one unfolded equation in the \( AdS \) space is
\[ D_{tw}^{\text{ads}} C(y, \bar{y}|x) = 0 . \]

## 3 Rank-two equations and covariant oscillators in \( AdS_4 \)

In this paper we are specifically interested in the case of rank–two field \( |J(Y|x)\rangle \) because, as shown in [13], it is associated with 4d conformal conserved currents built from massless fields of all spins. It satisfies the rank–two current equation
\[ D^{(2)}|J(Y|x)\rangle := (d + \sum_{i=1,2} W(Y_i, Z_i|X))|J(Y|x)\rangle = 0 \]
with \( W(Y_i, Z|x) \) (2.6).

In the two-component spinor notation (2.18), after the rescaling the oscillator variables \( y_2, \bar{y}_2 \rightarrow i y_2, i \bar{y}_2 \) and \( z_2, \bar{z}_2 \rightarrow i z_2, i \bar{z}_2 \) for the future convenience, this gives
\[ D^{(2)}_{\text{tw}}|J(Y|x)\rangle = 0 , \]
\[ D^{(2)\text{tw}}_{\text{ads}} = D^{(2)L} + \tilde{W}^{(2)} , \]
\[ \tilde{W}^{(2)} = \lambda (e^{\alpha \alpha'} y_\alpha y_{\alpha'}^{\dagger} - e^{\alpha \alpha'} z_\alpha z_{\alpha'}^{\dagger}) - \lambda (e^{\alpha \alpha'} y_\alpha y_{\alpha'}^{\dagger} - e^{\alpha \alpha'} z_\alpha z_{\alpha'}^{\dagger}) , \]
\[ D^{(2)L} = d + (\omega^L_{\alpha \beta} (y_1 z_{\beta}^{\dagger} + y_2 z_{\beta}^{\dagger}) + \bar{\omega}^L_{\alpha' \beta'} (\bar{y}_1 z_{\beta'}^{\dagger} + \bar{y}_2 z_{\beta'}^{\dagger}) ) . \]

Let us look for a three-form \( \omega \), closed by virtue of the rank-two equations, in the form
\[ \omega = \langle \Omega | J \rangle , \]
where $\langle \Omega \rangle$ is a three-form that obeys the equation
\[
d\langle \Omega \rangle + \langle \Omega \rangle \hat{W}^{(2)}(Y, Z|x) = 0,
\]
which, together with the current equation (3.1), implies that $\omega$ is closed
\[
d\omega = 0.
\]

On-shell closed forms generate conserved charges
\[
Q(\omega) = \int_{M^3} \omega.
\]

Conservation means that $Q$ is independent of local variations of the surface such as local variation of the time parameter. Exact $\omega = d\chi$ do not contribute to $Q$ for solutions of the field equations that decrease fast enough at space infinity. Hence, nontrivial charges $Q$ are associated with the cohomology of currents.

Clearly, any three-form
\[
\omega(\eta(A, B)) = \langle \Omega | \eta(A, B) | J \rangle,
\]
where $\eta(A, B)$ is any function of the oscillators $A_i^1\hat{A}$ and $B_i^1\hat{B}$ with $i = 1, 2$, is also closed. Hence, the full space of closed currents is the space of arbitrary functions of the oscillators (2.17). This freedom should encode the freedom in different HS charges. Indeed, as shown in [4], the realization of a rank–two field in terms of bilinears of rank–one fields gives rise to the full list of conformal gauge invariant conserved currents of all spins in four dimensions, which generalize the so-called generalized Bell-Robinson currents constructed by Berends, Burgers and van Dam [1]. However, the freedom in a function of two sets of oscillators $A_i^1\hat{A}$ and $B_i^1\hat{B}$ is far larger than that in HS symmetries of rank–one equations, parametrized by a function of the rank–one variables $A_i\hat{A}$ and $B_i\hat{B}$ (2.10). Hence, in [13] we conjectured that most of the closed forms (3.8) are exact, generating no nontrivial HS charges. The identification of nontrivial conserved charges in the flat space was done in [5] by a different approach. In this paper we extend these results to the $AdS_4$ geometry using the methods of unfolded dynamics.

Let us pack the oscillators $y_{\alpha i}, \bar{y}_{\alpha i}, z_{\alpha i}, \bar{z}_{\alpha i}$ into $\kappa_{\alpha n}^{\hat{n}}, \zeta_{\alpha n}^{\hat{n}}$ with $n = +, -$ and $\hat{n} = \hat{+}, \hat{-}$ by setting
\[
k_{\alpha n}^{\hat{+}} = y_{\alpha 2}, \quad k_{\alpha n}^{\hat{-}} = y_{\alpha 1}, \quad k_{\alpha n}^{\hat{\hat{+}}} = z_{\alpha 1}, \quad k_{\alpha n}^{\hat{\hat{-}}} = z_{\alpha 2};
\]
\[
\zeta_{\alpha n}^{\hat{+}} = z_{\alpha 2}, \quad \zeta_{\alpha n}^{\hat{-}} = z_{\alpha 1}, \quad \zeta_{\alpha n}^{\hat{\hat{+}}} = \bar{y}_{\alpha 1}, \quad \zeta_{\alpha n}^{\hat{\hat{-}}} = \bar{y}_{\alpha 2}.
\]

One can see that in these terms the nonzero commutation relations are
\[
[k_{\beta m}^{n}, k_{\alpha n}^{m\hat{n}}] = \varepsilon^{n m} \varepsilon^{\hat{n} \hat{m}} \varepsilon_{\beta \alpha}, \quad [\zeta_{\beta n}^{\hat{m}}, \zeta_{\alpha n}^{m\hat{n}}] = \varepsilon^{n m} \varepsilon^{\hat{m} \hat{n}} \varepsilon_{\beta \alpha},
\]
where indices are raised and lowered by $\varepsilon_{n m}, \varepsilon_{\hat{n} \hat{m}}, \varepsilon^{n m}$ and $\varepsilon^{\hat{n} \hat{m}}$ with
\[
\varepsilon_{-+} = -\varepsilon_{+ -} = 1, \quad \varepsilon_{-+} = -\varepsilon_{+ -} = 1, \quad \varepsilon_{\hat{-} \hat{+}} = -\varepsilon_{\hat{+} \hat{-}} = 1, \quad \varepsilon_{\hat{-} \hat{+}} = -\varepsilon_{\hat{+} \hat{-}} = 1.
\]
Evidently, from (2.3) it follows
\[ \langle 0 | \kappa_\alpha^+ \hat{m} = 0 , \quad \langle 0 | \zeta_\alpha^- \hat{m} = 0 , \quad \kappa_\alpha^- \hat{m} | 0 \rangle = 0 , \quad \zeta_\alpha^+ \hat{m} | 0 \rangle = 0 . \] (3.12)

In these terms, Eq. (3.3) takes the form
\[ \tilde{W}^{(2)}(\kappa, \zeta|x) = \lambda \epsilon^{\alpha\beta} \kappa_\alpha^m \zeta_\beta^m \hat{m} \hat{n} . \] (3.13)

Analogously, the covariantly constant oscillators \( \mathcal{A}, \mathcal{B} \) (2.17) are packed into
\[ \tau_a^n \hat{m}(\kappa, \zeta|x) , \quad \nu_a^n \hat{m}(\kappa, \zeta|x) , \quad a = 1, 2 ; \quad a' = 1, 2 \] (3.14)
so that
\[ \tau_a^m \hat{m}(\kappa, \zeta|0) = \kappa_a^m \delta_a^n , \quad \nu_a^m \hat{m}(\kappa, \zeta|0) = \zeta_a^m \delta_a^n . \] (3.15)

Eq. (3.13) guarantees that the oscillators \( \tau(x) \) and \( \nu(x) \) satisfy the canonical commutation relations analogous to (3.10) at any \( x \)
\[ [\tau_b^n \hat{m}(x) , \tau_a^m \hat{m}(x) = \varepsilon^{nm} \varepsilon_{ba'} , \quad [\nu_b^n \hat{m}(x) , \nu_a^n \hat{m}(x) = \varepsilon^{nm} \varepsilon_{ba'} , \quad [\tau_b^n \hat{m}(x) , \nu_a^m \hat{m}(x) = 0 . \] (3.16)

This can also be seen in terms of Killing spinors \( c^\beta(x) \) and \( s^\beta(x) \) of [13], that obey the equations
\[ D^L c^\alpha(x) + \lambda \epsilon^{\alpha\beta} s_\beta(x) = 0 , \quad D^L s^\beta(x) + \lambda \epsilon^{\alpha\beta} c_\alpha(x) = 0 . \] (3.17)

A basic of the space of solutions of this system is formed by four independent pairs of spinors \( (c_a^\beta(x), s_a^\beta(x)) \) and \( (c_a^{a'}(x), s_a^{a'}(x)) \) labeled by \( a = 1, 2 \) and \( a' = 1, 2 \) and obeying the following initial conditions at \( x = 0 \)
\[ c_a^\beta(0) = \delta_a^\beta , \quad s_a^\beta(0) = 0 , \quad c_a^{a'}(0) = 0 , \quad s_a^{a'}(0) = \delta_a^{a'} . \] (3.18)

where 0 denotes any point of space-time. From these conditions it follows that
\[ \overline{c_a^\beta(x)} = s_a^{a'}(x) , \quad \overline{s_a^\beta(x)} = c_a^{a'}(x) . \]

A specific form of the Killing spinors depends on a chosen coordinate system. The covariantly constant oscillators \( \tau, \nu \) are expressed via the Killing spinors as
\[ \tau_a^m \hat{m} = c_a^\gamma(x) \kappa_\gamma^m \hat{k} - s_a^\gamma(x) \zeta_\gamma^m \hat{k} , \quad \nu_a^m \hat{m} = -c_a^\gamma(x) \kappa_\gamma^m \hat{k} + s_a^\gamma(x) \zeta_\gamma^m \hat{k} . \] (3.19)

We observe that the operators
\[ f^{nm} = \frac{1}{4} \{ \kappa_\beta^m \hat{m} , \kappa_\beta^m \hat{m} \} + \frac{1}{4} \{ \zeta_\beta^m \hat{m} , \zeta_\beta^m \hat{m} \} , \] (3.20)
\[ g^{nm} = \frac{1}{4} \{ \kappa_\beta^m \hat{n} , \kappa_\beta^m \hat{n} \} + \frac{1}{4} \{ \zeta_\beta^m \hat{n} , \zeta_\beta^m \hat{n} \} \] (3.21)
are covariantly constant with respect to the rank-two covariant derivative $D^{(2)\text{tw}}_{ads}$ and form two mutually commuting $\mathfrak{sl}_2$ algebras. The algebras $\mathfrak{sl}_2$ and $\mathfrak{h}_2$ will be referred to as vertical $\mathfrak{sl}_2$ and horizontal $\mathfrak{h}_2$, respectively.

The action of $\mathfrak{sl}_2$ and $\mathfrak{h}_2$ on $\kappa$ and $\zeta$ is

$$
[f^{mn}, \kappa^k \hat{m}] = \frac{1}{2} \varepsilon^{mk} \kappa^j \hat{n} + \frac{1}{2} \varepsilon^{nj} \kappa^m \hat{k}, \quad [f^{mn}, \zeta^k \hat{m}] = \frac{1}{2} \varepsilon^{mk} \zeta^j \hat{n} + \frac{1}{2} \varepsilon^{nj} \zeta^m \hat{k},
$$

$$
[g^{\hat{m} \hat{n}}, \kappa^k] = \frac{1}{2} \varepsilon^{k \hat{m}} \kappa^j \hat{n} + \frac{1}{2} \varepsilon^{j \hat{n}} \kappa^k \hat{m}, \quad [g^{\hat{m} \hat{n}}, \zeta^k] = \frac{1}{2} \varepsilon^{k \hat{m}} \zeta^j \hat{n} + \frac{1}{2} \varepsilon^{j \hat{n}} \zeta^k \hat{m}.
$$

From here it follows that $\mathfrak{h}_2$ and $\mathfrak{sl}_2$ act analogously on the oscillators $\tau$, $\nu$ and $\kappa$.

$$
[f^{mn}, \tau^k \hat{m}] = \frac{1}{2} \varepsilon^{mk} \tau^j \hat{n} + \frac{1}{2} \varepsilon^{nj} \tau^m \hat{k}, \quad [f^{mn}, \nu^k \hat{m}] = \frac{1}{2} \varepsilon^{mk} \nu^j \hat{n} + \frac{1}{2} \varepsilon^{nj} \nu^m \hat{k},
$$

$$
[g^{\hat{m} \hat{n}}, \tau^k] = \frac{1}{2} \varepsilon^{k \hat{m}} \tau^j \hat{n} + \frac{1}{2} \varepsilon^{j \hat{n}} \tau^k \hat{m}, \quad [g^{\hat{m} \hat{n}}, \nu^k] = \frac{1}{2} \varepsilon^{k \hat{m}} \nu^j \hat{n} + \frac{1}{2} \varepsilon^{j \hat{n}} \nu^k \hat{m}.
$$

This is not surprising, since, being covariantly constant, the operators $f \in \mathfrak{sl}_2$ and $g \in \mathfrak{h}_2$ keep the same form both in terms of $\kappa^m \hat{m}$, $\zeta^m \hat{m}$ and in terms of $\tau^m \hat{m}$, $\nu^m \hat{m}$, i.e.,

$$
f^{nm} = \frac{1}{4} \{\tau^b \hat{m}, \tau^b \hat{m}\} + \frac{1}{4} \{\nu^b \hat{m}, \nu^b \hat{m}\},
$$

$$
g^{\hat{m} \hat{n}} = \frac{1}{4} \{\tau \hat{m}, \tau \hat{m}\} + \frac{1}{4} \{\nu \hat{m}, \nu \hat{m}\}.
$$

### 4 On-shell de Rham cohomology of $AdS_4$ currents

Let us show that the three-forms $\Omega^{\hat{m} \hat{k}}$ that satisfy (3.3) can be chosen in the form

$$
\langle \Omega^{\hat{m} \hat{k}} \rangle = \langle 0 | \Omega^{\hat{m} \hat{k}} \rangle, \quad \Omega^{\hat{m} \hat{k}} = \hat{H} \wedge \Omega^{\hat{m} \hat{k}} = \mathcal{H}^{\alpha \beta} \{\kappa_\alpha \hat{m} \zeta_\beta - \hat{k} + \kappa_\alpha \hat{k} \zeta_\beta - \hat{m}\},
$$

Using (3.12) and

$$
\hat{H}^{a \beta} e^{\gamma \gamma'} = e^{\alpha \beta} e^{\gamma \gamma'} \hat{H}, \quad \hat{H} := \frac{1}{4} e_{\alpha \beta} \hat{H}^{\alpha \beta},
$$

Eq. (3.13) yields

$$
d\langle \Omega^{\hat{m} \hat{k}} \rangle = \langle 0 | \langle \Omega^{\hat{m} \hat{k}} | \hat{W}^{(2)}(\kappa, \zeta) x \rangle = \langle 0 | \hat{W}^{(2)}(\kappa, \zeta) \Omega^{\hat{m} \hat{k}} \rangle + \langle 0 | \hat{W}^{(2)}(\kappa, \zeta) \Omega^{\hat{m} \hat{k}} \rangle = \lambda \mathcal{H} \langle 0 | \kappa_{\beta - \hat{m} \zeta_{\beta'} - \hat{k}} + \kappa_{\beta - \hat{m} \zeta_{\beta'} - \hat{k}} + \kappa_{\beta - \hat{m} \zeta_{\beta'} - \hat{k}} \rangle = 0
$$

since the last term is symmetric in $\hat{k}$ and $\hat{m}$ while because all indices take two values it has to be proportional to $\epsilon_{\hat{k} \hat{m}}$. Then for an arbitrary polynomial $\eta_{\hat{m} \hat{k}}$ the form

$$
\omega = \langle \Omega^{\hat{m} \hat{k}} | \eta_{\hat{m} \hat{k}}(\tau, \nu) | J(Y) x \rangle
$$

(4.4)
with $\langle \Omega^{\tilde{m}} \tilde{k} | (4.1) \rangle$ is closed provided that $J(Y|x)$ satisfies the current equation (3.2).

The central fact of the analysis of the on-shell cohomology is that each of the forms

$$\omega^m_{\alpha} \tilde{m} = \langle \Omega^{\tilde{m}} \tilde{k} | \tau^m_{\alpha} \tilde{k} \eta(\tau, v) | J(Y|x) \rangle, \quad \omega^m_{\alpha} \tilde{m} = \langle \Omega^{\tilde{m}} \tilde{k} | \nu^{\alpha}_{\beta} \tilde{k} \eta(\tau, \nu_{\alpha}) | J(Y|x) \rangle$$

is exact provided that $|J(Y|x)\rangle$ solves (3.2).

To prove this fact the following simple formulae are useful

$$[\Omega^{\tilde{m}} \tilde{k}, \tau^m_{\alpha} \tilde{k}] = [H^{\alpha\beta} \{K^\mu_{\alpha} \tilde{m} \zeta^\nu_{\alpha} \tilde{k} + K^\nu_{\alpha} \tilde{k} \zeta^\mu_{\alpha} \tilde{m} \}, \tau^m_{\alpha} \tilde{k}] = -3H^{\alpha\beta} s_a \alpha \tau(x) K^a_{\alpha} \tilde{m},$$

and, as a consequence of (3.12),

$$2\langle 0 | c_{ba} = \langle 0 | c_{b} \gamma_{\alpha} \gamma_{\beta} K^a_{\alpha} K^a_{\beta} = \langle 0 | \tau^m_{\alpha} \tilde{m} K^a_{\alpha} \tilde{m}. \quad (4.8)$$

Let us prove that $\omega^m_{\alpha} \tilde{m}$ (3.3) is exact (other cases are analogous). By virtue of (3.19)

$$\tau^m_{\alpha} \tilde{m} = c_{a} \gamma_{\alpha} \gamma_{\beta} K^a_{\alpha} \tilde{m} - s_a \alpha \gamma_{\beta} \zeta^a_{\alpha} \tilde{m}. \quad (4.9)$$

Then, by virtue of the Eqs. (3.17) and (3.18), (3.12) along with (4.6), (4.7), (4.8), using that

$$H^{\alpha\beta} \epsilon_{\gamma\delta} = \epsilon_{\alpha\gamma} H^{\beta\delta} + \epsilon_{\beta\gamma} H^{\alpha\delta}$$

for

$$H^{\alpha\beta} = \epsilon_{\alpha\beta} e^{\beta\alpha},$$

we obtain

$$d\langle 0 | H^{\alpha\beta} c_{ba} K^a_{\beta} \tilde{m} | J(Y|x) \rangle = \langle 0 | H^{\alpha\beta} (d c_{ba}) K^a_{\beta} \tilde{m} - H^{\alpha\beta} c_{ba} K^a_{\beta} \tilde{m} \tilde{W}(2)(\kappa, \zeta|x) | J(Y|x) \rangle$$

$$\langle 0 | \left(-3H^{\beta\gamma} e_{\gamma\delta} c_{ba} \gamma_{\beta} \gamma_{\alpha} K^a_{\gamma} \tilde{m} + \tilde{W}\alpha^{\beta\gamma} \tilde{W}\alpha^{\beta\gamma} c_{ba} K^a_{\beta} \tilde{m} \tilde{W}\alpha^{\beta\gamma} \tilde{W}\alpha^{\beta\gamma} \right) | J(Y|x) \rangle$$

$$\langle 0 | \left[\Omega^{\tilde{m}} \tilde{k}, \tau^m_{\alpha} \tilde{k}\right] + H^{\beta\gamma} c_{ba} \mu_{\beta} K^a_{\beta} \gamma_{\alpha} \gamma_{\gamma} = \kappa_{\mu} \kappa_{\gamma} \zeta_{\nu} \nu_{\gamma} \right] | J(Y|x) \rangle \rangle = \lambda \langle 0 | \left[\tau^m_{\alpha} \tilde{m}, \nu^{\beta}_{\gamma} \tilde{k}\right] \eta(\tau, \nu) | J(x) \rangle \right) = \lambda \langle 0 | \left[\tau^m_{\alpha} \tilde{m}, \nu^{\beta}_{\gamma} \tilde{k}\right] \eta(\tau, \nu) | J(x) \rangle \right).$$

Analogously one can show that the both of the forms in (4.6) that contain antisymmetrizations (contractions) of hatted indices in $\Omega^{\tilde{m}} \tilde{k} \tau^m_{\alpha} \tilde{k}$ and $\Omega^{\tilde{m}} \tilde{k} \nu^{\alpha}_{\beta} \tilde{k}$ are exact. An important consequence of this fact along with the commutation relations (3.10) is that any of the forms

$$\langle \Omega^{\tilde{m}} \tilde{k} | \{\tau^m_{\alpha} \tilde{k}, \nu^{\alpha}_{\beta} \tilde{k}\} \eta(\tau, v) | J \rangle, \quad \langle \Omega^{\tilde{m}} \tilde{k} | \{\nu^{\alpha}_{\beta} \tilde{k}, \nu^{\alpha}_{\beta} \tilde{k}\} \eta(\tau, v) | J \rangle \right)$$

are also exact.

To prove this for the first expression in Eq. (1.12) we observe that

$$0 \sim \langle \Omega^{\tilde{m}} \tilde{n} | (\tau^m_{\alpha} \tilde{n} \nu^{\alpha}_{\beta} \tilde{k} - \nu^{\alpha}_{\beta} \tilde{k} \nu^{\alpha}_{\beta} \tilde{k}) \eta(\tau, v) | J \rangle = \frac{1}{2} \langle \Omega^{\tilde{m}} \tilde{n} | \tilde{e} \tilde{n} \tilde{k} \nu^{\alpha}_{\beta} \tilde{k} \eta(\tau, v) | J \rangle \rangle = \frac{1}{2} \langle \Omega^{\tilde{m}} \tilde{n} | \tilde{e} \tilde{n} \tilde{k} \nu^{\alpha}_{\beta} \tilde{k} \eta(\tau, v) | J \rangle \rangle.$$
The proof for the other forms in (4.12) is analogous.

This fact admits the following interpretation. The bilinears in \(\tau_a^m \bar{\eta}^\hat{k}\) and \(\nu^b \bar{\eta}^\hat{k}\) form a Lie algebra \(\mathfrak{sp}(16)\) while

\[
G^{m,k}_{a b} = \frac{1}{2}\{\tau_a^m \bar{\eta}^\hat{k}, \tau_b^k \bar{\eta}^\hat{k}\}, \quad G^{m,k}_{a b'} = \frac{1}{2}\{\tau_a^m \bar{\eta}^\hat{k}, \nu^b \bar{\eta}^\hat{k}\}, \quad G^{m,k}_{a' b'} = \{\nu^a' \bar{\eta}^\hat{k}, \nu^b' \bar{\eta}^\hat{k}\}
\]

form a Lie algebra \(\mathfrak{o}(8)\) that commutes with the horizontal \(\hbar\mathfrak{s}\mathfrak{l}_2\) (3.21) acting on the hatted indices. For the space of parameters \(\eta\) polynomial in oscillators, factorization of the generators (4.14) allows us to factor out any combination of oscillators with antisymmetrization of a pair of the hatted Latin indices.\(^1\)

Since the forms (4.12) and (4.3) that contain antisymmetrized hatted indices are exact, the leftover forms belong to the space of differential forms \(\hat{\omega}\) (4.4) with totally symmetrized hatted indices. To describe such forms consider expressions

\[
\eta_a(\eta) = \Omega^{\hat{n}, \hat{m}} \eta(\tau_a^m \bar{\eta}^\hat{k}, \nu_a^n \bar{\eta}^\hat{k})
\]

polynomial in \(\tau_a^m \bar{\eta}^\hat{k}\) and \(\nu_a^n \bar{\eta}^\hat{k}\). Let \(P_{AdS}\) be the space of polynomials (4.13) with symmetrized hatted indices. Clearly, \(\hbar\mathfrak{s}\mathfrak{l}_2\) leaves \(P_{AdS}\) invariant. Since any \(\hbar\mathfrak{s}\mathfrak{l}_2\)-highest vector has the form

\[
\Omega^{\hat{+}, \hat{+}} \eta(\tau_a^m \bar{\eta}^\hat{k}, \nu_a^n \bar{\eta}^\hat{k})
\]

for some polynomial \(\eta\), \(P_{AdS}\) is a span of vectors

\[
\Lambda_n(\eta) = \frac{1}{n!} \text{ad}^n_g \left( \Omega^{\hat{+}, \hat{+}} \eta(\tau_a^m \bar{\eta}^\hat{k}, \nu_a^n \bar{\eta}^\hat{k}) \right), \quad \text{ad}_x(y) = [x, y].
\]

Now we observe that the Cartan elements \(g^{\hat{+}, \hat{+}} \in \hbar\mathfrak{s}\mathfrak{l}_2\) (3.21) and \(f^{\hat{-}, \hat{+}} \in \mathfrak{u}\mathfrak{s}\mathfrak{l}_2\) (3.20) annihilate both \(|0\rangle\) and \(|\emptyset\rangle\). Hence,

\[
\langle 0 \mid [g^{\hat{+}, \hat{+}}, \Lambda_n(\eta)J] \mid 0 \rangle = 0, \quad \langle 0 \mid [f^{\hat{-}, \hat{+}}, \Lambda_n(\eta)J] \mid 0 \rangle = 0
\]

for any \(\Lambda_n(\eta)\) (4.17) and \(J\). This implies that \(\Lambda_n(\eta)\) can give nonzero form only if

\[
[g^{\hat{+}, \hat{+}}, \Lambda_n(\eta)J] = 0, \quad [f^{\hat{-}, \hat{+}}, \Lambda_n(\eta)J] = 0.
\]

Note that \([f^{\hat{-}, \hat{+}}, \bar{\eta}^{\hat{m}} \bar{\eta}^\hat{n}] = 0\), \([f^{\hat{-}, \hat{+}}, \Omega^{\hat{m}} \bar{\eta}^\hat{n}] = 0\). Hence for a given current field \(J = J_{sh,sv}^\tau\) such that

\[
[g^{\hat{+}, \hat{+}}, J_{sh,sv}^\tau] = s_h J_{sh,sv}^\tau, \quad [f^{\hat{-}, \hat{+}}, J_{sh,sv}^\tau] = s_v J_{sh,sv}^\tau
\]

and polynomial \(\eta_{sh,sv}(\tau, \nu)\) such that

\[
[g^{\hat{+}, \hat{+}}, \eta_{sh,sv}] = e_h \eta_{sh,sv}, \quad [f^{\hat{-}, \hat{+}}, \eta_{sh,sv}] = e_v \eta_{sh,sv},
\]

\(^1\)Beyond the space of polynomials the situation is different because, formally, it follows that all nontrivial \(\mathfrak{o}(8)\)-modules should be factored out. However, this ‘exact’ representation for conserved currents turns out to be space-time non-local, containing infinitely many derivatives, giving rise to quasi-exact representations analogous to those considered in [18].
Eq. (4.19) holds for

\[ \eta = \eta_{e_h,e_v}, \quad \text{and} \quad J = J_{s_h,s_v} \]  (4.21)

provided that

\[ e_h + 2 - 2n + s_h = 0, \quad e_v + s_v = 0. \]  (4.22)

Hence, the condition \( \langle 0| \Lambda_n(\eta_{e_h,e_v}) J_{s_h,s_v}(Y|x)|0 \rangle \neq 0 \) determines \( e_h \) via \( n \) and \( s_h \) via Eq. (4.22).

Decomposing \( J \) into \( J = \sum_{s_h,s_v} J_{s_h,s_v} \) according to (4.20), a generating function for elements of \( P_{AdS} \) can be written as

\[ \Lambda^+_{gen} = \sum_{e_h,e_v} \Lambda_n(\eta_{e_h,e_v}) \tilde{c}_+(g^{\sim^+}) \equiv \sum_{e_h,e_v} \exp \left( \text{ad}_{g^{\sim^+}} \left( \Omega^{\sim^+} \eta_{e_h,e_v}(\tau^{m^+_a}, v^{n^+_a}) \right) \right) \tilde{c}_+(g^{\sim^+}), \]  (4.23)

with an arbitrary function \( \tilde{c}_+ \) distinguishing between \( J_{s_h,s_v} \) with different \( s_h \).

To distinguish between odd and even currents with respect to spinorial variables it is convenient to consider another generation function

\[ \Lambda^-_{gen} = \sum_{e_h,e_v} \exp \left( -\text{ad}_{g^{\sim^-}} \left( \Omega^{\sim^-} \eta_{e_h,e_v}(\tau^{m^-_a}, v^{n^-_a}) \right) \right) \tilde{c}_-(g^{\sim^-}). \]  (4.24)

Using the Taylor expansion \( f(x + y) = \exp(x \frac{\partial}{\partial y})f(y) \), Eqs. (4.23), (4.24) give

\[ \Lambda^{\pm}_{gen} = \tilde{\Omega}^{\pm} \eta_{\pm}(\tau^{m^+_a} \pm \tau^{m^-_a}, v^{n^+_a} \pm v^{n^-_a}|g^{\sim^\pm}), \]  (4.25)

where \( \eta_{\pm}(a^{m^+_a}, b^{n^+_a}|g) = \sum_{e_h,e_v} \eta_{e_h,e_v}(a^{m^+_a}, b^{n^+_a})\tilde{c}_{\pm}(g) \), while

\[ \tilde{\Omega}^{\pm} = \Omega^{\sim^+} + \Omega^{\sim^-} \pm \left( \Omega^{\sim^+} + \Omega^{\sim^-} \right). \]  (4.26)

To make contact with [13], let us reformulate the result in the notations analogous (modulo some rescalings of variables) to those of [13] with \( \partial_{\pm\alpha} = z^\pm_{\alpha} \pm z_{\alpha}^2, y_{\pm\alpha} = y_{1\alpha} \pm y_{2\alpha} \) etc. In these terms, nontrivial charges are represented by the closed three-forms

\[ H^{\alpha\alpha'} \partial_{-\alpha} \partial_{-\alpha'} \eta(g, \bar{g}, H_1 - H_2) J(y_{\pm}, \bar{y}_{\pm}|x) \bigg|_{y_{\pm}=\bar{y}_{\pm}=0}, \]  (4.27)

\[ H^{\alpha\alpha'} \partial_{+\alpha} \partial_{+\alpha'} \eta(\epsilon, \bar{\epsilon}, H_1 - H_2) J(y_{\pm}, \bar{y}_{\pm}|x) \bigg|_{y_{\pm}=\bar{y}_{\pm}=0}, \]  (4.28)

where \( J \) satisfies the current equation (3.2),

\[ g^{\sim^+} = H_1 - H_2, \quad H_1 = y^{1\alpha} \frac{\partial}{\partial y^{1\alpha}} - \bar{y}^{1\alpha'} \frac{\partial}{\partial \bar{y}^{1\alpha'}}, \quad H_2 = y^{2\alpha} \frac{\partial}{\partial y^{2\alpha}} - \bar{y}^{2\alpha'} \frac{\partial}{\partial \bar{y}^{2\alpha'}}, \]

\[ \epsilon^-_{\alpha} = c_{\alpha}(x)y_{\alpha} + s_{\alpha'}(x)\bar{y}_{\alpha'}, \quad \epsilon^+_{\alpha} = c_{\alpha}(x)\partial_{+\alpha} + s_{\alpha'}(x)\bar{y}^-_{\alpha'}, \]

\[ \bar{\epsilon}^-_{\alpha'} = s_{\alpha'}(x)\bar{y}_{\alpha'} + c_{\alpha}(x)\partial_{+\alpha}, \quad \bar{\epsilon}^+_{\alpha'} = s_{\alpha'}(x)\bar{y}^-_{\alpha'} + c_{\alpha}(x)y^-_{\alpha}, \]
\[ \varrho_{-\alpha} = c_{a}^{\alpha}(x)\partial_{-\alpha} + s_{a}^{\alpha'}(x)\bar{y}^{+}\alpha', \quad \varrho_{+\alpha} = c_{a}^{\alpha}(x)y_{+\alpha}^{\alpha} + s_{a}^{\alpha'}(x)\bar{\partial}_{-\alpha'}, \]
\[ \bar{\varrho}_{-\alpha'} = s_{a}^{\alpha'}(x)\bar{\partial}_{-\alpha'} + c_{a}^{\alpha'}(x)y_{+\alpha}, \quad \bar{\varrho}_{+\alpha'} = s_{a}^{\alpha'}(x)y_{+\alpha'}^{\alpha} + c_{a}^{\alpha'}(x)\partial_{-\alpha}. \]

For bilinear currents
\[ J(y^{\pm}\bar{y}^{\pm}|x) = C_{p}^{+}(y^{+} + y^{-}, \bar{y}^{+} + \bar{y}^{-}|x)C_{q}^{-}(y^{+} - y^{-}, \bar{y}^{+} - \bar{y}^{-}|x), \]
where rank–one fields \( C^{\pm}(y, \bar{y}|x) \) solve \( (2.31) \)
\[ D_{\text{ads}}^{tw} C_{p}^{+}(y, \bar{y}|x) = 0, \quad D_{\text{ads}}^{tw} C_{q}^{-}(iy, i\bar{y}|x) = 0 \]
and have helicities \( \frac{1}{2}p, \frac{1}{2}q \) respectively, \( s_{h}(J) = p - q \), Eqs. (4.27) give two generating functions for bilinear conserved currents in \( AdS \) space
\[ H^{\alpha\alpha'} \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial \bar{y}^{\alpha'}}\eta(\varrho, \bar{\varrho}|p - q)C_{p}^{+}(y^{+} + y^{-}, \bar{y}^{+} + \bar{y}^{-}|x)C_{q}^{-}(y^{+} - y^{-}, \bar{y}^{+} - \bar{y}^{-}|x)|_{y^{\pm} = \bar{y}^{\pm} = 0}, \]
\[ H^{\alpha\alpha'} \frac{\partial}{\partial y^{+\alpha}} \frac{\partial}{\partial \bar{y}^{+\alpha'}}\eta(\epsilon, \bar{\epsilon}|p - q)C_{p}^{+}(y^{+} + y^{-}, \bar{y}^{+} + \bar{y}^{-}|x)C_{q}^{-}(y^{+} - y^{-}, \bar{y}^{+} - \bar{y}^{-}|x)|_{y^{\pm} = \bar{y}^{\pm} = 0}, \]
announced in [13], which represent gauge invariant conformal HS current cohomology in \( AdS_{4} \).

## 5 Minkowski current cohomology

Analogous results for the flat Minkowski space announced in [13] can be obtained as follows.

In the limit \( \lambda \to 0 \), appropriately rescaled Minkowski current equations, resulting from (3.2) along with (3.3) and (3.4), take the form (see, e.g., [13])
\[ D_{\text{Mnk}}^{(2)tw} J(Y|x) = 0, \quad D_{\text{Mnk}}^{(2)tw} = d + \tilde{W}_{\text{Mnk}}^{(2)}, \]
\[ \tilde{W}_{\text{Mnk}}^{(2)} = -e^{\alpha\alpha'}\tilde{\epsilon}_{1}\tilde{\epsilon}_{2} + e^{\alpha\alpha'}\epsilon_{1}\epsilon_{2} = e^{\alpha\beta'}\kappa_{\alpha}^{-}\tilde{n}_{\gamma}^{\beta'} - \tilde{n}_{\gamma}^{\beta'}. \]

One can see that \( f^{+} \in \mathfrak{s}\mathfrak{l}_{2} \) (3.20) and the full algebra \( h\mathfrak{s}\mathfrak{l}_{2} \) (3.20) commute with \( D_{\text{Mnk}}^{(2)tw} \),

Minkowski Killing spinors
\[ \tilde{c}_{a}^{\alpha} = \delta_{\alpha}^{a}, \quad \tilde{z}_{a}^{\alpha'} = \delta_{a}^{\alpha'}, \quad \tilde{c}_{a}^{\alpha} = -x^{\alpha\alpha'}\epsilon_{\alpha\alpha'}, \quad \tilde{z}_{a}^{\alpha'} = -x^{\alpha\alpha'}\epsilon_{\alpha\alpha}, \]
obey
\[ \text{d}\tilde{c}_{a}^{\alpha} = 0, \quad \text{d}\tilde{s}_{a}^{\alpha'} + e^{\alpha\beta'}\tilde{c}_{a}^{\alpha} = 0, \quad \text{d}\tilde{z}_{a}^{\alpha'} = 0, \quad \text{d}\tilde{c}_{a}^{\alpha} + e^{\alpha\beta'}\tilde{s}_{a}^{\beta'} = 0. \]

Hence, a basis of covariantly constant oscillators can be chosen as
\[ \vartheta_{\gamma}^{+\kappa} = \tilde{c}_{a}^{\gamma}(x)\kappa_{\gamma}^{-}\kappa + \tilde{s}_{a}^{\gamma'}(x)\zeta_{\gamma}^{'}^{\kappa} + \kappa, \quad \vartheta_{\gamma}^{-\kappa} = \tilde{c}_{a}^{\gamma}(x)\kappa_{\gamma}^{-}\kappa + \tilde{s}_{a}^{\gamma'}(x)\zeta_{\gamma}^{'}^{-}\kappa, \quad \varphi_{\alpha'}^{+\kappa} = \tilde{s}_{a}^{\alpha'}(x)\kappa_{\alpha'}^{+}\kappa. \]
Proceeding analogously to the $AdS_4$ case this gives that the Minkowski nontrivial charges are fully represented by the following closed three-forms

$$\mathcal{H}^{\alpha\alpha'} \partial_{-\alpha} \partial_{-\alpha'} \eta(\xi, \bar{\xi}, H_1 - H_2) J(y^\pm, \bar{y}^\pm | x) \bigg|_{y^\pm = \bar{y}^\pm = 0}, \quad (5.7)$$

$$\mathcal{H}^{\alpha\alpha'} \partial_{+\alpha} \partial_{+\alpha'} \eta(\chi, \bar{\chi}, H_1 - H_2) J(y^\pm, \bar{y}^\pm | x) \bigg|_{y^\pm = \bar{y}^\pm = 0},$$

where $J$ satisfies Minkowski current equations (5.1), $H_i$ are defined in (4.28) and

$$\chi^+ = \frac{\partial}{\partial y^+}, \quad \bar{\chi}^+ = \frac{\partial}{\partial \bar{y}^+}, \quad \chi^- = y^- - x^\alpha \beta^\alpha \partial_{\bar{y}^-} \beta^\alpha, \quad \bar{\chi}^- = \bar{y}^- - x^\alpha \beta^\alpha \partial_y \beta^\alpha, \quad \chi^{-\alpha} = y^{-\alpha} - x^\alpha \beta^\alpha \partial_{\bar{y}^-} \beta^\alpha, \quad \bar{\chi}^{-\alpha} = \bar{y}^{-\alpha} - x^\alpha \beta^\alpha \partial_y \beta^\alpha, \quad (5.8)$$

$$\xi^{-\alpha} = \frac{\partial}{\partial y^-}, \quad \bar{\xi}^{-\alpha} = \frac{\partial}{\partial \bar{y}^-}, \quad \xi^{+\alpha} = y^{+\alpha} - x^\alpha \beta^\alpha \partial_{\bar{y}^+} \beta^\alpha, \quad \bar{\xi}^{+\alpha} = \bar{y}^{+\alpha} - x^\alpha \beta^\alpha \partial_y \beta^\alpha. \quad (5.9)$$

In agreement with the result announced in [13], Eq. (5.7) represents the nontrivial current cohomology in Minkowski space.

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