A Note on Decay Rates of the Local Energy for Wave Equations with Lipschitz Wavespeeds

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Abstract

We consider the Cauchy problem for wave equations with variable coefficients in the whole space \( \mathbb{R}^n \). We improve the rate of decay of the local energy, which has been recently studied by J. Shapiro \[12\], where he derives the log-order decay rates of the local energy under stronger assumptions on the regularity of the initial data.

1 Introduction

We consider in this work the Cauchy problem associated to the wave equation with variable coefficient in \( \mathbb{R}^n \) \((n \geq 1)\) as follow

\[
\begin{align*}
    u_{tt}(t, x) - c(x)^2 \Delta u(t, x) &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
    u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \((u_0, u_1)\) are initial data chosen as

\[
    u_0 \in H^1(\mathbb{R}^n), \quad u_1 \in L^2(\mathbb{R}^n),
\]

and the function \(c : \mathbb{R}^n \to \mathbb{R}\) satisfies the two assumptions below:

(A-1) \(c(x) > 0 \quad (x \in \mathbb{R}^n), \quad c, c^{-1} \in L^\infty(\mathbb{R}^n), \quad \nabla c \in (L^\infty(\mathbb{R}^n))^n,\)

(A-2) there exists a constant \(L > 0\) such that \(c(x) = 1\) for \(|x| > L\).

In particular, the condition (A-1) implies \(c \in C^{0,1}(\mathbb{R}^n)\) (see e.g., [1, Theorem IX.12]). Here, we have set

\[
    u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}, \quad x = (x_1, \cdots, x_n).
\]

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Note that solutions and/or functions considered in this paper are all real valued except for some parts concerning the Fourier transform.

Considering the previous assumptions on the initial data and $c(x)$ it is known that the problem (1.1)-(1.2) has a unique weak solution

$$u \in C([0, \infty); H^1(R^n)) \cap C^1([0, \infty); L^2(R^n)) =: C^0_t,$$

satisfying the energy conservation property:

$$E_u(t) = E_u(0), \quad (1.3)$$

where the total energy $E_u(t)$ to the equation (1.1) is defined by

$$E_u(t) := \frac{1}{2} \int_{R^n} \left( \frac{1}{c(x)^2} |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right) dx.$$

Furthermore, the local energy $E_R(t)$ on the zone $\{|x| \leq R\}$ $(R > 0)$ corresponding to the solution $u(t, x)$ of (1.1)-(1.2) is defined by

$$E_R(t) := \frac{1}{2} \int_{|x| \leq R} \left( \frac{1}{c(x)^2} |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right) dx.$$

Also, we set

$$B(x, R) := \{ y \in R^n : |y - x| < R \}.$$

Our main concern of this paper is to obtain a local energy decay estimate with an algebraic decay order. For related important results concerning the local energy decay, one can cite several celebrated papers due to Morawetz [8], Lax-Phillips [7], Morawetz-Ralston-Strauss [9], Ralston [11], Vainberg [14], and the references therein.

By the way, concerning local energy decay results, quite recently Shapiro [12] announces the following interesting result. It should be mentioned that the decay rate of (1.4) below was first obtained by Burq [2] for smooth perturbations of the Laplacian outside an obstacle.

**Theorem 1.1** (Shapiro [12]) Let $n \geq 2$, and assume (A-1) and (A-2). Suppose that the supports of $u_0$ and $u_1$ are contained in $B(0, R_1)$, and $\nabla u_0 \in (H^1(R^n))^n$ and $u_1 \in H^1(R^n)$. Then for any $R_2 > 0$, there exists $C > 0$ such that the solution $u$ to (1.1)-(1.2) satisfies for $t \geq 0$,

$$E_{R_2}(t) \leq \left( \frac{C}{\log(2 + t)} \right)^2 \left( \|\nabla u_0\|_{H^1(R^n)}^2 + \|u_1\|_{H^1(R^n)}^2 \right). \quad (1.4)$$

Our observation is that Shapiro [12] imposes rather stronger hypothesis on the regularity of the initial data such as

(I) the supports of initial data are compact, and as a result $[u_0, u_1] \in H^2(R^n) \times H^1(R^n)$.

Furthermore, in a sense, (I) the obtained decay order $(\log t)^{-2}$ of the local energy seems to be rather slow.

In this paper, under weaker regularity assumptions on the initial data to modify (I), one obtains faster algebraic decay rate which improves (I) in the case when the coefficient $c(x)$ and the parameter $L$ have a special relation.

Our method is based on the so-called Morawetz identity [8], so we never use the spectral analysis like resolvent estimates. In order to state our results, we introduce the following weighted functional spaces.

$$L^{p, \gamma}(R^n) := \left\{ f \in L^p(R^n) \mid \|f\|_{p, \gamma} := \left( \int_{R^n} (1 + |x|^{\gamma}) |f(x)|^p dx \right)^{1/p} < +\infty \right\}.$$

Our main results read as follows.
**Theorem 1.2** Let \( n \geq 3 \), and assume (A-1) and (A-2). If the initial data \([u_0, u_1] \in H^1(\mathbb{R}^n) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))\) further satisfies

\[
\int_{\mathbb{R}^n} (1 + |x|) \left( \frac{1}{c(x)^2} |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) dx < +\infty,
\]

then the unique solution \( u \in C^\infty_1 \) to problem (1.1)-(1.2) satisfies

\[
E_R(t) = O(t^{-(1-\eta)}) \quad (t \to \infty),
\]

for each \( R > L \) provided that \( \eta := 2L \| \frac{1}{c(x)^2} \|_\infty \| \nabla c \|_\infty \in [0, 1] \).

**Theorem 1.3** Let \( n = 2 \), and assume (A-1) and (A-2). Let \( \gamma \in (0, 1) \). If \([u_0, u_1] \in H^1(\mathbb{R}^n) \times (L^2(\mathbb{R}^n) \cap L^{1,\gamma}(\mathbb{R}^n))\) further satisfies

\[
\int_{\mathbb{R}^2} (1 + |x|) \left( \frac{1}{c(x)^2} |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) dx < +\infty,
\]

and

\[
\int_{\mathbb{R}^2} \frac{u_1(x)}{c(x)^2} dx = 0,
\]

then the unique solution \( u \in C^\infty_1 \) to problem (1.1)-(1.2) satisfies

\[
E_R(t) = O(t^{-(1-\eta)}) \quad (t \to \infty),
\]

for each \( R > L \) provided that \( \eta := 2L \| \frac{1}{c(x)^2} \|_\infty \| \nabla c \|_\infty \in [0, 1] \).

**Theorem 1.4** Let \( n = 1 \), and assume (A-1) and (A-2). If \([u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) further satisfies

\[
\int_{\mathbb{R}} (1 + |x|) \left( \frac{1}{c(x)^2} |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) dx < +\infty,
\]

then the unique solution \( u \in C^\infty_1 \) to problem (1.1)-(1.2) satisfies

\[
E_R(t) = O(t^{-(1-\eta)}) \quad (t \to \infty),
\]

for each \( R > L \) provided that \( \eta := 2L \| \frac{1}{c(x)^2} \|_\infty \| \nabla c \|_\infty \in [0, 1] \).

**Remark 1.1** Our gain is that the \( n = 1 \) dimensional case is included in our results, and we do not assume any compactness of the supports of initial data, and weaker regularity assumptions such as \([u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) are imposed. Instead of stronger regularity as in [12] we have to pay a price to assume various weighted conditions on the initial data in some functional spaces, and the parameter \( \eta \) must be chosen to satisfy \( \eta \in [0, 1] \). This condition on \( \eta \) is crucial in this paper. In this connection, if \( c(x) = 1 \) for all \( x \in \mathbb{R}^n \), then \( \| \nabla c \|_\infty = 0 \), so that \( \eta = 0 \), and in this case the obtained results remind us of those of [5] studied in an exterior domain with a star-shaped compliment set (star-shaped obstacle).

**Remark 1.2** For example, if \( L > 0 \) is small, \( \inf_{x \in \mathbb{R}^n} c(x) \) is sufficiently far from 0, and \( \| \nabla c \|_\infty \) is small, then we can realize the hypothesis \( \eta \in [0, 1] \). The smallness of \( L \) implies \( -c(x)^2 \Delta = -\Delta \) for \( x \in \mathbb{R}^n \setminus B(0, \varepsilon) \) with small \( \varepsilon > 0 \). Note that \( \inf_{x \in \mathbb{R}^n} c(x) > 0 \) under the assumption (A-1).
This paper is organized as follows. In section 2 after preparing several propositions and lemmas we shall prove Theorems 1.2, 1.3 and 1.4 at a stroke. The key tool is already prepared in [3].

**Notation.** Throughout this paper, $\| \cdot \|_q$ stands for the usual $L^q(\mathbb{R}^n)$-norm. For simplicity of notation, in particular, we use $\| \cdot \|$ instead of $\| \cdot \|_2$. Furthermore, we denote $\| \cdot \|_{H^1}$ as the usual $H^1$-norm. On the other hand, we denote the Fourier transform of $f$ by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \left(\frac{1}{2\pi}\right)^\frac{n}{2} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx$$

as usual with $i := \sqrt{-1}$. As the $L^2$-inner product, one employs the following notation:

$$(f,g) := \int_{\mathbb{R}^n} f(x)g(x) dx, \quad f,g \in L^2(\mathbb{R}^n).$$

## 2 $L^2$-bounds of solutions

In order to prove the previous theorems one first prepare in this section the so-called Morawetz identity. This is our starting point.

**Proposition 2.1** Let $n \geq 1$. Under the assumption (A-1), the (unique) weak solution $u \in C^0_1$ to problem (1.1)-(1.2) satisfies

$$tE_u(t) = \frac{n-1}{2} \left( \frac{1}{c(-t)^2} u_1, u_0 \right) + \left( \frac{1}{c(-t)^2} u_1, x \cdot \nabla u_0 \right)$$

$$- \frac{n-1}{2} \left( \frac{1}{c(-t)^2} u_t(t, \cdot), u(t, \cdot) \right) - \left( \frac{1}{c(-t)^2} u_t(t, \cdot), x \cdot \nabla u(t, \cdot) \right)$$

$$+ \int_0^t \int_{\mathbb{R}^n} \frac{1}{c(x)^2} (x \cdot \nabla c(x)) |u_s(s,x)|^2 dx ds \quad (t \geq 0).$$

The proof of the Morawetz identity can be derived first to the smooth solution with $u(t, x)$ for initial data with compact support, say $[u_0, u_1] \in C^0_\infty(\mathbb{R}^n) \times C^0_\infty(\mathbb{R}^n)$, by relying on the multiplier

$$M(u) := tu_t + x \cdot \nabla u + \frac{n-1}{2}u,$$

the finite speed of propagation property, integration by parts, and then by the density arguments. The final identity can be established to the desired weak solution $u \in C^0_1$. Note that $c \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ under the assumption (A-1) (cf. [1] Theorem IX.12).

As a second work, we derive several $L^2$-bounds of solutions under non-compact support conditions on the initial data. For this purpose, we rely on an improvement version of an original idea established in [4] because we can now use the Fourier transform appropriately to obtain them. We have the following significant propositions. These results will be used when one estimates the term $\left( \frac{1}{c(-t)^2} u_t(t, \cdot), u(t, \cdot) \right)$ in Proposition 2.1.

**Proposition 2.2** Let $n \geq 3$. If $[u_0, u_1] \in H^1(\mathbb{R}^n) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))$, then the unique solution $u \in C^0_1$ to problem (1.1)-(1.2) satisfies

$$\|u(t, \cdot)\| \leq C \|c^{-1}\|_{L^\infty}^2 \left( \|u_1\| + \|u_1\|_1 \right) + C \|c^{-1}u_0\|,$$

with some constant $C > 0$. 

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follows. For any \( \varepsilon > 0 \) use the idea from [4] in the Fourier space. Let

\[
\int_{\mathbb{R}^n} \frac{u_1(x)}{c(x)^2} \, dx = 0,
\]

then the unique solution \( u \in C^2_1 \) to problem (1.1)-(1.2) satisfies

\[
\|u(t, \cdot)\| \leq C\|c^{-1}\|_\infty^2 (\|u_1\| + \|u_1\|_{1, \gamma}) + C\|c^{-1}u_0\|,
\]

with some constant \( C > 0 \).

In the course of proofs of Propositions 2.2 and 2.3, the next inequality concerning the Fourier image of the Riesz potential plays an crucial role. This comes from [3, (ii) of Proposition 2.1].

**Proposition 2.4** Let \([n, \gamma, \theta] \) satisfy \( n \geq 1, \gamma \in [0, 1] \) and \( \theta \in [0, \gamma + \frac{n}{2}) \). Then, for all \( f \in L^2(\mathbb{R}^n) \cap L^{1, \gamma}(\mathbb{R}^n) \) satisfying

\[
\int_{\mathbb{R}^n} f(x) \, dx = 0
\]

it is true that

\[
\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|^2}{|\xi|^{2\theta}} \, d\xi \leq C(\|f\|_{1, \gamma}^2 + \|f\|^2)
\]

with some constant \( C = C_{n, \theta, \gamma} > 0 \).

While, the following result is well-known (cf., B. Muckenhoupt [10, Theorem 1])

**Proposition 2.5** Let \([n, \gamma, \theta] \) satisfy \( n \geq 1, \gamma \in [0, 1] \) and \( \theta \in [0, \frac{n}{2}) \). Then, for all \( f \in L^2(\mathbb{R}^n) \cap L^{1, \gamma}(\mathbb{R}^n) \) it is true that

\[
\int_{\mathbb{R}^n} \frac{|\hat{f}(\xi)|^2}{|\xi|^{2\theta}} \, d\xi \leq C(\|f\|_{1, \gamma}^2 + \|f\|^2 + \int_{\mathbb{R}^n} f(x) \, dx^2)
\]

with some constant \( C = C_{n, \theta, \gamma} > 0 \).

**Proof of Propositions 2.2 and 2.3.** Let us prove Propositions 2.2 and 2.3 at a stroke. We use the idea from [4] in the Fourier space. Let \( v(t, x) := \int_0^t u(s, x) \, ds \). Then, the function \( v \) satisfies

\[
\frac{1}{c(x)^2} v_t(t, x) - \Delta v(t, x) = \frac{1}{c(x)^2} u_1(x),
\]

\[
v(0, x) = 0, \quad v_t(0, x) = u_0(x), \quad x \in \mathbb{R}^n.
\]

Multiplying both sides of (2.1) by \( v_t \), and integrating over \([0, t]\) we derive that

\[
E_v(t) = \frac{1}{2} \|c^{-1}u_0\|^2 + \int_{\mathbb{R}^n} w(x) v(t, x) \, dx.
\]

where

\[
w(x) := \frac{u_1(x)}{c(x)^2}.
\]

Note that \( w \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) because of the assumption (A-1) and the condition that \( u_1 \in L^1 \cap L^2 \). By using the Plancherel theorem, the last term of (2.3) can be estimated as follows. For any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that for all \( t \geq 0 \) one has

\[
\left| \int_{\mathbb{R}^n} w(x) v(t, x) \, dx \right| = \left| \int_{\mathbb{R}^n} \hat{w}(\xi) \hat{v}(t, \xi) \, d\xi \right|
\]

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The following properties of the function $\psi$ from (2.4), (2.5) or (2.6) that one has just used the assumption that $n > 2$ small enough. Thus, one has from (2.3)

$$\frac{1}{2} \|v(t, \cdot)\|^2 + \frac{1}{2} \epsilon \|\nabla v(t, \cdot)\|^2 \leq C \int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, d\xi + \epsilon \|v(t, \cdot)\|^2.$$

(2.4)

Now, when $n \geq 3$, by using Proposition 2.5 with $\gamma = 0$, one has

$$\int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, d\xi \leq C(\|w\|^2 + \|w\|^2)$$

$$\leq C \|c\|^4 \|u_1\|^2_1 + \|u_1\|^2_2).$$

(2.5)

In the case when $n = 2$, by relying on Proposition 2.4 with $\gamma \in (0, 1]$ for $n = 2$, one can have

$$\int_{\mathbb{R}^n} |\hat{w}(\xi)|^2 \, d\xi \leq C \|c\|^4 \|u_1\|^2_1, \|u_1\|^2_2),$$

(2.6)

where one has just used the assumption $\int_{\mathbb{R}^n} w(x) \, dx = 0$ in Proposition 2.3. Therefore, it follows from (2.4), (2.5) or (2.6) that

$$\frac{1}{2} \|v(t, \cdot)\|^2 + (1 - \epsilon) \|\nabla v(t, \cdot)\|^2$$

$$\leq \frac{1}{2} \|c^{-1} u_0\|^2 + C \|c^{-1}\|^4 \|u_1\|^2_1, \|u_1\|^2_2), \quad (n \geq 3),$$

and

$$\frac{1}{2} \|v(t, \cdot)\|^2 + (1 - \epsilon) \|\nabla v(t, \cdot)\|^2$$

$$\leq \frac{1}{2} \|c^{-1} u_0\|^2 + C \|c^{-1}\|^4 \|u_1\|^2_1, \|u_1\|^2_2), \quad (n = 2, \gamma \in (0, 1]).$$

The desired estimates of the Propositions 2.2 and 2.3 can be derived because of $v_t = u$ by taking $\epsilon > 0$ small enough.

The next step is to handle with the term $(\frac{1}{c} x u_t(t, \cdot), x \cdot \nabla u(t, \cdot))$ which appears in Morawetz identity. To do that we prepare an important weighted energy estimate below. For this purpose we define a weight function $\psi : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\psi(t, x) = \begin{cases} (1 + |x| - t) & (|x| \geq t), \\ (1 + t - |x|)^{-1} & (|x| < t). \end{cases}$$

The following properties of the function $\psi \in C^1([0, \infty) \times \mathbb{R}^n)$ are a direct calculation

$$\frac{\partial \psi}{\partial t}(t, x) < 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

(2.7)

$$|\nabla \psi(t, x)|^2 - (\frac{\partial \psi}{\partial t}(t, x))^2 = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,$$

(2.8)
The equation (2.8) is the so-called Eikonal equation to the free wave equation
\[ u_{tt}(t, x) - \Delta u(t, x) = 0. \]

How to choose \( \psi(t, x) \) has its origin in [5]. Based on the (modified) weighted energy estimates originated in [13], one can get the following lemma.

**Lemma 2.1** Let \( n \geq 1 \) and assume (A-2). If the initial data \([u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) further satisfies
\[
I_0^2 := \int_{\mathbb{R}^n} (1 + |x|) \left( \frac{1}{c(x)^2} |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) dx < +\infty, \tag{2.10}
\]
for each \( R > L \), then the solution \( u \in C^1 \) to the problem (1.1)-(1.2) satisfies
\[
\int_{|x| \geq R} \psi(t, x) \left( \frac{1}{c(x)^2} |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right) dx \leq CI_0^2 \quad (t \geq 0),
\]
where the constant \( C = 2 + L > 0 \).

Since one cannot apply the style itself of the weighted function \( \psi(t, x) \) to the equation (1.1) with variable coefficient \( c(x) \), one has to introduce a new auxiliary function \( \phi(t) \) of \( \psi(t, x) \) defined by
\[
\phi(t) = \begin{cases} 
(1 + L - t) & (L > t), \\
(1 + t - L)^{-1} & (L \leq t). 
\end{cases}
\]

Note that \( \phi \in C^1([0, \infty)) \), and
\[ \phi_t(t) < 0. \tag{2.11} \]

It is easy to check that \( \psi(t, x) \) and \( \phi(t) \) satisfy the following identities.
\[
0 = (\psi u_t) \left( \frac{1}{c(x)^2} u_{tt} - \Delta u \right) = \frac{d}{dt}(\psi(t, x) E(t, x)) - \nabla \cdot (\psi u_t \nabla u)
\]
\[
- \frac{1}{2\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 + \frac{u_t^2}{2c(x)^2 \psi_t} \left( c(x)^2 |\nabla \psi|^2 - \psi_t^2 \right), \tag{2.12}
\]
and
\[
0 = (\phi u_t) \left( \frac{1}{c(x)^2} u_{tt} - \Delta u \right) = \frac{d}{dt}(\phi(t) E(t, x)) - \phi(t) \nabla \cdot (u_t \nabla u)
\]
\[
- \frac{\phi_t(t)}{2c(x)^2} \left( c(x)^2 |\nabla u|^2 + u_t^2 \right), \tag{2.13}
\]
where
\[ E(t, x) := \frac{1}{2} \left( \frac{1}{c(x)^2} |u_t(t, x)|^2 + |\nabla u(t, x)|^2 \right). \tag{2.14} \]

**Proof of Lemma 2.1.** The proof can be proceeded similarly to [6, Lemma 3.3] with a constant \( r_0 \) replaced by \( L \). However, for making this paper self-contained we shall draw its full proof.

It follows from (2.7) and (2.12) that
\[
0 \geq \frac{d}{dt}(\psi(t, x) E(t, x)) - \nabla \cdot (\psi u_t \nabla u)
\]
\[ + \frac{u_t^2}{2c(x)^2\psi_t^2} (c(x)^2|\nabla \psi|^2 - \psi_t^2) \]. \tag{2.15}

Integrating (2.15) over \([0, t] \times (\mathbb{R}^n \setminus B(0, L))\) one can get
\[
\int_{|x| \geq L} \psi(0, x)E(0, x)dx + \int_0^t \int_{|x| \geq L} \nabla \cdot (\psi(s, x)u_s(s, x)\nabla u(s, x))dxds \\
\geq \int_{|x| \geq L} \psi(t, x)E(t, x)dx + \int_0^t \int_{|x| \geq L} \frac{u_s(s, x)^2}{2\psi(s, x)}(|\nabla \psi(s, x)|^2 - \psi_s(s, x)^2)dxds,
\]
where one has just used the assumption \((A-2)\). By applying (2.8) one obtains
\[
\int_{|x| \geq L} \psi(0, x)E(0, x)dx + \int_0^t \int_{|x| \geq L} \nabla \cdot (\psi(s, x)u_s(s, x)\nabla u(s, x))dxds \\
\geq \int_{|x| \geq L} \psi(t, x)E(t, x)dx. \tag{2.16}
\]

On the other hand, by integrating (2.13) over \([0, t] \times B(0, L)\), because of (2.11) one can get
\[
\int_{|x| \leq L} \phi(0)E(0, x)dx + \int_0^t \int_{|x| \leq L} \phi(s)\nabla \cdot (u_s(s, x)\nabla u(s, x))dxds \\
\geq \int_{|x| \leq L} \phi(t)E(t, x)dx. \tag{2.17}
\]

Now, since \(\phi(t) = \psi(t, x)\) on the sphere \(|x| = L\), it follows from the divergence formula that
\[
\int_0^t \int_{|x| \geq L} \nabla \cdot (\psi(s, x)u_s(s, x)\nabla u(s, x))dxds + \int_0^t \int_{|x| \leq L} \phi(s)\nabla \cdot (u_s(s, x)\nabla u(s, x))dxds = 0.
\]

Thus, by summing up (2.16) and (2.17) one can arrived at the desired estimate
\[
\int_{|x| \geq R} \psi(t, x)E(t, x)dx \leq \int_{|x| \geq L} (1 + |x|)E(0, x)dx + (1 + L) \int_{|x| \leq L} E(0, x)dx,
\]
where one has used the fact that \(R > L\) and \(\phi(t) > 0\). \(\Box\)

Based on Lemma 2.1, one can also obtain the following lemma. This is re-stated version of \(6\) Lemma 3.4 with \(r_0\) replaced by \(L\). Note that from the assumption on \(c(x)\), one has
\[c_m := \inf_{x \in \mathbb{R}^n} c(x) > 0.\]

**Lemma 2.2** Let \(R > L, t > R\), and \(c(x)\) satisfies the assumptions \((A-1)\) and \((A-2)\). Then it is true that
\[
\left|\left(\frac{1}{c(\cdot)^2} u_t(t, \cdot), x \cdot \nabla u(t, \cdot)\right)\right| \leq \frac{R}{c_m}E_R(t) + C\frac{T_0^2}{2} + t \int_{|x| \geq R} E(t, x)dx,
\]
where \(C > 0\) is a constant independent from initial data.

**Remark 2.1** When one checks the proof of Lemma 2.2 above, one has to use the assumption \((A-2)\) such that \(c(x) = 1\) for \(x \in \mathbb{R}^n\) satisfying \(|x| > L\), and in this case one notices that (see (2.14))
\[E(t, x) = \frac{1}{2} \left(|u_t(t, x)|^2 + |\nabla u(t, x)|^2\right), \ |x| > L, \ t \geq 0.\]
3 Proof of Theorems 1.2, 1.3 and 1.4

Now, in this section, we are in a position to prove Theorems 1.2, 1.3 and 1.4 based on Lemmas 2.1 and 2.2.

Note that it suffices to check only the case for \( n \geq 3 \), and the case for \( n = 2 \) is similar by using Proposition 2.3 in place of Proposition 2.2. Note also that for the case of \( n = 1 \), one does not need to use Proposition 2.3 because of the existence of the coefficient \( \frac{n-1}{2} \) in Proposition 2.1.

Let \( R > L \). We first start with Proposition 2.1 under the assumption (A-2):

\[
tE_u(t) = J_0^2 - \frac{n-1}{2} \left( \frac{1}{c(\cdot)} u_t(t, \cdot), u(t, \cdot) \right) - \frac{1}{c(\cdot)} u_t(t, \cdot), x \cdot \nabla u(t, \cdot) \right) \]

\[
+ \int_0^t \int_{|x| \leq L} \frac{1}{c(x)^3} (x \cdot \nabla c(x)) |u_s(s, x)|^2 dx ds \quad (t \geq 0),
\]

where

\[
J_0^2 := \frac{n-1}{2} \left( \frac{1}{c(\cdot)} u_1, u_0 \right) + \frac{1}{c(\cdot)} u_1, x \cdot \nabla u_0.
\]

We observe that \( J_0^2 \) is not necessarily positive.

Note that under the assumption on the regularity imposed on the initial data, one can check that (see (2.10))

\[
\left| \frac{1}{c(\cdot)} u_1, x \cdot \nabla u_0 \right| \leq \frac{1}{c(\cdot)} \|u_1\|_\infty \int_{\mathbb{R}^n} \left( \frac{1}{c(x)} \sqrt{|x|} \|u_1(x)\| (\sqrt{|x|} \|\nabla u_0(x)\|) \right)
\]

\[
\leq \frac{1}{2} \frac{1}{c(\cdot)} \|u_1\|_\infty \int_{\mathbb{R}^n} \left( \frac{1}{c(x)^2} |x| \|u_1(x)\|^2 + |x| \|\nabla u_0(x)\|^2 \right) dx < +\infty.
\]

Then, it follows from the Schwarz inequality, (A-1) and (A-2) that

\[
\int_0^t \int_{|x| \leq L} \frac{1}{c(x)^3} (x \cdot \nabla c(x)) |u_s(s, x)|^2 dx ds \leq 2L \|c(\cdot)\|_\infty \|\nabla c\|_\infty \int_0^t \int_{|x| \leq L} \frac{1}{c(x)^2} |u_s(s, x)|^2 dx ds
\]

\[
\leq 2L \|c(\cdot)\|_\infty \|\nabla c\|_\infty \int_0^t E_R(s) ds.
\]

Since

\[
tE_u(t) = tE_R(t) + t \int_{|x| \geq R} E(t, x) dx,
\]

it follows from (3.1) and (3.2) that

\[
tE_R(t) + t \int_{|x| \geq R} E(t, x) dx \leq J_0^2 + \frac{n-1}{2} \left( \frac{1}{c(\cdot)^2} u_t(t, \cdot), u(t, \cdot) \right)
\]

\[
+ \left( \frac{1}{c(\cdot)^2} u_t(t, \cdot), x \cdot \nabla u(t, \cdot) \right) + \eta \int_0^t E_R(s) ds,
\]

where

\[
\eta := 2L \|c(\cdot)\|_\infty \|\nabla c\|_\infty.
\]

This parameter \( \eta \) is quite important.
While, by Lemma 2.2 and (3.3) one can obtain
\[
t E_R(t) + t \int_{|x| \geq R} E(t,x)dx \leq J_0^2 + \frac{n-1}{2} \left| \frac{1}{c(\cdot)^2} u_t(t,\cdot), u(t,\cdot) \right| \\
+ \frac{R}{c_m} E_R(t) + C \frac{J_0^2}{2} + t \int_{|x| \geq R} E(t,x)dx + \eta \int_0^t E_R(s)ds,
\]
which implies
\[
(t - \frac{R}{c_m}) E_R(t) \leq J_0^2 + C \frac{J_0^2}{2} + \frac{n-1}{2} \left| \frac{1}{c(\cdot)^2} u_t(t,\cdot), u(t,\cdot) \right| + \eta \int_0^t E_R(s)ds.
\]
Here, let us estimate the term
\[
\left| \frac{1}{c(\cdot)^2} u_t(t,\cdot), u(t,\cdot) \right|,
\]
by relying on Propositions 2.2 and 2.3.

We estimate only the case for \( n \geq 3 \). Based on Proposition 2.2, (A-1) and the Schwarz inequality we can derive
\[
\left| \frac{1}{c(\cdot)^2} u_t(t,\cdot), u(t,\cdot) \right| \leq c_m^{-1} \left\| \frac{1}{c(\cdot)} u_t(t,\cdot) \right\| \left\| u(t,\cdot) \right\|
\leq \frac{1}{2c_m^2} \left\| \frac{1}{c(\cdot)} u_t(t,\cdot) \right\|^2 + \frac{1}{2} \left\| u(t,\cdot) \right\|^2
\leq \frac{1}{2c_m^2} \left\| \frac{1}{c(\cdot)} u_t(t,\cdot) \right\|^2 + C \left( \left\| u_1 \right\| + \left\| u_1 \right\|^2 \right) + C \left\| u_0 \right\|^2.
\]
Since \( 2^{-1} \left\| \frac{1}{c(\cdot)} u_t(t,\cdot) \right\|^2 \leq E_u(t) = E_u(0) \) (see (1.3)), one has
\[
\left| \frac{1}{c(\cdot)} u_t(t,\cdot), u(t,\cdot) \right| \leq \frac{1}{c_m} E_u(0) + C \left( \left\| u_1 \right\| + \left\| u_1 \right\|^2 \right) + C \left\| u_0 \right\|^2.
\]
Because of (3.5) and (3.6) one can arrive at the significant inequality of the Gronwall type:
\[
(t - \frac{R}{c_m}) E_R(t) \leq J_0^2 + C \frac{J_0^2}{2} + \frac{n-1}{2} \left| \frac{1}{c(\cdot)^2} u_t(t,\cdot), u(t,\cdot) \right|
+ \eta \int_0^t E_R(s)ds \leq K_0^2 + \eta \int_0^t E_R(s)ds,
\]
where
\[
K_0^2 := J_0^2 + C \frac{J_0^2}{2} + \frac{n-1}{2 c_m^2} E_u(0) + C \frac{n-1}{2} \left( \left\| u_1 \right\| + \left\| u_1 \right\|^2 \right) + C \left\| u_0 \right\|^2.
\]
Now, let us solve the integral inequality (3.7) under the assumption \( \eta \in [0,1) \). This is rather standard. For completeness we write its full proof.

To do this we consider the function
\[
\xi(t) := (t - \frac{R}{c_m})^{-\eta} \int_0^t E_R(s)ds \quad (t > R/c_m).
\]
Then, it follows from (3.7) that
\[
\xi'(t) = (t - \frac{R}{c_m})^{-1-\eta} \left\{ \left( t - \frac{R}{c_m} \right) E_R(t) - \eta \int_0^t E_R(s)ds \right\} \leq K_0^2 (t - \frac{R}{c_m})^{-1-\eta}, \quad (t > R/c_m).
\]
Integrating over $[t_0, t]$ with large $t_0 \gg 1$, one can get
\[
\xi(t) \leq \xi(t_0) + K_0^2 \int_{t_0}^{t} (s - \frac{R}{c_m})^{-1-\eta} ds \leq \xi(t_0) + K_0^2 \eta^{-1} (t_0 - \frac{R}{c_m})^{-\eta} =: M_0. \tag{3.8}
\]

By (3.7) and (3.8) one can obtain the desired estimate
\[
(t - \frac{R}{c_m})E_R(t) \leq K_0^2 + \eta \int_0^t E_R(s) ds \leq K_0^2 + \eta M_0 (t - \frac{R}{c_m})^{\eta} \quad (t > t_0 \gg 1).
\]

This completes the proof of Theorems 1.2, 1.3 and 1.4. \hfill \Box

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References

[1] H. Brezis, Analyse Fonctionnelle, Théorie et applications, Dunod, Paris, 1999.
[2] N. Burq, Décroissance de l’énergie locale de L’équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel, Acta Math. 180 (1998), 1-29.
[3] R. Ikehata, Fast energy decay for wave equations with a localized damping in the $n$-D half space, Asymptotic Anal. 103 (2017), 77-94.
[4] R. Ikehata and T. Matsuyama, $L^2$-behavior of solutions to the linear heat and wave equations in exterior domains, Sci. Math. Japon. 55 (2002), 33-42.
[5] R. Ikehata and K. Nishihara, Local energy decay for wave equations with initial data decaying slowly near infinity, Gakuto International Series, The 5th East Asia PDE Conf., Math. Sci. Appl. 22 (2005), 265-275.
[6] R. Ikehata and G. Sobukawa, Local energy decay for some hyperbolic equations with initial data decaying slowly near infinity, Hokkaido Math. J. 36 (2007), 53-71.
[7] P. D. Lax and R. S. Phillips, Scattering theory, Revised Edition. Academic Press, New York, 1989.
[8] C. Morawetz, The decay of solutions of the exterior initial-boundary value problem for the wave equation, Comm. Pure Appl. Math. 14 (1961), 561-568.
[9] C. Morawetz, J. Ralston and W. Strauss, Decay of solutions of the wave equation outside nontrapping obstacles, Comm. Pure Appl. Math. 30 (1977), 447-508.
[10] B. Muckenhoupt, Weighted norm inequalities for the Fourier transform, Transactions of AMS. 276 (1983), 729-742. doi:10.1090/S0002-9947-1983-0688974-X.
[11] J. Ralston, Solutions of the wave equation with localized energy, Comm. Pure Appl. Math. 22 (1969), 807-823.
[12] J. Shapiro, Local energy decay for Lipschitz wavespeeds, Communications in Partial Differential Eqns, DOI: 10.1080/03605302.2018.1475491.
[13] G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Diff. Eqns 174 (2001), 464-489.

[14] B. R. Vainberg, On the short wave asymptotic behavior of solutions of stationary problems and the asymptotic behavior as $t \to \infty$ of solutions of nonstationary problems, Russian Math. Survey 30 (1975), 1-58.