A DICHOTOMY FOR THE STABILITY OF ARITHMETIC PROGRESSIONS

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Abstract. Let \( H \) stand for the set of homeomorphisms \( \phi : [0, 1] \to [0, 1] \). We prove the following dichotomy for Borel subsets \( A \subset [0, 1] \):

- either there exists a homeomorphism \( \phi \in H \) such that the image \( \phi(A) \) contains no 3-term arithmetic progressions;
- or, for every \( \phi \in H \), the image \( \phi(A) \) contains arithmetic progressions of arbitrary finite length.

In fact, we show that the first alternative holds if and only if the set \( A \) is meager (a countable union of nowhere dense sets).

1. Definitions

Let \( \mathbb{R}, \mathbb{Q} \) denote the sets of real and rational numbers, respectively. By an AP (arithmetic progression) we mean a finite strictly increasing sequence in \( \mathbb{R} \) of the form \( x = (x + kd)_{k=0}^{n-1} \), with \( d > 0 \) and \( n \geq 3 \). The convention is sometimes abused by identifying the sequence \( x \) with the set of its elements. An AP is completely determined by its first term \( x = \min x \), its length \( n = |x| \) and its step (difference) \( d > 0 \).

A subset \( S \subset \mathbb{R} \) is called FAP (free of APs) if it does not contain 3-term APs.

A subset \( S \subset \mathbb{R} \) is called RAP (rich in APs) if it contains APs of arbitrary large finite length.

Denote by \( H \) the set of homeomorphisms \( \phi : [0, 1] \to [0, 1] \) of the unit interval. The result presented in the abstract can be restated as follows.

Theorem 1. Let \( S \subset [0, 1] \) be a Borel subset. Then exactly one of the following two assertions holds:

1. (either) there exists a \( \phi \in H \) such that \( \phi(S) \) is FAP;
2. (or) \( \phi(S) \) is RAP for every \( \phi \in H \).

Moreover, (1) holds if and only if \( S \) is meager.

Recall some basic relevant definitions. Let \( S \subset \mathbb{R} \). A set \( S \) is called nowhere dense if its closure \( \overline{S} \subset \mathbb{R} \) has empty interior. \( S \) is called meager (a set of first category), if it is a countable union of nowhere dense sets. \( S \) is called residual, or co-meager, if \( \mathbb{R}\setminus S \) is meager; \( S \) is called residual in a subinterval \( X \subset \mathbb{R} \) if the complement \( X \setminus S \) is meager. Finally, \( S \) is called the set of second category if it is not meager.

The following proposition lists some “largeness” properties of a set \( A \subset \mathbb{R} \) which force it to be RAP. Denote by \( \lambda \) the Lebesgue measure on \( \mathbb{R} \).

Proposition 1 (Classes of RAP sets). Let \( A \subset \mathbb{R} \). Then \( S \) is RAP if \( S \) belongs to at least one of the following four classes:

- \( E_1 = \{ S \subset \mathbb{R} \mid S \text{ is Lebesgue measurable with } 0 < \lambda(S) \leq \infty \} \),
- \( E_2 = \{ S \subset \mathbb{R} \mid S \text{ is residual in some interval } X \subset \mathbb{R} \text{ of positive length} \} \),
- \( E_3 = \{ S \subset \mathbb{R} \mid S \text{ is winning in Schmidt’s game} \} \),
- \( E_4 = \{ S \subset \mathbb{R} \mid S \text{ is Borel and not meager} \} \).

Date: March 2013.

Key words and phrases. Meager subset, arithmetic progressions, homeomorphism.

The authors were supported in part by NSF Grants DMS-1102298, DMS-1004372.
Proof. For a set $S \in \mathcal{E}_1$, one easily produces APs near any its Lebesgue density point. The argument for the classes $E_2$ and $E_3$ is even easier because residual subsets and the class $E_3$ are closed under finite (and even countable) intersections.

Finally, the sets $S \in \mathcal{E}_4$ are RAP because $\mathcal{E}_4 \subset \mathcal{E}_2$. (A Borel subset $S \subset \mathbb{R}$ of second category must be residual in some subinterval, see e.g. Proposition 3.5.6 and Corollary 3.5.2 in [5] page 108)).

Note that the the problems of finding finite or countable configurations $F$ in sets $S \subset \mathbb{R}$, under various “largeness” metric assumptions on $S$, has been considered by several mathematicians.

Following Kolountzakis [3], a set $F$ is called universal for a class $E$ of subsets of reals if $F \ll S$ for all $S \in \mathcal{E}$. Henceforth $F \ll S$ means that $S$ contains an affine image of $F$, i.e. that $aF + b \subset S$, for some $a, b \in \mathbb{R}$, $a > 0$. For example, $S$ is RAP iff $\{1, 2, \ldots, n\} \ll S$ for all $n \geq 1$; $S$ is FAP iff $\{1, 2, 3\} \ll S$.

Every finite subset of reals is universal for all the classes $\mathcal{E}_k$, $1 \leq k \leq 4$. Every bounded countable subset is universal for the classes $\mathcal{E}_k$, $2 \leq k \leq 4$.

An old question of Erdős is whether there is an universal infinite set $F \subset \mathbb{R}$ for the class $\mathcal{E}_1$ (of sets of positive measure). The question is still open even though some families of countable sets $F$ are shown not to contain universal functions, see Kolountzakis [3], Paul and Laczkovich [6] and references there. In [6] an elegant combinatorial characterization of universal sets $F$ (for the class $\mathcal{E}_1$) is given which reproduces earlier results in the subject.

Keleti [2] constructed a compact set $A \subset [0, 1]$ of Hausdorff dimension 1 which is FAP; on the other hand, Laza and Pramanik in [4] showed that under certain assumptions (on the Fourier transform of supported measure) compact sets of fractional dimension close to 1 must contain 3-term APs (i.e., cannot be FAP). We refer to [4] for survey of related questions.

The central result of the paper, Theorem 1, completely characterizes the topological (rather than metric) properties of a Borel set $S \subset \mathbb{R}$ which guarantee it to be RAP. This theorem is an immediate consequence of the following proposition and the fact that the sets $S \in \mathcal{E}_1$ must be RAP (Proposition 1).

Proposition 2. For every meager subset $C \subset [0, 1]$, there is a map $\phi \in \mathcal{H}$, $\phi : [0, 1] \to [0, 1]$, such that $\phi(C)$ is FAP.

A stronger version of Proposition 2 (Proposition 3) is presented and proved in the next section.

2. Proofs of Propositions 2 and 3

Denote by $\mathcal{C}$ the Banach space of continuous maps $f : [0, 1] \to \mathbb{R}$ equipped with the norm

$$\|f\| = \|f\|_\infty = \max_{x \in [0, 1]} |f(x)|.$$  

Denote by $\mathcal{F}$ and $\mathcal{H}^+$ the following subsets of $\mathcal{C}$:

$$\mathcal{F} = \{f \in \mathcal{C} \mid f \text{ is non-decreasing with } f(0) = 0; f(1) = 1\},$$  

$$\mathcal{H}^+ = \{f \in \mathcal{F} \mid f \text{ is injective}\} = \{f \in \mathcal{H} \mid f \text{ is increasing on } [0, 1]\}.$$  

The set $\mathcal{F}$ is a closed subset of $\mathcal{C}$, while $\mathcal{H}^+$ is residual in $\mathcal{F}$. (Indeed,

$$\mathcal{H}^+ = \bigcap_{0 < a < b < 1} F_{a,b}, \quad F_{a,b} = \{f \in \mathcal{F} \mid f(a) < f(b)\}$$

where $\mathbb{Q}$ stands for the set of rationals, and $F_{a,b}$ are open dense subsets of $\mathcal{F}$).

The following proposition is a stronger version of Proposition 2

Proposition 3. Let $C \subset [0, 1]$ be a meager subset. Then, for residual subset of $\phi \in \mathcal{H}^+$, the image $\phi(C)$ is FAP (has no 3-term APs).

Since a meager set is a countable union of nowhere dense sets, it is enough to prove the above proposition under the weaker assumption that $C$ is nowhere dense. Indeed, a meager set $C$ has a representation in the form $C = \bigcup_{i=1}^\infty C_i$ where $C_i$ are nowhere dense. Then the unions $U_k = \bigcup_{i=1}^k C_i$ form a nested sequence of nowhere dense sets, and $\phi(U_k)$ are.
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Let

(2.4) \( \mathcal{H}_\varepsilon(C) = \{ \phi \in \mathcal{H}^+ \mid \phi(C) \) has no 3-term APs of step \( d \geq \varepsilon \}. \)

In the proof of Proposition 3 we need the following lemma. Its proof is provided in the end of the next section.

**Lemma 1.** Let \( C \subset [0, 1] \) be a nowhere dense subset and \( \varepsilon > 0 \). Then \( \mathcal{H}_\varepsilon(C) \) contains a dense open subset of \( \mathcal{H}^+ \). In particular, \( \mathcal{H}_\varepsilon(C) \) is residual in \( \mathcal{H}^+ \).

**Proof of Proposition 3 assuming Lemma 1.** We may assume that \( C \) is nowhere dense (see the sentence following Proposition 3). We may also assume that \( C \) is compact (otherwise replacing \( C \) by its closure \( \overline{C} \)).

By Lemma 1 each of the sets \( \mathcal{H}_\varepsilon(C), \varepsilon > 0, \) is residual in \( \mathcal{H}^+ \). It follows that the set \( \mathcal{H}_0(C) = \bigcap_{k=1}^{\infty} \mathcal{H}_{1/k}(C) \) is residual. It is also clear that, for \( \phi \in \mathcal{H}_0(C) \), the images \( \phi(C) \) are FAP.

This completes the proof of Proposition 3. \( \square \)

3. **Proof of Lemma 1**

First we prepare some auxiliary results.

**Lemma 2.** Let \( C \subset [0, 1] \) be a nowhere dense set, let \( f \in \mathcal{H}^+ \) and let \( \varepsilon > 0 \) be given. Then there exists \( g \in \mathcal{H}^+ \) such that \( \|g - f\| < \varepsilon \) and the set \( g(C) \) has no 3-term APs with step \( d \geq \varepsilon \).

**Proof.** Without loss of generality, we assume that \( \varepsilon < 1/2 \). Pick an integer \( r \geq 3 \) such that \( r\varepsilon > 1 \).

Since \( C \) is nowhere dense, so is \( f(C) \), and one can select \( r-1 \) points \( x_1, x_2, \ldots, x_{r-1} \in (0, 1) \setminus f(\overline{C}) \),

\[
0 = x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r = 1,
\]

partitioning the unit interval into \( r \) subintervals \( X_k = (x_{k-1}, x_k) \), each shorter than \( \varepsilon \):

\[
0 < |X_k| = x_{k+1} - x_k < \varepsilon \quad (1 \leq k \leq r).
\]

Then one selects non-empty open subintervals \( Y_k = (y_k^-, y_k^+) \subset X_k \), \( 1 \leq k \leq r \), in such a way that the following four conditions are met:

\[
(3.1) \quad \begin{align*}
& (c1) \quad f(C) \subset \bigcup_{k=1}^r \bar{Y}_k, \\
& (c2) \quad x_{k-1} < y_k^- < y_k^+ < x_k \quad \text{(i.e., \( \bar{Y}_k \subset X_k \)), \ for \( 2 \leq k \leq r-1 \)}, \\
& (c3) \quad 0 = x_0 = y_1^- < y_1^+ < x_1, \ \text{and} \\
& (c4) \quad x_{r-1} < y_r^- < y_r^+ = x_r = 1.
\end{align*}
\]

That is, between \( Y_j \) and \( Y_{j+1} \) there exists \( x_j \notin f(\overline{C}) \) and \( |Y_j| < |X_j| < \varepsilon \) for all \( j \).

Set \( p_1 = 0, p_r = 1 \) and then select the \( r-2 \) points \( p_k \in Y_k, \ 2 \leq k \leq r-1, \) so that the set \( P = \{p_k\}_{k=1}^r \) contain no 3-term APs. Then the sequence \( \{p_k\}_{k=1}^r \) is strictly increasing, and

\[
d = \min_{1 \leq m < n < k \leq r} |p_m + p_k - 2p_n| > 0.
\]

Next, for \( 1 \leq k \leq r \), we select open subintervals \( Z_k \subset Y_k \), each shorter than \( \frac{d}{4} \), with \( p_k \subset \bar{Z}_k \).

Define \( u \in \mathcal{H} \) to be the homeomorphism \( [0, 1] \to [0, 1] \) which affinely contracts \( \bar{Y}_k \) to \( \bar{Z}_k \) and affinely expands the gaps between the intervals \( \bar{Y}_k \) to fill it in. Note that

\[
(3.2) \quad |u(x) - x| < \varepsilon, \quad \text{for} \quad x \in \bigcup_{k=1}^r \bar{Y}_k,
\]

because \( x \in \bar{Y}_k \) implies \( u(x) \in \bar{Y}_k \) and hence \( |u(x) - x| \leq |Y_k| < |X_k| < \varepsilon \).

Since \( u(x) - x \) is linear on each of the \( (r-1) \) gaps between the intervals \( \bar{Y}_k \), the inequality extends to the whole unit interval: \( \|u(x) - x\| < \varepsilon \).

Define \( g \in \mathcal{H} \) as the composition \( g(x) = (u \circ f)x = u(f(x)) \). Then

\[
\|g - f\| = \|u \circ f - f\| = \|u(x) - x\| < \varepsilon.
\]
It remains to show that \( g(C) \) has no 3-term APs with step \( d \geq \varepsilon \). In view of (3.1),

\[
\bigcup_{k=1}^{r} \tilde{Z}_k = h\left( \bigcup_{k=1}^{r} \tilde{Y}_k \right) \supset h(f(C)) = g(C),
\]

so it would suffice to proof that \( \bigcup_{k=1}^{r} \tilde{Z}_k \) has no 3-term APs with step \( d \geq \varepsilon \).

Assume to the contrary that such an AP exists, say \( a_1, a_2, a_3 \), with \( d = a_2 - a_1 = a_3 - a_2 \geq \varepsilon \). Let \( a_i \in \tilde{Z}_{k_i} \), for \( i = 1, 2, 3 \). These \( k_i \) are uniquely determined, and since \(|Z_{k_i}| < |X_{k_i}| < \varepsilon \leq d\), we have \( k_1 < k_2 < k_3 \). Taking in account that \(|a_i - p_{k_i}| \leq |Z_{k_i}| < \delta/4\), we obtain

\[
|a_1 + a_3 - 2a_2| \geq |p_{k_1} + p_{k_3} - 2p_{k_2}| - (|a_1 - p_{k_1}| + |a_3 - p_{k_3}| + 2|a_2 - p_{k_2}|) > \delta - 4 \cdot \frac{\delta}{4} = 0,
\]
a contradiction with the assumption that \( a_1, a_2, a_3 \) forms an AP. □

**Corollary 1.** Let \( C \subset [0, 1] \) be a nowhere dense set. Then for all \( \varepsilon > 0 \), the sets \( \mathcal{H}_\varepsilon(C) \) (defined by (2.4)) are dense in \( \mathcal{H}^+ \).

**Proof.** Note that the sets \( \mathcal{H}_\varepsilon(C) \) are monotone in \( \varepsilon > 0 \): \( \mathcal{H}_{\varepsilon_2}(C) \subset \mathcal{H}_{\varepsilon_1}(C) \) if \( 0 < \varepsilon_2 < \varepsilon_1 \).

By the previous lemma (Lemma 2), all sets \( \mathcal{H}_\varepsilon(C) \) are \( \varepsilon \)-dense. Then, for a given \( \varepsilon > 0 \), the set \( \mathcal{H}_\varepsilon(C) \) is \( \delta \)-dense for every positive \( \delta < \varepsilon \) (because even the smaller set \( \mathcal{H}_{\delta_3}(C) \subset \mathcal{H}_\varepsilon(C) \) is \( \delta \)-dense). This argument completes the proof of Corollary 1. □

**Lemma 3.** Let \( C \subset [0, 1] \) be a compact nowhere dense set, let \( g \in \mathcal{H} \) and let \( \varepsilon > 0 \) be given. Assume that the set \( g(C) \) has no 3-term APs with step \( d \geq \varepsilon \). Then there exists a \( \delta > 0 \) such that for all \( h \in \mathcal{H} \) such that \( \|h - g\| < \delta \) the sets \( h(C) \) have no 3-term APs with step exceeding \( 2\varepsilon \).

**Proof.** Let

\[
M = \{(x_1, x_2, x_3) \in g(C)^3 \mid x_2 - x_1 \geq \varepsilon \text{ and } x_3 - x_2 \geq \varepsilon \}.
\]

Then \( M \) is compact, and \( F: M \to \mathbb{R} \) defined by \( F(x_1, x_2, x_3) = |x_1 + x_3 - 2x_2| \) assumes its minimum

\[
\gamma = \min_{x \in M} F(x) > 0
\]

which is positive because \( g(C) \) has no 3-term APs with step \( d \geq \varepsilon \). Take \( \delta = \min(\varepsilon/2, \gamma/5) \).

Assume to the contrary that for some \( h \in \mathcal{H} \) with \( \|h - g\| < \delta \), the set \( h(C) \) contains an AP with step \( d' > 2\varepsilon \), i.e. that there are \( c_1, c_2, c_3 \in C \) such that

\[
h(c_3) - h(c_2) = h(c_2) - h(c_1) > 2\varepsilon.
\]

Then, for both \( i = 1, 2 \), we have

\[
g(c_{i+1}) - g(c_i) > h(c_{i+1}) - h(c_i) - 2\delta > 2\varepsilon - 2\delta \geq \varepsilon,
\]

whence \( (g(c_1), g(c_2), g(c_3)) \in M \) and hence

\[
\gamma \leq F(g(c_1), g(c_2), g(c_3)) = \|g(c_1) + g(c_3) - 2g(c_2)\| \leq \|h(c_1) + h(c_3) - 2h(c_2)\| + 4\delta = 0 + 4\delta \leq \frac{\delta}{\varepsilon} \gamma < \gamma,
\]
a contradiction. □

**Proof of Lemma 1.** It follows from Lemma 3 that there is an (intermediate) open subset \( U \subset \mathcal{H}^+ \) such that

\[
\mathcal{H}_\varepsilon(C) \subset U \subset \mathcal{H}_{2\varepsilon}(C) \subset \mathcal{H}^+.
\]

This set \( U \) is dense in \( \mathcal{H}^+ \) because its subset \( \mathcal{H}_\varepsilon(C) \) is (by Corollary 1). Thus the set \( \mathcal{H}_{2\varepsilon}(C) \) contains an open dense subset \( U \subset \mathcal{H}^+ \). Since \( \varepsilon > 0 \) is arbitrary, the proof is complete. □
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