First-order equivalent to
Einstein-Hilbert Lagrangian

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Abstract
A first-order Lagrangian $L^\nabla$ variationally equivalent to the second-order
Einstein-Hilbert Lagrangian is introduced. Such a Lagrangian depends
on a symmetric linear connection, but the dependence is covariant under
diffeomorphisms. The variational problem defined by $L^\nabla$ is proved to be
regular and its Hamiltonian formulation is studied, including its covariant
Hamiltonian attached to $\nabla$.

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1 Introduction

Let $p: \mathfrak{M} \to M$ be the bundle of pseudo-Riemannian metrics of a given signature $(n^+, n^-)$, $n^+ + n^- = n = \dim M$, over a connected $C^\infty$ manifold oriented by a volume form $\mathfrak{v} \in \Omega^n(M)$. The Einstein-Hilbert (or E-H for short) functional is the second-order Lagrangian density $L_{EH} \mathfrak{v}$ on $\mathfrak{M}$ defined along a metric $g$ by $s^g \mathfrak{v}_g$, where $s^g$ denotes the scalar curvature of $g$ and $\mathfrak{v}_g$ its Riemannian volume form; namely,

$$L_{EH} \circ j^2 g = \sqrt{\det(g_{ab})} \left[ \frac{\partial (\Gamma^g i_{jk})}{\partial x^i} - \frac{\partial (\Gamma^g i_{kl})}{\partial x^l} + (\Gamma^g i_{jk}) (\Gamma^g i_{kl}) - (\Gamma^g i_{jk}) (\Gamma^g i_{kl}) \right],$$

where $(\Gamma^g i_{jk})$ are the Christoffel symbols of the Levi-Civita connection $\nabla^g$ of the metric $g$. As is known (e.g., see [1, 3.3.1–3.3.2]), the first-order Lagrangian $L_1$ defined along $g$ by $\sqrt{\det(g_{ab})} \left[ g^{jk} (\Gamma^g i_{ij}) (\Gamma^g i_{kl}) - (\Gamma^g i_{jk}) (\Gamma^g i_{kl}) \right]$ differs from $L_{EH}$ by a divergence term, but unfortunately $L_1$ is not an invariantly defined quantity.

Below, a geometrically defined first-order Lagrangian $L^\nabla$ (depending on an auxiliary symmetric linear connection $\nabla$ on $M$) is introduced, which is variationally equivalent to E-H Lagrangian $L_{EH}$ and, consequently, it has the same Euler-Lagrange equations, namely Einstein’s field equations in the vacuum for arbitrary signature. In particular, this explains why the E-H Lagrangian admits a true first-order Hamiltonian formalism.

Although $L^\nabla$ depends on an auxiliary symmetric linear connection, this dependence is natural with respect to the action of diffeomorphisms of $M$ on connections and on Lagrangian functions, as proved in section 4 below. This fact justifies the construction of such a Lagrangian and the interest of its existence.

Furthermore, the Lagrangian $L^\nabla$ is seen to be regular and its Hamiltonian formulation is studied, computing explicitly its momenta functions and the covariant Hamiltonian attached to $\nabla$ in the sense of [10].

2 The equivalent Lagrangian $L^\nabla$ defined

The difference tensor field between the Levi-Civita connection $\nabla^g$ of a metric $g$ and a given symmetric linear connection $\nabla$ on $M$ is the 2-covariant 1-contravariant tensor given by,

$$T^g \nabla = \nabla^g - \nabla = \left( (\Gamma^g i_{ij}) - (\Gamma^g i_{ij}) \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k},$$

where $(\Gamma^g i_{jk})$ (resp. $\Gamma^g i_{jk}$) are the Christoffel symbols of the connection $\nabla^g$ (resp. $\nabla$). A Lagrangian function $L^\nabla$ on the bundle of metrics $p: \mathfrak{M} \to M$ is defined as follows:

$$L^\nabla \left( j^2_\mathfrak{v} g \right) \mathfrak{v}_x = \left\{ s^g(x) + c \left( \text{alt}_{23} \left( \nabla^g T^g \nabla \right) \right)_x \right\} \left( \mathfrak{v}_g \right)_x, \quad \forall j^2_\mathfrak{v} g \in J^2 \mathfrak{M},$$
where we confine ourselves to consider coordinate systems \((x^1, \ldots, x^n)\) on \(M\) adapted to \(\mathfrak{v}\), i.e.,

\[
\mathfrak{v} = dx^1 \wedge \ldots \wedge dx^n, \quad \mathfrak{v}_g = \sqrt{|\det (g_{uv})|} \mathfrak{v}, \quad g = g_{uv}dx^u \otimes dx^v,
\]

\(\text{alt}_{23} : \otimes^3 T^* M \otimes T M \to \otimes^3 T^* M \otimes T M\) denotes the alternation of the second and third covariant indices, \(\sharp: \otimes^3 T^* M \otimes T M \to \otimes^2 T^* M \otimes^2 T M\) is the isomorphism induced by \(g\),

\[
w_1 \otimes w_2 \otimes w_3 \otimes X \mapsto w_1 \otimes w_2 \otimes (w_3)^\sharp \otimes X, \quad \forall X \in T_x M, \quad \forall w_1, w_2, w_3 \in T_x^* M,
\]

and, finally, \(c: \otimes^2 T^* M \otimes^2 T M \to \mathbb{R}\) denotes the (total) contraction of the first and second covariant indices with the first and second contravariant ones, respectively. We write \(L^V\) in order to emphasize the fact that the Lagrangian depends on the auxiliary symmetric linear connection \(\nabla\) previously chosen.

If \(y_{ij} = y_{ji}, \ i, j = 1, \ldots, n\), are the coordinates on the fibres of \(p\) induced from a coordinate system \((x^h)_{h=1}^n\) on \(M\), namely, \(g_x = y_{ij}(g_x)dx^i \otimes dx^j\) for every metric \(g_x\) over \(x \in M\), and \((x^h, y_{ij}, y_{ij,k}, y_{ij,kl} = y_{ij,kl})\) denotes the coordinate system induced on \(J^2 M\), then \(L_{EH}\) is locally given by,

\[
L_{EH} = \rho \left( y^{ac} y^{bd} - y^{ab} y^{cd} \right) y_{ab,cd} + L_0,
\]

where

\[
\rho = \sqrt{|\det (y_{ij})|},
\]

\[
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\]

and \(G^i_{ij}: J^1 \mathfrak{M} \to \mathbb{R}\) are defined by,

\[
G^i_{ij} = \frac{1}{2} y^{ks} \left( y_{rs,ij} + y_{js,ri} - y_{rj,si} \right).
\]

If \(L^V\) is the second-order Lagrangian on \(\mathfrak{M}\) determined by the second summand of the right-hand side in the formula (2), namely

\[
L^V(j^2 g) = c \left( (\text{alt}_{23} (\nabla^g T^{g,V}))^3 \right),
\]

then (2) can equivalently be rewritten as follows: \(L^V = L_{EH} + \rho L^V\) and as a calculation shows,

\[
L^V \circ j^2 g = g^{ij} \left\{ \frac{\partial (T^{g,V})^i}{\partial x^j} - \frac{\partial (T^{g,V})^i}{\partial x^j} \right\} + \left( T^{g,V} \right)_{ij} + \left( T^{g,V} \right)_{ai} \right\}.
\]

Lemma 2.1. The Lagrangian \(L^V\) is of first order.

Proof. Taking the definition of \(T^{g,V}\) and the formulas (5), (11) into account, one
obtains

\[ \sqrt{|\text{det}(g_{uv})|}(L^{\nabla^g} \circ j^2 g) = -L_{EH} \circ j^2 g \]
\[ + \sqrt{|\text{det}(g_{uv})|}g^{jr} \left\{ (\Gamma^a_i)^a_{ji} \left( (\Gamma^g)^r_a - (\Gamma^g)^a_r \Gamma^g_{jr} \right) \right\} \]
\[ - \sqrt{|\text{det}(g_{uv})|}g^{jr} \left\{ \frac{\partial \Gamma^a_i}{\partial x^j} - \frac{\partial \Gamma^a_i}{\partial x^j} + (\Gamma^g)^a_i \right\} \]
\[ - (\Gamma^g)^a_i \Gamma^g_{ai} + (\Gamma^g)^a_r \Gamma^g_{aj} - (\Gamma^g)^a_i \Gamma^g_{ij} \}. \]

Hence \((\rho L^{\nabla^g} + L_{EH}) \circ j^2 g\) depends on the values of the metric \(g\) and its first derivatives only. \[\square\]

In fact, the following local expression is readily deduced:

\[ L^\nabla = \rho g^{jr} \left\{ G^a_{ji} T^i_{ra} - G^a_{ai} T^i_{jr} + G^a_{ij} \Gamma^a_{ai} - G^a_{ir} \Gamma^a_{aj} - \frac{\partial \Gamma^a_i}{\partial x^j} + \frac{\partial \Gamma^a_j}{\partial x^i} \right\}, \]

\(T^i_{jk} : J^1 \mathbb{M} \to \mathbb{R}\) being the functions defined by, \(T^b_{ij} = G^b_{ij} - \Gamma^b_{ij}\).

**Remark 2.1.** As \(L^\nabla\) has a geometrical definition, the local expression above actually provides a global Lagrangian. Moreover, if \(\nabla\) is a flat linear connection and one considers an adapted coordinate system to \(\nabla\) (i.e., a coordinate system on which all the Christoffel symbols of \(\nabla\) vanish), then the local expression for \(L^\nabla\) coincides with the local Lagrangian \(L_1\) defined in the introductory section.

### 3 \(L^\nabla\) and \(L_{EH}\) are variationally equivalent

As a computation shows, the second summand in the definition of \(L^\nabla\) can be rewritten in terms of the metric \(g\) and the auxiliary connection \(\nabla\) only, as follows:

\[
\begin{align*}
    c \left( \left( \text{alt}_{23} \left( \text{Nabla}^g T^g \nabla^g \right) \right)^2 \right) &= (g^{js} g^{ir} - g^{jr} g^{is}) g_{ri, sj} \\
    &+ \frac{1}{2} \left\{ (2g^{js} g^{br} - g^{bi} g^{rj} - g^{br} g^{ij}) g^{as} \\
    &+ (g^{ar} g^{jb} + g^{as} g^{ra} - 2g^{ir} g^{ab}) g^{js} \\
    &- (g^{sr} g^{jb} - g^{br} g^{aj}) g^{ai} \\
    &- (g^{ar} g^{sb} - g^{sr} g^{ab}) g^{ij} \right\} g_{ab, i} g_{rs, i} \\
    &- g^{jr} \left( \frac{\partial \Gamma^a_i}{\partial x^j} - \frac{\partial \Gamma^a_j}{\partial x^i} \right) \\
    &+ \frac{1}{2} \left\{ (2g^{js} g^{ar} - g^{jr} g^{as}) g_{rj, a} \Gamma^a_i \\
    &+ (g^{jr} g^{ab} - 2g^{ar} g^{jb}) g_{ab, i} \Gamma^a_j \right\}.
\end{align*}
\]

**Lemma 3.1.** If \(D_i\) denotes the total derivative with respect to \(x^i\), then

\[
c \left( \left( \text{alt}_{23} \left( \text{Nabla}^g T^g \nabla^g \right) \right)^2 \right) v_g = - (D_i ((L_{EH})^g) \circ j^2 g) v,
\]

\section*{4 Conclusion}

The results presented in this paper provide a rigorous framework for understanding the relationship between the geometric properties of a manifold and the variationally equivalent Lagrangians. This framework is particularly useful in contexts where the underlying geometry is not flat, as it allows for a more direct computation of the Lagrangian's variation with respect to the metric and its derivatives. The geometric interpretation of these results is crucial for applications in theoretical physics, particularly in the study of gravitational theories and the construction of variational principles for field equations. The use of local expressions and flat connections in the adapted coordinate system further simplifies the computation, making it more accessible for both theoretical and practical applications. The equivalence of the Lagrangians \(L^\nabla\) and \(L_{EH}\) is a key result that underpins the geometric formulation of (possibly non-flat) gravitational theories, providing a solid foundation for further investigations into the variational principles governing the dynamics of spacetime.
where

\[(L_{EH})^i_\gamma = \sum_{c \leq r} \frac{1}{2-\alpha_c} \frac{\partial L_{EH}}{\partial y_{cr,ib}} (y_{cr,b} - (\Gamma_{bc} y_{ar} + \Gamma_{br} y_{ac})).\]

From this lemma it follows that \(L^\nabla\) and \(L_{EH}\) are variationally equivalent as, according to the formula (2), one has

\[(L^\nabla \circ f^2) v = (L_{EH} \circ f^2) v + c \left( (\text{alt}_{23} (\nabla^g T^g) \nabla) \right) v_g = \left\{ (L_{EH} - D_i \left( (L_{EH})^i_\gamma \right)) \circ f^2 \right\} v.

Hence \(L^\nabla = L_{EH} - D_i \left( (L_{EH})^i_\gamma \right)\) and therefore, \(L^\nabla\) and \(L_{EH}\) differ in a total divergence.

The proof of Lemma 3.1 follows by computing \(D_i \left( (L_{EH})^i_\gamma \right)\) using (3) and (7), taking the identity \(D_i \rho = \frac{\partial}{\partial y} y^r y_{rs,i}\) into account, after a simple—but rather long—computation.

### 4 Dependence on \(\nabla\)

Below, the dependence of the Lagrangian \(L^\nabla\) with respect to the symmetric linear connection \(\nabla\), is analysed. First, some geometric preliminaries are introduced.

The image of a linear connection \(\nabla\) by a diffeomorphism \(\phi: M \to M\) is defined to be \((\phi \cdot \nabla)_X Y = \phi \cdot (\nabla_{\phi^{-1} X} (\phi^{-1} \cdot Y)), \forall X, Y \in \mathfrak{X}(M)\). As is well known (e.g., see [4, p. 643]), the Levi-Civita connection of a metric transforms according to the rule: \(\phi^{-1} \cdot \nabla^g = \nabla^{\phi^* g}\). Hence the following formulas hold:

\[\phi^{-1} \cdot T^g \nabla = T^{\phi^* g \cdot \phi^{-1} \nabla}, \quad S^g \nabla = (\phi^{-1})^* S^\nabla = \phi \cdot S^\nabla, \quad s^g = s^{\phi^* g},\]

where \(S^\nabla(X, Y) = \text{trace}(Z \mapsto R^\nabla(Z, X)Y)\) is the Ricci tensor of \(\nabla\) (e.g., see [7 VI, p. 248]). Moreover, the lift of \(\phi\) to the bundle of metrics \(\mathfrak{M} \to M\) is given by \(\tilde{\phi}(g_x) = (\phi^{-1})^* g_x, \forall g_x \in p^{-1}(x)\) (cf. [11]); hence \(p \circ \tilde{\phi} = \phi \circ p\), and the mapping \(\phi: \mathfrak{M} \to \mathfrak{M}\) has an extension to the \(r\)-jet bundle \(\tilde{\phi}(r): J^r \mathfrak{M} \to J^r \mathfrak{M}\) defined by, \(\tilde{\phi}(r)(j^r g) = f_{\phi(x)}(\tilde{\phi}(g)).\)

Let \(v_{\mathfrak{M}}\) be the nowhere-vanishing \(p\)-horizontal \(n\)-form on \(\mathfrak{M}\) defined as follows: \(v_{2\mathfrak{M}}|_{g_x} = v_{g_x}, \forall g_x \in \mathfrak{M}\), where, as above, \(v_{g_x}\) denotes the Riemannian volume form attached to \(g_x\). Hence \(v_{\mathfrak{M}} = \rho v\), where \(\rho\) is as in (11). Every \(r\)-th order Lagrangian density \(\Lambda\) on \(\mathfrak{M}\) can thus be written as \(\Lambda = L v_{\mathfrak{M}}\) for a certain Lagrangian function \(L \in C^\infty(J^r \mathfrak{M})\) and \(\Lambda\) is invariant under diffeomorphisms, i.e., \((\tilde{\phi}(r))^* \Lambda = \Lambda, \forall \phi \in \text{Diff} M\), if and only if \(L\) is, i.e., \(L \circ \tilde{\phi}(r) = L\), as \((\tilde{\phi}(r))^* \Lambda = (L \circ \tilde{\phi}(r))(\tilde{\phi}^* v_{\mathfrak{M}})\) and, according to [12 Proposition 7], \(v_{\mathfrak{M}}\) is invariant under diffeomorphisms, i.e., \(\tilde{\phi}^* v_{\mathfrak{M}} = v_{\mathfrak{M}}\).

The E-H Lagrangian density \(L_{EH} v\) is known to be invariant under diffeomorphisms, i.e., \((\tilde{\phi}(2))^* (L_{EH} v) = L_{EH} v, \forall \phi \in \text{Diff} M\). In fact, there exists a
classical result by Hermann Weyl ([14, Appendix II], also see [5], [8]), according to which the only \( \text{Diff} M \)-invariant Lagrangians on \( J^2 M \) depending linearly on the second-order coordinates \( y_{ab,ij} \) are of the form \( \lambda L_{EH} \pm \mu \), for scalars \( \lambda, \mu \).

Therefore, transforming the equation \( L_{\nabla} v = L_{EH} v + L_{\nabla}^\circ \phi v_{\nabla} \) by a diffeomorphism \( \phi \), one obtains \( (\bar{\phi})(1) \ast (L_{\nabla} v) = L_{EH} v + (L_{\nabla}^\circ \phi v_{\nabla}) \), and one is led to compute \( L_{\nabla}^\circ \phi v_{\nabla} \), which, by using the formulas above, is proved to transform according to the following rule:

\[
L_{\nabla}^\circ \phi v_{\nabla} = L_{\nabla}^\circ \phi^{-1} v_{\nabla}.
\]

5 Hamiltonian formalism

5.1 Regularity of \( L_{\nabla} \)

**Proposition 5.1.** For \( \text{dim} M = n \geq 3 \), the Lagrangian \( L_{\nabla} \) is regular, namely, the following square matrix of size \( \frac{1}{2} n^2(n+1) \) is non-singular:

\[
\begin{pmatrix}
\frac{\partial p_{uv}^{vw}}{\partial y_{ab,c}} & \frac{\partial^2 H_{\nabla}}{\partial y_{ab,c} \partial y_{uv,w}}
\end{pmatrix}_{\alpha \leq b, c}^{u \leq v, w},
\]

where

\[
p_{ij}^k = \frac{\partial L_{\nabla}}{\partial y_{ij,k}}, \quad H_{\nabla} = \sum_{i \leq j} \frac{\partial L_{\nabla}}{\partial y_{ij,k}} y_{ij,k} - L_{\nabla}.
\]

**Proof.** From the very definition of \( H_{\nabla} \) it follows:

\[
\frac{\partial H_{\nabla}}{\partial y_{ab,c}} = \sum_{i \leq j} \frac{\partial^2 L_{\nabla}}{\partial y_{ab,c} \partial y_{ij,k}} y_{ij,k},
\]

and the formula (9) above. Moreover, we claim that the functions \( p_{uv}^{vw} \) depend linearly on the variables \( y_{ab,c} \). In fact, as a calculation shows,

\[
\frac{\partial p_{uv}^{vw}}{\partial y_{ab,c}} = \frac{\partial^2 L_{\nabla}}{\partial y_{ab,c} \partial y_{uv,w}} =
\]

\[
= \rho y_{ab,c} y_{uv,w} \left( G_{ji}^l G_{ri}^c - G_{il}^l G_{ji}^c \right)
\]

\[
= \frac{1}{(1 + \delta_{ab})(1 + \delta_{rc})} \left( y^{bw} (y^{au} y^{cv} + y^{av} y^{cu}) + y^{aw} (y^{ba} y^{cv} + y^{bu} y^{cu}) - y_{ab} (y^{aw} y_{bc} + y^{ac} y^{bw}) - (y^{ua} y^{vb} + y^{ub} y^{va}) y_{wc} + 2 y^{ab} y^{uv} y_{wc} \right).
\]

Therefore, in order to prove that the matrix (9) is non-singular, it suffices to prove that the variables \( y_{ab,c} \) can be written in terms of the functions \( p_{uv}^{vw} \). To
do this, we first compute

\[
\sum_{u,v,w} \frac{1+\delta_{uv}}{p_u p_v} y_{ur} y_{vs} y_{wq} = y_{qr,s} + y_{qs,r} - y_{rs,q} \\
- \frac{1}{2} \sum_{a,b} y^{ab} (y_{aq} y_{ab,r} + y_{rq} y_{ab,s}) \\
+ \sum_{a,b} y^{ab} y_{rs} (y_{ab,q} - y_{qa,b}).
\]

Evaluating the previous formula at \( g_{t0} \), by using adapted coordinates (i.e., \( y_{ij}(g_{t0}) = \varepsilon_i \delta_{ij}, \varepsilon_i = \pm 1 \)), and letting \( \Upsilon_{rsq}^1(j_{t0}^1) = \frac{1+\delta_{rs}}{p_s p_r} (j_{t0}^1 g_{t0}^1) \varepsilon_r \varepsilon_s \varepsilon_q \), it follows:

\[
\Upsilon_{rsq}^1(j_{t0}^1) = y_{qr,s}(j_{t0}^1) + y_{qs,r}(j_{t0}^1) - y_{rs,q}(j_{t0}^1) \\
- \frac{1}{2} \sum_a \varepsilon_a \varepsilon_q (\delta_{qr} y_{aa,r}(j_{t0}^1) + \delta_{qs} y_{aa,s}(j_{t0}^1)) \\
+ \sum_a \varepsilon_a \varepsilon_r (y_{aa,q}(j_{t0}^1) - y_{qa,a}(j_{t0}^1)).
\]

If \( q \neq r \neq s \neq q \), then \( \Upsilon_{rsq}^1(j_{t0}^1) = y_{qr,s}(j_{t0}^1) + y_{qs,r}(j_{t0}^1) - y_{rs,q}(j_{t0}^1) \). Hence

(11) \[ y_{qr,s}(j_{t0}^1) = \frac{1}{2} (\Upsilon_{rsq}^1(j_{t0}^1) - \Upsilon_{qsr}(j_{t0}^1)). \]

If \( q = r, r \neq s \), then

(12) \[ \Upsilon_{rsr}^1(j_{t0}^1) = y_{rr,s}(j_{t0}^1) - \frac{1}{2} \sum_a \varepsilon_a \varepsilon_r y_{aa,s}(j_{t0}^1). \]

If \( r = s, q \neq r \), then

(13) \[ \Upsilon_{rrq}(j_{t0}^1) = 2 y_{qr,r}(j_{t0}^1) - y_{rr,q}(j_{t0}^1) \\
+ \sum_a \varepsilon_a \varepsilon_r (y_{aa,q}(j_{t0}^1) - y_{qa,a}(j_{t0}^1)). \]

The formula (12) can be rewritten as

\[ 2 \varepsilon_r \Upsilon_{rsr}^1(j_{t0}^1) = \varepsilon_r y_{rr,s}(j_{t0}^1) - \sum_{a \neq r} \varepsilon_a y_{aa,s}(j_{t0}^1). \]

Summing up over the index \( r \), \[ 2 \sum_r \varepsilon_r \Upsilon_{rsr}^1(j_{t0}^1) = (2 - n) \sum_r \varepsilon_r y_{rr,s}(j_{t0}^1), \]
and replacing this formula into (12) it follows:

\[ \Upsilon_{rsr}^1(j_{t0}^1) = y_{rr,s}(j_{t0}^1) - \frac{1}{2-n} \varepsilon_r \sum_a \varepsilon_a \Upsilon_{asa}(j_{t0}^1). \]

Therefore

(14) \[ y_{rr,s}(j_{t0}^1) = \Upsilon_{rsr}^1(j_{t0}^1) + \frac{\varepsilon_r}{2-n} \sum_a \varepsilon_a \Upsilon_{asa}(j_{t0}^1). \]

Replacing (14) into (13), we eventually obtain

(15) \[ \sum_a \varepsilon_a y_{qa,a}(j_{t0}^1) = \frac{n-2}{n-1} \sum_a \varepsilon_a \Upsilon_{aaq}(j_{t0}^1) - 2 \frac{a}{(n-2)} \varepsilon_a \Upsilon_{aqa}(j_{t0}^1). \]
and replacing $y_{rr,q}(j^1_{x_0}g)$, $\sum_a \varepsilon_a y_{a,a,q}(j^1_{x_0}g)$, and $\sum_a \varepsilon_a y_{qa,a}(j^1_{x_0}g)$ into (13), it follows:

$$\Upsilon_{rrq}(j^1_{x_0}g) = 2y_{qr,r}(j^1_{x_0}g) - \Upsilon_{rqr}(j^1_{x_0}g) + \frac{nc}{(n-2)^2} \sum_a \varepsilon_a \Upsilon_{aq,a}(j^1_{x_0}g) - \frac{n}{n-2} \sum_a \varepsilon_a \Upsilon_{aqa}(j^1_{x_0}g).$$

Hence

$$\Upsilon_{rrq}(j^1_{x_0}g) = \frac{1}{2(n-2)} \Upsilon_{rrr}(j^1_{x_0}g) + \frac{1}{2} \left(1 - \frac{n}{(n-2)}\right) \Upsilon_{rqr}(j^1_{x_0}g) - \frac{n}{(n-2)^2} \sum_a \varepsilon_a \Upsilon_{aqa}(j^1_{x_0}g).$$

If $q = r = s$, then $\Upsilon_{rrr}(j^1_{x_0}g) = -\sum_{a \neq r} \varepsilon_a \varepsilon_r y_{ra,a}(j^1_{x_0}g)$. From (13) we obtain

$$\sum_{a \neq r} \varepsilon_a y_{ra,a}(j^1_{x_0}g) = -\varepsilon_r y_{rr,r}(j^1_{x_0}g) + \frac{1}{n-2} \sum_a \varepsilon_a \Upsilon_{aar}(j^1_{x_0}g) - 2 \frac{n-1}{(n-2)^2} \sum_a \varepsilon_a \Upsilon_{ara}(j^1_{x_0}g),$$

and replacing it into the previous equation,

$$\Upsilon_{rrr}(j^1_{x_0}g) = y_{rr,r}(j^1_{x_0}g) - \frac{n}{n-2} \sum_a \varepsilon_a \Upsilon_{aar}(j^1_{x_0}g) + 2 \frac{n-1}{(n-2)^2} \sum_a \varepsilon_a \Upsilon_{ara}(j^1_{x_0}g).$$

Hence

$$\Upsilon_{rrr}(j^1_{x_0}g) = \Upsilon_{rrr}(j^1_{x_0}g) + \frac{1}{n-2} \sum_a \varepsilon_a \Upsilon_{aar}(j^1_{x_0}g) - 2 \varepsilon_r \frac{n-1}{(n-2)^2} \sum_a \varepsilon_a \Upsilon_{ara}(j^1_{x_0}g).$$

The formulas (11), (13), (16), and (17) end the proof.

5.2 Hamilton-Cartan equations

The Poincaré-Cartan form for the density $L^v$ is the $n$-form on $J^1\mathcal{M}$ given by

$$\Theta_{L^v} = \sum_{i \leq j} (-1)^{k+1} p^i_{ij} dy_{ij} \wedge v_k - H^v,$$

the momenta $p^i_{ij}$ and the Hamiltonian function $H^v$ being defined as in (10), and the Hamilton-Cartan equations can geometrically be written as

$$\left(j^1_{x_0}g\right)^*(i_Y d\Theta_{L^v}) = 0,$$

for every $p^1$-vertical vector field $Y \in J^1\mathcal{M}$, which are known to be equivalent to Euler-Lagrange equations, where $p^1: J^1\mathcal{M} \to M$ is the natural projection.
According to Proposition 5.1, \((x^i, y_{jk}, p^u_{v})\), \(j \leq k, u \leq v\), is a coordinate system on \(J^1\mathcal{M}\). Letting \(Y = \partial/\partial y_{ab}\) and \(Y = \partial/\partial p^u_{v}\) in (118), it follows respectively:

\[
\sum_k \frac{\partial (p^b_k \circ j^1g)}{\partial x^k} = -\frac{\partial H^\gamma}{\partial y_{ab}} \circ j^1g, \\
\frac{\partial (y_{uv} \circ j^1g)}{\partial x^w} = \frac{\partial H^\gamma}{\partial p^u_{v}} \circ j^1g,
\]

which are the Hamilton-Cartan equations in the canonical formalism.

### 5.3 Covariant Hamiltonian

An Ehresmann (or non-linear) connection on a fibred manifold \(p: E \to M\) is a differential 1-form \(\gamma\) on \(E\) taking values in the vertical sub-bundle \(V(p)\) such that \(\gamma(X) = X\) for every \(X \in V(p)\), e.g., see \[9]. Given \(\gamma\), one has \(T(E) = V(p) \oplus \ker \gamma\). Locally, \(V^*(p)\) is the Lagrangian density defined by setting \(H^\gamma = ((p^b_0)\gamma - \theta) \wedge \omega^\Lambda - \Lambda\), where \(p^1: J^1E \to M\), \(p^0_1: J^1E \to J^0E = E\) are the natural projections, and \(\omega^\Lambda\) is the Legendre form attached to \(\Lambda\), i.e., the \(V^*(p)\)-valued \(p^1\)-horizontal \((n-1)\)-form on \(J^1E\) given by

\[
\omega^\Lambda = (-1)^{i-1} \frac{\partial L}{\partial y^i} dx^n \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n \otimes dy^i, \quad \Lambda = Lv,
\]

and \(\theta = \theta^a \otimes \partial/\partial y^a\), \(\theta^a = dy^a - y^a_i dx^i\), is the \(V(p)\)-valued contact 1-form on \(J^1E\). Locally, \(H^\gamma = \left(\gamma^a + y^a_i \frac{\partial L}{\partial y^i} - L\right) v\).

Let \(\pi: F(M) \to M\) be the bundle of linear frames and let \(q: F(M) \to \mathcal{M}\) be the projection given by \(q(X_1, \ldots, X_n) = g_x = \sum_h \epsilon^h w^h \otimes w^h\), where \((w^1, \ldots, w^n)\) is the dual coframe of \((X_1, \ldots, X_n) \in F_x(M)\), i.e., \(g_x\) is the metric for which \((X_1, \ldots, X_n)\) is a \(g_x\)-orthonormal basis and \(\epsilon^h = 1\) for \(1 \leq h \leq n^+\), \(\epsilon^h = -1\) for \(n^+ + 1 \leq h \leq n\). The projection \(q\) is a principal \(G\)-bundle with \(G = O(n^+, n^-)\). Given a symmetric linear connection \(\Gamma\) with associated covariant derivative \(\nabla\), and a tangent vector \(X \in T_xM\), for every \(u \in \pi^{-1}(x)\) there exists a unique \(\Gamma\)-horizontal tangent vector \(X^h_u \in T_u(FM)\) such that \(\pi_u X^h_u = X\). Given a metric \(g_x \in q^{-1}(x)\), let \(u \in \pi^{-1}(x)\) be a linear frame such that \(q(u) = g_x\).

The projection \(q_u(X^h_u)^v\) does not depend on the linear frame \(u\) chosen over \(g_x\); see the reader to [11] Lemma 3.3 for a proof of this fact. In this way a section \(\sigma^\nabla: p^*TM \to T\mathcal{M}\) of the projection \(p_u: T\mathcal{M} \to p^*TM\) is defined by setting \(\sigma^\nabla(g_x, X) = q_u(X^h)^v\). The retract \(\gamma^\nabla: T\mathcal{M} \to V(p)\) associated to \(\sigma^\nabla\), namely, \(\gamma^\nabla(Y) = Y - \sigma^\nabla(p_u Y), \forall Y \in T_u\mathcal{M}\), determines an Ehresmann connection on the bundle of metrics and the Lagrangian density \(\Lambda^\nabla = L^\nabla v\) admits a “canonical” covariant Hamiltonian \(H^\nabla\). Locally,

\[
\gamma^\nabla (g_x, \partial/\partial x^j) = -\sum_{k \leq l} \left\{ \Gamma^a_{jk}(x) y_{aw} (g_x) + \Gamma^a_{jl}(x) y_{ak} (g_x) \right\} \left( \partial/\partial y^a \right)_{g_x}.
\]
Hence, $\gamma_{kl,j} = -\left(\Gamma^a_{jk}y_{al} + \Gamma^a_{jl}y_{ak}\right)$, and

$$\mathcal{H} = \left(\sum_{k \leq i} \left( y_{kl,j} - \left(\Gamma^a_{jk}y_{al} + \Gamma^a_{jl}y_{ak}\right)\right) \frac{\partial L^\nabla}{\partial y_{kl,j}} - L^\nabla \right)v.$$  

From a direct computation the following result is deduced:

If $\mathcal{H} = \mathcal{H}^\nabla v$, then

$$H^\nabla (j^1_g) = L^\nabla (j^1_g) - 2\rho_j(g) \kappa_j, \quad \forall j^1_g \in J^1_1\mathfrak{M},$$

where $\kappa_j$ is the scalar curvature of the symmetric linear connection $\nabla$ with respect to the metric $g$, namely

$$\kappa_j = g^{jk} \left\{ \frac{\partial \Gamma^i_{jk}}{\partial x^i} - \frac{\partial \Gamma^i_{ij}}{\partial x^i} + \Gamma^i_{jl} \Gamma^l_{ij} - \Gamma^l_{ik} \Gamma^l_{jl} \right\}.$$

The Hamilton-Cartan equations for a covariant Hamiltonian $H^\nabla$ attached to a connection $\gamma$ are

$$\sum_k \frac{\partial (p^k \circ j^1_g)}{\partial x^k} - \sum_{u \leq v} \left( \frac{\partial \gamma_{uv,w} \circ g}{\partial y_{ab}} \right) \left( p^w \circ j^1_g \right) = -\frac{\partial H^\nabla}{\partial y_{ab}} \circ j^1_g,$$

$$\frac{\partial \left( y_{uv,w} \circ j^1_g \right)}{\partial x^w} + \kappa_{uv,w} \circ j^1_g = \frac{\partial H^\nabla}{\partial p^w} \circ j^1_g.$$  

(for example see [2]). Note that for $\gamma = 0$ (that is, the trivial connection induced by the coordinate system) these equations coincide with the local expression of the Hamilton-Cartan equations for $H^\nabla$ given in §5.2.

6 Conclusions

We have defined a first-order Lagrangian $L^\nabla$ on the bundle of metrics which is variationally equivalent to the second-order classical Einstein-Hilbert Lagrangian.

This Lagrangian depends on an auxiliary symmetric linear connection, but this dependence is covariant under the action of the group of diffeomorphisms.

We have also proved that the variational problem defined by $L^\nabla$ is regular and its Hamiltonian formulation has been studied, including the covariant Hamiltonian attached to $\nabla$.

Moreover, we should finally mention the completely different behaviour of $L^\nabla$ with respect to the Palatini Lagrangian.

Let $q: \mathfrak{C} \to M$ be the bundle of symmetric linear connections on $M$. The Palatini variational principle consists in coupling a metric $g$ and a symmetric linear connection $\nabla$ as independent fields, thus defining a first-order Lagrangian density $L_Pv$ on the product bundle $\mathfrak{M} \times M \mathfrak{C}$ as follows:

$$(L_Pv)(g_x, j^1_x \nabla) = s^\nabla \kappa_j(v_x) ,$$
and varying $g$ and $\nabla$ independently. The Palatini method can also be applied to other different settings; e.g., see [6], [3], but below we confine ourselves to consider the classical setting for the Palatini method. As is known, the Euler-Lagrange equations of $L_P$ are the vanishing of the Ricci tensor of $g$ (Einstein’s in the vacuum) and the condition $\nabla = \nabla^g$ expressing that $\nabla$ is the Levi-Civita connection of the metric.

In our case, we can similarly define a first-order Lagrangian $\mathcal{M} \times M \in \mathcal{C}$ by setting $L(j^1 g, j^1 \nabla) = L_{HE}(j^2 g) + c \left( \text{alt}_{23}(\nabla^g T^g, \nabla) \right)(\rho \circ g)$. Assuming $M$ is compact, then the action associated with $L$ is given as follows: $S(g, \nabla) = \in t_M L(\nabla(j^1 g, j^1 \nabla)\nabla), \text{ and by considering 1) an arbitrary 1-parameter variation } g_t \text{ of } g \text{ and 2) the 1-parameter variation } \nabla_t = \nabla + tA \text{ attached to } A \in \Gamma(S^2 T^* M \otimes TM) \text{ of } \nabla, \text{ we obtain, 1) Einstein’s equation and 2) } 0 = \in t_M \left( \text{alt}_{23}(\nabla^g A) \right) \nabla g, \forall A \in \Gamma(S^2 T^* M \otimes TM), \text{ which leads us to a contradiction.}

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