On the Structure of QFT in the Particle Picture of the Path Integral Formulation

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In quantum field theory (QFT), the path integral is usually formulated in the wave picture, i.e., as a sum over field evolutions. This path integral is difficult to define rigorously because of analytic problems whose resolution may ultimately require knowledge of non-perturbative or even Planck scale physics. Alternatively, QFT can be formulated directly in the particle picture, namely as a sum over all multi-particle paths, i.e., over Feynman graphs. This path integral is well-defined, as a map between rings of formal power series. This suggests a program for determining which structures of QFT are provable for this path integral and thus are combinatorial in nature, and which structures are actually sensitive to analytic issues. For a start, we show that the fact that the Legendre transform of the sum of connected graphs yields the effective action is indeed combinatorial in nature and is thus independent of analytic assumptions. Our proof also leads to new methods for the efficient decomposition of Feynman graphs into n-particle irreducible (nPI) subgraphs.

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At the heart of the path integral formulation of quantum field theory, e.g., on flat space, is the integral over fields,

$$Z[J] = \mu \int e^{i[S]\{\Phi\} + i\int J\Phi \, d^dx} \, D[\Phi] ,$$  \hspace{1cm} (1)

i.e., the Fourier transform of $e^{iS}$. Here, $S$ is the classical action, $\mu = e^{i\beta}$ corresponds to the cosmological constant, $J, \Phi$ stand for (a collection of), e.g., real bosonic fields and their corresponding sources, and $c = h = 1$. We assume suitable ultraviolet and infrared cutoffs so that the space of fields, equipped with the inner product $\langle J, \Phi \rangle = \int J(x)\Phi(x) \, d^dx$, is of finite dimension, say $N$. Choosing an orthonormal basis, $ \{b_n\}_{n=1}^N$, in the space of fields, we have $\Phi = \Phi(b, a), J = J_a b_a, \langle J, \Phi \rangle = J_a \Phi(b)$, and:

$$S[\Phi] = \sum_{n \geq 2} \frac{1}{n!} S^{(n)}_{a_1 \ldots a_n} \Phi_{a_1} \ldots \Phi_{a_n} .$$  \hspace{1cm} (2)

Twice occurring indices are to be contracted, i.e., summed over. $S^{(2)}$ is assumed to contain a Feynman $i\epsilon$ term, and we assume $S$ does not lead to a non-zero vacuum field expectation value. Then, $Z[J] = \mu \int_{\mathbb{R}^N} e^{i[S]\{\Phi\} + i\int J \Phi \, d^dx} \prod J_a b_a, \Phi_{a}, \langle J, \Phi \rangle = J_a \Phi(b)$, and:

$$Z[J] = \mu' \sum_{n \geq 2} \frac{1}{n!} S^{(n)}_{a_1 \ldots a_n} \theta_{(i\epsilon J_a \ldots i\epsilon J_a)} e^{(i\epsilon J_a) S^{(2)-1}(i\epsilon J_a)}$$

Thus, $Z[J]$ is the generating functional of all Feynman graphs $\gamma$ built from the Feynman rules $edge = i(S^{(2)})^{-1}$, and $n$-vertex $= iS^{(n)}$, with at least one edge. We can view $Z[J]$ also as a sum of all graphs with the additional Feynman rule 1-vertex $= iJ_a$, where each graph $\gamma$ has a symmetry factor $\omega(\gamma) = 2^{-\ell} k^{-1}$. Here, $\ell$ is the number of edges of $\gamma$ joining a vertex with itself and $k$ is the number of automorphisms of $\gamma$. Note that if $\gamma$ is a tree graph with labelled ends (i.e., no 1-vertices) then $\omega(\gamma) = 1$.

Correspondingly, let us denote the sum of only the connected graphs by $iW[J]$. When exponentiated, it yields the sum of all graphs, i.e., $\exp(iW[J]) = Z[J]$, as is easy to see combinatorially. Further, as is well-known:

**Theorem 1**: For a given action, $S$, let $iW[J]$ denote the sum of all connected graphs. Assume that the power series $W[J]$ converges to a function which is convex. The definition $\varphi_a = \partial W[J]/\partial J_a$ can, therefore, be inverted to obtain $\hat{J}[\varphi] = (J[\varphi])$. Then, the Legendre transform of $W[J]$, namely $\Gamma[\varphi] = -\mathcal{J}[\varphi] J_a \varphi_a + W[J[\varphi]]$, yields $i\Gamma[\varphi]$, which is the generating functional of the sum of n-point 1-particle irreducible (1PI) graphs for n > 2, and $i\Gamma^{(2)} = iS^{(2)} + \sum (2$-point 1PI graphs). Thus, overall:

$$e^{iS[\Phi]} \xrightarrow{\text{Fourier}} Z[J] \xrightarrow{\text{log/exp}} iW[J] \xrightarrow{\text{Legendre}} i\Gamma[\varphi].$$

We will here question the assumptions underlying Thm.1. First, let us consider Theorem 1’s broad significance:

**A)** The practical calculation of Feynman graphs. Any connected graph can be viewed as consisting of maximal 1PI subgraphs that are connected by edges whose deletion would disconnect the graph. For practical calculations of Feynman graphs, this conveniently identifies the 1PI graphs as building blocks. After renormalizing them, the 1PI graphs may be glued together to form connected graphs with no further loop integrations needed. Later we shall discuss a strategy for the further decomposition of 1PI graphs by extending Thm.1 through higher order, i.e., multi-field Legendre transforms.

**B)** The action and the generating functional of tree graphs are related by Legendre transform. Continuing the
discussion of the structure of connected graphs in (A), we notice that any connected graph is a tree graph whose vertices are 1PI graphs connected by strings of edges and 2-point 1PI graphs. Thus, the sum of connected graphs, \( iW[J] \), is also the sum of all tree graphs made from new Feynman rules. The new Feynman rules’ \( n \)-vertex is the sum of all \( n \)-point 1PI graphs, while the new rules’ edge is given by \( - + - \circ \cdots - \circ \circ \cdots - \circ \circ - \cdots = ((-1)^{-1} + 0)^{-1} \), where \( \circ \) is the sum of 2-point 1PI graphs and we summed a geometric series. Clearly, these are the Feynman rules generated by \( \Gamma[\varphi] \) if viewed as an action. Thus, the Legendre transform maps the sum of trees, \( W \), into the action \( \Gamma \). Now every generating functional of tree graphs, \( iT[K] \), is the generating functional of connected graphs for some action, \( F[\Psi] \), since Fourier transform and exponentiation are invertible. Thus:

**Theorem 1 (2nd formulation):** Let \( iT[K] \) denote a sum of all tree graphs built from the Feynman rules of some action, \( F[\Psi] \). Assume that the power series \( T[K] \) converges to a function which is convex, so that the definition \( \Psi_a = \partial T[K]/\partial K_a \) is invertible, to obtain \( K[\Psi]_a \). Then, \( F[\Psi] \) and \( T[K] \) are related by Legendre transform:

\[
F[\Psi] = -K[\Psi]_a \Psi_a + T[K[\Psi]].
\]

(3)

Below, we will prove Thm.1 in this formulation but with weaker assumptions.

**C) The perturbative solution to the classical equations of motion can be obtained from the sum of the tree graphs.** To see this, consider the action, \( F[\Psi] + \int K \Psi d^3x \), of a classical system coupled linearly to a source field, or driving force, \( K \). The equations of motion, \( \delta F/\delta \Psi = -K \), are to be solved for the field \( \Psi[K] \) as a functional of the applied source \( K \). By Thm.1 (2nd formulation), the inverse Legendre transform \( T[K] = F[\Psi] + \int K \Psi d^3x \) of \( F[\Psi] \) yields the generating functional, \( iT[K] \), of trees. From the properties of Legendre transforms we have:

\[ \Psi[K] = \delta T[K]/\delta K. \]

Thus, \(-i\times\) the sum of the tree graphs, \( iT[K] \), once differentiated by \( K \), yields the perturbative solution to the classical equations of motion in powers of the perturbing source field \( K \).

**D) Effective action.** \( \Gamma[\varphi] \) plays the rôle of a quantum effective action because it is that action which when treated classically yields the correct quantum theoretic answer: any \( n \)-point function can be calculated as a sum of all connected graphs using the Feynman rules of the action \( S \) or also, as if classical, \( i.e. \), as a sum of all tree graphs only, when adopting the Feynman rules generated by the effective action \( \Gamma \).

**E) Duality of problems and solutions.** One usually defines a problem by specifying an action, \( S \), and the classical and quantum solutions are then obtained by calculating \( T \) and \( W \) respectively. \( T \) and \( e^{iW} \) are the Legendre and Fourier transforms of \( S \) and \( e^{iS} \), respectively. Both transforms are invertible. Thus, one may also define a problem by specifying, say, \( T \) (or \( W \)). The problem’s solution is then the action, \( S \). In fact, since the Legendre and Fourier transforms are involutive (up to a trivial sign), \( S \) can be calculated in the same way by using new, “dual” Feynman rules: a given \( W \) is viewed as an action, the dual Feynman rules are read off, and \( iS \) is obtained as the sum of all connected graphs. This duality was first noticed in the context of statistical physics, in [1]. Here we add that, similarly, a given \( T \) can be viewed as an action, dual Feynman rules can be read off, and \( S \) can then be calculated from the sum of all tree graphs. For example, in cosmology, there are efforts to reconstruct the potential in the inflaton action, \( S \), from the inflaton correlation functions in \( W \) obtained \( \text{via} \) measurements of the cosmic microwave background [2]. In principle, it should be possible to view \( W \) (as far as \( W \) is known) as an action, read off the dual Feynman rules and calculate \( iS \) as the sum of connected graphs. Similarly, by summing up only the tree graphs, one should, in principle, obtain the inflaton’s quantum effective action.

It appears that every theory specified by an action possesses a Fourier dual as well as a Legendre dual theory. Thus, \( e.g., \) in addition to the Dyson-Schwinger equation \( (\delta S/\delta \Phi)(-i\delta/\delta J) + J)e^{iW[\Psi]} = 0 \), the involutive property of the Fourier transform implies a dual Dyson-Schwinger equation: \( (\delta W/\delta J[i\delta/\delta \Phi] - \Phi)e^{iS[\Psi]} = 0 \). Similarly, there are, \( e.g., \) dual Slavnov-Taylor identities for gauge theories. We note that an instance where the Legendre transform of an effective action is itself the effective action of a known theory was found in [3] in the context of S-duality and weak versus strong coupling regimes. Finally, we notice that the involutive property of the Legendre transform implies that the “sum of the trees of trees” must reduce to the original sum of the Feynman rules. The involutive property of the Fourier transform implies a corresponding statement for connected graphs.

**From the wave picture to the particle picture.** All of the above considerations appear to hinge on analytic assumptions. Namely, it appears that \( S, Z, W \) and \( \Gamma \) should be series that converge to well-defined functions which possess Fourier and Legendre transforms, respectively. For example, the power series \( W \) and \( \Gamma \) would seem to have to converge to convex functions in order to possess Legendre transforms. As is well known, however, not even their convergence can be assumed in QFT, a problem whose solution, it is thought, may require knowledge of non-perturbative or even Planck scale physics.

The fact that perturbative QFT is nevertheless very successful in practice suggests that it should be possible to make the formalism of QFT mathematically well-defined without analytic assumptions such as convexity or even convergence. Within such a framework, it should be possible to prove key theorems combinatorially, such as Thm.1, the involution properties of the Fourier and Legendre transforms, or the Dyson-Schwinger equations.

To this end, we define \( S, Z, W, \Gamma, T \) and \( F \) as elements in a ring of formal power series, for bosons as for fermions.
All physically relevant information is encoded in the individual coefficients. For rings of formal power series, see, e.g., [4]. For any formal power series $F$, with $F^{(1)} = 0$ and $F^{(2)}$ invertible, (which would require only local convexity), we then define a “combinatorial Legendre transform”, $T$, namely as the following map: view $F[Ψ]$ as an action, read off the Feynman rules and then obtain $iT[K]$ as the power series generating all tree graphs. We also define a “combinatorial Fourier transform”, $e^{iW[J,z]}$ of $e^{iS[Ψ]}$: read off Feynman rules from $z^{-1}S$, where $z$ is an indeterminate, and set $iW = z \sum$ connected graphs with the combinatorial factors $ω(g)$. While the edge and vertices are proportional to $z$ and $z^{-1}$, no negative powers of $z$ occur in $W$. This is because for any connected graph, $g$, the numbers of edges and vertices, $E(g)$ and $V(g)$, obey $E(g) - V(g) \geq -1$. Note that $z$ counts powers in $h$ (and thus loops). Indeed, the combinatorial Legendre transform is contained in $iW[J,z]$ as the term proportional to $z^0$. This is because exactly for tree graphs, as is easy to verify:

$$1 = V(g) - E(g). \tag{4}$$

Within this framework, $Z$ is defined not through Eq.1, i.e., as a sum over all field evolutions (the wave picture) but instead through $Z = e^{iW}$ as a sum over all multi-particle paths (the “particle picture”), where the term “path” means graph. Notice that the principle that a particle’s classical path is, in a suitable measure, the shortest path, while quantum theory requires a sum over all paths, persists in second quantization: while the classical solutions are obtained from the tree graphs only, QFT requires summing over all graphs. Indeed, tree graphs are the shortest graphs in terms of the number of edges for any given number of leaves of the graph, i.e., for any given perturbation order. Also, the free propagator, i.e., the edge, can itself be viewed as a sum over paths. A “path” in QFT is, then, a graph of paths.

The QFT path integral is mathematically well-defined through the combinatorial Fourier transform because the calculation of each coefficient involves only a finite number of terms. This suggests the program of trying to prove key equations of QFT combinatorially, for example the Dyson-Schwinger equations, or Eq.3. This is non-trivial because, where successful, it shows that the equation in question is fundamentally combinatorial in nature and does not hinge upon analytic assumptions - such as assumptions of convergence and convexity in Eq.2 of Thm.1, or, in the case of the Dyson-Schwinger equations, the assumption that boundary terms can be neglected when path integrating a total derivative. While one aim is to reveal the robustness or fragility of the key equations of QFT with respect to analytic assumptions, any deeper understanding of the key equations in QFT has of course the potential to reveal useful new structures.

Starting this program, we here give a transparent and bare-bones combinatorial proof of Thm.1 which shows that the theorem is robust against issues of analyticity. Our proof shows that the Legendre transform in QFT can be understood, more deeply, as a simple statement (namely Eq.4 about tree graphs. This insight then leads to useful new results, namely about the decomposition of Feynman graphs into their nPI components.

**Theorem 1 in the new framework.** In the second formulation of Thm.1 above, $F$ may or may not be an effective action. Our aim is to prove Thm.1 in this general form, but for the combinatorial Legendre transform.

**Theorem 1 (3rd, combinatorial formulation):** Let $F[Ψ] = \sum_{n \geq 2} \frac{1}{n!} Ψ^{(n)} Ψ_{a_1} \cdots Ψ_{a_n}$ be an element of a ring of formal power series in commutative indeterminates $Ψ_a$. Assuming that the coefficient matrix $F^{(2)}$ is invertible, $F[Ψ]$ can be viewed as an action that defines Feynman rules. The sum of their tree graphs yields a formal power series, say $iT[K]$, in variables $K_a$. By definition, we relate the variables $Ψ$ and $K$ through the algebraic derivative $Ψ[K]_a = iT[K] / δK_a$, which is a well-defined operation in the ring, so that $Ψ[K]_a$ is a formal power series in the $K_a$. Then, the formal power series $F[Ψ]$ and $T[K]$ obey the Legendre transform equation:

$$T[K] = K_a Ψ[K]_a + F[Ψ[K]]. \tag{5}$$

We remark that $F[Ψ[K]]$ is well-defined as a formal power series since $Ψ[K]$ has no constant term. The theorem covers the special case when the usual analytic Legendre transform of $T[K]$ is well-defined, i.e., the case in which $T[K]$ obeys the analytic conditions of Thm.1 in its second formulation. This is because in this case the transformed variable obeys $Ψ^{\text{analytic}}[a] = δT[K] / δK_a = Ψ_a$ and therefore with $F^{\text{analytic}} := −Ψ_a K_a + T$ and Eq.5 we have $F = F^{\text{analytic}}$. We can then conclude that the Feynman rules underlying $T$ are generated by the Legendre transform $F^{\text{analytic}}$. Notice that Thm.1 in its 1st and 2nd formulations makes no claim when the sum of the tree graphs does not converge to a function, or does converge but the function is not convex. Our generalized Thm.1 (3rd formulation) shows that Eq.5 holds even then.

In the literature, Thm.1 is proven in the first formulation above, see [2, 6, 7]. The proof by Weinberg, [3], essentially addresses Thm.1 in its second formulation, i.e., directly as a map between any action and its sum of tree graphs. However, that proof relies on analytic assumptions and requires the taking of a subtle limit.

**Combinatorial proof of Thm.1.** Our proof strategy is to show that $i \times$ Eq.5 is term by term equivalent to a much simpler statement, namely Eq.4. To this end, let us prove the power series equation $i \times$ Eq.5 for the coefficients of each $m$-power of $K$ for $m \geq 2$. That is,

$$\frac{∂}{∂K_{a_1}} \cdots \frac{∂}{∂K_{a_m}} iT[K]_{K=0} = \left. \frac{∂}{∂K_{a_1}} \cdots \frac{∂}{∂K_{a_m}} (iK_a Ψ[K]_a + iF[Ψ[K]]) \right|_{K=0}. \tag{6}$$

By definition of $T[K]$, on the LHS of Eq.6 we obtain the sum of all tree graphs $g$ with $m$ ends, labelled by
$a_1, \ldots, a_m$, with each such $g$ occurring exactly once and with $\omega(g) = 1$. We will complete the proof by showing that the RHS consists of all such graphs with multiplicity $V(g) - E(g)$. We begin by writing the RHS of $i \times \text{Eq.5}$ in terms of tree graphs. We have $iK_a = 1$-vertex, and

$$iF[\Psi[K]] = \sum_{n \geq 2} \frac{1}{n!} F^{(n)}_{a_1, \ldots, a_n} \Psi[K]_{a_1} \cdots \Psi[K]_{a_n},$$

contains $iF^{(n)}_{a_1, a_2} = - \text{(edge)}^{-1}$ and $iF^{(n)}_{a_1, \ldots, a_n} = n$-vertex for $n > 2$. Thus, the RHS of $i \times \text{Eq.5}$ takes the form:

$$(\text{1-vertex})_a \Psi[K]_a - \frac{1}{2} \Psi[K]_{b_1} (\text{edge})^{-1} \Psi[K]_{b_2} + \sum_{n > 2} \frac{1}{n!} (\text{n-vertex})_{a_1 \cdots a_n} \Psi[K]_{a_1} \cdots \Psi[K]_{a_n}.$$ 

After differentiating $m$ times and setting $K = 0$, we obtain Eq.6, which contains only graphs with $m$ labelled ends. Since, by definition, $\Psi[K]_{b_i} = \partial T[K]/\partial (iK_a)$ is the sum of trees with one end vertex removed, the RHS of Eq.6 is the sum of all tree graphs with $m$ labelled ends obtained by taking a term of the action, i.e., $- \text{(edge)}^{-1}$ or any $n$-vertex, and attaching the sum of all tree graphs at each of its free indices. After simplification, this means that Eq.5 reads, schematically:

$$\sum_{\text{trees}} = \sum_{\text{trees}} - \frac{1}{2} \sum_{\text{trees}} + \frac{1}{3!} \sum_{\text{trees}} + \cdots$$

Let us consider an arbitrary tree graph, $g$, with $m$ labelled ends. On the LHS, it occurs exactly once. To count its occurrences on the RHS, we choose an arbitrary edge $e$ of $g$, and let $l$ and $r$ arbitrarily denote the two subtrees to either side of the edge. In the second term on the RHS, the edge $e$ occurs twice, and because of the $1/2$ in the action, $g$ occurs with weight $-E(g)$.

We now choose an arbitrary $n$-vertex $v$ of $g$, where $n \in \{1, 3, 4, \ldots\}$. Let $\{t_j\}_{j=1}^n$ arbitrarily denote the sub-trees emanating from its legs. From the remaining terms on the RHS of Eq.5, our vertex $v$ with the attached subtrees $\{t_j\}_{j=1}^n$ arises $n!$ times, which is cancelled by the $1/n!$ in the action. Thus, $g$ occurs $V(g)$ times in the remaining terms and therefore indeed with overall weight $V(g) - E(g)$ on the RHS.

**Outlook.** In a follow-up paper, we will show how insight from our combinatorial proof of Thm.1 yields a strategy for a more efficient decomposition of Feynman graphs into nPI graphs for practical calculations. Namely, recall that we reduced the Legendre transform, Eq.4 to the simple equation Eq.4 which is Euler’s formula, a special case of the general Euler-Poincaré formula for the homology of graphs. As we will show in the follow-up paper, the so-called cactus representation $H(g)$ of nPI graphs $g$, see [9], can be used to obtain a generalization of Eq.4 to $1 = V(H(g)) - E(H(g)) + C(H(g))$ (where $C$ is the number of cycles) which allows us to generalize the diagrammatic analysis of [9] and improve on the results on higher order Legendre transforms and the analysis of Dyson-Schwinger equations of [9]. We will also show that the involutive property of the Legendre transform can be proven purely combinatorially. This means that a) the sum of trees of trees indeed always reduces to the original sum of Feynman rules and b) that every theory has a Legendre dual whose dual is the original theory. We will also study the involutive property of the Fourier transform.

There are a number of further key equations of QFT which are usually derived using analytic arguments and assumptions, and it should be very interesting to use the combinatorial Fourier transform to investigate which of these equations are actually of a purely combinatorial nature and therefore robust against analytic difficulties. For example, the origins of anomalies and of ghost fields are usually traced, analytically, to the measure in the wave picture path integral, Eq.11. Is there a combinatorial derivation of these, perhaps involving what could be viewed as a “measure” in the particle picture path integral? The usual derivation of the Dyson Schwinger equations assumes that boundary terms in Eq.11 can be neglected. What is the combinatorial analog of derivatives and boundary terms for the combinatorial Fourier transform of the particle picture path integral? Also, certain discontinuities in QFT renormalization can be traced to functional analytic restrictions of the domain of the wave picture path integration, see e.g., [10]. What is the analog in the particle picture path integral?

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