The Fermat cubic, elliptic functions, continued fractions, and a combinatorial excursion

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1 "ALGEBRAIC" CONTINUED FRACTIONS

\[ S \text{ (Stieltjes)} \]
\[ \frac{1}{z \tan z} = \frac{1}{1 - \frac{z^2}{1 - \frac{z^2}{3 - \frac{z^2}{5 - \ldots}}}} \]

\[ J \text{ (Jacobi)} \]
\[ \sum_{n \geq 0} n! \cdot z^n = \frac{1}{1 - 1 \cdot z - \frac{1^2 \cdot z^2}{1 - 3 \cdot z - \frac{2^2 \cdot z^2}{\ldots}}} \]

CF: Iterate \( X \mapsto 1/X; \quad X = [X] + \{X\}. \) Here: \( f = f(0) + z f'(0) + z^2 \{f\} \).

(Irrationality of \( \pi \) (Lambert) and summation of divergent series (Euler). Also related to orthogonal polynomials Padé approximants, moment problems, etc.)

Explicit CFs are very rare: From Perron, Wall, Chihara, etc, perhaps less than 100 continued fractions are known for special functions.
Theorem (Apéry 1978): \( \zeta(3) = \sum 1/n^3 \) is irrational.

\[
\zeta(3) = \frac{6}{1^6}, \quad \varpi(0) = \frac{1}{1^6}, \quad \varpi(1) = \frac{2}{2^6}, \quad \varpi(2) = \frac{3}{3^6}, \ldots,
\]

with \( \varpi(n) := (2n + 1)(17n(n + 1) + 5) \).

(Stieltjes) \[
\sum_{n \geq 0} \frac{1}{(n + z)^3} = \frac{1}{1^6}, \quad \sigma(0) = \frac{1}{1^6}, \quad \sigma(1) = \frac{2}{2^6}, \quad \sigma(2) = \frac{3}{3^6}, \ldots,
\]

with \( \sigma(n) = (2n + 1)(2z(z + 1) + n(n + 1) + 1) \). Cf Berndt/Ramanujan.
Theorem (Conrad 2002): For a certain function $sm$:

$$
\int_0^\infty sm(u)e^{-u/x} du = \frac{x^2}{1 + b_0x^3 - \frac{1 \cdot 2^2 \cdot 3^2 \cdot 4 x^6}{1 + b_1x^3 - \frac{4 \cdot 5^2 \cdot 6^2 \cdot 7 x^6}{1 + b_2x^3 - \frac{7 \cdot 8^2 \cdot 9^2 \cdot 10 x^6}{\ldots}}}},
$$

where $b_n = 2(3n + 1)((3n + 1)^2 + 1)$, and

$$
sm(z) = \text{Inv} \int_0^z \frac{dt}{(1 - t^3)^{2/3}} = \text{Inv} z \cdot _2F_1 \left[\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; z^3\right].
$$
Plan: some cute combinatorics surrounding the functions

— The Fermat cubic $x^3 + y^3 = 1$ and Dixonian functions

— A first model related to Pólya urns and branching processes

— A second model of Dixonian function by permutations
  * based on parity constraints (cf Viennot, F., Dumont)

— A third model of Dixonian function by weighted Dyck paths, related to continued fractions, and permutations
  * based on patterns of order 3 (cf F.-Françon)

Side effects: An analytic-combinatorial approach to urn processes that are $2 \times 2$ balanced.
The Fermat curve $F_m$ is the complex algebraic curve

$$x^m + y^m = 1.$$ 

**Circle $F_2$:** Consider $[s' = c, \quad c' = -s]$, with $s(0) = 0, \quad c(0) = 1$.

The transcendental functions $s, c$ do parameterize the circle,

$$s(z)^2 + c(z)^2 = 1, \quad \text{since} \quad (s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0.$$ 

Also: inversion from abelian integral $\int R(z, y) \, dz$ on $F_2$:

$$\int_0^{\sin z} \frac{dt}{(1 - t^2)^{1/2}} = z, \quad \cos(z) = \sqrt{1 - \sin(z)^2}$$

For combinatorialists: $\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z}$ enumerate alternating (aka up-and-down, zig-zag) permutations (Désiré André, 1881).
The “complexity” of integral calculus over an algebraic curve depends on its (topological) genus.

Sphere with 3 holes, $g = 3$

For Fermat curve $F_p$, genus is $\frac{1}{2}(p - 1)(p - 2)$.

- $F_2 \implies g = 0$;
- $F_3 \implies g = 1$; Normal forms of Weierstraß and Jacobi + Dixon;
- $F_4 \implies g = 3, \ldots$
A clever generalization of \( \sin, \cos \): the \textit{nonlinear} system

\[ s' = c^2, \quad c' = -s^2 \quad \text{with} \quad s(0) = 0, \ c(0) = 1. \]

We have: \( s(z)^3 + c(z)^3 = 1 \): the pair \( \langle s(z), c(z) \rangle \) \textit{parametrizes} \( F_3 \).

Follow \textbf{Dixon (1890)} and set: \( \text{sm}(z) \equiv s(z), \quad \text{cm}(z) \equiv c(z). \)

(See \( sn, cn \) by Jacobi, \( sl, cl \) for lemniscate.)

\[
\begin{aligned}
\text{sm}(z) &= z - 4 \frac{z^4}{4!} + 160 \frac{z^7}{7!} - 20800 \frac{z^{10}}{10!} + 6476800 \frac{z^{13}}{13!} - \cdots \\
\text{cm}(z) &= 1 - 2 \frac{z^3}{3!} + 40 \frac{z^6}{6!} - 3680 \frac{z^9}{9!} + 8880000 \frac{z^{12}}{12!} - \cdots .
\end{aligned}
\]
Alfred Cardew Dixon
Born: 22 May 1865 in Northallerton, Yorkshire, England
Died: 4 May 1936 in Northwood, Middlesex, England

Generally, ACD considers $X^3 + Y^3 - 3\alpha XY = 1$. 
2.1 A hypergeometric connection.

One can make \( s \equiv s_m \) and \( c \equiv c_m \) somehow “explicit”. Start from the defining system and differentiate

\[
\begin{align*}
    s' &= c^2 \quad \Rightarrow \quad s'' = 2cc' \\
    s'' &= -2cs^2 \quad \Rightarrow \quad s'' = -2c\sqrt{s'}.
\end{align*}
\]

Then “cleverly” multiply by \( \sqrt{s'} \) to integrate (\( \int \)):

\[
\begin{align*}
    s'' \sqrt{s'} &= -2s^2s' \quad \Rightarrow \quad \int \frac{2}{3} (s')^{3/2} = -\frac{2}{3} s^3 + K.
\end{align*}
\]

\[
\int_0^{\text{sm}(z)} \frac{dt}{(1 - t^3)^{2/3}} = z, \quad \text{cm}(z) = \sqrt[3]{1 - \text{sm}(z)^3}
\]

= Abelian integral over \( F_3 \) + incomplete Beta integral + hypergeometric
Classical hypergeometric function:

\[ 2F_1[\alpha, \beta, \gamma; z] := 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{z}{1!} + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{z^2}{2!} + \cdots. \]

\( \text{Inv}(f) \) is the inverse of \( f \) w.r.t. composition: \( \text{Inv}(f) = g \) if \( f \circ g = g \circ f = \text{Id} \).

**Proposition:** Function \( \text{sm} \) is defined by inversion,

\[ \text{sm}(z) = \text{Inv} \int_0^z \frac{dt}{(1 - t^3)^{2/3}} = \text{Inv} z \cdot 2F_1 \left[ \begin{array}{c} 1/3, 2/3, 4/3; z^3 \end{array} \right]. \]

The function \( \text{cm} \) is then defined near 0 by

\[ \text{cm}(z) = \sqrt[3]{1 - \text{sm}^3(z)}. \]
A STARTLING FRACTION.

From Eric van Fossen Conrad, PhD Columbus, OH, 2002.

\[
\int_0^\infty \text{sm}(u)e^{-u/x} \, du = \frac{x^2}{1 + b_0 x^3 - \frac{1 \cdot 2^2 \cdot 3^2 \cdot 4 x^6}{1 + b_1 x^3 - \frac{4 \cdot 5^2 \cdot 6^2 \cdot 7 x^6}{1 + b_2 x^3 - \frac{7 \cdot 8^2 \cdot 9^2 \cdot 10 x^6}{\cdots}}}},
\]

where \( b_n = 2(3n + 1)((3n + 1)^2 + 1) \).
Proof: Follow Stieltjes and Rogers. Cleverly introduce

\[ S_n := \int_0^\infty \sm^n(u)e^{-u/x} \, du. \]

Then integration by parts shows that

\[ \frac{S_n}{S_{n-3}} = \frac{n(n-1)(n-2)x^3}{1 + 2n(n^2 + 1)x^3 - n(n + 1)(n + 2)x^3 \frac{S_{n+3}}{S_n}}. \]

⇒ “Pump” out the continued fraction.

**Six J-fractions:** sm, sm², sm³, cm, cm·sm, cm·sm²:

\[ 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, \ldots \]

+ **Three S-fractions:** sm, cm, sm·cm.
4 BALLS GAMES

Cf. Théorie analytique des probabilités Laplace (1812).

**Pólya urn model.** An urn contains black and white balls. At each epoch, a ball in the urn is chosen at random.

Described by a placement matrix. Here:

$$\mathcal{M}_{12} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}, \quad \circ \rightarrow \bullet \bullet, \quad \bullet \rightarrow \circ \circ$$

A *history* of length $n$ (Françon78) is any description of a legal sequence of $n$ moves of the Pólya urn. For instance ($n = 5$):

$$\times \rightarrow \y \rightarrow \yx \rightarrow \yxy \rightarrow \xyxy \rightarrow \xxyxy \rightarrow \xxyyxx,$$

What are the “history numbers”? The sequence for $(1, 0) \mapsto (0, \star)$ starts as $0, 1, 0, 0, 4, 0, 0, 160$. Cf sm?
4.1 Urns and Dixonian functions.

Take the (autonomous, nonlinear) ordinary differential system

\[ \Sigma : \quad \frac{dx}{dt} = y^2, \quad \frac{dy}{dt} = x^2, \quad \text{with} \quad x(0) = x_0, \quad y(0) = y_0, \]

\[ \langle x(t), y(t) \rangle \] parameterizes the “Fermat hyperbola”: \( y^3 - x^3 = 1 \).

For \( x_0 = 0, y_0 = 1 \), get trivial variants: \( \text{smh}(z) = -\text{sm}(-z), \text{cmh}(z) = \text{cm}(-z) \).
Define a linear transformation \( \delta \) acting on polynomials \( \mathbb{C}[x, y] \):

\[
\delta[x] = y^2, \quad \delta[y] = x^2, \quad \delta[u \cdot v] = \delta[u] \cdot v + u \cdot \delta[v],
\]

(Cf the elegant presentation of Chen grammars by (Dumont96) and the "combinatorial integral calculus" of Leroux–Viennot.)

(i) **Combinatorially**, the \( n \)th iterate \( \delta^n [x^a y^b] \) is such that

\[
\text{# histories from } (a_0, b_0) \text{ to } (k, \ell) = \text{coeff}[x^k y^\ell] \delta^n [x^{a_0} y^{b_0}],
\]

(ii) **Algebraically**, the operator \( \delta \) describes the "logical consequences" of the differential system \( \Sigma = \{ \dot{x} = y^2, \dot{y} = x^2 \} \):

\[
\delta^n [x^a y^b] = \frac{d^n}{dt^n} x(t)^a y(t)^b \quad \text{expressed in} \quad x(t), y(t),
\]

\( \heartsuit \text{ Taylor } \implies H(x(t), y(t); z) = x(t + z)^{a_0} y(t + z)^{b_0} ; \text{ set } t = 0 \ldots \)
Combinatorial Interpretation I

**Proposition:** The EGFs of histories of the urn $\mathcal{M}_{12}$ starting with one ball: and ending with balls . . .

All of the other colour: \[
\frac{\text{sm}(z)}{\text{cm}(z)} = -\text{sm}(-z).
\]

All of the original colour: \[
\frac{1}{\text{cm}(z)} = \text{cm}(-z).
\]

Homogeneous monomial **differential systems** $\iff k \times k$ balanced urns.

Note: Get full composition (=Gaussian), large deviations, etc.

\[
\mathbb{P}(X_n = 0) \sim c\rho^{-n}, \quad \rho = \frac{\sqrt{3}}{6\pi} \Gamma \left( \frac{1}{3} \right)^3, \quad n \equiv 1 \pmod{3}.
\]
Note: The knight’s moves of Bousquet-Melou & Petkovšek.

\[
\begin{array}{c}
\circ-----.
|     |
|     |
| .   .
|     |
|     |
P=(p,q) \circ----------.
|     |
|     |
| .   .
|     |
|     |
\end{array}
\text{multiplicity } p

\[
\begin{array}{c}
\circ----------.
|     |
|     |
| .   .
|     |
|     |
P \circ----------.
|     |
|     |
| .   .
|     |
|     |
\end{array}
\text{multiplicity } q

The OGF of walks that start at \((1,0)\) and end on the horizontal axis is

\[
G(x) = \sum_{i \geq 0} (-1)^i \left( \xi^{(i)}(x) \xi^{(i+1)}(x) \right)^2,
\]

where \(\xi\), a branch of the (genus 0) cubic \(x\xi - x^3 - \xi^3 = 0\) is

\[
\xi(x) = x^2 \sum_{m \geq 0} \binom{3m}{m} \frac{x^{3m}}{2m+1}.
\]
4.2 **Continuous-time branching = Yule process.**

Foatons and Viennons live an exponential time and disintegrate . . .

**Proposition.** Consider the Yule process with two types of particles. The probabilities that particles are all of the second type at time $t$ are

$$X(t) = e^{-t} \text{smh}(1 - e^{-t}), \quad Y(t) = e^{-t} \text{cmh}(1 - e^{-t}),$$

depending on whether the system at time 0 is initialized with one particle of the first type ($X$) or of the second type ($Y$).
Remarks on urn processes. For urn

\[
\begin{pmatrix}
-\alpha & \beta \\
\gamma & -\delta
\end{pmatrix}, \quad -\alpha + \beta = \gamma - \delta,
\]

associate a partial differential operator:

\[
\Gamma = x^{1-\alpha} y^\beta \frac{\partial}{\partial x} + x^\gamma y^{1-\delta} \frac{\partial}{\partial y}.
\]

Develop a general theory of Pólya Urn Processes (FlDuPu06).

Can characterize all six matrices such that $e^{z\Gamma}$ is expressible by elliptic functions (FlGaPe05). One such model $\in \{\text{sm, cm}\}$. 
$A = \begin{pmatrix} -2 & 3 \\ 4 & -3 \end{pmatrix}$,  
$B = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix}$,  
$C = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$,  
$D = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$,  
$E = \begin{pmatrix} -1 & 3 \\ 5 & -3 \end{pmatrix}$,  
$F = \begin{pmatrix} -1 & 4 \\ 5 & -2 \end{pmatrix}$.

Figure 7. The six elliptic cases in order $A, B, C, D, E, F$: The diagrams formed by the fundamental polygon together with its rotated images. (The elementary kite is darkened.)
5 FIRST PERMUTATION MODEL

A permutation can always be represented as a tree, which is binary, rooted, and increasing.

\[
\text{Tree}(w) = \langle \xi, \text{Tree}(w'), \text{Tree}(w'') \rangle
\]

Level of node ≡ distance to root. Type of node ∼ Peak, Valley, db-rise, db-fall.

| Peaks          | Valleys         | Double rises | Double falls |
|----------------|-----------------|--------------|--------------|
| \(\sigma_{j-1} < \sigma_j > \sigma_{j+1}\) | \(\sigma_{j-1} > \sigma_j < \sigma_{j+1}\) | \(\sigma_{j-1} < \sigma_j < \sigma_{j+1}\) | \(\sigma_{j-1} > \sigma_j > \sigma_{j+1}\) |
Combinatorial Interpretation II

Proposition: Consider the class $\mathcal{X}$ (resp. $\mathcal{Y}$) of permutations such that elements at any odd (resp. even) level are valleys only. Then the exponential generating functions are

$$X(z) = \text{sm}h(z) = -\text{sm}(-z), \quad Y(z) = \text{cm}h(z) = \text{cm}(-z).$$

(Follows from standard combinatorics, reading off $X' = Y^2, Y' = X^2$.)

Other interpretations based on parity:
- Viennot, a first in 1980: Jacobi permutations, alternate reverse.
- Flajolet: alternating permutations, parity of peaks.
- Dumont on Schett, based on cycle structure.
The Second Permutation Model

Inspired by Fl-Françon (1989) = a model for Jacobi $sn, cn$ when $r = 2$.

Definition: An $r$–repeated permutation of size $rn$ is a permutation such that for each $j$, the (existing) elements $jr + 1, jr + 2, \ldots, jr + r − 1$ are all of the same ordinal type ($P, V, DR, DF$).

Proposition: Ordinary generating function for $r$–repeated is:

$$
\sum_{n \geq 0} R_{rn} z^n = \frac{1}{1 - 2 \cdot 1^r z - \frac{1 \cdot 2^2 \cdots r^2 \cdot (r + 1) \cdot z^2}{1 - 2 \cdot (r + 1)^r z - \frac{(r + 1) \cdot (r + 2)^2 \cdots (2r)^2 \cdot (2r + 1) \cdot z^2}{\ddots}}},
$$

Numerators of degree $2r$; denominators of degree $r$. 
6.1 Combinatorial aspects of continued fractions.

A lattice path aka Motzkin path is a sequence $s = (s_0, s_1, \ldots, s_n)$:

$s_0 = s_n = 0$, $s_j \in \mathbb{Z}_{\geq 0}$, $|s_{j+1} - s_j| \in \{-1, 0, +1\}$.

Let $P(a, b, c)$ be the infinite-variable generating function of lattice paths with ascent $\leftrightarrow a_k$, descent $\leftrightarrow b_k$, level $\leftrightarrow c_k$.

**Theorem** (what Foata calls “the shallow Flajolet Theorem”):

$$P(a, b, c) = \frac{1}{1 - c_0 - \frac{a_0 b_1}{1 - c_1 - \frac{a_1 b_2}{1 - c_2 - \frac{a_2 b_3}{\ddots}}}}.$$ 

$m(\varpi) = a_0 a_1 c_2 b_2 a_1 a_3 b_3 b_1 a_0 b_1$ 

$= a_0^2 a_1 a_2 b_1 b_2^2 b_3 c_1.$
6.2 **Lattice paths and permutations.**

- A bijection due to Françon-Viennot (1979);
- What V.I. Arnold (2000) calls *snakes*

Consider piecewise monotonic smooth functions from $\mathbb{R}$ to $\mathbb{R}$, such that all the critical values are different, and take the equivalence classes up to orientation preserving maps of $\mathbb{R}^2$.

Clearly an equivalence class is an alternating permutation, and by André’s theorem the EGFs are $\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\sec(z) = \frac{1}{\cos(z)}$. 
The sweepline algorithm: a snake and its associated Dyck path.

An encoding is obtained by the system of possibilities:

$$\Pi^\text{odd} : \quad \alpha_j = (j + 1), \quad \beta_j = (j + 1), \quad \gamma_j = 0.$$
\[\int_0^\infty \tan(zt)e^{-t} \, dt = \frac{z}{1 - \frac{1}{1 \cdot 2} z^2}, \quad \int_0^\infty \sec(zt)e^{-t} \, dt = \frac{1}{1 - \frac{1}{1^2} z^2}.\]

All perms: encode double rises and double falls by level steps.

\[\sum_{n=1}^\infty n!z^n = \frac{z}{1 - 2z - \frac{1}{2} \cdot 3 z^2}, \quad \sum_{n=0}^\infty n!z^n = \frac{1}{1 - z - \frac{1}{1^2} z^2}.\]

Stieltjes + Euler
6.3 The model of 3–repeated permutations.

Combinatorial Interpretation III

**Proposition:** The exponential generating function of 3–repeated polarized permutations bordered by \((-\infty, -\infty)\) is

\[ \text{smh}(z). \]

**Notes:** By Fl-Françon, 2–repeated + recording rises (cf Eulerian #’s) gives Jacobian \(sn, cn, dn\).

**Corollary:** Combinatorial proofs of Conrad’s fractions.

Also: \(\wp(z - \zeta_0; 0, -1)\) expanded near its real zero, \(\zeta_0 = \frac{1}{3\pi} \Gamma \left( \frac{1}{3} \right)^3\), has CF expansion with cubic denominators and sextic numerators.

\[ \wp(z - \zeta_0; 0, -1) \equiv \text{smh}(z) \cdot \text{cmh}(z) = \text{Inv} Y \cdot \text{2F1} \left[ \frac{1}{3}, \frac{1}{2}, \frac{4}{3}; -4Y^2 \right]. \]
7 PERSPECTIVES & QUESTIONS

Worth looking at nonlinear differential systems associated to algebraic curves and their Abelian integrals?

• **Q.** Three types of balls? (cf Schett-Dumont for a special elliptic case) $A \rightarrow BC$, $B \rightarrow CA$, $C \rightarrow AB$ is elliptic ($sn$, $cn$); hyperelliptic generalizations.

• **Q.** What about numerators like $k^6$ and such in CF? Combinatorics?

• **Q.** Anything to say about orthogonal polynomials (cf Carlitz for $sn$, $cn$)? Cf Galiano Valent et al. — very intriguing!

• **Q.** Any possibility of enumerating directly $r$-repeated perms for $r \geq 4$?

• **Q.** Anything (combinatorially) interesting regarding higher order systems associated to $\mathbb{F}_p$ for $p > 3$?

At least consequences for urn models (FIDuPu06).