THE SPIEGELUNGSSATZ FOR THE CARLITZ MODULE; AN ADDENDUM TO: ON A PROBLEM À LA KUMMER-VANDIVER FOR FUNCTION FIELDS

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1. Introduction

Let \( p \) be an odd prime number. Let \( \mu_p \) be the set of \( p \)th roots of unity in an algebraic closure of \( \mathbb{Q} \) and \( \text{Cl}(\mathbb{Q}(\mu_p)) \) be the ideal class group of \( \mathbb{Q}(\mu_p) \). Then, using class field theory and Kummer theory, one can show (see \[8, \text{p. 188–191}\]) that there exists a Galois-equivariant morphism:

\[
\text{Hom}_\mathbb{Z}(\text{Cl}(\mathbb{Q}(\mu_p)), \mu_p) \to \text{Cl}(\mathbb{Q}(\mu_p))[p],
\]

such that its kernel is a cyclic \( \mathbb{F}_p[\text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})] \)-module which is Galois-isomorphic to a subgroup of \( \mathbb{Z}[[\mu_p]]*/(\mathbb{Z}[[\mu_p]]^*)^p \). Leopoldt’s Spiegelungssatz \[8, \text{Theorem 10.9}\] implies that also the cokernel of the above map is a cyclic \( \mathbb{F}_p[\text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})] \)-module.

In this note we prove a kind of analogue of Leopoldt’s Spiegelungssatz for cyclotomic function fields. Such a result is implicitly contained in \[7\] but not explicitly formulated.

The authors thank David Goss for helpful comments and suggestions.

2. Notation

We use the same notation as in \[7\]. So \( k \) is a finite field of \( q \) elements, \( p \) is its characteristic, \( A = k[T] \), \( K = k(T) \) and \( K_\infty = k((T^{-1})) \). Let

\[
\phi: A \to \text{End}_k \mathbb{G}_{a,A}, \ T \mapsto (x \mapsto Tx + x^q)
\]

be the Carlitz module. If \( R \) is an \( A \)-algebra we denote by \( C(R) \) the \( k \)-vector space \( R \) equipped by the \( A \)-module structure induced by \( \phi \).

We now fix a maximal ideal \( \mathfrak{p} \subset A \) of degree \( d \). Let \( L/K \) be the splitting field of the \( p \)-torsion of \( C \) and let \( R \subset L \) be the integral closure of \( A \) in \( L \). We write \( \Delta = \text{Gal}(L/K) \). This is a cyclic group of order \( q^d - 1 \). In fact, let \( \Lambda \) be the module of \( p \)-torsion points in \( C(L) \). We have \( \Lambda \cong A/\mathfrak{p} \) non-canonically, and \( \Delta \) acts on \( \Lambda \) via the “Teichmüller” character \( \omega: \Delta \to (A/\mathfrak{p})^\times \), which is an isomorphism. We refer the reader to \[4, \text{Ch. 7}\] or to \[3, \text{Ch. 12}\] for the basic properties of the abelian extension \( L/K \).
Set $L_\infty := K_\infty \otimes_K L$. Then the Carlitz exponential (see [4] Chapter 3), exp$_C$, induces an $A[\Delta]$-morphism exp$_C: L_\infty \to C(L_\infty)$. The class module attached to $R$ and $\phi$ is defined by:

$$H(R) = \frac{C(L_\infty)}{\text{exp}_C(L_\infty) + C(R)}.$$ 

It is shown in [6] that the $A[\Delta]$-module $H(R)$ is finite.

### 3. Spiegelungssatz

As a consequence of the results obtained in [7] in order to prove an analogue of the Herbrand-Ribet theorem (using different methods, this result has recently been refined in [3]), we have the following analogue of Leopoldt’s Spiegelungssatz.

**Theorem 1.** There is a natural $(A/p)[\Delta]$-morphism

$$\text{Hom}_A(H(R), \Lambda) \to A/p \otimes_{F_p}(\text{Pic} R)[p]$$

whose kernel and cokernel are cyclic $(A/p)[\Delta]$-modules.

A cyclic module is a (possibly trivial) module generated by one element.

Note that the map in the theorem relates the dual of the $p$-part of the class module to the $p$-part of the class group!

**Proof.** We show how to obtain this from the results in [7].

Let $q$ be the unique prime of $R$ above $p$. Combining Theorems 2 and 6 from [7] we find an exact sequence of $(A/p)[\Delta]$-modules

$$(1) \quad 0 \to \text{Hom}_A(H(R), \Lambda) \xrightarrow{f} A/p \otimes_k \Omega_R^{-1} \to \Omega_R/q^c\Omega_R,$$

where $\Omega_R$ is the module of Kähler differentials on $R$ over $k$ and $c$ is the $(k$-linear) $q$-Cartier operator. Kummer theory gives a short exact sequence of $(A/p)[\Delta]$-modules

$$(2) \quad 0 \to A/p \otimes_{\mathbb{Z}} R^\times \xrightarrow{g} A/p \otimes_k \Omega_R^{-1} \to A/p \otimes_{F_p}(\text{Pic} R)[p] \to 0,$$

see [7], exact sequence (2) and section 6.

Combining the exact sequences (1) and (2) we find an equivariant $A/p$-linear map

$$\alpha: \text{Hom}_A(H(R), \Lambda) \to A/p \otimes_{F_p}(\text{Pic} R)[p].$$

Observe that we have an isomorphism of $(A/p)[\Delta]$-modules:

$$\text{Ker} \alpha \cong \text{Im} f \cap \text{Im} g.$$

Thus the kernel of $\alpha$ is a isomorphic to a submodule of the cyclic $(A/p)[\Delta]$-module $A/p \otimes_{\mathbb{Z}} R^\times$, and therefore cyclic.
It remains to show that the cokernel is cyclic. Observe that we have a \((A/p)[\Delta]\)-isomorphism:

\[
\text{Coker } \alpha \cong \frac{A/p \otimes_k \Omega_{R_i}^{-1}}{\text{Im } f + \text{Im } g}.
\]

In particular, the cokernel of \(\alpha\) is a quotient of the cokernel of \(f\). Note that the cokernel of \(f\) is a submodule of the kernel of the following map of \((A/p)[\Delta]\)-modules

\[
(3) \quad 1 - c^d : \Omega_R/q^d\Omega_R \to \Omega_R/q\Omega_R.
\]

So it suffices to show that the kernel of the above map \(1 - c^d\) is a cyclic \((A/p)[\Delta]\)-module.

The \((A/p)[\Delta]\)-module \(\Omega_R/q^d\Omega_R\) has a natural decreasing filtration

\[
\Omega_R/q^d\Omega_R \supset \cdots \supset q^i\Omega_R/q^{i+1}\Omega_R \supset \cdots \supset 0.
\]

The intermediate quotient \(q^i\Omega_R/q^{i+1}\Omega_R\) is generated by \(\chi d\lambda\). It isomorphic with \(A/p\) with \(\Delta\) acting via \(\omega^{i+1}\). In particular, we have

\[
\dim_{A/p} \frac{\Omega_R}{q^d\Omega_R}(\chi) = \begin{cases} 
1 & \text{if } \chi \neq \omega \\
2 & \text{if } \chi = \omega
\end{cases}.
\]

Now consider the element \(d\lambda\) of \(\Omega_R/q^d\Omega_R(\omega)\). We have \(c(d\lambda) = 0\) and \(d\lambda\) is nonzero in \(\Omega_R/q\Omega_R\). So we find that \((c^d - 1)(d\lambda)\) is nonzero in \(\Omega_R/q\Omega_R\) and conclude that the \(\omega\)-part of the kernel of the map (3) is one-dimensional. In particular, we find that the kernel of (3) is a cyclic \((A/p)[\Delta]\)-module, and hence that the cokernel of \(\alpha\) is cyclic. \(\square\)

Let \(\chi\) be a character \(\Delta \to (A/p)^\times\) with \(\chi(k^\times) \neq \{1\}\). Assume that \((A/p \otimes_k H(R_P))(\omega\chi^{-1}) = \{0\}\). The above theorem implies that \((A/p \otimes_k \text{Cl}^p(L))(\chi)\) is a cyclic \(A/p\)-module. This is a key ingredient in the construction of counterexamples to the analogue of the Kummer-Vandiver conjecture for the class module. In [3] it is shown that these also yield counterexamples to Anderson’s conjecture [1].

**References**

[1] G. Anderson, Log-Algebraicity of Twisted A-Harmonic Series and Special Values of L-Series in Characteristic p, *J. Number Theory* 60 (1996), 165–209.

[2] B. Anglès, L. Taelman, On a problem à la Kummer-Vandiver for function fields, *J. Number Theory* 133 (2013), 830–841.

[3] B. Anglès, L. Taelman (with an appendix by V. Bosser), Arithmetic of characteristic \(p\) special L-values, arXiv:1205.2794 (2012).

[4] D. Goss, *Basic Structures of Function Field Arithmetic*, Springer, Berlin, 1996.

[5] M. Rosen, *Number Theory in Function Fields*, Springer, 2002.

[6] L. Taelman, A Dirichlet unit theorem for Drinfeld modules, *Math. Ann.* 348 (2010), 899–907.
[7] L. Taelman, A Herbrand-Ribet theorem for function fields, *Invent. Math.* **188** (2012), 253–275.

[8] L. Washington, *Introduction to Cyclotomic Fields*, Second Edition, Springer, 1997.

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