A CERTAIN RECIPROCAL POWER SUM IS NEVER AN INTEGER

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Abstract. By \((\mathbb{Z}^+)^\infty\) we denote the set of all the infinite sequences \(S = \{s_i\}_{i=1}^\infty\) of positive integers (note that all the \(s_i\) are not necessarily distinct and not necessarily monotonic). Let \(f(x)\) be a polynomial of nonnegative integer coefficients. For any integer \(n \geq 1\), one lets \(S_n := \{s_1, ..., s_n\}\) and \(H_f(S_n) := \sum_{i=1}^n \frac{1}{f(i)}\). When \(f(x)\) is linear, it is proved in [Y.L. Feng, S.F. Hong, X. Jiang and Q.Y. Yin, A generalization of a theorem of Nagell, Acta Math. Hungari, to appear] that for any infinite sequence \(S\) of positive integers, \(H_f(S_n)\) is never an integer if \(n \geq 2\). Now let \(\text{deg} f(x) \geq 2\). Clearly, \(0 < H_f(S_n) < \zeta(2) < 2\). But it is not clear whether the reciprocal sum \(H_f(S_n)\) can take 1 as its value. In this paper, with the help of a result of Erdős, we use the analytic and \(p\)-adic method to show that for any infinite sequence \(S\) of positive integers and any positive integer \(n \geq 2\), \(H_f(S_n)\) is never equal to 1. Furthermore, we use a result of Kakeya to show that if \(\frac{1}{\pi} \leq \sum_{i=1}^n \frac{1}{f(i)}\) holds for all positive integers \(k\), then the union set \(\bigcup_{S \in (\mathbb{Z}^+)^\infty} \{H_f(S_n)\} \in \mathbb{Z}^+\) is dense in the interval \((0, \alpha_f)\) with \(\alpha_f := \sum_{i=1}^\infty \frac{1}{f(i)}\). It is well known that \(\alpha_f = \frac{1}{2}(\pi^2/6 - 1) \approx 1.076674\) when \(f(x) = x^2 + 1\). Our dense result infers that when \(f(x) = x^2 + 1\), for any sufficiently small \(\varepsilon > 0\), there are positive integers \(n_1\) and \(n_2\) and infinite sequences \(S^{(1)}\) and \(S^{(2)}\) of positive integers such that \(1 - \varepsilon < H_f(S^{(1)}_{n_1}) < 1\) and \(1 < H_f(S^{(2)}_{n_2}) < 1 + \varepsilon\).

1. Introduction

Let \(\mathbb{Z}\), \(\mathbb{Z}^+\) and \(\mathbb{Q}\) be the set of integers, the set of positive integers and the set of rational numbers, respectively. Let \(n \in \mathbb{Z}^+\). In 1915, Theisinger [11] showed that the \(n\)-th harmonic sum \(1 + \frac{1}{2} + ... + \frac{1}{n}\) is never an integer if \(n > 1\). In 1923, Nagell [10] extended Theisinger’s result by showing that if \(a\) and \(b\) are positive integers and \(n \geq 2\), then the reciprocal sum \(\sum_{i=0}^{n-1} \frac{1}{i+1}\) is never an integer. Erdős and Niven [3] generalized Nagell’s result by considering the integrality of the elementary symmetric functions of \(\frac{1}{a}, \frac{1}{a+b}, ..., \frac{1}{a+(n-1)b}\). In the recent years, Erdős and Niven’s result [4] was extended to the general polynomial sequence, see [11, 15, 16] and [17]. Another interesting and related topic is presented in [14].

By \((\mathbb{Z}^+)^\infty\) we denote the set of all the infinite sequence \(\{s_i\}_{i=1}^\infty\) of positive integers (note that all the \(s_i\) are not necessarily distinct and not necessarily monotonic). For any given \(S = \{s_i\}_{i=1}^\infty \in (\mathbb{Z}^+)^\infty\), we let \(S_n := \{s_1, ..., s_n\}\). Associated to the infinite sequence \(S\) of positive integers and a polynomial \(f(x)\) of nonnegative integer coefficients, one can form an infinite sequence \(\{H_f(S_n)\}_{n=1}^\infty\) of positive rational fractions with \(H_f(S_n)\) being

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defined as follows:

\[ H_f(S_n) := \sum_{k=1}^{n} \frac{1}{f(k)^n}. \]

Very recently, Feng, Hong, Jiang and Yin [4] showed that when \( f(x) \) is linear, the reciprocal power sum \( H_f(S_n) \) is never an integer if \( n \geq 2 \). It is natural to ask the following interesting question: Is the similar result still true when \( f(x) \) is of degree at least two and nonnegative integer coefficients?

In this paper, our main goal is to study this question. In fact, by using the analytic and \( p \)-adic method and with the help of Erdős theorem [2] on the distribution in the arithmetic progression \( \{4n + 1\}_{n=1}^{\infty} \), we will show the following result that is the first main result of this paper.

**Theorem 1.1.** Let \( f(x) \) be a polynomial of nonnegative integer coefficients and of degree at least two. Then for any infinite sequence \( S \) of positive integers and for any positive integer \( n \geq 2 \), the reciprocal power sum \( H_f(S_n) \) is never an integer.

Clearly, Theorem 1.1 gives an affirmative answer to the above mentioned question.

Associated to any given infinite sequence \( S \) of positive integers, we let

\[ H_f(S) := \{H_f(S_n) | n \in \mathbb{Z}^+\} \]

and

\[ \alpha_f(S) := \sum_{k=1}^{\infty} \frac{1}{f(k)^{s_k}}. \]

Put

\[ \alpha_f := \sum_{k=1}^{\infty} \frac{1}{f(k)}. \]  

(1)

Note that \( \alpha_f \) may be +\( \infty \). Then \( \alpha_f(S) \leq \alpha_f \) and \( H_f(S) \subseteq (\inf H_f(S), \alpha_f(S)) \). It is clear that \( H_f(S) \) is not dense (nowhere dense) in the interval \( (\inf H_f(S), \alpha_f(S)) \). However, if we put all the sets \( H_f(S) \) together, then one arrives at the following interesting dense result that is the second main result of this paper.

**Theorem 1.2.** Let \( f(x) \) be a polynomial of nonnegative integer coefficients and let \( U_f \) be the union set defined by

\[ U_f := \bigcup_{S \in (\mathbb{Z}^+)^\infty} H_f(S). \]

(i). If \( \deg f(x) = 1 \), then \( U_f \) is dense in the interval \( (\delta, +\infty) \) with \( \delta = 1 \) if \( f(x) = x \), and \( \delta = 0 \) otherwise.

(ii). If \( \deg f(x) \geq 2 \) and

\[ \frac{1}{f(k)} \leq \sum_{i=1}^{\infty} \frac{1}{f(k+i)} \]

holds for all positive integers \( k \), then \( U_f \) is dense in the interval \( (0, \alpha_f) \) with \( \alpha_f \) being given in (1).

Let \( H(S_n) := H_f(S_n) \) and \( H(S) := H_f(S) \) if \( f(x) = x^2 + 1 \). It is well known that (see, for instance, [2])

\[ \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} = \frac{1}{2} \left( \pi^2 + 1 - 1 \right) := \alpha. \]

(3)
A certain reciprocal power sum is never an integer 3

Furthermore, \( \alpha \approx 1.076674 \). Evidently, for any positive integer \( n \), we have

\[
0 < H(S_n) \leq \sum_{k=1}^{n} \frac{1}{k^2 + 1} < \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} < 2.
\]

Theorem 1.1 tells us that \( H(S_n) \) is never equal to 1 for any infinite sequence \( S \) of positive integers and any positive integer \( n \). This extends the corresponding result in [9] and [14] which states that for the infinite sequence \( S \) with \( s_i = s_1 \) for all integers \( i \geq 1 \), \( H(S_n) \) is not equal to 1. On the other hand, one can easily check that (2) is true when \( f(x) = x^2 + 1 \). So Theorem 1.2 infers that for any sufficiently small \( \varepsilon > 0 \), there are positive integers \( n_1 \) and \( n_2 \) and infinite sequences \( S^{(1)} \) and \( S^{(2)} \) of positive integers such that \( 1 - \varepsilon < H(S^{(1)}_{n_1}) < 1 \) and \( 1 < H(S^{(2)}_{n_2}) < 1 + \varepsilon \).

This paper is organized as follows. First, in Section 2, we recall the early results due to Erdős [2] and Kakeya [6], respectively, and then show some preliminary lemmas which are needed in the proofs of Theorems 1.1 and 1.2. Then in Sections 3 and 4, we supply the proofs of Theorems 1.1 and 1.2, respectively. The final section is devoted to some remarks. Actually, a conjecture on the case of integer coefficients polynomial is proposed there.

2. Auxiliary Lemmas

In this section, we present several auxiliary lemmas that are needed in the proofs of Theorems 1.1 and 1.2. We begin with a well-known result due to Erdős.

Lemma 2.1. [2] For any real number \( \xi \geq 7 \), there exists a prime \( p \in (\xi, 2\xi] \) such that \( p \equiv 1 \pmod{4} \).

For any given prime \( p \) with \( p \equiv 1 \pmod{4} \), the congruence \( x^2 + 1 \equiv 0 \pmod{p} \) is solvable, and in the remaining part of this paper, we use \( r_p \) to stand for the smallest positive root of \( x^2 + 1 \equiv 0 \pmod{p} \). In the conclusion of this section, we use Lemma 2.1 to show the following result that is vital in the proof of Theorem 1.1.

Lemma 2.2. For any integer \( n \geq 2 \), there is a prime \( p \equiv 1 \pmod{4} \) such that \( r_p \leq n < p \).

Proof. If \( n = 2, 3 \) or 4, then letting \( p = 5 \) gives us that \( r_p = 2 \). So Lemma 2.2 is true in this case.

If \( n = 5 \) or 6, then picking \( p = 13 \) gives us that \( r_p = 5 \). Lemma 2.2 holds in this case.

Now let \( n \geq 7 \). At this moment, Lemma 2.1 guarantees the existence of a prime \( p \) such that \( p \equiv 1 \pmod{4} \) and \( n < p < 2n \). Since \( p - r_p \) is another positive root of \( x^2 + 1 \equiv 0 \pmod{p} \) and \( r_p < p - r_p \), it follows that

\[
r_p \leq \frac{p - 1}{2} < \frac{p}{2} < n < p.
\]
as required. Hence Lemma 2.2 is proved.

Now let us state a result obtained by Kakeya in 1914.

Lemma 2.3. [6] Let \( \sum_{k=1}^{\infty} a_k \) be an absolutely convergent infinite series of real numbers and let the set, denoted by \( SPS \), of all the partial sums of the series \( \sum_{k=1}^{\infty} a_k \) be defined by

\[
SPS := \left\{ \sum_{i=1}^{m} a_k \mid m \in \mathbb{Z}^+ \cup \{\infty\}, 1 \leq k_1 < \ldots < k_m \right\}.
\]
Let \( u := \inf \SP S \) and \( v := \sup \SP S \) (note that \( u \) may be \(-\infty\) and \( v \) may be \(+\infty\)). Then the set \( U \) consists of all the values in the interval \((u, v)\) if and only if
\[
|a_k| \leq \sum_{i=1}^{\infty} |a_{k+i}|
\]
holds for all \( k \in \mathbb{Z}^+ \).

Using Lemma 2.3, we can prove the following two useful results that play key roles in the proof of Theorem 1.2.

**Lemma 2.4.** Let \( \sum_{k=1}^{\infty} a_k \) be a convergent infinite series of positive real numbers and
\[
V := \left\{ \sum_{i=1}^{m} a_{k_i} \mid m \in \mathbb{Z}^+, 1 \leq k_1 < \cdots < k_m \right\}.
\]
If
\[
a_k \leq \sum_{i=1}^{\infty} a_{k+i}
\]
holds for all \( k \in \mathbb{Z}^+ \), then the set \( V \) is dense in the interval \((0, v)\) with \( v := \sum_{k=1}^{\infty} a_k \).

**Proof.** From the condition (4) and Lemma 2.3 we know that the set \( \SP S = \left\{ \sum_{i=1}^{m} a_{k_i} \mid m \in \mathbb{Z}^+ \cup \{\infty\}, 1 \leq k_1 < \cdots < k_m \right\} \) consists of all the values in the interval \((0, v)\) since here \( \inf \SP S = 0 \). Let \( r \) be any given real number in \((0, v)\) and \( \varepsilon \) be any sufficiently small positive number (one may let \( \varepsilon < \min(r, v-r) \)). Then \( r \in \SP S \) which implies that there is an integer \( m \in \mathbb{Z}^+ \cup \{\infty\} \) and there are \( m \) integers \( k_1, \ldots, k_m \) with \( 1 \leq k_1 < \cdots < k_m \) such that \( r = \sum_{i=1}^{m} a_{k_i} \).

If \( m \in \mathbb{Z}^+ \), then \( r \) is in \( V \). Lemma 2.4 is true in this case.

If \( m = \infty \), then \( r = \sum_{i=1}^{\infty} a_{k_i} \). That is, \( \lim_{n \to \infty} \sum_{i=1}^{n} a_{k_i} = r \). Thus there is a positive integer \( m' \) such that \( |r - \sum_{i=1}^{m'} a_{k_i}| < \varepsilon \). Noticing that all \( a_{k_i} \) are positive, we deduce that \( r - \varepsilon < \sum_{i=1}^{m'} a_{k_i} < r \) as desired.

This completes the proof of Lemma 2.4. \( \square \)

**Lemma 2.5.** Let \( \sum_{k=1}^{\infty} a_k \) be a divergent infinite series of positive real numbers with \( a_k \) decreasing as \( k \) increasing and \( a_k \to 0 \) as \( k \to \infty \). Define
\[
V := \left\{ \sum_{i=1}^{m} a_{k_i} \mid m \in \mathbb{Z}^+, 1 \leq k_1 < \cdots < k_m \right\}.
\]
Then the set \( V \) is dense in the interval \((0, +\infty)\).

**Proof.** Let \( r \) be any given real number in \((0, +\infty)\) and \( \varepsilon \) be any sufficiently small positive number (one may let \( \varepsilon < r \)). Let \( a_0 := 0 \) and \( m_0 = 0 \). Since the series \( \sum_{k=0}^{\infty} a_k \) is divergent, there exists a unique integer \( m_1 \geq 0 \) such that
\[
\sum_{k=m_0}^{m_1} a_k < r
\]
and
\[
\sum_{k=m_0}^{m_1} a_k + a_{m_1+1} \geq r.
\]
On the one hand, since \( a_k \) decreases as \( k \) increases and \( a_k \to 0 \) as \( k \to \infty \), there is an integer \( m_2 \) with \( m_2 > m_1 + 1 \) and

\[
a_{m_2} < r - \sum_{k=m_0}^{m_1} a_k \leq a_{m_1+1}.
\]

Moreover, there exists an integer \( m_3 \) with \( m_3 \geq m_2 \) and

\[
\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k < r
\]

and

\[
\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + a_{m_3+1} \geq r
\]

since \( \sum_{k=m_2}^{\infty} a_k \) also diverges.

Continuing in this way, we can form an increasing sequence \( \{m_k\}_{k=0}^{\infty} \) such that

\[
\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t+1}}^{m_{2t+2}} a_k < r
\]

but

\[
\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t+1}}^{m_{2t+2}} a_k + a_{m_{2t+2}+1} \geq r
\]

for any nonnegative integer \( t \). Obviously, one has

\[
\sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t}}^{m_{2t+1}} a_k \in V.
\]

On the other hand, since \( \lim_{k \to +\infty} a_k = 0 \), it follows that there exists a nonnegative integer \( t_0 \) such that \( a_{m_{2t_0+1}+1} < \varepsilon \). That is, we have

\[
r - \varepsilon < r - a_{m_{2t_0+1}+1} \leq \sum_{k=m_0}^{m_1} a_k + \sum_{k=m_2}^{m_3} a_k + \cdots + \sum_{k=m_{2t_0}}^{m_{2t_0+1}} a_k < r.
\]

Hence \( V \) is dense in the interval \((0, +\infty)\).

This concludes the proof of Lemma 2.5.

3. Proof of Theorem 1.1

As usual, for any prime \( p \) and for any integer \( x \), we let \( v_p(x) \) stand for the \( p \)-adic valuation of \( x \), i.e., \( v_p(x) \) is the biggest nonnegative integer \( r \) with \( p^r \) dividing \( x \). If \( x = \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \neq 0 \), then define \( v_p(x) := v_p(a) - v_p(b) \).

We can now prove Theorem 1.1 as follows.

Proof of Theorem 1.1 We just need to prove that \( H_f(S_n) \) is between two adjacent integers or \( v_p(H_f(S_n)) < 0 \) for some prime \( p \). Let \( f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 \) with \( m \geq 2 \) and \( a_m \geq 1 \).

If there is some \( a_i \neq 0 \) with \( 1 \leq i \leq m - 1 \), then \( f(k) \geq a_m k^m + a_i k^i \geq k^2 + k \) for all \( k \geq 1 \). Therefore,

\[
H_f(S_n) = \sum_{k=1}^{n} \frac{1}{f(k)^{n_k}} \leq \sum_{k=1}^{n} \frac{1}{k^2 + k} = \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1} < 1.
\]
If $a_i = 0$ for all $0 \leq i \leq m - 1$, then $f(x) = a_m x^m$. Furthermore,

$$1 < H_f(S_n) = \sum_{k=1}^{n} \frac{1}{(k^m)_{s_k}} \leq \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6} < 2$$

when $a_m = 1$, and

$$0 < H_f(S_n) = \sum_{k=1}^{n} \frac{1}{(a_m k^m)_{s_k}} \leq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2} < 1.$$ 

when $a_m \geq 2$.

If $a_i = 0$ for all $1 \leq i \leq m - 1$ and $a_0 \neq 0$, then $f(x) = a_m x^m + a_0$. Moreover, if $a_m \geq 2$ or $a_0 \geq 2$, then $f(k) = k^m + (a_m - 1)k^m + a_0 \geq k^2 + 2$ for all $k \geq 1$. So

$$0 < H_f(S_n) \leq \sum_{k=1}^{n} \frac{1}{k^2 + 2} \leq \sum_{k=1}^{n} \frac{1}{k^2 + 1} - \frac{1}{2} - \frac{1}{5} + \frac{1}{3} + \frac{1}{6} < \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} - \frac{1}{5} < 1.$$

If $m \geq 3$, then $f(k) \geq k^3 + 1$ for all $k \geq 1$. So

$$0 < H_f(S_n) \leq \sum_{k=1}^{n} \frac{1}{k^3 + 1} \leq \sum_{k=1}^{n} \frac{1}{3} - 1 - \frac{1}{8} + \frac{1}{2} + \frac{1}{9} < \zeta(3) - \frac{37}{72} < 1.$$

In what follows, we let $f(x) = x^2 + 1$.

By Lemma 2.2 there is a prime $p$ such that $p \equiv 1 \pmod{4}$ and $r_p \leq n < p$ where $r_p$ is the smallest positive root of $x^2 + 1 \equiv 0 \pmod{p}$. Since $p \mid (v_2^2 + 1)$, one has $r_p^2 \geq p - 1$. Noticing that $p \geq 5$, it follows that

$$2 \leq \sqrt{p-1} \leq r_p \leq \frac{p-1}{2} < p - r_p.$$

Therefore

$$0 < r_p^2 + 1 < (p - r_p)^2 + 1 \leq (p - 2)^2 + 1 < p^2.$$

This infers that $v_p(r_p^2 + 1) = 1$ and $v_p((p - r_p)^2 + 1) = 1$. So we have

$$v_p\left(\frac{1}{(r_p^2 + 1)^{s_{r_p}}}\right) = -s_{r_p} < 0,$$

$$v_p\left(\frac{1}{((p - r_p)^2 + 1)^{s_{p-r_p}}}\right) = -s_{p-r_p} < 0$$

and

$$v_p\left(\frac{1}{(k^2 + 1)^{s_k}}\right) = 0$$

for any integer $k$ with $1 \leq k \leq p$ and $k \not\in \{r_p, p-r_p\}$.

Now we divide the proof into the following two cases.

**CASE 1.** $r_p \leq n < p - r_p$. Since

$$v_p\left(\sum_{k=1}^{n} \frac{1}{(k^2 + 1)^{s_k}}\right) \geq 0,$$

it follows from the isosceles triangle principle (see, for example, [3]) that

$$v_p(H(S_n)) = v_p\left(\frac{1}{(r_p^2 + 1)^{s_{r_p}}} + \sum_{k=1}^{n} \frac{1}{(k^2 + 1)^{s_k}}\right) = v_p\left(\frac{1}{(r_p^2 + 1)^{s_{r_p}}}\right) = -s_{r_p} < 0.$$

Namely, $H(S_n) \not\in \mathbb{Z}$. So Theorem 1.1 is proved in this case.
CASE 2. \( p - r_p \leq n < p \). Let \( H(S_n) = A + B \), where

\[
A := \frac{1}{(r_p^2 + 1)^s r_p} + \frac{1}{((p - r_p)^2 + 1)^s p - r_p}
\]

and

\[
B := \sum_{k \neq r_p, k \neq p - r_p} \frac{1}{(k^2 + 1)^s}.
\]

Evidently, one has \( v_p(B) \geq 0 \). We claim that \( v_p(A) < 0 \). Then by the claim and the isosceles triangle principle again, we obtain that

\[ v_p(H(S_n)) = v_p(A + B) = v_p(A) < 0, \]

which implies that \( H(S_n) \notin \mathbb{Z} \) as desired. It remains to show the truth of the claim.

If \( s_{r_p} \neq s_{p - r_p} \), then it is obvious that \( v_p(A) = \min(-s_{r_p}, -s_{p - r_p}) < 0 \). So the claim is true if \( s_{r_p} \neq s_{p - r_p} \).

Now let \( s_{r_p} = s_{p - r_p} := s \). Then

\[
A = \frac{((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s}{(r_p^2 + 1)^s ((p - r_p)^2 + 1)^s}.
\]

We introduce an auxiliary function \( g(x) \) as follows:

\[
g(x) := \left( \left( \frac{p}{2} + x \right)^2 + 1 \right)^s + \left( \left( \frac{p}{2} - x \right)^2 + 1 \right)^s.
\]

Then the derivative of \( f(x) \) is

\[
g'(x) = s \left( \left( \frac{p}{2} + x \right)^2 + 1 \right)^{s-1} (p + 2x) - s \left( \left( \frac{p}{2} - x \right)^2 + 1 \right)^{s-1} (p - 2x).
\]

So \( g'(x) > 0 \) if \( x \in (0, \frac{p}{2}] \). This implies that \( g(x) \) is increasing if \( x \in (0, \frac{p}{2}] \).

Since \( 2 \leq r_p < \frac{p}{2} \), one derives that \( 0 < \frac{p}{2} - r_p \leq \frac{p}{2} - 2 < \frac{p}{2} \). Hence

\[
((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s = \frac{g \left( \frac{p}{2} - r_p \right)}{\left( \frac{p}{2} - 2 \right)^s} = (2^2 + 1)^s + ((p - 2)^2 + 1)^s < 5^s + (p - 1)^{2s} < p^{2s},
\]

where the last inequality follows from the fact that \( p \geq 5 \) implies that

\[
p^{2s} - (p - 1)^{2s} = (p^s + (p - 1)^s)(p^s - (p - 1)^s) > 5^s.
\]

Thus

\[
v_p\left( ((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s \right) < 2s.
\]

Therefore

\[
v_p(A) = v_p\left( ((p - r_p)^2 + 1)^s + (r_p^2 + 1)^s \right) - v_p\left( (r_p^2 + 1)^s ((p - r_p)^2 + 1)^s \right)
\]  
\[
< 2s - 2s = 0.
\]

The claim holds if \( s_{r_p} = s_{p - r_p} \). The claim is proved.

This finishes the proof of Theorem 1.1.
4. Proof of Theorem 1.2

In the section, we present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let

\[ V_f := \left\{ \sum_{i=1}^{m} \frac{1}{f(k_i)} \mid m \in \mathbb{Z}^+, 1 \leq k_1 < \ldots < k_m \right\} \]

and

\[ \bar{V}_f := \left\{ \sum_{i=1}^{m} \frac{1}{f(k_i)} \mid m \in \mathbb{Z}^+, 2 \leq k_1 < \ldots < k_m \right\}. \]

Pick any given real number \( r \) in \((\inf U_f, \sup U_f)\) and let \( \varepsilon \) be any sufficiently small positive number (one may let \( \varepsilon < \min(r - \inf U_f, \sup U_f - r) \)).

(i). Since \( f(x) \) is a polynomial of nonnegative integer and degree one, it follows that \( \sum_{k=1}^{\infty} \frac{1}{f(k)} \) (resp. \( \sum_{k=2}^{\infty} \frac{1}{f(k)} \)) is a divergent infinite series of positive real numbers with \( \left\{ \frac{1}{f(k)} \right\}_{k=1}^{\infty} \) (resp. \( \left\{ \frac{1}{f(k)} \right\}_{k=2}^{\infty} \)) directly decreasing to 0 as \( k \) increases. By Lemma 2.5, we know that \( V_f \) (resp. \( \bar{V}_f \)) is dense in the interval \((0, +\infty)\). Clearly, we have \( \sup U_f = \sup V_f = +\infty \).

If \( f(1) = 1 \), then \( f(x) = x \) which implies that \( f(2) > 1 \), \( \inf U_f = 1 \) and \( r \in (\inf U_f, \sup U_f) = (1, +\infty) \). Since \( \bar{V}_f \) is dense in the interval \((0, +\infty)\), there is an element

\[ \sum_{i=1}^{m} \frac{1}{f(k_i)} \in \left( r - \varepsilon, r - 1 - \frac{\varepsilon}{2} \right) \]

with \( 2 \leq k_1 < \ldots < k_m \). Now let \( s_k = 1 \) for \( k \in \{k_1, \ldots, k_m\} \) and \( s_k > \frac{\log 2km}{\log f(2)} \) for \( k \in \{2, 3, \ldots, k_m\} \setminus \{k_1, \ldots, k_m\} \). Then

\[ 0 \leq \sum_{k=2}^{k_m} \frac{1}{f(k)} < \frac{k_m}{f(2)^{\frac{\log 2km}{\log f(2)}}} = \varepsilon. \]

It follows from (5) and (6) that

\[ \sum_{k=1}^{k_m} \frac{1}{f(k)} + \sum_{k=2}^{k_m} \frac{1}{f(k)} + \sum_{i=1}^{m} \frac{1}{f(k_i) s_{k_i}} \in (r - \varepsilon, r). \]

That is, \( U_f \) is dense in the interval \((\inf U_f, \sup U_f) = (1, +\infty)\) in this case.

If \( f(1) > 1 \), then \( \inf U_f = 0 \) and \( r \in (\inf U_f, \sup U_f) = (0, +\infty) \). Since \( V_f \) is dense in the interval \((0, +\infty)\), there is an element

\[ \sum_{i=1}^{m} \frac{1}{f(k_i)} \in \left( r - \varepsilon, r - \frac{\varepsilon}{2} \right) \]

with \( 1 \leq k_1 < \ldots < k_m \). Now, let \( s_k = 1 \) for \( k \in \{k_1, \ldots, k_m\} \) and \( s_k > \frac{\log 2km}{\log f(1)} \) for \( k \in \{1, 2, \ldots, k_m\} \setminus \{k_1, \ldots, k_m\} \). One has

\[ 0 \leq \sum_{k=1}^{k_m} \frac{1}{f(k)^s_k} < \frac{k_m}{f(1)^{\frac{\log 2km}{\log f(1)}}} = \varepsilon. \]
and so by (7) and (8),
\[ \sum_{k=1}^{m} \frac{1}{f(k)^{s_k}} = \sum_{k \in \{1, \ldots, m\}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^{m} \frac{1}{f(k_i)^{s_{k_i}}} \in (r - \varepsilon, r). \]
Namely, \( U_f \) is dense in the interval \( (\inf U_f, \sup U_f) = (0, +\infty) \) in this case.

(ii). First of all, since \( f(x) \) is a polynomial of nonnegative integer and \( \deg f(x) \geq 2 \), we know that \( \sum_{k=1}^{\infty} \frac{1}{f(k)^{s_k}} \) is a convergent infinite series of positive real numbers. With the hypothesis \( \frac{1}{f(k)} \leq \sum_{k=1}^{\infty} \frac{1}{f(k)^{s_k}} \) for any positive integer \( k \), Lemma 2.4 yields that \( V_f \) is dense in the interval \( (0, \sup V_f) \).

We claim that \( f(1) > 1 \). Otherwise, \( f(1) = 1 \). Then \( f(x) = x^m \) with \( m \geq 2 \). However,
\[ \frac{1}{f(1)} = 1 > \frac{\pi^2}{6} - 1 = \sum_{i=1}^{\infty} \frac{1}{(1+i)^2} \geq \sum_{i=1}^{\infty} \frac{1}{(1+i)}, \]
which contradicts with our hypothesis. So we must have \( f(1) > 1 \). The claim is proved.

In the following, we let \( f(1) > 1 \). Then \( \inf U_f = 0, \sup U_f = \sup V_f = \alpha_f \) and \( r \in (\inf U_f, \sup U_f) = (0, \alpha_f) \). Since \( V_f \) is dense in the interval \( (0, \sup V_f) = (0, \alpha_f) \), there is an element
\[ \sum_{i=1}^{m} \frac{1}{f(k_i)} \in (r - \varepsilon, r - \frac{\varepsilon}{2}) \]
with \( 1 \leq k_1 < \cdots < k_m \). Then letting \( s_k = 1 \) for \( k \in \{k_1, \ldots, k_m\} \) and \( s_k > \frac{\log 2k_m}{\log f(1)} \) for \( k \in \{1, 2, \ldots, k_m\} \setminus \{k_1, \ldots, k_m\} \) gives us that
\[ 0 \leq \sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} < \frac{k_m}{\log f(1)} = \frac{\varepsilon}{2}. \]
It infers that
\[ \sum_{k=1}^{k_m} \frac{1}{f(k)^{s_k}} = \sum_{k \in \{1, \ldots, k_m\}}^{k_m} \frac{1}{f(k)^{s_k}} + \sum_{i=1}^{m} \frac{1}{f(k_i)^{s_{k_i}}} \in (r - \varepsilon, r). \]
In other words, \( U_f \) is dense in the interval \( (0, \alpha_f) \). So part (ii) is proved.

The proof of Theorem 1.2 is complete. \( \square \)

5. Concluding Remarks

1. Let \( \mathcal{T} := ((0, \alpha) \cap \mathbb{Q}) \setminus \bigcup_{S \in (\mathbb{Z} \setminus \mathbb{Q})^*} H(S) \). Then Theorem 1.1 tells us that \( 1 \in \mathcal{T} \).
But it is well known that if \( p \) is a prime, then the congruence \( x^2 + 1 \equiv 0 \pmod{p} \) is solvable if and only if either \( p = 2 \), or \( p \equiv 1 \pmod{4} \). Thus for any infinite sequence \( S \) of positive integers and for any positive integer \( n \), if one writes \( H(S_n) = \frac{H_1(S_n)}{H_2(S_n)} \), where \( H_1(S_n), H_2(S_n) \in \mathbb{Z}^+ \) and gcd\( (H_1(S_n), H_2(S_n)) = 1 \), then \( H_2(S_n) \) is not divisible by any prime \( p \) with \( p \equiv 3 \pmod{4} \). It follows that \( \mathcal{L} := \left\{ \left( \frac{a}{b}, \frac{c}{d} \right) \in (0, \alpha) | a, b \in \mathbb{Z}, (a, b) = 1, b \right\} \subset \mathcal{T} \). An interesting question naturally arises: Are there other elements in the set \( \mathcal{T} \) except for the elements in \( \{1\} \cup \mathcal{L} \)? Further, one would like to determine the set \( \mathcal{T} \). This problem is kept open so far.

2. We let \( f(x) \) be a polynomial of nonnegative integer coefficients and of degree at least two, and let \( U_f \) be the union set given in Theorem 1.2. Then part (ii) of Theorem
Conjecture 4.1 of [4] and Conjecture 3.1 of [9].

concluding this paper, we suggest the following more general conjecture that generalizes the sufficient and necessary condition for the union set $U$ to be dense in the interval $(0, \alpha f)$.

3. Now let $f(x)$ be a nonzero polynomial of integer coefficients. Let $Z_f := \{x \in \mathbb{Z} : f(x) = 0\}$ be the set of integer roots of $f(x)$ and $\{a_i\}_{i=1}^\infty := \mathbb{Z}^+ \setminus Z_f$ be arranged in the increasing order. Then $f(a_i) \neq 0$ for all integers $i \geq 1$. Let $n$ and $k$ be integers such that $1 \leq k \leq n$ and let $H_f^{(k)}(S_n)$ stand for the $k$-th elementary symmetric functions of

$$\frac{1}{f(a_1)^{s_1}} \cdot \frac{1}{f(a_2)^{s_2}} \cdots \frac{1}{f(a_n)^{s_n}}.$$ 

That is,

$$H_f^{(k)}(S_n) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^k \frac{1}{f(a_{i_j})^{s_{i_j}}}.$$ 

Then $H_f^{(1)}(S_n) = H_f(S_n)$. Let

$$\bar{H}_f^{(k)}(S_n) := \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \prod_{j=1}^k \frac{1}{f(a_{i_j})^{s_{i_j}}}.$$ 

When $f(x)$ is of nonnegative integer coefficients and $s_i = 1$ for all integers $i \geq 1$, the integrality of $H_f^{(k)}(S_n)(1 \leq k \leq n)$ was previously investigated in [1], [3], [5], [9] and [12]. But such integrality problem has not been studied when $f(x)$ contains negative coefficients. On the one hand, for any given integer $N_0 \geq 1$, one can easily find a polynomial $f_0(x)$ of integer coefficients such that for all integers $n$ and $k$ with $1 \leq k \leq n \leq N_0$, both of $H_f^{(k)}(S_n)$ and $\bar{H}_f^{(k)}(S_n)$ are integers. Actually, letting

$$f_0(x) = \prod_{i=1}^{N_0} (x - i) \pm 1$$

gives us the expected result. On the other hand, for any given nonzero polynomial $f(x)$ of integer coefficients, we believe that the similar integrality result is still true. So in concluding this paper, we suggest the following more general conjecture that generalizes Conjecture 4.1 of [1] and Conjecture 3.1 of [3].

Conjecture 5.1. Let $f(x)$ be a nonzero polynomial of integer coefficients and $S = \{s_i\}_{i=1}^\infty$ be an infinite sequence of positive integers (not necessarily increasing and not necessarily distinct). Then there is a positive integer $N$ such that for any integer $n \geq N$ and for all integers $k$ with $1 \leq k \leq n$, both of $H_f^{(k)}(S_n)$ and $\bar{H}_f^{(k)}(S_n)$ are not integers.

By Theorem 1.1, one knows that Conjecture 5.1 holds when $k = 1$. It is clear that Conjecture 5.1 is true when $k = n$. Thus we need just to look at the case $2 \leq k \leq n - 1$. Obviously, the results presented in [1], [3]-[5], [9]-[13] and Theorem 1.1 of this paper supply evidences to Conjecture 5.1.

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