INVENTORY CONTROL FOR SPECTRALLY POSITIVE LÉVY DEMAND PROCESSES

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ABSTRACT. We revisit the single item continuous-time inventory model. In the same problem setting as in Bensoussan et al. [8], we show the optimality of an \((s, S)\)-policy for a general spectrally positive Lévy demand process. Using the fluctuation theory of spectrally one-sided Lévy processes, we express the value function analytically using the scale function. Numerical examples under a Lévy process in the \(\beta\)-family with jumps of infinite activity are provided to confirm the analytical results. The case with no fixed ordering costs is also studied.

Key words: inventory model; impulse control; \((s, S)\)-policy; spectrally one-sided Lévy processes; scale functions

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1. INTRODUCTION

In this paper, we study the inventory control problem where the objective is to minimize the sum of the inventory cost incurred over time and the ordering cost incurred at each replenishment. The latter consists of the fixed and proportional costs. In existing literature, the demand process is typically modeled as a combination of the following components: (i) a constant continuous process, (ii) a (compound) Poisson process and (iii) a Wiener process. This spans the set of spectrally positive Lévy processes with jumps of finite activity, or equivalently Lévy processes with i.i.d. positive-valued compound Poisson jumps. The optimal policy is in many cases shown to be of \((s, S)\)-type; namely, it is optimal to bring the inventory level up to \(S\) whenever it goes below \(s\). In particular, in a seminal paper by [8], they studied the case where the demand is any combination of the above; the optimality of the \((s, S)\)-policy was proved with a complement by [5]. We refer the reader to [8, 9, 34] and references therein for a detailed review of inventory control problems. See also a recent book by Bensoussan [6].

In this paper, we generalize the existing models by solving the inventory control problem for a general spectrally positive Lévy demand process. This generalization encompasses jumps of infinite activity and/or infinite variation that had not been studied previously in inventory models. It accommodates important processes such as the (spectrally positive versions of) variance gamma, CGMY and normal...
inverse Gaussian (NIG) processes as well as classical examples such as the gamma process and a subset of stable processes. The ones perturbed by Wiener processes are also accommodated.

Recently, there has been new demand and trend of introducing jumps of infinite activity/variation in other stochastic models. In finance, for example, empirical evidence shows that it is appropriate to model frequent small fluctuations of asset prices in terms of jumps of infinite activity as opposed to modeling them by Wiener processes; see, e.g., the introduction of [11]. For this reason, variance gamma, CGMY and NIG processes with jumps of infinite activity are now popular tools for realistic financial models. In optimal dividend problems, the classical compound Poisson model (with or without a Wiener process) has been generalized to a spectrally negative Lévy model [24, 27, 28, 29] thanks to the fluctuation theories of the reflected Lévy process developed by, e.g., [2, 33]. See also [3, 4] for its spectrally positive Lévy version, and [1, 26] under tax payments.

Inventory models are certainly related to these problems. Indeed, the demand is the most important determinant of the price, and hence the above-mentioned reason for introducing jumps of infinite activity/variation applies to our case. Moreover, while a Wiener process is easy to handle analytically, there is no evidence that small fluctuations are Gaussian. This motivates us to work on the inventory control problem for a general spectrally positive Lévy demand process.

Until recently, the inclusion of jumps of infinite activity/variation has been difficult both analytically and numerically. However, in the last decade, there has been a great development in the fluctuation theory of Lévy processes (see e.g. [10, 13, 23]). In particular, for the case of spectrally one-sided Lévy processes, the so-called scale function can be used efficiently to express many quantities of interest such as hitting time probabilities, resolvent (potential) measures, and overshoots/undershoots. Its analytical properties such as log-convexity and smoothness/continuity has been used successfully in various stochastic control problems.

In this paper, we take advantage of these results to show that an $(s, S)$-policy is optimal for appropriate choice of $s$ and $S$. This differs from the approach in [8] and other literature where quantities of interest are written as solutions to certain integro-differential equations. In addition to the generalization of the underlying process discussed above, our approach has the following advantages.

First, because the expected total costs under the $(s, S)$-policy is written explicitly in terms of the scale function, the derivation and verification of optimal policy turn out to be more direct and simple. The values of $s$ and $S$ are determined by the continuous (resp. smooth) fit at $s$ when the demand process is of bounded (resp. unbounded) variation, together with another usual condition of the slope at $S$. The optimality is verified using the martingale and some known properties of the scale function, and also the results by [5]. The optimal levels $(s^*, S^*)$ are the zero of the function $G(s, S)$ define in (4.9) below and that of its derivative with respect to the second argument. The resulting value function can be written concisely via the scale function.

Second, thanks to the analytical form of the value function, we can see clearly the sensitivity with respect to the parameters of the problem. In particular, the limiting case with no fixed ordering cost can
be easily solved. We shall show, in this case, the optimal policy is of barrier type. Using the fluctuation theory of reflected Lévy processes as in [2, 33], the value function can again be written using the scale function.

Finally, computation of the value function can be easily carried out. The scale function has analytical expressions for many cases (see, among others, [20, 23]) and in general can be approximated efficiently by, e.g., [17, 35]. In order to confirm the analytical results obtained in this paper, we give numerical examples with a quadratic inventory cost and a demand process in the $\beta$-family of [21] with jumps of infinite activity. We see that the optimal levels $(s^*, S^*)$ and the value function can be computed instantaneously. We analyze the results and also confirm the convergence as the fixed ordering cost decreases to zero.

To our best knowledge, the inventory control problem has not been solved in this generality. As related literature, the optimal dividend problem under transaction costs is solved for spectrally negative and positive Lévy processes by [29] and [4], respectively. Their techniques are similar to ours; the optimality of the $(s, S)$-policy in their settings are shown via the fluctuation theory and the scale function. For a detailed review on impulse control, see [7, 30, 31].

The rest of the paper is organized as follows. Section 2 gives a mathematical model of the problem. Section 3 reviews spectrally one-sided Lévy processes and scale functions. In Section 4, we compute the expected total costs under the $(s, S)$-policy via the scale function. Section 5 shows via the continuous/smooth fit principle to obtain a candidate policy and show its existence. Section 6 verifies its optimality. Section 7 studies the case without a fixed ordering cost. We conclude the paper with numerical results in Section 8. Proofs of lemmas are deferred to the appendix.

### 2. Inventory Models

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting a stochastic process $D = \{D_t; t \geq 0\}$ with $D_0 = 0$, which represents the demand of a single item. Under the conditional probability $\mathbb{P}_x$, the initial level of inventory is given by $x \in \mathbb{R}$ (in particular, we let $\mathbb{P} \equiv \mathbb{P}_0$). Hence, the inventory in the absence of control follows the stochastic process

\[(2.1) \quad X_t := x - D_t, \quad t \geq 0.\]

We shall consider the case $D$ is a spectrally positive Lévy process, or equivalently $X$ is a spectrally negative Lévy process; we will define these processes formally in the next section. Let $\mathcal{F} := \{\mathcal{F}_t : t \geq 0\}$ be the filtration generated by $X$ (or equivalently by $D$).

An (ordering) policy $\pi := \{L_t^\pi; t \geq 0\}$ is given in the form of an impulse control $(T_1^\pi, u_1^\pi; T_2^\pi, u_2^\pi; \cdots)$ with $L_0^\pi = 0$ and $L_t^\pi = \sum_{i: T_i^\pi \leq t} u_i^\pi$, $t \geq 0$, where $\{T_i; i \geq 1\}$ is an increasing sequence of $\mathcal{F}$-stopping times and $u_i > 0$ is an $\mathcal{F}_{T_i}$-measurable random variable for $i \geq 1$. Corresponding to every policy $\pi$, the (controlled) inventory process is given by $U_\pi = \{U_t^\pi; t \geq 0\}$ where $U_{-}^\pi = 0$ and

\[U_t^\pi := X_t + L_t^\pi, \quad t \geq 0.\]
The problem is to compute, for a given discount factor \( q > 0 \), the total expected total costs given by
\[
v_\pi(x) := \mathbb{E}_x \left[ \int_0^\infty e^{-qt} f(U^\pi_t) dt + \sum_{i=1}^{\infty} e^{-qT^\pi_i} g(u^\pi_i) \right]
\]
with \( \Delta L^\pi_t := L^\pi_t - L^\pi_{t-}, \ t \geq 0 \), and to obtain an admissible policy that minimizes it, if such a policy exists.

Here, \( f: \mathbb{R} \to \mathbb{R} \) corresponds to the cost of holding and shortage when \( x > 0 \) and \( x < 0 \), respectively. Regarding \( g \), we assume
\[
g(y) := Cy + K, \quad y > 0,
\]
for some unit (proportional) cost of the item \( C \geq 0 \) and fixed ordering cost \( K > 0 \). We shall study the case \( K = 0 \) separately in Section 7. As in [5, 8], we assume the following.

**Assumption 2.1.**
1. \( f \) is a piecewise continuously differentiable function with \( f(0) = 0 \), and grows (or decreases) at most polynomially in the tail.
2. There exists a number \( a \) such that the function
\[
\tilde{f}(x) := f(x) + Cqx, \quad x \in \mathbb{R},
\]
is increasing on \((a, \infty)\) and decreasing and convex on \((\infty, a)\).
3. There exist a \( c_0 > 0 \) and an \( x_0 \geq a \) such that \( \tilde{f}'(x) \geq c_0 \) for \( x \geq x_0 \).

As in [5, 8], this is a crucial assumption for our analysis. This implies Lemma 5.1 below, which will be used to verify the existence of the optimal policy. For example, any convex function that first decreases and then increases clearly satisfies Assumption 2.1 (2) and (3).

Finally, the (optimal) value function is written as
\[
v(x) := \inf_{\pi \in \Pi} v_\pi(x), \quad x \in \mathbb{R},
\]
where \( \Pi \) is the set of all admissible policies. If the infimum is attained by some admissible policy \( \pi^* \in \Pi \), then we call \( \pi^* \) an optimal policy.

### 3. Spectrally Negative Lévy Processes and Scale Functions

Throughout this paper, we assume that the demand process \( D \) is a general spectrally positive Lévy process. Equivalently, the process \( X \) as in (2.1) is a spectrally negative Lévy process. Specifically, we assume that \( X \) has a Laplace exponent given by
\[
\psi(s) := \log \mathbb{E} [e^{sX_1}] = \gamma s + \frac{1}{2} \sigma^2 s^2 + \int_{(0,\infty)} (e^{-sz} - 1 + sz1_{(0<z<1)}) \nu(dz), \quad s \geq 0,
\]
where $\nu$ is a Lévy measure with the support $(0, \infty)$ that satisfies the integrability condition $\int_{(0,\infty)} (1 \wedge z^2) \nu(\mathrm{d}z) < \infty$. It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(0,1)} z \nu(\mathrm{d}z) < \infty$; in this case, we write (3.1) as

$$\psi(s) = \delta s + \int_{(0,\infty)} (e^{-sz} - 1) \nu(\mathrm{d}z), \quad s \geq 0,$$

with $\delta := \gamma + \int_{(0,1)} z \nu(\mathrm{d}z)$. We exclude the case in which $X$ is a subordinator (i.e., $X$ has monotone paths a.s.). This assumption implies that $\delta > 0$ when $X$ is of bounded variation.

For the problem to be nontrivial and make sense, we assume that $X_1$ has a finite moment.

**Assumption 3.1.** We assume that $\mu \in (-\infty, \infty)$ where we define

(3.2)

$$\mu := \mathbb{E}[X_1] = \psi'(0+).$$

### 3.1. Scale functions.

Fix $q > 0$. For any spectrally negative Lévy process, there exists a function called the $q$-scale function

$$W^{(q)} : \mathbb{R} \to [0, \infty),$$

which is zero on $(-\infty, 0)$, continuous and strictly increasing on $[0, \infty)$, and is characterized by the Laplace transform:

$$\int_{0}^{\infty} e^{-sx} W^{(q)}(x) \mathrm{d}x = \frac{1}{\psi(s) - q}, \quad s > \Phi(q),$$

where

$$\Phi(q) := \sup \{ \lambda \geq 0 : \psi(\lambda) = q \}.$$

Here, the Laplace exponent $\psi$ in (3.1) is known to be zero at the origin and convex on $[0, \infty)$; therefore $\Phi(q)$ is well defined and is strictly positive as $q > 0$. We also define, for $x \in \mathbb{R}$,

$$\overline{W}^{(q)}(x) := \int_{0}^{x} W^{(q)}(y) \mathrm{d}y,$$

$$Z^{(q)}(x) := 1 + q \overline{W}^{(q)}(x),$$

$$\overline{Z}^{(q)}(x) := \int_{0}^{x} Z^{(q)}(z) \mathrm{d}z = x + q \int_{0}^{x} \int_{0}^{z} W^{(q)}(w) \mathrm{d}w \mathrm{d}z.$$

Because $W^{(q)}$ is uniformly zero on the negative half line, we have

(3.3) $$Z^{(q)}(x) = 1 \quad \text{and} \quad \overline{Z}^{(q)}(x) = x, \quad x \leq 0.$$

Let us define the first down- and up-crossing times, respectively, of $X$ by

(3.4) $$\tau_{b}^{-} := \inf \{ t \geq 0 : X_t < b \} \quad \text{and} \quad \tau_{b}^{+} := \inf \{ t \geq 0 : X_t > b \}, \quad b \in \mathbb{R}.$$
Then, for any $b > 0$ and $x \leq b$,
\[
\mathbb{E}_x \left[ e^{-q T_b} 1 \{ T_b < T_0 \} \right] = \frac{W^{(q)}(x)}{W^{(q)}(b)}.
\] (3.5)
\[
\mathbb{E}_x \left[ e^{-q T_b} 1 \{ T_b > T_0 \} \right] = Z^{(q)}(x) - Z^{(q)}(b) \frac{W^{(q)}(x)}{W^{(q)}(b)},
\]
\[
\mathbb{E}_x \left[ e^{-q T_0} \right] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x).
\]

Fix $\lambda \geq 0$ and define $\psi_\lambda(\cdot)$ as the Laplace exponent of $X$ under $\mathbb{P}^\lambda$ with the change of measure
\[
\frac{d\mathbb{P}^\lambda}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = \exp \left( \lambda X_t - \psi(\lambda) t \right), \quad t \geq 0;
\]
see page 213 of [23]. Suppose $W^{(q)}_\lambda$ and $Z^{(q)}_\lambda$ are the scale functions associated with $X$ under $\mathbb{P}^\lambda$ (or equivalently with $\psi_\lambda(\cdot)$). Then, by Lemma 8.4 of [23], $W^{(q-\psi_\lambda)}_\lambda(x) = e^{-\lambda x} W^{(q)}(x)$, $x \in \mathbb{R}$, which is well defined even for $q \leq \psi(\lambda)$ by Lemmas 8.3 and 8.5 of [23]. In particular, we define
\[
W^{(q)}_{\psi(q)}(x) := W^{(q)}_{\psi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x), \quad x \in \mathbb{R},
\]
which is known to be an increasing function.

**Remark 3.1.**

1. If $X$ is of unbounded variation or the Lévy measure is atomless, it is known that $W^{(q)}$ is $C^1(\mathbb{R}\setminus \{0\})$; see, e.g., [12]. Hence,
   (a) $Z^{(q)}$ is $C^1(\mathbb{R}\setminus \{0\})$ and $C^0(\mathbb{R})$ for the bounded variation case, while it is $C^2(\mathbb{R}\setminus \{0\})$ and $C^1(\mathbb{R})$ for the unbounded variation case, and
   (b) $\overline{Z}^{(q)}$ is $C^2(\mathbb{R}\setminus \{0\})$ and $C^1(\mathbb{R})$ for the bounded variation case, while it is $C^3(\mathbb{R}\setminus \{0\})$ and $C^2(\mathbb{R})$ for the unbounded variation case.

2. Regarding the asymptotic behavior near zero, as in Lemmas 4.3 and 4.4 of [25],
\[
W^{(q)}(0) = \begin{cases} 
0, & \text{if } X \text{ is of unbounded variation,} \\
\frac{1}{\sigma}, & \text{if } X \text{ is of bounded variation,}
\end{cases}
\] (3.6)
\[
W^{(q)}(0+) := \lim_{x \downarrow 0} W^{(q)}(x) = \begin{cases} 
\frac{2}{\sigma}, & \text{if } \sigma > 0, \\
\infty, & \text{if } \sigma = 0 \text{ and } \nu(0, \infty) = \infty, \\
\frac{q + \nu(0, \infty)}{\sigma}, & \text{if } \sigma = 0 \text{ and } \nu(0, \infty) < \infty.
\end{cases}
\] (3.7)

3. As in (8.18) and Lemma 8.2 of [23],
\[
\frac{W^{(q)}(y)}{W^{(q)}(y)} \leq \frac{W^{(q)}(x)}{W^{(q)}(x)}, \quad y > x > 0.
\]

In all cases, $W^{(q)}(x-) \geq W^{(q)}(x+)$ for all $x \in \mathbb{R}$.
As necessary tools for subsequent discussions, we shall define, for all \( x > 0 \),
\[
\Theta^{(q)}(x) := W^{(q)'}(x+) - \Phi(q)W^{(q)}(x) = e^{\Phi(q)x}W^{(q)'}(x+) > 0, \\
\Theta^{(q)}(x) := W^{(q)}(x) - \Phi(q)W^{(q)}(x) > 0.
\] (3.8)

Here the positivity of the former holds because \( W^{(q)}(x+) \geq 0 \) for \( x > 0 \) and that of the latter holds because it is an integral of the former. Their positivity will be important in deriving the existence of the optimal solution and the verification of optimality.

4. THE \((s, S)\)-POLICY

We aim to prove that the \((s^*, S^*)\)-policy is optimal for some \(-\infty < s^* < S^* < \infty\). For arbitrary \(-\infty < s < S < \infty\), an \((s, S)\)-policy, \( \pi_{s,S} := \{L_t^{s,S}; t \geq 0\} \), brings the level of the inventory process \( U_{s,S} := X + L^{s,S} \) up to \( S \) whenever it goes below \( s \). In Figure 1, we show sample paths of the control process \( L^{s,S} \) and its corresponding controlled process \( U_{s,S} \) for \( s = -1, S = 0 \) and the starting value \( x = 0 \). Due to the negative jumps, it can jump to a level strictly below \( s \) (and is then immediately pushed to \( S \)).

Let us define the corresponding expected total costs as
\[
v_{s,S}(x) := \mathbb{E}\left[ \int_0^\infty e^{-qt} f(U_t^{s,S})dt + \sum_{0 \leq t < \infty} e^{-qt} g(\Delta L_t^{s,S})1_{\{\Delta L_t^{s,S} > 0\}} \right], \quad x \in \mathbb{R},
\] (4.1)
and also its “tilted version” (with respect to the proportional cost \( C \)) as
\[
\tilde{v}_{s,S}(x) := v_{s,S}(x) + Cx, \quad x \in \mathbb{R}.
\] (4.2)

As has been already seen as in, e.g., [5, 8], the algebra in the following arguments gets simpler if we deal with the latter rather than the former. The objective of this section is to express (4.2) in terms of the scale function.

Recall the down-crossing time \( \tau_{s}^- \) as in (3.4). By (2.2) and the strong Markov property, the expectation (4.1) must satisfy, for every \( S > s \) and \( x > s \),
\[
v_{s,S}(x) = \mathbb{E}_x \left[ \int_0^{\tau_{s}^-} e^{-qt} f(X_t)dt + \mathbb{E}_x \left[ e^{-q\tau_{s}^-} \left( C(S - X_{\tau_{s}^-}) + K \right) \right] \right] + \mathbb{E}_x \left[ e^{-q\tau_{s}^-} \right] v_{s,S}(S).
\] (4.3)
Define
\[
k(s, x) := \mathbb{E}_x \left[ \int_0^{\tau_{s}^-} e^{-qt} f(X_t)dt \right] - CE_x \left[ e^{-q\tau_{s}^-} X_{\tau_{s}^-} \right] + KE_x \left[ e^{-q\tau_{s}^-} \right] + Cx, \quad x > s.
\] (4.4)
Then, using (3.5), we can modify (4.3) and write (4.2) as
\[
\tilde{v}_{s,S}(x) = k(s, x) + \mathbb{E}_x \left[ e^{-q\tau_{s}^-} \right] \tilde{v}_{s,S} = k(s, x) + \left( 1 - \frac{q}{\Phi(q)} \frac{\Theta^{(q)}}{\Theta^{(q)}}(x-s) \right) \tilde{v}_{s,S}, \quad x > s.
\] (4.5)
Figure 1. Sample paths of the control process $L_{s,S}$ and its corresponding controlled process $U_{s,S}$ for $s = -1$, $S = 0$. The circles indicate the “pre-exercise” points $U_{\tau_s} + \Delta X_{\tau_s}$ when it gets at or below $s$.

where

\[ \tilde{v}_{s,S} := \tilde{v}_{s,S}(S) = \Phi(q) \frac{k(s, S)}{q - \Theta(q)(S - s)}, \quad S > s. \]

(4.6)

Here the last equality holds by solving (4.5) for $x = S$. Hence, once we identify the expression for $k(\cdot, \cdot)$, we can compute $\tilde{v}_{s,S}$ and consequently the whole function (4.2) as well.

Before stating their expressions, we define short-hand notations, for any measurable function $h$ and $s \in \mathbb{R}$:

\[ \Psi(s; h) := \int_{0}^{\infty} e^{-\Phi(q)y} h(y + s) dy = \int_{s}^{\infty} e^{-\Phi(q)(y-s)} h(y) dy, \]

\[ \varphi_s(x; h) := \int_{s}^{x} W'(q)(x - y) h(y) dy, \quad x \in \mathbb{R}. \]

(4.7)

Here $\varphi_s(x; h) = 0$ for any $x \leq s$ because $W'(q)$ is uniformly zero on $(-\infty, 0)$.

Remark 4.1. By Assumption 2.1(1), the functions $\Psi(s; f)$, $\Psi(s; f')$, $\Psi(s; \tilde{f})$ and $\Psi(s; \tilde{f}')$ are finite for any $s \in \mathbb{R}$.

By the following lemma, we can write (4.7) interchangeably for $f$ and $\tilde{f}$. 
Lemma 4.1. For $s \in \mathbb{R}$, we define
\[
\Psi(s; f) = \Psi(s; \tilde{f}) - \frac{Cq}{\Phi(q)} \left[ \frac{1}{\Phi(q)} + s \right],
\]
\[
\varphi_s(x; f) = \varphi_s(x; \tilde{f}) - C \left[ sZ^{(q)}(x-s) + \overline{Z}^{(q)}(x-s) - x \right], \quad x \in \mathbb{R}.
\]

The main result of this section is given below.

Proposition 4.1. (1) For any $x > s$,
\[
k(s, x) = \Theta^{(q)}(x-s) \left[ \Psi(s; \tilde{f}) - \frac{q}{\Phi(q)} \left( K + \frac{C\mu}{q} \right) \right] + G(s, x),
\]
where we define
\[
G(s, x) := \Phi(q)\Psi(s; \tilde{f})W^{(q)}(x-s) + K - \varphi_s(x; \tilde{f}), \quad x > s.
\]

(2) We have
\[
\tilde{v}_{s,S}(x) = -\frac{\Theta^{(q)}(x-s)}{\Theta^{(q)}(S-s)}G(s, S) + G(s, x) + \tilde{v}_{s,S}, \quad x > s.
\]

The proof of the first claim consists of evaluating the three expectations in (4.4). The first expectation can be obtained using the $q$-resolvent measure written in terms of the scale function; for its proof, see, e.g., [15, 23].

Lemma 4.2. For any $x, s \in \mathbb{R}$, we have $E_x \left[ \int_0^{\tau} e^{-qt} f(X_t) \, dt \right] = W^{(q)}(x-s)\Psi(s; f) - \varphi_s(x; f)$, which is well defined by Remark 4.1.

Using Lemmas 4.1 and 4.2, we have, for $x, s \in \mathbb{R}$,
\[
E_x \left[ \int_0^{\tau} e^{-qt} f(X_t) \, dt \right] = \Theta^{(q)}(x-s) \left[ \Psi(s; \tilde{f}) - \frac{Cq}{\Phi(q)} \left( \frac{1}{\Phi(q)} + s \right) \right] + C \left[ \overline{Z}^{(q)}(x-s) - (x-s) \right]
\]
\[
+ \Phi(q)W^{(q)}(x-s) \left[ \Psi(s; \tilde{f}) - \frac{Cq}{\Phi(q)^2} \right] - \varphi_s(x; \tilde{f}).
\]

Regarding the second expectation of (4.4), Proposition 2 of [2] gives $E_x \left[ e^{-q\tau_0} X_{\tau_0} \right] = \overline{Z}^{(q)}(x) - \mu \overline{W}^{(q)}(x) - (q - \mu \Phi(q))W^{(q)}(x)/(\Phi(q)^2)$. This together with (3.5) gives the following.

Lemma 4.3. For $x, s \in \mathbb{R}$,
\[
E_x \left[ e^{-q\tau_s} X_{\tau_s} \right] = \overline{Z}^{(q)}(x-s) - \left( s - \frac{\mu}{q} \right) \frac{q}{\Phi(q)} \overline{Z}^{(q)}(x-s) + s - \frac{q}{\Phi(q)^2}W^{(q)}(x-s).
\]
In (4.4), substituting (3.5), (4.11) and the term in Lemma 4.3, we have the expression for $k(\cdot, \cdot)$. This proves the first claim of Proposition 4.1.

For the second claim, by (4.6) and (4.8),

$$
\frac{q}{\Phi(q)} \tilde{v}_{s,S} = \frac{k(s, S)}{\Theta(q)(S - s)} = \Psi(s; \tilde{f}) - \frac{q}{\Phi(q)} \left( K + \frac{C\mu}{q} \right) + \frac{G(s, S)}{\Theta(q)(S - s)}.
$$

Substituting this and (4.8) in (4.5), the second claim of Proposition 4.1 is proved.

5. CANDIDATE POLICIES

In this section, we obtain the candidates of $(s,S)$ for the optimal policy. We shall identify them as roots of two equations: $G(s, S) = 0$ and $H(s, S) = 0$ where $G$ is defined as in (4.9) and $H$ is the derivative of $G$ with respect to the second argument, i.e.,

$$
H(s, x) := \frac{\partial}{\partial x} G(s, x), \quad x > s.
$$

We shall then show the existence of the pair $(s^*, S^*)$ that simultaneously satisfy these two equations.

5.1. Continuous/smooth fit. We will see that the condition $G(s, S) = 0$ is equivalent to the so-called continuous/smooth fit condition at $s$. It is well known, in the existence of a diffusion component, that smooth fit holds for impulse control problems; see, e.g., [18]. On the other hand, for a Lévy process of bounded variation, continuous fit may be used alternatively. This is well-studied particularly for optimal stopping problems; see, e.g., [16, 25]. Here we apply continuous fit for the case $X$ is of bounded variation and smooth fit for the case it is of unbounded variation.

Once $G(s, S) = 0$ is satisfied, then the second condition $H(s, S) = 0$ turns out to be equivalent to the condition that the slope of the value function at $S$ equals the negative of the proportional cost $C$, i.e., the slope of $\tilde{v}_{s,S}(\cdot)$ at $S$ is zero. Note that this condition is typically used in impulse control when an $(s,S)$-policy is shown to be optimal.

For all $x > s$, the function (5.1) can be written

$$
H(s, x) = \Phi(q) \Psi(s; \tilde{f}) W^{(q)}(x - s) - \varphi_s'(x; \tilde{f}) = \Psi(s; \tilde{f}') W^{(q)}(x - s) - \varphi_s(x; \tilde{f}'),
$$

where the last equality holds because integration by parts gives

$$
\varphi_s(x; \tilde{f}) = \overline{W}^{(q)}(x - s) \tilde{f}(s) + \int_s^x \overline{W}^{(q)}(x - y) \tilde{f}'(y) dy,
$$

and $\Psi(y; \tilde{f}) = [\tilde{f}(y) + \Psi(y; \tilde{f}')] / \Phi(q)$.

**Proposition 5.1.** Suppose $(s, S)$ are such that $G(s, S) = H(s, S) = 0$. Then

1. $\tilde{v}_{s,S}$ is continuous (resp. differentiable) at $s$ when $X$ is of bounded (resp. unbounded) variation,
2. $\tilde{v}_{s,S}'(S) = 0$. 

The proof of Proposition 5.1 can be carried out by a straightforward differentiation of the scale function and its asymptotic behavior near zero as in Remark 3.1(2).

By taking $x \downarrow s$ in (4.10) and because
\[
\tilde{v}_{s,S}(s+) = -\frac{\Theta(q)(0+)}{\Theta(q)(S-s)} G(s,S) + K + \tilde{v}_{s,S} = -\frac{\Theta(q)(0+)}{\Theta(q)(S-s)} G(s,S) + \tilde{v}_{s,S}(s-).
\]

Note that $\Theta(q)(0+) = 0$ if and only if $X$ is of unbounded variation in view of Remark 3.1(2). Hence, the continuity at $x = s$ holds if and only if $G(s,S) = 0$ for the case of bounded variation while it holds automatically for the unbounded variation case.

For the case of unbounded variation, we further pursue the differentiability at $x = s$. Differentiating (4.10), we get
\[
\tilde{v}'_{s,S}(x) = -\frac{\Theta(q)(x-s)}{\Theta(q)(S-s)} G(s,S) + \mathcal{H}(s,x), \quad x > s.
\]

Because $\mathcal{H}(s,s+) = 0$ for the case of unbounded variation, we see that the differentiability holds if and only if $G(s,S) = 0$ as well.

We now turn our attention to the slope at $S$. If we impose $G(s,S) = 0$ in (5.3), we have $\tilde{v}'_{s,S}(S) = \mathcal{H}(s,S)$. Hence $\tilde{v}'_{s,S}(S) = 0$ if and only if $\mathcal{H}(s,S) = 0$. This completes the proof of Proposition 5.1.

5.2. **Existence of $(s^*, S^*)$.** We shall show that there indeed exists a pair $(s^*, S^*)$ such that $G(s^*, S^*) = \mathcal{H}(s^*, S^*) = 0$; in the next section we show that this requirement is sufficient to prove the optimality of the $(s^*, S^*)$-policy. We refer the reader to [14] and [19] of stochastic games where similar arguments are used to identify a pair of parameters that describe the optimal strategy.

Recall that $\Psi(\cdot; \tilde{f}')$ is equivalent to (4.23) of [8] (times a positive constant). By Assumption 2.1, this satisfies the following. These results are due to Proposition 5.1 of [8] and hence the proof is omitted.

**Lemma 5.1.** (1) There exists a unique number $a_0 < a$ such that $\Psi(a_0; \tilde{f}') = 0$, $\Psi(x; \tilde{f}') < 0$ if $x < a_0$ and $\Psi(x; \tilde{f}') > 0$ if $x > a_0$.
(2) $\Psi'(x; \tilde{f}') > 0$ for $x \leq a$.
(3) $\Psi(x; \tilde{f}') \geq c_0/\Phi(q)$ for $x \geq x_0$.

With $a_0$ defined above, we show that the desired pair $(s^*, S^*)$ exist, and in particular $s^*$ lies on the left-hand side and $S^*$ lies on the right-hand side of $a_0$.

**Proposition 5.2.** There exists $s^* < a_0$ and $S^* > a_0$ such that
\[
s^* := \sup \left\{ s < a_0 : \inf_{S \geq s} G(s,S) = 0 \right\} \quad \text{and} \quad S^* \in \arg \inf_{S \geq a_0} G(s,S),
\]
with \( \mathcal{H}(s^*, S^*) = \mathcal{G}(s^*, S^*) = 0 \).

We prove Proposition 5.2 for the rest of this section. Toward this end, we first rewrite (4.9) and (5.1) as integrals of \( \Psi(\cdot, \tilde{f}^\prime) \) so as to use Lemma 5.1 efficiently.

**Lemma 5.2.** For \( x > s \),
\[
\mathcal{G}(s, x) = \int_s^x \Psi(y; \tilde{f}^\prime) \Theta(q)(x - y)dy + K,
\]
\[
\mathcal{H}(s, x) = \Psi(x; \tilde{f}^\prime) W(q)(0) + \int_s^x \Psi(y; \tilde{f}^\prime) \Theta(q)(x - y)dy.
\]

Second, we obtain the asymptotic behavior of \( \mathcal{G} \) as follows.

**Lemma 5.3.**
1. For every fixed \( s \in \mathbb{R} \), \( \lim_{S \uparrow \infty} \mathcal{G}(s, S) = \infty \).
2. For every fixed \( S \in \mathbb{R} \), \( \lim_{s \downarrow -\infty} \mathcal{G}(s, S) = -\infty \).

We are now ready to show the existence of \((s^*, S^*)\). While this is shown analytically below, numerical plots of \( \mathcal{G} \) and \( \mathcal{H} \) in Figure 3 of Section 8 are certainly helpful in understanding the proof.

First recall the definition of \( a_0 \) as in Lemma 5.1(1). For any \( S > s \geq a_0 \), Lemmas 5.1(1) and 5.2 imply that \( \mathcal{H}(s, S) > 0 \) uniformly. This together with (5.2) implies that the function \( \mathcal{G}(s, \cdot) : S \mapsto \mathcal{G}(s, S) \) starts at \( K > 0 \) (at \( S = s \)) and increases monotonically as \( S \uparrow \infty \) while never touching the zero line.

Let us now start at \( s = a_0 \) and consider decreasing the value of \( s \) toward \(-\infty \). By Lemma 5.3(1) and because \( \mathcal{H}(s, S) < 0 \) for any \( S < a_0 \), there exists a global minimizer \( S(s) \in \arg \inf_{s \geq a_0} \mathcal{G}(s, S) \) larger than \( a_0 \) and it also attains a local minimum (hence \( \mathcal{H}(s, S(s)) = 0 \)). Moreover, Lemma 5.3(2) implies, for sufficiently small \( s \), \( \inf_{S \geq a_0} \mathcal{G}(s, S) < 0 \). Hence, there must exist \((s^*, S^*)\) such that (5.4) holds and \( \mathcal{G}(s^*, S^*) = \mathcal{H}(s^*, S^*) = 0 \). Note that while \( s^* \) is unique by construction, \( S^* \) may not be unique. Moreover, by construction, \( s^* < a_0 \) and \( S^* > a_0 \) (because \( \mathcal{H}(s^*, S) \) is negative for \( S \in (s^*, a_0) \)). This completes the proof of Proposition 5.2.

### 6. Verification of Optimality

With the \((s^*, S^*)\) obtained in Proposition 5.2, we shall show that the \((s^*, S^*)\)-policy is optimal. By substituting \( \mathcal{G}(s^*, S^*) = 0 \) in (4.10),
\[
\bar{\nu}_{s^*, S^*}(x) = \mathcal{G}(s^*, x) + \bar{\nu}_{s^*, S^*}, \quad x > s^*.
\]
Notice from (4.12) that
\[
\bar{\nu}_{s^*, S^*} = \frac{\Phi(q)}{q} \Psi(s^*; \tilde{f}) - K - \frac{C\mu}{q}.
\]
Substituting this and by the definition of \( \mathcal{G}(\cdot, \cdot) \) as in (4.9),
\[
\bar{\nu}_{s^*, S^*}(x) = \frac{\Phi(q)}{q} \Psi(s^*; \tilde{f}) Z(q)(x - s^*) - \varphi_{s^*}(x; \tilde{f}) - \frac{C\mu}{q}, \quad x > s^*.
\]
The function \( v_{s^*,S^*} \) can be recovered by (4.2). By Lemma 4.1, we can also write
\[
(6.3) \quad v_{s^*,S^*}(x) = \left( \frac{\Phi(q)}{q} \right) \Psi(s^*; f) + C \left( \frac{Z(q)}{q} (x - s^*) - C \left( \frac{Z(q)}{q} (x - s^*) + \frac{\mu}{q} \right) - \varphi_{s^*}(x; f). \right.
\]
It is clear from (3.3) that this expression is also valid for \( x \leq s^* \).

We state the main result of the paper for the case \( K > 0 \).

**Theorem 6.1.** The \((s^*, S^*)\)-policy is optimal and the value function is given by \( v_{s^*,S^*}(\cdot) \) as in (6.3).

In order to show this theorem, we verify that the function \( v_{s^*,S^*} \) as in (6.3) satisfies the QVI (quasi-variational inequality); see [7].

Let \( L \) be the infinitesimal generator associated with the process \( X \) applied to a sufficiently smooth function \( h \)
\[
Lh(x) := \gamma h'(x) + \frac{1}{2} \sigma^2 h''(x) + \int_{[0,\infty)} \left[ h(x - z) - h(x) + h'(x) z 1_{\{0 < z < 1\}} \right] \nu(dz), \quad x \in \mathbb{R}.
\]
By Remark 3.1(1) and Proposition 5.1, the function \( v_{s^*,S^*} \) is \( C^0(\mathbb{R}) \) and \( C^1(\mathbb{R}\setminus\{s^*\}) \) (resp. \( C^1(\mathbb{R}) \) and \( C^2(\mathbb{R}\setminus\{s^*\}) \)) when \( X \) is of bounded (resp. unbounded) variation. Moreover, the integral part is well defined and finite by Assumption 3.1 and because \( v_{s^*,S^*} \) is linear below \( s^* \). Hence, \( L v_{s^*,S^*}(\cdot) \) makes sense anywhere on \( \mathbb{R}\setminus\{s^*\} \).

We first show the first part of the QVI, which can be shown easily thanks to the martingale property of the scale function and the fact that \( v_{s^*,S^*} \) is linear below \( s^* \).

**Lemma 6.1.**
\[
(1) \quad (L - q) v_{s^*,S^*}(x) + f(x) = 0 \text{ for } x > s^*,
\]
\[
(2) \quad (L - q) v_{s^*,S^*}(x) + f(x) \geq 0 \text{ for } x < s^*.
\]

The second part of the QVI is given as follows.

**Proposition 6.1.** For every \( x \in \mathbb{R} \), we have \( v_{s^*,S^*}(x) \leq K + \inf_{u \geq 0} [C u + v_{s^*,S^*}(x + u)] \).

It is clear that showing Proposition 6.1 is equivalent to showing \( \tilde{v}_{s^*,S^*}(x) \leq K + \inf_{u \geq 0} \tilde{v}_{s^*,S^*}(x + u) \). Toward this end, we use the following lemma.

**Lemma 6.2.** The following holds true.
\[
(1) \quad \tilde{v}_{s^*,S^*}(S^*) = \inf_{x \in \mathbb{R}} \tilde{v}_{s^*,S^*}(x).
\]
\[
(2) \quad \tilde{v}_{s^*,S^*}(x) \text{ is decreasing on } [s^*, a_0].
\]
\[
(3) \quad \lim_{x \to -\infty} \tilde{v}_{s^*,S^*}(x) = \infty.
\]

For \( x \leq s^* \), Proposition 6.1 holds immediately by Lemma 6.2(1) and because \( \tilde{v}_{s^*,S^*}(x) = \tilde{v}_{s^*,S^*}(s^*) = \tilde{v}_{s^*,S^*}(S^*) + K \). For \( s^* \leq x \leq a_0 \), it also holds by Lemma 6.2(2). The proof for \( x > a_0 \) is the most difficult part of the verification. However, thanks to Lemmas 6.1 and 6.2, we can apply Theorem 1 of [5] where they used the concept of the non-\( K \)-decreasing function. While they assume a mixture of a Wiener process and a compound Poisson process, the arguments are still valid for a general spectrally
negative Lévy process because small frequent jumps would not affect their arguments. Hence, we have Proposition 6.1.

By Lemma 6.1 and Proposition 6.1, we have shown that \( v_{s^*,S^*} \) satisfies the QVI. The rest of the proof of optimality (the verification lemma) is omitted because this is a well-studied problem and the results hold by applying Itô’s formula to \( v_{s^*,S^*}(U_t^\pi) \) for an arbitrary policy \( \pi \in \Pi \). Here one needs to apply an appropriate version of Itô’s formula (e.g. Theorem 3.2 of [32]) since the value function \( v_{s^*,S^*} \) is not smooth enough at \( s^* \) to apply the usual version. However, thanks to the continuous/smooth fit as in Proposition 5.1, the local time terms that might accumulate around \( s^* \) will vanish and hence have no effects after all. See also Theorem 6.2 of [31] for the verification lemma for impulse control problems.

7. The Case with No Fixed Ordering Costs

In this section, we study the variant of the problem with \( K = 0 \). Here, we widen the set of admissible policies to accommodate also the processes containing diffuse increments; we consider the set of \( \pi := \{ L_t^\pi; t \geq 0 \} \) given by a nondecreasing, right-continuous and \( \mathbb{F} \)-adapted process such that \( L_{0-} = 0 \). With \( U_t^\pi := X_t + L_t^\pi, t \geq 0 \), the problem is to compute the total costs:

\[
v_\pi(x) := \mathbb{E}_x \left[ \int_0^\infty e^{-qt} \left( f(U_t^\pi) + C dL_t^\pi \right) \right], \quad x \in \mathbb{R},
\]

for some \( C \geq 0 \) and to obtain an admissible policy that minimizes it, if such a policy exists.

From the results in the previous section, the optimal policy is easily conjectured. We have seen, for the case \( K > 0 \), the \((s^*,S^*)\)-policy is optimal for some \( s^* < a_0 < S^* \). Because the distance between \( s^* \) and \( S^* \) is expected to shrink as \( K \) decreases, it is a reasonable guess for the case \( K = 0 \) that a barrier policy with lower barrier \( a_0 \) is optimal. We shall show that it is indeed so.

Define, for \( s \in \mathbb{R} \),

\[ L_t^s := \sup_{0 \leq t' \leq t} (s - X_{t'}) \vee 0, \quad t \geq 0. \]

The corresponding inventory process \( U_t^s := X_t + L_t^s \) becomes a reflected Lévy process that always stays at or above \( s \). Figure 2 shows sample paths of these processes. Our objective is to show that \( v_{a_0}(x) = \inf_{\pi \in \Pi} v_\pi(x) \) for all \( x \in \mathbb{R} \) where

\[
(7.1) \quad v_s(x) := \mathbb{E}_x \left[ \int_0^\infty e^{-qt} \left( f(U_t^s) + C dL_t^s \right) \right], \quad x, s \in \mathbb{R},
\]

and \( \Pi \) is the set of all admissible policies.

The fluctuation theory of the reflected Lévy process has been well studied as in, e.g., [2, 33]. The expression (7.1) can be computed easily via the scale function.

**Lemma 7.1.**

1. We have \( \mathbb{E}_x \left[ \int_0^\infty e^{-qt} dL_t^s \right] = -Z(q)(x - s) - \mu/q + Z'(q)(x - s)/\Phi(q) \) for any \( x, s \in \mathbb{R} \).
2. We have \( \mathbb{E}_x \left[ \int_0^\infty e^{-qt} f(U_t^s) dt \right] = Z(q)(x - s)\Phi(q)\Psi(s; f)/q - \varphi_s(x; f) \) for any \( x, s \in \mathbb{R} \).
Combining the two results above, we can now write

\[ v_{a_0}(x) = -C \left( \frac{Z^{(q)}(x - a_0) + \mu}{q} \right) + Z^{(q)}(x - a_0) \phi \left( \frac{\Phi(q)}{\Phi(q)} \right) \psi \left( a_0; f \right) - \varphi_{a_0}(x; \tilde{f}), \]

which holds also for \( x < s \) by (3.3).

For the rest of this section, we show the following.

**Theorem 7.1.** The barrier strategy \( L^{a_0} \) is optimal and the value function is given by \( v_{a_0}(\cdot) \) as in (7.2).

By Lemma 4.1 and because \( \psi(a_0; \tilde{f}) = \tilde{f}(a_0)/\Phi(q) \) by the definition of \( a_0 \), we can write

\[ \tilde{v}_{a_0}(x) := v_{a_0}(x) + C x = -\frac{C \mu}{q} + \frac{Z^{(q)}(x - a_0)}{q} \tilde{f}(a_0) - \varphi_{a_0}(x; \tilde{f}), \quad x \in \mathbb{R}. \]

It turns out that our choice of \( s = a_0 \) guarantees the smoothness of (7.2) (or equivalently (7.3)) at \( a_0 \) that is even stronger than the case \( K > 0 \) as in Section 5. First, it is clearly continuous. Second, by taking a derivative and because \( \varphi'_{a_0}(x; \tilde{f}) = W^{(q)}(x - a_0) \tilde{f}(a_0) + \varphi_{a_0}(x; \tilde{f}) \),

\[ \tilde{v}'_{a_0}(x) = W^{(q)}(x - a_0) \tilde{f}(a_0) - \varphi'_{a_0}(x; \tilde{f}) = -\varphi_{a_0}(x; \tilde{f}) \xrightarrow{x \downarrow a_0} 0. \]

In particular, for the unbounded variation case,

\[ \tilde{v}''_{a_0}(x) = -\int_{a_0}^{x} W^{(q)}(x - y) \tilde{f}'(y) \, dy \xrightarrow{x \downarrow a_0} 0. \]

These results together with the smoothness of the scale function as in Remark 3.1(1) show the following.
Lemma 7.2. The function $v_{a_0}$ is $C^1(\mathbb{R})$ (resp. $C^2(\mathbb{R})$) for the case $X$ is of bounded (resp. unbounded) variation.

We shall now prove that $v_{a_0}$ solves the variational inequality. By Lemma 7.2 and because the integral part is well defined and finite by Assumption 3.1 and the linearity of $v_{a_0}$ below $a_0$, we confirm that $\mathcal{L}v_{a_0}(x)$ makes sense for all $x \in \mathbb{R}$.

Lemma 7.3. (1) $(\mathcal{L} - q)v_{a_0}(x) + f(x) = 0$ for $x > a_0$,
(2) $(\mathcal{L} - q)v_{a_0}(x) + f(x) \geq 0$ for $x \leq a_0$.

Lemma 7.4. We have $\tilde{v}_{a_0}'(x) = 0$ for every $x \leq a_0$ and $\tilde{v}_{a_0}'(x) \geq 0$ for every $x > a_0$.

By Lemmas 7.3 and 7.4 and thanks to the smoothness as in Lemma 7.2, a standard verification lemma (see, e.g., Proposition 4 of [2]) shows the optimality. This concludes the proof of Theorem 7.1.

8. Numerical Results

In this section, we conduct numerical experiments using, for $X$, the spectrally negative Lévy process in the $\beta$-family introduced by [21]. The following definition is due to Definition 4 of [21].

Definition 8.1. A spectrally negative Lévy process is said to be in the $\beta$-family if

$$\psi(z) = \hat{\delta}z + \frac{1}{2}\sigma^2z^2 + \frac{\varpi}{\beta}\left\{B(\alpha + \frac{z}{\beta}, 1 - \lambda) - B(\alpha, 1 - \lambda)\right\}$$

for some $\hat{\delta} \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $\varpi \geq 0$, $\lambda \in (0, 3)\setminus\{1, 2\}$ and the beta function $B(x, y) := \Gamma(x)\Gamma(y)/\Gamma(x + y)$.

This process has been receiving much attention recently due to many analytical properties that make many computations possible. In particular, it can approximate tempered stable (or CGMY) processes and is hence suitable to model the price of an asset. As we discussed in the introduction, the demand is a main determinant of the price; hence it is a natural choice for our inventory process $X$.

The $\beta$-family is a subclass of the meromorphic Lévy process [22] and hence the Lévy measure can be written

$$\nu(dz) = \sum_{j=1}^{\infty} p_j \eta_j e^{-\eta_j z}dz, \quad z \geq 0,$$

for some $\{p_k, \eta_k; k \geq 1\}$. The equation $\psi(\cdot) = q$ has a countable negative real-valued roots $\{-\xi_{k,q}; k \geq 1\}$ that satisfy the interlacing condition:

$$\cdots < -\eta_k < -\xi_{k,q} < \cdots < -\eta_2 < -\xi_{2,q} < -\eta_1 < -\xi_{1,q} < 0.$$

By using a similar argument as in [17], the scale function can be written as

$$(8.1) \quad W^q(q)(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} - \sum_{i=1}^{\infty} B_{i,q} e^{-\xi_{i,q}x}, \quad x \geq 0,$$
where

\[ B_{i,q} := \frac{\Phi(q)}{q} \xi_{i,q} A_{i,q} \quad \text{and} \quad A_{i,q} := \left(1 - \frac{\xi_{i,q}}{\eta_i}\right) \prod_{j \neq i} \frac{1 - \xi_{i,q}}{1 - \frac{\xi_{i,q}}{\xi_{j,q}}}, \quad i \geq 1. \]

We suppose \( \delta = 0.1, \lambda = 1.5, \alpha = 3, \beta = 1 \) and \( \varpi = 0.1 \). With this specification, the process has jumps of infinite activity (and of bounded variation), which is not covered in the framework of [8]. We consider \( \sigma = 0 \) and \( \sigma = 0.2 \) so as to study both the bounded and unbounded variation cases.

We let \( q = 0.03 \), and, for the inventory cost, we consider the quadratic case \( f(x) = x^2, x \in \mathbb{R} \). By straightforward calculation, \( a = -Cq/2, a_0 = -Cq/2 - \Phi(q)^{-1} \),

\[ \Psi(s; \tilde{f}) = \frac{2}{\Phi(q)^3} + \frac{2s + Cq}{\Phi(q)^2} + \frac{s^2 + Cqs}{\Phi(q)}, \quad \Psi(s; \tilde{f}^r) = \frac{2}{\Phi(q)^2} + \frac{Cq + 2s}{\Phi(q)}, \quad s \in \mathbb{R}, \]

and for \( x \geq s \)

\[ \varphi_s(x; \tilde{f}) = \frac{e^{\Phi(q)x}}{\psi^r(\Phi(q))} (Cq \kappa^{(1)}(s, x; -\Phi(q)) + \kappa^{(2)}(s, x; -\Phi(q))) \\
- \sum_{i=1}^{\infty} B_{i,q} e^{-\xi_{i,q}x} (Cq \kappa^{(1)}(s, x; \xi_{i,q}) + \kappa^{(2)}(s, x; \xi_{i,q})), \]

\[ \varphi_s(x; \tilde{f}^r) = \frac{e^{\Phi(q)x}}{\psi^r(\Phi(q))} (Cq \kappa^{(0)}(s, x; -\Phi(q)) + 2\kappa^{(1)}(s, x; -\Phi(q))) \\
- \sum_{i=1}^{\infty} B_{i,q} e^{-\xi_{i,q}x} (Cq \kappa^{(0)}(s, x; \xi_{i,q}) + 2\kappa^{(1)}(s, x; \xi_{i,q})), \]

where we define \( \kappa^{(n)}(t, t'; \zeta) := \int_t^{t'} e^{\psi y} y^n dy \) for \( t' > t, \zeta \in \mathbb{R} \) and \( n \geq 0 \).

8.1. Results. For the case \( K > 0 \), the first step is to obtain the pair \((s^*, S^*)\) as in Proposition 5.2. As is discussed in Section 5.2, starting at \( s = a_0 \) and as we decrease the value of \( s \), we arrive at the desired \((s^*, S^*)\) that makes the function \( G(s^*, \cdot) \) tangent to the \( x \)-axis at \( S^* \). Figure 3 plots \( G(s, \cdot) \) and \( H(s, \cdot) \) for \( s = a_0, (a_0 + s^*)/2, s^*, -a_0/2 + 3s^*/2, -a_0 + 2s^* \). The lines in red correspond to the desired curve; the starting point becomes \( s^* \) and the point touching zero becomes \( S^* \). Here we assume \( C = K = 10 \). As it turns out, \( H(s, \cdot) \) appears to be strictly convex in this example. Moreover, as is already clear analytically, it starts at zero (for the unbounded variation case) or below zero (for the bounded variation case). Hence \( G(s, \cdot) \) has a unique global minimum over \( S \in [s, \infty) \), which also appears to be increasing in \( s \). Hence, we applied a bisection method to obtain \( s^* \) and \( S^* \).

With \((s^*, S^*)\) computed instantaneously using the technique addressed above, the value function is calculated using (6.2). In Figure 4, we first plot it against the initial value \( x \) for the proportional cost \( C = 30, 20, 10, 5, 1, 0 \) with the common fixed cost \( K = 10 \). The plus signs indicate \((s^*, v_{s^*, S^*}(s^*))\) and \((S^*, v_{s^*, S^*}(S^*))\) for each choice of \( C \). It can be confirmed that the value function is increasing in \( C \) uniformly in \( x \in \mathbb{R} \). Moreover, \( s^* \) tends to increase as \( C \) decreases. This is consistent with our intuition.
that one is more eager to replenish as the ordering cost decreases. We also see in this plot that $S^*$ also tends to increase as $C$ decreases.

We now consider decreasing the value of the fixed cost $K$ and confirm the convergence to the case $K = 0$ as studied in Section 7. In Figure 5, we plot the value functions for $K = 100, 50, 10, 5, 1$ (dotted) along with that for the no-fixed cost case given by (7.2) (solid) with the common proportional cost $C = 10$. The circle signs indicate $(a_0, v_{a_0}(a_0))$. We can confirm that, as the value of $K$ decreases, the value function converges decreasingly to the one for $K = 0$. The convergence of the points $(s^*, S^*)$ to $a_0$ is also confirmed. Regarding the smoothness of the value function, it appears indeed that it is continuous
at $s^*$ for the case of bounded variation while it is differentiable for the case of unbounded variation. Furthermore, the smaller the value of $K$, the smoother the value function gets. This is consistent with Lemma 7.2 where the smoothness holds in a higher order for $K = 0$. 

**Figure 4.** The value functions for various values of the proportional cost $C$. 

**Figure 5.** The value functions for various values of the fixed cost $K$. 

unbounded variation case ($\sigma > 0$)  
bounded variation case ($\sigma = 0$)
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APPENDIX A. PROOFS

A.1. Proof of Lemma 4.1. The first claim holds by integration by parts. For the second claim, we have

\[ \varphi_s(x; f) - \varphi_s(x; \tilde{f}) = -C q \int_s^x y W^{(q)}(x - y) \, dy \]

\[ = -C q \left[ s W^{(q)}(x - s) + \int_0^{x-s} W^{(q)}(y) \, dy \right] = -C \left[ -x + s Z^{(q)}(x - s) + Z^{(q)}(x - s) \right], \]

as desired.

A.2. Proof of Lemma 5.2. We shall show the former because the latter is simply a derivative of the former. By integration by parts and because \( \Psi(y; \tilde{f}) = [\tilde{f}(y) + \Psi(y; \tilde{f}')] / \Phi(q), \)

\[ \Psi(s; \tilde{f}) W^{(q)}(x - s) = \int_s^x \left[ \Psi(y; \tilde{f}) W^{(q)}(x - y) - \Psi(y; \tilde{f}') W^{(q)}(x - y) \right] \, dy \]

\[ = \int_s^x \left[ \frac{\Psi(y; \tilde{f}') W^{(q)}(x - y) - \Psi(y; \tilde{f}) W^{(q)}(x - y)}{\Phi(q)} \right] \, dy + \varphi_s(x; \tilde{f}) \Phi(q) \]

\[ = \frac{1}{\Phi(q)} \left[ \int_s^x \Psi(y; \tilde{f}') \overline{\Theta}^{(q)}(x - y) \, dy + \varphi_s(x; \tilde{f}) \right]. \]

Substituting this in (4.9), we have the claim.

A.3. Proof of Lemma 5.3. (1) By Lemmas 5.1(3) and 5.2, for any \( S \geq s \lor x_0, \)

\[ G(s, S) \geq \frac{c_0}{\Phi(q)} \int_{s \lor x_0}^S \overline{\Theta}^{(q)}(S - y) \, dy + \int_s^{x_0} \Psi(y; \tilde{f}') \overline{\Theta}^{(q)}(S - y) \, dy + K. \]

The claim is now immediate because, for \( \epsilon > 0 \) and sufficiently large \( S, \)

\[ \int_s^S \overline{\Theta}^{(q)}(S - y) \, dy = \int_0^{S-x_0} \overline{\Theta}^{(q)}(y) \, dy \geq \int_{S-x}^{S-x_0} \overline{\Theta}^{(q)}(\epsilon) \, dy \xrightarrow{S \uparrow \infty} \infty, \]

where the inequality holds because \( \overline{\Theta}^{(q)} \) is positive and monotonically increasing. Notice here that \( \overline{\Theta}^{(q)} \) is bounded in view of (3.5).

(2) Similarly, if we choose \( b < a_0, \) we have, for any \( s < S \land b, \)

\[ G(s, S) = \int_s^{S \land b} \Psi(y; \tilde{f}') \overline{\Theta}^{(q)}(S - y) \, dy + \int_{S \land b}^S \Psi(y; \tilde{f}') \overline{\Theta}^{(q)}(S - y) \, dy + K. \]
By Lemma 5.1(1) and (2) and because $\Psi(b; \tilde{f}') < 0$, we have, for any $\epsilon > 0$ and sufficiently small $s$,
\[
\int_s^{s\wedge B} \Psi(y; \tilde{f}') \Theta^{(q)}(S - y) dy \leq \Psi(b; \tilde{f}') \int_s^{s\wedge B} \Theta^{(q)}(S - y) dy
\]
\[
= \Psi(b; \tilde{f}') \int_{S - s \wedge B}^{s - s} \Theta^{(q)}(y) dy \leq \Psi(b; \tilde{f}') \int_{S - s \wedge B + \epsilon}^{s - s} \Theta^{(q)}(S - S \wedge b + \epsilon) dy \xrightarrow{\epsilon \to 0} -\infty.
\]

A.4. **Proof of Lemma 6.1.** (1) By Proposition 2 of [2] and as in the proof of Theorem 8.10 of [23], the processes
\[
e^{-q(t \wedge \tau^+_B \wedge \tau^+_B)} Z^{(q)}(X_{t \wedge \tau^+_B \wedge \tau^+_B} - s^*) \quad \text{and} \quad e^{-q(t \wedge \tau^+_B \wedge \tau^+_B)} R^{(q)}(X_{t \wedge \tau^+_B \wedge \tau^+_B} - s^*), \quad t \geq 0,
\]
for any $B > s^*$ and $R^{(q)}(x) := Z^{(q)}(x) + \mu/q, x \in \mathbb{R}$, are martingales. Thanks to the smoothness of $Z^{(q)}$ and $\overline{Z}^{(q)}$ on $(0, \infty)$ as in Remark 3.1(1), we obtain $(L - q) R^{(q)}(y - s^*) = (L - q) Z^{(q)}(y - s^*) = 0$ for any $y > s^*$. This step is similar to the proof of Theorem 2.1 in [3].

On the other hand, as in the proof of Lemma 4.5 of [15],
\[
(L - q) \varphi_{s^*}(x; f) = f(x).
\]

Hence in view of (6.3), (1) is proved.

(2) For $x < s^*$, because $L v_{s^*} \varphi_{s^*}$ is a constant and by (3.3) and (6.3), we have
\[
(\frac{\partial}{\partial x}) [(L - q) v_{s^*} \varphi_{s^*}(x) + f(x)] = q C + f'(x) = \tilde{f}'(x) \leq 0, \quad \text{a.e. } x < s^*,
\]
where the last inequality holds because $x < s^* < a_0 \leq a$ and by the definition of $a$. Hence $(L - q) v_{s^*} \varphi_{s^*}(x) + f(x)$ is decreasing on $(-\infty, s^*)$. Therefore it is sufficient to show that
\[
(L - q) v_{s^*} \varphi_{s^*}(s^* -) + f(s^*) \geq 0.
\]

Suppose $X$ is of bounded variation. Because $s^* < a_0$ (and hence $\Psi(s^*; \tilde{f}') < 0$), we have
\[
\tilde{v}'_{s^*} \varphi_{s^*}(s^* +) = H(s^*, s^* +) = \Psi(s^*; \tilde{f}') \Theta^{(q)}(0) \leq 0.
\]

This implies, together with (1), continuous fit as in Proposition 5.1 and $\delta > 0$, shows (A.2) for the case of bounded variation.

Suppose $X$ is of unbounded variation. Then
\[
\tilde{v}''_{s^*} \varphi_{s^*}(x) = H'(s^*, x) = \frac{\partial}{\partial x} \int_{s^*}^x \Psi(y; \tilde{f}') \Theta^{(q)}(x - y) dy, \quad x > s^*.
\]

Taking limits and notice again that $s^* < a_0$,
\[
\tilde{v}''_{s^*} \varphi_{s^*}(s^* +) = \Psi(s^*; \tilde{f}') \Theta^{(q)}(0+) \leq 0.
\]

This implies together with (1) and smooth fit as in Proposition 5.1 shows (A.2).
A.5. **Proof of Lemma 6.2.** By (6.1) and because $S^*$ minimizes $G(s^*, x)$ over $x$ as in Proposition 5.2, the first claim holds. The second claim holds because $\tilde{v}'_{s^*, s^*}(x) = H(s^*, x) < 0$ on $[s^*, a_0]$ in view of Lemmas 5.1(1) and 5.2. The third claim holds by Lemma 5.3.

A.6. **Proof of Lemma 7.1.** (1) As in the proof of Theorem 1 of [2], if we define $\tau_B^r := \inf\{ t \geq 0 : U_t^0 > B \}$ for $B \in \mathbb{R}$,

$$\mathbb{E}_x \left[ \int_{0}^{\tau_B^r} e^{-q t} dL_t^0 \right] = -l(x) + Z^{(q)}(x) \frac{l(B)}{Z^{(q)}(B)}$$

with $l(x) := Z^{(q)}(x) + \mu/q - Z^{(q)}(x)/\Phi(q)$. From Exercise 8.5 of [23] and l’Hôpital’s rule, $Z^{(q)}(B)/Z^{(q)}(B) \to \Phi(q)^{-1}$ as $B \uparrow \infty$. Hence, by the monotone convergence theorem,

$$\mathbb{E}_x \left[ \int_{0}^{\tau_B^r} e^{-q t} dL_t^0 \right] = -l(x) + Z^{(q)}(x) \lim_{B \uparrow \infty} \frac{l(B)}{Z^{(q)}(B)} = -l(x).$$

By shifting the initial position of $X$, the proof is complete.

(2) By Theorem 1(i) of [33], for every $B > x$

$$\mathbb{E}_x \left[ \int_{0}^{\tau_B^r} e^{-q t} f(U_t^0) dt \right] = \int_{(s, \infty)} \left[ Z^{(q)}(x-s) \frac{W^{(q)}(B-y)}{Z^{(q)}(B-s)} - W^{(q)}(x-y) \right] f(y) dy$$

$$= \int_{(s, \infty)} \left[ Z^{(q)}(x-s) \frac{W^{(q)}(B-y)}{Z^{(q)}(B-s)} e^{-\Phi(q)(y-s)} \frac{W_{\Phi(q)}(B-y)}{W_{\Phi(q)}(B-s)} - W^{(q)}(x-y) \right] f(y) dy.$$

Thanks to Assumption 2.1(1), the boundedness of $W^{(q)}(\cdot)/Z^{(q)}(\cdot)$ in view of (3.5) and because $W_{\Phi(q)}$ is increasing, this is bounded uniformly in $B$. Hence, the dominated convergence theorem along with the convergence $W^{(q)}(B)/Z^{(q)}(B) \to \Phi(q)/q$ (by Exercise 8.5 of [23]) yields the result.

A.7. **Proof of Lemma 7.3.** (1) In view of (7.2), the proof is similar to Lemma 6.1.

(2) Using $x < a_0 \leq a$ and the definition of $a$ as in Assumption 2.1, the same steps as in the proof of Lemma 6.1(2) yield that $(\mathcal{L} - q)v_{a_0}(x) + f(x)$ is decreasing on $(-\infty, a_0)$. Now the claim holds immediately by (1) and Lemma 7.2 because $(\mathcal{L} - q)v_{a_0}(a_0-0) + f(a_0) = (\mathcal{L} - q)v_{a_0}(a_0+0) + f(a_0) = 0$.

A.8. **Proof of Proposition 7.4.** Recall (7.4). When $x \leq a_0$, it is clear that $\varphi_{a_0}(x; \tilde{f}') = 0$. When $a_0 < x \leq a$, because $\tilde{f}'(y) \leq 0$ for every $a_0 \leq y \leq a$, the claim holds in view of the definition of $\varphi_{a_0}(x; \tilde{f}')$ as in (4.7).

Now suppose $x > a$. We write

(A.3) $$\varphi_{a_0}(x; \tilde{f}') = e^{\Phi(q)x} \int_{a_0}^{x} e^{-\Phi(q)y} W_{\Phi(q)}(x-y) \tilde{f}'(y) dy.$$ 

We shall show that the integral is negative. Because $\tilde{f}'(y) \geq 0$ for any $y \geq x \geq a$,

(A.4) $$\int_{a_0}^{x} e^{-\Phi(q)y} \tilde{f}'(y) dy \leq \int_{a_0}^{\infty} e^{-\Phi(q)y} \tilde{f}'(y) dy = e^{-\Phi(q)a_0} \Psi(a_0, \tilde{f}') = 0.$$
Because $\Phi(x - y)$ is decreasing in $y$, the inequality of (A.4) implies that the integral in (A.3) must be negative. Indeed, in the integral, $\Phi(x - y)$ puts less weight for $y \in [a, x)$ where $f'(y) \geq 0$ and more weight for $y \in [a_0, a)$ where $f'(y) \leq 0$. This completes the proof.

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