Energy-Momentum Distribution in Static and Non-static Cosmic String Space-times

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In this paper, we elaborate the problem of energy-momentum in GR by energy-momentum prescriptions theory. In this regard, we calculate Möller, Landau-Lifshitz, Papapetrou, Einstein, Bergmann, Tolman, and Weinberg’s energy-momentum complexes in static and non-static cosmic string space-times. We obtain strong coincidences between the results. These coincidences can be considered as an extension of Virbhadra’s viewpoint that different energy-momentum prescriptions may provide some basis to define a unique quantity. In addition, our results disagree with Lessner’s belief about Möller’s prescription, and support the Virbhadra’s conclusion about power of Einstein’s prescription.

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1. INTRODUCTION

In classical mechanics and even in special relativity, we can always introduce a two-indices, symmetric tensorial quantity, i.e. $T^b_{a,b}$, which is named as energy-momentum tensor and represents the energy and momentum of matter and non-gravitational fields sources. Besides mentioned properties (being tensorial and symmetric), it has an important special characteristic: it is localized. This means that in every point of the manifold the quantity of energy-momentum is conserved. In the other words, energy-momentum tensor is a divergenceless quantity. In fact, in any local point of manifold no contribution of this quantity produces and no eliminates. We have:

$$T^b_{a,b} = 0$$

Eq. (1) is definition of energy-momentum conservation and known as conservation laws. Since the energy and momentum are two important, conserved quantities in physics, people are interested to keep it (as usual form of conservation laws) unchanged in all fields of physics, especially in general relativity’s theory. But in GR, ordinary derivatives transform to covariant derivatives. So we have [1]:

$$T^b_{a,b} = \frac{1}{\sqrt{-g}} (\sqrt{-g} T^b_a)_b - \Gamma^b_{ac} T^c_b = 0$$

It is obvious from Eq. (2) that $T^b_a$ no longer satisfies $T^b_{a,b} = 0$, but as noted before, we are interested to have a similar equation in GR. We add an additional term to $T^b_a$, e.g. $t^b_a$, so that the summation of these two terms remains divergenceless. In reality, the quantity that is actually conserved in the sense of Eq. (1) is some effective quantity which is given (in one variant) by Eq. (20.18) of MTW [2] as $\varepsilon_{eff} T^b_a = (T^b_a + t^b_a)$. In other variants, we obtain

$$\varepsilon_{eff} T^b_a = (-g)^{\frac{3}{2}} (T^b_a + t^b_a)$$

where $g = det(g_{ab})$ and $n$ is a positive integer that indicates the weight. For each of these $\varepsilon_{eff} T^b_a$, Eq. (2) can be rewritten as:

$$\varepsilon_{eff} T^b_{a,b} = 0$$

Conserved quantity $\varepsilon_{eff} T^b_a$ refers to the flux and density of energy and momentum of gravitational systems. In fact, coming from SR to GR, we add a contribution of gravitational fields, $t^b_a$, to the contribution of matter and all non-gravitational fields, $T^b_a$. Einstein, himself, proposed the first prescription for $\varepsilon_{eff} T^b_a$ just after GR’s formulation in 1916. Then, many other persons such as Möller [3], Landau-Lifshitz [4], Papapetrou [5], Bergmann [6], Tolman [7], and Weinberg [1] gave different prescriptions. All proposed expressions are called energy-momentum complexes, because they can be expressed as a combination of a tensor, $T^b_a$, and a pseudo-tensor, $t^b_a$. However, by using this (adding $t^b_a$ to $T^b_a$) we could solve the problem of being non-zero divergence of energy-momentum tensor, but some serious problems arise. Actually, it can be shown that $t^b_a$ does not obey tensor transformations. This non-tensorial property of $\varepsilon_{eff} T^b_a$ has caused that these complexes not to satisfy the required covariance and be coordinate dependent; it is the main problem of using energy-momentum complexes. Some authors tried to introduce new coordinate independent prescriptions. In fact, except few prescriptions including Penrose [8], Möller [3], and Komar’s [9] prescriptions, for other energy-momentum complexes one gets physically meaningful results only in Cartesian coordinate system.

Next problem is that it is not necessary for $\varepsilon_{eff} T^b_a$ to be
symmetric in all prescriptions. We can define conserved angular momentum quantity only for symmetric prescriptions [1]. In this regard, anti-symmetric characteristic of Einstein’s prescription was the main motivation for Landau and Lifshitz to look for an alternative prescription for energy-momentum which is symmetric. We have listed some prevalent and well-known prescriptions (that we have used in the next sections) and their properties in table I.

For making the subject clearer, it should be noted that $a^bT^a_a$ can be written as the divergence of some super-potential $H^{[bc]}_a$ that is anti-symmetric in its two upper indices [10] as

$$a^bT^a_a = H^{[bc]}_a$$

In addition, a new function like $U^{bc}_a$ can also play the role of $H^{[bc]}_a$ if it has the following conditions:

$$U^{bc}_a = H^{[bc]}_a + \Psi^{bc}_a (\Psi^{bc}_a, a, c \equiv 0, \text{ or } \Psi^{bc}_a, a, c)$$

Then, the quantity $\Theta^{bc}_a$ which is defined by this new super-potential remains conserved locally:

$$\Theta^{bc}_a = U^{bc}_a \Rightarrow \Theta^{bc}_a = 0$$

Using this freedom on the choice of the super-potential, authors like Einstein and Tolman arrived through different methods at the following super-potentials [11]:

$$H^{[bc]}_a = \frac{1}{2\kappa} \tilde{g}^{ab}(\tilde{g}^{cd}g^{de} - \tilde{g}^{cd}g^{de}) \psi, \text{ (Einstein)}$$

$$\tau^{bc}_a = H^{[bc]}_a + \frac{1}{2\kappa} (\delta^{ca}g^{db} - \delta^{da}g^{cb}) \psi, \text{ (Tolman)}$$

where $\tilde{g}^{ab} = \sqrt{-g^{ab}}$.

Considering above discussion, there are many prescriptions for new energy-momentum density $\omega^{a}_T$ [1,3-9] which their differences are in a curl term. Each of them has its own advantages and disadvantages and it has not proved any preferences between them. However, Palmer [12] and Virbhadra [13] discussed the importance of Einstein’s energy-momentum prescription and Lessner [14] believed that Møller’s prescription is a powerful tool for calculating the energy-momentum in GR.

The problems associated with concept of energy-momentum complexes resulted in some researchers even doubting the concept of energy-momentum localization. Misner et al. [2] argued that to look for a local energy-momentum is looking for the right answer to the wrong question. He showed that the energy can be localized only in systems which have spherical symmetry. Cooperstock and Sarracino [15] proved that if energy is localizable for spherical systems, then it can be localized in any system. In 1990, Bondi [16] argued that a non-localizable form of energy is not allowed in GR. Recently, besides energy-momentum prescriptions theory, it was suggested another solution for energy problem in GR that is in agreement with energy-momentum prescriptions theory about localization of energy, i.e. Tele-Parallel Gravity (for example see [17]). On the other hand, some people do not believe in localization of energy and momentum in GR. In addition, some physicists propose a new concept in this regard: quasi-localization (for example see [18]). Unlike energy-momentum prescriptions theory, quasi-localization theory does not restrict one to use particular coordinate system, but this theory has also its drawbacks [19,20]. In general, there is no generally accepted definition for energy and momentum in GR until now. Chang et al. in ref. [21] showed that every energy-momentum complex can be associated with distinct boundary term which gives the quasi local energy-momentum. By this way, he dispels doubts expressed about the physical meaning of energy-momentum complexes.

For a long time, there have been an uncertainty that different energy-momentum complexes would give different results for a given space-time. Many researches considered different prescriptions and obtained interesting results. Virbhadra et al. [13,22-25] investigated several examples of the space-times and showed that different prescriptions could provide exactly the same results for a given space-time. Aguirregabiria et al. [26] proved the consistency of the results obtained by using the different energy-momentum complexes for any Kerr-Schild class metric and revived the energy-momentum prescriptions theory after a long period of time.

In this paper we extend the previous works by calculating the energy of static and non-static cosmic string space-times in a specific region by seven well-known energy-momentum complexes. We obtain encouraging results which show interesting coincidences between the results calculated by different prescriptions. Our results about Møller’s prescription disagree with Lessner’s viewpoint but support the Virbhadra’s conclusion that Einstein’s prescription is the best available method for computing energy-momentum in a given space-time. The rest of the paper is organized as follows: in section (2) we introduce energy-momentum complexes those we use in next sections. Section (3)

| Prescription       | Coordinate System | Symmetry    |
|--------------------|-------------------|-------------|
| Møller             | any coordinate system | anti-symmetric |
| Landau-Lifshitz    | cartesian | symmetric |
| Papapetrou         | cartesian | symmetric |
| Einstein           | cartesian | anti-symmetric |
| Bergmann           | cartesian | non-symmetric |
| Tolman             | cartesian | non-symmetric |
| Weinberg           | cartesian | symmetric |
contains an introduction to static and non-static cosmic string space-times. The method and results of calculations are written in section (4). In section (5) we summarize and conclude with some remarks and discussions. Conventions: We use geometrized units in which the speed of light in vacuum \( c \) is taken to be equal to 1 and the metric has signature \( (+−−) \). Latin indices take values 0...3.

2. ENERGY-MOMENTUM PRESCRIPTIONS

Among many different forms proposed for energy-momentum pseudo-tensors, in this article we shall use Møller, Landau-Lifshitz, Papapetrou, Einstein, Bergmann, Tolman, and Weinberg’s prescriptions. Just in the same direction as the previous works in the literature here we try to show the compatibilities and to find out any existing discrepancies between predictions of these prescriptions when applying to static and non-static cosmic string space-times. Specific forms of each energy-momentum pseudo-tensor, conservation laws, and energy-momentum 4-vectors are listed in Table II briefly. Interested readers can refer to the mentioned references for details. In the last column of Table II Gauss’ theorem is used. In the surface integrals \( n_a \) represent the components of a normal one form over an infinitesimal surface element \( ds \). The results of calculations according to each of the individual forms will be shown in the following sections.

3. COSMIC STRING SPACE-TIMES

Very early universe is one of the hot and interesting subjects of theoretical physics that its structure has remained as a mystery until today. Cosmologists are generally assumed that at very early stages of its evolution, the universe has gone through a number of phase transitions. One of the immediate consequences of this phase transitions is the formation of defects or mismatches in the orientation of the Higgs field in causally disconnected regions [27]. Cosmic strings are one of remarkable topological defects that have received particular attention because of their cosmological implications. The double quasar problem can be explained by strings and galaxy formation might also be generated by density fluctuation in the early universe due to strings [28].

Suppose an infinitely long, thin, straight, static string lying along \( z \) axis with the following stress-energy tensor

\[
T^b_a = \mu \delta(x) \delta(y) diag(1, 0, 0, 1)
\]

where \( \mu \) is the mass per unit length of the string in the \( z \) direction. Considering space-time symmetries, Einstein’s field equations lead to well-known solution for case \( \Lambda = 0 \) in polar cylindrical coordinate system \((t, \rho, \phi, z)\) [29-31]:

\[
ds^2 = dt^2 - dz^2 - d\rho^2 - (1 - 4G\mu)^2 \rho^2 d\phi^2 \quad (11)
\]

For \( \Lambda \neq 0 \) Einstein’s field equations lead to general form of static cosmic string space-time with the following line element in polar cylindrical coordinate system \((t, \rho, \phi, z)\) [32]:

\[
ds^2 = \cos^4 \left(\frac{\sqrt{3\Lambda}}{2} \rho\right) (dt^2 - dz^2) - d\rho^2 - \frac{4(1 - 4G\mu)}{3\Lambda} \cos^4 \left(\frac{\sqrt{3\Lambda}}{2} \rho\right) \times \tan^2 \left(\frac{\sqrt{3\Lambda}}{2} \rho\right) d\phi^2
\]

where for \( \Lambda \rightarrow 0 \) reduces to the previous metric, Eq. (11). Investigating the non-static solution of the cosmic strings Einstein’s field equations lead to non-static cosmic string space-time with the following line element in polar cylindrical coordinate system \((t, \rho, \phi, z)\) [32]:

\[
ds^2 = dt^2 - e^{2\sqrt{3\Lambda}t} \left[d\rho^2 + (1 - 4G\mu)^2 \rho^2 d\phi^2 + dz^2\right] \quad (13)
\]

where for \( \Lambda \rightarrow 0 \) or \( t \rightarrow 0 \) reduces to Eq. (11). In the next section we calculate the energy of these space-times by different energy-momentum prescriptions.

4. CALCULATIONS

4.1. Method

As mentioned in section (2), for calculating the energy of a given space-time in a specific region by energy-momentum prescriptions we should integrate energy-momentum super-potentials over a suitable surface in space-time. So, we should calculate the super-potential components, and then integrate normal vector over infinitesimal surface element \( d\Sigma \). In the two next subsections integrations are over a cylindrical surface surrounding the length \( L \) from the string symmetrically with radius \( \rho \).

It should be noted that in Cartesian coordinate system (considering \( \theta = \arctan(\frac{y}{x}) \), and \( \rho = \sqrt{x^2 + y^2} \)) Eq. (12) and Eq. (13) transform to the following line elements (Eq. (14), (15)) respectively:

\[
ds^2 = \cos^4 \alpha dt^2 - \frac{1}{3} \frac{\alpha}{\Lambda} x^2 (x^2 + y^2) \cos^4 \alpha + 4a^2 y^2 \sin^2 \alpha \frac{dx^2}{x^2 + y^2}\\
- \frac{2}{3} \frac{\alpha}{\Lambda} (x^2 + y^2)^2 \cos^4 \alpha - 4a^2 y^2 \sin^2 \alpha \frac{xy dx dy}{x^2 + y^2}\\
- \frac{1}{3} \frac{\alpha}{\Lambda} x^2 (x^2 + y^2) \cos^4 \alpha + 4a^2 x^2 \sin^2 \alpha \frac{dy^2}{x^2 + y^2}\\
- \cos^4 \alpha dz^2
\]

where \( a = (1 - 4G\mu) \) and \( \alpha = \frac{\sqrt{3\Lambda}}{2} \rho \).

\[
ds^2 = dt^2 - e^{\alpha t} \frac{x^2 + a^2 y^2}{x^2 + y^2} dx^2\\
+ 2 e^{\alpha t} \frac{a^2 - 1}{x^2 + y^2} xy dx dy - e^{\alpha t} y^2 + a^2 x^2 \frac{y^2 dy^2}{x^2 + y^2} - e^{\alpha t} dz^2
\]
TABLE II: Energy-momentum Prescriptions

| Prescription | Energy-momentum pseudo-tensor | Conservation Laws | Energy-momentum 4-vector |
|--------------|-------------------------------|------------------|--------------------------|
| Moller [3]   | $\chi_{ik}^l = \frac{1}{8\pi} \frac{\partial}{\partial x^l} \left[ \phi \eta^k \eta^p \right]$ | $\frac{\partial M^k}{\partial x^k} = 0$ | $P_i = \int \int M^0 dx^i dx^2 dx^3 = \frac{1}{16\pi} \int \chi^l_{ikm} n_a ds$ |
| Landau-Lifshitz[4] | $\lambda^{iklm} = \frac{1}{16\pi} \frac{\partial}{\partial x^l} \left[ \delta^{ikm} \eta^j \right]$ | $\frac{\partial L^k}{\partial x^k} = 0$ | $P_i = \int \int L^{0i} dx^i dx^2 dx^3 = \frac{1}{16\pi} \int \lambda^l_{ikm} n_a ds$ |
| Papapetrou[5] | $\sigma_{iklm} = \frac{1}{16\pi} \frac{\partial}{\partial x^l} \left[ \delta^{ikm} \eta^j \right]$ | $\frac{\partial P^k}{\partial x^k} = 0$ | $P_i = \int \int \Sigma^{0i} dx^i dx^2 dx^3 = \frac{1}{16\pi} \int \sigma^l_{ikm} n_a ds$ |
| Einstein[3] | $H^i_{kl} = -H^i_{kl} = \frac{\alpha_{ikm}}{\sqrt{-g}} \left[ g^{ik} g^{jm} - g^{lm} g^{km} \right]$ | $\frac{\partial \Theta^k}{\partial x^k} = 0$ | $P_i = \int \int \Theta^0_{ikm} dx^i dx^2 dx^3 = \frac{1}{16\pi} \int H^0_{ikl} n_a ds$ |
| Bergman[6] | $B^i_{jk} = \frac{1}{16\pi} \frac{\partial}{\partial x^l} \left[ \delta^{ikm} \eta^j \right]$ | $\frac{\partial B^k}{\partial x^k} = 0$ | $P_i = \int \int B^{0i} dx^i dx^2 dx^3 = \frac{1}{16\pi} \int \beta^l_{ikm} n_a ds$ |
| Tolman[7] | $T^i_{kl} = \frac{1}{16\pi} \frac{\partial}{\partial x^l} \left[ \delta^{ikm} \eta^j \right]$ | $\frac{\partial T^k}{\partial x^k} = 0$ | $P_i = \int \int T^0_{ikl} dx^i dx^2 dx^3 = \frac{1}{16\pi} \int U^0_{ikl} n_a ds$ |
| Weinberg[1] | $W^{ik} = \frac{\partial h_{ik}}{\partial x^i} \eta^k - \frac{\partial h_{ik}}{\partial x^k} \eta^i$ | $\frac{\partial W^k}{\partial x^k} = 0$ | $P_i = \int \int W^{0i} dx^i dx^2 dx^3 = \frac{1}{16\pi} \int D^{0i} n_a ds$ |

where $\alpha = (1 - 4G\mu) \sigma \frac{1}{3} (remember that $\alpha$ in Eq. (14) is different from that is defined in Eq. (15)).

Everywhere we use $ds = \rho d\phi dz$ as infinitesimal surface element. In polar cylindrical coordinate system (allowed in Møller’s Prescription) we have $n_a = (0, 1, 0, 0)$, and in Cartesian coordinate system (all prescriptions) we have $n_a = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. Summarize all above, for calculating the energy, after extracting needed super-potential components, we must calculate surface integrals over a cylindrical surface with suitable normal vector $n_a$ that depends on used coordinate system. Following this method, in the next subsections we bring needed non-zero components of super-potentials, and final energy results (i.e. $P^0 \text{ or } P_0$) which are calculated by different energy-momentum prescriptions. Exact expressions of energy (except for Møller prescription) are very well-defined but long and complicated. So, we restricted ourselves to study the manner of energy around $\Lambda = 0$. Calculations for static($\Lambda \neq 0$) and non-static cosmic string space-times are classified in two separate subsections. Meanwhile, it should be noticed that we have done similar calculations by using static($\Lambda = 0$) cosmic string space-time (line element Eq. (11)) that their results are presented in Table III directly.
4.2. Static Cosmic String ($\Lambda \neq 0$)

Defining $\alpha = \frac{\sqrt{3}}{2} \rho$ and $a = 1 - 4G\mu$ energy can be calculated by different energy-momentum prescriptions as follows:

4.2.1. Möller prescription

Using Table II and Eq. (12) we find that non-zero needed components of $\lambda^l_i$ are:

\[
\lambda^l_t = -\frac{4}{3} a \sin^2 \alpha \frac{x}{x^2 + y^2} \tag{16}
\]

\[
\lambda^l_y = -\frac{4}{3} a \sin^2 \alpha \frac{y}{x^2 + y^2} \tag{17}
\]

Using surface integral (Table II) energy can be obtained as

\[
M E = -\frac{1}{3} a L \sin^2 \alpha \tag{18}
\]

after Taylor expansion around $\Lambda = 0$ we have:

\[
M E = -\frac{1}{4} a \rho^2 L \Lambda + \frac{1}{16} a \rho^4 L \Lambda^2 - \frac{1}{160} a \rho^6 L \Lambda^3 + ... \tag{19}
\]

That for $\Lambda = 0$ vanishes immediately. In addition, as we expect, same energy expression (Eq. (18)) are obtained after Taylor expansion around $\Lambda = 0$ we have:

\[
M E = -\frac{1}{4} a \rho^2 L \Lambda + \frac{1}{16} a \rho^4 L \Lambda^2 - \frac{1}{160} a \rho^6 L \Lambda^3 + ... \tag{20}
\]

4.2.2. Landau-Lifshitz prescription

Using Table II and Eq. (14) we find that non-zero needed components of $N^{\text{lin}}_{\text{lin}}$ are:

\[
N^{ttxx} = -\frac{1}{6} \sqrt{3} \Lambda a \cos \alpha \sin \alpha \frac{x(x^2 + y^2)}{a \rho^2 L \Lambda^2} \tag{21}
\]

\[
N^{ttyy} = -\frac{1}{6} \sqrt{3} \Lambda a \cos \alpha \sin \alpha \frac{y(x^2 + y^2)}{a \rho^2 L \Lambda^2} \tag{22}
\]

\[
N^{ttzz} = -\frac{1}{6} \sqrt{3} \Lambda a \cos \alpha \sin \alpha \frac{z(x^2 + y^2)}{a \rho^2 L \Lambda^2} \tag{23}
\]

\[
N^{ttyx} = N^{ytt} = \frac{\cos \alpha}{6} \Lambda a \cos \alpha \sin \alpha \frac{x(y^2 + x^2) - a^2 \sin^2 \alpha}{a \rho^2 L \Lambda^2} \tag{24}
\]

\[
N^{tttx} = N^{ttty} = \frac{\cos \alpha}{6} \Lambda a \cos \alpha \sin \alpha \frac{x(x^2 + y^2) - a^2 \sin^2 \alpha}{a \rho^2 L \Lambda^2} \tag{25}
\]

After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain:

\[
\rho E = \frac{(1 - a^2)}{8a} L + \frac{(3a^2 - 1)}{16a} L \rho^2 L - \frac{(17a^2 - 2)}{320a} L \rho^4 L^2 + ... \tag{26}
\]

4.2.3. Papapetrou prescription

Using Table II and Eq. (14), we find that non-zero needed components of $N^{\text{lin}}_{\text{lin}}$ are:

\[
N^{ttxx} = -\frac{1}{6} \sqrt{3} \Lambda a \cos \alpha \sin \alpha \frac{x(x^2 + y^2)}{a \rho^2 L \Lambda^2} \tag{27}
\]

\[
N^{ttyy} = -\frac{1}{6} \sqrt{3} \Lambda a \cos \alpha \sin \alpha \frac{y(x^2 + y^2)}{a \rho^2 L \Lambda^2} \tag{28}
\]

\[
N^{ttzz} = -\frac{1}{6} \sqrt{3} \Lambda a \cos \alpha \sin \alpha \frac{z(x^2 + y^2)}{a \rho^2 L \Lambda^2} \tag{29}
\]

\[
N^{ttyx} = N^{ytt} = \frac{\cos \alpha}{6} \Lambda a \cos \alpha \sin \alpha \frac{x(y^2 + x^2) - a^2 \sin^2 \alpha}{a \rho^2 L \Lambda^2} \tag{30}
\]

\[
N^{tttx} = N^{ttty} = \frac{\cos \alpha}{6} \Lambda a \cos \alpha \sin \alpha \frac{x(x^2 + y^2) - a^2 \sin^2 \alpha}{a \rho^2 L \Lambda^2} \tag{31}
\]

4.2.4. Einstein prescription

Using Table II and Eq. (14), we find that complicated quantities of $H^{tx}$ and $H^{ty}$ are only non-zero needed components of super-potential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain:

\[
E E = \frac{(1 - a^2)}{8a} L + \frac{(3a^2 - 1)}{16a} L \rho^2 L - \frac{(17a^2 - 2)}{320a} L \rho^4 L^2 + ... \tag{32}
\]

4.2.5. Bergmann prescription

Using Table II, and Eq. (14), we find that complicated quantities of $B^{ttx}$ and $B^{ttty}$ are only non-zero needed components of super-potential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain:

\[
\mu E = \frac{(1 - a^2)}{8a} L + \frac{L \rho^2 a \Lambda + (11a^2 - 1)}{640a} L \rho^4 L^2 + ... \tag{33}
\]
components of super-potential. After surface integration (Table II) and Taylor expansion around $\Lambda = 0$ we obtain:

$$TE = \frac{(1-a^2)}{8a}L + \frac{(3a^2 - 1)}{16a}Lp^2\Lambda - \frac{(17a^2 - 2)}{320a}Lp^4\Lambda^2 + ...$$

(32)

4.2.7. Weinberg prescription

Using Table II and Eq.(14), we find that non-zero components of $N^{kl}$ are:

$$N^{tt} = -\frac{(y^2 + a^2x^2)}{a(x^2 + y^2)}e^{\frac{1}{2}\alpha t}$$

(41)

$$N^{yy} = -\frac{(y^2 + a^2y^2)}{a(x^2 + y^2)}e^{\frac{1}{2}\alpha t}$$

(42)

$$N^{zz} = -(1 + e^{\alpha t})ae^{\frac{1}{2}\alpha t}$$

(43)

After surface integration (Table II) we obtain:

$$pE = \frac{(1-a^2)}{8a}e^{\frac{1}{2}\alpha t}L$$

(45)

4.3. Non-static Cosmic String

With $\alpha = 2\sqrt{\frac{A}{3}}$ and $a = 1 - 4G\mu$ different energy-momentum prescriptions can be evaluated as follows.

4.3.1. Möller prescription

Using Table II and Eq.(15) we find that non-zero components of $\chi^{kl}_i$ are:

$$\chi^{xy}_x = -\chi^{yx}_x = \frac{(a-1)e^{\frac{1}{2}\alpha t}}{a(x^2 + y^2)}y$$

(34)

$$\chi^{yx}_y = -\chi^{xy}_y = \frac{(a-1)e^{\frac{1}{2}\alpha t}}{a(x^2 + y^2)}x$$

(35)

After surface integration, we find that integral of energy vanishes. As we expect, in polar cylindrical coordinate system we obtain same result for the energy integral as we have obtained in cartesian coordinate system i.e. Möller $E = 0$.

4.3.2. Landau-Lifshitz prescription

Using Table II and Eq.(15) we find that non-zero needed components of $\chi^{kl}$ are:

$$\chi^{txx} = -e^{2\alpha t}\frac{y^2 + a^2x^2}{x^2 + y^2}$$

(36)

$$\chi^{tzz} = -e^{2\alpha t}a^2$$

(38)

$$\chi^{txy} = \chi^{ttx} = e^{2\alpha t}\frac{1 - a^2}{x^2 + y^2}xy$$

(39)

After surface integration (Table II) we obtain:

$$LL = \frac{(1-a^2)}{8}e^{2\alpha t}L$$

(40)

4.3.3. Papapetrou prescription

Using Table II and Eq.(14), we find that non-zero needed components of $N^{kl}$ are:

$$N^{ttx} = -\frac{(y^2 + a^2x^2)}{a(x^2 + y^2)}e^{\frac{1}{2}\alpha t}$$

(41)

$$N^{tyy} = -\frac{(y^2 + a^2y^2)}{a(x^2 + y^2)}e^{\frac{1}{2}\alpha t}$$

(42)

$$N^{tzz} = -(1 + e^{\alpha t})ae^{\frac{1}{2}\alpha t}$$

(43)

After surface integration (Table II) we obtain:

$$pE = \frac{(1-a^2)}{8a}e^{\frac{1}{2}\alpha t}L$$

(45)

4.3.4. Einstein prescription

Using Table II and Eq.(15) we obtain non-zero needed components of $H^{kl}_{i}$:

$$H^{txx} = \frac{(1-a^2)e^{\frac{1}{2}t}}{a(x^2 + y^2)}x$$

(46)

$$H^{tyy} = \frac{(1-a^2)e^{\frac{1}{2}t}}{a(x^2 + y^2)}y$$

(47)

After surface integration (Table II) we obtain:

$$E = \frac{(1-a^2)}{8a}e^{\frac{1}{2}\alpha t}L$$

(48)
Using Table II, Eq. (13) and Eq. (15), we find non-zero needed components of $B^{kl}$ as:

$$B^{tx} = \frac{(1 - a^2)e^{\frac{\alpha t}{2}}}{a(x^2 + y^2)} x$$

(49)

$$B^{ty} = \frac{(1 - a^2)e^{\frac{\alpha t}{2}}}{a(x^2 + y^2)} y$$

(50)

After surface integration (Table II) we obtain:

$$BE = \frac{(1 - a^2)}{8a} e^{\frac{\alpha t}{2}} a L$$

(51)

### 4.3.6. Tolman prescription

Using Table II and Eq. (15), we find that non-zero needed components of $U^{kl}_i$ are:

$$U^{tx}_i = \frac{(1 - a^2)e^{\frac{\alpha t}{2}}}{2a(x^2 + y^2)} x$$

(52)

$$U^{ty}_i = \frac{(1 - a^2)e^{\frac{\alpha t}{2}}}{2a(x^2 + y^2)} y$$

(53)

After surface integration (Table II) we obtain:

$$TE = \frac{(1 - a^2)}{8a} e^{\frac{\alpha t}{2}} a L$$

(54)

### 4.3.7. Weinberg prescription

Using Table II and Eq. (15), we obtain that non-zero needed components of $D^{kl}_i$ are:

$$D^{tx} = \frac{(1 - a^2)(e^{\alpha t}a^2 - 1 - a^2)}{a^2(x^2 + y^2)} e^{-2\alpha t} x$$

(55)

$$D^{ty} = \frac{(1 - a^2)(e^{\alpha t}a^2 - 1 - a^2)}{a^2(x^2 + y^2)} e^{-2\alpha t} y$$

(56)

After surface integration (Table II) we obtain:

$$WE = \frac{(1 - a^2)(-e^{\alpha t}a^2 + 1 + a^2)}{4a^2} e^{-2\alpha t} L$$

(57)

### 5. CONCLUSIONS AND REMARKS

In previous section we calculated the energy of static and non-static cosmic string space-times in a cylinder with length $L$ and radius $\rho$ surrounding the string symmetrically. We have summarized all obtained results in table III.

Regarding contents of table III:

- It is concluded that the energy is turn out to be finite and well-defined in all these prescriptions for these space-times.

- Substituting $a = 1$ in first column all prescriptions give energy equal to zero that is completely consistent. Because, if $a = 1$, Eq. (11) reduces to the Minkowski line element which its energy is equal to zero in any arbitrary region.

- For static ($\Lambda \neq 0$) cosmic string space-time Einstein, Tolman, and Papapetrou’s prescriptions lead to the same results. In addition, when $\Lambda \rightarrow 0$, Bergmann prescription joins to this list. For the non-static case Einstein, Papapetrou, Tolman, and Bergmann prescriptions have the same result. This coincidence supports and extends Virbhadra’s viewpoint [26] that different energy-momentum prescriptions may provide some basis to define a unique quantity. However, the remaining prescriptions give different energy densities (because of non-covariant property of pseudo-tensors).

- As we expect, for $\Lambda \rightarrow 0$ energy expressions in the second and third columns reduce to their corresponding expressions in the first column. It should be noticed that we calculated the components of the second column separately, by using line element Eq. (11) in energy-momentum prescriptions.

- Unlike other prescriptions, Møller’s prescription leads to a zero quantity for energy. This shortcoming is in contradiction with Lessner’s viewpoint and supports Virbhadra’s conclusion. Lessner [14] believed that Møller’s prescription is a powerful tool for calculating the energy-momentum pseudo-tensors in GR, and Virbhadra [13] concluded that Einstein’s energy-momentum prescription is still the best available method for computing energy-momentum in a given space-time.

- Reviewing the results shows that adding a factor $a$ in denominators of Landau-Lifshitz’s results causes that this prescription also give equivalent results (in comparison with Einstein, Tolman, and Papapetrou’s prescriptions). In other words Landau-Lifshitz’s results are different with other similar results (Einstein, Tolman, and Papapetrou) just in a factor $a$ in denominator. This dilemma is due to the fact that the conserved quantity in Landau-Lifshitz prescription is $\omega^a T^b_a = (-g)(T^b_a + \lambda^b_a) \text{ (weight +2)}$ instead of $\omega^a T^b_a = \sqrt{-g}(T^b_a + \lambda^b_a) \text{ (see Eq. (3) and Reference [4])}$ which its weight is +1. So, we should be careful about using this expression with weight +2 in our integration (see [33] chapter 7). Calculating energy with using a correction to Landau-Lifshitz prescription i.e. $\lambda^{iklm} = \sqrt{-g}(g^{ik}g^{lm} - g^{il}g^{km})$ instead of $\lambda^{iklm} = (-g)(g^{ik}g^{lm} - g^{il}g^{km})$ (Table II) leads to consistent results.
TABLE III: Energy of three different cosmic string space-times in a cylinder with length $L$ and radius $\rho$ surrounding symmetrically the string ($\alpha = 2\sqrt{\frac{\Lambda}{3}}$ and $\alpha = 1 - 4G\mu$)

| Prescriptions | Static ($\Lambda = 0$) | Static ($\Lambda \neq 0$) | Non-static ($\Lambda \neq 0$) |
|---------------|------------------------|---------------------------|----------------------------|
| Moller        | 0                      | $-\frac{1}{4}\pi\alpha^2 L \Lambda + \frac{1}{160}\pi\alpha^4 L \Lambda^2$ | 0                          |
| Landau-Lifshitz | $\frac{L(\alpha^2)}{8}$ | $\frac{(\alpha^2)}{8} + \frac{(3\alpha^2 - 16)}{16} L \rho^2 \Lambda - \frac{(7\alpha^2 - 1)}{126} L \rho^4 \Lambda^2 + \ldots$ | $\frac{(\alpha^2)}{8} e^{2\alpha t} L$ |
| Papapetrou    | $\frac{L(\alpha^2)}{8a}$ | $\frac{(\alpha^2)}{8a} + \frac{(3\alpha^2 - 16)}{16a} L \rho^2 \Lambda - \frac{(7\alpha^2 - 1)}{126a} L \rho^4 \Lambda^2 + \ldots$ | $\frac{(\alpha^2)}{8a} e^{\frac{1}{2}\alpha t} L$ |
| Einstein      | $\frac{L(\alpha^2)}{8a}$ | $\frac{(\alpha^2)}{8a} + \frac{(3\alpha^2 - 16)}{16a} L \rho^2 \Lambda - \frac{(7\alpha^2 - 1)}{126a} L \rho^4 \Lambda^2 + \ldots$ | $\frac{(\alpha^2)}{8a} e^{\frac{1}{2}\alpha t} L$ |
| Bergmann      | $\frac{L(\alpha^2)}{8a}$ | $\frac{(\alpha^2)}{8a} + \frac{(3\alpha^2 - 16)}{16a} L \rho^2 \Lambda - \frac{(7\alpha^2 - 1)}{126a} L \rho^4 \Lambda^2 + \ldots$ | $\frac{(\alpha^2)}{8a} e^{\frac{1}{2}\alpha t} L$ |
| Tolman        | $\frac{L(\alpha^2)}{4a^2}$ | $\frac{(\alpha^2)}{4a^2} + \frac{(3\alpha^2 - 16)}{16a^2} L \rho^2 \Lambda - \frac{(7\alpha^2 - 1)}{126a^2} L \rho^4 \Lambda^2 + \ldots$ | $\frac{(\alpha^2)}{4a^2} e^{\alpha t} a^2 + \frac{(\alpha^2)}{4a^2} e^{-2\alpha t} L$ |
| Weinberg      | $\frac{L(\alpha^2)}{4a^2}$ | $\frac{(\alpha^2)}{4a^2} + \frac{(3\alpha^2 - 16)}{16a^2} L \rho^2 \Lambda - \frac{(7\alpha^2 - 1)}{126a^2} L \rho^4 \Lambda^2 + \ldots$ | $\frac{(\alpha^2)}{4a^2} e^{\alpha t} a^2 + \frac{(\alpha^2)}{4a^2} e^{-2\alpha t} L$ |

In the final remark we would like to raise some points on the validity of the metric [12]. Eq.[12] with $\mu \rightarrow 0$ faces with some problems to present the deSitter space-time (dSS). It has intrinsic singularities at $\rho = \frac{\alpha}{\sqrt{\Lambda}}$, $n = odd$ [34], while dSS is free of them. The standard form of dSS is:

$$ds^2 = dt^2 - \frac{3}{\Lambda} \cosh^2(\frac{3}{\Lambda} t) \times$$

$$(d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2))$$

where $-\infty < t < +\infty$, $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$. By the following transformations:

$$\dot{t} = \sqrt{\frac{3}{\Lambda}} \log(\sinh(\frac{3}{\Lambda} t) + \cosh(\frac{3}{\Lambda} t) + \cos \chi)$$

(59)

$$\dot{\chi} = \frac{\cosh(\frac{3}{\Lambda} t) \sin \chi \cos \theta}{\sinh(\frac{3}{\Lambda} t) + \cosh(\frac{3}{\Lambda} t) + \cos \chi}$$

(60)

$$\dot{\theta} = \frac{\cosh(\frac{3}{\Lambda} t) \sin \theta \cos \phi}{\sinh(\frac{3}{\Lambda} t) + \cosh(\frac{3}{\Lambda} t) + \cos \chi}$$

(61)

$$\dot{\phi} = \frac{\cosh(\frac{3}{\Lambda} t) \sin \theta \sin \phi}{\sinh(\frac{3}{\Lambda} t) + \cosh(\frac{3}{\Lambda} t) + \cos \chi}$$

(62)

The metric [58] transforms to the steady state form or the so-called half deSitter metric:

$$ds^2 = d\tilde{t}^2 - \exp(2\frac{3}{\Lambda} t) (d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2).$$

(63)

In polar coordinate systems it takes the form:

$$ds^2 = d\tilde{t}^2 - \exp(2\frac{3}{\Lambda} t) \times$$

$$(d\tilde{r}^2 + \tilde{v}^2(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2))$$

(64)

Then by transformation:

$$r = \exp(\frac{3}{\Lambda} t) \tilde{r}$$

(65)

$$t = \frac{1}{2} \sqrt{\frac{3}{\Lambda} \ln(1 - \frac{\Lambda}{3} r^2)}$$

(66)

The metric [65] can be transformed to the static form:

$$ds^2 = (1 - \frac{\Lambda}{3} r^2) dt^2 - \frac{1}{(1 - \frac{\Lambda}{3} r^2)^2} dr^2$$

$$- r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

(67)
Writing the metric (67) in cylindrical coordinate systems \((t, \rho, \varphi, z)\) we have:

\[
ds^2 = (1 - \frac{\Lambda}{3} r^2) dt^2 - \frac{1 - \frac{\Lambda}{3} z^2}{1 - \frac{\Lambda}{3} r^2} d\rho^2 - \frac{2 \frac{\Lambda}{3} \rho \varphi}{1 - \frac{\Lambda}{3} z^2} d\rho dz - \frac{\rho^2}{1 - \frac{\Lambda}{3} r^2} d\varphi^2 - \frac{1 - \frac{\Lambda}{3} \rho^2}{1 - \frac{\Lambda}{3} r^2} dz^2
\]

(68)

Now the difference between Eq.(68) and Eq.(12) when \(\mu \to 0\) is quite evident. This means although the static metric (12) is an exact solution for Einstein equations but there is some doubt as to whether it actually fulfills precisely the requirements of the space-time associated with a cylindrical cosmic string located in a cosmological constant background.

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