Conjecture on a classification criterion for holonomies for the $U(1)$ group

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Abstract

Our main goal is to find a criterion to classify if a given group have a structure similar to a $U(1)$ Lie group.

1 The Infintesimal Logarithmic Spiral

In the course of seeking such a criterion, lets start with the analisy s of another problem. Lets start with a problem of reversibility. The Bernoulli application

$$A_{i+1} = A_i \| 1 \| \quad (1.1)$$

where $\| K \|$ stands for $\text{mod}(K)$.

This is an example of an irreversible transformation. In first place, the inverse transformation have an attractor in 0. Moreover, information is always lost in the module process. More generally, a transformation as

$$A_{i+\epsilon} = f(A_i) \| 2K \pi \| \quad (1.2)$$

is readily indentified as an irreversible transformation. Comparing with the function $f$, our transformation may be visualised as taking the domain line and wrapping it around a circle with radius $K$. This procedure makes the transformation irreversible, even disregarding possible atractors of the function $f$ when submitted to the transformation. This happens because many points of the image of $f$ are projected over the same point in the circumference. In this case, a possible tentative of eliminating the irreversibility would be to transform our circle in an spiral wich have an infinitesimal change in its curvature ratio. The most obvious choice is the infinitesimal logarithmic spiral. So we could transform our circle in a spiral using the following diffeomorfism, in polar coordinates.
\[ R(\theta) = K \implies R(\theta) = e^{i\theta} \] (1.3)

The neighborhood of a point in the circle would be transferred to the neighborhood of a point of the spiral in a bijective way, but analytically, our transformation would be no more \textit{a priori} irreversible.

![Figure 1: Diffeomorphism between a circle and a logarithmic spiral](image)

Speaking objectively, we want to study if makes sense to regard the application of a logarithmic spiral as a criterion for classifying groups of coordinate transformations as being similar to a \( U(1) \) group or not.

An element of the \( U(1) \) group, the group of rotations in the internal space with one real parameter, may be exponentiated as \( g = e^{i\theta} \). The basic criterions to form a \( U(1) \) group are:

\[ g_i g_j \in U(1) \iff e^{i\theta_i} e^{i\theta_j} = e^{i(\theta_i + \theta_j)} = e^{i\theta_k} \in U(1) \] (1.4)

\[ g g^\dagger = g^\dagger g = 1 \iff e^{i\theta} e^{-i\theta} = e^{-i\theta} e^{i\theta} = 1 \] (1.5)

as well as to obey to the Jacobi identity

\[ [[g_i, g_j], g_k] + [[g_k, g_i], g_j] + [[g_j, g_k], g_i] = 0 \] (1.6)

where \([x, y]\) stands for an anticommutative bilinear in \( x \) and \( y \).

If we multiply our element \( g \) by \( e^{i\theta} \) we would have an element of the form

\[ e^{i\theta} e^{i\theta} = e^{(e+i)i\theta} \] (1.7)

in general, our new element is out of the \( U(1) \) group, because

\[ g g^\dagger = e^{(e+i)i\theta} e^{(e-i)i\theta} = e^{2i\theta} \neq 1 \implies g \notin U(1) \] (1.8)
In a specific case, however, our element stays in the $U(1)$ group. The Lie groups are continuous groups, specifically continuous transformations groups. The discrete transformations are obtained from the continuous transformation. In the case where \( \theta \) is infinitesimal, our group element stays in the $U(1)$ group, once we are using a first order theory.

\[
    gg^\dagger = \lim_{\theta \to 0} e^{2\theta} = e^0 = 1 \implies g \in U(1)
\]  

(1.9)

Here we can find clearly a symmetry break between the major parts, the discrete transformations and the minor parts, the infinitesimal transformations from which the former are composed. Due to the definition of a continuous group, we can regard the discrete transformations as composed by the infinitesimal ones. Obviously the discrete elements can overlap, etc. So we have a symmetry break between two scales of transformations which can be said as following.

The application of an infinitesimal logarithmic spiral to an element of the group, defines if we are dealing with an infinitesimal element or a discrete one. In the case of an infinitesimal element, it stays in the group, otherwise if in the case of a finite one it gets away from the group.

The question is - Can the application of the infinitesimal logarithmic spiral transformation to a group of coordinate transformations define if is possible to make an holonomy of the group with the $U(1)$ group?

Another question is - In a coordinate transformation group, what stands for the element in what the criterion will be applied? Our first choice is to test the Jacobian. More specifically, the determinant of the Jacobian of the coordinate transformation. In the case of a finite element it would take away from the group, otherwise, in the case of an infinitesimal element it would stay in the group. If the group don’t follow this rule it could not be considered an holonomy between the group and the $U(1)$ group. In specific cases other methods may be used to apply this criterion directly on the coordinate transformation group.

Finally, is there something that deserves analysis in the symmetry breaking between the finite and infinitesimal elements of the $U(1)$ and similar (in our sense) groups when operated by the infinitesimal logarithmic spiral?