Abstract. Let \( R \) be a commutative ring, and let \((A, \{\cdot, \cdot\})\) be a Poisson algebra over \( R \). We construct a structure of an \((R, A)\)-Lie algebra in the sense of Rinehart on the \( A \)-module of Kähler differentials of \( A \) depending naturally on \( A \) and \( \{\cdot, \cdot\} \). This gives rise to suitable algebraic notions of Poisson homology and cohomology for an arbitrary Poisson algebra. A geometric version thereof includes the ‘canonical homology’ and ‘Poisson cohomology’ of a Poisson manifold introduced by Brylinski, Koszul, and Lichnerowicz, and absorbs the latter in standard homological algebra by expressing them as Tor and Ext groups, respectively, over a suitable algebra of differential operators. Furthermore, the Poisson structure determines a closed 2-form \( \pi_{\{\cdot, \cdot\}} \) in the complex computing Poisson cohomology. This 2-form generalizes the 2-form \( \sigma \) defining a symplectic structure on a smooth manifold \( N \); moreover, the class of \( \pi_{\{\cdot, \cdot\}} \) in Poisson cohomology generalizes the class \([\sigma] \in H^2_{\text{deRham}}(N, \mathbb{R})\) of a symplectic structure \( \sigma \) on a smooth manifold \( N \) and appears as a crucial ingredient for the construction of suitable linear representations of \((A, \{\cdot, \cdot\})\), viewed as a Lie algebra; representations of this kind occur in quantum theory. To describe this class and to construct the representations, we relate formal concepts of connection and curvature generalizing the classical ones with extensions of Lie algebras. We illustrate our results with a number of examples of Poisson algebras and with a quantization procedure for a relativistic particle with zero rest mass and spin zero.

0. Introduction

The concept of a Poisson manifold is currently of much interest, see e.g. Berger [7], Bhaskara-Viswanath [8], Bracconier [10], [11], Brylinski [12], Coste-Dazord-Weinstein [17], Conn [18], [19], De Wilde-Le Comppe [21], Gelfand-Dorfman [27] – [29], Karasev [44], Kassel [45], Koszul [52], Lichnerowicz [56] – [62], Magri-Morosi [66], Magri-Morosi-Ragnisco [67], Mikami-Weinstein [74], Stasheff [95], [96], Tulczyjew [97], [98], Vinogradov-Krasil’shchik [100], Weinstein [102] – [106]. A Poisson structure on a smooth manifold \( N \) is a Lie
bracket \{\cdot, \cdot\} on the (multiplicative) algebra of smooth functions on \(N\) satisfying
the additional condition \(\{fg, h\} = f\{g, h\} + \{f, h\}g\). More generally, an algebra \(A\)
over a commutative ring \(R\) together with a Lie bracket \{\cdot, \cdot\} on \(A\) satisfying
the formal analogue of the above additional condition is called a Poisson algebra. The
significance of Poisson structures in physics is classical, see \(\textit{LIE}\) [63] and \(\textit{DIRAC}\) [22],
[23].

For a symplectic manifold \((N, \sigma)\), the rule \(\{f, g\} = \sigma(X_f, X_g)\), where \(X_f\) is the
Hamiltonian vector field corresponding to \(f\), defines a Poisson structure on \(N\). However,
this is not the only way in which a Poisson structure on a manifold arises, see e. g. \textit{Weinstein} [103].

In [58] \textsc{Lichnerowicz} introduced what he called “Poisson cohomology” of a
Poisson manifold; when the Poisson structure comes from a symplectic one, this
Poisson cohomology coincides with de Rham cohomology [58], cf. (3.15) below. In [52] \textsc{Koszul}
troduced a notion of homology for a Poisson manifold which was christened “canonical homology” by \textsc{Brylinski} [12]. In the present paper we introduce
the corresponding notions of Poisson homology and cohomology for an arbitrary Poisson
algebra \((A, \{\cdot, \cdot\})\). We now explain briefly and informally our approach:

The key idea is that a Poisson structure \{\cdot, \cdot\} on an arbitrary algebra \(A\) over a
commutative ring \(R\) gives rise to a structure of an \((R, A)\)-Lie algebra in the sense of \textsc{Rinehart} [80] on the \(A\)-module \(D_A\) of Kähler differentials for \(A\) in a natural
fashion. An \((R, A)\)-Lie algebra is a Lie algebra over \(R\) which acts on \(A\) and is also
an \(A\)-module satisfying suitable compatibility conditions which generalize the usual
properties of the Lie algebra of smooth vector fields on a smooth manifold viewed as
a module over its ring of smooth functions; these objects have been introduced by
\textsc{Herz} [37] under the name “pseudo-algèbre de Lie” and were examined by \textsc{Palais}
[77] under the name “\(d\)-Lie ring”. Any \((R, A)\)-Lie algebra \(L\) gives rise to a complex
\(\text{Alt}_A(L, A)\) of alternating forms which generalizes the usual de Rham complex of a
manifold and the usual complex computing \textsc{Chevalley-Eilenberg} [16] Lie algebra
cohomology. This observation is again due to \textsc{Palais} [77]. Moreover, extending
earlier work of \textsc{Hochschild}, \textsc{Kostant} and \textsc{Rosenberg} [39], \textsc{Rinehart} [80] has
shown that, when \(L\) is projective as an \(A\)-module, the homology of the complex
\(\text{Alt}_A(L, A)\) may be identified with \(\text{Ext}^*_{U(A, L)}(A, A)\) over a suitably defined universal
algebra \(U(A, L)\) of differential operators; see Section 1 below for details. In particular,
when \(A\) is the algebra of smooth functions on a smooth manifold \(N\) and \(L\) the
Lie algebra of smooth vector fields on \(N\), then \(U(A, L)\) is the algebra of (globally
defined) differential operators on \(N\).

In Section 3 below we construct, for any Poisson algebra \((A, \{\cdot, \cdot\})\), a natural
structure of an \((R, A)\)-Lie algebra on the \(A\)-module \(D_A\) of Kähler differentials for
\(A\); we write \(D_{\{\cdot, \cdot\}}\) for the resulting \((R, A)\)-Lie algebra. We can then apply the
machinery of \textsc{Palais} [77] and \textsc{Rinehart} [80]. In this vein we define the Poisson
cohomology \(H^*_\text{Poisson}(A, \{\cdot, \cdot\}; A)\) of \((A, \{\cdot, \cdot\})\) as the homology of \(\text{Alt}_A(D_{\{\cdot, \cdot\}}, A)\). The
Poisson structure \{\cdot, \cdot\} then determines a natural closed 2-form \(\pi_{\{\cdot, \cdot\}} \in \text{Alt}_{\text{Poisson}}^2(D_{\{\cdot, \cdot\}}, A)\) and hence a natural class \([\pi_{\{\cdot, \cdot\}}] \in H^2_{\text{Poisson}}(A, \{\cdot, \cdot\}; A)\), see (3.10) for details; we refer
to this class as the Poisson class of \((A, \{\cdot, \cdot\})\). These notions of Poisson homology
and cohomology are entirely algebraic. In the special case where the ground ring \(R\)
is that of the reals (or complex numbers) and \(A\) is the ring of smooth functions
on a smooth finite dimensional Poisson manifold \((N,\lbrace\cdot,\cdot\rbrace)\), a variant of the above construction yields a natural structure of an \((R,A)\)-Lie algebra on the \(A\)-module \(D^\geo_A\) of smooth 1-forms on \(N\); we write \(D^\geo_{\lbrace\cdot,\cdot\rbrace}\) for the resulting \((R,A)\)-Lie algebra. Again we can then apply the machinery of PALAIS [77] and RINEHART [80]. We define geometric Poisson cohomology \(H^*_\mathrm{Poisson}(N,\lbrace\cdot,\cdot\rbrace;R)\) of \((N,\lbrace\cdot,\cdot\rbrace)\) as the homology of \(\mathrm{Alt}_A(D^\geo_{\lbrace\cdot,\cdot\rbrace},A)\). The Poisson structure \(\lbrace\cdot,\cdot\rbrace\) then determines a natural closed 2-form \(\pi^\geo_{\lbrace\cdot,\cdot\rbrace} \in \mathrm{Alt}^2_A(D^\geo_{\lbrace\cdot,\cdot\rbrace},A)\) and hence a natural class \([\pi^\geo_{\lbrace\cdot,\cdot\rbrace}] \in H^2_\mathrm{Poisson}(N,\lbrace\cdot,\cdot\rbrace;R)\), see (3.12) for details; we refer to this class as the Poisson class of \((N,\lbrace\cdot,\cdot\rbrace)\). The complex \(\mathrm{Alt}_A(D^\geo_{\lbrace\cdot,\cdot\rbrace},A)\) is precisely the one introduced by LICHNEROWICZ [58]. In this way Lichnerowicz’ Poisson cohomology appears as \(\mathrm{Ext}^*_U(A,D^\geo_{\lbrace\cdot,\cdot\rbrace})(A,A)\). Furthermore, it turns out that the obvious morphism \(D_{\lbrace\cdot,\cdot\rbrace} \longrightarrow D^\geo_{\lbrace\cdot,\cdot\rbrace}\) is one of \((R,A)\)-Lie algebras and induces an isomorphism \(H^*_\mathrm{Poisson}(N,\lbrace\cdot,\cdot\rbrace;R) \cong H^*_\mathrm{Poisson}(A,\lbrace\cdot,\cdot\rbrace;A)\) on cohomology. Hence in the smooth case there is no need to distinguish between algebraic and geometric Poisson cohomology; see (3.12.13) below. Moreover, when the Poisson structure \(\lbrace\cdot,\cdot\rbrace\) comes from a symplectic structure \(\sigma\) on \(N\), this structure induces an isomorphism \(\sigma^* : H^*_{\text{deRham}}(N,\mathbb{R}) \cong H^*_\mathrm{Poisson}(N,\lbrace\cdot,\cdot\rbrace;R)\), and under this isomorphism the class \([\sigma] \in H^2_{\text{deRham}}(N,\mathbb{R})\) goes to the Poisson class \([\pi_{\lbrace\cdot,\cdot\rbrace}] \in H^2_\mathrm{Poisson}(N,\lbrace\cdot,\cdot\rbrace;R)\).

Likewise, inspection shows that a suitable complex computing \(\mathrm{Tor}_U(A,D^\geo_{\lbrace\cdot,\cdot\rbrace})(A,A)\) is exactly the one used by KO SZUL [52] and BRYLINSKI [12] to define canonical homology. It admits an obvious generalization to arbitrary Poisson algebras; this leads to our notion of (algebraic) Poisson homology, see Section 3 for details. However, in the smooth case the two notions of Poisson homology differ.

The concept of an \((R,A)\)-Lie algebra has a geometric analogue which is nowadays called a Lie algebroid, see COSTE-DAZORD-WEINSTEIN [17], MACKENZIE [64], PRADINES [78], WEINSTEIN [105]. To our knowledge the first ones to notice that a Poisson structure on a smooth manifold gives rise to a Lie bracket on the space of its 1-forms were MAGRI AND MOROSI [66], see also (2.2) in MAGRI-MOROSI-RAGNISCO [67]; however, it seems that only in (3.1) of WEINSTEIN [105] and (III.2.1) of COSTE-DAZORD-WEINSTEIN [17] is it pointed out that the bracket yields in fact a structure of a Lie algebroid. The relationship of a Poisson structure with the work of PALAIS [77] and RINEHART [80] does not seem to have been observed in the literature so far.

In Section 1 of the present paper we extend some of the results in RINEHART’s paper [80]. In Section 2 we introduce formal concepts of connection and curvature which over a smooth manifold boil down to the usual ones. Among others, a connection appears as a section (of the underlying modules) of a suitable extension of Lie algebras, and its curvature is a corresponding (in general non-abelian) 2-cocycle; for example, the Bianchi identity is then nothing else than the cocycle condition. In the geometric situation such a description goes back to ATIYAH [5] and has recently been reworked and elaborated upon by MACKENZIE [64]. In Section 3 we introduce Poisson homology and cohomology, relate it with the earlier notions, and give some examples. Section 4 deals with the problem of constructing a linear representation of the Lie algebra underlying a Poisson algebra \((A,\lbrace\cdot,\cdot\rbrace)\) in such a way that the elements of the ground ring \(R\) act as scalar operators, i.e. by the usual multiplication; here the class \([\pi_{\lbrace\cdot,\cdot\rbrace}]\) plays a crucial role. This is motivated
by geometric quantization theory I. Segal [84], Kostant [48], Souriau [93], see also Woodhouse [108] and the literature there. The problem of quantizing Poisson algebras which are not associated with a symplectic manifold really arises in physics, see e.g. Gotay [32], Śniatycki-Weinstein [92], and Section 5 below. In these two papers certain singular systems are treated in analogy with the symplectic case. Our approach pushes the analogy further. Below we shall show that the usual prequantization construction carries over to arbitrary Poisson algebras, with the roles of the second integral cohomology group and the symplectic class being played by the Picard group and the Poisson class introduced in the present paper, respectively. Thus our approach offers tools to handle singular systems with non-trivial Poisson class. We only mention at this stage that, for any real Poisson algebra \((A, \{\cdot, \cdot\})\), the usual notions of a polarization (of a symplectic structure) and of quantizability can be rephrased and generalized in terms of the \((\mathbb{R}, A)\)-Lie algebra structure on \(D_\{\cdot, \cdot\}\) or \(D^\text{geo}_\{\cdot, \cdot\}\) (as appropriate); see Sections 4 and 5 for details. For example, when \(A\) is the Poisson algebra of smooth functions on a real Poisson manifold \((N, \{\cdot, \cdot\})\) and when \(H_1, \ldots, H_n\) are smooth functions on \(N\) that Poisson commute pairwise, their differentials \(dH_1, \ldots, dH_n\) generate a sub \((\mathbb{R}, A)\)-Lie algebra of \(D^\text{geo}_\{\cdot, \cdot\}\) that is isotropic with respect to \(\pi^\text{geo}_\{\cdot, \cdot\}\). A special case is that of a symplectic \(2n\)-dimensional manifold \((N, \sigma)\) with \(n\) “independent integrals of motion”. We believe that our description in terms of differentials is somewhat closer to the old idea of separation of variables than the usual description in terms of Hamiltonian vector fields. The obvious advantage of our description is its applicability to situations where arguments involving a symplectic structure are not available. For illustration, we apply our methods to a relativistic particle with zero rest mass and spin zero in Minkowski space \(Q\) in Section 5. The resulting Poisson algebra arises from what is called Poisson reduction of the cotangent bundle \(T^*Q\) with respect to a singular constraint in Śniatycki-Weinstein [92], and the underlying meaningful phase space has a singularity; the Poisson algebra of this system is not associated with a symplectic manifold.

In view of its complete generality, we believe that our approach to Poisson structures will prove useful for other singular systems and for infinite dimensional geometrical theories of the kind that have recently arisen in physics, cf. e.g. Marsden [70] or Chernoff and Marsden [15]. Also our approach may well apply to Poisson algebras arising in different ways, cf. Berger [7]. Furthermore, if one wants to investigate local questions more thoroughly, our approach can be reworked in the language of sheaves, cf. Berger [7], Conn [18], [19], and Kamber-Tondeur [43]. It is likely that our methods can then be extended to yield among others a quantization procedure for symplectic varieties including singular ones; this has recently become an interesting topic of research in view of Witten’s [107] topological quantum field theory. We do not pursue this here and likewise we leave aside any analytical considerations – which may be delicate – since this would only add unnecessary complications to the formal aspects we are about to study.

We include a rather long bibliography. We all wish to believe that we are forerunners but it is also important not to lose contact with the past.

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of a relativistic particle with zero rest mass and spin zero, and to K. Mackenzie for
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1. \((R,A)\)-Lie algebras

Let \(R\) be a commutative ring, fixed throughout; the unadorned tensor product symbol
\(\otimes\) will always refer to the tensor product over \(R\). Recall that a Lie algebra \((L,[\cdot,\cdot])\)
over \(R\) consists of an \(R\)-module \(L\) and a pairing \([\cdot,\cdot]:L \otimes L \to L\) of \(R\)-modules,
called a Lie bracket, which satisfies the relations of antisymmetry and Jacobi identity.

For \(x,y \in L\), as usual, we also write \((ad(x))(y) = [x,y]\). Given two Lie algebras \(L\) and \(L'\), a morphism \(\phi:L \to L'\) of Lie algebras over \(R\) is the obvious thing, i.e. it is a
morphisms of \(R\)-modules which is compatible with the Lie brackets.

Let \(A\) be an algebra over \(R\), not necessarily with 1. Recall that a derivation of \(A\)
(over \(R\)) is a morphism \(\delta:A \to A\) of \(R\)-modules so that \(\delta(ab) = (\delta(a))b + a\delta(b)\).

It is well known that the \(R\)-module \(\text{Der}(A)\) of derivations of \(A\), viewed as a submodule
of \(\text{Hom}_R(A,A)\), and with bracket \([\cdot,\cdot]\) given by \([\alpha,\beta](a) = \alpha(\beta(a)) - \beta(\alpha(a))\), where
\(\alpha,\beta \in L\), \(a \in A\), is a Lie algebra over \(R\). If \(L\) is a Lie algebra over \(R\), as usual, an
action of \(L\) on \(A\) is a morphism \(\omega:L \to \text{Der}(A)\) of Lie algebras over \(R\); henceforth
we shall write \((\omega(\alpha))(a) = \alpha(a), \alpha \in L, a \in A\). Given two algebras \(A\) and \(A'\)
over \(R\), two Lie algebras \(L\) and \(L'\) over \(R\), and actions of \(L\) and \(L'\) on \(A\) and \(A'\),
respectively, a morphism \((\phi,\psi):(A,L) \to (A',L')\) (of actions) is the obvious thing,
i.e. it consists of a morphism \(\phi:A \to A'\) of \(R\)-algebras and a morphism \(\psi:L \to L'\)
of Lie algebras over \(R\) so that, for every \(a \in A\), \(\alpha \in L\), \(\phi(\alpha(a)) = (\psi(\alpha))(\phi(a))\).

For an algebra \(U\) over \(R\), the associated Lie algebra over \(R\), written \(LU\) or, with
an abuse of notation, just \(U\), has the same underlying \(R\)-module as \(U\), while for
\(u,v \in U\), as usual the bracket \([u,v]\) is given by \([u,v] = uv - vu\). If \(L\) is a Lie algebra
over \(R\) and if \(M\) is an \(R\)-module, as usual, an action of \(L\) on \(M\) is a morphism
\(\omega:L \to L\text{End}_R(M)\) of Lie algebras over \(R\), and \(M\) is then said to be a \((left)\)
\(L\)-module; henceforth we shall write \((\omega(\alpha))(x) = \alpha(x), \alpha \in L, x \in M\). The precise
definition of the concept of a morphism of such structures is clear and is left to the
reader.

Let now \(A\) be a commutative \(R\)-algebra, not necessarily with 1. Let \(L\) be a Lie algebra
over \(R\), let \(\mu:A \otimes A \to L\) be a structure of a left \(A\)-module on \(L\) – as usual
we shall write \(\mu(\alpha \otimes \alpha) = a\alpha - \alpha a\) and let \(\omega:L \to \text{Der}(A)\) be an action of \(L\) on \(A\). As
in Rinehart [80] we shall refer to \(L\) as an \((R,A)\)-Lie algebra, provided

\[
(1.1.a) \quad (a \alpha)(b) = a (\alpha(b)), \quad \alpha \in L, a,b \in A, \\
(1.1.b) \quad [\alpha, \alpha \beta] = \alpha [\alpha, \beta] + \alpha(\alpha) \beta, \quad \alpha, \beta \in L, a \in A.
\]

For an \(R\)-algebra \(A\) and an \((R,A)\)-Lie algebra \(L\), we shall occasionally refer to the pair
\((A,L)\) as a Lie-Rinehart algebra. Given two Lie-Rinehart algebras \((A,L)\) and
\((A',L')\), a morphism \((\phi,\psi):(A,L) \to (A',L')\) of Lie-Rinehart algebras is a morphism
of actions so that, furthermore, \(\psi:L \to L'\) is a morphism of \(A\)-modules where \(A\) acts
on \(L'\) via \(\phi\). It is clear that, with this notion of morphism, Lie-Rinehart algebras
constitute a category. A useful example of a morphism of Lie-Rinehart algebras will
be given in (3.8.4) below.

An example of an \((R,A)\)-Lie algebra is the \(R\)-module \(\text{Der}(A)\) of derivations of a
commutative algebra \(A\) with the obvious \(A\)-module structure; here the commutativity
of $A$ is crucial. Indeed, a little thought reveals that for a non-commutative algebra $U$ over $R$ the $R$-module of derivations of $U$ does not inherit a structure of a $U$-module. Another example is classical: Let $R$ be the real numbers, let $N$ be a smooth manifold, let $A$ be the algebra of smooth functions on $N$, and let $L$ be the Lie algebra of smooth vector fields on $N$; then, with the obvious structures, $L$ is an $(R, A)$-Lie algebra. This works for an arbitrary smooth Banach manifold, modelled on a Banach space, see e. g. SCHWARTZ [83], LANG [56]. We note that in general for a smooth Banach manifold a derivation need not even locally come from a vector field, see e. g. p. 105 of SCHWARTZ [83]. Similar examples arise in the analytic and algebraic setting. We leave the details to the reader.

Given an $(R, A)$-Lie algebra $L$ and an $R$-module $M$ having the structures of a left $A$-module and that of a left $L$-module $\omega: L \to \text{End}(M)$, we shall refer to $M$ as an $$(A, L)$$-module, provided the actions are compatible, i. e. for $\alpha \in L$, $a \in A$, $m \in M$,

(1.2.a) \[ (a \alpha)(m) = a(\alpha(m)) , \]

(1.2.b) \[ \alpha(a m) = a \alpha(m) + \alpha(a)m. \]

For example, let $A$ be the algebra of smooth functions on a smooth finite dimensional manifold $N$, let $L$ be the Lie algebra of smooth vector fields on $N$, let $\xi: E \to N$ be a smooth vector bundle on $N$, and let $M$ be the $A$-module of smooth sections of $\xi$; then an $(A, L)$-module structure on $M$ is precisely a (linear) connection on $\xi$ with zero curvature. We shall elaborate on this in the next Section.

Recall that, for a Lie algebra $L$ over $R$ and an $R$-module $M$ which is also an $L$-module, the $R$-multilinear functions from $L$ into $M$ with the Cartan-Chevalley-Eilenberg differential $d$ given by

(1.3) \[ (df)(\alpha_1, \ldots, \alpha_n) = (-1)^n \sum_{i=1}^{n} (-1)^{i-1} \alpha_i(f(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n)) \]

\[ + (-1)^n \sum_{j<k} (-1)^{j+k} f([\alpha_j, \alpha_k], \alpha_1, \ldots, \widehat{\alpha_j}, \ldots, \widehat{\alpha_k}, \ldots, \alpha_n) \]

constitute a chain complex $\text{Alt}_R(L, M)$ where as usual ‘$\widehat{\ }$’ indicates omission of the corresponding term. If $L$ is projective as an $R$-module this chain complex computes the usual Lie algebra cohomology $H^*(L, M)$; we recall that the latter is defined as usual by $H^*(L, M) = \text{Ext}^*_U(L, R)(R, M)$ where $U(L, R)$ denotes the corresponding universal algebra (= the universal enveloping algebra if $L$ is projective as an $R$-module). The sign $(-1)^n$ in (1.3) has been introduced according to the usual Eilenberg-Koszul convention in differential homological algebra for consistency with what is said in a subsequent paper [40 II]; in the classical approach such a sign does not occur.

Likewise, given $L$-modules $M'$ and $M''$, the usual formula

(1.4.1) \[ \alpha(x \otimes y) = \alpha(x) \otimes y + x \otimes \alpha(y), \quad \alpha \in L, \ x \in M', \ y \in M'' , \]

endows the tensor product $M' \otimes M''$ with a structure of an $L$-module; if $M$ is another $L$-module, a pairing $\mu: M' \otimes M'' \to M$ of $R$-modules which is a morphism of $L$-modules (with respect to (1.4.1)) will be said to be a a pairing of $L$-modules.
Given such a pairing \( \mu \) of \( L \)-modules, the standard shuffle multiplication of alternating maps given by

\[
(\alpha \wedge \beta)(x_1, \ldots, x_{p+q}) = \sum_{\sigma} \text{sign}(\sigma)\mu(\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)})) \otimes \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)})
\]

induces a pairing

\[
(1.5) \quad \text{Alt}_R(L, M') \otimes \text{Alt}_R(L, M'') \longrightarrow \text{Alt}_R(L, M)
\]
of chain complexes which is associative in the obvious sense; here \( \sigma \) runs through \((p, q)\)-shuffles and \( \text{sign}(\sigma) \) refers to the sign of \( \sigma \). In particular, for an \( R \)-algebra \( B \), the chain complex \( \text{Alt}_R(L, B) \) inherits a structure of a differential graded algebra which is graded commutative if \( B \) is commutative, and this structure induces a ring structure on cohomology as usual.

As before, let \( A \) be a commutative \( R \)-algebra and let \( L \) be an \((R, A)\)-Lie algebra. It is not hard to see that, as observed first by Palais [77], for an \((A, L)\)-module \( M \), the differential on \( \text{Alt}_R(L, M) \) passes to an \( R \)-linear differential on the graded \( A \)-submodule \( \text{Alt}_A(L, M) \) of \( A \)-multilinear functions; we note that the differential will not be \( A \)-linear unless \( L \) acts trivially on \( A \). Before we proceed further we mention that a distinction between graded \( A \)-algebras and differential graded \( R \)-algebras will persist throughout. We shall carry out most constructions over \( A \); however, in view of the non-trivial action of \( L \) on \( A \), most resulting differential graded algebras will be over the ground ring \( R \) only.

Given \((A, L)\)-modules \( M' \) and \( M'' \), a little thought reveals that the formula \((1.4.1)\) endows the tensor product \( M' \otimes_A M'' \) with a structure of an \((A, L)\)-module; we refer to \( M' \otimes_A M'' \) with this structure as the tensor product of \( M' \) and \( M'' \) in the category of \((A, L)\)-modules. Given \((A, L)\)-modules \( M, M' \), and \( M'' \), a pairing \( \mu_A: M' \otimes_A M'' \longrightarrow M \) of \( A \)-modules which is compatible with the \( L \)-structures will be said to be a pairing of \((A, L)\)-modules. Given a pairing \( \mu_A: M' \otimes_A M'' \longrightarrow M \) of \((A, L)\)-modules, let

\[
\mu = \mu_{A, \text{pr}}: M' \otimes_R M'' \longrightarrow M'
\]

be the indicated pairing of \( L \)-modules where ‘\( \text{pr} \)’ refers to the obvious projection map; inspection shows that under these circumstances the corresponding pairing \((1.5)\) with respect to the present \( \mu \) induces a pairing

\[
(1.5') \quad \text{Alt}_A(L, M') \otimes_R \text{Alt}_A(L, M'') \longrightarrow \text{Alt}_A(L, M)
\]
of chain complexes over \( R \). In particular, \( \text{Alt}_A(L, A) \) inherits a structure of a differential graded commutative algebra over the ground ring \( R \) (but not over \( A \) unless \( L \) acts trivially on \( A \)). We now elaborate on a conceptual explanation of these facts which is due to Rinehart [80].

Given an \((R, A)\)-Lie algebra \( L \), its universal object \((U(A, L), \iota_L, \iota_A)\) is an \( R \)-algebra \( U(A, L) \) together with a morphism \( \iota_A: A \longrightarrow U(A, L) \) of \( R \)-algebras and a morphism \( \iota_L: L \longrightarrow U(A, L) \) of Lie algebras over \( R \) having the properties

\[
\iota_A(\alpha)\iota_L(\alpha) = \iota_L(\alpha)\iota_A(\alpha), \quad \iota_L(\alpha)\iota_A(\alpha) - \iota_A(\alpha)\iota_L(\alpha) = \iota_A(\alpha(\alpha)),
\]

and \((U(A, L), \iota_L, \iota_A)\) is universal among triples \((B, \phi_L, \phi_A)\) having these properties. More precisely:
1.6. Given

(i) another $\mathbb{R}$-algebra $B$, viewed at the same time as a Lie algebra over $\mathbb{R}$,
(ii) a morphism $\phi_L: L \to B$ of Lie algebras over $\mathbb{R}$, and
(iii) a morphism $\phi_A: A \to B$ of $\mathbb{R}$-algebras,

so that, for $\alpha \in L, a \in A$,

(1.6.1) $\phi_A(a)\phi_L(\alpha) = \phi_L(a\alpha),$

(1.6.2) $\phi_L(\alpha)\phi_A(a) - \phi_A(a)\phi_L(\alpha) = \phi_A(\alpha(a)),$

there is a unique morphism $\Phi: U(A,L) \to B$ of $\mathbb{R}$-algebras so that $\Phi \iota_A = \phi_B$ and $\Phi \iota_L = \phi_L$.

For example, when $A$ is the algebra of smooth functions on a smooth manifold $N$ and $L$ the Lie algebra of smooth vector fields on $N$, then $U(A,L)$ is the algebra of (globally defined) differential operators on $N$.

The universal property is not spelled out in Rinehart [80]. A universal property equivalent to the above one is given in Malliavin [69] where it is attributed to Feld’man [26]. An explicit construction for the $\mathbb{R}$-algebra $U(A,L)$ is given in Rinehart [80]. For convenience, we now give a new alternate construction which employs the Massey-Peterson [73] algebra. Let $(U(R,L), \iota_L, \iota_R)$ be the usual universal algebra for $L$ over $R$, for $\alpha \in L$, write $\overline{\alpha} = \iota_L(\alpha)$, and consider the algebra

(1.7) $A \odot U(R,L) = (A \odot_R U(R,L), \mu),$

whose underlying left $A$-module is the one induced from $U(R,L)$ as indicated, and whose multiplication on the generators is defined by

$$a \overline{\alpha} = a \otimes \overline{\alpha}, \quad \overline{\alpha} a = a \overline{\alpha} + \alpha(a), \quad \alpha \in L, a \in A.$$ 

The universal property of $(U(R,L), \iota_L, \iota_R)$ implies that this is well defined, i.e. that $\alpha(a)$ depends only on $\overline{\alpha}$ and $a$. Furthermore, the Jacobi identity implies that $\mu$ is associative, i.e. that $A \odot U(R,L)$ is indeed an $\mathbb{R}$-algebra. However, there is a more conceptual way to understand this algebra structure: It is clear that the diagonal map $\Delta: L \to L \oplus L$ is a morphism of Lie algebras. As is well known, the universal property of $(U(R,L), \iota_L, \iota_R)$ yields a multiplicative extension to a diagonal map $\Delta$ for $U(R,L)$ which endows the latter with the structure of a cocommutative Hopf algebra. Furthermore, since on the generators the diagonal map $\Delta$ is given by $\Delta(\overline{\alpha}) = \overline{\alpha} \otimes 1 + 1 \otimes \overline{\alpha}$ as a Hopf algebra, $U(R,L)$ is primitively generated. With the obvious structure, $A \otimes A$ is an $(U(R,L) \otimes U(R,L))$-module, and the diagonal map on $U(R,L)$ induces on $A \otimes A$ a structure of a left $U(R,L)$-module; the above requirement that $L$ acts on $A$ by derivations means precisely that $A$ is an algebra over $U(R,L)$, i.e. that the structure map $\mu: A \otimes A \to A$ is a morphism of left $(U(R,L))$-modules. The algebra $A \odot U(R,L)$ is the corresponding Massey-Peterson algebra [73]; its structure map $\mu$ is given by

$$(a \otimes u)(b \otimes v) = ab \otimes uv + \sum a u'_i(b) \otimes u''_i v, \quad a, b \in A, u, v \in U(R,L),$$
where $\Delta(u) = \sum u_i^I \otimes u_i^I \in U(R, L) \otimes U(R, L)$. Notice that the Hopf algebra structure of $U(R, L)$ organizes the requisite combinatorics needed to prove associativity of the algebra $A \circ U(R, L)$. We note that the algebra $A \circ U(R, L)$ together with the obvious morphisms $i_A^R: A \longrightarrow A \circ U(R, L)$ of $R$-algebras and $i_L^R: L \longrightarrow A \circ U(R, L)$ of Lie algebras over $R$ is what is called the algebra of differential operators of the representation $\omega: L \longrightarrow \text{Der}(A)$ of $L$ in $A$ on p.175 of JACOBSON [42]. Our construction differs from the one in [42].

To complete the construction of the universal object, let $J$ be the right ideal in $A \circ U(R, L)$ generated by the elements $ab \otimes \alpha - a \otimes b \alpha$, $a,b \in A$, $\alpha \in L$, where the term $b \alpha$ in $a \otimes b \alpha$ refers to the left $A$-module structure on $L$. A straightforward calculation shows that for $a,b,c \in A$, $\alpha,\beta \in L$,

$$(c \otimes \beta)(ab \otimes \alpha - a \otimes b \alpha) = (cab \otimes \beta \alpha - c \otimes ab \beta \alpha) + ca\beta(b) \otimes \alpha - c \otimes a\beta(b)\alpha,$$

whence $J$ is a two-sided ideal in $A \circ U(R, L)$. Let

$$(1.8.1) \quad U(A, L) = (A \circ U(R, L))/J,$$

and let $i_A$ and $i_L$ be the obvious morphisms. By construction it is then clear that $(U(A, L), i_L, i_A)$ has the universal property (1.6).

We mention in passing that when $A = R$ with trivial $L$-action the object $(U(R, L), i_L, i_A)$ is the usual universal algebra of $L$ (over $R$).

It is obvious that there is a one-one correspondence between $(A, L)$-modules and $U(A, L)$-modules; this correspondence is in fact an equivalence of categories. In particular, the obvious $(A, L)$-module structure on $A$ mentioned above induces on $A$ that of a left $U(A, L)$-module; the corresponding structure map is given by

$$(1.8.2) \quad \mu: U(A, L) \otimes A \longrightarrow A, \quad \mu(\alpha \otimes a) = \alpha(a),$$

where $\alpha \in L$, $a \in A$. In particular, let $\varepsilon: U(A, L) \longrightarrow A$ be the obvious morphism given by

$$(1.8.3) \quad \varepsilon(a) = a, \quad \varepsilon(aa) = 0, \quad \varepsilon(aa) = \alpha(a).$$

It is manifestly a morphism of left $U(A, L)$-modules, but not one of algebras unless $L$ acts trivially on $A$, and its kernel is the left ideal in $U(A, L)$ generated by $L$. In particular, the composite $\varepsilon i_A$ is the identity map of $A$ whence $i_A$ is injective. Henceforth we shall identify $A$ with its image in $U(A, L)$, and we shall not distinguish in notation between the elements of $A$ and their images in $U(A, L)$. Furthermore, it is clear that given two Lie-Rinehart algebras $(A, L)$ and $(A', L')$, a morphism $(\phi, \psi): (A, L) \longrightarrow (A', L')$ of Lie-Rinehart algebras induces a morphism $U(\phi, \psi): U(A, L) \longrightarrow U(A', L')$ of $R$-algebras. Hence $U(\cdot, \cdot)$ is a functor from the category of Lie-Rinehart algebras into the category of $R$-algebras. We mention in passing that, in view of the universal property of $U(A, L)$, $A$ also inherits a structure of a right $U(A, L)$-module given by

$$(1.8.4) \quad a \cdot \alpha = -\alpha(a), \quad a \in A, \alpha \in L.$$
REMARK after publication. This construction does not work; in general there is no such right module $U(A,L)$-module structure. This does not cause any difficulty, though. The general right module $U(A,L)$-module structure is never used in the paper. The special case where the right $U(A,L)$-module structure arises from a Poisson structure as explained between (3.8.6) and (3.8.7) below is correct, that is, the construction given there exhibits a right $U(A,L)$-module structure.

The universal algebra $U(A,L)$ admits an obvious a filtered algebra structure, cf. Rinehart [80], with $U_{-1}(A,L) = 0$ and $U_p(A,L)$ the left $A$-submodule of $U(A,L)$ generated by products of at most $p$ elements of the image $\mathcal{T}$ of $L$ in $U(A,L)$; further, for $a \in A$ and $z \in U_p(A,L)$ we have $az - za \in U_{p-1}(A,L)$ whence the inherited left and right $A$-module structures on the associated graded object $E^0(U(A,L))$ coincide, and $E^0(U(A,L))$ inherits a structure of a commutative graded $A$-algebra. The Poincaré-Birkhoff-Witt Theorem for $U(A,L)$ then assumes the following form, cf. (3.1) of Rinehart [80], where $S_A[L]$ denotes the symmetric $A$-algebra on $L$.

**Theorem 1.9 [Rinehart].** For an $(R,A)$-Lie algebra $L$ which is projective as an $A$-module, the canonical $A$-epimorphism $S_A[L] \rightarrow E^0(U(A,L))$ is an isomorphism of $A$-algebras.

**Corollary 1.10.** For an $(R,A)$-Lie algebra $L$ which is projective as an $A$-module, the morphism $\iota_L: L \rightarrow U(A,L)$ is injective.

The usual construction of the Koszul complex computing Lie algebra cohomology (see e. g. Chevalley-Eilenberg [16]) carries over as well: Let $\Lambda_A(sL)$ be the exterior Hopf algebra over $A$ on the suspension $sL$ of $L$, where “suspension” means that $sL$ is $L$ except that its elements are regraded by 1. We shall write typical elements in the form

\[(1.11) \quad \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle = (s\alpha_1)(s\alpha_2)\ldots s(\alpha_n) \in \Lambda_A(sL), \quad \alpha_1, \alpha_2, \ldots, \alpha_n \in L.\]

For $u \in U(A,L)$ and $\alpha_i \in L$ let

\[d(u \otimes \langle \alpha_1, \ldots, \alpha_n \rangle) = \sum_{i=1}^{n} (-1)^{(i-1)}u\alpha_i \otimes \langle \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n \rangle + \sum_{j < k} (-1)^{(j+k)}u \otimes \langle [\alpha_j, \alpha_k], \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \hat{\alpha}_k, \ldots, \alpha_n \rangle,\]

where $A$ acts on the right of $U(A,L)$ by means of the embedding $\iota_A: A \rightarrow U(A,L)$. This yields an operator $d: U(A,L) \otimes_A \Lambda_A(sL) \rightarrow U(A,L) \otimes_A \Lambda_A(sL)$. We shall refer to

\[K(A,L) = (U(A,L) \otimes_A \Lambda_A(sL), d),\]

as the *Koszul complex* for $(A,L)$. It is proved in Rinehart [80] that $d$ is an $U(A,L)$-linear differential whence $K(A,L)$ is indeed a chain complex. It is manifest that the Koszul complex is functorial in $(A,L)$. Given a morphism $(\phi, \psi): (A,L) \rightarrow (A',L')$ of Lie-Rinehart algebras, we shall denote the induced morphism by $K(\phi, \psi): K(A,L) \rightarrow K(A',L')$. It is a morphism of $U(A,L)$-modules where $K(A',L')$ is viewed as an $U(A,L)$-module via $U(\phi, \psi)$. Furthermore, when $L$
is projective or free as a left $A$-module, $K(A,L)$ is a projective or free resolution of $A$ in the category of left $U(A,L)$-modules according as $L$ is a projective or free left $A$-module; details may be found in Rinehart [80].

For an $(R,A)$-Lie algebra $L$ and an $(A,L)$-module $M$, as in Rinehart [80], we shall write $H^{*}_{A}(L,M) = \text{Ext}_{U(A,L)}^{*}(A,M)$, and we shall refer to this as the $A$-Lie algebra cohomology of $L$ with coefficients in $M$. In the guise of standard homological algebra we view $\text{Ext}_{U(A,L)}^{*}(A,M)$ as the primary object; it is always defined, whether or not $L$ is projective as an $A$-module. It is clear that the above establishes the following.

**Proposition 1.14.** Let $L$ be an $(R,A)$-Lie algebra, and let $M$ be an $(A,L)$-module. Then the $A$-multilinear functions $\text{Alt}_{A}(L,M)$ form a subcomplex of $\text{Alt}_{R}(L,M)$, and, if the $L$ underlying $A$-module is projective, $\text{Alt}_{A}(L,M)$ computes

$$H^{*}_{A}(L,M)(= \text{Ext}_{U(L,A)}^{*}(A,M)).$$

Furthermore, for $M = A$, the usual shuffle product (1.4) induces a structure of a differential graded commutative algebra on $\text{Alt}_{A}(L,A)$ which induces that of a graded commutative algebra on $H^{*}_{A}(L,A)$.

For example, when $A$ is the algebra of smooth functions on a smooth manifold $N$ and $L$ the Lie algebra of smooth vector fields on $N$, then $\text{Hom}_{U(A,L)}(K(L,A),A)$ is the usual de Rham complex of $N$ and the de Rham cohomology of $N$ is $\text{Ext}_{U(A,L)}^{*}(A,A)$ over the algebra $U(A,L)$ of differential operators on $N$. Likewise, for a Lie algebra $L$ over $R$ acting trivially on $R$ and an $L$-module $M$, the object $K(R,L)$ is the usual Koszul complex; in particular, when $L$ is projective as an $R$-module, $K(R,L)$ is the usual Koszul resolution computing Lie algebra homology and cohomology.

A partial converse to (1.14) is given by (1.15) below, whose proof is routine and left to the reader:

**Theorem 1.15.** Let $A$ be a commutative $R$-algebra, let $L$ be a left $A$-module, let $\omega: L \rightarrow \text{End}_{R}(A)$ be a morphism of $R$-modules, and let $[\cdot, \cdot]: L \otimes_{R} L \rightarrow L$ be a skew symmetric pairing as indicated. For $\alpha \in L$ and $a \in A$, write $\alpha(a) = (\omega(\alpha))(a)$, define an operator $d$ on the graded commutative algebra $\text{Alt}_{R}(L,A)$ of $R$-multilinear alternating functions by means of (1.3), and suppose that $d$ passes to an operator on the subalgebra $\text{Alt}_{A}(L,A) \subseteq \text{Alt}_{R}(L,A)$ which endows $\text{Alt}_{A}(L,A)$ with a structure of a differential graded commutative algebra. Then $\omega$ factors through $\text{Der}(A)$ so that

\begin{align*}
(1.15.1) \quad (a \alpha)(b) &= a(\alpha(b)), \quad \alpha \in L, \ a, b \in A, \\
(1.15.2) \quad \alpha(a_2(a)) - \alpha_2((\alpha_1)(a)) &= [\alpha_1, \alpha_2](a), \ \alpha_1, \alpha_2 \in L, a \in A.
\end{align*}

Furthermore, when $L$ is projective as an $A$-module, $\omega$ and $[\cdot, \cdot]$ yield an $(R,A)$-Lie algebra structure on $L$.

The next aim is to introduce what will be called “induced structures”; their significance for the present paper will emerge in (3.18) below. I am indebted to K. Mackenzie for suggesting the following description of induced structures which replaces a more clumsy one given in an earlier version of the paper.

Suppose the following data are given:
\(-\) \(R\)-algebras \(A, A',\)
- an \((R, A)\)-Lie algebra \(L,\) with structure maps \(\omega : L \to \text{Der}(A)\) and 
\([\cdot , \cdot] : L \otimes_R L \to L,\)
- an action \(\tilde{\omega} : L \to \text{Der}(A')\) of \(L\) on \(A'\) (but \(L\) is not assumed to admit an \(A'\)-module structure),
- a morphism \(\phi : A \to A'\) of algebras which is also a morphism of \(L\)-modules.

Write \(L' = A' \otimes_A L;\) it is an \(A'\)-module in an obvious fashion. Our aim is to endow \(L'\) with a structure of an \((R, A')\)-Lie algebra. To this end, we consider the obvious pairings

\[A' \otimes_R L \otimes_R A' \otimes_R L \to A' \otimes_A L,\]

given by \(u \otimes \alpha \otimes v \otimes \beta \mapsto uv \otimes [\alpha, \beta] - (v \beta(u)) \otimes \alpha + (\alpha \alpha(v)) \otimes \beta,\) where \(u, v \in A',\) \(\alpha, \beta \in L,\)

and

\[A' \otimes_R L \otimes_R A' \to A',\quad u \otimes \alpha \otimes v \mapsto u \cdot \alpha(v),\quad u, v \in A', \alpha \in L.\]

**Proposition 1.16.** Under the above circumstances, suppose that, for every \(a \in A\) and for every \(\alpha \in L,\)

\[(\phi, \phi \otimes \text{Id}) : (A, L) \to (A', L')\]

is a morphism of \(\text{Lie-Rinehart algebras.}\)

**Proof.** This comes down to routine checking and is therefore left to the reader. \(\square\)

In the situation of this Proposition we shall say that \((L', [\cdot , \cdot], \omega')\) is *induced from \(\phi;* often we shall then write \(L'\) rather than \((L', [\cdot , \cdot], \omega').\) Moreover, we then have the two morphisms

\[(\text{Id}, \phi_*) : \text{Alt}_A(L, A) \to \text{Alt}_A(L, A')\]

and

\[(\phi \otimes \text{Id})^* : \text{Alt}_A'(L', A') \to \text{Alt}_A(L, A')\]

doing differential graded algebras. However, the latter is a standard adjointness isomorphism and hence \(\phi\) induces a morphism

\[\text{Alt}_A(L, A) \to \text{Alt}_A'(A' \otimes_A L, A')\]

of differential graded algebras.

**Example 1.16.4.** Let \(A\) be an algebra over \(R,\) let \(g\) be a Lie algebra over \(R,\) and let \(\omega : g \to \text{Der}(A)\) be an action of \(g\) on \(A.\) Then with the obvious change in notation, (1.16.1) and (1.16.2) endow \(A \otimes g\) with a structure of an \((R, A)\)-Lie algebra. It is referred to as a *crossed product* (produit croisé) in Malliavin [69]. We shall have to say more about this \((R, A)\)-Lie algebra in (3.18) below.

A geometric analogue of the above notion of induced structure may be found in Higgins-Mackenzie [38].
2. Extensions, connection, curvature

In this Section we relate extensions of \((R,A)\)-Lie algebras with formal concepts of connection and curvature. This extends earlier work of Atiyah [5] and others. Historical comments will be given at the end of this Section.

Let \(A\) be an algebra, and let \(L', L, L''\) be \((R,A)\)-Lie algebras. Extending common notation, we refer to a short exact sequence

\[ e: 0 \to L' \to L \to L'' \to 0 \]

in the category of \((R,A)\)-Lie algebras as an extension of \((R,A)\)-Lie algebras; notice in particular that the Lie algebra \(L'\) necessarily acts trivially on \(A\). If also \(\bar{e}: 0 \to \bar{L}' \to \bar{L} \to \bar{L}'' \to 0\) is an extension of \((R,A)\)-Lie algebras, as usual, \(e\) and \(\bar{e}\) are said to be congruent, if there is a morphism \((\text{Id}, \cdot, \text{Id}): e \to \bar{e}\) of extensions of \((R,A)\)-Lie algebras.

An extension of the kind (2.1) may be represented by a 2-cocycle, provided the extension (2.1) splits in the category of \(A\)-modules, e. g. if \(L''\) is projective as an \(A\)-module. More precisely, let \(\omega: L'' \to L\) be a section of \(A\)-modules for the projection \(L \to L''\). We shall occasionally refer to \(\omega\) as an \(e\)-connection. Given an \(e\)-connection, define the corresponding \((e-)\) curvature \(\Omega: L'' \otimes_A L'' \to L'\) as the morphism \(\Omega\) of \(A\)-modules satisfying

\[ [\omega(\alpha), \omega(\beta)] = \omega[\alpha, \beta] + \Omega(\alpha, \beta) \]

for every \(\alpha, \beta \in L''\); a little thought reveals that \(\Omega\) is indeed well defined as an alternating \(A\)-bilinear 2-form on \(L''\) with values in \(L'\). Since in \(L\) the Jacobi identity holds, the morphism \(\Omega\) must satisfy the following 2-cocycle condition: The Lie algebra \(L\) acts on \(L''\) via the adjoint representation \(\text{ad}: L \to \text{End}(L'')\). Define an operator \(D^\omega\) on the complex \(\text{Alt}_A(L', L'')\) by means of the formal analogue of (1.3), i. e. by (2.3)

\[ (D^\omega f)(\alpha_1, \ldots, \alpha_n) = (-1)^n \sum_{i=1}^{n} (-1)^{i-1} \text{ad}(\omega(\alpha_i))(f(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)) \]

\[ + \ (-1)^n \sum_{j<k} (-1)^{j+k} f([\alpha_j, \alpha_k], \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \hat{\alpha}_k, \ldots, \alpha_n). \]

Then the Jacobi identity in \(L\) boils down to

\[ D^\omega(\Omega) = 0. \]

We refer to the latter as the generalized Bianchi identity.

We mention in passing that these concepts of \(e\)-connection and \(e\)-curvature generalize the notions of principal connection and curvature; details will be explained in a follow up paper [41 I].

As one would expect, the above 2-cocycle is unique up to a coboundary. More precisely, let \(\omega': L'' \to L\) be another section of \(A\)-modules, and let \(\Omega': L'' \otimes L'' \to L'\) be the corresponding morphism of \(A\)-modules so that

\[ [\omega'(\alpha), \omega'(\beta)] = \omega'[\alpha, \beta] + \Omega'(\alpha, \beta). \]
Then $\omega - \omega'$ factors through an $A$-linear morphism $u: L'' \to L'$ and, if we view $u$ as an element of $\text{Alt}_A(L'', L')$, we obtain $\Omega' - \Omega$ as the 1-coboundary of $u$ in the sense that, for $\alpha, \beta \in L$,

$$\tag{2.5} (\Omega' - \Omega)(\alpha, \beta) = \text{ad}(\omega(\alpha))(u(\beta)) + \text{ad}(\omega'(\beta))(u(\alpha)) - u([\alpha, \beta]).$$

It is well known that when $L'$ is abelian which under the present circumstances means that $L'$ is just an $A$-module with trivial Lie bracket, the adjoint action of $L$ on $L'$ induces an action $\rho: L'' \to \text{End}(L')$ of $L''$ on $L'$. Inspection shows that this then endows $L'$ in fact with the structure of an $(A, L'')$-module. Furthermore, the cocycle condition then boils down to $d\Omega = 0 \in \text{Alt}_A(L'', L')$ with respect to the action $\rho$ – which is now independent of a section of $A$-modules of the kind $L'' \to L$, and the classical argument in Eilenberg-Mac Lane cohomology, cf. e. g. VII.3 in Hilton-Stammbach [39], may easily be extended to a proof of the following.

**Theorem 2.6.** Let $L'$ and $L''$ be $(R, A)$-Lie algebras, assume that $L'$ is abelian, and let $\rho: L'' \to \text{End}(L')$ be a structure of an $(A, L'')$-module on $L'$. Then the assignment of a 2-cocycle $\Omega \in \text{Alt}_A(L'', L')$ to its extension (2.1) of $(A, L'')$-Lie algebras yields a bijection between the congruence classes of extensions of $L'$ by $L''$ whose underlying extension of $A$-modules split and the classes in $H^2(\text{Alt}_A(L'', L')) \ (= H^2_A(L'', L')$ if $L''$ is $A$-projective).

In the case where $L'$ is non-abelian, $\Omega$ is a non-abelian 2-cocycle in a suitable sense and hence it does not lead to a cohomology class in the above sense. We shall explain elsewhere a formal analogue of the classical Chern-Weil construction [41 I]; (2.6) will then be a special case thereof.

We now relate the above material to the classical notions of connection and curvature. We pursue this here only as far as needed for the study of Poisson algebras in the next Section. We shall elaborate further on these ideas in [41].

Let $L$ be an $(R, A)$-Lie algebra and let $M$ be a left $A$-module. Then an $R$-linear morphism $\omega: L \to \text{End}(M)$ will be referred to as a (linear) $L$-connection on $M$, if for $\alpha \in L$, $a \in A$, $m \in M$,

$$\tag{2.7.a} (a \alpha)(m) = a(\alpha(m)), \quad (2.7.b) \quad \alpha(a m) = a \alpha(m) + \alpha(a) m.$$

Here is another way to say this: Write $D^\omega: M \to \text{Hom}(L, M)$ for the indicated adjoint of $\omega$, so that

$$\tag{2.7.c} D^\omega_\alpha(m) = (\omega(\alpha))(m), \quad \alpha \in L, \ m \in M;$$

then instead of (2.7.a) we may require that $D^\omega$ factors through $\text{Hom}_A(L, M)$, i. e. that it may be displayed as

$$\tag{2.7.a'} D^\omega: M \to \text{Hom}_A(L, M),$$

and the rule (2.7.b) then assumes the formally well known form

$$\tag{2.7.b'} D^\omega_\alpha(am) = (\alpha(a))m + a D^\omega_\alpha(m).$$
Given a linear $L$-connection $\omega: L \to \text{End}(M)$ on $M$, as usual, extend $D\omega$ to the operator of covariant derivative on $\text{Alt}_A(L, M)$ again by means of the formal analogue of (1.3), i.e. by

\begin{equation}
(D\omega f)(\alpha_1, \ldots \alpha_n) = (-1)^n \sum_{i=1}^{n} (-1)^{(i-1)}(\omega(\alpha_i))(f(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n))
+ (-1)^n \sum_{j<k} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \ldots, \widehat{\alpha_j}, \ldots, \widehat{\alpha_k}, \ldots, \alpha_n),
\end{equation}

and define the curvature $\Omega: L \otimes_A L \to \text{End}(M)$ of $\omega$ as the adjoint of $D\omega D\omega$:

\begin{equation}
D\omega D\omega: M \to \text{Alt}_A^2(L, M).
\end{equation}

Explicitly, with $D = D\omega$, this is the formally well known formula

\[ \Omega(\alpha, \beta) = D\alpha D\beta - D\beta D\alpha - D_{[\alpha, \beta])}. \]

The standard argument shows that $\Omega$ is a “tensor”, i.e. that it is a 2-form with values in $\text{End}_A(M)$. Furthermore, we now literally have the Bianchi identity

\begin{equation}
D\omega(\Omega) = 0 \in \text{Alt}_A(L, \text{End}_A(M)).
\end{equation}

An $L$-connection will be said to be flat, if its curvature is zero. We note that when $A$ is the algebra over the reals $\mathbb{R}$ of smooth functions on a smooth finite dimensional manifold $N$, when $L$ is the $(\mathbb{R}, A)$-Lie algebra of smooth vector fields on $N$, and when $M$ is the $A$-module of smooth sections of a vector bundle over $N$, the above notions boil down to the usual ones.

Let $M$ be an $A$-module, and let $L$ be an $(R, A)$-Lie algebra. We now introduce an $(R, A)$-Lie algebra $\text{DO}(A, L, M)$ that acts on $M$ by the analogue of infinitesimal gauge transformations. I am indebted to A. Weinstein for asking whether there is such an object in general since it is well known to exist in the special case where $M$ is the module of sections of a vector bundle. The elements of $\text{DO}(A, L, M)$ may be viewed as acting as “differential operators” on $M$, whence the notation ‘DO’. A related (geometric) object for a smooth vector bundle $E$, denoted $\text{CDO}(E)$, is introduced on p. 103 of Mackenzie [64].

Consider the direct sum $\text{End}_R(M) \oplus L$, equipped with the obvious componentwise Lie algebra structure so that, for every $\beta, \beta' \in \text{End}_R(M)$ and every $\alpha, \alpha' \in L$,

\begin{equation}
[(\beta, \alpha), (\beta', \alpha')] = (\beta \beta' - \beta' \beta, [\alpha, \alpha']) \in \text{End}_R(M) \oplus L,
\end{equation}

and let $\text{DO}(A, L, M) \subseteq \text{End}_R(M) \oplus L$ be the $R$-submodule consisting of those pairs $(\beta, \alpha) \in \text{End}_R(M) \oplus L$ that satisfy

\begin{equation}
\beta(am) = (\alpha(a))m + a(\beta(m)), \quad a \in A, \ m \in M.
\end{equation}

Further, for $a \in A$, $m \in M$, and $\beta \in \text{End}_R(M)$, define $a\beta \in \text{End}_R(M)$ by

\begin{equation}
(a\beta)(m) = a(\beta(m)).
\end{equation}
Then a little thought reveals that, in view of (1.2.a) and the commutativity of $A$, for every $(\beta, \alpha) \in \DO(A, L, M)$ and $a, b \in A$, $m \in M$,

\[(2.11.4) \quad (b\beta)(am) = ((b\alpha)(a))(m) + a(b\beta)(m),\]

whence the rule

\[(2.11.5) \quad (b, \beta, \alpha) \mapsto (b\beta, b\alpha), \quad (\beta, \alpha) \in \DO(A, L, M), b \in A,\]

endows $\DO(A, L, M)$ with a structure of a (left) $A$-module. Moreover, an easy calculation shows that, for every $(\beta, \alpha)$, $(\beta', \alpha') \in \DO(A, L, M)$, and $a \in A$, $m \in M$,

\[(2.11.6) \quad (\beta\beta' - \beta'\beta)(am) = ([\alpha, \alpha'](a)m + a(\beta\beta' - \beta'\beta)(m)) \in M,\]

whence $\DO(A, L, M)$ inherits a structure of a Lie algebra over $R$ from $\text{End}_R(M) \oplus L$. Finally, the obvious morphism

\[(2.11.7) \quad \DO(A, L, M) \rightarrow L\]

is manifestly a morphism of $A$-modules and of Lie algebras over $R$, and this morphism, combined with the given action of $L$ on $A$, induces an action of $\DO(A, L, M)$ on $A$ in such a way that $\DO(A, L, M)$ is an $(R, A)$-Lie algebra. It is clear that the obvious inclusion $\text{End}_A(M) \subseteq \text{End}_R(M)$ induces an injection $\text{End}_A(M) \subseteq \DO(A, L, M)$ of $(R, A)$-Lie algebras. Now, with the zero morphism $\text{End}_A(M) \rightarrow \text{Der}(A)$ and the obvious Lie algebra structure, the object $\text{End}_A(M)$ is an $(R, A)$-Lie algebra, and by construction, we have an exact sequence

\[(2.11.8) \quad 0 \rightarrow \text{End}_A(M) \rightarrow \DO(A, L, M) \rightarrow L \rightarrow 0\]

of $(R, A)$-Lie algebras. Moreover, it is clear that by construction the obvious morphism

\[(2.11.9) \quad \DO(A, L, M) \rightarrow \text{End}_R(M)\]

induces a structure of an $(A, \DO(A, L, M))$-module. We refer to the $(R, A)$-Lie algebra $\DO(A, L, M)$ as the infinitesimal gauge algebra of $M$ with respect to $L$, and to the endomorphisms $\beta$ of $M$ coming from $\DO(A, L, M)$ as infinitesimal gauge transformations of $M$ with respect to $L$. The reason for this terminology will be given in (2.16)(2) below. We note that $\DO(A, L, M)$ is an invariant of $M$ and $L$. Moreover, borrowing some terminology from Galois theory, we shall say that $M$ is $L$-normal, if the morphism (2.11.7) is surjective, i. e. if for every $\alpha \in L$ there is an $R$-linear endomorphism $\beta$ of $M$ so that (2.11.2) holds. Thus, an $L$-normal $A$-module $M$ has an exact sequence

\[(2.11.10) \quad e_M: 0 \rightarrow \text{End}_A(M) \rightarrow \DO(A, L, M) \rightarrow L \rightarrow 0\]

of $(R, A)$-Lie algebras that is natural in terms of the given data. It is clear that an $A$-module $M$ admits an $L$-connection if and only if it is $L$-normal and if the corresponding extension (2.11.10) splits in the category of $A$-modules. Moreover, if this happens to be the case, after a connection $\omega$ for $M$ has been chosen, the
corresponding curvature $\Omega$ for the connection is just a corresponding 2-cocycle for
the extension (2.11.10) defined by (2.2) above.

Once an $L$-connection has been chosen, there is a more direct (and less invariant)
construction of the extension (2.11.10). We shall need it later and therefore give the
details now:

Let $\omega: L \to \text{End}(M)$ be an $L$-connection on $M$ with curvature $\Omega$. As an $A$-module, let
\begin{equation}
(2.12.1) \quad \text{DO}(A, \omega) = \text{End}_A(M) \oplus L;
\end{equation}
 furthermore, we extend the bracket on $\text{End}_A(M)$ to one on $\text{DO}(A, \omega)$ by means of
\begin{equation}
(2.12.2) \quad [[(0, \alpha), (0, \beta)] = (\Omega(\alpha, \beta), [\alpha, \beta]), \quad \alpha, \beta \in L,
\end{equation}
where at first the commutator $[\mu, \omega(\alpha)]$ is taken in $\text{End}(M)$. A little thought reveals
that (1.2.b) entails indeed $[\mu, \omega(\alpha)] \in \text{End}_A(M) \subseteq \text{End}(M)$, and that the Bianchi
identity (2.10) says that this bracket satisfies the Jacobi identity, whence we obtain
a Lie algebra. Furthermore, (1.2.b) implies that $\text{End}_A(M)$ is a Lie ideal in $\text{DO}(A, \omega)$ 
– in fact, these two properties are equivalent –, and with the obvious morphisms,
\begin{equation}
(2.12) \quad 0 \to \text{End}_A(M) \to \text{DO}(A, \omega) \to L \to 0
\end{equation}
then constitutes an extension of Lie algebras. We next define an action of $\text{DO}(A, \omega)$
on $A$ through the projection onto $L$; then it is not hard to see that (2.12) is indeed
an extension of $(R, A)$-Lie algebras. For example, (1.2.a) implies that there is no
problem with the $A$-module structures. Finally, the obvious morphism
\begin{equation}
(2.12.3) \quad \iota + \omega: \text{DO}(A, \omega) = \text{End}_A(M) \oplus L \to \text{End}(M),
\end{equation}
where $\iota: \text{End}_A(M) \to \text{End}(M)$ refers to the obvious embedding, endows $M$ with the
structure of an $(A, \text{DO}(A, \omega))$-module, and it is clear that if we take the obvious
section $L \to \text{DO}(A, \omega)$ of $A$-modules and define the corresponding 2-cocycle by (2.2),
this 2-cocycle is just the curvature $\Omega$ of our $L$-connection.

It is clear that a choice of $\omega$ induces a congruence isomorphism
\[(\text{Id}, \cdot, \text{Id}): (2.11.10) \longrightarrow (2.12),\]
and hence, up to congruence of extensions of $(R, A)$-Lie algebras, an extension of the
kind (2.12) depends only on $A$ and $M$ and not on a particular choice of $\omega$. This can
also be seen in the following more direct and less invariant way: Let $\omega': L \to \text{End}(M)$
be another $L$-connection on $M$, and let
\begin{equation}
(2.12') \quad 0 \to \text{End}_A(M) \to \text{DO}(A, \omega') \to L \to 0
\end{equation}
be the corresponding extension of $(R, A)$-Lie algebras. The rule (1.1.b) implies at
once that the difference $u = \omega - \omega'$ factors through $\text{End}_A(M)$, i. e. that $u$
may be displayed as $u: L \to \text{End}_A(M)$. It is then straightforward to check that the
rule $(0, \alpha) \mapsto (u(\alpha), \alpha)$ induces a congruence isomorphism $(\text{Id}, \cdot, \text{Id}): (2.12') \longrightarrow (2.12)$. Summarizing we spell out the following.
Proposition 2.13. For an \((R,A)\)-Lie algebra \(L\) and an \(A\)-module \(M\) there is, up to congruence of extensions, at most one extension of the kind (2.12).

Next we indicate how the classical argument (see e.g. Koszul [53]) yields the existence of connections for projective \(A\)-modules \(M\), indeed relatively projective ones: Let \(M\) be a relatively free \(A\)-module, and write \(M = M^2 \otimes A\). Then an \(L\)-connection \(\omega: L \to \text{End}(M)\) (with respect to \(M^2\)) is given by

\[
(\omega(\alpha))(b) = 0, \quad (\omega(\alpha))(ab) = \alpha(a)b, \quad a \in A, b \in M^2.
\]

Likewise, if \(M\) is relatively projective, let \(N\) be an \(A\)-module so that \(M \oplus N\) is relatively free, let \(D^\oplus: M \oplus N \to \text{Hom}_A(L,M \oplus N)\) be the adjoint (2.8.a') of a connection of the kind (2.14.1), and let \(D\) be the composite

\[
D: M \to M \oplus N \overset{D^\oplus}{\longrightarrow} \text{Hom}_A(L,M \oplus N) \to \text{Hom}_A(L,M),
\]

where the unlabelled arrows are the obvious morphisms. Then the adjoint

\[
\omega: L \to \text{End}(M)
\]

is an \(L\)-connection on \(M\).

Before we spell out the following, we note that for a projective rank one module \(M\) the algebras \(\text{End}_A(M)\) and \(A\) are canonically isomorphic. As usual, we denote the Picard group of \(A\) by \(\text{Pic}(A)\); we remind the reader that it consists of classes of projective rank one modules, with addition being induced by the tensor product.

Theorem 2.15. Let \(L\) be an \((R,A)\)-Lie algebra. Then the assignment to the class \([M] \in \text{Pic}(A)\) of a projective rank one module \(M\) of the class \([\Omega_M] \in H^2(\text{Alt}_A(L,A))\) of the curvature of an \(L\)-connection on \(M\) is a homomorphism

\[
(2.15.1) \quad \text{Pic}(A) \longrightarrow H^2(\text{Alt}_A(L,A))
\]

of \(R\)-modules.

REMARK after publication. The map (2.15.1) is only a homomorphism of abelian groups. This is enough to validate what follows.

Proof. Theorem 2.13 implies that (2.15.1) is well defined. To see that it is a homomorphism of \(R\)-modules, let \(M_1\) and \(M_2\) be projective rank one modules, and let \(\omega_1: L \to \text{End}(M_1)\) and \(\omega_2: L \to \text{End}(M_2)\) be \(L\)-connections, with curvatures \(\Omega_1\) and \(\Omega_2\), respectively. For \(\alpha \in L, x_1 \in M_1, x_2 \in M_2\), let

\[
(\omega(\alpha))(x_1 \otimes x_2) = ((\omega_1(\alpha))(x_1)) \otimes x_2 + x_1 \otimes ((\omega(\alpha))(x_2)).
\]

This defines an \(L\)-connection on \(M_1 \otimes M_2\). It is easy to see that it has curvature \(\Omega_1 + \Omega_2\). □

For illustration, let \(N\) be a smooth real manifold, let \(A\) be the algebra of smooth complex functions on \(N\), and let \(L\) be the Lie algebra of smooth complex vector
fields on $N$, i.e. that of smooth sections of the complexified tangent bundle $T^{\mathbb{C}}N$, where $\mathbb{C}$ refers to the complex numbers. Then $\text{Pic}(A)$ is the group of classes of complex line bundles, the assignment of its first Chern class to a line bundle yields an isomorphism $\text{Pic}(A) \rightarrow H^2(N, \mathbb{Z})$, and the morphism (2.15.1) is part of the corresponding Chern-Weil map.

**Remarks 2.16.** Some historical comments about the algebraic approach to connection theory seem appropriate.

1. Let $R$ be the ring of the reals or that of the complex numbers, let $N$ be a manifold, either smooth or complex analytic, with tangent bundle $TN$, let $A$ be the algebra of smooth or analytic functions on $N$, let $L$ be the Lie algebra of vector fields on $N$, either smooth or complex analytic, let $\zeta$ be a smooth or complex analytic vector bundle on $N$, and let $M = \Gamma(\zeta)$ be the $A$-module of sections of $\zeta$, either smooth or complex analytic. Moreover, let $\xi: P \rightarrow N$ be a principal bundle for $\zeta$, with structure group $G$ and Lie algebra $g$, and consider the corresponding extension

$$0 \rightarrow V \rightarrow TP \rightarrow TN \rightarrow N \rightarrow P \rightarrow 0$$

of vector bundles over $P$, where $V$ is the vertical subbundle; the latter is well known to be canonically isomorphic to the trivial bundle $g \times P$. A treatment of the notions of connection and curvature by pure algebra goes back at least to Cartan [13], and Cartan’s notions of algebraic connections and curvature are derived from formal properties of (2.16.1) and its spaces of sections.

2. The sequence (2.16.1) inherits an obvious $G$-action; when we divide it out we obtain an extension

$$0 \rightarrow g(\xi) \rightarrow TP/G \rightarrow TN \rightarrow 0$$

of vector bundles over $N$, where $g(\xi) = (V/G)$ is the bundle associated to the principal bundle by the adjoint representation. Then the spaces $\Gamma(g(\xi))$ and $\Gamma(TP/G)$ of sections inherit obvious structures of Lie algebras, in fact of $(R,A)$-Lie algebras, and the corresponding sequence of sections

$$0 \rightarrow \Gamma(g(\xi)) \rightarrow \Gamma(TP/G) \rightarrow \Gamma(TN) \rightarrow 0$$

is an extension of $(R,A)$-Lie algebras; moreover, the obvious action of $\Gamma(TP/G)$ on the space $\Gamma(\zeta)$ of sections for $\zeta$ endows the latter with a structure of an $(A,\Gamma(TP/G))$-module in our sense. Indeed, when $G$ is the full linear group on the fibre of $\zeta$, the sequence (2.16.3) is just the above sequence (2.11.10), and the sections of $TP/G$ are exactly the infinitesimal gauge transformation of $\zeta$ covering infinitesimal diffeomorphisms of $N$ in the usual sense (which correspond bijectively to the $G$-invariant vector fields on $P$). In general, the principal bundle comes with a linear representation of $G$ on the fibre, and the structure induces in particular a morphism

$$\begin{array}{cccccc}
0 & \rightarrow & \Gamma(g(\xi)) & \rightarrow & \Gamma(TP/G) & \rightarrow & \Gamma(TN) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow & & \\
0 & \rightarrow & \text{End}_A(M) & \rightarrow & \text{DO}(A,L,M) & \rightarrow & L & \rightarrow & 0
\end{array}$$
of extensions of \((R, A)\)-Lie algebras.

The sequence (2.16.2) was introduced by Atiyah [5] (Theorem 1) and is now usually called the Atiyah sequence of the principal bundle \(\xi\). Furthermore, the classical notions of linear connection and curvature have been described by Atiyah [5] in formally the same way as in the present Section. In particular, when we write \(L = \Gamma(TN)\), we see that for any vector bundle \(\zeta\) the \(A\)-module \(\Gamma(\zeta)\) of sections is \(L\)-normal in the above sense in both the smooth and complex analytic context; however, while the exact sequence (2.16.3) will always split in the smooth case, it will in general not split in the complex analytic case; it will split e. g. over a Stein manifold, but a counterexample is given e. g. in Proposition 22 of Atiyah [5]. For a complete account to Atiyah sequences see App. A in Mackenzie [64].

(3) Notions of algebraic connections and curvature of the above kind (corresponding to the linear notions in the classical case) have been introduced by Koszul [53]. However, we do not know whether our extension theory, the infinitesimal gauge algebra, and the related material are already in the literature.

(4) The formal properties of the concept of an Atiyah sequence have been incorporated in the more general concept of a Lie algebroid; this is the geometric analogue of an \((R, A)\)-Lie algebra. Pradines [78] associated to any differentiable groupoid a Lie algebroid as a first order infinitesimal invariant. This generalizes the construction of the Atiyah sequence of a principal bundle (reproduced above). For a complete account to differentiable groupoids, Lie algebroids, and connection theory while staying in the category of finite rank vector bundles see Mackenzie [64]; his geometric notions are similar to the above algebraic ones of connection and curvature. Further relevant references are Almeida-Molino [2], Coste-Dazord-Weinstein [17], Weinstein [102] – [106].

3. Poisson algebras

As before, let \(R\) be a commutative ring with 1, and let \(A\) be a commutative \(R\)-algebra. Recall that a Poisson algebra structure on \(A\) (over \(R\)) is a Lie bracket \(\{\cdot, \cdot\}: A \otimes A \rightarrow A\) on \(A\) so that, for every \(a, b, c \in A\),

\[
\{ab, c\} = a \{b, c\} + \{a, c\} b.
\]

We note that this implies at once that, for \(a \in A\) and \(r \in R\), we have \(\{a, r\} = 0\). For example, the ring of smooth functions on a smooth Poisson manifold, see e. g. Weinstein [102], [103], is a Poisson algebra. In the present Section we shall characterize such a structure on an arbitrary \(R\)-algebra \(A\) in terms of a suitable \((R, A)\)-Lie algebra structure together with an appropriate closed 2-form on the \(A\)-module \(D_A\) of Kähler differentials for \(A\).

For convenience we review briefly the construction of Kähler differentials: Let \(A\) be a commutative \(R\)-algebra, and let \(I = \ker(\mu): A \otimes A \rightarrow A\) so that, with the obvious \(A \otimes A\)-module structures, \(0 \rightarrow I \rightarrow A \otimes A \xrightarrow{\mu} A \rightarrow 0\) is an extension of \(A \otimes A\)-modules. As an \(A \otimes A\)-module, \(I\) is generated by the elements \(a \otimes 1 - 1 \otimes a \in A \otimes A\); hence as an \(R\)-module, \(I\) is generated by elements of the form \(b(a \otimes 1 - 1 \otimes a)c\), \(a, b, c \in A\). Let \(D_A\) be the \(A\)-module of formal differentials or Kähler differentials for \(A\), cf. e. g.
Kunz [55]; as an $A$-module, it is generated by elements $da, a \in A$, subject to the relations

\[(3.2.1) \quad d(bc) = (db)c + bdc, \quad dr = 0,\]

where $b, c \in A$, $r \in R$. It is well known that the rule $a \otimes 1 - 1 \otimes a \mapsto da$ induces an isomorphism $\text{Tor}^1_A(A, A) = I/I^2 \rightarrow D_A$ of $A$-modules. In the standard way, $D_A$ represents the functor $\text{Der}(A, -)$ from the category of $A$-modules to itself. More precisely, given an $A$-module $M$ and an element $h \in \text{Hom}_A(I, M)$, the $R$-endomorphism $d_h: A \rightarrow A$ defined by $d_h(a) = h(a \otimes 1 - 1 \otimes a)$ is a derivation $A \rightarrow M$, and it is well known that the rule $h \mapsto d_h$ induces a natural isomorphism

\[(3.2.2) \quad \text{Hom}_A(D_A, M) \rightarrow \text{Der}(A, M)\]

of $A$-modules; in fact, this property characterizes the Kähler differentials. In particular, the $A$-module $\text{Hom}_A(\text{Der}(A), A)$ may canonically be identified with the double dual $D_A^{**}$ of $D_A$ and there is a canonical map

\[(3.2.3) \quad D_A \rightarrow \text{Hom}_A(\text{Der}(A), A)\]

of $A$-modules. Moreover, with the usual differential $d$ given by $d(ad_1 \cdots db_k) = dad_1 \cdots db_k$, the graded exterior algebra $\Lambda_A[D_A]$ is the standard differential graded commutative algebra of Kähler forms; it is natural in $A$. Henceforth we write $\Lambda A = (\Lambda_A[D_A], d)$. It is well known that (3.2.3) induces a morphism

\[(3.2.4) \quad \Lambda A \rightarrow \text{Alt}_A(\text{Der}(A), A)\]

of differential graded algebras. It is proved in Hochschild-Kostant-Rosenberg [40] that, for a regular affine algebra $A$ over a perfect field, (3.2.3) and hence (3.2.4) are isomorphisms. Hence there is then no need to distinguish between formal differentials and differential forms. On the other hand, when $A$ is the algebra of smooth functions on a smooth finite dimensional manifold $N$, the algebra $\Lambda A$ of formal differentials will be much bigger than the usual algebra $\text{Alt}_A(\text{Der}(A), A)$ of differential forms. For example, in the case $N = \mathbb{R}$, the formal differential $df - f'dt \in D_A$ will be non-zero when $f$ and $t$ are algebraically independent, see e. g. p. 27 of Krasilsh'chik, Lychagin, and Vinogradov [54]. We shall say more about this in (3.12) below.

Let $L$ be an $(R, A)$-Lie algebra. As in the classical case, by functoriality, we can extend the action of $L$ on $A$ to one of $L$ on $D_A$, in fact on $\Lambda A$, cf. e. g. Lang [56]. Explicitly, given $X \in L$ and $\alpha \in D_A$, for $a, b \in A$, let

\[(3.3) \quad \lambda_X(bda) = (X(b))da + bd(X(a)).\]

This endows $D_A$ with a structure of an $(A, L)$-module and it is clear that this extends to a structure $\omega: L \rightarrow \text{End}(\Lambda A)$ of an $(A, L)$-module on the differential graded algebra $\Lambda A$ of Kähler differentials, in fact to that of an algebra over $(A, L)$.

REMARK after publication. This yields only $L$-module, not $(A, L)$-module structures, on $D_A$ and $\Lambda A$. This oversight is irrelevant for the rest of the paper.
Given $X \in L$ and $\alpha \in \Lambda A$, we refer to $\lambda_X(\alpha)$ as the Lie derivative of $\alpha$ with respect to $X$. It is clear that when $A$ is the ring of smooth functions on a smooth finite dimensional manifold $N$ and when $L$ is the $(R, A)$-Lie algebra of smooth vector fields on $N$ so that $\text{Alt}_A(\text{Der}(A), A)$ coincides with the usual de Rham complex, the morphism (3.2.4) is compatible with Lie derivatives, where on the right hand side $\text{Alt}_A(\text{Der}(A), A)$ the usual Lie derivative is understood. It is also clear that, for an arbitrary algebra $A$ and an arbitrary $(R, A)$-Lie algebra $L$, the usual formula yields an operation of Lie derivative $\omega: L \to \text{Alt}_A(\text{Der}(A), A)$ for multilinear alternating forms on $D_A$ with values in $A$; for example, for $X \in L$ and a bilinear alternating form $\Omega: D_A \otimes A D_A \to A$ this formula reads

$$\lambda_X(\Omega)(\alpha, \beta) = X(\Omega(\alpha, \beta)) - \Omega(\lambda_X(\alpha), \beta) - \Omega(\alpha, \lambda_X(\beta)).$$

**Lemma 3.5.** Let $\{\cdot, \cdot\}: A \otimes A \to A$ be a Poisson structure on $A$, and, for $a, b, u, v \in A$, let

$$\pi(adu \otimes bdv) = ab\{u, v\} \in A.$$  

Then $\pi$ is an alternating $A$-bilinear form

$$\pi: D_A \otimes_A D_A \to A.$$  

When we wish to emphasize the dependence of $\pi$ on $\{\cdot, \cdot\}$ we shall write $\pi_{\{\cdot, \cdot\}}$. We refer to $\pi_{\{\cdot, \cdot\}}$ as the Poisson (2-) form of $(A, \{\cdot, \cdot\})$. In view of (3.15) below, this form generalizes the symplectic form of a symplectic manifold.

**Proof of 3.5.** As an $A$-module, $D_A$ is generated by the elements $da, a \in A$, subject to the relations (3.2.1). On the other hand, by definition, for $a, b, c \in A$, $r \in R$, we have

$$\pi(d(ab) \otimes dc) = \{ab, c\} = a\{b, c\} + b\{a, c\} = \pi((bda + adb) \otimes dc) \in A,$$

$$\pi(da \otimes dr) = \{a, r\} = 0,$$

whence $\pi$ is indeed well defined. □

Given a Poisson structure $\{\cdot, \cdot\}: A \otimes A \to A$ on $A$, let $\pi: D_A \otimes_A D_A \to A$ be the corresponding form (3.5.2), and let

$$\pi^*: D_A \to \text{Hom}_A(D_A, A) = \text{Der}(A)$$

be the indicated adjoint of $\pi$; then $\pi^*$ is a morphism of $A$-modules. The corresponding adjoint $D_A \otimes A \to A$ will be denoted by

$$\pi^b: D_A \otimes A \to A.$$ 

We shall say that (3.5.1) is non-degenerate, if $\pi^*$ is injective. For $\alpha \in D_A$, we shall often write $\alpha^* = \pi^*(\alpha) \in \text{Der}(A)$. Further, given $a \in A$, we refer to the derivation $(da)^* = \{a, -\}: A \to A$ as the corresponding Hamiltonian element.
**Theorem 3.8.** Let $(A,\{\cdot,\cdot\})$ be a Poisson algebra, let $\pi^\sharp:D_A \to \text{Der}(A)$ be the morphism (3.6), and, for $a,b,u,v \in A$, let

\[ (3.8.1) \quad [adu,bdv] = a\{u,b\}dv + b\{a,v\}du + abd\{u,v\} \in D_A. \]

Then $(\pi^\sharp,\{\cdot,\cdot\})$ together with the $A$-module structure endows $D_A$ with a structure of an $(R,A)$-Lie algebra in such a way that $\pi^\sharp$ is a morphism of $(R,A)$-Lie algebras.

**Proof.** We indicate at first that $\{\cdot,\cdot\}$ is well defined. To this end, consider

\[ d(ab),cdv = \{ab,c\}dv + c\{ab,v\}db = b\{a,c\}dv + a\{b,c\}dv + c\{a,v\}db + cad\{b,v\} + c\{b,v\}da. \]

On the other hand,

\[ [adb,cdv] = a\{b,c\}dv + c\{a,v\}db + acd\{b,v\} \]
\[ [bda,cdv] = b\{a,c\}dv + c\{b,v\}da + bcd\{a,v\} \]

whence indeed $[d(ab),cdv] = [adb + bda,cdv]$ as desired.

It is clear that the identities (1.1.a) and (1.1.b) hold. Furthermore, a calculation shows that the Jacobi identity for $\{\cdot,\cdot\}$ boils down to

\[ (3.8.2) \quad [[du,dv],dw] + [[dv,dw],du] + [[dw,du],dv] = 0, \]

and it is clear that (3.8.2) is equivalent to the Jacobi identity for $\{\cdot,\cdot\}$. Hence $(\pi^\sharp,\{\cdot,\cdot\})$ together with the $A$-module structure is indeed a structure of an $(R,A)$-Lie algebra on $D_A$.

Finally, to see that $\pi^\sharp$ is a morphism of $(R,A)$-Lie algebras, let $a,b,c,u,v \in A$, and consider

\[ [adu,bdv]^\sharp(c) = a\{u,b\}\{v,c\} + b\{v,a\}\{u,c\} + ab\{u,v\},c\}. \]

On the other hand,

\[ (adu)^\sharp(bdv)^\sharp(c) = a\{u,b\}{v,c} = a\{u,b\}\{v,c\} + ab\{u,v,c\}, \]
\[ (bdv)^\sharp(adu)^\sharp(c) = b\{v,a\}\{u,c\} = b\{v,a\}\{u,c\} + ab\{v,u,c\}. \]

In view of the Jacobi identity $\{\{u,v\},c\} = \{u,\{v,c\}\} - \{v,\{u,c\}\}$ we conclude that

\[ [adu,bdv]^\sharp(c) = (adu)^\sharp(bdv)^\sharp(c) - (bdv)^\sharp(adu)^\sharp(c) \]

whence $\pi^\sharp$ is indeed a morphism of $(R,A)$-Lie algebras. □

**Addendum 3.8.3.** In terms of the above notion (3.3) of Lie derivative the formula (3.8.1) may be rewritten

\[ (3.8.1') \quad [\alpha,\beta] = \lambda_{\alpha^\sharp}\beta - \lambda_{\beta^\sharp}\alpha - d(\pi\{\cdot,\cdot\}(\alpha,\beta)) \in D_A, \quad \alpha,\beta \in D_A. \]

This is straightforward and left to the reader. □
Addendum 3.8.4. Given two Poisson algebras \((A, \{\cdot, \cdot\})\) and \((A', \{\cdot, \cdot\}')\) and a morphism

\[(3.8.5)\]
\[
\phi: (A, \{\cdot, \cdot\}) \to (A', \{\cdot, \cdot\}')
\]
of Poisson algebras, let \(\phi_*: D_A \to D_{A'}\) be the induced \(A\)-linear morphism between the \(K\)ähler differentials as indicated where \(A\) acts on \(D_{A'}\) through \(\phi\). Then \((\phi, \phi_*)\) is a morphism

\[(3.8.6)\]
\[
(\phi, \phi_*): (A, D_{\{\cdot, \cdot\}}) \to (A', D_{\{\cdot, \cdot\}'})
\]
of Lie-Rinehart algebras (cf. Section 1). More precisely, \(\phi_*\) is a morphism \(\phi_*: D_{\{\cdot, \cdot\}} \to D_{\{\cdot, \cdot\}'})\) of Lie algebras over \(R\) (and one of \(A\)-modules), and the diagram

\[
\begin{array}{ccc}
D_A \otimes A & \xrightarrow{\pi_{\{\cdot, \cdot\}}} & A \\
\phi_* \otimes \phi \downarrow & & \downarrow \phi \\
D_{A'} \otimes A' & \xrightarrow{\pi_{\{\cdot, \cdot\}'}} & A'
\end{array}
\]
is commutative.

This is again straightforward and left to the reader.

Henceforth we write \(D_{\{\cdot, \cdot\}}\) for \(D_A\) together with the \((R, A)\)-Lie algebra structure given in (3.8) above. It is clear that the machinery of Section 1 applies to \(D_{\{\cdot, \cdot\}}\). In particular, we have the differential graded commutative algebra \(\text{Alt}_A(D_{\{\cdot, \cdot\}}, A)\); we refer to its cohomology as the \textit{Poisson cohomology of} \((A, \{\cdot, \cdot\})\), written \(H^*_\text{Poisson}(A, \{\cdot, \cdot\}; A)\). More generally, for any \((A, D_{\{\cdot, \cdot\}})\)-module \(M\), we have the differential graded \(\text{Alt}_A(D_{\{\cdot, \cdot\}}, A)\)-module \(\text{Alt}_A(D_{\{\cdot, \cdot\}}, M)\); we refer to its cohomology as the \textit{Poisson cohomology of} \((A, \{\cdot, \cdot\})\) \textit{with values in} \(M\), written \(H^*_\text{Poisson}(A, \{\cdot, \cdot\}; M)\). In view of (1.14), when \(D_A\) is projective as an \(A\)-module, we have \(H^*_\text{Poisson}(A, \{\cdot, \cdot\}; M) = \text{Ext}^*_U (A, D_{\{\cdot, \cdot\}})(A, M)\), where \(U(A, D_{\{\cdot, \cdot\}})\) refers to the corresponding universal algebra of differential operators introduced in Section 1. A cochain, cocycle etc. in \(\text{Alt}_A(D_{\{\cdot, \cdot\}}, M)\) will be referred to as a \textit{Poisson} cochain or cocycle etc. as appropriate. Notice that Poisson cohomology depends only on the \((R, A)\)-Lie algebra structure on the \(A\)-module \(D_A\) of \(K\)ähler differentials which is derived from \(\{\cdot, \cdot\}\) by means of (3.6) and (3.8.1); see also (3.11.3) below. Likewise, for any right \((A, D_{\{\cdot, \cdot\}})\)-module \(M\), we have the chain complex \(M \otimes_{U(A, D_{\{\cdot, \cdot\}})} K(A, D_{\{\cdot, \cdot\}})\) where \(K(A, D_{\{\cdot, \cdot\}})\) refers to the corresponding Koszul complex (1.13). We refer to its homology as the \textit{Poisson homology of} \((A, \{\cdot, \cdot\})\) \textit{with values in} \(M\), written \(H^*_\text{Poisson}(A, \{\cdot, \cdot\}; M)\). In particular, with respect to the right \(U(A, D_{\{\cdot, \cdot\}})\)-module structure (1.8.4) on \(A\), we have the Poisson homology of \((A, \{\cdot, \cdot\})\) with values in \(A\).

Remark after publication. As pointed out after (1.8.4), the construction given there does not work. Under the present circumstances, the requisite right \(U(A, D_{\{\cdot, \cdot\}})\)-module structure is given by the association

\[a \otimes (bdu) \mapsto \{ab, u\}\]
and this is perfectly o.k. This construction has been elaborated upon in follow up papers:

Lie-Rinehart algebras, Gerstenhaber algebras, and Batalin-Vilkovisky algebras, Annales de l’institut Fourier 48 (1998) 425-440, dg-ga/9704005

Duality for Lie-Rinehart algebras and the modular class, J. für die reine und angew. Mathematik 510 (1999) 103–159, dg-ga/9702008

Differential Batalin-Vilkovisky algebras arising from twilled Lie-Rinehart algebras, Banach center publications 51 (2000) 87–102

Twilled Lie-Rinehart algebras and differential Batalin-Vilkovisky algebras, J.H.25

For an arbitrary right \((A, D)\)-module \(M\), when \(D_A\) is projective as an \(A\)-module, we have \(H^n_{\text{Poisson}}(A, \{\cdot, \cdot\}; M) = \text{Tor}^U_{A, D(\cdot, \cdot)}(M, A)\).

In view of the naturality explained in (3.8.4), these notions of Poisson cohomology and homology are natural in the following sense: Let \(\phi: (A, \{\cdot, \cdot\}) \rightarrow (A', \{\cdot, \cdot\}')\) be a morphism of Poisson algebras. By (3.8.4) it induces a morphism

\[(\phi, \phi_*) : (A, D_{\{\cdot, \cdot\}}) \rightarrow (A', D_{\{\cdot, \cdot\}'})\]

of Lie-Rinehart algebras, and hence morphisms

(3.8.7) \[U(\phi, \phi_*): U(A, D_{\{\cdot, \cdot\}}) \rightarrow U(A', D_{\{\cdot, \cdot\}'})\]

of algebras and

(3.8.8) \[K(\phi, \phi_*): K(A, D_{\{\cdot, \cdot\}}) \rightarrow K(A', D_{\{\cdot, \cdot\}'})\]

of chain complexes, the second one being \(U(A, D_{\{\cdot, \cdot\}})\)-linear in the obvious sense. It is clear that, given right \((A, D_{\{\cdot, \cdot\}})\) and \((A', D_{\{\cdot, \cdot\}'})\)-modules \(M\) and \(M'\) and a morphism \(\psi: M \rightarrow M'\) of right \((A, D_{\{\cdot, \cdot\}})\)-modules (where the pair \((A, D_{\{\cdot, \cdot\}})\) acts on \(M'\) through \((\phi, \phi_*)\)), these combine to a morphism

(3.8.9) \[M \otimes_{U(A, D_{\{\cdot, \cdot\}})} K(A, D_{\{\cdot, \cdot\}}) \rightarrow M' \otimes_{U(A', D_{\{\cdot, \cdot\}'})} K(A', D_{\{\cdot, \cdot\}'})\]

and hence induce a morphism

(3.8.10) \[(\phi, \psi)_* : H^n_{\text{Poisson}}(A, \{\cdot, \cdot\}; M) \rightarrow H^n_{\text{Poisson}}(A', \{\cdot, \cdot\}; M')\]

in Poisson homology. Notice the special case where \(M = A\) and \(M' = A'\). Likewise, given left \((A, D_{\{\cdot, \cdot\}})\) and \((A', D_{\{\cdot, \cdot\}'})\)-modules \(M\) and \(M'\) and a morphism \(\psi: M' \rightarrow M\) (backwards) of left \((A, D_{\{\cdot, \cdot\}})\)-modules (where again the pair \((A, D_{\{\cdot, \cdot\}})\) acts on \(M'\) through \((\phi, \phi_*)\)), these combine to a morphism

(3.8.11) \[\text{Alt}_{A'}(D_{\{\cdot, \cdot\}'}, M') \rightarrow \text{Alt}_A(D_{\{\cdot, \cdot\}}, M)\]

of chain complexes and hence induce a morphism

(3.8.12) \[(\phi^*, \psi_*) : H^n_{\text{Poisson}}(A', \{\cdot, \cdot\}; M') \rightarrow H^n_{\text{Poisson}}(A, \{\cdot, \cdot\}; M)\]

in Poisson cohomology.

We have seen that a Poisson structure \(\{\cdot, \cdot\}\) on \(A\) determines a structure of an \((R, A)\)-Lie algebra on \(D_A\). We now examine the question to what extent the latter determines the former.
Lemma 3.9. The Poisson 2-form $\pi_{\{\cdot, \cdot\}}: D_A \otimes_A D_A \to A$ of a Poisson algebra $(A, \{\cdot, \cdot\})$ given by (3.5.1) is a Poisson 2-cocycle, i.e., a 2-cocycle in $\text{Alt}_A(D_{\{\cdot, \cdot\}}, A)$. Moreover, $\pi_{\{\cdot, \cdot\}}$ is natural in the obvious sense.

Proof. This is an easy consequence of the Jacobi identity for $\{\cdot, \cdot\}$. □

Hence for any Poisson algebra $(A, \{\cdot, \cdot\})$, its Poisson 2-form $\pi_{\{\cdot, \cdot\}}$ determines a class

$$\pi_{\{\cdot, \cdot\}} \in H^2_{\text{Poisson}}(A, \{\cdot, \cdot\}; A)$$

which is natural in an obvious sense; we refer to it as the Poisson class of $(A, \{\cdot, \cdot\})$. This class generalizes the class $[\sigma] \in H^2(N, \mathbb{R})$ of a symplectic structure $\sigma$ on a smooth real manifold $N$ to arbitrary Poisson algebras, see (3.15) below. Furthermore, we write $L_{\{\cdot, \cdot\}}$ for the corresponding $(R, A)$-Lie algebra whose underlying $A$-module looks like $A \oplus D_A$ and whose Lie structure is given by

$$[(0, \alpha), (0, \beta)] = (\pi_{\{\cdot, \cdot\}}(\alpha, \beta), [\alpha, \beta]),
[(a, 0), (0, \alpha)] = (-\alpha(a), 0),$$

where $a \in A$, $\alpha, \beta \in D_A$, so that $L_{\{\cdot, \cdot\}}$ fits into an extension

$$0 \to A \to L_{\{\cdot, \cdot\}} \to D_{\{\cdot, \cdot\}} \to 0$$

of $(R, A)$-Lie algebras which, in view of (2.6), is classified by $[\pi_{\{\cdot, \cdot\}}]$. The extension (3.10.3) is natural in Poisson structures. When the class $[\pi_{\{\cdot, \cdot\}}]$ is zero, a Poisson 1-form $\vartheta: D_{\{\cdot, \cdot\}} \to A$ so that $d\vartheta = \pi_{\{\cdot, \cdot\}}$ will be referred as a Poisson potential. In view of (3.15) below, this generalizes the notion of a symplectic potential, see e.g. p. 9 of Woodhouse [108]. It admits yet another interpretation: Let $\vartheta: D_A \to A$ be a 1-form on $D_A$, viewed as a derivation $X = X\vartheta: A \to A$, cf. (3.2.2). Then the Lie derivative (3.4) of the Poisson 2-form $\pi_{\{\cdot, \cdot\}}$ with respect to $X$, which is an element of the $(R, A)$-Lie algebra $\text{Der}(A)$, boils down to

$$\lambda_X(\pi_{\{\cdot, \cdot\}}) = -d\vartheta \in \text{Alt}_A(D_{\{\cdot, \cdot\}}, A).$$

In fact,

$$\lambda_X(\pi_{\{\cdot, \cdot\}})(da, db) = X(\{a, b\} - \{X(a), b\} - \{a, X(b)\})
= \vartheta(d\{a, b\} - \{\vartheta(da), b\} - \{a, \vartheta(db)\})
= -(d\vartheta)(da, db).$$

In particular, when $\vartheta$ is a Poisson potential so that $d\vartheta = \pi_{\{\cdot, \cdot\}}$, we obtain

$$\lambda_{-\vartheta}(\pi_{\{\cdot, \cdot\}}) = \pi_{\{\cdot, \cdot\}} \in \text{Alt}_A(D_{\{\cdot, \cdot\}}, A),$$

i.e. up to sign the Lie derivative of $\pi_{\{\cdot, \cdot\}}$ with respect to its Poisson potential is just $\pi_{\{\cdot, \cdot\}}$. I am indebted to A. Weinstein for informing me that this holds for a Poisson manifold and hence for prompting me to examine the general case. In
the manifold case a vector field $X$ corresponding to a Poisson potential is called a *conformal Poisson* or *Liouville* vector field.

**Remark 3.10.6.** In view of (3.8.4), a morphism $\phi: (A, \{\cdot, \cdot\}) \to (A', \{\cdot, \cdot\}')$ of Poisson algebras induces the two morphisms

$$\phi^* : H^*_\text{Poisson}(A', \{\cdot, \cdot\}; A') \to H^*_\text{Poisson}(A, \{\cdot, \cdot\}; A')$$

and

$$\text{Id} \circ \phi : H^*_\text{Poisson}(A, \{\cdot, \cdot\}; A) \to H^*_\text{Poisson}(A, \{\cdot, \cdot\}; A')$$

in Poisson cohomology. It is clear that the classes $[\pi_{\{\cdot, \cdot\}}] \in H^2_\text{Poisson}(A, \{\cdot, \cdot\}; A)$ and $[\pi_{\{\cdot, \cdot\}'}] \in H^2_\text{Poisson}(A', \{\cdot, \cdot\}'; A')$ go to the same class

$$(\phi^* \circ \text{Id})[\pi_{\{\cdot, \cdot\}}] = (\text{Id} \circ \phi)[\pi_{\{\cdot, \cdot\}'}] \in H^*_\text{Poisson}(A, \{\cdot, \cdot\}; A').$$

In this sense the Poisson class of a Poisson algebra is natural; notice, however, we cannot in general directly relate $[\pi_{\{\cdot, \cdot\}}]$ and $[\pi_{\{\cdot, \cdot\}'}]$ by means of a map sending one class to the other.

**Theorem 3.11.** Let $\pi : D_A \otimes_A D_A \to A$ be an alternating $A$-bilinear form on $D_A$, let $\omega : D_A \to \text{Hom}_A(D_A, A) = \text{Der}(A)$ be its adjoint, for $a, b \in A$, let $\{a, b\} = \pi(da \otimes db)$, for $a, b, u, v \in A$, let

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\} \in D_A,$$

and assume that $(\omega, [\cdot, \cdot])$ together with the $A$-module structure endows $D_A$ with a structure of an $(R, A)$-Lie algebra. Then $\{\cdot, \cdot\}$ endows $A$ with that of a Poisson algebra if and only if $\pi$ is a 2-cocycle in $(\text{Alt}_A(D_A, A), d)$.

**Proof.** We have already seen that the condition is necessary. To see that it is also sufficient, assume that $\pi$ is a 2-cocycle, and let

$$0 \to A \to A \oplus_{-\pi} D_A \to D_A \to 0$$

be the corresponding extension of $(R, A)$-Lie algebras, cf. Section 2 above, in particular (2.6). Inspection shows that the morphism

$$\iota_\pi = (\text{Id}, d) : A \to A \oplus_{-\pi} D_A$$

is compatible with the bracket operations, i. e., given $a, b \in A$, $\iota_\pi \{a, b\} = [\iota_\pi a, \iota_\pi b]$. Since $A \oplus_{-\pi} D_A$ is a Lie algebra, and since $\iota_\pi$ is manifestly injective, we conclude that $\{\cdot, \cdot\}$ satisfies the Jacobi identity. Moreover, the defining relations (3.2.1) for the Kähler differentials and the $A$-bilinearity of $\pi$ imply at once that $A$ satisfies (3.1). Since $\{\cdot, \cdot\}$ is obviously skew symmetric, we are done. □

**Remark 3.11.3.** Under the circumstances of (3.11), when $\pi$ is not a 2-cocycle, we can only conclude that $\{\cdot, \cdot\}$ induces on $dA \subseteq D_A$ a structure of a Lie algebra. On the other hand, the differential graded commutative algebra $\text{Alt}_A(D_A, A)$ over $R$ and hence its cohomology are still defined.
3.12. THE GEOMETRIC VERSION

Our notions of Poisson homology and cohomology are entirely algebraic. In the special case where \( A \) is the ring of smooth real functions on a smooth finite dimensional Poisson manifold suitable geometric notions of Poisson homology and cohomology may be found in the literature BRYLINSKI [12], KOSZUL [52], LICHNEROWICZ [58].

To relate the various approaches we need some preparations.

Recall that for an \( R \)-algebra \( A \) the \( A \)-module \( D_A \) of Kähler differentials represents the functor \( \text{Der}(A, -) \) on the category of \( A \)-modules. However, in the smooth case the appropriate category to work with is that of “geometric modules”: Let \( N \) be a smooth finite dimensional manifold and let \( A \) be its ring of smooth functions. Using a terminology introduced on p. 27 of KRASILSH'CHIK, Lychagin, and Vinogradov [54] we shall say that an \( A \)-module \( M \) is geometric, if \( \bigcap_{x \in N} \mu_x M = 0 \), where \( \mu_x \subseteq A \) denotes the maximal ideal of functions on \( N \) that vanish on \( x \in N \). The \( A \)-module \( D_A \) of Kähler differentials is apparently not geometric; however, its geometric analogue is the \( A \)-module of smooth 1-forms on \( N \) which, in view of the isomorphism (3.2.2), may be identified with the double dual \( D_A^{\ast\ast} \); henceforth we denote this \( A \)-module by \( D_A^{\text{geo}} \) – the superscript ‘\text{geo}’ stands for ‘geometric’. The obvious morphism

\[(3.12.1) \quad q: D_A \rightarrow D_A^{\text{geo}} \]

of \( A \)-modules which sends a formal differential to the corresponding differential form is manifestly surjective. For example, in the case \( N = \mathbb{R} \), the formal differential \( df - f'dt \in D_A \) will be non-zero when \( f \) and \( t \) are algebraically independent but goes to zero in \( D_A^{\text{geo}} \) under (3.12.1).

Let \( N \) be a smooth finite dimensional manifold, let \( A \) be its ring of smooth functions, and let \( L \) be the \((\mathbb{R}, A)\)-Lie algebra of smooth vector fields on \( N \). Roughly speaking, the geometric approach to Poisson structures and Poisson cohomology proceeds in formally the same way as above with the Kähler differentials replaced by the smooth 1-forms on \( N \). To explain it we recall at first LICHNEROWICZ\’ [58] characterization of a Poisson structure on \( N \) in terms of a certain 2-tensor: Let \( [\cdot, \cdot]: \Lambda^L \to \Lambda^L \otimes \Lambda^L \) be the “Schouten product” or “Lagrangian concomitant” SCHOUTEN [81], [82], Nijenhuis [76], Kirillov [46], KOSZUL [52]. Given a 2-tensor \( G \in \Lambda^2_L \), let \( \{\cdot, \cdot\}: A \otimes A \to A \) be defined by

\[(3.12.2) \quad \{f, g\} = \pi_G(df, dg), \quad f, g \in A, \]

where \( \pi_G \) denotes the image of \( G \) under the obvious isomorphism

\[(3.12.3) \quad \phi: \Lambda_A[L] \rightarrow \text{Alt}_A(D_A^{\text{geo}}, A) \]

of graded \( A \)-algebras; conversely, given \( \{\cdot, \cdot\}: A \otimes A \to A \), define \( G \) by (3.12.2). Then \( \{\cdot, \cdot\} \) is a Poisson structure on \( N \) if and only if \( [G, G] = 0 \), see LICHNEROWICZ [58].

Given a Poisson structure \( \{\cdot, \cdot\} \), we shall refer to \( \pi_G \) as the geometric Poisson (2-)form on \( N \), and we denote by \( \pi_G^\sharp: D_A^{\text{geo}} \rightarrow L \) the indicated adjoint. Furthermore, for \( \alpha \in D_A^{\text{geo}} \), we write

\[ \alpha^\sharp = \pi_G^\sharp(\alpha). \]

The following is known (where as before \( \lambda \) refers to the usual Lie derivative):
**Proposition 3.12.4.** Let \( N \) be a smooth Poisson manifold, let \( G \in \Lambda^2_A[L] \) be its 2-tensor, and, for \( \alpha, \beta \in D^\text{geo}_A \), let

\[
[\alpha, \beta] = \lambda_{\alpha} \beta - \lambda_{\beta} \alpha - d(\pi_G(\alpha, \beta)) \in D^\text{geo}_A, \quad \alpha, \beta \in D^\text{geo}_A.
\]

Then \((\pi^*_G, [\cdot, \cdot])\) together with the \(A\)-module structure endows \( D^\text{geo}_A \) with a structure of an \((R, A)\)-Lie algebra in such a way that \(\pi^*_G\) is a morphism of \((R, A)\)-Lie algebras.

**Remark 3.12.6.** For an arbitrary commutative algebra \( A \) over an arbitrary ground ring \( R \), the obvious morphism

\[
\Lambda_A[\text{Der}(A)] \longrightarrow \text{Alt}_A(D_A, A)
\]

is always defined, but it will in general not be an isomorphism; furthermore, given a Poisson structure \(\{\cdot, \cdot\}\) on \(A\), its (algebraic) closed 2-form \(\pi_{\{\cdot, \cdot\}} \in \text{Alt}^2_A(D_A, A)\) introduced in (3.5) is always defined but need not come from \(\Lambda_A[\text{Der}(A)]\). In the situation of (3.12.4) the morphism (3.12.7) is just the composite of (3.12.3) with the morphism

\[
q^*: \text{Alt}_A(D^\text{geo}_A, A) \longrightarrow \text{Alt}_A(D_A, A),
\]

induced by (3.1.1), and under (3.12.7) the 2-tensor \(G\) is mapped to \(\pi_{\{\cdot, \cdot\}}\). Moreover, we shall see in (3.12.13) below that (3.12.8) is indeed an isomorphism; N. B. that such a remark makes sense since we are in the geometric case. This explains in which sense the algebraic 2-form \(\pi_{\{\cdot, \cdot\}}\) generalizes Lichnerowicz’ geometric 2-tensor for smooth finite dimensional Poisson manifolds.

**Proof of 3.12.4.** It is shown in (3.1) of WEINSTEIN [105] and (III.2.1) of COSTE-DAZORD-WEINSTEIN [17] that \(G\) yields a morphism \(T^*N \to TN\) of vector bundles inducing \(\pi^*_G\) and that this structure together with the bracket yields in fact a structure of a Lie algebroid on the cotangent bundle \(T^*N\). This is just another way to spell out the assertion. \(\Box\)

Henceforth we write \(D^\text{geo}_{\{\cdot, \cdot\}}\) for \(D^\text{geo}_A\) together with the \((R, A)\)-Lie algebra structure given in (3.12.5) above. It is clear that the machinery of Section 1 applies to \(D^\text{geo}_{\{\cdot, \cdot\}}\) as well and (3.8) – (3.11) have precise geometric analogues. In particular, for a smooth Poisson manifold \(N\) we refer to the cohomology of the differential graded commutative algebra \(\text{Alt}_A(D^\text{geo}_{\{\cdot, \cdot\}}, A)\) as the geometric Poisson cohomology of \((N, \{\cdot, \cdot\})\), written \(H^*_\text{Poisson}(N, \{\cdot, \cdot\}; R)\). More generally, for any vector bundle \(\zeta\) over \(N\) with a flat \(D^\text{geo}_{\{\cdot, \cdot\}}\)-connection we have the differential graded \(\text{Alt}_A(D^\text{geo}_{\{\cdot, \cdot\}}, A)\)-module \(\text{Alt}_A(D^\text{geo}_{\{\cdot, \cdot\}}, \Gamma(\zeta))\); we refer to its cohomology as the geometric Poisson cohomology of \((N, \{\cdot, \cdot\})\) with values in \(\zeta\), written \(H^*_\text{Poisson}(N, \{\cdot, \cdot\}; \zeta)\). In view of (1.14), since \(D^\text{geo}_A\) is projective as an \(A\)-module, we have \(H^*_\text{Poisson}(N, \{\cdot, \cdot\}; \zeta) = \text{Ext}^*_U(A, D^\text{geo}_{\{\cdot, \cdot\}})(A, \Gamma(\zeta))\), where \(U(A, D^\text{geo}_{\{\cdot, \cdot\}})\) refers to the corresponding universal algebra of differential operators introduced in Section 1. Virtually the same argument as in (3.9) shows that the geometric 2-form \(\pi_G\) of \(N\) is a 2-cocycle and represents a class \([\pi_G] \in H^2_\text{Poisson}(N, \{\cdot, \cdot\}; R)\). Likewise, with respect to the right \(U(A, D^\text{geo}_{\{\cdot, \cdot\}})\)-module structure (1.8.4) on \(A\), we have the chain complex \(A \otimes U(A, D^\text{geo}_{\{\cdot, \cdot\}}) K(A, D^\text{geo}_{\{\cdot, \cdot\}})\) where \(K(A, D^\text{geo}_{\{\cdot, \cdot\}})\) refers to
the corresponding Koszul complex (1.13); this chain complex computes what we call the geometric Poisson homology of \((N,\{\cdot,\}\}) with values in \(\mathbb{R}\). We write \(H^{\mathrm{Poisson}}_\bullet(N,\{\cdot,\};\mathbb{R})\) for the latter. Since \(D^\mathrm{geo}_A\) is projective as an \(A\)-module, we have \(H^{\mathrm{Poisson}}_\bullet(N,\{\cdot,\};\mathbb{R}) = \text{Tor}_\bullet(U(A,D^\mathrm{geo}_\{\cdot,\})^\vee(A,A)\). We note that there is a more general notion of geometric Poisson homology with values in a suitable vector bundle, exactly analogous to that for the algebraic case explained earlier; we refrain from giving the details.

It has been noted by several people that, for the ring \(A\) of smooth functions on a smooth finite dimensional Poisson manifold \(N\), the formula (3.12.5) yields a Lie bracket on the space of 1-forms of \(N\). To our knowledge the first ones to spell out such a bracket were GELFAND-DORFMAN [29] (p. 243), but it is not shown, however, that this bracket satisfies the Jacobi identity. The bracket occurs also in MAGRI AND MOROSI [67], see also (2.2) in MAGRI-MOROSI-RAGNISIO [68], on p. 266 of KOZUL [52] (denoted by \([\cdot,\cdot]_w = [\cdot,\cdot]_\Delta\), in (3.22) of KARASEV [44] (for a special class of Poisson manifolds), in (2.1) of BHASKARA AND VISWANATH [8], in COSTE-DAZORD-WEINSTEIN [17], in WEINSTEIN [105], and perhaps work of others; however, it seems that only in (3.1) of WEINSTEIN [105] and (III.2.1) of COSTE-DAZORD-WEINSTEIN [17] is it pointed out that the bracket yields in fact a structure of a Lie algebroid over \(N\) which is the geometric analogue of an \((\mathbb{R},A)\)-Lie algebra.

Next we relate the algebra with the geometry by means of the following.

**Lemma 3.12.9.** Let \(N\) be a smooth Poisson manifold, and let \((A,\{\cdot,\})\) be its real Poisson algebra. Then the surjection \(q:D_A \to D^\mathrm{geo}_A\) is a morphism \(q:D_{\{\cdot,\}} \to D_{\{\cdot,\}}\) of \((\mathbb{R},A)\)-Lie algebras. In particular, the adjoint \(\pi^2_{\{\cdot,\}}:D_A \to \text{Der}(A)\) factors through \(q\). Moreover, the corresponding algebraic and geometric Poisson 2-forms \(\pi_{\{\cdot,\}}:D_A \otimes_A D_A \to A\) and \(\pi_G:D^\mathrm{geo}_A \otimes_A D^\mathrm{geo}_A \to A\) are related by

\[\pi_G(q \otimes q) = \pi_{\{\cdot,\}}:D_A \otimes_A D_A \to A.\]

**Proof.** Inspection of the definitions of \(\pi_{\{\cdot,\}}\) in (3.5) and \(\pi_G\) in (3.12.2) reveals that \(\pi_G(q \otimes q) = \pi_{\{\cdot,\}}\). Moreover this implies that the adjoint \(\pi^2_{\{\cdot,\}}:D_A \to \text{Der}(A)\) factors through \(q\). Finally, since the bracket on \(D_A\) may as well be defined by (3.8.1) and since the bracket on \(D^\mathrm{geo}_A\) is defined by (3.12.5), the morphism \(q:D_A \to D^\mathrm{geo}_A\) is in fact a morphism \(q:D_{\{\cdot,\}} \to D^\mathrm{geo}_{\{\cdot,\}}\) of \((\mathbb{R},A)\)-Lie algebras. □

**Corollary 3.12.10.** Let \(N\) be a smooth Poisson manifold, let \((A,\{\cdot,\})\) be its real Poisson algebra, and let \(\zeta\) be a smooth vector bundle on \(N\) with a flat \(D^\mathrm{geo}_{\{\cdot,\}}\)-connection. Then the morphism \(q:D_A \to D^\mathrm{geo}_A\) induces morphisms

\[
H^\mathrm{Poisson}_\bullet(N,\{\cdot,\};\zeta) \to H^\mathrm{Poisson}_\bullet(A,\{\cdot,\};\Gamma(\zeta))
\]

and

\[
H^\mathrm{Poisson}_\bullet(A,\{\cdot,\};A) \to H^\mathrm{Poisson}_\bullet(N,\{\cdot,\};\mathbb{R})
\]

of graded real vector spaces. Furthermore, the class \([\pi_G] \in H^2_{\text{Poisson}}(N,\{\cdot,\};\mathbb{R})\) goes to the class \([\pi_{\{\cdot,\}}] \in H^2_{\text{Poisson}}(A,\{\cdot,\};\mathbb{A})\).

Somewhat surprisingly, we have the following.
Theorem 3.12.13. Let $N$ be a smooth Poisson manifold, let $(A,\{\cdot,\cdot\})$ be its real Poisson algebra, and let $\zeta$ be a smooth vector bundle on $N$ with a flat $D_{\{\cdot,\cdot\}}$-connection. Then the morphism

\[ q^*: \text{Alt}_A(D_{\{\cdot,\cdot\}}^\text{geo}, \Gamma(\zeta)) \to \text{Alt}_A(D_{\{\cdot,\cdot\}}, \Gamma(\zeta)) \]

induced by $q: D_A \to D_A^\text{geo}$ is an isomorphism of real chain complexes. Consequently (3.12.11) is an isomorphism of real graded vector spaces.

Hence in the smooth case there is no need to distinguish between geometric and algebraic Poisson cohomology with coefficients in a smooth vector bundle.

Proof. Since the morphism $q: D_A \to D_A^\text{geo}$ of $A$-modules is surjective, the induced morphism (3.12.14) is an injective morphism of graded $A$-modules and in view (3.12.9) is compatible with the differentials. We show it is also surjective.

To see this we observe first that it suffices to consider the case where $\zeta$ is a trivial line bundle and hence, as an $A$-module, $\Gamma(\zeta) \cong A$. In fact, since $\Gamma(\zeta)$ is a finitely generated projective $A$-module, we can easily reduce to the case of a finitely generated free $A$-module, and from this we reduce further to the case of a free $A$-module of dimension 1.

To see that $\text{Alt}_A(D_A^\text{geo}, A) \to \text{Alt}_A(D_A, A)$ is surjective, we observe first that we already know that

\[ \text{Hom}_A(D_A^\text{geo}, A) \to \text{Hom}_A(D_A, A) \]

is surjective, in fact an isomorphism. Indeed, cf. (3.2.2), $\text{Hom}_A(D_A, A)$ is canonically isomorphic to the $A$-module $\text{Der}(A)$ of derivations of $A$, i.e. to the smooth vector fields on $N$, while $\text{Hom}_A(D_A^\text{geo}, A)$ is the dual of the (projective) $A$-module of smooth 1-forms which is again canonically isomorphic to the smooth vector fields on $N$. In other words, in degree one the isomorphism amounts to the classical fact that on a smooth finite dimensional manifold a derivation is always induced from a smooth vector field. The general case is only slightly more complicated than this. In fact, let

\[ \phi: (D_A)^{\otimes k} \to A \]

be an $A$-multilinear $k$-form on $D_A$ with values in $A$, and let

\[ \phi^\#: D_A \to \text{Hom}_A((D_A)^{\otimes (k-1)}, A) \]

be its adjoint. By induction we may assume that the induced morphism

\[ \text{Hom}_A((D_A^\text{geo})^{\otimes (k-1)}, A) \to \text{Hom}_A((D_A)^{\otimes (k-1)}, A) \]

is an isomorphism, and hence we can identify $\text{Hom}_A((D_A)^{\otimes (k-1)}, A)$ with the space of sections of the $(k-1)$’th tensor power $TN^{\otimes (k-1)}$ of the tangent bundle $TN$. In other words, $\phi^\#$ may be viewed as a differential operator between the trivial line bundle on $N$ and $TN^{\otimes (k-1)}$. However, such a differential operator is a section in the smooth vector bundle $\text{Hom}(T^*N, TN^{\otimes (k-1)})$ over $N$, and hence $\phi^\#$ factors through $D_A \to D_A^\text{geo}$. Consequently $\phi$ factors through $(D_A^\text{geo})^{\otimes Ak}$, and hence the induced morphism

\[ \text{Hom}_A((D_A^\text{geo})^{\otimes Ak}, A) \to \text{Hom}_A((D_A)^{\otimes Ak}, A) \]

is an isomorphism. This completes the proof. □

Next we relate our notions of Poisson homology and cohomology with what is in the literature.
Theorem 3.13. Let \( N \) be a smooth finite dimensional Poisson manifold, and let \( A \) be its ring of smooth real functions with Poisson structure \( \{ \cdot, \cdot \} \). Then the geometric Poisson cohomology of \( (N, \{ \cdot, \cdot \}) \) with values in \( \mathbb{R} \) coincides with the Poisson cohomology of \( N \) introduced by Lichnerowicz [58]. In other words, Lichnerowicz’ Poisson cohomology is naturally isomorphic to \( \Ext^{*}_{U(A,D^\text{geo}_{\{\cdot,\cdot\}})}(A,A) \) with respect to the corresponding universal algebra \( U(A,D^\text{geo}_{\{\cdot,\cdot\}}) \) of differential operators. Furthermore, this cohomology also coincides with the (algebraic) Poisson cohomology of \( (A,\{\cdot,\cdot\}) \) with values in \( A \).

A brief comparison of the definitions shows that the first statement holds. In fact, in the present special case, the chain complex \( \Hom_{U(A,D^\text{geo}_{\{\cdot,\cdot\}})}(K(A,D^\text{geo}_{\{\cdot,\cdot\}}),A) \) which computes \( H_{\text{Poisson}}^*(N,\{\cdot,\cdot\};\mathbb{R}) = \Ext^{*}_{U(A,D^\text{geo}_{\{\cdot,\cdot\}})}(A,A) \) is the same as that introduced by Lichnerowicz. Thus our approach yields a description of this cohomology in terms of standard homological algebra, i.e. as an \( \Ext \) over a suitable ring. The ‘furthermore’ statement is an immediate consequence of Theorem 3.12.13. It shows that our notion of Poisson cohomology generalizes Lichnerowicz’ notion for smooth Poisson manifolds to arbitrary Poisson algebras.

Theorem 3.14. Let \( N \) be a smooth finite dimensional Poisson manifold, and let \( A \) be the ring of smooth functions on \( N \). Then the geometric Poisson homology of \( (N,\{\cdot,\cdot\}) \) with values in \( \mathbb{R} \) coincides with the canonical homology of \( N \) introduced by Brylinski [12]. In other words, Brylinski’s canonical homology is naturally isomorphic to \( \Tor^{*}_{U(A,D^\text{geo}_{\{\cdot,\cdot\}})}(A,A) \) with respect to the corresponding universal algebra \( U(A,D^\text{geo}_{\{\cdot,\cdot\}}) \) of differential operators.

This is again seen by a brief comparison of the definitions. In the present special case, the chain complex \( A \otimes_{U(A,D^\text{geo}_{\{\cdot,\cdot\}})} K(A,D^\text{geo}_{\{\cdot,\cdot\}}) \) coincides with the chain complex introduced by Koszul [52] and used in Brylinski [12] for the definition of canonical homology. Thus our approach yields a description of Brylinski’s canonical homology in terms of standard homological algebra, i.e. as a \( \Tor \) over a suitable ring. However, unlike the Poisson cohomology case, there is no reason why the two notions of Poisson homology should coincide. More precisely, algebraic Poisson homology is computed by \( \Lambda_{A}[D_{A}] \), equipped with a suitable differential, while geometric Poisson homology is computed by \( \Lambda_{A}[D^\text{geo}_{A}] \) equipped with a suitable differential, and the surjection \( \Lambda_{A}[D_{A}] \twoheadrightarrow \Lambda_{A}[D^\text{geo}_{A}] \) induced by (3.12.1) is compatible with the differentials. Notice that as a graded \( A \)-algebra \( \Lambda_{A}[D^\text{geo}_{A}] \) is just the de Rham complex of \( N \). Further, the obvious morphism \( \iota: R \rightarrow A \) gives rise to the morphism \( (\iota, \Id):(R,D^\text{geo}_{\{\cdot,\cdot\}}) \rightarrow (A,D^\text{geo}_{\{\cdot,\cdot\}}) \) which, cf. Section 1, induces a morphism \( K(\iota, \Id):K(R,D^\text{geo}_{\{\cdot,\cdot\}}) \rightarrow K(A,D^\text{geo}_{\{\cdot,\cdot\}}) \) of the corresponding Koszul complexes. This yields an obvious morphism from ordinary Chevalley-Eilenberg [16] Lie algebra homology to geometric Poisson homology whose existence was observed by Brylinski [12].

Let \( N \) be a smooth manifold, let \( A \) be the algebra of smooth real valued functions on \( N \), and let \( L \) be the \((\mathbb{R}, A)\)-Lie algebra of smooth vector fields. As in the classical case, call an \( A \)-bilinear alternating 2-form \( \sigma:L \otimes_{A} L \rightarrow A \) a symplectic structure if its adjoint \( \sigma^{\sharp}:L \rightarrow \Hom_{A}(L,A) \) is an isomorphism of \( A \)-modules, cf. e.g. p. 28 of Marsden [70]. If this is the case, and if \( N \) is finite dimensional so that (i)
We now study this example in the light of our notion of Poisson cohomology: Write $S_{\{\cdot,\cdot\}}$ is a Poisson potential for $\sigma$ the adjoint algebraic approach to Poisson structures works in complete generality.

Proposition 3.15. Let $(N, \sigma)$ be a smooth finite dimensional symplectic manifold, let $A$ be its algebra of smooth real functions, let $L$ be the $(R, A)$-Lie algebra of smooth vector fields on $N$, and let $\{\cdot,\cdot\}$ the the corresponding Poisson structure on $A$. Then the adjoint $\sigma^b: L \to D^\text{geo}_{\{\cdot,\cdot\}}$ of $\sigma$ is an isomorphism of $(R, A)$-Lie algebras.

This entails at once the following.

Addendum to 3.13 (Lichnerowicz [58]). For a smooth finite dimensional symplectic manifold the de Rham and Poisson cohomologies coincide.

Furthermore, (3.15) explains the $^*$-operation introduced in (2.2.2) of Brylinski [12]: In fact, this operation is just the composite of the Poincaré duality map with the isomorphism $\text{Tor}_*^{U(A, D^\text{geo}_{\{\cdot,\cdot\}})}(A, A) \to \text{Tor}_*^{U(A,L)}(A, A) = H_*(N, R)$ induced by the inverse of $\sigma^b$ – where we have to keep in mind that a symplectic manifold is orientable. We note that, for an arbitrary algebra $A$ over a commutative ring $R$, the concept of a symplectic structure does not seem to extend properly whereas our algebraic approach to Poisson structures works in complete generality.

We conclude this Section with a number of examples:

Example 3.16. Let $A$ be a commutative algebra, let $L$ be an $(R, A)$-Lie algebra, and let $S_A[L]$ be the symmetric algebra on $L$ over $A$. Then the $L$-action on $A$ and the bracket operation on $L$ induce an obvious Poisson structure

(3.16.1) \[ \{\cdot,\cdot\}: S_A[L] \otimes S_A[L] \to S_A[L] \]

on $S_A[L]$; explicitly, this structure is determined by

(3.16.2.1) \[ \{\alpha, \beta\} = [\alpha, \beta], \quad \alpha, \beta \in L, \]

(3.16.2.2) \[ \{\alpha, a\} = \alpha(a) \in A, \quad a \in A, \alpha \in L, \]

(3.16.2.3) \[ \{u, vw\} = \{u, v\}w + v\{u, w\}, \quad u, v, w \in S_A[L]. \]

We now study this example in the light of our notion of Poisson cohomology: Write $S = S_A[L]$, and let $\pi: D_S \otimes_{S} D_S \to S$ be the 2-form (3.5.1); as before, we write $D_{\{\cdot,\cdot\}}$ for $D_S$ together with the $(R, S)$-Lie algebra structure. We assert that $\pi$ is a Poisson coboundary, i. e. $\{\cdot,\cdot\}$ admits a Poisson potential. As an algebra, $S$ is generated by the elements of $A$ and those of $L$. Hence as an $S$-module, $D_S$ is generated by the formal differentials $da, a \in A$, and $d\alpha, \alpha \in L$, and there is an obvious surjection $S \otimes_A D_A \otimes S \otimes_A L \to D_S$ given by $1 \otimes da \mapsto da, a \in A$, and $1 \otimes \alpha \mapsto d\alpha, \alpha \in L$, with a slight abuse of notation. We assert that the 1-form $\vartheta: D_S \to S$ given by

(3.16.3) \[ \vartheta(da) = 0, \quad a \in A, \quad \vartheta(d\alpha) = \alpha, \alpha \in L, \]

is a Poisson potential for $\{\cdot,\cdot\}$. Indeed, from the formula

\[
(d\vartheta)(du, dv) = (du)^\sharp(\vartheta(dv)) - (dv)^\sharp(\vartheta(du)) - \vartheta[du, dv] = \{u, \vartheta(dv)\} - \{v, \vartheta(du)\} - \vartheta(d\{u, v\})
\]
we conclude at once

\[(3.16.4) \quad \pi = d\theta \in \text{Alt}^2_S(D_{\{\cdot,\cdot\}}, S).\]

When \(L\) is projective as an \(A\)-module, so that, in view of Rinehart’s result reproduced as (1.9) above, the obvious map \(S_A[L] \to E^0(U(A,L))\) is an isomorphism, the non-commutative Poisson algebra \(U(A,L)\) of differential operators is a deformation in the sense of Gerstenhaber [29] of the commutative Poisson algebra \((S_A[L], \{\cdot,\cdot\})\). In the even more special case where \(N\) is a smooth finite dimensional manifold, \(A\) the ring of smooth functions on \(N\), and \(L\) the Lie algebra of smooth vector fields on \(N\), \(S_A[L]\) is the algebra of smooth functions on the cotangent bundle of \(N\) which are polynomial on each fibre; these are called polynomial observables, see e. g. p. 84 of Woodhouse [108]. Furthermore, the Poisson structure (3.16.1) then comes from the classical one on the smooth functions on the cotangent bundle of \(N\); this Poisson structure is the “symmetric concomitant” of Schouten [81], [82]. For these matters, see also p. 180 of Vinogradov and Krasil’shchik [100] and (3.3) of Braconnier [10], [11]. When \(N\) is finite dimensional, the relation (3.16.4) is the algebraic analogue of the usual local formula expressing the symplectic structure of the cotangent bundle \(T^*N\) as a coboundary \(\sigma = dpdq\) of the 1-form \(pdq\) written out in local coordinates \((q,p)\). The relationship between \(S_A[L]\) and \((U(A,L)\) is the heart of the geometric quantization program initiated by I. Segal [84], Kostant [48], and Souriau [93].

Example 3.17. We now modify the above example: Under the circumstances of (3.16), let \(\psi: D_{\{\cdot,\cdot\}} \to S \otimes_A L\) be the morphism of \(S\)-modules determined by \(\psi(da) = 0, a \in A,\) and \(\psi(da) = \alpha, \alpha \in L,\) and let \(\psi^*: \text{Alt}_S(S \otimes_A L, S) \to \text{Alt}_S(D_{\{\cdot,\cdot\}}, S)\) be the induced morphism of graded algebras. Inspection shows its composite

\[(3.17.1) \quad \text{Alt}_A(L, A) \to \text{Alt}_S(D_{\{\cdot,\cdot\}}, S)\]

with the obvious map \(\text{Alt}_A(L, A) \to \text{Alt}_A(L, S) \cong \text{Alt}_S(S \otimes_A L, S)\) is a morphism of differential graded algebras. We note that (1.16.1) and (1.16.2) endow \(S \otimes A L\) in fact with a structure of an \((R,S)\)-Lie algebra, but \(\psi\) is not a morphism of \((R,S)\)-Lie algebras. In particular, let \(\chi: L \otimes_A L \to A\) be a closed alternating 2-form and let \(\chi^b: D_S \otimes_S D_S \to S\) be its image in \(\text{Alt}_S^2(D_{\{\cdot,\cdot\}}, S)\). Then \(\pi_{\{\cdot,\cdot\}} + \chi^b\) is a closed alternating \(S\)-bilinear 2-form on \(D_S\). Moreover, let \(\omega: D_S \to \text{Hom}_S(D_S, S)\) be its adjoint, and define a new bracket \([\cdot,\cdot]: D_S \otimes_S D_S \to D_S\) by (3.11.0). Inspection shows that this yields a structure of an \((R,S)\)-Lie algebra on \(D_S\) and hence, by virtue of (3.11), a new Poisson algebra structure \(\{\cdot,\cdot\}\) on \(S = S_A[L]\). Explicitly, this structure is determined by

\[(3.17.2) \quad \{\alpha, \beta\} = [\alpha, \beta] + \chi(\alpha, \beta), \quad \alpha, \beta \in L,\]

together with (3.16.2.2) and (3.16.2.3). This yields examples of Poisson algebras with non-trivial Poisson class.

To explain the geometric analogue of this class of examples, let \(N\) be a smooth finite dimensional manifold, let \(\tau: T^*N \to N\) be its cotangent bundle, let \(\sigma = dpdq\) be the standard symplectic form on \(T^*N\), and let \(\chi\) an arbitrary closed 2-form on
$N$. Then $\sigma + \tau^*(\chi)$ is again a symplectic structure on $T^*N$. The above 2-form $\pi_{\{.,\}} + \chi^b$ is the formal analogue of the image of $\sigma + \tau^*(\chi)$ under the isomorphism $\text{Alt}_C(L_C, C) \to \text{Alt}_C(D_C^{\text{geo}}, C)$ in the complex $\text{Alt}_C(D_C^{\text{geo}}, C)$ computing the Poisson cohomology of the algebra $C = C^\infty(T^*N)$ of smooth functions on $T^*N$; here $L_C$ refers to the smooth vector fields on $T^*N$, and $\text{Alt}_C(L_C, C) \to \text{Alt}_C(D_C^{\text{geo}}, C)$ to the isomorphism induced by the adjoint of $\sigma + \tau^*(\chi)$, cf. (3.15). In particular, the cohomology class $[\sigma + \tau^*(\chi)] \in H^2(T^*N)$ is non-zero if $[\chi] \in H^2(N)$ is non-zero. It is clear that this yields examples of Poisson algebras with non-trivial Poisson class.

**Example 3.18.** Another variant of (3.16) yields an example which in a special case goes back to Lie [63]: Let $g$ be a Lie algebra over $R$, let $S = S[g]$ be the symmetric algebra on $g$, and let $\{\cdot, \cdot\}: S \otimes S \to S$ be the corresponding Poisson structure (3.16.1) (with the obvious change in notation). We assume that $g$ is projective as an $R$-module. Then the morphism

\[(3.18.1) \quad S \otimes g \to D_S, \quad s \otimes x \mapsto sdx, \quad s \in S, \ x \in g,\]

is an isomorphism of $S$-modules. Furthermore, we assert that, when $S \otimes g$ is endowed with the induced $(R, S)$-Lie algebra structure illustrated in (1.16.4), the morphism (3.18.1) is an isomorphism $S \otimes g \to D_{\{\cdot, \cdot\}}$ of $(R, S)$-Lie algebras, where as before $D_{\{\cdot, \cdot\}}$ refers to $D_S$ with the $(R, S)$-Lie algebra structure given by (3.8.1) above. In fact, for $x, y \in g \subseteq S$, the Poisson bracket $\{x, y\} \in S$ is defined by $\{x, y\} = [x, y] \in g \subseteq S$. Now, on the one hand, the bracket (3.8.1) on $D_S$ is given by

\[adx, bdy = a\{x, b\}dy + b\{a, y\}dx + ab\{x, y\}, \quad a, b \in S, \ x, y \in g,\]

while on the other hand the bracket which is part of the induced $(R, S)$-Lie algebra structure (1.16.4) on $S \otimes g$ is given by

\[ax, by = ax(b)y - by(a)x + ab[x, y], \quad a, b \in S, \ x, y \in g,\]

where we have discarded the tensor product symbol and written $ax = a \otimes x$ etc. Hence (3.18.1) is indeed an isomorphism of $(R, S)$-Lie algebras. In this way, for an arbitrary Poisson algebra $A$, the $(R, A)$-Lie algebra structure (3.8.1) on the $A$-module $D_A$ of formal differentials appears as a generalization of the concept of induced structure in (1.16.4).

Under the present circumstances we can identify Poisson cohomology with a well known object: It is clear that the obvious injection map $g \to S \otimes g$ induces an isomorphism

\[(3.18.2) \quad \text{Alt}_S(S \otimes g, S, d) \longrightarrow (\text{Alt}_R(g, S), d)\]

of chain complexes where $d$ refers to the corresponding Cartan-Chevalley-Eilenberg differentials (1.3). In view of the isomorphism (3.18.2) we conclude:

**3.18.3.** For any $g$-module $M$, the Poisson cohomology of $S[g]$ with values in $M$ (with the obvious $(S[g], D_{\{\cdot, \cdot\}})$-module structure on $M$) is isomorphic to the usual Lie algebra cohomology of $g$ with values in $M$.
We note that in view of what was said in (3.16) above, the Poisson class of \((S,\{\cdot,\cdot\}); S)\) in \(H^2_{\text{Poisson}}(S,\{\cdot,\cdot\}; S)\) is trivial. In fact, in view of (3.16.3) a Poisson potential \(\vartheta : D_S \to S\) is given by \(\vartheta(dx) = x, x \in g\).

An algebra closely related to \(S[g]\) is classical and has been studied extensively in the literature: Assume the ground ring is that of the reals \(\mathbb{R}\), let \(g\) be a finite dimensional real Lie algebra, and let \(g^*\) be its dual. Then the obvious modification of the above construction yields a Poisson algebra structure on the algebra of smooth, analytic, or some other class functions on \(g^*\) (the prior construction yields the polynomial functions). In this way \(g^*\) becomes a Poisson manifold (in the appropriate category), even though it is not in general a symplectic manifold. Such structures were studied by Lie [63] and others Berezin [6], Kostant [50], Kirillov [46], Souriau [93], [94], Lichnerowicz [61], Weinstein [102] – [106], Koszul [52], Conn [18], [19]. Furthermore, the obvious modifications of (3.18.1) and (3.18.2) yield at once the following:

**3.18.4.** The Poisson cohomology of \(C^\infty(g^*)\) with values in \(C^\infty(g^*)\) is isomorphic to the usual Lie algebra cohomology of \(g\) with values in \(C^\infty(g^*)\) with respect to the coadjoint representation.

This cohomology plays a significant role in the “linearization problem” for real Poisson manifolds Weinstein [102], [103], Conn [18], [19].

**Example 3.19.** Let \(A = R[u_1, u_2]\) be the polynomial algebra in \(u_1\) and \(u_2\) as indicated, let \(p \in A\) be an arbitrary polynomial, and define a Poisson structure \(\{\cdot,\cdot\}\) on \(A\) by \(\{u_1, u_2\} = p\). Then a little thought reveals that if this Poisson structure admits a Poisson potential \(\vartheta\), for degree reasons we must have \(\vartheta(du_1) = \alpha u_1\) and \(\vartheta(du_2) = \beta u_2\) for suitable constants \(\alpha, \beta \in R\). Now

\[
d\vartheta(du_1, du_2) = \{u_1, \vartheta(du_2)\} + \{\vartheta(du_1), u_2\} - \vartheta(d\{u_1, u_2\})
\]

\[
= (\alpha + \beta)p - \alpha \frac{\partial p}{\partial u_1} u_1 - \beta \frac{\partial p}{\partial u_2} u_2,
\]

and from this it is straightforward to decide whether \((A,\{\cdot,\cdot\})\) admits a Poisson potential. For example, when \(p\) is a non-trivial homogeneous quadratic polynomial \(p = au_1^2 + bu_1u_2 + cu_2^2\), (3.19.1) entails

\[
d\vartheta(du_1, du_2) = (\alpha - \beta)(cu_2^2 - au_1^2),
\]

and hence the Poisson class \([\pi_{\{\cdot,\cdot\}}]\) \(\in H^2_{\text{Poisson}}(A,\{\cdot,\cdot\}; A)\) will be non-trivial unless

1. \(b = 0, \alpha = 0, \alpha - \beta = 1\), or
2. \(b = 0, c = 0, \beta - \alpha = 1\).

It was pointed out to me by A. Weinstein that the question whether or not \((A,\{\cdot,\cdot\})\) admits a Poisson potential is closely related with the question whether or not the vector field \(X = (u_1, u_2)\) generating dilations of the plane leaves invariant the Poisson structure, i.e. whether or not the Lie derivative \(\lambda_X(p)\) equals \(p\).

**4. Linear representations of the underlying Lie algebra**

Let \((A,\{\cdot,\cdot\})\) be Poisson algebra, and let \(D_{\{\cdot,\cdot\}}\) be the corresponding \((R,A)\)-Lie algebra introduced in (3.8). In this Section we relate the Poisson class (3.10.1) of \((A,\{\cdot,\cdot\})\)
with the representation theory of the underlying Lie algebra. This is motivated by the Dirac quantization problem [22], [23], see also Kostant [48], Kostant-Sternberg [51], Simms-Woodhouse [87], Woodhouse [108]. The problem is to construct an $R$-linear representation so that the elements of the ground ring $R$ act by scalar multiplication; this is non-trivial since under the adjoint representation the elements of the ground ring act trivially.

Let $\pi = \pi_{\{\cdot,\cdot\}}$ be the Poisson 2-form (3.5.1) of $(A,\{\cdot,\cdot\})$, and let $\bar{L}_{\{\cdot,\cdot\}} = A \oplus -\pi D_{\{\cdot,\cdot\}}$ be the indicated $(R,A)$-Lie algebra whose structure is given by (3.10.2) except that $-\pi_{\{\cdot,\cdot\}}$ comes into play instead of $\pi_{\{\cdot,\cdot\}}$, so that the corresponding extension

\begin{equation}
0 \to A \to \bar{L}_{\{\cdot,\cdot\}} \to D_{\{\cdot,\cdot\}} \to 0
\end{equation}

of $(R,A)$-Lie algebras is classified by $-\{\pi_{\{\cdot,\cdot\}}\} \in H^2_{\text{Poisson}}(A,\{\cdot,\cdot\};A)$, cf. (2.6). Moreover, let

\begin{equation}
\iota_{\{\cdot,\cdot\}} = (\text{Id},d): A \longrightarrow \bar{L}_{\{\cdot,\cdot\}}.
\end{equation}

It is clear that $dA \subseteq DA$ inherits a Lie algebra structure from that on $A$. Moreover, in view of (3.8.1), for $u,v \in A$,

$$d\{u,v\} = [du, dv],$$

whence $dA \subseteq D_{\{\cdot,\cdot\}}$ is a sub Lie algebra over $R$; we denote $dA$ together with this Lie algebra structure by $\text{Ham}_{\{\cdot,\cdot\}}$. We note that the corresponding morphism (3.6) maps the elements of $\text{Ham}_{\{\cdot,\cdot\}}$ to the Hamiltonian elements in $\text{Der}(A)$, cf. what was said just before (3.8). Direct inspection proves the following:

**Proposition 4.2.** Let $(A,\{\cdot,\cdot\})$ be a Poisson algebra. Then the morphism (4.2.1) is one of Lie algebras over $R$ and makes commutative the diagram

\begin{equation}
\begin{array}{cccccc}
0 & \longrightarrow & R & \longrightarrow & A & \longrightarrow & \text{Ham}_{\{\cdot,\cdot\}} & \longrightarrow & 0 \\
& \downarrow & & \downarrow_{\iota_{\{\cdot,\cdot\}}} & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A & \longrightarrow & \bar{L}_{\{\cdot,\cdot\}} & \longrightarrow & D_{\{\cdot,\cdot\}} & \longrightarrow & 0
\end{array}
\end{equation}

in the category of Lie algebras over $R$. Furthermore, $\iota = \iota_{\{\cdot,\cdot\}}$ satisfies the formula

\begin{equation}
\iota(ab) = a\iota(b) + b\iota(a) - (ab,0),
\end{equation}

where $\iota = \iota_{\{\cdot,\cdot\}}$.

We shall say that $(A,\{\cdot,\cdot\})$ is representable if there is a projective rank one $A$-module $M$ and a $(D_{\{\cdot,\cdot\}})$-connection $\nabla: D_{\{\cdot,\cdot\}} \longrightarrow \text{End}(M)$ with curvature $-\pi_{\{\cdot,\cdot\}}$. In view of (2.15), $(A,\{\cdot,\cdot\})$ is representable if and only if its Poisson class $\{\pi_{\{\cdot,\cdot\}}\} \in H^2_{\text{Poisson}}(A,\{\cdot,\cdot\};A)$ (cf. (3.10)) lies in the image of the corresponding morphism (2.15.1). We note that $\text{End}_A(M) \cong A$ canonically.
Theorem 4.3. Let \((A,\{\cdot,\cdot\})\) be a representable Poisson algebra, and let \(M\) be a projective rank one \(A\)-module together with a \((D_{\{\cdot,\cdot\}})\)-connection \(\nabla:D_{\{\cdot,\cdot\}} \rightarrow \text{End}(M)\) with curvature \(-\pi_{\{\cdot,\cdot\}}\). Then the composite of \(i_{\{\cdot,\cdot\}}:A \rightarrow L_{\{\cdot,\cdot\}}\) with the flat \((L_{\{\cdot,\cdot\}})\)-connection \(L_{\{\cdot,\cdot\}} \rightarrow \text{End}(M)\) on \(M\) is a representation of the \(R\)-Lie algebra underlying \(A\) on \(M\) viewed as an \(R\)-module having the property that the “constants”, i. e. the elements of \(R\), act by multiplication. Explicitly, the representation is given by \(a \mapsto \hat{a}\) where, for \(a \in A\), \(\hat{a}\) refers to the operator given by
\[
\hat{a}x = \nabla_{da}x + ax, \quad x \in M.
\]
When the form \((3.5.1)\) is non-degenerate, the representation is faithful.

Proof. This follows at once from \((4.2)\). \(\Box\)

In particular, let \((A,\{\cdot,\cdot\})\) be a Poisson algebra with zero Poisson class \([\pi_{\{\cdot,\cdot\}}]\), and let \(\vartheta:D_{\{\cdot,\cdot\}} \rightarrow A\) be a Poisson potential for \((A,\{\cdot,\cdot\})\) so that \(d\vartheta = \pi_{\{\cdot,\cdot\}}\). Define a \(D_{\{\cdot,\cdot\}}\)-connection \(\nabla:D_{\{\cdot,\cdot\}} \rightarrow \text{End}(A)\) on \(A\), viewed as a free \(A\)-module with basis 1, by
\[
(4.3.1) \quad \nabla_{\alpha}(a) = \alpha^{2}(a) - \alpha\vartheta(\alpha).
\]
Then \(\nabla\) has curvature \(-\pi_{\{\cdot,\cdot\}}\), whence \((A,\{\cdot,\cdot\})\) is representable. Notice this applies in particular to the example in \((3.18)\).

We now take the ground ring \(R\) to be that of the reals \(\mathbb{R}\) and introduce a concept of “prequantization” for real Poisson algebras by means of a variant of the above. The problem of quantizing Poisson algebras which are not associated with a symplectic manifold arises in the physics of singular constrained systems, see e. g. Gotay [31], Śniatycki-Weinstein [92], and Section 5 below.

We write \(\mathbb{C}\) for the complex numbers. Consider the complexified algebra \(A \otimes \mathbb{C}\), with the obvious Poisson structure, still denoted by \(\{\cdot,\cdot\}\), and let \(D_{\{\cdot,\cdot\}}^{\mathbb{C}}\) and \(\pi_{\{\cdot,\cdot\}}^{\mathbb{C}}\) be the corresponding \((\mathbb{C},A \otimes \mathbb{C})\)- Lie algebra and Poisson 2-form, respectively. We note that these objects arise from the former ones by merely an extension of “scalars". We shall say that the real Poisson algebra \((A,\{\cdot,\cdot\})\) is quantizable if there is a projective rank one \((A \otimes \mathbb{C})\)-module \(M\) with a \((D_{\{\cdot,\cdot\}}^{\mathbb{C}})\)-connection \(\nabla\) having curvature
\[
K_{\nabla} = -i\pi_{\{\cdot,\cdot\}}^{\mathbb{C}} \in \text{Alt}^{2}_{A \otimes \mathbb{C}}(D_{\{\cdot,\cdot\}}^{\mathbb{C}}, A \otimes \mathbb{C}).
\]
In view of \((3.12.13)\) and \((3.15)\), when \((A,\{\cdot,\cdot\})\) is the algebra of smooth real valued functions on a finite dimensional symplectic manifold \((N,\sigma)\), then \((A,\{\cdot,\cdot\})\) is quantizable in our sense if and only if there is a prequantum bundle for \((N,\sigma)\), i. e. a complex line bundle \(\lambda:E \rightarrow N\) with a connection having curvature \(-i\sigma\). The reader will note that some of the numerical constants used here differ from those in the literature but this is of course of no account.

Let \(M\) be an \((A \otimes \mathbb{C})\)-module. An \(A\)-linear pairing \(\{\cdot,\cdot\}:M \otimes_{A} M \rightarrow A \otimes \mathbb{C}\) on \(M\) will be said to be a Hermitian structure on \(M\) if it has the usual Hermitian properties. Further, we shall say that a \((D_{\{\cdot,\cdot\}}^{\mathbb{C}})\)-connection \(\nabla\) on \(M\) preserves the Hermitian structure \(\{\cdot,\cdot\}\) if for \(\alpha \in D_{\{\cdot,\cdot\}}^{\mathbb{C}}\) and \(s,s' \in M\), \(\alpha(s,s') = (\nabla_{\alpha}s,s') + (s,\nabla_{\alpha}s')\) where as usual the symbol “\(-\)" refers to complex conjugation. Extending common notation, we shall say that an \(\mathbb{R}\)-linear operator \(\vartheta:M \rightarrow M\) is symmetric, if for \(s,s' \in M\), \((\vartheta s,s') = (s,\vartheta s')\). The symmetric operators on \(M\) constitute a Lie algebra over \(\mathbb{R}\) with Lie bracket \(\{\cdot,\cdot\}\), given by \([\vartheta,\xi]_{s} = i[\vartheta,\xi]\), where \(\vartheta\) and \(\xi\) are arbitrary symmetric operators and where \([\vartheta,\xi] = \vartheta\xi - \xi\vartheta\) as usual.
Theorem 4.4. Let \((A, \{\cdot, \cdot\})\) be a quantizable Poisson algebra, let \(M\) be a projective rank one \((A \otimes \mathbb{C})\)-module with a \((D^\mathbb{C}_{\{\cdot, \cdot\}})\)-connection \(\nabla\) having curvature \(K\nabla = -i\pi^\mathbb{C}_{\{\cdot, \cdot\}}\), and, for \(\alpha \in D_{\{\cdot, \cdot\}}\) and \(a \in A\), let

\[
\omega(\alpha) = -i \nabla_\alpha, \quad \omega(a) = \mu_a,
\]

where \(\mu_a\) refers to multiplication by \(a \in A\). This defines a representation \(\omega: \bar{L}_{\{\cdot, \cdot\}} \to \text{End}_R(M)\) of the \((R, A)\)-Lie algebra \(\bar{L}_{\{\cdot, \cdot\}}\) by \(R\)-linear operators on \(M\) so that, for \(a \in A\) and \(u, v \in \bar{L}_{\{\cdot, \cdot\}}\),

\[
\omega(a) = \mu_a, \quad i[\omega(u), \omega(v)] = \omega([u, v]), \quad \omega(au) = (\mu_a \omega)(u),
\]

where \(\mu_a \omega\) refers to composition of operators. When \(M\) has a Hermitian structure in such a way that \(\nabla\) is compatible with the Hermitian structure, the representation is a homomorphism into the real Lie algebra of symmetric operators on \(M\).

Proof. This is left to the reader. □

We note that, in contrast to (4.3), the morphism \(\omega\) in (4.4) does not endow \(M\) with a structure of a \((A, \bar{L}_{\{\cdot, \cdot\}})\)-module. We also note that, as usual, the factor \(i\) in (4.4.1) has the effect that, when \(M\) has a Hermitian structure and \(\nabla\) is compatible with the Hermitian structure, a differential operator is represented by a symmetric operator rather than an antisymmetric one.

When we write out the composite of \(\iota_{\{\cdot, \cdot\}}: A \to \bar{L}_{\{\cdot, \cdot\}}\) with \(\nabla\), we arrive at a proof of the following:

Corollary 4.5. Let \((A, \{\cdot, \cdot\})\) be a quantizable Poisson algebra, and let \(M\) be a projective rank one \((A \otimes \mathbb{C})\)-module with a \((D^\mathbb{C}_{\{\cdot, \cdot\}})\)-connection \(\nabla\) having curvature \(K\nabla = -i\pi^\mathbb{C}_{\{\cdot, \cdot\}}\). Then, for \(a \in A\), the formula

\[
\hat{a}s = -i \nabla_{\hat{a}} s + as, \quad s \in M,
\]

yields a representation of the \(R\)-Lie algebra underlying the Poisson algebra \(A\) by \(R\)-linear operators on \(M\) so that

1. the constants act by multiplication, and
2. for \(a, b \in A\), \(\{a, b\} = i[\hat{a}, \hat{b}]\).

When \(M\) has a Hermitian structure in such a way that \(\nabla\) is compatible with the Hermitian structure, the representation is a homomorphism into the real Lie algebra of symmetric operators on \(M\).

In Dirac’s terms [22], [23], the Poisson bracket \(\{\cdot, \cdot\}\) is thus the classical counterpart of the quantum commutator \(i[\cdot, \cdot]\). When \((A, \{\cdot, \cdot\})\) is the algebra of smooth real valued functions on a finite dimensional symplectic manifold \((N, \sigma)\), up to constants, the Corollary yields precisely Kostant’s prequantization construction [48], see also Simms-Woodhouse [87], Woodhouse [108]. In this case, the significance of the corresponding Atiyah sequence analogous to (4.1) (cf. 2.16.1) for quantization was noticed already by Almeida-Molino [2]. Corollary 4.5 goes beyond traditional prequantization since it yields a prequantization for e. g. a Poisson algebra of the
kind in (3.18) (involving a Lie algebra $g$ over the ground ring and the symmetric algebra $S[g]$ on $g$).

The usual completion of the geometric quantization procedure as explained e. g. in Woodhouse [108] points the way to the completion of the present kind of quantization. Here we confine ourselves with some remarks: The usual notions of an isotropic and Lagrangian subspace make perfectly good sense for $D_{\{\cdot,\cdot\}}^{\text{red}}$ and the 2-form $\pi_{\{\cdot,\cdot\}}$, and hence so does the concept of a polarization. Indeed, given a real Poisson algebra $(A,\{\cdot,\cdot\})$, a complex polarization $P$ may be defined as a $(\mathbb{C}, A \otimes \mathbb{C})$-Lie subalgebra $P \subseteq D_{\{\cdot,\cdot\}}^{\mathbb{C}}$ which is maximally isotropic with respect to the complexified 2-form $\pi_{\{\cdot,\cdot\}}^{\mathbb{C}}$. However, since $D_{\{\cdot,\cdot\}}^{\mathbb{C}}$ will in general not act faithfully on $A$, it remains to be seen what the appropriate generalization of a polarization should be, cf. what will be said in the next Section. Whatever choice of polarization then has been made, one can then introduce analogues of the cohomology spaces introduced e. g. on p. 216 of Woodhouse [108] and the usual inner product problem arises, cf. Rawnsley-Schmid-Wolf [79]. In the next Section we shall employ the above to quantize a system described in terms of a Poisson algebra that is not associated with a symplectic manifold.

5. Poisson reduction and a non-classical example

In this Section we extend a standard construction to the present setting. This will yield examples of Poisson algebras of a kind different from those in (3.16) – (3.19).

Let $(A,\{\cdot,\cdot\})$ be a Poisson algebra over $R$, let $g$ be a Lie algebra over $R$, and let $\delta: g \to A$ be a morphism of Lie algebras (over $R$). Let $I \subseteq A$ be the ideal in $A$ generated by the image $\delta(g) \subseteq A$. Since $\delta$ is compatible with the Lie structures, $I$ is closed under the Poisson bracket; note, however, that $I$ is not necessarily a Lie ideal. Further, with the obvious structure explained in (1.18), $A \otimes g$ is an $(R,A)$-Lie algebra, $A$ is an $(A, A \otimes g)$-module, and the obvious map $\delta^g: A \otimes g \to A$ given by $\delta^g(a \otimes y) = a \delta(y) \subseteq A$ is a morphism of $(A, A \otimes g)$-modules whence the quotient $A/I$ inherits a $g$-action. Let $A_{\text{red}} = (A/I)^g$ be the sub algebra of $g$-invariants. To describe it we recall that for an arbitrary Lie algebra $k$ and a sub Lie algebra $h \subseteq k$, the normalizer $k^h \subseteq k$ of $h$ in $k$ consists of all $\alpha \in k$ having the property that $[\alpha, x] \in h$ for every $x \in h$. Now $A_{\text{red}} = (A/I)^g$ it consists of all classes of elements $a \in A$ for which $\{a, I\} \subseteq I$, i. e. $A_{\text{red}}$ is the image $I^A/I$ in $A/I$ of the normalizer $I^A \subseteq A$ of $I$ in the sense of Lie algebras. Inspection shows that $I^A$ inherits a Poisson algebra structure, and hence so does $A_{\text{red}} = I^A/I$. We write $\{\cdot,\cdot\}_{\text{red}}$ for the latter structure and call $(A_{\text{red}},\{\cdot,\cdot\}_{\text{red}})$ the reduced Poisson algebra of $(A,\{\cdot,\cdot\})$ (with respect to $\delta$). Special cases of this kind of reduction are the reduction procedures of Marsden and Weinstein [72], and of Śniatycki and Weinstein [92], see also Weinstein [103], p. 51 of Kostant and Sternberg [51], and Stasheff [95], [96]. In fact, let $A$ be the ring of smooth functions on a Poisson manifold $N$, let $\delta: g \to A$ be a morphism of real Lie algebras, let $I \subseteq A$ be the ideal in $A$ generated by $\delta(g) \subseteq A$, and define the corresponding moment mapping $J: N \to g^*$ as usual by

$$(J(x))(\xi) = (\delta(\xi))(x), \quad x \in N, \xi \in g.$$
The ideal \( \text{Whitney} \) the sense of \( \text{quantization} \) procedure itself will not make use of the Lemma.

**5.1.** For \( X \in g, \{ \delta(X), f \} \) vanishes on \( J^{-1}(0) \).

Geometrically this means that \( f \) is constant along the restriction to \( J^{-1}(0) \) of any integral curve in \( N \) of the vector field \( \{ \delta(X), - \} \). Hence the reduced Poisson algebra \( A_{\text{red}} = (A/I)^\theta \) then appears as the algebra of classes of smooth functions on \( N \) having the property (5.1), where two such functions are identified if they assume the same values on \( J^{-1}(0) \). In the situation of symplectic reduction the assumptions are made that (i) \( N \) is a finite dimensional symplectic manifold with a Hamiltonian \( G \)-action, where \( G \) is a Lie group with Lie algebra \( g \), that (ii) the map \( J \) is the corresponding moment mapping, that (iii) 0 is a regular value thereof, and that (iv) the foliation on \( J^{-1}(0) \) comes from a principal \( G \)-bundle so that the space of leaves is the base \( B \) of the bundle. Then \( B \) inherits a symplectic structure \( \sigma_B \), the algebra of smooth functions on \( B \) may be identified with \( A_{\text{red}} \), and the reduced Poisson structure \( \{ \cdot, \cdot \}_{\text{red}} \) is induced from \( \sigma_B \). When 0 is not a regular value of \( J \), the Śniatycki-Weinstein-reduction [92] is formally the same as the above algebraic procedure except that in the description in [92] still a Lie group \( G \) with Lie algebra \( g \) comes into play. However, the question whether the ideal \( I \) contains all smooth functions on \( N \) that vanish on \( J^{-1}(0) \) then becomes a delicate problem. When this is so, the algebra \( A/I \) may be identified with the algebra of smooth functions in the sense of Whitney on \( J^{-1}(0) \), see e. g. Malgrange [68], and the normalizer \( I^A \) still consists of the smooth functions \( f \) on \( N \) having the property (5.1).

We now illustrate the algebraic reduction procedure with an example which also occurs in Gotay [31]. This example has quadratic singularities. We note that Arms, Marsden and Moncrief [3] have shown that singular momentum mappings typically have quadratic singularities.

Let \( Q \) be four dimensional Minkowski space-time, let \( T^*Q \) be its cotangent bundle, with the usual coordinates \((x_0, x_1, x_2, x_3, p_0, p_1, p_2, p_3)\) and symplectic form \( \sigma = \sum dp_j \wedge dx_j \), let \( A \) be its ring of smooth functions, and let \( J: T^*Q \to R \) be the moment mapping \( J(x, p) = p_0^2 - p_1^2 - p_2^2 - p_3^2 \). Then \( J = m^2 \) describes the constrained system, see e. g. p. 256 of Woodhouse [108], for a spinless relativistic particle with rest mass \( m \). Henceforth we write \( C(c) \) for the zero locus of \( J - c = 0 \). Notice that, geometrically, for \( c > 0 \), \( C(c) \) is a product of an \( R^4 \) with a 2-sheeted hyperboloid \( H(c) \) (each copy of which is topologically a cone on \( S^2 \), for \( c < 0 \), \( C(c) \) is a product of an \( R^4 \) with a 1-sheeted hyperboloid \( H(c) \) (which is topologically an \( S^2 \times R \)), and for \( c = 0 \) these degenerate to a product of an \( R^4 \) with a cone \( C \) (= the zero locus of \( p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0 \)).

**Lemma 5.2.** [Gotay]. The ideal \( I \) in \( A \) of smooth functions that vanish on \( C(0) \) coincides with the ideal in \( A \) generated by \( J \).

**Proof.** See (4.1) of Gotay [31]. □

Hence the algebra \( A/I \) may be identified with the algebra of smooth functions in the sense of Whitney on \( J^{-1}(0) \). We reproduced the statement of the Lemma since it will enable us to illustrate geometrically the quantization result obtained below. The quantization procedure itself will *not* make use of the Lemma.
We view $\mathbb{R}$ as an abelian Lie algebra, with the single basis element $1$. By construction, for every $c \in \mathbb{R}$, the adjoint $\delta$ of the corresponding moment mapping is given by $\delta(1) = J - c \in A$, and, whatever value $c$ assumes, the vector field $\{\delta(1), -\} = \{J, -\}$ may be described by

$$\{J, -\} = -2p_0 \frac{\partial}{\partial x_0} + 2p_1 \frac{\partial}{\partial x_1} + 2p_2 \frac{\partial}{\partial x_2} + 2p_3 \frac{\partial}{\partial x_3}.$$  

In view of what was said above, whatever value $c$ assumes, the reduced Poisson algebra $A_{\text{red}}$ consists of classes of smooth functions $f$ on $T^*\mathbb{Q}$ having the property that $\{J, f\}$ vanishes on $C(c)$, where two functions are identified whenever they coincide on $C(c)$. These are precisely classes of functions that are constant along the flow lines

$$\{t, (x, p)\} \mapsto (x_0 + tp_0, x_1 - tp_1, x_2 - tp_2, x_3 - tp_3, p_0, p_1, p_2, p_3), \quad t \in \mathbb{R},$$

with $p_0^2 - p_1^2 - p_2^2 - p_3^2 = c$. In Dirac’s terms [23] the elements of $A_{\text{red}}$ are the corresponding true (classical) observables.

The rule (5.2.2) defines a smooth action

$$G \times T^*\mathbb{Q} \rightarrow T^*\mathbb{Q}$$

of the additive group $G$ of the real numbers on $T^*\mathbb{Q}$, which is free on $T^*\mathbb{Q} - \mathbb{R}^4 \times \{0\}$ and fixes the singular set $\mathbb{R}^4 \times \{0\}$ pointwise. Since for $c \neq 0$, $(p_0, p_1, p_2, p_3) = (0, 0, 0, 0) \notin C(c)$, (5.2.3) yields a structure of a principal $G$-bundle on $C(c)$; hence, by symplectic reduction Marsden-Weinstein [72], the quotient $B(c) = C(c)/G$ then inherits a symplectic structure. Likewise, when $c = 0$, (5.2.3) yields a structure of a principal $G$-bundle on $C_0 = C(0) - \mathbb{R}^4 \times \{0\}$ and again by symplectic reduction the quotient $B_0 = C_0/G$ inherits a symplectic structure. In both cases, the reduced Poisson algebra in our sense then manifestly coincides with the standard Poisson algebra of smooth functions on $B(c)$ or $B_0$ (as appropriate). However, such a statement makes no sense for $C(0)$.

The problem we wish to study next is the quantization of the reduced Poisson algebra $(A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}})$. Since for $c \neq 0$ the quotient $B(c)$ inherits a smooth symplectic structure this case can be handled by standard geometric quantization theory, and we assume henceforth that $c = 0$. In Gotay [31] the corresponding particle is referred to as a photon and treated by a method different from that to be given below. We shall not use the name photon since in the physics literature a photon means something else (i. e. an irreducible representation of the Poincaré group with mass zero and helicity $\pm 1$).

We denote by $D_{\text{red}}$ the $A_{\text{red}}$-module of Kähler differentials for $A_{\text{red}}$ with the corresponding $(\mathbb{R}, A_{\text{red}})$-Lie algebra structure spelled out in (3.8), and we write $\pi_{\text{red}}: D_{\text{red}} \otimes A_{\text{red}} D_{\text{red}} \rightarrow A_{\text{red}}$ for its Poisson 2-form. Since the Poisson structure on $T^*\mathbb{Q}$ is that induced from the cotangent bundle symplectic structure, the morphism $\partial_{\text{geo}}: D^\text{geo}_{\{\cdot, \cdot\}} \rightarrow A$ of $A$-modules given by

$$\partial_{\text{geo}}(dp_j) = p_j, \quad \partial_{\text{geo}}(dx_j) = 0, \quad 0 \leq j \leq 3,$$
is a geometric Poisson potential for the geometric Poisson 2-form $\pi_{\text{geo}}^{\cdot,\cdot}$ associated with the real Poisson algebra $A$ of real smooth functions on $T^*Q$. We note that, with $M=T^*Q$, under the isomorphism $T^*M \to TM$ of vector bundles induced by the symplectic structure, this geometric Poisson potential corresponds to the usual symplectic potential $\Theta = \sum p_i dx_i$, viewed as a 1-form on $M$. In fact the isomorphism $T^*M \to TM$ identifies $dp_i$ with the vector field $\frac{\partial}{\partial x_i}$ etc.; cf. what was said in (3.16).

Theorem 5.4. The geometric Poisson potential $\vartheta_{\text{geo}}$ induces an algebraic Poisson potential $\vartheta_{\text{red}}: D_{\text{red}} \to A_{\text{red}}$ for the Poisson algebra $A_{\text{red}}$, i.e. a 1-form $\vartheta_{\text{red}}$ so that

$$d(\vartheta_{\text{red}}) = \pi_{\text{red}} \in \text{Alt}^2(D_{\text{red}}, A_{\text{red}}).$$

Proof. Let $\pi_{\cdot,\cdot}^{\cdot,\cdot} : D_{\cdot,\cdot} \otimes_A D_{\cdot,\cdot} \to A$ be the corresponding algebraic Poisson 2-form for $\langle A, \{\cdot,\cdot\} \rangle$. It is the composite of the obvious surjection

$$D_{\cdot,\cdot} \otimes_A D_{\cdot,\cdot} \to D_{\text{geo}}^{\cdot,\cdot} \otimes_A D_{\text{geo}}^{\cdot,\cdot}$$

with the geometric Poisson 2-form $\pi_{\text{geo}}^{\cdot,\cdot}$. Hence the composite of the obvious surjection

$$D_{\cdot,\cdot} \to D_{\text{geo}}^{\cdot,\cdot}$$

with the geometric Poisson potential $\vartheta_{\text{geo}}$ is an algebraic Poisson potential

$$\vartheta: D_{\cdot,\cdot} \to A$$

for $\langle A, \{\cdot,\cdot\} \rangle$. We assert that $\vartheta$ passes to a Poisson potential for $I^A$. To see this we observe first that, by construction, for $f \in A$,

$$\vartheta(df) = \left( \frac{\partial f}{\partial p_0} p_0 + \frac{\partial f}{\partial p_1} p_1 + \frac{\partial f}{\partial p_2} p_2 + \frac{\partial f}{\partial p_3} p_3 \right) \in A. \tag{5.4.1}$$

We note, for clarity, that here $df$ refers to the formal differential $df \in D_A$. From this and the description (5.2.1) of the vector field $\{J, -\}$ we deduce that, for $f \in A$,

$$\{J, \vartheta(df)\} = \vartheta d\{J, f\} - \{J, f\}. \tag{5.4.2}$$

In fact, a straightforward calculation shows that, for $0 \leq j \leq 3$,

$$\left\{ J, \frac{\partial f}{\partial p_j} \right\} = p_j \frac{\partial}{\partial p_j} \left\{ J, f \right\} - p_j \left\{ \frac{\partial J}{\partial p_j}, f \right\}. \tag{5.4.3}$$

From the description (5.4.1) of $\vartheta(df)$ we conclude at once that (5.4.2) holds. We now recall that the normalizer $I^A$ of $I$ in $A$ in the sense of Lie algebras consists of all $f \in A$ having the property that $\{J, f\}$ lies in $I$. However, this implies that, for $f \in I^A$, $\{J, \vartheta(df)\}$ lies in $I$ as well, i.e. that for $f \in I^A$ the value $\vartheta(df)$ lies in $I^A$. In fact, given $f \in I^A$, there is an $h \in A$ so that

$$\{J, f\} = hJ.$$
However,

\[ \vartheta(dJ) = \vartheta(d(p_0^2 - p_1^2 - p_2^2 - p_3^2)) = p_0 \vartheta(dp_0) - p_1 \vartheta(dp_1) - p_2 \vartheta(dp_2) - p_3 \vartheta(dp_3) = 2(p_0^2 - p_1^2 - p_2^2 - p_3^2) = 2J \in A. \]

Moreover

\[ d(hJ) = (dh)J + h dJ. \]

Consequently

\[ \vartheta(d\{J, f\}) = (\vartheta(dh))J + 2hJ \in I, \]

whence, in view of (5.4.2), \( \{J, \vartheta(df)\} \in I \) and hence \( \vartheta(df) \in I^A \) as asserted. Therefore the algebraic Poisson potential \( \vartheta \) induces a morphism \( \vartheta_I \) of \( I^A \)-modules so that the diagram

\[
\begin{array}{ccc}
D_{I^A} & \longrightarrow & D_A \\
\vartheta_I \downarrow & & \vartheta \downarrow \\
I^A & \longrightarrow & A
\end{array}
\]

is commutative, and \( \vartheta_I \) is an algebraic Poisson potential for the Poisson algebra \( I^A \), with corresponding Poisson 2-form

\[ \pi_I: D_{I^A} \otimes_{I^A} D_{I^A} \longrightarrow I^A. \]

By the general theory of Kähler differentials, see e. g. KUNZ [55], we know that there is an exact sequence

\[ I/I^2 \longrightarrow A_{\text{red}} \otimes_{I^A} D_{I^A} \longrightarrow D_{\text{red}} \longrightarrow 0 \]

of \( A_{\text{red}} \)-modules where the first arrow is obtained by sending a class \( f \) mod \( I^2 \) to the class of its differential \( df \) and where the second one is the obvious morphism of \( A_{\text{red}} \)-modules. We now assert that the morphism \( A_{\text{red}} \otimes_{I^A} D_{I^A} \longrightarrow A_{\text{red}} \) induced by \( \vartheta_I \) vanishes on the image of \( I/I^2 \) and hence induces an algebraic Poisson potential

\[ \vartheta: D_{\text{red}} \longrightarrow A_{\text{red}} \]

for \( \{\cdot, \cdot\}_{\text{red}} \). To see this, let \( f \in I \), so that \( f = hJ \) for some \( h \in A \). Then the image of the class \( f \) mod \( I^2 \) in \( A_{\text{red}} \otimes_{I^A} D_{I^A} \) is \([h] \otimes dJ\) where \([h] \in A_{\text{red}} = I^A/I\) denotes the class of \( h \in I^A \). However, cf. what was said above, \( \vartheta_I(dJ) = 2J = 0 \in A_{\text{red}} \), and hence \( \vartheta \) induces an algebraic Poisson potential

\[ \vartheta_{\text{red}}: D_{\text{red}} \longrightarrow A_{\text{red}} \]

as asserted. \( \Box \)

Since \([\pi_{\text{red}}] = 0 \in H^2_{\text{Poisson}}(A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}}; A_{\text{red}})\), the Poisson algebra \((A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}})\) is representable: Let \( M = A_{\text{red}}\langle b \rangle \) be the free \( A_{\text{red}} \)-module with a single basis element \( b \), and define a \( D_{\text{red}} \)-connection \( D \) on \( M \) by

\[ D_{\alpha}(b) = -(\vartheta_{\text{red}}(\alpha))b, \quad \alpha \in D_{\text{red}}. \]
Then Theorem 4.3 yields an $\mathbb{R}$-linear representation of $(A_{\text{red}}, \{\cdot, \cdot\})$, viewed as a real Lie algebra, on $M$.

Likewise, the Poisson algebra $(A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}})$ is quantizable. In fact, let $A_{\text{red}}^C = A_{\text{red}} \otimes \mathbb{C}$, $D_{\text{red}}^C = D_{\text{red}} \otimes \mathbb{C}$, etc., and define a complex $D_{\text{red}}^C$-connection $\nabla$ on $M^C = M \otimes \mathbb{C}$ by

$$\nabla_\alpha(b) = -i(\vartheta_{\text{red}}(\alpha)) b, \quad \alpha \in D_{\text{red}}^C.$$  

Its curvature $K_\nabla$ is manifestly given by

$$K_\nabla = -i\pi_{\text{red}}^C,$$  

since $d(i\vartheta_{\text{red}}) = i\pi_{\text{red}}$. In view of (4.5), the corresponding representation of the $A_{\text{red}}$ underlying real Lie algebra on $M^C = A_{\text{red}} \otimes \mathbb{C}\langle b \rangle$ is given by the formula

$$\hat{a} s = -i \nabla_{da}s + as, \quad a \in A_{\text{red}}, s \in M^C.$$  

Our prequantization construction for $(A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}}$ is now complete. To construct the quantum state space we must introduce a “polarization” and a Hilbert space structure. Because of the singularity in the classical picture the situation is rather subtle and we shall give all requisite details.

First we introduce an analogue of the “horizontal polarization” in $T^*Q$. Since $D_{\text{red}}$ does not act faithfully on $A_{\text{red}}$ some care is needed here, and we proceed as follows:

We write $(A_0, \{\cdot, \cdot\}_0), D_0,$ and $\pi_0; D_0 \otimes A_0 \to A_0$ for the corresponding structures arising from the reduced manifold $B_0 = C_0/G$. As a smooth manifold, $B_0$ is a disjoint union of two copies of the product $(\mathbb{R}^3 - 0) \times \mathbb{R}^3$. We can in fact identify $B_0$ with the cotangent bundle on the two component space $P = P_+ \cup P_-$ where $P_+(\cong \mathbb{R}^3 - 0)$ and $P_-(\cong \mathbb{R}^3 - 0)$ are the subspaces of the cone $C$ given by $p_0 > 0$ and $p_0 < 0$, respectively. To see this we introduce coordinates as follows: For $1 \leq j \leq 3$, let

$$\xi_j = \frac{x_0}{p_0} p_j + x_j,$$

where $p_0 = \pm \sqrt{p_1^2 + p_2^2 + p_3^2}$ according as we are on $P_+$ or $P_-$. In these coordinates the Poisson brackets in $A_0$ assume the form

$$\{\xi_i, p_j\} = \delta_{i,j}, \quad \{\xi_i, \xi_j\} = 0, \quad \{p_i, p_j\} = 0.$$  

This implies at once that, as a Poisson manifold, $B_0$ is the cotangent bundle $T^*P$ on $P$. In particular, $B_0$ is a symplectic manifold. We denote the symplectic structure by $\sigma$. Let $D_0^{\text{geo}}$ be the corresponding $(\mathbb{R}, A_0)$-Lie algebra of smooth 1-forms on $B_0$. Then, cf. (5.3) above, the formulas

$$\vartheta^{\text{geo}}_0(dp_j) = p_j, \quad \vartheta^{\text{geo}}_0(d\xi_j) = 0, \quad 1 \leq j \leq 3,$$  

yield a geometric Poisson potential $\vartheta_0^{\text{geo}}: D_0^{\text{geo}} \to A_0$ for the induced geometric Poisson 2-form on $B_0$. Let $L_0$ be the $(\mathbb{R}, A_0)$-Lie algebra of smooth vector fields on $B_0$, ...
and let \( \sigma^b: D_0^{\text{geo}} \to L_0 \) be the isomorphism of \((\mathbb{R}, A_0)\)-Lie algebras induced by the symplectic structure \( \sigma \) on \( B_0 \), cf. (3.15). Under \( \sigma^b \) the geometric Poisson potential \( \vartheta_0^{\text{geo}} \) corresponds to the symplectic potential \( \sum_{1 \leq j \leq 3} p_j d\xi_j \) on \( B_0 \). We note that, since in the present description of \( B_0 \) the coordinates \( p_j \) are ‘positions’ and the \( \xi_j \) ‘momenta’, this is not the standard symplectic potential \( pdq \) of a cotangent bundle.

To relate the Poisson algebras \( A_{\text{red}} \) and \( A_0 \) we observe that the inclusion \( C_0 \to C(0) \) induces a morphism \( A_{\text{red}} \to A_0 \) of Poisson algebras and hence, as explained in (3.8.4), a morphism \( (A_{\text{red}}, D_{\text{red}}) \to (A_0, D_0) \) of Lie-Rinehart algebras. Furthermore, the composite of \( \vartheta_0^{\text{geo}} \) with the obvious surjection \( D_0 \to D_0^{\text{geo}} \) is an algebraic Poisson potential \( \vartheta_0: D_0 \to A_0 \). By construction, the diagram

\[
\begin{array}{ccc}
D_{\text{red}} & \longrightarrow & D_0 \\
\vartheta \downarrow & & \vartheta_0 \downarrow \\
A_{\text{red}} & \longrightarrow & A_0
\end{array}
\]

is commutative.

Let \( H_0^{\text{geo}} \subseteq D_0^{\text{geo}} \) be the \((\mathbb{R}, A_0)\)-Lie subalgebra generated by the 1-forms \( dp_1, dp_2, dp_3 \). Under the isomorphism \( \sigma^b: D_0^{\text{geo}} \to L_0 \) of \((\mathbb{R}, A_0)\)-Lie algebras \( H_0^{\text{geo}} \) is mapped isomorphically to the vertical polarization of \( B_0 \), viewed as the cotangent bundle on the space \( P \). Notice, however, that on the cotangent bundle \( T^*Q \) of Minkowski space \( Q \) the 1-forms \( dp_0, dp_1, dp_2, dp_3 \) generate an object corresponding to the horizontal polarization of \( T^*Q \). Let \( H_0 \subseteq D_0 \) be the pre-image of \( H_0^{\text{geo}} \) under the obvious surjection \( D_0 \to D_0^{\text{geo}} \) of \((\mathbb{R}, A_0)\)-Lie algebras, and let \( H_{\text{red}} \subseteq D_{\text{red}} \) be the pre-image of \( H_0 \) under the above morphism \( (A_{\text{red}}, D_{\text{red}}) \to (A_0, D_0) \) of Lie-Rinehart algebras. A little thought reveals that \( H_{\text{red}} \) is indeed an \((\mathbb{R}, A_{\text{red}})\)-Lie algebra. It is clear that the formal differentials \( d[p_0], d[p_1], d[p_2], d[p_3] \) lie in \( H_{\text{red}} \) where \( [p_i] \in A_{\text{red}} \) denotes the class of \( p_i \in I^A \). Our philosophy is that an object of the kind \( H_{\text{red}} \) is the correct generalization of the concept of a polarization.

The object \( H_{\text{red}} \) acts on \( M^C \) in the usual way through the connection (5.5), and the \( H_{\text{red}} \)-invariants \((M^C)^{H_{\text{red}}} \subseteq M^C \) consist of elements \( ab \in M^C \), \( a \in A_{\text{red}}^C \), satisfying

\[
\nabla_{d[p_j]}(ab) = 0, \quad 0 \leq j \leq 3.
\]

To evaluate \( \nabla_{d[p_j]}(ab) \), we pick a representative \( \phi \in A \otimes C \) of \( a \in A_{\text{red}}^C \) and compute

\[
\nabla_{d[p_j]}(ab) = \left[ (dp_j)^2 \phi - i\theta(dp_j, \phi)b \right] = \left[ (dp_j, \phi) - ip_j \phi \right] b = -\left[ \frac{\partial \phi}{\partial x_j} + ip_j \phi \right] b.
\]

Hence \((M^C)^{H_{\text{red}}} \) may be identified with classes of smooth complex functions \( \phi \) on \( T^*Q \) having the property \( \{J, \phi\} \in I \otimes C \) and which, for \( 0 \leq j \leq 3 \), satisfy

\[
(5.9.1) \quad \frac{\partial \phi}{\partial x_j} + ip_j \phi \in I \otimes C;
\]

in view of Gotay’s Lemma 5.2 this means precisely that these \( \phi \) satisfy the equations

\[
(5.9.2) \quad \frac{\partial \phi}{\partial x_j} + ip_j \phi = 0, \quad 0 \leq j \leq 3,
\]
on \( C(0) \). These classes of functions \( \phi \) restrict to honest “wave functions” on \( B_0 \) in the polarization \( F = \text{im}(H_0^{\text{geo}}) \subseteq L_0 \). In fact, for a function \( \phi \) on \( T^*Q \) constant along the flow lines (5.2.2) with \( p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0 \), let \( \phi_0 \) be the function on \( B_0 \) given by

\[
\phi_0(\xi_1, \xi_2, \xi_3, p_1, p_2, p_3) = \phi(0, \xi_1, \xi_2, \xi_3, p_0, p_1, p_2, p_3),
\]

where \( p_0 = \pm \sqrt{p_1^2 + p_2^2 + p_3^2} \) according as \( (p_1, p_2, p_3) \in P_+ \) or \( (p_1, p_2, p_3) \in P_- \). If \( \phi \) satisfies (5.9.1), \( \phi_0 \) satisfies the equations

\[
(5.9.3) \quad \frac{\partial \phi_0}{\partial \xi_j} + ip_j\phi_0 = 0, \quad 1 \leq j \leq 3.
\]

To see that \( (M^C)^{H_{\text{red}}} \subseteq M^C \) is non-empty we recall that each ‘wave function’ on \( T^*Q \) in the horizontal polarization is a solution \( \phi \) of the equations

\[
\frac{\partial \phi}{\partial x_j} + i p_j \phi = 0, \quad 0 \leq j \leq 3.
\]

Hence such a \( \phi \) satisfies

\[
\{J, \phi\} = 2i J \phi \in I \otimes C
\]

and therefore represents an element of \( A_{\text{red}}^C \), in fact of \( (M^C)^{H_{\text{red}}} \subseteq M^C \), whence the latter is non-empty.

The next task is to construct the requisite Hilbert space. To this end we play the same game with \( B_0, \pi_0, \vartheta_0 \), etc. Virtually the same formula as in (5.5) above then yields a \( D_0^C \)-connection \( \nabla_0 \) on \( M_0 = (A_0 \otimes C) \langle b \rangle \) having curvature \(-i\pi_0 \). Now \( B_0 \) is the cotangent bundle on the space \( P = P_+ \cup P_- \), and \( M_0 \) is the space of smooth sections of the trivial complex line bundle \( \Lambda = B_0 \times C \) over \( B_0 \), with connection \( \nabla_0 \). This situation can be handled by standard geometric quantization theory, cf. Śniatycki [88] and Woodhouse [108]: Pick a metalinar structure on the (real) polarization \( F = \text{im}(H_0^{\text{geo}}) \subseteq L_0 \). By the general theory, a metalinar structure (on the vertical polarization of a cotangent bundle \( T^*P \)) exists if and only if \( 0 = w_1^2(P) \in H^2(P, \mathbb{Z}/2) \), where \( w_1(P) \in H^1(P, \mathbb{Z}/2) \) is the first Stiefel-Whitney class; the set of all metalinar structures, if non-empty, is parametrized up to equivalence by \( H^1(P, \mathbb{Z}/2) \). Since in our case \( P \) is homotopy equivalent to a disjoint union of two 2-spheres, a metalinar structure exists and is unique up to equivalence. Let \( \sqrt{\wedge^3 F} \) be the corresponding half form bundle over \( P \); it carries a canonical partial flat connection covering \( F \). Under the present circumstances the bundle \( \sqrt{\wedge^3 F} \) is trivial. Let \( \nu \) be a section thereof, suitably normalized (with respect to its values on appropriately chosen metatframes). The corresponding space of smooth wave functions consists of the polarized sections \( \psi = \phi_0 b \otimes \nu \) of \( \Lambda \otimes \sqrt{\wedge^3 F} \) where \( \phi_0 \) is a function on \( B_0 \). We note that ‘polarized’ amounts to the requirement that \( \phi_0 \) satisfies the equation (5.9.3). Moreover, for a smooth complex function \( \alpha \) on \( P \), let \( \phi^\alpha \) be the function on \( B_0 \) given by

\[
\phi^\alpha(\xi, p) = \alpha(p)e^{i(\xi, p)}.
\]

Then the association \( \alpha \mapsto \phi^\alpha b \otimes \nu \) defines an isomorphism between the space of smooth complex functions on \( P \) and the space of smooth wave functions. Moreover,
perhaps up to constants, the inner product between two such wave functions \( \psi \) and \( \psi' \) is given by

\[
\langle \psi, \psi' \rangle = \int_P \alpha \overline{\alpha} \, \varepsilon,
\]

where \( \varepsilon = dp_1 dp_2 dp_3 \) refers to the natural volume element on \( P \). We write \( Y_0 \) for the Hilbert space arising from this construction. Now given two functions \( \phi \) and \( \phi' \) on \( T^* Q \) representing elements of \( (MC)^{H_{\text{red}}} \subseteq MC \), we define their inner product by

\[
\langle [\phi], [\phi'] \rangle = \langle \phi_0 b \otimes \nu, \phi'_0 b \otimes \nu \rangle,
\]

where \( \phi_0 \) and \( \phi'_0 \) are the restrictions to \( B_0 \) of the classes \([\phi] \) and \([\phi'] \) in \( A_{\text{red}} \). This yields a pre-Hilbert space; its completion \( Y \) will be our quantum state space for \((A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}})\). We note that \( Y \) may be viewed as a sub Hilbert space of \( Y_0 \).

Our final task is to construct the quantum operators on \( Y \) corresponding to the classical observables in \( A_{\text{red}} \). As in standard geometric quantization theory, the polarization restricts the observables in \( A_{\text{red}} \) that eventually become “quantized”, and the formula (5.7) describing the quantum operators must be modified according to the Lie derivative of the chosen half forms. Now in general, if \( g \) is just a Lie algebra, \( h \subseteq g \) a sub Lie algebra, and if \( U \) is a \( g \)-module, with structure map \( \phi: g \to \text{End}(U) \), the action of \( g \) on \( U \) passes to an action of the pre-image \( \phi^{-1}(N_{\mathfrak{g}}(h)) \subseteq g \) of the normalizer \( N_{\mathfrak{g}}(h) = \phi(h) \) in \( \mathfrak{g} = \phi(g) \) on the invariants \( U^h \). In our situation, we recall the corresponding extension (4.1) of \((\mathbb{R}, A_{\text{red}})\)-Lie algebras which now looks like

\[
0 \to A_{\text{red}} \to \mathcal{L}_{\{\cdot, \cdot\}_{\text{red}}} \to D_{\text{red}} \to 0
\]

and take \( g = \mathcal{L}_{\{\cdot, \cdot\}_{\text{red}}} \), \( h = H_{\text{red}} \), viewed as a sub Lie algebra of \( \mathcal{L}_{\{\cdot, \cdot\}_{\text{red}}} \) through the obvious \( D_{\text{red}} \)-connection

\[
D_{\text{red}} \to A_{\text{red}} \oplus D_{\text{red}} = \mathcal{L}_{\{\cdot, \cdot\}_{\text{red}}}.
\]

Moreover, we take \( U = MC \). Our construction will directly quantize the pre-image in \( A_{\text{red}} \) of the corresponding normalizer \( N_{\mathfrak{g}}(h) \) with respect to the injection

\[
\iota_{\text{red}}: A_{\text{red}} \to \mathcal{L}_{\{\cdot, \cdot\}_{\text{red}}},
\]

cf. (4.2) above. This pre-image is the sub Lie algebra of \( A_{\text{red}} \) consisting of those \( a = [f] \in A_{\text{red}} \) which satisfy the conditions

\[
(5.10) \quad [df, dp_k] \in H_0^{\text{geo}}, \quad 0 \leq k \leq 3,
\]

where \( f \) is a function representing \( a \in A_{\text{red}} \), and where \( df \) and \( dp_k \) refer to smooth 1-forms on \( B_0 \). To understand what this really means, for \( 0 \leq k \leq 3 \), we compute

\[
[df, dp_k] = d\{f, p_k\} = d\left( \frac{\partial f}{\partial x_k} \right)
= \frac{\partial^2 f}{\partial x_0 \partial x_k} \, dx_0 + \cdots + \frac{\partial^2 f}{\partial x_3 \partial x_k} \, dx_3 + \frac{\partial^2 f}{\partial p_0 \partial x_k} \, dp_0 + \cdots + \frac{\partial^2 f}{\partial p_3 \partial x_k} \, dp_3 \in D_0^{\text{geo}},
\]
where $d$ still refers to the usual exterior derivative of smooth functions. We conclude from this that (5.10) is equivalent to the requirement that, for every $a \in A_{\text{red}}$, each representative $f \in A$ of $a$ has the property that, for $0 \leq j, k \leq 3$, the second partial derivatives

\begin{equation}
\frac{\partial^2 f}{\partial x_j \partial x_k}
\end{equation}

lie in the ideal $I$, i.e., in view of Gotay’s Lemma 5.2, vanish on the constraint $C(0)$. Hence (5.7) induces a representation of the sub Lie algebra $A_{\text{red}}^H$ of $(A_{\text{red}}, \{\cdot,\cdot\}_{\text{red}})$ consisting of classes of smooth functions $f$ on $T^*Q$ that (i) satisfy $\{J, f\} \in I$ and (ii) have the additional property that the second partial derivatives $\frac{\partial^2 f}{\partial x_j \partial x_k}$ lie in $I$.

To complete the construction we recall that on the space of smooth wave functions over $B_0$, for a function $k \in A_0$ whose Hamiltonian vector field $(dk)^\sharp$ preserves the polarization $F$ (i.e. $[(dk)^\sharp, F] \subseteq F$), according to standard geometric quantization theory, the corresponding operator $\hat{k}$ is given by

\begin{equation}
\hat{ks} = -i (\nabla_0)_{dk} s + ks - \frac{1}{2} i \text{div}_\varepsilon((dk)^\sharp)s, \quad k \in A_0, s \in M_0^C.
\end{equation}

Here for a vector field $Z$ on $B_0$ preserving the polarization $F$ the expression $\text{div}_\varepsilon(Z)$ is defined by

$$\lambda_Z \varepsilon = \text{div}_\varepsilon(Z) \varepsilon,$$

where as before $\lambda_Z$ refers to the operation of Lie derivative. Now a vector field

\[ Z = \alpha_1 \frac{\partial}{\partial \xi_1} + \alpha_2 \frac{\partial}{\partial \xi_2} + \alpha_3 \frac{\partial}{\partial \xi_3} + \beta_1 \frac{\partial}{\partial p_1} + \beta_2 \frac{\partial}{\partial p_2} + \beta_3 \frac{\partial}{\partial p_3} \]

preserves the polarization $F$ if and only if

$$\frac{\partial \beta_1}{\partial \xi_1} = 0, \quad \frac{\partial \beta_2}{\partial \xi_2} = 0, \quad \frac{\partial \beta_3}{\partial \xi_3} = 0,$$

and a straightforward calculation yields

$$\lambda_Z \varepsilon = \left( \frac{\partial \beta_1}{\partial p_1} + \frac{\partial \beta_2}{\partial p_2} + \frac{\partial \beta_3}{\partial p_3} \right) \varepsilon.$$

For $f \in A$ we now write

$$\text{div}_\varepsilon(f) = \left( \frac{\partial^2 f}{\partial x_1 \partial p_1} + \frac{\partial^2 f}{\partial x_2 \partial p_2} + \frac{\partial^2 f}{\partial x_3 \partial p_3} \right) \in A.$$

A calculation shows that, whenever the second partial derivatives $\frac{\partial^2 f}{\partial x_j \partial x_k}$ of $f$ lie in $I^2$,

$$\{J, \text{div}_\varepsilon(f)\} \in I,$$
whence then \( \text{div}_\varepsilon(f) \in I^A \). For \( a \in A_{\text{red}} \) having the property that the second partial derivatives \( \frac{\partial^2 f}{\partial x_j \partial x_k} \) of each representative \( f \) lies in \( I^2 \) and for \( s \in M^C \) we now define

\[
\hat{a}s = -i \nabla_{da}s + as - \frac{1}{2} i [\text{div}_\varepsilon(f)]s.
\]

It is manifest that for such an \( a \in A_{\text{red}} \) the restriction \( a_0 \in A_0 \) to \( B_0 \) acts on \( Y_0 \) via (5.12) and hence the injection \( Y \rightarrow Y_0 \) is compatible with the actions. Consequently \( \hat{a} \) is a symmetric operator on \( Y \); it is self-adjoint whenever the Hamiltonian vector field on \( B_0 \) induced by \( a \) is complete. For example, for \( a \in A_{\text{red}} \) represented by a function \( f \) of the kind

\[
f(x,p) = u(p) + x_0v_0(p) + x_1v_1(p) + x_2v_2(p) + x_3v_3(p),
\]

and for a state represented by a ‘wave function’ \( \phi \) on \( T^*Q \) of the kind

\[
\phi(x,p) = \alpha(p)e^{-i\langle x,p \rangle}
\]

where \( \alpha \) is a function depending on \( p_0, \ldots, p_3 \) only as indicated, this amounts to

\[
\hat{a}[\phi] = [\hat{\alpha}e^{-i\langle \cdot, \cdot \rangle}],
\]

where

\[
\hat{\alpha} = u\alpha - i \sum_0^3 v_j \frac{\partial \alpha}{\partial p_j} - \frac{1}{2} i \sum_1^3 \alpha \frac{\partial v_j}{\partial p_j}.
\]

For illustration, we mention that any function \( f \) independent of \( x_0, \ldots, x_3 \) trivially satisfies \( \{J,f\} \in I \) and trivially has second partial derivatives \( \frac{\partial^2 f}{\partial x_j \partial x_k} \) in \( I^2 \), and these requirements are met by the boosts \( \beta_j, \ 1 \leq j \leq 3 \), defined by

\[
\beta_j(x_0, x_1, x_2, x_3, p_0, p_1, p_2, p_3) = x_0p_j + x_jp_0, \quad 1 \leq j \leq 3,
\]

and by the components \( \alpha_{k,j}, \ 1 \leq k < j \leq 3 \), of angular momentum given by

\[
\alpha_{k,j}(x_0, x_1, x_2, x_3, p_0, p_1, p_2, p_3) = xkp_j - x_jp_k, \quad 1 \leq k < j \leq 3,
\]

too. The corresponding operators look like

\[
\widehat{[\beta_j]}[\phi] = -i [\{\beta_j, \phi\}], \quad [\widehat{\alpha_{k,j}}][\phi] = -i [\{\alpha_{k,j}, \phi\}],
\]

where \( \phi \in A \) represents \([\phi] \in A^C_{\text{red}}\). Notice the Poisson brackets \( \{\beta_k, \beta_j\} = \alpha_{k,j} \). This shows that indeed some observables are quantized by our construction. In particular, it yields among others a precise description of those quantum observables on \( Y_0 \) that arise from quantum observables of the system classically described by \((A_{\text{red}}, \{\cdot, \cdot\}_{\text{red}})\).

We leave it to the experts to decide whether ours is a physically meaningful quantization of a spinless relativistic particle with zero rest mass.

**Remark 5.15.** To get more results a machinery is needed which enables one to handle geometric versions of objects of the kind \( D_{A/I} \) etc. For example, let \( A \) be the
algebra of smooth functions on a smooth finite dimensional manifold $N$, let $K \subseteq N$ be compact, and let $I \subseteq A$ be the ideal of functions that vanish on $K$. When we view e. g. $A/I$ as the smooth functions $C^\infty(K)$ in the sense of Whitney on $K$, see e. g. Malgrange [68], a candidate for the corresponding geometric object would be

$$D_{C^\infty(K)}/\left(\cap_{x \in K} \mu_x D_{C^\infty(K)}\right),$$

where $\mu_x \subseteq C^\infty(K)$ refers to the maximal ideal of functions that vanish on $x \in K$. An alternate approach would be to introduce Poisson varieties and/or schemes with singularities, thereby staying entirely within algebraic geometry. We intend to address both issues elsewhere.

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