Greed Works: An Improved Analysis of Sampling
Kaczmarz-Motkzin

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Abstract

Stochastic iterative algorithms have gained recent interest in machine learning and signal processing for solving large-scale systems of equations, $A \mathbf{x} = \mathbf{b}$. One such example is the Randomized Kaczmarz (RK) algorithm, which acts only on single rows of the matrix $A$ at a time. While RK randomly selects a row of $A$ to work with, Motzkin’s Method (MM) employs a greedy row selection. Connections between the two algorithms resulted in the Sampling Kaczmarz-Motzkin (SKM) algorithm which samples a random subset of $\beta$ rows of $A$ and then greedily selects the best row of the subset. Despite their variable computational costs, all three algorithms have been proven to have the same theoretical upper bound on the convergence rate. In this work, an improved analysis of the range of random (RK) to greedy (MM) methods is presented. This analysis improves upon previous known convergence bounds for SKM, capturing the benefit of partially greedy selection schemes. This work also further generalizes previous known results, removing the theoretical assumptions that $\beta$ must be fixed at every iteration and that $A$ must have normalized rows.

1 Introduction

Large-scale systems of equations arise in many areas of data science, including in machine learning and as subroutines of several optimization methods \cite{9}. We consider solving these large systems of linear equations, $A \mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^{m}$, and $m \gg n$. Iterative methods which use a small portion of the data in each iteration are typically employed in this
Figure 1: Several iterations of a Kaczmarz method. The iterate $x_{j+1}$ is the orthogonal projection of $x_j$ onto the solution set of the selected equation (represented by a line).

domain. These methods offer a small memory footprint and good convergence guarantees. The Kaczmarz method \cite{26} is such an iterative method that consists of sequential orthogonal projections towards the solution set of a single equation (or subsystem). Given the system $Ax = b$, the method computes iterates by projecting onto the hyperplane defined by the equation $a_i^T x = b_i$ where $a_i^T$ is a selected row of the matrix $A$ and $b_i$ is the corresponding entry of $b$. The iterates are recursively defined as

$$x_{j+1} = x_j + \frac{b_i - a_i^T x_j}{\|a_i\|^2} a_i.$$

We assume that $A$ is full-rank and system $Ax = b$ has unique solution $x^*$. We will use $r_j := Ax_j - b$ to represent the $j$th residual and $e_j := x_j - x^*$ to represent the $k$th error term. Additionally, we let $\sigma_{\text{min}}(A)$ be the smallest singular value of $A$ and unless otherwise noted, we let $\|\cdot\|$ represent the Euclidean norm. We let $\|\cdot\|_F$ denote the Frobenius norm and $\|\cdot\|_\infty$ denote the $\ell_\infty$ norm. A visualization of several iterations of a Kaczmarz method are shown in Figure 1.

The Kaczmarz method was originally proposed in the late 30s \cite{27} and rediscovered in the 1970’s under the name *algebraic reconstruction technique (ART)* as an iterative method for reconstructing an image from a series of angular projections in computed tomography \cite{20, 25}. This method has seen
popularity among practitioners and researchers alike since the beginning of the digital age [11, 24], but saw a renewed surge of interest after the elegant convergence analysis of the Randomized Kaczmarz (RK) method in [44]. In [44], the authors showed that for a consistent system with unique solution, RK (with specified sampling distribution) converges at least linearly in expectation with the guarantee
\[ E\|e_k\|^2 \leq \left(1 - \frac{\sigma_{\min}^2(A)}{\|A\|_F^2}\right)^k \|e_0\|^2. \] (1)

Many variants and extensions followed, including convergence analyses for inconsistent and random linear systems [33, 12], connections to other popular iterative algorithms [30, 35, 40, 41, 16], block approaches [36, 42], acceleration and parallelization strategies [17, 28, 31], and techniques for reducing noise and corruption [48, 23].

Another popular Kaczmarz method extension is greedy (rather than randomized) row selection, which has been rediscovered several times in the literature as the “most violated constraint control” or the “maximal-residual control” [10, 38, 39]. This method was proposed in the 1950’s as an iterative relaxation method for linear programming by Agmon, Motzkin, and Schoenberg under the name Motzkin’s relaxation method for linear inequalities (MM) [32, 1]. In [1], the author showed that MM converges at least linearly (deterministically) with the convergence rate of (1). The bodies of literature studying this greedy strategy have remained somewhat disjoint, with analyses for linear systems of equations in the numerical linear algebra community and analyses for linear systems of inequalities in the operations research and linear programming community [18, 19, 45, 3, 6, 7, 13]. There has been recent work in analyzing variants of this greedy strategy [15, 4, 5, 43]. In [43], the authors analyze MM on a system to which a Gaussian sketch has been applied. In [4, 5], the authors analyze variants of MM in which the equation selected in each iteration is chosen randomly amongst the set whose residual values are sufficiently near the maximal residual value. In [15], the authors provide a convergence analysis for a generalized version of MM in which the equation chosen in each iteration is that which has the maximal weighted residual value which are the residual values divided by the norm of the corresponding row of the measurement matrix. In [14], the authors illustrated the connection between MM and RK and proposed a family of algorithms that interpolate between the two, known as the Sampling Kaczmarz-Motzkin (SKM) methods.

The SKM methods operate by randomly sampling a subset of the system of equations, computing the residual of this subset, and projecting onto the
equation corresponding to the largest magnitude entry of this sub-residual. The family of methods (parameterized by the size of the random sample of equations, $\beta$) interpolates between MM, which is SKM with $\beta = m$, and RK, which is SKM with $\beta = 1$. In [14], the authors prove that the SKM methods converge at least linearly in expectation with the convergence rate specified in [1]. Meanwhile, the empirical convergence of this method is seen to depend upon $\beta$; however, increasing $\beta$ also increases the computational cost of each iteration so the per iteration gain from larger sample size may be outweighed by the in-iteration cost. This is reminiscent of other methods which use sampled subsets of data in each iteration, such as the block projection methods [2, 36, 37].

Like SKM, the randomized block Kaczmarz (RBK) methods use a subset of rows $\tau \subseteq [m]$ to produce the next iterate; rather than forcing the next iterate to satisfy the single sampled equation as in RK, block iterates satisfy all the equations in the randomly sampled block. The $(k+1)$st RBK iteration is given by

$$x_{k+1} = x_k + (A_{\tau})^\dagger(b_{\tau} - A_{\tau}x_k),$$

where $A_{\tau}$ and $b_{\tau}$ represent the restriction onto the row indices in $\tau$ and $(A_{\tau})^\dagger$ denotes the Moore-Penrose inverse of the matrix $A_{\tau}$. In [36], the authors prove that on a system with a row-normalized measurement matrix with a well-conditioned row-paving RBK converges at least linearly in expectation with the guarantee

$$E\|e_k\|^2 \leq \left(1 - \frac{\sigma_{\min}^2(A)}{C\|A\|^2\log(m + 1)}\right)^k\|e_0\|^2. \tag{2}$$

where $C$ is an absolute constant and $\|A\|$ denotes the operator norm of the matrix. This can be a significant improvement over the convergence rate of [1] when $\|A\|^2 \gg \|A\|^2\log(m + 1)$. However, the cost per iteration scales with the size of the blocks. In [37], the authors generalize this result to inconsistent systems and show that, up to a convergence horizon, RBK converges to the least-squares solution. In [34], the authors, inspired by the sketching framework in [21], construct a block-type method which iterates by projecting onto a Gaussian sketch of the equations. They show that this method converges at least linearly in expectation with the guarantee

$$E\|e_k\|^2 \leq \left(1 - \left[\frac{\sqrt{s}\sigma_{\min}(A)}{9\sqrt{s}\|A\| + C\|A\|_F}\right]^2\right)^k\|e_0\|^2. \tag{3}$$
where $C$ is an absolute constant and $s$ is the number of rows in the resulting sketched system. This result requires a Gaussian sketch which is a costly operation, however the authors suggest using a Gaussian sketch of only a subset of the equations. This result is most related to SKM and to our main result.

2 Previous Results

This section focuses on the convergence behavior of the RK, MM, and SKM methods. Each of these projection methods is a special case of Algorithm 1 with a different selection rule (Line 4). In iteration $j$, RK uses the randomized selection rule that chooses $t_j = i$ with probability $\|a_i\|^2/\|A\|^2_F$, MM uses the greedy selection rule $t_j = \arg\max_i |a_i^\top x_{j-1} - b_i|$, and SKM uses the hybrid selection rule that first samples a subset of $\beta$ rows, $\tau_j \sim \text{unif}(\binom{m}{\beta})$, and then chooses $t_j = \arg\max_{i \in \tau_j} |a_i^\top x_{j-1} - b_i|$. As previously mentioned, RK and MM are special cases of the SKM method when the sample size $\beta = 1$ and $\beta = m$, respectively. Each of the methods converge linearly when the system is consistent with unique solution (RK and SKM converge linearly in expectation, MM converges linearly deterministically). In Table 1, we present the selection rules and convergence rates for RK, MM, and SKM. Note that under the assumption that $A$ has been normalized so that $\|a_i\|^2 = 1$, each of these upper bounds on the convergence rate is the same since $\|A\|^2_F = m$. Thus, these results do not reveal any advantage the more computationally expensive methods (MM, SKM with $\beta \gg 1$) enjoy over RK. There are, in fact, pathological examples on which RK, MM, and SKM exhibit nearly the same behavior (e.g., consider the system defining two lines that intersect at one point in $\mathbb{R}^2$), so it is not possible to prove significantly different convergence rates without leveraging additional properties of the system.

In [22], the authors demonstrate that MM can converge faster than RK or SKM and that the convergence rate depends on the structure of the residual terms of the iterations, $r_k = Ax_k - b$. In particular, they prove that

$$\|e_k\|^2 \leq \Pi_{j=0}^{k-1} \left(1 - \frac{\sigma_{\min}(A)}{4\gamma_j}\right) \|e_0\|^2,$$

where $\gamma_j$ is the dynamic range of the $i$th residual, $\gamma_j := \frac{\|r_k\|^2}{\|r_k\|_\infty}$. Our main contribution in this paper is to prove that the SKM methods can exhibit a similarly accelerated convergence rate and the advantage scales with the size
Algorithm 1 Generic Kaczmarz Method

1: procedure KACZ(A,b,x₀) 
2: \( k = 1 \) 
3: repeat 
4: \( \) Choose \( t_k \in [m] \) according to selection rule. 
5: \( x_k = x_{k-1} - \frac{a_{t_k}^T x_{k-1} - b_{t_k}}{\| a_{t_k} \|_2^2} a_{t_k} \). 
6: \( k = k + 1 \) 
7: until stopping criterion reached 
8: return \( x_k \) 
9: end procedure

Selection Rule Convergence Rate

RK \[44]\( \mathbb{P}(t_j = i) = \frac{\| a_i \|^2}{\| A \|_F^2} \) \( \mathbb{E}\| e_k \|^2 \leq (1 - \frac{\sigma_{\min}^2(A)}{\| A \|_F^2})^k \| e_0 \|^2 \)

SKM \[14]\( t_j \sim \text{unif}(\binom{[m]}{\beta}) \) \( t_j = \arg \max_{i \in \tau_j} |a_i^T x_j - 1 - b_i| \) \( \mathbb{E}\| e_k \|^2 \leq (1 - \frac{\sigma_{\min}^2(A)}{m})^k \| e_0 \|^2 \)

MM \[1]\( t_j = \arg \max_{i} |a_i^T x_j - 1 - b_i| \) \( \mathbb{E}\| e_k \|^2 \leq (1 - \frac{\sigma_{\min}^2(A)}{m})^k \| e_0 \|^2 \)

Table 1: The selection rules and convergence rates of RK, SKM, and MM. The presented results for MM and SKM assume that \( A \) has been normalized so that \( \| a_i \|^2 = 1 \).

Now, we let \( \mathbb{E}_{\tau_j} \) denote expectation with respect to the random sample \( \tau_j \) conditioned upon the sampled \( \tau_i \) for \( i < j \), and \( \mathbb{E} \) denote expectation with respect to all random samples \( \tau_i \) for \( 1 \leq i \leq j \) where \( j \) is understood to be the last iteration in the context in which \( \mathbb{E} \) is applied. We state our main result below in Corollary 1; this is a corollary of our generalized result which will be discussed and proven later.

**Corollary 1.** Let \( A \) be normalized so \( \| a_i \| = 1 \) for all rows \( i = 1, \ldots, m \). Let \( x^* \) denote the unique solution to the system of equations \( A x = b \). Then
SKM converges at least linearly in expectation and the bound on the rate depends on the dynamic range, $\gamma_k$ of the random sample of $\beta$ rows of $A$, $\tau_k$. Precisely, in the $k$th iteration of SKM, we have

$$E_{\tau_k}\|x_k - x^*\|^2 \leq \left(1 - \frac{\beta \sigma_{\min}^2(A)}{\gamma_k m}\right)\|x_{k-1} - x^*\|^2,$$

so applying expectation with respect to all iterations, we have

$$E\|x_k - x^*\|^2 \leq \prod_{j=1}^{k} \left(1 - \frac{\beta \sigma_{\min}^2(A)}{\gamma_j m}\right)\|x_0 - x^*\|^2.$$

Corollary 1 shows that SKM experiences at least linear convergence where the contraction term is a product of terms that are less than one and dependent on the sub-sample size $\beta$. When $\beta = 1$, as in RK, $\gamma_k = 1$, so Corollary 1 recovers the upper bound for RK shown in [44]. However, when $\beta = m$ for MM, Corollary 1 offers an improved upper bound on the error over [22]; specifically

$$\|e_k\|^2 \leq \left(1 - \frac{\sigma_{\min}^2(A)}{\gamma_k}\right)\|e_{k-1}\|^2.$$

Our result illustrates that the progress made by an iteration of the SKM algorithm depends upon the dynamic range of the residual of that iteration. The dynamic range of each iteration, $\gamma_j$, satisfies

$$1 \leq \gamma_j \leq \beta.$$

Note that the upper bound, $\gamma_j = \beta$, is achieved by a constant residual where $|a_i^T x_j - b_i| = |a_{i'}^T x_j - b_{i'}|$ for all $i, i' \in [m]$, while the lower bound is achieved by the residual with one nonzero entry. As smaller $\gamma_j$ provides a smaller upper bound on the new error $e_j$, we consider the situation with one nonzero entry in the residual as the “best case” and the situation with a constant residual as the “worst case.” We now compare our single iteration result in the best and worst cases to the previously known single iteration results of [1, 14, 22, 44]. These are summarized in Table 2; we present only the contraction terms $\alpha$ such that

$$E_{\tau_k}\|e_k\|^2 \leq \alpha\|e_{k-1}\|^2,$$

for each upper bound in the case that $A$ is normalized so that $\|a_i\|^2 = 1$ for $i \in [m]$. In particular, note that the worst case residual provides the same upper bound rate as those of [44, 14, 1].
|          | Best Case            | Worst Case           | Previous Best Case | Previous Worst Case |
|----------|----------------------|----------------------|--------------------|---------------------|
| MM       | $1 - \sigma_{\text{min}}^2(A)$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ |
| SKM      | $1 - \frac{\beta \sigma_{\text{min}}(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ |
| RK       | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ | $1 - \frac{\sigma_{\text{min}}^2(A)}{m}$ |

Table 2: Contraction terms $\alpha$ such that $E_{\tau_k} \| e_k \|^2 \leq \alpha \| e_{k-1} \|^2$ for the best and worst case bounds of MM, SKM, and RK.

3 Main Results

Corollary [1] is a specialization of our general result to SKM with a fixed sample size $\beta$ and systems that are row-normalized. Our main result requires neither row-normalization nor a static sample size. However, we must additionally generalize the SKM sampling distribution for systems that are not row-normalized. We now consider the general SKM method which samples $\beta_k$ many rows of $A$ in the $k$th iteration (according to probability distribution $p_{\beta_{k-1}}$ defined in (5)) and projects onto the hyperplane associated to the largest magnitude entry of the sampled sub-residual. The generalized probability distribution over the subset of rows of $A$ of size $\beta_k$ is denoted $p_x : \binom{|m|}{\beta_k} \rightarrow [0, 1)$. The sampled subset of rows of $A$, $\tau_k \sim p_x$ where

$$p_x(\tau_k) = \frac{\| a_{t(\tau, x)} \|^2}{\sum_{\tau \in \binom{|m|}{\beta_k}} \| a_{t(\tau, x)} \|^2},$$

and $t(\tau, x) = \arg \max_{t \in \tau} (a_t^\top x - b_t)^2$. Thus, our generalized SKM method is Algorithm [4] with selection rule $\tau_j \sim p_{\beta_{j-1}}$ and $t_j = t(\tau_j, x_{j-1})$. Similar to the RK probability distribution of [44], the computation of (5) is utilized here simply to theoretically analyze the SKM algorithm without requiring normalized rows. We do not suggest that this probability distribution be implemented in a real world setting.

Our main result shows that the generalized SKM converges at least linearly in expectation with a bound that depends on the dynamic range of the sampled sub-residual, the size of the sample, and the minimum squared singular value of $A$, $\sigma_{\text{min}}^2(A)$. In the event that there are multiple rows within the sub-residual which achieve $\max_{t \in \tau} (a_t^\top x - b_t)^2$, an arbitrary choice can be made amongst those rows and the main result will not be affected by this choice.
Theorem 1 provides theoretical convergence guarantees for the generalized SKM method. Whereas previous guarantees for SKM required normalized rows or fixed sample sizes \( \beta \) \cite{14, 22}, the guarantees presented here do not require either assumption. In addition, the contraction term of the generalized SKM method shows dependence on the dynamic range, another feature lacking in previous works. Following the statement of the theorem, we use standard techniques in the Kaczmarz literature to prove our main result. We also remark on the special case in which rows have equal norm (and thus subsets \( \tau \) are selected uniformly at random), the case where \( \beta_k \) is fixed, and the case in which \( \beta_k = 1 \) in order to make connections to previous results.

**Theorem 1.** Let \( x^* \) denote the unique solution to the system of equations \( Ax = b \). Then generalized SKM converges at least linearly in expectation and the bound on the rate depends on the dynamic range, \( \gamma_k \) of the random sample of \( \beta_k \) rows of \( A \), \( \tau_k \). Precisely, in the \( k \)th iteration of generalized SKM, we have

\[
E_{\tau_k} \| x_k - x^* \|^2 \leq \left( 1 - \frac{\beta_k \begin{pmatrix} m \end{pmatrix} \sigma_{\text{min}}^2 (A)}{\gamma_k m \sum_{\tau \in \begin{pmatrix} m \end{pmatrix}} \| a_t(\tau, x_{k-1}) \|^2} \right) \| x_{k-1} - x^* \|^2.
\]

**Proof.** We begin by rewriting the iterate \( x_k \) and simplifying the resulting expression which yields

\[
\| x_k - x^* \|^2 = \| x_{k-1} - \frac{a_t(\tau_k, x_{k-1}) x_{k-1} - b_t(\tau_k, x_{k-1})}{\| a_t(\tau_k, x_{k-1}) \|^2} a_t(\tau_k, x_{k-1}) - x^* \|^2
\]

\[
= \| x_{k-1} - x^* \|^2 - \frac{\| a_t(\tau_k, x_{k-1}) x_{k-1} - b_t(\tau_k, x_{k-1}) \|^2}{\| a_t(\tau_k, x_{k-1}) \|^2}
\]

\[
= \| x_{k-1} - x^* \|^2 - \| A_{\tau_k} x_{k-1} - b_{\tau_k} \|_\infty^2 / \| a_t(\tau_k, x_{k-1}) \|^2.
\]

Now, we take expectation of both sides (with respect to the sampled \( \tau_k \).
according to the distribution (5)). This gives

\[ E_{\tau_k} \| x_k - x^* \|^2 = \| x_{k-1} - x^* \|^2 - \frac{\| A_{\tau_k} x_{k-1} - b_{\tau_k} \|_2^2}{\| a_{\tau_k(x_{k-1})} \|^2} \]

\[ = \| x_{k-1} - x^* \|^2 - \sum_{\tau \in [m]_{\beta_k}} p_{x_{k-1}(\tau)} \| A_{\tau} x_{k-1} - b_{\tau} \|_2^2 \]

\[ = \| x_{k-1} - x^* \|^2 - \sum_{\tau \in [m]_{\beta_k}} \sum_{\pi \in [m]_{\beta_k}} \| a_{t(\pi,x_{k-1})} \|^2 \frac{\| A_{\tau} x_{k-1} - b_{\tau} \|_2^2}{\| a_{\tau(x_{k-1})} \|^2} \]

\[ = \| x_{k-1} - x^* \|^2 - \frac{1}{\gamma_k} \sum_{\pi \in [m]_{\beta_k}} \| a_{t(\pi,x_{k-1})} \|^2 \sum_{\tau \in [m]_{\beta_k}} \| A_{\tau} x_{k-1} - b_{\tau} \|^2 \]

\[ = \| x_{k-1} - x^* \|^2 - \frac{(m_{\beta_k}) \beta_k}{\gamma_k m \sum_{\pi \in [m]_{\beta_k}} \| a_{t(\pi,x_{k-1})} \|^2} \|| A_{x_{k-1}} - b ||^2 \]

where the last line follows from standard properties of singular values. This completes our proof. \( \square \)

One may be assured that this contraction term is always strictly positive. We prove this simple fact in Lemma 1.

**Lemma 1.** For any matrix \( A \) defining consistent system with unique solution \( x^* \), we have

\[ \gamma_j \geq \frac{\beta_j (m_{\beta_j}) \sigma^2_{\min}(A)}{m \sum_{\tau \in [m]_{\beta_j}} \| a_{t(\tau,x_{j-1})} \|^2}. \]
**Proof.** Beginning with the definition of $\gamma_j$, we have

\[
\sum_{\tau_j \in (^{m}_{\beta_j})} \| A_{\tau_j} x_{j-1} - b_{\tau_j} \|^2 = \frac{\beta_j (^{m}_{\beta_j}) \| A (x_{j-1} - x^*) \|^2}{\sum_{\tau_j \in (^{m}_{\beta_j})} \| a^{\top}_{\tau_j} (x_{j-1} - x^*) \|^2} \geq \frac{\beta_j (^{m}_{\beta_j}) \sigma^2_{\min}(A)}{\sum_{\tau_j \in (^{m}_{\beta_j})} \| a^{\top}_{\tau_j} (x_{j-1} - x^*) \|^2} \| x_{j-1} - x^* \|^2 \leq \frac{\beta_j (^{m}_{\beta_j}) \sigma^2_{\min}(A)}{m \sum_{\tau \in (^{m}_{\beta_j})} \| a^{\top}_{\tau} (x_{j-1} - x^*) \|^2},
\]

where the inequality follows from properties of singular values and Cauchy-Schwartz.

The next remarks make simplifying assumptions on the generalized SKM algorithm and our main result to provide better context for comparison with previous works.

**Remark 1.** *(Recovery of RK guarantees)* If all of the rows of $A$ have equal norm (not necessarily unit norm), then our result specializes to

\[
\mathbb{E}_{\tau_k} \| x_k - x^* \|^2 \leq \left( 1 - \frac{\beta_k \sigma^2_{\min}(A)}{\gamma_k \| A \|^2_F} \right) \| x_{k-1} - x^* \|^2. \tag{6}
\]

Additionally, when $\beta_k = 1$, the sampling distribution [5] and theoretical error upper bound [6] simplifies to the probability distribution and error guarantees of [44].

**Remark 2.** *(Improvement of MM guarantees)* Corollary [7] is obtained from Theorem [8] when rows of $A$ have unit norm and $\beta_k = \beta$. When $\beta_k = m$, then an improved convergence rate of

\[
\| x_k - x^* \|^2 \leq \left( 1 - \frac{\sigma^2_{\min}(A)}{4 \gamma_k} \right) \| x_{k-1} - x^* \|^2,
\]

for MM over that shown in [22],

\[
\| x_k - x^* \|^2 \leq \left( 1 - \frac{\sigma^2_{\min}(A)}{4 \gamma_k} \right) \| x_{k-1} - x^* \|^2,
\]

is obtained.
Remark 3. (Connection to Block RK) Note that this bound on the convergence rate of SKM additionally provides a bound on the convergence rate of a block Kaczmarz variant. This variant is distinct from the block Kaczmarz method considered in [36]. The analysis of [36] requires a pre-partitioned row paving, while the variant considered here allows the blocks to be sampled randomly and not pre-partitioned. Consider the block Kaczmarz variant which in each iteration selects a block of $\beta_k$ rows of $A$, $\tau_k$, and projects the previous iterate into the solution space of the entire block of $\beta_k$ equations. This variant necessarily converges faster than SKM as it makes more progress in each iteration. In particular, note that $\{x \mid A_{\tau_k}x = b_{\tau_k}\} \subset \{x \mid a^T_{(\tau_k,y)}x = b_{(\tau_k,y)}\}$.

Given iterate $x_{k-1}$ and sample of rows $\tau_k$, let $x_{SKM}^k$ denote the iterate produced by SKM and $x_{BK}^k$ denote the iterate produced by this block Kaczmarz variant. Note that $x_{SKM}^k$ is the closest point to $x_{k-1}$ on the hyperplane associated to equation $t(\tau_k, x_{k-1})$ so, since $x_{BK}^k$ also lies on this hyperplane, we have

$$\|x_{SKM}^k - x_{k-1}\|^2 \leq \|x_{BK}^k - x_{k-1}\|^2.$$

Now, we note that by orthogonality of the projections, we have

$$\|x_{BK}^k - x^*\|^2 \leq \|x_{SKM}^k - x^*\|^2.$$

A visualization of this situation is presented in Figure 2. Thus, the progress made by BK in any fixed iteration is at least as large as the progress made by SKM, so it must converge at least as quickly.

Because Theorem 1 shows that the contraction coefficient for generalized SKM is dependent by the dynamic range, the following section discusses bounds on the dynamic range for special types of linear systems.

4 Analysis of the Dynamic Range

Since the dynamic range plays an integral part in the convergence behavior for generalized SKM, the dynamic range is analyzed here for different types of specialized linear systems. Note that the dynamic range has also appeared in other works, although not under the guise of “dynamic range”. For example, in [4] the authors proposed a Greedy Randomized Kaczmarz (GRK) algorithm that finds a subset of indices to randomly select the next
Figure 2: The SKM and BK iterates, $x_k^{SKM}$ and $x_k^{BK}$, generated by one iteration starting at $x_{k-1}$ satisfy $\|x_k^{BK} - x^*\|^2 \leq \|x_k^{SKM} - x^*\|^2$.

row to project onto. The operation of finding this subset relies on a ratio between the $\ell_\infty$ and $\ell_2$ norms of the residual at the current iteration, essentially using a proxy of the dynamic range. In this section, we analyze the dynamic range for random Gaussian linear systems and remark on the extension to other random linear systems.

4.1 Gaussian Matrices

When entries of the measurement matrix $A$ are drawn i.i.d. from a standard Gaussian distribution, it can be shown that the dynamic range is upper bounded by $O(n\beta/\log \beta)$. The proof of the upper bound of $\gamma_k$ is similar to Lemma 2 of [22], where the authors analyze the dynamic range for $\beta = m$. Here, we generalized the bound for varying samples sizes $\beta_k$.

Lemma 2. Let $A \in \mathbb{R}^{m \times n}$ be a random Gaussian matrix with $a_{ij} \sim \mathcal{N}(0, \sigma^2)$. For each subset $\tau \in \binom{[m]}{\beta_k}$, let $I_\tau \subseteq \tau$ denote the set of rows in $\tau$ that are independent of $x$ and note $|I_\tau| \leq \beta_k$. Assuming there is at least $n'$ rows in $[m]$ which are independent of $x$, the dynamic range can be upper bounded as:

$$\gamma_j = \frac{\sum_{\tau \in \binom{[m]}{\beta_k}} E_{a} \|A_{\tau} x\|^2}{\sum_{\tau \in \binom{[m]}{\beta_k}} E_{a} \|A_{\tau} x\|_{\infty}^2} \leq \frac{(\binom{m}{\beta_k}) (\beta_k n + \sum_{i \in \tau \setminus I_\tau} \|a_i\|^2 / \sigma^2)}{(\binom{m}{\beta_k}) \log(\beta_k)}. \quad (7)$$
Remark 4. Note that the factor \( \frac{m!}{m'!} \) is \( O(1) \) as \( m \to \infty \) since
\[
\frac{m!}{m'!} = \frac{\beta_k!(m-j-\beta_k)!}{\beta_k!(m-\beta_k)!} = \prod_{i=0}^{j} \frac{m-i}{m-\beta_k-i}.
\]

Thus, we conclude that the expected dynamic range for any iteration \( k \) is \( O(n\beta_k/\log(\beta_k)) \).

Proof. Without loss of generality, we let the solution to the system \( x^* = 0 \) so that \( b = 0 \). We are then interested in finding an upper bound on the dynamic range \([4]\) in expectation. Here, the expectation is taken with respect to the random i.i.d. draws of the entries of \( A \). To that end, we derive upper bounds and lower bounds on the numerator and denominator of \([4]\). Starting with the upper bound on the numerator we have
\[
\sum_{\tau \in \binom{[m]}{\beta_k}} E_a \| A_{\tau} x \|^2 \leq \sum_{\tau \in \binom{[m]}{\beta_k}} \sum_{i \in \tau} E_a (\| a_i \|^2 \| x \|^2)
\]
\[
= \sum_{\tau \in \binom{[m]}{\beta_k}} \left( \sum_{i \in I_{\tau}} E_a \| a_i \|^2 \| x \|^2 + \sum_{i \in \tau \setminus I_{\tau}} \| a_i \|^2 \| x \|^2 \right)
\]
\[
= \sum_{\tau \in \binom{[m]}{\beta_k}} \left( n\sigma^2 \| x \|^2 + \sum_{i \in \tau \setminus I_{\tau}} \| a_i \|^2 \| x \|^2 \right)
\]
\[
\leq \sum_{\tau \in \binom{[m]}{\beta_k}} \left( \beta_k n\sigma^2 + \sum_{i \in \tau \setminus I_{\tau}} \| a_i \|^2 \right) \| x \|^2
\]
\[
= \left( \frac{m}{\beta_k} \right) \left( \beta_k n\sigma^2 + \sum_{i \in \tau \setminus I_{\tau}} \| a_i \|^2 \right) \| x \|^2
\]

where the first inequality follows from the Cauchy-Schwartz inequality and remaining computation uses the fact that \( E_a \| a_i \|^2 = n\sigma^2 \) and simplifies the
expression. The lower bound follows from

\[ \sum_{\tau \in \binom{[m]}{\beta}} E_a \| A_\tau x \|_2^2 = \sum_{\tau \in \binom{[m]}{\beta}} E_a \max_{i \in \tau} \langle a_i, x \rangle^2 \geq \sum_{\tau \in \binom{[m]}{\beta}} E_a \max_{i \in I_\tau} \langle a_i, x \rangle^2 \]

\[ \geq \sum_{\tau \in \binom{[m]}{\beta}} \left( E_a \max_{i \in I_\tau} \langle a_i, x \rangle \right)^2 \geq \sum_{\tau \in \binom{[m]}{\beta}} \left( E_a \max_{i \in I_\tau} \langle a_i, x \rangle \right)^2 \]

\[ \geq \sum_{\tau \in \binom{[m]}{\beta}} \sigma^2 \| x \|^2 \log(\beta_k) \]

\[ \geq \left( \frac{m'}{\beta_k} \right) \sigma^2 \| x \|^2 \log(\beta_k), \]

where the second to last inequality uses the fact that for i.i.d. Gaussian random variables \( g_1, g_2, \ldots, g_N \sim \mathcal{N}(0, \sigma^2) \), we have that \( E(\max_{i \in [N]} g_i) \gtrsim \sigma \sqrt{\log N} \) and that \( \langle a_i, x \rangle \sim \mathcal{N}(0, \sigma^2 \| x \|^2) \). Therefore, we have

\[ \gamma_k = \frac{\sum_{\tau \in \binom{[m]}{\beta}} E_a \| A_\tau x \|^2}{\sum_{\tau \in \binom{[m]}{\beta}} E_a \| A_\tau x \|_\infty^2} \]

\[ \leq \left( \frac{m'}{\beta_k} \right) \left( \beta_k n \sigma^2 + \sum_{i \in \tau \setminus I_\tau} \| a_i \|^2 \right) \left( \frac{m'}{\beta_k} \right) \sigma^2 \log(\beta_k). \]

Dividing all terms by \( \sigma^2 \) attains the desired result (7).

We conjecture that the true bound is actually \( O(\beta_k / \log(\beta_k)) \) and that the \( n \) is an artifact of our proof technique. We have plotted \( \gamma_k \) and the corresponding conjectured bound in the left of Figure 3 for a Gaussian matrix of size 50000 \( \times \) 500.

**Remark 5.** To extend to other distributions, one can simply note that as the signal dimension \( n \) gets large, the Law of Large numbers can be invoked and a similar computation can be used to show an upper bound on the dynamic range of the system.

### 4.2 Incidence Matrices

In the previous subsection, we analyzed the dynamic range for systems with measurement matrices that are randomly generated. Deterministically generated matrices are additionally of interest. In this subsection,
we analyze the dynamic range associated to incidence matrices of directed graphs, $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. The incidence matrix $Q$ associated to a directed graph is of size $|\mathcal{A}| \times |\mathcal{V}|$. For each arc, $(i, j) \in \mathcal{A}$ which connects vertex $i$ to vertex $j$, the associated row of $Q$ is all zeros with a one and negative one in the $i$th and $j$th entries. These types of matrices arise in the average consensus problem. The gossip methods that solve this problem are generalized by the Kaczmarz methods [29]. RK specializes to the randomized gossip method in which the pair of nodes which update are selected at random [47, 8]. SKM specializes to a variant of greedy gossip with eavesdropping (GGE) in which the nodes are selected from amongst a random sample to maximize the update [46]. Thus, our analysis provides an alternate convergence rate for GGE.

Now, we consider the dynamic range for an incidence matrix. We can derive a simple bound on the dynamic range in each iteration that depends
only upon the entries of the current error vector, $y_k := x_k - x^*$. In particular,

$$
\gamma_k = \frac{\sum_{\tau \in (\beta_k)} \|Q_\tau y_k\|^2}{\sum_{\tau \in (\beta_k)} \|Q_\tau y_k\|^2} = \frac{\left(\frac{\beta_k}{m}\right) \sum_{(i,j) \in A} (y_k^{(i)} - y_k^{(j)})^2}{\sum_{\tau \in (\beta_k)} \max_{(i,j) \in \tau} (y_k^{(i)} - y_k^{(j)})^2} \leq \frac{\beta_k (m - \beta_k + 1) \sum_{(i,j) \in A} (y_k^{(i)} - y_k^{(j)})^2}{m \sum_{n=\beta_k} \|y_k^{(n)} - y_k^{(n)}\|^2}, \quad (8)
$$

where $n_i$ and $n_j$ denote the vertices connected by the $n$th smallest magnitude difference across an arc. This bound improves for iterates with a sufficient amount of variation in the coordinates. We have plotted $\gamma_k$ and the corresponding bounds in the right of Figure 3. We calculate these values for the incidence matrix $Q \in \mathbb{R}^{44850 \times 300}$ of the complete graph $K_{300}$ in the cases when the error is a Gaussian vector (red) and a Bernoulli vector (blue).

**Remark 6.** If the right-hand-side vector associated to the system $Qx = b$ is $b = 0$, as in the average consensus problem, then this bound on the dynamic range is easily computed from the current iterate,

$$
\gamma_k \leq \frac{\beta_k (m - \beta_k + 1) \sum_{(i,j) \in A} (x_k^{(i)} - x_k^{(j)})^2}{m \sum_{n=\beta_k} (x_k^{(n)} - x_k^{(n)})^2}.
$$

## 5 Experiments

In this section, we present simulated and real world experiments using SKM for varying sample sizes $\beta$. In the simulated experiments, we compare the theoretical convergence guarantees to the empirical performance of SKM averaged over 20 random trials. The number of rows $m = 50000$ and number of columns $n = 500$ are fixed for all simulated experiments. The solution to the system is a vector $x^* \in \mathbb{R}^n$ where each entry is drawn i.i.d. from a standard Gaussian distribution. In each experiment, the systems are consistent so that $b = Ax^*$. The sample sizes considered for this experiment are $\beta = \{1, 100, 200, 500, 1000\}$. For practical reasons, we normalize the rows of $A$ and utilize the bound shown in Corollary 1.

Figure [4] and Figure [5] show the results for Gaussian and Uniform random matrices $A$ respectively. For Gaussian random matrices, each entry of $A$ is drawn i.i.d. from a standard Gaussian distribution. For Uniform random matrices, entries of $A$ are drawn i.i.d. uniformly from the interval $[0, 1]$. In
Figure 4: Comparison of SKM for various choices of fixed $\beta$ values on linear system with entries of $A$ drawn from i.i.d. from $\mathcal{N}(0, 1)$. (left) Iteration vs Approximation Error with dashed lines representing average empirical performance of SKM and solid lines representing theoretical upper bounds for SKM. (middle) FLOPS vs Approximation Error. (right) CPU time vs Approximation Error.

For linear systems with Gaussian random matrices, we see in Figure 4 that the convergence upper bound proven in this work closely matches the behavior of the SKM algorithm regardless of the choice of sample size $\beta$. To compare this result to previous works, note that when $\beta = 1$, the upper bound provided in Corollary 1 simply recovers the previous known upper bound for SKM with normalized rows, a bound which was completely independent of $\beta$. In other words, the solid red line is the comparative previous known SKM upper bound for all $\beta$. Of course, choices of large sample sizes $\beta$ come at a cost, which are captured in the middle and right most subplots of Figure 4. When measuring efficiency, it seems that $\beta = 1$ makes the most progress with minimal FLOPS while $\beta = 100$ is optimal amongst the tested sample sizes with respect to CPU time. This difference is typically explained by the programming and computer architecture (e.g., it may be more efficient to work on batches of rows as opposed to single rows at a time).

Figure 5 uses a uniform random matrix $A$ instead of a Gaussian random matrix. While the algorithm efficiency with respect to FLOPS and CPU time have similar conclusions to the Gaussian measurement matrix case (as
one would expect), the iteration vs approximation error plot now tells a different story. Unlike in the Gaussian case, the theoretical upper bound no longer closely tracks the approximation error of SKM. The looseness here comes from lower bounding the norm of \( \|Ax\|_2^2 \) with the magnitude of \( x \) times the smallest singular value of \( A \) squared. Empirically, we have seen that this lower bound is tighter for Gaussian systems than Uniform systems. It should be noted that even though our theoretical bounds do not track the approximation error for SKM as tightly, they are still a slight improvement over the previous known bounds for SKM (\( \beta = 1 \)).

In addition to being an improvement over the previously known SKM bound, the convergence bound shown in this work enjoys the flexibility of being amendable to a dynamically selected sample size \( \beta_k \). Figure 6 shows the empirical results from experiments where \( \beta_k \) is allowed to change at every iteration. Here, the measurement matrix \( A \) is again a random Gaussian matrix. We consider three sampling regimes that change \( \beta_k \) at every iteration: ‘useDynRng’ which allocates \( \beta_k \) as a function of the dynamic range, ‘slowInc’ which increases \( \beta_k \) at every iteration until \( \beta_k = m \), and finally ‘rand’ which uniformly at random selects a \( \beta_k \in [m] \) at every iteration. Even though \( \beta_k \) changes at each iteration, we see that the theoretical guarantees proven in this work still tracks the progress of SKM. This indeed opens up new and interesting avenues of research including how one can compute an optimal \( \beta_k \) at every iteration. Since the focus of this work is the improvement of the convergence bound of SKM, we leave this for future work.

Figure 7 employs the upper bound on the dynamic range derived in
Lemma 1 to approximate an upper bound for the error of SKM iterates when \( \beta = 100 \). Here, we compare the empirical performance of SKM with its previous known upper bound using the contraction term \( 1 - \frac{\sigma^2_{\text{max}}}{m} \) and \( 1 - \frac{\log(\beta)\sigma^2_{\text{max}}}{m} \). Note that we drop the factor of \( n \) apparent in Lemma 1 as we suspect it to be an artifact of the proof technique used and conjecture that the true upper bound of the dynamic range is actually \( O(\beta/\log(\beta)) \).

6 Conclusion

This work unifies the spectrum between the randomized Kaczmarz and a greedy variant of the Kaczmarz (Motzkin’s Method) algorithm by improving the convergence bound of SKM, a hybrid randomized-greedy algorithm. We show that the behavior of SKM depends on the sample parameter \( \beta_k \) and the dynamic range of the linear system. This result improves upon previous work showing only the linear convergence of SKM. In presenting an
improved convergence bound for SKM that highlights the impact of the sub-sample size $\beta_k$, we have opened up new and exciting avenues for SKM-type algorithms. Future directions of this work include finding optimal sample sizes for different types of linear systems and designing adaptive sample size selection schemes.

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