HAUSDORFF OPERATORS ON LEBESGUE SPACES WITH POSITIVE DEFINITE PERTURBATION MATRICES ARE NON-RIESZ

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Abstract. We consider generalized Hausdorff operators with positive definite and permutable perturbation matrices on Lebesgue spaces and prove that such operators are not Riesz operators provided they are non-zero.

Key words and phrases. Hausdorff operator, Riesz operator, quasinilpotent operator, compact operator.

1 Introduction and preliminaries

The one-dimensional Hausdorff transformation

\[(\mathcal{H}_1 f)(x) = \int_{\mathbb{R}} f(xt) d\chi(t),\]  

where \(\chi\) is a measure on \(\mathbb{R}\) with support \([0, 1]\), was introduced by Hardy [1, Section 11.18] as a continuous variable analog of regular Hausdorff transformations (or Hausdorff means) for series. Its modern \(n\)-dimensional generalization looks as follows:

\[(\mathcal{H} f)(x) = \int_{\mathbb{R}^n} \Phi(u)f(A(u)x)du,\]  

where \(\Phi : \mathbb{R}^m \to \mathbb{C}\) is a locally integrable function, \(A(u)\) stands for a family of non-singular \(n \times n\)-matrices, \(x \in \mathbb{R}^n\), a column vector. See survey articles [2], [3] for historical remarks and the state of the art up to 2014.

To justify this definition the following approach may be suggested. Hardy [1, Theorem 217] proved that (if \(\chi\) is a probability measure) the transformation (1) gives rise to a regular generalized limit at infinity of the function \(f\) in a sense that if \(f\) is continuous on \(\mathbb{R}\), and \(f(x) \to l\) then \(\mathcal{H}_1 f(x) \to l\) when \(x \to \infty\). Note that the map \(x \mapsto xt (t \neq 0)\) is the general form of
automorphisms of the additive group \( \mathbb{R} \). This observation leads to the definition of a (generalized) Hausdorff operator on a general group \( G \) via the automorphisms of \( G \) that was introduced and studied by the author in \([4]\), and \([5]\). For the additive group \( \mathbb{R}^n \) this definition looks as follows.

**Definition 1.** Let \((\Omega, \mu)\) be some \( \sigma \)-compact topological space endowed with a positive regular Borel measure \( \mu \), \( \Phi \) a locally integrable function on \( \Omega \), and \((A(u))_{u \in \Omega}\) a \( \mu \)-measurable family of \( n \times n \)-matrices that are nonsingular for \( \mu \)-almost every \( u \) with \( \Phi(u) \neq 0 \). We define the **Hausdorff operator** with the kernel \( \Phi \) by \((x \in \mathbb{R}^n \text{ is a column vector})\)

\[
(H_{\Phi,A}f)(x) = \int_{\Omega} \Phi(u)f(A(u)x)d\mu(u).
\]

The general form of a Hausdorff operator given by definition 1 (with an arbitrary measure space \((\Omega, \mu)\) instead of \( \mathbb{R}^m \)) gives us, for example, the opportunity to consider (in the case \( \Omega = \mathbb{Z}^n \)) discrete Hausdorff operators \([7]\), \([8]\).

As was mentioned above Hardy proved that the Hausdorff operator (1) possesses some regularity property. For the operator given by the definition 1 the multidimensional version of his result is also true as the next proposition shows.

**Proposition 1.** \([8]\) Let the conditions of definition 1 are fulfilled. In order that the transformation \( H_{\Phi,A} \) should be regular, i.e. that \( f \) is measurable and locally bounded on \( \mathbb{R}^n \), \( f(x) \to l \) when \( x \to \infty \) should imply \( H_{\Phi,A}f(x) \to l \), it is necessary and sufficient that \( \int_{\Omega} \Phi(u)d\mu(u) = 1 \).

So, as for the classic transformation considered by Hardy the Hausdorff transformation in the sense of the definition 1 gives rise to a new family (for various \((\Omega, \mu), \Phi, \) and \( A(u) \)) of regular generalized limits at infinity for functions of \( n \) variables.

(For a different approach to justify the definition (2) see \([6]\).)

The problem of compactness of Hausdorff operators was posed by Liflyand \([9]\) (see also \([2]\)). There is a conjecture that nontrivial Hausdorff operator in \( L^p(\mathbb{R}^n) \) is non-compact. For the case \( p = 2 \) and for commuting \( A(u) \) this hypothesis was confirmed in \([7]\) (and for the diagonal \( A(u) \) — in \([4]\)). Moreover, we conjecture that every nontrivial Hausdorff operator in \( L^p(\mathbb{R}^n) \) is non-Riesz.

Recall that a **Riesz operator** \( T \) is a bounded operator on some Banach space with spectral properties like those of a compact operator; i.e., \( T \) is a non-invertible operator whose nonzero spectrum consists of eigenvalues of finite multiplicity with no limit points other than 0. This is equivalent to the fact that \( T - \lambda \) is Fredholm for every \( \lambda \neq 0 \) \([10]\). For example, a sum of
a quasinilpotent and compact operator is Riesz [11, Theorem 3.29]. Other interesting characterizations for Riesz operators one can also find in [11].

In this note we prove the above mentioned conjecture for the case where the family \( A(u) \) consists of permutable and positive (negative) definite matrices.

2 The main result

We shall employ three lemmas to prove our main result.

Lemma 1 [4] (cf. [1, (11.18.4)], [12]). Let \(|\det A(u)|^{-1/p}\Phi(u) \in L^1(\Omega)\). Then the operator \( H_{\Phi,A} \) is bounded in \( L^p(\mathbb{R}^n) \) (1 \( \leq p \leq \infty \)) and

\[
\|H_{\Phi,A}\| \leq \int_\Omega |\Phi(u)||\det A(u)|^{-1/p}d\mu(u).
\]

This estimate is sharp (see theorem 1 in [8]).

Lemma 2 [8] (cf. [12]). Under the conditions of Lemma 1 the adjoint for the Hausdorff operator in \( L^p(\mathbb{R}^n) \) has the form

\[
(H_{\Phi,A}^*f)(x) = \int_\Omega \Phi(v)|\det A(v)|^{-1}f(A(v)^{-1}x)d\mu(v).
\]

Thus, the adjoint for a Hausdorff operator is also Hausdorff.

Lemma 3. Let \( S \) be a ball in \( \mathbb{R}^n \), \( q \in [1, \infty) \), and \( R_{q,S} \) denotes the restriction operator \( L^q(\mathbb{R}^n) \to L^q(S) \), \( f \mapsto f|S \). If we as usual identify the dual of \( L^q \) with \( L^p \) \((1/p + 1/q = 1)\), then the adjoint \( R_{q,S}^* \) is the operator of natural embedding \( L^p(S) \hookrightarrow L^p(\mathbb{R}^n) \).

Proof. For \( g \in L^p(S) \) let

\[
g^*(x) = \begin{cases} g(x) & \text{for } x \in S, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus S. \end{cases}
\]

Then the map \( g \mapsto g^* \) is the natural embedding \( L^p(S) \hookrightarrow L^p(\mathbb{R}^n) \).

By definition, the adjoint \( R_{q,S}^* : L^q(S)^* \to L^q(\mathbb{R}^n)^* \) acts according to the rule

\[
(R_{q,S}^*\Lambda)(f) = \Lambda(R_{q,S}f) \quad (\Lambda \in L^q(S)^*, f \in L^q(\mathbb{R}^n)).
\]

If we (by the Riesz theorem) identify the dual of \( L^q(S) \) with \( L^p(S) \) via the formula \( \Lambda \leftrightarrow g \), where

\[
\Lambda(h) = \int_S g(x)h(x)dx \quad (g \in L^p(S), h \in L^q(S)),
\]

3
and analogously for the dual of \( L^q(\mathbb{R}^n) \), then the definition of \( R_{q,S}^* \) takes the form

\[
\int_{\mathbb{R}^n} (R_{q,S}^* g)(x) f(x) dx = \int_S g(x) (f|S)(x) dx.
\]

But

\[
\int_S g(x) (f|S)(x) dx = \int_{\mathbb{R}^n} g^*(x) f(x) dx \quad (f \in L^q(\mathbb{R}^n)).
\]

The right-hand side of the last formula is the linear functional from \( L^q(\mathbb{R}^n)^* \). If we (again by the Riesz theorem) identify this functional with the function \( g^* \), the result follows. \( \square \)

**Theorem 1.** Let \( A(v) \) be a commuting family of real positive definite \( n \times n \)-matrices (\( v \) runs over the support of \( \Phi \) ), and \((\det A(v))^{-1/n} \Phi(v) \in L^1(\Omega)\). Then every nontrivial Hausdorff operator \( \mathcal{H}_{\phi,A} \) in \( L^p(\mathbb{R}^n) \) \((1 \leq p \leq \infty)\) is a non-Riesz operator (and in particular it is non-compact).

Proof. Assume the contrary. Since \( A(u) \) form a commuting family, there are an orthogonal \( n \times n \)-matrix \( C \) and a family of diagonal non-singular real matrices \( A'(u) \) such that \( A'(u) = C^{-1} A(u) C \) for \( u \in \Omega \). Consider the bounded and invertible operator \( \hat{C} f(x) := f(Cx) \) in \( L^p(\mathbb{R}^n) \). Because of the equality \( \hat{C} \mathcal{H}_{\phi,A} \hat{C}^{-1} = \mathcal{H}_{\phi,A'} \), operator \( \mathcal{H} := \mathcal{H}_{\phi,A'} \) is Riesz and nontrivial, too.

Note that each open hyperoctant in \( \mathbb{R}^n \) is an \( A(u) \)-invariant. Chose such an open \( n \)-hyperoctant \( U \) that \( \mathcal{K} := \mathcal{H}|L^p(U) \neq 0 \). Then \( L^p(U) \) is a closed \( \mathcal{K} \)-invariant subspace of \( L^p(\mathbb{R}^n) \) and \( \mathcal{K} \) is a nontrivial Riesz operator in \( L^p(U) \) by \([11] \) p. 80, Theorem 3.21.

Let \( 1 \leq p < \infty \). To get a contradiction, we shall use the modified \( n \)-dimensional Mellin transform for the \( n \)-hyperoctant \( U \) in the form

\[
(\mathcal{M} f)(s) := \frac{1}{(2\pi)^{n/2}} \int_U |x|^{-\frac{n}{q}+is} f(x) dx, \quad s \in \mathbb{R}^n
\]

Here and below we assume \( |x|^{-\frac{n}{q}+is} = \prod_{j=1}^n |x_j|^{-\frac{1}{q}+is_j} \) where \( |x_j|^{-\frac{1}{q}+is_j} := \exp((-\frac{1}{q} + is_j) \log |x_j|) \). The map \( \mathcal{M} \) is a bounded operator between \( L^p(U) \) and \( L^q(\mathbb{R}^n) \) for \( 1 \leq p \leq 2 \) \((1/p + 1/q = 1)\). It can be easily obtained from the Hausdorff–Young inequality for the \( n \)-dimensional Fourier transform by using the exponential change of variables (see \([13] \)). Let \( f \in L^p(U) \). First assume that \( |y|^{-1/q} f(y) \in L^1(U) \). Then as in the proof of theorem 1 from \([7] \), using the Fubini–Tonelli’s theorem and integrating by substitution \( x = A(u)^{-1} y \), yield the following

\[
(\mathcal{MK} f)(s) = \varphi(s)(\mathcal{M} f)(s) \quad (s \in \mathbb{R}^n),
\]
where the function $\varphi$ ("the symbol of the the Hausdorff operator" [7]) is bounded and continuous on $\mathbb{R}^n$.

Thus,

$$\mathcal{M}Kf = \varphi Mf.$$  \hspace{1cm} (3)

By continuity the last equality is valid for all $f \in L^p(U)$.

Let $1 \leq p \leq 2$. There exists a constant $c > 0$, such that the set \{ $s \in \mathbb{R}^n : |\varphi(s)| > c$ \} contains an open ball $S$. Formula (3) implies that

$$M\psi R_{q,S} MK = R_{q,S} M,$$

where $\psi = (1/\varphi)|S$, $M\psi$ denotes the operator of multiplication by $\psi$, and $R_{q,S} : L^q(\mathbb{R}^n) \rightarrow L^q(S)$, $f \mapsto f|S$ — the restriction operator. Let $T = R_{q,S} M$. Passing to the conjugates gives

$$\mathcal{K}^* T^* M^* \psi = T^*.$$

By [14, Theorem 1] this implies that the operator $T^* = \mathcal{M}^* R_{q,S}^*$ has finite rank. By Lemma 3 $R_{q,S}^*$ is the operator of natural embedding $L^p(S) \hookrightarrow L^p(\mathbb{R}^n)$.

For $g \in L^p(\mathbb{R}^n)$ consider the operator

$$(\mathcal{M}'g)(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |x|^{-\frac{1}{q} + i\xi} g(s)ds, \ x \in U.$$ 

This is a bounded operator taking $L^p(\mathbb{R}^n)$ into $L^q(U)$. Indeed, since

$$|x|^{-\frac{1}{q} + i\xi} = \prod_{j=1}^n |x_j|^{-\frac{1}{q}} \exp(is_j \log |x_j|),$$

we have

$$(\mathcal{M}'g)(x) = |x|^{-\frac{1}{q}} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \exp(is \cdot \log |x|) g(s)ds, \ x \in U,$$

where $|x| := |x_1| \ldots |x_n|$, $\log |x| := (\log |x_1|, \ldots, \log |x_n|)$, and the dot denotes the inner product in $\mathbb{R}^n$. Thus, we can express the function $\mathcal{M}'g$ via the Fourier transform $\hat{g}$ of $g$ as follows: $$(\mathcal{M}'g)(x) = |x|^{-1/q} \hat{g}(-\log |x|), \ (x \in U)$$

and therefore

$$\|\mathcal{M}'g\|_{L^q(U)} = \left( \int_U |x|^{-1} |\hat{g}(-\log |x|)|^q dx \right)^{1/q}.$$ 

5
Putting here \( y_j := -\log |x_j| \) \((j = 1, \ldots, n)\) and taking into account that the Jacobian of this transformation is
\[
\frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)} = \det \text{diag}(e^{-y_1}, \ldots, e^{-y_n}) = \exp \left( -\sum_{j=1}^{n} y_j \right),
\]
we get by the Hausdorff–Young inequality that
\[
\|M'g\|_{L^q(U)} = \|\hat{g}\|_{L^q(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)}.
\]
If \( f \in L^p(U) \), and \( f(x)|x|^{-1/q} \in L^1(U) \), \( g \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) the Fubini–Tonelli’s theorem implies
\[
\int_{\mathbb{R}^n} (Mf)(s)g(s)ds = \int_U f(x)(M'g)(x)dx.
\]
Since the bilinear form \((\varphi, \psi) \mapsto \int \varphi \psi d\mu\) is continuous on \( L^p(\mu) \times L^q(\mu) \), the last equality is valid for all \( f \in L^p(U) \), \( g \in L^p(\mathbb{R}^n) \) by continuity. So, \( M' = M^* \).

It was shown above that the restriction of the operator \( M^* \) to \( L^p(S) \) has finite rank. Since \( M^* \) can be easily reduced to the Fourier transform, this is contrary to the Paley–Wiener theorem on the Fourier image of the space \( L^2(S) \) \( (L^2(S) \subset L^p(S)) \) (see, i. g., [16, Theorem III.4.9]).

Finally, if \( 2 < p \leq \infty \) one can use duality arguments. Indeed, by lemma 2 the adjoint operator \( H_{\Phi,A'} \) (as an operator in \( L^q(\mathbb{R}^n) \)) is also of Hausdorff type. More precisely, it equals to \( H_{\Phi,B} \), where \( B(u) = A(u)^{-1} = \text{diag}(1/a_1(u), \ldots, 1/a_n(u)), \; \Psi(u) = \Phi(u)|\det A(u)^{-1}| = \Phi(u)/a(u) \). It is easy to verify that \( H_{\Phi,B} \) satisfies all the conditions of theorem 1 (with \( q, \Psi \) and \( B \) in place of \( p, \Phi \) and \( A \) respectively). Since \( 1 \leq q < 2 \), the operator \( H_{\Phi,B} \) is not a Riesz operator in \( L^q(\mathbb{R}^n) \), and so is \( H_{\Phi,A} \), because \( T \) is a Riesz operator if only if its conjugate \( T^* \) is a Riesz operator [11, p. 81, Theorem 3.22].

### 3 Corollaries and examples

For the next corollary we need the following

**Lemma 4.** Let \( J : X \rightarrow X \) be a linear isometry of a Banach space \( X \). A bounded operator \( T \) on \( X \) which commutes with \( J \) is a Riesz operator if and only if such is \( JT \).

**Proof.** We use the fact that an operator \( T \) is a Riesz operator if and only if it is asymptotically quasi-compact [10] (see also [11, Theorem 3.12]). This means that
\[
\lim_{n \to \infty} \left( \inf_{C \in \mathcal{K}(X)} \|T^n - C\|^{1/n} \right) = 0,
\]

6
where $\mathcal{K}(X)$ denotes the ideal of compact operators in $X$ (Ruston condition).

Since $(UT)^n = U^nT^n$ and

$$\inf_{C \in \mathcal{K}(X)} \|(UT)^n - C\|^{1/n} = \inf_{C \in \mathcal{K}(X)} \|T^n - U^{-n}C\|^{1/n} = \inf_{C' \in \mathcal{K}(X)} \|T^n - C'\|^{1/n},$$

the result follows. □

**Corollary 1.** Let $A(v)$ be a commuting family of real negative definite $n \times n$-matrices ($v$ runs over the support of $\Phi$), and $(\det A(v))^{-1/p} \Phi(v) \in L^1(\Omega)$. Then every nontrivial Hausdorff operator $\mathcal{H}_{\Phi,A}$ in $L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) is non-Riesz (and in particular it is non-compact).

Proof. Let $Jf(x) := f(-x)$. Since $-A(v)$ form a commuting family of real positive definite $n \times n$-matrices, and $\mathcal{H}_{\Phi,A} = J\mathcal{H}_{\Phi,-A}$, this corollary follows from lemma 4 and theorem 1. □

**Corollary 2.** Under the conditions of theorem 1 or corollary 1 Hausdorff operator $\mathcal{H}_{\Phi,A}$ is not the sum of the quasinilpotent and compact operators. Indeed, as was mentioned in the introduction, the sum of the quasinilpotent and compact operators is a Riesz operator.

**Corollary 3.** Let $n = 1$, $\phi: \Omega \to \mathbb{C}$ and let $a(v)$ be a real and positive (negative) function on $\Omega$ ($v$ runs over the support of $\phi$), and $|a(v)|^{-1/p} \phi(v) \in L^1(\Omega)$. Then every nontrivial Hausdorff operator

$$\mathcal{H}_{\phi,a}f(x) = \int_{\Omega} \phi(u)f(a(u)x)d\mu(u) \quad (x \in \mathbb{R})$$

in $L^p(\mathbb{R})$ ($1 \leq p \leq \infty$) is a non-Riesz operator (and in particular it is non-compact).

**Example 1.** Let $t^{-1/q}\psi(t) \in L^1(0, \infty)$. Then by corollary 3 the operator

$$\mathcal{H}_{\psi}f(x) = \int_0^\infty \frac{\psi(t)}{t} f\left(\frac{x}{t}\right) dt$$

is a non-Riesz operator in $L^p(\mathbb{R})$ ($1 \leq p \leq \infty$) provided it is non-zero.

**Example 2.** Let $(t_1t_2)^{-1/p}\psi_2(t_1, t_2) \in L^1(\mathbb{R}^2_+)$. Then by theorem 1 the operator

$$\mathcal{H}_{\psi_2}f(x_1, x_2) = \frac{1}{x_1x_2} \int_0^\infty \int_0^\infty \psi_2\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) f(t_1, t_2) dt_1 dt_2$$

is a non-Riesz operator in $L^p(\mathbb{R}^2_+) (1 \leq p \leq \infty)$ provided it is non-zero.

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