Small-Essentially Pseudo-Injective Modules

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Abstract: Let R be associative ring with unit element and X be unitary right R-module. In this work, we introduce the definition of the concept small-essentially pseudo injective module (shortly, S-Ess-pseudo injective). Many properties of this concept are introduced and also we are consider some of their characterizations. Furthermore, we are studied the relation between our concept and some known R-modules and give some results on their endomorphism rings.

Keywords: Injective module; pseudo--injective module; essentially --pseudo injective; S-Ess- pseudo injective; small-essential sub module.

1. Introduction

Through this introduction, we will mention some concepts related to our concept as well as some well-known concepts that we need to complete this work.

"Let X and Y be two R-module, Y is called, (pseudo-) X-injective if for every sub module D of X, any R-homomorphism (R-monomorphism) h:D→Y can be extended" "to, an R-homomorphism, α:X→Y . An R-module Y is called, injective, if it is X-injective for each R-module X".

"An R-module X is said to be: quasi-injective (pseudo--injective), if it is (pseudo) X--injective". see[8],[10],[4]. "Let A be a sub module of a right R-module X" . "Then we say that A is essential in X (shortly, A ≤ₚ X) if, for every non-zero sub module H of X, H∩A ≠ {0}" ,see,[9],[6].

In [5],"A non-zero R-module X is called uniform, if every non-zero sub module of X is essential in X". "A sub module, D of a right R-module X is said to be small in X (shortly ,D ≤ₜ X) if for every sub module H of X, D+H = X implies H = X" , [9].

"The idea of small-essential sub modules was given by D. X. Zhou and X. R. Zhang in[12]”. A sub module W of" a R-module X is said to be small-essential or s-essential in X (shortly, W ≤ₜ X), if" W∩A = 0 with A ≤ₜ X implies A = 0 . "It is clear that every essential sub module of X is s-essential in X, but the converse may be not true in general",[12].
Inaam,M.Hadi and H. K.Marhoon in [7]" are presented the concept of small-uniform (shortly S-uniform)" modules. "An R-module E is called S-uniform if every non-zero small sub module of E is essential" in E .It is clear that every uniform R-module is s-uniform R-module ,but in general the converse, is not true",[7].

Al-Ahmadi and N.Er in [2], was introduced the notion of essentially, pseudo-Y-injective . Let X, and Y, be two R--modules. "Then, X is said to be, essentiallypseudo Y-injective, if for any essential sub module W of" Y, any R-monomorphism, h:W→X can be extended to some ∅ ∈ Hom. (Y,X). X is called essentially pseudo-injective if X is essentially pseudo-X-injective. So we have the following implications:

Injective → quasi-injective, → pseudo-injective, → essentiallypseudo-injective.

These information motivate us to give and study in this work the notion of small-essentially pseudo injective modules which is a proper generalization of pseudo injective modules and in this time is weaker than of essentially-pseudo injective.

The following symbols : D, ≤ X, D≤ X; D≤ X, D≤ X, are denotes to that D is sub module, essential sub module, small sub module and s-essential sub module respectively.

2. Small-essentially pseudo-injective

Definition 2.1 : Let X and Y be two R-module . Then Y is called small-essentially pseudo-X-injective .(shortly, S-Ess-pseudo-X-injective), if for any smallessential sub module W of X, any R--

An R-module Y is called small-essentially pseudo-injective if Y is a smallessentially pseudo-Y

Remark and Examples 2.2 :

(1) Every pseudo.injective is S-Ess pseudo.injective, but the converse is not true, in general .For example: The Z-module Z/4Z is not pseudo--(Z ⊕ Z/8Z)-injective since the obvious isomorphism i: 2Z/8Z→Z/4Z can not be extended to any element of Hom(Z ⊕ Z/8Z, Z /4Z), but it is S-Ess-pseudo-

(2) Every S-Ess-pseudo-X.injective module, is essentiallypseudo-X-injective, but the converse, may not true in general.

Proof : Suppose that Y is S-Ess-pseudo-X-injective. Let W ≤ X ,and α:W→Y be R-monomorphism. Since every essential sub module of X is small-essential in X then W ≤ X and by S-Ess-pseudo-X-injectivity of Y, there exists ∅:X→Y be R-homomorphism such that α = ∅ o i. Hence Y is essentially pseudo-X-injective.

(3) Q and Zn as Z-module is S-Ess-pseudo-injective module,( in fact, Q and Zn are pseudo injective ,where Q is injective and Zn is quasi injective).

(4) Z as Z-module is not essentially pseudo-injective, and hence by remark (2). Z as Z-module is not S-Ess-pseudo-

jective.
Proof: Let \( i:2\mathbb{Z} \rightarrow \mathbb{Z} \) be inclusion map (it is clearly \( 2\mathbb{Z} \) is essential sub module of \( \mathbb{Z} \)). Define the \( \mathbb{Z} \)-isomorphism \( h:2\mathbb{Z} \rightarrow \mathbb{Z} \) such that \( h(2u) = u \), \( u \in \mathbb{Z} \). Suppose that \( \mathbb{Z} \) is essentially pseudo-injective, then there exists an \( \mathbb{R} \)-homomorphism \( \emptyset: \mathbb{Z} \rightarrow \mathbb{Z} \) such that \( \emptyset \) is an extension of \( h \). For each \( u \in \mathbb{Z} \), we get, \( u = h(2u) = \emptyset(2u) = \emptyset(1).2u \), and hence \( \emptyset(1) = \frac{1}{2} \) which is a contradiction. Therefore \( \mathbb{Z} \) is not essentially pseudo-injective and hence \( \mathbb{Z} \) as \( \mathbb{Z} \)-module is not \( S \)-Ess-pseudo-injective.

So we obtain from above the following, implications for \( R \)-modules:

\[ \text{pseudo--injective} \rightarrow \text{small-essentially \ pseudo-injective} \rightarrow \text{essentially-pseudo-injective}. \]

Proposition 2.3: Let \( X \) be auniform \( R \)-module. Then the following, statements are equivalent:

(i) \( Y \) is \( S \)-Ess-pseudo-\( X \)-injective .
(ii) \( Y \) is essentially pseudo-\( X \)-injective.

Proof: \( i \Rightarrow ii \): It is clear by Remark (2.2,2).

\( ii \Rightarrow i \): Let \( H \leq_{se} X \) and \( \beta:H \rightarrow Y \) be a monomorphism. Since \( X \) is uniform \( R \)-module, then \( H \leq X \).

Thus by essentially pseudo-\( X \)-injectivity of \( Y \), there exists \( \psi:X \rightarrow Y \) such that \( \psi \circ i = \beta \). Therefore, \( Y \) is \( S \)-Ess-pseudo-\( X \)-injective.

Proposition 2.4: Let \( X \) be \( s \)-uniform \( R \)-module. Then the following statements are equivalent:

(i) \( Y \) is pseudo-\( X \)-injective .
(ii) \( Y \) is \( S \)-Ess-pseudo-\( X \)-injective.

Proof: \( i \Rightarrow ii \): It is clear .

\( ii \Rightarrow i \): Let \( K \leq X \) and \( \beta:K \rightarrow Y \) be a monomorphism. Since \( X \) is \( s \)-uniform \( R \)-module, then \( K \leq_{se} X \).

And by \( S \)-Ess-pseudo-\( X \)-injectivity of \( Y \) there exists \( h:X \rightarrow Y \) such that \( h \circ i = \beta \). Therefore \( Y \) is pseudo-\( X \)-injective.

According to [7], every uniform \( R \)-module is \( s \)-uniform. Then by proposition (2.4) directly, we get
the proof of the following corollary.

Corollary 2.5: Let \( X \) be uniform \( R \)-module. Then, the following statements are equivalent

(i) \( Y \) is pseudo-\( X \)-injective .
(ii) \( Y \) is \( S \)-Ess-pseudo-\( X \)-injective.

Theorem 2.6: Let \( X \) be a uniform \( R \)-module. Then, the following statements are equivalent:

(i) \( Y \) is pseudo-\( X \)-injective .
(ii) \( Y \) is \( S \)-Ess-pseudo-\( X \)-injective .
(iii) \( Y \) is essentially pseudo-\( X \)-injective.

Proof: \( i \Rightarrow ii \Rightarrow iii \): It is clear .

\( ii \Rightarrow i \) directly by (corollary 2.6).

\( iii \Rightarrow ii \) directly by (proposition 2.3).
iii ⇒ i  

Let $D \leq X$ and $\beta: W \to Y$ be a monomorphism. Now, by uniformity we get $D \leq X$ and since $Y$ is essentially --pseudo--X-injective, then exists. $f:X \to Y$ such that $f \circ i = \beta$. Therefore $Y$ is pseudo--X-injective.

Following [12], let $X$ and $Y$ be modules. A monomorphism $f:X \to Y$ is small-essential in case $\text{Im} f \leq s.e X$.

Now, we can introduce a characterization of S-Ess pseudo--X-injective.

**Proposition 2.7** :-

Let $X$ and $Y$ be two an $R$-module. Then the following statements are equivalent:

(i) $Y$ is S-Ess pseudo--X-injective.

(ii) For any $R$-module $B$, any small-essential $R$-monomorphism $\beta:B \to X$ and any $R$-monomorphism $h:B \to Y$ there exists an $R$-homomorphism $\psi:X \to Y$ such that $h = \beta \psi$.

**Proof** :-

i ⇒ ii  

Let $B$ be an $R$-module, $\alpha:B \to Y$ be an monomorphism and let $\beta:B \to X$ be a $s$-essential monomorphism, then $\beta(B) \leq s.e X$. Define $h_1: \beta(B) \to Y$ suchthat, $h_1(\beta(b)) = h(b)$ for each $b \in B$. Clearly, $h_1$ is a monomorphism. Now, since $Y$ is S-Ess pseudo--X-injective then there exists, an $R$-homomorphism, $\lambda:X \to Y$ such that; $\lambda |_{\beta(B)} = h$. Thus $\lambda \circ \beta(b) = \lambda(b) = h(h(b))$ for each $b \in B$. Thus $\lambda = \lambda \circ \beta$.

ii ⇒ i  

It is directly from the definition (2.1).

Following [12]. The properties of the $s$-essential sub module was gave as in the following results, but without prove and here we present the proofs for them.

**Proposition 2.8**:

Let $X$, be a module.

(1) Assume that; $Y, D, A$ are sub modules of $X$ with $D \leq Y$.

(a) If $D \leq s.e X$, then $D \leq s.e Y$ and $Y \leq s.e X$.

(b) $Y \cap A \leq s.e X$ if and, only if, $Y \leq s.e X$ and $A \leq s.e X$.

(2) If $D \leq s.e Y$ and $h:X \to Y$ is a homomorphism, then $h^{-1}(D) \leq s.e X$.

(3) Assume that $D_1 \leq X_1 \leq X$, $D_2 \leq X_2 \leq X$ and $X = X_1 \oplus X_2$, then $D_1 \oplus D_2 \leq s.e X_1 \oplus X_2$ if and only if $D_1 \leq s.e X_1$ and $D_2 \leq s.e X_2$.

**Proof** :-

(a) Let $D \leq Y \leq X$ and $A \leq Y$ such that $D \cap A = 0$. To prove that $A = 0$. Since $A \leq s.e Y$, then $A \leq s.e X$ by [9; proposition (5.1.3)], but by hypothesis $D \leq s.e X$. Hence $A = 0$.

Now, let $Y \leq X$ and $A \leq X$ such that $Y \cap A = 0$. To prove that $A = 0$. Since, $D \leq Y$ then $D \cap A = 0$.

(b) By definition of small-essential sub module. For, $H \leq s.e X$, we have $Y \cap A \leq s.e X$ if and, only if, $(Y \cap A) \cap H = 0$ implies $H = 0$ if and, only if, $(Y \cap H) = 0$ and $(A \cap H) = 0$ implies $H = 0$ if and, only if, $Y \leq s.e X$ and $A \leq s.e X$.
(2) Assume that, \( A \leq s X \), such that \( h^{-1}(D) \cap A = 0 \). We must show that \( A = 0 \). Clearly, \( D \cap h(A) = 0 \) and by [9 : (5.1.3)], \( h(A) \leq s Y \). But \( D \leq s Y \), thus \( h(A) = 0 \) implies \( A \leq \ker(h) = h^{-1}(0) \). But \( h^{-1}(0) \leq h^{-1}(D) \) implies \( A \leq h^{-1}(D) \), so we get \( A = h^{-1}(D) \cap A \). Hence by assumption we get \( A = h^{-1}(D) \cap A = 0 \) implies \( A = 0 \). Therefore \( h^{-1}(D) \leq s_e X \).

(3) Assume that \( D_1 \leq s X_1 \leq X, D_2 \leq s X_2 \leq X \) and \( X = X_1 \oplus X_2 \).

**First direct:** Let \( H_1 \leq s X_1 \) and \( H_2 \leq s X_2 \) such that \( D_1 \cap H_1 = 0 \) and \( D_2 \cap H_2 = 0 \) implies \( (D_1 \cap H_1) \oplus (D_2 \cap H_2) = 0 \) implies \( (D_1 \oplus D_2) \cap (H_1 \oplus H_2) = 0 \). But, according to [9], we have, \( H_1 \oplus H_2 \leq s X_1 \oplus X_2 \) and by hypothesis \( D_1 \oplus D_2 \leq s_e X_1 \oplus X_2 \) thus \( H_1 \oplus H_2 = 0 \) implies \( H_1 = 0 \) and \( H_2 = 0 \).

Therefore, \( D_1 \leq s_e X_1 \) and \( D_2 \leq s_e X_2 \).

**Second direct:** Assume that \( D_1 \leq s_e X_1 \) and \( D_2 \leq s_e X_2 \). Now let \( H_1 \oplus H_2 \leq s X_1 \oplus X_2 \) such that \( (D_1 \oplus D_2) \cap (H_1 \oplus H_2) = 0 \). Now, by [9], we have \( H_1 \leq s X_1 \) and \( H_2 \leq s X_2 \) then by assumption, \( H_1 = 0 \) and \( H_2 = 0 \) implies \( H_1 \oplus H_2 = 0 \). Therefore, \( D_1 \oplus D_2 \leq s_e X_1 \oplus X_2 \).

In analogous the proposition (2.1) in [4], we can introduce the following proposition.

**Proposition 2.9:** Let \( X \) be an \( R \)-module and \( Y \leq s X \). If \( Y \) is \( S \)-\( \text{Ess-pseudo-} X \)-injective then any \( R \)-monomorphism \( f : Y \to X \) splits.

**Proof:** Let \( h : Y \to X \) be monomorphism and \( h^{-1} : h(Y) \to Y \) be the inverse of \( h \). By proposition [2.8(2)], \( h(Y) \leq s_e X \)

Now, consider the following diagram:-

where \( i : h(Y) \to X \) is the inclusion map. From the \( S \)-\( \text{Ess-pseudo-} X \)-injectivity of \( Y \), certainly there exists; an \( R \)-homomorphism \( \phi : X \to Y \) such that \( \phi \circ i = h \) put \( \mu = \phi \circ h \), it is obvious \( \mu \) is an identity map of \( Y \). Thus by [9; corollary (3.4.11)], \( h \) splits.

**Proposition 2.10:** \( Y \) is injective if and only if \( Y \) is \( S \)-\( \text{Ess-pseudo-} X \)-injective, for all \( X \).

**Proof:** By (proposition 2.9), if \( Y \) is \( X \)-pseudo-injective, for all \( X \), then each monomorphism \( Y \to X \) splits, hence, \( Y \) is injective. Thus \( Y \) is injective, if and only if \( Y \) is \( X \)-injective, for all \( X \), if and only if \( Y \) is \( X \)-pseudo-injective for all \( X \) and only if \( Y \) is \( X \)-\( \text{Ess-pseudo-} \)-injective for all \( X \).

**Proposition 2.11:** Let \( X \) be small-uniform. If \( Y \) is \( S \)-\( \text{Ess-pseudo--} X \)-injective then \( Y \) is \( S \)-\( \text{Ess-pseudo-} W \)-injective for every sub-module \( W \) of \( X \).

**Proof:**
Let $B \leq W$ and $h: B \rightarrow Y$ any $R$-monomorphism, $i_B: B \rightarrow W$ and $i_w: W \rightarrow M$ be the inclusion map of $B$ in $W$ and $W$ in $X$ respectively. Consider the following diagram:

Since $X$ is small-uniform then $B \leq X$ and hence by S-Ess-pseudo-X-injectivity of $Y$, surely there exists; an homomorphism $\phi: X \rightarrow Y$ such that $\phi(i_w \circ i_B) = h$. Define $\phi_1 = \phi \circ i_w$, then $\phi_1$ is a homomorphism of $W$ into $Y$. Note that $\phi_1$ extends $h$, that is, for $a \in X$, $(g_1 \circ i_B)(a) = \phi_1(i_B(a)) = \phi_1(a) = (\phi \circ i_w)(a) = \phi(i_w(a)) = (\phi \circ i_w(i_B(a))) = (\phi \circ i_w \circ i_B)(a) = h(a).

**Proposition 2.12**: Every direct summand of, S-Ess-pseudoX-injective module is S--Ess-pseudo-X-injective.

**Proof**: Let $Y$ be S-Ess-pseudoX-injective and $Y = W \oplus V$.

Consider the following diagram:

Where $B \leq X$, $i: B \rightarrow X$ is the inclusion map, $h: B \rightarrow Y$ is any $-monomorphism. Let $j: W \rightarrow Y$ and $p: Y \rightarrow W$ be the injection and projection homomorphism. Obviously, $j \circ h$ is an $R$-monomorphism. Now since $Y$ is S-Ess pseudo-X-injective, $\exists \phi: X \rightarrow Y$ such that $j \circ h = \phi \circ i$. Define $q: X \rightarrow W$ by $q = p \circ \phi$. Now $q \circ i = p \circ \phi \circ i = p \circ j \circ h = I \circ h = h$. Hence $W$ is S-Ess-pseudo-X-injective.

Following [4], R-module $W$ and $V$ are called; relatively injective if $W$ is $V$-injective and $V$ is $W$-injective.

**Definition 2.13**: Let $W$ and $V$ be an R-module. $W$ and $V$ are called relatively S-Ess-pseudo injective, if $W$ is S-Ess-pseudo $V$-injective and $V$ is S-Ess-pseudo$W$-injective.

**Theorem 2.14**: If $X_1 \oplus X_2$ is S-Ess-pseudo injective, then $X_1$ and $X_2$ relatively S-Ess-pseudo injective.
Proof: Suppose that $X_1 \oplus X_2$ is S-Ess-pseudo injective module. To show that $X_1$ is S-Ess-pseudo $X_2$-injective. Let $W \leq X_2$, $h: W \to X_1$ be any $-\text{monomorphism}$ and the injection and projection homomorphism are define respectively as follows $j: X_1 \to X_1 \oplus X_2$, $p: X_1 \oplus X_2 \to X_1$ be. Define $T: W \to X_1 \oplus X_2$ by $T(w) = (h(w), (w))$ for all $w \in W$. It is easy, to show that $T$ is $R$-monomorphism.

Consider the following diagram:

Since $X_1 \oplus X_2$ is small-essentially pseudo($X_1 \oplus X_2$) -injective and $X_2 \leq X_1 \oplus X_2$, then by (proposition 2.11). $X_1 \oplus X_2$ is small-essentially $X_2$-pseudo-injective. Then there exists $\phi: X_2 \to X_1 \oplus X_2$ such that $\phi \circ i = T$, put $q = p \circ \phi \circ i = p \circ T$. Hence $(p \circ \phi \circ i)(w) = (p \circ T)(w) = p(T(w)) = p(h(w), (w)) = h(w)$.

**Corollary 2.15** : Let $\{X_i\}_{i \in I}$ be a family of $R$-modules, where $I$ is an index set. If $\bigoplus_{i \in I} X_i$ is S-Ess-pseudo injective, then $X_j$ is s-essentially $X_k$ -pseudo injective for all distinct $j, k \in I$.

**Proof** : By (Theorem 2.14)

**Corollary 2.16** : $Y$ is quasiinjective $R$-module if and only if $Y^2$ is S-Ess-pseudo-$Y$-injective.

**Proof** : ($\Rightarrow$) It is clear.

($\Leftarrow$) If $Y^2$ is S-Ess-pseudo-$Y$-injective, thus by (Theorem 2.14), $Y$ is $Y$-injective, this mean $Y$ is quasi-injective.

**Proposition 2.17** : Let $Y \leq X$. If $Y$ is S-Ess-pseudo-$X$-injective, then $Y$ is a direct-summand of $X$.

**Proof** : Let $I_Y : Y \to Y$ be the identity $-\text{homomorphism}$. Since $Y$ is S-Ess--pseudo-$X$-injective $R$-module, thus $\exists \alpha: X \to Y$ such that $\alpha(y) = I_Y(y)$, for all $y \in Y$. Hence $(\alpha \circ i)(y) = y$, $\forall a \in Y$. Where $i$ is the inclusion $-\text{homomorphism}$ of $Y$ in $X$. Thus by [9; corollary (3.4.11)] $i: Y \to X$ is split $R$-homomorphism and hence, by [9; corollary (3.4.8)] $Y$ is directsummand, of $X$.

3. **Endomorphism of s-essentially pseudo injective modules**

**Theorem 3.1** : Let $X$ and $Y$ be two $R$-modules and $S = \text{End}_R(X)$. For let $X$ small-uniform module $Y$. Then the following statements are equivalent :

(i) $X$ is S-Ess-pseudo-$Y$-injective.

(ii) For each $x \in X$ and $y \in Y$, such that $\text{ann}_R(y) = \text{ann}_R(x)$, there exists an $R$-homomorphism $g: X \to Y$ such that $g(y) = x$. 

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(iii) For each $x \in X$, $y \in Y$ such that $\text{ann}(y) = \text{ann}(x)$, we have $Sx \subseteq \text{Hom}_R(Y,X)_y$.

(iv) For each R-monomorphism $f:W \to X$ (where $W$ be sub module of $Y$) and each $w \in W$, there exists an $R$-homomorphism $g:Y \to X$ such that $g(w) = f(w)$.

**Proof:**

i $\Rightarrow$ ii Spouse that, $X$ be a $S$-Ess-pseudo-$Y$-injective $R$-module. Now, let $x \in X$, $y \in Y$ such that $\text{ann}(y) = \text{ann}(x)$. Define $f:yR \to X$ by $f(yr) = xr$, for all $r \in R$. Clearly, $f$ is a well-defined $R$-monomorphism. Since $X$ is $S$-Ess-pseudo-$Y$-injective $R$-module, thus $\exists g:Y \to X$ be $R$-homomorphism, such that $g(y) = f(y)$ for all $y \in yR$. Therefore $g(y) = f(y) = x$.

ii $\Rightarrow$ iii Let $x \in X$, $y \in Y$ such that $\text{ann}_R(y) = \text{ann}_R(x)$. By hypothesis, $\exists g:Y \to X$ be $R$-homomorphism such that $g(y) = m$. Let $\alpha \in S$, thus $\alpha(x) = \alpha(g(y)) = (\alpha \circ g)(y)$. Since $\alpha \circ g \in \text{Hom}_R(Y,X)$, therefore $\alpha(x) \in \text{Hom}_R(Y,X)_y$. Therefore $Sx \subseteq \text{Hom}_R(Y,X)_y$.

iii $\Rightarrow$ iv Let $f:W \to X$ be any $R$-monomorphism where $W$ be sub module of $Y$, and let $w \in W$. Put $x = f(w)$, since $x \in X$ and $\text{ann}_R(x) = \text{ann}_R(w)$, thus by hypothesis we have $Sx \subseteq \text{Hom}_R(Y,X)_x$. Let $I_x:X \to X$ be the identity $R$-homomorphism. Since $I_x \in S$, thus there exists an $R$-homomorphism $g \in \text{Hom}_R(Y,X)$ such that $I_x(x) = g(w)$. Thus $g(w) = x = f(w)$.

iv $\Rightarrow$ i Let $W = wR \subseteq Y$ and $f:W \to X$ be any $R$-monomorphism. Since $w \in W$, thus by hypothesis, $\exists g:Y \to X$ be $R$-homomorphism; such that $g(w) = f(w)$. For each $v \in A$, $v = wr$ for some $r \in R$, we have that $g(v) = g(wr) = g(w)r = f(w)r = f(wr) = f(v)$. Therefore $X$ is $S$-Ess-pseudo-$Y$-injective $R$-module.

As an immediate consequence of Theorem (3.1) we have the following corollary in which we get many characterizations of $S$-Ess-pseudo-injective modules.

**Corollary 3.2:** The following statements are equivalent for an $R$-module $X$:

(i) $X$ is $S$-Ess-pseudo-injective.

(ii) For each $n,m \in X$ such that $\text{ann}_R(n) = \text{ann}_R(m)$, there exists an $R$-homomorphism $h:X \to Y$ such that $h(n) = m$.

(iii) For each $n,m \in X$ such that $\text{ann}_R(n) = \text{ann}_R(m)$, we have $Sn \subseteq Sm$ where $S = \text{End}(X)$.

(iv) For each $R$-monomorphism $q:W \to X$ (where $W \subseteq Y$) and each $w \in W$, there exists an $R$-homomorphism: $J:X \to X$ such that $J(w) = q(w)$.

**Proposition 3.3:** If $X$ is $S$-Ess-pseudo-injective $R$-module and $S = \text{End}_R(X)$, then $SW = SV$, for any isomorphic $R$-sub modules $W,V$ of $X$.

**Proof:** Since $W$ isomorphic to $V$, then there exists. an $R$-isomorphism $\alpha:W \to V$. Let $v \in V$, since $\alpha$ is $R$-epimorphism, thus there exists; an element $w \in W$ such that $\alpha(w) = v$.

It is clear that $\text{ann}_R(w) = \text{ann}_R(v)$ Since $X$ is $S$-Ess-pseudo-injective $R$-module, then by (corollary 3.2) $Sw \subseteq Sw$ and so $Sw \subseteq SV$ for all $v \in V$. Then $SV \subseteq SW$. Similarly we can prove that $SW \subseteq SV$. Therefore $SW = SV$.

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