Interaction of Subharmonic Light Modes with Three-Level Atom; Perfect Squeezing in the Subharmonic Light Modes

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Abstract

We have analyzed the squeezing and statistical properties of the light generated by the interaction of a three-level atom with subharmonic light modes emerging from a nonlinear crystal pumped by coherent light. We have found that the maximum quadrature squeezing of the superposed subharmonic light modes is 100% when the top and bottom levels of the three-level atom are not coupled by the coherent light emerging from the nonlinear crystal. However, this coupling increases the mean photon number of the superposed subharmonic light modes.

keywords: Quadrature squeezing, Mean photon number, Subharmonic light modes

1 Introduction

Squeezing is one of the nonclassical features of light that has attracted a great deal of interest. Several authors have carried out the analysis of the quantum properties of the squeezed light generated by various optical systems [1-10]. In squeezed light the noise in one quadrature is below the vacuum state level with the product of the uncertainties in the two quadratures satisfying the uncertainty relation [9-11]. Squeezed light has potential applications in low-noise optical communications, precision measurements, and weak signal detections [2, 3]. Hence it is vital to find new optical devices or to combine the existing ones to generate highly squeezed and bright light.

It has been predicted that three-level atoms in a cascade configuration and available in a cavity coupled to a vacuum reservoir via a single-port mirror can produce squeezed light [9, 10]. The three-level atoms are pumped to the top level by electron bombardment or coherent light. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons, which are highly correlated, are generated. The two photons may have the same or different frequency. Fesseha [10] has studied the squeezing and statistical properties of the light produced by three-level atoms in a closed cavity and driven by electron bombardment at a constant rate. He has found that the generated light is in a squeezed state, with the maximum quadrature squeezing being 50% below the vacuum-state level.
On the other hand, it has been shown theoretically [12-23] and subsequently confirmed experimentally [24-26] that a subharmonic generator produces squeezed light. It has been studied the twin light beams produced by the process of subharmonic generation taking place inside a nonlinear crystal driven by coherent light and in a cavity coupled to a vacuum reservoir via a single-port mirror. It has been shown that the superposition of these light beams is in a squeezed state, with a maximum quadrature squeezing at threshold and at steady state being 50% below the vacuum-state level [10]. Some authors have studied the statistical and squeezing properties of the light generated by three-level atoms interacting with subharmonic light beams produced by a nonlinear crystal driven by coherent light in a cavity, using the usual commutation relation [27-33]. However, it appears to be difficult to believe that the results obtained in this manner to be correct.

In this paper we consider the interaction of a three-level atom with the light beams generated by a nonlinear crystal driven by coherent light and available in a closed cavity coupled to a vacuum reservoir via a single-port mirror. The top and bottom levels of the three-level atom are coupled by the coherent light emerging from the nonlinear crystal. Our interest is to analyze the squeezing and statistical properties of the superposed subharmonic light modes. We carry out our analysis applying the quantum Langevin equations for the cavity mode operators and the Heisenberg equations for the expectation values of atomic operators. The large-time approximation is used to obtain decoupled equations of evolution for the expectation values of the atomic operators. Finally, employing the steady-state solutions of the quantum Langevin and Heisenberg equations, we calculate the mean photon number and the quadrature squeezing.

2 Operator Dynamics

We consider here the case in which a three-level atom and a nonlinear crystal driven by coherent light are available in a closed cavity coupled to a vacuum reservoir via a single-port mirror. The three-level atom interacts with the coherent light emerging from the nonlinear crystal. We denote the top, intermediate, and bottom levels of the atom by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. We prefer to call the light emitted from the top level light mode $a$ and the one emitted from the intermediate level light mode $b$. We carry out our analysis with light modes $a$ and $b$ having the same or different frequencies. In addition, we assume the cavity modes to be at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$, with direct transition between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden.

The process of subharmonic generation taking place inside the nonlinear crystal is described by the Hamiltonian

$$\hat{H}' = i\lambda (\hat{c}^{\dagger} \hat{a} \hat{b} - \hat{c} \hat{a}^{\dagger} \hat{b}^{\dagger}),$$

where the operators $\hat{a}$ and $\hat{b}$ represent subharmonic light modes, $\lambda$ is the coupling constant between the coherent light and light mode $a$ or $b$ and $\hat{c}$ is the annihilation operator for the coherent light. Upon replacing the operator $\hat{c}$ by $\gamma$ which is taken to be real, positive, and
constant, we can write the Hamiltonian as

\[ \hat{H}' = i\varepsilon (\hat{a}\hat{b} - \hat{a}^\dagger \hat{b}^\dagger), \]  

(2)

where \( \varepsilon = \lambda \gamma \). These operators satisfy at the initial time the standard commutation relation [34]

\[ [\hat{a}(0), \hat{a}^\dagger(0)] = [\hat{b}(0), \hat{b}^\dagger(0)] = 1 \]  

(3)

and commute with each other. In addition, the Hamiltonian describing the interaction between the coherent light emerging from the nonlinear crystal and the three-level atom can be expressed as

\[ H'' = \frac{i\Omega}{2} (\hat{\sigma}_c^\dagger - \hat{\sigma}_c), \]  

(4)

where \( \Omega = 2g'\gamma \), with \( g' \) being the coupling constant between the coherent light and the atom. Moreover, the interaction of the cavity modes with the three-level atom can be described at resonance by the Hamiltonian

\[ \hat{H}''' = ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{b} - \hat{b}^\dagger \hat{\sigma}_b), \]  

(5)

where \( g \) is the coupling constant between the atom and light mode \( a \) or \( b \) and \( \hat{\sigma}_a, \hat{\sigma}_b, \) and \( \hat{\sigma}_c \) are lowering atomic operators defined by

\[ \hat{\sigma}_a = |b\rangle\langle a|, \]  

(6)

\[ \hat{\sigma}_b = |c\rangle\langle b|, \]  

(7)

\[ \hat{\sigma}_c = |c\rangle\langle a|. \]  

(8)

The total Hamiltonian describing the system is then given by

\[ \hat{H} = \frac{i\Omega}{2} (\hat{\sigma}_c^\dagger - \hat{\sigma}_c) + i\varepsilon (\hat{a}\hat{b} - \hat{a}^\dagger \hat{b}^\dagger) + ig(\hat{\sigma}_a^\dagger \hat{a} - \hat{a}^\dagger \hat{\sigma}_a + \hat{\sigma}_b^\dagger \hat{b} - \hat{b}^\dagger \hat{\sigma}_b). \]  

(9)
We carry out our calculation by putting the noise operators in normal order. Then the noise operators will not have any effect on the dynamics of the cavity mode operators. We can thus drop the noise operators and write the quantum Langevin equations for the operators $\hat{a}$ and $\hat{b}$ as
\[
\frac{d}{dt} \hat{a}(t) = -\frac{\kappa}{2} \hat{a}(t) - i [\hat{a}(t), \hat{H}(0)]
\] (10)
and
\[
\frac{d}{dt} \hat{b}(t) = -\frac{\kappa}{2} \hat{b}(t) - i [\hat{b}(t), \hat{H}(0)],
\] (11)
where following Ref. [34], we have fixed the Hamiltonian given by Eq. (9) at the initial time and $\kappa$ is the cavity damping constant. Then with the aid of Eq. (9), Eqs. (10) and (11) can be written as
\[
\frac{d}{dt} \hat{a}(t) = -\frac{\kappa}{2} \hat{a}(t) - \varepsilon \hat{b}^\dagger(t) - g\hat{\sigma}_a(t)
\] (12)
and
\[
\frac{d}{dt} \hat{b}(t) = -\frac{\kappa}{2} \hat{b}(t) - \varepsilon \hat{a}^\dagger(t) - g\hat{\sigma}_b(t).
\] (13)
Furthermore, applying the Heisenberg equation
\[
\frac{d}{dt} \langle \hat{A}(t) \rangle = -i \langle [\hat{A}(t), \hat{H}(0)] \rangle
\] (14)
along with Eq. (9), we readily get
\[
\frac{d}{dt} \langle \hat{\sigma}_a(t) \rangle = \frac{\Omega}{2} \langle \hat{\sigma}_b(t) \rangle + g \langle (\hat{\eta}_b(t) - \hat{\eta}_a(t))\hat{a}(t) + \hat{b}^\dagger(t)\hat{\sigma}_c(t) \rangle,
\] (15)
\[
\frac{d}{dt} \langle \hat{\sigma}_b(t) \rangle = -\frac{\Omega}{2} \langle \hat{\sigma}_a(t) \rangle + g \langle -\hat{a}^\dagger(t)\hat{\sigma}_c(t) + (\hat{\eta}_c(t) - \hat{\eta}_b(t))\hat{b}(t) \rangle,
\] (16)
\[
\frac{d}{dt} \langle \hat{\sigma}_c(t) \rangle = \frac{\Omega}{2} \langle \hat{\sigma}_b(t) - \hat{\eta}_a(t) \rangle + g \langle \hat{\sigma}_b(t)\hat{a}(t) - \hat{\sigma}_a(t)\hat{b}(t) \rangle,
\] (17)
\[
\frac{d}{dt} \langle \hat{\eta}_a(t) \rangle = \frac{\Omega}{2} \langle \hat{\sigma}_c(t) + \hat{\eta}_c(t) \rangle + g \langle \hat{\sigma}_a^\dagger(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{\sigma}_a(t) \rangle,
\] (18)
\[
\frac{d}{dt} \langle \hat{\eta}_b(t) \rangle = g \langle \hat{\sigma}_b^\dagger(t)\hat{b}(t) + \hat{b}^\dagger(t)\hat{\sigma}_b(t) - (\hat{\sigma}_a^\dagger(t)\hat{a}(t) + \hat{a}^\dagger(t)\hat{\sigma}_a(t) \rangle,
\] (19)
\[
\frac{d}{dt} \langle \hat{\eta}_c(t) \rangle = -\frac{\Omega}{2} \langle \hat{\sigma}_a^\dagger(t) + \hat{\sigma}_a(t) \rangle - g \langle \hat{\sigma}_b^\dagger(t)\hat{b}(t) + \hat{b}^\dagger(t)\hat{\sigma}_b(t) \rangle.
\] (20)
where
\[
\hat{\eta}_a = |a\rangle \langle a|,
\] (21)
\[
\hat{\eta}_b = |b\rangle \langle b|,
\] (22)
\[
\hat{\eta}_c = |c\rangle \langle c|.
\] (23)
We see that Eqs. (15)-(20) are nonlinear differential equations and hence it is not possible to find the exact time-dependent solutions of these equations. To overcome this problem, we apply the large-time approximation scheme to Eqs. (12) and (13) and get the following...
approximately valid relations

\[ \dot{a}(t) = -\frac{2\varepsilon}{\kappa} \hat{a}^\dagger(t) - \frac{2g}{\kappa} \hat{\sigma}_a(t) \]  

(24)

and

\[ \dot{b}(t) = -\frac{2\varepsilon}{\kappa} \hat{a}^\dagger(t) - \frac{2g}{\kappa} \hat{\sigma}_b(t). \]  

(25)

Applying Eqs. (24) and (25), we easily find

\[ \dot{a}(t) = \frac{4\varepsilon \kappa g}{\kappa^2 - 4\varepsilon^2} \left( \frac{\hat{a}^\dagger(t)}{\kappa} - \frac{\hat{\sigma}_a(t)}{2\varepsilon} \right) \]  

(26)

and

\[ \dot{b}(t) = \frac{4\varepsilon \kappa g}{\kappa^2 - 4\varepsilon^2} \left( \frac{\hat{b}^\dagger(t)}{\kappa} - \frac{\hat{\sigma}_b(t)}{2\varepsilon} \right). \]  

(27)

Now upon substituting Eqs. (26) and (27) together with their adjoint into Eqs. (15)-(20), we readily obtain

\[ \frac{d}{dt} \langle \hat{\sigma}_a(t) \rangle = \frac{\Omega}{2} \langle \hat{a}^\dagger \rangle + \frac{\kappa^2 \varepsilon \gamma_c}{4\varepsilon^2 - \kappa^2} \left( \frac{\hat{\sigma}_a(t)}{\varepsilon} - \frac{\hat{\sigma}_b(t)}{\kappa} \right), \]  

(28)

\[ \frac{d}{dt} \langle \hat{\sigma}_b(t) \rangle = -\frac{\Omega}{2} \langle \hat{b}^\dagger \rangle + \frac{\kappa^2 \varepsilon \gamma_c}{4\varepsilon^2 - \kappa^2} \left( \frac{\hat{\sigma}_b(t)}{2\varepsilon} \right), \]  

(29)

\[ \frac{d}{dt} \langle \hat{\sigma}_c(t) \rangle = \frac{\Omega}{2} \langle \hat{a}^\dagger \rangle + \frac{\kappa^2 \varepsilon \gamma_c}{4\varepsilon^2 - \kappa^2} \left( \frac{\hat{\sigma}_c(t)}{2\varepsilon} \right) + \frac{1}{\varepsilon} \langle \hat{\eta}_c(t) - \hat{\eta}_c(t) \rangle, \]  

(30)

\[ \frac{d}{dt} \langle \hat{\eta}_a(t) \rangle = \frac{\Omega}{2} \langle \hat{a}^\dagger \rangle + \frac{\kappa^2 \varepsilon \gamma_c}{4\varepsilon^2 - \kappa^2} \left( -\frac{1}{\varepsilon} \langle \hat{\sigma}_a(t) + \hat{\sigma}_c(t) \rangle + \frac{\hat{\eta}_a}{\varepsilon} \right), \]  

(31)

\[ \frac{d}{dt} \langle \hat{\eta}_b(t) \rangle = \frac{\Omega}{2} \langle \hat{b}^\dagger \rangle + \frac{\kappa^2 \varepsilon \gamma_c}{4\varepsilon^2 - \kappa^2} \left( \frac{\hat{\eta}_b(t)}{\varepsilon} \right), \]  

(32)

\[ \frac{d}{dt} \langle \hat{\eta}_c(t) \rangle = -\frac{\Omega}{2} \langle \hat{b}^\dagger \rangle + \frac{\kappa^2 \varepsilon \gamma_c}{4\varepsilon^2 - \kappa^2} \left( \frac{\hat{\eta}_c(t)}{\varepsilon} \right), \]  

(33)

where \( \gamma_c = \frac{4\varepsilon^2}{\kappa^2} \) is the stimulated emission decay constant.

We note that the steady-state solutions of Eqs. (30)-(32) have the form

\[ \langle \hat{\sigma}_c \rangle = \frac{\kappa \Omega \left( \frac{4\varepsilon^2}{\kappa^2} - 1 \right) \langle \hat{\eta}_a \rangle - \langle \hat{\eta}_b \rangle + 2\varepsilon \gamma_c \langle \hat{\eta}_c \rangle - \langle \hat{\eta}_b \rangle}{\gamma_c \kappa}, \]  

(34)

\[ \langle \hat{\sigma}_c^\dagger \rangle + \langle \hat{\sigma}_c \rangle = \langle \hat{\eta}_a \rangle \left( \frac{2\gamma_c \kappa}{2\gamma_c \varepsilon - \kappa \Omega \left( \frac{4\varepsilon^2}{\kappa^2} - 1 \right)} \right), \]  

(35)

\[ \langle \hat{\eta}_a \rangle - \langle \hat{\eta}_b \rangle = \frac{\varepsilon}{\kappa} \left( \langle \hat{\sigma}_c^\dagger \rangle + \langle \hat{\sigma}_c \rangle \right). \]  

(36)

Now from Eqs. (35) and (36), one can easily get

\[ \langle \hat{\eta}_b \rangle = \langle \hat{\eta}_a \rangle \left( 1 - \frac{2 \varepsilon \gamma_c}{2\gamma_c \varepsilon - \kappa \Omega \left( \frac{4\varepsilon^2}{\kappa^2} - 1 \right)} \right), \]  

(37)
Employing Eqs. (34) and (35), we have
\[
\frac{\kappa \Omega (\frac{4\varepsilon^2}{\kappa^2} - 1) (\langle \hat{\eta}_a \rangle - \langle \hat{\eta}_c \rangle) + 2\varepsilon \gamma_c (\langle \hat{\eta}_c \rangle - \langle \hat{\eta}_b \rangle)}{\gamma_c \kappa} = \langle \hat{\eta}_c \rangle \left( \frac{\gamma_c \kappa}{2\gamma_c \varepsilon - \kappa \Omega \left( \frac{4\varepsilon^2}{\kappa^2} - 1 \right)} \right)
\] (38)
and on taking into account Eq. (37), we find
\[
\langle \hat{\eta}_c \rangle = \langle \hat{\eta}_a \rangle \beta^2 \left( \beta^2 - 4\varepsilon^2 \gamma_c^2 + \kappa^2 \gamma_c^2 \right),
\] (39)
where
\[
\alpha = \kappa \Omega \left( \frac{4\varepsilon^2}{\kappa^2} - 1 \right)
\] (40)
and
\[
\beta = 2\gamma_c \varepsilon - \alpha.
\] (41)
Now applying the completeness relation
\[
\langle \hat{\eta}_a \rangle + \langle \hat{\eta}_b \rangle + \langle \hat{\eta}_c \rangle = 1
\] (42)
along with Eqs. (37) and (39), we arrive at
\[
\langle \hat{\eta}_a \rangle = \frac{\beta^2}{2 \beta^2 - \alpha \beta - 4\varepsilon^2 \gamma_c^2 + \kappa^2 \gamma_c^2}.
\] (43)
Thus in view of this result, Eqs. (37) and (39) take the form
\[
\langle \hat{\eta}_b \rangle = -\frac{\alpha \beta}{2 \beta^2 - \alpha \beta - 4\varepsilon^2 \gamma_c^2 + \kappa^2 \gamma_c^2}
\] (44)
and
\[
\langle \hat{\eta}_c \rangle = \frac{\beta^2 - 4\varepsilon^2 \gamma_c^2 + \kappa^2 \gamma_c^2}{2 \beta^2 - \alpha \beta - 4\varepsilon^2 \gamma_c^2 + \kappa^2 \gamma_c^2}.
\] (45)
Finally, combination of Eqs. (34), (43), (44), and (45) leads to
\[
\langle \hat{\sigma}_c \rangle = \frac{1}{\gamma_c \kappa q} \left( \alpha (\beta^2 - p) + 2\varepsilon \gamma_c (p - \alpha \beta) \right),
\] (46)
in which
\[
p = \beta^2 - 4\varepsilon^2 \gamma_c^2 + \kappa^2 \gamma_c^2
\] (47)
and
\[
q = p + \beta^2 - \alpha \beta.
\] (48)
The steady-state solutions of Eqs. (28) and (29) have the form
\[
\hat{\sigma}_a = \hat{\sigma}_b^\dagger \left( \frac{\beta}{2\gamma_c \kappa} \right)
\] (49)
and
\[ \hat{\sigma}_b = \hat{\sigma}_a^\dagger \left( \frac{\alpha}{\gamma_c \kappa} \right). \] (50)

Employing the adjoint of Eqs. (49) and (50), Eqs. (26) and (27) can be readily put in the form
\[ \hat{a} = \frac{2\Omega g}{\alpha \gamma_c \kappa^2} \left( \gamma_c \kappa^2 - 2\varepsilon \alpha \right) \hat{\sigma}_a \] (51)
and
\[ \hat{b} = \frac{2\Omega g}{\alpha \gamma_c \kappa^2} \left( \gamma_c \kappa^2 - \varepsilon \beta \right) \hat{\sigma}_b. \] (52)

3 Global Quadrature Squeezing

We now seek to calculate the mean photon number, the quadrature variance and quadrature squeezing for the two-mode cavity light. We represent the two-mode cavity light by
\[ \hat{c} = \hat{a} + \hat{b}. \] (53)

The mean photon number of the two-mode cavity light is defined by
\[ \bar{n} = \langle \hat{c}^\dagger \hat{c} \rangle. \] (54)

Then employing Eq. (53) and its adjoint, we arrive at
\[ \bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle + \langle \hat{a}^\dagger \hat{b} \rangle + \langle \hat{b}^\dagger \hat{a} \rangle. \] (55)

Using Eqs. (51) and (52), we easily find
\[ \langle \hat{a}^\dagger \hat{b} \rangle = \langle \hat{b}^\dagger \hat{a} \rangle = 0, \] (56)
so that Eq. (55) becomes
\[ \bar{n} = \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle. \] (57)

We see that the mean photon number of the two-mode cavity light is the sum of the mean photon numbers of light modes a and b. In addition, taking into account Eqs. (51) and (52) along with Eqs. (43) and (44), one readily obtains
\[ \bar{n} = \frac{\Omega^2}{q \alpha^2 \kappa^3 \gamma_c} \left[ \beta^2 \left( \gamma_c \kappa^2 - 2\varepsilon \alpha \right)^2 - \alpha \beta \left( \gamma_c \kappa^2 - \varepsilon \beta \right)^2 \right]. \] (58)

We now consider the special case in which the nonlinear crystal is removed from the cavity, with the top and bottom levels of the atom coupled by the coherent light. Thus upon setting \( \lambda = 0 \) in Eq. (58), we get
\[ \bar{n} = \frac{\gamma_c}{\kappa} \left( \frac{2\Omega^2}{3\Omega^2 + \gamma_c^2} \right). \] (59)
This represents the mean photon number of the light emitted at steady-state by the three-level atom pumped by the coherent light [10].

From the plots in Fig. 2, we observe that the mean photon number of the two-mode cavity light increases due to the coupling of the top and bottom levels of the three-level atom by the coherent light.

We now proceed to calculate the quadrature squeezing. To this end, applying the large-time approximation scheme to Eq. (29), we get

\[ \langle \hat{\sigma}_b(t) \rangle = \frac{\alpha}{\kappa \gamma_c} \langle \hat{\sigma}_a^\dagger(t) \rangle \]  

(60)

Then employing the complex conjugate of this result, one can put Eq. (28) in the form

\[ \frac{d}{dt} \langle \hat{\sigma}_a(t) \rangle = -\frac{\mu}{2} \langle \hat{\sigma}_a(t) \rangle, \]  

(61)

in which

\[ \mu = \frac{\Omega}{\alpha \kappa \gamma_c} \left( \alpha \beta - 2 \gamma_c^2 \kappa^2 \right). \]  

(62)

With the atom considered to be initially in the bottom level, the solution of Eq. (61) turns out to be

\[ \langle \hat{\sigma}_a(t) \rangle = 0. \]  

(63)
Moreover, using the same approximation scheme, we obtain from Eq. (28)

\[
\langle \hat{\sigma}_a(t) \rangle = \left( \frac{\beta}{2\kappa \gamma_c} \right) \langle \hat{\sigma}_b^\dagger(t) \rangle.
\] (64)

Then with the aid of the complex conjugate of this result, we can put Eq. (29) in the form

\[
\frac{d}{dt} \langle \hat{\sigma}_b \rangle = -\frac{\mu}{4} \langle \hat{\sigma}_b \rangle.
\] (65)

With the atom considered to be initially in the bottom level, the solution of Eq. (65) is found to be

\[
\langle \hat{\sigma}_b(t) \rangle = 0.
\] (66)

Now substituting the adjoint of Eq. (27) into Eq. (12), we get

\[
\frac{d}{dt} \hat{a}(t) = -\hat{a}(t) \left( \kappa^2 - 4\varepsilon^2 \right) + g \left( \frac{2\varepsilon}{\kappa} \hat{\sigma}_b(t) - \hat{\sigma}_a(t) \right)
\] (67)

and on taking into account Eq. (60), we have

\[
\frac{d}{dt} \hat{a}(t) = -\hat{a}(t) \left( \kappa^2 - 4\varepsilon^2 \right) - \frac{g}{\gamma_c \kappa^2} \left( \gamma_c \kappa^2 - 2\varepsilon \alpha \right) \hat{\sigma}_a(t).
\] (68)

The solution of this equation is a well-behaved function provided that

\[
\frac{\kappa^2 - 4\varepsilon^2}{2\kappa} > 0.
\] (69)

It then follows that

\[
\varepsilon < \frac{\kappa}{2}.
\] (70)

We realize that the expectation value of the solution of Eq. (68) is expressible as

\[
\langle \hat{a}(t) \rangle = \langle \hat{a}(0) \rangle e^{\int_0^t e^{\frac{\gamma_c}{\kappa^2} \left( \kappa^2 - 4\varepsilon^2 \right) \left( t - t' \right)}} \langle \hat{\sigma}_a(t') \rangle dt'.
\] (71)

Hence in view of the assumption that the cavity light is initially in a vacuum state and on the basis of Eq. (63), Eq. (71) turns out to be

\[
\langle \hat{a}(t) \rangle = 0.
\] (72)

Using the adjoint of Eq. (26) along with Eq. (64), Eq. (13) can be written as

\[
\frac{d}{dt} \hat{b}(t) = -\hat{b}(t) \left( \kappa^2 - 4\varepsilon^2 \right) - \frac{g}{\gamma_c \kappa^2} \left( \gamma_c \kappa^2 - \varepsilon \beta \right) \hat{\sigma}_b(t)
\] (73)
In view of Eq. (66) and the assumption that the cavity light is initially in a vacuum state, the expectation value of the solution of Eq. (73) becomes

\[ \langle \hat{b}(t) \rangle = 0. \]  

(74)

On account of Eqs. (72) and (74), Eq. (53) takes the form

\[ \langle \hat{c}(t) \rangle = 0. \]  

(75)

In addition, on adding Eqs. (68) and (73), we find

\[ \frac{d}{dt} \hat{c}(t) = -\frac{\hat{c}(t)}{2\kappa} \left( \kappa^2 - 4\varepsilon^2 \right) - g\hat{\sigma}(t), \]  

(76)

where

\[ \hat{\sigma}(t) = \frac{1}{\gamma_c\kappa^2} \left( (\gamma_c\kappa^2 - 2\varepsilon\alpha)\hat{a}(t) + (\gamma_c\kappa^2 - \varepsilon\beta)\hat{b}(t) \right). \]  

(77)

Therefore on the basis of Eqs. (76) and (75), we see that the operator \( \hat{c} \) is a Gaussian variable with zero mean. We note that the steady-state solution of Eq. (76) has the form

\[ \hat{c} = -\frac{2\kappa g}{\kappa^2 - 4\varepsilon^2}\hat{\sigma}. \]  

(78)

Hence employing Eqs. (77) and (78) along with Eqs. (44) and (45), we readily obtain

\[ \langle \hat{c}\hat{c}^\dagger \rangle = \frac{\Omega^2}{\alpha^2 q\gamma_c\kappa^3} \left( \alpha(\beta^2 - p) + 2\varepsilon\gamma_c(p + \alpha\beta) \right) \left( \gamma_c\kappa^2 - 2\varepsilon\alpha \right)^2. \]  

(79)

and

\[ \langle \hat{c}^2 \rangle = \langle \hat{c}^\dagger \rangle = \frac{\Omega^2}{\alpha^2 q\gamma_c^2\kappa^4} \left( \alpha(\beta^2 - p) + 2\varepsilon\gamma_c(p + \alpha\beta) \right) \left( \gamma_c\kappa^2 - 2\varepsilon\alpha \right) \left( \gamma_c\kappa^2 - \varepsilon\beta \right). \]  

(80)

The squeezing properties of the two-mode cavity light are described by two quadrature operators defined by

\[ \hat{c}_+ = \hat{c}^\dagger + \hat{c} \]  

(81)

and

\[ \hat{c}_- = i(\hat{c}^\dagger - \hat{c}). \]  

(82)

Applying Eqs. (81) and (82), it can be readily established that

\[ [\hat{c}_+, \hat{c}_-] = -2i(\hat{c}\hat{c}^\dagger - \hat{c}\hat{c}^\dagger). \]  

(83)

It then follows that [10]

\[ \Delta c_+ \Delta c_- \geq |\langle \hat{c}\hat{c}^\dagger \rangle - \langle \hat{c}\hat{c}^\dagger \rangle|. \]  

(84)
and with the aid of Eqs. (58) and (79), we easily find

$$
\Delta c_+ \Delta c_- \geq \frac{\Omega^2}{\alpha^2 q \gamma_c \kappa^3} \left| (\gamma_c \kappa^2 - 2 \varepsilon \alpha)^2 (\beta^2 + \alpha \beta) - (\gamma_c \kappa^2 - \varepsilon \beta)^2 (\alpha \beta + p) \right|. \quad (85)
$$

The variance of the quadrature operators is expressible as

$$
(\Delta c_\pm)^2 = \pm ((\hat{c}^\dagger \pm \hat{c})^2) = (\langle \hat{c}^\dagger \rangle \pm \langle \hat{c} \rangle)^2 \quad (86)
$$

and on account of Eq. (75), we have

$$
(\Delta c_\pm)^2 = \langle \hat{c}^\dagger \hat{c} \rangle + \langle \hat{c}^2 \rangle \pm (\langle \hat{c}^2 \rangle + \langle \hat{c}^3 \rangle). \quad (87)
$$

Now on applying Eqs. (58), (79) and (80), Eq. (87) can be put in the form

$$
(\Delta c_+)^2 = \frac{\Omega^2}{\alpha^2 q \gamma_c \kappa^3} \left[ (\gamma_c \kappa^2 - 2 \varepsilon \alpha)^2 (\beta^2 - \alpha \beta) + (\gamma_c \kappa^2 - \varepsilon \beta)^2 (p - \alpha \beta) + \frac{2}{\gamma_c \kappa} \left( \alpha (\beta^2 - p) + 2 \varepsilon \gamma_c (p + \alpha \beta) \right) \right. \left. \left( \gamma_c \kappa^2 - 2 \varepsilon \alpha \right) \left( \gamma_c \kappa^2 - \varepsilon \beta \right) \right]. \quad (88)
$$
Figure 4: Plots of quadrature squeezing [Eq. (92)] versus $\varepsilon$ for $\kappa = 0.8$, and $\gamma_c = 0.5$.

and

$$
(\Delta c_-)^2 = \frac{\Omega^2}{\alpha^2 \gamma_c \kappa^3} \left[ (\gamma_c \kappa^2 - 2\varepsilon \alpha)^2 (\beta^2 - \alpha \beta) + (\gamma_c \kappa^2 - \varepsilon \beta)^2 (p - \alpha \beta) - \frac{2}{\gamma_c \kappa} \left( (\alpha (\beta^2 - p) + 2\varepsilon \gamma_c (p + \alpha \beta)) (\gamma_c \kappa^2 - 2\varepsilon \alpha) (\gamma_c \kappa^2 - \varepsilon \beta) \right) \right].
$$

(89)

From the plots in Fig.3, we observe that the two-mode cavity light is in a squeezed state for $\Omega = 0$ and for $0 < \varepsilon < 0.355$. In addition, the two-mode cavity light is in a squeezed state for $\Omega = 0.2$ and for $0 < \varepsilon < 0.275$.

We note that for $\Omega = \varepsilon = 0$, Eqs. (88) and (89) reduce to

$$
(\Delta c_+)_v^2 = (\Delta c_-)_v^2 = \frac{\gamma_c}{\kappa}.
$$

(90)

This indeed represents the quadrature variance of a two-mode cavity vacuum state. Now we calculate the quadrature squeezing of the two-mode cavity light relative to the quadrature variance of the two-mode cavity vacuum state. We then define the quadrature squeezing of the two-mode cavity light by [10]

$$
S = \frac{(\Delta c_-)_v^2 - (\Delta c_-)_v^2}{(\Delta c_-)_v^2}.
$$

(91)
Hence employing Eqs. (89) and (90) in Eq. (91), we get
\[
S = 1 - \frac{\Omega^2}{\alpha^2 q \gamma_c K^2} \left[ \left( \gamma_c k^2 - 2 \varepsilon \alpha \right) \left( \beta^2 - \alpha \beta \right) + \left( \gamma_c k^2 - \varepsilon \beta \right) \left( p - \alpha \beta \right) - \frac{2}{\gamma_c K} \left( \alpha \left( \beta^2 - p \right) + 2 \varepsilon \gamma_c \left( p + \alpha \beta \right) \right) \left( \gamma_c k^2 - 2 \varepsilon \alpha \right) \left( \gamma_c k^2 - \varepsilon \beta \right) \right].
\] (92)

We notice from the plots in Fig. 4 that the coupling of the top and bottom levels of the three-level atom by coherent light leads to a decrease in the quadrature squeezing of the two-mode cavity light. On the other hand, the maximum global quadrature squeezing for \( \Omega = 0 \) and \( \Omega = 0.02 \) are 100% and 90% for values of \( \varepsilon \) close to 0.3 and 0.25, respectively. The condition \( \Omega = 0 \) is physically realized by covering the right-side of the nonlinear crystal by a screen which can absorb the coherent light.

4 Conclusion

In this research work, we have studied the squeezing and statistical properties of the light produced by the interaction of a three-level atom with the subharmonic light modes generated by a nonlinear crystal driven by the coherent light. Applying the steady-state solutions of the quantum Langevin equations and the Heisenberg equations, we have calculated the mean photon number and the quadrature squeezing of the superposed subharmonic light modes. We have found that the two-mode cavity light is in a squeezed state and the squeezing occurs in the minus quadrature. The maximum quadrature squeezing of the two-mode cavity light turns out to be 100% for \( \Omega = 0 \) and for values of \( \varepsilon \) close to 0.3. In addition, we have seen that the coupling of the bottom and top levels of the three-level atom by the coherent light in general increases the mean photon number and decrease the quadrature squeezing of the superposed light modes.

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