Supersensitive estimation of the coupling rate in cavity optomechanics with an impurity-dopped Bose-Einstein condensate

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We propose a scheme to implement a supersensitive estimation of the coupling strength in a hybrid optomechanical system which consists of a cavity-Bose-Einstein condensate system coupled to an impurity. This method can dramatically improve the estimation precision even when the involved photon number is small. The quantum Fisher information indicates that the Heisenberg scale sensitivity of the coupling rate could be obtained when the photon loss rate is smaller than the corresponding critical value in the input of either coherent state or squeezed state. The critical photon decay rate for the coherent state is larger than that of the squeezed state, and the coherent state input case is more robust against the photon loss than the squeezed state case. We also present the measurement scheme which can saturate the quantum Cramér-Rao bound.

I. INTRODUCTION

Cavity optomechanics studies the radiation pressure coupling between the optical modes and the mechanical oscillation [1, 2]. Cavity optomechanical systems intrinsically integrate the advantages of precision detection (optical mode) and signal sensing (mechanical oscillation), and thus the optomechanical systems provide a useful platform for the study of high-precision metrology [3–7]. Recently, a lot of investigation has been devoted to the estimation of physical quantities based on various optomechanical systems [8–10]. These measurement schemes include weak force detection [11–13], rotation sensing [14, 15], mass sensing [16, 17], gravity estimation [18], and gravitational wave detection [19].

The cavity optomechanical systems can be implemented with various physical setups. In particular, the system of ultracold atoms has some advantage over other systems to realize the optomechanical interaction: (i) The thermal noise in this system can be largely suppressed because the thermal effect in ultracold atoms is negligible. (ii) The strong coupling regime at the level of single quanta can be realized owing to the collective enhancement effect in the interaction between the cavity field and the atoms. (iii) There are good and tested experimental techniques in the fields of atomic Bose-Einstein condensates (BECs) and cavity quantum electrodynamics [20–27]. These features motivate a variety of applications based on the utilizing of genuine quantum optomechanical effects.

In this paper, we propose to study the supersensitive estimation of the coupling parameters related to the optical and oscillation degrees of freedom in a hybrid optomechanical system. This system is formed by an optical cavity, a cigar-shaped BEC, and a two-level impurity atom immersed in the BEC. Here, both the BEC and the impurity atom are coupled to the cavity field in the dispersive regime. The coupling between the BEC atoms and the cavity mode takes a form as the optomechanical-type interaction, and the coupling between the two-level impurity and the BEC is described by the spin-boson model [28–32]. We consider the optical cavity in either a coherent state or a squeezed vacuum state. To estimate the coupling rate, we calculate the quantum Fisher information (QFI) [33–35] of the impurity part in the final state, which is related to the quantum Cramér-Rao (QCR) bound on the precision of estimator. We find that the injection of both the coherent and squeezed light into the optomechanical cavity can improve the estimation precision to the Heisenberg limit (HL) when the photon loss rate is smaller than the critical values. The squeezed vacuum state has advantage over that of the coherent state under small photon decay rate. As a tradeoff, however, the squeezed state scheme is fragile to the photon loss. This is because the squeezing scheme is not only sensitive to the change of the estimated parameters, but also to the noise. In this sense, the coherent state is more robust against the photon loss of the cavity.

The rest of this work is organized as follows. In Sec. II, we introduce the physical model and present the Hamiltonian of the system. In Secs. III and IV, we calculate the dynamics of the impurity atom and the Fisher information corresponding to the coherent state and squeezed state inputs of the cavity field, respectively. In Sec. V, we study how to realize the supersensitive estimation of the coupling parameters. A conclusion will be presented in Sec. VI. Finally, we add an appendix to show a de-
tailed derivation of the effective Hamiltonian in the large detuning case.

II. MODEL AND HAMILTONIAN

We consider a hybrid optomechanical system which is formed by a cigar-shaped BEC of $N$ two-level atoms placed in a cavity and a two-level impurity atom immersed inside the BEC (as depicted in Fig. 1). We study the case where the BEC atoms are coupled to the cavity field via the dispersive coupling, and the coupling between these atoms are described by the $s$-wave scattering interaction. In the frame rotating at the cavity frequency, the coupling between the cavity field and the BEC can be described by the Hamiltonian [22, 23]

$$H_1 = \int dx \Psi^\dagger(x) \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \hbar U_0 \cos^2(kx)a^\dagger a + \frac{g_1D}{2}|\Psi(x)|^2 \right] \Psi(x),$$ (1)

where $a^\dagger$ and $a$ are the creation and annihilation operators of the cavity with the resonance frequency $\omega_c$, $\Psi^\dagger(x)$ and $\Psi(x)$ are the creation and annihilation operators of the Schrödinger field describing the atomic BEC, $m$ is the mass of a single atom. In the dispersive coupling region, the BEC atom-cavity photon interaction induces an additional potential $U_0 \cos^2(kx)a^\dagger a$ for atoms with the wave vector $k$. Here $U_0 = -\hbar^2/\Delta_{ac}$ is the maximal light shift per photon that an atom may experience, with $g$ being the atom-photon coupling constant and $\Delta_{ac} = \omega_a - \omega_c$ the detuning between the BEC atomic resonance frequency $\omega_a$ and cavity frequency $\omega_c$.

As we are interested in the case of weak atom-atom interactions $g_1D \sim 0$, which is tunable through the Feshbach resonance, and therefore the macroscopically occupied zero momentum state is coupled to the symmetric superposition of the $|\pm 2\hbar k\rangle$ momentum states via absorption and stimulated emission of cavity photons [22, 23]. The corresponding Schrödinger field can be expanded as the following single-mode quantum field

$$\Psi(x) = b_0 \sqrt{1/L} + b \cos(2kx) \sqrt{2/L}$$ (2)

with $L$ being the length of the BEC. Here $b$ is the annihilation operator of Bogoliubov mode corresponding to the quantum fluctuation of the atomic field about the condensate mode $b_0$. Assuming that depletion of the zero-momentum component of the condensate is small, we adopt the classical treatment via the replacement $b_0 \rightarrow \sqrt{N}$. Substituting them into Hamiltonian (1) and neglecting unimportant constant terms, we then get [23]

$$H'_1 \approx \hbar \omega_m b^\dagger b + \hbar \Omega_c a^\dagger a + \hbar \chi a^\dagger a(b + b^\dagger),$$ (3)

where $\omega_m = \hbar(2k)^2/(2m) = 4\omega_{rec}$ with $\omega_{rec}$ being the recoil frequency of the condensate atoms. Besides, $\Omega_c = U_0\sqrt{N}/2$ is the effective Stark-shifted detuning. The last term of Eq. (3) can be interpreted as an optomechanical interaction and the coupling strength is $\chi = U_0\sqrt{N}/8$.

In the low temperature limit, we assume that the impurity atom is in the ground state of harmonic trap $V(x) = m_A\omega_A^2 x^2/2$ that is independent of the internal states, and hence the spatial wave function is

$$\varphi(x) = \frac{1}{\pi^{1/4}\ell^{1/2}} \exp(-x^2/2\ell^2),$$ (4)

where $\ell = \sqrt{\hbar/(m_A\omega_A)} \ll L$ with $\omega_A$ being the trap frequency and $m_A$ the mass of the impurity. In a rotating frame with respect to $\hbar\omega_c\langle \epsilon \rangle\langle \epsilon \rangle$, the Hamiltonian of the impurity atom becomes

$$H_2 = \hbar \Delta \langle \epsilon \rangle\langle \epsilon \rangle,$$ (5)

with $\Delta = \Omega - \omega_c$ being the detuning between the cavity mode and the impurity resonance frequency. Here, $\hbar\Omega$ is level splitting between the ground ($\langle \gamma \rangle$) and excited ($\langle \epsilon \rangle$) states.

For the case of large detuning, the atomic transition to the excited state is suppressed, then the interaction between the cavity and the impurity can be obtained as

$$H_3 = \hbar \delta_{ac}(4 \langle \epsilon \rangle\langle \epsilon \rangle a^\dagger a + \langle \epsilon \rangle^2 - 2a^\dagger a),$$ (6)

where $\delta_{ac} = g_{ac}^2/\Delta$ and $g_{ac} = g_0 \int dx \varphi(x) \cos(kx)$ with $g_0$ being the maximum impurity-cavity coupling strength.

Finally, we assume that the impurity atom undergoes $s$-wave collisions with the BEC atoms only when the impurity atom is in the excited state, and the corresponding scattering length is $a_{AB}$. The impurity-BEC interaction can then be described by the following Hamiltonian [30]

$$H_4 = \frac{4\hbar^2 g_{AB}}{m_{AB}(E_{AB}^2 + E_{B1}^2)} \langle \epsilon \rangle\langle \epsilon \rangle \int dx |\varphi(x)|^2 \psi^\dagger(x)\psi(x) - \hbar \delta_{ab}\langle \epsilon \rangle\langle \epsilon \rangle + \hbar g_{ab}\langle \epsilon \rangle\langle \epsilon \rangle^2 (b + b^\dagger).$$ (7)
Here \( \delta_{ab} = \frac{4N \hbar}{m_{AB}(\ell_{A\perp} + \ell_{B\perp})} \) is the level shift due to the collision with \( m_{AB} = m_A m/(m_A + m) \) being the reduced mass, \( \ell_{A\perp} \) and \( \ell_{B\perp} \) are the harmonic oscillator lengths of the impurity and BEC in the \( y, z \)-directions, and \( g_{ab} = \frac{4 \sqrt{2} \hbar}{m_{AB}(\ell_{A\perp} + \ell_{B\perp})} \) is the impurity-BEC coupling parameter.

Based on the above analysis, the effective Hamiltonian of the hybrid system can be written as (see the Appendix for details)

\[
H_{\text{eff}} = H_1' + H_2 + H_3 + H_4. \tag{8}
\]

By phenomenologically introducing the cavity photon loss \( \kappa \) to Eq. (8), we have

\[
H_{\text{eff}}' = \hbar \omega_m b^\dagger b + \hbar (\Omega_c - i \kappa) a^\dagger a + \hbar G |e\rangle \langle e| a^\dagger a + \hbar \Omega_A |e\rangle \langle e| + \hbar (\chi a^\dagger a + g_{ab} |e\rangle \langle e|)(b^\dagger + b), \tag{9}
\]

where \( \Omega_c = \bar{\Omega}_c - 2 \delta_{ac} \) and \( \Omega_A = \Delta + \delta_{ac} + \delta_{ab} \) denote the effective oscillation frequency for cavity and impurity, respectively, and \( G = 4 \delta_{ac} \) is the coupling strength between cavity and impurity. Note that because the cavity decay \( \kappa \) dominates over the spontaneous decay of all atoms, thus we omit the effect of atomic decay.

III. DYNAMICS OF THE IMPURITY ATOM

By using Magnus expansion [39–41], the time evolution operator governed by Hamiltonian (9) can be obtained as (hereafter we let \( \hbar = 1 \))

\[
U(t) = e^{-i \left( \Omega_c a^\dagger a + \chi a^\dagger a + \hbar \Omega_A |e\rangle \langle e| + G |e\rangle \langle e| a^\dagger a \right) t} 
\times e^{i \left( \chi a^\dagger a + g_{ab} |e\rangle \langle e| \right) \omega_m t - \sin(\omega_m t)} e^{-\kappa a^\dagger a t} 
\times e^{i \left( \chi a^\dagger a + g_{ab} |e\rangle \langle e| \right)(b^\dagger - b)}, \tag{10}
\]

where \( \bar{\chi} \equiv \chi/\omega_m \) and \( \bar{g}_{ab} \equiv g_{ab}/\omega_m \) are rescaled dimensionless coupling strengths, and \( \Lambda = 1 - e^{-\omega_m t} \).

To proceed, we consider that the initial state of the total system as

\[
\rho^T(0) = |\phi\rangle \langle \phi| \otimes \rho^B \otimes |\psi\rangle_{\text{imp}} \langle \psi|, \tag{11}
\]

in which \( |\phi\rangle \) is an arbitrary input states of the cavity photon and \( \rho^B \) is the thermal equilibrium state of the Bogoliubov mode, defined by \( \rho^B = \left[1 - \exp(-\beta \omega_m)\right] \exp(-\beta \omega_m b^\dagger b) \) with \( \beta \) the inverse temperature. Here, the impurity atom is prepared as \( |\psi\rangle_{\text{imp}} = c_e |e\rangle + c_g |g\rangle \) involving two-level state \( |e\rangle \) and \( |g\rangle \), and the superposition coefficients \( c_e, c_g \) satisfy the normalization condition \( |c_e|^2 + |c_g|^2 = 1 \).

The estimation of the coupling strengths \( \chi \) and \( g \), equivalent to estimate \( \bar{\chi} \) and \( \bar{g}_{ab} \) when \( \omega_m \) is known, can be realized by implementing the estimation in the internal state of the impurity atom. In terms of Eqs. (10) and (11), we can get the final state of the total system at time \( t \), and the reduced density matrix elements of the impurity atom can be obtained by tracing out the cavity photon and Bogoliubov mode, which read as

\[
\rho_{eg}(t) = \rho_{ge}^*(t) = c_e c_g \sum_{n=0}^{\infty} e^{-2\kappa n} e^{-i(\Omega_A + G n)t} \langle n| \phi \rangle^2 
\times \exp \left\{ i(2\bar{\chi} \bar{g}_{ab} n + \bar{g}_{ab}^2) [\omega_m t - \sin(\omega_m t)] \right\} 
\times \exp \left\{ -\bar{g}_{ab}^2 \beta \omega_m^2 [1 - \cos(\omega_m t)] \right\} \coth \left( \frac{\beta \omega_m}{2} \right), \tag{12}
\]

with \( \langle n| \phi \rangle \) being the overlap between \( |\phi\rangle \) and number state \( |n\rangle \). At time \( \tau = 2\pi/\omega_m \), the last term of \( \rho_{eg}(t) \) vanishes, and then Eq. (12) can be reduced as

\[
\rho_{eg}(\tau) = \rho_{ge}^*(\tau) = c_e c_g e^{2\pi(\bar{g}_{ab}^2 - \Omega_A)} \sum_{n=0}^{\infty} e^{-4\pi \kappa n} e^{-2\pi(\bar{G} - 2\bar{\chi} \bar{g}_{ab}) n} \langle n| \phi \rangle^2, \tag{13}
\]

where we also have defined rescaled dimensionless parameters \( \tilde{x} \equiv x/\omega_m \) \( x \in \{ \bar{\Omega}_A, G, \kappa \} \). Based on Eq. (13), we can find that \( \rho_{eg} \) carries the information of the coupling strength \( \tilde{\chi} \) and \( \tilde{g}_{ab} \) via the relation \( O = \tilde{\chi} \tilde{g}_{ab} \). In what follows, we directly consider the precision of \( O \) for estimating the coupling strengths.

IV. QUANTUM FISHER INFORMATION FOR COHERENT STATE AND SQUEEZED VACUUM STATE INPUTS

We now introduce the QFI to study the best sensitivity of the estimated parameters in our hybrid system. The QFI gives a theoretical-achievable limit on the precision of an unknown parameter \( \theta \) via QCR bound, \( \delta \theta \geq 1/\sqrt{\nu F(\theta)} \) with \( \nu \) being the number of repeated measurements. To evaluate the QFI, we should first normalize and diagonalize \( \rho(\tau) \) as \( \rho = \sum_i \lambda_i |\varphi_i\rangle \langle \varphi_i| \) with \( \sum_i \lambda_i = 1 \). Without loss of generality, hereafter we will choose \( \nu = 1 \) and \( c_e = c_g = 1/\sqrt{2} \), then the corresponding eigenvalues and eigenvectors are given by

\[
\lambda_1 = \frac{1}{2} \left[ \frac{\rho_{pe}}{2\rho_{ee}} \right] \quad \lambda_2 = \frac{1}{2} \left[ \frac{\rho_{eg}}{2\rho_{ee}} \right], \tag{14}
\]

and

\[
|\varphi_1\rangle = \frac{1}{\sqrt{2}} \left[ \left( \frac{\rho_{pe}}{\rho_{eg}} \right) |e\rangle + |g\rangle \right], \quad |\varphi_2\rangle = \frac{1}{\sqrt{2}} \left[ \left( \frac{\rho_{pe}}{\rho_{eg}} \right) |e\rangle - |g\rangle \right]. \tag{15}
\]
Therefore, the QFI with respect to $O$ can be explicitly expressed as \cite{ref1,ref2,ref3}

\[
F_O = \frac{1}{\lambda_1(1-\lambda_1)} + 4\left(1-2\lambda_1\right)^2 |\langle\varphi_1|\partial O\varphi_1\rangle|^2,
\]

(16)

which contains two terms. The first term is regarded as the classical contribution, whereas the second term contains the truly quantum contribution \cite{ref1}. Using Eqs. (14)-(17), we have $F = F_c + F_q$,

\[
F_c = \frac{(\partial O\lambda_1)^2}{\lambda_1(1-\lambda_1)} = \frac{\text{Re}[\mathcal{A}]|^2}{\rho_{ee} - |\rho_{eg}|^2},
\]

\[
F_q = 4\left(1-2\lambda_1\right)^2 |\langle\varphi_1|\partial O\varphi_1\rangle|^2 = \frac{|\rho_{eg}|^2(\text{Im}[\mathcal{A}])^2}{\rho_{ee}}.
\]

(18)

where $\text{Re}[\mathcal{A}]$ (Im[$\mathcal{A}$]) is the real (imaginary) part of $\mathcal{A}$, defined as

\[
\mathcal{A} = \frac{\partial O\rho_{eg}}{\rho_{eg}}.
\]

(19)

In terms of the above equations, we find that the QFI relies on the values of $\rho_{ee}, |\rho_{eg}|$ and $\mathcal{A}$ which depend on the form of input optical states $|\phi\rangle$. Below we will consider two common input states: the coherent state $|\alpha\rangle$ and the squeezed vacuum state $|\xi\rangle$.

**A. Coherent state $|\alpha\rangle$**

Suppose that the cavity photons are in a coherent state \cite{ref4}

\[
|\phi\rangle \equiv |\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\]

(20)

and substitute it into Eq. (13), we get the corresponding matrix elements as

\[
\rho_{eg}(\tau) = \frac{e^{-i2\pi(\tilde{n}_{|\alpha\rangle} - \tilde{\tilde{n}}_{|\alpha\rangle})} \exp\left[\tilde{n}_{|\alpha\rangle} e^{-4\pi \tilde{n}} e^{-i2\pi(\tilde{\tilde{n}} - 2)}\right]}{2e^{\tilde{n}_{|\alpha\rangle}}},
\]

\[
\rho_{ge}(\tau) = \rho_{eg}^*(\tau),
\]

\[
\rho_{ee}(\tau) = \frac{1}{2} \exp\left[\tilde{n}_{|\alpha\rangle} (e^{-4\pi \tilde{n}} - 1)\right],
\]

(21)

where $\tilde{n}_{|\alpha\rangle} = |\alpha|^2$ is the average photon number.

After deriving $\rho_{eg}$ with $O$, the expression of $\mathcal{A}$ can be written as

\[
\mathcal{A}_{|\alpha\rangle} = i4\pi \tilde{n}_{|\alpha\rangle} e^{-4\pi \tilde{n}} e^{-i2\pi(\tilde{\tilde{n}} - 2)}.
\]

(22)

Based on Eqs. (16) and (17), we can obtain the QFI assisted by the case of coherent state input (see Tab. 1).

**B. Squeezed vacuum state $|\xi\rangle$**

We now turn to the case of squeezed vacuum state input the cavity. The definition of squeezed vacuum state
TABLE I. The average photon number \( \bar{n} \) in the cavity, the optimal working points \( O^* \), the optimal classical contribution \( F^*_c \), the optimal quantum contribution \( F^*_q \) of the QFI, and the total QFI in the optimal working points for coherent state and squeezed vacuum state when \( \kappa \to 0 \).

| \( |\phi\rangle \) | \( \bar{n} \) | optimal points \( O^* \) | classical contribution \( F^*_c \) | quantum contribution \( F^*_q \) | total QFI \( F^*_{\text{total}} = F^*_c + F^*_q \) |
|---|---|---|---|---|---|
| \( |\alpha\rangle \) | \( |\alpha|^2 \) | 0.5(\( \bar{G} \pm 0.5m \)), \( m = 0, 1, 2, ... \) | \( 16\pi^2\bar{n}e^{-4\kappa} \) | \( \frac{32\pi^2\alpha(n+1)}{(n+1)e^{8\pi\kappa} - b} \) | \( 16\pi^2\bar{n}e^{-8\pi\kappa} \) | \( 48\pi^2\bar{n}(\bar{n} + 2/3) > \bar{n}^2 \) |
| \( |\xi\rangle \) | \( \sinh^2(r) \) | 0.5(\( \bar{G} \pm 0.5m \)), \( m = 0, 1, 2, ... \) | \( \frac{16\pi^2\bar{n}e^{-8\pi\kappa}}{(n+1)e^{8\pi\kappa} - b} \) | \( 16\pi^2\bar{n}e^{-8\pi\kappa} \) | \( 48\pi^2\bar{n}(\bar{n} + 2/3) > \bar{n}^2 \) |

As is shown, in our hybrid system both the coherent state and squeezed vacuum state can induce very large QFI which depends on \( \bar{G} \) and the maximum value is larger than \( \bar{n}^2 \) (the HL). For coherent state the optimal values of QFI can be found when \( O \to 0.5(\bar{G} \pm m) \), \( (m = 0, 1, 2, ...) \), while for squeezed state the optimal points are \( O \to 0.5(\bar{G} \pm 0.5m) \), \( (m = 0, 1, 2, ...) \). As expected, when \( \kappa \) is not too large, as a useful resource for quantum metrology, the squeezed state outperforms that of coherent state. However, for large \( \kappa \) the situation becomes quite different.

In Fig. 3, we plot the optimal rates of the HL \( F/\bar{n}^2 \) with respect to mean photon number \( \bar{n} \) for different values of loss rate \( \kappa \). When \( \kappa \to 0 \) and \( \bar{n} = 1 \), for squeezed state the optimal rate is \( F/\bar{n} \to 90\pi^2 \), which gains an advantage over that of the coherent state case \( F/\bar{n} \to 32\pi^2 \) (see Tab. 1). With the average photon number \( \bar{n} \) increasing, both the rates will decrease and then tend to the steady values, \( F/\bar{n}^2 = 48\pi^2 \) and \( F/\bar{n} = 16\pi^2 \) (see Fig. 3(a)). When photon loss rate \( \kappa \) increasing, the advantage of squeezed state gradually loses, as shown in Fig. 3(a) and (b). These results mean that we can obtain supersensitive estimation of the coupling rates in our hybrid optomechanical system by using either coherent state or squeezed vacuum state even when small number of photons input. For small \( \kappa \), squeezed state shows advantage, but with the increase of \( \kappa \) the coherent state is more robust against the photon loss.

According to Eqs. (16) and (17), we know that the total QFI involves two parts: the classical \( (F_c) \) and quantum \( (F_q) \) contribution. In Fig. 4(a) and (b), we compare \( F_c \) and \( F_q \) induced by coherent state and squeezed state input cases. Here we fix the average photon number \( \bar{n} = 5 \), that is \( \alpha = \sqrt{5} \) and \( r \approx 1.545 \). As shown in Fig. 4, for coherent state input case the QFI main originate from quantum contribution \( F_q \). In fact, we can check that when \( \kappa = 0 \) the maximal values of \( F_q \) and \( F_c \) are \( F_q \approx 16\pi^2\bar{n}^2 \) and \( F_c \approx 16\pi^2\bar{n} \), respectively. Besides, the optimal values of \( F_q \) are the same for the two different input states. It means that the advantage of squeezed state for quantum metrology in our system is attributed to the classical contribution. In addition, we can clearly see that \( F_c > F_q \) for squeezed state.

Let us now investigate the influence of the photon loss on the estimation precision. As the QFI depends on average cavity photon number, the decrease of the average cavity photon number will result in the estimation sen-
sitivity reduce. To understand that coherent state can perform better for parameter estimation than squeezed vacuum state in the presence of photon loss, we first calculate the average cavity photon number for the above input states with photon loss and then give the critical conditions of HL for them.

At time $\tau = 2\pi/\omega_m$, by trace over the impurity part and the Bogoliubov mode from the total density matrix, the matrix elements of the cavity optical modes are

$$
\rho_{nn}(\tau) = \frac{1}{2} e^{-2\pi i(n+m+1)} e^{-i\tilde{\Omega} \tau} e^{i\frac{\pi}{2} (m^2 - n^2)} |\langle n|\phi\rangle|^2 \times \left[ e^{-i2\pi \tilde{G}(m-n)} e^{i\pi \tilde{G} \delta_{nm} (m-n) + 1}\right].
$$

(27)

Then the average photon number $\bar{n}$ in the cavity for coherent state and squeezed vacuum state can be obtained as

$$
\bar{n}_{\alpha} = \sum_n n \rho_{nn} = |\alpha|^2 \exp \left[ |\alpha|^2 (e^{-4\pi \tilde{k}} - 1) - 4\pi \tilde{k}\right],
$$

$$
\bar{n}_\xi = \sum_n 2n \rho_{2n2n} = \frac{\text{sech}(r) \tanh^2(r) e^{4\pi \tilde{k}}}{|\rho\tilde{k} - \tanh^2(r)|^{3/2}}.
$$

(28)

Figure 5(a) shows the average photon number $\bar{n}$ in the cavity as a function of loss rate $\tilde{k}$. It indicates that with the increase of $\tilde{k}$, $\bar{n}_{\xi}$ demonstrates faster decay than that of $\bar{n}_{\alpha}$.

To maintain the Heisenberg-limited sensitivity, $F \sim \bar{n}^2$, the values of photon loss $\tilde{k}$ should be below a critical condition. By calculating equation $F(\tilde{k}^*) = \bar{n}^2$, we can obtain the critical values for coherent state and squeezed vacuum state, respectively, as

$$
\tilde{k}^*_{\alpha} = \frac{1}{4\pi} \ln \left[ \frac{4\pi \bar{n}}{\sqrt{4\pi^2 + \bar{n}^2} - 2\pi}\right],
$$

$$
\tilde{k}^*_\xi = \frac{1}{8\pi} \ln \left[ \frac{\bar{n}^2 + 4\pi \sqrt{3\bar{n}(3\bar{n} + 2)}}{\bar{n}(\bar{n} + 1)}\right].
$$

(29)

We consider a fixed measurement that can lead to the highest precision of the estimated parameter $O$. Here, we consider a fixed measurement $\{\{|\pm\rangle\} = \langle \pm | \rho | \pm \rangle / \sqrt{2}$ to extract the information of the estimated parameter $O$ [38], which is quantified by the classical Fisher information (CFI) $F_{cl}$

$$
F_{cl} = \sum_\pm |\partial O \rho(\pm|O)\rangle|^2 / P(\pm|O),
$$

(30)

where $P(\pm|O) = \langle \pm | \rho | \pm \rangle$ is the probability of getting the measurement result at the eigenvalues of $|\pm\rangle$. Through some calculations, we have

$$
P(\pm|O) = \frac{1}{2} \pm \frac{\text{Re}[\rho_{\alpha\alpha}]}{2\rho_{ee}},
$$

(31)

and the CFI can be obtained as

$$
F_{cl} = \frac{(\text{Re}[A] \text{Re}[\rho_{\alpha\alpha}] - \text{Im}[A] \text{Im}[\rho_{\alpha\alpha}])^2}{\rho_{ee} - (\text{Re}[\rho_{\alpha\alpha}])^2}.
$$

(32)

Then the standard deviation for the coefficient $O$ is given by the Cramér-Rao (CR) bound

$$
\delta O_{CR} \geq \frac{1}{\sqrt{F_{cl}}} = \frac{\sqrt{\rho_{ee} - (\text{Re}[\rho_{\alpha\alpha}])^2}}{\text{Re}[A] \text{Re}[\rho_{\alpha\alpha}] - \text{Im}[A] \text{Im}[\rho_{\alpha\alpha}]}.
$$

(33)
deviation for the coefficient $O$ is given by
\[ \delta O = \frac{\Delta \sigma_x}{|d(\sigma_x)/dO|} = \frac{1}{\sqrt{F_{\text{cr}}}}, \] (36)
which exactly agrees with Eq. (33).

By calculating Eq. (33), as shown in Fig. 6, we find that the coupling strength sensitivity $\delta O_{\text{CR}}$ can be in accordance with the best sensitivity, i.e., $\delta O_{\text{QCR}} \equiv 1/\sqrt{F}$, when omitting the phasing $e^{i2\pi(\bar{\xi}_a - \bar{\xi}_x)}$ in Eq. (13).

**VI. CONCLUSION**

We have presented a scheme to obtain HL coupling-rate estimation in a hybrid cavity optomechanics formed by a cavity-BEC system coupled to an impurity. By calculating the QFI of the impurity, we found that the parameter sensitivity can be remarkably improved by using either coherent state or squeezed state fed into the cavity. For small photon decay rate, compared with the case of coherent state the squeezed vacuum state has some advantage owing to the classical contribution of the QFI. We also found that the HL can maintain even in the presence of cavity photon loss. Through comparing the critical values of photon loss rate, we found that the coherent state is more robust against the photon loss than squeezed state. Furthermore, we have demonstrated that the superprecision given by the QCR bound can be realized by implementing the fixed measurement $\{\{+,|\rangle\langle \cdot|\},\{-,|\rangle\langle \cdot|\}\}$ on the impurity atom.

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**Appendix: Derivation of the effective Hamiltonian in the large detuning case**

In this Appendix, we present a detailed derivation of the effective Hamiltonian of the system in the large-detuning case. In the frame rotating at the cavity frequency $\omega_c$, the many-body Hamiltonian of the quasi-1D BEC reads [24]

\[ H_{\text{BEC}} = \int dx \left[ \Psi_g^\dagger(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right) \Psi_g(x) + \Psi_e^\dagger(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \hbar \Delta_{ac} \right) \Psi_e(x) \right] + \frac{g_{1D}}{2} \int dx \Psi_g^\dagger(x) \Psi_g^\dagger(x) \Psi_g(x) \Psi_g(x), \] (A.1)

where $\Delta_{ac} = \omega_a - \omega_c$ is the detuning between the BEC atoms and the cavity field, $\Psi_g(x)$ and $\Psi_e(x)$ denote the atomic field operators, which describe the annihilation of an atom at position $x$ in the ground and excited states, respectively.
They obey the usual bosonic commutation relations
\[ [\Psi_f(x), \Psi_{f'}(x')] = \delta(x - x')\delta_{ff'}, \quad [\Psi_f(x), \Psi_f(x')] = [\Psi_{f'}(x), \Psi_{f'}(x')] = 0 \] (A.2)
for \( f, f' \in \{e, g\} \). The interaction between the BEC and the cavity is described by the Hamiltonian
\[ H_{\text{cav-BEC}}^\text{int} = \hbar g \cos(kx) \int dx \left[ \Psi_g^\dagger(x) a^\dagger \Psi_g(x) + \Psi_e^\dagger(x) a \Psi_g(x) \right] . \] (A.3)
In this rotating frame, the Hamiltonian of the two-level impurity atom reads \( H_{\text{imp}} = \hbar \Delta |e\rangle \langle e| \). The Hamiltonian describing the cavity-impurity coupling is written as
\[ H_{\text{cav-imp}} = \hbar g_{ac}(a^\dagger \sigma_a + a \sigma_a^+) , \] (A.4)
where \( \sigma_a^+ = (\sigma_a^-)^\dagger = |e\rangle \langle g| \) are the ladder operators for the impurity, and \( g_{ac} = g_0 \int dx \varphi(x) \cos(kx) \) with \( g_0 \) being the maximum impurity-cavity coupling strength.

We assume that the BEC atoms are only coupled with the excited state \(|e\rangle\) of the impurity atom via the Raman transition
\[ H_{\text{BEC-imp}} = \hbar \tilde{g}_{\text{BEC-imp}}^D |e\rangle \langle e| \int dx |\varphi|^2 \Psi_g^\dagger(x) \Psi_g(x) \] (A.5)
where \( \tilde{g}_{\text{BEC-imp}}^D \) is the 1D interaction strength, which can be obtained by integrating out the \( y \) and \( z \) variables of the atomic field operators
\[ \tilde{g}_{\text{BEC-imp}}^D = \frac{4\pi \hbar a_{AB}}{m_{AB}} \int dy dz |\varphi_\perp(y, z)|^2 \Psi_\perp^\dagger(y, z) \Psi_\perp(y, z) = \frac{4\hbar a_{AB}}{m_{AB}(\ell_{A\perp}^2 + \ell_{B\perp}^2)} , \] (A.6)
where we have assumed that \( \Psi_\perp(y, z) = (\pi \ell_{B\perp}^2)^{-1/2} e^{-(y^2+z^2)/(2\ell_{B\perp}^2)} \) and \( \varphi_\perp(y, z) = (\pi \ell_{A\perp}^2)^{-1/2} e^{-(y^2+z^2)/(2\ell_{A\perp}^2)} \).

Based on the above analyses, the Hamiltonian of the whole system can be expressed as
\[ H = H_{\text{BEC}} + H_{\text{cav-BEC}}^\text{int} + H_{\text{imp}} + H_{\text{cav-imp}} + H_{\text{BEC-imp}} \] (A.7)
Accordingly, the Heisenberg equations for the various field operators are obtained as
\[ \frac{\partial \Psi_e(x)}{\partial t} = -i \frac{\hbar}{\Delta} [\Psi_e(x), H] = i \left( -\hbar^2 \frac{d^2}{2m} - \Delta_{ac} \right) \Psi_e(x) - ig(x) a \Psi_g(x) \] (A.8)
and
\[ \frac{\partial \sigma_a^+}{\partial t} = -i \left( \sigma_a^+, H \right) = i \Delta \sigma_a^+ + i g_{ac} a^\dagger \tilde{g} \int dx |\varphi|^2 \Psi_g^\dagger(x) \Psi_g(x) - ig_{ac} \sigma_a a^\dagger . \] (A.9)

Below, we assume that the temperature of the BEC is low enough to neglect the thermal excitation. In particular, we consider the case where both the BEC-atom-cavity coupling and the impurity-cavity coupling work in the large-detuning regime, and then the quantum coherence transition of the impurity and the BEC atoms can be eliminated adiabatically. To this end, we neglect the kinetic energy term of the BEC atoms in the zero-temperature limit, because it is much smaller than the internal energy term. Then the excited-state atomic field operator and the transition operator of the impurity atom can be expressed as
\[ \Psi_e(x) = \frac{g(x)a \Psi_g(x)}{\Delta_{ac}} , \quad \sigma_a^+ = \frac{g_{ac} a^\dagger \sigma_a^z}{\Delta + \tilde{g}_{\text{BEC-imp}}^D |e\rangle \langle e| \int dx |\varphi|^2 \Psi_g^\dagger(x) \Psi_g(x)} \approx \frac{g_{ac} a^\dagger \sigma_a^z}{\Delta + \delta_{ab}|e\rangle \langle e| + g_{ac} |e\rangle \langle e|} \approx \frac{g_{ac} a^\dagger \sigma_a^z}{\Delta} , \] (A.10)
the “≈” in the above equation based on the fact that \( \Delta \gg \tilde{g}_{\text{BEC-imp}}^D, \delta_{ac} \) and \( g_{ab} \). Inserting the above equations for \( \Psi_e(x) \) and \( \sigma_a^+ \) into \( H \), we arrive at
\[ H_{\text{eff}} = \int dx \Psi_g^\dagger(x) \left( -\hbar^2 \frac{d^2}{2m} + \hbar U_0 \cos^2(kx) a^\dagger a + \frac{\tilde{g}_{\text{BEC-imp}}^D |e\rangle \langle e| \int dx |\varphi|^2 \Psi_g^\dagger(x) \Psi_g(x)}{2} \right) \Psi_g(x) + H_{\text{imp}} + \hbar \tilde{g}_{\text{BEC-imp}}^D |e\rangle \langle e| \int dx |\varphi|^2 \Psi_g^\dagger(x) \Psi_g(x) + \hbar \delta_{ac} \frac{|e\rangle \langle e| a^\dagger a - 2a^\dagger a + |e\rangle \langle e|}{2} , \] (A.11)
where \( U_0 = -g^2/\Delta_{ac} \). In the main text, we denote \( \Psi_g(x) \) with \( \Psi(x) \).

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