Geometry of canonical bases and mirror symmetry

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Abstract

A decorated surface $S$ is an oriented surface with boundary and a finite, possibly empty, set of special points on the boundary, considered modulo isotopy. Let $G$ be a split reductive group over $\mathbb{Q}$.

A pair $(G, S)$ gives rise to a moduli space $\mathcal{A}_{G,S}$, closely related to the moduli space of $G$-local systems on $S$. It is equipped with a positive structure [FG1]. So a set $\mathcal{A}_{G,S}(\mathbb{Z}^t)$ of its integral tropical points is defined. We introduce a rational positive function $W$ on the space $\mathcal{A}_{G,S}$, called the potential. Its tropicalisation is a function $W^t : \mathcal{A}_{G,S}(\mathbb{Z}^t) \to \mathbb{Z}$. The condition $W^t \geq 0$ defines a subset of positive integral tropical points $\mathcal{A}_{G,S}^+(\mathbb{Z}^t)$. For $G = SL_2$, we recover the set of positive integral $A$-laminations on $S$ from [FG1].

We prove that when $S$ is a disc with $n$ special points on the boundary, the set $\mathcal{A}_{G,S}^+(\mathbb{Z}^t)$ parametrises top dimensional components of the fibers of the convolution maps. Therefore, via the geometric Satake correspondence $L$-cycles and $L$-local systems for $G = GL_2$, we recover the set of positive integral $A$-laminations on $S$ from [FG1].

We define our main example. We prove that the set $\mathcal{A}_{GL_2,S}^+(\mathbb{Z}^t)$ with Knutson-Tao’s hives [KT] characterises the families of $A$-laminations. Our result generalises a theorem of Kamnitzer [K1], who used hives to parametrise top components of convolution varieties for $G = GL_m, n = 3$. For $G = GL_m, n > 3$, we prove Kamnitzer’s conjecture [K1]. Our parametrisation is naturally cyclic invariant. We show that for any $G$ and $n = 3$ it agrees with Berenstein-Zelevinsky’s parametrisation [BZ], whose cyclic invariance is obscure.

We define more general positive spaces with potentials $(\mathcal{A}, W)$, parametrising mixed configurations of flags. Using them, we define a generalization of Mirković-Vilonen cycles [MV], and a canonical basis in $V_m \otimes \cdots \otimes V_n$, generalising the Mirković-Vilonen basis in $V_m$. Our construction comes naturally with a parametrisation of the generalised MV cycles. For the classical MV cycles it is equivalent to the one discovered by Kamnitzer [K].

We prove that the set $\mathcal{A}_{G,S}^+(\mathbb{Z}^t)$ parametrises top dimensional components of a new moduli space, surface affine Grassmannian, generalising the fibers of the convolution maps. These components are usually infinite dimensional. We define their dimension being an element of a $Z$-torsor, rather then an integer. We define a new moduli space $Loc_{G,L,S}$, which reduces to the moduli spaces of $G$-local systems on $S$ if $S$ has no special points. The set $\mathcal{A}_{G,S}^+(\mathbb{Z}^t)$ parametrises a basis in the linear space of regular functions on $Loc_{G,L,S}$.

We suggest that the potential $W$ itself, not only its tropicalization, is important – it should be viewed as the potential for a Landau-Ginzburg model on $\mathcal{A}_{G,S}$. We conjecture that the pair $(\mathcal{A}_{G,S}, W)$ is the mirror dual to $Loc_{G,L,S}$. In a special case, we recover Givental’s description of the quantum cohomology connection for flag varieties and its generalisation [GLO2]. [RZ]. We formulate equivariant homological mirror symmetry conjectures parallel to our parametrisations of canonical bases.

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2
1 Introduction

1.1 Geometry of canonical bases in representation theory

1.1.1 Configurations of flags and parametrization of canonical bases

Let G be a split semisimple simply-connected algebraic group over \( \mathbb{Q} \). There are several basic vector spaces studied in representation theory of the Langlands dual group \( G^L \):

1. The weight \( \lambda \) component \( U(\mathcal{N}^L)(\lambda) \) in the universal enveloping algebra \( U(\mathcal{N}^L) \) of the maximal nilpotent Lie subalgebra in the Lie algebra of \( G^L \).

2. The weight \( \mu \) subspace \( V_{\lambda}^\mu(\mu) \) in the highest weight \( \lambda \) representation \( V_{\lambda} \) of \( G^L \).

3. The tensor product invariants \( (V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^{G^L} \).

4. The weight \( \mu \) subspaces in the tensor products \( V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \).

Calculation of the dimensions of these spaces, in the cases 1)-3), is a fascinating classical problem, which led to Weyl’s character formula and Kostant’s partition function.

The first examples of special bases in finite dimensional representations are Gelfand-Tsetlin’s bases [GT1], [GT2]. Other examples of special bases were given by De Concini-Kazhdan [DCK].

The canonical bases in the spaces above were constructed by Lusztig [L1], [L3], [L5]. Another approach to canonical bases was developed by Kashiwara [Ka]. Canonical bases in representations of \( GL_3, Sp_4 \) were defined by Gelfand-Zelevinsky-Retakh [GZ], [RZ].
Closely related, but in general different bases were considered by Nakajima [N1, N2], Malkin [M], Mirković-Vilonen [MV], and extensively studied afterwords. Abusing terminology, we also call them canonical bases.

It was discovered by Lusztig [L] that, in the cases 1)-2), the sets parametrising canonical bases in representations of the group G are intimately related to the Langlands dual group $G^L$.

Kashiwara discovered in the cases 1)-2) an additional crystal structure on these sets, and Joseph proved a rigidity theorem [J] asserting that, equipped with the crystal structure, the sets of parameters are uniquely determined.

One of the results of this paper is a uniform geometric construction of the sets parametrizing all of these canonical bases, which leads to a natural uniform construction of canonical bases parametrized by these sets in the cases 2)-4). In particular, we get a new canonical bases in the case 4), generalizing the Mirković-Vilonen (MV) basis in $V_A$. To explain our set-up let us recall some basic notions.

A positive space $\mathcal{Y}$ is a space, which could be a stack whose generic part is a variety, equipped with a positive atlas. The latter is a collection of rational coordinate systems with subtraction free transition functions between any pair of the coordinate systems. Therefore the set $\mathcal{Y}(\mathbb{Z})$ of the integral tropical points of $\mathcal{Y}$ is well defined. We review all this in Section 2.1.1.

Let $(\mathcal{Y}, \mathcal{W})$ be a positive pair given by a positive space $\mathcal{Y}$ equipped with a positive rational function $\mathcal{W}$. Then one can tropicalize the function $\mathcal{W}$, getting a $\mathbb{Z}$-valued function

$$\mathcal{W}^t : \mathcal{Y}(\mathbb{Z}) \longrightarrow \mathbb{Z}.$$ 

Therefore a positive pair $(\mathcal{Y}, \mathcal{W})$ determines a set of positive integral tropical points:

$$\mathcal{Y}^t_W(\mathbb{Z}) := \{ l \in \mathcal{Y}(\mathbb{Z}) \mid \mathcal{W}^t(l) \geq 0 \}. \quad (2)$$

We usually omit $\mathcal{W}$ in the notation and denote the set by $\mathcal{Y}^+(\mathbb{Z})$.

To introduce the positive pairs $(\mathcal{Y}, \mathcal{W})$ which play the basic role in this paper, we need to review some basic facts about flags and decorated flags in $G$.

**Decorated flags and associated characters.** Below $G$ is a split reductive group over $\mathbb{Q}$. Recall that the flag variety $B$ parametrizes Borel subgroups in $G$. Given a Borel subgroup $B$, one has an isomorphism $B = G/B$.

Let $G'$ be the adjoint group of $G$. The group $G'$ acts by conjugation on pairs $(U, \chi)$, where $\chi : U \rightarrow \mathbb{A}^1$ is an additive character of a maximal unipotent subgroup $U$ in $G'$. The subgroup $U$ stabilizes each pair $(U, \chi)$. A character $\chi$ is non-degenerate if $U$ is the stabilizer of $(U, \chi)$. The principal affine space $G/U$ parametrizes pairs $(U, \chi)$ where $\chi$ is a non-degenerate additive character of a maximal unipotent group $U$. Therefore there is an isomorphism

$$i_\chi : A_{G'} \rightarrow G'/U.$$  

This isomorphism is not canonical: the coset $[U] \in G'/U$ does not determine a point of $A_{G'}$. To specify a point one needs to choose a non-degenerate character $\chi$. One can determine uniquely the character by using a pinning, see Sections 2.1.2, 2.1.3. So writing $A_{G'} = G'/U$ we abuse notation, keeping in mind a choice of the character $\chi$, or a pinning.

Having said this, one defines the principal affine space $A_G$ for the group $G$ by $A_G := G/U$. We often write $A$ instead of $A_G$. The points of $A$ are called decorated flags in $G$. The group $G$ acts on $A$ from the left. For each $A \in A$, let $U_A$ be its stabilizer. It is a maximal unipotent subgroup of $G$. There is a canonical projection

$$\pi : A \longrightarrow B, \quad \pi(A) := \text{the normalizer of } U_A. \quad (3)$$

The projection $G \rightarrow G'$ gives rise to a map $p : A_G \longrightarrow A_{G'}$ whose fibers are torsors over the center of $G$. Let $p(A) = (U_A, \chi_A)$. Here $U_A$ is a maximal unipotent subgroup of $G'$. It is identified with a similar subgroup of $G$, also denoted by $U_A$. So a decorated flag $A$ in $G$ provides a non-degenerate character of the maximal unipotent subgroup $U_A$ in $G$:

$$\chi_A : U_A \longrightarrow \mathbb{A}^1. \quad (4)$$

\[\text{Inspite of the name, it is not an affine variety.}\]
Clearly, if \( u \in U_A \), then \( gug^{-1} \in U_{gA} \), and

\[
\chi_A(u) = \chi_{gA}(gug^{-1}).
\]  
(5)

**Example.** A flag for \( \text{SL}_m \) is a nested collection of subspaces in an \( m \)-dimensional vector space \( V_m \) equipped with a volume form \( \omega \in \det V_m^* \):

\[
F_0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m, \quad \dim F_i = i.
\]

A decorated flag for \( \text{SL}_m \) is a flag \( F_\bullet \) with a choice of non-zero vectors \( f_i \in F_i/F_{i-1} \) for each \( i = 1, \ldots, m-1 \), called *decorations*. For example, \( A_{\text{SL}_2} \) parametrises non-zero vectors in a symplectic space \( (V_2, \omega) \). The subgroup preserving a vector \( f \in V_2 - \{0\} \) is given by transformations \( u_f(a) : v \mapsto v + a\omega(v, f)v \). Its character \( \chi_f \) is given by \( \chi_f(u_f(a)) = a \).

Our basic geometric objects are the following three types of configuration spaces:

\[
\text{Conf}_n(\mathcal{A}) = G \setminus \mathcal{A}^n, \quad \text{Conf}(\mathcal{A}^n, \mathcal{B}) := G \setminus (\mathcal{A}^n \times \mathcal{B}), \quad \text{Conf}(\mathcal{B}, \mathcal{A}^n, \mathcal{B}) := G \setminus (\mathcal{B} \times \mathcal{A}^n \times \mathcal{B}).
\]  
(6)

**The potential \( \mathcal{W} \).** A key observation is that there is a natural rational function

\[
\chi^o : \text{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}) = G \setminus (\mathcal{B} \times \mathcal{A} \times \mathcal{B}) \longrightarrow \mathbb{A}^1.
\]

Let us explain its definition. A pair of Borel subgroups \( \{B_1, B_2\} \) is *generic* if \( B_1 \cap B_2 \) is a Cartan subgroup in \( G \). A pair \( \{A_1, B_2\} \in \mathcal{A} \times \mathcal{B} \) is generic if the pair \( (\pi(A_1), B_2) \) is generic. Generic pairs \( \{A_1, B_2\} \) form a principal homogeneous \( G \)-space. Thus, given a triple \( \{B_1, A_2, B_3\} \in \mathcal{B} \times \mathcal{A} \times \mathcal{B} \) such that \( \{A_2, B_3\} \) and \( \{A_2, B_1\} \) are generic, there is a unique \( u \in U_{A_2} \) such that

\[
\{A_2, B_3\} = u \cdot \{A_2, B_1\}.
\]  
(7)

So we define \( \chi^o(B_1, A_2, B_3) := \chi_{A_2}(u) \). Using it as a building block, we define a positive rational function \( \mathcal{W} \) on each of the spaces (6).

For example, to define the \( \mathcal{W} \) on the space \( \text{Conf}_n(\mathcal{A}) \) we start with a generic collection \( \{A_1, \ldots, A_n\} \in \mathcal{A}^n \), set \( B_i := \pi(A_i) \), and define \( \mathcal{W} \) as a sum, with the indices modulo \( n \):

\[
\mathcal{W} : \text{Conf}_n(\mathcal{A}) \longrightarrow \mathbb{A}^1, \quad \mathcal{W}(A_1, \ldots, A_n) := \sum_{i=1}^{n} \chi^o(B_{i-1}, A_i, B_{i+1}).
\]  
(8)

Note that the potential \( \mathcal{W} \) is well-defined when each adjacent pair \( \{A_i, A_{i+1}\} \) is generic, meaning that \( \{\pi(A_i), \pi(A_{i+1})\} \) is generic. Assigning the (decorated) flags to the vertices of a polygon, we picture the potential \( \mathcal{W} \) as a sum of the contributions \( \chi_A \) at the \( A \)-vertices (shown boldface) of the polygon, see Fig 1.

By construction, the potential \( \mathcal{W}_G \) on the space \( \text{Conf}_n(\mathcal{A}_G) \) is the pull back of the potential \( \mathcal{W}_{G'} \) for the adjoint group \( G' \) via the natural projection \( p_{G \rightarrow G'} : \text{Conf}_n(\mathcal{A}_G) \rightarrow \text{Conf}_n(\mathcal{A}_{G'}) \):

\[
\mathcal{W}_G = p_{G \rightarrow G'}^* \mathcal{W}_{G'}.
\]  
(9)

Potentials for the other two spaces in (6) are defined similarly, as the sums of the characters assigned to the decorated flags of a configuration. A formula similar to (9) evidently holds.

**Parametrisations of canonical bases.** It was shown in [FGH] that all of the spaces (6) have natural positive structures. We show that the potential \( \mathcal{W} \) is a positive rational function.

We prove that the sets parametrizing canonical bases admit a uniform description as the sets \( \mathcal{Y}_g \setminus \mathcal{V}'(Z') \) of positive integral tropical points assigned to the following positive pairs \( (\mathcal{Y}, \mathcal{W}) \). To write the potential \( \mathcal{W} \) we use an abbreviation \( \chi_{A_i} := \chi^o(B_{i-1}, A_i, B_{i+1}) \), with indices mod \( n \):

1. The canonical basis in \( U(\mathcal{A}^L) \):

\[
\mathcal{Y} = \text{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}), \quad \mathcal{W}(B_1, A_2, B_3) := \chi_{A_2}.
\]
Figure 1: The potential $\mathcal{W}$ is a sum of the contributions $\chi_A$ at the A-vertices (boldface).

2. The canonical basis in $V_\lambda$:

$$\mathcal{Y} = \text{Conf}(A, A, B), \quad \mathcal{W}(A_1, A_2, B_3) := \chi_{A_1} + \chi_{A_2}.$$ 

3. The canonical basis in invariants of tensor product of $n$ irreducible $G_L$-modules:

$$\mathcal{Y} = \text{Conf}_n(A), \quad \mathcal{W}(A_1, \ldots, A_n) := \sum_{i=1}^{n} \chi_{A_i}. \quad (10)$$

4. The canonical basis in tensor products of $n$ irreducible $G_L$-modules:

$$\mathcal{Y} = \text{Conf}(A^{n+1}, B), \quad \mathcal{W}(A_1, \ldots, A_{n+1}, B) := \sum_{i=1}^{n+1} \chi_{A_i}. \quad (11)$$

Natural decompositions of these sets, like decompositions into weight subspaces in 1) and 2), are easily described in terms of the corresponding configuration space, see Section 2.3.2.

Let us emphasize that the canonical bases in tensor products are not the tensor products of canonical bases in irreducible representations. Similarly, in spite of the natural decomposition

$$V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} = \oplus_{\lambda} V_{\lambda} \otimes (V_{\lambda_1}^* \otimes V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^{G_L},$$

the canonical basis on the left is not a product of the canonical bases on the right.

Descriptions of the sets parametrizing the canonical bases were known in different but equivalent formulations in the following cases:

In the cases 1)-2) there is the original parametrization of Lusztig [L].

In the case 3) for $n = 3$, there is Berenstein-Zelevinsky’s parametrization [BZ], referred to as the BZ data. We produce in Appendix an isomorphism between our parametrization and the BZ data. The cyclic symmetry, evident in our approach, is obscure for the BZ data.

The description in the $n > 3$ case in 3) seems to be new.

The cases 1), 2) and 4) were investigated by Berenstein and Kazhdan [BK1, BK2], who introduced and studied geometric crystals as algebraic-geometric avatars of Kashiwara’s crystals. In particular, they describe the sets parametrizing canonical bases in the form [2], without using, however, configuration spaces. We show in Appendix A that the space of generic configurations $\text{Conf}^*(A^n, B)$ with the potential $\mathcal{W}$ is a positive decorated geometric crystal in the sense of [BK3]. Interpretation of geometric crystals relevant to representation theory as moduli spaces of mixed configurations of flags makes, to our opinion, the story more transparent.

To define canonical bases in representations, one needs to choose a maximal torus in $G_L$ and a positive Weyl chamber. Usual descriptions of the sets parametrizing canonical bases require the same choice. Unlike this, working with configurations we do not require such choices.

---

2 We would like to stress that the positive structures and potentials on configuration spaces which we employ for parametrization of canonical bases do not depend on any extra choices, like pinning etc., in the group. See Section 6.3.

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Most importantly, our parametrization of the canonical basis in tensor products invariants leads immediately to a similar set which parametrizes a linear basis in the space of functions on the moduli space \( \text{Loc}_{G,l, S} \) of \( G \)-local systems on a decorated surface \( S \). Here the approach via configurations of decorated flags, and in particular its transparent cyclic invariance, are essential. See the example when \( G = SL_2 \) in Section 1.3.1

Summarizing, we understood the sets parametrizing canonical bases as the sets of positive integral tropical points:

We call it the space of positive integral tropical points:

\[ \text{Conf}_n \times \text{Loc} \]

Let us now go to canonical bases in invariants of tensor products of \( G \)-local systems on a decorated surface \( S \). Here the approach via configurations of decorated flags, and in particular its transparent cyclic invariance, are essential. See the example when \( G = SL_2 \) in Section 1.3.1

1.1.2 Constructing canonical bases in tensor products invariants

We start with a simple general construction. Let \( Y \) be a positive space, understood just as a collection of split tori glued by positive birational maps \([FG1]\). Since it is a birational notion, there is no set of \( F \)-points of \( Y \), where \( F \) is a field. Let \( K := \mathbb{C}(t) \). In Section 2.2.1 we introduce a set \( Y^\circ(K) \). We call it the set of transcendental \( K \)-points of \( Y \). It is a set making sense of “generic \( K \)-points of \( Y \)”.

Let us now go to canonical bases in invariants of tensor products of \( G \)-local systems on a decorated surface \( S \). Here the approach via configurations of decorated flags, and in particular its transparent cyclic invariance, are essential. See the example when \( G = SL_2 \) in Section 1.3.1

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1. A transcendental cell \( C^\circ \) of \( \text{Conf}_n(A) \) is contained in \( \text{Conf}_{n}^O(A) \) if and only if it corresponds to a positive tropical point. Moreover, given a point \( l \in \text{Conf}_n(A)(\mathbb{Z}^t) \), one has

\[ l \in \text{Conf}_n^+(A)(\mathbb{Z}^t) \iff C^\circ \subset \text{Conf}_n^O(A) \iff C^\circ \cap \text{Conf}_n^O(A) \neq \emptyset. \quad (14) \]

2. Let \( \text{Gr} := G(K)/G(\mathcal{O}) \) be the affine Grassmannian. It follows immediately from the very definition of the subspace \([14]\) that there is a canonical map

\[ \kappa : \text{Conf}^O_n(A) \rightarrow \text{Conf}_n(\text{Gr}) := G(K)/(\text{Gr})^n. \]

These two properties of \( \text{Conf}^O_n(A) \) allow to transport points \( l \in \text{Conf}^+_n(A)(\mathbb{Z}^t) \) into the top components of the stack \( \text{Conf}_n(\text{Gr}) \). Namely, given a point \( l \in \text{Conf}^+_n(A)(\mathbb{Z}^t) \), we define a cycle

\[ M_l := \text{closure of } M^\circ_l \subset \text{Conf}_n(\text{Gr}), \quad \text{where } M^\circ_l := \kappa(C^\circ_l). \]
1.2.1 Our main example

1.2 Positive tropical points and top components

The cycle $C^+_l$ is defined for any $l \in \text{Conf}_n(A)(\mathbb{Z}^l)$. However, as it clear from (14), the map $\kappa$ can be applied to it if and only if $l$ is positive: otherwise $C^+_l$ is not in the domain of the map $\kappa$.

We prove that the map $l \mapsto M_l$ provides a bijection

$$\text{Conf}_n(A)(\mathbb{Z}^l) \xrightarrow{\sim} \{ \text{closures of the top dimensional components of the stack Conf}_n(\text{Gr}) \}. \tag{15}$$

Here the very notion of a “top dimensional” component of a stack requires clarification. For now, we will bypass this question in a moment by passing to more traditional varieties.

We use a very general argument to show the injectivity of the map $l \mapsto M_l$. Namely, given a positive rational function $F$ on $\text{Conf}_n(A)$, we define a $\mathbb{Z}$-valued function $D_F$ on $\text{Conf}_n(\text{Gr})$. It generalizes the function on the affine Grassmannian for $G = \text{GL}_m$ and its products defined by Kamnitzer [K], [K1]. We prove that the restriction of $D_F$ to $M_l^n$ is equal to the value $F^t(l)$ of the tropicalization $F^t$ of $F$ at the point $l \in \text{Conf}_n^+(A)(\mathbb{Z}^l)$. Thus the map (15) is injective.

Let us reformulate our result in a more traditional language. The orbits of $G(O)$ acting on $G \times \text{Gr}$ are labelled by dominant weights of $G^L$. We write $L_1 \sim L_2$ if $(L_1, L_2)$ is in the orbit labelled by $\lambda$. Let $[1]$ be the identity coset in $\text{Gr}$. A set $\Lambda = (\lambda_1, \ldots, \lambda_n)$ of dominant weights of $G^L$ determines a cyclic convolution variety, better known as a fiber of the convolution map:

$$\text{Gr}_c(\Lambda) := \{ (L_1, \ldots, L_n) \mid L_1 \xrightarrow{\lambda_1} L_2 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_n} L_{n+1}, L_1 = L_{n+1} = [1] \} \subset [1] \times \text{Gr}^{n-1}. \tag{16}$$

These varieties provide a $G(O)$-equivariant decomposition

$$[1] \times \text{Gr}^{n-1} = \bigsqcup_{\lambda = (\lambda_1, \ldots, \lambda_n)} \text{Gr}_c(\lambda). \tag{17}$$

Since $G(O)$ is connected, it preserves each component of $\text{Gr}_c(\lambda)$. Thus the components of $\text{Gr}_c(\lambda)$ live naturally on the stack

$$\text{Conf}_n(\text{Gr}) = G(O)\backslash([1] \times \text{Gr}^{n-1}).$$

We prove that the cycles $M_l$ assigned to the points $l \in \text{Conf}_n^+(A)(\mathbb{Z}^l)$ are closures of the top dimensional components of the cyclic convolution varieties. The latter, due to the geometric Satake correspondence, give rise to a canonical basis in (1). We already know that the map (15) is injective. We show that the $\lambda$-components of the sets related by the map (15) are finite sets of the same cardinality, respected by the map. Therefore the map (15) is an isomorphism.

Our result generalizes a theorem of Kamnitzer [K1], who used hives [KT] to parametrize top components of convolution varieties for $G = \text{GL}_m$, $n = 3$.

Our construction generalizes Kamnitzer’s construction of parametrizations of Mirković-Vilonen cycles [K]. At the same time, it gives a coordinate free description of Kamnitzer’s construction.

When $G = \text{GL}_m$, there is a special coordinate system on the space $\text{Conf}_3(A)$, introduced in Section 9 of [FG1]. We show in Section 3 that it provides an isomorphism of sets

$$\text{Conf}_3^+(A)(\mathbb{Z}^l) \xrightarrow{\sim} \{ \text{Knutson-Tao's hives} \ [KT] \}. \tag{15}$$

Using this, we get a one line proof of Knutson-Tao-Woodward’s theorem [KTW] in Section 2.1.6.

For $G = \text{GL}_m$, $n > 3$, we prove Kamnitzer conjecture [K1], describing the top components of convolution varieties via a generalization of hives – we identify the latter with the set $\text{Conf}_n^+(A)(\mathbb{Z}^l)$ via the special positive coordinate systems on $\text{Conf}_n(A)$ from [FG1].

1.2 Positive tropical points and top components

1.2.1 Our main example

Denote by $\text{Conf}_n^+(A)$ the subvariety of $\text{Conf}_n(A)$ parametrizing configurations of decorated flags $(A_1, \ldots, A_n)$ such that the flags $(\pi(A_i), \pi(A_{i+1}))$ are in generic position for each $i = 1, \ldots, n$ modulo $n$. The potential $\mathcal{W}$ was defined in (5). It is evidently a regular function on $\text{Conf}_n^+(A)$.
Let $P^+$ be the cone of dominant coweights. There are canonical isomorphisms
\[ \alpha : \text{Conf}_2^+ (A) \xrightarrow{\sim} H, \quad \text{Conf}_2(\text{Gr}) = P^+. \] (18)

Configurations $(A_1, \ldots, A_n)$ sit at the vertices of a polygon, as on Fig 2. Let $\pi_E : \text{Conf}_n(A) \to \text{Conf}_2(A)$ be the projection corresponding to a side $E$ of the polygon. Denote by $\pi^+_E$ its restriction to $\text{Conf}_n^+(A)$. The collection of the maps $\{\pi^+_E\}$, followed by the first isomorphism in (18) provides a map
\[ \pi : \text{Conf}_n^+(A) \longrightarrow \text{Conf}_2^+(A)^n \cong H^n. \]

Using similarly the second isomorphism in (18), we get a map
\[ \pi_{\text{Gr}} : \text{Conf}_n(\text{Gr}) \longrightarrow \text{Conf}_2(\text{Gr})^n = (P^+)^n. \]

Let $\{\omega_i\}$ be a basis of the cone of positive dominant weights of $H$. The functions $\pi^+_E \omega_i$ are equations of the irreducible components of the divisor $D := \text{Conf}_n(A) - \text{Conf}_n^+(A)$:
\[ D := \text{Conf}_n(A) - \text{Conf}_n^+(A) = \cup_{E,i} D^E_{i}. \]

Equivalently, the component $D^E_{i}$ is determined by the condition that the pair of flags at the endpoints of the edge $E$ belongs to the codimension one $G$-orbit corresponding to the simple reflection $s_i \in W$. \[ \text{Fig 2: Going from an } \mathcal{O}\text{-integral configuration of decorated flags to a configuration of lattices.} \]

The space $\text{Conf}_n(A)$ has a cluster $\mathcal{A}$-variety structure, described for $G = \text{SL}_n$ in [FG1], Section 10. An important fact [FG3] is that any cluster $\mathcal{A}$-variety $\mathcal{A}$ has a canonical cluster volume form $\Omega_{\mathcal{A}}$, which in any cluster $\mathcal{A}$-coordinate system $(A_1, \ldots, A_n)$ is given by
\[ \Omega_{\mathcal{A}} = \pm d\log A_1 \wedge \ldots \wedge d\log A_n. \]

The functions $\pi^+_E \omega_i$ are the frozen $\mathcal{A}$-cluster coordinates in the sense of Definition [FG3]. This is equivalent to the claim that the canonical volume form $\Omega_{\mathcal{A}}$ on $\text{Conf}_n(A)$ has non-zero residues precisely at the irreducible components of the divisor $D^E_{i}$.

All this data is defined for any split semi-simple group $G$ over $\mathbb{Q}$. Indeed, the form $\Omega$ on $\text{Conf}_n(A)$ for the simply-connected group is invariant under the action of the center of the group, and thus its integral multiple descends to a form on $\text{Conf}_n(\mathcal{A}_G)$. The potential $W_G$ is defined by pulling back the potential $W_{G'}$ for the adjoint group $G'$. We continue discussion of this example in Section [FG4] where it is casted as an example of the mirror symmetry.

The simplest example. Let $(V_2, \omega)$ be a two dimensional vector space with a symplectic form. Then $\text{SL}_2 = \text{Aut}(V_2, \omega)$, and $\mathcal{A}_{\text{SL}_2} = V_2 - \{0\}$. Next, $\text{Conf}_n(\mathcal{A}_{\text{SL}_2}) = \text{Conf}_n(V_2)$ is the space of configuration

---

3 Indeed, $\omega_i(\alpha(A_1, A_2)) = 0$ if and only if the corresponding pair of flags belongs to the codimension one $G$-orbit corresponding to a simple reflection $s_i$.

4 Indeed, it follows from Lemma [FG3] and an explicit description of cluster structure on $\text{Conf}_n(A)$ that the form $\Omega_{\mathcal{A}}$ can not have non-zero residues anywhere else the divisors $D^E_{i}$. One can show that the residues at these divisors are non-zero.
(l_1, ..., l_n) of n non-zero vectors in V_2. Set \( \Delta_{i,j} := \langle \omega, l_i \wedge l_j \rangle \). Then the potential is given by the following formula, where the indices are mod n:

\[
\mathcal{W} := \sum_{i=1}^{n} \frac{\Delta_{i,i+2}}{\Delta_{i,i+1} \Delta_{i+1,i+2}}.
\]

The boundary divisors are given by equations \( \Delta_{i,i+1} = 0 \). To write the volume form, pick a triangulation \( T \) of the polygon whose vertices are labeled by the vectors. Then, up to a sign,

\[
\Omega := \bigwedge E \log \Delta_E.
\]

where \( E \) are the diagonals and sides of the n-gon, and \( \Delta_E := \Delta_{i,j} \) if \( E = (i,j) \). The function (19) is invariant under \( l_i \rightarrow -l_i \), and thus descends to \( \text{Conf}_n(\mathcal{A}_{\text{PGL}_2}) = \text{Conf}_n(V_2/ \pm 1) \).

1.2.2 The general framework

Let us explain main features of the geometric picture underlying our construction in most general terms, which we later on elaborate in details in every particular situation. First, there are three main ingredients:

1. A positive space \( Y \) with a positive rational function \( W \) called the potential, and a volume form \( \Omega_Y \) with logarithmic singularities. This determines the set \( Y_+^\pm W(\mathbb{Z}_t) \) of positive integral tropical points – the set parametrizing a canonical basis.

2. A subset of \( \mathcal{O} \)-integral points \( Y^\mathcal{O} \subset Y^0(\mathcal{K}) \). Its key feature is that, given an \( l \in Y^0(\mathbb{Z}'_t) \),

\[
l \in Y_+^\mathcal{W}(\mathbb{Z}') \iff C_l^\mathcal{O} \subset Y^\mathcal{O} \iff C_l^\mathcal{O} \cap Y^\mathcal{O} \neq \emptyset.
\]

3. A moduli space \( \text{Gr}_{Y,W} \), together with a canonical map

\[
\kappa : Y^\mathcal{O} \rightarrow \text{Gr}_{Y,W}.
\]

These ingredients are related as follows:

- Any positive rational function \( F \) on \( Y \) gives rise to a \( \mathbb{Z} \)-valued function \( D_F \) on \( \text{Gr}_{Y,W} \), such that for any \( l \in Y_+^\mathcal{W}(\mathbb{Z}') \), the restriction of \( D_F \) to \( \kappa(C_l^\mathcal{O}) \) equals \( F(l) \).

So we arrive at a collection of irreducible cycles

\[
\mathcal{M}_l^\mathcal{O} := \kappa(C_l^\mathcal{O}) \subset \text{Gr}_{Y,W}, \quad \mathcal{M}_l := \text{closure of } \mathcal{M}_l^\mathcal{O}, \quad l \in Y_+^\mathcal{W}(\mathbb{Z}').
\]

Thanks to the \( \bullet \), the assignment \( l \mapsto \mathcal{M}_l \) is injective.

Consider the set \( \{D_c\} \) of all irreducible divisors in \( Y \) such that the residue of the form \( \Omega_Y \) at \( D_c \) is non-zero. We call them the boundary divisors of \( Y \). We define

\[
Y^\times := Y - \bigcup D_c.
\]

By definition, the form \( \Omega_Y \) is regular on \( Y^\times \). In all examples the potential \( W \) is regular on \( Y^\times \).

There is a split torus \( \mathbb{H} \), and a positive regular surjective projection

\[
\pi : Y^\times \rightarrow \mathbb{H}.
\]

The map \( \pi \) is determined by the form \( \Omega_Y \). For example, assume that each boundary divisor \( D_c \) is defined by a global equation \( \Delta_c = 0 \). Then the regular functions \( \{\Delta_c\} \) define the map \( \pi \), i.e.

\[
\pi(y) = \{\Delta_c(y)\}.
\]

The set \( Y(\mathbb{Z}') \), the tropicalization \( \mathcal{W}^\mathcal{O} \), and thus the subset \( Y_+^\mathcal{W}(\mathbb{Z}') \) can also be determined by the volume form \( \Omega_Y \), without using the positive structure on \( Y \).
Next, there is a semigroup $\mathbb{H}^\mathcal{O} \subset \mathbb{H}(\mathcal{K})$ containing $\mathbb{H}(\mathcal{O})$, defining a cone
$$\mathcal{P} := \mathbb{H}^\mathcal{O}/\mathbb{H}(\mathcal{O}) \subset \mathbb{H}(\mathcal{Z}^t) := \mathbb{H}(\mathcal{K})/\mathbb{H}(\mathcal{O}) = X_*^s(\mathbb{H}),$$
such that the tropicalization of the map $\pi$ provides a map $\pi^t : \mathcal{Y}^+_W(\mathcal{Z}^t) \to \mathcal{P}$, and there is a surjective map $\pi_{Gr} : \text{Gr}_{\mathcal{Y},W} \to \mathcal{P}$. Denote by $\pi^\mathcal{O}$ restricting of $\pi \otimes \mathcal{K}$ to $\mathcal{Y}^\mathcal{O}$. These maps fit into a commutative diagram

$$\begin{array}{ccc}
\mathcal{Y}^+_W(\mathcal{Z}^t) & \xleftarrow{\text{val}} & \mathcal{Y}^\mathcal{O} & \xrightarrow{\kappa} & \text{Gr}_{\mathcal{Y},W} \\
\pi^t \downarrow & & \pi^\mathcal{O} \downarrow & & \downarrow \pi_{Gr} \\
\mathcal{P} & \xleftarrow{\text{val}} & \mathbb{H}^\mathcal{O} & \xrightarrow{\text{val}} & \mathcal{P}
\end{array}$$

We define $\text{Gr}_{\mathcal{Y},W}^{(\lambda)}$ and $\mathcal{Y}^+_W(\mathcal{Z}^t)_\lambda$ as the fibers of the maps $\pi_{Gr}$ and $\pi^t$ over a $\lambda \in \mathcal{P}$. So we have

$$\text{Gr}_{\mathcal{Y},W} = \prod_{\lambda \in \mathcal{P}} \text{Gr}_{\mathcal{Y},W}^{(\lambda)}, \quad \mathcal{Y}^+_W(\mathcal{Z}^t) = \prod_{\lambda \in \mathcal{P}} \mathcal{Y}^+_W(\mathcal{Z}^t)_\lambda. \quad (24)$$

The following is a key property of our picture:

- The map $l \mapsto \mathcal{M}_l$ provides a bijection
  $$\mathcal{Y}^+_W(\mathcal{Z}^t)_\lambda \longleftrightarrow \{\text{Closures of top dimensional components of } \text{Gr}_{\mathcal{Y},W}^{(\lambda)}\}.$$ 

Although the space $\text{Gr}_{\mathcal{Y},W}$ is usually infinite dimensional, it is nice. The map $\pi_{Gr} : \text{Gr}_{\mathcal{Y},W} \to \mathcal{P}$ slices it into highly singular and reducible pieces. However the slicing makes the perverse sheaves geometry clean and beautiful. It allows to relate the positive integral tropical points to the top components of the slices.

**Example.** In our main example, discussed in Section 1.1 we have

$$\mathcal{Y} = \text{Conf}_n(A), \quad \mathcal{Y}^\mathcal{X} = \text{Conf}_n^\mathcal{X}(A), \quad \mathcal{Y}^\mathcal{O} = \text{Conf}_n^\mathcal{O}(A), \quad \text{Gr}_{\mathcal{Y},W} = \text{Conf}_n(\text{Gr}), \quad \mathcal{H} = \mathbb{H}^n, \quad \mathcal{P} = (\mathbb{P}^+)^n.$$ 

The potential $\mathcal{W}$ is defined in [8], and decomposition (24) is described by cyclic convolution varieties [17].

### 1.2.3 Mixed configurations and a generalization of Mirković-Vilonen cycles

Let us briefly discuss other examples relevant to representation theory. All of them follow the set-up of Section 1.2. The obtained cycles $\mathcal{M}_l$ can be viewed as generalisations of Mirković-Vilonen cycles. Let us list first the spaces $\mathcal{Y}$ and $\text{Gr}_{\mathcal{Y},W}$. The notation $\text{Conf}_{w_0}$ indicates that the pair of the first and the last flags in configuration is in generic position.

i) **Generalized Mirković-Vilonen cycles:**

$$\mathcal{Y} := \text{Conf}_{w_0}(A, A^n, B), \quad \text{Gr}_{\mathcal{Y},W} := \text{Conf}_{w_0}(A, \text{Gr}^n, B) = \text{Gr}^n.$$ 

If $n = 1$, we recover the Mirković-Vilonen cycles in the affine Grassmannian [MV].

ii) **Generalized stable Mirković-Vilonen cycles:**

$$\mathcal{Y} := \text{Conf}_{w_0}(B, A^n, B), \quad \text{Gr}_{\mathcal{Y},W} := \text{Conf}_{w_0}(B, \text{Gr}^n, B) = H(\mathcal{K}) \setminus \text{Gr}^n.$$ 

If $n = 1$, we recover the stable Mirković-Vilonen cycles in the affine Grassmannian. In our interpretation they are top components of the stack

$$\text{Conf}_{w_0}(B, \text{Gr}, B) = H \setminus \text{Gr}.$$ 

iii) **The cycles providing canonical bases in tensor products**

$$\mathcal{Y} := \text{Conf}(A^{n+1}, B), \quad \text{Gr}_{\mathcal{Y},W} := \text{Conf}(\text{Gr}^{n+1}, B) = B^-(\mathcal{O}) \setminus \text{Gr}^n.$$ 

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The spaces \( Y \) in examples i) and iii) are essentially the same. However the potentials are different: in the case iii) it is the sum of contributions of all decorated flags, while in the case i) we skip the first one. Passing from \( Y \) to \( G_r Y \) we replace those \( A \)'s which contribute to the potential by \( Gr \)'s, but keep the \( B \)'s and the \( A \)'s which do not contribute to the potential intact.

We picture configurations at the vertices of a convex polygon, as on Fig 1. Some of the \( A \)-vertices are shown boldface. The potential \( W \) is a sum of the characters assigned to the boldface \( A \)-vertices, generalizing (8). The decorated polygons in the cases ii) and iii) are depicted on the right of Fig 8 and on Fig 6. We discuss these examples in detail in Sections 2.3 - 2.4.

1.3 Examples related to decorated surfaces

1.3.1 Laminations on decorated surfaces and canonical basis for \( G = SL_2 \)

1. Canonical basis in the tensor products invariants. This example can be traced back to XIX century. We relate it to laminations on a polygon.

Definition 1.1. An integral lamination \( l \) on an \( n \)-gon \( P_n \) is a collection \( \{ \beta_j \} \) of simple nonselfintersecting intervals ending on the boundary of \( P_n - \{ \text{vertices} \} \), modulo isotopy.

Figure 3: An integral lamination on a pentagon of type \((4, 4, 1, 6, 3)\).

Pick a vertex of \( P_n \), and number the sides clockwise. Given a collection of positive integers \( a_1, \ldots, a_n \), consider the set \( L_n(a_1, \ldots, a_n) \) of all integral laminations \( l \) on the polygon \( P_n \) such that the number of endpoints of \( l \) on the \( k \)-th side is \( a_k \). Let \((V_2, \omega)\) be a two dimensional \( \mathbb{Q} \)-vector space with a symplectic form. Let us assign to an \( l \in L_n(a_1, \ldots, a_n) \) an \( SL_2 \)-invariant map

\[
\mathbb{I}_l : (\otimes^{a_1} V_2) \otimes \ldots \otimes (\otimes^{a_n} V_2) \rightarrow \mathbb{Q}.
\]

We assign the factors in the tensor product to the endpoints of \( l \), so that the order of the factors match the clockwise order of the endpoints. Then for each interval \( \beta \) in \( l \) we evaluate the form \( \omega \) on the pair of vectors in the two factors of the tensor product labelled by the endpoints of \( \beta \), and take the product over all intervals \( \beta \) in \( l \). Recall that the \( SL_2 \)-modules \( S^a V_2 \), \( a > 0 \), provide all non-trivial irreducible finite dimensional \( SL_2 \)-modules up to isomorphism.

Theorem 1.2. Projections of the maps \( \mathbb{I}_l \), \( l \in L_n(a_1, \ldots, a_n) \), to \( S^{a_1} V_2 \otimes \ldots \otimes S^{a_n} V_2 \) form a basis in \( \text{Hom}_{SL_2}(S^{a_1} V_2 \otimes \ldots \otimes S^{a_n} V_2, \mathbb{Q}) \).

2. Canonical basis in the space of functions on the moduli space of \( SL_2 \)-local systems.

Definition 1.3. Let \( S \) be a surface with boundary. An integral lamination \( l \) on \( S \) is a collection of simple, mutually non intersecting, non isotopic loops \( \alpha_i \) with positive integral multiplicities

\[
l = \sum_i n_i [\alpha_i], \quad n_i \in \mathbb{Z}_{>0},\]

considered modulo isotopy. The set of all integral laminations on \( S \) is denoted by \( \mathcal{L}_Z(S) \).

\(^{\ddagger}\)Laminations on decorated surfaces were investigated in [FG1], Section 12, and [FG3]. However the two types of laminations considered there, the \( A\) - and \( X\)-laminations, are different then the ones in Definition 1.3. Indeed, they parametrise canonical bases in \( \mathcal{O}(X_{PGL_2,S}) \) and, respectively, \( \mathcal{O}(A_{SL_2,S}) \), while the latter parametrise a canonical basis in \( \mathcal{O}(Loc_{SL_2,S}) \). Notice that a lamination in Definition 1.3 can not end on a boundary circle.
In the case when $S$ is a surface without boundary we get Thurston’s integral laminations.

Given an integral lamination $l$ on $S$, let us define a regular function $M_l$ on the moduli space $\text{Loc}_{\text{SL}_2, S}$ of $\text{SL}_2$-local systems on $S$. Denote by $\text{Mon}_\alpha(L)$ the monodromy of an $\text{SL}_2$-local system $L$ over a loop $\alpha$ on $S$. The value of the function $M_l$ on $L$ is given by

$$M_l(L) := \prod_i \text{Tr}(\text{Mon}_{\alpha_i}^n(L)).$$

**Theorem 1.4.** ([FG1], Proposition 12.2). The functions $M_l, \ l \in \mathcal{L}(S)$, form a linear basis in the space $\mathcal{O}(\text{Loc}_{\text{SL}_2, S})$.

Recall that a **decorated surface** $S$ is an oriented surface with boundary, and a finite, possibly empty, collection $\{s_1, ..., s_n\}$ of **special** points on the boundary, considered modulo isotopy.

We define a moduli space $\text{Loc}_{\text{SL}_2, S}$ for any decorated surface $S$, so that laminations on $S$ provide a canonical basis $\mathcal{O}(\text{Loc}_{\text{SL}_2, S})$, generalising both Theorem 1.2 (when $S$ is a polygon) and Theorem 10.14, see Section 10.3.

Let us discuss now how to generalize constructions of Section 1.1.2 to the decorated surfaces.

### 1.3.2 Positive $G$-laminations and top components of surface affine Grassmannians

A pair $(G, S)$ gives rise to a moduli space $\mathcal{A}_{G, S}$ ([FG1]). Here are two basic examples.

- When $S$ is a disc with $n$ special points on the boundary, we recover the space $\text{Conf}_n(A)$.
- When $S$ is just a surface, without special points, the moduli space $\mathcal{A}_{G, S}$ is a twisted version of the moduli space of $G$-local systems with unipotent monodromy around boundary components on $S$ equipped with a covariantly constant decorated flag near every boundary component of $S$.

The space $\mathcal{A}_{G, S}$ has a positive structure ([FG1]). We define in Section 10 a **potential** $W$ on the space $\mathcal{A}_{G, S}$. It is a rational positive function, with the tropicalization $W^t : \mathcal{A}_{G, S}(\mathbb{Z}^t) \rightarrow \mathbb{Z}$.

The condition $W^t \geq 0$ determines a subset of **positive integral** $G$-laminations on $S$:

$$\mathcal{A}^+_t(S)(\mathbb{Z}^t) := \{ l \in \mathcal{A}_{G, S}(\mathbb{Z}^t) \mid W^t(l) \geq 0 \}.$$ (25)

For any decorated surface $S$, the set $\mathcal{A}^+_t(S)(\mathbb{Z}^t)$ is canonically isomorphic to the set of integral laminations on $S$, see Section 10.3. An interesting approach to a geometric definition of laminations for $G = \text{SL}_m$, which employs the affine Grassmannian, was suggested by Ian Le [Le].

There is a canonical volume form $\Omega$ on the space $\mathcal{A}_{G, S}$, which can be defined by using an ideal triangulation of $S$ and the volume forms on $\text{Conf}_n(A)$. When $G$ is simply-connected, it is also the cluster volume form $\Omega_A$.

We also assign to a pair $(G, S)$ a stack $\text{Gr}_{G, S}$, which we call the **surface affine Grassmannian**. When $S$ is a disc with $n$ special points on the boundary, we recover the stack $\text{Conf}_n(\text{Gr})$. In general it is an infinite dimensional stack.

The components of the punctured boundary $\partial S \setminus \{s_1, ..., s_n\}$ isomorphic to intervals are called boundary intervals. We define the torus $\mathbb{H}$ and the lattice $\mathbb{P}$ by

$$\mathbb{H} := \mathbb{H}^t(\text{boundary intervals on } S), \quad \mathbb{P} := \mathbb{P}^t(\text{boundary intervals on } S).$$
The map $\pi$ is defined by assigning to a boundary interval $I$ the element $i(A_{+},A_{-}) \in H$, see (18), where $(A_{-},A_{+})$ are the decorated flags at the ends of the interval $I$, ordered by the orientation of $S$, provided by the very definition of the space $A_{G,S}$.

Given a point $l \in \mathcal{A}_{G,S}^{+}(\mathbb{Z}^t)$, we define a cycle $\mathcal{M}^o_l \subset \text{Gr}_{G,S}$. Given an element $\lambda \in \mathbb{P}$, we prove that the map $l \mapsto \mathcal{M}^o_l$ gives rise to a bijection of sets

$$\mathcal{A}_{G,S}(\mathbb{Z}^t)_{\lambda} \xrightarrow{\sim} \{\text{closures of top dimensional components of } \text{Gr}_{G,S}^{(\lambda)}\}. \quad (26)$$

However in this case we can no longer bypass the question what are the “top components” of an infinite dimensional stack, as we did in Section 1.1.2 So we define in Section 10.5.1 “dimensions” of certain relevant stacks with values in certain dimension $\mathbb{Z}$-torsors. As a result, although the “dimension” is no longer an integer, the difference of two “dimensions” from the same dimension $\mathbb{Z}$-torsor is an integer, and so the notion of “top dimensional components” does make sense.

To define the analog of the space of tensor product invariants for a decorated surface $S$, we introduce in Section 10 a moduli space $\text{Loc}_{G,S}$. If $S$ has no special points, it is the moduli space of $G^L$-local systems on $S$. If $S$ is a disc with $n$ points on the boundary, it is the space $\text{Conf}_n A_{GL}$. We prove there that the set $\mathcal{A}_{G,S}^{+}(\mathbb{Z}^t)$ parametrizes a linear basis in $\mathcal{O}(\text{Loc}_{G,S})$.

### 1.4 Canonical bases, canonical pairings, and homological mirror symmetry

Below we write $A$ for $A_G$ etc., and use notation $A_L$ for $A_{GL}$ etc.

For any split reductive group $G$, the space $\mathcal{O}(A_L)$ of regular functions on the principal affine space $A_L$ of $G^L$ is a model of representations of $G^L$: every irreducible $G^L$-module appears there once. This allows us to organize the direct sum of all vector spaces of a given kind where the canonical bases live into a vector space of regular functions on a single space. For example:

$$\bigoplus_{(\lambda_1,\ldots,\lambda_n) \in (P^+)^n} V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} = \mathcal{O}(A^0_L). \quad (27)$$

$$\bigoplus_{(\lambda_1,\ldots,\lambda_n) \in (P^+)^n} \left(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}\right)^{G_L} = \mathcal{O}(A^0_L)^{G_L} = \mathcal{O}(\text{Conf}_n(A_L)). \quad (28)$$

Using this, let us interpret the statement that a canonical basis of a given kind is parametrized by positive integral tropical points of a certain space as existence of a canonical pairing.

#### 1.4.1 Tensor product invariants and mirror symmetry

For any split reductive group $G$, the set $\text{Conf}_n^+(A)(\mathbb{Z}^t)$ parametrizes a canonical basis in the space (28). So there are canonical pairings

$$I_G : \text{Conf}_n^+(A)(\mathbb{Z}^t) \times \text{Conf}_n(A_L) \rightarrow \mathbb{A}^1. \quad (29)$$

$$I_{GL} : \text{Conf}_n(A) \times \text{Conf}_n^+(A_L)(\mathbb{Z}^t) \rightarrow \mathbb{A}^1. \quad (30)$$

So the story becomes completely symmetric. The idea that the set parametrizing canonical bases in tensor product invariants is a subset of $\text{Conf}_n(A)(\mathbb{Z}^t)$ goes back to Duality Conjectures from [FG1]. It is quite surprising that taking into account the potential we get a canonical basis in the space of regular functions on the same kind of space, $\text{Conf}_n(A_L)$, for the Langlands dual group.

To picture this symmetry, consider a convex $n$-gon $P_n$ on the left of Fig 5 and assign a configuration $(A_1,\ldots,A_n) \in \text{Conf}_n(A)$ to its vertices. The potential $W$ is a sum of the vertex contributions; so the vertices are shown boldface. The pair of decorated flags at each side is generic; so all sides are dashed. Tropicalizing the data at the vertices, and using the isomorphism $\text{Conf}_n(A)(\mathbb{Z}^t) = P^+$, we assign a dominant weight $\lambda_k$ of $G^L$ to each side of the left polygon. Consider now the dual $n$-gon $*P_n$ on the right, and a configuration of decorated flags $(A_1^*,\ldots,A_n^*)$ in $G^L$ at its vertices. The dominant weight $\lambda_k$ on the left corresponds to the irreducible representation $V_{\lambda_k}$, realised in the model $\mathcal{O}(A_L)$ assigned to the dual vertex of $*P_n$. 




of real positive points. It is a Lagrangian submanifold for the symplectic form \( \omega \) on \( (M,\omega) \). In our examples the category \( FS(M) \) is Hamiltonian. Let \( \pi: M \rightarrow H \) be a complex affine variety. So it has an affine embedding \( i: M \hookrightarrow \mathbb{C}^N \). The Kahler form \( \sum z_i \bar{z}_i \) on \( \mathbb{C}^N \) induces a Kahler form on \( M(\mathbb{C}) \) with an exact symplectic form \( \omega \). The wrapped Fukaya category \( FS(M,\omega) \) does not depend on the embedding \( i \). We denote it by \( FS(M) \). A potential \( W \) allows to define the wrapped Fukaya-Seidel category \( FS(M) = FS(M,\omega,W) \). The case of a potential with only Morse singularities is treated in \( [AS] \). It also does not depend on the choice of affine embedding. A volume form \( \Omega \) provides a \( \mathbb{Z} \)-grading on \( FS(M) \) \( [S00] \).

The positive A-brane. In our examples \( M \) is a positive space over \( \mathbb{Q} \). So it has a submanifold \( M(\mathbb{R}_{>0}) \) of real positive points. It is a Lagrangian submanifold for the symplectic form \( \omega \) induced by any affine embedding. The form \( \Omega \) is defined over \( \mathbb{Q} \), and so \( M(\mathbb{R}_{>0}) \) is a special Lagrangian submanifold since it restricts to a real volume form on \( M(\mathbb{R}) \). The potential \( W \) is a positive function on \( M \). So the special Lagrangian submanifold \( M(\mathbb{R}_{>0}) \) should give rise to an object of the wrapped Fukaya-Seidel category of \( M \), which we call the positive A-brane, denoted by \( \mathcal{L}_+ \).

The projection / action data. In all our examples we have a mirror dual pair \( M \leftrightarrow M_L \) equipped with the following data: a projection \( \pi: M \rightarrow H \) onto a split torus \( H \), an action of the split torus \( T \) on \( M \) preserving the volume form and the potential, and a similar pair of tori \( H_L, T_L \) for \( M_L \). These tori are in duality:

\[
X_*(T_L) = X^*(H), \quad X_*(H_L) = X^*(T).
\]

This projection / action data gives rise to the following additional structures on the categories.

i) The group \( \text{Hom}(X_*(H),C^*) = \mathbb{H}(C) \) of \( C^* \)-local systems on the complex torus \( \mathbb{H}(C) \) acts on the category \( FS(M) \). Namely, we assume that the objects of the category are given by Lagrangian submanifolds in \( M(\mathbb{C}) \) with \( U(1) \)-local systems. Then a \( U(1) \)-local system \( \mathcal{L} \) on \( \mathbb{H}(C) \) acts by the tensor product with \( \pi^*(\mathcal{L}) \), providing an action of the subgroup \( \text{Hom}(X_*(H),U(1)) \) on the category. We assume that the action extends to an algebraic action of the complex torus

\[
\text{Hom}(X_*(H),C^*) = X^*(\mathbb{H}) \otimes C^* = X_*(\mathbb{H}) \otimes C^* = \mathbb{H}(C).
\]

ii) Let \( T_K \) be the maximal compact subgroup of the torus \( T(\mathbb{C}) \). We assume that the action of the group \( T_K \) on the symplectic manifold \( (M,\omega) \) is Hamiltonian. Then any subgroup \( S^1 \subset T_K \) provides a family of symplectic maps \( r_t \), \( t \in \mathbb{R}/\mathbb{Z} = S^1 \). The map \( r_1 \) provides an invertible functorial automorphism of \( \text{Hom} \)'s of

\[
\text{In our main examples the symplectic structure is exact, } \omega = da. \text{ So averaging the form } \alpha \text{ by the action of the compact group } T_K \text{ we can assume that it is } T_K \text{-invariant, Therefore the action is Hamiltonian: the Hamiltonian at } x \text{ for a one parametric subgroup } g^t \text{ is given by the formula } \alpha(g^t(x))\text{.}
\]

Figure 5: Duality between configurations spaces of decorated flags for \( G \) and \( G^L \). The potential is a sum of contributions at the boldface vertices. Pairs of decorated flags at the dashed sides are in generic position. No condition on the pairs of decorated flags at the solid sides.
by the pairs \((E, i)\) over the vertices \((32)\) match the irreducible components of the divisor at infinity \((33)\) on the dual space.

The cycle class \([D]_\text{obtained by adding to the symplectic form on } M\).

It was anticipated by Hori-Vafa [HV] and Auroux [Au1] that adding

\[\text{a potential on a space } M\]

\[\text{denote by } D\]

as the divisors on \(M\).

Z acts as the divisors on \(M\).

\[\text{components } W\]

\[\text{on Conf}\]

\[\text{match the sides of the dual polygon } P\].

By construction, the potential is a sum \(W = \sum_c W_c\), \(M_L - M_L^x = \cup_c D_c\), \(W_c \leftrightarrow D_c\).

There are several ways to explain how this correspondence should work.

i) The potential \(W_c\) determines an element \([W_c] \in HH^0(M)\), which defines a deformation of the category \(D^b\text{Coh}(M)\) as a \(\mathbb{Z}/2\mathbb{Z}\)-category. On the dual side it corresponds to a deformation of the Fukaya category obtained by adding to the symplectic form on \(M\) a multiple of the 2-form \(\omega_c\), whose cohomology class is the cycle class \([D_c] \in H^2(M_L, \mathbb{Z}(1))\) of the divisor \(D_c\).

ii) The Landau-Ginzburg potential \(W_c\) should be obtained by counting the holomorphic discs touching the divisor \(D_c\), as was demonstrated by Auroux [Au1] in examples.

iii) In the cluster variety set up the correspondence is much more precise, see Section 11.

**Example.** To illustrate the set-up, let us specify the data on the moduli space Conf\(_n\)(A).

- A regular positive function, the potential \(W : \text{Conf}\_n^x(A) \to \mathbb{A}^1\).
- A regular volume form \(\Omega\) on \(\text{Conf}\_n^x(A)\), with logarithmic singularities at infinity.
- A regular projection \(\pi : \text{Conf}\_n^x(A) \to \mathbb{H}\) onto a torus \(\mathbb{H} := \mathbb{H}^{(\text{sides of the } n\text{-gon } P_n)}\).
- An action \(r\) of the torus \(T := \mathbb{H}^{(\text{vertices of } P_n)}\) on \(\text{Conf}\_n(A)\) by rescaling decorated flags.

Changing \(G\) to \(G^L\) we interchanges the action with the projection:

- The torus \(T_L\) is dual to the torus \(\mathbb{H}\), i.e. there is a canonical isomorphism \(X_*(T_L) = X^*(\mathbb{H})\).

By construction, the potential is a sum

\[W = \sum_v \sum_{i \in I} W_v^i\]

over the vertices \(v\) of the polygon \(P_n\), parametrising configurations \((A_1, \ldots, A_n)\), and the set \(I\) of simple positive roots for \(G\). Indeed, a non-degenerate character \(\chi\) of \(U\) is naturally a sum \(\chi = \sum_i \chi_i\).

On the other hand, the set of irreducible components of the divisor \(\text{Conf}^x_n(A_L) - \text{Conf}^x_n(A_L)\) is parametrised by the pairs \((E, i)\) where \(E\) are the edges of the dual polygon \(*P_n\), see Section 1.2.1

\[\text{Conf}_n(A_L) - \text{Conf}_n^x(A_L) = \cup_{E} \cup_{i \in I} D_i^E\].

Since vertices of the polygon \(P_n\) match the sides of the dual polygon \(*P_n\), the components of the potential \([32]\) match the irreducible components of the divisor at infinity \([33]\) on the dual space.

We start with the most basic form of our mirror conjectures, which does not involve the potential.
Conjecture 1.5. For any split semisimple group $G$ over $\mathbb{Q}$, there is a mirror duality
\[ (\text{Conf}_n^w(A), \Omega) \text{ is mirror dual to } (\text{Conf}_n^w(A_L), \Omega_L). \]  

This means in particular that one has an equivalence of $A_\infty$-categories
\[ \mathcal{F}_{wr}(\text{Conf}_n^w(A)(\mathbb{C})) \simto D^b\text{Coh}(\text{Conf}_n^w(A_L)). \]  

This equivalence maps the positive $A$-brane $\mathcal{L}_+$ to the structure sheaf $\mathcal{O}$.

It identifies the action of the group $\mathbb{H}(\mathbb{C})$ on the category $\mathcal{F}_{wr}(\text{Conf}_n^w(A)(\mathbb{C}))$ with the action of the group $T_L(\mathbb{C})$ on $D^b\text{Coh}(\text{Conf}_n^w(A_L))$, and identifies the subalgebras
\[ \mathcal{O}(\mathbb{H}) \subset \text{Center}(\mathcal{F}_{wr}(\text{Conf}_n^w(A)(\mathbb{C}))) \text{ and } \mathcal{O}(\mathbb{H}_L) \subset \text{Center}(D^b\text{Coh}(\text{Conf}_n^w(A_L))). \]

The projection / action data for the pair $[34]$ is given by
\[ \mathbb{H} = H^n, \quad \mathbb{H}_L = H^L, \quad T = H^n, \quad T_L = H^L. \]

The pair $[34]$ is symmetric: interchanging the group $G$ with the Langlands dual group $G^L$ amounts to exchanging the $A$-model with the $B$-model.

Using the mirror pair $[34]$ as a starting point, we can now turn on the potentials at all vertices of the left polygon $P_n$. This amounts to a partial compactification of the dual space. Namely, we take the space $\text{Conf}_n(A_L)$, and consider its affine closure $\text{Conf}_n(A_L)_a := \text{Spec} \left( \mathcal{O}(A_n^w)_{\mathbb{Q}_L} \right)$.

Since the action of the group $H^n$ on $\text{Conf}_n^w(A)$ alters the potential $W$, and the projection $\pi_L$ onto $H^L$ does not extend to $\text{Conf}_n(A_L)_a$, the projection / action data for the pair $[35]$ is
\[ \mathbb{H} = H^n, \quad \mathbb{H}_L = \{e\}, \quad T = \{e\}, \quad T_L = H^L. \]

Therefore by turning on the potentials we arrive at the following Mirror Conjecture:

Conjecture 1.6. For any split semisimple group $G$ over $\mathbb{Q}$, there is a mirror duality
\[ (\text{Conf}_n^w(A), W, \Omega) \text{ is mirror dual to } \text{Conf}_n(A_L)_a. \]

This means in particular that there is an equivalence of $A_\infty$-categories
\[ \mathcal{F}_{\mathcal{S}wr}(\text{Conf}_n^w(A)(\mathbb{C}), W, \Omega) \simto D^b\text{Coh}(\text{Conf}_n(A_L)_a). \]  

It maps the positive $A$-brane $\mathcal{L}_+$ to the structure sheaf $\mathcal{O}$, and identifies the action of the group $\mathbb{H}(\mathbb{C})$ on the category $\mathcal{F}_{\mathcal{S}wr}(\text{Conf}_n^w(A)(\mathbb{C}))$ with the action of $T_L(\mathbb{C})$ on $D^b\text{Coh}(\text{Conf}_n(A_L)_a)$.

The geometry of mirror dual objects in Conjectures 1.5 and 1.6 is essentially dictated by representation theory. Indeed, the tropical points are determined by birational types of the spaces, and canonical bases tell the algebras of functions on the dual affine varieties:

The set $\text{Conf}_n^w(A)(\mathbb{Z}^L)$ parametrises a canonical basis in $\mathcal{O}(\text{Conf}_n(A_L))$.

The set $\text{Conf}_n(A_L)(\mathbb{Z}^L)$ should parametrise a canonical basis in $\mathcal{O}(\text{Conf}_n^w(A))$.

The potential $W$ and the projection $\pi$ define a regular map $(\pi, W) : \text{Conf}_n^w(A) \to \mathbb{H} \times \mathbb{A}^1$. The form $\Omega$ on $\text{Conf}_n^w(A)$ and the canonical volume forms on $\mathbb{H}$ and $\mathbb{A}^1$ provide a volume form $\Omega^{(a,c)}$ at the fiber $F_{a,c}$ of this map over a generic point $(a, c) \in \mathbb{H} \times \mathbb{A}^1$.

More generally, we can turn on only partial potentials at the vertices of the polygon $P_n$, which amounts on the dual side to taking partial compactifications, and then considering their affine closures. This way we get an array of conjecturally dual pairs, described as follows.

\[ 8 \text{ Although the claim } 39 \text{ is not addressed in the paper, it can be deduced from } 38. \]
For each vertex \( v \) of the polygon \( P_n \) parametrising configurations \( (A_1, ..., A_n) \) choose an arbitrary subset \( I_v \subset I \) of the set parametrising the simple positive roots of \( G \). It determines a partial potential

\[
\mathcal{W}_{\{I_v\}} = \sum_v \mathcal{W}_v, \quad \mathcal{W}_v := \sum_{i \in I_v} \mathcal{W}_i^v.
\]  

(40)

On the dual side, subsets \( \{I_v\} \) determine a partial compactification of the space \( \text{Conf}_n^*(\mathcal{A}_L) \), obtained by adding the divisors \( D_i^{E_v} \) where \( i \in I_v \). Here \( E_v \) is the side of the polygon \( *P_n \) dual to the vertex \( v \) of \( P_n \):

\[
\text{Conf}_n(\mathcal{A}_L)_{\{I_v\}} := \text{Conf}_n^*(\mathcal{A}_L) \bigcup \bigcup_{v \in I_v} D_i^{E_v}.
\]

(41)

For each vertex \( v \) of \( P_n \) there is a subgroup \( H_{L_v} \subset H \) preserving the partial potential \( \mathcal{W}_{1_v} \) at \( v \). On the dual side, let \( H^*_L \) be the dual quotient of the Cartan group \( H_L \). So we arrive at the projection / action data

\[
H = H^n, \quad H_L = \prod_v H^*_L, \quad T = \prod_v H_v, \quad T_L = H^*_L.
\]

(42)

So turning on partial potentials we arrive at Conjecture \ref{c1.7}, interpolating Conjectures \ref{c1.5} and \ref{c1.6}

\textbf{Conjecture 1.7.} For any split semisimple group \( G \) over \( \mathbb{Q} \), there is a mirror duality

\[
(\text{Conf}_n^*(\mathcal{A}), \mathcal{W}_{\{I_v\}}, \Omega) \quad \text{is mirror dual to the affine closure of} \quad \text{Conf}_n(\mathcal{A}_L)_{\{I_v\}}.
\]

(43)

Its action / projection data is given by \ref{43}.

Needless to say, the positive integral tropical points of the left space parametrise a basis in the space of functions on the right space.

Here is another general principle to generate new mirror dual pairs. We start with a mirror dual pair \((\mathcal{M}, \Omega, \mathcal{W}) \leftrightarrow \mathcal{M}_L\), equipped with the projection / action data which involves a dual pair \((T, H_L)\). So \( T \) acts by automorphisms of the triple \((\mathcal{M}, \Omega, \mathcal{W})\), and there is a dual projection \( \pi_L : \mathcal{M}_L \to H_L \).

Choose any subgroup \( T' \subset T \), and consider the corresponding \( T' \)-equivariant category. If the group \( T' \) acts freely, this amounts to taking the quotient of the space with potential \((\mathcal{M}, \mathcal{W})\) by the action of \( T' \). A volume form on \( T' \) gives rise to a volume form on the quotient, obtained by contracting the volume form \( \Omega \) with the dual polyvector field on \( T' \). The subgroup \( T'' \subset T \) determines by the duality a quotient group \( H_L \to H''_L \), and therefore a projection \( \pi'_L : \mathcal{M}_L \to H''_L \).

- The quotient stack \((\mathcal{M}/T', \mathcal{W})\) is mirror dual to the family \( \pi'_L : \mathcal{M}_L \to H''_L \).

In the examples below \((\mathcal{M}/T', \mathcal{W})\) is just dual to a fiber \( \pi_L^{-1}(a) \subset \mathcal{M}_L, a \in H'_L \).

In particular, starting from a mirror dual pair \ref{43}, we can choose any subgroup \( T'' \subset T = \prod_v H_v \) acting on the space with potential on the left. All examples below are obtained this way.

\textbf{Example.} We start with the space \( \text{Conf}_n^*(A^{n+1}) \) with the potential \( \mathcal{W}_{1,...,n} \) given by the sum of the full potentials at all vertices but one, the vertex \( A_{n+1} \). The action of the group \( H \) on the decorated flag \( A_{n+1} \) preserves the potential \( \mathcal{W}_{1,...,n} \). Applying the above principle, we get a dual pair illustrated on Fig \ref{fig4}. The fiber over \( a \), illustrated by the middle picture on Fig \ref{fig4} is canonically isomorphic to the less symmetrically defined space illustrated on the right.

In the next Section we consider this example from a different point of view, starting from representation-theoretic picture, just as we did with our basic example, and arrive to the same dual pairs.

1.4.2 Tensor products of representations and mirror symmetry

The set \( \text{Conf}_n(A^{n+1}) \mathcal{B}(\mathbb{Z}^1) \) defined using the potential \( \mathcal{W} \) from \ref{11} parametrises canonical bases in \( n \)-fold tensor products of simple \( G^L \)-modules. So using \ref{27} we arrive at a canonical pairing

\[
I : \text{Conf}_n^*(A^{n+1}) \mathcal{B}(\mathbb{Z}^1) \times A_L^n \to \mathbb{A}^1.
\]

(44)

18
Figure 6: Dual pairs \((\text{Conf}^x(A^3, B), \mathcal{W}_{1,2,3})\) and \(\text{Conf}_{w_0}(A^3_L, B_L) = A^2_L\). The H-components of the projection \(\lambda\) sit at the A-decorated blue dashed edges on the left. The projection \(\mu\) is assigned to the red \(A_3B_4A_1\).

Let us present \(A^2_L\) as a configuration space. Recall that \(\text{Conf}_{w_0}(A^{n+1}_L, B_L)\) parametrises configurations \((A_1, \ldots, A_{n+1}, B_{n+2})\) such that the pair \((A_{n+1}, B_{n+2})\) is generic. Generic pairs \((A, B)\) form a \(G^L\)-torsor. Let \(\{A^+, B^-\}\) be a standard generic pair. Then there is an isomorphism

\[
A^n_L \xrightarrow{\pi} \text{Conf}_{w_0}(A^{n+1}_L, B_L), \quad \{A_1, \ldots, A_n\} \mapsto (A_1, \ldots, A_n, A^+, B^-).
\]

The subspace \(\text{Conf}^x(A^{n+1}_L, B_L)\) parametrises configurations \((A_1, \ldots, A_{n+1}, B_{n+2})\) such that the consecutive pairs of flags are generic. It is the quotient of \(\text{Conf}_{n+2}^r(A)\) by the action of the group \(H\) on the last decorated flag. The projection \(\text{Conf}_{n+2}^r(A) \rightarrow H^{n+2}\) induces a map, see [18],

\[
\pi = (\lambda, \mu) : \text{Conf}^x(A^{n+1}, B) \rightarrow H^n \times H.
\]

Then the symmetry is restored, and we can view (44) as a manifestation of a mirror duality:

\[
(\text{Conf}^x(A^{n+1}, B), \mathcal{W}, \Omega, \pi) \text{ is mirror dual to } (\text{Conf}_{w_0}(A^{n+1}_L, B_L), \Omega_L, \pi_L).
\]

Here \(\pi_L\) is the action of \(H^{n+1}_L\) by rescaling of the decorated flags. The projection/action data is

\[
\mathbb{H} = H^{n+1}, \quad \mathbb{H}_L = \{e\}, \quad T = \{e\}, \quad T_L = H^{n+1}_L,
\]

The analog of mirror dual pair (44) and its projection/action data are given by, see Fig 7,

\[
(\text{Conf}^x(A^{n+1}, B), \Omega) \text{ is mirror dual to } (\text{Conf}^x(A^{n+1}_L, B_L), \Omega_L).
\]

Here \(\mathbb{H} = H^{n+1}\) and \(\mathbb{H}_L = H^{n+1}_L\), \(T = H^{n+1}\), \(T_L = H^{n+1}_L\).

So we arrived at the two dual pairs (44) and (45) using canonical pairings as a guideline.

As discussed in the Example in Section 1.4.3, we can get them from the basic dual pairs \(\mathcal{A}\) and \(\mathcal{B}\) using the action / projection duality \(\ast\), which in this case tells that the quotient by the action of the group \(H\) on one side is dual to a fiber of the family of spaces over the dual group \(H_L\) over a point \(a \in H_L\).

In particular, the dual pair \(\mathcal{B}\) leads to the dual pair illustrated on Fig 4. Notice that configurations \((A_1, \ldots, A_{n+2})\) with \(\alpha(A_{n+1}, A_{n+2}) = a \in H\) are in bijection with configurations \((A_1, \ldots, A_{n+1}, B_{n+2})\) where the pair \((A_{n+1}, B_{n+2})\) is generic. So the two diagrams on the right of Fig 7 represent isomorphic configuration spaces, and we get the dual pair \(45\) from \(44\). The dual pair \(47\) is obtained from \(48\) by adding potentials at the A-vertices, thus allowing arbitrary pairs of flags on the dual sides.

We conjecture that the analogs of Conjectures 1.5 and 1.6 hold for the pairs \(48\) and \(47\).

1.4.3 Landau-Ginzburg mirror of a maximal unipotent group \(U\) and it generalisations

We view Lusztig’s dual canonical basis in \(\mathcal{O}(U^L)\) as a canonical pairing, and hence as a mirror duality:

\[
I : U^+_L(Z^L) \times U^L \rightarrow A^1, \quad (U^*, \chi) \text{ is mirror dual to } U^L.
\]
To define $U^*$, we realise a maximal unipotent subgroup $U$ as a big Bruhat cell in the flag variety, and intersect it with the opposite big Bruhat cell. The $\chi$ is a non-degenerate additive character of $U$, restricted to $U^*$. This example is explained and generalised using configurations as follows.

Let $\text{Conf}_{\lambda_0}(B, A^n, B)$ be the space parametrising configurations $(B_1, A_2, \ldots, A_{n+1}, B_{n+2})$ such that the pairs $(B_1, B_{n+2})$ and $(A_{n+1}, B_{n+2})$ are generic, see the right picture on Fig 8. There is an isomorphism

$$U_L \times A_{L}^{n-1} = \text{Conf}_{\lambda_0}(B_L, A_{L}^n, B_L), \quad \{B_1, A_2, \ldots, A_n\} \mapsto (B_1, A_2, \ldots, A_n, A^+, B^-).$$

The group $H^0_L$ acts on $\text{Conf}_{\lambda_0}(B_L, A_{L}^n, B_L)$ by rescaling decorated flags.

The subspace $\text{Conf}_L^{\infty}(B, A^n, B)$ parametrises configurations where each consecutive pair of flags is generic. It is depicted on the left of Fig 8. It is the quotient of $\text{Conf}_{\lambda_0}(A)$ by the action of $H \times H$ on the first and last decorated flags. Thus there is a map $\pi$, defined similarly to (10):

$$\pi = (\lambda, \mu) : \text{Conf}_L^{\infty}(B, A^n, B) \to H^{n-1} \times H. \quad (51)$$

So the projection / action data in this case is

$$H = H^0, \quad H_L = \{e\}, \quad T = \{e\}, \quad T_L = H^0_L,$$

For example, $\text{Conf}_L^{\infty}(B, A, B) = U^*$, in agreement with $U^*$ in (19).

**Conjecture 1.8.** The set $\text{Conf}_L^{\infty}(B, A^n, B)(Z^\nu)$ parametrises a canonical basis in $O(U_L \times A_{L}^{n-1})$. The subset $(\lambda^t, \mu^t)^{-1}(\lambda_1, \ldots, \lambda_{n-1}; \nu)$ parametrises a canonical basis in the weight $\nu$ subspace of $U(N^L) \otimes V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_{n-1}}$.

The analogs of Conjectures 7.5 and 7.6 hold for the following mirror dual pairs:

$$\text{Conf}_L^{\infty}(B, A^n, B, \Omega) \text{ is mirror dual to } \text{Conf}_L^{\infty}(B_L, A_{L}^n, B_L, \Omega_L),$$

$$\text{Conf}_L^{\infty}(B, A^n, B, W, \Omega, \pi) \text{ is mirror dual to } \text{Conf}_{\lambda_0}(B_L, A_{L}^n, B_L, r_L)$$

These mirror pairs can be obtained from the basic mirror pairs (64) and (66) by trading, using the action / projection principle $\bullet$, the quotient by $H^2_L$ to the fiber over $(a, b) \in H^2$ on the dual side, see Fig 8.

### 1.4.4 Landau-Ginzburg mirror of a simple split group $G$

In this Section we interpret a split simple group $G$ as a configuration space, and using this deduce its Landau-Ginzburg mirror from Conjecture 1.6 by using our standard toolbox. The companion conjecture tells that the mirror of the maximal double Bruhat cell for $G$ is the maximal double Bruhat cell for $G^L$.

Denote by $\text{Conf}_L^{\infty}(B, A, B, A)$ the space parametrising configurations $(B_1, A_2, B_3, A_4)$ where all four consecutive pairs are generic. There is a potential given by the sum of the potentials at the $A$-vertices:

$$W_{2,4}(B_1, A_2, B_3, A_4) := \chi_{A_2}(B_1, A_2, B_3) + \chi_{A_4}(B_3, A_4, B_1).$$
Let us describe its mirror. Recall the isomorphism $\alpha : \operatorname{Conf}^\times (A, A) \to H$. Consider the moduli space of configurations

$$(A_1, A_2, A_3, A_4) \in \operatorname{Conf}_4(A_L) \mid (A_1, A_2), (A_3, A_4) \text{ are generic; } \alpha(A_1, A_2) = \alpha(A_3, A_4) = e. \quad (52)$$

The picture on the right of Fig 9 illustrates this moduli space.

**Lemma 1.9.** The moduli space $\operatorname{Conf}_4(A_L)$ is isomorphic to the group $G^L$.

**Proof.** Pick a generic pair $\{A_1, A_2\}$ with $\alpha(A_1, A_2) = e$. Then for each $G^L$-orbit in $\operatorname{Conf}_4(A_L)$ there is a unique representative $\{A_1, A_2, A_3, A_4\}$ where $\{A_1, A_2\}$ is the chosen pair. There is a unique $g \in G^L$ such that $g\{A_1, A_2\} = \{A_3, A_4\}$. The map $(A_1, A_2, A_3, A_4) \to g$ provides the isomorphism. \[\square\]

**Conjecture 1.10.** The mirror to a split semisimple algebraic group $G^L$ over $\mathbb{Q}$ is the pair

$$(\operatorname{Conf}^\times (B, A, B, A), W_{2,4}). \quad (53)$$

**Example.** Let $G^L = PGL_2$, so $G = SL_2$. Then $A = k^2 - \{0\}$, $B = \mathbb{P}^1$, and

$$\operatorname{Conf}^\times (B, A, B, A) = \{(L_1, v_2, L_3, v_4)\}/SL_2. \quad (54)$$

Here $L_1, L_3$ are one dimensional subspaces in a two dimensional vector space $V_2$, and $v_2, v_4$ are non-zero vectors in $V_2$. The pairs $(L_1, v_2), (v_2, L_3), (L_3, v_4), (v_4, L_1)$ are generic, i.e. the corresponding pairs of lines are distinct. Pick non-zero vectors $l_1 \in L_1$ and $l_3 \in L_3$. Then

$$W_{2,4} = \frac{\Delta(l_1, l_3)}{\Delta(l_1, v_2)\Delta(v_2, l_3)} + \frac{\Delta(l_1, l_3)}{\Delta(l_3, v_4)\Delta(l_1, v_4)}.
$$

It is a regular function on $(54)$, independent of the choice of vectors $l_1, l_3$. To calculate it, set

$$l_1 = (1, 0), \quad v_2 = (x, 1/p), \quad l_3 = (1, y/p), \quad v_4 = (0, 1). \quad (55)$$

Then

$$\operatorname{Conf}^\times (B_L, A_L, B_L, A_L) = \{(x, y, p) \in \mathbb{A}^1 \times \mathbb{A}^1 \times \mathbb{G}_m - (xy - 1 = 0)\}. \quad (56)$$
\[ W_{2,4} = \frac{y/p}{1/p \cdot (xy/p - 1/p)} + \frac{y/p}{1 \cdot 1} = \frac{yp}{xy - 1} + \frac{y}{p}. \]  

(56)

The case \( G = PGL_2, G^L = SL_2 \) is similar, except that now \( \mathcal{A}_{PGL_2} = A^2 - \{0\} / \pm 1. \)

Let us explain how this conjecture can be deduced from our general conjecture.

**Step 1.** Conjecture 1.6 tells us mirror duality, illustrated on Fig 10:

\[ \text{Conf}_4^g(A, W_{1,2,3,4}) \leftrightarrow \text{Conf}_4(A_L). \]

**Step 2.** We alter the pair \( \text{Conf}_4^g(A, W_{1,2,3,4}) \) by removing the potentials at the vertices \( A_1 \) and \( A_3 \). This reduces the potential \( W_{1,2,3,4} \) to a new potential:

\[ W_{2,4}(A_1, A_2, A_3, A_4) := \chi_{A_2}(B_1, A_2, B_3) + \chi_{A_4}(B_3, A_4, B_1). \]

In the dual picture this amounts to removing two divisors from \( \text{Conf}_4(A_L) \), illustrated by two punctured edges on the right of Fig 11 dual to the vertices \( A_1 \) and \( A_3 \) on the left. Precisely, we introduce a subspace \( \widetilde{\text{Conf}}_4(A_L) \) such that the pairs of decorated flags at punctured sides are generic. The obtained dual pair is illustrated on Fig 11 In particular there is a projection provided by the two punctured sides:

\[ \widetilde{\text{Conf}}_4(A_L) \to H^2_L. \]  

(57)

**Step 3.** There is an action of the group \( H \times H \) on \( \text{Conf}_4^g(A) \) preserves the potential \( W_{2,4} \), given by \( (A_1, A_2, A_3, A_4) \to (h_1 \cdot A_1, A_2, h_1 \cdot A_3, A_4). \) The quotient is the space \( (\text{Conf}_4^g(A, W_{2,4}))/\langle H \times H \rangle = (\text{Conf}(B, A, B, A), W_{2,4}). \)
Step 4. The action of the group $H \times H$ is dual to the projection (12). The quotient by the $H \times H$-action is dual to the fiber over $e \in H_L \times H_L$. The fiber is just the space (12). On the level of pictures, this is how we go from Fig 11 to Fig 9. This way we arrived at Conjecture 1.14.

Canonical basis motivation. Let us explain how the positive integral tropical points of the space from Conjecture 1.14 parametrise a canonical basis in $O(G^L)$. One has $O(G^L) = \bigoplus_{\lambda \in P^+} V_\lambda \otimes V_\lambda^*$. 

Figure 12: The (tropicalised) Landau-Ginzburg model dual to $G^L$ is obtained by gluing the two LG models dual to $A^L$ along their “vertical sides”, as shown on the left.

Recall that $O(A_L) = \bigoplus_{\lambda \in P^+} V_\lambda$. The decomposition of $O(A_L)$ into irreducible $G^L$-modules is provided by the $H_L$-action on $A_L$. According to our general picture, $A_L = \text{Conf}_{w_0}(B_L, A_L, A_L)$ is mirror dual to $(\text{Conf}(B, A, A), W_{2,3})$. Therefore the tensor product of canonical basis in $V_\lambda \otimes V_\lambda^*$ is parametrised by the fiber over $\lambda$ of the tropicalisation of the positive rational map $\text{Conf}(B, A, B, A) \longrightarrow \text{Conf}(A, A)$. 

Lemma 1.11. The space $\text{Conf}^\times(B, A, B, A)$ is isomorphic to the open double Bruhat cell of $G$.

Proof. Note that $\text{Conf}^\times(B, A, B, A)$ is isomorphic to the moduli space parametrizing the configurations $(A_1, A_2, A_3, A_4) \in \text{Conf}_4(A)$ such that $\alpha(A_1, A_2) = \alpha(A_4, A_3) = e$ and each consecutive pair $(A_i, A_{i+1})$ is generic. There is a unique element $g \in G$ such that $\{g \cdot A_1, g \cdot A_2\} = \{A_4, A_3\}$. Let $\pi(A_1) = B$ and $\pi(A_2) = B^-$. Then we have

\[
\{A_1, A_4\} = \{A_1, g \cdot A_1\} \text{ is generic } \iff \ g \in B_{w_0}B,
\]
\[
\{A_2, A_3\} = \{A_2, g \cdot A_2\} \text{ is generic } \iff \ g \in B^- w_0B^-.
\]

So the space is isomorphic to the open double Bruhat cell $B_{w_0}B \cap B^- w_0 B^-$. 

Conjecture 1.12. The open double Bruhat cell of $G$ is mirror to the open double Bruhat cell of $G^L$.

Below we investigate the case when $G = \text{SL}_2$. 

\[\text{B_2} \quad \text{A_2}\]
\[\text{A_4} \quad \text{B_3}\]
\[\text{A_4} \quad \text{A_3}\]
\[\text{A_3}\]
\[\text{B_1}\]
An example: the open double Bruhat cell of $\text{SL}_2$. It consists of elements
\[
\begin{bmatrix} x & p \\ q & y \end{bmatrix} \in \text{SL}_2, \quad xy - 1 = pq, \quad p, q \in \mathbb{G}_m.
\]

Let
\[
D = \{xy - 1 = 0\} \subset \mathbb{A}^2, \quad X = \mathbb{A}^2 \setminus D.
\]
The open Bruhat cell of $\text{SL}_2$ is isomorphic to $X \times \mathbb{G}_m$. It is a cluster $\mathcal{A}$-variety with two seeds $p \leftarrow x \rightarrow q$ and $p \leftarrow y \rightarrow q$. The $\{p, q\}$ are the frozen variables. The seeds are related by the cluster transformation
\[
x = \frac{1 + pq}{y}.
\]

Note that the coordinates here are compatible with (55). In particular, the potential function (56) becomes
\[
\mathcal{W} = \frac{y}{q} + \frac{1}{p}.
\]

An example: the open double Bruhat cell of $\text{PGL}_2$. It consists of elements
\[
\begin{bmatrix} px & 1 \\ p & y \end{bmatrix} \in \text{GL}_2, \quad xy - 1 \neq 0, \quad p \in \mathbb{G}_m.
\]

It is again isomorphic to $X \times \mathbb{G}_m$. So we expect that $X \times \mathbb{G}_m$ is mirror to itself.

This open double Bruhat cell admits a cluster $\mathcal{X}$-structure. We set
\[
x_1 = px, \quad x_2 = xy - 1, \quad x_3 = x.
\]

It gives rise to a cluster $\mathcal{X}$-structure $x_1 \leftarrow x_2 \rightarrow x_3$. Mutation at $x_2$ delivers $x'_1 \leftarrow x'_2 \rightarrow x'_3$. Here
\[
x'_1 = x_1(1 + x_2)^{-1} = p/y,
\]
\[
x'_2 = x_2^{-1} = \frac{1}{xy - 1},
\]
\[
x'_3 = x_3(1 + x_2)^{-1} = 1/y.
\]

The example $X = \mathbb{A}^2 \setminus D$ is considered by Auroux in Section 5 of [Au1]. See also Section 2 of [P].

Finally, let us remind that in general, when $G$ is simply connected, then the open double Bruhat cell is a cluster $\mathcal{A}$-variety. The open double Bruhat cell of $G^L$ is a cluster $\mathcal{X}$-variety.

1.4.5 Landau-Ginzburg mirror of $G^n$

Since $G^n$ is a semi-simple group, the previous discussion applies. However our general approach leads to a slightly different mirror dual, which has an additional symmetry: the group $\mathbb{Z}/(n + 1)\mathbb{Z}$ acts naturally on each of the spaces. It starts from the dual pair
\[
\text{Conf}^X_{2n+2}(A) \text{ mirror dual to } \text{Conf}^\mathcal{X}_{2n+2}(A_L).
\]

Let $\text{Conf}^X_{2n+2}(A, \mathcal{B}, \ldots, A, \mathcal{B})$ be the space parametrising configurations $(A_1, B_2, A_3, B_4, \ldots, A_{2n+1}, B_{2n+2})$ such that any consecutive pair is generic. There is a potential
\[
\mathcal{W}_{1,3,\ldots,2n+1}(A_1, B_2, \ldots, A_{2n+1}, B_{2n+2}) := \sum_{i=1,3,\ldots,2n+1} \chi_{A_i}(B_{i-1}, A_i, B_{i+1}).
\]

The dual space parametrises configurations
\[
(A_1, \ldots, A_{2n+2}) \in \text{Conf}^\mathcal{X}_{2n+2}(A_L) \text{ such that } (A_{2k+1}, A_{2k+2}) \text{ are generic, and } \alpha(A_{2k+1}, A_{2k+2}) = 1. \quad (58)
\]

The group $\mathbb{Z}/(n + 1)\mathbb{Z}$ acts by automorphisms of this pair of spaces. The dual pair is illustrated on Fig [13].
Lemma 1.13. The space (58) is isomorphic to \((G^L)^n\).

Proof. For any given collection \(\{A_1, \ldots, A_{2n+2}\}\) representing a point in the moduli space (58) there is unique \(g_k \in G^L\) such that \(\{A_{2k+1}, A_{2k+2}\} = g_k\{A_1, A_2\}\). So picking a representative with the first pair \(\{A_1, A_2\}\) provided by a pinning in \(G^L\), we get an isomorphism with \((G^L)^n\).

Conjecture 1.14. The mirror to \((G^L)^n\) is the pair

\[
(\text{Conf}_{2n+2}(A, B, \ldots, A, B), W_{1, 2, \ldots, 2n+1}).
\] (59)

As in the \(n = 1\) case, Conjecture 1.14 can be deduced from Conjecture 1.10 telling that

\[
(\text{Conf}_{2n+2}^x(A), W_{1, 2, \ldots, 2n+2}) \text{ mirror dual to } \text{Conf}_{2n+2}(A_L).
\] (60)

Indeed, starting with duality (60), we turn off the potentials at the even vertices. Then the group \(H^{n+1}\) acts by automorphisms of the space with the new potential. The quotient is the pair (59).

On the other hand, turning off the potentials at the even vertices amounts on the dual side to removing the divisors from \(\text{Conf}_{2n+2}(A_L)\) assigned to the sides of the \((2n+2)\)-gon dual to those vertices. The obtained space is fibered over \(H_L^{n+1}\). The fiber over \(e\) is the space (58).

Figure 13: Getting the mirror dual for \(G^n\) from configurations.

1.5 Representation theory and examples of homological mirror symmetry for stacks

As soon as our space \(\mathcal{M}\) is fibered over a split torus \(H\), the mirror dual space \(\mathcal{M}_L\) acquires an action of the dual torus \(T_L\). Thus we want to find the mirror of the stack \(\mathcal{M}_L/T_L\). Let us discuss two examples corresponding to the examples in Section 1.4.2 and 1.4.3.

Let us look first at the dual pair (47). The subgroup \(1 \times H^n_L\) acts freely on the last \(n\) decorated flags in \(\text{Conf}_{w_0}(B_L, A_L^{n+1})\), and the quotient is \(B_L^n\). So one has

\[
\text{Conf}_{w_0}(B_L, A_L^{n+1})/(H_L \times H^n_L) = H_L \backslash B_L^n.
\] (61)

We start with the problem reflecting the A-model to this stack.

1. Equivariant quantum cohomology of products of flag varieties. There is a way to understand mirror symmetry as an isomorphism of two modules over the algebra of \(h\)-differential operators \(\mathcal{D}_h\): one provided by the quantum cohomology connection, and the other by the integral for the mirror dual Landau-Ginzburg model:

The quantum cohomology \(\mathcal{D}_h\)-module of a projective (Fano) variety \(\mathcal{M} = \)
The $D_h$-module for the Landau-Ginzburg mirror $(\pi: \mathcal{M}^\vee \to \mathbb{H}, \mathcal{W}, \Omega)$, defined by $\int e^{-W/h}\Omega$.

Here the space $\mathcal{M}^\vee$ is fibered over a torus $\mathbb{H}$, the $\Omega$ is a volume form on $\mathcal{M}^\vee$, and $\mathcal{W}$ is a function on $\mathcal{M}^\vee$, called the Landau-Ginzburg potential. The form $\Omega$ and the canonical volume form on the torus $\mathbb{H}$ define a volume form $\Omega^{(a)}$ on the fiber of the map $\pi$ over an $a \in \mathbb{H}$. The integrals $\int e^{-W/h}\Omega^{(a)}$ over cycles in the fibers are solutions of the $D_h$-module $\pi_*(e^{-W/h}\Omega)$ on $\mathbb{H}$.

This approach to mirror symmetry was originated by Givental [Gi], see also Witten [W] and [EHX], and developed further in [HV] and many other works. See [AU1], [AU2] for a discussion of examples of mirrors for the complements to anticanonical divisors on Fano varieties.

In our situation $\mathcal{M}$ is a positive space and $\mathcal{W}$ is a positive function, so there is an integral

$$F_\mathcal{M}(a; h) := \int_{\gamma^+(a)} e^{-W/h}\Omega^{(a)}$$

If it converges, it defines a function on $\mathbb{H}(\mathbb{R}_>0)$. This function as well as its partial Mellin transforms is a very important object to study. It plays a key role in the story. Below we elaborate some examples related to representation theory.

Let $\psi_s$ be the character of $H_{\mathbb{R}_>0}$ corresponding to an element $s \in H_L(\mathbb{R}_>0)$. Recall the projection $\mu: \text{Conf}^\times(A^{n+1}, B) \to H$ from [FG1]. Consider the integral

$$F_{\text{Conf}^\times(A^{n+1}, B)}(a, s; h) := \int_{\gamma^+(a)} \mu^*(\psi_s) e^{-W/h}\Omega^{(a)}, \quad (a, s) \in (H^n \times H_L)(\mathbb{R}_>0).$$

It is the Mellin transform of the function [62] along the torus $1 \times H \subset H^{n+1}$. If $n = 1$, one can identify integral [63] with an integral presentation for the Whittaker-Bessel function of the principal series representation of $G(\mathbb{R})$ corresponding to the character $\psi_s$. The latter solves the quantum Toda lattice integrable system [Ko].

Therefore it provides, generalising Givental’s work [Gi2] for $G = GL_m$ in non-equivariant setting, the integral presentation of the special solution of equivariant quantum cohomology $D_h$-module for the flag variety $B_L$ studied in [GK], [Gi3], [GKLO], [GLO1], [GLO3], [L], [R1], [R2].

Recall the special cluster coordinate system on $\text{Conf}_3(A)$ for $G = GL_m$ from [FG1]. It has a slight modification providing a rational coordinate system on $\text{Conf}_{w_0}(B, A, A)$, see Section 3.

**Theorem 1.15.** i) Let $G = GL_m$. Then the potential $\mathcal{W}$, expressed in the special cluster coordinate system on $\text{Conf}_{w_0}(B, A, A)$, is precisely Givental’s potential from [Gi2].

The value of the integral $F_{\text{Conf}^\times(B, A, A)}(a; s, h)$ at $s = e$ coincides with Givental’s integral for a solution of the quantum cohomology $D_h$-module $QH^*(B_L)$ [Gi2].

ii) For any group $G$, the integral $F_{\text{Conf}^\times(B, A, A)}(a; s)$ is a solution of the $D_h$-module $QH^*_H(B_L)$.

**Proof.** i) It is proved in Section 6.2.

ii) Since integral [63] provides an integral presentation for the Whittaker function, it is equivalent to the results of [GLO1], [R1]. Observe that the parameter $a \in H(\mathbb{C})$ is interpreted as the parameter on $H^2(B_L, \mathbb{C}^*)$, which is the base of the small quantum cohomology connection, while the parameter $s \in H_L(\mathbb{R}_>0)$ is the parameter of the $H_L$-equivariant cohomology.

For arbitrary $n$, integral [63] determines the equivariant quantum cohomology $D_h$-module of $B^n_L$. The latter lives on $H^n \times H_L$, it is a $D_h$-module on $H^n$, but only $O$-module along $H_L$. Integral [63] is a solution of this $D_h$-module.

### 2. Mirror of equivariant $B$-model on $B^n_L$. The integral [63] admits analytic continuation in $s$ provided by the analytic continuation of the character $\psi_s$ in the integrand. The complex integrand lives on an analytic space defined as follows. Let $\tilde{H}(\mathbb{C})$ is the universal cover of $H(\mathbb{C})$. Denote by $(B \times \ldots \times B)^n_{a}$ the fiber of the map $\lambda$ in [40] over an $a \in H^n$. It is a Zariski open subset of $B^n$. Consider the fibered product

$$\begin{array}{ccc}
(B \times \ldots \times B)^n_{a}(\mathbb{C}) & \xrightarrow{\exp} & (B \times \ldots \times B)^n_{a}(\mathbb{C}) \\
\mu \downarrow & & \mu \downarrow \\
\tilde{H}(\mathbb{C}) & \xrightarrow{\exp} & H(\mathbb{C})
\end{array}$$

26
Let $\tilde{W}$ and $\tilde{\Omega}$ be the lifts of $W$ and $\Omega$ by the map $\text{exp}$. We get a locally constant family of categories $\mathcal{FS}_{\text{wr}}((B \times \ldots \times B)_n, \tilde{W}, \tilde{\Omega})$ over $H^n(C)$. So the fundamental group $\pi_1(H^n(C))$ acts on the category for any given $a$. The group $\pi_1(H(C))$ also acts on it by the deck transformations induced from the universal cover $\tilde{H}(C) \rightarrow H(C)$.

On the other hand, the Picard group of the stack $H_L \backslash B^n_L$,

$$\text{Pic}(H_L \backslash B^n_L) = X^*(H_L) \times \text{Pic}(B^n_L) = X^*(H_L) \times X^*(H^n_L)$$

acts by autoequivalences of the category $D^b\text{Coh}_{H_L}(B^n_L)$.

**Conjecture 1.16.** There is an equivalence of $\mathcal{A}_\infty$-categories

$$\mathcal{FS}_{\text{wr}}((B \times \ldots \times B)_n, \tilde{W}, \tilde{\Omega}) \sim D^b\text{Coh}_{H_L}(B^n_L). \quad (64)$$

It intertwines the deck transformation action of $\pi_1(H(C)) \times$ the monodromy action of $\pi_1(H^n(C))$ on the Fukaya-Seidel category with the action of $X^*(H_L) \times \text{Pic}(B^n_L)$ on the category $D^b\text{Coh}_{H_L}(B^n_L)$.

The integral \((63)\) over Lagrangian submanifolds supporting objects of the Fukaya-Seidel category is a central charge for a stability condition on the category.

Kontsevich argued \([K13]\) that there is a smaller class of stability conditions, which he called “physical stability conditions”. Stability conditions above should be from that class.

**Examples.** 1. Let $n = 1$. Then $B_1^t$ is the intersection $B^t$ of two big Bruhat cells in the flag variety $B$. It parametrising flags in generic position to two generic flags, say $(B^+, B^-)$.

2. Let $G = SL_2$, $n = 1$. Then $B_1^t = C^*$ with the coordinate $u$, $\tilde{B}_1^t = C$ with the coordinate $t$, $u = e^t$, and $\tilde{W} = a^{-1}(e^t + e^{-t})$ where $a \in C^*$ is a parameter. Next, $B_L = \mathbb{C}P^1$, with the natural $C^*$-action preserving $0, \infty$. Conjecture 1.16 predicts an equivalence

$$\mathcal{FS}_{\text{wr}}(C; a^{-1}(e^t + e^{-t}), dt) \sim D^b\text{Coh}_{C^*}(\mathbb{C}P^1), \quad a \in C^*.$$ The equivalence is a trivial exercise for the experts. It can be checked by using the Kontsevich combinatorial model \([K09]\, [A09]\, [STZ]\, [DK]\) for the Fukaya-Seidel category as a category of locally constant sheaves on the Lagrangian skeleton for a surface with potential in the case of $(C, e^t + e^{-t})$, shown on Fig 14.

Varying the parameter $a \in C^*$ in the potential we get a locally constant family of the Fukaya-Seidel categories. Its monodromy is an autoequivalence corresponding to the action of a generator of the group $\text{Pic}(\mathbb{C}P^1)$. The translation $t \mapsto t + 2\pi i$ is another autoequivalence corresponding to the action of a generator of the character group $X^*(C^*) = \mathbb{Z}$ on $D^b\text{Coh}_{C^*}(\mathbb{C}P^1)$.

![Figure 14: Horizontal rays are the rays of fast decay of the potential. Together with the vertical line, they form the Lagrangian skeleton of the Kontsevich model of $\mathcal{FS}_{\text{wr}}(C; a^{-1}(e^t + e^{-t}), dt)$.](image)

Let us consider now the oscillatory integral

$$\int_{L} \exp\left(\frac{1}{\hbar}(-a^{-1}(e^t + e^{-t}) - st)\right) dt = \int_{\exp(L)} \frac{e^{-a^{-1}(u+u^{-1})/\hbar} u^{g/\hbar} du}{u}.$$  

Here $L$ is a path which goes to infinity along the line of fast decay of the integrand. This is an integral for the Bessel function. It defines a family of stability conditions on the Fukaya-Seidel category depending on $s \in C$ – it is the value of the central charge on the $K_0$-class of the object supported on $L$. The parameter $s$ reflects the equivariant parameter for the $C^*$-action.
3. Mirror of equivariant B-model on $B^n_L \times U_L$. There is an integral very similar to (63):

$$F_{\text{Conf}}(\gamma(s)) = \int \gamma(s)^+ \mu^*(\psi) e^{-W/\hbar} \Omega^{[a]}, \quad (a, s) \in (H^{n-1} \times H_L)(\mathbb{R}_{>0}).$$

(65)

Denote by $\lambda^{\frac{1}{[51]}}$ the map $\lambda$ onto $H^{n-1}_L$ from (51). The integrand has an analytic continuation in $s$ which lives on the fibered product

$$\tilde{\lambda}^{-1}([51])(a)(\mathbb{C}) \xrightarrow{\exp} \lambda^{-1}([51])(a)(\mathbb{C})$$

$\mu \downarrow$ $\tilde{H}(\mathbb{C}) \xrightarrow{\exp} \text{H}(\mathbb{C})$

There is a conjecture similar to Conjecture 1.16 describing the category $D^b\text{Coh}_{H_L}(B^n_L \times U_L)$. For example, when $n = 1$ it reads as follows.

**Conjecture 1.17.** There is an equivalence of $A_{\infty}$-categories

$$FSw(U^*(\mathbb{C}), W, \Omega) \sim D^b\text{Coh}_{H_L}(U_L).$$

(66)

It intertwines the deck transformation action of $\pi_1(H(\mathbb{C}))$ on the Fukaya-Seidel category with the action of $X^*(H_L)$ on the category $D^b\text{Coh}_{H_L}(U_L)$.

**Example.** If $G = \text{SL}_2$ and $n = 1$, then $\text{Conf}_{w_{[0]}}(\mathcal{B}, \mathcal{A}, \mathcal{B}) = \mathbb{C}$ with the $\mathbb{C}^*$-action. On the dual side, $\text{Conf}^*(\mathcal{B}, \mathcal{A}, \mathcal{B}) = \mathbb{C}^*$, $\pi = \mu$ is the identity map, $W = u$, $\Omega = du/u$. The universal cover of $\mathbb{C}^*$ is $\mathbb{C}$ with the coordinate $t$ such that $u = e^t$. The integral is

$$F(s) = \int_0^\infty e^{-u} u^s du/u = \Gamma(s).$$

The equivalence of categories predicted by Conjecture 1.17 is

$$FSw(\mathbb{C}, e^t, dt) \sim D^b\text{Coh}_{\mathbb{C}^*}(\mathbb{C}).$$

(67)

It can be checked by using the Kontsevich combinatorial model for the Fukaya-Seidel category.

1.6 Concluding remarks

1. Mirror dual of the moduli spaces of $G^L$-local systems on $S$. The true analog of the moduli space of $G^L$-local systems for a decorated surface $S$ is the moduli space $\text{Loc}_{G^L,S}$. We view the function $W$ on the space $A_{G,S}$ as the Landau-Ginzburg potential on $A_{G,S}$, and suggest

**Conjecture 1.18.**

$$(A_{G,S}^\times, W, \Omega, \pi)$$

is mirror dual to $$(\text{Loc}_{G^L,S}, \Omega_L, r_L).$$

(68)

It would be interesting to compare this mirror duality conjecture with the mirror duality conjectures of Kapustin-Witten [KW] and Gukov-Witten [GW], which do not involve a potential, and refer to families of moduli spaces, which are somewhat different then the moduli spaces we consider.

Notice also that if each boundary component of $S$ has at least one special point, then $\text{Loc}_{G^L,S} = A_{G^L,S}$, and so in this case we have a more symmetric picture:

$$(A_{G,S}^\times, W, \Omega, \pi)$$

is mirror dual to $$(A_{G,L}^\times, \Omega_L, r_L).$$

(69)

$$(A_{G,S}^\times, \Omega, \pi)$$

is mirror dual to $$(A_{G,L}^\times, \Omega, r_L).$$

(70)
2. Oscillatory integrals. The analog of integral $[62]$ in the surface case is an integral
\[
\mathcal{F}_{G,S}(a) := \int_{\gamma(a)/\Gamma_S} e^{-W/\hbar} Q(a).
\] (71)
Since the integrand is $\Gamma_S$-invariant, the integration cycles are defined by intersecting the fibers with $A_{G,S}(\mathbb{R}_{>0})/\Gamma_S$. Notice that $A_{G,S}(\mathbb{R}_{>0})$ is the decorated Higher Teichmuller space $[FG1]$. If $G = SL_2$, the integral converges.

For other groups convergence is a problem.

Notice also that the three convergent oscillatory integrals
\[
\mathcal{F}_{\text{Conf}^+_S(A,B)}(s), \quad \mathcal{F}_{\text{Conf}^+_S(A,A,B)}(a; s), \quad \mathcal{F}_{\text{Conf}^+_S(A)}(a_1, a_2, a_3), \quad a_i \in H(\mathbb{R}_{>0}), \ s \in H_L(\mathbb{R}_{>0})
\]
are continuous analogs of the Kostant partition function, weight multiplicities and dimensions of triple tensor product invariants for the Langlands dual group $G^L(\mathbb{R})$.

3. Relating our dualities to cluster Duality Conjectures $[FG2]$. The latter study dual pairs $(A, \mathcal{X}^\vee)$, where $A$ is a cluster $\mathcal{A}$-variety, and $\mathcal{X}^\vee$ is the Langlands dual cluster $\mathcal{X}$-variety:
\[
A \text{ is dual to } \mathcal{X}^\vee.
\]

There is a discrete group $\Gamma$ acting by automorphisms of each of the spaces $A$ and $\mathcal{X}^\vee$, called the cluster modular group. So it acts on the sets of tropical points $\mathcal{A}(\mathbb{Z}^t)$ and $\mathcal{X}^\vee(\mathbb{Z}^t)$. Cluster Duality Conjectures predict canonical $\Gamma$-equivariant pairings
\[
I_A : A(\mathbb{Z}^t) \times \mathcal{X}^\vee \to \mathbb{A}^1, \quad I_X : A \times \mathcal{X}^\vee(\mathbb{Z}^t) \to \mathbb{A}^1.
\] (72)

As the work $[GHK13]$ shows, in general the functions assigned to the tropical points may exist only as formal universally Laurent power series rather then universally Laurent polynomials.

We suggest that, in a rather general situation, there is a natural $\Gamma$-invariant positive potential $W_A$ on the space $A$, a similar potential $W_X$ on the space $X$, and a certain “alterations” $\mathcal{X}^\vee$ and $\mathcal{A}$ of the spaces $\mathcal{X}^\vee$ and $A^\vee$ providing mirror dualities underlying canonical pairings (72):
\[
(A, W_A, \Omega_A, \pi_A) \text{ is mirror dual to } (\mathcal{X}^\vee, \Omega_{X^\vee}, r_{X^\vee}).
\] (73)

Canonical pairings (72) should induce canonical pairings related to the potentials and alterations:
\[
I_{(A, W_A)} : A^+_W(\mathbb{Z}^t) \times \mathcal{X}^\vee \to \mathbb{A}^1, \quad I_{(X, W_X)} : \mathcal{X}^+_W(\mathbb{Z}^t) \times \mathcal{A} \to \mathbb{A}^1.
\]

This should provide a cluster generalisation of our examples. For instance, there is a split torus $\mathbb{H}_A$ associated to a cluster variety $A$, coming with a canonical basis of characters, given by the frozen $A$-coordinates. They describe the projection $\pi_A : A \to \mathbb{H}_A$, see Section 11.

4. Conclusion. A parametrisation of a canonical basis, casted as a canonical pairing $I$, should be understood as a manifestation of a mirror duality between a space with a Landau-Ginzburg potential and a similar space for the Langlands dual group.

Our main evidence is that canonical pairing $[11]$ describing a parametrisation of canonical basis in tensor products of $n$ irreducible $G^L$-modules is related via an integral presentation to the $D_r$-module describing the equivariant quantum cohomology of $(B_r)^n$.

There is a remarkable mirror conjecture of Gross-Hacking-Keel $[GHK11]$, who start with a maximally degenerate log Calabi-Yau $Y$ and conjecture that the Gromov-Witten theory of $Y$ gives rise to a commutative ring $R(Y)$, with a basis. Its spectrum is an affine variety which is conjectured to be the mirror of $Y$. 29
Notice that in our conjectures we give an \textit{a priori} description of the mirror dual pair of spaces, while in \cite{GHK11} the mirror space is encrypted in the conjecture. For example, mirror conjecture \cite{GHK11} is expected to be an example of the Gross-Hacking-Keel conjecture, but we do not know how to deduce, starting from the pair $(\text{Conf}_n^\times(A), \Omega)$, the former from the latter, and in particular why the Langlands dual group appears in the description of the mirror.

We want to stress that in our mirror conjectures we usually deal with mirror dual pairs where at least one is a Landau-Ginzburg model, i.e. is represented by a space with a potential. In particular canonical bases in representation theory and their generalisations related to moduli spaces of $G$-local systems on decorated surfaces $S$ always require the dual space to be a space with a non-trivial potential, unless $S$ is a closed surface without boundary.

Finally, in applications to representation theory we are forced to deal with stacks rather then varieties, as discussed in Section 1.5. This is a less explored chapter of the homological mirror symmetry. See also a recent paper of C. Teleman \cite{Te14} in this direction.

The space $\mathcal{M}(K)$ of $K$-points of a space $\mathcal{M}$ is a cousin of the loop space $\Omega \mathcal{M}(\mathbb{C})$. Heuristically, the quantum cohomology $D_\hbar$-module is best seen in the (ill defined) $S^1$-equivariant Floer cohomology of the loop space $\Omega \mathcal{M}(\mathbb{C})$ \cite{Ch}, which are sort of “semi-infinite cohomology” of the loop space. It would be interesting to relate this to the infinite dimensional cycles $C_{\infty} \subset \mathcal{M}^0(K)$.

It would be very interesting to relate our approach to the construction of canonical bases via cycles $\mathcal{M}_{\infty}^0$ to the work in progress of Gross-Hacking-Keel-Kontsevich on construction of canonical bases on cluster varieties via scattering diagrams.

\textbf{Organization of the paper.} In Section 2 we present main definitions and results relevant to representation theory. We start from a detailed discussion of the geometry of the tensor product invariants in Sections 2.1-2.2. We discuss more general examples in Sections 2.3. In Section 2.4 we construct a canonical basis in tensor products of finite dimensional $G^L$-modules, and its parametrization. In Sections 2 we give all definitions and complete descriptions of the results, but include a proof only if it is very simple. The only exception is a proof of Theorem 2.38 in Section 2.4. The rest of the proofs occupy the next Sections. In Section 10 we discuss the general case related to a decorated surface. In the Section 11 we discuss the volume form and the potential in the cluster set-up.

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\section{Main definitions and results: the disc case}

\subsection{Configurations of decorated flags, the potential $W$, and tensor product invariants}

\subsubsection{Positive spaces and their tropical points}

Below we recall briefly the main definitions, following \cite[Section 1]{FG2}.

\textbf{Positive spaces.} A positive rational function on a split algebraic torus $T$ is a nonzero rational function on $T$ which in a coordinate system, given by a set of characters of $T$, can be presented as a ratio of two polynomials with positive integral coefficients.

A \textit{positive rational morphism} $\varphi : T_1 \to T_2$ of two split tori is a morphism such that for each character $\chi$ of $T_2$ the function $\chi \circ \varphi$ is a positive rational function.
A positive atlas on an irreducible space (i.e. variety / stack) \( Y \) over \( \mathbb{Q} \) is given by a non-empty collection \( \{c\} \) of birational isomorphisms over \( \mathbb{Q} \)

\[
\alpha_c : T \rightarrow Y,
\]
where \( T \) is a split algebraic torus, satisfying the following conditions:

- For any pair \( c, c' \) the map \( \varphi_{c,c'} := \alpha_c^{-1} \circ \alpha_{c'} \) is a positive birational isomorphism of \( T \).
- Each map \( \alpha_c \) is regular on a complement to a divisor given by positive rational function.

A positive space is a space with a positive atlas. A split algebraic torus \( T \) is the simplest example of a positive space. It has a single positive coordinate system, given by the torus itself.

A positive rational function \( F \) on \( Y \) is a rational function given by a subtraction free rational function in one, and hence in all coordinate systems of the positive atlas on \( Y \).

A positive rational map \( Y \rightarrow Z \) is a rational map given by positive rational functions in one, and hence in all positive coordinate systems.

**Tropical points.** The tropical semifield \( \mathbb{Z}^t \) is the set \( \mathbb{Z} \) equipped with tropical addition and multiplication given by

\[
a + t b = \min\{a, b\}, \quad a \cdot b = a + b, \quad a, b \in \mathbb{Z}.
\]

This definition can be motivated as follows. Consider the semifield \( \mathbb{R}_+((t)) \) of Laurent series \( f(t) \) with positive leading coefficients: there is no “-” operation in \( \mathbb{R}_+((t)) \). Then the valuation map \( f(t) \mapsto \text{val}(f) \) is a homomorphism of semifields \( \text{val} : \mathbb{R}_+((t)) \rightarrow \mathbb{Z}^t \).

Denote by \( X_*(T) = \text{Hom}(\mathbb{G}_m, T) \) and \( X^*(T) = \text{Hom}(T, \mathbb{G}_m) \) the lattices of cocharacters and characters of a split algebraic torus \( T \). There is a pairing \( \langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \mathbb{Z} \).

The set of \( \mathbb{Z}^t \)-points of a split torus \( T \) is defined to be its lattice of cocharacters:

\[
T(\mathbb{Z}^t) := X_*(T).
\]

A positive rational function \( F \) on \( T \) gives rise to its tropicalization \( F^t \), which is a \( \mathbb{Z} \)-valued function on the set \( T(\mathbb{Z}^t) \). Its definition is clear from the following example:

\[
F = \frac{x_1 x_2^2 + 3 x_2 x_3^2}{x_2 x_4}, \quad F^t = \min\{x_1 + 2 x_2, x_2 + 5 x_3\} - \min\{x_2 + x_4\}.
\]

Similarly, a positive morphism \( \varphi : T \rightarrow S \) of two split tori gives rise to a piecewise linear morphism \( \varphi^t : T(\mathbb{Z}^t) \rightarrow S(\mathbb{Z}^t) \).

There is a unique way to assign to a positive space \( Y \) a set \( \mathcal{Y}(\mathbb{Z}^t) \) of its \( \mathbb{Z}^t \)-points such that

- Each of the coordinate systems \( c \) provides a canonical isomorphism
  \[
  \alpha^t_c : T(\mathbb{Z}^t) \rightarrow \mathcal{Y}(\mathbb{Z}^t),
  \]
- These isomorphisms are related by piecewise-linear isomorphisms \( \varphi^t_{c,c'} : \alpha^t_c(l) = \alpha^t_c \circ \varphi^t_{c,c'}(l) \).

We raise the above process to the category of positive spaces. It gives us a functor called tropicalization from the category of positive spaces to the category of sets of tropical points. For each positive morphism \( f : Y \rightarrow Z \), denote by \( f^t : \mathcal{Y}(\mathbb{Z}^t) \rightarrow Z(\mathbb{Z}^t) \) its corresponding tropicalized morphism.

Pick a basis of cocharacters of \( T \). Then, assigning to each positive coordinate system \( c \) a set of integers \((l^c_1, \ldots, l^c_n) \in \mathbb{Z}^n \) related by piecewise-linear isomorphisms \( \varphi^t_{c,c'} \), we get an element

\[
l = \alpha^t_c(l^c_1, \ldots, l^c_n) \in \mathcal{Y}(\mathbb{Z}^t).
\]

For a variety \( Y \) with a positive atlas, the set \( \mathcal{Y}(\mathbb{Z}^t) \) can be interpreted as the set of transcendental cells of the infinite dimensional variety \( \mathcal{Y}(\mathcal{C}((t))) \), as we will explain in Section 2.2.1.
The set of positive tropical points. Let $(\mathcal{Y}, \mathcal{W})$ be a pair given by a positive space $\mathcal{Y}$ equipped with a positive rational function $\mathcal{W}$. Let us tropicalize this function, getting a map

$$\mathcal{W}^t : \mathcal{Y}(\mathbb{Z}) \to \mathbb{Z}.$$ 

We define the set of positive tropical points:

$$\mathcal{Y}^+_W(\mathbb{Z}) := \{ l \in \mathcal{Y}(\mathbb{Z}) \mid \mathcal{W}^t(l) \geq 0 \}.$$

Example. The Cartan group $H$ of $G$ is a split torus and hence has a standard positive structure. The set $H(\mathbb{Z}) = X_*(H)$ is the coweight lattice of $G$. Let $\{\alpha_i\}$ the set of simple positive roots indexed by $I$. We define

$$\mathcal{W} : H \to \mathbb{A}^1, \quad h \mapsto \sum_{i \in I} \alpha_i(h).$$

The set of positive tropical points is the positive Weyl chamber in $X_*(H)$:

$$H^+(\mathbb{Z}) := H^+_W(\mathbb{Z}) = \{ \lambda \in X_*(H) \mid \langle \lambda, \alpha_i \rangle \geq 0, \forall i \in I \}.$$

2.1.2 Basic notations for a split reductive group $G$

Denote by $H$ the Cartan group of $G$, and by $H^\mathbb{Z}$ the Cartan group of the Langlands dual group $G^\mathbb{L}$. There is a canonical isomorphism $X^\vee(\mathbb{H}^\mathbb{Z}) = X_*(H)$. Denote by $\Delta^+ \subset X^\vee(H)$ the set of positive roots for $G$, and by $\Pi := \{\alpha_i\} \subset \Delta^+$ the subset of simple positive roots, indexed by a finite set $I$. We sometimes use $P$ instead of $X_*(H)$. Denote by $P^+$ the positive Weyl chamber in $P$. It is also the cone of dominant weights for the dual group $G^\mathbb{L}$. Denote by $V_\lambda$ the irreducible finite dimensional $G^\mathbb{L}$-modules parametrized by $\lambda \in P^+$.

Let $U_i^\pm (i \in I)$ be the simple root subgroup of $U^\mathbb{A}$. Let $\alpha_i^\vee : \mathbb{G}_m \to H$ be the simple coroot corresponding to the root $\alpha_i : H \to \mathbb{G}_m$. For all $i \in I$, there are isomorphisms $x_i : \mathbb{G}_a \to U_i^+$ and $y_i : \mathbb{G}_a \to U_i^-$ such that the maps

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_i(a), \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mapsto y_i(b), \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto \alpha_i^\vee(t)$$

(76)

provide homomorphisms $\phi_i : \text{SL}_2 \to G$.

Let $s_i (i \in I)$ be the simple reflections generating the Weyl group. Set $\tau_i := y_i(1)x_i(-1)y_i(1)$. The elements $\tau_i$ satisfy the braid relations. So we can associate to each $w \in W$ its representative $\overline{w}$ in such a way that for any reduced decomposition $w = s_{i_1} \ldots s_{i_k}$, one has $\overline{w} = \overline{\tau}_{i_k} \ldots \overline{\tau}_{i_1}$.

Denote by $w_0$ be the longest element of the Weyl group. Set $s_G := w_0^2$. It is an order two central element in $G$. For $G = \text{SL}_2$ it is the element $-\text{Id}$. For an arbitrary reductive $G$ the element $s_G$ is the image of the element $s_{\text{SL}_2}$ under a principal embedding $\text{SL}_2 \hookrightarrow G$. For example, $s_{\text{SL}_m} = (-1)^{m-1}\text{Id}$. See [FG1, Section 2.3] for proof.

2.1.3 Lusztig’s positive atlas of $U$ and the character $\chi_A$

Let $w_0 = s_{i_1} \ldots s_{i_m}$ be a reduced decomposition. It is encoded by the sequence $i = (i_1, i_2, \ldots, i_m)$. It provides a regular map

$$\phi_i : (\mathbb{G}_m)^m \to U, \quad (a_1, ..., a_m) \mapsto x_{i_1}(a_1) \ldots x_{i_m}(a_m).$$

(77)

The map $\phi_i$ is an open embedding [4], and a birational isomorphism. Thus it provides a rational coordinate system on U. It was shown in loc.cit. that the collection of these rational coordinate systems form a positive atlas of $U$, which we call Lusztig’s positive atlas. There is a similar positive atlas on $U^-$ provided by the maps $y_i$.

The choice of the maps $x_i$, $y_i$ in [40] provides the standard character:

$$\chi : U \to \mathbb{A}^1, \quad x_{i_1}(a_1) \ldots x_{i_m}(a_m) \mapsto \sum_{j=1}^m a_j.$$ 

(78)
It is evidently a positive function in Lusztig’s positive atlas. Moreover it is independent of the the sequence \(i\) chosen. Similarly, there is a character \(\chi^- : U^- \to \mathbb{A}^1\), \(y_{i_1}(b_1)\ldots y_{i_m}(b_m) \mapsto \sum_{j=1}^m b_j\), which is positive in the positive atlas on \(U^-\).

Let \(A := g \cdot U\) be a decorated flag. Its stabilizer is \(U_A = g U g^{-1}\). The associated character is \(\chi_A : U_A \to \mathbb{A}^1, \ u \mapsto \chi(g^{-1}ug)\).

For example, for an \(h \in H\), the character \(\chi_{A,h,U}\) is given by \(x_{i_1}(a_1)\ldots x_{i_m}(a_m) \mapsto \sum_{j=1}^m a_j/\alpha_j(h)\).

2.1.4 The potential \(W\) on the moduli space \(\text{Conf}_n(A)\).

Given a group \(G\) and \(G\)-sets \(X_1, \ldots, X_n\), orbits of the diagonal \(G\)-action on \(X_1 \times \ldots \times X_n\) are called configurations. Denote by \(\{x_1, \ldots, x_n\}\) a collection of points, and by \(\{x_1, \ldots, x_n\}\) its configuration.

We usually denote a decorated flag as \(A\) and the corresponding flag \(\pi(A_i)\) by \(B_i\). Denote the set \(\{1, \ldots, n\}\) of consecutive integers by \([1, n]\).

**Definition 2.1.** A pair \(\{B_1, B_2\} \in B \times B\) of Borel subgroups is generic if \(B_1 \cap B_2\) is a Cartan subgroup in \(G\). A collection \(\{A_1, \ldots, B_{m+n}\} \in A^n \times B^m\) is generic if for any distinct \(i, j \in [1, m+n]\), the pair \(\{B_i, B_j\}\) is generic.

Set \(\text{Conf}(A^n, B^m) := G \setminus (A^n \times B^m)\). Note that if \(\{A_1, \ldots, B_{m+n}\}\) is generic, then so is \(g \cdot \{A_1, \ldots, B_{m+n}\}\) for any \(g \in G\). Denote by \(\text{Conf}^*(A^n, B^m)\) the subset of generic configurations.

**Definition 2.2.** A frame for a split reductive algebraic group \(G\) over \(\mathbb{Q}\) is a generic pair \(\{A, B\} \in A \times B\). Denote by \(\mathcal{F}_G\) the moduli space of frames.

The space \(\mathcal{F}_G\) is a left \(G\)-torsor. If \(G = \text{SL}_m\), then a \(K\)-point of \(\mathcal{F}_G\) is the same thing as a unimodular frame in a vector space over \(K\) of dimension \(m\) with a volume form. If \(G\) is an adjoint group, then a frame is the same thing as a pinning.

Let \(\{A_1, \ldots, A_n\}\) be a generic collection of decorated flags. For each \(j \in [1, n]\), take the triple \(\{B_{j-1}, A_j, B_{j+1}\}\). Since \(\mathcal{F}_G\) is a \(G\)-torsor, there is a unique \(u_j \in U_{A_j}\) such that

\[
\{A_j, B_{j+1}\} = u_j \cdot \{A_j, B_{j-1}\}. \tag{79}
\]

Consider the following rational function on \(A^n\), whose definition is illustrated on Fig 15

\[
W(A_1, \ldots, A_n) := \sum_{j=1}^n \chi_{A_j}(u_j). \tag{80}
\]

**Lemma 2.3.** For any \(g \in G\), we have \(W(gA_1, \ldots, gA_n) = W(A_1, \ldots, A_n)\).

**Proof.** Clearly \(\{gA_j, gB_{j+1}\} = gu_jg^{-1} \cdot \{gA_j, gB_{j-1}\}\). The Lemma follows from [4]. \(\square\)

![Figure 15: The potential is a sum of the contribution at the vertices.](image)

Since \(W\) is invariant under the \(G\)-diagonal action on \(A^n\), we define
Definition 2.4. The potential $W$ is a rational function on $\text{Conf}_n(A)$, given by \[ W \].

Theorem 2.5. The potential $W$ is a positive rational function on the space $\text{Conf}_n(A)$, $n > 2$.

Theorem 2.5 is a non-trivial result. It is based on two facts: the character $\chi$ is a positive function on $U$, and the positive structure on $\text{Conf}_n(A)$ is twisted cyclic invariant, see Section 2.1.6. We prove Theorem 2.5 in Section 6.3.

Therefore we arrive at the set of positive tropical points of $\text{Conf}_n(A)$:

\[
\text{Conf}_n^+(A)(\mathbb{Z}^t) := \{ l \in \text{Conf}_n(A)(\mathbb{Z}^t) \mid W^t(l) \geq 0 \}, \quad n > 2. \tag{81}
\]

Example. Let $G = \text{SL}_2$. The space $\text{Conf}_3(A)$ parametrizes configurations $(v_1, v_2, v_3)$ of vectors in a two dimensional vector space with a volume form $\omega$. Set $\Delta_{i,j} := \langle v_i \wedge v_j, \omega \rangle$. Then

\[
W(v_1, v_2, v_3) = \frac{\Delta_{1,3}}{\Delta_{1,2} \Delta_{2,3}} + \frac{\Delta_{1,2}}{\Delta_{2,3} \Delta_{1,3}} + \frac{\Delta_{2,3}}{\Delta_{1,3} \Delta_{1,2}}. \tag{82}
\]

Therefore tropicalizing the function (82) we get

\[
\text{Conf}_n^+(A_{\text{SL}_2})(\mathbb{Z}^t) = \{ a, b, c \in \mathbb{Z} \mid a \geq b + c, \ b \geq a + c, \ c \geq a + b \}.
\]

Notice that the inequalities imply $a, b, c \in \mathbb{Z}_{\leq 0}$.

2.1.5 Parametrization of a canonical basis in tensor products invariants

By Bruhat decomposition, for each $(A_1, A_2) \in \text{Conf}_2^+(A)$, there is a unique $h_{A_1, A_2} \in H$ such that

\[(A_1, A_2) = (U, h_{A_1, A_2} \cdot \omega_0 \cdot U).\]

It provides an isomorphism, which induces a positive structure on $\text{Conf}_2(A)$:

\[ \alpha : \text{Conf}_2^+(A) \xrightarrow{\sim} H, \quad (A_1, A_2) \mapsto h_{A_1, A_2}. \tag{83} \]

We extend definition (81) to $n = 2$ using the potential (75), so that one has an isomorphism

\[ \alpha^t : \text{Conf}_n^+(A)(\mathbb{Z}^t) \xrightarrow{\sim} H^+(\mathbb{Z}^t) = \mathbb{P}^+. \]

See more details in Section 6.3, formula (191), and [FG1].

The restriction maps $\pi_{ij}$. We picture configurations $(A_1, ..., A_n)$ at the labelled vertices $[1, n]$ of a convex $n$-gon $P_n$. Each pair of distinct $i, j \in [1, n]$ gives rise to a map

\[ \pi_{ij} : \text{Conf}_n(A) \rightarrow \text{Conf}_2(A), \quad (A_1, ..., A_n) \rightarrow \begin{cases} (A_i, A_j) & \text{if } i < j, \\ (s_G \cdot A_i, A_j) & \text{if } i > j. \end{cases} \]

The maps $\pi_{ij}$ are positive [FG1], and therefore can be tropicalized:

\[
\text{Conf}_n(A)(\mathbb{Z}^t) \cup \text{Conf}_2(A)(\mathbb{Z}^t) = \mathbb{P}
\]

\[
\text{Conf}_n^+(A)(\mathbb{Z}^t) \cup \text{Conf}_2^+(A)(\mathbb{Z}^t) = \mathbb{P}^+
\]

The fact that $\pi_{ij}(\text{Conf}_n^+(A)(\mathbb{Z}^t)) \subseteq \mathbb{P}^+$ is due to Lemma 6.1.

In particular, the oriented sides of the polygon $P_n$ give rise to a positive map

\[ \pi = (\pi_{12}, \pi_{23}, \ldots, \pi_{n,1}) : \text{Conf}_n(A) \rightarrow (\text{Conf}_2(A))^n \simeq H^n. \tag{84} \]
A decomposition of $\text{Conf}^+_n(A)(Z^t)$. Given $\Delta := (\lambda_1, \ldots, \lambda_n) \in (\mathbb{P}^+)^n$, define

$$C_{\Delta} = \{ l \in \text{Conf}^+_n(A)(Z^t) \mid \pi^t(l) = \Delta \}. \quad (85)$$

The weights $\Delta$ of $G^L$ are assigned to the oriented sides of $P_n$, as shown on Fig 16. Such sets provide a canonical decomposition

$$\text{Conf}^+_n(A)(Z^t) = \bigsqcup_{\Delta \in (\mathbb{P}^+)^n} C_{\Delta}. \quad (86)$$

Tensor products invariants. Here is one of our main results.

**Theorem 2.6.** Let $\lambda_1, \ldots, \lambda_n \in \mathbb{P}^+$. The set $C_{\lambda_1, \ldots, \lambda_n}$ parametrizes a canonical basis in the space of invariants $(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n})^{G_L}$.

Theorem 2.6 follows from Theorem 2.20 and geometric Satake correspondence, see Section 2.2.4.

Alternatively, there is a similar set, defined by reversing the order of the side $(1, n)$:

$$C_{\lambda_1, \ldots, \lambda_{n-1}} := \{ l \in \text{Conf}^+_n(A)(Z^t) \mid \pi^t_{1,i+1}(l) = \lambda_i, \ i = 1, \ldots, n-1, \ \pi^t_{1,n}(l) = \lambda_n \}. \quad (87)$$

Then

$$C_{\lambda_1, \ldots, \lambda_n} = C_{\lambda_1, \ldots, \lambda_{n-1}}^{-w_0(\lambda_n)}. \quad (88)$$

The set $C_{\lambda_1, \ldots, \lambda_{n-1}}$ parametrizes a basis in the space of tensor product multiplicities

$$\text{Hom}_{G_L}(\Lambda_n, V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_{n-1}}). \quad (89)$$

2.1.6 Some features of the set $\text{Conf}^+_n(A)(Z^t)$.

Here are some features of the set $\text{Conf}^+_n(A)(Z^t)$. All of them follow immediately from the definition of the potential $W$ and basic facts about the positive structure on $\text{Conf}^+_n(A)$. One of the most crucial is twisted cyclic invariance, so we start from it.

The twisted cyclic shift. It was proved in [FG1, Section 8] that the defined there positive atlas on $\text{Conf}^+_n(A)$ is invariant under the twisted cyclic shift

$$t : \text{Conf}^+_n(A) \rightarrow \text{Conf}^+_n(A), \ (A_1, \ldots, A_n) \mapsto (A_2, \ldots, A_n, A_1 \cdot s_G).$$

Its tropicalization is a cyclic shift on the space of the tropical points:

$$t : \text{Conf}^+_n(A)(Z^t) \rightarrow \text{Conf}^+_n(A)(Z^t). \quad (90)$$

- **Twisted cyclic shift invariance.** The potential $W$ is evidently invariant under the twisted cyclic shift. Therefore the set $\text{Conf}^+_n(A)(Z^t)$ is invariant under the tropical cyclic shift $t$. 

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Given a triangle $t = \{i_1 < i_2 < i_3\}$ inscribed into the polygon $P_n$, there is a positive map
\[
\pi_t : \text{Conf}_n(A) \to \text{Conf}_3(A), \quad (A_1, \ldots, A_n) \mapsto (A_{i_1}, A_{i_2}, A_{i_3}).
\]
Each triangulation $T$ of $P_n$ gives rise to a positive injection $\pi_T : \text{Conf}_n(A) \to \prod_{t \in T} \text{Conf}_3(A)$, where the product is over all triangles $t$ of $T$. Set its image
\[
\text{Conf}_T(A) := \text{Im} \pi_T \subset \prod_{t \in T} \text{Conf}_3(A).
\]
For each pair $(t, d)$, where $t \in T$ and $d$ is a side of $t$, there is a map given by obvious projections
\[
p(t, d) : \prod_{t \in T} \text{Conf}_3(A) \xrightarrow{\text{pr}_t} \text{Conf}_3(A) \xrightarrow{\text{pr}_d} \text{Conf}_2(A).
\]
For each diagonal $d$ of $T$, there are two triangles, $t^d_1$ and $t^d_2$, sharing $d$. A point $x$ of $\text{Conf}_T(A)$ is described by the condition that $p(t^d_1, d)(x) = p(t^d_2, d)(x)$ for all diagonals $d$ of $T$.

**Proposition 2.7.** [FG1] There is an isomorphism of positive moduli spaces
\[
\pi_T : \text{Conf}_n(A) \xrightarrow{\sim} \text{Conf}_T(A).
\]
It leads to an isomorphism of sets of their $\mathbb{Z}$-tropical points:
\[
\pi_T^\mathbb{Z} : \text{Conf}_n(A)(\mathbb{Z}) \xrightarrow{\sim} \text{Conf}_T(A)(\mathbb{Z}).
\]
Some important features of the potential $W$ are the following:

- **Scissor congruence invariance.** For any triangulation $T$ of the polygon, the potential $W_n$ on $\text{Conf}_n(A)$ is a sum over the triangles $t$ of $T$:
\[
W_n = \sum_{t \in T} W_3 \circ \pi_t.
\]
This follows immediately from the fact that $\chi_A$ is a character of the subgroup $U_A$. Combining this with the isomorphism (91) we get

- **Decomposition isomorphism.** Given a triangulation $T$ of $P_n$, one has an isomorphism
\[
t^+_T : \text{Conf}_n^+(A)(\mathbb{Z}) \xrightarrow{\sim} \text{Conf}_T^+(A)(\mathbb{Z}).
\]

So one can think of the data describing a point of $\text{Conf}_n^+(A)(\mathbb{Z})$ as of a collection of similar data assigned to triangles $t$ of a triangulation $T$ of the polygon, which match at the diagonals. Therefore each triangulation $T$ provides a further decomposition of the set $\text{Conf}_n^+(A)(\mathbb{Z})$. By Lemma 6.14, the weights of $G^L$ assigned to the sides and edges of the polygon are dominant.

Consider an algebra with a linear basis $e_\lambda$ parametrized by dominant weights $\lambda$ of $G^L$ with the structure constants given by the cardinality of the set $C^\mu_{\lambda_1, \lambda_2}$:
\[
e_{\lambda_1} \ast e_{\lambda_2} = \sum_{\mu \in P^+} |C^\mu_{\lambda_1, \lambda_2}| e_\mu.
\]
The following basic property is evident from our definition of the set $C^\mu_{\lambda_1, \lambda_2}$:

- **Associativity.** The product $\ast$ is associative.

The associativity is equivalent to the fact that there are two different decompositions of the set $\text{Conf}_n^+(A)(\mathbb{Z})$ corresponding to two different triangulations of the 4-gon.

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A simple proof of Knutson-Tao-Woodward’s theorem [KTW]. That theorem asserts the associativity of the similar $*$-product whose structure constants are given by the number of hives. The associativity in our set-up, where the structure constant are given by the number of positive integral tropical points, is obvious for any group $G$. So to prove the theorem we just need to relate hives to positive integral tropical points for $G = \text{GL}_m$, which is done in Section 3.

2.2 Parametrization of top components of fibers of convolution morphisms

2.2.1 Transcendental cells and integral tropical points

For a non-zero $C = \sum_{k \geq p} c_k t^k \in \mathbb{K}$ such that $c_p$ is not zero, we define its valuation and initial term:

$$\text{val}(C) := p, \quad \text{in}(C) := c_p.$$

A decomposition of $T(\mathbb{K})$. For each split torus $T$, there is a natural projection, which we call the valuation map:

$$\text{val} : T(\mathbb{K}) \twoheadrightarrow T(\mathbb{K})/T(\mathcal{O}) = T(\mathbb{Z}^l).$$

Given an isomorphism $T = (\mathbb{G}_m)^k$, the map is expressed as $(C_1, \ldots, C_k) \mapsto (\text{val}(C_1), \ldots, \text{val}(C_k))$.

Each $l \in T(\mathbb{Z}^l)$ gives rise to a cell

$$T_l := \{ x \in T(\mathbb{K}) \mid \text{val}(x) = l \}.$$

It is a projective limit of irreducible algebraic varieties: each of them is isomorphic to $(\mathbb{G}_m)^k \times \mathbb{A}^N$. Therefore $T_l$ is an irreducible proalgebraic variety, and $T(\mathbb{K})$ is a disjoint union of them:

$$T(\mathbb{K}) = \coprod_{l \in T(\mathbb{Z}^l)} T_l.$$

Transcendental $\mathbb{K}$-points of $T$. Let us define an initial term map for $T(\mathbb{K})$ in coordinates:

$$\text{in} : T(\mathbb{K}) \longrightarrow T(\mathbb{C}), \quad (C_1, \ldots, C_k) \longmapsto (\text{in}(C_1), \ldots, \text{in}(C_k)).$$

A subset $\{c_1, \ldots, c_q\} \subset \mathbb{C}$ is algebraically independent if $P(c_1, \ldots, c_q) \neq 0$ for any $P \in \mathbb{Q}(X_1, \ldots, X_q)^*$.

**Definition 2.8.** A point $C \in T(\mathbb{K})$ is transcendental if its initial term $\text{in}(C)$ is algebraically independent as a subset of $\mathbb{C}$. Denote by $T^\circ(\mathbb{K})$ the set of transcendental points in $T(\mathbb{K})$. Set

$$T^\circ_l := T_l \cap T^\circ(\mathbb{K}).$$

**Lemma 2.9.** Let $F$ be a positive rational function on $T$. For any $C \in T^\circ(\mathbb{K})$, we have

$$\text{val}(F(C)) = F^t(\text{val}(C)).$$

**Proof.** It is clear.  

Transcendental $K$-cells of a positive space $\mathcal{Y}$.

**Definition 2.10.** A birational isomorphism $f : \mathcal{Y} \to Z$ of positive spaces is a positive birational isomorphism if it is a positive morphism, and its inverse is also a positive morphism.

**Theorem 2.11.** Let $f : T \to S$ be a positive birational isomorphism of split tori. Then

$$f(T_l^0) = S_{\beta(l)}^l.$$  

We prove Theorem 2.11 in Section 5. It is crucial that the inverse of $f$ is also a positive morphism. As a counterexample, the map $f : \mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x + 1$ is a positive morphism, but its inverse $x \mapsto x - 1$ is not. Let $l \in \mathbb{G}_m(\mathbb{Z}^l) = \mathbb{Z}$. If $l > 0$, then Theorem 2.11 fails: the points in $f(T_l^0)$ are not transcendental since $\text{in}(f(T_l^0)) \equiv 1$.

**Definition 2.12.** Let $\alpha_c : T \to \mathcal{Y}$ be a coordinate system from a positive atlas on $\mathcal{Y}$. The set of transcendental $K$-points of $\mathcal{Y}$ is

$$\mathcal{Y}^0(K) := \alpha_c(T^0(K)).$$

For each $l \in \mathcal{Y}(\mathbb{Z}^l)$, the transcendental $l$-cell of $\mathcal{Y}$ is

$$C_l^0 := \alpha_c(T^0_{\beta(l)}), \quad \text{where } \beta = \alpha_c^{-1}.$$  

By Theorem 2.11, this definition is independent of the coordinate system $\alpha_c$ chosen. Similarly one can upgrade the valuation map to positive spaces: given a positive space $\mathcal{Y}$, there is a unique map

$$\text{val} : \mathcal{Y}^0(K) \to \mathcal{Y}(\mathbb{Z}^l)$$

such that

$$C_l^0 = \{ y \in \mathcal{Y}^0(K) \mid \text{val}(y) = l \}.$$  

The valuation map is functorial under positive birational isomorphisms of positive spaces. Therefore the transcendental cells are also functorial under positive birational isomorphisms.

Thus there is a canonical decomposition parametrized by the set $\mathcal{Y}(\mathbb{Z}^l)$:

$$\mathcal{Y}^0(K) = \bigsqcup_{l \in \mathcal{Y}(\mathbb{Z}^l)} C_l^0.$$  

Thanks to the following Lemma, one can identify each tropical point $l$ with $C_l^0$.

**Lemma 2.13.** Let $F$ be a positive rational function on $\mathcal{Y}$. For any $C \in \mathcal{Y}^0(K)$, we have

$$\text{val}(F(C)) = F^\#(\text{val}(C)).$$

**Proof.** It follows immediately from Lemma 2.10 and Theorem 2.11. 

\[\square\]

### 2.2.2 $\mathcal{O}$-integral configurations of decorated flags and the affine Grassmannian

Recall the affine Grassmannian $\text{Gr}$. Recall the moduli space $\mathcal{F}_G$ of frames from Definition 2.2.

**Lemma–Construction 2.14.** There is a canonical onto map

$$L : \mathcal{F}_G(K) \to \text{Gr}, \quad \{ A_1, B_2 \} \mapsto L(A_1, B_2).$$

**Proof.** Let $\{ U, B^- \} \in \mathcal{F}_G(\mathbb{Q})$ be a standard frame. There is a unique $g_{\{ A_1, B_2 \}} \in G(K)$ such that

$$\{ A_1, B_2 \} = g_{\{ A_1, B_2 \}} \cdot \{ U, B^- \}.$$  

It provides an isomorphism $\mathcal{F}_G(K) \overset{\sim}{\to} G(K)$. Composing it with the projection $[\cdot] : G(K) \to \text{Gr},$

$$L(A_1, B_2) := [g_{\{ A_1, B_2 \}}] \in \text{Gr}.$$  

Note that $\mathcal{F}_G(\mathbb{Q})$ is a $G(\mathbb{Q})$-torsor. So choosing a different frame in $\mathcal{F}_G(\mathbb{Q})$ we get another representative of the coset $g_{\{ A_1, B_2 \}} \cdot G(\mathbb{Q})$. Since $G(\mathbb{Q}) \subset G(\mathcal{O})$, the resulting lattice will be the same. Therefore the map $L$ is canonical. 

\[\square\]
Symmetric space and affine Grassmannian. The affine Grassmannian is the non-archimedean version of the symmetric space $G(\mathbb{R})/K$, where $K$ is a maximal compact subgroup in $G(\mathbb{R})$. A generic pair of flags $\{B_1, B_2\}$ over $\mathbb{R}$ gives rise to an $H(\mathbb{R}_{>0})$-torsor in the symmetric space – the projection of $B_1 \cap B_2$. Notice that $H(\mathbb{R}_{>0}) = H(\mathbb{R})/(H(\mathbb{R}) \cap K)$. A generic pair $\{A_1, B_2\}$ determines a point $Q(A_1, B_2) \in G(\mathbb{R})/K$. So we get the archimedean analog of the map 11:

$$Q : \mathcal{F}_G(\mathbb{R}) \to G(\mathbb{R})/K, \quad \{A_1, B_2\} \mapsto Q(A_1, B_2). \quad (97)$$

Decorated flags and horospheres. For the adjoint group $G'$, the principal affine space $A$ can be interpreted as the moduli space of horospheres in the symmetric space $G'(\mathbb{R})/K$ in the archimedean case, or in the affine Grassmannian $Gr$. The horosphere $\mathcal{H}_A$ assigned to a decorated flag $A$ is an orbit of the maximal unipotent subgroup $U_A$. Let $B^*_A$ be the open Schubert cell of flags in generic position to a given decorated flag $A$. Then there is an isomorphism

$$B^*_A \to \mathcal{H}_A, \quad B \mapsto L(A, B) \text{ or } B \mapsto Q(A, B).$$

![Figure 18: The metric $q(h, y)$ determined by a horocycle $h$ and a boundary point $y$.](image)

Examples. 1. Let $G(\mathbb{R}) = \text{SL}_2(\mathbb{R})$. Its maximal compact subgroup $K = SO_2(\mathbb{R})$. The symmetric space $\text{SL}_2(\mathbb{R})/SO_2(\mathbb{R})$ is the hyperbolic plane $\mathcal{H}^2$. A decorated flag $A_1 \in \mathcal{A}_{PGL_2}(\mathbb{R})$ corresponds to a horocycle $h$ based as a point $x$ at the boundary. A flag $B_2$ corresponds to another point $y$ at the boundary. Let $q(x, y)$ be the geodesic connecting $x$ and $y$. The point $Q(A_1, B_2)$ is the intersection of $h$ and $g(x, y)$, see Fig. 18

$$q(h, y) := h \cap g(x, y) \in \mathcal{H}^2.$$

2. Let $G = \text{GL}_n$. Recall that a flag $F_*$ in an $n$-dimensional vector space $V_n$ over a field is a data $F_1 \subset \ldots \subset F_n$, $\dim F_i = i$. A generic pair of flags $(F_*, G_*)$ in $V_n$ is the same thing as a decomposition of $V_n$ into a direct sum of one dimensional subspaces

$$V_n = L_1 \oplus \ldots \oplus L_n, \quad (98)$$

where $L_i = F_i \cap G_{n+1-i}$. Conversely, $F_\alpha = L_1 \oplus \ldots \oplus L_\alpha$ and $G_\beta = L_{n-\beta+1} \oplus \ldots \oplus L_n$.

Over the field $\mathbb{R}$, this decomposition gives rise to a $\mathbb{R}^n_{>0}$-torsor in the symmetric space, given by a family of positive definite metrics on $V_n$ with the principal axes $(L_1, \ldots, L_n)$:

$$a_1x_1^2 + \ldots + a_nx_n^2, \quad a_i > 0. \quad (99)$$

Here $(x_1, \ldots, x_n)$ is a coordinate system for which the lines $L_i$ are the coordinate lines.

A decorated flag $A$ in $V_n$ is a flag $F_*$ plus a collection of non-zero vectors $l_i \in F_i/F_{i-1}$. A frame in $\mathcal{F}_{GL_n}$ is equivalent to a generic pair of flags $(F_*, G_*)$ and a decorated flag $A$ over the flag $F_*$. It determines a basis $(e_1, \ldots, e_n)$ in $V_n$ and vice versa. Here $e_i \in L_i$ and $e_i = l_i$ under the projection $L_i \to F_i/F_{i-1}$. This basis determines a metric – the positive definite metric with the principal axes $L_i$ such that the vectors $e_i$ are unit vectors.

---

10Here is a non-archimedean analog: A generic pair of flags $\{B_1, B_2\}$ over $K$ gives rise to an $H(K)/H(O)$-torsor in the affine Grassmannian – the projection of $B_1(\mathbb{K}) \cap B_2(\mathbb{K})$ to $G(\mathbb{K})/G(O)$.

11In the archimedean case, a maximal compact subgroup $K$ is defined by using the Cartan involution. A generic pair $\{A, B\}$ determines a pinning, and hence a Cartan involution.
3. Over the field $\mathcal{K}$, decomposition ([8] gives rise to an $\mathcal{H}(\mathcal{K})/\mathcal{H}(\mathcal{O}) = \mathbb{Z}^n$-torsor in $\text{Gr}$, given by the following collection of lattices in $V_n$.

$$\mathcal{O}t^{k_1}e_1 \oplus \ldots \oplus \mathcal{O}t^{k_n}e_n, \ k_i \in \mathbb{Z}.$$ These lattices are the non-archimedean version of the unit balls of the metrics ([9]).

**$\mathcal{O}$-integral configurations of decorated flags.**

**Definition 2.15.** A collection of decorated flags $\{A_1, \ldots, A_n\}$ over $\mathcal{K}$ is $\mathcal{O}$-integral if it is generic and for any $i \in [1, n]$ the lattice $L(A_i, B_j)$ does not depend on the choice of $j$ different then $i$.

Let $g \in G(\mathcal{K})$. Note that $L(gA_i, gB_j) = g \cdot L(A_i, B_j)$. Therefore if $\{A_1, \ldots, A_n\}$ is $\mathcal{O}$-integral, so is $g \cdot \{A_1, \ldots, A_n\}$. Thus we define

**Definition 2.16.** A configuration in $\text{Conf}_n(\mathcal{A})(\mathcal{K})$ is $\mathcal{O}$-integral if it is a $G(\mathcal{K})$-orbit of an $\mathcal{O}$-integral collection of decorated flags. Denote by $\text{Conf}_n^{\mathcal{O}}(\mathcal{A})$ the space of such configurations.

The archimedean version of Definition 2.16 is trivial. For example, let $G = \text{SL}_2(\mathbb{R})$. Then there are no horocycles $(h_1, h_2, h_3)$ such that their boundary points $(x_1, x_2, x_3)$ are distinct, and the intersection of the horocycle $h_i$ with the geodesic $g(x_i, x_j)$ do not depend on $j \neq i$.

In contrast with this, we demonstrate below that the non-archimedean version is very rich. The difference stems from the fact that in the archimedean case the intersection $K \cap U = e$ is trivial, while in the non-archimedean $G(\mathcal{O}) \cap U(\mathcal{K}) = U(\mathcal{O})$.

**Transcendental cells and $\mathcal{O}$-integral configurations.** The following fact is crucial.

**Theorem 2.17.** If $l \in \text{Conf}_n^+(\mathcal{A})(\mathbb{Z}')$, then there is an inclusion $\mathcal{C}_l^o \subset \text{Conf}_n^{\mathcal{O}}(\mathcal{A})$. Otherwise $\mathcal{C}_l^o \cap \text{Conf}_n^{\mathcal{O}}(\mathcal{A})$ is an empty set.

Theorem 2.17 gives an alternative conceptual definition of the set of positive integral tropical points of the space $\text{Conf}_n(\mathcal{A})$, which refers neither to the potential $W$, nor to a specific positive coordinate system. However to show that the set $\text{Conf}_n^+(\mathcal{A})(\mathbb{Z}')$ is “big”, or even non-empty, we use the potential $W$ and its properties, which imply, for example, that the set $\text{Conf}_n^+(\mathcal{A})(\mathbb{Z}')$ is obtained by amalgamation of similar sets assigned to triangles of a triangulation of the polygon. We prove Theorem 2.17 in Section 6.3.

### 2.2.3 The canonical map $\kappa$ and cycles on $\text{Conf}_n(\text{Gr})$

**The canonical map $\kappa$.** Recall the configuration space

$$\text{Conf}_n(\text{Gr}) := G(\mathcal{K}) \backslash (\text{Gr} \times \ldots \times \text{Gr}).$$

Given an $\mathcal{O}$-integral collection $\{A_1, \ldots, A_n\}$ of decorated flags, we get a collection of lattices $\{L_1, \ldots, L_n\}$ by setting $L_i := L(A_i, B_j)$ for some $j \neq i$. By definition, the lattice $L_i$ is independent of $j$ chosen. This construction descends to configurations, providing a canonical map

$$\kappa : \text{Conf}_n^{\mathcal{O}}(\mathcal{A}) \rightarrow \text{Conf}_n(\text{Gr}), \quad (A_1, \ldots, A_n) \mapsto (L_1, \ldots, L_n). \quad (100)$$

The map is evidently cyclic invariant, and commutes with the restriction to subconfigurations:

$$\kappa(A_{i_1}, \ldots, A_{i_k}) = (L_{i_1}, \ldots, L_{i_k}) \quad \text{for any } 1 \leq i_1 < \ldots < i_k \leq n.$$ 

**The cycles $\mathcal{M}_l$ in $\text{Conf}_n(\text{Gr})$.** Let $l \in \text{Conf}_n^+(\mathcal{A})(\mathbb{Z}')$. Thanks to Theorem 2.17, we can combine the inclusion there with the canonical map (100):

$$\mathcal{C}_l^o \subset \text{Conf}_n^{\mathcal{O}}(\mathcal{A}) \xrightarrow{\kappa} \text{Conf}_n(\text{Gr}). \quad (101)$$
Definition 2.18. The cycle $\mathcal{M}_l \subset \text{Conf}_n(\text{Gr})$ is a substack given by the closure of $\kappa(\mathcal{C}_l^\circ)$:

$$\mathcal{M}_l := \overline{\mathcal{M}_l^\circ}, \quad \mathcal{M}_l^\circ := \kappa(\mathcal{C}_l^\circ) \subset \text{Conf}_n(\text{Gr}), \quad l \in \text{Conf}_n^+(\mathcal{A})(Z^l). \quad (102)$$

Lemma 2.19. The cycle $\mathcal{M}_l$ is irreducible.

Proof. For a split torus $T$, the cycle $T_1$ is irreducible. So the cycles $\mathcal{C}_l^\circ$ and $\mathcal{M}_l$ are irreducible. \hfill \Box

In other words, $\mathcal{M}_l$ is a $G(\mathbb{K})$-invariant closed subspace in $\text{Gr}^n$. There is a bijection

$$\{G(\mathbb{K}) \text{-orbits in } \text{Gr}^n\} \xleftrightarrow{\sim} \{G(\mathcal{O}) \text{-orbits in } [1] \times \text{Gr}^{n-1}\}. \quad (103)$$

Therefore one can also view the cycles $\mathcal{M}_l$ as $G(\mathcal{O})$-invariant closed subspaces in $[1] \times \text{Gr}^{n-1}$. Let us describe them using this point of view.

2.2.4 Top components of the fibers of the convolution morphism

Given $\mathbf{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{P}^+)^n$, recall the cyclic convolution variety

$$\text{Gr}_{c(\mathbf{\lambda})} := \{(l_1, \ldots, l_n) \in \text{Gr}^n \mid l_1 \xrightarrow{\lambda_1} l_2 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_n} l_{n+1}, \; l_1 = l_{n+1} = [1]\}.$$ 

It is a finite dimensional reducible variety of top dimension

$$\text{ht}(\mathbf{\lambda}) := (\rho, \lambda_1 + \ldots + \lambda_n).$$

It is the fiber of the convolution morphism, and therefore, thanks to the geometric Satake correspondence \[L4\] \[G\] \[MV\], there is a canonical isomorphism

$$\text{IH}^{\text{ht}(\mathbf{\lambda})}(\text{Gr}_{c(\mathbf{\lambda})}) = \left(V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}\right)^{G_L}. \quad (104)$$

Each top dimensional component of $\text{Gr}_{c(\mathbf{\lambda})}$ provides an element in the space $\text{IH}^{\text{ht}(\mathbf{\lambda})}$. These elements form a canonical basis in $\{104\}$. Let $T_\mathbf{\lambda}$ be the set of top dimensional components of $\text{Gr}_{c(\mathbf{\lambda})}$. Recall the set $C_\mathbf{\lambda}$ of positive tropical points $\{53\}$, and the cycle $\mathcal{M}_l$ from Definition 2.18.

Theorem 2.20. Let $l \in C_\mathbf{\lambda}$. Then the cycle $\mathcal{M}_l$ is the closure of a top dimensional component of $\text{Gr}_{c(\mathbf{\lambda})}$. The map $l \mapsto \mathcal{M}_l$ provides a canonical bijection from $C_\mathbf{\lambda}$ to $T_\mathbf{\lambda}$.

Theorem 2.20 is proved in Section 9.4. It implies Theorem 2.6.

2.2.5 Constructible equations for the top dimensional components

We have defined the cycles $\mathcal{M}_l$ as the closures of the images of the cells $\mathcal{C}_l^\circ$. Now let us define the cycles $\mathcal{M}_l$ by equations, given by certain constructible functions on the space $\text{Conf}_n(\text{Gr})$. These functions generalize Kamnitzer’s functions $H_{l_1, \ldots, l_n}$ for $G = \text{GL}_m$ (K1).

Constructible function $D_F$. Let $R$ be a reductive algebraic group over $\mathbb{C}$. We assume that there is a rational left algebraic action of $R$ on $\mathbb{C}^n$. Let $\mathbb{C}(x_1, \ldots, x_n)$ be the field of rational functions on $\mathbb{C}^n$. We get a right algebraic action of $R$ on $\mathbb{C}(x_1, \ldots, x_n)$ denoted by $\circ$.

Let $\mathbb{K}(x_1, \ldots, x_n)$ be the field of rational functions with $\mathbb{K}$-coefficients. The valuation of $\mathbb{K}^\times$ induces a natural valuation map

$$\text{val} : \mathbb{K}(x_1, \ldots, x_n)^\times \rightarrow \mathbb{Z}.$$ 

Let $F, G \in \mathbb{K}(x_1, \ldots, x_n)^\times$. The valuation map has two basic properties

$$\text{val}(FG) = \text{val}(F) + \text{val}(G), \quad (105)$$
$$\text{val}(F + G) = \text{val}(F), \quad \text{if } \text{val}(F) < \text{val}(G). \quad (106)$$

The group $R(\mathbb{K})$ acts on $\mathbb{K}(x_1, \ldots, x_n)$ on the right. We have the following
Lemma 2.21. Let $F \in \mathcal{K}(x_1, \ldots, x_n)^\times$. If $h \in R(O)$, then $\text{val}(F \circ h) = \text{val}(F)$.

Proof. For any $k \in \mathcal{K}^\times$, we have $(kF) \circ h = k(F \circ h)$. Therefore it suffices to prove the case when $\text{val}(F) = 0$.

Note that the group $R$ is reductive. It is generated by

$$x_i(a) \in U, \quad y_i(b) \in U^-, \quad \alpha(c) \in H, \quad \text{where } i \in I \text{ and } \alpha \in \text{Hom}(\mathbb{G}_m, H).$$

Since the action of $R$ is algebraic, for any $f \in \mathbb{C}(x_1, \ldots, x_n)^\times$, we have $f \circ x_i(a) \in \mathbb{C}(x_1, \ldots, x_n, a)^\times$. Note that $f \circ x_i(0) = f$. Therefore we get

$$f \circ x_i(a) = \frac{f + af_1 + \ldots + af_i}{1 + a g_1 + \ldots + a^m g_m}, \quad \text{where } f_j, g_j \in \mathbb{C}(x_1, \ldots, x_n). \quad (107)$$

If $a \in \mathbb{C}$, then $f \circ x_i(a) \in \mathbb{C}(x_1, \ldots, x_n)$. Moreover $f \circ x_i(a)$ is non-zero. Otherwise, $f = (f \circ x_i(a)) \circ x_i(-a) = 0$.

If $a \in tO$, then by the basic property, we get $\text{val}(f \circ x_i(a)) = \text{val}(f) = 0$.

Let $a = a_0 + b = a_0 + a_1 t + a_2 t^2 + \ldots \in O$. Then $f \circ x_i(a) = (f \circ x_i(a_0)) \circ x_i(b)$. Note that $f \circ x_i(a_0) \in \mathbb{C}(x_1, \ldots, x_n)^\times$ and $b \in tO$. Combining the above arguments we get

$$\text{val}(f \circ x_i(a)) = \text{val}(f \circ x_i(a_0)) = 0 = \text{val}(f), \quad \forall a \in O. \quad (108)$$

Now let $F \in \mathbb{K}(x_1, \ldots, x_n)^\times$ such that $\text{val}(F) = 0$. Then $F$ can be expressed as

$$F = f_0 + b_1 f_1 + \ldots + b_k f_k, \quad \text{where } f_i \in \mathbb{C}(x_1, \ldots, x_n).$$

Here $f_0, f_p, g_q \in \mathbb{C}(x_1, \ldots, x_n)^\times$, $b_p, c_q \in \mathbb{K}^\times$, $\text{val}(b_p) > 0$, $\text{val}(c_q) > 0$. By definition, we have

$$F \circ x_i(a) = \frac{f_0 \circ x_i(a) + b_1 f_1 \circ x_i(a) + \ldots + b_k f_k \circ x_i(a)}{1 + c_1 g_1 \circ x_i(a) + \ldots + c_m g_m \circ x_i(a)}.$$ 

Let $a \in O$. By $\text{109}$, we get

$$\text{val}(f_0 \circ x_i(a)) = 0, \quad \text{val}(b_p) \circ x_i(a) = \text{val}(b_p) + \text{val}(f_p \circ x_i(a)) = \text{val}(b_p) > 0, \quad \text{val}(c_q g_q \circ x_i(a)) = \text{val}(c_q) + \text{val}(g_q \circ x_i(a)) = \text{val}(c_q) > 0.$$

By the basic property $\text{109}$, we get $\text{val}(F \circ x_i(a)) = \text{val}(f_0 \circ x_i(a)) = 0$. Hence we prove that

$$\text{val}(F \circ x_i(a)) = \text{val}(F), \quad \forall a \in O.$$

By the same argument, we show that

$$\text{val}(F \circ y_i(b)) = \text{val}(F), \quad \forall b \in O, \quad \text{val}(F \circ \alpha(c)) = \text{val}(F), \quad \forall c \in O^\times.$$

Note that $R(O)$ is generated by the elements $x_i(a), y_i(b), \alpha(c), a, b \in O, c \in O^\times$. The Lemma is proved. \hfill $\square$

Let $\mathbb{X}$ be rational space over $\mathbb{C}$, i.e., $\mathbb{C}(\mathbb{X}) \cong \mathbb{C}(x_1, \ldots, x_n)$. Similarly, there is a valuation map $\text{val} : \mathcal{K}(\mathbb{X})^\times \rightarrow \mathbb{Z}$. We assume that there is left algebraic action of $R$ on $\mathbb{X}$. Lemma 2.21 implies

Lemma 2.22. Let $F \in \mathcal{K}(\mathbb{X})^\times$. If $h \in R(O)$, then $\text{val}(F \circ h) = \text{val}(F)$.

Constructible equations for top components. Let $\mathbb{X} := \mathcal{A}^n$ and let $R := G^n$. Let $F \in \mathbb{C}(\mathcal{A}^n)$ and let $(g_1, \ldots, g_n) \in G^n$. Then $G^n$ acts on $\mathbb{C}(\mathcal{A}^n)$ on the right:

$$(F \circ (g_1, \ldots, g_n))(A_1, \ldots, A_n) := F(g_1 \cdot A_1, \ldots, g_n \cdot A_n), \quad \forall(A_1, \ldots, A_n) \in \mathcal{A}^n. \quad (109)$$

By definition, a nonzero rational function $F \in \mathbb{C}($Conf$_n(\mathcal{A}))$ is also a $G$-diagonal invariant function on $\mathcal{A}^n$

$$F(gA_1, \ldots, gA_n) = F(A_1, \ldots, A_n).$$

There is a $\mathbb{Z}$-valued function

$$D_F : G(K)^n \rightarrow \mathbb{Z}, \quad D_F(g_1(t), \ldots, g_n(t)) := \text{val}\left(\frac{F(g_1(t), \ldots, g_n(t))}{1 + g_1(t) + \ldots + g_n(t)}\right). \quad (110)$$
Lemma–Construction 2.23. The function \( D_F \) is invariant under the left diagonal action of the group \( G(\mathbb{K}) \) on \( G(\mathbb{K})^n \), and the right action of the subgroup \( G(O)^n \subset G(\mathbb{K})^n \). Therefore \( D_F \) descends to a function \( \text{Conf}_n(\text{Gr}) \to \mathbb{Z} \) which we also denote by \( D_F \).

Proof. The first property is clear since \( F \in \mathbb{C}(\mathcal{A}^n)^G \). The second property is by Lemma 2.22. \( \square \)

Let \( \mathbb{Q}_+(\text{Conf}_n(\mathcal{A})) \) be the semifield of positive rational functions on \( \text{Conf}_n(\mathcal{A}) \). Take a non-zero function \( F \in \mathbb{Q}_+(\text{Conf}_n(\mathcal{A})) \). Therefore it gives rise to a function \( D_F \) on \( \text{Conf}_n(\text{Gr}) \). Meanwhile, its tropicalization \( F' \) is a function on \( \text{Conf}_n(\mathcal{A})(\mathbb{Z}) \).

Theorem 2.24. Let \( l \in \text{Conf}_n^+(\mathcal{A})(\mathbb{Z})^l \) and \( F \in \mathbb{Q}_+(\text{Conf}_n(\mathcal{A})) \). Then \( D_F(\kappa(\mathcal{C}_l^+)) \equiv F'(l) \).

Theorem 2.24 is proved in Section 8. It implies that the map in Theorem 2.20 is injective. It can be reformulated as follows:

For any \( l \) and \( F \) as above, the generic value of \( D_F \) on the cycle \( \mathcal{M}_l \) is \( F'(l) \).

When \( G = \text{GL}_m \), one can describe the set \( \mathcal{C}_\mathcal{A} \) by using the special collection of functions on the space \( \text{Conf}_n(\mathcal{A}) \) defined in Section 9 of [FG1]. The obtained description coincides with Kamnitzer’s generalization of hives [KL]. He conjectured in [KL] that the latter set parametrizes the components of the convolution variety for \( \text{GL}_m \). Therefore Theorems 2.20 and 2.24 imply Conjecture 4.3 in [KL].

2.3 Mixed configurations and a generalization of Mirković-Vilonen cycles

In this Section we discuss several other examples. Each of them fits in the general scheme of Section 1.2. We show how to encode all the data in a polygon.

2.3.1 Mixed configurations and the map \( \kappa \)

Definition 2.25. i) Given a subset \( I \subset [1, n] \), the moduli space \( \text{Conf}_1(\mathcal{A}; \mathcal{B}) \) parametrizes configurations \( (x_1, ..., x_n) \), where \( x_i \in \mathcal{A} \) if \( i \in I \), otherwise \( x_i \in \mathcal{B} \).

ii) Given subsets \( J \subset I \subset [1, n] \), the moduli space \( \text{Conf}_{\leq 1}(\text{Gr}; \mathcal{A}, \mathcal{B}) \) parametrizes configurations \( (x_1, ..., x_n) \) where \( x_i \in \text{Gr} \) if \( i \in J \), \( x_i \in \mathcal{A}(\mathcal{K}) \) if \( i \in I - J \), \( x_i \in \mathcal{B}(\mathcal{K}) \) otherwise.

We set \( \text{Conf}_1(\text{Gr}; \mathcal{B}) := \text{Conf}_{\leq 1}(\text{Gr}; \mathcal{B}) \).

A positive structure on the space \( \text{Conf}_1(\mathcal{A}; \mathcal{B}) \) is defined in Section 6.3. This positive structure is invariant under a cyclic twisted shift. See Lemma 6.7 for the precise statement.

Definition 2.26. Let \( J \subset I \subset [1, n] \). A configuration in \( \text{Conf}_1(\mathcal{A}; \mathcal{B})(\mathcal{K}) \) is called \( O \)-integral relative to \( J \) if

1. For all \( j \in J \) and \( k \neq j \), the pairs \( (A_j, B_k) \) are generic. Here \( B_k = \pi(A_k) \) if \( k \in I \).
2. The lattices \( L_j := L(A_j, B_k) \) given by the above pairs only depend on \( j \).

Denote by \( \text{Conf}_{\leq 1}^O(\mathcal{A}; \mathcal{B}) \) the moduli space of such configurations.

By the very definition, there is a canonical map

\[
\kappa : \text{Conf}_{\leq 1}^O(\mathcal{A}; \mathcal{B}) \to \text{Conf}_{\leq 1}(\text{Gr}; \mathcal{A}, \mathcal{B}).
\] (112)

It assigns to \( A_j \) the lattice \( L_j \) when \( j \in J \) and keeps the rest intact.

Recall \( u_j \in U_{A_j} \) in (104). The potential \( W_j \) on \( \text{Conf}_1(\mathcal{A}; \mathcal{B}) \) is a function

\[
W_j := \sum_{j \in J} \chi_{A_j}(u_j).
\] (113)

Positivity of \( W_j \) is proved in Section 6.4. Next Theorem generalizes Theorem 2.17. Its proof is the same. See Section 6.4.
Theorem 2.27. Let \( l \in \text{Conf}_I(\mathcal{A}; \mathcal{B})(\mathbb{Z}^I) \). A configuration in \( \mathcal{C}_l^0 \) is \( \mathcal{O} \)-integral relative to \( J \) if and only if \( \mathcal{W}_J^l(l) \geq 0 \).

Denote by \( \text{Conf}_{I \subset I}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^I) \) the set of points \( l \in \text{Conf}_I(\mathcal{A}; \mathcal{B})(\mathbb{Z}^I) \) such that \( \mathcal{W}_J^l(l) \geq 0 \). Set
\[
\mathcal{M}_I^l := \kappa(\mathcal{C}_l^0) \subset \text{Conf}_{I \subset I}(\mathcal{A}; \mathcal{B})(\mathbb{Z}^I).
\]

These cycles generalize the Mirković-Vilonen cycles, as we will see in Section 2.3.3.

2.3.2 Basic invariants

Recall the isomorphism (83):
\[
\alpha : \text{Conf}^\ast(\mathcal{A}, \mathcal{A}) \xrightarrow{\sim} H, \quad \alpha(A_1 \cdot h_1, A_2 \cdot h_2) = h_1^{-1}w_0(h_2)\alpha(A_1, A_2).
\]

Given a generic triple \( \{A_1, B_2, A_3\} \), we choose a decorated flag \( A_2 \) over the flag \( B_2 \), and set
\[
\mu(A_1, B_2, A_3) := \alpha(A_1, A_2)\alpha(A_3, A_2)^{-1} \in H.
\]

Due to (115), it does not depend on the choice of \( A_2 \). We illustrate the invariant \( \mu \) by a pair of red dashed arrows on the left in Fig 19.

Given a generic configuration \( (A_1, B_2, B_3, A_4) \), see the right of Fig 19 choose decorated flags \( A_2, A_3 \) over the flags \( B_2, B_3 \), and set
\[
\mu(A_1, B_2, B_3, A_4) := \alpha(A_2, A_1)\alpha_2(A_2, A_3)^{-1}\alpha(A_4, A_3) \in H.
\]

These invariants coincide with a similar \( H \)-valued \( \mu \)-invariants from Section 1.4.

Figure 19: The invariants \( \mu(A_1, B_2, A_3) \in H \) and \( \mu(A_1, B_2, B_3, A_4) \in H \).

There are canonical isomorphisms:
\[
\begin{align*}
\pi_{\text{Gr}} : \text{Conf}(\text{Gr}, \text{Gr}) & \xrightarrow{\sim} P^+, \\
\alpha_{\text{Gr}} : \text{Conf}(\mathcal{A}, \text{Gr}) & \xrightarrow{=} P, \\
\alpha'_{\text{Gr}} : \text{Conf}(\text{Gr}, \mathcal{A}) & \xrightarrow{=} P.
\end{align*}
\]

The first map uses the decomposition \( G(\mathcal{K}) = G(\mathcal{O}) \cdot H(\mathcal{K}) \cdot G(\mathcal{O}) \):
\[
\text{Conf}(\text{Gr}, \text{Gr}) = G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) = W\backslash H(\mathcal{K})/H(\mathcal{O}) = P^+.
\]

The second map uses the Iwasawa decomposition \( G(\mathcal{K}) = U(\mathcal{K}) \cdot H(\mathcal{K}) \cdot G(O) \):
\[
\text{Conf}(\mathcal{A}, \text{Gr}) = G(\mathcal{K})\backslash \left( G(\mathcal{K})/U(\mathcal{K}) \times G(\mathcal{K})/G(\mathcal{O}) \right) = U(\mathcal{K})\backslash G(\mathcal{K})/G(\mathcal{O}) = H(\mathcal{K})/H(\mathcal{O}) = P.
\]

The third map is a cousin of the second one:
\[
\alpha'_{\text{Gr}}(L, A) := -w_0(\alpha_{\text{Gr}}(A, L)).
\]
Remark. These isomorphisms parametrize \(G(\mathcal{O}), U(\mathcal{K})\) and \(U^{-}(\mathcal{K})\)-orbits of \(\text{Gr}\). Each coweight \(\lambda \in \mathcal{P} = H(\mathbb{Z}^{l}) = H(\mathcal{K})/H(\mathcal{O})\) corresponds to an element \(t^{\lambda}\) of \(\text{Gr}\). Then
\[
\pi_{\text{Gr}}([1, g \cdot t^{\lambda}]) = \lambda, \quad \forall g \in G(\mathcal{O});
\]
\[
\alpha_{\text{Gr}}(U, u \cdot t^{\lambda}) = \lambda, \quad \forall u \in U(\mathcal{K});
\]
\[
\alpha'_{\text{Gr}}(v \cdot t^{-\lambda}, \mathcal{P} \cdot U) = \lambda, \quad \forall v \in U^{-}(\mathcal{K}).
\]

We define Grassmannian versions of \(\mu\)-invariants:
\[
\mu_{\text{Gr}} : \text{Conf}(\text{Gr}, \mathcal{B}, \text{Gr}) \rightarrow \mathcal{P}, \quad \mu_{\text{Gr}} : \text{Conf}(\text{Gr}, \mathcal{B}, \mathcal{B}, \text{Gr}) \rightarrow \mathcal{P}
\]
\[
\mu_{\text{Gr}}(L_{1}, B_{2}, B_{3}, L_{4}) := \alpha_{\text{Gr}}(L_{1}, A_{2}) - \alpha'_{\text{Gr}}(L_{3}, A_{2}) \in \mathcal{P}.
\]
Let \(pr : B^{-}(\mathcal{K}) \rightarrow H(\mathcal{K}) \rightarrow \mathcal{P}\) be the composite of standard projections. The first map has an equivalent description:
\[
\mu_{\text{Gr}}([b_{1}], B^{-}, [b_{2}]) = pr(b_{1}^{-1}b_{2}), \quad b_{1}, b_{2} \in B^{-}(\mathcal{K}).
\]

More generally, take a chain of flags starting and ending by a decorated flag, pick an alternating sequence of arrows, and write an alternating product of the \(\alpha\)-invariants. We get regular maps
\[
\mu : \text{Conf}^{*}(\mathcal{A}, \mathcal{B}^{2n+1}, \mathcal{A}) \rightarrow H, \quad (A_{1}, B_{2}, ..., B_{2n+2}, A_{2n+3}) \rightarrow \frac{\alpha(A_{1}, A_{2}) \alpha(A_{3}, A_{4}) \cdots \alpha(A_{2n+1}, A_{2n+2})}{\alpha(A_{3}, A_{2}) \alpha(A_{5}, A_{4}) \cdots \alpha(A_{2n+3}, A_{2n+2})}.
\]
\[
\mu : \text{Conf}^{*}(\mathcal{A}, \mathcal{B}^{2n}, \mathcal{A}) \rightarrow H, \quad (A_{1}, B_{2}, ..., B_{2n+1}, A_{2n+2}) \rightarrow \frac{\alpha(A_{2}, A_{1}) \alpha(A_{4}, A_{3}) \cdots \alpha(A_{2n+2}, A_{2n+1})}{\alpha(A_{2}, A_{3}) \alpha(A_{4}, A_{5}) \cdots \alpha(A_{2n}, A_{2n+1})}.
\]

Given a cyclic collection of an even number of flags, there is an invariant which for \(n = 2\) and \(G = \text{SL}_{2}\) recovers the cross-ratio:
\[
\text{Conf}^{*}_{2n}(\mathcal{B}) \rightarrow H, \quad (B_{1}, ..., B_{2n}) \rightarrow \frac{\alpha(A_{1}, A_{2}) \alpha(A_{3}, A_{4}) \cdots \alpha(A_{2n-1}, A_{2n})}{\alpha(A_{3}, A_{2}) \alpha(A_{5}, A_{4}) \cdots \alpha(A_{2n}, A_{2n+1})}.
\]

One gets Grassmannian versions by replacing \(\mathcal{A}\) by \(\text{Gr}\), and \(\alpha\) by one of the maps \(\pi_{\text{Gr}}\).

These invariants provide decompositions for both spaces in \([114]\).

Let us encode all the data in a polygon, as illustrated on Fig \([\ref{fig:generalized-MV-cycles}]\). Let \(l \in \text{Conf}_{l \subset 1}(\mathcal{A}; B)(\mathbb{Z}^{l})\). We show on the left an element of \(\mathcal{C}_{l}^{\top}\). Flags or decorated flags are assigned to the vertices of a convex polygon. The vertices labeled by \(J\) are boldface. Note that although we order the vertices by choosing a reference vertex, due to the twisted cyclic invariance the story does not depend on its choice.

The solid blue sides are labeled by a pair of decorated flags. There is an invariant \(\lambda_{E} \in \mathcal{P}\) assigned to such a side \(E\). It is provided by the tropicalization of the isomorphism \([115]\) evaluated on \(l\). The collection of dashed edges determines an invariant \(\mu \in \mathcal{P}\).

Recall the cone \(\mathbb{R}^{+} \subset \mathcal{P}\) generated by positive coroots. The \(\mathcal{O}\)-integrality imposes restrictions on basic invariants, summarized in Lemma \([28]\) and illustrated on Fig \([\ref{fig:generalized-MV-cycles}a]\).
**Lemma 2.28.** i) Let \( (A_1, A_2, B_3) \in C_\lambda^\circ \subset \text{Conf}^O(A, A, B) \). Then \( \text{val} \circ \alpha(A_1, A_2) \in P^+ \).

ii) Let \( (B_1, A_2, B_3) \in C_\lambda^\circ \subset \text{Conf}^O(B, A, B) \). Then \( \text{val} \circ \mu(A_2, B_1, B_3, A_2) \in R^+ \).

*Proof.* Here i) follows from Lemma 6.14 and ii) follows from Lemmas 5.3 & 6.4(4).

Applying the map \( \kappa \), we replace the decorated flag at each boldface vertex by the corresponding lattice. Others remain intact. We use the notation \( \lambda \) for the decorated flags which do not contribute the character \( \chi \) to the potential – they are assigned to the unmarked vertices. For example, we associate to the polygons on Fig. 20 the following maps

\[
\kappa : C_\lambda^\circ \longrightarrow \text{Conf}(A, \text{Gr}^3, B), \quad l \in \text{Conf}^+(\lambda, A^3, B)(\mathbb{Z}).
\]

\[
\pi : \text{Conf}^+(\lambda, A^3, B) \longrightarrow H^3, \quad \mu : \text{Conf}^+(\lambda, A^3, B) \longrightarrow H,
\]

\[
(\pi^\lambda, \mu^\lambda) : \text{Conf}^+(\lambda, A^3, B)[\mathbb{Z}] \longrightarrow P \times (P^+)^2 \times P,
\]

\[
(\pi_{Gr}, \mu_{Gr}) : \text{Conf}(\lambda, \text{Gr}^3, B) \longrightarrow P \times (P^+)^2 \times P.
\]

\[ (120) \]

It is easy to check that the targets of the invariants assigned to configurations of flags are the same as the targets of their Grassmannian counterparts.

### 2.3.3 Generalized Mirković-Vilonen cycles

Let us recall the standard definition of *Mirković-Vilonen cycles* following [MV], [A], [K].

For \( w \in W \), let \( U_w = wUw^{-1} \). For \( w \in W \) and \( \mu \in P \) define the *semi-infinite cells*

\[
S^\mu_w := U_w(K)\bar{t}^\mu.
\]

Let \( \lambda, \mu \in P \). The closure \( S^\lambda_w \cap S^\mu_w \) is non-empty if and only if \( \lambda - \mu \in R^+ \). In that case, it is also well known that \( S^\lambda_w \cap S^\mu_w \) has pure dimension \( \text{ht}(\lambda - \mu) := \langle \rho, \lambda - \mu \rangle \).

**Definition 2.29.** A component of \( S^\lambda_w \cap S^\mu_w \subset \text{Gr} \) is called an *MV cycle of coweight* \((\lambda, \mu)\).

Since \( H \) normalizes \( U_w \), for each \( h \in H(K) \) such that \( [h] = t^\nu \), we have \( h \cdot S^\mu_w = S^\mu_{w^\nu} \). Therefore if \( V \) is an MV cycle of coweight \((\lambda, \mu)\), then \( h \cdot V \) is an MV cycle of coweight \((\lambda + \nu, \mu + \nu)\). The \( H(K) \)-orbit of an MV cycle of coweight \((\lambda, \mu)\) is called a *stable* MV cycle of coweight \( \lambda - \mu \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in (P^+)^n \). Consider the convolution variety

\[
\text{Gr}_\lambda = \{ (L_1, L_2, \ldots, L_n) \mid [1] \xrightarrow{\lambda_1} L_1 \xrightarrow{\lambda_2} \ldots \xrightarrow{\lambda_n} L_n \} \subset \text{Gr}^n.
\]

\[ (122) \]

Let \( pr_n : \text{Gr}^n \to \text{Gr} \) be the projection onto the last factor. Set

\[
\text{Gr}^\mu_\lambda := \text{Gr}_\lambda \cap pr_n^{-1}(S^\mu_w).
\]

\[ (123) \]

When \( n=1 \), under the geometric Satake correspondence, the components of \( \text{Gr}^\mu_\lambda \) give a basis (the MV basis) for the weight space \( V^\mu(\lambda) \), see [MV] Corollary 7.4. It is easy to see that they are precisely MV cycles of coweight \((\lambda, \mu)\) contained in \( \text{Gr}_\lambda \), see [A] Proposition 3.

Now we restrict constructions in preceding subsections to four main examples associated to an \((n+2)\)-gon. The \( n = 1 \) case recovers the above three versions of MV cycles. In this sense, the following can be viewed as a generalization of MV cycles.
**Example 1:** $J = [2, n + 1] \subset I = [1, n + 1]$. Let $\text{Conf}_{w_0}(A, Gr^n, B) \subset \text{Conf}_{J \subset I}(Gr; A, B)$ be the substack parametrizing configurations $(A_1, L_2, \ldots, L_{n+1}, B_{n+2})$ where $(A_1, B_{n+2})$ is generic.

Recall $\mathcal{F}_G$ in Definition 2.2. Then

$$\text{Conf}_{w_0}(A, Gr^n, B) = G(K) \setminus (\mathcal{F}_G(K) \times Gr^n).$$

Since $\mathcal{F}_G$ is a $G$-torsor, we get an isomorphism

$$i : Gr^n \xrightarrow{\sim} \text{Conf}_{w_0}(A, Gr^n, B), \quad (L_1, \ldots, L_n) \mapsto (U, L_1, \ldots, L_n, B^-). \quad (124)$$

From now on we identify $Gr^n$ with $\text{Conf}_{w_0}(A, Gr^n, B)$.

There is a map, whose construction is illustrated on the right of Fig 20,

$$\pi_{Gr} : \text{Conf}_{w_0}(A, Gr^n, B) \to \mathbb{P} := P \times (P^+)^{n-1} \times P.$$

Its fibers are finite dimensional subvarieties $Gr^\mu_{\lambda, \Delta}$:

$$Gr^n = \bigsqcup Gr^\mu_{\lambda, \Delta}, \text{ where } (\lambda, \Delta, \mu) \in P \times (P^+)^{n-1} \times P. \quad (125)$$

By (117) we see that

$$Gr^\mu_{\lambda, \Delta} = \{(L_1, \ldots, L_n) \in Gr^n \mid L_1 \xrightarrow{\lambda_1} \ldots \xrightarrow{\lambda_n} L_n, \ \lambda_1 \in S^\lambda_e, \ L_n \in S^n_{w_0}, \ \Delta := (\lambda_2, \ldots, \lambda_n).$$

When $n = 1$, it is the intersection $S^\lambda_e \cap S^n_{w_0}$. Note that the very notion of MV cycles depends on the choice of the pair $H \subset B$. We transport the MV cycles to $\text{Conf}_{w_0}(A, Gr, B)$ by the isomorphism (124). It is then independent of the pair chosen. In general we define

**Definition 2.30.** The irreducible components of $\overline{\text{Gr}^\mu_{\lambda, \Delta}}$ are called the generalized Mirković-Vilonen cycles of coweight $(\lambda, \Delta, \mu)$.

Similarly the left of Fig 20 provides a map

$$\pi^t : \text{Conf}^+(A, A^n, B)(Z^t) \to \mathbb{P} \times (P^+)^{n-1} \times P. \quad (126)$$

Let $P^\mu_{\lambda, \Delta} := \text{Conf}^+(A, A^n, B)(Z^t)^{\mu}_{\lambda, \Delta}$ be the fiber of map (126) over $(\lambda, \Delta, \mu)$. Then

$$\text{Conf}^+(A, A^n, B)(Z^t) = \bigsqcup P^\mu_{\lambda, \Delta}, \text{ where } (\lambda, \Delta, \mu) \in P \times (P^+)^{n-1} \times P. \quad (127)$$

By definition $\pi^t \circ \text{val}$ and $\pi_{Gr} \circ \kappa$ deliver the same map from $C^\mu_t$ to $\mathbb{P}$. Thus we arrive at

$$\mathcal{M}_l := \mathcal{M}_l^\mu \subset Gr^\mu_{\lambda, \Delta}, \quad l \in P^\mu_{\lambda, \Delta} := \text{Conf}^+_w(A, A^n, B)(Z^t)^{\mu}_{\lambda, \Delta}. \quad (128)$$

**Theorem 2.31.** The cycles (128) are precisely the generalized MV cycles of coweight $(\lambda, \Delta, \mu)$.

**Example 2:** $J = I = [2, n + 1]$. Let $\text{Conf}_{w_0}(B, Gr^n, B) \subset \text{Conf}_{J \subset I}(Gr; A, B)$ be the substack parametrizing configurations $(B_1, L_2, \ldots, L_{n+1}, B_{n+2})$ where $(B_1, B_{n+2})$ is generic.

Similarly, we get an isomorphism of stacks

$$i_s : H(K) \setminus Gr^n \xrightarrow{\sim} \text{Conf}_{w_0}(B, Gr^n, B), \quad (L_1, \ldots, L_n) \mapsto (B, L_1, \ldots, L_n, B^-). \quad (129)$$

Here the group $H(K)$ acts diagonally on $Gr^n$. Let $b \in H(K)$. If $[b] = t^\mu$, then $h \cdot \overline{Gr^\mu_{\Delta, \lambda}} = \overline{Gr^\mu_{\Delta, \lambda + \nu}}$. It provides an isomorphism between the sets of components of both varieties.

**Definition 2.32.** The $H(K)$-orbit of a generalized MV cycle of coweight $(\lambda, \Delta, \nu)$ is called a generalized stable MV cycle of coweight $(\lambda, \Delta, \lambda - \nu)$. 

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When \( n = 1 \), it recovers the usual stable MV cycles. The generalized stable MV cycles live naturally on the stack \( H(\mathcal{K})/\text{Gr}^n \). The isomorphism \( \{1\} \) transports them to \( \text{Conf}_{w_0}(B, \text{Gr}^n, B) \).

The solid blue arrows and the triple of dashed reds on Fig 22 provide a canonical projection
\[
(\pi^t, \mu^t) : \text{Conf}^+(B, \mathcal{A}^n, B)(\mathcal{Z}^t) \longrightarrow (P^+)^{n-1} \times P.
\]

Let \( A^\mu_\lambda : = \text{Conf}^+(B, \mathcal{A}^n, B)(\mathcal{Z}^t)^\mu_\lambda \) be its fiber over \((\lambda, \mu)\). Then
\[
\text{Conf}^+(B, \mathcal{A}^n, B)(\mathcal{Z}^t) = \bigsqcup \lambda \in (P^+)^{n-1}, \mu \in P. \quad (130)
\]

On the other hand, our general construction provides us with the irreducible cycles
\[
M_l : = \text{Conf}^+(B, \mathcal{A}^n, B)(\mathcal{Z}^t)^\mu_\lambda , \quad l \in A^\mu_\lambda . \quad (131)
\]

**Theorem 2.33.** The cycles \( (131) \) are precisely the generalized stable MV cycles of coweight \((\lambda, \mu)\).

![Figure 22: Generalized stable MV cycles \( \mathcal{M}_l \subset \text{Conf}(B, \text{Gr}^3, B) = H(\mathcal{K})/\text{Gr}^3 \).](image)

**Example 3:** \( J = I = [1, n + 1] \). By Iwasawa decomposition we get an isomorphism
\[
i_b : B^-(\mathcal{O})/\text{Gr}^n \longrightarrow \text{Conf}(\text{Gr}^{n+1}, B), \quad (L_1, \ldots, L_n) \longmapsto ([1], L_1, \ldots, L_n, B^-). \quad (132)
\]

There are two projections, illustrated on Fig 23
\[
(\pi_\text{Gr}, \mu_\text{Gr}) : \text{Conf}(\text{Gr}^{n+1}, B) \longrightarrow (P^+)^n \times P, \quad (133)
\]
\[
(\pi^t, \mu^t) : \text{Conf}^+(\mathcal{A}^{n+1}, B)(\mathcal{Z}^t) \longrightarrow (P^+)^n \times P. \quad (134)
\]

Their fibers over \((\lambda, \mu) \in (P^+)^n \times P\) provide decompositions
\[
\text{Conf}(\text{Gr}^{n+1}, B) = \bigsqcup \lambda \in (P^+)^{n-1}, \mu \in P. \quad (135)
\]
\[
\text{Conf}^+(\mathcal{A}^{n+1}, B)(\mathcal{Z}^t) = \bigsqcup \lambda \in (P^+)^{n-1}, \mu \in P. \quad (136)
\]

By definition, these decompositions are compatible under the map \( \kappa \). We get irreducible cycles
\[
\mathcal{M}_l := \overline{\mathcal{M}_l} \subset B^-(\mathcal{O})/\text{Gr}^n = \text{Conf}(\text{Gr}^{n+1}, B), \quad l \in B^\mu_\lambda := \text{Conf}^+(\mathcal{A}^{n+1}, B)(\mathcal{Z}^t)^\mu_\lambda . \quad (137)
\]

The connected group \( B^-(\mathcal{O}) \) acts diagonally \( \text{Gr}^n \). It preserves components of subvarieties \( \text{Gr}^\lambda \) in \( (133) \). Hence these components live naturally on the stack \( B^-(\mathcal{O})/\text{Gr}^n \). We transport them to \( \text{Conf}(\text{Gr}^{n+1}, B) \) by \( (132) \).

**Theorem 2.34.** The cycles \( (137) \) are precisely the components of \( B^-(\mathcal{O})/\text{Gr}^\lambda \).
Figure 23: Generalized MV cycles \( \mathcal{M}_l \subset \text{Conf}(\text{Gr}^4, \mathcal{B}) = B^- (\mathcal{O}) \setminus \text{Gr}^3 \).

Example 4: \( J = I = [1, n + 2] \). There is an isomorphism
\[
i_g : G(\mathcal{O}) \setminus \text{Gr}^{n+1} \xrightarrow{\sim} \text{Conf}_{n+2}(\text{Gr}), \quad (L_1, \ldots, L_{n+1}) \mapsto ([1], L_1, \ldots, L_{n+1}) .
\] (138)

We arrive at irreducible cycles defined in Definition 2.18
\[
\mathcal{M}_l := \overline{\mathcal{M}_l^0} \subset G(\mathcal{O}) \setminus \text{Gr}^{n+1} = \text{Conf}_{n+2}(\text{Gr}) \quad l \in \mathcal{C}_\lambda := \text{Conf}^+_{n}(\mathcal{A})(\mathcal{Z}_t^1)_{\lambda} .
\]

This example recovers Theorem 2.20

![Figure 24: Mirković-Vilonen cycles \( \mathcal{M}_l \subset \text{Conf}_{w_0}(\mathcal{A}, \text{Gr}, \mathcal{B}) = \text{Gr} \).](image)

Figure 24: Mirković-Vilonen cycles \( \mathcal{M}_l \subset \text{Conf}_{w_0}(\mathcal{A}, \text{Gr}, \mathcal{B}) = \text{Gr} \).

![Figure 25: Stable Mirković-Vilonen cycles \( \mathcal{M}_l \subset \text{Conf}_{w_0}(\mathcal{B}, \text{Gr}, \mathcal{B}) = H(\mathcal{K}) \setminus \text{Gr} \).](image)

Figure 25: Stable Mirković-Vilonen cycles \( \mathcal{M}_l \subset \text{Conf}_{w_0}(\mathcal{B}, \text{Gr}, \mathcal{B}) = H(\mathcal{K}) \setminus \text{Gr} \).

Specializing Theorems 2.31-2.34 to \( n = 1 \), we get

**Theorem 2.35.** 1) Mirković-Vilonen cycles of coweight \( (\lambda, \mu) \) are precisely the cycles
\[
\mathcal{M}_l \subset \text{Gr}, \quad l \in \mathcal{P}_\lambda^\mu := \text{Conf}^+(\mathcal{A}, \mathcal{A}, \mathcal{B})(\mathcal{Z}_t^1)^\mu_{\lambda} \quad \text{for} \quad \mathcal{W} = \chi_{\lambda_2} .
\]

2) Stable Mirković-Vilonen cycles of coweight \( \mu \) are precisely the cycles
\[
\mathcal{M}_l \subset H(\mathcal{K}) \setminus \text{Gr}, \quad l \in \mathcal{A}_\mu := \text{Conf}^+(\mathcal{B}, \mathcal{A}, \mathcal{B})(\mathcal{Z}_t^1)^\mu_{\lambda} \quad \text{for} \quad \mathcal{W} = \chi_{\lambda_2} .
\]

3) Mirković-Vilonen cycles of coweight \( (\lambda, \mu) \) which lie in \( \overline{\text{Gr}_\lambda} \subset \text{Gr} \) are precisely the cycles
\[
\mathcal{M}_l \subset B^- (\mathcal{O}) \setminus \text{Gr}, \quad l \in \mathcal{B}_\lambda^\mu := \text{Conf}^+(\mathcal{A}, \mathcal{A}, \mathcal{B})(\mathcal{Z}_t^1)^\mu_{\lambda} \quad \text{for} \quad \mathcal{W} = \chi_{\lambda_1} + \chi_{\lambda_2} .
\]
Figure 26: MV cycles which lie in $\text{Gr}_\lambda$ are the cycles $\mathcal{M}_l^\circ \subset \text{Conf}(\text{Gr}, \text{Gr}, \mathcal{B})_\lambda = \mathcal{B}^{-}(\mathcal{O}) \setminus \text{Gr}_\lambda$.

Figure 27: Generalized MV cycles $\mathcal{M}_l \subset \text{Conf}(\text{Gr}, \text{Gr}, \text{Gr})$.

Theorem 2.35 is proved in Section 9.1.

Note that there is a positive birational isomorphism $\text{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B}) \cong U$. Thus we identify $\text{Conf}^+(\mathcal{B}, \mathcal{A}, \mathcal{B})(\mathbb{Z})[\mathbb{Z}]$ with the subset of $U(\mathbb{Z})$ used by Lusztig [L]. [L1] to parametrize the canonical basis in Lemma 5.1. Then Theorem 2.35 is equivalent to the main results of Kamnitzer’s paper [K]. Our approach, using the moduli space $\text{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B})$ rather than $U$, makes parametrization of the MV cycles more natural and transparent, and puts it into the general framework of this paper.

To summarize, there are four different versions of the cycles relevant to representation theory related to mixed configurations of triples, as illustrated on Fig 24-27.

2.3.4 Constructible equations for the cycles $\mathcal{M}_l^\circ$

Let $F$ be a rational function on the stack $\text{Conf}_1(\mathcal{A}; \mathcal{B})$. We generalize the construction of $D_F$ from Section 2.2.5. As an application, it implies that the cycles $\mathcal{M}_l^\circ$ in (114) are disjoint.

Given $J \subset I \subset [1, n]$, let $m$ be the cardinality of $J$. We assume $J = \{j_1, \ldots, j_m\}$.

Consider the space

$$
\mathcal{X} := X_1 \times \ldots \times X_n, \quad \text{where } X_i = \begin{cases} 
G & \text{if } i \in J, \\
\mathcal{A} & \text{if } i \in I - J, \\
\mathcal{B} & \text{otherwise}.
\end{cases}
$$

Let $\mathcal{X}_*$ be its subset consisting of collections $\{x_1, \ldots, x_n\}$ whose subcollections $\{x_{i_1}, \ldots, x_{i_{n-m}}\}$, $i_s \notin J$, are generic.

Given a rational function $F$ on $\text{Conf}_1(\mathcal{A}; \mathcal{B})$, each $x = \{x_1, \ldots, x_n\} \in \mathcal{X}_*(\mathcal{K})$ provides a function $F_x$ on $\mathcal{A}^m$, whose value on $\{A_{j_1}, \ldots, A_{j_m}\} \in \mathcal{A}^m$ is

$$
F_x(A_{j_1}, \ldots, A_{j_m}) := F(x'_1, \ldots, x'_n) \in \mathcal{K}, \quad x'_i = \begin{cases} 
x_j \cdot A_j & \text{if } j \in J, \\
x_i & \text{otherwise}.
\end{cases}
$$

Then $F_x \in \mathcal{K}(\mathcal{A}^m)$.

Recall the map $\text{val} : \mathcal{K}(\mathcal{A}^m)^\times \to \mathbb{Z}$. We get a $\mathbb{Z}$-valued function

$$
D_F : \mathcal{X}_*(\mathcal{K}) \to \mathbb{Z}, \quad D_F(x) := \text{val}(F_x).
$$
Recall the right action of $G^n$ on $\mathbb{C}(A^m)$. Thanks to Lemma 2.22 and the fact that $F \in \mathbb{Q}(\text{Conf}_1(A;B))$, we have
\[
\forall g \in G(k), \forall h \in G(O)^m, \quad \text{val}(F_{g \times} \circ h) = \text{val}(F_x).
\]
Thus $D_F$ descends to
\[
D_F : \text{Conf}_1^+(Gr;A,B) \rightarrow \mathbb{Z}.
\]
Here $\text{Conf}_1^+(Gr;A,B)$ is a subspace of $\text{Conf}_1^+(Gr;A,B)$ consisting of the configurations whose subconfigurations of flags and decorated flags are generic.

By definition, $M^n_\mu$ in (114) are contained in $\text{Conf}_1^+(Gr;A,B)$. The following Theorem is a generalization of Theorem 2.24. See Section 8 for its proof.

*Theorem 2.36.* Let $l \in \text{Conf}_1^+(A;B)(\mathbb{Z})$. Let $F \in \mathbb{Q}_+(\text{Conf}_1(A;B))$. Then $D_F(M^n_\mu) \equiv F^l(l)$.

### 2.4 Canonical bases in tensor products and $\text{Conf}_1(A^n;B)$

Recall that a collection of dominant coweights $\lambda = (\lambda_1, \ldots, \lambda_n)$ gives rise to a convolution variety $\text{Gr}_{\lambda} \subset \text{Gr}^n$. It is open and smooth. Its dimension is calculated inductively:

\[
\dim \text{Gr}_{\lambda} = 2\text{ht}(\lambda) := 2(\rho, \lambda_1 + \ldots + \lambda_n).
\]

The subvarieties $\text{Gr}_{\lambda}$ form a stratification $S$ of $\text{Gr}^n$. Let $IC_{\lambda}$ be the IC-sheaf of $\text{Gr}_{\lambda}$. By the geometric Satake correspondence,

\[
H^*(IC_{\lambda}) = V_{\lambda} = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}.
\]

Let $pr_n : \text{Gr}^n \rightarrow \text{Gr}$ be the projection onto the last factor. Recall the point $t^\mu \in \text{Gr}$. Set

\[
S_\mu := pr_n^{-1}(U(\mathbb{K})t^\mu) \subset \text{Gr}^n, \quad T_\mu := pr_n^{-1}(U^-(\mathbb{K})t^\mu) \subset \text{Gr}^n.
\]

The sum of positive coroots is a cocharacter $2\rho^\vee : \mathbb{G}_m \rightarrow H$. It provides an action of the group $\mathbb{G}_m$ on $\text{Gr}^n$ given by the action on the last factor. The subvarieties $S_\mu$ and $T_\mu$ are attracting and repulsing subvarieties for this action. Set

\[
\text{Gr}^\mu := \text{Gr}_{\lambda} \cap S_\mu.
\]

*Lemma 2.37.* If $\text{Gr}^\mu$ is non-empty, then it is a subvariety of pure dimension

\[
\dim \text{Gr}^\mu = \text{ht}(\lambda; \mu) := (\rho, \lambda_1 + \ldots + \lambda_n + \mu).
\]

Denote by $\text{Irr}(X)$ the set of top dimensional components of a variety $X$, and by $\mathbb{Q}[\text{Irr}(X)]$ the vector space with the bases parametrised by the set $\text{Irr}(X)$.

*Theorem 2.38.* There are canonical isomorphisms

\[
H^*(\text{Gr}^n, IC_{\lambda}) = \oplus_\mu H^*_{c,\text{IC}}(S_\mu, IC_{\lambda}) = \oplus_\mu \mathbb{Q}[\text{Irr}(\text{Gr}^\mu_{\lambda})].
\]

*Proof.* Theorem 2.38 for $n = 1$ is proved in [MV Section 3]. The proof for arbitrary $n$ follows the same line. For convenience of the reader we provide a complete proof.

Let $m : \mathbb{C}^* \times X \rightarrow X$ be a map defining an action of the group $\mathbb{C}^*$ on $X$. Let $D(X)$ be the bounded derived category of constructible sheaves on $X$. An object $F \in D(X)$ is weakly $\mathbb{C}^*$-equivariant, if $m^*F = L \boxtimes F$ for some locally constant sheaf $L \subset \mathbb{C}^*$.

Recall the action of $G_m$ on $\text{Gr}^n$ defined above. Denote by $P_S(\text{Gr}^n)$ the category of weakly $\mathbb{C}^*$-equivariant perverse sheaves on $\text{Gr}^n$ which are constructible with respect to the stratification $S$.

*Lemma 2.39.* The sheaf $IC_{\lambda}$ is locally constant along the stratification $S$. It belongs to the category $P_S(\text{Gr}^n)$.

*Proof.* Given a subgroup $G' \subset G$, denote by $G'_{[k,n]} \subset \text{Gr}^n$ the subgroup of elements $(e, \ldots, e, g_1, \ldots, g)$, with $(n-k+1)$ of $g \in G'$. Denote by $G(L)$ the subgroup stabilising a point $L \in \text{Gr}$. The group $G(L)_{[k,n]}$ preserves the category $P_S(\text{Gr}^n)$. Take two collections $(L_1, \ldots, L_n), (M_1, \ldots, M_n) \in \text{Gr}^n$, with $L_1 = M_1 = [1]$ and in the same stratum. We can move $(L_1, \ldots, L_n)$ by an element of $G(L_1)_{[1,n]}$, getting $(M_1, M_2, L_3', \ldots, L_n')$. Then we move it by an element of $G(M_2)_{[2,n]}$, getting $(M_1, M_2, M_3, \ldots, L_n')$, and so on, using subgroups $G(L_n)_{[k,n]}$ for $k = 3, 4, \ldots, n - 1$. In the last step we get $(M_1, \ldots, M_n)$. The $\mathbb{C}^*$-equivariance is evident. \qed
Proposition 2.40. For all $\mathcal{P} \in \mathcal{P}_S(\text{Gr}^n)$ we have a canonical isomorphism
\[
H^k_c(S_\mu, \mathcal{P}) \cong H^k_{\mathcal{T}_\mu}(\text{Gr}^n, \mathcal{P}).
\] (146)
Both sides vanish if $k \neq 2\text{ht}(\mu)$. The functors $F_\mu := H^{2\text{ht}(\mu)}_c(S_\mu, -) : \mathcal{P}_S(\text{Gr}^n) \rightarrow \text{Vect}$ are exact.

Proof. Isomorphism (146) follows from the hyperbolic localisation theorem of Braden [Br]. Let us briefly recall how it works.

Let $X$ be a normal complex variety on which the group $\mathbb{C}^*$ acts. Let $F$ be the stable points variety. It is a union of components $F_1, ..., F_k$. Consider the attracting and repulsing subvarieties
\[
X^+_k = \{ x \in X | \lim_{t \rightarrow 0} t \cdot x \in F_k \}, \quad X^-_k = \{ x \in X | \lim_{t \rightarrow \infty} t \cdot x \in F_k \},
\]
Let $X^+$ (resp. $X^-$) be the disjoint union of all the $X^+_k$ (resp. $X^-_k$). There are projections
\[
\pi^+: X^+ \rightarrow F; \quad \pi^+(x) = \lim_{t \rightarrow 0} t \cdot x, \quad \pi^-(x) = \lim_{t \rightarrow \infty} t \cdot x.
\]
Let $g^\pm : X^\pm \hookrightarrow X$ be the natural inclusions. Given an object $\mathcal{F} \in D(X)$, define hyperbolic localisation functors
\[
\mathcal{F}^\uparrow_{\uparrow\downarrow} := (\pi^+)_!(g^+)^* \mathcal{F}, \quad \mathcal{F}^\downarrow_{\uparrow\downarrow} := (\pi^-)_!(g^-)^* \mathcal{F}.
\]
Combining Theorem 1 and Section 3 of [Br], we have the following result, which implies (146).

Proposition 2.41. If $\mathcal{F}$ is weakly $\mathbb{C}^*$-equivariant, the natural map $\mathcal{F}_{\uparrow\downarrow} \rightarrow \mathcal{F}^\downarrow_{\uparrow\downarrow}$ is an isomorphism.

Let us prove the vanishing. One has $H^k_c(\text{Gr}^n, \mathbb{Q}) = 0$ for $k > 2\text{dimGr}^n = 2\text{ht}(\lambda; \mu)$. Due to perversity, the restriction of any $\mathcal{P} \in \mathcal{P}_S(\text{Gr}^n)$ to $\text{Gr}^n_\lambda$ lies in degrees $\leq -\text{dimGr}^n_\lambda = -2\text{ht}(\lambda)$. So
\[
H^k_c(\text{Gr}^n_\lambda, \mathcal{P}) = 0 \quad \text{if } k > 2\text{ht}(\mu).
\] (147)
Although $S_\mu$ is infinite dimensional, we can slice it by its intersections with the strata $\text{Gr}^n_\lambda$. Since the estimate (147) on each strata does not depend on $\lambda$, a devissage using exact triangles $j_i^* j_i^* \mathcal{A} \rightarrow \mathcal{A} \rightarrow i_i^* i_i^* \mathcal{A}$ tells that
\[
H^k_c(S_\mu, \mathcal{P}) = 0 \quad \text{if } k > 2\text{ht}(\mu).
\]
Applying the duality, and using the fact that $* \mathcal{P} = \mathcal{P}$, we get the dual estimate
\[
H^k_{\mathcal{T}_\mu}(\text{Gr}^n, \mathcal{P}) = 0 \quad \text{if } k < 2\text{ht}(\mu).
\]
Combining with the isomorphism (146), we get the proof. The last claim is then obvious. $\square$

Proposition 2.42. We have natural equivalence of functors
\[
H^* \cong \oplus_{\mu \in \mathcal{P}} H^{2\text{ht}(\mu)}_c(S_\mu, -) : \mathcal{P}_S(\text{Gr}^n) \rightarrow \text{Vect}.
\]
Proof. The proof of Theorem 3.6 in [MV] works in our case. Namely, the two filtrations of $\text{Gr}^n$ by the closures of $S_\mu$ and $T_\mu$ give rise to two filtrations of $H^*$, given by the kernels of $H^* \rightarrow H^*_c(S_\mu, -)$ and the images of $H^*_c(T_\mu)(\text{Gr}^n, -) \rightarrow H^*$. The vanishing implies $H^{2\text{ht}(\mu)}_c(S_\mu, -) \rightarrow H^{2\text{ht}(\mu)}_c(S_\mu, -)$ and $H^{2\text{ht}(\mu)}_c(S_\mu, -) \rightarrow H^{2\text{ht}(\mu)}_c(S_\mu, -)$ is an isomorphism. So the two filtrations split each other. $\square$

Corollary 2.43. The global cohomology functor $H^* : \mathcal{P}_S(\text{Gr}^n) \rightarrow \text{Vect}$ is faithful and exact.

Denote by $H^0_{\text{per}} \mathcal{F}$ the cohomology of an $\mathcal{F} \in D^B_c(\text{Gr}^n)$ for the the perverse $t$-structure. Let $j : \text{Gr}_\lambda \hookrightarrow \overline{\text{Gr}_\lambda}$ be the natural embedding, $j_{\lambda}(\lambda, Q) := H^0_{\text{per}} (j_! Q[\text{dimGr}^n_\lambda])$, and $j_{\lambda}(\lambda, Q) := H^0_{\text{per}} (j_! Q[\text{dimGr}^n_\lambda])$. The following Lemma is a generalisation of Lemma 7.1 of [MV].

Lemma 2.44. The category $\mathcal{P}_S(\text{Gr}^n)$ is semi-simple. The sheaves $j_{\lambda}(\lambda, Q)$, $j_*(\lambda, Q)$, and $j_{*}(\lambda, Q)$ are isomorphic.
Proof. Let us prove first the parity vanishing for the stalks of the sheaf $\mathcal{J}_n(\underline{\lambda}, \mathbb{Q})$: the stalks could have non-zero cohomology only at even degrees. For $n = 1$ it is proved in [L4]. It can also be proved by using the Bott-Samelson resolution of the Schubert cells in the affine (i.e. Kac-Moody) case, as was explained to us by A. Braverman. Let $\mathcal{F}$ be a Kac-Moody flag variety. Take an element $w = w_1 \ldots w_n$ of the affine Weyl group such that $l(w) = l(w_1) + \ldots + l(w_n)$. Denote by $\mathcal{F}_{w_1, \ldots, w_n}$ the variety parametrising flags $(F_1 = [1], F_2, \ldots, F_n)$ such that the pair $(F_i, F_{i+1})$ is in the incidence relation $w_i$. Choose reduced decompositions $[w_1], \ldots, [w_n]$ of the elements $w_1, \ldots, w_n$. Their product is a reduced decomposition $[w]$ of $w$. It gives rise to the Bott-Samelson variety $X_{[w]}$. By its very definition, it is a tower of fibrations

$$X([w_1], \ldots, [w_n]) \longrightarrow X([w_1], \ldots, [w_{n-1}]) \longrightarrow \ldots \longrightarrow X([w_1]).$$

The Bott-Samelson resolution of the affine Schubert cell $\text{Gr}_{\lambda}$ is a smooth projective variety $X_{\lambda}$ with a map $\beta_{\lambda}: X_{\lambda} \to \text{Gr}_{\lambda}$ which is $1:1$ at the open stratum, and which, according to [Gau1], [Gau2], has the following property. For each of the strata $\text{Gr}_{\mu} \subset \text{Gr}_{\lambda}$, there exists a point $p_{\mu} \in \text{Gr}_{\mu}$ such that the fiber $\beta_{\lambda}^{-1}(p_{\mu})$ of the Bott-Samelson resolution has a cellular decomposition with the cells being complex vector spaces. Therefore the stalk of the push forward $\beta_{\lambda}^*\mathcal{Q}_{X_{\lambda}}$ of the constant sheaf on $X_{\lambda}$ at the point $p_{\mu}$ satisfies the parity vanishing. By the decomposition theorem [BBD], the sheaf $\mathcal{I}_C$ is a direct summand of the push forward $\beta_{\lambda*}\mathcal{Q}_{X_{\lambda}}$ of the constant sheaf on $X_{\lambda}$. Indeed, the latter is a direct sum of shifts of perverse sheaves, and it is the constant sheaf over the open stratum. Therefore the stalk of the sheaf $\mathcal{I}_C$ at the point $p_{\mu}$ satisfies the parity vanishing. Since the cohomology of $\mathcal{I}_C$ is locally constant over each of the stratum $\text{Gr}_{\mu}$, we get the parity vanishing. The general case of $\text{Gr}_{\lambda}$ is treated very similarly to the case of $\text{Gr}_{\lambda}$.

The rest is pretty standard, and goes as follows. The strata $\text{Gr}_{\lambda}$ are simply connected: this is well known for $n = 1$, and the strata $\text{Gr}_{\lambda}$ is fibered over $\text{Gr}_{\lambda}$ with the fiber $\text{Gr}_{\lambda_\mu}$, where $\underline{\lambda} = (\lambda, \lambda_\mu)$. Since the strata are even dimensional over $\mathbb{R}$, this plus the parity vanishing implies that there are no extensions between the simple objects in $P_\mathcal{S}(\text{Gr}^n)$. Indeed, by devissage this claim reduces to calculation of extensions between constant sheaves concentrated on two open strata. Thus there are no extensions in the category $P_\mathcal{S}(\text{Gr}^n)$, i.e. it is semi-simple.

Let us show now that $\mathcal{J}_n(\underline{\lambda}, \mathbb{Q}) = \mathcal{J}_n(\underline{\lambda}, \mathbb{Q})$. Since $H^p_{\text{per}}(j_!\mathcal{Q}_{\text{Gr}_{\lambda}}) = 0$ for $p > 0$, there is a map $j_!\mathcal{Q}_{\text{Gr}_{\lambda}} \to H^0_{\text{per}}(j_!\mathcal{Q}_{\text{Gr}_{\lambda}}) = \mathcal{J}_n(\underline{\lambda}, \mathbb{Q})$. If $\mathcal{J}_n(\underline{\lambda}, \mathbb{Q}) \neq \mathcal{J}_n(\underline{\lambda}, \mathbb{Q})$, since the category $P_\mathcal{S}(\text{Gr}^n)$ is semisimple, there is a non-zero direct summand $B$ of $\mathcal{J}_n(\underline{\lambda}, \mathbb{Q})$ supported at a lower stratum. Composing these two maps, we get a non-zero map $j_!\mathcal{Q}_{\text{Gr}_{\lambda}} \to B$. On the other hand, given a space $X$ and complexes of sheaves $\mathcal{A}$ and $\mathcal{B}$ supported at disjoint subsets $A$ and $B$ respectively, one has $\text{Hom}(j_!\mathcal{A}, \mathcal{B}) = 0$, where $j: A \hookrightarrow X$. Contradiction. The statement about $\mathcal{J}_n$ follows by the duality.

Lemma 2.45. There are canonical isomorphisms

$$F_\mu[\mathcal{J}_n(\underline{\lambda}, \mathbb{Q})] = \mathbb{Q}[\text{Irr}(\text{Gr}_{\underline{\lambda}}^\mu)] = F_\mu[\mathcal{J}_n(\underline{\lambda}, \mathbb{Q})].$$

Proof. We prove the first claim. The second is similar. We follow closely the proof of Proposition 3.10 in [MV]. Set $\mathcal{F} := \mathcal{J}_n(\underline{\lambda}, \mathbb{Q})$. Let $\text{Gr}_{\lambda}$ be a stratum in the closure of $\text{Gr}_{\lambda}$. Let $i_{\underline{\lambda}}: \text{Gr}_{\underline{\lambda}} \hookrightarrow \text{Gr}_{\lambda}$ be the natural embedding. Then $i_{\underline{\lambda}}^*\mathcal{F} \in D^{\leq -\dim \text{Gr}_{\underline{\lambda}} - 2}(\text{Gr}_{\lambda})$. Indeed, we use $i_{\underline{\lambda}}^*j_!\mathcal{Q} = 0$, and $H^p_{\text{per}}j_!\mathcal{Q}_{\dim \text{Gr}_{\lambda}} = 0$ for $p > 0$ and apply $i_{\underline{\lambda}}^*$ to the exact triangle

$$\longrightarrow \tau^{\leq -1}_{\text{per}}(j_!\mathcal{Q}[\dim \text{Gr}_{\lambda}]) \to j_!\mathcal{Q}[\dim \text{Gr}_{\lambda}] \to H^0_{\text{per}}(j_!\mathcal{Q}[\dim \text{Gr}_{\lambda}]) \to \ldots.$$
Parametrization of a canonical basis. Since the group $B(O)$ is connected, the projection
$$p : Gr^0 \Delta \rightarrow B(O) \setminus Gr^0 \Delta = \text{Conf}(Gr^{m+1}, B) \Delta$$
identifies the top components. So Theorem 2.34 tells that the cycles $p^{-1}(M^\mu_l)$, $l \in B^\mu_n$, see (137), are the top components of $Gr^\mu \Delta$. Theorem 2.38 plus (141) implies that they give rise to classes $[p^{-1}(M^\mu_l)] \in V^\mu \Delta$. Moreover, the $\mu$ is the weight of the class in $V^\mu \Delta$. So we get the following result.

Theorem 2.46. The set $B^\mu_n$ parametrises a canonical basis in the weight $\mu$ part $V^\mu(O)$ of the representation $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}$ of $GL^n$. This basis is given by the classes $[p^{-1}(M^\mu_l)]$, $l \in B^\mu_n$.

3 The potential $\mathcal{W}$ in special coordinates for $GL_m$

3.1 Potential for $\text{Conf}_3(A)$ and Knutson-Tao’s rhombus inequalities

Recall that a flag $F$ for $GL_m$ is a collection of subspaces in an $m$-dimensional vector space $V_m$:

$$F = F_0 \subset F_1 \subset \ldots \subset F_{m-1} \subset F_m, \quad \dim F_i = i.$$ A decorated flag for $GL_m$ is a flag $F_\bullet$ with a choice of non-zero vectors $f_i \subset F_i/F_{i-1}$ for each $i = 1, \ldots, m$, called decorations. It determines a collection of decomposable $k$-vectors

$$f_{(1)} := f_1, \quad f_{(2)} := f_1 \wedge f_2, \quad \ldots, \quad f_{(m)} := f_1 \wedge \ldots \wedge f_m.$$ A decorated flag is determined by a collection of decomposable $k$-vectors such that each divides the next one. A linear basis $(f_1, \ldots, f_m)$ in the space $V_m$ determines a decorated flag by setting $F_k := (f_1, \ldots, f_k)$, and taking the projections of $f_k$ to $F_k/F_{k-1}$ to be the decorations.

Recall the notion of an $m$-triangulation of a triangle [FG1 Section 9]. It is a graph whose vertices are parametrized by a set

$$\Gamma_m := \{(a, b, c) \mid a + b + c = m, \quad a, b, c \in \mathbb{Z}_{\geq 0}\}.$$ Let $(F, G, H) \in \text{Conf}_3(A)$ be a generic configuration of three decorated flags, described by a triple of linear bases in the space $V_m$:

$$F = (f_1, \ldots, f_m), \quad G = (g_1, \ldots, g_m), \quad H = (h_1, \ldots, h_m).$$ Let $\omega \in \text{det} V_m^*$ be a volume form. Then each vertex $(a, b, c) \in \Gamma_m$ gives rise to a function

$$\Delta_{a,b,c} := (f(a) \wedge g(b) \wedge h(c), \omega).$$ There is a one dimensional space $L_{a,b,c} := F_{a+1} \cap (G_b \oplus H_c)$.

Let $e_{a,b,c} \in L_{a,b,c}$ such that $e_{a,b,c}^2 - f_{a+1} \in F_a$. It is easy to see that $e_{a,b,c}^{b+1,c-1} - e_{a,b,c}^{b+1,c} \in L_{a,c-1}$. Therefore there exists a unique scalar $a_{a,b,c}^{b,c}$ such that $e_{a,b,c}^{b+1,c-1} - e_{a,b,c}^{b+1,c} = a_{a,b,c}^{b,c} e_{a,c-1}^{b,c}$.

Lemma 3.1. One has

$$a_{a,b,c}^{b,c} = \frac{\Delta_{a-1,b+1,c} \Delta_{a+1,b,c-1}}{\Delta_{a,b,c} \Delta_{a,b+1,c-1}}.$$ Proof. Set

$$\alpha := a_{a,b,c}^{b,c}, \quad \beta := \frac{\Delta_{a,b,c}}{\Delta_{a+1,b,c}}, \quad \gamma := \frac{\Delta_{a+1,b+1,c-1}}{\Delta_{a+1,b,c-1}}.$$ By definition,

$$f(a) = f(a-1) \wedge e_{a-1}^{b,c+1},$$
$$f(a+1) = f(a) \wedge e_{a,c} = f(a) \wedge e_{a-1,c+1},$$
$$g(b) \wedge h(c) = \beta e_{a,c} \wedge g(b) \wedge h(c),$$
$$g(b+1) \wedge h(c-1) = \gamma e_{a,c}^{b+1,c-1} \wedge g(b) \wedge h(c-1).$$
Therefore,
\[ g(b+1) \land h(c) = \gamma e_a^{b+1,c-1} \land g(b) \land h(c) \]
\[ = \beta \gamma e_a^{b+1,c-1} \land e_a^{b,c} \land g(b) \land h(c-1) \]
\[ = \beta \gamma (e_a^{b+1,c-1} - e_a^{b,c}) \land e_a^{b,c} \land g(b) \land h(c-1) \]
\[ = \beta \gamma \alpha e_a^{b+1,c} \land e_a^{b,c} \land g(b) \land h(c-1). \]

So
\[ f(a-1) \land g(b+1) \land h(c) = \alpha \beta \gamma f(a+1) \land g(b) \land h(c-1). \]

Therefore,
\[ \alpha \beta \gamma = \Delta_a^{-1,b+1,c} \Delta_a^{-1,b,c} \Delta_a^{-c+1,b,c}. \]

Go back to (150), the Lemma is proved.

As shown on Fig 28 each zig-zag path \( p \) provides a basis \( E_p \) for \( F \). For example,
\[ E_l := \{ e_0^{0,n}, e_1^{0,n-1}, \ldots, e_{n-1}^{0,1} \}, \quad E_r := \{ e_0^{n,0}, e_1^{n-1,1}, \ldots, e_{n-1}^{1,0} \} \]
are the bases provided by the very left and very right paths.

Given two zig-zag paths, say \( p \) and \( q \), there is a unique unipotent element \( u_{pq} \) stabilizing \( F \), transforming \( E_p \) to \( E_q \). Recall the character \( \chi_F \) in section 1. For each triple \((p,q,r)\) of zig-zag paths, we have
\[ \chi_F(u_{pq}) = -\chi_F(u_{qp}), \]
\[ \chi_F(u_{pr}) = \chi_F(u_{pq}) + \chi_F(u_{qr}). \]

If \( p, q \) are adjacent paths, see the right of Fig 28 then by Lemma 3.1
\[ \chi_F(u_{pq}) = \alpha_{a}^{b,c} = \frac{\Delta^{-1,b+1,c}}{\Delta^{-b,c} \Delta^{-1,b,c}}. \]

One can transform the very left path to the very right by a sequence of adjacent paths. Let \( u \in U_F \) transform \( E_l \) to \( E_r \). Then
\[ \chi_F(u) = \sum_{(a,b,c) \in \Gamma_m, a \neq 0, c \neq 0} \alpha_{a}^{b,c} = \sum_{(a,b,c) \in \Gamma_m, a \neq 0, c \neq 0} \frac{\Delta^{-1,b+1,c} \Delta^{a+1,b,c}}{\Delta^{-b,c} \Delta^{-1,b,c}}. \]
Its tropicalization

\[ \chi_F^t = \min_{(a,b,c) \in \Gamma, a \neq 0, c \neq 0} \left\{ \Delta_{a-1,b+1,c}^t + \Delta_{a+1,b,c-1}^t - \Delta_{a,b,c}^t - \Delta_{a-1,b,c}^t - \Delta_{a,b,c+1}^t \right\} \]

delivers 1/3 of Knutson-Tao rhombus inequalities. Clearly, same holds for the other two directions. By definition,

\[ \mathcal{W}(F,G,H) = \chi_F + \chi_G + \chi_H. \]

Our set Conf^+_3(A)(\mathbb{Z}) coincides with the set of hives in [KT].

Figure 29: Calculating the potential \( W \) on Conf(\( A, A, B \)) in the special coordinates for GL_4.

In Sections 3.2-3.3 we show that the potential on the space Conf(\( A, A, B \)) for GL_m, written in the special coordinates there, recovers Givental’s potential and, after tropicalization, Gelfand-Tsetlin’s patterns.

### 3.2 The potential for Conf(\( A, A, B \)) and Givental’s potential for GL_m

Let \( G = GL_m \). Recall the set \( \Gamma_m \), see (148). For each triple \( (a,b,c) \in \Gamma_m \), there is a canonical function \( \Delta_{a,b,c} : \text{Conf}_3(\mathcal{A}) \to \mathbb{A}^1 \). Consider the functions \( \Delta_{a,b,c} \) with \( (a,b,c) \in \Gamma_m - (0,0,m) \), illustrated by the \( \bullet \)-vertices on Fig 30. For each triple \( (a,b,c) \in \Gamma_m - 1 \), let us set

\[ R_{a,b,c} := \frac{\Delta_{a+1,b,c}}{\Delta_{a,b+1,c}}. \]

The functions \( R_{a,b,c} \) are assigned naturally to the \( \circ \)-vertices on Fig 30. Each of them is the ratio of the \( \Delta \)-functions at the ends of the slant edge centered at a \( \circ \)-vertex. They are functions on Conf(\( A, A, B \)) since \( R_{a,b,c}(A_1,A_2,A_3 \cdot h) = R_{a,b,c}(A_1,A_2,A_3) \) for any \( h \in H \). The functions \( R_{a,b,c} \) form a coordinate system on Conf(\( A, A, B \)), referred to as the special coordinate system.

The functions \( \{R_{a,b,0}\} \) provide the canonical map

\[ \text{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B}) \rightarrow \text{Conf}(\mathcal{A}, \mathcal{A}) = (\mathbb{G}_m)^{m-1}. \]
Consider now the Givental quiver $\Gamma_{m-1}$, whose vertices are the $\circ$-vertices, parametrised by the set $\Gamma_{m-1}$, with the arrows going down and to the right, as shown on Fig 30. For each arrow connecting two vertices, take the source/tail ratio of the corresponding functions. For example, see Fig 29, the vertical arrow $\alpha$ connecting $(a+1, b-1, c)$ and $(a, b-1, c+1)$ provides

$$Q_{\alpha} = \frac{\Delta_{a,b,c+1} \Delta_{a+2,b-1,c}}{\Delta_{a,b-1,c+1} \Delta_{a+1,b,c}}.$$  \hspace{1cm} (153)

Recall the function $\chi_{A_1}$, $\chi_{A_2}$ on $\text{Conf}(A, A, B)$. Taking the sum of $Q_{\alpha}$ over the vertical arrows $\alpha$, and a similar sum over the horizontal arrows $\beta$, and using (153), we get

$$\chi_{A_1} = \sum_{\alpha \text{ vertical}} Q_{\alpha}, \quad \chi_{A_2} = \sum_{\beta \text{ horizontal}} Q_{\beta}.$$  

Relating to Givental’s work. Givental [Gi2], pages 3-4, introduced parameters $T_{i,j}$, $0 \leq i \leq j \leq m$, matching the vertices of the Givental quiver:

$$\begin{array}{cccc}
T_{00} & T_{01} & T_{02} & T_{03} \\
T_{10} & T_{11} & T_{12} & T_{13} \\
T_{20} & T_{21} & T_{22} & T_{23} \\
T_{30} & T_{31} & T_{32} & T_{33}
\end{array}$$

He treats the entries on the main diagonal $a = (T_{00}, T_{11}, \ldots, T_{m,m})$ as parameters, and defines the potential as a sum over the oriented edges of the quiver:

$$W_a = \sum_{0 \leq i < j \leq m} \left( \exp(T_{i,j} - T_{i,j-1}) + \exp(T_{i,j} - T_{i+1,j}) \right).$$

Let $Y_a$ be the subvariety with a given value of $a$. Then Givental’s integral is

$$F(a, \hbar) = \int_{Y_a} \exp(-W_a/\hbar) \prod_{i=1}^{n} \prod_{j=0}^{i-1} dT_{i,j}.$$  

Givental’s variables $T_{i,j}$ match our coordinates $R_{a,b,c}$ where $a + b + c = m - 1$:

$$R_{m-i-1,j-i} = \exp(T_{i,j}).$$

Observe that $Y_a$ is the fiber of the map [152] over a point $a = (R_{m-1,0}, R_{m-2,1}, \ldots, R_{0,m-1})$. Givental’s potential coincides with $\chi_{A_1} + \chi_{A_2}$. Givental’s volume form on $Y_a$ coincides, up to a sign, with ours since

$$\left( \prod_{i=1}^{n} \prod_{j=0}^{i-1} dT_{i,j} \right) = \pm \prod_{\text{a+b+c=m-1, c}>0} d\log R_{a,b,c}.$$  

3.3 The potential for $\text{Conf}(A, A, B)$ and Gelfand-Tsetlin’s patterns for $\text{GL}_m$

Gelfand-Tsetlin’s patterns for $\text{GL}_m$ [GT1] are arrays of integers $\{p_{i,j}\}$, $1 \leq i \leq j \leq m$, such that

$$p_{i,j+1} \leq p_{i,j} \leq p_{i+1,j+1}.$$  \hspace{1cm} (154)

Theorem 3.2. The special coordinate system on $\text{Conf}(A_{\text{GL}_m}, A_{\text{GL}_m}, B_{\text{GL}_m})$ together with the potential $W = \chi_{A_1} + \chi_{A_2}$ provide a canonical isomorphism

$$\{\text{Gelfand-Tsetlin’s patterns for GL}_m\} = \text{Conf}^+(A_{\text{GL}_m}, A_{\text{GL}_m}, B_{\text{GL}_m})(\mathbb{Z}^t).$$

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Figure 31: Gelfand-Tsetlin patterns for GL₄ and the special coordinates for Conf( пом, пом).

Proof. The space Conf( пом, пом) of GLₘ-orbits on пом × detₘ* has dimension \( \frac{(m+1)(m+2)}{2} \). It has a coordinate system given by the functions \( \Delta_{a,b,c} \), \( a + b + c = m \), parametrized by the vertices of the graph \( \Gamma_m \), shown on the left of Fig 31. The coordinates on Conf( пом, пом, пом) are parametrized by the edges \( E \) of the graph parallel to the edge \( A_1A_2 \) of the triangle. They are little red segments on the right of Fig 31. They are given by the ratios of the coordinates at the ends of the edge \( E \), recovering formula (151). Notice that the edges \( E \) are oriented by the orientation of the side \( A_1A_2 \). The monomials of the potential \( \chi_{A_1} + \chi_{A_2} \) are parametrized by the blue edges, that is by the edges of the graph inside of the triangle parallel to either side \( B_3A_1 \) or \( B_3A_2 \). We claim that the monomials of potential \( \chi_{A_1} + \chi_{A_2} \) are in bijection with Gelfand-Tsetlin’s inequalities. Indeed, a typical pair of inequalities (154) is encoded by a part of the graph shown on Fig 32. The three coordinates \( (P_1, P_2, Q) \) on Conf( пом, пом, пом) assigned to the red edges are expressed via the coordinates \( (A, B, C, D, F) \) at the vertices:

\[ P_1 = \frac{B}{A}, \quad P_2 = \frac{C}{B}, \quad Q = \frac{E}{D}. \]

The monomials of the potential at the two blue edges are \( \frac{EA}{DB} \) and \( \frac{DC}{EB} \). Their tropicalization delivers the inequalities \( p_1 \leq q, q \leq p_2 \).

4 Proof of Theorem 2.11

Let \( T \) be a split torus. Let \( g := \sum_{\alpha \in X^*(T)} g_\alpha X^\alpha \) be a nonzero positive polynomial on \( T \), i.e. its coefficients \( g_\alpha \geq 0 \) are non-negative. The integral tropical points \( l \in T(\mathbb{Z}^I) = X_*^I(T) \) are cocharacters of \( T \). The tropicalization of \( g \) is a piecewise linear function on \( T(\mathbb{Z}^I) \):

\[ g^I(l) = \min_{\alpha \mid g_\alpha > 0} \{ \langle l, \alpha \rangle \}. \]

Fix an \( l \in T(\mathbb{Z}^I) \). Set

\[ \Lambda_{g,l} := \{ \alpha \in X^I(T) \mid g_\alpha > 0, \langle l, \alpha \rangle = g^I(l) \}, \quad g_l := \sum_{\alpha \in \Lambda_{g,l}} g_\alpha X^\alpha. \]

The set \( \Lambda_{g,l} \) is non-empty. Therefore \( g_l \) is a nonzero positive polynomial. If \( f \) and \( g \) are two such polynomials, so is the product \( f \cdot g \). We have \( (f \cdot g)_l = f_l \cdot g_l \) for all \( l \in T(\mathbb{Z}^I) \).
Let \( h \) be a nonzero positive rational function on \( T \). It can be expressed as a ratio \( f/g \) of two nonzero positive polynomials. Set \( h_1 := f/g \). Let \( h = f'/g' \) be another expression. Then

\[
f/g = f'/g' \implies f \cdot g' = f' \cdot g \implies f_i \cdot g_i = f'_i \cdot g_i \implies f_i/g_i = f'_i/g'_i.
\]

Hence \( h_1 \) is well defined.

**Lemma 4.1.** Let \( h, l \) be as above. For each \( C \in T_l \) such that \( h_l(\text{in}(C)) \in \mathbb{C}^* \), we have

\[
\text{val}(h(C)) = h^l(l), \quad \text{in}(h(C)) = h_l(\text{in}(C)). \tag{155}
\]

**Proof.** Assume that \( h \) is a nonzero positive polynomial. By definition

\[
\forall C \in T_l, \quad h(C) = h_l(\text{in}(C))^{h^l(l)} + \text{terms with higher valuation}.
\]

If \( h_l(\text{in}(C)) \in \mathbb{C}^* \), then (155) follows. The argument for a positive rational function is similar. \( \square \)

Let \( f = (f_1, \ldots, f_k) : T \to S \) be a positive birational isomorphism of split tori. Let \( l \in T(\mathbb{Z}^l) \). We generalize the above construction by setting \( f_i := (f_{1,i}, \ldots, f_{k,i}) : T \to S \).

**Lemma 4.2.** Let \( f, l \) be as above. Let \( C \in T^?_l \). Then

\[
\text{val}(f(C)) = f^l(l), \quad \text{in}(f(C)) = f_l(\text{in}(C)). \tag{156}
\]

Let \( h \) be a nonzero positive rational function on \( S \). Then

\[
\text{in}(h \circ f(C)) = h_{f^l(l)}(\text{in}(f(C))). \tag{157}
\]

**Proof.** Here (156) follows directly from Lemma 4.1. Note that \( h_{f^l(l)} \circ f_l \) is a nonzero positive rational function on \( T \). Since \( C \) is transcendental, we get

\[
h_{f^l(l)}(\text{in}(f(C))) = h_{f^l(l)} \circ f_l(\text{in}(C)) \in \mathbb{C}^*.
\]

Thus (157) follows from Lemma 4.1. \( \square \)

**Proof of Theorem 2.11.** It suffices to prove \( f(T^?_l) \subseteq S^?_{f^l(l)} \). The other direction is the same.

Let \( C = (C_1, \ldots, C_k) \in T^?_l \). Let \( f(C) := (C'_1, \ldots, C'_k) \). By (156), we get \( f(C) \in S^?_{f^l(l)} \) and the field extension \( \mathbb{Q}(\text{in}(C'_1), \ldots, \text{in}(C'_k)) \subseteq \mathbb{Q}(\text{in}(C_1), \ldots, \text{in}(C_k)) \).

Let \( g = (g_1, \ldots, g_k) : S \to T \) be the inverse morphism of \( f \). Then \( C_j = g_j \circ f(C) \) for \( j \in [1, k] \). The functions \( g_j \) are nonzero positive rational functions on \( S \). Therefore

\[
\text{in}(C_j) = \text{in}(g_j \circ f(C)) \implies g_j \circ f^l(l)(\text{in}(f(C))) \in \mathbb{Q}(\text{in}(C'_1), \ldots, \text{in}(C'_k)).
\]

Therefore \( \mathbb{Q}(\text{in}(C_1), \ldots, \text{in}(C_k)) \subseteq \mathbb{Q}(\text{in}(C'_1), \ldots, \text{in}(C'_k)) \). Summarizing, we get

\[
\mathbb{Q}(\text{in}(C_1), \ldots, \text{in}(C_k)) = \mathbb{Q}(\text{in}(C'_1), \ldots, \text{in}(C'_k)). \tag{158}
\]

Therefore \( f(C) \) is transcendental. Theorem 2.11 is proved.

## 5 Positive structures on the unipotent subgroups \( U \) and \( U^- \)

### 5.1 Lusztig’s data and MV cycles

**Lusztig’s data.** Fix a reduced word \( i = (i_m, \ldots, i_1) \) for \( w_0 \). There are positive functions

\[
F_{i,j} : U \to \mathbb{A}^1, \quad x_{i_m}(a_m) \ldots x_{i_1}(a_1) \mapsto a_j. \tag{159}
\]

\( \text{Every transcendental point } C \in T^?_l \text{ automatically satisfies such conditions.} \)
Their tropicalizations induce an isomorphism $f_1 : U(Z^t) \xrightarrow{\sim} Z^m$, $p \mapsto \{F_{i,j}^t(p)\}$.

Let $N = \{0, 1, 2, \ldots\}$. Lusztig proved \[\text{Lemma 5.1}\] that the subset

$f_i^{-1}(N^m) \subset U(Z^t)$

(160)

does not depend on $i$, and parametrizes the canonical basis in the quantum enveloping algebra of the Lie algebra of a maximal unipotent subgroup of the Langlands dual group $G^L$.

**Lemma 5.1.** The subset $U^+(Z^t) := \{l \in U(Z^t) \mid \chi^t(l) \geq 0\}$ is identified with the set \[\text{Lusztig's set parametrizing the canonical basis of weight } \lambda \text{ in the quantum enveloping algebra of the Lie algebra of } G.\]

Proof. Note that $\chi = \sum_{j=1}^m F_{i,j}$. It tropicalization is $\min_{1 \leq j \leq m} \{F_{i,j}^t\}$. Let $l \in U(Z^t)$. Then

$\chi^t(l) \geq 0 \iff F_{i,j}^t(l) \geq 0, \forall j \in [1, m] \iff f_i(l) \in N^m$.  

\[\square\]

Let $l \in U(Z^t)$. Recall the transcendental cell $C^t \subset U(K)$.

**Lemma 5.2.** Let $u \in C^t$. Then $u \in U(O)$ if and only if $l \in U^+(Z^t)$.

Proof. Set $u = x_{i_{m}}(a_{m}) \ldots x_{i_{1}}(a_{1}) \in C^t$. Note that $u$ is transcendental. Using \[\text{Lemma 2.13}\] we get

$\chi^t(l) = \val(\chi(u)); \quad F_{i,j}^t(l) = \val(a_j), \forall j \in [1, m].$

If $l \in U^+(Z^t)$, then $\val(a_j) = F_{i,j}^t(l) \geq 0$. Therefore $a_j \in O$. Hence $u \in U(O)$.

Note that $\chi$ is a regular function of $U$. So $\chi(0) \in U(O)$. Therefore $\chi^t(l) = \val(\chi(u)) \geq 0$. Hence $l \in U^+(Z^t)$.

\[\square\]

**The positive morphism $\beta$.** Let $[g]_0 := h$ if $g = u_hh$, where $u_h \in U^\pm$, $h \in H$. Define

$\beta : U \rightarrow H, \quad u \mapsto [\overline{\pi}u]_0.$

(161)

Let $i = (i_m, \ldots, i_1)$ as above. Let $w_k^i := s_{i_1} \ldots s_{i_k} \in W$. Let $\beta_k^i := w_{k-1}^i(\alpha_k^\vee) \in P$. The following Lemma shows that $\beta$ is a positive map.

**Lemma 5.3** ([KZ, Lemma 6.4]). For each $u = x_{i_{m}}(a_{m}) \ldots x_{i_{1}}(a_{1}) \in U$, we have $[\overline{\pi}u]_0 = \prod_{k=1}^m \beta_k^i(a_k^{-1})$.

Let $l \in U(Z^t)$. The tropicalization $\beta^t$ becomes $\beta^t(l) = -\sum_{k=1}^m F_{i,k}^t(l)\beta_k^{-1}$.

Note that $\beta_k \in P$ are positive coroots. If $l \in U^+(Z^t)$, then $-\beta^t(l) \in R^+$. Hence

$U^+(Z^t) = \bigsqcup_{\lambda \in R^+} A^\lambda. \quad A^\lambda := \{l \in U^+(Z^t) \mid -\beta^t(l) = \lambda\}. $

(162)

The set $A^\lambda$ is identified with Lusztig’s set parametrizing the canonical basis of weight $\lambda$ \[\text{Lusztig's data (i.e. } U^+(Z^t) \text{ in our set-up) and the set of stable MV cycles.} \]

Let us briefly recall Kamnitzer’s result for future use.

Let $U_+ := U \cap B^{-w_0B}$. There is an well-defined isomorphism

$\eta : U_+ \rightarrow U_+, \quad u \mapsto \eta(u).$

(163)

such that $\eta(u)$ is the unique element in $U^{-} \cap Bw_0B$. The map $\eta$ was used in \[\text{FZ}\]. Set

$\kappa_{\text{Kam}} : U_+(K) \rightarrow \text{Gr}, \quad u \mapsto [\eta(u)]$.  

(164)

Let $l \in U(Z^t)$. Then $C^t \subset U_+(K)$. Define

$\text{MV}_i := \kappa_{\text{Kam}}(C^t) \subset \text{Gr}$.  

(165)

The following Theorem is a reformulation of Kamnitzer’s result.

**Theorem 5.4** ([K, Theorem 4.5]). Let $l \in A^\lambda$. Then $\text{MV}_i$ is an MV cycle of coweight $\lambda$. It gives a bijection between $A^\lambda$ and the set of such MV cycles.

A stable MV cycle of coweight $\lambda$ has a unique representative of coweight $(\lambda, 0)$. Therefore Theorem 5.4 tells that the set $A^\lambda$ parametrizes the set of stable MV cycles of coweight $\lambda$.  

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5.2 Positive functions $\chi_i, \mathcal{L}_i, \mathcal{R}_i$ on $U$.

Let $i \in I$. We introduce positive rational functions $\chi_i, \mathcal{L}_i, \mathcal{R}_i$ on $U$, and $\chi_i^-, \mathcal{L}_i^-, \mathcal{R}_i^-$ on $U^-$.

Let $i = (i_1, \ldots, i_m)$ be a reduced word for $w_0$. Let

$$x = x_{i_1}(a_1) \cdots x_{i_m}(a_m) \in U, \quad y = y_{i_1}(b_1) \cdots y_{i_m}(b_m) \in U^-.$$ 

Using above decompositions of $x$ and $y$, we set

$$\chi_i(x) := \sum_{p \mid i_p = i} a_p, \quad \chi_i^-(y) := \sum_{p \mid i_p = i} b_p.$$ 

By definition the characters $\chi$ and $\chi^-$ have decompositions $\chi = \sum_{i \in I} \chi_i$ and $\chi^- = \sum_{i \in I} \chi_i^-$. We take $i$ which starts from $i_1 = i$. Define the “left” functions:

$$\mathcal{L}_i(x) := a_1, \quad \mathcal{L}_i^-(y) := b_1.$$ 

We take $i$ which ends by $i_m = i$. Define the “right” functions:

$$\mathcal{R}_i(x) := a_m, \quad \mathcal{R}_i^-(y) := b_m.$$ 

It is easy to see that the above functions are well-defined and independent of $i$ chosen.

For each simple reflection $s_i \in W$, set $s_i^*$ such that $w_0 s_i^* = s_i w_0$. Set $\text{Ad}_w(g) := vgv^{-1}$. For any $u \in U$, set $\tilde{u} := \text{Ad}_m(u^{-1}) \in U^-$.

Lemma 5.5. The map $u \mapsto \tilde{u}$ is a positive birational isomorphism from $U$ to $U^-$. Moreover,

$$\chi_i(u) = \chi_i^-(\tilde{u}), \quad \mathcal{L}_i(u) = \mathcal{R}_i^-(\tilde{u}), \quad \mathcal{R}_i(u) = \mathcal{L}_i^-(\tilde{u}) \quad \forall i \in I. \tag{166}$$

Proof. Note that $\text{Ad}_m \left(x_i(a)\right) = y_i^-(a)$. Let $u = x_{i_1}(a_1) \cdots x_{i_m}(a_m) \in U$. Then

$$\tilde{u} = \text{Ad}_m(u^{-1}) = y_{i_m}^-(a_m) \cdots y_{i_1}^-(a_1).$$

Clearly it is a positive birational isomorphism. Identities in (166) follow by definition. \hfill $\Box$

Lemma 5.6. Let $h \in H$, $x \in U$ and $y \in U^-$. For any $i \in I$, we have

$$\chi_i(\text{Ad}_h(x)) = \chi_i(x) \cdot \alpha_i(h), \quad \mathcal{L}_i(\text{Ad}_h(x)) = \mathcal{L}_i(x) \cdot \alpha_i(h), \quad \mathcal{R}_i(\text{Ad}_h(x)) = \mathcal{R}_i(x) \cdot \alpha_i(h). \tag{167}$$

$$\chi_i^-(\text{Ad}_h(y)) = \chi_i^-(y) / \alpha_i(h), \quad \mathcal{L}_i^-(\text{Ad}_h(y)) = \mathcal{L}_i^-(y) / \alpha_i(h), \quad \mathcal{R}_i^-(\text{Ad}_h(y)) = \mathcal{R}_i^-(y) / \alpha_i(h). \tag{168}$$

Proof. Follows from the identities $\text{Ad}_h(x_i(a)) = x_i(a \alpha_i(h))$ and $\text{Ad}_h(y_i(a)) = y_i(a / \alpha_i(h))$. \hfill $\Box$

5.3 The positive morphisms $\Phi$ and $\eta$

We show that each $\chi_i$ is closely related to $\mathcal{L}_i^-$ by the following morphism.

Definition 5.7. There exists a unique morphism $\Phi : U^- \rightarrow U$ such that

$$u_-B = \Phi(u_-)w_0B. \tag{169}$$

Lemma 5.8. For each $i \in I$, one has

$$\frac{1}{\mathcal{L}_i} = \chi_i \circ \Phi, \quad \frac{1}{\chi_i^-} = \mathcal{L}_i \circ \Phi \tag{170}$$
Example. Let $G = \text{SL}_3$. We have
\[ y = y_1(b_1)y_2(b_2)y_1(b_3) = y_2(b_2b_3/b_1 + b_3)y_1(b_1 + b_3)y_2(b_1b_2/b_1 + b_3). \]
\[ \Phi(y) = x_1(1/b_1 + b_3)x_2(1/b_1)x_1(1/b_1) = x_1(b_1 + b_3), \]
\[ 1/L^{-1}_1(y) = \chi_1(\Phi(y)) = \frac{1}{b_1}, \quad 1/L^{-2}_2(y) = \chi_2(\Phi(y)) = \frac{b_1 + b_3}{b_2b_3}. \]
\[ 1/\chi_1^{-1}(y) = L_1(\Phi(y)) = b_1 + b_3, \quad 1/\chi^{-2}_2(y) = L_2(\Phi(y)) = b_1b_2. \]

The proof was suggested by the proof of Proposition 3.2 of [L2].

Proof. We prove the first formula. The second follows similarly by considering the inverse morphism $\Phi^{-1} : U \to U^+$ such that $uB^- = \Phi^{-1}(u)v_0B$.

Let $i \in I$. Let $w \in W$ such that its length $l(w) < l(s_iw)$. We use two basic identities:
\[ y_i(b)x_i(a) = x_i(a/(1 + ab))y_i(b(1 + ab))\alpha_i^\vee(1/(1 + ab)). \quad (171) \]
\[ y_i(b)wB = x_i(1/b)s_iwB. \quad (172) \]

By (172), one can change $y_i(b)$ on the most right to $x_i(1/b)$. By (171), one can “move” $y_i(b)$ from left to the right. After finite steps, we get
\[ y_{i_1}(b_1)y_{i_2}(b_2)\ldots y_{i_m}(b_m)B = y_{i_1}(b_1)x_{i_1}(a_m)x_{i_{m-1}}(a_{m-1})\ldots x_{i_2}(a_2)s_{i_2}\ldots s_{i_m}B. \quad (173) \]

The last step is to move the very left term $y_{i_1}(b_1)$ to the right. Let
\[ f_s(c_1, c_2, \ldots, c_m) = x_{i_1}(c_m)x_{i_{m-1}}(c_{m-1})\ldots x_{i_2}(c_2)s_{i_2}\ldots s_{i_m}B. \]

We will need the relations between $\{c_i\}$ and $\{c'_i\}$ such that
\[ f_s(c_1, c_2, \ldots, c_m) = f_{s-1}(c'_1, c'_2, \ldots, c'_m). \]

By (171)–(172), if $i_1 \neq i_s$, then $c_p = c'_p$ for all $p$. If $i_1 = i_s$, then
\[ c'_p = c_p \quad \text{for } p = s + 1, \ldots, m; \]
\[ c'_s = c_s/(1 + c_1c_s), \quad c'_s = c_1(1 + c_1c_s); \]
\[ c'_p = c_p(1 + c_1c_s)^{- (\alpha_i^\vee, \alpha_p)} \quad \text{for } p = 2, \ldots, s - 1. \]

For each $q = f_s(c_1, c_2, \ldots, c_m)$, we set
\[ h(q) := \frac{1}{c_1} + \sum_{p \mid i_p=i_1, \ p>s} c_p. \quad (174) \]

If $i_s = i_1$, then
\[ \frac{1}{c_1} + \sum_{p \mid i_p=i_1, \ p>s-1} c'_p = \frac{1}{c_1(1 + c_1c_s)} + \frac{c_s}{1 + c_1c_s} + \sum_{p \mid i_p=i_1, \ p>s} c_p = \frac{1}{c_1} + \sum_{p \mid i_p=i_1, \ p>s} c_p. \]

Same is true for $i_s \neq i_1$. Therefore the function (174) does not depend on $s$.

Back to (173), we have
\[ uB = y_{i_1}(b_1)y_{i_2}(b_2)\ldots y_{i_m}(b_m)B \]
\[ = y_{i_1}(b_1)x_{i_1}(a_m)\ldots x_{i_2}(a_2)s_{i_2}\ldots s_{i_m}B \]
\[ = x_{i_1}(c_m)\ldots x_{i_2}(c_2)y_{i_1}(c_1)s_{i_2}\ldots s_{i_m}B \]
\[ = x_{i_1}(c_m)\ldots x_{i_2}(c_2)x_{i_1}(1/c_1)s_{i_2}\ldots s_{i_m}B \]
\[ = \Phi(u)v_0B \]

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Hence $\Phi(u) = x_i(a_1)x_2(1/a_1)$. Then
\[
\chi_{x_i}(\Phi(u)) = \frac{1}{c_1} + \sum_{p \mid i_p = i_1, p > 1} c_p = h(uB) = \frac{1}{b_1} = \frac{1}{L_{c_1}(u)}. 
\]

**Lemma 5.9.** The morphism $\Phi : U^+ \to U$ is a positive birational isomorphism with respect to Lusztig’s positive atlases on $U^+$ and $U$.

**Proof.** According to the algorithm in the proof of Lemma 5.8 clearly $\Phi$ is a positive morphism. By the same argument, one can show that $\Phi^{-1}$ is a positive morphism. The Lemma is proved.

The morphism $\eta$ in (163) is the right hand side version of $\Phi$, i.e. $B^−u = B^−w_0\eta(u)$. Similarly,

**Lemma 5.10.** The morphism $\eta : U \to U^−$ is a positive birational isomorphism. Moreover,
\[
\forall i \in I, \quad 1/R_i = \chi^−_i \circ \eta, \quad 1/\chi_i = R^−_i \circ \eta.
\]

### 5.4 Birational isomorphisms $\phi_i$ of $U$

Let $i \in I$. Define
\[
z_i(a) := \alpha_i^+(a)y_i(-a), \quad z_i^+(a) := \alpha_i^+(1/a)y_i(1/a).
\]

Clearly $z_i(a)z_i^+(a) = 1$.

**Lemma–Construction 5.11.** There is a birational isomorphism
\[
\phi_i : U \overset{\sim}{\longrightarrow} U, \quad u \mapsto \bar{s}_i \cdot u \cdot z_i(\chi_i(u)).
\]

**Remark.** The map $\phi_i$ is not a positive birational isomorphism.

**Proof.** We need the following identities:
\[
z_i(a)x_i(a)z_i(a) = x_i(-1/a). \tag{177}
\]
\[
z_i^+(a)x_i(b-a)z_i(b) = x_i(1/a - 1/b). \tag{178}
\]

If $j \neq i$, then
\[
z_i^+(a)x_j(b)z_i(a) = x_j(ba^{-\langle \alpha_i^+, \alpha_j \rangle}). \tag{179}
\]

Let $i = (i_1, i_2, \ldots, i_m)$ be a reduced word for $w_0$ such that $i_1 = i$. For each $s \in [1, m]$, define
\[
I_s^i := \{ p \in [1, s] \mid i_p = i \}.
\]

Let $u = x_{i_1}(a_1) \cdots x_{i_m}(a_m) \in U$. Set $d_s := \sum_{k \in I_s^i} a_k$. In particular, $d_1 = a_1$, $d_m = \chi_i(u)$.

Let us assume that $u \in U$ is generic, so that $d_s \neq 0$ for all $s \in [1, m]$. By (177), (179), we get
\[
\phi_i(u) = \bar{s}_i \cdot x_i(a_1)x_{i_2}(a_2) \cdots x_{i_m}(a_m) \cdot z_i(\chi_i(u))
\]
\[
= \bar{s}_i \cdot x_i(a_1)z_i(d_1) \cdot z_i^+(d_1)x_{i_2}(a_2)z_i(d_2) \cdots \cdot (z_i^+(d_{m-1})x_{i_m}(a_m)z_i(d_m))
\]
\[
= x_{i_1}(a'_1)x_{i_2}(a'_2) \cdots x_{i_m}(a'_m). \tag{180}
\]

Here $a'_1 = -d_1$. For $s > 1$,
\[
a'_s = \begin{cases} 
1/d_{s-1} - 1/d_s, & \text{if } i_s = i, \\
\frac{a_s}{d_s - \langle \alpha_i^+, \alpha_i \rangle}, & \text{if } i_s \neq i.
\end{cases} \tag{181}
\]

Thus $\phi_i(u) \in U$. The map $\phi_i$ is well-defined. By (181), we have $\chi_i(\phi_i(u)) = -1/\chi_i(u)$. Therefore
\[
\phi_i \circ \phi_i(u) = \bar{s}_i \cdot \bar{s}_i \cdot u \cdot z_i(\chi_i(u)) \cdot z_i(-1/\chi_i(u)) = \bar{s}_i^2 \cdot u \cdot \bar{s}_i^2.
\]

Since $\bar{s}_i^4 = 1$, we get $\phi_i^4 = id$. Therefore $\phi_i$ is birational. \hfill \square
Let $\lambda \in \mathbb{P}^+$. Recall $t^\lambda \in \text{Gr}$. Recall the $G(O)$-orbit $\text{Gr}_\lambda$ of $t^\lambda$ in $\text{Gr}$.

**Lemma 5.12.** Let $l \in U(\mathbb{Z}^t)$. For any $u \in \mathbb{C}_l^*$, the element $u \cdot t^\lambda \in \overline{\text{Gr}}_\lambda$ if and only if $l \in U^+(\mathbb{Z}^t)$.

**Proof.** If $l \in U^+(\mathbb{Z}^t)$, by Lemma 5.2 we see that $u \in U(\mathcal{O})$. Hence $u \cdot t^\lambda \in \overline{\text{Gr}}_\lambda$.

If $\chi^t(l) = \min_{i \in I} \{\chi_i^t(l)\} < 0$, then pick $i$ such that $\chi_i^t(l) < 0$. Set $\mu := \lambda - \chi_i^t(l) \cdot \alpha_i^\vee$. Since $y_i(t(\alpha_i^\vee, \lambda)/\chi_i(u)) \in G(O)$, we get

$$z_i^t(\chi_i(u)) \cdot t^\lambda = \alpha_i^\vee(1/\chi_i(u)) \cdot t^\lambda \cdot y_i(t(\alpha_i^\vee, \lambda)/\chi_i(u)) = \alpha_i^\vee(1/\chi_i(u)) \cdot t^\lambda = t^\mu.$$  

Recall the $U_w(K)$-orbit $S^s_w$ of $t^\nu$ in $\text{Gr}$. We have

$$u \cdot t^\lambda = uz_i(\chi_i(u)) \cdot z_i^t(\chi_i(u)) t^\lambda \overset{\text{LEMMA 6.1}}{=} uz_i(\chi_i(u)) \cdot t^\nu = uz_i(\chi_i(u)) \cdot t^\nu = s_i t^\nu(\mu) \in S^s(\nu).$$  

It is well-known that the intersection $S^s_w \cap \overline{\text{Gr}}_\lambda$ is nonempty if and only if $t^\nu \in \overline{\text{Gr}}_\lambda$. In this case $t^\nu(\mu) \notin \overline{\text{Gr}}_\lambda$. Therefore $S^s(\nu) \cap \overline{\text{Gr}}_\lambda$ is empty. Hence $u \cdot t^\lambda \notin \overline{\text{Gr}}_\lambda$. \hfill \Box

### 6 A positive structure on the configuration space $\text{Conf}_1(\mathcal{A}; \mathcal{B})$

#### 6.1 Left $G$-torsors

Let $G$ be a group. Let $X$ be a left principal homogeneous $G$-space, also known as a left $G$-torsor. Then for any $x, y \in X$ there exists a unique $g_{x,y} \in G$ such that $x = g_{x,y}y$. Clearly,

$$g_{x,y}g_{y,z} = g_{x,z}, \quad g_{g_{x,y}, y} = g_{g_{x,y}, y}g_{x,y}g_{x,y}^{-1}, \quad g \in G.$$  

(184)

Given a reference point $p \in X$, one defines a “$p$-distance from $x$ to $y$”:

$$g_p(x, y) := g_{p, x}g_{y, p} \in G.$$  

(185)

If $i_p : X \to G$ is a unique isomorphism of $G$-sets such that $i_p(p) = e$, then $g_p(x, y) = i_p(x)^{-1}i_p(y)$.

**Lemma 6.1.** One has:

$$g_p(x, y)g_p(y, z) = g_p(x, z).$$  

(186)

$$g_p(g_{x, y}) = g_p(x, y), \quad g \in G.$$  

(187)

$$y = g_p(p, y) \cdot p.$$  

(188)

**Proof.** Indeed,

$$g_p(x, y)g_p(y, z) = g_{p, x}g_{y, p}g_{p, y} g_{z, p} = g_{p, x}g_{z, p} = g_p(x, z),$$

$$g_p(g_{x, y}) = g_{p, x}g_{y, p} = g_{p, x}g_{y, p}^{-1}g_{y, p} = g_{p, x}g_{y, p} = g_p(x, y),$$

$$y = g_{y, p} \cdot p = g_{p, y, p} \cdot p = g_p(p, y) \cdot p.$$  

\hfill \Box

Recall $\mathcal{F}_G$ in Definition 2.2. From now on, we apply the above construction in the set-up

$$X = \mathcal{F}_G, \quad p = \{\text{U, B}^-\}.$$  

Pick a collection $\{A_1, ..., A_n\}$ representing a configuration in $\text{Conf}_n(\mathcal{A})$. We assign $A_i$ to the vertices of a convex $n$-gon, so that they go clockwise around the polygon. Each oriented pair $\{A_i, A_j\}$ provides a frame $\{A_i, B_j\}$, shown on Fig 6.33 by an arrow with a white dot.

#### 6.2 Basic invariants associated to a generic configuration

We introduce several invariants that will be useful in the rest of this paper. We employ $\cdot$ to denote the action of $G$ on (decorated) flags.
The invariant $u_{B_1,B_3}^{A_2} \in U$. Let $(B_1, A_2, B_3) \in \text{Conf}(\mathcal{B}, \mathcal{A}, \mathcal{B})$ be a generic configuration. Set

$$u_{B_1,B_3}^{A_2} := g_{\{U,B^-\}}(\{A_2, B_1\}, \{A_2, B_3\}).$$

(189)

By (187), the invariant $u_{B_1,B_3}^{A_2}$ is independent of the representative chosen. Clearly, $u_{B_1,B_2}^{A_1} \in U$.

The invariant $h_{A_1,A_2} \in H$. Let $(A_1, A_2)$ be a generic configuration. There is a unique element $h_{A_1,A_2} \in H$ such that

$$(A_1, A_2) = (U, h_{A_1,A_2} \overline{w}_0 \cdot U).$$

(190)

Using the notation (185), we have

$$h_{A_1,A_2} \overline{w}_0 = g_{\{U,B^-\}}(\{A_1, B_2\}, \{A_2, B_1\}).$$

(191)

The invariant $b_{B_3}^{A_1,A_2} \in B^-$. Let $(A_1, A_2, B_3)$ be a generic configuration. Define

$$b_{B_3}^{A_1,A_2} := g_{\{U,B^-\}}(\{A_1, B_3\}, \{A_2, B_3\}) \in B^-.$$

Relations between basic invariants. Let $(A_1, \ldots, A_n) \in \text{Conf}_n^*(\mathcal{A})$. Set

$$h_{ij} := h_{A_i,A_j} \in H, \quad u_{jk}^i := u_{B_i,B_k}^{A_j} \in U, \quad b_{B_k}^{A_i,A_j} \in B^-.$$

(192)

We denote these invariants by dashed arrows, see Fig. 34.

![Diagram of invariants of a configuration](image-url)
Lemma 6.2. The data (192) satisfy the following relations:

1. $h_{12}w_0h_{21}w_0 = 1$.
2. $u_{13}u_{34} = u_{14}$, in particular $u_{23}u_{32} = 1$.
3. $b_4^{12}b_4^{23} = b_4^{13}$.
4. $b_4^{13} = u_{13}h_{12}w_0u_{13} = h_{13}w_0u_{12}w_0^{-1}h_{23}^{-1}$.
5. $u_{13}h_{12}w_0u_{13}h_{23}w_0u_{21}h_{31}w_0 = 1$.

Proof. We prove the first identity of 4. The others follow similarly. Let $P = \{U, B^-\}$. Let $x_1 = \{A_1, B_3\}$, $x_2 = \{A_1, B_2\}$, $x_3 = \{A_2, B_1\}$, $x_4 = \{A_2, B_3\}$.

As illustrated by Figure 34,

$$b_{32}^{12} = g_{p(x_1, x_4)}$, $u_{32} = g_{p(x_1, x_2)}$, $h_{12}w_0 = g_{p(x_2, x_3)}$, $u_{13}^2 = g_{p(x_3, x_4)}$.

By (193), we get $g_{p(x_1, x_4)} = g_{p(x_1, x_2)}g_{p(x_2, x_3)}g_{p(x_3, x_4)}$.

Lemma 6.3. Let $x \in \text{Conf}(A, A, B)$ be a generic configuration. Then it has a unique representative $\{A_1, A_2, B_3\}$ with $\{A_1, B_3\} = \{U, B^-\}$. Such a representative is

$$\{U, u_{32}h_{12}w_0 \cdot U, B^-\}.$$ (193)

Proof. The existence and uniqueness are clear. It remains to show that it is (193). By Fig 35

$$g_{p(U, B^-)}(\{A_1, B_3\}, \{A_2, B_1\}) = u_{32}h_{12}w_0.$$ (194)

If $\{A_1, B_3\} = \{U, B^-\}$, then by (188), we get

$$\{A_2, B_1\} = g_{p(U, B^-)}(\{A_1, B_3\}, \{A_2, B_1\}) \cdot \{U, B^-\} = \{u_{32}h_{12}w_0 \cdot U, B\}.$$
Each $b \in B^-$ can be decomposed as $b = y_l \cdot h = h \cdot y_r$, where $h \in H$, $y_l, y_r \in U^-$. Thus $B^-$ has a positive structure induced by positive structures on $U^-$ and $H$. There are three positive maps

$$\pi_l, \pi_r : B^- \rightarrow U^-, \; \pi_h : B^- \rightarrow H, \; \pi_l(b) = y_l, \; \pi_r(b) = y_r, \; \pi_h(b) = h. \quad (195)$$

These maps give rise to three more invariants.

**The invariant** $\mu_{B_3}^{A_1, A_2} \in H$. For each generic $(A_1, A_2, B_3)$, we define

$$\mu_{B_3}^{A_1, A_2} := \pi_h(b_{B_3}^{A_1, A_2}). \quad (196)$$

**The invariant** $r_{B_3}^{B_1, A_2} \in U^-$. For any $h \in H$, we have

$$b_{B_3}^{A_1, h^{-1}, A_2} = h \cdot b_{B_3}^{A_1, A_2} \quad (197)$$

Thus we can define

$$r_{B_3}^{B_1, A_2} := \pi_r(b_{B_3}^{A_1, h^{-1}, A_2}) = \pi_r(b_{B_3}^{A_1, A_2}) \in U^- \quad (198)$$

**The invariant** $l_{B_3}^{A_1, B_2} \in U^-$. For any $h \in H$, we have

$$b_{B_3}^{A_1, A_2 - h} = b_{B_3}^{A_1, A_2} \cdot h \quad (199)$$

Define

$$l_{B_3}^{A_1, B_2} := \pi_l(b_{B_3}^{A_1, A_2 - h}) = \pi_l(b_{B_3}^{A_1, A_2}) \in U^- \quad (200)$$

For simplicity, we set

$$\mu^{i,j}_k := \mu_{B_k}^{A_i, A_j} \in H, \; r^{i,j}_k := r_{B_k}^{B_i, A_j} \in U^-, \; l^{i,j}_k := l_{B_k}^{A_i, B_j} \in U^- \quad (201)$$

Recall that $\tilde{u} = \overline{w}_0 \overline{u}^{-1} \overline{w}_0^{-1}$. By Relations 3, 4 of Lemma 6.2, we get

$$\mu_4^{l, 23} = \mu_4^{13}. \quad (202)$$

$$b_3^{l, 12} = t_3^{l, 12} \mu_3^{l, 12} = \mu_3^{r, 12} = u_3^{l, 12} h_1^{l, 12} \overline{w}_0 u_3^{l, 12} = h_1^{l, 12} h_1^{l, 12} \quad (203)$$

Recall the morphisms $\Phi$, $\eta$ and $\beta$ in Section 5. By the definition of these morphisms, we get

**Lemma 6.4.** We have

1. $u_{32}^{l, 12} = \Phi(t_3^{l, 12})$.
2. $r_3^{l, 12} = \eta(u_3^{l, 12})$.
3. $\tilde{u}_3^{l, 12} = \mathrm{Ad}_{h_2^{l, 12}}(l_3^{l, 12}) = \mathrm{Ad}_{h_2^{l, 12}}(r_3^{l, 12})$.
4. $\mu_3^{l, 12} = h_1^{l, 12} \beta(u_3^{l, 12}) = h_1^{l, 12} h_2^{l, 12} h_2^{l, 12}$.

**Proof.** By (203), we have

$$l_3^{l, 12} \mu_3^{l, 12} = u_{32}^{l, 12} (h_1^{l, 12} \overline{w}_0 u_3^{l, 12}).$$

The first identity follows. Similarly, the second identity follows from

$$\mu_3^{l, 12} r_3^{l, 12} = (u_3^{l, 12} h_1^{l, 12} \overline{w}_0) u_3^{l, 12}$$

The third identity follows from

$$l_3^{l, 12} \mu_3^{l, 12} = h_1^{l, 12} u_3^{l, 12} h_2^{l, 12} = \mathrm{Ad}_{h_2^{l, 12}}(u_3^{l, 12}) h_1^{l, 12} h_2^{l, 12}, \quad \mu_3^{l, 12} r_3^{l, 12} = h_1^{l, 12} h_2^{l, 12} \mathrm{Ad}_{h_2^{l, 12}}(r_3^{l, 12}).$$
The identity \( \mu_3^{12} = h_{12} \beta(u_{13}^2) \) follows from
\[
\mu_3^{12} \iota_3^{12} = u_{42}^1 h_{12} \cdot (\overline{w_0 u_{13}^2}).
\]

The identity \( \mu_3^{12} = h_{13} h_{23}^{-1} \) follows from
\[
\iota_3^{12} \mu_3^{12} = \text{Ad}_{h_{13}}(u_{21}^3) h_{13} h_{23}^{-1}.
\]

\[\square\]

**Lemma 6.5.** We have
\[
\chi(u_{21}^3) = \sum_{i \in I} \frac{\alpha_i(h_{13})}{\mathcal{L}_i(u_{32}^1)} = \sum_{i \in I} \frac{\alpha_i(h_{21})}{\mathcal{K}_i(u_{13}^1)}.
\] (204)
\[
\alpha_i(h_{12}) = \alpha_{i'}(h_{21}), \quad \forall i \in I.
\] (205)

**Proof.** Use Lemmas 5.5, 6.4, 5.6 and 5.8, we get
\[
\chi(u_{21}^3) = \chi_{-}(u_{21}^3) = \chi_{-}^{\text{Ad}}_{h_{13}}(l_{13}^{12}) = \sum_{i \in I} \alpha_i(h_{13}) \chi_i^{l_{13}^{12}} = \sum_{i \in I} \frac{\alpha_i(h_{13})}{\mathcal{L}_i(u_{32}^1)}.
\]

By the same argument, we get the other identity in (204). By Relation 1 of Lemma 5.8 we get
\[
h_{12} = \overline{w_0 h_{21}^{-1} w_0^{-1} \cdot s_G}.
\]

Then (205) follows. \[\square\]

### 6.3 A positive structure on \( \text{Conf}_I(\mathcal{A}; \mathcal{B}) \)

Let \( I \subset [1, n] \) be a nonempty subset of cardinality \( m \). Following [FG1] Section 8, there is a positive structure on the configuration space \( \text{Conf}_I(\mathcal{A}; \mathcal{B}) \). We briefly recall it below.

Let \( x = (x_1, \ldots, x_n) \in \text{Conf}_I(\mathcal{A}; \mathcal{B}) \) be a generic configuration such that
\[
x_i = A_i \in \mathcal{A} \text{ when } i \in I, \text{ otherwise } x_i = B_i \in \mathcal{B}.
\] (206)

Set \( B_j := \pi(A_j) \) when \( j \in I \). Let \( i \in I \). For each \( k \in [2, n] \), set
\[
u^i_k(x) := u^A_{i'k}, \quad \text{where the subscript is modulo } n.
\] (207)

For each pair \( i, j \in I \), recall
\[
\pi_{ij}(x) := \begin{cases} h_{A_i A_j}, & \text{if } i < j; \\ h_{s_G A_i A_j}, & \text{if } i > j. \\
\end{cases}
\] (208)

**Lemma 6.6.** Fix \( i \in I \). The following morphism is birational
\[
\alpha_i : \text{Conf}_I(\mathcal{A}; \mathcal{B}) \to H^{m-1} \times U^{n-2}, \quad x \mapsto (\{\pi_{ij}(x)\}, \{u^i_k(x)\}), \quad j \in I - \{i\}, \ k \in [2, n - 1].
\]

**Example.** Fig. [70] illustrates the map \( \alpha_1 \) for \( I = \{1, 3, 5\} \subset [1, 6] \).

**Proof.** Assume that \( i = 1 \in I \). Clearly \( \alpha_1 \) is well defined on the subspace
\[
\overline{\text{Conf}_I(\mathcal{A}; \mathcal{B})} := \{(x_1, \ldots, x_n) \mid (x_1, x_k) \text{ is generic for all } k \in [2, n]\}.
\]

Note that \( \overline{\text{Conf}_I(\mathcal{A}; \mathcal{B})} \) is dense in \( \text{Conf}_I(\mathcal{A}; \mathcal{B}) \). We prove the Lemma by showing that \( \alpha_1 \) is a bijection from \( \overline{\text{Conf}_I(\mathcal{A}; \mathcal{B})} \) to \( H^{m-1} \times U^{n-2} \).

Let \( y = (\{h_j\}, \{u_k\}) \in H^{m-1} \times U^{n-2} \). Set \( u_n := 1 \). Set \( u^i_k := u_{i-1} \ldots u_k \) for \( k \in [2, n - 1] \). Let \( x = (x_1, \ldots, x_n) \in \overline{\text{Conf}_I(\mathcal{A}; \mathcal{B})} \) such that
\[
x_1 := U; \quad x_j := u^i_{h_j} \overline{w_0} \cdot U \in \mathcal{A}, \quad j \in I - \{1\}; \quad x_k := u^i_k \cdot B^{-} \in \mathcal{B}, \ k \notin I.
\] (209)
Clearly $\alpha_1(x) = y$. Hence $\alpha_1$ is a surjection.

Let $x \in \text{Conf}_I(A; B)$ such that $\alpha_1(x) = y$. Note that $x$ has a unique representative $\{x_1, \ldots, x_n\}$ such that $\{x_1, x_n\} = \{U, B\}$ if $n \notin I$, and $\{x_1, \pi(x_n)\} = \{U, B^-\}$ if $n \in I$. By Lemma 6.3 each $x_i$ is uniquely expressed by (209). The injectivity of $\alpha_1$ follows.

The product $H^{m-1} \times U^{n-2}$ has a positive structure induced by the ones on $H$ and $U$.

When $I = [1, n]$, we first introduce a positive structure on $\text{Conf}_n(A; B)$ such that the map $\alpha_1$ is a positive birational isomorphism. Such a positive structure is twisted cyclic invariant:

**Theorem 6.7** ([FG1 Section 8]). The following map is a positive birational isomorphism

$$t : \text{Conf}_n(A) \xrightarrow{\sim} \text{Conf}_n(A), \quad (A_1, \ldots, A_n) \mapsto (A_2, \ldots, A_n, A_1 \cdot s_G).$$

Each $\alpha_i$ determines a positive structure on $\text{Conf}_n(A; B)$. Theorem 6.7 tells us that these positive structures coincide. We prove the same result for $\text{Conf}_I(A; B)$, using the following Lemmas.

**Lemma 6.8.** Let $Y$ be a space equipped with two positive structures denoted by $Y_1$ and $Y_2$. If for every rational function $f$ on $Y_1$ we have

$$f \text{ is positive on } Y_1 \iff f \text{ is positive on } Y_2,$$

then $Y_1$ and $Y_2$ share the same positive structure.

**Proof.** It is clear.

**Lemma 6.9.** Let $Y, Z$ be a pair of positive spaces. If there are two positive maps $\gamma : Y \to Z$ and $\beta : Z \to Y$ such that $\beta \circ \gamma = \text{id}_Y$, then for every rational function $f$ on $Y$ we have

$$f \text{ is positive on } Y \iff \beta^*(f) \text{ is positive on } Z.$$

**Proof.** If $f$ is positive on $Y$, since $\beta$ is a positive morphism, then $\beta^*(f)$ is positive on $Z$.

If $\beta^*(f)$ is positive on $Z$, since $\gamma$ is a positive morphism, then $\gamma(\beta^*(f)) = f$ is positive.

**Lemma 6.10.** Every $\alpha_i$ $(i \in I)$ determines the same positive structure on $\text{Conf}_I(A; B)$.

**Remark.** Lemma 6.10 is equivalent to say that for any pair $i, j \in I$, the map $\phi_{i,j} := \alpha_i \circ \alpha_j^{-1}$ is a positive birational isomorphism of $H^{m-1} \times U^{n-2}$.

**Proof.** Let us temporary denote the positive structure on $\text{Conf}_I(A; \gamma)$ by $\text{Conf}_I(A; B)$ such that $\alpha_i$ is a positive birational isomorphism.

There is a projection $\beta : \text{Conf}_n(A) \to \text{Conf}_I(A; B)$ which maps $A_k$ to $A_k$ if $k \in I$ and maps $A_k$ to $\pi(A_k)$ otherwise. By Lemma 6.7 $\beta$ is a positive morphism for all $\text{Conf}_I(A; B)$. 

Figure 36: The map $\alpha_1$ for $I = \{1, 3, 5\} \subset [1, 6]$. 
Fix \( i \in I \). Each generic \( x = (x_1, \ldots, x_n) \in \text{Conf}_1(A; B) \) has a unique preimage \( \gamma^i(x) := (A_1, \ldots, A_n) \in \text{Conf}_n(A) \) such that

\[
A_j = x_j \text{ when } j \in I, \text{ otherwise } A_j \text{ is the preimage of } x_j \text{ such that } \pi_{ij}(\gamma^i(x)) = 1.
\]

Clearly \( \gamma^i \) a positive morphism from \( \text{Conf}_1(A; B) \) to \( \text{Conf}_n(A) \). By definition \( \beta \circ \gamma^i = \text{id} \).

Let \( f \) be a rational function on \( \text{Conf}_1(A; B) \). Let \( i, j \in I \). By Lemma 6.8

\[
f \text{ is positive on } \text{Conf}_1(A; B) \iff \beta^*(f) \text{ is positive on } \text{Conf}_n(A) \iff f \text{ is positive on } \text{Conf}_1(A; B).
\]

This Lemma follows from Lemma 6.10. \( \square \)

Thanks to Lemma 6.10 we introduce a canonical positive structure on \( \text{Conf}_1(A; B) \). From now on, we view \( \text{Conf}_1(A; B) \) as a positive space.

Given \( k \in \mathbb{Z}/n \), we define the \( k \)-shift of the subset \( I \) by setting \( I(k) := \{i \in [1, n] \mid i+k \in I\} \). The following Lemma is clear now.

**Lemma 6.11.** The following map is a positive birational isomorphism

\[
t : \text{Conf}_1(A; B) \xrightarrow{\sim} \text{Conf}_{1(1)}(A; B), \quad (x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n, x_1 \cdot s_G).
\]

**An invariant definition of positive structures.** We have defined above positive structures on the configuration spaces using pinning in \( G \), which allows to make calculations. Let us explain now how to define positive structures on the configurations spaces without choosing a pinning. When \( G \) is of type \( A_m \), such a definition is given in [FG1, Section 9]. In general, given a decomposition of the longest Weyl group element \( w_0 = s_{i_1} \ldots s_{i_n} \), for each generic pair \( \{B, B'\} \) of flags, there exists a unique chain

\[
B = B_0 \xrightarrow{i_1} B_1 \xrightarrow{i_2} \ldots \xrightarrow{i_{n-1}} B_{n-1} \xrightarrow{i_n} B_n = B'.
\]

Here \( B_{k-1} \xrightarrow{i_k} B_k \) indicates that \( \{B_{k-1}, B_k\} \) is in the position \( s_{i_k} \). The positive structure of \( \text{Conf}(B, A, B) \) can be defined via the birational map

\[
\text{Conf}(B, A, B) \to (G_m)^n, \quad (B, A, B') \mapsto (\chi^\gamma(B_0, A, B_1), \chi^\gamma(B_1, A, B_2), \ldots, \chi^\gamma(B_{n-1}, A, B_n)).
\]

Each generic pair \( \{A, A'\} \in \mathcal{A}^2 \) uniquely determines a pinning for \( G \) such that

\[
x_i(a) \in U_A, \quad \chi_A(x_i(a)) = a, \quad y_i(a) \in U_{A'}, \quad i \in I.
\]

The pinning gives rise to a representative \( \overline{w}_0 \in G \) of \( w_0 \). There is a unique element \( h \in \pi(A) \cap \pi(A') \) such that

\[
A' = h\overline{w}_0 \cdot A.
\]

Such an element \( h \) gives rise to a birational map from \( \text{Conf}_2(A) \) to the Cartan group of \( G \), determining a positive structure of \( \text{Conf}_2(A) \). The positive structures of general configuration spaces are defined via the positive structures of \( \text{Conf}_2(A) \) and \( \text{Conf}(B, A, B) \).

**6.4 Positivity of the potential \( \mathcal{W}_J \) and proof of Theorem 2.27.**

Let \( J \subset I \subset [1, n] \). Consider the ordered triples \( \{i, j, k\} \subset [1, n] \) such that

\[
j \in J, \text{ and } i, j, k \text{ seated clockwise.} \quad (210)
\]

Let \( x \in \text{Conf}_1(A; B) \) be presented by \( \{200\} \). Define \( p_{j;i,k}(x) := u^A_{B_i,B_k} \). In particular, we are interested in the triples \( \{j-1, j, j+1\} \). Set

\[
p_j(x) := p_{j;j-1,j+1} = u^A_{B_j-1,B_{j+1}}, \quad \forall j \in J. \quad (211)
\]
Lemma 6.12. The following morphisms are positive morphisms

1. \( \pi_{ij} : \text{Conf}_1(A; B) \to H, \forall i, j \in I \)

2. \( p_{j;i,k} : \text{Conf}_1(A; B) \to U, \forall \{i, j, k\} \in \{210\} \)

Proof. The positivity of \( \pi_{ij} \) is clear. By Relation 2 of Lemma 6.2, we get

\[
u^{A_j}_{B_i, B_k} = u^{A_j}_{B_i, B_{i-1}} u^{A_j}_{B_{i-1}, B_{i-2}} \cdots u^{A_j}_{B_{k+1}, B_k}.
\]

The product map \( U \times U \to U, (u_1, u_2) \mapsto u_1 u_2 \) is positive. The positivity of \( p_{j;i,k} \) follows.

Positivity of the potential \( W_j \). Recall the positive function \( \chi \) on \( U \). Let \( x \in \text{Conf}_1(A; B) \) be a generic configuration presented by \( (206) \). By Lemma 6.3, each generic triple \((B_j, A_j, B_{j+1})\) has a unique representative \( (B^-, U, u^{A_j}_{B_{j-1}, B_{j+1}} \cdot B^-) \). In this case \( u_j \) in \( (209) \) becomes \( p_j(x) \). Therefore \( \chi_{A_j}(u_j) = \chi \circ p_j(x) \). The potential \( W_j \) of \( \text{Conf}_1(A; B) \) becomes

\[
W_j = \sum_{j \in J} \chi \circ p_j
\]

(212)

Since \( p_j \) are positive morphisms, the positivity of \( W_j \) follows.

By Relation 2 of Lemma 6.2, we get

\[
\chi \circ p_j = \chi \circ p_{j;j-1,i} + \chi \circ p_{j;i,k} + \chi \circ p_{j;k,j+1}
\]

(213)

All summands on right side are positive functions. By \( (212) \), the set \( \text{Conf}^O_{j;1}(A; B)(Z^\ell)(Z^\ell') \) of tropical points such that \( W_j \geq 0 \) is the set

\[
\{l \in \text{Conf}_1(A; B)(Z^\ell) \mid p^l_{j;i,k}(l) \in U^+(Z^\ell) \text{ for all } \{i, j, k\} \in \{210\}\}
\]

(214)

Proof of Theorem 2.27. Recall the moduli space \( \text{Conf}^O_{j;1}(A; B) \) in Definition 2.26

Lemma 6.13. A generic configuration in \( \text{Conf}_1(A; B)(K) \) is \( O \)-integral relative to \( J \) if and only if \( u^{A_j}_{B_i, B_k} \in U(O) \) for all \( \{i, j, k\} \in \{210\} \).

Proof. By definition \( L(A_j, B_k) = [g_{(A_j, B_k)}(U, B^-), \cdots] \in \text{Gr.} \) Let \( \{i, j, k\} \in \{210\} \). Then

\[
L(A_j, B_k) = L(A_j, B_i) \iff g_{(A_j, B_k)}^{-1}(U, B^-) g_{(A_j, B_k)}^{-1}(U, B^-) = u^{A_j}_{B_i, B_k} \in G(O).
\]

The Lemma is proved.

Let \( l \in \text{Conf}_1(A; B) \). Let \( x \in C^\ell_l \) be presented by \( (206) \). By Lemma 6.2, \( u^{A_j}_{B_i, B_k} \in U(O) \) if and only if \( p^l_{j;i,k}(l) \in U^+(Z^\ell) \). Theorem 2.27 follows from Lemma 6.13 and \( (214) \).

Tropicalizing the morphism \( (208) \), we get \( \pi^l_{ij} : \text{Conf}_1(A; B)(Z^\ell) \to H(Z^\ell) = P \).

Lemma 6.14. Let \( i, j \in J \). If \( l \in \text{Conf}^O_{j;1}(A; B)(Z^\ell') \), then \( \pi^l_{ij}(l) \in P^+ \).

Proof. Since \( \pi^l_{ij}(l) = -u_0(\pi^l_{ij}(l)) \), we can assume that there exists \( k \) such that \( \{i, j, k\} \in \{210\} \). Otherwise we switch \( i \) and \( j \). Set \( \lambda := \pi^l_{ij}(l), u_1 := p^l_{j;k,j}(l), u_2 := p^l_{j;i,k}(l) \). We tropicalize \( (204) \):

\[
\chi^l(u_2) = \min_{\ell' \in I} \{\langle \lambda, \alpha_\ell \rangle - R^l_{\ell'}(u_1)\}
\]

(215)

If \( l \in \{214\} \), then \( \chi^l(u_1) \geq 0, \chi^l(u_2) \geq 0 \). By the definition of \( R^l \) and \( \chi \), we get \( R^l(u_1) \geq \chi^l(u_1) \). Therefore \( R^l(u_1) \geq 0 \). Hence

\[
\forall \ell \in I, \quad \langle \lambda, \alpha_\ell \rangle \geq \langle \lambda, \alpha_\ell \rangle - R^l_{\ell'}(u_1) \geq \chi^l(u_2) \geq 0 \implies \lambda \in P^+.
\]

\( \square \)
7 Main examples of configuration spaces

As discussed in Section 1, the pairs of configuration spaces especially important in representation theory are:

\{Conf_\omega(A), Conf_\omega(\text{Gr})\}, \{Conf(\mathcal{A}^n, \mathcal{B}), Conf(\mathcal{G}r^n, \mathcal{B})\}, \{Conf(\mathcal{B}, \mathcal{A}^n, \mathcal{B}), \text{Conf}(\mathcal{B}, \mathcal{G}r^n, \mathcal{B})\}.

In Section 7 we express the potential \( \mathcal{W} \) and the map \( \kappa \) in these cases under explicit coordinates.

7.1 The configuration spaces Conf_\omega(A) and Conf_\omega(\text{Gr})

Recall \( h_{ij}, u_{ij}^k \) in [92]. Recall the positive birational isomorphism

\[ \alpha_1 : \text{Conf}_\omega(A) \xrightarrow{\sim} \mathbb{H}^{n-1} \times \mathbb{U}^{n-2} , \quad (A_1, \ldots, A_n) \mapsto (h_{12}, \ldots, h_{1n}, u_{3,2}^1, \ldots, u_{n,n-1}^1). \]  

The potential \( \mathcal{W} \) on \( \text{Conf}_\omega(A) \) induces a positive function \( \mathcal{W}_{\alpha_1} := \mathcal{W} \circ \alpha_1^{-1} \) on \( \mathbb{H}^{n-1} \times \mathbb{U}^{n-2} \).

Theorem 7.1. The function

\[ \mathcal{W}_{\alpha_1}(h_2, \ldots, h_n, u_2, \ldots, u_{n-1}) = \sum_{j=2}^{n-1} (\chi(u_j) + \sum_{i \in I} \frac{a_i(h_j)}{R_i(u_j)} + \sum_{i \in I} \frac{a_i(h_{j+1})}{\mathcal{L}_i(u_j)}). \]  

Proof. By the scissor congruence invariance [92], we get \( \mathcal{W}(A_1, \ldots, A_n) = \sum_{j=2}^{n-1} \mathcal{W}(A_1, A_j, A_{j+1}) \). The rest follows from [212] and Lemma 6.5.

Let us choose a map without stable points which is not necessarily a bijection:

\[ \alpha : [1, n] \rightarrow [1, n], \quad \alpha(k) \neq k. \]

Let \( x = (A_1, \ldots, A_n) \in \text{Conf}_\omega^\alpha(A) \). Define

\[ \omega_k(x) := [g(\{U, B\}, \{A_1, B_n\}, \{A_k, B_{R(k)}\})] \in \text{Gr}. \]

By the definition of \( \text{Conf}_\omega^\alpha(A) \), the map \( \omega_k \) is independent of the map \( \alpha \) chosen. Define

\[ \omega := (\omega_2, \ldots, \omega_n) : \text{Conf}_\omega^\alpha(A) \rightarrow \text{Gr}^{n-1}, \quad x \mapsto (\omega_2(x), \ldots, \omega_n(x)). \]

Consider the projection

\[ i_1 : \text{Gr}^{n-1} \rightarrow \text{Conf}_\omega(\text{Gr}), \quad \{I_2, \ldots, I_n\} \mapsto ([1], I_2, \ldots, I_n) \]

Lemma 7.2. The map \( \kappa \) in [100] is \( i_1 \circ \omega \).

Proof. Here \( \omega_k(x) = g(\{U, B\}, \{A_1, B_n\}, L(A_k, B_{\alpha(k)})) \). In particular \( \omega_1(x) = [1] \). The Lemma follows.

Below we give two explicit expressions of \( \omega \) based on different choices of the map \( \alpha \). We emphasize that although the expressions look entirely different from each other, they are the same map. As before, set \( x = (A_1, \ldots, A_n) \in \text{Conf}_\omega^\alpha(A) \).

1. Let \( \alpha(k) = k - 1 \). It provides frames \( \{A_1, B_{i-1}\} \), see the first graph of Fig 37. Set

\[ g_k := g(\{U, B\}, \{A_k, B_{k-1}\}, \{A_{k+1}, B_k\}) \ast u_{B_{k-1}, B_{k+1}} A_k, B_{k+1}, \theta_0. \]

See Fig 35 for proof of \( \ast \). By [186], we get

\[ \omega_k(x) = [g(\{U, B\}, \{A_1, B_n\}, \{A_k, B_{k-1}\})] = [g_1 \ldots g_{k-1}], \quad k \in [2, n] \]

Therefore

\[ \omega(x) = ([g_1], \ldots, [g_1 \ldots g_{n-1}]) \in \text{Gr}^{n-1}. \]

2. Let \( \alpha(k) = n \) when \( k \neq n \). Let \( \alpha(n) = 1 \). See the second graph of Fig 37. Set

\[ b_k := b_{A_{k-A_{k+1}}}, \quad k \in [1, n-2]; \quad h_n := h_{A_1, A_n}. \]

Then

\[ \omega_k(x) = [g(\{U, B\}, \{A_1, B_n\}, \{A_k, B_{n}\})] = [b_1 \ldots b_{k-1}], \quad k \in [2, n-1]; \quad \omega_n(x) = [h_n]. \]

Therefore

\[ \omega(x) = ([b_1], \ldots, [b_1 \ldots b_{n-2}], [h_n]) \in \text{Gr}^{n-1}. \]
7.2 The configuration spaces $\text{Conf}(A^n, B)$ and $\text{Conf}(\text{Gr}^n, B)$

Consider the scissoring morphism

$$s : \text{Conf}(A^{m+n+1}, B) \to \text{Conf}(A^{m+1}, B) \times \text{Conf}(A^{n+1}, B),$$

$$(A_1, \ldots, A_{m+n+1}, B_0) \mapsto (A_1, \ldots, A_{m+1}, B_0) \times (A_{m+1}, \ldots, A_{m+n+1}, B_0).$$

By Lemmas 6.6, 6.10, the morphism $s$ is a positive birational isomorphism.

In fact, the inverse map of $s$ can be defined by “gluing” two configurations:

$$s^*: \text{Conf}^*(A^{m+1}, B) \times \text{Conf}^*(A^{n+1}, B) \to \text{Conf}(A^{m+n+1}, B),$$

$$(a, b) \mapsto a \ast b.$$ (226)

By Lemma 6.3, $a$ has a unique representative $\{A_1, \ldots, A_m, U, B^-\}$, $b$ has a unique representative $\{U, A'_1, \ldots, A'_n, B^-\}$. We define the convolution product $a \ast b := (A_1, \ldots, A_m, U, A'_1, \ldots, A'_n, B^-)$. The associativity of the convolution product is clear.

Recall $b_k^{ij}$ in (192). Recall the morphisms $\pi_r, \pi_l$ in (195).

**Theorem 7.3.** The following morphism is a positive birational isomorphism

$$c : \text{Conf}(A^n, B) \to (B^-)^{n-1},$$

$$(A_1, \ldots, A_n, B_{n+1}) \mapsto (b_{n+1}^{1,2}, \ldots, b_{n+1}^{i+1}, \ldots, b_{n+1}^{n-1,n}).$$

**Proof.** Scissoring the convex $(n+1)$-gon along diagonals emanating from $n+1$, see Fig 38, we get a positive birational isomorphism $\text{Conf}(A^n, B) \to (\text{Conf}(A^2, B))^{n-1}$. The Theorem is therefore reduced to $n = 2$. Recall $\alpha_2$ in Lemma 6.6. By Lemma 6.3, it is equivalent to prove that $H \times U \to H \times U^-$, $(h, u) \mapsto (\beta(u)h, \eta(u))$ is a positive birational isomorphism. Since $\eta$ is a positive birational isomorphism, and $\beta$ is a positive map, the Theorem follows.

The potential $W$ on $\text{Conf}(A^n, B)$ induces a positive function $W_c = W \circ c^{-1}$ on $(B^-)^{n-1}$. 

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Lemma 7.4. The function

$$W_c(b_1, \ldots, b_{n-1}) = \sum_{j=1}^{n-1} \sum_{i \in I} \left( \frac{1}{L_i - \pi_l(b_j)} + \frac{1}{R_i - \pi_r(b_j)} \right)$$  \hspace{1cm} (228)$$

Proof. Note that

$$W(A_1, \ldots, A_n, B_{n+1}) = \sum_{j=1}^{n-1} W(A_j, A_{j+1}, B_{n+1}) = \sum_{j=1}^{n-1} \left( \chi(u_{n+1,j+1}) + \chi(u_{j,n+1}^{j+1}) \right).$$

The Lemma follows directly from Lemma 5.8, (175) and Lemma 6.4. \qed

Figure 39: Frames assigned to \((A_1, \ldots, A_n, B_{n+1})\).

Define

$$\tau : \text{Conf}^\mathcal{O}(A^n, B) \rightarrow \text{Gr}^{n-1}, \quad (A_1, \ldots, A_n) \mapsto \{[b_{n+1}^{1,2}], \ldots, [b_{n+1}^{1,n}]\}. \hspace{1cm} (229)$$

Consider the projection

$$\iota_b : \text{Gr}^{n-1} \rightarrow \text{Conf}(\text{Gr}^n, B), \quad \{L_2, \ldots, L_n\} \mapsto ([1], L_2, \ldots, L_n, B^-).$$

Recall the map \(\kappa\) in (112). As illustrated by Fig 39 we get

Lemma 7.5. When \(J = I = [1, n] \subset [1, n+1]\), we have \(\kappa = \iota_b \circ \tau\).

7.3 The configuration spaces \(\text{Conf}(B, A^n, B)\) and \(\text{Conf}(B, \text{Gr}^n, B)\)

Recall \(r^{ij}_k\) in (201). Similarly, there is a positive birational isomorphism

$$p : \text{Conf}(B, A^n, B) \rightarrow U^- \times (B^-)^{n-1}, \quad (B_1, A_2, \ldots, A_{n+1}, B_{n+2}) \mapsto (r^{1,2}_{n+2}, b_{n+2}^{2,3}, \ldots, b_{n+2}^{n,n+1}). \hspace{1cm} (230)$$

The potential \(W\) on \(\text{Conf}(B, A^n, B)\) induces a positive function \(W_p := W \circ p^{-1}\) on \(U^- \times (B^-)^{n-1}\). We have

$$W_p(r_1, b_2, \ldots, b_n) = \sum_{i \in I} \frac{1}{R_i \circ (r_1)} + \sum_{2 \leq j \leq n} \sum_{i \in I} \left( \frac{1}{L_i - \pi_l(b_j)} + \frac{1}{R_i - \pi_r(b_j)} \right). \hspace{1cm} (231)$$

Recall the map \(\kappa\) in (112). Define

$$\tau_\kappa : \text{Conf}^\mathcal{O}_w(A, B^n, A) \rightarrow \text{Gr}^n, \quad (B_1, A_2, \ldots, A_{n+1}, B_{n+2}) \mapsto ([r_{n+2}^{1,2}], r_{n+2}^{1,2} b_{n+2}^{2,3}, \ldots, [r_{n+2}^{1,2} b_{n+2}^{2,n+1}]). \hspace{1cm} (232)$$
Consider the projection

\[ i : Gr^n \to \text{Conf}_{w_0}(B, Gr^n, B), \quad \{L_2, \ldots, L_{n+1}\} \mapsto (B, L_2, \ldots, L_{n+1}, B^-). \]

Let \( x = (B_1, A_2, \ldots, A_{n+1}, B_{n+2}) \in \text{Conf}_{w_0}^o(A, B^n, A). \) Let \( A_i^* \in A \) be the preimage of \( B_1 \) such that \( b_{B_{n+2}}^{A_i^*A_2} = r_{n+2}^{1,2}. \) As illustrated by Fig 40 we get

Lemma 7.6. When \( J = I = [2, n+1] \subset [1, n+2], \) we have \( \kappa = i_s \circ \tau_s. \)

8 Proof of Theorems 2.24 and 2.36

8.1 Lemmas

Let \( \mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_k \) be a product of positive spaces. The positive structure on \( \mathcal{Y} \) is induced by positive structures on \( \mathcal{Y}_i. \) Let \( y_i \in \mathcal{Y}_i^o(K). \) Let \( (y_{i,1}, \ldots, y_{i,n}) \) be the coordinate of \( y_i \) in a positive coordinate system \( c_i. \) Define the field extension

\[ Q(y_1, \ldots, y_k) := Q(\text{in}(y_{1,1}), \ldots, \text{in}(y_{i,n}), \ldots, \text{in}(y_{k,n})). \] (233)

Thanks to [158], such an extension is independent of the positive coordinate systems chosen.

Recall the morphisms \( \pi_s, \pi_r \) in [195].

Lemma 8.1. Fix \( i \in I. \) Let \((b, c) \in (B^- \times \mathbb{G}_m)^o(K). \) Recall \( y_i(c) \in U^- (K). \) Then \( b' := b \cdot y_i(c) \in (B^-)^o(K). \)

Moreover, if \( \text{val}(R_i^- \circ \pi_r(b)) \leq \text{val}(c), \) then \( \text{val}(b') = \text{val}(b) \) and \( Q(b', c) = Q(b, c). \)

Proof. Let \( b = h \cdot y. \) Fix a reduced word for \( w_0 \) which ends with \( i_m = i. \) It provides a decomposition \( y = y_{i_1}(c_1) \ldots y_{i_m}(c_m). \) Then \( b' = h \cdot y_{i_1}(c_1) \ldots y_{i_m}(c_m + c). \) The rest is clear. \( \square \)

Lemma 8.2. Let \((b, h) \in (B^- \times H)^o(K). \) Then \( b' := b \cdot h \in (B^-)^o(K). \)

Moreover, if \( h \in H(K), \) then \( \text{val}(b') = \text{val}(b) \) and \( Q(b', h) = Q(b, h). \)

Proof. Let \( b = y \cdot h_0. \) The rest is clear. \( \square \)

Lemma 8.3. Let \((b, p) \in (B^- \times B^-)^o(K). \) Assume \( p \in B^-(C). \)

1. If \( \text{val}(R_i^- \circ \pi_r(b)) \leq 0 \) for all \( i \in I, \) then \( b \cdot p \) is a transcendental point. Moreover

\[ \text{val}(b \cdot p) = \text{val}(b), \quad Q(b \cdot p, p) = Q(b, p). \]
2. If \( \text{val}(\mathcal{L}_i^- \circ \pi_l(b)) \leq 0 \) for all \( i \in I \), then \( p^{-1} \cdot b \) is a transcendental point. Moreover
\[
\text{val}(p^{-1} \cdot b) = \text{val}(b), \quad \mathbb{Q}(p^{-1} \cdot b, p) = \mathbb{Q}(b, p).
\]

**Proof.** Combining Lemmas 8.1, 8.2, we prove 1. Analogously 2 follows. \( \square \)

8.2 Proof of Theorem 2.36.

Our first task is to prove Theorem 2.36 for the cases when \( I = [1, n] \subset [1, n + 1] \).

Let \( J = \{j_1, \ldots, j_m\} \subset I \). Recall \( \mathcal{W}_j \) in \([13]\). Let \( l \in \text{Conf}(\mathcal{A}^n, \mathcal{B})(\mathbb{Z}) \) be such that \( \mathcal{W}_j(l) \geq 0 \).

Let \( x \in C_\mathcal{B}^\mathcal{P} \). Recall the map \( c \) in Theorem \([76]\). Set \( c(x) := (b_1, \ldots, b_{n-1}) \in (B^-)^{n-1}(\mathcal{K}) \).

**Lemma 8.4.** For every \( i \in I \), we have

1. \( \text{val}(\mathcal{L}_i^- \circ \pi_l(b_j)) \leq 0 \) if \( j \in [1, n - 1] \cap J \),
2. \( \text{val}(\mathcal{R}_i^- \circ \pi_r(b_{k-1})) \leq 0 \) if \( k \in [2, n] \cap J \).

**Proof.** Let \( j \in [1, n - 1] \cap J \). By definition \( b_j = b_{l+1}^{A_j} \cdot b_{l+1}^{A_j+1} \). By Lemmas 8.3, 8.4 we get
\[
\text{val}(\mathcal{L}_i^- \circ \pi_l(b_j)) = -\text{val}(\chi_1(u_{b_{l+1}, B_{l+1}})) \leq -\chi_{A_j}(l) \leq 0.
\]

The second part follows similarly. \( \square \)

As illustrated by Fig 39, we see that
\[
x = (g_1 \cdot U, g_2 \cdot U, \ldots, g_n \cdot U, B^-), \quad g_1 := 1, \quad g_j := b_1 \ldots b_{j-1}, \quad j \in [2, n].
\]

If \( j \in J \), then \( L_j := L(g_j \cdot U, B^-) = \{g_j\} \in \text{Gr} \). Therefore
\[
\kappa(x) = (x_1, \ldots, x_n, B^-), \quad x_j = \begin{cases} 
[g_j]_j \cdot U & \text{if } j \in J, \\
\text{otherwise}
\end{cases}
\]

Let \( \{A_j, \ldots, A_{j_m}\} \in \mathcal{A}_m(\mathcal{C}) \) be a generic point in the sense of algebraic geometry. Define
\[
y := (A'_1, A'_2, \ldots, A'_n, B^-) \in \text{Conf}(\mathcal{A}^n; \mathcal{B}), \quad A'_j = \begin{cases} g_j \cdot A_j & \text{if } j \in J, \\
g_j \cdot U & \text{otherwise}.
\end{cases}
\]

Let \( F \in \mathbb{Q}_+(\text{Conf}(\mathcal{A}^n; \mathcal{B})) \). By the very definition of \( D_F \), we have \( D_F(\kappa(x)) = \text{val}(F(y)) \).

Since \( \{A_j, \ldots, A_{j_m}\} \) is generic, it can be presented by
\[
\{A_1, \ldots, A_{j_m}\} := \{p_1 \cdot U, \ldots, p_{j_m} \cdot U\}, \quad \mathbf{p} = \{p_1, \ldots, p_{j_m}\} \in (B^-)^m(\mathcal{C}).
\]

We can also assume that \((x, \mathbf{p})\) is a transcendental point, so that
\[
(c(x), \mathbf{p}) \in ((B^-)^{m+n-1})^\circ(\mathcal{K}).
\]

Set \( p_j = 1 \) for \( j \notin J \). Keep the same \( p_j \) for \( j \in J \). Then
\[
y = (g_1 p_1 \cdot U, \ldots, g_n p_n \cdot U, B^-); \quad c(y) = (\tilde{b}_1, \ldots, \tilde{b}_{n-1}), \quad \tilde{b}_j := p_j^{-1} b_j p_{j+1} \in B^-(\mathcal{K}).
\]

By Lemmas 8.3, 8.4, we get
\[
\mathbb{Q}(c(x), \mathbf{p}) = \mathbb{Q}(b_1, \ldots, b_{n-1}, p_1, \ldots, p_{j_m}) = \mathbb{Q}(\tilde{b}_1, \ldots, b_{n-1}, p_i, \ldots, p_{j_m}) = \cdots = \mathbb{Q}(\tilde{b}_1, \ldots, b_{n-1}, p_i, \ldots, p_{j_m}) = \mathbb{Q}(c(y), \mathbf{p}).
\]

\[
\text{val}(b_j) = \text{val}(\tilde{b}_j), \quad \forall j \in [1, n - 1].
\]
Recall the map $u: \nu^\circ_9$. Configurations and generalized Mircović-Vilonen cycles. The second identity is due to the special cases discussed before. The rest are by definition.

Proof of Theorem 2.35. Let $l \in \text{Conf}_{\nu^\circ_9}(A; B)(Z')$. There exists $l' \in \text{Conf}(A^n; B)(Z')$ such that $W_1^l(l') \geq 0$ and $d_1^l(l') = l$.

Proof. We prove the case when $J$ contains $\{1, n\}$. In fact, the other cases are easier. Let $x = (A_1, \ldots, A_n, B_{n+1})$. Consider a map $u: \text{Conf}(A^n; B) \to U$ given by $x \mapsto u_{B_{n+1}, B_n}^A$. Then

$$W_1(x) = W_1(A_1, \ldots, A_n) + W(A_1, A_n, B_{n+1}) = W_1(d_1(x)) + \chi(u_{B_{n+1}, B_n}^A) + \chi(u_{B_1, B_{n+1}}^A) = W_1(d_1(x)) + \chi(u(x)) + \sum_{i \in I} \pi_{I, n}(d_1(x)).$$

(238)

By Lemma 6.14 we have $\lambda := \pi_{I, n}(l) \in P^+$. Clearly there exists $l' \in \text{Conf}(A^n; B)(Z')$ such that $d_1^l(l') = l$ and $u^l(l') = 0 \in U(Z')$. We tropicalize (238).

$$W_1^l(l') = \min\{W_1^l(l), \chi^0(0), \min\{\lambda, \alpha_i - R_i^0(0)\}\} = \min\{W_1^l(l), 0, \min\{\lambda, \alpha_i\}\} = 0.$$

Let $l, l'$ be as above. Let $x \in C^\circ_9$. Clearly there exists $z \in C^\circ_9$ such that $d_1(z) = x$. For any $F \in \mathcal{Q}_+(\text{Conf}(A; B))$, we have

$$D_F(\kappa(x)) = D_{F \circ d_1}(\kappa(z)) = (F \circ d_1)^l(l') = F^l \circ d_1^l(l') = F^l(l).$$

The second identity is due to the special cases discussed before. The rest are by definition.

9 Configurations and generalized Mircović-Vilonen cycles

9.1 Proof of Theorem 2.35

In this Section we use extensively the notation from Section 6.2 such as $u_{B_1, B_2}^{A_2}$, $r_{B_1, A_2}^{B_2} \in U^\circ$. We identify the subset $A_{i,} \subset \mathbf{A}_i$ in Theorem 2.35 with the subset $A_{i,} \subset U^\circ(Z^l)$ in (162) by tropicalizing

$$\alpha: \text{Conf}(B, A, B) \to U, \ (B_1, A_2, B_3) \mapsto u_{B_1, B_2}^{A_2}.$$ (239)

Thanks to identity 4 of Lemma 6.4 the index $\nu$ for both definitions match.

Proof of Theorem 2.35 2). Let $l \in A_{i,}$. Let $x = (B_1, A_2, B_3) \in C^\circ_9$. By Lemma 6.4 $r_{B_1, A_2}^{B_2} = \eta(u_{B_1, B_2}^{A_2})$. Recall $\kappa_{Kam}$ in (164). Recall $i_s$ in (129). By Lemma 146 we get

$$\kappa(x) = (B, [i_s^{B_1, A_2}], B^-) = (B, \kappa_{Kam}(u_{B_1, B_2}^{A_2}), B^-) = i_s(\kappa_{Kam}(\alpha(x))).$$ (240)

Recall $MV_l$ in (165). Then $M_l = i_s(MV_l)$. Thus 2) is a reformulation of Theorem 5.4

1). Recall the map

$$p_i: \text{Conf}(A, A, B) \to U, \ (A_1, A_2, B_3) \mapsto u_{B_1, B_2, B_3}^{A_i}, \ i = 1, 2.$$ (241)

Recall the map $\tau$ defined by (229)

$$\tau: \text{Conf}(A, A, B)(K) \to Gr, \ (A_1, A_2, B_3) \mapsto [i_s^{A_1, A_2}].$$ (242)
Note that $p_2^\mu$ induces a bijection from $P_\lambda^\mu$ to $A_{\lambda-\mu}$. The MV cycles of coweight $(\lambda-\mu,0)$ are

$$\kappa_{\text{Kam}} \circ p_2(C_2^\mu) = \kappa_{\text{Kam}} (C_2^\mu(t)) = \text{MV}_{p_2}(t), \quad l \in P_\lambda^\mu.$$ 

Let $x = (A_1, A_2, B_3) \in C_0^\mu$. Note that

$$\tau(x) = [b_{A_1,A_2}] = [b_{B_3, A_2}, b_{B_3, A_1}] = \mu(x) \cdot \kappa_{\text{Kam}}(p_2(x)), \quad [\mu(x) = [b_{A_1,A_2}] = t^\mu.$$ 

We get $\tau(C_0^\mu) = t^\mu \cdot \text{MV}_{p_2}(t)$. They are precisely MV cycles of coweight $(\lambda, \mu)$. Recall the isomorphism $i$ in [124]. Clearly $M_l = i(\tau(C_0^\mu))$. Thus 1) is proved.

3). The set $B_\lambda^\mu$ is a subset of $P_\lambda^\mu$ such that $p_2^\mu(B_\lambda^\mu) \subset U_\lambda^+(Z^l)$. By Lemma [6,11] $B_\lambda^\mu$ is empty unless $\lambda \in P^+$. So we assume $\lambda \in P^+$. Let $l \in P_\lambda^\mu$. Let $x = (A_1, A_2, B_3) \in C_0^\mu$. By Lemma [6,2]

$$\tau(x) = [b_{A_1,A_2}] = [u_{B_3,A_2} b_{A_1,A_2} u_{B_1,B_3}] = p_1(x) \cdot t^\lambda. $$

The last identity is due to $p_2^\mu(l) \in U_\lambda^+(Z^l)$ (hence $u_{B_1,B_3} \in U(\mathcal{O})$).

By Lemma [6.2] $\tau(x) \in Gr_\lambda$ if and only if $p_1^\mu(l) \in U_\lambda^+(Z^l)$. Therefore

$$\tau(C_0^\mu) \subset Gr_\lambda \iff p_1^\mu(l) \in U_\lambda^+(Z^l) \iff l \in B_\lambda^\mu.$$ 

The rest follows from Lemma [7,33].

### 9.2 Proof of Theorems 2.31, 2.33, 2.34

By Theorem 2.33 we have

$$S_{u_0}^\mu \cap S_\mu^\lambda = \bigcup_{t \in P_\lambda^\mu} N_t, \quad S_{u_0}^\mu \cap Gr_\lambda = \bigcup_{t \in B_\lambda^\mu} M_t, $$

where $N_t = \tau(C_0^\mu)$ are MV cycles.

#### Proof of Theorem 2.31

Scissoring the convex $(n+2)$-gon along diagonals emanating from the vertex labelled by $n+2$, see Fig [33]; we get a positive birational isomorphism between $\text{Conf}(\mathcal{A}^{n+1}, \mathcal{B})$ and $(\text{Conf}(\mathcal{A}^2, \mathcal{B}))^\nu$. Its tropicalization provides a decomposition

$$P_\lambda^\mu = \bigcup_{\mu_1 + \ldots + \mu_n = \mu} P_{\lambda_1}^{\mu_1} \times P_{\lambda_2}^{\mu_2} \times \ldots \times P_{\lambda_n}^{\mu_n}, \quad \Delta = (\lambda_2, \ldots, \lambda_n) \in (P^+)^{n-1}. $$

Let $l = (l_1, \ldots, l_n) \in P_\lambda^\mu$. We construct an irreducible subset

$$C_l := \{(b_1, \ldots, b_n) \in Gr^n \mid b_1 \in B^-(K), \quad \{b_1\} \in N_{l_1}, \quad \{b_i\} \in M_{l_i}, \quad i \in \{2, n\}\}.$$ 

By induction, $C_l$ is of dimension $\langle \rho, \lambda + \lambda_2 + \ldots + \lambda_n - \mu \rangle$.

#### Lemma 9.1

Recall the subvariety $Gr_\lambda^\mu$ in $[125]$. We have $Gr_\lambda^\mu = \cup C_l$ where $l \in P_\lambda^\mu$.

**Proof.** Thanks to the isomorphism $B^{-}(K)/B^{-}(\mathcal{O}) \sim Gr$, each $x \in Gr_\lambda^\mu$ can be presented as $([b_1], [b_1 b_2], \ldots, [b_1 \ldots b_n])$, where $b_i \in B^-(K)$ for all $i \in \{1, n\}$. By the definition of $Gr_\lambda^\mu$ we have

$$\{b_i\} \in Gr_{\lambda_i}, \quad \forall i \in \{2, n\}; \quad \{b_1\} \in S_{\mu_1}, \quad \{b_1 \ldots b_n\} \in S_{u_0}^\mu.$$ 

Let $pr : B^{-}(K) \to H(K) \to H(K)/H(\mathcal{O}) = P$ be the composite of standard projections. Set $pr(b_i) := \mu_i$. Then $[b_i] \in S_{u_0}^\mu.$

When $i = 1$, $[b_1] \in S_{u_0}^\mu \cap S_{\mu_1}$. Thus $[b_1] \in N_{l_1}$ for some $l_1 \in P_{\lambda_1}^{\mu_1}$. When $i > 1$, $[b_i] \in S_{u_0}^\mu \cap Gr_{\lambda_i}$. Thus $[b_i] \in M_{l_i}$ for some $l_i \in P_{\lambda_i}^{\mu_i}$. Note that $\mu_1 + \ldots + \mu_n = pr(b_1) + \ldots + pr(b_n) = pr(b_1 \ldots b_n) = \mu$. Thus $l := (l_1, \ldots, l_n) \in P_\lambda^\mu$. By definition $x \in C_l$. Therefore $Gr_\lambda^\mu \subset \cup_{l \in P_\lambda^\mu} C_l$. The other direction follows similarly. □
Let \( l \in \mathbf{P}^n_{\lambda, \omega} \). Recall the map

\[ \tau : \text{Conf}(A^{n+1}, B) \rightarrow \text{Gr}^n, \quad (A_1, \ldots, A_{n+1}, B_{n+2}) \mapsto ([b_{B_1}A_2], \ldots, [b_{B_{n+2}}A_{n+1}]). \]

Clearly \( \tau(C^\mu_{\lambda}) \) is a dense subset of \( C_l \). Recall the isomorphism \( i \) in (124). Following Lemma 7.5, the isomorphism \( i \) identifies \( \tau(C^\mu_{\lambda}) \) with \( \mathcal{M}_l^\mu \). By Theorem 2.36 the cells \( \mathcal{M}_l^\mu \) are disjoint. Theorem 2.31 follows from Lemma 9.1.

**Proof of Theorem 2.33** The group \( H(K) \) acts diagonally on \( \text{Gr}^n \). Let \( h \in H(K) \) be such that \( [h] = t^\mu \). Then \( h \cdot \text{Gr}_{\lambda, \omega} = \text{Gr}_{\lambda + t^\mu \omega} \). One can choose \( h \) such that \( [h] = t^{-\mu} \). The rest follows by the same argument in the proof of Theorem 2.31.

**Proof of Theorem 2.34** By definition \( \textbf{B}^l_{\lambda_1, \lambda_2, \ldots, \lambda_n} \subset \mathbf{P}^n_{\lambda_1, \lambda_2, \ldots, \lambda_n} \). The Theorem follows by the same argument in the proof of Theorem 2.31.

### 9.3 Components of the fibers of convolution morphisms

Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbf{P}^+)^n \). Recall the convolution variety \( \text{Gr}_{\lambda} \) in (122). By the geometric Satake correspondence, \( \text{IH}(\text{Gr}_{\lambda}) = V_{\lambda} := V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \).

Set \( |\lambda| := \lambda_1 + \ldots + \lambda_n \). Set \( ht(|\lambda|, \mu) := (\rho, |\lambda| - \mu) \). The **convolution morphism** \( m_{\lambda} : \text{Gr}_{\lambda} \rightarrow \text{Gr}_{|\lambda|} \) projects \( (L_1, \ldots, L_n) \) to \( L_n \). It is semismall, i.e. for any \( \mu \in \mathbf{P} \) such that \( t^\mu \in \text{Gr}_{|\lambda|} \), the fiber \( m_{\lambda}^{-1} (t^\mu) \) is of top dimension \( ht(|\lambda|, \mu) \). See [MV] for proof.

By the decomposition theorem [BBD], we have

\[
\text{IH}(\text{Gr}_{\lambda}) = \bigoplus_{\mu} F_{\mu} \otimes \text{IH}(\text{Gr}_{|\lambda|, \mu}).
\]

Here the sum is over \( \mu \in \mathbf{P} \) such that \( t^\mu \subseteq \text{Gr}_{|\lambda|} \), and \( F_{\mu} \) is the vector space spanned by the fundamental classes of top dimensional components of \( m_{\lambda}^{-1} (t^\mu) \). As a consequence, the number of top components of \( m_{\lambda}^{-1} (t^\mu) \) equals the tensor product multiplicity \( c_{\lambda, \mu} \) of \( V_{\mu} \) in \( V_{\lambda} \).

Recall the subsets \( C^\mu_{\lambda} \) in (87). By Lemma 6.13 the set \( C^\mu_{\lambda} \) is empty unless \( (\mu, \lambda) \in (\mathbf{P}^+)^{n+1} \). Recall the map \( \omega \) in (219). In this subsection we prove

**Theorem 9.2.** Let \( T^\mu_{\lambda} \) be the set of top components of \( m_{\lambda}^{-1} (t^\mu) \). For each \( l \in C^\mu_{\lambda} \), the closure \( \overline{\omega(C^\mu_{\lambda})} \in T^\mu_{\lambda} \). It gives a bijection between \( C^\mu_{\lambda} \) and \( T^\mu_{\lambda} \).

First we prove the case when \( n = 2 \). In this case, the fiber \( m_{\lambda_1, \lambda_2}^{-1} (t^\mu) \) is isomorphic to \( \{ L \in \text{Gr} | (L, t^\mu) \in \text{Gr}_{\lambda_1, \lambda_2} \} = \text{Gr}_{\lambda_1} \cap t^\mu \text{Gr}_{\lambda_2} \).

Here \( \lambda_2 := -w_0(\lambda_2) \in \mathbf{P}^+ \). The following Theorem is due to Anderson.

**Theorem 9.3 ([A]). The top components of \( \text{Gr}_{\lambda_2} \cap t^\mu \text{Gr}_{\lambda_1} \) are precisely the MV cycles of coweight \( (\lambda_1, \mu - \lambda_2) \) contained in \( \text{Gr}_{\lambda_1} \cap t^\mu \text{Gr}_{\lambda_2} \).**

Recall the positive morphisms

\[ p_1 : \text{Conf}_3(A) \rightarrow U, \quad (A_1, A_2, A_3) \mapsto u_{B_{i-2}, B_{i+2}}, \quad i \in \mathbf{Z} / 3. \]

Let us put the potential condition on two vertices, see the left of Fig 11, getting

\[
\textbf{B}^\mu_{\lambda_1, \lambda_2} := \{ l \in \text{Conf}_3(A)(\mathbf{Z}^l) | (\pi_{12}, \pi_{23}, \pi_{13})^l(l) = (\lambda_1, \lambda_2, \mu), \quad p_1^l(l) \in U^+_X(\mathbf{Z}^l), \quad p_2^l(l) \in U^+_X(\mathbf{Z}^l) \}.
\]
Due to the scissoring congruence invariance, the map \( \pi_3 \) induces a bijection \( \pi_3^l : \tilde{B}_\lambda^\mu \rightarrow B_{\lambda_1}^{\mu - \lambda_2} \).

Consider the projection \( \pi_3 : \text{Conf}_3(\mathcal{A}) \rightarrow \text{Conf}(\mathcal{A}^2, \mathcal{B}) \) which maps \((A_1, A_2, A_3)\) to \((A_1, A_2, B_3)\). Its tropicalization \( \pi_3^t \) induces a bijection \( \tilde{B}_\lambda^\mu \rightarrow B_{\lambda_1}^{\mu - \lambda_2} \). Recall \( \omega_2 \) in (223). By (224), the cycles

\[
\omega_2(C_l) = \tau(C_{\pi_3^l(l)}) = l \in \tilde{B}_\lambda^\mu
\]

are precisely MV cycles of coweight \((\lambda_1, \mu - \lambda_2)\) contained in \( \text{Gr}_{\lambda_1}^\mu \).

Let \( l \in \tilde{B}_\lambda^\mu \). Let \( x = (A_1, A_2, A_3) \in C_l^\mu \). By identity 2 of Lemma 6.2,

\[
\omega_2(x) = [\pi_{13}(x) \overrightarrow{0} \pi_3(x)]^{-1} \pi_{32}(x), \quad \text{where} \quad [\pi_{13}(x)] = t^\mu, \quad [\pi_{32}(x)] = t^{\lambda_1^\vee}.
\]

Therefore

\[
\omega_2(x) \in t^\mu \text{Gr}_{\lambda_1}^\mu \iff t^{-\mu} \omega_2(x) \in \text{Gr}_{\lambda_1}^\mu \iff t^{-\mu} \pi_{13}(x) \overrightarrow{0} \pi_3(x)[(p_3(x))^{-1} \pi_{32}(x)] \in \text{Gr}_{\lambda_1}^\mu \iff (p_3(x))^{-1} \cdot t^{\lambda_1^\vee} \in \text{Gr}_{\lambda_1}^\mu.
\]

Here the last equivalence is due to the fact that \( t^{-\mu} \pi_{13} \overrightarrow{0} \pi_3 \in G(\mathcal{O}) \). Therefore for any \( l \in \tilde{B}_\lambda^\mu \),

\[
\omega_2(C_l^\mu) \subset t^\mu \text{Gr}_{\lambda_1}^\mu \iff (p_3(C_l^\mu))^{-1} \cdot t^{\lambda_1^\vee} \subset \text{Gr}_{\lambda_1}^\mu.
\]

By Lemma 6.2, Lemma 5.12 and the definition of \( C_{\lambda_1, \lambda_2}^\mu \), we get

\[
(p_3(C_l^\mu))^{-1} \cdot t^{\lambda_1^\vee} \in \text{Gr}_{\lambda_1}^\mu \iff (p_3(C_l^\mu))^{-1} \in U(\mathcal{O}) \iff p_3(C_l^\mu) \in U(\mathcal{O}) \iff p_3^\prime(l) \in U_{\lambda_1}^\vee(\mathcal{O}) \iff l \in C_{\lambda_1, \lambda_2}^\mu.
\]

Let \( l \in C_{\lambda_1, \lambda_2}^\mu \). Let \( x = (A_1, A_2, A_3) \in C_l^\mu \). Note that \( \omega_3(x) = [h_{A_1, A_3}] = t^\mu \). Therefore \( \omega(x) = (\omega_2(x), \omega_3(x)) \in m_{\lambda_1, \lambda_2}^{-1}(t^\mu) \). The rest is due to Theorem 9.3.

Now let us prove the general case. Consider the scissoring morphism

\[
c = (c_1, c_2) : \text{Conf}_{n+1}(\mathcal{A}) \rightarrow \text{Conf}_n(\mathcal{A}) \times \text{Conf}_3(\mathcal{A}),
\]

\[
(A_1, \ldots, A_{n+1}) \rightarrow (A_1, \ldots, A_{n-1}, A_{n+1}) \times (A_{n-1}, A_n, A_{n+1})
\]

Due to the scissoring congruence invariance, the map \( c^l \) induces a decomposition

\[
C_{\lambda_1, \ldots, \lambda_n}^\mu = \bigsqcup_{\nu \in P^+} C_{\lambda_1, \ldots, \lambda_{n-2}, \nu}^\mu \times C_{\lambda_{n-1}, \lambda_n}^\nu.
\]

**Proposition 9.4.** The cardinality of \( C_{\lambda}^\mu \) is the tensor product multiplicity \( c_{\lambda}^\mu \) of \( V_{\mu} \) in \( V_{\lambda} \).

\(^{13}\)There is a positive Cartan group action on \( \text{Conf}_3(\mathcal{A})(Z) \) defined via

\[
H \times \text{Conf}_3(\mathcal{A}) \rightarrow \text{Conf}_3(\mathcal{A}), \quad h \times (A_1, A_2, A_3) \mapsto (A_1, A_2, A_3 \cdot h).
\]

Its tropicalization determines a free \( H(Z) \)-action on \( \text{Conf}_3(\mathcal{A})(Z) \). By definition, one can thus identify the \( H(Z) \)-orbits of \( \text{Conf}_3(\mathcal{A})(Z) \) with points of \( \text{Conf}(\mathcal{A}, \mathcal{A}, \mathcal{B})(Z) \). Note that each element in \( B_{\lambda_1, \lambda_2}^{\mu - \lambda_2} \) has a unique representative in \( \tilde{B}_\lambda^\mu \). Hence the map \( \pi_3^l \) is a bijection.
Proof. Decomposing the last tensor products in $V_{\lambda_1} \otimes \ldots \otimes (V_{\lambda_{n-1}} \otimes V_{\lambda_n})$ into a sum of irreducibles, and tensoring then each of them with $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_{n-2}}$, we get

$$c''_{\mu_1, \ldots, \mu_n} = \sum_{\nu \in P^+} c'_{\lambda_1, \ldots, \lambda_{n-2}, \nu} c''_{\lambda_{n-1}, \lambda_n}.$$ 

As a consequence of $n = 2$ case, $|C_{\lambda, \mu}| = c''_{\lambda, \mu}$. The Lemma follows by induction and (248). □

**Lemma 9.5.** For $l \in C^\mu_{\Delta_{\lambda}}$ the cycles $\omega(C^\nu_i)$ are disjoint.

**Proof.** By Lemma (22) $\kappa(C^\varpi_i) = i_1 \circ \omega(C^\varpi_i)$. The Lemma follows from Theorem (2.21). □

**Lemma 9.6.** For any $l \in C^\mu_{\Delta_{\lambda}}$, we have $\omega(C^\nu_i) \subset m^{-1}_{\Delta_{\lambda}}(t^\mu)$.

**Proof.** Let $x = (a_1, \ldots, a_{n+1}) \in C^\nu_i$. Recall the expression (222). We have

$$[g_i] := [u_{A_1, B_{n+1}} a h_{A_1, A_{n+1}} w_0] = u_{A_1, B_{n+1}} \cdot t^{\lambda_i} \in Gr_{\lambda_i}, \quad i \in [1, n].$$

Thus $\omega(x) \in Gr_{\Delta_{\lambda}}$. Meanwhile $m^{-1}_{\Delta_{\lambda}} \circ \omega(x) = [h_{A_1, A_{n+1}}] = t^\mu$. The Lemma is proved. □

**Lemma 9.7.** Let $l \in C^\mu_{\Delta_{\lambda}}$. The closure $\overline{\omega(C^\nu_i)}$ is an irreducible variety of dimension $ht(\Delta_{\lambda}; \mu)$.

**Proof.** By construction, $\overline{\omega(C^\nu_i)}$ is irreducible. Note that $m^{-1}_{\Delta_{\lambda}}(t^\mu)$ is of top dimension $ht(\Delta_{\lambda}; \mu)$. By Lemma 9.6, $\dim \overline{\omega(C^\nu_i)} \leq ht(\Delta_{\lambda}; \mu)$. To show that $\dim \overline{\omega(C^\nu_i)} = ht(\Delta_{\lambda}; \mu)$, we use induction.

Set $\pi_{n-1, n+1}(l) := v$. Recall $c = (c_1, c_2)$ in (247). Then $c^\backslash_i(l) \in C^\mu_{\lambda_1, \ldots, \lambda_{n-2}, \nu}, c_i(l) \in C^\nu_{\lambda_1, \ldots, \lambda_n}$. Consider the projection

$$pr : \omega(C^\nu_i) \longrightarrow Gr^{n-1}, \quad (L_1, \ldots, L_{n-1}, L_n) \longrightarrow (L_1, \ldots, L_{n-2}, L_{n}).$$

Its image $pr(\omega(C^\nu_i)) = \omega(C^\nu_{c^\backslash_i}(l))$. Let $b = (L_1, \ldots, L_{n-2}, L_n) \in \omega(C^\nu_{c^\backslash_i}(l))$. The fiber over $b$ is

$$pr^{-1}(b) := \{L \in Gr_n : (L_1, \ldots, L_{n-2}, L_n) \in \omega(C^\nu_{c^\backslash_i}(l))\}. $$

Let $y = (a_1, \ldots, a_{n-1}, a_{n+1}) \in C^\mu_{c^\backslash_i}(l)$ such that $\omega(y) = b$. Set $b_y := b_{A_1, A_{n+1}}$. For any $x \in C^\nu_{c^\backslash_i}(l)$ such that $c_1(x) = y$, we have $pr(\omega(x)) = \omega(y) = b$. By (223), we have

$$\omega_{n-1}(x) = [b_{A_1, A_{n+1}}^A] = b_y \cdot \omega_2(c_2(x)) \in pr^{-1}(b).$$

Then it is easy to see that $b_y \cdot \omega_2(c^\nu_{c^\backslash_i}(l)) \subset pr^{-1}(b)$. Therefore

$$\dim \overline{\omega(C^\nu_i)} \geq \dim \overline{\omega(C^\nu_{c^\backslash_i}(l))} + \dim \overline{\omega(C^\nu_{c_i}(l))}.$$ 

The case when $n = 2$ is proved above. The Lemma follows by induction. □

**Proof of Theorem 2.20** By Lemmas 9.6, 9.7 the map $C^\mu_{\Delta_{\lambda}} \longrightarrow T^\mu_{\Delta_{\lambda}}, l \longrightarrow \omega(C^\nu_i)$ is well-defined. By Lemma 9.5 and the very construction of the cell $C^\nu_i$, it is injective. Since $|C^\mu_{\Delta_{\lambda}}| = |T^\mu_{\Delta_{\lambda}}| = c''_{\mu, \lambda}$ the map is a bijection. □

### 9.4 Proof of Theorem 2.20

We focus on the case when $\mu = 0$ for $C^\mu_{\Delta_{\lambda}}$. Consider the scissoring morphism

$$c = (c_1, c_2) : Conf_{n+1}(A) \longrightarrow Conf_n(A) \times Conf_3(A),$$

$$(A_1, \ldots, A_n, A_{n+1}) \longrightarrow (A_1, \ldots, A_n) \times (A_1, A_n, A_{n+1}).$$

Due to the scissoring congruence invariance, the morphism $(c_1, c_2)$ induces a decomposition

$$C^\nu_{\Delta_{\lambda}} = \bigcup_{\nu} C^\nu_{\lambda_1, \ldots, \lambda_{n-1}, \nu} \times C^0_{\nu, \lambda_n}. $$

Note that $C^0_{\nu, \lambda_n}$ is empty if $\nu \neq \lambda_n$. Moreover $|C^0_{\nu, \lambda_n}| = 1$. Thus $c^\backslash_1 : C^\nu_{\Delta_{\lambda}} \longrightarrow C_{\Delta_{\lambda}}$ is a bijection.

Consider the shifted projection

$$p_{\nu} : Gr^\nu \longrightarrow Conf_n(Gr), \quad \{L_1, \ldots, L_n\} \longrightarrow (L_n, L_1, \ldots, L_{n-1}).$$
Lemma 9.8. Let \( l \in C^0_\Delta \). Then \( p_s \circ \omega(C_d^0) = \kappa(C_d^0) \).

Proof. Let \( x = (A_1, \ldots, A_{n+1}) \in C_d^0 \). Then \( u := w_{B_{n+1}, B_n} \in U(O) \). Let \( y := c_1(x) \in C_d^0 \).

Recall \( \omega_i \) in (218). Then \( \omega_{i+1}(x) = [1] \). For \( i \in [2, n] \), we have

\[
\omega_i(x) = [g_{(U,B)}(\{A_1, B_{n+1}\}, \{A_i, B_1\})] = u \cdot [g_{(U,B)}(\{A_1, B_n\}, \{A_i, B_1\})] = u \cdot \omega_i(y).
\]

Therefore

\[
p_s \circ \omega(x) = (\omega_{n+1}(x), u \cdot \omega_2(y), \ldots, u \cdot \omega_n(y)) = ([1], \omega_2(y), \ldots, \omega_n(y)) = \kappa(y).
\]

Here the last step is due to Lemma 7.2. Since \( c_1(C_d^0) = C_d^0 \), the Lemma is proved.

Recall \( Gr_c(\Delta) \) and the set \( T_\Delta \) of its top components in Theorem 2.20. The connected group \( G(O) \) acts on \( Gr_c(\Delta) \). It preserves each component of \( Gr_c(\Delta) \). So these components live naturally on the stack \( Conf_n(Gr) = G(O) \backslash (\{1\} \times Gr^{n-1}) \).

Recall the fiber \( m_\Delta^{-1}(1) \) and the set \( T_0^\Delta \) in Theorem 9.2. Note that \( p_s(m_\Delta^{-1}(1)) = G(O) \backslash Gr_c(\Delta) \subset Conf_n(Gr) \). It induces a bijection \( T_0^\Delta \rightarrow T_\Delta \).

Proof of Theorem 2.20. By Theorem 9.2 and above discussions, there is a chain of bijections: \( C_\Delta \rightarrow C_d^0 \rightarrow T_0^\Delta \rightarrow T_\Delta \). By Lemma 9.8 this chain is achieved by the map \( \kappa \). The Theorem is proved.

10. Positive G-laminations and surface affine Grassmannians

A decorated surface \( S \) comes with an unordered collection \( \{s_1, \ldots, s_n\} \) of special points, defined up to isotopy. Denote by \( \partial S \) the boundary of \( S \). We assume that \( \partial S \) is not empty. We define punctured boundary

\[
\widehat{\partial}S := \partial S - \{s_1, \ldots, s_n\}.
\]

Its components are called boundary circles and boundary intervals.

Let us shrink all holes without special points on \( S \) into punctures, getting a homotopy equivalent surface. Abusing notation, we denote it again by \( S \). We say that the punctures and special points on \( S \) form the set of marked points on \( S \):

\[
\{\text{marked points}\} := \{\text{special points} s_1, \ldots, s_n\} \cup \{\text{punctures}\}.
\]

Pick a point \( *s_i \) in each of the boundary intervals. The dual decorated surface \( *S \) is given by the same surface \( S \) with the set of special points \( \{*s_1, \ldots, *s_n\} \). We have a duality: \( *S = S \).

Observe that the marked points are in bijection with the components of the punctured boundary \( \widehat{\partial}(S) \).

10.1 The space \( A_{G,S} \) with the potential \( \mathcal{W} \)

Twisted local systems and decorations. Let \( T'S \) be the complement to the zero section of the tangent bundle on a surface \( S \). Its fiber \( T'_y \) at \( y \in S \) is homotopy equivalent to a circle. Let \( x \in T'_y S \). The fundamental group \( \pi_1(T'_y S, x) \) is a central extension:

\[
0 \rightarrow \pi_1(T'_y S, x) \rightarrow \pi_1(T'S, x) \rightarrow \pi_1(S, y) \rightarrow 0, \quad \pi_1(T'_y S, x) = \mathbb{Z}.
\]

Let \( L \) be a G-local system on \( T'S \) with the monodromy \( s_G \) around a generator of \( \pi_1(T'_y S, x) \). Let us assume that \( G \) acts on \( L \) on the right. We call \( L \) a twisted G-local system on \( S \). It gives rise to the associated decorated flag bundle \( L_A := L \times_G A \).

Let \( C \) be a component of \( \widehat{\partial}(S) \). There is a canonical up to isotopy section \( \sigma : C \rightarrow T'C \) given by the tangent vectors to \( C \) directed according to the orientation of \( C \). A decoration on \( L \) over \( C \) is a flat section of the restriction of \( L_A \) to \( \sigma(C) \).

Definition 10.1 [FG1]. A twisted decorated G-local system on \( S \) is a pair \( (L, \alpha) \), where \( L \) is a twisted G-local system on \( S \), and \( \alpha \) is given by a decoration on \( L \) over each component of \( \widehat{\partial}(S) \).

The moduli space \( A_{G,S} \) parametrizes twisted decorated G-local systems on \( S \).

Abusing terminology, a decoration is given by decorated flags at the marked points.
Remark. Since the boundary $\partial S$ of $S$ is not empty, the extension \( \pi_1(T^*S,x) \) splits:
\[
\pi_1(T^*S,x) \cong \pi_1(T_1^*S,x) \times \pi_1(S,y).
\]
However the splitting is not unique. As a space, $\mathcal{A}_{G,S}$ is isomorphic, although non canonically if $s_G \neq 1$, to its counterpart of usual unipotent G-locals on $S$ with decorations. The mapping class group $\Gamma_S$ acts differently on the two spaces. For example, when $S$ is a disk $D_n$ with $n$ special points on the boundary, then $\Gamma_{D_n} = \mathbb{Z}/n\mathbb{Z}$. Both moduli spaces are isomorphic to the configuration space $\text{Conf}_n(A)$. The mapping class group $\mathbb{Z}/n\mathbb{Z}$ acts on the untwisted moduli space is by the cyclic rotation $\mathcal{A}_{G,D^n}$ is given by the “twisted” rotation
\[
(A_1, A_2, \ldots, A_n) \mapsto (A_n, A_1, \ldots, A_{n-1}).
\]

**Theorem 10.2** (loc.cit.). The space $\mathcal{A}_{G,S}$ admits a natural positive structure such that the mapping class group $\Gamma_S$ acts on $\mathcal{A}_{G,S}$ by positive birational isomorphisms.

Below we give two equivalent definitions of the potential $\mathcal{W}$ on $\mathcal{A}_{G,S}$.

**Potential via generalized monodromy.** A decorated flag $A$ provides an isomorphism
\[
i_A : U_A/[U_A, U_A] \xrightarrow{\cong} \oplus_{\alpha \in \Pi} A^1.
\] (251)

Let $\Sigma : \oplus_{\alpha \in \Pi} A^1 \to A^1$ be the sum map. Then $\chi_A = \Sigma \circ i_A$. This characterizes the map $i_A$.

Let us assign to each component $C$ of $\tilde{\partial}(\ast S)$ a canonical rational map, called generalized monodromy at $C$:
\[
\mu_C : \mathcal{A}_{G,S} \to \oplus_{\alpha \in \Pi} A^1.
\] There are two possible cases.

(i) The component $C$ is a boundary circle. The decoration over $C$ is a decorated flag $A_C$ in the fiber of $\mathcal{L}_A$ on $C$, invariant under the monodromy around $C$. It defines a conjugacy class in the unipotent subgroup $U_{A_C}$ preserving $A_C$. So we get a regular map
\[
\mu_C : \mathcal{A}_{G,S} \to U_{A_C}/[U_{A_C}, U_{A_C}] \cong \oplus_{\alpha \in \Pi} A^1.
\]

(ii) The component $C$ is a boundary interval on a hole $h$. The universal cover of $h$ is a line. We get an infinite sequence of intervals on this line projecting to the boundary interval(s) on $h$. There are decorated flags assigned to these intervals. Take an interval $C'$ on the cover projecting to $C$. Let $C'_-$ and $C'_+$ be the intervals just before and after $C'$. We get a triple of decorated flags $(A_-, A, A_+)$ sitting over these intervals. There is a unique $u \in U_A$ such that $B_+ = u \cdot B_-$, where $B_{\pm} = \pi(A_{\pm}) \in B$. Projecting $u$ to $U_A/[U_A, U_A]$, we get a map $\mu_C : \mathcal{A}_{G,S} \to \oplus_{\alpha \in \Pi} A^1$. It is clear that $\mu_C$ does not depend on the choice of $C'$.

Composing the generalized monodromy $\mu_C$ with the sum map $\oplus_{\alpha \in \Pi} A^1 \to A^1$, we get
\[
\mathcal{W}_C := \Sigma \circ \mu_C : \mathcal{A}_{G,S} \to A^1,
\] (252) called the potential associated with $C$.

**Definition 10.3.** The potential $\mathcal{W}$ on the space $\mathcal{A}_{G,S}$ is defined as
\[
\mathcal{W} := \sum_{\text{components } C \text{ of } \tilde{\partial}(\ast S)} \mathcal{W}_C.
\] (253)

**Potential via ideal triangulations.**

**Definition 10.4.** An ideal triangulation of a decorated surface $S$ is a triangulation of the surface whose vertices are the marked points of $S$.

Let $T$ be an ideal triangulation of $S$. Pick a triangle $t$ of $T$. The restriction to $t$ provides a projection $\pi_t$ from $\mathcal{A}_{G,S}$ to $\text{Conf}_3(A)$. Recall the potential $\mathcal{W}_t$ on the latter space.

---

14 If the vertices of $t$ coincide, one can first pull back to a sufficient big cover $\tilde{S}$ of $S$, and then consider the restriction to a triangle $t \subset \tilde{S}$ which projects onto $t$. Clearly the result is independent of the pair $t \subset \tilde{S}$ chosen.
Definition 10.5. The potential on the space $\mathcal{A}_{G,S}$ is defined as

$$W := \sum_{\text{triangles } t \text{ of } T} W_3 \circ \pi_t.$$ (254)

Changing $T$ by a flip we do not change the sum (254) since the potential on a quadrilateral is invariant under a flip (Section 2). Since any two ideal triangulations are related by a sequence of flips, the potential (254) is independent of the ideal triangulation $T$ chosen.

The above definitions are equivalent. There is a natural bijection between the marked points, that is the vertices of $T$, and the components of $\hat{\partial}^\ast$, this bijection is $\pi_3 \circ \pi_2$. Working with definition (254), the sum over all angles of the triangles shared by a puncture is the potential $W_C$ assigned to the corresponding boundary circle. A similar sum over all angles shared by a special point is the potential $W_C$ assigned to the corresponding boundary interval. Thus the potentials (254) and (255) coincide.

Positivity of the potential $W$. In the positive structure of $\mathcal{A}_{G,S}$ introduced in [FG1], the projection $\pi_t : \mathcal{A}_{G,S} \to \text{Conf}_3(A)$ is a positive morphism. By Theorem 2.5 and (254), we get

Theorem 10.6. The potential $W$ is a positive function on the space $\mathcal{A}_{G,S}$.

Positive integral $G$-laminations. We define the set of positive integral $G$-laminations on $S$:

$$\mathcal{A}_{G,S}^+(Z^f) = \{ l \in \mathcal{A}_{G,S}(Z^f) \mid W^t(l) \geq 0 \}. \quad (255)$$

By tropicalization, the mapping class group $\Gamma_S$ acts on $\mathcal{A}_{G,S}(Z^f)$. The potential $W$ is $\Gamma_S$-invariant. Thus $\Gamma_S$ acts on the subset $\mathcal{A}_{G,S}^+(Z^f)$.

Partial potentials. Given any simple positive root $\alpha$, there is a component $\chi_{A,\alpha}$ of the character $\chi_A$ so that $\chi_A = \sum_{\alpha \in A} \chi_{A,\alpha}$. Let $S$ be a decorated surface. Then to each boundary component $C \in \partial^\ast S$ one associates a function $W_{C,\alpha}$. It is evidently invariant under the action of the mapping class group $\Gamma_S$ of $S$.

Theorem 10.7. Let $S$ be a surface with $n$ holes and no special points. Then the algebra of regular $\Gamma_S$-invariant functions on the space $\mathcal{A}_{G,S}$ is a polynomial algebra in $\text{rk}(G)$ variables freely generated by the partial potentials $W_{C,\alpha}$, where $C$ run through all boundary circles on $S$, and $\alpha$ are simple positive roots.

Proof. It is well known that the action of the mapping class group $\Gamma_S$ on the moduli space $\text{Loc}_{G,S}^{\text{un}}$ of unipotent $G$-local systems on a surface $S$ with holes is ergodic. So there are no non-constant $\Gamma_S$-invariant regular functions on this space. On the other hand, there is a canonical $\Gamma_S$-invariant projection given by the generalised monodromy around the holes:

$$\mathcal{A}_{G,S} \rightarrow \prod_{\text{holes of } S} (A^1)^{\Pi},$$

Its fiber over zero is the space $\text{Loc}_{G,S}^{\text{un}}$. $\square$

10.2 Duality Conjectures for decorated surfaces

Definition 10.8. The moduli space $\text{Loc}_{G,S}$ parametrizes pairs $(L, \gamma)$, where $L$ is a twisted $G$-local system on $S$, and $\gamma$ assigns a decoration on $L$ to each boundary interval of $\partial^\ast S$.

It is important to consider several different types of twisted $G$-local system on $S$ which differ by the data assigned to the boundary. Recall that components of the punctured boundary $\partial^\ast S$ are in bijection with the marked points of $S$. There are three options for the data at a given marked point, which could be either a special point, or a puncture:

1) No data.
2) A decoration, that is a flat section of the associated decorated flag bundle $L_A$ near $m$.
3) A framing, that is a flat section of the associated flag bundle $L_B$ near $m$.

In accordance to this, there are five different moduli spaces:
• $A_{G,S}$: decorations at both special points and punctures.
• $\mathcal{L}_{G,S}$: no extra data.
• $\mathcal{L}_{G,S}$: decorations at the special points only. No extra data at the punctures.
• $\mathcal{P}_{G,S}$: decorations at the special points, framings at the punctures.
• $\chi_{G,S}$: framings at the special points and punctures.

If $S$ has no special points, it is silly to consider $\mathcal{L}_{G,S}$ since it ignores them.
If $S$ has no punctures, then (besides $\mathcal{L}_{G,S}$) there are three different moduli spaces:
$$A_{G,S} = \mathcal{L}_{G,S}, \quad \mathcal{P}_{G,S}, \quad \chi_{G,S}.$$ 

If $S$ has no special points, i.e. it is a punctured surface, there are three different moduli spaces:
$$A_{G,S}, \quad \mathcal{L}_{G,S} = \mathcal{L}_{G,S}, \quad \mathcal{P}_{G,S} = \chi_{G,S}.$$ 

Duality Conjectures interchange a group $G$ with the Langlands dual group $G^L$, and a decorated surface $S$ with the dual decorated surface $*S$.

Generalisations. Let us assign to each marked point $m$ of $S$ a subset $I_m \subset I$, possibly empty.

First, let us define a new potential on the space $A_{G,S}$. Observe that any non-degenerate additive character $\chi$ of $U$ is naturally decomposed into a sum of characters parametrised by the set of positive simple roots:
$$\chi = \sum_{i \in I} \chi_i.$$ Then, replacing in the definition of the potential at a given marked point $m$ the nondegenerate character $\chi$ by the character $\sum_{i \in I_m} \chi_i$, we get a new function $W_{m,I_m}$ at $m$, and set
$$W_{\{I_m\}} := \sum_{\text{marked points } m \text{ on } S} W_{m,I_m}.$$ (256)

Next, let us define a modified moduli space $P^{(I_m)}_{G^L,*S}$.

Recall that for each simple positive root $\alpha_i$ there is a $G$-invariant divisor in $B \times B$. Let $D_i$ be its preimage in $A \times A$. We say that a pair $(A_1, A_2) \in A \times A$ is in position $I - I_m$ if $(A_1, A_2) \in A \times A \cup \cup_{i \in I - I_m} D_i$.

Recall that $C_m$ is the boundary component of $*S$ matching a marked point $m$ on $S$.

Definition 10.9. The moduli space $P^{(I_m)}_{G^L,*S}$ parametrizes twisted $GL$-local systems on $S$ plus
a) A reduction of the structure group $G^L$ near each puncture $m$ to the parabolic subgroup of type $I - I_m$.
b) A decoration at every boundary interval $C_m$ of $*S$ such that

• The decorated flags at the ends of the boundary interval $C_m$ are in the position $I - I_m$.

So if $I = I_m$, the data a) is empty, and the condition b) is vacuous.

Finally, we consider the largest subspace
$$A^{(I_m)}_{G,S} \subset A_{G,S}$$
on which the potential $W_{\{I_m\}}$ is regular. This condition is vacuous at punctures, and boils down to the
•-condition from Definition 10.9 at boundary intervals of $*S$. So if $I_m = \emptyset$ at every special point $m$, then
$$A^{(I_m)}_{G,S} = A_{G,S}.$$ 

\[13\text{Although the decorated surface } *S \text{ is isomorphic to } S, \text{ the isomorphism is not quite canonical.}\]
Conjecture 10.10. \((A^I_{G.S}, W^I_{I_m})\) is dual to \((P^I_{G.S})\).

Let us now formulate what the Duality Conjecture tells about canonical bases for the most interesting moduli space \(L_{G.S.S}\), leaving similar formulations in other cases as a straightforward exercise.

Duality Conjecture for the space \(L_{G.S.S}\). The group \(\Gamma\) acts on the set \(A^+_S(\mathbb{Z}^I)\), and on the space \(O(L_{G.S.S})\) of regular functions on \(L_{G.S.S}\).

Conjecture 10.11. There is a canonical basis in the space \(O(L_{G.S.S})\) parametrized by the set \(A^+_S(\mathbb{Z}^I)\). This parametrization is \(\Gamma\)-equivariant.

Example. If \(S\) is a disc \(D_n\) with \(n\) special points on the boundary, then \(\Gamma_{D_n} = \mathbb{Z}/n\mathbb{Z}\). Theorem 2.6 provides a \(\Gamma_{D_n}\)-equivariant canonical basis. Thus Conjecture 10.11 is proved.

If \(G = SL_2\) (or \(G = PGL_2\)), then [FG1] provides a concrete construction of the \(\Gamma\)-equivariant parametrization, using laminations.

The following Theorem tells that the set \(A^+_S(\mathbb{Z}^I)\) is of the right size.

Theorem 10.12. Given an ideal triangulation \(T\) of a decorated surface \(S\), there is a linear basis in \(O(L_{G.S.S})\) parametrized by the set \(A^+_S(\mathbb{Z}^I)\).

Remark. The parametrization depends on the choice of the ideal triangulations. In particular, it is not \(\Gamma\)-equivariant.

Proof. The graph \(\Gamma\) dual to the triangulation \(T\) is a ribbon trivalent graph homotopy equivalent to \(S\). An end vertex of \(\Gamma\) is a univalent vertex of the graph. It corresponds to a boundary interval of \(\partial S\). Let \(\text{Loc}_{G.S.S}\) be the moduli space of pairs \((\mathcal{L}, \gamma)\), where \(\mathcal{L}\) is a \(G^S\)-local system on \(\Gamma\), and \(\gamma\) is a flat section of the restriction of the local system \(\mathcal{L}\) on \(\partial S\).

Choose an orientation of the edges of \(\Gamma\). Let \(V(\Gamma)\) and \(E(\Gamma)\) be the sets of vertices and edges of \(\Gamma\). Pick an edge \(E = (v_1, v_2)\) of \(\Gamma\), oriented from \(v_1\) to \(v_2\). Given a function \(\lambda : E(\Gamma) \rightarrow P^+\), we assign irreducible \(G^S\)-modules to the two flags of \(E\), denoted \(V_{v,E}\):

\[
V_{v, E} := V_{\lambda(E)}, \quad V_{v, E} := V_{-w_0(\lambda(E))}.
\]

According to [FG1] Section 12.5, (12.30)]], there is a canonical isomorphism

\[
O(\text{Loc}_{G.S.S}) = \bigoplus_{\{\lambda : E(\Gamma) \rightarrow P^+\}} \bigotimes_{v \in V(\Gamma)} \bigotimes_{(v, E)} V_{\lambda(v, E)}^{G^S}.
\]

The second tensor product is over all flags incident to a given vertex \(v\) of \(\Gamma\). By Applying Theorem 2.6 parametrizing a basis in the \(G^S\)-invariants of the tensor product for each vertex of \(\Gamma\), it follows that \(O(\text{Loc}_{G.S.S})\) admits a linear basis parametrized by \(A^+_S(\mathbb{Z}^I)\). Note that the central extension (250) is split. Following the remark after Definition 10.1, the moduli space \(L_{G.S.S}\) is isomorphic to \(L_{G.S.S}\). The Theorem is proved.

10.3 Canonical basis in the space of functions on \(L_{G.S.S}\)

Given any decorated surface \(S\), there is a generalisation of integral laminations on \(S\).

Definition 10.13. Let \(S\) be a decorated surface. An integral lamination \(l\) on \(S\) is a formal sum

\[
l = \sum_i n_i[a_i] + \sum_j m_j[\beta_j], \quad n_i, m_j \in \mathbb{Z}_{>0}.
\]

where \(\{a_i\}\) is a collection of simple nonisotopic loops, \(\{\beta_j\}\) is a collection of simple nonisotopic intervals ending inside of boundary intervals on \(\partial S\) – \(\{s_1, ..., s_n\}\), such that the curves do not intersect, considered modulo isotopy. The set of integral laminations on \(S\) is denoted by \(\mathcal{L}(S)\).
Let $\text{Mon}_\alpha(\mathcal{L}, \alpha)$ be the monodromy of a twisted $\text{SL}_2$-local system $(\mathcal{L}, \alpha)$ over a loop $\alpha$ on $S$.

Let us show that a simple path $\beta$ on $S$ connecting two points $x$ and $y$ on $\partial S$ gives rise to a regular function $\Delta_\beta$ on $\text{Loc}_{\text{SL}_2, S}$.

Let $(\mathcal{L}, \alpha)$ be a decorated $\text{SL}_2$-local system on $S$. The associated flat bundle $\mathcal{L}_A$ is a two dimensional flat vector bundle without zero section. Let $v_x$ and $v_y$ be the tangent vectors to $\partial S$ at the points $x, y$. The decoration $\alpha$ at $x$ and $y$ provides tangent vectors $l_x$ and $l_y$ in the fibers of $\mathcal{L}_A$ over $v_x$ and $v_y$. The set $S_\beta$ of non-zero tangent vectors to $\beta$ is homotopy equivalent to a circle. Let us connect $v_x$ and $v_y$ by a path $p$ in $S_\beta$, and transform the vector $l_x$ at $v_x$ to the fiber of $\mathcal{L}_A$ over $v_y$, getting there a vector $l'_x$. We claim that $\Delta(l'_x, l_y)$ is independent of the choice of $p$. This uses crucially the fact that $\mathcal{L}$ is a twisted local system. So we arrive at a well defined number $\Delta(l'_x, l_y)$ assigned to $(\mathcal{L}, \alpha)$. We denote by $\Delta_\beta$ the obtained function on $\text{Loc}_{\text{SL}_2, S}$.

Given an integral lamination $l$ on $S$ as in [258], we a regular function $M_l$ on $\text{Loc}_{\text{SL}_2, S}$ by

$$M_l(\mathcal{L}, \alpha) := \prod_i \text{Tr}((\text{Mon}_l)_i(\mathcal{L}, \alpha)) \prod_j \Delta_{\beta_j}(\mathcal{L}, \alpha).$$

**Theorem 10.14.** The functions $M_l, l \in \mathcal{L}_Z(S)$, form a linear basis in the space $\mathcal{O}(\text{Loc}_{\text{SL}_2, S})$.

**Theorem 10.15.** For any decorated surface $S$, there is a canonical isomorphism

$$\mathcal{A}^+_{\text{PGL}_2, S}(\mathbb{Z}) = \mathcal{L}_Z(S).$$

Theorem [10.14] is proved similarly to Theorem 12.1 in [FG1]. Notice that $\mathcal{A}^+_{\text{PGL}_2, S}$ is a positive space for the adjoint group $\text{PGL}_2$, the potential $W$ lives on this space and is a positive function there. Theorem [10.14] is proved by using arguments similar to the proof of Theorem [10.12] and [FG1 Proposition 12.2].

Combining Theorem [10.14] and Theorem [10.15] we arrive at a construction of the canonical basis predicted by Conjecture [10.11] for $G = \text{PGL}_2$.

### 10.4 Surface affine Grassmannian and amalgamation.

**The surface affine Grassmannian** $\text{Gr}_{G, S}$. Given a twisted right $G(K)$-local system $\mathcal{L}$ on $S$, there is the associated flat affine Grassmannian bundle $\mathcal{L}_{\text{Gr}} := \mathcal{L} \times_{G(K)} \text{Gr}$. Similarly to Definition [10.1] we define

**Definition 10.16.** Let $S$ be a decorated surface. The moduli space $\text{Gr}_{G, S}$ parametrizes pairs $(\mathcal{L}, \nu)$ where $\mathcal{L}$ is a twisted right $G(K)$-local system on $S$, and $\nu$ a flat section of the restriction of $\mathcal{L}_{\text{Gr}}$ to the punctured boundary $\partial^* S$.

Abusing terminology, the data $\nu$ is given by the lattices $L_m$ at the marked points $m$ on $S$.

The moduli space $\text{Gr}_{G, S}$ parametrizes similar data $(\widetilde{\mathcal{L}}, \nu)$, where $\widetilde{\mathcal{L}}$ is a twisted $G(K)$-local system on $S$ trivialized at a given point of $S$. So one has $\text{Gr}_{G, S} = G\backslash \text{Gr}_{G, S}$.

**Example.** Let $D_n$ be a disc with $n$ special points on the boundary. Then a choice of a special point provides isomorphisms

$$\text{Gr}_{G, D_n} = \text{Conf}_n(\text{Gr}), \quad \text{Gr}_{G, D_n} = \text{Gr}^n.$$
Cutting and amalgamating decorated surfaces. Let I be an ideal edge on a decorated surface $S$, i.e. a path connecting two marked points. Cutting $S$ along the edge I we get a decorated surface $S^*$. Denote by $I'$ and $I''$ the boundary intervals on $S^*$ obtained by cutting along I.

Conversely, gluing boundary intervals $I'$ and $I''$ on a decorated surface $S^*$, we get a new decorated surface $S$. We assume that the intervals $I'$ and $I''$ on $S^*$ are oriented by the orientation of the surface, and the gluing preserves the orientations.

More generally, let $S$ be a decorated surface obtained from decorated surfaces $S_1, ..., S_n$ by gluing pairs $\{I_1', I_1''\}, ..., \{I_m', I_m''\}$ of oriented boundary intervals. We say that $S$ is the amalgamation of decorated surfaces $S_1, ..., S_n$, and use the notation $S = S_1 * ... * S_n$. Abusing notation, we do not specify the pairs $\{I_1', I_1''\}, ..., \{I_m', I_m''\}$.

Amalgamating surface affine Grassmannians. There is a moduli space $Gr_{G, I}$ related to an oriented closed interval I, so that there is a canonical isomorphism of stacks

$$Gr_{G, I} = Conf_2(Gr).$$

**Definition 10.17.** Let $I'$, $I''$ be boundary intervals on a decorated surface $S^*$, perhaps disconnected. The amalgamation stack $Gr_{G, S^*}(I' * I'')$ parametrises triples $(L, \gamma, g)$, where $(L, \gamma)$ is the data parametrised by $Gr_{G, S^*}$, and $g$ is a gluing data, given by an equivalence of stacks

$$g : Gr_{G, I} \sim \rightarrow Gr_{G, I'}. \quad (259)$$

This immediately implies that there is a canonical equivalence of stacks:

$$Gr_{G, S} \sim \rightarrow Gr_{G, S^*}(I' * I''). \quad (260)$$

Given decorated surfaces $S_1, ..., S_n$ and a collection $\{I_1', I_1''\}, ..., \{I_m', I_m''\}$ of pairs of boundary intervals, generalising the construction from Definition 10.17 we get the amalgamation stack

$$Gr_{G, S_1 * ... * S_n} = Gr_{G, S_1 * ... * S_n}(I_1' * I_1'', ..., I_m' * I_m'').$$

Applying equivalences (260) we get

**Lemma 10.18.** There is a canonical equivalence of stacks:

$$Gr_{G, S} \sim \rightarrow Gr_{G, S_1 * ... * S_n}(I_1' * I_1'', ..., I_m' * I_m''). \quad (261)$$

Let $T$ be an ideal triangulation of a decorated surface $S$. Let $t_1, ..., t_n$ be the triangles of the triangulation. Abusing notation, denote by $t_i$ the decorated surface given by the triangle $t_i$, with the special points given by the vertices. Denote by $I_i'$ and $I_i''$ the pair of edges obtained by cutting an edge $I_i$ of the triangulation $t_i$, $i = 1, ..., m$. Then one has an isomorphism of stacks

$$Gr_{G, t_1 * ... * t_n}(I_1' * I_1'', ..., I_m' * I_m''). \quad (262)$$

10.5 Top components of the surface affine Grassmannian

10.5.1 Regularised dimensions

Recall that if a finite dimensional group $A$ acts on a finite dimensional variety $X$, we define the dimension of the stack $X/A$ by

$$\dim X/A := \dim X - \dim A.$$ 

Our goal is to generalise this definition to the case when $X$ and $A$ could be infinite dimensional.
**Dimension torsors** \( t^n \). Let us first define a rank one \( \mathbb{Z} \)-torsor \( t \). The kernel \( \mathbb{N} \) of the evaluation map \( G(O) \to G(\mathbb{C}) \) is a prounipotent algebraic group over \( \mathbb{C} \). Let \( N \) be its finite codimension normal subgroup. We assign to each such an \( N \) a copy \( \mathbb{Z}(N) \) of \( \mathbb{Z} \), and for each pair \( N_1 \subset N_2 \) such that \( N_2/N_1 \) is a finite dimensional, an isomorphism of \( \mathbb{Z} \)-torsors

\[
i_{N_1,N_2} : \mathbb{Z}(N_1) \to \mathbb{Z}(N_2), \quad x \mapsto x + \dim N_2/N_1. \tag{263}
\]

**Definition 10.19.** A \( \mathbb{Z} \)-torsor \( t \) is given by the collection of \( \mathbb{Z} \)-torsors \( \mathbb{Z}(N) \) and isomorphisms \( i_{N_1,N_2} \). We set \( t^n := t^\otimes n \) for any \( n \in \mathbb{Z} \).

In particular, \( t^0 = \mathbb{Z} \). To define an element of \( t^n \) means to exhibit a collection of integers \( d_N \) assigned to the finite codimension subgroups \( N \) of \( \mathbb{N} \) related by isomorphisms (263).

**Example.** There is an element \( \text{dim} \ G(O) \in t \), given by an assignment

\[
\text{dim} \ G(O) := \{ N \mapsto \dim G(O)/N \in \mathbb{Z}(N) \} \in t.
\]

More generally, there is an element

\[
n \dim G(O) := \{ N \mapsto \dim (G(O)/N)^n \in \mathbb{Z}(N) \} \in t^n.
\]

For example, the stack \( */G(O)^n \), where \( * = \text{Spec}(\mathbb{C}) \) is the point, has dimension

\[
\text{dim} \ */G(O)^n = -n \dim G(O) \in t^{-n}.
\]

If \( X \) and \( Y \) have dimensions \( \text{dim} \ X \in t^n \) and \( \text{dim} \ Y \in t^m \), then \( \text{dim} \ X \times Y \in t^{n+m} \).

**Dimension torsors** \( t^n_A \). We generalise this construction by replacing the group \( G(O) \) by a pro-algebraic group \( A \), which has a finite codimension prounipotent normal subgroup. Then there are the dimension torsor \( t_A \), its tensor powers \( t^n_A \), \( n \in \mathbb{Z} \), and an element \( \text{dim} \ A \in t_A \). One has \( t_A^n = t_A^m \). Moreover,

\[
n \dim A \in t^n_A, \quad t^n_A = \{ m + n \dim A \}, m \in \mathbb{Z}.
\]

**Regularised dimension.** Given such a group \( A \), we can define the dimension of a stack \( \mathcal{X} \) under the following assumptions.

1. There is a finite codimension prounipotent subgroup \( N \subset A \) such that \( N^n \) acts freely on \( \mathcal{X} \).
2. There is a finite dimensional stack \( \mathcal{Y} \) and an action of the group \( A^m \) on \( \mathcal{Y} \) such that
   \[
   \mathcal{Y}/A^m = \mathcal{X}/N^n. \tag{264}
   \]
3. There exists a finite codimension normal prounipotent subgroup \( M \subset A \) such that the action of \( A^m \) on \( \mathcal{Y} \) restricts to the trivial action of the subgroup \( M^m \) on \( \mathcal{Y} \).

The last condition implies that we have a finite dimensional stack \( \mathcal{Y}/(A/M^m) \). The stack \( \mathcal{Y}/A^m \) is the quotient of the stack \( \mathcal{Y}/(A/M^m) \) by the trivial action of the group \( M^m \).

In this case we define an element of the torsor \( t^n_A - m \) by the assignment

\[
(N,M) \mapsto \text{dim}(\mathcal{Y}/A^m) + \text{dim} (N^n) := (n-m) \dim A + \dim \mathcal{Y} - n \dim(A/N) \in t^n_A - m. \tag{265}
\]

**Definition 10.20.** Assuming 1) – 2), the assignment (263) defines the regularised dimension

\[
\text{dim} \ \mathcal{X} \in t^n_A - m.
\]

**Remark.** Often an infinite dimensional stack \( \mathcal{X} \) does not have a canonical presentation (263), but rather a collection of such presentations. For instance such a presentation of the stack \( \mathcal{M}_T^n \) defined below depends on a choice of an ideal triangulation \( T \) of \( S \). Then we need to prove that the regularised dimension is independent of the choices.\(^{10}\)

\(^{10}\)Taking the quotient by a unipotent group does not affect the category of equivariant sheaves. This is why we require the prounitiveness condition here.
10.5.2 Top components of the stack $\text{Gr}_{G,S}$

Suppose that a decorated surface $S$ is an amalgamation of decorated surfaces:

$$S = S_1 \ast \ldots \ast S_n.$$  \hfill (266)

**Definition 10.21.** Given an amalgamation pattern $(266)$, define the amalgamation

$$\mathcal{A}_{G,S_1}(Z') * \ldots * \mathcal{A}_{G,S_n}(Z') := \{ (l_1, \ldots, l_n) \in \mathcal{A}_{G,S_1}(Z') \times \ldots \times \mathcal{A}_{G,S_n}(Z') \mid (267) \text{ holds} \} :$$

$$\pi^l_{i_1}(l_i) = \pi^l_{i_2}(l_j) \text{ for any boundary intervals } I_k \subset S_i \text{ and } I'_k \subset S_j \text{ glued in } S.$$  \hfill (267)

**Lemma 10.22.** Given an amalgamation pattern $(266)$, there are canonical isomorphism of sets

$$\mathcal{A}_{G,S}(Z') = \mathcal{A}_{G,S_1}(Z') * \ldots * \mathcal{A}_{G,S_n}(Z').$$

$$\mathcal{A}_{G,S}^+(Z') = \mathcal{A}_{G,S_1}^+(Z') * \ldots * \mathcal{A}_{G,S_n}^+(Z').$$

In this case we say that $l$ is presented as an amalgamation, and write $l = l_1 * \ldots * l_n$.

Let us pick an ideal triangulation $T$ of $S$, and present $S$ as an amalgamation of the triangles:

$$S = t_1 * \ldots * t_n.$$  \hfill (268)

By Lemma 10.22 any $l \in \mathcal{A}_{G,S}^+(Z')$ is uniquely presented as an amalgamation

$$l = l_1 * \ldots * l_n, \quad l_i \in \mathcal{A}_{G,t_i}(Z').$$  \hfill (269)

Recall that given a polygon $D_n$, there are cycles

$$M^o_l := \kappa(C^2_l) \subset \text{Gr}_{G,D_n} \quad l \in \mathcal{A}_{G,D_n}^+(Z').$$

**Definition 10.23.** Given an ideal triangulation $T$ of $S$ and an $l \in \mathcal{A}_{G,S}^+(Z')$ we set, using amalgamations $(268)$ and $(269)$,

$$M^o_{l,t} = M^o_{l_1,t_1} * \ldots * M^o_{l_n,t_n}, \quad M_{T,l} := \text{Zariski closure of } M^o_{l,t}.$$

Thanks to Lemma 6.14 the restriction to the boundary intervals of $S$ leads to a map of sets

$$\mathcal{A}_{G,S}^+(Z') \to \mathbb{P}(\text{boundary intervals of } S).$$

It assigns to a point $l \in \mathcal{A}_{G,S}^+(Z')$ a collection of dominant coweights $\lambda_1, \ldots, \lambda_n \in \mathbb{P}$ at the boundary intervals $I_1, \ldots, I_n$ of $S$.

For any decorated subsurface $i : S' \subset S$ there is a projection given by the restriction map for the surface affine Grassmannian: $r_{Gr} : \text{Gr}_{G,S} \to \text{Gr}_{G,S'}$. There are two canonical projections:

$$\mathcal{A}_{G,S}^+(Z') \quad \text{Gr}_{G,S}$$

$$r^+ \downarrow \quad \downarrow r_{Gr}$$

$$\text{Conf}_{G,S}^+(\mathcal{A})(Z') \quad \text{Gr}_{G,S'}$$

10.24 Let $S$ be a decorated surface.

i) The stack $M_{T,I}$ does not depend on the triangulation $T$. We denote it by $M_I$.

ii) Let $l \in \mathcal{A}_{G,S}^+(Z')$. Let $\{I_1, \ldots, I_n\}$ be the set of boundary intervals of $S$, and $\lambda_1, \ldots, \lambda_n$ are the dominant coweights assigned to them by $l$. Then

$$\dim M_l = \langle \rho, \lambda_1 + \ldots + \lambda_n \rangle - \chi(S) \dim \text{G(O)} \in \mathbb{t}^{-\chi(S)}.$$  \hfill (271)

iii) The stacks $M_I, l \in \mathcal{A}_{G,S}^+(Z')$, are top dimensional components of $\text{Gr}_{G,S}$.

iv) The map $l \mapsto M_l$ provides a bijection

$$\mathcal{A}_{G,S}^+(Z') \sim \{ \text{top dimensional components of the stack } \text{Gr}_{G,S} \}.$$

This isomorphism commutes with the restriction to decorated subsurfaces of $S$.

**Proof.** Let us calculate first dimensions of the stacks $M_{T,I}$, and show that they are given by formula (271).

We present first a heuristic dimension count, and then fill the necessary details.
Heuristic dimension count. Let us present a decorated surface $S$ as an amalgamation of a (possible disconnected) decorated surface along a pair of boundary intervals $I, I''$, as in Definition 10.17. The space of isomorphisms $g$ from $L_0 \to L_1'$ (respectively $L_0'' \to L_1''$) be a pair of lattices assigned to the vertices of the interval $I$ (respectively $I''$). Then the glueing data is a map $g : (L_0', L_1') \to (L_0'', L_1'')$. Let $G_\lambda$ be the subgroup stabilising the pair $L_0', L_1'$. The space of gluings is a $G_\lambda$-torsor. The group $G_\lambda$ is a subgroup of codimension 2 in $G$. So

$$\dim G_\lambda = \dim G(O) - 2\langle \rho, \lambda \rangle = \dim G(O) - \dim Gr_{\lambda, \lambda'}.$$

Take the stack $M_{t,l}^O$ assigned to a triangle $t$ and a point $l \in \text{Conf}_+(A)(\mathbb{Z}')$. Let $\lambda_1, \lambda_2, \lambda_3$ be the dominant coweights assigned to the sides of the triangle by $l$. Then $M_{t,l}^O$ is an open part of a component of the stack $Gr_{\lambda_1, \lambda_2, \lambda_3}/G(O)$. Thus

$$\dim M_{t,l}^O = \langle \rho, \lambda_1 + \lambda_2 + \lambda_3 \rangle - \dim G(O) \in t^{-1}. \tag{272}$$

Let us calculate now the dimension of the stack $M_{T,T''}^O$. Let $|T|$ be the number of triangles, and $E_{\text{int}}$ (respectively $E_{\text{ext}}$) the set of the internal (respectively external) edges of the triangulation $T$. Then the dimension of the product of stacks assigned to the triangles is

$$\sum_{E \in E_{\text{int}}} \langle \rho, \lambda_E \rangle + 2 \sum_{E \in E_{\text{ext}}} \langle \rho, \lambda_E \rangle - |T| \dim G(O) \in t^{-|T|}.$$

Gluing two boundary intervals into an internal edge $E$, with the dominant weights $\lambda_E$ associated to it, we have to add the dimension of the corresponding glueing data torsor, that is

$$\dim G(O) - 2\langle \rho, \lambda_E \rangle \in t.$$

So, gluing all the intervals, we get

$$\sum_{E \in E_{\text{int}}} \langle \rho, \lambda_E \rangle + (|E_{\text{int}}| - |T|) \dim G(O) = \sum_{E \in E_{\text{ext}}} \langle \rho, \lambda_E \rangle - \chi(S) \dim G(O) = \dim G(O) \tag{271}.$$

Notice that $|E_{\text{int}}| - |T| = -\chi(S)$. Indeed, the triangles $t$ with external sides removed cover the surface $S$ minus the boundary, which has the same Euler characteristic as $S$.

Rigorous dimension count. For each of the triangles $t$ of the triangulation $T$ there are three dominant coweights $\lambda(t) := \lambda_1(t), \lambda_2(t), \lambda_3(t)$ assigned by $l$ to the sides of $t$. Pick a vertex $v(t)$ of the triangle $t$. We present the stack $Gr_{G,t}$ as a quotient of the convolution variety

$$Gr_{G,t} = Gr_{\lambda(t)}/G(O). \tag{273}$$

Namely, choose the lattice $L_{v(t)}$ at the vertex $v(t)$ to be the standard lattice $L_{v(t)} = G(O)$.

There exists a finite codimension normal pronipotent subgroup $N_{t,l} \subset G(O)$ acting trivially on $Gr_{\lambda(t)}$. It depends on the choice of coweights $\lambda(t)$, and, via them, on the choice of the $t$ and $l$. We assign to each finite codimension normal subgroup $N_{t,l} \subset N_{t,l}$ a finite dimensional stack

$$\frac{Gr_{\lambda(t)}}{G(O)/N_{t,l}}.$$

Its dimension is $\langle \rho, \lambda_1 + \lambda_2 + \lambda_3 \rangle - \dim G(O)/N_{t,l}$. This just means that we have formula (272).

There is a canonical surjective map of stacks

$$Gr_{G,S} \to \prod_{t \in T} Gr_{G,t} = \prod_{t \in T} Gr_{\lambda(t)}/G(O). \tag{274}$$
Its fibers are torsors over the product over the set $E_{\text{int}}$ of internal edges $E$ of $T$ of certain groups $G_{\lambda(E)}$ defined as follows. Let $\lambda(E)$ be the dominant coweight assigned to $E$ by $l$. Consider the pair $E',E''$ of edges of triangles glued into the edge $E$. For each of them, there is a pair of the lattices assigned to its vertices. We get two pairs of lattices:

$$(L_E^-, \lambda(E) \rightarrow L_E^+) \text{ and } (L_{E''}^-, \lambda(E) \rightarrow L_{E''}^+).$$

Choose one of the edges, say $E'$. Set $G_{\lambda(E)} := \text{Aut} (L_{E'}^-, \lambda(E) \rightarrow L_{E'}^+)$. Therefore we conclude that

The fibers of the map $274$ are torsors over the group $\prod_{E \in E_{\text{int}}} G_{\lambda(E)}$.

For each $E$, choose a finite codimension subgroup $N_{\lambda(E)} \subset G_{\lambda(E)}$. Then we are in the situation discussed right before Definition 10.20 where

$$\mathcal{X} = \mathcal{M}^T_{ij}, \quad A = G(\mathcal{O}), \quad N := \cap_{E \in E_{\text{int}}} N_{\lambda(E)}, \quad M = \cap_{t=1} N'_{t,t}, \quad n = |E_{\text{int}}|, \quad m = |T|.$$

So we get the expected formula for the regularised dimension of $\mathcal{M}^T_{\pi(t)}$. The resulting regularised dimension does not depend on the choice of ideal triangulation $T$ – the triangulation does not enter to the answer.

Alternatively, one can see this as follows. Any two ideal triangulations of $S$ are related by a sequence of flips. Let $T \rightarrow T'$ be a flip at an edge $E$. Let $R_E$ be the unique rectangle of the triangulation $T$ with the diagonal $E$. Consider the restriction map $\pi : \text{Gr}_{G,S} \rightarrow \text{Gr}_{R_E,S}$. So one can fiber $\mathcal{M}^T_{ij}$ over the component $\mathcal{M}^T_{\pi(t)}$. The dimension of the latter does not depend on the choice of the triangulation of the rectangle.

A similar argument with a flip of triangulation proves i). Combining with the formula for the regularised dimension of $\mathcal{M}^T_{\pi(t)}$ we get ii).

iii), iv). Present $S$ as an amalgamation of the triangles of an ideal triangulation. It is known that the cycles $\mathcal{M}_l$ are the top dimensional components of the convolution variety, and thus the stack $\text{Gr}_{G,t}$, assigned to the triangle. It remains to use Lemma 10.18.

\[\square\]

### 11 Cluster varieties, frozen variables and potentials

#### 11.1 Basics of cluster varieties

**Definition 11.1.** A quiver $q$ is described by a data $(\Lambda, \Lambda_0, \{e_i\}, (\ast, \ast))$, where

1. $\Lambda$ is a lattice, $\Lambda_0$ is a sublattice of $\Lambda$, and $\{e_i\}$ is a basis of $\Lambda$ such that $\Lambda_0$ is generated by a subset of frozen basis vectors;

2. $(\ast, \ast)$ is a skew-symmetric $\frac{1}{2}\mathbb{Z}$-valued bilinear form on $\Lambda$ with $(e_i, e_j) \in \mathbb{Z}$ unless $e_i, e_j \in \Lambda_0$.

Any non-frozen basis element $e_k$ provides a mutated in the direction $e_k$ quiver $q'$. The quiver $q'$ is defined by changing the basis $\{e_i\}$ only. The new basis $\{e'_i\}$ is defined via halfreflection of the $\{e_i\}$ along the hyperplane $(e_k, \cdot) = 0$:

$$e'_i := \begin{cases} e_i + [e_{ik}]e_k & \text{if } i \neq k, \\ -e_k & \text{if } i = k. \end{cases}$$

(275)

Here $[\alpha]_+ := \alpha$ if $\alpha \geq 0$ and $[\alpha]_+ := 0$ otherwise. The frozen/non-frozen basis vectors of the mutated quiver are the images of the ones of the original quiver. The composition of two mutations in the same direction $k$ is an isomorphism of quivers.

Set $\varepsilon_{ij} := (e_i, e_j)$. A quiver can be described by a data $q = (I, I_0, \varepsilon)$, where $I$ (respectively $I_0$) is the set parametrising the basis vectors (respectively frozen vectors). Formula $275$ amounts then to the Fomin-Zelevinsky formula telling how the $\varepsilon$-matrix changes under mutations.

$$\varepsilon'_{ij} := \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i,j\}, \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \ k \notin \{i,j\}, \\ \varepsilon_{ij} + |\varepsilon_{ik}| \cdot \varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0, \ k \notin \{i,j\}. \end{cases}$$

(276)
We assign to every quiver \( q \) two sets of coordinates, each parametrised by the set \( I \): the \( X \)-coordinates \( \{X_i\} \), and the \( A \)-coordinates \( \{A_i\} \). Given a mutation of quivers \( \mu_k : q \rightarrow q' \), the cluster coordinates assigned to these quivers are related as follows. Denote the cluster coordinates related to the quiver \( q' \) by \( \{X'_i\} \) and \( \{A'_i\} \). Then

\[
A'_k A'_k := \prod_{j \mid \varepsilon_{kj} > 0} A_j^{\varepsilon_{kj}} + \prod_{j \mid \varepsilon_{kj} < 0} A_j^{-\varepsilon_{kj}}; \quad A'_i = A_i, \quad i \neq k.
\]  

(277)

If any of the sets \( \{j \mid \varepsilon_{kj} > 0\} \) or \( \{j \mid \varepsilon_{kj} < 0\} \) is empty, the corresponding monomial is 1.

\[
X'_i := \begin{cases} 
X_k^{-1} 
\quad \text{if } i = k \\
X_i (1 + X_k^{-\text{sgn}(\varepsilon_{ik})})^{-\varepsilon_{ik}} 
\quad \text{if } i \neq k,
\end{cases}
\]  

(278)

The tropicalizations of these transformations are

\[
a'_k := -a_k + \min \left\{ \sum_{j \mid \varepsilon_{kj} > 0} \varepsilon_{kj} a_j, \sum_{j \mid \varepsilon_{kj} < 0} -\varepsilon_{kj} a_j \right\}; \quad a'_i = a_i, \quad i \neq k.
\]  

(279)

\[
x'_i := \begin{cases} 
x_k 
\quad \text{if } i = k \\
x_i - \varepsilon_{ik} \min \{0, -\text{sgn}(\varepsilon_{ik}) x_k\} 
\quad \text{if } i \neq k,
\end{cases}
\]  

(280)

Cluster transformations are transformations of cluster coordinates obtained by composing mutations. Cluster \( A \)-coordinates and mutation formulas (278) and (277) are main ingredients of the definition of cluster algebras [FZI]. Cluster \( X \)-coordinates and mutation formulas (277) describe a dual object, introduced in [FG2] under the name cluster \( X \)-variety.

**The cluster volume forms [FG5].** Given a quiver \( q \), consider the volume forms

\[
\text{Vol}_{A}^q := d \log A_1 \wedge \ldots \wedge d \log A_n, \quad \text{Vol}_{X}^q := d \log X_1 \wedge \ldots \wedge d \log X_n.
\]

Cluster transformations preserve them up to a sign: given a mutation \( q \rightarrow q' \), we have

\[
\text{Vol}_{A}^{q'} = -\text{Vol}_{A}^q, \quad \text{Vol}_{X}^{q'} = -\text{Vol}_{X}^q.
\]

Denote by \( \text{Or}_A \) the two element set of orientations of a rank \( n \) lattice \( A \), given by expressions \( l_1 \wedge \ldots \wedge l_n \) where \( \{l_i\} \) form a basis of \( A \). An orientation \( \text{or}_A \) of \( A \) is a choice of one of its elements. Given a basis \( \{e_i\} \) of \( A \), we define its sign \( \text{sign}(e_1, \ldots, e_n) \) by \( e_1 \wedge \ldots \wedge e_n = \text{sign}(e_1, \ldots, e_n) \text{or}_A \). A quiver mutation changes the sign of the basis, and the sign of each of the cluster volume forms. So there is a definition of the cluster volume forms invariant under cluster transformations.

**Definition 11.2.** Choose an orientation \( \text{or}_A \) for a quiver \( q \). Then in any quiver obtained by from \( q \) by mutations, the cluster volume forms are given by

\[
\text{Vol}_{A} = \text{sign}(e_1, \ldots, e_n) d \log A_1 \wedge \ldots \wedge d \log A_n, \quad \text{Vol}_{X} = \text{sign}(e_1, \ldots, e_n) d \log X_1 \wedge \ldots \wedge d \log X_n.
\]

**Residues of the cluster volume form \( \text{Vol}_{A} \) and frozen variables.** Take a space \( M \) equipped with a cluster \( A \)-coordinate system \( \{A_i\} \).

**Lemma 11.3.** Let us assume that \( k \in 1 - I_0 \) is nonfrozen, and \( \varepsilon_{kj} \neq 0 \) for some \( j \). Then

\[
\text{Res}_{A_k=0}(\text{Vol}_{A}) = 0.
\]

(281)

**Proof.** We have \( \text{Res}_{A_k=0}(\text{Vol}_{A}) = \pm \wedge_{i \neq k} d \log A_i \). Since \( k \) is nonfrozen, there is an exchange relation (277). It implies a monomial relation on the locus \( A_k = 0 \): \( \prod_j A_j^{\varepsilon_{kj}} = -1 \). Since \( \varepsilon_{kj} \) is not identically zero, this monomial is nontrivial. Thus \( \wedge_{i \neq k} d \log A_i = 0 \) at the \( A_k = 0 \) locus. \( \square \)
Corollary 11.4. A coordinate \( A_k \), with \( \varepsilon_{kj} \neq 0 \) for some \( j \), can be nonfrozen only if we have \( |287| \), i.e. the functions \( A_1, \ldots, \hat{A}_k, \ldots, A_n \) become dependent on every component of the \( A_k = 0 \) locus.

If we define a cluster algebra axiomatically, without referring to a particular space on which it is realised, then any subset of an initial quiver can be declared to be the frozen subset. However if a cluster algebra is realised geometrically, we do not have much freedom in the definition of frozen variables, as Corollary 11.4 shows. This leads to the following geometric definition of the frozen coordinates.

**Definition 11.5.** Let \( M \) be a space equipped with a cluster \( A \)-coordinate system. Then a cluster variable \( A \) is a frozen variable if and only if the residue form \( \text{Res}_A(\text{Vol}_A) \) is not zero.

**Non-negative real points for a cluster algebra.** The space of positive real points of any positive space is well defined. Let us define the space of non-negative real points for a cluster algebra.

Let \( \{ A^q_i \} \), \( i \in I \), be the set of all cluster coordinates in a given quiver \( q \). The cluster algebra \( \mathcal{O}_{\text{aff}}(A) \) is the algebra generated by the formal variables \( \{ A^q_i \} \), for all quivers \( q \) related by mutations to a given one, modulo the ideal generated by exchange relations \( |277| \):

\[
\mathcal{O}_{\text{aff}}(A) := \frac{\mathbb{Z}[A^q_i]}{\text{(exchange relations)}}. \tag{282}
\]

This ring is not necessarily finitely generated. Let \( A_{\text{aff}} \) be its spectrum. Then the points of \( \mathcal{O}_{\text{aff}}(\mathbb{R}_{\geq 0}) \) are just the collections of positive real numbers \( \{ a^q_i \in \mathbb{R}_{\geq 0} \} \) satisfying the exchange relations. The positive boundary is defined as the complement to the set of positive real points:

\[
\partial A_{\text{aff}}(\mathbb{R}_{\geq 0}) := A_{\text{aff}}(\mathbb{R}_{\geq 0}) - A_{\text{aff}}(\mathbb{R}_{> 0}).
\]

Let \( A_f \) be a frozen variable. Then \( \{ A_f = 0 \} \cap \partial A_{\text{aff}}(\mathbb{R}_{\geq 0}) \) is of real codimension one in \( A_{\text{aff}}(\mathbb{R}_{\geq 0}) \). Indeed, the frozen \( A \)-cluster coordinates do not mutate, and so the codimension one domain given by the points with the coordinates \( A_{f_j} = 0, A^q_j > 0 \) where \( j \) is different then \( f_j \) is a part of the intersection.

Let \( A^q_k \) be a non-frozen variable. It is likely, although we did not prove this, that in many cases

\[
\{ A^q_k = 0 \} \cap \partial A_{\text{aff}}(\mathbb{R}_{\geq 0}) \text{ is of real codimension } \geq 2 \text{ in } A_{\text{aff}}(\mathbb{R}_{\geq 0}). \tag{283}
\]

Indeed, the exchange relation for the \( A^q_k \), restricted to the \( A^q_k = 0 \) hyperplane, reads

\[
0 \cdot A^q_k = \prod_{j|\varepsilon_{kj} > 0} A^q_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj} < 0} A^q_j^{-\varepsilon_{kj}}.
\]

So both monomials on the right, being non-negative, are zero, and each of them is non-empty: the empty one contributes 1, violating 0 on the left. So we get at least two different cluster coordinates equal to zero. It is easy to see that then in any cluster coordinate system at least two of cluster coordinates are zero.

### 11.2 Frozen variables, partial compactification \( \hat{A} \), and potential on the \( X \)-space

**Potential on the \( X \)-space**

**Lemma 11.6.** Any frozen \( f \in I_0 \) gives rise to a tropical point \( l_f \in A(\mathbb{Z}^l) \) such that in any cluster \( A \)-coordinate system all tropical \( A \)-coordinates except \( a_f \) are zero, and \( a_f = 1 \).

**Proof.** Pick a cluster \( A \)-coordinate system \( \alpha = \{ A_f, \ldots \} \) starting from a coordinate \( A_f \). Consider a tropical point in \( A(\mathbb{Z}^l) \) with the coordinates \( (1, 0, \ldots, 0) \). It is clear from \(|279| \) that the coordinates of this point are invariant under mutations at non-frozen vertices. Indeed, at least one of the two quantities we minimize in \( |279| \) is zero, and the other must be non-negative. \( \square \)
The potential. Let us assume that there are canonical maps, implied by the cluster Duality Conjectures for the dual pair \((A, X^\vee)\) of cluster varieties:

\[
I_A : \mathcal{A}(\mathbb{Z}^I) \rightarrow \mathbb{L}_+(X^\vee), \quad I_X : X^\vee(\mathbb{Z}^I) \rightarrow \mathbb{L}_+(A).
\]

Here \(\mathbb{L}_+(A)\) and \(\mathbb{L}_+(X)\) are the sets of universally Laurent functions.

Definition 11.7. Let us assume that for each frozen \(f \in I_0\) there is a function

\[
W_{X^\vee,f} := I_A(l_f) \in \mathbb{L}_+(X^\vee)
\]

predicted by the Duality Conjectures. Then the potential on the space \(X\) is given by the sum

\[
W_{X^\vee} := \sum_{f \in I_0} W_{X^\vee,f}.
\]

Partial compactifications of the \(A\)-space. Given any subset \(I_0 \subset I_0\), we can define a partial completion \(\mathcal{A}_I := \mathcal{A} \bigcup_{f \in I_0} D_f\) of \(\mathcal{A}\) by attaching to \(\mathcal{A}\) the divisor \(D_f\) corresponding to the equation \(A_f = 0\) for each \(f \in I_0\).

The duality should look like

\[
(A \bigcup_{f \in I_0} D_f) \leftrightarrow (X^\vee, \sum_{f \in I_0} W_f).
\]

The order of pole of \(I_{X}(l)\) at the divisor \(D_f\) should be equal to \(W_f(l)\). In particular, \(I_{X}(l)\) extends to \(A \bigcup_{f \in I_0} D_f\) if and only if it is in the subset \(\{ l \in X^\vee(\mathbb{Z}^I) | W_f(l) \geq 0 \} \subset X^\vee(\mathbb{Z}^I)\).

Canonical tropical points of the \(X^\vee\)-space. Let \(i \in I\). Given a cluster \(X\)-coordinate system, consider a point \(t_i \in X(\mathbb{Z}^I)\) with the coordinates \(\varepsilon_{ji}, j \in I\).

Lemma 11.8. The point \(t_i\) is invariant under mutations of cluster \(X\)-coordinate systems. So there is a point \(t_i \in X(\mathbb{Z}^I)\) which in any cluster \(X\)-coordinate system has coordinates \(\varepsilon_{ji}, j \in I\).

Proof. Given a mutation in the direction of \(k\), let us compare, using \(280\), the rule how the \(X\)-coordinates \(\{\varepsilon_{ji}\}, j \in I\) change with the mutation formulas \(276\) for the matrix \(\varepsilon_{ij}\).

Let us assume that \(k \notin \{i, j\}\). Then, due to formula \(280\) for mutation of tropical \(X\)-points, we have to prove that

\[
\varepsilon_{ji}' = \varepsilon_{ji} - \varepsilon_{jk} \min \{0, -\text{sgn}(\varepsilon_{jk}) \varepsilon_{ki}\}.
\]

Let us assume now that \(\varepsilon_{jk} \varepsilon_{ki} < 0\). Then \(\text{sgn}(\varepsilon_{jk} \varepsilon_{ki}) > 0\). So \(\min \{0, \text{sgn}(\varepsilon_{jk} \varepsilon_{ki})\} = 0\), and the right hand side is \(\varepsilon_{ji}\). This agrees with \(276\), in this case.

If \(\varepsilon_{jk} \varepsilon_{ki} > 0\), then \(\text{sgn}(\varepsilon_{jk} \varepsilon_{ki}) < 0\). So the right hand side is

\[
\varepsilon_{ji} - \varepsilon_{jk} \min \{0, \text{sgn}(\varepsilon_{jk} \varepsilon_{ki})\} = \varepsilon_{ji} - \varepsilon_{jk} \text{sgn}(\varepsilon_{jk} \varepsilon_{ki}) = \varepsilon_{ji} + |\varepsilon_{jk}| \varepsilon_{ki}.
\]

Comparing with \(276\), we see that in both cases we get the expected formula \(284\).

Finally, if \(k \in \{i, j\}\), then \(\varepsilon_{ij}' = -\varepsilon_{ij}\), and by formula \(280\), we also get \(-\varepsilon_{ij}\). □

Let us assume that, for each frozen \(f \in I_0\), there is a function \(I_X(t_f) \in \mathbb{L}_+(A^\vee)\), predicted by the duality conjectures. Then we conjecture that in many situations there exist monomials \(M_f\) of frozen \(A\)-coordinates such that the potential on the space \(A\) is given by

\[
W_{A^\vee} := \sum_{f \in I_0} M_f \cdot I_X(t_f).
\]
A Geometric crystal structure on \( \text{Conf}(\mathcal{A}^n, \mathcal{B}) \)

We construct a geometric crystal structure on \( \text{Conf}(\mathcal{A}^n, \mathcal{B}) \). See [BK3] for definition of geometric crystals. By the positive birational isomorphism \( p : \text{Conf}(\mathcal{A}^2, \mathcal{B}) \to \mathcal{B}^- \), it recovers the crystal structure on \( \mathcal{B}^- \) defined in Example 1.10 of loc.cit.

Using machinery of tropicalization, the subset \( \mathcal{B}_A^n \subset \text{Conf}(\mathcal{A}^2, \mathcal{B})(\mathbb{Z}^t) \) becomes a crystal basis. We refer the reader to Section 2 of loc.cit. for details concerning the tropicalization of geometric crystals. Braverman and Gaitsgory [BG] construct a crystal structure for the MV basis. As a direct consequence of this paper, the MV basis are parametrized by the set \( \mathcal{B}_A^n \). Therefore we give a direct isomorphism between the crystals of [BK2 Theorem 6.15] and [BG] without using the uniqueness theorem from [J].

The tensor product of crystals can be interpreted as tropicalizations of the convolution product \( \ast \) from Section 4. Given the geometric background of the configuration spaces, definitions/proofs become simple.

A.1 Geometric crystal structure on \( \text{Conf}(\mathcal{A}^n, \mathcal{B}) \)

Let \( x = (A_1, \ldots, A_n, B_{n+1}) \in \text{Conf}(\mathcal{A}^n, \mathcal{B}) \). Let \( i \in I \). Recall the following positive maps:

- \( p : \text{Conf}(\mathcal{A}^n, \mathcal{B}) \to \mathcal{B}^- \) such that \( p(x) = b_{B_{n+1}}^{A_1, A_n} \).
- \( \mu : \text{Conf}(\mathcal{A}^n, \mathcal{B}) \to \mathcal{H} \) such that \( \mu(x) = \mu_{B_1, A_n}^{A_1, A_n} \).
- \( \varphi_i, \varepsilon_i, W : \text{Conf}(\mathcal{A}^n, \mathcal{B}) \to \mathbb{A}^1 \) such that
  \[
  \varphi_i(x) = \mathcal{L}_i(u_{B_{n+1}, B_n}^{A_1}), \quad \varepsilon_i(x) = \mathcal{R}_i(u_{B_1, B_{n+1}}^{A_n}), \quad W(x) = \sum_{k=1}^{n} \chi(u_{B_k-1, B_{k+1}}^{A_k}).
  \]

Here \( k \) is modulo \( n+1 \).

- \( e_i : \mathbb{G}_m \times \text{Conf}(\mathcal{A}^n, \mathcal{B}) \to \text{Conf}(\mathcal{A}^n, \mathcal{B}) \), where \( e_i^c(A_1, \ldots, A_n, B_{n+1}) = (A_1, \ldots, A_n, \tilde{B}_{n+1}) \) is such that
  \[
  e_i^c(A_1, A_n, B_{n+1}) = (A_1, A_n, \tilde{B}_{n+1})
  \]

The action \( e_i \) on \( \text{Conf}(\mathcal{A}^2, \mathcal{B}) \) is defined by (??).

**Theorem A.1.** The 6-tuple \( \{\text{Conf}(\mathcal{A}^n, \mathcal{B}), \mu, \varphi_i, \varepsilon_i, W, e_i \mid i \in I \} \) is a positive decorated geometric crystal.

**Warning.** The maps \( \varphi_i, \varepsilon_i \) are the inverse of those \( \varphi_i, \varepsilon_i \) in [BK3 Definition 1.3].

**Proof.** By [BK3] Definitions 1.3 & 2.7, it remains to show the following Lemmas.

**Lemma A.2.** Let \( \alpha_i^\vee \) and \( \alpha_i \) be the simple coroot and simple root corresponding to \( i \in I \). Let \( x = (A_1, \ldots, A_n, B_{n+1}) \). Let \( c \in \mathbb{G}_m \). Then

1. \( \mu(e_i^c(x)) = \alpha_i^\vee(c)\mu(x) \).
2. \( \varepsilon_i(x)\alpha_i(\mu(x)) = \varphi_i(x) \).
3. \( \varphi_i(e_i^c(x)) = c\varphi_i(x), \varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x) \).
4. \( W(e_i^c(x)) = W(x) + (c-1)\varphi_i(x) + (c^{-1}-1)e_i(x) \).

**Proof.** We pick \( A_{n+1} \) such that \( \pi(A_{n+1}) = B_{n+1} \). By [203],

\[
\chi_i^\ast(u_{B_k, B_{k+1}}^{A_{n+1}}) = \frac{\alpha_i(h_{A_1, A_{n+1}})}{\mathcal{L}_i(u_{B_{n+1}, B_n}^{A_1})} = \frac{\alpha_i(h_{A_n, A_{n+1}})}{\mathcal{R}_i(u_{B_1, B_{n+1}}^{A_n})}
\]

Therefore,

\[
\frac{\varphi_i(x)}{\varepsilon_i(x)} = \frac{\mathcal{L}_i(u_{B_{n+1}, B_n}^{A_1})}{\mathcal{R}_i(u_{B_1, B_{n+1}}^{A_n})} = \alpha_i(h_{A_1, A_{n+1}}h_{A_n, A_{n+1}}^{-1}) = \alpha_i(\mu(x)). \quad (285)
\]
Similarly, by (??) we have

\[ \mu(x) = h_{A_1, A_2} \beta(u_{B_1, B_2}^{A_3}) = h \prod_{k=1}^{m} \beta_k(b_k^{-1}). \]

Thus 1 follows. By (??) and definitions of the functions \( \varepsilon_i, \varphi_i, \mathcal{W} \), we get 3 and 4.

**Lemma A.3.** For two different \( i, j \in I \), let \( a_{ij} = \langle \alpha_i, \alpha_j \rangle \). We have the following relation

\[
\begin{align*}
  e_i^c e_j^z &= e_j^z e_i^c & &\text{if } a_{ij} = 0; \\
  e_i^c e_j^c e_i^z &= e_j^z e_i^c e_j^z e_i^c & &\text{if } a_{ij} = a_{ji} = -1; \\
  e_i^c e_j^c e_i^c e_j^z &= e_j^z e_i^c e_j^c e_i^z & &\text{if } a_{ij} = -1, a_{ji} = -2; \\
  e_i^c e_j^z e_i^c e_j^z &= e_j^z e_i^c e_j^c e_i^c & &\text{if } a_{ij} = -1, a_{ji} = 3.
\end{align*}
\]

**Proof.** By the definition of the action \( e_i \), it is enough to prove the case when \( n = 2 \), i.e. \( \text{Conf}(A_1, A_2, B) \). By (??), we reduce the Lemma to the case when \( G \) is of rank 2. The first identity is clear. For the second identity, one can check for \( G = \text{PGL}_3 \) case directly. The third and the fourth identities can be reduced to simple-laced case by “folding”. See [BK1] Section 5.2 for details.

**Theorem A.4.** Let \( a \in \text{Conf}^+(A^{n+1}, B) \) and let \( b \in \text{Conf}^+(A^{n+1}, B) \). Recall the convolution product \(*\) from Section 7.2. The following identities hold

1. \( p(a * b) = p(a)p(b), \mu(a * b) = \mu(a)\mu(b) \).
2. \( \mathcal{W}(a * b) = \mathcal{W}(a) + \mathcal{W}(b) \).
3. \( \varphi_i(a * b) = \varphi_i(a)\varphi_i(b), \varepsilon_i(a * b) = \varepsilon_i(a)\varepsilon_i(b) \).
4. \( e_i^c(a * b) = e_i^c(a) * e_i^c(b) \), where \( c_1 = \frac{\varepsilon_i(a) + \varepsilon_i(b)}{e_i(a) + b_i(b)} \), \( c_2 = \frac{\varepsilon_i(a) + \varepsilon_i(b)}{e_i(a) + b_i(b)} \).

**Proof.** 1-2. Follow from Lemma 6.1.

3. We prove the second formula. The first one follows similarly. By Figure 13 it suffices to prove the case when \( a = (A_1, A_2, B_4), b = (A_2, A_3, B_4) \in \text{Conf}(A^2, B) \). Then \( a * b = (A_1, A_2, A_3, B_4) \). Pick \( A_4 \in A \) such that \( \pi(A_4) = B_4 \). By (285), \( \alpha_i(h_{A_4, A_4}) = \alpha_i(h_{A_3, A_4})\varphi_i(b)/\varepsilon_i(b) \). By 204, we have

\[
\chi_i^*(u_{A_4}^{B_3, B_2}) = \frac{\alpha_i(h_{A_4, A_4})}{\varepsilon_i(b)}, \quad \chi_i^*(u_{B_2, B_4}^{A_4}) = \frac{\alpha_i(h_{A_4, A_4})\varphi_i(b)}{\varepsilon_i(a)\varepsilon_i(b)}, \quad \chi_i^*(u_{B_3, B_2}^{A_4}) = \frac{\alpha_i(h_{A_3, A_4})}{\varepsilon_i(a + b)}.
\]

Since \( \chi_i^*(u_{B_3, B_2}^{A_4}) = \chi_i^*(u_{B_3, B_2}^{A_4}) + \chi_i^*(u_{B_3, B_2}^{A_4}) \), the formula follows.

4. It suffices to prove the case when \( a = (A_1, A_2, B_4), b = (A_2, A_3, B_4) \). Let \( e_i^c(a * b) = (A_1, A_2, A_3, B_4) \). Recall the definition of \( e_i \) by cross ratio in Section 6.1. Let \( P \in \mathcal{P}_1 \) be the parabolic subgroup containing \( B_4 \) and \( B'_4 \). Let \( B'_1, B'_2, B'_3 \in B_P \) such that

\[
B_4 \xrightarrow{s_i} B_k \xrightarrow{s_i w_0} \pi(A_k), \quad k = 1, 2, 3.
\]
We have
\[ c = r(B_4, B'_4; B'_1, B'_3) = r(B_4, B'_4; B'_1, B'_2)r(B_4, B'_4; B'_2, B'_3) = c_1 c_2. \] (290)

Note that
\[ \varepsilon_i(a) + \varphi_i(b) = \chi(u_{B'_1,B'_4}) + \chi(u_{B'_1,B'_3}) = \chi(u_{B'_1,B'_3}) + \chi(u_{B'_1,B'_3}) = c_1^{-1} \varepsilon_i(a) + c_2 \varphi_i(b). \] (291)

Combining (290) and (291), we get 4. \( \square \)

**Remark.** This Theorem recovers the properties in [BK1, Lemma 3.9]. It is analogous to the tensor product of Kashiwara’s crystals.

A.2 A simple illustration of tensor product of crystals.

Recall the subsets \( B_{\mu}^{\lambda} \) and \( C_{\lambda}^{\nu} \). We tropicalize the map
\[ c_{1,n+1} : \text{Conf}(A^{n+1}, B) \to \text{Conf}_{n+1}(A) \times \text{Conf}(A^2, B), \]
\[ (A_1, \ldots, A_{n+1}, B_{n+2}) \mapsto (A_1, \ldots, A_{n+1}) \times (A_1, A_{n+1}, B_{n+2}). \]

It provides a canonical decomposition
\[ B_{\lambda}^{\mu} = \bigsqcup_{\nu} C_{\lambda}^{\nu} \times B_{\nu}^{\mu}. \] (292)

Recall the decomposition (240). As illustrated by Figure 44, we get

**Theorem A.5.** There is a canonical bijection
\[ \bigsqcup_{\mu_1 + \ldots + \mu_n = \mu} B_{\lambda_1}^{\mu_1} \times \ldots \times B_{\lambda_n}^{\mu_n} = \bigsqcup_{\nu} C_{\lambda_1,\ldots,\lambda_n}^{\nu} \times B_{\nu}^{\mu}. \]

We show that \( B_{\lambda}^{\mu} \) parametrizes a crystal basis. Figure 44 illustrates the tensor product of crystals. When \( n = 2 \), it recovers [BC, Theorem 3.2].

**B Weight multiplicities and tensor product multiplicities**

The following Theorem is due to [L, Section 8]. We provide a simple proof by using the set \( B_{\lambda}^{\mu} \).

**Theorem B.1 (L).** The weight multiplicity \( b_{\lambda}^{\mu} := \dim V_{\lambda}^{\mu} \) is equal to the cardinality of the subset
\[ \{ l \in A_{\lambda-\mu} \mid \mathcal{L}_i^l(l) \leq \langle \lambda, \alpha_i^* \rangle, \forall i \in I \} \subset U_{\lambda}^+(\mathbb{Z}). \] (293)
Proof. Recall the positive birational isomorphism

\[ \alpha_2 = (\pi_{12}, p_2) : \text{Conf}(A^2; B) \to H \times U, \quad (A_1, A_2, B_3) \mapsto (h_{A_1, A_2}, u_{B_1, B_3}^A). \]

The potential \( W \) on \( \text{Conf}(A^2; B) \) induces a positive function \( W_{\alpha_2} = W \circ \alpha_2^{-1} : H \times U \to \mathbb{A}^1 \). By Lemma 6.5, we get

\[ W_{\alpha_2}(h, u) = \sum_{i \in I} \alpha_{i^*}(h) L_i(u) + \chi(u). \]

Its tropicalization becomes

\[ W_{\alpha_2}^t(\lambda, l) = \min_{i \in I} \{ \langle \lambda, \alpha_{i^*} \rangle - L_i^t(l), \chi^t(l) \}. \]

Therefore, under the map \( p_{2i} \), the set \( B_{\mu}^\lambda \) is identified with \( (293) \). Note that the set \( B_{\mu}^\lambda \) parametrized the MV basis of the weight space \( V^\mu_{\lambda} \) of \( V_{\lambda} \). Therefore the weight multiplicity \( b_{\mu}^\lambda \) is equal to the cardinality of the set \( B_{\lambda}^\mu \).

Below we give a (rather simple) proof of [BZ, Corollary 3.4] based on Proposition 9.4.

**Theorem B.2** ([BZ]). The tensor product multiplicity \( c_{\lambda, \nu}^\mu \) is equal to the cardinality of the set

\[ \{ l \in A_{\lambda+\nu-\mu} \mid L_i^t(l) \leq \langle \lambda, \alpha_{i^*} \rangle \text{ and } R_i^t(l) \leq \langle \nu, \alpha_i \rangle \text{ for all } i \in I \} \subset U^*_\chi(Z^t) \]  

(294)

**Proof.** Recall the positive birational isomorphism

\[ \alpha_2 = (\pi_{12}, p_2, \pi_{23}) : \text{Conf}_3(A) \to H \times U \times H, \quad (A_1, A_2, A_3) \mapsto (h_{A_1, A_2, A_3}, u_{B_1, B_2, B_3}^A). \]

By Theorem 7.1, the function \( W_{\alpha_2} = W \circ \alpha_2^{-1} \) becomes

\[ W_{\alpha_2}(h_1, u, h_2) = \sum_{i \in I} \frac{\alpha_i(h_1)}{L_i^t(u)} + \chi(u) + \sum_{i \in I} \frac{\alpha_i(h_2)}{R_i^t(u)}. \]  

(295)

We tropicalize (295):

\[ W_{\alpha_2}^t(\lambda, l, \nu) = \min \{ \min_{i \in I} \{ \langle \lambda, \alpha_{i^*} \rangle - L_i^t(l), \chi^t(l) \}, \min_{i \in I} \{ \langle \nu, \alpha_i \rangle - R_i^t(l) \} \}. \]

Therefore, under \( p_{2i}^\mu \), the set \( C_{\lambda, \nu}^\mu \) is identified with (294). The rest is due to Proposition 9.4.

**C Cycles assigned to \( \text{Conf}_w^+(\mathbb{A}^n, 
B, B)(\mathbb{Z}^t) \)**

Given an element \( w \in W \), there are moduli spaces

\[ \text{Conf}_w(\mathbb{A}^n, B, B) \subset \text{Conf}(\mathbb{A}^n, B, B), \quad \text{Conf}_w^+(\mathbb{A}^n, B, B) \subset \text{Conf}^+(\mathbb{A}^n, B, B) \]
determined by the condition that the last two flags belong to the G-orbit \((B \times B)_w \subset B \times B\) parametrised by \(w\). So there are decompositions into disjoint unions

\[
\Conf(A^n, B, B) = \bigoplus_{w \in W} \Conf_w(A^n, B, B), \quad \Conf^O(A^n, B, B) = \bigoplus_{w \in W} \Conf^O_w(A^n, B, B).
\]

Similarly there are moduli space \(\Conf_w(Gr^n, B, B)\), and a canonical map

\[
\kappa_w : \Conf^O_w(A^n, B, B) \longrightarrow \Conf_w(Gr^n, B, B).
\]

The subgroup \(B_w := B \cap wBw^{-1}\) is a stabiliser of the group \(G\) acting on \((B \times B)_w\). So one has

\[
\Conf_w(Gr^n, B, B) = B_w(\K) \setminus Gr^n, \quad B_w := B \cap wBw^{-1}.
\]

For \(w = e\) we get

\[
\Conf_e(Gr^n, B, B) = \Conf(Gr^n, B, B).
\]

The space \(\Conf_w(A, B, B)\) is birational isomorphic to a subgroup \(U_{(w)} := U \cap w^{-1}B^{-1}w \subset U\). The latter has a positive structure defined by Lusztig [L] using reduced decompositions of \(w\). Combined with the standard construction, we arrive at a positive atlas on \(\Conf_w(A^n, B, B)\).

There is a potential \(W\) on \(\Conf_w(A^n, B, B)\), defined by restriction of the usual one. So the set \(\Conf^+_w(A^n, B, B)(\Z^t)\) is defined. The canonical map \(\kappa_w\) provides a collection of cycles

\[
\mathcal{M}^+_l \subset \Conf_w(Gr^n, B, B), \quad l \in \Conf^+_w(A^n, B, B)(\Z^t).
\]

Notice that our approach makes the map \(\kappa\) transparent, and allows to avoid any kind of explicit parametrisations in its definition. It makes obvious a parametrisation of generalized MV cycles, defined as components of \(\overline{S^n_w} \cap S^n_w\) for arbitrary \(w \in W\) – one needs to use the whole configuration space \(\Conf(A, B, B)\), not only its generic part.

**Constructible functions** \(D_F\). Let \(F\) be a rational function on \(A^n \times (B \times B)_w\), invariant under the left diagonal action of \(G\). Using the isomorphism \(\mathbb{Q}(A^n \times (B \times B)_w)^G = \mathbb{Q}(A^n)^B_w\), we realize \(F\) as an \(B_w\)-invariant rational function on \(A^n\). Define a function \(D_F\) on \(G(\K)^n\) by

\[
D_F(g_1(t), ..., g_n(t)) := \text{val} F(g_1(t)A_1, ..., g_n(t)A_n) \quad \text{for some } A_1, ..., A_n \in A(\mathbb{C}).
\]

It is left \(B_w(\K)\)-equivariant, and right \(G(\O)^n\)-equivariant, and so descends to a function

\[
D_F : B_w(\K) \setminus Gr^n \longrightarrow \Z.
\]

**Remark.** The function \(D_F\) assigned to a positive rational function \(F\) on \(\Conf(A^n, B, B)\) is not a function on the whole space \(\Conf(Gr^n, B, B)\), only on its generic part. One has

\[
\Conf(Gr^n, B, B) = \bigoplus_{w \in W} \Conf_w(Gr^n, B, B),
\]

and one needs to use the positive rational functions on the strata \(\Conf_w(A^n, B, B)\) to define constructible functions on the strata \(\Conf_w(Gr^n, B, B)\).

**Theorem C.1.** Let \(l \in \Conf^+_w(A^n, B, B)(\Z^t)\), and \(F \in \mathbb{Q}_+(\Conf_w(A^n, B, B))\). Then we have

\[
D_F(\mathcal{M}^+_l) = F^t(l).
\]
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