Investigation of $\Xi^- nn (S = -2)$ Hypernucleus in Low-energy Pionless Halo Effective Theory

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Abstract

In the strangeness $S = -2$ sector, we study the $\Xi^- nn (I = 3/2, J^P = 1/2^+)$ three-body system using pionless halo effective field theory (EFT), which provides a systematic model independent framework for assessing the feasibility of light particle-stable three-body bound states, utilizing low-energy universality. Here we take recourse to a simplistic speculation of the three-body system by eliminating the repulsive spin-singlet $\Xi^- n$ sub-system, while retaining the predominantly attractive (possibly bound) spin-triplet $\Xi^- n$ and the virtual bound spin-singlet $nn$ sub-system. In particular, a qualitative leading order EFT investigation by introducing a sharp momentum (ultraviolet) cut-off parameter $\Lambda_c$ into the Faddeev-like coupled integral equations, indicates a discrete scaling behavior akin to a renormalization group limit cycle, thereby suggesting the formal existence of Efimov states in the unitary limit, as $\Lambda_c \to \infty$. Our subsequent non-asymptotic analysis indicates that the three-body binding energy $B_3$ is sensitively dependent on the cut-off without the inclusion of three-body contact interactions. Furthermore, our analysis reproduces several values of the binding energy $B_3 \sim 3-4$ MeV, predicted in context of existing potential models, with the regulator $\Lambda_c$ in the range, $\sim 350 - 460$ MeV. Finally, based on these model inputs for $B_3$, a ballpark estimate of the three-body scattering length in the range, $2.6 - 4.9$ fm, is naively constrained by our EFT analysis, ostensibly demonstrating the universal nature of three-body correlations that is likely to manifest themselves in a halo-bound $\Xi^- nn$ system.

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I. INTRODUCTION

Over the last two decades the physics of hypernuclei has gained considerable attention in the strangeness nuclear physics community through numerous studies of exotic hypernuclei (for recent reviews, see e.g., Refs. [1–4]), and in astrophysical studies of neutron stars where hyperonic matter is expected to appear at their cores (e.g., see Refs. [5–7]). In particular, the strangeness $S = -2$ sector of late has elicited great interests behind ideas, such as the existence of putative bound states like the $H$-dibaryon and light exotic $\Xi$-hypernuclei, and to seek possible resolution of the well-known hyperon puzzle [8, 9]. For example, with regard to the feasibility of the $H$-particle, as conjectured by Jaffe more that 40 years ago [10], no definite conclusion has been reached till date, despite the extensive theoretical [11–18] and experimental investigations [19, 20]. It is envisaged that a thorough understanding of the underlying character of and role played by two-body hyperon-nucleon ($YN$) and hyperon-hyperon ($YY$) interactions, as well as hyperon-nucleon-nucleon ($YNN$) and hyperon-hyperon-nucleon ($YYN$) three-body forces (3BF), may be vital towards arriving at definite conclusions in regard to these contentious issues. Especially, in the context of the stability of neutron stars, it has been realized that the sole inclusion of the two-body interactions becomes questionable since they lead to considerable softening of the equation-of-state (EoS) to support stellar masses typically close to or larger than twice the solar mass ($2M_\odot$) against gravitational collapse. The answer in this case probably lies in the inclusion of an admixture of $NNN$, $\Lambda NN$, $\Xi NN$ and $\Lambda\Lambda N$ 3BFs that may potentially be the key in estimating the correct stiffness of the EoS of dense baryonic matter governing the stability of the cores. Notably, the Quantum Monte Carlo calculations by Lonardini et al [21, 22] have already shown encouraging indication that the inclusion of $\Lambda NN$ 3BF compensates the excessive overbinding due to the $\Lambda N$ interactions, ostensibly resolving the “$B_\Lambda$-overbinding” problem. More of such work in this direction would be worthwhile toward a comprehensive understanding of the 2017 observation of gravitational waves from two-neutron star mergers by the LIGO Scientific Collaboration [23].

In 2001, the NAGARA event [24] from the KEK E373 emulsion experiment undoubtedly provided the first evidence of the light double-$\Lambda$ hypernuclei $^6\Lambda\Lambda$He, demonstrating that the $S = -2$ $\Lambda\Lambda$ interactions are less attractive than $S = -1$ $\Lambda N$ interactions. On the other hand, the feasibility of a light $\Xi$-hypernuclei based on the state-of-the-art experimental [25–
and theoretical studies remains largely equivocal, since they were first claimed in the 1959 experimental work of Wilkinson et al. This is primarily due to the current acute scarcity of $S = -2$ empirical information needed to determine the underlying character of hypernuclear interactions, caused by the non-availability of any $\Xi$-hyperon scattering data, either current or obtainable soon in near future. All that one finds in the literature are a few scattered upper bounds for $\Xi^{-}p$ elastic and inelastic cross sections from emulsion experiments. Thus, it is no surprise that different existing model analyses lead to substantially contrasting views regarding the nature of the $\Xi N$ potentials, ranging from moderately or weakly attractive, even vanishing, to weakly repulsive. In fact, the KISO event from the KEK E373 experiment in 2015, which undeniably confirmed the particle-stable $\Xi$-hypernucleus $^{15}_{\Xi}C$ (interpreted as the ground state of a deeply bound $\Xi^{-}^{14}N$ cluster system with binding energy $4.38 \pm 0.25$ MeV), at the least corroborated that the constituent $\Xi N$ channels are attractive. Specifically, the updated Extended-soft-core (ESC08c) Nijmegen potential model has predicted that the $\Xi N$ two-body system in the maximal spin-isospin ($i = 1, j = 1$) channel is stable with $^3S_1$ scattering length $a_{\Xi n}^{(j=1)} = 4.911$ fm, while the ($i = 1, j = 0$) channel is mainly repulsive with $^1S_0$ scattering length $a_{\Xi n}^{(j=0)} = 0.579$ fm. With the former attractive $\Xi N$ channel having such a large scattering length, the two-body system was found to be bound, the so-called $D^*$ state, with an estimated binding energy of $1.56$ MeV ($1.67$ MeV) with (without) taking into consideration the latter repulsive $\Xi N$ channel. In a contrasting scenario, the recent SU(3) chiral effective field theory (EFT) based predictions from the relativistic calculations, as well as non-relativistic in-medium $G$-matrix analysis have practically ruled out the possibility of a stable $\Xi N$ bound state in the $(1,1)$ channel, being constrained respectively by the recent HAL QCD lattice results of Ref. and the aforementioned empirical upper bounds from $\Xi^{-}p$ cross sections data. Nevertheless, the Faddeev calculation analyses on the $(I = 3/2, J^P = 1/2^+)$ $\Xi NN$ three-body system, relying on the same updated $\Xi N$ Nijmegen ESC08c potential as input, have hinted at the feasibility of a deeply bound $\Xi NN$ state implying a strongly attractive nature of the $\Xi N$ interaction.
II. PIONLESS EFT (π EFT): A BRIEF SURVEY

Prompted by this unresolved scenario, we present in this work an alternative qualitative assessment regarding the viability of a putative Ξ⁻nn two-neutron halo-bound state in the $I = 3/2, J^P = 1/2^+$ channel. In particular, the reasons motivating our study of the ΞNN system in this maximal spin-isospin channel are as follows:

- first, the decoupling of this channel from strong decay into the open ΛΛN channel via $\Xi N \rightarrow \Lambda \Lambda$ is forbidden by isospin conservation, while Pauli principle works favorably, thereby supporting stable ΞNN bound states, and

- second, the absence of Coulomb effects to a great extent simplifies the construction of the Faddeev-like integral equations.

Our treatment is based on a low-energy pionless EFT (π EFT) where explicit pion exchanges are integrated out at scales much smaller than the pion mass. A speciality of such an approach is that the results are obtained following a general perturbative scheme without underlying model assumptions by utilizing principles of low-energy universality with controlled error estimates; observables are expressed as an expansion of a small low-energy parameter $\epsilon = Q/\Lambda_H$, with $Q$ being the typical momentum scale of dynamics within the system in question, and $\Lambda_H \sim m_\pi$, the hard or breakdown scale of the EFT which is identified with the pion mass $m_\pi$. Such a methodology is complementary to ab initio approaches, where the universal couplings or low-energy constants (LECs) in the effective Lagrangian are fixed from phenomenology (e.g., available scattering data and information on binding energies) could be used to make predictions on various few-body observables. Such universal aspects of π EFT have been successfully exploited to investigate the dynamics of finely tuned systems driven arbitrarily close to the unitary limit, either artificially, as in Feshbach resonances in ultra-cold atoms, or even naturally, as in nuclear systems with large two-body scattering lengths leading to formation of threshold two-body bound states, such as the deuteron or the aforementioned putative $D^*$ state. More interestingly, interacting three-body S-wave systems when driven to the unitary limit, lead to the well-known Efimov phenomenon, associated with an infinite tower of geometrically spaced three-body level states emerging from zero-energy (i.e., three-particle or particle-dimer breakup) threshold (see, e.g., and reference therein for a detailed review of Efimov...
physics and its applications in atomic and nuclear physics). In that case a modified \(^7\)EFT power counting scheme was suggested by Bedaque et al \[64, 65\], in which a non-derivatively coupled (naively subleading) three-body contact interactions was required to be promoted to leading order (LO) due to the appearance of a renormalization group (RG) limit cycle behavior with the onset of a discrete scaling symmetry in its couplings.

An important variant based on the generalization of the standard \(^7\)EFT, the so-called halo/cluster EFT \[66\], was introduced to study light halo nuclei and their clustering phenomena, characterized by multi-scale threshold dynamics at scales typically lower than in standard \(^7\)EFT. Such modified \(^7\)EFT analyses have been successfully extended to study light hypernuclei, since the very first in 2002 by Hammer \[67\] on the LO investigation of hypertriton (\(\Lambda H\)) as a \((I = 0, J = 1/2)\) \(\Lambda NN\) Efimov-like bound system. Subsequently, a number of halo/cluster EFT works based on similar LO analysis appeared in the literature, both in the \(S = −1\) \[68, 69\] and \(S = −2\) \[70–72\] strangeness sectors, in the search for exotic single and double-\(\Lambda\) hypernuclear states, e.g., \(nn\Lambda,^4\Lambda^4\text{He}, ^5\Lambda^5\text{H}, ^6\Lambda^6\text{He}\) and \(^6\Lambda^6\text{He}\). It is worth mentioning here that a novel \(ab\ initio\) LO \(^7\)EFT technique using few-body stochastic variational method of calculation was initially suggested in Refs. \[74–77\] for the study of lattice nuclei, complimentary to the halo \(^7\)EFT approach. The same framework was later used by Contessi et al. \[78\] in the \(S = −1\) sector to seek a solution to the \(B\Lambda\)-overbinding problem, and more recently in the feasibility studies of several light \(S = −2\) double-\(\Lambda\) hypernuclei \[79\].

\[1\] The \(^7\)EFT analysis by Ando et al. \[68\] attempted to investigate the feasibility of the putative \(nn\Lambda\) bound state, as reported by the HypHI Collaboration \[73\] in 2013. In that analysis a coupled system of integral equation was constructed in the physical basis involving only the spin projected two-couplings but without considering isospin projections. This led to a slightly different estimation of the asymptotic RG limit cycle scaling exponent \(s_{nn\Lambda}^0 = 0.80339\), which is inconsistent with the expected universal scaling based on the the relative three-particle mass ratios \[58\]. As recently elucidated by Hildenbrand and Hammer \[64\], the correct scaling could be restored by a proper reformulation of the integral equations in the spin-isospin basis leading to value, \(s_{nn\Lambda}^0 = 1.0076\), which is identical to that obtained in the study of hypertriton, and also reproduces the well-known asymptotic scaling, \(s_0 = 1.00624\), for identical masses \[58, 63\]. Furthermore, with their revised \(nn\Lambda\) analysis \[69\] a threshold ground state appeared at a momentum cut-off scale \(\Lambda_c \sim 600\) MeV, whereby the likelihood of a physically realizable Efimov-like bound/resonance state may not be excluded outright, unlike in Ref. \[68\] where such a possibility was practically ruled out with \(\Lambda_c \gtrsim 1.5\) GeV.
III. HALO $^\#$EFT OF $\Xi^{-}nn$

In this work, we use halo $^\#$EFT at LO to assess the feasibility of a $(S = 1/2, I = 3/2)$ $\Xi^{-}nn$ bound state primarily on the basis of Efimov universality as reflected through the sharp momentum cut-off scale ($\Lambda_c$) dependence of the RG limit cycle exhibited by the three-body coupling $g_3(\Lambda_c)$ (see details in the following subsection). Despite the significant uncertainties regarding the exact nature of the $\Xi N$ interactions in existing approaches, we preferably assume the following scenario for the purpose of an exploratory study. Our approach is motivated by the results from a series of recent potential model analyses [30, 32–34] based on Faddeev calculations, which rely on the basic assumption that the $^3S_1 \Xi^{-}n$ sub-system is dominantly attractive and bound with the aforementioned large and positive S-wave scattering length $a_{\Xi^{-}n}^{(1)}$, as predicted by the Nijmegen ESC08c model [48, 49]. It was especially noted in some of these model studies that if one included the real bound $^3S_1$ channel as the only $\Xi^{-}n$ sub-system channel (other than the virtual bound $^1S_0$ nn sub-system with large and negative S-wave scattering length $a_{nn} = -18.63$ fm [80]), the $\Xi^{-}nn$ system exhibited a three-body deeply bound state. On the other hand, no bound state was obtained with the $^3S_1$ channel replaced by the repulsive $^1S_0$ $\Xi^{-}n$ sub-system channel with the aforementioned small but positive S-wave scattering length $a_{\Xi^{-}n}^{(0)}$ [48, 49]. A similar situation happens with our framework as well, which primarily motivated the work in this paper. The Faddeev-type coupled system of integral equations in the momentum space [58] (also termed as the Skornyakov-Ter-Martirosyan or STM equations [81, 82]) does not exhibit an RG limit cycle with the repulsive $^1S_0$ $\Xi^{-}n$ channel included as the only $\Xi^{-}n$ sub-system channel. This implies an unbound $\Xi^{-}nn$ system. On the other hand, we have checked that an asymptotic RG limit cycle in the (two-body) unitary limit is always found to manifest itself in the presence of the $\Xi^{-}n$ triplet channel, irrespective of the inclusion of the $\Xi^{-}n$ singlet channel. However, a full-fledged numerical evaluation of the integral equations in our context of the so-called dibaryon formalism [57, 64, 65] becomes technically demanding, especially when dealing with the repulsive $\Xi^{-}n$ sub-system with a small and positive scattering length. The problem is attributed to the presence of a deep pole in the $^1S_0$ $\Xi^{-}n$ dibaryon propagator [cf. Eq. 6] corresponding to an unphysically large binding momentum, $\gamma_{\Xi^{-}n}^{(0)} \approx 1/a_{\Xi^{-}n}^{(0)} \approx 340$ MeV (estimate based on the Nijmegen ESC08c model [48, 49]). Since there is no straightforward way of “renormalizing” the effect of such a (two-body) pole, evidently the EFT breaks down
leading to erroneous estimation of the Efimov spectrum that turns out to be anomalously deep. Consequently, in this work we take recourse to a simplistic study of the $\Xi^-nn$ system to look for possible emergence of physically realizable Efimov-like trimer states by completely excluding the repulsive $^1S_0\Xi^-n$ channel. Moreover, in our halo EFT formalism the same scale associated with the triplet dibaryon pole defines the $n-(\Xi^-n)_t$ particle-triplet dimer break-up threshold, only beyond which the $\Xi^-nn$ trimer levels are expected to appear with binding energies given by the eigensolutions to the integral equations [38].

In the ensuing EFT analysis, we present a qualitative investigation of the regulator scale ($\Lambda_c$) dependence of the a priori undetermined three-body coupling $g_3$ introduced in the integral equation for the purpose of renormalization. Through this study we hope to establish a correspondence with existing potential model analyses on the same system [30, 32–34], thereby serving as a consistency check between both the approaches. In particular, based on existing model estimates for the three-body binding energy $B_3$, we naively predict a window of possible values of the S-wave three-body scattering length $a_3$, associated with the elastic $n-(\Xi^-n)_t$ scattering process, leading to three-body universal correlations such as the Phillips line [83]. Hence, the rationale behind our preliminary study is to bring to light the inherent universal features, albeit approximately, which could yield valuable insights for facilitating more rigorous future analyses to assess the fate of the realistic $\Xi^-nn$ system.

A. Effective Lagrangian and Formalism

In a simplified picture, the $\Xi^-nn$ system may be visualized as a two-neutron halo with the two loosely bound neutrons orbiting about the $\Xi^-$-hyperon “elementary” core forming a shallow bound state with a diffuse structure. Such universal class of systems exploits the distinct separation of scale between the typical dynamical scale, $Q \sim \gamma^{(1)}_{\Xi n} \sim 40$ MeV, associated with the “attractive pole” momentum of the $^3S_1\Xi^-n$ dibaryon propagator (ignoring possible artifact due to the deep pole at $\gamma^{(0)}_{\Xi n} \sim 340$ MeV associated with the repulsive $^1S_0\Xi^-n$ sub-system), and the breakdown scale, $\Lambda_H \sim m_\pi$ of standard $\pi$EFT. This implies that $\epsilon \sim Q/m_\pi \sim 1/3$ defines a reasonable expansion parameter that is amenable to an halo EFT treatment. The effective Lagrangian is constructed on the basis of available low-energy symmetries and degrees of freedom. The interaction vertices are represented by local contact interactions and the Lagrangian is expressed in a derivative expansion of the fundamental
Table I: PDG values of particle masses considered in the analysis.

| Particle         | Mass Symbol | Numerical Value (MeV) |
|------------------|-------------|-----------------------|
| Ξ-Hyperon        | $M_\Xi$     | 1321.710              |
| Neutron (n)      | $M_n$       | 939.565               |

fields. For our system, the fundamental degrees of freedom consist of the $\Xi^-$-hyperon and neutron ($n$) fields. The LO non-relativistic Lagrangian is free from derivative terms and expressed as a sum of one-, two-, and three-body parts, namely,

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{1\text{-body}} + \mathcal{L}_{2\text{-body}} + \mathcal{L}_{3\text{-body}}.$$  

(1)

Below we consider each of the components of the effective Lagrangian separately.

*One-body part.* The terms $\mathcal{L}_\Xi$ and $\mathcal{L}_n$ constitute the one-body Lagrangian $\mathcal{L}_{1\text{-body}}$ corresponding to the kinetic part of the $\Xi^-$-hyperon and neutron fields respectively, and are expressed in the physical basis as

$$\mathcal{L}_{1\text{-body}} = \mathcal{L}_\Xi + \mathcal{L}_n ,$$

where,

$$\mathcal{L}_\Xi = \Xi \left[ i \mathbf{v} \cdot \partial + \frac{(v \cdot \partial)^2 - \partial^2}{2M_\Xi} \right] \Xi ,$$

$$\mathcal{L}_n = n \left[ i \mathbf{v} \cdot \partial + \frac{(v \cdot \partial)^2 - \partial^2}{2M_n} \right] n ,$$

(2)

where, $M_\Xi,n$ are the physical masses of the $\Xi^-$-hyperon and neutron fields, as given in Table II and $\mathbf{v}^\mu = (1, \mathbf{0})$ is the four-velocity vector which is used to express the Lagrangian in a manifestly covariant manner akin to the heavy-baryon formalism. It follows that the non-relativistic propagators associated with these fundamental fields are given by

$$iS_{\Xi}(p_0, \mathbf{p}) = \frac{i}{p_0 - \frac{\mathbf{p}^2}{2M_\Xi} - i\eta} ,$$

$$iS_{n}(p_0, \mathbf{p}) = \frac{i}{p_0 - \frac{\mathbf{p}^2}{2M_n} - i\eta} ; \quad \eta \rightarrow 0 ,$$

(3)

where, $p_0$ and $\mathbf{p}$ are temporal and spatial parts of the generic four-momentum $p^\mu$.

*Two-body part.* In $\pi/\text{EFT}$, to deal with the formation of shallow S-wave bound states one needs to *unitarize* the two-body sector by employing the so-called Kaplan-Savage-Wise
(KSW) power counting rule \[51–55\]. To efficiently capture such two-body physics in the vicinity of a non-trivial fixed-point described by the RG of the two-body contact interactions, it was suggested to introduce auxiliary dimer fields in the effective Lagrangian \[56–58, 64, 65\]. Thus, in our case we introduce the dimer fields, namely, the spin-isospin triplet \((i = 1, j = 1)\) \(\Xi^- n\) dibaryon field \(u_1\), and the isospin-triplet spin-singlet \((i = 1, j = 0)\) \(nn\) dibaryon field \(u_0\). Here we re-emphasize that the spin-singlet isospin-triplet \((i = 1, j = 0)\) \(\Xi^- n\) sub-system is considered decoupled from the picture as it is not amenable to our EFT formalism. The corresponding two-body LO Lagrangian (physical basis) in terms of the dibaryon fields is given as

\[
\mathcal{L}_{2\text{-body}} = \mathcal{L}_{u_0} + \mathcal{L}_{u_1},
\]

\[
\mathcal{L}_{u_0} = - (u_0)^a \overrightarrow{\partial} \left( iv \cdot \partial + \frac{(v \cdot \partial)^2 - \partial^2}{4M_n} \right) (u_0)^a - y_0 (u_0)^a \overrightarrow{\partial} \left( n^T \hat{\mathcal{P}}_{(nn)} (n) + \text{h.c.} \right),
\]

\[
\mathcal{L}_{u_1} = - (u_1)^k \overrightarrow{\partial} \left( iv \cdot \partial + \frac{(v \cdot \partial)^2 - \partial^2}{2(M_n + M_{\Xi})} \right) (u_1)^k - y_1 (u_1)^k \overrightarrow{\partial} \left( n^T \hat{\mathcal{P}}_{(\Xi n)} (n) + \text{h.c.} \right),
\]

noting that the “wrong sign” in front the respective kinetic terms suggest the non-dynamical or quasi-particle nature of the dibaryon fields. Here, \(\hat{\mathcal{P}}_{(nn)} = \frac{1}{\sqrt{8}} \tau^a \tau^a \sigma_k\) and \(\hat{\mathcal{P}}_{(\Xi n)} = \frac{1}{\sqrt{8}} \tau^a \tau^a \sigma_k\) are the spin-isospin projection operators, with \(\sigma_k\) and \(\tau^a\) \((k, a = 1, 2, 3)\) being the Pauli matrices in the spin and isospin spaces respectively. The two-body non-derivatively coupled LO contact interactions or LECs \(y_{0,1}\) between the respective dibaryons and their constituent elementary fields encode all UV physics that remain unresolved in the EFT. These LO couplings are easily fixed as \[86\]

\[
y_1 = \sqrt{\frac{2\pi}{\mu}}, \quad \text{and} \quad y_0 = \sqrt{\frac{4\pi}{M_n}},
\]

where \(\mu = M_n M_{\Xi}/(M_n + M_{\Xi}) = 549.174\) MeV is the reduced mass of \(\Xi^- n\) two-body subsystem. Next, we spell out the “dressed” (unitarized) non-relativistic propagators for the \(nn\) and \(\Xi^- n\) dibaryon fields (cf. Fig[1], consistent with the KSW power counting scheme \[51–55\]) after being renormalized using the power-divergence subtraction (PDS) \[52\]:

\[
i \mathcal{D}_0(p_0, \mathbf{p}) = \frac{4\pi}{y_0^2 M_n} \gamma_{(0)} \frac{i}{\sqrt{-M_n (p_0 - \frac{p^2}{4M_n}) - i\eta - i\eta}},
\]

\[
i \mathcal{D}_1(p_0, \mathbf{p}) = \frac{2\pi}{y_1^2 \mu} \gamma_{(1)} \frac{i}{\sqrt{-2\mu (p_0 - \frac{p^2}{2(M_n + M_{\Xi})}) - i\eta - i\eta}}; \quad \eta \to 0,
\]
Figure 1: The renormalized dressed propagators for (A) \(^1S_0\) \(nn\), and (B) \(^3S_0\) \(\Xi^- n\) dibaryon fields. The dashed lines represent the \(\Xi^-\)-hyperon field propagator and the solid lines represent the neutron field propagator.

where at LO in \(\pi/E\)FT, \(\gamma_{nn}^{(0)} = 1/a_{nn}\) and \(\gamma_{\Xi n}^{(1)} = 1/a_{\Xi n}^{(1)}\), are the \((1,0)\) \(nn\) and \((1,1)\) \(\Xi^- n\) binding momenta respectively.

**Three-body part.** Formally the \(\Xi^- nn\) three-body system with a possible fine-tuned two-body sector exhibits the well-known Efimov effect at the unitary limit. This is reflected by the fact that the integral equations for the system with two-body interactions only become ill-defined in the asymptotic UV limit. The inherent reason for this anomalous UV behavior is the partial breakdown of the expected fixed-point scaling invariance of the system in the vicinity of bound states at the unitary limit into a discrete scaling invariance. A possible remedy to this problem is obtained by introducing a sharp momentum UV cut-off regulator \(\Lambda_c\) in the integral equations, which simultaneously necessitates the introduction of LO 3BF contact terms with scale dependent couplings \(g_3(\Lambda_c)\) to renormalize the artificial cut-off dependence. The resulting atypical scaling behavior gets reflected through the emergence of a RG limit cycle behavior in the 3BF couplings (for a pedagogical review on this topic, see Ref. [58]). Here we present a certain choice of the LO three-body contact interaction (counterterm) Lagrangian in the \(I = 3/2, J = 1/2\) channel, given by

\[
\mathcal{L}_{3\text{-body}} = -\frac{g_3(\Lambda_c)}{\Lambda_c^2} \left[ \frac{M_{\Xi n} y_1^2}{2} \{ (u_1)_l^a \hat{P}_{ab}^1 n \} \{ (u_1)_k^c \hat{P}_{cb}^1 n \} + \frac{\sqrt{3} M_{\Xi n} y_0 y_1}{\sqrt{2}} \{ (u_1)_l^a \hat{P}_{ab}^1 n \} \{ u_0^c \hat{P}_{cb} \Xi \} + \text{h.c.} \right],
\]  

(7)
where the spin-isospin projection operators have the following forms:

\[
[\mathcal{P}^{cb}_k]_{\alpha\beta} = \frac{1}{3\sqrt{3}} \left[ (\tau^c \tau^b)_{\alpha\beta} + \delta_{cb} \delta_{\alpha\beta} \right] \sigma_k ,
\]

\[
[\mathcal{P}^{cb}]_{\alpha\beta} = \frac{1}{3} \left[ (\tau^c \tau^b)_{\alpha\beta} + \delta_{cb} \delta_{\alpha\beta} \right] ,
\]

with \( \alpha, \beta = 1, 2 \) being the isospin-1/2 SU(2) indices \[87\]. As mentioned earlier, the cut-off dependence of \( g_3 \) is \textit{a priori} undetermined in the EFT and can be fixed only using a three-body datum, e.g., three-body binding energy \( B_3 \) or the corresponding scattering length \( a_3 \), none of which is currently available from experimental data. Thus, in the absence of such empirical information, we rely on the \( \Xi^{-}nn \) binding energy estimates from the Faddeev calculations provided by erstwhile potential model analyses, as in Refs. \[29, 30, 32-34\].

**B. Coupled Integral Equations**

For the sake of theoretical analysis, we study the \( \Xi^{-}nn \) system in both the kinematical three-body bound and scattering domains using a representative choice of \( 1 + 2 \rightarrow 1 + 2 \) elastic reaction channel, given by the following low-energy scattering process with dominant S-wave contributing:

\[
n + (\Xi^{-}n)_t \rightarrow n + (\Xi^{-}n)_t .
\]

In this regard, we must emphasize that above “reference” scattering process is chosen solely for the sake of demonstrating our theoretical methodology, irrespective of the impracticability of performing such experimental studies at current facilities. The chosen reaction channel yields a set of coupled Faddeev-type (Fredholm) integral equations in the momentum space, whose eigenvalues yield all possible values of binding energies \( B_3 \) of trimer level states and the corresponding eigenvectors yield the scattering amplitudes. In Fig. \[2\] we display the relevant Feynman diagrams for the scattering process in question, in term of the two \textit{half-off-shell} S-wave scattering amplitudes, namely, \( t_A(k, p; E) \) representing the elastic process, \( n + u_1 \rightarrow n + u_1 \), and \( t_B(k, p) \) representing the open inelastic channel, \( n + u_1 \rightarrow \Xi^{-} + u_0 \).

Here \( k = |k| \) (\( p = |p| \)) denotes the on-shell (off-shell) incoming (outgoing) relative three-momentum in the center-of-mass (CM) system, and \( E \) is the total CM energy of the three-body system given by

\[
E = \pm \frac{k^2}{2\mu_{n(n\Xi)}} - B_2 ; \quad B_2 = \frac{\left(\frac{\gamma_{\Xi^{-}n}}{2\mu}\right)^2}{2\mu} = 1.47 \text{ MeV} , \quad (10)
\]
Figure 2: Feynman diagrams for the representative coupled channel elastic scattering process, \( n + (\Xi^- n)_t \rightarrow n + (\Xi^- n)_t \), where “\( t \)” is used to denotes the \( ^3S_1 \Xi^- n \) sub-system. The solid (dash) line represents neutron (\( \Xi^- \)-hyperon) propagator. The off-shell double lines with insertions of the small empty oval (square) blobs represent the renormalized dressed \( ^1S_0 \) \( nn \) \( (u_0) \) and \( ^3S_0 \Xi^- n \) \( (u_1) \) dibaryon field propagators. The large blob \( t_A \) \( (t_B) \) denotes the elastic (inelastic) half-off-shell scattering amplitude corresponding to the \( n + u_1 \rightarrow n + u_1 \) \( (n + u_1 \rightarrow \Xi^- + u_0) \) scattering. The dark blobs represent three-body contact interaction.

where, the “−” sign is to be used for the kinematical three-body bound state domain and the “+” sign for the corresponding scattering domain. \( B_2 \) is the CM binding energy of a possible threshold bound \( (\Xi^- n) \), sub-system which also sets the particle-triplet dimer \( (n + u_1) \) break-up threshold energy, \(^2\) and \( \mu_{n(\Xi)} = M_n(M_\Xi + M_n)/(2M_n + M_\Xi) = 663.768 \) MeV is the corresponding reduced mass of three-body (particle-dimer) system. The construction methodology of these integrals equation is similar to those employed in several earlier \#EFT works [68, 70–72, 88]. The renormalized S-wave projected coupled channel elastic and inelastic integral equations with the introduced UV sharp momentum cut-off \( \Lambda_c \) are respectively given as

\(^2\) The value \( B_2 \) which corresponds to the the pole position of the \( u_1 \) dibaryon propagator in our halo EFT formalism may be compared to the binding energy of the putative \((1,1) \Xi N\) bound state \((D^+)\) predicted in the potential model analyses in Refs. [33, 34, 48, 49].
\[ t_A^{(R)}(p, k; E) = Z_{\Xi n} \left( \frac{y_1^2 M_{\Xi}}{2} \left[ K_{(a)}(p, k; E) - \frac{g_3(\Lambda_c)}{\Lambda_c^2} \right] \right. \]
\[ \left. - \frac{M_{\Xi}}{2\pi \mu} \int_0^{\Lambda_c} dq q^2 \left[ K_{(a)}(p, q; E) - \frac{g_3(\Lambda_c)}{\Lambda_c^2} \right] \phi_1 \left( E - \frac{q^2}{2M_n}, q \right) t_A^{(R)}(q, k; E) \right. \]
\[ + \frac{\sqrt{6}y_1}{\pi y_0} \int_0^{\Lambda_c} dq q^2 \left[ K_{(b_2)}(p, q; E) - \frac{g_3(\Lambda_c)}{\Lambda_c^2} \right] \phi_0 \left( E - \frac{q^2}{2M_{\Xi}}, q \right) t_B^{(R)}(q, k; E) \],
\[ (11) \]

and
\[ t_B^{(R)}(p, k; E) = -Z_{\Xi n} \sqrt{\frac{3}{2}} \left( y_1 y_0 M_n \right) \left[ K_{(b_1)}(p, k; E) - \frac{g_3(\Lambda_c)}{\Lambda_c^2} \right] \]
\[ + \sqrt{\frac{3}{2}} \frac{M_n y_0}{\mu \pi y_1} \int_0^{\Lambda_c} dq q^2 \left[ K_{(b_1)}(p, q; E) - \frac{g_3(\Lambda_c)}{\Lambda_c^2} \right] \phi_1 \left( E - \frac{q^2}{2M_n}, q \right) t_A^{(R)}(q, k; E), \]
\[ (12) \]

where the renormalized amplitudes
\[ t_{A,B}^{(R)}(p, k; E) = \sqrt{Z_{\Xi n}} t_{A,B}(p, k; E) \sqrt{Z_{\Xi n}} \]
\[ (13) \]
are obtained by multiplying the corresponding half-off-shell amplitudes \( t_{A,B}(p, k; E) \) by the wavefunction renormalization factor \( Z_{\Xi n} \) associated with the possible bound \((\Xi^- n)_1\) subsystem, given as
\[ Z_{\Xi n}^{-1} = \frac{y_1^2 \mu^2}{2\pi \gamma_{\Xi n}^{(1)}}. \]
\[ (14) \]

Finally, the renormalized elastic amplitude could be used to calculate the S-wave \( n - (\Xi^- n)_1 \) three-body scattering length by considering the threshold limit of the on-shell momentum, i.e., \( k \to 0 \), namely,
\[ a_3 = -\lim_{k \to 0} \frac{\mu_{n(\Xi)}}{2\pi} t_A^{(R)}(k, k). \]
\[ (15) \]

In the above integral equations, the term \( K_{(a)} \) denotes the S-wave projected one-\( \Xi^- \)-exchange interaction kernel, while \( K_{(b_1, b_2)} \) are the two possible variants of the one-neutron-exchange interaction kernel, namely,
\[ K_{(a)}(p, \kappa; E) = \frac{1}{2pk} \ln \left( \frac{p^2 + \kappa^2 + apk - 2\mu E}{p^2 + \kappa^2 - apk - 2\mu E} \right), \]
\[ (16) \]
\[ K_{(b_1)}(p, \kappa; E) = \frac{1}{2pk} \ln \left( \frac{bp^2 + \kappa^2 + pk - M_n E}{bp^2 + \kappa^2 - pk - M_n E} \right), \]
\[ (17) \]
\[ K_{(b_2)}(p, \kappa; E) = \frac{1}{2pk} \ln \left( \frac{p^2 + bk^2 + pk - M_n E}{p^2 + bk^2 - pk - M_n E} \right), \]
\[ (18) \]
where, the generic momentum $\kappa$ denotes either the incoming on-shell relative momentum $(k)$ or the loop momentum $(q)$. Also, $a = 2\mu/M_\Xi$ and $b = M_n/(2\mu)$ are mass dependent parameters.

C. Asymptotic Analysis

In order to assess that the coupled system of integral equation indeed has the potentiality to yield three-body bound state solutions, one needs to check for possible manifestation of Efimov effect at the asymptotic unitary limit as $\Lambda_c \to \infty$. In this case, all other low-energy scales in the problem (e.g., $E, \gamma_{01}, k \ll p, q \sim \Lambda_c$) become irrelevant, and the integral equations can be well approximated by considering only the homogeneous parts, as well as dropping all the $k$ dependence and three-body interactions terms with $g_3$. Thus, with no other relevant scales in the theory, the integral equations become dilation invariant and symmetric under the inversion transformation $q \to 1/q$, such that the half-off-shell channel amplitudes exhibit a power-law scaling, namely, $t_{A,B}(\kappa) \sim \kappa^{s-1}$, with $\kappa \sim \Lambda_c$ and exponent $s$ is an undetermined three-body parameter. By performing a sequence of Mellin transformation the integral equations can be converted into a single transcendental equation which could be used to solve for the exponent $s$, namely,

$$1 = \frac{M_\Xi}{2\mu C_1} \left[ \frac{\sin[s \sin^{-1}(a/2)]}{s \cos(\pi s/2)} \right] + \frac{3M_n}{\mu C_1 C_2} \left[ \frac{\sin[s \cot^{-1}\sqrt{4b-1}]}{s \cos(\pi s/2)} \right]^2,$$

where

$$C_1 = \sqrt{\frac{\mu}{\mu_{n(\Xi)}}}, \quad C_2 = \sqrt{\frac{M_n}{2\mu_{n(\Xi)}}},$$

and $\mu_{\Xi(nn)} = 2M_nM_\Xi/(2M_n + M_\Xi) = 775.942$ MeV is the reduced mass of the $\Xi^-+u_0$ particle-dimer system. Solving Eq. (19) yield an imaginary solution, i.e., $s = \pm is_0^\infty = \pm i0.803391 \cdots$. The solution immediately suggests the existence of an asymptotic UV RG limit cycle with a discrete scaling symmetry associated with the scale factor, $\lambda_\infty = e^{\pi/s_0^\infty} = 49.919712 \cdots$. Consequently, our LO EFT formally manifests Efimov effect in the unitary limit of the $\Xi^-nn$ system. Thus, it becomes imperative to introduce scale dependent 3BF as counterterms in the effective Lagrangian to renormalize the ill-defined asymptotic limit of the two-body sector. Such 3BF terms which are otherwise considered subleading in the naive dimensional counting must be promoted to the LO for consistency of renormalization (see Ref. [58]) for a
detailed discussion on asymptotic analysis of integral equations in the context of EFT). Here we must, however, mention that the asymptotic scaling exponent \( s_0^\infty \) we obtain considerably differs from the expected value, \((s_0^\infty)^{\text{exp}} \sim 1.01\), that follows from the universal RG limit cycle scaling depending on the relative three-particle mass ratios \( \Xi^\pm n \) sub-system channel from the integral equations, whose dynamics are not easily captured in our present low-energy EFT description.

IV. RESULTS AND DISCUSSION

In this section we present the results of our preliminary investigation of the cut-off regulator \((\Lambda_c)\) dependence of the integral equations, Eq. (12), at non-asymptotic low-energy scales. For the sake of numerical evaluations, we use the particle masses as presented in Table I while the S-wave scattering lengths \(a_{nn} = -18.63\) fm \[80\] and \(a_{\Xi n} = 4.911\) fm \[48, 49\] constitute the principal input two-body parameters in our LO EFT framework. In the last section, our asymptotic analysis demonstrated the evidence of Efimov states at the unitary limit of the \(\Xi^-nn\) system with an RG limit cycle discrete scaling symmetry determined by the multiplicative factor \(e^{\pi/s_0^\infty} \sim 50\). With \(\kappa^{(1)} \equiv \gamma_{\Xi n}^{(1)} \sim 40\) MeV as the typical momentum scale of the problem, it is envisaged that the next higher momentum scale appears at \(\kappa^{(2)} \equiv \lambda_{\Xi n} \gamma_{\Xi n}^{(1)} \sim 2\) GeV \(\gg \Lambda_H \sim m_\pi\), which is well beyond the accessibility of our low-energy EFT description. Hence, it is likely that only one Efimov-like state emerges as a plausible bound \(\Xi^-nn\) hypernucleus, if at all. Figure 3 shows the cut-off dependence of the three-body contact interaction coupling \(g_3(\Lambda_c)\). In the absence of a three-body datum to constrain this \textit{a priori} unknown coupling, we at the very outset of our analysis assume that the \(\Xi^-nn\) system already exhibits at the least one three-body bound state whose eigenenergy \(E = -B_3\), corresponds to the one of possible binding energy predictions from the erstwhile Faddeev analysis based potential models, as in Refs. [30, 32–34]. Hence, we choose our benchmark range of the input three-body binding energy, namely, between \(B_3 = 2.886\) MeV taken from Ref. [34], and \(B_3 = 4.06\) MeV taken from Ref. [30], both relying on the same two-body inputs, e.g., S-wave scattering length \(a_{\Xi n}^{(1)} = 4.911\) fm, provided by the updated
Figure 3: The first three branches corresponding to the approximate RG limit cycle behavior of the three-body coupling $g_3$ for the $\Xi^-nn$ system, as a function of cut-off scale $\Lambda_c$, obtained by solving the integral equations, Eq. (11) & (12). The input three-body binding energies $B_3 = 2.886$ MeV and 4.06 MeV, are predictions from the Faddeev based potential models [30, 34], and the input S-wave spin-isospin triplet $\Xi^-n$ scattering length $a^{(1)}_{\Xi^-n} = 4.911$ fm, provided by the ESC08c Nijmegen potential model analyses [48, 49].

ESC08c Nijmegen potential model analysis [48, 49]. The figure displays the typical quasi-periodic logarithmic singularities of the approximate RG limit cycles for the two limiting $B_3$ inputs. The corresponding non-asymptotic scale factor, $\lambda_n \lesssim \lambda_\infty$, may be obtained by considering the ratio of two successive cut-offs where the three-body coupling vanishes, i.e., if $g_3(\Lambda_c^{(n)}) = g_3(\Lambda_c^{(n+1)}) = 0$, then

$$\lambda_n = \frac{\Lambda_c^{(n+1)}}{\Lambda_c^{(n)}} ; \quad n = 1, 2, \ldots, \infty,$$

(21)

where $\Lambda_c^{(n)}$ is the cut-off corresponding to $n^{th}$ zero of $g_3$. For example, Table III displays the estimated non-asymptotic scale factors $\lambda_n$ corresponding to successive pairs of zeros of the $g_3$ function.

3 It is notable that the value $B_3 = 2.886$ MeV, as obtained in model analysis of Ref. [34], resulted from considering both the repulsive (1,0) and attractive (1,1) $\Xi^-N$ channels in the Feddeev calculations, whereas the value $B_3 = 4.06$ MeV in model analysis of Ref. [30] resulted from considering only the attractive channel. However, irrespective of the details we consider their predicted three-body binding energies as input to our EFT analysis.
Table II: The approximate RG limit cycle behavior with scale factor, $\lambda_n \to \lambda_\infty$, obtained by solving the coupled integral equations, Eqs. (11) & (12) for the $\Xi^-nn$ system. Here, results for $n = 1 - 4$ are only displayed which indicate a rapid convergence toward the asymptotic limit $\lambda_\infty = 49.919712 \cdots$.

The input three-body binding energies $B_3 = 2.886$ MeV and 4.06 MeV, are predictions from the Faddeev based potential models [30, 34], and the input S-wave spin-isospin triplet $\Xi^-n$ scattering length $a^{(1)}_{\Xi n} = 4.911$ fm, provided by the ESC08c Nijmegen potential model analyses [48, 49].

| Binding Energy $B_3$ (MeV) | $n \in \mathbb{Z}$ | $n^{th}$ zero of $g_3$ | $(n + 1)^{th}$ zero of $g_3$ | Scale factor $\lambda_n = \Lambda^{(n+1)}/\Lambda^{(n)}$ |
|---------------------------|----------------|---------------------|---------------------|------------------------|
|                           | $\Lambda^{(n)}_n$ (MeV) | $\Lambda^{(n+1)}_n$ (MeV) |                     |
| 2.886 [34]                | 1   | 334.283            | 16344.134           | 48.893105 \cdots     |
|                           | 2   | 16344.134          | 815412.631          | 49.890232 \cdots     |
|                           | 3   | 815412.631         | 40704680.527        | 49.919119 \cdots     |
|                           | 4   | 40704680.527       | 2031965537.021      | 49.919702 \cdots     |
| 4.06 [30]                 | 1   | 465.937            | 22919.007           | 49.189069 \cdots     |
|                           | 2   | 22919.007          | 1143628.429         | 49.898690 \cdots     |
|                           | 3   | 1143628.429        | 57089119.370        | 49.919290 \cdots     |
|                           | 4   | 57089119.370       | 2849872042.899      | 49.919706 \cdots     |

RG limit cycles, each for the two input values of the binding energy $B_3$. In each case the scale factor $\lambda_n$ is obtained close to but less than the asymptotic value $\lambda_\infty$. But, however, by progressively choosing larger pairs of the successive zeros of $g_3$, i.e., with $n \to \infty$, $\lambda_n$ is found to converge rapidly to $\lambda_\infty$.

Next, in Fig. [4] we display the cut-off variation of the binding energy $B_3$ excluding the 3BF term, i.e., with $g_3 = 0$ in the integral equations. In particular, due to the ambiguities regarding the precise nature of the (1,1) $\Xi N$ sub-system interactions between the different existing phenomenological analyses [25, 33, 38–49], we consider here two representative scenarios with contrasting perspectives, either of which formally may generate an Efimov-like bound state emerging from the respective thresholds, as elucidated below:

• First, the scenario with $a^{(1)}_{\Xi n} = 4.911$ fm, as predicted by the updated ESC08c Nijmegen potential model analyses in Ref. [48, 49], suggests a strongly attractive $^3S_1 \Xi^-n$ sub-
system commensurate with the likely existence of a two-body threshold bound state, such as the $D^*$ [33, 34]. Consequently, in the three-body sector with a pair of likely bound $(\Xi^- n)_t$ sub-systems and a virtual bound $nn$ sub-system, the $\Xi^- nn$ assumes a halo-bound (samba configuration [89]) structure emerging from the particle-dimer break-up threshold at the CM energy $E = -B_2$. This corresponds to the solid line curve (left panel) in Fig. 4 representing the regulator dependence of the relative three-body binding energy $B_d = B_3 - B_2$, for an Efimov-like ($n = 0$) ground state which appear at the critical cut-off scale $\Lambda_{\text{crit}}^{(0)} \approx 80$ MeV.

- Second, the scenario with $a_{(1)}^{(1)} = -1.17$ fm, as predicted by the chiral EFT analysis up to next-to-leading-order (NLO) in Ref. [45], suggests a weakly attractive $^3S_1 \Xi^- n$ sub-
system that is unlikely to exhibit a two-body threshold bound state. Consequently, in the three-body sector with none of the two-body sub-systems bound, the $\Xi^{-}nn$ assumes a bound *borromean* structure [89] emerging from the three-particle break-up threshold $E = 0$. This corresponds to the dashed line curve (right panel) in Fig. 4 representing the cut-off variation of $B_3$ for the corresponding Efimov-like ground state which appears at the scale $\Lambda_{\text{crit}}^{(0)} \approx 1940$ MeV.

Evidently, the large value of the critical cut-off in the latter scenario is not amenable to our EFT framework, and hence the resulting Efimov-like state is unlikely to physically manifest as a bound $\Xi$-hypernucleus. In contrast, the small value of the critical cut-off in the former scenario lies well within the realm of our EFT, indicating an encouraging prospect of a potentially feasible $\Xi^{-}nn$ Efimov-bound state. Thus, we shall henceforth discuss our results only pertaining to the former choice of $\Xi^{-}nn$ system.

In the absence of the three-body contact interactions for renormalization our results for the three-body binding energy exhibit considerable sensitivity to the cut-off variations. Figure 4 also compares our results with the regulator independent predictions for the $\Xi^{-}nn$ binding energy from the potential models [30, 34] which also rely on the two-body inputs from the Nijmegen ESC08c model analyses [48, 49]. Our EFT eigenenergies reproduce the model predictions $B_3 = 2.886$ MeV of Ref. 34 and $B_3 = 4.06$ MeV of Ref. 30 at the cut-off scales $\Lambda_c \approx 334$ MeV and $\Lambda_c \approx 465$ MeV respectively. The situation is clearly elucidated in Fig. 5 where we plot the variation of $B_3$ as function of the input scattering length $a_{\Xi n}^{(1)}$ for several fixed cut-off values. The potential model predicted binding energy $B_3$, namely, in the range $2.886 - 4.06$ MeV, as demarcated by the horizontal band in the figure, is well constrained within the regulator range of $\Lambda_c \approx 334 - 465$ MeV. In particular, Table III summarizes the $\Lambda_c$ values at which our EFT solutions reproduce several of the existing Faddeev calculations based model predictions for the $\Xi^{-}nn$ binding energy [30, 32–34]. Although such a regulator range apparently seems well beyond the expected EFT hard scale, $\Lambda_H \sim m_\pi$, the potential model results may still be accommodated within the framework of a halo EFT having an extended domain of validity than standard EFT. Consequently, such a modified EFT should have a larger breakdown scale, say, $\tilde{\Lambda}_H \lesssim 500$ MeV, where interactions between the $\Xi$-hyperon and neutron are possibly dominated by two-pion ($\pi\pi$) or sigma ($\sigma$) meson exchange mechanisms, since one-pion-exchanges are naturally ruled out by isospin
invariance in strong processes.

Figure 5: Variation of the three-body binding energy $B_3$ of the $(I = 3/2, J = 1/2) \Xi^- nn$ system as a function of input values of the S-wave $^3S_1 \Xi^- n$ scattering length, $a_{\Xi n}^{(1)} > 0$, for fixed cut-offs $\Lambda_c$ excluding three-body interactions. The horizontal shaded band represents our benchmark range of values of $B_3$ considered between the limits, $B_3 = 2.886$ MeV and 4.06 MeV, predicted by the Faddeev calculations based potential model analyses [30, 34]. The vertical dotted line indicates the S-wave spin-isospin triplet $\Xi^- n$ scattering length $a_{\Xi n}^{(1)} = 4.911$ fm, predicted by the updated ESC08c Nijmegen model analyses [48, 49].

Finally as a simple demonstration of predictability of our EFT framework, we attempt a naive estimation of the $(I = 3/2, J = 1/2) \Xi^- nn$ three-body scattering length or more precisely the $n - (\Xi^- n)_t$ elastic S-wave scattering length $a_3$ by utilizing the three-body information on the binding energies predicted by the erstwhile potential models [30, 32-34]. Here we need to solve our coupled integral equations, Eqs. (11) and (12), in the kinematical scattering domain for the elastic on-shell renormalized amplitude $t_{A}^{(R)}$ and then consider its threshold limit (i.e., $k \rightarrow 0$) [cf. Eq. (15)]. Solving the integral equations with the 3BF terms excluded (i.e., with $g_3 = 0$) leads to strong regulator dependence with the resulting amplitude displaying quasi-periodic singularities akin to the limit cycle behavior (see left panel of Fig. 6). Such divergences are renormalized by introducing the scale-dependent 3BF counterterms with running coupling $g_3(\Lambda_c)$ which is already fixed from the RG limit cycle...
Figure 6: Regulator ($\Lambda_c$) dependence of the ($I = 3/2, J = 1/2$) $\Xi^-nn$ or $n - (\Xi^-n)_t$ S-wave elastic three-body scattering length $a_3$. They are obtained by solving the coupled integral equation, Eqs. (11) & (12), excluding the three-body coupling, i.e., $g_3 = 0$ (left panel) and including the three-body coupling, i.e., $g_3(\Lambda_c) \neq 0$ (right panel). For the later, $g_3$ is determined by the respective RG limit cycles (cf. Fig. 3) corresponding to the two three-body inputs $B_3 = 2.886$ MeV and $4.060$ MeV, taken from the potential models [30, 34]. The S-wave spin-isospin triplet $\Xi^-n$ scattering length $a_{\Xi n} = 4.911$ fm, is taken from the updated ESC08c Nijmegen model [48, 49]. Our predictions, namely, $a_3^\infty = 4.860$ fm and $2.573$ fm (right panel), correspond to the asymptotic limits of the renormalized plots.

behavior (cf. Fig. 3) once the binding energy ($B_3$) model inputs are considered.

Figure 6 (right panel) depicts the regulator dependence of the three-body scattering length $a_3(\Lambda_c)$ renormalized by the 3BF terms. As mentioned, the scale dependence of $g_3$ is fixed using the RG limit cycles corresponding to the model inputs for the three-body binding energy [30, 34]. Each renormalized plot still displays a residual regulator dependence stemming from the scale dependent counterterms, especially at low cut-off scales owing to the decoupling of most underlying physics. However, for large enough values of the cut-off, say $\Lambda_c \gtrsim 400$ MeV, most of the underlying low-energy three-body dynamics are well captured by our solutions to the integral equations. Consequently, the renormalization of $a_3$ by the counterterms becomes more effective at large $\Lambda_c$ leading to a well-defined asymptotic
\[ a_3^\infty = \lim_{\Lambda_c \to \infty} a_3(\Lambda_c). \]  

Thus, in each case of a different three-body input \( B_3 \), a constant value \( a_3^\infty \) is obtained, as demanded by renormalization invariance, which represents our predicted three-body scattering length. In particular, our limiting inputs \( B_3 = 2.886 \text{ MeV} \) and \( 4.06 \text{ MeV} \), lead to \( a_3^\infty = 4.860 \text{ fm} \) and \( 2.573 \text{ fm} \) respectively. In addition, Table III displays the other intermediate results corresponding to two more existing model inputs, namely, \( B_3 = 3.89 \text{ MeV} \) and \( 3.00 \text{ MeV} \), from the Faddeev analyses, Refs. [32, 33]. In this context, we note that for a possible negative choice of the \( ^3S_1 \Xi^-n \) scattering length, e.g., the NLO chiral EFT prediction \( a_\Xi^{(1)}_n = -1.17 \text{ fm} \) [45], the \((\Xi^-n)_t\) sub-system is unbound with an undefined kinematical particle-dimer scattering domain, i.e., \(-B_2 \leq E \leq 0\). In this case the scattering domain is only definable above the three-particle break-up threshold, i.e., \( E > 0 \), which is beyond the interest of our work. It is also noteworthy from the table that \( a_3^\infty \) decreases with increasing \( B_3 \), as anticipated by our naive expectation \( a_3^\infty \sim 1/\sqrt{2\mu_{n(\Xi n)}B_3}. \)

Our predicted range of three-body scattering lengths represents a naive ballpark estimate based on the induced universal three-body correlations which is expected to manifest themselves in a halo-bound \( \Xi^-nn \) system, despite the approximations considered in this analysis. To further elucidate the universal aspects, it is worth demonstrating the \( B_3 \) versus \( a_3^\infty \) correlations with our preferred choice of the input \( \Xi^-n \) scattering length, namely, \( a_\Xi^{(1)}_n = 4.911 \text{ fm} \) [48, 49]. This yields the well-known Phillips line plot for the \( \Xi^-nn \) system, as depicted in Fig. 7. The curve diverges as \( B_3 \to B_2 = 1.47 \text{ MeV}, \) the \( n-(\Xi^-n)_t \) particle-dimer threshold, whenever an Efimov-like bound state emerges at zero-energy threshold (i.e., \( B_d = B_3 - B_2 = 0 \)). A second virtual bound three-body state with large negative value of \( a_3^\infty \) is expected to emerge around \( B_3^{(\text{virt})} = e^{\pi/s_0}B_2 \approx 70 \text{ MeV}, \) where the Phillips line diverges again. However, the latter state lies outside the domain of validity of standard \( ^3\text{EFT} \) with an estimated breakdown energy \( B_3 \approx 14 \text{ MeV} \), as determined by the three-body binding momentum of the order of the pion mass, i.e., \( \gamma_3 = \sqrt{2\mu_{n(\Xi n)}B_3} \gtrsim m_\pi \). The four data points displayed in the figure correspond to the \( B_3 \) predictions from the potential model analyses [30, 32–34], all of which rely on the same input two-body scattering length \( a_\Xi^{(1)}_n = 4.911 \text{ fm} \) [48, 49] (cf. Table III). Evidently, the Phillips plot clearly demonstrates the compatibility of the model inputs with our EFT description.
Table III: Summary of our EFT results with two different input S-wave spin-isospin triplet $\Xi^-n$ scattering lengths, namely, $a^{(1)}_{\Xi^-n} = 4.911 \text{ fm}$, taken from the updated ESC08c Nijmegen model analyses [48, 49], and $a^{(1)}_{\Xi^-n} = -1.17 \text{ fm}$, taken from the NLO chiral EFT analysis [45]. Displayed are the cut-off scales ($\Lambda_c$) at which the level energy for the Efimov-like ground state (excluding three-body contact interactions) reproduce several existing potential model predictions of the three-body binding energies $B_3$ of the $(I = 3/2, J = 1/2) \Xi^-nn$ system [30, 32–34]. Also summarized are our predicted three-body scattering length $a^{\infty}_3$ corresponding to each model input for $B_3$, with the three-body coupling $g_3(\Lambda_c)$ determined by the respective RG limit cycles. Our results corresponding to the case, $a^{(1)}_{\Xi^-n} = -1.17 \text{ fm}$, has an undefined particle-dimer scattering domain and is unlikely to support a $\Xi^-nn$ bound state.

| Two-body scattering length $a^{(1)}_{\Xi^-n}$ (fm) | Three-body binding energy $B_3$ (MeV) | Cut-off $\Lambda_c$ (MeV) | Three-body scattering length $a^{\infty}_3$ (fm) |
|-----------------------------------------------|---------------------------------|-------------------|---------------------|
| 4.911 (ESC08c) [48, 49]                       | 2.886 [34]                      | 334               | 4.860               |
|                                               | 2.89 [32]                       | 335               | 4.849               |
|                                               | 3.00 [33]                       | 348               | 4.562               |
|                                               | 4.06 [30]                       | 465               | 2.573               |
| -1.17 (Chiral EFT) [45]                       | 2.886 [34]                      | 2333              | -                   |
|                                               | 2.89 [32]                       | 2334              | -                   |
|                                               | 3.00 [33]                       | 2345              | -                   |
|                                               | 4.06 [30]                       | 2440              | -                   |

V. SUMMARY AND CONCLUSIONS

A knowledge of few-body dynamics in light $(S = -2)$ $\Xi$-hypernuclei can serve as essential input to determine the correct EoS for neutron star matter for possible explanation of the stability of neutron stars with masses close to or above $2M_\odot$. In this regard, the $(I = 3/2, J^P = 1/2^+) \Xi^-nn$ three-body system is one of the simplest systems to investigate the nature of the underlying 3BF, owing to the stability of this channel against strong decays and to the Coulomb-free dynamics. However, the impracticability of performing $\Xi$-hyperon scattering experiments and the lack of empirical data thereof have so far eluded rigorous
Figure 7: Phillips line correlation plot for the \((I = 3/2, J = 1/2)\) \(\Xi^{-}nn\) system corresponding to the input \(^3S_1\ \Xi^{-}n\) scattering length \(a^{(1)}_{\Xi n} = 4.911\) fm, as predicted by updated ESC08c Nijmegen model analyses [48, 49]. The data points correspond to the input values of the three-body binding energy \(B_3 = 2.886\) MeV, 2.89 MeV, 3.00 MeV and 4.06 MeV, predicted by the potential model analyses [30, 32–34]. The vertical dotted line on the left represents the \(n - (\Xi^{-}n)_{t}\) particle-dimer threshold at \(B_3 = B_2 = 1.47\) MeV, while the hashed region, \(B_3 \gtrsim 14\) MeV, represents the expected breakdown region of our EFT description.

determination of essential observables. Thus, a general qualitative approach relying solely on low-energy universality is demanded to illuminate specific characteristics of the underlying interactions that may reflect the emergence of exotic bound states.

Here we used the framework of leading order halo \(\pi\)-EFT in a speculative study to explore the feasibility of \(\Xi^{-}nn\) to be Efimov-bound. Notably, the presence of the predominantly repulsive \(^1S_0\ \Xi^{-}n\) sub-system channel potentially leads to the generation of anomalously deep two- and three-body bound states beyond the breakdown scale of the \(\pi\)-EFT. In our current simplistic approach, as the first step such unphysical contributions are avoided, albeit non-systematically, by explicitly decoupling this channel in the construction of our coupled integral equations in momentum space. In a more systematic (realistic) approach which includes both the repulsive \(^1S_0\) and attractive \(^3S_1\ \Xi^{-}n\) sub-system channels, one perhaps requires a modification of the dibaryon formalism, \textit{vis-a-vis} the halo \(\pi\)-EFT power
counting \[64, 65\]. For example, a simultaneous perturbative and non-perturbative expansion schemes in the singlet and triplet \(\Xi^-n\) channels respectively, may be incorporated. However, such a modified EFT analysis is beyond the scope of the current work and must be explored in the future as a possible next step. Our asymptotic analysis of the \(\Xi^-nn\) integral equations revealed the formal appearance of Efimov states in the unitary limit associated with a RG limit cycle with a discrete scale factor, \(\lambda_\infty = e^{\pi/s_\infty} \approx 50\). This factor is, however, quite different from the expected scaling based on the universality of three-particle mass ratios \([58]\), namely, \(\lambda_{\text{exp}} = e^{\pi/(s_\infty)^{\text{exp}}} \approx 20\), owing to the neglect of the \(1S_0\) \(\Xi^-n\) channel. Such scaling differences are certainly going to affect our numerical predictions in the low-energy non-asymptotic domain, but the general qualitative features are unlikely to change substantially even with both the \(\Xi^-n\) spin channels included.\(^4\)

Evidently, the absence of any three-body datum to fix the scale dependence of the three-body coupling \(g_3(\Lambda_c)\) is a major drawback of our approach which prevents robust predictions. Thus, we relied on several Faddeev calculations based models \([30, 32–34]\) for the input three-body binding energy \(B_3\) in the range, \([2.886 – 4.06\text{ MeV}]\). Moreover, the \#EFT formalism also requires two-body inputs, for which we considered the following S-wave scattering lengths: the \(1S_0\) \(nn\) scattering length \(a_{nn} = -18.63\) \([84]\), and the \(3S_1\) \(\Xi^-n\) scattering length \(a_{\Xi^-n}^{(1)}\). For the latter, we considered two contrasting choices, namely, \(a_{\Xi^-n}^{(1)} = 4.911\) fm \([48, 49]\) and \(-1.17\) fm \([45]\), given the current uncertainty regarding the exact nature of the spin-isospin triplet \((1,1)\) \(\Xi^-n\) interaction. However, with the input, \(a_{\Xi^-n}^{(1)} = -1.17\) fm, our investigations hinted at a predominantly unbound \(\Xi^-nn\) system. While the input, \(a_{\Xi^-n}^{(1)} = 4.911\) fm, indicated favourable prospects of the \(\Xi^-nn\) system to be physically realized as an Efimov-like ground state, with the proviso that the validity domain of our EFT could be augmented (with \(\tilde{\Lambda}_H \gtrsim 500\text{ MeV}\)) to accommodate interactions mediated by \(\pi\pi\) or \(\sigma\)-meson exchanges. Specifically, our leading order solutions to the integral equations excluding the 3BF terms could reproduce the aforementioned range of model binding energies \((B_3)\) for cut-offs in the

\(^4\) In either cases we expect to find remnant universal features like the quasi-periodic RG limit cycle nature of the leading order three-body coupling \(g_3(\Lambda_c)\). However, the inclusion of the repulsive \(1S_0\) \(\Xi^-n\) channel interactions is likely to shift the RG limit cycles towards higher cut-off values. Although, this reduces the likelihood of a viable Efimov-like state in the valid EFT domain, this does not pre-empt the possibility that the aforementioned scale shift toward higher values is partially or wholly compensated by the corresponding decrease in the scale factor \(\lambda_\infty\) from 50 to 20.
range $\Lambda_c \approx 335 - 464$ MeV.

Finally, as a simple demonstration of the predictive power of our EFT formalism, we evaluated the three-body S-wave $n-(\Xi^-n)_t$ scattering length in the range, $a_3^{\infty} \approx 2.6-4.9$ fm, by including our estimated $g_3(\Lambda_c)$ fixed from the RG limit cycles corresponding to the model $B_3$. Given the very speculative nature of this work and the indeterminable scattering length $a_3^{\infty}$ from present day emulsion or production experiments, the numerical figures displayed in this work are by all means ballpark estimates. Nevertheless, they are indicative of the emergent features of a prospective bound $\Xi^-nn$ system which induce universal three-body correlations like the Phillips line. Such three-body correlations are apparently robust against ambiguities in the two-body description provided the $B_3$ inputs do not change substantially. Thus, it is not entirely wishful to surmise similar qualitative features to arise from a more rigorous future investigation even including the $^1S_0$ $\Xi^-n$ sub-system channel. Needless to say that the above conclusions tacitly rely on our assumed halo-bound structure of the $\Xi^-nn$ system, viz. a sufficiently attractive $^3S_1$ $\Xi^-n$ sub-system that is ostensibly bound, and a repulsive $^1S_0$ $\Xi^-n$ sub-system with small but positive scattering length.

VI. ACKNOWLEDGEMENTS

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