Twists on the torus equivariant under the 2-dimensional crystallographic point groups

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Abstract

A twist is a datum playing a role of a local system for topological $K$-theory. In equivariant setting, twists are classified into four types according to how they are realized geometrically. This paper lists the possible types of twists for the torus with the actions of the point groups of all the 2-dimensional space groups (crystallographic groups), or equivalently, the torus with the actions of all the possible finite subgroups in its mapping class group. This is carried out by computing Borel's equivariant cohomology and the Leray-Serre spectral sequence. As a byproduct, the equivariant cohomology up to degree three is determined in all cases.

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1 Introduction

Topological $K$-theory is recently recognized as a useful tool for a classification of topological insulators in condensed matter physics. In Kitaev’s 10-fold way [11], the usual complex $K$-theory and also $KO$ or Atiyah’s $KR$-theory are of use. These classifications are in some sense the most simple case, and a recent study of topological insulators focuses on more complicated case. Such complicated cases arise when we take the symmetry of quantum systems into account. Then equivariant $K$-theory and its twisted version naturally fit into the classification scheme of such systems [5]. Actually, as will be explained in Section 2, a certain quantum system on the $d$-dimensional space $\mathbb{R}^d$ invariant under its space group provides a $K$-theory class on the $d$-dimensional torus $T^d$ equivariant under the point group of the space group. If the space group is of nonsymmetric, then the equivariant $K$-class is naturally twisted. In the case of $d = 2$, such (twisted) equivariant $K$-theories are computed for the 17 classes of 2-dimensional space groups, in view of the classification of topological crystalline insulators [19].

A part of this result computing a twisted equivariant $K$-theory leads to the discovery of topological insulators which are classified by $\mathbb{Z}_2$ but do not require the so-called time-reversal symmetry nor the particle-hole symmetry [18]. This type of topological insulators is new in the sense that the known topological insulators classified by $\mathbb{Z}_2$ so far require the time-reversal symmetry or the particle-hole symmetry.

Upon the understanding of the importance of twisted equivariant $K$-theory in condensed matter physics, a mathematically natural issue is to determine the possibility of ‘twists’ for equivariant $K$-theory. To explain this issue more concretely, let us recall that twisted $K$-theory [2, 15] is in some sense a $K$-theory with ‘local coefficients’. The datum playing the role of a ‘local system’ admits various geometric realization. In this paper, we realize them by twists in the sense of [4]. If a compact Lie group $G$ acts on a space $X$, then graded twists on $X$ are classified by the Borel equivariant cohomology $H^1_G(X; \mathbb{Z}_2) \times H^3_G(X; \mathbb{Z}_2)$.

Similarly, ungraded twists are classified by $H^3_G(X; \mathbb{Z}_2)$, on which we focus for a moment. (Sometimes $H^0_G(X; \mathbb{Z}_2)$ may be included in the twists, but we regard it as the degree of the $K$-theory.)

By definition, the Borel equivariant cohomology $H^3_G(X; \mathbb{Z}_2)$ is the usual cohomology $H^n(EG \times_G X; \mathbb{Z}_2)$ of the Borel construction $EG \times_G X$, which is the quotient of $EG \times X$ by the diagonal $G$-action, where $EG$ is the total space of the universal $G$-bundle $EG \to BG$. Associated to the Borel construction is the fibration $X \to EG \times_G X \to BG$, and hence the Leray-Serre spectral sequence
that converges to the graded quotient of a filtration

\[ H^n_G(X; \mathbb{Z}) \supset F^1 H^n_G(X; \mathbb{Z}) \supset F^2 H^n_G(X; \mathbb{Z}) \supset \cdots \supset F^{n+1} H^n_G(X; \mathbb{Z}) = 0. \]

The subgroups \( F^p H^n_G(X; \mathbb{Z}) \subset H^n_G(X; \mathbb{Z}) \) can be interpreted geometrically along the classification of twists, and there are four types (see Section 3 for detail):

(i) Twists which can be represented by group 2-cocycles of \( G \) with coefficients in the trivial \( G \)-module \( U(1) \). These twists are classified by \( F^3 H^3_G(X; \mathbb{Z}) \).

(ii) Twists which can be represented by group 2-cocycles of \( G \) with coefficients in the group \( C(X, U(1)) \) of \( U(1) \)-valued functions on \( X \) regarded as a (right) \( G \)-module by pull-back. These twists are classified by \( F^2 H^3_G(X; \mathbb{Z}) \).

(iii) Twists which can be represented by central extensions of the groupoid \( X//G \). These twists are classified by \( F^1 H^3_G(X; \mathbb{Z}) \).

(iv) Twists which cannot be represented by central extensions of \( X//G \).

The equivariant twists on \( T^d \) arising from quantum systems on \( \mathbb{R}^d \) as will be explained in Section 2 belong to \( F^2 H^3_P(T^d; \mathbb{Z}) \) with \( P \) the point group of a \( d \)-dimensional space group \( S \), and so are the twists considered in [19]. Now, the mathematical issue is whether the twists arising in this way cover all the possibility or not. The present paper answers this question by a theorem in the case of \( d = 2 \).

To state the theorem, let \( S \) be a 2-dimensional space group, which is also known as a 2-dimensional crystallographic group, a plane symmetry group, a wallpaper group, and so on. It is a subgroup of the Euclidean group \( O(2) \times \mathbb{R}^2 \) of isometries of \( \mathbb{R}^2 \), and is an extension of a finite group \( P \subset O(2) \) called the point group by a rank 2 lattice \( \Pi \cong \mathbb{Z}^2 \) of translations of \( \mathbb{R}^2 \):

\[
\begin{align*}
1 & \longrightarrow \mathbb{R}^2 \longrightarrow O(2) \times \mathbb{R}^2 \longrightarrow O(2) \longrightarrow 1 \\
& \quad \cup \quad \cup \quad \cup \\
1 & \longrightarrow \Pi \longrightarrow S \longrightarrow P \longrightarrow 1.
\end{align*}
\]

Naturally the point group \( P \cong S/\Pi \) acts on the torus \( T^2 = \mathbb{R}^2/\Pi \). Since \( P \) is a finite subgroup of \( O(2) \), its possibility is the cyclic group \( \mathbb{Z}_n \) of order \( n \) or the dihedral group \( D_n = \langle C, \sigma \mid C^n, \sigma^2, \sigma C \sigma C \rangle \) of degree \( n \) and order \( 2n \). The classification of 2-dimensional space groups has long been known, and there are 17 types [17, 8], which we label following [16]. Notice that some space groups share the same point groups, and there arise 13 distinct finite group actions on the torus. These actions realize essentially all the possible finite subgroups in the mapping class group of the torus [13], which is isomorphic to \( GL(2; \mathbb{Z}) \) as is well-known [14].

**Theorem 1.1.** Let \( P \cong S/\Pi \) be the point group of one of the 2-dimensional space groups \( S \), acting on \( T^2 = \mathbb{R}^2/\Pi \) naturally. Then, it holds that

\[ H^3_P(T^2; \mathbb{Z}) = F^0 H^3_P(T^2; \mathbb{Z}) = F^1 H^3_P(T^2; \mathbb{Z}). \]

This cohomology group and its subgroups \( F^p H^3_P(T^2; \mathbb{Z}) \) are as in Figure 1.1.
### Table: Space Group and Subgroups

| Space group $S$ | $P$ | ori | $H^3_P(T^2; \mathbb{Z})$ | $F^2$ | $F^3$ | $E^3_{p,3}$ | $E^4_{p,4}$ |
|-----------------|-----|-----|--------------------------|-------|-------|------------|------------|
| p1              | 1   | +   | 0                        | 0     | 0     | 0          | 0          |
| p2              | $\mathbb{Z}_2$ | +   | 0                        | 0     | 0     | 0          | 0          |
| p3              | $\mathbb{Z}_3$ | +   | 0                        | 0     | 0     | 0          | 0          |
| p4              | $\mathbb{Z}_4$ | +   | 0                        | 0     | 0     | 0          | 0          |
| p6              | $\mathbb{Z}_6$ | +   | 0                        | 0     | 0     | 0          | 0          |
| pm/pg           | $\mathbb{Z}_2$ | −   | $\mathbb{Z}_2^{m_3}$    | $\mathbb{Z}_2$ | 0     | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| cm              | $\mathbb{Z}_2$ | −   | $\mathbb{Z}_2$          | 0     | 0     | $\mathbb{Z}_2$ | 0          |
| pmm/pg/pgg      | $D_2$ | −   | $\mathbb{Z}_2^{m_3}$    | $\mathbb{Z}_2^{m_3}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^{m_3}$ |
| cmm             | $D_2$ | −   | $\mathbb{Z}_2^{m_3}$    | $\mathbb{Z}_2^{m_3}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0          |
| p3m1            | $D_3$ | −   | $\mathbb{Z}_2$          | 0     | 0     | $\mathbb{Z}_2$ | 0          |
| p3lm            | $D_3$ | −   | $\mathbb{Z}_2$          | 0     | 0     | $\mathbb{Z}_2$ | 0          |
| p4m/p4g         | $D_4$ | −   | $\mathbb{Z}_2^{m_3}$    | $\mathbb{Z}_2^{m_3}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^{m_3}$ |
| p6m             | $D_6$ | −   | $\mathbb{Z}_2^{m_3}$    | $\mathbb{Z}_2^{m_3}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0          |

Figure 1: The list of $H^3_P(T^2; \mathbb{Z})$ and its subgroups $F_P = F_P H^3_P(T^2; \mathbb{Z})$ for the point group $P$ of each space 2-dimensional space group $S$. The $E_{p,3}$-term of the Leray-Serre spectral sequence is related to these subgroups by $E_{p,3} \cong F_P / F_{P+1}$. The column “ori” indicates “+” if $P$ preserves the orientation of $T^2$ and “−” if not. Nonsymmorphic groups are pg, pmg, pgg and p4g.

#### Corollary 1.2

Under the same hypothesis as in Theorem 1.1, it holds that:

(a) All the twists can be represented by central extensions of $T^2$. In particular, there are no non-trivial twists if $P$ preserves the orientation of $T^2$.

(b) If $P$ does not preserve the orientation of $T^2$, then there are twists which can be represented by central extensions of $T^2//S$ but not by group 2-cocycles.

(c) The subgroup $F^2 H^3_P(T^2; \mathbb{Z})$ is generated by the twists represented by:

- the group 2-cocycle with values in $C(T^2, U(1))$ induced from the nonsymmorphism of $S$; and
- the group 2-cocycle with values in $U(1)$.

As a result, all the twists classified by $F^2 H^3_P(T^2; \mathbb{Z})$ have relationship to topological insulators, whereas there actually exist other twists which cannot be realized by group cocycles. At present their roles in condensed matter theory seem to be open.

Theorem 1.1 follows from case by case computations of the equivariant cohomology $H^3_P(T^2; \mathbb{Z})$ and the Leray-Serre spectral sequence. These computations contain enough information to determine the equivariant cohomology $H^3_P(T^2; \mathbb{Z})$, ($n \leq 2$) of the torus acted by the possible finite subgroups in the mapping class group $GL(2, \mathbb{Z})$.  

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Theorem 1.3. Let $P$ be the point group of one of the 2-dimensional space groups $S$, acting on $T^2 = \mathbb{R}^2/\Gamma$ naturally. For $n \leq 3$, the $P$-equivariant cohomology $H^n_P(T^2; \mathbb{Z})$ is as in Figure 1.3.

| Space group $S$ | $P$ | ori | $H^0_P(T^2)$ | $H^1_P(T^2)$ | $H^2_P(T^2)$ | $H^3_P(T^2)$ |
|----------------|-----|-----|---------------|---------------|---------------|---------------|
| p1             | 1   | +   | $\mathbb{Z}$ | $\mathbb{Z}$ | 0             | 0             |
| p2             | $\mathbb{Z}_2$ | +   | $\mathbb{Z}$ | 0             | $\mathbb{Z} \oplus \mathbb{Z}_2^{\times 4}$ | 0             |
| p3             | $\mathbb{Z}_3$ | +   | $\mathbb{Z}$ | 0             | $\mathbb{Z} \oplus \mathbb{Z}_2^{\times 2}$ | 0             |
| p4             | $\mathbb{Z}_4$ | +   | $\mathbb{Z}$ | 0             | $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ | 0             |
| p6             | $\mathbb{Z}_6$ | +   | $\mathbb{Z}$ | 0             | $\mathbb{Z} \oplus \mathbb{Z}_6$ | 0             |
| pm/pg          | $\mathbb{Z}_2$ | -   | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2^{\times 2}$ | $\mathbb{Z}_2^{\times 2}$ |
| cm             | $\mathbb{Z}_2$ | -   | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| pmm/pgm/gpg    | $D_2$ | -   | $\mathbb{Z}$ | 0             | $\mathbb{Z}_2^{\times 4}$ | $\mathbb{Z}_2^{\times 4}$ |
| cmm            | $D_2$ | -   | $\mathbb{Z}$ | 0             | $\mathbb{Z}_2^{\times 4}$ | $\mathbb{Z}_2^{\times 4}$ |
| p3m1           | $D_3$ | -   | $\mathbb{Z}$ | 0             | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| p3lm           | $D_3$ | -   | $\mathbb{Z}$ | 0             | $\mathbb{Z}_3 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| p4m/p4g        | $D_4$ | -   | $\mathbb{Z}$ | 0             | $\mathbb{Z}_2^{\times 4}$ | $\mathbb{Z}_2^{\times 4}$ |
| p6m            | $D_6$ | -   | $\mathbb{Z}$ | 0             | $\mathbb{Z}_2^{\times 2}$ | $\mathbb{Z}_2^{\times 2}$ |

Figure 2: The list of equivariant cohomology up to degree 3.

So far we focused on ungraded twists. To complete the classification of $P$-equivariant twists on $T^2$, we need to compute the equivariant first cohomology with coefficients in $\mathbb{Z}_2$, which provides the information of ‘grading’ of a twist. But, the computation is immediately completed by a simple application of the universal coefficient theorem to Theorem 1.3. Notice that the equivariant cohomology $H^1_P(T^2; \mathbb{Z}_2)$ also admits a filtration

$$H^1_P(T^2; \mathbb{Z}_2) = F^0 H^1_P(T^2; \mathbb{Z}_2) \supset F^1 H^1_P(T^2; \mathbb{Z}_2) \supset F^2 H^1_P(T^2; \mathbb{Z}_2) = 0.$$  

Because the degree in question is 1, the degeneracy of the Leray-Serre spectral sequence gives the identification

$$F^1 H^1_P(T^2; \mathbb{Z}_2) = \text{Hom}(P, \mathbb{Z}_2) = H^1_P(\text{pt}; \mathbb{Z}_2),$$

which is a direct summand of $H^1_P(T^2; \mathbb{Z}_2)$ and is also computed immediately by using the knowledge of the equivariant cohomology of pt in Subsection 4.1.

Corollary 1.4. Let $P$ be the point group of one of the 2-dimensional space groups $S$, acting on $T^2 = \mathbb{R}^2/\Gamma$ naturally. Then the $P$-equivariant cohomology $H^1_P(T^2; \mathbb{Z}_2)$ is as in Figure 1.4.

The grading of twists classified by $F^1 H^1_P(T^2; \mathbb{Z}_2) = \text{Hom}(P, \mathbb{Z}_2)$ plays a role in a quantum system with symmetry (see Remark 2.2). However, there are
Figure 3: The list of first equivariant cohomology with coefficients $\mathbb{Z}_2$. The quotient $H^1_p(T^2; \mathbb{Z}_2)/F^1 H^1_p(T^2; \mathbb{Z}_2)$ is denoted with $E_{1,0}^{1,0}$.

other gradings generally, and their role in condensed matter theory seems to be also open.

The outline of this paper is as follows: In Section 2, we explain how a certain quantum system leads to a twist and defines a twisted $K$-class, mainly based on a formulation in [5]. In Section 3, we review the Leray-Serre spectral sequence for Borel equivariant cohomology and the notion of twists. The geometric interpretations of the filtration of the degree 3 equivariant cohomology is also provided here, after a general property of the spectral sequence is established. Then, in Section 4, we carry out the case-by-case computations to prove Theorem 1.1 and Theorem 1.3. In the case of $p2$ and $pm/pg$, the computation of the $\mathbb{Z}_2$-equivariant cohomology is based on the Gysin exact sequence for ‘Real’ circle bundles [6], whereas the other computations are carried out in an elementary manner.

Throughout, familiarity with basic algebraic topology [1, 7] will be supposed.

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2 From quantum system to twisted $K$-theory

We here illustrate how twisted equivariant $K$-theory arises from a quantum system with symmetry, mainly based on a formulation in [5]. (We refer the reader to [20] for a $C^*$-algebraic approach.)
2.1 Setting

Let us consider the following mathematical setting:

- A lattice $\Pi \subset \Pi \otimes \mathbb{Z} = \mathbb{R}^d$ of rank $d$.
- A symmetric bilinear form $\langle , \rangle : \Pi \times \Pi \to \mathbb{Z}$ such that the induced bilinear form on $\mathbb{R}^d$ is positive definite.
- A subgroup $S$ of the Euclidean group $O(d) \ltimes \mathbb{R}^d$ which is an extension of a finite group $P \subset O(d)$ by:
  
  $1 \to \mathbb{R}^d \to O(d) \times \mathbb{R}^d \to O(d) \to 1$
  
  $1 \to \Pi \to S \xrightarrow{\pi} P \to 1$.

We assume $P$ preserves $\Pi$ and $\langle , \rangle : \Pi \times \Pi \to \mathbb{Z}$.

- A unitary representation $U : P \to U(V)$ on a Hermitian vector space $V$.

The group $S$ is nothing but a $d$-dimensional space group, and $P$ is called the point group of $S$. When $S$ is the semi-direct product of $P$ and $\Pi$, it is called ‘symmorphic’, otherwise ‘nonsymmorphic’.

Based on the mathematical setting above, we can introduce a quantum system on $\mathbb{R}^d$ which has $S$ as its symmetry and $V$ as its internal freedom:

- The ‘quantum Hilbert space’ consisting of ‘wave functions’ is the $L^2$-space $L^2(\mathbb{R}^d, V)$, on which $g \in S$ acts by
  
  $\psi(x) \mapsto (\rho(g)\psi)(x) = U(\pi(g))\psi(g^{-1}x)$.

- The ‘Hamiltonian’ is a self-adjoint operator $H$ on $L^2(\mathbb{R}^d, V)$ invariant under the $S$-action: $H \circ \rho(g) = \rho(g) \circ H$. A typical form of $H$ is $H = \Delta + \Phi$, where $\Delta = \sum \partial^2 / \partial x_i^2$ is the Laplacian and $\Phi : \mathbb{R}^d \to \text{End}(V)$ is a potential term.

2.2 Bloch transformation

Even if the Hamiltonian $H$ is invariant under the translation of $\Pi$, a solution $\psi$ to the ‘time-independent Schrödinger equation’ $H\psi = E\psi$ with $E \in \mathbb{R}$ is not necessarily $S$-invariant. The so-called ‘Bloch transformation’ allows us to handle such a situation.

Let $\hat{\Pi} = \text{Hom}(\Pi, U(1))$ denote the Pontryagin dual of $\Pi$. Because of the bilinear form $\langle , \rangle$, we can identify $\hat{\Pi}$ with $\mathbb{R}^d / \Pi$ by associating the homomorphism $m \mapsto \exp 2\pi i(m, k)$ to $k \in \mathbb{R}^d / \Pi$. We define $L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V)$ by

$L^2_{\Pi}(\hat{\Pi} \times \mathbb{R}^d, V) = \{ \psi \in L^2(\hat{\Pi} \times \mathbb{R}^d, V) | \hat{\psi}(k, x + m) = e^{2\pi i(m, k)}\hat{\psi}(k, x) \ (m \in \Pi) \}$. 

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We also define transformations $\mathcal{B}$ and $\mathcal{B}$, inverse to each other:

$$\mathcal{B} : L^2(\mathbb{R}^d, V) \to L^2(\tilde{\Pi} \times \mathbb{R}^d, V), \quad (\mathcal{B}\psi)(k, x) = \sum_{n \in \tilde{\Pi}} e^{-2\pi i(n,k)} \psi(x + n),$$

$$\mathcal{B} : L^2(\tilde{\Pi} \times \mathbb{R}^d, V) \to L^2(\mathbb{R}^d, V), \quad (\mathcal{B}\hat{\psi})(x) = \int_{k \in \tilde{\Pi}} \hat{\psi}(k, x) dk.$$

As is detailed in [5], the space $L^2(\tilde{\Pi} \times R^d; V)$ can be identified with the space $L^2(\tilde{\Pi} \times \mathbb{R}^d; V)$ of $L^2$-sections of a vector bundle $E \otimes V \to \tilde{\Pi}$. The infinite dimensional vector bundle $E \to \tilde{\Pi}$ is given by

$$E = \bigcup_{k \in \tilde{\Pi}} L^2(\mathbb{R}^d / \Pi, \mathcal{L}|_{\{k\} \times \mathbb{R}^d / \Pi}),$$

where $\mathcal{L} \to \tilde{\Pi} \times \mathbb{R}^d / \Pi$ is the Poincaré line bundle, the quotient of the product line bundle $\tilde{\Pi} \times \mathbb{R}^d \times \mathbb{C} \to \tilde{\Pi} \times \mathbb{R}^d$ by the following $\Pi$-action:

$$\Pi \times (\tilde{\Pi} \times \mathbb{R}^d \times \mathbb{C}) \to \tilde{\Pi} \times \mathbb{R}^d \times \mathbb{C}, \quad (m, k, x, z) \mapsto (k, x + m, e^{2\pi i(m,k)} z).$$

In sum, we get an identification of $L^2$-spaces

$$L^2(\mathbb{R}^d, V) \cong L^2(\tilde{\Pi} \times \mathbb{R}^d, V) \cong L^2(\tilde{\Pi} \times \mathbb{R}^d \otimes V).$$

The Hamiltonian $H$ on $L^2(\mathbb{R}^d, V)$ then induces an operator $\hat{H}$ on $L^2(\tilde{\Pi} \times \mathbb{R}^d, V) \cong L^2(\tilde{\Pi} \times \mathbb{R}^d \otimes V)$ by $\hat{H} \circ \mathcal{B} = \mathcal{B} \circ H$. If for instance $H$ is of the form $H = \Delta + \Phi$, then $\hat{H}$ preserves the fiber of $E \otimes V$. Generally, this is a consequence of the translation invariance of the Hamiltonian. In the case where the present quantum system is supposed to be an ‘insulator’, a finite number of discrete spectra of $\hat{H}(k)$ would be confined in a compact region in $\mathbb{R}$ as $k \in \tilde{\Pi}$ varies. Then the corresponding eigenfunctions form a finite rank subbundle $E \subset E \otimes V$, called the ‘Bloch bundle’. The $K$-class of this vector bundle $E \to \tilde{\Pi}$ is regarded as an invariant of the quantum system under study.

### 2.3 Nonsymorphic group and twisted $K$-theory

We now take the symmetry into account. From the extension

$$1 \to \Pi \to S \xrightarrow{\nu} P \to 1,$$

we can associate a ‘twisted $P$-equivariant vector bundle’ on $\tilde{\Pi}$ to the $S$-module $L^2(\mathbb{R}^d, V)$. This is essentially an avatar of the so-called Mackey machine.

Recall that the Euclidean group $O(d) \times \Pi$ is the semi-direct product of the orthogonal group $O(d)$ and the lattice $\Pi$. Hence any $p \in P$ is expressed as

$$(p, a_p) \in O(d) \times \Pi$$

by means of a map $a : P \to \mathbb{R}^d$. For $p_1, p_2 \in P$ we put

$$\nu(p_1, p_2) = a_{p_1} + p_1 a_{p_2} - a_{p_1, p_2}.\quad 8$$
Since $S$ is a subgroup of $O(d) \times \Pi$, we see $\nu(p_1, p_2) \in \Pi$ and also $\nu : P \times P \to \Pi$ is a group 2-cocycle of $P$ with values in $\Pi$. This group 2-cocycle measures the failure for $S$ to be symmorphic.

By means of the $S$-action $\rho$ on $L^2(\mathbb{R}^d, V)$, we define an ‘action’ of $p \in P$ by
\[ \rho(p) : L^2(\mathbb{R}^d, V) \to L^2(\mathbb{R}^d, V), \quad \rho(p) = \rho(a_p) \]
The explicit description on $\psi \in L^2(\mathbb{R}^d, V)$ is
\[ (\rho(p)\psi)(x) = U(p)\psi(p^{-1}x + a_{p^{-1}}). \]
The Bloch transformation then induces the following ‘action’ of $P$:
\[ \hat{\rho}(p) : L^2_\Pi(\tilde{\Pi} \times \mathbb{R}^d, V) \to L^2_\Pi(\tilde{\Pi} \times \mathbb{R}^d, V), \quad \hat{\rho}(p) \circ \hat{B} = \hat{B} \circ \rho(p), \]
whose explicit formula on $\hat{\psi} \in L^2_\Pi(\tilde{\Pi} \times \mathbb{R}^d, V)$ is
\[ (\hat{\rho}(p)\hat{\psi})(k, x) = U(p)\hat{\psi}(p^{-1}k, p^{-1}x + a_{p^{-1}}). \]
Notice that $\rho$ and $\hat{\rho}$ can be made honest actions of $P$ in the case of symmorphic $S$, but not in the case of nonsymmorphic $S$, for the usual composition rule is violated. It actually holds that:
\[ (\hat{\rho}(p_1)(\hat{\rho}(p_2)\hat{\psi}))(k, \xi) = e^{2\pi i (\nu(p_2^{-1}, p_2^{-1}) + p_2^{-1}p_2^{-1})}(\hat{\rho}(p_1p_2)\hat{\psi})(k, \xi). \]

To interpret the ‘action’ $\rho(p)$ in terms of the vector bundle $\mathcal{E} \otimes V$ through $L^2_\Pi(\tilde{\Pi} \times \mathbb{R}^d, V) \cong L^2(\tilde{\Pi}, \mathcal{E} \otimes V)$, recall that the fiber of $\mathcal{E} \otimes V$ at $k \in \tilde{\Pi}$ is
\[ \mathcal{E}|_k \otimes V = L^2(\mathbb{R}^d/\Pi, \mathcal{L}|_{(k) \times \mathbb{R}^d/\Pi} \otimes V), \]
and $\hat{\psi} \in L^2_\Pi(\tilde{\Pi} \times \mathbb{R}^d, V)$ corresponds to the following section $\Psi \in L^2(\tilde{\Pi}, \mathcal{E} \otimes V)$:
\[ \Psi(k) : \mathbb{R}^d/\Pi \to \mathcal{L}|_{(k) \times \mathbb{R}^d/\Pi} \otimes V, \quad x \mapsto [k, x, \hat{\psi}(k, x)]. \]
Define for $p \in P$ and $k \in \tilde{\Pi}$ a linear map
\[ \rho_{\mathcal{E} \otimes V}(p ; k) : \mathcal{E}|_k \otimes V \to \mathcal{E}|_{pk} \otimes V \]
by the assignment of the sections:
\[ \rho_{\mathcal{E} \otimes V}[x \mapsto [k, x, \hat{\psi}(k, x)] = [k, x, U(p)\hat{\psi}(k, p^{-1}x + a_{p^{-1}})]. \]
These maps assemble to give a vector bundle map $\rho_{\mathcal{E} \otimes V}(p) : \mathcal{E} \otimes V \to \mathcal{E} \otimes V$ covering the translation $k \mapsto pk$ on $\tilde{\Pi}$:
\[ \begin{array}{ccc}
\mathcal{E} \otimes V & \xrightarrow{\rho_{\mathcal{E} \otimes V}(p)} & \mathcal{E} \otimes V \\
\downarrow \quad \downarrow & & \downarrow \\
\tilde{\Pi} & \xrightarrow{p} & \tilde{\Pi}.
\end{array} \]
This is a \( \tau \)-twisted \( P \)-action, in the sense that the formula
\[
\rho_{E \otimes V}(p_1; p_2 k) \rho_{E \otimes V}(p_2; k) \xi = \tau(p_1, p_2; k) \rho_{E \otimes V}(p_1, p_2; k) \xi
\]
holds for \( p_1, p_2 \in P \), \( k \in \hat{P} \) and \( \xi \in \mathcal{E}[k] \otimes V \). Here \( \tau : P \times P \times \hat{P} \to U(1) \) is defined by
\[
\tau(p_1, p_2; k) = \exp 2\pi i (\nu(p_2^{-1}, p_1^{-1}), k),
\]
and is regarded as a group 2-cocycle of \( P \) with its coefficients in the group \( C(\hat{P}, U(1)) \) of \( U(1) \)-valued functions on \( \hat{P} \) thought of as a right \( P \)-module through the pull-back under the left action \( k \mapsto pk \) of \( p \in P \) on \( k \in \hat{P} \). The map \( \rho_{E \otimes V}(p) \) on the vector bundle induces the transformation on the sections
\[
\rho_{E \otimes V}(p) : L^2(\hat{P}, \mathcal{E} \otimes V) \to L^2(\hat{P}, \mathcal{E} \otimes V)
\]
by \( (\rho_{E \otimes V}(p) \Psi)(k) = \rho_{E \otimes V}(p; p^{-1}k) \Phi(p^{-1}k) \). One can verify that: if \( \Psi \in L^2(\hat{P}, \mathcal{E} \otimes V) \) corresponds to \( \psi \in L^2_{\hat{P}}(\hat{P} \times \mathbb{R}^d, \mathcal{V}) \), then \( \rho_{E \otimes V}(p) \Psi \) corresponds to \( \hat{\rho}(p) \hat{\psi} \). Hence the ‘action’ \( \hat{\rho}(p) \) on \( L^2(\hat{P}, \mathcal{E} \otimes V) \cong L^2(\hat{P}, \mathcal{E} \otimes V) \) agrees with that induced from the \( \tau \)-twisted \( P \)-action on \( \mathcal{E} \otimes V \).

Now, under the assumption that \( \hat{H} \) describes an insulator, the Bloch bundle \( E \subset \mathcal{E} \otimes V \) inherits a \( \tau \)-twisted \( P \)-action from \( \mathcal{E} \otimes V \). This is a consequence of the invariance of the Hamiltonian under the space group action. Therefore the Bloch bundle, being a \( \tau \)-twisted \( P \)-equivariant vector bundle of finite rank, defines a class in the \( \tau \)-twisted \( P \)-equivariant \( K \)-theory \( K^{	au \cdot 0}_P(\hat{P}) \), which is regarded as an invariant of the insulating system under study.

As is obvious from the construction, we can adapt the construction of the group 2-cocycle \( \tau \) to symmorphic space groups. However, in the symmorphic case, the cocycle \( \nu \) and hence \( \tau \) can be trivialized.

So far a linear representation of \( P \) on \( V \) is considered. We can relax this representation to be a projective representation of \( P \) with its group 2-cocycle \( c : P \times P \to U(1) \). In this case, the resulting Bloch bundle defines a class in the twisted equivariant \( K \)-theory \( K^{	au \cdot c + 0}_P(\hat{P}) \).

**Remark 2.1.** The phase factor in the composition rule of \( \hat{\rho} \):
\[
\tau_R(k; p_1, p_2) = \exp 2\pi i (\nu(p_2^{-1}, p_1^{-1}), p_2^{-1} p_1^{-1} k)
\]
defines a group 2-cocycle of \( P \) with coefficients in \( C(\hat{P}, U(1)) \) thought of as a left \( P \)-module through the right action \( k \mapsto kp = p^{-1}k \) of \( p \in P \) on \( k \in \hat{P} \). The 2-cocycles \( \tau \) and \( \tau_R \) are related by \( \tau_R(k; p_1, p_2) = \tau(p_1, p_2; (p_1 p_2)^{-1} k) \). This generalizes to a cochain bijection of group cochains with coefficients in the left/right \( P \)-modules \( C(\hat{P}, U(1)) \). Thus, the information of \( \tau \) and \( \tau_R \) are cohomologically the same. We also remark that \( \tau \) and \( \tau_R \) are respectively cohomologous to the following 2-cocycles:
\[
\tau'(p_1, p_2; k) = e^{-2\pi i (\nu(p_1, p_2; p_1 p_2 k)}, \quad \tau'_R(k; p_1, p_2) = e^{-2\pi i (\nu(p_1, p_2; k)}.
\]

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Remark 2.2. Using a homomorphism $\epsilon : P \to \mathbb{Z}_2$ given, we can impose that the Hamiltonian $H$ and the symmetry $\rho(g)$ with $g \in S$ are graded commutative:

$$H \circ \rho(g) = \epsilon(\pi(g))\rho(g) \circ H.$$  

Then the quantum system with symmetry in question leads to an element of the twisted equivariant $K$-theory $K^+_T(\mathbb{H}; \mathbb{Z}_2)$. It should be noticed that the construction of the element uses Karoubi’s formulation of $K$-theory [10] and requires a reference quantum system. These points of discussion, which will not be detailed in this paper, are implicit in the absence of the graded twist.

3 Spectral sequence and twist

This section gives a geometric interpretation of the filtration of $H^*_G(X; \mathbb{Z})$ for the Leray-Serre spectral sequence through types of twists. This is carried out by identifying the Leray-Serre spectral sequence with another natural spectral sequence computing the Borel equivariant cohomology.

Throughout this section, we assume $G$ is a finite group acting by left on a ‘reasonable’ space $X$, such as a locally contractible, paracompact and regular topological space as in [4], or a $G$-CW complex [12].

3.1 Spectral sequences

The Borel equivariant cohomology $H^*_G(X; \mathbb{Z})$ is defined to be the (singular) cohomology of the quotient space $EG \times_G X$ of $EG \times X$ under the diagonal $G$-action $(\xi, x) \mapsto (\xi g, g^{-1}x)$, where $EG$ is the total space of the universal $G$-bundle $EG \to BG$. Associated to the fibration $X \to EG \times_G X \to B$ is the Leray-Serre spectral sequence

$$E^{p,q}_r \Rightarrow H^*_{G}(X; \mathbb{Z})$$

converging to the graded quotient of a filtration

$$H^*_G(X; \mathbb{Z}) = F^0H^*_G(X; \mathbb{Z}) \supset F^1H^*_G(X; \mathbb{Z}) \supset \cdots \supset F^{n+1}H^*_G(X; \mathbb{Z}) = 0,$$

that is, $E^{p,q}_\infty = F^pH^*_G(X; \mathbb{Z})/F^{p+1}H^*_G(X; \mathbb{Z})$. The $E_2$-term is given by the group cohomology of $G$:

$$E^{p,q}_2 = H^p_{\text{group}}(G; H^q(X; \mathbb{Z})),$$

where the coefficient $H^q(X; \mathbb{Z})$ is regarded as a right $G$-module by the pull-back action: As a convention of this paper, the group of $p$-cochains with values in a right $G$-module $M$ is defined by

$$C^p_{\text{group}}(G; M) = C(G^p, M) = \{ \tau : G^p \to M \},$$

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The cohomology $\text{H}^n(X;\mathbb{Z})$ consists of a sequence of spaces $\{\text{H}^i(X;\mathbb{Z})\}_{i\geq 0}$ together with the fact that the geometric realization $|G^\bullet \times X|$ of $G^\bullet \times X$ is identified with $EG \times_G X$.

The simplicial space $G^\bullet \times X$ is associated to the left $G$-action on $X$, and consists of a sequence of spaces $\{G^p \times X\}_{p\geq 0}$ together with the face map $\partial_i : G^p \times X \rightarrow G^{p-1} \times X$, $(i = 0, \ldots, p)$ and the degeneracy map $s_i : G^p \times X \rightarrow G^{p+1} \times X$, $(i = 0, \ldots, p)$ given by

$$\partial_i(g_1, \ldots, g_p, x) = \begin{cases} (g_2, \ldots, g_p, x), & (i = 0) \\ (g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_p, x), & (i = 1, \ldots, p-1) \\ (g_1, \ldots, g_{p-1}, g_p x), & (i = p) \end{cases}$$

$$s_i(g_1, \ldots, g_p, x) = (g_1, \ldots, g_{i-1}, 1, g_i, \ldots, g_p, x).$$

The cohomology $H^n(G^\bullet \times X;\mathbb{Z})$ is then defined to be the total cohomology of the double complex $(C^\bullet(G^\bullet \times X;\mathbb{Z}), \partial, \delta)$, where $(C^\bullet(G^\bullet \times X;\mathbb{Z}), \delta)$ is the complex computing the cohomology of $G^\bullet \times X$ with coefficients in $\mathbb{Z}$ and $\partial : C^i(G^\bullet \times X;\mathbb{Z}) \rightarrow C^i(G^{i+1} \times X;\mathbb{Z})$ is $\partial = \sum_{i=0}^{j+1} (-1)^{i} \partial^i$. The double complex admits a natural filtration $\{\delta_i \geq 0, C^\bullet(G^\bullet \times X;\mathbb{Z})\}_{p\geq 0}$. The associated spectral sequence agrees with the Leray-Serre spectral sequence $E^{p,q}$. 

Now, let us consider the standard exponential exact sequence of sheaves on the simplicial space:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow U(1) \rightarrow 0,$$

where $\mathbb{R}$ consists of the sheaf of $\mathbb{R}$-valued functions on $G^p \times X$ and $U(1)$ consists of the sheaf of $U(1)$-valued functions on $G^p \times X$. We can readily show that $H^n(G^\bullet \times X;\mathbb{R}) = 0$ for $n > 0$. This vanishing together with the associated long exact sequence leads to the following isomorphism for $n \geq 1$:

$$H^n(G^\bullet \times X;U(1)) \cong H^{n+1}_G(X;\mathbb{Z}).$$

The cohomology $H^n(G^\bullet \times X;U(1))$ can be defined exactly in the same way as in the case of $H^n(G^\bullet \times X;\mathbb{Z})$ by using a double complex. Therefore we have a spectral sequence

$$\text{E}^{p,q}_r \Rightarrow H^{p+q}(G^\bullet \times X;\overline{U(1)})$$
converging to the graded quotient of a filtration

\[ 0 \to F^0 H^n(G^\bullet \times X; U(1)) \to H^n(G^\bullet \times X; U(1)) \supset \cdots, \]

whose $E_2$-term is

\[ E_2^{p,q} = H^p_{\text{group}}(G; H^q(X; U(1))), \]

where $H^q(X; U(1))$ is regarded as a right $G$-module by pull-back. Apparently

$H^0(X; U(1)) \cong \mathbb{C}(X, U(1))$ and $H^n(X; U(1)) \cong H^{n+1}(X; \mathbb{Z})$ for $n \geq 1$. Since $E_2^{p,q}$ involves the group cohomology with coefficients in $C(X; U(1))$, its computation seems to be more complicated than that of $E_2^{p,q}$. However, the spectral sequence is useful from a geometric viewpoint, as will be seen shortly.

In view of the exponential exact sequence, the filtrations of $H^n_G(X; \mathbb{Z}) \cong H^{n+1}(G^\bullet \times X; U(1))$ for $n \geq 1$ are related as follows:

\[
\begin{align*}
H^n(G^\bullet \times X; U(1)) &= E_2^{p,q} \supset E^1 H^n \supset \cdots \\
H_G^{n+1}(X; \mathbb{Z}) &= F^0 H^{n+1} \supset F^1 H^{n+1} \supset \cdots 
\end{align*}
\]

The spectral sequences are related by a map $E_2^{p,q} \to E_2^{p,q+1}$. In particular, the $E_2$-terms $E_2^{p,0}$ and $E_2^{p,1}$ are related by a map $C(X; U(1)) \to H^1(X; \mathbb{Z})$ fitting into the exact sequence:

\[ 0 \to H^0(X; \mathbb{Z}) \to C(X, \mathbb{R}) \to C(X, U(1)) \to H^1(X; \mathbb{Z}) \to 0. \]

As is mentioned, because of the isomorphism $H^q(X; U(1)) \cong H^{q+1}(X; \mathbb{Z})$, we have $E_2^{p,q} \cong E_2^{p,q+1}$ for $q \geq 1$. A more detailed relation between these spectral sequences will be given later under some hypothesis.

### 3.2 Twist

We here provide the definition of twists in [4, 5] for convenience. We mainly consider ungraded twists, and refer the reader [4] for the full account of graded twists (see also Remark 3.7). Recall that associated to an action of a finite group $G$ on a space $X$ is the groupoid $X//G$ such that its set of objects is $X$ and the set of morphisms is $G \times X$.

**Definition 3.1.** A central extension $(L, \tau)$ of the groupoid $X//G$ consists of the following data:

- a complex line bundle $L_g \to X$, $(g \in G)$
- isomorphisms of line bundles $\tau_{g,h}(x) : L|_{x \cdot h} \otimes L|_x \to L|_{x \cdot gh}$, $(g, h \in G)$ making the following diagram commutative:

\[
\begin{array}{ccc}
L_g|_{h \cdot x} \otimes L_h|_{x \cdot k} \otimes L_k|_x & \xrightarrow{1 \otimes \tau_{g,h}(x)} & L_g|_{h \cdot x} \otimes L_{h \cdot x} \\
\tau_{g,h}(x) \otimes 1 & \downarrow & \tau_{g,h}(x) \\
L_{gh}|_{x \cdot k} & \xrightarrow{\tau_{gh,k}(x)} & L_{gh}|_{x \cdot k}
\end{array}
\]
Notice that if $L_g$ is the product line bundle, then the central extension is just a group 2-cocycle of $G$ with coefficients in $C(X, U(1))$.

**Definition 3.2.** An isomorphism of central extensions of $X//G$,

$$(K, \beta_g) : (L_g, \tau_{g,h}) \longrightarrow (L'_g, \tau'_{g,h})$$

consists of the following data:

- a complex line bundle $K \to X$,
- An isomorphism $\beta_g(x) : L_g|_x \otimes K|_x \to K|_{gx} \otimes L'_g|_x$ of line bundles making the following diagram commutative:

$$
\begin{array}{ccc}
L_g|_{hx} \otimes L_h|_x \otimes K|_x & \xrightarrow{1 \otimes \beta_h(x)} & L_g|_{hx} \otimes K|_{hx} \otimes L'_h|_x \\
\downarrow \tau_{g,h}(x) \otimes 1 & & \downarrow \beta_h(hx) \otimes 1 \\
L_{gh}|_x \otimes K|_x & & K|_{ghx} \otimes L'_g|_{hx} \otimes L'_h|_x \\
\|
\downarrow \beta_{gh}(x) & & \downarrow 1 \otimes \tau'_{g,h}(x) \\
L_{gh}|_x \otimes K|_x & \xrightarrow{\beta_{gh}(x)} & K|_{ghx} \otimes L'_g|_{hx} \otimes L'_h|_x.
\end{array}
$$

The isomorphisms $(K, \beta_g)$ and $(K', \beta'_g)$ from $(L_g, \tau_{g,h})$ to $(L'_g, \tau'_{g,h})$ are identified if there is an isomorphism $f : K \to K'$ making the following diagram commutative:

$$
\begin{array}{ccc}
L_g|_x \otimes K|_x & \xrightarrow{\beta_g(x)} & K|_{gx} \otimes L'_g|_x \\
\downarrow 1 \otimes f(x) & & \downarrow f(gx) \otimes 1 \\
L_g|_x \otimes K'|_x & \xrightarrow{\beta'_g(x)} & K'|_{gx} \otimes L'_g|_x.
\end{array}
$$

Without loss of generality, we often assume $L_g \to X$ is a Hermitian line bundle and $\tau_{g,h}$ preserves the Hermitian structures. In this case, the isomorphisms of central extensions are also formulated in a consistent way.

**Definition 3.3.** An **ungraded** $G$-equivariant twist of $X$, or a **twist** for short, is a central extension of a groupoid $\tilde{X}$ which has a local equivalence to $X//G$.

A point in this definition is that a twist needs an extra groupoid $\tilde{X}$. A central extension of $X//G$ is a special type of a twist such that $\tilde{X} = X//G$. Taking the extra groupoids into account, we can introduce a notion of isomorphisms to twists. We refer the reader to [4] for the details of the isomorphisms and the following classification:

**Proposition 3.4.** The isomorphisms classes of ungraded $G$-equivariant twists of $X$ form an abelian group isomorphic to $H^3_G(X; \mathbb{Z})$.

A key to the classification is the isomorphism $H^3_G(X; \mathbb{Z}) \cong H^2(G^\bullet \times X; U(1))$. A close look at the proof of the classification leads to:
Lemma 3.5. The following holds true:

(i) $'F^1 H^2(G^* \times X; U(1))$ classifies twists represented by central extensions of the groupoid $X//G$.

(ii) $'F^2 H^2(G^* \times X; U(1))$ classifies twists represented by group 2-cocycles of $G$ with coefficients in the $G$-module $C(X, U(1))$.

Remark 3.6. In [5], an isomorphism of central extensions of $X//G$ is formulated only by using the product line bundle $K = X \times \mathbb{C}$. The reason of the difference in these definitions is that we are considering an isomorphism of central extensions of $X//G$ regarded as twists. By the same reasoning, group cocycles which are not cohomologous can be isomorphic as twists.

Remark 3.7. The modification needed to define a graded twist is to replace the line bundle $L$ constituting a central extension $(L, \tau)$ with a $\mathbb{Z}_2$-graded line bundle. Since $L$ is of rank 1, its $\mathbb{Z}_2$-grading amounts to specifying the degree of $L$ to be even or odd. With the suitable modification of the notion of isomorphisms, we can eventually classify graded twists by $H^1_G(X; \mathbb{Z}) \times H^3_G(X; \mathbb{Z})$.

3.3 Comparison of two spectral sequences

The relation between the spectral sequences $E^{p,q}_r$ and $'E^{p,q}_r$ can be made more clear under a simple assumption. To present this here, we begin with a key lemma: Recall that the exponential exact sequence of sheaves on $X$ induces a natural exact sequence of right $G$-modules:

$$0 \rightarrow H^0(X; \mathbb{Z}) \rightarrow C(X, \mathbb{R}) \rightarrow C(X, U(1)) \rightarrow H^1(X; \mathbb{Z}) \rightarrow 0.$$ 

Let us fold this into a short exact sequence:

$$0 \rightarrow C(X, \mathbb{R})/H^0(X; \mathbb{Z}) \rightarrow C(X, U(1)) \rightarrow H^1(X; \mathbb{Z}) \rightarrow 0.$$ 

In general, this does not split as an exact sequence of $G$-modules. (Such an example is provided by the circle $S^1 \subset \mathbb{R}^2$ with the action of $D_2 \subset O(2)$.) Notice that if $X$ is path connected, then $H^0(X; \mathbb{Z}) = \mathbb{Z}$.

Lemma 3.8. If a finite group $G$ acts on a compact and path connected space $X$ fixing a point $pt \in X$, then the following exact sequence of $G$-modules splits:

$$0 \rightarrow C(X, \mathbb{R})/\mathbb{Z} \rightarrow C(X, U(1)) \rightarrow H^1(X; \mathbb{Z}) \rightarrow 0.$$ 

Proof. For notational convenience, we adapt the identification $U(1) \cong \mathbb{R}/\mathbb{Z}$ in this proof. Let $C(X, pt, \mathbb{R}) \subset C(X, \mathbb{R})$ be the subgroup consisting of functions taking 0 at $pt$. The inclusion $\iota : pt \rightarrow X$ induces an isomorphism of $G$-modules

$$C(X, \mathbb{R}) \rightarrow C(X, pt, \mathbb{R}) \oplus \mathbb{R}, \quad f \mapsto (f - \iota^* f, \iota^* f).$$

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Similarly, we have an isomorphism \( C(X, \mathbb{R}/\mathbb{Z}) \cong C(X, \text{pt}, \mathbb{R}/\mathbb{Z}) \oplus \mathbb{R}/\mathbb{Z} \) of \( G \)-modules. Thus the exact sequence of \( G \)-modules in question is equivalent to:

\[
0 \rightarrow C(X, \text{pt}, \mathbb{R}/\mathbb{Z}) \rightarrow C(X, \text{pt}, \mathbb{R}/\mathbb{Z}) \overset{\delta}{\rightarrow} H^1(X; \mathbb{Z}) \rightarrow 0.
\]

Since \( X \) is supposed to be compact, \( H^1(X; \mathbb{Z}) \) is a free abelian group of finite rank. Let us choose a basis \( H^1(X; \mathbb{Z}) \cong \bigoplus \mathbb{Z}a_i \), and also \( \varphi_i : X \rightarrow \mathbb{R}/\mathbb{Z} \) such that \( \delta \varphi_i = a_i \) and \( \varphi_i(\text{pt}) = 0 \). Modifying the splitting \( a_i \mapsto \varphi_i \) of the exact sequence of abelian groups, we construct a splitting of the exact sequence of \( G \)-modules, which will complete the proof.

For the modification, we introduce a square matrix \( A(g) = (A_{ij}(g)) \) with integer coefficients to each \( f_i \) that \( \sum_{j} A_{ij}(g)a_j \). It holds that \( A(gh) = A(g)A(h) \). Because of the exact sequence, there are functions \( f_i \in C(X, \text{pt}, \mathbb{R}) \) such that the following holds in \( C(X, \text{pt}, \mathbb{R}/\mathbb{Z}) \):

\[
g^*\varphi_i = \sum_j A_{ij}(g)\varphi_j + (f_i \mod \mathbb{Z}).
\]

This can be expressed as \( g^*\Phi = A(g)\Phi + F_g \) by using the vectors \( \Phi = (\varphi_i) \) and \( F_g = (F_i) \). It then holds that \( F_{gh} = A(g)F_h + h^*F_g \) in \( C(X, \text{pt}, \mathbb{R}) \). Since \( A(g) \) is invertible, this is equivalent to:

\[
A(gh)^{-1}F_{gh} = A(h)^{-1}F_h + (h^*A(g)^{-1}F_g).
\]

Write \( |G| \) for the order of \( G \), and put \( \mathcal{F} = \frac{1}{|G|} \sum_{g \in G} A(g)^{-1}F_g \). Taking the average over \( g \in G \) in the formula above, we get:

\[
\mathcal{F} = A(h)^{-1}F_h + (h^*\mathcal{F}),
\]

which is equivalent to \( F_g = A(g)\mathcal{F} - g^*\mathcal{F} \). Now \( g^*(\Phi + \mathcal{F}) = A(g)(\Phi + \mathcal{F}) \). Thus, under the expression \( \mathcal{F} = (\mathcal{F}) \) by using \( \mathcal{F} \in C(X, \text{pt}, \mathbb{R}) \), the assignment \( a_i \mapsto \varphi_i + (\mathcal{F} \mod \mathbb{Z}) \) defines a splitting \( H^1(X; \mathbb{Z}) \rightarrow C(X, \text{pt}, \mathbb{R}/\mathbb{Z}) \) compatible with the \( G \)-module structures.

**Lemma 3.9.** Let \( G \) be a finite group acting on a compact and path connected space \( X \) fixing a point \( \text{pt} \in X \). Then, for \( n \geq 1 \), there is an isomorphism:

\[
H^n_{\text{group}}(G; C(X, U(1))) \cong H^n_{\text{group}}(G; U(1)) \oplus H^n_{\text{group}}(G; H^1(X; \mathbb{Z})),
\]

where \( U(1) \) is the trivial \( G \)-module, and \( H^1(X; \mathbb{Z}) \) is regarded as a \( G \)-module through the action of \( G \) on \( X \).

**Proof.** Lemma 3.8 implies

\[
H^n_{\text{group}}(G; C(X, \text{pt}, U(1))) \cong H^n_{\text{group}}(G; C(X, \mathbb{R}/\mathbb{Z})) \oplus H^n_{\text{group}}(G; H^1(X; \mathbb{Z})),
\]

for all \( n \geq 1 \). By the \( G \)-module isomorphism \( C(X, \mathbb{R})/\mathbb{Z} \cong C(X, \text{pt}, \mathbb{R}) \oplus \mathbb{R}/\mathbb{Z} \) utilized in Lemma 3.8, we have

\[
H^n_{\text{group}}(G; C(X, \mathbb{R}/\mathbb{Z})) \cong H^n_{\text{group}}(G; C(X, \text{pt}, \mathbb{R})) \oplus H^n_{\text{group}}(G; \mathbb{R}/\mathbb{Z}).
\]

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Since \( C(X, \text{pt}, \mathbb{R}) \) is a vector space over \( \mathbb{R} \), we can prove the vanishing
\[
H^n_{\text{group}}(G; C(X, \text{pt}, \mathbb{R})) = 0
\]
for \( n \geq 1 \) by an average argument. \( \square \)

**Proposition 3.10.** Suppose that a finite group \( G \) acts on a compact and path connected space \( X \) fixing a point \( \text{pt} \in X \). Then for \( r \geq 2 \) it holds that
\[
'E_r^{p,0} \cong E_r^{p,1} \oplus E_r^{p+1,0}, \quad (p \geq 1), \quad 'E_r^{p,q} \cong E_r^{p,q+1}, \quad (p \geq 0, q \geq 1).
\]

**Proof.** Recall that the exponential exact sequence induces the connecting homomorphism \( \delta : H^q(X; U(1)) \to H^{q+1}(X; \mathbb{Z}) \) and this induces a natural homomorphism \( \delta : 'E_r^{p,q} \to 'E_r^{p,q+1} \) compatible with the differentials \( 'd_r \) and \( d_r \). In the case of \( r = 2 \), the homomorphism \( \delta : 'E_2^{p,q} \to 'E_2^{p,q+1} \) is bijective for \( q \geq 1 \) and \( p \geq 0 \), and we have \( 'E_2^{p,0} \cong E_2^{p,1} \oplus E_2^{p+1,0} \) for \( p \geq 1 \) as a consequence of Lemma 3.9. Notice that, under this isomorphism, \( \delta : 'E_2^{p,0} \to 'E_2^{p,1} \) for \( p \geq 1 \) restricts to the identity on the direct summand \( E_2^{p,1} \subset 'E_2^{p,0} \). Note also that \( E_2^{p,0} = E_\infty^{p,0} \) for any \( p \), because
\[
E_2^{p,0} = H^p_{\text{group}}(G; \mathbb{Z}) = H^p(BG; \mathbb{Z}) = H_r^p(\text{pt}; \mathbb{Z})
\]
is a direct summand of \( H^p(BG; \mathbb{Z}) \cong H_r^p(\text{pt}; \mathbb{Z}) \oplus \tilde{H}_r^p(X; \mathbb{Z}) \), where \( \tilde{H}_r^p(X; \mathbb{Z}) \) is the reduced cohomology. Thus, for \( p \geq 1 \), the map \( \delta : 'E_2^{p,0} \to 'E_2^{p,1} \) is the projection onto \( E_2^{p,1} \) and the image of the differential \( 'd_2 : 'E_2^{p,1} \to 'E_2^{p,0} \) is in the direct summand \( E_2^{p,0} \). This leads to
\[
'E_3^{p,0} \cong E_3^{p,0} \oplus E_3^{p+1,1}, \quad (p \geq 1), \quad 'E_3^{p,q} \cong E_3^{p,q+1}, \quad (p \geq 0, q \geq 1).
\]
The calculation above can be repeated inductively on \( r \). \( \square \)

**Corollary 3.11.** Let \( G \) and \( X \) be as in Proposition 3.10. Then, for any \( n \geq 1 \) and \( p = 0, \ldots, n \), there is a natural isomorphism
\[
'F^n H^n(G^* \times X; U(1)) \cong F^n H^{n+1}_G(X; \mathbb{Z}).
\]
In addition, we have the decomposition
\[
'F^n H^n(G^* \times X; U(1)) \cong F^n H^{n+1}_G(X; \mathbb{Z}) \cong E_\infty^{n+1} \oplus F^{n+1} H^{n+1}_G(X; \mathbb{Z}),
\]
in which \( F^{n+1}_G H^{n+1}_G(X; \mathbb{Z}) \cong H^{n+1}_G(\text{pt}; \mathbb{Z}) \cong H^{n+1}_{\text{group}}(G; U(1)) \).

**Proof.** Write \( 'F^n H^n = 'F^n H^n(G^* \times X; U(1)) \) and \( F^n H^{n+1} = F^n H^{n+1}_G(X; \mathbb{Z}) \) for short. The exponential exact sequence induces a homomorphism of short exact sequences:
\[
\begin{array}{cccccc}
0 & \longrightarrow & 'F^{p+1} H^n & \longrightarrow & 'F^p H^n & \longrightarrow & 'E_r^{p,n-p} & \longrightarrow & 0 \\
\delta & & \delta & & \delta & & \delta & & \\
0 & \longrightarrow & F^{p+1} H^{n+1} & \longrightarrow & F^p H^{n+1} & \longrightarrow & E_\infty^{n+1-p} & \longrightarrow & 0.
\end{array}
\]
In the case of $p = n$, the diagram above becomes:

\[
\begin{array}{c c c c c}
0 & \rightarrow & 0 & \rightarrow & \Gamma F^n H^n & \rightarrow & \Gamma E_{\infty}^{n,0} & \rightarrow & 0 \\
\delta & & \delta & & \delta & & \delta & & \\
0 & \rightarrow & F^{n+1} H^{n+1} & \rightarrow & F^n H^{n+1} & \rightarrow & E_{\infty}^{n+1} & \rightarrow & 0.
\end{array}
\]

Notice that $F^{n+1} H^{n+1} \cong E^{n+1,0}_{\infty} \cong E_{2}^{n+1,0}$ since $E_2^{n+1,0} \cong H_2^{n+1}(pt; \mathbb{Z})$ must survive into $H_2^{n+1}(X; \mathbb{Z}) \cong H_2^{n+1}(pt; \mathbb{Z}) \oplus H_2^{n+1}(X; \mathbb{Z})$. Hence $F^n H^{n+1} \cong E^{n+1,0}_{\infty} \oplus E_{n+1}^{n,1}$. If $n \geq 1$, then this isomorphism is seen to be compatible with the isomorphism $\Gamma E_{\infty}^{n,0} \cong E_{n+1}^{n,0} \oplus E_{n+1}^{n,1}$ in Proposition 3.10 through $\delta$, so that

\[\Gamma F^n H^n \cong \Gamma E_{\infty}^{n+1,0} \cong F^{n+1} H^{n+1} \cong E^{n+1,0}_{\infty} \oplus E^{n,1}_{\infty}.
\]

For $p = n-1, n-2, \ldots, 1, 0$, we know $\delta : \Gamma E^{p,n-p}_{\infty} \rightarrow \Gamma E^{p,n-p+1}_{\infty}$ is bijective by Proposition 3.10. Therefore $\Gamma F^n H^n \cong \Gamma F^n H^{n+1}$ inductively. \qed

Combining the above corollary with Lemma 3.5, we get the interpretations of $F^n H^{n+1}(X; \mathbb{Z})$ by twists presented in Introduction:

**Corollary 3.12.** Let $G$ and $X$ be as in Proposition 3.10.

(i) $F^1 H^1_G(X; \mathbb{Z})$ classifies twists which can be represented by central extensions of the groupoid $X/G$.

(ii) $F^2 H^2_G(X; \mathbb{Z})$ classifies twists which can be represented by 2-cocycles of $G$ with coefficients in the $G$-module $C(X, U(1))$.

(iii) $F^3 H^3_G(X; \mathbb{Z}) = H^3_{\text{group}}(X; U(1))$ classifies twists which can be represented by 2-cocycles of $G$ with its coefficients in the trivial $G$-module $U(1)$.

**Remark 3.13.** The coincidence $\Gamma F^n H^n(G^\bullet \times X; U(1)) = F^n H^{n+1}(X; \mathbb{Z})$ in Corollary 3.11 holds true for $n \geq 0$ without the assumption that $G$ fixes a point on $X$. This is because $\Gamma E_{\infty}^{n,0}$ and $E_{0,n+1}^{0,1}$ are subgroups of $H^{n}(X; U(1)) \cong H^{n+1}(X; \mathbb{Z})$ and it holds that

\[\Gamma F^n H^n(G^\bullet \times X; U(1)) = F^n H^{n+1}(X; \mathbb{Z})
= \ker[f : H^n(G^\bullet \times X; U(1)) \cong H_0^{n+1}(X; \mathbb{Z}) \rightarrow H^{n+1}(X; \mathbb{Z})],
\]

where $f$ is the homomorphism of “forgetting the group actions”.

### 4 The proof of Theorem 1.1 and Theorem 1.3

Theorem 1.1 and Theorem 1.3 are proved here based on case-by-case computations of the equivariant cohomology and the Leray-Serre spectral sequence.
4.1 Some generality

The cohomology $H^n(T^2; \mathbb{Z})$ of the torus is well-known, so that nothing remains to prove in the case of $p=1$.

For the point group $P$ of any 2-dimensional space group, the vanishing $H^3(T^2; \mathbb{Z}) = 0$ implies $E_{\infty}^{0,3} = 0$, so that

$$H^n_{P}(T^2; \mathbb{Z}) = F^n_{P}(T^2; \mathbb{Z}) = F^n_{P}(T^2; \mathbb{Z}).$$

Note also each point group $P$ fixes a point on $T^2$, so that

$$F^3_{P}(T^2; \mathbb{Z}) = H^3_{P}(pt; \mathbb{Z}) = H^3_{\text{group}}(P; \mathbb{Z}) = H^2_{\text{group}}(P; U(1)).$$

Then the main task for the proof of Theorem 1.1 is to compute $H^2_{P}(T^2; \mathbb{Z})$ and $H^3_{P}(T^2; \mathbb{Z})$, since in the case where $P$ is the cyclic group $\mathbb{Z}_n$ or the dihedral group $D_n$, the cohomology $H^n_{P}(pt; \mathbb{Z})$ is summarized as follows:

| $P$ | $H^n_{P}(pt; \mathbb{Z})$ | $H^n_{P}(pt; \mathbb{Z})$ | $H^n_{P}(pt; \mathbb{Z})$ | $H^n_{P}(pt; \mathbb{Z})$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| $\mathbb{Z}_n$ | $\mathbb{Z}$ | $0$ | $\{\mathbb{Z}_2, (n : \text{odd})\}$ | $\{0, (n : \text{odd})\}$ |
| $D_n$ | $\mathbb{Z}$ | $0$ | $\{\mathbb{Z}_2^0, (n : \text{even})\}$ | $\{\mathbb{Z}_2, (n : \text{even})\}$ |

That $H^n_{P}(pt; \mathbb{Z}) = H^0(BP; \mathbb{Z}) = \mathbb{Z}$ is clear. Since $P$ is finite, $H^1_{P}(pt; \mathbb{Z}) \cong \text{Hom}(P, \mathbb{Z})$ gets trivial. The cohomology $H^2_{P}(pt; \mathbb{Z}) \cong \text{Hom}(P, U(1))$ can be seen by the classification of irreducible representations. Finally, $H^2_{P}(pt; \mathbb{Z}) \cong H^2_{\text{group}}(P; U(1))$ for $P = \mathbb{Z}_n, D_n$ can be found in [9].

In the sequel, we may use a structure of $T^2$ as a $P$-$CW$ complex: In general, for a compact Lie group $G$, a $G$-$CW$ complex is an analogue of a CW complex made of $G$-cells. A $d$-dimensional $G$-cell is a $G$-space of the form $G/H \times e^d$, where $H \subset G$ is a closed subgroup and $e^d$ is the standard $d$-dimensional cell. The action of $G$ on $G/H$ is by the left translation, whereas that on $e^d$ is trivial. For the details, we refer the reader to [12].

We later compute a group cohomology via cohomology of a $G$-$CW$ complex:

**Lemma 4.1.** Let $G$ be a finite group acting on a path connected space $Y$ fixing a point $pt$. Suppose further that $Y$ is a $G$-$CW$ complex consisting of only $G$-cells of dimension less or equal to 1. Then the following holds true for all $n \geq 0$.

$$H^n_{\text{group}}(G; H^1(Y; \mathbb{Z})) \cong \tilde{H}^{n+1}_{G}(Y; \mathbb{Z}).$$

**Proof.** Consider the Leray-Serre spectral sequence

$$E^2_{pq} = H^p_{\text{group}}(G; H^q(Y; \mathbb{Z})) \Rightarrow H^n(G; \mathbb{Z}).$$

Note that $H^q(Y; \mathbb{Z}) = 0$ for $q \neq 0, 1$. The $E_2$-term $E^0_{2q} = H^p_{\text{group}}(G; \mathbb{Z})$ must survive into the direct summand $H^p_{\text{group}}(pt; \mathbb{Z})$ in $H^p_G(Y; \mathbb{Z}) \cong H^p_G(pt; \mathbb{Z}) \oplus H^p_G(\mathbb{Z}; \mathbb{Z})$. Therefore it must hold that:

$$\tilde{H}^p_G(Y; \mathbb{Z}) \cong E^p_{-1,1} = E^p_{2-1,1} \cong H^p_{\text{group}}(G; H^1(Y; \mathbb{Z})).$$

which completes the proof. \qed
We also prepare a simple fact about group cohomology: Let $G$ be a finite group, $\epsilon : G \to \mathbb{Z}_2 = \{\pm 1\}$ a surjective homomorphism, and $\hat{\mathbb{Z}}$ the $G$-module such that its underlying group is $\mathbb{Z}$ and $G$ acts (by right) through $m \mapsto m\epsilon(g)$. A typical example is a finite subgroup $P \subset O(2)$ such that $P \not\subset SO(2)$ with $\epsilon$ the composition of the inclusion $P \to O(2)$ and the determinant $O(2) \to \mathbb{Z}_2$.

**Lemma 4.2.** Let $G$, $\epsilon$ and $\hat{\mathbb{Z}}$ be as above. Then, $H^1_{\text{group}}(G; \hat{\mathbb{Z}}) = 0$. If $n \in C^0_{\text{group}}(G; \hat{\mathbb{Z}}) = \mathbb{Z}$, its coboundary $\partial n : G \to \mathbb{Z}$ is $(\partial n)(g) = n(1 - \epsilon(g))$. Thus, the assumption that $\epsilon$ is surjective implies the vanishing of the 0th cohomology. The inclusion $\text{Ker}(\epsilon) \subset G$ induces an injection on 1-cocycles: $Z^1_{\text{group}}(G; \hat{\mathbb{Z}}) \to Z^1_{\text{group}}(\text{Ker}(\epsilon); \hat{\mathbb{Z}}) = \text{Hom}(\text{Ker}(\epsilon), \mathbb{Z}) = 0$.

Thus, given a group 1-cocycle $\phi \in Z^1_{\text{group}}(G; \hat{\mathbb{Z}})$, it holds that $\phi(g) = 0$ for all $g \in \text{Ker}(\epsilon)$. If $g, h \not\in \text{Ker}(\epsilon)$, then the cocycle condition $(\partial \phi)(g, h) = 0$ implies $\phi(h) = \phi(g)$ always of the form $\phi(g) = n(1 - \epsilon(g))/2$ for some $n \in \mathbb{Z}$. This provides the identification $Z^1_{\text{group}}(G; \hat{\mathbb{Z}}) \cong \mathbb{Z}$ as well as $B^1_{\text{group}}(G; \hat{\mathbb{Z}}) \cong 2\mathbb{Z}$, which completes the proof.

An application of the lemma above is the following lemma, which we use in the computations in the case of $cm$ and $pmm/pmg/pgg$.

**Lemma 4.3.** Let $G \subset O(2)$ be a finite subgroup such that $G \not\subset SO(2)$. We let $G$ act on $\mathbb{R}^2$ through the inclusion $G \subset O(2)$. Then, it holds that $H^0_{\text{group}}(D(\mathbb{R}^2), S(\mathbb{R}^2); \mathbb{Z}) \cong \begin{cases} 0, & (n \leq 2) \\ \mathbb{Z}_2, & (n = 3) \end{cases}$

where $D(\mathbb{R}^2)$ and $S(\mathbb{R}^2)$ are the unit disk and the unit circle in $\mathbb{R}^2$.

Proof. We have a relative version of the Leray-Serre spectral sequence

$E^2_{p,q} = H^p_{\text{group}}(G; H^q(D(\mathbb{R}^2), S(\mathbb{R}^2); \mathbb{Z})) \Rightarrow H^p_{\text{group}}(D(\mathbb{R}^2), S(\mathbb{R}^2); \mathbb{Z})$.

By the dimensional reason, the spectral sequence degenerates at $E_2$, and

$H^p_{\text{group}}(D(\mathbb{R}^2), S(\mathbb{R}^2); \mathbb{Z}) \cong E^2_{p,2} = H^p_{\text{group}}(G; H^2(D(\mathbb{R}^2), S(\mathbb{R}^2); \mathbb{Z}))$.

Since $H^2(D(\mathbb{R}^2), S(\mathbb{R}^2); \mathbb{Z}) \cong \hat{\mathbb{Z}}$, Lemma 4.2 completes the proof.

The following lemma will also be used repeatedly.

**Lemma 4.4.** Let $G$ be a finite group $G$ acting on the torus $T^2$ such that:

- there is a fixed point $\text{pt} \in T^2$,
- the $G$-action does not preserve the orientation of $T^2$. 

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Then the following holds true about the Leray-Serre spectral sequence:

(a) \( F^2 H^n_\mathbb{Z}(T^2; \mathbb{Z}) \cong E^{2,0}_2 \oplus E^{3,0}_2 \).

(b) \( H^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \cong \bigoplus_{p+q=n} E^{p,q}_2 \) for \( n \leq 2 \).

Proof. In the Leray-Serre spectral sequence

\[
E^{p,q}_2 = H^n_{\text{group}}(G; H^q(T^2; \mathbb{Z})),
\]

the coefficient in the group cohomology \( H^0(T^2) \cong \mathbb{Z} \) is identified with the trivial \( G \)-module, and \( H^2(T^2) \cong \mathbb{Z} \) with the \( G \)-module in Lemma 4.2. Then the relevant \( E_2 \)-terms can be summarized as follows:

| \( q \) | \( p = 0 \) | \( p = 1 \) | \( p = 2 \) | \( p = 3 \) |
|---|---|---|---|
| 3 | 0 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 |
| 1, 0 | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| 1, 1 | \( E^{2,1}_2 \) | \( E^{2,1}_2 \) | \( E^{2,1}_2 \) |
| 0 | \( E^{p,0}_2 \) | \( E^{1,0}_2 \) | \( E^{2,0}_2 \) |
| \( E^{p,0}_2 \) | \( p = 0 \) | \( p = 1 \) | \( p = 2 \) | \( p = 3 \) | \( p = 4 \) |

Since \( G \) fixes \( \text{pt} \in T^2 \), we have the decomposition \( H^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \cong H^n_{\mathbb{Z}}(\text{pt}; \mathbb{Z}) \oplus \tilde{H}^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \), where \( \tilde{H}^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \) is the reduced cohomology. Therefore the \( E_2 \)-term \( E^{p,0}_2 = H^n_{\text{group}}(G; \mathbb{Z}) \cong H^n_{\mathbb{Z}}(\text{pt}; \mathbb{Z}) \) must survive into the direct summand \( H^n_{\mathbb{Z}}(\text{pt}; \mathbb{Z}) \) in \( H^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \). This implies that \( E^{p,0}_2 = E^{p,0}_\infty \) is always a direct summand of the subgroups \( F^p H^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \subset H^n_{\mathbb{Z}}(T^2; \mathbb{Z}) \) and that \( d_2 : E^{p,-2,1}_2 \to E^{p,0}_2 \) is trivial. As a result, we get \( E^{2,1}_2 = E^{2,1}_\infty \) and the expression (a). Also \( E^{p,q}_2 = E^{p,q}_\infty \) for \( p + q \leq 2 \), and the expression (b). \( \square \)

### 4.2 \( p2 \)

The lattice \( \Pi \subset \mathbb{R}^2 \) is the standard one \( \Pi = \mathbb{Z} \oplus \mathbb{Z} \) and the point group \( P = \mathbb{Z}_2 = \{ \pm 1 \} \) acts on \( \Pi \) and \( \mathbb{R}^2 \) by \( (x, y) \mapsto (-x, -y) \).

**Theorem 4.5 (p2).** The \( \mathbb{Z}_2 \)-equivariant cohomology of \( T^2 \) is as follows:

| \( n \) | \( n = 0 \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) |
|---|---|---|---|
| \( H^n_{\mathbb{Z}_2}(T^2; \mathbb{Z}) \) | \( \mathbb{Z} \) | \( 0 \) | \( \mathbb{Z} \oplus \mathbb{Z}_2 \) | \( 0 \) |

Proof. We use the Gysin exact sequence for ‘Real’ circle bundles in [6]: We write \( H^n_{\mathbb{Z}_2}(X) = H^n_{\mathbb{Z}_2}(X; \mathbb{Z}) \) for the equivariant cohomology and \( H^n_{\mathbb{Z}_2}(X) \cong H^n_{\mathbb{Z}_2}(X; \mathbb{Z}(1)) \) for a variant of the equivariant cohomology, which can be formulated by the equivariant cohomology with local coefficients. The torus \( T^2 \) is the product of two copies of \( \tilde{S}^1 \), where \( \tilde{S}^1 = U(1) \) is the circle with the involution \( z \mapsto z^{-1} \). We can think of \( \tilde{S}^1 \times \tilde{S}^1 \) as the trivial ‘Real’ circle bundle on \( \tilde{S}^1 \). Similarly, \( \tilde{S}^1 \) is the trivial ‘Real’ circle bundle on pt. The Gysin exact sequences for these ‘Real’ circle bundles are splitting, and we find:

\[
H^n_{\mathbb{Z}_2}(T^2) \cong H^n_{\mathbb{Z}_2}(\tilde{S}^1) \oplus H^{n-1}_{\mathbb{Z}_2}(\tilde{S}^1) \cong H^n_{\mathbb{Z}_2}(\text{pt}) \oplus H^{n-1}_{\mathbb{Z}_2}(\text{pt}) \oplus H^{n-1}_{\mathbb{Z}_2}(\text{pt}) \oplus H^{n-2}_{\mathbb{Z}_2}(\text{pt}).
\]

As given in [6], the cohomology \( H^n_{\mathbb{Z}_2}(\text{pt}) \) is isomorphic to \( \mathbb{Z}_2 \) if \( n > 0 \) is odd, and is trivial otherwise. We already know \( H^n_{\mathbb{Z}_2}(\text{pt}) \), and get \( H^n_{\mathbb{Z}_2}(T^2) \) easily. \( \square \)
4.3 \textbf{p3}

The lattice $\Pi = \mathbb{Z}a \oplus \mathbb{Z}b \subset \mathbb{R}^2$ is spanned by

\[ a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}. \]

The point group $P = \mathbb{Z}_3 = \langle C \rangle \mathbb{C}^3$ acts on $\Pi$ and $\mathbb{R}^2$ through

\[ C = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}. \]

Under the standard identification $\mathbb{R}^2 = \mathbb{C}$, we have $a = 1$ and $b = \tau = \exp 2\pi i / 6$. Hence $\Pi = \mathbb{Z} \oplus \mathbb{Z} \tau$ and $C$ is the multiplication of $\tau^2$. A fundamental domain is $\{s + t \tau | 0 \leq s, t \leq 1 \}$.

To compute the equivariant cohomology, we consider a $\mathbb{Z}_3$-CW decomposition of $T^2$. The $\mathbb{Z}_3$-cells are listed as follows:

| 0-cell     | 1-cell     | 2-cell     |
|------------|------------|------------|
| $e_j^0 = \text{pt}$ | $e_j^1 = \mathbb{Z}_3 \times e^1$ | $e_j^2 = \mathbb{Z}_3 \times e^2$ | $(j = 0, 1, 2)$ | $(j = 0, 1, 2)$ | $(j = 1, 2)$ |

- (0-cell) There are three fixed points on $T^2$:

\[ e_0^0 = 0, \quad e_1^0 = \frac{1 + \tau}{3}, \quad e_2^0 = \frac{2(1 + \tau)}{3}, \]

each of which gives a 0-dimensional $\mathbb{Z}_3$-cell $(\mathbb{Z}_3/\mathbb{Z}_3) \times e^0 = \text{pt}$.

- (1-cell) $T^2$ has three 1-dimensional $\mathbb{Z}_3$-cells, each of which is of the form $\mathbb{Z}_3 \times e^1$. For $j = 0, 1, 2$, the 1-cell $e_j^1 = \mathbb{Z}_3 \times e^1$ consists of the three segments connecting $e_0^j$ and $e_2^j$. 

\[ \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} \]
• (2-cell) For $j = 1, 2$, the 2-dimensional $\mathbb{Z}_3$-cell $\bar{e}_j^2 = \mathbb{Z}_3 \times e^2$ consists of the three small triangle regions surrounded by the 1-cells $\bar{e}_j^1$ and $\bar{e}_j^1$.

Let $Y \subset T^2$ be the $\mathbb{Z}_3$-invariant subspace $Y = \bar{e}_0^0 \cup \bar{e}_0^1$.

**Lemma 4.6.** The $\mathbb{Z}_3$-equivariant cohomology of $Y$ in low degree is as follows:

| $H^n_{\mathbb{Z}_3}(Y; \mathbb{Z})$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|-----------------------------------|---------|---------|---------|---------|
|                                   | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_3$ | 0      |

**Proof.** We can readily identify $\mathcal{Y}$ with $(\mathbb{Z}_3 \times e^1)/(\mathbb{Z}_3 \times \partial e^1)$, the space obtained by collapsing $\mathbb{Z}_3 \times \partial e^1 \subset \mathbb{Z}_3 \times e^1$ to a point. Hence the reduced equivariant cohomology of $Y$ can be computed as follows:

$$\tilde{H}^n_{\mathbb{Z}_3}(Y; \mathbb{Z}) \cong H^0_{\mathbb{Z}_3}(\mathbb{Z}_3 \times e^1, \mathbb{Z}_3 \times \partial e^1; \mathbb{Z}) \cong H^n(e^1, \partial e^1; \mathbb{Z}) \cong H^{n-1}(pt; \mathbb{Z}).$$

Now the lemma is completed by $H^0_{\mathbb{Z}_3}(Y; \mathbb{Z}) \cong H^0_{\mathbb{Z}_3}(pt; \mathbb{Z}) \oplus \tilde{H}^0_{\mathbb{Z}_3}(Y; \mathbb{Z})$. □

**Lemma 4.7.** There are an exact sequence of $\mathbb{Z}_3$-modules:

$$0 \to H^1(T^2; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \xrightarrow{\pi} \mathbb{Z} \to 0$$

and a module homomorphism $s : \mathbb{Z} \to H^1(Y; \mathbb{Z})$ such that $\pi \circ s = 3$, where $\mathbb{Z}$ is the trivial $\mathbb{Z}_3$-module.

**Proof.** Let $\gamma_1, \gamma_2 \in H_1(T^2)$ be the homology classes of the loops going along the vectors $1$ and $\tau$ respectively in the fundamental domain, which form a basis of $H_1(T^2) \cong \mathbb{Z}^2$. Also, let $\eta_1, \eta_2, \eta_3 \in H_1(Y)$ be the homology classes of loops along $1$, $\tau$, and $\tau - 1$, which form a basis of $H_1(Y) \cong \mathbb{Z}^3$. The inclusion map $i : Y \to T^2$ relates these bases by $i_* \eta_1 = \gamma_1$, $i_* \eta_2 = \gamma_2$ and $i_* \eta_3 = \gamma_2 - \gamma_1$. The actions of $C \in \mathbb{Z}_3$ on these bases are:

\[
\begin{align*}
C_* \eta_1 &= \eta_2 - \eta_1, \\
C_* \eta_2 &= -\eta_1, \\
C_* \eta_3 &= \gamma_3, \\
C_* \gamma_1 &= \gamma_3, \\
C_* \gamma_2 &= -\gamma_1, \\
C_* \gamma_3 &= \gamma_2.
\end{align*}
\]

Let $\{h_1, h_2\} \subset H^1(T^2)$ and $\{g_1, g_2, g_3\} \subset H^1(T^2)$ be dual to the homology bases. They are related by $i^* h_1 = g_1 - g_3$ and $i^* h_2 = g_2 + g_3$, and the $\mathbb{Z}_3$-actions are:

\[
\begin{align*}
C^* h_1 &= -h_1 - h_2, \\
C^* h_2 &= h_1, \\
C^* g_1 &= -g_2, \\
C^* g_2 &= -g_3, \\
C^* g_3 &= g_1.
\end{align*}
\]

It is now clear that $\text{Coker}(i^*) \cong \mathbb{Z}$ as a $\mathbb{Z}_3$-module, and we get the short exact sequence in question. In $H^1(Y)$ is the submodule that is generated by $g_1 - g_2 + g_3$ and is isomorphic to $\mathbb{Z}$. If we define the homomorphism $s : \mathbb{Z} \to H^1(Y)$ by $s(1) = g_1 - g_2 + g_3$, then $\pi \circ s = 3$. □
Lemma 4.8. The group cohomology of $\mathbb{Z}_3$ with coefficients in $H^1(T^2; \mathbb{Z})$ is:

| $n$ | $H^n_{\text{group}}(\mathbb{Z}_3; H^1(T^2; \mathbb{Z}))$ |
|-----|--------------------------------|
| 0   | 0                                |
| 1   | $\mathbb{Z}_3$                    |
| 2   | 0                                |
| 3   | $\mathbb{Z}_3$                    |

Proof. We use the long exact sequence in group cohomology induced from Lemma 4.7. By Lemma 4.1 and Lemma 4.6, we have

$$H^n_{\text{group}}(\mathbb{Z}_3; H^1(Y; \mathbb{Z})) \cong H^{n+1}_{\text{group}}(Y) \cong H^n(\text{pt}; \mathbb{Z}).$$

The relevant group cohomology can be summarized as follows:

| $n$ | $H^n_{\text{group}}(\mathbb{Z}_3; H^1(T^2))$ | $H^n_{\text{group}}(\mathbb{Z}_3; H^1(Y))$ | $H^n_{\text{group}}(\mathbb{Z}_3; \mathbb{Z})$ |
|-----|-----------------------------------|---------------------------------|----------------------------------|
| 3   | 0                                 | 0                               | 0                                |
| 2   | $\mathbb{Z}_3$                     | 0                               | $\mathbb{Z}_3$                   |
| 1   | 0                                 | $\mathbb{Z}_3$                   | 0                                |
| 0   | 0                                 | 0                               | $\mathbb{Z}_3$                   |

Note that 0th cohomology consists of invariants in each $\mathbb{Z}_3$-module. Using the bases in the proof of Lemma 4.7, we can see that the homomorphism $H^0_{\text{group}}(\mathbb{Z}_3; H^1(Y)) \to H^0_{\text{group}}(\mathbb{Z}_3; \mathbb{Z})$ is the map $\mathbb{Z} \to \mathbb{Z}$ multiplying with 3. This determines the 0th and the 1st group cohomology with coefficients in $H^1(T^2)$. The remaining cohomology groups are obvious. 

Theorem 4.9 (p3). The $\mathbb{Z}_3$-equivariant cohomology of $T^2$ is as follows:

| $n$ | $H^n_{\mathbb{Z}_3}(T^2; \mathbb{Z})$ |
|-----|----------------------------------|
| 0   | $\mathbb{Z}$                     |
| 1   | $\mathbb{Z} \oplus \mathbb{Z}_3^{\oplus 2}$ |
| 2   | 0                                |
| 3   | 0                                |

Proof. In the $E_2$-term of the Leray-Serre spectral sequence:

$$E_2^{p,q} = H^p_{\text{group}}(\mathbb{Z}_3; H^q(T^2; \mathbb{Z})), $$

the coefficients $H^0(T^2)$ and $H^2(T^2)$ are the trivial $\mathbb{Z}_3$-modules. Therefore the relevant part of the $E_2$-terms is summarized as follows:

| $q$ | $E_2^{p,q}$ |
|-----|-------------|
| 3   | 0           |
| 2   | $\mathbb{Z}$|
| 1   | $\mathbb{Z}_3$|
| 0   | 0           |

This leads to $H^3_{\mathbb{Z}_3}(T^2) \cong \mathbb{Z}$ and $H^1_{\mathbb{Z}_3}(T^2) = 0$ immediately. We can see $E_2^{p,q} = E_\infty^{p,q}$ for $p+q = 2$. Because $E_2^{2,0}$ must survive into the direct summand $H^2_{\mathbb{Z}_3}(\text{pt})$ of $H^2_{\mathbb{Z}_3}(T^2)$, it holds that $F^1 H^2_{\mathbb{Z}_3}(T^2) \cong E_\infty^{1,0} \oplus E_\infty^{2,0} \cong \mathbb{Z}^{\oplus 2}$. Since the abelian group $E_\infty^{0,2} \cong \mathbb{Z}$ is free, we eventually get $H^2_{\mathbb{Z}_3}(T^2) \cong E_\infty^{2,0} \oplus F^1 H^2_{\mathbb{Z}_3}(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}_3^{\oplus 2}$. Finally, $E_\infty^{p,q} = E_2^{p,q} = 0$ for $p+q = 3$ and $H^3_{\mathbb{Z}_3}(T^2) = 0$. 

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4.4 p4

The lattice $\Pi = \mathbb{Z}^2 \subset \mathbb{R}^2$ is the standard one. The point group $P = \mathbb{Z}_4 = \langle C \mid C^4 \rangle$ acts on $\Pi$ and $\mathbb{R}^2$ through

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. $$

Under the standard identification $\mathbb{R}^2 = \mathbb{C}$, we have $\Pi = \mathbb{Z} \oplus \mathbb{Z}i$ and $C$ is the multiplication of $i$. A fundamental domain is $\{s + ti \mid 0 \leq s, t \leq 1\}$.

 Decomposing this domain, we provide a decomposition of $T^2$ as a $\mathbb{Z}_4$-CW complex. The list of $\mathbb{Z}_4$-cells is as follows:

| 0-cell | 1-cell | 2-cell |
|--------|--------|--------|
| $e_0^0 = \text{pt}$ | $e_0^0 = \mathbb{Z}_4 \times e^1$ | $e^2 = \mathbb{Z}_4 \times e^2$ |
| $e_1^0 = \text{pt}$ | $e_1^0 = \mathbb{Z}_4 \times e^1$ | |
| $e_2^0 = \mathbb{Z}_4/\mathbb{Z}_2$ | | |

- (0-cell) The two fixed points on $T^2$ define the 0-cells:

  $$e_0^0 = 0, \quad e_1^0 = \frac{1 + i}{2},$$

  The remaining 0-cell $e_2^0 = (\mathbb{Z}_4/\mathbb{Z}_2) \times e^0 = \mathbb{Z}_4/\mathbb{Z}_2$ consists of $1/2$ and $i/2$.

- (1-cell) For $j = 0, 1$, the 1-dimensional $\mathbb{Z}_4$-cell $e_j^1 = \mathbb{Z}_4 \times e^1$ consists of the four segments connecting $e_j^0$ and $e_2^0$.
The 2-cell \( \tilde{e}^2 = \mathbb{Z}_4 \times \tilde{e}^2 \) consists of the four small square regions surrounded by the 1-cells.

Let \( Y \subset T^2 \) be the 1-dimensional subcomplex \( Y = \tilde{e}_0^1 \cup \tilde{e}_1^1 \cup \tilde{e}_0^0 \).

**Lemma 4.10.** The \( \mathbb{Z}_4 \)-equivariant cohomology of \( Y \) is as follows:

\[
\begin{array}{c|cccc}
H^n_{\mathbb{Z}_4}(Y; \mathbb{Z}) & n = 0 & n = 1 & n = 2 & n = 3 \\
\hline
\mathbb{Z} & 0 & \mathbb{Z}_2 \oplus \mathbb{Z}_4 & 0 & 0
\end{array}
\]

**Proof.** We cover \( Y \) by two invariant subspaces \( U \) and \( V \) which have the following \( \mathbb{Z}_4 \)-equivariant homotopy equivalences:

\[ U \cong \text{pt}, \quad V \cong \mathbb{Z}_4/\mathbb{Z}_2, \quad U \cap V \cong \mathbb{Z}_4. \]

These spaces have the following equivariant cohomology groups:

\[
\begin{array}{cccc}
n = 3 & 0 & 0 & 0 \\
n = 2 & \mathbb{Z}_4 \oplus \mathbb{Z}_2 & 0 & 0 \\
n = 1 & 0 & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \\
n = 0 & H^n_{\mathbb{Z}_4}(Y) & H^n_{\mathbb{Z}_4}(U) \oplus H^n_{\mathbb{Z}_4}(V) & H^n_{\mathbb{Z}_4}(U \cap V)
\end{array}
\]

The homomorphism \( \Delta \) in the Mayer-Vietoris exact sequence for \( \{U, V\} \):

\[ \cdots \rightarrow H^n_{\mathbb{Z}_4}(Y) \rightarrow H^n_{\mathbb{Z}_4}(U) \oplus H^n_{\mathbb{Z}_4}(V) \xrightarrow{\Delta} H^n_{\mathbb{Z}_4}(U \cap V) \rightarrow H^{n+1}_{\mathbb{Z}_4}(Y) \rightarrow \cdots \]

is expressed as \( \Delta(u, v) = j_U^*(u) - j_V^*(v) \) by using the inclusions \( j_U : U \cap V \rightarrow U \) and \( j_V : U \cap V \rightarrow V \). This allows us to determine \( H^n_{\mathbb{Z}_4}(Y) \) for \( n \leq 3 \).

**Theorem 4.11 (p4).** The \( \mathbb{Z}_4 \)-equivariant cohomology of \( T^2 \) is as follows:

\[
\begin{array}{cccc}
n = 0 & n = 1 & n = 2 & n = 3 \\
\hline
H^n_{\mathbb{Z}_4}(T^2; \mathbb{Z}) & \mathbb{Z} & 0 & \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 & 0
\end{array}
\]

**Proof.** In the \( E_2 \)-term of the Leray-Serre spectral sequence

\[ E_2^{p,q} = H^p_{\text{group}}(\mathbb{Z}_4; H^q(T^2; \mathbb{Z})) \]

the coefficients \( H^0(T^2) \) and \( H^2(T^2) \) are the trivial \( \mathbb{Z}_4 \)-modules. We also have the isomorphism \( H^1(T^2) \cong H^1(Y) \) of \( \mathbb{Z}_4 \)-modules induced by the inclusion \( Y \rightarrow T^2 \). By using Lemma 4.1, we can summarize the relevant \( E_2 \)-terms as follows:

\[
\begin{array}{cccc}
q = 3 & 0 & 0 & 0 & 0 \\
q = 2 & \mathbb{Z} & 0 & \mathbb{Z}_4 & 0 \\
q = 1 & Z_2 & 0 & \mathbb{Z}_4 & 0 \\
q = 0 & \mathbb{Z} & 0 & \mathbb{Z}_4 & 0 \\
E_2^{p,q} & p = 0 & p = 1 & p = 2 & p = 3
\end{array}
\]

We immediately get \( H^0_{\mathbb{Z}_4}(T^2) \cong \mathbb{Z} \) and \( H^1_{\mathbb{Z}_4}(T^2) = 0 \). We also get \( E_2^{p,q} = E_\infty^{p,q} \) for \( p + q = 2 \). Note that \( E_2^{2,0} \) must survive into the direct summand \( H^2_{\mathbb{Z}_4}(\text{pt}) \) in \( H^2_{\mathbb{Z}_4}(T^2) \), and that \( E_2^{0,2} \cong \mathbb{Z} \) is free. This implies \( H^2_{\mathbb{Z}_4}(T^2) \cong E_2^{0,2} \oplus E_2^{1,1} \oplus E_2^{2,0} \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \). Clearly, \( E_\infty^{p,q} = E_2^{p,q} = 0 \) for \( p + q = 3 \) implies \( H^3_{\mathbb{Z}_4}(T^2) = 0 \).
4.5 \textbf{p6}

The lattice $\Pi \subset \mathbb{R}^2$ is the same as that in p3. The point group $P = \mathbb{Z}_6 = \langle C \mid C^6 \rangle$ acts on $\Pi$ and $\mathbb{R}^2$ through

$$C = \begin{pmatrix} 0.5 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0.5 \end{pmatrix}.$$ 

Under the standard identification $\mathbb{R}^2 = \mathbb{C}$, we have $\Pi = \mathbb{Z} \oplus \mathbb{Z} \tau$ with $\tau = \text{exp} \, 2\pi i/6$ and $C$ is the multiplication of $\tau$. Decomposing a fundamental domain \{s + t\tau | 0 \leq s, t \leq 1\}, we introduce a $\mathbb{Z}_6$-CW structure to $T^2$. The $\mathbb{Z}_6$-cells are listed as follows:

| 0-cell | 1-cell | 2-cell |
|--------|--------|--------|
| $e_0^0 = \text{pt}$ | $e_1^1 = \mathbb{Z}_6 \times e^1$ | $e_2^2 = \mathbb{Z}_6 \times e^2$ |
| $e_0^0 = (\mathbb{Z}_6/\mathbb{Z}_2) \times \text{pt}$ | $e_1^1 = \mathbb{Z}_6 \times e^1$ | $e_2^2 = \mathbb{Z}_6 \times e^2$ |
| $e_0^0 = (\mathbb{Z}_6/\mathbb{Z}_3) \times \text{pt}$ | $e_1^1 = \mathbb{Z}_6 \times e^1$ | $e_2^2 = \mathbb{Z}_6 \times e^2$ |

- (0-cell) The 0-cell $e_0^0 = (\mathbb{Z}_6/\mathbb{Z}_6) \times e^0 = \text{pt}$ is the unique fixed point $0 \in T^2$. The remaining 0-cells are as follows:

$$e_1^0 = \left\{ \frac{1}{2}, \frac{\tau}{2}, \frac{1 + \tau}{2} \right\} \cong (\mathbb{Z}_6/\mathbb{Z}_2) \times e^0 \cong \mathbb{Z}_3,$$

$$e_2^0 = \left\{ \frac{1 + \tau}{3}, \frac{2(1 + \tau)}{3} \right\} \cong (\mathbb{Z}_6/\mathbb{Z}_3) \times e^0 \cong \mathbb{Z}_2.$$

- (1-cell) For $j = 1, 2$, the 1-cell $e_j^1 = \mathbb{Z}_6 \times e^1$ consists of the six segments attached to $e_0^0$ and $e_j^0$.

- (2-cell) The 2-cell $e_2^2 = \mathbb{Z}_6 \times e^2$ consists of the six small triangles surrounded by the 1-cells.

We define an invariant subspace $Y \subset T^2$ by setting $Y = e_0^0 \cup e_1^0 \cup e_1^1$. 

\textbf{Lemma 4.12.} The $\mathbb{Z}_6$-equivariant cohomology of $Y$ is as follows:

| $n$ | $H^n_{\mathbb{Z}_6}(Y; \mathbb{Z})$ |
|-----|---------------------------------|
| 0   | $\mathbb{Z}$                    |
| 1   | $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ |
| 2   | 0                               |
| 3   | 0                               |
Proof. We cover $Y$ by its invariant subspaces $U$ and $V$ which have the following equivariant homotopy equivalences:

$$
U \simeq \text{pt}, \quad V \simeq \mathbb{Z}_6/\mathbb{Z}_2, \quad U \cap V \simeq \mathbb{Z}_6.
$$

The equivariant cohomology of these spaces are summarized as follows:

| $n$ | $H^0_{\mathbb{Z}_6}(Y)$ | $H^0_{\mathbb{Z}_6}(U)$ $\oplus$ $H^0_{\mathbb{Z}_6}(V)$ | $H^0_{\mathbb{Z}_6}(U \cap V)$ |
|-----|----------------------|---------------------------------|-----------------|
| 3   | 0                    | 0                               | 0               |
| 2   | $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ | 0                               | 0               |
| 1   | 0                    | 0                               | 0               |
| 0   | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$                   | $\mathbb{Z}$    |

In the Mayer-Vietoris exact sequence for $\{U, V\}$:

$$
H^0_{\mathbb{Z}_6}(Y) \to H^0_{\mathbb{Z}_6}(U) \oplus H^0_{\mathbb{Z}_6}(V) \xrightarrow{\Delta} H^0_{\mathbb{Z}_6}(U \cap V) \to H^1_{\mathbb{Z}_6}(Y) \to \cdots,
$$

we have $\Delta(u, v) = u - v$. Now $H^0_{\mathbb{Z}_6}(Y)$ is clear for $n \leq 3$.

Let $\hat{\mathbb{Z}}$ be the $\mathbb{Z}_6$-module such that its underlying group is $\mathbb{Z}$ and $\mathbb{Z}_6 = \langle C, C^6 \rangle$ acts by $C : n \mapsto -n$ for $n \in \hat{\mathbb{Z}}$.

**Lemma 4.13.** There are an exact sequence of $\mathbb{Z}_6$-module:

$$
0 \to H^1(T^2; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \xrightarrow{\pi} \hat{\mathbb{Z}} \to 0
$$

and a module homomorphism $s : \hat{\mathbb{Z}} \to H^1(Y; \mathbb{Z})$ such that $\pi \circ s = 3$.

**Proof.** The proof is similar to the case of $p^3$. We take the same bases as in the proof of Lemma 4.7. The generator $C \in \mathbb{Z}_6$ acts on the homology bases by

$$
\left\{ \begin{array}{l}
C_* \eta_1 = \eta_2, \\
C_* \eta_2 = \eta_2 - \eta_1,
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
C_* \gamma_1 = \gamma_2, \\
C_* \gamma_2 = \gamma_3, \\
C_* \gamma_3 = -\gamma_1.
\end{array} \right.
$$

Dually, the action on the cohomology bases are expressed as follows:

$$
\left\{ \begin{array}{l}
C^* h_1 = -h_2, \\
C^* h_2 = h_1 + h_2,
\end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l}
C^* g_1 = -g_3, \\
C^* g_2 = g_1, \\
C^* g_3 = g_2.
\end{array} \right.
$$

These cohomology bases are related by $i^* h_1 = g_1 - g_3$ and $i^* h_2 = g_2 + g_3$ through the inclusion $i : Y \to T^2$. A direct computation proves $\text{Coker}(i^*) \cong \mathbb{Z}$, and we get the exact sequence. The module homomorphism $s : \hat{\mathbb{Z}} \to H^1(Y)$ is given by $s(1) = g_1 - g_2 + g_3$.

**Lemma 4.14.** $H^n_{\text{group}}(\mathbb{Z}_6; H^1(T^2; \mathbb{Z})) = 0$ for $n = 0, 1, 2$. 

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Proof. We use the long exact sequence in group cohomology induced from the exact sequence of coefficients $0 \to H^1(T^2) \to H^1(Y) \xrightarrow{s} \hat{Z} \to 0$. The group cohomology with coefficients in $H^1(Y)$ can be computed by using Lemma 4.1, and that with $\hat{Z}$ by Lemma 4.2. The relevant results are summarized as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
n & 0 & n = 2 & n = 1 \\
\hline
H^0_{\text{group}}(Z_6; H^1(T^2)) & Z_2 & Z_2 & 0 \\
H^1_{\text{group}}(Z_6; H^1(Y)) & 0 & 0 & 0 \\
H^2_{\text{group}}(Z_6; \hat{Z}) & 0 & 0 & 0 \\
\hline
\end{array}
\]

Apparently $H^0_{\text{group}}(D_6; H^1(T^2)) = 0$. The homomorphism in group cohomology induced from $\pi : H^1(Y) \to \hat{Z}$ is surjective in degree 1, because $\pi \circ s = 3$ and 3 in inverted in $\mathbb{Z}_2$. This leads to the remaining vanishing in degree 1 and 2.

Theorem 4.15 (p6). The $\mathbb{Z}_6$-equivariant cohomology of $T^2$ is as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
n & 0 & n = 1 & n = 2 & n = 3 \\
\hline
H^n_{\mathbb{Z}_6}(T^2; \mathbb{Z}) & \mathbb{Z} & 0 & \mathbb{Z} \oplus \mathbb{Z}_6 & 0 \\
\hline
\end{array}
\]

Proof. We use the Leray-Serre spectral sequence, whose $E_2$-term is:

\[E_2^{p,q} = H^p_{\text{group}}(Z_6; H^q(T^2; \mathbb{Z})).\]

We can identify the coefficients $H^0(T^2)$ and $H^2(T^2)$ with the trivial $\mathbb{Z}_6$-modules. Then the $E_2$-terms are summarized as follows:

\[
\begin{array}{|c|c|c|c|c|}
\hline
q & 3 & 2 & 0 & 0 \\
\hline
p & 0 & 0 & 0 & 0 \\
\hline
E_2^{p,q} & Z & Z & Z_6 & 0 \\
\hline
\end{array}
\]

This clearly leads to $H^0_{\mathbb{Z}_6}(T^2) \cong \mathbb{Z}$ and $H^1_{\mathbb{Z}_6}(T^2) = 0$. It is also clear that $E_2^{0,q} = E_2^{p,q}$ for $p + q = 2$. Because $E_2^{0,2} \cong \mathbb{Z}$ is free, we get $H^2_{\mathbb{Z}_6}(T^2) \cong E_2^{0,2} \oplus E_2^{1,1} \oplus E_2^{2,0} \cong \mathbb{Z} \oplus \mathbb{Z}_6$. Finally $H^3_{\mathbb{Z}_6}(T^2) = 0$ is obvious.

4.6 $\text{pm/pg}$

The lattice is the standard one $\Pi = \mathbb{Z}^2 \subset \mathbb{R}^2$. The point group $\mathbb{Z}_2 = \langle \sigma | \sigma^2 \rangle$ acts on $\Pi$ and $\mathbb{R}^2$ by the following reflection:

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\]

Theorem 4.16 (pm/pg). The $\mathbb{Z}_2$-equivariant cohomology of $T^2$ is as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
n & 0 & n = 1 & n = 2 & n = 3 \\
\hline
H^n_{\mathbb{Z}_2}(T^2; \mathbb{Z}) & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
\hline
\end{array}
\]
Proof. In the notation of the proof of Theorem 4.5 in the case of $p_2$, the present torus is identified with $T^2 = S^1 \times S^1$, which is the trivial ‘Real’ circle bundle on $S^1$. Then the Gysin exact sequence for this ‘Real’ circle bundle and that for the equivariant circle bundle $S^1 \to \text{pt}$ provide us

$$H^q_{\mathbb{Z}_2}(T^2) \cong H^q_{\mathbb{Z}_2}(S^1) \oplus H^{q-1}_{\mathbb{Z}_2}(S^1) \cong H^q_{\mathbb{Z}_2}(pt) \oplus H^{q-1}_{\mathbb{Z}_2}(pt) \oplus H^{q-2}_{\mathbb{Z}_2}(pt).$$

The theorem now follows from the knowledge of $H_{\mathbb{Z}_2}(pt)$ and $H_{\mathbb{Z}}(pt)$. \qed

**Theorem 4.17** (pm/pg). $F^2 H^2_{\mathbb{Z}_2}(T^2) \cong \mathbb{Z}_2$.

**Proof.** The $E_2$-term of the Leray-Serre spectral sequence is

$$E^{p,q}_{2} = H^p_{\text{group}}(\mathbb{Z}_2; H^q(T^2; \mathbb{Z})).$$

We have the following identifications of $\mathbb{Z}_2$-modules:

$$H^0(T^2) \cong \mathbb{Z}(0) = \mathbb{Z}, \quad H^1(T^2) \cong \mathbb{Z}(1) \oplus \mathbb{Z}(0), \quad H^2(T^2) \cong \mathbb{Z}(1),$$

where $\mathbb{Z}(e)$ is the group $\mathbb{Z}$ with the $\mathbb{Z}_2$-action $m \mapsto (\pm 1)^{\mathbb{Z}(e)} m$. (In particular $\mathbb{Z}(1) = \hat{\mathbb{Z}}$ in Lemma 4.2.) The group cohomology $H^n_{\text{group}}(\mathbb{Z}_2; \mathbb{Z}(1))$ is isomorphic to $\mathbb{Z}_2$ if $n > 0$ is odd, and is trivial otherwise. (Actually, $H^n_{\text{group}}(\mathbb{Z}_2; \mathbb{Z}(1)) \cong H^n_{\mathbb{Z}_2}(pt)$ in the notation of [6]). Now, the $E_2$-terms are:

| $q$ | 0 | 0 | 0 | 0 |
|----|---|---|---|---|
| $q = 3$ | 0 | 0 | 0 | 0 |
| $q = 2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ |
| $q = 1$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $q = 0$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | 0 |

Hence Lemma 4.4 completes the proof. \qed

### 4.7 cm

The lattice is $\Pi = \mathbb{Z} a \oplus \mathbb{Z} b \subset \mathbb{R}^2$, where

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$ 

The point group $P = \mathbb{Z}_2$ acts on $\Pi$ and $\mathbb{R}^2$ by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

A fundamental domain is $\{sa + tb | 0 \leq s, t \leq 1\}$. 

```
      b
     /
    /  
   /    
  a
```
We introduce a $\mathbb{Z}_2$-CW structure on $T^2$ as follows:

| 0-cell | 1-cell          | 2-cell          |
|-------|---------------|---------------|
| $\bar{e}^0 = \text{pt}$ | $e_1^1 = \mathbb{Z}_2 \times e^1$ | $\bar{e}^2 = \mathbb{Z}_2 \times e^2$ |

- (0-cell) The 0-cell $\bar{e}^0$ is the fixed point $(0, 0) \in T^2 = \mathbb{R}^2/\Pi$.

- (1-cell) The 1-cell $\bar{e}_1^1 = \mathbb{Z}_2 \times e^1$ consists of two segments connecting the 0-cell along the lattice vectors $a$ and $b$. The other 1-cell $\bar{e}_2^1 = \mathbb{Z}_2/\mathbb{Z}_2 \times e^1$ consists of the segment connecting the 0-cell along the vector $a + b$.

- (2-cell) The 2-cell $\bar{e}^2 = \mathbb{Z}_2 \times e^2$ consists of the two triangles surrounded by the 1-cells.

We put $Y = \bar{e}^0 \cup \bar{e}_1^1$.

**Lemma 4.18.** The equivariant cohomology of $Y$ in low degree is as follows:

| $H^n_{\mathbb{Z}_2}(Y; \mathbb{Z})$ | $n = 0$ | $n = 1$ | $n = 2$ | $n = 3$ |
|-------------------------------|---------|---------|---------|---------|
| $\mathbb{Z}$                   | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0       |

**Proof.** We can cover $Y$ by $\mathbb{Z}_2$-invariant subspaces $U$ and $V$ which have the following equivariant homotopy equivalences:

$U \simeq \text{pt}$, $V \simeq \mathbb{Z}_2$, $U \cap V \simeq \mathbb{Z}_2 \sqcup \mathbb{Z}_2$.

The equivariant cohomology groups of these spaces are:

| $n$   | $H^n_{\mathbb{Z}_2}(Y)$ | $H^n_{\mathbb{Z}_2}(U) \oplus H^n_{\mathbb{Z}_2}(V)$ | $H^n_{\mathbb{Z}_2}(U \cap V)$ |
|-------|-------------------------|-----------------------------------|--------------------------------|
| $n = 3$ | 0                       | $\mathbb{Z}_2 \oplus 0$           | 0                             |
| $n = 2$ | $\mathbb{Z}_2 \oplus 0$ | 0                                 | 0                             |
| $n = 1$ | 0                       | 0                                 | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $n = 0$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus 0$           | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
In the Mayer-Vietoris exact sequence for \( \{U, V\} \):

\[ \cdots \rightarrow H^n_{\mathbb{Z}_2}(Y) \rightarrow H^n_{\mathbb{Z}_2}(U) \oplus H^n_{\mathbb{Z}_2}(V) \xrightarrow{\Delta} H^n_{\mathbb{Z}_2}(U \cap V) \rightarrow H^{n+1}_{\mathbb{Z}_2}(Y) \rightarrow \cdots, \]

we have \( \Delta(u, v) = u - v \) in degree 0. Then the lemma immediately follows. \( \square \)

**Theorem 4.19 (cm).** The \( \mathbb{Z}_2 \)-equivariant cohomology of \( T^2 \) is as follows:

\[
\begin{array}{|c|c|c|c|c|}
\hline
H^n_{\mathbb{Z}_2}(T^2; \mathbb{Z}) & n = 0 & n = 1 & n = 2 & n = 3 \\
\hline
\mathbb{Z} & \mathbb{Z} & \mathbb{Z}_2 & \mathbb{Z}_2 \ 
\end{array}
\]

**Proof.** We are to use the exact sequence for the pair \( (T^2, Y) \):

\[ \cdots \rightarrow H^n_{\mathbb{Z}_2}(T^2, Y) \rightarrow H^n_{\mathbb{Z}_2}(T^2) \rightarrow H^n_{\mathbb{Z}_2}(Y) \xrightarrow{\delta} H^n_{\mathbb{Z}_2}(T^2, Y) \rightarrow \cdots. \]

By the excision axiom, we have \( \delta \), \( H^n_{\mathbb{Z}_2}(T^2, Y) \cong H^n_{\mathbb{Z}_2}(D(\mathbb{R}^2), S(\mathbb{R}^2)) \), where \( D(\mathbb{R}^2) \) and \( S(\mathbb{R}^2) \) are the unit disk and the unit circle in \( \mathbb{R}^2 \) on that \( \mathbb{Z}_2 \) acts through the inclusion \( \mathbb{Z}_2 \rightarrow O(2) \). Recalling Lemma 4.3, we can summarize the relevant equivariant cohomology groups as follows:

\[
\begin{array}{|c|c|c|c|}
\hline
n & H^2_{\mathbb{Z}_2}(T^2; \mathbb{Z}) & H^2_{\mathbb{Z}_2}(T^2) & H^2_{\mathbb{Z}_2}(Y) \\
\hline
3 & \mathbb{Z}_2 & 0 & 0 \\
2 & 0 & \mathbb{Z}_2 & 0 \\
1 & 0 & \mathbb{Z} & 0 \\
0 & 0 & \mathbb{Z} & 0 \\
\hline
\end{array}
\]

Notice that \( Y \) and \( T^2 \) have fixed points. Hence \( H^2_{\mathbb{Z}_2}(Y) \cong H^2_{\mathbb{Z}_2}(pt) \) is isomorphic to the direct summand \( H^2_{\mathbb{Z}_2}(pt) \subset H^2_{\mathbb{Z}_2}(T^2) \). This means that \( \delta : H^2_{\mathbb{Z}_2}(Y) \rightarrow H^2_{\mathbb{Z}_2}(T^2, Y) \) in the exact sequence is trivial. Hence \( H^2_{\mathbb{Z}_2}(T^2) = H^2_{\mathbb{Z}_2}(Y) \) for \( n \leq 2 \) and \( H^3_{\mathbb{Z}_2}(T^2) \cong H^3_{\mathbb{Z}_2}(T^2, Y) \cong \mathbb{Z}_2 \).

**Theorem 4.20 (cm).** \( F^2H^3_{\mathbb{Z}_2}(T^2; \mathbb{Z}) = 0. \)

**Proof.** We compute the Leray-Serre spectral sequence:

\[ E_2^{p,q} = H^p_{\text{group}}(\mathbb{Z}_2; H^q(T^2; \mathbb{Z})). \]

Clearly, \( H^0(T^2; \mathbb{Z}) \cong \mathbb{Z}(0) \) and \( H^2(T^2; \mathbb{Z}) \cong \mathbb{Z}(1) \) as \( \mathbb{Z}_2 \)-modules, where \( \mathbb{Z}(c) \) is the same as in the proof of Theorem 4.17 in the case of \( \mathbb{R}^n / \mathbb{R}^m \). The inclusion \( Y \rightarrow T^2 \) provides an isomorphism \( H^1(T^2; \mathbb{Z}) \cong H^1(Y; \mathbb{Z}) \) of \( \mathbb{Z}_2 \)-modules. Now, using Lemma 4.1, we can find the following \( E_2 \)-terms:

\[
\begin{array}{|c|c|c|c|c|}
\hline
q & 0 & 0 & 0 & 0 \\
\hline
p = 0 & & & & \\
\hline
p = 1 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z}_2 \\
\hline
p = 2 & \mathbb{Z} & 0 & 0 & \mathbb{Z}_2 \\
\hline
\end{array}
\]

Now Lemma 4.4 completes the proof. \( \square \)
4.8 pmm/pmg/pgg

We consider the standard lattice $\Pi = \mathbb{Z}^2 \subset \mathbb{R}^2$. The point group is $P = D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. We let the following matrices $\sigma_x$ and $\sigma_y$ generate $D_2$, and act on $\Pi$ and $\mathbb{R}^2$:

$$
\sigma_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

A fundamental domain is $\{s(1,0) + t(0,1) \in \mathbb{R}^2 | 0 \leq s, t \leq 1\}$.

We introduce a $D_2$-CW decomposition of $T^2$ as follows:

| 0-cell | 1-cell | 2-cell |
|-------|-------|-------|
| $e_{ij}^0 = \text{pt}$ $\quad (i, j = 0, 1)$ | $e_{x}^1 = D_2 / \{1, \sigma_y\} \times e^1$ | $e_{y}^3 = D_2 \times e^3$ |
| $e_{y}^3 = D_2 / \{1, \sigma_y\} \times e^1$ | $e_{y}^3 = D_2 / \{1, \sigma_x\} \times e^1$ | $e_{y}^3 = D_2 / \{1, \sigma_x\} \times e^1$ |

- (0-cell) For $i, j = 0, 1$, the 0-cell $e_{ij}^0$ is the flexed point $(i/2, j/2) \in T^2$.

- (1-cell) The 1-cell $e_{x}^1 = D_2 / \{1, \sigma_y\} \times e^1$ consists of the two segments connecting $e_{00}^0$ and $e_{10}^0$, and $e_{y}^3 = D_2 / \{1, \sigma_y\} \times e^1$ consists of the two segments connecting $e_{11}^0$ and $e_{01}^0$. Similarly, $e_{y}^3 = D_2 / \{1, \sigma_x\} \times e^1$ consists of the two segments connecting $e_{00}^0$ and $e_{01}^0$, and $e_{y}^3 = D_2 / \{1, \sigma_x\} \times e^1$ consists of the two segments connecting $e_{11}^0$ and $e_{10}^0$. 

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• (2-cell) The 2-cell $\tilde{e}^2 = D_2 \times e^2$ consists of the four small square regions surrounded by the 1-cells.

We define $D_2$-invariant subspaces $Y_x$ and $Y_y$ in $T^2$ by

$$Y_x = e^0_{00} \cup e^0_{10} \cup e^1_{x}, \quad Y_y = e^0_{00} \cup e^0_{01} \cup e^1_{y}.$$  

We also define $Y \subset T^2$ to be $Y = Y_x \vee Y_y = e^0_{00} \cup e^0_{10} \cup e^0_{01} \cup e^1_{x} \cup e^1_{y}$.

**Lemma 4.21.** The equivariant cohomologies of $Y_x, Y_y$ and $Y$ are as follows:

| $n$ | $H^n_{D_2}(Y_x; \mathbb{Z})$ | $H^n_{D_2}(Y_y; \mathbb{Z})$ | $H^n_{D_2}(Y; \mathbb{Z})$ |
|-----|-----------------|-----------------|-----------------|
| 0   | $\mathbb{Z}$   | $\mathbb{Z}$   | $\mathbb{Z}$   |
| 1   | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ |
| 2   | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^{2^2}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 3   | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^{2^2}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |

**Proof.** First of all we compute $Y_x$ by the Mayer-Vietoris exact sequence: We cover $Y_x$ by invariant subspaces $U$ and $V$ such that $U$ and $V$ are $D_2$-equivariantly contractible and $U \cap V \simeq D_2/\{1, \sigma_y\} \cong \{1, \sigma_x\}$. The relevant equivariant cohomology groups of $U$, $V$ and $U \cap V$ are as follows:

$$\begin{array}{c|ccc}
  n & 0 & 1 & 2 & 3 \\
  \hline
  H^n_{D_2}(U) & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z}_2^2 \\
  H^n_{D_2}(V) & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z}_2^2 \\
  H^n_{D_2}(U \cap V) & \mathbb{Z} & 0 & \mathbb{Z}_2 & \mathbb{Z}_2^2 \\
\end{array}$$

Notice the inclusion $j_U : U \cap V \to U$ induces $j_U^!: H^2_{D_2}(U) \to H^2_{D_2}(U \cap V)$ by pull-back. Under the natural identifications

$$H^2_{D_2}(U) \cong \text{Hom}(D_2, U(1)), \quad H^2_{D_2}(U \cap V) \cong \text{Hom}(\{1, \sigma_y\}, U(1)),$$

the homomorphism corresponding to $j_U^!$ is induced from the inclusion $\{1, \sigma_y\} \to D_2$. Therefore $j_U^!$ turns out to be surjective. The same is true for the inclusion $j_V : U \cap V \to V$. In the Mayer-Vietoris exact sequence

$$\cdots \to H^n_{D_2}(U \cup V) \to H^n_{D_2}(U \cap V) \xrightarrow{\Delta} H^n_{D_2}(U \cap V) \to \cdots,$$

the homomorphism $\Delta$ is given by $\Delta(u, v) = j_U^!(u) - j_V^!(v)$. We can now solve the Mayer-Vietoris exact sequence to determine $H^n_{D_2}(Y_x)$ for $n \leq 3$ as claimed.

Since $Y_x$ and $Y_y$ are equivariantly homeomorphic, their equivariant cohomology groups are isomorphic. The equivariant cohomology of $Y = Y_x \vee Y_y$ now follows from a general formula in equivariant cohomology:

$$H^n_{D_2}(Y_x \vee Y_y) \cong H^n_{D_2}(pt) \oplus H^n_{D_2}(Y_x) \oplus H^n_{D_2}(Y_y),$$

which is a simple consequence of the Mayer-Vietoris exact sequence. \qed
**Theorem 4.22** (pmm/pmg/pgg). The $D_2$-equivariant cohomology of $T^2$ is:

| $n$ | $H^0_{D_2}(T^2; \mathbb{Z})$ | $H^1_{D_2}(T^2; \mathbb{Z})$ | $H^2_{D_2}(T^2; \mathbb{Z})$ | $H^3_{D_2}(T^2; \mathbb{Z})$ |
|-----|------------------|------------------|------------------|------------------|
| 0   | $\mathbb{Z}$    | 0                | $\mathbb{Z}_2^{\oplus 4}$ | $\mathbb{Z}_2^{\oplus 4}$ |
| 1   | $\mathbb{Z}$    | 0                | $\mathbb{Z}_2^{\oplus 4}$ | $\mathbb{Z}_2^{\oplus 4}$ |
| 2   | $\mathbb{Z}$    | 0                | 0                | $\mathbb{Z}_2^{\oplus 4}$ |
| 3   | 0                | 0                | 0                | $\mathbb{Z}_2^{\oplus 4}$ |

**Proof.** We consider the exact sequence for the pair $(T^2, Y)$:

$$
\cdots \to H^{n-1}(Y) \to H^n_{D_2}(T^2, Y) \to H^n_{D_2}(T^2) \xrightarrow{\iota^*} H^n_{D_2}(Y) \to \cdots
$$

The equivariant cohomology group of $Y$ is already computed. The cohomology of $(T^2, Y)$ is also computed by Lemma 4.3, because the excision axiom gives us $H^n_{D_2}(T^2, Y) \cong H^n_{D_2}(D(\mathbb{R}^2), S(\mathbb{R}^2))$. As a result, the theorem will be established once we prove that the exact sequence splits:

$$
H^n_{D_2}(T^2) \cong H^n_{D_2}(T^2, Y) \oplus H^n_{D_2}(Y).
$$

We prove this by showing that the map $i^*: H^n_{D_2}(T^2) \to H^n_{D_2}(Y)$ induced from the inclusion $i: Y \to T^2$ is surjective. We actually show that $i^*: H^n_{D_2}(T^2) \to \widetilde{H}^n_{D_2}(Y)$ is surjective: Recall that $Y = Y_x \cup Y_y$, and hence the inclusions $i_x: Y_x \to Y$ and $i_y: Y_y \to Y$ induce the bijection $i_x^* \oplus i_y^*: H^n_{D_2}(Y) \to \widetilde{H}^n_{D_2}(Y_x) \oplus \widetilde{H}^n_{D_2}(Y_y)$.

We here let $S^1_x = \mathbb{R}/\mathbb{Z}$ and $S^1_y = \mathbb{R}/\mathbb{Z}$ be the circles with the $D_2$-actions:

$$
D_2 \times S^1_x \to S^1_x, \quad \left\{ \begin{array}{l}
\sigma_x x = -x, \\
\sigma_y x = x,
\end{array} \right. \quad D_2 \times S^1_y \to S^1_y, \quad \left\{ \begin{array}{l}
\sigma_x y = y, \\
\sigma_y y = -y.
\end{array} \right.
$$

Now, the projection $\pi_x: T^2 \to S^1_x$, $(\pi_x(x, y) = x)$ is $D_2$-equivariant. Under the obvious identification $Y_x \cong S^1_x$, the composition $\pi_x \circ i_x$ agrees with the identity map on $Y_x$. This means that $(i \circ i_x)^*: H^n_{D_2}(T^2) \to H^n_{D_2}(Y_x)$ is surjective, and similarly $(i \circ i_y)^*$ is. Because $i_x^* \oplus i_y^*$ is bijective, $i^*$ is surjective.

**Theorem 4.23** (pmm/pmg/pgg). $F^2 H^3_{D_2}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2^{\oplus 3}$.

**Proof.** In the $E_2$-term of the Leray-Serre spectral sequence

$$
E_2^{p,q} = H^p_{\text{group}}(D_2; H^q(T^2; \mathbb{Z})),
$$

the coefficient $D_2$-module $H^q(T^2; \mathbb{Z})$ is identified as follows: The $D_2$-module $H^0(T^2) = \mathbb{Z}$ is trivial, $H^1(T^2) \cong \mathbb{Z}_2^{\oplus 4}$ is identified with $H^1(Y)$, and $H^2(T^2) \cong \mathbb{Z}$ with $\mathbb{Z}$. By means of Lemma 4.1 and Lemma 4.2, we get:

| $q$ | 3 | 2 | 1 | 0 |
|-----|---|---|---|---|
| $E_2^{p,q}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2^{\oplus 2}$ | $\mathbb{Z}_2^{\oplus 2}$ |
| $p = 0$ | $p = 1$ | $p = 2$ | $p = 3$ |

Now the theorem follows from Lemma 4.4. \qed
4.9 \textbf{cmm}

The lattice $\mathbb{L} = \mathbb{Z}a \oplus \mathbb{Z}b \subset \mathbb{R}^2$ is the same as in the case of \textit{cm}:

$$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$ 

The point group is $P = D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The following matrices $\sigma_x$ and $\sigma_y$ generate $D_2$, and let it act on $\mathbb{L}$ and $\mathbb{R}^2$:

$$\sigma_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

A fundamental domain is \{sa + tb \; | \; 0 \leq s, t \leq 1\}.

We introduce a $D_2$-CW structure on $T^2$ as follows:

| 0-cell       | 1-cell                      | 2-cell                      |
|--------------|-----------------------------|-----------------------------|
| $e^0_0 = \text{pt}$ | $e^1 = D_2 \times e^1$ | $e^2 = D_2 \times e^2$ |
| $e^0_0 = \text{pt}$ | $e^1_x = D_2/\{1, \sigma_x\} \times e^1$ |  |
| $e^0_2 = D_2/\{1, \sigma_x, \sigma_y\}$ | $e^1_y = D_2/\{1, \sigma_x\} \times e^1$ |  |

- (0-cell) For $j = 0, 1$, the 0-cell $e^0_j$ is the fixed point $(0, j) \in T^2 = \mathbb{R}^2/\mathbb{L}$.
The 0-cell $e^0_2 = D_2/\{1, \sigma_x, \sigma_y\}$ consists of the two points $(\pm 1/2, 1/2) \in T^2$.

- (1-cell) The 1-cell $e^1 = D_2 \times e^1$ consists of the four segments connecting the 0-cells $e^0_0$ and $e^0_2$. The 1-cell $e^1_x = D_2/\{1, \sigma_y\} \times e^1$ consists of two segments connecting $e^0_0$ and $e^1_1$ parallel to the vector $a - b$. The $e^1_y = D_2/\{1, \sigma_x\} \times e^1$ consists of two segments connecting $e^0_0$ and $e^1_1$ parallel to $a + b$. 

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• (2-cell) The 2-cell $e^2 = D_2 \times e^2$ consists of the four triangle regions surrounded by the 1-cells.

We define an invariant subspace $Y \subset T^2$ to be $Y = e_0^2 \cup e_1^2 \cup e_1^3$.

**Lemma 4.24.** The equivariant cohomology of $Y$ in low degree is as follows:

| $n$   | $Z$       | $\mathbb{Z}_2^0$ | $\mathbb{Z}_2$ |
|-------|-----------|------------------|----------------|
| $0$   | $\mathbb{Z}$ | $0$              | $\mathbb{Z}_2$ |
| $1$   | $\mathbb{Z}$ | $0$              | $\mathbb{Z}_2$ |
| $2$   | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $3$   | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |

**Proof.** We can cover $Y$ by $\mathbb{Z}_2$-invariant subspaces $U$ and $V$ which have the following equivariant homotopy equivalences:

$U \simeq \text{pt}$, $V \simeq D_2/[1, \sigma_x \sigma_y]$, $U \cap V \simeq D_2$.

The cohomology groups of these spaces can be summarized as follows:

| $n$   | $H^0_{D_2}(Y)$ | $H^0_{D_2}(U) \oplus H^0_{D_2}(V)$ | $H^0_{D_2}(U \cap V)$ |
|-------|----------------|----------------------------------|---------------------|
| $0$   | $\mathbb{Z}$    | $\mathbb{Z}_2 + 0$            | $\mathbb{Z}_2$       |
| $1$   | $\mathbb{Z}_2$  | $\mathbb{Z}_2$                | $\mathbb{Z}$         |
| $2$   | $\mathbb{Z}_2$  | $\mathbb{Z}_2$                | $\mathbb{Z}_2$       |
| $3$   | $\mathbb{Z}_2$  | $\mathbb{Z}_2$                | $\mathbb{Z}$         |

In the Mayer-Vietoris exact sequence for $\{U, V\}$:

$H^0_{D_2}(Y) \to H^0_{D_2}(U) \oplus H^0_{D_2}(V) \xrightarrow{\Delta} H^0_{D_2}(U \cap V) \to H^1_{D_2}(Y) \to \cdots$,

the homomorphism $\Delta$ is of the form $\Delta(u, v) = u - v$. With these information, we can determine $H^n_{D_2}(Y)$ for $n \leq 3$.

Let $X_1$ be the 1-skeleton of $T^2$, namely, the union of all the 0-cells and 1-cells in the $D_2$-CW complex $T^2$.

**Lemma 4.25.** $H^3_{D_2}(X_1; \mathbb{Z}) \cong \mathbb{Z}_2^0$.

**Proof.** We can cover $X_1$ by $D_2$-invariant subspaces $U$ and $V$ which have the following equivariant homotopy equivalences:

$U \simeq Y$, $V \simeq \text{pt}$, $U \cap V \simeq D_2/[1, \sigma_x] \sqcup D_2/[1, \sigma_y]$.

In the Mayer-Vietoris exact sequence for $\{U, V\}$,

$H^2_{D_2}(U) \oplus H^2_{D_2}(V) \xrightarrow{\Delta} H^2_{D_2}(U \cap V) \to H^3_{D_2}(U) \oplus H^3_{D_2}(V)$,

we have the expression $\Delta(u, v) = j^*_U(u) - j^*_V(v)$, where $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ are the inclusions. Under the natural isomorphisms

$H^3_{D_2}(V) \cong \text{Hom}(D_2, U(1))$,$\quad H^3_{D_2}(U \cap V) \cong \text{Hom}([1, \sigma_x], U(1)) \oplus \text{Hom}([1, \sigma_y], U(1))$,

the inclusions $[1, \sigma_x] \to D_2$ and $[1, \sigma_y] \to D_2$ induce the homomorphism $j^*_\Delta$. Accordingly, we can see that $j^*_\Delta$ is surjective, and so is $\Delta$. Since $H^3_{D_2}(U \cap V) = 0$, it follows that $H^3_{D_2}(T^2) \cong H^3_{D_2}(U) \oplus H^3_{D_2}(V) \cong \mathbb{Z}_2^0$. 

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Theorem 4.26 (cmm). $H^3_{D_2}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2^\oplus 2$.

Proof. By the excision axiom, we have
\[ H^3_{D_2}(T^2, X_1) \cong H^3_{D_2}(D_2 \times \varepsilon^2, D_2 \times \partial \varepsilon^2) \cong H^n(e^2, \partial e^2) \cong H^{n-2}(pt). \]
By the exact sequence for the pair $(T^2, X_1)$,
\[ H^3_{D_2}(T^2, X_1) \to H^3_{D_2}(T^2) \to H^3_{D_2}(X_1) \to H^4_{D_2}(T^2, X_1), \]
we get $H^3_{D_2}(T^2) \cong H^3_{D_2}(X_1) \cong \mathbb{Z}_2^\oplus 2$.

Theorem 4.27 (cmm). The following holds true:

(a) $F^2 H^1_{D_2}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2$.

(b) The $D_2$-equivariant cohomology of $T^2$ in low degree is as follows:
\[ H^0_{D_2}(T^2; \mathbb{Z}) = \mathbb{Z}, \quad H^1_{D_2}(T^2; \mathbb{Z}) = 0, \quad H^2_{D_2}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2^\oplus 2. \]

Proof. In the Leray-Serre spectral sequence
\[ E_2^{p,q} = H^p_{\text{group}}(D_2; H^q(T^2; \mathbb{Z})), \]
the $D_2$-modules $H^0(T^2)$, $H^1(T^2)$ and $H^2(T^2)$ are respectively identified with the trivial $D_2$-module $\mathbb{Z}$, $H^1(Y)$ and $\mathbb{Z}$. Thus, thanks to Lemma 4.1 and Lemma 4.2, the $E_2$-terms can be summarized as follows:

| $q$   | 0 | 0 | 0 | 0 |
|-------|---|---|---|---|
| $q = 3$ | 0 | 0 | 0 | 0 |
| $q = 2$ | 0 | $\mathbb{Z}_2$ | 0 | 0 |
| $q = 1$ | 0 | $\mathbb{Z}_2$ | 0 | 0 |
| $q = 0$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2^\oplus 2$ | $\mathbb{Z}_2$ |

Now Lemma 4.4 leads to the theorem.

4.10 p4m/p4g

The lattice $\Pi = \mathbb{Z}_2^2 \subset \mathbb{R}^2$ is standard. The point group is
\[ P = D_4 = \langle C_4, \sigma_x \rangle \langle C_4^2, \sigma_z, \sigma_y C_4 \sigma_x C_4 \rangle. \]
For $D_4$ acts on $\Pi$ and $\mathbb{R}^2$, we fix the following matrix presentation:
\[ C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]
In the sequel, we will use the following notations to indicate elements in $D_4$:
\[ 1, \quad C_4, \quad C_2 = C_4^2, \quad C_4^3 = C_4^{-1}, \quad \sigma_x, \quad \sigma_d = \sigma_x C_4, \quad \sigma_y = C_2 \sigma_x, \quad \sigma_y = C_4 \sigma_x. \]

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Based on a fundamental domain \( \{ s(1,0) + t(0,1) \in \mathbb{R}^2 \mid 0 \leq s, t \leq 1 \} \), we introduce a \( D_4 \)-CW decomposition of \( T^2 \) as follows:

| 0-cell | 1-cell | 2-cell |
|--------|--------|--------|
| \( \bar{e}_0^0 = \text{pt} \) | \( \bar{e}_0^1 = D_4/(1, \sigma_y) \times e^1 \) | \( \bar{e}_1^2 = D_4 \times e^2 \) |
| \( \bar{e}_1^1 = \text{pt} \) | \( \bar{e}_1^1 = D_4/(1, \sigma_y) \times e^1 \) | \( \bar{e}_1^1 = D_4/(1, \sigma_y) \times e^1 \) |
| \( \bar{e}_2^2 = D_4/(1, C_2, \sigma_x, \sigma_y) \) | \( \bar{e}_2^2 = D_4/(1, C_2, \sigma_x, \sigma_y) \times e^1 \) | \( \bar{e}_2^2 = D_4/(1, C_2, \sigma_x, \sigma_y) \times e^1 \) |

- **(0-cell)** Two of the 0-cells are the fixed points \( \bar{e}_0^0 = (0,0) \) and \( \bar{e}_1^1 = (1/2, 1/2) \). The remaining 0-cell \( \bar{e}_2^2 \) is defined by:
  \[
  \bar{e}_2^2 = \{(1/2,0),(0,1/2)\} \\
  \cong D^4/\{1,C_2,\sigma_x,\sigma_y\} = \{1,C_2,\sigma_x,\sigma_y\},\{C_4,C_4',\sigma_d,\sigma_d'\}.
  \]

- **(1-cell)** Among the three 1-cells, we let \( \bar{e}_1^1 \) consist of the four segments connecting \( \bar{e}_0^0 \) and \( \bar{e}_2^2 \), and \( \bar{e}_1^1 \) of the four segments connecting \( \bar{e}_0^1 \) and \( \bar{e}_2^2 \), both of which are of the form:
  \[
  D_4/(1,\sigma_y) \times e^1 \cong \{1,\sigma_y\},\{C_4,\sigma_d\},\{\sigma_x,C_2\},\{C_4',\sigma_d'\} \times e^1.
  \]

The remaining 1-cell \( \bar{e}_1^1 \) consists of the four segments connecting \( \bar{e}_0^0 \) and \( \bar{e}_1^1 \), which is of the form:

\[
D_4/(1,\sigma_d) \times e^1 \cong \{1,\sigma_d\},\{C_4,\sigma_x\},\{C_2,\sigma_d'\},\{C_4',\sigma_y\} \times e^1.
\]

- **(2-cell)** The 2-cell \( \bar{e}_2^2 = D_4 \times e^2 \) consists of the eight triangle regions surrounded by the 1-cells.

Define an invariant subspace \( Y \subset T^2 \) by \( Y = \bar{e}_0^0 \cup \bar{e}_0^1 \cup \bar{e}_2^2 \).

**Lemma 4.28.** The equivariant cohomology of \( Y \) is as follows:

| \( H^n_{D_4}(Y; \mathbb{Z}) \) | \( n = 0 \) | \( n = 1 \) | \( n = 2 \) | \( n = 3 \) |
|----------------|---------|---------|---------|---------|
| \( \mathbb{Z} \) | \( 0 \) | \( \mathbb{Z}^{2^3} \) | \( \mathbb{Z}^{2^2} \) |
Proof. We use the Mayer-Vietoris exact sequence: Cover $Y$ by invariant subspaces $U$ and $V$ with the following $D_4$-equivariant homotopy equivalences:

\[ U \simeq \text{pt}, \quad V \simeq D_4/D_2^{(v)}, \quad U \cap V \simeq D_4/Z_2^{(v)}, \]

where $D_2^{(v)} = \{1, C_2, \sigma_x, \sigma_y\} \cong D_2$ and $Z_2^{(v)} = \{1, \sigma_y\} \cong Z_2$. We can summarize the equivariant cohomology of these spaces in low degree as follows:

\[
\begin{array}{|c|c|c|}
\hline
n = 3 & Z_2 \oplus Z_2 & 0 \\
\hline
n = 2 & Z_2^2 \oplus Z_2^2 & Z_2 \\
\hline
n = 1 & 0 & 0 \\
\hline
n = 0 & Z \oplus Z & Z \\
\hline
\end{array}
\]

In the Mayer-Vietoris exact sequence

\[
\cdots \to H^n_{D_4}(Y) \to H^n_{D_4}(U) \oplus H^n_{D_4}(V) \xrightarrow{\Delta} H^n_{D_4}(U \cap V) \to H^{n+1}_{D_4}(Y) \to \cdots,
\]

the map $\Delta : H^n_{D_4}(U) \oplus H^n_{D_4}(V) \to H^n_{D_4}(U \cap V)$ is expressed as $\Delta(u, v) = j_U^* (u) - j_V^* (v)$, where $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ are the inclusions. Under the natural identifications:

\[
H^1_{D_4}(V) \cong H^2_{Z_2^{(v)}}(\text{pt}) \cong \text{Hom}(D_2^{(v)}, U(1)),
\]

\[
H^1_{D_4}(U \cap V) \cong H^2_{Z_2^{(v)}}(\text{pt}) \cong \text{Hom}(Z_2^{(v)}, U(1)),
\]

the map $j_V^*$ agrees with that induced from the inclusion $Z_2^{(v)} \to D_2^{(v)}$. This implies that $j_U^*$ is surjective, and so is $\Delta$ in degree 2. Apparently, $\Delta : H^0_{D_4}(U) \oplus H^1_{D_4}(V) \to H^0_{D_4}(U \cap V)$ is identified with the homomorphism $Z \oplus Z \to Z$ given by $(m, n) \mapsto m - n$. Hence we can solve the Mayer-Vietoris exact sequence for $\{U, V\}$ to get the result claimed in this lemma. \(\square\)

Let $X_1$ be the 1-skeleton.

**Lemma 4.29.** $H^1_{D_4}(X_1; Z) \cong Z_{2^3}$.  

**Proof.** We cover $X_1$ by invariant subspaces $U$ and $V$ which have the following $D_4$-equivariant homotopy equivalences:

\[ U \simeq Y, \quad V \simeq \text{pt}, \quad U \cap V \simeq (D_4/Z_2^{(v)}) \cup (D_4/Z_2^{(d)}), \]

where $Z_2^{(v)} = \{1, \sigma_y\}$ and $Z_2^{(d)} = \{1, \sigma_d\}$. In the Mayer-Vietoris sequence

\[
\cdots \to H^0_{D_4}(X_1) \to H^0_{D_4}(U) \oplus H^0_{D_4}(V) \xrightarrow{\Delta} H^0_{D_4}(U \cap V) \to H^1_{D_4}(X_1) \to \cdots,
\]

the map $\Delta : H^0_{D_4}(U) \oplus H^0_{D_4}(V) \to H^0_{D_4}(U \cap V)$ is expressed as $\Delta(u, v) = j_U^* (u) - j_V^* (v)$ by using the inclusions $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$, and $j_U^*$ is identified with the homomorphism

\[
\text{Hom}(D_4, U(1)) \to \text{Hom}(Z_2^{(v)}, U(1)) \oplus \text{Hom}(Z_2^{(d)}, U(1))
\]

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induced from the inclusions $\mathbb{Z}_2^v \to D_4$ and $\mathbb{Z}_2^d \to D_4$. Let $\rho_1$ and $\rho_2$ be the following 1-dimensional irreducible representations of $D_4$:

\[
\begin{align*}
\rho_1(C_4) &= 1, & \rho_2(C_4) &= -1, \\
\rho_1(\sigma_x) &= -1, & \rho_2(\sigma_x) &= 1,
\end{align*}
\]

which form a basis $\{\rho_1, \rho_2\}$ of $\text{Hom}(D_4, U(1)) \cong \mathbb{Z}_2^\oplus 2$. We can see that $j_V^*$ is surjective, and so is $\Delta$ on the second cohomology. Now we get

\[
H^3_{D_4}(X_1) \cong H^3_{D_4}(U) \oplus H^3_{D_4}(V) \cong H^3_{D_4}(Y) \oplus H^3_{D_4}(pt) \cong \mathbb{Z}_2^\oplus 3,
\]

since $H^3_{D_4}(U \cap V) = 0$. 

**Theorem 4.30 (p4m/p4g).** $H^3_{D_4}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2^\oplus 3$.

**Proof.** Since $H^n_{D_4}(T^2, X_1) \cong H^{n-2}(pt)$ by the excision axiom, the exact sequence for the pair $(T^2, X_1)$,

\[
H^3_{D_4}(T^2, X_1) \to H^3_{D_4}(T^2) \to H^3_{D_4}(X_1) \to H^4_{D_4}(T^2, X_1),
\]

shows $H^3_{D_4}(T^2) \cong H^3_{D_4}(X_1) \cong \mathbb{Z}_2^\oplus 3$. 

**Theorem 4.31 (p4m/p4g).** The following holds true:

(a) $F^2 H^3_{D_4}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2^\oplus 2$.

(b) The $D_4$-equivariant cohomology of $T^2$ in low degree is as follows:

\[
H^0_{D_4}(T^2; \mathbb{Z}) \cong \mathbb{Z}, \quad H^1_{D_4}(T^2; \mathbb{Z}) = 0, \quad H^2_{D_4}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2^\oplus 3.
\]

**Proof.** In the Leray-Serre spectral sequence

\[
E^p_{2,q} = H^p_{\text{group}}(D_4; H^q(T^2; \mathbb{Z})),
\]

the $D_4$-modules $H^0(T^2)$, $H^1(T^2)$ and $H^2(T^2)$ are identified with the trivial $D_4$-module $\mathbb{Z}$, $H^1(Y)$ and $\mathbb{Z}_2$, respectively. Using Lemma 4.1 and Lemma 4.2, we can summarize the $E_2$-terms as follows:

| $q$ | 3 | 2 | 1 | 0 | 0 | 0 | 0 |
|-----|---|---|---|---|---|---|---|
| $p=0$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $p=1$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}_2$ |
| $p=2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |

Now the proof is completed by Lemma 4.4.
4.11 p3m1

Let \( \Pi = \mathbb{Z}a \oplus \mathbb{Z}b \subset \mathbb{R}^2 \) be the same lattice as in the case of p3 and p6. The point group is

\[
P = D_3 = (C, \sigma_x | C^3, \sigma_2^2, \sigma_x C \sigma_2 C).
\]

We let \( D_3 \) act on \( \Pi \) and \( \mathbb{R}^2 \) through the inclusion \( D_3 \subset O(2) \) given by

\[
C = \left( \begin{array}{cc} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{array} \right), \quad \sigma_x = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right).
\]

We may adapt the identification \( a = 1 \) and \( b = \tau = \exp 2\pi i/6 \) under \( \mathbb{R}^2 = \mathbb{C} \). A fundamental domain is \( \{sa+tb | 0 \leq s, t \leq 1\} \) or \( \{s + t\tau | 0 \leq s, t \leq 1\} \). Based on this, we introduce a \( D_3 \)-CW decomposition of \( T^2 \) as follows:

| 0-cell | 1-cell | 2-cell |
|--------|--------|--------|
| \( \tilde{e}_j^0 = pt \) | \( \tilde{e}_j^1 \) \( D_3/\mathbb{Z}_2 \times e^1 \) | \( \tilde{e}_j^2 = D_3 \times e^2 \) |
| \( j = 0, 1, 2 \) | \( (0 \leq i < j \leq 2) \) | |

- (0-cell) The three 0-cells are the following fixed points on \( T^2 \):

\[
\tilde{e}_0^0 = 0, \quad \tilde{e}_1^0 = \frac{1 + \tau}{3}, \quad \tilde{e}_2^0 = \frac{2(1 + \tau)}{3}.
\]

- (1-cell) For \( 0 \leq i < j \leq 2 \), the 1-cell \( \tilde{e}_j^1 = D_3/\mathbb{Z}_2 \times e^1 \) consists of the three segments connecting \( \tilde{e}_j^0 \) and \( \tilde{e}_j^0 \). (Notice that we write \( \mathbb{Z}_2 \subset D_3 \) for one of the three subgroups of order 2, which are all conjugate to each other.)

- (2-cell) The 2-cell \( \tilde{e}_j^2 = D_3 \times e^2 \) consists of the six regular triangle regions surrounded by the 1-cells.

Let \( X_1 \) be the 1-skeleton with respect to the \( D_3 \)-CW decomposition of \( T^2 \).

Lemma 4.32. \( H^3_{D_3}(X_1; \mathbb{Z}) \cong \mathbb{Z}_2 \).
Proof. We can find invariant subspaces $U$ and $V$ of $X_1$ having the following $D_3$-equivariant homotopy equivalences:

\[ U \simeq \bigsqcup_{j=0}^{2} \text{pt}, \quad V \simeq \bigsqcup_{0 \leq i < j \leq 2} D_3/\mathbb{Z}_2, \quad U \cap V \simeq \bigsqcup_{k=1}^{6} D_3/\mathbb{Z}_2. \]

Then we consider the following part of the Mayer-Vietoris exact sequence:

\[
\begin{array}{c}
H^3_{D_3}(U \cup V) \xrightarrow{\Delta} H^3_{D_3}(U \cap V) \longrightarrow H^3_{D_3}(X_1) \longrightarrow H^4_{D_3}(T^2, Y) \mapsto 0.
\end{array}
\]

We can express $\Delta : \mathbb{Z}_2^{\oplus 3} \oplus \mathbb{Z}_2^{\oplus 3} \to \mathbb{Z}_2^{\oplus 6}$ as follows:

\[ \Delta(u_0, u_1, u_2, v_0, v_2, v_0, v_2, v_1, v_2) = (u_0 - v_0, u_0 - v_0, u_1 - v_0, u_1 - v_0, u_2 - v_2, u_2 - v_2, u_2 - v_2). \]

We can readily see $\ker \Delta \cong \mathbb{Z}_2$ and hence $H^3_{D_3}(X_1) \cong \text{coker} \Delta \cong \mathbb{Z}_2$. \hfill \Box

**Theorem 4.33** (p3m1). $H^3_{D_3}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2$.

**Proof.** Notice $H^3_{D_3}(T^2, X_1) \cong H^{n-2}(pt)$ by the excision axiom. By the exact sequence for the pair $(T^2, X_1)$:

\[ H^3_{D_3}(T^2, Y) \to H^3_{D_3}(T^2) \to H^3_{D_3}(X_1) \to H^4_{D_3}(T^2, Y), \]

we get $H^3_{D_3}(T^2) \cong H^3_{D_3}(X_1) \cong \mathbb{Z}_2$. \hfill \Box

Let $Y \subset T^2$ be the invariant subspace given by the three segments connecting the fixed point $c_0^3 \in T^2$:

If we forget about the $D_3$-action, then $Y$ is just a bouquet of three circles. We can think of $Y$ as a $D_3$-CW complex: It has two 0-cells, one of which is $c_0^3$ and the other is of the form $D_3/\mathbb{Z}_2$. The 1-cell of $Y$ is then of the form $D_3 \times e^1$. 

```
Lemma 4.34. The equivariant cohomology of $Y$ is as follows:

| $n$  | $H^0_{D_3}(Y; Z)$ | $H^1_{D_3}(Y; Z)$ | $H^2_{D_3}(Y; Z)$ | $H^3_{D_3}(Y; Z)$ |
|------|------------------|------------------|------------------|------------------|
| $n = 0$ | $Z$              | 0                | 0                | 0                |
| $n = 1$ | 0                | $Z_2$            | 0                | 0                |
| $n = 2$ | 0                | 0                | $Z_2$            | 0                |
| $n = 3$ | 0                | 0                | 0                | $Z_2$            |

Proof. We can find invariant subspaces $U$ and $V$ in $T^2$ which admit the following equivariant homotopy equivalences:

$U \simeq \text{pt}$, \hspace{1cm} $V \simeq D_3/Z_2$, \hspace{1cm} $U \cap V \simeq D_3$.

The equivariant cohomology of these spaces relevant to the computation of the Mayer-Vietoris exact sequence for $\{U, V\}$ are as follows:

| $n$  | $H^0_{D_3}(Y)$ | $H^1_{D_3}(U) \oplus H^1_{D_3}(V)$ | $H^2_{D_3}(U \cap V)$ |
|------|----------------|----------------------------------------|------------------------|
| $n = 3$ | 0              | 0                                      | 0                      |
| $n = 2$ | 0              | $Z_2 \oplus Z_2$                       | 0                      |
| $n = 1$ | 0              | 0                                      | $Z_2$                  |
| $n = 0$ | 0              | 0                                      | 0                      |

Since the homomorphism $\Delta$ in the Mayer-Vietoris exact sequence for $\{U, V\}$:

$H^0_{D_3}(Y) \to H^0_{D_3}(U) \oplus H^0_{D_3}(V) \xrightarrow{\Delta} H^1_{D_3}(U \cap V) \to H^1_{D_3}(Y) \to \cdots$

has the expression $\Delta(u, v) = u - v$, the lemma is seen immediately.

We write $\tilde{Z}$ for the $D_3$-module whose underlying group is $Z$ and $D_3$ acts through the inclusion $D_3 \to O(2)$ and $\det : O(2) \to Z_2$.

Lemma 4.35. There are an exact sequence of $D_3$-modules

$$0 \to H^1(T^2; Z) \to H^1(Y; Z) \xrightarrow{\pi} \tilde{Z} \to 0,$$

and a module homomorphism $s : \tilde{Z} \to H^1(Y; Z)$ such that $\pi \circ s = 3$.

Proof. We use the same bases as used in the proof of Lemma 4.7. On the homology bases, the action of the generator $C \in D_3$ is the same as in the case of $p_3$, while the action of $\sigma_x \in D_3$ is expressed as:

$$\begin{cases}
\sigma_x \eta_1 = -\eta_1, \\
\sigma_x \eta_2 = \eta_2 - \eta_1,
\end{cases} \quad \begin{cases}
\sigma_x \gamma_1 = -\gamma_1, \\
\sigma_x \gamma_2 = \gamma_3, \\
\sigma_x \gamma_3 = \gamma_2.
\end{cases}$$

Accordingly, the action of $C \in D_3$ on the dual basis of cohomology is the same as in the case of $p_3$, and that of $\sigma_x \in D_3$ is expressed as:

$$\begin{cases}
\sigma^*_x h_1 = -h_1 - h_2, \\
\sigma^*_x h_2 = h_2,
\end{cases} \quad \begin{cases}
\sigma^*_x g_1 = -g_1, \\
\sigma^*_x g_2 = g_3, \\
\sigma^*_x g_3 = g_2.
\end{cases}$$

We can now directly verify that the cokernel of $i^* : H^1(T^2) \to H^1(Y)$ induced from the inclusion $i : Y \to T^2$ is isomorphic to $\tilde{Z}$ as a $D_3$-module, and hence get the exact sequence in question. The proof is completed by defining $s : \tilde{Z} \to H^1(Y)$ to be $s(1) = g_1 - g_2 + g_3$. 

\qed
Lemma 4.36. \( H^n_{\text{group}}(D_3; H^1(T^2; \mathbb{Z})) = 0 \) for \( n = 0, 1, 2 \).

Proof. The exact sequence in Lemma 4.35 induces the long exact sequence of group cohomology. By using Lemma 4.1 and Lemma 4.2, we can summarize relevant cohomology groups as follows:

| \( n = 2 \) | \( n = 1 \) | \( n = 0 \) |
| --- | --- | --- |
| \( H^n_{\text{group}}(D_3; H^1(T^2)) \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( H^n_{\text{group}}(D_3; H^1(Y)) \) | \( 0 \) | \( 0 \) |
| \( H^n_{\text{group}}(D_3; \mathbb{Z}) \) | \( 0 \) | \( 0 \) |

Because 3 can be inverted in \( \mathbb{Z}_2 \), the homomorphism in degree 1 group cohomology induced from \( \pi : H^1(Y) \to \mathbb{Z} \) is surjective, and hence bijective. Therefore we get \( H^n_{\text{group}}(D_3; H^1(T^2; \mathbb{Z})) = 0 \) for \( n = 0, 1, 2 \).

Theorem 4.37 (p3m1). The following holds true:

(a) \( F^2 H^0_{D_3}(T^2; \mathbb{Z}) = 0 \).

(b) The \( D_3 \)-equivariant cohomology of \( T^2 \) in low degree is as follows:

\[
H^0_D(T^2; \mathbb{Z}) \cong \mathbb{Z}, \quad H^1_D(T^2; \mathbb{Z}) = 0, \quad H^2_D(T^2; \mathbb{Z}) \cong \mathbb{Z}_2.
\]

Proof. Consider the \( E_2 \)-term of the Leray-Serre spectral sequence

\[
E_2^{p,q} = H^p_{\text{group}}(D_3; H^q(T^2; \mathbb{Z})).
\]

In the coefficients of the group cohomology, \( H^0(T^2) \) is identified with the trivial \( D_3 \)-module \( \mathbb{Z} \). The group cohomology with coefficients in the \( D_3 \)-module \( H^2(T^2) \cong \mathbb{Z} \) is computed in Lemma 4.2, and that with coefficients in \( H^1(T^2) \) is also computed already. The \( E_2 \)-terms are now summarized as follows:

| \( q = 3 \) | \( q = 2 \) | \( q = 1 \) | \( q = 0 \) |
| --- | --- | --- | --- |
| \( p = 0 \) | \( 0 \) | \( Z_2 \) | \( Z \) |
| \( p = 1 \) | \( 0 \) | \( 0 \) | \( 0 \) |
| \( p = 2 \) | \( Z_2 \) | \( 0 \) | \( 0 \) |

Now Lemma 4.4 establishes the theorem.

4.12 p31m

Let \( \Pi = \mathbb{Z}a \oplus \mathbb{Z}b \subset \mathbb{R}^2 \) be the same lattice as in the case of p3, p6 and p3m1. The point group is

\[
P = D_3 = \langle C, \sigma_y \rangle \cap C^3, \sigma_y^2, \sigma_y C \sigma_y C \rangle.
\]

Let \( D_3 \) act on \( \Pi \) and \( \mathbb{R}^2 \) through the inclusion \( D_3 \subset O(2) \) given by

\[
C = \begin{pmatrix}
-1/2 & -\sqrt{3}/2 \\
\sqrt{3}/2 & -1/2
\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

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A fundamental domain is \{sa + tb | 0 \leq s, t \leq 1\}. If we adapt the identifications \(a = 1\) and \(b = \tau = \exp 2\pi i/6\) under \(\mathbb{R}^2 = \mathbb{C}\), then the fundamental domain is \{s + t\tau | 0 \leq s, t \leq 1\}. We introduce a \(D_3\)-CW decomposition of \(T^2\) as follows:

| 0-cell | 1-cell | 2-cell |
|--------|--------|--------|
| \(\widetilde{e}_0^0 = \text{pt}\) | \(\widetilde{e}_0^1 = (D_3/\mathbb{Z}_2) \times e^1\) | \(\widetilde{e}_2^1 = D_3 \times e^2\) |
| \(\widetilde{e}_1^1 = D_3/\mathbb{Z}_3\) | \(\widetilde{e}_1^1 = D_3 \times e^1\) | |

- **(0-cell)** The 0-cell \(\widetilde{e}_0^0 = (D_3/D_3) \times \text{pt} = \text{pt}\) is the unique fixed point on \(T^2\). The other 0-cell is:
  \[\widetilde{e}_1^0 = \left\{\frac{1 + \tau}{3}, \frac{2(1 + \tau)}{3}\right\} \cong (D_3/\mathbb{Z}_3) \times e^0.\]

- **(1-cell)** The 1-cell \(\widetilde{e}_0^1 = (D_3/\mathbb{Z}_2) \times e^1\) consists of the three segments attached only to \(\widetilde{e}_0^0\), where \(\mathbb{Z}_2 = \{1, \sigma y\}\). The other 1-cell \(\widetilde{e}_1^1 = D_3 \times e^1\) consists of the six segments connecting \(\widetilde{e}_0^0\) and \(\widetilde{e}_1^0\).

- **(2-cell)** The 2-cell \(\widetilde{e}_2^0 = D_3 \times e^2\) consists of the six triangle regions surrounded by the 1-cells.

Let \(Y \subset T^2\) be the invariant subspace \(Y = \widetilde{e}_0^0 \cup \widetilde{e}_1^0\).

**Lemma 4.38.** The equivariant cohomology of \(Y\) is as follows:

| \(n\) | \(H_{D_3}(Y; \mathbb{Z})\) |
|------|-----------------|
| 0    | \(\mathbb{Z}\)          |
| 1    | \(\mathbb{Z}\)          |
| 2    | \(\mathbb{Z}_2\)        |
| 3    | \(\mathbb{Z}_2\)        |

**Proof.** We can cover \(Y\) by invariant subspaces \(U\) and \(V\) such that there are the following equivariant homotopy equivalences:

\[U \simeq \text{pt}, \quad V \simeq D_3/\mathbb{Z}_2, \quad U \cap V \simeq (D_3/\mathbb{Z}_2) \cup (D_3/\mathbb{Z}_2).\]

We can summarize the equivariant cohomology of these spaces as follows:

| \(n\) | \(H_{D_3}(Y)\) | \(H_{D_3}(U)\) | \(H_{D_3}(U \cap V)\) |
|------|-----------------|-----------------|-----------------|
| 0    | \(0\)          | \(0\)          | \(0\)          |
| 1    | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) |
| 2    | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) |
| 3    | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) |
In the Mayer-Vietoris exact sequence

\[ \cdots \rightarrow H^n_D(Y) \rightarrow H^n_D(U) \oplus H^n_D(V) \xrightarrow{\Delta} H^n_D(U \cap V) \rightarrow H^{n+1}_D(Y) \rightarrow \cdots \]

the homomorphism \( \Delta : H^0_D(U) \oplus H^0_D(V) \rightarrow H^0_D(U \cap V) \) is expressed as \( \Delta(u, v) = (u-v, u-v) \). This allows us to determine \( H^0_D(Y) \cong \mathbb{Z} \) and \( H^1_D(Y) \cong \mathbb{Z} \). Similarly, \( \Delta : H^2_D(U) \oplus H^2_D(V) \rightarrow H^2_D(U \cap V) \) is expressed as \( \Delta(u, v) = (u-v, u-v) \), which leads to \( H^2_D(Y) \cong \mathbb{Z}_2 \) and \( H^3_D(Y) \cong \mathbb{Z}_2 \).

Let \( Z \subset T^2 \) be the invariant subspace \( Z = \mathbb{Z}_0^c \cup \mathbb{Z}_1^c \cup \mathbb{Z}_1^c \).

**Lemma 4.39.** \( H^3_D(Z; \mathbb{Z}) = 0. \)

**Proof.** We can find invariant subspaces \( U \) and \( V \) in \( Z \) which have the following equivariant homotopy equivalences:

\[ U \cong \text{pt}, \quad V \cong D_3/\mathbb{Z}_3, \quad U \cap V \cong D_3. \]

A part of the Mayer-Vietoris exact sequence for \( \{U, V\} \) is

\[ H^3_D(U \cap V) \rightarrow H^3_D(Z) \rightarrow H^3_D(U) \oplus H^3_D(V), \]

in which \( H^3_D(U \cap V) = 0 \), \( H^3_D(U) = 0 \) and \( H^3_D(V) = 0 \).

Let \( X_1 \) be the 1-skeleton of \( T^2 \) with respect to the \( D_3 \)-CW complex structure.

**Lemma 4.40.** \( H^3_D(X_1; \mathbb{Z}) \cong \mathbb{Z}_2. \)

**Proof.** The 1-skeleton is given by identifying \( Y \) and \( Z \) at the fixed point: \( X_1 = Y \lor Z \). Then \( H^3_D(X_1) \cong H^3_D(\text{pt}) \oplus H^3_D(Y) \oplus H^3_D(Z) \) completes the proof.

**Theorem 4.41 (p31m).** \( H^3_D(T^2; \mathbb{Z}) \cong \mathbb{Z}_2. \)

**Proof.** Let us see the exact sequence for the pair \( (T^2, X_1) \):

\[ H^3_D(T^2, X_1) \rightarrow H^3_D(T^2) \rightarrow H^3_D(X_1) \rightarrow H^3_D(T^2, X_1). \]

By excision \( H^n_D(T^2, X_1) \cong H^{n-2}(\text{pt}) \), so that \( H^3_D(T^2) \cong H^3_D(X_1) \cong \mathbb{Z}_2 \).

**Lemma 4.42.** There are an exact sequence of \( D_3 \)-modules:

\[ 0 \rightarrow H^1(T^2; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z}) \xrightarrow{\pi} \mathbb{Z} \rightarrow 0 \]

and a homomorphism \( s : \mathbb{Z} \rightarrow H^1(Y; \mathbb{Z}) \) such that \( \pi \circ s = 3 \), where \( \mathbb{Z} \) stands for the trivial \( D_3 \)-module.

**Proof.** We use the same bases as in the proof of Lemma 4.7 in the case of \( D_3 \). The action of \( C \subset D_3 \) is then the same as in the case of \( D_3 \), while that of \( \sigma_y \in D_3 \) on the homology bases is as follows:

\[
\begin{align*}
\sigma_y \eta_1 &= \eta_1, \\
\sigma_y \eta_2 &= \eta_1 - \eta_2.
\end{align*}
\]

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Thus, on the dual cohomology bases, the action of $C \mathbb{Z}_2 \mathbb{D}_3$ is the same as in the case of $p3$, and that of $\sigma_y \in \mathbb{D}_3$ is:

\[
\begin{align*}
\sigma^*_y h_1 &= h_1 + h_2, \\
\sigma^*_y h_2 &= -h_2, \\
\sigma^*_y g_1 &= g_1, \\
\sigma^*_y g_2 &= -g_3, \\
\sigma^*_y g_3 &= -g_2.
\end{align*}
\]

We can find an isomorphism of $\mathbb{D}_3$-modules $\text{Coker}(i^*) \cong \mathbb{Z}$, where $i^* : H^1(T^2) \to H^1(Y)$ is induced from the inclusion $i : Y \to T^2$. This gives the exact sequence. Defining $s : \mathbb{Z} \to H^2(Y)$ by $s(1) = g_1 - g_2 + g_3$, we complete the proof. 

**Lemma 4.43.** The group cohomology $H_n^\text{group}(\mathbb{D}_3; H^1(T^2))$ is as follows:

| $n$  | $n = 0$ | $n = 1$ | $n = 2$ |
|------|---------|---------|---------|
|      | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |          |

**Proof.** We consider the long exact sequence in group cohomology induced from the short exact sequence in Lemma 4.42. By Lemma 4.1, we have:

| $n = 2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
|---------|----------------|----------------|
| $n = 1$ | $\mathbb{Z}$  | $\mathbb{Z}$  |

We can identify $H^0_\text{group}(\mathbb{D}_3; H^1(Y)) \to H^0_\text{group}(\mathbb{D}_3; \mathbb{Z})$ with the map $\mathbb{Z} \to \mathbb{Z}$ multiplying 3 because of $s : \mathbb{Z} \to H^1(Y)$. Using $s$ again, we can see that $H^2_\text{group}(\mathbb{D}_3; H^1(Y)) \to H^2_\text{group}(\mathbb{D}_3; \mathbb{Z})$ is bijective. These information determines the group cohomology in question.

**Theorem 4.44** (p31m). The following holds true:

(a) $F^2H^2_{D_3}(T^2; \mathbb{Z}) = 0$.

(b) The $\mathbb{D}_3$-equivariant cohomology of $T^2$ in low degree is as follows:

$H^0_{D_3}(T^2; \mathbb{Z}) \cong \mathbb{Z}$, $H^1_{D_3}(T^2; \mathbb{Z}) = 0$, $H^2_{D_3}(T^2; \mathbb{Z}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_2$.

**Proof.** Recall the $E_2$-term of the Leray-Serre spectral sequence

$E_2^{p,q} = H^q_\text{group}(\mathbb{D}_3; H^p(T^2; \mathbb{Z}))$.

The coefficient $H^0(T^2)$ is identified with the trivial $\mathbb{D}_3$-module $\mathbb{Z}$, and $H^2(T^2)$ with $\mathbb{Z}$ in Lemma 4.2. The group cohomology with its coefficients in $H^1(T^2)$ is already seen. The $E_2$-term is now summarized as follows:

| $q = 3$ | $0$  | $0$  | $0$  | $0$  |
|---------|------|------|------|------|
| $q = 2$ | $0$  | $\mathbb{Z}_2$ |      |      |
| $q = 1$ | $0$  | $\mathbb{Z}_3$ | $0$  |      |
| $q = 0$ | $\mathbb{Z}$ | $0$  | $\mathbb{Z}_2$ | $0$  |

The theorem now follows from Lemma 4.4.
4.13 \textit{p6m}

We let $\mathbb{Z}_a \oplus \mathbb{Z}_b \subset \mathbb{R}^2$ be the same lattice as in the case of \textit{p3}, \textit{p6}, \textit{p3m1} and \textit{p31m}. The point group $P$ is

$$D_6 = \langle C, \sigma_1 \mid C^6, \sigma_1^2, \sigma_1 C \sigma_1 C \rangle = \{1, C, C^2, C^3, C^4, C^5, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\},$$

where $\sigma_6 = C^{6-1}\sigma_1$. This group acts on $\Pi$ and $\mathbb{R}^2$ through the inclusion $D_6 \subset O(2)$ defined by

$$C = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If we adapt the identification $a = 1$ and $b = \tau = \exp 2\pi i/6$ under $\mathbb{R}^2 = \mathbb{C}$, then the action of $C \in D_6$ is the multiplication of $\tau$ and $\sigma_1$ is the complex conjugation. A fundamental domain is $\{sa + tb \mid 0 \leq s, t \leq 1\}$ or equivalently $\{s + t\tau \mid 0 \leq s, t \leq 1\}$. A $D_6$-CW decomposition of $T^2$ is as follows:

| 0-cell | 1-cell | 2-cell |
|--------|--------|--------|
| $\vec{e}_0^0 = \text{pt}$ | $\vec{e}_0^1 = (D_6/\{1, \sigma_1\}) \times e^1$ | $\vec{e}^2 = D_6 \times e^2$ |
| $\vec{e}_0^1 = D^6/\{1, C^3, \sigma_1, \sigma_4\}$ | $\vec{e}_0^2 = (D_6/\{1, \sigma_2\}) \times e^1$ | |
| $\vec{e}_0^2 = D^6/\{1, C^2, C^4, \sigma_2, \sigma_4, \sigma_6\}$ | $\vec{e}_{12}^0 = (D_6/\{1, \sigma_4\}) \times e^1$ | |

- (0-cell) The 0-cell $\vec{e}_0^0 = (D_6/D_6) \times e^0 = \text{pt}$ is the unique fixed point on $T^2$. The other 0-cells are defined as follows:

  $\vec{e}_0^1 = \left\{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\right\} \cong (D_6/\{1, C^3, \sigma_1, \sigma_4\}) \times e^0$,
  $\vec{e}_0^2 = \left\{\frac{1+\tau}{3}, \frac{2(1+\tau)}{3}\right\} \cong (D_6/\{1, C^2, C^4, \sigma_2, \sigma_4, \sigma_6\}) \times e^0$.

- (1-cell) For $0 \leq i < j \leq 2$, the 1-cell $\vec{e}_{ij}^1$ consists of the six segments connecting $\vec{e}_0^0$ and $\vec{e}_0^j$. They are of the forms $\vec{e}_{01}^1 = (D_6/\{1, \sigma_1\}) \times e^1$, $\vec{e}_{12}^1 = (D_6/\{1, \sigma_2\}) \times e^1$ and $\vec{e}_{12}^1 = (D_6/\{1, \sigma_4\}) \times e^1$. 

\[
\begin{array}{c}
\includegraphics[width=0.8\textwidth]{figure1.png}
\end{array}
\]
• (2-cell) The 2-cell \( \tilde{c}^2 = D_6 \times e^2 \) consists of the twelve small triangle regions surrounded by the 1-cells.

Let \( Y \subset T^2 \) be the invariant subspace \( Y = \tilde{c}^0_0 \cup \tilde{c}^0_1 \cup \tilde{c}^1_{01} \).

**Lemma 4.45.** The equivariant cohomology of \( Y \) is as follows:

| \( n \) | \( H^n_{D_6}(Y; \mathbb{Z}) \) |
|---|---|
| \( n = 0 \) | \( \mathbb{Z} \) |
| \( n = 1 \) | \( \mathbb{Z}_2 \) |
| \( n = 2 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \) |
| \( n = 3 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \) |
| \( n = 4 \) | \( 0 \) |

**Proof.** We can find \( D_6 \)-invariant subspaces \( U \) and \( V \) in \( Y \) which have the following equivariant homotopy equivalences:

\[
U \simeq \tilde{c}^0_0 = pt, \quad V \simeq \tilde{c}^0_1 = D_6/\mathbb{Z}_2, \quad U \cap V \simeq \tilde{c}^1_{01} \simeq D_6/\mathbb{Z}_2^{(1)},
\]

where \( D_2 = \{1, C^3, \sigma_1, \sigma_4\} \) and \( \mathbb{Z}_2^{(1)} = \{1, \sigma_1\} \). The equivariant cohomology groups of these spaces can be summarized as follows:

| \( n \) | \( H^n_{D_6}(U) \) | \( H^n_{D_6}(U) \oplus H^n_{D_6}(V) \) | \( H^n_{D_6}(U \cap V) \) |
|---|---|---|---|
| \( n = 0 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2 \) |
| \( n = 1 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2 \) |
| \( n = 2 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \) | \( 0 \) | \( 0 \) |
| \( n = 3 \) | \( \mathbb{Z}_2 \oplus \mathbb{Z}_2^2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( n = 4 \) | \( 0 \) | \( 0 \) | \( 0 \) |

In the Mayer-Vietoris exact sequence:

\[
\cdots \to H^n_{D_6}(Y) \to H^n_{D_6}(U) \oplus H^n_{D_6}(V) \xrightarrow{\Delta} H^n_{D_6}(U \cap V) \to H^{n+1}_{D_6}(Y) \to \cdots,
\]

the homomorphism \( \Delta \) is expressed as \( \Delta(u, v) = j_U^* (u) - j_V^* (v) \) with \( j_U : U \cap V \to U \) and \( j_V : U \cap V \to V \) the inclusions. This immediately determines \( H^n_{D_6}(Y) \cong \mathbb{Z} \) and \( H^1_{D_6}(Y) = 0 \). To complete the proof, we recall the identifications

\[
H^2_{D_6}(U) \cong \text{Hom}(D_6, U(1)) \cong \mathbb{Z}_2^{(1)},
\]

\[
H^2_{D_6}(V) \cong \text{Hom}(D_2, U(1)) \cong \mathbb{Z}_2^{(2)},
\]

\[
H^2_{D_6}(U \cap V) \cong \text{Hom}(\mathbb{Z}_2^{(1)}, U(1)) \cong \mathbb{Z}_2,
\]

under which \( j_U^* \) and \( j_V^* \) are induced from the inclusions \( D_2 \to D_6 \) and \( \mathbb{Z}_2^{(1)} \to D_6 \).

As a basis of \( H^2_{D_6}(U) \) we can choose the following 1-dimensional representations \( \rho_1 : D_6 \to U(1) \) of \( D_6 \):

\[
\rho_1 : \begin{cases} C \mapsto 1, \\ \sigma_1 \mapsto -1. \end{cases}
\]

\[
\rho_2 : \begin{cases} C \mapsto -1, \\ \sigma_1 \mapsto 1. \end{cases}
\]

Similarly, we can choose the following 1-dimensional representations \( \rho'_1 \) of \( D_2 = \{1, \sigma_1, C^3, \sigma_4\} \) as a basis of \( H^2_{D_6}(V) \):

\[
\rho'_1 : \begin{cases} C^3 \mapsto 1, \\ \sigma_1 \mapsto -1. \end{cases}
\]

\[
\rho'_2 : \begin{cases} C^3 \mapsto -1, \\ \sigma_1 \mapsto 1. \end{cases}
\]
Now, we can see $H^2_{D_6}(Y) \cong \ker \Delta \cong \mathbb{Z}_2^{\oplus 3}$, and it has the following basis

$$\{(\rho_1, \rho'_1), (\rho_2, \rho'_2), (0, \rho'_2)\} \subset \text{Hom}(D_6, U(1)) \oplus \text{Hom}(D_4, U(1)).$$

We can also see $\Delta$ is surjective, and $H^3_{D_6}(Y) \cong \mathbb{Z}_2^{\oplus 2}$. \hfill \Box

Let $Y_1$ be the 1-skeleton of the $D_6$-CW complex $T^2$.

**Lemma 4.46.** $H^3_{D_6}(X_1; \mathbb{Z}) \cong \mathbb{Z}_2^{\oplus 2}$.

**Proof.** We cover $X_1$ by invariant subspaces $U'$ and $V'$ which admit the following equivariant homotopy equivalences:

$$U' \cong Y, \quad V' \cong e^3_2 = D_6/D_3, \quad U' \cap V' \cong e^1_{02} \sqcup e^1_{12} \cong D_6/Z_2^{(2)} \sqcup D_6/Z_2^{(4)},$$

where $D_3 = \{1, C^2, C^4, \sigma_2, \sigma_4, \sigma_4\}$, $Z_2^{(2)} = \{1, \sigma_2\}$ and $Z_2^{(4)} = \{1, \sigma_4\}$. The equivariant cohomology groups of these spaces are summarized as follows:

| $n$  | $H^n_{D_6}(X_1)$ | $H^n_{D_6}(U') \oplus H^n_{D_6}(V')$ | $H^n_{D_6}(U' \cap V')$ |
|------|------------------|------------------------------------|---------------------|
| $n = 3$ | $\mathbb{Z}_2^{\oplus 2} \oplus 0$ | $0$ | $0$ |
| $n = 2$ | $\mathbb{Z}_2^{\oplus 2} \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| $n = 1$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $n = 0$ | $H^0_{D_6}(X_1)$ | $H^0_{D_6}(U') \oplus H^0_{D_6}(V')$ | $H^0_{D_6}(U' \cap V')$ |

The homomorphism $\Delta$ in the Mayer-Vietoris exact sequence

$$\cdots \rightarrow H^2_{D_6}(X_1) \rightarrow H^2_{D_6}(U') \oplus H^2_{D_6}(V') \xrightarrow{\Delta} H^2_{D_6}(U' \cap V') \rightarrow H^3_{D_6}(X_1) \rightarrow \cdots$$

is expressed as $\Delta(u, v) = j^*_U(u) - j^*_V(v)$ by using the inclusions $j^*_U : U' \cap V' \rightarrow U'$ and $j^*_V : U' \cap V' \rightarrow V'$. An inspection proves that $j^*_U$ agrees with the composition of the following two homomorphisms:

(i) The inclusion that follows from the calculation of $H^2_{D_6}(Y)$:

$$H^2_{D_6}(U') \cong H^2_{D_6}(Y) \rightarrow \text{Hom}(D_6, U(1)) \oplus \text{Hom}(D_2, U(1)).$$

(ii) The sum $i_2 \oplus i_4$ of the homomorphisms

$$i_2 : \text{Hom}(D_6, U(1)) \rightarrow \text{Hom}(Z_2^{(2)}, U(1)),$$

$$i_4 : \text{Hom}(D_2, U(1)) \rightarrow \text{Hom}(Z_2^{(4)}, U(1)),$$

induced from the inclusions $i_2 : Z_2^{(2)} \rightarrow D_6$ and $Z_2^{(4)} \rightarrow D_2$.

Then, using the basis presented in the calculation of $H^2_{D_6}(Y)$, we find

$$j^*_U(\rho_1, \rho'_1) = (\rho, \rho), \quad j^*_U(\rho_2, \rho'_2) = (\rho, \rho), \quad j^*_U(0, \rho'_2) = (0, \rho),$$

where $\rho : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ is the identity map generating $\text{Hom}(Z_2, U(1)) \cong \mathbb{Z}_2$. Hence $j^*_U$, as well as $\Delta$ are surjective, and $H^3_{D_6}(X_1) \cong H^3_{D_6}(Y) \cong \mathbb{Z}_2^{\oplus 2}$. \hfill \Box
Theorem 4.47 (p6m). \( H^2_{D_6}(T^2; Z) \cong Z_2^\oplus 2 \).

Proof. The relevant part of the exact sequence for the pair \((T^2, X_1)\) is:

\[
H^3_{D_6}(T^2, X_1) \to H^3_{D_6}(T^2) \to H^3_{D_6}(X_1) \to H^3_{D_6}(T^2, X_1).
\]

By means of the excision axiom, it holds that \( H^3_{D_6}(T^2, X_1) \cong H^{n-2}(pt) \). Therefore we get \( H^3_{D_6}(T^2) \cong H^3_{D_6}(X_1) \cong Z_2^\oplus 2 \).

Let \( \hat{Z} \) be the \( D_6 \)-module such that its underlying group is \( Z \) and \( D_6 \) acts by \( C : n \mapsto -n \) and \( \sigma_1 : n \mapsto n \) for \( n \in \hat{Z} \).

Lemma 4.48. There are an exact sequence of \( D_6 \)-module:

\[
0 \to H^1(T^2; Z) \to H^1(Y; Z) \xrightarrow{\pi} \hat{Z} \to 0
\]

and a homomorphism \( s : \hat{Z} \to H^1(Y; Z) \) such that \( \pi \circ s = 3 \).

Proof. We consider the same bases as in the proof of Lemma 4.7. The action of \( C \in D_6 \) on these bases has the same expression as in the proof of Lemma 4.13, and that of \( \sigma_1 = \sigma_y \in D_6 \) is in Lemma 4.42. These expressions allow us to prove that the cokernel of the homomorphism \( i^* : H^1(T^2) \to H^2(Y) \) induced from the inclusion \( i : Y \to T^2 \) is isomorphic to \( \hat{Z} \), yielding the exact sequence. The homomorphism \( s : \hat{Z} \to H^1(Y) \) is given by \( s(1) = g_1 - g_2 + g_3 \).

Lemma 4.49. The group cohomology of \( D_6 \) with coefficients in \( \hat{Z} \) is:

\[
\begin{array}{c|c|c|c}
\text{Group Cohomology} & n = 0 & n = 1 & n = 2 \\
\hline
H^n_{D_6}(\hat{Z}; Z_2) & 0 & Z_2 & Z_2 \\
\end{array}
\]

Proof. Let \( \hat{S}^1 \) be the unit circle in \( \mathbb{C} \) equipped with an action of \( D_6 \) defined by \( C(u) = u^{-1} \) and \( \sigma_1(u) = u \) for \( u \in \hat{S}^1 \). Clearly, \( \hat{S}^1 \) admits the structure of a \( D_6 \)-CW complex. Also, \( H^1(\hat{S}^1) \cong \hat{Z} \) as \( D_6 \)-modules. Now, we have \( H^n_{D_6}(\hat{S}^1; Z) \cong H^{n+1}(\hat{S}^1; Z) \) by Lemma 4.1, so that it suffices to compute the equivariant cohomology of \( \hat{S}^1 \). We can find \( D_6 \)-invariant subspaces \( U \) and \( V \) in \( \hat{S}^1 \) having the following homotopy equivalences:

\[
U \simeq pt,, \quad V \simeq pt, \quad U \cap V \simeq D_6/D_3 \times e^1,
\]

where \( D_3 = \{1, C^2, C^4, \sigma_1, \sigma_3, \sigma_5\} \subset D_6 \). The cohomology of these spaces are:

\[
\begin{array}{c|c|c|c}
n & 3 & 2 & 1 & 0 \\
\hline
H^n_{D_6}(\hat{S}^1) & Z_2 \oplus Z_2 & Z_2 & Z_2 & 0 \\
\end{array}
\]

In the Mayer-Vietoris exact sequence:

\[
\cdots \to H^n_{D_6}(\hat{S}^1) \to H^n_{D_6}(U) \oplus H^n_{D_6}(V) \xrightarrow{\Delta} H^n_{D_6}(U \cap V) \to H^{n+1}_{D_6}(\hat{S}^1) \to \cdots
\]

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the homomorphism \( \Delta \) is expressed as \( \Delta(u, v) = j_U^+(u) - j_V^+(v) \), where \( j_U : U \cap V \to U \) and \( j_V : U \cap V \to V \) are the inclusions. This immediately determines 

\[
H^0_{\text{group}}(D_6; \tilde{Z}) \cong \tilde{H}^1_{D_6}(\tilde{S}^1) = 0.
\]

Under the identifications,

\[
\begin{align*}
H^2_{D_6}(U) & \cong \text{Hom}(D_6, U(1)), & \quad H^2_{D_6}(U \cap V) & \cong \text{Hom}(D_6', U(1)), \\
\end{align*}
\]

the homomorphism \( j^+_U \) is induced from the inclusion \( D_0 \to D_6 \). This implies that \( j^+_U \) is surjective, and so is \( \Delta \). This allows us to get 

\[
H^1_{\text{group}}(D_6; \tilde{Z}) \cong \tilde{H}^2_{D_6}(\tilde{S}^1) \cong \mathbb{Z}_2 \quad \text{and} \quad H^2_{\text{group}}(D_6; \tilde{Z}) \cong \tilde{H}^3_{D_6}(\tilde{S}^1) \cong \mathbb{Z}_2.
\]

**Lemma 4.50.** \( H^n_{\text{group}}(D_6; H^1(T^2; \mathbb{Z})) = 0 \) for \( n = 0, 1, 2 \).

**Proof.** We use the long exact sequence in group cohomology induced from the exact sequence \( 0 \to H^1(T^2) \to H^1(Y) \to \tilde{Z} \to 0 \) in coefficients. By Lemma 4.1 and Lemma 4.49, we get the following:

\[
\begin{array}{c|c|c}
 n = 2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
 n = 1 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
 n = 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
 H^n_{\text{group}}(D_6; H^1(T^2)) & H^n_{\text{group}}(D_6; H^1(Y)) & H^n_{\text{group}}(D_6; \mathbb{Z}) \\
\end{array}
\]

Apparently, \( H^0_{\text{group}}(D_6; H^1(T^2)) = 0 \). The homomorphism in group cohomology induced from \( \pi : H^1(Y) \to \tilde{Z} \) is surjective in degree 1 and 2, because \( \pi \circ s = 3 \). This leads to the remaining vanishing.

**Theorem 4.51 (p6m).** The following holds true:

(a) \( H^2_{D_6}(T^2; \mathbb{Z}) \cong \mathbb{Z}_2 \).

(b) The \( D_6 \)-equivariant cohomology of \( T^2 \) in low degree is as follows:

\[
H^0_{D_6}(T^2; \mathbb{Z}) \cong \mathbb{Z}, \quad H^1_{D_6}(T^2; \mathbb{Z}) = 0, \quad H^2_{D_6}(T^2; \mathbb{Z}) \cong \mathbb{Z}^2.
\]

**Proof.** In the \( E_2 \)-term of the Leray-Serre spectral sequence:

\[
E_2^{p,q} = H^p_{\text{group}}(D_6; H^q(T^2; \mathbb{Z}))
\]

the coefficient \( H^0(T^2) \) is identified with the trivial \( D_6 \)-module \( \mathbb{Z} \), and \( H^2(T^2) \) with \( \tilde{Z} \). The group cohomology with coefficients in \( H^1(T^2) \) is already computed. The \( E_2 \)-terms are summarized as follows:

\[
\begin{array}{c|c|c|c|c|c|c}
 q = 3 & 0 & 0 & 0 & 0 \\
 q = 2 & 0 & \mathbb{Z}_2 & & \\
 q = 1 & 0 & 0 & 0 & \\
 q = 0 & \mathbb{Z} & 0 & \mathbb{Z}^2 & \mathbb{Z}_2 \\
 E_2^{p,q} & p = 0 & p = 1 & p = 2 & p = 3 \\
\end{array}
\]

This list and Lemma 4.4 lead to the theorem.
4.14 The proof of Corollary 1.2

The only non-trivial point in the corollary is (c), which we prove here. Let $P$ be the point group of one of the 2-dimensional space groups. We can assume that $P$ does not preserve the orientation of $T^2$. Then we have

$$F^2 H_P^3(T^2; \mathbb{Z}) \cong E_2^{2,1} \oplus E_2^{3,0}$$

by Lemma 4.4, in which the direct summands are

$$E_2^{2,1} = H_{\text{group}}^3(P; H^1(T^2; \mathbb{Z})), \quad E_2^{3,0} = H_{\text{group}}^3(P; \mathbb{Z}) \cong H_{\text{group}}^2(P; U(1)).$$

Thus, it suffices to prove that the group cocycles induced from the nonsymmorphic 2-dimensional space groups as in Section 2 generate $E_2^{2,1}$.

Let us recall the construction of a group 2-cocycle in Section 2: For the point group $P$ of a 2-dimensional space group, we get a map $a : P \to \mathbb{R}^2$ by expressing the composition of the inclusions $P \to O(2)$ and $O(2) \to O(2) \times \mathbb{R}^2$ as $p \mapsto (p, a_p)$. We put $\nu(p_1, p_2) = a_{p_1} + p_1 a_{p_2} - a_{p_1 p_2}$ and define the group 2-cocycle $\tau \in Z_{\text{group}}^2(P; C(T^2; \mathbb{Z}))$ by $\tau(p_1, p_2, k) = e^{2\pi i (\nu(p_2, a_{p_1}^{-1} - 1), k)}$ for $p_1, p_2 \in P$ and $k \in T^2 = \mathbb{R}^2/\mathbb{Z}$. The invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ can be taken to be the standard one in every case. To simplify the notation, we adapt the identification $\mathbb{R}^2/\mathbb{Z} \cong S^1 \times S^1$ by $(k_1, k_2) \mapsto (e^{2\pi i k_1}, e^{2\pi i k_2})$, and write $u = e^{2\pi i k_1}$ and $v = e^{2\pi i k_2}$. Then, for the nonsymmorphic 2-dimensional space groups $pg$, $pmg$, $pgg$ and $p4g$, the map $a : P \to \mathbb{R}^2$ and the group 2-cocycle are as follows.

- **(pg)** The point group is $\mathbb{Z}_2 = (\sigma|\sigma^2)$ and:

  $$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_\sigma = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

  The values of the induced 2-cocycle $\tau_{pg} \in Z_{\text{group}}^2(\mathbb{Z}_2; C(T^2, U(1)))$ are:

  $$
  \begin{array}{c|cc}
  \tau_{pg}(p_1, p_2; u, v) & p_2 = 1 & p_2 = \sigma \\
  \hline
  p_1 = 1 & 1 & 1 \\
  p_1 = \sigma & 1 & v \\
  \end{array}
  $$

- **(pmg)** The point group is $D_2 = \{1, \sigma_x, \sigma_y, \sigma_x \sigma_y\}$ and $a : D_2 \to \mathbb{R}^2$ is:

  $$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_{\sigma_x} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \quad a_{\sigma_y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_{\sigma_x \sigma_y} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

  The values of the 2-cocycle $\tau_{pmg} \in Z_{\text{group}}^2(D_2; C(T^2, U(1)))$ are:

  $$
  \begin{array}{c|cccc}
  \tau_{pmg}(p_1, p_2; u, v) & p_2 = 1 & p_2 = \sigma_x & p_2 = \sigma_y & p_2 = \sigma_x \sigma_y \\
  \hline
  p_1 = 1 & 1 & 1 & 1 & 1 \\
  p_1 = \sigma_x & 1 & v & \bar{v} & 1 \\
  p_1 = \sigma_y & 1 & 1 & 1 & 1 \\
  p_1 = \sigma_x \sigma_y & 1 & v & \bar{v} & 1 \\
  \end{array}
  $$

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• (pgg) The point group is $D_2 = \{1, \sigma_x, \sigma_y, \sigma_x \sigma_y\}$ and:

$$a_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad a_{\sigma_x} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \quad a_{\sigma_y} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \quad a_{\sigma_x \sigma_y} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$ 

The values of $\tau_{\text{pgg}} \in Z^2_{\text{group}}(D_2; C(T^2, U(1)))$ are:

| $\tau_{\text{pgg}}(p_1, p_2; u, v)$ | $p_2 = 1$ | $p_2 = \sigma_x$ | $p_2 = \sigma_y$ | $p_2 = \sigma_x \sigma_y$ |
|---------------------------------|-----------|----------------|----------------|----------------|
| $p_1 = 1$                       | 1         | 1              | 1              | 1              |
| $p_1 = \sigma_x$                | 1         | $v$            | $\bar{v}$      | 1              |
| $p_1 = \sigma_y$                | 1         | $\bar{u}$      | $u$            | 1              |
| $p_1 = \sigma_x \sigma_y$       | 1         | $\bar{u}v$     | $uv$           | 1              |

• (p4g) The point group is $D_4$ and $a : D_4 \to \mathbb{R}^2$ is given by:

$$a_p = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}$$

The values of $\tau_{\text{p4g}} \in Z^2_{\text{group}}(D_4; C(T^2, U(1)))$ are:

| $\tau_{\text{p4g}}(p_1, p_2; u, v)$ | $C_4$ | $C_4^2$ | $C_4^3$ | $\sigma_x$ | $\sigma_d$ | $\sigma_y$ | $\sigma_d'$ |
|---------------------------------|------|---------|---------|-----------|------------|------------|----------|
| $p_1$                           | 1    | 1       | 1       | 1         | 1          | 1          | 1        |
| $C_4$                           | $\bar{v}$ | 1     | $v$     | $\bar{u}$ | 1          | $u$        | 1        |
| $C_4^2$                         | 1    | $u\bar{v}$ | 1       | $\bar{u}v$ | $uv$       | 1          | $uv$     |
| $C_4^3$                         | 1    | $u$    | 1       | $\bar{u}$ | $v$        | $\bar{v}$  | 1        |
| $\sigma_x$                      | 1    | $u$    | 1       | $\bar{u}$ | $v$        | $\bar{v}$  | 1        |
| $\sigma_d$                      | 1    | 1      | 1       | 1          | 1          | 1          | 1        |
| $\sigma_y$                      | 1    | $\bar{v}$ | 1     | $\bar{u}$ | 1          | $u$        | 1        |
| $\sigma_d'$                     | 1    | $u\bar{v}$ | 1       | $\bar{u}v$ | $uv$       | 1          | $uv$     |

Lemma 4.52. The group cohomology

$$H^2_{\text{group}}(\mathbb{Z}_2; \mathbb{Z}) \cong \mathbb{Z}_2$$

with its coefficients in the trivial $\mathbb{Z}_2$-module $\mathbb{Z}$ is generated by the group cocycle $c : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}$ whose values are:

$$c(1, \pm 1) = c(\pm 1, 1) = 0, \quad c(-1, -1) = 1.$$ 

Proof. For any group 2-cochain $c$, its coboundary is:

$$\partial c(1, 1, 1) = 0, \quad \partial c(-1, 1, 1) = c(1, 1) - c(-1, 1),$$

$$\partial c(1, 1, -1) = c(1, 1) - c(1, -1), \quad \partial c(-1, 1, -1) = c(-1, 1) - c(-1, -1),$$

$$\partial c(1, -1, 1) = 0, \quad \partial c(-1, -1, 1) = c(-1, -1) - c(-1, 1),$$

$$\partial c(1, -1, -1) = c(1, 1) - c(-1, 1), \quad \partial c(-1, -1, -1) = c(-1, 1) - c(1, -1).$$

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Hence the cocycle condition $\partial c = 0$ is equivalent to:

$$c(1, 1) = c(1, -1) = c(-1, 1).$$

In particular, there is no constraint on $c(-1, -1)$. For a 1-cochain $b$, we have:

$$(\partial b)(1, 1) = b(1), \quad (\partial b)(-1, 1) = 2b(-1) - b(1),$$

Thus, we can always normalize a 2-cocycle $c$ so as to be $c(1, 1) = 0$. The normalized 2-cocycles $c$ are characterized by $c(-1, -1)$, and constitute a group isomorphic to $\mathbb{Z}$. In this group, the 2-coboundaries compatible with the normalization form the subgroup $2\mathbb{Z} \subset \mathbb{Z}$. This implies the second group cohomology, which is isomorphic to $\mathbb{Z}$, is represented by the 2-cocycle as stated.

Now, we are in the position to complete the proof of the corollary. In the following, we consider the image of the group 2-cocycle $\tau$ under the homomorphism

$$\delta : H^2_{\text{group}}(P; C(T^2, U(1))) \to H^2_{\text{group}}(P; H^1(T^2; \mathbb{Z})) = E^2_{2,1}$$

induced from the natural surjection $C(T^2, U(1)) \to H^1(T^2; \mathbb{Z})$.

- **In the case of $pg$,** notice that

  $$H^2_{\text{group}}(\mathbb{Z}_2; H^1(T^2; \mathbb{Z})) \cong H^2_{\text{group}}(\mathbb{Z}_2; \mathbb{Z}) \cong \mathbb{Z}_2,$$

  where the coefficient $\mathbb{Z}$ in the group cohomology is the trivial $\mathbb{Z}_2$-module. Comparing the values of $\tau_{pg}$ and the generator given in Lemma 4.52, we conclude that the image of the cohomology class of $\tau_{pg}$ under $\delta$ generates $E^2_{2,1} \cong \mathbb{Z}_2$.

- **In the case of $p_{mg}$ and $pgg$,** we consider the subgroups $\mathbb{Z}_2^{(x)} \cong \{1, \sigma_x\}$ and $\mathbb{Z}_2^{(y)} \cong \{1, \sigma_y\}$ in $D_2 \cong \mathbb{Z}_2^{(x)} \times \mathbb{Z}_2^{(y)}$. The inclusions of these subgroups induce the homomorphism

  $$H^2_{\text{group}}(D_2; H^1(T^2)) \to H^2_{\text{group}}(\mathbb{Z}_2^{(x)}; H^1(T^2)) \oplus H^2_{\text{group}}(\mathbb{Z}_2^{(y)}; H^1(T^2)),$$

  which is $E^2_{2,1} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thanks to their explicit values of $\tau_{p_{mg}}$ and $\tau_{pgg}$, the same argument as in the case of $pg$ shows that the images of $\delta([\tau_{p_{mg}}])$ and $\delta([\tau_{pgg}])$ under the homomorphism $E^2_{2,1} \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ are $(1, 0)$ and $(1, 1)$. Hence the group 2-cocycles generate $E^2_{2,1} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

- **In the case of $p_{4mg}$,** the inclusion $\mathbb{Z}_2 = \{1, \sigma_x\} \to D_4$ induces

  $$H^2_{\text{group}}(D_4; H^1(T^2)) \to H^2_{\text{group}}(\mathbb{Z}_2; H^1(T^2)),$$

  under which $\delta([\tau_{p_{4mg}}])$ goes to $\delta([\tau_{pg}]) \neq 0$, as can be seen by the explicit values of $\tau_{p_{4mg}}$. Hence this 2-cocycle generates $E^2_{2,1} \cong \mathbb{Z}_2$. 

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