NEW REALISATIONS OF MINIMAL MODELS
AND THE STRUCTURE OF W-STRINGS

C. M. HULL

Physics Department, Queen Mary and Westfield College,
Mile End Road, London E1 4NS, United Kingdom.

and

Institute for Theoretical Physics, University of
California, Santa Barbara, CA 93106, USA.

ABSTRACT

The quantization of a free boson whose momentum satisfies a cubic constraint leads to a $c = \frac{1}{2}$ conformal field theory with a BRST symmetry. The theory also has a $W_\infty$ symmetry in which all the generators except the stress-tensor are BRST-exact and so topological. The BRST cohomology includes states of conformal dimensions $0, \frac{1}{16}, \frac{1}{2}$, together with ‘copies’ of these states obtained by acting with picture-changing and screening operators. The 3-point and 4-point correlation functions agree with those of the Ising model, suggesting that the theory is equivalent to the critical Ising model. At tree level, the $W_3$ string can be viewed as an ordinary $c = 26$ string whose conformal matter sector includes this realisation of the Ising model. The two-boson $W_3$ string is equivalent to the Ising model coupled to two-dimensional quantum gravity. Similar results apply for other W-strings and minimal models.
1. Introduction

There are an infinite number of extended conformal algebras – super-conformal algebras, W-algebras, topological conformal algebras, fractional supersymmetry algebras etc – and it seems that corresponding to each of these one has the possibility of constructing a generalisation of string theory. An important question is whether these are really fundamentally new string theories, or whether they are equivalent to ones that are already known. The first step in the construction of such theories is to promote the semi-local extended conformal symmetry of some matter system to a fully local symmetry by coupling to world-sheet gauge fields (supergravity, W-gravity, topological gravity etc). For the case of the $W_3$ algebra [1], the corresponding W-gravity theory was constructed in [2] and it was subsequently shown that this construction will work for any extended conformal algebra [3,4]. Quantising this system defines a generalised string theory (superstring, W-string, fractional superstring etc) and the string theory will be critical if the matter system is chosen in such a way that all anomalies in the local gauge symmetries are cancelled by ghost contributions. For non-linearly realised symmetries of the type that arise in W-gravity, the cancellation of anomalies is much more delicate than in the usual string and superstring theories [5]. For the $W_3$ algebra, it was finally shown [6] that all anomalies will cancel if the matter system is a $c = 100$ realisation of $W_3$, and the physical states correspond to the cohomology classes of the BRST operator constructed in [7], as anticipated in [8]. A $c = 100$ realisation of $W_3$ can be constructed by taking any conformal field theory with $c = 25\frac{1}{2}$ and adding to it a single free boson with background charge [9], and all $c = 100$ realisations that are known take this form.

A similar construction works for many other extended conformal algebras. For example, to construct a critical $W_N$ string theory, it is necessary to find a matter theory with $W_N$ symmetry whose central charge takes the critical value $c_N^* = 2(N-1)(2N^2 + 2N + 1)$. One way of attaining such a realisation is given by taking
an arbitrary conformal field theory with central charge

\[ \tilde{c} = 26 - \left( 1 - \frac{6}{N(N+1)} \right) \]  

(1.1)

and adding \( N - 2 \) free bosons \( \phi^a \) \((a = 1, \ldots, N - 2)\) with background charge.

Remarkably, \( \tilde{c} = 26 - c_{N-min} \) where \( c_{N-min} \) is the central charge of the \( N'\)th minimal model. In the last year or so, a large amount of effort has been devoted to the spectra and interactions of such theories [11-21]; in the case of the \( W_3 \) string, the spectrum is now known explicitly for the first few levels, suggesting some interesting conjectures which have so far not been proven (beyond the first few levels). It appears that in each case the physical states can be constructed from the primary fields of the effective theory with central charge \( \tilde{c} \) whose conformal dimensions are

\[ \Delta = 1 - h_{r,s}^N \]

(1.2)

where \( h_{r,s}^N \) are the weights of the \( N'\)th minimal model. This suggests a close relation between the \( W_N \) string and the \( N'\)th minimal model [10-21], and there are generalisations of this construction which correspond to minimal series for other \( W \)-algebras [13].

These results suggest that there might be a sense in which such \( W \)-strings could be regarded as ordinary \( c = 26 \) strings in which part of the matter sector is taken to be a minimal model [21], but it has not been clear how this could be done.

The purpose of this paper is to show that it is indeed the case that \( W \)-strings can be thought of as ordinary strings with minimal-model sectors, and to investigate the new realisations of minimal models that arise in these theories. The realisation of the Ising model that occurs in the \( W_3 \) string will be analysed in detail, drawing heavily on the results for the spectrum and interactions of the \( W_3 \) string derived in [11-21]; indeed, some of what follows is implicit in the work of

\[ \star \]

Note that the values of the central charges and intercepts given by (1.1),(1.2) are the only ones for which it is possible to have a unitary conformal field theory with \( 25 < c < 26 \) [28].
[11-21], but the reformulation given here considerably clarifies the structure of W-strings and will be used to analyse the complete spectrum of the theory. A key ingredient is the fact that, after a field redefinition, the structure of W$_3$ strings based on the W$_3$ representation of [9] simplifies dramatically and the BRST charge for W$_3$ strings can be written as $\hat{Q} = Q + Q_0$ where $Q_0$ is the standard bosonic string BRST operator and $Q$ acts only on the ghost system for the spin-three symmetry and the single free boson [19,23]. The result of the analysis indicates that, at least at tree level, the $W_N$ strings that have been constructed so far can be viewed as ordinary string theories in which the matter sector consists of a minimal model plus an arbitrary conformal field theory with central charge (1.1).

This can be understood as follows. The W$_3$ string is constructed from a matter system $M$ with $c = 100$ that has W$_3$ symmetry, together with the usual $c = -26$ $b,c$ ghost system, and a $c = -74$ W-ghost system $\beta,\gamma$. These fields can be re-organised into the $bc$ ghosts plus the $c = 26$ system $(M, \beta, \gamma)$, so that at tree level the theory is a conventional string which happens to have a particular type of matter system with extra symmetry (in the same sense in which a superstring can be regarded as a conventional $(c = 26)$ string in which the matter sector consists of a $c = 11$ commuting $\beta\gamma$ system, plus a $c = 15$ conformal field theory which has extra super-conformal symmetry). However, the only $c = 100$ conformal field theories with W$_3$ symmetry that are known are constructed from a $c = 25\frac{1}{2}$ conformal matter system $X$, which need have no extra symmetry, plus a single boson $\phi$ with background charge and $c = 74\frac{1}{2}$. The $(\phi, \beta, \gamma)$ system has $c = \frac{1}{2}$ and will be shown in this paper to be essentially the Ising model at its critical point. The spin-three symmetry acts entirely in the $(\phi, \beta, \gamma)$ system, and no restrictions are imposed on the $c = 25\frac{1}{2}$ conformal matter system $X$. Thus the reducibility of the theory is a consequence of the reducibility of the particular representation of the $c = 100$ W$_3$ algebra that is used, and would not hold for a matter system that did not reduce in this way. Even for the reducible cases, it may often be convenient to consider such models from the viewpoint of W-strings, so that the extra symmetry that arises for these particular cases of the bosonic string is more
manifest and can be used to organise the analysis. It remains to be seen whether this viewpoint can be applied at higher genus or whether, when moduli are taken into account, the theories are really distinct.

2. Free Field Realisation of the Ising Model

Consider a conformal theory consisting of a free boson $\phi$ and a ghost/anti-ghost system $\beta, \gamma$ where $\beta, \gamma$ are anti-commuting and have spins 3 and $-2$, respectively. The stress-tensor is

$$T = T_\phi + T_{\beta\gamma}$$

where

$$T_\phi = -\frac{1}{2} \partial \phi \partial \phi - q \partial^2 \phi, \quad T_{\beta\gamma} = -3 \beta \partial \gamma - 2 \partial \beta \gamma$$

and $q$ is a background charge. The central charge of the $\beta, \gamma$ system is $c_{\beta\gamma} = -74$, while the scalar field central charge is $c_\phi = 1 + 12q^2$. Choosing $q^2 = \frac{49}{8}$ gives $c_\phi = 74\frac{1}{2}$ so that the total central charge of the $\phi, \beta, \gamma$ system is $\frac{1}{2}$. This free field theory can be quantised in the conventional way to give a non-unitary reducible conformal field theory with $c = \frac{1}{2}$. The theory has a conserved primary spin-three current given by

$$W = W^\phi + W^{\beta\gamma}$$

$$W^\phi = \frac{1}{3} (\partial \phi)^3 + q \partial^2 \phi \partial \phi + \left( \frac{q^2}{3} - \frac{5}{4} \right) \partial^3 \phi$$

$$W^{\beta\gamma} = \partial Y \gamma + 2 Y \partial \gamma, \quad Y \equiv \frac{1}{2} (3 \partial \phi \beta + q \partial \beta)$$

The commutator $[W(z), W(w)]$ can be written in terms of $T$ and a spin-four current $W^{(4)}$, and $W^{(4)}$ can be written as a sum of terms each of which involves either $W$ or $\beta$. Indeed, the currents $T, W, \beta$ generate a closed W-algebra with field-dependent structure functions. Alternatively, one can regard $W^{(4)}$ as an independent new current and closing the algebra leads to an infinite set of currents $W^{(n)} = \frac{1}{n} (\partial \phi)^n +$
... for $n = 2, 3, 4, \ldots$, with $W^{(2)} = T$ and $W^{(3)} = W$, and these satisfy a $W_\infty$ algebra with $c = \frac{1}{2}$.

Consider the BRST operator

\[
Q = \int dz \gamma \left[ W^\phi + \frac{1}{2} W^{\beta \gamma} \right] = \int dz \gamma \left[ \frac{1}{3} (\partial \phi)^3 + 2q \phi^2 \partial \phi + \left( \frac{q^2}{3} - \frac{5}{4} \right) \partial^3 \phi + \frac{3}{2} \partial \phi \beta \partial \gamma + \frac{q}{2} \partial \beta \partial \gamma \right]
\] (2.4)

This is nilpotent if $q^2 = 49/8$, $Q^2 = 0$, and commutes with the stress-tensor $T$, the spin-three current $W$ and hence with all of the $W_\infty$ currents $W^{(n)}$. We shall define the physical states of our model to be the cohomology classes of $Q$. The stress-tensor is BRST non-trivial, but $W$ is BRST exact:

\[
\{ Q, \beta \} = W
\] (2.5)

as are all of the higher-spin currents $W^{(n)}$ for $n \geq 3$. Thus the $W$-sector of the theory is ‘topological’, while the ordinary sector is not (correlation functions are independent of any background spin-three gauge field $B$ but not of the metric etc).

This theory can be thought of as arising from a model with Lagrangian

\[
L = \partial \phi \bar{\partial} \phi + B \tilde{W}, \quad \tilde{W} = \frac{1}{3} (\partial \phi)^3
\] (2.6)

corresponding to a free boson subject to the constraint $\tilde{W} = 0$ imposed by the Lagrange multiplier $B$, which can be viewed as a spin-three gauge field for the $W$-gravity based on the $W$-algebra with the single generator $\tilde{W}$ (which closes since $[\tilde{W}(z), \tilde{W}(w)]$ can be written in terms of $\tilde{W}$, with field-dependent structure functions). Choosing the gauge $B = 0$ and introducing ghosts $\beta, \gamma$ leads to the naive (‘classical’) BRST current $\frac{1}{3} (\partial \phi)^3 + \frac{3}{2} \partial \phi \beta \partial \gamma$ [23]. A nilpotent quantum BRST operator is obtained by adding quantum corrections to this to obtain (2.4) [23], and including a background charge $q$ in the stress tensor ensures that $[Q, T] = 0$ provided $q^2 = 49/8$. These quantum corrections can be systematically derived as in [22] using the methods of Batalin and Vilkovisky [24].
As \([Q, T] = 0\), the cohomology classes can be organised into representations of the Virasoro algebra so that it is sufficient to restrict attention to those classes represented by primary fields, as all other ones will be represented by descendents of these. (In fact, they can be arranged into representations of the \(W_\infty\) algebra, as will be discussed in the next section.)

The cohomology classes represented by primary fields all have conformal dimension \(\Delta\) given by \(0, \frac{1}{2}\) or \(\frac{1}{16}\). The operators

\[
V_0 = \partial \gamma e^{-\frac{q}{2} \phi}, \quad V_{\frac{1}{16}} = \partial \gamma e^{-\frac{q}{2} \phi}, \quad V_{\frac{1}{2}} = e^{-\frac{q}{2} \phi}
\]

all represent non-trivial cohomology classes and have conformal dimension \(0, \frac{1}{16}\) and \(\frac{1}{2}\) respectively. These will be identified with the identity operator \(1\), the spin operator \(\sigma\) and the energy operator \(\varepsilon\) of the Ising model. The operator \(V_{\frac{1}{16}}\) is self-conjugate, while the conjugate of \(V_0\) is \(V_0 = \partial \gamma e^{-\frac{q}{2} \phi}\), since \(a_0^\dagger = a_0 - 2i q\) [14].

The picture-changing operator

\[
P = [Q, \phi] = -\frac{19}{24} \partial^2 \gamma - \frac{3}{2} \beta \partial \gamma \gamma - \partial \phi \partial \phi \gamma + q \partial \phi \partial \gamma
\]

clearly satisfies \([Q, P] = 0\) and has dimension 0. As usual, this is not regarded as being BRST trivial as \(\phi\) is not a proper conformal field. Thus given a primary cohomology class represented by \(V\), there is another one of the same conformal dimension represented by \(PV =: PV\), and the classes can be organised into doublets \([V, PV]\) (note that \(P^2 V\) is BRST trivial).

In addition, there are cohomology classes represented by non-local operators; if a field \(U(z)\) satisfies

\[
[Q, U] = \partial Y
\]

for some \(Y\), then \(S_U = \int dz U\) will satisfy \([Q, S_U] = 0\) and so will represent a (possibly trivial) cohomology class. Of particular interest are such classes for which
$U$ has dimension one, so that the corresponding $S_U$ are well-defined and have dimension zero; these are the screening operators of the theory and play a role similar to the screening operators in the Coulomb gas representation of the minimal models [25,26].

The standard screening operators formed from the spin-one Virasoro primary fields $e^{iq_\pm \phi}$

$$A_\pm = \oint dz e^{iq_\pm \phi}, \quad q_\pm(q_\pm - 2q^2) + 2 = 0 \quad (2.10)$$

do not commute with the BRST operator and so do not play a role here; the screening operators of this theory all take the form $\oint dz K(\beta, \gamma, \partial \phi)e^{-p\phi}$ for some momentum $p$ and some operator $K$ constructed from $\partial \phi, \beta$ and $\gamma$. The basic screening operators of the theory are [20]

$$S = \oint dz \beta e^{\frac{2}{7}q_\phi} \quad (2.11)$$

and

$$R = \oint dz \gamma e^{-\frac{6}{7}q_\phi}, \quad \bar{R} = \oint dz \gamma e^{-\frac{8}{7}q_\phi} \quad (2.12)$$

There are in fact an infinite number of screening operators $S(n), \bar{S}(n)$ ($n = 0, \pm 1, \pm 2, \ldots$) with $S(0) = R, \bar{S}(0) = \bar{R}, S(1) = S$, which will be constructed presently. Given any local physical operator $V(z)$, new physical operators can be constructed by acting with the screening charges $S(n), \bar{S}(n)$ and the picture changing operator $P$. For a set of screening charges charges $S_i = \oint dw U_i(z)$, the product is defined by its action on $V$:

$$\left( \prod_{i=1}^n S_i V \right)(z) = \oint_{C_1} dw_1 \oint_{C_2} dw_2 \ldots \oint_{C_n} dw_n U(w_1)U(w_2)\ldots U(w_n)V(z) \quad (2.13)$$

where the contour $C_n$ surrounds the point $z$, the contour $C_i$ surrounds $C_{i+1}$ and all the contours are taken to touch each other in precisely one point, as in [26,20]. If $U_i$ contains a factor $exp(-a\phi)$ and $V$ contains a factor $exp(-b\phi)$, then $S_iV$
will only be a well-defined local operator if the product of the ‘momenta’ \( ab \) is an integer, and there are similar restrictions on the momenta of the \( U_i \) and \( V \) for the construction (2.13) to be well-defined [20]. When a new operator obtained in this way is well-defined and non-zero, it will represent a non-trivial BRST cohomology class if the original operator \( V \) did, and will have the same conformal dimension as the original operator, but different momentum and ghost number in general.

Acting with \( S \) and \( P \) on the three operators (2.7), the only local operators that can be constructed are defined by [20]

\[
\begin{align*}
\bar{V}(0, n) &= SPV(0, n), \\
V(0, n) &= S^3P\bar{V}(0, n - 1), \\
V(0, 0) &= V_0 \\
V(\frac{1}{16}, n) &= S^2PV(\frac{1}{16}, n - 1), \\
V(\frac{1}{16}, 0) &= V_\frac{1}{16} \\
V(\frac{1}{2}, n) &= S^3P\bar{V}(\frac{1}{2}, n - 1), \\
\bar{V}(\frac{1}{2}, n) &= SPV(\frac{1}{2}, n), \\
V(\frac{1}{2}, 0) &= V_\frac{1}{2}
\end{align*}
\]

(2.14)

plus the picture-changed partners given by acting with \( P \) on each of these. The operator \( V(\Delta, n) \) has conformal dimension \( \Delta \). This series can be extended to negative \( n \) by acting with \( R, \bar{R} \):

\[
\begin{align*}
V(0, n - 1) &\propto RPV(0, n - 1) \\
V(\frac{1}{16}, n - 2) &\propto RPV(\frac{1}{16}, n) \\
V(\frac{1}{2}, n - 1) &\propto RPV(\frac{1}{2}, n - 1)
\end{align*}
\]

(2.15)

For example, \( V(0, -1) \) is a new physical vertex operator of conformal weight zero and ghost number four given by \( \gamma \partial \gamma \partial^2 \gamma \partial^7 \gamma exp(-16q\phi/7) \). The operators \( V(0, n), V(\frac{1}{16}, n)V(\frac{1}{2}, n) \) have ghost numbers given respectively by \( 2 - 2n, 2 - n, 1 - 2n \) and momenta given by \( p = iqk/7 \) with \( k \) given by \( 8 - 8n, 7 - 4n, 4 - 8n \) respectively. Thus there are operators for arbitrarily large positive or negative ghost number and imaginary momentum.

The operator \( V(0, 1) \) is the identity operator \( V(0, 1) = 1 \), and so the whole family of dimension zero operators \( V(0, n), \bar{V}(0, n) \) are obtained by acting on \( 1 \).
with \( P \) and screening operators. The operator \( x = \bar{V}(0,1) \) also has dimension zero and ghost-number zero, and acts on \( \bar{V}_0 = \bar{V}(0,0) \) to give \( x\bar{V}_0 = \bar{V}_0 \). The product \( x^n \) will be well-defined if \( n = 4m \) or \( n = 4m + 1 \) and the operators \( 1, x^4 \) generate the ground ring of the theory \([15]\). (The operator \( V(\frac{1}{16}, 2) \) also has ghost number zero.)

Given any screening operator \( S = \oint dz U \), the BRST variation of \( U \) is \([Q, U] = \partial V \) for some \( V \). However, this \( V \) must have dimension zero and be BRST-invariant, and conversely given any of the dimension-zero BRST-invariant operators \( V(0, n), \bar{V}(0, n) \) defined by (2.14),(2.15), one can solve

\[
[Q, U(n)] = \partial V(0, n), \quad [Q, \bar{U}(n)] = \partial \bar{V}(0, n)
\]

(2.16)

to obtain dimension-one operators \( U(n), \bar{U}(n) \). The screening charges \( S(n), \bar{S}(n) \) are defined as the integrals of the operators \( U(n), \bar{U}(n) \), which are determined by the descent equation (2.16). Acting on the set of operators (2.7) with these screening operators gives no further operators; note in particular that \( S(2) V(0, 0) \) gives the identity operator, 1. Note also that not all of these screening operators are non-trivial e.g. \( U(1) \) is trivial since \( V(0, 1) = 1 \).

Corresponding to each of the three basic operators (2.7), there is an infinite set of ‘screened and picture-changed’ versions of the same physical operator, all of which have the same conformal dimension, but have different ghost-numbers and different discrete values of the momenta in general. As in the Coulomb gas models, these are to be interpreted as physically equivalent versions of the three original operators, and the different screened versions are needed to construct amplitudes. All the operators are generated by the action of \( S, R, \bar{R}, P \) on three basic operators, which can be taken to be those given by (2.7), although for some purposes it is more natural to represent the dimension zero operators by 1. As a result of (2.14),(2.15), the action of a string of \( S \)’s and \( P \)’s, when defined, can be inverted by the action of appropriate \( R \)’s and \( P \)’s, and vice versa, and this gives a strong indication that these should be all the independent local operators and screening operators, since
given any vertex operator of ghost number \( N \), acting on it with a suitable string of \( S \)'s, \( R \)'s and \( P \)'s should bring it to one of low ghost number, and this should be one of the ones already found. Then we can invert the action of the \( S \)'s, \( R \)'s and \( P \)'s by a further such string to regain the original operator, so the original operator can be written in terms of the action of \( S \)'s, \( R \)'s and \( P \)'s on one of the operators in (2.7), and so must be one of the operators already constructed. To complete this argument, it would be necessary to show that there is no operator that can’t be brought to one of low ghost number by the action of the screening and picture operators, and to complete the classification of operators of low ghost number begun in [10-21].

In the next section, further evidence will be given that the operators of the theory are all obtained from basic operators using screening and picture operators, so that the spectrum should be that of the Ising model (regarding different screened and picture-changed versions of a given operator as being physically equivalent). In section 4 the correlation functions will be considered, and shown to be consistent with the assertion that this constrained free field theory is equivalent to the Ising model.

3. BRST Cohomology

In this section, the BRST cohomology of the \((\phi, \beta, \gamma)\) theory will be analysed; further details will be given elsewhere. As all the \( W_\infty \) generators are \( Q \)-closed, \([Q, W^{(m)}] = 0\), the cohomology classes of physical operators can be organised into representations of the \( W_\infty \) algebra, so that it is sufficient to restrict attention to those classes represented by \( W_\infty \)-primary fields, as all other ones will be represented by descendents of these. Such primary fields correspond to states \( |\Psi > \) that are annihilated by the modes \( W_n^{(m)} \) for \( n > 0, m \geq 2 \) and are eigenstates of \( W_0^{(m)} \). Since for spins \( m > 2 \) the \( W_\infty \) generators are exact, \( W_0^{(m)} = \{Q, V_0^{(m)}\} \) for some \( V_0^{(m)} \), it follows that any \( Q \)-closed state with non-zero \( W_0^{(m)} \) eigenvalue \( \lambda^{(m)} \) for
$m > 2$ must be BRST trivial:

$$W_0^{(m)} |\Psi > = \lambda^{(m)} |\Psi > \quad \Rightarrow \quad |\Psi > = \frac{1}{\lambda^{(m)}} QV^{(m)} |\Psi >$$

(3.1)

Thus for a state $|\Psi >$ to represent a non-trivial cohomology class, it must satisfy $W_0^{(m)} |\Psi > = 0$ for all $m > 2$, but the Virasoro weight (the eigenvalue of $L_0$) can be non-zero. Furthermore, the descendents obtained by acting on such a physical highest weight state with polynomials in $W_0^{(m)}$ for $m > 2$ are all BRST trivial, so that the classes can be represented by $W_\infty$ highest-weight states or by the Virasoro descendents obtained by acting with polynomials in $L_{-n}$. The representatives of the classes can also be chosen to have definite ghost-number and momentum $p$ (the eigenvalue of $a_0$, where $a_n$ are the modes of $i\partial\phi$). The spin-three constraint $W_0 |\Psi > = 0$ then implies that the momentum $p$ must satisfy a cubic equation, so that for, a fixed level and ghost number etc, the momentum must be frozen to one of three discrete values, and for physical states this turns out to be imaginary and quantised in units of $q/7$: $p = i q N/7$ for some integer $N$.

Modes are defined as usual by

$$\beta = \sum_n \beta_n z^{-n-3}, \quad \gamma = \sum_n \gamma_n z^{-n+2},$$

$$T = \sum_n L_n z^{-n-2}, \quad W = \sum_n W_n z^{-n-3},$$

$$i\partial\phi = \sum_n a_n z^{-n-1}, \quad P = \sum_n P_n z^{-n}$$

(3.2)

Note that since each of the operators $\beta_n, \gamma_n$ is nilpotent, there is an associated cohomology for each and the interplay between these, the de Rham cohomology on the world-sheet and the $Q$-cohomology leads to a series of descent equations that play an interesting part in the analysis.

The state space of the theory is spanned by the states

$$a_{-n_1} a_{-n_2} \ldots a_{-n_r} \beta_{-m_1} \beta_{-m_2} \ldots \beta_{-m_s} \gamma_{-p_1} \gamma_{-p_2} \ldots \gamma_{-p_t} |p >$$

(3.3)
where $|p>\rangle$ is the Fock vacuum with momentum $p$ and zero ghost-number satisfying

$$a_n|p>\rangle = \beta_n|p>\rangle = \gamma_n|p>\rangle = L_n|p>\rangle = W_n^{(m)}|p>\rangle = 0$$

$$(a_0 - p)|p>\rangle = \beta_0|p>\rangle = (L_0 - \Delta)|p>\rangle = 0$$

(3.4)

for $n \geq 0$. For generic momenta $p$, we can make the following change of basis for $n > 0$:

$$a_{-n} \rightarrow \hat{a}_{-n} \equiv L_{-n} = a_{-n} P(n) + \ldots,$$

$$\gamma_{-n} \rightarrow \hat{\gamma}_{-n} \equiv [Q, a_{-n}] = n(\partial P)_{-n} = G(n)\gamma_{-n} + \ldots,$$

$$\gamma_0 \rightarrow \hat{\gamma}_0 \equiv P_0$$

(3.5)

where

$$P(n) = (a_0 + i(n + 1)q),$$

$$G(n) = (n + 2) \left[ a_0^2 + (n + 1) \left( ia_0 q - \frac{19}{24} n \right) \right] + \ldots$$

(3.6)

(note that $G(n)$ contains ghost-dependent terms). For certain special discrete values of the momentum $p$, $P(n)$ or $G(n)$ will vanish for some $n$ and for those momenta this change of basis is singular and extra discrete states can arise. However, for any value of $p$ not lying in this discrete set, a basis of states with that momentum is given by those of the form

$$\hat{a}_{-n_1}\hat{a}_{-n_2}...\hat{a}_{-n_r}\beta_{-m_1}\beta_{-m_2}...\beta_{-m_s}\hat{\gamma}_{-p_1}\hat{\gamma}_{-p_2}...\hat{\gamma}_{-p_t}|p>\rangle$$

(3.7)

where $|p>\rangle$ is now taken to satisfy (3.4) with $a, \gamma$ replaced by $\hat{a}, \hat{\gamma}$. The advantage of this basis is that $\hat{a}_n$ and $\hat{\gamma}_n$ (anti-)commute with $Q$. There will also be certain discrete values of the momenta at which $W$-algebra singular vectors occur, as we shall see below. However, first we will deal with the case of generic momenta at which the change of basis is non-singular and there are no $W$-algebra singular vectors. For such generic momenta, $\hat{\gamma}_{-n}$ is BRST exact for $n > 0$ and $\{Q, \beta_{-n}\} = W_{-n}$ is a non-singular operator, and it is then straightforward to argue (using e.g. the methods of [27]) that the BRST classes can be represented by states
involving no $\beta_{-n}, \gamma_{-n}$ oscillators for $n > 0$. This leaves the states $|p>$ and the descendents obtained by acting on these with the $\hat{a}_n = L_{-n}$ and the picture-changing operator $\hat{\gamma}_0 = P_0$. Acting with $P_0$ more than once gives a BRST trivial state, so the physical states fit into picture-changed doublets $\{ |\Psi>, P|\Psi> \}$. These descendents will represent non-trivial $Q$-cohomology classes if and only if the state $|p>$ does. The constraint $W_0|p>= 0$ gives a cubic equation for $p$ whose solutions are $p = iq, 6iq/7, 8iq/7$ and these are the states $|p>$ given by acting on the $SL(2)$ invariant vacuum $|\Omega> = \beta_{-1}\beta_{-2}|0>$ with $\hat{\gamma}_{-16}, \hat{\gamma}_0, \hat{\gamma}_0$ and their $P$-doubles. They are BRST invariant and represent non-trivial cohomology classes.

All other physical states must occur at those discrete values of the momentum at which either the change of basis is singular, or there are singular vectors. (Note that the values of these discrete physical momenta depend on the ghost number in general.) In each case the extra discrete states fit into multiplets consisting of a highest weight state, together with the descendents obtained by acting with the $L_{-n}$, plus the doubles obtained by acting once with $P_0$.

The Jacobian for the change of basis $a_{-n} \rightarrow L_{-n}$ will vanish when $P(n)$ does, i.e. for momenta given by $p = -(n + 1)iq$ for some integer $n$. For these momenta, there are Virasoro singular vectors and so there are states in the scalar Fock space that are annihilated by linear combinations of the Virasoro generators, so that the Verma module generated by the $L_{-n}$ does not fill the Fock space. For momentum $p = -(n+1)iq$, this means that the argument eliminating the oscillator $a_{-n}$ breaks down and there is the possibility of extra physical states involving that oscillator.

The next possibility is that the change of basis $\gamma \rightarrow \hat{\gamma}$ is singular, will occur for momenta and ghost numbers at which $G(n)$ vanishes for some $n$. This will occur if there are states that are annihilated by some function of the $\hat{\gamma}_{-n}$ in a non-trivial way. For example, given a $Q$-closed state $|\Psi>$ satisfying $\hat{\gamma}_{-n}|\Psi> = 0$, the state $a_{-n}|\Psi>$ will also be $Q$-closed. More generally, given a state $|\Psi>$ which is annihilated by some function $f$ of the $\hat{\gamma}$ given by $f = \hat{\gamma}_{-n} + ...$ for some $n$ and which is not of the form $|\Psi> = \hat{\gamma}_{-n}|\Theta> + ...$, there should be a
physical state of the form $a_{-n}|\Psi > +...$ If $G(n) \neq 0$, then $a_{-n}|\Psi > = Q|\Theta > +...$ where $|\Theta > = \frac{1}{2}G(n)^{-1}a^2_{-n}\beta_n|\Psi > +...$ so such states can only be non-trivial when $G(n) = 0$, and reflect the singularity of the change of basis.

Finally, there are the momenta at which the argument eliminating the ghost, anti-ghost or scalar modes might break down, due to the presence of singular vectors of the $W$-algebra. Consider, then, the momenta at which $W_\infty$ singular vectors occur. These correspond to states in the Fock-space which are annihilated by polynomials in the $W_\infty$ generators $W_{-n}^{(m)}$. Consider first the simple case of momenta at which there is a state $|\Psi >$ annihilated by one of the modes of the spin-three current, $W_{-n}|\Psi > = 0$. Then the argument leading to the elimination of the corresponding anti-ghost $\beta_{-n}$ breaks down and the state $\beta_{-n}|\Psi >$ represents a new non-trivial cohomology class. However, since $\beta_{-n}$ is nilpotent, it is not necessary for $|\Psi >$ to be annihilated by $Q$ for this to work; it is sufficient that the state satisfy the descent equation

$$Q|\Psi > = \beta_{-n}|\Phi >$$

for some $|\Phi >$, as this implies $Q\beta_{-n}|\Psi > = W_{-n}|\Psi >$ which vanishes.

The general situation is that in which a state is annihilated by some polynomial $f(W_{-n}, \beta_{-m}, \gamma_{-r}, L_{-s})$; nothing extra is gained by including the modes of the higher spin currents $W^{(s)}$, as these can all be rewritten in terms of multiple commutators of the modes of $W$, as $W^{(s)}$ arises in the commutator $[W_n^{(3)}, W_{n-m}^{(s-1)}]$. For a state $|\Psi >$ annihilated by a polynomial $f = W_{-n} + ...$, it is expected that there should be a physical state of the form $\beta_{-n}|\Psi > +...$ and in general such states can be constructed perturbatively.

As a first example, the (non-physical) state $|p >$ with $p = 4iq/7$ satisfies

$$(W_{-1} - \lambda L_{-1})|p > = 0$$

with $\lambda = -12q/7$, suggesting that we consider the state $\beta_{-1}|4iq/7 >$; this is in fact
physical without any further modification as $Q|4i\eta/7 >= \lambda \gamma_0|4i\eta/7 >$. This gives the physical state $\beta_{-1}|4i\eta/7 >= V_2|\Omega >$.

As another example, the (non-physical) zero-momentum state $|0 >$ is annihilated by an operator of ghost-number $-1$, as it satisfies

$$(\beta_{-1}W_{-2} - \beta_{-2}W_{-1} - 12q\beta_{-1}\beta_{-2}\gamma_0)|0 >= 0 \quad (3.10)$$

This implies that there is a physical state at ghost-number $-2$ given by $\beta_{-1}\beta_{-2}|0 >$, which is precisely the $SL(2)$ invariant vacuum. The constraint on the ghost-number zero state $|0 >$ appears to have non-trivial ghost-dependence, but the situation becomes clarified by rewriting (3.10) in the form

$$(W_{-2} + \frac{1}{2}W_{-1}L_{-1} - \frac{15q}{4}L_{-1}^2)\beta_{-1}|0 >= 0 \quad (3.11)$$

which is a conventional (ghost-independent) constraint on the state $\beta_{-1}|0 >$ of ghost number $-1$. From (3.11), we see that there should be a physical state of the form $\beta_{-2}\beta_{-1}|0 > + ..., and again it turns out that no further corrections are needed, so that $|\Omega >= \beta_{-2}\beta_{-1}|0 >$ is a physical state, the $SL(2)$ invariant vacuum. More generally, given a W-algebra singular vector of ghost number $N$, there should be a physical state at ghost number $N - 1$ and it appears that the physical states constructed in the last section can be viewed as arising in this way.

In this section it has been seen that extra physical states arise only from singular vectors of the W-algebra, or from singular vectors of the $\gamma_{-n}$ operators. From the last section, it was seen that the same physical states can be constructed using screening operators, so that there should be a screening charge construction of the W-algebra singular vectors, generalising the screening charge or vertex operator construction of Virasoriso singular vectors [28]. To complete the analysis requires a classification of such singular vectors or the corresponding screening operators and work in that direction is currently in progress. However, the picture that is emerging supports the conjecture that all the physical operators are obtained by
acting on the three basic operators (2.7) with screening and picture-changing operators, so that there are just three physical operators in the theory. Note that one might also regard the theory as the Ising model tensored with a \( c = 0 \) conformal theory whose spectrum consists of an infinite number of discrete ground states of zero conformal weight, giving an infinite number of copies of each of the three Ising operators, and that the Ising model emerges on identifying the copies of each Ising operator, \textit{i.e.} by factoring out the discrete theory. It is important to check that identifying the different operators of a given spin gives a sensible well-defined theory; in the next section we will check that the correlation functions are well-defined and agree with those of the Ising model.

4. Correlation Functions

Since the model in question is a free field theory, it is straightforward to calculate correlation functions. Consider first the partition function. Choosing one representative for each of the three physical operators, the analysis of [16] suggests that the partition function of the \( \phi, \beta, \gamma \) theory should be precisely that of the Ising model. Consider next a tree-level \( N \)-point correlation function involving \( N_0 \) vertices \( V_0 \), \( N_{16} \) vertices \( V_{\frac{1}{16}} \) and \( N_{\frac{1}{2}} \) vertices \( V_{\frac{1}{2}} \) with \( N = N_0 + N_{\frac{1}{16}} + N_{\frac{1}{2}} \). This will vanish unless one includes appropriate insertions of the picture-changing operator \( P \) (as in fermionic strings) and of the screening charges (as in the Coulomb gas construction). For any amplitude, it is sufficient to include \( N_P \) picture-changing operator insertions and \( N_S \) insertions of the screening operator \( S \) to obtain the free-field correlation function

\[
< \Omega | V_0(z_1)...V_0(z_{N_0})V_{\frac{1}{16}}(w_1)...V_{\frac{1}{16}}(w_{N_{\frac{1}{16}}})V_{\frac{1}{2}}(y_1)...V_{\frac{1}{2}}(y_{N_{\frac{1}{2}}})P^{N_P}S^{N_S}|\Omega > \quad (4.1)
\]

where \( |\Omega> \) is the product of the zero-momentum \( \phi \) Fock-space vacuum and the \( SL(2, R) \) invariant vacuum of the ghost system. The operators \( V_0, V_{\frac{1}{16}}, V_{\frac{1}{2}}, S, P \)
have momenta $8i\eta/7, i\eta, 4i\eta/7, -2i\eta/7, 0$ and ghost-numbers $2, 2, 1, -1, 1$ respectively. Momentum conservation\textsuperscript{*} in the presence of a background charge $q$ implies that the amplitude vanishes unless \( \frac{1}{2}(8N_0 + N_{\frac{1}{16}} + 4N_{\frac{1}{2}} - 2N_S) = 2 \), so that the number of screening charges is fixed to be

\[
N_S = \frac{1}{2}(8N_0 + N_{\frac{1}{16}} + 4N_{\frac{1}{2}}) - 7 \quad (4.2)
\]

For any operator $X$, the expectation value $\langle 0 | X | 0 \rangle$ will vanish unless the ghost number of $X$ is 5; this can be thought of as the condition that the insertion of $X$ cancels the effect due to the anomaly in the conservation of ghost-number. This will be the case if $2N_0 + 2N_{\frac{1}{16}} + N_{\frac{1}{2}} - N_S + N_P = 5$, which together with (4.2) fixes the number of picture-changing operators to be

\[
N_P = 2N_0 + \frac{3}{2}N_{\frac{1}{16}} + 3N_{\frac{1}{2}} - 2 \quad (4.3)
\]

The amplitude is independent of the positions at which the picture-changing operators are inserted, as

\[
P(z_2) - P(z_1) = [Q, K], \quad K = \int_{z_1}^{z_2} dw \partial \phi(w) \quad (4.4)
\]

and the insertion of a BRST-exact operator into a correlation function of physical operators gives a vanishing result. Thus all that remains to determine the amplitude (4.1) is the choice of contours for the screening charges $S$. Following [25,20], this is done by acting with as many as possible of the factors of $S$ to convert the operators (2.7) to ones of the form (2.14), and then using crossing symmetry and unitarity to fix the remaining ones.

\textsuperscript{*} Strictly speaking, the integration over the bosonic zero mode does not give a momentum-conserving delta function since the momenta are imaginary. For the present purposes, we restrict ourselves to the resonant amplitudes which conserve momentum.
The calculations of all three-point and four-point correlation functions are contained as special cases of the calculations given in [17,18,19,20] and in all cases they agree with the corresponding Ising model correlation functions. The Ising correlation function for $N_0$ identity operators, $N_1$ spin operators and $N_2$ energy operators is given by (4.1) where the number of screening charges and picture changing operator insertions is fixed by (4.2),(4.3). No other kinds of screening operators appear to be necessary. In particular, the three-point functions give the correct fusion rules for the Ising model [17]

$$1 \times 1 = 1, \quad \sigma \times \sigma = 1 + \varepsilon, \quad \sigma \times \varepsilon = \sigma, \quad \varepsilon \times \varepsilon = 1.$$  \hspace{1cm} (4.5)

As an example, consider the correlation function of four spin-half operators. This requires the insertion of one screening operator $S$ and two picture changing operators. Using one $\tilde{V}(\frac{1}{2},0) = SP(V_1)$ operator and one $PV_1$ operator, it is straightforward to bosonise the ghosts and calculate the correlation function as in [17] to give

$$\langle \Omega |V_1(z_1)V_1(z_2)(SPV_1)(z_3)(PV_1)(z_4)\Omega \rangle = 2(z_{12}z_{13}z_{14}z_{23}z_{24}z_{34})^{-1/3}x^{-2/3}(1-x)^{-2/3}(1-x+x^2)$$  \hspace{1cm} (4.6)

where

$$z_{ij} = z_i - z_j, \quad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$  \hspace{1cm} (4.7)

and this agrees with the corresponding Ising model correlation function.
5. \(W_3\)-Strings

To attempt to construct a critical string theory, one can now add to the \(\phi, \beta, \gamma\) realisation of the Ising model an extra conformal matter sector (whose fields we denote generically by \(X\)) with stress tensor \(T_X\) and the usual ghosts \(b, c\) with stress tensor \(T_{b,c}\). The standard BRST charge

\[
Q_0 = \int dz \ c \left( T_X + T_\phi + T_{\beta,\gamma} + \frac{1}{2} T_{b,c} \right)
\]

will then be nilpotent provided that the total matter central charge is 26, \(i.e.\) provided that the central charge \(c_X\) for the matter system is \(c_X = 25\frac{1}{2}\). Moreover, this BRST charge anti-commutes with \(Q\) so that one can define the total BRST charge

\[
\hat{Q} = Q_0 + Q
\]

and this is itself nilpotent. The theory given by the matter sector \(X, \phi\), the ghost sector \(b, c, \beta, \gamma\) and the BRST charge (5.2) is precisely the \(W_3\) string theory of [6] (after a canonical change of variables given in [19]), with physical states identified with the cohomology classes of (5.2). Moreover, the matter system \((X, \phi)\) is a conformal field theory with \(c = 100\) and is a realisation of the \(c = 100\) \(W_3\) algebra constructed by Romans [9]; this is the most general \(c = 100\) realisation that has so far been constructed and so all known critical \(W_3\) string theories arise in this way. One convenient choice of matter system is to take the fields \(X\) to be \(D\) free bosons with a background charge \(a\), chosen so that \(D + 12a^2 = 25\frac{1}{2}\).

If \(V_\Delta\) is any of the physical vertex operators of the Ising system with dimension \(\Delta = 0, \frac{1}{16}, \frac{1}{2}\) and \(V_h\) is a primary field of the \(X, b, c\) system with conformal dimension \(h\), then the operator

\[
\mathcal{V} = cV_h(X, b, c)V_\Delta(\phi, \beta, \gamma)
\]

will be \(\hat{Q}\)-BRST invariant provided \(h + \Delta = 1\), so that with \(\Delta = 0, \frac{1}{16}, \frac{1}{2}\) one can construct physical vertex operators using vertices of the \(c = 25\frac{1}{2}\) matter theory.
corresponding to states with intercepts 1, \(\frac{15}{16}\) and \(\frac{1}{2}\). There are also copies with \(c\) replaced with \(c\partial c\). In the case in which the \(X\) are free bosons, all the physical states of the \(W_3\) string in which the \(X\) fields can have continuous momenta are obtained in this way, and this has been verified explicitly for the first few levels [14-20].

However, this does not exhaust the physical states of the theory. Tensoring together the \(X\) system and this realisation of the Ising model leads to extra physical states at discrete values of the momenta which are cohomology classes of \(Q\) but do not factor into products of Ising model states and \(X\)-states [17,14,15]. These arise in exactly the same way as the extra states that were found in the Ising model and can be understood in terms of singular vectors for the algebra generated by \(W\) and the total stress-tensor \(T\) (instead of just \(T_\phi + T_{\beta,\gamma}\)).

The identity operator 1 is clearly BRST invariant, as is its conjugate \(c\partial c\partial^2 c\gamma\partial \gamma\partial^2 \gamma\partial^3 \gamma\partial^4 \gamma\). The picture-changing operator given by (2.8) is not invariant under the total BRST charge \(\hat{Q}\) and must be modified to become \(P' = [\hat{Q}, \phi] = P + c\partial \phi - q\partial c\). It follows from (2.15) that the operator \(V_0 = V(0,0)\) can be constructed from the identity operator by \(V(0,0)(z) = RP(1) = \oint dw \gamma(w)e^{-8q\phi(w)/7}P(z)\) and \(cV_1(X)V_0\) will be BRST invariant for any \(X\)-space vertex operator \(V_1\) with weight \(h = 1\). (Note that \(cRP(1) = cRP'(1)\) as the extra term involves \(cc = 0\).) However, one would expect to be able to construct physical operators using the identity operator from the \(X, b, c\) system and a dimension zero operator from the Ising sector. The operator \(V(0,0) = RP(1)\) is not invariant under the total BRST charge, but is invariant if \(P\) is replaced by \(P'\). This gives a new physical operator \(RP'(1) = (c\gamma + \frac{2}{58}\sqrt{58i\partial \gamma})e^{-8q/7}\), which is one of the discrete state vertex operators found in [15]. An infinite sequence of physical discrete states involving the identity operator from the \(X, b, c\) sector and a dimension zero operator from the Ising sector can be constructed from the identity operator by acting with \(S, P', R, \bar{R}\) on the identity operator.
\[ V'(0, n) = [S^3 P' S P']^{n-1}(1), \quad \bar{V}'(0, n) = S P' V(0, n) \]
\[ V'(0, -n) = [R P']^{n+1}(1) \] (5.4)

These reduce to \( V(0, n), \bar{V}(0, n) \) on replacing \( P' \) with \( P \), and are manifestly invariant under the total BRST charge \( \hat{Q} \), since \( S, R, P' \) are. The identity operator is a discrete state of the \( X, b, c \) system and if the \( X, b, c \) system has extra discrete states, as will be the case if the \( X \) system consists of a single boson with background charge, then there will be extra discrete states constructed by replacing 1 in (5.4) with the operators corresponding to these discrete states. All of the discrete states so far constructed for the \( W_3 \) string arise in this way.

Of particular interest is the case in which the extra matter fields consist of a single extra boson \( X \) with background charge. The theory so constructed is now seen to be precisely the Ising model coupled to two-dimensional gravity (i.e. Liouville theory) and this is known to have an infinite number of extra discrete physical states. This model can also be viewed as the critical \( W_3 \) string constructed from the two-boson \( c = 100 \) realisation of \( W_3 \); this model and its discrete state structure has been analysed from the \( W_3 \) point of view in [15]. From the present point of view, these arise because the \( X, b, c \) system has an infinite number of discrete states in its own right, and new discrete states are obtained by tensoring together discrete states from the \( X, b, c \) sector with ones from the \( \phi, \beta, \gamma \) sector, after replacing \( P \) with \( P' \).

For \( D \) bosons \( X \), the model becomes equivalent to a system of \( D - 1 \) free bosons without background charge (so their contribution to the central charge is \( D - 1 \)) plus the \( c = \frac{1}{2} \) Ising model coupled to two-dimensional gravity, so that it could well be the case that some of the work on \( W_3 \) strings could shed light on the coupling of two-dimensional gravity to \( c > 1 \) matter; in particular, the \( W_3 \) string analysis shows that this theory must also have extra discrete states, but not as many as for the \( c \leq 1 \) matter theories.
6. Other $W$-Strings and Other Minimal Models

The discussion of the relation between the $W_3$ string and the Ising model generalises to other $W$-string theories [13,21] and minimal models. A simple generalisation of (2.6) is the lagrangian

$$L = \partial \phi \bar{\partial} \phi + B \bar{W}, \quad \bar{W} = \frac{1}{s} (\partial \phi)^s$$

(6.1)

for a free boson subject to the constraint $(\partial \phi)^s \sim 0$, with $B$ a spin-$s$ gauge field. Quantising the theory in the gauge $B = 0$ requires the introduction of an anti-commuting ghost $\gamma$ of spin $1 - s$ and an anti-ghost $\beta$ of spin $s$. The naive BRST charge is of the form $Q = \int dz \gamma \bar{W} + ...$ and as in the $s = 3$ case it must be modified by higher derivative terms for $Q$ to be nilpotent. The results of [21] give an explicit construction of a nilpotent BRST charge for $s = 4, 5, 6$ and there is an implicit construction of $Q$ from the BRST charge for $W_s$ strings for all $s$. The stress-tensor is of the form (2.1) where

$$T_\phi = -\frac{1}{2} \partial \phi \bar{\partial} \phi - q \partial^2 \phi, \quad T_{\beta \gamma} = -s \beta \partial \gamma - (s - 1) \partial \beta \gamma$$

(6.2)

and $q$ is a background charge, which is determined by requiring $[Q, T] = 0$ and given by [21]

$$q^2 = \frac{(s - 1)(2s + 1)^2}{4(s + 1)}$$

(6.3)

As in the spin-3 case, there is a $W_\infty$ symmetry generated by currents $W^{(n)} = (\partial \phi)^n + ...$, $n = 2, 3, 4, ...$ which commute with the BRST operator. In this case, the currents of spin $n \geq s$ are BRST trivial, and so topological, while the currents of spins $2, 3, ..., s - 1$ are BRST non-trivial. In particular, $W^{(s)} = \{Q, \beta\}$.

The total central charge of the $\phi, \beta, \gamma$ system is

$$c_s = 1 + 12q^2 - 2(6s^2 - 6s + 1) = \frac{2(s - 2)}{s + 1}$$

(6.4)
The $N$'th minimal model $M_{M,N}$ of the $W_M$ algebra has central charge

$$c_{M,N} = (M - 1) \left( 1 - \frac{M(M + 1)}{N(N + 1)} \right)$$

(6.5)

and so (6.4) is $c_{s-1,s}$, suggesting that this model could be the first minimal model of the $W_{s-1}$ algebra. Defining the physical states to be the cohomology classes of the BRST operator $Q$ gives a model with a discrete spectrum which the analysis of [21] shows to be consistent with the spectrum of the first $W_{s-1}$ minimal model. Much of the analysis of this paper carries over straightforwardly to this case.

The quantisation of the system (6.1) outlined above is not unique; for $s = 4$ there are three nilpotent BRST charges with the correct classical limit, for $s = 6$ there are four, while for $s = 3$ or $s = 5$ there is only one [21]. The different BRST charges correspond to different values of the background charge and hence of the central charge $c$. In addition to the models described above which exist for any $s$, there are then certain extra models which are not yet understood, but which may correspond to non-unitary minimal models.

As in the case $s = 3$, a critical string theory can be constructed by adding an effective matter system $X$ and ghosts $b, c$ to construct a total BRST charge (5.1),(5.2) which will be nilpotent provided the central charge of the effective matter system is $26 - c_s$. This gives the $W$-string theories of [21], based on the quantization of certain $W$-gravity theories first constructed in [3], which are seen to consist of $c = 26$ strings which include a $W_{s-1}$ minimal model in its conformal matter sector.

Another interesting class of models is found by considering the construction of $W_N$ strings [13]. The $W_N$ algebra is generated by the spin-two stress tensor $T$ and currents $W^{(s)}$ of spins $s$, $s = 3, \ldots, N$. The construction of critical $W_N$ string theories requires a matter sector which is a realisation of the $W_N$ algebra whose central charge takes the critical value

$$c_N^* = 2(N - 1)(2N^2 + 2N + 1)$$

(6.6)

together with the usual conformal ghosts $b, c$ plus the $W$-ghosts $\beta^{(s)}, \gamma^{(s)}$ of spins
One $W_N$ matter realisation with critical central charge (6.6) [13] is given by taking an arbitrary conformal field theory $X$ with central charge

$$\tilde{c} = 26 - \left(1 - \frac{6}{N(N+1)}\right) = 26 - c_{N-min} \tag{6.7}$$

and adding $N - 2$ free bosons $\phi_s \,(s = 3, \ldots, N)$ with background charge. The system given by these $N - 2$ scalars and the ghosts $\beta^{(s)}, \gamma^{(s)}$ for $s = 3, 4, \ldots, N$ has central charge $c_{N-min}$. A BRST charge has not been constructed explicitly for $N > 3$, but it seems reasonable to expect that such charges exist, and that they can be written in the form $\hat{Q} = Q_0 + Q_N$, where $Q_0$ is the usual Virasoro BRST operator (5.1) and $Q_N$ acts only on the $(\phi_s, \beta_s, \gamma_s)$ system. Then the model defined by restricting the $(\phi_s, \beta_s, \gamma_s)$ system to the cohomology classes of $Q_N$ is a conformal field theory with the central charge of the $N$'th Virasoro minimal model and the results of [13] for the spectrum of this model are consistent with the identification of this model with the $N$'th Virasoro minimal model. Then the $W_N$ string based on this realisation of $W_N$ is, at tree level, a $c = 26$ string whose matter sector consists of the $N$'th Virasoro minimal model, plus an arbitrary conformal field theory with central charge (6.7).

There are undoubtedly many similar free boson and ghost models which correspond to minimal models. As far as I am aware, all known free-field representations of $W$-algebras are of the form $(X, \phi)$, where $X$ are the fields of some conformal field theory on which the higher-spin $W$-generators do not act, plus some extra bosons $\phi$, which have the property that combining the bosons $\phi$ with the $W$-ghosts $\beta, \gamma$ (and truncating using a BRST operator) gives a minimal model. Using such a $W$-representation to construct a $W$-string theory will always give a normal Virasoro string which includes a minimal model sector, with the $W$-symmetry acting only within the minimal model sector. It would be interesting to find representations of $W$-algebras which are not reducible in this way, as they would offer the prospect of truly new $W$-strings.

---

$\star$ i.e. there is a non-primary basis for the generators of the $W$-algebra for which this is so.
Acknowledgements:

I would like to thank J. Distler, B. Spence and C. Thorn for helpful discussions and the ITP for its hospitality. This work was supported by NSF grant no. PHY 89-04035.

REFERENCES

1. A.B. Zamolodchikov, Teor. Mat. Fiz. 65 (1985) 1205.
2. C.M. Hull, Phys. Lett. 240B (1990) 110.
3. C.M. Hull, Nucl. Phys. 353 B (1991) 707.
4. C.M. Hull, Phys. Lett. 259B (1991) 68.
5. C.M. Hull, Nucl. Phys. B367 (1991) 731; Phys. Lett. 259B (1991) 68; QMW preprint QMW/PH/91/14 (1991); K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B364 (1991) 584 and B371 (1992) 315; in proceedings of 1991 Miami Workshop (Plenum, New York, (1991)).
6. C.N. Pope, L.J. Romans and K.S. Stelle, Phys. Lett. 268B (1991) 167 and 269B (1991) 287.
7. J. Thierry-Mieg, Phys. Lett. 197B (1987) 368.
8. A. Bilal and J.-L. Gervais, Nucl. Phys. B326 (1989) 222.
9. L.J. Romans, Nucl. Phys. B352 829.
10. S.R. Das, A. Dhar and S.K. Rama, Mod. Phys. Lett. A6 (1991) 3055; Int. J. Mod. Phys. A7 (1992) 2295; S.K. Rama, Mod. Phys. Lett. A6 (1991) 3531.
11. C.N. Pope, L.J. Romans, E. Sezgin and K.S. Stelle, Phys. Lett. B274 (1992) 298.
12. H. Lu, C.N. Pope, S. Schrans and K.W. Xu, Nucl. Phys. B385 (1992) 99.
13. H. Lu, C.N. Pope, S. Schrans and X.J. Wang, Nucl. Phys. **B379** (1992) 47.

14. H. Lu, B.E.W. Nilsson, C.N. Pope, K.S. Stelle and P.C. West, “The low-level spectrum of the $W_3$ string,” preprint CTP TAMU-64/92, [hep-th/9212017](http://arxiv.org/abs/hep-th/9212017).

15. C.N. Pope, E. Sezgin, K.S. Stelle and X.J. Wang, Phys. Lett. **B299** (1993) 247.

16. P.C. West, “On the spectrum, no ghost theorem and modular invariance of $W_3$ strings,” preprint KCL-TH-92-7, [hep-th/9212016](http://arxiv.org/abs/hep-th/9212016).

17. H. Lu, C.N. Pope, S. Schrans and X.J. Wang, “The interacting $W_3$ string,” preprint CTP TAMU-86/92, KUL-TF-92/43, [hep-th/9212117](http://arxiv.org/abs/hep-th/9212117).

18. M. Freeman and P. West, “$W_3$ string scattering,” preprint, KCL-TH-92-4, [hep-th/9210134](http://arxiv.org/abs/hep-th/9210134).

19. H. Lu, C.N. Pope, S. Schrans and X.J. Wang, “On the spectrum and scattering of $W_3$ strings,” preprint CTP TAMU-4/93, KUL-TF-93/2, [hep-th/9301099](http://arxiv.org/abs/hep-th/9301099).

20. M.D. Freeman and P.C. West, “The covariant scattering and cohomology of $W_3$ strings,” preprint, KCL-TH-93-2, [hep-th/9302114](http://arxiv.org/abs/hep-th/9302114).

21. H. Lu, C.N. Pope and X.J. Wang, ‘On Higher-spin Generalisations of String Theory’, preprint CTP TAMU–22/93, (1993) [hep-th/9304115](http://arxiv.org/abs/hep-th/9304115).

22. E. Bergshoeff, A. Sevrin, and X. Shen, Phys. Lett. **B296** (1992) 95.

23. E. Bergshoeff, H.J. Boonstra, M. de Roo, S. Panda and A. Sevrin, “On the BRST operator of $W$ strings,” preprint, UG-2/93, UCB-PTH-93/05,LBL-33737.

24. I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **102B** (1981) 27 and Phys. Rev. **D28** (1983) 2567.

25. V. Dotsenko and V. Fateev, Nucl. Phys. **B240** (1984) 312; **B251** (1985) 691.

26. G. Felder, Nucl. Phys. **B317** (1989) 215.
27. C. Thorn, Nucl. Phys. B286 (1987) 61; Phys. Rep. 175 (1989) 1.

28. C. Thorn, Nucl. Phys. B248 (1984) 551.