Universal properties of the near-horizon geometry

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We derive universal properties of the near-horizon geometry of spherically symmetric black holes. These properties follow solely from the existence of an apparent horizon and its regularity. Only two types of solutions are possible, and both appear at different stages of the black hole formation. If semiclassical gravity is valid, then accretion after horizon formation inevitably leads to a firewall that violates the quantum energy inequalities. Consequently, physical black holes can only evaporate once a horizon has formed. We describe how these results extend to modified theories of gravity, including Einstein–Cartan theories. Comparison of the required energy and time scales with the known semiclassical results suggests that the observed astrophysical black holes are horizonless ultra-compact objects, and the presence of a horizon is associated with currently unknown physics.

I. INTRODUCTION

Models of astrophysical black hole candidates describe them as either horizonless ultra-compact objects (UCOs) or physical black holes [1]. Both are consistent with the current precision data [2, 3] and have theoretically appealing properties as well as arguably undesirable features. Future observations are expected to differentiate between the two classes by resolving differences in the geometry on the scale of the horizon [1, 4]. A good understanding of the near-horizon domain is therefore particularly important.

Spherical symmetry considerably simplifies the analysis. Nonetheless, definite results can be obtained only if the solutions under consideration are either static or asymptotic and/or matter follows a prescribed evolution [5–9]. Numerical studies must assume the matter content and the equations of state [10]. As a result, despite spectacular successes in modeling the behavior of UCOs, the question of whether or not horizons exist is still open [4, 11].

Here we derive properties that inevitably follow from the existence of spherically symmetric horizons using a self-consistent approach. Working in the framework of semiclassical gravity [12–14], we use classical notions (horizons, trajectories, etc.) and describe dynamics via the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$, or modifications thereof.

There is no unanimously agreed upon definition of a black hole [11], but strong gravity that locally prevents light from escaping is a common characteristic. A physical black hole (PBH) [15] has a trapped region, i.e. it contains a spacetime domain where ingoing and outgoing future-directed null geodesics originating from a two-dimensional spacelike surface with spherical topology have negative expansion [7, 8, 16]. Its evolving outer boundary is the apparent horizon. In spherical symmetry it is unambiguously defined in all foliations that respect this symmetry. To be of physical relevance, the apparent horizon must form in finite time according to the clock of a stationary observer situated at spacelike infinity (Bob) [17].

We investigate the consequences of the existence of PBHs, assuming only that the apparent horizon is a regular surface in the sense that the curvature invariants there are finite. Within any given theory, we do not assume any specific matter content nor a quantum state $\omega$ that produces the expectation values of the energy-momentum tensor (EMT) $T_{\mu\nu} = \langle T_{\mu\nu} \rangle_\omega$.

Note that this EMT describes the total matter content — both the original collapsing matter and the produced excitations. We do not assume the presence of Hawking-like radiation. A PBH may possess an event horizon and singularity, or may be a regular black hole.

The results that we establish follow from the construction of finite invariants from divergent quantities. A general spherically symmetric metric in Schwarzschild coordinates is given by

$$ds^2 = -e^{2h(t,r)}f(t,r)dt^2 + f(t,r)^{-1}dr^2 + r^2d\Omega^2,$$  \hspace{1cm} (1)

where $r$ denotes the areal radius. These coordinates provide geometrically preferred foliations with respect to Kodama time, a natural divergence-free preferred vector field [16, 18, 19]. Using the advanced null coordinate $v$ the metric is written as

$$ds^2 = -e^{2h_+}(1 - \frac{C_+}{r}) dv^2 + 2e^{h_+}dvdr + r^2d\Omega^2.$$  \hspace{1cm} (2)

The Misner–Sharp (MS) mass [16, 20] $C(t,r)$ is invariantly defined via

$$f(t,r) := 1 - C/r := \partial_{\mu}r\partial^\mu r,$$  \hspace{1cm} (3)

and thus $C(t,r) = C_+(v(t,r),r)$. The functions $h(t,r)$ and $h_+(v,r)$ play the role of integrating factors in coordinate transformations, such as

$$dt = e^{-h}(e^{h_+}dv - f^{-1}dr).$$  \hspace{1cm} (4)

The apparent horizon is located at the Schwarzschild radius $r_s(t) = r_+(v)$ that is the largest root of $f(t,r) = 0$ [16, 21].

We use the invariants $T := T_{\mu}^{\mu}$ and $\Xi := T^{\mu\nu}T_{\mu\nu}$ to express the regularity of the apparent horizon [17] and assume that the dynamics is governed by the standard Einstein equations of general relativity (GR) unless stated otherwise. Then $T \equiv -R/8\pi$ and $\Xi \equiv R^\mu_{\nu\rho\sigma}R_{\mu\rho}/64\pi^2$, where $R_{\mu\nu}$ and $R$ are the Ricci tensor and Ricci scalar, respectively. It is convenient to introduce

$$\tau_t := e^{2h}T_{tt}, \quad \tau_r := T^{rr}, \quad \tau^r := e^{-h}T_{t}^{r}.$$  \hspace{1cm} (5)

In this notation the three Einstein equations for $G_{tt}, G_{rr}$, and
$G^{rr}$ are

$$\frac{\partial_{r} C}{r^2} = 8\pi \tau^{\alpha}/f, \quad (6)$$

$$\frac{\partial_{\tau} C}{r^2} = 8\pi e^{\lambda} \tau^{\alpha}, \quad (7)$$

$$\frac{\partial_{h} h}{r} = 4\pi (\tau^{\alpha} + \tau^{\beta}), \quad (8)$$

This manuscript is structured as follows: first, we review properties of the near-horizon geometry for a general dynamic solution. We then show that the so-called non-singular solutions are either static or describe the PBH dynamic only in the extreme case. We indicate how our results can be extended to several modified theories of gravity. Finally, we outline the only PBH formation scenario that is consistent with our findings and discuss its implications.

II. BLACK HOLE SOLUTIONS

A. Generic solution

The two scalars

$$T = (\tau^{\alpha} - \tau^{\beta})/f, \quad \Xi = ((\tau^{\alpha})^2 + (\tau^{\beta})^2 - 2(\tau^{\alpha})^2)/f^2, \quad (9)$$

are required to be finite at the apparent horizon, i.e. the leading terms in the functions $\tau_{a}$, $a \in \{t, r, \tau\}$, must scale as $f^k$ for some $k$ as $r \to r_g$. In GR, the term $T^0_0 = T_{\Xi}$ and does not affect the convergence properties of Eq. (9). A priori, there are infinitely many functions that satisfy these requirements and potentially describe the near-horizon geometry. However, it was shown that only two classes of dynamic solutions (with $k = 0, 1$) satisfy the regularity conditions [22]. We demonstrate below that the case $k = 0$ with

$$\tau_{t} \approx \tau^{\alpha} = -\Xi^{2} + O(\sqrt{x}), \quad \tau_{t}^{\tau} = -\Xi^{2} + O(\sqrt{x}), \quad (10)$$

for some $\Xi(t)$, where $x := r - r_g$ is the coordinate distance from the apparent horizon, is the only appropriate description of the EMT near an evolving apparent horizon beyond formation of the first marginally trapped surface.

If the EMT components satisfy Eq. (10), then the metric functions that solve Eq. (6) and Eq. (8) are

$$C = r_g - 4\pi r_g^{3/2} \sqrt{x} + O(x), \quad h = \frac{1}{2} \ln \frac{x}{\xi} + O(\sqrt{x}), \quad (11)$$

where $\xi(t)$ is determined by the choice of time variable and the higher-order terms depend on the higher-order terms in the EMT expansion [23]. Eq. (7) must then hold identically. Both sides contain terms that diverge as $1/\sqrt{x}$, and their identification results in the consistency condition

$$r_g/\sqrt{\tau} = \pm 4\sqrt{\pi} r_g \Xi, \quad (12)$$

where the lower (upper) sign corresponds to evaporation (accretion), analogous to the signature of $\tau^{\alpha}$. The invariants $T$ and $\Xi$ are finite by construction, and direct calculations show that all remaining algebraically independent scalars are finite [24]. Taking $\tau_{t} \to \tau^{\alpha} \to +\tau^2$ yields complex-valued solutions (see Ref. [17], Appendix C). On the other hand, a negative sign of $\tau_{t}$ and $\tau^{\alpha}$ results in violation of the null energy condition (NEC) [7, 9, 25, 26] in the vicinity of the apparent horizon for both signs of $\tau_{t}$. A future-directed outward (inward) pointing radial null vector $k^\mu$ satisfies $T_{\mu\nu} k^\mu k^{\nu} < 0$ for the contracting (expanding) Schwarzschild radius $r_g$ [17].

This result should be compared with the conclusions of Sec. 9.2 of Ref. [7] that in general asymptotically flat spacetimes with an asymptotically predictable future the trapped surface cannot be visible from future null infinity unless the weak energy condition is violated. Here, we are only considering the spherically symmetric setting, but without making any assumptions about the asymptotic structure of spacetime.

Useful information can be obtained by working with retarded and advanced null coordinates. If $r_g > 0$, it is convenient to use the retarded null coordinate $u$. For $r_g < 0$, the advanced null coordinate $v$ is particularly useful,

$$C_{+}(v, r) = r_{+}(v) + \sum_{i\geq 1} w_i(v)(r - r_+)^i, \quad (13)$$

$$h_{+}(v, r) = \sum_{i\geq 1} \chi_i(v)(r - r_+)^i, \quad (14)$$

for some functions $w_i(v), \chi_i(v)$, where $w_1 \leq 1$ due to the definition of $r_+$. This is the general form of the metric functions in $(v, r)$ coordinates that ensures finite curvature scalars at the apparent horizon. It allows to eliminate the majority of candidates for the functions $\tau$. In this case the components of the EMT are related by

$$\theta_v := e^{-2\Xi + \Theta_{v\nu}} = \tau_{t}, \quad (15)$$

$$\theta_{vr} := e^{-\Xi + \Theta_{vr}} = (\tau_{t}^{\tau} - \tau_t)/f, \quad (16)$$

$$\theta_r := \Theta_{rr} = (\tau^{\alpha} + \tau_t - 2\tau^{\alpha} + \tau_{t}^{\tau})/f^2, \quad (17)$$

where $\Theta_{\mu\nu}$ denotes the EMT components in $(v, r)$ coordinates [22].

A static observer finds that the energy density $\rho = T_{\mu\nu} u^\mu u^\nu = -T^0_0$, pressure $p = T_{\mu\nu} n^\mu n^\nu = T^r_r$, and flux $\phi := T_{\mu\nu} u^\mu n^\nu$, where $u^\mu$ is the four-velocity and $n^\mu$ is the outward-pointing radial spacelike vector, diverge at the apparent horizon. The experience of a radially-infalling observer Alice moving on the trajectory $x^\mu_A(\tau) = (t_A, r_A, 0, 0)$ is different, and also differs from the infall into a classical eternal black hole.

First, horizon crossing happens not only at some finite proper time $\tau_0$, but due to the form of the metric also at a finite time $t_0(\tau_0), r_g(t_0(\tau_0)) = r_A(\tau_0)$ according to the clock of a distant observer Bob. This is particularly easy to see for ingoing null geodesics, where

$$\frac{dt}{dr} = -\frac{e^{-\Xi(t, r)}}{f(t, r)} \to \pm \frac{1}{r_g}, \quad (18)$$

at $r = r_g$, the rhs is obtained by using Eqs. (11) and (12) [23] (see Appendix A for details), and the upper sign corresponds
to evaporation. This is consistent with the coordinate transformation of Eq. (4) that results in the leading order expansion $t(v, r_g + \delta r) = t(r_g) + \delta r/r_g^2$.

For an evaporating black hole ($r_g' < 0$), energy density, pressure, and flux in Alice’s frame are finite. For example, if the geometry is approximately Vaidya ($w_1 = 0, \chi_1 = 0$), then

$$\rho_A^r = p_A^r = \phi_A^r = -\frac{\gamma^2}{32\pi r_A^3},$$

(19)

at $r_+ = r_A$ [24], where $r_A \equiv dr_A/d\tau$. However, upon crossing the apparent horizon of an accreting PBH, Alice encounters a firewall,

$$\rho_A^r = -\frac{2r_A^2}{gX} + \mathcal{O}(1/\sqrt{X}),$$

(20)

where $X \equiv r_A(\tau) - r_g(t_A(\tau))$ [24].

Violations of the NEC are bounded by quantum energy inequalities (QEIs) [26, 27]. For spacetimes of small curvature, explicit expressions that bound the time-averaged energy density for a geodesic observer were derived in Ref. [28]. This bound is violated by the $1/f^2$ divergence of the energy density. Thus we are faced with the following conundrum: either accretion to an UCO can only occur before the first marginally trapped surface appears, and PBHs, once formed, can only evaporate, or semiclassical physics breaks down at the horizon scale. We restrict our discussion to evaporating PBHs in what follows.

The triple limit $\tau_a \to -\gamma^2$ was observed in ab initio calculations of the renormalized EMT on a Schwarzschild background [29]. Transformation to the orthonormal basis shows that the EMT of these solutions is similar to type II in the Segre–Hawking–Ellis classification scheme [7, 25]. At $r \sim r_g$, the EMT coincides with that of a perfect exotic (i.e. NEC-violating) null fluid only if the metric is sufficiently close to Vaidya metrics (see Appendix A). However, it is the key ingredient of matter near the apparent horizon and becomes dominant as $r \to r_g$ for all $\tau_a \sim f^0$ solutions.

### B. Extreme solution. Static solutions

The static solution with $k = 0$ is impossible, as in this case $\mathcal{S}$ would diverge at the apparent horizon. Consequently, EMT components that allow for static solutions must behave differently. Many models of static non-singular black holes assume a finite-valued energy density and pressure at the apparent horizon [15, 30, 31]. With respect to the invariants of Eq. (9), this is the $k = 1$ solution, with

$$\tau_t \to E(t)f, \quad \tau^r \to P(t)f, \quad \tau_t^r \to \Phi(t)f,$$

(21)

where $\rho = E$ and $p = P$ at the apparent horizon. Any two functions can be expressed algebraically in terms of the third and $8\pi T_T^T E \leq 1$ (Appendix B provides a brief summary of their properties and gives explicit expressions for the metric functions $C \approx r_g + 8\pi T_T^T x$ and $h$).

We now show that only a unique dynamic case with the extreme value of $E$ is possible. From Eqs. (15) and (13) it follows that $w_1 = 1$. As a result, as $C_+(v, r) - r$ changes sign at $r = r_+$, the leading terms in the expansion of the MS mass in Eq. (13) are $C_+ = r_+ + y + w_3 y^3$, where $w_3 \leq 0$ and $y := r_+ - r$. If $w_3 = 0$ the nonlinear terms begin from a higher odd power.

This expression for the mass must coincide with $C(t(v, r_+ + y), r_+ + y)$. We use Eq. (18) to obtain the expansion parameter $x := r - r_g$ as $x(v, y) = -r_g^2 y^3/(2r_g^5) + \mathcal{O}(y^5)$. This implies that

$$C_+ = C = r_g + y + (1 - 8\pi T_T^T E)\frac{y^3}{2r_g^2} + \mathcal{O}(y^5),$$

(22)

and thus $E \equiv 1/(8\pi T_T^T)$. Using the next (half-integral) terms in the expansion of $\tau_a$ leads to $f \approx c_{32}(t)x^{3/2}/r_g$ for some coefficient $c_{32}(t) > 0$, setting via Eq. (21) the scaling of other leading terms in the EMT. Consistency of Eqs. (7) and (8) implies $P = -E = -1/(8\pi T_T^T)$ and $\Phi = 0$. From the next order expansion we obtain that $h = -\frac{3}{4} \ln(x/\xi) + \mathcal{O}(\sqrt{x})$ and the relation $r_g' = -c_{32}^{3/2}/r_g$ (Appendix B presents the details of the calculations).

On the other hand, solutions with a time-independent apparent horizon or general static solutions do not require $w_1 = 1$ to satisfy Eqs. (15)–(17). Since $r_+(v) = r_g(t) = const$ it is possible to have non-extreme solutions. Then Eq. (7) implies $\Phi = 0$ and the identity $E = -P$ follows from Eq. (17), leading to a regular function $h(t, r)$. However, in this case Eq. (4) indicates that the apparent horizon cannot be reached in a finite time $t$.

### III. PHYSICAL BLACK HOLES IN MODIFIED GRAVITY

There are numerous arguments as to why a classical theory of gravity may or should differ from GR [32]. Strong fields in the vicinity of UCOs are one of the regimes where the effects of modified gravity are expected to be discernable. Mathematically, these theories are typically more involved than GR, and exact and approximate black hole solutions are used both to test the consistency of such theories and also to differentiate between models of horizonless UCOs and PBHs [33].

One group of models includes various additional curvature-dependent terms in the gravitational Lagrangian, $\mathcal{L}_g = R + \lambda 3(g^{\mu\nu}, R_{\mu
u\rho\sigma})$, where $\lambda$ is a small dimensionless parameter [32, 34, 35]. The Einstein equations are modified by fourth or higher-order terms, $G_{\mu\nu} + \lambda \mathcal{E}_{\mu\nu} = 8\pi T_{\mu\nu}$, where the terms $\mathcal{E}_{\mu\nu}$ result from the variation of $\mathcal{F}$ [36]. The most general spherically symmetric metric is still given by Eq. (1), and the requirements of finiteness of $T$ and $\mathcal{S}$ are still meaningful. However, they are no longer directly related to the finiteness of the curvature scalars. For example, in $f(R)$ theories [35],

$$\mathcal{L} = f(R),$$

$$f'(R)R + 2f(R) + 3\square f'(R) = 8\pi T,$$

(23)

and unlike in GR the finiteness of $T^\mu_\nu$ is not guaranteed a priori. It is conceivable that the metric is such that the curvature
invariants are finite, but $\Box R$ and thus $T$ diverge at the apparent horizon.

Nevertheless, the two types that were discussed above are the only perturbatively possible classes of solutions in spherical symmetry. While their existence must be established separately for each theory, it is clear that divergences stronger than those allowed in GR are not permitted at any order of $T_{\mu\nu} = \bar{T}_{\mu\nu} + \lambda T^{(1)}_{\mu\nu} + \cdots$, where $\bar{T}_{\mu\nu}$ denotes the unperturbed GR expression. Such terms will contribute stronger singularities to the functions $C$ and $h$, and thus invalidate the perturbative expansion close to the apparent horizon.

The Einstein–Cartan theory of gravity is a modification of GR in which spacetime can have torsion in addition to curvature [34, 37]. The torsion tensor is expressed as the antisymmetric part of the connection $Q^\mu_{\nu\eta} = \frac{1}{2} (\Gamma^\mu_{\nu\eta} - \Gamma^\mu_{\eta\nu})$. Despite having a non-metric part of the connection, it is still assumed that $\nabla g_{\mu\nu} = 0$. The full set of equations now consists of the equations for $G_{\mu\nu}$ that are related to the EMT, and the equations for $Q^\mu_{\nu\eta}$ that relate the torsion to the density of intrinsic angular momentum.

However, it is possible to represent this system by a single set of Einstein equations with an effective EMT on the rhs,

$$\dot{G}_{\mu\nu} = 8\pi T^{\text{eff}}_{\mu\nu}, \tag{24}$$

where $\dot{G}_{\mu\nu}$ is derived from the metric alone and the effective EMT includes terms that are quadratic in spin [37, 38]. Requiring now that $\dot{R}^\nu_\nu$ and $\dot{R}^{\mu\nu}R_{\mu\nu}$ are finite at the apparent horizon $r = r_g$ leads to the same types of PBH solutions.

### IV. IMPLICATIONS FOR BLACK HOLE FORMATION

Consider now possibilities for horizon formation. Assume that the first marginally trapped surface appears at some $v_S$ at $r = r_+(v_S)$. For $v \leq v_S$ the MS mass in its vicinity can be described by modifying Eq. (13) as

$$C(v, r) = \sigma(v) + r_+(v) + \sum_{i \geq 1} w_i(v)(r - r_+)^i,$$  \tag{25}

where the deficit function $\sigma(v) \leq 0$, and $r_+(v)$ corresponds to the maximum of $\Delta_v(r) := C(v, r) - r$. At the advanced time $v_S$ the location of the maximum corresponds to the first marginally trapped surface, $r_+(v_S) = r_+(v_S)$ and $\sigma(v_S) = 0$. For $v \geq v_S$ the MS mass is described by Eq. (13). For $v \leq v_S$ the (local) maximum of $\Delta_v$ satisfies $d\Delta_v/dr = 0$, hence $w_1(v) - 1 \equiv 0$. Before the PBH is formed there are no a priori restrictions on the evolution of $r_+$. However, since an accreting PBH leads to a firewall, $r_+(v_S) \leq 0$. Since the trapped region is of a finite size for $v > v_S$ the maximum of $C(v, r)$ does not coincide with $r_+(v)$. As a result, $w_1(v) < 1$ for $v > v_S$.

This scenario means that at its formation a PBH is described by a $k = 1$ solution that is necessarily extreme. It immediately switches to the $k = 0$ solution. Since the energy density and pressure are negative in the vicinity of the apparent horizon and positive in the vicinity of the inner horizon [15, 24], density and pressure jump at the intersections of the two horizons. However, an abrupt transition from $f^1$ to $f^0$ behavior is only of conceptual importance: this aspect of the evolution is continuous in $(v, r)$ coordinates and there will be no discontinuity according to observers crossing the $r = r_+$ and subsequently $r = r_g$ surfaces.

The universal properties of PBHs follow from the existence and regularity of the apparent horizon. The NEC is violated in the vicinity of the apparent horizon and the matter content is dominated by a null fluid. The near-horizon EMT is characterized by two functions of time and does not depend on the properties of the collapsing matter. If the semiclassical picture is valid, then accretion leads to a firewall that violates the bounds on the violation of the NEC. As a result, accretion can occur only before a PBH is formed. This firewall is not an artifact of spherical symmetry. The same effect was demonstrated for the Kerr-Vaidya metric [39]. While a sufficiently slow massive test particle can be prevented from crossing the horizon, the crossing generally happens in finite time according to a distant observer. Taking the proper radial velocity to be of the order of one (for a test particle falling from infinity with zero initial velocity into a Schwarzschild black hole $\dot{r}(r) = -3/4$ at $r = r_g$), we see that the time dilation for a non-stationary Alice is $d\hat{t}_A/dr \sim |r_g|^{-1}$ at the apparent horizon.

On the other hand, it is still not clear how the collapsing matter actually behaves. Violation of the NEC in some vicinity of the apparent horizon is incompatible with the preservation of the normal character of the collapsing matter. Thus we must presume some mechanism that converts the original matter into the exotic matter present in the vicinity of the forming apparent horizon, thereby creating something akin to a shock wave to restore the normal behavior near the inner horizon. Alternatively, the observed UCOs may actually be horizonless — not due to some exotic supporting matter or dramatic variation in the laws of gravity, but simply because the conditions for the formation of a PBH have not been met at the present moment of $t$. Indeed, emission of Hawking-like radiation does not require the formation of an event or even an apparent horizon [13, 40–42]. However, it is a non-violent process that approaches at latter times the Hawking radiation and Page’s evaporation law $r_+^\nu \sim \kappa/r_g^2$, $\kappa \sim 10^{-3} \sim 10^{-4}$ [8, 29, 43]. It is also conceivable that the conditions are not met before evaporation is complete or before effects of quantum gravity become dominant [8, 31].

Moreover, even if the necessary NEC violation occurs in nature, the process may be too slow to transform the UCOs that we observe into PBHs. Eq. (18) sets the time scale of the last stages of infall according to Bob. Assuming that it is applicable through the radial interval of the order of $r_g$, we have $t_{\text{inf}} \sim r_g/r_g^\nu$. For an evaporating macroscopic PBH, this is of the same order of magnitude as the Hawking process decay time $t_{\text{evp}} \sim 10^3 r_g^3$. Such behavior was found in thin shell collapse models, where the exterior geometry is modeled by a pure outgoing Vaidya metric [23]. For a solar mass black hole this time is about $10^{64}$ yr, indicating that it is simply too early for the horizon to form.

The possibility that exotic new physics is only needed for the formation of black holes, but not for the formation of hori-
zonless objects has interesting consequence for the information loss paradox [8, 44, 45]. Its formulation is ineluctably linked to the existence of the event horizon and singularity [46]. Horizon avoidance [47] may thus occur due to the absence of new physics, and not because of it. A better understanding of the near-horizon geometry of PBHs will improve models developed to take full advantage of the new era of multi-messenger astronomy [4, 45], using observations not only to learn about the true nature of astrophysical black holes, but also to obtain new insights into fundamental physics.

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Appendix A: Some properties of $k = 0$ solutions

The EMT expansion for $k = 0$ solutions is given by

$$\tau_t = -\Upsilon^2 + \sum_{j \geq 1} \alpha_{j/2} x^{j/2},$$

(A1)

$$\tau^r = -\Upsilon^2 + \sum_{j \geq 1} \beta_{j/2} x^{j/2},$$

(A2)

$$\tau^r = -\Upsilon^2 + \sum_{j \geq 1} \gamma_{j/2} x^{j/2}.$$  

(A3)

The first constraint can be read off from Eq. (17), giving

$$\alpha_{12} + \gamma_{12} - 2\beta_{12} = 0,$$

(A4)

where we have omitted the dash from fractional indices to reduce clutter. Additional constraints follow from the higher-order expansion of Eq. (7).

The higher-order terms in the expansion

$$t(v, r_+, y) = t(v, r_+) + y/r_+^f + O(y^2)$$  

(A5)

are obtained by taking the limit $r \to r_g$ of the corresponding derivatives of $dt/dr = -(e^h f)^{-1}$. As a result

$$x = 0 \to x = y - r_g(v, r_+, y) = -r_g^y y^2 + O(y^3).$$  

(A6)

Invariance of the MS mass, $C_+ = C$, then allows to identify the terms via

$$r_+ + (1 + w_1) y + w_2 y^2 + \cdots$$

$$= r_g + \left(1 - r_g \sqrt{-r_g} \frac{\sqrt{2\xi}}{2\xi} \right) y + \frac{1}{3} r_g'' y^2 + \cdots,$$

(A7)

where we have used Eq. (12) to simplify the coefficient of $y$ on the rhs.

Curvature scalars can be conveniently evaluated using the expression for the Riemann tensor in the orthonormal frame that is based on the normalisation $(\partial_t, \partial_r, \partial_{\theta_a}, \partial_{\phi_a})$ [19]. In particular, the Kretschmann scalar $K := R_{\mu
\nu\gamma
\delta} R^{\mu
\nu\gamma
\delta}$ satisfies the simple expression

$$K = 4R^2_{0101} + 8R^2_{0202} - 16R^2_{0212} + R^2_{1212} + 4R^2_{2323}. $$  

(A8)

The EMT in this orthonormal basis has the form

$$T_{ab} = \begin{pmatrix} q + \mu_1 q + \mu_2 & 0 & 0 \\ 0 & \mu_2 q + \mu_3 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix},$$

(A9)

where

$$q = \frac{\Upsilon}{4\sqrt{\pi r_g^2}},$$

(A10)

and the remaining coefficients are finite at the apparent horizon. Analogous expressions can be obtained using $(v, r)$ coordinates.

Appendix B: Some properties of the fully regular solution

For non-extreme solutions the EMT expansion is

$$\tau_t = Ef + \sum_{j \geq 3} \alpha_{j/2} x^{j/2},$$  

(B1)

$$\tau^r = \Phi f + \sum_{j \geq 3} \beta_{j/2} x^{j/2},$$  

(B2)

$$\tau^r = Pf + \sum_{j \geq 3} \gamma_{j/2} x^{j/2}.$$  

(B3)

The leading terms of the metric functions are

$$C = r_g + 8\pi r_g^2 E x + O(x^{3/2}),$$

(B4)

$$h = -\ln \frac{x}{\xi} + O(\sqrt{x}),$$

(B5)

where $8\pi r_g^2 E \leq 1$ due to the definition $C(t, r_g) = r_g, f > 0$ for $r > r_g$. The functions $P$ and $\Phi$ can be expressed as

$$P = \frac{-1 + 4r_g^2 E}{4\pi r_g^2}, \quad \Phi = \pm \frac{1 - 8\pi r_g^2 E}{8\pi r_g^2}.$$  

(B6)

However, only the extreme case $E = 1/(8\pi r_g^2)$ is consistent for an evolving apparent horizon. Since the limit of Eq. (16) results in

$$-\frac{w_1}{8\pi r_g} = \Phi - E,$$

(B7)

we have $\Phi = 0$ and by Eq. (17) $P = -E = -1/(8\pi r_g^2)$. As a result $f \approx c_{32} x^{3/2}$ near $r = r_g$, and the next term in the expansion of $\tau_t$ is $\alpha_2$. Solving Eq. (6) results in

$$C(t, r) = r - 4\sqrt{-\pi \alpha_2 / 3} r_g^{3/2} x^{3/2} + O(x^2).$$  

(B8)
Therefore \( \partial_t C = \frac{\alpha}{2} r_g \hat{\sigma}_{32} \sqrt{x} \). On the other hand, the leading term of the flux \( \tau^r_r \) is \( \beta x^2 \), and to satisfy Eq. (7) the function \( h \) that results from Eq. (8) should satisfy

\[
\partial_x h = -\frac{3}{2x} = 4\pi \left( \frac{\alpha_2 + \gamma_2}{r_g^3} \right) \left( \frac{x}{c_{32}^2 x} \right)
\]  

at leading order. Since according to Eq. (B8)

\[
\alpha_2 = -\frac{3c_{32}^2}{16\pi r_g^3}
\]  

we have \( \gamma_2 = \alpha_2 \), and according to Eq. (17) \( \beta_2 = \alpha_2 \). At leading order

\[
h = -\frac{3}{2} \ln \frac{x}{\xi} + \mathcal{O}(\sqrt{x}),
\]

and Eq. (7) leads to

\[
r_g' = -c_{32} \xi^{3/2}/r_g,
\]

Direct evaluation of \( R, R_{\mu\nu} R^{\mu\nu} \) and \( K \) shows that this condition suffices to ensure their finite values on the apparent horizon.

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