The ‘non-Kerrness’ of domains of outer communication of black holes and exteriors of stars

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In this paper, we construct a geometric invariant for initial datasets for the vacuum Einstein field equations \((S, h_{ab}, K_{ab})\), such that \(S\) is a three-dimensional manifold with an asymptotically Euclidean end and an inner boundary \(\partial S\) with the topology of the 2-sphere. The hypersurface \(S\) can be thought of being in the domain of outer communication of a black hole or in the exterior of a star. The geometric invariant vanishes if and only if \((S, h_{ab}, K_{ab})\) is an initial dataset for the Kerr spacetime. The construction makes use of the notion of Killing spinors and of an expression for a Killing spinor candidate, which can be constructed out of concomitants of the Weyl tensor.

Keywords: black holes; Kerr spacetime; Killing spinors

1. Introduction

Let \((S, h_{ab}, K_{ab})\) be an initial dataset for the vacuum Einstein field equations such that \(S\) has two asymptotically Euclidean ends, but otherwise trivial topology.¹ In Bäckdahl & Valiente Kroon (2010a), a geometric invariant for this type of initial datasets has been constructed—see also Bäckdahl & Valiente Kroon (2010b) for a detailed discussion. This invariant is a non-negative number having the property that it vanishes if and only if the initial dataset corresponds to data for the Kerr spacetime. Thus, the invariant measures the non-Kerrness of the initial data.

In view of possible applications of the non-Kerrness to the problem of the uniqueness of stationary black holes and the nonlinear stability of the Kerr spacetimes, a different type of initial hypersurface is of more interest: a three-dimensional hypersurface with the topology of the complement of an open ball in \(\mathbb{R}^3\), \(S \approx (\mathbb{R}^3 \setminus B_1)\). This type of 3-manifold can be thought of as a Cauchy hypersurface in the domain of outer communication of a black hole or the exterior of a star. In the present paper, we discuss the construction of a geometric invariant measuring the non-Kerrness of this type of initial hypersurface.

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¹More precisely, \(S \approx (\mathbb{R}^3 \setminus B_1) \# (\mathbb{R}^3 \setminus B_1)\) where \(B_1\) denotes an open ball of radius 1 and \# indicates that the boundaries of the two copies of \((\mathbb{R}^3 \setminus B_1)\) are identified in the trivial way.
(a) Outline of the article

In §2, we provide a brief summary of the theory of non-Kerrness invariants developed in Bäckdahl & Valiente Kroon (2010a,b). This is provided for quick reference and contains the essential ingredients required in the construction of the present paper. Section 3 contains a discussion of properties of vacuum Petrov-type D spacetimes that are relevant for our discussion. In particular, it provides a formula of a Killing spinor candidate written entirely in terms of concomitants of the Weyl tensor. For a spacetime that is exactly of Petrov-type D, this expression provides a Killing spinor of the spacetime. This expression is used in the sequel to provide the boundary value of an elliptic problem. Section 4 provides a discussion of a boundary value problem for the approximate Killing spinor equation. Section 5 makes use of the solution to the boundary value problem to construct the non-Kerrness invariant. Finally, in §6 we provide some conclusions and outlook. This paper also includes two appendices. The first one provides a summary of the results on boundary value problems for elliptic systems used in our construction. The second appendix contains an improved theorem characterizing the Kerr spacetime in terms of Killing spinors. This theorem removes some technical assumptions made in Bäckdahl & Valiente Kroon (2010a,b).

(b) Notation

Throughout, \((\mathcal{M}, g_{\mu\nu})\) will denote an orientable and time orientable, globally hyperbolic vacuum spacetime. It follows that the spacetime admits a spin structure (Geroch 1968, 1970). In what follows, \(\mu, \nu, \ldots\) will denote abstract four-dimensional tensor indices. The metric \(g_{\mu\nu}\) will be taken to have a signature \((+,-,-,-)\). Let \(\nabla_\mu\) denote the Levi–Civita connection of \(g_{\mu\nu}\). The triple \((\mathcal{S}, h_{ab}, K_{ab})\) will denote initial data on a hypersurface of the spacetime \((\mathcal{M}, g_{\mu\nu})\). The symmetric tensors \(h_{ab}\) and \(K_{ab}\) will correspond, respectively, to the 3-metric and the extrinsic curvature of the 3-manifold \(\mathcal{S}\). The metric \(h_{ab}\) will be taken to be negative definite. The indices \(a, b, \ldots\) will denote abstract three-dimensional tensor indices, while \(i, j, \ldots\) will denote three-dimensional tensor coordinate indices. Let \(D_a\) denote the Levi–Civita covariant derivative of \(h_{ab}\). Spinors will be used systematically. We follow the conventions of Penrose & Rindler (1984). In particular, \(A, B, \ldots\) will denote abstract spinorial indices, while \(a, b, \ldots\) will be indices with respect to a specific frame.

A space spinor formalism will be used throughout. A very brief introduction to this formalism is given in what follows—see Sommers (1980), Bäckdahl & Valiente Kroon (2010b) for more detailed expositions. Let \(\tau^{AA'}\) denote the spinorial counterpart of the normal \(\tau^\mu\) to the surface \(\mathcal{S}\), with normalization \(\tau_{AA'}\tau^{AA'} = 2\). Given the spacetime solder forms \(\sigma^A_{AA'}\) and \(\sigma^\mu_{AA'}\) satisfying

\[
\sigma_{\mu AB} = \epsilon_{AB} \epsilon_{A'B'} \sigma^A_{\mu A'} \sigma^{BB'}_{\nu A'} \quad \text{and} \quad \sigma^A_{\mu A'} \sigma^\nu_{AA'} = \delta^\nu_{\mu},
\]

the relation \(\tau^A_{A'} \tau^{BA'} = \epsilon^{AB}\) allows us to introduce the spatial solder forms

\[
\sigma^A_{\mu AB} \equiv \sigma^A_{\mu (A'} \tau^{B)A'} \quad \text{and} \quad \sigma^\mu_{AB} \equiv \tau^A_{(B} \sigma^\mu_{A'A')}. \]

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One has that
\[ \sigma^\mu_{AB} \sigma_{\nu}^{AB} = h^\mu_{\nu}, \quad \tau_\mu \sigma^\mu_{AB} = 0, \quad h_{\mu \nu} \sigma_{AB}^{\mu} \sigma_{CD}^{\nu} = -\epsilon_{A(C \epsilon D)B}. \]

Any spatial tensor has a space-spinor counterpart. For example, if \( T_{\mu}^\nu \) is a spatial tensor (i.e. \( \tau^\mu T_{\mu}^\nu = 0 \) and \( \tau_\nu T_{\mu}^\nu = 0 \)), then its space-spinor counterpart is given by \( T_{AB}^{CD} = \sigma_{AB}^{\mu} \sigma_{CD}^{\nu} T_{\mu}^\nu \).

Let \( \nabla_{AA'} \) denote the spinorial counterpart of the spacetime connection \( \nabla_\mu \). Besides the connection \( \nabla_{AA'} \), two other spinorial connections will be used: \( D_{AB} \), the spinorial counterpart of the Levi–Civita covariant derivative \( D_a \), and \( \nabla_{AB} \), the Sen covariant derivative of \( (S, h_{ab}, K_{ab}) \). The Sen connection is defined by \( \nabla_{AB} \equiv \sigma_{AB}^\mu \nabla_\mu \).

### 2. Killing spinors and non-Kerrness

In this section, we provide a brief account of the theory of non-Kerrness developed in Bäckdahl & Valiente Kroon (2010a,b).

(a) *Killing spinors and Killing spinor initial data*

The starting point of the construction in Bäckdahl & Valiente Kroon (2010a,b) is the space-spinor decomposition of the Killing spinor equation

\[ \nabla_{A'(A\kappa_{BC})} = 0, \quad (2.1) \]

where \( \kappa_{AB} = \kappa_{(AB)} \) and the spinorial conventions of Penrose & Rindler (1984) are being used.

Important for our purposes is the idea of how to encode that the development of an initial dataset \( (S, h_{ab}, K_{ab}) \) admits a solution to the Killing spinor equation (2.1). This question can be addressed by means of the space-spinor formalism discussed in the previous section.

The space-spinor decomposition of equation (2.1) renders a set of three conditions intrinsic to the hypersurface \( S \):

\[ \xi_{ABCD} = 0, \quad (2.2a) \]
\[ \Psi_{(ABC)^F \kappa_D)F} = 0 \quad (2.2b) \]

and

\[ 3\kappa_{(A}^E \nabla_B^F \Psi_{CD)EF} + \Psi_{(ABC)^F \xi_D)F} = 0, \quad (2.2c) \]

where we have written

\[ \xi_{ABCD} \equiv \nabla_{(AB} \kappa_{CD)}, \quad \xi_{AB} \equiv \frac{3}{2} \nabla_{(A}^D \kappa_{B)D}, \quad \xi \equiv \nabla_{PQ} \kappa_{PQ}, \]

and \( \nabla_{AB} \) denotes the spinorial version of the Sen connection associated with the pair \( (h_{ab}, K_{ab}) \) of intrinsic metric and extrinsic curvature. It can be expressed in terms of the spinorial counterpart, \( D_{AB} \) of the Levi–Civita connection of the
3-metric $h_{ab}$, and the spinorial version, $K_{ABCD} = K_{(AB)(CD)} = K_{CDAB}$, of the second fundamental form $K_{ab}$. For example, given a valence 1 spinor $\pi_A$ one has that
\[ \nabla_A \pi_C = D_A \pi_C + \frac{1}{2} K_{ABC} Q \pi_Q, \]
with the obvious generalizations to higher valence spinors. In equations (2.2b) and (2.2c), the spinor $\Psi_{ABCD}$ denotes the restriction to the hypersurface $S$ of the self-dual Weyl spinor. Crucially, the spinor $\Psi_{ABCD}$ can be written entirely in terms of initial data quantities via the relations
\[ \Psi_{ABCD} = E_{ABCD} + iB_{ABCD}, \]
with
\[ E_{ABCD} = -r_{(ABCD)} + \frac{1}{2} \Omega_{(AB}^{PQ} \Omega_{CD)PQ} - \frac{1}{6} \Omega_{ABCD} K \]
and
\[ B_{ABCD} = -i D^Q (A \Omega_{BCD})Q, \]
and where $\Omega_{ABCD} \equiv K_{(ABCD)}$, $K \equiv K_{PQ}^{PQ}$. Furthermore, the spinor $r_{ABCD}$ is the Ricci tensor, $r_{ab}$, of the 3-metric $h_{ab}$.

The key property of equations (2.2a)–(2.2c) is contained in the following result proved in Bäckdahl & Valiente Kroon (2010b)—see also García-Parrado & Valiente Kroon (2008).

**Proposition 2.1.** Let equations (2.2a)–(2.2c) be satisfied for a symmetric spinor $\tilde{k}_{AB}$ on an open set $U \subset S$. Then the Killing spinor equation (2.1) has a solution, $\kappa_{AB}$, on the future domain of dependence $D^+(U)$.

(b) Approximate Killing spinors

The spatial Killing spinor equation (2.2a) can be regarded as a (complex) generalization of the conformal Killing vector equation. It will play a special role in our considerations. As in the case of the conformal Killing equation, equation (2.2a) is clearly overdetermined. However, one can construct a generalization of the equation which under suitable circumstances can always be expected to have a solution. One can do this by composing the operator in equation (2.2a) with its formal adjoint—see Bäckdahl & Valiente Kroon (2010a). This procedure renders the equation
\[ L K_{CD} \equiv \nabla A B \nabla_{(AB} K_{CD)} - \Omega_{(A}^{ABF} (\nabla_{[DF]} K_{B)C} - \Omega_{(A}^{ABF} (\nabla_B) F K_{CD} = 0, \]
which will be called the approximate Killing spinor equation. One has the following result proved in Bäckdahl & Valiente Kroon (2010b).

**Lemma 2.2.** The operator $L$ defined by the left-hand side of equation (2.3) is a formally self-adjoint elliptic operator.

In Bäckdahl & Valiente Kroon (2010a,b) it has been shown that if $S$ has the same topology as Cauchy slices of the Kerr spacetime, and if the pair $(h_{ab}, K_{ab})$ is suitably asymptotically Euclidean, then there exists a certain asymptotic behaviour at infinity for the spinor $\kappa_{AB}$ for which the approximate Killing spinor equation always admits a solution.
If one wants to extend the construction discussed in the previous paragraphs to a 3-manifold on, say, the domain of outer communication of a black hole or the exterior of a star so that \( \mathcal{S} \approx (\mathbb{R}^3 \setminus \mathcal{B}_1) \), then in addition to prescribing the asymptotic behaviour of the spinor \( \kappa_{AB} \) at infinity, one also has to prescribe the behaviour at the inner boundary \( \partial \mathcal{S} \). One wants to prescribe this information in such a way that \( \kappa_{AB} \) has the right Killing behaviour at the boundary whenever all of the Killing spinor data equations (2.2a)–(2.2c) are satisfied. In this paper, we discuss how this can be done, and as a result we construct the non-Kerrness for 3-manifolds with topology \( (\mathbb{R}^3 \setminus \mathcal{B}_1) \). These 3-manifolds can be interpreted as slices in the domain of outer communication of a black hole or slices in the exterior of a star. It is expected that this construction will be of use in the reformulation of problems involving the Kerr spacetime: the uniqueness of stationary black holes, the construction of an interior for the Kerr solution, and possibly also the evolution of nonlinear perturbations of the Kerr spacetime.

3. Petrov-type D spacetimes

In order to analyse what is the right initial data to be prescribed on the boundary \( \partial \mathcal{S} \) of our initial 3-manifold \( \mathcal{S} \), we will look at some properties of vacuum spacetimes of Petrov-type D.

(a) The canonical form for type D

Let \( \Psi_{ABCD} \) denote the Weyl spinor of a vacuum spacetime \( (\mathcal{M}, g_{\mu\nu}) \). We shall consider the following invariants of \( \Psi_{ABCD} \)

\[
\mathcal{I} \equiv \frac{1}{2} \Psi_{ABCD} \Psi^{ABCD}
\]

and

\[
\mathcal{J} \equiv \frac{1}{6} \Psi_{ABCD} \Psi^{CDEF} \Psi^{EFAB}.
\]

The Petrov type of the spacetime is determined as a solution of the eigenvalue problem

\[
\Psi_{ABCD} \eta^{CD} = \lambda \eta_{AB},
\]

e.g. Stephani et al. (2003). The eigenvalues \( \lambda \) satisfy the equation

\[
\lambda^3 - \mathcal{I} \lambda - 2 \mathcal{J} = 0.
\]

Let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) denote the roots of the above polynomial. The invariants \( \mathcal{I} \) and \( \mathcal{J} \) can be expressed in terms of the eigenvalues by

\[
\mathcal{I} = \frac{1}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)
\]

and

\[
\mathcal{J} = \frac{1}{2} \lambda_1 \lambda_2 \lambda_3.
\]
In what follows we assume $\lambda_1, \lambda_2, \lambda_3 \neq 0$. The Petrov-type D is characterized by the condition $\lambda_1 = \lambda_2$. Using expressions (3.2a) and (3.2b), one has that the remaining root satisfies the equation

$$\lambda_3^3 - 2I \lambda_3 + 4J = 0. \quad (3.3)$$

Combining equations (3.1) and (3.3) one finds that

$$\lambda_3 = 6JI^{-1}.$$ 

For a Petrov-type D spacetime, there exist spinors (the principal spinors) $\alpha_A$, $\beta_A$ satisfying the normalization $\alpha_A \beta_A^* = 1$ such that

$$\Psi_{ABCD} = \psi(\alpha_A \alpha_B \beta_C \beta_D) \quad \text{and} \quad \psi = -3\lambda_3 = -18JI^{-1}. \quad (3.4)$$

It will be convenient to define the spinor $v_{AB} = \alpha(A \beta_B)$. Observe that because of our normalization conditions one has that $v_{AB} v^{AB} = -\frac{1}{2}$. Using the spinor $v_{AB}$ one obtains the following alternative expression for $\Psi_{ABCD}$:

$$\Psi_{ABCD} = \psi(v_{AB} v_{CD} + \frac{1}{6} h_{ABCD}) \quad \text{and} \quad h_{ABCD} \equiv -\epsilon_A(\epsilon_C \epsilon_D) B. \quad (3.5)$$

The $h_{ABCD}$ term is chosen to compensate for the traces of $v_{AB} v_{CD}$ so the right-hand side is tracefree—and thus completely symmetric. The formula can be verified by making an irreducible decomposition or by substituting

$$\epsilon_{AB} = \alpha_A \beta_B - \beta_A \alpha_B \quad \text{and} \quad v_{AB} = \alpha(A \beta_B)$$

in equation (3.5).

The expression (3.5) can be used to obtain a formula for the spinor $v_{AB}$ in terms of the Weyl spinor $\Psi_{ABCD}$. Let $\zeta_{AB}$ denote a non-vanishing symmetric spinor. Contracting (3.5) with an arbitrary spinor $v_{CD}^*$ one obtains

$$\Psi_{ABCD} \zeta_{CD} = \psi(v_{AB} v_{PQ} \zeta_{PQ}^* + \frac{1}{6} \zeta_{AB}) \quad (3.6a)$$

and

$$\Psi_{ABCD} \zeta_{AB} = \psi((v_{PQ} \zeta_{PQ}^*)^2 + \frac{1}{6} v_{PQ} \zeta_{PQ}^*). \quad (3.6b)$$

Using equation (3.6a) to solve for $v_{AB}$ and equation (3.6b) to solve for $v_{PQ} \zeta_{PQ}^*$ one obtains the following formula for $v_{AB}$ in terms of $\Psi_{ABCD}$ and the arbitrary spinor $\zeta_{AB}$

$$v_{AB} = \Xi^{-1/2}(\psi^{-1} \Psi_{ABPQ} \zeta_{PQ} - \frac{1}{6} \zeta_{AB}) \quad (3.7)$$

with

$$\Xi \equiv \psi^{-1} \Psi_{PQRS} \zeta_{RS}^{PQ} - \frac{1}{6} \zeta_{PQ} \zeta_{PQ}^*. \quad (3.8)$$

In the last formulae it is assumed that $\zeta_{AB}$ is chosen such that

$$\Xi \neq 0.$$
(b) The Killing spinor of a Petrov-type D spacetime

Let $\kappa_{AB}$ be a solution to the Killing spinor equation (2.1). An important property of a Killing spinor is that

$$\xi_{AA'} \equiv \nabla^Q A' \kappa_{AQ}$$

satisfies the (spinorial version of the) Killing vector equation

$$\nabla_{AA'} \xi_{BB'} + \nabla_{BB'} \xi_{AA'} = 0.$$ 

In general, the Killing vector $\xi_{AA'}$ given by formula (3.9) is complex—that is, it encodes the information of two real Killing vectors. This property is closely related to the fact that all vacuum-type D spacetimes admit, at least, a pair of commuting Killing vectors—e.g. Kinnersley (1969). Vacuum spacetimes of Petrov-type D for which $\xi_{AA'}$ is real are called generalized Kerr-NUT (Newman–Unti–Tamburino) spacetimes.

Every vacuum spacetime of Petrov-type D has a Killing spinor—see Penrose & Rindler (1986) and references therein. Indeed, in the notation of the previous section, one has that

$$\kappa_{AB} = \psi^{-1/3} v_{AB}$$

satisfies equation (2.1). Using formula (3.7), one obtains the following result.

Proposition 3.1. Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum spacetime. If on $\mathcal{U} \subset \mathcal{M}$, the spacetime is of Petrov-type D and $\zeta_{AB}$ is a symmetric spinor satisfying

$$\zeta_{AB} \neq 0, \quad \psi^{-1} \Psi_{PQRS} \zeta^{PQ} \zeta^{RS} - \frac{1}{6} \zeta_{PQ} \zeta^{PQ} \neq 0$$
on $\mathcal{U}$,

then

$$\kappa_{AB} = \psi^{-1/3} \Xi^{-1/2} (\psi^{-1} \Psi_{ABPQ} \zeta^{PQ} - \frac{1}{6} \zeta_{AB}),$$

with $\Xi$ given by equation (3.8) is a Killing spinor on $\mathcal{U}$. The formula (3.11) is independent of the choice of $\zeta_{AB}$.

That expression (3.10) is independent of the choice of $\zeta_{AB}$ can be verified by writing

$$\zeta_{AB} = \zeta_0 \alpha_A \alpha_B + \zeta_1 (\alpha_A \beta_B) + \zeta_2 \beta_A \beta_B,$$

where $\{\alpha_A, \beta_A\}$ is the dyad given by equation (3.4). Substituting the latter into equation (3.11) one readily obtains (3.10).

Observation 3.2. Formula (3.11) can be evaluated for any vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$. In general, of course, it will not give a solution to the Killing spinor equation (2.1). The resulting spinor $\kappa_{AB}$ will depend upon the choice of $\zeta_{AB}$. We make the following definition:

Definition 3.3. Let $(\mathcal{M}, g_{\mu\nu})$ be a vacuum spacetime. Consider $\mathcal{U} \subset \mathcal{M}$ and on $\mathcal{U}$ a symmetric spinor $\zeta_{AB}$ satisfying

$$\zeta_{AB} \neq 0, \quad \psi^{-1} \Psi_{PQRS} \zeta^{PQ} \zeta^{RS} - \frac{1}{6} \zeta_{PQ} \zeta^{PQ} \neq 0$$
on $\mathcal{U}$.

The symmetric spinor given by

$$\tilde{\kappa}_{AB} = \psi^{-1/3} \Xi^{-1/2} (\psi^{-1} \Psi_{ABPQ} \zeta^{PQ} - \frac{1}{6} \zeta_{AB})$$

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with
\[ \Xi \equiv \psi^{-1} \psi_{PQRS} \zeta^{PQ} \zeta^{RS} - \frac{1}{6} \zeta_{PQ} \zeta^{PQ}, \]
will be called the \( \zeta_{AB} \)-Killing spinor candidate on \( \mathcal{U} \).

**Remark 3.4.** Although the choice of \( \zeta_{AB} \) is essentially arbitrary, as it will be seen, in many applications there is a natural choice.

**Remark 3.5.** The choice of branch cut for the square root of \( \Xi \) can be chosen to be \( \{-re^{i\theta} : r > 0\} \), where \( \theta \) is the argument of \( \psi^{-1} \psi_{PQRS} \zeta^{PQ} \zeta^{RS} = \frac{1}{6} \zeta_{PQ} \zeta^{PQ} \).

### 4. A boundary value problem for the approximate Killing spinor equation

In this section, we formulate a boundary value problem for the approximate Killing spinor equation (2.3) on a 3-manifold \( S \approx \mathbb{R}^3 \setminus B_1 \). As discussed in §1, this type of 3-manifold can be thought of as a Cauchy hypersurface in the domain of outer communication of a black hole or the exterior of a star. For simplicity of the presentation, it will be assumed that the initial data \((S, h_{ab}, K_{ab})\) satisfy in its asymptotic region the behaviour
\[
h_{ij} = -\left(1 + \frac{2m}{r}\right) \delta_{ij} + o_\infty(r^{-3/2}) \quad (4.1a)
\]
and
\[
K_{ij} = o_\infty(r^{-5/2}), \quad (4.1b)
\]
with \( r = ((x^1)^2 + (x^2)^2 + (x^3)^2)^{1/2} \), and \((x^1, x^2, x^3)\) are asymptotically Cartesian coordinates and \( m \) denotes the Arnowitt–Deser–Misner mass. Our present discussion could be extended at the expense of more technical details to include the case of boosted initial datasets (e.g. Bäckdahl & Valiente Kroon (2010b)). Here, and in what follows, the fall off conditions of the various fields will be expressed in terms of weighted Sobolev spaces \( H^s_\beta \), where \( s \) is a non-negative integer and \( \beta \) is a real number. Here, we use the conventions for these spaces given in Bartnik (1986)—see also Bäckdahl & Valiente Kroon (2010b). We say that \( \eta \in H^s_\beta \) if \( \eta \in H^s_\beta \) for all \( s \). Thus, the functions in \( H^\infty_\beta \) are smooth over \( S \) and have a fall off at infinity such that \( \partial^l \eta = o(r^{-|l|}) \). We will often write \( \eta = o_\infty(r^\beta) \) for \( \eta \in H^\infty_\beta \) at the asymptotic end.

Following the ideas of Bäckdahl & Valiente Kroon (2010a,b), we shall look for solutions to the approximate Killing spinor equation (2.3) which expressed in terms of an asymptotically Cartesian frame and coordinates has an asymptotic behaviour and is given by
\[
\kappa_{AB} = -\frac{\sqrt{2}}{3} \left(1 + \frac{2m}{r}\right) x_{AB} + o_\infty(r^{-1/2}) \quad (4.2)
\]
with
\[
x_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix}
-x^1 + ix^2 & x^3 \\
x^3 & x^1 + ix^2
\end{pmatrix}.
\]
(a) Behaviour at the inner boundary

The ideas of §3 will be used to prescribe the value of the spinor $\kappa_{AB}$ on the boundary $\partial S$. In general, one could choose to prescribe any value for $\kappa_{AB}$ on the boundary $\partial S$ as long as this choice coincides with the correct value in the exact Kerr case. However, one would like to have a coordinate independent choice. This requires constructing the candidate from geometrical objects. Equation (3.12) gives a good choice with these properties provided its normalization is adjusted so it coincides with the correct value in the exact Kerr case. Note that equation (3.12) contains an arbitrary non-vanishing spinor $\zeta_{AB}$. In principle, one could choose any values for $\zeta_{AB}$ as long as the conditions in definition 3.2 are satisfied. If the choice of $\zeta_{AB}$ is to depend only on the geometry of the problem, then the spinorial counterpart, $n_{AB}$, of the normal to the hypersurface $\partial S$, $n_a$, is the only natural choice. By convention, $n_a$ is assumed to point outside $S$ (outward pointing). Note that because of the use of a negative definite 3-metric one has that $n_{PQ}n^{PQ} = -1$. Hence, the condition $\zeta_{AB} \neq 0$ is trivially satisfied if one uses $\zeta_{AB} = n_{AB}$.

**Remark 4.1.** It is, in principle, natural to suspect that there are other choices of Dirichlet boundary conditions that allow us to define a geometric invariant of the type we are interested in. However, we are not aware of any other explicit choice besides the one discussed here.

It will be convenient to define the following set

$$Q \equiv \{ z \in \mathbb{C} \mid z = \Xi(p), \quad p \in \partial S \}$$

with

$$\Xi = \psi^{-1} \psi_{PQRS} n^{PQ} n^{RS} - \frac{1}{6} n_{PQ} n^{PQ}.$$

We shall make the following technical assumption on the initial dataset $(S, h_{ab}, K_{ab})$:

**Assumption 4.2.** The initial dataset $(S, h_{ab}, K_{ab})$ is such that $\Xi$ is a smooth function over $\partial S$ satisfying

(i) $0 \notin Q$;
(ii) $Q$ does not encircle the point $z = 0$.

As a consequence of assumption 4.2 one can choose a cut of the square root function on the complex plane such that $\Xi^{-1/2}(p)$ is smooth for all $p \in \partial S$.

**Remark 4.3.** The conditions in assumption 4.2 are satisfied by standard Kerr data (in Boyer–Lindquist coordinates) at the horizon. Furthermore, by construction if the data $(S, h_{ab}, K_{ab})$ are data for the Kerr spacetime, then the boundary data given by the Killing spinor candidate formula in definition 3.2 give the right boundary behaviour for the restriction of its Killing spinor to $S$.

**Remark 4.4.** In order to match the asymptotic behaviour of the $n_{AB}$-Killing spinor candidate given by definition 3.3 with that given by equation (4.2) one needs to incorporate a normalization factor to equation (3.12). To this end,
it is noticed that the decay conditions (4.1a) and (4.1b) imply the asymptotic expansions

\[ \psi_{ABCD} = \frac{3mx_{AB}x_{CD}}{r^5} + o_\infty(r^{-7/2}) \text{ and } \psi = \frac{6m}{r^3} + o_\infty(r^{-7/2}). \]

Now, if the 2-surface \( \partial S \) is sent to infinity in such a way that

\[ n_{AB} = \frac{x_{AB}}{r} + o_\infty(r^{-1/2}), \]

so that \( \partial S \) becomes more and more like a 2-sphere, one finds that

\[ \Xi = \frac{1}{3} + o_\infty(r^{-1/2}). \]

Hence, the leading term of the \( n_{AB} \)-Killing spinor candidate shown in equation (3.12) is given by

\[ \tilde{k}_{AB} = -\frac{x_{AB}}{6^{1/3}3^{1/6}m^{1/3}} + o_\infty(r^{1/2}). \]

Thus, in order to have a Killing spinor candidate whose asymptotic behaviour agrees with that of equation (4.2), one needs to consider the normalized expression

\[ \tilde{k}'_{AB} = -\left( \frac{2^{5/6}m^{1/3}}{3^{1/6}} \right) \psi^{-1/3} \Xi^{-1/2} \left( \psi^{-1}\psi_{ABPQ}n^{PQ} - \frac{1}{6} n_{AB} \right). \]

(b) Existence of solutions to the approximate Killing spinor equation

Following the strategy put forward in Bäckdahl & Valiente Kroon (2010a,b), we provide an Ansatz for a solution to the approximate Killing spinor equation (2.3) that encodes the desired behaviour at infinity. To this end, let

\[ \tilde{k}'_{AB} \equiv -\frac{\sqrt{2}}{3} \left( 1 + \frac{2m}{r} \right) x_{AB}\phi_R(r), \]

where \( \phi_R \) is a smooth cut-off function such that for \( R > 0 \) large enough

\[ \phi_R(r) = 1, \quad r \gg R \]

and

\[ \phi_R(r) = 0, \quad r < R. \]

One then has the following result:

**Theorem 4.5.** Let \( (S, h_{ab}, K_{ab}) \) be an initial dataset for the Einstein vacuum field equations such that \( S \) is a manifold with a smooth boundary \( \partial S \approx S^2 \) satisfying assumption 4.1. Assume that \( (h_{ab}, K_{ab}) \) satisfy the asymptotic conditions (4.1a) and (4.1b) with \( m \neq 0 \). Then, there exists a unique smooth solution, \( \kappa_{AB} \), to the approximate Killing equation (2.3) with behaviour at the asymptotic end of the form (4.2) and with boundary value at \( \partial S \) given by the \( n_{AB} \)-Killing spinor candidate \( \tilde{k}'_{AB} \) of equation (4.3).
Proof. Following the procedure described in Bäckdahl & Valiente Kroon (2010a,b), we consider the Ansatz

\[ \kappa_{AB} = \tilde{\kappa}_{AB} + \theta_{AB}, \quad \theta_{AB} \in H^2_{-1/2}. \]  

(4.5)

The substitution of the latter into equation (2.3) renders the following equation for the spinor \( \theta_{AB} \):

\[ \mathbf{L}\theta_{CD} = -\mathbf{L}^\#\kappa_{CD}. \]  

(4.6)

In view that \( \tilde{\kappa}_{AB} \) vanishes outside the asymptotic region, then the value of \( \theta_{AB} \) at \( \partial S \) coincides with that of \( \kappa_{AB} \). That is, we set

\[ \theta_{AB}|_{\partial S} = \tilde{\kappa}_{AB}'. \]  

(4.7)

By construction it follows that

\[ \nabla_{(AB}\kappa_{CD)} \in H^{\infty}_{-3/2}, \]  

so that

\[ F_{CD} \equiv -\mathbf{L}^\#\kappa_{CD} \in H^{\infty}_{-5/2}. \]

The operator associated with the Dirichlet elliptic boundary value problem (4.6) and (4.7) is given by \((\mathbf{L}, \mathbf{B})\), where \( \mathbf{B} \) denotes the Dirichlet boundary operator on \( \partial S \). As discussed in Bäckdahl & Valiente Kroon (2010a,b), under assumptions (4.1a) and (4.1b), the operator \( \mathbf{L} \) is asymptotically homogeneous—see appendix A for a concise summary of the ideas and results of the theory elliptic systems being used here. Now, elliptic boundary value problems with Dirichlet boundary conditions satisfy the Lopatinski–Shapiro compatibility conditions—see Wloka et al. (1995). Consequently, the operator \((\mathbf{L}, \mathbf{B})\) is L-elliptic and the map

\[ (\mathbf{L}, \mathbf{B}) : H^2_{-1/2}(S) \rightarrow H^0_{0} \times H^{3/2}_{-5/2}(\partial S) \]

is Fredholm—see theorem A.1 of appendix A. The rest of the proof is an application of the Fredholm alternative. Using proposition A.2 with \( \delta = -1/2 \), one concludes that equation (4.6) has a unique solution if \( F_{AB} \) is orthogonal to all \( \nu_{AB} \in H^0_{-1/2} \) in the Kernel of \( \mathbf{L}^\# = \mathbf{L} \) with \( \nu_{AB} = 0 \) on \( \partial S \). If \( \mathbf{L}\nu_{AB} = 0 \), then an integration by parts shows that

\[ \int_{S} \nabla^{(AB}\nu_{CD)} \nabla_{AB}\nu_{CD} \, d\mu = \int_{\partial S} n^{AB}\nu_{CD} \nabla_{(AB}\nu_{CD)} \, dS + \int_{\partial S_{\infty}} n^{AB}\nu_{CD} \nabla_{(AB}\nu_{CD)} \, dS, \]

where \( \partial S_{\infty} \) denotes the sphere at infinity. The boundary integral over \( \partial S \) vanishes because of \( \nu_{AB} \in \text{Ker}(\mathbf{L}, \mathbf{B}) \), so that \( \nu_{AB} = 0 \) on \( \partial S \). As \( \nu_{AB} \in H^2_{-1/2} \) by assumption, it follows that \( \nabla_{(AB}\nu_{CD)} \) \( \in H^{\infty}_{-3/2} \) and furthermore that \( n^{AB}\nu_{CD} \nabla_{(AB}\nu_{CD)} = o(r^{-2}) \).

An integral over a finite sphere will then be of type \( o(1) \). Thus, the integral over \( \partial S_{\infty} \) vanishes. Hence, one concludes that

\[ \nabla_{(AB}\nu_{CD)} = 0, \quad \text{on } S. \]

Using the same methods as in Bäckdahl & Valiente Kroon (2010b, proposition 21) one finds that there are no non-trivial solutions to the spatial Killing spinor equation that go to 0 at infinity. Thus, there are no restrictions on \( F_{AB} \) and

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equation (4.6) has a unique solution as desired. Owing to elliptic regularity, any $H_{-1/2}^2$ solution to equation (4.6) is in fact a $H_{-1/2}^\infty$ solution —cf. lemma A.3. Thus, $\theta_{AB}$ is smooth. ■

**Remark 4.6.** It is worth mentioning that similar methods can be used to obtain solutions to the approximate Killing spinor equation on annular domains of the form $\mathcal{A} = \overline{B_{R_2}} \setminus \overline{B_{R_1}}$, where $R_2 > R_1$. Again, one would use the Killing spinor candidate of definition 3.2 to provide boundary value data on the two components of $\partial \mathcal{A}$. This type of construction is of potential relevance in the nonlinear stability of the Kerr spacetime and in the numerical evaluation of the non-Kerrness.

### 5. The geometric invariant

In this section, we show how the approximate Killing spinor $\kappa_{AB}$ obtained from theorem 4.5 can be used to construct an invariant measuring the non-Kerrness of the 3-manifold with boundary $\mathcal{S}$. To this end, we recall the following lemma from Bäckdahl & Valiente Kroon (2010a):

**Lemma 5.1.** The approximate Killing spinor equation (2.3) is the Euler–Lagrange equation of the functional

$$J \equiv \int_{\mathcal{S}} \nabla_{(AB}k_{CD)} \nabla^{AB}k^{CD} \, d\mu.$$  

In what follows, it will be assumed that $\kappa_{AB}$ is the solution to equation (2.3) given by theorem 4.4. Furthermore, let

$$I_1 \equiv \int_{\mathcal{S}} \Psi_{(ABC}^{\xi F}k_{D)F} \hat{\Psi}^{ABC}G^{\xi} k^{CD} \, d\mu \tag{5.2a}$$

and

$$I_2 \equiv \int_{\mathcal{S}} (3k_{(A}^{E} \nabla_{B}^{F} \Psi_{CD)EF} + \Psi_{(ABC}^{F} \xi_{D)F})$$

$$\times (3k_{AP}^{AB} \nabla_{BQ} \Psi_{CD)PQ} + \hat{\Psi}^{ABCP} \xi_{D)P}) \, d\mu. \tag{5.2b}$$

The geometric invariant is then defined by

$$I \equiv J + I_1 + I_2. \tag{5.3}$$

**Remark 5.2.** It can be verified that $I$ is coordinate independent. Furthermore, if the initial dataset satisfies the decay conditions (4.1a) and (4.1b), then $I$ is finite.

The desired characterization of Kerr data on 3-manifolds $\mathcal{S}$ with boundary and one asymptotic end is given by the following theorem.

**Theorem 5.3.** Let $(\mathcal{S}, h_{ab}, K_{ab})$ be an initial dataset for the Einstein vacuum field equations such that $\mathcal{S}$ is a manifold with boundary $\partial \mathcal{S} \approx \mathbb{S}^2$ satisfying Assumption 4.1. Furthermore, assume that $\mathcal{S}$ has only one asymptotic end and that the asymptotic conditions (4.1a) and (4.1b) are satisfied with $m \neq 0$. Let $I$ be the invariant defined by equations (5.1), (5.2a), (5.2b) and (5.3), where $\kappa_{AB}$ is given as the only solution to equation (2.3) with asymptotic behaviour given by
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(4.4) and with boundary value at \( \partial S \) given by the \( n_{AB} \)-Killings spinor candidate \( \kappa'_{AB} \) of equation (4.3), where \( n_{AB} \) is the outward pointing normal to \( \partial S \). The invariant \( I \) vanishes if and only if \( (S, h_{ab}, K_{ab}) \) is an initial dataset for the Kerr spacetime.

The proof of this result is analogous to the one given in Bäckdahl & Valiente Kroon (2010a,b) and will be omitted.

**Remark 5.4.** In the previous theorem, for an initial dataset for the Kerr spacetime it will be understood that \( D(S) \) (the union of the past and future domains of dependence of \( S \)) is isometric to a portion of the Kerr spacetime. In order to make stronger assertions about \( D(S) \), one needs to provide more information about \( \partial S \). For example, if it can be asserted that \( \partial S \) coincides with the intersection of the past and future components of a non-expanding horizon, then theorem 5.3 will give that \( D(S) \) is the domain of outer communication of the Kerr spacetime.

6. Conclusions and outlook

Theorem 5.3 and the methods developed in the present articles are expected to be of relevance in several outstanding problems concerning the Kerr spacetime: a proof of the uniqueness of stationary black holes that does not assume analyticity of the horizon, and whether the Kerr solution can describe the exterior of a rotating star. The boundary value problem discussed in the present paper might also play a role in applications of Killing spinor methods to the nonlinear stability of the Kerr spacetime and in the evaluation of the non-Kerarness in slices of numerically computed black hole spacetimes. The motivation for some of these claims is briefly discussed in the next paragraphs.

For the problem of the uniqueness of stationary black holes, as mentioned in the remark after theorem 5.3, one would like to consider slices in the domain of outer communication of a stationary black hole that intersects the intersection of the two components of the non-expanding horizon. One then would have to analyse the consequences that the existence of this type of boundary have on the Killing spinor candidate constructed out of the normal to \( \partial S \)—the Weyl tensor is known to be of type D on non-expanding horizons (Ionescu & Klainerman 2009). The main challenge in this approach is to find a convenient way of relating the \textit{a priori} assumption about stationarity made in the problem of uniqueness of black holes with the \textit{Killing vector initial data candidate} \( (\xi, \xi_{AB}) \) provided by the solution, \( \kappa_{AB} \), to the approximate Killing spinor equation (2.3).

With regards to the problem of the existence of an interior solution for the Kerr spacetime, the key question to be analysed is what kind of conditions on the boundary \( \partial S \) needs to be prescribed to ensure that the solution to the approximate Killing spinor equation (2.3) given by theorem 4.4 renders a vanishing invariant \( I \). It is to be expected that these conditions will impose strong restrictions to the type of matter models describing a hypothetical interior solution.

Finally, in what concerns numerical simulations of black hole spacetimes and the non-linear stability of the Kerr solution, the key issue is the behaviour of the geometric invariant upon time evolution. If some type of monotonic behaviour along a foliation of spacetime can be established, then our invariant could be a valuable tool for the investigation of the dynamics of the gravitational field.
The ideas touched upon in the previous paragraphs will be further elaborated in future works.

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Appendix A. Elliptic results for slices in the domain of outer communication of a black hole

In this appendix, we summarize the results on the theory of boundary value problems for elliptic systems that have been used in the present paper. The presentation is adapted from Lockhardt & McOwen (1985).

As in the main text, let \( S \) denote a three-dimensional manifold with the topology of \( \mathbb{R}^3 \setminus B_1 \), where \( B_1 \) denotes the open ball of radius 1. Note that \( S \) is closed. Assume \( \partial S \approx S^2 \) to be \( C^\infty \). In what follows, let \( u \) denote an \( N \)-dimensional vector-valued function over \( S \). Following Cantor (1981) and Lockhart (1981), a second-order elliptic operator \( L \) acting on \( u \) will be said to be asymptotically homogeneous if it can be written in the form

\[
L u(x) = (a_{ij}^{\infty} + a_{ij}^i(x)) D_i D_j u(x) + a^i(x) D_i u(x) + a(x) u(x), \quad x \in S,
\]

where \( a_{ij}^{\infty} \) denotes a matrix with constant coefficients while \( a_{ij}^i, a^i \) and \( a \) are matrix-valued functions of the coordinates such that

\[
a_{ij}^i \in H^\infty_{-1/2}(S), \quad a^i \in H^\infty_{-3/2}(S), \quad a \in H^\infty_{-5/2}(S).
\]

On \( \partial S \) we will consider the homogeneous Dirichlet operator \( B \) given by

\[
B u(y) = u(y), \quad y \in \partial S.
\]

The combined operator \((L, B)\) is said to be \( L\)-elliptic if \( L \) is elliptic on \( S \) and \((L, B)\) satisfies the Lopatinski–Shapiro compatibility conditions—see Wloka et al. (1995) for detailed definitions. Crucial for our purposes is that if \( L \) is elliptic and \( B \) is the Dirichlet boundary operator, then the Lopatinski–Shapiro conditions are satisfied and thus \((L, B)\) is \( L\)-elliptic—see again Wloka et al. (1995, theorem 10.7).

The Fredholm properties for the combined operator \((L, B)\) follow from Lockhardt & McOwen (1985, theorem 6.3)—cf. similar results in Klenk (1991); Reula (1989). Bartnik’s conventions are used for the weights of the Sobolev spaces \( H^s_\delta \)—see Bartnik (1986).

**Theorem A.1.** Let \( L \) denote a smooth second-order asymptotically homogeneous operator on \( S \approx \mathbb{R}^3 \setminus B_1 \). Furthermore, let \( \partial S \) be smooth, and let \( B \) denote the Dirichlet boundary operator. Then for \( \delta \) not a negative integer, \( s \geq 2 \) the map

\[
(L, B) : H^s_\delta(S) \to H^{s-2}_\delta(S) \times H^{s-1/2}(\partial S)
\]

is Fredholm.

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The same arguments used in Cantor (1981, theorem 6.3) then allow us to provide the following version of the Fredholm alternative:

**Proposition A.2.** Let \((\mathbf{L}, \mathbf{B})\) as in theorem A.1. Given \(\delta\) not a negative integer, the boundary value problem

\[
\mathbf{L}u(x) = f(x), \quad f \in H^0_{\delta-2}(\mathcal{S}), \quad x \in \mathcal{S}
\]

and

\[
u(y) = g(y), \quad g \in H^0(\partial \mathcal{S}), \quad y \in \partial \mathcal{S}
\]

has a solution \(u \in H^2_\delta(\mathcal{S})\) if

\[
\int_{\mathcal{S}} f \cdot v \, d\mu = 0,
\]

for all \(v \in H^0_{1-\delta}(\mathcal{S})\) such that

\[
\mathbf{L}^* v(x) = 0, \quad x \in \mathcal{S}
\]

and

\[
v(y) = 0, \quad y \in \partial \mathcal{S},
\]

where \(\mathbf{L}^*\) denotes the formal adjoint of \(\mathbf{L}\).

Finally, we note the following lemma—cf. Lockhardt & McOwen (1985, eqn (1.13)).

**Lemma A.3.** Let \((\mathbf{L}, \mathbf{B})\) as in theorem A.1. Then for any \(\delta \in \mathbb{R}\) and any \(s \geq 2\), there exists a constant \(C\) such that for every \(u \in H^{s}_{\text{loc}} \cap H^0_{\delta}(\mathcal{S})\), the following inequality holds

\[
||u||_{H^s_\delta(\mathcal{S})} \leq C(||\mathbf{L}u||_{H^{s-2}_{\delta-2}(\mathcal{S})} + ||\mathbf{B}u||_{H^{s-1/2}(\partial \mathcal{S})} + ||u||_{H^{s}_{\delta-2}(\mathcal{S})}).
\]

In this lemma, \(H^s_{\text{loc}}\) denotes the local Sobolev space. That is, \(u \in H^s_{\text{loc}}\) if for an arbitrary smooth function \(v\) with compact support, \(uv \in H^s\).

**Remark A.4.** If \(\mathbf{L}\) has smooth coefficients and \(\mathbf{L}u = 0\), then it follows that all the \(H^s_{\delta}(\mathcal{S})\) norms of \(u\) are bounded by the \(H^0_{\delta}(\mathcal{S})\) and the \(H^{s-1/2}(\partial \mathcal{S})\) norms. Thus, it follows that if a solution to the boundary value problem exists and the boundary data are smooth, then the solution must be, in fact, smooth—elliptic regularity.

**Appendix B.** An improved characterization of the Kerr spacetime by means of Killing spinors

In Bäckdahl & Valiente Kroon (2010b) a characterization of the Kerr spacetime by means of Killing spinors was given. This characterization contains an \textit{a priori} assumption on the Weyl tensor—namely, that it is nowhere of type N or D. The purpose of the present appendix is to show that these assumptions can be removed.

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As in the main text, let $\kappa_{AB}$ denote a totally symmetric spinor. Let

$$\xi_{AA'} \equiv \nabla^Q A' \kappa_{AQ}.$$ 

If $\kappa_{AB}$ is a solution to the Killing spinor equation (2.1), then $\xi_{AA'}$ satisfies the Killing equation

$$\nabla_{AA'} \xi_{BB'} + \nabla_{BB'} \xi_{AA'} = 0.$$ 

In general, $\xi_{AA'}$ will be a complex Killing vector. The Killing form $F_{AA'BB'}$ associated with $\xi_{AA'}$ is defined by

$$F_{AA'BB'} \equiv \frac{1}{2} (\nabla_{AA'} \xi_{BB'} - \nabla_{BB'} \xi_{AA'}).$$

In the cases where $\xi_{AA'}$ is real, we will consider the self-dual Killing form $\mathcal{F}_{AA'BB'}$ defined by

$$\mathcal{F}_{AA'BB'} \equiv \frac{1}{2} (F_{AA'BB'} + iF^*_{AA'BB'}),$$

where $F^*_{AA'BB'}$ is the Hodge dual of $F_{AA'BB'}$. Owing to the symmetries of the self-dual Killing form one has that

$$\mathcal{F}_{AA'BB'} = \mathcal{F}_{AB} \varepsilon_{A'B'} \quad \text{and} \quad \mathcal{F}_{AB} \equiv \frac{1}{2} F_{ABQ} Q' = \mathcal{F}_{BA}.$$ 

The characterization of the Kerr spacetime discussed in Bäckdahl & Valiente Kroon (2010b) is, in turn, based on the following characterization proved by Mars (2000).

**Theorem B.1 (Mars 1999, 2000).** Let $(M, g_{\mu\nu})$ be a smooth vacuum spacetime with the following properties:

(i) $(M, g_{\mu\nu})$ admits a real Killing vector $\xi_{AA'}$ such that the spinorial counterpart of the Killing form of $\xi_{AA'}$ satisfies

$$\Psi_{ABCD} \mathcal{F}^{PQ} = \varphi \mathcal{F}_{AB},$$ 

with $\varphi$ a scalar;

(ii) $(M, g_{\mu\nu})$ contains a stationary asymptotically flat 4-end, and $\xi_{AA'}$ tends to a time translation at infinity and the Komar mass of the asymptotic end is non-zero.

Then $(M, g_{\mu\nu})$ is locally isometric to the Kerr spacetime.

**Remark B.2.** A stationary asymptotically flat 4-end is an open submanifold $M_\infty \subset M$ diffeomorphic to $I \times (\mathbb{R}^3 \setminus B_R)$, where $I \subset \mathbb{R}$ is an open interval and $B_R$ is a closed ball of radius $R$ such that in local coordinates $(t, x^i)$ defined by the diffeomorphism, the metric satisfies

$$|g_{\mu\nu} - \eta_{\mu\nu}| + |r \partial_t g_{\mu\nu}| \leq Cr^{-\alpha}, \quad \partial_t g_{\mu\nu} = 0,$$

with $C, \alpha \geq 1$ constants, $\eta_{\mu\nu}$ the Minkowski metric and

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

In this context, the notions of Komar and ADM mass coincide.
We want to relate the notion of Killing form and that of Killing spinors. As discussed in Bäckdahl & Valiente Kroon (2010b), if $\xi_{AA'}$ is real, the commutators for a vacuum spacetime readily yield that

$$\mathcal{F}_{AB} = \frac{3}{4} \Psi_{ABPQ} k^{PQ}. \quad (B\,2)$$

Now, vacuum spacetimes admitting a Killing spinor, $\kappa_{AB}$, can only be of Petrov-type D, N or O. If the spacetime is of type O at some point (so that $\Psi_{ABCD} = 0$), then theorem B.2 shows that $\mathcal{F}_{AB} = 0$, and the relation (B1) is trivially satisfied. If the spacetime is of Petrov-type N, then $\kappa_{AB}$ has a repeated principal spinor that coincides with the repeated principal spinor of $\Psi_{ABCD}$—e.g. Jeffryes (1984). Hence, again one has that $\mathcal{F}_{AB} = 0$, and theorem B.1 is satisfied trivially. For Petrov-type D spacetimes with a Killing spinor such that $\xi_{AA'}$ is real, it has already been shown in Bäckdahl & Valiente Kroon (2010b) that theorem B.1 is satisfied.

From the discussion in the previous paragraph, we obtain the following characterization of the Kerr spacetime in terms of Killing spinors.

**Theorem B.3.** A smooth vacuum spacetime $(\mathcal{M}, g_{\mu\nu})$ is locally isometric to the Kerr spacetime if and only if the following conditions are satisfied:

(i) there exists a Killing spinor $\kappa_{AB}$ such that the associated Killing vector $\xi_{AA'}$ is real;

(ii) the spacetime $(\mathcal{M}, g_{\mu\nu})$ has a stationary asymptotically flat 4-end with non-vanishing mass in which $\xi_{AA'}$ tends to a time translation.

As a consequence of this theorem, the *a priori* conditions on the Petrov type of the Weyl required in Bäckdahl & Valiente Kroon (2010b, theorem 28) can be dropped.

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