The membership problem for constant-sized quantum correlations is undecidable

Honghao Fu *1, Carl A. Miller †1,2, and William Slofstra ‡3

1Joint Institute for Quantum Information and Computer Science, University of Maryland, College Park, MD, 20740, USA
2National Institute of Standards and Technology, 100 Bureau Dr., Gaithersburg, MD 20899, USA
3Institute for Quantum Computing and Department of Pure Mathematics, University of Waterloo, Waterloo, Canada

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Abstract

When two spatially separated parties make measurements on an unknown entangled quantum state, what correlations can they achieve? How difficult is it to determine whether a given correlation is a quantum correlation? These questions are central to problems in quantum communication and computation. Previous work has shown that the general membership problem for quantum correlations is computationally undecidable. In the current work we show something stronger: there is a family of constant-sized correlations — that is, correlations for which the number of measurements and number of measurement outcomes are fixed — such that solving the quantum membership problem for this family is computationally impossible. Thus, the undecidability that arises in understanding Bell experiments is not dependent on varying the number of measurements in the experiment. This places strong constraints on the types of descriptions that can be given for quantum correlation sets. Our proof is based on a combination of techniques from quantum self-testing and undecidability results for linear system nonlocal games.

* h7fu@umd.edu
† camiller@umd.edu
‡ weslofst@uwaterloo.ca
1 Introduction

Suppose two spatially separated parties, say Alice and Bob, are each able to perform different measurements on their local system. If Alice can perform \( n_A \) different measurements, each with \( m_A \) outcomes, and Bob can perform \( n_B \) different measurements, each with \( m_B \) outcomes, then from the point of view of an outside observer, their behaviour is captured by the collection

\[
P = \{ P(a, b|x, y) : 0 \leq a < m_A, 0 \leq b < m_B, 0 \leq x < n_A, 0 \leq y < n_B \}
\]

where \( P(a, b|x, y) \) is the probability that Alice measures outcome \( a \) and Bob measures outcome \( b \), given that Alice performs measurement \( x \) and Bob performs measurement \( y \). The collection \( P \) is called a correlation (matrix) or behaviour \([\text{Tsi93}]\). Colloquially, the size of a correlation is given by the tuple \((n_A, n_B, m_A, m_B)\).

It is natural to ask which correlations can occur in nature. Suppose measurement \( x \) on Alice’s system always gives outcome \( c_x \), and measurement \( y \) on Bob’s system always gives outcome \( d_y \). Then the corresponding correlation is

\[
P(a, b|x, y) = \delta_{a,c_x}\delta_{b,d_y},
\]

where \( \delta \) is the Kronecker delta. Correlations of this form are called deterministic correlations. The convex hull of the set of deterministic correlations is denoted by \( C_c(n_A, n_B, m_A, m_B) \), or \( C_c \) when the tuple \((n_A, n_B, m_A, m_B)\) is clear. Correlations in \( C_c \) are called classical correlations. All deterministic correlations obviously occur in nature, and if Alice and Bob have access to shared randomness, they can also achieve all correlations in \( C_c \). It is a fundamental fact of
quantum mechanics, first observed theoretically by John Bell and now verified in many experiments, that Alice and Bob can achieve correlations outside of $C_c$ by using quantum entanglement [Bel64].

Bell’s theorem leads to the question of which correlations can be achieved in quantum mechanics. To study this question, Tsirelson introduced the set of quantum correlations [Tsi93]. There are actually several ways to define the set of quantum correlations, depending on whether we assume that all Hilbert spaces are finite-dimensional, and whether we use the tensor-product axiom or commuting-operator axiom for joint systems. This leads to four different choices for the set of quantum correlations: the finite-dimensional quantum correlations $C_q$, the quantum-spatial correlations $C qs$, the quantum-approximate correlations $C qa$, and the commuting-operator correlations $C qc$. We use the same convention as for classical correlations, in that $C_t$ refers to $C_t(n_A, n_B, m_A, m_B)$ when the tuple $(n_A, n_B, m_A, m_B)$ is clear. Tsirelson suggested that all four sets should be equal, but we now know that (for large enough $n_A, n_B, m_A, m_B$) all four sets are different, and hence give a strictly increasing sequence

\[ C_c \subsetneq C_q \subsetneq C qs \subsetneq C qa \subsetneq C qc \]

[Slo19, CS20, JNV+20, DPP19, Col20]. The last inequality $C qa \subsetneq C qc$ is a very exciting consequence of the recent proof [JNV+20] that $\text{MIP}^* = \text{RE}$ by Ji, Natarajan, Vidick, Wright, and Yuen, and following [Fri12, JNP+11], this inequality gives a negative resolution to the Connes embedding problem.

As the convex hull of a finite set, $C_c$ is a polytope in $\mathbb{R}^N$, where $N = n_A n_B m_A m_B$. The sets $C_t, t \in \{q, qs, qa, qc\}$, are also convex subsets of $\mathbb{R}^N$ (in addition, $C qa$ and $C qc$ are closed), but it follows from a result of Tsirelson [Tsi87] that these sets are not polytopes. Following up on this point in [Tsi93, Problem 2.10], Tsirelson asks whether the sets of quantum correlations might still have nice geometric descriptions, specifically by analytic or even polynomial inequalities. This question is significant for two reasons: practical, in that the quantum correlation set captures what is possible with quantum entanglement, and thus a description of this set tells us what is theoretically achievable in experiments and quantum technologies; and conceptual, in that a nice description of the set of quantum correlations could improve our conceptual understanding of quantum entanglement, similarly to how the description of $C_c$ as the convex hull of deterministic correlations is central to our understanding of classical correlations.

Due to the significance of this question, describing the set of quantum correlations has been a central question in the field. On the geometric side, Tsirelson’s original results show that when $m_A = m_B = 2$, a certain linear slice of the quantum correlation set is the elliptope, a convex set described by quadratic inequalities ([Tsi87], see also [Lan88, WW01, Mas03, Pit08] for subsequent work on the special case that $n_A = n_B = 2$, and [TVC19] for a description as the elliptope). The convex geometry of $C_q(2, 2, 2, 2)$ is studied in detail in [GKW+18]. The case of $C_q(2, 2, 2, 2)$ benefits from a dimension reduction argument: by Jordan’s lemma, any correlation in $C_q(2, 2, 2, 2)$ can be expressed as a convex combination of correlations from two-qubit systems. In general, we might ask whether there is a bound on the dimension of Hilbert spaces needed to realize correlations in $C_q(n_A, n_B, m_A, m_B)$. There are several different proofs that, as the number of questions and outcomes increases, there
are correlations which require Hilbert spaces of arbitrarily high dimension [Tsi93, BPA+08, Slo11]. If we fix \(n_A, n_B, m_A, m_B\) such that \(C_q(n_A, n_B, m_A, m_B)\) is not closed (which again, happens if \(n_A, n_B, m_A, m_B\) are large enough), then \(C_q(n_A, n_B, m_A, m_B)\) contains correlations which require Hilbert spaces of arbitrarily large dimension. The first author gives an explicit family of correlations \(\{P_d \mid d \geq 1\}\) with \(n_A, n_B, m_A, m_B\) fixed requiring maximally entangled states of dimension \(d\) in [Fu22]. Thus the methods used to study \(C_q(2,2,2,2)\) do not work for more measurements or outcomes. Using a different dimension reduction argument, Russell describes another linear slice of \(C_q\), the synchronous correlations, in \(C_q(3,3,2,2)\), but again this description does not extend to other numbers of measurements and outcomes [Rus20].

In another line, a number of authors have considered whether it’s possible to give a conceptual, rather than geometric, description of the quantum correlation sets. The first result in this line comes from Tsirelson’s original definition of quantum correlations, where he observes that quantum correlations belong to the set of nonsignalling correlations, which are those correlations \(P\) for which the sums

\[
\sum_b P(a, b|x, y) \quad \text{and} \quad \sum_a P(a, b|x, y)
\]

are independent of \(y\) and \(x\) respectively. This condition captures the fact that, when spatially separated, Alice and Bob cannot communicate with each other. Since the set of nonsignalling correlations is strictly larger than the commuting-operator correlations \(C_{qc}\), the fact that Alice and Bob cannot communicate does not identify the set of quantum correlations among all correlations. But it is natural to ask whether there might not be additional principles which would suffice to identify the set of correlations. Some examples of conditions which further restrict the set of nonsignalling correlations and which are satisfied by quantum correlations can be found in [BBL+06, PPK+09, NW09, FSA+13, SGAN18], but so far these do not give a complete description of the set of quantum correlations.

Based on the apparent difficulty of describing the set of quantum correlations, there has also been a line of work studying the computational complexity of problems related to these sets. The main line of inquiry, initiated in [CHTW04], has been to consider the difficulty of determining the quantum and commuting-operator values of a nonlocal game. For example, one can consider the problem of determining whether a given nonlocal game has a perfect strategy.

**Problem** PerfectStrategy. Given a tuple of natural numbers \((n_A, n_B, m_A, m_B)\) and a nonlocal game \(G\) with \(n_A\) and \(n_B\) questions and \(m_A\) and \(m_B\) answers, does \(G\) have a perfect strategy in \(C_t\)?

From the point of view of convex geometry, the quantum (resp. commuting-operator) value of a nonlocal game is the maximum a certain linear functional on the set \(C_{qa}\) (resp. \(C_{qc}\)). Asking whether such a nonlocal game has a perfect strategy corresponds to asking whether this maximum is equal to 1. Leading up to [JNV+20], there was a series of deep works showing that even the approximate version of this optimization problem is indeed very difficult [IV12, RUV13, Ji17, NV18, NW19]. These results have implications
in computational complexity theory, as they imply lower bounds on the complexity class MIP* of multiprover proofs with entangled provers. In the exact (rather than approximate) case, previous results by the last author of the current paper imply that the problems PerfectStrategy, are undecidable for \( t \in \{ q, qs, qa, qc \} \) [Slo19, Slo20, FJVY19]. To understand the difficulty of approximating the quantum and commuting-operator values, we can look at a gapped variant of PerfectStrategy:

**Problem** GappedPerfectStrategy, Given a tuple of natural numbers \((n_A, n_B, m_A, m_B)\) and a nonlocal game \( G \) with \(n_A\) and \(n_B\) questions and \(m_A\) and \(m_B\) answers, decide whether \( G \) has a perfect strategy in \( C_t \), or the quantum value of \( G \) is \( \leq 1/2 \), given that one of the two is the case.

The result by Ji, Natarajan, Vidick, Wright, and Yuen mentioned above shows that GappedPerfectStrategy is also undecidable for \( t \in \{ q, qs, qa \} \) [JNV+20].

Rather than looking at nonlocal games, a more straightforward way to study the difficulty of describing quantum correlation sets is to look at the membership problem for these sets. Specifically, we can look at the decision problems for \( t \in \{ q, qs, qa, qc \} \) and subfields \( \mathbb{K} \subseteq \mathbb{R} \).

**Problem** Membership, Given a tuple \((n_A, n_B, m_A, m_B)\), and a correlation \( P \in \mathbb{K}^{n_An_Bm_Am_B} \), is \( P \in C_t(n_A, n_B, m_A, m_B) \)?

The point of restricting to correlations in \( \mathbb{K}^{n_An_Bm_Am_B} \) rather than \( \mathbb{R}^{n_An_Bm_Am_B} \) is that it is not possible to describe all real numbers in a finite fashion. We are primarily interested in subfields of \( \mathbb{R} \) such as \( \mathbb{Q} \), where it is practical to work with elements of the field on a computer. For our results we actually need to take a larger field than \( \mathbb{Q} \), so in what follows we’ll set \( \mathbb{K} = \overline{\mathbb{Q}} \cap \mathbb{R} \) unless otherwise noted, where \( \overline{\mathbb{Q}} \) is the algebraic closure of the rationals.\(^1\)

The problems Membership are a very general way of studying descriptions of the sets \( C_t \) for \( t \in \{ q, qs, qa, qc \} \), since we don’t restrict to any particular form of description, but instead just look at a basic functionality that we would hope to have from any nice description, namely a way of being able to distinguish elements inside the set from those outside. The decision problems Membership are not equivalent to the problems PerfectStrategy or GappedPerfectStrategy, since nonlocal games do not necessarily have unique perfect strategies in \( C_t \). Nonetheless, the two families of decision problems are closely related. Indeed, the methods used in [Slo20] to show the undecidability of PerfectStrategy are adapted in [CS19] to show the undecidability of Membership \( \mathbb{K} \) [CS19]. The methods of [Slo19] can be adapted to show the undecidability of Membership \( \mathbb{K} \) for \( t \in \{ q, qs, qa \} \) in similar fashion (although some work is needed for the case \( t = q \)). The undecidability of GappedPerfectStrategy can be used (in a blackbox fashion, without referring to the proof methods) to get the stronger result that Membership is undecidable for \( t \in \{ q, qs, qa \} \) [JNV+20].

The above undecidability results put very strong restrictions on what descriptions of the quantum correlation sets are possible. For instance, they imply that there is no

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\(^1\)Since \( \overline{\mathbb{Q}} \) is computable, it is possible to work with \( \overline{\mathbb{Q}} \) and \( \overline{\mathbb{Q}} \cap \mathbb{R} \) on a computer, and indeed support for this is included in Mathematica and other computer algebra packages.
There is an integer $T$ such that a Turing machine which takes tuples $(n_A, n_B, m_A, m_B)$ as inputs, and outputs a description of $C_t(n_A, n_B, m_A, m_B)$ in terms of a finite list of polynomial inequalities, since such a Turing machine would allow us to decide Membership$_{t,K}$ (such a description is sometimes called a uniform polynomial description). Similarly, these results also imply that there can be no finite set of principles, independent of $(n_A, n_B, m_A, m_B)$, such that we can decide whether a correlation satisfies every principle, and such that a correlation satisfies all the principles if and only if it belongs to $C_t(n_A, n_B, m_A, m_B)$.

However, we note that the reasoning in the last two paragraphs depends crucially on the fact that the parameters $(n_A, n_B, m_A, m_B)$ can vary. The papers [Slo20, Slo19, JNV+20] all involve games with unbounded alphabet size. Hence these results leave open the possibility that every set $C_t(n_A, n_B, m_A, m_B)$ has a nice description, but that it is just not possible to have a Turing machine which outputs these descriptions as a function of $(n_A, n_B, m_A, m_B)$ (in other words, they might have a non-uniform description). Thus it is natural to ask what happens to the complexity of Membership$_{t,K}$ when $(n_A, n_B, m_A, m_B)$ is held constant. This question motivates our main result.

**Problem** Membership$_{(n_A, n_B, m_A, m_B)_t,K}$. Given a correlation $P \in \mathbb{K}^{n_An_Bm_Am_B}$, is $P \in C_t(n_A, n_B, m_A, m_B)$?

**Theorem 1.1.** There is an integer $\alpha$ such that the decision problem Membership$_{(n_A, n_B, m_A, m_B)_t,K}$ is undecidable for $t \in \{qa, qc\}$ and $n_A, n_B, m_A, m_B > \alpha$.

This result shows that the undecidability of membership in $C_{qa}$ and $C_{qc}$ is not only a consequence of varying the size of the correlation, but is in fact embedded into the shape of a single set $C_t(n_A, n_B, m_A, m_B)$ for large enough $(n_A, n_B, m_A, m_B)$. As a practical consequence, the result shows that there is no description of the set $C_t(n_A, n_B, m_A, m_B)$ (e.g. by polynomial inequalities) that would allow us to decide membership in that set.

As mentioned above, in this theorem $\mathbb{K}$ is the intersection $\overline{\mathbb{Q}} \cap \mathbb{R}$. However, the proof of this theorem does not rely on writing down very complicated elements of $\overline{\mathbb{Q}}$. In fact, $\mathbb{K}$ could be replaced with $\mathbb{K}_0 \cap \mathbb{R}$, where $\mathbb{K}_0$ is the subfield of $\overline{\mathbb{Q}}$ generated by roots of unity.

In this way, the theorem is similar to the undecidability results for (Membership$_{t,K}$) that follow from [Slo19, Slo20, CS19]. However, in those results, if the correlations are defined in terms of observables instead of measurements, then it is possible to take $\mathbb{K} = \mathbb{Q}$. In our case, even if we work with correlations defined in terms of observables, we still need to use roots of unity. We also note that the correlations constructed in the proof of Theorem 1.1 are synchronous (see Definition 3.3), so Theorem 1.1 holds for the subsets of synchronous correlations.

It is interesting to also consider upper bounds on the complexity of the problem Membership$_{(n_A, n_B, m_A, m_B)_t,K}$. When $t = qc$, this problem is contained in coRE, and Theorem 1.1 actually shows that this problem is coRE-complete (for large enough $n_A, n_B, m_A, m_B$). When $t = q$ or $t = qs$, this problem is contained in $\text{RE}$, but when $t = qa$, the best known upper bound on this decision problem is $\Pi^0_2$. In this case, Theorem 1.1 only shows that Membership$_{(n_A, n_B, m_A, m_B)_{qa,K}}$ is coRE-hard, so this lower bound is not necessarily tight.

Recently, Mousavi, Nezhadi, and Yuen have shown that PerfectStrategy$_{qa}$ is $\Pi^0_2$-complete [MNY21], and it seems reasonable to conjecture that Membership$_{(n_A, n_B, m_A, m_B)_{qa,K}}$ is also $\Pi^0_2$-complete for large enough $n_A, n_B, m_A, m_B$. We leave this for future research.
1.1 Paper overview

We summarize the technical content of this paper. The starting point for the proof of Theorem 1.1 is the fact that the halting problem for Minsky machines is undecidable [Min67]. Minsky machines, which we review in Section 5, are a model of universal computation similar to Turing machines. To relate Minsky machines to correlations, we go through group theory: specifically the Kharlampovich-Myasnikov-Sapir (KMS) groups [KMS17], also described in Section 5. To construct correlations from these groups, we use the machinery of [Slo19], described in Section 4. Section 2 and Section 3 contain some basic background on group theory and quantum correlations, respectively.

For the proof of Theorem 1.1, we pick a Minsky machine $MM$ with an undecidable halting problem. For each $n \geq 1$, we then write down a finite set of correlations $F_n$ such that $F_n \cap C_{qa} \neq \emptyset$ if MM does not accept $n$, and $F_n \cap C_{qc} = \emptyset$ otherwise. The correlations in $F_n$ can be thought of as a combination of two subcorrelations. The first subcorrelation is constructed from MM using the method of [Slo19], and is independent of $n$. The second subcorrelation encodes the input $n$ using the method of [Fu22] to keep the number of measurements and outcomes fixed. This is described in Section 6. In this way, the number of measurements and measurement outcomes for correlations in $F_n$ will depend only on MM, not on $n$. The proof of Theorem 1.1 is completed in Section 7.

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2 Notation and group theory background

In this section we give a brief description of some of the notation and group theory concepts we’ll use throughout the paper. For basic notation, we denote the set $\{0, 1, \ldots, n - 1\}$ by $[n]$ and the set $\{x \in \mathbb{R} \mid x \geq c\}$ by $\mathbb{R}_{\geq c}$. We index vectors in $\mathbb{C}^n$ starting from 0, so $\mathbb{C}^n = \mathbb{C}^{[n]}$. The $n$-th root of unity is denoted by $\omega_n := e^{i2\pi/n}$. For a Hilbert space $\mathcal{H}$, we let $\mathcal{L}(\mathcal{H})$ be the set of all bounded linear operators acting on $\mathcal{H}$, and $\mathcal{U}(\mathcal{H})$ be the group of unitaries acting on $\mathcal{H}$. We let $\| \cdot \|_{op}$ denote the operator norm on $\mathcal{L}(\mathcal{H})$. For finite-dimensional Hilbert spaces, we also work with the normalized Hilbert-Schmidt norm, which is defined by

$$\|M\| = \sqrt{\frac{\text{Tr}(M^\dagger M)}{d}},$$

for $M \in \mathcal{L}(\mathbb{C}^d)$. Note that we don’t use any subscript to distinguish this from other norms, as this will be our default norm. We also let $\hat{\text{Tr}}(M)$ be the normalized trace $\text{Tr}(M)/d$ of $M$. When working with a group $G$, we use $e$ for the identity, and let $[g, h]$ be the commutator.
g^{-1}h^{-1}gh of g, h ∈ G. We let g^h denote the conjugate h^{-1}gh of g by h. This notation matches with [KMS17].

If S is a set, we let F(S) be the free group generated by S. If R is a subset of F(S), then we let ⟨S : R⟩ be the quotient of F(S) by the normal subgroup generated by R. The pair S, R is called a presentation of ⟨S : R⟩, and as usual we use ⟨S : R⟩ to refer to the presentation and the group defined by the presentation interchangeably. We also will write ⟨S⟩ for the presentation and the group defined by the presentation. Let G be a group, H a subgroup of G, and let \( \phi : H \to G \) be an injective homomorphism, then the HNN-extension of \( (G, H) \) is defined by \( G_1 \ast_\phi G_2 := G_1 \ast G_2 / \langle h \phi(h)^{-1} | h \in H \rangle \). An example of a finitely-presented group that we’ll use is the dihedral group

\[ D_n = \langle t_1, t_2 : t_1^2 = t_2^2 = (t_1t_2)^n = e \rangle. \]

This group has order 2n, and the elements are \( (t_1t_2)^j, (t_1t_2)^j t_1 \), and \( (t_1t_2)^j t_2 \) for \( j, n \in \mathbb{Z} \).

The free product of a group G with a group H is denoted by \( G \ast H \). Note that if \( G = \langle S_G : R_G \rangle \) and \( H = \langle S_H : R_H \rangle \), then \( G \ast H = \langle S_G \cup S_H : R_G \cup R_H \rangle = \langle G, S_H : R_H \rangle \), where the unions of \( S_G \) and \( S_H \) are disjoint. A more general notion of the free product of groups is the free product with amalgamation. Let \( G_1 \) and \( G_2 \) be two groups with subgroups \( H_1 \) and \( H_2 \) respectively such that there exists an isomorphism \( \phi : H_1 \to H_2 \). Then the free product of \( G_1 \) and \( G_2 \) with amalgamation is defined by \( G_1 \ast_\phi G_2 := G_1 \ast G_2 / \langle h \phi(h)^{-1} | h \in H \rangle \).

Another way to construct new groups from a given group is by Higman-Neumann-Neumann extension (HNN-extension) [HNN49]. If \( H \) is a subgroup of \( G \) and \( \phi : H \to H \) is an injective homomorphism, then the HNN-extension of \( G \) is \( \hat{G} = \langle G, t : t\phi^{-1}ht = \phi(h), h \in H \rangle \). By [Rot12, Theorem 11.70], the natural homomorphism sending \( g \in G \) to its image in \( \hat{G} \) is injective, meaning that we can regard \( G \) as a subgroup of \( \hat{G} \). We shall introduce other important properties of the free product with amalgamation and the HNN-extension later when they are needed. For more background on these concepts, we refer to [Rot12].

When \( \phi : H \to H \) is an isomorphism of order \( n \), we similarly define the \( \mathbb{Z}_n \)-HNN extension of \( G \) by \( \hat{G} = \langle G, t : t^n = e, t^{-1}ht = \phi(h) \rangle \). As in the case of the ordinary HNN-extension, \( G \) is embedded in \( \hat{G} \):

**Lemma 2.1.** Let \( G \) be a group, \( H \) a subgroup of \( G \), \( \phi : H \to H \) an isomorphism of order \( n \), and \( \hat{G} := \langle G, t : t^n = e, t^{-1}ht = \phi(h) \rangle \) the \( \mathbb{Z}_n \)-HNN extension. Then the inclusion \( G \to \hat{G} : g \mapsto g \) is injective, and \( t \) has order \( n \).

**Proof.** Let \( G^{*n} \) denote the free product of \( G \) with itself \( n \) times, where we index the factors by elements of \( \mathbb{Z}_n \). Let \( i_k : G \hookrightarrow G^{*n} \) be the inclusion of the \( k \)th factor, \( k \in \mathbb{Z}_n \), and let \( \psi : G^{*n} \to G^{*n} \) be the cyclic shift, so \( \psi(i_k(g)) = i_{k+1}(\psi(g)) \) for \( k \in \mathbb{Z}_n \).

Let \( N \) be the normal subgroup of \( G^{*n} \) generated by \( i_k(h^{-1})i_{k+1}(\psi(h)) \) for all \( h \in H \) and \( 0 \leq k \leq n - 2 \). Since we only include these relations for \( k \leq n - 2 \), we can check (for instance, by looking at presentations) that \( G^{*n} / N \) is an iterated amalgamated product of
We conclude that \( \hat{\psi} \) is injective when considered as a homomorphism \( \hat{G} \to \hat{K} \). Consequently \( \psi(N) = N \), so \( \psi \) induces an automorphism \( \hat{\psi} \) of \( G^n/\langle \gamma \rangle \). We claim that \( \hat{\psi} \) is isomorphic to \( K := G^n/\langle \gamma \rangle \{1\} \). Indeed, suppose \( t \) is the generator of \( \mathbb{Z}_n \), so that

\[
t^k \cdot x = \hat{\psi}^k(x) \cdot t^k \quad \text{for all} \quad x \in G^n/\langle \gamma \rangle.
\]

Then

\[
t^{-1}i_0(h)Nt = \hat{\psi}^{-1}(i_0(h)N) = i_{n-1}(h)N = i_0(\phi(h))N
\]

for all \( h \in H \), so there is a homomorphism \( \alpha : \hat{G} \to K \) sending \( g \mapsto i_0(g)N \), \( g \in G \), and \( t \mapsto t \).

Going the other way, there is a homomorphism \( \beta : G^n \to \hat{G} \) sending \( i_k(g) \mapsto t^kgt^{-k} \). This homomorphism sends

\[
i_k(h^{-1})i_{k+1}(\phi(h)) \mapsto t^k(h^{-1}t^{-k}) \cdot i_k+1(\phi(h))t^{-(k+1)} = t^{k+1}(t^{-1}h^{-1}t)\phi(h)t^{-(k+1)} = e,
\]

so \( \beta \) descends to a homomorphism \( \hat{\beta} : G^n/N \to \hat{G} \). If \( g \in G \) and \( a \in \mathbb{Z}_n \) then

\[
t^a \hat{\beta}(i_k(g)N)t^{-a} = t^a + kgt^{-(a+k)} = \hat{\beta}(i_{a+k}(g)N) = \hat{\beta}(\hat{\psi}^a(i_k(g)))
\]

We conclude that \( \hat{\beta}(\hat{\psi}^a(x)) = t^a \hat{\beta}(x)t^{-a} \) for all \( x \in G^n/N \), and hence there is a homomorphism \( K \to \hat{G} \) sending \( x \in G^n/N \) to \( \hat{\beta}(x) \) and \( t \mapsto t \). Since, in particular, this homomorphism sends \( i_0(g)N \) to \( g \), it is an inverse to \( \alpha \), proving the claim that \( \hat{G} \) and \( K \) are isomorphic.

Since \( G^n/N \) is an iterated amalgamated free product, the homomorphism \( G \to G^n/N \) sending \( g \mapsto i_0(g)N \) is injective. Since \( G^n/N \) is a subgroup of \( K \), \( g \mapsto i_0(g)N \) is still injective when considered as a homomorphism \( G \to K \). Composing with the isomorphism \( K \cong \hat{G} \), we get that the homomorphism \( G \to \hat{G} \) is injective. Finally, as the generator of \( \mathbb{Z}_n \), \( t \) has order \( n \).

A unitary representation \( \rho \) of a group \( G \) on the Hilbert space \( \mathcal{H} \) is a homomorphism \( \rho : G \to \mathcal{U}(\mathcal{H}) \). For any set \( X \), we let \( \ell^2X \) denote the Hilbert space with Hilbert basis \( \{ |x\rangle : x \in X \} \). The left regular representation \( L : G \to \mathcal{U}(\ell^2G) \) of a group \( G \) is defined by \( L(g)|h\rangle = |gh\rangle \), and right regular representation \( R : G \to \mathcal{U}(\ell^2G) \) is defined by \( R(g)|h\rangle = |hg^{-1}\rangle \) for all \( g, h \in G \). Note that \( L(g) \) and \( R(g') \) commute for all \( g, g' \in G \).

To construct correlations in \( C_\alpha \) (which, recall from the introduction, are limits of finite-dimensional correlations), we use finite-dimensional approximate representations of groups. The norm we use for these approximate representations is the normalized Hilbert-Schmidt norm.

**Definition 2.2** (Definition 5 of [Slo19]). Let \( G = \langle S : R \rangle \) be a finitely-presented group, and let \( \mathcal{H} \) be a finite-dimensional Hilbert space. A finite-dimensional \( \epsilon \)-approximate representation of \( G \) is a homomorphism \( \phi : \mathcal{F}(S) \to \mathcal{U}(\mathcal{H}) \) such that \( \| \phi(r) - 1 \| \leq \epsilon \) for all \( r \in R \).
An element $g \in G = \langle S : R \rangle$ represented by a word $w \in \mathcal{F}(S)$ is nontrivial in approximate representations of $G$ if there exists some $\delta > 0$ such that, for all $\epsilon > 0$, there is an $\epsilon$-approximate representation $\phi : \mathcal{F}(S) \to \mathcal{U}(\mathcal{H})$ such that $\|\phi(w) - 1\| \geq \delta$ (this does not depend on the choice of word $w$). Otherwise we say that $g$ is trivial in approximate representations. If $F$ is a finite subset of elements which are non-trivial in approximate representations, then we can find $\epsilon$-representations where all the elements of $F$ are bounded away from the identity, and in fact we can do this by having the trace of all the elements of $F$ be close to zero.

**Proposition 2.3.** Let $G = \langle S : R \rangle$ and $W$ be a finite subset of $\mathcal{F}(S)$. Then, for every $\epsilon, \zeta > 0$, there is an $\epsilon$-approximate representation $\phi$ such that for all $w \in W$,

1. if $w$ is trivial in approximate representations of $G$, then $1 - \zeta \leq \text{Tr}(\phi(w)) \leq 1$, and
2. if $w$ is nontrivial in approximate representations of $G$, then $0 \leq \text{Tr}(\phi(w)) \leq \zeta$.

The proof of the proposition above is very similar to the proof of [Slo19, Lemma 12], so we omit it here. The next two well-known lemmas are useful when working with approximate representations.

**Lemma 2.4.** Let $G = \langle S : R \rangle$ be a finitely presented group. Suppose $f \in \mathbb{C}[\mathcal{F}(S)]$. Then there is a constant $c$ (depending on $f$) such that for any $\epsilon$-approximate representation $\phi : \mathcal{F}(S) \to \mathcal{U}(\mathcal{H})$, $\|\phi(f)\|_{op} \leq c$. Furthermore, if $f = 0$ in $\mathbb{C}[G]$, then there is a constant $c'$ (also depending on $f$) such that for any $\epsilon$-approximate representation $\phi : \mathcal{F}(S) \to \mathcal{U}(\mathcal{H})$, $\|\phi(f)\| \leq c' \epsilon$.

**Proof.** Suppose $f = \sum_{i \in [k]} a_i u_i$ where $a_i \in \mathbb{C}$ and $u_i \in \mathcal{F}(S)$. Then

$$\|\phi(f)\|_{op} \leq \sum_{i \in [k]} |a_i| \|\phi(u_i)\|_{op} = \sum_{i \in [k]} |a_i|.$$ 

If $f = 0$ in $\mathbb{C}[G]$, we can write

$$f = \sum_{i \in [k]} b_i x_i (e - r_i) y_i,$$

where $b_i \in \mathbb{C}, x_i, y_i \in \mathcal{F}(S)$ and $r_i \in R$. Then

$$\|\phi(f)\| \leq \sum_{i \in [k]} |b_i| \|\phi(x_i)(1 - \phi(r_i))\phi(y_i)\| = \sum_{i \in [k]} |b_i| \|1 - \phi(r_i)\| \leq \sum_{i \in [k]} |b_i| \epsilon,$$

by the unitary invariance of the normalized Hilbert-Schmidt norm. The lemma follows from taking $c = \sum_{i \in [k]} |a_i|$ and $c' = \sum_{i \in [k]} |b_i|$.

**Lemma 2.5.** There is a nondecreasing function $\Delta : \mathbb{R}_{\geq 1} \times \mathbb{N} \to \mathbb{R}_{\geq 1}$ such that if $\{P_i \mid i \in [n]\} \subset \mathcal{L}(\mathbb{C}^d)$ is a set of matrices such that

$$\|P_i\|_{op} \leq c, \quad \|P_i^2 - P_i\| \leq \epsilon, \quad \|P_i^* - P_i\| \leq \epsilon, \quad \|P_i P_j\| \leq \epsilon, \text{ and } \|\sum_{k \in [n]} P_k - 1\| \leq \epsilon$$

for all $i, j \in [n], i \neq j$ and some $c \in \mathbb{R}$, then there is a projective measurement $\{\Pi_i \mid i \in [n]\} \subset \mathcal{L}(\mathbb{C}^d)$ such that $\|\Pi_i - P_i\| \leq \Delta(c, n) \epsilon$ for all $i \in [n]$. 

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Proof. When \( c = 1 \) and \( P_i \) is positive for all \( i \in [n] \), this is shown in Lemma 3.5 of [KPS18], with function \( \Delta(1, n) = \Delta_{pos}(n) \), where \( \Delta_{pos}(n) \) is defined recursively by \( \Delta_{pos}(n + 1) = (40n + 3)\Delta_{pos}(n) \) and \( \Delta_{pos}(1) = 2\sqrt{2} \). To reduce to this case, suppose that \( \{ P_i \mid i \in [n] \} \) satisfy the conditions of the lemma. If \( \{ Q_i \mid i \in [n] \} \subseteq \mathcal{L}(\mathbb{C}^d) \) are self-adjoint operators such that \( \| Q_i \|_{op} \leq c' \) and \( \| P_i - Q_i \| \leq \delta \) for all \( i \in [n] \), then

\[
\| Q_i^2 - Q_i \| \leq \| Q_i(Q_i - P_i) \| + \| (Q_i - P_i)P_i \| + \| Q_i - P_i \| + \| P_i^2 - P_i \|
\leq \| Q_i \|_{op} \| Q_i - P_i \| + \| P_i \|_{op} \| Q_i - P_i \| + \| Q_i - P_i \| + \| P_i^2 - P_i \|
\leq (c + c')\delta + \epsilon.
\]

Similarly, \( \| Q_iQ_j \| \leq (c + c')\delta + \epsilon \) and \( \| \sum_{i \in [n]} Q_i - 1 \| \leq n\delta + \epsilon \).

If we take \( Q_i = (P_i + P_i^*)/2 \), then \( Q_i \) is self-adjoint, \( \| Q_i \|_{op} \leq c \) and \( \| Q_i - P_i \| \leq \epsilon/2 \). Let \( \chi_{[1/2, \infty)} \) be the indicator function of the interval \([1/2, \infty)\). Since \( |\chi_{[1/2, \infty)}(t) - t| \leq 2|t^2 - t| \) for all \( t \in \mathbb{R} \),

\[
\| \chi_{[1/2, \infty)}(Q_i) - Q_i \| \leq 2\| Q_i^2 - Q_i \| \leq (2c + 3)\epsilon.
\]

Hence the self-adjoint projections \( Q_i' = \chi_{[1/2, \infty)}(Q_i) \) satisfy the conditions

\[
\| Q_i'Q_j' \| \leq (c + 1)(2c + 3)\epsilon + \| Q_iQ_j \| \leq (2c^2 + 7c + 4)\epsilon \text{ and }
\| \sum_{i \in [n]} Q_i' - 1 \| \leq (2c + 3)n\epsilon + \| \sum_{i \in [n]} Q_i - 1 \| \leq (2c + 7/2)n\epsilon + \epsilon.
\]

Applying Lemma 3.5 of [KPS18] to \( \{ Q_i' \mid i \in [n] \} \) yields \( \{ \Pi_i \mid i \in [n] \} \) such that

\[
\| \Pi_i - Q_i' \| \leq \Delta_{pos}(n)(2c^2 + 7c + 5)n\epsilon.
\]

Since

\[
\| \Pi_i - P_i \| \leq \| \Pi_i - Q_i' \| + \| Q_i' - Q_i \| + \| Q_i - P_i \|
\leq \Delta_{pos}(n)(2c^2 + 7c + 5)n\epsilon + (2c + 3)\epsilon + 1/2\epsilon,
\]

the lemma is true with \( \Delta(c, n) = \Delta_{pos}(n)(2c^2 + 7c + 5)n + 2c + 4 \).

\[\square\]

There are variants of Lemma 2.5 that reduce the dependence on \( n \) (see, e.g., [dlS21]). In this paper, \( c \) and \( n \) are fixed, so \( \Delta(c, n) \) is a constant.

By the definition of the normalized Hilbert-Schmidt norm, the set of elements of \( G \) that are trivial in finite-dimensional approximate representations forms a normal subgroup of \( G \), denoted by \( N^{fa} \). For a group \( G \), we define

\[G^{fa} := G/N^{fa}.\]

If \( \phi : G \to H \) is a homomorphism between finitely-presented groups and \( x \in G \) is trivial in approximate representations of \( G \), then \( \phi(x) \) is trivial in approximate representations of \( H \), so there is an induced homomorphism \( G^{fa} \to H^{fa} \).
**Definition 2.6** (Definition 14 of [Slo19]). For finitely-presented groups $G$ and $H$, a homomorphism $\phi : G \to H$ is an **fa-embedding** if the induced map: $G^{fa} \to H^{fa}$ is injective.

In other words $\phi : G \to H$ is an fa-embedding if whenever $x$ is non-trivial in approximate representations of $G$, then $\phi(x)$ is non-trivial in approximate representations of $H$.

A finitely-presented group $G$ is said to be **hyperlinear** if every non-trivial element is non-trivial in approximate representations. Although we’ve defined hyperlinearity only for finitely-presented groups, whether a group is hyperlinear is independent of the presentation. To show that groups are hyperlinear, we use the stronger properties of solvability, amenability, and soficity. Recall that a group $G$ is **solvable** if it has subgroups $G_0 = \{e\}, G_1, \ldots, G_{k-1}$ and $G_k = G$ such that $G_{j-1}$ is normal in $G_j$ and $G_j/G_{j-1}$ is an abelian group, for $1 \leq j \leq k$. For the purposes of our paper, the definitions of amenable and sofic groups are irrelevant; we just need the following well-known properties of these classes of groups (see [CLP15, Proposition 2.4.1]):

1. Solvable groups are amenable, amenable groups are sofic, and sofic groups are hyperlinear.

2. If $H$ is an amenable subgroup of a sofic group $G$, and $\alpha : H \to H$ is an injective homomorphism, then the HNN-extension of $G$ by $\alpha$ is sofic.

3. If $H_1$ and $H_2$ are amenable subgroups of sofic groups $G_1$ and $G_2$, and $\alpha : H_1 \to H_2$ is an isomorphism, then the free product of $G_1$ and $G_2$ with amalgamation, $G_1 \ast_\alpha G_2$, is sofic.

4. If $N$ is a normal subgroup of $G$ such that $N$ is sofic and $G/N$ is amenable, then $G$ is sofic.

To these properties we can add:

**Lemma 2.7.** If $H$ is an amenable subgroup of a sofic group $G$, and $\phi : H \to H$ is an isomorphism of order $n$, then the $\mathbb{Z}_n$-HNN-extension of $G$ by $\phi$ is sofic.

**Proof.** We continue with the notation from the proof of Lemma 2.1. As $G^{*n}/N$ is an iterated amalgamated free product over the amenable group $H$, $G^{*n}/N$ is hyperlinear. Since $K/(G^{*n}/N) \cong \mathbb{Z}_n$ and $\mathbb{Z}_n$ is amenable, $K$ is sofic.

**3 Quantum correlations**

In this section, we now introduce our main object of study. Consider a scenario with two parties or players, Alice and Bob, and a referee. The referee chooses questions to send to Alice and Bob from finite sets $\mathcal{X}$ and $\mathcal{Y}$ respectively, and they return answers from finite sets $\mathcal{A}$ and $\mathcal{B}$. As mentioned in the introduction, this can also be thought of as a scenario in which Alice and Bob perform measurements labelled by the elements of $\mathcal{X}$ and $\mathcal{Y}$, and
receive outcomes from $A$ and $B$ respectively. Alice and Bob’s behaviour in this scenario can be described by the function

$$P : A \times B \times X \times Y \to \mathbb{R}_{\geq 0} : (k, \ell, i, j) \mapsto P(k, \ell|i, j),$$

where $P(k, \ell|i, j)$ is the probability of answers $(k, \ell) \in A \times B$ with questions $(i, j) \in X \times Y$. We call a tuple $(X, Y, A, B)$ of finite sets a nonlocal scenario, and a function $P : A \times B \times X \times Y \to \mathbb{R}_{\geq 0}$ such that

$$\sum_{k \in A, \ell \in B} P(k, \ell|i, j) = 1$$

for all $i \in X, j \in Y$ a bipartite correlation for the scenario $(X, Y, A, B)$.

With quantum correlations, we want to capture what Alice and Bob can do in a nonlocal scenario when they cannot communicate. Even though Alice and Bob cannot communicate, the rules of quantum mechanics do allow them to share entanglement. We can visualize this scenario as in Figure 1.

Recall that a vector state in a Hilbert space $\mathcal{H}$ is a unit vector, and a projective measurement (with $m$ outcomes) is a collection $\{P(i) \mid i \in [m]\}$ of orthogonal projections on $\mathcal{H}$ such that

$$\sum_{i \in [m]} p^{(i)} = 1.$$ 

There are two ways in quantum mechanics to handle the restriction that Alice and Bob cannot communicate. In the first, we require that their joint Hilbert space be a tensor product of subspaces:

**Definition 3.1.** A bipartite correlation $P$ for the scenario $(X, Y, A, B)$ is a quantum correlation if there are

(a) finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$,

(b) a vector state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,

(c) a collection of projective measurements $\{M_i^{(k)} \mid k \in A\}$ on $\mathcal{H}_A$ for every $i \in X$, and

(d) a collection of projective measurements $\{N_j^{(\ell)} \mid \ell \in B\}$ on $\mathcal{H}_B$ for every $j \in Y$,
such that

\[ P(k, \ell|i, j) = \langle \psi | M_i^{(k)} \otimes N_j^{(\ell)} | \psi \rangle \]

for all \( i \in X, j \in Y, k \in A, \ell \in B \).

The set \( C_q(X, Y, A, B) \) is the set of all quantum correlations for scenario \((X, Y, A, B)\), and the set \( C_{qa}(X, Y, A, B) \) is the closure of \( C_q(X, Y, A, B) \) in \( \mathbb{R}^{A \times B \times X \times Y} \). If \( n_A, n_B, m_A, \) and \( m_B \) are positive integers, and \( t \in \{q, qa\} \), then we set \( C_t(n_A, n_B, m_A, m_B) := C_t([n_A], [n_B], [m_A], [m_B]) \).

If \( |X| = n_A, |Y| = n_B, |A| = m_A, \) and \( |B| = m_B \), then \( \mathbb{R}^{A \times B \times X \times Y} \cong \mathbb{R}^{m_A m_B n_A n_B} \), and this is used to define the closure of \( C_q(X, Y, A, B) \). In fact, this isomorphism identifies \( C_t(X, Y, A, B) \) with \( C_t(n_A, n_B, m_A, m_B) \) for \( t \in \{q, qa\} \). So although it’s convenient to be able to use arbitrary labels for questions and answers (and we’ll use non-integer labels in this paper), we could just work with the sets \( C_t(n_A, n_B, m_A, m_B) \) if we wanted.

Some additional terminology we’ll use: a collection of Hilbert spaces, vector state, and projective measurements as in parts (a)-(d) is called a quantum strategy. Also, although we don’t study this set in this paper, the set \( C_{qs}(X, Y, A, B) \) of quantum-spatial correlations is defined similarly to the set of quantum correlations, but the Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) in the strategy are allowed to be infinite-dimensional. The closure of \( C_{qs} \) is also equal to \( C_{qa} \) [SW08].

Moving on, a more general way to handle the restriction that Alice and Bob cannot communicate is to drop the requirement that Alice and Bob’s projective measurements act on different Hilbert spaces, and instead just require that their projective measurements commute:

**Definition 3.2.** A bipartite correlation \( P \) for a scenario \((X, Y, A, B)\) commuting-operator correlation if there is

(a) a Hilbert space \( \mathcal{H} \),

(b) a vector state \( |\psi\rangle \in \mathcal{H} \),

(c) a collection of projective measurements \( \{M_i^{(k)} | k \in A\} \) on \( \mathcal{H} \) for every \( i \in X \), and

(d) a collection of projective measurements \( \{N_j^{(\ell)} | \ell \in B\} \) on \( \mathcal{H} \) for every \( j \in Y \),

such that

\[ M_i^{(k)} N_j^{(\ell)} = N_j^{(\ell)} M_i^{(k)} \]

and

\[ P(k, \ell|i, j) = \langle \psi | M_i^{(k)} \cdot N_j^{(\ell)} | \psi \rangle \]

for all \( i \in X, j \in Y, k \in A, \ell \in B \).

The set \( C_{qc}(X, Y, A, B) \) is the set of all commuting-operator correlations for the scenario \((X, Y, A, B)\), and if \( n_A, n_B, m_A, \) and \( m_B \) are positive integers, then \( C_{qc}(n_A, n_B, m_A, m_B) := C_{qc}([n_A], [n_B], [m_A], [m_B]) \).

As with quantum correlations, we refer to a Hilbert space, vector state, and projective measurements as in (a)-(d) as a commuting-operator strategy. Note that the Hilbert space \( \mathcal{H} \) in a commuting-operator strategy does not have to be finite-dimensional.

We define one more subtype of correlation that we’ll use:
Definition 3.3. A bipartite correlation $P$ for scenario $(X, Y, A, B)$ is synchronous if $X = Y$, $A = B$, and

$$\sum_{k \in A} P(k, k|i, i) = 1$$

for all $i \in X$.

Equivalently, a correlation with $X = Y$ and $A = B$ is synchronous if $P(k, \ell|i, i) = 0$ for all $k \neq \ell$ and $i$, or in other words if Alice and Bob always return the same answer when given the same question. The following fact about synchronous correlations is well-known:

Proposition 3.4 (Theorem 5.5(i) in [PSS+16]). Let $P$ be a synchronous correlation for $(X, X, A, A)$, and let $H_i, |\psi\rangle, \{M_j^{(k)} | k \in A\}, \{N_{i}^{(\ell)} | \ell \in A\}$ be a commuting operator strategy for $P$. Then

$$M_j^{(k)}|\psi\rangle = N_{i}^{(k)}|\psi\rangle$$

for all $i \in X, k \in A$.

Proposition 3.4 is an immediate consequence of the following lemma, which is contained in the proof of [PSS+16, Theorem 5.5(i)].

Lemma 3.5. Let $|\psi\rangle \in H$ be a quantum state, and let $\{M_j | j \in [n]\}$ and $\{N_{i} | j \in [n]\}$ be two projective measurements on $H$ for some $n \geq 2$, such that $M_j N_k = N_k M_j$ for all $j, k \in [n]$. If $\langle \psi | M_j N_k | \psi \rangle = 0$ for all $j \neq k \in [n]$, then

$$M_j |\psi\rangle = N_{i} |\psi\rangle$$

for each $j \in [n]$.

The next two conclusions of Lemma 3.5 will help us work with correlations.

Lemma 3.6. Let $|\psi\rangle \in H$ be a quantum state, and let $\{M_0^{(k)} | k \in [m_A]\}$ and $\{M_1^{(k)} | k \in [m_A]\}$ be two projective measurements on $H$, both of which commute with the projective measurement $\{N^{(l, l')} | l, l' \in [m_A]\}$ on $H$. If

$$\langle \psi | M_0^{(k)} N^{(l, l')} | \psi \rangle = \langle \psi | M_1^{(k')} N^{(l, l')} | \psi \rangle = 0$$

for any $k \neq 1$ and $k' \neq l'$, then

$$M_0^{(k)} M_1^{(k')} |\psi\rangle = M_1^{(k')} M_0^{(k)} |\psi\rangle$$

for any $k, k' \in [m_A]$.

Proof. The condition implies that the two measurement pairs

$$\{M_0^{(k)} | k \in [m_A]\}, \left \{ \sum_{l' \in [m_A]} N^{(k, l')} | k \in [m_A] \right \}$$


and
\[ \{ M_1^{(k')} | k' \in [m_A] \}, \{ \sum_{l \in [m_A]} N^{(l,k')} | k' \in [m_A] \} \]
both satisfy the condition of Lemma 3.5 with respect to \( |\psi\rangle \). We conclude that
\[ M_0^{(k)}|\psi\rangle = \sum_{l' \in [m_A]} N^{(k,l')}|\psi\rangle \quad \text{and} \quad M_1^{(k')}|\psi\rangle = \sum_{l \in [m_A]} N^{(l,k')}|\psi\rangle \]
for each \( k, k' \in [m_A] \). Then,
\[ M_0^{(k)} M_1^{(k')}|\psi\rangle = M_0^{(k)} \sum_{l \in [m_A]} N^{(l,k')}|\psi\rangle = \sum_{l \in [m_A]} N^{(l,k')} M_0^{(k)}|\psi\rangle = \sum_{l' \in [m_A]} N^{(k,l')}|\psi\rangle \sum_{l \in [m_A]} N^{(l,k')} = \sum_{l' \in [m_A]} N^{(l',k)}|\psi\rangle = M_1^{(k')} M_0^{(k)}|\psi\rangle, \]
for each \( k, k' \in [m_A] \).

Lemma 3.7 (Substitution Lemma). Let \( |\psi\rangle \in \mathcal{H} \) be a quantum state. Suppose there exist sequences of unitaries \( \{ V \}, \{ V_i | i \in [k] \} \) and \( \{ M_i | i \in [n] \} \) on \( \mathcal{H} \) commuting with another sequence of unitaries \( \{ N_i | i \in [n] \} \) on \( \mathcal{H} \), such that
\[ M_i|\psi\rangle = N_i|\psi\rangle \]
for each \( i \in [n] \), and
\[ V|\psi\rangle = \prod_{i \in [k]} V_i|\psi\rangle. \]
Then
\[ V \prod_{i \in [n]} M_i|\psi\rangle = \left( \prod_{i \in [k]} V_i \right) \left( \prod_{i \in [n]} M_i \right) |\psi\rangle. \]

Proof. We prove this lemma by induction on \( n \). The \( n = 0 \) case follows the condition that \( V|\psi\rangle = \prod_{i \in [k]} V_i|\psi\rangle \). Assume the conclusion holds for \( j = m \) and consider the case
which completes the proof.

Finally we recall how to embed an arbitrary finitely presented group in a solution group. and use the solution group to show that every linear system has a perfect correlation. In this section, we introduce the notion of a perfect correlation associated with a binary linear system. We then recall the notion of a solution group associated to a linear system, and use the solution group to show that every linear system has a perfect correlation.

\[ V \prod_{i \in [m+1]} M_i |\psi\rangle = V \left( \prod_{i \in [m]} M_i \right) M_m |\psi\rangle = V \left( \prod_{i \in [m]} M_i \right) N_m |\psi\rangle = N_m V \left( \prod_{i \in [m]} M_i \right) |\psi\rangle = N_m \left( \prod_{i \in [k]} V_i \right) \left( \prod_{i \in [m]} M_i \right) |\psi\rangle = \left( \prod_{i \in [k]} V_i \right) \left( \prod_{i \in [m]} M_i \right) |\psi\rangle, \]

which completes the proof.

When working with correlations, it sometimes simplifies arguments if we restrict to strategies with the following property:

**Definition 3.8.** A commuting operator strategy

\[ |\psi\rangle \in \mathcal{H}, \{ M_i^{(k)} \mid k \in \mathcal{A} \}, i \in \mathcal{X}, \{ N_j^{(\ell)} \mid \ell \in \mathcal{B} \}, j \in \mathcal{Y} \]

is good if

(a) for all \( i \in \mathcal{X} \) and \( k \in \mathcal{A} \), if \( \langle \psi | M_i^{(k)} | \psi \rangle = 0 \), then \( M_i^{(k)} = 0 \), and

(b) for all \( j \in \mathcal{Y} \) and \( \ell \in \mathcal{B} \), if \( \langle \psi | N_j^{(\ell)} | \psi \rangle = 0 \), then \( N_j^{(\ell)} = 0 \).

**Proposition 3.9.** If \( P \in \mathcal{C}_{\psi_c}(\mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}) \), then \( P \) has a good commuting-operator strategy.

**Proof.** Suppose \( (|\psi\rangle, \{ M_i^{(k)} \}, \{ N_j^{(\ell)} \}) \) is a strategy for \( P \). If \( P(k, \ell | i, j) \neq 0 \) then

\[ \langle \psi | M_i^{(k)} | \psi \rangle = \sum_{\ell' \in \mathcal{B}} P(k, \ell' | i, j) \neq 0. \]

If \( \langle \psi | M_i^{(k)} | \psi \rangle = 0 \), then \( \langle \psi | M_i^{(k)} N_j^{(\ell)} | \psi \rangle = P(k, \ell | i, j) = 0 \) for all \( j \in \mathcal{Y} \) and \( \ell \in \mathcal{B} \). Since \( \sum_{k' \in \mathcal{A}} M_i^{(k')} = 1 \), there must be some \( k' \) such that \( \langle \psi | M_i^{(k')} | \psi \rangle \neq 0 \). If we replace \( M_i^{(k')} \) with \( M_i^{(k')} + M_i^{(k)} \) and \( M_i^{(k)} \) with 0, we get another commuting-operator strategy for \( P \), and doing this for all \( k \in \mathcal{A} \), \( i \in \mathcal{X} \) with \( \langle \psi | M_i^{(k)} | \psi \rangle = 0 \), and similarly for all \( \ell \in \mathcal{B} \), \( j \in \mathcal{Y} \) with \( \langle \psi | N_j^{(\ell)} | \psi \rangle = 0 \), gives a good strategy for \( P \).

4 Quantum correlations and group theory

In this section, we introduce the notion of a perfect correlation associated with a binary linear system. We then recall the notion of a solution group associated to a linear system, and use the solution group to show that every linear system has a perfect correlation. Finally we recall how to embed an arbitrary finitely presented group in a solution group.
4.1 Solution groups and correlations

Definition 4.1. Let $Ax = 0$ be an $m \times n$ binary linear system, so $A$ is an $m \times n$ matrix over $\mathbb{Z}_2$, and $0 \in \mathbb{Z}_2^n$. Suppose that each row of $A$ has $\kappa$ non-zero entries. For each $i \in [m]$, let

$$I_i = \{ j \in [n] \mid A_{ij} = 1 \}$$

and let $\phi_i : I_i \rightarrow [\kappa]$ be the unique order-preserving bijection (so the smallest element of $I_i$ maps to 0, the largest maps to $\kappa - 1$, and so on). Let $\mathcal{X}_{\text{var}} := \{ x_i \mid i \in [n] \}$, and let $\mathcal{X} := [m] \cup \mathcal{X}_{\text{var}}$. Let $S := \{ v \in \mathbb{Z}_2^\kappa : \sum_{i \in [\kappa]} v_i = 0 \}$. A correlation $P$ for the scenario $(\mathcal{X}, \mathcal{X}, \mathbb{Z}_2^\kappa, \mathbb{Z}_2^\kappa)$ is a perfect correlation for $Ax = 0$ if $P(a, b|x, y) = 0$ whenever

1. $x \in [m]$ and $a \notin S$ or $y \in [m]$ and $b \notin S$;
2. $x \in \mathcal{X}_{\text{var}}$ and $(a_0, \ldots, a_{\kappa - 2}) \neq (0, 0, \ldots, 0)$, or $y \in \mathcal{X}_{\text{var}}$ and $(b_0, \ldots, b_{\kappa - 2}) \neq (0, 0, \ldots, 0)$;
3. $x, y \in [m]$ and $a_{\phi_x(k)} \neq b_{\phi_y(k)}$ for some $k \in I_x \cap I_y$;
4. $x \in [m], y = x_i \in \mathcal{X}_{\text{var}}$ for some $i \in I_x$, and $a_{\phi_x(i)} \neq b_{\kappa - 1}$ in $\mathbb{Z}_2$;
5. $x = x_i \in \mathcal{X}_{\text{var}}$ for some $i \in I_y$, $y \in [m]$, and $a_{\kappa - 1} \neq b_{\phi_y(i)}$ in $\mathbb{Z}_2$; or
6. $x = y \in \mathcal{X}_{\text{var}}$, and $a_{\kappa - 1} \neq b_{\kappa - 1}$.

In perfect correlations as defined here, the questions that Alice and Bob receive are either variables in $\mathcal{X}_{\text{var}}$, or indices from $[m]$. When Alice or Bob gets an index $x \in [m]$, condition (1) requires them to return an element $a \in S$. This element should be thought of as an assignment to the $x$th equation, where variable $x_k$, $k \in I_x$, receives value $a_{\phi_x(k)}$. When Alice or Bob gets a variable $x_i \in \mathcal{X}_{\text{var}}$, condition (2) forces them to return an element $a \in \mathbb{Z}_2^n$ with $a_j = 0$ for $0 \leq j \leq \kappa - 2$ and $a_{\kappa - 1} \in \{0, 1\}$. This should be thought of as an assignment to $x_i$ from $\mathbb{Z}_2$. The remaining conditions state that if Alice and Bob are asked about the same variable (either as part of an equation or directly) then their assignments to that variable must agree. Note that conditions (3) and (6) imply that every perfect correlation for $Ax = 0$ is synchronous.

Definition 4.1 is stated for linear systems with a constant number $\kappa$ of non-zero entries in each row. This allows us to use answer sets $\mathbb{Z}_2^\kappa$. It is possible to define perfect correlations for systems with a varying number of entries in each row, by either using answer sets which vary with the question, or by using larger answer sets $\mathbb{Z}_2^n$ as in [KPS18]. However, all the linear systems we work with have a constant number of non-zero entries in each row, and making this assumption in Definition 4.1 simplifies later analysis.

The point of perfect correlations is that any strategy for a perfect correlation yields a representation of a certain group associated with $Ax = 0$.

Definition 4.2 (Definition 17 of [Slo19]). Let $Ax = 0$ be an $m \times n$ linear system over $\mathbb{Z}_2$, so $A$ is an $m \times n$ matrix with entries in $\mathbb{Z}_2$ and $0 \in \mathbb{Z}_2^n$. For $j \in [m]$, define $I_j = \{ k \in [n] \mid A_{jk} = 1 \}$. 

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The homogeneous solution group of $Ax = 0$ is

$$\Gamma(A) := \langle x_0, x_1, \ldots, x_{n-1} : x_j^2 = e \text{ for all } j \in [n],$$

$$\prod_{k \in I} x_k = e \text{ for all } i \in [m],$$

$$[x_j, x_k] = e \text{ if } j, k \in I_i \text{ for some } i \rangle.$$ 

**Proposition 4.3.** Let $Ax = 0$ be a binary linear system with $\kappa$ non-zero entries in each row. Suppose $P$ is a perfect correlation for $Ax = 0$, and that $\mathcal{H}, |\psi\rangle, \{M_x^{(a)} | a \in \mathbb{Z}_2^\kappa\}, x \in \mathcal{X}, \{N_y^{(b)} | b \in \mathbb{Z}_2^\kappa\}, y \in \mathcal{X}$ is a good commuting-operator strategy for $P$. Let $\mathcal{H}_0 := \mathcal{A} \cdot \langle \psi \rangle$, the closure of $\mathcal{A}|\psi\rangle$ in $\mathcal{H}$, where $\mathcal{A}$ is the algebra generated by $M_x^{(a)}$ and $N_x^{(a)}$ for $x \in \mathcal{X}, a \in \mathbb{Z}_2^\kappa$. For $i \in [n]$, let

$$M(x_i) := M_{x_i}^{(0,\ldots,0,0)} - M_{x_i}^{(0,\ldots,0,1)} \text{ and } N(x_i) := N_{x_i}^{(0,\ldots,0,0)} - N_{x_i}^{(0,\ldots,0,1)}.$$ 

Then

(1) $M(x_i)$ and $N(x_i)$ are binary observables such that $M(x_i)|\psi\rangle = N(x_i)|\psi\rangle$ for all $i \in [n]$,

(2) there are unitary representations $\Phi_M$ and $\Phi_N$ of $\Gamma(A)$ on $\mathcal{H}_0$ sending $x_i \in \Gamma(A)$ to $M(x_i)|\mathcal{H}_0\rangle$ and $N(x_i)|\mathcal{H}_0\rangle$ respectively, and

(3) if $\Phi_M(r)|\psi\rangle = |\psi\rangle$ for some $r \in \Gamma(A)$, then $\Phi_M(r) = 1_{\mathcal{H}_0}$.

Proposition 4.3 is similar to [CLS17, Lemma 8]. We include a full proof for completeness.

**Proof of Proposition 4.3.** Since perfect correlations are synchronous, $M_x^{(a)}|\psi\rangle = N_x^{(a)}|\psi\rangle$ for all $x \in \mathcal{X}, a \in \mathbb{Z}_2^\kappa$ by Proposition 3.4. As a result, we have $M(x_i)|\psi\rangle = N(x_i)|\psi\rangle$ for all $i \in [n]$. For the second part of the proposition, define projections

$$M_{i,k}^{(c)} := \sum_{a \in \mathbb{Z}_2^\kappa |_{\Phi_i(k)=c}} M_i^{(a)}, i \in [m], k \in I_i, c \in \mathbb{Z}_2.$$

Observe that by condition (2) of Definition 4.1, if $i \in [n]$ and $b \in \mathbb{Z}_2^\kappa$ with $(b_0, \ldots, b_{\kappa-2}) \neq (0, \ldots, 0)$, then

$$\langle \psi | N_{x_i}^{(b)} | \psi \rangle = \sum_{a \in \mathbb{Z}_2^\kappa} \langle \psi | M_1^{(a)} N_{x_i}^{(b)} | \psi \rangle = \sum_{a \in \mathbb{Z}_2^\kappa} P(a, b|1, x_i) = 0,$$

and hence $N_{x_i}^{(b)}|\psi\rangle = 0$. Since the strategy is a good strategy, by Definition 3.8, $N_{x_i}^{(b)} = 0$ for $(b_0, \ldots, b_{\kappa-2}) \neq (0, \ldots, 0)$. Thus $\{N_{x_i}^{(0)}, N_{x_i}^{(1)}\}$ is a complete measurement, where for convenience we write $N_{x_i}^{(c)}$ for $N_{x_i}^{(0,\ldots,0,c)}$ for all $i \in [n]$ and $c \in \mathbb{Z}_2$. Then $N(x_i) = N_{x_i}^{(0)} - N_{x_i}^{(1)}$ is a binary observable on $\mathcal{H}$. Similarly, condition (1) of Definition 4.1 and Definition 3.8 imply that $M_i^{(a)} = 0$ for all $i \in [m]$ and $a \not\in S$. 

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By condition (4) of Definition 4.1,
\[
\langle \psi | M^{(0)}_{i,k} \cdot N^{(1)}_{x_k} | \psi \rangle = \sum_{a \in \mathbb{Z}_2^k} \langle \psi | M^{(a)}_{i,k} N^{(1)}_{x_k} | \psi \rangle = \sum_{a,b \in \mathbb{Z}_2^k} P(a, (0, \ldots, 0, 1)|i, x_k) = 0,
\]
and \( \langle \psi | M^{(1)}_{i,k} \cdot N^{(0)}_{x_k} | \psi \rangle = 0 \) similarly. Applying Lemma 3.5 to the projective measurements \( \{ M^{(0)}_{i,k}, M^{(1)}_{i,k} \} \) and \( \{ N^{(0)}_{x_k}, N^{(1)}_{x_k} \} \), we get that \( M^{(c)}_{i,k} | \psi \rangle = N^{(c)}_{x_k} | \psi \rangle \) for all \( i \in [m], k \in I_i \), and \( c \in \mathbb{Z}_2 \).

Now let \( M_{i,k} := M^{(0)}_{i,k} - M^{(1)}_{i,k} \) for some \( i \in [m] \). Since the projections \( M^{(a)}_i, a \in \mathbb{Z}_2^k \) commute, \( [M_{i,k}, M_{i,l}] = 0 \) for all \( i \in [m] \) and \( k, l \in I_i \). If \( k_0, \ldots, k_{\ell-1} \) is a sequence in \( I_i \), and \( \sigma \) is a permutation of \( [\ell] \), then
\[
N(x_{k_0}) \cdots N(x_{k_{\ell-1}}) | \psi \rangle = M_{i,k_{\ell-1}} M_{i,k_{\ell-2}} \cdots M_{i,k_0} | \psi \rangle = M_{i,k_{\ell-1}} \cdots M_{i,k_0} | \psi \rangle
= N(x_{\sigma(k_0)}) \cdots N(x_{\sigma(k_{\ell-1})}) | \psi \rangle,
\]
so the operators \( N(x_k), k \in I_i \) commute on \( | \psi \rangle \). When we take the product across all of \( I_i \),
\[
\prod_{k \in I_i} N(x_k) | \psi \rangle = \prod_{k \in I_i} M_{i,k} | \psi \rangle = \prod_{k \in I_i} \left( \sum_{a \in \mathbb{Z}_2^k} (-1)^{a_{\phi(k)}} M^{(a)}_{i} \right) | \psi \rangle = \sum_{a \in \mathbb{Z}_2^k} (-1)^{\sum k_{\in [x]} a_k} M^{(a)}_{i} | \psi \rangle
= \sum_{a \in S} M^{(a)}_{i} | \psi \rangle = \sum_{a \in \mathbb{Z}_2^k} M^{(a)}_{i} | \psi \rangle = | \psi \rangle,
\]
where we use that \( M^{(a)}_{i} = 0 \) for \( a \not\in S \) and \( \sum_k a_k = 0 \) for \( a \in S \). Finally, \( N(x_k)^2 = N^{(0)}_{x_k} + N^{(1)}_{x_k} = 1_\mathcal{H} \).

To finish the proof, let \( \mathcal{A}_0 \) (resp. \( \mathcal{A}_1 \)) be the algebra generated by \( M^{(a)}_x \) (resp. \( N^{(a)}_x \)) for \( x \in \mathcal{X} \) and \( a \in \mathbb{Z}_2^k \). A standard result about synchronous correlations, following immediately from the fact that \( M^{(a)}_x | \psi \rangle = N^{(a)}_x | \psi \rangle \), is that \( \mathcal{A} | \psi \rangle = \mathcal{A}_0 | \psi \rangle = \mathcal{A}_1 | \psi \rangle \). If \( R \in \mathcal{A}_1 \) satisfies \( R | \psi \rangle = 0 \), then \( RT | \psi \rangle = TR | \psi \rangle = 0 \) for all \( T \in \mathcal{A}_0 \), and thus \( RV = 0 \) for all \( v \in \mathcal{H}_0 = A | \psi \rangle = A_0 | \psi \rangle \). Define \( \Phi_N : \mathcal{F}(x_1, \ldots, x_n) \to U(\mathcal{H}_0) : x_i \mapsto N(x_i) | \mathcal{H}_0 \rangle \), and suppose \( r \in \mathcal{F}(x_1, \ldots, x_n) \) is a defining relation for \( \Gamma(A) \) from Definition 4.2. We’ve shown that \( \Phi_N(r) | \psi \rangle = | \psi \rangle \), and hence \( 1 - \Phi_N(r) \) is 0 on \( \mathcal{H}_0 \). It follows that \( \Phi_N \) induces a representation of \( \Gamma(A) \) on \( \mathcal{H}_0 \) sending \( x_i \mapsto N(x_i) | \mathcal{H}_0 \rangle \). Switching \( M \) and \( N \) in the above argument, we see that there is also a representation of \( \Gamma(A) \) on \( \mathcal{H}_0 \) sending \( x_i \mapsto M(x_i) | \mathcal{H}_0 \rangle \). The same argument shows that if \( \Phi_M(r) | \psi \rangle = | \psi \rangle \) for \( r \in \Gamma(A) \), then \( \Phi_M(r) = 1_\mathcal{H}_0 \).

We can also construct perfect correlations for \( Ax = 0 \) from representations of the solution group \( \Gamma(A) \). We do this for particular representations in Section 7.

### 4.2 Embedding groups in solution groups

To embed KMS groups into solution groups, we use the embedding results of [Slo19]. Recall the following technical definition:
Definition 4.4 (Definition 32 of [Slo19]). Let $A$ be an $m \times n$ matrix over $\mathbb{Z}_2$, $C_0 \subseteq [n] \times [n]$, $C_1 \subseteq [\ell] \times [n] \times [n]$ for some $\ell \geq 1$, and $L$ be an $\ell \times \ell$ lower-triangular matrix with non-negative integer entries. Let

$$
E \Gamma(A, C_0, C_1, L) := \langle \Gamma(A), y_0, \ldots, y_{\ell-1} : x_i x_j x_i = x_k \text{ for all } (i, j, k) \in C_0, \quad y_i^{-1} x_j y_i = x_k \text{ for all } (i, j, k) \in C_1, \quad y_i^{-1} y_j y_i = y_j^{l_{ij}} \text{ for all } i > j \text{ with } L_{ij} > 0 \rangle.
$$

We say a group $G$ is an $m \times n \times \ell$ extended homogeneous-linear-plus-conjugacy group if it has a presentation of this form.

By [Slo19, Propositions 27 and 33], extended homogenous-linear-plus-conjugacy groups can be fa-embedded into solution groups. We use the following version of that result:

**Proposition 4.5.** Let $G = E \Gamma(A, C_0, C_1, L)$ be an $m \times n \times \ell$ extended homogeneous linear-plus-conjugacy group. Then there is an $m' \times n'$ matrix $A'$ over $\mathbb{Z}_2$ for some $m' \geq m$ and $n' \geq n$, and a homomorphism

$$
\tilde{\phi} : \mathcal{F}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{\ell-1}) \rightarrow \mathcal{F}(x_0, \ldots, x_{n'-1}),
$$

such that:

(a) Each row in $A'$ has only three non-zero entries.

(b) $\tilde{\phi}(x_i) = x_i$ for all $i \in [n]$.

(c) For all $i \in [\ell]$ there are $j, k \in [n']$ such that $\tilde{\phi}(y_i) = x_j x_k$.

(d) If $r$ is a defining relation of $G$, then $\tilde{\phi}(r)$ is in the normal subgroup generated by the defining relations of $\Gamma(A')$, so $\tilde{\phi}$ induces a homomorphism $\phi : G \rightarrow \Gamma(A')$.

(e) There are integers $n_1, n_2, n_3$ such that if $\gamma$ is an $e$-representation of $G$, then there is an $O(e)$-representation $\alpha$ of $\Gamma$ with

$$
\alpha(\tilde{\phi}(g)) = \gamma(g)^{\oplus n_1} \oplus \gamma(g)^{\oplus n_2} \oplus 1_{n_3}
$$

for all $g \in \mathcal{F}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{\ell-1})$. As a result, the homomorphism $\phi : G \rightarrow \Gamma(A')$ is an fa-embedding.

Furthermore, both $A'$ and $\tilde{\phi}$ are constructible, in the sense that there's a Turing machine which, given $A$, $C_0$, $C_1$, and $L$, will output $A'$ and $\tilde{\phi}(y_i)$ for all $i \in [\ell]$.

Although we don't need this fact, $A'$ and $\tilde{\phi}$ can be constructed in polynomial time.

**Proof.** Parts (b)-(e) follow from the proofs of Propositions 27 and 33 in [Slo19]. Proposition 27 in [Slo19] is actually about non-homogeneous solution groups $\Gamma(A, b)$, which have an additional central element $J$ of order 2 representing a scalar in $\mathbb{Z}_2$. However, the non-homogeneous solution group $\Gamma(A, b)$ associated with an $m \times n$ linear system $Ax = b$ can
be regarded as a homogeneous solution group by adding variables $x_n, \ldots, x_{2n}$, replacing $J$ with $x_{2n}$ wherever it occurs in the presentation, and adding linear relations $x_i x_{n+i} x_{2n} = e$ for all $0 \leq i < n$ to force $x_{2n}$ to be central. Alternatively, the proof of Proposition 27 can be adapted to the homogeneous case by replacing $J$ with $e$ wherever it occurs in group presentations, and ignoring assignments to $J$ in approximate representations.

For part (a), given an $m_0 \times n_0$ matrix $A_0$ over $\mathbb{Z}_2$, we can find an $m_1 \times n_1$ matrix $A_1$, where $m_1 \geq m_0$ and $n_1 \geq n_0$, such that there is an isomorphism $\Gamma(A_0) \to \Gamma(A_1)$ sending $x_i \mapsto x_i$ for all $i \in [n_0]$, and $A_1$ has exactly three non-zero entries in each row. Indeed, suppose the $i$th row has non-zero entries in columns $j_1, \ldots, j_r$, where $r > 3$. Adding variables $z_{1i}, z_{2i}, z_3$ and equations $z_3 = x_{j_1} x_{j_2}$, $z_1 x_{j_i} x_{j_i} = e$, and $z_2 x_{j_2} x_{j_3} = e$ for all $t = 3, \ldots, r$, and replacing the $i$th row of $A$ with the equation $z_3 x_{j_3} \cdots x_{j_r} = e$, we get an isomorphic solution group where equation $i$ has the number of non-zero entries reduced by one, and all the added equations have length exactly three (the equations $z_{ij} x_{j_i} x_{j_i}$ are needed to force $x_{j_i}$ and $x_{j_i}$ to commute). If row $i$ of $A_0$ has exactly one non-zero entry in column $j$, then we can add variables $z_{ij}, z_{j_2}, z_3$ and replace the $i$th equation with equations $x_j z_1 z_2 = x_j z_1 z_3 = x_j z_2 z_3 = z_1 z_2 z_3 = e$ (which together force $x_j = z_1 = z_2 = z_3 = e$). If row $i$ has exactly two non-zero entries in columns $j$ and $k$, then we can add variables $z_{1i}, z_{2i}$, and replace the $i$th equation with equations $x_j z_1 z_2 = x_k z_1 z_2 = e$. Iterating these steps, we eventually get $A_1$ as desired.

\[\Box\]

5 Minsky machines and Kharlampovich-Myasnikov-Sapir groups

5.1 Minsky machines

A $k$-glass Minsky Machine [Min67], denoted by MM, consists of $k$ glasses where each glass can hold arbitrarily many coins, a set of states $[N]$, and a finite list of commands. Just like a Turing machine, a configuration of MM describes which state the machine is in and how many coins are in each of the glasses. A computation running on MM is a sequence of commands, where each command maps one configuration to another. Commands can leave a glass unchanged, add a coin to a glass, or remove a coin from a non-empty glass, as well as change the state. Glasses are numbered starting from 1. In formal language, this means that a configuration of MM is an element $(i; n_1, n_2, \ldots, n_k) \in [N] \times (\mathbb{Z}_{\geq 0})^k$. The state 0 is regarded as a final halt state, and 1 is regarded as a start state. The accept configuration is $(0; 0, 0, \ldots, 0)$ and the starting configuration with input $m$ is $(1; m, 0, \ldots, 0)$. There are four types of commands:

1. Adding coins: When the state is $i$, add a coin to each of the glasses numbered $j_1, j_2, \ldots, j_\ell$ where $\ell \leq k$, and go to state $j$. This command is encoded as

   \[i; \rightarrow j; Add(j_1, j_2, \ldots, j_\ell).\]

2. Removing coins: When the state is $i$, if the glasses numbered $j_1, j_2, \ldots, j_\ell$, $\ell \leq k$, are all nonempty, then remove a coin from each of the glasses numbered $j_1, j_2, \ldots, j_\ell$, and go
3. Empty check: When the state is $i$, if the glasses numbered $j_1, j_2, \ldots, j_\ell$, $\ell \leq k$, are empty, go to state $j$. This command is encoded as 

$$i; n_{j_1} = 0, n_{j_2} = 0, \ldots, n_{j_\ell} = 0 \rightarrow j.$$

4. Stop: When the state is $i$, change state to 0. This command is encoded as 

$$i; \rightarrow 0.$$

In addition, the input state $i$ to each command must be non-zero, so there are no commands leaving the halt state. As can be seen from the descriptions, commands can change a configuration, but not every command can be applied to every configuration. Some configurations may not have any applicable commands, while some configurations can have more than one applicable command. If for every configuration, there is at most one command that can be applied, the Minsky machine is deterministic. Otherwise, the Minsky machine is non-deterministic. A $k$-glass Minsky machine $MM$ accepts an input $n \in \mathbb{Z}_{\geq 0}$ if there is a sequence of configurations

$$(1; n, 0, \ldots, 0) \equiv_{MM} C_0 \rightarrow C_1 \rightarrow \ldots \rightarrow C_N := (0; 0, \ldots, 0)$$

from the input configuration to the accept configuration such that for every $0 \leq i < N$, there is a command of $MM$ that applies to $C_i$ and transforms it to $C_{i+1}$. In general, we let $\equiv_{MM}$ be the equivalence relation on configurations generated by the relations $C \equiv_{MM} C'$, where $C$ and $C'$ are configurations for which there is a command of $MM$ which applies to $C$ and transforms it to $C'$. Note that the equivalence relation generated by these relations also includes the relations $C' \equiv_{MM} C$ whenever there is a command transforming $C$ to $C'$, as well as the transitive closure of both types of relations. So $C \equiv_{MM} C'$ does not necessarily mean that $C$ can be transformed to $C'$ by applying commands from $MM$, but rather that $C$ can be transformed to $C'$ by applying commands or the inverses of commands. If $MM$ is deterministic, then it’s easy to see that $C_1 \equiv_{MM} C_2$ if and only if $C_1$ and $C_2$ can both be transformed to the same configuration $C_3$ by the operations of $MM$. Since there are no commands with input state 0, a deterministic Minsky machine $MM$ accepts an input $n$ if and only if $(1; n, 0, \ldots, 0) \equiv_{MM} (0; 0, \ldots, 0)$.

Recall that a subset $S$ of natural numbers is recursively enumerable, or RE, if there is an algorithm such that the algorithm halts on input $s$ if and only if $s \in S$. Minsky machines can recognize any recursively enumerable set (or in other words, Minsky machines are Turing complete):

**Theorem 5.1** (Theorem 2.7, part (a) of [KMS17]). $X$ is recursively enumerable if and only if there exists a 3-glass deterministic Minsky machine $MM$ such that $n \in X$ if and only if $MM$ accepts $n$. 

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Proof. In Theorem 2.7, part (a) of [KMS17], the theorem is stated for 2-glass Minsky machines with a different encoding of the input. However, the statement above for 3-glass Minsky machines is used as part of the proof of the 2-glass case.

If we change the input encoding, then 2-glass Minsky machines are also sufficiently powerful to recognize all RE sets. However, we use 3-glass Minsky machines as the starting point for our construction, since the simpler input encoding is necessary to connect with other parts of the construction.

Lemma 5.2. Let $MM$ be a $k$-glass deterministic Minsky machine. Then there is a $(k+1)$-glass deterministic Minsky machine $MM'$ such that

(a) $n$ is accepted by $MM'$ if and only if $n$ is accepted by $MM$, and

(b) if $n$ is not accepted by $MM$, then $(1; n, 0, \ldots, 0) \equiv_{MM'} (1; m_1, \ldots, m_{k+1})$ if and only if $(n, 0, \ldots, 0) = (m_1, \ldots, m_{k+1})$.

Proof. To construct $MM'$, we add six new states $0', 1', 2', 3', 4', 5'$ and $6'$ to $MM$. We keep the commands of $MM$ the same, except that we replace states 0 and 1 with $0'$ and $1'$ wherever they occur in commands. We then add commands

$$
1; n_2 = 0, n_3 = 0, \ldots = n_{k+1} = 0 \rightarrow 2'
$$

$$
2'; n_1 > 0 \rightarrow 3'; Sub(1)
$$

$$
3'; \rightarrow 2'; Add(2, k + 1)
$$

$$
2'; n_1 = 0 \rightarrow 4'
$$

$$
4'; n_2 > 0 \rightarrow 5'; Sub(2)
$$

$$
5'; \rightarrow 4'; Add(1)
$$

$$
4'; n_2 = 0 \rightarrow 1'
$$

$$
0'; n_1 = 0, n_2 = 0, \ldots, n_k = 0 \rightarrow 6'
$$

$$
6'; n_{k+1} > 0 \rightarrow 6'; Sub(k + 1)
$$

$$
6'; n_{k+1} = 0 \rightarrow 0.
$$

To explain these commands, suppose we start in configuration $(1; m_1, \ldots, m_{k+1})$ for $MM'$. We are allowed to move onto state $2'$ if and only if $m_2 = \ldots = m_{k+1} = 0$, or in other words we are in an input configuration $(1; n, 0, \ldots, 0)$. In states $2'$ and $3'$, we copy the 1st glass to the 2nd and $k + 1$th glass. Once the configuration is $(2'; 0, n, 0, \ldots, 0, n)$, we can move on to state $4'$, where we copy back the 2nd glass to the 1st glass to get configuration $(4'; n, 0, \ldots, 0, n)$. At this point, we can move on to state $1'$, where we can now start applying commands from $MM$. Note that commands from $MM$ do not affect the $k + 1$th glass. If input $n$ is accepted by $MM$, then at some point $MM'$ will end up in configuration $(0'; 0, \ldots, 0, n)$. At this point the state changes to $6'$, and we empty out the $k + 1$th glass and then move to the accept configuration $(0; 0, \ldots, 0)$.

Since 0 is not an allowed input state for commands of $MM$, the commands coming from $MM$ in $MM'$ do not share input states with any of the added commands above, so $MM'$ is deterministic. The explanation of the commands above shows that if $MM$ accepts
$n$, then so does $MM'$. If $MM$ does not accept $n$, then $MM'$ will never reach configuration $(0';0,\ldots,0,n)$, and thus cannot proceed to state 0, so $MM'$ does not accept $n$ either.

Finally, suppose $c_1 \equiv_{MM'} c_2$, where $c_1 := (1;n,0,\ldots,0)$ and $c_2 := (1;m_1,\ldots,m_{k+1})$, and $n$ is not accepted by $MM'$. Since $MM'$ is deterministic, this means that $MM'$ will take $(1;n,0,\ldots,0)$ and $(1;m_1,\ldots,m_{k+1})$ to a common configuration $c_3 = (s;p_1,\ldots,p_{k+1})$. If $s = 1$ then $c_1 = c_3 = c_2$. Otherwise $m_2 = \ldots = m_{k+1} = 0$. If $s \in \{1',2',3',4',5'\}$ then $MM'$ will send $c_1$ and $c_2$ to $(1';n,0,\ldots,0,n) = (1';m_1,0,\ldots,0;1$. Since $MM$ does not accept $n$, we cannot have $s \in \{6',0\}$. Since commands from $MM$ do not change the $k+1$th glass of $MM'$, if $s \not\in \{1',2',3',4',5'\}$ then we must have $n = p_{k+1} = m_1$. In each case, we conclude that $(n,0,\ldots,0) = (m_1,m_2,\ldots,m_{k+1})$.

\[\square\]

### 5.2 Kharlampovich-Myasnikov-Sapir groups

For a $k$-glass Minsky machine $MM$, deterministic or non-deterministic, the **Kharlampovich-Myasnikov-Sapir (KMS) group** $G(MM)$ is a finitely presented group defined in [KMS17], based on an earlier construction of Kharlampovich [Kha82]. The definition of this group depends on a parameter $p$, which we always take to be 2. We let $S(MM)$ and $R(MM)$ denote the generating set and relations for the presentation given in [KMS17, Section 4.1].

The point of KMS groups is the following theorem:

**Theorem 5.3** (Properties 3.1 and 3.2 and Theorem 4.3 of [KMS17]). Let $MM$ be a Minsky machine. Then $G(MM)$ is solvable, and there is a computable function $w$ from configurations $c$ of $MM$ to words $w(c)$ in the free group $F(S(MM))$, such that

$$w(c) = w(c') \text{ in } G(MM) \text{ if and only if } c \equiv_{MM} c'.$$

In particular, if $w(n) := w((1;n,0,\ldots,0))$ is the word for input configuration and $w_{accept} := w((0;0,\ldots,0))$ is the word for the accept configuration, then

$$w(n) = w_{accept} \text{ in } G(MM) \text{ if and only if } (1;n,0,\ldots,0) \equiv_{MM} (0;0,\ldots,0).$$

If $MM$ is deterministic, then we can replace this last condition with the condition that $MM$ accepts $n$.

For our purposes, we need some details of the definition of $G(MM)$ and the function $w$ (we also include some additional details for context). We use the notation from [KMS17] for ease of reference. Suppose $MM$ has $k$-glasses and state set $[N+1]$. The generating set $S(MM)$ is divided into subsets $L_0$, $L_1$, and $L_2$, where

$$L_0 = \{x(q_iA_i \cdots A_{i_m}) | i \in [N+1], 0 \leq m \leq k, 0 \leq i_1 < i_2 < \ldots < i_m \leq k\},$$

$$L_1 = \{A_i | 0 \leq i \leq k\},$$

and

$$L_2 = \{a_i, a'_i, \bar{a}_i, \bar{a}'_i | 1 \leq i \leq k\}.$$

Intuitively, the generators $x(u)$ are used to keep track of state, the generators $A_i$ represent the bottom of the $i$th glass, and the generators $a_i$ are used to keep track of coins in the $i$th glass.
The relations $R(MM)$ include a number of relations (marked as (G1)-(G7) in [KMS17, Section 4.1]) which are common to all $k$-glass Minsky machines with state set $[N + 1]$, and then a number of relations (marked as (G8) in the same section) for the commands. For the purpose of discussion, we’ll call these common relations and command relations. The common relations include relations specifying that the elements of $L_0$ and $L_1$ have order two, and that $[x, y] = e$ for all $x, y \in L_i$, $i = 0, 1, 2$. The order of the generators in $L_2$ is unspecified. The common relations also have the property that if $MM'$ has state set $[N' + 1]$ with $N \leq N'$, then the common relations of $R(MM)$ are common relations of $R(MM')$.

For the command relations, there is one relation for each command. To specify these relations, the following notation is used: if $f \in \mathcal{F}(S(MM))$ and $1 \leq j \leq k$, let

$$f \circ a_j := f^{-1}f^{a_j}(f^{-1})^{a_j^{-1}}f^{a_j^{-1}},$$

and

$$f \circ A_j := [f, A_j]$$

(recall that $f^a := a^{-1}fa$). Also, let

$$t_1 \circ t_2 \cdots \circ t_m := (\ldots(t_1 \circ t_2) \circ \ldots) \circ t_m$$

and

$$t_1 \circ t_2^n := t_1 \circ (t_2 \circ \ldots \circ t_2),$$

Then with this notation, the relation for $i; \rightarrow j; Add(j_1, \ldots, j_\ell)$ is

$$x(q_iA_0) = x(q_jA_0) \circ a_{j_1} \circ \cdots \circ a_{j_\ell},$$

the relation for $i; n_{j_1} > 0, \ldots, n_{j_\ell} > 0 \rightarrow j; Sub(j_1, \ldots, j_\ell)$ is

$$x(q_iA_0) \circ a_{j_1} \circ \cdots \circ a_{j_\ell} = x(q_jA_0),$$

the relation for $i; n_{j_1} = 0, n_{j_2} = 0, \ldots, n_{j_\ell} = 0 \rightarrow j$ is

$$x(q_iA_0) \circ A_{j_1} \circ \cdots \circ A_{j_\ell} = x(q_jA_0) \circ A_{j_1} \circ \cdots \circ A_{j_\ell},$$

and the relation for $i; \rightarrow 0$ is

$$x(q_iA_0) = x(q_0A_0).$$

For $n \in \mathbb{Z}_{\geq 0}$, the word corresponding to the input configuration for $n$ is

$$w(n) := x(q_1A_0) \circ a_1^{\circ n} \circ A_1 \circ \cdots \circ A_k.$$

In particular, $w(0) := x(q_1A_0) \circ A_1 \circ \cdots \circ A_k$. For the accept configuration, the group element is

$$w_{accept} := x(q_0A_0) \circ A_1 \circ \cdots \circ A_k.$$

By [KMS17, Relations (G5a) and (G1)], we can actually reduce these two words to

$$w(0) = x(q_1A_0A_1 \cdots A_k)$$

and

$$w_{accept} = x(q_0A_0A_1 \cdots A_k).$$
In particular, \(w_{\text{accept}}^2 = w(0)^2 = [w(0), w_{\text{accept}}] = e\).

Suppose we start with a \(k\)-glass Minsky machine \(\text{MM}_0\), and add states and commands to form a \(k\)-glass Minsky machine \(\text{MM}_1\). Then the generating set \(\text{S}(\text{MM}_0) \subseteq \text{S}(\text{MM}_1)\). As previously mentioned, all the common relations for \(\text{R}(\text{MM}_0)\) belong to \(\text{R}(\text{MM}_1)\), and since all the commands of \(\text{MM}_0\) are commands of \(\text{MM}_1\), the same is true for the command relations. Hence, we also have \(\text{R}(\text{MM}_0) \subseteq \text{R}(\text{MM}_1)\). This leads immediately to the following lemma:

**Lemma 5.4.** Let \(\text{MM}_0\) and \(\text{MM}_1\) be \(k\)-glass Minsky machines with state sets \([N_0]\) and \([N_1]\) respectively, where \(N_0 \leq N_1\). If every command of \(\text{MM}_0\) is a command of \(\text{MM}_1\), then there is a homomorphism \(G(\text{MM}_0) \rightarrow G(\text{MM}_1)\) sending \(x \mapsto x\) for all \(x \in \text{S}(\text{MM}_0) \subseteq \text{S}(\text{MM}_1)\). In particular, this homomorphism sends the elements \(w(0)\) and \(w_{\text{accept}}\) for \(G(\text{MM}_0)\) to the same elements for \(G(\text{MM}_1)\).

We finish this section by stating how KMS groups connect to correlations:

**Lemma 5.5 (Lemma 42 of [Slo19]).** Let \(\text{MM}\) be a Minsky machine, and let \(\text{S}(\text{MM}) = L_0 \cup L_1 \cup L_2\) be the partition described above. Then the KMS group \(G(\text{MM})\) has a presentation as an \(m \times n \times \ell\) extended homogeneous linear-plus-conjugacy group \(\Gamma(A, C_0, C_1, L)\), in which:

(a) the generators in \(L_0\) and \(L_1\) and the elements \(x(q_1A_0) \otimes a_1\) and \(w(0)w_{\text{accept}}\) all belong to the generating set \(\{x_0, \ldots, x_{n-1}\}\), and

(b) the generators in \(L_2\) belong to \(\{y_0, \ldots, y_{\ell-1}\}\).

**Proof.** By Lemma 42 of [Slo19], \(G(\text{MM})\) has a presentation as an \(m' \times n' \times \ell\) extended homogeneous linear-plus-conjugacy group, in which \(x(q_1A_0) \otimes a_1\) belongs to the generating set \(\{x_1, \ldots, x_{n'}\}\) (in the notation of [Slo19], \(x(q_1A_0) \otimes a_1\) belongs to \(\mathcal{N}(L_0, S)\), which is abelian [Slo19, Theorem 40, part (c)]. Looking at the proof of Lemma 42, this presentation is constructed by starting with the generators for \(G(\text{MM})\), and then adding additional generators so that all the relations for \(G(\text{MM})\) can be rewritten as either linear or conjugacy relations. The generators from \(L_0\) and \(L_1\) end up in the generating set \(\{x_1, \ldots, x_{n'}\}\) for the constructed presentation, and the generators \(L_2\) end up in the generating set \(\{y_1, \ldots, y_{\ell}\}\).

The element \(w(0)w_{\text{accept}}\) also belongs to \(\mathcal{N}(L_0, S)\), and hence we can also add additional generators and relations to include this in the generating set of our presentation, following the procedure detailed in the proof of Lemma 42. Alternatively, since \(w(0) = x(q_1A_0A_1 \ldots A_k)\) and \(w_{\text{accept}} = x(q_0A_0A_1 \ldots A_k)\) are already generators in the presentation, and \(w(0)\) and \(w_{\text{accept}}\) commute, we can also just add one additional generator \(x_{n'+1}\), along with the linear relation \(x_{n'+1}w(0)w_{\text{accept}} = e\), to get a presentation with \(w(0)w_{\text{accept}}\) in the generating set as required. \(\Box\)

### 5.3 An extension of the Kharlampovich-Myasnikov-Sapir group

This section is devoted to proving the following proposition.

**Proposition 5.6.** Let \(p(n), n \geq 1\) be an increasing sequence of prime numbers, where the function \(p : \mathbb{Z}_{\geq 1} \rightarrow \mathbb{Z}_{\geq 1}\) is computable, let \(X\) be a recursively enumerable set of positive
integers, and let \( r \) be a positive integer which is coprime to \( p(n) \) for all \( n \geq 1 \). Then there exists an \( m \times n' \times \ell \) extended homogeneous linear-plus-conjugacy group \( H = E \Gamma(A, C_0, C_1, L) \) and generators \( x \in \{x_0, \ldots, x_{n' - 1}\} \) and \( u, t \in \{y_0, \ldots, y_{\ell - 1}\} \) satisfying the following properties:

(a) \( u^{-1}tu = t' \) in \( H \),

(b) \( H / \langle t^{p(n)} = e \rangle \) is sofic for all \( n \geq 1 \),

(c) \( x = e \) in \( H / \langle t^{p(n)} = e \rangle \) if and only if \( n \in X \), and

(d) \( t \) has order \( p(n) \) in \( H / \langle t^{p(n)} = e \rangle \).

The function of the different generators \( x, t, \) and \( u \) will be explained in the next section. We prove Proposition 5.6 in a number of steps, starting with:

**Lemma 5.7.** Let \( p(n), n \geq 1 \) be an increasing sequence of prime numbers, where the function \( p : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} \) is computable, and let \( X \) be a recursively enumerable set of positive integers. Then the set

\[
P_X := \{p(n) \mid n \in X\}
\]

is recursively enumerable.

**Proof.** Let \( A_X \) be the Turing machine that accepts \( x \in \mathbb{N} \) if and only if \( x \in X \). Consider the Turing machine which does the following: Given \( q \in \mathbb{N} \), it computes \( p(n) \) for all \( n \leq q \). If \( q = p(k) \) for some \( k \leq q \), then it runs \( A_X \) on \( k \) and accept if \( A_X \) accepts. If \( q \neq p(k) \) for all \( k \leq q \), then it runs forever without accepting. Since \( p(n) \) is an increasing sequence, if \( q = p(k) \) for some \( k \in \mathbb{Z}_{\geq 1} \) then \( k \leq q \). As a result, this algorithm accepts \( q \) if and only if \( q = p(k) \) for \( k \in X \).

Let \( \text{MM}_0 \) be a 3-glass deterministic Minsky machine that accepts \( q \in \mathbb{N} \) if and only if \( q \in P_X \) (where \( P_X \) is the set from Lemma 5.7), and let \( \text{MM} \) be a 4-glass deterministic Minsky machine satisfying parts (a) and (b) of Lemma 5.2 with respect to \( \text{MM} = \text{MM}_0 \). Let \( G(\text{MM}) = \langle S(\text{MM}) : R(\text{MM}) \rangle \) be the KMS group of \( \text{MM} \). Although \( w(p(n)) = w_{\text{accept}} \) in \( G(\text{MM}) \) if and only if \( n \in X \), for the group \( H \) in Proposition 5.6 we need elements \( x \) and \( t \) which do not depend on \( n \), such that \( x = e \) in \( H / \langle t^{p(n)} = e \rangle \) if and only if \( n \in X \). To get these elements, we’ll start with the group

\[
\overline{G(\text{MM})} := \langle G(\text{MM}), t : [t, a_1] = [t, a_1'] = e, t^{-1}x(q_1A_0)t = x(q_1A_0) \otimes a_1 \rangle,
\]

where \( a_1, a_1' \), and \( x(q_1A_0) \) are the generators of \( G(\text{MM}) \) described above. We’ll then construct \( H \) at the end of the subsection by adding \( u \) to \( \overline{G(\text{MM})} \). For our construction, we also want to consider two other groups

\[
G_{p(n)}(\text{MM}) := \langle G(\text{MM}) : x(q_1A_0) \otimes a_1^{p(n)} = x(q_1A_0) \rangle \text{ and }
\]

\[
\overline{G_{p(n)}(\text{MM})} := \langle G(\text{MM}) : x(q_1A_0) \otimes a_1^{p(n)} = x(q_1A_0), tp(n) = e \rangle,
\]

defined for every \( n \geq 1 \). The group \( \overline{G_{p(n)}(\text{MM})} \) is the quotient of \( \overline{G(\text{MM})} \) by \( \langle t^{p(n)} = e \rangle \). To show this, we need to explain the definition of \( \overline{G(\text{MM})} \) a little more:
**Lemma 5.8.** Let MM be a Minsky machine, and let K be the subgroup of G(MM) generated by $x(q_1A_0), a_1, \text{and } a_1'$. Then there is a homomorphism $\alpha : K \rightarrow K$ sending $a_1 \mapsto a_1, a_1' \mapsto a_1'$, and $x(q_1A_0) \mapsto x(q_1A_0) \otimes a_1$.

**Proof.** As we’ve already noted, the generators of K satisfy the relations $x(q_1A_0)^2 = [a_1, a_1'] = e$. Let

$$\psi : F(S(MM)) \rightarrow \langle b_1, b_2 : [b_1, b_2] = e \rangle = \mathbb{Z} \times \mathbb{Z}$$

be the homomorphism defined by $\psi(a_1) = b_1, \psi(a_1') = b_2$, and $\psi(s) = e$ for all $s \in S(MM) \setminus \{a_1, a_1'\}$. Checking the relations in [KMS17], we see that $\psi(r) = e$ for all $r \in R(MM)$. Hence $\psi$ descends to a homomorphism $G(MM) \rightarrow \mathbb{Z} \times \mathbb{Z}$, and this homomorphism restricts to a surjective homomorphism $\psi : K \rightarrow \mathbb{Z} \times \mathbb{Z}$. Let $\langle x(q_1A_0) \rangle^K$ be the normal subgroup generated by $x(q_1A_0)$ in K. Since $\psi(x(q_1A_0)) = e$, this normal subgroup is contained in the kernel of $\psi$, and hence there is a surjective homomorphism $K/\langle x(q_1A_0) \rangle^K \rightarrow \mathbb{Z} \times \mathbb{Z}$. Since $[a_1, a_1'] = e$, we conclude that there is also a homomorphism $\mathbb{Z} \times \mathbb{Z} \rightarrow K$ sending $b_1 \mapsto a_1$ and $b_2 \mapsto a_1'$. Hence $K/\langle x(q_1A_0) \rangle^K \cong \mathbb{Z} \times \mathbb{Z}$, and thus $\langle x(q_1A_0) \rangle^K$ is the kernel of $\psi$ in K. We conclude that $K \cong \langle x(q_1A_0) \rangle^K \times (\mathbb{Z} \times \mathbb{Z})$, and in particular every element of K can be written uniquely as $ta_1^n(a_1')^m$ for some $t \in \langle x(q_1A_0) \rangle^K$ and $n, m \in \mathbb{Z}$.

By [KMS17, Lemma 4.1], $\langle x(q_1A_0) \rangle^K$ is abelian. Hence the functions $\langle x(q_1A_0) \rangle^K \rightarrow \langle x(q_1A_0) \rangle^K$ sending $f \mapsto f^{-1}, f \mapsto f^{a_1}, f \mapsto (f^{-1})^{a_1}$, and $f \mapsto f^{(a_1')^{-1}}$ are all homomorphisms, and so we conclude that

$$\langle x(q_1A_0) \rangle^K \rightarrow \langle x(q_1A_0) \rangle^K : f \mapsto f \otimes a_1$$

is a homomorphism. Using the relation $[a_1, a_1'] = e$ again, we also see that

$$f^{a_1^n} \otimes a_1 = (f \otimes a_1)^{a_1^n} \text{ and } f^{a_1^n} \otimes a_1 = (f \otimes a_1)^{a_1^n}$$

for all $n, m \in \mathbb{Z}$. Using the fact that $K \cong \langle x(q_1A_0) \rangle^K \times (\mathbb{Z} \times \mathbb{Z})$, we see that there is a homomorphism $\alpha : K \rightarrow K$ sending $a_1 \mapsto a_1, a_1' \mapsto a_1'$, and $f \in \langle x(q_1A_0) \rangle^K$ to $f \otimes a_1$ as desired.

Let $K$ and $\alpha$ be the group and homomorphism from Lemma 5.8, with $MM = \overline{MM}$. The presentation of $G(MM)$ states that $t^{-1}ft = \alpha(f)$ for $f \in \{a_1, a_1', x(q_1A_0)\}$, so this identity holds for all $f \in K$. In other words, $G(MM)$ is analogous to the HNN-extension of $G(MM)$ by $\alpha$, although strictly speaking we do not know if $G(MM)$ is an HNN extension, since we do not know if $\alpha$ is injective. This is enough to show:

**Corollary 5.9.** $G(MM)/\langle t^{p(n)} = e \rangle \cong G_{p(n)}(MM)$.

**Proof.** Let $K$ and $\alpha$ be the group and homomorphism from Lemma 5.8, with $MM = \overline{MM}$. Since $t^{-1}ft = \alpha(f)$ for all $f \in K$, $t^{-k}ft^k = \alpha^k(f)$ for all $k \geq 0$. If $f \in \langle x(q_1A_0) \rangle^K$, then the proof of Lemma 5.8 shows that $\alpha(f) = f \otimes a_1 \in \langle x(q_1A_0) \rangle^K$, so $\alpha^k(f) = f \otimes a_1^{\otimes k}$ for all $k \geq 0$. Thus, in $G(MM)/\langle t^{p(n)} = e \rangle$ we have

$$x(q_1A_0) \otimes a_1^{\otimes p(n)} = t^{-p(n)}x(q_1A_0)t^{p(n)} = x(q_1A_0).$$

So $G(MM)/\langle t^{p(n)} = e \rangle = \langle G(MM) : t^{p(n)} = e, x(q_1A_0) \otimes a_1^{\otimes p(n)} = x(q_1A_0) \rangle = G_{p(n)}(MM)$. 

\[ \square \]
Lemma 5.10. \(\overline{G_{p(n)}(MM)}\) is a \(\mathbb{Z}_{p(n)}\)-HNN-extension of \(G_{p(n)}(MM)\) over the subgroup \(K_{p(n)}\) generated by \(x(q_1A_0)\), \(a_1\), and \(a_1'\) in \(G_{p(n)}(MM)\). In addition, \(K_{p(n)}\) is amenable.

Proof. We continue with the notation from the proof of Lemma 5.8, with \(MM = MM\). Note that the map \(\psi : F(S(MM)) \to \langle b_1, b_2 : [b_1, b_2] = e \rangle\) sending \(a_1 \mapsto b_1, a_1' \mapsto b_2\), and \(s \mapsto e\) for all \(s \in S(MM) \setminus \{a_1, a_1'\}\) sends \(r \mapsto e\) for all \(r\) in the normal subgroup of \(F(S(MM))\) generated by \(S(MM) \setminus \{a_1, a_1'\}\). If \(f\) belongs to this subgroup, then \(f \odot a_1\) also belongs to this subgroup, and hence

\[
\psi(x(q_1A_0) \odot a_1^{\odot p(n)}) = e = \psi(x(q_1A_0)).
\]

We conclude that \(\psi\) induces a homomorphism \(G_{p(n)}(MM) \to \mathbb{Z} \times \mathbb{Z}\). We can then follow the proof of Lemma 5.8 exactly to show that \(K_{p(n)} = \langle x(q_1A_0) \rangle^{K_{p(n)}} \times (\mathbb{Z} \times \mathbb{Z})\), and that there is a homomorphism \(\tilde{\alpha} : K_{p(n)} \to K_{p(n)}\) such that \(\tilde{\alpha}(a_1) = a_1, \tilde{\alpha}(a_1') = a_1',\) and \(\tilde{\alpha}(f) = f \odot a_1\) for all \(f \in \langle x(q_1A_0) \rangle^{K_{p(n)}}\).

If \(f \in \langle x(q_1A_0) \rangle^{K_{p(n)}}\), then \(\tilde{\alpha}(f) = f \odot a_1 \in \langle x(q_1A_0) \rangle^{K_{p(n)}}\), so \(\tilde{\alpha}^k(f) = f \odot a_1^{\odot k}\) for all \(k \geq 0\). Hence

\[
\tilde{\alpha}^{p(n)}(x(q_1A_0)) = x(q_1A_0) \odot a_1^{\odot p(n)} = x(q_1A_0).
\]

Since \(\tilde{\alpha}^{p(n)}(a_1) = a_1\) and \(\tilde{\alpha}^{p(n)}(a_1') = a_1'\) as well, we conclude that \(\tilde{\alpha}^{p(n)} = 1\) on \(K_{p(n)}\). In other words, \(\tilde{\alpha}\) is an automorphism of order \(p(n)\). Looking at the presentations, we see that \(\overline{G_{p(n)}(MM)}\) is the \(\mathbb{Z}_{p(n)}\)-HNN extension of \(G_{p(n)}(MM)\) by \(\tilde{\alpha}\).

To see that \(K_{p(n)}\) is amenable, observe that \(\langle x(q_1A_0) \rangle^{K_{p(n)}}\) is a subgroup of the group \(T\) from [KMS17, Lemma 4.5], and hence is abelian. As a semidirect product of two abelian groups, \(K_{p(n)}\) is solvable, and hence amenable.

Towards proving part (b) of Proposition 5.6, we get:

Corollary 5.11. The group \(\overline{G_{p(n)}(MM)}\) is sofic.

Proof. Since \(G(MM)\) is solvable and \(G_{p(n)}(MM)\) is a quotient of \(G(MM)\), \(G_{p(n)}(MM)\) is solvable. Since \(\overline{G_{p(n)}(MM)}\) is a \(\mathbb{Z}_{p(n)}\)-HNN-extension of \(G_{p(n)}(MM)\) over the amenable subgroup \(K_{p(n)}\), \(\overline{G_{p(n)}(MM)}\) is sofic by Lemma 2.7.

For the proof of Proposition 5.6, we’ll take \(x = w(0)w_{\text{accept}}\). We already have the ingredients to show that \(x = e\) in \(G_{p(n)}(MM)\) if \(n \in X\). However, we also need to ensure that \(x \neq e\) if \(n \notin X\). For this, we introduce a non-deterministic modification of \(MM\), denoted by \(MM(p(n))\). To construct \(MM(p(n))\) from \(MM\), we add \(p(n) - 1\) additional states, which we denote by \(2', \ldots, p(n)\). We include all the commands of \(MM\) in \(MM(p(n))\), and we add \(p(n)\) new commands:

\[
\begin{align*}
1; \ &\to \ 2'; \text{Add}(1) \\
i'; \ &\to \ (i + 1)' \text{; Add}(1) \text{ for } 2 \leq i < p(n) \\
p(n)'; \ &\to \ 1; \text{Add}(1).
\end{align*}
\]
In other words, in configuration \((1;k,0,0,0)\) in \(\text{MM}^{p(n)}\), we have two choices. We can either apply commands from \(\text{MM}\), or add a coin to the first glass and proceed to state \(2'\). After this choice, we are forced to go through the states \(i'\) for \(i = 2, \ldots, p(n)\), adding a coin to the first glass each time, until we return to state 1 in configuration \((1;k + p(n),0,0,0)\). Since \(\text{MM}\) was constructed using Lemma 5.2, we can show:

**Lemma 5.12.** Let \(n \geq 1\). Then \((1;0,0,0,0) \equiv_{\text{MM}^{p(n)}} (0;0,0,0,0)\) if and only if \(n \in X\).

**Proof.** If \(n \in X\), then \((1;p(n),0,0,0)\) is accepted by \(\text{MM}\). So in \(\text{MM}^{p(n)}\) there is a computation path going from \((1;0,0,0,0)\) to \((1;p(n),0,0,0)\), and from there to the accept configuration. Hence \((1;0,0,0,0) \equiv_{\text{MM}^{p(n)}} (0;0,0,0,0)\).

For the other direction, suppose \((1;0,0,0,0) \equiv_{\text{MM}^{p(n)}} (0;0,0,0,0)\). Let \(N\) be the smallest integer such that there is a sequence of configurations

\[
(1;0,0,0,0) =: C_0 \rightarrow C_1 \rightarrow \ldots \rightarrow C_N := (0;0,0,0,0),
\]

where for all \(i = 1, \ldots, N\), either \(C_{i-1}\) can be transformed to \(C_i\) by a command of \(\text{MM}^{p(n)}\), or \(C_i\) can be transformed to \(C_{i-1}\). Let \(k\) be the largest integer such that \(C_0 \rightarrow C_1 \rightarrow \ldots \rightarrow C_k\) does not use any commands from \(\text{MM}\). The states of the configurations \(C_i\) for \(0 \leq i \leq k\) must belong to \(\{1,2',3',\ldots, p(n)\}'\), so in particular \(k < N\). In addition, since \(C_k \rightarrow C_{k+1}\) must use a command from \(\text{MM}\), \(C_k\) must be in state 1. The sequence \(C_0 \rightarrow \ldots \rightarrow C_k\) shows that \((1;0,0,0,0) \equiv_{\text{MM}'} C_k\) in the Minsky machine \(\text{MM}'\) with states \(\{1,2',\ldots, p(n)\}'\) and commands

\[
\begin{align*}
1; & \rightarrow 2'; \text{Add}(1) \\
i' & \rightarrow (i+1)'; \text{Add}(1) \text{ for } 2 \leq i < p(n) \\
p(n)' & \rightarrow 1; \text{Add}(1).
\end{align*}
\]

\(\text{MM}'\) is deterministic and sends \((1;a,0,0,0)\) to \((1;a + mp(n),0,0,0)\) for \(m \geq 0\). Thus \((1;0,0,0,0) \equiv_{\text{MM}'} (1;b,0,0,0)\) if and only if \(a = b \mod p(n)\). We conclude that \(C_k = (1;mp(n),0,0,0)\) for some \(m \geq 0\).

Let \(\ell \leq k\) be the largest integer such that \(C_k \rightarrow \ldots \rightarrow C_\ell\) involves only commands from \(\text{MM}\). If \(\ell = N\), then \((1;mp(n),0,0,0) \equiv_{\text{MM}} (0;0,0,0,0)\). Since \(\text{MM}\) is deterministic, that would mean that \(mp(n)\) is accepted by \(\text{MM}\), so \(m = 1\) and \(n \in X\) as desired.

Suppose \(\ell < N\), so \(C_\ell \rightarrow C_{\ell+1}\) involves a command not in \(\text{MM}\). The configurations \(C_i\) for \(k \leq i \leq \ell\) cannot be in states \(\{2',\ldots, p(n)\}'\), so \(C_\ell\) must be in state 1. The sequence \(C_k \rightarrow \ldots \rightarrow C_\ell\) shows that \(C_\ell \equiv_{\text{MM}} C_k\). If \(n \notin X\), then \(mp(n)\) is not accepted by \(\text{MM}\) for all \(m \geq 0\), so \(C_\ell = C_k\) by Lemma 5.2. But then \(C_0 \rightarrow \ldots \rightarrow C_k \rightarrow C_{\ell+1} \rightarrow \ldots \rightarrow C_N\) is a sequence showing that \((1;0,0,0,0) \equiv_{\text{MM}^{p(n)}} (0;0,0,0,0)\). Since \(k < \ell\) by the definition of \(k\), this contradicts the minimality of \(N\). So again, we conclude that \(n \in X\) as desired. 

We are now ready to show:

**Lemma 5.13.** In \(G_{p(n)}(\text{MM})\), \(w(0) = w_{\text{accept}}\) if and only if \(n \in X\).
Proof. Because $x(q_1 A_0) \otimes a_1^{p(n)} = x(q_1 A_0)$ in $G_{p(n)}(\MM)$,

$$w(0) = x(q_1 A_0) \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_4 = x(q_1 A_0) \otimes a_1^{p(n)} \otimes A_1 \otimes A_2 \otimes A_3 \otimes A_4 = w(p(n)).$$

If $n \in X$, then $p(n)$ is accepted by $\MM$, so $w(p(n)) = w_{\text{accept}}$ in $G(\MM)$, and hence the same is true in $G_{p(n)}(\MM)$. Thus we conclude that $w(0) = w_{\text{accept}}$ in $G_{p(n)}(\MM)$ when $n \in X$.

For the other direction, we want to show that there is a homomorphism from $G_{p(n)}(\MM)$ to $G(\MM)^{p(n)}$. By Lemma 5.4, there is a homomorphism $G(\MM) \rightarrow G(\MM)^{p(n)}$ which is the identity on generators. From the states and commands added to $\MM^{p(n)}$, we have generators $x(q_i A_0)$ for $2 \leq i \leq p(n)$, and relations

$$x(q_1 A_0) = x(q_2 A_0) \otimes a_1,$$

$$x(q_i A_0) = x(q_{i+1} A_0) \otimes a_1 \text{ for all } 2 \leq i < p(n),$$

$$x(q_{p(n)} A_0) = x(q_1 A_0) \otimes a_1.$$

Combining these relations, we see that

$$x(q_1 A_0) \otimes a_1^{p(n)} = x(q_1 A_0)$$

in $G(\MM)^{p(n)}$, so the homomorphism $G(\MM) \rightarrow G(\MM)^{p(n)}$ descends to a homomorphism $G_{p(n)}(\MM) \rightarrow G(\MM)^{p(n)}$. If $n \not\in X$, then by Lemma 5.12, $(1;0,0,0,0) \not\equiv_{\MM^{p(n)}} (0;0,0,0,0)$, so $w(0) \neq w_{\text{accept}}$ in $G(\MM)^{p(n)}$. Hence $w(0) \neq w_{\text{accept}}$ in $G_{p(n)}(\MM)$.

The relations between $\overline{G(\MM)} / \langle t^{p(n)} = e \rangle$, $G_{p(n)}(\MM)$, $\overline{G_{p(n)}(\MM)}$ and $G(\MM)^{p(n)}$ are summarized in the following equation:

$$G(\MM)^{p(n)} \leftarrow G_{p(n)}(\MM) \leftrightarrow \overline{G_{p(n)}(\MM)} \cong \overline{G(\MM)} / \langle t^{p(n)} = e \rangle.$$

We are finally ready to prove:

Proof of Proposition 5.6. Let

$$H = \langle \overline{G(\MM)}, u : u^{-1}tu = t^r \rangle.$$

We take $x = w(0)w_{\text{accept}} \in H$, and $u$ and $t$ to be the generators already defined in $H$. Part (a) follows immediately from the definition. For parts (b)-(d), observe that

$$H / \langle t^{p(n)} = e \rangle = \langle \overline{G_{p(n)}(\MM)}, u : u^{-1}tu = t^r \rangle$$

by Corollary 5.9. By Lemma 5.10, $\overline{G_{p(n)}(\MM)}$ is a $\mathbb{Z}_{p(n)}$-HNN extension of $G_{p(n)}(\MM)$. Hence $t$ has order $p(n)$ in $\overline{G_{p(n)}(\MM)}$ by Lemma 2.1. Since $r$ is coprime to $p(n)$, $t^i \mapsto t^{ri}$ is an automorphism of the subgroup generated by $t$ in $\overline{G_{p(n)}(\MM)}$, so $H / \langle t^{p(n)} = e \rangle$ is an HNN-extension of $\overline{G_{p(n)}(\MM)}$ over the subgroup generated by $t$. Since the subgroup generated by $t$ is finite (and hence amenable), $H / \langle t^{p(n)} = e \rangle$ is sofic by [CLP15, Proposition
2.4.1], proving part (b). We also get that \( \overline{G_{p(n)}}(\mathbb{M}) \) is a subgroup of \( H / \langle t^{p(n)} = e \rangle \), so \( t \) has order \( p(n) \) in \( H / \langle t^{p(n)} = e \rangle \) as well, proving part (d). And since \( G_{p(n)}(\mathbb{M}) \) is a subgroup of \( \overline{G_{p(n)}}(\mathbb{M}) \) by Lemma 2.1, \( x = e \) in \( H / \langle t^{p(n)} = e \rangle \) if and only if \( n \in X \) by Lemma 5.13, proving part (c).

It remains to show that \( H \) is an extended homogeneous linear-plus-conjugacy group, with \( u, t, \) and \( x \) in the generating set. By Lemma 5.5, \( G(\mathbb{M}) \) has a presentation as an \( m \times n' \times \ell' \) extended homogeneous linear-plus-conjugacy group \( E\Gamma(A, C_0, C_1, L) \), in which \( x(q_1A_0), x = w(0)w_{\text{accept}} \) and \( x(q_1A_0) \otimes a_1 \) belong to the generating set \( \{x_0, \ldots, x_{n'-1}\} \), and \( a_1 \) and \( a'_1 \) belong to the generating set \( \{y_0, \ldots, y_{\ell'-1}\} \). To present \( H \) as an extended homogeneous linear-plus-conjugacy group, we can add two additional generators \( y_{\ell'} \) and \( y_{\ell'+1} \) for \( t \) and \( u \) respectively, and add the conjugacy relations

\[
y_{\ell'}^{-1}x(q_1A_0)y_{\ell'} = x(q_1A_0) \otimes a_1, y_{\ell'}^{-1}a_1y_{\ell'} = a_1, y_{\ell'}^{-1}a'_1y_{\ell'} = a'_1, \text{ and } y_{\ell'+1}^{-1}y_{\ell'}y_{\ell'+1} = y_{\ell'}.
\]

This gives a presentation of \( H \) as an \( m \times n' \times \ell \) extended homogeneous linear-plus-conjugacy group, where \( \ell = \ell' + 2 \). \( \square \)

### 5.4 Embedding KMS groups in solution groups

**Proposition 5.14.** Let \( p(n) \), \( n \geq 1 \) be an increasing sequence of prime numbers, where the function \( p : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1} \) is computable, let \( X \) be a recursively enumerable set of positive integers, and let \( r \) be a positive integer which is coprime to \( p(n) \) for all \( n \geq 1 \). Then there is an \( m \times n \) solution group \( \Gamma(A) \) with generators \( x, t_1, t_2, u_1, u_2 \in \{x_0, \ldots, x_{n-1}\} \) such that

(a) \( u_2u_1t_1t_2u_1u_2 = (t_1t_2)^r \) in \( \Gamma(A) \),

(b) \( x = e \) in \( \Gamma(A) / \langle (t_1t_2)^{p(n)} = e \rangle \) if and only if \( n \in X \),

(c) if \( n \notin X \), then \( x \) is non-trivial in approximate representations of \( \Gamma(A) / \langle (t_1t_2)^{p(n)} = e \rangle \),

(d) \( t_1t_2 \) has order \( p(n) \) in \( \Gamma(A) / \langle (t_1t_2)^{p(n)} = e \rangle \),

(e) \( w \in \langle t_1, t_2 \rangle \setminus \{e\} \) is nontrivial in approximate representations of \( \Gamma(A) / \langle (t_1t_2)^{p(n)} = e \rangle \), and

(f) each row of \( A \) only has three nonzero entries.

**Proof.** Let \( H \) be the \( m \times n \times \ell \) extended homogeneous linear-plus-conjugacy group from Proposition 5.6, with elements \( x, t, u \in H \). Let \( \Gamma(A') \) be the \( m' \times n' \) solution group corresponding to \( H \) from Proposition 4.5, and let

\[
\tilde{\phi} : \mathcal{F}(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{\ell-1}) \to \mathcal{F}(x_0, \ldots, x_{n'-1}) \text{ and } \phi : H \to \Gamma(A')
\]

be the homomorphisms from that proposition. By parts (b) and (c) of Proposition 4.5, \( \tilde{\phi}(x) = x, \tilde{\phi}(t) = t_1t_2, \) and \( \tilde{\phi}(u) = u_1u_2 \) for some generators \( x, t_1, t_2, u_1, u_2 \in \{x_0, \ldots, x_{n'-1}\} \). Since \( \phi \) is a homomorphism,

\[
u_2u_1t_1t_2u_1u_2 = \phi(u^{-1}tu) = \phi(t') = (t_1t_2)^r,
\]
proving part (a). Let \( \Gamma_n := \Gamma(A') / \langle (t_1 t_2)^{p(n)} = e \rangle \). If \( n \in X \), then \( x = e \) in \( H / \langle t^{p(n)} = e \rangle \), so \( x = e \) in \( \Gamma_n \).

By part (e) of Proposition 4.5, there are integers \( n_1, n_2, n_3 \) such that for any \( \epsilon \)-representation \( \psi \) of \( H \), there is an \( O(\epsilon) \)-representation \( \alpha \) of \( \Gamma(A') \) such that

\[
\alpha(\tilde{\phi}(g)) = \psi(g)^{\otimes n_1} \oplus \tilde{\psi}(g)^{\otimes n_2} \oplus 1_{n_3} \text{ for all } g \in F(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{\ell-1}).
\]

If \( \psi \) is an \( \epsilon \)-representation of \( H / \langle t^{p(n)} = e \rangle \), then \( \| \psi(t^{p(n)}) - 1 \| \leq \epsilon \), so

\[
\| \alpha((t_1 t_2)^{p(n)}) - 1 \| = \| \alpha(\tilde{\phi}(t^{p(n)})) - 1 \| = \| \psi(t^{p(n)})^{\otimes n_1} \oplus \tilde{\psi}(t^{p(n)})^{\otimes n_2} \oplus 1_{n_3} - 1 \| \leq \epsilon
\]
as well, and \( \alpha \) is an \( O(\epsilon) \)-representation of \( \Gamma_n \).

Suppose \( w \neq e \) in \( H / \langle t^{p(n)} = e \rangle \). Since \( H / \langle t^{p(n)} = e \rangle \) is sofic, \( w \) is non-trivial in approximate representations of \( H / \langle t^{p(n)} = e \rangle \), which means that there is \( \delta > 0 \) such that for all \( \epsilon > 0 \), there is an \( \epsilon \)-representation \( \psi \) with \( \| \psi(w) - 1 \| \geq \delta \). If \( \alpha \) is an \( O(\epsilon) \)-representation of \( \Gamma_n \) as above, then

\[
\| \alpha(\tilde{\phi}(w)) - 1 \| \geq \sqrt{\frac{n_1 d + n_2 d}{n_1 d + n_2 d + n_3}} \geq \sqrt{\frac{n_1 + n_2}{n_1 + n_2 + n_3}} \delta.
\]

Since \( n_1, n_2, \) and \( n_3 \) are independent of \( \epsilon \), \( \tilde{\phi}(w) \) is non-trivial in approximate representations of \( \Gamma_n \).

If \( n \not\in X \), then \( x \neq e \) in \( H / \langle t^{p(n)} = e \rangle \), so \( x \) is non-trivial in approximate representations of \( \Gamma_n \). In particular, \( x \neq e \) in \( \Gamma_n \), so parts (b) and (c) hold. Similarly, \( t^i \neq e \) in \( H / \langle t^{p(n)} = e \rangle \) for all \( 0 < i < p(n) \), and thus \( (t_1 t_2)^i \neq e \) in \( \Gamma_n \), proving part (d). This also shows that \( (t_1 t_2)^i \) is nontrivial in approximate representations of \( \Gamma_n \) for \( 0 < i < p(n) \), or in other words that \( (t_1 t_2)^i \neq e \) in \( \Gamma_n^{f_a} \). This means that \( t_1 \) and \( t_2 \) must have order 2 in \( \Gamma_n^{f_a} \). Thus \( \langle t_1, t_2 \rangle \Gamma_n^{f_a} \cong D_p(n) \). Since \( \langle t_1, t_2 \rangle \Gamma_n \cong D_p(n) \) as well, this proves part (e). Finally, part (f) follows immediately from part (a) of Proposition 4.5.

\[\Box\]

6 Constant-sized correlations \( \mathcal{C}_p \) for the dihedral groups

In Section 4, we showed that any strategy for a perfect correlation of a solution group must come from a representation of that group. The dihedral group \( D_n \) has a presentation as an extended homogeneous linear-plus-conjugacy group, and thus can be embedded in solution groups. However, the size of this solution group will depend on \( n \). In this section we write down a constant-sized correlation \( \mathcal{C}_p \) for the dihedral group \( D_p, \) \( p \) a prime, such that any commuting-operator strategy (meeting a condition which we will enforce in the next section using perfect correlations for the solution group from Proposition 5.14) induces a representation of the dihedral group. The construction we use comes from [Fu22], although we modify the construction slightly so that the correlation comes from the regular representation, rather than the representation used in [Fu22]. This modification is necessary for the next section, where we use the correlation from this section in conjunction with perfect correlations for the solution group in Proposition 5.14.
To define \( C_p \), recall that \( D_p \) is generated by \( t_1 \) and \( t_2 \), and consists of the elements \((t_1 t_2)^j \) and \( t_2 (t_1 t_2)^j \) for \( j \in [p] \). As in Section 2, let \( L \) and \( R \) denote the left and right regular representations of \( D_p \) on

\[
\ell^2 D_p = \text{span}(\{(t_1 t_2)^j, t_2 (t_1 t_2)^j \mid j \in [p]\}).
\]

Define elements

\[
\begin{align*}
\pi_0^{(0)} &= \frac{1}{p} \sum_{j \in [p]} (t_1 t_2)^j, \\
\pi_0^{(1)} &= \frac{2}{p} \sum_{j \in [p]} \cos\left(\frac{2j\pi}{p}\right)(t_1 t_2)^j, \\
\pi_0^{(2)} &= e - \pi_0^{(0)} - \pi_0^{(1)}, \\
\pi_1^{(0)} &= \frac{1}{2} \pi_0^{(1)} + \frac{1}{p} \sum_{j \in [p]} \cos\left(\frac{(2j+1)\pi}{p}\right)t_2 (t_1 t_2)^j, \\
\pi_1^{(1)} &= \pi_0^{(1)} - \pi_1^{(0)}, \\
\pi_1^{(2)} &= e - \pi_0^{(1)}, \\
\pi_2^{(0)} &= \frac{1}{2} \pi_0^{(1)} + \frac{1}{p} \sum_{j \in [p]} \sin\left(\frac{(2j+1)\pi}{p}\right)t_2 (t_1 t_2)^j, \\
\pi_2^{(1)} &= \pi_0^{(1)} - \pi_2^{(0)}, \quad \text{and} \\
\pi_2^{(2)} &= e - \pi_0^{(1)} \\
\end{align*}
\]

in the group algebra \( \mathbb{C}[D_p] \). These elements are all projections. For instance, \( \pi_0^{(0)} \) is the sum of central projections for the trivial and sign representations. \( \pi_0^{(1)} \) is the central projection for the 2-dimensional irreducible representation \( V^{(1)} \) sending

\[
t_1 \mapsto \begin{pmatrix} 0 & \omega_p \\ \omega_p^{-1} & 0 \end{pmatrix} \quad \text{and} \quad t_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The projections \( \pi_1^{(0)} \) and \( \pi_2^{(0)} \) are more complicated: they correspond to the rank-one projections

\[
\begin{pmatrix} 1 & \omega_2 \omega_p \\ \omega_2 \omega_p^{-1} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -i\omega_2 \omega_p \\ i\omega_2 \omega_p^{-1} & 1 \end{pmatrix}
\]

in \( V^{(1)} \).

Before we define \( C_p \), we first consider the correlation \( C'_p \) for the scenario \([(5), (5), (3), (3)]\) defined by

\[
C'_p(a, b \mid x, y) = \frac{(e | L(e - \pi_0^{(0)}) P^{(a)}_x Q^{(b)}_y L(e - \pi_0^{(0)}) | e)}{||L(e - \pi_0^{(0)})|e||^2}, \quad x, y \in [5], a, b \in [3],
\]

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where

\[
\hat{P}_x^{(a)} = \begin{cases} 
L(\pi_0^{(a+1)}) & \text{if } x = 0, a \in \{0, 1\} \\
\frac{L(e) + (-1)^a L(t_x)}{2} & \text{if } x \in \{1, 2\}, a \in \{0, 1\} \\
L(\pi_1^{(a)}) & \text{if } x = 3 \\
L(\pi_2^{(a)}) & \text{if } x = 4 \\
0 & \text{otherwise}
\end{cases},
\]

\[
\hat{Q}_y^{(b)} = \begin{cases} 
R(\pi_0^{(b+1)}) & \text{if } y = 0, b \in \{0, 1\} \\
\frac{R(e) + (-1)^b R(t_y)}{2} & \text{if } y \in \{1, 2\}, b \in \{0, 1\} \\
R(\pi_1^{(b)}) & \text{if } y = 3 \\
R(\pi_2^{(b)}) & \text{if } y = 4 \\
0 & \text{otherwise}
\end{cases}.
\]

Direct calculation gives us the following.

**Lemma 6.1.** The correlation \( \mathfrak{C}_p \) is the correlation \( \hat{P}_{-\pi/p} \) defined in [Fu22, Definition 3.2].

One important property of \( \hat{P}_{-\pi/p} \) is summarized in the following proposition.

**Proposition 6.2** (Proposition 3.3 of [Fu22]). Let \( |\psi\rangle, \{|P_x^{(a)}\rangle\}, \{|Q_y^{(b)}\rangle\} \) be an inducing strategy of \( \hat{P}_{-\pi/p} \), and let

\[
|\phi_1\rangle = \frac{1}{2} \left( P_3^{(0)} + IP_3^{(1)} - iP_3^{(0)} + P_3^{(1)} \right) |\psi\rangle,
\]

where \( P_4 = P_4^{(0)} - P_4^{(1)} \). Then

\[
|||\phi_1\rangle||^2 = 1/(p - 1), \quad P_1P_2|\phi_1\rangle = \omega_p^{-1}|\phi_1\rangle, \quad \text{and} \quad Q_1Q_2|\phi_1\rangle = \omega_p|\phi_1\rangle,
\]

where \( P_x = P_x^{(0)} - P_x^{(1)} \) and \( Q_y = Q_y^{(0)} - Q_y^{(1)} \) for \( x, y \in \{1, 2\} \).

Note that the proof of Proposition 3.3 of [Fu22] also tells us that

\[
|\phi_1\rangle = \frac{1}{2} \left( Q_3^{(0)} - iQ_4Q_3^{(1)} + iQ_4Q_3^{(0)} + Q_3^{(1)} \right) |\psi\rangle,
\]

where \( Q_4 = Q_4^{(0)} - Q_4^{(1)} \).

Now we are ready to define \( \mathfrak{C}_p \). Let \( I := \{0, 1, 2, t_1, t_2, (0, t_1), (0, t_2)\} \). We define \( \mathfrak{C}_p \) as a correlation for the scenario \( (I, I, [3] \times [2], [3] \times [2]) \) by

\[
\mathfrak{C}_p(a, b|x, y) = \langle e|\bar{M}_{x}^{(a)}N_{y}^{(b)}|e\rangle, \quad x, y \in I, a, b \in [3] \times [2],
\]

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where \(|e\rangle \in L_2D_p\),

\[
\tilde{M}_{x}^{(a_0,a_1)}(t) = \begin{cases} 
L(t x^{(a_1)}) & \text{if } x \in \{0,1,2\}, a_1 = 0, \\
L(e) + (-1)^{a_1}L(x) & \text{if } x \in \{t_1, t_2\}, a_0 = 0, \text{ and}\\n\tilde{M}_0^{(a_0,0)} \tilde{M}_t^{(0,a_1)} & \text{if } x = (0, t)\\n0 & \text{otherwise}
\end{cases}
\]

\[
\tilde{N}_{y}^{(b_0,b_1)}(t) = \begin{cases} 
R(t y^{(b_1)}) & \text{if } y \in \{0,1,2\}, b_1 = 0, \\
R(e) + (-1)^{b_1}R(y) & \text{if } y \in \{t_1, t_2\}, b_0 = 0, \text{ and}\\n\tilde{N}_0^{(b_0,0)} \tilde{N}_t^{(0,b_1)} & \text{if } y = (0, t)\\n0 & \text{otherwise}
\end{cases}
\]

It’s easy to see that the families \( \{ \tilde{M}_x^{(a_0,a_1)} \} \) and \( \{ \tilde{N}_y^{(b_0,b_1)} \} \) are projective measurements for all \( x,y \in \{0,1,2,t_1,t_2\} \). For \( x,y \in \{(0,t_1),(0,t_2)\} \), this follows from the fact that \( t x^{(a_1)} \) and \( t y^{(b_1)} \) are projective measurements for all \( a,b \in [3] \). So \( \mathfrak{C}_p \) is in \( C_qc(I,I,[3] \times [2],[3] \times [2]) \). Some of the most important entries of \( \mathfrak{C}_p \) are shown in Table 1. We summarize some additional properties in the following lemma:

**Lemma 6.3.**

(a) \( \mathfrak{C}_p \) is a synchronous correlation in \( C_qc(I,I,[3] \times [2],[3] \times [2]) \).

(b) The entries of \( \mathfrak{C}_p \) are computable elements of \( \overline{Q} \).

**Proof.** Let \( \iota : \mathbb{C}[D_p] \rightarrow \mathbb{C}[D_p] \) be the linear involution sending \( g \mapsto g^{-1} \) for all \( g \in D_p \). It is not hard to see that \( \pi_j^{(i)} = \iota(\pi_j^{(i)}) \) for all \( i,j \). In addition, \( R(\alpha)|e\rangle = L(\iota(\alpha))|e\rangle \) for all \( \alpha \in \mathbb{C}[D_p] \), so \( \tilde{N}_y^{(b)}|e\rangle = \tilde{M}_y^{(b)}|e\rangle \) for all \( y \in I \) and \( b \in [3] \times [2] \). So

\[
\langle e|\tilde{M}_x^{(a)} \tilde{N}_x^{(b)}|e\rangle = \langle e|\tilde{M}_x^{(a)} \tilde{M}_x^{(b)}|e\rangle = 0
\]

for all \( x \in I \) and \( a \neq b \in [3] \times [2] \). We conclude that \( \mathfrak{C}_p \) is synchronous. Part (b) follows from the fact that the elements \( \pi_j^{(i)} \) belong to \( \overline{Q}[D_p] \).

We now come to the main theorem of this section. To state this theorem, recall that an integer \( r \) is a primitive root of a prime \( p \) if all the integers between \( 1 \) and \( p-1 \) are congruent modulo \( p \) to some power of \( r \).
Theorem 6.4. Let \( S = (|\psi\rangle, \{M_x^{(a_0,a_1)}\}, \{N_y^{(b_0,b_1)}\}) \) be a good strategy for \( \mathcal{C}_p \) and let \( r \) be a primitive root of \( p \). Suppose there exist unitaries \( U_A \) and \( U_B \) such that

\[
U_A U_B |\psi\rangle = U_B U_A |\psi\rangle, \\
U_A N_y^{(b_0,b_1)} |\psi\rangle = N_y^{(b_0,b_1)} U_A |\psi\rangle \text{ for all } y, b_0, b_1, \\
U_B M_x^{(a_0,a_1)} |\psi\rangle = M_x^{(a_0,a_1)} U_B |\psi\rangle \text{ for all } x, a_0, a_1, \\
U_A U_B |\psi\rangle = |\psi\rangle, \\
(N_{t_1} N_{t_2}) U_B |\psi\rangle = U_B (N_{t_1} N_{t_2})^r |\psi\rangle, \text{ and} \\
(M_{t_1} M_{t_2}) U_A |\psi\rangle = U_A (M_{t_1} M_{t_2})^r |\psi\rangle,
\]

where \( M_x = M_x^{(0,0)} - M_x^{(0,1)} \) and \( N_y = N_y^{(0,0)} - N_y^{(0,1)} \) for \( x, y \in \{t_1, t_2\} \). Then

\[
(M_{t_1} M_{t_2})^p |\psi\rangle = |\psi\rangle.
\]

When we use this theorem in the next section, the existence of \( U_A \) and \( U_B \) will be guaranteed by Proposition 5.14.

The rest of the section is devoted to the proof of Theorem 6.4, so for the remainder of the section, we will assume that we have a good strategy \( S \) and unitaries \( U_A \) and \( U_B \) satisfying the conditions of the theorem. Note that in a good strategy, \( M_x^{(a_0,a_1)} = N_x^{(a_0,a_1)} = 0 \) if \( x \in \{0,1,2\} \) and \( a_1 \neq 0 \), or if \( x \in \{t_1,t_2\} \) and \( a_0 \neq 0 \). In particular, \( M_{t_i} \) and \( N_{t_i} \) are binary observables.

The basic idea of the proof is to find a decomposition of \( |\psi\rangle \) as \( |\psi\rangle = \sum_{j=0}^p |\psi_j\rangle \), where \( \|\|\psi_0\|\|^2 = \|\|\psi_p\|\|^2 = 1/2p \), \( \|\|\psi_j\|\|^2 = 1/p \) for \( 1 \leq j \leq p - 1 \), and \( |\psi_j\rangle \) is an eigenvector of \( M_{t_1} M_{t_2} \) with eigenvalue \( \omega_j^p \). Intuitively, \( |\psi_0\rangle \) and \( |\psi_p\rangle \) are in the 1-dimensional irreducible representation of \( D_p \), and \( |\psi_j\rangle \) and \( |\psi_{p-j}\rangle \) in the 2-dimensional irreducible representation of \( D_p \), in which

\[
t_1 t_2 \mapsto \begin{pmatrix} \omega_j^p & 0 \\ 0 & \omega_p^{-j} \end{pmatrix}
\]

for \( 1 \leq j \leq (p-1)/2 \). The norms of the vectors are chosen because the multiplicity of the 2-dimensional irreducible representations in the regular representation of \( D_p \) is 2, and the multiplicity of the 1-dimensional irreducible representations is 1.

The vectors \( |\psi_0\rangle \) and \( |\psi_p\rangle \) are defined as

\[
|\psi_0\rangle = M_{t_1}^{(0,0)} M_{0}^{(0,0)} |\psi\rangle, \text{ and } |\psi_p\rangle = M_{t_1}^{(0,1)} M_{0}^{(0,0)} |\psi\rangle. \tag{5}
\]

It follows immediately from the definition of \( M_{t_1} \) that

\[
M_{t_1} |\psi_0\rangle = |\psi_0\rangle, \text{ and } M_{t_1} |\psi_p\rangle = -|\psi_p\rangle. \tag{6}
\]

To see that

\[
\|\|\psi_0\|\|^2 = \|\|\psi_p\|\|^2 = \frac{1}{2p}, \tag{7}
\]

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|        | $y = 0$                              |
|--------|-------------------------------------|
|        | $b = (0, 0)$ | $b = (1, 0)$ | $b = (2, 0)$ |
| $x = 0$| $a = (0, 0)$ | $1/p$        | $0$          | $0$          |
|        | $a = (1, 0)$ | $0$          | $2/p$        | $0$          |
|        | $a = (2, 0)$ | $0$          | $0$          | $(p - 3)/p$  |

(a) $\mathcal{C}_p$: the correlation values for $x = y = 0$.

|        | $y = 1$                              |
|--------|-------------------------------------|
|        | $b = (0, 0)$ | $b = (1, 0)$ |
| $x = t_1$| $a = (0, 0)$ | $\cos^2(\pi/2p)$ | $\sin^2(\pi/2p)$ | $1 - \sin(\pi/p)$ | $1 + \sin(\pi/p)$ |
|        | $a = (0, 1)$ | $\sin^2(\pi/2p)$ | $\cos^2(\pi/2p)$ | $1 + \sin(\pi/p)$ | $1 - \sin(\pi/p)$ |
| $x = t_2$| $a = (0, 0)$ | $\cos^2(\pi/2p)$ | $\sin^2(\pi/2p)$ | $1 - \sin(\pi/p)$ | $1 - \sin(\pi/p)$ |
|        | $a = (0, 1)$ | $\sin^2(\pi/2p)$ | $\cos^2(\pi/2p)$ | $1 + \sin(\pi/p)$ | $1 - \sin(\pi/p)$ |

(b) $\mathcal{C}_p$: the correlation values for $x \in \{t_1, t_2\}$ and $y \in \{1, 2\}$.

|        | $x = 1$ | $x = 2$ | $x = 0$ |
|--------|--------|--------|--------|
|        | $a_0 = 0$ | $a_0 = 1$ | $a_0 = 2$ |
| $y = 1$| $b_0 = 0$ | $1/p$ | $0$ | $1/p$ | $0$ |
|        | $b_0 = 1$ | $1/p$ | $0$ | $1/p$ | $0$ |
|        | $b_0 = 2$ | $0$ | $p^{-2}/p$ | $0$ | $p^{-2}/p$ |
| $y = 2$| $b_0 = 0$ | $1/p$ | $1/p$ | $0$ | $1/p$ | $0$ |
|        | $b_0 = 1$ | $1/p$ | $0$ | $1/p$ | $0$ |
|        | $b_0 = 2$ | $0$ | $p^{-2}/p$ | $0$ | $p^{-2}/p$ |
| $y = 0$| $b_0 = 1$ | $1/p$ | $0$ | $1/p$ | $0$ |
|        | $b_0 \neq 1$ | $0$ | $p^{-2}/p$ | $0$ | $p^{-2}/p$ |

(c) $\mathcal{C}_p$: the correlation values for $x, y \in \{0, 1, 2\}$.

|        | $y = (0, t_1)$ |
|--------|----------------|
|        | $b = (0, 0)$ | $b = (0, 1)$ |
| $x = 0$| $a_0 = 0$ | $1/2p$ | $0$ | $0$ | $0$ |
|        | $a_0 = 1$ | $0$ | $1/2p$ | $0$ | $0$ |
|        | $a_0 = 2$ | $0$ | $0$ | $1/2p$ | $0$ |
| $x = t_1$| $a_0 = 0$ | $1/2p$ | $0$ | $0$ | $0$ |
|        | $a_0 = 1$ | $0$ | $1/2p$ | $0$ | $0$ |
|        | $a_0 = 2$ | $0$ | $0$ | $1/2p$ | $0$ |

(d) $\mathcal{C}_p$: the correlation values for the commutation test for Alice’s questions 0 and $t_1$.

|        | $y = (0, t_2)$ |
|--------|----------------|
|        | $b = (0, 0)$ | $b = (0, 1)$ |
| $x = (0, t_1)$| $a = (0, 0)$ | $1/p$ | $0$ |
|        | $a = (0, 1)$ | $0$ | $1/p$ |

(e) $\mathcal{C}_p$: some of the values for $x = (0, t_1), y = (0, t_2), a_0 = b_0 = 0$.

Table 1: Some important values of $\mathcal{C}_p$. 

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we need the following identities. Lemma 3.6 applied to Table 1d implies that

\[ M_{x}^{(0,a_1)} M_{0}^{(a_0,0)} |\psi\rangle = N_{(0,x)}^{(a_0,a_1)} |\psi\rangle = M_{0}^{(a_0,0)} M_{x}^{(0,a_1)} |\psi\rangle \]  

(8)

for \( a_0 \in [3], a_1 \in [2] \) and \( x \in \{t_1, t_2\} \). Then

\[ ||\psi_0||^2 = \langle \psi | M_{0}^{(0,0)} M_{t_1}^{(0,0)} M_{0}^{(0,0)} |\psi\rangle = \langle \psi | M_{0}^{(0,0)} N_{(0,t_1)}^{(0,0)} |\psi\rangle = \frac{1}{2p} \]

as shown in Table 1d. The derivation of \( ||\psi_p||^2 = \frac{1}{2p} \) is similar. Next, Lemma 3.5 applied to Table 1e implies that

\[ M_{(0,t_1)}^{(0,a_1)} |\psi\rangle = N_{(0,t_2)}^{(0,a_1)} |\psi\rangle \]

for each \( a_1 \in [2] \). Hence,

\[ |\psi_0\rangle = M_{t_1}^{(0,0)} M_{0}^{(0,0)} |\psi\rangle = N_{(0,t_2)}^{(0,0)} |\psi\rangle = M_{t_2}^{(0,0)} M_{0}^{(0,0)} |\psi\rangle, \]

and similarly, \( |\psi_p\rangle = M_{t_2}^{(0,1)} M_{0}^{(0,0)} |\psi\rangle \). Thus

\[ M_{t_2} |\psi_0\rangle = |\psi_0\rangle, \text{ and } M_{t_2} |\psi_p\rangle = -|\psi_p\rangle, \]

(9)

implying that \( |\psi_0\rangle \) and \( |\psi_p\rangle \) are 1-eigenvectors of \( M_{t_1} M_{t_2} \).

The vectors \( |\psi_j\rangle \) for \( 2 \leq j \leq p - 1 \) can be constructed from \( |\psi_1\rangle \), but the construction of \( |\psi_1\rangle \) is more complicated and requires Proposition 6.2. We first show a strategy for \( \mathcal{C}_p' \) can be extracted from any strategy for \( \mathcal{C}_p \).

**Proposition 6.5.** Suppose \( (|\psi\rangle, \{M_x^{(a)} | a \in [3] \times [2]\}, x \in I, \{N_y^{(b)} | b \in [3] \times [2]\}, y \in I) \) is a good strategy for \( \mathcal{C}_p \). Let \( |\psi'\rangle = (1 - M_{0}^{(0,0)}) |\psi\rangle / ||(1 - M_{0}^{(0,0)}) |\psi\rangle || \), and define projective measurements \( \{P_x^{(a)} | a \in [3]\}, x \in [5], \text{ and } \{Q_y^{(b)} | b \in [3]\}, y \in [5] \) by

\[
\begin{align*}
P_0^{(0)} &= M_0^{(1,0)}, & Q_0^{(0)} &= N_0^{(1,0)}, \\
P_0^{(1)} &= M_0^{(2,0)}, & Q_0^{(1)} &= N_0^{(2,0)}, \\
P_0^{(2)} &= 0, & Q_0^{(2)} &= 0, \\
for x = 1, 2: & \quad P_x^{(a)} = \begin{cases} M_{t_x}^{(0,a)} & \text{if } a = 0, 1 \\
0 & \text{otherwise} \end{cases}, & Q_x^{(a)} = \begin{cases} N_{t_x}^{(0,a)} & \text{if } a = 0, 1 \\
0 & \text{otherwise} \end{cases}, \\
P_3^{(a)} &= M_1^{(a,0)}, & Q_3^{(a)} &= N_1^{(a,0)}, \\
P_4^{(a)} &= M_2^{(a,0)}, & Q_4^{(a)} &= N_2^{(a,0)}. 
\end{align*}
\]

Then \( (|\psi'\rangle, \{P_x^{(a)} | a \in [3]\}, x \in [5], \{Q_y^{(b)} | b \in [3]\}, y \in [5] \) is a strategy for \( \mathcal{C}_p' \).
Proof. For this proof, the first key observation is that
\[ \| (1 - M_0^{(0,0)}) |\psi\| \|^2 = (p - 1) / p, \]
which follows from the fact that \( \| M_0^{(0,0)} |\psi\| \|^2 = 1 / p. \) We also need the following identities:
\[ (M_0^{(0,0)} + M_0^{(2,0)}) |\psi\| = M_1^{(2,0)} |\psi\| = M_2^{(2,0)} |\psi\| = N_1^{(2,0)} |\psi\| = N_2^{(2,0)} |\psi\| = (N_0^{(0,0)} + N_0^{(2,0)}) |\psi\|, \]
and
\[ (M_1^{(0,0)} + M_1^{(1,0)}) |\psi\| = (M_2^{(0,0)} + M_2^{(1,0)}) |\psi\| = M_0^{(1,0)} |\psi\| = N_0^{(1,0)} |\psi\| = (N_1^{(0,0)} + N_1^{(1,0)}) |\psi\|. \]
To prove Equation (10), first observe that \( \{ M_i^{(0,0)}, M_i^{(1,0)}, M_i^{(2,0)} \} \) and \( \{ N_i^{(0,0)}, N_i^{(1,0)}, N_i^{(2,0)} \} \) are projective measurements for \( i = 0, 1, 2. \) By Table 1d, the two projective measurements \( \{ M_0^{(1,0)}, M_0^{(0,0)} + M_0^{(2,0)} \} \) and \( \{ N_1^{(0,0)} + N_1^{(1,0)}, N_1^{(2,0)} \} \) satisfy the conditions of Lemma 3.5 with respect to \( |\psi\|, \) and hence we have \( (M_0^{(0,0)} + M_0^{(2,0)}) |\psi\| = N_0^{(2,0)} |\psi\|. \) For the same reason, \( N_1^{(2,0)} |\psi\| = M_1^{(2,0)} |\psi\| = (N_2^{(0,0)} + N_2^{(1,0)}) |\psi\| \) and so on. Equation (11) follows from Equation (10) as \( 1 - (M_0^{(0,0)} + M_0^{(2,0)}) = M_0^{(1,0)}, 1 - N_1^{(2,0)} = N_1^{(0,0)} + N_1^{(1,0)} \) and so on.

Since \( \| M_x^{(a,0)} N_0^{(2,0)} |\psi\| \|^2 = \langle \psi | M_x^{(a,0)} N_0^{(2,0)} |\psi\| \rangle = 0 \) from Table 1d, we have \( M_x^{(a,0)} N_0^{(2,0)} |\psi\| = 0. \) Using the identities in Equations (10) and (11), we can prove that, for \( x = 1, 2 \) and \( a \in \{0, 1\}, \)
\[ M_x^{(a,0)} (1 - M_0^{(0,0)}) |\psi\| = M_x^{(a,0)} (N_0^{(1,0)} + N_0^{(2,0)}) |\psi\| = M_x^{(a,0)} N_0^{(1,0)} |\psi\| = M_x^{(a,0)} (M_x^{(0,0)} + M_x^{(1,0)}) |\psi\| = M_x^{(a,0)} |\psi\|. \]
That is,
\[ M_x^{(a,0)} |\psi\| = M_x^{(a,0)} (1 - M_0^{(0,0)}) |\psi\|. \]
(12)
The same argument can also give us that, for \( x = 1, 2 \) and \( a \in \{0, 1\}, \)
\[ N_x^{(a,0)} |\psi\| = N_x^{(a,0)} (1 - M_0^{(0,0)}) |\psi\|. \]
(13)
Following the definitions of \( M_0^{(a,0)} \) and \( N_0^{(a,0)} \) and using \( N_0^{(0,0)} |\psi\| = M_0^{(0,0)} |\psi\|, \) we can also see that
\[ M_0^{(a,0)} |\psi\| = M_0^{(a,0)} (1 - M_0^{(0,0)}) |\psi\| \) and \( N_0^{(a,0)} |\psi\| = N_0^{(a,0)} (1 - M_0^{(0,0)}) |\psi\| \)
(14)
for \( a = 1, 2. \)

To see the extracted strategy induces \( \mathcal{C}_p' \), we need to determine \( \langle \psi | (1 - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} (1 - M_0^{(0,0)}) |\psi\| \rangle \) for \( x, y \in [5] \) and \( a, b \in [3] \) from the values of \( \mathcal{C}_p. \) If \( x, y \in \{0, 3, 4\}, \) it follows from the definitions of \( P_x^{(a)} \) and \( Q_y^{(b)} \) and Equations (12) to (14) that
\[ \langle \psi | (1 - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} (1 - M_0^{(0,0)}) |\psi\| \rangle = \langle \psi | P_x^{(a)} Q_y^{(b)} |\psi\| \rangle. \]
If \( x \in \{1, 2\} \) and \( y \in \{0, 3, 4\} \), Equations (8), (13) and (14) imply that
\[
\langle \psi | (I - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} (I - M_0^{(0,0)}) | \psi \rangle = \langle \psi | P_x^{(a)} Q_y^{(b)} (I - M_0^{(0,0)})^2 | \psi \rangle = \langle \psi | P_x^{(a)} Q_y^{(b)} | \psi \rangle.
\]
The same identity holds for \( x \in \{0, 3, 4\} \) and \( y \in \{1, 2\} \) by symmetry. Finally, when \( x, y \in \{1, 2\} \), Equation (8) implies that
\[
\langle \psi | (I - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} (I - M_0^{(0,0)}) | \psi \rangle = \langle \psi | P_x^{(a)} (I - M_0^{(0,0)})^2 Q_y^{(b)} | \psi \rangle = \langle \psi | (I - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} | \psi \rangle.
\]
If \( a = 0 \),
\[
P_x^{(0)} (I - M_0^{(0,0)}) | \psi \rangle = M_{tx}^{(0,0)} (I - M_0^{(0,0)}) | \psi \rangle = M_{tx}^{(0,0)} | \psi \rangle - | \psi_0 \rangle,
\]
where \( | \psi_0 \rangle = M_{tx}^{(0,0)} M_0^{(0,0)} | \psi \rangle \). Observe that, because \( \mathcal{C}_p \) is synchronous,
\[
| \psi_0 \rangle = M_{tx}^{(0,0)} (M_0^{(0,0)})^2 | \psi \rangle = M_{tx}^{(0,0)} M_0^{(0,0)} N_0^{(0,0)} | \psi \rangle = N_0^{(0,0)} | \psi_0 \rangle.
\]
Hence,
\[
\langle \psi | (I - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} | \psi \rangle = \langle \psi | M_{tx}^{(0,0)} Q_y^{(b)} | \psi \rangle - \langle \psi_0 | Q_y^{(b)} | \psi \rangle
\]
\[
= \langle \psi | M_{tx}^{(0,0)} Q_y^{(b)} | \psi \rangle - \langle \psi_0 | N_0^{(0,0)} Q_y^{(b)} | \psi \rangle
\]
\[
= \langle \psi | M_{tx}^{(0,0)} Q_y^{(b)} | \psi \rangle - \langle \psi_0 | \psi_1 \rangle,
\]
where \( j = 0 \) if \( b = 0 \) and \( j = p \) otherwise. Since \( | \psi_0 \rangle \) and \( | \psi_p \rangle \) are 1 and \(-1\)-eigenvectors of \( M_{t1} \) by Equation (6), \( \langle \psi_0 | \psi_p \rangle = 0 \). If \( a = 1 \), the same calculation holds except that \( \langle \psi_0 | \) is replaced by \( \langle \psi_p | \) and \( M_{tx}^{(0,0)} \) is replaced by \( M_{tx}^{(0,1)} \). Thus in all cases, we can calculate
\[
\langle \psi | (I - M_0^{(0,0)}) P_x^{(a)} Q_y^{(b)} (I - M_0^{(0,0)}) | \psi \rangle
\]
from \( \mathcal{C}_p \), and direct calculation shows that the induced correlation is \( \mathcal{C}_p' \).

We can now define \( | \psi_1 \rangle \) based on Propositions 6.2 and 6.5.

**Corollary 6.6.** Let \( \{ | \psi \rangle, \{ M_{x}^{(a)} \} | a \in [3] \times [2] \}, x \in I, \{ N_{y}^{(b)} \} | b \in [3] \times [2] \}, y \in I \) be a good strategy for \( \mathcal{C}_p \), and let
\[
| \psi_1 \rangle = \frac{1}{2} (M_1^{(0,0)} + i M_2 M_1^{(1,0)} - i M_2 M_1^{(0,0)} + M_1^{(1,0)}) | \psi \rangle,
\]
where \( M_2 = M_2^{(0,0)} - M_2^{(1,0)} \). Then
\[
| \psi_1 \rangle = \frac{1}{2} (N_1^{(0,0)} - i N_2 N_1^{(1,0)} + i N_2 N_1^{(0,0)} + N_1^{(1,0)}) | \psi \rangle,
\]
\[
|| | \psi_1 \rangle ||^2 = \frac{1}{p} = \langle \psi | \psi_1 \rangle,
\]
\[
M_{t1} M_{t2} | \psi_1 \rangle = \omega_p | \psi_1 \rangle, \quad M_{t1} | \psi_1 \rangle = \omega_p^{-1} | \psi_1 \rangle,
\]
\[
N_{t1} N_{t2} | \psi_1 \rangle = \omega_p^{-1} | \psi_1 \rangle,
\]
where \( M_{tx} = M_{tx}^{(0,0)} - M_{tx}^{(0,1)} \) and \( N_{tx} = N_{tx}^{(0,0)} - N_{tx}^{(0,1)} \) for \( x = 1, 2 \).
Proof. Suppose \(|\psi\rangle, \{M^{(a)}_x \mid a \in [3] \times [2]\}, x \in I, \{N^{(b)}_y \mid b \in [3] \times [2]\}, y \in I)\) is a good strategy for \(C_p\). Then

\[ \langle \psi|\psi_1 \rangle = \frac{1}{2} \langle \psi|N_1^{(0,0)} + i(M_2^{(0,0)} - M_2^{(1,0)})N_1^{(1,0)} - i(M_2^{(0,0)} - M_2^{(1,0)})N_1^{(0,0)} + N_1^{(1,0)}|\psi \rangle = \frac{1}{p} \]

by Table 1c. Let \(|\psi'\rangle, \{P^{(a)}_x \mid a \in [3]\}, x \in [5]\{Q^{(b)}_y \mid b \in [3]\}, y \in [5]\) be the strategy for \(C'_p\) from Proposition 6.5, and let \(|\phi_1\rangle = \frac{1}{2} (P^{(0)}_3 + iP^{(1)}_3 - iP^{(0)}_4 + P^{(1)}_3)|\psi\rangle\) as in Proposition 6.2. Expanding \(P^{(a)}_x\), we see that

\[ \| (1 - M^{(0,0)}_0) |\psi\rangle \| \cdot |\phi_1\rangle = \frac{1}{2} (M^{(0,0)}_1 - iM^{(1,0)}_2 + iM^{(0,0)}_2 + M^{(1,0)}_1)(1 - M^{(0,0)}_0)|\psi\rangle \]

\[ = \frac{1}{2} (M^{(0,0)}_1 - iM^{(1,0)}_2 + iM^{(0,0)}_2 + M^{(1,0)}_1)|\psi\rangle \]

\[ = |\phi_1\rangle, \]

where the second equality follows from Equation (12). Equation (16) follows from expanding Equation (4) similarly. Equations (17) to (19) follow immediately from Proposition 6.2 and the fact that \(\| (1 - M^{(0,0)}_0) |\psi\rangle \|^2 = (p - 1)/p\).

We can now finish the proof of Theorem 6.4.

Proof of Theorem 6.4. We have already defined \(|\psi_0\rangle\) and \(|\psi_p\rangle\) in Equation (5), and \(|\psi_1\rangle\) in Corollary 6.6. Next we define \(|\psi_j\rangle\) for \(2 \leq j \leq p - 1\) as

\[ |\psi_j\rangle = (U_AU_B)^{\log_2 j} |\psi_1\rangle \]

where \(\log_r j\) is the discrete log of \(j\) modulo \(p\) (in other words, \(\log_r j = a\) where \(r^a \equiv j \pmod{p}\)). The discrete log is defined for all \(1 \leq j \leq p - 1\) because \(r\) is a primitive root of \(p\). Since \(U_A\) and \(U_B\) are unitary, \(\| |\psi_j\rangle \|^2 = 1/p\). To prove

\[ (M_{t_1}M_{t_2})|\psi_j\rangle = \omega_p^j |\psi_j\rangle, \quad (N_{t_1}N_{t_2})|\psi_j\rangle = \omega_{p^{-j}} |\psi_j\rangle, \tag{20} \]

observe that since \(M_{t_1}|\psi\rangle = N_{t_1}|\psi\rangle, (M_{t_1}M_{t_2})^{-n}|\psi\rangle = (N_{t_1}N_{t_2})^{-n}|\psi\rangle,\) and similarly \(U_A^n|\psi\rangle = U_B^n|\psi\rangle.\) Thus

\[ (M_{t_1}M_{t_2})^{-n}U_A|\psi\rangle = (M_{t_1}M_{t_2})^{-n-1}U_A(M_{t_1}M_{t_2})^{-r}|\psi\rangle \]

\[ = (N_{t_1}N_{t_2})^{-r}(M_{t_1}M_{t_2})^{-n-1}U_A|\psi\rangle \]

\[ \cdots \]

\[ = (N_{t_1}N_{t_2})^{-nr}U_A|\psi\rangle \]

\[ = U_A(M_{t_1}M_{t_2})^{nr}|\psi\rangle. \]
Hence
\[
(M_t M_{l_2}) U^n_A |\psi\rangle = (M_t M_{l_2}) U_A (U_B^t)^{n-1} |\psi\rangle \\
= (U_B^t)^{n-1} U_A (M_t M_{l_2})^t |\psi\rangle \\
= (U_B^t)^{n-2} U_A (M_t M_{l_2})^t U_A |\psi\rangle \\
= (U_B^t)^{n-2} U_A^2 (M_t M_{l_2})^t^2 |\psi\rangle \\
\ldots \\
= U_A^n (M_t M_{l_2})^t^n |\psi\rangle.
\]

Then
\[
(M_t M_{l_2}) |\psi_j\rangle = (M_t M_{l_2}) (U_A U_B)^{log,j} |\psi_1\rangle \\
= \frac{1}{2} U_B^{log,j} (N_1^{(0,0)} - iN_2 N_1^{(1,0)} + iN_2 N_1^{(0,0)} + N_1^{(1,0)}) (M_t M_{l_2}) U_A^{log,j} |\psi\rangle \\
= \frac{1}{2} U_B^{log,j} (N_1^{(0,0)} - iN_2 N_1^{(1,0)} + iN_2 N_1^{(0,0)} + N_1^{(1,0)}) U_A^{log,j} (M_t M_{l_2}) |\psi\rangle \\
= (U_A U_B)^{log,j} (M_t M_{l_2}) |\psi_1\rangle = \omega_p^j (U_A U_B)^{log,j} |\psi_1\rangle = \omega_p^j |\psi_j\rangle,
\]
as desired.

Let
\[
|\psi'\rangle = \sum_{j=0}^p |\psi_j\rangle.
\]

Since eigenvectors with different eigenvalues are orthogonal, \( \langle \psi_j | \psi_k \rangle = 0 \) for \( 0 \leq j \neq k \leq p \). As a result, \( |||\psi'\|| = 1 \). If \( j = 0 \) or \( p \), then \( \langle \psi | \psi_j \rangle = |||\psi'\||^2 = 1/2p \). If \( 1 \leq j \leq p - 1 \), then
\[
\langle \psi | \psi_j \rangle = \langle \psi | (U_A U_B)^{log,j} |\psi_1\rangle = \langle \psi | \psi_1 \rangle = 1/p
\]
using Corollary 6.6 and the fact that \( U_A U_B |\psi\rangle = |\psi\rangle \).

Thus,
\[
\langle \psi | \psi' \rangle = \langle \psi | \psi_0 \rangle + \langle \psi | \psi_p \rangle + \sum_{j=1}^{p-1} \langle \psi | \psi_j \rangle \\
= \frac{1}{2p} + \frac{1}{2p} + (p - 1) \frac{1}{p} = 1,
\]
implying that \( |\psi\rangle = |\psi'\rangle \). We conclude that
\[
(M_t M_{l_2})^p |\psi\rangle = (M_t M_{l_2})^p \left( \sum_{j=0}^p |\psi_j\rangle \right) = \sum_{j=0}^p \omega_p^j |\psi_j\rangle = |\psi\rangle,
\]
which completes the proof.
7 Membership problems

Recall that Membership\((n_A, n_B, m_A, m_B)_t, K\), where \(t \in \{q, qs, qa, qc\}\) and \(K\) is a subfield of \(\mathbb{R}\), is defined in Section 1 as the problem of deciding if a correlation \(P \in K^{n_A n_B m_A m_B}\) is in the correlation set \(C_t(n_A, n_B, m_A, m_B)\). In this section, we let \(K = K_0 \cap \mathbb{R}\), where \(K_0\) is the subfield of \(\mathbb{C}\) generated by the roots of unity \(\omega_n^k\) for \(k, n \in \mathbb{Z}\). We then drop the subscript \(K\) when referring to membership problems. The hardness of Membership\((n_A, n_B, m_A, m_B)_t\) is related to the hardness of a more general problem:

**Problem** Intersection\((n_A, n_B, m_A, m_B)_t\). Given a set of correlations \(F \subset K^{n_A n_B m_A m_B}\) with constants \(n_A, n_B, m_A\) and \(m_B\), is \(F \cap C_t(n_A, n_B, m_A, m_B) \neq \emptyset\)?

**Proposition 7.1.** For fixed \(n_A, n_B, m_A, m_B \in \mathbb{N}\) and \(t \in \{q, qs, qa, qc\}\),

\[
\text{Intersection}(n_A, n_B, m_A, m_B)_t \text{ and } \text{Membership}(n_A, n_B, m_A, m_B)_t
\]

are equivalent under Cook reduction.

**Proof.** If \(D_M\) is a decider for Membership\((n_A, n_B, m_A, m_B)_t\), we can decide if \(F \cap C_t(n_A, n_B, m_A, m_B) = \emptyset\) by running \(D_M\) on all the elements of \(F\). If \(D_I\) is a decider for Intersection\((n_A, n_B, m_A, m_B)_t\), we can decide if a correlation \(P \in C_t(n_A, n_B, m_A, m_B)\) by running \(D_I\) on \(\{P\}\).

The main result of this section is the following.

**Theorem 7.2.** For every recursively enumerable set \(X\) of positive integers, there exists \(N \in \mathbb{N}\) and a computable family of finite sets of correlations \(\{F_n \mid n \in \mathbb{N}\}\), where \(F_n \subset K^{N^2 \times 8^2}\), such that

\[
F_n \cap C_{qc}(N, N, 8, 8) = \emptyset \text{ if } n \in X, \text{ and } \\
F_n \cap C_{qa}(N, N, 8, 8) \neq \emptyset \text{ if } n \notin X.
\]

Before proving Theorem 7.2, we first observe that Theorem 1.1 follows directly from Theorem 7.2.

**Proof of Theorem 1.1.** Let \(X\) be a RE-complete set of positive integers, and let \(N\) and \(F_n \subset K^{N^2 \times 8^2}\) be as in Theorem 7.2. Set \(\alpha = \max(N, 8)\), and suppose \(n_A, n_B, m_A, m_B \geq \alpha\). For any \(n \in \mathbb{N}\) and \(C \in F_n\), define \(C' \in K^{n_A n_B m_A m_B}\) by

\[
C'(a, b \mid x, y) = \begin{cases} 
C(a, b \mid \delta(x), \delta(y)) & \text{if } a, b < 8 \\
0 & \text{otherwise}
\end{cases}
\]

where \(\delta : \mathbb{N} \to \mathbb{N}\) is defined by

\[
\delta(i) = \begin{cases} 
i & \text{if } i < N \\
N & \text{otherwise}
\end{cases}
\]

It follows easily from the definitions that

\(C' \in C_t(n_A, n_B, m_A, m_B)\) if and only if \(C \in C_t(N, N, 8, 8)\).
Hence if \( F'_n = \{ C' \mid C \in F_n \} \), then
\[
F'_n \cap C_t(n_A, n_B, m_A, m_B) \neq \emptyset \text{ if and only if } F_n \cap C_t(N, N, 8, 8) \neq \emptyset.
\]
Since \( C_qa(n_A, n_B, m_A, m_B) \subseteq C_qc(n_A, n_B, m_A, m_B) \), Theorem 7.2 implies that
\[
F'_n \cap C_t(n_A, n_B, m_A, m_B) \neq \emptyset \text{ if and only if } n \notin X
\]
for both \( t = qa \) and \( t = qc \). Thus Intersection\( (n_A, n_B, m_A, m_B)_t \) is coRE-hard for \( t = qa, qc \). By Proposition 7.1, Membership\( (n_A, n_B, m_A, m_B)_t \) is also coRE-hard.

Although we take \( m_A, m_B \geq \alpha \) in the proof, note that it is sufficient to choose \( m_A, m_B \geq 8 \). Also, it has been shown that Membership\( (n_A, n_B, m_A, m_B)_q \) is in coRE [NPA08]. Hence, Membership\( (n_A, n_B, m_A, m_B)_q \) is coRE-complete for \( n_A, n_B \geq N \) and \( m_A, m_B \geq 8 \).

To prove Theorem 7.2, we first construct \( F_n \) from \( X \). Recall that an integer \( r \) is a primitive root of a prime \( p \) if all the integers between 1 and \( p - 1 \) are congruent modulo \( p \) to some power of \( r \). By a result of Gupta and Murty [GM84], there are integers \( r \) which are primitive roots of infinitely many primes. We use a version of this result due to Heath-Brown.

**Lemma 7.3** ([HB86], see also [Mur88]). There exists \( r \in \{2, 3, 5\} \) such that \( r \) is a primitive root of infinitely many primes.

For the construction of \( F_n \), fix \( r \in \{2, 3, 5\} \) such that \( r \) is a primitive root of infinitely many primes. Let \( p(n) \) be the \( n \)-th prime greater than \( r \) for which \( r \) is a primitive root. Since we can decide if \( r \) is a primitive root of a given prime, the sequence of primes \( p(1) < p(2) < \ldots \) is computable. Let \( \Gamma(A) \) be the solution group from Proposition 5.14 for the function \( p \), set \( X \) and integer \( r \). Let \( m \) and \( \ell \) be the number of rows and columns of \( A \) respectively, and note that each row of \( A \) has three nonzero entries. Recall that the generating set \( \{ x_i \mid i \in [\ell] \} \) has special generators \( x, t_1, t_2, u_1 \) and \( u_2 \). By reordering the generators, we can take \( x = x_0 \), which lets us use \( x \) for other things. Recall that for perfect correlations associated with \( Ax = 0 \), we use question set \( [m] \cup X_{var} \), where \( X_{var} = \{ x_i \mid i \in [\ell] \} \) is the set of variables in the system, and answer set \( Z_2^3 \). For correlations in \( F_n \), the question set is \( X = X_{var} \cup [m] \cup \{ m, m + 1, m + 2, (m, t_1), (m, t_2) \} \), and the answer set is \( A = Z_2^3 \). The questions \( m, m + 1, m + 2, (m, t_1) \) and \( (m, t_2) \) will correspond to questions 0, 1, 2, (0, t_1) and (0, t_2) from the correlation \( \mathfrak{c}_{p(n)} \).

To define the entries of the correlations in \( F_n \), we use the notations from Definition 4.1. In particular, \( I_j = \{ k \in [\ell] \mid A(j, k) \neq 0 \} \) for \( j \in [m] \). Let
\[
G_n = \langle x_0, x_1, \ldots, x_{\ell - 1} : x_j^2 = e \text{ for all } j \in [\ell], [x_j, x_k] = e \text{ if } j, k \in I_i \text{ for some } i, (t_1 t_2)^{p(n)} = e \rangle,
\]
so that \( \Gamma(A)/\langle (t_1 t_2)^{p(n)} = e \rangle \) is a quotient of \( G_n \). \( G_n \) is a Coxeter group, so its word problem is decidable [Hum90, Chapter 5]. Specifically, two words \( w_0 \) and \( w_1 \) over the generators of \( G_n \) are equal in \( G_n \) if they can both be transformed into a third word using
the transformations
\[ x_j^2 \to e, \]
\[ x_j x_k \to x_k x_j \text{ if } j, k \in I_i \text{ for some } i, \]
\[ \frac{t_1 t_2 \ldots t_1}{\text{length } p(n)} \to \frac{t_2 t_1 \ldots t_2}{\text{length } p(n)} \text{ and } \]
\[ \frac{t_2 t_1 \ldots t_2}{\text{length } p(n)} \to \frac{t_1 t_2 \ldots t_1}{\text{length } p(n)}. \]

Since the transformations never increase the length of a word, determining if two words are equal is a finite problem.

Recall that a group algebra like \( \mathbb{C}[G_n] \) is a \(*\)-algebra under the operation \((\sum_g a_g g)^* = \sum_g \overline{a_g} g^{-1}\). We define a mapping \( \sigma : \mathcal{X} \times \mathcal{A} \to \mathbb{C}[G_n] \) from question-answer pairs to self-adjoint projections in \( \mathbb{C}[G_n] \) as follows. For \((a_0, a_1) \in \mathbb{Z}_2^2\), let \#(a_0, a_1) be the element of \([4]\) with binary representation \((a_0, a_1)\).

- When \( x \in \mathcal{X}_{\text{var}}, \)
  \[ \sigma(x, a) = \begin{cases} \frac{e + (-1)^{a_2} x}{2} & \text{if } (a_0, a_1) = (0, 0), \\ 0 & \text{otherwise} \end{cases} \]

- When \( x = i \in [m], \)
  \[ \sigma(i, a) = \prod_{k \in I_i} \frac{e + (-1)^{a_{\phi_i(k)}} x_k}{2}. \]

- When \( x \in \{m, m+1, m+2\}, \)
  \[ \sigma(x, a) = \begin{cases} \pi_0^{(#(a_0, a_1))} & \text{if } x = m \text{ and } #(a_0, a_1) \leq 2, a_2 = 0, \\ \pi_1^{(#(a_0, a_1))} & \text{if } x = m + 1 \text{ and } #(a_0, a_1) \leq 2, a_2 = 0, \\ \pi_2^{(#(a_0, a_1))} & \text{if } x = m + 2 \text{ and } #(a_0, a_1) \leq 2, a_2 = 0, \\ 0 & \text{otherwise} \end{cases} \]

where \( \pi_i^{(a)} \in \mathbb{C}[[t_1, t_2]] \cong \mathbb{C}[D_{p(n)}] \) is defined in Section 6, Equation (2).

- When \( x = (m, t_i) \) for \( i = 1, 2, \)
  \[ \sigma((m, t_1), (a_0, a_1, a_2)) = \begin{cases} \pi_0^{(#(a_0, a_1))} \left( \frac{e + (-1)^{a_2} t_1}{2} \right) & \text{if } #(a_0, a_1) < 3 \text{ and } \\ 0 & \text{otherwise} \end{cases} \]
  \[ \sigma((m, t_2), (a_0, a_1, a_2)) = \begin{cases} \pi_0^{(#(a_0, a_1))} \left( \frac{e + (-1)^{a_2} t_2}{2} \right) & \text{if } #(a_0, a_1) < 3 \text{ and } \\ 0 & \text{otherwise} \end{cases} \]

where again \( \pi_0^{(a)} \in \mathbb{C}[[t_1, t_2]] \cong \mathbb{C}[D_{p(n)}] \) is defined in Section 6, Equation (2).
If \( z = \sum g a_g g \), let
\[
\text{supp}(z) = \{ g \in \mathbb{C}_n \ | \ a_g \neq 0 \}.
\]

Define
\[
W_n = \bigcup_{x,y \in \mathcal{X}, a,b \in A} \text{supp}(\sigma(x,a)\sigma(y,b)),
\]
and let
\[
\mathcal{F}_n = \{ f : W_n \to \{0,1\} \ | \ f(e) = 1, f(x_0) = 0, f(g) = 0 \text{ for } g \in \langle t_1, t_2 \rangle \setminus \{e\} \}.
\]

Functions \( f : W_n \to \{0,1\} \) can be regarded as linear functions spanned \( C(W_n) \to \mathbb{C} \) by extending linearly. Hence, given a function \( f \in \mathcal{F}_n \), we can define a bipartite correlation \( C_f \) for the scenario \((\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A})\) by \( C_f(a,b|x,y) = f(\sigma(x,a)\sigma(y,b)) \). These correlations contain a copy of the correlation \( \mathcal{E}_{p(n)} \) from Section 6.

**Proposition 7.4.** Let \( Q = \{t_1, t_2, m, m+1, m+2, (m, t_1), (m, t_2)\} \subseteq \mathcal{X} \) and let \( I = \{t_1, t_2, 0, 1, 2, (0, t_1), (0, t_2)\} \) as in Section 6. Let \( \alpha : Q \to I \) be the bijection \( \alpha(t_i) = t_i, \alpha((m, t_i)) = (0, t_i) \) for \( i = 1, 2, \) and \( \alpha(m+j) = j \) for \( j \in [3] \). If \( x, y \in Q \) and \( a, b \in A \) such that \( \#(a_0, a_1), \#(b_0, b_1) < 3 \), then
\[
C_f(a,b|x,y) = \mathcal{E}_{p(n)}((\#(a_0, a_1), a_2), (\#(b_0, b_1), b_2) | x(a), x(y)),
\]
for all \( f \in \mathcal{F}_n \). Furthermore, if \( S = (|\psi\rangle \in \mathcal{H}, \{M_x^{(a)}, N_x^{(a)} | a \in A\}, x \in \mathcal{X}) \) is a good strategy for \( C_f \), then \( S' = (|\psi\rangle \in \mathcal{H}, \{\tilde{M}_x^{(a)}, \tilde{N}_x^{(a)} | a \in [3] \times [2]\}, x \in I) \) is a good strategy for \( \mathcal{E}_{p(n)} \), where
\[
\tilde{M}_x^{(\#(a_0, a_1), a_2)} = M_{\alpha^{-1}(x)}^{(a)} \text{ and } \tilde{N}_x^{(\#(a_0, a_1), a_2)} = N_{\alpha^{-1}(x)}^{(a)} \text{ for } x \in I, a \in A, \#(a_0, a_1) < 3.
\]

**Proof.** Since the subgroup generated by \( t_1 \) and \( t_2 \) in \( G_n \) is a parabolic subgroup, \( \langle t_1, t_2 \rangle \subseteq \mathbb{C}_n \) is isomorphic to \( D_{p(n)} \). By construction, if \( f \in \mathcal{F}_n \) and \( g \in \langle t_1, t_2 \rangle \) then \( f(g) = 1 \) if and only if \( g = e \). Hence when \( x, y \in Q \) and \( \#(a_0, a_1), \#(b_0, b_1) < 3 \),
\[
f(\sigma(x,a)\sigma(y,b)) = \langle e|L(\sigma(x,a))R(\sigma(y,b))|e\rangle,
\]
where \( |e\rangle \in \ell^2D_{p(n)} \) and \( L : \mathbb{C}[D_{p(n)}] \to \mathcal{U}(\ell^2D_{p(n)}) \) and \( R : \mathbb{C}[D_{p(n)}] \to \mathcal{U}(\ell^2D_{p(n)}) \) are the left and right regular representations of \( \mathbb{C}[D_{p(n)}] \). Since \( L(\sigma(x,a)) = M_{\alpha(x)}^{(\#(a_0, a_1), a_2)} \) and \( R(\sigma(y,b)) = N_{\alpha(y)}^{(\#(b_0, b_1), b_2)} \) from the definition of \( \mathcal{E}_p \) in Section 6, Equation (21) follows.

If \( a \in A \) with \( \#(a_0, a_1) = 3 \), then \( C_f(a,b|x,y) = C_f(b,a|x,y) = 0 \) for all \( x \in Q, y \in \mathcal{X} \) and \( b \in A \). Hence \( \langle \psi|M_x^{(a)}|\psi\rangle = \langle \psi|N_x^{(a)}|\psi\rangle = 0 \) for all \( x \in Q \). Since \( S \) is a good strategy, \( M_x^{(a)} = N_x^{(a)} = 0 \). We conclude that \( \{\tilde{M}_x^{(a)} | a \in [3] \times [2]\} \) and \( \{\tilde{N}_x^{(a)} | a \in [3] \times [2]\} \) are projective measurements for all \( x \in I \), and thus \( S' \) is a good strategy for \( \mathcal{E}_{p(n)} \). \( \square \)
Finally, we are ready to define $F_n$:

$$F_n = \{ C_f, f \in \mathcal{F}_n \mid C_f | \mathcal{A} \times \mathcal{A} \times (\mathcal{X}_{\text{var}} \cup \{m\}) \times (\mathcal{X}_{\text{var}} \cup \{m\}) \text{ is a perfect correlation for } Ax = 0 \},$$

where $C_f | \mathcal{A} \times \mathcal{A} \times (\mathcal{X}_{\text{var}} \cup \{m\}) \times (\mathcal{X}_{\text{var}} \cup \{m\})$ is the restriction of $C_f$ to the question set $\mathcal{X} \cup \{m\}$, and $A$ is the matrix fixed above. Since support sets are finite, $W_n$ and $\mathcal{F}_n$ are finite, and hence $F_n$ is finite. Since the word problem of $G_n$ is decidable, the sets $W_n$ and $\mathcal{F}_n$ are computable from $n$. Whether $C_f | \mathcal{A} \times \mathcal{A} \times (\mathcal{X}_{\text{var}} \cup \{m\}) \times (\mathcal{X}_{\text{var}} \cup \{m\})$ is a perfect correlation for $Ax = 0$ is also decidable, and therefore $F_n$ is computable from $n$.

**Proposition 7.5.** If $n \in X$, $F_n \cap C_{qc}(\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}) = \emptyset$.

**Proof.** Assume $C_f \in F_n \cap C_{qc}(\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A})$ for some $f$. By Proposition 3.9, there is a good commuting-operator strategy

$$S = (|\psi\rangle \in \mathcal{H}, \{M_x^{(a), N_x^{(a)}}, a \in \mathcal{A}\}, x \in \mathcal{X})$$

for $C_f$. Since $C_f | \mathcal{A} \times \mathcal{A} \times (\mathcal{X}_{\text{var}} \cup \{m\}) \times (\mathcal{X}_{\text{var}} \cup \{m\})$ is a perfect correlation for $Ax = 0$, and $S' = (|\psi\rangle, \{M_x^{(a), N_x^{(a)}}, a \in \mathcal{A}\}, x \in \mathcal{X}_{\text{var}} \cup \{m\})$ is a strategy for this perfect correlation, Proposition 4.3 states that there exists a subspace $\mathcal{H}_0$ of $\mathcal{H}$ containing $|\psi\rangle$ and unitary operators $M(x_i)$ and $N(x_i)$ for $x_i \in \mathcal{X}_{\text{var}}$ on $\mathcal{H}$ inducing commuting representations $\Phi_M$ and $\Phi_N$ of $\Gamma(A)$ on $\mathcal{H}_0$. The operators $M(x_i)$ and $N(x_i)$ are defined as

$$M(x_i) = M_{x_i}^{0,0,0} - M_{x_i}^{0,0,1} \quad \text{and} \quad N(x_i) = N_{x_i}^{0,0,0} - N_{x_i}^{0,0,1}.$$  

By Proposition 5.14, the generators $t_1, t_2, u_1$ and $u_2$ of $\Gamma(A)$ satisfy the relation $u_2u_1t_1t_2u_1u_2 = (t_1t_2)^r$, so

$$\begin{align*}
(M(t_1)M(t_2))(M(u_1)M(u_2))|\psi\rangle &= (M(u_1)M(u_2))(M(t_1)M(t_2))^r|\psi\rangle \quad \text{and} \\
(N(t_1)N(t_2))(N(u_1)N(u_2))|\psi\rangle &= (N(u_1)N(u_2))(N(t_1)N(t_2))^r|\psi\rangle.
\end{align*}$$

Let $S' = (|\psi\rangle \in \mathcal{H}, \{\tilde{M}_x^{(a)}, \tilde{N}_x^{(a)} \mid a \in [3] \times [2], x \in I\})$ be the strategy for $C_{p(n)}$ from Proposition 7.4. Notice that $\tilde{M}_{t_i} := \tilde{M}_{t_i}^{0,0} - \tilde{M}_{t_i}^{0,1} = M_{t_i}^{0,0,0} - M_{t_i}^{0,0,1} = M(t_i)$ for $i = 1, 2$. Similarly $\tilde{N}_{t_i} := \tilde{N}_{t_i}^{0,0} - \tilde{N}_{t_i}^{0,1} = N(t_i)$ for $i = 1, 2$. Let $U_A = M(u_1)M(u_2)$ and $U_B = N(u_1)N(u_2)$. By Proposition 4.3, $M(x_i)N(x_i)|\psi\rangle = |\psi\rangle$, so $U_AU_B|\psi\rangle = |\psi\rangle$. Hence $U_A$ and $U_B$ satisfy the conditions of Theorem 6.4 with the strategy $S'$. We conclude that

$$\Phi_M((t_1t_2)^{p(n)}|\psi\rangle) = (M(t_1)M(t_2))^{p(n)}|\psi\rangle = |\psi\rangle.$$  

By part (3) of Proposition 4.3, $\Phi_M$ descends to a representation of $\Gamma(A) / \langle (t_1t_2)^{p(n)} = e \rangle$ on $\mathcal{H}_0$. On the other hand,

$$\langle \psi | M(x_0) | \psi \rangle = \langle \psi | M_{x_0}^{0,0,0} - M_{x_0}^{0,0,1} | \psi \rangle = f(\sigma(x_0, (0,0,0)) - \sigma(x_0, (0,0,1))) = f(x_0) = 0$$

by the definition of $\mathcal{F}_n$. Hence $\Phi_M(x_0) \neq 1_{\mathcal{H}_0}$, which implies that $x \neq e$ in $\Gamma(A) / \langle (t_1t_2)^{p(n)} = e \rangle$. By part (b) of Proposition 5.14, $n \notin X$. 

\qed
Proposition 7.6. If \( n \not\in X \), \( F_n \cap C_{qa}(\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}) \neq \emptyset \).

Proof. For this proof, let \( \Gamma_n := \Gamma(A) / \langle (t_1 t_2)^{p(n)} = e \rangle \) and let \( W_n^{fa} = \{ g \in W_n \mid g \neq e \text{ in } \Gamma_n^{fa} \} \). Suppose \( n \not\in X \), and define \( f : W_n \rightarrow C \) by \( f(g) = 0 \) if \( g \in W_n^{fa} \) and \( f(g) = 1 \) if \( g \in W_n \setminus W_n^{fa} \). (The function \( f \) is the pullback of the canonical trace on \( \Gamma_n^{fa} \) to \( G_n \) and restricted to \( W_n \).) Since \( n \not\in X \), \( x_0 \in W_n^{fa} \) by part (c) of Proposition 5.14, so \( f(x_0) = 0 \). By parts (d) and (e) of Proposition 5.14, the set \( \langle t_1, t_2 \rangle \setminus \{ e \} \subseteq W_n^{fa} \) as well. Thus \( f \in F_n \).

In the rest of proof, we show that \( C_f \in C_{qa}(\mathcal{X}, \mathcal{X}, \mathcal{A}, \mathcal{A}) \). Let \( \pi : F(\mathcal{X}_{var}) \rightarrow G_n \) be the quotient homomorphism. For each \( x \in \mathcal{X} \) and \( a \in \mathcal{A} \), let \( \tilde{\sigma}(x, a) \in C[F(\mathcal{X}_{var})] \) be a lift of \( \sigma(x, a) \), i.e. an element such that \( \pi(\tilde{\sigma}(x, a)) = \sigma(x, a) \). Let

\[
\tilde{W}_n = \bigcup_{x, y \in \mathcal{X}, a, b \in \mathcal{A}} \text{supp}(\tilde{\sigma}(x, a) \tilde{\sigma}(y, b))
\]

and

\[
\tilde{W}_n^{fa} = \{ g \in \tilde{W}_n \mid g \neq e \text{ in } \Gamma_n^{fa} \}.
\]

Note that if \( w \in \tilde{W}_n \), then \( \pi(w) \in W_n \), and \( w \in \tilde{W}_n^{fa} \) if and only if \( \pi(w) \in W_n^{fa} \). The polynomial \( \tilde{\sigma}(x, a)^2 - \tilde{\sigma}(x, a) \) is not necessarily 0 in \( C[F(\mathcal{X}_{var})] \), but is 0 in \( C[\Gamma_n] \), and the same is true for the polynomials \( \tilde{\sigma}(x, a)^* - \tilde{\sigma}(x, a) \), \( a \in \mathcal{A} \), \( x \in \mathcal{X} \), \( \tilde{\sigma}(x, a) \tilde{\sigma}(x, b) \), \( a \neq b \), \( x \in \mathcal{X} \), and \( \sum_{a \in \mathcal{A}} \tilde{\sigma}(x, a) - e \), \( x \in \mathcal{X} \). By Lemma 2.4, there is a constant \( c \) such that for all \( \epsilon \)-approximate representation \( \rho : F(\mathcal{X}_{var}) \rightarrow U(\mathbb{C}^d) \) of \( \Gamma_n \)

\[
\| \rho(\tilde{\sigma}(x, a)) \|_\text{op} \leq c, \quad \| \rho(\tilde{\sigma}(x, a)^2 - \tilde{\sigma}(x, a)) \| \leq c, \quad \| \rho(\tilde{\sigma}(x, a)^* - \tilde{\sigma}(x, a)) \| \leq c,
\]

\[
\| \rho(\tilde{\sigma}(x, a) \tilde{\sigma}(x, b)) \| \leq c, \quad \text{and} \quad \| \rho(\sum_{a \in \mathcal{A}} \tilde{\sigma}(x, a') - e) \| \leq c
\]

for all \( x \in \mathcal{X} \) and \( a \neq b \in \mathcal{A} \). By Proposition 2.3, for any \( \epsilon, \zeta > 0 \) there is an \( \epsilon \)-approximate representation \( \rho : F(\mathcal{X}_{var}) \rightarrow U(\mathbb{C}^d) \) of \( \Gamma_n \), where \( \delta \) depends on \( \epsilon \) and \( \zeta \), such that

\[
0 \leq \text{Tr}(\rho(w)) \leq \zeta \quad \text{for each } w \in W_n^{fa}, \quad \text{and} \quad 1 - \zeta \leq \text{Tr}(\rho(w)) \leq 1 \quad \text{for each } w \in W_n \setminus W_n^{fa}.
\]

For \( x \in \mathcal{X} \) and \( a \in \mathcal{A} \), let

\[
\tilde{M}_x^a = \rho(\tilde{\sigma}(x, a)).
\]

For all \( x, y \in \mathcal{X} \) and \( a, b \in \mathcal{A} \), we can write \( \tilde{\sigma}(x, a) \tilde{\sigma}(y, b) = \sum_{g \in \tilde{W}_n} u_g g \) for some \( u_g \in \mathbb{R} \), where \( |u_g| < 1 \) for all \( g \in \tilde{W}_n \) by definition of \( \tilde{\sigma}(x, a) \). Thus

\[
|C_f(a, b \mid x, y) - \text{Tr}(\tilde{M}_x^a \tilde{M}_y^b)| \leq \sum_{g \in \tilde{W}_n} |u_g| |f(\pi(g)) - \text{Tr}(\rho(g))|
\]

\[
= \sum_{g \in \tilde{W}_n^{fa}} |u_g||0 - \text{Tr}(\rho(g))| + \sum_{g \in \tilde{W}_n \setminus \tilde{W}_n^{fa}} |u_g||1 - \text{Tr}(\rho(g))|
\]

\[
\leq |\tilde{W}_n| \zeta.
\]

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Unfortunately, \( \{ \tilde{M}_a^x | a \in A \} \) may not be a measurement. However, by Lemma 2.5 there are projective measurements \( \{ M_a^x | a \in A \}, x \in X \), such that
\[
\| M_a^x - \tilde{M}_a^x \| \leq \Delta(c,8)c \epsilon
\]
for all \( x \in X \) and \( a \in A \) (where 8 comes from the size of \( A \)). Then
\[
| \langle \psi | M_a^x \otimes N_b^y | \psi \rangle - \tilde{C}_f (a,b \mid x,y) | \leq | \tilde{\text{Tr}}(M_a^x M_b^y) - \tilde{\text{Tr}}(\tilde{M}_a^x \tilde{M}_b^y) | + | \tilde{\text{Tr}}(\tilde{M}_a^x \tilde{M}_b^y) - C_f (a,b \mid x,y) | \leq (1 + c) \cdot \Delta(c,8)c \epsilon + | \tilde{W}_n | \zeta.
\]
The correlation defined by \( (|\psi\rangle, \{ M_a^x | a \in A \}, \{ N_a^x | a \in A \}, x \in X \) \) belongs to \( \mathcal{C}_q(\mathcal{X},\mathcal{X}',\mathcal{A},\mathcal{A}) \). Since \( \epsilon \) and \( \zeta \) can be arbitrarily small, we see that \( \tilde{C}_f \in \mathcal{C}_q(\mathcal{X},\mathcal{X}',\mathcal{A},\mathcal{A}) = \mathcal{C}_{qa}(\mathcal{X},\mathcal{X}',\mathcal{A},\mathcal{A}) \).

The proof of Theorem 7.2 follows immediately from Propositions 7.5 and 7.6.

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