Maximal Ideal Spaces of Invariant Function Algebras on Compact Groups

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Abstract—Let $G$ be a compact group and $A$ be a closed subalgebra of $C(G)$ which is invariant under the left and right shifts in $G$. We consider maximal ideal spaces (spectra) $M_A$ of these algebras. They can be defined as closed sub-bialgebras of $C(G)$. There is a natural semigroup structure in $M_A$ that admits an involutive anti-automorphism and a polar decomposition. If $M_A \neq G$ then $M_A$ has a nontrivial analytic structure. If $G$ is a Lie group then every idempotent in $M_A$ is the identity element of a complex Lie semigroup embedded to $M_A$. The semigroup $M_A$ admits an analogue of Cartan’s decomposition $KA^*K$, namely, $M_A = G\hat{T}G$, where $\hat{T}$ is an abelian semigroup that is a hull of the maximal torus $T$.

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INTRODUCTION

Let $G$ be a compact Hausdorff group, $C(G)$ be the Banach algebra of all continuous functions on $G$ with the sup-norm. We say that $A$ is an invariant algebra on $G$ if it is a closed subalgebra of $C(G)$ which is invariant under left and right shifts in $G$ and contains the constant functions.

In this paper, we consider maximal ideal spaces (spectra) $M_A$ of such algebras. They have a natural analytic structure and a semigroups structure. The multiplication is induced by the convolution of representing measures. An example is the closure $\overline{D}$ of the open unit disc $D$ in the complex plane $C$. It is the maximal ideal space of the disc-algebra $A(\overline{D})$ that consists of continuous functions on the circle $T$ which can be analytically extended to $D$. The polynomially convex hull $\hat{G}$ of any compact matrix group $G$ is a semigroup of the same type for matrix multiplication. It is the maximal ideal space of the uniform closure $\mathcal{G}$ of the algebra of all holomorphic polynomials. For example, the hull $\widehat{U(n)}$ of the group $U(n)$ of unitary matrices in the space of $n \times n$-matrices is the unit matrix ball for the standard operator norm. This is the maximal ideal space of the algebra of all functions holomorphic in the open ball and continuous up to its boundary.

In the examples above, the invariant algebras are finitely generated, i.e., they are generated as Banach algebras by their finite-dimensional bi-invariant subspaces. The general case is more difficult. For example, let $G = T^n$ and the homomorphism $\phi : \mathbb{R} \to G$ defines an irrational winding of $G$. Then $\phi$ can be extended from $\mathbb{R}$ to a homomorphism of the closed upper half-plane in $\mathbb{C}$ to $M_A$, where $A$ consists of functions that are continuous on $G$ and admit analytic extension onto this half-plane and its shifts. One can consider $A$ as an algebra of analytic and continuous functions on this half-plane which are also almost periodic on $\mathbb{R}$.

The semigroup $M_A$ admits an involutive anti-automorphism $\ast$ that on functions from $A$ acts as the composition of inversion in the group $G$ and complex conjugation. It is anti-holomorphic with respect to the natural analytic structure and defines a polar decomposition of $M_A$.

In a sense, the backbone of $M_A$ is the set of idempotents $J_A$ in it and the family $\mathcal{R}$ of one-parameter $\ast$-symmetric semigroups, which connect the points of $J_A$. For brevity, we will call these semigroups rays. Any idempotent $j \in J_A$ is associated with several objects:

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the stable subgroup \(G_j \subseteq G\) of \(j\) for the left action of the group \(G\), actually coinciding with the analogous group for the right action,

- its normalizer \(N_j\) and the quotient \(N_j/G_j\), which can be identified with the subgroup \(G^j = N_j j\) of the semigroup \(M_A\),

- the family \(\mathcal{R}_j\) of rays starting at \(j\).

Each ray admits the unique extension to a complex ray, i.e. a homomorphism \(\gamma : \mathbb{C}^+ \to M_A\) of the closed right half-plane \(\mathbb{C}^+\) in the plane \(\mathbb{C}\) to \(M_A\). It defines the analytic structure in \(M_A^j\); the composition of \(\gamma\) with any function \(f \in A\) is holomorphic in the open right half-plane of \(\mathbb{C}\). The boundary line \(i\mathbb{R}\) is one parameter group in \(G^j\). The vectors tangent to \(\gamma(i\mathbb{R})\) at \(j\) form a convex cone in the Lie algebra \(g^j\) of the group \(G^j\).

Let the group \(G\) be abelian and let \(S\) be the set of those one-dimensional characters \(G\), which are contained in \(A\). Then the linear span of \(S\) is dense in \(A\). The set \(S\) is a subsemigroup of the dual to \(G\) group and \(M_A\) can be identified with the family \(\text{Hom}(S, \mathbb{D})\) of homomorphisms from \(S\) to \(\mathbb{D}\), where \(\mathbb{D}\) is considered as a multiplicative semigroup with unit and zero.

In the case of Lie groups, the structure of \(M_A\) is determined by the maximal ideal space of the closure of restriction \(A T\) of the algebra \(A\) onto the maximal torus \(T\) of the group \(G\). Any element of \(M_A\) is conjugate by the action of \(G\) with a point from the \(A\)-hull \(\overline{T}\) of \(T\) in \(M_A\) (the hull is defined below in (8)). In other words, for \(M_A\), there is an analogue of the Cartan decomposition \(KAK\) of a real semisimple Lie group.

Invariant algebras have been studied since the 1950s. Initially, invariant algebras on abelian groups were considered as a natural generalization of the algebra analytic functions in \(\mathbb{D}\). In [2], Arens and Singer proved some analogues of classical results, for example, a generalization of the Poisson integral. They found a realization of \(M_A\) as \(\text{Hom}(S^\mathbb{A}, \mathbb{D})\) and defined the polar decomposition in \(\text{Hom}(S^\mathbb{A}, \mathbb{D})\).

The latter appeared also in MacKey’s [27] articles. The paper [25] by de Leeuw and Mirkil contains basic information about invariant algebras on locally compact abelian groups, mainly within the framework of abelian harmonic analysis. They also proved in [26] that any \(\text{SO}(n)\)-invariant algebra on the sphere \(S^{n-1}\), \(n > 2\) is self-adjoint with respect to the complex conjugation. Similar results were obtained by Wolf [37] and Gangolli in [8]. Wolf characterized compact groups \(G\) that have the property that every invariant algebra on \(G\) is self-adjoint. Gangolli proved that this property holds for connected compact semisimple Lie groups. It follows from the Stone-Weierstrass theorem that a self-adjoint invariant algebra \(A\) consists of all continuous functions on \(G\) which are constant on cosets of some normal subgroup of \(G\). The Wolf condition is that the image of any one-dimensional character \(G\) is finite. Further, Rider [30] proved that a compact group admitting an antisymmetric invariant Dirichlet algebra is connected and abelian. By definition, a function algebra is a Dirichlet algebra if it does not contain nontrivial real orthogonal measure. The algebra of functions is antisymmetric if it does not contain non-constant real functions. Antisymmetric invariant algebras were characterized by Rosenberg in [32] by several conditions, which we do not formulate here because they are rather complicated. The article [32] also contains a generalization of the result of [30]. In [16] and [10] it was noted that an invariant algebra \(A\) is antisymmetric if and only if the Haar measure of \(G\) is multiplicative on \(A\) (see section 5). Glicksberg’s article [16] also contains a construction of analytic discs in the maximal ideal spaces of invariant algebras on locally compact abelian groups. Also, we mention the papers [1], [23], [24], [21], [29] that clarify the structure of invariant algebras but don’t consider their spectra.

The approach based on the observation that \(M_A\) has a natural semigroup structure, was used in [10], where the case of non-compact groups was mainly considered. Initial results on compact groups are announced in [11], the proofs are published in the hard-to-reach article [12]. This paper contains an extended exposition of them.

An obvious continuation of the topic of this article is invariant algebras on homogeneous spaces. Their family is much wider and more diverse. Some results in this direction have been obtained in the papers [13], [14], [24], [21].

The paper is organized as follows. Section 1 contains the notation, definitions and auxiliary information. Moreover, it is proved in it that an invariant algebra is the same as a sub-bialgebra of the bialgebra \(C(G)\). In Section 2 we consider the main properties of \(M_A\) as a semigroup with involution.

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It can be realized as the semigroup of all endomorphisms of \( A \) that commute with every left shift. They are bounded in the \( L^2(G) \) norm and therefore can be extended to the closure \( H^2 \) of the algebra \( A \) in \( L^2(G) \). Moreover, the involution coincides with the conjugation of operators in Hilbert space \( H^2 \). For an abelian \( G \), there is another natural method to define the structure of a semigroup with involution in \( M_A \). It is proved that they coincide.

The closure of the restriction of an invariant algebra \( A \) to a closed subgroup \( H \subseteq G \) is also an invariant algebra. In Section 3, we prove that its maximal ideal space can be identified with the \( A \)-hull of \( H \) in \( M_A \). Averaging over left shifts by elements of \( H \) we obtain a projection onto the algebra \( A^H \) of all functions from \( A \) that are left-invariant with respect to \( H \). It is proved that the dual mapping \( \mathcal{M}_A \rightarrow \mathcal{M}_A^H \) is surjective. The algebra \( A^H \) can be viewed as a \( G \)-invariant algebra of functions on the homogeneous space \( M = H \setminus G \), where \( G \) acts from the right. This action extends to the semigroup \( \mathcal{M}_A \). It is transitive in the following sense: \( p\mathcal{M}_A = \mathcal{M}_A^H \) for all \( p \in M \). Furthermore, it is proved that the averaging over normal subgroups makes it possible to realize the compact semigroup \( \mathcal{M}_A \) for an invariant algebra \( A \) on any compact Hausdorff group \( G \) as the projective limit of maximal ideal spaces of invariant algebras on Lie groups.

In Section 4, we study the set \( \mathcal{J}_A \) of all idempotents in \( \mathcal{M}_A \) applying the previous results. There is the standard partial order in \( \mathcal{J}_A \): \( j \preceq k \) if and only if \( jk = kj = j \). We prove that \( \mathcal{J}_A \) is a complete lattice. This section also contains a proof of several idempotent-related properties of \( \mathcal{M}_A \), which are permanently used in the subsequent text. Section 5 is short but it contains an important result concerning the maximal ideal spaces. A function algebra is said to be antisymmetric if every real valued function in it is constant. It is proved that \( A \) is antisymmetric if and only if the Haar measure of \( G \) is multiplicative on \( A \). This is true if and only if \( \mathcal{M}_A \) is a semigroup with zero. Polar decomposition in \( \mathcal{M}_A \), rays (i.e., one parameter symmetric semigroups of endomorphisms), and one dimensional analytic structure are constructed in Section 6. Sections 6 and 7 contain the results that have been described above. Let us mention a few more.

- Any idempotent in \( \mathcal{M}_A \) can be connected with the unit of \( G \) by a chain of rays.
- There is a criterion for antisymmetry in the case of connected Lie groups that is distinct from the criterion of Section 5: \( A \) is antisymmetric if and only if the Haar measure of the maximal torus \( T \) is multiplicative on \( A \). This is equivalent to the antisymmetry of the restriction of \( A \) to \( T \).
- The hull \( \hat{T} \) of the group \( T \) is an abelian subsemigroup of \( \mathcal{M}_A \). For any symmetric \( s \in \mathcal{M}_A \), there is \( g \in G \) such that \( g^{-1}sg \in \hat{T} \). In particular, this is true for idempotents. Moreover, this implies an analogue of the Cartan decomposition \( KAK \) of reductive Lie groups for the maximal ideal spaces: \( \mathcal{M}_A = G \hat{T} G \).

The abelian semigroup \( \hat{T} \) contains an important information about \( \mathcal{M}_A \). This is an analogue of the Cartan subalgebra of a Lie algebra. There is no detailed description of \( \hat{T} \) yet.

Section 8 is devoted to the study of \( \mathcal{M}_A \) for invariant algebras on tori. This is motivated basically by the previous section. In this case one dimensional characters of \( G = \mathbb{T}^n \) form the semigroup \( S \) in the group \( \mathbb{Z}^n \) dual to \( G \) and \( \mathcal{M}_A = \text{Hom}(S, \mathbb{C}) \). An idempotent corresponds to the characteristic function of a semigroup \( P \subseteq S \) such that \( S \setminus P \) is a semigroup ideal. If \( A \) is finitely generated then there exists a bijection between \( \mathcal{J}_A \) and the family of all faces of the convex cone generated by \( S \) in \( \mathbb{R}^n \). The rays can be identified with the exponents \( e^{-\lambda(x)} \), where \( \lambda \) is a linear functional that is non-negative on \( S \). If \( S \) is not finitely generated then the structure of family of rays is more complicated. Also, this section contains several examples of maximal ideal spaces that illustrate the results of the article.

1. AUXILIARY MATERIAL

The notation of this section will be used throughout the article. In particular, \( G \) is a compact Hausdorff topological group. The space \( C(G) \) of all continuous functions on \( G \) is endowed with the uniform norm \( \|f\| = \sup_{x \in G} |f(x)| \). An index may be added to indicate the set over which the supremum is taken. By \( e \) we denote the unit of the group \( G \) and its subgroups, the units of algebras are denoted as \( 1 \). They can be provided with indices.
If $L$ and $M$ are subsets of a group then $LM$ or $L \cdot M$ stands for the set of products $uv$, where $u \in L$, $v \in M$. If $L, M$ are subspaces of an algebra then $LM$ denotes the linear span of the products.

The term “weak” refers to the $*$-weak topology of the dual space, and the term “strong” to the strong operator topology in the space $BL(X)$ of bounded linear operators in the Banach space $X$.

All representations are assumed to be strongly continuous. For a representation $\rho$ or its space $V_\rho$, $\text{Sp} \rho$ denotes the set of its irreducible components (without multiplicities).

The dual to $X$ space of linear continuous functionals on $X$ is denoted by $X^*$ since the symbol * is overloaded. The latter will be used for the involution in the semigroup $\mathcal{M}_A$ and for convolution of measures and will not be used in the cases of $*$-weak topology and duality in linear spaces. By $Y^\perp$, where $Y \subseteq X$, we denote the set $\lambda \in X^*$ such that $\lambda(y) = 0$ for all $y \in Y$. The several facts from the functional analysis which we use in this article can be found in [22] and [33].

1.1. Representation of compact groups. The object $\hat{G}$ dual to $G$ consists of classes of equivalent irreducible unitary representations of $G$. In what follows, “representation” means “finite-dimensional unitary representation”. Let $V_\tau$ be the space of the representation $\tau$. A pair $\xi, \eta \in V_\tau$ relates to the matrix element $\tau$,

$$m_{\xi,\eta}(g) = \langle \tau(g)\xi, \eta \rangle.$$  

Denote by $M_\tau$ the linear span of all such $m_{\xi,\eta}$, and by $\chi_\tau(g) = \text{Tr} \, \tau(g)$ the character of $\tau$. For all finite-dimensional representations $\tau, \sigma$, the following identities hold:

$$M_{\tau \otimes \sigma} = M_\tau M_\sigma,$$

$$\chi_{\tau \otimes \sigma} = \chi_\tau \chi_\sigma. \quad \quad \quad (1)$$

Let $\sigma$ be the Haar measure, i.e., the invariant probability measure on $G$. If $\tau \in \hat{G}$ and $\rho$ is a continuous representation of $G$ then the formula

$$P_\tau = \int_G \rho(g)\chi_\tau(g) \, d\sigma(g)$$

gives the orthogonal projection $P_\tau$ onto the isotypic component $\tau$ in $V_\rho$, that is, the sum of the corresponding to $\tau$ irreducible subspaces of $V_\rho$.

The convolution $\mu * \nu$ of finite regular Borel measures $\mu, \nu$ on $G$ can be defined by the equality

$$\int_G f(g) \, d(\mu * \nu)(g) = \int_{G \times G} f(gh) \, d\mu(g)d\nu(h), \quad \quad \quad (2)$$

where $f$ runs over $C(G)$ and the space $M(G)$ of finite regular Borel measures is treated as the dual to $C(G)$ Banach space. The convolution defines the structure of a Banach algebra in $M(G)$. For functions, i.e., for measures of the type $f\sigma$, the convolution can be rewritten as

$$u * v(g) = \int_G u(gh^{-1})v(h) \, d\sigma(h).$$

The spaces $C(G)$, $L^p(G, \sigma)$, $p \geq 1$, are embeded naturally into $L^1(G)$ as subalgebras, and $L^1(G)$ can be identified with the ideal of absolutely continuous with respect to $\sigma$ measures in $M(G)$. Here and in the sequel we drop $\sigma$ in the notation $L^p(G, \sigma)$.

The spaces $M_\tau$, $\tau \in \hat{G}$, are minimal ideals of $M(G)$. They are finite-dimensional. Here and henceforth “ideal” means “two-sided ideal”. By the Peter–Weyl theorem,

$$L^2(G) = \sum_{\tau \in \hat{G}} \bigoplus M_\tau,$$  \quad \quad \quad (3)

where the sum is orthogonal with respect to the standard scalar product in $L^2(G)$

$$\langle u, v \rangle = \int_G u(g)\overline{v(g)} \, dg. \quad \quad \quad (4)$$
the bar stands for complex conjugation. Throughout the paper, a space of functions on $G$ is said to be invariant if it is bi-invariant, i.e., if it is invariant under left and right shifts defined by the equalities

$$L_g f (h) = f (gh), \quad R_g f (h) = f (hg),$$

respectively. Note that $L$ is not a representation. For a closed invariant function space $F$ on $G$, set

$$Sp \ F = \{ \tau \in \hat{G} : M_\tau \subseteq F \},$$

where the sum is algebraic. The space $F_{\text{fin}}$ is characterized by the following property: $f \in F_{\text{fin}}$ if and only if the linear space generated by the set of all $L_g f$, where $g$ runs over $G$, is finite-dimensional. Replacing $L_g$ with $R_g$, we get the same space $F_{\text{fin}}$. It follows from the Peter–Weyl theorem that $F_{\text{fin}}$ is dense in $F$, since otherwise $F$ would have a non-trivial orthogonal complementary to the closure of $F_{\text{fin}}$ subspace which however must contain at least one of spaces $M_\tau$. Therefore, the correspondence between $F$ and $F_{\text{fin}}$ is one-to-one.

Put

$$f^*(g) = \overline{f(g^{-1})}. \quad \quad \quad (6)$$

This is an antilinear involutive automorphism of the algebra $C(G)$, which induces an antiautomorphism of the algebra $M(G)$. A simple calculation shows that

$$m_{\xi,\eta}^* = m_{\eta,\xi}.$$

Therefore, $M^*_\tau = M_\tau$ for all $\tau \in \hat{G}$ and, according to (5), $F_{\text{fin}}^* = F_{\text{fin}}$.

1.2. Uniform algebras. The space $C(Q)$ of all continuous functions on a Hausdorff compact $Q$ with the uniform norm $||f|| = \sup \{|f(g)| : g \in Q\}$ and pointwise multiplication is a commutative Banach algebra. The latter means that $||uv|| \leq ||u||\,||v||$ for every its elements $u, v$. The unit 1 of the algebra $C(Q)$ is the constant function equal to 1. A uniform algebra is a closed subalgebra of $C(Q)$ that contains 1. We need some information about uniform algebras. A detailed presentation can be found in [9, Ch. 2].

Every proper maximal ideal in a commutative Banach algebra $A$ is closed and has codimension 1. Denote by $M_A$ the set of all such ideals. Let $\varphi \in A^*$ be such that $\ker \varphi \in M_A$ and $\varphi(1) = 1$. Then the linear functional $\varphi$ is multiplicative, i.e., $\varphi(uv) = \varphi(u)\varphi(v)$ for all $u, v \in A$. It follows that

$$M_A \cong \text{Hom}(A, \mathbb{C}),$$

where the set on the right-hand side is the set of all homomorphisms of algebras with unit. For brevity, we use the notation $M_A$ for both sides of this equality and call $M_A$ the maximal ideal space. Thus, $M_A \subset A^*$. It is a weakly closed subset of the unit ball in $A^*$ and consequently it is weakly compact. Moreover, the norm of each $\varphi \in M_A$ is equal to 1, i.e., $M_A$ is contained in the unit sphere.

The space $M(Q)$ of all finite regular complex measures on $Q$ is dual to $C(Q)$ for any Hausdorff compact set $Q$. Since $||\varphi||_{A^*} = 1$ and $\varphi(1) = 1$, any norm-preserving extension of $\varphi$ from $A$ to $C(Q)$ is a positive measure of the total mass 1. Such measures are called representing. The set of all representing measures for $\varphi \in M_A$ is denoted by $M_\varphi$. It is a weakly compact convex subset of $M(Q)$.

A measure $\mu \in M_\varphi$ is multiplicative on $A$,

$$\int uv \, d\mu = \int u \, d\mu \int v \, d\mu$$

for all $u, v \in A$, since $\varphi(f) = \int f \, d\mu$ for all $f \in A$.

Every point $x \in Q$ corresponds the functional $ev_x$, which calculates values in it: $ev_x(f) = f(x)$. The atomic measure $\delta_x$ at the point $x$ is representing for $ev_x$. The image of $Q$ under the mapping $ev : x \mapsto ev_x$ can be identified with the factor set of $Q$ by the following equivalence:

$$x \sim_A y \iff f(x) = f(y) \quad \text{for all} \quad f \in A. \quad \quad \quad (7)$$

An algebra $A$ separates points if $ev$ is injective (we will also say that $A$ is separating). Then $Q$ can be considered as a subset of $M_A$. If $A$ is a closed separating subalgebra of $C(Q)$, where $Q$ is a Hausdorff
compact set then the Gelfand transform \( \hat{\cdot} \): \( A \to C(M_A) \) defined by \( \hat{f}(\varphi) = \varphi(f) \), can be treated as an extension of \( f \) from \( Q \) to \( M_A \).

Let \( A \subseteq C(Q) \) be a uniform algebra. The set \( E \subseteq Q \) is called a peak set for \( A \) if there exists a function \( f \in A \) such that

\[
\begin{cases}
  f(x) = 1, & x \in E, \\
  |f(x)| < 1, & x \notin E.
\end{cases}
\]

Then \( f \) is said to be a peak function for \( E \). A peak point is the peak set consisting of a single point. Intersection of a family of peak sets is called the generalized peak set or \( p \)-set, and \( p \)-point is a point with this property. It is clear that \( p \)-sets are closed. If \( A \) is separating then for any \( f \in A \) the function \( |f| \) reaches its maximum value at some \( p \)-point. The atomic measure \( \delta_x \) at a \( p \)-point \( x \) is the unique representing measure for \( e_v \).

Let \( E \subseteq Q \) be closed and \( B \) be the closure in \( C(E) \) of the restriction \( A|_E \) of \( A \) to \( E \). Then \( \varphi \in M_B \) can be considered as a functional on \( A \). This defines an embedding \( M_B \to M_A \) whose image consists of those \( \varphi \in M_A \) that admits a continuous extension to \( B \). This set is called the \( A \)-hull of \( E \). It is denoted by \( \hat{E} \) and can be defined as follows:

\[
\hat{E} = \{ \varphi \in M_A \mid |\varphi(f)| \leq \|f\|_E \text{ for all } f \in A \}.
\]

Note that \( \hat{G} \) is not the same as \( \hat{G} \). Since \( A \) can be considered as a subalgebra of \( C(M_A) \), the definition can be applied to subsets of \( M_A \). The set \( E \subseteq M_A \) is called \( A \)-convex if \( \hat{E} = E \). Replacing in (8) \( A \) with the algebra \( P(V) \) of all polynomials on a complex linear space \( V \) and \( M_A \) with \( V \), we get the definition of the polynomially convex hull \( \hat{E} \) of \( E \subseteq V \).

In Lemma 1.1 and afterwards, \( \text{supp} \mu \) denotes the support of the measure \( \mu \).

**Lemma 1.1.** Let \( E \) be a closed subset of \( Q \). Then

(a) \( \varphi \in \hat{E} \) if and only if there exists a measure \( \mu \in M_\varphi \) with support in \( E \),

(b) if \( E \) is a \( p \)-set and \( \varphi \in \hat{E} \) then \( \text{supp} \mu \subseteq E \) for any \( \mu \in M_\varphi \).

**Proof.** If \( \mu \in M_\varphi \) and \( \text{supp} \mu \subseteq E \) then the inequality \( |\varphi(f)| \leq \|f\|_E \) is obvious. Hence, \( \varphi \in \hat{E} \). If \( \varphi \in \hat{E} \) then (8) and the Hahn–Banach theorem imply the existence of a norm-preserving extension \( \varphi \) to \( C(E) \). This is a measure \( \mu \) such that \( \mu(E) = \varphi(1) = 1 \). Thus (a) is true.

In (b), we can assume without loss of generality that \( E \) is a peak set. Let \( f \) be a peak function for \( E \). Then \( \varphi(f) = 1 \) since \( f|_E \) is the unit of the algebra \( A|_E \). If \( x \notin E \) then \( |f(x)| < 1 \) by definition of the peak point. Hence, \( \text{supp} \mu \subseteq E \) and \( \mu \in M_\varphi \) imply \( 1 = |\varphi(f)| = |\int_E f \, d\mu| < 1 \). Thus \( \text{supp} \mu \subseteq E \).

Let \( D \) be a domain in a complex manifold. An analytic structure in \( M_A \) is a mapping

\[
\lambda : D \to M_A
\]

such that the function \( f(\lambda(z)) \) is holomorphic on \( D \) for all \( f \in A \).

The group of inner automorphisms of the group \( G \), \( x \to g^{-1}xg \), with \( g \in G \), will be denoted as \( \text{Aut}_0(G) \).

1.3. Invariant algebras. We say that \( A \subseteq C(G) \) is an invariant algebra if it is a closed subalgebra of \( C(G) \) that contains \( 1 \) and is preserved under all left and right shifts. For an invariant algebra \( A \) on the group \( G \), the relation \( \sim_A \) on \( G \) defined by (7) is bi-invariant. Hence, the equivalence class of \( e \) in \( G \) is a closed normal subgroup \( N \) and the algebra \( A \) can be considered as an invariant algebra on the group \( G/N \) which separates its points (so we don’t assume that \( A \) separates the points of \( G \)). It is clear that the invariant subspace \( A \subseteq C(G) \) is an algebra if and only if \( A_{\text{fin}} \) is algebra. It follows from (1) that \( A_{\text{fin}} \) is an algebra if and only if \( \text{Sp} A \) contains all irreducible components of \( \tau \otimes \sigma \) for every \( \tau, \sigma \in \text{Sp} A \).

The proof of the following assertion is omitted because the facts presented in it are well known (see, for example, [17, Ch. 7]).
**Proposition 1.1.** A weakly closed subspace $F$ in $M(G)$ is an ideal of the convolution algebra $M(G)$ if and only if it is invariant. For any closed invariant space $F$ in $C(G)$ or $L^2(G)$,

$$F_{\text{fin}} = F \cap C(G)_{\text{fin}} \tag{9}$$

and the space $F_{\text{fin}}$ is dense in $F$.

The first assertion follows from two facts: first, both left and right shifts by $g \in G$, are equivalent to convolutions from the relating side with the atomic measure $\delta_g$ at the point $g$, and second, finite linear combinations of atomic measures are weakly dense in $M(G)$. Second is true because linear functionals on $X^*$ that are continuous in the weak topology the space $X^*$ dual to the Banach space $X$ have the form $\ell_x(\xi) = \xi(x)$, where $x \in X$, $\xi \in X^*$. The equality (9) follows from (5).

Proposition 1.1, (6), and (5) imply the following assertion.

**Corollary 1.1.** If $F$ is a closed invariant subspace of $C(G)$ then $F^* = F$.

An element $x$ of a semigroup or an algebra with unit $X$ is called invertible if there exist $y, z \in X$ such that $xy = zx = 1$. Then $y = z$. The set of all invertible elements in $X$ is a group.

Henceforth, $H^2$ denotes the closure of the invariant algebra $A$ in $L^2(G)$. Since $M_\tau$ are minimal ideals, it follows from (3) and (5) that

$$H^2 = \sum_{\tau \in \text{Sp} A} \oplus M_\tau,$$

where the sum is orthogonal.

Let $\nu \in M(Q)$, where $Q$ is a Hausdorff compact set, $\phi$ is a continuous map from $Q$ to a Banach space $X$, and $\Phi$ is a strongly continuous map $Q \to \text{BL}(X)$. The integrals of $\phi, \Phi$ in the situations arising in this paper can be defined by the equalities

$$\xi\left(\int_Q \phi(q) \, d\nu(q)\right) = \int_Q \xi(\phi(q)) \, d\nu(q) \quad \text{for all } \xi \in X^*, \tag{10}$$

$$\left(\int_Q \Phi(q) \, d\nu(q)\right)x = \int_Q \Phi(q)x \, d\nu(q) \quad \text{for all } x \in X. \tag{11}$$

It is easy to prove that the right-hand side of the equality (10) is weakly continuous in $\xi$ and therefore is given by an element of the space $X$ predual to $X^*$. Also, the right-hand side of (11) defines a bounded operator in $X$.

Formula (11) extends the strongly continuous representation $\tau$ of the group $G$ to the convolution algebra $M(G)$:

$$\tau(\nu) = \int_G \tau(g) \, d\nu(g). \tag{12}$$

It is known that, for any separating uniform algebra on a compact $Q$, the set of $p$-points is not empty and the same is true for the peak points if $Q$ is metrizable (see [9, Theorem 11.6, Theorem 12.10]). Modulo these facts, the following lemma is obvious.

**Lemma 1.2.** Let $A$ be a separating $G$-invariant algebra on a homogeneous space $M = G/H$ of a compact group $G$. Then every point of $M$ is a $p$-point. Furthermore, if $G$ is a Lie group then all points of $M$ are peak points.

1.4. **Semigroups.** A topological semigroup is a topological space $S$ with an associative multiplication $S \times S \to S$ if the multiplication is continuous. We always assume the topology to be Hausdorff.

The unit circle and the open (or closed) unit disc in $C$ will be denoted by $T$, $D$ (or $\overline{D}$), respectively. They are usually considered as a subgroup and subsemigroups of the multiplicative semigroup $C$. The closed right half-plane $C^+$ in $C$ and the half-line $\mathbb{R}^+ = [0, \infty)$ are additive semigroups. A one parameter semigroup in a topological semigroup $S$ is a continuous homomorphism $\mathbb{R}^+ \to S$.

We use only elementary facts concerning topological semigroups and convolution semigroups of measures. Let us mention one of them: Any convolution idempotent positive measure of the total mass 1 on a compact Hausdorff group $G$ is the Haar measure of some closed subgroup of $G$ (see, e.g., [18, Theorem 1.2.10]). The condition of being positive is essential.
An element $\epsilon$ of a semigroup $S$ is called its zero if $\epsilon x = x\epsilon = \epsilon$ for all $x \in S$. For a subset $X \subseteq S$, put
\begin{align*}
Z_S(X) &= \{ z \in S : zx = xz \text{ for all } x \in X \}, \\
N_S(X) &= \{ z \in S : zX = Xz \}.
\end{align*}
We omit the index if $S = M_A$.

1.5. **Convex cones.** **Relative interior** of the set $X$ in a real vector space $V$ is the interior in its affine span. It will be denoted as $\text{Int}(X)$. In what follows, by a cone $C$ in $V$ we mean a subsemigroup of the vector group $V$, which persists dilations $v \to tv, t \geq 0$. We consider only convex cones. The cone $C$ has a natural preorder:
\begin{equation}
\text{u \preceq v} \iff -\varepsilon u + (1 + \varepsilon)v \in C \text{ for some } \varepsilon > 0.
\end{equation}
It defines an equivalence whose classes are called faces. For $x \in C$, denote by $F_x$ or $F_x, C$ the face containing $x$. It’s clear that every face is open in its linear span. The closure of $F_x$ in $C$ coincides with the union of those faces of $C$, which are contained in the linear span of $F_x$, as well as with the set
\begin{equation}
\mathcal{T}_x = \{ y \in C : y \preceq x \}.
\end{equation}
We say that $\mathcal{T}_x$ is a closed face and denote by $\mathcal{F}_C$ the family of all closed faces of $C$.

A closed cone $C$ is called pointed if $C \cap (-C) = 0$ (in other words, if $\{0\}$ is a face of $C$). An arbitrary cone is called pointed if its closure has this property.

The condition (13) for cones is obviously equivalent to the following:
\begin{equation}
\text{u \preceq v} \iff -ku + lv \in C \text{ for some } k, l \in \mathbb{N}.
\end{equation}
The group $\mathbb{Z}^n$ is always assumed to be canonically embedded into $\mathbb{R}^n$. Any semigroup $S \subseteq \mathbb{Z}^n$ has the asymptotic cone
\begin{equation}
\alpha(S) = \text{clos} \bigcup_{n=1}^{\infty} \frac{1}{n} S \subseteq \mathbb{R}^n,
\end{equation}
where clos means closure. It is clear that $S \subseteq \alpha(S)$ and $\alpha(S)$ is the smallest convex closed cone, which contains $S$. Being convex and closed, $\alpha(S)$ coincides with its bi-dual. The cone dual to a set $X \subseteq \mathbb{R}^n$ is defined as follows:
\begin{equation}
X^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \geq 0 \text{ for all } x \in X \},
\end{equation}
where $\langle , \rangle$ is the standard inner product in $\mathbb{R}^n$. It’s clear that $X^* = (\text{clos } X)^*$ and $X^* = \left(\frac{1}{n} X\right)^*$ for all $n \in \mathbb{N}$. Hence, $S^* = \alpha(S)^*$ and this gives another definition of $\alpha(S)$:
\begin{equation}
\alpha(S) = S^{**}.
\end{equation}

1.6. **A characterization of the invariant algebras as sub-bialgebras of $C(G)$**. Let $X, Y$ be compact. Algebraic tensor product $C(X) \otimes C(Y)$ over $\mathbb{C}$ can be identified with the linear span of functions of the form $u(x)v(y)$ on $X \times Y$, where $u \in C(X), v \in C(Y)$. It follows from the Stone–Weierstrass theorem that $C(X) \otimes C(Y)$ is dense in $C(X \times Y)$. Thus, we can assume that $C(X \times Y)$ is a completion of $C(X) \otimes C(Y)$ and denote it as $C(X) \hat{\otimes} C(Y)$. If $X = Y = G$, where $G$ is a compact group then the group multiplication $G \times G \to G$ defines the comultiplication $\Delta : C(G) \to C(G) \hat{\otimes} C(G) = C(G \times G)$ by the rule
\begin{equation}
\Delta(f) (g, h) = f(gh), \quad f \in C(G).
\end{equation}
The definition of convolution of measures (2) can be rewritten as
\begin{equation}
\mu \ast \nu(f) = (\mu \otimes \nu)(\Delta(f)).
\end{equation}
For $L \subseteq C(G)$, denote by $L \hat{\otimes} L$ the closure of $L \otimes L$ in $C(G) \hat{\otimes} C(G)$. 

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Theorem 1.2. A closed linear subspace $A \subseteq C(G)$ is an invariant algebra if and only if it satisfies the following conditions:

$$1 \in A,$$

$$A \cdot A \subseteq A,$$

$$\Delta(A) \subseteq A \hat{\otimes} A.$$  

Proof. Since (19) and (20) mean that $A$ is a uniform algebra, it is sufficient to prove that (21) holds if and only if $A$ is invariant.

Let $A$ be an invariant algebra. According to Proposition 1.1, it is sufficient to prove (21) for $A_{\text{fin}}$. Thus (21) holds by (1) and (9).

Let (21) be true. Then, according to the definition of $A \hat{\otimes} A$, for any $f \in A$, the function $f(gh)$ can be approximated uniformly on $G \times G$ arbitrary close by finite sums of products of the type $u(g)v(h)$, where $u, v \in A$. For any such product and every couple of measures $\mu, \nu \in M(G)$,

$$\mu \otimes \nu(u \otimes v) = \mu(u)\nu(v).$$

If at least one of the measures $\mu, \nu$ is contained in $A^\perp$ then the right-hand side equals zero. Hence, $\mu \otimes \nu(\Delta(f)) = 0$ for any such $\mu, \nu$ and all $f \in A$. According to (18), this means that $\mu * \nu \in A^\perp$ if $\mu \in A^\perp$ or $\nu \in A^\perp$. In other words, $A^\perp$ is an ideal of $M(G)$. Since the space $A^\perp$ is weakly closed, it is invariant by Proposition 1.1. Hence, $A$ is also invariant. This completes the proof of the theorem.

It is clear that $\Delta$ is an isometric and isomorphic embedding of $A$ into $A \hat{\otimes} A$. This defines multiplication in $A^*$:

$$\phi * \psi(f) = (\phi \otimes \psi)(\Delta(f)).$$  

(22)

Corollary 1.3. Formula (22) defines the structure of an algebra in $A^*$ and, moreover, $A^\perp$ is an ideal in $M(G)$ and $A^*$ is canonically isomorphic to $M(G)/A^\perp$.

Proof. It follows from (2), (18), (21) that the restriction of linear functionals from $C(G)$ to $A$ is a homomorphism $M(G) \rightarrow A^*$ with the kernel $A^\perp$.

Note that

$$\|\psi * \phi\| \leq \|\psi\|\|\phi\|$$

(23)

according to (17) and (2). Hence, $A^*$ is a Banach algebra.

The space $C(G)_{\text{fin}}$ endowed with the usual pointwise multiplication of functions, comultiplication $\Delta$, unit $1$, co-unit $\delta_e$, and antipode $f(g) \rightarrow f(g^{-1})$ is a Hopf algebra. This means that a certain set of axioms must hold, and $C(G)_{\text{fin}}$ is one of the canonical examples for them. In the definition of a bialgebra, the antipode is excluded. Theorem 1.2 implies that for any invariant algebra $A$ the space $A_{\text{fin}}$ is a sub-bialgebra of $C(G)_{\text{fin}}$ and the closure of any sub-bialgebra of $C(G)_{\text{fin}}$ is an invariant algebra. The latter implies that $A = A^*$, i.e., it has an involution, which however is not complex linear. We do not give definitions related to Hopf algebras here since they are rather complicated, the proof of the theorem above does not need them, and they will not be used in this paper. There is extensive literature on Hopf algebras (see, for example, [6], [7], [28], [36]).

2. MAXIMAL IDEAL SPACE AS A SEMIGROUP

In this section, we consider the simplest properties of $M_A$ as a semigroup. Afterwards, they will be considered in more detail.

2.1. Basic properties of the semigroup $M_A$. 

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Lemma 2.1. For any $\varphi \in A^*$, there is the unique continuous linear operator $R_\varphi : A \to A$ that satisfies the equality
\[ \psi(R_\varphi f) = (\psi \ast \varphi)(f) \] (24)
for all $\psi \in A^*$ and $f \in A$. The mapping $R : \varphi \to R_\varphi$ extends the right regular representation of $G$ onto $\mathcal{M}_A$. It can be defined by the formula
\[ R_\varphi = \int_G R_g \, d\mu(g), \] (25)
where $\mu \in M(G)$ is an arbitrary extension of $\varphi \in A^*$ from $A$ to $C(G)$.

Proof. It suffices to prove the first assertion for $f \in A_\mathrm{fin}$ since $A_\mathrm{fin}$ is dense in $A$ in the norm. Then $\Delta(f) = \sum_k u_k \otimes v_k$, where the sum is finite and $u_k, v_k \in A_\mathrm{fin}$. Therefore, (24) uniquely defines $R_\varphi f \in A_\mathrm{fin}$ for fixed $\varphi$ and $f$. By (23) and the obvious equality $\|\Delta f\| = \|f\|$ we have $|\psi(R_\varphi f)| \leq \|\psi\|\|\varphi\|\|f\|$. Thus, the operator $R_\varphi$ is bounded and
\[ \|R_\varphi\| \leq \|\varphi\|. \] (26)
By Theorem 1.2, $A^\perp$ is an ideal in $M(G)$. It follows from (24), (10), and (11) that the integral in (25) is independent of the choice of the extension $\mu$ of the functional $\varphi$. A simple calculation with (18) shows that (24) holds for the right-hand side of (25). This proves (25) and the lemma.

Corollary 2.1. The operators $R_\varphi$ and $L_g$ commute for all $\varphi \in A^*$ and $g \in G$.

Proof. This is true by (25).

Set
\[ \mathcal{B} = \{T \in \mathrm{BL}(A) : \|T\| \leq 1, TL_g = L_g T \text{ for all } g \in G\}. \]
We endow the closed unit ball $\mathcal{B}$ in $A^*$ and the set $\mathcal{B}$ with the weak topology of $A^*$ and the strong operator topology of $\mathrm{BL}(A)$, respectively.

Lemma 2.2. The mapping $R : \varphi \to R_\varphi$ is a homeomorphism between $\mathcal{B}$ and $\mathcal{B}$. The inverse mapping $R^{-1}$ relates to $T \in \mathcal{B}$ the functional
\[ f \to Tf(e). \] (27)
Moreover, $R$ is an algebraic isomorphism.

Proof. Since $\mathcal{B}$ is compact, it is sufficient to prove that $R$ is a continuous bijection between $\mathcal{B}$ and $\mathcal{B}$. If $\|\varphi\| \leq 1$ then $\|R_\varphi\| \leq 1$ by (24) and Corollary 2.1. Thus $R$ maps $\mathcal{B}$ to $\mathcal{B}$. Let $T \in \mathcal{B}$ and $\varphi(f) = Tf(e)$. Applying Corollary 2.1 again, we get
\[ \varphi(L.gf) = TL_g f(e) = L_g Tf(e) = Tf(g). \] (28)
On the other hand, $\varphi(L.gf) = (\delta_g \ast \varphi)(f) = \delta_g(R_\varphi f) = R_\varphi(g)$ according to (24). Hence, $T = R_\varphi$. Thus, (27) is true and $R$ is bijective.

Let us prove that $R$ is continuous. Put
\[ U_{f,\varphi,\varepsilon} = \{\psi \in \mathcal{B} : \|R_\psi f - R_\varphi f\| < \varepsilon\}, \]
where $f \in A, \varphi \in \mathcal{B}$ and $\varepsilon > 0$. Finite intersections of the sets
\[ \{R_\psi : \psi \in U_{f,\varphi,\varepsilon}\} \]
form a base for the filter of neighborhoods of $R_\varphi$ in $\mathcal{B}$. Thus, it is sufficient to find a weak neighborhood of $\varphi$ in $\mathcal{B}$ that is contained in $U_{f,\varphi,\varepsilon}$. Let the neighborhood $V$ of the point $e$ in $G$ and $g_1, \ldots, g_n \in G$ be such that
\[ \|L_h f - f\| < \varepsilon \text{ for all } h \in V \]
and the family $Vg_k, k = 1, \ldots, n$, covers $G$. Then for any $g \in G$, there are $k \in \{1, \ldots, n\}$ and $h_k \in V$ such that $g = h_k g_k$. For brevity, denote $\eta = \varphi - \psi$. Then $R_\psi f(g) - R_\varphi f(g) = R_\eta f(g)$. Thus $\|\eta\| \leq 2$ and
\[ |R_\eta f(g)| = |L_{h_k} R_\eta f(g_k)| = |(R_\eta(L_{h_k} f - f) + R_\eta f)(g_k)| \]
Hence, the inequalities $|R_\eta f(g_k)| < \varepsilon$, $k = 1, \ldots, n$ define a weak neighborhood of $\varphi$ in $\mathcal{B}$, which $R$ maps to $U_{f, \varphi, 3\varepsilon}$. Hence, $R$ is continuous.

Equality $R_\varphi R_\varphi = R_{\varphi \ast \varphi}$ follows from Corollary 1.3 and a similar property of the algebra $M(G)$. Hence, $R$ is a semigroup homomorphism. This completes the proof of the lemma.

Let $\mathfrak{M}_A$ be the set of all continuous nonzero endomorphisms of the algebra $A$ that commute with all left shifts endowed with the strong operator topology. The set $\mathfrak{M}_A$ is a semigroup, which is contained in the unit ball $\mathcal{B}$. Otherwise, there exist $T \in \mathfrak{M}_A$ and $f \in A$ such that $\|Tf\| > 1$, $\|f\| < 1$, and we get a contradiction comparing $\|Tf^n\| = \|(Tf)^n\| = \|Tf\|^n$ with $\|f^n\| = \|f\|^n$.

**Theorem 2.2.** Let $A$ be a separating invariant algebra on a compact group $G$. Then the following assertions hold:

1. The space $M_A$ with multiplication (22) is a compact topological semigroup. The mapping $R$ is a topological isomorphism between the semigroups $M_A$ and $\mathfrak{M}_A$.
2. The evaluation mapping $ev : G \to M_A$ is an isomorphic and homeomorphic embedding and $ev_{e}$ is the unit of $M_A$.
3. If $\varphi, \psi \in M_A$ and $\varphi \ast \psi \in ev(G)$ then $\varphi, \psi \in ev(G)$. In particular, $ev(G)$ coincides with the group of invertible elements $M_A$.

**Proof.** By Lemma 2.2, $R$ is a homeomorphism and an algebraic isomorphism between $M_A$ and $\mathfrak{M}_A$. For any $\varphi', \psi' \in M_A$ and all $f \in C(G)$, we have $\|(R_{\varphi' \ast \psi'} - R_{\varphi \ast \psi})f\| \leq \|(R_{\psi'} - \psi_f)f\| + \|(R_{\varphi'} - \varphi_f)\| f$ since $\|R_{\varphi_f}\| = 1$. Hence, $M_A$ is a topological semigroup. This proves (1).

The mapping $ev$ is injective because $A$ separates points. All other statements of (2) are obvious.

For every $g \in G$, the Dirac measure $\delta_g$ is the only representing measure for $ev(g)$ by Lemma 1.2. Let $\varphi, \psi \in M_A$, $\mu \in M_{\varphi}$ and $\nu \in M_{\psi}$. Since $\mu$ and $\nu$ are positive, we have

$$\text{supp } \mu \ast \nu = \text{supp } \mu \ast \text{supp } \nu.$$  

(29)

If $\varphi \ast \psi \in G$ then $\text{supp } \mu \ast \nu$ is a point. By (29), the same is true for $\text{supp } \mu$ and $\text{supp } \nu$. Thus, $\varphi, \psi \in ev(G)$. This completes the proof of the theorem.

In what follows, we assume that $G$ is embedded into $M_A$ and write $G$ instead of $ev(G)$. This assumes that $A$ separates points. In addition, we omit $*$ in the notation for multiplication in $M_A$, if this does not lead to confusion.

2.2. **Action of $M_A$ in the Hilbert space $H^2$.** By Corollary 1.1, $A^* = A$. The involution $*$ is an antilinear automorphism of $A$. It defines an involution in $M_A$ by the formula

$$\varphi^*(f) = \overline{\varphi(f')}.$$  

(30)

where the bar stands for the complex conjugation. This is also an anti-automorphism of the semigroup $M_A$: $(\varphi \psi)^* = \psi^* \varphi^*$ for all $\varphi, \psi \in M_A$. On the group $G$ it coincides with the inversion $g \rightarrow g^{-1}$ according to (6) and (30).

**Theorem 2.3.** The following assertions hold.

1. For any $\varphi \in M_A$, $R_\varphi$ admits the unique continuous extension onto $H^2$. This defines a homeomorphic and isomorphic embedding of $M_A$ to the unit ball in $BL(H^2)$.
2. For all $\varphi \in M_A$, the equality $R_{\varphi^*} = R_{\varphi}^*$ is valid, where $R_{\varphi}^*$ is the adjoin to $R_{\varphi}$ operator in the Hilbert space $H^2$.

**Proof.** The equality (25) extends $R$ to $H^2$ and, moreover, it implies that $\|R_\varphi\|_{BL(H^2)} \leq 1$. The map $R : M_A \to BL(H^2)$ is strongly continuous because $A_{\text{fin}}$ is dense in $H^2$. It is homeomorphic since $M_A$ is compact, and the embedding $M_A \to BL(H^2)$ is injective. This proves (1).

The operator $R_g$ is unitary on $H^2$ if $g \in G$ since $H^2$ is a $G$-invariant subspace of $L^2(G)$ and $R_{g^{-1}} = R_g^{-1} = R_{g}^*$ on $L^2(G)$. The equality $R_{\varphi^*} = R_{\varphi}^*$ holds for $\varphi = ev_g$, where $g \in G$, because $M_\varphi = \{\delta_g\}$. It follows from (25) that $R_{\varphi^*} = R_{\varphi}^*$ if the measure $\mu$ in (25) is real. By definition of $M_\varphi$, every $\mu \in M_\varphi$ is positive. This completes the proof of the theorem.
2.3. The case of abelian groups. There is another way to define the same structure of a topological semigroup in $M_A$ if $G$ is abelian. Then the dual to object $\hat{G}$ is the group of one dimensional characters $\text{Hom}(G, \mathbb{T})$, which can be treated as a subgroup of the multiplicative group of invertible elements in $C(G)$. One dimensional characters can be defined by the equality

$$\Delta(\chi) = \chi \otimes \chi.$$  \hfill (31)

The set $\text{Sp}A = A \cap \hat{G}$ is a semigroup in $\hat{G}$. The converse is also true: If $S \subseteq \hat{G}$ is a semigroup and $e \in S$ then the closure in $C(G)$ of its linear span is an invariant algebra. The restriction to $\text{Sp}A$ of any $\varphi \in M_A$ is a homomorphism $\text{Sp}A \to \mathbb{D}$ such that $\varphi(1) = 1$, where the unit disc $\mathbb{D}$ is considered as a topological semigroup with the complex conjugation as an involution.

The set $\text{Hom}(\text{Sp}A, \mathbb{D})$ of all nonzero homomorphisms $\text{Sp}A \to \mathbb{D}$ is a topological semigroup with the pointwise multiplication and the topology of pointwise convergence, which is generated by the sets of $\eta \in \text{Hom}(\text{Sp}A, \mathbb{D})$ such that $|\eta(\chi) - \eta(\chi_0)| < \varepsilon$ indexed by $\chi_0 \in \text{Hom}(\text{Sp}A, \mathbb{D})$ and $\varepsilon > 0$.

The essential part of the following theorem had been proved in [2].

**Theorem 2.4.** If $G$ is abelian then the restriction of linear functionals from $M_A$ to $\text{Sp}A \subset A$ defines an isomorphism $\rho : M_A \to \text{Hom}(\text{Sp}A, \mathbb{D})$ of topological semigroups. Moreover, for all $\varphi \in M_A$,

$$\rho(\varphi^*) = \overline{\rho(\varphi)}.$$  \hfill (32)

**Proof.** Since $S = \text{Sp}A \subset A$, the topology of pointwise convergence on $S$ coincides with the weak topology. Hence, $\rho$ is continuous. In addition, $\rho$ is one-to-one since the linear span of $S$ equals $A_{\text{fin}}$, which is dense in $A$. The equality (31) together with (18) means that $\rho$ is a homomorphism.

It is clear that $\chi^* = \chi$ for any $\chi \in \hat{G}$. Thus (32) follows from (30). It remains to prove that $\rho$ is surjective (recall that $M_A$ is compact). Every $\varphi \in \text{Hom}(S, \mathbb{D})$ can be extended to a continuous linear functional on $l^1(S)$ because $\varphi$ is bounded on $S$. The spaces $l^1(\hat{G})$ and $l^1(S)$ are convolution Banach algebras. Obviously, $\varphi$ is multiplicative on $l^1(S)$. It is known that the maximal ideal space of $l^1(\hat{G})$ equals $G$ (see, for example, [5, Ch. 2]). By the Gelfand–Naimark formula $\lim_{n \to \infty} \|f^n\|_{l^1(\hat{G})} = \|f\|_{C(G)}$ for any $f \in l^1(\hat{G})$. Thus, the relations $|\varphi(f)| = |\varphi(f^n)|^{1/n}$ and $|\varphi(f^n)| \leq \|f^n\|_{l^1(\hat{G})}$ imply $|\varphi(f)| \leq \|f\|_{C(G)}$. Since $l^1(S)$ contains the dense subalgebra $A_{\text{fin}}$ of $A$, $\varphi$ admits a continuous extension to $A$. This completes the proof of theorem. \hfill \Box

2.4. Examples. In this subsection we give two simple illustrating examples.

**Example 5.** Algebra $A(\mathbb{D})$ of all functions analytic in $\mathbb{D}$ and continuous in $\overline{\mathbb{D}}$, which is often called the disc algebra, can be considered as a subalgebra of $C(\mathbb{T})$ due to the maximum modulus principle. This is an invariant algebra on $\mathbb{T}$. The group $\mathbb{T}$ is abelian and $\hat{\mathbb{T}} = \mathbb{Z}$, where $n \in \mathbb{Z}$ refers to the character $\chi_n(e^{it}) = e^{int}$. It is clear that $\text{Sp}(A(\mathbb{D})) = \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. According to Theorem 2.4, $M_{A(\mathbb{D})} \cong \text{Hom}(\mathbb{Z}_+, \mathbb{D}) \cong \mathbb{D}$. The latter means that the homomorphism $\varphi : \mathbb{Z}_+ \to \mathbb{D}$ is uniquely determined by the equality $\varphi(1) = \lambda$, where $\lambda \in \mathbb{D}$. The multiplication in $\mathbb{D}$ is the usual multiplication of numbers, the involution is the complex conjugation.

**Example 6.** Similarly, the algebra $A(B_n)$ of all functions continuous on the matrix ball $B_n = \{Z \in \text{BL}(C^n) : 1 - Z^*Z \geq 0\}$ and analytic in its interior can be considered as an invariant algebra on the group $\text{U}(n)$. The inequality means that the matrix on the left is non-negative definite. The multiplication (22) is the usual matrix multiplication, the involution (30) is the Hermitian transposition. We cannot prove these claims by a reference to the previous material. However, it is easy to prove directly that $B_n \cong M_{A(B_n)}$ assigning to a multiplicative linear functional on $A(B_n)$ its values on the coordinate functions of matrices and checking the coincidence of the multiplication and involution operations in $M_{A(B_n)}$ with the standard matrix operations in $B_n$.

We conclude this section with a characterization of the center of $M_A$. Note that the action of group $\text{Aut}_0(G)$ defined on $G$ by the formula $x \to g^{-1}xg$ naturally extends to $M_A$. 

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Proposition 2.1. Let \( \zeta \in \mathcal{M}_A \). The following assertions are equivalent:

1. \( \zeta \in Z(\mathcal{M}_A) \),
2. \( \zeta \in Z(G) \),
3. \( \mathcal{M}_\zeta \) contains an \( \text{Aut}_0(G) \)-invariant measure.

Proof. The implication (1) \( \Rightarrow \) (2) is trivial. If (2) is true then the group \( \text{Aut}_0(G) \) preserves a convex weakly compact set \( \mathcal{M}_\zeta \). Consequently, it contains an \( \text{Aut}_0(G) \)-fixed point, that is, an \( \text{Aut}_0(G) \)-invariant measure. Hence, (3) follows from (2). Since the center of the convolution algebra \( M(G) \) consists of such measures, the implication (3) \( \Rightarrow \) (1) is true by Corollary 1.3.

3. AVERAGING OVER SUBGROUP AND RESTRICTION TO SUBGROUP

In this section, \( H \) denotes a closed subgroup of \( G \).

3.1. Subgroup hulls. Recall that the \( A \)-hull \( \widehat{X} \) of the set \( X \subseteq \mathcal{M}_A \) is defined by (8).

Proposition 3.1. Let \( H \) be a closed subgroup of \( G \). Then

1. \( \widehat{H} \) is a closed subsemigroup of \( \mathcal{M}_A \),
2. the closure of the restriction \( A|_H \) is an invariant algebra \( B \) on \( H \),
3. the mapping dual to the restriction homomorphism \( A \rightarrow B \) defines topological isomorphism of \( * \)-semigroups \( \mathcal{M}_B \) and \( \widehat{H} \).

Proof. According to Corollary 1.3, multiplication in \( \mathcal{M}_A \) and in \( \mathcal{M}_B \) agree with convolution of representing measures. By Lemma 1.1 there are measures \( \mu \in \mathcal{M}_\varphi \) and \( \nu \in \mathcal{M}_\psi \) with supports on \( H \) if \( \varphi, \psi \in \widehat{H} \). Then

\[ \text{supp } \mu \ast \nu = \text{supp } \mu \cdot \text{supp } \nu \subseteq H. \]

Applying Lemma 1.1 again, we get \( \varphi \psi \in \widehat{H} \). The remaining assertions follow from this fact and the natural identification \( \mathcal{M}_B = \widehat{H} \) that holds for any closed subset \( H \subseteq G \) by (8).

Replacing \( A \) with the algebra of all holomorphic polynomials on \( C^n \) in the definition of \( A \)-hulls, we obtain the definition of a polynomially convex hull.

Corollary 3.1. Polynomially convex hull of a compact linear group \( G \subseteq \text{GL}(n, \mathbb{C}) \) is a semigroup.

Proof. We may assume that \( G \subseteq \text{U}(n) \) since \( G \) admits a positive definite Hermitian invariant form being compact. Then Proposition 3.1 can be applied to the algebra of Example 6.

3.2. Averaging over \( H \). The left averaging operator over the subgroup \( H \) is defined as

\[ L_H = \int_H L_h \, dh. \]  \hfill (33)

It is clear that \( L_H \) is well defined in any left-invariant closed subspace of \( C(G) \), it commutes with \( L_h \) for any \( h \in H \) and with \( R_\varphi \) for all \( \varphi \in \mathcal{M}_A \) and, moreover, \( L_H^2 = L_H \).

Proposition 3.2. Let \( B = L_H A \) and \( \tilde{H} = \{ h \in G : h \mathbin{\overset{\sim}{\sim}} e \} \). Then

1. The class \( \tilde{H} \) is a closed subgroup of \( G \) that contains \( H \). It is normal if \( H \) is normal. The space \( B \) consists of all functions \( f \in A \) such that \( L_h f = f \) for arbitrary \( h \in \tilde{H} \) and is a closed right invariant subalgebra of \( A \).
2. The group \( \tilde{H} \) is a \( p \)-set for \( A \), which is a peak set if \( G \) is a Lie group.
3. We have \( \tilde{H} = H \) if and only if \( H \) is a \( p \)-set for \( A \).
In general, $\gamma = R$ with $\alpha$ in some proper maximal ideal of $\mathcal{M}$, contradictory to the assumption that $\alpha$ is a separating $G$-invariant algebra of functions on the homogeneous space $M = \tilde{H} \setminus G$, where $G$ acts on $M$ from the right. Hence, all assertions of (2) follow from Lemma 1.2.

If $H$ is normal then the Haar measure on $L_H$ is $\text{Auto}(G)$-invariant. By Proposition 2.1 $L_H$ commutes with $L_g$ and $R_g$ for all $g \in G$. Hence, $B$ and $L_B$ are left invariant, and $\tilde{H}$ is normal. This completes the proof of (1).

Thus, $B$ is the set of all functions from $A$ that are constant on the classes $Hg$ for all $g \in G$. Therefore, it can be considered as a separating $G$-invariant algebra of functions on the homogeneous space $M = \tilde{H} \setminus G$, where $G$ acts on $M$ from the right. Hence, all assertions of (2) follow from Lemma 1.2.

The equivalence $\bowtie$ extends to $\alpha(\mathcal{M}_A) = \mathcal{M}_B$. Let $H$ be an ideal of $B$, $I \neq B$. If $I$ does not generate a proper ideal in $A$ then there are functions $b_1, \ldots, b_n \in I$ and $a_1, \ldots, a_n \in A$ such that $a_1b_1 + \cdots + a_nb_n = 1$. It follows from (34) that

$$1 = (L_H a_1)b_1 + \cdots + (L_H a_n)b_n \in I$$

contadictory to the assumption that $I$ is proper. Therefore, every maximal ideal of $B$, $I \neq B$, is contained in some proper maximal ideal of $A$.

Let $M = H \setminus G$ be a right homogeneous space $G$. In the following theorem, the algebra $A^H = L_H A$ is considered as a subalgebra of $C(M)$ as well as a subalgebra in $C(G)$.

**Theorem 3.2.** Let $H$ be a $p$-set. The action of the group $G$ on $M$ extends to the right action of the semigroup $\mathcal{M}_A$ on $\mathcal{M}_A^H$ by the formula

$$f(\psi \varphi) = R_\varphi f(\psi), \quad (35)$$

where $\varphi \in \mathcal{M}_A$ and the right side determines the left side for any $f \in A^H$, $\psi \in \mathcal{M}_A^H$. For all $p \in M$,

$$p\mathcal{M}_A \cong \mathcal{M}_A^H. \quad (36)$$

**Proof.** Obviously, $A^H$ is right invariant. Since $A^H \subseteq A$, every irreducible component of the right regular representation $R$ of the group $G$ in $A^H$ is contained in $Sp A$. By (25) $R$ extends to a representation of $\mathcal{M}_A$ in $A^H$. This is equivalent to (35) and defines an action of $\mathcal{M}_A$ on $\mathcal{M}_A^H$ that commutes with the above mapping $\alpha$. Since $g\mathcal{M}_A = \mathcal{M}_A$ for any $g \in G$, (36) follows from the Proposition 3.3.

**Proposition 3.3.** Let $A$, $B$ be as in Proposition 3.2. Then $\alpha(\mathcal{M}_A) = \mathcal{M}_B$.

**Proof.** Let $I$ be an ideal of $B$, $I \neq B$. If $I$ does not generate a proper ideal in $A$ then there are functions $b_1, \ldots, b_n \in I$ and $a_1, \ldots, a_n \in A$ such that $a_1b_1 + \cdots + a_nb_n = 1$. It follows from (34) that

$$1 = (L_H a_1)b_1 + \cdots + (L_H a_n)b_n \in I$$

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3.3. The maximal ideal space as a projective limit. By the well-known structure theorem for compact groups, every neighborhood of $e$ contains a closed normal subgroup $H$ such that $G/H$ is a Lie group (see, for example, [20, Corollary 2.36]). For any couple of normal subgroups $H_a \subseteq H_b$, there exists a natural surjective homomorphism $\gamma_{ab} : G_a \to G_b$, where $G_a = G/H_a$ and $G_b = G/H_b$. Let $\mathfrak{N} = \{H_a\}_{a \in I}$, where $I$ is the set of indices, be a family of closed normal subgroups in $G$ such that

$$H, P \in \mathfrak{N} \implies H \cap P \in \mathfrak{N}, \quad (37)$$

$$\bigcap_{H \in \mathfrak{N}} H = \{e\}. \quad (38)$$
Then $G$ is topologically isomorphic to the projective limit of Lie groups $G/H$ over $\mathcal{H}$, i.e., to the subgroup of the topological Cartesian product of these groups, consisting of sets $(g_\alpha)_{\alpha \in I}$ such that $g_\beta = \gamma_{\alpha\beta}(g_\alpha)$ if $H_\alpha \subseteq H_\beta$. According to the following theorem, the semigroup $\mathcal{M}_A$ can also be obtained in this way. We define $\mathfrak{P} = \mathfrak{P}(G, A)$ as a family of normal subgroups of $H \subseteq G$ such that that $G/H$ is a Lie group and $H$ is a $p$-set for $A$.

**Theorem 3.3.** Let $A$ be a separating invariant algebra on a compact group $G$ and let $\mathfrak{P}$ satisfies the above conditions and, in addition, satisfies (38). Then the following assertions hold:

1. For any $H \in \mathfrak{P}$, the averaging $L_H A = A^H$ is a separating invariant algebra on the Lie group $G/H$.

2. The projection $\alpha_H : \mathcal{M}_A \rightarrow \mathcal{M}_{AH}$ dual to the embedding $A^H \rightarrow A$, defines a continuous surjective semigroup homomorphism.

3. As a compact topological semigroup, $\mathcal{M}_A$ is a projective limit of $\mathcal{M}_{AH}$ over $\mathfrak{P}$.

**Proof.** Since $G/H$ is a Lie group and $H$ is a $p$-set, there exists a peak function $u \in A$ for $H \in \mathcal{H}$. Then the unit of $G/H$ is a peak point for $A^H$ and $L_H u$ is the peak function for it. Hence, the equivalence $A^H \simeq (G/H)$ is trivial and the algebra $A^H$ separates points of $G/H$. This proves (1).

The group $Aut_0(G)$ preserves the Haar measure $\sigma_H$ of the group $H$ because it is normal. Therefore $\sigma_H$ belongs to the center of the convolution algebra $M(G)$ and $\sigma_H \ast \sigma_H = \sigma_H$. Let $\varphi, \psi \in \mathcal{M}_A$, $\mu \in \mathcal{M}_\varphi$, $\nu \in \mathcal{M}_\psi$. Then $\mu \ast \nu \in \mathcal{M}_{\varphi \psi}$, $\mu \ast \sigma_H$ and $\nu \ast \sigma_H$ are the representing measures for $\varphi$ and $\psi$ restricted to $A^H$, respectively, and

$$\mu \ast \nu \ast \sigma_H = (\mu \ast \sigma_H) \ast (\nu \ast \sigma_H).$$

Hence, the projection $\mathcal{M}_A \rightarrow \mathcal{M}_{AH}$ is a continuous homomorphism of semigroups. It is surjective by Proposition 3.3. Thus (2) is true.

Let $H, P \in \mathfrak{P}$ and let $u$ and $v$ be peak functions for $H$ and $P$, respectively. Then $uv$ has peak on $H \cap P$. Thus, (37) holds for $\mathfrak{P}$. If $H \subseteq P$ then we can apply the assertion (2) to $A^H$. Therefore the homomorphism $\alpha_{H,P} : \mathcal{M}_{AH} \rightarrow \mathcal{M}_{AP}$ is well defined. According to (38), for any $f \in A$ we have

$$\lim_{H \in \mathfrak{P}} \| f - L_H f \| = 0$$

due to the uniform continuity of $f$, where $\mathfrak{P}$ is ordered by inclusion. The operation of restriction of $\varphi \in \mathcal{M}_A$ to $A^H$ defines the set $\{ \varphi_H \}_{H \in \mathfrak{P}}$, where $\varphi_H \in \mathcal{M}_{AH}$. This set can be extended to the family $\{ \varphi_H \circ L_H \}_{H \in \mathfrak{P}}$ in $\mathcal{A}^*$ whose weak limit is $\varphi$. This allows us to complete the proof of the theorem by the standard checking of the definition of the projective limit of semigroups.

**Corollary 3.4.** The equality $\mathcal{M}_A = G$ holds if and only if $\mathcal{M}_{AH} = G/H$ for all $H \in \mathfrak{P}$.

**Proof.** If $\mathcal{M}_A = G$ then $\mathcal{M}_{AH} = G/H$ for all $H \in \mathfrak{P}$ by Theorem 3.3, (2) and Theorem 2.2, (3). The converse is true according to the Theorem 3.3, (3).

Theorem 3.3 reduces some problems of invariant algebras to the case of Lie groups. For example, if $G$ is a Lie group then $\mathcal{M}_A = G$ implies $A = C(G)$ due to known results on polynomial approximation on polynomially convex manifolds. Corollary 3.4 extends this assertion to the class of compact groups.

## 4. IDEMPOTENTS IN $\mathcal{M}_A$

The set $\mathfrak{I}_A$ of all idempotents is an important ingredient of the semigroup $\mathcal{M}_A$. In this section, we assume that $G$ is embedded to $\mathcal{M}_A$. Thus $A$ separates points of $G$.

### 4.1. Groups associated with idempotents

There are three natural actions of the group $G$ on $\mathcal{M}_A$:

1. Left, right, and adjoint. Every $j \in \mathfrak{I}_A$ associates with its stable groups for these actions. Put

$$G_j = \{ g \in G : gj = jg = j \}$$

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and let $G^l_j$ and $G^r_j$ be the stable subgroups of $j$ for the left and right actions $G$. They are defined by the equalities $gj = j$ and $jg = j$, respectively. Hence, $G_j = G^l_j \cap G^r_j$. We will see soon that these three groups coincide. The group

$$N_j = \{ g \in G : gj = jg \}$$

is the centralizer $Z(j)$ of $j$ in $G$, but we denote it by $N_j$ because it is the normalizer of $G_j$ in $G$ by the lemma below. Set

$$G^l_j = jN_j = N_jj \cong N_j/G_j,$$

$$M_j = \{ \varphi \in M_A : \varphi j = j\varphi = \varphi \} = jM_A j,$$

and denote by $\sigma_j$ the Haar measure of the group $G_j$.

The next lemma combines several simple properties of these objects. We will often refer to them later truncating to (x) references of the type Lemma 4.1, (x) with Roman numerals x.

Recall that the set of idempotents of any semigroup admits a natural order:

$$j \preceq k \iff jk = kj = j.$$  

**Lemma 4.1.** Let $j \in \mathfrak{J}_A$. Then

(i) $G_j$ is a $p$-set for $A$,

(ii) $\sigma_j \in M_j$,

(iii) $j^* = j$,

(iv) $G_j = G^l_j = G^r_j$,

(v) for every $k \in \mathfrak{J}_A$, $j \preceq k$ is equivalent to $G_j \supseteq G_k$,

(vi) $N_j = N_G(G_j)$, in particular, $j \in Z(M_A)$ if and only if $G_j$ is normal,

(vii) $\varphi, \psi \in M_j$ and $\varphi \psi \in G^l_j$ imply $\varphi, \psi \in G^l_j$.

**Proof.** If $j \in \mathfrak{J}$ then $R^2_j = R_j$ and $R_j$ projects $A$ onto its image $B = R_j A$. Moreover, $R_j$ is an endomorphism of $A$ commuting with left shifts. Hence, $B$ is a closed left invariant subalgebra of $A$. By (35) the equality $f(gj) = R_j f(g)$ holds for all $f \in A$ and $g \in G$. If $f \in B$ then $f(gj) = f(g)$ and $f(gh) = f(g)$ for arbitrary $g \in G$ and $h \in G^l_j$. Hence, $B$ can be identified with the restriction of $A$ to the left orbit $O^l_j = \{ gj : g \in G \}$ of the idempotent $j$ in $M_A$. It is the homogeneous space $G^l_j/G^l_j$. Since $A$ separates the points of $M_A$, the algebra $B$ is a closed separating subalgebra of $C(O^l_j)$. According to Lemma 1.2 every point of $O^l_j$ is a $p$-point for $B$. Hence, $G^l_j$ is a $p$-set for $A$. Thus, (i) holds for $G^l_j$. Together with Lemma 1.1 this implies

$$\text{supp}\, \mu \subseteq G^l_j$$  

(39)

for any measure $\mu \in M_j$. Since the set $M_j$ is weakly compact, convex and left $G^l_j$-invariant, it follows from (39) that $M_j$ contains the Haar measure $\sigma_j$ of the group $G^l_j$. Therefore, (ii) holds for the Haar measure $G^l_j$.

The inversion preserves the measure $\sigma_j$. Hence, $\sigma_j \in M_j^*$ and (iii) is true. The equality $(gj)^* = j^* g^{-1}$ shows that $G^r_j = G^l_j$. Together with (iii) this proves (iv). Thus we can replace $G^l_j$ with $G_j$ in the above statements. The inclusion $G_j \supseteq G_k$ is true if and only if $\sigma_j * \sigma_k = \sigma_k * \sigma_j = \sigma_j$. This is equivalent to $jk = kj = j$ and proves (v). If $g \in N_G(G_j)$ then $\delta_g * \sigma_j = \sigma_j * \delta_g$ whence $gj = jg$. Conversely, if $gjg^{-1} = j$ then

$$\text{supp}(\delta_g * \sigma_j * \delta_g^{-1}) = ggjg^{-1} \subseteq G_j$$

according to (39) and Lemma 1.1. Thus $g \in N_G(G_j)$ and (vi) is true.

It remains to prove (vii). Note that $M_j$ is a semigroup and $G^l_j \subseteq M_j$. For some $g \in N_j$, we have $g\varphi \psi = j$. We can assume $\varphi \psi = j$ replacing $\varphi$ with $g\varphi$. Let $\mu \in M_\varphi$ and $\nu \in M_\psi$. Then $g = \mu * \nu \in M_j$.  

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Due to the equalities \(j\varphi = \varphi\) and \(\psi j = \psi\) we have \(\sigma_j * \mu \in \mathcal{M}_j\) and \(\nu * \sigma_j \in \mathcal{M}_j\). Therefore, we can assume \(\sigma_j \ast \mu = \mu\) and \(\nu * \sigma_j = \nu\). Then \(\sigma_j * \varrho = \varrho * \sigma_j = \varrho\). By (i) and Lemma 1.1, \(\text{supp} \varrho \subseteq G_j\). Hence, \(\varrho = \sigma_j\).

Let \(h \in \text{supp} \mu\). Set \(\mu' = \mu * \delta_{h^{-1}}\) and \(\nu' = \delta_h * \nu\). Then \(\mu' * \nu' = \mu * \nu = \varrho\), \(\sigma_j * \mu' = \mu'\) and \(\nu' * \sigma_j = \nu'\). Furthermore,

\[
\text{supp} \mu' \cdot \text{supp} \nu' \subseteq G_j
\]

and \(e \in \text{supp} \mu'\). Therefore, \(\mu' = \nu = \sigma_j\). Thus \(\varphi = jh\) and \(\psi = h^{-1} j\). By the definition of \(\mathcal{M}_j\), we have \(j \varphi j = \varphi\). Hence, \(j \psi j = jh\) and \(j(hjh^{-1}) = j\). It follows that \(G_j(hG_jh^{-1}) = G_j\) and \(hG_jh^{-1} \subseteq G_j\).

The latter implies \(h \in N_j\). This completes the proof of the lemma.

**Lemma 4.2.** The closed subsemigroup \(S \subseteq \mathcal{M}_A\) generated by a subset \(I \subseteq \mathcal{J}_A\) is a semigroup with zero.

**Proof.** Due to Theorem 2.3, it is sufficient to prove the lemma for the semigroup \(\mathcal{M}_A \subseteq \text{BL}(H^2)\) acting on \(H^2\). Let \(\mathcal{G}\) be the corresponding to \(S\) subsemigroup \(\mathcal{M}_A\). It is strongly closed, commutes with left shifts and is \(*\)-invariant because \(R_j^* = R_j\) for all \(j \in \mathcal{J}\) due to (iii) and Theorem 2.3. Thus, every \(R_j\) is an orthogonal projection.

Set \(L = \bigcap_{j \in I} R_j H^2\). The semigroup \(\mathcal{G}\) obviously preserves the Peter-Weyl decomposition. Hence, \(L\) is the direct orthogonal sum of the spaces \(L_\tau = L \cap M_\tau\), where \(\tau\) runs through \(\hat{G}\). The same is true for its orthogonal complement \(L^\perp\), which is a direct sum of the spaces \(L_\tau^\perp = L_\tau \cap M_\tau\).

Since \(L\) is the set of all fixed points of \(\mathcal{G}\) and \(\mathcal{G}\) preserves \(L_\tau^\perp\), it is sufficient to prove that the restriction of \(\mathcal{G}\) to \(L^\perp\) contains the zero of \(\text{BL}(L^\perp)\). Then the orthogonal projection onto \(L^\perp\) is the zero of \(\mathcal{G}\). Thus it is sufficient to prove that for any \(f \in L^\perp\) there exists a sequence \(S_n\) in \(\mathcal{G}\) such that \(\|S_n f\| \to 0\) as \(n \to \infty\) and, moreover, to prove this for every finite dimensional space \(L_j^\perp\) because the number of nonzero components in the Peter-Weyl expansion of the function \(f\) is at most countable. Thus the problem reduces to the following one: We need to prove that a compact linear semigroup \(\mathcal{G} \subseteq \text{BL}(\mathbb{R}^d)\) that is generated by a family \(I\) of orthogonal projections, which do not have non-zero common fixed points contains the zero of \(\text{BL}(\mathbb{R}^d)\). Clearly, we can assume \(I\) finite. Then the product \(S\) of all projections in \(I\) satisfies the inequality \(\|S\| < 1\) and therefore \(S^n \to 0\) as \(n \to \infty\).

It remains to note that the family \(I\) in the statement of the lemma preserves every space \(L^\perp\) and have no common nontrivial fixed point in each of them by definition of these spaces.

**Corollary 4.1.** The set \(\mathcal{J}_A\) contains its least lower bound.

**Proof.** The zero of the semigroup generated by \(\mathcal{J}_A\) is obviously an idempotent.

An ordered set is called a complete lattice if every its bounded subset \(I\) has the least upper bound \(\sup I\) and the greatest lower bound \(\inf I\). It follows from (v) that \(e\) is the upper bound of \(\mathcal{J}_A\), and it has a lower bound by Corollary 4.1. Thus, in the case of \(\mathcal{J}_A\) we need not assume that \(I\) is bounded in the definition of a complete lattice.

**Theorem 4.2.** The set \(\mathcal{J}_A\) is a complete lattice. In particular, \(\mathcal{J}_A\) contains the least idempotent \(\epsilon\) and we have the following inclusions:

\[
\hat{G}_\epsilon \supseteq \mathcal{J}_A,
\]

\[
\epsilon \in Z(\mathcal{M}_A).
\]

**Proof.** Let \(I \subseteq \mathcal{J}_A\) and \(S\) be the closed semigroup generated by \(I\). According to Lemma 4.2, \(S\) contains its zero \(\zeta\). It is evidently a lower bound for \(I\). Let \(j \in \mathcal{J}_A\) be an arbitrary lower bound for \(I\). Then \(jk = kj = j\) for all \(k \in I\). Hence, \(j \varphi = \varphi j = j\) for all \(\varphi \in S\) and, in particular, for \(\zeta\). Therefore, \(j \leq \zeta\) and the bound is exact, i.e., \(\zeta = \inf I\).

Let \(J = \{j \in \mathcal{J}_A : \forall k \in I, \exists jk = kj = k\}\) be the set of all upper bounds for \(I\) and \(S\) be a semigroup generated by \(J\). Obviously, the same equalities hold for the products of such \(j\). According to Lemma 4.2, \(\inf J \in S \cap \mathcal{J}_A\). Hence, \(\inf J = \sup I\).

It follows from (v) that \(G_\epsilon \supseteq G_j\) for all \(j \in \mathcal{J}_A\). Together with (ii) and Lemma 1.1 this proves (40) and, moreover, this implies inclusion \(g^{-1}G_\epsilon g \subseteq G_{\epsilon}\) for all \(g \in G\) because \(g^{-1} \epsilon g \in \mathcal{J}_A\). Hence, the subgroup \(G_\epsilon\) is normal and (41) is true due to (iv) and Proposition 2.1.
5. ANTISYMMETRIC INVARIANT ALGEBRAS

This short section is an important part of the paper. We prove in it a simple property, which characterizes the Haar measures of closed subgroups of \( G \) that are multiplicative on \( A \).

An algebra of functions \( B \subset C(Q) \) is called antisymmetric if any real valued function from \( B \) is constant. The set \( E \) is called the set of antisymmetry if every real on \( E \) function \( f \in B \) is constant on \( E \). If two sets of antisymmetry have a common point then their union is also a set of antisymmetry. Therefore, each set of antisymmetry is contained in a maximal set of antisymmetry.

By the Bishop–Shilov theorem, a continuous function \( f \) is contained in \( B \) if and only if \( f|_E \subset B|_E \) for every maximal set of antisymmetry \( E \). Let \( B^A \) be the algebra of all real functions from \( B \). According to the Stone–Weierstrass theorem, \( B^A \) can be identified with the algebra of all real continuous functions on \( Q/\sim^A \). We will show that the classes of the equivalence \( \sim^A \) are maximal sets of antisymmetry if \( B \) is an invariant algebra. In general, this is not true.

For more information on this and related topics, see [9]. We need one more fact.

**Theorem 5.1.** Let \( A \) be a separating invariant algebra on a compact group \( G \). If \( \mathcal{M}_A = G \) then \( A = C(G) \).

This follows from [4, Theorem 6.1]. For Lie groups, the theorem is a consequence of known results on polynomial approximation on polynomially convex submanifolds in \( C^n \). It can be extended to the class of compact groups due to Theorem 3.3, (3).

**Lemma 5.1.** If \( \mathcal{M}_A \neq G \) then \( \epsilon \neq e \).

**Proof.** Due to Theorem 4.2, it is sufficient to prove that \( \mathcal{M}_A \) contains at least one idempotent distinct from \( e \) if \( \mathcal{M}_A \neq G \). Let \( \varphi \in \mathcal{M}_A \setminus G \) and \( s = \varphi^* \varphi \). Then \( s \notin G \) due to Theorem 2.2, (3). The operator \( R_s \) is self-adjoint in \( H^2 \) and is not identical. Since all components \( M_r \) in the Peter–Weil expansion are finite-dimensional and \( R_s \) preserves them, \( R_s \) has a basis of eigenfunctions in the spaces \( M_r \), and the corresponding eigenvalues are located in the segment \([0,1]\). Therefore, the sequence \( R_s^n \) converges strongly to some projection in \( H^2 \). Since \( \mathfrak{M}_A \) is strongly compact, the limit relates to a non-trivial idempotent in \( \mathcal{M}_A \).

**Theorem 5.2.** Let \( A \) be an invariant algebra on a compact group \( G \). The group \( G_\epsilon \), where \( \epsilon \) is the least idempotent in \( \mathcal{M}_A \), is a maximal set of antisymmetry for \( A \). This is a normal subgroup whose cosets exhaust the family of all maximal sets of antisymmetry, and the algebra \( A^\epsilon \) is the algebra of all real continuous functions on \( G \) that are constant on them.

The following statements are equivalent:

(a) \( A \) is antisymmetric,
(b) \( \mathcal{M}_A \) is a semigroup with zero,
(c) the Haar measure of \( G \) is multiplicative on \( A \).

**Proof.** The subgroup \( G_\epsilon \) is normal by Theorem 4.2 and (vi). Note that for every \( \varphi \in \mathcal{M}_A \) and any \( \mu \in \mathcal{M}_\varphi \), the set \( \text{supp} \mu \) is a set of antisymmetry. Indeed, if \( f \in A \) is a real non-constant on \( \text{supp} \mu \) function then \( \varphi(f) \) is real and

\[
\varphi((f - \varphi(f))^2) = \int (f - \varphi(f))^2 \, d\mu > 0. \quad (42)
\]

On the other hand, \( \varphi((f - \varphi(f))^2) = \varphi((f - \varphi(f))^2) = 0 \), since the functional \( \varphi \) multiplicative. Thus, \( G_\epsilon \) is a set of antisymmetry by (ii).

Put \( H = G_\epsilon \). It follows from (i) and item (1) of Theorem 3.3 that \( A^H \) is an invariant separating subalgebra of \( C(G/H) \). We claim that \( \mathcal{M}_{A^H} = G/H \). Let \( \tilde{\epsilon} \) be the least idempotent of \( \mathcal{M}_{A^H} \). By Lemma 5.1, \( \tilde{\epsilon} \) cannot be the unit of \( \mathcal{M}_{A^H} \) if \( \mathcal{M}_A \neq G/H \). Let the latter hold. The operator \( R_{\tilde{\epsilon}} \) is a projection of \( A \) onto \( A^H \). Hence, the composition \( R_{\tilde{\epsilon}} R_{\epsilon} \) is well defined. It is a projection onto a subspace of \( A \), which is smaller than \( A^H \). Since the endomorphisms \( R_{\tilde{\epsilon}} \) and \( R_{\epsilon} \) commute with the left shifts, their composition has the same properties and, consequently, corresponds to some idempotent in \( \mathcal{M}_A \), which is strictly less than \( \epsilon \). We get a contradiction, which proves the equality \( \mathcal{M}_{A^H} = G/H \).
According to Theorem 5.1, \( A^H = C(G/H) \). Therefore, \( G_e \) is a maximal set of antisymmetry as well as every its coset \( gG_e, g \in G \). There are no other such sets because they cover \( G \) and the maximal sets of antisymmetry are pairwise disjoint. Moreover, this proves that \( A^e \) is the algebra of all real continuous functions on \( G \) that are constant on the cosets of \( G_e \).

Thus \( A \) is antisymmetric if and only if \( G_e = G \), i.e. if (b) is true. Hence, (a) and (b) are equivalent.

Together (ii) and (b) imply (c). Since the support of any positive multiplicative measure is an antisymmetry set, (c) implies (a). This completes the proof. 

For abelian groups \( G \), there is another simple criterion of antisymmetry in terms of the semigroup \( S = \text{Sp} A \) (we use the notation of Subsection 2). Namely, \( A \) is antisymmetric if and only if

\[
S \cap (-S) = \{0\} \tag{43}
\]

(in additive notation). Indeed, if \( \chi, \overline{\chi} \in A \) then \( \text{Re} \chi \in A \). Conversely, \( A \) contains all characters in the support of the Fourier transform of any \( f \in A \), but for real \( f \) the support is invariant under the inversion \( x \to -x \) in the dual group \( \hat{G} \).

We conclude this section with the following proposition, which makes it possible to consider the restriction of \( A \) to \( G_j \) independently of the ambient space \( \mathcal{M}_A \).

**Proposition 5.1.** Let \( j \in \mathcal{J}_A \). Then the restriction \( A_j \) of \( A \) to \( G_j \) is closed in \( C(G_j) \) and, moreover, it is an antisymmetric invariant algebra on \( G_j \). We have

\[
\mathcal{M}_{A_j} \cong \hat{G}_j = \{ \varphi \in \mathcal{M}_A : \varphi j = j \varphi = j \}.
\]

**Proof.** By Proposition 3.1, (2) the closure of \( A_j \) in \( C(G_j) \) is an invariant algebra on \( G_j \). By (i), \( G_j \) is a \( p \)-set for \( A \). Hence, \( A_j \) is closed by [9, 12.3 and 12.7]. The antisymmetry of \( A_j \) follows from (ii) and the equivalence of (a) and (c) in Theorem 5.2.

Proposition 3.1, (3) implies \( \mathcal{M}_{A_j} \cong \hat{G}_j \). According to (i), (ii) and Lemma 1.1, each of the equalities \( j \varphi = j \) and \( \varphi j = j \) hold if and only if \( \text{supp} \mu \subseteq G_j \) for every \( \mu \in \mathcal{M}_\varphi \). The latter is equivalent to \( \varphi \in \mathcal{M}_j \) by Lemma 1.1 and Lemma 4.1 (item (i)).

6. ONE PARAMETER SEMIGROUPS, ANALYTIC STRUCTURE, AND POLAR DECOMPOSITION

This section partially overlaps with the article [10].

6.1. Continuity of endomorphisms. We show that an algebraic endomorphism that is bounded in the \( L^2 \)-norm is also continuous in the uniform norm.

**Lemma 6.1.** Let \( Q \) be a compact set, \( \mu \) be a regular probability measure on \( Q \), and let \( B \) be a subalgebra of \( C(Q) \), possibly non-closed. Let \( \text{supp} \mu = Q \) and \( E \) be an endomorphism of \( B \), which is bounded as a linear operator from \( B \) to \( B \) in the norm of \( L^2(Q, \mu) \). Then \( E \) is bounded in \( B \) with respect to the \( \text{sup} \)-norm and, moreover, \( \|E\| \leq 1 \).

**Proof.** If \( E \) is unbounded in the \( \text{sup} \)-norm or \( \|E\| > 1 \) then there exists \( f \in B \) such that \( \|f\| < 1 \) and \( \|Ef\| > 1 \). Then the set

\[
U = \{ q \in Q : |Ef(q)| > 1 \}
\]

is open and non-empty. Hence, \( \mu(U) > 0 \). Therefore,

\[
\|Ef^n\|_{L^2(Q, \mu)} = \|(Ef)^n\|_{L^2(Q, \mu)} > \sqrt{\mu(U)},
\]

\[
\|f^n\|_{L^2(Q, \mu)} \leq \|f\|^n \to 0 \quad \text{at } n \to \infty,
\]

contradictory to the assumption of the lemma. Hence, \( \|E\| \leq 1 \).

Let \( B \subseteq C(Q) \) be a subalgebra that admits an orthogonal grading by the commutative semigroup \( \Lambda \), i.e.

\[
B = \sum_{\eta \in \Lambda} B_\eta, \tag{44}
\]

\[
B_\tau \cdot B_\eta \subseteq B_{\tau \eta} \quad \text{for all } \tau, \eta \in \Lambda, \tag{45}
\]

\[
B_\tau \perp B_\eta \text{ in } L^2(Q, \mu), \quad \text{if } \tau \neq \eta, \tag{46}
\]

where the sum is algebraic. We do not assume that \( B_\tau \) is finite-dimensional and that \( B_\tau \neq 0 \).
Lemma 6.2. Let $B$ satisfy the conditions (44)–(46), $\chi : \Lambda \to \mathbb{R}$ be a semigroup homomorphism, and the linear operator $E : B \to B$ be defined by $Ef = \chi(\tau)f$ for $f \in B_\tau$. Then $E$ is bounded in $B$ with respect to the uniform norm and, moreover, $\|E\| \leq 1$.

Proof. A calculation shows that $E$ is an endomorphism of $B$. Since the characters $\chi$ are bounded and $B_\tau$ are pairwise orthogonal by (46), then $E$ is bounded on $B$ in the norm of $L^2(Q, \mu)$ and the assertion follows from Lemma 6.1.

6.2. One parameter semigroups in $M_A$. In what follows, we consider $\mathbb{R}^+ = [0, \infty)$ and the closed right half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re } z \geq 0\}$ as additive semigroups. Put

$$S_A = \{s \in M_A : s^* = s, \ R_s \geq 0 \text{ on } H^2\},$$

where the inequality means that $R_s$ is nonnegative definite in the Hilbert space $H^2$. According to Theorem 2.3, $\varphi^* \varphi \in S_A$ for any $\varphi \in M_A$. A continuous homomorphism $\gamma : \mathbb{R}^+ \to M_A$ will be called a ray if $\gamma(\mathbb{R}^+) \subseteq S_A$. Note that $\gamma(0) \in J_A$ and $J_A \subseteq S_A$. Let $M$ be the family of all rays in $M_A$. For $j \in J_A$ set

$$M^j = \{\gamma \in M : \gamma(0) = j\},$$

$$S^j_A = \bigcup_{\gamma \in M^j} \gamma(\mathbb{R}^+).$$

The continuous homomorphism $\gamma : \mathbb{C}^+ \to M_A$ will be called the complex ray, if its restriction to $\mathbb{R}^+$ is a ray and the function $f(\gamma(z))$ is holomorphic in the open right half-plane for any $f \in A$.

Theorem 6.1. The following assertions hold:

(1) For any $s \in S_A$ there is the unique ray $\gamma$ such that $\gamma(1) = s$.

(2) Every ray $\gamma$ extends uniquely to a complex ray. The extension satisfies the equality $\gamma(z)^* = \gamma(\overline{z})$.

(3) Any complex ray has a limit

$$\lim_{\text{Re } z \to -\infty} \gamma(z) =: \gamma(\infty) \in J_A.$$ 

(4) The restriction of $\gamma$ to the boundary line $i\mathbb{R}$ in $\mathbb{C}^+$ is one parameter group in $G^j$, where $j = \gamma(0) \in J_A$.

Proof. The operator $R_s$ preserves the Peter–Weyl decomposition. Hence, $R_sA_{\text{fin}} = (R_sA)_{\text{fin}} = \sum_{\tau \in \mathbb{C}} M_\tau \cap R_sA$. Since these spaces are finite-dimensional and the operator $R_s$ is symmetric, the linear span of its eigenfunctions is dense in $A$. Moreover, we can assume that $H^2$ has an orthogonal basis $b$ that consists of eigenfunctions of $R_s$, and every function in $b$ is contained in $M_\tau$ for some $\tau \in \text{Sp } A$.

Let $A$ be the spectrum of $R_s$ and $A_\lambda$ be the corresponding to $\lambda \in \Lambda$ eigenspace in $A$. We claim that the spaces $A_\lambda$ satisfy the conditions of the Lemma 6.2 with $A_\text{fin}$ as $B$. The condition (46) holds because $R_s$ is symmetric. Any minimal bi-invariant subspace in $H^2$ is $R_s$-invariant and finite dimensional. This implies $A_\text{fin} = \sum_{\tau \in \mathbb{C}} A_\tau$ as well as (44). If $u \in A_\tau$, $v \in A_\eta$ then

$$R_s(uv) = (R_su)(R_sv) = \tau \eta uv,$$

i.e., $uv \in A_{\tau \eta}$. Thus (45) is true. This completes the proof of the claim.

By Lemma 6.1 we have $\|R_s\| \leq 1$ and, consequently, $A \subseteq [0, 1]$.

The operator $R_s$ admits the unique non-negative square root $R_s^{\frac{1}{2}}$, which has the same eigenfunctions. They correspond to the non-negative square roots of the eigenvalues of $R_s$. Therefore, the equalities (51) hold for $R_s^{\frac{1}{2}}$ with $\sqrt{\tau}, \sqrt{\eta}$ instead of $\tau, \eta$. Hence, $R_s^{\frac{1}{2}}$ is an endomorphism of $A_{\text{fin}}$. It is bounded as an operator in $H^2$. By Lemma 6.1, $R_s^{\frac{1}{2}}$ extends continuously to $A$. Obviously, $R_s^{\frac{1}{2}}$ commutes with left shifts. Thus, $R_s^{\frac{1}{2}} \in M_A$ and $s$ has the unique square root in $S_A$. Denote it by $\gamma(\frac{1}{2})$. Applying this procedure repeatedly, we get $\gamma(2^{-n})$ for all $n \in \mathbb{N}$. Together with their powers $\gamma(\frac{m}{2^n})$ we obtain a homomorphism of the binary rational numbers to $\mathbb{R}^+$. Due to compactness of $M_A$ and existence of the basis $b$, this

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well-known construction extends $\gamma(t)$ to $\mathbb{R}^+$ to a continuous mapping $\gamma : \mathbb{R}^+ \rightarrow S_A$, which is obviously unique. It is a homomorphism $\mathbb{R}^+ \rightarrow M_A$ with values in $S_A$. This proves (1).

Let $\gamma$ be a ray such that $\gamma(1) = s$. Then $A_s$ is the eigenspace of $R_{\gamma(t)}$ for any $t \geq 0$. The corresponding eigenvalue is $\lambda^t$ for all $t \geq 0$ and 0 if $\lambda = 0$. This easily extends to the case $t \in \mathbb{C}^+$. The extended map from $\mathbb{C}^+$ to $\mathbb{D}$ is a homomorphism. According to Lemma 6.2, $R_{\gamma(t)} \in M_A$ for any $t \in \mathbb{C}^+$. The uniqueness of holomorphic extension of the eigenvalues $R_{\gamma(t)}$ is obvious. For any $f \in A_{\text{fin}}$,

$$f(\gamma(z)) = \gamma(z)(f) = R_{\gamma(z)}f(e) = \sum_{\lambda \in \Lambda} \lambda^zf_\lambda(e),$$

where $f_\lambda \in A_\lambda$ and the sum is finite. Approximating $f \in A$ uniformly by elements of $A_{\text{fin}}$ to $G$, we obtain a sequence of bounded holomorphic functions on $C^+$, continuous up to the boundary $i\mathbb{R}$ in $C^+$, which converges uniformly to $f(\gamma(z))$ on $i\mathbb{R}$. Therefore, it converges uniformly on $C^+$ and $f(\gamma(z))$ is holomorphic inside it. The equality $\gamma(z)^* = \gamma(\overline{z})$ holds on $\mathbb{R}^+$ and admits a holomorphic extension onto $\mathbb{C}^+$, which is necessarily unique. This proves (2).

Let $\pi_\lambda$ be the orthogonal projection onto $A_\lambda$. Then $R_s = \sum_{\lambda \in \Lambda} \lambda\pi_\lambda$. One parameter semigroup $\gamma$ such that $\gamma(1) = s$ can be defined as

$$R_{\gamma(z)} = \sum_{\lambda \in \Lambda} \lambda^z\pi_\lambda,$$

where $\Re z > 0$. If $t = 0, \infty$ then $R_{\gamma(0)}, R_{\gamma(\infty)}$ are orthogonal projections onto the closure of $R_sA$ and $R_sA_1$ respectively. Hence, $\gamma(0) \neq \gamma(\infty)$ if $\gamma$ is non-trivial. For $\gamma(\infty)$, this means that

$$\lim_{\Re z \to \infty} \|R_{\gamma(z)}f - \pi_1f\|_{H^2} = 0$$

for any $f \in A$. Thus, $R_z \to R_{\gamma(\infty)}$ as $Re z \to \infty$ in $M_A$ and (3) follows.

Set $j = \gamma(0)$. For any $t \in \mathbb{R}$, we have $j\gamma(it) = \gamma(it)j = \gamma(it)$ and $\gamma(it)\gamma(-it) = j$. By Lemma 4.1, (vii), $\gamma(t) \in G^j$. This completes the proof of the theorem.

**Corollary 6.2.** If $A$ is a separating invariant algebra distinct from $C(G)$ then $M_A$ admits a non-trivial analytic structure.

**Proof.** By Theorem 5.1, $M_A \neq G$. Let $\varphi \in M_A \setminus G$. Then the complex ray passing through $\varphi^*\varphi$ is non-trivial.

We will often denote $\gamma(z)$ by $s^z$, bearing in mind that $s = \gamma(1)$. Thus, $s^0, s^\infty \in j_A$.

**Corollary 6.3.** For any $s \in S_A$ and $z \in \mathbb{C}^+$,

$$Z_G(s) \subseteq Z_G(s^z).$$

Moreover, if $\Re z > 0$ then $Z_G(s) = Z_G(s^z)$.

**Proof.** Let $s$ be a fixed point of the automorphism $\varphi \rightarrow \varphi^{-1}\varphi g$, where $g \in G$. Then $s^z$ is also its fixed point by Theorem 6.1, (2). This proves the inclusion (52). If $Re z > 0$ then the reverse inclusion is true due to the equality $\gamma(z)\gamma(z)^* = \gamma(2\Re z)$. This proves the second statement.

**Corollary 6.4.** If $\gamma \in M^j$ then $\gamma(\mathbb{C}^+) \subseteq \gamma(i\mathbb{R})$.

**Proof.** For every $f \in A$, the function $u(z) = f(\gamma(z))$ is holomorphic and bounded on $\mathbb{C}^+$. Therefore $|u(z)| \leq \sup_{t \in \mathbb{R}} |u(it)|$ if $z \in \mathbb{C}^+$.

### 6.3. The polar decomposition.

**Theorem 6.5.** For any $\varphi \in M_A$, there are $s \in S_A$ and $g \in G$ such that $\varphi = gs$. In this decomposition $s$ is uniquely defined, and $gs = hs$ for $h \in G$ if and only if $gG_j = hG_j$, where $j = s^0$. In particular, the decomposition is unique if $j = e$. 

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\textbf{Proof.} Let \( R_\varphi = US \) be the polar decomposition of the operator \( R_\varphi \) in the Hilbert space \( H^2 \). Then
\[ S^2 = R_\varphi^* R_\varphi = R_\varphi^* R_\varphi = R_{\varphi^* \varphi}. \]
Hence, \( S = R_s \) for \( s = (\varphi^* \varphi)^{\frac{1}{2}} \). This proves the uniqueness of the component \( s \in S_A \) in the polar decomposition, if it exists.

The operators \( R_\varphi \) and \( R_s \) preserve every finite-dimensional space \( M_\tau \), where \( \tau \) is an irreducible representation of \( G \). The operator \( R_s \) has the same kernel as the orthogonal projection \( R_j \), it maps \( H^2 \) to \( R_j H^2 \) and its image is dense in \( R_j H^2 \). The latter space is the direct orthogonal sum of the spaces \( M_\tau \cap R_j H^2 \). Let \( B \) be the algebraic sum of them. Then \( R_{s_1}^{-1} \) is well defined on \( B \) for any \( t > 0 \). The operator on the right in the equality
\[ R_\varphi R_{s_1}^{-1} = UR_{s_1-t} \]
is bounded on \( B \) in the norm \( H^2 \) if \( 0 < t < 1 \). The operator on the left is an algebraic endomorphism \( B \to B \). Lemma 6.1 implies that it is bounded in the sup-norm and its norm does not exceed 1. Computing the limit as \( t \to 1 \), we obtain the inclusion \( UR_j \subseteq M_A \). This reduces the problem to the case \( s = j \).

If \( s = j \) then \( j = j^2 = \varphi^* \varphi \) according to (iii). Choose \( \mu \in M_{\varphi^*} \), \( g \in \supp \mu \) and put \( \kappa = g^{-1} \varphi \), \( \nu = \delta_{g^{-1}} \ast \mu \in M_\kappa \). Then \( \kappa^* \kappa = \varphi^* \varphi = j \) and \( e \in \supp \nu \). Therefore,
\[ \supp(\nu^* \ast \nu) \subseteq G \]
where the first inclusion follows from (ii) and Lemma 1.1 since \( \nu^* \ast \nu \) is a representing measure for \( \kappa^* \kappa \). Hence, \( \supp \nu \subseteq G \). Together with (ii) this implies \( \kappa^* \kappa = \varphi \). The decomposition \( \varphi = g \) exists and \( s \) is uniquely defined by \( \varphi \).

Due to (ii) we can assume that \( \supp \mu = g \supp \sigma_j = gG \). The above arguments can be applied to any \( h \in gG \). Hence, \( \varphi = gj = hj \) if \( hG \). Conversely, if \( gj = hj \) for some \( g \in G \) then \( h^{-1} g \in G \).

The last assertion of the theorem is obvious. \hfill \( \square \)

If \( G \) is abelian then \( M_A = \text{Hom}(\text{Sp} \ A, \mathbb{D}) \) by Theorem 2.4 and the polar decomposition of \( \chi \in \text{Hom}(\text{Sp} \ A, \mathbb{D}) \) corresponds to the natural polar decomposition of functions: \( \chi(x) = \rho(x)|\chi(x)| \), where \( \rho \in \text{Hom}(\text{Sp} \ A, \mathbb{T}) \). This equality uniquely defines \( \rho(x) \) if \( \chi(x) \neq 0 \). In general, \( \rho \) must be extended to \( \text{Sp} \ A \). Existence of the extension follows from Theorem 6.5.

7. INVARIANT ALGEBRAS ON LIE GROUPS

In this section, \( G \) is a Lie group. We do not assume it connected and assume \( A \) separating unless the contrary is explicitly stated. We expand the notation of Section 4, adding to the family of groups related to an idempotent \( j \in J_A \) the identity components \( Z^j, Z_j \) of the centers of the groups \( G^j, G_j \), respectively. The Lie algebras of groups will be denoted by the corresponding lowercase Gothic letters with the same indices. For example, \( g_j, n_j, z^j \) are the Lie algebras of the groups \( G_j, N_j, Z^j \). By \( T \) we denote a maximal torus in \( G \). If \( G \) is connected then \( T \) is a maximal abelian subgroup of \( G \).

7.1. Invariant cones in the Lie algebras \( g^j \). Let \( \tau \in \text{Sp} \ A \) be a unitary representation of the group \( G \) in the space \( V_{\tau} \). It can be extended to a representation of \( M_A \) by formula (12). For any \( j \in J_A \), this defines a representation of \( G^j \) and its Lie algebra \( g^j \) in \( \tau(j)V_{\tau} \). We keep for the extension the same notation \( \tau \). The natural homomorphism \( N_j \to G^j = N_j/G_j \) is denoted by \( \Phi_j \) as well as the relating homomorphism \( n_j \to g^j \) of their Lie algebras. Recall that \( N_j, S^j_A \) in Theorem 7.1 are defined by (48) and (49).
**Theorem 7.1.** Let \( \gamma \in \mathfrak{R} \) and \( j = \gamma(0) \). There is the unique \( \xi \in \mathfrak{g}^j \) such that
\[
\gamma(it) = \exp(t \xi) = j \exp(t \xi) = \exp(t \xi)j,
\]
for any \( t \in \mathbb{R} \) and \( \xi \in \Phi_j^{-1} \xi \subseteq n_j \). The set
\[
C^j = \{ \xi \gamma : \gamma \in \mathfrak{R}^j \}
\]
is a closed convex pointed \( \text{Ad}(G^j) \)-invariant cone \( C^j \subset \mathfrak{g}^j \), which can also be defined as
\[
C^j = \{ \xi \in \mathfrak{g}^j : i \tau(\xi) \leq 0 \text{ for all } \tau \in \text{Sp} \, A \}.
\]
The relation between \( \xi \), and \( \gamma \) as well as between \( \xi \) and \( \gamma(1) \) are one-to-one on \( C^j \). They map \( C^j \) to \( \mathfrak{R}^j \) and to \( S_A^j \), respectively.

**Proof.** The first assertion is a consequence of the theorem 6.1, (4), since the mapping \( g \to jg \) on \( N_j \) is a homomorphism \( \Phi_j : N_j \to G^j \).

The cone \( C^j \) defined by (55) is obviously \( \text{Ad}(G^j) \)-invariant, convex, and closed. Thus we have to prove that the definitions (54) and (55) are equivalent. If \( \xi \in C^j \), \( \tau \in A \) then \( \exp(iz\tau(\xi)) \) is uniformly bounded in \( V_\tau \) over \( z \in \mathbb{C}^+ \). Therefore, the group \( R_{\exp(z \xi)} \), \( t \in \mathbb{R} \), of operators in \( R_j M_\tau \) admits the analytic extension onto \( \mathbb{C}^+ \). This is true for \( \tau \in \text{Sp} \, A \) and consequently for \( R_j A_{\text{fin}} \). The extension is bounded in the \( L^2(G) \) norm and, moreover, it is an endomorphism of the algebra \( A_j = R_j A_{\text{fin}} \) since \( R_{\exp(z \xi)} \in \text{Hom}(A_j, A_j) \) for \( z \in \mathbb{R} \) and the analytic extension agrees with the multiplication of functions. Thus \( R_{\exp(it \xi)} \) is well defined for \( t \geq 0 \). This is one parameter semigroup of endomorphisms of \( A_{\text{fin}} \), which commute with the left shifts and are symmetric in \( M_\tau, \tau \in \text{Sp} \, A \). Hence, it defines a ray \( \gamma \in \mathfrak{R}^j \), which can be written as \( \gamma(t) = \exp(it \xi)j \). Conversely,
\[
\frac{d}{dt} \tau(\gamma(t)) \bigg|_{t=0} = i \tau(\xi) \leq 0
\]
since \( \tau(\gamma(t)) \) is a bounded and symmetric semigroup in \( V_\tau \) for every \( \gamma \in \mathfrak{R}^j \) and any \( \tau \in \text{Sp} \, A \). Thus, (54) defines the same cone as (55).

If \( \xi, -\xi \in C^j \) then \( f(\exp(z \xi)j) \) is bounded on \( \mathbb{C} \) for every \( f \in A_{\text{fin}} \). Since this the function is continuous on \( \mathbb{C} \) and holomorphic on \( \mathbb{C} \setminus \mathbb{R} \), it is entire. By Liouville’s theorem \( f(\exp(z \xi)j) = f(j) \). This implies \( \exp(z \xi)j = j \) since \( A_{\text{fin}} \) separates the points \( M_A \). Hence, \( \xi = 0 \). Therefore, the cone \( C^j \) is pointed.

The remaining assertions follow from Theorem 6.1, (1). \( \square \)

**Proposition 7.1.** If \( C^j \neq 0 \) then the center of \( G^j \) is not discrete and there exists a nontrivial \( j \)-central ray \( \gamma \) lying in \( \hat{Z}^j \).

**Proof.** If \( \xi \in C^j \setminus \{0\} \) then
\[
\zeta = \int \text{Ad}(h)\xi \, d\sigma_j(h) \neq 0
\]
since \( C^j \) is pointed. Clearly, \( \zeta \) is a fixed point of the group \( \text{Ad}(G^j) \). Hence, \( \exp(t \zeta) \) lies in the center of \( G^j \) and \( R_{\exp(it \zeta)} R_j \) commutes with \( R_\varphi \) for all \( t \geq 0 \) and \( \varphi \in G^j \). Thus \( \gamma(i \mathbb{R}) \subseteq Z_{G^j}(G^j) \) and, moreover, \( \gamma(i \mathbb{R}) \subseteq Z^j \) because \( \gamma(i \mathbb{R}) \) is connected. By Corollary 6.4, \( \gamma(\mathbb{R}^+) \subseteq \widehat{\gamma(\mathbb{R}^+)} \). Thus \( \gamma(\mathbb{R}^+) \subseteq \hat{Z}^j \). By Lemma 1.1, every \( \varphi \in \hat{Z}^j \) has a representing measure with support in \( Z^j \). It follows from (25) that \( \varphi \) commutes with all points in \( G^j \). Hence, the ray \( \gamma \) is \( j \)-central. \( \square \)

**7.2. Complex Lie semigroups in \( M_A \).** Denote by \( G^c \) the complexification of the group \( G \). We can assume that \( G \) is a matrix group since every finite-dimensional representation of \( G \) can be uniquely extended to \( G^c \). Then \( G^c = G \exp(ig) \).
For $j \in \mathcal{J}$, let $\mathcal{V}$ be the real linear span of $C_j$ in $\mathcal{G}^j$. This is an ideal in $\mathcal{G}^j$ since $C_j$ is $\text{Ad}(\mathcal{G}^j)$-invariant. The latter also implies the equality

$$G^j \exp(i C_j) = \exp(i C_j) G^j.$$  \hfill (56)

Let $L^j$ and $L^{j,c}$ be the respective connected subgroups of $G^j$ and $G^{j,c}$ corresponding to $\mathcal{V}$ and $\mathcal{V}^c$, respectively. In the theorem below, they are provided with the topology defined by the smooth structure of a Lie group. Set

$$P^j = G^j \exp(i C_j).$$

By Theorem 7.1 we can consider $P^j$ as a subset of $\mathcal{M}_A$, namely,

$$P^j = G^j S^j_A.$$  \hfill (57)

Let $D^j$ be the interior of $L^{j,c} \cap P^j$ in $L^{j,c}$.

**Theorem 7.2.** (Multidimensional Analytic Structure). For any $j \in \mathcal{J}$, the set $P^j$ is a subsemigroup of $\mathcal{M}_A$ consisting of $g \in G^{j,c}$ such that

$$\|\tau(g)\|_{\text{BL}(V_j)} \leq 1$$  \hfill (58)

for all $\tau \in \text{Sp} A$. The set $D^j$ is an open semigroup in $L^{j,c}$ and the natural embedding $D^j \to \mathcal{M}_A$ defines an analytic structure in $\mathcal{M}_A$, which is non-trivial if $C_j \neq 0$.

**Proof.** Let $g = h \exp(i \xi) \in G^{j,c}$, where $h \in G^j$, $\xi \in \mathcal{G}^j$. Then $\tau(\exp(i \xi)) = \tau(g) \tau(h)^{-1}$. Hence, the inclusion $\tau(g) \in \tau(\mathcal{M}_A)$ is equivalent to $\tau(\exp(i \xi)) \in \tau(\mathcal{M}_A)$. Since $\tau(h)$ is unitary in $V_j$, (58) holds for $\exp(i \xi)$ and $g$ simultaneously. Further, (58) holds for $\exp(i \xi)$ if and only if

$$\|\tau(\exp(i t \xi))\|_{\text{BL}(V_j)} \leq 1$$

for all $t > 0$ and $\tau$ the norm of a symmetric non-negative operator is equal to its largest eigenvalue, and this is equivalent to the inequality in (55). By Theorem 7.1 $P^j$ is defined by the inequalities (58). Hence, $P^j$ is a semigroup in $G^{j,c}$. It is clear that $P^j \cap L^{j,c} = L^j \exp(i C_j)$ and this set has a non-empty interior $D^j$ in $(L^j)^c$, which is also a semigroup. The embedding $D^j \to \mathcal{M}_A$ defines an analytic structure in $\mathcal{M}_A$, since the mapping $g \to \tau(g)$ is holomorphic on $L^{j,c}$ for all $\tau \in \text{Sp} A$ and the uniform closure preserves this property. It is clear that $C_j \neq 0$ implies $\mathcal{V} \neq 0$ and $D^j \neq \emptyset$. This completes the proof. \hfill $\square$

7.3. Chains of rays.

**Lemma 7.1.** The set $P^e$ in (57) can be defined by the equality $(\varphi^* \varphi)^0 = e$, where $\varphi \in \mathcal{M}_A$. This is an open subsemigroup of $\mathcal{M}_A$ that contains $G$ and does not contain any idempotent distinct from $e$.

**Proof.** Since $A_{\text{fin}}$ is dense in $A$ and $A$ separates the points of $G$, there exists a finite-dimensional representation $\tau$ of the group $G$ such that $\text{Sp} \tau \subseteq \text{Sp} A$ and $\tau$ is exact on $G$. Formula (12) extends it to $\mathcal{M}_A$. It follows from (12), the lemma 4.1, (ii), and the choice of $\tau$ that $\tau(j) \neq 1$ for any $j \neq e$ in $\mathcal{J}$. Therefore, $\det \tau(j) = 0$. Put

$$U = \{ \varphi \in \mathcal{M}_A : \det \tau(\varphi) \neq 0 \}.$$ 

We claim that $U = P^e$ and $U$ satisfies the lemma. Indeed, $U$ is an open subsemigroup of $\mathcal{M}_A$ since $\varphi \to \det \tau(\varphi)$ is a continuous homomorphism $\mathcal{M}_A \to \mathbb{C}$. It contains $G$ because $\tau(g)$ is invertible for all $g \in G$. If $\det \tau(\varphi) \neq 0$ then $\det \tau(\varphi^* \varphi) \neq 0$, whence $\det (\varphi^* \varphi)^0 \neq 0$ and, consequently, $(\varphi^* \varphi)^0 = e$. Conversely, this equality implies $\det \tau(\varphi) \neq 0$. Hence, it defines $U$.

If $(\varphi^* \varphi)^0 = e$ then the component $s$ in the polar decomposition of $\varphi$ belongs to $S^e_A$ by definition of this set. This is equivalent to inclusion $\varphi \in G S^e_A = P^e$.

The remaining assertions of the lemma are either already proved or are evident. \hfill $\square$

**Proposition 7.2.** If $A \neq C(G)$ then $C^e \neq 0$. 

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Proof. Suppose $U \cap (\mathcal{M}_A \setminus G) = \emptyset$ for some neighborhood $U$ points $e$ in $\mathcal{M}_A$. Then $gU$ has the same property for all $g \in G$. Therefore $G$ is closed and open in $\mathcal{M}_A$ and, by Shilov's idempotent theorem (see [9, Corollary 6.5]), $A$ contains the characteristic function $\mathcal{M}_A \setminus G$, which is 0 on $G$ and 1 on $\mathcal{M}_A \setminus G$. It can't be true by definition of $A$. Thus, $U \cap (\mathcal{M}_A \setminus G) \neq \emptyset$ for any neighborhood $U$ of the point $e$.

If $U$ is small enough then $\varphi \in U$ implies $(\varphi^* \varphi)^0 = e$ by Lemma 7.1. Let $\varphi = gs$ be the polar decomposition of $\varphi$. If $\varphi \notin G$ then $s \neq e$, because we have $\varphi = ge \in G$ otherwise. The ray $\gamma$ such that $\gamma(1) = s$ is non-trivial and $\gamma(0) = s^0 = e$. Hence, $C^e \neq 0$ by Theorem 7.1.

**Corollary 7.3.** If $A \neq C(G)$ then there exists $\zeta \in \text{Int}(C^e)$ such that the corresponding one parameter group in $G$ is a subgroup of $Z(G)$ and $\exp(i\zeta t), t \geq 0$, is a nontrivial central ray in $\mathcal{M}_A$.

**Proof.** Proposition 7.2 and Proposition 7.1 together imply this assertion.

**Lemma 7.2.** If $j, k \in j_A, j < k$, and $j \neq k$ then $\dim G_j > \dim G_k$.

**Proof.** If $\dim G_j = \dim G_k$ then the homogeneous space $G_j/G_k$ is finite and the algebra $R_k A|G_j$ on it is separating by Proposition 3.2 and Lemma 4.1, (v). Every separating uniform algebra on a finite set is obviously the algebra of all functions on it. Hence, the measure $\sigma_j$ cannot be multiplicative on $A$ contradictory to Lemma 4.1, (ii).

**Lemma 7.3.** Let $j \in j_A$. If $j \neq e$ then $Z(G^j)$ contains a non-trivial ray starting at $j$. Moreover, if $j \in Z(\mathcal{M}_A)$ then such a ray exists in $Z(\mathcal{M}_A)$.

**Proof.** We first prove the lemma assuming that $j \in Z(\mathcal{M}_A)$. Then $N_j = G$. Hence, the algebra $A_j = R_j A$ can be considered as the algebra of all functions from $A$ that are constant on the cosets of $G_j$. On the other hand, it can be treated as the restriction of $A$ to $G^j = G_j \cong G/G_j$. Thus $\mathcal{M}_A \cong \hat{G}_j$ by Proposition 3.1. The assumption $j \neq e$ and Lemma 4.1, (ii) together imply that the restriction of $A_j$ to the group $G_j$ is non-trivial. Since the measure $\sigma_e$ is multiplicative on $A_j$, we have $A_j \neq C(G^j)$. By Corollary 7.3, $\hat{G}_j$ contains a non-trivial ray $\gamma$ that is central in this semigroup. Since $G^j = G_j$, $j \in Z(\mathcal{M}_A)$, $\gamma = \gamma_j$ and $j\varphi = \varphi_j \in j\mathcal{M}_A j$ for all $\varphi \in \mathcal{M}_A$, the ray $\gamma$ is also central in $\mathcal{M}_A$. Thus, the lemma is true for central $j$.

Let us prove the lemma in the general case. Set

$$k = \inf\{k' \in j_A \cap Z(\mathcal{M}_A) : j \leq k'\}.$$ 

The set on the right is non-empty because it contains $e$. We can assume $j \notin Z(\mathcal{M}_A)$. Then $j \neq k$ and $jk = jk = j$. Let $\gamma$ be a central ray starting at $k$ and $\bar{k} = \gamma(\infty)$. It is clear that $\bar{k} \in Z(\mathcal{M}_A)$ and $\bar{k} < k$. The latter implies that $\bar{k}$ is not an upper bound for $j$ and therefore $\bar{k}j = j\bar{k} \neq j$. Thus, $j\gamma(t)$ is a non-trivial ray starting at $j$.

**Corollary 7.4.** The equality $C^j = 0$ holds if and only if $j = e$.

We say that the finite sequence $\gamma_1, \ldots, \gamma_n$ is a chain of rays if all rays are non-trivial and

$$\gamma_l(\infty) = \gamma_{l+1}(0), \quad l = 1, \ldots, n - 1.$$ 

The number $n$ is called the length of the chain. Also, we say that the chain joins $\gamma_1(0)$ and $\gamma_n(\infty)$.

**Theorem 7.5.** For any $j, k \in j_A$ such that $j \leq k$ and $j \neq k$, there exists a chain of rays, which joins them. Its length does not exceed $\dim G_j - \dim G_k$.

**Proof.** It is sufficient to prove the theorem for the restriction of $A$ to $G_j$. Obviously, it is a separating invariant algebra on $G_j$. It is closed in $C(G_j)$ according to [9, Lemma 12.3 and Theorem 12.7] since $G_j$ is the peak set for $A$ due to Lemma 4.1, (i). Thus we can assume $j = e$. Then the assertion is almost obvious: By Lemma 7.3, there is a chain of rays starting at $k$, and it can be continued if its endpoint is not the smallest idempotent.

The chain length estimate follows from Lemma 7.2.

7.4. Representing measures with supports in abelian subgroups. Let $\omega_j$ denote the Haar measure of the group $Z_j$.

**Proposition 7.3.** For any $j \in j_A$, we have $\omega_j \in \mathcal{M}_j$.
If any of these assertions hold then that \( \tau \). Therefore, there exists \( \bar{\tau} \) antisymmetry for \( \epsilon \). Corollary 7.3, on some step we get \( \delta \). Hence, we can assume \( s \) word for word, we obtain the inclusion \( \omega \). According to Theorem 6.1, \( (4) \), assuming \( M \) to the algebra \( A \) is connected and proves the proposition.

Thus \( s \) has a representing measure with support on the closure \( H \) of the set \( Z_j \exp(\mathbb{R}\xi) \). Since the set \( Z_j \) is obviously \( \text{Aut}_0(N_j) \)-invariant and \( \xi \in n_j \), \( H \) is a group and the subgroup \( Z_j \) is normal in it. Since \( Z_j \) is connected and abelian, \( H \) is connected and solvable. This implies that \( H \) is abelian because it is compact and proves the proposition.

Proposition 7.4 makes it possible to prove a stronger version of Theorem 5.2 for connected Lie groups.

**Theorem 7.6.** Let \( G \) be connected, \( A \) be an invariant algebra on \( G \), \( T \) be a maximal torus in \( G \), and \( \tau \) be the Haar measure of \( T \). The following assertions are equivalent:

(a) \( A \) is antisymmetric,

(b) the closure in \( C(T) \) of the restriction \( A|_T \) is antisymmetric,

(c) \( \tau \) is multiplicative on \( A \).

If any of these assertions hold then \( \tau \in \mathcal{M}_\epsilon \), where \( \epsilon \) is the zero of \( \mathcal{M}_A \).
Proof. Let \(A\) be antisymmetric. Then \(G = G_e\) by Theorem 5.2 and \(e\) is the zero of \(\mathcal{M}_A\). The group \(Z_e\) is contained in every maximal torus in \(G\) because \(Z_e \subseteq Z(G)\). According to Proposition 7.3, \(\omega_\epsilon \in \mathcal{M}_e\). Hence, \(\delta_h + \omega_\epsilon \in \mathcal{M}_e\) for all \(h \in T\). The convex weakly closed hull of the family of these measures contains \(\tau\). Hence, \(\tau \in \mathcal{M}_e\). Thus, (a) implies (c) as well as the last assertion of the theorem.

If \(A\) is not antisymmetric then it contains a non-constant real-valued function \(f\). Then \(f(g) \neq f(e)\) for some \(g \neq e\). It is well known that every orbit Aut_0(G) has a common point with any maximal torus \(G\) if \(G\) is connected. Hence, we may assume that \(g \in T\). Then \(f \in A\) is a real function that is non-constant on \(T\). Thus (a) follows from (b).

It remains to note that (b) and (c) are equivalent since Theorem 5.2 can be applied to \(T\). \(\square\)

7.5. The Cartan subsemigroup of \(\mathcal{M}_A\). In the case of Lie groups, the ordered set \(\mathcal{J}_A\) has some special properties.

**Proposition 7.5.** Every \(j \in \mathcal{J}_A\) has a neighborhood \(U\) in \(\mathcal{M}_A\) such that \(j \leq k\) for any \(k \in \mathcal{J}_A \cap U\) that commutes with \(j\).

**Proof.** Due to Lemma 7.1 we can assume \(j \neq e\). Let \(\rho\) be a finite-dimensional representation of \(G\) such that \(\rho\) is exact on \(G^3\) and \(\text{Sp} \rho \subseteq \text{Sp} A\). Set

\[
W = \{ \varphi \in \mathcal{M}_A : \text{rank} \rho(\varphi) \geq \text{rank} \rho(j) \}.
\]

Then the interior of \(W\) contains \(j\). For any nontrivial ray \(\gamma \in \mathcal{W}_j\) one parameter semigroup \(\rho(\gamma(t))\) is non-trivial by Theorem 7.1. Since the ray consists of symmetric matrices with eigenvalues in \([0,1]\), due its non-triviality we have \(\text{rank} \rho(\gamma(\infty)) < \text{rank} \rho(j)\). Hence, \(\gamma(\infty) \notin W\). Together with Theorem 7.5 this implies that \(W\) does not contain \(k \in \mathcal{J}_A\) if \(k \leq j\) and \(k \neq j\). Let \(k \in \mathcal{J}_A \cap Z(j)\). Then \(kj = jk \in \mathcal{J}_A\) and \(kj \leq j\). Hence, \(kj \neq j\) implies \(kj \notin W\). This implies that \(kj = j\) if \(kj \in W\). Thus, the proposition holds for any neighborhood \(U \subseteq W\) of the idempotent \(j\) such that \(U^2 \subseteq W\). \(\square\)

**Lemma 7.4.** Let \(j \in \mathcal{J}_A\) and \(T\) be a maximal torus in \(G\). Then

\[
j \in \hat{T} \iff j \in Z(T) \iff T \subseteq N_j \iff Z_j \subseteq T.
\]

If any of the above inclusions holds then

\[
\{ g \in G : gj \in Z(T) \} \subseteq N_j
\]

and \(Tj\) is a maximal torus in \(G^j\).

**Proof.** If \(j \in \hat{T}\) then by Lemma 1.1 there exists a representing measure for \(j\) with support in \(T\). This implies \(j \in Z(T)\) because \(T\) is abelian.

It follows from \(j \in Z(T)\) that \(hj = jh\) for all \(h \in T\). Thus \(T \subseteq N_j\) by definition of \(N_j\).

Let \(T \subseteq N_j\). Then \(T\) is a maximal torus in \(N_j\). Being the identity component of the center of \(G_j\), the group \(Z_j\) is Aut_0(N_j)-invariant. Hence, \(Z_jT = TZ_j\) and \(Z_jT\) is a group. Let \(T_j\) be the identity component of \(T \cap G_j\). Then \(Z_jT_j\) is a normal abelian subgroup in \(Z_jT\). The groups \(Z_jT_j\) and \((Z_jT)/(Z_jT_j)\) are abelian, compact, and connected. Hence, \(Z_jT\) is solvable, compact and connected. It follows that \(Z_jT\) is abelian. Since \(T\) is a maximal abelian connected subgroup of \(N_j\), we have \(Z_jT = T\), whence \(Z_j \subseteq T\).

Due to Proposition 7.3 we have \(\omega_j \in \mathcal{M}_j\). The inclusion \(Z_j \subseteq T\) means that \(j\) has a representing measure with support in \(T\) since \(\text{supp} \omega_j = Z_j\). Thus \(j \in \hat{T}\) by Lemma 1.1. This completes the proof of all implications in (60).

Let \(g \in G\) and \(gj \in Z(T)\). Then \(h gj = gjh\) for all \(h \in T\). If \(Z_j \subseteq T\) then, for any \(f \in A\),

\[
f(gj) = \int f(h gj) d\omega_j(h) = \int f(gjh) d\omega_j(h) = f(gj)
\]

since \(\omega_j \in \mathcal{M}_j\) by Proposition 7.3. This implies \(gj = gj, g^{-1}gj = j\) and, due to (ii), \(\delta_{g^{-1}} * \sigma_j * \delta_g * \sigma_j \in \mathcal{M}_j\). According to (i) and Lemma 1.1, the support of the measure on the left is contained in \(G_j\). Hence, \(g^{-1}G_jg \subseteq G_j\) and \(g \in N_j\). Thus, (61) is true.
It remains to prove that $T_j$ is a maximal torus in $G^j$. By (60), the inclusions $Z_j \subseteq T$ and $T \subseteq N_j$ are equivalent. The mapping $g \to gj$ is the epimorphism $N_j \to G^j$ whose kernel is equal to $G_j$. Further, $T_j$ is the identity component of $T \cap G_j$ by definition. Since $T$ is compact and connected, it is sufficient to prove that $t/t_j$ is a maximal abelian subalgebra of $g^j$.

Since $T \subseteq N_j$, the spaces $t + g_j$ and $n_j$ are evidently $\text{Ad}(T)$-invariant. Let $n_j = (t + g_j) \oplus l$, where $l$ is $\text{Ad}(T)$-invariant. Then $\text{ad}(t_j)l = 0$ since $t_j \subseteq g_j$ and $g_j$ is an ideal of $n_j$. On the other hand, $\text{ad}(x)$ is non-degenerate on $l$ for a generic $x \in t_k$ because $t_k$ is a maximal abelian subalgebra of $n_j$. Thus $\text{ad}(x)$ is well defined and non-degenerate on $l$ for a generic $x \in t/t_j$. This completes the proof of the lemma.

**Proposition 7.6.** Let $T$ be a maximal torus in $G$. Then

$$S_A \cap Z(T) = S_A \cap \hat{T}.$$  

**Proof.** Note first that $Z(T) = Z(\hat{T})$. Indeed, the inclusion $T \subseteq \hat{T}$ implies $Z(\hat{T}) \subseteq Z(T)$. Let $\varphi \in Z(T)$. Then $\varphi h = h\varphi$ for all $h \in T$. If $\psi \in \hat{T}$ then it admits a representing measure $\mu$ concentrated in $T$. Thus

$$f(\varphi \psi) = \int f(h\varphi)\,d\mu(h) = \int f(\varphi h)\,d\mu(h) = f(\varphi \psi)$$

for all $f \in A$. Hence, $\varphi \psi = \psi \varphi$ for any $\psi \in \hat{T}$ and $\varphi \in Z(\hat{T})$.

Let $s \in S_A \cap Z(T)$. Then $s^k \in Z(T)$ for all $k \in \mathbb{Z}^+$ due to Corollary 6.3. In particular, $j = s^0 \in Z(T)$.

By Lemma 7.4 this implies $j \in \hat{T}$. The one parameter group $\gamma(it) = s^it, t \in \mathbb{R}$, also lies in $Z(T)$. Since $\gamma(it)$ commutes with $j = \gamma(0)$, we have $\gamma(it) \in G^j$. The set $\gamma(it)$ is connected and contains $j$. Therefore, in this inclusion we can replace the group $G^j$ with its identity component $G^j_1$. If $j \in \hat{T}$ then $T_j \subseteq \hat{T}$ and, consequently, $Z(\hat{T}) \subseteq Z(T_j)$. Hence,

$$\gamma(it) \subseteq G^j_1 \cap Z(T) \subseteq G^j_1 \cap Z(T) = T_j,$$

where the equality holds because $T_j$ is a maximal torus in $G^j$ by Lemma 7.4, hence, it is a maximal abelian subgroup of $G^j_1$ since $G^j_1$ is connected. According to Corollary 6.4, $s \in \gamma(\mathbb{R})$. This proves the inclusion $S_A \cap Z(T) \subseteq S_A \cap \hat{T}$. The inverse one is obvious.

The following theorem reduces some problems concerning invariant algebras to the case of abelian group $G$. We say that the semigroup $\hat{T}$, where $T$ is a maximal torus in $G$, is a Cartan subsemigroup of $M_A$.

**Theorem 7.7.** Let $T$ be a maximal torus in $G$.

(a) For every $s \in S_A$, there exists $g \in G$ such that $g^{-1}sg \in \hat{T}$.

(b) For every $\varphi \in M_A$ there are $g, h \in G$ and $a \in \hat{T}$ such that $\varphi = gah$. In other words, $M_A = G\hat{T}G$.

(c) If $G$ is connected then $P^e \cap Z(T) = P^e \cap \hat{T}$, where $P^e = G^e_{\gamma\in\mathbb{A}}$.

It is worth noting that $P^e$ is an open subsemigroup of $M_A$, which includes $G$ and contains no idempotent other than $e$ by Lemma 7.1.

**Proof.** By Proposition 7.4 there exists a connected abelian subgroup $H$ of $G$ such that $s \in \hat{H}$. According to Lemma 1.1, $s$ has a representing measure on $H$. The group $H$ is contained in some maximal torus and, consequently, it is conjugate to a subgroup of $T$ by an inner automorphism of the group $G$. This proves (a).

Let $\varphi = us$ be the polar decomposition of $\varphi$ and $s = h^{-1}ah, h \in G$ and $a \in \hat{T}$. Then $\varphi = gah$ for $g = uh^{-1}$. Thus (a) implies (b).

If $G$ is connected then $T$ is a maximal abelian subgroup of $G$. It follows that $Z(T) \cap G = T$. Let $\varphi \in P^e \cap Z(T)$ and $\varphi = gs$ be the polar decomposition of $\varphi$. By definition of $P^e$ we have $s^0 = e$. Thus $\ker R_s = 0$ and the component $g \in G$ of this decomposition is uniquely determined. If $\varphi \in Z(T)$ then $\varphi^* \in Z(T)$ since $T^* = T$ and $s = (\varphi^*\varphi)^{1/2} \in Z(T)$ by Corollary 6.3. This implies $g \in Z(T)$ because $R_{\varphi^*}^{-1}$ is well defined on $A_{\text{fin}}$. Thus, $g, s \in Z(T)$. By Proposition 7.6, $s \in \hat{T}$. The equality $Z(T) \cap G = T$ implies $g \in T$. Hence, $\varphi = gs \in \hat{T}$. This proves (c).
In general, $\hat{T} \neq Z(T)$ and $\hat{T}$ is not a maximal abelian subsemigroup in $M_A$. This can be true if $G$ is abelian and disconnected. Here is a simple example.

**Example 7.** Let $A$ be the algebra of pairs of functions $f_1, f_2$ from $A(\mathbb{D})$ such that $f_1(0) = f_2(0)$, where $A(\mathbb{D})$ is the algebra of all analytic functions in the disc $\mathbb{D}$ that are continuous up to to its boundary $T$. We can consider it as an invariant algebra on the group $G = \mathbb{T} \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm 1\}$. The space $M_A$ is the union of a couple of discs glued at the centers. The maximal torus in $G$ is the group $\mathbb{T} \times \{1\}$, the maximal abelian subgroup is $G$. The characters of $G$ has the form $\chi_{n,m}(\lambda, \varepsilon) = \varepsilon^m \lambda^n$, where $|\lambda| = 1, \varepsilon = \pm 1, n \in \mathbb{Z}, m = 0, 1$. The corresponding to $A$ semigroup can be written as

$$S = \{(n, 0) : n \in \mathbb{Z}, n \geq 0\} \cup \{(n, 1) : n \in \mathbb{Z}, n > 0\}.$$  

The character $\chi_{0,1}$ is orthogonal to $A$.

8. INVARIANT ALGEBRAS ON TORI

Theorem 2.4 identifies $M_A$ and $\text{Hom}(\text{Sp} A, \mathbb{D})$ for an abelian $G$. We assume $G$ connected. Then $G = \mathbb{T}^n, \hat{G} = \mathbb{Z}^n$, $\text{Sp} A$ is a semigroup in $\mathbb{Z}^n$, and the geometry of $\text{Sp} A$ defines the algebraic structure of $\text{Hom}(\text{Sp} A, \mathbb{D})$. We use the notation of Subsection 1, in particular, $S^*$ is a dual to $S$ cone. Everywhere in this section, $S$ stands for $\text{Sp} A$. It can be an arbitrary semigroup in $\mathbb{Z}^n$. The corresponding invariant algebra is separating if and only if $S - S = \mathbb{Z}^n$.

8.1. One parameter semigroups and idempotents. Note that $\chi^* = \chi$ for any $\chi \in S$ since $\hat{G} = \text{Hom}(G, \mathbb{T})$. Thus the antilinear involution $^*$ on finite weighted sums of characters reduces to the complex conjugation of coefficients:

$$(c_1\chi_1 + \cdots + c_n\chi_n)^* = \overline{c}_1\chi_1 + \cdots + \overline{c}_n\chi_n.$$  

Since the semigroup $M_A$ is abelian, the set $S_A$ of symmetric elements of $M_A$ is also a semigroup. According to Theorem 2.4, it can be identified with $\text{Hom}(S, \mathbb{I})$, where $\mathbb{I} = [0, 1]$ with the usual multiplication. Thus, each idempotent in $\text{Hom}(S, \mathbb{I})$ corresponds to the characteristic function $\kappa_X$ of some set $X \subseteq S$:

$$\kappa_X(x) = \begin{cases} 1, & x \in X, \\ 0, & x \notin X. \end{cases}$$  

In what follows, 1 denotes the function $\kappa_S$ and $I_S$ stands for the set of all $P \subseteq S$ that relate to some idempotent of $M_A$.

**Lemma 8.1.** Let $\gamma$ be one parameter semigroup in $\text{Hom}(S, \mathbb{I})$, $\gamma(0) = 1$. Then $\gamma(t)(x) = e^{-t\lambda(x)}$, where $\lambda \in \alpha(S)^*$, $t \geq 0, x \in S$ and, moreover,

$$\lim_{t \to \infty} \gamma(t)(x) = \kappa_P(x), \quad (63)$$  

where $P = S \cap F$ for some $F \in \mathfrak{I}_{\alpha(S)}$.

**Proof.** We can assume without loss of generality that $S$ linearly generates $\mathbb{R}^n$. Let $x \in S$. The assumption $\gamma(0)(x) = 1$ implies $\gamma(t)(x) > 0$ for all $t \geq 0$ due to continuity and the semigroup property of $\gamma$. Therefore, the real valued function

$$\lambda_t(x) = -\log \gamma(t)(x)$$  

is defined correctly. It is clear that $\lambda_t$ is additive and non-negative on $S$. Thus, it extends to an additive functional on the cocompact discrete group $S - S$ and, consequently, to a linear functional on $\mathbb{R}^n$. Moreover, $\lambda_t = t\lambda_1$, because $\lambda_0(x) = 0$ and $\lambda_t(x)$ is an additive and continuous function of $t$ for any $x \in S$. Obviously, $\lambda_t \in \alpha(S)^* = S^*$, where $S^*$ is the cone dual to $S$ (cf. Section 1). Hence, $\gamma(t) = e^{-t\lambda}$, where $\lambda = \lambda_1 \in S^*$. This implies (63) with $P = S \cap F$, where $F = \alpha(S) \cap \lambda^{-1}(0)$. It remains to note that $F \in \mathfrak{I}_{\alpha(S)}$ because $\lambda \in \alpha(S)^*$.
Corollary 8.1. The set $S^e_A = \exp(iC^e)$ is open in $S_A$ and, moreover, for any $x \in \text{Int} \, \alpha(S)$,
\[ S^e_A = \{ \chi \in \text{Hom}(S, \mathbb{I}) : \chi(x) \neq 0 \} \].

Proof. Let $\chi \in \text{Hom}(S, \mathbb{I}) = S_A$. If $\chi \not\in S^e_A$ then $\chi \kappa_P = \chi$ for some $P \neq S$. By Lemma 8.1 and Theorem 7.5, this implies $\chi = 0$ on $\text{Int}(\alpha(S)) \cap S$.

We omit the proof of the following proposition because it is easy.

Proposition 8.1. For the set $P \subseteq S$, the inclusion $P \in \mathcal{F}$ is equivalent to every of the following conditions:

1. $P$ is a semigroup, and $S \setminus P$ is an ideal of the semigroup $S$,
2. $\kappa_P \in \text{Hom}(S, \mathbb{I})$.

Moreover, if $P \in \mathcal{F}$ and $P \neq S$ then $\alpha(P)$ is a closed cone, which is contained in some face $F \in \mathcal{F}_\alpha(S)$, $F \neq \alpha(S)$.

8.2. The case of a finitely generated semigroup $S$. The semigroup $S$ uniquely determines $\mathcal{F}_\alpha(S)$, but the converse is not true. For example, the semigroup in $\mathbb{Z}$ generated by 0, 2, and 3 corresponds to the same maximal ideal space as for the semigroup of all non-negative integers, namely, to the disc $\overline{D}$.

We describe $\mathcal{F}_\alpha(S)$ in the simplest case of a finitely generated semigroup $S$. In general, the structure of $\mathcal{F}_\alpha(S)$ is more complicated. In the next subsection we give several illustrative examples. Further, we use the notation of Theorem 7.2.

Recall that an invariant algebra $A$ is finitely generated if there is its finite dimensional invariant subspace that generates $A$ as a Banach algebra. For abelian $G$, this is equivalent to the assumption that $S$ is finitely generated.

Theorem 8.2. Let the invariant algebra $A$ be finitely generated. Then

(a) the correspondence between $\mathcal{F}$ and $\mathcal{F}_\alpha(S)$ defined by the relation
\[ P = F \cap \alpha(S), \]
where $P \in \mathcal{F}$ and $F \in \mathcal{F}_\alpha(S)$, is a bijection,

(b) $S^e_A$ is dense and open in $S_A$, and the set $P^{e} = GS^e_A$ is dense and open in $\mathcal{F}_\alpha(S)$,

(c) if $A$ is antisymmetric and separating then $G = L^e$, and $\mathcal{F}_\alpha(S)$ is a compactification of $D^e \subseteq C^e$.

Proof. Let $C = \alpha(S)$ and let $X = \{x_1, \ldots, x_m\}$ generate $S$. Then
\[ S = \mathbb{Z}^+ x_1 + \cdots + \mathbb{Z}^+ x_m, \]
\[ C = \mathbb{R}^+ x_1 + \cdots + \mathbb{R}^+ x_m, \] (64)
where $\mathbb{Z}^+ = \{0\} \cup \mathbb{N}$ and $\mathbb{R}^+ = [0, \infty)$. The first equality is obvious. The cone on the right in (64) contains $\alpha(S)$ since it contains $S$. The reverse inclusion is true because $\mathbb{R}^+ x \subseteq \alpha(S)$ for any $x \in X$. Thus (64) is true.

Every closed face $F$ of a convex closed cone $C$ has the form $F = \lambda^+ \cap C$ for some $\lambda \in C^*$ (more precisely, for $\lambda \in \text{Int}(F^\perp \cap C^*)$). Put $X_\lambda = \lambda^+ \cap X$ and $X^\perp_\lambda = X \setminus X_\lambda$. Then
\[ X^\perp_\lambda = \{ x \in X : \lambda(x) > 0 \}, \] (65)
\[ F = \sum_{x \in X_\lambda} \mathbb{R}^+ x. \] (66)
Due to (65) and (66) each $x \in C$ admits a decomposition $x = x_0 + x^+$, where $x_0 \in F$ and $x^+ = \sum_{x \in X^\perp_\lambda} t_x x_\lambda$, where $t_x \geq 0$. Hence, $F = \alpha(P)$ for $P = F \cap S$ and
\[ \mathcal{F}_F \subseteq \mathcal{F}_C. \] (67)
Thus the mapping $P \rightarrow \alpha(P)$ is the inverse of $F \rightarrow F \cap S$ and (a) is true.

It follows from (65) and (66) that
\[ \lim_{t \rightarrow \infty} e^{-t\lambda(x)} = \kappa_F(x) \]

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for all $x \in C$. Therefore, any closed face $C$ is associated with the idempotent $j \in \mathcal{J}_a$, which is the limit of some one parameter semigroup starting at $e$. According to (a), these idempotents exhaust the family $\mathcal{J}_A$.

Moreover, this implies the equality $R^j = j R^e$ since any non-negative linear functional $\lambda$ defined on $F$ extends to some $\lambda \in \mathcal{C}^*$. Assuming $\lambda$ to be positive on $F \setminus \{0\}$ and using the expansion $x = x_0 + x^+$, we get the equality $\tilde{\gamma} = j \gamma$, where $\tilde{\gamma}(t) = e^{-t \lambda}$ on $F$ and $\gamma(t) = e^{-t \lambda}$. Thus, it follows from Corollary 8.1, (67), Theorem 6.1, and Theorem 7.5 that $S^e_A$ is open and dense in $S_A$. By Theorem 6.5, the set $P^e = G S^e_A$ is dense and open in $\mathcal{M}_A$ (in fact, this is the set $U$ of Lemma 7.1). This proves (b).

Let $A$ be antisymmetric. Then the cone $C$ is pointed according to (43) and (64). Hence, the dual cone $C^* = C^e \subset g$ has non-empty interior in $g$ and $I = g$. Since $G$ is connected and $A$ is separating, we have $G = L^e$ and $D^e$ is the interior of $P^e$ in the group $G^2$. The polar decomposition on the set $P^e$ is bijection. Hence, it is locally homeomorphic. Thus $D^e$ is dense in $P^e$ and $P^e$ is dense in $\mathcal{M}_A$. This implies (c) and completes the proof of the theorem.

**Corollary 8.3.** If $S$ is finitely generated then the set $\mathcal{J}_A$ is finite.

8.3. **Examples.** It is easy to prove that an invariant algebra $A$ is finitely generated if and only if there exist a finite-dimensional Hilbert space $V$ and isomorphic and continuous embedding of $G$ to $U(V)$ such that $A_{fin} = \mathcal{P}(V)_{G}$. Then $\mathcal{M}_A$ coincides with the polynomially convex hull $\hat{G}$ of the group $G$ in $U(V)$. Recall that the polynomial hull of $Q \subseteq \mathcal{C}^n$ is defined as

$\hat{Q} = \{ z \in \mathbb{C}^n : |p(z)| \leq \max_{w \in Q} |p(w)| \text{ for all } p \in \mathcal{P}(V) \}$.

The example below illustrates some typical effects that occur in this setting.

**Example 8.** Put

$S = \{(p, q) \in \mathbb{Z}^2 : p, q \geq 0 \text{ and } p \text{ is even if } q = 0 \}$.

This semigroup is generated by $x_1 = (2, 0)$, $x_2 = (0, 1)$, $x_3 = (1, 1)$. The generators satisfy the relation $x_1 + x_2 = 2x_2$. The algebra $A$ is isomorphic to the uniform closure on $\mathbb{T}^2$ of the linear span of all holomorphic monomials $z_1^p z_2^q$ in $\mathbb{C}^2$ except for odd powers of $z_1$.

By Theorem 8.2 $\mathcal{J}_A$ consists of 4 idempotents corresponding to 4 faces of the asymptotic cone

$\alpha(S) = \mathbb{R}^+ \oplus \mathbb{R}^+$.

Denote by $e, j, k, e$ the idempotents corresponding to the faces $\alpha(S)$, $\mathbb{R}^+x_1$, $\mathbb{R}^+x_2$ and $\{0\}$ respectively. The first is the unit of $\mathcal{M}_A$, the last is its zero.

The element $(\lambda, \mu) \in \mathbb{T}^2$ of the group $G = \mathbb{T}^2$ corresponds to the function $\chi^p \mu^q$ on $\mathbb{Z}^2$. Here $p, q \in \mathbb{Z}$. The above idempotents correspond to functions $\kappa_P$, where $P = S, (2\mathbb{Z}^+, 0), (0, \mathbb{Z}^+), (0, 0)$, respectively. Thus, the equality $gk = k$, which defines the group $G_k$, holds if and only if $\mu = 1$, and can be identified with $\mathbb{T} \times \{1\} \subset \mathbb{T}^2$. Equality $gj = j$ is true if and only if $\lambda^2 = 1$, i.e. $\lambda = \pm 1$. Hence, $G_j = \{\pm 1\} \times \mathbb{T}$ and $G_j$ is not connected.

The space $\mathcal{M}_A$ can be realized as a bidisc $\mathbb{D}^2$ with identified points $(\pm z, 0)$. Setting $z_k = \chi(x_k)$, where $\chi \in \text{Hom}(S, \mathbb{D})$ and $k = 1, 2, 3$, we get an embedding of $\mathcal{M}_A$ to the variety $z_1 z_2^2 = z_3^2$ in $\mathbb{C}^3$. Its image is equal to the image of the mapping $(\zeta_1, \zeta_2) \mapsto (\zeta_1^2, \zeta_2, \zeta_1 \zeta_2)$ from $\mathbb{D}^2$ to $\mathbb{C}^3$. It can also be defined by the inequalities $|z_1|, |z_2| \leq 1$ and the equality $z_1 z_2^2 = z_3^2$.

The following three examples illustrate some properties of infinitely generated algebras, which a finitely generated algebra cannot have. The first of them shows that $S_A^e$ need not be dense in $S_A$. This is a modification of the example in [3, §5].

**Example 9.** We use the notation of Subsection 7. Put

$S = \{(p, q, r) \in \mathbb{Z}^3 : r > 0 \text{ or } r = 0 \text{ and } p, q \geq 0 \}$.

The algebra $A$ is antisymmetric according to (43). Let $e_1, e_2, e_3$ be the standard basis in $\mathbb{R}^3$. There are five idempotents in $\mathcal{M}_A$. Two of them are trivial: $e = \kappa_S$ and $e = \kappa_{(0)}$. The remaining $l, j, k$ are the same as in the previous example. By Proposition 8.1, they refer to the sets $\mathbb{Z}^+e_1 + \mathbb{Z}^+e_2, \mathbb{Z}^+e_1$, and $\mathbb{Z}^+e_2$, respectively.
There is only one ray $\gamma$ starting at $e$. Its endpoint is the idempotent $l$. Thus, the set $S^e$ is one-dimensional. The Lie algebra $g$ of the group $T^3$ can be identified with $\mathbb{R}^3$, and its subalgebra $P^e$ with $\mathbb{R}e_3$. The functional $\lambda$ corresponding to $\gamma$ by Lemma 8.1 is the third coordinate of a point $x \in S$. It takes integer values on $S$. Thus, $D^e = \mathbb{D}$ and, moreover, $P^e = T^3D^e$ is a $T^3$-invariant bundle with the fiber $\mathbb{D}$ and the base $T^2$, which is the orbit of the center of the disc $\mathbb{D}$.

The rest of $M_A$ corresponds to the semigroup $\mathbb{Z}^+e_1 + \mathbb{Z}^+e_2$, which defines the bidisk $\mathbb{D} \times \mathbb{D}$ in $\mathbb{C}^2$. The set $S_A$ is equivalent to the union of the square $|x| + |y| \leq 1$ in $\mathbb{R}^2$ and the segment $[1, 2]$ that represents $S^e$, where 2 corresponds to $e$. Hence, $S_A^e$ is not dense in $S_A$ and the semigroup $S$ is not finitely generated by Theorem 8.2, (b).

It is easy to check that $M_A$ is the union of the $T^3$-orbit of the set $(0, 0, \mathbb{D})$ and the bidisc $(\mathbb{D}, \mathbb{D}, 0)$. Both parts are $T^3$-invariant and have real dimension 4. The algebra $A$ coincides with the algebra of continuous functions analytic on $z_3$ in the first part and on $z_1, z_2$ in the second one.

Corollary 8.3 can also fail if $S$ is not finitely generated. Note first that the set $J_A$ is always at most countable since every $j \in J$ relates the closed group $G_j \subseteq T^3$, which uniquely corresponds to its annihilator $G_j^\perp$ in $\mathbb{Z}^3$. The latter family is countable since every subgroup of $\mathbb{Z}^n$ can be generated by $n$ its elements.

**Example 10.** The light cone $r^2 - p^2 - q^2 = 0$ in $\mathbb{R}^3 \setminus 0$ contains infinitely many lines passing through the points of $\mathbb{Z}^3$. Let the semigroup $S \subset \mathbb{Z}^3$ be defined by the inequalities $r \geq 0$ and $r^2 - p^2 - q^2 \geq 0$. Then $J_A$ is infinite. The algebra $A$ is antisymmetric. The only limit point of $J_A$ is the least idempotent $e$ that corresponds to the characteristic function of the point 0. Indeed, any sequence of distinct idempotents in $\text{Hom}(S, \mathbb{D})$ converges pointwise to zero on $S \setminus \{0\}$.

The last example also illustrates a significant difference between finitely and infinitely generated algebras.

**Example 11.** Let $\alpha \in \mathbb{R}$ be irrational and $S = \{(p, q) \in \mathbb{Z}^2 : p + q\alpha > 0\}$. Then $A$ can be realized as an algebra of almost periodic functions on $\mathbb{R}$ that are bounded and analytic in the upper half-plane in $\mathbb{C}$. The algebra $A$ is antisymmetric, has no orthogonal real measures and is a maximal subalgebra of $C(T^2)$. The real line $\mathbb{R}$ naturally embeds to $T^2$ as an irrational winding. The natural mapping $\mathbb{R}$ to $T^2$ extends to the upper half-plane in $\mathbb{C}$ and defines an analytic structure in $M_A$. The latter is true for every its shift. In [9, Ch. 7], the algebras of this kind are considered in detail.

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