SMOOTHED PROJECTIONS OVER WEAKLY LIPSCHITZ DOMAINS

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Abstract. We develop finite element exterior calculus over weakly Lipschitz domains. Specifically, we construct commuting projections from $L^p$ de Rham complexes over weakly Lipschitz domains onto finite element de Rham complexes. These projections satisfy uniform bounds for finite element spaces with bounded polynomial degree over shape-regular families of triangulations. Thus we extend the theory of finite element differential forms to polyhedral domains that are weakly Lipschitz but not strongly Lipschitz. As new mathematical tools, we use the collar theorem in the Lipschitz category, and we show that the degrees of freedom in finite element exterior calculus are flat chains in the sense of geometric measure theory.

1. Introduction

The aim of this article is to contribute to the understanding of finite element methods for partial differential equations over domains of low regularity. For partial differential equations associated to a differential complex, projections that commute with the relevant differential operators are central to the analysis of mixed finite element methods. In particular, smoothed projections from Sobolev de Rham complexes to finite element de Rham complexes are used in finite element exterior calculus (FEEC) [1, 3]. This was researched when the underlying domain is a Lipschitz domain. In this article, we regard more generally finite element exterior calculus when the underlying domain is merely a weakly Lipschitz domain. Specifically, we construct and analyze smoothed projections. Thus we enable the abstract Galerkin theory of finite element exterior calculus within that generalized geometric setting.

It is easy to provide motivation for considering the class of weakly Lipschitz domains in the context of finite element methods. A domain is called weakly Lipschitz if its boundary can be flattened locally by a Lipschitz coordinate transformation. This generalizes the classical notion of (strongly) Lipschitz domains, whose boundaries, by definition, can be written locally as Lipschitz graphs. Although Lipschitz domains are a common choice for the geometric ambient in the theoretical and numerical analysis of partial differential equations, they exclude several practically relevant domains. It is easy to find three-dimensional polyhedral domains that are not Lipschitz domains, such as the “crossed bricks domain” [25, p.39, Figure 3.1].

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But as we show in this article, every three-dimensional polyhedral domain is still a weakly Lipschitz domain.

Moreover, weakly Lipschitz domains have attracted interest in the theory of partial differential equations because basic results in vector calculus, well-known for strongly Lipschitz domains, are still available in this geometric setting [19, 20, 20, 14, 6, 4]. For example, one can show that the differential complex

\[ H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \]

over a bounded three-dimensional weakly Lipschitz domain \( \Omega \) satisfies Poincaré-Friedrichs inequalities, and realizes the Betti numbers of the domain on cohomology. Furthermore, a vector field version of a Rellich-type compact embedding theorem is valid, and the scalar and vector Laplacians over \( \Omega \) have a discrete spectrum. Recasting this in the calculus of differential forms, one can more generally establish the analogous properties for the \( L^2 \) de Rham complex

\[ H^0(\Omega) \xrightarrow{d} H^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H^n(\Omega) \]

over a bounded weakly Lipschitz domain \( \Omega \subset \mathbb{R}^n \).

It is therefore of interest to develop finite element analysis over weakly Lipschitz domains. Since the analytical theory is formulated within the calculus of differential forms, we wish to adopt this calculus on the discrete level. Specifically, we use the framework of finite element exterior calculus, and our agenda is to extend that framework to numerical analysis on weakly Lipschitz domains. The foundational idea is to study a finite element de Rham complex

\[ \Lambda^0(\mathcal{T}) \xrightarrow{d} \Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\mathcal{T}) \]

that mimics the \( L^2 \) de Rham complex. Here, each \( \Lambda^k(\mathcal{T}) \) is a subspace of \( H\Lambda^k(\Omega) \) whose members are piecewise polynomial with respect to a fixed triangulation \( \mathcal{T} \) of the domain. Arnold, Falk, and Winther [1] have classified

A central component of finite element exterior calculus are uniformly bounded smoothed projections. Our main contribution (Theorem 7.13) in this article is to devise such a projection when the domain is merely weakly Lipschitz. A condensed version of our result is the following theorem.

**Theorem 1.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded weakly Lipschitz domain, and let \( \mathcal{T} \) be a simplicial triangulation of \( \Omega \). Let \( \ref{eq:deRham} \) be a differential complex of finite element spaces of differential forms as in finite element exterior calculus (1). Then there exist bounded linear projections \( \pi^k : L^2\Lambda^k(\Omega) \rightarrow \Lambda^k(\mathcal{T}) \) such that the following diagram commutes:

\[
\begin{array}{cccc}
H\Lambda^0(\Omega) & \xrightarrow{d} & H\Lambda^1(\Omega) & \xrightarrow{d} \cdots & \xrightarrow{d} & H\Lambda^n(\Omega) \\
\pi^0 & & \pi^1 & & \pi^n & \\
\Lambda^0(\mathcal{T}) & \xrightarrow{d} & \Lambda^1(\mathcal{T}) & \xrightarrow{d} \cdots & \xrightarrow{d} & \Lambda^n(\mathcal{T}).
\end{array}
\]

Moreover, \( \pi^k \omega = \omega \) for \( \omega \in \Lambda^k(\mathcal{T}) \). The operator norm of \( \pi^k \) is bounded uniformly in terms of the maximum polynomial degree of \( \ref{eq:deRham} \), the shape measure of the triangulation, and geometric properties of \( \Omega \).

As an immediate consequence, the a priori error estimates of finite element exterior calculus are applicable over weakly Lipschitz domains. Commuting projection operators have been approached from different perspectives in the theory of finite
Let us outline the construction of the smoothed projection and the new tools which we employ. We largely follow ideas in published literature [1, 9] but introduce significant technical modifications in this article. Given a differential form over the domain, the smoothed projection is constructed in several steps.

We first extend the differential form beyond the original domain by reflection along the boundary, using a parameterized tubular neighborhood of the boundary. For strongly Lipschitz domains, such a parametrization can be constructed using the flow along a smooth vector field transversal to the boundary [1, 9], but for weakly Lipschitz domains such a transversal vector field does not necessarily exist. Instead we obtain the desired parameterized tubular neighborhood via a variant of the collaring theorem in Lipschitz topology [22].

Next, a mollification operator smooths the extended differential form. In order to guarantee uniform bounds for shape-regular families of meshes, the mollification radius is locally controlled by a smoothed mesh size function. This is similar to [9], but we elaborate the details of the construction and make a minor correction; see also Remark 7.9. We find that the mollified differential form has well-defined degrees of freedom.

We then apply the canonical finite element interpolator to the mollified differential form. The resulting smoothed interpolator commutes with the exterior derivative and satisfies uniform bounds, but it is generally not idempotent. We can, however, control the interpolation error over the finite element space. If the smoothed interpolator is sufficiently close to the identity over the finite element space, then a commuting and uniformly bounded discrete inverse exists. Following an idea of Schöberl [27], the composition of this discrete inverse with the smoothed interpolator yields the desired smoothed projection.

In order to derive the aforementioned interpolation error estimate over the finite element space, we call on geometric measure theory [12, 29]. The principle motivation in utilizing geometric measure theory is the low regularity of the boundary, which requires new techniques in finite element theory. A key observation, which we believe to be of independent interest, is the identification of the degrees of freedom as flat chains in the sense of geometric measure theory. The desired estimate of the interpolation error over the finite element space is proven eventually with distortion estimates on flat chains. To the author’s best understanding, the results in this article also provide non-trivial details for some proofs in the aforementioned references, hitherto not available in literature; see also Remark 7.12.

Most of literature on commuting projections focuses on the $L^2$ theory (but see also [11]). We consider differential forms with coefficients in general $L^p$ spaces, following [15]. This article moreover prepares future research on smoothed projections which preserve partial boundary conditions.

The remainder of this work is structured as follows. In Section 2 we introduce weakly Lipschitz domains and a collar theorem. We recapitulate the calculus of differential forms in Section 3. We briefly review triangulations in Section 4. The relevant background in geometric measure theory is given in Section 5. Then we introduce finite element spaces, degrees of freedom, and interpolation operators in Section 6. In Section 7 we finally construct the smoothed projection.
2. Geometric Setting

We begin by establishing the geometric background. We review the notion of weakly Lipschitz domains and prove the existence of a closed two-sided Lipschitz collar along the boundaries of such domains. We refer to [22] for further background in Lipschitz topology.

Throughout this article, and unless stated otherwise, we let finite-dimensional real vector spaces $\mathbb{R}^n$ and their subsets be equipped with the canonical Euclidean metrics. We let $B_r(U)$ be the closed Euclidean $r$-neighborhood, $r > 0$, of any set $U \subseteq \mathbb{R}^n$, and we write $B_r(x) := B_r(\{x\})$.

We introduce some basic notions of Lipschitz analysis. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$, and let $f : X \to Y$ be a mapping. For a subset $U \subseteq X$, we let the Lipschitz constant $\text{Lip}(f, U) \in [0, \infty]$ of $f$ over $U$ be the minimal $L \in [0, \infty]$ that satisfies
\[ \forall x, x' \in U : \|f(x) - f(x')\| \leq L\|x - x'\|. \]
We call $f$ Lipschitz if $\text{Lip}(f, X) < \infty$. We call $f$ locally Lipschitz or LIP if for each $x \in X$ there exists a relatively open neighborhood $U \subseteq X$ of $x$ such that $f|_U : U \to Y$ is Lipschitz. If $f$ is invertible, then we call $f$ bi-Lipschitz if both $f$ and $f^{-1}$ are Lipschitz, and we call $f$ a homeomorphism if both $f$ and $f^{-1}$ are locally Lipschitz. If $f : X \to Y$ is locally Lipschitz and injective such that $f : X \to f(X)$ is a homeomorphism, then we call $f$ a LIP embedding. The composition of Lipschitz mappings is again Lipschitz, and the composition of locally Lipschitz mappings is again locally Lipschitz. If $X$ is compact, then every locally Lipschitz mapping is also Lipschitz.

Let $\Omega \subseteq \mathbb{R}^n$ be open. We call $\Omega$ a weakly Lipschitz domain if for all $x \in \partial \Omega$ there exist a closed neighborhood $U_x$ of $x$ in $\mathbb{R}^n$ and a bi-Lipschitz mapping $\phi_x : U_x \to [-1, 1]^n$ such that $\phi_x(x) = 0$ and such that
\begin{align*}
(2.1a) \quad & \phi_x(\Omega \cap U_x) = [-1, 1]^{n-1} \times [-1, 0], \\
(2.1b) \quad & \phi_x(\partial \Omega \cap U_x) = [-1, 1]^{n-1} \times \{0\}, \\
(2.1c) \quad & \phi_x(\overline{\Omega}^c \cap U_x) = [-1, 1]^{n-1} \times (0, 1].
\end{align*}

The closed sets $\{\partial \Omega \cap U_x \mid x \in \partial \Omega\}$ cover $\partial \Omega$ and the mappings $\phi_x|_{\partial \Omega \cap U_x} : \partial \Omega \cap U_x \to [-1, 1]^{n-1}$ are bi-Lipschitz. Note that $\Omega$ is a weakly Lipschitz domain if and only if $\overline{\Omega}^c$ is a weakly Lipschitz domain.

Remark 2.1. In other words, a weakly Lipschitz domain is a domain whose boundary can be flattened locally by a bi-Lipschitz coordinate transformation. The notion of weakly Lipschitz domain contrasts with the classical notion of Lipschitz domain, then also called strongly Lipschitz domain. A strongly Lipschitz domain is an open subset $\Omega$ of $\mathbb{R}^n$ whose boundary $\partial \Omega$ can be written locally as the graph of a Lipschitz function in some orthogonal coordinate system. Strongly Lipschitz domains are weakly Lipschitz domains, but the converse is generally false.

A different access towards the idea originates from differential topology: a weakly Lipschitz domain is a locally flat Lipschitz submanifold of $\mathbb{R}^n$ in the sense of [22]. This idea has motivated the notion of weakly Lipschitz domains inside general Lipschitz manifolds [13].
Example 2.2. Every bounded domain $\Omega \subset \mathbb{R}^3$ with a finite triangulation is a weakly Lipschitz domain. We will make this statement precise, and provide a proof, in Section 4 after having formally introduced triangulations. At this point, let us consider a concrete and well-known example, namely the crossed bricks domain, which we already mentioned in the introduction. Let

$$\Omega_{CB} := (-1, 1) \times (0, 1) \times (0, -1) \cup (0, 1) \times (0, -1) \times (-1, 1) \cup (0, 1) \times \{0\} \times (0, -1).$$  

At the origin, $\partial \Omega_{CB}$ is not the graph of a Lipschitz function in any coordinate system. But whereas $\Omega_{CB}$ is not a strongly Lipschitz domain, it is still a weakly Lipschitz domain. To see this, we first observe that near every non-zero $z \in \partial \Omega_{CB}$ we can write $\partial \Omega$ as a Lipschitz graph, from which we can easily construct a suitable Lipschitz coordinate chart around $z$. Finally, to obtain a bi-Lipschitz coordinate chart at the origin, we use a bi-Lipschitz mapping to transform $\Omega_{CB}$ into a domain that is a Lipschitz graph in a neighborhood of the origin. See also Figure 1.

The remainder of this section builds up a key notion of this article. We show that weakly Lipschitz domains allow for a two-sided Lipschitz collar. This result will serve for the construction of a commuting extension operator later in Section 7.

**Theorem 2.3.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded weakly Lipschitz domain. Then there exists a LIP embedding $\Psi : \partial \Omega \times [-1, 1] \to \mathbb{R}^n$ such that $\Psi(x, 0) = x$ for $x \in \partial \Omega$, and

$$\Psi(\partial \Omega, [-1, 0]) \subseteq \Omega, \quad \Psi(\partial \Omega, (0, 1]) \subseteq \overline{\Omega^c}.$$  

**Proof.** We first prove a one-sided version of the result. From definitions we deduce that there exists a collection $\{V_i\}_{i \in \mathbb{N}}$ of relatively open subsets of $\partial \Omega$ that constitute a covering of $\partial \Omega$, and a collection $\{\psi_i\}_{i \in \mathbb{N}}$ of LIP embeddings $\psi_i : V_i \times [0, 1) \to \overline{\Omega}$ such that for each $i \in \mathbb{N}$ we have $\psi_i(x, 0) = x$ for each $x \in \partial \Omega$. It follows that $\{(V_i, \psi_i)\}_{i \in \mathbb{N}}$ is a local LIP collar in the sense of Definition 7.2 in [22]. By Theorem 7.4 in [22], and a successive reparametrization, there exists a LIP embedding $\Psi^-(x, t) : \partial \Omega \times [0, 1] \to \overline{\Omega}$ such that $\Psi^-(x, 0) = x$ for all $x \in \partial \Omega$.

Recall that also $\overline{\Omega^c}$ is a weakly Lipschitz domain. By the same arguments, there exists a LIP embedding $\Psi^+(x, t) : \partial \Omega \times [0, 1) \to \Omega^c$ such that $\Psi^+(x, 0) = x$ for all $x \in \partial \Omega$. 

- **Figure 1.** Left: polyhedral domain in 3D that is not the graph of a Lipschitz function at the marked point. Right: bi-Lipschitz transformation of that domain into a strongly Lipschitz domain.
\(x \in \partial \Omega\). We combine these two LIP embeddings. Let

\[ \Psi : \partial \Omega \times [-1,1] \to \mathbb{R}^n, \quad (x,t) \mapsto \begin{cases} 
\Psi^{-}(x,-t) & \text{if } t \in [-1,0), \ x \in \partial \Omega, \\
x & \text{if } t = 0, \ x \in \partial \Omega, \\
\Psi^{+}(x,t) & \text{if } t \in (0,1], \ x \in \partial \Omega. 
\end{cases} \]

Then \(\Psi\) is well-defined, bijective, and \((\ref{eq:lip_embedding})\) holds. Moreover, we have finite constants

\[ C^- := \text{Lip}(\Psi, \partial \Omega \times [-1,0]), \quad C^+ := \text{Lip}(\Psi, \partial \Omega \times [0,1]), \]

\[ c^- := \text{Lip}(\Psi^{-1}, \partial \Omega \times [-1,0]), \quad c^+ := \text{Lip}(\Psi^{-1}, \partial \Omega \times [0,1]). \]

It remains to show that \(\Psi\) is a LIP embedding. Let \(x_1, x_2 \in \partial \Omega\) and let \(t_1, t_2 \in [-1,1]\). It suffices to show that

\[ \|x_1 - x_2\| + c|t_2 - t_1| \leq \|\Psi(x_1, t_1) - \Psi(x_2, t_2)\| \leq \|x_1 - x_2\| + C|t_2 - t_1| \]

for \(c = \max(c^+, c^-)^{-1}\) and \(C = \max(C^+, C^-)\). If \(t_1\) and \(t_2\) are both non-negative or both non-positive, then the both inequalities follow directly from the properties of \(\Psi^+\) or \(\Psi^-\). Hence we consider the case \(t_1 < 0 < t_2\). We first observe that

\[ \|\Psi(x_1, t_1) - \Psi(x_2, t_2)\| \leq \|\Psi(x_1, t_1) - x_1\| + \|x_1 - x_2\| + \|x_2 - \Psi(x_2, t_2)\| \]

\[ \leq C^-|t_1| + C^+|t_2| + \|x_1 - x_2\| \]

\[ \leq \max(C^+, C^-)|t_1 - t_2| + \|x_1 - x_2\|. \]

Furthermore, there exists \(x \in \partial \Omega\) on the straight line segment from \(\Psi(x_1, t_1)\) to \(\Psi(x_2, t_2)\). We then have

\[ \|\Psi(x_1, t_1) - \Psi(x_2, t_2)\| = \|\Psi(x_1, t_1) - x\| + \|x - \Psi(x_2, t_2)\| \]

\[ \geq |t_1| + \|x_1 - x\| + \|x_2 - x\| + |t_2| \]

\[ \geq \max(c^+, c^-)^{-1}|t_1 - t_2| + \|x_1 - x_2\|. \]

This completes the proof. \(\square\)

**Remark 2.4.** Our Theorem \((\ref{eq:lip_embedding})\) realizes an idea from differential topology in a Lipschitz setting: if a surface is locally bi-collared, then it is also globally bi-collared. Such a result is well-known in the topological or smooth sense, but it seems to be only folklore in the Lipschitz sense. Notably, the result is mentioned in the unpublished preprint \([13]\). We have provided a proof for formal completeness.

For a strongly Lipschitz domain, it is well-known that a Lipschitz collar can be defined using transversal vector fields near the boundary \([28, 8, 15]\).

### 3. Differential forms

In this section we review the calculus of differential forms in a setting of low regularity. Particular attention is given to differential forms with coefficients in \(L^p\) spaces and their transformation properties under bi-Lipschitz mappings. We adopt the notion of \(L^{p,q}\) differential form of \([15]\), to which we also refer for further details on Lebesgue spaces of differential forms. Further details can be found in \([15]\), from which the notion of \(L^{p,q}\) differential form is adopted. An elementary introduction to the calculus of differential forms is given in \([21]\).

Let \(U \subseteq \mathbb{R}^n\) be an open set. We let \(M(U)\) denote the vector space of locally integrable functions over \(U\). For \(k \in \mathbb{Z}\) we let \(MA^k(U)\) be the vector space of locally
integrable differential $k$-forms over $U$. We denote by $\omega \land \eta \in M\Lambda^{k+1}(U)$ the exterior product of $\omega \in M\Lambda^k(U)$ and $\eta \in M\Lambda^1(U)$, and we recall that $\omega \land \eta = (-1)^{ki} \eta \land \omega$.

Let $e_1, \ldots, e_n$ be the canonical orthonormal basis of $\mathbb{R}^n$. The constant 1-forms $dx^1, \ldots, dx^n \in M\Lambda^1(U)$ are uniquely defined by $dx^i(e_j) = \delta_{ij}$, where $\delta_{ij} \in \{0, 1\}$ denotes the Kronecker delta. In the sequel, we let $\Sigma(k,n)$ denote the set of strictly ascending mappings from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$. Note that $\Sigma(0,n) = \emptyset$. The basic $k$-alternators are the exterior products

$$dx^\sigma := dx^{\sigma(1)} \land \cdots \land dx^{\sigma(k)} \in M\Lambda^k(U), \quad \sigma \in \Sigma(k,n),$$

and $dx^0 := 1$. Every $\omega \in M\Lambda^k(U)$ can be written uniquely as

$$\omega = \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma dx^\sigma,$$

where $\omega_\sigma = \omega(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$. For every $n$-form $\omega \in M\Lambda^n(U)$ there exists a unique $\omega_n \in M(U)$ such that $\omega = \omega_n \mathrm{vol}^n$, where $\mathrm{vol}^n := dx^1 \land \cdots \land dx^n$ is the canonical volume $n$-form of $\mathbb{R}^n$. We define the integral of $\omega \in M\Lambda^n(U)$ over $U$ as

$$\int_U \omega := \int_U \omega_n \, dx$$

whenever $\omega_n \in M(U)$ is integrable. If $\omega, \eta \in M\Lambda^k(U)$, then we define

$$\langle \omega, \eta \rangle := \sum_{\sigma \in \Sigma(k,n)} \omega_\sigma \eta_{\sigma} \in M(U).$$

For $\omega \in M\Lambda^k(U)$ we let $|\omega| = \sqrt{\langle \omega, \omega \rangle}$. We let $L^p(U)$ denote the Lebesgue space with exponent $p \in [1, \infty]$, and let $L^p M\Lambda^k(U)$ denote the Banach space of differential $k$-forms with coefficients in $L^p(U)$. The topology of $L^p M\Lambda^k(U)$ is generated by the norm

$$\|\omega\|_{L^p M\Lambda^k(U)} := \left\|\sqrt{\langle \omega, \omega \rangle}\right\|_{L^p(U)}, \quad \omega \in L^p M\Lambda^k(U).$$

We let $C\Lambda^k(\overline{U})$ be the Banach space of continuous differential $k$-forms over $\overline{U}$, equipped with the maximum norm. We let $C^\infty \Lambda^k(U)$ be the space of smooth differential $k$-forms over $U$, we let $C^\infty \Lambda^k(\overline{U})$ be the subspace of $C^\infty \Lambda^k(U)$ whose members can be extended smoothly onto $\mathbb{R}^n$, and we let $C^\infty_{\mathrm{loc}} \Lambda^k(U)$ be the subspace of $C^\infty \Lambda^k(U)$ whose members have compact support in $U$.

The exterior derivative $d : C^\infty \Lambda^k(\overline{U}) \to C^\infty \Lambda^{k+1}(\overline{U})$ over smooth differential forms is defined by

$$d\omega = \sum_{\sigma \in \Sigma(k,n)} \sum_{i=1}^n (\partial_i \omega_{\sigma} dx^i) \land dx^\sigma, \quad \omega \in C^\infty \Lambda^k(\overline{U}),$$

where we use the representation (3.1). One can show that $d$ is linear, that $dd = 0$, and that

$$d(\omega \land \eta) = d\omega \land \eta + (-1)^{ki} \omega \land d\eta, \quad \omega \in C^\infty \Lambda^k(\overline{U}), \quad \eta \in C^\infty \Lambda^1(\overline{U}).$$

We are interested in defining the exterior derivative in a weak sense over differential forms of low regularity. If $\omega \in M\Lambda^k(U)$ and $\xi \in M\Lambda^{k+1}(U)$ such that

$$\int_U \xi \land \eta = (-1)^{k+1} \int_U \omega \land d\eta, \quad \eta \in C^\infty \Lambda^{n-k-1}(U),$$
then \( \xi \) is the only member of \( M\Lambda^{k+1}(U) \) with this property, up to equivalence almost everywhere, and we call \( d\omega := \xi \) the \textit{weak exterior derivative} of \( \omega \). Note that \( d\omega \) has vanishing weak exterior derivative, since

\[
(3.7) \quad \int_U d\omega \wedge d\eta = (-1)^k \int_U \omega \wedge dd\eta = 0, \quad \eta \in C_\infty^\infty \Lambda^{n-k-1}(U).
\]

Moreover, \( \text{(3.5)} \) generalizes in the obvious manner to the weak exterior derivative, provided all expressions are well-defined.

Next we introduce a notion of Sobolev differential forms. For \( p,q \in [1,\infty] \), we let \( L^p,q\Lambda^k(U) \) be the space of differential \( k \)-forms in \( L^p\Lambda^k(U) \) whose members have a weak exterior derivative in \( L^q\Lambda^{k+1}(U) \). We equip \( L^p,q\Lambda^k(U) \) with the norm

\[
(3.8) \quad \|\omega\|_{L^p,q\Lambda^k(U)} = \|\omega\|_{L^p\Lambda^k(U)} + \|d\omega\|_{L^q\Lambda^{k+1}(U)}.
\]

Moreover, we have the following density result \cite{15} Lemma 1.3.

**Lemma 3.1.** If \( p,q \in [1,\infty] \), then \( C^\infty \Lambda^k(\overline{U}) \) is dense in \( L^p,q\Lambda^k(U) \).

Note that, by definition, \( dL^p,q\Lambda^k(U) \subset L^{q,r}\Lambda^{k+1}(U) \) for \( p,q,r \in [1,\infty] \). Hence one may study de Rham complexes of the form

\[
\cdots \longrightarrow L^p,q\Lambda^k(U) \xrightarrow{d} L^{q,r}\Lambda^{k+1}(U) \xrightarrow{d} \cdots
\]

The choice of the Lebesgue exponents determines analytical and algebraic properties of these de Rham complexes. This is not a subject of the present article, but we refer to \cite{16} for corresponding results over smooth manifolds. De Rham complexes of the above form with a Lebesgue exponent \( p \) fixed are known as \( L^p \) de Rham complexes \( \text{e.g.} \ \cite{24} \). Two examples of such de Rham complexes are of specific relevance to us.

**Example 3.2.** The space \( \Lambda^k(U) = L^2,2\Lambda^k(U) \) is a Hilbert space, consisting of those \( L^2 \) differential \( k \)-forms whose exterior derivative has \( L^2 \) integrable coefficients.

\[
\langle \omega, \eta \rangle_{H^\Lambda^k(U)} = \langle \omega, \eta \rangle_{L^2\Lambda^k(U)} + \langle d\omega, d\eta \rangle_{L^2\Lambda^{k+1}(U)}, \quad \omega, \eta \in \Lambda^k(U).
\]

induces the topology of \( \Lambda^k(U) \). In particular, the norms \( \| \cdot \|_{L^2,2\Lambda^k(U)} \) and \( \| \cdot \|_{\Lambda^k(U)} \) are equivalent. These spaces constitute the \( L^2 \) de Rham complex

\[
\cdots \longrightarrow H^\Lambda^k(U) \xrightarrow{d} H^\Lambda^{k+1}(U) \xrightarrow{d} \cdots
\]

which has received considerable attention in global and numerical analysis.

**Example 3.3.** The space \( L^\infty,\infty \Lambda^k(U) \) of \textit{flat differential forms} is spanned by those differential forms with essentially bounded coefficients whose exterior derivative has essentially bounded coefficients. This coincides with the notion of flat differential form in geometric integration theory \cite{29}; see also Theorem 1.5 of \cite{15}. These spaces constitute the flat de Rham complex

\[
\cdots \longrightarrow L^\infty,\infty \Lambda^k(U) \xrightarrow{d} L^\infty,\infty \Lambda^{k+1}(U) \xrightarrow{d} \cdots
\]

which has been studied extensively in geometric integration theory.

We conclude this section with some basic results on the behavior of differential forms and their integrals under pullback by bi-Lipschitz mappings. For the remainder of this section, we let \( U,V \subseteq \mathbb{R}^n \) be open sets, and let \( \Phi : U \to V \) be a bi-Lipschitz mapping.
We first gather some facts on the Jacobians of bi-Lipschitz mappings. It follows from Rademacher’s theorem [12, Theorem 3.1.6] that the Jacobians
\[ D\Phi : U \to \mathbb{R}^{n \times n}, \quad D\Phi^{-1} : V \to \mathbb{R}^{n \times n} \]
exist almost everywhere. One can show that
\[ \|D\Phi\|_{L^\infty(U)} \leq \text{Lip}(\Phi, U), \quad \|D\Phi^{-1}\|_{L^\infty(V)} \leq \text{Lip}(\Phi^{-1}, V). \]
According to [12, Lemma 3.2.8], the identities
\[ (3.10) \quad D\Phi^{-1}_{\Phi(x)} \cdot D\Phi_x = \text{Id}_U, \quad D\Phi_{\Phi^{-1}(y)} \cdot D\Phi_y^{-1} = \text{Id}_V \]
hold true almost everywhere over \( U \) and \( V \), respectively. In particular, the Jacobians have full rank almost everywhere. Moreover, by [12, Corollary 4.1.26] the signs of the Jacobians are essentially locally constant: under the condition that \( U \) and \( V \) are simply connected, there exists \( o(\Phi) \in \{-1, 1\} \) such that
\[ (3.11) \quad o(\Phi) = \text{sgn} \det D\Phi, \quad o(\Phi) = o(\Phi^{-1}) \]
amost everywhere over \( U \), respectively. It follows from [12, Theorem 3.2.3] that
\[ (3.12) \quad \int_U \omega(\Phi(x)) |\det D\Phi_x| \, dx = \int_V \omega(y) \, dy \]
for \( \omega \in M(V) \) if at least one of the integrals exists.

The pull-back \( \Phi^*\omega \in MA^k(U) \) of \( \omega \in MA^k(V) \) under \( \Phi \) is defined as
\[ \Phi^*\omega_x(v_1, \ldots, v_k) := \omega_{\Phi(x)}(D\Phi_x \cdot v_1, \ldots, D\Phi_x \cdot v_k). \]
By the discussion at the beginning of Section 2 of [12], the algebraic identity
\[ \Phi^*(\omega \wedge \eta) = \Phi^*\omega \wedge \eta + (-1)^k \omega \wedge \Phi^*\eta \]
holds for \( \omega \in MA^k(V) \) and \( \eta \in MA^l(V) \). Next we show how the integral of \( n \)-forms transforms under pullback by bi-Lipschitz mappings:

**Lemma 3.4.** If \( \Phi : U \to V \) is a bi-Lipschitz mapping between simply connected open subsets of \( \mathbb{R}^n \), then
\[ (3.13) \quad \int_U \Phi^*(\omega \text{vol}_V^n) = o(\Phi) \int_V \omega \text{vol}_V^n, \quad \omega \in M(V). \]

**Proof.** Using (3.11) and (3.12), we find
\[ \int_U \Phi^*(\omega \text{vol}_V^n) = \int_U \omega \circ \Phi(x) \det(D\Phi_x) \, dx = \int_U \omega \circ \Phi(x) \cdot \text{sgn} \det D\Phi_x \cdot |\det D\Phi_x| \, dx = o(\Phi) \int_V \omega \text{vol}_V^n. \]
This shows the desired identity. \( \square \)

It can be shown that the pullback under bi-Lipschitz mappings commutes with the exterior derivative and preserves the \( L^p \) and \( L^{p,q} \) classes of differential forms.

**Lemma 3.5** (Theorem 2.2 of [12]). Let \( \Phi : U \to V \) be a bi-Lipschitz mapping between simply-connected open subsets of \( \mathbb{R}^n \), and let \( p, q \in [1, \infty] \). If \( \omega \in L^p\Lambda^k(V) \), then \( \Phi^*\omega \in L^p\Lambda^k(U) \), and if moreover \( \omega \in L^{p,q}\Lambda^k(V) \), then \( \Phi^*\omega \in L^{p,q}\Lambda^k(U) \) and \( \Phi^*\omega = d\Phi^*\omega \).

We refine the preceding statement and give an explicit estimate for the norm of the pullback operation. Here and in the sequel, \( n/\infty = 0 \) for \( n \in \mathbb{N} \).
Theorem 3.6. Let $\Phi : U \to V$ be a bi-Lipschitz mapping between open sets $U, V \subseteq \mathbb{R}^n$, and let $p \in [1, \infty]$. Then

\[
\|\Phi^*\omega\|_{LP^k(U)} \leq \|D \Phi\|_{L^\infty(U)} \|\det D \Phi^{-1}\|_{L^\infty(V)} \|\omega\|_{LP^k(V)}
\]

(3.14)

for $\omega \in LP^k(U)$.

Proof. Let $\Phi : U \to V$ and $p \in [1, \infty]$ be as in the statement of the theorem, and let $\omega \in LP^k(U)$. If $x \in U$ such that $D \Phi|_x$ exists, then we observe

\[
\|\Phi^*\omega\|_{LP^1(U)}^2 = \sum_{\rho \in \Sigma(k,n)} \left\langle \Phi^*\omega|_x, dx^\rho \right\rangle^2
\]

\[
= \sum_{\sigma \in \Sigma(k,n)} \left(\omega_{\sigma|\Phi(x)}\right)^2 \sum_{\rho \in \Sigma(k,n)} \left\langle \Phi^*dx^\rho, dx^\rho \right\rangle^2
\]

\[
= \sum_{\sigma \in \Sigma(k,n)} \left(\omega_{\sigma|\Phi(x)}\right)^2 \|\Phi^*dx^\sigma\|
\]

\[
\leq \|D \Phi\|_{2k}^2 \sum_{\sigma \in \Sigma(k,n)} \left(\omega_{\sigma|\Phi(x)}\right)^2 = \|D \Phi|_x\|_{2k}^2 \|\omega_{\Phi(x)}\|^2.
\]

From this we easily infer that

\[
\|\Phi^*\omega\|_{LP^k(U)} \leq \|D \Phi\|_{L^\infty(U)} \|\det D \Phi^{-1}\|_{L^\infty(V)} \|\omega\|_{LP^k(V)}.
\]

If $p = \infty$, then the desired statement follows trivially, and if $p \in [1, \infty)$, then Lemma 3.4 provides

\[
\int_U \|\omega\|_{\Phi(x)}^p dx \leq \|\det D \Phi^{-1}\|_{L^\infty(V)} \int_U \|\omega\|_{\Phi(x)}^p \det D \Phi|_x| dx
\]

\[
\leq \|\det D \Phi^{-1}\|_{L^\infty(V)} \int_{\Phi(U)} \|\omega\|^p dx.
\]

We note that $\Phi(U) = V$. This shows the first inequality, and the second inequality follows by Hadamard’s inequality. $\square$

4. Triangulations

In this section we review simplicial triangulations of domains and related notions, of which most is standard in literature.

Let us assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set such that $\overline{\Omega}$ is a topological manifold with boundary and $\Omega$ is its interior. A finite triangulation of $\overline{\Omega}$ is a finite set $\mathcal{T}$ of closed simplices such that the union of the elements of $\mathcal{T}$ equals $\overline{\Omega}$, such that for any $T \in \mathcal{T}$ and any subsimplex $S \subseteq T$ we have $S \in \mathcal{T}$, and such that for all $T, T' \in \mathcal{T}$ the set $T \cap T'$ is either empty or a common subsimplex of both $T$ and $T'$. We write

\[
\Delta(T) := \{S \in \mathcal{T} \mid S \subseteq T\}, \quad \mathcal{T}(T) := \{S \in \mathcal{T} \mid S \cap T \neq \emptyset\}.
\]

With some abuse of notation, we let $\mathcal{T}(T)$ also denote the closed set that is the union of the simplices of $\mathcal{T}$ adjacent to $T$. We write $\mathcal{T}^m$ for the set of $m$-dimensional simplices in $\mathcal{T}$.
Having formally introduced triangulations, we make precise and prove the introduction’s claim that all polyhedral domains in \( \mathbb{R}^3 \) are weakly Lipschitz domains.

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^3 \) be an open set such that \( \overline{\Omega} \) is a topological manifold with interior \( \Omega \). If there exists a finite triangulation \( T \) of \( \Omega \), then \( \Omega \) is a weakly Lipschitz domain.

**Proof.** Let \( x \in \partial \Omega \). We need to find a compact neighborhood \( U_x \subset \mathbb{R}^3 \) of \( x \) and a bi-Lipschitz mapping \( \phi_x : U_x \rightarrow [-1,1]^3 \) such that \( \phi_x(x) = 0 \) and (4.1). It is easy to find such \( U_x \) and \( \phi_x \) if \( x \notin T^0 \).

It remains to consider the case \( x \in T^0 \). Let \( r > 0 \) be so small that \( B_r(x) \) intersects \( T \) in \( T^3 \) if and only if \( x \in T \). We observe that \( \partial B_r(x) \cap \partial \Omega \) is a simple closed curve in \( \partial B_r(x) \) composed of finitely many spherical arcs. By Theorem 7.8 of [22], there exists a bi-Lipschitz mapping

\[
\phi_0^3 : \partial B_r(x) \rightarrow \partial B_1(0) \subset \mathbb{R}^3
\]

which maps \( \partial B_r(x) \cap \partial \Omega \) onto \( \partial B_1(0) \cap \{ x \in \mathbb{R}^3 \mid x_3 = 0 \} \). By radial continuation, we obtain a bi-Lipschitz mapping

\[
\phi_1^3 : B_r(x) \rightarrow B_1(0) \subset \mathbb{R}^3
\]

which maps \( B_r(x) \cap \partial \Omega \) onto \( B_1(0) \cap \{ x \in \mathbb{R}^3 \mid x_3 = 0 \} \) and with \( \phi_1^3(x) = 0 \).

Moreover, there exists a bi-Lipschitz mapping

\[
\phi_2^1 : B_1(0) \rightarrow [-1,1]^3
\]

which maps \( B_1(0) \cap \{ x \in \mathbb{R}^3 \mid x_3 = 0 \} \) onto \( [-1,1]^2 \times \{0\} \) with \( \phi_2^1(0) = 0 \). The theorem follows with \( U_x := B_r(x) \) and \( \phi_x := \phi_2^1 \phi_1^3 \). \( \square \)

The remainder of this section is devoted to notions of regularity of triangulations. Let us fix a finite triangulation \( T \) of \( \overline{\Omega} \). When \( T \in T^m \) is any simplex of the triangulation, then we write \( h_T = \text{diam}(T) \) for the diameter of \( T \), and \( |T| = \text{vol}^m(T) \) for the \( m \)-dimensional volume of \( T \). If \( V \in T^0 \), then \( |V| = 1 \), and \( h_V \) is defined, by convention, as the average length of all \( n \)-simplices of \( T \) that are adjacent to \( V \).

We define the *shape constant* of \( T \) as the minimal \( C_{\text{mesh}} > 0 \) that satisfies

\[
\forall T \in T^m : h_T^2 \leq C_{\text{mesh}} |T|, \tag{4.1}
\]

\[
\forall T \in T, S \in T(T) : h_T \leq C_{\text{mesh}} h_S. \tag{4.2}
\]

Intuitively, (4.1) describes a bound on the flatness of the simplices, while (4.2) describes that the diameter of adjacent simplices are comparable. In applications, we consider families of triangulations, such as generated by successive uniform refinement [5] or newest vertex bisection [23], whose shape constants are uniformly bounded.

We can bound some important quantities in terms of \( C_{\text{mesh}} \) and the geometric ambient. There exists a constant \( C_N > 0 \), depending only on \( C_{\text{mesh}} \) and the ambient dimension \( n \), such that

\[
\forall T \in T : |T(T)| \leq C_N. \tag{4.3}
\]

This bounds the numbers of neighbors of any simplex. There exists a constant \( \epsilon_h > 0 \), depending only on \( C_{\text{mesh}} \) and \( \Omega \), such that

\[
\forall T \in T : B_{\epsilon_h h_T}(T) \subseteq \Omega^c, \tag{4.4a}
\]

\[
\forall T \in T : B_{\epsilon_h h_T}(T) \cap \overline{\Omega} \subseteq T(T). \tag{4.4b}
\]
In the sequel, we use affine transformations to a reference simplex. Let
\[ \Delta^n = \text{convex}\{0, e_1, \ldots, e_n\} \subseteq \mathbb{R}^n \]
be the \( n \)-dimensional reference simplex. For each \( n \)-simplex \( T \in \mathcal{T}^n \) of the triangulation, we fix an affine transformation \( A_T(x) = M_T x + b_T \) where \( b_T \in \mathbb{R}^n \) and \( M_T \in \mathbb{R}^{n \times n} \) are such that \( A_T(\Delta^n) = T \). Each matrix \( M_T \) is invertible, and
\[ \|M_T\| \leq c_M h_T, \quad \|M_T^{-1}\| \leq C_M h_T^{-1} \]
for constants \( c_M, C_M > 0 \) that depend only on \( C_{mesh} \) and \( n \).

5. Elements of Geometric Measure Theory

This section gives an outline of relevant ideas from geometric measure theory, for which we use Whitney’s monograph [29] as the main reference. Our motivation to consider geometric measure theory lies in proving Theorem 7.11 later in this paper. A key observation for this purpose is that the degrees of freedom of finite element exterior calculus are flat chains (Lemma 5.1). Analogously, we identify finite element differential forms as flat forms. This allows us to estimate Lipschitz deformations of flat chains (Lemma 5.2) in a finite element setting.

We begin with basic notions of chains and cochains. The space \( P_k(\mathbb{R}^n) \) of polyhedral \( k \)-chains is the vector space of functions which can be written as finite sums \( \sum_{i=1}^l a_i \chi_{S_i} \), where \( a_i \in \mathbb{R} \) and \( \chi_{S_i} \) is the indicator function of a closed simplex \( S_i \). We consider two such functions as equivalent if they are identical almost everywhere with respect to the \( k \)-dimensional Hausdorff measure.

We fix an arbitrary orientation for each \( k \)-simplex \( S \subseteq \mathbb{R}^n \), henceforth called positive orientation. We identify \( S \) in positive orientation with \( \chi_S \), and \( S \) in negative orientation with \( -\chi_S \). The boundary operator \( \partial : P_k(\mathbb{R}^n) \rightarrow P_{k-1}(\mathbb{R}^n) \) is now defined as follows. If \( S \) is an oriented \( k \)-simplex, then \( \partial S \) is the formal sum \( F_0 + \cdots + F_k \) of its faces equipped with the outward orientation induced by \( S \). This defines \( \partial \) over \( P_k(\mathbb{R}^n) \) by linear extension.

The mass \( |S|_k \) of a polyhedral \( k \)-chain is its \( L^1 \)-norm with respect to the \( k \)-dimensional Hausdorff measure. We write \( \overline{P_k(\mathbb{R}^n)} \) for the completion of \( P_k(\mathbb{R}^n) \) by the mass norm. We define the flat norm of \( S \in P_k(\mathbb{R}^n) \) as
\[ \|S\|_{k, \flat} := \inf_{Q \in \mathcal{P}_{k+1}(\mathbb{R}^n)} \|S - \partial Q\|_k + |Q|_{k+1}. \]
One can show that \( \|\cdot\|_{k, \flat} \) is a norm on \( P_k(\mathbb{R}^n) \). We define the Banach space \( C^\flat_k(\mathbb{R}^n) \), the space of flat \( k \)-chains in \( \mathbb{R}^n \) as the completion of \( P_k(\mathbb{R}^n) \) with respect to the flat norm. We have \( \|S\|_{k, \flat} \leq |S|_k \) for \( S \in P_k(\mathbb{R}^n) \), so \( \overline{P_k(\mathbb{R}^n)} \) embeds densely into \( C^\flat_k(\mathbb{R}^n) \). Moreover, we have \( \|\partial S\|_{k-1, \flat} \leq \|S\|_{k, \flat} \) for \( S \in P_k(\mathbb{R}^n) \), so the boundary operator extends to a bounded linear operator \( \partial : C^\flat_k(\mathbb{R}^n) \rightarrow C^\flat_{k-1}(\mathbb{R}^n) \). Note, however, that \( \partial \) is densely-defined but unbounded over \( \overline{P_k(\mathbb{R}^n)} \).

We now study the duality of flat chains and flat differential forms. Flat forms were studied in [29], there mainly as representations of flat cochains, and in [15]. For the following facts, we refer to Section 2 of [15] and Chapters IX and X of [29].

Flat differential forms have well-defined traces on simplices. More precisely, for each \( m \)-simplex \( S \subset \mathbb{R}^n \) there exists a bounded linear mapping \( \text{tr} : L^{\infty, \infty}(\Lambda^k(\mathbb{R}^n)) \rightarrow L^{\infty, \infty}(\Lambda^k(S)) \), which extends the trace of smooth forms. In particular, \( \text{tr}_{S} \omega \) does only
depend on the values of $\omega$ near $S$. We write $\int_S \omega$ for the integral of $\omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n)$ over a $k$-simplex $S$. This pairing extends by linearity to $S \in \mathcal{P}^k(\mathbb{R}^n)$. We have

$$\left| \int_S \omega \right| \leq |S|_k \|\omega\|_{L^{\infty, \infty} \Lambda^k(\mathbb{R}^n)}, \quad \omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n), \quad S \in \mathcal{P}^k(\mathbb{R}^n).$$

This pairing furthermore extends to flat chains. We have

$$\left| \int_\alpha \omega \right| \leq \|\alpha\|_{k, \beta} \|\omega\|_{L^{\infty, \infty} \Lambda^k(\mathbb{R}^n)}, \quad \alpha \in C^\alpha_k(\mathbb{R}^n), \quad \omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n).$$

The exterior derivative between spaces of flat forms is dual to the boundary operator between spaces of flat chains. We have

$$\int_{\partial \alpha} \omega = \int_\alpha d\omega, \quad \alpha \in C^\alpha_k(\mathbb{R}^n), \quad \omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n),$$

as a generalized Stokes' theorem.

We consider pushforwards of chains and pullbacks of differential forms along Lipschitz mappings. Assume that $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a Lipschitz mapping. Then there exists a mapping $\phi_* : C^\alpha_k(\mathbb{R}^m) \rightarrow C^\alpha_k(\mathbb{R}^n)$, the pushforward along $\phi$, such that

$$\partial \phi_* \alpha = \phi_* \partial \alpha, \quad \alpha \in C^\alpha_k(\mathbb{R}^m),$$

$$\|\phi_* \alpha\|_{k, \beta} \leq \sup \{\text{Lip}(\phi, \mathbb{R}^m)^k, \text{Lip}(\phi, \mathbb{R}^m)^{k+1}\} \|\alpha\|_{k, \beta}, \quad \alpha \in C^\alpha_k(\mathbb{R}^m),$$

$$|\phi_* |S|_k \leq \text{Lip}(\phi, \mathbb{R}^m)^k |S|_k, \quad S \in \mathcal{P}^k(\mathbb{R}^m).$$

Dually, there exists a mapping $\phi^* : L^{\infty, \infty} \Lambda^k(\mathbb{R}^n) \rightarrow L^{\infty, \infty} \Lambda^k(\mathbb{R}^m)$, called the pullback along $\phi$, which satisfies

$$d\phi^* \omega = \phi^* d\omega,$$

$$\|\phi^* \omega\|_{L^{\infty, \infty} \Lambda^k(\mathbb{R}^n)} \leq \text{Lip}(\phi, \mathbb{R}^m)^k \|\omega\|_{L^{\infty, \infty} \Lambda^k(\mathbb{R}^m)},$$

$$\|\phi^* d\omega\|_{L^{\infty, \infty} \Lambda^{k+1}(\mathbb{R}^m)} \leq \text{Lip}(\phi, \mathbb{R}^m)^{k+1} \|d\omega\|_{L^{\infty, \infty} \Lambda^{k+1}(\mathbb{R}^m)},$$

for $\omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n)$. The pushforward and the pullback are related by the identity

$$\int_{\phi_* \alpha} \omega = \int_\alpha \phi^* \omega, \quad \omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n), \quad \alpha \in C^\alpha_k(\mathbb{R}^m).$$

Having outlined basic concepts of geometric measure theory, we provide a new result which makes these notions productive for finite element theory: the degrees of freedom in finite element exterior calculus are flat chains.

**Lemma 5.1.** Let $F \subset \mathbb{R}^n$ be a closed oriented $m$-simplex and let $\eta \in C^\infty \Lambda^{m-k}(F)$. Then there exists a flat chain $\alpha_\eta \in C^\alpha_k(\mathbb{R}^n)$ such that

$$\int_F \text{tr}_F \omega \wedge \eta = \int_{\alpha_\eta} \omega, \quad \omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n).$$

Moreover, $\alpha_\eta \in \mathcal{P}^k(\mathbb{R}^n)$ and $\partial \alpha_\eta \in \mathcal{P}^{k-1}(\mathbb{R}^n)$.

**Proof.** We first assume that $\dim F = n$, and that $F$ is positively oriented. As is well-known, there exists $\star \eta \in C^\infty \Lambda^k(F)$ such that

$$\int_F \omega \wedge \eta = \int_F (\omega, \star \eta), \quad \omega \in L^{\infty, \infty} \Lambda^k(\mathbb{R}^n),$$
and \( \| \eta \|_{L^1(\Lambda^{n-k}(F))} = \| \star \eta \|_{L^1(\Lambda^k(F))} \). We use Theorem 15A of \cite{29} Chapter IX to deduce the existence of \( \alpha_\eta \in C^*_k(\mathbb{R}^n) \) such that

\[
\int_{\alpha_\eta} \omega = \int_F \langle \omega, \star \eta \rangle, \quad \omega \in L^1(\Lambda^k(\mathbb{R}^n)),
\]

and \( |\alpha_\eta|_k = \| \star \eta \|_{L^1(\Lambda^k(F))} \). In particular, \( \alpha_\eta \in P^k(\mathbb{R}^n) \).

Now assume that \( \dim F = m < n \). There exists a simplex \( F_0 \subset \mathbb{R}^m \) and an isometric inclusion \( \phi : \mathbb{R}^m \to \mathbb{R}^n \) which maps \( F_0 \) onto \( F \). Recall that the pullback of a flat form along a Lipschitz mapping is well-defined. We have

\[
\int_F tr_F \omega \wedge \eta = \int_{\phi F_0} tr_{F_0} \phi^* \omega \wedge \phi^* \eta = \int_{F_0} \phi^* tr_F \omega \wedge \phi^* \eta = \int_{\alpha_\phi \eta} \phi^* tr_F \omega = \int_{\phi \alpha_\phi \eta} \omega
\]

for \( \omega \in L^1(\Lambda^k(\mathbb{R}^n)) \). Thus we may choose \( \alpha_\eta = \phi_* \alpha_{\phi \eta} \in P^k(\mathbb{R}^n) \).

It remains to show that \( \partial \alpha_\eta \in P^{k-1}(\mathbb{R}^n) \). For \( \omega \in L^1(\Lambda^{k-1}(\mathbb{R}^n)) \), we have

\[
\int_{\partial \alpha_\eta} \omega = \int_{\alpha_\eta} d\omega = \int_F \eta \wedge tr_F d\omega = (-1)^{k+1} \int_F d\eta \wedge tr_F \omega + \sum_{G \subset \partial F} \int_G tr_G \eta \wedge tr_G \omega.
\]

Here, the sum is taken over all faces of \( F \) of dimension \( m-1 \), equipped with the outward orientation. This is a sum of functionals as in the statement of the theorem. Since \( P^{k-1}(\mathbb{R}^n) \) is a Banach space, we have \( \partial \alpha \in P^{k-1}(\mathbb{R}^n) \). The proof is complete. \( \square \)

We finish this section with an estimate on the deformation of flat chains by Lipschitz mappings. The following result is implied by Theorem 13A in Chapter X and the discussions in Paragraphs 1 and 2 of Chapter VIII in \cite{29}.

**Lemma 5.2.** Let \( F \subseteq \mathbb{R}^n \) be an \( m \)-simplex and let \( \eta \in C^\infty \Lambda^{m-k}(F) \). Let \( \alpha \in C^0_k(\mathbb{R}^n) \) be the associated flat chain in the manner of Lemma 5.1A. Let \( U \subseteq \mathbb{R}^n \) be open and convex with \( F \subset U \), and let \( \phi : U \to \mathbb{R}^n \) be Lipschitz. Then

\[
\| \phi_* \alpha - \alpha \|_{k,\mathbb{R}^n} \leq \| \phi - \text{Id} \|_{L^\infty(U,\mathbb{R}^n)} (l^k|\alpha|_k + l^{k-1}|\partial \alpha|_{k-1}) ,
\]

where \( l := \sup \{ \text{Lip}(\phi, U), 1 \} \).

### 6. Finite Element Spaces, Degrees of Freedom, and Interpolation

In this section we outline the discretization theory of finite element exterior calculus. We summarize basic facts on the finite element spaces and their spaces of degrees of freedom. The most important construction is the canonical finite element interpolator \( I^k \). Moreover we consider several inverse inequalities. The reader is assumed to be familiar with the background in \cite{2} and \cite{1} Section 3–5]. We outline this background and additionally apply geometric measure theory in the perspective of the preceding section.

For the duration of this section, we fix a bounded weakly Lipschitz domain \( \Omega \subset \mathbb{R}^n \) and a finite triangulation \( \mathcal{T} \) of \( \Omega \).
The essential idea is to consider a differential complex of finite element spaces that mimics the de Rham complex on a discrete level. The finite element spaces are finite-dimensional spaces of piecewise polynomial differential forms.

Let $T \in \mathcal{T}^n$ be an $n$-simplex, and let $r, k \in \mathbb{Z}$. We define $\mathcal{P}_r\Lambda^k(T)$ as the space of differential $k$-forms whose coefficients are polynomials over $T$ of degree at most $r$. We define $\mathcal{P}_r^-\Lambda^k(T) := \mathcal{P}_{r-1}\Lambda^k(T) + \tilde{X} \mathcal{P}_{r-1}\Lambda^{k+1}(T)$, where $\tilde{X}$ denotes contraction with the source vector field $\tilde{X}(x) = x$. One can show that $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ are invariant under pullback by affine automorphisms of $T$. For any sub-simplex $F \in \Delta(T)$ of $T$ we set

$$\mathcal{P}_r\Lambda^k(F) = \text{tr}_{T,F} \mathcal{P}_r\Lambda^k(T), \quad \mathcal{P}_r^-\Lambda^k(F) = \text{tr}_{T,F} \mathcal{P}_r^-\Lambda^k(T),$$

which do not depend on $T$. Some basic properties of these spaces are

$$\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}_{r+1}\Lambda^k(T), \quad \mathcal{P}_r^-\Lambda^k(T) \subseteq \mathcal{P}_r\Lambda^k(T),$$

$$d\mathcal{P}_r\Lambda^k(T) \subseteq \mathcal{P}_{r-1}\Lambda^{k+1}(T), \quad d\mathcal{P}_r^-\Lambda^k(T) = d\mathcal{P}_r^-\Lambda^k(T),$$

$$\mathcal{P}_r\Lambda^0(T) = \mathcal{P}_r^-\Lambda^0(T), \quad \mathcal{P}_r\Lambda^n(T) = \mathcal{P}_{r+1}\Lambda^n(T).$$

We define the finite element spaces

$$\mathcal{P}_r\Lambda^k(T) := \{ \omega \in L^\infty,\infty \Lambda^k(\Omega) \mid \forall T \in \mathcal{T}^n : \omega|_T \in \mathcal{P}_r\Lambda^k(T) \},$$

$$\mathcal{P}_r^-\Lambda^k(T) := \{ \omega \in L^\infty,\infty \Lambda^k(\Omega) \mid \forall T \in \mathcal{T}^n : \omega|_T \in \mathcal{P}_r^-\Lambda^k(T) \}.$$

These are spaces of piecewise polynomial differential forms. The regularity $L^\infty,\infty \Lambda^k(\Omega)$ enforces that their members feature tangential continuity along simplex boundaries. In particular, if $\omega \in \mathcal{P}_r\Lambda^k(T)$ and $T, T' \in \mathcal{T}$ are neighbouring simplices, then the restrictions of $\omega$ to $T$ and $T'$ have the same trace on their common sub-simplex $T \cap T'$. We have recovered precisely the finite element spaces of [1].

From $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ we can construct finite element de Rham complexes, but the combination of spaces is not arbitrary. We single out a class of differential complexes that we call FEEC-complexes in this article. A FEEC-complex is a differential complex

$$0 \to \Lambda^0(T) \xrightarrow{d} \Lambda^1(T) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(T) \to 0$$

such that for all $k \in \mathbb{Z}$ there exists $r \in \mathbb{Z}$ with

$$\Lambda^k(T) \in \{ \mathcal{P}_r\Lambda^k(T), \mathcal{P}_r^-\Lambda^k(T) \}$$

and that for all $k \in \mathbb{Z}$ we have

$$\Lambda^k(T) \in \{ \mathcal{P}_r\Lambda^k(T), \mathcal{P}_r^-\Lambda^k(T) \} \quad \Rightarrow \quad \Lambda^{k+1}(T) \in \{ \mathcal{P}_{r-1}\Lambda^{k+1}(T), \mathcal{P}_r^-\Lambda^{k+1}(T) \}.$$ 

These are the finite element de Rham complexes discussed in [1].

We next introduce the degrees of freedom of finite element exterior calculus. They are represented by taking the trace of a differential form onto a simplex of $\mathcal{T}$ and then integrating against a smooth differential form. By virtue of Lemma [5.1] we introduce the degrees of freedom as chains of finite mass. Specifically, when
$F \in \mathcal{T}$ and $m = \dim(F)$, then we define

$$\mathcal{P}_r C^F_k := \left\{ S \in C^k_\mathcal{T}(\mathbb{R}^n) \mid \exists \eta_S \in \mathcal{P}^r_{r+k-m} \Lambda^{m-k}(F) : \int_S \eta_S \cdot : = \int_F \eta_S \cdot : \right\},$$

$$\mathcal{P}^- r C^F_k := \left\{ S \in C^k_\mathcal{T}(\mathbb{R}^n) \mid \exists \eta_S \in \mathcal{P}^-_{r+k-m} \Lambda^{m-k}(F) : \int_S \eta_S \cdot : = \int_F \eta_S \cdot : \right\}.$$

We furthermore obtain by Lemma 5.1 that the degrees of freedom are flat chains of finite mass with boundaries of finite mass. One can show that we have direct sums

$$\mathcal{P}_r \mathcal{C}_k(T) := \sum_{F \in \mathcal{T}} \mathcal{P}_r C^F_k, \quad \mathcal{P}^- r \mathcal{C}_k(T) := \sum_{F \in \mathcal{T}} \mathcal{P}^- r C^F_k$$

and that we have the inclusions

$$\partial \mathcal{P}_r \mathcal{C}_k(T) \subseteq \mathcal{P}^-_{r+1} \mathcal{C}_{k-1}(T), \quad \mathcal{P}_r \mathcal{C}_k(T) \subseteq \mathcal{P}^-_{r+1} \mathcal{C}_k(T), \quad \mathcal{P}^- r \mathcal{C}_k(T) \subseteq \mathcal{P}_r \mathcal{C}_k(T).$$

With respect to a given FEEC-complex 6.1, we then define

$$\mathcal{C}_k(T) = \left\{ \begin{array}{ll}
\mathcal{P}_r \mathcal{C}_k(T) & \text{if } \Lambda^k(T) = \mathcal{P}_r \Lambda^k(T), \\
\mathcal{P}^- r \mathcal{C}_k(T) & \text{if } \Lambda^k(T) = \mathcal{P}^- r \Lambda^k(T).
\end{array} \right.$$

for $k \in \mathbb{Z}$. Note that $\partial \mathcal{C}_{k+1}(T) \subseteq \mathcal{C}_k(T)$ by construction. We have a well-defined complex of degrees of freedom

$$0 \leftarrow \mathcal{C}_0(T) \leftarrow \mathcal{C}_1(T) \leftarrow \cdots \leftarrow \mathcal{C}_n(T) \leftarrow 0.$$

We can prove a duality between the finite element complex (6.1) and the complex of degrees of freedom (6.5), following [2, Section 5]. One can show that

$$\forall S \in \mathcal{C}_k(T) : S \neq 0 \implies \exists \omega \in \Lambda^k(T) : \int_S \omega \neq 0,$$

$$\forall \omega \in \Lambda^k(T) : \omega \neq 0 \implies \exists S \in \mathcal{C}_k(T) : \int_S \omega \neq 0.$$

We conclude that $\mathcal{C}_k(T)$, restricted to $\Lambda^k(T)$, spans the dual space of $\Lambda^k(T)$. Notably, the last implication can be strengthened to the following “local” result. When $T \in \mathcal{T}^n$ and $\omega \in \Lambda^k(T)$, then

$$\omega|_T = 0 \iff \forall F \in \Delta(T) : \exists S \in \mathcal{C}^F_k : \int_S \omega = 0.$$

So the value of $\omega \in \Lambda^k(T)$ is determined uniquely by the values of the degrees of freedom associated with that simplex.

**Remark 6.1.** At this point it is helpful to recall the role of degrees of freedom in finite element theory. On the one hand, they are functionals which span the dual space of a finite element space, and on the other hand, the degrees of freedom are used in the construction of the canonical finite element interpolator. Corresponding to these two applications, we treat the degrees of freedom as functionals both over $L^\infty \Lambda^k(\Omega^c)$ and $CA^k(\Omega^c)$, and we define the canonical finite element interpolator over both spaces. This is possible because the degrees of freedom are flat chains of finite mass.

We introduce the **canonical finite element interpolator.** This linear mapping is well-defined and bounded both over $CA^k(\Omega^c)$ and $L^\infty \Lambda^k(\Omega^c)$. We define

$$I^k : CA^k(\Omega^c) + L^\infty \Lambda^k(\Omega^c) \rightarrow \Lambda^k(T)$$

(6.8)
by requiring that

\begin{equation}
\int_S \omega = \int_S I^k \omega, \quad S \in C_k(T), \quad \omega \in C^{\lambda^k}(\Omega^c) + L^{\infty, \infty} \Lambda^k(\Omega^c).
\end{equation}

The finite element interpolator commutes with the exterior derivative, which follows easily from (6.9) and (5.4). We have

\begin{equation}
\int_S I^k+1 d\omega = \int_S d\omega = \int_S I^k \omega = \int_S dI^k \omega
\end{equation}

for all $\omega \in L^{\infty, \infty} \Lambda^k(\Omega^c)$ and $S \in C_{k+1}(T)$. In particular, the following diagram commutes:

\begin{equation}
\cdots \longrightarrow L^{\infty, \infty} \Lambda^k(\Omega^c) \xrightarrow{d} L^{\infty, \infty} \Lambda^{k+1}(\Omega^c) \longrightarrow \cdots
\end{equation}

Furthermore $I^k$ is idempotent, i.e.

\begin{equation}
I^k \omega = \omega, \quad \omega \in \Lambda^k(T),
\end{equation}

as follows directly from (6.7).

In the remainder of this section, we introduce a number of inverse inequalities. These rely on the equivalence of norms over finite-dimensional vector spaces.

We note that, by construction, the pullbacks $A_T^* \omega|_T$ lie in a common finite-dimensional vector space as $\omega \in \Lambda^k(T)$ and $T \in T^n$ vary. For example, this can be a fixed space of differential forms with polynomial coefficients of sufficiently high degree. Hence for each $p \in [1, \infty]$ there exists a constant $C_{y,p,n} > 0$ such that

\begin{equation}
\|A_T^* \omega\|_{L^{\infty, \infty} \Lambda^k(\Delta^n)} \leq C_{y,p,n} \|A_T^* \omega\|_{L^p \Lambda^k(\Delta^n)}, \quad \omega \in \Lambda^k(T), \quad T \in T^n.
\end{equation}

The constant $C_{y,p,n}$ depends only on $n$ and the maximal polynomial degree in the finite element de Rham complex.

Another inverse inequality applies to the degrees of freedom. By Lemma 5.1, each degree of freedom can be identified with a flat chain of finite mass whose boundary is again a flat chain of finite mass. In general, the boundary operator is an unbounded operator as a mapping between spaces of polyhedral chains with respect to the mass norm. But in the present setting, the pushforward of the degrees of freedom onto the reference simplex takes values in a finite-dimensional vector space. We conclude that there exists $C_\partial > 0$ such that

\begin{equation}
\|A_T^* \partial S\|_{\Lambda^k(\Delta^n)} \leq C_\partial |A_T^* S|_k, \quad S \in C^F_k, \quad F \in \Delta(T), \quad T \in T^n.
\end{equation}

Again, the constant $C_\partial$ depends only on $n$ and the maximal polynomial degree in the finite element de Rham complex.

Finally, we observe that there exists $C_I > 0$ such that

\begin{equation}
\|A_T^* I^k \omega\|_{L^{\infty, \infty} \Lambda^k(\Delta^n)} \leq C_I \sup_{F \in \Delta(T)} \|A_T^* S\|_k \int_{A_T^* S} A_T^* \omega
\end{equation}
for all $T \in \mathcal{T}^n$ and $\omega \in CA^k(\Omega)$. Similar as before, the constant $C_\partial$ depends only on $n$ and the maximal polynomial degree in the finite element de Rham complex. Note that this inequality immediately implies

$$
\|A^*_T I^k \omega\|_{L^\infty(\Delta^n)} \leq C_I \|A^*_T \omega\|_{CA^k(\Delta^n)}, \quad \omega \in CA^k(\Omega).
$$

To see why (6.15) holds true, recall that $A^*_T I^k A^{-s}$ defines a linear mapping from $CA^k(A^{-1}_T \Omega)$ onto a space of polynomial differential forms over the reference simplex $\Delta^n$. By construction,

$$
\int_{A^{-s}_T S} A^*_T I^k \omega = \int_S I^k A^{-s}_T \omega = \int_S A^{-s}_T \omega = \int_{A^{-s}_T S} \omega
$$

when $S \in C_k(\mathcal{T})$ and $A^{-s}_T \omega \in CA^k(\Omega)$. Since the pushforwards of degrees of freedom and the pullbacks of finite element differential forms to the reference simplex vary within finite dimensional vector spaces, the existence of $C_I > 0$ follows.

Remark 6.2. The existence of constants $C_{\delta,p,n}$, $C_{\partial}$, and $C_I$ as above follows trivially if the triangulation $\mathcal{T}$ and the sequences (6.1) and (6.5) are fixed. But in applications we consider families of triangulations with associated sequences (6.1) and (6.5), and demand uniform bounds for those constants. Such uniform bounds hold if the triangulations have uniformly bounded shape constants and the finite element spaces have uniformly bounded polynomial degree. The results of this article do not attend to estimates that are uniform in the polynomial degree, as would be relevant for $p$- and $hp$-methods.

7. Smoothed Projection

In this section, we construct the smoothed projection in several stages. First, we devise an extension operator $E^k$, applying the two-sided Lipschitz collar discussed in Section 2. We then formulate a mollification operator $R^kh$, where we use a smooth mesh size function $h$ as an auxiliary construction. Successive composition with the canonical finite element interpolator $I^k$ from Section 6 yields an uniformly bounded commuting mapping $Q^k_\epsilon$, the smoothed interpolator, from differential forms with coefficients in $L^p$ onto finite element differential forms. $Q^k_\epsilon$ is generally not idempotent on the finite element space, but the interpolation error can be controlled. After a small modification, we obtain the desired smoothed projection $\pi^k_\epsilon$.

Throughout this section we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded simply-connected weakly Lipschitz domain and that $\mathcal{T}$ is a finite triangulation of $\overline{\mathcal{T}}$. We additionally assume that we have fixed a FEEC-complex (6.1) and a corresponding complex of degrees of freedom (6.5). In the sequel, we adhere to the convention of stating each result accompanied by explicit estimates of the various constants and parameter ranges. We call a quantity uniformly bounded if it can be bounded in terms of the shape constant, the geometric ambient, and the polynomial degree of the finite element space.

7.1. Extension. Since $\Omega$ is a bounded weakly Lipschitz domain, we may apply Theorem 2.3 to fix a compact neighborhood $\mathcal{C} \Omega$ of $\partial \Omega$ in $\mathbb{R}^n$ and a bi-Lipschitz mapping

$$
\Psi : \partial \Omega \times [-1,1] \to \mathcal{C} \Omega
$$
such that \( \Psi(x, 0) = x \) for \( x \in \partial \Omega \), and such that
\[
\Psi(\partial \Omega \times [-1, 0)) = \mathcal{C} \Omega \cap \Omega, \quad \Psi(\partial \Omega \times (0, 1]) = \mathcal{C} \Omega \cap \overline{\Omega}.
\]

Additionally we write
\[
\text{Lemma 7.1.}
\]

for the interior collar part \( C^- \Omega \), the exterior collar part \( C^+ \Omega \), and the extended domain \( \Omega^e \), respectively. Eventually, we have a well-defined bi-Lipschitz mapping
\[
\text{Lemma 7.2.}
\]

from the outer collar part into the inner collar part, called \textit{collar reflection}. We define the extension operator using the pullback along the collar reflection. If \( \omega \in M\Lambda^k(\Omega) \) is a locally integrable \( k \)-form over \( \Omega \), then
\[
\text{Lemma 7.1.}
\]

is the locally integrable differential \( k \)-form constructed by extending \( \omega \) onto \( C^+ \Omega \) using the pullback along \( \mathcal{A} \). We first show that the linear mapping \( \mathcal{E}^k \) satisfies local estimates:
\[
\text{Lemma 7.1.}
\]

\( \mathcal{E}^k : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\Omega^e), \quad \omega \mapsto \mathcal{E}^k \omega. \)

Moreover, there exists \( C_{A,p} > 0 \), depending only on \( \mathcal{A} \) and \( p \), such that for \( 0 \leq s \leq t \leq 1 \) and \( G \subseteq \partial \Omega \) closed we have
\[
\text{Lemma 7.1.}
\]

Constants: we may assume that \( C_{A,q} \leq C_{A,p} \) for \( 1 \leq p \leq q \leq \infty \).

\( \text{Proof.} \) Let \( p \in [1, \infty) \), let \( G \subseteq \partial \Omega \) be closed, let \( 0 \leq s \leq t \leq 1 \), and let \( \omega \in L^p\Lambda^k(\Omega) \).

We apply Lemma 3.6 to find
\[
\text{Lemma 7.6.}
\]

Hence (7.4) holds for some \( C_{A,p} > 0 \). For \( G \times [s, t] = \partial \Omega \times [0, 1] \) we find
\[
\text{Lemma 7.6.}
\]

so \( \mathcal{E}^k \) is bounded from \( L^p\Lambda^k(\Omega) \) to \( L^p\Lambda^k(\Omega^e) \).

The local bound in the preceding lemma can be refined:
\[
\text{Lemma 7.2.}
\]

There exist \( \delta_0 > 0 \) and \( L_\Psi \geq 1 \), depending only on \( \Psi \), such that for all \( \delta \in [0, \delta_0) \), \( p \in [1, \infty] \), and all closed sets \( A \subseteq \Omega \) we have
\[
\text{Lemma 7.2.}
\]

\[\begin{align*}
\| \mathcal{E}^k \omega \|_{L^p\Lambda^k(B_{\delta}(A) \cap \Omega^e)} & \leq (1 + C_{A,p}) \| \omega \|_{L^p\Lambda^k(B_{L_\Psi \delta}(A) \cap \overline{\Omega})}, \quad \omega \in L^p\Lambda^k(\Omega).
\end{align*}\]
Proof. Let $\delta \geq 0$, $p \in [1, \infty]$, and let $A \subset \overline{\Omega}$ be closed. Then
\[
\| E^k \omega \|_{L^p A^k (B_\delta (A) \cap \Omega^c)} \leq \| \omega \|_{L^p A^k (B_\delta (A) \cap \Omega)} + \| E^k \omega \|_{L^p A^k (B_\delta (A) \cap \partial \Omega)}.
\]
We set $H^+ := B_\delta (A) \cap \partial \Omega$. There exists $H^- \subseteq \overline{\Omega}$ such that $H^- = A (H^+)$. By the definition of $E^k$ and Lemma 3.6 we have
\[
\| E^k \omega \|_{L^p A^k (H^+)} \leq \| D A^k \|_{L^\infty (\Omega^c)} \| D A^{-1} \|_{L^\infty (\partial \Omega)} \| \omega \|_{L^p A^k (H^-)}.
\]
Let $x \in B_\delta (A) \cap \partial \Omega$. There exists $z \in B_\delta (A) \cap \partial \Omega$ with $\| x - z \| \leq \delta$, since every $x \in A$ and $z \in \partial \Omega$ with $\| x - z \| \leq \delta$ are connected by a straight line segment of length at most $\delta$ which intersects $\partial \Omega$ at least once. Furthermore there exist $x_0 \in \partial \Omega$ and $t \in [0, 1]$ with $x = \Phi (x_0, t)$. It is easily seen that $\| A (x) \| \leq C_1 t$ for some constant $C_1$ that depends only on $\Psi$. Since $\Psi$ is a LIP embedding, we also know that $\| x_0 - z \|^2 + | t |^2 \leq C_2 \| x - z \|$ for some constant $C_2$ that depends only on $\Psi$. In combination we have $H^- \subseteq B_{(1 + C_1 C_2 \delta)} (A)$. This completes the proof. \qed

Corollary 7.3. Let $p \in [1, \infty]$ and let $\epsilon > 0$ be small enough. There exists $C_{E,p} > 0$ such that for $\omega \in L^p A^k (\Omega)$ and $F \in T$ we have
\[
\| E^k \omega \|_{L^p A^k (B_{(1+\epsilon \delta)} (F) \cap \Omega^c)} \leq C_{E,p} \| \omega \|_{L^p A^k (B_{(\epsilon \delta)} (F) \cap \Omega^c)}.
\]
Constants: we may assume $C_{E,p} := (1 + C_{A,p})$. It suffices that $\epsilon \delta < \delta_0$ for all $F \in T$.

Next, we show that $E^k$ commutes with the exterior derivative.

Lemma 7.4. Let $p, q \in [1, \infty]$. If $\omega \in L^p A^k (\Omega)$, then $E^k \omega \in L^p A^k (\Omega^c)$ and $E^k d \omega = d E^k \omega$.

Proof. It suffices to consider the case $p = q = 1$. Let $\omega \in L^{1,1} A^k (\Omega)$. Lemma 7.1 implies that $E^k \omega \in L^{1,1} A^k (\Omega^c)$ and $E^k d \omega \in L^{1,1} A^{k+1} (\Omega^c)$. To show that $E^k \omega \in L^{1,1} A^k (\Omega^c)$, it suffices to show that there exists a covering $(U_i)_{i \in N}$ of $\Omega^c$ by relatively open subsets $U_i \subseteq \Omega^c$ such that $E^k \omega_{| U_i} \in L^{1,1} A^k (U_i)$ and $E^k d \omega_{| U_i} = d E^k \omega_{| U_i}$.

From the definition of weakly Lipschitz domains we easily see that there exists a family $\{ (\theta_i)_{i \in N} \}$ of LIP embeddings $\theta_i : (-1,1)^n \rightarrow \partial \Omega$ whose images cover $\partial \Omega$. Consequently, the mappings $\phi_i : (-1,1)^n \rightarrow \Psi (\partial \Omega, (-1,1))$ defined by $\phi_i (\theta (x, t)) = \Psi (x, t)$ are a finite family of LIP embeddings whose images $U_i := \phi_i ((-1,1)^n)$ cover $\partial \Omega$. Together with $\Omega$ we thus have a finite covering of $\Omega^c$.

We recall that $E^k \omega_{| \Omega} \in L^{1,1} A^k (\Omega)$ and $E^k d \omega_{| \Omega} = d E^k \omega_{| \Omega}$. Next we define
\[
\omega_i := \phi_i^* (E^k \omega_{| U_i}), \quad \xi_i := \phi_i^* (E^{k+1} d \omega_{| U_i}).
\]
It remains to show that $E^k \omega_{| U_i} \in L^{1,1} A^k (U_i)$ and $E^k d \omega_{| U_i} = d E^k \omega_{| U_i}$, which is equivalent to $\omega_i \in L^{1,1} A^k ((-1,1)^n)$ and $d \omega_i = \xi_i$. To see this, we let $\mathcal{S} : (-1,1)^{n-1} \times (0,1) \rightarrow (-1,1)^{n-1} \times (-1,0)$ be the reflection by the $n$-th coordinate. It is evident that
\[
\omega_i ((-1,1)^{n-1} \times (0,1)) = \mathcal{S}^* \omega_i ((-1,1)^{n-1} \times (-1,0))
\]
and
\[
\xi_i ((-1,1)^{n-1} \times (0,1)) = \mathcal{S}^* \xi_i ((-1,1)^{n-1} \times (-1,0)) = \mathcal{S}^* d \omega_i ((-1,1)^{n-1} \times (-1,0))
\]
By Lemma 3.1 there exists a sequence $\{ \omega_i^\delta \}_{\delta > 0}$ of smooth differential $k$-forms that converge to $\omega_i$ over $(-1,1)^{n-1} \times (-1,0)$ in the $L^{1,1}$ norm for $\delta \rightarrow 0$. We let each $\omega_i^\delta$ be extended to $(-1,1)^{n-1} \times (0,1)$ by pullback along $\mathcal{S}$. With this extension,
\( \omega^\delta \) converges to \( \omega^0 \) in \( L^1 \Lambda^k ((-1,1)^n) \) and \( d\omega^\delta \) converges to \( \xi_i \) in \( L^1 \Lambda^k ((-1,1)^n) \) for \( \delta \to 0 \). Hence \( \omega_i \in L^1 \Lambda^k ((-1,1)^n) \) with \( d\omega_i = \xi_i \).

The proof is complete. \( \square \)

7.2. Mesh size functions and Mollification. The next step is constructing a commuting mollification operator. We let the mollification radius vary over the domain, so the operator satisfies local estimates uniformly for shape-regular families of triangulations. A key component is a smooth function that indicates the local domain, so the operator satisfies local estimates uniformly for shape-regular families commuting mollification operator. We let the mollification radius vary over the mesh size.

Recall the standard mollifier. This is a smooth function

\[
\mu : \mathbb{R}^n \to [0,1], \quad y \mapsto \begin{cases} C_\mu \exp \left( -\frac{1}{|y|^2} \right) & \text{if } |y| \leq 1, \\ 0 & \text{if } |y| > 1, \end{cases}
\]

with compact support, where \( C_\mu > 0 \) is chosen such that \( \mu \) has unit integral. We set \( \mu_r(y) := r^{-n} \mu(y/r) \) for \( y \in \mathbb{R}^n \) and \( r > 0 \).

First we prove the existence of a mesh size function \( H \) with Lipschitz regularity, and then the existence of a mesh size function \( h \) that is smooth.

**Lemma 7.5.** There exists \( L_\Omega > 0 \), only depending on \( \Omega \), and a Lipschitz continuous function \( H : \overline{\Omega} \to \mathbb{R}_0^+ \) such that

\[
(7.6) \quad \forall F \in \mathcal{T} : C_{\text{mesh}}^{-1} h_F \leq H_F \leq C_{\text{mesh}} h_F,
\]

\[
(7.7) \quad \text{Lip}(H, \overline{\Omega}) \leq C_{\text{mesh}} L_\Omega.
\]

**Proof.** Let the function \( H : \overline{\Omega} \to \mathbb{R}_0^+ \) be defined as follows. If \( V \in \mathcal{T}^0 \), then we set \( H(V) = h_V \). We then extend \( H \) to each \( T \in \mathcal{T} \) by affine interpolation between the vertices of \( T \). With this definition, \( H \) is continuous, and \( (7.7) \) follows from \( (1.2) \). It remains to prove \( (7.8) \). Obviously, \( \text{Lip}(H, T) \leq C_{\text{mesh}} \) for \( T \in \mathcal{T}^0 \).

Since \( \Omega \) is a bounded weakly Lipschitz domain, there exists be a finite family \( (U_i)_{1 \leq i \leq N} \) of relatively open sets \( U_i \subseteq \overline{\Omega} \) such that such that the union of all \( U_i \) equals \( \overline{\Omega} \), and such that there exist \( \phi_i : \overline{U_i} \to [-1,1]^n \) bi-Lipschitz for each \( 1 \leq i \leq N \). By Lebesgue’s number lemma, we may pick \( \gamma > 0 \) so small that for each \( x \in \overline{\Omega} \) there exists \( 1 \leq i \leq N \) such that \( B_\gamma(x) \cap \overline{\Omega} \subseteq U_i \).

First assume that \( x, y \in \Omega \) with \( 0 < \|x - y\| \leq \gamma \). Then there exists \( 1 \leq i \leq N \) with \( x, y \in U_i \). For \( M \in \mathbb{N} \), consider a partition of the line segment in \([-1,1]^n\) from \( \phi(x) \) to \( \phi(y) \) into \( M \) subsegments of equal length with points \( \phi_i(x) = z_0, z_1, \ldots, z_M = \phi_i(x) \). Let \( x_m := \phi_i^{-1}(z_m) \in U_i \). For \( M \) large enough, the straight line segment between \( x_{m-1} \) and \( x_m \) is contained in \( U_i \) for all \( 1 \leq m \leq M \).

After a further subpartitioning, not necessarily equidistant, we may assume to have a sequence \( x = w_0, \ldots, w_M = y \) for some \( M' \in \mathbb{N} \) such that for all \( 1 \leq m \leq M' \) the points \( w_{m-1} \) and \( w_m \) are connected by a straight line segment in \( U_i \) and such
that there exists $F_m \in \mathcal{T}$ with $w_{m-1}, w_m \in F_m$. We observe

$$|H(y) - H(x)| \leq \sum_{m=1}^{M'} |H(w_m) - H(w_{m-1})|$$

$$\leq C_{mesh} \sum_{m=1}^{M'} \|w_m - w_{m-1}\|$$

$$= C_{mesh} \sum_{m=1}^{M} \|x_m - x_{m-1}\|$$

$$\leq C_{mesh} \text{Lip}(\phi_i^{-1}) \sum_{m=1}^{M} \|\phi_i(x_m) - \phi_i(x_{m-1})\|$$

$$\leq C_{mesh} \text{Lip}(\phi_i^{-1}) : \|\phi_i(y) - \phi_i(x)\|$$

$$\leq C_{mesh} \text{Lip}(\phi_i^{-1}) \text{Lip}(\phi_i) : \|y - x\|.$$  

If we instead assume that $x, y \in \Omega$ with $\|x - y\| \geq \gamma$, then

$$|H(y) - H(x)| \leq \gamma^{-1} \text{diam}(\Omega) \cdot |H(x) - H(y)| \leq \gamma^{-1} \text{diam}(\Omega) \cdot C_{mesh} \cdot \|y - x\|,$$

since $\gamma < \text{diam}(\Omega)$. Hence $\text{Lip}(H, \Omega) \leq C_{mesh} L_{\Omega}$ with

$$L_{\Omega} := \sup \{ \gamma^{-1} \text{diam}(\Omega), \text{Lip}(\phi_1^{-1}) \text{Lip}(\phi_1), \ldots, \text{Lip}(\phi_N^{-1}) \text{Lip}(\phi_N) \}.$$  

Thus $\text{Lip}(H, \overline{\Omega}) \leq C_{mesh} L_{\Omega}$ because any Lipschitz continuous function is Lipschitz continuous over the closure of its domain with the same Lipschitz constant. □

**Remark 7.6.** The preceding result was used before in literature, but estimating $\text{Lip}(H)$ did not receive much attention. An interesting observation is that $\text{Lip}(H)$ is the product of $C_{mesh}$, which only depends on the shape of the simplices, and $L_{\Omega}$, which depends only the geometry. Conceptually, $L_{\Omega}$ compares the inner path metric of $\overline{\Omega}$ to the Euclidean metric over $\overline{\Omega}$. The equivalence of these two metrics is non-trivial in general, but holds true for bounded weakly Lipschitz domains.

**Lemma 7.7.** There exist a smooth function $h : \Omega^e \rightarrow \mathbb{R}^+_0$ and uniformly bounded constants $C_h > 0$ and $L_h > 0$ such that

$$h \in T : \forall x \in F : C_h^{-1} h_F \leq h(x) \leq C_h h_F,$$

$$\text{Lip}(h, \overline{\Omega}) \leq L_h.$$  

**Constants:** we may choose $C_h = C_{mesh}^2$ and $L_h = (1 + \text{Lip}(A)) C_{mesh} L_{\Omega}$.

**Proof.** Let $H : \overline{\Omega} \rightarrow \mathbb{R}^+_0$ be as in Lemma 7.5. We observe that $E^0 H$ is Lipschitz with

$$\text{Lip} \left( E^0 H, \Omega^e \right) \leq (1 + \text{Lip}(A)) \text{Lip} \left( H, \overline{\Omega} \right).$$

Let $r > 0$ and define $h := \mu_r * E^0 H$ be the convolution of $E^0 H$ with the scaled mollifier $\mu_r$. For $r$ small enough it is easily verified that

$$\text{Lip} \left( h, \overline{\Omega} \right) \leq \text{Lip} \left( E^0 H, \Omega^e \right).$$

By standard results, $h$ is smooth. Moreover we have

$$h(x) = \int_{B_r(x) \cap \Omega} \mu_r(y) E^0 H(x + y) \, dy + \int_{B_r(x) \cap \partial^+ \Omega} \mu_r(y) E^0 H(x + y) \, dy.$$
If \( z \in B_r(x) \cap \overline{\Omega} \), then \( A(z) \in \Omega \) has at most distance \( r + \text{Lip}(A)r \) from \( x \). We conclude that \( h(x) \) lies in the convex hull of values of \( h \) over \( B_{r+\text{Lip}(A)r}(x) \cap \overline{\Omega} \). Let \( r > 0 \) so small that \( r + \text{Lip}(A)r < h_{\min, \epsilon_h} \), where \( h_{\min} \) is the minimal diameter of the simplices in \( \mathcal{T} \) and \( \epsilon_h \) is as in \((4.4)\). Then \( B_{r+\text{Lip}(A)r}(x) \cap \overline{\Omega} \subseteq \mathcal{T}(T) \), and the desired statement follows.

We use the mesh size function to define a family of LIP embeddings of \( \Omega \) into \( \Omega^\epsilon \). For \( \epsilon > 0 \) we introduce

\[
(7.11) \quad \Phi_{\epsilon_h} : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n, \quad \Phi_{\epsilon_h}(x,y) = x + \epsilon_h(x)y.
\]

We abbreviate \( \Phi_{\epsilon_h,y} = \Phi_{\epsilon_h}(x,y) \). Note that \( \Phi_{\epsilon_h} \) is smooth:

\[
(7.12) \quad D_x \Phi_{\epsilon_h}(x,y) = 1d + \epsilon \cdot y \otimes dh_x.
\]

It is easy to see that for \( \epsilon > 0 \) small enough, \( \Phi_{\epsilon_h,y} \) is a LIP embedding with \( \Phi_{\epsilon_h}(\Omega, B_1) \subseteq \Omega^\epsilon \) for any \( y \in B_1(0) \). We now define the mollification operator \( R_{\epsilon_h}^k \).

For \( \omega \in L^1(\Lambda_k(\Omega^\epsilon)) \) we let

\[
(7.13) \quad R_{\epsilon_h}^k \omega |_{x} := \int_{\mathbb{R}^n} \mu(y)(\Phi_{\epsilon_h,y}^k \omega) |_{x} dy, \quad x \in \Omega.
\]

We first observe that \( R_{\epsilon_h}^k \) is a bounded linear operator from \( L^p(\Lambda_k(\Omega^\epsilon)) \) into \( CA_k(\Omega) \). For the mollification operator to yield continuous differential forms, it is crucial that \( \Phi_{\epsilon_h} \) has continuous first derivatives.

**Lemma 7.8.** Let \( \epsilon > 0 \) be small enough. The operator

\[
R_{\epsilon_h}^k : L^p(\Lambda_k(\Omega^\epsilon)) \to CA_k(\Omega), \quad p \in [1, \infty],
\]

is well-defined and linear, and we have

\[
(7.14) \quad \| R_{\epsilon_h}^k \omega \|_{CA_k(T)} \leq (1 + \epsilon L_h)^k \text{vol}^n(B_1(0)) C_{\epsilon_h}^{\frac{n}{2}} \epsilon^{-\frac{n}{2} k} \| \omega \|_{L^p(\Lambda_k(B_{\epsilon_h, \eta_T}(T)))}
\]

for every \( p \in [1, \infty] \), \( T \in \mathcal{T}^n \) and \( \omega \in L^p(\Lambda_k(\Omega^\epsilon)) \). Moreover, for \( \omega \in L^p(\Lambda_k(\Omega^\epsilon)) \) with \( p, q \in [1, \infty] \) we have

\[
R_{\epsilon_h}^{k+1} d\omega \in CA^{k+1}(\Omega), \quad dR_{\epsilon_h}^k \omega = R_{\epsilon_h}^{k+1} d\omega.
\]

**Constants:** it suffices that \( C_h \epsilon < \epsilon_h \).

**Proof.** Let \( p \in [1, \infty] \) and let \( \omega \in L^p(\Lambda_k(\Omega^\epsilon)) \). If \( \epsilon < \epsilon_h C_{\epsilon_h}^{-1} \) and \( \epsilon < 1/2L_h \), then \( \Phi_{\epsilon_h,y} \) is a LIP embedding from \( \Omega \) to \( \Omega^\epsilon \) for every \( y \in B_1(0) \). Hence \( \mu(y)(\Phi_{\epsilon_h,y}^k \omega) |_{x} \) is measurable in \( y \) for every \( x \in \overline{\Omega} \), and the integral \((7.13)\) is well-defined.

We prove the estimate \((7.14)\) pointwise. Let \( x \in T \) for some \( T \in \mathcal{T}^n \). Similar as in the proof of Lemma 3.0 we observe

\[
\| R_{\epsilon_h}^k \omega \|_x \leq \int_{\mathbb{R}^n} \mu(y) \| D_x \Phi_{\epsilon_h,y}^k \|_x \| \omega \|_{\Phi_{\epsilon_h,y}(x)}.
\]

Since integrand vanishes for \( y \notin B_1(0) \), we may use that

\[
\| D_x \Phi_{\epsilon_h,y}^k \|_x \leq \text{Lip}(\Phi_{\epsilon_h,y}, \overline{\Omega}) \leq (1 + \epsilon L_h)
\]

and, via a substitution of variables and Hölder’s inequality, use

\[
\int_{\mathbb{R}^n} | \mu(y) | \cdot | \omega |_{x+\epsilon_h(x)y} dy \leq \text{vol}^n(B_1(0)) \cdot \epsilon^{-\frac{n}{2} k} \| \omega \|_{L^p(B_{\epsilon_h, \eta_T}(x))}.
\]

Both estimates in combination deliver \((7.14)\).
Now we show that $R_{ch}^k \omega$ is continuous over $\Omega$. Let $\omega$ be written as in (3.1), and $x \in \Omega$. Then

$$R_{ch}^k \omega |_x = \int_{R^n} \mu(y) (\Phi_{ch,y}^* \omega) |_x \, dy$$

$$= \sum_{\sigma \in \Sigma(k,n)} \int_{R^n} \mu(y) \omega_{\sigma} (x + ch(x)y) (\Phi_{ch,y}^* dx^\sigma) |_x \, dy$$

$$= \epsilon^{-n} h(x)^{-n} \sum_{\sigma \in \Sigma(k,n)} \int_{R^n} \mu \left( \epsilon^{-1} h(x)^{-1} (y - x) \right) \omega_{\sigma} (y) \cdot W_{x,y}^\sigma \, dy,$$

where we have written

$$W_{x,y}^\sigma := (\Phi_{ch,\epsilon^{-1} h(x)^{-1} (y - x)}^* dx^\sigma) |_x.$$

We recall that $h$ and $\Phi$ are smooth, that $\omega_{\sigma} \in L^1(\Omega)$, and that $\Omega$ is compact. In particular, the derivatives of $\Phi$ are continuous. The desired continuity is now a simple consequence of the dominated convergence theorem.

It remains to show the commutativity property. Let $\eta \in C_c^\infty \Lambda^{n-k-1}(\Omega)$. By Fubini’s theorem we have

$$\int_{\Omega} R_{ch}^k \omega \wedge d\eta = \int_{\Omega} \int_{R^n} \mu(y) \Phi_{ch,y}^* \omega \wedge d\eta \, dy = \int_{R^n} \mu(y) \int_{\Omega} \Phi_{ch,y}^* \omega \wedge d\eta \, dy,$$

$$\int_{\Omega} R_{ch}^k d\omega \wedge \eta = \int_{\Omega} \int_{R^n} \mu(y) \Phi_{ch,y}^* d\omega \wedge \eta \, dy = \int_{R^n} \mu(y) \int_{\Omega} \Phi_{ch,y}^* d\omega \wedge \eta \, dy.$$

When $\epsilon > 0$ is small enough, then $\Phi_{ch,y} : \Omega \to \Omega^e$ is a LIP embedding for every $y \in B_1(0)$. Hence by Lemma 3.5 we find

$$\int_{\Omega} \Phi_{ch,y}^* \omega \wedge d\eta = (-1)^{k+1} \int_{\Omega} d\Phi_{ch,y}^* \omega \wedge \eta = (-1)^{k+1} \int_{\Omega} \Phi_{ch,y}^* d\omega \wedge \eta.$$

By definition, $dR_{ch}^k \omega = R_{ch}^{k+1} d\omega$. The proof is complete. $\square$

**Remark 7.9.** Our Lemma 7.8 is analogous to prior findings in literature. Let us briefly review the situation. The smoothed projection constructed in 1 applies to quasi-uniform families of triangulations. A family of triangulations is called quasi-uniform if for each triangulation $T$ in that family we have

$$\forall T \in T^n : h_T^p \leq C_{mesh} |T|,$$

$$\forall T, S \in T : h_T \leq C_{mesh} h_S,$$

with a common constant $C_{mesh} > 0$. In that case, a classical mollification operator can be used instead of our $R_{ch}^k$. That result was expanded in 9 to include shape-uniform families of triangulations, which means that the conditions (4.1) and (4.2) are satisfied for all triangulations $T$ in that family with a common constant $C_{mesh}$. The Lipschitz continuous mesh size function of Lemma 7.5 was introduced first in 9. But simple examples show that, contrarily to the statement in 9, a regularization operator with that mesh size function does not yield a continuous differential form. This is due to the differential of the mesh size function being discontinuous in general. As a remedy, we explicitly construct a mesh size function that is smooth.

The Lipschitz continuous mesh size function in Lemma 7.5 is the limit of the smoothed mesh size function in Lemma 7.7 for decreasing mollification radius. It is natural to ask how this limit process is reflected in the regularization operator. It
is easily seen that the gradient of the original mesh size function features tangential continuity. Using this additional property, one can show that the regularization operator of [9] does yield differential forms that are piecewise continuous with respect to the triangulation and that are single-valued along simplex boundaries. Consequently, the regularized differential form, though not continuous, still has well-defined degrees of freedom, and the finite element interpolator can be applied as intended. We emphasize that the main result of [9] remains unchanged.

7.3. Smoothed Interpolation and Smoothed Projection. Combining the extension operator, the mollification operator, and the finite element interpolator, we provide the smoothed interpolator

\begin{equation}
Q^k: L^p\Lambda^k(\Omega) \to L^p\Lambda^k(T), \quad \omega \mapsto I^kR^k_{ch}E^k\omega, \quad p \in [1, \infty].
\end{equation}

We show that \(Q^k\) satisfies local bounds and commutes with the exterior derivative:

**Theorem 7.10.** Let \(\epsilon > 0\) be small enough. For \(p \in [1, \infty]\), the operator \(Q^k: L^p\Lambda^k(\Omega) \to L^p\Lambda^k(T)\) is linear and bounded, and there exists uniformly bounded \(C_{Q,p} > 0\) such that

\begin{align}
\|Q^k\omega\|_{L^p\Lambda^k(T)} &\leq C_{Q,p}e^{-\frac{\epsilon}{h}}\|\omega\|_{L^p\Lambda^k(\partial T)}, \quad \omega \in L^p\Lambda^k(\Omega), \quad T \in \mathcal{T}^n, \\
\|Q^k\omega\|_{L^p\Lambda^k(\Omega)} &\leq C_NC_{Q,p}e^{-\frac{\epsilon}{h}}\|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in L^p\Lambda^k(\Omega).
\end{align}

Moreover, we have

\begin{equation}
\text{d}Q^k\omega = Q^k\text{d}\omega, \quad \omega \in L^{p,q}\Lambda^k(\Omega), \quad p, q \in [1, \infty].
\end{equation}

**Proof.** Let \(\omega \in L^p\Lambda^k(\Omega)\) and let \(T \in \mathcal{T}^n\). Then

\begin{align}
\|Q^k\omega\|_{L^p\Lambda^k(T)} &\leq \|I^kR^k_{ch}E^k\omega\|_{L^p\Lambda^k(T)} \\
&\leq |T|^\frac{1}{p}\|I^kR^k_{ch}E^k\omega\|_{L^\infty\Lambda^k(T)} \leq h^\frac{1}{p}\|I^kR^k_{ch}E^k\omega\|_{L^\infty\Lambda^k(T)}.
\end{align}

Estimate (6.10) gives

\begin{align}
\|I^kR^k_{ch}E^k\omega\|_{L^\infty\Lambda^k(T)} &\leq C_Mh^{-k}\|A^*_T I^kR^k_{ch}E^k\omega\|_{L^\infty\Lambda^k(\Delta^*o)} \\
&\leq C_T^kC_Mh^{-k}\|A^*_T I^kR^k_{ch}E^k\omega\|_{L^\infty\Lambda^k(\Delta^*o)} \\
&\leq C_T h^kC_M\|R^k_{ch}E^k\omega\|_{C^0\Lambda^k(T)}.
\end{align}

Assuming that \(\epsilon > 0\) is small enough, we apply Lemma 7.8

\begin{equation}
\|R^k_{ch}E^k\omega\|_{C^0\Lambda^k(T)} \leq (1 + \epsilon L_h)^k\text{vol}^n(B_1(0))\epsilon^{-\frac{1}{p}}h^\frac{1}{p}C_n^p\|E^k\omega\|_{L^p\Lambda^k(B_{\epsilon h}(T))},
\end{equation}

and find with (4.4) and Corollary 7.3 that

\begin{align}
\|E^k\omega\|_{L^p\Lambda^k(B_{\epsilon h}(T))} &= \|E^k\omega\|_{L^p\Lambda^k(B_{\epsilon h}(T)\cap \Omega)} \\
&\leq C_{E,p}\|\omega\|_{L^p\Lambda^k(B_{\epsilon h}(T)\cap \Omega)} \\
&\leq C_{E,p}\|\omega\|_{L^p\Lambda^k(B_{\epsilon h}(T)} \leq C_{E,p}\|\omega\|_{L^p\Lambda^k(T)}.
\end{align}
Thus the local bound (7.18) follows. The global bound (7.19) is obtained via
\[
\|Q^k \omega\|_{L^p(A_k^k)_{\lambda}} = \sum_{T \in \mathcal{T}^n} \|Q^k \omega\|_{L^p(A_k^k(T))} \leq C_{Q,p} \sum_{T \in \mathcal{T}^n} \|\omega\|_{L^p(A_k^k(T))} \\
\leq C_{Q,p}^k C_N \sum_{T \in \mathcal{T}^n} \|\omega\|_{L^p(A_k^k(T))} \leq C_{Q,p}^k C_N \|\omega\|_{L^p(A_k^k)}
\]
for \( p \in [1, \infty) \), and for \( p = \infty \) similarly. Finally, (7.20) follows from Theorem 7.14, Theorem 7.3, and our assumptions on \( I^k \). The proof is complete. \( \square \)

The smoothed interpolator \( Q^k \) is local and satisfies uniform bounds. Although \( Q^k \) generally does not reduce to the identity over \( A_k^k(T) \), we can show that, for \( \epsilon > 0 \) small enough, it is close to the identity and satisfies a local error estimate.

**Theorem 7.11.** For \( \epsilon > 0 \) small enough, there exists uniformly bounded \( C_{e,p} > 0 \) for every \( p \in [1, \infty) \) such that
\[
\|\omega - Q^k \omega\|_{L^p(A_k^k(T))} \leq \epsilon C_{e,p} \|\omega\|_{L^p(A_k^k(T))}, \quad \omega \in A_k^k(T), \quad T \in \mathcal{T}^n.
\]

**Constants:** It suffices that \( \epsilon > 0 \) is small enough such that Theorem 7.10 is applicable and \( L\Phi M C h_{e} < \epsilon_{h} \). We may choose
\[
C_{e,p} = C^{2k+1} M C^{2k+2+\eta} C_t C_h (1 + c_{M} C_{M} L_{h} \epsilon)^k (1 + C_{H}) C_{E,\infty} C_{b,p,k}.
\]

**Proof:** We prove the statement by a series of inequalities. Let \( \omega \in A_k^k(T) \) and let \( T \in \mathcal{T}^n \). Then
\[
\|\omega - Q^k \omega\|_{L^p(A_k^k(T))} = \|I^k (E^k \omega - R_{ch} E^k \omega)\|_{L^p(A_k^k(T))} \\
\leq h_{T}^{\frac{1}{2}} \|E^k \omega - R_{ch} E^k \omega\|_{L^\infty(A_k^k(T))} \\
\leq C_{M} h_{T}^{\frac{1}{2}} \|A_{T} I^k (E^k \omega - R_{ch} E^k \omega)\|_{L^\infty(A_{T-1}^k)}
\]
as follows from (3.9). By (6.15) and (5.11), we have
\[
\|A_{T} I^k (E^k \omega - R_{ch} E^k \omega)\|_{L^\infty(A_{T-1}^k)} \leq C_{I} \sup_{\substack{F \in \Delta(T) \setminus \mathcal{C}_k \setminus S \\ S \subseteq \mathcal{C}_k}} |A_{T_F}^{-1} S|_{k}^{-1} \int_{S} E^k \omega - R_{ch} E^k \omega.
\]

We need to bound the last expression. Fix \( F \in \Delta(T) \) and \( S \in \mathcal{C}_k^F \). We see that
\[
\int_{S} E^k \omega - R_{ch} E^k \omega = \int_{S} \int_{\mathbb{R}^n} \mu(y) (E^k \omega - \Phi_{ch,y}^* E^k \omega) \, dy.
\]

By assumption on \( S \), both integrals are taken in the sense of measure theory, and we may apply Fubini’s theorem:
\[
\int_{S} \int_{\mathbb{R}^n} \mu(y) \Phi_{ch,y}^* E^k \omega \, dy = \int_{\mathbb{R}^n} \mu(y) \int_{S} A_{T_F}^{-1} S - A_{F_T}^{-1} \Phi_{ch,y} \omega \, dy.
\]

Using these observations and (5.11) again, we have
\[
\int_{\mathbb{R}^n} \mu(y) \int_{S} E^k \omega - \Phi_{ch,y}^* E^k \omega \, dy = \int_{\mathbb{R}^n} \mu(y) \int_{A_{T_F}^{-1} S - A_{F_T}^{-1} \Phi_{ch,y} \omega} A_{T} E^k \omega \, dy.
\]
With \([5.3]\), it follows that
\[
\int_{\mathbb{R}^n} \mu(y) \int_{A^*_T S - A^*_T \Phi_{e,y,0} S} A^*_T E^k \omega \, dy \, ds \leq \sup_{y \in B_1(0)} \| A^*_T S - A^*_T \Phi_{e,y,0} S \|_{k,b} \cdot \| A^*_T E^k \omega \|_{L^\infty A^k(B_{C_M C_M^\epsilon}(\Delta^n))}. \]

We need to bound this product. On the one hand, we observe that \( \epsilon > 0 \) be small enough and apply Corollary \( [7.3] \) to obtain the interpolation error estimate over simplices \( T \). By Lemma \( [5.2] \), the authors of \( [7.4] \) gives
\[
\| \partial A^*_T S \|_{k-1} \leq C_\beta \| A^*_T S \|_k.
\]

The inverse inequality \( [6.13] \) gives
\[
\| \partial A^*_T S \|_{k-1} \leq C_\beta \| A^*_T S \|_k.
\]

On the other hand, we observe
\[
\| A^*_T E^k \omega \|_{L^\infty A^k(B_{C_M C_M^\epsilon}(\Delta^n))} \leq C_E, k \leq C^k_M \| A^*_T \omega \|_{L^\infty A^k(A^*_T S)}.
\]

To see this, we let \( \epsilon > 0 \) be small enough and apply Corollary \( [7.3] \) to obtain
\[
\| A^*_T E^k \omega \|_{L^\infty A^k(B_{C_M C_M^\epsilon}(\Delta^n))} \leq C_E, k \leq C^k_M \| E^k \omega \|_{L^\infty A^k(B_{C_M C_M^\epsilon}(\Delta^n))}.
\]

We treat \( A^*_T E^k \omega \) similarly. The inverse inequality \( [6.13] \) gives
\[
\| A_T^* \omega \|_{L^\infty A^k(A^*_T S)} \leq C_\beta, k \| A_T^* \omega \|_{L^p A^k(A^*_T S)}.
\]

In combination, it follows that
\[
\| A^*_T I^k (\omega - R^{h^k} E^k \omega) \|_{L^\infty A^k(\Delta^n)} \leq C_T C_M (1 + C_M L_k) \| A_T^* \omega \|_{L^p A^k(T)}.
\]

We finally recall that
\[
\| A^*_T \omega \|_{L^p A^k(A^*_T S)} \leq C_M^\beta C_M^\beta \| \omega \|_{L^p A^k(T)}.
\]

This completes the proof. \( \Box \)

**Remark 7.12.** Our Theorem \( [7.11] \) resembles Lemma 5.5 in \( [1] \) and Lemma 4.2 in \( [9] \). Let us briefly motivate why we use a different method of proof. In order to obtain the interpolation error estimate over simplices \( T \in \mathcal{T} \), the authors of the aforementioned references suppose that finite element differential forms are piecewise Lipschitz near \( T \). This holds true if \( T \) is an interior simplex but not if \( T \) touches the boundary of \( \Omega \), and it is not clear how their method applies for such \( T \). The reason is that their extension operator, like ours, involves a pullback
along a bi-Lipschitz mapping, so the extended finite element differential form is not necessarily Lipschitz continuous anywhere outside of \( \Omega \). The extended differential form, however, is still a flat form, and this motivates our utilization of geometric measure theory to prove the desired estimate for the interpolation error.

For strongly Lipschitz domains, Lipschitz collars with stronger regularity may provide an alternative remedy, but we do explore this idea further in this article.

We are now in a position to prove the main result of this article. For \( \epsilon > 0 \) small enough, we can correct the error of the smoothed interpolation over the finite element space. The resulting smoothed projection is, however, non-local.

**Theorem 7.13.** Let \( \epsilon > 0 \) be small enough. There exists a bounded linear operator

\[
\pi^k_\epsilon : L^p\Lambda^k(\Omega) \to L^p\Lambda^k(\mathcal{T}), \quad p \in [1, \infty],
\]

such that

\[
\pi^k_\epsilon \omega = \omega, \quad \omega \in \Lambda^k(\mathcal{T}),
\]

such that

\[
d\pi^k_\epsilon \omega = \pi^k_\epsilon d\omega, \quad \omega \in L^{p,q}\Lambda^k(\Omega), \quad p, q \in [1, \infty],
\]

and such that for all \( p \in [1, \infty] \) there exist uniformly bounded \( C_{\pi,p} > 0 \) with

\[
\|\pi^k_\epsilon \omega\|_{L^p\Lambda^k(\mathcal{T})} \leq C_{\pi,p} \epsilon^{\frac{-k}{p-q}} \|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in L^p\Lambda^k(\Omega).
\]

Constants: it suffices that \( \epsilon > 0 \) is so small that Theorem 7.10 and Theorem 7.11 apply, and that \( C_{e,p} \epsilon < 2 \). We may assume \( C_{\pi,p} = 2C_{Q,p}C_{N} \).

**Proof.** If \( \epsilon > 0 \) is small enough and \( p \in [1, \infty] \), then Theorem 7.11 implies that

\[
\|\omega - Q^k_\epsilon \omega\|_{L^p\Lambda^k(\Omega)} \leq \frac{1}{2} \|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in \Lambda^k(\mathcal{T}).
\]

By standard results, the operator \( Q^k_\epsilon : L^p\Lambda^k(\mathcal{T}) \to L^p\Lambda^k(\mathcal{T}) \) is invertible. Let \( J^k_\epsilon : L^p\Lambda^k(\mathcal{T}) \to L^p\Lambda^k(\mathcal{T}) \) be its inverse. \( J^k_\epsilon \) does not depend on \( p \), since \( Q^k_\epsilon \) does not depend on \( p \). The construction of \( J^k_\epsilon \) via a Neumann series reveals that

\[
\|J^k_\epsilon \omega\|_{L^p\Lambda^k(\Omega)} \leq 2 \|\omega\|_{L^p\Lambda^k(\Omega)}, \quad \omega \in \Lambda^k(\mathcal{T}).
\]

So \( J^k_\epsilon \) is bounded. Moreover, \( J^k_\epsilon \) commutes with the exterior derivative because

\[
dJ^k_\epsilon \omega = J^k_\epsilon Q^k_\epsilon dJ^k_\epsilon \omega = J^k_\epsilon dQ^k_\epsilon J^k_\epsilon \omega = J^k_\epsilon d\omega, \quad \omega \in \Lambda^k(\mathcal{T}).
\]

The theorem follows with \( \pi^k_\epsilon := J^k_\epsilon Q^k_\epsilon \). \( \square \)

**Remark 7.14.** Several quantities in this section depend on a Lebesgue exponent \( p \in [1, \infty] \), but it suffices to consider only the case \( p = 1 \). We carefully observe that

\[
C_{A,p} \leq C_{A,1}, \quad C_{E,p} \leq C_{E,1}, \quad C_{Q,p} \leq C_{Q,1}, \quad C_{e,p} \leq C_{e,1}, \quad C_{\pi,p} \leq C_{\pi,1},
\]

for all \( p \in [1, \infty] \). Hence a sufficiently small choice of \( \epsilon > 0 \) is sufficient to enable Theorem 7.13 for all \( p \in [1, \infty] \) simultaneously.

**Remark 7.15.** Throughout this section, we have provided explicit formulas for the admissible ranges of \( \epsilon \) and the various constants. With the exception of \( C_{A,p} \) and \( L_\Omega \), the quantities in those formulas depend only on the ambient dimension, the polynomial degree, and the mesh regularity. If bounds for \( C_{A,p} \) and \( L_\Omega \) are known, then all constants in this section are effectively computable.
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