Supersymmetric Harmonic Maps into Lie Groups

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Abstract

We look at the supersymmetric generalization of harmonic maps into Lie
groups, known to physicists as the chiral model. Explicit solutions to the
equations are found and examined using Backlund transformations.

I. INTRODUCTION

A. Motivation

Harmonic maps into Lie groups have been of interest to physicists since the 1970’s under
the name of chiral models (we use the terms interchangeably) and studied them as toy
models for gauge theories. The concept of a harmonic map generalises the concept of a
geodesic.

The harmonic maps in this paper are from a simply connected 2-dimensional Riemannian
domain to a Lie group and are therefore two dimensional analogues of geodesics.

Harmonic maps in two dimensions are special because they have the property of being
an ”integrable system”. This term is used loosely in the literature, and no precise definition
exists. Various fundamental properties of an integrable system are recognised, but normally
it is a differential equation, with a large symmetry group, which is solvable by algebraic
means. We will assume for our purposes the existence of a Lax pair to be sufficient.

The chiral model is an example of a relativistically invariant integrable non-linear p.d.e.,
and as such is important to physicists as a physical model. It is related to the chiral
symmetries which feature in particle physics, such as the study of strong interactions at low
energies.

Most of the known integrable theories are two dimensional. The classic examples are
the Korteweg-de Vries and Sine-Gordon models, and the integrability problem for two-
dimensional field theories has been studied systematically using Lax equations and the theory
of affine Lie algebras.

Harmonic maps in two dimensions have been studied in the past ten years using a
parametrised Lax equation. [1].

In our analysis, we draw heavily on the uniton solutions constructed using a parametrised
Lax pair by Uhlenbeck in 1989. She showed that harmonic maps into a unitary group can
be factorised into a product of these simple maps. Harmonic maps into unitary groups can
therefore be characterised by a certain number, the number of these factors, known as the uniton number.

The chiral model has a relationship to several different non-linear field theories, such as the non-linear $\sigma$ model. They are members of a class of models constructed on special coset spaces or homogeneous spaces, the symmetric spaces.

Supersymmetric field theories are the most promising candidates to extend the standard model of strong and electroweak interactions and were first studied in the context of string theory. The question then arises as to what are the analogous results for the supersymmetric chiral model.

B. Outline

The outline of this paper is as follows: In section II, we define harmonic maps, in sections III and IV we describe the connections between harmonic maps and other integrable systems, and in sections V through VII we study supersymmetric harmonic maps.

In section II, we give a very brief introduction to the theory of harmonic maps. We look at the particular case of harmonic maps into Lie groups, and review fundamental results of Uhlenbeck, including the uniton solutions.

In section III, we begin our development of a theory of supersymmetric harmonic maps or superharmonic maps with a brief review of the formalism of $N=1$ supergeometry. We present our definition of the supercurvature and we prove that the definition permits the construction of unique super Lax pair solutions and extended solutions. In section IV, we find the supersymmetric analogue of Uhlenbeck’s uniton solutions and write down explicit formulae for these special simple superharmonic maps into unitary supergroups. We conclude in section V by developing Backlund transformations between these maps as a tool to help analyse them. In doing so, we construct a representation on extended superharmonic maps and use this write down some example Backlund transformations for the superuniton case, highlighting some unique features.

II. HARMONIC MAPS

The most common definition of the harmonic map is as the critical point of the energy functional, the integral of the energy density over the domain.

There are some simple examples of harmonic maps. If the target manifold is the space of real numbers, then $f$ is just a harmonic function on $M$, i.e. it vanishes under the Laplacian operator. If the domain is an interval on the Real line, the equation becomes that of a geodesic.

After elaborating on these definitions and looking at a few examples, we consider in detail the case of harmonic maps into Lie groups or the chiral model. In later sections, we will describe the supersymmetric version.
A. The energy of a harmonic map

The energy density of a map \( f : M \rightarrow N \) between two Riemannian manifolds with metrics \( g \) and \( h \) respectively is defined as half the trace of the pull-back \( f^*h \) with respect to the metric \( g \) of \( M \):

\[
e(f) = \frac{1}{2} \text{Tr}_g(f^*h)
\]

(1)

where \( \{e_i\}, i = 1, ..., \dim M \). The integral of the energy density over \( M \) yields the energy of the map:

\[
E(f) = \frac{1}{2} \int_M \sqrt{g} \text{Tr}_g(f^*h)
\]

(2)

When \( M \) is compact, the energy is finite.

**Definition** A map \( f : M \rightarrow N \) is harmonic if and only if it is an extremal of the energy integral \( E \)

B. Harmonic maps into Lie groups

At this point we look at maps \( f : M \rightarrow G \), where \( G \) is the real form of a complex Lie group and \( M \) is a Riemann surface. The Maurer-Cartan form \( \theta \) maps left-invariant vector fields to their corresponding Lie algebra element, and because of identities relating the structure constants of the group, satisfies the Maurer-Cartan structure equation:

\[
d\theta + \frac{1}{2} [\theta \wedge \theta] = 0
\]

(3)

The pull-back \( B = f^*\theta \) is an \( L(G) \) valued 1-form on \( M \) satisfying

\[
dB + \frac{1}{2} [B \wedge B] = 0
\]

(4)

If we view this as the equation for a flat connection on \( M \), we can write \( A = f^{-1}df \). Write the energy of a map \( f : M \rightarrow G \), using (2) as

\[
E(f) = \frac{1}{2} \int_M \sqrt{g} \sum_i |f^{-1} \frac{\partial f}{\partial x_i}|^2
\]

(5)

If we find the Euler-Lagrange equations for this integral, or use the result [13] that a map \( f \) into a Lie group is harmonic if and only if \( d^*(f^*\theta) = 0 \), we arrive at the harmonic map equations in this case as:

\[
\frac{\partial}{\partial x} \left( f^{-1} \frac{\partial f}{\partial x} \right) + \frac{\partial}{\partial y} \left( f^{-1} \frac{\partial f}{\partial y} \right) = 0
\]

(6)

Let \( 2A = B \). The above equation can be written in characteristic co-ordinates as:

\[
\partial_x A + \partial_y \tilde{A} = 0
\]

(7)

By the definition of \( A \) we have
\[ \partial z A_z - \partial \bar{z} A_{\bar{z}} + 2 [A_z, A_{\bar{z}}] = 0. \]  

(8)

The above two equations (7) and (8) are equivalent to a single equation (and its dual):

\[ \partial \bar{z} A_z + [A_z, A_{\bar{z}}] = 0 \]  

(9)

We need the following theorem from [1] in all that follows:

**Theorem 2.1** Let \( \Omega \) be simply connected and \( A : \Omega \rightarrow T^*(\Omega) \otimes g \). Let \( 2A = f^{-1}df \). Then \( f \) is harmonic if and only if the curvature of the connection in \( \Omega \times \mathbb{C}^N \)

\[ D_\lambda = (\partial \bar{z} + (1 - \lambda)A, \partial z + (1 - \lambda^{-1})A_z) \]  

(10)

vanishes for all \( \lambda \in \mathbb{C}^* = \mathbb{C} \setminus 0 \)

**Proof:**

This is proved by writing out the curvature equations and expanding in \( \lambda \).

The problem is also described as a pair of simultaneous linear equations which trivialise the connection:

\[ \partial \bar{z} g_\lambda = (1 - \lambda)g_\lambda A_{\bar{z}}, \quad \partial z g_\lambda = (1 - \lambda^{-1})g_\lambda A_z \]  

(11)

The \( g \) which satisfy these equations can be uniquely chosen to satisfy the properties described in the following theorem [1]:

**Theorem 2.2** Let \( f \) be harmonic and \( f_0(p) \equiv I \) for some \( p \in \Omega \). Then there exists a unique \( g_\lambda : C \times \Omega \rightarrow G \) satisfying [14] with

\[ \begin{align*}  
(a) & \quad g_1 \equiv I \\
(b) & \quad g_{-1} = f \\
(c) & \quad g_\lambda(p) = I
\end{align*} \]  

(12)

Furthermore, if \( f \) is unitary, \( g_\lambda \) is unitary for \( |\lambda| = 1 \).

**Theorem** The converse is also true: if \( g_\lambda : C \times \Omega \rightarrow G \) is analytic and holomorphic in the first variable, \( g_1 \equiv I \) and the expressions

\[ \frac{\partial \bar{z} g_\lambda(g_\lambda)^{-1}}{1 - \lambda} \quad \frac{\partial z g_\lambda(g_\lambda)^{-1}}{1 - \lambda^{-1}} \]  

(13)

are constant in \( \lambda \), then \( f = g_{-1} \) is harmonic.

These extended solutions \( g_\lambda(x) \) can be expanded in a Laurent series in \( \lambda \) about either zero or \( \infty \). Harmonic maps can be classified according to their **uniton number**.

**Definition** The **uniton number** is the highest power of \( \lambda \) in this Laurent series expansion, possibly \( \infty \).

The maps we look at are local to \( \Omega \subset \mathbb{C} \). We assume that these maps in \( \mathbb{C} \) can be extended over the point at \( \infty \) to maps into \( S^2 = \mathbb{C} \cup \infty \). Then these equations have finite-energy solutions.

**Definition** An **n-uniton** is a harmonic map \( f : \Omega \rightarrow U(N) \) which has an extended solution

\[ g_\lambda : C^* \times \Omega \rightarrow GL(N) \]  

(14)
with

(a) \( g_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha \) for \( T_\alpha : \Omega \rightarrow L(G) \) \hspace{1cm} (15)

(b) \( g_1 = 1 \)

(c) \( g_{-1} = q f^{-1} \) for \( q \in U(N) \) constant \hspace{1cm} (16)

(d) \( (g_\lambda)^* = (g_{\lambda^{-1}})^{-1} \) \hspace{1cm} (17)

For \( n = 0 \), \( g_\lambda \equiv I \) is the only extended solution, which represents \( f \equiv q^{-1} \) or the constant harmonic maps.

We will also take note of the fact, also shown in [1], that all these harmonic maps are defined into totally geodesic Grassmanian submanifolds. We must consider maps into

\[ G_{k,N} \subset G = U(N) \] \hspace{1cm} (19)

given by

\[ G_{k,N} = \{ \phi \in U(N) : \phi^2 = I \text{ with } k \text{-dimensional } + 1 \text{ eigenspace} \} \] \hspace{1cm} (20)

We identify \( \phi \) with the \( k \)-dimensional subspace corresponding to the \( +1 \) eigenspace. Since the embedding is geodesic, we can identify maps \( s \) into \( U(N) \) satisfying \( s^2 = 1 \) as harmonic because they represent hermitian projections \( \pi \) on a \( k \)-dimensional sub-bundle in \( \Omega \times \mathbb{C}^N \). These harmonic maps are given by

\[ s = \pi - \pi^\perp = 2\pi - I \] \hspace{1cm} (21)

So ends our accelerated review of the theory of harmonic maps, at least insofar as it applies in this work.

III. SUPERHARMONIC MAPS INTO SUPER LIE GROUPS

In this section we make a study of the chiral model in \( N=1 \) superspace. After reviewing some elementary principles of supergeometry, we construct a superspace version of the Lax pair construction. This allows us to examine integrable systems in superspace, and in particular we define a superharmonic map into a (super) Lie group. Extended solutions to the associated linear problem are defined, and this will lead us into a study of the simplest superharmonic maps and their solutions in the next section.

A. Supergeometry

We work throughout in complexified superspace with co-ordinates \( (z, \bar{z}; \theta, \bar{\theta}) \). \( (\theta, \bar{\theta}) \) are anticommuting variables. Thus, a sign change occurs when moving such quantities past each other. For convenience, we will denote by a hat \( \hat{X} \) the operation of flipping the sign of the anti-commuting or 'odd' part of any quantity \( X \).

The derivates in the anticommuting variables are themselves anticommuting quantities, as are the superderivates
\[D_\theta = \partial_\theta + \theta \partial_z, \quad D_{\bar{\theta}} = \partial_{\bar{\theta}} + \bar{\theta} \partial_{\bar{z}}\] (22)

The superderivates are related to the regular space derivatives in a simple and easily verifiable manner:

\[D_\theta^2 = \partial_\theta, \quad D_{\bar{\theta}}^2 = \partial_{\bar{\theta}}\] (23)

Any quantity defined over superspace can be expanded in a Taylor series over the anti-commuting co-ordinates

\[X = x_0 + x_+ \theta + x_- \bar{\theta} + x_2 \theta \bar{\theta}\] (24)

and it is convenient to denote this in vector notation as

\[X = \begin{pmatrix} x_0 \\ x_+ \\ x_- \\ x_2 \end{pmatrix}\] (25)

B. Lax pair for superspace

A unique feature of our analysis is the definition of the supercurvature for a connection \((A_\theta, A_{\bar{\theta}})\) defined over superspace, denoted \(F(A_\theta, A_{\bar{\theta}})\).

It is distinguished from the simple supercommutator by minor but essential sign changes, which are recorded with the hat notation we described above.

**Definition 5.1** The supercurvature of the connection \((A_\theta, A_{\bar{\theta}})\) is defined as

\[F(A_\theta, A_{\bar{\theta}}) = D_{\bar{\theta}}A_\theta + D_\theta A_{\bar{\theta}} + \hat{A}_{\bar{\theta}}A_\theta + \hat{A}_\theta A_{\bar{\theta}}\] (26)

This is also the definition we use when we refer to Lax pairs in superspace. Any analysis now requires that such Lax pairs make sense in the usual way of integrable systems, that they are equivalent to an associated linear problem. Hence the following theorem.

**Theorem 5.1** The vanishing of the supercurvature \(F(A_\theta, A_{\bar{\theta}})\) is a necessary and sufficient condition for the existence and uniqueness of a trivialization of the connection \((A_\theta, A_{\bar{\theta}})\) in connected superspace, that is to say, a solution to the equations

\[D_\theta G = A_\theta G, \quad D_{\bar{\theta}} G = A_{\bar{\theta}} G\] (27)

exists if and only if our connection is flat according to our definition, where \(G\) takes its values in some Lie group or supergroup and \(A\) in the corresponding algebra.

**Proof** Clearly, since the superderivative anticommutes, we have

\[D_\theta D_{\bar{\theta}} G + D_{\bar{\theta}} D_\theta G = 0\] (28)

so that, by (27), we get

\[D_\theta (A_{\bar{\theta}} G) + D_{\bar{\theta}} (A_\theta G) = 0\] (29)
Expanding this, and bearing in mind that moving the superderivative past a term changes the sign of the odd components, we find that the expression (26) vanishes.

It is more laborious to prove the theorem in the other direction, to show that flat superconnections can be trivialised but it can be shown that the problem reduces to a standard one for a trivial connection in the body component alone. By the existence theorem for PDE, a unique solution exists for \( g_0 \) and hence \( G \) once we prescribe its value at some base point, provided that the compatibility conditions for the second and seventh of (??) are satisfied, that is, that the curvature of this connection vanishes in the traditional sense.

It is clear on examination of these calculations that our definition of the Lax pair (the supercurvature) is fully determined and is not merely a convenient choice. With this understood we press on to define a superharmonic map from superspace into a super Lie group.

C. Superharmonic maps

Firstly, we will need a couple of identities which are easily checked. Again, the hat indicates that the odd components of the term are given opposite sign.

\[
\begin{align*}
D_\theta M^{-1} &= -\hat{M}^{-1} \cdot D_\theta M \cdot M^{-1} \\
D_\theta (MN) &= (D_\theta M)N + M(D_\theta N)
\end{align*}
\]

Definition A superharmonic map \( S : \Omega \to G \) from \( N = 1 \) superspace into a (super) Lie group \( G \) is a solution of the following equation:

\[
D_\theta (D_\bar{\theta} S \cdot S^{-1}) - D_{\bar{\theta}} (D_\theta S \cdot S^{-1}) = 0
\]  

(31)

Clearly, all harmonic maps are trivially superharmonic. A supersymmetric analysis of superharmonic maps analogous to the material presented in section II now follows. As in the non-supersymmetric case, we can introduce a spectral parameter to encode this definition of a superharmonic map in terms of a single Lax pair expression. We will need this result, expressed in the following theorem, in all that follows:

Theorem 5.2 Let \( \Omega \) be simply connected and \( A : \Omega \to T^*(\Omega) \otimes L(G) \). Then \( 2A_{\theta,\bar{\theta}} = D_{\theta,\bar{\theta}} S \cdot S^{-1} \), where \( S \) is superharmonic if and only if the supercurvature

\[
F \left( (1 - \lambda^{-1})A_{\theta}, (1 - \lambda)A_{\bar{\theta}} \right) = 0
\]  

(32)

vanishes for all \( \lambda \in C^* = C - 0 \)

In terms of coefficients of \( \lambda \), we have

\[
\begin{align*}
\lambda \left( D_\theta A_{\bar{\theta}} + \hat{A}_{\bar{\theta}} A_\theta + \hat{A}_\theta A_{\bar{\theta}} \right) + \lambda^0 \left( D_\theta A_{\bar{\theta}} + D_{\bar{\theta}} A_\theta + 2(\hat{A}_{\bar{\theta}} A_\theta + \hat{A}_\theta A_{\bar{\theta}}) \right) \\
+ \lambda^{-1} \left( D_{\bar{\theta}} A_{\theta} + \hat{A}_{\theta} A_\theta + \hat{A}_\theta A_{\bar{\theta}} \right) = 0
\end{align*}
\]  

(33)

Proof It is a simple matter to show that this is equivalent to the supersymmetric form of the chiral equation and an identity arising from the definition of \( A_{\theta}, A_{\bar{\theta}} \).

By theorem 1 and since \( \Omega \) is simply connected, we can explore solutions to these equations by solving the trivialisation problem. These are solutions \( G_\lambda \) to the simultaneous equations
\[ D_{\bar{\theta}}G_{\lambda} = (1 - \lambda)A_{\bar{\theta}}G_{\lambda}, \quad D_{\theta}G_{\lambda} = (1 - \lambda^{-1})A_{\theta}G_{\lambda} \]

which are valid for all values of the spectral parameter \( \lambda \), of which \( G_{\lambda} \) is a function.

Solutions are prescribed uniquely by setting the body component, \( (g_{\lambda})_0(p) \) at any base point \( p \in \Omega \). We can safely normalise our solution so that \( s_0(p) = I \) at \( p \). We then choose \( g_{\lambda}^0(p) = I(g_{\lambda})_0(p) = I \).

Also, if \( \lambda = 1 \), \( D_{\bar{\theta}}G^1 = D_{\theta}G^1 = 0 \), for which a Taylor expansion in \( \theta, \bar{\theta} \) reveals that \( dg_{1}^0 = 0 \), all other components vanishing identically. So \( G^1 \equiv I \).

At this point we can write down a theorem:

**Theorem 5.3** If \( S \) is superharmonic and \( s_0(p) \equiv I \), then there exists a unique \( G_{\lambda} : \mathbb{C} \times \Omega \to G \) satisfying (34) with

\[
\begin{align*}
(a) & \quad G^1 \equiv I \\
(b) & \quad G^{-1} = S \\
(c) & \quad G_{\lambda}(p) = I
\end{align*}
\]  

Also if \( S \) is unitary and a c-number (commuting quantity), \( G_{\lambda} \) is unitary for \( |\lambda| = 1 \).

**Proof** It only remains to check the final statement, which follows from the fact that

\[ -D_{\bar{\theta}}(G_{\lambda})^{-1} = (1 - \lambda)(\hat{G}_{\lambda})^{-1}A_{\bar{\theta}}, \quad -D_{\theta}(G_{\lambda})^{-1} = (1 - \lambda^{-1})(\hat{G}_{\lambda})^{-1}A_{\theta} \]

As in section II, the converse is also true. That is, if \( G^1 = I \) and the quantities

\[ \frac{D_{\bar{\theta}}G_{\lambda}(G_{\lambda})^{-1}}{1-\lambda}, \quad \frac{D_{\theta}G_{\lambda}(G_{\lambda})^{-1}}{1-\lambda^{-1}} \]

are constant in \( \lambda \), then \( S = G^{-1} \) is superharmonic.

The point of all this is that we can now study superharmonic maps using only the data encoded by the \( G_{\lambda} \). In particular, we can expand the \( G_{\lambda} \) in a Laurent series as a function of \( \lambda \)

\[ G_{\lambda} = \sum_{\alpha = -\infty}^{\infty} T_\alpha \lambda^\alpha \]

where \( T_\alpha \) are Lie algebra-valued functions on superspace.

We use these tools to find explicit solutions to a simple class of superharmonic maps in the following section.

**IV. SUPERHARMONIC MAP SOLUTIONS IN THE UNITARY CASE**

We seek to understand the relationships of superharmonic maps into Lie groups to the traditional harmonic maps and other integrable systems. For the case of the simplest finite-action superharmonic maps into \( SU(M/N) \), which we call superunitons, we find expressions for their solution in terms of a holomorphic projector and a related quantity.

By analogy with the non-supersymmetric case, any finite-energy superharmonic map can be factorised in terms of these maps, and for \( SU(1/1) \) it is known that these are the only such maps.

We first look at the construction of maps into superspace Grassmannians, which will give us the mathematical equipment we need for the rest of this analysis.
A. Superunitons

As in the non-supersymmetric theory, our study of superharmonic maps is made by examining maps into Grassmannians. In this case, we must be conscious of the presence of the superspace variables by looking at maps from superspace into supersymmetric Grassmannians, the space of $k/l$-planes in $M/N$ superspace.

Then consider the inclusion

$$G_{k,l;M,N} = \frac{U(M/N)}{U(k/l) \times U(M-k/N-l)} \subset U(M/N) \quad (39)$$

defined by

$$G_{k,l;M,N} = \{ \phi \in U(M/N) : \phi^2 = I \text{ with } k/l \text{ dimensional } + 1 \text{ eigenspace} \} \quad (40)$$

The embedding $G_{k,l;M,N} \subset U(M/N)$ is totally geodesic, and we have

**Proposition 6.1** The combined map $\phi S : \Omega \xrightarrow{S} G_{k,l;M,N} \xrightarrow{\phi} U(M/N)$ is superharmonic if and only if $S$ is superharmonic.

So we study superharmonic maps $S : \Omega \to U(M/N)$ for which $S = S^{-1}$. The maps are given by

$$S = (\Pi - \Pi^\perp) = (2\Pi - 1) \quad (41)$$

where $\Pi$ is the (super)Hermitian projection of rank $k,l$ on a $k/l$-dimensional sub-bundle at each $q \in \Omega$.

Algebraically we can identify $\Pi$ by noting that

(a) $\Pi^*(q) = \Pi(q)$ for all $q \in \Omega \quad (42)$

(b) $\Pi^2(q) = \Pi(q)$ for all $q \in \Omega \quad (43)$

(c) $\Pi(q)$ has rank $k/l$ at every point $q \in \Omega \quad (44)$

We also note that $\Pi^\perp = (1 - \Pi)$ is the Hermitian projection on the orthogonal bundle.

We now define a discrete series of elemental superharmonic maps, which we call superunitons, as a supersymmetric extension of the constructions of [1]. The level of complexity of these maps as well as the energies are defined by the minimum number of terms needed in the expansion an extended solution.

**Definition** An $n$-superuniton is a superharmonic map $S : \Omega \to U(M/N)$ which has an extended solution

$$G^\lambda : C^* \times \Omega \to GL(M/N) \quad (45)$$

with

(a) $G^\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ for $T_\alpha : \Omega \to L(G)$

(b) $G^1 = 1$

(c) $G^{-1} = QS^{-1}$ for $Q \in U(M/N)$ constant

(d) $(G^\lambda)^* = (G^{\lambda^{-1}})^{-1} \quad (46)$
For \( n = 0 \), \( G^λ \equiv I \) is the only extended solution, which represents \( S \equiv Q^{-1} \) or the constant superharmonic maps.

We now examine the 1-superuniton, the simplest example.

**Proposition 6.2** \( S : Ω \to U(M/N) \) is a one-S-uniton if and only if \( S = Q \prod - \prod^⊥ \) for \( Q \in U(M/N) \), where \( \prod \) is a Hermitian projector and \( \bar{D} \prod \cdot \prod^⊥ = 0 \).

**Proof.** Clearly, we have \( G^λ = T_0 + λ(1 - T_0) \)

The reality condition gives us

\[
(I - T_0)^* T_0 = 0, \quad T_0^* (I - T_0) = 0
\]

Combining these equations gives us \( T_0^* = T_0 \) and \( T_0^2 = T_0 \), so that \( T_0 = \prod \).

Apart from the final statement, the theorem is now apparent by inspection. The superharmonic condition from the end of the previous section is that expressions for

\[
\bar{D}G^λ \cdot (G^λ)^{-1}, \quad \bar{D}G^λ \cdot (G^λ)^{-1}
\]

be independent of \( λ \). The former can be written as

\[
\frac{\bar{D}(\prod + λ(1 - \prod)) \cdot (\prod + λ^{-1} \prod^⊥)}{(1 - λ)} = \frac{\bar{D}(\prod(1 - λ)(\prod - 1)) \cdot (\prod + λ^{-1} \prod^⊥)}{(1 - λ)}
\]

and the latter as

\[
\frac{-\bar{E}λ \cdot DE^{-1}}{(1 - λ^{-1})} = \frac{-\bar{E}(\prod + λ^⊥) \cdot D(\prod + λ^{-1}(1 - \prod))}{(1 - λ^{-1})} = \frac{\bar{E}(\prod + λ^⊥) \cdot D((1 - λ^{-1})(1 - \prod))}{(1 - λ^{-1})}
\]

so our condition for superharmonicity is

\[
\bar{D} \prod \cdot \prod^⊥ = (\bar{\prod} \cdot D\prod)^* = 0
\]

This concludes the proof.

One-unitons in the classical theory are identified as simply holomorphic maps due to the fact that Grassmannian manifolds are Kahler. We find in this case that some additional data is required to identify the simplest one-superunitons beyond the holomorphic body component.

**B. Explicit solutions**

We will again use the vector notation of the previous section to calculate the superuniton condition in terms of the superspace components.

We expand the projector \( \prod \) in terms of the supervariables \( (Θ, \bar{Θ}) \) as \( \prod = π_0 + π_+ θ + π_- \bar{θ} + π_2 θ \bar{θ} \) or in vector format as
\[ \Pi = \begin{pmatrix} \pi_0 \\ \pi_+ \\ \pi_- \\ \pi_2 \end{pmatrix} \]  

(52)

**Theorem 6.3** The one-S-uniton \( S : \Omega \rightarrow U(M/1) \) is given by a holomorphic projector \( p \) into \( U(M) \) and an ancillary \( m \)-vector \( v \) which are related by the equations

\[ v^T p = v^T (1 - p) = 0 \]

(53)

**Proof**

In superspace components, we write the superuniton condition as

\[ \bar{D} \Pi \cdot \Pi^\dagger = \begin{pmatrix} \pi_- \\ -\pi_2 \\ \pi_{0,\bar{z}} \\ -\pi_{+\bar{z}} \end{pmatrix} \cdot \begin{pmatrix} 1 - \pi_0 \\ -\pi_+ \\ -\pi_- \\ -\pi_2 \end{pmatrix} \]

(54)

\[ = \begin{pmatrix} \pi_- (1 - \pi_0) \\ -\pi_- \pi_+ - \pi_2 (1 - \pi_0) \\ -\pi_2^2 + \pi_{0,\bar{z}} (1 - \pi_0) \\ -\pi_- \pi_2 + \pi_2 \pi_- - \pi_{+\bar{z}} (1 - \pi_0) + \pi_{0,\bar{z}} \pi_+ \end{pmatrix} = 0 \]

(55)

Combining equations \( 55a \) and \( 55b \), we find

\[ \pi_{0,\bar{z}} (1 - \pi_0) = \pi_2^2 = 0 \]

(56)

and by duality

\[ \pi_2^+ = 0 \]

(57)

From this we learn three things: firstly, because \( \pi_{0,\bar{z}} (1 - \pi_0) = 0 \), we know that \( \pi_0 \) is a solution of the non-supersymmetric one-uniton equations described in section II – it is just a holomorphic projector. We also see that \( \pi_- \), \( \pi_+ \) are nilpotent.

If we now take our group \( G \) to be \( U(M/1) \), then we can write, without loss of generality, and using our knowledge of this group’s properties,

\[ \pi_0 = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \]

(58)

\[ \pi_+ = \begin{pmatrix} 0 & v^T \\ v^T & 0 \end{pmatrix} \]

(59)

\[ \pi_- = \pi_+^\dagger = \begin{pmatrix} 0 & i\bar{v} \\ 0 & 0 \end{pmatrix} \]

(60)

where \( p \) is a holomorphic projector into \( U(M) \), \( v \) is an \( m \)-vector, and \( \dagger \) denotes the super-hermitian adjoint.
Since $\Pi^2 = \Pi$, we have

$$
\begin{pmatrix}
\frac{\pi_0^2}{\pi_0} \\
\pi_0\pi_+ + \pi_+\pi_-
\pi_0\pi_- + \pi_-\pi_0
\pi_0\pi_2 + \pi_2\pi_0 + \pi_+\pi_- - \pi_-\pi_+
\end{pmatrix}
= 
\begin{pmatrix}
\pi_0 \\
\pi_+ \\
\pi_- \\
\pi_2
\end{pmatrix}
$$

(61)

so that, combining (61) with (55a) from before, we get

$$
\pi_0\pi_- = 0 = \pi_+\pi_0
$$

(62)

Explicitly,

$$
\begin{pmatrix}
0 & 0 \\
v^T & 0
\end{pmatrix}
\begin{pmatrix}
p & 0 \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0 \\
v^Tp & 0
\end{pmatrix}
= 0
$$

(63)

so that $v^T = 0$

Equation (55d) now reads, if we write $\pi_2$ as

$$
\begin{pmatrix}
a & 0 \\
0 & d
\end{pmatrix}
$$

and plug in for the other components,

$$
(D\Pi \cdot \Pi^T)_2 = 
\begin{pmatrix}
0 & -iv\bar{v}d + i\bar{a}\bar{v}

-v^T\bar{z}(1-p)
\end{pmatrix}
= 0
$$

(64)

so that our super-uniton is determined by the additional parameter $v$ and the condition

$$
v^Tp = v^T\bar{z}(1-p) = 0
$$

(65)

This ends the proof.

We can actually go further and construct an explicit solution for $\Pi$ in terms of the holomorphic projector $p$ and the new ancillary data. The complete super-uniton is written explicitly as

$$
\Pi = 
\begin{pmatrix}
p - iv\bar{v}^T\bar{\theta}\bar{\theta} & i\bar{v}\bar{\theta} \\
v^T\bar{\theta} & 1 - iv^T\bar{v}\bar{\theta}\bar{\theta}
\end{pmatrix}
$$

(66)

V. BACKLUND TRANSFORMATIONS FOR SUPERHARMONIC MAPS

Knowing extended solutions to the superharmonic map equations, we can generate others using Backlund transformations of the kind described in [1].

A Backlund transformation is a method of obtaining new solutions of partial differential equations from old solutions by solving ordinary differential equations. In this case, our transformation is obtained by using factorisations of solutions extended over maps on overlapping regions of the 2-sphere. We will see that the relationship between these factorisations allows us to transform between solutions 'of simplest type'.
A. A representation on holomorphic maps

The factorization we want to use is a modification of the standard Birkhoff factorization. In this case, the 2-sphere $C^* \cup 0 \cup \{\infty\}$ is divided into two overlapping regions:

$$S_+^\varepsilon = \{ \lambda : |\lambda| \geq (1 + \varepsilon)^{-1} \}, \quad S_-^\varepsilon = \{ \lambda : |\lambda| \leq (1 + \varepsilon) \}$$

We use a slightly different contour, a pair of circles around 0 and $\infty$, and ignore the anti-commuting variables which are irrelevant for understanding the analyticity of the map. We can now factor an analytic map $g$ from $N = 1$ superspace into a (super) Lie group $G$ into two maps, one of which is analytic away from 0 and $\infty$, the other meromorphic in neighbourhoods of $(0, \infty)$.

Let

$$X^k = \{ e : C^* \to G : e, e^{-1} \text{ have Laurent expansions} \} \text{ of order k and } e(1) = I \}$$

and

$$Y = \{ f : S^2 \to G \text{ meromorphic in neighbourhoods of } (0, \infty) \text{ and } f(1) = I \}$$

Any suitable map now has two factorisations,

$$g = f^X_L f^Y_R = f^Y_L f^X_R$$

where $f^X_L, f^X_R \in X^k$ and $f^Y_L, f^Y_R \in Y$.

We use this factorisation to write down a group transformation on the space $X^k$.

We write $f^#(e) = R \cdot e \cdot f$ for $R, f \in Y$ and $e, f^#(e) \in X^k$.

**Lemma** If $f^#(e)$ can be defined, there exits a unique $f^#(e)$ taking the value $I$ at 1. Also, if $f^#(e)$ and $g^#(f^#(e))$ are defined and normalised to be $I$ at $\lambda = 1$, then

$$(gf)^#(e) = g^#(f^#(e))$$

The proof follows as in Lemma 5.1 of [1].

**Definition** $f \in Y$ is of simplest type if $f(\lambda) = \Pi + \xi(\lambda)\Pi^\perp$, where $\Pi$ is super-Hermitian projection on complex superspace, $\Pi^\perp = 1 - \Pi$ is a projection onto the orthogonal subspace and $\xi(\lambda)$ is a rational complex function of degree one which is 1 at $\lambda = 1$.

**Theorem** If $f(\lambda) = \Pi + \xi_\alpha(\lambda)\Pi^\perp$ is of simplest type, then $e^# = f^#(e) = R_f e f$ is always defined.

To cancel with $f$, $R_f$ must be a similar sum of orthogonal components to cancel with the corresponding terms:

$$R = \bar{\Pi} + \xi(\lambda)^{-1}\bar{\Pi}^\perp$$

We need only check that the supersymmetric reality condition consistently satisfies the requirements of this condition at the zero and pole of $\xi$ which by the reality condition, are given by $\alpha$ and $\lambda = 1/\bar{\alpha}$ respectively.

At $\lambda = \alpha$, we should have
\[ \Pi e(\alpha) \tilde{\Pi}^\perp = 0 \]

and at \( \lambda = 1/\bar{\alpha} \) we need

\[ \Pi^\perp e(1/\bar{\alpha}) \tilde{\Pi} = 0 \]

These two equations are compatible, since:

\[
\begin{align*}
\left( \Pi^\perp e(1/\bar{\alpha}) \tilde{\Pi} \right)^\# & = J^{-1} \left( \Pi^\perp e(1/\bar{\alpha}) \tilde{\Pi} \right)^* J \\
& = J^{-1} \tilde{\Pi}^* e(1/\bar{\alpha})^* \Pi^\perp J \\
& = \tilde{\Pi} J^{-1} e(1/\bar{\alpha})^* J \Pi^\perp \\
& = \tilde{\Pi} \left( e(\alpha)^{-1} \right)^* \Pi^\perp \\
& = 0
\end{align*}
\]

We can now consistently define \( \tilde{\Pi} \) to be the projection on the subspace \( e(\alpha)^* V \), where \( V \) is the subspace \( \Pi \) projects onto.

**B. A representation on extended superharmonic maps**

We now have proved that \( f \rightarrow f^\# \) is a representation on \( X^k \). We can now use this to define a representation on the moduli space of extended solutions of superharmonic map equations \( E : C^* \times \Omega \rightarrow G \). We define \( \left( f^\#(E) \right)(q) = f^\#(E(q)) \) for all \( q \in \Omega \) and the only condition we need to check is the superharmonic condition:

\[
(1 - \lambda)^{-1} \bar{D} E \left( f^\# E \right)^{-1} = \bar{A}_\theta \]

\[
(1 - \lambda^{-1})^{-1} DE \left( f^\# E \right)^{-1} = \bar{A}_\theta
\]

for some \( \bar{A}_\theta, \bar{A}_\theta \) independent of \( \lambda \).

We use Liouville’s theorem to show that the quantities on the left are independent of \( \lambda \) as required. We look at the \( \bar{D} \) term only as the other follows. This quantity has no pole at 1 because \( \left( f^\# E \right)_1 = 1 \) from before, so that \( \bar{D} \left( f^\# E \right)_1 = 0 \). Now we can calculate that

\[
\begin{align*}
(1 - \lambda)^{-1} \bar{D} E \left( f^\# E \right)^{-1} & = (1 - \lambda)^{-1} \bar{D} \left( R Ef \right) \left( R Ef \right)^{-1} \\
& = (1 - \lambda)^{-1} \left( (\bar{D} R) E + \hat{R} \bar{D} E \right) \left( R E \right)^{-1} \\
& = (\bar{D} R)^{-1} \frac{(1 - \lambda)}{(1 - \lambda)} \hat{R} \left( \bar{D} E \right) E^{-1} R \\
& = (\bar{D} R)^{-1} + \hat{R} A_{\theta} R
\end{align*}
\]

The second term is independent of \( \lambda \) by Theorem 5.3 because it contains the superharmonic map term; the first term is holomorphic at \( \lambda = 0, \infty \) by construction, so by Liouville’s theorem the entire expression is constant in \( \lambda \).
Again by Theorem 5.3 and the equivalent result for the $D$ term, it follows that $(f^\#E)_{-1}$ is superharmonic. We must check the normalisation condition, but since $E(p) \equiv I$, it follows that $R(p) = f^{-1}$ so that $f^\#E = fI f^{-1} = I$, as required.

C. Example Backlund transformations

We can produce new solutions from known solutions by looking at just those of simplest type.

From above, using our representation on extended superharmonic solutions, we write for $f = \Pi + \xi(\lambda) \Pi^\perp$,

$$f^\#(E_\lambda) = \left( \Pi + \xi(\Pi^\perp) \right) E_\lambda \left( \Pi + \xi^{-1}\Pi^\perp \right)$$

(71)

Here $\Pi$ is the super-Hermitian projection on the subspace $E_\alpha^* V$, where $\alpha$ is the zero of $\xi$ and $V$ is the vector space image of $\Pi$.

The new solution is obtained algebraically from $E_\lambda$ but in general this is not trivial. We can however, find a special case in which the algebra can be calculated to give an easy relationship between the solutions – that is the case when our super-uniton solutions have no body, i.e., consist of anti-commuting variables only.

To simplify the calculations, we can use the relationship between the Hermitian projections defined in (69),

$$\tilde{\Pi}_w E_\alpha^{-1} \Pi^\perp_v = 0$$

where $\tilde{\Pi}_w$ is the Hermitian projection of the transformed solution given by $\Pi_v$.

From the previous section, we know that in general,

$$\Pi_v = \begin{pmatrix} p_v - i\bar{v}^\perp \theta & i\bar{\theta} \bar{v} \vphantom{\theta} \\ v^\perp \theta & I - iv^\perp \bar{v} \bar{\theta} \vphantom{\theta} \end{pmatrix}$$

(72)

where $p_v$ is a holomorphic projector into $U(M)$, and $v$ is the ancillary data described in the previous section. Because we are looking only at the supersymmetric data, $p_v$ and $p_w$ are zero.

The extended superuniton solution takes the form

$$E_\lambda = \begin{pmatrix} p_u + \lambda (1 - p_u) + O(\bar{\theta}) & (1 - \lambda) i\bar{u} \theta \\ (1 - \lambda) u^\perp \bar{\theta} & I + O(\bar{\theta}) \end{pmatrix}$$

(73)

In this case the terms in $\theta \bar{\theta}$ have no effect on the calculation as the anticommuting variables are nilpotent.

Setting $\tilde{\Pi}_w E_\alpha^{-1} \Pi^\perp_v = 0$ gives us some algebra to calculate, but in the end we get a consistent expression relating the two solutions:

$$v^\perp = w^\perp \cdot \left( p + \alpha^{-1} p^\perp \right) + u^\perp (1 - \alpha^{-1})$$

(74)

Using (6.25), and multiplying on the right by $\left( p + \alpha p^\perp \right)$, we can write down a direct expression for the Backlund transformation, which is just the inverse:
\[ w^\perp = v^\perp \cdot \left( p + \alpha^{-1} p^\perp \right) + u^\perp (1 - \alpha^{-1}) \]  

(75)

We thus have an expression for \( \Pi_w \) which we can use to write down another superharmonic map. We have therefore proved the following theorem:

**Theorem** When applied to superunitons, those maps \( f \) of simplest type defined only on superspace produce Backlund transformations which transform between superharmonic maps without body components.

**VI. CONCLUSION AND ACKNOWLEDGEMENTS**

In this paper, we have seen how the harmonic maps into Lie groups are fundamental theories in several respects, and we have developed a supersymmetric generalisation.

Using variables and derivates with anti-commuting components, we have written a super Lax pair and have found special unique solutions of the supersymmetric chiral model.

The goal in this project is to generalise results obtained in the classical chiral model theory to the supersymmetric case. The next step in this effort might be to examine how this theory relates to other known supersymmetric integrable systems, in particular the supersymmetric Toda models.

A physical analysis of the theory requires us as some point to Wick rotate the model from the Euclidean case to the Minkowski metric. While this raises global difficulties, it should still be possible to obtain meaningful results locally in the rotated model.

In particular, the Backlund transformations examined in the last section provide a way to transform between solutions. In addition, the explicit relationship to Toda theories demonstrated in ?? should allows us to calculate physically relevant quantities from the work in the area of Toda field theory.

Finally, the classification theorem presented here show that the finite-energy action in the supersymmetric case is discrete and quantized according to the superuniton number. The physical significance of this is yet to be teased out, but the tools provided here should assist any field-theoretic analysis.

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