On the heights of algebraic points on curves over number fields

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Let $X$ be a semi-stable regular curve over the spectrum $S$ of the integers in a number field $F$, and $\bar{L} = (L, h)$ an hermitian line bundle on $X$, i.e. $L$ is an algebraic line bundle on $X$ and $h$ is a smooth hermitian metric (invariant by complex conjugation) on the restriction of $L$ to the set $X(\mathbb{C})$ of complex points of $X$. In this paper we are interested in the height $h_{\bar{L}}(D)$ of irreducible divisors $D$ on $X$ which are flat over $S$, i.e. the arithmetic degree of the restriction of $\bar{L}$ to $D$.

First we assume that the degree $\deg(L)$ of $L$ on the generic fiber $X_F$ is positive and we denote by $\bar{L} \cdot \bar{L} \in \mathbb{R}$ the self-intersection of the first arithmetic Chern class of $\bar{L}$. Define

$$e(\bar{L}, d) = \inf_{\deg(D)=d} \frac{h_{\bar{L}}(D)}{d}.$$ 

Our first result (Theorem 2) is that

$$\liminf_{d} e(\bar{L}, d) \geq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)}.$$ 

This is a generalization of an inequality of S. Zhang (13, Th. 6.3).

Next, when $X_F$ has genus at least two and $\bar{\omega}$ denotes the relative dualizing sheaf of $X$ over $S$ with its Arakelov metric [1], we obtain in Theorem 3 explicit lower bounds for $e(\bar{\omega}, d)$.

We prove also some upper bounds. Assume that $\deg(L) > 0$ and that $\deg(L|E) \geq 0$ for every vertical irreducible divisor $E$ on $X$. For any integer $d_0 > 0$ we define

$$e'(\bar{L}, d_0) = \sup_{D_0} \inf_{D \cap D_0} \frac{h_{\bar{L}}(D)}{\deg(D)},$$

where $D_0$ runs over all irreducible horizontal divisors of degree $d_0$, and $D$ runs over all such divisors which meet $D_0$ properly. We prove in Theorem 4 that

$$\lim_{d_0} \sup e'(\bar{L}, d_0) \leq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)}.$$
and, when $X_F$ has genus at least two, we give in Theorem 5 explicit upper bounds for $e'(\omega, d_0)$.

The main tool in the proof of these inequalities is the lower bounds for successive minima of the lattice $H^1(X, M^{-1})$ with its $L^2$-metric which we obtained in previous papers [9] [10] [11]. From these lower bounds we deduce upper bounds for the successive minima of $H^0(X, M \otimes \omega)$ by using a transference theorem relating the successive minima of a lattice with those of its dual (Theorem 1).

## 1 Duality and successive minima:

### 1.1

Let $F$ be a number field, $\mathcal{O}_F$ its ring of integers and $S = \text{Spec}(\mathcal{O}_F)$. Consider an hermitian vector bundle $\bar{E} = (E, h)$ on $S$, i.e. $E$ is a finitely generated projective $\mathcal{O}_F$-module and, for every complex embedding $\sigma : F \to \mathbb{C}$, the corresponding extension $E_\sigma = E \otimes_{\mathcal{O}_F} \mathbb{C}$ of $E$ from $\mathcal{O}_F$ to $\mathbb{C}$ is equipped with an hermitian scalar product $h_\sigma$. Furthermore, we assume that $h = (h_\sigma)$ is invariant under complex conjugation.

We are interested in (the logarithm of) the successive minima of $\bar{E}$. Namely, for any positive integer $k \leq N$, where $N$ is the rank of $E$, we let $\mu_k(\bar{E})$ be the infimum of the set of real numbers $\mu$ such that there exist $k$ vectors $e_1, \ldots, e_k$ in $E$ which are linearly independent in $E \otimes F$ and such that, for every complex embedding $\sigma : F \to \mathbb{C}$ and for all $i = 1, \ldots, k$,

$$\|e_i\|_\sigma \leq \exp(\mu),$$

where $\|\cdot\|_\sigma$ is the norm defined by $h_\sigma$. We shall compare the successive minima of $\bar{E}$ with those of its dual $\bar{E}^*$.

Let $r_1$ (resp. $r_2$) be the number of real (resp. complex) places of $F$, $r = [F : \mathbb{Q}]$ the degree of $F$ over $\mathbb{Q}$, and $\Delta_F$ its absolute discriminant. We define

$$C(N, F) = \frac{1}{r} \log |\Delta_F| + \frac{3}{2} \log(N) + \frac{5}{2} \log(r) - \frac{r_2}{r} \log(\pi).$$

### 1.2

**Theorem 1.** For every $k \leq N$ the following inequalities hold:

$$0 \leq \mu_k(\bar{E}) + \mu_{N+1-k}(\bar{E}^*) \leq C(N, F).$$

To prove the first inequality in Theorem 1 we use a result of Borek [3] which compares the successive minima and the slopes of hermitian vector bundles over $S$. Namely, according to [3], Th. 1, if $\sigma_k(\bar{E})$ is the $k$-th slope of $\bar{E}$, the following inequality holds:

$$0 \leq \mu_k(\bar{E}) + \sigma_k(\bar{E}).$$
Similarly

\[ 0 \leq \mu_{N+1-k}(\bar{E}^*) + \sigma_{N+1-k}(\bar{E}^*) \, \cdot \]

On the other hand, we know that

\[ \sigma_k(\bar{E}) + \sigma_{N+1-k}(\bar{E}^*) = 0 \]

(see [6], 5.15(2)). So, by adding up, we get

\[ 0 \leq \mu_k(\bar{E}) + \mu_{N+1-k}(\bar{E}^*) \, . \]

1.3

The second inequality in Theorem 1 will be proved by reducing it to the case

\[ F = \mathbb{Q} \, . \]

For every positive integer \( k \leq Nr \) let \( \lambda_k \) be the infimum of the set of real numbers \( \lambda \) such that there exist \( k \) vectors \( e_1, \ldots, e_k \in E \) which are \( \mathbb{Q} \)-linearly independent in \( E \otimes E \) and such that, for every \( \sigma \in \Sigma \) and every \( i = 1, \ldots, k \),

\[ ||e_i||_\sigma \leq \exp(\lambda) . \]

The following lemma is used in [12].

**Lemma 1.** For every positive integer \( k \leq N \), the following inequality holds :

\[ \mu_{k+1}(\bar{E}) \leq \lambda_{kr+1} . \]

**Proof.** Let \( e_1, \ldots, e_{kr+1} \in E \) be vectors which are \( \mathbb{Q} \)-linearly independent, and \( V \) (resp. \( W \)) the \( F \)-vector space (resp. the \( \mathbb{Q} \)-vector space) spanned by these vectors. Since \( W \subset V \) and \( \dim_F(V) = r \dim_F(V) \) we get

\[ r \dim_F(V) \geq kr + 1 , \]

hence \( \dim_F(V) \geq k + 1 \). The lemma follows from this inequality and the definition of successive minima.

1.4

Let \( E^\vee = \text{Hom}_E(E, \mathbb{Z}) \) and \( \omega = \text{Hom}_E(O_F, \mathbb{Z}) \). The morphism

\[ \alpha : E^* \otimes_{O_F} \omega \to E^\vee \]

mapping \( u \otimes T \) to \( u \circ T \) is an isomorphism of \( O_F \)-modules. If \( Tr \in \omega \) is the trace morphism, we endow \( \omega \) with the hermitian metric such that \( |Tr|_\sigma = 1 \) (resp. \( |Tr|_\sigma = 2 \)) if \( \sigma = \bar{\sigma} \) (resp. \( \sigma \neq \bar{\sigma} \)). For every \( \sigma \in \Sigma \), the morphism

\[ E_{\omega}^\vee \to E_{\sigma}^* \]
induced by $\alpha$ is an isometry ([7], p. 354). For any positive integer $k \leq Nr$, let $\lambda_k^\vee$ be the infimum of the set of real numbers $\lambda$ such that there exist $k$ vectors $e_1, \ldots, e_k \in E^\vee$ which are linearly independent over $\mathbb{Q}$ and such that, for every $i = 1, \ldots, k$,

$$\sum_{\sigma \in \Sigma} \|e_i\|_{\sigma} \leq \exp(\lambda).$$

According to [2] Theorem 2.1 and section 3, we have, for $k = 1, \ldots, Nr$,

$$\lambda_k + \lambda_{Nr+1-k}^\vee \leq \frac{3}{2} \log(Nr). \quad (2)$$

1.5

Since $\omega$ is invertible we have

$$E^\ast \simeq E^\vee \otimes \omega^{-1}$$

and, for any $v \in \omega^{-1}$, $v \neq 0$,

$$\mu_k(E^\ast) \leq \mu_k(E^\vee) + \sup_{\sigma \in \Sigma} \log \|v\|_{\sigma}. \quad (3)$$

By Minkowski theorem we can choose $v$ such that, for every $\sigma \in \Sigma$,

$$r \log \|v\|_{\sigma} \leq r \log(2) + \log \text{covol}(\omega^{-1}) - \log \text{vol}(B),$$

where $\text{vol}(B)$ is the volume of the unit ball in the real vector space $\omega_{\mathbb{R}}^{-1}$ and $\text{covol}(\omega^{-1})$ is the covolume of the lattice $\omega^{-1}$. We have

$$\text{vol}(B) = 2^{r_1} \pi^{r_2}$$

and, according to [7] p. 355,

$$\log \text{covol}(\omega^{-1}) = \log |\Delta_F| - 2r_2 \log(2).$$

So we can choose $v \in \omega^{-1}$, $v \neq 0$, such that

$$\sup_{\sigma \in \Sigma} \log \|v\|_{\sigma} \leq \frac{1}{r} \log |\Delta_F| - \frac{r_2}{r} \log(\pi). \quad (4)$$

1.6

From Lemma 1 and the fact that

$$\sum_{\sigma \in \Sigma} \|x\|_{\sigma} \leq r \sup_{\sigma} \|x\|_{\sigma}$$

we get, for every $k \leq N$,

$$\mu_{k+1}(E^\vee) \leq \lambda_{kr+1}^\vee + \log(r). \quad (5)$$
Therefore, using (3) and (4), we get

\[
\mu_k(\bar{E}) + \mu_{N+1-k}(\bar{E}^*) \\
\leq \lambda_{(k-1)r+1} + \mu_{N+1-k}(\bar{E}^\vee) + \frac{r}{r} \log |\Delta_F| - \frac{r}{r} \log(\pi)
\]

\[
\leq \lambda_{k+1-r} + \lambda_{(N-k)r+1}^\vee + \log(r) + \frac{1}{r} \log |\Delta_F| - \frac{r}{r} \log(\pi).
\]

Since, by (2),

\[
\lambda_{k+1-r} + \lambda_{(N-k)r+1}^\vee \leq \lambda_{kr} + \lambda_{Nr - kr + 1}^\vee \leq \frac{3}{2} \log(Nr),
\]

Theorem 1 follows.

2 Lower bounds for the height of irreducible divisors

2.1

Let \( S = \text{Spec}(\mathcal{O}_F) \) be as above. Consider a semi-stable curve \( X \) over \( S \) such that \( X \) is regular and its generic fiber \( X_F \) is geometrically irreducible of genus \( g \). Let \( h_X \) be an hermitian metric, invariant under complex conjugation, on the variety \( X(\mathbb{C}) \) of complex points of \( X \). Let \( \omega_0 \) be the associated Kähler form, defined by the formula

\[
\omega_0 = \frac{i}{2\pi} h_X \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) dz d\bar{z}
\]

if \( z \) is any local holomorphic coordinate on \( X(\mathbb{C}) \). Let \( \bar{L} = (L, h) \) be an hermitian line bundle over \( X \) (with \( h \) invariant under complex conjugation). If \( L_C \) is the restriction of \( L \) to \( X(\mathbb{C}) \), the vector space \( H^0(X(\mathbb{C}), L_C) \) of holomorphic sections of \( L_C \) on \( X(\mathbb{C}) \) is equipped with the sup norm

\[
\|s\|_{\text{sup}} = \sup_{x \in X(\mathbb{C})} \|s(x)\|
\]

where \( \|\cdot\| \) is the norm defined by \( h \), and with the \( L^2 \)-norm

\[
\|s\|^2_{L^2} = \sup_{\sigma} \int_{X_\sigma} \|s(x)\|^2 \omega_0,
\]

where \( \sigma \) runs over all complex embeddings of \( F \) and \( X_\sigma = X \otimes \mathbb{C} \) is the corresponding complex variety. We let

\[
A(\bar{L}_C) = \sup_s \log(\|s\|_{\text{sup}}/\|s\|_{L^2}),
\]

where \( s \) runs over all sections of \( L_C \).
Consider the relative dualizing sheaf \( \overline{\omega}_{X/S} \) of \( X \) over \( S \), equipped with the
metric dual to \( h_X \), and let \( \overline{M} = \overline{L} \otimes \overline{\omega}_{X/S} \). We endow the \( \mathcal{O}_F \)-module
\[ H^1 = H^1(X, M^{-1}) \]
with the \( L^2 \)-metric and we denote by \( \mu_k(H^1) \) its successive minima, \( k = 1, \ldots, N = \dim_F H^1(X_F, M^{-1}) \).

Let now \( D \) be an irreducible divisor on \( X \), flat over \( S \), of degree \( d \) on \( X_F \). We are interested in the Faltings height \( h_{\overline{L}}(D) \) of \( D \) with respect to \( \overline{L} \). Recall [4] that \( h_{\overline{L}}(D) \in \mathbb{R} \) is the arithmetic degree of the restriction of \( \overline{L} \) to \( D \). Let \( t = \dim_F H^0(X_F, L(-D)) \) and assume that \( N > t \).

**Proposition 1.** The following inequality holds :
\[
\frac{h_{\overline{L}}(D)}{dr} \geq \mu_{N-t}(H^1) - A(\overline{L}_C) - C(N, F).
\]

**Proof.** To prove Proposition 1, let \( s \in H^0(X, L) \) be a section of \( L \) which does not belong to the vector space \( H^0(X_F, L(-D)) \). The restriction of \( s \) to \( D(\mathbb{C}) \) does not vanish hence, since \( D \) is irreducible, for any point \( P \) in \( D(\mathbb{C}) \) we have \( s(P) \neq 0 \). The height of \( D \) can be computed using \( s \) ([4] (3.2.2))
\[
h_{\overline{L}}(D) = h_{\overline{L}}(\text{div}(s|D)) - \sum_{\alpha} \log \|s(P_\alpha)\| \geq -\sum_{\alpha} \log \|s(P_\alpha)\|,
\]
where \( D(\mathbb{C}) = \sum_{\alpha} P_\alpha \). Next we have
\[
\sum_{\alpha} \log \|s(P_\alpha)\| \leq dr \log \|s\|_{\text{sup}} \leq dr(\log \|s\|_{L^2} + A(\overline{L}_C)).
\]

Let \( \overline{E} = (H^0(X, L), h_{L^2}) \). If \( t \) is the rank of \( H^0(X, L(-D)) \) we can choose \( s \) such that
\[
\log \|s\|_{L^2} \leq \mu_{t+1}(\overline{E}).
\]
By Theorem 1
\[
\mu_{t+1}(\overline{E}) \leq -\mu_{N-t}(\overline{E}^*) + C(N, F),
\]
and, by Serre duality, \( \overline{E}^* = H^1(X, M^{-1}) \) with the \( L^2 \)-metric. Therefore Proposition 1 follows from (6) and (7).

**2.2**

We keep the hypotheses of Proposition 1 and we denote by \( \overline{M} \cdot \overline{M} \in \mathbb{R} \) the self-intersection of the first arithmetic Chern class \( \overline{c}_1(\overline{M}) \in \text{CH}^1(X) \). Let \( \delta = \deg(L) \) be the degree of \( L \) on \( X_F \) and \( m = \deg(M) = \delta - 2g + 2 \).
Proposition 2. Assume that $\delta$ is even and that
\[2g + 1 \leq d \leq \delta \leq 2d - 2.\]
Then
\[
\frac{h_L(D)}{dr} \geq \frac{\bar{M} \cdot \bar{M}}{2mr} - A(\bar{L}_C) - C(N, F) - \log(\delta - g + 1).
\]

Proof. According to [11] Th. 2 and [11] 2.3.1, the inequality
\[
\mu_k(\bar{E}^*) \geq \frac{\bar{M} \cdot \bar{M}}{2mr} - \log(\delta - g + 1)
\]
holds when $k \geq \frac{m}{2} + g = \frac{\delta}{2} + 1$.

Consider the exact sequence of cohomology groups
\[
0 \rightarrow H^0(X_F, L(-D)) \rightarrow H^0(X_F, L) \rightarrow H^0(D_F, L|_D) \rightarrow H^1(X_F, L(-D)) \rightarrow H^1(X_F, L).
\]

We first assume that $\delta > d + 2g - 2$ i.e.
\[\text{deg}(L(-D)) > 2g - 2.\]
This implies $H^1(X_F, L(-D)) = 0$ and
\[N - t = \dim_F H^0(D_F, L|_D) = d.\]
Since $d \geq \frac{\delta}{2} + 1$, the proposition follows from Proposition 1 and (8).

Next, we assume that
\[d \leq \delta \leq d + 2g - 2,
\]
and we apply Clifford’s theorem to the Serre dual of $L(-D)$ on $X_F$. It is special unless $H^0(X_F, L(-D)) = 0$, in which case $t = 0$ hence
\[N - t = \delta - g + 1 \geq \frac{\delta}{2} + 1
\]
since $\delta \geq 2g$, and we can conclude as above.

When $H^0(X_F, L(-D))$ does not vanish, Clifford’s theorem says that
\[
\dim_F H^1(X_F, L(-D)) - 1 \leq \frac{1}{2} \deg(\omega_{X/S} \otimes L^{-1}(D)) = g - 1 - \frac{\delta}{2} + \frac{d}{2}.
\]
From (9) it follows that
\[N - t \geq d - \dim H^1(X_F, L(-D))\]
and therefore
\[ N - t \geq \frac{d}{2} + \frac{\delta}{2} - g. \]

Since \( d \geq 2g + 1 \) this implies
\[ N - t \geq \frac{\delta}{2} + 1 \]
and, since \( \delta \) is even, we get
\[ N - t \geq \frac{\delta}{2} + 1 \]
and the proposition follows from Proposition 1 and (8).

2.3

For any hermitian line bundle \( \bar{L} \) on \( X \), and any integer \( d \), we define
\[ e(\bar{L}, d) = \inf_{\deg(D) = d} \frac{h_{\bar{L}}(D)}{d} \]
and
\[ e(\bar{L}, \infty) = \lim_d e(\bar{L}, d). \]

**Theorem 2.** If \( \deg(L) \) is positive we have :
\[ e(\bar{L}, \infty) \geq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)}. \]

**Proof.** By definition
\[ e(\bar{L}, \infty) = \lim_{n \to \infty} \inf_{\deg(D) = d \geq n} \frac{h_{\bar{L}}(D)}{d}. \]
Assume that \( n \geq 2g + 1 \) and \( n \geq \deg(L) + 3 \). Then, for any \( d \geq n \), there exists an even integer \( k \) such that, if \( \delta = k \deg(L) \), the inequalities
\[ 2g + 1 \leq d \leq \delta \leq 2d - 2 \]
hold. Fix a Kähler metric \( h_X \) on \( X(\mathbb{C}) \) (invariant by complex conjugation) and let
\[ \bar{M} = \bar{L} \otimes \bar{\omega}^*. \]
From Proposition 2 applied to \( \bar{L} \otimes k \) we get, for any irreducible horizontal divisor \( D \) of degree \( d \),
\[ \frac{k h_{\bar{L}}(D)}{d} \geq \frac{\bar{M} \cdot \bar{M}}{2 \deg(M)} - A(\bar{L} \otimes k_{\mathbb{C}}) - C(N, F) - \log(\delta(g + 1)). \]
(10)
When \( n \) tends to infinity, the same is true for \( d \) and \( k \). Therefore
\[
\lim_{n \to \infty} \frac{\log(\delta(\delta - g + 1))}{k} = 0.
\] (11)

The rank \( N \) of \( H^0(X_F, L \otimes^k) \) is \( \delta - g + 1 \) so, by (1), we have
\[
\lim_{n \to \infty} \frac{C(N, F)}{k} = 0.
\] (12)

According to a result of Gromov \([8]\) Lemma 30 the quantity \( \exp A(\bar{L} \otimes^k) \) is bounded from above by a polynomial in \( k \). Therefore
\[
\lim_{n \to \infty} \frac{A(\bar{L} \otimes^k)}{k} = 0.
\] (13)

Finally
\[
\deg(M) = k \deg(L) - 2g + 2
\]
and
\[
\bar{M} \cdot \bar{M} = (k \bar{L} - \bar{\omega})^2,
\]
therefore
\[
\lim_{n \to \infty} \frac{\bar{M} \cdot \bar{M}}{k \deg(M)} = \frac{\bar{L} \cdot \bar{L}}{\deg(L)}.
\] (14)

The theorem follows from (10)–(14).

2.4

In \([13]\) S. Zhang defines
\[
e_L = \inf_D \frac{h_L(D)}{r \deg(D)}
\]
and
\[
e'_L = \lim_D \inf \frac{h_L(D)}{r \deg(D)},
\]
where \( D \) runs over all irreducible horizontal divisors on \( X \).

**Lemma 2.** When \( \deg(L) \) is positive we have
\[
e(\bar{L}, \infty) = r e'_L.
\]

**Proof.** By definition
\[
e(\bar{L}, \infty) = \lim_{n \to \infty} \inf_{\text{deg}(D) \geq n} \frac{h_L(D)}{\deg(D)}.
\] (15)

For any positive integer \( n \) let \( X(n) \) be the set of horizontal irreducible divisors \( D \) such that
\[
\deg(D) < n \quad \text{and} \quad h_L(D) \leq (e(\bar{L}, \infty) + 1) n.
\]
¿From [4], Cor. 3.2.5, we know that $X(n)$ is finite and we get

$$r e'(ar{L}) = \lim_{n \to \infty} \inf_{D \in X(n)} \frac{h_L(D)}{\deg(D)}.$$  \hspace{1cm} (16)

The complement of $X(n)$ consists of those $D$ such that either $\deg(D) \geq n$ or $\deg(D) \leq n$ and $h_L(D) > (e(L, \infty) + 1)n$. In the second case we have

$$\frac{h_L(D)}{\deg(D)} > e(L, \infty) + 1.$$  

Therefore (16) and (17) imply

$$r e'(ar{L}) = \inf(e(L, \infty), e(L, \infty) + 1) = e(L, \infty).$$

q.e.d.

When the first Chern form of $\bar{L}_C$ is semi-positive and $\deg(L_E) \geq 0$ for any vertical irreducible divisor $E$ on $X$, Theorem 6.3 in [13] states that

$$r e'_{\bar{L}} \geq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)}.$$  

Therefore Theorem 2 is not new in that case.

### 2.5

We come back to the situation of § 2.1 and 2.2, and we fix an integer $k \geq 1$. Furthermore we assume that the first Chern form of $M_C$ is positive and that $\deg(M_E) \geq 0$ for any vertical irreducible divisor $E$ on $X$. If $k > 1$ define

$$D(m, k) = (m + g) \sum_{\alpha=0}^{\inf(k-1, g)} \binom{m + g - k - \alpha}{k - 1 - \alpha} \binom{g}{\alpha},$$

and let $D(m, 1) = 1$.

**Proposition 3.** Assume that $\delta \geq d \geq k$ and that either $m > 2k > 2$ or $m > k = 1$. Then the following inequality holds:

$$\frac{h_L(D)}{dv} \geq \frac{k}{m^2 r} M^2 - \frac{2k}{m} e_M + e_M - A(\bar{L}_C) - C(N, F) - \frac{\log D(m, k)}{m^2} - 1.$$  

**Proof.** According to [10] Th. 4 i) (resp. [9] Th. 2) we have

$$1 + \mu_k(H^1) \geq \frac{k}{m^2 r} M \cdot \bar{M} - \frac{2k}{m} e_M + e_M - \frac{\log D(m, k)}{m^2} \hspace{1cm} (17)$$
as soon as \( m > 2k > 2 \) (resp. \( k = 1 \) and \( m > 1 \)). If we assume that \( \delta > d + 2g - 2 \) we have \( H^1(X_F, L(-D)) = 0 \) hence \( N - t = d \geq k \). Therefore

\[
\mu_{N - t}(H^1) \geq \mu_k(H^1)
\]

and the proposition follows from (18) and Proposition 1. When \( d \leq \delta \leq d + 2g - 2 \) we consider the Serre dual of \( L(-D) \) over \( X_F \). It is special unless \( t = 0 \), in which case

\[
N - t = \delta - g + 1 = m + g - 1 \geq k.
\]

When \( t \neq 0 \), Clifford’s theorem says that

\[
\dim H^1(X_F, L(-D)) - 1 \leq \frac{1}{2} \deg(\omega \otimes L^{-1}(D)) = g - 1 - \frac{\delta}{2} + \frac{d}{2},
\]

and

\[
N - t \geq \frac{\delta}{2} + \frac{d}{2} - g.
\]

But

\[
\frac{\delta}{2} - g = \frac{m}{2} - 1 \geq k - 1,
\]

hence

\[
N - t \geq k + \frac{d}{2} - 1
\]

and \( N - t \geq k \) since \( d \geq 1 \).

Again, the proposition follows from (18) and Proposition 1.

2.6

We now assume that \( g \geq 2 \) and we let \( \tilde{\omega} \) be the relative dualizing sheaf \( \omega_{X/S} \) of \( X \) over \( S \), equipped with its Arakelov metric \([1]\). As in \([2, 3]\) above we consider

\[
e(\tilde{\omega}, d) = \inf_{\deg(D) = d} \frac{h_{\tilde{\omega}}(D)}{d},
\]

(18)

**Theorem 3.** There is a constant \( C = C(g, r) \) such that the following inequalities hold:

\[
e(\tilde{\omega}, d) \geq \frac{\tilde{\omega} \cdot \tilde{\omega}}{4(g - 1)} \frac{dg + g - 1}{d + 2g - 2} - \frac{g - 1}{d + 2g - 2} \log |\Delta_F| - C \frac{\log(d)}{d},
\]

(19)

and, if \( d \geq 2g + 1 \),

\[
e(\tilde{\omega}, d) \geq \frac{\tilde{\omega} \cdot \tilde{\omega}}{4(g - 1)} \frac{d - 2g + 1}{d - g} - \frac{g - 1}{d - g} \log |\Delta_F| - C \frac{\log(d)}{d}.
\]

(20)

\(^1\)Theorem 4, i) in \([10]\) assumes that \( g \geq 2 \) and the metric on \( L_C \) is admissible in the sense of Arakelov \([1]\), but these extra hypotheses are not used in the proof of that statement.
Proof. To prove (19) we apply Proposition 3 to a power $\tilde{L} = \bar{\omega}^\otimes n$ of $\bar{\omega}$. We take $k = d$. When $d = 1$, (19) follows from the inequalities

$$e(\bar{\omega}, 1) \geq r e_{\bar{\omega}}$$

and

$$r e_{\bar{\omega}} \geq \frac{\bar{\omega} \cdot \bar{\omega}}{4g(g-1)}$$ (21)

(cf. [5]). When $d > 1$, the condition $m > 2k$ in Proposition 3 becomes

$$(n-1)(g-1) > d,$$

i.e.

$$n > \frac{d}{g-1} + 1.$$ We take

$$n = \left\lfloor \frac{d}{g-1} \right\rfloor + 2.$$ According to Proposition 3, for any irreducible horizontal divisor $D$ of degree $d$,

$$\frac{h_{\tilde{L}}(D)}{d} \geq k \frac{\bar{\omega} \cdot \bar{\omega}}{4(g-1)^2} + r e_{\bar{\omega}} \left( n - 1 - \frac{k}{g-1} \right) - r \left( A(\tilde{L}_C) + C(N, F) + \frac{\log D(m, k)}{m^2} + 1 \right).$$

Using the lower bound (21) for $e_{\bar{\omega}}$ and the fact that $h_{\tilde{L}}(D) = n h_{\bar{\omega}}(D)$

we get

$$e(\bar{\omega}, d) \geq \frac{\bar{\omega} \cdot \bar{\omega}}{4g(g-1)} \frac{k + n - 1}{n} - \frac{r}{n} \left( A(\tilde{L}_C) + C(N, F) + \frac{\log D(m, k)}{m^2} + 1 \right).$$ (22)

Since

$$n \leq 2 + \frac{d}{g-1}$$

we get

$$\frac{k + n - 1}{n} \geq \frac{dg + g - 1}{d + 2g - 2}.$$ (23)

Gromov's estimate for $A(\bar{\omega}^\otimes n)$ implies

$$\frac{A(\bar{\omega}^\otimes n)}{n} = O \left( \frac{\log(n)}{n} \right) = O \left( \frac{\log(d)}{d} \right).$$ (24)
From (1) we deduce that
\[ \frac{r}{n} C(N, F) = \frac{1}{n} \log |\Delta_F| + O\left(\frac{\log(n)}{n}\right). \] (25)

Finally, according to [10] § 3.8,
\[ \log D(m, k) = O(m \log(m)) = O(d \log(d)). \] (26)

The inequality (19) follows from (22)–(26).

To prove (20) we apply Proposition 2 to a power \( \tilde{L} = \tilde{\omega}^\otimes n \) of \( \tilde{\omega} \). We get
\[ e(\tilde{\omega}, d) \geq \frac{n - 1}{n} \frac{\tilde{\omega} \cdot \tilde{\omega}}{4(g - 1)} - \frac{r}{n} \left( A(L_C) + C(N, F) + \log(\delta(\delta - g + 1)) \right) \] (27)
as soon as
\[ 2g + 1 \leq d \leq (2g - 2)n \leq 2d - 2. \]

We choose
\[ n = \left\lfloor \frac{d - 1}{g - 1} \right\rfloor \geq \frac{d - g}{g - 1} \]
in which case
\[ \frac{n - 1}{n} \geq \frac{d - 2g + 1}{d - g}. \]
The second summand of the right-hand side of (27) is estimated as above. This proves (20).

3 Upper bounds for the height of irreducible divisors

3.1

Let \( X \) and \( h_X \) be as in § 2.1. Let \( \tilde{L} \) and \( \tilde{M} \) be two hermitian line bundles on \( X \). We assume that \( \deg(L) > 0 \) and \( \deg(L_E) \geq 0 \) for every vertical irreducible divisor \( E \) on \( X \). Let \( D_0 \) be an irreducible horizontal divisor,
\[ N = \dim_F H^0(X_F, M) \]

and
\[ t = \dim_F H^0(X_F, M(-D_0)). \]

We assume that \( N > t \). Denote by \( \mu_k(H^1) \), \( k = 1, \ldots, N \), the successive minima of \( H^1 = H^1(X, \omega_{X/S} \otimes M^{-1}) \) equipped with its \( L^2 \)-metric. We write \( \tilde{L} \cdot \tilde{M} \in \mathbb{R} \) for the arithmetic intersection of \( \tilde{c}_1(\tilde{L}) \) with \( \tilde{c}_1(\tilde{M}) \), and we write \( D \cap D_0 \) to mean that \( D \) is an irreducible horizontal divisor meeting \( D_0 \) properly.
Proposition 4. The following inequality holds:
\[
\inf_{D \in D \cap D_0} \frac{h_L(D)}{r \deg(D)} \leq \frac{\bar{L} \cdot M}{r \deg(M)} - \mu_{N-t}(H^1) \frac{\deg(L)}{\deg(M)} + \frac{\deg(L)}{\deg(M)} (A(\bar{M}_C) + C(N, F)).
\]

Proof. Let \( \bar{E} = (H^0(X, M), h_{L^2}) \) and choose a section \( s \in H^0(X, M) \) such that
\( s \notin H^0(X_F, M(-D_0)) \) and
\[
\log \| s \|_{L^2} \leq \mu_{t+1}(\bar{E}).
\]
If \( \text{div}(s) \) is the divisor of \( s \) we get (4) (3.2.2)
\[
\bar{L} \cdot M = h_L(\text{div}(s)) - \int_{X(C)} \log \| s \| c_1(\bar{L}_C)
\geq h_L(\text{div}(s)) - r \deg(L)(\mu_{t+1}(\bar{E}) + A(\bar{M}_C)). \tag{28}
\]
We can write
\[
\text{div}(s) = \sum \alpha D_\alpha + V
\]
where each \( D_\alpha \) is irreducible and flat over \( S \), and \( V \) is effective and vertical on \( X \). Therefore, by our assumption on \( L \), we have
\[
h_L(\text{div}(s)) \geq \sum \alpha h_L(D_\alpha)
\]
and
\[
\deg(\text{div}(s)) = \sum \alpha \deg(D_\alpha).
\]
Therefore, since each \( D_\alpha \) is transverse to \( D_0 \),
\[
\frac{h_L(\text{div}(s))}{\deg(M)} \geq \inf_{\alpha} \frac{h_L(D_\alpha)}{\deg(D_\alpha)} \geq \inf_{D \in D \cap D_0} \frac{h_L(D)}{\deg(D)}. \tag{29}
\]
From Theorem 1 we get
\[
\mu_{t+1}(\bar{E}) \leq -\mu_{N-1}(H^1) + C(N, F) \tag{30}
\]
and the proposition follows from (28), (29) and (30).

3.2

We keep the notation of the previous section and we let
\[
\bar{K} = \bar{M} \otimes \bar{\omega}_{X/S}, \; m = \deg(M) \; \text{and} \; d_0 = \deg(D_0).
\]
Proposition 5. Assume that $m$ is even and

$$2g + 1 \leq d_0 \leq m \leq 2d_0 - 2.$$ 

The following inequality holds:

$$\inf_{D \cap D_0} \frac{h_L(D)}{r \deg(D)} \leq \frac{\bar{L} \cdot \bar{M}}{m} - \frac{\bar{K} \cdot \bar{K}}{2r \deg(K)} \frac{\deg(L)}{m} + \frac{\deg(L)}{m}(A(M) + C(N, F) + \log(m(m - g + 1))).$$

Proof. The number $\mu_{N - \ell}(H^1)$ can be estimated from below using [11] exactly as in the proof of Proposition 2. Therefore the proposition follows from Proposition 4.

3.3

Let $\bar{L}$ be an hermitian line bundle on $X$ such that $\deg(L) > 0$ and $\deg(L|E) \geq 0$ for any irreducible vertical divisor $E$ on $X$. For any integer $d_0 \geq 1$ consider

$$e'(\bar{L}, d_0) = \sup_{D_0} \inf_{D \cap D_0} \frac{h_L(D)}{\deg(D)},$$

where $D_0$ runs over all irreducible horizontal divisors of degree $d_0$. Let

$$e'(\bar{L}, \infty) = \lim_{d_0} \sup e'(\bar{L}, d_0).$$

Theorem 4. The following inequality holds:

$$e'(\bar{L}, \infty) \leq \frac{\bar{L} \cdot \bar{L}}{2 \deg(L)}.$$ 

Proof. As in the proof of Theorem 2, when the integer $n$ is big enough, for any $d_0 \geq n$ we can choose an even power $\bar{M}$ of $\bar{L}$ such that, if $m = \deg(M)$, the following inequalities hold:

$$2g + 1 \leq d_0 \leq m \leq 2d_0 - 2.$$ 

Then we apply Proposition 5 to $\bar{L}$ and $\bar{M}$. If $\bar{K} = \bar{M} \otimes \bar{\omega}_{X/S}^*$ we get

$$\lim_{n \to \infty} \frac{\bar{K} \cdot \bar{K}}{\deg(K)} \frac{\deg(L)}{m} = \frac{\bar{L} \cdot \bar{L}}{\deg(L)} \quad (31)$$

and

$$\lim_{n \to \infty} \frac{\bar{L} \cdot \bar{M}}{m} = \frac{\bar{L} \cdot \bar{L}}{\deg(L)} \quad (32).$$
By the same estimates as in the proof of Theorem 2 we get
\[ \lim_{n \to \infty} \left( A(M_C) + C(N, F) + \log(m(m - g + 1)) \right)/m = 0. \] (33)

The theorem follows from (31), (32), (33) and Proposition 5.

**Remark.** For any \( d_0 \) we have
\[ re_L \leq \epsilon'(L, d_0). \]

Therefore Theorem 3 implies
\[ re_L \leq \frac{\bar{L} \cdot \bar{L} - \deg(L)}{2 \deg(L)}. \]

But it does not follow from [13], Th. 6.3.

### 3.4

We come back to the notation of 3.2 and we let
\[ k = \deg(K) = m - 2g + 2. \]

We fix an integer \( h \geq 1 \). We assume that the first Chern form of \( \bar{K}_C \) is positive and that \( \deg(K_{|E}) \geq 0 \) for every irreducible vertical divisor \( E \) on \( X \).

**Proposition 6.** Assume that \( m \geq d_0 \geq h \) and that either \( k > 2h > 2 \) or \( k > h = 1 \). Then the following inequality:
\[ \inf_{D \in D_0} \frac{h_L(D)}{r \deg(D)} \leq \frac{\bar{L} \cdot \bar{M} + \deg(L)}{m} - \frac{\deg(L)}{m} \left( \frac{h}{k^2} K^2 - \frac{2h}{k} e_K + e_K \right) \]
\[ + \frac{\deg(L)}{m} \left( A(M_C) + C(N, F) + \frac{\log D(h) h^2}{h^2} + 1 \right). \] (34)

**Proof.** This inequality follows from Proposition 4 by bounding \( \mu_{N - t}(H^1) \) from below in the same way as in the proof of Proposition 3.

### 3.5

Assume now that \( g \geq 2 \) and let \( \bar{\omega} \) be \( \omega_{X/S} \) with its Arakelov metric. Recall that
\[ e'(\bar{\omega}, d_0) = \sup_{\deg(D_0) = d_0} \inf_{D \in D_0} \frac{h_L(D)}{\deg(D)}. \]

**Theorem 5.** There exists a constant \( C = C(g, r) \) such that the following inequalities hold:
\[ e'(\bar{\omega}, d_0) \leq \frac{\bar{\omega} \cdot \bar{\omega}}{4(g - 1)} + \frac{2g - 1}{4g(d_0 + 2g - 2)} \bar{\omega} \cdot \bar{\omega} + \frac{g - 1}{d_0 + g - 1} \log |\Delta_F| + C \frac{\log(d_0)}{d_0}, \] (35)
and, when $d_0 \geq 2g + 1$,

$$e'(\bar{\omega}, d_0) \leq \frac{\bar{\omega} \cdot \bar{\omega}}{4(g - 1)} + \frac{\bar{\omega} \cdot \bar{\omega}}{4(d_0 - g)} + \frac{g - 1}{d_0 - g} \log |\Delta_F| + C \frac{\log(d_0)}{d_0}. \quad (36)$$

**Proof.** To prove (35) we apply Proposition 6 with $\bar{L} = \bar{\omega}$, $\bar{M} = \bar{\omega} \otimes n$ and $h = d_0$. When $d_0 = 1 < k$ we have $n(g - 1) \geq g$. When $d_0 > 1$ and

$$k = n(2g - 2) - 2g + 2 > 2d_0$$

we get $n(g - 1) > d_0 + g - 1$.

In both cases we choose

$$n = 2 + \left\lfloor \frac{d_0}{g - 1} \right\rfloor.$$

The right hand side of (34) (Proposition 6) becomes $X_1 + X_2$, with

$$X_1 = \frac{n \bar{\omega} \cdot \bar{\omega}}{rn(2g - 2)} - \frac{1}{n} \left( \frac{d_0 \bar{\omega} \cdot \bar{\omega}}{(2g - 2)^2 r} + \left( 1 - \frac{2d_0}{n(2g - 2)} \right) (n - 1) e_\bar{\omega} \right)$$

and

$$X_2 = \frac{\text{deg}(L)}{m} \left( A(M_C) + C(N, F) + \frac{\log D(k, h)}{h^2} + 1 \right).$$

As in the proof of Theorem 3 we get

$$X_2 \leq C \frac{\log(d_0)}{d_0} + \frac{1}{nr} \log |\Delta_F|$$

and

$$\frac{1}{n} \leq \frac{g - 1}{d_0 + g - 1}.$$

On the other hand, since

$$re_\bar{\omega} \geq \frac{\bar{\omega} \cdot \bar{\omega}}{4g(g - 1)},$$

we get

$$r X_1 \leq \bar{\omega} \cdot \bar{\omega} \left( \frac{1}{2g - 2} - \frac{d_0}{n(2g - 2)^2} - \frac{n - 1}{4g(g - 1)n} + \frac{d_0}{4ng(g - 1)^2} \right)$$

$$= \bar{\omega} \cdot \bar{\omega} \left( \frac{2g - 1 - d_0}{4g(g - 1)} \frac{1}{d_0} \right).$$

Since $n \leq 2 + \frac{d_0}{g - 1}$ we get

$$r X_1 \leq \bar{\omega} \cdot \bar{\omega} \left( \frac{2g - 1 - (d_0 - 1)(g - 1)}{2g - 2 + d_0} \right)$$

$$= \bar{\omega} \cdot \bar{\omega} \left( \frac{2g - 1}{4g(d_0 + 2g - 2)} \right) \bar{\omega} \cdot \bar{\omega}.$$
This proves (35).

To prove (36) we apply Proposition 5 when \( \bar{L} = \bar{\omega} \) and \( \bar{M} = \bar{\omega}^\otimes n \). If \( d_0 \leq m \leq 2d_0 - 2 \) we get

\[
e(\bar{L}, d_0) \leq rY_1 + rY_2
\]

where

\[
Y_2 = \frac{\deg(L)}{m} \left( A(\bar{M}_C) + C(N, F) + \log(m(m - g + 1)) \right)
\]

\[
\leq C \frac{\log(d_0)}{d_0} + \frac{1}{nr} \log |\Delta_F|
\]

as in the proof of Theorem 3, and

\[
r Y_1 = \frac{\bar{L} \cdot \bar{M}}{m} - \frac{\bar{K} \cdot \bar{K}}{2 \deg(K)} \frac{\deg(L)}{m}
\]

\[
= \frac{\bar{\omega} \cdot \bar{\omega}}{2g - 2} - \frac{n - 1}{4n(g - 1)} \bar{\omega} \cdot \bar{\omega}
\]

\[
= \frac{\bar{\omega} \cdot \bar{\omega}}{4(g - 1)} + \frac{\bar{\omega} \cdot \bar{\omega}}{4n(g - 1)}.
\]

Since \( n(g - 1) \leq d_0 - 1 \) we can assume that

\[
n = \left\lfloor \frac{d_0 - 1}{g - 1} \right\rfloor,
\]

hence \( n \geq \frac{d_0 - 1}{g - 1} - 1 \). This implies

\[
\frac{1}{n} \log |\Delta_F| \leq \frac{g - 1}{d_0 - g} \log |\Delta_F|
\]

and

\[
r Y_1 \leq \frac{\bar{\omega} \cdot \bar{\omega}}{4(g - 1)} + \frac{\bar{\omega} \cdot \bar{\omega}}{4(d_0 - g)},
\]

from which (36) follows.
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