On the DLM/FD methods for simulating neutrally buoyant swimmers moving in non-Newtonian shear thinning fluids

Ang Li1 | Tsorng-Whay Pan2 | Roland Glowinski2

1Department of Mathematics, Lane College, Jackson, Tennessee, USA
2Department of Mathematics, University of Houston, Houston, Texas, USA

Correspondence
Tsorng-Whay Pan, Department of Mathematics, University of Houston, Houston, TX 77204, USA.
Email: pan@math.uh.edu

Abstract
In this article we discuss the generalization of a Lagrange multiplier based fictitious domain (DLM/FD) method to simulating the motion of neutrally buoyant particles of non-symmetric shape in non-Newtonian shear-thinning fluids. Numerical solutions of steady Poiseuille flow of non-Newtonian shear-thinning fluids are compared with the exact solutions in a two-dimensional channel. Concerning a self-propelled swimmer formed by two disks, the effect of shear-thinning makes the swimmer moving faster and decreases the critical Reynolds number (for the moving direction changing to the opposite one) when decreasing the value of the power index $n$ in the Carreau-Bird model.

KEYWORDS
Carreau-Bird model, fictitious domain method, finite element approximations, neutrally buoyant particle, operator splitting, self-propelled swimmer

1 | INTRODUCTION

The study of micro-scale robots swimming in viscous fluids is a phenomenon not only of fundamental theoretical interest, but it also has the potential to dramatically change many aspects of medicine by navigating bodily fluids to perform tasks such as targeted drug delivery, intravenous tumor detection, and minimally invasive micro-surgical operations.1 Micro-robots, like microorganisms, swimming in a low Reynolds number ($Re$) regime, encounter stringent constraints due to the dominance of viscous over inertial forces.2,3 As a result of kinetic reversibility, Purcell’s scallop theorem2 rules out reciprocal motion (i.e., strokes with time-reversal symmetry) for effective locomotion in the absence of inertia in Newtonian fluids. Another important fact is that many biological fluids, including blood, respiratory, and cervical mucus, have non-Newtonian properties, such as viscoelasticity and shear-thinning viscosity (i.e., the viscosity decreases non-linearly with the shear rate4,5). The physics governing micro-swimmer locomotion at low Reynolds numbers in Newtonian fluids is relatively well understood, while low Reynolds propulsion in non-Newtonian fluids remains largely unexplored.3 Since the scallop theorem does not hold in non-Newtonian fluids,6,7 it should be possible to design and build novel micro-scale robots that can move with reciprocal motion in non-Newtonian fluids. A study on a reciprocal sliding sphere swimmer in a shear-thinning (non-elastic) fluid8 suggests that propulsion is achievable by reciprocal motion, in which backward and forward strokes occur at different rates. Similarly, another study in Reference 9 reported that a symmetric micro-scallop, a single-hinge micro-swimmer, can propel in shear-thickening and shear-thinning (non-elastic) fluids at low Reynolds number. Thus reciprocal swimming mechanism can help in designing biomedical micro-devices that can propel by a simple actuation scheme in biological fluids. A recent experimental work reported in Reference 10 was about the spontaneous symmetry-breaking propulsion of rotating spherical microparticles within viscoelastic shear-thinning fluids. Two equal and opposite propulsion states along the sphere’s rotation axis can be obtained, and the
motion along the rotating axis is surprising since it is symmetry-prohibited. This spontaneous symmetry-breaking shows us another possible approach to circumvent other restrictions on propulsion, revising notions of micro-robotic design and control.

In this article, we have focused on the effect of shear-thinning on the swimmer motion in a two-dimensional channel. To study such effect by direct numerical simulation, we have adapted the Carreau-Bird viscosity model for shear-thinning property. For example, in References 11 and 12, particle settling in generalized Oldroyd-B viscoelastic fluids with shear-thinning property was studied numerically by using the Carreau-Bird viscosity model. Direct simulation of the motion of particles has been carried out by using an arbitrary Lagrangian–Eulerian moving mesh technique with finite element method. The power law, which is the other commonly used model, was used to investigate the effects of both shear-thickening and thinning on the motion of scallop swimmers numerically in Reference 9. A finite element method with the dynamic mesh-adaptation for moving boundaries was used to study how the scallop swimmer can swim by reciprocal motion at low (not zero) Reynolds numbers. For simulating the swimmer motion in non-Newtonian shear-thinning fluids, we have extended a Lagrange multiplier based fictitious domain (DLM/FD) method developed for disks in Reference 13 to simulate the motion of a neutrally buoyant non-symmetric swimmer in non-Newtonian shear-thinning fluids. This non-symmetric swimmer of two disks with different radii is formed by connecting their mass centers with a massless spring like the one in, for example, References 14 and 15. To keep the main flavor of the fictitious domain methods, that is, the usage of fast solvers to solve the resulting linear systems, we have split the diffusion term in our proposed methodologies. Accurate computational solutions are obtained by the proposed methods for Poiseuille flow of non-Newtonian shear-thinning fluids in a two-dimensional channel (thanks to the recently published exact solutions for such Poiseuille flow by Griffiths in Reference 16). Then we have further validated our proposed numerical methods by studying the migration of 56 disks in a two-dimensional channel, which was considered in Reference 17. For the effect of shear-thinning on the swimmer motion, we have obtained that, via direct numerical simulation, the swimmer moves faster and the critical Reynolds number (for the moving direction changing to the opposite one) decreases when decreasing the value of the power index in the Carreau-Bird model. The content of this article is as follows: In Section 2, we introduce a fictitious domain formulation of the model problem associated with the neutrally buoyant particle cases; then in Section 3 we discuss the time and space discretizations and the related numerical methodology. We then present and discuss the numerical results in Section 4. Concluding remarks are given in Section 5.

2 | A FICTITIOUS DOMAIN FORMULATION OF THE MODEL PROBLEM

In Reference 13, Pan and Glowinski developed a distributed Lagrange multiplier/fictitious domain (DLM/FD) method to simulate the motion of neutrally buoyant disks freely moving in a Newtonian viscous incompressible fluid in a two dimensional channel. Later they extended in References 18-20 such DLM/FD method to simulate and investigate the motion of neutrally buoyant balls in circular Poiseuille flows. In this article, we have extended the above DLM/FD method to study the motion of a self-propelled swimmer in non-Newtonian shear-thinning fluids. This non-symmetric swimmer of two disks with different radii is formed by connecting their mass centers with a massless spring like the one in, for example, References 14 and 15. In this section, we will address the DLM/FD formulation for such problem. Let Ω ⊂ ℝ² be a rectangular region. We suppose that Ω is filled with a non-Newtonian viscous incompressible fluid of density ρf and contains a moving neutrally buoyant rigid particle B centered at \( G = \{G_1, G_2\}^T \) of density \( \rho_f \) as shown in Figure 1; the flow is modeled by the Navier–Stokes equations and the motion of \( B \) is described by the Euler–Newton’s equations. Following the DLM/FD formulation developed in Reference 13, we define

\[
W_{0,R} = \{v | v \in (H^1(\Omega))^2, v = 0 \text{ on the top and bottom boundary of } \Omega \text{ and } v \text{ is periodic in the } x_1 \text{ direction} \},
\]

\[
L_2 = \{ q | q \in L^2(\Omega), \int_\Omega q \, dx = 0, \},
\]

\[
\Lambda_0(t) = \{ \mu | \mu \in (H^1(B(t)))^2, < \mu, e_i >_{B(t)} = 0, \; i = 1, 2, < \mu, \frac{\partial G}{\partial x} >_{B(t)} = 0 \}
\]

with \( e_1 = \{1, 0\}^T, e_2 = \{0, 1\}^T, \frac{\partial G}{\partial x} = \{-(x_2 - G_2), x_1 - G_1\}^T \) and \( < \cdot, \cdot >_{B(t)} \) an inner product on \( \Lambda_0(t) \) which can be the standard inner product on \( (H^1(B(t)))^2 \) (see Reference 21, Section 5, for further information on the choice of \( < \cdot, \cdot >_{B(t)} \).
FIGURE 1  An example of two-dimensional flow region with one rigid body

Then the fictitious domain formulation with distributed Lagrange multipliers for flow around a freely moving neutrally buoyant particle is as follows

For a.e. \( t > 0 \), find \( \mathbf{u}(t) \in W_{0,p}, \ p(t) \in L^2_0, \ \mathbf{V}_G(t) \in \mathbb{R}^2, \ G(t) \in \mathbb{R}^2, \ \omega(t) \in \mathbb{R}, \ \lambda(t) \in \Lambda_0(t) \) such that

\[
\begin{aligned}
\rho_f \int_{\Omega} \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] \cdot \mathbf{v} \, dx + 2 \int_{\Omega} \eta \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) \, dx - \int_{\Omega} p \nabla \cdot \mathbf{v} \, dx \\
- < \lambda, \mathbf{v} >_{B(t)} = \rho_f \int_{\Omega} \mathbf{g} \cdot \mathbf{v} \, dx + \int_{\Omega} \mathbf{F} \cdot \mathbf{v} \, dx, \ \forall \mathbf{v} \in W_{0,p},
\end{aligned}
\]

\[
\int_{\Omega} q \nabla \cdot \mathbf{u}(t) \, dx = 0, \ \forall q \in L^2(\Omega),
\]

\[
< \mu, \mathbf{u}(t) >_{B(t)} = < \mu, \mathbf{u}_p(t) >_{B(t)}, \ \forall \mu \in \Lambda_0(t),
\]

\[
\frac{d\mathbf{G}}{dt} = \mathbf{V}_G.
\]

\[
\mathbf{V}_G(0) = \mathbf{V}_G^0, \ \omega(0) = \omega^0, \ G(0) = G^0 = \{ G_1^0, G_2^0 \}^T,
\]

\[
\mathbf{u}(x, 0) = \mathbf{u}_0(x) = \begin{cases} \mathbf{u}_0(x), & \forall x \in \Omega \setminus \overline{B(0)}, \\ \mathbf{V}_G^0 + \omega^0 [-(x_2 - G_2^0), x_1 - G_1^0]^T, & \forall x \in \overline{B(0)}, \end{cases}
\]

where \( \mathbf{u} \) and \( p \) denote velocity and pressure, respectively, \( \eta(\dot{\gamma}) \) is the fluid viscosity, \( \lambda \) is a Lagrange multiplier, \( \mathbf{D}(\mathbf{v}) = (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)/2 \), \( \mathbf{g} \) is gravity, \( \mathbf{F} \) is the pressure gradient pointing in the \( x_1 \) direction for particle moving in a Poiseuille flow, \( \mathbf{V}_G \) is the translation velocity of the particle \( B \), and \( \omega \) is the angular velocity of \( B \). We suppose that the no-slip condition holds on \( \partial B \). We also use, if necessary, the notation \( \varphi(t) \) for the function \( x \rightarrow \varphi(x, t) \). In Equation (3), \( \mathbf{u}_p(t) \) is a part of actual particle motion given as follows

\[
\mathbf{u}(x, t) = \mathbf{V}_G(t) + \omega(t) [-(x_2 - G_2^0), x_1 - G_1^0]^T + \mathbf{u}_p(x, t).
\]

Thus we have \( \mathbf{u}_p = 0 \) if \( B \) is a single piece neutrally buoyant rigid particle like a disk suspended in fluid; but for \( B \) as a swimmer formed by two neutrally buoyant disks considered in this article, \( \mathbf{u}_p(t) \) is a given reciprocal motion of the two disks with respect to the swimmer mass center (see Section 4). Shear thinning can be easily added to the above model by using the Carreau-Bird viscosity model

\[
\frac{\eta - \eta_\infty}{\eta_0 - \eta_\infty} = (1 + (\lambda_3 \dot{\gamma})^3)^{(n-1)/2},
\]

where \( \dot{\gamma} \) is the shear rate.
where $\eta_0$ (resp., $\eta_\infty$) is the fluid viscosity at zero shear rate (resp., infinite shear rate), $\lambda_3$ is the relaxation time, $\dot{\gamma}$ is determined by the second invariant of the rate of strain tensor $(\dot{\gamma})^2 = (\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}))/2$, and the power index $n$ is in $(0, 1]$ (if $n = 1$, the fluid is a Newtonian fluid with constant viscosity $\eta_0$).

Remark 1. In (3), the rigid body motion in the region occupied by the particle is enforced via Lagrange multipliers $\lambda_i$. To recover the translation velocity $\mathbf{V}_G(t)$ and the angular velocity $\omega(t)$, we solve the following equations

$$
\begin{align*}
&\langle \mathbf{e}_i, \mathbf{u}(t) - \mathbf{V}_G(t) - \omega(t) \mathbf{G}^\perp \rangle_{B(t)} = 0, \text{ for } i = 1, 2, \\
&\langle \mathbf{G}^\perp, \mathbf{u}(t) - \mathbf{V}_G(t) - \omega(t) \mathbf{G}^\perp \rangle_{B(t)} = 0.
\end{align*}
$$

3 SPACE APPROXIMATION AND TIME DISCRETIZATION

Concerning the finite element based space approximation of problem (1)–(6), we use $P_1$-iso-$P_2$ and $P_1$ finite elements for the velocity field and pressure, respectively. (see, e.g., References 22 and 23(ch.5)). More precisely with $h$ a space discretization step, we introduce a finite element triangulation $T_h$ of $\Omega$ and then $T_{2h}$ a triangulation twice coarser. Then we approximate $W_{0,p}, L^2$ and $L^0$ by the following finite dimensional spaces

$$W_{0,h} = \{ \mathbf{v}_h | \mathbf{v}_h \in (C^0(\Omega))^2, \quad \mathbf{v}_h|_T \in P_1 \times P_1, \quad \forall T \in T_h, \quad \mathbf{v}_h = 0 \text{ on the top and bottom boundary of } \Omega \text{ and } \mathbf{v} \text{ is periodic in the } x_1 \text{ direction} \},$$

$$L^2_h = \{ q_h | q_h \in C^0(\Omega), \quad q_h|_T \in P_1, \quad \forall T \in T_{2h}, \quad q_h \text{ is periodic in the } x_1 \text{ direction} \},$$

and

$$L^0_{0,h} = \{ q_h | q_h \in L^0_h, \quad \int_\Omega q_h \, dx = 0 \},$$

respectively; in (9)–(11), $P_1$ is the space of polynomials in two variables of degree $\leq 1$.

A finite dimensional space approximating $\Lambda_0(t)$ is defined in the following: let $\{ \mathbf{x}_i \}_{i=1}^N$ be a set of points covering $\overline{B(t)}$; we define then

$$\Lambda_h(t) = \{ \mu_h | \mu_h = \sum_{i=1}^N \mu_i \delta(\mathbf{x} - \mathbf{x}_i), \quad \mu_i \in \mathbb{R}^2, \quad \forall i = 1, \ldots, N \},$$

where $\delta(\cdot)$ is the Dirac measure at $\mathbf{x} = 0$. Then, instead of the inner product of $(H^1(B(t)))^2$ we shall use $\langle \cdot, \cdot \rangle_{B(t)}$ defined by

$$\langle \mu_h, \mathbf{v}_h \rangle_{B(t)} = \sum_{i=1}^N \mu_i \cdot \mathbf{v}_h(\mathbf{x}_i), \quad \forall \mu_h \in \Lambda_h(t), \quad \forall \mathbf{v}_h \in W_{0,h}.$$  

Then we approximate $\Lambda_0(t)$ by

$$\Lambda_{0,h}(t) = \{ \mu_h | \mu_h \in \Lambda_h(t), \langle \mu_h, \mathbf{e}_i \rangle_{B(t)} = 0, \quad i = 1, 2, \quad \langle \mu_h, \mathbf{G}^\perp \rangle_{B(t)} = 0 \}.$$  

A typical choice of points for defining (12) is to take the grid points of the velocity mesh internal to the particle $B$ and whose distance to the boundary of $B$ is greater than, for example, $h$, and to complete with selected points from the boundary of $B(t)$ (see Figure 2).
Using the above finite dimensional spaces leads to the following approximation of problem (1)–(6):

For a.e. $t > 0$, find $u(t) \in W_{0,h}$, $p(t) \in L^{2}_{0,h}$, $V_{G}(t) \in \mathbb{R}^{2}$, $G(t) \in \mathbb{R}^{2}$, $\omega(t) \in \mathbb{R}$, $\lambda_{h}(t) \in \Lambda_{0,h}(t)$ such that

$$\begin{cases}
\rho \int_{\Omega} \frac{\partial u_{h}}{\partial t} + (u_{h} \cdot \nabla) u_{h} \cdot v \, dx + 2 \int_{\Omega} \eta D(u_{h}) : D(v) \, dx \\
- \int_{\Omega} p_{h} \nabla \cdot v \, dx - \lambda_{h} v >_{B_{h}(t)} = \int_{\Omega} F \cdot v \, dx, \quad \forall v \in W_{0,h},
\end{cases}$$

$$\int_{\Omega} q \nabla \cdot u_{h}(t) dx = 0, \quad \forall q \in L^{2}_{h},$$

$$< \mu, u_{h}(t) >_{B_{h}(t)} = < \mu, u_{p}(t) >_{B_{h}(t)}, \quad \forall \mu \in \Lambda_{0,h}(t),$$

$$\frac{dG}{dt} = V_{G},$$

$$V_{G}(0) = V^{0}_{G}, \quad \omega(0) = \omega^{0}, \quad G(0) = G^{0} = \{G^{0}_{1}, G^{0}_{2}\}^T,$$

$$u_{h}(x, 0) = \bar{u}_{0,h}(x) \text{ (with } \nabla \cdot \bar{u}_{0,h} = 0).$$

Applying a first order operator splitting scheme, namely the Lie scheme^24 (see also References 23 and 25(ch.2)), to discretize Equations (15)–(20) in time, we obtain (after dropping some of the subscripts $h$):

$$u^{0} = \bar{u}_{0,h}, \quad V^{0}_{G}, \quad \omega^{0}, \quad \text{and } G^{0} \text{ are given;}$$

For $n \geq 0$, knowing $u^{n}$, $V^{n}_{G}$, $\omega^{n}$ and $G^{n}$, compute $u^{n+1/6}$ and $p^{n+1/6}$ via the solution of

$$\begin{cases}
\rho \int_{\Omega} \frac{u^{n+1/6} - u^{n}}{\Delta t} \cdot v \, dx - \int_{\Omega} p^{n+1/6} \nabla \cdot v \, dx = 0, \quad \forall v \in W_{0,h}.
\end{cases}$$

$$\int_{\Omega} q \nabla \cdot u^{n+1/6} \, dx = 0, \quad \forall q \in L^{2}_{h}, \quad u^{n+1/6} \in W_{0,h}, \quad p^{n+1/6} \in L^{2}_{0,h}.$$
Then compute \( u^{n+2/6} \) via the solution of

\[
\begin{align*}
\int_{\Omega} \frac{\partial u}{\partial t} \cdot v \, dx + \int_{\Omega} (u^{n+1/6} \cdot \nabla)u \cdot v \, dx &= 0, \quad \forall v \in W_{0,h}, \quad \text{on } (t^n, t^{n+1}), \\
\left( u^n \right) = u^{n+1/6}, \quad u(t) \in W_{0,h},
\end{align*}
\]

(23)

\[
\left( u^{n+2/6} \right) = u(t^{n+1}).
\]

(24)

Next, compute \( u^{n+3/6} \) via the solution of

\[
\begin{align*}
\rho \int_{\Omega} \frac{\partial u^{n+3/6} - u^{n+2/6}}{\Delta t} \cdot v \, dx + 2\alpha \int_{\Omega} \eta(u^{n+2/6}) D(u^{n+3/6}) : D(v) \, dx &= 0, \\
\forall v \in W_{0,h}, \quad u^{n+3/6} \in W_{0,h}.
\end{align*}
\]

(25)

Now predict the position and the translation velocity of the center of mass of the particles as follows: Take \( V_{G}^{n+2/0} = V_{G}^{n} \) and \( G_{n+2/0} = G_{n}^{n} \), then predict the new position of the particle via the following sub-cycling and predicting-correcting technique:

For \( k = 1, \ldots, N \), compute

\[
\begin{align*}
V_{G}^{n+3/4, k} &= V_{G}^{n+3/4, k-1} + F(G_{n+3/4, k-1}) \Delta t/2N, \\
G_{n+3/4, k} &= G_{n+3/4, k-1} + (V_{G}^{n+3/4, k} + V_{G}^{n+3/4, k-1}) \Delta t/4N, \\
V_{G}^{n+3/4, k} &= V_{G}^{n+3/4, k-1} + (F(G_{n+3/4, k}) + F(G_{n+3/4, k-1})) \Delta t/4N, \\
G_{n+3/4, k} &= G_{n+3/4, k-1} + (V_{G}^{n+3/4, k} + V_{G}^{n+3/4, k-1}) \Delta t/4N, \\
\end{align*}
\]

endo;

and let \( V_{G}^{n+3/4} = V_{G}^{n+3/4, N} \), \( G_{n+3/4} = G_{n+3/4, N} \).

(30)

Now, compute \( u^{n+5/6} \), \( \lambda^{n+5/6} \), \( V_{G}^{n+5/6} \), and \( o^{n+5/6} \) via the solution of

\[
\begin{align*}
\rho \int_{\Omega} \frac{u^{n+5/6} - u^{n+3/6}}{\Delta t} \cdot v \, dx + 2\beta \int_{\Omega} \eta(u^{n+3/6}) D(u^{n+5/6}) : D(v) \, dx &= 0, \\
\forall v \in W_{0,h}, \\
\mu \frac{u^{n+5/6} - \mu(t^{n+1})}{\beta_{h, n+1/2}} = 0, \quad \forall \mu \in \Lambda_{0,h}^{n+4/6}, \quad u^{n+5/6} \in W_{0,h}, \quad \lambda^{n+5/6} \in \Lambda_{0,h}^{n+4/6},
\end{align*}
\]

(31)

and solve for \( V_{G}^{n+5/6} \) and \( o^{n+5/6} \) from

\[
\begin{align*}
< e, u^{n+5/6} - V_{G}^{n+5/6} - o^{n+5/6} >_{H^{1/2}} & = 0, \quad \text{for } i = 1, 2, \\
< e, u^{n+5/6} - V_{G}^{n+5/6} - o^{n+5/6} >_{H^{1/2}} & = 0.
\end{align*}
\]

(32)

Finally, take \( V_{G}^{n+1/0} = V_{G}^{n+5/6} \) and \( G_{n+1/0} = G_{n+4/6} \); then predict the final position and translation velocity as follows:
For $k = 1, \ldots , N$, compute

$$
\nabla \rho_{G}^{n+1,k} = \nabla \rho_{G}^{n+1,k-1} + \mathbf{F}(\nabla \rho_{G}^{n+1,k-1}) \Delta t / 2N,
$$
(33)

$$
\mathbf{G}^{n+1,k} = \mathbf{G}^{n+1,k-1} + (\nabla \rho_{G}^{n+1,k} + \nabla \rho_{G}^{n+1,k-1}) \Delta t / 4N,
$$
(34)

$$
\nabla \omega_{G}^{n+1,k} = \nabla \omega_{G}^{n+1,k-1} + (\mathbf{F}\prime(\nabla \omega_{G}^{n+1,k}) + \mathbf{F}\prime(\nabla \omega_{G}^{n+1,k-1})) \Delta t / 4N,
$$
(35)

$$
\mathbf{G}^{n+1,k} = \mathbf{G}^{n+1,k-1} + (\nabla \omega_{G}^{n+1,k} + \nabla \omega_{G}^{n+1,k-1}) \Delta t / 4N,
$$
(36)

and let $\nabla \rho_{G}^{n+1} = \nabla \rho_{G}^{n+1,N}$, $\mathbf{G}^{n+1} = \mathbf{G}^{n+1,N}$, and set $\mathbf{u}^{n+1} = \mathbf{u}^{n+5/6}$, $\omega^{n+1} = \omega^{n+5/6}$.

In algorithm (21)–(36), we have $t^{n+s} = (n + s) \Delta t$, $\lambda_{0,h}^{n+s} = \lambda_{0,h}(t^{n+s})$, $B_{0}^{n+s}$ is the region occupied by the particle centered at $\mathbf{G}^{n+s}$, and $\mathbf{F}\prime$ is a short range repulsion force which prevents the particle/particle and particle/wall penetration.\(^{21,26}\) Finally, $\alpha$ and $\beta$ verify $\alpha + \beta = 1$; we have chosen $\alpha = 1$ and $\beta = 0$ in the numerical simulations discussed later.

The methodologies used to solve subproblems (22), (23), (25), and (31) numerically are briefly discussed in the following. The degenerated quasi-Stokes problem (22) is solved by a preconditioned conjugate gradient method introduced in Reference 27, in which the discrete elliptic problems from the preconditioning are solved by a matrix-free fast solver from FISHPAK by Adams et al. in Reference 28. The advection problem (23) for the velocity field is solved by a wave-like equation method as in References 29, 30, which is a time matching scheme. Since the advection subproblem is decoupled from others, a proper time step satisfied the stability condition can be chosen easily. To enforce the rigid body motion inside the region occupied by the particles, we employ the conjugate gradient method discussed in Reference 13 for problem (31). Problem (25) is a classical discrete elliptic problem. To solve problem (25) by the same matrix-free fast solver, we have modified the equation as follows:

$$
\rho \int_{\Omega} \nabla \mathbf{u}^{n+3/6} - \mathbf{u}^{n+2/6} \cdot \mathbf{v} dx + a\hat{\eta} \int_{\Omega} \nabla \mathbf{u}^{n+3/6} \cdot \nabla \mathbf{v} dx = \int_{\Omega} \mathbf{F} : \mathbf{v} dx
$$

$$
+ a\hat{\eta} \int_{\Omega} \nabla \mathbf{u}^{n+2/6} \cdot \nabla \mathbf{v} dx - 2a \int_{\Omega} \eta(\mathbf{u}^{n+2/6}) \mathbf{D}(\mathbf{u}^{n+2/6}) : \mathbf{D}(\mathbf{v}) dx,
$$
(37)

In problem (37), the value of $\nabla \mathbf{u}^{n+2/6}$ at each mesh node is obtained via a super-convergent scheme discussed in Reference 31, and constant viscosity $\hat{\eta}$ can be $\eta_{0}$ or the one at the maximum or mean shear rate of the flow. In this article, all computational results of shear-thinning fluids were obtained by algorithm (21)–(36) with the above subproblem (37) instead of problem (25) and $\hat{\eta} = \eta_{0}$.

## 4 | Numerical Experiments and Discussion

### 4.1 | Poiseuille flow of non-Newtonian shear thinning fluids

In the first validation case, we have considered the pressure driven Poiseuille flow of a non-Newtonian shear-thinning fluid. The computational results are validated by comparing them with the exact solutions recently reported in Reference 16. Since there is no particle involved in the computation, the computational results are obtained from steps (21)–(24), and (37) in algorithm (21)–(36). In this section, all dimensional quantities are in the centimeter-gram-second system.

The computational domain is $\Omega = (0, L) \times (-H, H) = (0, 8\, \text{cm}) \times (-1\, \text{cm}, 1\, \text{cm})$. We have used structured triangular meshes in all computations. The mesh sizes for the velocity field are $h_{v} = 1/32\, \text{cm}$, $1/64\, \text{cm}$, and $1/96\, \text{cm}$, while the mesh size for the pressure is $h_{p} = 2h_{v}$. The time step is chosen to be $\Delta t = 0.001\, \text{s}$. The pressure gradient pointing in the $x_{1}$ direction is $(F_{1}, F_{2})^{T} = (-G, 0)^{T}$ with $G = 1\, \text{g} \text{cm}^{-2} \text{s}^{-1}$ so that the flow moves from left to right. The flow
velocity is periodic in the $x_1$ (horizontal) direction and zero at the top and bottom boundary of the domain. The initial flow velocity is 0 cm s$^{-1}$. The fluid density is 1 g cm$^{-3}$ and the fluid viscosities are $\eta_0 = 1$ g cm$^{-1}$ s$^{-1}$ and $\eta_\infty = 0$ g cm$^{-1}$ s$^{-1}$. The three values of power index $n$ considered in this section are 1/3, 1/2, and 2/3. Then as obtained in Reference 16, for $n = 1/3$, 1/2, and 2/3, the values of relaxation time $\lambda_3$ in Equation (7) are $\sqrt{2}/\sqrt{3}$ s, $\sqrt{2}/\sqrt{3}$ s, and $\sqrt{3}/\sqrt{2}$ s, respectively. The numerical results obtained with $h = 1/96$ cm and the actual profiles from the exact solutions obtained in Reference 16 are presented in Figures 3–5 for the steady state horizontal profile of Poiseuille flow.

The curves agree very well in those figures. The maximal errors of steady numerical solutions shown in Figures 3–5 are $4.513406 \times 10^{-2}$, $2.268202 \times 10^{-3}$, and $1.736727 \times 10^{-3}$, respectively, for $n = 1/3$, 1/2, and 2/3. The $L^2$-errors of steady numerical solutions in Table 1 show that the expected error order for a $P_1$ finite element approximation has been obtained.
FIGURE 5  Normalized horizontal velocity profiles obtained for $h = 1/96$ cm (red dashed line) and exact solution (blue solid line) for $n = 2/3$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 $L^2$-errors for the steady state velocity field of Poiseuille flow for the power indices $n = 1/3$, $1/2$, and $2/3$

| $n$  | $h$  | $L^2$-error | order | $n$  | $h$  | $L^2$-error | order |
|------|------|-------------|-------|------|------|-------------|-------|
| 1/3  | 1/32 | 8.510003368×$10^{-3}$ |       | 1/2  | 1/32 | 1.405723487×$10^{-4}$ |       |
| 1/3  | 1/64 | 2.475437454×$10^{-3}$ | 1.833 | 1/2  | 1/64 | 3.667227424×$10^{-5}$ | 1.938 |
| 1/3  | 1/96 | 1.157654939×$10^{-3}$ | 1.815 | 1/2  | 1/96 | 1.633156504×$10^{-5}$ | 1.959 |
| 2/3  | 1/32 | 7.847740427×$10^{-5}$ |       |      |      |             |       |
| 2/3  | 1/64 | 1.990780506×$10^{-5}$ | 2.008 |      |      |             |       |
| 2/3  | 1/96 | 8.704140007×$10^{-6}$ |       |      |      |             | 2.001 |

4.2  On the migration of 56 disks in the Poiseuille flow of a shear thinning fluid

The second validation case we consider concerns the motion and cross stream migration of 56 neutrally buoyant identical circular cylinders in the pressure driven Poiseuille flow of a non-Newtonian shear-thinning fluid, which was a test problem studied numerically in Reference 17. In this section, all dimensional quantities are in the centimeter-gram-second system. The proposed numerical scheme has been validated by comparing our results with those reported in Reference 17. The computational domain is $\Omega = (0, 21 \text{ cm}) \times (0, 10 \text{ cm})$. The initial flow velocity and particle velocities are 0 cm/s. The pressure drop is $F_1 = -2 \text{ g cm}^{-2} \text{ s}^{-1}$ so that the maximum horizontal speed is 25 cm s$^{-1}$ when there is no particle for the Poiseuille flow of a Newtonian fluid with the viscosity $\eta_0 = 1 \text{ g cm}^{-1} \text{ s}^{-1}$. For non-Newtonian shear-thinning fluids modeled by equation (7), the viscosity at infinity shear rate is $\eta_\infty = 0.1 \eta_0 \text{ g cm}^{-1} \text{ s}^{-1}$ and the relaxation time is $\lambda_3 = 1 \text{ s}$. The values of the power index $n$ are 0.4, 0.5, and 0.7. The fluid and particle densities, $\rho_f$ and $\rho_p$, are 1 g cm$^{-3}$. The disk diameter $d$ is 1 cm and $u_p(t)$ is 0 cm s$^{-1}$ for each disk. Thus the solid fraction for the cases considered in this section is 20.94%. The mesh size for the velocity field is $h_v = 1/16 \text{ cm}$, while the mesh size for the pressure is $h_p = 2h_v$. The time step is $\Delta t = 0.001 \text{ s}$. Finally, the particle Reynolds number is defined as $Re_p = \rho_0 U_p a / \eta_0$ where $U_p$ is the average particle speed and $a$ is the disk radius.

For the case of a Newtonian fluid without shear-thinning ($n = 1$), the migration of the disks is shown in Figure 6. There are no disks at the center line and the disks tend to accumulate a distance of 0.6 from the center line of the channel, which is known as the Segre-Silberberg effect. In general, lubrication forces move the particles away from the channel walls while the curvature of the velocity profile moves the particles away from the centerline of the channel. For a small
Migration of 56 neutrally buoyant identical particles in the pressure driven flow of a Newtonian fluid without shear thinning ($n = 1.0$). (A) Velocity profile of the fluid without particles (top curve) and velocities of particles (bottom curve); (B) particle positions in the channel at time $t = 100$ s [Colour figure can be viewed at wileyonlinelibrary.com]

neutrally buoyant ball in a plane Poiseuille flow via the matched asymptotic expansion methods, it was found that the of the fluid without particles and the velocities of the particles at $t = 100$ s are shown in Figure 6. At $t = 100$ s, the maximum particle velocity is 14.0864 cm s$^{-1}$, while the maximum fluid velocity without particles is 25 cm s$^{-1}$ so the slip velocity is 10.9136 cm s$^{-1}$ near the centerline. In Reference 17 the maximum particle velocity is 15 cm s$^{-1}$ near the centerline and the maximum slip velocity is 10 cm s$^{-1}$. Both results are in good agreement. Also, at $t = 100$ s, the average particle speed is 10.8278 cm s$^{-1}$. Thus the particle Reynolds number is $Re_p = 5.4139$.

The velocity profile with shear-thinning is quite different from the one at $n = 1$ as shown in Figure 7. The effects of shear-thinning can be increased by decreasing the power index $n$. The maximum particle velocities are 23.0664, 29.9960, and 30.1694 cm s$^{-1}$, while the maximum fluid velocities are 47.3956, 84.5356, and 111.9036 cm s$^{-1}$ for $n = 0.7, 0.5$, and 0.4, respectively. Hence we obtain the slip velocities in Table 2 by following the approach given in Reference 17. Their slip velocities are 23, 54, and 81.4 cm s$^{-1}$ for $n = 0.7, 0.5$, and 0.4, respectively, implying our numerical results are in a good agreement with those in Reference 17. The average particle speeds are 24.9323, 24.1436, and 18.1764 cm s$^{-1}$ for $n = 0.7, 0.5$, and 0.4, respectively, and thus the associated particle Reynolds numbers are 12.46615, 12.0718, and 9.0822.

Due to the Segre-Silberberg effect, the disks separate into two groups (see Figures 7 and 8), the top group and the bottom one, according to the particle positions shown in Figure 8. Following these particle separations, there are three
FIGURE 7  Velocity profile of the fluid without particles and velocities of the 56 particles: $n = 0.7$ at $t = 42.46$ s (top), $n = 0.5$ at $t = 31.78$ s (middle), and $n = 0.4$ at $t = 27.41$ s (bottom) [Colour figure can be viewed at wileyonlinelibrary.com]
FIGURE 8 Positions of the 56 neutrally buoyant particles in the pressure driven flow of a shear thinning fluid: \( n = 0.7 \) at \( t = 42.46 \) s (top), \( n = 0.5 \) at \( t = 31.78 \) s (middle), and \( n = 0.4 \) at \( t = 27.41 \) s (bottom) [Colour figure can be viewed at wileyonlinelibrary.com]

particle free zones, the one in the middle (called middle zone), the one next to the top wall (called the top boundary zone), and the one next to the bottom wall (the bottom boundary zone), which can be identified easily from the particle velocity plots in Figure 7. The sizes of these particle-free zones are called the gaps and are shown in Table 2, in which the average boundary gap is the mean of the two boundary gaps. As pointed out in Reference 17, the disk accumulation is enhanced so that the middle gap of particle-free zone increases and the average boundary gap decreases, when decreasing the value of the power index \( n \). Our results are consistent with those findings obtained in Reference 17 qualitatively.

4.3 On the motion of a self-propelled swimmer

Even Purcell’s scallop theorem\(^2\) rules out the reciprocal motion (i.e., strokes with time-reversal symmetry) for effective locomotion in the absence of inertia \((Re = 0)\) in Newtonian fluids, the motion of a simple reciprocal model swimmer (an asymmetric dumbbell) in a Newtonian fluid at intermediate Reynolds numbers has been studied extensively in References 14 and 15. In this section, we have considered the case of a neutrally buoyant self-propelled swimmer freely moving in a non-Newtonian shear-thinning fluid. Such swimmer is formed by two disks of different sizes connecting by a massless
### TABLE 2

The slip velocity (cm/s), average of gap sizes (cm) next to two walls, and gap size (cm) in the center of the channel for the power indices \( n = 0.4, 0.5, 0.7, \text{and} 1 \)

| \( n \) | Slip velocity | Average boundary gap | Middle gap |
|---|---|---|---|
| 1 | 10.9136 | 0.95167 | 1.60403 |
| 0.7 | 24.3292 | 0.85906 | 1.67752 |
| 0.5 | 54.5396 | 0.83388 | 1.80585 |
| 0.4 | 81.7342 | 0.83313 | 2.00206 |

**FIGURE 9**

An example of the relative position of the two disks: The reciprocal motion with respect to the swimmer mass center at different time over one period is \( d(t) = d_0 + A \sin(\omega t) \). Disk centers of mass (blue *) are indicated [Colour figure can be viewed at wileyonlinelibrary.com]

spring at their mass centers (see Figure 9). Hence it is a long asymmetric body, but symmetric with respect to the line segment connecting two disk mass centers. The reciprocal motion of two disks with respect to the swimmer mass center is prescribed (see Figure 9 and Equation (38)). Due to the asymmetric disk motion with respect to the swimmer mass center, a swimmer can move in either directions parallel to the string connecting the two disk mass centers. It is interesting to find out how the shear-thinning impacts the swimmer motion. Following the units used in Reference 14, we assume in this section all dimensional quantities are in the physical meter, kilogram, and second units.

The computational domain is \( \Omega = (-4 \text{ m}, 4 \text{ m}) \times (-4 \text{ m}, 4 \text{ m}) \). The flow velocity is \( 0 \text{ m/s} \) initially and zero at the boundary of domain \( \Omega \). The densities of fluid and swimmer are \( 10^3 \text{ kg m}^{-3} \), which is the water density. The initial position of the swimmer mass center is at \( (0, 0) \). The distance between the two disk centers is \( d(t) = d_0 + A \sin(\omega t) \) as in Reference 14 where the equilibrium distance between the two disk centers is \( d_0 = 10r_0 \), the amplitude is \( A = r_0 \), and \( r_0 = 0.15 \text{ m} \). The angular frequency is \( \omega = 20\pi \text{ s}^{-1} \) (so the frequency is 10). The massless spring connecting the two disks is located in the direction parallel to the \( x_1 \) direction. The radius of the larger disk \( D_M \) is \( R_M = 2r_0 = 0.3 \text{ m} \) and that of the smaller disk \( D_m \) is \( R_m = r_0 = 0.15 \text{ m} \). The amplitude \( A \) is \( A = A_M + A_m \) where the component for the larger disk is \( A_M = \frac{AR_m^2}{R_m^2 + R_M^2} \) and the one for the smaller disk is \( A_m = \frac{AR_m^2}{R_m^2 + R_M^2} \). The Reynolds number is defined by \( Re = A_mR_m\omega/\nu \) (as in Reference 14) where \( \nu \) is the kinematic viscosity of the fluid at zero shear rate. We vary the value of \( Re \) to obtain the fluid viscosity at zero shear rate. We have used structured triangular meshes in all simulations. The mesh size for the velocity field is \( h_v = 1/128 \text{ m} \), while the mesh size for pressure is \( h_p = 2h_v \). The time step is \( \Delta t = 0.001 \text{ s} \). Concerning the shear-thinning, the ratio of viscosity at infinity shear rate and the one at zero shear rate is 0.1 and the relaxation time is \( \lambda_1 = 1 \text{ s} \). The values of the power index \( n \) are 0.5, 0.6, 0.7, 0.8, and 0.9.

In this section, we have mainly studied the effect of the power index \( n \) and of the Reynolds number on the moving direction and speed of the swimmer. In order to focus on the swimmer moving direction, the swimmer is only allowed to freely move in the \( x_1 \)-direction. To accommodate this restricted motion and the Dirichlet boundary condition, we have to
modify the spaces defined previously as follows

\[ W_0 = \{ \mathbf{v} \mid \mathbf{v} \in (H^1(\Omega))^2, \mathbf{v} = \mathbf{0 \ on \ the \ boundary \ of \ } \Omega \}, \]
\[ \Lambda_0(t) = \{ \mu \mid \mu \in (H^1(B(t)))^2, \langle \mu, \mathbf{e}_1 \rangle_{B(t)} = 0 \}, \]
\[ W_{0,h} = \{ \mathbf{v}_h \mid \mathbf{v}_h \in (C^0(\overline{\Omega}))^2, \mathbf{v}_h |_{T} \in P_1 \times P_1, \forall T \in \mathcal{T}_h, \]
\[ \mathbf{v}_h = \mathbf{0 \ on \ the \ boundary \ of \ } \Omega \}, \]
\[ L_{0,h}^2 = \{ q_h | q_h \in C^0(\overline{\Omega}), q_h |_{T} \in P_1, \forall T \in \mathcal{T}_{2h}, \int_{\Omega} q_h \, dx = 0 \}, \]
\[ \Lambda_{0,h}(t) = \{ \mu_h \mid \mu_h \in \Lambda_h(t), \langle \mu_h, \mathbf{e}_1 \rangle_{B_h(t)} = 0 \}. \]

Following the above given restricted motion, the reciprocal motion with respect to the swimmer mass center is

\[
\mathbf{u}_p(t) = \begin{cases} 
(A_{M} \omega \cos(\omega t), 0)^T, \forall \mathbf{x} \in D_M, \\
(-A_{m} \omega \cos(\omega t), 0)^T, \forall \mathbf{x} \in D_m.
\end{cases}
\]

Applying algorithm (21)–(36) (with the subproblem (37) instead of problem (25) for shear-thinning fluids) and above discrete spaces, we have studied numerically the motion of two disk swimmer in the \( x_1 \) direction in Newtonian and non-Newtonian fluids. We first show the average velocities of swimmer for different values of \( Re \) and two power index values, \( n = 1 \) and 0.8, obtained by the velocity mesh sizes \( h = 1/128 \) m, 1/256 m, and 1/512 m and time step \( \Delta t = 0.001 \) s.
Each average value was computed from the last twenty periods of horizontal velocity for $0 \leq t \leq 20$ s. All those average velocities for each Reynolds number are in a good agreement as shown in Figure 10.

Let us consider a swimmer of two same size disks, that is, $R_m = R_M = 2r_0 = 0.3$ m, suspended in a Newtonian fluid ($n = 1$). Due to the symmetry of reciprocal motion of the two same size disks (see Figure 11), the swimmer mass center does not move as expected, at least for $Re = 1, 3, 5, 7, 9, 11, 13,$ and $15$, the cases we have considered. But for the case of two different size disks in a Newtonian fluid with $R_M = 2r_0$ and $R_m = r_0$ for $r_0 = 0.15$ m, the swimmer can move in the positive or negative $x_1$-direction up to the Reynolds number. For example, in Figure 12, we have shown these two typical swimmer motions. For both, the swimmer mass center oscillates; but its average horizontal velocity is positive (resp., negative) for $Re = 15$ (resp., $Re = 5$). There is a critical Reynolds number $Re_c$ so that the swimmer moves to the left (resp., right) for $Re < Re_c$ (resp., $Re > Re_c$) if the larger disk is located on the right side (see the plots in Figures 13 and 14). The critical Reynolds number is about $Re_c = 12.5$ for the swimmer shown in Figure 13.

However when swimming in a non-Newtonian shear-thinning fluid, the average moving speed ($V_1$) per period in the positive-$x_1$ direction increases when decreasing the value of the power index $n$ as shown in Figures 14 for the power index
**Figure 13** Swimmer position at $t = 0$ s (top), velocity field next to the swimmer and swimmer position at $t = 20$ s for $Re = 5$ (middle) and 15 (bottom) and $n = 1$

**Figure 14** Average horizontal speed $V_1$ of a swimmer versus $Re$ for the power index $n = 0.5, 0.6, 0.7, 0.8, 0.9$, and 1 [Colour figure can be viewed at wileyonlinelibrary.com]
values, \( n = 0.5, 0.6, 0.7, 0.8, 0.9, \) and 1. Interestingly the average swimming velocity is also slowed down when the swimmer moves in the minus-\( x_1 \) direction when decreasing the value of the power index \( n \). Thus the swimmer position is further to the right when comparing its position at \( t = 20 \) s as in Figure 15. Our computational results show that the shear-thinning does have a strong effect on the motion of a swimmer made of two different size disks, and the critical Reynolds number decreases for a non-Newtonian shear-thinning fluid as the power index value decreases. Another interesting observation from the results shown in Figure 14 is that for a 2-disk swimmer moving in the negative-\( x_1 \) direction, its velocity \( V_1 \) reaches its fastest speed first and then tends to slow down to zero when decreasing \( Re \) to negligible value. This result is consistent with the one for \( n = 1 \) obtained in Reference 14.

5 | CONCLUSIONS

In this article, a distributed Lagrange multiplier/fictitious domain method has been developed for simulating a non-symmetric (two-disk) particle moving freely in non-Newtonian shear-thinning fluids. The Carreau-Bird model is adapted for the fluid shear-thinning property. Our numerical methods have been validated by comparing our computed solutions with the recently published exact solutions for Poiseuille flows of generalized Newtonian fluids in two dimensions. Accurate numerical solutions have been obtained and the \( L^2 \)-errors of the velocity field show the expected second order for \( P_1 \) finite elements. For many neutrally buoyant particle cross stream migration in a Poiseuille flow of non-Newtonian shear-thinning fluids, the particles migrate away from the wall and center of the channel. The size of the boundary gap decreases and that of the middle gap increases when reducing the power index value. These results are consistent qualitatively with those reported in Reference 17. Concerning the two-disk swimmer, the effect of shear-thinning does let the swimmer move faster when moving in the positive \( x_1 \)-direction, that is, from the smaller to larger disk, and
the critical Reynolds number (for changing the moving direction from the left to the right) decreases for smaller value of the power index. In a near future, we want to apply the above methodologies to study the motion of micro-robots, such as two-disk swimmers, in generalized Newtonian and viscoelastic fluids with different specified disk motions at the lower Reynolds numbers.

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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

ORCID
Tsorng-Whay Pan https://orcid.org/0000-0002-1526-2615

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