Kirszbraun’s extension theorem fails for Almgren’s multiple valued functions

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ABSTRACT. We show that there is no analog of Kirszbraun’s extension theorem for Almgren’s multiple valued functions.

KEYWORDS: Kirszbraun’s extension theorem, multiple valued functions, geometric measure theory.

MSC (2010): 54C20, 49Q20.

1. Introduction

Almgren’s multiple valued functions play a key role in geometric measure theory since they are employed in the analysis of the branching behaviour of minimal surfaces in codimension larger than or equal to 2 (see [1] and [3]).

We recall basic definitions for multiple valued functions. Let $Q$ be a positive integer, then

$$A_Q(\mathbb{R}^n) = \left\{ \sum_{i=1}^{Q} [P_i] : P_i \in \mathbb{R}^n, 1 \leq i \leq Q \right\},$$

where $[P]$ denotes the Dirac measure at $P$. This space is endowed with the $L^2$-Wasserstein distance: for $T_1 = \sum_{i=1}^{Q} [P_i]$ and $T_2 = \sum_{i=1}^{Q} [S_i]$ we define

$$\mathcal{G}(T_1, T_2) = \min_{\sigma \in P_Q} \left\{ \sum_{i=1}^{Q} |P_i - S_{\sigma(i)}|^2 \right\},$$

where $P_Q$ denotes the group of permutations of $\{1, \ldots, Q\}$.

One of the main ingredients in the theory of multiple valued functions is the following extension theorem (see Theorem 1.7 in [3]).

**1.1. Theorem.** Let $B \subset \mathbb{R}^m$ be a measurable set and let $f : B \to A_Q(\mathbb{R}^n)$ be Lipschitz. Then there exists a constant $C = C(m, Q) > 0$ and an extension $\bar{f} : \mathbb{R}^m \to A_Q(\mathbb{R}^n)$ of $f$ such that

$$\text{Lip}(\bar{f}) \leq C\text{Lip}(f).$$

In the Euclidean case, the classical Kirszbraun’s extension theorem (see Theorem 2.10.43 in [2]) states that an analogous result holds with $C = 1$. More precisely, Kirszbraun’s theorem states that Lipschitz functions defined on a subset of $\mathbb{R}^m$ with values in $\mathbb{R}^n$ (both endowed with the Euclidean distance) can be extended to all of $\mathbb{R}^m$ without increasing the Lipschitz constant. The
conclusion may fail as soon as $\mathbb{R}^m$ or $\mathbb{R}^n$ is remetrized by a metric which is not induced by an inner product, as shown in 2.10.44 in [2].

In §2 we prove that the conclusion also fails in the setting of multiple valued functions, by exhibiting a $\sqrt{2}/3$-Lipschitz function $f$ defined on a subset of $\mathbb{R}^2$ with values in $A_2(\mathbb{R}^2)$ with the property that any Lipschitz extension $\bar{f}$ to $\mathbb{R}^2$ has Lipschitz constant at least 1.

2. Construction of the counterexample

Let $A = (0, 1), B = (-\sqrt{3}/2, -1/2), C = (\sqrt{3}/2, -1/2)$ and let $P_1, \ldots, P_6$ be the vertices of a regular hexagon centered at 0, with side length 1: $P_1 = (0, 1), P_2 = (\sqrt{3}/2, 1/2), P_3 = (\sqrt{3}/2, -1/2), P_4 = (0, 1), P_5 = (-\sqrt{3}/2, -1/2)$ and $P_6 = (-\sqrt{3}/2, 1/2).

Consider the map $f : \{A, B, C\} \subset \mathbb{R}^2 \to A_2(\mathbb{R}^2)$ given by

\[
\begin{align*}
  f(A) &= \llbracket P_1 \rrbracket + \llbracket P_4 \rrbracket, \\
  f(B) &= \llbracket P_2 \rrbracket + \llbracket P_5 \rrbracket, \\
  f(C) &= \llbracket P_3 \rrbracket + \llbracket P_6 \rrbracket.
\end{align*}
\]

The Lipschitz constant of $f$ is $\sqrt{2}/3$. In fact, $|A - B| = |A - C| = |B - C| = \sqrt{3}$ and

$G(f(A), f(B)) = G(f(A), f(C)) = G(f(B), f(C)) = \sqrt{2}$.

Now consider a map $\bar{f} : \{A, B, C\} \cup \{0\} \to A_2(\mathbb{R}^2)$. We will prove that if $\bar{f}$ is an extension of $f$, then the Lipschitz constant of $\bar{f}$ is at least 1. Indeed, let $\bar{f}(0) = \llbracket S_1 \rrbracket + \llbracket S_2 \rrbracket$. Assume by contradiction $\text{Lip}(\bar{f}) < 1$, then $S_1$ and $S_2$ should lie on different sides of the perpendicular bisector of the line segment $P_1 P_4$ (see Figure 1). In fact, if for example $S_1$ and $S_2$ both lie in the half plane \{ $y \leq 0$ \} then $|P_i - S_i| \geq 1$ for $i = 1, 2$ which implies $G(\bar{f}(0), f(A)) \geq 1$. The latter contradicts the assumption since $|A| = 1$.

![Figure 1. S_1 and S_2 must lie on different sides of y = 0](image)

Arguing analogously for $P_2 P_5$ and $P_3 P_6$ we deduce that $S_1$ and $S_2$ must lie on opposite sectors among the six determined by the three perpendicular bisectors.
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Figure 2.

Without loss of generality we can assume that $S_1$ belongs to the intersection of the sector containing $P_1$ and the first orthant (see Figure 2).

Since $|S_1 - P_6| \leq |S_1 - P_3|$ and $|S_2 - P_6| \geq |S_2 - P_3|$, we can estimate the distance between $\bar{f}(0)$ and $f(C)$ and get

$$G(\bar{f}(0), f(C))^2 = |S_1 - P_6|^2 + |S_2 - P_3|^2 \geq \frac{3}{4} + \frac{1}{4} = 1,$$

which contradicts our assumption since $|C| = 1$.

2.1. Remark. Following the proof of Theorem 1.1 in [3] one can explicitly determine the growth of the constant $C$ depending on $m$ and $Q$. It would be desirable to understand if the sharp constant has the same growth (or at least if $C(m, Q)$ goes to infinity as either $m$ or $Q$ goes to infinity). Clearly, just considering one-point extensions cannot lead to an answer to this question as the following general argument shows.

Let $(M, d_M)$ and $(N, d_N)$ be two complete metric spaces such that $M$ has the Heine-Borel property, $A$ a subset of $M$ and $f : A \to N$ be Lipschitz continuous. Then for every $P \in M \setminus A$ there exists a Lipschitz extension $\bar{f} : A \cup \{P\} \to N$ such that

$$\text{Lip}(\bar{f}) \leq 2\text{Lip}(f).$$

In fact, let $S \in \bar{A}$ be a point realizing the distance between $P$ and $\bar{A}$. Let $\bar{f}(P)$ be the value at $S$ of the unique continuous extension of $f$ to $\bar{A}$, denoted by $f(S)$. Then for every $y \in A \setminus \{S\}$ we get

$$\frac{d_N(\bar{f}(P), f(y))}{d_M(P, y)} = \frac{d_N(f(S), f(y))}{d_M(S, y)}\frac{d_M(S, y)}{d_M(P, y)} \leq 2\text{Lip}(f),$$

because $d_M(S, y) \leq d_M(S, P) + d_M(P, y)$ and $d_M(S, P) \leq d_M(P, y)$ by the definition of $S$.

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