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Rate-equation theory of a feedback insensitive unidirectional semiconductor ring laser

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Abstract: For our recently designed continuous-wave and single-frequency ring laser with intra-cavity isolator, we have formulated a rate-equation theory which accounts for two sources of mutual back-scattering between the clockwise and counterclockwise modes, i.e. induced by side-wall irregularities and due to inversion-grating-induced spatial hole burning. With this theory we first confirm that for a ring laser without intra-cavity isolation, from sufficiently large pumping strength on, the inversion-grating-induced bistable operation (i.e. either clockwise or counterclockwise) will overrule the back-reflection-induced coupled-mode operation (i.e. both clockwise and counterclockwise). We then analyze the robustness of unidirectional operation in case of intra-cavity isolation against the intra-cavity back-reflection mechanism and grating-induced mode coupling and derive for this case an explicit expression for the directionality in the presence of external optical feedback, valid for sufficiently strong isolation. The predictions posed in the second reference remain unaltered in the presence of the mode coupling mechanisms here considered.

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1. Introduction

In state-of-the-art photonic integrated circuits (PICs) it is desirable to integrate one or more lasers on a single chip together with components such as modulators, splitters and filters. Such a device needs only electrical inputs and one or more optical outputs that allow direct coupling to optical fibers. Compared to fiber-optical or free-space configurations, the stability of the complete system can be increased and the cost greatly reduced. However, a fundamental problem is the sensitivity of a semiconductor laser to external optical feedback (EOF), which can easily lead to unstable dynamics. The EOF could easily originate from a on-chip reflection. In order to protect the laser from EOF inside the PIC, it is not possible to use the conventional solution in fiber- or free-space optics, based on magnetic materials, since a suitable integrated optical isolator is not available [1].

Recently we proposed an integrated unidirectional ring laser for which reduced sensitivity to EOF up to $-1$ dB was predicted [2]. In general, monolithic semiconductor bulk and quantum-well ring lasers can already exhibit reduced sensitivity to EOF up to $-30$ dB even in the absence of any special precautions, due to intrinsic bistability with respect to clock-wise (CW) and counter-clock-wise (CCW) operation [3]. However, the actual mode of operation is not predetermined and external optical perturbations could induce switching to the other mode, even for EOF as small as $-40$ dB [4]. This kind of sensitivity can be reduced considerably by applying a one-sided strong reflector [5], but EOF exceeding 30 dB can still lead to unstable behavior. In this respect, we mention [6], who developed a “snail” laser consisting of a ring laser with the CW output blocked by a reflector. This laser operates unidirectionally, but the EOF sensitivity is not discussed. It is our aim to almost completely eliminate the effect of EOF by including an intra-cavity isolator in the ring.

In our design [2], the isolator is inspired by the phase-modulator cascade proposed by Doerr et al. [7], an integrated version of which was analyzed and characterized [8]. The RF-driven modulators cause side peaks at (multiples of) the RF-modulation frequency in the light propagating in the backward direction. The side peaks are suppressed by the filter included in the ring laser. The isolator thereby introduces a loss difference between the CW and CCW modes of the laser, increasing the lasing threshold for one mode with respect to the other and forcing the laser to operate in the mode with lowest loss. As illustrated in the example of Fig. 1, the lasing mode with lowest loss is the CW and the EOF light will return to the non-lasing CCW mode.

![Fig. 1. Sketch of a feedback insensitive ring laser subjected to EOF. Lasing in a single longitudinal mode is ensured by the filter and by the cavity itself, while the isolator ensures unidirectional laser operation. EOF is modeled by a point-reflection that reflects a fraction $R$ of the optical power. $\tau$ is the feedback delay time.](image)

A set of coupled differential equations describe the evolutions of optical intensities and phases for both modes as well as the inversion in the SOA. It is assumed that the spectral
filter width allows for only one longitudinal mode to be considered for both propagation
directions. The slowly-varying-envelope approximation is used and the effects of EOF are
accounted for in the usual way [9]. The on-chip isolation is modeled by a roundtrip-loss
difference between CW and CCW modes, derived from the corresponding transmission
roundtrip differences.

2. Formulation of the rate equations

We have extended our previous theory [2] so as to account for the effects of back-scattering
(BS) due to spurious reflections [5] and side-wall irregularities [10] and the inversion-grating
induced by spatial hole burning [11]. The system is fully described by the coupled rate
equations:

\[ E_{cw} = \frac{1}{2} \left( 1 + i \alpha \right) \xi \left( N_0 E_{cw} + N_1 E_{ccw} \right) + K e^{i \phi_b} E_{cw}; \]  
\[ E_{ccw} = \frac{1}{2} \Delta \Gamma E_{cw} + \frac{1}{2} \left( 1 + i \alpha \right) \xi \left( N_0 E_{ccw} + N_1 E_{cw} \right) + K e^{i \phi_b} E_{cw} + \]  
\[ \frac{1}{2} \left( \gamma + \xi \phi_b \right) E_{cw}(t - \tau) \]  
\[ N_0 = \Delta J - \frac{N_0}{T_0} \left( \Gamma + \xi N_0 \right) I - \xi (N_1 E_{cw} E_{ccw}^* + c.c.); \]  
\[ N_1 = -\frac{N_1}{T_1} - \left( \Gamma + \xi N_0 \right) E_{cw}^* E_{ccw}^* - \xi N_1 I. \]

All parameters are listed in Table 1. The variables \( E_{cw}, E_{ccw}, N_0, N_1, I \) are taken at time \( t \)
unless explicitly indicated, where \( E_{cw} \) and \( E_{ccw} \) are the weakly time-dependent complex
mode amplitudes; \( N_0 \) is the inversion with respect to its value at laser threshold; \( N_1 \) the
(complex) amplitude of the inversion-grating and \( I \) the total intensity, \( I \equiv |E_{cw}|^2 + |E_{ccw}|^2 \).

Table 1. Parameter listing.

| Parameter symbol | Name | Typical value |
|------------------|------|---------------|
| \( \alpha \)     | Linewidth enhancement factor | 2.5 |
| \( \xi \)        | Differential gain | \( 1 \times 10^4 \) s\(^{-1} \) |
| \( \Gamma \)      | Cavity loss | \( 1 \times 10^4 \) s\(^{-1} \) |
| \( \Delta \Gamma \) | Isolator-induced loss difference | \( 1 \times 10^3 \) s\(^{-1} \) |
| \( K \)          | Back scatter coupling rate CCW \( \leftrightarrow \) CW | \( 1 \times 10^6 \) s\(^{-1} \) |
| \( \phi_{ccw} \)  | CCW \( \leftrightarrow \) CW back scatter phase | Arbitrary |
| \( \gamma \)     | Feedback rate | \( 1 \times 10^8 \) s\(^{-1} \) |
| \( \tau \)       | Feedback delay time | \( 3 \times 10^{-9} \) s |
| \( \omega_0 \)    | Optical frequency at threshold | \( 2 \pi \times 10^{14} \) s\(^{-1} \) |
| \( \phi_b \)     | Feedback phase; \( -\omega_0 \tau \) | Arbitrary |
| \( \Delta J \)    | Injection current w.r.t. threshold | \( 10^7 \) s\(^{-1} \) |
| \( T_0 \)        | Inversion lifetime | \( 1 \times 10^{-9} \) s |
| \( T_1 \)        | Inversion grating lifetime | \( 1 \times 10^{-14} \) s |
Table 2. Variables listing.

| Variable | Name |
|----------|------|
| $E_{cw}$ | Weakly time-dependent CW complex mode amplitude; $E_{cw} = \sqrt{I_{cw}} e^{i\phi_{cw}}$ |
| $E_{ccw}$ | Weakly time-dependent CCW complex mode amplitude; $E_{ccw} = \sqrt{I_{ccw}} e^{i\phi_{ccw}}$ |
| $N_0$ | Inversion w.r.t. threshold; real. (# of e-h pairs) |
| $N_1$, $\mathcal{M}$ | Inversion grating amplitude; complex. (# of e-h pairs) |
| $I_{cw}$ | CW mode intensity; # of CW photons |
| $I_{ccw}$ | CCW mode intensity; # of CCW photons |
| $\epsilon$ | Directionality, $\epsilon \equiv \sqrt{I_{cw}/I_{ccw}}$. |
| $I$ | Total intensity; $I \equiv I_{cw} + I_{ccw}$; total # of photons |
| $\phi_{cw}$ | CW mode phase |
| $\phi_{ccw}$ | CCW mode phase |
| $\mathcal{A}$ | Phase difference, $\mathcal{A} \equiv \phi_{ccw} - \phi_{cw}$ |

It is convenient to introduce the phase difference $\mathcal{A}$ between the CCW and CW modes:

$$\mathcal{A} \equiv \phi_{ccw} - \phi_{cw}, \quad (5)$$

and the phase-shifted inversion grating amplitude $\mathcal{M}$ as

$$\mathcal{M} \equiv N_1 e^{-i\mathcal{A}}. \quad (6)$$

With Eqs. (5) and (6), the inversion rate equations can be expressed as

$$N_0 = \Delta J - \frac{N_0}{T_0} (\Gamma + \xi N_0) I - 2\xi \sqrt{I_{cw} I_{ccw}} \text{Re}\mathcal{M}; \quad (7)$$

$$\mathcal{M} = -i \mathcal{A} \mathcal{M} - \frac{\mathcal{M}}{T_1} - (\Gamma + \xi N_0) \sqrt{I_{cw} I_{ccw}} - \xi \mathcal{M} I. \quad (8)$$

From Eqs. (1) and (2), rate equations for the mode intensities $I_{cw} \equiv |E_{cw}|^2$, $I_{ccw} \equiv |E_{ccw}|^2$ and the phase difference can be derived

$$I_{cw} = \xi N_0 I_{cw} + 2K \sqrt{I_{cw} I_{ccw}} \cos(\phi_{cw} + \mathcal{A}) + \xi \sqrt{I_{cw} I_{ccw}} (\text{Re}\mathcal{M} + \alpha \text{Im}\mathcal{M}); \quad (9)$$

$$I_{ccw} = -\eta I_{cw} + 2\xi \sqrt{I_{cw} I_{ccw}} \cos(\phi_{cw} - \mathcal{A}) + \xi \sqrt{I_{cw} I_{ccw}} (\text{Re}\mathcal{M} + \alpha \text{Im}\mathcal{M}) + \eta \sqrt{I_{cw} I_{ccw}} \cos(\phi_{cw} + \phi_{ccw} - (t - \tau)) - \phi_{cw}. \quad (10)$$

$$\mathcal{A} = \frac{1}{2} \xi \left[ \frac{I_{cw}}{I_{ccw}} (\alpha \text{Re}\mathcal{M} + \text{Im}\mathcal{M}) - \frac{I_{cw}}{I_{ccw}} (\alpha \text{Re}\mathcal{M} - \text{Im}\mathcal{M}) \right] + K \sqrt{I_{ccw}} \sin(\phi_{cw} - \mathcal{A}) - \frac{1}{2} \sqrt{I_{ccw}} \sin(\phi_{cb} + \mathcal{A}) + \frac{\sqrt{I_{ccw} (t - \tau)}}{I_{cw}} \sin(\phi_{cb} - \mathcal{A} + \phi_{cw} (t - \tau) - \phi_{ccw}). \quad (11)$$

Equations (7) to (11) form an independent set of coupled equations, which describe the dynamical evolution of the ring laser with inversion grating, back scattering and EOF from the CW mode. We introduce as a measure for the degree of directionality of the light in the ring laser the variable $\epsilon$ as
\[
\epsilon(t) \equiv \frac{I_{cw}}{I_{cw}}. 
\] (12)

From this definition it is straightforward to derive from Eqs. (9) and (10) the differential equation,

\[
\dot{\epsilon} + \frac{\Delta \Gamma}{2} \epsilon = \frac{1}{2} \xi \left\{ \epsilon \left( \text{Re} M - \alpha \text{Im} M \right) - \epsilon \left( \text{Re} M + \alpha \text{Im} M \right) \right\} \epsilon + K \frac{1}{\epsilon} \cos (\varphi_{in} - A) - \epsilon \cos (\varphi_{in} + A) \epsilon + \frac{1}{2} \gamma \sqrt{\frac{I_{cw}(t - \tau)}{I_{cw}}} \cos (\varphi_{bs} - A + \varphi_{cw} (t - \tau) - \varphi_{cw}). 
\] (13)

The first terms in Eqs. (11) and (13) are inversion-grating induced terms; in the absence of the other terms this term in Eq. (11) describes a frequency splitting between the CW and CCW mode. The second terms describe coupling of the modes due to sidewall-induced backscattering; these terms by themselves lead to frequency locking of the two modes. The third terms are induced by the EOF.

Our model described by Eqs. (7) to (13) is different in several aspects from models studied by others. We mention in chronological order M. Sorel et al [12], Pérez et al [13], Ermakov et al [15] and Morthier and Mechet [5] who do include back scattering, but not the hole-burning-induced inversion grating amplitude. Instead, they introduce self and cross saturation parameters phenomenologically. The theory by Dontsov [4] does not consider back scattering, but includes the spatial hole-burning effect on the inversion grating from first principles in a traveling wave approach. We will show that the intrinsic bistable behavior with respect to CW and CCW operation is a direct consequence of the hole-burning-induced inversion grating for which additional introduction of cross and self saturation is not required.

3. Analysis close to steady state

We assume that the system evolves towards a steady state with constant intensities, i.e. \( I_{cw} = I_{ccw} = 0 \), and a constant small value for \( \varphi_{in} \). This assumption implies that the CW and CCW frequencies are close together. We then find from Eq. (8), using \( \sqrt{I_{cw}I_{ccw}} = I \epsilon / (1 + \epsilon^2) \), \( T_i^{-1} \gg |iA + \xi I| \) and \( \xi N_0 \ll \Gamma \) in good approximation,

\[
\mathcal{M} = -T_i \Gamma I \epsilon / (1 + \epsilon^2). 
\] (14)

For the inversion-grating induced frequency shift \( \Delta \omega \) given by the first term in Eq. (11), we obtain

\[
\Delta \omega = \frac{1}{2} \xi \alpha \mathcal{M} \left( \frac{1}{\epsilon} - \epsilon \right) = -\frac{1}{2} \alpha \xi T_i \Gamma I (1 - \epsilon^2) / (1 + \epsilon^2). 
\] (15)

The second term in Eq. (11) is the back-scattering contribution and can be rewritten as

\[
K \left\{ \frac{1}{\epsilon} \sin (\varphi_{in} - A) - \epsilon \sin (\varphi_{in} + A) \right\} = W_{bs} \sin (\Psi_{bs} - A), 
\] (16)

where

\[
\Psi_{bs} \equiv \text{Arc} \tan \left( (1 + \epsilon^2) \cos \varphi_{in}, (1 - \epsilon^2) \sin \varphi_{in} \right); 
\] (17)

\[
W_{bs} \equiv K \sqrt{\epsilon^2 + \epsilon^2 + 2 \cos 2 \varphi_{in}}, 
\] (18)
and \( \text{Arctan}(x,y) \) denotes the arc tangent of \( y/x \) taking account which quadrant the point \((x,y)\) is in.

With Eq. (16), we can write the phase evolution Eq. (11) in very good approximation as (ignore the time delay in \( I_{cw} \), i.e. valid close to steady state)

\[
\mathcal{A} = \Delta \omega + W_{bs} \sin (\Psi_{bs} - A) + W_{fb} \sin (\Psi_{fb} - A),
\]

(19)

where

\[
W_{fb} = \frac{\gamma}{2e},
\]

(20)

and the effective feedback phase \( \Psi_{fb} \) is defined as

\[
\Psi_{fb} = \varphi_{fb} - \Delta \omega_{op} \tau.
\]

(21)

With \( \Delta \omega_{op} \), the steady-state operation frequency shift with respect to \( \omega_b \),

\[
\Delta \omega_{op} \equiv \varphi_{cw} = \psi_{cw},
\]

(22)

where the last equality follows from \( \mathcal{A} = 0 \) in the steady state. Equation (19) can be written in a form equivalent to the Adler’s equation:

\[
\mathcal{A} = \Delta \omega + C_{eff} \sin \left( \Psi_{eff} - A \right),
\]

(23)

where

\[
C_{eff} \equiv \sqrt{W_{bs}^2 + W_{fb}^2 + 2W_{bs}W_{fb} \cos (\Psi_{bs} - \Psi_{fb})} \cos (\Psi_{bs} - \Psi_{fb}) \times \sin (\Psi_{bs} - \Psi_{fb}) \times W_{bs} \sin (\Psi_{bs} - \Psi_{fb}) \times W_{fb} \sin (\Psi_{fb} - \Psi_{fb}).
\]

(24)

Equation (23) is in a suitable form to analyze the behavior of the phase difference near steady state in terms of the quantities \( C_{eff} \) and \( \Psi_{eff} \), where it should be realized that they are functions of the directionality \( \epsilon \), explicitly via \( \Psi_{bs} \), \( W_{bs} \) and \( W_{fb} \) (see the definitions Eqs. (17), (18) and (20)) and implicitly via \( \Psi_{fb} \) and \( \Delta \omega_{op} \) (see Eqs. (21) and (22)). According to Eq. (23) the phase difference \( \mathcal{A} \) will lock to a time-independent value whenever

\[
C_{eff} > |\Delta \omega|.
\]

(27)

with the stable solution

\[
\mathcal{A} = \Psi_{eff} + \text{Arcsin} \left( \frac{\Delta \omega}{C_{eff}} \right).
\]

(28)

It is convenient to rewrite Eq. (13) in the form (ignore the time delay in \( I_{cw} \), i.e. close to steady state)

\[
\epsilon = -\left( \frac{\Delta \Gamma}{2} - \frac{\Delta \omega}{\alpha} \right) \epsilon + K \{ \cos (\varphi_{bs} - A) - \epsilon^2 \cos (\varphi_{bs} + A) \} + \frac{\gamma}{2} \cos (\Psi_{fb} - A),
\]

(29)

with \( \Delta \omega \) and \( \Psi_{fb} \) given by Eqs. (15) and (21), respectively.
The general structure of the theory can now be summarized as the Adler Eq. (23) with solution Eq. (28) in terms of “parameters” that depend on that solution. This implies that the solution process requires a self-consistent method, which can be achieved, for instance, by iteration. Substitution of Eq. (28) into the right hand side of Eq. (29) and equating $\epsilon'$ to zero yields an equation for $\epsilon$ that can be solved numerically. The stable solutions for $\epsilon$ are the zero crossings of the right-hand side as function of $\epsilon$ with negative slope. Before analyzing in Sec.5 the case of CW-dominant operation, we will investigate in the next section the case in which the spatial hole burning effect $\Delta \omega$ dominates all other terms in Eq. (29).

4. Bistable operation without intra-cavity isolation and EOF

Throughout this section we put $\Delta \Gamma = \gamma = 0$. First, assume no back scattering, i.e. $K = 0$. In this case, Eq. (29) reduces to \( \Delta = 0, I = (1 + \epsilon^2) I_{cw} \)

\[
\epsilon' = \frac{\Delta \omega}{a} \epsilon - \frac{\xi T \Gamma I}{2} 1 - \epsilon^2 \epsilon, \quad (30)
\]

with the stable steady-state ($t \to \infty$) solutions

\[
\epsilon = 0, \infty \quad (31)
\]

and the unstable solution

\[
\epsilon = 1. \quad (32)
\]

According to Eq. (31) an ideal ring laser will exhibit bistability with respect to CW and CCW operation. Here we have shown explicitly how this intrinsic bistable behavior is related to the spatial hole burning of the inversion. Note that no additional assumptions of self and cross saturation in the differential gain $\xi$ as was made in [5,12,13,15].

The stability eigenvalues for each stable solution Eq. (31) is $\frac{\xi T \Gamma I}{2}$, and the eigenvalue for the unstable solution is $\frac{\xi T \Gamma I}{4}$. Note that these eigenvalues are proportional to the lifetime of the inversion grating $T_i$ and the pump strength (through the intensity $I$). Hence we predict that for lasers with slower carrier diffusion and thus longer grating life time, the respective stability eigenvalues will be accordingly larger.

We will now investigate the effect of coherent back scattering. It can be seen from Eq. (27) that the feedback term has the same structure as one of the back-scatter terms. This property was used in [5] to treat the feedback as an effective back-scatter term, absorbing $\gamma$ in the corresponding $K$.

By substituting the stable solution Eq. (28) in Eq. (29) with $\Delta \Gamma = \gamma = 0$ and realizing that Eq. (28) gives $A$ in terms of $\epsilon$, the zero crossings of the right-hand side of Eq. (27) with negative slope correspond to stable steady-state solutions. This was analyzed in a recent conference paper [14] where for all values of the pump strength above threshold the locked symmetric state with $\epsilon = 1$ was found to be a steady-state solution, always stable except for some back scatter phase values. From a certain pump strength onward, the basin of attraction was seen to shrink rapidly, while at the same time, the system has two stable symmetry-broken phase-locked solutions, corresponding to CW/CCW net flux operation. Likely, in an experimental situation the symmetric state may then no longer be observable. Numerical time integration of Eq. (7) to Eq. (11) revealed also a region of intermediate pump strengths with coexisting oscillating behavior of the coupled CW and CCW modes, in agreement with experimental findings in [3] and [12].
5. Unidirectional operation with strong isolation

The steady-state of the laser is characterized by equating the right hand side of Eq. (29) to zero. In case of sufficiently large isolation, i.e. for $\Delta \Gamma \gg |\Delta \omega|$, this yields a quadratic equation for $\epsilon$ with two solutions given by

$$
\epsilon_\pm = \frac{\Delta \Gamma}{2} \pm \sqrt{\left(1 + \frac{16 \cos(\phi_{in} + A) \left[K \cos(\phi_{in} - A) + \frac{\gamma}{2} \cos(\Psi_{\beta} - A)\right]}{-2K \cos(\phi_{in} + A)}\right)}.
$$

(33)

Since $|\Delta \omega| \sim 10^6 \text{s}^{-1}$ and $\Delta \Gamma \sim 10^8 - 10^{10} \text{s}^{-1}$, the approximation leading to Eq. (33) is well satisfied. Only the root in Eq. (33) with negative sign has physical significance, since that one behaves well for $K \to 0$. The other root yields either a negative value for $\epsilon$ or a large value $\epsilon \gg 1$, in both cases an unphysical situation with the CCW mode below threshold.

If in addition the isolation $\Delta \Gamma$ is much larger than $4 \sqrt{K(K + \frac{\gamma}{2})}$, we can approximate the physically acceptable solution as

$$
\epsilon = \frac{2K \cos(\phi_{in} - A) + \gamma \cos(\Psi_{\beta} - A)}{\Delta \Gamma},
$$

(34)

i.e. yielding a small value for $\epsilon \ll 1$ and valid for

$$
\Delta \Gamma \gg \max\left(|\Delta \omega|, 4 \sqrt{K(K + \frac{\gamma}{2})}\right).
$$

(35)

The value for $A$ should follow from Eq. (28), which in case of $\epsilon \downarrow 0$ reduces to $A = \Psi_{\text{eff}}$. Using Eq. (25) and some trigonometric relations, Eq. (34) can be cast in the following appealing form:

$$
\epsilon = \frac{2K + \gamma}{\Delta \Gamma} \sqrt{\frac{1 - K \gamma}{(K + \frac{\gamma}{2})^2} [1 - \cos(\phi_{in} - \Psi_{\beta})]},
$$

(36)

which combines all relevant quantities in one single analytic expression. The isolation strength $\Delta \Gamma$ is order $10^{10} \text{s}^{-1}$, the side-wall backscatter rate $K$ is order $10^6 \text{s}^{-1}$ and the feedback rate $\gamma$ order $10^8 \text{s}^{-1}$. It then follows from Eq. (36) that $\epsilon \leq \frac{2K + \gamma}{\Delta \Gamma} = 10^{-2}$, meaning that the condition Eq. (35) for the validity under which Eq. (36) was derived is well met. It is shown in [2] that with the above-derived directionality $\epsilon \leq 10^{-2}$ full insensitivity to external feedback in terms of RIN and linewidth. This concludes our analysis of a unidirectional ring laser with built-in isolator and subject to back scattering and external optical feedback.

6. Conclusion

We have theoretically analyzed the dynamical behavior of a unidirectional ring laser with built-in optical isolator to prevent one direction from lasing. Especially, we investigated the robustness of unidirectional operation against coherent back scattering and hole-burning-induced gain saturation.
Our theory applied to the case of a laser without isolator revealed how the intrinsic tendency for bistable behavior with respect to CW/CCW operation is a natural consequence of the spatial hole-burning-induced inversion grating. On the other hand, in case of back scattering such as due to side wall irregularities, the device tends towards symmetric coupled-mode operation $\epsilon = 1$ (i.e. both CW and CCW). Since the tendency for bistability scales with the pump strength whereas the tendency for symmetric $\epsilon = 1$ operation is pump-strength independent, the competition of the two tendencies leads to symmetry-broken bistable operation for high enough pump strength. Such scenario has indeed been observed by references [3] and [12].

Next, an analytic expression for the directionality in the presence of external optical feedback is derived, valid for sufficiently strong isolation. This expression shows that the effects of EOF as well as back scattering are similar in structure, and lead to correspondingly similar effects on the directionality of operation. Our findings for a unidirectional, or rather quasi-unidirectional, ring laser with built-in isolator to suppress one of the circular modes, corroborate the predictions in [2] concerning the external feedback sensitivity of the laser. The theory in [2] did not take into account intrinsic back scattering such as due to the inversion grating or side-wall irregularities.

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