Extenders under ZF and
constructibility of rank-to-rank embeddings

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Abstract
Assume ZF (without the Axiom of Choice). Let $j : V_\delta \to V_\delta$ be a non-
trivial $\Sigma_1$-elementary embedding, where $\delta$ is a limit ordinal. We prove
some basic restrictions on the constructibility of $j$ from $V_\delta$; in particular,
if $j \in L(V_\delta)$ then $\text{cof}(\delta) > \omega$. We show that, however, assuming an $I_3$-
embedding, with the appropriate $\delta, j$, it is possible to have $j \in L(V_\delta)$.
Assuming Dependent Choice and that $\delta$ has countable cofinality (but not
assuming $V = L(V_\delta)$), and $j$ is as above, we show that the collection
of such embeddings is of high complexity, and that there are “perfectly
many” such embeddings. We also show that a ZF theorem of Suzuki, that
no elementary $j : V \to V$ is definable from parameters, actually follows
from a theory weaker than ZF. The main results rely on a development
of extenders under ZF, which we also give.

1 Introduction
Large cardinal axioms are typically exhibited by a class elementary embedding

$$j : V \to M$$

where $V$ is the universe of all sets and $M \subseteq V$ some transitive inner model.
Stronger large cardinal notions are obtained by demanding more resemblance
between $M$ and $V$, and by demanding that more of $j$ gets into $M$. William
Reinhardt, in [8] and [9], took this notion to its extreme by setting $M = V$;
that is, he considered embeddings of the form

$$j : V \to V.$$

The critical point of such an embedding became known as a Reinhardt cardinal.

But Kunen showed in [7] that if $V \models \text{ZFC}$ then there is no such $j$, and in fact
no ordinal $\lambda$ and elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. Given this violation

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†The least ordinal $\kappa$ such that $j(\kappa) > \kappa$.

‡If there is $j : V \to V$ then there is a proper class of ordinals $\alpha$ such that $j : V_\alpha \to V_\alpha$,
and then $j : V_{\alpha+n} \to V_{\alpha+n}$ for each $n < \omega$. 

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of AC, most research in large cardinals following Kunen’s discovery has been focused below the AC threshold. In this paper, we drop the AC assumption, and investigate embeddings \( j : V_\delta \to V_\delta \), assuming only ZF in \( V \). (We mention it explicitly when we use some DC.) Thus, Kunen’s objections are not valid here, and moreover, no one has succeeded in refuting \( j : V_{1+2} \to V_{\lambda+2} \), or even \( j : V \to V \), in ZF alone, except for in the sense we describe next.

The following theorem is Suzuki \[13\] Theorem 2.1:

**Fact 1.1** (Suzuki). Assume ZF and let \( j \subseteq V \) be a class which is definable from parameters.\(^3\) Then it is not the case that \( j : V \to V \) is a \( \Sigma_1 \)-elementary embedding.\(^4\)

So if we interpret “class” as requiring definability from parameters – which is one standard interpretation – then Suzuki’s theorem actually disproves the existence of Reinhardt cardinals from ZF. However, there are other interpretations of “class” which are more liberal, and for which this question is still open. (And significantly, note that this theorem doesn’t seem to say anything about the existence of an elementary \( j : V_{\lambda+2} \to V_{\lambda+2} \).)

What can “classes” be, if not definable from parameters? One way to consider this question, is to shift perspective: instead of considering \( V \), let us consider some rank segment \( V_\delta \) of \( V \), where \( \delta \) is an ordinal. The sets \( X \subseteq V_\delta \) such that \( X \) is definable from parameters over \( V_\delta \), are exactly those sets \( X \in L_1(V_\delta) \), the first step in Gödel’s constructibility hierarchy above base set \( V_\delta \). So by Suzuki’s theorem, if \( V_\delta \models ZF \), then there is no \( \Sigma_1 \)-elementary \( j : V_\delta \to V_\delta \) with \( j \in L_1(V_\delta) \). Viewed this way, we now have the obvious question, as to whether we can have such a \( j \in L_\alpha(V_\delta) \), for larger \( \alpha \). And the subsets of \( V_\delta \) which are in \( L(V_\delta) \), for example, provide another reasonable notion of “class” with respect to \( V_\delta \).

So, wanting to examine these issues, this paper focuses mostly on the question of the constructibility of such \( j \) from \( V_\delta \), looking for generalizations of Suzuki’s theorem, and examining the consequences of having such \( j \in L(V_\delta) \). We remark that the papers \[4, 2\] develop some general analysis of rank-to-rank embeddings \( j : V_\delta \to V_\delta \), for arbitrary ordinals \( \delta \), where we include all \( \Sigma_\delta \)-elementary maps in this notion. But in this paper, we consider only the case that \( \delta \) is a limit.\(^5\) For convenience, we write \( \mathcal{E}_m(V_\delta) \) for the set of all \( \Sigma_m \)-elementary \( j : V_\delta \to V_\delta \), and \( \mathcal{E}(V_\delta) = \mathcal{E}_\omega(V_\delta) \).

The first basic result here is the following (in the case that \( \delta \) is inaccessible, the result is due independently and earlier to Goldberg, who used other methods): 

**Theorem** \[6,3\]. Assume ZF + \( V = L(V_\delta) \) where \( \text{cof}(\delta) > \omega \). Then \( \mathcal{E}_1(V_\delta) = \emptyset \); that is, there is no \( \Sigma_1 \)-elementary \( j : V_\delta \to V_\delta \).

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3Here given a structure \( M \) and \( X \subseteq M \), we say that \( X \) is definable from parameters over \( M \) iff there is \( \bar{p} \in M^{<\omega} \) and a formula \( \varphi \) such that for all \( x \in M \), we have \( x \in X \) if and only if \( M \models \varphi(x, \bar{p}) \).
4Actually, because \( V \models ZF \), \( \Sigma_1 \)-elementarity is equivalent to full elementarity in this case.
5That is, we start with \( V_\delta \) as the first set in the hierarchy, instead of \( L_0 = \emptyset \), and then proceed level-by-level as for \( L \).
6Most of the results in this paper first appeared in the author’s informal notes \[12\]. Those notes were broken into pieces according to theme, and this paper is one of those pieces. There are also some further observations here not present in \[12\], mainly in \[41\]. We also corrected an error in the definition of extender given in \[12\]; see \( \S3 \) of the present paper, and in particular Footnote \[5\].

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Allowing \( \text{cof}(\delta) = \omega \) in \( L(V_\delta) \), we can still establish the following:

**Theorem 7.6.** Assume ZF, \( \delta \in \text{Lim} \) and \( j \in \mathcal{E}_1(V_\delta) \). Let \( \kappa \in \text{OR} \) be least with \( L_\kappa(V_\delta) \) admissible. Then \( j \notin L_\kappa(V_\delta) \), \( j \) is not \( \cap \sum_{\lambda \in \kappa} L_\lambda(V_\delta) \), and not \( \bigcup_{\lambda} L_\lambda(V_\delta) \).

We also show in Theorem 7.8 assuming an \( I_3 \)-embedding, that it is consistent that \( j \) is \( I_3 \) and \( j \in L_{\kappa+1}(V_\delta) \) (for the appropriate \( \delta \)).

The question of constructibility of embeddings is also related to questions on uniqueness of rank-to-rank embeddings. That is, consider an \( I_3 \) rank-to-rank embedding \( j \) (so \( j: V_\delta \to V_\delta \) where \( \delta \) is a limit and \( \delta = \kappa_\omega(j) = \lim_{n<\omega} \kappa_n \) where \( \kappa_0 = \text{cr}(j) \) and \( \kappa_{n+1} = j(\kappa_n) \)). Each finite iterate \( j_n \) is also such an embedding, and \( \lim_{n<\omega} \text{cr}(j_n) = \delta \). It follows easily that for each \( \alpha < \delta \), there are multiple elementary \( k: V_\delta \to V_\delta \) such that \( k \models V_\alpha = j \models V_\alpha \) (just consider \( j_n(j) \) for some sufficiently large \( n \)). But what if instead, \( j: V_{\lambda+\omega} \to V_{\lambda+\omega} \) is elementary where \( \lambda = \kappa_\omega(j) \)? Can there be \( n < \omega \) such that \( j \) is the unique elementary \( k: V_{\lambda+\omega} \to V_{\lambda+\omega} \) extending \( j \models V_{\lambda+n} \)? The \( I_3 \) argument clearly doesn’t work here, since an elementary \( k: V_{\lambda+\omega} \to V_{\lambda+\omega} \) must have \( \text{cr}(k) < \lambda \).

The answer is “no” under DC:

**Theorem 7.11.** Assume ZF and let \( \delta \) be a limit ordinal where \( \mathcal{E}(V_\delta) \neq \emptyset \), and \( m \in [1,\omega] \). Then:

1. In a set-forcing extension of \( V \), for each V-amenable\(^7\) \( j \in \mathcal{E}_m(V_\delta) \) and \( \alpha < \delta \) there is a V-amenable \( k \in \mathcal{E}_m(V_\delta) \) with \( k \models V_\alpha = j \models V_\alpha \) but \( k \neq j \).

2. If DC holds and \( \text{cof}(\delta) = \omega \) then for each \( j \in \mathcal{E}_m(V_\delta) \) and \( \alpha < \delta \) there is \( k \in \mathcal{E}_m(V_\delta) \) with \( k \models V_\alpha = j \models V_\alpha \) but \( k \neq j \).

More generally, given a \( \Sigma_1 \)-elementary \( j \in \mathcal{E}_1(V_\delta) \) with limit \( \delta \), one can ask whether \( j \models V_\delta \in \text{HOD}(V_\delta) \). (The I_3 example above does give an instance of this, since \( L(V_\delta) \subseteq \text{HOD}(V_\delta) \).)

We also prove the following related result, which is another kind of strengthening of Suzuki’s theorem above, and also a strengthening of [4] 5.7 part 1.\(^{***}\). The theory RL (Rank Limit, see Definition 2.2) is a sub-theory of ZF, which holds in \( V_\delta \) for limit \( \eta \):

**Theorem 2.3.** Assume RL. Then there is no \( \Sigma_1 \)-elementary \( j: V \to V \) which is definable from parameters.

As a useful tool, and also for more general use elsewhere, we develop the theory of extenders and ultrapowers under ZF (see [14]). Of course a key fact when considering ultrapowers is Loé's theorem, which is related to AC. However, it turns out that a proper class of weak Löwenheim-Skolem (wLS) cardinals (due to Usubu [14]; see Definition 5.3) gives enough choice to secure Loé's theorem for the kinds of extenders over \( V \) we will consider here. A proper class of wLS cardinals is not known to be inconsistent with choiceless large cardinals, and is in fact implied by a super Reinhardt; it also holds if AC is forceable with a set-forcing (see [14]). Under this large cardinal assumption, we will prove the following generalization that measurability classifies critical points under ZFC; we say that an ordinal \( \kappa \) is V-critical iff there is an elementary \( j: V \to M \) with \( \text{cr}(j) = \kappa \):
1.1 Background and terminology

We now list some background and terminology, which is pretty standard. Our basic background theory throughout is ZF (in fact, many of the results require much less, but we leave that to the reader), with additional hypotheses stated as required. However, in $\mathsf{ZF}$ we work in the weaker theory $\mathsf{RL}$. In the $\mathsf{ZF}$ (or weaker) context, certain familiar definitions from the $\mathsf{ZFC}$ context need to be modified appropriately.

Work in $\mathsf{ZF}$. $\mathsf{OR}$ denotes the class of ordinals and $\mathsf{Lim}$ the class of limit ordinals. Let $\delta \in \mathsf{Lim}$. The cofinality of $\delta$, regularity, singularity are defined as usual (in terms of cofinal functions between ordinals). We say $\delta$ (or $\langle \delta \rangle$) is inaccessible iff there is no $\langle \gamma, f \rangle$ such that $\gamma < \delta$ and $f : \gamma \rightarrow \delta$ is cofinal in $\delta$. A norm on a set $X$ is a surjective function $\pi : X \rightarrow \omega$ for some $\alpha \in \mathsf{OR}$. The associated prewellorder on $X$ is the relation $R \subseteq X^2$ where $xRy$ iff $\pi(x) < \pi(y)$. This can of course be inverted. If $\delta \in \mathsf{OR}$ is regular but non-inaccessible, then the Scott ordertype $\mathsf{scot}(\delta)$ is the set $P$ of all prewellorders of $\mathsf{V}_{\alpha+1}$ in ordertype $\delta$, where $\alpha$ is least admitting such. See [4, 5.2, 5.3***?] for some basic facts about this. A partial function $f$ from (some subset of) $X$ to $Y$ is denoted $f : p X \rightarrow Y$.

Let $M = (U, \in^M, A_1, \ldots, A_n)$ be a first-order structure with universe $U$ and $A \subseteq U$. (We normally abbreviate this by just writing $A \subseteq M$.) We say that $A$ is definable over $M$ from parameters if there is a first-order formula $\varphi \in \mathcal{L}_A, \ldots, A_n$ (with symbols $\in, =, A_1, \ldots, A_n$) and some $\bar{x} \in M^{<\omega}$ such that for all $y \in M$, we have $y \in A$ iff $M \models \varphi(\bar{x}, y)$. This is naturally refined by $\Sigma_n$-definable from parameters, if we require $\varphi$ to be $\Sigma_n$, and by from parameters in $X$, if we restrict to $\bar{x} \in X^{<\omega}$ (where $X \subseteq M$), and by from $\bar{x}$, if we may use only $\bar{x}$.

For an extensional structure $M = ([M], \in^M, =^M)$, the wellfounded part $\mathsf{wfp}(M)$ of $M$ is the class of all transitive isomorphs of elements of $M$. That is, the class of all transitive sets $x$ such that $(x, \in \upharpoonright x, = \upharpoonright x)$ is isomorphic to $(y, \in^M, =^M, y)$ for some $y \in [M]$. The illfounded part $\mathsf{illfp}(M)$ of $M$ is $[M] \setminus \mathsf{wfp}(M)$. We have $\mathsf{wfp}(M), \mathsf{illfp}(M) \in \mathsf{L}(M)$, which results by ranking the elements of $M$ as far as is possible. Note that $\mathsf{illfp}(M)$ is the largest $X \subseteq [M]$ such that for every $x \in X$ there is $y \in X$ such that $y \in M^x$. If $M$ models enough set theory that it has a standard rank function, but $M$ is illfounded, then $\mathsf{OR}^M$ (the collection of all $x \in [M]$ such that $M \models \text{“}x$ is an ordinal”$)$ is illfounded, because if $x \in M$ then $\text{rank}_M(x) \in^M \text{rank}_M(y)$.

Given a structure $M$ and $k \in [1, \omega]$, $\delta_k(M)$ denotes the set of all $\Sigma_k$-elementary $j : M \rightarrow M$, and $\delta(M)$ denotes $\delta_{\omega}(M)$. Let $\delta \in \mathsf{Lim}$ and $j \in \delta_k(V_\delta)$. For $C \subseteq V_\delta$, define $j^+(C) = \bigcup_{\alpha < \delta} j(C \cap V_\alpha)$. Let $\kappa_0 = \text{cr}(j)$ and $\kappa_{n+1} = j(\kappa_n)$ and $\kappa_\omega(j) = \sup_{n < \omega} \kappa_n$ (note $\kappa_\omega(j) \leq \delta$). We write $j_0 = j$ and $j_{n+1} = (j_n)^+ (j_{n})$ for $n < \omega$; then $j_{\omega} \in \delta_k(V_\delta)$ (see [4, Theorem 5.6***?]) and $\kappa_n = \text{cr}(j_n)$.

$\mathsf{ZF}_2$ denotes the two-sorted theory, with models of the form $M = (V, E, P)$, where $(V, E) \models \mathsf{ZF}$ and $P$ is a collection of “subsets” of $V$. The elements of $V$ are the sets of $M$ and the elements of $P$ the classes. The axioms are like those of $\mathsf{ZF}$, but include Separation for sets and Collection for sets with respect
to all formulas of the two-sorted language, and also Separation for classes with respect to such formulas. When we “work in ZF₂”, we mean that we work in such a model \( M \), and all talk of proper classes refers to elements of \( P \). Note that the theory is first-order; there can be countable models of ZF₂. One has \((V_\alpha, \in, V_{\alpha+1}) \models ZF_2\) if \( \alpha \) is inaccessible.

The language of set theory with predicate \( L_{\in, A} \) is the first order language with binary predicate symbol \( \in \) and predicate symbol \( A \). The theory ZF(\( A \)) is the theory in \( L_{\in, A} \) with all ZF axioms, allowing all formulas of \( L_{\in, A} \) in the Separation and Collection schemes (so \( A \) represents a class). ZFR (ZF + Reinhardt) is the theory ZF(j) + “\( j : V \rightarrow V \) is \( \Sigma_1 \)-elementary”. By [6] Proposition 5.1, ZFR proves that \( j : V \rightarrow V \) is fully elementary (as a theorem scheme).

Work in ZF₂. Then \( \kappa \in OR \) is Reinhardt iff there is a class \( j \) such that \((V, j) \models ZFR \) and \( \kappa = \text{cr}(j) \). Following [1], \( \kappa \in OR \) is super Reinhardt iff for every \( \lambda \in OR \) there is a class \( j \) such that \((V, j) \models ZFR \) and \( \text{cr}(j) = \kappa \) and \( j(\kappa) \geq \lambda \).

### 2 \( RL \Rightarrow \) no parameter-definable \( j : V \rightarrow V \)

**Fact 2.1** (Suzuki). Assume ZF. Then no class \( j \) which is definable from parameters is an elementary \( j : V \rightarrow V \).

Of course, the theorem is really a theorem scheme, giving one statement for each possible formula \( \varphi \) being used to define \( j \) (from a parameter). We generalize here the proof of [4] Theorems 5.6, 5.7[***?], in order to show Suzuki’s fact above is actually a consequence of lesser theory RL, which is the basic first order theory modelled by \( V_\lambda \) for limit \( \lambda \) (without choice):

**Definition 2.2.** RL (for Rank Limit) is the theory in \( \mathcal{L} \) consisting of Empty Set, Extensionality, Foundation, Pairing, Union, Power Set, Separation (for all formulas, from parameters), together with the statements “For every ordinal \( \alpha \), \( \alpha + 1 \) exists, \( V_\alpha \) exists and \( (V_\beta)_{\beta < \alpha} \) exists”, and “For every \( x \), there is an ordinal \( \alpha \) such that \( x \in V_\alpha \).”

Since RL lacks Collection, the two extra statements regarding the cumulative hierarchy at the end are important. Clearly a model \( M \models RL \) can contain objects \( R \) such that \( M \models \text{“}R \text{ is a wellorder”} \), but such that there is no \( \alpha \in OR^M \) such that \( M \models \text{“}\alpha \text{ is the ordertype of } R \text{”} \).

**Theorem 2.3.** Assume RL. Then there is no \( \Sigma_1 \)-elementary \( j : V \rightarrow V \) which is definable from parameters.

**Proof Sketch.** The proof is a refinement of those of [4] Theorems 5.6, 5.7[***?], with which we assume that reader is familiar. By Suzuki’s fact, we may assume that ZF fails. So either (i) OR is \( \Sigma_m \)-singular for some \( m \in N \); that is, there is an ordinal \( \gamma \) and \( \Sigma_m \) formula \( \varphi \) and parameter \( p \) such that \( \varphi(p, \xi, \beta) \) defines a cofinal map \( \gamma \rightarrow OR \), via \( \xi \mapsto \beta \), and we take then \( (\gamma, k) \) then lexicographically least, so \( \gamma \) is regular; or (ii) otherwise, and there is \( \eta \in OR \) and \( m \in N \) and a \( \Sigma_m \) formula \( \varphi \) and \( p \in V \) such that \( \varphi(p, x, \beta) \) defines a cofinal map \( f : V_\eta \rightarrow OR \), via \( x \mapsto \beta \), and we take \( \eta \) least such, which as in [4] Remark 5.3[***?] implies that \( \eta = \alpha + 1 \) and that \( \text{rg}(f) \) has ordertype OR, and then we may in fact take \( f \) to be surjective, and obtain prewellorders of \( V_{\alpha+1} \) having ordertype OR,
and so define \( P = \text{scot}(\text{OR}) \) as before. (RL suffices here; for example, starting with an arbitrary definable cofinal \( f : V_{\alpha+1} \to \text{OR} \), for each \( \beta \in \text{OR} \), \( \text{rg}(f) \cap \beta \) is a set, by Separation. And the fact that a prewellorder has ordertype OR is a first-order statement under RL.) Let \( x_0 \) be this \( \gamma \) or \( P \) respectively. Note that \( x_0 \) is definable without parameters (but maybe not \( \Sigma_1 \)-definable without parameters).

Now as before, given a \( \Sigma_1 \)-elementary \( j : V \to V \) which is definable from parameters, we can define finite iterate \( j_n \), and

\[
j_n : (V, j_n) \to (V, j_{n+1})
\]

is \( \epsilon \)-cofinal \( \Sigma_1 \)-elementary. Note here that \( \mathbb{N} \) above denotes the standard integers, which might not be in \( V \), so \( m \) is standard, but when we mention “\( \omega \)” at present, we are talking about the \( \omega \) of the model \( V \) of RL we are working in, which might be illfounded from outside. Now the sequence \( \langle j_n \rangle_{n<\omega} \) is in fact a definable class (show by induction on \( n < \omega \) that for each \( \alpha \in \text{OR}^V \), there is \( \beta \in \text{OR}^V \) such that \( j_n \upharpoonright V_\beta \) is computable from \( j \upharpoonright V_\beta \). So basically as in \([4]\), for each \( \alpha \in \text{OR}^V \), there is \( n < \omega \) such that \( j_n(\alpha) = \alpha \) (and hence \( j_m(\alpha) = \alpha \) for \( m \geq n \)); however, we must use here the definability of \( \langle j_n \rangle_{n<\omega} \) from parameters, and Separation, to get that the relevant sequence of sets \( \langle A_n \rangle_{n<\omega} \) is a set; hence we can consider \( A = \bigcap_{n<\omega} A_n \) and \( j(A) \). Similarly, if OR is regular to class functions definable from parameters and \( P \) is as above, then \( j_n(P) = P \) for some \( n \).

Now assume there is some \( j : V \to V \) which is \( \Sigma_1 \)-elementary and definable from parameters, and fix a formula \( \varphi \) and parameter \( p \) which defines \( j \). So there is such a \( (\varphi, p, j) \) satisfying \( j(x_0) = x_0 \) (of course, if we need to pass to \( j_n \) with \( n \) non-standard, we can incorporate \( n \) into \( p \)). For \( q \in V \) let

\[
j_q = \{(x, y) \mid \varphi(q, x, y)\}.
\]

Let \( \kappa_0 \in \text{OR} \) be least such that for some \( q, j_q : V \to V \) is \( \Sigma_1 \)-elementary and \( \text{cr}(j_q) = \kappa_0 \) and \( j_q(x_0) = x_0 \). Since \( x_0 \) is outright definable, so is \( \kappa_0 \).

Fix \( p_0 \) witnessing this. Since \( j_{p_0} \) is \( \Sigma_1 \)-elementary and \( j_{p_0}(x_0) = x_0 \), \( j_{p_0} \) is fully elementary. But \( j_{p_0}(\kappa_0) > \kappa_0 \), so \( \kappa_0 \notin \text{rg}(j_{p_0}) \), a contradiction. ☐

### 3 Low-level definability over \( L_1(V_\delta) \)

We saw in \([2]\) a generalization of the fact, shown in \([4]\) Theorem 5.7***?, that if \( j \in \mathcal{E}_1(V_\delta) \) where \( \delta \in \text{Lim} \), then \( j \) is not definable from parameters over \( V_\delta \). The sets \( X \subseteq V_\delta \) which are definable from parameters over \( V_\delta \) are exactly those which are in \( L_1(V_\delta) \), or equivalently, in terms of Jensen’s hierarchy, those which are in \( \mathcal{J}(V_\delta) \) (the rudimentary closure of \( V_\delta \cup \{V_\delta\} \)). Beyond this, it is natural to consider whether one might get such a \( j \in \mathcal{E}_1(V_\delta) \) which is constructible from \( V_\delta \), i.e., in \( L(V_\delta) \). This also naturally ramifies: given an ordinal \( \alpha \), can there be
such a $j \in J_\alpha(V_\delta)$ (or $\in L_\alpha(V_\delta)$)? The question can also of course be extended beyond $L(V_\delta)$.

Later in the paper, we will find some quite precise answers to some of these questions. But in this section, for a warm-up and for some motivation, we consider the simplest instance not covered by [13] or [4]; that is, the question of whether there can be a $j \in \mathcal{E}_1(V_\delta)$ which is at the simplest level of definability beyond definability from parameters over $V_\delta$.

**Remark 3.1.** We use Jensen’s refinement of the $J$-hierarchy into the $S$-hierarchy. Here is a summary of the features we need; the reader can refer to [5] or [10] for more details. Recall that for a transitive set $X$,

$$J(X) = \{ f(V_\lambda, \bar{x}) \mid f \text{ is a rudimentary function and } \bar{x} \in X^{<\omega} \} = \bigcup_{n<\omega} S_n(X),$$

where $S$ is Jensen’s $S$-operator, and $S_0(X) = X$ and $S_{n+1}(X) = S(S_n(X))$. Taking $S$ defined appropriately (maybe not exactly how Jensen originally defined it), each $S_n(X)$ is transitive, $S_n(X) \subseteq S_{n+1}(X)$ and $S_n(X) \subseteq S_{\alpha}(X)$ for all transitive $X$.

The truth of $\Sigma_1$ statements over $J(X)$ also reduces uniformly recursively to countable disjunctions of statements over $X$. That is, there is a recursive function $(\varphi, \vec{f}, n) \mapsto \psi = \psi_{\varphi, \vec{f}, n}$ such that for all $\Sigma_1$ formulas $\varphi$, tuples $\vec{f}$ of (terms for) rudimentary functions and $n < \omega$, then $\psi = \psi_{\varphi, \vec{f}, n}$ is a formula in $L$, and for all transitive $X$ and $\bar{x} \in X^{<\omega}$,

$$J(X) \models \varphi(\vec{f}(X, \bar{x})) \iff \exists n < \omega \ [X \models \psi_{\varphi, \vec{f}, n}(\bar{x})].$$

Also, for each rudimentary $f$, the graph

$$\{(\vec{x}, y) \mid \vec{x} \in X^{<\omega} \text{ and } y = f(X, \vec{x})\}$$

is $\Sigma_1$-definable over $J(X)$ from the parameter $X$, uniformly in $X$. For $A \subseteq X$, we have $A \in J(X)$ iff $A$ is definable from parameters over $X$.

**Lemma 3.2. (ZF) Let $\lambda \in \text{Lim}$ and $j \in \mathcal{E}(V_\lambda)$. There is a unique $j' \in \mathcal{E}_1(J(V_\lambda))$ with $j \subseteq j'$.**

**Proof.** We must set $j'(\lambda) = \lambda$ and $j'(V_\lambda) = V_\lambda$. Because

$$J(V_\lambda) = \bigcup_{n<\omega} S_n(V_\lambda) = \{ f(V_\lambda, \bar{x}) \mid \bar{x} \in V_\lambda^{<\omega} \},$$

and since for rudimentary $f$, the graph $\{(\vec{x}, y) \mid \vec{x} \in V_\lambda^{<\omega} \text{ and } y = f(V_\lambda, \vec{x})\}$ is $\Sigma_1$ over $J(V_\lambda)$ in the parameter $V_\lambda$, we must set $j'(f(V_\lambda, \bar{x})) = f(V_\lambda, j(\bar{x}))$, giving uniqueness.

But this definition gives a well-defined $j' \in \mathcal{E}_1(J(V_\lambda))$. This is by [5]: all rudimentary functions are simple, and hence for each $\Sigma_0$ formula $\varphi$ and all rud functions $f_0, \ldots, f_{k-1}$, there is a formula $\varphi'$ such that for all $\bar{x} \in V_\lambda^{<\omega}$ and $y_i = f_i(V_\lambda, \bar{x})$, we have

$$J(V_\lambda) \models \varphi(\vec{y}) \iff V_\lambda \models \varphi'(\bar{x}).$$
But by the elementarity of $j$,

$$V_\lambda \models \varphi'(\bar{x}) \iff V_\lambda \models \varphi'(j(\bar{x})) \iff \mathcal{J}(V_\lambda) \models \varphi(j(y_0), \ldots, j(y_{k-1})).$$

Note $j'$ is also $\epsilon$-cofinal, and hence $\Sigma_1$-elementary. And $j \subseteq j'$.

Note that in the following, $j \upharpoonright V_\alpha \in V_\lambda$ for each $\alpha < \lambda$. The following theorem strengthens [4, Theorem 5.7 part 1**?] (in a different manner than 2.3). We will actually prove more later (Theorems 6.3 and 7.6), but we start here for some motivation:

**Theorem 3.3. (ZF)** Let $\lambda \in \text{Lim}$ and $j \in \mathcal{E}_1(V_\lambda)$. Then $\tilde{j} = \{ j \upharpoonright V_\alpha \mid \alpha < \lambda \}$ is not $\Sigma^j_1(V_\lambda)$.

**Proof.** Suppose otherwise. Note that each finite iterate $j_n$ is then likewise definable, so by [4, Theorem 5.6**?] we may assume that $j \in \mathcal{E}(V_\lambda)$. Let $\kappa$ be the least critical point among all such fully elementary $j$, select $j$ witnessing the choice of $\kappa$, and choose $p \in V_\lambda$ and a $\Sigma_1$ formula $\varphi$ such that

$$\tilde{j} = \{ k < \lambda \mid \mathcal{J}(V_\lambda) \models \varphi(k, p, V_\lambda) \}.$$

For $n < \omega$, let $\eta_n = \bigcup \{ \alpha < \lambda \mid \mathcal{S}_n(V_\lambda) \models \varphi(j \upharpoonright V_\alpha, p, V_\lambda) \}$. By Remark 3.1 and since $j$ is not definable from parameters over $V_\lambda$ (by Theorem 2.3 or [4, Theorem 5.7**?]), it follows that $\eta_n < \lambda$. Note $\eta_0 \leq \eta_{n+1}$ and $\sup_{n<\omega} \eta_n = \lambda$.

Let $(q, \mu) \in V_\lambda \times \lambda$ and $m < \omega$. Say that $(q, \mu)$ is $m$-good iff for each $n \leq m$, there are $(\xi_n, \ell_n) \in \lambda^2$ and $\ell_n \in V_\lambda$ such that:

1. $\ell_n : V_{\xi_n} \to V_{\xi_n}^\mu$ is $\Sigma_0$-elementary and cofinal (where if $\xi_n$ is a successor $\gamma + 1$, cofinality means that $\xi_n'$ is also a successor and $\ell_n(V_{\xi_n-1}) = V_{\xi_n-1}$)
2. for all $\Sigma_m$ formulas $\psi$ and $x \in V_{\xi_n}$ $[V_\lambda \models \psi(x) \iff V_\lambda \models \psi(\ell_n(x)),$
3. $\mathcal{S}_n(V_\lambda) \models "\ell_n \text{ is the union of all } k \text{ such that } \varphi(k, q, V_\lambda),"
4. $\ell_i \subseteq \ell_n$ for each $i \leq n$,
5. $\text{cr}(\ell_n) = \mu$.

If $(q, \mu)$ is $n$-good, write $\ell_n^q = \ell_n$ and $\xi_n^q = \xi_n$.

Let also $n < \omega$. Say that $(q, \mu)$ is $(m, n)$-strong iff $(q, \mu)$ is $m$-good and $\eta_n \leq \xi_n^q$. Say that $(q, \mu)$ is $n$-strong iff $\exists m < \omega \ [ (q, \mu) \text{ is } (m, n)\text{-strong}]$.

Recall $j \in \mathcal{E}(V_\lambda)$ with $\text{cr}(j) = \kappa$, etc. Let $\alpha_0 < \lambda$ be such that $p \in j(\alpha_0)$. Let

$$A_m = \{(q, \mu) \in V_{\alpha_0} \times \kappa \mid (q, \mu) \text{ is } m\text{-good}\}.$$

Let $A_\omega = \bigcap_{m<\omega} A_m$. Note that $(A_m)_{m<\omega} \in V_\lambda$.

By Lemma 3.2 because $j \in \mathcal{E}(V_\lambda)$, it extends uniquely to a $\tilde{j} \in \mathcal{E}(\mathcal{J}(V_\lambda))$ with $\tilde{j}(V_\lambda) = V_\lambda$. This gives that $j(\mathcal{S}_n(V_\lambda)) = \mathcal{S}_n(V_\lambda)$ for each $n < \omega$, and we have a fully elementary map

$$j_\lambda^\mathcal{S} : \mathcal{S}_n(V_\lambda) : \mathcal{S}_n(V_\lambda) \to \mathcal{S}_n(V_\lambda).$$

**Claim 1.** $A_\omega \neq \emptyset$, and moreover, $(p, \kappa) \in j(A_\omega)$.
Proof. Because $j_n^*$ is fully elementary and $A_n \in V_\lambda$, note
\[ j(A_n) = j_n^*(A_n) = \{(q, \mu) \in V_{\omega_1} \times j(\kappa) \mid (q, \mu) \text{ is } n\text{-good}\}. \]
But $(p, \kappa)$ is $n$-good, so $(p, \kappa) \in j(A_n)$. Also $(A_n)_{n < \omega} \in V_\lambda$, and
\[ j(A_\omega) = j\left( \bigcap_{n < \omega} A_n \right) = \bigcap_{n < \omega} j(A_n), \]
so $(p, \kappa) \in j(A_\omega) \neq \emptyset$, so $A_\omega \neq \emptyset$. \hfill \qed

Let $B_{(m, n)} = \{(q, \mu) \in V_\alpha \times \kappa \mid (q, \mu) \text{ is } (m, n)\text{-strong}\}$, and
\[ B_n = \{(q, \mu) \in V_\alpha \times \kappa \mid (q, \mu) \text{ is } n\text{-strong}\} \]
and $B_\omega = \bigcap_{n < \omega} B_n$. These sets are in $V_\lambda$.

Claim 2. $A_\omega \cap B_\omega \neq \emptyset$, and moreover, $(p, \kappa) \in j(A_\omega \cap B_\omega)$.

Proof. By the previous claim and like in its proof, it suffices to see that $(p, \kappa) \in j(B_n)$ for each $n < \omega$. But $B_n = \bigcup_{m < \omega} B_{(m, n)}$, so
\[ j(B_n) = \bigcup_{m < \omega} j(B_{(m, n)}), \]
and $j(B_{(m, n)}) = j_n^*(B_{(m, n)}) = \{(q, \mu) \in V_{\omega_1} \times j(\kappa) \mid (q, \mu) \text{ is } m\text{-good, with } j(\eta_m) \leq \xi^m_n\}$. But there is $m < \omega$ with $j(\eta_m) \leq \eta_m = \xi^p_m$. Then $(p, \kappa) \in j(B_{(m, n)})$, so $(p, \kappa) \in j(B_n)$, as required. \hfill \qed

Now by the claims, we may pick $(q, \mu) \in A_\omega \cap B_\omega$. Let $\ell = \bigcup_{n < \omega} \ell_n^q$.

Claim 3. $\ell : V_\lambda \rightarrow V_\lambda$ is fully elementary, $\cr(\ell) = \mu < \kappa$, and $\ell$ is $\Sigma^1_2(V_\lambda)$.

Proof. Because $(q, \mu) \in A_\omega$, for each $n < \omega$, $\ell_n^q : V_\lambda \rightarrow V_\lambda$ is cofinal $\Sigma_0$-elementary with $\cr(\ell_n^q) = \mu$, and $\ell_n^q \subseteq \ell_{n+1}^q$. So $\ell$ is a function with domain $V_\lambda = \bigcup_{n < \omega} V_\lambda$, the equality is because $(q, \mu)$ is $n$-strong for each $n < \omega$. So $\ell : V_\lambda \rightarrow V_\lambda$. But because $(q, \mu)$ is $m$-good for each $m < \omega$, $\ell$ is fully elementary. And $\cr(\ell) = \mu$. Finally note that $\ell$ is appropriately definable from the parameter $(\eta_0, V_\lambda)$. \hfill \qed

But $\mu < \kappa$, so the claim contradicts the minimality of $\kappa$, QED. \hfill \qed

One can directly generalize the foregoing argument, showing that an elementary $j : V_\delta \rightarrow V_\delta$ cannot appear in $J_\alpha(V_\delta)$, for some distance. But especially once we get to $\alpha \geq \kappa = \cr(j)$ (or worse, $\alpha \geq \kappa_\omega(j)$), things are clearly more subtle, because in order to extend $j$ to $j : J_\alpha(V_\delta) \rightarrow J_\alpha(V_\delta)$, $j$ must move ordinals $\geq \delta$. But a natural and general way to extend $j$ is through taking an ultrapower by the extender of $j$. So we treat this topic in the next section.
4 Ultrapowers and extenders under ZF

In order to help us analyse the model $L(V_3)$ further, and for more general purposes, we want to be able to deal with ultrapowers via extenders. This will, for example, assist us in extending embeddings of the form $j : V_3 \rightarrow V_3$ to larger models, such as $L(V_3)$.

Without AC, there are some small technical difficulties here in the definitions, which we will work through first. The more significant issues are those of wellfoundedness of the ultrapower and (generalized) Łoś’ theorem; when we apply extenders, we will usually want to know that these properties hold of the ultrapowers we form.

**Definition 4.1. (ZF)** Write $\approx_{\text{rank}}$ for the equivalence relation determined by rank; that is, $x \approx_{\text{rank}} y$ iff rank($x$) = rank($y$). A set $r$ is **rank-extensional** iff for all $x, y \in r$ with $x \neq y$ but $x \approx_{\text{rank}} y$, there is $z \in r$ with $z \in x \iff z \notin y$.

Let $r$ be rank-extensional. The **membership-rank diagram** of $r$ is the structure $\text{mr}(r) = (r, \in, \approx_{\text{rank}} r)$.

An easy induction on rank gives:

**Lemma 4.2. (ZF)** Let $r$ be rank-extensional. Then:

- For all $\alpha \in \text{OR}$, $r \cap V_\alpha$ is also rank-extensional.
- There is no non-trivial automorphism of $\text{mr}(r)$.

**Definition 4.3. (ZF)** We say that $r$ is an **index** iff $r$ is finite and rank-extensional.

Note that we have an equivalence relation on the class of all indices given by $a \approx b$ iff $\text{mr}(a) \cong \text{mr}(b)$. The **signature** $\text{sig}(a)$ of an index $a$ is the equivalence class of $a$ with respect to $\approx$. Note that every signature is represented by some element of $V_\omega$. By selecting in some natural way the minimal such representative, we may consider $\text{sig}(a)$ as being this element of $V_\omega$. A signature is a set $\text{sig}(a)$ for some index $a$. Given a transitive set $X$, we write $\langle X \rangle^{< \omega}$ for the set of indices $r \subseteq X$, and given an index $b$, $\langle X \rangle^b$ denotes $\{ r \in \langle X \rangle^{< \omega} \mid \text{sig}(r) = \text{sig}(b) \}$.

If $a \approx b$ then $\pi^{ab} : \text{mr}(a) \rightarrow \text{mr}(b)$ denotes the unique isomorphism. Let $a, \tilde{a}, \tilde{b}$ be indices with $a \subseteq \tilde{a}$ and $\text{sig}(\tilde{a}) = \text{sig}(\tilde{b})$. Then $\tilde{b}^{\tilde{a}a}$ denotes $\pi^{\tilde{a}a} \circ \pi^{ab}$.

**Lemma 4.4. (ZF)** For every finite set $c$ there is an index $b$ with $c \subseteq b$ and $\text{rank}(c) = \text{rank}(b)$.

---

8 One might have expected (as did the author initially) that for every finite extensional set $a$, there is a finite extensional set $b$ with $a \subseteq b$ (where extensional means that for all $x, y \in a$ with $x \neq y$, there is $z \in a$ with $z \in x$ if $z \notin y$). In fact, in the first draft of the notes [12] (v1 on arxiv.org), which contained the first version of the development of extenders here, we defined index with extensionality replacing rank-extensionality, and we made precisely that claim. But that claim is false; here is a counterexample. Define sets $n^i$ as follows:

- $0^i = \emptyset,$
- $1^i = \{0^i\}$ and $2^i = \{1^i\},$
- $3^i = \{0^i, 2^i\}$ and $4^i = \{1^i, 3^i\},$
- $(2n + 1)^i = \{0^i, 2^i, \ldots, (2n)^i\}$ and $(2n + 2)^i = \{1^i, 3^i, \ldots, (2n + 1)^i\}.$

Let $x = \{2n^i \mid n \in \omega\}$ and $y = \{(2n + 1)^i \mid n \in \omega\}$. Then $p = \{x, y\}$ is finite but there is no finite extensional $q$ with $p \subseteq q$. (Let $p \subseteq q$ with $q$ finite and consider the largest $k \in \omega$ such that $k^i \in q$. Observe that either $x \cap q = k^i \cap q$ or $y \cap q = k^i \cap q$, and hence $q$ is not extensional.)
Proof. This is an induction on rank (recall though that we don’t assume AC). Assume \( c \neq \emptyset \). Let \( \alpha \) be the maximum rank of elements of \( c \), so \( \text{rank}(c) = \alpha + 1 \).

Let \( c_{\text{max}} \) be the set of elements of \( c \) of rank \( \alpha \).

First choose a finite set \( c' \) such that \( c \subseteq c' \), all elements of \( c' \) have rank < \( \alpha \), and with \( c' \) extensional with respect to \( c_{\text{max}} \) (that is, for all \( x,y \in c_{\text{max}} \) with \( x \neq y \), there is \( z \in c' \) with \( z \in x \Delta y \)). Note that \( \text{rank}(c' \setminus c_{\text{max}}) \leq \alpha \). So by induction we can fix an index \( b' \) with \( c' \setminus c_{\text{max}} \subseteq b' \) and \( \text{rank}(b') \leq \alpha \). Now note that \( b = b' \cup c_{\text{max}} \) is as desired. \( \square \)

**Definition 4.5** *(Extender, Ultrapower)*. \((\text{ZF})\) Let \( M \models \text{RL} \) be transitive. We say that \( A \) is an amenable to \( M \), and write \( A \subseteq \text{amb} M \), iff \( A \subseteq M \) and \( A \cap x \in M \) for each \( x \in M \). We also write \( A \subseteq \text{amb} \langle M \rangle^{< \omega} \) to mean \( A \subseteq \text{amb} M \) and \( A \subseteq \langle M \rangle^{< \omega} \).

Let \( M,N \) be transitive, satisfying RL, and \( j : M \rightarrow N \) be \( \Sigma_1 \)-elementary and \( \in \)-cofinal. The *extender derived from \( j \)* is the set \( E \) of all pairs \((A,a)\) such that \( a \in \langle N \rangle^{< \omega} \), \( A \subseteq \text{amb} M \) and \( a \in j(A) \). Given \( a \in \langle N \rangle^{< \omega} \), let \( E_a = \{ A \mid (A,a) \in E \} \).

Let \( A \subseteq \text{amb} \langle M \rangle^{< \omega} \) and \( a, \bar{a} \) be indices with \( a \subseteq \bar{a} \). Then \( A^{aa} \) denotes the set of all \( u \in \langle M \rangle^{\bar{a}} \) such that \( u^{\bar{a}} \in A \), and \( A^{\bar{a}a} \) denotes the set of all \( u \in \langle M \rangle^a \) such that there is \( v \in A \cap \langle M \rangle^{\bar{a}} \) and \( u = v^{\bar{a}} \).

Let \( f : \langle M \rangle^{< \omega} \rightarrow V \). Let \( a, \bar{a} \) be indices with \( a \subseteq \bar{a} \). We define a function \( f^{aa} : \langle M \rangle^{< \omega} \rightarrow V \) as follows. Let \( u \in \langle M \rangle^{< \omega} \). If \( \text{sig}(u) \neq \text{sig}(\bar{a}) \) then \( f^{aa}(u) = \emptyset \), and if \( \text{sig}(u) = \text{sig}(\bar{a}) \) then \( f^{aa}(u) = f(u^{\bar{a}}) \).

Given a transitive rudimentarily closed structure \( P \) in the language of set theory (or possibly in a larger language), we say that \( E \) is an *extender over \( P \)* iff \( \text{OR}_M < \text{OR}_P \) and \( V_{\text{or}_M}^M = M \). Suppose \( E \) is over \( P \). A *\( P \)-relevant pair* (with respect to \( E \)) is a pair \((a,f)\) such that \( a \in \langle N \rangle^{< \omega} \) and \( f \in P \) and \( f : \langle M \rangle^{< \omega} \rightarrow P \). We define the (internal) *ultrapower \( \text{Ult}_0(P,E) \) of \( P \) by \( E \).* We first define an equivalence relation \( \approx_E \) on the class of \( P \)-relevant pairs, by setting \((a,f) \approx_E (b,g) \) iff for some/all \( c \in \langle N \rangle^{< \omega} \) with \( a \cup b \subseteq c \), we have

\[
\{ u \in \langle M \rangle^{< \omega} \mid f^{ac}(u) = g^{bc}(u) \} \in E_c.
\]

We write \([a,f]_{E}^{P,0} \) for the equivalence class of \((a,f)\). We define the relation \( \in_E \) likewise, replacing the condition “\( f^{ac}(u) = g^{bc}(u) \)” with “\( f^{ac}(u) \in f^{bc}(u) \)”. Let \([U] \) be the class of equivalence classes of \( P \)-relevant pairs with respect to \( \approx_E \), and \( c' \) be the relation on \([U] \) induced by \( \in_E \). Then the (internal) *ultrapower \( \text{Ult}_0(P,E) \) of \( P \) by \( E \) is the structure \( U = ([U],\in_E) \). If \( U \) is extensional and wellfounded then we identify it with its transitive collapse. We define the associated *ultrapower embedding \( i_{E}^{P,0} : P \rightarrow U \) by*

\[
i_{E}^{P,0}(x) = ([\emptyset,c_x])_{E}^{P,0}
\]

\(^9\)Clearly \((A,a) \in E \) iff \((A \cap V_c^M,a) \in E \) where \( \xi \) is any ordinal in \( M \) such that \( a \in j(V^M_\xi) \).

And given the manner in which \( \hat{E} \) will be used, we could have actually added the extra demand that \( A \in M \) to the requirements specifying when \((A,a) \in E \), and in terms of information content and cardinality, it would be more natural to do so. But it is convenient in other ways to allow more arbitrary amenable sets \( A \).

\(^10\)The “sub-0” in “\( \text{Ult}_0 \)” and the “super-0” in “\( i_{E}^{P,0} \)” denotes the internality of the ultrapower, i.e. that the functions \( f \) used in forming the ultrapower all belong to \( P \). This is an artefact of related notation in inner model theory, where one can have \( \text{Ult}_n \) for \( n \leq \omega \).
where $c_x : \langle M \rangle ^{<\omega} \to P$ is the constant function $c_x(u) = x$. We often abbreviate this by $i_E^P$ or $i_E$. For any set $y$, let $\text{ind}(y)$ be the unique index with universe $\{y\}$, and let $\text{elmt}(\text{ind}(y)) = y$ and $\text{elmt}(u) = \emptyset$ if $u$ is not of form $\text{ind}(y)$. We write $\text{spt}(E) = N$.

Of course, if $P \not\models \text{AC}$ then the proof of Łos’ theorem does not go through in the usual manner, so in general Ulث$_0(P, E)$ might not even be extensional.

**Lemma 4.6. (ZF)** With notation as in Definition 4.4, we have:

1. Let $a, \tilde{a} \in \langle N \rangle ^{<\omega}$ with $a \subseteq \tilde{a}$ and $A \subseteq \text{ambl} \langle M \rangle ^{<\omega}$. Then:
   
   (a) If $A \subseteq \langle M \rangle^a$ and $B = \langle M \rangle^a \setminus A$ then $B^{\tilde{a}} = \langle M \rangle^{\tilde{a}} \setminus A^{\tilde{a}}$.
   
   (b) $E_a$ is an ultrafilter over the set of all $A \subseteq \text{ambl} \langle M \rangle ^{<\omega}$ and $\langle M \rangle^a \in E_a$,
   
   and in fact, $\langle V^M_\xi \rangle^a \in E_a$ for each $\xi < \text{OR}^M$ with $a \subseteq j(V^M_\xi)$.
   
   (c) $A \in E_a$ iff $A^{\tilde{a}} \in E_{\tilde{a}}$.
   
   (d) If $A \in E_{\tilde{a}}$ then $A^{\tilde{a}} \in E_a$.

2. $f^{ac} = (f^{ab})^{bc}$ for all $a, b, c \in \langle N \rangle ^{<\omega}$ with $a \subseteq b \subseteq c$ and all functions $f$.

3. In the definition of $\approx_E$ and $\in_E$, the choice of $c$ is irrelevant.

4. $\approx_E$ is an equivalence relation on the $P$-relevant pairs.

5. $\in_E$ respects $\approx_E$.

6. $N \subseteq \text{wp}(U)$ and for each $\beta < \text{OR}^N$ we have $V^U_\beta = V^N_\beta$. Moreover, for $x \in V^N_\beta$, we have $x = [\text{ind}(x), \text{elmt}]^{P,0}_E$.

**Proof.** Parts (1) and (2) are straightforward. Part (3) Consider $\approx_E$, and pairs $(a, f), (b, g)$. Take $c, c' \in \langle N \rangle ^{<\omega}$ with $a \cup b \subseteq c, c'$. Note we may assume $c \subseteq c'$.

We must see $A \in E_c$ iff $A' \in E_{c'}$ where

- $A = \{ u \in \langle V^M_a \rangle^c \mid f^{ac}(u) = g^{bc}(u) \}$,
- $A' = \{ u \in \langle V^M_a \rangle^{c'} \mid f^{ac}(u) = g^{bc}(u) \}$.

As $c \subseteq c'$, part (2) gives that $A' = A^{c'}$, so $A \in E_c$ iff $A' \in E_{c'}$ by part (1). The rest of parts (3–5) is similar or follows easily.

Part (6) Easily from the definitions, for $x, y \in N$ we get

\[
\text{ind}(x), \text{elmt}) \approx_E \text{(ind}(y), \text{elmt}) \iff x = y,
\]

\[
\text{ind}(x), \text{elmt}) \in_E \text{(ind}(y), \text{elmt}) \iff x \in y.
\]

So let $(a, f)$ be a $P$-relevant pair and $x \in N$ be such that

\[(a, f) \in_E \text{(ind}(x), \text{elmt}).\]

Note we may assume $x \in a$ and there is $\xi < \text{OR}^M$ such that $a \in j(V^M_\xi)$ and $\text{rg}(f) \subseteq V^M_\xi$, and so $g = f | V^M_\xi \in M$. But then $(a, f) \approx_E (a, g)$ and

\[j(g)(a) \in j(\text{elmt} | V^M_\xi)(\text{ind}(x)) = x,
\]

so $j(g)(a) = y$ for some $y \in x$. But then $(a, f) \approx_E \text{(ind}(y), \text{elmt})$, as desired. □
Definition 4.7. (ZF) Let $E$ be an extender over a transitive rudimentarily closed structure $M$. We say that $\Sigma_0$-Loš criterion holds for $\text{Ult}_0(M,E)$ iff for all $n < \omega$, for all $f_0, f_1, \ldots, f_n \in M$, for all $a \in \langle \text{spt}(E) \rangle^{< \omega}$, and all $\Sigma_0$ formulas $\varphi$, if there are $E_\varphi$-measure one many $u \in \langle M \rangle^{< \omega}$ such that

$$M \models \exists y \in f_0(u)[\varphi(f_1(u), \ldots, f_n(u), y)]$$

then there is $b \in \langle \text{spt}(E) \rangle^{< \omega}$ and $g \in M$ such that $a \subseteq b$ and for $E_b$-measure one many $v$, we have

$$M \models g(v) \in f_0^b(v) \text{ and } \varphi(f_1^b(v), \ldots, f_n^b(v), g(v)).$$

We define Loš criterion for $\text{Ult}_0(M,E)$ analogously, but we allow arbitrary formulas $\varphi$, and the $\exists$ quantifier is unbounded. 

Theorem 4.8 (Generalized Loš Theorem). (ZF) Let $M$ be a transitive rudimentarily closed structure and $E$ be an extender over $M$. Suppose that $\Sigma_0$-Loš criterion holds for $\text{Ult}_0(M,E)$. Then for all $n < \omega$, all $f_0, f_1, \ldots, f_n \in M$, all $a_1, \ldots, a_n \in \langle \text{spt}(E) \rangle^{< \omega}$ and all $\Sigma_0$ formulas $\varphi$, letting $a \in \langle \text{spt}(E) \rangle^{< \omega}$ be such that $a_i \subseteq a$ for each $i$, we have

$$\text{Ult}_0(M,E) \models \varphi([a_1, f_1], \ldots, [a_n, f_n])$$

iff there are $E_\varphi$-measure one many $v \in \langle M \rangle^{< \omega}$ such that

$$M \models \varphi(f_1^{a_1}(v), \ldots, f_n^{a_n}(v)).$$

Therefore, the ultrapower embedding $i_{E,0}^M$ is $\in$-cofinal and $\Sigma_1$-elementary. Moreover, if Loš criterion holds for $\text{Ult}_0(M,E)$, then the above equivalence holds for arbitrary formulas $\varphi$, and $i_{E,0}^M$ is fully elementary.

Proof. This is basically the usual induction to prove Loš theorem under AC, except that we use Loš requirement instead of appealing to AC. One difference, however, is that we need to allow enlarging $a$ to $b$ in order to find an element $[b, g]$ of the ultrapower witnessing the $a$ statement; in the usual proof of Loš theorem, one can take $a = b$.

5 Definability of $V$-criticality and wLS cardinals

Under ZFC, the fact that $\kappa = \text{cr}(j)$ for some elementary $j : V \rightarrow M$ with $M$ transitive, is equivalent to the measurability of $\kappa$. Therefore this “$V$-criticality” of $\kappa$ is first-order. We make a brief digression to consider this question under ZF.

Definition 5.1. (ZF) An ordinal $\kappa$ is $V$-critical iff there is an elementary $j : V \rightarrow M$ with $\text{cr}(j) = \kappa$, where $M \subseteq V$ is transitive.

Lemma 5.2. (ZF) Let $\kappa$ be $V$-critical. Then $\kappa$ is inaccessible.

Proof. Suppose not and let $\alpha < \kappa$ and $f : V_\alpha \rightarrow \kappa$ be cofinal. Let $j : V \rightarrow M$ be elementary with $\text{cr}(j) = \kappa$. Then $j(f) = j \circ f = f$, although by elementarity, $j(f) : V_\alpha \rightarrow j(\kappa)$ is cofinal, a contradiction.
We do not know whether ZF$_2$ proves that V-criticality is a first-order property. But we will show that ZF$_2$+“There is a proper class of weak Löwenheim-Skolem cardinals” does prove this. Recall this notion from [14] Definition 4:

**Definition 5.3** (Usaba). (ZF) A cardinal $\kappa$ is **weak Löwenheim-Skolem (wLS)** if for every $\gamma < \kappa$ and $\alpha \in [\kappa, \text{OR}]$ and $x \in V_\alpha$, there is $X \subset V_\gamma$ with $X \equiv V_\gamma$ and the transitive collapse of $X$ in $V_\kappa$.

**Remark 5.4.** Usaba also defines **Löwenheim-Skolem (LS) cardinals**, which is at least superficially stronger. As Usaba mentions in [14], ZFC proves that there is a proper class of LS cardinals, and that (as a result of Woodin is that) under just ZF, every supercompact cardinal is an LS cardinal. Thus, assuming ZF and that there is a super Reinhardt cardinal then there is a proper class of LS cardinals, and hence wLS cardinals. The next lemma is immediate, due to Usaba:

**Lemma 5.5.** (ZF) We have:

1. The class of wLS cardinals is closed.

2. Suppose there is a proper class of wLS cardinals and let $\gamma \in \text{OR}$ be regular. Then there is a proper class of wLS cardinals $\delta$ such that $\text{col}(\delta) = \gamma$.

**Definition 5.6.** (ZF) A V-criticality pre-witness is a tuple $(\kappa, \delta, N, j)$ such that $\delta$ is a weak Löwenheim-Skolem cardinal, $N \models \text{RL}$ is transitive, $j : V_\delta \to N$ is $\in$-cofinal and $\Sigma_1$-elementary and $\text{cr}(j) = \kappa$.

**Theorem 5.7.** (ZF) Let $(\kappa, \delta, N, j)$ be a V-critical pre-witness. Let $U = \text{Ult}(V, E_j)$. Then:

1. Łoś’ criterion holds for $U$, so $U$ is extensional and $i_E$ is elementary, and

2. If $\text{col}(\delta) > \omega$ then $U$ is wellfounded and $j \subseteq i_E$ and $\kappa = \text{crit}(i_E)$ is V-critical.

**Proof.** Let $a \in \langle N \rangle <^\omega$ and $f : \langle V_\delta \rangle <^\omega \to V$ and $\varphi$ be a formula and suppose that for $E_\alpha$-measure many $\alpha$, there is $y$ such that $\varphi(f(u), y)$. Let $n < \omega$ be large and $\alpha \in \text{OR}$ be large and with $V_\alpha \equiv_n V$. Let $\gamma < \delta$ be such that $a \subseteq j(V_\gamma)$. So $\langle V_\gamma \rangle <^\omega \in E_\alpha$. Applying weak Löwenheim-Skolemness at $\delta$, let $X \equiv V_\alpha$ with $V_\gamma \cup \{\gamma, f, \delta, j, N, E\} \subseteq X$ and such that the transitive collapse $\bar{X}$ of $X$ is in $V_\delta$. Let $\pi : \bar{X} \to X$ be the uncollapse map. Let $\pi(f) = f$, etc.

By the elementarity, for each $u \in \langle V_\gamma \rangle <^\omega$, and each $v \in \bar{X}$, we have that $\bar{X} \models \varphi(u, v)$ if $V \models \varphi(f(u), \pi(v))$.

Note we can fix $y \in j(\bar{X})$ such that $j(\bar{X}) \models \varphi(\bar{f}(a), y)$. Let $\xi \in (\gamma, \delta)$ with $X \subseteq V_\xi$ and $b \in \langle j(V_\xi) \rangle <^\omega$ with $a \cup \{a, y\} \subseteq b$. Then for $E_\delta$-measure one many $w \in \langle V_\xi \rangle <^\omega$, letting $(a^w, y^w) = \pi^{\text{bw}}(a, y)$, we have that $a^w \in \langle V_\gamma \rangle <^\omega$ and $y^w \in \langle \bar{X} \rangle <^\omega$ and $X \models \varphi(f(a^w), y^w)$, and hence $V \models \varphi(f(a^w), \pi(y^w))$.

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[12] In [12] v1, it asserted that ZF $+$ a proper class of Reinhardt cardinals proves there is a proper class of LS cardinals, but this should have been a super Reinhardt (which easily implies a proper class of the same).
So define \( g : (V_\xi)^{<\omega} \rightarrow V \) by setting \( g(w) = \pi(y^w) \) for all such \( w \) (and \( g(w) = \emptyset \) otherwise). Then for \( E_\xi \)-measure one many \( w \in (V_\xi)^{<\omega} \), we have \( V \models \varphi(f^{ab}(w), g(w)) \), as desired.

Part 2 Suppose not. So \( \text{cof}(\delta) > \omega \). For limit ordinals \( \xi < \delta \), let \( E_\xi \) be the extender derived from

\[ j \upharpoonright V_\xi : V_\xi \rightarrow V_{\sup j \upharpoonright \xi}. \]

Let \( U_\xi = \text{Ult}_0(V, E_\xi) \) and \( k_\xi : U_\xi \rightarrow \text{Ult}_0(V, E) \) the natural factor map and \( k_{\xi \xi} : U_\xi \rightarrow U_\zeta \) likewise. We have not verified Lo"{s}' criterion for these partial ultrapowers, so we do not claim elementarity of the maps; nor do we claim that \( U_\xi \) is extensional. But note that \( k_\xi \) and \( k_{\xi \xi} \) are well-defined and respect "\( \varepsilon \)" and "\( =\)" (that is, in the sense of the ultrapowers, even if they fail extensionality), and commute; that is, \( k_{\xi \beta} = k_\xi \circ k_{\xi \beta} \).

Let \( \xi \leq \delta \) be a limit. Let \( \mathcal{O}_\xi = \text{OR}^{U_\xi} = \text{Ult}_V(\text{OR}, E_\xi) \), with the notation meaning that we use all functions (in \( V \)) which map into \( \text{OR} \) to form the ultrapower. Each \( \mathcal{O}_\xi \) is a linear order. Now \( U_\xi \) is wellfounded if \( \mathcal{O}_\xi \) is wellfounded (see [11]). In particular, \( \mathcal{O}_\delta \) is illfounded. By restricting \( k_{\xi \xi} \), we get a commuting system of order-preserving maps \( \ell_{\xi \xi} : \mathcal{O}_\xi \rightarrow \mathcal{O}_\zeta \) (so \( \ell_{\xi \xi} \subseteq k_{\xi \xi} \) and \( \ell_{\xi \beta} = \ell_{\xi \beta} \circ \ell_{\xi \beta} \)). Note that the direct limit of the \( \mathcal{O}_\xi \) under the maps \( \ell_{\xi \xi} \), \( \xi < \delta \), is isomorphic to \( \mathcal{O}_\delta \), \( \ell_{\xi \xi} \) is the direct limit map. Note that each \( \ell_{\xi \xi} \) is cofinal.

We claim there is \( \xi < \delta \) such that \( \mathcal{O}_\xi \) is illfounded (here we use \( \text{cof}(\delta) > \omega \)). For suppose not. Then \( \mathcal{O}_\xi \equiv \text{OR} \). Define a sequence \( \langle \xi_n, \eta_n \rangle_{n<\omega} \) of pairs of ordinals. Let \( \xi_0 = 0 \). Now \( \ell_{\xi \beta} \) is cofinal. Let \( \eta_0 \) be the least \( \eta \) with \( \ell_{\xi \beta}(\eta) \) in the illfounded part of \( \mathcal{O}_\delta \). Given \( \langle \xi_n, \eta_n \rangle \) with \( \ell_{\xi_n \delta}(\eta_n) \) in the illfounded part, there is a pair \( \langle \xi, \eta \rangle \) such that \( \ell_{\xi \xi}(\eta_n) > \eta \) and \( \ell_{\xi \beta}(\eta) \) in the illfounded part. Let \( \ell_{\xi_{n+1}, \eta_{n+1}} \) be lexicographically least such. Let \( \xi = \sup_{n<\omega} \xi_n \). Because \( \text{cof}(\delta) > \omega \), we have \( \xi < \delta \). But the sequence just constructed exhibits that \( \mathcal{O}_\xi \) is illfounded, a contradiction.

So fix \( \xi < \delta \) with \( \mathcal{O}_\xi \) illfounded. Let \( n < \omega \) be large, let \( \alpha \in \text{OR} \) be large with \( V_\alpha \preceq_n V \); hence, for some \( \beta < \alpha \), we have \( V_\alpha \models \text{"Ult}_0(V_{\beta}, E_\xi) \text{ is illfounded"} \).

Let \( \xi' = \sup j \upharpoonright \xi \). Using the weak Löwenheim-Skolemness of \( \delta \), let \( X \preceq V_\alpha \) with

\[ V_\xi \cup V_{\xi'}^{\beta} \cup \{ N, j, \xi, E_\xi, \beta \} \subseteq X \]

and the transitive collapse \( \tilde{X} \) of \( X \) in \( V_{\beta} \). So letting \( \pi : \tilde{X} \rightarrow X \) be the uncollapse map, we have \( \pi(E_\xi) = E_{\xi'} \), \( \pi \upharpoonright V_\xi = \text{id} \), etc. And \( X \models \text{"Ult}_0(V_{\beta}, E_\xi) \text{ is illfounded"} \), where \( \pi(\beta) = \beta \). As \( X \) is transitive and models enough of \( \text{ZF} \), it follows that \( U = \text{Ult}_0(V_{\beta}, E_\xi) \) is illfounded.

Now define \( \sigma : U \rightarrow N \) by setting \( \sigma([a, f]) = j(f)(a) \). (This makes sense, as \( f \in \tilde{X} \subseteq V_{\beta} = \text{dom}(j) \).) Then note that (since \( E_{\xi'} \) is derived from \( j \)), \( \sigma \) is \( \varepsilon \)-preserving. But then \( N \) is illfounded, contradicting our assumption that \( N \) is transitive. So \( U \) is wellfounded, as desired. This completes the proof of the theorem.

Of course under \( \text{ZFC}_2 \), \( V \)-criticality is equivalent to measurability, and has a first-order formulation. We can now generalize this result:

**Theorem 5.8.** (\( \text{ZF}_2 \)) Assume a proper class of \( w \text{LS} \) cardinals. Let \( \kappa \in \text{OR} \).

Then the following are equivalent:

- \( \kappa \) is \( V \)-critical
there is a $V$-criticality pre-witness $(\kappa, \delta, N, j)$ with $\text{cof}(\delta) > \omega$.

- there is a $V$-criticality pre-witness $(\kappa, \delta, N, j)$ such that $\text{Ult}_0(V, E_j)$ is well-founded, where $E_j$ is the extender derived from $j$.

In particular, $V$-criticality is first-order definable over $V$.

Proof. Suppose first that $\kappa$ is $V$-critical, and let $k : V \to M$ be elementary with $\text{cr}(k) = \kappa$. By Lemma 5.2, $\kappa$ is regular. So by hypothesis and Lemma 5.5, we can fix a Lowenheim-Skolem cardinal $\delta > \kappa$ with $\text{cof}(\delta) > \omega$ (in fact $\text{cof}(\delta) = \kappa$ is possible). Let $\xi = \sup k^* \delta$ and $N = V_\xi^M$ and $j = k \restriction V_\delta$. Then $(\kappa, \delta, N, j)$ is a $V$-criticality pre-witness with $\text{cof}(\delta) > \omega$. We will show below that it follows that $\text{Ult}_0(V, E_j)$ is wellfounded, but here it is easier: define

$$\ell : \text{Ult}_0(V, E_j) \to M, \quad \ell([a, f]_{E_j}) = k(f)(a),$$

which, directly from the definition of “$\in$” and “$=$” in the ultrapower, preserves membership and equality. But $M$ is transitive, so $\text{Ult}_0(V, E_j)$ is wellfounded. Note that we have not yet proved that Los’ theorem holds, or even that $\text{Ult}_0(V, E_j)$ is extensional.

Now suppose that $(\kappa, \delta, N, j)$ is any $V$-criticality pre-witness such that either $U = \text{Ult}_0(V, E)$ is wellfounded, where $E = E_j$, or $\text{cof}(\delta) > \omega$. Then by Theorem 5.7, $U$ is wellfounded and Los’ criterion holds for this ultrapower, and hence the ultrapower map $k : V \to U$ is elementary by Generalized Los’ Theorem, so $U$ is extensional, so we take $U$ transitive, and $j \subseteq k$, so $\text{cr}(k) = \kappa$.

Corollary 5.9. (ZF) Let $\kappa \in \text{OR}$. Then $\kappa$ is “definably $V$-critical” (that is, witnessed by some definable-from-parameters class $k : V \to M$) iff there is a $V$-criticality pre-witness $(\kappa, \delta, N, j)$ such that $\text{cof}(\delta) > \omega$.

Proof. Just use the proof of Theorem 5.8, but now all relevant classes are definable from parameters.

Corollary 5.10. (ZF$_2$) If $\kappa$ is super Reinhardt, then there is a normal measure on $\kappa$ concentrating on $V$-critical ordinals.

Proof. If there is a super Reinhardt cardinal then there is a proper class of weak Lowenheim Skolem cardinals, by Remark 5.4, so the theorem applies, and easily yields the corollary.

Question 5.11. (ZF$_2$) Suppose $\kappa$ is Reinhardt. Is $V$-criticality first-order? Must there be a $V$-critical ordinal $< \kappa$?

Of course if $V$-criticality is first-order and $\kappa$ is Reinhardt, then like before, there are unboundedly many $V$-critical ordinals $< \kappa$.

Remark 5.12. One can now easily observe that if $(V, j) \models \text{ZFR}$ then there is no set $X$ such that $V = L(X)$. Actually much more than this is known but here

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13 Assume ZFR. Goldberg showed a few years ago (unpublished at the time) that every set has a sharp. The author showed in [12] that $V \neq \text{HOD}(X)$ for every set $X$. Goldberg [3] and Usuba [15] then both (independently) sent proofs to the author that there is no set-forcing extension satisfying AC. The author then showed that $\mathcal{M}_n^\exists (X)$ exists for every set $X$, but the proof is not yet published. Much more local proofs of sharp existence can now be seen in [2, ***where?] and [11, ***where?].
is the proof of this simpler fact: Suppose otherwise. Then there is a proper class of wLS cardinals. Let $\delta \in \text{OR}$ be such that $\text{cof}(\delta) > \omega$ and $X \in V_\delta$ and $\bar{\eta} \subseteq \delta$.

Let $E$ be the extender derived from $j \upharpoonright V_\delta : V_\delta \rightarrow V_\delta$. Then ($\text{cr}(j), \delta, V_\delta, j \upharpoonright V_\delta$) is a $V$-criticality pre-witness. So by Theorem 5.3, $U = \text{Ult}_0(V, E)$ is extensional and wellfounded, the ultrapower map $k : V \rightarrow U$ is elementary, and $V_\delta \subseteq U$. So in fact $U = L(V_\delta) = V$. But $k$ is definable from the parameter $E$, contradicting Suzuki [2, 11].

6 $L(V_\delta)$ and uncountable cofinality

Lemma 6.1. (ZF) Assume $\delta$ is inaccessible and $V = \text{HOD}(V_\delta)$. Then $\delta$ is wLS.

In fact, for all $\alpha \in (\delta, \text{OR})$ and $p \in V_\alpha$ and $\beta < \delta$ there is $(X, \delta, \pi)$ such that $\delta, p \in X \not\in V_\alpha$ and $\beta \leq \delta < \delta$ and $X \cap V_\delta = V_\delta$ and $\pi : V_\delta \rightarrow X$ is a surjection.

Proof. Given $n < \omega$, we may assume that $V_\alpha \not\subseteq V_n$, by increasing $\alpha$ and then subsuming the old $\alpha$ into the parameter $p$. So since $V = \text{HOD}(V_\delta)$, we may assume

$$V_\alpha = \text{Hull}^{V_\delta}_{\Sigma_2}(\text{OR} \cup V_\delta).$$

(1)

Let $X' = \text{Hull}^{V_\delta}(V_\delta \cup \{p, \delta\})$ and $\delta = \sup(X' \cap \delta)$ and $X = \text{Hull}^{V_\delta}(V_\delta \cup \{\delta, \delta\})$.

By the inaccessibility of $\delta$, we have $\delta < \delta$. So it suffices to see that $X \not\subseteq V_\alpha$ and $X \cap V_\delta = V_\delta$, as clearly we get an appropriate surjection $\pi$.

We first show that $X \cap V_\delta = V_\delta$. Given $n < \omega$ and $\xi < \delta$, let

$$\varepsilon_n(\xi) = \sup(\text{Hull}^{V_\delta}_{\Sigma_n}(V_\delta \cup \{p, \delta\}) \cap \delta).$$

By inaccessibility of $\delta$, we have $\varepsilon_n(\xi) < \delta$. Note $\varepsilon_n(\xi)$ is definable over $V_\alpha$ from the parameter $(\xi, p, \delta)$ (since $n$ is fixed). Therefore $\varepsilon_n^{\alpha} \subseteq \delta$, which gives $X \cap V_\delta = V_\delta$.

Now for elementarity. First note that for each $n < \omega$ and $\xi < \delta$, there is $\eta < \delta$ such that for each $\Sigma_n$ formula and $\bar{x} \in V_\delta^{< \omega}$, if $V_\alpha \models \exists y \varphi(y, \bar{x}, p, \delta)$ then there is $\bar{y} \in V_\delta$ with $V_\alpha \models \varphi(y, \bar{x}, p, \delta)$ and $y$ being $\Sigma_{n+3}$-definable from elements of $V_\eta \cup \{p, \delta\}$. This is an easy consequence of line (1) (using the usual trick of minimizing on ordinal parameters to get rid of them) and the inaccessibility of $\delta$. Let $\eta_n(\xi)$ be the least $\eta$ that witnesses this for $\xi$. Note that $\eta_n^{\alpha} \subseteq \delta$. But then $X \not\subseteq V_\alpha$ and $X \cap V_\delta = V_\delta$, as desired.

Theorem 6.2. (ZF) Assume $V = \text{HOD}(V_\delta)$ where $\text{cof}(\delta) > \omega$. Let $j \in \mathcal{E}_1(V_\delta)$. Then for all sufficiently large $m < \omega$, Lo"{s}' criterion holds for $\text{Ult}(V, E)$, where $E = E_{j_m}$, and this ultrapower is wellfounded.

We can easily deduce the main result of this section:

Theorem 6.3. (ZF) Assume $V = L(V_\delta)$ where $\text{cof}(\delta) > \omega$. Then $\mathcal{E}_1(V_\delta) = \emptyset$.

Proof. Suppose $j \in \mathcal{E}_1(V_\delta)$. By the theorem, we may assume Lo"{s}' theorem and wellfoundedness for $\text{Ult}(V, E_j)$. But then as in Remark 5.12 we get $i_{E_j} : V \rightarrow V$ is definable from $E_j$, a contradiction.

\[\text{[12]}\] There is actually a more efficient argument available here, which we had used in version 1 of [12]: Instead of arguing via Theorem 5.7, one can directly establish Lo"{s}' theorem and wellfoundedness for the ultrapower, using that it is a factor of $j$; for this, we don't need to take $\text{cof}(\delta) > \omega$.

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Proof of Theorem 6.2. If $\delta$ is inaccessible then by Lemma 6.1, $\delta$ is wLS, so by
Theorem 5.7, we can take $j$ itself (i.e. $m \geq 1$). So assume $\delta$ is non-inaccessible.

Let $\gamma = \text{cof}(\delta)$. Now by [4 Theorem 5.6**?], and replacing $j$ with $j^m$ with
a sufficiently large $m$, we may assume:

- $j : (V_\delta, A) \rightarrow (V_\delta, j^+(A))$ is fully elementary for every $A \subseteq V_\delta$,
- if $\gamma < \delta$ then $j(\gamma) = \gamma$, and
- if $\gamma = \delta$ is regular but non-inaccessible, then $j(P) = P$, where $P = \text{scot}(\delta)$.

Claim 4. Los’ criterion holds.

Proof. Let $\varphi$ be a $\Sigma_k$ formula and let $\Omega \in \text{OR}$ be such that $V_\Omega \equiv k+2 V$;
in particular, $V_\Omega = \text{Hull}_{\Sigma_2}(V_\delta \cup \Omega)$. Let $\alpha < \delta$ and $f : \langle V_\alpha \rangle^{< \omega} \rightarrow V$ and
$a \in \langle V_\alpha \rangle^{< \omega}$ be such that for $E_\alpha$-measure one many $u \in \langle V_\alpha \rangle^{< \omega}$, we have

$$ V_\Omega \models \exists y [\varphi(f(u), y)]. $$

For $u \in \langle V_\alpha \rangle^{< \omega}$, let $\beta_u$ be the least $\beta < \delta$ such that there is $z \in V_\beta$ and $y \in V_\beta$ such that $V_\beta \models \varphi(f(u), y)$ and $y$ is $\Sigma_2$-definable from ordinals and $z$ over $V_\Omega$.

Given $\beta \leq \delta$, let $A_\beta$ be the set of all $u \in \langle V_\alpha \rangle^{< \omega}$ such that $\beta_u \leq \beta$. Note that
$A_\beta$ has $E_\alpha$-measure one, if $\beta_0 < \beta_1 \leq \delta$ then $A_{\beta_0} \subseteq A_{\beta_1}$, and $A_\beta = \bigcup_{\beta \leq \delta} A_\beta$.

Subclaim 1. There is $\beta < \delta$ such that $A_\beta$ is $E_\alpha$-measure one.

Proof. Suppose not. Suppose first that $\delta$ is singular, so $\gamma = \text{cof}(\delta) < \delta$. Let
$f : \gamma \rightarrow \delta$ be cofinal. Then

$$ A_\delta = \bigcup_{\xi < \gamma} A_{f(\xi)}. $$

The sequence $\langle A_{f(\xi)} \rangle_{\xi < \gamma} \in V_\delta$. But recall $j(\gamma) = \gamma$, so it follows that

$$ j(A_\delta) = j\left( \bigcup_{\xi < \gamma} A_{f(\xi)} \right) = \bigcup_{\xi < \gamma} j(A_{f(\xi)}). $$

But $a \in j(A_\delta)$, so there is $\xi < \gamma$ such that $a \in j(A_{f(\xi)})$, so $A_{f(\xi)}$ is $E_\alpha$-measure
one, a contradiction.

Now suppose instead that $\delta$ is regular. So $P = \text{scot}(\delta) \in V_\delta$ and $j(P) = P$.
(see [13, and 4, 5.2, 5.3]). We mimic the previous argument. For each $R \in P$, let
$\pi_R : V_{\alpha+1} \rightarrow \delta$ denote the corresponding norm (that is, $\pi_R$ is a surjection
and $xRy$ if $\pi_R(x) \leq \pi_R(y)$). Define an equivalence relation $\equiv$ on $P \times V_{\alpha+1}$ by
setting $(R, y) \equiv (R', y')$ if $\pi_R(y) = \pi_{R'}(y')$. Let $[R, y]$ denote the equivalence
class of $(R, y)$. Let $\mathcal{E}$ be the set of equivalence classes. Define the prewellorder $\leq^*$ on $\mathcal{E}$ by $(R, y) \leq^* (R', y')$ if $\pi_{R'}(y) \leq \pi_{R'}(y')$. So $\leq^*$ has ordertype $\delta$. So

$$ \bar{A} = \langle A_{\pi_R(y)} \rangle_{[R, y] \in \mathcal{E}} \in V_\delta. $$

Since $j(P) = P$, $j$ also fixes $\alpha, \mathcal{E}, \leq^*, \bar{A}$, and $j^*\mathcal{E}$ is cofinal in $\leq^*$. Since $A_\delta = \bigcup \bar{A}$ and this union is increasing in $\leq^*$, it follows as in the previous case that
there is $[R, y] \in \mathcal{E}$ such that $a \in j(A_{\pi_R(y)})$. \qed
So fix $\beta$ as in the subclaim. Let $A_\beta'$ be the set of pairs $(u, z)$ such that $u \in (V_n)_0^{<\omega}$ and $z \in V_\beta$ and $(u, z)$ are as above. Since $a \in j(A_\beta)$, we have some $b$ with $(a, b) \in j(A_\beta')$, and note we may assume that $a \cup \{a\} \subseteq b \in (V_\delta)^{<\omega}$, by increasing $\beta$ if needed. Now for $z \in (A_\beta)^{b}$ with $(z^{ba}, z) \in A_\beta'$, let $g(z)$ be the least $y$ which is $\Sigma_2$-definable over $V_\Omega$ from ordinals and $z$, and such that $V_\Omega \models \varphi(f(z^{ba}), y)$. (Here we minimize on the $\Sigma_2$ formula and ordinal parameters in order to specify the least $y$.) Then note that for $E_2$-measure one many $z$, we have $V_\Omega \models \varphi(f^{ab}(z), g(z))$, as desired. □

Claim 5. $\text{Ult}(V, E)$ is wellfounded.

Proof. We argue much as in the proof of Theorem 5.7. Fix some $\Omega \in (\delta, \text{OR})$ and some large enough $k < \omega$ with $V_\Omega \equiv_{k+2} V$. As before, it suffices to see that $\text{Ult}_0^{V_\Omega}(\Omega, E)$ is wellfounded, where the notation means we use all functions in $V_\Omega$ to form the ultrapower.

Now $V_\Omega = \text{Hull}_{\Sigma_2}(V_\delta \cup \Omega)$ and (because $k$ is large enough) $V_\Omega$ computes the ultrapower, and its wellfounded (and illfounded) parts. Therefore, this reflects into $X = \text{Hull}_1^{V_\delta}(V_\delta \cup \{E\})$ (note we have dropped the parameters in $\Omega \setminus \delta$), and $X \equiv_{V_\Omega} V_\Omega$. Let $H$ be the transitive collapse of $X$. Then it suffices to see that $\mathcal{O} = \text{Ult}_0^H(\text{OR}^H, E)$ is wellfounded.

Given $\eta \leq \delta$, let $H_\eta = \text{Hull}^H(V_\delta \cup \{E\})$. Note we do not claim that $H_\eta \equiv H$. But we have $H_\eta = H \cap H_\eta$. Let $\mathcal{O}_\eta$ be the substructure of $\mathcal{O}$ given by elements of the form $[a, f]$ where $a \in (V_\delta)_0^{<\omega}$ and $f \in H_\eta$. So if $\eta_0 < \eta_1$ then $\mathcal{O}_\eta_0$ is just the restriction of $\mathcal{O}_\eta_1$ to its domain, and $\mathcal{O} = \bigcup_{\eta \in \delta} \mathcal{O}_\eta$.

Now suppose that $\mathcal{O}$ is illfounded. Then there is a $\eta < \delta$ such that $\mathcal{O}_\eta$ is illfounded. For suppose otherwise. We argue like in the proof of Theorem 5.7. Let $\eta_0$ be the least $\eta$ such that some illfp($\mathcal{O}$) $\cap \mathcal{O}_\eta \neq \emptyset$, and let $x_0$ be the $<\mathcal{O}_\eta$ least $x \in \text{illfp}(\mathcal{O}) \cap \mathcal{O}_\eta$. Then given $\eta_n, x_n$, let $\eta_{n+1}$ be the least $\eta \in (\eta_n, \delta)$ such that there is $x \in \text{illfp}(\mathcal{O}) \cap \mathcal{O}_\eta$ with $x < \mathcal{O}_{\eta_n}, x_n$. Setting $\eta = \sup_{n < \omega} \eta_n$, we get a contradiction.

Now fix such an $\eta < \delta$ and let $\gamma$ be the ordertype of $\text{OR} \cap H_\eta$ and $\pi : \gamma \to \text{OR} \cap H_\eta$ the uncollapse map. Given $f : (V_\eta)_0^{<\omega} \to \text{OR}$ with $f \in H_\eta$, let $\text{rg}(f) \subseteq \text{rg}(\pi)$. Let $\tilde{f} : (V_\eta)_0^{<\omega} \to \gamma$ be the natural collapse (so $\pi \circ \tilde{f} = f$). We have a surjection $\sigma : V_\eta \to \gamma$. Given $f$ as above, let $f'$ be the corresponding collapse; that is, $f' : (V_\eta)_0^{<\omega} \to V_{\eta+1}$.

$$f'(u) = \{ x \in V_\eta \mid \sigma(x) = \tilde{f}(u) \}.$$  

Let

$$\mathcal{F} = \{ f' \mid f \in H_\eta \text{ and } f : (V_0)_0^{<\omega} \to \text{OR} \}.$$  

Let $\leq_\sigma$ be the prewellorder of $V_\eta$ induced by $\sigma$. Clearly then $U = \text{Ult}_0^{\mathcal{F}}(\leq_\sigma, E)$ is illfounded, as it is in fact isomorphic to $\mathcal{O}_\eta$. But by the $\Sigma_1$-elementarity of $j$, $j(\leq_\sigma) \in V_\delta$ is a prewellorder of $V_{j(\eta)}$ (in particular wellfounded), and we can absorb $U$ into $j(\leq_\sigma)$ as usual, by mapping

$$[a, f']_E \mapsto j(f')(a).$$  

So $j(\leq_\sigma)$ is illfounded, a contradiction, proving the claim. □

By the two claims, we are done. □
7 Admissible $L_\kappa(V_\delta)$ and countable cofinality

Lemma 7.1. (ZF) Assume $V = L(V_\delta)$ where $\delta \in \text{Lim}$. Let $j \in \mathcal{E}_1(V_\delta)$. Then:

1. $\text{col}(\delta) = \omega$ and $j$ is fully elementary.

2. For all $\alpha \leq \text{OR}, \Sigma_0$-Loś’ criterion holds for $\text{Ult}_0(\mathcal{J}_\alpha(V_\delta), E_j)$.

3. $\text{Ult}_0(L(V_\delta), E)$ is illfounded.

Proof Sketch. We write $\mathcal{J}_\alpha$ for $\mathcal{J}_\alpha(V_\delta)$. Part 1 is by Theorem 6.3 and Theorem 5.6. Note that trivially then, $j(\text{cof}(\delta)) = \text{cof}(\delta)$.

Part 2. This is much as in the proof of Theorem 6.2, using that

$$\mathcal{J}_\alpha = \text{Hull}_{\Sigma_1}(V_\delta \cup (\delta + \beta))$$

for all $\beta \leq \alpha$. Suppose for example that $\alpha = \beta + 1$. Let $f, h \in \mathcal{J}_\alpha$ with $f, h : (V_\delta)^{\leq \omega} \to V$. Suppose that for $E_\alpha$-measure one many $u$, we have

$$\mathcal{J}_\alpha \models \exists y \in h(u) \ [\varphi(f(u), y)],$$

where $\varphi$ is $\Sigma_0$. We have $m < \omega$ such that $f, g \in S_m(\mathcal{J}_\beta)$, where $S$ denotes Jensen’s $\mathcal{S}$-operator (so $\mathcal{J}_\alpha = \bigcup_{k<\omega} S_k(\mathcal{J}_\beta)$). Fix a surjection

$$\pi : (V_\delta \times \beta^{< \omega}) \to S_m(\mathcal{J}_\beta)$$

with $\pi \in \mathcal{J}_\alpha$. Then arguing as before, using that $\text{cof}(\delta) = \omega$, we can find $\xi < \delta$ such that for $E_\alpha$-measure one many $u$, there is $y \in \pi^{\alpha}(V_\xi \times \beta^{< \omega})$ such that

$$\mathcal{J}_\alpha \models y \in h(u) \ & \ \varphi(f(u), y).$$

(2)

Now for pairs $(u, v) \in (V_\delta)^{< \omega} \times V_\xi$, let $g'(u, v)$ be the least $y \in \pi^{\alpha}(\{v\} \times \beta^{< \omega})$ such that line (2) holds, if there is such a $y$. Then we find an appropriate index $b$ and convert $g'$ into a function $g$, with $(b, g)$ witnessing $\Sigma_0$-Loś’ criterion, like before.

Part 3. Since $V = L(V_\delta) \models \text{ZF}$ and by part 2, $i_{\mathcal{J}_\alpha}^{V_{\mathcal{J}_\alpha}} : V \to V$ is fully elementary, so this is like before.

\[\square\]

Remark 7.2. Let $M$ be a transitive set. Recall that $M$ is admissible iff $M$ satisfies Pairing, Infinity, $\Sigma_0$-Separation, and whenever $d, p \in M$ and $\varphi$ is a $\Sigma_1$ formula and $M \models \forall x \in d \ \exists y \ \varphi(x, y, p)$ then there is $e \in M$ such that $M \models \forall x \in d \ \exists y \in e \ \varphi(x, y, p)$.

Definition 7.3. Given a transitive set $X$, let $\kappa_X$ denote the least $\kappa \in \text{OR}$ such that $\mathcal{J}_\kappa(X)$ is admissible.

Recall the notation wfp and illfp from [1]. A well-known fact is:

Fact 7.4. (ZF) Let $M$ be an extensional structure in the language of set theory, let $X \in \text{wfp}(M)$ (and we assume $M$ is transitive below $X$). Suppose $M \models \"V = L(X)\"$ (but $M$ might not satisfy ZF). If $M$ is illfounded then $\mathcal{J}_{\kappa_X}(X) \subseteq M$. 

\[\square\]
Proof. Let \( \lambda = \text{OR} \cap \text{wfp}(M) \). Because \( \lambda \subseteq \text{OR}^M \) but \( \lambda \notin M \), and \( M \models \neg \text{V = L}(X) \), and hence, \( M \models \text{I am rudimentarily closed} \), it is easy to see that \( \lambda \) is closed under ordinal addition and multiplication. Moreover, it is easy to see that \( J_\lambda(X) \subseteq M \), and hence, \( J_\lambda(X) \subseteq \text{wfp}(M) \).

Now suppose that \( \lambda < \kappa_X \). Then we can fix a \( \Sigma_1 \) formula \( \varphi \) and \( d, p \in J_\lambda(X) \) such that \( \lambda \) is the least \( \lambda' \) such that

\[
J_{\lambda'}(X) \models \forall x \in d \exists y \varphi(x, y, p).
\]

Note then that for all \( \alpha \in \text{OR}^M \setminus \lambda \),

\[
M \models \neg J_\alpha(X) \models \forall x \in d \exists y \varphi(x, y, p).
\]

But then for such \( \alpha \),

\[
M \models \neg \forall \alpha \in \text{OR} \mid J_\alpha(X) \models \forall x \in d \exists y \varphi(x, y, p) \in J_{\alpha+1}(X).
\]

But note that this set is exactly \( \lambda \), so \( \lambda \in M \), a contradiction. \( \square \)

**Fact 7.5.** (ZF) Let \( X \) be transitive. Then for every \( \alpha \leq \kappa_X \), we have

\[
J_\alpha(V_\delta) = \text{Hull}_{\Sigma_1} J_\alpha(V_\delta) \cap \{V_\delta\}.
\]

Therefore (i) \( \mathcal{P}(V_\delta) \cap J_{\alpha+1}(V_\delta) \not\subseteq J_\alpha(V_\delta) \) and (ii) for every \( x \in J_\alpha(V_\delta) \) there is a surjection \( \pi : V_\delta \to x \) with \( \pi \in J_\alpha(V_\delta) \).

Proof. Let \( H = \text{Hull}_{1} J_\alpha(V_\delta) \cap \{V_\delta\} \) and \( \beta = \text{sup}(H \cap \text{OR}) \). Note then that \( H = J_\beta(V_\delta) \). So it suffices to see that \( \beta = \alpha \), so suppose \( \beta < \alpha \). Then \( J_\beta(V_\delta) \) is inadmissible. So let \( p, d \in J_\beta(V_\delta) \) and \( \varphi \) be \( \Sigma_1 \), such that \( \beta \) is least such that

\[
J_\beta(V_\delta) \models \forall x \in d \exists y \varphi(x, y, p).
\]

Then

\[
J_\alpha(V_\delta) \models \exists \beta' \in \text{OR} \mid J_{\beta'}(V_\delta) \models \forall x \in d \exists y \varphi(x, y, p).
\]

But \( \beta \) is the least such \( \beta' \), and since \( p, d \in H \), it follows that \( \beta \in H \), a contradiction.

Part (i) of the “therefore” clause now follows by a standard diagonalization. For part (ii), if \( \alpha = \beta + 1 \), use that \( J_\alpha = \bigcup_{n<\omega} S_n(J_\beta) \) (where \( S_n \) is the \( n \)th iterate of Jensen’s \( S \)-operator) and for each \( n \in \{1, \omega\} \),

\[
S_n(J_\beta) = \text{Hull}_{\Sigma_1} S_n(J_\beta)(V_\delta \cup \{V_\delta, \beta\}).
\]

We now prove the promised strengthening of Theorem 3.3.

**Theorem 7.6.** (ZF) Let \( \delta \in \text{Lim} \) and \( j : V_\delta \to V_\delta \) be \( \Sigma_1 \)-elementary. Let \( \theta = \kappa_{V_\delta} \) (see \( \natural \)). Then \( j \notin J_\theta(V_\delta) \). In fact, \( j \) is not \( \Sigma_1^j \), and \( \text{not } \Pi_1^j \).

Proof. We write \( J_\alpha \) for \( J_\alpha(V_\delta) \). Suppose first \( j \notin J_\alpha \), and we deduce the rest. Let \( \varphi \) be \( \Sigma_1 \) and \( p \in J_\theta \), and suppose for all \( x, y \in V_\delta \), we have \( j(x) = y \) iff \( J_\theta \models \varphi(x, y, p) \). Then note that

\[
J_\theta \models \forall x \in V_\delta \exists \alpha \in \text{OR} \mid J_\alpha \models \exists y \in V_\delta \varphi(x, y, p),
\]

and
and so by admissibility, there is \( \lambda < \theta \) such that

\[ J_\lambda \models \forall x \in V_\delta \exists y \in V_\delta \ \varphi(x, y, p). \]

But then for \( x, y \in V_\delta \), we have \( j(x) = y \) iff \( J_\delta \models \varphi(x, y, p) \), so \( j \in J_\delta \), contradiction.

Now suppose that for all \( x, y \in V_\delta \), we have \( j(x) = y \) iff \( J_\delta \models \neg \varphi(x, y, p) \). Note for each \( x \in V_\delta \), letting \( y = j(x) \),

\[ J_\delta \models \forall y' \in V_\delta \setminus \{ y \} \ \varphi(x, y', p), \]

and so by admissibility, there is (a least) \( \alpha_x < \theta \) such that

\[ J_{\alpha_x} \models \forall y' \in V_\delta \setminus \{ y \} \ \varphi(x, y', p). \]

Then \( J_{\alpha} \models \forall x \in V_\delta \exists \alpha \in \text{OR} \ [J_\alpha \models \exists y \in V_\delta \ \forall y' \in V_\delta \setminus \{ y \} \ \varphi(x, y', p)] \), but then by admissibility, we get \( \text{sup}_{x \in V_\delta} \alpha_x < \theta \), but then \( j \in J_\delta \), a contradiction.

So we need to see \( j \notin J_\delta \). Suppose otherwise. By Lemma 7.1, \( \text{cof}^{L(V_\delta)}(\delta) = \omega \) and \( j : V_\delta \to V_\delta \) is fully elementary. Let \( E = E_j \) and \( \alpha < \theta \) with \( E \in J_\alpha \).

Let \( \kappa = \text{cr}(E) \). Let \( M = J_{\alpha + \kappa + 1} \) and \( U = \text{Ult}_\kappa(M, E) \). By Lemma 7.1, \( \Sigma_\alpha^0 \)-Loś’ criterion holds for \( U \), so \( i_E^M \) is \( \epsilon \)-cofinal and \( \Sigma_1 \)-elementary. Because \( M \models \forall V = L(V_\delta) \), therefore \( U \models \forall V = L(i_E^M(V_\delta)) \).

**Claim 6.** \( i_E^M(V_\delta) = V_\delta \).

**Proof.** It suffices to see that \( i_E^M \) is continuous at \( \delta \). So let \( f \in M \) and \( \alpha < \delta \) with \( f : (V_\alpha)^{< \omega} \to \delta \), and let \( a \in (V_j(\alpha))^{< \omega} \). We want to see that \( [a, f]_E^M < \delta \). But \( \text{cof}(\delta) = \omega \), so fix \( g : \omega \to \delta \) cofinal, and for \( n < \omega \) let

\[ A_n = \{ u \in (V_\alpha)^{< \omega} \mid f(u) < g(n) \}. \]

Then \( (V_\alpha)^{< \omega} = \bigcup_{n < \omega} A_n \), so the usual argument gives \( A_n \in E_\alpha \) for some \( n < \omega \), which suffices.

By the claim, \( V_\delta \in \text{wfp}(U) \) and \( U \models \forall V = L(V_\delta) \).

**Claim 7.** \( U \) is illfounded.

**Proof.** Suppose \( U \) is wellfounded. Then note that \( i_E^M(\alpha + \kappa) > \alpha + \kappa \), and by the previous claim, that \( J_{i_E^M(\alpha + \kappa)} \subseteq U \), so

\[ \mathcal{P}(V_\delta) \cap J_{\alpha + \kappa + 2} \subseteq U. \]

But \( \mathcal{P}(V_\delta) \cap U \subseteq M \), because given any \( A \in \mathcal{P}(V_\delta) \cap U \), we can find some pair \( (a, f) \) such that \( [a, f]_E^M = A \), with \( f \in M \) and \( a \in V_\delta \), and since \( E \in M \), it easily follows that \( A \in M \). Putting the \( \subseteq \)-statements together, we contradict Fact 7.5.

By the above claim and Fact 7.4, we have \( J_\delta \subseteq U \), so \( \mathcal{P}(V_\delta) \cap J_\delta \subseteq U \). But then we reach a contradiction like in the proof of the claim. This completes the proof.

We next observe that the preceding result is optimal, at least in the case where we have a lot of \( \text{AC} \):
Fact 7.7 (Corazza). (ZFC) Suppose \( j \in \mathcal{U}(V_\delta) \), where \( \delta \in \text{Lim} \) (so \( \delta = \kappa_\omega(j) \) and \( V_\delta \models \text{ZFC} \)). Then there is a set-forcing \( \mathbb{P} \) which forces (i) \( V_\delta \models \text{ZFC} + "V = \text{HOD}" \) and (ii) there is \( k \in \mathcal{U}(V_\delta) \) with \( j \subseteq k \).

Theorem 7.8. (ZF) Let \( \delta \in \text{OR} \) with \( V_\delta \models \text{ZFC} + "V = \text{HOD}" \) and \( \mathcal{U}(V_\delta) \neq \emptyset \), and take \( \delta \) least such. Let \( \theta = \kappa_{V_\delta} \) and \( M = \mathcal{F}_0(V_\delta) \). Then there is \( j \in \mathcal{U}(V_\delta) \) which is \( \Sigma_1 \{ (V_\delta) \} \) and \( \Pi_1^M \{ (V_\delta) \} \), meaning there are \( \Sigma_1 \) formulas \( \varphi, \psi \) such that

\[
\forall x, y \in V_\delta \ [j(x) = y] \iff M \models [\varphi(x, y, V_\delta) \land \neg \psi(x, y, V_\delta)].
\]

Proof. Note \( \text{cof}(\delta) = \omega \). The \( j \) satisfying these requirements is just the left-most branch through the natural tree searching for such an embedding. That is, let \( T \) be the tree whose nodes are finite sequences

\[(j_0, \alpha_0, \beta_0), (j_1, \alpha_1, \beta_1), \ldots, (j_{n-1}, \alpha_{n-1}, \beta_{n-1})\]

such that for each \( i < n, j_i : V_{\alpha_i} \rightarrow V_{\beta_i} \) is elementary and \( \kappa = \text{cr}(j_i) \) exists, \( V_\delta \models "V = \text{HOD}" \), and if \( i + 1 < n \) then \( \beta_i < \alpha_{i+1} \) and \( j_i \subseteq j_{i+1} \).

Now any \( \Sigma_1 \)-elementary \( j : V_\delta \rightarrow V_\delta \) determines an infinite branch through \( T \) (we can take \( \alpha_0 = \text{cr}(j) \) and \( \beta_0 = j(\alpha_n) \) and \( \alpha_{n+1} = \beta_n + 1 \)). This is clear enough except for the fact that \( V_\delta \models "V = \text{HOD}" \) where \( \kappa = \text{cr}(j) \). But because \( V_\delta \models \text{ZFC} \), we must have \( \kappa_{V_\delta}(j) = \delta \), and since \( V_\delta \models "V = \text{HOD}" \), it follows that \( V_\delta \models "V = \text{HOD}" \) also. Conversely, let \( ((j_i, \alpha_i, \beta_i))_{i<\omega} \) be an infinite branch through \( T \), and \( \lambda = \bigcup_{i<\omega} \alpha_i = \bigcup_{i<\omega} \beta_i \). Then \( \lambda \in \text{Lim} \) and \( j \in \mathcal{U}(V_\lambda) \), and since \( V_\lambda \models "V = \text{HOD}" \) where \( \kappa = \text{cr}(j) \), therefore \( V_\lambda \models \text{ZFC} + "V = \text{HOD}" \). In fact \( j \) is fully elementary (either since \( V_\lambda \models \text{ZF} \), or because \( \text{cof}(\lambda) = \omega \) and by \[4\] Theorem 5.6[?])], so \( \lambda = \delta \) by minimality.

Note that \( T \) is definable over \( V_\delta \). Now the rank analysis of \( T \) is computed over \( M. \) That is, given a node \( t \in T \), let

\[T_t = \{ s \in T \mid t \leq s \text{ or } s \leq t \} \]

Then there is a rank function for \( T_t \) (in \( V \)) iff there is one in \( M \); this is a standard consequence of admissibility. Let \( <^* \) be the standard wellorder of \( V_\delta \) resulting from the fact that \( V_\delta \models "V = \text{HOD}" \). Let \( b = (t_i)_{i<\omega} \) be the left-most branch of \( T \) with respect to \( <^* \). That is, \( t_0 = \emptyset, t_1 = ((j_0, \alpha_0, \beta_0)) \) is the \( <^* \)-least node of \( T \) of length 1 such that there is no rank function for \( T_{t_1} \) (in \( M \)), and then \( t_2 = ((j_1, \alpha_1, \beta_1))_{i<\omega} \) is the \( <^* \)-least node of \( T \) of length 2, extending \( t_1 \), such that there is no rank function for \( T_{t_2} \) (in \( M \)), etc. Note here that because \( T_{t_\omega} \) has no rank function (in \( M \)), \( t_{n+1} \) does exist. This determines our branch \( b \), and hence a \( \Sigma_1 \)-elementary \( j : V_\delta \rightarrow V_\delta \).

Finally note that \( b \) is appropriately definable.

Definition 7.9. (ZF) Let \( \delta \in \text{Lim} \) and \( m \leq \omega \). Then \( T = T^{\delta,m} \) denotes the following tree of attempts to build a (possibly partial) \( \Sigma_m \)-elementary \( j : V_\delta \rightarrow V_\delta \). The nodes in \( T \) are finite sequences

\[t = ((j_0, \alpha_0, \beta_0), \ldots, (j_n, \alpha_n, \beta_n))\]

such that \( j_0 : V_{\alpha_0} \rightarrow V_{\beta_0} \) is \( \Sigma_1 \)-elementary and cofinal, \( j_1 : V_\delta \rightarrow V_\delta \) is \( \Sigma_m \)-elementary on its domain \( V_{\alpha_1}, \beta_1 < \alpha_{i+1} \) and \( j_0 \subseteq j_0 \alpha_1 \). Write \( j_t = j_n \).

\[15\]In both statements here the \( V_\delta \) is in the sense of the forcing extension.
For $\alpha \in \text{OR}$, let $T_\alpha$ be the $\alpha$th derivative of $T$, defined as follows. Set $T_0 = T$, and for limit $\lambda$ set $T_\lambda = \bigcap_{\beta < \lambda} T_\beta$. Given $T_\alpha$, $T_{\alpha+1}$ is the set of all $t \in T_\alpha$ such that for every $\beta < \delta$ there is an extension $s$ of $t$ with $s \in T_\alpha$ such that $\beta \in \text{dom}(j_s)$. Let $T_\infty = T_\text{OR}$. We say that $T_\infty$ is perfect iff for every $t \in T_\infty$ there are $s_1, s_2 \in T_\infty$ both extending $t$, such that $j_{s_1} \not\subseteq j_{s_2} \not\subseteq j_{s_1}$.

We write $[T]$ for the set of infinite branches through $T$. Clearly each $b \in [T]$ determines a $\Sigma_1$-elementary map $j_b : V_\lambda \to V_\lambda$ for some limit $\lambda \leq \delta$, and if $\lambda = \delta$ then $j$ is $\Sigma_m$-elementary.

We say that an embedding $k : V_\lambda \to V_\lambda$, for limit $\lambda$, is $V$-amenable iff $k \upharpoonright V_\alpha \in V$ for each $\alpha < \lambda$. Clearly $j_b$ above is $V$-amenable.

Note $(T_\alpha)_{\alpha \in \text{OR}} \in L(V_\delta)$, where $T = T^{4,m}$. The following lemma shows that if $T_\infty \neq \emptyset$ then $T_\infty$ is, by a certain natural measure, of maximal complexity:

**Lemma 7.10.** (ZF) Let $\delta \in \text{Lim}$. Let $\gamma$ be least such that $T_\gamma = T_\infty$. Then:

1. If $\varepsilon_m(V_\delta) \neq \emptyset$ then $T_\infty \neq \emptyset$.
2. If $T_\infty \neq \emptyset$ then $T_\infty$ is perfect and $\gamma = \kappa_{V_\delta}$.
3. If $T_\infty = \emptyset$ then $\gamma < \kappa_{V_\delta}$.

**Proof.** If $\gamma \in \varepsilon_m(V_\delta)$ then easily $T_\infty \neq \emptyset$, and in fact, if we force over $V$ to collapse $\aleph_1$ to become countable, then in $V[G]$, there is an infinite branch $b \in [T_\infty]$ with $j_b = j$.

The fact that $\gamma \leq \kappa = \kappa_{V_\delta}$ is standard: Suppose not and let $t \in T_\infty \setminus T_{\gamma+1}$. Then we can fix $\alpha < \delta$ such that no $s \in T_\alpha$ extending $t$ has $\alpha \in \text{dom}(j_s)$. But then $J_\kappa(V_\delta) \models \text{"For every } s \in T \text{ extending } t \text{ with } \alpha \in \text{dom}(j_s) \text{ there is } \beta \in \text{OR} \text{ such that } s \notin T_\beta."$ By admissibility, it follows that there is $\xi < \kappa$ such that $J_\xi(V_\delta)$ satisfies this. But then note that $t \notin T_\xi$, contradiction. The same argument (but slightly simpler) shows that if $T_\infty = \emptyset$ then $\gamma < \kappa$.

Now suppose that $T_\infty \neq \emptyset$ but $T_\infty$ is not perfect. Then fix $t \in T_\infty$ such that for all $s_1, s_2 \in T_\infty$ extending $t$, we have $j_{s_1} \not\subseteq j_{s_2} \not\subseteq j_{s_1}$. Let $\mathcal{F}$ be the set of all $s \in T_\infty$ extending $t$, and $j = \bigcup_{s \in \mathcal{F}} j_s$. Then note that $j : V_\delta \to V_\delta$ is a well-defined function and is $\Sigma_m$-elementary, and $j \in L(V_\delta)$.

In fact, $j \in J_\kappa$, contradicting Theorem 7.6. For fix $\alpha < \delta$ with $\alpha > \text{dom}(j_t)$; so $j \upharpoonright V_\alpha \in V_\delta$. Then for all nodes $s \in T$ extending $t$ with $V_\alpha \subseteq \text{dom}(j_s)$ and $j \upharpoonright V_\alpha \neq j_s \upharpoonright V_\alpha$, there is $\xi < \kappa$ with $s \notin T_\xi$. So by admissibility, there is $\xi < \kappa$ such that $s \notin T_\xi$ for all such $s$. Therefore, for each $\alpha < \delta$ such that $\alpha > \text{dom}(j_t)$, there is $\beta < \delta$ and a map $k : V_\alpha \to V_\beta$ (actually $k = j \upharpoonright V_\alpha$) with $j_t \subseteq k$ and there is $\xi < \kappa$ such that $s \notin T_\xi$ for all $s$ as above. So by admissibility, there is $\xi < \kappa$ which works simultaneously for all $\alpha < \delta$. But then clearly $j$ is definable from parameters over $J_\xi$, so $j \in J_\kappa$, contradicting Theorem 7.6.

It remains to see that if $T_\infty \neq \emptyset$, hence perfect, then $\gamma = \kappa_{V_\delta}$. Suppose otherwise. Let $\gamma < \xi < \kappa_{V_\delta}$. Then $T_\infty \in J_\xi$, and we can force over $L(V_\delta)$ with $T_\infty$, in the obvious manner, with the generic filter $G$ being an infinite branch $b$ through $T_\infty$, and note that by genericity, $j_b : V_\delta \to V_\delta$ is $\Sigma_m$-elementary (that is, genericity ensures that $\text{dom}(j_b) = V_\delta$). Note that $j_b$ is $V$-amenable.

Now we will proceed through basically the argument from before, but just need to see that things adapt alright to the generic embedding $j_\delta$. We first consider the finite iterates $(j_b)_n$ of $j_b$ and the eventual fixedness of ordinals (that is, whether $(j_b)_n(\alpha) = \alpha$ for some $n$). If $\alpha < \delta$ is a limit and $j_b(\alpha) = \alpha$
then \( j_b \upharpoonright V_\alpha \) determines \( (j_b)_n \upharpoonright V_\alpha \) for each \( n < \omega \) as usual, and this is all in \( V \), so all ordinals \( \alpha \) are eventually fixed. So we may assume that there is a bound \( < \delta \) on such ordinals \( \alpha \). If \( \delta = \alpha + \omega \) for some limit \( \alpha \) then clearly \( j_b(\alpha + n) = \alpha + n \) for all \( n < \omega \), so we are also done in this case. So we are left with the case that \( \delta \) is a limit of limits, and there is \( \alpha < \delta \) such that \( j_b \) fixes no ordinal in \( [\alpha, \delta) \), and take \( \alpha \) least such. In particular, \( j_b(\alpha) > \alpha \). Letting \( \alpha_0 = \alpha \) and \( \alpha_{n+1} = j_b(\alpha_n) \), note that \( \sup_{n<\omega} \alpha_n = \delta \) (for if \( \eta = \sup_{n<\omega} \alpha_n < \delta \) then \( j_b \upharpoonright V_\eta \in V \), so \( \langle \alpha_n \rangle_{n<\omega} \in V \), so \( \text{cof}(\eta) = \omega \), so \( j_b(\eta) = \eta \), contradiction). But \( j_b \) has unboundedly many fixed points \( < \alpha_0 \). Therefore \( (j_b)_n \) has unboundedly many \( < \alpha_1 \), etc, \( (j_b)_n \) has unboundedly many \( < \alpha_n \). But then \( (j_b)_n \upharpoonright V_\alpha \) is enough to determine \( (j_b)_{n+k} \upharpoonright V_\alpha \) for \( k < \omega \) working in \( V \), so we get that all points \( < \alpha_\alpha \) are eventually fixed. So if \( \delta \) is singular in \( V \), we can find \( n < \omega \) such that \( (j_b)_n(\text{cof}^V(\delta)) = \text{cof}^V(\delta) \). Similarly, if \( \delta \) is regular but non-inaccessible, we can find \( n < \omega \) such that \( (j_b)_n(\text{scot}^V(\delta)) = \text{scot}^V(\delta) \) (for this, argue as before to first find \( n < \omega \) and a limit \( \alpha < \delta \) such that \( (j_b)_n \) has cofinally many fixed points \( < \alpha \) and \( \text{scot}^V(\delta) \in V_\alpha \), and then proceed in \( V \).

So fix \( n < \omega \) such that \( k = (j_b)_n \) is like this. Then for each \( \eta \in \text{OR} \), \( \Sigma_0 \)-Loś’s theorem holds for \( \text{Ult}_0(\mathcal{J}_\eta, E_k) \). The proof is just like before – the fact that \( k \notin V \) does not matter. (The ultrapower is formed using only functions in \( V \), so all the calculations with partitioning measure one sets is done in \( V \), and because either \( V_\delta \) is inaccessible in \( V \) or \( k \) fixes the relevant objects, the argument goes through.)

So consider \( U = \text{Ult}_0(\mathcal{J}_\chi, E_k) \) where \( \chi = (\omega(\delta + \xi)) + cr(k) + 1 \) (we had \( P = T_\infty \in \mathcal{J}_\xi \)). We claim that \( U \) is illfounded. For otherwise \( \text{OR}^U \supset \chi \), so as before, we get

\[
\tau = \text{Th}^{\mathcal{J}_\chi}_{\Sigma_\eta}(V_\delta \cup \{V_\delta\}) \in \mathcal{J}_\chi \upharpoonright G.
\]

Let \( \tau \in \mathcal{J}_\chi \) be a \( T_\infty \)-name such that \( \tau_G = t \). Now \( t \in L(V_\delta) \), and since \( G \) is \( L(V_\delta) \)-generic, there is \( p \in G \) such that \( L(V_\delta) \models \langle \tau \rangle \). But then for \( (\varphi, x) \in V_\delta \) we have

\[
(\varphi, x) \in t \iff \mathcal{J}_\chi \models p \upharpoonright \mathcal{P}(\varphi, x) \in \tau,
\]

because for example if \( (\varphi, x) \in t \) but there is \( q \leq p \) and \( \mathcal{J}_\chi \models q \upharpoonright \mathcal{P}(\varphi, x) \notin \tau \), then \( p \) cannot have forced \( \check{t} \models \tau \); moreover here this forcing relation is definable over \( \mathcal{J}_\chi \), and in fact, it is definable from parameters over \( \mathcal{J}_{\chi-1} \). For the \( \Sigma_0 \) forcing relation over \( \mathcal{J}_{\omega}(\delta + \xi) \) is \( \Delta^{\mathcal{J}_{\omega}(\delta + \xi)}_{\chi}(\{P\}) \), because we have enough closure at this stage, and this is then maintained level by level, and the \( \Sigma_0 \) forcing relation over \( \mathcal{J}_{\chi-1} \) is \( \Delta^{\mathcal{J}_{\chi-1}}_{\chi}([\mathcal{P}]_P) \), and the \( \Sigma_\eta \) forcing relation for \( \Sigma_\eta^{\mathcal{J}_{\chi-1}} \)-definable names \( \subseteq \mathcal{J}_{\chi-1} \) is definable from \( P \) over \( \mathcal{J}_{\chi-1} \), but \( \tau \) can be taken to be such a name. Hence we get \( t \in \mathcal{J}_\chi \), which is a contradiction.

So \( U \) is illfounded, and hence \( \mathcal{J}_{\kappa_{\omega3}} \subseteq \text{wfp}(U) \). But then we again get \( t \in \mathcal{J}_\chi \upharpoonright G \), which is again a contradiction, completing the proof.

**Theorem 7.11. (ZF)** Let \( \delta \in \text{Lim} \) with \( \text{cf}(V_\delta) \neq 0 \), where \( m \in [1, \omega] \). Then:

1. In a set-forcing extension of \( V \), for each \( V \)-amenable \( j \in \text{E}_0(V_\delta) \) and \( \alpha < \delta \) there is a \( V \)-amenable \( k \in \text{E}_0(V_\delta) \) with \( k \upharpoonright V_\alpha = j \upharpoonright V_\alpha \) but \( k \neq j \).

---

Since \( j_b \notin V \), it seems that \( \delta \) might not have cofinality \( \omega \) in \( V \) here.
2. If DC holds and \( \text{cof}(\delta) = \omega \) then for each \( j \in \mathcal{E}_m(V_\delta) \) and \( \alpha < \delta \) there is \( k \in \mathcal{E}_m(V_\delta) \) with \( k \upharpoonright V_\alpha = j \upharpoonright V_\alpha \) but \( k \neq j \).

Proof. By Lemma 7.10, \( T_\infty \) is perfect (notation as there), which immediately gives the theorem (of course we can take the generic extension to be \( V[G] \) where \( G \) collapses \( V_\delta \) to become countable).

We now show that the kind of embedding defined in Theorem 7.8 cannot be extended to the whole of \( \mathcal{J}_0(V_\delta) \):

**Theorem 7.12.** (ZF) Let \( \delta \in \text{Lim} \) and \( j \in \mathcal{E}(V_\delta) \). Let \( \theta = \kappa_{V_\delta} \). Suppose that \( j \) is definable over \( \mathcal{J}_0(V_\delta) \). Then there is \( \alpha < \theta \) such that \( \text{Ult}_0(\mathcal{J}_\alpha(V_\delta), E_j) \) is illfounded.

Proof. Write \( \mathcal{J}_\alpha \) for \( \mathcal{J}_\alpha(V_\delta) \). Since \( j \in L(V_\delta) \), we have \( \text{cof}(\delta) = \text{cof}^{L(V_\delta)}(\delta) = \omega \).

**Claim 1.** Let \( \alpha < \theta \) with \( \alpha \) either a successor or \( \text{cof}^V(\alpha) < \delta \) and \( j \) continuous at \( \text{cof}(\alpha) \). Let \( f : (V_\delta)^{<\omega} \to \mathcal{J}_\alpha \) and \( a \in (V_\delta)^{<\omega} \). Then there is \( g \in \mathcal{J}_\alpha \) such that \( g(u) = f(u) \) for \( E_a \)-measure one many \( u \).

Proof. By the usual calculations using the continuity of \( j \), we can assume that \( \text{rg}(f) \subseteq x \) for some \( x \in \mathcal{J}_\alpha \). But by Fact 7.4 there is a surjection \( \pi : V_\delta \to x \) with \( \pi \in \mathcal{J}_\alpha \). Therefore, we can assume that \( x = V_\delta \). (That is, given \( y \in x \), let \( Z_y = \{ w \in V_\delta \mid \pi(w) = y \} \), and then let \( z_y = Z_y \cap V_\delta \) where \( \beta \) is least such that this intersection is \( \neq \emptyset \), and then let \( f' : (V_\delta)^{<\omega} \to V_\delta \) be \( f'(u) = z_{f(u)} \). Then clearly \( f' \) codes \( f \) modulo \( \pi \).) But since \( \text{cof}(\delta) = \omega \), there is then \( \beta < \delta \) such that \( f(u) \in V_\beta \) for \( E_a \)-measure one many \( u \). But then restricting to this set, we get a function \( g \in V_\delta \).

So let \( M = \mathcal{J}_\theta \) and \( U = \text{Ult}_0(M, E) \). Suppose now that the theorem fails.

**Claim 2.** \( U = M \) is wellfounded and \( j_+ = i^M_\delta : M \to U \) is cofinal and \( \Sigma_1 \)-elementary, with \( j_+(V_\delta) = V_\delta \) and \( j \subseteq j_+ \).

Proof. We also have \( \Sigma_0 \)-Loś’ criterion for the ultrapower, by Lemma 7.4. This gives the \( \Sigma_1 \)-elementarity of \( i^M_\delta \). And note that by the previous claim and our contradictory hypothesis, \( U \) is wellfounded. The fact that \( j_+(V_\delta) = V_\delta \) follows from the continuity of \( j_+ \) at \( \delta \), which holds because \( \text{cof}(\delta) = \omega \), like in the proof of the previous claim.

It remains to see that \( U = M \). Note that \( U = \mathcal{J}_\gamma(V_\delta) \) for some \( \gamma \), and certainly \( \theta \leq \gamma \). But \( M \models \text{"There is no } \alpha > \delta \in \text{OR such that } \mathcal{J}_\alpha \text{ is admissible"} \), so by \( \Sigma_1 \)-elementarity and as \( j_+(V_\delta) = V_\delta \), \( U \) satisfies this statement. Therefore \( \gamma \leq \theta \), so we are done.

**Claim 3.** Given any \( \Sigma_1 \)-elementary \( k : V_\delta \to V_\delta \), there is at most one extension of \( k \) to a \( \Sigma_1 \)-elementary \( k_+ : \mathcal{J}_0 \to \mathcal{J}_0 \) with \( k_+(V_\delta) = V_\delta \). Moreover, if \( k \) is definable from parameters over \( \mathcal{J}_0 \) then so is \( k_+ \).

Proof. Let us first observe that \( k_+ \upharpoonright \theta \) is uniquely determined. Given \( \alpha < \theta \), there is a \( \Sigma_1 \) formula \( \varphi \) and \( z \in V_\delta \) such that \( \alpha \) is the least \( \alpha' \in \text{OR} \) such that \( \mathcal{J}_{\alpha'} \models \psi(z, V_\delta) \), where

\[
\psi(\bar{z}, \bar{v}) = \forall x \in \bar{v} \exists y \varphi(x, y, \bar{z}, \bar{v}).
\]
Hence $k_+(\alpha)$ must be the least $\alpha' \in \text{OR}$ such that $\mathcal{J}_{\alpha'} \models \psi(k(z), V_\delta)$.

But now $\mathcal{J}_\alpha = \text{Hull}^{T_\alpha}(V_\delta \cup \{V_\delta\})$, and since $k_+|_{\mathcal{J}_\alpha} : \mathcal{J}_\alpha \rightarrow \mathcal{J}_{k_+(\alpha)}$ must be $\Sigma_1$-elementary and have $k_+(V_\delta) = V_\delta$, $k$ determines $k_+|_{\mathcal{J}_\alpha}$.

The “moreover” clause clearly follows from the manner in which we have computed $k_+$ above from $k$.

**Claim 4.** Let $k_+ : \mathcal{J}_\theta \rightarrow \mathcal{J}_\theta$ be $\Sigma_1$-elementary with $k_+(V_\delta) = V_\delta$. Then $k_+$ is fully elementary.

**Proof.** Given $\gamma \leq \delta$ and $\alpha \leq \theta$, let $t_\gamma^\alpha = \text{Th}_{\mathcal{J}_\alpha}^{T_\alpha}(V_\gamma \cup \{V_\delta\})$. Let $\tilde{t}_\gamma^\alpha$ code $t_\gamma^\alpha$ as a subset of $V_\gamma$. We claim that $k_+(\tilde{t}_\gamma^\alpha) = \tilde{t}_\gamma^\alpha$. For given $\gamma < \delta$, since $\tilde{t}_\gamma^\alpha \in V_\delta$, by admissibility there is $\alpha_\gamma < \theta$ such that $t_\gamma^\alpha = \tilde{t}_\gamma^\alpha$ for all $\beta \in [\alpha_\gamma, \theta]$. But then it easily follows that $k(\tilde{t}_\gamma^\alpha) = \tilde{t}_\gamma^\alpha$. But then it follows that $k(\tilde{t}_\gamma^\alpha) = \tilde{t}_\gamma^\alpha$.

Also, if $\delta$ is singular in $\mathcal{J}_\theta$ then $k$ is continuous at $\text{cof}^{\mathcal{J}_\theta}(\delta)$, because $k(\delta) = \delta$.

Similarly if $\mathcal{J}_\theta \models \delta$ is regular but not inaccessible”.

From here we can argue as in the proof of [3] Theorem 5.6***?.

Using the preceding claims, we can now derive the usual kind of contradiction, considering the least critical point $\kappa$ of any $\Sigma_1$-elementary $k_+ : \mathcal{J}_\theta \rightarrow \mathcal{J}_\theta$ such that $k_+(\delta) = \delta$ and $k_+$ is $\Sigma_n$-definable from parameters over $\mathcal{J}_\theta$ (for some appropriate $n$). This completes the proof.

**Remark 7.13.** The argument above shows that if $\delta$ is a limit and $\kappa = \kappa_{V_\delta}$, then $\text{cof}^{L(V_\delta)}(\kappa) = \text{cof}^{L(V_\delta)}(\delta)$, definably over $\mathcal{J}_\kappa(V_\delta)$. It also shows that, with $T_\infty$ as before, for each $\alpha < \delta$, since $T_\infty \cap V_\alpha \in V_\delta$, there is $\gamma < \kappa$ such that $T_\gamma \cap V_\alpha = T_\infty \cap V_\alpha$. Also note that, for example, if $\delta = \lambda + \omega$, then $T_\infty$ cannot be just finitely splitting beyond some node $t$ (counting here the number of nodes beyond $t$ in each $V_{\lambda+n}$, for $n < \omega$). For otherwise (taking $t$ with $\text{dom}(j_t) = V_{\lambda+n}$ for some $n$), the tree structure of $(T_\infty)_t$ is coded by a real, hence is in $V_\delta$, and an admissibility argument easily gives that $(T_\infty)_t$ is computed by some $\gamma < \kappa$, which leads to a contradiction as before. However, such an argument doesn’t seem to work in the case that $(T_\infty)_t$ is just $\omega$-splitting.

**Remark 7.14.** In the only example we have where $\delta \in \text{Lim}$ and $j \in \delta(V_\delta) \cap L(V_\delta)$, we have $V_\delta \models \text{ZFC}$+¬$V = \text{HOD}$, so $L(V_\delta) \models \text{AC}$ and $\delta = \kappa_\omega(j)$. Is it possible to have $j \in L(V_\delta) \cap \delta(V_\delta)$ with $\kappa_\omega(j) < \delta$? Or even with $\kappa_\omega(j) = \delta$ but $L(V_\delta) \models \text{¬AC}$? Is it possible to have this with $(V_\delta, j) \models \text{ZFR}$? We know that we need $\text{cof}^{L(V_\delta)}(\delta) = \omega$ for this. We have the tree $T_\infty \in L(V_\delta)$. But without left-most branches, it is not clear to the author how to get an embedding in $L(V_\delta)$ from this. Relatedly, is it possible for $j$ to be generic over $L(V_\delta)$ and have $V_\delta = V_\delta^L(V_\delta)(j)$?

**References**

[1] Joan Bagaria, Peter Koellner, and W. Hugh Woodin. Large cardinals beyond choice. *Bulletin of Symbolic Logic*, 25, 2019.

[2] Gabriel Goldberg. Even ordinals and the Kunen inconsistency. arXiv:2006.01084.

[3] Gabriel Goldberg. Personal communication, 2020.
[4] Gabriel Goldberg and Farmer Schlutzenberg. Periodicity in the cumulative hierarchy. arXiv: 2006.01103.

[5] R. Björn Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 4(3), 1972.

[6] Akihiro Kanamori. *The higher infinite: large cardinals in set theory from their beginnings*. Springer monographs in mathematics. Springer-Verlag, second edition, 2005.

[7] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *Journal of Symbolic Logic*, 36(3), 1971.

[8] W. N. Reinhardt. *Topics in the metamathematics of set theory*. PhD thesis, UC Berkeley, 1967.

[9] W. N. Reinhardt. Remarks on reflection principles, large cardinals, and elementary embeddings. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 189–205. Amer. Math. Soc., Providence, R. I., 1974.

[10] Ralf Schindler and Martin Zeman. Fine structure. In *Handbook of set theory*, pages 605–656. Springer, 2010.

[11] Farmer Schlutzenberg. On the consistency of ZF with an elementary embedding from $V_{\lambda+2}$ into $V_{\lambda+2}$. Submitted. arXiv: 2006.01077.

[12] Farmer Schlutzenberg. Reinhardt cardinals and non-definability (draft 1). Notes. arXiv: 2002.01215v1.

[13] Akira Suzuki. No elementary embedding from $V$ into $V$ is definable from parameters. *Journal of Symbolic Logic*, 64(4), 1999.

[14] Toshimichi Usuba. Choiceless Löwenheim-Skolem property and uniform definability of grounds. arXiv: 1904.00895, 2019.

[15] Toshimichi Usuba. Personal communication, 2020.