Tracking differentiator based on improved Gaussian distribution

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Abstract: Concerning the problems of the existing tracking differentiator, such as large computation, complex parameter tuning and output chattering, a nonlinear tracking differentiator (GTD) based on the improved Gaussian distribution was proposed. It is a simple calculating nonlinear tracking differentiator with easy parameters tuning and strong ability of noise restraint. Firstly, the amplitude gain and exponential factor are introduced to improve the Gaussian distribution, then the acceleration function is constructed by utilizing the improved Gaussian distribution. Secondly, the global asymptotic stability of the designed TD is proved by constructing Lyapunov function. And its singularity type and phase trajectory distribution are analyzed by the phase plane analysis method, and the qualitative law of parameter adjustment is also given. Finally, simulations are performed and results are compared with linear differentiator and anti-hyperbolic sinusoidal based TD. It concludes that the differential signal of the improved Gaussian-distribution-based nonlinear tracking differentiator does not have the high frequency flutter behavior in steady state. Moreover, the GTD not only has good dynamic response but also has strong filtering ability.

1. Introduction

In the design of control system, it is very important to extract control signal accurately. In recent years, with the rise of robots, hypersonic vehicles and other fields, the tracking and differential effects of classical differentiators are not ideal, so extracting accurate differential signals from interference signals has become an important research direction. The concept of tracking differentiator was first proposed by Han Jingqing, an expert of system and control in China, in 1994[1]. It is an important part of ADRC[2]. It is mainly used to extract continuous filtering tracking signal and differential signal from continuous or noise-contaminated measurement signal in practical engineering problems. Subsequently, many scholars have proposed a variety of new tracking differentiators. Sun Biao[3] designed the discretized synthesis function of the fastest speed control, which is a non-linear, non-smooth and discontinuous function. Its tracking and differential effect is ideal, but the parameter adjustment is complicated. Wang Xinhua[4] proposed a fast tracking differentiator, which uses the form of weighted power function to design the differentiator. It is simple in form and has good rapidity. However, when the initial value of the system state deviates from the input signal, there is tremor. Therefore, it is necessary to design a tracking differentiator with simple parameter adjustment and strong noise suppression ability.

Based on the Gauss-based tracking differentiator (GTD) in probability theory, this paper proposes a new nonlinear tracking differentiator (GTD) with strong noise suppression ability. The improved GTD has the characteristics of both linear and non-linear links, and the curve shape is approximately a smooth
saturation function with adjustable linear working interval. In practical engineering applications, according to the characteristics of Gauss distribution, it can be converted into standard normal distribution, which can be quickly calculated by table lookup method with the characteristics of small amount of calculation. In addition, the control parameters of the tracking differentiator are less, and the regulation rules are given. The parameter tuning is simple and feasible.

2. The improvement of Gaussian distribution
Gauss distribution is a very important probability distribution in the fields of mathematics, physics and engineering. It also has great influence in many aspects of statistics. The concrete expression is

\[ F(x) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx. \]  \hspace{1cm} (1)

From the knowledge of higher mathematics and probability theory, it can be seen that the function is smooth and strictly monotone increasing, and its range is \([0,1]\). Now the mathematical expectation is \(\mu = 0\), and the function is transformed into a simple linear transformation so that the range of variation is \([-0.5a, 0.5a]\).

\[ F(x; a, \sigma) = a \left( \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2\sigma^2}} \, dx - 0.5 \right). \]  \hspace{1cm} (2)

Among them, parameter \(a > 0\) is the gain of amplitude, which is used to adjust the amplitude of \(F(x; a, \sigma)\) function; parameter \(\sigma > 0\) is the exponential factor, which is used to regulate the approximate linear working range of \(F(x; a, \sigma)\) function. Fig. 1 shows the function curves corresponding to different exponential factors when the amplitude gain is 1.

![Function curves with different exponential factors](image)

Fig. 1 F(x; a, \sigma) function plots with different exponential factors

In conclusion, the function \(F(x; a, \sigma)\) has not only linear properties but also non-linear properties in \(x \in \mathbb{R}\). When the absolute value of the independent variable increases, the change of the function value tends to be smooth and gradually reaches the saturation value; when the absolute value of the independent variable decreases, the function value gradually approaches the linear change. This function is not only saturated and strictly monotonic, but also singular, and has continuity and smoothness which general saturation function does not possess. It can be used to design state feedback of second-order system, which can effectively weaken the high-frequency flutter of the system and has good dynamic performance.

3. TD based on improved Gaussian distribution

Theorem 1 for the following system \(\Sigma_0\)

\[
\begin{align*}
\dot{z}_1(t) &= z_2(t) \\
\dot{z}_2(t) &= -F(z_1(t); a_1, \sigma_1) - F(z_2(t); a_2, \sigma_2).
\end{align*}
\]  \hspace{1cm} (3)

If \(a_1, a_2, \sigma_1, \sigma_2\) are positive, and the system satisfies Lyapunov asymptotic stability condition at the origin, then
\[
\lim_{t \to \infty} z_1(t) = 0, \quad \lim_{t \to \infty} z_2(t) = 0.
\]  

(4)

**Proof of Theorem 1** The Lyapunov function is constructed as follows:

\[
V(z_1, z_2) = \int_0^{z_1} F(\tau; a_1, \sigma_1) d\tau + \frac{1}{2} z_2^2.
\]  

(5)

Because \( F(\tau; a, \sigma) \) is an odd function of the first and third quadrants in the definition field \( x \in \mathbb{R} \). From the integral mean value theorem, so we can get:

\[
\int_0^{z_1} F(\tau; a_1, \sigma_1) d\tau = F(\xi; a_1, \sigma_1) * z_1 > 0.
\]  

(6)

So there is \( V(z_1, z_2) > 0 \). Then, the total derivative of the equation (5) is obtained along the solution time \( t \) of the system \( \Sigma_0 \), so we can get:

\[
\dot{V}(z_1, z_2) = z_2 * F(z_1(t); a_1, \sigma_1) + z_2 * \dot{z}_2
\]

\[
= -z_2 * F(z_2(t); a_2, \sigma_2).
\]  

(7)

Similarly, here is \( z_2 * F(z_2(t); a_2, \sigma_2) \geq 0 \). Therefore, we can get \( \dot{V}(z_1, z_2) \leq 0 \).

Now we investigate the case that equation (7) takes the equal sign: in the neighbourhood of the origin, the satisfiable formula (7) takes the point set of the equal sign as \( z_2 = 0 \), that is, the point on the transverse axis of the phase plane. For \( \Sigma_3 \), on the horizontal axis of phase plane, if \( z_1 \neq 0 \), then \( \dot{z}_2 = -F(z_1; a_1, \sigma_1) \neq 0 \), we can see that \( z_2 \) will not remain zero, that is to say, the set of points with equal sign of satisfying equation (7) does not contain any other trajectory of \( \Sigma_0 \) except the origin. Therefore, according to LaSalle's invariant set theorem, it can be inferred that when \( t \to \infty \), \( z_1 \to 0, z_2 \to 0 \).

**Theorem 2** for the following system \( \Sigma_3 \)

\[
\begin{cases}
\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = -R^2 \left( F(x_1(t) - v(t)) + F(x_2(t)/R) \right)
\end{cases}
\]  

(8)

If \( a_1, a_2, \sigma_1, \sigma_2 \) are positive constants, then for any bounded measurable function \( v(t) \) and any constant \( T > 0 \), the solution \( x_1(t) \) of the system satisfies:

\[
\lim_{R \to \infty} \int_0^T |x_1(t) - v(t)| \, dt = 0.
\]  

(9)

**Lemma 1**[1][6] If the system \( \Sigma_4 \)

\[
\begin{cases}
\dot{z}_1(t) = z_2(t) \\
\dot{z}_2(t) = f(z_1(t), z_2(t))
\end{cases}
\]  

If \( \lim_{t \to \infty} (z_1(t), z_2(t)) = 0 \), then for any bounded measurable function \( v(t) \) and any constant \( T > 0 \), when the fast adjusting factor \( R \) is large enough, the system:

\[
\begin{cases}
\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = R^2 f(x_1(t) - v(t), x_2(t)/R)
\end{cases}
\]

Solution \( x_1(t) \) satisfies:

\[
\lim_{R \to \infty} \int_0^T |x_1(t) - v(t)| \, dt = 0.
\]

Combining Theorem 1 and Lemma 1, according to the principle of system equivalence, the conclusion of Easy Proof Theorem 2 is established, which is not discussed here.

Furthermore, theorem 2 shows that when \( R \) is large enough, the solution \( x_1(t) \) of \( \Sigma_3 \) can approximate the input signal \( v(t) \) sufficiently in any finite time. Thus, the differential of \( x_1(t) \) can be obtained, which is the approximate differential signal of input signal.

4. Phase plane analysis

Phase-plane method is a time-domain-based analysis method. It can not only analyze the stability and
self-oscillation of the system, but also reflect the influence of different initial conditions and parameters on the system motion more intuitively and accurately. This method is an important tool for qualitative research and analysis of second-order nonlinear time-invariant control systems. The phase plane method is used to investigate the system and find out the influence of four parameters in the acceleration function on the tracking differentiator.

Consider the system, let \( \dot{z}_1(t) = 0 \) and \( \dot{z}_2(t) = 0 \), and get the unique singularity \((0, 0)\) of the system. The Jacobian matrix of \( \Sigma_0 \) at the origin is

\[
A = \begin{bmatrix}
0 & 1 \\
-a_1/(\sigma_1\sqrt{2\pi}) & -a_2/(\sigma_2\sqrt{2\pi})
\end{bmatrix}.
\]

(10)

Obviously, all the eigenvalues of matrix \( A \) have non-zero real parts, and the origin is a hyperbolic singular point of system \( \Sigma_0 \). According to the knowledge of ordinary differential equation, for the hyperbolic singular point of non-linear system, the linear approximation system obtained by Jacobian matrix can be used to analyze the type of singular point and the phase trajectory distribution in the neighborhood of the singular point. System \( \Sigma_0 \) can be linearized into system \( \Sigma_5 \):

\[
\begin{align*}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -a_1z_1/(\sigma_1\sqrt{2\pi}) - a_2z_2/(\sigma_2\sqrt{2\pi})
\end{align*}
\]

(11)

For approximate linear system \( \Sigma_5 \), the eigenvalues of matrix \( A \) are

\[\lambda_{1,2} = -\frac{a_2/(\sigma_2\sqrt{2\pi}) \pm \sqrt{\Delta}}{2}\]

Among them, \( \Delta = a_2/(\sigma_2\sqrt{2\pi})^2 - 4a_1/(\sigma_1\sqrt{2\pi}) \). When \( \Delta \geq 0 \) can be obtained \( a_2/(\sigma_2\sqrt{2\pi}) \geq 2\sqrt{a_1/(\sigma_1\sqrt{2\pi})} \). The singular point \((0, 0)\) is the stable node of the system \( \Sigma_5 \), and the corresponding non-linear system \( \Sigma_0 \) is also the stable node at the origin. Its phase trajectory is shown in Fig. 2. The phase trajectories in the graph all approach the origin along the same two directions, so the corresponding system of the node is directly convergent, and the oscillation of the transition process is relatively small. When \( \Delta < 0 \), the singularity \((0, 0)\) is the stable focus of the system \( \Sigma_5 \), and the corresponding non-linear system \( \Sigma_0 \) is also the stable focus at the origin. Its phase trajectory is shown in Fig. 3. The phase trajectory in the graph approaches to the singularity in a spiral manner, and the oscillation of the transition process is relatively large. Therefore, the singularity should be stabilized when the system is controlled, that is to say, the condition of \( \Delta \geq 0 \) should be satisfied.

Combining the phase plane method with the previous analysis, we can see that the system \( \Sigma_3 \) contains five parameters: \( R, a_1, \sigma_1, a_2, \sigma_2 \), in which \( R \) is related to the tracking effect. Increasing \( R \) can improve the tracking speed of the system, but increase the high-frequency noise of the system differential signal too much; \( a_1 \) is related to the tracking effect, and its effect is similar to \( R \); \( a_2 \) is related to the differential effect, and increasing \( a_2 \) can suppress the noise of the differential signal. But too much will slow down the tracking; \( \sigma_1 \) is related to the tracking effect, its reciprocal function is
similar to \(a_1\), but it is achieved by changing the transition process between the linear region and the saturated region; \(\sigma_2\) is related to the differential effect, its reciprocal function is similar to \(a_2\), and it is also achieved by changing the transition process between the linear region and the saturated region.

5. Simulation

To verify the tracking and estimation effect of GTD, the following comparisons are made with linear differentiator (LD) and anti-hyperbolic sinusoidal tracking differentiator (TD). In order to ensure the scientificity of case comparison, the parameters of the three tracking differentiators are pre-adjusted before the simulation experiment, and the performances of the three tracking differentiators are compared on the premise of keeping the same rapidity. The selected tracking differentiator system and parameters are as follows:

1) **Linear differentiator (LD)**\(^7\)
   
   \[
   \begin{align*}
   \dot{x}_1(t) &= x_2(t) \\
   \varepsilon^2 x_2(t) &= -a_1(x_1(t) - v(t)) - a_2 \varepsilon x_2(t)
   \end{align*}
   \]
   
   Among them: \(\varepsilon = 0.005, a_1 = 2, a_2 = 1\).

2) **Tracking differentiator in anti-hyperbolic sinusoidal form (TD)**\(^8\)
   
   \[
   \begin{align*}
   \dot{x}_1(t) &= x_2 \\
   \dot{x}_2(t) &= -R^2(a_1 \text{arsinh}[b_1(x_1(t) - v(t))] + a_2 \text{arsinh}[b_2 x_2(t)/R])
   \end{align*}
   \]
   
   Among them: \(R = 200, a_1 = 3, b_1 = 1, a_2 = 3, b_2 = 2\).

The specific parameters of the tracking differentiator (GTD) proposed in this paper are as follows:

\(R = 200, a_1 = 1, \sigma_1 = 0.8, a_2 = 2, \sigma_2 = 0.5\).

5.1 Tracking response of step signal

The input signal is a unit step and the step is 0.001s. The simulation results are shown in Fig. 4-5.

![Fig. 4 The tracking responses](image1)

![Fig. 5 The differential responses](image2)

As can be seen from the above figure, the tracking and differential results of linear differentiator (LD) flutter in the initial phase of input signal jump, which is the defect of linear high gain differentiator. That is, when there is a large error between the initial state value \(x_1(0)\) and the initial input information \(v(0)\), the flutter phenomenon becomes more obvious.

At the same time, it is not difficult to find that the high frequency chatter can be completely avoided because the anti-hyperbolic sinusoidal tracking differentiator (TD) and the tracking differentiator (GTD) in this paper both approach the origin in a linear way. In addition, the approximation effects of TD and GTD are almost the same, and they can track and estimate step signals without overshoot and error.

5.2 Tracking response of sinusoidal signal with colored noise

The input signal is a sinusoidal signal sequence \(v(kT_0) = \sin(kT_0) + n(kT_0)\), where \(n(kT_0)\) is a sequence of colored noise and the sampling step \(T_0\) is 0.001 s. In order to ensure that the tracking
differentiator has enough ability to suppress noise, the speed regulation factor R is set to 15, and the other parameters are consistent with the previous section. In the simulation, the selected sequence of colored noise is as follows:

\[ n(k) = \frac{1 + 0.6z^{-1} + 0.3z^{-2}}{1 - 1.2z^{-1} + 0.8z^{-2} + 0.1z^{-3}} \xi(k). \]

Among them, \( \xi(k) \) is a Gauss white noise sequence with a mean value of 0 and a variance of 0.01. The tracking results of the simulated input signal and tracking differentiator are shown in Fig. 6-8.

From Fig 7-8, it can be seen that the tracking effect of TD and GTD on sinusoidal signals with colored noise is basically the same, and they can track ideal signals without overshoot and quickly. On the premise of consistency of speed, the differential tracking differentiator (GTD) proposed in this paper has stronger ability to suppress noise, and the effect of differential tracking is obviously better than that of TD.

6. Conclusion

In this paper, based on the Gauss distribution in probability theory, a non-linear tracking differentiator (GTD) with strong noise suppression ability is proposed. The improved Gauss distribution function has the characteristics of both linear and non-linear links. The simulation results show that the tracking differentiator does not have the high frequency flutter phenomenon of differential signal in steady state, and has good dynamic response and strong filtering ability, taking into account the requirements of system speed and accuracy. It has important reference significance and reference value for the design of control law in control system, such as the extraction of feedback and differential information in ADRC.
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