Convexified Open-Loop Stochastic Optimal Control for Linear Non-Gaussian Systems

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Abstract—We consider stochastic optimal control of linear dynamical systems with additive non-Gaussian disturbance. We propose a novel, sampling-free approach, based on Fourier transformations and convex optimization, to cast the stochastic optimal control problem as a difference-of-convex program. In contrast to existing moment based approaches, our approach invokes higher moments, resulting in less conservatism. We employ piecewise affine approximations and the well-known convex-concave procedure, to efficiently solve the resulting optimization problem via standard conic solvers. We demonstrate that the proposed approach is computationally faster than existing particle based and moment based approaches, without compromising probabilistic safety constraints.

I. INTRODUCTION

Stochastic optimal control requires enforcement of chance constraints, which permit violation of the state constraints with a probability below a specified threshold [1], [2], [3], [4]. Chance constraints trade off constraint violation with the objective cost. However, such constraints are hard to implement in a computationally tractable manner, especially for systems with non-Gaussian disturbances. In this paper, we propose a method for stochastic optimal control of linear systems with arbitrary disturbances, that results in a scalable solution based in convex programming.

Enforcing probabilistic safety constraints in stochastic optimal control problems is difficult because it typically requires high dimensional integrals that are hard to compute and enforce. The two main approaches to tackle chance constraints are based in sampling or risk allocation [5]. Sampling based approaches approximate the uncertainty distribution using a finite number of samples (particles), and formulate a mixed-integer optimization problem [4]. This approach is independent of the particular distribution, and has well characterized lower bounds on the number of particles needed to achieve high quality solutions [6], [7]. However, these bounds typically require a large number of particles, resulting in computationally expensive, mixed-integer optimization problems.

In contrast, risk allocation based approaches are sampling-free approaches that compute open-loop or affine-feedback controllers [8], [9], [10], [11]. They utilize Boole’s inequality to decompose joint chance constraints into simpler, individual chance constraints, and optimize for violation probability thresholds present in the constraints. For a fixed risk allocation, the control synthesis problem is convex for Gaussian disturbances [8], [9]. On the other hand, non-Gaussian disturbances admit convex but conservative enforcement of the chance constraints using concentration inequalities [11], [12]. Unfortunately, simultaneous risk allocation and controller synthesis renders the optimal control problem non-convex. Therefore, existing approaches leverage coordinate descent algorithms to approximately solve the stochastic optimal control problem.

Our main contribution is a computationally efficient and numerically robust solution for stochastic optimal control of linear dynamical systems with non-Gaussian disturbances, based in risk allocation, Fourier transformations, and convex optimization. Our approach simultaneously performs risk allocation and open-loop controller synthesis, without compromising on computational tractability or relying on conservative enforcement of chance-constraints. The key to this is 1) the use of characteristic functions (Fourier transformations of the probability density function) to enforce chance constraints involving non-Gaussian random vectors exactly, and 2) reformulation of the risk allocation problem as a difference-of-convex program, which can be solved locally efficiently via convex optimization [13]. In combination with tight, conic, piecewise affine approximations of the non-conic convex constraints, we can leverage standard off-the-shelf conic solvers to solve the stochastic optimal control problem.

The main limitation of this approach is that it requires open-loop controller synthesis, which results in more conservative solutions than with a closed-loop controller. Open-loop control synthesis are commonplace in stochastic model predictive control [5], [14], and essential in applications with hard computational constraints or sensing constraints that preclude feedback control. Consider hypersonic vehicles, which suffer from computing and sensing limitations at their operational speeds and temperatures [15], [16], or space applications in harsh environments, such as on Mars [17], in which production and testing of sensors that work reliably is difficult.

The organization of the paper is as follows: We present the problem formulation in Section II. Reformulation of the stochastic optimal control problem using risk allocation, piecewise affine approximation, and difference-of-convex program-
ming is presented in Section III. Specialization to Gaussian disturbances, and to random initial conditions are presented in Section IV. We demonstrate our approach on two motion planning examples in Section V, and summarize our contribution in Section VI.

II. PROBLEM STATEMENT

We employ the following notation throughout the paper: The discrete-time interval \( \mathbb{N}_{[a,b]} \) enumerates all natural numbers from integers \( a \) to \( b \). Random vectors are denoted with a bold case \( \mathbf{v} \), non-random vectors are denoted with an overline \( \overline{v} \), and the trace operator is denoted by \( \text{tr}() \).

Consider a stochastic, linear, time-varying system

\[
x(k+1) = A(k)x(k) + B(k)\pi(k) + w(k)
\]

with state \( x(k) \in \mathbb{R}^n \), input \( \pi(k) \in \mathcal{U} \subset \mathbb{R}^m \), and disturbance \( w(k) \in \mathbb{R}^p \). For a time horizon of \( N \in \mathbb{N} \), we assume knowledge of the disturbance probability density \( \psi_W \) describing the stochasticity of the concatenated disturbance random vector \( W = [w(0)\top, w(1)\top, \ldots, w(N-1)\top] \in \mathbb{R}^pN \). For example, for an independent and identical random disturbance process \( w(k) \sim \psi_w \) with \( k \in \mathbb{N}_{[0,N-1]} \), \( \psi_W = \prod_{k=0}^{N-1} \psi_w \).

Throughout the paper, we will assume that \( \psi_W \) is log-concave. Log-concave probability densities form a wide class of unimodal densities [18], including Gaussian and exponential disturbances, and disturbances with convex finite support like triangular and uniform disturbances over convex sets. Recall that a function \( f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R} \) is log-concave, if \( \log(f) \) is a concave [19, Sec. 3.5.1]. We follow the convention that \( \log(0) \triangleq -\infty \). Since log-concavity is preserved under products, log-concavity of \( \psi_W \) is sufficient for log-concavity of \( \psi_W \).

Given a fixed initial state \( \pi(0) \in \mathbb{R}^n \), we define the concatenated (stochastic) state vector and concatenated (deterministic) input vector associated with the dynamics (1) as follows:

\[
X = \begin{bmatrix} x(1)\top & \cdots & x(N)\top \end{bmatrix}\top \in \mathbb{R}^{nN},
\]

\[
U = \begin{bmatrix} \pi(0)\top & \cdots & \pi(N-1)\top \end{bmatrix}\top \in \mathcal{U}^N \subset \mathbb{R}^{mN}.
\]

From (1) and (2), we have

\[
X = \tilde{A}\pi(0) + HU + GW
\]

where the matrices \( \tilde{A} \in \mathbb{R}^{nN \times n} \), \( H \in \mathbb{R}^{nN \times mN} \), and \( G \in \mathbb{R}^{mN \times pN} \) are obtained from the dynamics (1). Due to the linearity of (3), the mean and the covariance vector of \( X \) admit closed-form expressions.

\[
\overline{X}_\mathcal{U} = \tilde{A}\overline{\pi}(0) + H\overline{U} + G\overline{\pi}_W
\]

\[
C_X = GCWG\top.
\]

We are interested in solving a stochastic optimal control problem that minimizes a quadratic cost in \( X \) and \( U \) with pre-specified positive semi-definite matrices \( Q \in \mathbb{R}^{(nN)\times (nN)} \) and \( R \in \mathbb{R}^{(mN)\times (mN)} \), while satisfying hard constraints on the input \( \mathcal{U}^N \subset \mathbb{R}^{mN} \), and soft constraints on the state with high probability. We assume that the input and state constraints are polytopic. Given a probabilistic constraint violation threshold \( \Delta \in [0,1] \) and a desired trajectory \( \overline{X}_d \in \mathcal{S} \), we wish to solve the following stochastic optimal control problem,

\[
\begin{align}
\text{minimize} & \quad \mathbb{E}_X [(X - \overline{X}_d)\top Q(X - \overline{X}_d) + U\top R U] \quad (5a) \\
\text{subject to} & \quad \mathcal{U} \in \mathcal{U}^N, (4a), (4b) \quad (5b) \\
& \quad \mathbb{P}_X \{X_U \in \mathcal{S} \} \geq 1 - \Delta \quad (5c)
\end{align}
\]

with decision variable \( U \in \mathbb{R}^{mN} \). The cost function is convex quadratic in \( U \) since \( \text{tr}(QCU\tau) \) is independent of \( U \) by (4b). We define \( \mathcal{S} = \{ \overline{X} \in \mathbb{R}^{nN} : P\overline{X} \leq \overline{q} \} \) with \( P = [\overline{p}_1 \cdots \overline{p}_{Lx}]\top \in \mathbb{R}^{Lx \times nN} \) and \( \overline{q} = [q_1 \cdots q_{Lx}]\top \in \mathbb{R}^{Lx} \) defining the number of hyperplanes in the polytope.

For a \( \psi_W \) that is Gaussian, risk allocation is an established approach to conservatively assure (5c) [8], [9], [2], [10], [20]. By exploiting the properties of a Gaussian random variable, in conjunction with Boole’s inequality, (5c) can be reformulated as a collection of linear or second order cone constraints. This results in a convex program which enables efficient controller synthesis via standard solvers.

However, non-Gaussian disturbances do not admit similar reformulations. For non-Gaussian disturbances, particle based and moment based approaches are the two main approaches to solve (5). However, these approaches have significant drawbacks. Particle based approaches use sampling to approximate (9a), and rely on computationally expensive, mixed integer, linear program solvers for controller synthesis [4], [7]. Moment based approaches use concentration inequalities and risk allocation to enforce (5c). Even though the moment based approaches enable controller synthesis via convex optimization, the resulting reformulation is typically conservative [5], [11], [21]. The conservativeness arises from the fact that only few lower-order moments are used to tractably enforce the chance constraints, ignoring the available, higher-order moment information.

To address the computationally expensive nature of the particle based control and the conservativeness of the moment based approach, we present a Fourier transform based approach to solve (5), which uses all the moments of the underlying distribution. We propose to solve two problems:

Problem 1. Extend the risk-allocation technique for non-Gaussian disturbances using Fourier transforms and piecewise affine approximations.

Problem 2. Solve (5) for an arbitrary, log-concave, stochastic disturbance \( W \) using convex optimization and piecewise affine approximation of the chance constraint from Problem 1.

III. CONVEXIFICATION OF NON-GAUSSIAN JOINT CHANCE CONSTRAINTS

A. Risk-allocation for log-concave disturbances

The standard risk-allocation approach [2], [8], [9], [10], [20] transforms the joint chance constraints (5c) into a set
of individual chance constraints via Boole’s inequality,
\[ P \{ P X^T \leq \eta \} \geq 1 - \Delta \]
(6)
\[ \iff \{ \int_{s=1}^{L_X} \{ p_i^T G W \leq q_i - p_i^T (A\pi(0) + HU) \} \} \geq 1 - \Delta \]
\[ \iff \{ \int_{s=1}^{L_X} \{ p_i^T G W \geq q_i - p_i^T (A\pi(0) + HU) \} \} \leq \Delta \]
\[ \iff \sum_{i=1}^{L_X} \{ p_i^T G W \geq q_i - p_i^T (A\pi(0) + HU) \} \leq \Delta \]
\[ \forall x \in X \]
(7)
Here, \( \delta_i \in [0, 1) \) are auxiliary decision variables that represent
the risk of violating the constraint \( p_i^T X \leq q_i, i \in \mathbb{N}[1, L_X] \). We
have \( \delta_i \leq \Delta \) since \( \sum_{i=1}^{L_X} \delta_i \leq \Delta \) and \( \delta_i \) are non-negative.
Let \( \Phi_{p_i^T GW} : \mathbb{R} \rightarrow [0, 1] \) denote the cumulative distribution function of the random variable \( p_i^T GW \),
\[ \Phi_{p_i^T GW} (q') = \{ p_i^T GW \leq q' \}, \]
(8)
for any scalar \( q' \in \mathbb{R} \). We use \( \Phi_{p_i^T GW} \) to rewrite the constraints (7) as
\[ \Phi_{p_i^T GW} (d_i - p_i^T HU) \geq 1 - \delta_i \]
(9a)
\[ \sum_{i=1}^{L_X} \delta_i \leq \Delta, \delta_i \in [0, \Delta], \forall i \in \mathbb{N}[1, L_X] \]
(9b)
with scalar constants
\[ d_i = q_i - p_i^T \Phi (0), \forall i \in \mathbb{N}[1, L_X] \].

Any feasible controller \( \overline{U} \in \mathcal{U}^N \) with a feasible risk allocation \( \delta = [\delta_1, \ldots, \delta_{L_X}] \in [0, 1]^{L_X} \) that satisfies (9) automatically satisfies (5c).

B. Enforcing chance constraints using characteristic functions

The characteristic function of the disturbance vector \( W \) with probability density function \( \psi_W (z) \) is defined as
\[ \Psi_W (\beta) = \mathbb{E}_W [ \exp (j (\beta^T W) ) ] = \int \exp (j (\beta^T z)) \psi_W (z) d\bar{z} = \mathcal{F} \{ \psi_W \} (\mathbb{R}^N) \]
(10)
where \( \mathcal{F} \{ \cdot \} \) denotes the Fourier transformation operator and \( \beta \in \mathbb{R}^{nN} \). Furthermore, from [22, Eq. 22.6.3], the characteristic function of the random variable \( p_i^T GW \) is given by
\[ \Psi_{p_i^T GW} (\beta) = \Psi_W (\{ G^T p_i \} \beta) \]
(11)
for some \( \beta \in \mathbb{R} \).

A key insight we use in this paper is that the evaluation of the cumulative distribution function in (9a) is given by a one-dimensional integration, i.e., for any \( s \in \mathbb{R} \),
\[ \Phi_{p_i^T GW} (s) = \frac{1}{2} \int_{0}^{\infty} \text{Im} \left( \frac{\exp (j s \beta) \Psi_W (\{ G^T p_i \} \beta)}{j \beta} \right) d\beta, \]
(12)
where \( \text{Im} (z) \) denotes the imaginary component of a complex number \( z \). Equation (12) enables enforcing the chance constraint in (9a) using only \( \Psi_W \) as opposed to using the probability density function, the known characteristic function of the concatenated disturbance random vector \( W \). Equation (12) follows from the inversion of characteristic functions [23, 24, 25]. We implement (12) using quadrature techniques [26].

**Lemma 1** ([27, Thm. 4.2.1]). Under the assumption of log-concavity, \( \Phi_{p_i^T GW} \) is log-concave over \( \mathbb{R} \).

Using (9) and Lemma 1, we approximate (5) as follows,
\[ \min_{\overline{U}, \overline{\delta}} (\overline{p}_X - \overline{X})^T Q (\overline{p}_X - \overline{X}) + \overline{U}^T \overline{R} \overline{U} \]
(13a)
subject to \( \overline{U} \in \mathcal{U}^N \)
(13b)
\[ \forall i \in \mathbb{N}[1, L_X], \overline{p}_i^T HU + \Phi^{-1}_{p_i^T GW} (\epsilon) \leq d_i \]
(13c)
\[ \forall i \in \mathbb{N}[1, L_X], \log \left( \Phi_{p_i^T GW} (d_i - p_i^T HU) \right) \geq t_i \]
(13d)
\[ \forall i \in \mathbb{N}[1, L_X], \log \left( \sum_{i=1}^{L_X} \exp (t_i) \right) \geq \log (L_X - \Delta). \]
(13f)
for a small scalar \( \epsilon > 0 \) and a change of variables
\[ t_i = \log (1 - \delta_i), \forall i \in \mathbb{N}[1, L_X] \]
(14)
with \( \overline{l} = [l_1, \ldots, l_{L_X}] \in \mathbb{R}^{L_X} \).

We now establish the relationship between (5) and (13), and show that (13) is a non-convex program with a reverse convex constraint. Recall that reverse-convex constraints are optimization constraints of the form \( f \geq 0 \), where \( f \) is a convex function.

**Theorem 1.** Assuming that the underlying distribution is log-concave, the following statements hold for any \( \Delta \in [0, 1) \) and any \( \epsilon > 0 \):

1. Every feasible solution of (13) is feasible for (5), and
2. The cost and the constraints (13b)–(13d) are convex.
   However, (13f) is a reverse convex constraint.

**Proof:** 1) We observe that the constraints (5b) and (13b) are identical. We need to show that satisfaction of (13c)–(13f) satisfies (5c). Recall that the collection of constraints (9) tighten (5c). Therefore, it is sufficient to show that the satisfaction of constraints (13c)–(13f) guarantee satisfaction of (9).

The constraint (13c) ensures that the constraint (13d) is well-defined, since the satisfaction of (13c) ensures that \( \Phi_{p_i^T GW} (d_i - p_i^T HU) \) is positive. The satisfaction of (13d) implies satisfaction of (9a). The satisfaction of (13e) implies that \( \delta_i \in [0, \Delta] \) by (14). Finally, we show that (13f) and (9b) are equivalent via simple algebraic manipulations,
\[ \sum_{i=1}^{L_X} \delta_i \leq \Delta \iff \log (L_X - \Delta) \]
\[ \log \left( \sum_{i=1}^{L_X} \exp (t_i) \right) \geq \log (L_X - \Delta) \]
(15a)
(15b)
In other words, every feasible solution \( (\overline{U}, \overline{\delta}) \) of (13) maps to a feasible solution to (9) with \( \overline{U} \in \mathcal{U}^N \), and thereby is feasible for (5).
Proof of 2) We already know that the cost (13a) is a convex quadratic function of \( \overline{U} \). By construction, the constraints (13b), (13c), and (13e) are linear constraints in \( \overline{U} \) and \( \overline{t} \). The convexity of (13d) follows from Lemma 1 and the definition of log-concavity. Recall that log concavity of (13d) follows from Lemma 1 and the definition of log-concavity. The constraint (13f) is a reverse-convex constraint.

**C. Conic reformulation of (13d) via piecewise affine approximation**

We now focus on enforcing the convex constraint (13d). Despite its convexity, the constraint (13d) is not a conic constraint, which prevents the use of standard conic solvers in its current form. We present a tight conic reformulation of (13d) using piecewise affine approximations.

Given a concave function \( f : \mathcal{D} \to \mathcal{R} \) for bounded intervals \( \mathcal{D}, \mathcal{R} \subset \mathbb{R} \), we define its piecewise affine underapproximation as \( \ell_f^-(x) = \min_{j \in \mathbb{N}[1,N_f]} (m_j^- x + c_j^-) \).

For a user specified approximation error \( \eta > 0 \), Appendix B describes the sandwich algorithm [28] that computes \( \ell_f^- \) for a concave \( f \) such that

\[
\ell_f^- (x) \leq f(x) \leq \ell_f^+(x) + \eta.
\]

In (13), we use the piecewise affine underapproximation of the concave functions \( f_i = \log( p_{f_i}^{-1} \Phi_{GW} ) \) with \( N_i \in \mathbb{N} \) distinct pieces for every \( i \in [1,L] \) to conservatively enforce (13d). The functions \( f_i \) have bounded domain and range in \( \mathbb{R} \) due to (13c). We evaluate \( f_i \) using the one-dimensional numerical integration of characteristic functions, as discussed in (12). We obtain the following optimization problem,

\[
\begin{align*}
\min_{\overline{U}} & \quad (\overline{p}_X \overline{v} - \overline{X}_d) ^\top Q (\overline{p}_X \overline{v} - \overline{X}_d) + \overline{U} ^\top R \overline{U} + \text{tr}(Q C_{X,\overline{v}}) \\
\text{subject to} & \quad \overline{U} \in \mathcal{U}^N, \ s \geq 0 \quad \forall i \in [1,L], \ \overline{p}_i ^\top H \overline{U} + \Phi_{f_i}^{-1} \Phi_{GW} (\epsilon) \leq d_i \quad \forall \epsilon \in [0,1], \quad m_{i,j}^- (d_i - \overline{p}_i ^\top H \overline{U}) + c_{i,j}^- \geq t_i \\
& \quad \log \left( \sum_{i=1}^{L_X} \exp(t_i) \right) \geq \log(L_X - \Delta) \quad \forall i \in [1,L], \ t_i \in [\log(1 - \Delta), 0], \ s \geq 0
\end{align*}
\]

By Theorem 1 and the use of piecewise affine underapproximations of \( \log( \Phi_{f_i}^{-1} \Phi_{GW} ) \), every feasible solution of (18) is feasible for (13), and thereby (5).

**D. Solving (18) via difference of convex programming**

The optimization problem (18) has a quadratic cost (23a), linear constraints (18b)–(18e) in the decision variables \( U, s, R \), and a single reverse-convex constraint (18f). We now discuss a tractable solution to (18) using difference of convex programming [13].

Difference of convex programs are non-convex optimization problems of the form,

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) - g(x) \\
\text{subject to} & \quad f_i(x) - g_j(x) \leq 0, \quad \forall i \in [1, M], \ \forall j \in [1, N]
\end{align*}
\]

where \( f, g \) and \( f_i, g_j \) are convex for \( i \in [1,M], \ M \in \mathbb{N} \). The penalty based convex-concave procedure [13] solves (19) in a sequential convex optimization based approach starting from a potentially infeasible initial guess. See Appendix A and [13], [29] for more details.

Given the current estimate for the risk allocation \( \tau = [r_1 \ldots r_{L_X}] \in [0,1]^{L_X} \), the penalty based convex-concave procedure solves the following convex approximation of (18) at every iteration,

\[
\begin{align*}
\min_{\overline{U}, s} & \quad (\overline{p}_X \overline{v} - \overline{X}_d) ^\top Q (\overline{p}_X \overline{v} - \overline{X}_d) + \overline{U} ^\top R \overline{U} + \text{tr}(Q C_{X,\overline{v}}) + \tau_k s \\
\text{subject to} & \quad \overline{U} \in \mathcal{U}^N, \ s \geq 0 \quad \forall i \in [1,L], \ \overline{p}_i ^\top H \overline{U} + \Phi_{f_i}^{-1} \Phi_{GW} (\epsilon) \leq d_i \quad \forall \epsilon \in [0,1], \ m_{i,j}^- (d_i - \overline{p}_i ^\top H \overline{U}) + c_{i,j}^- \geq t_i \\
& \quad \log \left( \sum_{i=1}^{L_X} \exp(t_i) \right) \geq \log(L_X - \Delta) + \frac{1}{\sum_{i=1}^{L_X} \exp(r_i)} \sum_{i=1}^{L_X} \exp(r_i) (t_i - r_i) + s \geq 0
\end{align*}
\]

where \( \tau_k \geq 0 \) for \( k \in \mathbb{N} \) are optimization hyperparameters. The constraint (20f) corresponds to the first-order approximation of the reverse-convex constraint (18f), which is relaxed by a scalar slack variable \( s \). We penalize the slack variable \( s \).
in the objective (20a). We know (20) is convex, since (20f) is a linear constraint in \( \tau \) and \( s \), and all other constraints and the objective are convex (Theorem 1.b).

Starting with an arbitrary risk allocation \( \tau_0 \in [0,1]^n \), we iteratively solve (20) with monotonically increasing values of \( \tau_k \) to promote feasibility. In the numerical experiments, we chose a uniform risk allocation \( \tau_0 = \frac{1}{n} \mathbf{1}_n \), where \( \mathbf{1}_n \) is a \( n \)-dimensional vector of ones. See Appendix A for more details on the sequence \( \{\tau_k\}_{k \geq 1} \) and the stopping conditions for the penalty based convex-concave procedure.

In summary, we have decomposed the original stochastic optimal control problem presented in (5) into a convex quadratic program (13), then we converted the non-conic optimal control problem presented in (5) into a convex optimization problem, that can be solved via penalty based approximations, and piecewise affine approximations to formulate an optimal control problem presented in (5). Figure 2 summarizes the resulting convex optimization problems.

![Flowchart](image)

Fig. 2. Flow resulting in the convexified problem to solve original problem using standard solvers.

IV. EXTENSIONS AND SPECIAL CASES

A. Random initial state

We now consider the effect of a random initial state \( x(0) \), which is assumed to be statistically independent from \( W \). Similar to (18), we can use risk allocation, Fourier transformations, and piecewise affine approximations to formulate an optimization problem, that can be solved via penalty based convex-concave procedure.

Let \( \Psi_x \) be the characteristic function of \( x(0) \). Define a new random vector \( Z = Ax(0) + GW \). We have the characteristic function of \( Z \) in closed-form with the Fourier variable \( \beta \) in \( \mathbb{R}^{nN} \),

\[
\Psi_Z(\beta) = \Psi_x (\tilde{A}^T \beta) \Psi_W (G^T \beta).
\] (21)

Next, we formulate the risk-allocation based constraints on \( Z \) to conservatively enforce the soft state constraint (5c),

\[
\Phi_{\tilde{P}^T \beta} (q_i - \tilde{p}_i^T HU) \geq 1 - \delta_i, \quad \forall i \in [1,L_X],
\] (22a)

\[
\sum_{i=1}^{L_X} \delta_i \leq \Delta, \quad \delta_i \in [0,\Delta], \quad \forall i \in [1,L_X].
\] (22b)

Here, we compute \( \Phi_{\tilde{P}^T \beta} \) using (12) and (21). The satisfaction of (22) for any feasible controller \( U \in \mathcal{U}^N \) and risk allocation \( \delta \) implies that \( \Pr_{\tilde{X}} \{ X \in \mathcal{S} \} \geq 1 - \Delta. \) In contrast to (9a), (22a) has a different term in the left hand side since the initial state is now random.

Finally, we complete the optimization problem formulation using characteristic functions (Sections III-B) and piecewise affine underapproximations (Section III-C),

\[
\begin{align*}
\min_{\tilde{U},\tau} & \quad (\tilde{p}_x^T - \tilde{x}_d) \mathcal{Q}(\tilde{p}_x^T - \tilde{x}_d) + \tilde{U}^T R \tilde{U} \\
\text{s.t.} & \quad \Phi_{\tilde{P}^T \beta} (q_i - \tilde{p}_i^T HU) \geq 1 - \delta_i, \quad \forall i \in [1,L_X] \\
& \quad \sum_{i=1}^{L_X} \delta_i \leq \Delta, \quad \delta_i \in [0,\Delta], \quad \forall i \in [1,L_X].
\end{align*}
\] (23a)

For a Gaussian disturbance \( W \), existing literature solves the optimal control problem (5) via the following approximation,

\[
\begin{align*}
\min_{\tilde{U},\delta} & \quad (\tilde{p}_x^T - \tilde{x}_d) \mathcal{Q}(\tilde{p}_x^T - \tilde{x}_d) + \tilde{U}^T R \tilde{U} \\
\text{s.t.} & \quad \delta_i \geq 0, \quad \forall i \in [1,L_X] \\
& \quad \sum_{i=1}^{L_X} \delta_i \leq \Delta, \quad \delta_i \in [0,\Delta], \quad \forall i \in [1,L_X].
\end{align*}
\] (24a)

B. Gaussian disturbance \( W \): Risk allocation and controller synthesis via a single quadratic program for \( \Delta \leq 0.5 \)

For a Gaussian disturbance \( W \), existing literature solves the optimal control problem (5) via the following approximation,

\[
\begin{align*}
\min_{\tilde{U},\tau} & \quad (\tilde{p}_x^T - \tilde{x}_d) \mathcal{Q}(\tilde{p}_x^T - \tilde{x}_d) + \tilde{U}^T R \tilde{U} \\
\text{s.t.} & \quad \Phi_{\tilde{P}^T \beta} (q_i - \tilde{p}_i^T HU) \geq 1 - \delta_i, \quad \forall i \in [1,L_X] \\
& \quad \sum_{i=1}^{L_X} \delta_i \leq \Delta, \quad \delta_i \in [0,\Delta], \quad \forall i \in [1,L_X].
\end{align*}
\] (25a)
restrict $z \in [\delta_{sh}, \Delta]$ for some small $\delta_{sh} > 0$ to ensure bounded domain and range for $f$. We can construct $f_{I_{sa}}$ using the sandwich algorithm (Appendix B, Algorithm 2) since $-\Phi_{\text{LogNorm}}^{-1}(1 - z)$ is concave for $z \in [\delta_{sh}, \Delta]$. Consequently, any $\mathbf{U} \in \mathbb{R}^{mN}$ and $\mathbf{d} \in \mathbb{R}^{LX}$ that satisfies

$$\mathbf{p}^T H \mathbf{U} \leq q_i + \|C^2_{\mathbf{X}} \mathbf{p}_{\mathbf{i}}\|^2_2 (m_j^2 \delta_i + c_j^2),$$

for every $i \in \mathbb{N}_{[1,L_X]}$ and $\mathbf{d} \in \mathbb{R}^{1,N_i}$ satisfies (24d). We obtain a conservative solution to (5) for a Gaussian disturbance $\mathbf{W}$ by solving the following quadratic program,

$$\min_{\mathbf{U}, \mathbf{s}} \langle \mathbf{p}_{\mathbf{X}, \mathbf{U}} - \mathbf{X}_d \rangle^T \mathbf{Q}(\mathbf{p}_{\mathbf{X}, \mathbf{U}} - \mathbf{X}_d) + \mathbf{U}^T R \mathbf{U}$$

subject to $\Delta \geq \sum_{i=1}^{L_X} \delta_i$, (4a), (4b), $\mathbf{U} \in \mathcal{U}^N$ \hspace{1cm} (27a)

$$\delta_i \in [\delta_{sh}, \Delta], \hspace{1cm} \forall i \in \mathbb{N}_{[1,L_X]}$$

$$\mathbf{p}^T H \mathbf{U} \leq q_i + \|C^2_{\mathbf{X}} \mathbf{p}_{\mathbf{i}}\|^2_2 (m_j^2 \delta_i + c_j^2), \hspace{1cm} \forall i \in \mathbb{N}_{[1,L_X]}, \forall j \in \mathbb{N}_{[1,N_i]}.$$ 

(27d)

In contrast to existing coordinate-descent based approaches, we can now use standard quadratic program solvers to solve (27) efficiently. See our prior work [20] for more details.

V. NUMERICAL EXAMPLES

We apply the proposed approach on two examples: 1) a stochastic double integrator, and 2) a quadrotor in a harsh environment, with crosswind. We also compare the performance of the controller produced by our approach to: 1) a particle based approach [4], and 2) a moment based approach [11]. We measure the performance of the controllers based on the attained cost, probability of constraint satisfaction, and computational time. We also used a Monte-Carlo simulation with $10^5$ samples for validation.

All computations are done with MATLAB on an Intel Xeon CPU with 3.80 GHz clock rate and 32GB RAM. We implemented our algorithm and the particle based approach in CVX [30] with Gurobi [31]. We used fmincon and CVX to implement the moment based approach. We used MPT [32] and SReachTools [33] for the stochastic optimal control problem formulation.

For the implementation of the proposed approach via difference-of-convex programming, we set $\tau_{\text{max}} = 10000$, $\tau_0 = 0.1$, and $\epsilon_{\text{eod}} = 1.2$, and for the termination criteria we used 100 iterations or $\epsilon_{dc} = 1 \times 10^{-6}$. For the sandwich algorithm, we chose $\eta = 0.1$.

The particle based approach constructs an open-loop controller that solves (5) approximately via mixed-integer programming [4]. Specifically, we draw samples (particles) of the disturbance random vector $\mathbf{W}$ and utilize the particle based approximation of the state constraint probability as well as the expected cost to construct a particle based approximation of (5). This approach recovers the optimal open-loop controller for (5) as the number of particles considered increases, at the penalty of increased computational time. In the numerical experiments, we used 50 particles, and reported the average from three separate runs.

The moment based approach constructs an affine-feedback controller via coordinate-descent based optimization [11]. It enforces the chance constraint on the state via concentration inequalities, specifically the Chebyshev-Cantelli inequality. The moment based approach utilizes only the first and the second moment of the disturbance random vector $\mathbf{W}$, resulting in a high-degree of conservatism compared to the proposed approach. We also use the moment based approach to generate an open-loop controller by setting the gain matrix (a decision variable) to zero.

A. Constrained control of a stochastic double integrator

We first consider a double integrator system,

$$\begin{bmatrix} x(k+1) \\ \dot{x}(k) \end{bmatrix} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ \dot{x}(k) \end{bmatrix} + \begin{bmatrix} \frac{R}{T_s^2} \\ \frac{R}{T_s} \end{bmatrix} \mathbf{w}(k)$$

with state $x(k) \in \mathbb{R}^2$, input set $\mathcal{U} = [-20, 20]$, exponential disturbance $\mathbf{w}(k)$ with scale $\bar{\mathbf{w}}(k) \in \mathbb{R}^2$, sampling time $T_s = 0.25$s, and initial position $\mathbf{x}(0) = [-1, 0]^T$.

We seek to solve a constrained optimal control problem subject to dynamics (28), with quadratic cost (5a) that encodes our desire to track $\mathbf{X}_d \in \mathbb{R}^{mN}$, penalize high velocities, and minimize control effort. Specifically, we choose $Q = \text{diag}([10 \ 1]) \otimes I_{(mN) \times (mN)}$, $R = 10^{-3}I_{(mN) \times (mN)}$, $(\mathbf{X}_d)_{c_1} = [m_1 t + c_1, 0]^T$, $\forall t \in \mathbb{N}_{[0,N]}$, and set problem parameters $m_1, m_2, m_1 + c_1, c_2, c_r$, as 0.222, −0.222, −0.111, −5.222, 5.222, and 2.111 respectively. We define the time varying state constraints as

$$\mathcal{F} = \{(t, \mathbf{x}) \in \mathbb{N}_{[0,N]} \times \mathbb{R}^2 : m_1 t + c_1 \leq x_1 \leq m_2 t + c_2 \}.$$

and wish to maintain constraint satisfaction of $90\%$, i.e. $\Delta = 0.1$.

![Fig. 3. Mean trajectories from our approach, the particle based approach, and the affine feedback moment based approach. All approaches compute a controller that maintains the constraint violation threshold ($\Delta = 0.1$). The affine feedback moment approach tracks the desired trajectory closest, while our approach computes the controller fastest, and has lower constraint violation (Table I). The open-loop moment based approach failed to find a controller.](image-url)
While all the methods generated similar trajectories, the key differences can be seen in Table I, which compares the computed values of the cost and probability of satisfaction to their Monte Carlo estimates for $10^5$ simulated trajectories. The particle based approach is able to compute an open-loop controller the fastest using 50 particles, but the constraint violation is lower than the Monte Carlo (MC) estimate of violation. On the other hand, the affine feedback moment based approach computes a constraint violation of 0.907, but the Monte Carlo estimate of the constraint violation is 1. This is can be seen in Figure 5, which shows a fraction of the Monte Carlo trajectories for all the approaches, where the majority of the affine feedback moment based approach trajectories are well contained in the set. The benefit of affine feedback is clearly seen in the control effort in Figure 4, where the moment based affine approach provides an input at $k = 9$ to maintain a majority of the trajectories.

Our chance constrained approach obtains an open-loop controller that exceeds the computed constraint satisfaction in Monte Carlo evaluation, with very little increase in computation time compared to the particle based approach. In addition, computation time of our approach is comparable to the affine feedback moment based approach, while having a Monte Carlo estimate of 0.98, providing a balance between a high constraint satisfaction while being cheap to compute.

### Table I

| Method         | Cost (Comp) | Cost (MC) | $1 - \Delta$ (Comp) | $1 - \Delta$ (MC) | Time (s) |
|----------------|-------------|-----------|---------------------|-------------------|----------|
| Chance Open    | 124.599     | 124.507   | 0.90                | 0.981             | 2.468 s  |
| Particle [4]   | 108.21      | 108.24    | 1.00                | 0.973             | 1.07 s   |
| Moment Closed  | 105.628     | 109.482   | 0.907               | 1.00              | 6.88 s   |

While the particle based approach does better than our approach in solve time as for time horizons between 0 and 35 time steps, with an exponential disturbance with scale $\lambda_\text{w}(k) = [10 100] ^\top$ (Figure 6).

Moment based approaches fail for large time horizons, possibly due to their reliance on coordinate descent optimization. The open-loop approach fails for time horizons 10 and larger, and the affine approach fails for time horizons 15 and larger.

While the particle based approach does better than our approach in solve time as the time horizon is increased, branch and bound based approaches are solver specific hence the solve time can vary depending on the solver. In addition, as seen for the constant horizon case in Table I, while the probability of constraint satisfaction for the particle control was noted to be 1, the Monte Carlo estimate was lower. In comparison, our approach has consistent solve times with Monte Carlo constraint violation greater than what was reported from the computation.
B. Quadrotor in crosswind of a harsh environment

We consider a rigid-body quadcopter model,

\[
\begin{align*}
\dot{p}_x &= \frac{u_1}{m} \left( \cos \psi \sin \theta + \cos \theta \sin \phi \sin \psi \right) \quad (29a) \\
\dot{p}_y &= \frac{u_1}{m} \left( \sin \psi \sin \theta - \cos \theta \sin \phi \cos \psi \right) \quad (29b) \\
\dot{p}_z &= \frac{u_1}{m} \left( \cos \phi \cos \theta \right) - g \quad (29c) \\
\dot{\phi} &= \frac{I_{yy} - I_{zz} \dot{\theta}}{I_{xx}} \dot{\psi} + \frac{u_2}{I_{xx}} \quad (29d) \\
\dot{\theta} &= \frac{I_{zz} - I_{xx} \dot{\phi}}{I_{yy}} \dot{\psi} + \frac{u_3}{I_{yy}} \quad (29e) \\
\dot{\psi} &= \frac{I_{xx} - I_{yy} \dot{\phi} \dot{\theta}}{I_{zz}} + \frac{u_4}{I_{zz}} \quad (29f)
\end{align*}
\]

where the state variables \(p_x, p_y, \) and \(p_z\) define the translational motion and \(\phi, \theta, \) and \(\psi\) define the roll, pitch, and yaw respectively. The state is a 12-dimensional vector, \(\mathbf{x} = [p_x \ p_y \ p_z \ \dot{p}_x \ \dot{p}_y \ \dot{p}_z \ \phi \ \theta \ \psi \ \dot{\phi} \ \dot{\theta} \ \dot{\psi}]^\top.\) The net thrust is described by \(u_1,\) and the moments around the \(p_x, p_y, \) and \(p_z\) axes created by the difference in the motor speeds are described by \(u_2, u_3, \) and \(u_4.\) We use the following parameters for the quadcopter: mass \(m = 0.478\) kg and moment of inertia \(I_{xx} = I_{yy} = 0.0117\) kg m², and \(I_{zz} = 0.00234\) kg m² [34].

We linearize the nonlinear dynamics (29) in a hovering operation point (zero state and input \([4.6892, 0, 0, 0]^\top\)), and discretize the continuous-time dynamics via a zero-order hold with sampling time \(T_s = 0.25.\) We incorporate the effect of wind into the quadcopter model with an additive stochastic disturbance,

\[
x(k + 1) = A x(k) + B \pi(k) + w(k). \quad (30)
\]

We presume a time-invariant triangle distribution to model the wind via the disturbance \(w(k),\) to characterize the best, worst, and nominal values of the wind (Figure 7). The wind is assumed to directly influence only the translational motion \(p_x, p_y, \) and \(p_z.\) The distribution changes in the later half of the control interval as shown in Figure 7.

We solve the stochastic optimal control problem (5) for a time horizon of \(N = 10\) with \(Q = \text{diag}([10 \ 10 \ 10 \ 1 \times 9 N]) \otimes I_{N \times N} \) and \(R = 10^{-3} I_{N \times 4N}.\) We specify the desired trajectory \(\mathbf{X}_d\) between \((20, 50, 25)\) and \((50, 20, 25)\) via waypoints spread uniformly in time. The limits on the input are \(\mathcal{U} = [-5, 5]^4.\) The constraint set \(\mathcal{J}\),

\[\mathcal{J} = \{ \mathbf{x} \in \mathbb{R}^{12} : |p_x| \leq 100, |p_y| \leq 100, |p_z| \leq 100 \},\]

imposes restrictions on the translational motion. The initial condition is \(x(0) = [10 \ 0 \ 0 \ 0 \ldots 0] ^\top.\) The probability of constraint satisfaction required is 90% (\(\Delta = 0.1).\) Figure 8 shows the computed trajectories by our approach and the particle based approach. Both the moment based approaches failed to compute a controller due to numerical issues. While the trajectories look similar for both our approach and the particle based approach, Table II shows that our approach meets the desired constraint satisfaction (0.92) via Monte Carlo but the particle based control does not (0.767) even though it determined that constraint satisfaction of its controller is 1. The constraint violations can be seen in Figure 9, which shows a fraction of the Monte Carlo trajectories on bottom of the red constraint set. In addition, while the cost at each time-step (stage cost) of each approach is similar (Figure 10), the particle based approach utilizes some net thrust \(u_1\) whereas our approach uses none.

VI. CONCLUSION

We presented a convex optimization based approach for the constrained, optimal control of a linear dynamical system with additive, non-Gaussian disturbance. Our formulation utilizes a novel Fourier transformation based risk allocation technique to assure probabilistic safety for a non-Gaussian disturbance.
TABLE II
Quadcopter example: Cost and constraint satisfaction ($1 - \Delta$) for both computed (Comp) and Monte-Carlo (MC) simulation for $10^3$ samples of the disturbance trajectory for our approach and the particle based approach. The open-loop and affine moment based approaches did not compute an optimal controller.

| Method         | Cost ($\times 10^3$) | $1 - \Delta$ Comp | $1 - \Delta$ MC | Time (s) |
|----------------|-----------------------|--------------------|-----------------|----------|
| Chance - Open  | 84.79                 | 0.90               | 0.92            | 15.25    |
| Particle [4]   | 77.60                 | 1.00               | 0.767           | 237.85   |

Fig. 8. Mean trajectories for our approach and the particle based approach. Only our approach computes a controller that meets the constraint satisfaction when evaluated with sample trajectories (Table II).

Fig. 9. Monte Carlo trajectories of our approach (blue) and the particle based approach (orange). The constraint violation is apparent at the bottom of the shaded region. As seen from Table II, the particle based approach has more constraint violations compared to our approach. Both moment based approaches failed to find a controller.

Our approach solves a tractable difference-of-convex program to synthesize the desired controller. We make our problem amenable to standard conic solvers via the use of piecewise affine approximations. Numerical experiments show the efficacy of our approach over existing state of the art approaches, particle control and moment based approaches, in handling non-Gaussian disturbances.

APPENDIX

A. Difference of convex programming

We now briefly review the convex-concave procedure used to solve difference-of-convex program (19). Difference-of-convex programs can be solved to global optimality via general branch-and-bound methods [29]. However, these methods typically require additional computational effort. The penalty based convex-concave procedure (Algorithm 1) is a successive convexification based method to find local optima of (19) using convex optimization [13, Alg. 3.1]. Algorithm 1 relies on the observation that replacing $g_i$ with their first order Taylor series approximations in (19) yields a convex subproblem, which can then be solved iteratively. To accommodate a potentially infeasible starting point, we relax the DC constraints using slack variables $s_i(k) = [s_1^{(k)} \ s_2^{(k)} \ ... \ s_L^{(k)}] \in \mathbb{R}^L$, and penalize the value of the slack variables for each iteration $k$.

A possible exit condition, apart from $\tau > \tau_{\text{max}}$, is

$$\left| f_0(\bar{z}_k) - g_0(\bar{z}_k) \right| - \left( f_0(\bar{z}_{k+1}) - g_0(\bar{z}_{k+1}) \right) + \tau_k \sum_{i=1}^L (s_i^k - s_i^{k+1}) \leq \epsilon_{\text{dc}}$$

$$\sum_{i=1}^L s_i^{k+1} \leq \epsilon_{\text{viol}} \approx 0$$

(31a)

(31b)
where $\epsilon_{dc} > 0$ and $\epsilon_{viol} > 0$ are (small) user-specified tolerances. Here, (31a) checks if the algorithm has converged (in the value of the objective), and (31b) checks if $\tau_{k+1}$ is feasible. See [13] for more details, such as convergence guarantees of Algorithm 1.

Algorithm 1 Local optimization of (19) [13, Alg. 3.1]

Input: Initial point $\bar{x}_0$, $\gamma_0 > 0$, $\tau_{\text{max}}$, $\gamma > 1$

Output: Local optima of (19)

1. $k \leftarrow 0$
2. $\delta_g(\bar{x}_k) \leftarrow g(\bar{x}_k) + \nabla g(\bar{x}_k)^\top (\bar{x} - \bar{x}_k)$, $\forall i \in \{1, \ldots, L\}$
3. Solve the following convex problem for $\tau_{k+1}, \pi (k)$:
   minimize $f_0(\bar{x}_{k+1}) - \hat{g}_0(\bar{x}_{k+1}; \bar{x}_k) + \tau_k \sum_{i=1}^{L} s_i(k)$
   subject to $\pi(k) \geq 0$
   $\forall i \in \{1, \ldots, L\}$:
   $f_i(\bar{x}_{k+1}) - \hat{g}_i(\bar{x}_{k+1}; \bar{x}_k) \leq s_i(k)$
4. Update $\tau_{k+1} \leftarrow \min(\gamma \tau_k, \tau_{\text{max}})$ and $k \leftarrow k + 1$
5. while $\tau \leq \tau_{\text{max}}$ and (31) is not satisfied

B. Piecewise affine underapproximations for concave functions

Let $f : \mathcal{D} \rightarrow \mathcal{R}$ be a concave, differentiable function defined for bounded, closed, convex, intervals $\mathcal{D}, \mathcal{R} \subset \mathbb{R}$. Given a user specified approximation error $\eta > 0$, we seek a piecewise affine underapproximation $\ell_f^\eta$ which satisfies (17),

$$\ell_f^\eta(x) \leq f(x) \leq \ell_f^\eta(x) + \eta.$$

We use $\nabla f : \mathcal{D} \rightarrow \mathcal{R}$ to denote the derivative of $f$.

The sandwich algorithm (Algorithm 2) constructs such an underapproximation via bisection, specifically the slope-bisection rule [28]. The slope-bisection rule bisects a given interval $[l, u]$ at the point $x_m$ such that $\nabla f(x_m) = m = \frac{f(u) - f(l)}{u - l}$. Due to the concavity of $f$, the maximum error of underapproximating $f$ using a line $y = mx + c$ with $c = f(l) - ml$ over the interval $[l, u]$ occurs at $x_m$.

Algorithm 2 uses two stacks, which are last-in first-out data structures [35]. Recall that stacks have two operations: push to add an element to the top of the stack, and pop to retrieve (and delete) the element from the top of the stack. Here, we use the stack $\mathcal{I}$ to store the tuples associated with intervals that must be processed to satisfy the user-specified maximum underapproximation error $\eta$, and the stack $\mathcal{F}$ to store the resulting slope and intercept pairs that together define $\ell_f^\eta$.

To illustrate the use of Algorithm 2, we compute a piecewise affine underapproximation of the log of the cumulative distribution of a non-Gaussian random variable $\nu$. Such piecewise affine underapproximations admit conservative enforcement of
the chance constraints, as seen in (18). Figure 11 shows the approximations for the affine transformation of an exponential disturbance \(a^\top w_t\) where \(w_t = [w_1 \ w_2 \ w_3]^\top \in \mathbb{R}^3\) and \(a = [1.0 \ 0.75 \ 0.75]^\top\) where the scale parameters are \(\mathcal{N}(k) = [0.5 \ 0.25 \ 0.1667]^\top\). Note that the derivative of the cumulative distribution function \(\nabla \log(\Phi(w(x))) = \frac{\psi(w(x))}{\Phi(w(x))}\) where \(\psi(w(x))\) is the probability density function. Both the cumulative distribution function and the probability density function can be evaluated from the characteristic function via Fourier inversion [25].

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