Global existence of the classical solutions for the inhomogeneous relativistic Boltzmann equation near vacuum in the Robertson–Walker space-time

Etienne Takou 1 · Fidèle L. Ciake Ciake 2

Received: 3 January 2018 / Accepted: 3 September 2018 / Published online: 8 September 2018
© Springer Science+Business Media, LLC, part of Springer Nature 2018

Abstract
In this paper, we consider the Cauchy problem for the spatially inhomogeneous relativistic Boltzmann equation with near vacuum initial data. The collision kernel considered here is for the hard potentials case and the background space-time in which the study is done is the Robertson–Walker space-time. Unique global (with respect to the direction of time corresponding to the expansion of the universe) classical solution is obtained. This is done in a suitable weighted space.

Keywords Relativistic Boltzmann equation · Robertson–Walker space time · Inhomogeneous · Classical solution

Mathematics Subject Classification 76P05 · 35Q20

1 Introduction

The Boltzmann equation describes the evolution of the statistical distribution of particles in a suitable space-time. One of the biggest challenge while dealing with the Boltzmann equation is the study of the classical solutions; that are solutions which have first order derivatives with respect to each variables. The main difficulty is that each derivative of the distribution function creates singularities. This is due to the dependence of the scattering kernel on the relative momentum. This problem is widely
studied in the non-relativistic case. For interested readers, we refer to [1] and the references therein.

About the relativistic case, several authors studied this problem by taking the Minkowski space-time as background. Most of results available concern the study of mild solutions. A mild solution to the initial-value problem associated to the Boltzmann equation (2.2) is a continuous function \( f \) such that the function denoted \( f^\# \) satisfies the time-integrated form (2.21) of the Boltzmann equation; see below. Even though the Boltzmann is an integro-differential equation, with the differential part being described by a first order partial differential operator, for the mild solutions the differentiability in each variable is not required. Glassey [6] studied for some appropriate classes of scattering cross-section a global mild solution to the Cauchy problem for the relativistic Boltzmann equation with small data. The authors extended in [18] the Glassey’s result to the Robertson–Walker (FRW) space-time. The authors also proved in [19] the existence of mild solution of the relativistic Boltzmann equation on a spherically symmetric gravitational field.

Now for the classical solutions, one of the most interesting aspect while working in the Minkowski space-time is the existence of an equilibrium solution. The steady states of this model are the well known Jüttner solution, also known as the relativistic Maxwellian. This allows to define the perturbation of the distribution function to the relativistic Maxwellian. Using this splitting, Strain [15] proved that in the case of special relativity, unique classical solutions to the relativistic Boltzmann equation exists for all time and decay with any polynomial rate towards their steady state. This result was carried out in the case of a spatially periodic box and with collision kernel for soft potentials. The main technique in [15] is to study the linearized equation and then the equation.

One interesting question when dealing with the relativistic Boltzmann equation is the possibility of finding analytic solutions. Such solutions are possible only under very restrictive assumptions of relaxation time approximation (RTA); the total distribution function is then split into one symmetric term (which is generally large) and one asymmetric term (which is small). Recently, analytic solutions of the RTA Boltzmann equation for a system with Gubser flow; i.e a flow pattern that combines boost-invariant longitudinal expansion with fast azimuthally symmetric transverse flow were presented in [4,5,9].

In the present paper, instead of using the Minkowski space-time, we consider as background the Robertson–Walker space-time with a given metric (2.1). In Cosmology, it is the basic model for the study of the expanding Universe. To our knowledge, this space-time was not yet used for the study of the classical solution of the inhomogeneous relativistic Boltzmann equation. One of the main difference compared with what happens in the Minkowski space-time is that the Maxwellian can no longer be solution of the Boltzmann equation. In fact, even though the Maxwellian belongs to the null space of the collision operator, it is not in the null space of the transport part of this equation. We recall that the homogeneous Boltzmann equation was widely studied in the RW space-time and in the Bianchi type 1 space-time. For interest reader, we refer to [12–14,17] and references therein. Even though the Robertson–Walker space-time is spatially homogeneous, we allow in this work the density functions to depend on
the space variable; this can be considered as a generalization of the work of Strain [15] when the background is no longer the Minkowski space-time.

It is our purpose in this paper to provide an existence result for a unique global classical solution for the inhomogeneous relativistic Boltzmann equation in the Robertson–Walker space-time for initial data near vacuum in an appropriate weighted framework. In the present work, the only matter contents are the massive particles statistically described in terms of their distribution function, denoted $f$, which is a non-negative real valued function, of both the time $t \in \mathbb{R}^+$, the position $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ and the momentum $p = (p^1, p^2, p^3)$ of the particles. The main argument of the paper is to use the methods of Glassey [6] and Guo [7], the global in time existence of the solution will be proved using a standard iteration method. As the partial derivatives of post-collision momenta sometimes have singularities we will use two different parametrizations to overcome these singularities.

The paper is organized as follows: in Sect. 2, we recall the form of the equation, we give the main assumptions of the paper and we write the equation in new variables. In Sect. 3, we present some preliminary estimates on the terms allowing to define the collision operator. In Sect. 4, we establish the energy estimates and provide the global existence theorem.

2 The inhomogeneous Boltzmann equation

2.1 Notation and collision operator

The notation $a \lesssim b$ means that a positive constant $C$ exists such that $a \leq Cb$ holds uniformly over the range of parameters which are present in the inequality and moreover that the precise magnitude of the constant is unimportant. In the sequel, we let sometimes $C$ and $c$ denote generic and positive inessential constants whose values may change from line to line.

In this paper, Greek indices will be assumed to run from 0 to 3, while Latin indices run from 1 to 3. We adopt the Einstein summation convention $a_\alpha b^\alpha = \sum a_\alpha b^\alpha$ unless otherwise specified. We consider as background the spatially flat RW space-time where the metric tensor with signature (–, +, +, +) can be written as:

$$ds^2 = -dt^2 + R^2(t) \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right]$$

In which $R(t)$ is a strictly positive function of $t$, called the cosmological expansion factor. Using the Christoffel symbols computed in [13], the relativistic Boltzmann equation in the RW space-time can be written as follows:

$$\partial_t f + \hat{p} \cdot \nabla_x f - 2\frac{\dot{R}}{R} p \cdot \nabla_p f = Q(f, f)$$

(2.2)

where $\hat{p}$ is defined by $\hat{p} = \frac{p}{|p|^2}$, with $p^0 = \sqrt{1 + R^2 |p|^2}$. For more informations about the relativistic Boltzmann equation, we refer interested readers to [2–4]. In (2.2), $Q$
is a non-linear operator called “collision operator” which is specified as follows. In instantaneous, binary and elastic scheme due to Lichnerowicz and Chernikov [11], we consider that at a given position $x$, two particles (or two beams of particles) of momenta $p^\alpha = (p^0, p)$ and $q^\alpha = (q^0, q)$ collide without destroying each other. The collision affecting only their momenta that change after the collision. Let $p^\alpha = (p^0, p')$ and $q^\alpha = (q^0, q')$ be their momenta after the collision. By conservation of the energy-momentum principle, one has:

$$p^\alpha + q^\alpha = p'^\alpha + q'^\alpha. \quad (2.3)$$

The operator $Q$ is henceforth defined by $Q(f, g) = Q_g(f, g) - Q_l(f, g)$ where:

$$Q_g(f, g)(t, x, p) = R^3(t) \int_{\mathbb{R}^3} \int_{S^2} \frac{g \sqrt{s}}{p^0 q^0} \sigma(g, \omega) f(t, x, p') g(t, x, q') d\omega dq, \quad (2.4)$$

$$Q_l(f, g)(t, x, p) = R^3(t) \int_{\mathbb{R}^3} \int_{S^2} \frac{g \sqrt{s}}{p^0 q^0} \sigma(g, \omega) f(t, x, p) g(t, x, q) d\omega dq, \quad (2.5)$$

correspond to the gain term and the loss term respectively. For simplicity, we sometimes abbreviate $f(t, x, p)$ by $f(p)$. In that expression of $Q$:

- $S^2$ is the unit sphere of $\mathbb{R}^3$;
- $v_\phi := \frac{g \sqrt{s}}{p^0 q^0}$ is called Møller velocity;
- $s$ and $g$ are called respectively the square of the energy in the “center of momentum” system $p + q = 0$ and the relative momentum. They are defined by:

$$s = -(p_\alpha + q_\alpha)(p^\alpha + q^\alpha), \quad g = \sqrt{(p_\alpha - q_\alpha)(p^\alpha - q^\alpha)}. \quad (2.6)$$

- $\sigma(g, \omega)$ is called the differential cross-section or scattering kernel; it measures interaction’s effects between particles during the collision process and depends on the relative momentum and the scattering angle $\theta$ defined by the relation

$$\cos \theta = \frac{(p^\alpha - q^\alpha)(p'_\alpha - q'_\alpha)}{g^2}. \quad (2.7)$$

The authors proved in [18] that the parameter $\omega$ over the unit sphere and the scattering angle $\theta$ are precisely linked by the relation

$$\sin^2 \frac{\theta}{2} = \frac{4R^2 (p^0 q^0)^2 (\omega \cdot (\hat{p} - \hat{q}))^2}{g^2 \left((p^0 + q^0)^2 - R^2 (\omega \cdot (p + q))^2\right)}. \quad (2.8)$$

With this relation one can either write $\sigma(g, \omega)$ or $\sigma(g, \theta)$. 

\[ \text{Springer} \]
2.2 Equation in new variables

In the remainder of this paper, we consider (2.2) with covariant variables. So, the distribution function \( f \) will be considered as a function of \( t, x \) and \( p_k = g_{k\beta} p^\beta = R^2 p^k \), with \( k = 1, 2, 3 \). This change of variable was previously used in [10,18]. In what follows, for simplicity, we set

\[
v = (v^1, v^2, v^3) \quad \text{where} \quad v^k = R^2 p^k \quad \text{and} \quad v^0 = \sqrt{1 + R^{-2} |v|^2} = p^0. \quad (2.9)
\]

With these new variables, we use \( v'^k = R^2 p'^k \) and \( u'^k = R^2 q'^k \) for the post-collisional momenta.

One of the main problems while dealing with the relativistic Boltzmann equation is the parametrization of post-collisional momenta. Unlike in the non-relativistic case, the post-collisional momenta can usefully be parametrized in many different ways in the relativistic case.

- The authors [18] parametrized the post-collisional momentum as

\[
\begin{align*}
v' &= v - a(v, u, \omega) \omega \\
u' &= u + a(v, u, \omega) \omega; \quad \omega \in S^2
\end{align*}
\]

(2.10)

where \( \hat{v} = \frac{v}{v^0} \) and \( \hat{u} = \frac{u}{u^0} \), the real-valued function \( a \) is given by

\[
a(v, u, \omega) = \frac{2 v^0 u^0 e \omega \cdot (\hat{v} - \hat{u})}{e^2 - R^{-2}(\omega \cdot (v + u))^2},
\]

(2.11)

where \( e = v^0 + u^0 \).

- Lee [10] parametrized the post-collisional momentum as

\[
\begin{align*}
v'^0 &= \frac{v^0 + u^0}{2} + \frac{g}{2R} \frac{1}{\sqrt{s}} (\hat{v} + \hat{u}), \\
v'^k &= \frac{v^k + u^k}{2} + \frac{R_g}{2} \left( \hat{\omega}^k = \frac{(v + u) \cdot \hat{\omega}}{|v + u|^2} + \frac{e}{\sqrt{s}} \left( \frac{(v + u) \cdot \hat{\omega}}{|v + u|^2} \right) \right), \quad \hat{\omega} \in S^2
\end{align*}
\]

(2.12)

The post-collisional momenta in (2.10) and (2.12) are written with different unit vectors \( \omega \) and \( \hat{\omega} \). For a fixed \( v' \), one can find a relation between \( \omega \) and \( \hat{\omega} \). In this paper, we will use the two parametrizations of post-collisional momentum; this will depend on the region in the tangent bundle. This argument was originally suggested by Guo and Strain in [8] and later used by Ho Lee [10].

Now, we write (2.2) in new variables. Let \( \tilde{f}(t, x, v) = f(t, x, p) \). Then

\[
\partial_t \tilde{f} = \partial_t f - 2 \frac{\dot{R}}{R} p \cdot \nabla_p f, \quad \text{and} \quad \partial_{x_i} \tilde{f} = \partial_{x_i} f.
\]

(2.13)
Straightforward computation leads to \( dp = R^{-6} dv \). In the sequel, by abuse of notation, we will still write \( f \) instead of \( \tilde{f} \). So, with the new variables, still using \( Q \) to denote the collision operator, we have

\[
Q(f, f)(t, x, v) = R^{-3}(t) \int_{S^2} d\omega \int_{\mathbb{R}^3} du \nu_\omega \sigma(g, \omega)[f(v')f(u') - f(v)f(u)]
\]

\[
= Q_g(f, f)(t, x, v) - Q_l(f, f)(t, x, v). \quad (2.14)
\]

Taking into account (2.13), (2.2) becomes

\[
\partial_t f + \frac{1}{R^2} \hat{v}.\nabla_x f = Q_g(f, f)(t, x, v) - Q_l(f, f)(t, x, v). \quad (2.15)
\]

Since \( Q \) involves an integral of \( f \), (2.15) is an integro-differential equation. For any fixed \((x, v) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3\), the characteristics \( X^t(x, v) \) are defined by the following relations

\[
\begin{cases}
\frac{d}{dt} X^t(x, v) = R^{-2}(t) \hat{v}, \\
X^t(x, v)|_{t=0} = x.
\end{cases} \quad (2.16)
\]

From (2.16), we have

\[
X^t(x, v) = x + \int_0^t R^{-2}(\tau) \hat{v} d\tau = x + \left( \int_0^t \frac{R^{-2}(\tau)d\tau}{\sqrt{1 + R^{-2}(\tau)|v|^2}} \right) v. \quad (2.17)
\]

Let us now introduce the standard notation in the Boltzmann equation

\[
f^#(t, x, v) = f(t, X^t(x, v), v). \quad (2.18)
\]

(2.18) is the relation between \( f^# \) and \( f \). Using this relation, we have

\[
\frac{d}{dt} f^#(t, x, v) = \partial_t f + \frac{R^{-2}(t)}{\sqrt{1 + R^{-2}(t)|v|^2}} v.\nabla_x f = \partial_t f + R^{-2}(t) \hat{v}.\nabla_x f. \quad (2.19)
\]

By (2.19), (2.15) becomes

\[
\frac{d}{dt} f^#(t, x, v) = Q^#(f, f)(t, x, v). \quad (2.20)
\]

This leads to

\[
f^#(t, x, v) = f_0(x, v) + \int_0^t Q^#(f, f)(\tau, x, v)d\tau \quad (2.21)
\]

where \( Q^#(f, f) \) is given by:

\[
Q^#(f, f)(\tau, x, v) = Q(f, f)(\tau, X^\tau(x, v), v). \quad (2.22)
\]
Remark 2.1 Let us define the vector function $b(t, u, v)$ by

$$b(t, u, v) := \int_0^t \left( \frac{R^{-2}(\tau)v}{\sqrt{1 + R^{-2}(\tau)|v|^2}} - \frac{R^{-2}(\tau)u}{\sqrt{1 + R^{-2}(\tau)|u|^2}} \right) d\tau. \quad (2.23)$$

the following relation holds

$$f(t, X^t(x, v), u) = f^#(t, x + b(t, u, v), u). \quad (2.24)$$

Our main goal in this work is to provide a regular function $f$, i.e, a function which admits first order partial derivatives with respect to each variable and satisfies (2.21). Before collecting preliminary results, we list assumptions of the paper.

2.3 Main assumption

In the present work, we suppose that there exists $0 < b < 3$ such that the scattering kernel $\sigma(g, \omega)$ satisfies the following growth/decay estimates:

$$\frac{g}{\sqrt{s}} \sigma_0(\omega) \lesssim \sigma(g, \omega) \lesssim (1 + g^{-b})\sigma_0(\omega), \quad (2.25)$$

$$|\partial_g \sigma(g, \omega)| \lesssim g^{-1-b} \sigma_0(\omega), \quad (2.26)$$

where $\sigma_0(\omega)$ is non-negative, bounded, continuous and satisfies

$$\int_{S^2} \sigma_0(\omega) e^{-|\omega.y|^2} d\omega \lesssim e^{-|y|^2}, \quad \forall y \in \mathbb{R}^3 \text{ such that } |y| \geq 1. \quad (2.27)$$

This case of scattering kernels falls into the class of hard potentials.

About the cosmological expansion factor, we assume that

$$R(0) = 1, \quad R'(t) > 1, \quad \lim_{t \to +\infty} R(t) = +\infty, \quad (2.28)$$

$$\int_{\mathbb{R}_+} R^{-2}(\tau) d\tau < +\infty, \quad \int_{\mathbb{R}_+} R^{b-3}(t) dt < +\infty. \quad (2.29)$$

Remark 2.2 The assumptions (2.28) and (2.29) occur when $R$ grows fast enough. This is the case when space-time is considered with a positive cosmological constant. In such situation, the $R$ grows exponentially with time [17].
3 Preliminary results

Lemma 3.1 The relative momentum \( g \) and the energy \( s \) satisfy
\[
\begin{align*}
g & \leq \sqrt{s} \leq 2\sqrt{v^0u^0}, \quad \text{and} \quad \frac{R^{-4}|v \times u|^2 + R^{-2}|v - u|^2}{v^0u^0} \leq g^2 \leq R^{-2}|v - u|^2, \\
\max \left( \sqrt{\frac{v^0}{u^0}} \right) & \leq \sqrt{s} \leq 2\sqrt{v^0u^0}.
\end{align*}
\] (3.1) (3.2)

The function \( a(v, u, \omega) \) defined in (2.11) and \( \omega.(\hat{v} - \hat{u}) \) satisfy the following estimates:
\[
\begin{align*}
|a(v, u, \omega)| & \leq \frac{R^{-2}e|v - u|}{\sqrt{e^2 - R^{-2}|v + u|^2}}, \\
|\omega.(\hat{v} - \hat{u})| & \lesssim \frac{ge}{v^0u^0} \lesssim \frac{e|v - u|}{v^0u^0}.
\end{align*}
\] (3.3) (3.4)

Proof See [18]. □

Lemma 3.2 Given a positive constant \( B \), for fixed \( v \) and \( u \), there exists \( t_0 \in \mathbb{R}_+ \) such that in \( [t_0, +\infty) \), \( \Omega(t) = a\omega.(v - u) \) is bounded from above by \( B \).

Proof See [18]. □

Lemma 3.3 Let \( v \) and \( u \) be given. Suppose that \( v' \) and \( u' \) are parametrized by (2.11) or (2.12) with an unit vector \( \omega \in S^2_+ = \{ \omega \in S^2, a\omega.(v - u) \leq B \} \) where \( B \) is a fixed positive constant. Then, the following estimate holds.
\[
|v|^2 + |u|^2 - |v'|^2 - |u'|^2 \leq B.
\] (3.5)

Proof See [18] for the representation (2.11) and [10] for (2.12). □

Remark 3.1 The restriction of the type \( S^2_+ \) on the set \( S^2 \) was previously used by Strain [16], Lee [10] and Takou and Ciake Ciake [18]. Note that \( S^2_+ \) depends on \( v, u \) and \( t \). From lemma 3.2, we can find a finite \( t_0 \) such that \( S^2_+ = S^2 \) for \( t \geq t_0 \). This means that the restriction on \( S^2 \) disappears for large \( t \). In the sequel, we consider the collision operator with the restriction \( S^2_+ \).

Lemma 3.4 We have the following inequalities:
\[
\begin{align*}
\int_{\mathbb{R}^3} v\phi g^{-b}e^{-|u|^2} du & \leq C \quad \text{for} \quad 0 \leq b \leq 1, \\
\int_{\mathbb{R}^3} v\phi g^{-b}e^{-|u|^2} du & \leq CR^{b-1} \quad \text{for} \quad 1 \leq b < 4, \\
\int_{\mathbb{R}^3} |v - u|^{-\alpha}e^{-|u|^2} du & \leq C_\alpha (1 + |v|^2)^{-\frac{\alpha}{2}} \quad \text{for} \quad 0 \leq \alpha < 3,
\end{align*}
\] (3.6) (3.7) (3.8)

where \( C_\alpha \) is a positive constant depending on \( \alpha \).
Proof See [10].

Lemma 3.5 For $0 \leq b < 3$ and $m \in \mathbb{R}_+$, we have the following estimate:

$$
\int_{\mathbb{R}^3}(u^0)^m \ g^{-b} \ e^{-|u|^2} \ du \leq CR^b.
$$

(3.9)

where $C$ is a constant depending on $m$ and $b$.

Proof Using the third inequality of (3.1), since $0 \leq b < 3$, by (3.8)

$$
\int_{\mathbb{R}^3}(u^0)^m \ g^{-b} \ e^{-|u|^2} \ du \leq \int_{\mathbb{R}^3} \frac{R^b(u^0)^{m+b/2}(v^0)^{b/2}}{|u - v|^b} \ e^{-|u|^2} \ du
$$

$$
\leq R^b(v^0)^{b/2} \int_{\mathbb{R}^3} \frac{(u^0)^{m+b/2}}{|u - v|^b} \ e^{-|u|^2} \ du
$$

$$
\lesssim R^b(v^0)^{b/2} \int_{\mathbb{R}^3} \frac{e^{-\frac{1}{2}|u|^2}}{|v - u|^b} \ du
$$

$$
\lesssim R^b(v^0)^{b/2} (1 + |v|^2)^{-\frac{b}{2}} \lesssim R^b.
$$

Lemma 3.6 Under assumptions (2.25) and (2.26), we have

$$
|\partial_v v^0| \leq \frac{1}{R},
$$

(3.10)

$$
|\partial_v g| \leq \frac{u^0 \sqrt{v^0 u^0}}{R},
$$

(3.11)

$$
|\partial_v \sqrt{s}| \leq \frac{u^0 \sqrt{v^0 u^0}}{R},
$$

(3.12)

$$
|\partial_v [v \phi \sigma (g, \omega)]| \leq c \frac{1}{R} u^0 (1 + g^{-b}) \sigma_0(\omega).
$$

(3.13)

Proof The proof of inequalities of (3.10), (3.11), (3.12) is found in [10]. For the inequality (3.13), one has

$$
\partial_v [v \phi \sigma (g, \omega)] = \left[ (\partial_v g) \frac{\sqrt{s}}{v^0 u^0} + (\partial_v \sqrt{s}) \frac{g}{v^0 u^0} \right. \sigma (g, \omega)
$$

$$
- (\partial_v v^0) \frac{g \sqrt{s}}{(v^0)^2 u^0} \sigma (g, \omega) + \frac{g \sqrt{s}}{v^0 u^0} (\partial_g \sigma (g, \omega)).
$$
Using (3.1), (3.10), (3.11) and (3.12) as well as assumption (2.26), we have

\[
| \partial_{v^i} [v_\phi \sigma(g, \omega)] | \\
\leq \frac{2}{R} \left[ \frac{u^0}{\sqrt{v^0 u^0}} (\sqrt{s} + g) \sigma(g, \omega) + |\partial_g \sigma(g, \omega)| \frac{u^0 g \sqrt{s}}{\sqrt{v^0 u^0}} \right] + \frac{1}{R} \frac{g \sqrt{s}}{(v^0)^2} \sigma(g, \omega) \\
\leq \frac{c u^0}{R} (\sigma(g, \omega) + g |\partial_g \sigma(g, \omega)|) \leq c R^{-1} u^0 \left(1 + g^{-b}\right) \sigma_0(\omega).
\]

[Lemma 3.7]

Let a defined by (2.11) and the post-collisional momenta parametrized by (2.10) and (2.11), we have the following estimates

\[
| \partial_{v^i} a | \leq C v^0 (u^0)^5, \quad \quad (3.14) \\
| \partial_{v^i} v^k | \leq C v^0 (u^0)^5 \quad \text{and} \quad | \partial_{v^i} u^k | \leq C v^0 (u^0)^5, \quad \quad (3.15)
\]

for some constant C which does not depends on R.

**Proof**

We set \( a = \frac{N}{D} \), then \( \partial_{v^i} a = \frac{1}{D} \partial_{v^i} N - \frac{N}{D^2} \partial_{v^i} D \). By direct computation

\[
\partial_{v^i} N = \frac{2u^0 v^i}{R^2 v^0} \omega \cdot (v - u) - \frac{4 v^i}{R^2} \omega \cdot u + 2u^0 (v^0 + u^0) \omega^i \\
\partial_{v^i} D = \frac{2(v^0 + u^0) v^i}{R^2 v^0} - \frac{2 \omega \cdot (v + u) \omega^i}{R^2}.
\]

Using the fact that \( \frac{|v|}{R} \leq v^0 \) and \( \frac{|u|}{R} \leq u^0 \), by (3.16) and (3.17), we deduce that

\[
| \partial_{v^i} N | \lesssim v^0 (u^0)^2 \quad \text{and} \quad | \partial_{v^i} D | \lesssim \frac{1}{R} (v^0 + u^0). \quad \quad (3.18)
\]

We note that

\[
|N| \leq 2v^0 u^0 (v^0 + u^0) \left( \frac{|v|}{v^0} + \frac{|u|}{u^0} \right) \leq 2R v^0 u^0 (v^0 + u^0) \quad \text{and} \quad D \geq s \geq \frac{v^0}{u^0}.
\]

By the above relations, we have

\[
\left| \frac{1}{D} \partial_{v^i} N \right| \lesssim \frac{u^0}{v^0} v^0 (u^0)^2 \quad \Rightarrow \quad \left| \frac{1}{D} \partial_{v^i} N \right| \lesssim (u^0)^3. \\
\left| \frac{N}{D^2} \partial_{v^i} D \right| \lesssim R v^0 u^0 (v^0 + u^0) \frac{v^0 + u^0}{R} \left( \frac{u^0}{v^0} \right)^2 \Rightarrow \left| \frac{N}{D^2} \partial_{v^i} D \right| \lesssim v^0 (u^0)^5.
\]

Combining the above relations, the fact that \( v^0 \geq 1 \) and \( u^0 \geq 1 \) implies

\[
|\partial_{v^i} a| \lesssim (u^0)^3 + v^0 (u^0)^5 \Rightarrow |\partial_{v^i} a| \lesssim v^0 (u^0)^5
\]

\( \Box \) Springer
Lemma 3.8 If the post-collisional momenta are parametrized by \(2.12\), we have the following estimates
\[
\begin{align*}
|\partial_{v_i} v^{ik}| &\leq C \left( \frac{R v_0}{|v-u|} + \frac{R v_0}{|v+u|} + \frac{R^2 (v^0)^2}{|v-u|^2} \right) (u^0)^2, \\
|\partial_{v_i} u^{ik}| &\leq C \left( \frac{R v_0}{|v-u|} + \frac{R v_0}{|v+u|} + \frac{R^2 (v^0)^2}{|v-u|^2} \right) (u^0)^2
\end{align*}
\]
for some constant \(C\) which does not depend on \(R\).

**Proof** See [10].

To be able to use the parameterization \(2.11\) together with \(2.12\), we split the integration domain into three different integration regions:
\[
A_0 = \{|v| \leq R\}, \quad A_1 = \{|v| \geq R, |v| \leq 2|u|\}, \quad A_c = \{|v| \geq R, |v| \geq 2|u|\}
\]
\[(3.20)\]

Corollary 3.1 For a fixed finite time \(t\), the derivatives of post-collisional momenta are estimated as follows:

- **On the set** \(A_0\), \(|\partial_{v_i} v^{ik}| \lesssim (u^0)^5\). \[(3.21)\]
- **On the set** \(A_1\), \(|\partial_{v_i} v^{ik}| \lesssim (u^0)^6\). \[(3.22)\]
- **On the set** \(A_c\), \(|\partial_{v_i} v^{ik}| \lesssim (u^0)^3\). \[(3.23)\]

**Proof** \((3.21)\) and \((3.22)\) come from \((3.15)\). \((3.23)\) comes from \((3.19)\).

Lemma 3.9 If we let \(<u> = \sqrt{1 + |u|^2}\), the function vector \(b\) defined in \(2.23\) satisfies
\[
\begin{align*}
|\partial_{v_i} b^k(t, u, v)| &\leq C; \quad \text{(3.24)} \\
|\partial_{v_i} b^k(t, v', v)| &\leq C + C <u>^6; \quad \text{(3.25)} \\
|\partial_{v_i} b^k(t, u', v)| &\leq C + C <u>^6. \quad \text{(3.26)}
\end{align*}
\]
where \(C\) is a constant which does not depend on \(t\).

**Proof** – About the first inequality, by direct computation, we have
\[
|\partial_{v_i} b^k(t, u, v)| = \left| \int_0^t \frac{1}{R^2 v^0} \left[ \delta^{ik} - \frac{v^i v^k}{R v^0} \right] d\tau \right| \leq 2 \int_0^{+\infty} \frac{1}{R^2(\tau)} d\tau.
\]
The estimate \((3.24)\) follows from the integrability of \(\frac{1}{R^2(\tau)}\) over \(\mathbb{R}_+\).
– Now for the second inequality,
\[ \partial_{v^i} b^k(t, v', v) = \int_0^t \frac{1}{R^2 v^0} \left[ \delta^{ik} - \frac{v^i v^k}{R^2 (v^0)^2} \right] d\tau - \int_0^t \left[ \frac{\partial_{v^i} v^k}{R^2 v^0} - \frac{v^k \partial_{v^i} v^0}{(R v^0)^2} \right] d\tau. \]

We need to estimate \(|\partial_{v^i} v^0|\). We have thanks to (3.21), (3.22) and (3.23)
\[ \partial_{v^i} v^0 = \frac{1}{R^2 v^0} \sum_{j=1}^3 v^j \partial_{v^j} v^0 \Rightarrow |\partial_{v^i} v^0| \leq C \frac{1}{R} \sum_{j=1}^3 \frac{v^j}{R v^0} (u^0)^6 \leq C \frac{1}{R} (u^0)^6. \]

Since \(u^0 \leq u\), \(v^0 \geq 1\) and \(\frac{1}{R^2}\) is integrable over \(\mathbb{R}^+_+\), we obtain
\[ |\partial_{v^i} b^k(t, v', v)| \leq C + C \int_0^t \left[ \frac{(u^0)^6}{R^2 v^0} + \frac{(u^0)^6}{R^2 v^0} \right] d\tau \leq C + C <u>^6. \]

The last estimate is done with the same method.

4 Existence theorem

Before we study the existence theorem, we collect a priori estimates.

4.1 Energy estimates

We define the weight function by \(\rho(x, v) = e^{(|v|^2 + |x \times v|^2)}\) and the norms:
\[ \|g(t)e\| = \sup_{(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3} (x, v)|g(t, x, v)|. \]
\[ ||g(t)||_e = \|g(t)e\| + \sum_{k=1}^3 \left( \|\partial_{v^k} g(t)e\| + \|\partial_{x^k} g(t)e\| \right). \]

Before we establish the energy estimates, we recall the following lemma.

Lemma 4.1 Consider the vector function \(b(t, u, v)\) defined in (2.23), then
\[ D = \left| (x + b(t, v', v)) \times v' \right|^2 + \left| (x + b(t, u', v)) \times u' \right|^2 \geq |\omega(x \times v)|^2. \]

Proof See [18] page 18.

Lemma 4.2 Let \(f^\#\) be a solution of the Boltzmann equation (2.21).
\[ \|f^\#(t)e\| \leq \|f(0)e\| + C \sup_{\tau \in [0, t]} \|f^\#(\tau)e\|^2. \]

for some constant \(C\) which does not depend on \(t\).
Proof We first give the following relations which are straightforward.

\[
\begin{align*}
    f(t, X^i(x, v), v') &= f^\#(t, x + b(t, v', v), v'), \\
    f(t, X^i(x, v), u') &= f^\#(t, x + b(t, u', v), u').
\end{align*}
\]  

We integrate (2.20) on \([0, t]\) and then we multiply the resulting equation by \(\rho(x, v)\) to obtain

\[
\rho(x, v) f^\#(t, x, v) = \rho(x, v) f_0(x, v) + \int_0^t \rho(x, v) Q^\#(f, f)(\tau, x, v)d\tau
\]

\[
\leq \sup_{(x,v)\in\mathbb{R}^3 \times \mathbb{R}^3} [\rho(x, v) f_0(x, v)] + \int_0^t \rho(x, v) Q^\#(f, f)(\tau, x, v)d\tau
\]

\[
\leq \|f(0)\|_e + \int_0^t \frac{1}{R^3} \int_{S_+^2 \times \mathbb{R}^3} \rho(x, v)v_\phi \sigma(g, \omega)
\]

\[
\times f(\tau, X^\tau(x, v), v') f(\tau, X^\tau(x, v), u')d\omega du.
\]  

where we ignored the loss term. Let consider the second term of the RHS of the previous inequality \(\int_0^t \frac{1}{R^3} A(\tau) d\tau\), where \(A(\tau)\) is defined for each \(\tau\) by

\[
A(\tau) = \int_{S_+^2 \times \mathbb{R}^3} \rho(x, v)v_\phi \sigma(g, \omega)
\]

\[
f^\#(\tau, x + b(\tau, v', v), v')
\]

\[
f^\#(\tau, x + b(\tau, u', v), u')d\omega du.
\]

Using the definition (4.3) of \(D\) and the assumption (2.25), we obtain for each \(\tau\)

\[
A(\tau) \leq \int_{S_+^2 \times \mathbb{R}^3} \frac{e^{-D} \rho(x, v)}{e^{(|v'|^2 + |u'|^2)}} v_\phi (1 + g^{-b})\sigma_0(\omega)
\]

\[
\times e^{(|v'|^2 + (x + b(\tau, v', v)) \times v')^2)} f^\#(\tau, x + b(\tau, v', v), v')
\]

\[
\times e^{(|u'|^2 + (x + b(\tau, u', v)) \times u')^2)} f^\#(\tau, x + b(\tau, u', v), u')d\omega du
\]

\[
:= I_1(\tau) + I_2(\tau).
\]  

Let us control \(I_1(\tau)\) and \(I_2(\tau)\). By (4.3), \(D \geq |\omega \cdot (x \times v)|^2\). From (2.27), we get

\[
\int_{S_+^2} e^{-D} \sigma_0(\omega) d\omega \lesssim \int_{S_+^2} \sigma_0(\omega)e^{-|\omega \cdot (x \times v)|^2} d\omega \lesssim e^{-|x \times v|^2}.
\]  

From (3.5), \(e^{|v'|^2 + |u'|^2} \lesssim e^{|v'|^2 + |u'|^2}\). Thus, using (4.8) and (3.6) we obtain
\[ I_1(\tau) \lesssim \int_{S_\tau^+ \times \mathbb{R}^3} e^{\Phi(\omega)} v_{\phi} \sigma_0(\omega) e^{\left( |v'|^2 + |(x+b(\tau,\nu,v) ) \times v'|^2 \right)} f^#(\tau, x+b(\tau, \nu, v), v') \]
\[ \times e^{\left( |v'|^2 + |(x+b(\tau,\nu,v) ) \times v'|^2 \right)} f^#(\tau, x+b(\tau, \nu, v), u') e^{-D} d\omega d\nu \]
\[ \lesssim \| f^#(\tau) \|_e^2 \int_{S_\tau^+ \times \mathbb{R}^3} e^{-|u|^2} e^{\Phi(x,v)} v_{\phi} \sigma_0(\omega) e^{-D} d\omega d\nu \]
\[ \lesssim \| f^#(\tau) \|_e^2 \int_{\mathbb{R}^3} e^{-|u|^2} v_{\phi} \|_e \lesssim \| f^#(\tau) \|_e^2. \]

In the last inequality we used (3.6).

We now move to estimate \( I_2(\tau) \). Using the same argument as for \( I_1(\tau) \) we get
\[ I_2(\tau) \lesssim \int_{S_\tau^+ \times \mathbb{R}^3} e^{\Phi(\omega)} v_{\phi} g^{-b} \sigma_0(\omega) e^{\left( |v'|^2 + |(x+b(\tau,\nu,v) ) \times v'|^2 \right)} f^#(\tau, x+b(\tau, \nu, v), v') \]
\[ \times e^{\left( |v'|^2 + |(x+b(\tau,\nu,v) ) \times v'|^2 \right)} f^#(\tau, x+b(\tau, \nu, v), u') e^{-D} d\omega d\nu \]
\[ \lesssim \| f^#(\tau) \|_e^2 \int_{S_\tau^+ \times \mathbb{R}^3} e^{-|u|^2} e^{\Phi(x,v)} v_{\phi} g^{-b} \sigma_0(\omega) e^{-D} d\omega d\nu \]
\[ \lesssim \| f^#(\tau) \|_e^2 \int_{\mathbb{R}^3} e^{-|u|^2} v_{\phi} g^{-b} e^{-|u|^2} d\nu \lesssim R^{b-1} \| f^#(\tau) \|_e^2 \]

The last inequality is obtained by using (3.7).

Finally, we collect the above estimates for \( I_1(\tau) \) and \( I_2(\tau) \) to get, for each \( \tau \)
\[ A(\tau) \lesssim \left( 1 + R^{b-1} \right) \| f^#(\tau) \|_e^2 \lesssim \left( 1 + R^{b-1} \right) \sup_{\tau \in [0,t]} \| f^#(\tau) \|_e^2 \] \[ (4.9) \]

Next, we insert (4.9) into (4.6) taking the supremum of the LHS for any fixed \( t \), to obtain
\[ \| f^#(t) \|_e \lesssim \| f(0) \|_e + \sup_{\tau \in [0,t]} \| f^#(\tau) \|_e^2 \int_0^t \left( \frac{1}{R^3(\tau)} + \frac{1}{R^{4-b}(\tau)} \right) d\tau. \] \[ (4.10) \]

The desired result follows from the integrability of \( \frac{1}{R^3(\tau)} + \frac{1}{R^{4-b}(\tau)} \) over \( \mathbb{R}_+ \). In fact the integrability of \( \frac{1}{R^3(\tau)} + \frac{1}{R^{4-b}(\tau)} \) implies that of \( \frac{1}{R^3(\tau)} + \frac{1}{R^{4-b}(\tau)} \).

\( \square \)
Lemma 4.3 Let $f^\#$ be a solution of the Boltzmann equation (2.21). The following energy estimate for $\partial_{v^i} f^\#$ holds for a fixed index $i \in \{1, 2, 3\}$.

$$
\left\| \partial_{v^i} f^\#(t) \right\|_E \leq \left\| \partial_{v^i} f(0) \right\|_E + C \sup_{\tau \in [0, t]} \left[ \left\| f^\#(\tau) \right\|_E \left( \left\| f^\#(\tau) \right\|_E + \sum_{k=1}^{3} \left( \left\| \partial_{v^k} f^\#(\tau) \right\|_E + \left\| \partial_{x^k} f^\#(\tau) \right\|_E \right) \right) \right].
$$

(4.11)

for some constant $C$ which does not depend on $t$.

**Proof** For $i \in \{1, 2, 3\}$. We take $\partial_{v^i}$ to the Boltzmann equation (2.20) to have

$$
\frac{d}{dt} (\partial_{v^i} f^\#(t, x, v)) = \partial_{v^i} Q^\#(f, f).
$$

(4.12)

We integrate on $[0, t]$ and then multiply the resulting equation by $\rho(x, v)$ to obtain

$$
\rho(x, v) \partial_{v^i} f^\#(t, x, v) = \rho(x, v) \partial_{v^i} f(0, x, v) + \int_0^t \rho(x, v) \partial_{v^i} Q^\#(f, f)(\tau, x, v) d\tau.
$$

For a fixed $(t, x, v)$, the above relation yields

$$
\rho(x, v) |\partial_{v^i} f^\#(t, x, v)| \leq \left\| \partial_{v^i} f(0) \right\|_E + \int_0^t \frac{d\tau}{R^3(\tau)} (J_1(\tau) + J_2(\tau) + J_3(\tau) + J_4(\tau))
$$

(4.13)

where $J_1(\tau)$, $J_2(\tau)$, $J_3(\tau)$, and $J_4(\tau)$ are defined for each $\tau$ by

$$
J_1(\tau) = \int \int_{S^2_1 \times \mathbb{R}^3} \rho(x, v) \partial_{v^i} [v \phi \sigma(g, \omega)] f^\#(\tau, x, v) f^\#(\tau, x + b(\tau, v, u), u) d\omega du;
$$

$$
J_2(\tau) = \int \int_{S^2_1 \times \mathbb{R}^3} \rho(x, v) \partial_{v^i} [v \phi \sigma(g, \omega)] f^\#(\tau, x + b(\tau, v, u), v') f^\#(\tau, x + b(\tau, v', u), u') d\omega du;
$$

$$
J_3(\tau) = \int \int_{S^2_1 \times \mathbb{R}^3} \rho(x, v) v \phi \sigma(g, \omega) \partial_{v^i} [f^\#(\tau, x, v) f^\#(\tau, x + b(\tau, v, u), u)] d\omega du;
$$

$$
J_4(\tau) = \int \int_{S^2_1 \times \mathbb{R}^3} \rho(x, v) v \phi \sigma(g, \omega) \partial_{v^i} [f^\#(\tau, x + b(\tau, v, u), v') f^\#(\tau, x + b(\tau, v', u'), u') d\omega du.
$$
Now we are going to control carefully $J_1(\tau)$, $J_2(\tau)$, $J_3(\tau)$, $J_4(\tau)$ and
\[ \Sigma(\tau) = \sum_{k=1}^{4} J_k(\tau) \] for each $\tau$.

**Estimate of $J_1(\tau)$:** Using (3.13) and the fact that the integration with respect to $\omega$ is finite leads to
\[
J_1(\tau) \lesssim \frac{1}{R} \int_{\mathbb{R}^3} |\rho(x, v)u^0(1 + g^{-b}) f^#(\tau, x, v) f^#(\tau, x + b(\tau, v, u), u)| du
\]
\[
\lesssim \frac{1}{R} \|f^#(\tau)\|_e \int_{\mathbb{R}^3} u^0(1 + g^{-b}) f^#(\tau, x + b(\tau, v, u), u) du. \tag{4.14}
\]

We are now going to estimate $\xi(\tau) = \int_{\mathbb{R}^3} u^0(1 + g^{-b}) f^#(\tau, x + b(\tau, v, u), u) du$.
\[
\xi(\tau) = \int_{\mathbb{R}^3} u^0 e^{[u^2 + |(x + b(\tau, v, u)) \times u|^2]} (1 + g^{-b}) f^#(\tau, x + b(\tau, v, u), u) du
\]
\[
\lesssim \|f^#(\tau)\|_e \int_{\mathbb{R}^3} u^0 e^{-|u|^2} du
\]
\[
\lesssim \|f^#(\tau)\|_e \left( \int_{\mathbb{R}^3} u^0 e^{-|u|^2} du + \int_{\mathbb{R}^3} u^0 g^{-b} e^{-|u|^2} du \right)
\]
\[
\lesssim (1 + R^b) \|f^#(\tau)\|_e. \tag{4.15}
\]

We used (3.9) to obtain the last line of the above inequality.
Next, we insert (4.15) into (4.14) to obtain using $R \geq 1$
\[
J_1(\tau) \lesssim \left( \frac{1}{R} + \frac{1}{R^{1-b}} \right) \|f^#(\tau)\|_e^2. \tag{4.16}
\]

**Estimate of $J_2(\tau)$:** Still using (3.13), we have
\[
J_2(\tau) \lesssim \frac{1}{R} \int_{S_2^+ \times \mathbb{R}^3} \int_{\mathbb{R}^3} |\rho(x, v)u^0 \sigma_0(\omega) f^#(\tau, x + b(\tau, v, v), v') f^#(\tau, x + b(\tau, v', v), u')| d\omega dv du
\]
\[
+ \frac{1}{R} \int_{S_2^+ \times \mathbb{R}^3} \int_{\mathbb{R}^3} |\rho(x, v)u^0 \sigma_0(\omega) g^{-b} f^#(\tau, x + b(\tau, v, v), v') f^#(\tau, x + b(\tau, v', v), u')| d\omega dv du.
\]

The two terms in the RHS of the above inequalities look like the expression $A(\tau)$ defined by (4.7). The difference is that the term $v_{\phi}(1 + g^{-b})\sigma_0(\omega)$ of $A(\tau)$ is replaced by $u_0^R (1 + g^{-b})\sigma_0(\omega)$ of $J_2(\tau)$. So, the same method used to establish the energy estimate (4.4) allows us to control $J_2(\tau)$. Note that instead of Lemma 3.4, we use Lemma 3.5. This leads to
\[ J_2(\tau) \lesssim \left( \frac{1}{R} + \frac{1}{R^{1-b}} \right) \| f^\#(\tau) \|_e^2 \]  

(4.17)

**Estimate of** \( J_3(\tau) \): By elementary properties of derivation, we have

\[
\partial_{v^i} \left[ f^\#(\tau, x, v) f^\#(\tau, x + b(\tau, v, u), u) \right] = \partial_{v^i} f^\#(\tau, x, v) f^\#(\tau, x + b(\tau, v, u), u) \\
+ f^\#(\tau, x, v) \sum_{k=1}^3 \partial_{v^i} b^k(\tau, v, u) \partial_{x^k} f^\#(\tau, x + b(\tau, v, u), u)
\]

By (3.24), \( |\partial_{v^i} b^k(\tau, v, u)| \leq C \) for some constant \( C \). \( J_3(\tau) \) is easily controlled as

\[
J_3(\tau) \lesssim (1 + R^b) \| f^\#(\tau) \|_e \sum_{k=1}^3 (\| \partial_{x^k} f^\#(\tau) \|_e + \| \partial_{x^k} f^\#(\tau) \|_e)
\]  

(4.18)

**Estimate of** \( J_4(\tau) \): By elementary properties of derivation, we have

\[
\partial_{v^i} \left[ f^\#(\tau, x + b(\tau, v', v), v') f^\#(\tau, x + b(\tau, u', v), u') \right] \\
= f^\#(\tau, x + b(\tau, v', v), v') \\
\times \sum_{k=1}^3 \left( \partial_{v^i} b^k(\tau, v', v) \partial_{x^k} f^\#(\tau, x + b(\tau, u', v), u') + \partial_{v^i} u'^k \partial_{v^i} f^\#(\tau, x + b(\tau, u', v), u') \right) \\
+ f^\#(\tau, x + b(\tau, u', v), u') \\
\times \sum_{k=1}^3 \left( \partial_{v^i} b^k(\tau, v', v) \partial_{x^k} f^\#(\tau, x + b(\tau, v', v), v') + \partial_{v^i} u'^k \partial_{v^i} f^\#(\tau, x + b(\tau, v', v), v') \right)
\]

By (3.21), (3.22), (3.23), (3.25), and (3.26), we have

\[
|\partial_{v^i} \left[ f^\#(\tau, x + b(\tau, v', v), v') f^\#(\tau, x + b(\tau, u', v), u') \right]| \\
\lesssim < u >^6 \sum_{k=1}^3 \left| f^\#(\tau, x + b(\tau, v', v), v') \partial_{x^k} f^\#(\tau, x + b(\tau, u', v), u') \right| \\
+ < u >^6 \sum_{k=1}^3 \left| f^\#(\tau, x + b(\tau, u', v), v') \partial_{v^i} f^\#(\tau, x + b(\tau, u', v), u') \right| \\
+ < u >^6 \sum_{k=1}^3 \left| f^\#(\tau, x + b(\tau, u', v), u') \partial_{v^i} f^\#(\tau, x + b(\tau, v', v), v') \right| \\
+ < u >^6 \sum_{k=1}^3 \left| f^\#(\tau, x + b(\tau, v', v), u') \partial_{x^k} f^\#(\tau, x + b(\tau, v', v), v') \right|.
\]  

(4.19)
Using (4.19), $J_4(\tau)$ is controlled by using the same method as $A(\tau)$. We obtain

$$J_4(\tau) \lesssim (1 + R^b) \| f^\#(\tau) \|_e \sum_{k=1}^{3} \left( \| \partial_{x^k} f^\#(\tau) \|_e + \| \partial_{x^k} f^\#(\tau) \|_e \right). \quad (4.20)$$

Summing up the relations (4.16), (4.17), (4.18) and (4.20), we obtain that

$$\Sigma(\tau) \lesssim (1 + R^b) \sup_{\tau \in [0, t]} \| f^\#(\tau) \|_e \left[ \| f^\#(\tau) \|_e + \sum_{k=1}^{3} \left( \| \partial_{x^k} f^\#(\tau) \|_e + \| \partial_{x^k} f^\#(\tau) \|_e \right) \right]. \quad (4.21)$$

Inserting (4.21) into (4.13) and taking the supremum of the LHS for a fixed $t$ with respect to $(x, v)$ yields

$$\| \partial_{v^i} f^\#(t) \|_e \leq \| \partial_{v^i} f(0) \|_e + C \sup_{\tau \in [0, t]} B(\tau) \quad (4.22)$$

where $B(\tau)$ is given by

$$B(\tau) = \| f^\#(\tau) \|_e \left[ \| f^\#(\tau) \|_e + \sum_{k=1}^{3} \left( \| \partial_{x^k} f^\#(\tau) \|_e + \| \partial_{x^k} f^\#(\tau) \|_e \right) \right]. \quad (4.23)$$

The desired results is due to the fact that $\frac{1}{R^3} + \frac{1}{R^{3-\sigma}}$ is integrable over $[0, +\infty[$ and $R(\tau) \geq 1$. $\square$

**Lemma 4.4** Let $f^\#$ be a solution of the Boltzmann equation (2.21). Then for a fixed index $i \in \{1, 2, 3\}$ the following estimate holds

$$\| \partial_{x^i} f^\#(t) \|_e \leq \| \partial_{x^i} f(0) \|_e + C \sup_{\tau \in [0, t]} \| f^\#(\tau) \|_e \| \partial_{x^i} f^\#(\tau) \|_e. \quad (4.24)$$

for some constant $C$ which does not depend on $t$.

**Proof** We take $\partial_{x^i}$ to (2.20) to have

$$\frac{d}{dt} \left( \partial_{x^i} f^\#(t, x, v) \right) = \partial_{x^i} Q^\#(f, f) \quad (4.25)$$

We integrate (4.25) on $[0, t]$ and then multiply the resulting equation by $\rho(x, v)$ to obtain

$$\rho(x, v) \partial_{x^i} f^\#(t, x, v) = \rho(x, v) \partial_{x^i} f(0, x, v) + \int_0^t \rho(x, v) \partial_{x^i} Q^\#(f, f)(\tau, x, v) d\tau.$$
This allows us to have
\[
\rho(x, v) |\partial_{x_i} f^#(t, x, v)| \leq \|\partial_{x_i} f(0)\|_e + \int_0^t \frac{1}{R^3(\tau)} (K_1(\tau) + K_2(\tau)) d\tau
\]  
(4.26)

where for each \( \tau \), \( K_1(\tau) \) and \( K_2(\tau) \) are defined by
\[
K_1(\tau) = \iint_{S^2_+ \times \mathbb{R}^3} \rho(x, v) v_\phi |\sigma(g, \omega) \partial_{x_i} [f^#(\tau, x, v) f^#(\tau, x + b(\tau, v, u), u)]| d\omega du;
\]
\[
K_2(\tau) = \iint_{S^2_+ \times \mathbb{R}^3} \rho(x, v) v_\phi |\sigma(g, \omega) \partial_{x_i} [f^#(\tau, x + b(\tau, v', v), v') f^#(\tau, x + b(\tau, u', v), u')]| d\omega du.
\]

Next, we control \( K_1(\tau) \) and \( K_2(\tau) \).

**Estimate of** \( K_1(\tau) \): By the elementary property, we have
\[
\partial_{x_i} [f^#(\tau, x, v) f^#(\tau, x + b(\tau, v, u), u)] = \partial_{x_i} f^#(\tau, x, v) f^#(\tau, x + b(\tau, v, u), u)
\]
\[
+ f^#(\tau, x, v) \partial_{x_i} f^#(\tau, x + b(\tau, v, u), u).
\]

Thus, the same method used to control the above term \( J_3(\tau) \) allows us to obtain
\[
K_1(\tau) \lesssim (1 + R^3) \| f^#(\tau) \|_e \| \partial_{x_i} f^#(\tau) \|_e.
\]  
(4.27)

**Estimate of** \( K_2(\tau) \): By the elementary property, we have
\[
\partial_{x_i} [f^#(\tau, x + b(\tau, v', v), v') f^#(\tau, x + b(\tau, u', v), u')]
\]
\[
= \partial_{x_i} f^#(\tau, x + b(\tau, v', v), v') f^#(\tau, x + b(\tau, u', v), u')
\]
\[
+ f^#(\tau, x + b(\tau, v', v), v') \partial_{x_i} f^#(\tau, x + b(\tau, u', v), u').
\]

Thus, the same method used to control the above term \( J_4(\tau) \) allows us to obtain
\[
K_2(\tau) \lesssim (1 + R^3) \| f^#(\tau) \|_e \| \partial_{x_i} f^#(\tau) \|_e.
\]  
(4.28)

Finally, we insert (4.27) and (4.28) into (4.26) to obtain the desired result by using once again the integration of \( \frac{1}{R^3} + \frac{1}{R^{1-\beta}} \) over \([0, +\infty[^\).  

\[ \square \]

### 4.2 Global in time existence theorem

**Lemma 4.5** If \( f \) is a local-in-time solution of the Boltzmann equation (2.15) with initial data \( f_0 \), then \( f \) is extended to a global-in-time solution, if initial data is given such that \( \| f(0) \|_e \) is sufficiently small.

**Proof** Using the energy estimates (4.4), (4.11) and (4.24), if \( f \) is a local-in-time solution of (2.15) with initial data \( f_0 \), on a (short) time interval we have
\[
\| f^#(t) \|_e \leq \| f(0) \|_e + C \sup_{\tau \in [0, t]} \| f^#(\tau) \|_e^2.
\]  
(4.29)
Since the norm $||f||_e$ contains first order derivatives with respect to $x$ and $v$ variables, (4.29) allows to bound all the derivatives of the local solution on each short time interval when the initial data is sufficiently small. In fact, if $[0, T]$ is the maximal interval of the local solution, by (4.29), we have

$$\sup_{\tau \in [0, T]} ||f^#(\tau)||_e \leq ||f(0)||_e + C \sup_{\tau \in [0, T]} ||f^#(\tau)||^2_e$$  \hspace{1cm} (4.30)$$

The relation (4.30) occurs if $1 - 4C ||f(0)||_e \geq 0$, that is with initial enjoying which enjoy the lillleness condition $||f(0)||_e \leq \frac{1}{4C}$. This proves that the solution is extended to a global-in-time solution, if initial data is given such that $||f(0)||_e$ is sufficiently small. \hfill \Box

Now, we turn to the construction of a unique local-in-time solution of the Boltzmann equation.

**Theorem 4.1** Consider a Robertson–Walker space-time where the metric tensor is such that $R = R(t)$ is given and satisfies assumptions (2.28) and (2.29). Let $f_0 = f(0, x, v)$ be the initial data of the Boltzmann equation (2.20), that is differentiable. Suppose that the scattering kernel satisfies (2.25), (2.26) and (2.27). Then there exists $M_0 > 0$ such that if $||f(0)||_e < \frac{M_0}{2}$, there exists a unique global solution to the corresponding equation. Moreover

$$\sup_{t \in [0, +\infty]} ||f^#(t)||_e \leq M_0.$$  \hspace{1cm} (4.31)$$

**Proof** Due to lemma 4.5, it suffices to prove a unique local existence theorem. The rest of the proof will be divided into two steps.

**First step: Local existence theorem.**

Let $f_0$ be the initial data for the Boltzmann equation (2.15). We define recursively the following sequence $(f^#_n)_{n \geq 0}$ by:

$$\partial_t f^#_{n+1} = Q_H(f^#_n, f^#_n) - Q_l(f^#_{n+1}, f^#_n).$$  \hspace{1cm} (4.32)$$

$$f^#_{n+1}(0, x, v) = f(0, x, v) \text{ and } f^#_0(t, x, v) = f(0, x, v).$$  \hspace{1cm} (4.33)$$

We note that for a given $f^#_n$, (4.32) is a linear partial differential equation with $f^#_{n+1}$ as unknown and initial data $f_0$. It is standard from the linear theory that the sequence $(f^#_n)_{n \geq 0}$ is locally well defined. Our main goal is to get an uniform in $n$ estimate for $||f^#_n(t)||_e$. Precisely, we look for some small $M_0$ such that

$$\forall n \in \mathbb{N}, ||f^#_n(t)||_e \leq M_0 \text{ on the local in time interval.}$$  \hspace{1cm} (4.34)$$

We are going to do it by induction. We multiply (4.32) by $\rho(x, v)$ and then integrate from 0 to $t$ to obtain
\[
\rho(x, v) f_{n+1}^#(t, x, v) = \rho(x, v) f_0(x, v) + \int_0^t \rho(x, v) (Q^#_g(f_n, f_n) - Q^#_l(f_{n+1}, f_n))(\tau, x, v) d\tau.
\]

The same argument as in Lemma 4.2 allows us to obtain

\[
\| f_{n+1}^#(t) \|_e \leq \| f_0(t) \|_e + C \sup_{\tau \in [0, t]} \left( \| f_{n+1}^#(\tau) \|_e \| f_n^#(\tau) \|_e + \| f_n^#(\tau) \|_e^2 \right).
\]

(4.36)

Next, we proceed to the estimate of the derivatives of \( f_{n+1}^# \) with respect to the momenta variable. Let \( i \in \{1, 2, 3\} \). We take \( \partial_{\psi_i} \)-derivative to (4.32) and multiply it by \( \rho(x, v) \). To the equation obtained, we integrate over \([0, t] \) to have

\[
\rho(x, v) \partial_{\psi_i} f_{n+1}^#(t, x, v) = \rho(x, v) (\partial_{\psi_i} f_0)(x, v) + \int_0^t \rho(x, v) \partial_{\psi_i} Q^#_g(f_n, f_n)(\tau, x, v) d\tau
\]

\[
- \int_0^t \rho(x, v) \partial_{\psi_i} Q^#_l(f_{n+1}, f_n)(\tau, x, v) d\tau.
\]

(4.37)

Following the proof of Lemma 4.3, we obtain the following estimate:

\[
\| \partial_{\psi_i} f_{n+1}^#(t) \|_e \leq \| \partial_{\psi_i} f_0 \|_e + C \sup_{\tau \in [0, t]} \left( \| f_{n+1}^#(\tau) \|_e \| f_n^#(\tau) \|_e + \| f_n^#(\tau) \|_e^2 \right).
\]

(4.38)

Next, we proceed to the estimate of the derivative of \( f_{n+1}^# \) with respect to the \( x \)-variable. Let \( i \in \{1, 2, 3\} \). We take \( \partial_x^i \)- to (4.32) and we multiply it by \( \rho(x, v) \). To the equation obtained, we take integration on \([0, t] \) to have

\[
\rho(x, v) \partial_x^i f_{n+1}^#(t, x, v) = \rho(x, v) \partial_x^i f_0(x, v) + \int_0^t \rho(x, v) \partial_x^i Q^#_g(f_n, f_n)(\tau, x, v) d\tau
\]

\[
- \int_0^t \rho(x, v) \partial_x^i Q^#_l(f_{n+1}, f_n)(\tau, x, v) d\tau.
\]

(4.39)

Following the proof of Lemma 4.4, we obtain the following estimate:

\[
\| \partial_x^i f_{n+1}^#(t) \|_e \leq \| \partial_x^i f_0 \|_e + C \sup_{\tau \in [0, t]} \left( \| f_{n+1}^#(\tau) \|_e \| f_n^#(\tau) \|_e + \| f_n^#(\tau) \|_e^2 \right).
\]

(4.40)

Summing up (4.38), (4.39) and (4.40) we obtain

\[
\| f_{n+1}^#(t) \|_e \leq \| f_0 \|_e + C \sup_{\tau \in [0, t]} \left( \| f_{n+1}^#(\tau) \|_e \| f_n^#(\tau) \|_e + \| f_n^#(\tau) \|_e^2 \right).
\]

(4.41)
Suppose now that there exists a positive $M_0$ such that $\|f_0\|_e \leq \frac{M_0}{2}$ and $\|f^\#_n(t)\|_e \leq M_0$ on the local-in-time interval $[0, T]$, then we obtain the desired result i.e. $\|f^\#_{n+1}(t)\|_e \leq M_0$ for $t \in [0, T]$, provided $M_0$ is sufficiently small; for example with $M_0$ such that $M_0 \leq \frac{1}{4C}$.

Finally, taking limit in (4.32) as $n$ goes to infinity, we have a local-in-time solution such that $\|f(t)\|_e \leq M_0$ on the local in time interval $[0, T]$. The previous lemma proves that if $\|f_0\|_e$ is sufficiently small, then the solution exists globally in time.

**Second step: Uniqueness.**
We now prove the uniqueness of the solution. We assume that there is another solution $h$ to (2.20) with the same initial data $f_0$ such that $\sup_{t \in [0, +\infty]} \|h^\#(t)\|_e \leq M_0$. The difference $f - h$ satisfies
\[
\partial_t(f^\# - h^\#) = Q^\#(f - h, f) + Q^\#(h, f - h).
\] (4.42)

We proceed as in the proof of the energy estimate. Since $f(0, x, v) = h(0, x, v)$,
\[
\|f^\#(t) - h^\#(t)\|_e \leq C \sup_{\tau \in [0, \infty]} \left( \|f^\#(\tau)\|_e + \|h^\#(\tau)\|_e \right) \|f^\#(\tau) - h^\#(\tau)\|_e
\leq 2CM_0 \sup_{\tau \in [0, \infty]} \|f^\#(\tau) - h^\#(\tau)\|_e.
\] (4.43)

Since $M_0 \leq \frac{1}{4C}$, taking the supremum in (4.43) on the interval $[0, \infty]$, we obtain
\[
\sup_{t \in [0, \infty]} \|f^\#(t) - h^\#(t)\|_e \leq \frac{1}{2} \sup_{\tau \in [0, \infty]} \|f^\#(\tau) - h^\#(\tau)\|_e.
\] So $f^\# = h^\#$ on $\mathbb{R}_+$. \qed

**5 Conclusion**
We have studied the inhomogeneous relativistic Boltzmann equation in the spatially flat Robertson–Walker space-time. We prove the global (with respect to the direction of time corresponding to the expansion of the universe) existence of classical solutions for small initial data in a suitable weighted space for some collisional kernel which falls in the class of hard potentials. Such kernels are closer to those which naturally arise in physical problems. Our result extends existing results such as the one of [18] for mild solutions and also that of [10] for the spatially homogeneous case.

The Boltzmann equation is usually coupled to the Einstein equations through the energy-momentum tensor, and the energy-momentum tensor of the Boltzmann equation has the same form with that of the Vlasov equation. Since existence results is known in the case of Einstein–Vlasov equation, one interesting open question is to know whether the result of this paper can be extended to the Einstein–Boltzmann system when the distribution function is no longer spatially homogeneous. Another challenge could be the study of the properties of solutions. We also hope that this work can be applied for other situations such as soft potentials.
References

1. Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., Yang, T.: The Boltzmann equation without angular cut-off in a whole space: I, Global existence for soft potential. J. Funct. Anal. 262(3), 915–1010 (2012)
2. Bancel, D.: Problème Cauchy pour l’équation de Boltzmann en relativité générale. Ann. Inst. H. Poincaré Sect. A, 263–284 (1973)
3. Bancel, D., Choquet-Bruhat, Y.: Existence, uniqueness and local stability for the Einstein–Maxwell–Boltzmann system. Commun. Math. Phys. 33, 83–96 (1973)
4. Bazow, D., Denicol, G.S., Heinz, U., Martinez, M., Noronha, J.: Nonlinear dynamics from the relativistic Boltzmann equation in the Friedmann–Lemaître–Robertson–Walker spacetime. Phys. Rev. D 94, 125006 (2016)
5. Denicol, G.S., Heinz, U., Martinez, M., Noronha, J., Strickland, M.: New exact solution of the relativistic Boltzmann equation and its hydrodynamic limit. Phys. Rev. Lett. 113(20), 202301 (2014)
6. Glassey, R.T.: Global solution to the Cauchy problem for the relativistic Boltzmann equation with near-vacuum data. Commun. Math. Phys. 264, 705–724 (2006)
7. Guo, Y.: The Vlasov–Poisson–Boltzmann system near vacuum. Commun. Math. Phys. 218, 293–313 (2001)
8. Guo, Y., Strain, R.M.: Momentum regularity and stability of the relativistic Vlasov–Maxwell–Boltzmann system. Commun. Math. Phys. 310(3), 649–673 (2012)
9. Heinz, U., Martinez, M.: Investigating the domain of validity of the Gubser solution to the Boltzmann equation. Nucl. Phys. A 943, 26–38 (2015)
10. Lee, H.: Asymptotic behaviour of the relativistic Boltzmann equation in the Robertson–Walker spacetime. J. Diff. Eq. 225(11), 4267–4288 (2013)
11. Lichnerowicz, A.: Theorie Relativiste de la Gravitation et de l’Electromagnetisme. Masson et Cie, Paris (1955)
12. Noutchegueme, N., Dongo, D., Takou, E.: Global existence of solutions for the relativistic Boltzmann equation with arbitrarily large initial data on a Bianchi type I space-time. Gen. Relativ. Gravit. 37(12), 2047–2062 (2005)
13. Noutchegueme, N., Takou, E.: Global existence of solutions for the Einstein–Boltzmann system with cosmological constant in the Robertson–Walker space-time for arbitrarily large initial data. Commun. Math. Sci. 4(2), 291–314 (2006)
14. Noutchegueme, N., Takou, E., Tchuengue, E.K.: The relativistic Boltzmann equation on Bianchi type I space time for hard potentials. Rep. Math. Phys. 80(1), 87–114 (2017)
15. Strain, R.M.: Asymptotic stability of the relativistic Boltzmann equation for the soft potentials. Commun. Math. Phys. 300(2), 529–597 (2010)
16. Strain, R.M.: Global Newtonian limit for the relativistic Boltzmann equation near vacuum. SIAM J. Math. Anal. 42(4), 1568–1601 (2010)
17. Takou, E.: Global properties of the solutions of the Einstein–Boltzmann system with cosmological constant in the Robertson–Walker space-time. Commun. Math. Sci. 7, 399–410 (2009)
18. Takou, E., Ciakie Ciakie, F.L.: Inhomogeneous relativistic Boltzmann equation near vacuum in the Robertson–Walker space-time. Ann. Inst. Fourier 67(3), 947–967 (2017)
19. Takou, E., Ciakie Ciakie, F.L.: The relativistic Boltzmann equation on a spherically symmetric gravitational field. Class. Quant. Grav. 34, 195006 (2017)