We present three groups of noncanonical quantum oscillators. The position and the momentum operators of each of the groups generate basic Lie superalgebras, namely $sl(1/3)$, $osp(1/6)$ and $osp(3/2)$. The $sl(1/3)$-oscillators have finite energy spectrum and finite-dimensions. The $osp(1/6)$-oscillators are related to the para-Bose statistics. The internal angular momentum $s$ of the $osp(3/2)$-oscillators takes no more than three (half)integer values. In a particular representation $s = 1/2$. 

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In 1950 Wigner published a paper entitled: "Do the equations of motion determine the quantum mechanical commutation relations?" [1]. The question to answer was whether for a (one dimensional) quantum system with a Hamiltonian

\[ H = \frac{p^2}{2m} + V(q) \]

one can derive the canonical commutation relations (CCR's)

\[ [p, q] = -i\hbar, \]

assuming that the Hamiltonian equations

\[ \dot{q} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial V}{\partial q} \]

and the Heisenberg equations (in the corresponding picture)

\[ \dot{q} = -\frac{i}{\hbar}[q, H], \quad \dot{p} = -\frac{i}{\hbar}[p, H] \]

hold. The point of Wigner was that (2) and (3) have a more immediate physical significance than (1). The inverse is known to be true [2]: from (1) and (2) [resp. (1) and (3)] one derives (3) [resp. (2)]. Therefore the question actually was whether one can generalize the concept of a quantum system in a logically consistent way. Considering as an example an one-dimensional harmonic oscillator \((m = \hbar = 1)\), \(H = \frac{1}{2}(q^2 + p^2)\), Wigner has shown that such a generalization is possible indeed and in fact he found a family of noncanonical solutions, labeled with an arbitrary nonnegative number \(E_0\), the energy of the ground state. In terms of the operators

\[ a^+ = \frac{1}{\sqrt{2}}(q - ip), \quad a^- = \frac{1}{\sqrt{2}}(q + ip) \]

the result of Wigner can be stated as follows: the Hamiltonian equations (2) are identical with the Heisenberg equation (3) for all (representations of the) operators \(a^\pm\), which satisfy the relations [3]

\[ \{a^\xi, a^\eta\} = (\epsilon - \xi)a^\eta + (\epsilon - \eta)a^\xi. \]

Here and throughout \(\xi, \eta, \epsilon = \pm \) or \(\pm 1\); \([x, y] = xy - yx, \{x, y\} = xy + yx\). The case \(E_0 = 1/2\) corresponds to the canonical case, i.e., only for this value of \(E_0\) \(a^\pm\) are ordinary Bose operators, \([a^-, a^+] = 1\).

Although the paper of Wigner has attracted some immediate attention [4], most of the investigations, following it, remained in the frame of the one-dimensional case [5] (see also [3] for other references). Certainly, one can generalize immediately the above ideas to any \(n\)-dimensional oscillator, and in particular to a 3-dimensional oscillator with a Hamiltonian
\[ H = \frac{(p_1)^2 + (p_2)^2 + (p_3)^2}{2m} + \frac{m\omega^2}{2} \sum_i (r_i)^2, \quad (6) \]

assuming simply that the coordinates and momenta corresponding to different degrees of freedom commute with each other, \([p_i, r_i], (p_j, r_j) = 0 \text{ for } i \neq j\). There exist however also other, nontrivial generalizations. One such 3-dimensional oscillator with quite unconventional properties was studied by one of us (T. D. P.) in Ref. 6 (see also below). In the present note we shall give an example of another noncanonical 3-dimensional Wigner oscillator, which has an interesting physical property: the spin of the oscillator is 1/2.

The oscillators considered in [6,7] and the one we are going to study here are particular cases of what we call Wigner quantum oscillators (and, more generally, Wigner quantum system). The oscillator is said to be a Wigner quantum oscillator if the following conditions are fulfilled.

1. The state space \( W \) is a Hilbert space. The physical observables are Hermitian operators in \( W \).
2. The Hamiltonian equations and the Heisenberg equations are identical (as operator equations) in \( W \).
3. The internal angular momentum (the spin) of the oscillator \( s = (s_1, s_2, s_3) \) is a linear function of the position operators \( r = (r_1, r_2, r_3) \) and the momentum \( p = (p_1, p_2, p_3) \), so that \( s, r \) and \( p \) transform as vectors: \([s_j, c_k] = i \sum_{\ell=1}^3 \epsilon_{jkl} c_\ell, \quad c_k = s_k, r_k, p_k, \quad i, j, k = 1, 2, 3\).
4. The spectrum of \( H \) is bounded from below.

The underlying mathematical structure of the oscillators, which we consider, is one and the same. It is related to the representation theory of some basic Lie superalgebras [8]. As we shall see, this is also the case for the canonical 3-dimensional oscillator. In order to outline the link with the Lie superalgebras (see also [6]), introduce in place of the unknown \( p \) and \( r \) new unknown operators

\[ a_k^\pm = \sqrt{\frac{m\omega}{2\hbar}} r_k \mp \frac{i}{\sqrt{2m\omega\hbar}} p_k, \quad k = 1, 2, 3. \quad (7) \]

In terms of \( a_k^\pm \), which we call creation and annihilation operators (CAO’s), the Hamiltonian (6) reads:

\[ H = \frac{1}{2} \omega \hbar \sum_{k=1}^3 \{a_k^+, a_k^-\}. \quad (8) \]

The condition 2 yields \((k = 1, 2, 3)\):

\[ \sum_{i=1}^3 \{a_i^+, a_i^-\}, a_k^\pm \} = \pm 2a_k^\pm. \quad (9) \]

The equations (9) are a unique consequence from the Hamiltonian equations

\[ \dot{p} = -m\omega^2 r, \quad \dot{r} = \frac{p}{m} \quad (10) \]

and the Heisenberg equations.
\[ \dot{p} = -\frac{i}{\hbar}[p, H], \quad \dot{r} = -\frac{i}{\hbar}[r, H], \]

independently of the properties of the unknown CAO’s \( a_{k}^{\pm} \). They are equal time relations, the time dependence being \( a_{k}^{\pm}(t) = e^{i\varepsilon t}a_{k}^{\pm}(0), \varepsilon = \pm \). Hence eqs. (9) hold, if they are fulfilled at, say, \( t = 0 \).

In order to be slightly more general, let us denote by \( F(n) \) the associative algebra with unity, generators \( a_{1}^{\pm}, \ldots, a_{n}^{\pm} \) and relations (9). Then any representation of \( F(3) \) is a candidate for a Wigner oscillator, or, more precisely, the CAO’s of any Wigner oscillator give a representation of \( F(3) \). In such a case the representation space of \( F(3) \) (= the corresponding \( F(3) \)-module) is a state space of the oscillator. For definiteness we call the algebra \( F(n) \) an \((n\text{-dimensional})\) free oscillator algebra. Thus, as a first step, one has to find the representations of \( F(3) \) and then select those of them, for which also conditions 1, 3, and 4 hold. It turns out this is not an easy problem and, in fact, it is unsolved so far. Here we list three classes of solutions.

**Class 1 solutions: \( osp(1/6) \) oscillators.**

Let \( F_{1}(n) \) be the (free unital) associative superalgebra with odd generators \( a_{1}^{\pm}, \ldots, a_{n}^{\pm} \) and relations

\[
\{[a_{i}^{\xi}, a_{j}^{\eta}], a_{k}^{\zeta}\} = \delta_{ik}(\varepsilon - \xi)a_{j}^{\eta} + \delta_{jk}(\varepsilon - \eta)a_{i}^{\xi}, \quad i, j, k = 1, \ldots, n, \varepsilon, \eta, \xi = \pm \text{ or } \pm 1.
\]

The operators (12) satisfy eqs. (9) and therefore \( F_{1}(n) \) is a factor algebra of \( F(n) \). Consequently any representation of \( F_{1}(n) \) is a representation of \( F(n) \). Observe that the solutions of Wigner belong to this class \((n = 1)\). The canonical solution, namely the one with CAO’s being Bose operators is also from this class. It is easily verified that the operators (12) are para-Bose (pB) operators [9]. Their main algebraic property stems from the observation that the subspace

\[ B_{1} = \text{lin.env.}\{a_{i}^{\xi}, a_{j}^{\eta}, a_{k}^{\zeta}\} \quad | i, j, k = 1, \ldots, n, \varepsilon, \eta, \xi = \pm \} \subset F_{1}(n) \]

is a Lie superalgebra [10] with odd generators the pB operators. This algebra is isomorphic to one of the basic Lie superalgebras in the classification of Kac [8], namely to the orthosymplectic LS \( osp(1/2n) \equiv B(0/n) \), whereas \( F_{1}(n) \) is its universal enveloping algebra [11]. As a result the representation theory of any \( n \) pairs of pB-operators is completely equivalent to the representation theory of the LS \( osp(1/2n) \). It is another question that for physical reasons one has to select a subclass of representations, which in the case \( n = 3 \) should satisfy the conditions 1-4. Unfortunately not much is known about the representations of \( osp(1/6) \) and, more generally, about \( osp(1/2n) \) (apart from the full classification of the finite-dimensional modules [8]). The only technique to construct new representations from the Fock representation was developed by Green [9] through the Green ansatz [12]. It leads however to reducible representations and is realized in tensor products of Fock spaces. The representation with statistics of order \( p \) corresponds to the irreducible representation of \( osp(1/2n) \), containing the highest weight vector (which is the vacuum) in the tensor product of \( p \) copies of Fock spaces (considered as \( osp(1/2n) \)-modules). There exists however no effective methods.

\( 4 \)
to extract this representation from the reducible tensor product representation. This may be the reason why the para-Bose oscillator of dimension higher than one was not considered so far. The important for us conclusion is that there exist solutions of the free oscillator algebra \( F(n) \) with operators, which generate the basic Lie superalgebras \( osp(1/2n) \), namely a LS from the class \( B \) in the classification of Cartan-Kac \[8\]. This naturally leads to the idea to try to find solutions of eqs. (9) with representations of other LS’s from the same class \( B \) or from the other classes of basic Lie superalgebras.

**Class 2 solutions: \( sl(1/3) \) oscillators \[6\].**

Let \( F_2(n) \) be the associative superalgebra with generators \( a_1^\pm, \ldots, a_n^\pm \) and relations

\[
\begin{align*}
\{ [a_i^+, a_j^-], a_k^- \} &= \delta_{ik} a_i^+ - \delta_{ij} a_k^+, \\
\{ [a_i^+, a_j^-], a_k^+ \} &= -\delta_{ik} a_j^- + \delta_{ij} a_k^+, \\
\{ a_i^+, a_j^- \} &= \{ a_i^-, a_j^- \} = 0.
\end{align*}
\]

These operators also satisfy eqs. (10) and therefore \( F_2(n) \) is another factor algebra of the free oscillator algebra \( F(n) \). The subspace

\[
A = \text{lin.env.} \{ a_i^+, \{ a_j^+, a_k^- \} | i, j, k = 1, \ldots, n \} \subset F_2(n)
\]

is a Lie superalgebra with odd generators \( a_1^\pm, \ldots, a_n^\pm \), which is isomorphic to the Lie superalgebra \( sl(1/3) \equiv A(0, n - 1) \) from the class \( A \) of the basic Lie superalgebras. \( F_2(n) \) is its universal enveloping algebra. Hence any representation of \( sl(1/3) \) gives a solution of eqs. (9). The condition \( 1^0 \) restricts the class of representations to only the finite-dimensional representations of \( sl(1/3) \), which are explicitly known \[13\]. The internal angular momentum (condition \( 3^0 \)) is \( s_i = -\sum_{k=1}^3 \epsilon_{ikt} \{ a_k^+, a_l^- \} \). A class of state spaces, labeled with any nonnegative integer \( p \) was studied in \[6\]. The corresponding oscillator, one can call it \( sl(1/3) \)-oscillator, is very unconventional. We mention some of its properties. The spectrum of the Hamiltonian is finite; it has no more than 4 different eigenvalues. The square distance operator \( r^2 = (r_1)^2 + (r_2)^2 + (r_3)^2 \) is an integral of motion. Its maximal eigenvalue is \( (r_{\text{max}})^2 = \frac{3p\hbar}{2m\omega} \). Therefore the oscillator is confined in the space. It resembles in this respect a wavelet (see \[14\] and the references therein). The spin of the oscillator is either 0 or 1. Finally, the coordinates \( r_1, r_2, r_3 \) do not commute with each other, so that the position of the oscillating particle cannot be localized. The particle is smeared with a certain probability along a sphere with a fixed radius.

**Class 3 solutions: \( osp(3/2) \) oscillators.**

Another new class of solutions of the compatibility equations (9), i.e. of condition \( 2^0 \), is given with the set of all possible representations of operators \( a_1^\pm, a_1^\pm, a_3^\pm \), which satisfy the following relations (\( \epsilon = \pm \) or \( \pm 1 \), \( i, j, k = 1, 2, 3 \)):
\[
\{a_+^i, a_-^j]\} = \frac{3}{4} \delta_{ik} a_+^j - \frac{2}{3} \delta_{jk} a_+^i + \frac{2}{3} \delta_{ij} e a_k^+, \\
\{a_+^i, a_-^j, a_k^\epsilon\} = -\frac{4}{9} e a_k^+, \\
\{a_+, a_-\} = 0, \\
\{a_+^i, a_-^j\} = 0, \ i \neq j, \\
\{a_+^i, a_-^j\} = -\{a_-^j, a_+^i\}, \ i \neq j, \\
\{a_+^1, a_-^1\} = \{a_+^2, a_-^2\} = \{a_+^3, a_-^3\}, \\
\{a_1^+, a_1^-\} = -\{a_2^+, a_2^-\} = \{a_3^+, a_3^-\}, \\
(a_1^+)^2 = (a_2^+)^2 = (a_3^+)^2. \\
\]

We denote by \(F_3(3)\) the infinite-dimensional associative superalgebra with generators \(a_+^i, a_-^i, a_3^\pm\) and relations (16). The grading on \(F_3(3)\) is induced from the requirement that the CAO’s are odd generators.

Consider the subspace \(B_3 = \text{lin.env.}\{a_\xi^i, \{a_\eta^j, a_\epsilon^k\}| i, j, k = 1, \ldots, n \ \xi, \eta, \epsilon = \pm\} \subset F_3(3)\) and turn it into a Lie superalgebra with the natural for any associative superalgebra supercommutator, namely \([a, b] = ab - (-1)^{\alpha\beta} ba\), where \(\alpha = \text{deg}(a)\), \(\beta = \text{deg}(b)\). Elsewhere we shall show that \(B_3\) is isomorphic to the orthosymplectic Lie superalgebra \(osp(3/2)\) and that \(F_3(3)\) is its universal enveloping algebra. Therefore we call this oscillator an \(osp(3/2)\) oscillator. The angular momentum, satisfying condition \(3^0\) reads:

\[
s_j = \frac{3i}{4} \sum_{k,l=1}^3 \epsilon_{jkl}\{a_-^l, a_+^k\}, \ \ j = 1, 2, 3. \\
\]

The \(osp(3/2)\) modules (=representation spaces) for which also the conditions \(1^0\), \(4^0\) hold are infinite-dimensional. They are labelled with all possible pairs \((p, q)\), where \(p\) is an arbitrary nonnegative half-integer, and \(q\) is any negative real number, such that \(p + 2q \leq 0\). All representation spaces \(W(p, q)\) (among others) have been described in [15]. The energy of the oscillator depends only on the value of \(q\) and is

\[
E_n = \omega \hbar (n - 2q), \ \ n = 0, 1, 2, \ldots. \\
\]

Depending on the representation, the ground energy can be arbitrarily close to zero (for \(p = 0\) and very small negative \(q\)), but never zero. The spin \(s\) depends mainly on \(p\) and takes at most three different values. More precisely, the spin content within each state space \(W(p, q)\) reads:

1. \(p = 0\) \(s = 0, 1\);
2. \(p = 1/2, \ p + 2q = 0\) \(s = 1/2\);
3. \( p = 1/2, \ p + 2q < 0 \quad s = 1/2, \ 3/2; \)
4. \( p = 1, \ p + 2q = 0 \quad s = p - 1, \ p; \)
5. \( p \geq 1, \ p + 2q < 0 \quad s = p - 1, \ p, \ p + 1. \)

The derivation of the above results, together with the multiplicities of the states will be given elsewhere. Here we consider explicitly the most simple and, may be, the most interesting representation, the one corresponding to the spin 1/2 (Case 2.). An orthonormed basis in this state space \( W(1/2, -1/4) \equiv W(1/2) \) is given with the set of all vectors \( |n, s_3\rangle \), where \( n = 0, 1, 2, \ldots \) is labelling the energy of the state and \( s_3 = \pm 1/2 \) is the value of the third projection of the spin.

The transformations of the basis states under the action of the CAO’s reads:

\[
\begin{align*}
  a_{-1}^\dagger |n, s_3\rangle &= \frac{2}{\sqrt{3}} (1)^n s_3 \sqrt{n} |n - 1, -s_3\rangle, \\
  a_{+1}^\dagger |n, s_3\rangle &= \frac{2}{\sqrt{3}} (1)^n s_3 \sqrt{n + 1} |n + 1, -s_3\rangle, \\
  a_{-2}^\dagger |n, s_3\rangle &= \frac{1}{\sqrt{3}} \sqrt{n} |n - 1, s_3\rangle, \\
  a_{+2}^\dagger |n, s_3\rangle &= \frac{1}{\sqrt{3}} \sqrt{n + 1} |n + 1, s_3\rangle, \\
  a_{-3}^\dagger |n, s_3\rangle &= \frac{1}{\sqrt{3}} \sqrt{n} |n - 1, s_3\rangle, \\
  a_{+3}^\dagger |n, s_3\rangle &= \frac{1}{\sqrt{3}} \sqrt{n + 1} |n + 1, s_3\rangle.
\end{align*}
\]

(20)

From (8) and (20) one derives \( H |n, s_3\rangle = \omega \hbar (n + 1/2) |n, s_3\rangle \). Thus, the energy spectrum of the oscillator in this particular representation is the same as for an one-dimensional harmonic oscillator:

\[ E_n = \omega \hbar (n + 1/2), \quad n = 0, 1, 2, \ldots \]

(21)

The eigensubspace \( W_n(1/2) \) of the Hamiltonian with energy \( E_n \) is spanned on \( |n, s_3\rangle \), \( s_3 = \pm 1/2 \) and it is closed under the action of the spin operators. It carries a two-dimensional irreducible representation of the spin \( su(2) \) algebra with generators \( s_1, s_2, s_3 \). The state space \( W(1/2) \) is an infinite direct sum of spin 1/2 modules,

\[ W(1/2) = \bigoplus_{n=0}^{\infty} W_n(1/2). \]

(22)

Clearly this particular \( osp(3/2) \) oscillator is very different from the canonical 3-dimensional oscillator. The next table demonstrates this. By \( W_n, \ n = 0, 1, 2, \ldots \) we denote the eigensubspace of the Hamiltonian with energy \( E_n \).

| Canonical oscillator | \( Osp(3/2) \) oscillator |
|----------------------|-----------------------------|
| Energy \( E_n = \omega \hbar (n + 3/2) \) | \( E_n = \omega \hbar (n + 1/2) \) |
| Spin content of \( W_n \) | \( s = n, n - 2, n - 4, \ldots, 1 \) (or 0) | \( s = 1/2 \) |

(23)

The purpose of the present note was to show on simple examples that the ideas of Wigner to study more general quantum systems, the Wigner quantum systems in our terminology, are very rich in their origin.
If, for instance, one considers a noncanonical two particle system with internal variables, which have the properties of an $sl(1/3)$ oscillator [6], then the two particle system has finite space dimensions, it behaves like a system of two (nonrelativistic) quarks, confined in the space. The $osp(3/2)$-oscillator, viewed in the same way, gives a model of a spin 1/2 system, which has a classical analog: two noncanonical point particles are curling around each other and the resulting angular momentum of the composite system is 1/2. Hence this is a model of a spin (among several others; see [16] and the references therein).

One may think that the freedom in constructing such more general quantum systems is very large. As far as the 3-dimensional oscillator are concerned, we may say that the oscillators considered here exhaust all Wigner oscillators, for which the position and the momentum operators generate simple Lie superalgebras [17]. If one goes beyond the harmonic potentials, it is an open question to find those interactions, for which (noncanonical) Wigner quantum systems exists.

Elsewhere we shall show that the ideas of Wigner can be extended for any number of particles. In particular for oscillator-like interactions between the constituents one finds solutions, for which the composite system has finite dimensions and therefore behaves very much like a nucleus.

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