On the instability of classical dynamics in theories with higher derivatives

V. V. Nesterenko

Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

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Abstract

The development of instability in the dynamics of theories with higher derivatives is traced in detail in the framework of the Pais-Uhlenbeck fourth order oscillator. For this aim the external friction force is introduced in the model and the relevant solutions to equations of motion are investigated. As a result, the physical implication of the energy unboundness from below in theories under consideration is revealed.

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I. INTRODUCTION

All fundamental physical theories are defined by differential equations at most of the second order. However, in the course of searching for new theories the models described by higher-oder derivative Lagrangians are considered too \[1, 2\]. It is worth mentioning here the gauge theories with higher derivatives \[3, 4\], the gravitation models with higher oder curvature corrections to the Einstein-Hilbert action \[5, 6, 7, 8, 9, 10, 11, 12\], the models of point particles with Lagrangians depending on curvature and torsion of the world trajectories \[13, 14\], rigid strings \[15\]. The higher derivative theories have some appealing properties, in particular, the convergence of the Feynman diagrams is improved \[16\].

An important peculiarity of higher derivative theories is the energy unboundness from below already at the classical level. Here the energy is defined as the conserved Noether quantity corresponding to the invariance of the theory under time translation or, that is the same, as the value of the Ostrogradski Hamiltonian on the solutions to the equations of motion \[17, 18\]. It is agreed-upon that the bottomless energy spectrum in theories with higher derivatives results in their instability. However the development of this instability was not traced in detail in literature yet.

If the theory with higher derivatives is conservative, then the negative value of its energy, which is preserved in time, does not formally lead to any contradictions because it is unobservable simply. In experiment we can detect only the energy changes. However when such a system experiences an external action by dissipative forces, the following situation is feasible. The system will accomplish a positive work against the external friction forces and, at the same time, the amplitude of internal motion, which gives a negative contribution to the energy, will not damp, as usual, but it will increase. Remarkably, the energy conservation law will formally hold during this process. Such a behaviour of a material system is obviously absurd from the physical standpoint, and the pertinent theory should be discarded without any references on the instability. Nevertheless it is instructive to trace the development of this instability.

Since the energy in theories with higher derivatives is not restricted from below, there are no reasons to stop the increase of the amplitude of internal motion mentioned above. Hence a weak external action can lead to a drastical change of the internal dynamics of the system under consideration. It is this behaviour of the dynamical system described by
higher-order Lagrangian function that should be kept in mind when saying about instability of such systems.

It is worth noting that at the quantum level this feature of the theories with higher derivatives is manifested usually through the appearance of the states with negative norm \[1, 12, 17, 19\] and, as a consequence, in violation of the unitarity. However the physical origin of this drawback is the bottomless energy spectrum in theories at hand.

This paper seeks to trace in detail the development of this instability considering the Pais-Uhlenbeck fourth order oscillator \[1\] which is the simplest theory of this kind.

The layout of the paper is as follows. In Sec. II the Lagrangian and Hamiltonian descriptions of the Pais-Uhlenbeck fourth order oscillator are presented. In Sec. III the classical dynamics of this oscillator with allowance for the friction force is investigated. The development of the instability is traced in detail. In Conclusion (Sec. IV) the obtained results are summarized and their relation to other studies in this field are noted.

II. PAIS-UHLENBECK FOURTH ORDER OSCILLATOR: LAGRANGIAN AND HAMILTONIAN DESCRIPTIONS

The simplest higher derivative theory of a scalar field \(\varphi(t, x)\) is defined by the equation \[1\]

\[
\left( \frac{\partial^2}{\partial t^2} - \Delta + m_1^2 \right) \left( \frac{\partial^2}{\partial t^2} - \Delta + m_2^2 \right) \varphi(t, x) = 0 ,
\]

where \(\Delta\) is the Laplace operator, \(m_1\) and \(m_2\) are the mass parameters. Upon passing to the momentum space

\[
\varphi(t, x) = \int e^{ipx} \tilde{\varphi}(t, p) d^3p
\]

equation \[2.1\] acquires the form

\[
\left( \frac{d^2}{dt^2} + \omega_i^2 \right) \left( \frac{d^2}{dt^2} + \omega_i^2 \right) \tilde{\varphi}(t, p) = 0 ,
\]

where \(\omega_i^2 = p^2 + m_i^2\), \(i = 1, 2\).

When the space has zero dimension, we obtain from \[2.3\] the equation governing the dynamics of the respective point mechanical analog of the initial field theory \[2.1\], i.e., the Pais-Uhlenbeck fourth order oscillator

\[
\frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2x}{dt^2} + \omega_1^2 \omega_2^2 x = 0 ,
\]

(2.4)
where \( x \equiv x(t) \) is the coordinate of this oscillator.

The general solution to this equation has the form

\[
x(t) = x_1(t) + x_2(t),
\]

where

\[
x_n(t) = a_n e^{i\omega_n t} + a_n^* e^{-i\omega_n t}, \quad n = 1, 2.
\]

Here \( a_n \) are complex amplitudes, the positive frequencies \( \omega_1 \) and \( \omega_2 \) are assumed to be different and \( \omega_2 > \omega_1 \). The case of equal frequencies \( \omega_1 = \omega_2 \) requires a special consideration [20, 21].

Equation (2.4) is the Euler equation

\[
\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} = 0
\]

for the Lagrangian

\[
L = -\frac{1}{2} \dot{x}^2 + \frac{1}{2} (\omega_1^2 + \omega_2^2) \dot{x}^2 = \frac{1}{2} \omega_1^2 \omega_2^2 x^2.
\]

Transition to the Hamiltonian description is accomplished by making use of the Ostrogradski method [22]. Upon introducing the canonical variables

\[
q_1 = x, \\
q_2 = \dot{x}, \\
p_1 = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{x}} = (\omega_1^2 + \omega_2^2) \dot{x} + \ddot{x}, \\
p_2 = \frac{\partial L}{\partial \ddot{x}} = -\ddot{x}
\]

the Ostrogradski Hamiltonian is constructed in the usual way

\[
H = p_1 \dot{q}_1 + p_2 \dot{q}_2 - L = \frac{p_1^2}{2(\omega_1^2 + \omega_2^2)} + \frac{1}{2} \omega_1^2 \omega_2^2 q_1^2 - \frac{1}{2} p_2^2 - \frac{1}{2} (\omega_1^2 + \omega_2^2) \left( q_2 - \frac{p_1}{\omega_1^2 + \omega_2^2} \right)^2.
\]

Obviously, this Hamiltonian is not positive definite.

The Hamiltonian equations of motion have the form

\[
\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_2, \\
\dot{q}_2 = \frac{\partial H}{\partial p_2} = -p_2, \\
\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -\omega_1^2 \omega_2^2 q_1, \\
\dot{p}_2 = -\frac{\partial H}{\partial q_2} = (\omega_1^2 + \omega_2^2) q_2 - p_1.
\]
Twice differentiating the second equation in (2.11) and making use of the rest Hamiltonian equations one deduces the Lagrangian equation (2.4)

\[
\dot{q}_2 = \frac{d^4x}{dt^4} = \frac{d}{dt} (-\dot{p}_2) = -\frac{d}{dt} \left[ (\omega_1^2 + \omega_2^2) q_2 - p_1 \right] = - (\omega_1^2 + \omega_2^2) \dot{q}_2 + \dot{p}_1 = - (\omega_1^2 + \omega_2^2) \ddot{x} - \omega_1^2 \omega_2^2 \dot{x}.
\]

(2.12)

The Ostrogradski Hamiltonian (2.12) can be rewritten in terms of the Lagrangian variables \(x, \dot{x}, \ddot{x}, \ldots\):

\[
E = \frac{1}{2} \omega_1^2 \omega_2^2 x^2 + \frac{1}{2} (\omega_1^2 + \omega_2^2) \dot{x}^2 - \frac{1}{2} \ddot{x}^2 + \dot{x} \ddot{x}.
\]

(2.13)

Substituting in this equation the solutions \(x_1(t)\) and \(x_2(t)\) from (2.6) we obtain the values of the energy for oscillations with frequencies \(\omega_1\) and \(\omega_2\), respectively

\[
E_1 = (a_1 a_1^* + a_1 a_1^*) \omega_1^2 (\omega_2^2 - \omega_1^2),
\]

(2.14)

\[
E_2 = (a_2 a_2^* + a_2 a_2^*) \omega_2^2 (\omega_1^2 - \omega_2^2).
\]

(2.15)

It is obvious that \(E_1\) and \(E_2\) have the opposite signs. If the inequality \(\omega_2 > \omega_1\) holds, as we have assumed above, then the contribution to the Ostrogradski energy of the oscillations with the frequency \(\omega_1\) is positive while the same with the frequency \(\omega_2\) is negative. The energy (2.13) is an integral of motion, and the fact that it can acquire negative values does not lead to any physical contradictions. A different situation arises when the system under consideration experiences external action.

III. PAIS-UHLENBECK OSCILLATOR WITH DAMPING

Let us introduce the friction force into the equation of motion (2.4)

\[
\frac{d^4x}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2x}{dt^2} + \omega_1^2 \omega_2^2 x + \gamma \frac{dx}{dt} = 0,
\]

(3.1)

where \(\gamma\) is a positive constant (friction coefficient). As before the time dependence of the solutions to Eq. (3.1) is described by the factor \(e^{i\omega t}\), where the frequency \(\omega\) is the root of the equation

\[
\omega^4 - (\omega_1^2 + \omega_2^2) \omega^2 + \omega_1^2 \omega_2^2 + 2i \gamma \omega = 0.
\]

(3.2)

In order to simplify the calculations we assume that the damping is weak, so the perturbation theory is applicable. Substituting in Eq. (3.2)

\[
\omega = \omega_n + i \Delta \omega_n, \quad n = 1, 2,
\]

(3.3)
we arrive at the equation for $\Delta \omega_n$. In the approximation linear in $\gamma$ it reads

$$2 \omega_n^2 \Delta \omega_n - (\omega_1^2 + \omega_2^2) \Delta \omega_n + \gamma = 0, \quad n = 1, 2. \quad (3.4)$$

Thus, the frequency shifts for $\omega_1$ and $\omega_2$ prove to be real and equal in the absolute value but they have the opposite signs

$$\Delta \omega_1 = \frac{\gamma}{\omega_2^2 - \omega_1^2} > 0, \quad (3.5)$$

$$\Delta \omega_2 = \frac{\gamma}{\omega_1^2 - \omega_2^2} < 0, \quad (3.6)$$

because we have assumed that $\omega_2 > \omega_1$. As a result, the solution

$$x_1(t) \sim e^{i \omega_1 t - \Delta \omega_1 t} \quad (3.7)$$

exponentially decreases in time, while the solution

$$x_2(t) \sim e^{i \omega_2 t + |\Delta \omega_2| t} \quad (3.8)$$

exponentially increases in time. The damping behaviour of the solution $x_1(t)$ is physically motivated, namely, due to the external friction force the amplitudes $a_1$ and $a_1^*$ in Eq. (2.14) decrease in time and the energy $E_1$ also decreases being positive. Unlike this, the amplitudes $a_2$ and $a_2^*$ in Eq. (2.15) exponentially grow up and there are no reasons to stop this process, since the energy $E_2$, being negative, decreases without bound. Obviously such a dynamical behaviour is not acceptable from the physical stand point. Remarkably, for the both solutions $x_1(t)$ and $x_2(t)$ the energy is conserved however in the latter case this holds only formally.

Now we consider the Pais-Uhlenbeck oscillator which experiences an arbitrary force $f(t)$:

$$\frac{d^4 x}{dt^4} + (\omega_1^2 + \omega_2^2) \frac{d^2 x}{dt^2} + \omega_1^2 \omega_2^2 x = f(t). \quad (3.9)$$

The general solution to this equation can be expressed in terms of the relevant Green function

$$x(t) = \int_{-\infty}^{\infty} G(t - t') f(t') dt' = \int_{-\infty}^{\infty} \bar{G}(\omega) \bar{f}(\omega) d\omega, \quad (3.10)$$

where

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \bar{G}(\omega) d\omega, \quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\omega} \bar{f}(\omega) d\omega. \quad (3.11)$$
From Eq. (3.9) we deduce in a straightforward way

\[
\bar{G}(\omega) = \frac{1}{\omega_1^2 - \omega_2^2} \left( \frac{1}{\omega_1^2 - \omega_2^2} - \frac{1}{\omega_2^2 - \omega_1^2} \right).
\] (3.12)

Now we see from Eqs. (3.10) and (3.12), that the forces with spectral densities localized around \(\omega_1^2\) and \(\omega_2^2\) give rise to displacement \(y\) of opposite signs. Obviously, it implies that one of these displacements is unphysical.

IV. CONCLUSION

By making use of a simple and clear example, tractable analytically, we have shown that the instability of the theories with higher derivatives implies the following. Introduction of arbitrary weak interaction of such systems with surroundings results, for sure, in appearance of unbounded rising solutions. Being negative, the Ostrogradski energy for these solutions grows up in absolute value without bound and there are no reasons to terminate this process.

It is worth noting that introduction of external nonpotential forces is crucial in our consideration. Indeed, in Ref. [23] an additional potential term was introduced in undamped Pais-Uhlenbeck oscillator. It was shown there that indefiniteness of the energy does not forbid the stability in this case.

Our inference does not imply at all that any theory with higher derivatives is unphysical one at the classical level already. Indeed, let us consider, as an example, a dynamical theory, which is described by two ordinary differential equations of the second order. Let the dynamics of this system be stable (a conservative system with energy bounded from below). In the general case, these equations can be transformed into one differential equation of the fourth order. It maybe that the Ostrogradski energy for this equation proves to be unbounded from below. But it does not imply that the initial system of two equations is unstable with respect to external action. The point is, the Ostrogradski energy for the one forth order equation does not coincides, in the general case, with the usual energy for two initial differential equations of the second order. The examples of such systems are discussed now in the literature. First of all, it is worth noting here the Timoshenko beam theory which describes the transverse vibrations of an elastic beam or rod with allowance for flexure bending and transverse shear deformation [17, 24, 25, 26, 27].
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