LINEAR SYSTEMS IN $\mathbb{P}^2$ WITH BASE POINTS OF BOUNDED MULTIPLICITY

STEPHANIE YANG

Abstract. We present a proof of the Harbourne-Hirschowitz conjecture for linear systems with multiple points of order 7 or less. This uses a well-known degeneration of the plane developed by Ciliberto and Miranda as well as a combinatorial game that arises from specializing points onto lines.

1. Introduction

This paper discusses two techniques for determining if general multiple points in $\mathbb{P}^2$ impose independent linear conditions on the space of plane curves of a given degree. A well-known conjecture, formulated independently by Harbourne, Gimigliano, and Hirschowitz, gives geometric meaning to when this is the case.

Let $m_1, \ldots, m_r$ be a sequence of positive integers corresponding to general points $p_1, \ldots, p_r \in \mathbb{P}^2$. Denote by $L = L_d(m_1, \ldots, m_k)$ the linear system of degree $d$ curves with multiplicity $m_i$ at $p_i$. Vanishing to order $m$ at a point $p$ is equivalent to the vanishing of all derivatives at $p$ of order at most $m - 1$. Thus an $m$-fold point imposes $\binom{m+1}{2}$ linear conditions on plane curves and the expected dimension of $L$ is given by the equation:

$$e(L) = \max \left\{ \left( \frac{d+2}{2} \right) - \sum_{i=1}^{k} \left( \frac{m_i+1}{2} \right) - 1, -1 \right\}.$$

This estimate is a sharp lower bound for the actual dimension of $L$. When equality holds, we say that $L$ is non-special, and otherwise, we say that $L$ is special.

Let $\pi: V \to \mathbb{P}^2$ be the blow-up of the projective plane at the points $p_1, \ldots, p_r$. A curve $C \subseteq \mathbb{P}^2$ is called a $(-1)$-curve if it is rational and its proper transform $\tilde{C} \subset V$ has self-intersection equal to $-1$. With this in our vocabulary, it is easy to state the Harbourne-Hirschowitz conjecture:

Conjecture (Harbourne-Hirschowitz). $L$ is special if and only if it contains a multiple $(-1)$-curve in its base locus.

While one direction (the “if” part) of this equivalence is elementary, the other direction remains open except for special cases.

The Harbourne-Hirschowitz conjecture has a variety of algebro-geometric consequences. First, a proof of the conjecture would settle the longstanding
Nagata conjecture, posed in 1959 by Nagata after he constructed a counterexample to Hilbert’s 14th problem \cite{11}. (In short, Nagata conjectured if \( n \geq 10 \), then any degree \( d \) curve with \( n \) points of multiplicity \( m \) must satisfy \( d > m\sqrt{n} \).) The Harbourne-Hirschowitz conjecture also implies that any integral curve of negative self-intersection in the blow-up of \( \mathbb{P}^2 \) at (any number of) general points must have self-intersection \(-1\), thus giving a complete description of the Mori cone of such surfaces.

One approach to this problem has a simple geometric description. Suppose we are given a linear system \( \mathcal{L} \) of plane curves with multiple base points. Choose a triangle of three lines in \( \mathbb{P}^2 \) meeting in three distinct points. We specialize the base points by moving them onto these points and sliding the multiple points along the three lines to collide them. Each collision creates a larger singularity in the base locus of the limiting linear system, and the class of singularities that arise can be completely described via a combinatorial game involving checkers on a triangular board.

The second technique is a modification of a planar degeneration exploited by Ciliberto and Miranda in \cite{2} and \cite{3}. Let \( \Delta \) be a one-parameter family, and denote by \( X \) the blow-up of the three-fold \( \mathbb{P}^2 \times \Delta \) at a point. The fibers of \( X \) over \( \Delta \) can be viewed as a family of projective planes which degenerate to a reducible surface comprised of two rational components. If we have a family of plane curves with multiple points in \( \mathbb{P}^2 \), we can use this degeneration to ‘break’ a family of plane curves into two families defined on each of the two rational components of the special fiber of \( X \). This gives a recursive bound for the dimension of the original family. A consequence of this degeneration is the following statement, made precise in Theorem \ref{theo:main}.

*For any positive integer \( M \), there exists a bound \( D = D(M) \) such that:

\[
\begin{align*}
\text{The Harbourne-Hirschowitz conjecture is true for all linear systems } \mathcal{L}_d(m_1, \ldots, m_k) \\
\text{with } d < D(M) \text{ and } m_i \leq M.
\end{align*}
\]

\[
\implies \\
\begin{align*}
\text{The Harbourne-Hirschowitz conjecture is true for all linear systems } \mathcal{L}_d(m_1, \ldots, m_k) \\
\text{with } m_i \leq M.
\end{align*}
\]

The base points above are allowed to have mixed multiplicity. (Most recent results have applied only to collections of base points with all or all but one points of equal multiplicity.) Also note that the list of linear systems for the left hand side is finite in length, while those on the right are infinite. The exact formula for \( D(M) \) is given by Equation \ref{eq:bound} in Section \ref{sec:results}.

In particular, \( D(7) = 29 \) and the number of possible linear systems 28 or less, with multiple points of order 7 or less, is approximately \( 10^8 \). One hundred million cases sounds daunting to all but the computer-minded. We wrote a program (in \texttt{C++}) to enumerate this long list of cases and play the combinatorial game (of checkers on a triangle) on each case. Remarkably, the game worked to prove the Harbourne-Hirschowitz conjecture in almost all of the cases, cutting the number down to 42 (listed in Table \ref{tab:cases}), which are then handled with ad hoc methods in the last section of this paper, to prove:
Theorem 1. The Harbourne-Hirschowitz conjecture is true for all linear systems of plane curves with base points having multiplicity at most 7.

2. Checkers on a triangular board

2.1. Combinatorial description. In this section, we describe the rules of a combinatorial game which involves placing and moving up to \( \sum \left( \frac{m_i+1}{2} \right) \) checkers on a triangular checkerboard with side length \( d + 1 \), containing a total of \( \frac{(d+2)^2}{2} \) squares. The ultimate goal of the game is to place as many checkers on the board as possible; this gives an upper bound for the dimension of a linear system \( \mathcal{L}_d(m_1, \ldots, m_r) \).

Given \( \mathcal{L}_d(m_1, \ldots, m_k) \), form a \( (d+1) \times (d+1) \) triangle of boxes. We may place checkers in these boxes using only two types of moves:

**Type A:** For any multiplicity \( m_i \), we place \( \left( \frac{m_i+1}{2} \right) \) checkers in one of the three corners of the box, forming an \( m_i \times m_i \) triangle. If no corner of the box has enough empty squares available, then our only options are to quit the game, or perform moves of type B in order to create more empty squares in a corner. Two examples of valid moves are:

![Type A example](image)

**Type B:** We may perform one of six ‘slides’ which move all of the checkers as far as possible and in the same direction along rows, columns, or diagonals. The checkers may not overlap or share squares. Two examples of valid moves of this type are:

![Type B example](image)

The dimension of \( \mathcal{L}_d(m_1, \ldots, m_r) \) is bounded above by one less than the number of uncheckered boxes after any sequence of moves, so long as we use each multiplicity \( m_i \) for moves of type A at most once. In other words, if all of the \( \sum \left( \frac{m_i+1}{2} \right) \) checkers can be fit into the triangle using only the two moves described above, then \( \mathcal{L}_d(m_1, \ldots, m_r) \) is non-special.

**Examples.** As a first example, consider linear system \( \mathcal{L}_5(3,2,2,2,2,2) \) of quintics with one triple point and five double points. When we perform the triangle algorithm as follows:
Step 1: Place six checkers (for the triple point) onto the lower right hand corner of boxes.

Steps 2–3: Place three checkers for a double point onto the lower left hand corner of boxes, and slide all the checkers to the right.

Steps 4–5: Place three checkers for a double point onto the upper corner of boxes, and slide all the checkers down.

Steps 6–8: Place three checkers for a double point on the top corner, slide the checkers down, and then slide them to the right.

Steps 9–10: Repeat steps 4 and 5.

Step 11: Place three more checkers, for the last double point, into the remaining empty boxes in the upper corner.

On the triangular checkerboard, the steps look like this:

![Triangular Checkerboard Steps](image)

The darker dots • represent the newly placed checkers, while the lighter dots • represent checkers from previous moves. All 21 checkers fit into the board, and thus \( L_5(3, 2, 2, 2, 2) \) is empty and non-special.

Now consider the special linear system \( L = L_2(2, 2) \) of conics through two double points. After placing first three checkers in any corner of the triangle, we cannot fit another triangle of three checkers onto the board, even after any sequence of slides. This of course is due to the fact that \( L_2(2, 2) \) is special.

2.2. **Algebraic description of the game.** We translate these moves into the language of algebra using pictures to guide us with bookkeeping. Choose homogeneous coordinates for \( \mathbb{P}^2 \) so that the vertices of the triangle are \([1: 0: 0] , [0: 1: 0] , [0: 0: 1] \). Using these coordinates, we can create a \((d + 1) \times (d + 1)\) triangle of boxes which represent the monomial basis for degree \( d \) curves, with \( X^d, Y^d, Z^d \) represented by the three corner boxes, and the other monomials interpolated between these in the usual way.

![Algebraic Diagrams](image)

Notice that imposing an \( m \)-tuple base point onto \([1: 0: 0] , [0: 1: 0] , [0: 0: 1] \) corresponds exactly to the vanishing of the monomials in the an
Consider the six maps below; from now on we will refer to them as slide transformations:

\[(X, Y, Z) \mapsto (X + t^{-1}Y, Y, Z) \quad (X, Y, Z) \mapsto (X + t^{-1}Z, Y, Z)\]
\[(X, Y, Z) \mapsto (X, Y + t^{-1}X, Z) \quad (X, Y, Z) \mapsto (X, Y + t^{-1}Z, Z)\]
\[(X, Y, Z) \mapsto (X, Y, Z + t^{-1}X) \quad (X, Y, Z) \mapsto (X, Y, Z + t^{-1}Y)\]

Let \( \alpha \) denote a subset of \( \{(i, j) : i + j \leq d\} \) and let \( I_\alpha \) denote the ideal generated by the set \( \{X^i Y^j Z^{d-i-j} : (i, j) \in \alpha\} \). Pictorially, we represent \( \alpha \) on our triangle by adding checkers to the monomials boxes not in \( I_\alpha \). By lemma 2, the flat limit of \( I_\alpha \) under a slide transformation as \( t \) vanishes is another monomial ideal; we wish to calculate this flat limit and record how this moves the checkers on our board.

Consider the slide transformation \((X, Y, Z) \mapsto (X + t^{-1}Y, Y, Z)\); since this preserves the \( Z \)-degree of any monomial, we can simply calculate how the slide transformation acts on the \( Z \)-homogeneous parts of \( I_\alpha \). For this, assume that the set \( \{(i, j) : i + j \leq d\} - \alpha \) consists of pairs \((i, j)\) for which \( i + j = N \) for some fixed integer \( N \leq d \). In other words, we require all of the checkered boxes on the triangle associated with \( \alpha \) lie on a diagonal, for example:

\[\begin{array}{ccc}
\&\&
\&\&
\&\&
\&\&
\end{array}\]

Lemma 2 states that the flat limit of \( I_\alpha \) under the slide transformation \((X, Y, Z) \mapsto (X + t^{-1}Y, Y, Z)\) as \( t \) vanishes is the ideal with all of the checkers boxes shifted along the \( Z \)-homogeneous lines towards the highest power of \( Y^d \).

\[\begin{array}{ccc}
\&\&
\&\&
\&\&
\&\&
\end{array}\]

This analysis is similar for all six slide transformations with the appropriate permutation of \( X, Y, Z \). As a corollary, we have that the flat limit
if any ideal $I_\alpha$ under any slide transformation results in sliding the checkers along lines parallel to the sides of the triangle.

**Lemma 2.** Suppose the ideal $I_\alpha$ is generated by $l$ homogeneous monomials of degree $k$ in their $Z$-degree, for some fixed $k$. Then the flat limit of the ideal $I_\alpha$ under the slide transformation $T_t : (X,Y) \mapsto (X + t^{-1}Y, Y, Z)$ as $t \to \infty$ vanishes is the monomial ideal $I_\beta$ where $\beta = \{(i, d-k-i, k) : i = 0, \ldots, l-1\}$

**Proof.** Assume that $\alpha$ consists of pairs $(a_i, d-k-a_i)$ for $i = 1, \ldots, l$, with $a_1 < a_2 < \ldots, a_l$. The $T_t I_\alpha$ is generated by the $l$ elements:

$$
(X + t^{-1}Y)^{a_i} Y^{d-k-a_i} Z^k = \sum_{j=0}^{a_i} \binom{a_i}{j} t^{j-a_i} X^j Y^{d-k-j} Z^k
$$

Thinking of the set $\{X^i Y^{d-k-i} Z^k\}_{i=0}^d$ as a basis for the vector space of degree $d$ monomials in $X, Y, Z$ with $Z$-degree equal to $k$, we can form $l \times d$ matrix representing the equations above:

$$
M = \begin{bmatrix}
\binom{a_1}{0} t^{-a_1} & \binom{a_1}{1} t^{1-a_1} & \cdots & \binom{a_1}{d} t^{d-a_1} \\
\binom{a_2}{0} t^{-a_2} & \binom{a_2}{1} t^{1-a_2} & \cdots & \binom{a_2}{d} t^{d-a_2} \\
& \vdots & \ddots & \vdots \\
\binom{a_l}{0} t^{-a_l} & \binom{a_l}{1} t^{1-a_l} & \cdots & \binom{a_l}{d} t^{d-a_l}
\end{bmatrix}
$$

The first $k$ columns of this matrix are linearly independent. To see this, we compute the determinant

$$
\begin{vmatrix}
\binom{a_0}{0} t^{-a_0} & \binom{a_0}{1} t^{1-a_0} & \cdots & \binom{a_0}{d} t^{d-a_0} \\
\binom{a_1}{0} t^{-a_1} & \binom{a_1}{1} t^{1-a_1} & \cdots & \binom{a_1}{d} t^{d-a_1} \\
& \vdots & \ddots & \vdots \\
\binom{a_l}{0} t^{-a_l} & \binom{a_l}{1} t^{1-a_l} & \cdots & \binom{a_l}{d} t^{d-a_l}
\end{vmatrix} = t^{-N} \begin{vmatrix}
\binom{a_0}{0} & \binom{a_0}{1} & \cdots & \binom{a_0}{l} \\
\binom{a_1}{0} & \binom{a_1}{1} & \cdots & \binom{a_1}{l} \\
& \vdots & \ddots & \vdots \\
\binom{a_l}{0} & \binom{a_l}{1} & \cdots & \binom{a_l}{l}
\end{vmatrix} = \frac{1}{1! \ldots l!} \prod_{i<j} (a_i - a_j)
$$

where $N = \sum a_i - \frac{k(k+1)}{2} \geq 0$. The last equality comes from the fact that we can factor the second matrix above as an upper-diagonal matrix times a Van der Monde matrix:

$$
\begin{bmatrix}
\binom{a_1}{0} & \binom{a_2}{0} & \cdots & \binom{a_l}{0} \\
\binom{a_1}{1} & \binom{a_2}{1} & \cdots & \binom{a_l}{1} \\
& \vdots & \ddots & \vdots \\
\binom{a_1}{l} & \binom{a_2}{l} & \cdots & \binom{a_l}{l}
\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & \cdots & 1 \\
0 & 1 & 0 & 0 & \cdots & 1 \\
0 & 0 & \frac{1}{2} & 0 & \cdots & 1 \\
0 & \frac{1}{3} & 0 & \frac{1}{2} & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \frac{1}{l} & \cdots & \frac{1}{2} & \cdots & 1 \\
0 & \frac{1}{l} & \cdots & \frac{1}{2} & \cdots & 1
\end{bmatrix} \begin{bmatrix}1 & 1 & 1 & \cdots & 1 \\
a_1 & a_2 & a_3 & \cdots & a_l \\
a_1^2 & a_2^2 & a_3^2 & \cdots & a_l^2 \\
a_1^3 & a_2^3 & a_3^3 & \cdots & a_l^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_1^l & a_2^l & a_3^l & \cdots & a_l^l
\end{bmatrix}
$$

Thus we are able to perform row operations on the matrix $M$ to obtain the identity matrix in the first $l$ rows. If this is done, all of the elements of $M$
outside the first \(k\) columns will contain a power of \(t\). In other words, we can re-express the basis of \(T_{I_{\alpha}}\):

\[
T_{I_{\alpha}} = \langle X + t^{-1}Y \rangle^{a_{i}}Y^{d-k-a_{i}}Z^{k} \big| i=1 \bigangle
\]

and the flat limit of this ideal as \(t\) vanishes is simply \(X^{i}Y^{d-k-i}Z^{k}\) as \(i = 0, \ldots, l-1\). \(\square\)

By upper-semicontinuity, the dimension of \(L_{d}(m_{1}, \ldots, m_{k})\) is bounded above by one less than the dimension of the monomial ideal which arises from playing the triangular checkers game with all of them multiplicities \(m_{i}\); this is exactly the number of white boxes left at the end of the game. Thus, if there is some order of moves which fits all of the checkers onto the triangular board, then \(L_{d}(m_{1}, \ldots, m_{k})\) is non-special.

3. Degenerating \(\mathbb{P}^{2}\)

3.1. Degenerating the plane. In this section, we describe a degeneration of \(\mathbb{P}^{2}\) used by Ciliberto and Miranda in [2, 3] and then by them with Cioffi and Orecchia in [5] to prove the Harbourne-Hirschowitz conjecture for homogeneous linear systems with \(m \leq 20\). The degeneration described here is only slightly modified from their degeneration for the purpose of including linear systems with base points of mixed multiplicity.

Let \(\Delta\) be a disc around the origin, and let \(\pi : X \to \Delta \times \mathbb{P}^{2}\) be the three-fold obtained by blowing-up the product \(\Delta \times \mathbb{P}^{2}\) at a point \(p\) in the plane \(\{0\} \times \mathbb{P}^{2}\).

Denote by \(X_{t} = \pi^{-1}(t)\) the fiber of \(X\) over \(t \in \Delta\). If \(t \neq 0\), then \(X_{t} \cong \mathbb{P}^{2}\). The special fiber \(X_{0}\) is the union of the exceptional divisor \(\mathbb{P}\), which is a copy of the projective plane, with the Hirzebruch space \(\mathbb{F}\), isomorphic to \(\mathbb{P}^{2}\) blown-up at a point. Via this isomorphism it is easy to see that the Picard group of \(\mathbb{F}\) is freely generated by two divisors \(H\) and \(E\), where \(H\) is the pullback of a class of a general line in \(\mathbb{P}^{2}\) and \(E\) is the class of the exceptional divisor of the blow-up. Let \(R \subseteq X_{0}\) denote the divisor \(\mathbb{P} \cap \mathbb{F}\).

\[
\mathbb{P} \cup_{R} \mathbb{F} = X_{0} \rightarrow X
\]

\[
\Delta \times \mathbb{P}^{2} \xrightarrow{\phi} X
\]

(9)

Let \(\phi : X \to \mathbb{P}^{2}\) denote the composition of \(\pi\) with the standard projection from the second factor of \(\Delta \times \mathbb{P}^{2}\). Denote by \(\mathcal{O}(d, a)\) the line bundle

(10) \(\mathcal{O}(d, a) = \phi^{*}\mathcal{O}_{\mathbb{P}^{2}}(d) \otimes \mathcal{O}_{X}(-a\mathbb{P})\).
For any $0 \leq a \leq d$, the line bundle $O(d,a)$ is a flat family of line bundles over $\Delta$. If $t \neq 0$, then this line bundle restricts to $X_t$ as $O_{\mathbb{P}^2}(d)$. If $t = 0$, then this line bundle restricts to the components of the special fiber $X_0$ as

\begin{align}
O(d,a)|_{\mathbb{F}} &= O_{\mathbb{F}}(a) \\
O(d,a)|_{\mathbb{P}} &= O_{\mathbb{F}}(dH - aE),
\end{align}

This follows from the fact that $O_X(\mathbb{P})$ restricts to $\mathbb{P}$ as $O_{\mathbb{P}^2}(-1)$ and to $\mathbb{F}$ as $O_{\mathbb{F}}(E)$. These two line bundles agree when restricted to $R$.

We now modify our notation to help us index collections of multiple points in $\mathbb{P}^2$. Let $L_d(m_1^{k_1}, \ldots, m_s^{k_s})$ denote the linear system of degree $d$ curves through $\sum k_i$ general points, of which $k_i$ have multiplicity $m_i$. (For example, $L_7(3,3,3,3,3)$ can be abbreviated to $L_7(3^5)$, and $L_5(2,2,2,3)$ can be written as $L_5(2^3,3)$.) Let $l_1, \ldots, l_s$ be another sequence of positive integers such that $l_i \leq k_i$ for $i = 1, \ldots, s$.

Consider $\sum k_i$ general points in the reducible fiber $X_0$ with $l_i$ of the $m_i$-fold points in $\mathbb{F}$, for $i = 1, \ldots, s$, and the rest of the points in $\mathbb{P}$, all in general position. These points can be considered as limits of a family of multiple points in general position in the nearby fibers of $\pi$. Denote by $L_0$ the linear system of divisors in $|O(d,a)|$ which vanish at these multiple points in $X_0$. By semicontinuity we have

\begin{equation}
\dim L_0 \geq \dim L_d(m_1^{k_1}, \ldots, m_s^{k_s}).
\end{equation}

Our goal is to find parameters $a$ and $l_i$ that make $\dim L_0$ is as small as possible.

We take advantage of the fact that $X_0$ is a reducible surface whose components are rational. Specifically, let $L_\mathbb{P}$ and $L_\mathbb{F}$ denote the restrictions of $L_0$ to $\mathbb{P}$ and $\mathbb{F}$. Then,

\begin{align}
L_\mathbb{P} &\cong L_a(m_1^{k_1-l_1}, \ldots, m_s^{k_s-l_s}) \\
L_\mathbb{F} &\cong L_d(m_1^{l_1}, \ldots, m_s^{l_s}, a).
\end{align}

The second equation comes from blowing down the $(-1)$-curve $E$ in $\mathbb{F}$. Of course, $a$ can be equal to any of the multiplicities $m_i$; for convenience of notation it is easier to leave it as is. Let $R_\mathbb{P}$ and $R_\mathbb{F}$ denote the restrictions of $L_\mathbb{P}$ and $L_\mathbb{F}$ to $R$, and denote by $\hat{L}_\mathbb{P}$ and $\hat{L}_\mathbb{F}$ denote the kernels of these restrictions. Then

\begin{align}
\hat{L}_\mathbb{P} &\cong L_{a-1}(m_1^{k_1-l_1}, \ldots, m_s^{k_s-l_s}) \\
\hat{L}_\mathbb{F} &\cong L_d(m_1^{l_1}, \ldots, m_s^{l_s}, a + 1).
\end{align}

Denote by $l_\mathbb{P}$, $\ell_\mathbb{P}$, $\ell_\mathbb{F}$, $r_\mathbb{P}$, and $r_\mathbb{F}$ the dimension of the respective linear systems $L_\mathbb{P}$, $L_\mathbb{F}$, $\hat{L}_\mathbb{P}$, $R_\mathbb{P}$, and $R_\mathbb{F}$. A study of these linear systems in $[3]$
LINEAR SYSTEMS IN $\mathbb{P}^2$ WITH BASE POINTS OF BOUNDED MULTIPLICITY

offers the following equations:

\begin{align*}
(18) \quad r_P &= \ell_P - \hat{\ell}_P - 1 \\
(19) \quad r_F &= \ell_F - \hat{\ell}_F - 1 \\
(20) \quad \ell_0 &= \dim(\mathcal{R}_P \cap \mathcal{R}_F) + \hat{\ell}_P + \hat{\ell}_F + 1.
\end{align*}

In the last equation, $\dim(\mathcal{R}_P \cap \mathcal{R}_F)$ refers to the vector space dimension of the intersection of two linear systems inside the $(a+1)$-dimensional $H^0\mathcal{O}_R(a)$. The first two equations are immediate from definitions. The idea behind equation (20) is that elements of $\mathcal{L}_0$ come from pairs of elements in $\mathcal{L}_P$ and $\mathcal{L}_F$ which agree on $\mathcal{R}$. If $\mathcal{L}_P = \mathbb{P} W_P$ and $\mathcal{L}_F = \mathbb{P} W_F$, then

\begin{equation}
\mathcal{L}_0 = \mathbb{P} \left( W_F \times H^0\mathcal{O}_R(a) W_F \right),
\end{equation}

and the result follows from a simple dimension count.

If the systems $\mathcal{R}_P$ and $\mathcal{R}_F$ intersect transversely, then $\dim(\mathcal{R}_P \cap \mathcal{R}_F)$ is immediate, and we would have a recursion for $\ell_0$. In fact, transverse intersection is always the case, thanks to a proof by Hirschowitz in [9] and again by Ciliberto and Miranda in [2]. This gives us the following proposition and corollary:

**Proposition 3.** With the notation as above,

1. If $r_P + r_F \leq a + 1$, then $\ell_0 = \hat{\ell}_P + \hat{\ell}_F + 1$.
2. If $r_P + r_F > a + 1$, then $\ell_0 = \ell_P + \ell_F - a$.

**Corollary 4.** If there exists $a$ and $l_i$ such that, with the notation above, $\mathcal{L}_P$ and $\mathcal{L}_F$ are non-special, and $\hat{\mathcal{L}}_P$ and $\hat{\mathcal{L}}_F$ are empty, then $\mathcal{L}_d(m_{k_1}^1, \ldots, m_{k_s}^s)$ is non-special.

**Proof.** We can assume that $e(\mathcal{L}_d(m_{k_1}^1, \ldots, m_{k_s}^s)) \leq 0$, otherwise, we can impose additional simple base points until this is the case. If $\hat{\ell}_P = \hat{\ell}_F = -1$, then that $r_P = \ell_P$ and $r_F = \ell_F$. Thus

\begin{align*}
(22) \quad r_P + r_F &= \left( a + \frac{d + 2}{2} \right) - \sum k_i \left( \frac{m_i + 1}{2} \right) + \sum l_i \left( \frac{m_i + 1}{2} \right) \\
&\quad + \left( a + \frac{d + 2}{2} \right) - \sum l_i \left( \frac{m_i + 1}{2} \right) - \left( \frac{a + 1}{2} \right) \\
(23) \quad &\leq a + 1 + e \left( \mathcal{L}_d(m_{k_1}^1, \ldots, m_{k_s}^s) \right).
\end{align*}

The condition for first part of Proposition 3 is satisfied, and consequently $\ell_0 = \hat{\ell}_P + \hat{\ell}_F + 1 = -1$ as expected. 

We end this section with a final useful observation, which will be used several times in section 4. Let $\Delta^* = \Delta - \{0\}$, and suppose $\mathcal{L}_d(m_{k_1}^1, \ldots, m_{k_s}^s)$ contains a curve which lies in some fiber $X_t \cong \mathbb{P}^2$ for some $t \in \Delta^*$. As we vary the base points of this curve, it sweeps out a surface in $\mathcal{C}^* \subseteq \mathbb{P}^2 \times \Delta^*$. The closure of $\pi^{-1}\mathcal{C}^* \subseteq X$ we denote by $\mathcal{C}$. 


Lemma 5. Let $B$ be a $(-1)$-curve in $X_0$ such that $B \cdot R = 1$. If $B \cdot C = -\sigma < 0$, then $C$ contains $B$ to order at least $\sigma$.

The details of this proof, as well as generalizations of the statement above, are discussed in [1] and [4]. The idea behind the proof is to blow up $X$ along the $r$ sections above $\Delta$ which are the general points in each $\mathbb{P}^2$, and then to blow up again along the proper transform of $B$. The exceptional divisor $G$ of the last blow-up is a quadric surface; a quick calculation expressing the restriction to $G$ of the proper transform of $C$ yields the result.

These “matching conditions” are the basis of a useful technique that relates the non-speciality of a linear system $\mathcal{L}_d(m^1_k, \ldots, m^s_k)$ to the non-speciality of a linear system of lower degree with points in special position. An example of this technique is carried out in Lemma 5 of Section 4.

3.2. The induction argument for large degree. Define the following rather unsightly functions,

\[
\text{dlow}(\gamma, h, m) = \left\lceil \frac{m^2}{2} + \frac{\gamma + 1}{2} + \frac{2h + 1}{2} \left( m + 1 \right) - m\gamma - 1 \right\rceil \quad (24)
\]

\[
\text{dhigh}(\gamma, h, m) = m + h + mh + \gamma h - 1 \quad (25)
\]

\[
S(M) = -\frac{3}{2} + \left( \sum_{m=1}^{M} \left( 2 \left\lceil \frac{m^2 - 1}{3m + 4} \right\rceil + 1 \right) m(m + 1) \right)^{\frac{1}{2}} \quad (26)
\]

In this section we prove the following theorem.

Theorem 6. Let $M = \max\{m_i\}$, and let

\[
D = D(M) = \max \left\{ 4M + 1, \text{dlow}(-1, \left\lceil \frac{M^2 - 1}{3M + 4} \right\rceil, M), S(M) \right\}. \quad (27)
\]

Suppose the Harbourne-Hirschowitz conjecture holds for all linear systems $\mathcal{L}_d(m^1_k, \ldots, m^s_k)$ with $m_i \leq M$ for $i = 1, \ldots, s$ and $d < D(M)$. Then it is true for all $\mathcal{L}_d(m^1_k, \ldots, m^s_k)$ with $m_i \leq M$ for $i = 1, \ldots, s$ and all values of $d$.

This is based on an induction on both the degree $d$ and the multiplicities $m_i$ of the points, starting with the fact that if and $d \geq 3m_i$ for all $i$, then $\mathcal{L}$ contains no $(-1)$-curves. Together with the Harbourne-Hirschowitz conjecture, this fact implies that any special linear system must have at least one base point of multiplicity $d/3$ or greater.

Lemma 7. If $\mathcal{L}_d(m_1, \ldots, m_r)$ contains a $(-1)$-curve in its base locus, then $d < 3\max\{m_i\}$.

Proof. Let $M = \max\{m_i\}$, and let $C$ denote a $(-1)$-curve in the base locus of $\mathcal{L}$. As before, let $V$ be the blow-up of $\mathbb{P}^2$ at the base points $p_i$ of $\mathcal{L}$, and suppose that $\tilde{C}$, the proper transform of $C$, represents the class $eE_0 - \sum n_i E_i$, where $E_0$ denotes the class of the pullback of a general line and $E_i$ denotes...
the class of the exceptional divisor over $p_i$. The adjunction formula gives us the equation

$$3e - \sum n_i = 1.$$  

(28)

If $|D|$ contains $\tilde{C}$, then $\tilde{C} \cdot D < 0$, and thus

$$de < \sum n_i m_i \leq M \sum n_i = M(3e - 1),$$  

(29)

$$d < 3M.$$  

(30)

□

A linear system is called quasi-homogeneous if all of its base points except one are of the same multiplicity; i.e, if it is of the form $\mathcal{L}_d(m^k, a)$. A study of quasi-homogeneous systems in [2] yields the following result.

**Lemma 8.** $\mathcal{L}_d(m^b, d - m + \gamma)$ is non-special if, $2 \leq m \leq d$, $b$ is odd, and $-1 \leq \gamma \leq 1$.

The proof of this lemma follows from analysis of the equations that arise from the definition of quasi-homogeneous (-1)-curves and those that arise from permuting the multiple points of the same order within in a quasi-homogeneous systems. The lemma for $\gamma = 0$ and $\gamma = 1$ can also be seen easily using the methods discussed in section 2; the details are left to the reader.

**Proof of Theorem** [4]. As in the proof of Corollary [4] we may assume that $e\left(\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})\right) \leq 0$; otherwise, we add general simple points until this is the case. Assume the Harbourne-Hirschowitz conjecture is true for all $\mathcal{L}_d(m_i^{k_i}, \ldots, m_s^{k_s})$ with $m_i < M$ and $d < D$. We will find parameters $a$ and $l_i$ so that the systems $\mathcal{L}_\mathcal{F}, \mathcal{L}_\hat{\mathcal{F}}$ and $\hat{\mathcal{L}}_\mathcal{F}$ satisfy the conditions of Corollary [4].

Let $l_i = 2h + 1$ be odd and $l_j = 0$ for $j \neq i$, and set $a = d - m_i - \gamma$, for $\gamma \in \{-1, 0, 1\}$. By Lemma 8 the linear system $\mathcal{L}_\mathcal{F} = \mathcal{L}_d(m_i^{l_i}, a)$ is non-special, and the linear system $\hat{\mathcal{L}}_\mathcal{F} = \mathcal{L}_d(m_i^{l_i}, a + 1)$ contains $\gamma + 1$ lines through the $(a + 1)$-fold point and the $m_i$-fold points. The residual system to these lines is

$$\mathcal{L}_{d-l_i(\gamma+1)} \left((m_i - \gamma - 1)^{l_i}, a + 1 - l_i(\gamma + 1)\right),$$  

(31)

which is also non-special by Lemma 8.

If $d \geq D$, then $a \geq 3M$, and thus $\mathcal{L}_\mathcal{F}$ and $\hat{\mathcal{L}}_\mathcal{F}$ are also non-special by the induction hypothesis and Lemma 7.
Thus all four linear systems \( \mathcal{L}_P, \mathcal{L}_F, \hat{\mathcal{L}}_P, \) and \( \hat{\mathcal{L}}_F \) are non-special, and their virtual dimensions are:

\[
\begin{align*}
\nu_P &= \left( \frac{a + 2}{2} \right) - \sum_{j=1}^{s} l_j \left( \frac{m_j + 1}{2} \right) + l_i \left( \frac{m_i + 1}{2} \right) - 1 \\

\nu_F &= \left( \frac{d + 2}{2} \right) - \left( \frac{a + 1}{2} \right) - l_i \left( \frac{m_i + 1}{2} \right) - 1 \\

\hat{\nu}_P &= \left( \frac{a + 1}{2} \right) - \sum_{j=1}^{s} l_j \left( \frac{m_j + 1}{2} \right) + l_i \left( \frac{m_i + 1}{2} \right) - 1 \\

\hat{\nu}_F &= \left( \frac{d + 2}{2} \right) - \left( \frac{a + 2}{2} \right) - l_i \left( \frac{m_i + 1}{2} \right) - 1.
\end{align*}
\]

For \( \hat{\mathcal{L}}_P \) and \( \hat{\mathcal{L}}_F \) to be empty, we need that \( d \) (and consequently \( a \)) be small enough that \( \hat{\nu}_P \leq 0 \), yet large enough that \( \hat{\nu}_F \leq 0 \). Imposing \( \hat{\nu}_P \leq 0 \) and \( \hat{\nu}_F \leq 0 \) yield, respectively, an upper and lower bounds for \( d \):

\[
\begin{align*}
d &\leq d_{\text{low}}(\gamma, h, m_i) \quad (36) \\

&\geq d_{\text{high}}(\gamma, h, m_i). \quad (37)
\end{align*}
\]

Thus for fixed \( a = d - m_i + \gamma \) and \( l_i = 2h + 1 \), we have an interval

\[
\text{dlow}(\gamma, h, m_i) \leq d \leq \text{dhigh}(\gamma, h, m_i) \quad (38)
\]
on which we can prove that \( \mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s}) \) is empty. We force these intervals to overlap by varying \( \gamma \) and \( h \).

It is a “remarkable fact” \([2]\) that \( \text{dhigh}(-1, h, m_i) = \text{dlow}(0, h, m_i) \) and \( \text{dhigh}(0, h, m_i) = \text{dlow}(1, h, m_i) \). This extends our induction interval to

\[
\text{dlow}(m_i, -1, h) \leq d \leq \text{dhigh}(m_i, 1, h). \quad (39)
\]

We now vary \( h \) so that these intervals overlap, which gives us the equations

\[
\begin{align*}
\text{dhigh}(m_i, 1, h) + 1 &\geq \text{dlow}(m_i, -1, h) \quad (40) \\

h &\geq \left\lceil \frac{m_i^2 - 1}{3m_i + 4} \right\rceil. \quad (41)
\end{align*}
\]

For this to work, we must have the existence of some \( i \) for which \( k_i \geq 2h+1 = 2 \left\lceil \frac{m_i^2 - 1}{3m_i + 4} \right\rceil + 1 \). Since \( \chi \left( \mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s}) \right) \leq 0 \), this last requirement is fulfilled for large \( d \).

\[
\begin{align*}
\binom{d + 2}{2} &\geq \sum_{i=1}^{M} \left( 2 \left\lceil \frac{i^2 + i}{3i + 4} \right\rceil + 1 \right) \binom{i + 1}{2} \quad (42) \\

&\geq S(M). \quad (43)
\end{align*}
\]
4. Proof of Theorem 4

Below, Table 1 shows the first few values of $D(M)$. For all but the first few values of $M$, the function $D(M)$ is determined by $S(M)$. To prove

| $M$ | $4M + 1$ | $\text{dlow} \left( -1, \left\lceil \frac{M^2 - 1}{3M + 4} \right\rceil, M \right)$ | $S(M)$ | $D(M)$ |
|-----|---------|---------------------------------|--------|--------|
| 2   | 9       | 3                               | 3      | 9      |
| 3   | 13      | 5                               | 6      | 13     |
| 4   | 17      | 7                               | 10     | 17     |
| 5   | 21      | 13                              | 15     | 21     |
| 6   | 25      | 16                              | 21     | 25     |
| 7   | 29      | 19                              | 26     | 29     |
| 8   | 33      | 29                              | 34     | 34     |
| 9   | 37      | 33                              | 42     | 42     |
| 10  | 41      | 37                              | 51     | 51     |
| 11  | 45      | 51                              | 61     | 61     |
| 12  | 49      | 56                              | 71     | 71     |

Table 1. Values of $D(M)$ for $M = 2, \ldots, 12$

Theorem 4, we programmed a computer to enumerate all linear systems $\mathcal{L}_d(m_1, \ldots, m_k)$ of degree 29 or less, with points of multiplicity 7 or less. There 125220076 of these, almost all of which were shown to satisfy the Harbourne-Hirschowitz conjecture via the game discussed in Section 2:

- 125220076 total linear systems
- 124850912 are empty via the combinatorial method
- 366691 are empty because otherwise there would appear “too many curves” in the base locus
- 2013 are special because of multiple (-1)-curves in the base locus
- 418 are empty using Proposition 3

42 systems remain and are listed on Table 2

Twelve of the linear systems in the Table 2 can be handled via a combination of the triangular checker game and an analysis of which curves are forced to appear in the base locus. For example, to show that $\mathcal{L}_{15}(3, 4, 5^8)$ contains no curves, we play the triangle game in the following manner:

Steps 1–15: Place fifteen checkers (for a 5-tuple point) on the top of the triangle, and slide them down, and then slide them again to the right. Repeat this four more times.

Steps 16–18: Place ten checkers (for a quadruple point) on top of the triangle, slide them down, and then slide them to the right.

Steps 19–21: Place six checkers (for a triple point) on top of the triangle, and slide them down, and then slide them to the right.

After these seven steps, a line $L$ splits off six times (since we have six full rows of checkers on the bottom of the triangle). Residual to the line, we are
| $\mathcal{L}_d(m_1^{k_1}, \ldots, m_s^{k_s})$ | Reason for being empty | Order of specialization |
|--------------------------------|------------------------|------------------------|
| $\mathcal{L}_{15}(3, 4, 5^5)$ | Triangular checker game | $5^5, 4, 3$ |
| $\mathcal{L}_{16}(2, 5^{10})$ | Triangular checker game | $5^5, 2$, then $4^2$ |
| $\mathcal{L}_{17}(1^2, 5^5, 6, 7)$ | Cremona transformation | |
| $\mathcal{L}_{17}(2, 5^7, 6^3)$ | Cremona transformation | |
| $\mathcal{L}_{17}(4, 5^7, 7^2)$ | Cremona transformation | |
| $\mathcal{L}_{18}(1, 3, 5, 6^8)$ | Triangular checker game | $6^5, 5, 3, 1$ |
| $\mathcal{L}_{18}(1, 5^7, 7^3)$ | Cremona transformation | |
| $\mathcal{L}_{18}(1^2, 5^6, 6^2, 7^2)$ | Cremona transformation | |
| $\mathcal{L}_{18}(2, 6^9)$ | Cubics in the base locus | |
| $\mathcal{L}_{18}(4^5, 7^5)$ | Cremona transformation | |
| $\mathcal{L}_{19}(1, 3, 6^7, 7^2)$ | Cremona transformation | |
| $\mathcal{L}_{19}(1, 6^{10})$ | Homogeneous | |
| $\mathcal{L}_{19}(2^2, 5, 6^9)$ | Triangular checker game | $5, 6^6, 2^2$ |
| $\mathcal{L}_{19}(3, 5, 6, 7^6)$ | Cremona transformation | |
| $\mathcal{L}_{19}(3, 5, 6^5, 7^3)$ | Cremona transformation | |
| $\mathcal{L}_{19}(3, 5, 6^9)$ | Triangular checker game | $6^6, 5, 3$ |
| $\mathcal{L}_{19}(4, 5^4, 7^5)$ | Cremona transformation | |
| $\mathcal{L}_{19}(5, 6^8, 7)$ | Triangular checker game | $7, 6^4, 5, 6$ |
| $\mathcal{L}_{19}(6^{10})$ | Homogeneous | |
| $\mathcal{L}_{20}(1, 3, 6^4, 7^5)$ | Cremona transformation | |
| $\mathcal{L}_{20}(1, 3, 6^8, 7^2)$ | Triangular checker game | $7^2, 6^5, 1, 3$ |
| $\mathcal{L}_{20}(1, 6^{11})$ | Homogeneous | |
| $\mathcal{L}_{20}(1, 6^7, 7^3)$ | Cremona transformation | |
| $\mathcal{L}_{20}(3, 5, 6^2, 7^6)$ | Cremona transformation | |
| $\mathcal{L}_{20}(3, 5, 6^6, 7^3)$ | Cremona transformation | |
| $\mathcal{L}_{20}(5, 6^5, 7^4)$ | Cremona transformation | |
| $\mathcal{L}_{20}(6^{11})$ | Homogeneous | |
| $\mathcal{L}_{20}(6^7, 7^3)$ | Cremona transformation | |
| $\mathcal{L}_{21}(1, 2, 4, 5, 7^8)$ | Triangular checker game | $7^4, 1, 7, 5, 2$ |
| $\mathcal{L}_{21}(1, 6^4, 7^6)$ | Lemma $[9]$ | |
| $\mathcal{L}_{21}(1^2, 3, 6, 7^8)$ | Lemma $[9]$ | |
| $\mathcal{L}_{21}(2, 7^9)$ | Cubics in the base locus | |
| $\mathcal{L}_{21}(5, 6^2, 7^7)$ | Lemma $[9]$ | |
| $\mathcal{L}_{22}(1, 2, 6, 7^9)$ | Implied by $\mathcal{L}_{22}(2, 6, 7^9)$ | |
| $\mathcal{L}_{22}(1^3, 6, 7^9)$ | Implied by $\mathcal{L}_{22}(2, 6, 7^9)$ | |
| $\mathcal{L}_{22}(2, 6, 7^9)$ | Lemma $[9]$ | |
| $\mathcal{L}_{22}(2, 6^{13})$ | Lemma $[10]$ | |
| $\mathcal{L}_{22}(4, 5, 7^9)$ | Triangular checker game | $5, 7^5, 4$ |
| $\mathcal{L}_{22}(4, 6^2, 7^8)$ | Triangular checker game | $7^3, 6, 7^2, 6, 4$ |
| $\mathcal{L}_{23}(6, 7^{10})$ | Triangular checker game | $7^7, 6$ |
| $\mathcal{L}_{26}(6^{18})$ | Homogeneous | |
| $\mathcal{L}_{27}(5, 7^{14})$ | Triangular checker game | $7^{11}, 5$ |

Table 2. The forty two linear systems leftover from the computer program.
left with the monomial ideal represented by the triangle below, and three unspecialized 5-tuple points:

Any degree 9 curve containing the three unspecialized 5-tuple points must contain the lines through any two of the points; thus the base locus of the new linear system contains three lines, as well as the line $L$ once again. Residual to these four lines, we have the linear system of degree 5 curves through three general triple points, with a tangency to $L$ of order 3.

Once again, the triangle of lines through the three general triple points and the line $L$ split off; the residual linear system is simply $\mathcal{L}_1(3)$ which is clearly empty.

A similar argument works for eleven other linear systems in the table which are empty via the triangular checker game; after specializing all but three of the multiple points in the order prescribed by the third column of Table 2, triples of lines begin to split off of the base locus. Only $\mathcal{L}_{16}(2,5^{10})$ is slightly different. In this case, after specializing five quintuple points and a double point, a conic appears through the five remaining quintuple points. Residual to that, we have the ideal
with five unspecialized quadruple points. We specialize two of these quadruple points, after triangles start to appear in the base locus as in the previous argument.

Sixteen of the linear systems in the table are subject to quadratic Cremona transformations; that is, suppose $m_1p_1$, $m_2p_2$, and $m_3p_3$ are the three points of highest multiplicity. We blow up $\mathbb{P}^2$ at the three points $p_1$, $p_2$, and $p_3$ and then blow down the resulting surface along the proper transforms of the lines $\overline{p_1p_2}$, $\overline{p_1p_3}$, and $\overline{p_2p_3}$. If $s = m_1 + m_2 + m_3 - d$, then the linear system $L_d(m_1, \ldots, m_k)$ is transformed to the linear system $L_{d-s}(m_1-s, m_2-s, m_3-s, m_4, \ldots, m_k)$. In particular, if $s > 0$, then the Harbourne-Hirschowitz conjecture for $L_d(m_1, \ldots, m_k)$ is equivalent to the conjecture for a linear system of lower degree (see [6]).

The last two lemmas of this section will introduce two more techniques to prove the non-speciality of linear systems. This will handle the rest of the non-homogeneous cases in the table.

**Lemma 9.** The linear systems $L_{21}(1, 6^4, 7^6)$, $L_{21}(1^2, 3, 6, 7^8)$, $L_{21}(5, 6^2, 7^7)$, and $L_{22}(2, 6, 7^8)$ are empty.

**Proof.** The first three cases are so similar that we will only prove that one of the linear systems is non-special; the rest follow by an almost identical argument. To show that $L_{21}(1, 6^4, 7^6)$ is empty, we degenerate $\mathbb{P}^2$ into a reducible surface $\mathbb{P} \cup F$ as described in Section 3, placing four of the 7-tuple points in $F$ and the rest of the points in $\mathbb{P}$ and setting $a = 17$. Five curves appear with multiplicity three in the base locus of $L_F$ (if $\pi : F \to \mathbb{P}^2$ is the blow-down of $F$ along its $(-1)$-curve $E$, these lines are the proper transforms of the lines through $\pi(E)$ and each of the images quadruple points, and the proper transform of the conic through $\pi(E)$ and the image of the four quadruple points.) The linear system $L_F$ must thus satisfy the following conditions:

1. $L_F$ consists of degree 17 curves
2. $L_F$ contains two general 7-tuple points, four general 6-tuple points, and a simple point
3. $L_F$ contains five triple points on a line.

The last condition is a result of Lemma 5.

We now must show that $L_F$, characterized above, must be empty. To do this, we degenerate the plane $\mathbb{P}$ into a reducible surface $\mathbb{P}' \cup F'$ just as before, this time letting the new component $F'$ contain one of the triple points on the line, the two general 7-tuple points, and one 6-tuple point (see Figure 4), and setting $a = 14$. $L'_F$ is then a degree 14 linear system with three general 6-tuple points, a general simple point, and eight points of multiplicity 3 and 4 distributed on two lines according to the second picture in Figure 1.

We repeat this twice more, according to the figure below. At this point we are left with a degree 8 linear system, with ten points distributed onto four lines and one general simple point according to the forth diagram in the figure. We specialize this simple point onto the line through the two
simple and two triple points; this line splits off exactly once leaving us with $\mathcal{L}_7(2^4, 3^4)$. This must contain in its base locus the $(-1)$-curve of degree 4 through three double points and five simple points. Residual to his curve we have $\mathcal{L}_3(1^7, 2)$ which is clearly empty.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{$\mathcal{L}_{21}(1, 6^4, 7^6)$ is empty}
\end{figure}

The case $\mathcal{L}_{22}(2, 6, 7^9)$ is only slightly different and is outlined in Figure 2. This time we degenerate the plane five times, using the nine 7-tuple points and the 6-tuple point. The final result ($d = 7$) has fourteen points, mostly simple, distributed among five lines. In the last step, we specialize two triple points and one double point onto a line, after which more are forced to appear in the base locus than the degree of the residual linear system. The original linear system must be empty. The details are not difficult and are left to the reader.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{$\mathcal{L}_{22}(2, 6, 7^9)$ is empty}
\end{figure}

\begin{lemma}
$\mathcal{L}_{22}(2, 6^{13})$ is empty.
\end{lemma}

\begin{proof}
Let $\pi : X \to \mathbb{P}^2$ be the blow-up of the projective plane at the thirteen 6-tuple points. By the triangular checker game, we know that $\mathcal{L}_{22}(6^{13})$ is non-special and so the linear system $|22H - \sum_{i=1}^{13} 6E_i|$ on $X$ gives rise to a map $\phi : X \to \mathbb{P}^2$.

Assume for a contradiction that $\mathcal{L} = \mathcal{L}_{22}(2, 6^{13})$ is non-empty. The differential of $\phi$ fails to be injective at a general point in $\mathbb{P}^2$, thus the image $f(X) \in \mathbb{P}^2$ is a curve, say of degree $d$. Since $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^2}(1)$, the general element of $\mathcal{L}_{22}(2, 6^{13})$ has $d$ components of the same degree. The degree $d$ must be a divisor of 22, and a case-by-case analysis shows that this is impossible. Clearly $d$ cannot be 1 or 22. If $d = 11$, then $\mathcal{L}_{22}(6^{13})$ is comprised of 11 conics, which is impossible. If $d = 2$, then $\mathcal{L}_{22}(6^{13})$ is comprised of two curves in $\mathcal{L}_{11}(1, 3^{13})$, which is empty by previous calculations.
\end{proof}
These last two lemmas suffice to prove that the remaining linear systems in Table 2 are non-special and empty.

References

[1] C. Bocci and R. Miranda, *Topics on interpolation problems in algebraic geometry*, Rend. Sem. Mat. Univ. Politec. Torino 62 (2004), no. 4, 279–334. MR2129797

[2] Ciro Ciliberto and Rick Miranda, *Degenerations of planar linear systems*, J. Reine Angew. Math. 501 (1998), 191–220. MR2000m:14005

[3] Ciro Ciliberto, Linear systems of plane curves with base points of equal multiplicity, Trans. Amer. Math. Soc. 352 (2000), no. 9, 4037–4050. MR 2000m:14006

[4] Ciro Ciliberto, Matching Conditions for Degenerating Plane Curves and Applications. in preparation.

[5] Ciro Ciliberto, Francesca Cioffi, Rick Miranda, and Ferruccio Orecchia, *Bivariate Hermite interpolation via computer algebra and algebraic geometry techniques*, Publicazioni del Dipartimento di Matematica e Applicazioni di Napoli (2002), no. 26.

[6] Alessandro Gimigliano, *Our thin knowledge of fat points*, The Curves Seminar at Queen’s, Vol. VI (Kingston, ON, 1989), 1989, pp. Exp. No. B, 50. MR 91a:14007

[7] Brian Harbourne, *The geometry of rational surfaces and Hilbert functions of points in the plane*, Proceedings of the 1984 Vancouver conference in algebraic geometry, 1986, pp. 95–111. MR846019 (87k:14041)

[8] Brian Harbourne, *Problems and progress: a survey on fat points in $P^2$*, Zero-dimensional schemes and applications (Naples, 2000), 2002, pp. 85–132. MR 2003f:13032

[9] André Hirschowitz, *Existence de faisceaux réflexifs de rang deux sur $P^1$ à bonne cohomologie*, Inst. Hautes Études Sci. Publ. Math. (1988), no. 66, 105–137 (French). MR932136 (89c:14019)

[10] André Hirschowitz, *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques*, J. Reine Angew. Math. 397 (1989), 208–213 (French). MR993223 (90g:14021)

[11] Masayoshi Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. 81 (1959), 766–772. MR 21 #4151