NO MORE THAN THREE FAVOURITE SITES FOR SIMPLE RANDOM WALK

BÁLINT TÓTH

Technical University Budapest, Institute of Mathematics
Egry J. u. 1, H-1111 Budapest, Hungary
e-mail: balint@math.bme.hu

Abstract. We prove that, with probability one, eventually there are no more than three favourite (i.e. most visited) sites of simple symmetric random walk. This partially answers a relatively long standing question of Pál Erdős and Pál Révész.

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1. Introduction and Main Result.

Let \( S(t), t \in \mathbb{Z}_+ \) be a simple symmetric random walk on \( \mathbb{Z} \) with initial state \( S(0) = 0 \). Its upcrossings, downcrossings and (site) local time are defined for \( t \in \mathbb{N} \) and \( x \in \mathbb{Z} \) as follows:

\[
U(t, x) := \# \{ 0 < s \leq t : S(s) = x, S(s - 1) = x - 1 \}, \quad (1.1)
\]

\[
D(t, x) := \# \{ 0 < s \leq t : S(s) = x, S(s - 1) = x + 1 \}, \quad (1.2)
\]

\[
L(t, x) := \# \{ 0 < s \leq t : S(s) = x \} = U(t, x) + D(t, x). \quad (1.3)
\]

The following identities are straightforward:

\[
U(t, x) - D(t, x - 1) = \mathbb{1}_{\{0 < x \leq S(t)\}} - \mathbb{1}_{\{S(t) < x \leq 0\}}. \quad (1.4)
\]

\[
D(t, x) - U(t, x + 1) = \mathbb{1}_{\{S(t) \leq x < 0\}} - \mathbb{1}_{\{0 \leq x < S(t)\}}. \quad (1.5)
\]

And from these it follows that

\[
L(t, x) = D(t, x) + D(t, x - 1) + \mathbb{1}_{\{0 < x \leq S(t)\}} - \mathbb{1}_{\{S(t) < x \leq 0\}} = U(t, x) + U(t, x + 1) + \mathbb{1}_{\{S(t) \leq x < 0\}} - \mathbb{1}_{\{0 \leq x < S(t)\}}. \quad (1.6)
\]

The set of favourite (or: most visited) sites of the random walk at time \( t \in \mathbb{N} \), are those sites where the local time attains its maximum value:

\[
\mathcal{K}(t) := \{ y \in \mathbb{Z} : L(t, y) = \max_{z \in \mathbb{Z}} L(t, z) \}. \quad (1.7)
\]
It is clear that the number of favourite sites changes in time as follows:

\[
\#\mathcal{K}(t+1) = \begin{cases} 
\#\mathcal{K}(t) & \text{if } S(t+1) \notin \mathcal{K}(t+1) \\
\#\mathcal{K}(t) + 1 & \text{if } \mathcal{K}(t+1) = \mathcal{K}(t) \cup \{S(t+1)\} \\
1 & \text{if } \mathcal{K}(t+1) = \{S(t+1)\} \subset \mathcal{K}(t).
\end{cases}
\]  

(1.8)

In plain words one of the following three possibilities can occur at each step of the walk: Either the currently occupied site is not favourite and \(\mathcal{K}\) remains unchanged. Or the currently occupied site becomes a new favourite beside the favourites of the previous stage, and thus the number of favourites increases by one. Or, finally, a favourite site is visited and so this site becomes now the only new favourite. No other possibility.

Clearly \(\#\mathcal{K}(t) \geq 1\) for all \(t \geq 1\), and it is easy to verify that for infinitely many times, \(t \geq 1\), there are at least two favourite sites: \(\#\mathcal{K}(t) \geq 2\). Pál Erdős and Pál Révész formulated and repeatedly raised the following

**Question:** Does it happen that \(\#\mathcal{K}(t) \geq r\) infinitely often (i.e. almost surely for infinitely many times \(t \geq 1\)) for \(r = 3, 4, \ldots\)?

See e.g. Erdős and Révész (1984), (1987), (1991), Erdős (1994) or Révész (1990) for an extended list of related questions and problems.

Questions related to the asymptotic behaviour of the favourite (or most visited) sites of a random walk were considered by many authors since the mid-eighties. We quote here a few relevant results, with no claim of exhaustiveness.

- Bass and Griffin (1985) prove that almost surely, the set of favourites is transient. More exactly: they prove that the distance of the set of favourite sites from the origin increases faster than \(\sqrt{n}/(\log n)^{11}\) but slower than \(\sqrt{n}/(\log n)\).
- Csáki and Shi (1998) prove that the distance between the edge of the range of the random walk and the set of favourite sites increases as fast as \(\sqrt{n}/(\log \log n)^{3/2}\).
- Csáki, Révész and Shi (2000) prove that the position of a favourite site can have as large jumps as \(\sqrt{2n \log \log n}\), i.e., comparable with the diameter of the full range of the random walk. They also extend a much earlier result of Kesten (1965), identifying the set of joint limit points of the set of favourite sites and the favourite values (i.e. max. values of local time), both rescaled by \(\sqrt{2n \log \log n}\).
- There are many papers dealing with similar questions in the context of symmetric stable processes rather than random walks (or Brownian motion). See, e.g., Eisenbaum (1997), Bass, Eisenbaum and Shi (2000) and other papers quoted there.

In the present paper we answer in the negative the question of Erdős and Révész quoted above, for \(r \geq 4\): we prove that with probability 1, there are at most finitely many times \(t \geq 1\) when there are four or more favourite sites of the random walk \(S(t)\). In Tóth and Werner (1997) a similar result was proved for the set of favourite edges rather than favourite sites. The present paper deals
with the original question of Erdős and Révész. The starting general ideas of the present paper (see Sections 1-3) are very close to those of Tóth and Werner (1997). However: the details of the proof require more refined estimates and arguments. On the technical level (see Sections 4-6) this proof is rather different.

For \( r \geq 1 \) denote by \( f(r) \) the (possibly infinite) number of steps, when the currently occupied site is one of the \( r \) actual favourites:

\[
f(r) := \# \{ t \geq 1 : [S(t) \in K(t)] \land [\#K(t) = r] \}. \tag{1.9}
\]

From (1.8) it follows that for any \( r \geq 1 \), \( f(r + 1) \leq f(r) \). (Both sides of the inequality could be infinite.)

The main result of this paper is the following

**Theorem 1.**

\[ \mathbb{E}(f(4)) < \infty. \tag{1.10} \]

**Remarks:** (1) From this theorem the negative answer to the question of Erdős and Révész clearly follows, for the cases \( r \geq 4 \).

(2) The case \( r = 3 \) remains open. From the proof of the above theorem it becomes clear that \( \mathbb{E}(f(3)) = \infty \). Nevertheless we conjecture that \( f(3) < \infty \), almost surely.

The paper is organized as follows: In Section 2 we perform some straightforward manipulations (essentially: rearrangements of sums). In Section 3 we recall the Ray-Knight Theorems for the local times of simple random walks. In Section 4 first we express our relevant probabilities and expectations (found in Section 2) in terms of the Galton-Watson processes arising with the Ray-Knight representation. Then we formulate Proposition 1, stating some upper bounds on these probabilities and expectations, and using these bounds we prove Theorem 1. The proof of Proposition 1 is postponed to the end of Section 5. In Section 5 four lemmas and, as their consequence, Proposition 1 are proved. Throughout the technical parts of the proofs smaller, quite plausible statements are invoked. These are called Side-lemmas (1 to 7). Their proofs are postponed to Section 6.

Throughout the paper, in various upper bounds, multiplicative constants, respectively, constants in exponential rates will be denoted generically by \( C \), respectively, by \( \gamma \). The values of these constants may vary even within one proof, but we hope there is no danger of confusion.

2. Preparations.

In the following transformations the *inverse local times*, defined below for \( k \in \mathbb{N} \) and \( x \in \mathbb{Z} \), will play an essential rôlé:

\[
T_U(k, x) := \inf \{ t \geq 1 : U(t, x) = k \}, \tag{2.1}
\]

\[
T_D(k, x) := \inf \{ t \geq 1 : D(t, x) = k \}. \tag{2.2}
\]
It turns out that questions related to the local time are easier to handle if the random walk is observed at the random stopping times $T_U(k, x)$ and $T_D(k, x)$ rather than at deterministic times $t \geq 1$. This ‘combinatorial trick’ has its origin in Knight (1963) and has been successfully applied in various contexts. See, e.g., König (1996), Tóth (1995), (1996) and references cited there.

We express $f(4)$ with the help of some straightforward rearrangements of summations:

$$f(4) = \sum_{x \in \mathbb{Z}} (u(x) + d(x)) \quad (2.3)$$

where

$$u(x) := \sum_{t=1}^{\infty} \mathbb{1}\{S(t) = x, S(t-1) = x-1, x \in \mathcal{K}(t), \#\mathcal{K}(t) = 4\}$$

$$= \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{1}\{T_U(k,x) = t, x \in \mathcal{K}(t), \#\mathcal{K}(t) = 4\}$$

$$= \sum_{k=1}^{\infty} \mathbb{1}\{x \in \mathcal{K}(T_U(k,x)), \#\mathcal{K}(T_U(k,x)) = 4\} \quad (2.4)$$

and

$$d(x) := \sum_{t=1}^{\infty} \mathbb{1}\{S(t) = x, S(t-1) = x+1, x \in \mathcal{K}(t), \#\mathcal{K}(t) = 4\}$$

$$= \sum_{t=1}^{\infty} \sum_{k=1}^{\infty} \mathbb{1}\{T_D(k,x) = t, x \in \mathcal{K}(t), \#\mathcal{K}(t) = 4\}$$

$$= \sum_{k=1}^{\infty} \mathbb{1}\{x \in \mathcal{K}(T_D(k,x)), \#\mathcal{K}(T_D(k,x)) = 4\}. \quad (2.5)$$

Clearly,

$$u(x) \overset{\text{law}}{=} d(-x) \quad (2.6)$$

and, consequently

$$\mathbb{E}\left(f(4)\right) = 2 \sum_{x=1}^{\infty} \mathbb{E}\left(u(x)\right) + 2 \sum_{x=0}^{\infty} \mathbb{E}\left(d(x)\right) \quad (2.7)$$

with

$$\mathbb{E}\left(u(x)\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(x \in \mathcal{K}(T_U(k,x)) \land \#\mathcal{K}(T_U(k,x)) = 4\right) \quad (2.8)$$

$$\mathbb{E}\left(d(x)\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(x \in \mathcal{K}(T_D(k,x)) \land \#\mathcal{K}(T_D(k,x)) = 4\right) \quad (2.9)$$
We shall show in details that
\[
\sum_{x=1}^{\infty} E\left( u(x) \right) = \sum_{x=1}^{\infty} \sum_{k=1}^{\infty} P \left( [x \in K(T_U(k,x))] \land [\#K(T_U(k,x)) = 4] \right) < \infty. \tag{2.10}
\]

The similar statement
\[
\sum_{x=0}^{\infty} E\left( d(x) \right) = \sum_{x=0}^{\infty} \sum_{k=1}^{\infty} P \left( [x \in K(T_D(k,x))] \land [\#K(T_D(k,x)) = 4] \right) < \infty \tag{2.11}
\]
can be proved in an identical way.

3. Ray-Knight Representation.

Throughout this paper we denote by \( Y_t \) a critical branching process with geometric offspring distribution (Galton-Watson process) and by \( Z_t \) a critical branching process with geometric offspring distribution and one intruder at each generation. \( Y_t \) and \( Z_t \) are Markov chains with state space \( \mathbb{Z}_+ \) and transition probabilities:

\[
P\left( Y_{t+1} = j \mid Y_t = i \right) = \pi(i,j) := \begin{cases} 
\delta_{0,j} & \text{if } i = 0, \\
2^{-i-j}(i+j-1)! & \text{if } i > 0. 
\end{cases} \tag{3.1}
\]

\[
P\left( Z_{t+1} = j \mid Z_t = i \right) = \rho(i,j) := 2^{-i-j-1}(i+j)! \frac{1}{i!j!}. \tag{3.2}
\]

Let \( k \geq 0 \) and \( x \geq 1 \) be fixed integers and define the following three processes:

- \( Z_t, 0 \leq t \leq x - 1 \), is a Markov chain with transition probabilities \( \rho(i,j) \) and initial state \( Z_0 = k \);
- \( Y_t, -1 \leq t < \infty \), is a Markov chain with transition probabilities \( \pi(i,j) \) and initial state \( Y_{-1} = k \);
- Finally, \( Y'_t, 0 \leq t < \infty \), is another Markov chain with the same transition probabilities \( \pi(i,j) \) and initial state \( Y'_{0} = Z_{x-1} \).

The three chains are independent, except for the fact that \( Y' \) starts from the terminal state of \( Z \). Using these three chains we patch together the process

\[
\Delta_{x,k}(y) := \begin{cases} 
Z_{x-y-1} & \text{if } 0 \leq y \leq x - 1, \\
Y_{y-x} & \text{if } x - 1 \leq y \leq \infty, \\
Y'_{-y} & \text{if } -\infty \leq y \leq 0.
\end{cases} \tag{3.3}
\]

We also define

\[
\Lambda_{x,k}(y) := \Delta_{x,k}(y) + \Delta_{x,k}(y-1) + \mathbb{1}_{\{0 < y \leq x\}}. \tag{3.4}
\]
According to the, by now classical, Ray-Knight Theorems on the local time process of simple symmetric random walks on \( \mathbb{Z} \) (cf. Knight (1963), Ray (1963)), for any integers \( x \geq 1 \) and \( k \geq 0 \):

\[
\left( \Delta_{x,k}(y), \ y \in \mathbb{Z} \right) \overset{\text{law}}{=} \left( D(T_U(k + 1, x), y), \ y \in \mathbb{Z} \right). \tag{3.5}
\]

Using (1.6) and (3.4), from this we get

\[
\left( \Delta_{x,k}(y), \ y \in \mathbb{Z} \right) \overset{\text{law}}{=} \left( L(T_U(k + 1, x), y), \ y \in \mathbb{Z} \right). \tag{3.6}
\]

4. Proof of Theorem 1.

Given the Markov chains \( Y_t, Z_t \) and \( Y'_t \) we define

\[
\bar{Z}_t := Z_t + Z_{t-1} + 1, \quad \bar{Y}_t := Y_t + Y_{t-1}, \quad \bar{Y}'_t := Y'_t + Y'_{t-1} \tag{4.1}
\]

and for \( h \in \mathbb{N} \) the following stopping times

\[
\sigma_h := \inf \{ t \geq 0 : Y_t \geq h \}, \tag{4.2}
\]
\[
\sigma'_h := \inf \{ t \geq 0 : Y'_t \geq h \}, \tag{4.3}
\]
\[
\omega := \inf \{ t \geq 0 : Y_t = 0 \}, \tag{4.4}
\]
\[
\omega' := \inf \{ t \geq 0 : Y'_t = 0 \}, \tag{4.5}
\]
\[
\tau_h := \inf \{ t \geq 0 : Z_t \geq h \}, \tag{4.6}
\]
\[
\bar{\sigma}_{h,0} := 0, \quad \bar{\sigma}_{h,i+1} := \inf \{ t > \bar{\sigma}_{h,i} : \bar{Y}_t \geq h \}, \quad \bar{\sigma}_h := \bar{\sigma}_{h,1}, \tag{4.7}
\]
\[
\bar{\sigma}'_{h,0} := 0, \quad \bar{\sigma}'_{h,i+1} := \inf \{ t > \bar{\sigma}'_{h,i} : \bar{Y}'_t \geq h \}, \quad \bar{\sigma}'_h := \bar{\sigma}'_{h,1}, \tag{4.8}
\]
\[
\bar{\tau}_{h,0} := 0, \quad \bar{\tau}_{h,i+1} := \inf \{ t > \bar{\tau}_{h,i} : \bar{Z}_t \geq h \}, \quad \bar{\tau}_h := \bar{\tau}_{h,1}. \tag{4.9}
\]

For \( h \geq 1, \ p \geq 0 \) and \( x \geq 1 \) fixed integers we define the following events:

\[
A_{h,0} := \left\{ \max \{ \bar{Y}_t : 1 \leq t < \infty \} < h \right\} = \left\{ \bar{\sigma}_h = \infty \right\}, \tag{4.10}
\]
\[
A_{h,p} := \left\{ \max \{ \bar{Y}_t : 1 \leq t < \infty \} = h \right\} \land \left\{ \# \{ 1 \leq t < \infty : \bar{Y}_t = h \} = p \right\} = \left\{ \bar{\sigma}_{h,p} < \infty = \bar{\sigma}_{h,p+1} \right\} \land \left\{ \bar{\bar{Y}}_{\bar{\sigma}_{h,i}} = h, \ i = 1, \ldots, p \right\}, \tag{4.11}
\]
\[
A'_{h,0} := \left\{ \max \{ \bar{Y}'_t : 1 \leq t < \infty \} < h \right\} = \left\{ \bar{\sigma}'_h = \infty \right\}, \tag{4.12}
\]
\[
A'_{h,p} := \left\{ \max \{ \bar{Y}'_t : 1 \leq t < \infty \} = h \right\} \land \left\{ \# \{ 1 \leq t < \infty : \bar{Y}'_t = h \} = p \right\} = \left\{ \bar{\sigma}'_{h,p} < \infty = \bar{\sigma}'_{h,p+1} \right\} \land \left\{ \bar{\bar{Y}}'_{\bar{\sigma}'_{h,i}} = h, \ i = 1, \ldots, p \right\}, \tag{4.13}
\]
\( B_{x,h,0} := \{ \max \{ \tilde{Z}_t : 1 \leq t < x \} < h \} \)
\[
= \{ \bar{\tau}_h \geq x \}, \tag{4.14}
\]
\( B_{x,h,p} := \{ \max \{ \tilde{Z}_t : 1 \leq t < x \} = h \} \cap \{ \# \{ 1 \leq t < x : \tilde{Z}_t = h \} = p \} \)
\[
= \{ \bar{\tau}_{h,p} < x \leq \bar{\tau}_{h,p+1} \} \cap \{ \tilde{Z}_{\bar{\tau}_{h,i}} = h, i = 1, \ldots , p \} \}. \tag{4.15}
\]

With the help of the Ray-Knight representation and the events introduced above we get the expression:
\[
\mathbb{E}(u(x)) = \sum_{p+q+r=3} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P(A_{h,p} \mid Y_0 = h - k - 1) \times \pi(k, h - k - 1) \times P(B_{x,h,q} \land \{ Z_{x-1} = l \} \mid Z_0 = k) \times P(A'_{h,r} \mid Y'_0 = l) \tag{4.16}
\]

which leads directly to
\[
\sum_{x=1}^{\infty} \mathbb{E}(u(x)) \leq \sum_{p+q+r=3} \sum_{h=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P(A_{h,p} \mid Y_0 = h - k - 1) \times \pi(k, h - k - 1) \times \left( \sum_{x=1}^{\infty} P(B_{x,h,q} \mid Z_0 = k) \right) \times \left( \sup_{l \geq 0} P(A'_{h,r} \mid Y'_0 = l) \right) \tag{4.17}
\]

The proof of Theorem 1 will follow directly from the bounds provided by

**Proposition 1.** For any \( \varepsilon > 0 \) there exists a finite constant \( C < \infty \) such that for any \( h \geq 1 \) and \( k \geq 0 \):

(i) Without any restriction on \( k \) or \( p \)
\[
\sum_{x=1}^{\infty} P(B_{x,h,p} \mid Z_0 = k) \leq Ch \tag{4.18}
\]

(ii) if either \( k \in [(h - h^{1/2+\varepsilon})/2, (h - h^{1/2+\varepsilon})/2] \) or \( p \geq 1 \) holds then
\[
P(A_{h,p} \mid Y_0 = k) \leq \left( Ch^{-1/2+\varepsilon} \right)^{p+1} \tag{4.19}
\]
\[
\sum_{x=1}^{\infty} P(B_{x,h,p} \mid Z_0 = k) \leq \left( Ch^{-1/2+\varepsilon} \right)^{p+1} h \tag{4.20}
\]

Remark: For \( k \geq h \) the left hand side of (4.18), (4.19) and (4.20), of course, vanish.
We postpone the proof of this Proposition to the end of the next section and proceed with the proof of Theorem 1. Using the bounds (4.18)-(4.20) we prove (2.10). As we already mentioned, (2.11) is proved in a completely identical way. Theorem 1 follows from (2.10), (2.11) via (2.7).

In the forthcoming argument \( \cdots \) will stand as abbreviation of the summand on the right hand side of (4.17). On the right hand side of (4.17) keep \( p, q, r \) and \( h \) fixed and decompose the sum over \( k \geq 0 \) as follows:

\[
\sum_k \cdots = \sum_{k:|h-2k| \leq h^{1/2+\varepsilon}} \cdots + \sum_{k:|h-2k| > h^{1/2+\varepsilon}} \cdots \tag{4.21}
\]

Similar decompositions will be applied a few more times throughout the paper.

**Side-lemma 1.** For any \( \varepsilon > 0 \) there exist constants \( C < \infty \) and \( \gamma > 0 \) such that for any \( h \geq 1 \)

\[
\sum_{k:|h-2k| > h^{1/2+\varepsilon}} \pi(k, h - 1 - k) < C \exp(-\gamma h^{2\varepsilon}). \tag{4.22}
\]

Side-lemmas are proved in Section 6.

Using (4.18)-(4.21) we bound the sum over \( k \) on the right hand side of (4.17) as follows:

If \( r = 0 \) and \( p + q = 3 \)

\[
\sum_k \cdots \leq \left(CH^{-1/2+\varepsilon}\right)^{p+q+2} h + (Ch) \left(C \exp(-\gamma h^{2\varepsilon})\right) \leq C'h^{-3/2+5\varepsilon} \tag{4.23}
\]

with some properly chosen \( C' < \infty \).

If \( r > 0 \) and \( p + q + r = 3 \)

\[
\sum_k \cdots \leq \left(CH^{-1/2+\varepsilon}\right)^{p+q+r+3} h + (Ch) \left(C \exp(-\gamma h^{2\varepsilon})\right) \leq C'h^{-2+6\varepsilon} \tag{4.24}
\]

with some properly chosen \( C' < \infty \).

In both cases the upper bound is summable over \( h \geq 1 \), if we choose \( \varepsilon < 1/10 \).

Hence (2.10) and the statement of Theorem 1.

\( \square \) (Theorem 1)

5. Technical Lemmas.

The present section is divided in five subsections. In subsections 5.1-5.4 we state and prove some lemmas of more technical nature, needed in the proof of Proposition 1, which is presented in subsection 5.5. Throughout this section \( \varepsilon > 0 \) is fixed.
5.1. The Maximal Jump.

We prove that the largest jump of the Markov chains $Y_t$ and $Z_t$, before reaching level $h$, is less than $h^{1/2+\varepsilon}$, with overwhelming probability. Define the maximal jumps of $Y_t$, respectively, $Z_t$ as follows:

$$M_h := \sup \{|Y_t - Y_{t-1}| : 1 \leq t \leq \sigma_h\}$$
$$N_h := \sup \{|Z_t - Z_{t-1}| : 1 \leq t \leq \tau_h\}. \quad (5.1)$$

By definition $M_h = 0$ if $Y_0 \geq h$ and $N_h = 0$ if $Z_0 \geq h$.

**Lemma 1.** There exist two constants, $C < \infty$ and $\gamma > 0$, such that for any $h \geq 1$ and $k \geq 0$ the following bounds hold:

$$P\left(M_h > h^{1/2+\varepsilon} \mid Y_0 = k\right) < C \exp(-\gamma h^{2\varepsilon}), \quad (5.3)$$
$$P\left(N_h > h^{1/2+\varepsilon} \mid Z_0 = k\right) < C \exp(-\gamma h^{2\varepsilon}). \quad (5.4)$$

**Proof.** We prove here (5.3) in details. The proof of (5.4) is essentially similar and it is left for the reader. For the moment let $\gamma$ be an arbitrary positive number. Its value will be fixed at the end of this proof.

$$P\left(M_h > h^{1/2+\varepsilon} \mid Y_0 = k\right) \leq P\left([M_h > h^{1/2+\varepsilon}] \cap [\sigma_h \wedge \omega \leq h^2 \exp(\gamma h^{2\varepsilon})] \mid Y_0 = k\right)$$
$$+ P\left(\sigma_h \wedge \omega > h^2 \exp(\gamma h^{2\varepsilon}) \mid Y_0 = k\right). \quad (5.5)$$

**Side-lemma 2.** There exists a constant $C < \infty$, such that for any $h \geq 1$ and $k \geq 0$

$$E\left(\sigma_h \wedge \omega \mid Y_0 = k\right) \leq Ch^2. \quad (5.6)$$

By using Markov’s inequality we get the following upper bound on the second term of the right hand side of (5.5):

$$P\left(\sigma_h \wedge \omega > h^2 \exp(\gamma h^{2\varepsilon}) \mid Y_0 = k\right) \leq C \exp(-\gamma h^{2\varepsilon}). \quad (5.7)$$

To bound the first term on the right hand side of (5.5) we use the following representation of the Markov chain $Y_t$: Let $(\xi_{t,i})_{t \geq 1, i \geq 1}$ be i.i.d random variables with common geometric distribution $P(\xi_{t,i} = k) = 2^{-k-1}$. The process $Y_t$ is realized as follows: fix $Y_0$ and put

$$Y_{t+1} = \sum_{j=1}^{Y_t} \xi_{t+1,j}. \quad (5.8)$$
Using this representation we note that
\[ P\left(\left[M_h > h^{1/2+\varepsilon}\right] \wedge [\sigma_h \wedge \omega \leq h^2 \exp(\gamma h^{2\varepsilon})]\right|Y_0 = k) \]
\[ \leq P\left(\max_{1 \leq j \leq h} \left|\sum_{i=1}^{j} (\xi_{t,i} - 1)\right| : 1 \leq t \leq h^2 \exp(\gamma h^{2\varepsilon}) > h^{1/2+\varepsilon}\right) \]
\[ = 1 - \left(1 - P\left(\max_{1 \leq j \leq h} \left|\sum_{i=1}^{j} (\xi_{1,i} - 1)\right| > h^{1/2+\varepsilon}\right)\right)^{h^2 \exp(\gamma h^{2\varepsilon})} \] (5.9)

**Side-lemma 3.** Let \( \xi_i \) be i.i.d. random variables with the common geometric distribution \( P(\xi_i = k) = 2^{-k-1} \). Then there is a constant \( \theta_0 > 0 \) such that for any \( \lambda > 0 \) and \( n \in \mathbb{N} \) satisfying \( \lambda / (4n) < \theta_0 \):
\[ P\left(\max_{1 \leq j \leq n} \left|\sum_{i=1}^{j} (\xi_i - 1)\right| > \lambda\right) \leq 2 \exp(-\lambda^2/(8n)). \] (5.10)

Using this bound we get
\[ P\left(\left[M_h > h^{1/2+\varepsilon}\right] \wedge [\sigma_h \wedge \omega \leq h^2 \exp(\gamma h^{2\varepsilon})]\right|Y_0 = k) \]
\[ \leq 1 - \left(1 - 2 \exp(-h^{2\varepsilon}/8)\right)^{h^2 \exp(\gamma h^{2\varepsilon})} \]
\[ \leq 2h^2 \exp\left((\gamma - 8^{-1})h^{2\varepsilon}\right). \] (5.11)

In the last inequality we use the fact that for \( 0 < \alpha < 1 < \beta, \ 1 - \alpha \beta < (1 - \alpha)^\beta \). We choose \( \gamma < 16^{-1} \). From (5.5), (5.7) and (5.11) we get (5.3), with an appropriately chosen constant \( C < \infty \).

□(Lemma 1)

5.2. Hitting exactly \( h \).

**Lemma 2.** There exists a constant \( C < \infty \) such that for any \( h \geq 1 \) and \( k \geq 0 \)
\[ P\left(\left[\sigma_h < \infty\right] \wedge \left[Y_{\sigma_h} = h\right]\right|Y_0 = k) < Ch^{-1/2+\varepsilon} \] (5.12)
\[ P\left(\sigma_h = h\right|Z_0 = k) < Ch^{-1/2+\varepsilon} \] (5.13)

*Proof.* Again, we give the details of the proof of (5.12), leaving the identical details of (5.13) for the reader.
\[ P\left(\left[\sigma_h < \infty\right] \wedge \left[Y_{\sigma_h} = h\right]\right|Y_0 = k) = \]
\[ \sum_{l=0}^{\infty} P\left(\left[\sigma_h < \infty\right] \wedge \left[Y_{\sigma_h-l} = l\right] \wedge \left[Y_{\sigma_h} = h-l\right]\right|Y_0 = k) \] (5.14)
We divide the sum in two parts, as in (4.21):

\[
\sum_{l:|h-2l|>h^{1/2+\varepsilon}} P\left(\tilde{\sigma}_h < \infty \land [Y_{\tilde{\sigma}_h - 1} = l] \land [Y_{\tilde{\sigma}_h} = h-l] \big| Y_0 = k\right) \leq P\left(M_h > h^{1/2+\varepsilon} \big| Y_0 = k\right) < C \exp(-\gamma h^{2\varepsilon}),
\]

(5.15)

by Lemma 1. On the other hand:

\[
\sum_{l:|h-2l| \leq h^{1/2+\varepsilon}} P\left(\tilde{\sigma}_h < \infty \land [Y_{\tilde{\sigma}_h - 1} = l] \land [Y_{\tilde{\sigma}_h} = h-l] \big| Y_0 = k\right) = \\
\sum_{l:|h-2l| \leq h^{1/2+\varepsilon}} P\left(\tilde{\sigma}_h < \infty \land [Y_{\tilde{\sigma}_h - 1} = l] \big| Y_0 = k\right) \frac{\pi(l, h-l)}{\sum_{m \geq h-l} \pi(l, m)}
\]

(5.16)

**Side-lemma 4.** There exists a constant \(C < \infty\), such that for any \(h \geq 1\) and \(l \in [(h - h^{1/2+\varepsilon})/2, (h + h^{1/2+\varepsilon})/2]\)

\[
\frac{\pi(l, h-l)}{\sum_{m \geq h-l} \pi(l, m)} < Ch^{-1/2+\varepsilon}
\]

(5.17)

From this we get

\[
\sum_{l:|h-2l| \leq h^{1/2+\varepsilon}} P\left(\tilde{\sigma}_h < \infty \land [Y_{\tilde{\sigma}_h - 1} = l] \land [Y_{\tilde{\sigma}_h} = h-l] \big| Y_0 = k\right) \leq C h^{-1/2+\varepsilon}. \quad (5.18)
\]

Finally, (5.15) and (5.18) yield (5.13).

\(\square\) (Lemma 2)

5.3. \(\tilde{Y}_t\) does not hit level \(\geq h\).

**Lemma 3.** There exists a constant \(C < \infty\) such that for any \(h \geq 1\) and \(k \in [(h - h^{1/2+\varepsilon})/2, (h + h^{1/2+\varepsilon})/2]\)

\[
P\left(\tilde{\sigma}_h = \infty \big| Y_0 = k\right) < C h^{-1/2+\varepsilon}.
\]

(5.19)

**Proof:**

\[
P\left(\tilde{\sigma}_h = \infty \big| Y_0 = k\right) \leq P\left(\tilde{\sigma}_h = \infty \land [M_h \leq h^{1/2+\varepsilon}] \big| Y_0 = k\right) \\
+ P\left(M_h > h^{1/2+\varepsilon} \big| Y_0 = k\right).
\]

(5.20)

To bound the first term on the right hand side, note that

\[
\{\tilde{\sigma}_h = \infty \land [M_h \leq h^{1/2+\varepsilon}]\} \subset \{\sigma_{(h + h^{1/2+\varepsilon})/2} = \infty\}.
\]

(5.21)
Side-lemma 5. There exists a constant $C < \infty$ such that for any $0 \leq k < h$

$$P\left(\sigma_h = \infty \Big| Y_0 = k\right) < \frac{h - k}{h} + Ch^{-1/2}.$$ \hspace{1cm} (5.22)

Thus

$$P\left(\sigma_h = \infty \bigwedge [M_h \leq h^{1/2+\varepsilon}] \big| Y_0 = k\right) \leq P\left(\sigma_{(h+h^{1/2+\varepsilon})/2} = \infty \big| Y_0 = k\right) \leq Ch^{-1/2+\varepsilon},$$ \hspace{1cm} (5.23)

for $k \in [(h - h^{1/2+\varepsilon})/2, (h + h^{1/2+\varepsilon})/2]$. This bound, together with (5.20) and (5.3) yield (5.19).

□ (Lemma 3)

5.4. Expectation of $\tilde{\tau}_h$.

Lemma 4. There exists a constant $C < \infty$ such that for any $h \geq 1$ the following bounds hold:

(i) Without any restriction on $k$:

$$E\left(\tilde{\tau}_h \big| Z_0 = k\right) < Ch.$$ \hspace{1cm} (5.24)

(ii) For $k \in [(h - h^{1/2+\varepsilon})/2, (h + h^{1/2+\varepsilon})/2]$

$$E\left(\tilde{\tau}_h \big| Z_0 = k\right) < Ch^{1/2+\varepsilon}.$$ \hspace{1cm} (5.25)

Proof.

$$E\left(\tilde{\tau}_h \big| Z_0 = k\right) = E\left(\tilde{\tau}_h \mathbf{1}_{(N_h \leq h^{1/2+\varepsilon})} \big| Z_0 = k\right) + E\left(\tilde{\tau}_h \mathbf{1}_{(N_h > h^{1/2+\varepsilon})} \big| Z_0 = k\right).$$ \hspace{1cm} (5.26)

We bound the first, respectively, the second term on the right hand side, by noting

$$\tilde{\tau}_h \mathbf{1}_{(N_h \leq h^{1/2+\varepsilon})} \leq \tau_{(h+h^{1/2+\varepsilon})/2},$$ \hspace{1cm} (5.27)

respectively,

$$\tilde{\tau}_h \leq \tau_h.$$ \hspace{1cm} (5.28)

Thus we get

$$E\left(\tilde{\tau}_h \big| Z_0 = k\right) \leq E\left(\tau_{(h+h^{1/2+\varepsilon})/2} \big| Z_0 = k\right) + E\left(\tilde{\tau}_h \big| Z_0 = k\right)^{1/2} P\left(\tau_{(h+h^{1/2+\varepsilon})/2} \big| Z_0 = k\right)^{1/2}.$$ \hspace{1cm} (5.29)
Side-lemma 6. There exists a constant $C < \infty$ such that for any $0 \leq k < h$

$$E(\tau_h | Z_0 = k) < (h - k) + Ch^{1/2}. \tag{5.30}$$

Side-lemma 7. There exists a constant $C < \infty$ such that for any $0 \leq k < h$

$$E(\tau_h^2 | Z_0 = k) < Ch^2. \tag{5.31}$$

Putting together (5.29), (5.30) and (5.31), we get (5.24) and (5.25).

$\square$ (Lemma 4)

5.5 Proof of Proposition 1.

First note that

$$P(A_{h,0} | Y_0 = k) = P(\tilde{\sigma}_h = \infty | Y_0 = k), \tag{5.32}$$

$$\sum_{x=1}^{\infty} P(B_{x,h,0} | Z_0 = k) = \sum_{x=1}^{\infty} P(\tilde{\tau}_h \geq x | Z_0 = k) = E(\tilde{\tau}_h | Z_0 = k), \tag{5.33}$$

and for $p \geq 1$, using the strong Markov property of $Y_t$, respectively, $Z_t$:

$$P(A_{h,p} | Y_0 = k) = \sum_{l=0}^{\infty} P(\tilde{\sigma}_h < \infty \land [Y_{\tilde{\sigma}_h-1} = h - l] \land [Y_{\tilde{\sigma}_h} = l] | Y_0 = k) \times$$

$$P(A_{h,p-1} | Y_0 = l), \tag{5.34}$$

$$\sum_{x=1}^{\infty} P(B_{x,h,p} | Z_0 = k) = \sum_{l=0}^{\infty} P([Z_{\tilde{\tau}_h-1} = h - l] \land [Z_{\tilde{\sigma}_h} = l] | Z_0 = k) \times$$

$$\left(\sum_{x=1}^{\infty} P(B_{x,h,p-1} | Z_0 = l)\right). \tag{5.35}$$

We prove the bounds of Proposition 1 by induction on $p$.

According to (5.32), (5.33), for $p = 0$, (4.18), (4.19) and (4.20) are just restatements of (5.24), (5.19) and (5.25), respectively. (See Lemma 3 and Lemma 4.)

Next we consider the case $p = 1$. Again, we divide the sum over $l$ in (5.34) and (5.35) in two parts, as it was done in (4.21). From (5.12) (Lemma 2) and (5.19) (Lemma 3)

$$\sum_{l: |h - 2l| \leq h^{1/2+\varepsilon}} P(\tilde{\sigma}_h < \infty \land [Y_{\tilde{\sigma}_h-1} = h - l] \land [Y_{\tilde{\sigma}_h} = l] | Y_0 = k) P(A_{h,0} | Y_0 = l)$$

$$\leq \left(Ch^{-1/2+\varepsilon}\right) \left(Ch^{-1/2+\varepsilon}\right). \tag{5.36}$$
From (5.3) (Lemma 1)
\[
\sum_{l: |h-2l|>h^{1/2+\epsilon}} \mathbb{P} \left( \bar{\sigma}_h < \infty \right) \land \left[ Y_{\bar{\sigma}_h-1} = h-l \right] \land \left[ Y_{\bar{\sigma}_h} = l \right] \mathbb{P} \left( A_{h,0} \mid Y_0 = l \right) \leq \mathbb{P} \left( M_h > h^{1/2+\epsilon} \mid Y_0 = k \right) < C \exp(-\gamma h^{2\epsilon})
\] (5.37)

From (5.36) and (5.37) we get (4.19) for \( p = 1 \).

Applying the same ideas to (5.35): from (5.13) (Lemma 2) and (5.25) (Lemma 4)
\[
\sum_{l: |h-2l|>h^{1/2+\epsilon}} \mathbb{P} \left( Z_{\bar{\tau}_h-1} = h-l \right) \land \left[ Z_{\bar{\tau}_h} = l \right] \mathbb{P} \left( B_{x,h,0} \mid Z_0 = l \right) \leq \left( C h^{-1/2+\epsilon} \right) \left( Ch^{1/2+\epsilon} \right)
\] (5.38)

From (5.4) (Lemma 1) and (5.24) (Lemma 4)
\[
\sum_{l: |h-2l|>h^{1/2+\epsilon}} \mathbb{P} \left( Z_{\bar{\sigma}_h-1} = h-l \right) \land \left[ Z_{\bar{\sigma}_h} = l \right] \mathbb{P} \left( B_{x,h,0} \mid Z_0 = l \right) \leq \mathbb{P} \left( N_h > h^{1/2+\epsilon} \mid Z_0 = k \right) \left( \sup_{l \geq 0} \sum_{x=1}^{\infty} \mathbb{P} \left( B_{x,h,0} \mid Z_0 = l \right) \right) < \left( C \exp(-\gamma h^{2\epsilon}) \right) (Ch)
\] (5.39)

(5.38) and (5.39) yield (4.20) for \( p = 1 \).

For \( p \geq 2 \) the induction follows from the same reasonings, just one does not have to split the sum over \( l \) as in (5.36), (5.37). After the previous arguments we may ignore these completely straightforward details.

\( \Box \) (Proposition 1)

6. Proof of the Side-Lemmas.

First we prove Side-lemmas 1 and 4. Then Side-lemma 3 follows, which relies on an exponential Kolmogorov inequality. These proofs are rather standard ‘classroom exercises’. Side-lemmas 2, 5, 6 and 7 follow from an estimate on the overshooting of level \( h \) by the processes \( Y_t \) and \( Z_t \) stopped at \( \sigma_h \land \omega \), respectively, \( \tau_h \) and from optional stopping arguments.

Proof of Side-lemma 1.
Assume \( h \geq 2 \) and denote \( h-2 =: n, k-1 =: l \). Then, using the explicit form (3.1) of \( \pi(i,j) \), the right hand side of (4.22) becomes
\[
\sum_{k: |h-2k|>h^{1/2+\epsilon}} \pi(k, h-1-k) = \frac{1}{2} \mathbb{P} \left( |2B_n-n| > (n+2)^{1/2+\epsilon} \right)
\] (6.1)
where $B_n$ is binomially distributed: $P(B_n = l) = \binom{n}{l}2^{-n}$. Using the fact that for any $\gamma < 1/2$

$$
\sup_n E \left( \exp \left\{ \gamma (2B_n - n)^2 / n \right\} \right) = C_\gamma < \infty,
$$

(6.2)

by Markov’s inequality we get (4.22).

□ (Side-lemma 1)

Proof of Side-lemma 4.

Note first that for $i \geq 1$ and $j \geq 0$

$$
\pi(i, j + 1) = \pi(i, j) \frac{i + j}{2(1 + j)}
$$

(6.3)

From this it follows that the distribution $j \mapsto \pi(i, j)$ is unimodular and for $i \geq 2$ fixed

$$
\pi(i, j) < \pi(i, j + 1), \quad \text{for } 0 \leq j \leq i - 3,
$$

$$
\pi(i, i - 2) = \pi(i, i - 1),
$$

$$
\pi(i, j) > \pi(i, j + 1) \quad \text{for } i - 1 \leq j < \infty.
$$

(6.4)

We treat separately the cases $l \in [h/2, (h + h^{1/2+\varepsilon})/2]$ and $l \in [(h - h^{1/2+\varepsilon})/2, h/2]$:

For $l \in [h/2, (h + h^{1/2+\varepsilon})/2]$ the following two facts imply (5.17)

(1) By (6.4),

$$
\pi(l, h - l) \leq \pi(l, l - 1) = \frac{1}{2} \left( \frac{2(l - 1)}{l - 1} \right) - \frac{1}{2} \left( \frac{2([h/2] - 1)}{[h/2] - 1} \right) - \frac{2((h/2) - 1)}{[h/2] - 1} \leq C h^{-1/2}
$$

(6.5)

(2) By the central limit theorem: $\lim_{l \to \infty} \sum_{m \geq l} \pi(l, m) = \frac{1}{2}$ and thus there exists a constant $c > 0$ such that for any $0 < h/2 \leq l$

$$
\sum_{m \geq h - l} \pi(l, m) \geq \sum_{m \geq l} \pi(l, m) \geq c
$$

(6.6)

Let now $l \in [(h + h^{1/2-\varepsilon})/2, h/2]$ and $k := [h - l + h^{1/2-\varepsilon}]$ Then:

$$
\frac{\pi(l, h - l)}{\sum_{m \geq h - l} \pi(l, m)} \leq (k - h + l + 1)^{-1} \frac{\pi(l, h - l)}{\pi(l, k)}
$$

$$
\leq (k - h + l + 1)^{-1} \left( \frac{\pi(l, k - 1)}{\pi(l, k)} \right)^{k - h + l} = \ldots
$$
\[ \cdots = (k - h + l + 1)^{-1} \left( \frac{2k}{l + k - 1} \right)^{k-h+l} \leq h^{-1/2+\varepsilon} \left( \frac{2(h - l + h^{1/2-\varepsilon})}{h + h^{1/2-\varepsilon} - 1} \right)^{h^{1/2-\varepsilon}} \leq h^{-1/2+\varepsilon} \left( \frac{h + h^{1/2+\varepsilon} + 2h^{1/2-\varepsilon}}{h + h^{1/2-\varepsilon} - 1} \right)^{h^{1/2-\varepsilon}} \leq h^{-1/2+\varepsilon} \left( 1 + 3h^{-1/2-\varepsilon} \right)^{h^{1/2-\varepsilon}} \leq e^3 h^{-1/2+\varepsilon} \] (6.7)

In the first inequality we used (6.1). In the second one we exploited the fact that, according to (6.2), for any \( i \geq 1 \) fixed \( \pi(i, j)/\pi(i, j + 1) \) is an increasing function of \( j \geq 0 \). In the next equality, (6.2) was explicitly used. In the third inequality we inserted the value of \( k = \lfloor h - l + h^{1/2-\varepsilon} \rfloor \). In the fourth inequality \( l = [ (h - h^{1/2-\varepsilon})/2 \] was inserted to maximize the expression. In the last but one inequality we used \( 1 \leq h^{1/2-\varepsilon} \leq h^{1/2+\varepsilon} \). Finally, in the last inequality we use the fact that \( \sup_{\alpha \geq 1} (1 + 3\alpha^{-1})^\alpha \leq e^3 \)

\( \square \) (Side-lemma 4)

**Proof of Side-lemma 3.**

The Exponential Kolmogorov Inequality follows directly from Doob’s maximal inequality. For its proof see e.g. page 139 of Williams (1991).

**Exponential Kolmogorov Inequality.** Let \( \xi_j, j \geq 1 \), be i.i.d. random variables with \( \mathbf{E} \left( \exp \left\{ \theta \left| \xi_j \right| \right\} \right) < \infty \) for some \( \theta > 0 \) and \( \mathbf{E} \left( \xi_j \right) = 0 \). Then for any \( \lambda \in (0, \infty) \) and \( n \in \mathbb{N} \)

\[ \mathbf{P} \left( \max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} \xi_i \right| > \lambda \right) \leq e^{-\lambda \theta} \left\{ \mathbf{E} \left( e^{\theta \xi_j} \right)^n + \mathbf{E} \left( e^{-\theta \xi_j} \right)^n \right\} \] (6.8)

We apply the Exponential Kolmogorov Inequality to \( \tilde{\xi}_i = \xi_i - 1 \), with \( \mathbf{P} \left( \xi_i = k \right) = 2^{-k-1}, k \geq 0 \). There exists a constant \( \theta_0 > 0 \) such that for \( 0 \leq \theta < \theta_0 \) we get:

\[ \mathbf{E} \left( e^{\theta (\xi_i - 1)} \right) = e^{-\theta} (2 - e^\theta)^{-1} = 1 + \theta^2 + \mathcal{O}(\theta^3) < e^{2\theta^2} \] (6.9)

\[ \mathbf{E} \left( e^{-\theta (\xi_i - 1)} \right) = e^{\theta} (2e^\theta - 1)^{-1} = 1 + \theta^2 + \mathcal{O}(\theta^3) < e^{2\theta^2} \] (6.10)

Inserting these bounds into the right hand side of (6.8) and choosing \( \theta = \lambda/(4n) \) we obtain (5.10).

\( \square \) (Side-lemma 3)

The proof of Side-lemmas 2, 5, 6 and 7 will follow from the forthcoming Overshooting Lemma and standard optional stopping considerations. The Overshooting Lemma and its Corollary are extended restatements of Lemmas 3.2 and 3.4 from Tóth and Werner (1997).
**Overshooting Lemma.** For any \(0 \leq k < h \leq u\) the following overshoot bounds hold:

\[
P(Y_{\sigma_k} \geq u \mid Y_0 = k) \land [\sigma_k < \infty] \leq P(Y_1 \geq u \mid Y_0 = h) \land [Y_1 \geq h]
\]

\[
= \frac{\sum_{w=h}^{\infty} \pi(h,v)}{\sum_{w=h}^{\infty} \pi(h,w)}.
\]  

(6.11)

\[
P(Z_{\tau_k} \geq u \mid Z_0 = k) \leq P(Z_1 \geq u \mid Z_0 = h) \land [Z_1 \geq h]
\]

\[
= \frac{\sum_{v=u}^{\infty} \rho(h,v)}{\sum_{w=h}^{\infty} \rho(h,w)}.
\]  

(6.12)

In particular it follows that

**Corollary.** There exists a constant \(C < \infty\) such that for any \(0 \leq k < h\):

\[
E(Y_{\sigma_k} \mid Y_0 = k) \land [\sigma_k < \infty] \leq \frac{\sum_{v=h}^{\infty} \pi(h,v)}{\sum_{w=h}^{\infty} \pi(h,w)} \leq h + Ch^{1/2}
\]  

(6.13)

\[
E(Y_{\sigma_k}^2 \mid Y_0 = k) \land [\sigma_k < \infty] \leq \frac{\sum_{v=h}^{\infty} \pi(h,v)^2}{\sum_{w=h}^{\infty} \pi(h,w)} \leq h^2 + Ch^{3/2}
\]  

(6.14)

\[
E(Z_{\tau_k} \mid Z_0 = k) \leq \frac{\sum_{v=h}^{\infty} \rho(h,v)}{\sum_{w=h}^{\infty} \rho(h,w)} \leq h + Ch^{1/2}
\]  

(6.15)

\[
E(Z_{\tau_k}^2 \mid Z_0 = k) \leq \frac{\sum_{v=h}^{\infty} \rho(h,v)^2}{\sum_{w=h}^{\infty} \rho(h,w)} \leq h^2 + Ch^{3/2}
\]  

(6.16)

The rightmost bounds in (6.13)-(6.16) follow from explicit computations.

**Proof of the Overshooting Lemma.**

Straightforward manipulations lead to the following identities for \(1 \leq h \leq v\):

\[
P(Y_{\sigma_k} = v \mid Y_0 = k) \land [\sigma_k < \infty] = \sum_{l=0}^{h-1} P(Y_{\sigma_k} = l \mid Y_0 = k) \land [\sigma_k < \infty] \frac{\pi(l,v)}{\sum_{w=h}^{\infty} \pi(l,w)}.
\]  

(6.17)

\[
P(Z_{\tau_k} = v \mid Z_0 = k) = \sum_{l=0}^{h-1} P(Z_{\tau_k} = l \mid Z_0 = k) \frac{\rho(l,v)}{\sum_{w=h}^{\infty} \rho(l,w)}.
\]  

(6.18)

Using the explicit form (3.1), respectively, (3.2) of the transition probabilities \(\pi(i,j)\), respectively, \(\rho(i,j)\), it is easy to check the following inequalities, which hold for any \(0 < l < h \leq v\), respectively, \(0 \leq l < h \leq v\):

\[
\frac{\pi(l+1,v)}{\pi(l,v)} = \frac{l+v+1}{2l} < \frac{l+v+1}{2l+1} = \frac{\pi(l+1,v+1)}{\pi(l,v+1)},
\]  

(6.19)

\[
\frac{\rho(l+1,v)}{\rho(l,v)} = \frac{l+v+1}{2(l+1)} < \frac{l+v+2}{2(l+1)} = \frac{\rho(l+1,v+1)}{\rho(l,v+1)}.
\]  

(6.20)
It follows that for any $0 \leq l < h \leq v < w$

\[
\pi(l + 1, v)\pi(l, w) < \pi(l, v)\pi(l + 1, w),
\]

\[(6.21)\]

\[\rho(l + 1, v)\rho(l, w) < \rho(l, v)\rho(l + 1, w).
\]

\[(6.22)\]

Hence, for any $0 \leq l < h \leq u$

\[
\sum_{v=h}^{\infty} \pi(l + 1, v) \sum_{w=u}^{\infty} \pi(l, w) < \sum_{v=h}^{\infty} \pi(l, v) \sum_{w=u}^{\infty} \pi(l + 1, w),
\]

\[(6.23)\]

\[
\sum_{v=h}^{\infty} \rho(l + 1, v) \sum_{w=u}^{\infty} \rho(l, w) < \sum_{v=h}^{\infty} \rho(l, v) \sum_{w=u}^{\infty} \rho(l + 1, w),
\]

\[(6.24)\]

which directly imply \((6.11)\), respectively, \((6.12)\).

\(\square\) (Overshooting Lemma)

**Proof of Side-lemmas 2 and 5.**
We apply the Optional Stopping Theorem to the martingales $Y_t$ (for Side-lemma 5), respectively, $Y_2^2 - 2 \sum_{s=0}^{\tau_h-1} Y_s$ (for Side-lemma 2), $t \geq 0$, both stopped at $\sigma_h \wedge \omega$.

First we prove Side-lemma 5:

\[
k = \mathbb{E}\left(Y_{\sigma_h \wedge \omega} \big| Y_0 = k\right) = \mathbb{E}\left(Y_{\sigma_h} \big| Y_0 = k\right) \wedge \mathbb{P}\left(\sigma_h < \infty \big| Y_0 = k\right)
\]

\[
\leq \left(h + C\sqrt{h}\right) \mathbb{P}\left(\sigma_h < \infty \big| Y_0 = k\right).
\]

\[(6.25)\]

Where, in the last inequality we applied \((6.13)\). Hence \((5.23)\).

\(\square\) (Side-lemma 5)

Next we prove Side-lemma 2:

\[
k^2 = \mathbb{E}\left(Y_{\sigma_h \wedge \omega}^2 \big| Y_0 = k\right) \leq \mathbb{E}\left(Y_{\sigma_h \wedge \omega}^2 \big| Y_0 = k\right) - 2\mathbb{E}\left(\sigma_h \wedge \omega \big| Y_0 = k\right),
\]

\[(6.26)\]

where in the last inequality we used the fact that $Y_s \geq 1$ for $s < \omega$. Hence

\[
2\mathbb{E}\left(\sigma_h \wedge \omega \big| Y_0 = k\right) \leq \mathbb{E}\left(Y_{\sigma_h}^2 \big| Y_0 = k\right) \wedge \mathbb{P}\left(\sigma_h < \infty \big| Y_0 = k\right) - k^2 < Ch^2.
\]

\[(6.27)\]

In the last inequality we used \((6.14)\).

\(\square\) (Side-lemma 2)

**Proof of Side-lemmas 6 and 7.**
We apply the Optional Stopping Theorem to the martingale $Z_t - t$ (for Side-lemma 6), respectively, to the supermartingale $t^2 - 2tZ_t$ (for Side-lemma 7), $t \geq 0$, both stopped at $\tau_h$. 
First Side-lemma 6:

\[ E\left( \tau \mid Z_0 = k \right) = E\left( Z_{\tau} \mid Z_0 = k \right) - k \leq h - k + C\sqrt{h}. \]  \hspace{1cm} (6.28)

In the last inequality (6.15) was used.

□ (Side-lemma 6)

Next, Side-lemma 7:

\[ E\left( \tau^2 \mid Z_0 = k \right) \leq 2E\left( \tau \tau_\cap Z_0 = k \right) \leq 2\sqrt{E\left( \tau^2 \mid Z_0 = k \right)} \sqrt{E\left( Z^2 \mid Z_0 = k \right)} \]  \hspace{1cm} (6.29)

Hence, using (6.16) we get

\[ E\left( \tau^2 \mid Z_0 = k \right) \leq 4E\left( Z^2 \tau \mid Z_0 = k \right) \leq Ch^2. \]  \hspace{1cm} (6.30)

□ (Side-lemma 7)

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