Goldbug Variations

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Jim Propp bugs me sometimes. I’m usually glad when he does.
Today, Jim’s bugs are trained to hop back and forth on the positive integers: place a bug at 1, and with each hop, a bug at \( i \) moves to either \( i + 1 \) or \( i - 2 \). Of course, it might jump off the left edge; we put two bug-catching cups at 0 and \(-1\). Once a bug lands in a cup, we start a new bug at 1.

What I haven’t mentioned is how the bugs decide whether to jump left or right. We could declare it a random walk, stepping in either direction with probability \( \frac{1}{2} \), but the U. of Wisconsin professor’s bugs are more orderly than that. At each location \( i \), there is a signpost showing an arrow: it can point either Inbound, towards \( i - 2 \) and the bug cups, or Outbound, towards infinity. The bugs are somewhat contrarian, so when a bug lands at \( i \), it first flips the arrow to point the opposite direction, and then hops that way (Fig. 1). Add an initial condition that all arrows begin pointing Outbound, and we have a deterministic system. Let bugs hop till they drop (Fig. 2).

Well, what happens? Or, for those who would like a more directed question: First, show that every bug lands in a cup (as opposed to going off

![Figure 1: The two bug bounces.](image-url)
to infinity, or bouncing around in some bounded region forever). Second, find what fraction of the bugs end their journeys in the cup at $-1$, in the many-bugs limit. Go ahead, I’ll wait. Do the first ten bugs by hand and look for the beautiful pattern. You can even skip ahead to the next section and read about another related bug, one with far more inscrutable behavior, and come back and read my solution another day.

I should mention, by way of a delaying tactic, that the analysis of this bug was first done by Jim Propp and Ander Holroyd. A previous and closely related bug of theirs, in which every third visit to $i$ leads to $i-1$, appears in Peter Winkler’s new book *Mathematical Puzzles: A Connoisseur’s Collection*, yet another delightful mathematical offering from publishers A K Peters [14]. The book itself is a gold-mine of the type of puzzles that I expect readers of this column would enjoy immensely. Winkler’s solutions are insightful, well-written, and often leave the reader with more to think about than before. The preceding is an unpaid endorsement.

Very well, enough filler; here’s my answer. If you solved the problem without developing something like the theory below, please do let me know how.

Before we get to the serious work, let’s answer the question, can a bug hop around in a bounded area forever? It cannot: let $i$ be the minimal place visited infinitely often by the bug — oops, half the time, that visit is followed by a trip to $i-2$, a contradiction. So each bug either lands in a cup or, as far as we know now, runs off to infinity.

To motivate what follows, let’s have a little inspiration: some carefully chosen experimental data. Perhaps you noticed that of the first five bugs, the cup at $-1$ catches three, while the cup at 0 catches two. If that is not sufficiently suggestive, let me mention that after bug eight the score is $5:3$, while after bug thirteen it is $8:5$. On the speculation that this golden ratio trend continues, let us refer to the bouncing insects as goldbugs.

Now the details. Suppose that $\varphi$ is — I can only imagine your surprise — a real number satisfying $\varphi^2 = \varphi + 1$; when I care to be specific about...
which root, I will write $\varphi_{\pm}$ for $(1 \pm \sqrt{5})/2$.

The goldbugs and signs are, in fact, a number written in base $\varphi$. Position $i$ has “place value” $\varphi^i$, and the digits are conveniently mnemonic: Outbound is 0, Inbound is 1, and the bug itself is the not entirely standard digit $\varphi$, which may appear in addition to the 0 or 1 in the “$\varphi^i$’s place,” where it contributes $\varphi^{i+1}$ to the total. Of course, numbers do not have a unique representation with this setup, but that’s quite deliberate: the total value is an invariant, unchanged by bug bounces.

- **Bounce left:** Suppose the bug arrives in position $i$ and there is an Outbound arrow. The value of this part of the configuration is $(\varphi^i \times \varphi) + (\varphi^i \times 0)$. After the bounce, position $i$ holds an Inbound arrow and the bug has bounced to position $i-2$, for a total value of $(\varphi^i \times 1) + (\varphi^{i-2} \times \varphi)$. And these are the same, since $\varphi^{i-1} + \varphi^i = \varphi^{i+1}$.

- **Bounce right:** Suppose the bug arrives in position $i$ and there is an Inbound arrow. The value before the bounce is $(\varphi^i \times \varphi) + (\varphi^i \times 1)$. After the bounce, the arrow points Outbound, value zero, and the bug is at $i+1$, for a value of $\varphi^{i+2}$, again unchanged.

Now let’s see what happens when we add a bug at 1, let it run its course, and then remove it after it lands in a bug cup. Placing a bug at 1 increases the system’s value by $\varphi^2$. If it eventually lands in the cup at 0 and is then removed, the value of the system drops by $\varphi$, while if it lands in the cup at $-1$ and is removed, the value of the system drops by $\varphi^0 = 1$. So the net effect of adding a bug at 1, running the system, and then removing the bug is that the value of the system increases by $\varphi^2 - \varphi = 1$ for cup 0, and by $\varphi^2 - 1 = \varphi$ for cup $-1$.

Now we are well-equipped to answer the original questions, and then some.

- **Can a bug run off to infinity?** It cannot: if we take $\varphi$ to mean $\varphi_+ \approx 1.61803$, then each bug can increase the net value of the system by at most $\varphi_+$, and the positions far to the right are inaccessible to the goldbugs, because they would make the value of the system too high. So every bug lands in a cup.

- **What is the ratio of bugs landing in the two cups?** This time for $\varphi$, think about $\varphi_- \approx -0.61803$. Between bugs, when the system consists of just Inbound and Outbound flags, its minimum possible value is $\varphi_-^1 + \varphi_-^3 + \varphi_-^5 + \cdots = -1$, while its maximum possible value is $\varphi_-^2 + \varphi_-^4 + \varphi_-^6 + \cdots$.
\[ \varphi^4 + \varphi^6 + \cdots = -\varphi_+ \approx .61803, \] which I will write instead as \(1/\varphi_+\) to help us remember its sign.

So the value of the system is trapped in the interval \([-1, 1/\varphi_+]\), and as each successive bug passes through the system, the value either increases by 1, or else increases by \(\varphi_-\), i.e. decreases by \(1/\varphi_+\). For the value to stay in any bounded region, it must, in the limit, step down \(\varphi_+\) times as often as it steps up, and so the bugs land in the \(-1\) cup \(1/\varphi_+\) of the time. Moreover, this approximation is very good: if \(a\) bugs have landed at \(-1\) and \(b\) have landed at 0, the system’s value \(b-a/\varphi_+\) must lie in our interval, so \(|b/a - 1/\varphi_+| < 1/a\). Every single approximation \(b/a\) is one of the two best possible given the denominator, and that denominator grows as \(n/\varphi_+\).

In fact, notice that the length of the interval \([-1, 1/\varphi_+]\) is the sum of the two jump sizes, the smallest we could possibly hope for. If the value lies in the bottom \(1/\varphi_+\) of the interval, it must increase by 1, while if it lies in the top 1 of the interval, it must decrease by \(1/\varphi_+\). This leaves a single point, \(1/\varphi_+ - 1 \approx -0.38297\), where the bug’s destination cup isn’t determined. But that value can be attained only with infinitely many Inbound signs: if we run a single bug through a system with that starting value, its ending value would be either \(1/\varphi_+\) or \(-1\) — and to attain those two bounds, we found above, you must sum the infinite series of all positive or negative place values. So for any initial configuration with only finitely many signs pointing Inbound, the configuration’s numerical value alone determines the destination cup of every single bug; you don’t need to keep track of the arrows at all.

- If you prefer integers to irrationals, consider instead the following invariant. Label the (bug cups and the) sites with the Fibonacci numbers \((0,1,1,2,3,5,8,\ldots)\), as in Figure 3. Given an Inbound arrow the value \(F_i\) of its site, and give the bug there the value \(F_{i+1}\) of the site to its right. This invariant, of course, is an appropriate linear combination of the \(\varphi_+\) and \(\varphi_-\) ones above.

Adding a bug to the system now increases the value by 2, but what’s special is that removing the bug at either 0 or \(-1\) subtracts 1. So the bugs implement an accumulator: after the \(n\)th bug lands in its cup, we can look at the signs it has left behind, and read off the number \(n\), written in base Fibonacci!

…Hold the presses! Matthew Cook has pointed out to me that this
Figure 3: The signs after bug 117 passes through the system; $117 = 2 + 5 + 8 + 13 + 34 + 55$. Representations in base Fibonacci are not unique; ours is characterized by a lack of two consecutive zeros.

point of view can be taken further. We still get an invariant if we shift all those Fibonacci labels one site to the right. Now adding a bug increases the system’s total value by 1, and removing it from the right bug cup decreases the value by 1 — but removing it from the left bug cup decreases the score by 0. So after $n$ bugs pass through, the value of the Inbound arrows they leave behind counts the number of bugs that ended their journeys in the left bug cup.

Furthermore we can shift the labels right a second time. Now when the bug lands in the left cup, its value must be the $-1$st Fibonacci number, before 0 — which is 1 again, if the Fibonacci recurrence relation is still to hold. So with these labels, the Inbound arrows will count the number of bugs which landed in the right cup. As an exercise, decode the arrows to learn that the 117 bugs leading to Figure 3 split 72:45.

Once we know that the same set of arrows simultaneously count the total, left-cup, and right-cup bugs, it’s straightforward to see that the ratios of these three quantities are the same as the ratios of three consecutive Fibonacci numbers, in the limit.

These arrow-directed goldbugs are doing a great job of what what Jim Propp calls “derandomization.” It’s straightforward to analyze the corresponding random walk, in which bugs hop left or right each with probability $1/2$: Let $p_i$ be the probability that a bug at place $i$ eventually ends up in the cup at $-1$, and solve the recurrence $p_i = (p_{i-1} + p_{i+1})/2$ with $p_{-1} = 1$, $p_0 = 0$ — oh, and make up for one too few initial conditions by remembering that all the $p_i$ are probabilities, so no larger than 1. Here too we get $p_1 = 1/\phi_+$. So the deterministic goldbugs have the same limiting behavior as their random-walking cousins. But if we run the experiment with $n$ random walkers and count the number of bugs in a cup, we’ll generally see variation on the order of $\sqrt{n}$ around the expected number. Remember the sharp results of the $\varphi_-$ invariant: $n$ goldbugs, by contrast, simulate the expected behavior with only constant error!
There are more results on these and related one-dimensional not-very-random walks. But let’s move on — these bugs long for some higher-dimensional elbow room.

**The Rotor-Router**

This time, we will set up a system with bugs moving around the integer points in the plane. It will differ from the above in that this will be an aggregation model. Bugs still get added repeatedly at one source, but instead of falling into sinks (bug cups), the bugs will walk around until they find an empty lattice point, then settle down and live there forever.

Generalizing the Inbound and Outbound arrows of the goldbugs, we decree that each lattice site is equipped with an arrow, or rotor, which can be rotated so that it points at any one of the four neighbors. (Propp uses the word rotor to refer to the two-state arrows in dimension 1 as well.) The first bug to arrive at a particular site occupies it forever, and we decree that it sets the arrow there pointing to the East. Any bug arriving at an occupied site rotates the arrow one quarter turn counterclockwise, and then moves to the neighbor at which the rotor now points — where it may find an empty site to inhabit, or may find a new arrow directing its next step. In short, the first bug to reach \((i, j)\) lives there, and bugs which arrive thereafter are routed by the rotor: first to the North, then West, South, and East, in that order, and then the cycle begins with North again (Fig 4).

Once a bug finds an empty site to inhabit, we drop a new bug at the

Figure 4: A bug’s ramble through some rotors: before and after.
origin, and this one too meanders through the field of rotors, both directed by the arrows and changing them as a result of its visits. Every bug does indeed find a home eventually, and the proof is the same as for the goldbugs: the set of all sites which a particular bug visits infinitely often cannot have any boundary.

And so we ask, as we add more and more bugs, what does the set of occupied sites look like?

Let's take a look at the experimental answer. The beautiful image you see in Figure 5 is a picture of the set of occupied sites after three million bugs have found their permanent homes. The sites in black are vacant, still awaiting their first visitor. Other sites are colored according to the direction of their rotor, red/yellow for East/West and green/blue for North/South. On the cover is a close-up of a part of the boundary of the occupied region. At http://www.math.wisc.edu/~propp/rotor-router-1.0/ you can find a Java applet by Hal Canary and Francis Wong, if you'd like to experiment yourself.

As you can see, the edge of the occupied region is extraordinarily round: with three million bugs, the occupied site furthest from the origin is at distance $\sqrt{956609}$, while the unoccupied site nearest the origin is at distance $\sqrt{953461}$, a difference of about 1.6106.

And the internal coloration puts on a spectacular display of both large-scale structure and intricate local patterns. When Ed Pegg featured the rotor-router on his Mathpuzzle web site, he dubbed the picture a Propp Circle, and to this day I am jealous that I didn’t think of the name first: it connotes precisely the right mixture of aesthetic appreciation and conviction that there must be something deep and not fully understood at work.

Recall, for emphasis, that this was formed by bugs walking deterministically on a square lattice, not a medium known for growing perfect circles. Moreover, the rule that governed the bugs’ movement is inherently asymmetric: every site’s rotor begins pointing East, so there was no guarantee that the set of occupied sites would even appear symmetric under rotation by $90^\circ$, much less by unfriendly angles. On the other hand, while the overall shape seems to have essentially forgotten the underlying lattice, the internal structure revealed by the color-coded rotors clearly remembers it.

Lionel Levine, now a graduate student at U.C. Berkeley, wrote an undergraduate thesis with Propp on the rotor-router and related models [11]. It contains the best result so far on the roundness of the rotor-router blob: after $n$ bugs, it contains a disk whose radius grows as $n^{1/4}$. Below I report on two remarkable theorems which do not quite prove that the rotor-router blob is round, but which at least make me feel that it ought to be. I have...
Figure 5: The rotor-router blob after three million bugs. If you are viewing this image on-line, try magnifying to the point where you can see individual pixels as small boxes. Also, for best appearance of the entire image, use a viewer that will smooth/dither/antialias graphics.
less to offer on the intricate internal structure, but there is a connection to something a bit better understood. Let’s get to work.

**IDLA**

Internal Diffusion Limited Aggregation is the random walk–based model which Propp de-randomized to get the rotor-router. Most everything is as above — the plane starts empty, add bugs to the origin one at a time, each bug occupies the first empty site it reaches. But in IDLA, a bug at an occupied site walks to a random neighbor, each with probability $\frac{1}{4}$.

The idea underlying IDLA comes from a paper by Diaconis and Fulton [6]. They define a ring structure on the vector space whose basis is labeled by the finite subsets of a set $X$ equipped with a random walk. To calculate the product of subsets $A$ and $B$, begin with the set $A \cup B$, place bugs at each point of $A \cap B$, and allow each bug to execute a random walk until it reaches an outside point, which is then added to the set. The product of $A$ and $B$ records the probability distribution of possible outcomes. This appears to depend on the order in which the bugs do their random walks, but in fact it does not — we’ll explore this theme soon.

Consider the special case where $X$ is the $d$-dimensional integer lattice with the random walk choosing uniformly from among the nearest neighbors. Then repeatedly multiplying the singleton $\{0\}$ by itself is precisely the random-walk version of the rotor-router. A paper of Lawler, Bramson and Griffeath [9] dubbed this Internal Diffusion Limited Aggregation, to emphasize similarity with the widely-studied Diffusion Limited Aggregation model of Witten and Sandler [13]. DLA simulates, for example, the growth of dust: successive particles wander in “from infinity” and stick when they reach a central growing blob. The resulting growths appear dendritic and fractal-like, but rigorous results are hard to come by.

In contrast, the growth behavior of IDLA has been rigorously established. It is intuitive that the growing blob should be generally disk-shaped, since the next particle is more likely to fill in an unoccupied site close to the origin than one further away. But the precise statement in [9] is still striking: the random walk manages to forget the anisotropy of the underlying lattice entirely!

**Theorem (Lawler–Bramson–Griffeath).** Let $\omega_d$ be the volume of the $d$-dimensional unit sphere. Given any $\epsilon > 0$, it is true with probability 1 that for all sufficiently large $n$, the $d$-dimensional IDLA blob of $\omega_d n^d$ particles will contain every point in a ball of radius $(1-\epsilon)n$, and no point outside of a ball of radius $(1+\epsilon)n$. 

9
To be more specific, we could hope to define inner and outer error terms such that, with probability 1, the blob lies between the balls of radius $n - \delta_I(n)$ and $n + \delta_O(n)$. In a subsequent paper [10], Lawler proved that these could be taken on the order of $n^{1/3}$. Most recently, Blachère [3] used an induction argument based on Lawler’s proof to show that these error terms were even smaller, of logarithmic size. The precise form of the bound changes with dimension; when $d = 2$ he shows that $\delta_I(n) = O((\ln n \ln(\ln n))^{1/2})$ and $\delta_O(n) = O((\ln n)^2)$. Errors on that order were observed experimentally by Moore and Machta [12].

So how does the random walk–based IDLA relate to the deterministic rotor-router? We start drawing the connection with one key fact.

**It’s abelian!**

Here’s a possibly unexpected property of the rotor-router model: it’s abelian. There are several senses in which this is true.

Most simply, take a state of the rotor-router system — a set of occupied sites and the directions all the rotors point — and add one bug at a point $P_0$ (not necessarily the origin now) and let it run around and find its home $P_1$. Then add another at $Q_0$ and let it run until it stops at $Q_1$. The end state is the same as the result of adding the two bugs in the opposite order. This relies on the fact that the bugs are indistinguishable. Consider the (next-to-)simplest case, in which the paths of the $P$ and $Q$ bugs cross at exactly one point, $R$. If bug $Q$ goes first instead, it travels from $Q_0$ to $R$, and then follows the path the $P$ bug would have, from $R$ to $P_1$. The $P$ bug then goes from $P_0$ to $R$ to $Q_1$. At the place where their paths would first cross, the bugs effectively switch identities. For paths whose intersections are more complicated, we need to do a bit more work, but the basic idea carries us through.

Taking this to an extreme, consider the “rotor-router swarm” variant, where traffic is still directed by rotors at each lattice site, but any number of bugs can pass through a site simultaneously. The system evolves by choosing any one bug at random and moving it one step, following the usual rotor rule. Here too the final state is independent of the order in which bugs move; read on for a proof. To create our rotor-router picture, we can place three million bugs at the origin simultaneously, and let them move one step at a time, following the rotors, in whatever order they like.

In fact, even strictly following the rotors is unnecessary. The rotors control the order in which the bugs depart for the various neighbors, but in the end, we only care about how many bugs head in each direction.
Imagine the following setup: we run the original rotor-router with three million bugs as first described, but each time a bug leaves a site, it drops a card there which reads “I went North” or whichever direction. Now forget about the bugs, and look only at the collection of cards that were left behind at each site. This certainly determines the final state of the system: a site ends up occupied if and only if one of its neighbors has a card pointing towards it.

Now we could re-run the system with no rotors at all. When a bug needs to move on, it may pick up any card from the site it’s on and move in the indicated direction, eating the card in the process. No bug can ever “get stuck” by arriving at an occupied site with no card to tell it a way to leave: the stack of cards at a given site is just the right size to take care of all the bugs that can possibly arrive there coming from all of the neighbors. (There is, however, no guarantee that all the cards will get used; left-overs must form loops.) A version of this “stacks of cards” idea appeared in Diaconis and Fulton’s original paper, in the proof that the random walk version is likewise abelian — i.e. that their product operation is well-defined.

If the bugs are so polite as to take the cards in the cyclic N-W-S-E order in which they were dropped, then we simulate the rotor-router exactly. If we start all the bugs at the origin at once and let them move in whatever order they want — but insist that they always use the top card from the site’s stack — we get the rotor-router swarm variant above; QED.

Rotor-roundness

Now let me outline a heuristic argument that the rotor-router blob ought to be round, letting the Lawler–Bramson–Griffeath paper do all the heavy lifting.

I’d like to say that, for any $c < 1$, the $n$-bug rotor-router blob contains every lattice site in the disk of area $cn$ — as long as $n$ is sufficiently large. My strategy is easy to describe. Just as we did four paragraphs ago, think of each lattice site as holding a giant stack of cards: one card for each time a bug departed that site while the $n$-bug rotor-router blob grew. Now we start running IDLA: we add bugs at the origin, one at a time, and let them execute their random walk. But each time a bug randomly decides to step in a given direction, it must first look through the stack of cards at its site, find a card with that direction written on it, and destroy it.

As long as the randomly-walking bugs always find the cards they look for, the IDLA blob that they generate must be a subset of the rotor-router blob whose growth is recorded in the stacks of cards. This key fact follows
directly from the abelian nature of the models.

So the central question is, how long will this IDLA get to run before a bug wants to step in a particular direction and finds that there is no corresponding card available? Philosophically, we expect the IDLA to run through “almost all” the bugs without hitting such a snag: for any $c < 1$, we expect $cn$ bugs to aggregate, as long as $n$ is sufficiently large. If we can show this, we are certainly done: the rotor-router blob contains an IDLA blob of nearly the same area, which in turn contains a disk of nearly the same area, with probability one.

To justify this intuition, we clearly need to examine the function $d(i, j)$ which counts the number of departures from each site. This is a nonnegative integer-valued function on the lattice which is almost harmonic, away from the origin: the number of departures from a given site is about one quarter of the total number of visits to its four neighbors.

$$d(i, j) \approx \frac{d(i + 1, j) + d(i - 1, j) + d(i, j + 1) + d(i, j - 1)}{4} - b(i, j)$$

Here $b(i, j) = 1$ if $(i, j)$ is occupied and 0 otherwise, to account for the site’s first bug, which arrives but never departs. When $(i, j)$ is the origin, of course, the right-hand side should be increased by the number of bugs dropped into the system. Matthew Cook calls this the “tent equation”: each site is forced to be a little lower than the average of its neighbors, like the heavy fabric of a circus tent; it’s all held up by the circus pole at the origin — or perhaps by a bundle of helium balloons which can each lift one unit of tent fabric, since we do not get to specify the height of the origin, but rather how much higher it is than its surroundings.

For the rotor-router, the approximation sign above hides some rounding error, the precise details of which encapsulate the rotor-router rule. For IDLA, this is exact if we replace $d$ by $\hat{d}$, the expected number of departures, and replace $b$ by $\hat{b}$, the probability that a given site ends up occupied. (The results of [9] even give an approximation of $\hat{d}$.)

Now, I’d like to say that at any particular site, the mean number of departures for an IDLA of $cn$ bugs (for any $c < 1$ and large $n$) should be less than the actual number of departures for a rotor-router of $n$ bugs. If so, we’d be nearly done, with a just a bit of easy calculation to show that the $\sqrt{d}$-sized error terms at each site in the random walk is thoroughly swamped by the $(1 - c)n$ extra bugs in the rotor-router.

But this begs the question of showing that the rotor-router’s function $d$ and the IDLA’s function $\hat{d}$ are really the same general shape. Their difference is an everywhere almost-harmonic function with zero at the boundary —
but to paraphrase Mark Twain, the difference between a harmonic function and an almost-harmonic function is the difference between lightning and a lightning bug.

**Simulation with constant error**

After I wrote the preceding section, I learned of a brand-new result of Joshua Cooper and Joel Spencer. It doesn’t turn my hand-waving into a genuine proof, but it gives me hope that doing so is within reach. Their paper [4] contains an amazing result on the relationship between a random walk and a rotor-router walk in the $d$-dimensional integer lattice $\mathbb{Z}^d$.

Generalizing the rotor-router bugs above, consider a lattice $\mathbb{Z}^d$ in which each point is equipped with a rotor — that is to say, an arrow which points towards one of the $2d$ neighboring points, and which can be incremented repeatedly, causing it to point to all $2d$ neighbors in some fixed cyclic order. The initial states of the rotors can be set arbitrarily.

Now distribute some finite number of bugs arbitrarily on the points. We can let this distribution evolve with the bugs following the rotors: one step of evolution consists of every bug incrementing and then following the rotor at the point it is on. (Our previous bugs were content to stay put if they were at an uninhabited site, but in this version, every bug moves on.) Given any initial distribution of bugs and any initial configuration of the rotors, we can now talk about the result of $n$ steps of rotor-based evolution.

On the other hand, given the same initial distribution of bugs, we could just as well allow each bug to take an $n$-step random walk, with no rotors to influence its movement. If you believe my heuristic babbling above, then it is reasonable to hope that $n$ steps of rotor evolution and $n$ steps of random walk would lead to similar ending distributions.

With one further assumption, this turns out to be true in the strongest of senses. Call a distribution of bugs “checkered” if all bugs are on vertices of the same parity — that is, the bugs would all be on matching squares if $\mathbb{Z}^d$ were colored like a checkerboard.

**Theorem (Cooper–Spencer).** There is an absolute constant bounding the divergence between the rotor and random walk evolution of checkered distributions in $\mathbb{Z}^d$, depending only on the dimension $d$. That is, given any checkered initial distribution of a finite number of bugs in $\mathbb{Z}^d$, the difference between the actual number of bugs at a point $p$ after $n$ steps of rotor-based evolution, and the expected number of bugs at $p$ after an $n$-step random walk, is bounded by a constant. This constant is independent of the number of steps $n$, the initial states of the rotors, and the initial distribution of bugs!
I am enchanted by the reach of this result, and at the same time intrigued by the subtle “checkered” hypothesis on distributions. (Not only initial distributions: since each bug changes parity at each time step, a configuration can never escape its checkered past.) The authors tell me that without this assumption, one can cleverly arrange squadrons of off-parity bugs to reorient the rotors and steer things away from random walk simulation.

Thus the rotor-router deterministically simulates a random walk process with constant error — better than a single instance of the random process usually does in simulating the average behavior. Recall that we saw a similar outcome in one dimension, with the goldbugs.

There are other results which likewise demonstrate that derandomizing systems can reduce the error. Lionel Levine’s thesis [11] analyzed a type of one-dimensional derandomized aggregation model, and showed that it can compute quadratic irrationals with constant error, again improving on the $\sqrt{n}$-sized error of random trials. Joel Spencer tells me that he can use another sort of derandomized one-dimensional system to generate binomial distributions with errors of size $\ln n$ instead of $\sqrt{n}$. Surely the rotor-router should be able to cut IDLA’s already logarithmic-sized variations down to constant ones. Right?

**Coda: Sandpiles**

All of the preceding discussion addresses the overall shape of the rotor-router blob, but says nothing at all about the compelling internal structure that’s visible when we four-color the points according to the directions of the rotors. When we introduced the function $d(i, j)$, counting the number of departures from the $(i, j)$ lattice site, we were concerned with its approximate large-scale shape, which exhibits some sort of radial symmetry. The direction of the rotor tells you the value of $d(i, j) \mod 4$, and the symmetry of these least significant bits of $d$ is an entirely new surprise.

I can’t even begin to explain the fine structure — if you can, please let me know! But I can point out a surprising connection to another discrete dynamical system, also with pretty pictures.

Consider once again the integer points in the plane. Each point now holds a pile of sand. There’s not much room, so if any pile has five or more grains of sand, it collapses, with four grains sliding off of it and getting dumped on the point’s four neighbors. This may, in turn, make some neighboring piles unstable and cause further topplings, and so on, until each pile has size at most four.
Our question: what happens if you put, say, a million grains of sand at the origin, and wait for the resulting avalanche to stop? I won’t keep you hanging; a picture of the resulting rubble appears as Figure 6. Pixels are colored according to the number of grains of sand there in the final configuration. The dominant blue color corresponding to the largest stable pile, four grains. (This makes some sense, as the interior of such a region is stable, with each site both gaining and losing four grains, while evolution happens around the edges.)

This type of evolving system now goes under the names “chip-firing model” and “abelian sandpile model;” the adjective abelian is earned because the operations of collapsing the piles at two different sites commute. In full generality, this can take place on an arbitrary graph, with an excessively large sandpile giving any number of grains of sand to each of its neighbors, and some grains possibly disappearing permanently from the system. Variations have been investigated by combinatorists since about 1991 [2]; they adopted it from the mathematical physics community, which had been developing versions since around 1987 [1, 5]. This too was a rediscovery, as it seems that the mechanism was first described, under the name “the probabilistic abacus,” by Arthur Engel in 1975 in a math education journal [7, 8].

I couldn’t hope to survey the current state of this field here, or even give proper references. The bulk of the work appears to be on what I think of as steady-state questions, far from the effects of initial conditions: point-to-point correlation functions, the distribution of sizes of avalanches, or a marvelous abelian group structure on a certain set of recurrent configurations.

Our question seems to have a different flavor. For example, in most sandpile work, one can assume without loss of generality that a pile collapses as soon as it has enough grains of sand to give its neighbors what they are owed, leaving itself vacant. The version I described above is what I’ll call a “greedy sandpile,” in which each site hoards its first grain of sand, never letting it go. The shape of the rubble in Figure 6 does depend on this detail; Figure 7 is the analogue where a pile collapses as soon as it has four grains, leaving itself empty.

Most compelling to me is the fine structure of the sandpile picture. I’m amazed by the appearance of fractalish self-similarity at different scales despite the single-scale evolution rule; I think this is related to what the mathematical physics people call “self-organizing criticality,” about which I know nothing at all. But personally, in both pictures I am drawn to the eight-petalled central rosette, the boundary of some sort of phase change in their internal structures.
Figure 6: The greedy sand-pile with three million grains.
Figure 7: A non-greedy sand-pile. Here the dominant color is yellow, which again indicates the maximal stable site, now with three grains. It is hard to see the interior pixels colored black, indicating sites which were once filled but are now empty, impossible in the greedy version.
Bugs in the sand

So what is the connection between the greedy sandpile and the rotor-router? Recall the swarm variant of rotor-router evolution: we can place all the bugs at the origin simultaneously, and let them take steps following the rotor rule in any order, and still get the same final state.

Since we get to choose the order, what if we repeatedly pick a site with at least four bugs waiting to move on, and tell four of them to take one step each? Regardless of its state, the rotor directs one to each neighbor, and we mimic the evolution rule of the greedy sandpile perfectly. If we keep doing this until no such sites remain, we realize the sandpile final state as one step along one path to the rotor-router blob.

Note, in particular, that the $n$-bug rotor-router blob must contain all sites in the $n$-grain greedy sandpile. Surely it should therefore be possible to show that both contain a disk whose radius grows as $\sqrt{n}$.

More emphatically, the sandpile performs precisely that part of the evolution of the rotor-router that can take place without asking the rotors to break symmetry. If we define an energy function which is large when multiple bugs share a site, then the sandpile is the lowest-energy state which the rotor-router can get to in a completely symmetric way.

When we invoke the rotors, we can get to a state with minimal energy but without the a priori symmetry that the sandpile evolution rule guarantees. And yet, empirically, the rotor-router final state looks much rounder than that of the sandpile, whose boundary has clear horizontal, vertical, and slope $\pm 1$ segments.

At best, this only hints at why the sandpile and rotor-router internal structures seem to have something in common. For now, these hints are the best I can do.

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