COARSE FREE PRODUCTS

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Abstract. We define a notion of free product for coarse spaces that generalizes the corresponding notion of a free product for groups. We show that free products preserve coarse properties such as coarse property C, finite coarse decomposition complexity, and coarse property A. We also give an upper bound estimate on the dimension of a coarse free product in terms of the dimension of its factors.

1. Introduction

The free product of groups arises in the computation of fundamental groups of path-connected spaces expressed as a union of path-connected subspaces with simply connected intersection. Given groups \( A = \langle S_A \mid R_A \rangle \) and \( B = \langle S_B \mid R_B \rangle \), the free product is the group \( A \ast B \) with presentation \( \langle S_A \sqcup S_B \mid R_A \sqcup R_B \rangle \). Given a class of groups it is natural to ask whether this class is closed under the operation of free product. This motivated the study of asymptotic dimension (asdim) of free products [1], where the Bass-Serre theory was applied to prove that if \( A \) and \( B \) are two groups with \( \text{asdim} A \leq n \) and \( \text{asdim} B \leq n \), then \( \text{asdim} A \ast B \leq 2n + 1 \). This upper bound was also shown to hold for a metric notion of free product of metric spaces. Later, the upper bound estimate for the asymptotic dimension of a free product of groups was sharpened [6] and a formula was established for the asymptotic dimension of (the more general) free product with amalgamation [8].

Other coarse properties of groups, including finite decomposition complexity [12], property A [5], and asymptotic property C [7], are preserved by the operation of free product. In the cases of finite decomposition complexity and property A, the method of proof has become standard: one applies the Bass-Serre theory, union permanence, and fibering permanence, (see [11]). Other techniques were required in the case of asymptotic property C [7].

Motivated by our work with coarse direct products [2] and the metric notions of free product described by Bell and Dranishnikov [1] and Bell and Nagórko [7], we define a coarse free product of coarse spaces. We show that our coarse free product preserves finite dimension, coarse property A, coarse notions of decomposition complexity, and coarse property C. We also provide an upper bound for the dimension of a coarse free product in terms of the dimension of its factors. Our coarse free product is analogous to the free product of discrete metric spaces, but is distinct from the metric free product when applied to general metric spaces (see Example 3.7).

The paper is organized as follows. In the next section we give basic definitions related to the coarse category. We then define the coarse free product in Section 3.
In Section 4 we use techniques akin to Guentner’s [11] to exhibit permanence of many coarse properties with respect to the coarse free product. In Section 5 we provide an upper bound for the dimension of a coarse free product. We conclude by showing how the arguments of Bell and Nagórko [7] can be adapted to show that coarse property C is preserved by coarse free products in Section 6.

2. Preliminaries

To describe Roe’s coarse category, we first recall the operations of composition and inverse on the pair groupoid. Let $X$ be any set. The composition of two subsets $E \subset X \times X$ and $F \subset X \times X$ is the (possibly empty) set $E \circ F := \{ (x, z) \mid \exists y \in X, (x, y) \in E, (y, z) \in F \}$. The inverse of $E \subset X \times X$ is the set $E^{-1} := \{ (y, x) \mid (x, y) \in E \}$. The diagonal is the set $\Delta \subset X \times X$ defined by $\Delta := \{ (x, x) \in X \times X \}$.

**Definition 2.1.** [13] Definition 2.3] Let $X$ be a set and let $\mathcal{E}$ be a collection of subsets of $X \times X$. We say that $\mathcal{E}$ is a coarse structure on $X$ (and call the pair $(X, \mathcal{E})$ a coarse space) if $\mathcal{E}$ contains the diagonal, and is closed under inverses, finite unions, subsets, and compositions. The elements of $\mathcal{E}$ are called entourages.

**Definition 2.2.** Let $(X, \mathcal{E})$ be a coarse space. Let $\mathcal{F}$ be a collection of subsets of $X$. Let $E \in \mathcal{E}$. We say that $\mathcal{F}$ is $E$-disjoint if $(U \times U') \cap E = \emptyset$ whenever $U \neq U'$ are elements of $\mathcal{F}$. Let $K \in \mathcal{E}$. We say that $\mathcal{F}$ is $K$-bounded if $\cup_{U \in \mathcal{F}} U \cup U \subset K$. We call $\mathcal{F}$ bounded if it is $K$-bounded for some entourage $K$.

A subset $B$ of the coarse space $(X, \mathcal{E})$ will be called $K$-bounded if $\{B\}$ is $K$-bounded for some entourage $K$.

Given an entourage $E$, we write $E^k$ to mean the $k$-fold composition $E \circ E \circ \cdots \circ E$. By $E^0$, we mean $E \cap \Delta$. We define $D_E : X \times X \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ by $D_E(x, y) = \min\{k : (x, y) \in E^k \}$.

**Proposition 2.3.** [13] Proposition 3.6] Let $(X, \mathcal{E})$ be a coarse space and fix an entourage $E$ that is symmetric in the sense that $E = E^{-1}$. Then we have that

1. $D_E$ is symmetric;
2. $D_E(x, y) = 0$ iff $x = y$;
3. $D_E(x, y) \leq D_E(x, z) + D_E(z, y)$; and
4. for every $w, z \in X$ and each $A \subset X$ we have

$$|D_E(w, A) - D_E(z, A)| \leq D_E(w, z),$$

where $D_E(x, A) = \inf\{D_E(x, a) : a \in A\}$.

Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces and let $f$ be a map $f : X \to Y$. The map $f$ is said to be proper if the inverse image of every bounded set is bounded; it is said to be uniformly expansive if for each $E \in \mathcal{E}$ we have $(f \times f)(E) \in \mathcal{F}$. A map is called coarse if it is both proper and uniformly expansive. Two maps $f, f' : X \to Y$ are said to be close if $\{(f(x), f'(x)) \mid x \in X\} \in \mathcal{F}$. Finally, the coarse spaces $X$ and $Y$ are said to be coarsely equivalent if there are coarse maps $f : X \to Y$ and $g : Y \to X$ such that the compositions in both directions are close to the identity maps. A coarse fiber of $f$ at scale $F \in \mathcal{F}$ is any set $A \subset X$ that satisfies $(f \times f)(A \times A) \subset F$; i.e., for any $x, y \in A$ we have $(f(x), f(y)) \in F$.

It will be convenient to describe properties of families of coarse spaces that are satisfied uniformly. To achieve this, we use the notion of total space, following Guentner [11].
Definition 2.4. Let \((X, \mathcal{E})\) be a coarse space. Let \(\mathcal{U} = \{U_i\}_{i \in J}\) be a collection of subsets of \(X\). We define a coarse space called the total space \(T(\mathcal{U}, J)\) of the \(\{U_i\}_{i \in J}\) as follows. The underlying set is the disjoint union, \(\bigsqcup_{i \in J} U_i\). The entourages are the disjoint unions

\[
\bigsqcup_{i \in J} (E \cap (U_i \times U_i)) \subset \bigsqcup_{i \in J} (U_i \times U_i),
\]

where \(E\) ranges over \(\mathcal{E}\).

Let \(\mathcal{P}\) be a coarse property, i.e., a property of a coarse space that is invariant of coarse equivalence. We say that the family \(\{U_i\}_{i \in J}\) has property \(\mathcal{P}\) uniformly if the total space \(T(\mathcal{U}, J)\) has \(\mathcal{P}\).

3. The Coarse Free Product

Let \((X, \mathcal{E})\) be a coarse space. Fix a basepoint \(x_0 \in X\). Consider the collection \(*X\) of words in the alphabet \(X \setminus \{x_0\}\) along with the empty word \(\varepsilon\), which we identify with \(x_0\). By way of notation, letters set in bold type will be elements of \(*X\). We define the concatenation \(\mathbf{x} \cdot \mathbf{x}'\) of two words in the usual way, so that \(\mathbf{x} \cdot \varepsilon = \varepsilon \cdot \mathbf{x} = \mathbf{x}\). We will often write concatenation as juxtaposition unless extra emphasis is needed. We will also allow words in \(*X\) and elements \(x \in X\) to be concatenated to form new words; i.e., we do not distinguish between words consisting of a single letter and the letters themselves. For each distinct pair of elements \(\mathbf{x}\) and \(\mathbf{x}' \in *X\) there is a unique \(\mathbf{a} \in *X\) with the properties that \(\mathbf{x} = \mathbf{a} \cdot \varepsilon\) and \(\mathbf{x}' = \varepsilon \cdot \mathbf{c}\) with \(b \neq b'\), \(b, b'\) both in \(X\), and \(\mathbf{c}, \mathbf{c}'\) both in \(*X\). (Note that we allow the words \(\mathbf{a}, \mathbf{c}\) or \(\mathbf{c}'\) to be \(\varepsilon\) and at most one of \(b\) or \(b'\) may be \(x_0\) when the corresponding \(\mathbf{c}\) or \(\mathbf{c}'\) following it is \(\varepsilon\).) When necessary, we use the notation \(\mathbf{a} = \mathbf{x} \wedge \mathbf{x}'\) for this element.

Let \(E \in \mathcal{E}\) be given. Define \(\|\varepsilon\|_E = \|\varepsilon\|^E = 0\); for nonempty \(\mathbf{x} = x_1x_2\cdots x_k \in *X\) define \(\|\mathbf{x}\|_E = \sum_{i=1}^k D_E(x_0, x_i)\) and \(\|\mathbf{x}\|^E = \sum_{i=1}^k D_E(x_i, x_0)\). In the case that \(E\) is symmetric, \(\|\mathbf{x}\|_E = \|\mathbf{x}\|^E\). Moreover define \(D_E^*: *X \times *X \to \mathbb{Z}_{\geq 0} \cup \{\infty\}\) by

\[
D_E^*(\mathbf{x}, \mathbf{x}') = \begin{cases} 
0 & \mathbf{x} = \mathbf{x}' \\
D_E(b, b') + \|\mathbf{c}\|_E + \|\mathbf{c}'\|^E & \mathbf{x} \neq \mathbf{x}'.
\end{cases}
\]

Definition 3.1. Let \((X, \mathcal{E})\) be a coarse space, let \(E \in \mathcal{E}\), and let \(x \in X\). We define the symmetric ball of size \(E\) about \(x\) to be \(B(x, E) = E_x \cup E^x\) where \(E_x = \{y \in X \mid (x, y) \in E\}\) and \(E^x = \{y \in X \mid (y, x) \in E\}\).

Proposition 3.2. If \(E \in \mathcal{E}\) is symmetric, then \(D_E^*\) is an \(\infty\)-metric on \(*X*\).

Proposition 3.3. Let \(n \in \mathbb{Z}_{\geq 0}\) and put \((E, n) = \{(x, x') \in *X \times *X : D_E^*(x, x') \leq n\}\). Define the collection \(*\mathcal{E}\) to be the subset closure of \(\{(E, n) : E \in \mathcal{E}, n \in \mathbb{N}\}\). Then, \(*\mathcal{E}\) is a coarse structure on \(*X*\).

Proof. We must show that \(*\mathcal{E}\) (a) contains the diagonal, (b) is closed under inverses, (c) is closed finite under unions, (d) is closed under subsets, and (e) is closed under compositions.

(a) It is clear that \(\langle \Delta, 0 \rangle\) contains the diagonal in \(*X \times *X*\).

(b) Given \(L \in *\mathcal{E}\), we take a symmetric \(E\) such that \(\langle E, n \rangle\) contains \(L\). It is easy to see that \(\langle E^{-1}, n \rangle = \langle E, n \rangle^{-1}\).

(c) Let \(L\) and \(L'\) be given elements of \(*\mathcal{E}\). Find \(E, E' \in \mathcal{E}\) and \(n, n' \in \mathbb{N}\) such that \(L \subset \langle E, n \rangle\) and \(L' \subset \langle E', n' \rangle\). Then \(L \cup L' \subset \langle E, n \rangle \cup \langle E', n' \rangle \subset \langle E \cup E', n + n' \rangle\).

(d) This holds by definition.
(e) Let \( L \) and \( L' \) be given elements of \( \ast \mathcal{E} \). Find \( E, E' \in \mathcal{E} \) and \( n, n' \in \mathbb{N} \) such that \( L \subseteq \langle E, n \rangle \) and \( L' \subseteq \langle E', n' \rangle \). Then, \( \langle E, n \rangle \circ \langle E', n' \rangle \subseteq \langle E \cup E', n + n' \rangle \).

\[ \square \]

**Definition 3.4.** Let \((X, \mathcal{E})\) be a coarse space with basepoint \( x_0 \in X \). The coarse space \((X, \ast \mathcal{E})\) constructed above is called the **coarse free product of** \((X, \mathcal{E})\).

For ease of notation, we defined the unary free product, following Bell and Nagórkó [7]. One can obtain the free product \( X \ast Y \) of pointed spaces \((X, x_0)\) and \((Y, y_0)\) by forming \( \ast (X \cup Y) / x_0 \sim y_0 \).

**Definition 3.5.** Let \((X, \mathcal{E})\) be a coarse space with basepoint \( x_0 \in X \). Let \( x \neq e \). We say the **order (or length)** of \( x \) is \( n \) (and write \( \text{ord}(x) = n \)), if \( x \) can be expressed as \( x = x_1 x_2 \cdots x_n \in X \) with \( x_i \neq x_0 \), for all \( i \). We define \( \text{ord}(e) = 0 \).

Next, we wish to compare the coarse free product with the metric free product. A metric space \((X, d)\) carries a natural coarse structure called the **bounded coarse structure**, in which entourages are subsets of the form \( \{ (x, y) \in X \times X : \sup d(x, y) < \infty \} \) [13].

Suppose \((X, d)\) is a metric space and \( x_0 \in X \). The (metric) free product \( \ast X \) with respect to \( x_0 \) was defined by Bell and Nagórkó [4], see also [1]. They showed that the function \( d^* : \ast X \times \ast X \to [0, \infty) \) is a metric, where \( d^* \) is defined by \( d^*(x, x) = 0 \) and if \( x \neq x' \) are expressed as \( x = a_1 \cdots a_n \), \( x' = b_1' \cdots b_m' \), then

\[
\text{d}^*(a_1 \cdots a_n, b_1' \cdots b_m') = d(b_1', b_1) + \sum_{i=1}^{n} d(x_i, x_0) + \sum_{j=1}^{m} d(x_j, x_0).
\]

**Proposition 3.6.** Suppose \((X, d)\) is a metric space. Let \( \mathcal{B}_d \) represent the bounded coarse structure on \( X \) associated to \( d \), let \( d^* \) represent the metric on the free product \( \ast X \), and let \( \mathcal{B}_{d^*} \) represent the bounded coarse structure associated to \( d^* \). The coarse free product structure \( \ast \mathcal{B}_d \) is finer than the bounded coarse space \( \mathcal{B}_{d^*} \). In the case that \((X, d)\) is a discrete metric space, the two structures are equal.

**Proof.** Let \( L \in \ast \mathcal{B}_d \). We may assume \( L \) is a subset of an entourage of the form \( \langle E, n \rangle \) where \( E \subseteq \{ (x, y) \in X \times X : d(y, y') < R \} \) for some \( n \in \mathbb{N} \) and \( R \in \mathbb{R} \).

Suppose \( \langle x, x' \rangle \in L \). Then, with \( a = x \wedge x' \),

\[
n \geq D_E^*(a, x, x') = D_E^*(abc, ab'c') = D_E(b, b') + \|c\|_E + \|c'\|_E,
\]

where \( \text{ord}(c) < n \) and \( \text{ord}(c') < n \). But this means that \( d^*(x, x') < 3Rn \) thus \( L \in \ast \mathcal{B}_{d^*} \).

Suppose now that \((X, d)\) is discrete. Then there is some \( r > 0 \) such that \( r = \inf \{ d(y, y') \mid y \neq y' \in X \} \). Let \( F \in \mathcal{B}_{d^*} \); then there is some \( R \in \mathbb{R} \) such that \( F \subseteq \{ (x, x') \in \ast X \times \ast X : \ast^*(x, x') < R \} \). Take \( k \in \mathbb{N} \) such that \( (k-1)r \leq R < kr \). With this \( k \), we see that if \( \langle x, x' \rangle = (abc, ab'c') \in F \) then \( \text{ord}(bc) \leq k \) and \( \text{ord}(b'c') \leq k \).

Thus \( F \subseteq \langle E, k \rangle \in \ast \mathcal{B}_d \), where \( E = \{ (x, x') \in X \times X : d(x, x') < R \} \).

\[ \square \]

**Example 3.7.** Suppose that \((X, d)\) is a metric space with basepoint \( x_0 \) in which there exists a sequence of distinct points \( \{x_i\}_i \) converging to \( x_0 \in X \) with the property that \( \sum d(x_0, x_i) < 1 \). Then, all elements of the sequence of pairs \( \{ (x_0, x_0 \cdots x_1 \cdots x_i) \} \) belong to \( \langle x, x' \rangle \mid d^*(x, x') \leq 1 \), but there is no \( E \in \mathcal{B}_d \) such that \( \{ (x_0, x_0 \cdots x_1 \cdots x_i) \} \) is in \( \langle E, k \rangle \) for any fixed \( k \). Thus, the inclusion in Proposition 3.6 is strict.
The sequence in Example 3.7 shows that the identity map is not a coarse equivalence between the bounded coarse structure on the metric free product and the coarse free product taken with respect to the bounded coarse structure on $X$. We conjecture that in cases such as Example 3.7 no coarse equivalence exists.

**Question 3.8.** Let $(X, d)$ be a metric space. Suppose that the spaces $(X, \mathcal{B}_d)$ and $(X, \mathcal{B}_d')$ coarsely equivalent. Does it follow that $X$ is discrete?

### 4. Free-product permanence via unions and fibering

In this section, we show that several coarse properties are preserved by coarse free products. Our approach is similar to the one given by Guentner [11] for metric free products. We define a map $f : X \to Y$ to be a coarse map if, whenever $f : X \to Y$ is a uniformly expansive map to a coarse space $(Y, \mathcal{F})$ such that $Y$ has $\mathcal{P}$ and for each $F \in \mathcal{F}$ the coarse fibers of $f$ at scale $F$ have $\mathcal{P}$ uniformly, it follows that $X$ has $\mathcal{P}$.

**Definition 4.1.** Let $\mathcal{P}$ be a coarse property. Suppose that $(X, \mathcal{E})$ is a coarse space. We say that $\mathcal{P}$ satisfies **excisive union permanence** if, whenever $X$ is expressed as a union $X = \bigcup_{a \in J} X_a$ such that

1. $\{X_a\}_{a \in J}$ has uniform $\mathcal{P}$ and
2. for every $E \in \mathcal{E}$ there is a $Y_E \subset X$ with $\mathcal{P}$ such that $\{X_a \setminus Y_E\}_{a \in J}$ is $E$-disjoint,

it follows that $X$ has $\mathcal{P}$.

**Definition 4.2.** Let $\mathcal{P}$ be a coarse property. Suppose that $(X, \mathcal{E})$ is a coarse space. We say that $\mathcal{P}$ satisfies **fibering permanence** if, whenever $f : X \to Y$ is a uniformly expansive map to a coarse space $(Y, \mathcal{F})$ such that $Y$ has $\mathcal{P}$ and for each $F \in \mathcal{F}$ the coarse fibers of $f$ at scale $F$ have $\mathcal{P}$ uniformly, it follows that $X$ has $\mathcal{P}$.

**Theorem A.** Let $(X, \mathcal{E})$ be a coarse space with basepoint $x_0$. Let $\mathcal{P}$ be a coarse property of coarse spaces that satisfies excisive union permanence and fibering permanence. Suppose trees have $\mathcal{P}$. Then, the coarse free product $(X, \mathcal{E})$ has property $\mathcal{P}$ whenever $(X, \mathcal{E})$ does.

**Lemma 4.3.** Let $(X, \mathcal{E})$ be a coarse space with basepoint $x_0 \in X$ and consider the free product $(X, \mathcal{E})$. Let $\mathcal{P}$ be a coarse property. Suppose that $A \subset X$ has $\mathcal{P}$. Then, for any subset $Y \subset X$, the family $\{y \cdot A\}_{y \in Y}$ has property $\mathcal{P}$ uniformly.

**Proof.** We form the total space $T(A; Y)$ of the family $\{A\}_{y \in Y}$ (indexed by $y \in Y$) and observe that this total space has property $\mathcal{P}$. We define a map $f : T(A; Y) \to T(y \cdot A; Y)$ by $(a, y) \mapsto (y \cdot a, y)$. It remains to show that this is a coarse equivalence. Since $f$ is a bijection, it suffices to show it is a coarse map. We first show that $f$ is proper.

An entourage $F$ in the space $T(y \cdot A; Y)$ has the form $\bigsqcup_{y \in Y} (E \cap (y \cdot A \times y \cdot A))$, where $E \in \mathcal{E}$. Suppose that $K \in \mathcal{E}$ and $n \in \mathbb{N}$ have the property that $E \subset (K, n)$. We claim that $(f^{-1} \times f^{-1})(F) \subset \bigsqcup_{y \in Y} (K, n) \cap (A \times A)$, which is an entourage in $T(A; Y)$. Indeed, if $(x, x') \in F$, then there is some $y \in Y$ such that $x = y \cdot a$ and $x' = y \cdot a'$ for some $a$ and $a'$ in $A$. Since $E \subset (K, n)$, we see that $n \geq D^*_K(x, x') =
$D_R^*(a, a')$, so that $(a, a') \in (K, n)$. Thus, $(f^{-1} \times f^{-1})(x, x') \in \bigcup_{y \in Y} (K, n) \cap (A \times A)$, as required.

The proof that $f$ is uniformly expansive is similar. □

**Lemma 4.4.** Let $\mathcal{P}$ be a coarse property of coarse spaces that satisfies excisive union permanence. Let $(X, \mathcal{E})$ be a coarse space with basepoint $x_0 \in X$. Put $X^{(n)} = \{x \in *X: \text{ord}(x) = n\}$ (see Definition 3.5). If $X$ has property $\mathcal{P}$, then for each $n \in \mathbb{N}$, $X^{(n)}$ has property $\mathcal{P}$.

**Proof.** We apply induction. Put $X^* = X \setminus \{x_0\}$. Observe that $X^{(1)} = X^*$. Suppose the conclusion holds for $X^{(n-1)}$. Observe that $X^{(n)} = \bigcup_{x \in X^{(n-1)}} x \cdot X^*$. By Lemma 4.3, the collection $\{x \cdot X^*\}_{x \in X^{(n-1)}}$ has $\mathcal{P}$ uniformly.

Let $L \in \mathcal{E}$ be a symmetric entourage in $\mathcal{E}$ and a natural number $m$ such that $L \subset (E, m)$. Put $Y_{(E, m)} = X^{(n-1)} \cdot (B(x_0, E^m) \cap X^*)$. It is straightforward to show that $Y_{(E, m)}$ is a coarse set, hence $X^{(n-1)}$. Thus we see that $Y_{(E, m)}$ has property $\mathcal{P}$ by the inductive hypothesis. Finally, we show that the collection $\{x \cdot X^* \setminus Y_{(E, m)}: x \in X^{(n-1)}\}$ is $(E, m)$-disjoint. To this end if $w \neq w'$ are in this collection then we may write $w = a \cdot b \cdot c \cdot x$ and $w' = a \cdot b' \cdot c' \cdot x'$ where $a, c, c' \in *X$, $b, b', x, x' \in X^*$, $b \neq b'$ and $x, x' \notin E^m$. Then

$$D_E^*(w, w') = D_E(b, b') + \|c\|_E + \|c'\|_E + D_E(x_0, x) + D_E(x', x_0) \geq m$$

by assumption since $x, x' \notin E^m$. Therefore the collection is $(E, m)$-disjoint (hence $L$-disjoint).

We apply excisive union permanence to complete the proof. □

**Proof of Theorem A.** Let $T$ be a graph whose vertex set is in one-to-one correspondence with the elements of $*X$. We denote by $t_x$ the vertex of $T$ corresponding to the element $x \in *X$. We connect two vertices $t_{x_1}$ and $t_{x_2}$ of $T$ by an edge if and only if there is an $x \in X$ ($x \neq x_0$) for which $x_1 x = x_2$ or $x_2 x = x_1$ (as elements of $*X$). It is clear that $T$ is a tree. Give $T$ the bounded coarse structure it inherits as a metric space with the edge-length metric.

Define $f: *X \to T$ by $f(x) = t_x$. We claim that $f$ is uniformly expansive. To this end, let $L \in \mathcal{E}$ be given. Then, we can find an $E \in \mathcal{E}$ and an $n \in \mathbb{N}$ such that $L \subset (E, n)$. Suppose $x \neq x'$ and that $(x, x') \in L$. Put $a = x \land x'$, find $b \neq b'$ in $X$ and sequences $x_1, x_2, \ldots, x_m$ and $x'_1, x'_2, \ldots, x'_m$ of elements of $X^*$ such that $x = ax_1 \cdots x_m$ and $x' = ab'x'_1 \cdots x'_m$.

Then,

$$n \geq D_E^{*}(x, x') = D_E(b, b') + \sum_{i=1}^{m} \|x_i\|_E + \sum_{i=1}^{m'} \|x'_i\|_E \geq 1 + m + m' = d_T(t_x, t_{x'}) - 1.$$

Thus, for all pairs $(x, x') \in L$, we have $d_T(t_x, t_{x'}) \leq n + 1$. Therefore, the image $(f \times f)(L)$ is a uniformly bounded set, which means $f$ is uniformly expansive.

Let $L$ be the entourage in the bounded coarse structure on $T$. Then, there is a $t = t_y \in T$ (with $y \in *X$) and an $R > 0$ such that $F \subset B_R(t) \times B_R(t) \subset T \times T$, where $B_R(t)$ denotes the (open) metric $R$-ball around $t$. Then, the set $\{t_z: z = y \cdot x_1 \cdots x_k, x_i \in X^*, k \leq 2R\}$ contains $B_R(t)$. We observe that if $A \subset *X$ is a coarse fiber of $f$ at scale $F$, then $A \subset y \cdot X^{(\leq 2R)}$, where $X^{(\leq 2R)}$ denotes the collection of elements with order at most $2R$.

By Lemma 4.3 and Lemma 4.4, coarse fibers of $f$ have $\mathcal{P}$ uniformly. Since $\mathcal{P}$ is assumed to satisfy fiber permanence, we are done. □
We note that any tree (in the bounded coarse structure) has asymptotic dimension 1 \[10\] and therefore has coarse property A, coarse property C, as well as finite weak coarse decomposition complexity, finite coarse decomposition complexity, and straight finite decomposition complexity \[3\].

Guentner shows that coarse property A satisfies fibering and excisive union permanence \[11, \text{Theorem 6.5, Theorem 6.3}\]. Therefore, Theorem A immediately implies:

**Corollary 4.5.** Let \((X, \mathcal{E})\) be a coarse space with coarse property A. Let \(x_0\) be a basepoint. Then, the coarse free product \(*X\) has coarse property A.

Bell, Moran, and Nagórko show that \(P\) satisfies fibering permanence when \(P\) is finite weak coarse decomposition complexity, finite coarse decomposition complexity, or straight finite decomposition complexity \[3, \text{Theorem 4.14}\]. Moreover, each of these properties satisfy excisive union permanence \[3, \text{Theorem 4.18}\].

**Theorem 4.6.** Let \(P\) be one of the coarse properties: finite weak coarse decomposition complexity, finite coarse decomposition complexity, or straight finite decomposition complexity. Then, \(P\) satisfies fibering permanence and excisive union permanence.

Combining this with Theorem A immediately implies:

**Corollary 4.7.** Let \((X, \mathcal{E})\) be a coarse space with a property \(P\) among finite weak coarse decomposition complexity, finite coarse decomposition complexity, or straight finite decomposition complexity. Let \(x_0\) be a basepoint. Then, the coarse free product \(*X\) has \(P\).

It is not known whether coarse property C satisfies fibering permanence, so we cannot use Theorem A to show that coarse free products preserve coarse property C. We prove this using techniques similar to Bell and Nagórko \[7\] in Section 6.

5. Asymptotic dimension of a free product

By applying permanence results, we can show that finite asymptotic dimension is preserved by taking coarse free products as above. Instead, we apply the techniques of Theorem A to find an upper bound for the asymptotic dimension of a coarse free product.

The asymptotic dimension of a metric space was defined by Gromov \[10\]. Later, Roe \[13\] and then Grave \[9\] provided definitions of asymptotic dimension of coarse spaces as follows.

**Definition 5.1.** Let \((X, \mathcal{E})\) be a coarse space. We say that the asymptotic dimension of the coarse space \(X\) does not exceed \(n\) and write \(\text{asdim } X \leq n\) if for every \(E \in \mathcal{E}\) there are families \(\mathcal{U}_i (i = 0, 1, \ldots, n)\) of subsets of \(X\) such that

1. \(\bigcup_{i=0}^n \mathcal{U}_i\) covers \(X\);
2. each \(\mathcal{U}_i\) is \(E\)-disjoint; and
3. there is some \(K \in \mathcal{E}\) such that each \(\mathcal{U}_i\) is \(K\)-bounded.

The next definition can be deduced from previous ones, but we include it for clarity.

**Definition 5.2.** Let \(n \in \mathbb{Z}_{\geq 0}\). We say that coarse fibers of \(f\) have asymptotic dimension of at most \(n\) uniformly if for every \(L \in \mathcal{E}\) and \(F \in \mathcal{F}\) there is some \(K \in \mathcal{E}\)
such that whenever $A$ is a coarse fiber at scale $F$, there exist families of subsets, $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$, of $A$ that are $K$-bounded and $L$-disjoint, such that $\mathcal{U}_0 \cup \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$ covers $A$.

It is straightforward to show that the notion of uniform asymptotic dimension at most $k$ in the sense of Guentner agrees with the notion from Bell and Dranishnikov [1], when both are suitably translated to the coarse category.

**Proposition 5.3.** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a collection of subsets of $(X, \mathcal{E})$. Then $T(\mathcal{U}, J)$ has asymptotic dimension at most $k$ if and only if for every $E \in \mathcal{E}$ there is a $K \in \mathcal{E}$ and families $\mathcal{V}_0^i, \mathcal{V}_1^i, \ldots, \mathcal{V}_k^i$ such that for each $i$, $\cup_j \mathcal{V}_j^i$ covers $U_i$, each $\mathcal{V}_j^i$ is $E$-disjoint, and each $\mathcal{V}_j^i$ is $K$-bounded. □

We need the following union permanence result for asymptotic dimension.

**Theorem 5.4.** [3 Theorem 3.17] Let $(X, \mathcal{E})$ be a coarse space. Suppose that $X = \bigcup_n X_\alpha$, where $\text{asdim } X_\alpha \leq n$ uniformly and for each entourage $L \in \mathcal{E}$ there is a subset $Y_L \subseteq X$ with $\text{asdim } Y_L \leq n$ such that $\{X_\alpha \setminus Y_L\}$ forms an $L$-disjoint collection. Then, $\text{asdim } X \leq n$.

Theorem 5.4 immediately implies the following version of Lemma 4.4 for asymptotic dimension:

**Lemma 5.5.** Let $(X, \mathcal{E})$ be a coarse space with $\text{asdim } (X) \leq k$ and fix $x_0 \in X$. With $X^{(n)}$ as above, $\text{asdim } (X^{(n)}) \leq k$ for all $n \in \mathbb{N}$. □

**Lemma 5.6.** If $f : X \to Y$ is a uniformly expansive map of coarse spaces $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ with $\text{asdim } Y \leq k$ and if coarse fibers of $f$ have asymptotic dimension $n$ uniformly for some $n \in \mathbb{N}$ then $\text{asdim } X \leq (n + 1)(k + 1) - 1$

**Proof.** Let $L \in \mathcal{E}$ be given. Since $\text{asdim } Y \leq k$, we can cover $Y$ by $k + 1$-many $(f \times f)(L)$-disjoint families $\mathcal{V}_0, \ldots, \mathcal{V}_k$ of uniformly bounded subsets of $Y$.

Next, for each $V \in \cup_i \mathcal{V}_i$, since coarse fibers of $f$ have $\text{asdim } \leq n$ uniformly, there is a $K \in \mathcal{E}$ such that there are uniformly $K$-bounded, $L$-disjoint families $\mathcal{U}_0^V, \ldots, \mathcal{U}_n^V$ of subsets of $f^{-1}(V)$, whose union covers $f^{-1}(V)$.

Consider the collection $\mathcal{W}_{i,j} = \{U^V : V \in \mathcal{V}_j, U^V \in \mathcal{U}_i^V\}$. We claim that this collection (for $j = 0, \ldots, k$ and $i = 0, \ldots, n$) is a $K$-uniformly bounded, $L$-disjoint collection of subsets of $X$ that covers $X$.

Since the $\mathcal{V}_j$ cover $Y$ and the collections $\mathcal{U}_i^V$ cover $f^{-1}(V)$, it is clear that the collection $\mathcal{W}_{i,j}$ covers $X$.

Suppose now that we fix $i_0$ and $j_0$ and consider $\mathcal{W}_{i_0,j_0}$. We see that

$$\bigcup_{W \in \mathcal{W}_{i_0,j_0}} (W \times W) = \bigcup_{V \in \mathcal{V}_{j_0}} \bigcup_{U^V \in \mathcal{U}_{i_0}^V} (U^V \times U^V).$$

For each $V$, the union $\bigcup_{U^V \in \mathcal{U}_{i_0}^V} (U^V \times U^V)$ is a subset of $K$. Thus,

$$\bigcup_{W \in \mathcal{W}_{i_0,j_0}} (W \times W)$$

is a union of subsets of $K$ and hence a subset of $K$.

Suppose now that $W \neq W'$ in some $\mathcal{W}_{i,j}$. We can find $V$ and $V'$ in $\mathcal{V}_j$ such that $W \in \mathcal{U}_{i}^{V}$ and $W' \in \mathcal{U}_{i}^{V'}$. If $V = V'$ then $W \times W' \cap L = \emptyset$ by the assumptions on $\mathcal{V}_j$.

In the case that $V \neq V'$, then $W \times W' \cap L \subseteq (f \times f)^{-1}(V \times V') \cap (f \times f)^{-1}(f \times f)(L)$. Since $V \times V' \cap (f \times f)(L) = \emptyset$, we see that $(f \times f)^{-1}(V \times V') \cap L$ is also empty. □
Theorem 5.7. Let \((X, \mathcal{E})\) be a coarse space with \(\text{asdim}(X) \leq k\) and fix \(x_0 \in X\). Then \(\text{asdim}(\ast X) \leq 2k + 1\).

Proof. We take \(T\) and \(f : \ast X \to T\) as in the proof of Theorem \[\text{A}\] The map \(f\) was shown to be uniformly expansive. By Lemma 5.3, the coarse fibers of \(f\) have asymptotic dimension bounded by \(k\) uniformly and so, by Lemma 5.6 we are done. \[\square\]

Corollary 5.8. Let \((B, \mathcal{E})\) be a coarse space with basepoint \(x_0\). If \(B \times B \in \mathcal{E}\). Then \(\text{asdim} \ast B \leq 1\). \[\square\]

6. Property C

In this section, we adapt the argument that metric free products preserve asymptotic property C \[\text{[7]}\] to show that coarse property C \[\text{[3]}\] is preserved by coarse free products. The main obstacle in proving this is that while in the metric case one has a sequence of numbers \(R_1, R_2, \ldots\), which have meaning in the metric space \(X\) and the metric space \(\ast X\), the entourages in the coarse space \(X\) do not necessarily have straightforward analogs in the coarse space \(\ast X\).

Definition 6.1. \[\text{[3]}\] A coarse space \(X\) has coarse property C if and only if for any sequence \(E_1 \subset E_2 \subset \cdots\) of entourages there is a finite sequence \(U_1, U_2, \ldots, U_n\) of subsets of \(X\) such that

1. \(\bigcup_{i=1}^n U_i\) forms a cover for \(X\);
2. each \(U_i\) is \(E_i\)-disjoint; and
3. each \(U_i\) is bounded.

Theorem 6.2. Let \((X, E)\) be a coarse space. Assume that there is a \(k \geq 1\) such that for every infinite sequence \(E_1 \subset E_2 \subset \cdots\) of entourages there is a finite sequence \(U_1, U_2, \ldots, U_n\) of subsets of \(X\) such that

1. \(\bigcup U_i\) covers \(X\);
2. \(U_i\) is \(E_i\)-disjoint; and
3. \(U_i\) has asymptotic dimension bounded by \(k - 1\) uniformly.

Then \(X\) has coarse property C.

The proof is the same as \[\text{[7]}\] Lemma 6.1

Definition 6.3. Let \((X, \mathcal{E})\) be a coarse space with basepoint \(x_0\). A subset \(A \subset \ast X\) is said to be flat if there is some \(x \in \ast X\) such that \(A \subset x \cdot X\). Let \(E \in \mathcal{E}\) and \(A \subset \ast X\). Define the \(E\)-cone \(\text{con}_E(A) = A \cdot (\ast B(x_0, E))\), where \(B(x_0, E)\) is the symmetric ball \(\{x \in X : (x_0, x) \in E \cup E^{-1}\}\).

Lemma 6.4. Let \(\{A_\alpha\}_{\alpha \in J}\) be a collection of uniformly bounded flat subsets of \(\ast X\). Then, for each entourage \(L \in \mathcal{E}\), the collection \(\text{con}_L A_\alpha\) has asymptotic dimension bounded by 1 uniformly.

Proof. By assumption there is some \(K \in \mathcal{E}\) and an integer \(n\) such that \(\bigcup_{\alpha} A_\alpha \times A_\alpha \subset \langle K, n \rangle\). Since each \(A_\alpha\) is flat, there is (for each \(\alpha\)) an \(x_\alpha \in \ast X\) such that \(A_\alpha \subset x_\alpha \cdot X\). Therefore, \(A_\alpha \subset x_\alpha \cdot B(x_0, K^n)\) for each \(\alpha\).

Now, \(\text{con}_L A_\alpha \subset x_\alpha \cdot B(x_0, K^n) \cdot \ast B(x_0, L) \subset x_\alpha \cdot \ast B(x_0, K^n \cup L)\). We apply Corollary \[\text{5.8}\] and Lemma \[\text{4.3}\] to complete the proof. \[\square\]
Definition 6.5. Let $(X, \mathcal{E})$ be a coarse space and suppose $E \in \mathcal{E}$. A set $S \subset X$ is $E$-connected if for every $x, y \in S$ there is a finite sequence $x = s_0, s_1, \ldots, s_n = y$ of points of $S$ such that $(s_i, s_{i+1}) \in E$ for each $i$. An $E$-connected component of $X$ is a maximal $E$-connected subset of $X$.

Lemma 6.6. Let $(X, \mathcal{E})$ be a coarse space with $x_0 \in X$. Suppose that $E \in \ast \mathcal{E}$. Take some $L \in \mathcal{E}$ and $n$ such that $E \in \langle L, n \rangle$ and suppose that $A \in \ast X \cdot (X \setminus B(x_0, L^n))$ has the property that $E$-connected components of $A$ are uniformly bounded. Then, for each $M \in \mathcal{E}$, the $E$-connected components of $\operatorname{con}_M A$ have asymptotic dimension at most 1 uniformly.

Proof. We prove this first under the assumption that $E = \langle L, n \rangle$. The general case follows from the fact that $E$-connected components are contained in some $\langle L, n \rangle$-connected component and the fact that asymptotic dimension is monotonic on subsets.

Following the method of [7, Lemma 6.11], we can characterize the $E$-connected components of $\operatorname{con}_M A$ as follows: $C$ is an $E$-connected component of $\operatorname{con}_M A$ if and only if $X^{(\leq n)} \cap C$ is an $E$-connected component of $X^{(\leq n)} \cap \operatorname{con}_M A$.

Let $C$ be an $E$-connected component of $\operatorname{con}_M A$. Put $C_k = C \cap X^{(\leq k)}$. Let $k_0$ be the smallest integer for which $C_{k_0} \neq \emptyset$. We claim that $C_{k_0}$ is flat and uniformly bounded.

Take two words $x$ and $y$ in $C_{k_0}$. Then, $x$ and $y$ are in $\operatorname{con}_M A$ and there is an $E$-chain $(t_i)$ of elements of $C$ connecting them. By our observation above, we may take the $t_i$ in $C_{k_0}$.

Write $t_1 = a_{t_1} \cdot b_1 z_1 \cdots z_{k_0}$ with $a_{t_1}, b_1 \in A$, and $b_i \in X \setminus B(x_0, L^n)$. Similarly, write $x = a_x \cdot b_x x_1 \cdots x_{k_0}$. Since $(x, t_1) \in E$, we have $D_L^*(x, t_1) \leq n$. Since $\|b_x\| > n$ and $\|b_t\| > n$, we see that $a_x = a_{t_1}$, and in particular,

$$n \geq D_L^*(x, t_1) = D_L(b_x, b_t) + \sum_{j=l}^{k_0} (\|x_j\| L + \|z_j\| L).
$$

Next, we suppose that the words $z_1 \cdots z_{k_0}$ and $x_1 \cdots x_{k_0}$ are non-empty. Then, the minimality of $k_0$ means that $(a_{t_1} \cdot b_1 z_1 \cdots z_{k_0}, x) \notin E$. Thus, $\|x_{k_0}\| L > n$, contradicting Equation (1). We conclude that $C_{k_0} \subset A$ and is therefore uniformly bounded. We conclude also that $a_x = a_{t_1}$ and that $C_{k_0}$ is therefore flat.

We show by induction on $k$ that

$$C_k \subset \operatorname{con}_{M \cup L^n \cup D} C_{k_0}
$$

Let $x \in C_{k+1} \setminus C_k$, with $k \geq k_0$. Then, either $x \in \operatorname{con}_M C_k$ or by the argument above, $x$ lies in some $E$-connected component of $A$ that is also $E$-close to $\operatorname{con}_M C_k$. In the first case, we see that $x \in \operatorname{con}_{M \cup L^n \cup D} C_k$. In the second case, if $D \in \mathcal{E}$ is a bound on the diameter of $A$, $x \in \operatorname{con}_{M \cup L^n \cup D} C_k$. Since $\operatorname{con}_{M \cup L^n \cup D} \subset \operatorname{con}_{M \cup L^n \cup D} C_k = \operatorname{con}_{M \cup L^n \cup D} C_k$, we have proved our claim. By Lemma 6.4, the $E$-connected components of $\operatorname{con}_M A$ have asymptotic dimension at most 1 uniformly.

Theorem 6.7. Let $X$ be a coarse space with fixed basepoint $x_0$. If $X$ has coarse property $C$, then $\ast X$ has coarse property $C$.

Proof. Suppose $E_1 \subset E_2 \subset \cdots$ is a given sequence of entourages in $\ast \mathcal{E}$. For each $i$ find $L_i \in \mathcal{E}$ and an integer $n_i$ such that $E_i \subset \langle L_i, n_i \rangle$. Find a sequence
Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_p$ of $L^n_{p+1}$-disjoint, uniformly bounded subsets of $X$ whose union covers $X$. Put $n = \max\{n_i\}$.

Put $\mathcal{V}_i(x) = \{x \cdot (U \setminus B(x_0, L^n_{p+1})): U \in \mathcal{U}_i, x \in *X\}$ for each $i \in \{1, 2, \ldots, p\}$.

Put $\mathcal{V}_{p+1} = \{(x_0)\}$. We claim that

1. $\bigcup_{i=0}^{p+1} \con L^n_{p+1} \cup \mathcal{V}_i = *X$ and
2. $\{\mathcal{V}_i(x)\}_{i,x}$ is $E_i$-disjoint, uniformly bounded, and its elements are flat.

For (1), we consider an element $x \in *X$. Write $x = x_1x_2 \cdots x_k$, where each $x_i \in X$. If $(x_0, x_i) \in L^n_{p+1}$ for each $i$, then $x \in \con L^n_{p+1}(x_0)$. Otherwise, take $m$ to be the largest integer for which $(x_0, x_m) \notin L^n_{p+1}$. Find some $U$ in some $\mathcal{U}_i$ such that $x_m \in U$. Then, $x \in x_1x_2 \cdots x_{m-1} \cdot (U \setminus B(x_0, L^n_{p+1})) \cdot *B(x_0, L^n_{p+1})$. Thus, $x \in \con L^n_{p+1} \cup \mathcal{V}_i$.

For (2), we observe that each $\mathcal{V}_i$ is uniformly bounded and flat, so it remains only to show that these families are $E_i$-disjoint. Suppose that $\mathcal{V}_1$ and $\mathcal{V}_2$ are distinct elements of some $\mathcal{V}_i$. We can find $x_1, x_2 \in *X$ and subsets $U_1$ and $U_2$ in $\mathcal{U}_i$ for which $\mathcal{V}_i = x_1 \cdot U_1$ and $\mathcal{V}_2 = x_2 \cdot U_2$, where $U_i$ denotes the set $U_i$ with the ball $B(x_0, L^n_{p+1})$ removed. If $x_1 = x_2$, then we’re done since $\mathcal{U}_i$ is $L^n_{p+1}$-disjoint.

Otherwise, take $\mathcal{V}_1 = (v_1 \land v_2) b c u_1$ and $\mathcal{V}_2 = (v_1 \land v_2) b' c' u_2$. We compute

$$D_{L_i}(v_1, v_2) = D_{L^n_{p+1}}(b, b') + \|c\|_{L_i} + \|c'\|_{L_i} + D_{L_i}(x_0, u_1) + D_{L_i}(u_2, x_0)$$

$$\geq D_{L^n_{p+1}}(b, b') + \|c\|_{L_i} + \|c'\|_{L_i} + D_{L_i}(u_1, u_2)$$

$$\geq n_i.$$

Thus we see $\mathcal{V}_i$ is $(L_i, n_i)$-disjoint hence $E_i$-disjoint.

\[\square\]

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