Chaos Decomposition and Gap
Renormalization of Brownian Self-Intersection
Local Times

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Abstract

We study the chaos decomposition of self-intersection local times
and their regularization, with a particular view towards Varadhan’s
renormalization for the planar Edwards model.

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malization, white noise analysis

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1 Introduction

The self-intersection local time of \(d\)-dimensional Brownian motion, informally, is given as

\[
L = \int_0^T dt_2 \int_0^{t_2} dt_1 \delta(B(t_2) - B(t_1)).
\]  

We shall see that, while "reasonably well defined" for \(d = 1\), these local times become more and more singular as the dimension \(d\) increases. Intersections have thus been the object of extensive study by authors such as Dvoretzky, Erdös, Kakutani \[6, 7, 8\], Varadhan \[35\], Westwater \[30, 31, 32\], Le Gall \[20, 21\], Rosen \[24, 25, 26\], Dynkin \[9, 10, 11\], Watanabe \[29\], Yor \[33, 34\], Imkeller et al. \[18\], Albeverio et al. \[1, 2\]. For fractional Brownian motion there are papers e.g. by Rosen \[27\], Hu & Nualart \[17\], Grothaus et al. \[15\].

Apart from its intrinsic mathematical interest the self-intersection local time has played a role in constructive quantum field theory, and is a standard model in polymer physics for the self-repulsion ("excluded volume effect") of chain polymers in solvents \[28\].

Replacement of the Dirac delta function in (1) by a Gaussian

\[
\delta_\varepsilon(x) := \frac{1}{(2\pi \varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}, \quad \varepsilon > 0,
\]

leads to regularized local times

\[
L_\varepsilon := \int_0^T dt_2 \int_0^{t_2} dt_1 \delta_\varepsilon(B(t_2) - B(t_1))
\]

and for \(d = 1\) one can show \(L^2\) convergence w.r.t. white noise or Wiener measure space. But already for \(d = 2\) this fails since the expectation of \(L_\varepsilon\) will diverge in the limit, asymptotically

\[
E(L_\varepsilon) \approx -\frac{T}{2\pi} \ln \varepsilon.
\]

In this case it is sufficient to subtract the expectation, i.e. the centered regularized local time does have a well-defined \(L^2\) limit:

\[
L_{\varepsilon,c} := L_\varepsilon - E(L_\varepsilon) \to L_c.
\]
Apart from the Gaussian regularization above, others have been considered to remove the singularity at $t_1 = t_2$ in the integral [1]. The "staircase regularization" avoids the line $t_1 = t_2$ as in see e.g. Bolthausen [5] (Fig. 1).

![Fig. 1: Domain of integration for the staircase-regularized local time.](image)

The widely used "gap regularization" does the same by omitting the strip $t_2 - t_1 < \Lambda$ in the integral. In the modelling of chain polymers the gap size $\Lambda$ will be a ”microscopic” quantity, i.e. of the order of the inter-monomer distance, more precisely the ”Kuhn” or ”persistence” length. It plays an important role in renormalization group calculations [28]: critical parameters are obtained from the postulate that macroscopic quantities do not depend on microscopic length scales.
2 Tools from White Noise Analysis [16]

Based on a $d$-tuple of independent Gaussian white noises $\omega = (\omega_1, \ldots, \omega_d)$ one defines a $d$-dimensional Brownian motion $B$ through

$$B(t) \equiv \langle \omega, \mathbb{1}_{[0,t]} \rangle = \int_0^t ds \, \omega(s).$$

We shall use a multi-index notation

$$n = (n_1, \ldots, n_d), \quad n = \sum_{i=1}^d n_i, \quad n! = \prod_{i=1}^d n_i!$$

and for $d$-tuples of Schwartz test functions $f = (f_1, \ldots, f_d) \in S(\mathbb{R}, \mathbb{R}^d)$,

$$\langle f, f \rangle = \sum_{i=1}^d \int dt \, f_i^2(t)$$

and similarly for $\langle \omega^\otimes n, F_n \rangle$ where for $d$-tuples of white noise the Wick product $\cdots$ generalizes to

$$\omega^\otimes n := \bigotimes_{i=1}^d \omega_i^\otimes n_i :$$

The vector valued white noise $\omega$ has the characteristic function

$$C(f) := \mathbb{E}(e^{i\langle \omega, f \rangle}) = \int_{S^*(\mathbb{R}, \mathbb{R}^d)} d\mu(\omega) e^{i\langle \omega, f \rangle} = e^{-\frac{1}{2} \langle f, f \rangle},$$

where $\langle \omega, f \rangle = \sum_{i=1}^d \langle \omega_i, f_i \rangle$ and $f_i \in S(\mathbb{R}, \mathbb{R})$.

Writing

$$(L^2) := L^2(S^*(\mathbb{R}, \mathbb{R}^d), d\mu)$$

there is the Itô-Segal-Wiener isomorphism with the Fock space of symmetric square integrable functions:

$$(L^2) \simeq \left( \bigoplus_{k=0}^\infty \text{Sym} \ L^2(\mathbb{R}^k, k!d^k t) \right)^\otimes d.$$
This implies the chaos expansion

$$\varphi(\omega) = \sum_{n \in \mathbb{N}_0} \langle \omega^\otimes n, F_n \rangle$$

for $\varphi \in (L^2)$

with kernel functions $F_n$ in Fock space.

Generalized functionals are constructed via a Gel’fand triple

$$(S) \subset (L^2) \subset (S)^*.$$  

The generalized functionals in $(S)^*$ are conveniently characterized by their action on exponentials. In particular we use the

$$: \exp(\langle \omega, f \rangle) : = C(f) \exp(\langle \omega, f \rangle) \in (S)$$

to make the

**Definition 1**  
*The transformation defined for all test functions $f \in S(\mathbb{R}, \mathbb{R}^d)$ via the bilinear dual product on $(S)^* \times (S)$ by

$$\langle (S \Phi)(f) = \langle \langle \Phi, : \exp(\langle \cdot, f \rangle) : \rangle \rangle$$

is called the $S$-transform of $\Phi \in (S)^*$.**

The multilinear expansion of $S(\Phi)$

$$\langle (S \Phi)(f) = \sum_{n \in \mathbb{N}_0} \langle \varphi_n, f^\otimes n \rangle$$

extends the chaos expansion to $\Phi \in (S)^*$, with distribution valued kernels $\varphi_n$, such that

$$\langle \langle \Phi, F \rangle \rangle = \sum_{n \in \mathbb{N}_0} n! \langle \varphi_n, F_n \rangle.$$  

**Definition 2** *We shall indicate the projection onto chaos of order $n \geq k$ by a superscript $(k)$:

$$\langle \langle \Phi^{(k)}, F \rangle \rangle = \sum_{n, n \geq k} n! \langle \varphi_n, F_n \rangle.$$
Proposition 3 \[14\]

\[\delta^{(2N)}(B(t_2) - B(t_1)) \in (S)^*\]

with even kernel functions

\[\psi_{2n}(u_1, \ldots, u_{2n}; t_1, t_2) = \frac{1}{n!} (2\pi)^{-d/2} \left(\frac{1}{|t_2 - t_1|}\right)^{\frac{d}{2} - n} \left(-\frac{1}{2}\right)^n \prod_{k=1}^{2n} \mathbb{1}_{[t_1, t_2]}(u_k).\]

All the kernel functions with odd indices vanish.

Setting

\[v := \max(u_1, \ldots, u_{2n})\]
\[u := \min(u_1, \ldots, u_{2n})\]

one computes \[14\] the kernel functions of the truncated local time \(L^{(2N)}\) for \(2N > d - 2\) by integration over \(0 < t_1 < t_2 < T\):

\[\varphi_{2n}(u_1, \ldots, u_{2n}) = \frac{(2\pi)^{-d/2}}{n!} \left(-\frac{1}{2}\right)^n \int_0^T dt_2 \int_0^{t_2} dt_1 (t_2 - t_1)^{-n-d/2} \prod_{k=1}^{2n} \mathbb{1}_{[t_1, t_2]}(u_k)\]

\[= (-1)^n \left((n + \frac{d}{2} - 1)(n + \frac{d}{2} - 2)(2\pi)^{d/2} 2^n n!\right)^{-1} \cdot \Theta(u) \Theta(T - v) \cdot\]

\[\cdot (T^{-n-\frac{d}{2}+2} - v^{-n-\frac{d}{2}+2} - (T - u)^{-n-\frac{d}{2}+2})\]

except for \(2n = d = 2\) where

\[\varphi_2(u_1, u_2) = \frac{-1}{4\pi} \left(\ln v + \ln(T - u) - \ln(v - u) - \ln T\right) \cdot \Theta(u) \Theta(T - v).\]

The Heaviside function \(\Theta\) here is the indicator function of the positive half line.

3 Regularizations

Replacement of the Dirac delta function by a Gaussian

\[\delta_{\varepsilon}(x) = \frac{1}{(2\pi \varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}, \quad \varepsilon > 0,\]

6
leads to regularized local times

\[ L_\varepsilon = \int_0^T dt_2 \int_0^{t_2} dt_1 \delta_\varepsilon(B(t_2) - B(t_1)) \]  

(2)

with kernel functions \[ 14 \]

\[ \varphi_{\varepsilon,2n}(u_1, \ldots, u_{2n}) = \frac{(2\pi)^{-d/2}}{n!} \left( -\frac{1}{2} \right)^n \int_0^T dt_2 \int_0^{t_2} dt_1 (\varepsilon + |t_2 - t_1|)^{-n-d/2} \mathbb{I}_{[t_1, t_2]}^{\otimes 2n} (u_1, \ldots, u_{2n}) \]

\[ = (-1)^n \left( (n + \frac{d}{2} - 1)(n + \frac{d}{2} - 2)(2\pi)^{d/2} 2^n n! \right)^{-1} \cdot \Theta(u) \Theta(T - v) \cdot \]

\[ \cdot ((T + \varepsilon)^{-n-d/2 + 2} - (v + \varepsilon)^{-n-d/2 + 2} - (T - u + \varepsilon)^{-n-d/2 + 2} + (v - u + \varepsilon)^{-n-d/2 + 2}), \]

\[ \varphi_{\varepsilon,2}(u_1, u_2) = -\frac{1}{4\pi} \left( \ln(v + \varepsilon) + \ln(T - u + \varepsilon) - \ln(v - u + \varepsilon) - \ln(T + \varepsilon) \right) \cdot \Theta(u) \Theta(T - v). \]

3.1 Gap Regularization of Kernel Functions

In renormalization group studies of self-repelling Brownian motion another regularization is often used, see e.g. \[ 28 \] and the references there; it suppresses intersection in small time intervals \( t_2 - t_1 \) between intersections by setting informally

\[ L(\Lambda) := \int_{0 < t_1 < t_2 < T} d^2t \delta(B(t_2) - B(t_1)). \]

The expectation of \( L(\Lambda) \) is equal to

\[ \mathbb{E}(L(\Lambda)) = \int_0^T dt_2 \int_0^{t_2-\Lambda} dt_1 \psi_0(t_1, t_2) = (2\pi)^{-d/2} \int_0^T dt_2 \int_0^{t_2-\Lambda} dt_1 (t_2 - t_1)^{-d/2}. \]

We note that for in particular \( d = 2 \)

\[ \mathbb{E}(L(\Lambda)) = -\frac{T}{2\pi} \ln \Lambda + O(1). \]  

(3)

Recall \[ 14 \] that for \( \Lambda = 0 \) the kernel functions \( \varphi_{2n} \) of the truncated local time \( L^{(2N)} \) are obtained by integrating the kernel functions \( \psi_{2n}(u_1, \ldots, u_{2n}; t_1, t_2) \)
over the rectangle $0 < t_1 < u$ and $v < t_2 < T$, shaded grey in see Fig. 2. For
$\Lambda > 0$ the integration is further restricted to the light domain with $t_2 - t_1 > \Lambda$.
This restriction is non-trivial when $v - u < \Lambda$.

$$L^{(2N)} - L^{(2N)}(\Lambda)$$

thus has kernel functions $\rho_{2n}(u,v)$ with support on

$$\{0 < u < v < T\} \cap \{v - u < \Lambda\} .$$

Fig. 2 is pertinent to the case where $\Lambda < v$ and $u < T - \Lambda$. In this case the kernel functions $\rho_{2n}(u,v)$ are obtained by integrating the $\psi_{2n}(u_1, \ldots, u_{2n}; t_1, t_2)$ with respect to the $t_i$ over $v - \Lambda < t_1 < u$ and $v < t_2 < t_1 + \Lambda$. Excepting the kernel function $\rho_2(u,v)$ for $d = 2$, one finds,

$$\rho_{2n}(u,v) = \int_{v-\Lambda}^{u} dt_1 \int_{v}^{t_1+\Lambda} dt_2 \psi_{2n}(u_1, \ldots, u_{2n}; t_1, t_2)$$

$$= \frac{(2\pi)^{-d/2}}{n!} \left( -\frac{1}{2} \right)^n \frac{1}{d/2 + n - 1} \cdot \left( \frac{1}{d/2 + n - 2} \left( (v-u)^{-d/2-n+2} - \Lambda^{-d/2-n+2} \right) + (v-u-\Lambda)\Lambda^{-d/2-n+1} \right)$$

and for $2n = d = 2$

$$\rho_2(u,v) = \frac{1}{4\pi} \left( \ln(v-u) - \ln \Lambda + \frac{\Lambda - v + u}{\Lambda} \right) .$$
Using $\tau = t_2 - t_1$ we obtain the following estimate

$$|\rho_{2n}(u, v)| = \left| \int_{v-\Lambda}^{u} \int_{\tau}^{t_1+\Lambda} dt_1 \int_{\tau}^{t_2} dt_2 \psi_{2n}(u_1, \ldots, u_{2n}; t_1, t_2) \right|$$

$$= \frac{1}{2^n n! (2\pi)^{d/2}} \int_{v-\Lambda}^{u} dt_1 \int_{v-t_1}^{\Lambda} d\tau \left( \frac{1}{\tau} \right)^{d/2+n}$$

$$\leq \frac{1}{2^n n! (2\pi)^{d/2}} \frac{1}{d/2 + n - 1} \int_{v-\Lambda}^{u} dt_1 (v - t_1)^{-d/2-n+1}$$

$$\leq \frac{1}{2^n n! (2\pi)^{d/2}} \frac{1}{d/2 + n - 1} \frac{1}{d/2 + n - 2} (v - u)^{-d/2-n+2}$$

while for $d = 2n = 2$ one readily finds from (5)

$$|\rho_2(u, v)| \leq \frac{1}{4\pi} |\ln(v - u)|$$

when $v - u < \Lambda \ll 1$. 

**Fig. 2:** Domain of integration for kernels of the local time, light grey for the regularized local time, dark grey for the subtraction $\rho$ as in (4).
For very small or very large $u, v$, i.e. $0 < u < v < \Lambda$ or $T - \Lambda < u < v < T$, respectively, the range of integrations in (4) and (6) for $\psi_{2n}(u_1, \ldots, u_{2n}; t_1, t_2)$ will be $0 < t_1 < u$ and $v < t_2 < t_1 + \Lambda$, or $v < t_2 < T$ and $t_2 - \Lambda < t_1 < u$ respectively, see Fig. 3.

![Fig. 3](image.png)

**Fig. 3**: Modified integration domains for $\rho$ when $u, v$ are close to zero or to $T$ respectively.

Computations as in (4) are again straightforward, we note here only that the estimate of (6) is true also in these two cases.

**Theorem 4** For $N > 0$ and $0 < \Lambda \ll T$ the chaos expansion of the gap-regularized local time $L^{(2N)}(\Lambda)$ has the kernel functions

$$\varphi_{\Lambda,2n}(u_1, \ldots, u_{2n}) = \varphi_{2n}(u_1, \ldots, u_{2n}) - \Theta(\Lambda - (v - u))\rho_{2n}(u_1, \ldots, u_{2n}), \quad (8)$$

and zero otherwise, while

$$L^{(2N)} - L^{(2N)}(\Lambda)$$

has the kernel functions $\rho_{2n}$ for $0 < u < v < T$, $n \geq N$. 

10
The Heaviside function $\Theta$ exhibits the support property of the $\rho_{2n}$, i.e. in the gap regularization the kernel functions of the local time are only modified when all arguments $u_k$ are close to each other.

With these results one can in particular estimate the rate of convergence for the centered self-intersection local time in $d = 2$. Apart from the term $n = 1$ the sum

$$\| L^{(2)} - L^{(2)}(\Lambda) \|_{L^2}^2 = \sum_{n \geq 1} (2n)! \| \rho_{2n} \|_{L^2([0,T]^{2n})}^2$$

can be estimated as follows:

$$\sum_{n \geq 1} (2n)! \| \rho_{2n} \|_{L^2([0,T]^{2n})}^2 \leq (2\pi)^{-d} \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} \left( \frac{1}{2} \right)^{2n} \left( \frac{1}{d/2 + n - 1} \right)^2 \frac{1}{d/2 + n - 2} \int_0^T d^{2n} u_k \left( \frac{1}{v - u} \right)^{d+2n-4}.$$

We can integrate out the $2n-2$ variables $u_k$ with $u < u_k < v$ that lie between the smallest and the largest and obtain in this way

$$\sum_{n \geq 1} (2n)! \| \rho_{2n} \|_{L^2([0,T]^{2n})}^2 \leq (2\pi)^{-d} \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} \left( \frac{1}{2} \right)^{2n} \left( \frac{1}{n(n-1)} \right)^2 \cdot 2n(2n-1) \int_0^T dv \int_{v-u<\Lambda} du \int_0^v dv$$

$$\leq \frac{\Lambda T}{2\pi^2} \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} \frac{1}{2n-1} \frac{2n-1}{n(n-1)^2}.$$

The series is convergent (Stirling's formula). From (7) it is straightforward to estimate the remaining term with $n = 1$:

$$\| \rho_{2} \|_{L^2([0,T]^2)}^2 \leq \left( \frac{1}{4\pi} \right)^2 \int_0^T dt \int_0^\Lambda d\tau \ln^2 \tau = 0(T\Lambda \ln^2 \Lambda).$$

So we have shown

**Theorem 5** For $d = 2$

$$\| L^{(2)} - L^{(2)}(\Lambda) \|_{L^2}^2 = 0(T\Lambda \ln^2 \Lambda) \text{ as } \Lambda \downarrow 0.$$

A similar improvement of the rate of convergence has been found in the Gaussian regularization in [3].
4 Varadhan Renormalization

The model proposed by Edwards [12] for self-repelling Brownian motion suppresses self-crossings, modifying the Brownian path (or white noise) measure by a density function, informally

\[ \varphi = Z^{-1} \exp(-gL) \]

with \( g > 0 \)

\[ Z = \mathbb{E}(\exp(-gL)). \]

There is no problem for \( d = 1 \) since \( L \) is a positive random variable and hence \( \exp(-gL) < 1 \). For \( d = 2 \) however we should replace \( L \) by the centered \( L_c \) and this then is no more positive, so that \( \exp(-gL_c) \) is unbounded. The point of Varadhan’s theorem is to show that this happens only on small sets so that

**Theorem 6 (Varadhan [35])** For \( d = 2 \)

\[ \varphi = Z^{-1} \exp(-gL_c) \]

with

\[ Z = \mathbb{E}(\exp(-gL_c)) \]

is integrable.

Varadhan defines the centered local time as the limit of Gaussian approximations as in (2) and uses the Chebyshev inequality to show that \( \exp(-gL_c) \) is integrable for \( 0 < g < \pi \).

A similar slightly stronger result can be obtained using Varadhan’s technique with the approximation

\[ L^{(2)}(\Lambda) \to L^{(2)} = L_c. \]

Fix \( 0 < \Lambda < 1 \). By (3) there exists a positive constant \( k \) such that

\[ L^{(2)}(\Lambda) \geq -\mathbb{E}(L(\Lambda)) \geq -k - \frac{T}{2\pi} |\ln(\Lambda)|. \]

For any constant \( N \geq k + \frac{T}{2\pi} |\ln(\Lambda)| \) one has

\[ \mathbb{P}(L_c \leq -N) = \mathbb{P}(L_c - L^{(2)}(\Lambda) \leq -N - L^{(2)}(\Lambda)) \leq \mathbb{P}(\left|L^{(2)}(\Lambda) - L_c\right| \geq N - k - \frac{T}{2\pi} |\ln(\Lambda)|). \]
An application of Chebyshev’s inequality, using Theorem 5 then yields
\[
\Pr(L_c \leq -N) \leq \frac{\mathbb{E}(|L^{(2)}(\Lambda) - L_c|^2)}{(N - k - \frac{T}{2\pi} |\ln(\Lambda)|)^2} \leq K \frac{\Lambda \ln^2 \Lambda}{(N - k - \frac{T}{2\pi} |\ln(\Lambda)|)^2}.
\]
In particular, for
\[
\Lambda = \exp(-\alpha(N - k)), \quad 0 < \alpha < \frac{2\pi}{T}
\]
one obtains
\[
\Pr(L_c \leq -N) \leq \frac{K\alpha^2}{(1 - \frac{T}{2\pi} \alpha)^2} \exp(-\alpha(N - k)).
\]
Hence, \(\exp(-gL_c)\) is integrable for \(g < \frac{2\pi}{T}\).

For the Gaussian regularization an analogous result can be found in [3].

5 Concluding Remarks

[13], [14], [17], [18], [19], [22], [23], in the present context of regularizations and the rate of convergence see in particular [3]. Yet another regularization of the self-intersection local time is suggested by (8), namely
\[
L^{(2N)}_\Lambda := \sum_{n \geq N} \langle \omega_{2n}, \varphi^{(A)}_{2n} \rangle
\]
with kernel functions
\[
\varphi^{(A)}_{2n}(u_1, \ldots, u_{2n}) := \Theta(v - u - \Lambda) \varphi_{2n}(u_1, \ldots, u_{2n})
\]
where simply the range \(v - u < \Lambda\) is cut out. Details of this regularization will be discussed elsewhere.

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