Consensus versus persistence of disagreement in opinion formation: the role of zealots

Francesca Colaioiri\textsuperscript{1,2} and Claudio Castellano\textsuperscript{2,3}

\textsuperscript{1} Istituto dei Sistemi Complessi (ISC-CNR), UOS Sapienza, c/o Dipartimento di Fisica, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Rome, Italy
\textsuperscript{2} Dipartimento di Fisica, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Rome, Italy
\textsuperscript{3} Istituto dei Sistemi Complessi (ISC-CNR), Via dei Taurini 19, 00185 Roma, Italy
E-mail: claudio.castellano@roma1.infn.it

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Abstract. We consider a general class of three-state models where individuals hold one of two opposite opinions, or are neutral, and exchange opinions in generic pairwise interactions. We show that when opinions spread in a population where a fraction of individuals (zealots) unshakably maintain their view, one of four qualitatively distinct kinds of collective dynamics arises, depending on the specific rules governing the social interactions. Unsurprisingly, when their density is high, zealots drive the whole population to consensus on their opinion. For low densities a rich phase diagram emerges: a finite population of dissenters can survive and be the only stationary state or may need a critical initial mass of dissenters to be sustained; the critical mass may vanish or not as the density is reduced; the transition to the high density regime can be smooth or abrupt, and shows interesting hysteretic effects. For each choice of the interaction rules we calculate the critical density of zealots above which diverse opinions cannot survive.

Keywords: critical phenomena of socioeconomic systems, interacting agent models, opinion dynamics
1. Introduction

Consensus often develops spontaneously in societies, leading to the adoption of widespread opinions, ideologies and traditions. On the other hand disagreement among individuals is also very common, even on major issues. Disagreement is not always due to lack of communication, and in some cases it persists even when individuals are allowed to exchange opinions and debate for a long time. What are the mechanisms that lead to consensus, and which ingredients instead cause persistent disagreement?

Although broadcast media such as television and advertising have a significant impact on individuals making their opinion, macroscopic opinion shifts in social groups may be caused by peer-to-peer interactions occurring within the social network. People form and reconsider their opinion while constantly interacting with others, and are often exposed to considerable pressure to conform to the opinion of their friends and neighbors [1, 2]. Yet, on certain matters, some individuals are unshakable in their opinion, and therefore totally insensitive to peer pressure. The aim of this paper is to analyze the role of such ‘stubborn’ individuals, completely reluctant to change, in determining the onset of consensus or rather the persistence of disagreement.

The effect of stubborn individuals (denoted also as committed, zealots or inflexibles) on opinion dynamics has attracted considerable interest in the past years. Early
investigations analyzed the effect of a single [3] or multiple individuals [4, 5] in the context of the classical voter model on lattices. The presence of zealots has then been studied in many other contexts, including different types of dynamics, various interaction patterns, and the presence of single, multiple or competing zealots [6–14]. Among the most interesting and general results is the observation [7] of a transition occurring as the fraction of zealots in the system exceeds some threshold: when the density of zealots is above the threshold the system is driven to consensus; otherwise, alternative opinions survive in the population. Previous studies have generally focused on models which, in the absence of zealots, exhibit a symmetric dynamics, i.e. no opinion is a priori favored by the spontaneous evolution. Here we consider instead a very general class of three-state models where the dynamics may either favor or disfavor the opinion held by the zealots. In this wider context new non-trivial phenomena occur, including the counterintuitive possibility that an opinion disfavored by the dynamics and opposed by zealots survives and is even adopted by the majority of individuals. In a previous paper [15] we studied a similar general three-state opinion dynamics model focusing on the effect of media. The effect of media was schematized in the simplest possible way by assuming that people conform to the media recommendation at a constant rate and independently of their current state. The presence of inflexible individuals considered here can also be interpreted, at the mean-field level, as an effective external field, but more complex than the one considered in [15]: it induces spontaneous transitions either to the opinion supported by the media or to the neutral state, and the transition rates depend on the current state. The model presented here includes that of [15] as a special case.

The paper is structured as follows. After the description of the general model in section 2, we write down the mean-field equations for the dynamics in section 3. The position and stability of their stationary solutions, depending on the model parameters, and the consequent different shapes of the phase diagram are illustrated in section 4, while the detailed derivation of the results is deferred to section 5. Section 6 contains a discussion of the results and concluding remarks.

2. The model

We analyze a model of pairwise social influence for opinion dynamics in a society consisting of two types of agents: regular agents, who update their beliefs according to the information that they receive from their social neighbors; and stubborn agents, who never update their opinion. Stubborn agents might for example represent political activists or customers involved in the marketing process for a crowdsourced campaign. We include agents with no preferred opinion: neutral agents, who are uninformed or undecided. This allows significant change from models where only two opinions are allowed. Each agent can be in one of three states: holding opinion A, holding an opposite opinion B, or being undecided (U). A finite fraction p of the population forms a sub-population Z of agents committed to the opinion A: they never change state. We indicate with n_A, n_B, n_U the fraction of uncommitted agents in the A, B, and U state, respectively, so that n_A + n_B + n_U + p = 1. The interactions rules are specified in table 1. Interactions
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3. The dynamics on a complete graph

We now write the equations for the dynamics of this general model on a complete graph of infinite size (i.e. in mean field), describing the time evolution of the density of uncommitted agents in each state:

\[
\begin{align*}
\dot{n}_A &= 2\beta_1 p n_B + 2\varphi_1 n_A n_B + 2\varphi_2 (n_A + p) n_U \\
\dot{n}_B &= -2(\beta_1 + \beta_4) p n_B + 2\gamma_1 n_A n_B + 2\gamma_2 n_B n_U
\end{align*}
\]

where \( \varphi_1 = \alpha_1 - \alpha_2 - \alpha_3 - \alpha_5 \) and \( \gamma_1 = -\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 \) represent the net gain in \( A \) (\( B \)) states in an \( A-B \) interaction. The equation for \( \dot{n}_U \) can be derived from the normalization condition. Defining \( r = p/(1-p) \), and \( \bar{n}_i = n_i/(1-p) \) (\( i = A, B, U \)) as the fraction of uncommitted agents respectively in the \( A, B \) and \( U \) state, normalized with
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with respect to the total uncommitted population \((\bar{n}_A + \bar{n}_B + \bar{n}_U = 1)\), equations (1) can be rewritten as:

\[
\begin{align*}
\dot{\bar{n}}_A(1 + r) &= 2r\beta_1\bar{n}_B + 2r\varphi_2\bar{n}_U + 2\varphi_1\beta_1\bar{n}_B + 2\varphi_2\beta_1\bar{n}_U \\
\dot{\bar{n}}_B(1 + r) &= -2r(\beta_1 + \beta_4)\bar{n}_B + 2\gamma_1\beta_4\bar{n}_U + 2\gamma_2\beta_4\bar{n}_U \\
\end{align*}
\]

(2)

From equations (2) it is clear that, in terms of the densities of uncommitted sub-populations and with a proper rescaling of the time variable, the system maps exactly onto a system of uncommitted agents in an external field acting as an exogenous one-to-many broadcasting source. Interactions and the effect of the external field are specified as follows: the external field induces individuals holding opinion \(A\) to switch to opinion \(B\) at rate \(r\beta_1\), and to become undecided \((U)\) at rate \(r\beta_4\); undecided individuals acquire opinion \(A\) at rate \(r\phi_2\). The interactions are those described in table 1 for uncommitted agents. A similar case was studied in [15] where the focus was on the interplay between external media and interpersonal influence events occurring within the social network. In that case the effect of media was schematized in the simplest possible way by assuming that people conform to the media recommendation at a constant rate and independently of their current state. The model presented here can be seen as a generalization of [15]. The presence of zealots introduces in the dynamical equations additional terms, that can be interpreted as external fields inducing the spontaneous transitions \(B \rightarrow A\) and \(U \rightarrow A\) (but at different rates), as well as the transition \(B \rightarrow U\). The model studied in [15] is recovered as a special case of the model studied here, with \(\beta_1 = \phi_2\), \(\beta_4 = 0\), and \(r \rightarrow r/2\phi_2\), so that \(r_B \rightarrow A = r\), \(r_B \rightarrow U = 0\), \(r_U \rightarrow A = 2\phi_2\beta = r\). For \(r = 0\) the two models coincide.

4. Collective behavior at stationarity

Choosing \(x = \bar{n}_B\) and \(y = \bar{n}_A\) as independent variables, the stationarity condition in equation (2) reads:

\[
\begin{align*}
y^2 - y(1 - r + \phi x) + r\alpha x - r &= 0, \quad C_1, \\
x(\gamma y - x + 1 - r\beta) &= 0, \quad C_2, \\
\end{align*}
\]

(3)

where \(\phi = (\phi_1 - \phi_2)/\phi_2\), \(\gamma = (\gamma_1 - \gamma_2)/\gamma_2\), \(\alpha = \phi_2 - \beta_1)/\phi_2\), and \(\beta = (\beta_1 + \beta_4)/\gamma_2\), and we assume \(\phi_2 \neq 0\), \(\gamma_2 \neq 0\). The stationary solutions of the dynamic equations are therefore given by the intersections of the two conic sections \(C_1\) and \(C_2\) that fall within the physical region, which is the triangle delimited by the lines \(x = 0\), \(y = 0\) and \(y = 1 - x\). The first conic section, \(C_1\), is a non-degenerate hyperbola, while the second one, \(C_2\), factorizes into the product of the two straight lines:

\[
x = 0, \quad R_1, \\
y = \gamma^{-1}x + \gamma^{-1}(r\beta - 1), \quad R_2.
\]

(4)

Only one of the two intersections of \(R_1\) with \(C_1\), the one at \(P_1 = (0, 1)\), falls within the physical region; it is a stationary solution for any set of parameters values and for any \(r\). \(P_1\) corresponds to the trivial absorbing state of unanimous consensus on opinion

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A. The collective behavior of the system for any given set of parameters is fully determined by the possibility that other non-trivial fixed points (any possible intersections of $C_1$ and $R_2$) appear within the physical region as the control parameter $r$ is varied. The existence, position and stability of non-trivial fixed points depend on the specific values of the six parameters $(\gamma_1, \gamma_2, \varphi_1, \varphi_2, \beta_1, \beta_4)$ defining the model.

In what follows we classify each model, specified by a set of parameters, according to its collective behavior as the control parameter $r$ is varied ($r$ is a monotonic function of the fraction $p$ of committed agents). The behavior at large $r$ is independent of the parameter values; trivially, in that regime, the full consensus on opinion $A$ is the only stable stationary state. However, in the small $r$ regime, we observe four qualitatively distinct types of response to the variation of the control parameter. We therefore classify all possible specific models, each one specified by a set of parameters, in four classes. The phase diagram for each class is shown in figure 1, while the ‘meta-phase diagram’ giving the classification of the models according to their set of parameters is summarized in figure 2. We now describe the phenomenology for each one of the four classes, and postpone the detailed analysis of the solutions of equations (3) to the next section.

4.1. Class I: Finite critical mass (FCM) models

When $\varphi_1 < 0$, $\gamma_1 \leq 0$, i.e. in models where $A-B$ interactions produce on average an increase in undecided individuals, with no net gain in $A$ nor in $B$ states, the system undergoes a first order transition at a finite value $r = r_c$ of the external bias (see figure 1(a)). For large enough fraction of committed (large $r$), $P_1$ is the only fixed point and the system flows into the absorbing state of total consensus on opinion $A$, for any initial condition. At $r = r_c$ the system undergoes a saddle–node bifurcation [16]: one double solution appears. As $r$ is decreased below $r_c$ the double solution splits into one stable (the one with larger $n_B$) and one unstable solution. In the non-trivial stable fixed point the two opinions $A$ and $B$ coexist in the population, together with a fraction of undecided (see figure 1(a)). We call this state where disagreement persists ‘pluralism’.

The initial conditions determine whether consensus is asymptotically reached ($n_A = 1$),
or disagreement persists \((n_B > 0)\). The value of \(n_B\) at the unstable fixed point stays finite in the limit \(r \to 0\), implying that, no matter how small the fraction of individuals committed to opinion \(A\) is, a finite ‘critical mass’ [17] of dissenters (supporting opinion \(B\) in the initial state) is always necessary to reach the pluralistic state.

### 4.2. Class II: Vanishing critical mass (VCM) models

This class is identified by the range of parameters \(\varphi_1 < 0, 0 < \gamma_1 < \min(\gamma_1^* - \phi, 0)\) (see below for the definition of \(\gamma_1^*\)), and corresponds to models where \(A-B\) interactions cause on average a small increase of \(B\) states at the expense of \(A\) states. As in Class I models, lowering \(r\) below \(r_c\) the system undergoes a first order transition separating a regime \((r > r_c)\), where consensus on opinion \(A\) is the only stable state from a regime \((r \leq r_c)\) where a stable and an unstable additional fixed point appear through a saddle–node

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**Figure 2.** (a, c) Phase diagram in the \((\varphi_1, \gamma_1)\) plane. The curve \(\gamma_1^*\) separating regions VCM and ZCM depends on \(\varphi_2, \gamma_2, \beta_1\) and \(\beta_4\) In the upper panel (a) \(\varphi_2 = 0.4, \gamma_2 = 0.7, \beta_1 = 2, \text{ and } \beta_4 = 2\), so that \(\varphi_2 > \gamma_2\beta_4/(\beta_1 + \beta_4)\). In the lower panel (c) \(\varphi_2 = 0.05, \gamma_2 = 0.8, \beta_1 = 2, \text{ and } \beta_4 = 2\), so that \(\varphi_2 < \gamma_2\beta_4/(\beta_1 + \beta_4)\). (b, d) Plot of \(p_c\) (saddle–node bifurcation curve) and \(p^*\) (transcritical bifurcation line) as functions of \(\gamma_1\), and for \(\varphi_1 = -0.8\) (upper panel, (b)), \(\varphi_1 = -0.2\) (lower panel, (d)).
bifurcation (see next section). However, in this case, further reducing \( r \), the unstable fixed point collides with the point \( P_1 \) (consensus on \( A \)) at a finite value \( r^* (0 < r^* < r_c) \), and then exits the physical region. Correspondingly, the value of \( n_B \) at the unstable fixed point vanishes when \( r \to r^* \). This is a transcritical bifurcation [16]: when the two fixed points cross each other, they exchange stability; \( P_1 \) becomes unstable, so that below \( r^* \) the system, unless started with \( n_B = 0 \), always flows to the pluralistic state (figure 1(b)). Therefore in this case, the initial presence of even a few dissenters suffices for opinion \( B \) to survive. The curve \( \gamma_1 = \gamma_1^* \) separating Class II and III depends on all other model parameters. In a certain range of parameters \( \gamma_1^* \) becomes larger than \(-\varphi_1 \) so that all physical values of \( \gamma_1 \) are always below \( \gamma_1^* \), and Class III disappears (see figure 2(c)). The transcritical bifurcation line is \( r = r^* = \gamma_1/(\beta_1 + \beta_4) \), and always lies below the saddle–node bifurcation line.

4.3. Class III: Zero critical mass (ZCM) models

When \( \varphi_1 < 0, \gamma_1 > \gamma_1^* \), corresponding to models where \( A–B \) interactions give an increase of undecided and a large increase in \( B \) at the expense of \( A \), the system undergoes, at \( r = r^* \), a continuous transition (transcritical bifurcation) between total consensus on opinion \( A \) \((r > r^*) \) and pluralism \((r < r^*) \), see figure 1(c). In this class of behavior initial conditions do not play any role.

4.4. Class IV: Total consensus (TC) models

This class corresponds to the region \( \varphi_1 \geq 0 \), i.e. to models where \( A–B \) interactions result in a net increase of individuals holding opinion \( A \). The behavior is trivial: irrespectively of the value of all other parameters, for any initial condition, and no matter how small the density \( p \) of committed individuals is, the system always converges to the consensus state \( (P_1) \) (see figure 1(d)).

5. Derivation of the phase diagrams

We now show how the rich phenomenology described in the previous section and summarized in figure 1 is derived from equations (1). We need to find the solutions of equations (3) and look at how their behavior, as the control parameter \( r \) (or equivalently \( p \)) varies, depends on the parameters defining the model. Depending on the values of the six parameters \((\gamma_1, \gamma_2, \varphi_1, \varphi_2, \beta_1, \beta_4)\) controlling the interactions, different collective behaviors emerge.

In order to find the intersections of \( C_1 \) with \( R_1 \) and \( R_2 \) that correspond to physically meaningful fixed points, it is useful to first identify the position of \( C_1 \) with respect to the physical region. This can be done by locating the intersections of \( C_1 \) with the boundary of the physical region, delimited by the three lines \( x = 0, y = 0, y = 1 - x \). (i) The intersections of \( C_1 \) with \( x = 0 \) are in \( P_1 = (0, 1) \) and \( P_2 = (0, -r) \); only the first one belongs to the physical region. (ii) The only intersection of \( C_1 \) with \( y = 0 \) is in \((\alpha^{-1}, 0)\), which, since \( \alpha \leq 1 \), is always outside the physical region. (iii) The intersections of \( C_1 \) with \( y = 1 - x \)

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are in \( P_1 = (0, 1) \) and \( S_1 = (1 - r/\bar{r}, r/\bar{r}) \), with \( \bar{r} = (\alpha + 1)/(\alpha - 1) = -\varphi_1/\beta; \) \( S_1 \) belongs to the physical region only for \( \varphi_1 \leq 0 \) and \( r \leq \bar{r} \). Note that whenever \( S_1 \) belongs to the physical region (for \( \varphi_1 \leq 0 \) and \( r \leq \bar{r} \)) the upward concavity of \( C_1 \) guarantees that \( C_1 \) actually enters the physical region. The shape of \( C_1 \) is qualitatively represented in figure 3, where (a) corresponds to \( \varphi_1 \leq 0 \) and \( 0 < r \leq \bar{r} \), and (b) to all other cases where \( C_1 \) intersects the physical region in just one point (\( P_1 \)).

\( P_1 = (0, 1) \) always belongs both to \( C_1 \) and to \( R_1 \) (\( P_1 \) is in fact the only intersection of \( R_1 \) with the upper branch of \( C_1 \)), therefore \( P_1 \) is a fixed point for all possible choices of the parameters and for any \( r \). It corresponds to the trivial absorbing state of unanimous consensus on opinion A. For \( \varphi_1 \geq 0 \) (corresponding to Class IV models) \( C_1 \) intersects the physical region only in \( P_1 \) (see figure 3(b)), thus in this case there cannot be other fixed points besides \( P_1 \). Thus we restrict the following analysis to the non-trivial case \( \varphi_1 < 0 \): in this case, while \( P_1 \) still is the only fixed point for sufficiently large fraction of committed agents \( r \geq \bar{r}, \) see figure 3(b)), additional fixed points may arise for \( r < \bar{r} \) from the intersections of \( R_2 \) and \( C_1 \) within the physical region.

5.1. \( r = 0 \).

To understand how the fixed points move as the control parameter \( r \) is varied, it is useful to first understand what happens in the case \( r = 0 \), that is the limit of vanishing density of committed agents. For \( r = 0 \) the line \( R_2 \) always goes through \( Q_1 = (1, 0) \), that also belongs to the asymptote \( y = 0 \) of \( C_1 \), and is therefore a fixed point. It is clear that, since the slope of \( R_2 \) is \( \gamma^{-1} \), for \( \gamma > -1 \ (\gamma_1 > 0) \) \( Q_1 \) is the only point of \( R_2 \) that falls within the physical region (see figure 4). Let us rewrite \( C_1 \) as

\[
(\varphi y - r\alpha)(\varphi y - \varphi^2 x - \varphi + r(\varphi + \alpha)) = r(\varphi + \alpha)(\varphi - r\alpha)
\]

that allows us to identify the asymptotes. In the limit \( r \to 0 \) \( C_1 \) factorizes in the product of its asymptotes, that in the same limit are \( y = 0 \) and \( y = \varphi x + 1 \). Note that \( \varphi_1 < 0 \) implies \( \varphi < -1 \), so that both asymptotes cross the physical region at \( r = 0 \). When \( r = 0 \) the intersections of \( C_1 \) with \( R_2 \) are \( Q_1 = (1, 0) \) (consensus on opinion B, always within the physical region), and \( Q_2 = ((1 + \gamma)/\Gamma, (1 + \varphi)/\Gamma) \), where \( \Gamma = 1 - \gamma\varphi \). \( Q_2 \) falls within.
the physical region as long as \( \gamma_1 \leq 0 \) (that implies that \( 1 + \gamma \) and \( 1 + \varphi \) have the same sign, and therefore \( Q_2 \) belongs to the first quadrant), see figure 4(a). \( Q_2 \) coincides with \( P_1 \) for \( \gamma_1 = 0 \). For \( \gamma_1 > 0 \) \( Q_2 \) is always outside the physical region (to the left of \( P_1 \) for \( \Gamma < 0 \) (see figure 4(b)) or to the right of \( Q_1 \) for \( \Gamma > 0 \)). For \( \gamma > -1 \) \( \gamma_1 > 0 \) there are overall three fixed points in the physical region: \( P_1 \) (unanimous consensus on opinion A) at the intersection of \( C_1 \) with \( R_2 \); \( Q_1 \) (unanimous consensus on opinion B) at the intersection of \( C_1 \) with \( R_2 \); \( 0, 0 \), given by the other intersection of \( C_1 \) with \( R_1 \). For \( \gamma < -1 \) \( \gamma_1 < 0 \) instead, \( R_2 \) always crosses the physical region, and intersects the other asymptote in some \( Q_2 \) inside the physical region, giving another fixed point, besides the two corresponding to unanimous consensus, and the one in \( (0, 0) \). Note that for \( r = 0 \) the fixed point in \( (0, 0) \) is always unstable, and disappears as soon as \( r > 0 \).

5.2. \( r > 0 \).

As \( r \) increases, \( R_2 \) always moves leftward (the slope does not depend on \( r \) and the intersection with the \( x \) axis \( (1 - r\beta, 0) \) always moves leftward as \( r \) increases). In the meantime, the upper branch of \( C_1 \) moves rightward\(^4\). Therefore, the intersection \( Q_1 \) always moves leftward. We must now distinguish two cases:

1. \( C_1 \) and \( R_2 \) having one intersection on each branch: this occurs for \( \Gamma = 1 - \gamma \varphi > 0 \). In this case the situation is simple: \( Q_1 \) moves leftward as \( r \) increases and for some value \( r = r^* \) reaches \( P_1 \) and then leaves the physical region. The other intersection \( Q_2 \) is always on the unphysical branch of \( C_1 \) (the intersection between \( R_2 \) and the line \( y = 1 - x \), located in \( (1 - r/\tilde{r}, r/\tilde{r}) \), with \( \tilde{r} = \gamma_1/(\beta_1 + \beta_4) \) is always to the left of \( S_1 \), since \( \tilde{r} \leq \tilde{r} \) follows from the physical constraint \( \gamma_1 + \varphi_1 < 1 \), therefore \( Q_1 \) can only exit the physical region crossing \( P_1 \)).

\(^4\) For a given ordinate \( y = \mathcal{g} \), \( x = (1 - \mathcal{g})(\varphi + r)/(\rho \alpha - \varphi \mathcal{g}) \), so that \( \partial_x = -(\varphi + \alpha)\mathcal{g}(1 - \mathcal{g})/(\rho \alpha - \varphi \mathcal{g})^2 \), which is positive for \( \mathcal{g} \) in the physical region, being \( -(\varphi + \alpha) = (\beta_1 - \varphi_1)/\varphi_2 > 0 \).
Figure 5. Sketch of the motion of the curves $\mathcal{C}_1$ (red solid curve) and $\mathcal{R}_2$ (blue dashed line) when $r$ is increased. (a) The intersection point $Q_1$ moves leftward, while $Q_2$ moves rightward. For $r = r_c$ they collide and annihilate. (b) The intersection point $Q_1$ moves leftward, while $Q_2$ moves rightward. They may collide and annihilate either inside or outside the physical region.

(2) $\mathcal{C}_1$ and $\mathcal{R}_2$ having two intersections on the same branch: this occurs for $\Gamma = 1 - \gamma \varphi < 0$ (and small enough $r$) (see figure 5). In this case, $\mathcal{R}_2$ always has two intersections with $\mathcal{C}_1$ for $r = 0$: $Q_1$ at $(1, 0)$ that moves to the left as $r$ increases, and $Q_2$, that is within the physical region for $\gamma_1 < 0$, and outside the physical region (to the left of $P_1$) when $\gamma_1 > 0$. In both cases, as $r$ increases $Q_1$ moves leftward, and $Q_2$ moves rightward, and they collide at some point for some $r = r_c$. When $\gamma_1 < 0$ ($Q_2$ within the physical region for $r = 0$) $Q_1$ and $Q_2$ collide for some finite $r = r_c$ for $r > r_c$. In the other case, $\gamma_1 > 0$, again, as $r$ increases $Q_1$ and $Q_2$ get closer and closer ($Q_1$ moving leftward, and $Q_2$ moving rightward), but two different events may happen depending on the value of $\gamma_1$: $Q_1$ and $Q_2$ either collide within the physical region ($Q_2$ crosses $P_1$ for some $r = r^*$ entering the physical region, and then $Q_2$ collides with $Q_1$ for some $r_c > r^*$), or they collide outside the physical region ($Q_1$ crosses $P_1$ for some $r = r^*$, exiting the physical region and then $Q_1$ collides with $Q_2$ at some unphysical point to the left of $P_1$ for some $r_c > r^*$). A special value $\gamma_1 = \gamma_1^*$ separates the two behaviors: $\gamma_1^*$ is the value such that $Q_1$ and $Q_2$ collide exactly in $P_1$, on the boundary of the physical region.

5.3. Derivation of $r^*$ and $\gamma_1^*$.

To obtain the critical value $r^*$, we observe that $r^*$ is such that at least one intersection between $\mathcal{C}_1$ and $\mathcal{R}_2$ coincides with $P_1$, which requires that $\mathcal{R}_2$ goes through $P_1$, and therefore

$$r^* = \frac{\gamma_1}{(\beta_1 + \beta_2)}.$$  

A critical value $r_c$ such that $\mathcal{C}_1$ and $\mathcal{R}_2$ have a double intersection on the upper branch always exists under the necessary condition $\Gamma < 0$, however it is physically meaningful only as long as such double intersection falls within the physical region. We have seen that $Q_1$ and $Q_2$ certainly collide within the physical region for $\gamma_1 < 0$, and for small
enough positive $\gamma_1$ ($0 < \gamma_1 < \gamma_1^*)$. $\gamma_1^*$ is defined as the value of $\gamma_1$ such that $Q_1$ and $Q_2$ collide in $P_1$, on the boundary of the physical region. Imposing this condition we find that the value of $r_c$ at $\gamma_1^*$ is $r_c = r^* = \gamma_1^*/(\beta_1 + \beta_4)$, and that $\gamma_1^*$ is the positive solution of

$$\alpha(\gamma_1^*/\gamma_2)^2 + \gamma_1^*/\gamma_2(\varphi_1 + \alpha - 1) + \beta(1 + \varphi) = 0.$$  

(7)

One further piece is missing: while for any given $\varphi_1$ a positive value of $\gamma_1^*$ always exists, it could result to be unphysical in certain ranges of the other parameters $\varphi_2$, $\gamma_2$, $\beta_1$, and $\beta_4$. Indeed, besides being positive, physically meaningful values of $\gamma_1$ have to satisfy the condition $\gamma_1 + \varphi_1 \leq 1$. Imposing that $\gamma_1^* < -\varphi_1$ and after some lengthy but trivial calculation we find the condition $\varphi_2 = \gamma_2\beta_4/(\beta_1 + \beta_4)$, $\gamma_1^*$ becomes larger than the upper bound $-\varphi_1$, therefore physically accessible values of $\gamma_1$ are always smaller than $\gamma_1^*$, and the corresponding models are in Class III for any physical positive $\gamma_1$ ($0 < \gamma_1 < -\varphi_1$).

5.4. Wrap-up: building the phase diagram from the fixed points.

We now have all the elements to summarize the different classes of collective behavior, that depend crucially on the parameters $\varphi_1$ and $\gamma_1$. For large $r$ the trivial, stable absorbing state of unanimous consensus on opinion $A$ ($P_1$) is the only fixed point for all possible sets of parameter values. When $\varphi_1 > 0$ the state of unanimous consensus on opinion $A$ is the only fixed point for any $r$ (TC models). When $\varphi_1 \leq 0$, we can distinguish three qualitative different behaviours according to the value of $\gamma_1$: (i) For $\varphi_1 < 0$, $\gamma_1 < 0$ (FCM models), as $r$ becomes smaller than a critical value $r_c$, two additional fixed points are formed by means of a saddle–node bifurcation (discontinuous transition). The two fixed points remain physical for any value of $r$ down to $r = 0$. (ii) For $\varphi_1 < 0$, $0 \leq \gamma_1 \leq \min(\gamma_1^*, -\varphi_1)$ (VCM models) there is a discontinuous transition at $r = r_c$. When $r$ is further reduced one of the fixed points collides with $P_1$ for $r = r^*$ and then exits the physical region. At $r = r^*$ the fixed point $P_1$ becomes unstable and the non-trivial fixed point $Q_2$ remains the only stable stationary state, unavoidably reached by the dynamics. (iii) For $\varphi_1 < 0$, $\min(\gamma_1^*, -\varphi_1) \leq \gamma_1 \leq -\varphi_1$ (ZCM models), at $r = r^*$ the point $Q_1$ enters the physical region, thus becoming stable, while $P_1$ turns unstable. A continuous transition occurs between unanimous consensus on $A$ (for $r > r^*$) and a pluralistic state with $n_B > 0$ (for $r < r^*$).

For $\varphi_2 < \gamma_2\beta_4/(\beta_1 + \beta_4)$ $\gamma_1^*$ is always below $-\varphi_1$ corresponding to the phase diagram in figure 2(a). When $\varphi_2 \geq \gamma_2\beta_4/(\beta_1 + \beta_4)$ instead $\gamma_1^*$ crosses $-\varphi_1$ at $\varphi_1 = \varphi_1^* = -(\gamma_2\beta_4 - \varphi_2(\beta_1 + \beta_4))/(\varphi_2 + \beta_4)$. In this case, for $\varphi_2 \geq \varphi_1$ there is no continuous transition, and the phase diagram is as in figure 2(c).

6. Discussion and conclusions

In this paper we have introduced and solved analytically (in mean field) a very general model for opinions dynamics in the presence of committed agents. We have shown that when opinions spread in a population where a given fraction of individuals have
unshakable opinions, four qualitatively distinct kinds of collective dynamics emerge, depending on the specific rules governing the social interactions. We have categorized this very extensive set of opinion dynamics models accordingly within mean field. To keep the interaction rules very general the models considered depend on a large number of parameters. A high density of zealots trivially drives the system towards total consensus, irrespectively of all the parameters ruling the interactions. The qualitative behavior of the stationary states at smaller densities depends essentially only on the pair \((\gamma_1, \varphi_1)\), which encode information about interactions among individuals with opposite opinions. The other parameters have a role in locating the tricritical line, the saddle–point bifurcation line and the transcritical bifurcation line. In some parameters regions disagreement persists, while in others consensus is achieved. The final state could depend or not on initial conditions, and the behavior at low densities can change smoothly or abruptly to that at high \(p\). In one parameters region consensus is always achieved also at low densities. Our results allow one to precisely predict when these different behaviors occur.

Note that, although the parameters regulating the interactions with undecided agents have a marginal role in determining the collective behavior, the presence itself of the \(U\) state is crucial for a non-trivial behavior [18]5.

Above some critical density of zealots, the system always reaches asymptotically a state of unanimous consensus. The stability of the consensus obtained upon variations of the zealots density depends on the specific model. For models the FCM class consensus is very stable: once it is reached, it is maintained when the density of zealots is diminished, even down to zero. In models in the ZCM class dissenters nucleate as soon as the density of zealots is reduced below the threshold. Models in the VCM class shows an interesting hysteretic behavior: Once consensus is reached with a sufficiently high zealots density, it is maintained when the density goes below threshold, however, large fluctuations below \(p_c\) make the system unstable, and at \(p^\ast\) any infinitesimal perturbation causes an abrupt transition to a state with a finite density of \(B\) states.

We finally note a counterintuitive behavior that emerges in certain parameters ranges, where \(B\) states survive, and may even become the majority, although both the effect of the zealots and the interaction rules are biased against them6. The role of the neutral state is crucial in producing such behavior: although the peer interaction rules favor the \(A\) state, the rate at which they occur allows the \(B\) state to be favored on average [18].

For every single choice of the parameters our analysis identifies the value of the critical density of zealots \((\bar{p} = \max(p_c, p^\ast),\) with \(p_c = r_I/(1 + r_I)\) and \(p^\ast = r^\ast/(1 + r^\ast)\)) above which diverse opinions cannot survive. This critical fraction of committed individuals needed to achieve consensus has an upper bound in \(\bar{p} = \gamma/(1 + \gamma) = 1/(1 - \beta_1/\varphi_1)\). In order for this bound to be non-trivial it has to be \(\varphi_1 < 0\) (indeed for \(\varphi_1 > 0\) the system always reaches the consensus state). The bound \(\bar{p}\) only depends on the ratio \(\beta_1/\varphi_1\) that measures the relative net gain (or loss) of individuals in an \(A\) state after an \(A\)–\(B\) interaction.

\[5\] Eliminating the possibility for an individual to be undecided leads to a quite trivial behavior: either the system converge to total consensus on opinion \(A\) (when asymmetric interactions favors \(A\)), or it exhibit a continuous transition between consensus and pluralism (when asymmetric interactions favors \(B\)).

\[6\] Examples are systems in the FCM class that have \(\gamma_1 < \varphi_1\) and \(\varphi_2 > \gamma_2\) (the first condition means that more \(B\) than \(A\) states are lost in \(A\)–\(B\) interactions; the second that individuals in \(A\) state are more successful than those in \(B\) state in convincing undecided individuals). In this case, both the presence of zealots and the rules of peer interactions favor \(A\), yet for small densities of zealots and suitable initial conditions the system reaches a stationary state with finite density of dissenters.
when \( A \) is committed with respect to the case when \( A \) is uncommitted. When the average net gain in \( A-B \) interactions with committed \( A \) is larger than the average net loss in \( A \) due to \( A-B \) interactions with uncommitted \( A \), i.e. for \( \beta_1 > -\varphi_1 \), the bound ensures \( p < 1/2 \), meaning that a committed minority suffices to drive the system to consensus.

In our analysis, we have considered generic rules for the peer interactions, but we assumed symmetric roles for the two interacting partners. However, we point out that our analysis holds more generally, including cases where the interaction partners have distinct roles (e.g. speaker–listener), as often considered in the literature. This can easily be shown along the same lines followed in [15], where we have shown that a model allowing for asymmetric roles is always equivalent at the mean-field level to its symmetrized version, the outcome of an interaction in the symmetrized model being defined as the average result of two asymmetric interactions with exchanged roles.

Finally, we remind the reader that our results describe stationary properties of the general model under investigation. The time scales needed to reach the steady state may be strongly dependent on the different parameters in equations (2).

The present investigation allows us to recover in a unified framework various types of behavior previously uncovered in several other modeling attempts, thus allowing us to understand the physical ingredients behind the single observations. In particular some of the dynamics considered in [9] are special cases of our general model. For example, the ‘basic model’ of [9] corresponds to \( \gamma_1 = \varphi_1 = -1/2, \gamma_3 = \varphi_2 = 1/2, \beta_1 = 0, \beta_4 = 1/2 \). From our analytical treatment, it is easy to conclude that such a model exhibits a discontinuous transition, as \( p \) is decreased, from unanimous consensus to coexistence of individuals supporting both opinions.

An interesting direction for future work would be to test our prediction outside mean field on both real and synthetic networks. We expect that, unlike the case where the bias in favor of one opinion comes from a broadcast media source [15], the results in the presence of non-trivial topologies could depend on the location of the zealots, and deviate from mean field.

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