Exact four dimensional string solutions
and Toda-like sigma models
from ‘null-gauged’ WZNW theories

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We construct a new class of exact string solutions with a four dimensional target space metric of signature \((-,+,+,+)^{\cdot}\) by gauging the independent left and right nilpotent subgroups with ‘null’ generators of WZNW models for rank 2 non-compact groups \(G\). The ‘null’ property of the generators (\(\text{Tr}(N_n N_m) = 0\)) implies the consistency of the gauging and the absence of \(\alpha'\)-corrections to the semiclassical backgrounds obtained from the gauged WZNW models. In the case of the maximally non-compact groups \((G = SL(3), SO(2, 2), SO(2, 3), G_2)\) the construction corresponds to gauging some of the subgroups generated by the nilpotent ‘step’ operators in the Gauss decomposition. The rank 2 case is a particular example of a general construction leading to conformal backgrounds with one time-like direction. The conformal theories obtained by integrating out the gauge field can be considered as sigma model analogs of Toda models (their classical equations of motion are equivalent to Toda model equations). The procedure of ‘null gauging’ applies also to other non-compact groups. As an example, we consider the gauging of \(SO(1, 3)\) where the resulting metric has the signature \((-,-,+,+)^{\cdot}\) but admits two analytic continuations with Minkowski signature. The backgrounds we find have ‘2+2’ structure with two null Killing vectors. Their dual counterparts have one covariantly constant null Killing vector, i.e. are of ‘plane-wave’ type (with metric and dilaton depending only on transverse spatial coordinates) and also represent exact string solutions.

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1. Introduction

In trying to understand possible implications of string theory for gravitational physics it is important to study string solutions with physical dimension \( D = 4 \) and signature \((-++,+++)\). With a hope to be able to discuss issues of singularities and short distance structure one is mostly interested not just in solutions of the leading-order low-energy string effective equations but in the ones that are exact in \( \alpha' \) and/or which have an explicit conformal field theory interpretation. While the leading-order solutions are numerous, very few solutions of the second type are known. In addition to the obvious ‘direct product’ combinations of low-dimensional solutions (e.g. \( R \times SU(2) \) WZNW [1]) the known exact \( D = 4 \) ones include, in particular, spaces with a covariantly constant null Killing vector \( \mathbb{R} \times SU(2) \) WZNW [1], ‘black hole’-type [7] and cosmological [10] solutions based on \((G \times G')/(H \times H')\) gauged WZNW models (with \( G = SL(2, R) \), \( G' = SL(2, R) \) or \( SU(2) \) and \( H, H' = U(1) \) or \( SO(1, 1) \)) the \( SO(2, 3)/SO(1, 1) \) model of [16], a black hole solution of [17] and the solution [18] corresponding to the WZNW model for a non-semisimple \( D = 4 \) group which has a non-degenerate invariant bilinear form [1].

The aim of this paper is to present some new exact \( D = 4 \) solutions which correspond to gauged WZNW models and thus should have a direct conformal field theory

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1 For other superstring solutions see also [14] [15] and references there.
2 Since there is no dimension 4 simple Lie group one may try to obtain a \( D = 4 \) solution from a WZNW-type theory either by considering \( G/H \) gauged models or by using the construction [18] [19] [20] based on non-semisimple groups. It is possible to check explicitly [21] that there are no other (in addition to the algebra in [18]) non-trivial \( D = 4 \) solvable Lie algebras (for a classification see [22]) which have a non-degenerate invariant bilinear form and thus could lead to new WZNW-type models according to [23]. The existence of only one \( D = 4 \) non-abelian algebra with an invariant form is explained by a theorem in a recent paper [24]: since non-semisimple algebras with invariant forms are obtained by the procedure of “double extension” \( g \rightarrow g \oplus h \oplus h^* \) from an algebra \( g \) with an invariant form, to get a \( D = 4 \) algebra one needs to start with a \( D = 2 \) algebra as \( g \) (\( h \) must be one-dimensional to get \( D = 4 \)). Among the two \( D = 2 \) Lie algebras only the abelian one can have a non-degenerate invariant form; the corresponding \( D = 4 \) algebra is the central extension of the euclidean group in two dimensions, the algebra used in [18].
interpretation. The starting point will be what we shall call 'null gauged' WZNW models, i.e. gauged WZNW models \[25\][26][27][28] based on non-compact \[29\][30] groups with the generators of the gauged subgroup being 'null' (having zero Killing scalar products). The gauged subgroup will be thus chosen to be solvable (but need not be nilpotent in general).

We shall generalise the procedure of gauging WZNW models with nilpotent subgroups \[31\] to obtain conformal sigma models with Minkowski signature. The resulting sigma models will belong to the following class

\[
S = \frac{1}{\pi \alpha'} \int d^2 z [\partial x^i \bar{\partial} x_i + F(x) \partial u \bar{\partial} v] + \frac{1}{4\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} \phi(x),
\]

where the two functions \(F\) and \(\phi\) will be explicitly determined (in Section 2) by gauging of the nilpotent subgroups in WZNW models for rank \(n\) maximally non-compact groups. \(x^i (i = 1, ..., n)\) will be the linear combinations of the coordinates \(r^i \equiv r^{\alpha_i}\) corresponding to the simple roots \(\alpha_i,\)

\[
\partial x^i \bar{\partial} x_i = C_{ij} \partial r^i \bar{\partial} r^j, \quad \alpha_i \cdot x = K_{ij} r^j, \quad K_{ij} \equiv K_{\alpha_i \alpha_j} = \frac{2 \alpha_i \cdot \alpha_j}{|\alpha_j|^2} = \frac{1}{2} |\alpha_i|^2 C_{ij},
\]

where \(K_{ij}\) is the \(n \times n\) Cartan matrix. We shall find that

\[
F = \frac{1}{\sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x}}, \quad \phi = \frac{1}{2} \sum_{s=1}^m \alpha_s \cdot x - \frac{1}{2} \ln \left( \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x} \right) = \rho \cdot x + \frac{1}{2} \ln F,
\]

where the constants \(\epsilon_i\) can be chosen to be 0 or \(\pm 1\) and \(\rho = \frac{1}{2} \sum_{s=1}^m \alpha_s\) is half of the sum of all positive roots.\(^3\)

The case of rank 2 groups leading to four dimensional backgrounds will be studied in detail in Section 3.

In Section 4 we shall find the general conditions on the functions \(F, \phi\) which are necessary for conformal invariance and check that the functions obtained from gauged

\(^3\) If \(\alpha_1\) is a simple root corresponding to the generators \(E_{\pm \alpha_1}\) which are left ungauged (the remaining \(m - 1\) positive (negative) roots correspond to the generators of a left (right) nilpotent subgroup that was gauged) then \(\epsilon_1 = 1 (m = \frac{1}{2}(d - n), \ n = \text{rank } G, \ d = \dim G).\)
WZNW models are their solutions. We shall also consider the dual version of the above sigma model,
\[
\tilde{S} = \frac{1}{\pi \alpha'} \int d^2 z \left[ \partial x^i \partial \bar{x}_i + \tilde{F}(x) \partial \bar{u} \partial u - 2 \partial \bar{v} \partial u \right] + \frac{1}{4 \pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} \bar{\phi}(x),
\]
which is also an exact solution of conformal invariance conditions and discuss an apparent similarity to the Toda model (in particular, we shall find a direct relation between the solutions of the classical equations of motion). Some concluding remarks will be made in Section 5.

2. Null gauging of WZNW models

2.1. General scheme

The simplest possibility to construct a \( D = 4 \) solution using a \( G/H \) gauged WZNW model is to consider \( H \) to be a subgroup of a semisimple group \( G \) of a minimal possible dimension. Since we would like also to get a \( D = 4 \) space-time with a time-like direction the obvious candidates for \( G \) are non-compact groups, e.g. \( SL(2, R) \times SL(2, R) \) or \( SO(1, 3) \). The Killing form of the first algebra has the signature \((-,+,+,+,+,+)\) so that one can get the Minkowski signature of the \( D = 4 \) background by the standard (vector or axial) gauging of one compact and one non-compact generator.

There exist, however, another possibility which we shall exploit below. The indefinite signature of the Killing form for non-compact algebras implies that there is a number of ‘null’ generators \( T_n = N_n \) which have zero scalar products, \( \text{Tr} (N_n N_m) = 0 \). A subalgebra generated by such generators is thus solvable (but may not be nilpotent). \footnote{An example of a ‘null’ generator in the Lorentz group case is a sum of a spatial rotation with a boost. Note that a nilpotent \((N^2 = 0)\) generator is null but, in general, a null generator need not be nilpotent. Gauging of subgroups generated by nilpotent generators was previously discussed in [31] [32] [33] [34] [35] [36].} In this case
one can consider a left-right asymmetric gauging since the anomaly cancellation condition
\[ \text{Tr } T_L^2 = \text{Tr } T_R^2 \] is obviously satisfied.

Let us first recall the structure of the action of the standard vectorially gauged \( G/H \) WZNW model. The classical \( G/H \) gauged WZNW action \[ S_v = -k I_v(g, A) \] corresponds to the group \( G \) and the subgroup \( H \),
\[ I_v(g, A) = I(g) + \frac{1}{\pi} \int d^2z \text{ Tr } (-A \bar{A} gg^{-1} + \bar{A} g^{-1} \partial g + g^{-1} Ag \bar{A} - A \bar{A}) \equiv I_0(g, A) - \frac{1}{\pi} \int d^2z \text{ Tr } (A \bar{A}) , \] is invariant under the vector \( H \) - gauge transformations
\[ g \rightarrow w^{-1} g \bar{w} , \quad A \rightarrow w^{-1} (A + \partial) w , \quad \bar{A} \rightarrow \bar{w}^{-1} (\bar{A} + \bar{\partial}) \bar{w} , \quad \bar{w} = w = w(z, \bar{z}) . \] Parametrising \( A \) and \( \bar{A} \) in terms of \( h \) and \( \bar{h} \) from \( H \)
\[ A = h \partial h^{-1} , \quad \bar{A} = \bar{h} \bar{\partial} \bar{h}^{-1} , \quad h \rightarrow w^{-1} h , \quad \bar{h} \rightarrow \bar{w}^{-1} \bar{h} , \] one can use the Polyakov-Wiegmann identity \[ S_v = -k I_v(g, A) \] to represent the gauged action as the difference of the two manifestly gauge-invariant terms: the ungauged WZNW actions corresponding to the group \( G \) and the subgroup \( H \),
\[ I_v(g, A) = I(h^{-1} g \bar{h}) - I(h^{-1} \bar{h}) . \] Since
\[ I(h^{-1} g \bar{h}) = I_0(g, A) + I(h^{-1}) + I(\bar{h}) , \] \[ I(h^{-1} \bar{h}) = I(h^{-1}) + I(\bar{h}) + \frac{1}{\pi} \int d^2z \text{ Tr } (A \bar{A}) , \]
the non-local terms $I(h^{-1}) + I(\bar{h})$ cancel out in the classical action (4) but survive in the quantum effective one \[39\][40] since the coefficients $k$ of the two terms in (4) get different quantum corrections ($k \rightarrow k - \frac{1}{2} c_G$ and $k \rightarrow k - \frac{1}{2} c_H$). The presence of the two WZNW terms in (4) with the coefficients which renormalise in a different way implies that the target space background fields obtained after integrating out the gauge field \[41\][42] are modified by $k^{-1}(\sim \alpha')$ -corrections \[39\][40].

Suppose now that the parameters $w$ and $\bar{w}$ of the gauge transformation in (2) are not the same and belong to two different subgroups $H_+$ and $H_-$ of $G$. If these subgroups are generated by null generators ($\text{Tr} (N_nN_m) = 0$) we have the crucial property

$$I(h^{-1}) = I(\bar{h}) = 0 . \tag{6}$$

Then one may use the action

$$S_n = -kI_n , \quad I_n(g, A, \bar{A}) \equiv I(h^{-1}g\bar{h})$$

$$= I(g) + \frac{1}{\pi} \int d^2z \text{Tr} (-A \bar{\partial} g^{-1} + \bar{A} g^{-1} \partial g + g^{-1}Ag\bar{A}) , \tag{7}$$

which is manifestly gauge invariant and local when expressed in terms of $g$ and $A, \bar{A}$ as the gauged WZNW action.\[3]\] Assuming that the corresponding quantum theory is regularised in the ‘left-right decoupled’ way (so that the local counterterm $\text{Tr} (AA)$ does not appear) the only non-trivial renormalisation that can occur at the quantum level is the shift of the overall coefficient $k \rightarrow k - \frac{1}{2} c_G$ in front of the action (7). As a result, the couplings of

$^6$ $c_G$ is the value of the quadratic Casimir operator in adjoint representation. The negative sign of the shift is due to our choice of the ‘unphysical’ sign in the action (1) as usual in the non-compact case.

$^7$ $A$ and $\bar{A}$ should be considered as chiral projections of the two independent vector fields.

$^8$ Depending on a choice of the null subgroups $H_+$ and $H_-$ the trace of the product of their generators $\text{Tr} (N\bar{N})$ and hence $\text{Tr} (A\bar{A})$ may or may not vanish so that (7), in general, is different from (4) (which is not gauge invariant if $\text{Tr} (N\bar{N}) \neq 0$). In the special case of (6) the action (7) coincides with the action of the chiral gauged WZNW model \[43\][44][45] $I_c(g, A) = I(h^{-1}g\bar{h}) - I(h^{-1}) - I(\bar{h})$.  

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the sigma model obtained by integrating out the gauge fields $A, \bar{A}$ should not receive non-trivial $k^{-1}$ corrections, i.e. they should represent an exact solution of the sigma model conformal invariance equations. The central charge of the resulting gauged model will be equal to the central charge of the original WZNW model minus the dimension of the gauged subgroup.

2.2. Gauging of nilpotent subgroups in Gauss decomposition parametrisation

A particular case of such gauging (when the null subgroups are the nilpotent subgroups corresponding to the step generators in the Gauss decomposition) was considered previously [31][32] (see also [33][34]) in the context of Hamiltonian reduction [46][47] of WZNW theories related to Toda models. The approach based on gauging of any subgroup with null generators is more general since, in principle, we do not need to use the Gauss decomposition (which does not always exist for the real groups we are to consider to get a real WZNW action). The gauging based on the Gauss decomposition directly applies only to the groups with the algebras that are the ‘maximally non-compact’ real forms of the classical Lie algebras (real linear spans of the Cartan-Weyl basis), i.e. $sl(n+1, R)$, $so(n, n+1)$, $sp(2n, R)$, $so(n, n)$. The corresponding WZNW models can be considered as natural generalisations of the $SL(2, R)$ WZNW model. For these groups there exists a real group-valued Gauss decomposition

$$g = g_+g_0g_-, \quad g_+ = \exp(\sum_{\Phi_+} u^{\alpha} E_{\alpha}), \quad g_- = \exp(\sum_{\Phi_+} v^{\alpha} E_{-\alpha}), \quad g_0 = \exp(\sum_{\Delta} r^{\alpha} H_{\alpha}) = \exp(\sum_{i=1}^{n} x^i H_i).$$

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9 The WZNW model in the Gauss decomposition parametrisation was considered in [18]. The standard (vector or axial) gauging in the Gauss decomposition was also discussed in [35].

10 Since there exists a Cartan involution for every non-compact real form of complex simple Lie algebras [43] it may be possible to repeat the construction that follows for other non-compact groups using ‘generalised Gauss decomposition’ [50]. Our treatment of the $SO(1, 3)$ case below may be considered as a particular example of such a generalisation.
Here $\Phi^+$ and $\Delta$ are the sets of the positive and simple roots of a complex algebra with the Cartan-Weyl basis consisting of the step operators $E_\alpha$, $E_{-\alpha}$, $\alpha \in \Phi^+$ and $n (= \text{rank } G)$ Cartan subalgebra generators $H_\alpha$, $\alpha \in \Delta$. We shall use the following standard relations (we shall assume that a long root has $|\alpha|^2 = 2$) [51][52]

\[
[H_\alpha, E_\beta] = K_{\beta\alpha} E_\beta \quad (\alpha \in \Delta, \beta \in \Phi),
\]

\[
[H_\alpha, E_{-\alpha}] = H_\alpha \quad (\alpha \in \Delta), \quad [E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta},
\]

\[
\text{Tr} \ (H_\alpha H_\beta) \equiv C_{\alpha\beta} = \frac{2}{|\alpha|^2} K_{\alpha\beta}, \quad K_{\alpha\beta} = \frac{2\alpha \cdot \beta}{|\beta|^2}, \quad \text{Tr} \ (H_i H_j) = \delta_{ij},
\]

\[
H_\alpha = \tilde{\alpha}^i H_i, \quad x^i = \sum_{\alpha \in \Delta} \tilde{\alpha}^i r^\alpha, \quad K_{\alpha\beta} r^\beta = \alpha \cdot x, \quad \tilde{\alpha}^i \equiv \frac{2}{|\alpha|^2} \alpha^i,
\]

\[
\text{Tr} \ (E_\alpha E_{-\beta}) = \frac{2}{|\alpha|^2} \delta_{\alpha\beta}, \quad \text{Tr} \ (E_\alpha H_\beta) = 0,
\]

(9)

where $\alpha^i (i = 1, ..., n)$ are the components of the positive root vectors. It is clear that $E_\alpha$ and $E_{-\alpha}$ form sets of null generators so that some of the corresponding symmetries can be gauged according to (7). For example, we may take $w$, $A$ and $h$ in (2),(3) to belong to the one-dimensional subgroup generated by some $E_\gamma$ and $\bar{w}, \bar{A}$ and $\bar{h}$ – to the subgroup generated by $E_{-\gamma'}$ where $\gamma$ and $\gamma'$ are positive roots which may not necessarily be the same.

If one gauges the full left and right nilpotent subgroups $G_+$ and $G_-$ (dim $G_\pm = \frac{1}{2}(d - n) = m$, dim $G = d$) generated by all generators $E_\alpha$ and $E_{-\alpha}$ [31] one is left with the action for $n$ decoupled scalars $r^\alpha$ or $x^i$ which represent the free part of the Toda model action. Being interested in finding non-trivial conformal sigma models describing string solutions we are to consider the more general case of ‘partial’ gauging when only some subgroups $H_+$ and $H_-$ of $G_+$ and $G_-$ are gauged. $r^\alpha$ should correspond to spatial directions ($C_{\alpha\beta}$ in (9) is positive definite). Since the Killing form of the maximally non-compact groups has $m = \frac{1}{2}(d - n)$ time-like directions, to get a physical signature of the

\[\text{11} \text{ In what follows we shall assume that there is always a sum over repeated upper and lower indices. We shall also use } r^\alpha \text{ with understanding that } r^\alpha \neq 0 \text{ only if } \alpha \text{ is a simple root.}\]
resulting space-time we need to gauge away all but one pair of coordinates \( u, v \) in (8). Therefore the gauge groups \( H_\pm \) should have dimension \( m - 1 = \frac{1}{2}(d - n) - 1 \), i.e.

\[
\dim H_\pm = \dim G_\pm - 1.
\]

As we shall see below (in Sect.2.3) the ungauged generator(s) of \( G_\pm \) must be a simple root.

Moreover, to get the physical value \( D = 4 \) of the target space dimension we need to start with the rank 2 groups \( G \) \((D = n + 2 = 4)\).

Let us first consider the most general case when \( w, A \) and \( h \) in (2),(3) correspond to the subgroup \( H_+ \subset G_+ \) generated by some \( s \leq m \) linear combinations \( E_p = \lambda^\alpha_p E_\alpha \) of the generators \((E_\alpha, \alpha \in \Phi_+)\) of \( G_+ \) and \( \bar{w}, \bar{A} \) and \( \bar{h} \) to the subgroup \( H_- \subset G_- \) generated by some \( s' = s \) linear combinations \( \bar{E}_q = \bar{\lambda}^\alpha_q E_{-\alpha} \). Then it is straightforward to write down the resulting expression for the action (7) using the Polyakov-Wiegmann formula and (9) (i.e. \( I(g_+) = I(g_-) = 0 \), etc.)

\[
I_n = I(h^{-1}g\bar{h}) = I(g_0) + \frac{1}{\pi} \int d^2z \, \text{Tr} \left[ g_0^{-1}g_+^{-1}h\partial(h^{-1}g_+)g_0g_-\bar{h}\partial(\bar{h}^{-1}g_-^{-1}) \right]
\]

\[
= I(g_0) + \frac{1}{\pi} \int d^2z \, \text{Tr} \left[ g_0^{-1}(A + g_+^{-1}\partial g_+)g_0(\bar{A} - \partial g_-g_-^{-1}) \right].
\]

(10)

Setting

\[
A = \mathcal{E}_p B^p = \lambda^\alpha_p B^p E_\alpha, \quad \bar{A} = \bar{\mathcal{E}}_q \bar{B}^q = \bar{\lambda}^\alpha_q \bar{B}^q E_{-\alpha},
\]

\[
g_+^{-1}\partial g_+ \equiv J_u = U_{\alpha}^\beta(u)\partial u^\alpha E_\beta, \quad \bar{\partial} g_-g_-^{-1} \equiv J_v = J^\alpha_v E_{-\alpha} = V_{\alpha}^\beta(v)\bar{\partial} v^\alpha E_{-\beta},
\]

(11)

we get

\[
S_n = \frac{k}{\pi} \int d^2z \left[ \frac{1}{2} C_{\alpha\beta} \partial r^\alpha \bar{\partial} r^\beta + M_{\alpha\beta}(J_u^\alpha + \lambda^\alpha_p B^p)(J_v^\beta - \bar{\lambda}^\beta_q \bar{B}^q) \right]
\]

\[
= \frac{k}{2\pi} \int d^2z \left[ \partial x^i \bar{\partial} x_i + 2M_{\alpha\beta}(J_u^\alpha + \lambda^\alpha_p B^p)(J_v^\beta - \bar{\lambda}^\beta_q \bar{B}^q) \right],
\]

(12)

where

\[
M_{\alpha\beta} = \text{Tr} \left( g_0^{-1}E_\alpha g_0 E_{-\beta} \right) = f_\alpha(r)\delta_{\alpha\beta}, \quad f_\alpha(r) \equiv \frac{2}{|\alpha|^2} e^{-|\alpha|^2/2} = \frac{2}{|\alpha|^2} e^{-\alpha \cdot x}.
\]
The sums over $\alpha, \beta$ run over positive roots ($r^\alpha \neq 0$ for simple roots only). It is clear that when $H_\pm = G_\pm$, i.e. when $\lambda^\alpha_p$ and $\bar{\lambda}^\beta_q$ are non-degenerate we can eliminate $J_u$ and $\bar{J}_v$ from the action by redefining the gauge fields $B, \bar{B}$. One is then left with the free action for $r^\alpha$ plus the dilaton term $\phi = \phi_0 + \frac{1}{2} \sum_\alpha K_{\alpha \beta} r^\beta$ originating from the $B, \bar{B}$-determinant. More precisely, the latter determinant is given by the sum of the two terms \[53\] so that its contribution to the action is

$$\Delta S = -\frac{1}{2\pi} \int d^2 z \partial (\ln \det M) \bar{\partial} (\ln \det M) - \frac{1}{8\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} \ln \det M$$

$$= -\frac{1}{2\pi} \int d^2 z \sum_{\alpha, \beta \in \Phi_+} (\alpha \cdot \partial x)(\beta \cdot \bar{\partial} x) + \frac{1}{4\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} (\phi_0 + \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \cdot x)$$

$$= -\frac{2}{\pi} \int d^2 z (\rho \cdot \partial x)(\rho \cdot \bar{\partial} x) + \frac{1}{4\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} (\phi_0 + \rho \cdot x), \quad \rho \equiv \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha \cdot x. \quad (13)$$

Similar expression was found in \[48\] in the process of representing WZNW theory in terms of free fields. It was claimed in \[48\] that the first term combined with the free term in (12) produces the quantum renormalisation of the level coefficient $k$ leading to the correct expression for the total central charge. As it appears from (13), only the coefficient of the projection of $x$ on $\rho$ gets renormalised (this, in fact, is sufficient for reproducing the quantum value of the central charge of the WZNW model $C = k/(k - \frac{1}{2} c_G)$, $c_G = 24 \rho^2/d$).

In what follows we shall not include explicitly the derivative terms in similar gauge field determinant contributions anticipating that the quantum effective action of the resulting model has the shifted overall coefficient $\kappa = k - \frac{1}{2} c_G$.

Integrating out $B^p$ and $\bar{B}^q$ in (12) we get

$$S_n = \frac{k}{\pi} \int d^2 z \left[ \frac{1}{2} C_{\alpha \beta} \partial r^\alpha \bar{\partial} r^\beta + M_{\alpha \beta}(r) U^\alpha_\gamma(u) V^\beta_\delta(v) \partial u^\gamma \bar{\partial} v^\delta \right]$$

$$- \frac{1}{8\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} \ln \det M_{pq}(r), \quad (14)$$

$$M_{pq}(r) \equiv M_{\alpha \beta} \lambda^\alpha_p \bar{\lambda}^\beta_q = \sum_{\alpha} f_{\alpha}(r) \lambda^\alpha_p \bar{\lambda}^\alpha_q,$$

$$\mathcal{M}_{\alpha \beta}(r) = M_{\alpha \beta} - M^{-1}_{pq} \bar{\lambda}^\gamma_p \lambda^\delta_q M_{\alpha \gamma} M_{\beta \delta} = f_{\alpha \delta \beta} - f_{\alpha \beta} f_{\delta \epsilon} M^{-1}_{pq} \bar{\lambda}^\gamma_p \lambda^\delta_q. \quad (15)$$
For example, in the simplest case when the left and right null gauged subgroups are one-dimensional and generated by the opposite roots $E_\gamma, E_{-\gamma}$ we get

$$S_n = \frac{k}{2\pi} \int d^2z \left[ \partial x^i \bar{\partial} x_i + \sum_{\alpha \neq \gamma} \frac{4}{|\alpha|^2} e^{-\alpha \cdot x} U^\alpha_\sigma(u) V^\alpha_\delta(v) \partial u^\sigma \bar{\partial} v^\delta \right]$$

$$+ \frac{1}{4\pi} \int d^2z \sqrt{g(2)} R^{(2)}(\rho - \frac{1}{2} \gamma) \cdot x .$$

(16)

The derivatives of the corresponding coordinates $\partial u^\gamma$ and $\bar{\partial} v^\gamma$ are absent in the action (which is just the original ungauged WZNW action in (12) with $J^\gamma_u = J^\gamma_v = 0$) but $u^\gamma, v^\gamma$ themselves may still appear in (16) through $U^\alpha_\sigma(u) V^\alpha_\delta(v)$. This does not represent a problem for a sigma model interpretation since one can gauge fix $u^\gamma = v^\gamma = 0$ from the start (cf. [35]). The null gauging thus reduced the dimension of the WZNW model by two as expected since the left and right gauge groups are independent.

If the gauge fields in (12) belong to $\Phi/\Delta$ then after integrating them out we finish with the interaction term $2M_{\alpha\beta} J_\alpha^u J^\beta_v$ where $\alpha, \beta$ run over simple roots only. If one further adds the constraints [31] on these currents (which break manifest off-shell classical conformal invariance of the gauged WZNW model) one finds that (12) takes the form of the Toda model. As we shall see in Section 4.2 (see also Section 2.4), the imposition of constraints is not, in fact, necessary in order to make a connection to the Toda model.

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12 In general, $H_+$ and $H_-$ need not be the same so that $M_{pq}$ may be degenerate. The integrals over the corresponding zero eigenvalue combinations of $B, \bar{B}$ will produce $\delta$-function constraints on some of $J_u, J_v$. In the case when the gauge groups are one-dimensional and the corresponding left and right gauge generators are just different step operators $E_{\gamma}$ and $E_{-\gamma'}$, the $B\bar{B}$-term in (12) vanishes and the integrals over $B$ and $\bar{B}$ give the $\delta$-function constraints which set $J^\gamma_u$ and $J^{\gamma'}_u$ to zero ($M_{\alpha\beta}$ is diagonal) so that the final result for a sigma model action is

$$S_n = \frac{k}{2\pi} \int d^2z \left[ C_{\alpha\beta} \partial r^\alpha \bar{\partial} r^\beta + \sum_{\alpha \neq \gamma, \gamma'} f_\alpha(r) U^\alpha_\sigma(u) V^\alpha_\delta(v) \partial u^\sigma \bar{\partial} v^\delta \right] .$$

There is an extra dilaton term originating from the determinants which appear after integrating out the $\delta$-functions. This ‘degenerate’ gauging reduces the number of dimensions by four: $\partial u^\gamma, \bar{\partial} u^\gamma, \partial v^\gamma, \bar{\partial} v^\gamma$ are absent in the final action (14). In fact, the built-in gauge invariance of the original action (10) implies that $u^\gamma$ and $v^\gamma$ can be gauge-fixed to zero. Also, given that $\partial u^\gamma'$ and $\bar{\partial} v^\gamma$ appear in the final result only in the $\delta$-functions, it is natural to reduce the number of coordinates further by trying to integrate explicitly over $u^\gamma', v^\gamma$. 

10
In the case when gauged subgroups are non-abelian the resulting sigma models (14) may have non-abelian global symmetries and in particular cases have classical equations equivalent to equations in non-abelian Toda models (see, e.g., [54][55][50][56][57]). However, such models will contain more than one pair of \(u, v\) coordinates, i.e. more than one time-like direction and we shall not consider them here.

2.3. Models with one time-like coordinate

Let us now turn to the most interesting case when the dimensions of the gauge groups \(H_{\pm}\) are equal to \(\dim G_{\pm} - 1 = m - 1\) so that only one time-like coordinate appears in the resulting sigma model action. Let \(E_{\alpha_1}\) and \(E_{-\alpha_1}\) denote the generators of \(G_+\) and \(G_-\) which remain ungauged, i.e. which do not belong to \(H_+\) and \(H_-\). Since \(H_+\) must be a subgroup, \(E_{\alpha_1}\) cannot appear in the commutators of the generators of \(H_+\). According to (9) this is possible only if \(\alpha_1\) is a simple root, i.e. if it cannot be represented as a sum of two other positive roots. In fact, if we use the indices \(i, j\) to denote the simple roots \(\alpha = \alpha_i (i = (1, s) = 1, 2, ..., n)\) and indices \(a, b\) to denote the remaining positive roots \(\alpha_a (a = n + 1, ..., m)\) the commutators of the corresponding step operators are given by

\[
[E_i, E_j] \sim E_a \quad (\alpha_a = \alpha_i + \alpha_j),
\]

\[
[E_i, E_a] \sim E_b \quad (\alpha_b = \alpha_i + \alpha_a), \quad [E_a, E_b] \sim E_c \quad (\alpha_c = \alpha_a + \alpha_b).
\]

It is clear that one can also use linear combinations \(E'_s = E_s + \lambda_s E_1 (s = 2, ..., n)\) as the ‘simple’ part of the generators of \(H_+\) but one cannot mix the non-simple generators \(E_a\) with \(E_1\).

Let \(u^{\alpha_1} \equiv \frac{1}{\sqrt{2}} u, \ v^{\alpha_1} \equiv \frac{1}{\sqrt{2}} v;\) the remaining coordinates \(u^\sigma, v^\sigma (\sigma = (s, a)\) will be used to denote all ‘gauged’ \(m - 1\) positive roots) are transforming under the gauge group (with the leading-order term being just a shift) so that we can set them to zero as a gauge. In
this gauge \( J_u = \frac{1}{\sqrt{2}} \partial u E_{\alpha_1} \), \( J_v = \frac{1}{\sqrt{2}} \partial v E_{-\alpha_1} \) and the sigma model action (14) takes the form \( p, q = 1, \ldots, m - 1 \)\(^{13}\)

\[
S_n = \frac{k}{2\pi} \int d^2 z [\partial x^i \partial x_i + F(x) \partial u \partial v] + \frac{1}{4\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} \phi(x) ,
\]

\[
F(x) = f_{\alpha_1} - f_{\alpha_1}^2 M^{-1pq} \bar{\lambda}_p^\alpha \lambda_q^\alpha , \quad \phi(x) = -\frac{1}{2} \ln \det M_{pq} ,
\]

\[
M_{pq}(x) = \sum_\alpha f_\alpha(x) \lambda_p^\alpha \lambda_q^\alpha .
\]

The non-trivial elements of the ‘mixing’ matrix \( \lambda_\alpha^\alpha_p \) correspond to a possibility of changing the generators of \( H_+ \) by adding \( \lambda_\alpha^\alpha_p E_{\alpha_1} \). Without loss of generality the non-vanishing components of \( \lambda_\alpha^\alpha_p \) can be taken to be: \( \lambda_\sigma^\sigma_p = \delta_\sigma_p, \lambda_\alpha_1^\alpha_1 = \lambda_\sigma \delta_{ps} \) and similarly for \( \bar{\lambda}_q^\alpha \) (according to the remark above, only simple roots can be mixed with \( E_{\alpha_1} \)). Then \( s, t = 2, \ldots, n; a, b = n + 1, \ldots, m \)

\[
M_{pq} = \begin{pmatrix} M_{st} & 0 \\ 0 & M_{ab} \end{pmatrix} , \quad M_{st}(r) = f_s \delta_{st} + f_1 \lambda_s \bar{\lambda}_t , \quad M_{ab}(x) = f_a(x) \delta_{ab} ,
\]

\[
f_h(x) \equiv f_{\alpha_1} = \frac{2}{|\alpha_h|^2} e^{-\alpha_h \cdot x} , \quad h = (1, s, a) = 1, 2, \ldots, m , \quad m = \frac{1}{2}(d - n) .
\]

If we introduce \( \lambda_1 = \bar{\lambda}_1 = 1 \) in order to make the formulas look symmetric with respect to all simple roots, we find

\[
M_{st}^{-1} = f_s^{-1} \delta_{st} - \frac{f_s^{-1} f_t^{-1} \lambda_s \bar{\lambda}_t}{\sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i} , \quad \det M_{pq} = \left( \prod_{h=1}^m f_h \right) \left( \sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i \right) .
\]

As a result,

\[
F = f_1 - f_{\alpha_1}^2 M^{-1st} \bar{\lambda}_s \lambda_t = \frac{1}{\sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i} ,
\]

\[
F = \left( \sum_{i=1}^n \epsilon_i e^{\alpha_i \cdot x} \right)^{-1} , \quad \epsilon_i \equiv \frac{1}{2} |\alpha_i|^2 \lambda_i \bar{\lambda}_i ,
\]

\[
\phi = -\frac{1}{2} \sum_{h=1}^m \ln f_h - \frac{1}{2} \ln \sum_{i=1}^n f_i^{-1} \lambda_i \bar{\lambda}_i ,
\]

\(^{13}\) For notational convenience (to get rid of an extra factor of 2 in front of the \( F(x) \partial u \partial v \) term) we have redefined \( u \) and \( v \) by the factor of \( 1/\sqrt{2} \) as compared to (14).
\[
\phi = \phi_0 + \rho \cdot x - \frac{1}{2} \ln \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x}, \quad \rho = \frac{1}{2} \sum_{h=1}^{m} \alpha_h.
\] (22)

We discover the following relation between the basic function \( F \) and the dilaton

\[
F(x) = F_0 e^{2\phi(x)} e^{-2\rho \cdot x}.
\] (23)

This relation is not accidental, being necessary for the conformal invariance of the model (17) (see Section 4). Note that (23) implies that the string tree-level measure factor for the model (17) has a universal form which does not depend on a particular gauging but only on the sum of all positive roots for a given algebra

\[
\sqrt{G} e^{-2\phi} = F e^{-2\phi} = F_0 e^{-2\rho \cdot x}.
\] (24)

This factor, in fact, is the \( x \)-dependent part of the Haar measure of the original WZNW model expressed in the Gauss decomposition (see (14),(15)).

It is clear that non-equivalent models correspond only to \( \epsilon_s = 0, +1, -1 \) since if \( \epsilon_s \neq 0 \) we can make all \( |\epsilon_s| = 1 \) by constant shifts of \( x^i \). In the simplest case when all mixing parameters \( \lambda_s, \bar{\lambda}_s \) are equal to zero, i.e. \( \epsilon_s = 0, F = f_1, \phi(r) = \phi_0 - \frac{1}{2} \sum_{s=2}^{m} \ln f_h \), the action (17) becomes (cf.(16))

\[
S_n = \frac{k}{2\pi} \int d^2 z \left[ \partial x^i \bar{\partial} x_i + F_0 e^{-\alpha_1 \cdot x} \partial u \bar{\partial} v \right] + \frac{1}{4\pi} \int d^2 z \sqrt{g^{(2)}} R^{(2)} (\rho - \frac{1}{2} \alpha_{1i}) x^i,
\] (25)

where \( \alpha_{1i} \) are components of the ‘ungauged’ simple root. Using the rotational symmetry in \( x^i \) space we can make \( \alpha_1 = (\sqrt{2}, 0, ..., 0) \) so that the model factorises into a product of the \( SL(2, R) \) WZNW model and \( n - 1 \) free scalars with linear dilaton.

2.4. Summary

We have thus found the sigma model action (17),(21),(22), i.e.

\[
S_n = \frac{\kappa}{2\pi} \int d^2 z \left[ \partial x^i \bar{\partial} x_i + \left( \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x} \right)^{-1} \partial u \bar{\partial} v \right]
\]

\[\text{14} \] We absorb the prefactors \( 2/|\alpha_i|^2 \) of the exponential terms into a rescaling of \( u, v \) and \( \epsilon_i \). We also consider (26) as an effective action, including the quantum shift of \( k, \kappa \equiv k - \frac{1}{2} c_G \).
\[ +\frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} (\rho \cdot x - \frac{1}{2} \ln \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x}) , \quad (26) \]

with \( \alpha_i \) being the simple roots and \( \rho \) being half the sum of all positive roots. The values of the parameters \( \epsilon_i = 0, +1 \) or \(-1\) represent inequivalent gaugings of the original WZNW model or different conformal sigma models. Non-trivial models (not equivalent to direct products of \( SL(2, R) \) WZNW with free scalars) are found for non-vanishing values of the ‘mixing’ parameters \( \epsilon_2, ..., \epsilon_n \).

The metric of the corresponding \( D = n + 2 \) dimensional target space-time has two null Killing symmetries (in fact, the full 2d Poincare invariance in the \( u, v \) plane, \( u' = \rho u + a, \ v' = \rho^{-1} v + b \)). The non-trivial \((uv)\) components of the metric and the antisymmetric tensor and the non-linear part of the dilaton are all expressed in terms of a single function \( F(x) \) (21), which is the inverse of the sum of the exponentials of the spatial Cartan coordinates \( x^i \). The metric is non-singular if all \( \epsilon_i \) have the same sign.

The action (26) has a structure reminiscent of the Toda model (with the dimension 2 operator \( \partial u \bar{\partial} v \) instead of the dimension zero one in the interaction term). In general, the model (26) or its dual version which has the metric and dilaton given by (see Section 4.3)

\[ \tilde{F} = F^{-1} = \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x}, \quad \tilde{\phi} = \phi - \frac{1}{2} \ln F = \rho \cdot x, \quad (27) \]

can be considered as ‘sigma model analogs’ of the Toda model.

We are drawing an analogy with abelian Toda models (see, e.g., [58] [59] [60] [61] [31]). In non-abelian Toda models (see, e.g., [54] [55] [50] [57] [56]) the free Cartan subgroup \( x^i\)-part of the abelian Toda action is replaced by a WZNW model action (and, correspondingly, the potential term takes more complicated form). As noted at the end of Section 2.3, our discussion can be generalised to the case of non-abelian gauge subgroups so that the resulting sigma models (in the special cases of properly chosen subgroups as in the

\[ ^{15} \text{Inequivalent solutions corresponding to different possible choices of an ungauged simple root} \ 
\alpha_1 \text{ are easily included by assuming that} \ \epsilon_1 \text{ can also take values} \ 0 \text{ and} \ -1 \text{ but at least one of} \ \epsilon_i \text{ is non-vanishing. In general,} \ \epsilon_i \text{ taking arbitrary real values represent moduli of the solutions.} \]
generalised integral gradings gauging in \[55\] \[31\] \[50\]) have a structure similar to that of non-abelian Toda models (being equivalent to them at the level of classical equations of motion). However, such models have more than one time-like direction and do not lead to new examples of \(D = 4\) conformal invariant backgrounds (any non-trivial WZNW model has at least three degrees of freedom while a four dimensional space has only two transverse directions). In fact, one can show that the simplest model of this type obtained by starting with \(SL(3, R)\) WZNW model and gauging one linear combination of roots on the left and on the right gives a \(D = 6\) sigma model with the signature \((+, +, +, +, -, -)\) (for a particular choice of the gauged generator it has global \(SL(2, R)\) symmetry).

The reason for a connection to the Toda model can be understood, for example, by comparing our approach to that of \[31\] where additional ‘background’ terms linear in the gauge fields (implying constraints on the currents) were introduced into the action (10). They produced a potential term after the gauge fields were integrated out. We instead did not gauge the full \(m\)-dimensional nilpotent subgroups \(G_\pm\) and as a result got a sigma model - type interaction term for the ungauged root directions.

As we shall show in Section 4.2, the classical equations (on a flat 2d background) for the model (26) (or its dual) reduce, in fact, to the Toda model equations. The relation between the models is not, however, quite precise at the quantum level. Note that in contrast to the Toda model (and the dual model (27)) the dilaton in (26) is, in general, a non-linear function of \(x^i\). If we gauge the remaining nilpotent generator corresponding to \(\alpha_1\) or simply integrate over \(u, v\) in (26) we cancel the non-linear dilaton term\[14\] and get \(\phi = \rho \cdot x\). Though this expression may look similar to the dilaton of the Toda model in the simply-laced algebra case \[59\] \[61\] \[31\] \[63\] \[33\] it is actually different since it does not contain the second linear term present in the Toda model dilaton in the general non-simply-laced case \[55\] \[33\]. The reason for this disagreement can be traced back to the basic fact that

\[\Delta \phi = -\frac{1}{2} \ln F\]

which cancels (according to (23)) the non-linear term in \(\phi\).

\[\text{As in (13), the determinant resulting from the integration over } u, v \text{ gives the dilaton contribution } L_2 L_3 \Delta \phi = -\frac{1}{2} \ln F \text{ which cancels (according to (23)) the non-linear term in } \phi.\]
the interaction term in the action (26) has classical dimension zero while the potential
term of the Toda model has classical dimension two. We shall clarify this point in Section
4.2.

In Section 4 we shall also check explicitly that the background corresponding to (26)
satisfies the conformal invariance equations. In the next section we shall consider the
particular case of the rank $n = 2$ groups when the resulting target space has dimension
four.

3. Null gauging in the case of rank 2 groups: four dimensional space-times

Let us now illustrate the above discussion on the examples of null gauging of the
groups corresponding to the maximally non-compact real forms of the rank $n = 2$ algebras
$sl(3, R), so(2, 3) = sp(4, R), so(2, 2) = sl(2, R) \oplus sl(2, R)$ and $G_2$ which lead to $D = 4$
backgrounds. In the rank 2 case the action (26) is parametrised by a $2 \times 2$ Cartan matrix
$K_{ij}$ or by two simple roots with components $\alpha_{1i}$ and $\alpha_{2i}$ and one parameter $\epsilon = \epsilon_2/\epsilon_1$
with values $\pm 1$ (we assume that $\epsilon_1 \neq 0$ and also $\epsilon \neq 0$ to get a non-trivial, i.e., not a direct
product $SL(2, R) \times R^2$ model). It has the following explicit form ($\alpha' = 2/\kappa$; $x^i = (x, y)$)

$$S = \frac{1}{\pi \alpha'} \int d^2 z \left[ \partial x \bar{\partial} x + \partial y \bar{\partial} y + F(x, y) \partial u \bar{\partial} v \right] + \frac{1}{4\pi} \int d^2 z \sqrt{g^{(2)} R^{(2)}} \phi(x, y) , \quad (28)$$

$$F = \frac{1}{e^{\alpha_1 \cdot x} + e^{\alpha_2 \cdot x}} , \quad \phi = \rho \cdot x - \frac{1}{2} \ln \left( e^{\alpha_1 \cdot x} + e^{\alpha_2 \cdot x} \right) , \quad \rho = \frac{1}{2}(\alpha_1 + \ldots + \alpha_m) . \quad (29)$$

I addition to the Poincare symmetry in the $u, v$ plane this model is also invariant under a
correlated constant shift of $x^i$ and $u$ (or $v$). Below we shall consider the particular cases
of (28),(29) for all inequivalent choices of the Cartan matrices. In the last subsection we
shall discuss the backgrounds obtained by null gauging of the non-compact group $SO(1, 3)$
which does not admit the standard Gauss decomposition.
3.1. $SO(2,2) \simeq SL(2,R) \times SL(2,R)$

Let us start with the simplest case of $SO(2,2)$ or $SL(2,R) \times SL(2,R)$ when it is easy to repeat the above analysis explicitly from the very beginning. For $G = SL(2,R)$

$$g = e^{uE_+} e^{rH} e^{vE_-} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix},$$

(30)

$$S = \frac{\kappa}{\pi} \int d^2 z (\partial r \bar{\partial} r + e^{-2r} \partial u \bar{\partial} v).$$

(31)

Gauging the null generators of translations of $u$ and/or $v$ we get (the free part of) one-dimensional Liouville model [31][32] (see also [34]). Gauging independently the left and right null subgroups of $SL(2,R) \times SL(2,R)$ or of $SO(2,2)$ in a ‘twisted’ way we get the following action ($r_1, u_1, v_1$ and $r_2, u_2, v_2$ are the parameters of the two $SL(2,R)$ groups; $q = \kappa'/\kappa$ is the ratio of the corresponding $\kappa = k - 2$ factors for the two $SL(2,R)$ groups which may be considered as a free parameter of the Cartan matrix in this non-simple case)

$$S_n = \frac{\kappa}{\pi} \int d^2 z [\partial r_1 \bar{\partial} r_1 + e^{-2r_1}(\partial u_1 + \lambda B)(\bar{\partial} v_1 - \bar{\lambda} \bar{B})$$

$$+ q \partial r_2 \bar{\partial} r_2 + q e^{-2r_2}(\partial u_2 + B)(\bar{\partial} v_2 - \bar{B})].$$

(32)

This is the direct analog of (12), (13) in the $SL(2,R) \times SL(2,R)$ case with $\lambda, \bar{\lambda}$ being the parameters of the ‘mixing’ of the root operators in the generators of the left and right gauge subgroups. The action is invariant under $u_1' = u_1 + \lambda a$, $v_1' = v_1 + \bar{\lambda} b$, $u_2' = u_2 + a$, $v_2' = v_2 + b$, $B' = B - \partial a$, $\bar{B}' = \bar{B} + \bar{\partial} b$. Gauge fixing $u_2 = v_2 = 0$ and integrating over $B, \bar{B}$ we finish with the sigma model action with

$$F = \frac{q e^{-2r_1 - 2r_2}}{qe^{-2r_2} + \lambda e^{-2r_1}} = \frac{1}{e^{2r_1 + \epsilon e^{2r_2}}, \quad \epsilon = q\lambda\bar{\lambda}},$$

$$\phi = -\frac{1}{2} \ln (qe^{-2r_2} + \lambda\bar{\lambda} e^{-2r_1}) = \phi_0 + r_1 + r_2 + \frac{1}{2} \ln F.$$  

(33)

The whole group $SL(2,R)$ can be covered by four patches, i.e. a generic element is given by $g = e^{uE_+} e^{rH} e^{vE_-} \omega$, $\omega = \pm 1$, $\omega = \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Note that for $SL(2,R)$ $C_{\alpha\beta} = K_{\alpha\beta} = 2$, $|\alpha|^2 = 2$, $f_{\alpha} = e^{-2r}$.
To have the physical signature we are to assume \( q > 0 \).\(^{18}\)

When \( \lambda \) or \( \bar{\lambda} \) is zero we get the direct product of the free scalar with the \( SL(2,R) \) WZNW model. For \( \lambda\bar{\lambda} \neq 0 \) coefficient \( \epsilon = q^{-1}\lambda\bar{\lambda} \) can be made equal to \( \pm 1 \) by a shift \( r_2 \rightarrow r'_2 \) so we are left with only two non-trivial possibilities for the sigma model action (28) corresponding to \( \lambda\bar{\lambda} > 0 \) or \( \lambda\bar{\lambda} < 0 \) (\( x = \sqrt{2}r_1, \ y = q^{1/2}\sqrt{2}r'_2 \))

\[
F = \frac{1}{e^{\sqrt{2}x} + \epsilon e^{\sqrt{2}\mu y}}, \quad \mu \equiv q^{-1/2}, \quad \epsilon = \pm 1, \quad \tag{34}
\]

\[
\phi(x,y) = \phi_0 + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}\mu y - \frac{1}{2} \ln (e^{\sqrt{2}x} + \epsilon e^{\sqrt{2}\mu y}).
\]

As we shall explicitly check below, this background satisfies the conformal invariance conditions.

3.2. \( SL(3,R), \ SO(2,3), \ G_2 \)

Let us now consider the remaining cases of rank 2 maximally non-compact simple Lie groups. For \( SL(3,R) \) we have \( d = 8, \ n = 2 \), the number of positive roots \( m = 3 \) and all roots have equal length (\( |\alpha_i|^2 = 2 \)), i.e.\(^{19}\)

\[
C_{ij} = K_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},
\]

\[
\alpha_1 = (\sqrt{2},0), \quad \alpha_2 = (-\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}), \quad \rho = \alpha_1 + \alpha_2 = (\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}). \quad \tag{35}
\]

---

\(^{18}\) The model with \( q = -1 \) (with signature \((+,−,+,−)\)) can be transformed into another conformal sigma model with one time-like coordinate \( (r_2) \) by making the analytic continuation in \( t = \frac{1}{2}(u-v) \) to replace the \( F\partial u\partial v \) part of the action by \( F\partial(z+i\tau)\bar{\partial}(z-i\tau) \). Related models can be obtained by null gauging of the \( SO(1,3) \) group discussed below. The resulting metric is real but the antisymmetric tensor is, however, pure imaginary (so that the classical string equations and their solutions become complex). Because of the analytic continuation involved, this model may not also have a direct interpretation in terms of the \( SO(2,2)/H \) or \( SO(1,3)/H \) coset conformal theories.

\(^{19}\) We use the notation \( K_{ij} \equiv K_{\alpha_i,\alpha_j} \) where \( \alpha_i \) are simple roots.
Let the ungauged null generator which should correspond to a simple root have the index 1, another simple root – index 2 and their sum – index 3. Then the functions $F, \phi$ in (29), become

$$F = \frac{1}{e^{\sqrt{2}x} + e^{-\sqrt{2}x + \frac{3}{2}y}}, \quad \phi = \phi_0 + \frac{1}{2}x + \frac{3}{2}y - \frac{1}{2} \ln (e^{\sqrt{2}x} + e^{-\sqrt{2}x + \frac{3}{2}y}).$$

(36)

As in the $SO(2, 2)$ case considered above $\epsilon$ can be set equal to $\pm 1$ by shifting $y$ so that we get only two possible non-trivial models.

In the case of $SO(2, 3) \simeq Sp(4, R)$ or the algebra $B_2 = C_2 = so(5)$ there are $m = 4$ positive roots, two of which are simple and have lengths 1 and $\sqrt{2}$ (the other two positive roots also have lengths 1 and $\sqrt{2}$). The ‘kinetic’ matrix $C_{ij}$ and the Cartan matrix $K_{ij}$ are given by (we assume that the first simple root is the short one)

$$C_{ij} = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}, \quad K_{ij} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix},$$

$$\alpha_1 = (1, 0), \quad \alpha_2 = (-1, 1), \quad \rho = 2\alpha_1 + \frac{3}{2}\alpha_2 = \left(\frac{1}{2}, \frac{3}{2}\right).$$

(37)

If the ungauged direction corresponds to the first simple root we get the following expressions for the functions $F$ and $\phi$ in (29)

$$F = \frac{1}{e^x + e^{-x+y}}, \quad \phi = \phi_0 + \frac{1}{2}x + \frac{3}{2}y - \frac{1}{2} \ln (e^x + e^{-x+y}).$$

(38)

The case when the ungauged root is the long one corresponds to interchanging the two exponentials in the sums and is essentially equivalent to (38) (in the case when $\epsilon = -1$ we can also change the sign of $u$ or $v$).

For the maximally non-compact form of $G_2$ ($d = 14, \ n = 2, \ m = 6$) in addition to the short ($|\alpha_1|^2 = 2/3$) and long ($|\alpha_2|^2 = 2$) simple roots there are 4 other positive roots (2 short and 2 long ones) and

$$C_{ij} = \begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}, \quad K_{ij} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix},$$

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\[ \alpha_1 = \left( \sqrt{\frac{2}{3}}, 0 \right), \quad \alpha_2 = -\left( \sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}} \right), \quad \rho = 5\alpha_1 + 3\alpha_2 = \left( \frac{1}{\sqrt{6}}, \frac{3}{\sqrt{2}} \right), \] \hspace{1cm} (39)

so that we find (assuming that the short simple root is ungauged)

\[ F = \frac{1}{e^{\sqrt{\frac{2}{3}}x} + \epsilon e^{-\sqrt{\frac{2}{3}}x + \frac{1}{\sqrt{2}}y}}, \]

\[ \phi = \phi_0 + \frac{1}{\sqrt{6}}x + \frac{3}{\sqrt{2}}y - \frac{1}{2} \ln \left( e^{\sqrt{\frac{2}{3}}x} + \epsilon e^{-\sqrt{\frac{2}{3}}x + \frac{1}{\sqrt{2}}y} \right). \] \hspace{1cm} (40)

The choice of the long root as an ungauged one gives equivalent model. The models of this subsection do not seem to admit other analytic continuations with real metric of Minkowski signature.

3.3. **Null gauging of** $SO(1,3)$

As we have already mentioned above, the gauging based on the Gauss decomposition does not, however, exhaust the most general possible gauging of subgroups with null generators. To implement the prescription based on (7) one does not really need to use the Gauss decomposition which does not exist (at least in its standard version) for non-maximally non-compact real groups. To illustrate the procedure of the null gauging in this case let us consider the null gauging of $SO(1,3)$. The Killing form of the corresponding algebra has the signature $(-, -, -, +, +, +)$ implying that there exist null generators. Being restricted to the two-dimensional Cartan subalgebra the Killing form has the signature $(-, +)$ so that after gauging one left and one right null generator one should expect to find a four dimensional target space of the signature $(-, -, +, +)$. This background will turn out to be related by an analytic continuation to the one obtained above by gauging $SO(2,2)$ and will also have another interesting analytic continuation of the signature $(-, +, +, +)$.

Starting with the Weyl basis $(e_{mn})_i^k = \eta_{ml}\delta_n^k - \eta_{ml}\delta_m^k$, it is possible to represent the six generators of $so(1,3)$ as $4 \times 4$ real matrices $H_i, E_i, E_{-i}$ ($i = 1, 2$) with the following commutation relations and traces:\footnote{Alternatively, one may start with the $sl(2,C)$ algebra in the $\sigma$-matrix basis and construct its real form in terms of $4 \times 4$ matrices by replacing 1 and $i$ by $2 \times 2$ matrices $\delta_{ij}$ and $\epsilon_{ij}$.}

\[ [E_1, E_{-1}] = -[E_2, E_{-2}] = \frac{1}{2}H_1, \quad [E_2, E_{-1}] = [E_1, E_{-2}] = \frac{1}{2}H_2, \] \hspace{1cm} (41)
\[ [E_1, E_2] = [E_{-1}, E_{-2}] = 0, \quad [H_1, E_{\pm i}] = \pm E_{\pm i}, \quad [H_2, E_{\pm i}] = \pm \epsilon_{ij} E_{\pm j}, \]
\[ [H_i, H_j] = 0, \quad \text{Tr} \ H_1^2 = - \text{Tr} \ H_2^2 = 1, \]
\[ \text{Tr} \ E_{-1} E_1 = - \text{Tr} \ E_{-2} E_2 = \frac{1}{2}, \quad \text{Tr} \ H_i E_{\pm j} = \text{Tr} \ E_j = \text{Tr} \ E_{-i} E_{-j} = 0. \quad (42) \]

Note that (41),(42) are different from a Cartan basis relations. Still, the group element of \( SO(1, 3) \) can be parametrised in a way that mimics the Gauss decomposition for \( SL(2, C) \) or \( SO(2, 2) \)
\[
g = g_+ g_0 g_- = \exp(u_1 E_1 + u_2 E_2) \exp(x_1 H_1 + x_2 H_2) \exp(v_1 E_{-1} + v_2 E_{-2}). \quad (43)\]

We shall gauge the following left and right transformations generated by a pair of null generators\(^{21}\)
\[
g' = e^{a_+ E_1} g e^{a_- E_{-1}}. \quad (44)\]

The gauge-invariant action is given by (7), (10). Using (42),(43) we find in this particular case
\[
S_n = \frac{\kappa}{2\pi} \int d^2z (\partial x_1 \bar{\partial} x_1 - \partial x_2 \bar{\partial} x_2) \]
\[+ \frac{\kappa}{\pi} \int d^2z \text{Tr} \left[ g_0^{-1} (B E_1 + \partial u_1 E_1 + \partial u_2 E_2) g_0 \left( \bar{B} E_{-1} - \bar{\partial} v_1 E_{-1} - \bar{\partial} v_2 E_{-2} \right) \right], \quad (45)\]

where we have used that the gauge fields \( A, \bar{A} \) have the form \( B E_1, \bar{B} E_{-1} \). Evaluating the trace using (41),(42) and fixing the gauge \( u_1 = v_1 = 0 \) we finish with
\[
S_n = \frac{\kappa}{2\pi} \int d^2z \left[ \partial x_1 \bar{\partial} x_1 - \partial x_2 \bar{\partial} x_2 \right. \]
\[+ e^{-x_1} \left( \cos x_2 \bar{B} \bar{B} + \sin x_2 \partial u_2 \bar{B} - \sin x_2 \bar{\partial} v_2 \bar{B} + \cos x_2 \partial u_2 \bar{\partial} v_2 \right) \]. \quad (46)\]

After integrating out the gauge fields the action takes the sigma model form \((u \equiv u_2, \ v \equiv v_2)\)
\[
S_n = \frac{\kappa}{2\pi} \int d^2z (\partial x_1 \bar{\partial} x_1 - \partial x_2 \bar{\partial} x_2 + \frac{e^{-x_1}}{\cos x_2} \partial u \bar{\partial} v)\]

\(^{21}\) Gauging in the case when the left (right) groups are generated by linear combinations of \( E_i \) \( (E_{-i}) \) leads to equivalent results obtained by shifting \( x_2 \) in (47) by an arbitrary constant.
Here $x_1$ and $x_2$ can be shifted by arbitrary constants (e.g. $\cos x_2$ can be replaced by $\sin x_2$). The corresponding background represents an exact string solution with the signature $(-,-,+,+)$ The target space metric in (47) admits a straightforward analytic continuation which has a physical signature. If we set $x_2 = ix'_2$ we get a real action of the same structure as (28) with $F = e^{-x_1}(\cosh x'_2)^{-1}$. Furthermore, if we change the coordinates to $x = \frac{1}{\sqrt{2}}(x_1 - x'_2), \; y = \frac{1}{\sqrt{2}}(x_1 + x'_2)$ the model becomes equivalent to a particular version $(k = k', \; \mu = 1, \; \epsilon = 1)$ of the model (34) obtained above by gauging $SL(2, R) \times SL(2, R)$.

Two other interesting analytic continuations of this model are found by treating $x_1$ or $x_2$ as a time-like coordinate and rotating $u \pm v$. With $x_2 = t, \; x_1 = \chi, \; u = y - iz, \; v = y + iz$ the sigma model action (47) corresponds to a time-dependent background with the metric, antisymmetric tensor and dilaton given by

$$ds^2 = -dt^2 + dx^2 + \frac{e^{-x}}{\cos t}(dy^2 + dz^2), \; B_{yz} = i\frac{e^{-x}}{\cos t}, \; \phi = \frac{1}{2}x - \frac{1}{2}\ln \cos t. \quad (48)$$

If instead $x_2 = \chi, \; x_1 = t, \; u = -y + iz, \; v = y + iz$ and $\kappa \rightarrow -\kappa$ we get

$$ds^2 = -dt^2 + dx^2 + \frac{e^{-t}}{\cos x}(dy^2 + dz^2), \; B_{yz} = i\frac{e^{-t}}{\cos x}, \; \phi = \frac{1}{2}t - \frac{1}{2}\ln \cos x. \quad (49)$$

Up to a potential problem of imaginary nature of the antisymmetric tensor field (the $i$ factor in $B_{mn}$ drops out from the string effective equations but may not allow a physical interpretation of the solution since the string action and the string equations are complex) these backgrounds represent exact string solutions which may have a ‘cosmological’ interpretation.

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22 An extra factor of 2 in $F$ can be absorbed into $u, v$; one should also rescale the overall coefficient $k$ by 2 due to different normalisations of the traces. The model (34) with $\epsilon = -1$ can be also reproduced by starting with (47) and making appropriate complex shifts of $x_1$ and $x_2$.

23 While completing this paper we learned about a recent preprint [64] where a class of $D = 4$ backgrounds with 2 Killing vectors which solve the one-loop equations of conformal invariance was considered. The backgrounds (48),(49) are of the type discussed in [64]. The exact static backgrounds considered in the rest of our paper do not belong to the class of backgrounds $(ds^2 = f(t, z)(-dt^2 + dz^2) + g_{ab}(t, z)dx^a dx^b, \; x^a = (x, y))$ studied in [64].
4. A class of conformally invariant sigma models

4.1. Solutions of conformal invariance conditions

The models we have discussed above represent exact conformal sigma models belonging to the general class of sigma models (17), (26), i.e. \((i = 1, \ldots, n)\)

\[
S = \frac{1}{\pi \alpha'} \int d^2z [\partial x^i \bar{\partial} x_i + F(x) \partial u \bar{\partial} v] + \frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \phi(x). \tag{50}
\]

In addition to the boost symmetry in \(u, v\) plane (which guarantees the stability of the sigma model (50) under renormalisation) these models are invariant under the infinite-dimensional global symmetry \(u' = u + f(\bar{z}), \ v' = v + h(z)\) (these transformations are real on Minkowski world sheet) corresponding to two conserved chiral currents. It is straightforward to find the conditions of Weyl invariance of the sigma model (50) by computing the corresponding Weyl anomaly coefficients or \(\bar{\beta}\)-functions. The leading-order equations \(^{[65]}\)

\[
R_{mn} - \frac{1}{4} H_{mpq} H^{npq} + 2D_m D_n \phi = 0, \quad -\frac{1}{2} D_n H^n_{pq} + \partial_n \phi H^n_{pq} = 0,
\]

\[
\frac{1}{6\alpha'} (D - 26) - \frac{1}{2} D^2 \phi + (\partial \phi)^2 - \frac{1}{24} H_{mnk} H^{mnk} = 0, \quad D = n + 2,
\]

imply the following conditions on the two functions \(F\) and \(\phi\) in (17)

\[
-\partial_i \partial_j h + \partial_i \partial_j \phi = 0, \quad -\partial^i \partial_i h + 2\partial_i \phi \partial^i h = 0, \quad F \equiv e^{2h(x)}, \tag{51}
\]

\[
\frac{26 - D}{6\alpha'} = -\frac{1}{2} \partial^i \partial_i \phi + \partial^i \phi \partial_i \phi - \partial^i h \partial_i \phi + \partial^i h \partial_i h = \partial^i (h - \phi) \partial_i (h - \phi). \tag{52}
\]

As a consequence,

\[
\phi - h \equiv \tilde{\phi} = \phi_0 + \rho_i x^i, \quad \partial^i \partial_i h = 2\partial_i h \partial^i h + 2\rho_i \partial^i h, \quad D - 26 = -6\alpha' \rho_i \rho^i, \tag{53}
\]

where \(\rho_i\) is an arbitrary constant vector. The equation for \(h\) is equivalent to

\[
\partial^i \partial_i \tilde{F} = 2\rho_i \partial^i \tilde{F}, \quad \tilde{F} \equiv F^{-1} = e^{-2h}. \tag{54}
\]
A linear combination of solutions of (54) is

$$\tilde{F} = b_0 + b_i x^i + \sum_{r=1}^{N} \epsilon_r e^{\alpha_r \cdot x}, \quad b_i \rho^i = 0, \quad \alpha_r \cdot x = \alpha_r x^i, \quad (55)$$

where $\epsilon_r$ are arbitrary constants and a set of vectors $\alpha_r$ should satisfy (for each value of the index $r$ and for a fixed vector $\rho_i$)

$$|\alpha_r|^2 = 2 \rho \cdot \alpha_r. \quad (56)$$

A particular solution of (54),(56) is found by taking the number $N$ of the exponential terms in (55) to be equal to the number $n$ of coordinates $x^i$ and identifying $\alpha_r$ with the simple roots of a semisimple rank $n$ Lie algebra and $\rho_i$ with of the components of the vector equal to one half of the sum of all positive roots, or, equivalently, to the sum of the fundamental weights,

$$\rho = \frac{1}{2} \sum_{s=1}^{m} \alpha_s = \sum_{i=1}^{n} m_i, \quad m_i \cdot \alpha_j = \frac{1}{2} |\alpha_i|^2 \delta_{ij}, \quad \rho \cdot \alpha_i = \frac{1}{2} |\alpha_i|^2. \quad (57)$$

As a result,

$$F = \left( b_0 + b_i x^i + \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x} \right)^{-1}, \quad \phi = \phi_0 + \rho_i x^i - \frac{1}{2} \ln F. \quad (58)$$

We have thus checked that the model (26) and, in particular, all $D = 4$ backgrounds (28),(29) obtained by gauging the nilpotent subalgebras of maximally non-compact $n = 2$ algebras $sl(3,R)$, $so(2,3) = sp(4)$, $so(2,2)$ and $G_2$ correspond to the particular cases of the solution (58) (with $b_0 = b_i = 0$). The same conclusion is true for the backgrounds (47),(48),(49) obtained by null gauging of $SO(1,3)$.

The exact expression for the central charge of these models is found from (53) taking $\alpha' = 2/\kappa, \quad \kappa = k - \frac{1}{2} c_G$

$$C = n + 2 + 6 \alpha' |\rho|^2 = n + 2 + \frac{1}{2} c_G d = \frac{kd}{k - \frac{1}{2} c_G} - d + n + 2, \quad 24 |\rho|^2 = c_G d. \quad (59)$$

\footnote{Let us note also that (54) can be put into the form of a massive equation in flat $n$-dimensional Euclidean space $-\partial^i \partial_i \tilde{F}' + 2 |\rho|^2 \tilde{F}' = 0, \quad \tilde{F}' \equiv e^{\rho_i x^i} \tilde{F}'$. When $\rho = 0$ we get $\tilde{F} = b_0 + b_i x^i$. this solution corresponds to a non-trivial curved space which is dual (see Sect.4.3) to the flat space.}
i.e. is given by the central charge of the original WZNW model minus the number of
degrees of freedom that have been gauged away.

The background fields in (26) derived by the null gauging of the WZNW models
represent exact solutions of the conformal invariance conditions (to all orders in $\alpha'$ in
a particular scheme). It is natural to conjecture that, in general, there exists a scheme
in which the backgrounds of the type (50) satisfying the one-loop conformal invariance
condition (54) are actually the solutions to all orders in $\alpha'$. In fact, it is possible, to give
a path integral argument which demonstrates this [3]. Similar statement is known for
the plane wave - type backgrounds with one covariantly constant null Killing vector [3][4]
and also for the WZNW models or group spaces. This is certainly true for the model
(67) related to (50) by a duality transformation as discussed below. We shall also note
that the model (50) has two chiral currents, suggesting a possibility of a Sugawara-type
construction.

4.2. Relation to Toda model

The equation (54) can be interpreted as a condition of conformal invariance for a
massless (dimension two) perturbation in a linear dilaton background and is very similar
to the corresponding one for the tachyon (dimension zero) perturbation $T(x)$

$$-\partial^i \partial_i T + 2 \partial_i \phi_1 \partial^i T - 2\kappa T = 0, \quad \phi_1 = \phi_0 + \sigma_i x^i, \quad \frac{2}{\alpha'} = \kappa = k - \frac{1}{2}cG,$$

$$\partial^i \partial_i T = 2\sigma_i \partial^i T - 2\kappa T. \quad (60)$$

The potential of the Toda model

$$T(x) = \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x}, \quad (61)$$

is a particular solution of (60) if

$$|\alpha_i|^2 = 2\sigma \cdot \alpha_i - 2\kappa. \quad (62)$$
Eq. (62) is satisfied if (cf. (57))
\[ \sigma = \sum_{i=1}^{n} m_i + 2\kappa \sum_{i=1}^{n} \frac{2}{|\alpha_i|^2} m_i \equiv \rho + 2\kappa \delta . \] (63)

Comparing (59) with (54) or (62) with (56) we see that the origin of the second \( \delta \)-term in the Toda model dilaton \[55\] \[31\] \[33\] is in the presence of the ‘classical dimension’ or ‘mass’ term \( 2\kappa T \) in (59) which absent in the case of the sigma model - type interaction.

To demonstrate the equivalence to the Toda model at the classical level let us consider the classical equations for the model (50) (on a flat \( 2d \) background)
\[ \partial \bar{\partial} x_i - \frac{1}{2} \partial_i F \partial u \partial v = 0 , \quad \partial(F \bar{\partial} v) = 0 , \quad \bar{\partial}(F \partial u) = 0 . \] (64)
The model thus has two chiral currents. Integrating the last two equations and substituting the solutions in the first one we get
\[ \partial \bar{\partial} x_i + \frac{1}{2} \chi \partial_i F^{-1} = 0 , \quad F \bar{\partial} v = \nu(\bar{z}) , \quad F \partial u = \mu(z) , \quad \chi \equiv \nu(\bar{z})\mu(z) . \] (65)
Since \( \nu, \mu \) are chiral and \( \chi \) has a factorised form they can be made constant by the conformal transformations of \( z \) and \( \bar{z} \) (the sigma model (50) is always conformally invariant at the classical level). Equivalently, this can be considered as a gauge choice (alternative to the light cone gauge), i.e. \( u = a(\tau + \sigma) , \quad v = b(\tau - \sigma) \), for the conformal symmetry. The equation for \( x^i \) can be derived from the action
\[ S = \frac{1}{\pi \alpha'} \int d^2 z [\partial x^i \bar{\partial} x_i - \chi F^{-1}(x)] . \] (66)
With \( F \) given by (21) (i.e. \( F^{-1} = T = \sum_{i=1}^{n} \epsilon_i e^{\alpha_i \cdot x} \)) and in the conformal gauge with constant \( \chi \) the equation for \( x^i \) (65) and the action (66) are exactly those of the Toda model. This observation is, of course, related to the derivation of the Toda model from constrained WZNW model in \[31\].

As a consequence, the equations of the classical string propagation (including the constraints) on the backgrounds (26) discussed in the present paper are exactly integrable since their solutions can be directly expressed in terms of the Toda model solutions.

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25 Similar derivation of the Toda model from the ‘\( \beta \)-function’ conditions was given (in the simply-laced case) in \[83\] and in a general non-abelian Toda model case in \[57\]. The central charge of the Toda theory is thus (cf. (53)) \( C = n + 6\alpha' |\sigma|^2 = n + 12\kappa^{-1} |\rho + 2\kappa \delta|^2 \), in agreement with \[31\] \[33\] .
4.3. Dual sigma models

Another interesting property of the model (50) is that the dual model obtained by the standard abelian duality transformation in the $w = \frac{1}{2}(u + v)$ direction has a covariantly constant null Killing vector, i.e. has a ‘plane wave’ type structure (with the metric and dilaton depending only on transverse coordinates but not on a light-cone coordinate). This is a consequence of the fact that the $uv$-component of the metric in (50) is equal to the corresponding component of the antisymmetric tensor (what is also the reason for the existence of the two chiral currents). Following the standard steps [62][67] of gauging the symmetry $u' = u + a$, $v' = v + a$, i.e. adding the gauge field strength term with a Lagrange multiplier $\tilde{u}$ and integrating out the gauge field, we find for the dual model ($t = \frac{1}{2}(u - v) \equiv \tilde{v}$)

$$\tilde{S} = \frac{1}{\pi\alpha'} \int d^2z [\partial x^i \partial x_i + \tilde{F}(x) \partial \tilde{u} \partial \tilde{u} - 2\partial \tilde{v} \partial \tilde{v}] + \frac{1}{4\pi} \int d^2z \sqrt{g^{(2)}} R^{(2)} \tilde{\phi}(x) , \quad (67)$$

$$\tilde{F} = F^{-1}(x) , \quad \tilde{\phi} = \phi(x) - \frac{1}{2} \ln F(x) . \quad (68)$$

The dual space-time metric has one covariantly constant null Killing vector while the vanishing antisymmetric tensor is zero. Since the ‘transverse’ part of the metric is flat, the conditions of the Weyl invariance of the model (67) are given (to all orders in $\alpha'$) by the ‘one-loop’ conditions

$$-\partial^i \partial_i \tilde{F} + 2\partial_i \tilde{\phi} \partial^i \tilde{F} = 0 , \quad \partial_i \partial_j \tilde{\phi} = 0 , \quad (69)$$

$$\frac{26 - D}{6\alpha'} = -\frac{1}{2} \partial_i \partial^i \tilde{\phi} + \partial_i \tilde{\phi} \partial^i \tilde{\phi} .$$

$^{26}$ See, e.g., [68] for a general discussion. Since the dilaton is assumed to depend on the transverse coordinates, this model is a generalisation of the original ‘plane-wave’ models considered in [2][3][4][5]. Some special cases, in particular, the $D = 3$ case of such model – the duality rotation of the $SL(2, R)$ WZNW model – were already discussed (in connection with extremal black strings) in [69][70] (see also [30]; note that a ‘nilpotent duality’ discussed in [30] in the $SL(2, R)$ case is just the standard abelian duality if expressed in terms of the appropriate coordinates (30),(31)).
This system of equations is equivalent to the one in the original model (51), (52) (see (53), (54)) with a solution already given in (55), (53), (58)

\[ \tilde{F} = b_0 + b_i x^i + \sum_{i=1}^{n} \epsilon_i e^{\alpha_i x}, \quad \tilde{\phi} = \phi_0 + \rho_i x^i, \]  

(70)

where \( b_i, \alpha_i \) and \( \rho \) should satisfy the conditions in (55), (56). The backgrounds (58) and (70) thus provide another example of exact solutions related by the standard (leading-order) duality (cf. [71]). The classical equations of motion of the two dual models are, of course, also equivalent and can be represented in the Toda-like form (65).

There remains an open question if one can consider the isometric coordinate (with respect to which the duality is performed) to be periodic so that the two conformal field theories can be identified following the idea of [72]. This could open a possibility for a proof of unitarity of the models considered in this paper since the unitarity of the ‘plane wave’ model (67) is obvious in the light cone gauge.

5. Concluding remarks

In this paper we have found a new class of conformal sigma models which have Toda model-like structure and can be obtained by gauging of ‘null’ (or nilpotent) subgroups in WZNW theories based on non-compact groups. The corresponding target space fields represent exact string solutions. Their possible physical interpretation remains an open question. An interesting property of these solutions is that the classical string propagation is essentially described by the Toda model equations.

Let us briefly comment on the geometrical properties of the backgrounds discussed above. Computing the scalar curvature of the metric in (50) we find\(^{27}\)

\[ R = -2F^{-1}\partial^i\partial_i F + \frac{1}{2}F^{-2}\partial^i F \partial_i F = 2F \partial^i \partial_i F^{-1} - \frac{7}{2}F^2 \partial^i F^{-1} \partial_i F^{-1}. \]  

(71)

\(^{27}\) It should be noted that even though the string ‘feels’ a combination of the metric and the antisymmetric tensor (so that the geometrical properties of these backgrounds are better reflected in the string equations of motion) a point-like tachyonic state still follows the geodesics of the metric.
For $F$ in (26) or in (58) (with $b_0 = b_i = 0$) we obtain

$$R = \frac{\sum_{i,j=1}^{n} (4|\alpha_i|^2 - 7\alpha_i \cdot \alpha_j)\epsilon_i \epsilon_j e^{(\alpha_i + \alpha_j) \cdot x}}{2\sum_{i,j=1}^{n} \epsilon_i \epsilon_j e^{(\alpha_i + \alpha_j) \cdot x}}. \quad (72)$$

For comparison, in the case of the dual background (67) all scalar curvature invariants vanish while the $\tilde{u}\tilde{u}$-component of the Ricci tensor is given by

$$\tilde{R}_{\tilde{u}\tilde{u}} = -\frac{1}{2} \partial^i \partial_i \tilde{F} = -\frac{1}{2} \partial^i \partial_i F^{-1} = -\frac{1}{2} \sum_{i=1}^{n} \epsilon_i |\alpha_i|^2 e^{\alpha_i \cdot x}. \quad (73)$$

Eq. (72) gives $R = -6$ in the simplest case of $\epsilon_2 = \ldots = \epsilon_n = 0$ when the model is equivalent to the WZNW model for $SL(2, R) \times R^{n-1}$. It is clear that the curvature (72) is non-singular if $F^{-1}$ cannot vanish, i.e. if all $\epsilon_i$ have the same sign. For example, in the case of the four dimensional $SO(2, 2)$ background (34), i.e. $F^{-1} = e^{\sqrt{2}x} + \epsilon e^{\sqrt{2}\mu y}$, we have explicitly

$$R = \frac{-3e^{2\sqrt{2}x} + 4\epsilon(1 + \mu)e^{\sqrt{2}x + \sqrt{2}\mu y} - 3\mu^2 e^{2\sqrt{2}\mu y}}{(e^{\sqrt{2}x} + \epsilon e^{\sqrt{2}\mu y})^2}. \quad (74)$$

The curvature is regular for $\epsilon = +1$, approaching a constant value at large $x$ and $y$. Similar expressions are found for other metrics in Section 3.

It may be possible to construct new solutions by factorising over some discrete subgroups (i.e. by making some identifications of coordinates) using as a motivation a relation between the 3D black hole [74] and $SL(2, R)$ WZNW model.

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