A NONPOLYNOMIALLY CONVEX ISOTROPIC TWO-TORUS WITH NO ATTACHED DISCS

PURVI GUPTA

Abstract. We show — with the means of an example in $\mathbb{C}^3$ — that Gromov’s theorem on the presence of attached holomorphic discs for compact Lagrangian manifolds is not true in the isotropic (subcritical) case, even in the absence of an obvious obstruction, i.e., polynomial convexity.

A compact set $X \subset \mathbb{C}^n$ is called polynomially convex if, for every $z \notin X$, there is a holomorphic polynomial $P$ such that $|P(z)| > \sup_{x \in X} |P(x)|$. It is known that no real compact $n$-dimensional submanifold $M \subset \mathbb{C}^n$ (without boundary) can be polynomially convex. In the particular case when the inclusion $\iota : M \hookrightarrow \mathbb{C}^n$ is maximally isotropic (or Lagrangian) with respect to $\omega_{st} = i \sum_{1}^{n} dz_j \wedge d\bar{z}_j$, i.e., $\iota^*(\omega_{st}) = 0$, Gromov ([5]) proves a stronger statement: there is a holomorphic disc attached to $M$, i.e., there is a nonconstant holomorphic map from the unit disc $D$ to $\mathbb{C}^n$ that is continuous up to the boundary and maps $\partial D$ into $M$. Gromov’s result is not true in the subcritical case (when $\dim M < n$) as there are several examples of polynomially convex isotropic surfaces in $\mathbb{C}^3$.

In this note, we confirm that Gromov’s result is not true in the subcritical case, even in the absence of polynomial convexity. We produce an explicit nonpolynomially convex two-torus $T \subset \mathbb{C}^2$ that is isotropic with respect to $\omega_{st}$, but has no holomorphic discs attached to it. Note that the isotropicity of $T$ implies that it is both totally real and rationally convex (see [4]). Examples of totally real tori with no attached holomorphic discs have been given by Alexander ([1]) and Duval-Gayet ([3]) in $\mathbb{C}^2$, but such examples cannot be rationally convex in view of Duval-Sibony (see [4, Theorem 3.1]) and Gromov’s result. In the case of manifolds with boundary, Duval has constructed an example of a nonpolynomially convex Lagrangian surface in $\mathbb{C}^2$ that has no attached discs (see [2] or [4]).

Note. It was brought to the author’s attention — after the first version of this note was posted — that a stronger example was constructed in the $\mathcal{C}^\infty$-category in [6]. The example therein has a nontrivial hull that does not contain any analytic structure. To the best of our knowledge, no such example is known in the real-analytic category. The example in this note is real-analytic, but contains a holomorphic annulus in its hull.
**Theorem 1.** There is a real-analytic two-torus in $\mathbb{C}^3$ that is isotropic with respect to $\omega_{st}$, not polynomially convex, but has no holomorphic discs attached to it.

**Proof.** Let $p(z, w) := 1 - 4z^2 + 4w^2 - z^2w^2$ and 

$$T := \left\{ (z, w, \text{Re} p(z, w)) \in \mathbb{C}^3 : z, w \in \partial \mathbb{D} \right\}.$$ 

Being the graph of a real-valued function on the torus $T^2 := \partial \mathbb{D} \times \partial \mathbb{D}$, $T$ is isotropic with respect to $\omega_{st}$. We will show that $T$ is not polynomially convex, and its polynomial hull (defined below) consists of $T$ and an attached annulus.

Before we proceed, we fix some notation. If $A \subset \overline{\mathbb{D}}^2$ and $f : \overline{\mathbb{D}}^2 \to \mathbb{C}$, then $G_f(A) = \{(z, w, f(z, w)) \in \mathbb{C}^3 : (z, w) \in A\}$ denotes the graph of $f|_A$. If $\zeta \in \overline{\mathbb{D}}^2$, $G_f(\{\zeta\})$ is simplified to $G_f(\zeta)$. For a compact $X \subset \mathbb{C}^n$, the polynomial hull of $X$ is the set $\tilde{X} = \{z \in \mathbb{C}^n : |P(z)| \leq \sup_{x \in X} |P(x)|\}$.

Now, let $f(z, w) := \text{Re}(p(z, w))$. In our notation, $T = G_f(T^2)$. We first consider $T_1 := G_f(T^2)$. We claim that

\[(1) \quad \tilde{T}_1 = T_1 \cup G_p(\mathcal{Z}),\]

where $\mathcal{Z} = \{(z, w) \in \overline{\mathbb{D}}^2 : w^2 = \frac{4z^2 - 1}{4z^2} \}$. Since $p|_\mathcal{Z} \equiv 0$, $G_p(\mathcal{Z})$ is isomorphic to $\mathcal{Z}$. Moreover, by a computation due to Rudin (see [8, proof of Theorem B]) $\mathcal{Z}$ is a connected finite Riemann surface of genus 0 with two boundary components in $T^2$ — i.e, $G_p(\mathcal{Z})$ is an annulus attached to $T_1$.

To prove (1), we use a technique due to Jimbo (see [7]). Following the notation in [7], let

$$h(z, w) = (zw)^{-2}(z^2w^2 - 4w^2 + 4z^2 - 1);$$

$$L = (\overline{\mathbb{D}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{D}});$$

and

$$V = \{(z, w) \in \overline{\mathbb{D}}^2 \setminus (T^2 \cup L) : p(z, w) = h(z, w)\}.$$ 

Note that $h(z, w) = p(z, w)$ on $T^2$. Next, we compute

$$\Delta(z, w) = \begin{vmatrix} \frac{\partial p}{\partial z}(z, w) & \frac{\partial p}{\partial w}(z, w) \\ \frac{\partial h}{\partial z}(z, w) & \frac{\partial h}{\partial w}(z, w) \end{vmatrix} = \begin{vmatrix} -8z - 2zw^2 & 8w - 2z^2w \\ \frac{8}{z^3} + \frac{2}{z^3w^2} & -\frac{8}{w^3} - \frac{2}{w^3z^2} \end{vmatrix},$$

to obtain $\Delta(z, w) = -16(zw)^{-3}(z-iz)(z+iz)p(z, w)$. Setting $g_1 = (z-iz)$, $g_2 = z+iz$, $g_3 = p(z, w)$, and $Q_j := \{(z, w) \in \tilde{T}^2 : g_j(z, w) = 0\}$, $1 \leq j \leq 3$, we have that

\[(2) \quad Q_1 = \{(z, iz) \subset \tilde{T}^2 : z \in \partial \mathbb{D}\};\]

$$Q_2 = \{(z, -iz) \subset \tilde{T}^2 : z \in \partial \mathbb{D}\};$$

and

$$Q_3 = \mathcal{Z} \cap \tilde{T}^2 = \partial \mathcal{Z}.$$
In [7], Jimbo showed that if $\Delta(z, w) \neq 0$ on $\mathbb{D}^2 \setminus L$ and
\[
J := \{1 \leq j \leq 3 : \emptyset \neq Q_j \neq \hat{Q}_j, \hat{Q}_j \setminus (\mathbb{T}^2 \cup L) \subset V \} \neq \emptyset,
\]
then
\[
\mathcal{G}_{\mathbb{T}^2} = \mathcal{G}_{\mathbb{T}^2} \cup \bigcup_{j \in J} \{(z, w, p(z, w)) : (z, w) \in \hat{Q}_j\},
\]
and $p$ restricts to a constant on each $\hat{Q}_j$, $j \in J$. In view of (2), $J = \{3\}$, $\hat{Q}_3 = \mathcal{Z}$ and, since $p|_{\mathcal{Z}} = \overline{p}|_{\mathcal{Z}} = 0$, (1) holds — i.e., there is only one annulus attached to $T_1$. Since $\mathbb{T}^2$ is totally real and rationally convex, and $\overline{p}$ is smooth, $T_1 = \mathcal{G}_{\mathbb{T}^2}$ is totally real and rationally convex. Due to a result by Duval and Sibony (see [4]), $T_1$ is isotropic with respect to some Kähler form on $\mathbb{C}^3$. But, $\iota^*(\omega_{st}) \neq 0$, where $\iota : T_1 \hookrightarrow \mathbb{C}^3$ is the inclusion map.

We now return to $T := \mathcal{G}_{f}(\mathbb{T}^2)$. Note that the algebraic isomorphism
\[
F(z, w, \eta) \mapsto \left(z, w, \frac{1}{2}(\eta + p(z, w))\right)
\]
maps $T_1$ onto $T$ and fixes the variety $\mathcal{G}_p(\mathcal{Z})$. Thus, $\hat{T} = F(T_1) = T \cup \mathcal{G}_p(\mathcal{Z})$. As there are no nontrivial holomorphic discs attached to an annulus, there are none attached to $T$.

\section*{References}

[1] H. Alexander. Disks with boundaries in totally real and Lagrangian manifolds. \textit{Duke Math. J.}, 100(1):131–138, 1999.
[2] J. Duval. Convexité rationnelle des surfaces lagrangiennes. \textit{Invent. Math.}, 104(1):581–599, 1991.
[3] J. Duval and D. Gayet. Riemann surfaces and totally real tori. \textit{Comment. Math. Helv.}, 89(2):299–312, 2014.
[4] J. Duval and N. Sibony. Polynomial convexity, rational convexity, and currents. \textit{Duke Math. J.}, 79(2):487–513, 1995.
[5] M. Gromov. Pseudo-holomorphic curves in symplectic manifolds. \textit{Invent. Math.}, 82(2):307–347, 1985.
[6] A. J. Izzo, H. S. Kalm, and E. F. Wold. Presence or absence of analytic structure in maximal ideal spaces. \textit{Math. Ann.}, 369(1-2):459–478, 2016.
[7] T. Jimbo. Polynomial hulls of graphs on the torus in $\mathbb{C}^2$. \textit{Sci. Math. Jpn.}, 62(3):335, 2005.
[8] W. Rudin. Pairs of inner functions on finite Riemann surfaces. \textit{Trans. Amer. Math. Soc.}, 140:423–434, 1969.