Universal eigenstate entanglement of chaotic local Hamiltonians

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December 5, 2021

Abstract

In systems governed by “chaotic” local Hamiltonians, we conjecture the universality of eigenstate entanglement (defined as the average entanglement entropy of all eigenstates) by proposing an exact formula for its dependence on the subsystem size. This formula is derived from an analytical argument based on a plausible assumption, and is supported by numerical simulations.

1 Introduction

Entanglement, a concept of quantum information theory, has been widely used in condensed matter and statistical physics to provide insights beyond those obtained via “conventional” quantities. For ground states of local Hamiltonians, it characterizes quantum criticality [1–5] and topological order [6–11]. The scaling of entanglement reflects physical properties (e.g., correlation decay [12–14] and dynamical localization [15–17]) and is quantitatively related to the classical simulability of quantum many-body systems [18–22].

Besides ground states, it is also important to understand the entanglement of excited eigenstates. Significant progress has been made for a variety of local Hamiltonians [23–36]. In many-body localized systems [37–40], one expects an “area law” [41], i.e., the eigenstate entanglement between a subsystem and its complement scales as the boundary (area) rather than the volume of the subsystem [26–28]. In translationally invariant free-fermion systems, the average entanglement entropy of all eigenstates obeys a volume law with a coefficient depending on the subsystem size due to the integrability of the model [32].

In this paper we consider “chaotic” quantum many-body systems. We are not able to specify the precise meaning of being chaotic, for there is no clear-cut definition of quantum

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chaos. Intuitively, this class of systems should include non-integrable models in which energy is the only local conserved quantity. For such systems, there are some widely accepted opinions [23–25, 31, 35]:

1. The entanglement entropy of an eigenstate for a subsystem smaller than half the system size is (to leading order) equal to the thermodynamic entropy of the subsystem at the same energy density.

2. The entanglement entropy of an eigenstate at the mean energy density (of the Hamiltonian) is indistinguishable from that of a random (pure) state.

3. The entanglement entropy of a generic eigenstate is indistinguishable from that of a random state.

We briefly explain the reasoning behind these opinions. The eigenstate thermalization hypothesis (ETH) states that for expectation values of local observables, a single eigenstate resembles a thermal state with the same energy density [42–44]. Opinion 1 is a variant of ETH for entropy. Opinion 2 follows from Opinion 1 and the fact that the entanglement entropy of a random state is nearly maximal [45]. Opinion 3 follows from Opinion 2 because a generic eigenstate is at the mean energy density (Lemma 3 in Ref. [46]).

These opinions concern the scaling of the entanglement entropy only to leading order. A more ambitious goal is to find the exact value of eigenstate entanglement. We conjecture that the average entanglement entropy of all eigenstates is universal (model independent), and propose a formula for its dependence on the subsystem size. This formula is derived from an analytical argument based on an assumption that characterizes the chaoticity of the model. It is also supported by numerical simulations of a non-integrable spin chain.

The formula implies that by taking into account sub-leading corrections not captured in Opinion 3, a generic eigenstate is distinguishable from a random state in the sense of being less entangled. Indeed, this implication can be proved rigorously for any (not necessarily chaotic) local Hamiltonian. The proof also solves an open problem of Keating et al. [29].

The paper is organized as follows. Section 2 gives a brief review of random-state entanglement. Section 3 proves that for any (not necessarily chaotic) local Hamiltonian, the average entanglement entropy of all eigenstates is smaller than that of random states. Sections 4 and 5 provide an analytical argument and numerical evidence, respectively, for the universality of eigenstate entanglement in chaotic systems. The main text of this paper should be easy to read, for most of the technical details are deferred to Appendices A and B.

2 Entanglement of random states

We begin with a brief review of random-state entanglement. We use the natural logarithm throughout this paper.

Definition 1 (entanglement entropy). The entanglement entropy of a bipartite pure state $\rho_{AB} = |\psi\rangle\langle \psi|$ is defined as the von Neumann entropy

$$S(\rho_A) = - \text{tr}(\rho_A \ln \rho_A)$$ (1)
of the reduced density matrix $\rho_A = \text{tr}_B \rho_{AB}$. It is the Shannon entropy of $\rho_A$’s eigenvalues, which form a probability distribution as $\rho_A \geq 0$ (positive semidefinite) and $\text{tr} \rho_A = 1$ (normalization).

**Theorem 1** (conjectured and partially proved by Page [45]; proved in Refs. [47–49]). Let $\rho_{AB}$ be a bipartite pure state chosen uniformly at random with respect to the Haar measure. In average,

$$S(\rho_A) = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} - \frac{d_A - 1}{2 d_B} = \ln d_A - \frac{d_A}{2 d_B} + O(1),$$

where $d_A \leq d_B$ are the local dimensions of subsystems $A$ and $B$, respectively.

Let $\gamma \approx 0.5772$ be the Euler-Mascheroni constant. The second step of Eq. (2) uses the formula

$$\sum_{k=1}^{d_B} \frac{1}{k} = \ln d_B + \gamma + \frac{1}{2 d_B} + O(1/d_B^2).$$

(3)

3 Rigorous bounds on eigenstate entanglement

This section proves a rigorous upper bound on the average entanglement entropy of all eigenstates. The bound holds for any (not necessarily chaotic) local Hamiltonian, and distinguishes the entanglement entropy of a generic eigenstate from that of a random state.

For ease of presentation, consider a chain of $n$ spin-$1/2$’s governed by a local Hamiltonian

$$H = \sum_{i=1}^{n} H_i, \quad H_i := H_i' + H'_{i,i+1},$$

(4)

where $H_i'$ acts only on spin $i$, and $H'_{i,i+1}$ represents the nearest-neighbor interaction between spins $i$ and $i+1$. We use periodic boundary conditions by identifying the indices $i$ and $(i \mod n)$. Suppose $H_i'$ and $H'_{i,i+1}$ are linear combinations of one- and two-local Pauli operators, respectively, so that $\text{tr} H_i' = \text{tr} H_{i,i+1}' = 0$ (traceless) and $\text{tr}(H_i H_{i'}) = 0$ for $i \neq i'$. We assume translational invariance and $\|H_i\| = 1$ (unit operator norm). Let $d := 2^n$ and $\{|j\rangle\}_{j=1}^{d}$ be a complete set of translationally invariant eigenstates of $H$ with corresponding eigenvalues $\{E_j\}$.

**Lemma 1.** Consider the spin chain as a bipartite quantum system $A \otimes B$. Subsystem $A$ consists of spins $1, 2, \ldots, m$. Assume without loss of generality that $m$ is even and $f := m/n \leq 1/2$. Then,

$$S(\rho_{j,A}) \leq m \ln 2 - \frac{f E_j^2}{4n},$$

(5)

where $\rho_{j,A} := \text{tr}_B |j\rangle\langle j|$ is the reduced density matrix of $|j\rangle$ for $A$.

**Proof.** See Appendix A.

We are now ready to prove the main result of this section:
Theorem 2. In the setting of Lemma 1,
\[
\bar{S} := \frac{1}{d} \sum_{j=1}^{d} S(\rho_{j,A}) \leq m \ln 2 - \frac{f \langle H_1^2 \rangle}{4},
\]
where \( \langle \cdots \rangle := \frac{1}{d} \text{tr} \cdots \) denotes the expectation value of an operator at infinite temperature.

Proof. Theorem 2 follows from Lemma 1 and the observation that
\[
\frac{1}{d} \sum_{j=1}^{d} E_j^2 = \langle H^2 \rangle = \sum_{i,i'=1}^{n} \langle H_i H_{i'} \rangle = \sum_{i=1}^{n} \langle H_i^2 \rangle = n \langle H_1^2 \rangle.
\]
(7)

Recall that Theorem 2 assumes \( \|H_i\| = 1 \). Without this assumption, (6) should be modified to
\[
\bar{S} \leq m \ln 2 - \frac{f \langle H_1^2 \rangle}{4\|H_1\|^2}.
\]
(8)

For \( 2 \leq m = O(1) \), Theorem 2 gives the upper bound
\[
\bar{S} \leq m \ln 2 - \Theta(1/n).
\]
(9)

A lower bound can be easily derived from Theorem 1 in Ref. [29]
\[
\bar{S} \geq m \ln 2 - \Theta(1/n).
\]
(10)

Therefore, both bounds are tight. This answers an open question in Section 6.1 of Ref. [29].

Without translational invariance (e.g., in weakly disordered systems), a similar result is obtained by averaging over all possible ways of “cutting out” a region of length \( m \). Here, \( \|H_i\| \) may be site dependent but should be \( \Theta(1) \) for all \( i \).

Corollary 1. The average entanglement entropy \( \bar{S} \) of a random eigenstate for a random consecutive region of size \( m \) is upper bounded by \( m \ln 2 - \Theta(f) \).

Proof. First, we follow the proof of Lemma 1. Without translational invariance, (21) remains valid upon replacing \( \epsilon_{j,i} \) by \( \epsilon_{j,i}/\|H_i\| = \Theta(\epsilon_{j,i}) \). By the RMS-AM inequality and Eq. (7), we have
\[
\bar{S} \leq m \ln 2 - \frac{\Theta(f)}{4d} \sum_{j=1}^{d} \sum_{i=1}^{n} \epsilon_{j,i}^2 \leq m \ln 2 - \frac{\Theta(f)}{4dn} \sum_{j=1}^{d} E_j^2 = m \ln 2 - \frac{\Theta(f)}{4n} \sum_{i=1}^{n} \langle H_i^2 \rangle = m \ln 2 - \Theta(f).
\]
(11)

It is straightforward to extend all the results of this section to higher spatial dimensions.
4 Eigenstate entanglement of “chaotic” Hamiltonians

Suppose the Hamiltonian (4) is chaotic in a sense to be made precise below. This section provides an analytical argument for

**Conjecture 1** (universal eigenstate entanglement). Consider the spin chain as a bipartite quantum system $A \otimes B$. Subsystem $A$ consists of spins $1, 2, \ldots, m$. For a fixed constant $f := m/n \leq 1/2$, the average entanglement entropy of all eigenstates is

$$\bar{S} = m \ln 2 + \frac{\ln(1 - f)}{2} - \frac{2\delta f}{\pi}$$

in the thermodynamic limit $n \to +\infty$, where $\delta$ is the Kronecker delta.

We split the Hamiltonian (4) into three parts: $H = H_A + H_\partial + H_B$, where $H_A(B)$ contains terms acting only on subsystem $A(B)$, and $H_\partial = H'_{m,m+1} + H'_{n,1}$ consists of boundary terms. Let $\{|j\rangle_A\}_{j=1}^{2^m}$ and $\{|k\rangle_B\}_{k=1}^{2^{n-m}}$ be complete sets of eigenstates of $H_A$ and $H_B$ with corresponding eigenvalues $\{\epsilon_j\}$ and $\{\epsilon_k\}$, respectively. Since $H_A$ and $H_B$ are decoupled from each other, product states $\{|j\rangle_A|k\rangle_B\}$ form a complete set of eigenstates of $H_A + H_B$ with eigenvalues $\{\epsilon_j + \epsilon_k\}$. Due to the presence of $H_\partial$, a (normalized) eigenstate $|\psi\rangle$ of $H$ with eigenvalue $E$ is a superposition

$$|\psi\rangle = \sum_{j=1}^{2^m} \sum_{k=1}^{2^{n-m}} c_{jk} |j\rangle_A|k\rangle_B. \quad (13)$$

The locality of $H_\partial$ implies a strong constraint stating that the population of $|j\rangle_A|k\rangle_B$ is significant only when $\epsilon_j + \epsilon_k$ is close to $E$.

**Lemma 2.** There exist constants $c, \Delta > 0$ such that

$$\sum_{|\epsilon_j + \epsilon_k - E| \geq \Lambda} |c_{jk}|^2 \leq ce^{-\Lambda/\Delta}. \quad (14)$$

**Proof.** This is a direct consequence of Theorem 2.3 in Ref. [50].

In chaotic systems, we expect

**Assumption 1.** The expansion (13) of a generic eigenstate $|\psi\rangle$ is a random superposition subject to the constraint (14).

This assumption is consistent with, but goes beyond, the semiclassical approximation Eq. (16) of Ref. [33].

We now show that Assumption (1) implies Conjecture (1). Consider the following simplified setting. Let $M_j$ be the set of computational basis states with $j$ spins up and $n - j$ spins down, and $U_j \in \mathcal{U}(|M_j|) = \mathcal{U}(\binom{n}{j})$ be a Haar-random unitary on span $M_j$. Define $M'_j = \{U_j|\phi\rangle : \forall|\phi\rangle \in M_j\}$ so that $M := \bigcup_{j=0}^{n} M'_j$ is a complete set of eigenstates of the Hamiltonian

$$H = \sum_{i=1}^{n} \sigma_i^z. \quad (15)$$
The set $M$ captures the essentials of Assumption 1. Every state in $M$ satisfies
\[ \sum_{|\epsilon_j + \epsilon_k - E| \geq 1} |c_{jk}|^2 = 0, \] (16)
which is a hard version of the constraint (14). The random unitary $U_j$ ensures that Eq. (13) is a random superposition. Thus, we establish Conjecture 1 by

**Proposition 1.** The average entanglement entropy of all states in $M$ is given by Eq. (12).

### 5 Numerics

To provide numerical evidence for Conjecture 1, consider the spin-1/2 Hamiltonian $[51, 52]$
\[ H = \sum_{i=1}^{n} H_i, \quad H_i := \sigma_i^z \sigma_{i+1}^z + g \sigma_i^x + h \sigma_i^z, \] (17)
where $\sigma_i^x, \sigma_i^y, \sigma_i^z$ are the Pauli matrices at site $i (\leq n)$, and $\sigma_{n+1}^z := \sigma_1^z$ (periodic boundary condition). For generic values of $g, h$, this model is non-integrable in the sense of Wigner-Dyson level statistics $[51, 52]$. We compute the average entanglement entropy of all eigenstates by exact diagonalization in every symmetry sector.

Figure 1 shows the numerical results, which semiquantitatively support Conjecture 1. Noticeable deviations from Eq. (12) are expected due to significant finite-size effects. However, the trend appears to be that the difference between theory and numerics decreases as the system size increases.

### Acknowledgments and notes

We would like to thank Fernando G.S.L. Brandão and Xiao-Liang Qi for interesting discussions and Anatoly Dymarsky for insightful comments. We acknowledge funding provided by the Institute for Quantum Information and Matter, an NSF Physics Frontiers Center (NSF Grant PHY-1733907). Additional funding support was provided by NSF DMR-1654340.

After this paper appeared on arXiv, we became aware of a simultaneous work $[53]$ and a slightly later one $[54]$.

### A Proof of Lemma 1

Let $\epsilon_{j,i} := \langle j | H_i | j \rangle$ so that $|\epsilon_{j,i}| \leq 1$. Let $\rho_{j,i}$ be the reduced density matrix of $|j\rangle$ for spins $i$ and $i+1$. Let $I_4$ be the identity matrix of order 4. Let $\|X\|_1 := \text{tr} \sqrt{X^\dagger X}$ be the trace norm. Since $H_i$ is traceless, $|\epsilon_{j,i}|$ provides a lower bound on the deviation of $\rho_{j,i}$ from the maximally mixed state:
\[ |\epsilon_{j,i}| = |\text{tr}(\rho_{j,i} H_i)| = |\text{tr}((\rho_{j,i} - I_4/4) H_i)| \leq \|\rho_{j,i} - I_4/4\|_1 \|H_i\| = \|\rho_{j,i} - I_4/4\|_1 \]
\[ = \sum_{k=1}^{4} |\lambda_k - 1/4|, \] (18)
Figure 1: Numerical check of Conjecture 1 for two sets of parameters \((g, h)\). The horizontal axes are the fraction \(f\) of spins in subsystem \(A\). To be aesthetically pleasing, we allow \(0 < f < 1\) so that the plots are mirror symmetric with respect to \(f = 1/2\). Blue symbols represent corrections obtained by subtracting the average entanglement entropy \(\bar{S}\) of all eigenstates from the leading-order term \(\min\{f, 1 - f\}n \ln 2\). Different symbols correspond to different system sizes. Red curves are the theoretical prediction given by Eq. (12).

where \(\lambda_k\)'s are the eigenvalues of \(\rho_{j,i}\). An upper bound on \(S(\rho_{j,i})\) is

\[
\max \left\{-\sum_{k=1}^{4} p_k \ln p_k \right\}; \quad \text{s. t.} \quad \sum_{k=1}^{4} p_k = 1, \quad \sum_{k=1}^{4} |p_k - 1/4| \geq |\epsilon_{j,i}|. \tag{19}
\]

Since the Shannon entropy is Schur concave, it suffices to consider

- \(p_1 = p_2 = 1/4 + \epsilon_{j,i}/4, p_3 = p_4 = 1/4 - \epsilon_{j,i}/4;\)
- (if \(\epsilon_{j,i} \geq -1/2\) \(p_1 = 1/4 + \epsilon_{j,i}/2, p_2 = p_3 = p_4 = 1/4 - \epsilon_{j,i}/6;\)
- (if \(\epsilon_{j,i} \leq 1/2\) \(p_1 = 1/4 - \epsilon_{j,i}/2, p_2 = p_3 = p_4 = 1/4 + \epsilon_{j,i}/6.\)

For \(|\epsilon_{j,i}| \ll 1\), by Taylor expansion we can prove

\[
S(\rho_{j,i}) \leq 2 \ln 2 - \epsilon_{j,i}^2/2. \tag{20}
\]

We have checked numerically that this inequality remains valid for any \(|\epsilon_{j,i}| \leq 1\). Therefore,

\[
S(\rho_{j,A}) \leq \sum_{k=0}^{m/2-1} S(\rho_{j,2k+1}) \leq m \ln 2 - \frac{1}{2} \sum_{k=0}^{m/2-1} \epsilon_{j,2k+1}^2 \tag{21}
\]

due to the subadditivity [55] of the von Neumann entropy. We complete the proof using \(\epsilon_{j,i} = E_j/n\).
B Proof of Proposition

Assume without loss of generality that \( n \) is even. Let \( L_j \) (\( R_j \)) be the set of computational basis states of subsystem \( A \) (\( B \)) with \( j \) spins up and \( m-j \) \((n-m-j)\) spins down so that

\[
|L_j| = \binom{m}{j}, \quad |R_j| = \binom{n-m}{j}, \quad \text{and} \quad M_j = \bigcup_{k=\max\{0,m-n+j\}}^{\min\{m,j\}} L_k \times R_{j-k}.
\]

(22)

Thus, any (normalized) state \( |\psi\rangle \) in \( M_j' \) can be decomposed as

\[
|\psi\rangle = \sum_{k=\max\{0,m-n+j\}}^{\min\{m,j\}} c_k |\phi_k\rangle,
\]

(23)

where \( |\phi_k\rangle \) is a normalized state in \( \text{span} \ L_k \otimes \text{span} \ R_{j-k} \). Let \( \rho_A \) and \( \sigma_{k,A} \) be the reduced density matrices of \( |\psi\rangle \) and \( |\phi_k\rangle \) for \( A \), respectively. It is easy to see

\[
\rho_A = \bigoplus_{k=\max\{0,m-n+j\}}^{\min\{m,j\}} |c_k|^2 \sigma_{k,A} \quad \Rightarrow \quad S(\rho_A) = \sum_{k=\max\{0,m-n+j\}}^{\min\{m,j\}} |c_k|^2 S(\sigma_{k,A}) - |c_k|^2 \ln |c_k|^2.
\]

(24)

Since \( |\psi\rangle \) is a random state in \( \text{span} \ M_j \), each \( |\phi_k\rangle \) is a (Haar-)random state in \( \text{span} \ L_k \otimes \text{span} \ R_{j-k} \). Theorem \( 1 \) implies that in average,

\[
S(\sigma_{k,A}) = \ln \min\{|L_k|, |R_{j-k}|\} = \frac{\min\{\{L_k|, |R_{j-k}|\}}{2 \max\{|L_k|, |R_{j-k}|\}}.
\]

(25)

In average, the population \( |c_k|^2 \) is proportional to the dimension of \( \text{span} \ L_k \otimes \text{span} \ R_{j-k} \):

\[
|c_k|^2 = |L_k||R_{j-k}|/|M_j|.
\]

(26)

The deviation of \( |c_k|^2 \) (from the mean) for a typical state \( |\psi\rangle \in \text{span} \ M_j \) is exponentially small. In the thermodynamic limit, \( j, k \) can be promoted to continuous real variables so that \( |M_j|, |L_k| \) follow normal distributions with means \( n/2, fn/2 \) and variances \( n/4, fn/4 \), respectively. Let

\[
J := j/\sqrt{n} - \sqrt{n}/2, \quad K := k/\sqrt{n} - f\sqrt{n}/2.
\]

(27)

We have

\[
|L_k| = \sqrt{2d} e^{-2K^2/j}/\sqrt{f\pi n}, \quad |R_{j-k}| = \sqrt{2d} e^{-2(J-K)^2/(1-f)}/\sqrt{(1-f)\pi n},
\]

\[
|M_j| = \sqrt{2d} e^{-2J^2}/\sqrt{\pi n}, \quad |c_k|^2 = \sqrt{2d} e^{2J^2-2K^2/j-2(J-K)^2/(1-f)}/\sqrt{f(1-f)\pi n}.
\]

(28)

(29)

For any fixed constant \( f < 1/2 \), it is almost always the case that \( |L_k| \ll |R_{j-k}| \). Hence,

\[
S(\sigma_{k,A}) = \left(m + \frac{1}{2}\right) \ln 2 - \frac{\ln(f\pi n)}{2} = \frac{2K^2}{f}.
\]

(30)
Substituting Eqs. (29) and (30) into Eq. (24),

\[ S_j := S(\rho_A) = \int_{-\infty}^{+\infty} |c_k|^2 (S(\sigma_{k,A}) - \ln |c_k|^2) \, dk \]

\[ = \left( m + \frac{1}{2} \right) \ln 2 - \frac{\ln (f \pi n)}{2} + \frac{f - 4 f J^2 - 1}{2} + \frac{1 + \ln (f - f^2) + \ln (\pi n/2)}{2} \]

\[ = m \ln 2 + \frac{f (1 - 4 J^2)}{2} + \frac{\ln (1 - f)}{2}. \] (31)

Averaging over all states in \( M \),

\[ \overline{S} = \frac{1}{d} \sum_{j=0}^{n} |M_j| S_j \approx m \ln 2 + \frac{\ln (1 - f)}{2} + \frac{f}{\sqrt{2 \pi}} \int_{-\infty}^{+\infty} e^{-2 J^2} (1 - 4 J^2) \, dJ = m \ln 2 + \frac{\ln (1 - f)}{2}. \] (32)

For \( f = 1/2 \), we first assume that \( j \leq n/2 \) and \( k \leq j/2 \) (i.e., \( J \leq 0 \) and \( K \leq J/2 \)) so that \( |L_k| \leq |R_{j-k}| \). Hence,

\[ S(\sigma_{k,A}) = \left( \frac{n}{2} + 1 \right) \ln 2 - \frac{\ln (\pi n)}{2} - 4 K^2 - \frac{e^{4 J^2 - 8 J K}}{2}. \] (33)

Let

\[ \text{erfc} x := \frac{2}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-t^2} \, dt \] (34)

be the complementary error function. Substituting Eqs. (29) and (33) into Eq. (24),

\[ S_j := S(\rho_A) = 2 \int_{-\infty}^{j/2} |c_k|^2 S(\sigma_{k,A}) \, dk - \int_{-\infty}^{+\infty} |c_k|^2 \ln |c_k|^2 \, dk \]

\[ = \left( \frac{n}{2} + 1 \right) \ln 2 - \frac{\ln (\pi n)}{2} - \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2 J^2} \text{erfc}(-\sqrt{2} J)}{2} + \frac{1 + \ln (\pi n/8)}{2} \]

\[ = \frac{n - 1}{2} \ln 2 + \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2 J^2} \text{erfc}(-\sqrt{2} J)}{2}. \] (35)

This is the average entanglement entropy of a random state in span \( M_j \) for \( j \leq n/2 \). For \( j > n/2 \), Eq. (35) remains valid upon replacing \( J \) by \(-J\). Averaging over all states in \( M \),

\[ \overline{S} = \frac{1}{d} \sum_{j=0}^{n} |M_j| S_j \approx \frac{n - 1}{2} \ln 2 + \sqrt{\frac{8}{\pi}} \int_{-\infty}^{0} e^{-2 J^2} \left( \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2 J^2} \text{erfc}(-\sqrt{2} J)}{2} \right) \, dJ \]

\[ = \frac{(n - 1) \ln 2}{2} - \frac{2}{\pi}. \] (36)

Equation (12) follows from Eqs. (32) and (36).

References

[1] C. Holzhey, F. Larsen, and F. Wilczek. Geometric and renormalized entropy in conformal field theory. Nuclear Physics B, 424(3):443–467, 1994.
G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev. Entanglement in quantum critical phenomena. Physical Review Letters, 90(22):227902, 2003.

J. I. Latorre, E. Rico, and G. Vidal. Ground state entanglement in quantum spin chains. Quantum Information and Computation, 4(1):48–92, 2004.

P. Calabrese and J. Cardy. Entanglement entropy and quantum field theory. Journal of Statistical Mechanics: Theory and Experiment, 2004(06):P06002, 2004.

P. Calabrese and J. Cardy. Entanglement entropy and conformal field theory. Journal of Physics A: Mathematical and Theoretical, 42(50):504005, 2009.

A. Hamma, R. Ionicioiu, and P. Zanardi. Ground state entanglement and geometric entropy in the Kitaev model. Physics Letters A, 337(1-2):22–28, 2005.

A. Kitaev and J. Preskill. Topological entanglement entropy. Physical Review Letters, 96(11):110404, 2006.

M. Levin and X.-G. Wen. Detecting topological order in a ground state wave function. Physical Review Letters, 96(11):110405, 2006.

H. Li and F. D. M. Haldane. Entanglement spectrum as a generalization of entanglement entropy: identification of topological order in non-abelian fractional quantum Hall effect states. Physical Review Letters, 101(1):010504, 2008.

X. Chen, Z.-C. Gu, and X.-G. Wen. Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order. Physical Review B, 82(15):155138, 2010.

Y. Huang and X. Chen. Quantum circuit complexity of one-dimensional topological phases. Physical Review B, 91(19):195143, 2015.

F. G. S. L. Brandão and M. Horodecki. An area law for entanglement from exponential decay of correlations. Nature Physics, 9(11):721–726, 2013.

F. G. S. L. Brandão and M. Horodecki. Exponential decay of correlations implies area law. Communications in Mathematical Physics, 333(2):761–798, 2015.

D. Gosset and Y. Huang. Correlation length versus gap in frustration-free systems. Physical Review Letters, 116(9):097202, 2016.

M. Žnidarič, T. Prosen, and P. Prelovšek. Many-body localization in the Heisenberg XXZ magnet in a random field. Physical Review B, 77(6):064426, 2008.

J. H. Bardarson, F. Pollmann, and J. E. Moore. Unbounded growth of entanglement in models of many-body localization. Physical Review Letters, 109(1):017202, 2012.

Y. Huang. Entanglement dynamics in critical random quantum Ising chain with perturbations. Annals of Physics, 380:224–227, 2017.

F. Verstraete and J. I. Cirac. Matrix product states represent ground states faithfully. Physical Review B, 73(9):094423, 2006.

N. Schuch, M. M. Wolf, F. Verstraete, and J. I. Cirac. Entropy scaling and simulability by matrix product states. Physical Review Letters, 100(3):030504, 2008.
[20] T. J. Osborne. Hamiltonian complexity. *Reports on Progress in Physics*, 75(2):022001, 2012.

[21] S. Gharibian, Y. Huang, Z. Landau, and S. W. Shin. Quantum Hamiltonian complexity. *Foundations and Trends in Theoretical Computer Science*, 10(3):159–282, 2015.

[22] Y. Huang. *Classical simulation of quantum many-body systems*. PhD thesis, University of California, Berkeley, 2015.

[23] J. M. Deutsch. Thermodynamic entropy of a many-body energy eigenstate. *New Journal of Physics*, 12(7):075021, 2010.

[24] L. F. Santos, A. Polkovnikov, and M. Rigol. Weak and strong typicality in quantum systems. *Physical Review E*, 86(1):010102, 2012.

[25] J. M. Deutsch, H. Li, and A. Sharma. Microscopic origin of thermodynamic entropy in isolated systems. *Physical Review E*, 87(4):042135, 2013.

[26] D. A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S. L. Sondhi. Localization-protected quantum order. *Physical Review B*, 88(1):014206, 2013.

[27] B. Bauer and C. Nayak. Area laws in a many-body localized state and its implications for topological order. *Journal of Statistical Mechanics: Theory and Experiment*, 2013(09):P09005, 2013.

[28] Y. Huang and J. E. Moore. Excited-state entanglement and thermal mutual information in random spin chains. *Physical Review B*, 90(22):220202, 2014.

[29] J. P. Keating, N. Linden, and H. J. Wells. Spectra and eigenstates of spin chain Hamiltonians. *Communications in Mathematical Physics*, 338(1):81–102, 2015.

[30] Z.-C. Yang, C. Chamon, A. Hamma, and E. R. Mucciolo. Two-component structure in the entanglement spectrum of highly excited states. *Physical Review Letters*, 115(26):267206, 2015.

[31] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol. From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics. *Advances in Physics*, 65(3):239–362, 2016.

[32] L. Vidmar, L. Hackl, E. Bianchi, and M. Rigol. Entanglement entropy of eigenstates of quadratic fermionic Hamiltonians. *Physical Review Letters*, 119(2):020601, 2017.

[33] A. Dymarsky, N. Lashkari, and H. Liu. Subsystem eigenstate thermalization hypothesis. *Physical Review E*, 97(1):012140, 2018.

[34] Y. O. Nakagawa, M. Watanabe, H. Fujita, and S. Sugiura. Universality in volume-law entanglement of scrambled pure quantum states. *Nature Communications*, 9:1635, 2018.

[35] J. R. Garrison and T. Grover. Does a single eigenstate encode the full Hamiltonian? *Physical Review X*, 8(2):021026, 2018.

[36] Y. Huang and Y. Gu. Eigenstate entanglement in the Sachdev-Ye-Kitaev model. *Physical Review D*, 100(4):041901, 2019.
[37] R. Nandkishore and D. A. Huse. Many-body localization and thermalization in quantum statistical mechanics. *Annual Review of Condensed Matter Physics*, 6(1):15–38, 2015.

[38] E. Altman and R. Vosk. Universal dynamics and renormalization in many-body-localized systems. *Annual Review of Condensed Matter Physics*, 6(1):383–409, 2015.

[39] R. Vasseur and J. E. Moore. Nonequilibrium quantum dynamics and transport: from integrability to many-body localization. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(6):064010, 2016.

[40] Y. Huang, Y.-L. Zhang, and X. Chen. Out-of-time-ordered correlators in many-body localized systems. *Annalen der Physik*, 529(7):1600318, 2017.

[41] J. Eisert, M. Cramer, and M. B. Plenio. Colloquium: area laws for the entanglement entropy. *Reviews of Modern Physics*, 82(1):277–306, 2010.

[42] J. M. Deutsch. Quantum statistical mechanics in a closed system. *Physical Review A*, 43(4):2046–2049, 1991.

[43] M. Srednicki. Chaos and quantum thermalization. *Physical Review E*, 50(2):888–901, 1994.

[44] M. Rigol, V. Dunjko, and M. Olshanii. Thermalization and its mechanism for generic isolated quantum systems. *Nature*, 452(7189):854–858, 2008.

[45] D. N. Page. Average entropy of a subsystem. *Physical Review Letters*, 71(9):1291–1294, 1993.

[46] Y. Huang, F. G. S. L. Brandão, and Y.-L. Zhang. Finite-size scaling of out-of-time-ordered correlators at late times. *Physical Review Letters*, 123(1):010601, 2019.

[47] S. K. Foong and S. Kanno. Proof of Page’s conjecture on the average entropy of a subsystem. *Physical Review Letters*, 72(8):1148–1151, 1994.

[48] J. Sánchez-Ruiz. Simple proof of Page’s conjecture on the average entropy of a subsystem. *Physical Review E*, 52(5):5653–5655, 1995.

[49] S. Sen. Average entropy of a quantum subsystem. *Physical Review Letters*, 77(1):1–3, 1996.

[50] I. Arad, T. Kuwahara, and Z. Landau. Connecting global and local energy distributions in quantum spin models on a lattice. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(3):033301, 2016.

[51] M. C. Bañuls, J. I. Cirac, and M. B. Hastings. Strong and weak thermalization of infinite nonintegrable quantum systems. *Physical Review Letters*, 106(5):050405, 2011.

[52] H. Kim and D. A. Huse. Ballistic spreading of entanglement in a diffusive nonintegrable system. *Physical Review Letters*, 111(12):127205, 2013.

[53] L. Vidmar and M. Rigol. Entanglement entropy of eigenstates of quantum chaotic Hamiltonians. *Physical Review Letters*, 119(22):220603, 2017.

[54] T.-C. Lu and T. Grover. Renyi entropy of chaotic eigenstates. *Physical Review E*, 99(3):032111, 2019.
[55] H. Araki and E. H. Lieb. Entropy inequalities. *Communications in Mathematical Physics*, 18(2):160–170, 1970.
Universal entanglement of mid-spectrum eigenstates of chaotic local Hamiltonians

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December 5, 2021

Abstract

In systems governed by chaotic local Hamiltonians, my previous work [1] conjectured the universality of the average entanglement entropy of all eigenstates by proposing an exact formula for its dependence on the subsystem size. In this note, I extend this result to the average entanglement entropy of a constant fraction of eigenstates in the middle of the energy spectrum. The generalized formula is supported by numerical simulations of various chaotic spin chains.

1 Introduction

In systems governed by chaotic local Hamiltonians, my previous work [1] conjectured the universality of eigenstate entanglement by proposing an exact formula for its dependence on the subsystem size. This formula was derived from an analytical argument based on an assumption that characterizes the chaoticity of the system, and is supported by numerical simulations.

For simplicity, Ref. [1] only considered the average entanglement entropy of all eigenstates explicitly. Due to the recent interest [2], in this note I extend the result to the average entanglement entropy of a constant fraction of eigenstates in the middle of the energy spectrum. The extension is straightforward and does not require any essentially new ideas beyond those in Ref. [1].

For completeness and for the convenience of the reader, definitions and derivations are presented in full so that this note is technically self-contained, although this leads to substantial text overlap with the original paper [1]. It is not necessary to consult Ref. [1] before or during reading this note. However, in this note I do not discuss the conceptual aspects of the work. Such discussions are in Ref. [1].

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I recommend related works [3–5], which use a similar approach to study other aspects of eigenstate entanglement.

The rest of this note is organized as follows. Section 2 gives basic definitions and a brief review of random-state entanglement. Section 3 presents the main result. Section 4 provides numerical evidence for this analytical result in various chaotic spin chains. The main text of this note should be easy to read, for most of the technical details are deferred to Appendix A.

2 Preliminaries

Definition 1 (entanglement entropy). The entanglement entropy of a bipartite pure state $\rho_{AB}$ is defined as the von Neumann entropy $S(\rho_A) = -\text{tr}(\rho_A \ln \rho_A)$ of the reduced density matrix $\rho_A = \text{tr}_B \rho_{AB}$.

Theorem 1 (conjectured and partially proved by Page [6]; proved in Refs. [7–9]). Let $\rho_{AB}$ be a bipartite pure state chosen uniformly at random with respect to the Haar measure. In average,

$$S(\rho_A) = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} - \frac{d_A - 1}{2d_B} = \ln d_A - \frac{d_A}{2d_B} + O(1/d),$$

where $d_A \leq d_B$ are the dimensions of subsystems A and B, respectively, and $d = d_A d_B$ is the dimension of the total Hilbert space. For an equal bipartition $d_A = d_B$,

$$S(\rho_A) = \ln d_A - 1/2 + O(1/d).$$

Let $\gamma \approx 0.577216$ be the Euler-Mascheroni constant. The second step of Eq. (2) uses the formula

$$\sum_{k=1}^{d_B} \frac{1}{k} = \ln d_B + \gamma + \frac{1}{2d_B} + O(1/d^2).$$

Let $\text{erf} : \mathbb{R} \cup \{\pm \infty\} \rightarrow [-1, 1]$ be the error function

$$\text{erf} x := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Let $\text{erf}^{-1} : [-1, 1] \rightarrow \mathbb{R} \cup \{\pm \infty\}$ be the inverse error function such that both $\text{erf}^{-1} \circ \text{erf}$ and $\text{erf} \circ \text{erf}^{-1}$ are identity maps.

3 Universal eigenstate entanglement

Consider a chain of $N$ spin-1/2’s governed by a local Hamiltonian

$$H = \sum_{i=1}^{N-1} H_i,$$
where $H_i$ represents the nearest-neighbor interaction between spins at positions $i$ and $i+1$. For concreteness, we use open boundary conditions, but our argument also applies to other boundary conditions. Assume without loss of generality that $\text{tr} \, H_i = 0$ (traceless) so that the mean energy of $H$ is 0. We do not assume translational invariance. In particular, $\|H_i\|$ may be site dependent but should be $\Theta(1)$ for all $i$.

Suppose that the Hamiltonian (6) is chaotic in a sense to be made precise below. We provide an analytical argument for

**Conjecture 1 (universal eigenstate entanglement).** Assume without loss of generality that $N$ is even. Consider the spin chain as a bipartite quantum system $A \otimes B$. Subsystem $A$ consists of spins at positions $1, 2, \ldots, N/2$. For a constant $0 < \nu \leq 1$, let $\Lambda$ be such that $H$ has $\nu 2^N$ eigenvalues in the interval $[-\Lambda, \Lambda]$. The average entanglement entropy of the eigenstates in this energy interval is

$$\bar{S} = \frac{N-1}{2} \ln 2 + \frac{2(e^{-(\text{erf}^{-1} \nu)^2} - 1)}{\nu \pi} + \frac{(e^{-(\text{erf}^{-1} \nu)^2} + 2\nu - 2) \text{erf}^{-1} \nu}{2\nu \sqrt{\pi}}$$

in the thermodynamic limit $N \to +\infty$.

**Remark.** It is straightforward to extend Eq. (7) to the case where the subsystem size is an arbitrary constant fraction of the system size.

We split the Hamiltonian (6) into three parts: $H = H_A + H_B$, where $H_{A(B)}$ contains terms acting only on subsystem $A(B)$, and $H_B = H_{N/2}$ is the boundary term. Let $\{|j\rangle_A\}_{j=1}^{2^{N/2}}$ and $\{|k\rangle_B\}_{k=1}^{2^{N/2}}$ be complete sets of eigenstates of $H_A$ and $H_B$ with corresponding eigenvalues $\{\epsilon_j\}$ and $\{\epsilon_k\}$, respectively. Since $H_A$ and $H_B$ are decoupled from each other, product states $\{|j\rangle_A |k\rangle_B\}$ form a complete set of eigenstates of $H_A + H_B$ with eigenvalues $\{\epsilon_j + \epsilon_k\}$. Due to the presence of $H_B$, a (normalized) eigenstate $|\psi\rangle$ of $H$ with eigenvalue $E$ is a superposition

$$|\psi\rangle = \sum_{j,k=1}^{2^{N/2}} c_{jk} |j\rangle_A |k\rangle_B.$$  

The locality of $H_B$ implies a strong constraint stating that the population of $|j\rangle_A |k\rangle_B$ is significant only when $\epsilon_j + \epsilon_k$ is close to $E$.

**Lemma 1.** There exist constants $c, \Delta > 0$ such that

$$\sum_{|\epsilon_j + \epsilon_k - E| \geq \Lambda} |c_{jk}|^2 \leq c e^{-A/\Delta}.$$  

**Proof.** This is a direct consequence of Theorem 2.3 in Ref. [10].

In chaotic systems, we expect

**Assumption 1.** The expansion (8) of a generic eigenstate $|\psi\rangle$ is a random superposition subject to the constraint (9).
This assumption is consistent with, but goes beyond, the semiclassical approximation Eq. (16) of Ref. [11].

We now show that Assumption 1 implies Conjecture 1. Consider the following simplified setting. Let $M_j$ be the set of computational basis states with $j$ spins up and $N-j$ spins down, and $U_j \in \mathcal{U}(|M_j|) = \mathcal{U}((N))$ be a Haar-random unitary on span $M_j$. Define $M'_j = \{U_j|\phi\rangle : \forall|\phi\rangle \in M_j\}$ so that $M := \bigcup_{j=0}^{N} M'_j$ is a complete set of eigenstates of the Hamiltonian

$$H = \sum_{i=1}^{N} \sigma_i^z. \quad (10)$$

The energy of a state in $M$ is defined with respect to this Hamiltonian.

The set $M$ captures the essence of Assumption 1. Every state in $M$ satisfies

$$\sum_{|\epsilon_j + \epsilon_k - E| \geq 1} |c_{jk}|^2 = 0, \quad (11)$$

which is a hard version of the constraint (9). The random unitary $U_j$ ensures that Eq. (8) is a random superposition. Thus, we establish Conjecture 1 by

**Proposition 1.** The average entanglement entropy of the $\nu 2^N$ states in $M$ in the middle of the energy spectrum is given by Eq. (7) in the thermodynamic limit $N \to +\infty$.

## 4 Comparison with numerics

In this section, we compare Eq. (7) with the numerical results in the literature [2, 12, 13]. All these numerical results are obtained by exact diagonalization. They are limited to relatively small system sizes $N \leq 20$ and suffer from non-negligible finite-size effects. Although they cannot confirm Conjecture 1 conclusively, they are quite suggestive: Eq. (7) is supported by numerical simulations of various (not necessarily translation-invariant) chaotic spin chains for various values of $\nu$.

Sometimes an incorrect analytical formula with one or more fitting parameters can fit the numerical data well when the number of data points is limited. We do not need to worry about such false positives here, for Eq. (7) does not contain any fitting parameters.

### 4.1 $\nu = 1$

The original paper [1] considered the case $\nu = 1$ or the average entanglement entropy of all eigenstates. In this case, Eq. (7) becomes

$$\bar{S} = \frac{N-1}{2} \ln 2 - \frac{2}{\pi} \approx \frac{N}{2} \ln 2 - 0.983193. \quad (12)$$

This is the special case $f = 1/2$ of Eq. (12) in Ref. [1].

Let $\sigma_x^i, \sigma_y^i, \sigma_z^i$ be the Pauli matrices at site $i$. In the spin-1/2 chain

$$H = \sum_i \sigma_i^z \sigma_{i+1}^z + g \sigma_i^x + h \sigma_i^z \quad (13)$$
\[ \Delta S = \begin{array}{cccccc} \nu & 0^+ & 1/16 & 1/8 & 1/4 & 1/2 & 1 \\ \hline 0.096574 & 0.097582 & 0.100563 & 0.112324 & 0.160362 & 0.483193 \\ \end{array} \]

Table 1: \( \Delta S \), defined by subtracting the right-hand side of Eq. (7) from that of Eq. (3), as a function of \( \nu \).

with \((g, h) = (-1.05, 0.5) [14]\) and \(((5 + \sqrt{5})/8, (1 + \sqrt{5})/4) [15]\), the average entanglement entropy of all eigenstates was calculated up to the system size \( N = 18 \). The numerical results support Eq. (12). See Fig. 1 of Ref. [1].

4.2 \( \nu = 0^+ \)

In fact, the original paper [1] also presented the result of the case \( \nu = 0^+ \) or the entanglement entropy of the eigenstates at the mean energy of the Hamiltonian. In this case, Eq. (7) becomes

\[ \bar{S} = \frac{N - 1}{2} \ln 2 - \frac{1}{4} \approx \frac{N}{2} \ln 2 - 0.596574. \] (14)

This is the special case \( J = 0 \) of Eq. (35) in Ref. [1], and is slightly less than random-state entanglement [3].

In the spin-1/2 chain [13] with \((g, h) = (0.9045, 0.8090)\), the entanglement entropy of the eigenstate with energy closest to 0 was calculated for the system size \( N = 20 \) [13]. The numerical result is \( 10 \ln 2 - 0.635769 \) which is closer to Eq. (14) than to Eq. (3).

Let \( \{h_i\} \) be a set of independent random variables uniformly distributed on the interval \([-1, 1]\). In the spin-1/2 chain

\[ H = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z + 0.2 \sigma_i^x + h_i \sigma_i^z, \] (16)

the entanglement entropy of an eigenstate with energy close to 0 was calculated for the system size \( N = 16 \) [12]. The numerical result, averaged over 10 samples of \( \{h_i\} \), is \( 8 \ln 2 - 0.5733 \pm 0.0015 \), which is closer to Eq. (14) than to Eq. (3).

4.3 \( 0 < \nu < 1 \)

Let \( \Delta S \) be the difference between the right-hand sides of Eqs. (3) and (7). Its values as a function of \( \nu \) are listed in Table 1.

In the spin-1/2 chain

\[ H = \sum_i 5 \sigma_i^x \sigma_{i+1}^x + 15 \sigma_i^y \sigma_{i+1}^y + 9 \sigma_i^z \sigma_{i+1}^z + 5 \sigma_i^x \sigma_{i+2}^x + 15 \sigma_i^y \sigma_{i+2}^y + 9 \sigma_i^z \sigma_{i+2}^z + 4 \sigma_i^z + 16 \sigma_i^z, \] (17)

\[ \bar{S} = fN \ln 2 + \frac{f + \ln(1 - f)}{2} \] (15)

in the thermodynamic limit \( N \to +\infty \). This is the special case \( J = 0 \) of Eq. (31) in Ref. [1].

We thank the authors of Ref. [13] for sharing the exact value of the data point at \( \beta = 0.0 \) and \( L_A = 10 \) in their Fig. 3.
the average entanglement entropy $\bar{S}$ of the $\nu = 1/4, 1/8, 1/16$ fraction of eigenstates in the middle of the energy spectrum was calculated up to the system size $N = 16$. As shown in Fig. 1 the numerical results semi-quantitatively support Eq. (7).

### Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Acknowledgments

This work was supported by NSF grant PHY-1818914 and a Samsung Advanced Institute of Technology Global Research Partnership.

### A Proof of Proposition 1

Let $L_j$ ($R_j$) be the set of computational basis states of subsystem $A$ ($B$) with $j$ spins up and $N/2 - j$ spins down so that

$$|L_j| = |R_j| = \binom{N/2}{j}, \quad M_j = \min\{N/2, j\} \bigcup_{k=\max\{0, j-N/2\}} L_k \times R_{j-k}.$$  \hspace{1cm} (18)
Thus, any (normalized) state $|\psi\rangle$ in $M'_j$ can be decomposed as

$$|\psi\rangle = \sum_{k=\max\{0,j-N/2\}}^{\min\{N/2,j\}} c_k|\phi_k\rangle,$$

(19)

where $|\phi_k\rangle$ is a normalized state in span $L_k \otimes \text{span } R_{j-k}$. Let $\rho_A$ and $\sigma_{k,A}$ be the reduced density matrices of $|\psi\rangle$ and $|\phi_k\rangle$ for $A$, respectively. It is easy to see

$$\rho_A = \bigoplus_{k=\max\{0,j-N/2\}}^{\min\{N/2,j\}} |c_k|^2 \sigma_{k,A} \implies S(\rho_A) = \sum_{k=\max\{0,j-N/2\}}^{\min\{N/2,j\}} |c_k|^2 S(\sigma_{k,A}) - |c_k|^2 \ln |c_k|^2.$$

(20)

Since $|\psi\rangle$ is a random state in span $M_j$, each $|\phi_k\rangle$ is a (Haar-)random state in span $L_k \otimes \text{span } R_{j-k}$. Theorem 1 implies that in average,

$$S(\sigma_{k,A}) = \ln \min\{|L_k|, |R_{j-k}|\} - \frac{\min\{|L_k|, |R_{j-k}|\}}{2 \max\{|L_k|, |R_{j-k}|\}}.$$  

(21)

In average, the population $|c_k|^2$ is proportional to the dimension of span $L_k \otimes \text{span } R_{j-k}$:

$$|c_k|^2 = |L_k||R_{j-k}|/|M_j|.$$  

(22)

The deviation of $|c_k|^2$ (from the mean) for a typical state $|\psi\rangle \in \text{span } M_j$ is exponentially small. In the thermodynamic limit, $j,k$ can be promoted to continuous real variables so that $|M_j|, |L_k|$ follow normal distributions with means $N/2, N/4$ and variances $N/4, N/8$, respectively. Let

$$J := j/\sqrt{N} - \sqrt{N}/2, \quad K := k/\sqrt{N} - \sqrt{N}/4.$$  

(23)

We have

$$|L_k| = 2^{N/2+1}e^{-4K^2}/\sqrt{\pi N}, \quad |R_{j-k}| = 2^{N/2+1}e^{-(J-K)^2}/\sqrt{\pi N},$$  

(24)

$$|M_j| = 2^{(N+1)/2}e^{-2J^2}/\sqrt{\pi N}, \quad |c_k|^2 = \sqrt{8}e^{2J^2-4K^2}/\sqrt{\pi N}.$$  

(25)

Consider the case that $j \leq N/2$ and $k \leq j/2$ (i.e., $J \leq 0$ and $K \leq J/2$) so that $|L_k| \leq |R_{j-k}|$. Hence,

$$S(\sigma_{k,A}) = \left(\frac{N}{2} + 1\right) \ln 2 - \frac{\ln(\pi N)}{2} - 4K^2 - \frac{e^{4J^2-8JK}}{2}.$$  

(26)

Substituting Eqs. (25), (26) into Eq. (20),

$$S(\rho_A) = 2 \int_{-\infty}^{J/2} |c_k|^2 S(\sigma_{k,A}) \, dk - \int_{-\infty}^{+\infty} |c_k|^2 \ln |c_k|^2 \, dk$$

$$= \left(\frac{N}{2} + 1\right) \ln 2 - \frac{\ln(\pi N)}{2} - \frac{1}{4} + J\sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2J^2}(1 - \text{erf}(-\sqrt{2}J))}{2} + \frac{1 + \ln(\pi N/8)}{2}$$

$$= \frac{N-1}{2} \ln 2 + \frac{1}{4} + J\sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2J^2}(1 - \text{erf}(-\sqrt{2}J))}{2}.$$  

(27)
This is the average entanglement entropy of a random state in span $M_j$ for $j \leq N/2$. For $j > N/2$, Eq. (27) remains valid upon replacing $J$ by $-J$. We determine the energy cutoff $\Lambda$ such that $\nu 2^N$ states in $M$ have energies in the interval $[-\Lambda, \Lambda]$: 

$$
2\sqrt{\frac{2}{\pi}} \int_{-\Lambda}^{0} e^{-2J^2} \, dJ = \nu \implies \Lambda = \frac{\text{erf}^{-1} \nu}{\sqrt{2}}. \tag{28}
$$

Averaging over these $\nu 2^N$ states in $M$,

$$
\bar{S} = \frac{N-1}{2} \ln 2 + 2\nu \sqrt{\frac{2}{\pi}} \int_{-\Lambda}^{0} e^{-2J^2} \left( \frac{1}{4} + J \sqrt{\frac{2}{\pi}} - J^2 - \frac{e^{2J^2} (1 - \text{erf}(-\sqrt{2}J))}{2} \right) \, dJ \\
= \frac{N-1}{2} \ln 2 + \frac{2(e^{-(\text{erf}^{-1} \nu)^2} - 1)}{\nu \pi} + \frac{2(e^{-((\text{erf}^{-1} \nu)^2 + 2\nu - 2)} \text{erf}^{-1} \nu)}{2\nu \sqrt{\pi}}. \tag{29}
$$

References

[1] Y. Huang. Universal eigenstate entanglement of chaotic local Hamiltonians. *Nuclear Physics B*, 938:594–604, 2019.

[2] M. Haque, P. A. McClarty, and I. M. Khaymovich. Entanglement of mid-spectrum eigenstates of chaotic many-body systems – deviation from random ensembles. arXiv:2008.12782.

[3] L. Vidmar and M. Rigol. Entanglement entropy of eigenstates of quantum chaotic Hamiltonians. *Physical Review Letters*, 119(22):220603, 2017.

[4] T.-C. Lu and T. Grover. Renyi entropy of chaotic eigenstates. *Physical Review E*, 99(3):032111, 2019.

[5] C. Murthy and M. Srednicki. Structure of chaotic eigenstates and their entanglement entropy. *Physical Review E*, 100(2):022131, 2019.

[6] D. N. Page. Average entropy of a subsystem. *Physical Review Letters*, 71(9):1291–1294, 1993.

[7] S. K. Foong and S. Kanno. Proof of Page’s conjecture on the average entropy of a subsystem. *Physical Review Letters*, 72(8):1148–1151, 1994.

[8] J. Sánchez-Ruiz. Simple proof of Page’s conjecture on the average entropy of a subsystem. *Physical Review E*, 52(5):5653–5655, 1995.

[9] S. Sen. Average entropy of a quantum subsystem. *Physical Review Letters*, 77(1):1–3, 1996.

[10] I. Arad, T. Kuwahara, and Z. Landau. Connecting global and local energy distributions in quantum spin models on a lattice. *Journal of Statistical Mechanics: Theory and Experiment*, 2016(3):033301, 2016.

[11] A. Dymarsky, N. Lashkari, and H. Liu. Subsystem eigenstate thermalization hypothesis. *Physical Review E*, 97(1):012140, 2018.
[12] Z.-C. Yang, C. Chamon, A. Hamma, and E. R. Mucciolo. Two-component structure in the entanglement spectrum of highly excited states. *Physical Review Letters*, 115(26):267206, 2015.

[13] J. R. Garrison and T. Grover. Does a single eigenstate encode the full Hamiltonian? *Physical Review X*, 8(2):021026, 2018.

[14] M. C. Bañuls, J. I. Cirac, and M. B. Hastings. Strong and weak thermalization of infinite nonintegrable quantum systems. *Physical Review Letters*, 106(5):050405, 2011.

[15] H. Kim and D. A. Huse. Ballistic spreading of entanglement in a diffusive nonintegrable system. *Physical Review Letters*, 111(12):127205, 2013.