ON GOOD INFINITE FAMILIES OF TORIC CODES OR THE LACK THEREOF

MALLORY DOLORFINO, CORDELIA HORCH, KELLY JABBUSCH, AND RYAN MARTINEZ

Abstract. Toric codes, introduced by Hansen, are the natural extensions of Reed-Solomon codes. A toric code is a $k$-dimensional subspace of $\mathbb{F}_q^n$, determined by a toric variety or its associated integral convex polytope $P \subseteq [0, q-2]^n$, where $k = |P \cap \mathbb{Z}^n|$ (the number of integer lattice points of $P$). There are two relevant parameters that determine the quality of a code: the information rate, which measures how much information is contained in a single bit of each codeword; and the relative minimum distance, which measures how many errors can be corrected relative to how many symbols each codeword has. Soprunov and Soprunova defined a good infinite family of codes to be a sequence of codes of unbounded polytope dimension such that neither the corresponding information rates nor relative minimum distances go to 0 in the limit. We examine different ways of constructing families of codes by considering polytope operations such as the join and direct sum. In doing so, we give conditions under which no good family can exist and strong evidence that there is no such good family of codes.

1. Introduction

Toric codes are the natural extensions of Reed-Solomon codes. Fix a finite field $\mathbb{F}_q$ and let $P \subseteq \mathbb{R}^n$ be an integral convex polytope, which is contained in the $n$-dimensional box $[0, q-2]^n$ and has $k$ lattice points. The toric code $C_P(\mathbb{F}_q)$ is obtained by evaluating linear combinations of monomials corresponding to the lattice points of $P$ over the finite field $\mathbb{F}_q$. Toric codes were introduced by Hansen for $n = 2$, (see [1] and [2]), and subsequently studied by various authors including Joyner [3], Little and Schenck [4], Little and Schwarz [5], Ruano [7], and Soprunov and Soprunova [9], [10].

To determine if a given code is “good” from a coding theoretic perspective, one considers parameters associated to the code: the block length, the dimension, and the minimum distance (see Section 2 for the precise definitions). For the class of toric codes, the first two parameters are easy to compute. Given a toric code $C_P(\mathbb{F}_q)$, the block length is $N = (q-1)^n$, and the dimension of the code is given by the number of lattice points $k = |P \cap \mathbb{Z}^n|$ [7]. The minimum distance is not as easy to compute, but various authors have given formulas for computing (or bounding) minimum distances for toric codes coming from special classes of polytopes. In the case of toric surface codes (where $n = 2$), Hansen computed the minimum distance for codes coming from Hirzebruch surfaces ([1], [2]). Little and Schenck determined upper and lower bounds for the minimum distance of a toric surface code by examining Minkowski sum decompositions [4], and Soprunov and Soprunova improved these bounds for surface codes by examining the Minkowski length [9]. For toric codes arising from polytopes $P \subseteq \mathbb{R}^n$ with $n > 2$, Little and Schwarz used Vandermonde matrices to compute minimum distances of codes from simplices and rectangular polytopes [5], and Soprunov and Soprunova computed the minimum distance of a code associated to a product of polytopes and a code associated to a $k$-dilate of a pyramid over a polytope [10]. In general, an ideal code will have minimum distance and dimension large with respect to its block length. Restating these parameters with this in mind, we will consider the relative minimum distance, $\delta(C_P)$, of a toric code $C_P$, which is the ratio of the minimum distance, $d(C_P)$, to the block length: $\delta(C_P) = \frac{d(C_P)}{N}$, and the information rate, $R(C_P)$, which is the ratio of the dimension of the code to the block length: $R(C_P) = \frac{k}{N}$.

In this paper we investigate infinite families of toric codes, motivated by the work of Soprunov and Soprunova [10]. They define a good, infinite family of toric codes as a sequence of toric codes where neither the relative minimum distances nor information rates of the codes go to zero in the limit. In Section 3, we define this formally and explore methods of constructing infinite families through different polytope operations, in particular the join and the direct sum. Using these operations, we construct examples of infinite families of toric codes, but like the examples constructed in [10], none are good.
These examples lead us to conjecture that there are in fact no examples of such good infinite families of toric codes. In Section 4 we build intuition and give evidence towards Conjecture 4.9. In particular, we verify the conjecture for infinite families whose corresponding polytopes contain hypercubes of unbounded dimension or have certain Minkowski lengths.

**Theorem 1.1.** Let \{P_i\} be an infinite family of codes defined over the field \( \mathbb{F}_q \).

1. **Proposition 4.10:** If the \{P_i\} contain hypercubes of unbounded dimension, then \( \delta(P_i) \to 0 \) and the infinite family is not good.
2. **Proposition 4.11:** If the Minkowski lengths of the polytopes \( P_i \) are unbounded as \( i \to \infty \), then \( \delta(P_i) \to 0 \) and the infinite family is not good.

We conjecture that if the Minkowski length is bounded then the information rate, \( R(P_i) \), tends to 0, and discuss some special cases. We conclude the paper with some questions for future research if our conjecture is proved.

2. Preliminaries

To construct a toric code over a finite field \( \mathbb{F}_q \) for some prime power \( q \), let \( P \subseteq [0, q-2]^n \subseteq \mathbb{R}^n \) be an integral convex polytope with \( k \) lattice points. The toric code \( C_P(\mathbb{F}_q) \) is obtained by evaluating linear combinations of monomials corresponding to the lattice points of \( P \) over the finite field \( \mathbb{F}_q \). More concretely, \( C_P(\mathbb{F}_q) \) is given by a generator matrix with each row corresponding to a lattice point \( p = (p_1, p_2, \ldots, p_n) \in (P \cap \mathbb{Z}^n) \) and each column corresponding to an element \( a = (a_1, a_2, \ldots, a_n) \in (\mathbb{F}_q^\times)^n \) defined by:

\[
G = (a^p),
\]

where \( a^p = a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \). Equivalently, \( C_P(\mathbb{F}_q) \) is the image of the following evaluation map. Let

\[
\mathcal{L}_P = \text{span}_{\mathbb{F}_q}\{x^p | p \in P \cap \mathbb{Z}^n\}
\]

where \( x^p = x_1^{p_1} \cdots x_n^{p_n} \).

If we choose an ordering of the elements of \( (\mathbb{F}_q^\times)^n \), then the map

\[
\varepsilon : \mathcal{L}_P \to \mathbb{F}_q^{(q-1)^n}
\]

\[
\varepsilon(f) = (f(a)|a \in (\mathbb{F}_q^\times)^n)
\]

evaluates polynomials \( f \) in \( \mathcal{L}_P \) at each of the points of \( (\mathbb{F}_q^\times)^n \) to give a vector of \( \mathbb{F}_q^{(q-1)^n} \). The image of this map over all polynomials in \( \mathcal{L}_P \) gives the toric code. It can be verified that \( \varepsilon \) is a linear map between the vector spaces \( \mathcal{L}_P \) and \( \mathbb{F}_q^{(q-1)^n} \). Thus the toric code is a vector subspace of \( \mathbb{F}_q^{(q-1)^n} \), and is therefore a **linear code**. If we choose \( \{x^p | p \in P \cap \mathbb{Z}^n\} \) as our basis for \( \mathcal{L}_P \), and the standard basis of \( \mathbb{F}_q^{(q-1)^n} \), then the matrix of \( \varepsilon \) corresponds exactly with the generator matrix \( G \). For simplicity we will generally omit the reference to \( \mathbb{F}_q \) and refer to the toric code as \( C_P \).

**Example 2.1.** Let \( q = 5 \) and \( n = 2 \). Consider the polytope \( P \subseteq \mathbb{R}^2 \) with the \( k = 4 \) lattice points \( (0,0), (1,0), (0,1), (1,1) \) shown below.

\[
\begin{align*}
\text{Figure 1. The polytope } P. \\
\end{align*}
\]

The toric code \( C_P(\mathbb{F}_5) \) is given by the \( 4 \times 16 \) generator matrix \( G \) below. In particular, each row of \( G \) corresponds to a basis element of \( \mathcal{L}_P \), \( \{1, x, y, xy\} \) (corresponding to each of the four lattice points \( p \in P \)) and each column corresponds to a particular evaluation of the polynomial, one of the 16 elements \( a \in (\mathbb{F}_5^\times)^2 \).
Finally, the elements of the toric code $C_P(\mathbb{F}_5)$ are obtained by choosing the coefficients from $\mathbb{F}_5$ to use with the basis of $\mathcal{L}_P$. In particular,

$$C_P(\mathbb{F}_5) = \{aG|a \in \mathbb{F}_5^4\}.$$

The full generator matrix for our example is

$$G = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 & 2 & 4 & 1 & 3 & 1 & 4 & 2 & 4 & 3 & 2 & 1
\end{pmatrix}$$

We can see that the first row corresponds to the basis element 1 (the second to $x$ and so on), and each column represents evaluating the row’s basis element at a different member of $(\mathbb{F}_5^5)^2$. We immediately see from this generator matrix that $C_P(\mathbb{F}_5)$ is an at most 4 dimensional vector subspace of $\mathbb{F}_5^{16}$. It turns out that $C_P(\mathbb{F}_5)$ is exactly 4 dimensional since the polytope $P$ has exactly 4 lattice points.

In general the quality of a toric code $C = C_P(\mathbb{F}_q)$, where $P$ is a convex integral lattice polytope living in $[0,q-2]^n$ can be classified by the following parameters:

**Definition 2.2.** The **block length**, $N$, is the number of entries in each of $C$’s codewords. Based on the construction of toric codes, the block length of $C$ is exactly $(q-1)^n$.

Intuitively, the block length is the number of characters or symbols of $\mathbb{F}_q$ we write to represent an element of our code $C$. In the above example, Example 2.1, we have block length $(5-1)^2 = 16$.

**Definition 2.3.** The **dimension**, $k$, is $C$’s dimension over $\mathbb{F}_q$. If $P$ has $|P \cap \mathbb{Z}^n|$ lattice points, then the dimension of $C$ is exactly $k = |P \cap \mathbb{Z}^n|$ (the proof of this equality is given in [7]).

Intuitively, the dimension is the number of characters or symbols of $\mathbb{F}_q$ we need to represent an element of our code $C$. In the above example, Example 2.1, the dimension of $C_P$ is 4. That is, even though we write 16 symbols of $\mathbb{F}_5$ to represent each codeword, we are only storing 4 symbols of $\mathbb{F}_5$ worth in data.

The reason we choose to use this storage inefficiency is so that we can correct errors in our codewords. For example, suppose we are given the following codeword with an error from our running example above:

$$w = (1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,0).$$

If we operate under the assumption that most errors occur independently on individual symbols of the code with relatively small probability, then we can correct $w$ to

$$w' = (1,1,1,1, 1,1,1,1, 1,1,1,1, 1,1,1,1) = (1,0,0,0) G,$$

which is indeed in our code $C_P(\mathbb{F}_5)$. This notion is formalized by Hamming distance: the number of differences between two codewords. We see that the Hamming distance between $w$ and $w'$ is 1, and in fact there is no other $c \in C_P(\mathbb{F}_5)$ that is this close to $w$. As another example, consider

$$v = (0,0,0,0, 1,1,1,1, 0,0,0,0, 1,1,1,1).$$

We see that both

$$v' = (0,0,0,0, 1,1,1,1, 2,2,2,2, 3,3,3,3) = (4,1,0,0) G$$

and

$$v'' = (0,0,0,0, 0,0,0,0, 0,0,0,0, 0,0,0,0) = (0,0,0,0) G$$

are Hamming distance 8 away from $v$, so we don’t know what to correct to. It is customary to use the minimum distance to measure what the largest error we can correct is.

**Definition 2.4.** The **minimum distance** $d$ of a code $C$ is the minimum Hamming distance between any two codewords in $C$. Equivalently, the minimum distance is the minimum weight of a codeword in $C$, where the weight of a codeword $w$ is the Hamming distance between $w$ and $(0,0,\ldots,0) \in \mathbb{F}_q^N$. Because of this equivalence, we have for a toric code $C_P$ that the minimum distance is given by

$$d(C_P) = (q-1)^n - \max_{0 \neq f \in \mathcal{L}_P} |Z(f)|. $$

where $Z(f)$ is the set of $\mathbb{F}_q$ zeros of $f$. We’ll use $N(P)$ to denote the $\max_{0 \neq f \in \mathcal{L}_P} |Z(f)|$. 

Remark 2.5. The equivalence of the minimum Hamming distance between two codewords in $C$ and the minimum weight of a codeword in $C$ holds only for linear codes, and is thus valid for our considerations.

Suppose we have a code $C$ with minimum distance $d$, and we are given an erroneous codeword $w$ and valid codeword $w' \in C$ such that the Hamming distance between $w$ and $w'$ is $\ell \leq \lfloor (d-1)/2 \rfloor$. Since the Hamming distance defines a metric on $\mathbb{F}_q^N$, we know that for all other codewords $c \in C$ we have

$$\text{dist}(c, w) \geq \lfloor \text{dist}(c, w') - \text{dist}(w', w) \rfloor \geq d - \lfloor (d-1)/2 \rfloor = \lceil (d+1)/2 \rceil > (d-1)/2.$$

In other words, we know that $w'$ is the closest codeword to $w$ in $C$. Thus, for a code $C$ with minimum distance $d$ we know we can correct $\lfloor (d-1)/2 \rfloor$ errors.

Returning to Example 2.1, to compute the minimum distance we must count the number of $(\mathbb{F}_5^2)^2$ zeros of polynomials in $\mathcal{L}_P = \text{span}\{1, x, y, xy\}$. If $f(x, y) \in \mathcal{L}_P$ factors into two linear terms, say $f(x, y) = (x-a)(y-b)$, then if $x = a$, there are $(q-1)$ choices for $y$ and if $y = b, x \neq a$, then there are $(q-2)$ choices for $x$ such that $f$ evaluates to 0. Therefore $N(P) \geq (q-1) + (q-2) = 2q - 3$, which, when $q = 5$, is 7.

If $f(x, y) \in \mathcal{L}_P$ is an irreducible polynomial over $\mathbb{F}_5$, then the zero set of $f(x, y)$ is an affine conic section over $\mathbb{F}_5$, and hence contains at most six $\mathbb{F}_5$-rational points, by the Hasse-Weil bound. Thus an irreducible polynomial can have at most six zeros. Therefore, the maximum number of zeros of $f(x, y) \in \mathcal{L}_P$ is seven, and the minimum distance is

$$d(C_P) = (5-1)^2 - 7 = 9.$$

In other words, we are sure to be able to correct errors on $\lfloor (9-1)/2 \rfloor = 4$ or fewer symbols.

In general, an ideal code will have $d$ and $k$ large with respect to $N$. This means that the code will correct as many errors as possible and convey a lot of information while maintaining short codewords, which are easier to work with computationally.

Remark 2.6. One can perform certain operations on a polytope without changing the parameters of the code it yields. If $t: \mathbb{Z}^n \to \mathbb{Z}^n$ is a unimodular affine transformation, that is $t(x) = Mx + \lambda$, where $M \in \text{GL}(n, \mathbb{Z})$ is a unimodular matrix and $\lambda$ has integer entries, such that $t$ maps one polytope $P_1 \subseteq \mathbb{R}^n$ to another $P_2 \subseteq \mathbb{R}^n$, then we say that $P_1$ and $P_2$ are lattice equivalent. In this case, the codes $C_{P_1}$ and $C_{P_2}$ are monomially equivalent [5, Theorem 4]. Note that two lattice equivalent polytopes have the same number of lattice points, and monomially equivalent codes share the same parameters.

The motivation for this paper comes from the work of Soprunov and Soprunova [10] who construct classes of examples of higher dimensional toric codes and propose the problem of finding an infinite family of good toric codes. To analyze these families of codes, we need to slightly modify our parameters.

Definition 2.7. The information rate, $R(C_P)$, measures how large the dimension of the code is relative to the block length of the code:

$$R(C_P) = \frac{k}{N}.$$

Definition 2.8. The relative minimum distance, $\delta(C_P)$, measures how large the minimum distance is relative to the block length of the code:

$$\delta(C_P) = \frac{d(C_P)}{N}.$$

In the remainder of the paper we will use the simpler notation of $R(P)$ and $\delta(P)$ instead of $R(C_P)$ and $\delta(C_P)$.

Returning to Example 2.1 again, the information rate is $R(P) = \frac{4}{(q-1)^2}$ and the relative minimum distance is $\delta(P) = \frac{(q-1)^2 - (2q - 3)}{(q-1)^2}$. When $q = 5$, $R(P) = \frac{1}{4}$ and $\delta(P) = \frac{9}{16}$.

Given a polytope $P \subseteq \mathbb{R}^n$, we can view $P$ as sitting in a larger ambient space $\mathbb{R}^m$, for $m > n$. The next proposition shows that $\delta(P)$ is invariant under the dimension of the embedded space.

Proposition 2.9. Let $P \subseteq [0, q-2]^n \subseteq \mathbb{R}^n$ be an integral convex lattice polytope and let $m > n$. Let $P'$ be lattice equivalent to the natural embedding of $P$ in $\mathbb{R}^m$, $P \times \{0\}^{m-n}$. Then working over $\mathbb{F}_q$,

$$\delta(P) = \delta(P').$$
Proof. Since lattice equivalent polytopes have the same minimum distances [5, Theorem 4] it suffices to let \( P' = P \times \{0\}^{m-n} \). We will show that \( \delta(P') = \delta(P) \) by computing the minimum distance of \( C_{P'} \) and \( C_P \) directly.

Let \( Z_k(f) \) denote the zero set of the polynomial \( f \) over \((\mathbb{F}_q^*)^k\), where \( f \) has \( k \) variables. Notice that if \( f \in \mathbb{F}_q[x_1, \ldots, x_k] \) then
\[
|Z_{k+1}(f)| = (q-1)|Z_k(f)|
\]
This is because for each \((\mathbb{F}_q^*)^k \) zero of \( Z_k(f) \), we get \((q-1)(\mathbb{F}_q^*)^{k+1}\) zeros of \( Z_{k+1}(f) \): one for each value we set to the extra variable. Thus by induction we have for \( f \in \mathbb{F}_q[x_1, \ldots, x_n] \),
\[
|Z_m(f)| = (q-1)^{m-n}|Z_n(f)|
\]

Now, since each \( f \in \mathcal{L}_{P'} \) exclusively uses the first \( n \) of its \( m \) available variables by construction, we see that each \( f \in \mathcal{L}_{P'} \) is a member of \( \mathbb{F}_q[x_1, \ldots, x_n] \). Furthermore, since \( P' \) is a copy of \( P \) in a higher dimension, we have that \( \mathcal{L}_P = \mathcal{L}_{P'} \). Lastly, note that by definition
\[
\delta(P) = \frac{(q-1)^n - \max_{0 \neq f \in \mathcal{L}_P} |Z_n(f)|}{(q-1)^n} = 1 - \max_{0 \neq f \in \mathcal{L}_P} \frac{|Z_n(f)|}{(q-1)^n}
\]
and
\[
\delta(P') = \frac{(q-1)^m - \max_{0 \neq f \in \mathcal{L}_{P'}} |Z_m(f)|}{(q-1)^m} = 1 - \max_{0 \neq f \in \mathcal{L}_{P'}} \frac{|Z_m(f)|}{(q-1)^m}
\]
It follows that
\[
\delta(P') = 1 - \max_{0 \neq f \in \mathcal{L}_{P'}} \frac{|Z_m(f)|}{(q-1)^m} = 1 - \max_{0 \neq f \in \mathcal{L}_P} \frac{(q-1)^{m-n} |Z_n(f)|}{(q-1)^m} = 1 - \max_{0 \neq f \in \mathcal{L}_P} \frac{|Z_n(f)|}{(q-1)^n} = \delta(P).
\]
\[\square\]

Conversely, the information rate will change based on the embedded dimension.

**Proposition 2.10.** Let \( P \subseteq [0, q-2]^n \subseteq \mathbb{R}^n \) be an integral convex lattice polytope and let \( m > n \). Let \( P' \) be lattice equivalent to the natural embedding of \( P \) in \( \mathbb{R}^m \), \( P \times \{0\}^{m-n} \). Then working over \( \mathbb{F}_q \),
\[
R(P') = R(P)(q-1)^{n-m} = \frac{R(P)}{(q-1)^{m-n}}.
\]

**Proof.** This follows from the fact that lattice equivalent polytopes have the same number of lattice points and the definition of information rate. \[\square\]

Proposition 2.10 shows that to maximize the information rate for a given polytope, we should embed it into the smallest dimension possible. Moreover, since the relative minimum distance does not depend on the embedding, by Proposition 2.9, then in terms of both of our relevant parameters, the smaller dimension of the embedding, the better.

Finally, we note one further property of \( \delta \) that we will need later.

**Proposition 2.11.** Let \( P \) and \( Q \) be integral convex polytopes with \( P \subseteq Q \subseteq [0, q-2]^n \). Then \( \delta(P) \geq \delta(Q) \).

**Proof.** With respect to the field \( \mathbb{F}_q \), let \( d(C_P) \) and \( d(C_Q) \) denote the respective minimum distances. By definition, there exists a polynomial \( f \in \mathcal{L}_P \subseteq \mathbb{F}_q[x_1, \ldots, x_n] \) such that
\[
d(C_P) = (q-1)^n - |Z(f)|
\]
Now, since \( P \subseteq Q \) it follows that \( \mathcal{L}_P \subseteq \mathcal{L}_Q \) so that \( f \in \mathcal{L}_Q \). Thus by definition,
\[
d(C_Q) = (q-1)^n - \max_{g \in \mathcal{L}_Q} |Z(g)| \leq (q-1)^n - |Z(f)| = d(C_P).
\]
We then compute \( \delta \):
\[
\delta(Q) = \frac{d(C_Q)}{(q-1)^n} \leq \frac{d(C_P)}{(q-1)^n} = \delta(P).
\]
\[\square\]
Various authors have computed the minimum distance of toric codes arising from special polytopes. We record formulas for the minimum distance of toric codes defined by simplices and boxes, given in [5], and a generalization to the Cartesian product, given in [10].

**Theorem 2.12.** [5, Corollary 2] Let $\ell < q - 1$ and $P_i(n)$ be an $n$-dimensional simplex with side length $\ell$. Then the minimum distance of the toric code $C_{P_i(n)}$ is

$$d(C_{P_i(n)}) = (q - 1)^n - \ell(q - 1)^{n-1}.$$  

**Theorem 2.13.** [5, Theorem 3] Let $P_{\ell_1, \ldots, \ell_n}$ be the product of $n$-dimensional simplices $[0, \ell_1] \times \cdots \times [0, \ell_n] \subseteq \mathbb{R}^n$. Let $\ell_1, \ldots, \ell_n$ be small enough so that $P_{\ell_1, \ldots, \ell_n} \subseteq [0, q - 2]^n \subseteq \mathbb{R}^n$. Then the minimum distance of the toric code $C_{P_{\ell_1, \ldots, \ell_n}}$ is

$$d(C_{P_{\ell_1, \ldots, \ell_n}}) = \prod_{i=1}^{n} ((q - 1) - \ell_i).$$  

**Theorem 2.14.** [10, Theorem 2.1] Let $P \subseteq [0, q - 2]^n$ and $Q \subseteq [0, q - 2]^m$ be integral convex polytopes. Then

$$d(C_{P \times Q}) = d(C_P)d(C_Q).$$

3. Infinite Families of Toric Codes

We next investigate infinite families of toric codes, following Soprunov and Soprunova [10].

**Definition 3.1.** Let $\{P_i\}$ be a sequence of non-empty integral convex polytopes such that each $P_i \subseteq [0, q - 2]^{n_i}$ and $n_i \to \infty$ as $i \to \infty$. Let $C_{P_i}$ be the toric code associated with $P_i$. We say $\{C_{P_i}\}$ is an infinite family of toric codes.

Since every polytope corresponds to only one toric code, sometimes we will refer to $\{P_i\}$ as the infinite family of toric codes.

**Definition 3.2.** A good infinite family of toric codes is an infinite family of toric codes, $\{P_i\}$, such that $\delta(P_i) \to \delta$ and $R(P_i) \to R$ as $n_i \to \infty$, where $\delta, R \in (0, 1)$. In other words, an infinite family of toric codes is good if $\delta(P_i)$ and $R(P_i)$ approach positive constants as the dimension of the polytopes approaches infinity.

In [10], Soprunov and Soprunova construct various infinite families of toric codes; none of which are good families, and remark that it would be interesting to find such a good family.

3.1. Polytope Operations. With the intent of constructing a good infinite family of codes, we derive formulas for the relative minimum distance and information rate of codes corresponding to polytopes constructed using the join and direct sum operations.

3.1.1. The Join.

**Definition 3.3.** Let $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ be integral convex polytopes. The join of $P$ and $Q$, denoted by $P * Q$, is defined as

$$P * Q = \text{conv}((p, 0^m, 0) | p \in P) \cup ((0^n, q, 1) | q \in Q) \subseteq \mathbb{R}^{n+m+1}$$

**Example 3.4.** Let $P = [0, 2] \subseteq \mathbb{R}$ and $Q = [0, 3] \subseteq \mathbb{R}$ be line segments illustrated below:
Then the join of $P$ and $Q$ is the convex hull of the points \{(0,0,0), (2,0,0), (0,0,1), (0,3,1)\} given in Figure 3 below.

Figure 3. The Polytope $P \ast Q$

In general, the integral lattice points of the join of $P$ and $Q$ are exactly the disjoint union of the integral lattice points of $P$ and the integral lattice points of $Q$. This is because the last coordinate $z$ satisfies $0 \leq z \leq 1$. Hence every lattice point in the join has either $z = 0$ or $z = 1$. When $z = 0$ (resp. $z = 1$), the lattice points are in one-to-one correspondence with the lattice points in $P$ (resp. $Q$).

Proposition 3.5. Fix a finite field $\mathbb{F}_q$ and let $P \subseteq [0,q-2]^n \subset \mathbb{R}^n$ and $Q \subseteq [0,q-2]^m \subset \mathbb{R}^m$ be integral convex polytopes. Then the maximum number of zeros of $h \in L_{P \ast Q}$, is given by

$$N(P \ast Q) = \max \{(q-1)^{n+m}, N(P)(q-1)^{m+1}, N(Q)(q-1)^{n+1}, (q-1)N(P)N(Q) + ((q-1)^n - N(P))((q-1)^m - N(Q))\}$$

and the relative minimum distance of $C_{P \ast Q}$ is

$$\delta(P \ast Q) = \min \left\{ 1 - \frac{1}{q-1}, \delta(P), \delta(Q), \delta(P) + \delta(Q) - \delta(P)\delta(Q) \frac{q}{q-1} \right\}.$$ 

Proof. Every polynomial $h \in L_{P \ast Q}$ has the form $f + x_{n+m+1}g$ for some $f \in L_P$ and $g \in L_Q$, so we can count the number of $\mathbb{F}_q^n$-zeros in $(\mathbb{F}_q^n)^{n+m+1}$. If both $f$ and $g$ evaluate to zero, then $h$ evaluates to zero at all $(q-1)$ choices for $x_{n+m+1}$, and there are $|Z(f)|$ points in $(\mathbb{F}_q^n)^n$ such that $f$ evaluates to zero, and similarly, there are $|Z(g)|$ points in $(\mathbb{F}_q^m)$ such that $g$ evaluates to zero, so there are $(q-1)|Z(f)||Z(g)|$ points in...
Similarly, when \( F \delta \) we include these cases for completeness, so we have that \( f \). Note that when \( h \) does not evaluate to zero, and similarly there are \((q-1)^m - |Z(g)|\) points in \((\mathbb{F}_q^n)^n\) where \( g \) does not evaluate to zero, so there are \((1)((q-1)^n - |Z(f)|)((q-1)^m - |Z(g)|)\) additional points in \((\mathbb{F}_q^n)^{n+m+1}\) such that \( h \) evaluates to zero. Therefore, we have that for \( h = f + x_{n+m+1} g \in \mathcal{L}_{P*Q} \),

\[
|Z(h)| = (q-1)|Z(f)||Z(g)| + ((q-1)^n - |Z(f)|)((q-1)^m - |Z(g)|)
\]

In order to find \( N(P*Q) \), we need to determine what values of \( |Z(f)| \) and \( |Z(g)| \) maximize \( |Z(h)| \).

Regarding \( |Z(f)| \) and \( |Z(g)| \) as real variables \( x \) and \( y \) respectively, and \( |Z(h)| \) as a function of these two variables, \( F(x, y) \), we use calculus to determine the maximum value of \( F(x, y) = (q-1)xy + ((q-1)^n - x)((q-1)^m - y) \). Taking the partials of this equation, we have

\[
F_x = (q-1)y - (q-1)^m + y \quad \text{and} \quad F_y = (q-1)x - (q-1)^n + x.
\]

We find that \( F_x = 0 \) when \( y = \frac{(q-1)^m}{q} \) and \( F_y = 0 \) when \( x = \frac{(q-1)^n}{q} \). Computing second partial derivatives, we have:

\[
F_{xx} = 0, \quad F_{yy} = 0, \quad F_{xy} = q.
\]

Hence \((F_{xx})(F_{yy}) - (F_{xy})^2 = 0 - q^2 < 0\) so that by the second partial derivative test, \((\frac{(q-1)^m}{q}, \frac{(q-1)^n}{q})\) is a saddle point of \( F \). Note however that \( x \in [0, N(P)] \) and \( y \in [0, N(Q)] \), and \( F \) is continuous on this set \([0, N(P)] \times [0, N(Q)]\), so \( F \) attains its maximum value at some point in the set, and we know that since it is not at a critical point, it occurs on the boundary.

Next, we will show that the maximum value must occur at either the point \((0, 0)\) or the point \((N(P), N(Q))\). Note that \( F_x > 0 \) when \( y > \frac{(q-1)^m}{q} \) and \( F_y > 0 \) when \( x > \frac{(q-1)^n}{q} \). From this, we have that if \( x > \frac{(q-1)^n}{q} \) and \( y > \frac{(q-1)^m}{q} \) do not have the same sign, then \( F \) will not be at its maximum. Additionally, if \( x = N(P) \) and \( y \neq N(Q) \), and \( y > \frac{(q-1)^m}{q} \) then \( F \) is not at its maximum, since \( F \) will increase as \( y \) increases. In the same way, if \( x = 0 \) and \( y \neq 0 \), and \( |Z(g)| \) is not at \( \frac{(q-1)^m}{q} \) then \( F \) is not at its maximum, since \( F \) will increase as \( y \) decreases. A similar analysis holds for when \( y = 0 \) and \( x \neq 0 \) and when \( y = N(Q) \) and \( x \neq N(P) \). Therefore, \( F \) obtains its maximum value at either \((0, 0)\) or \((N(P), N(Q))\), so

\[
N(P*Q) = \max\{(q-1)^n(q-1)^m, (q-1)N(P)N(Q), ((q-1)^n - N(P))((q-1)^m - N(Q))\}.
\]

Note that when \( f \) is identically zero, then \( N(P) = (q-1)^n \) so

\[
(q-1)N(P)N(Q)+((q-1)^n-N(P))((q-1)^m-N(Q)) = (q-1)(q-1)^nN(Q)+(0)((q-1)^m-N(Q)) = (q-1)^n+1N(Q).
\]

Similarly, when \( g \) is identically zero, then \( N(Q) = (q-1)^m \) so

\[
(q-1)N(P)N(Q)+((q-1)^n-N(P))((q-1)^m-N(Q)) = (q-1)N(P)(q-1)^m+(q-1)^n-N(P)(0) = (q-1)^m+1N(P).
\]

We include these cases for completeness, so we have that

\[
N(P*Q) = \max\{(q-1)^n+m, N(P)(q-1)^{m+1}, N(Q)(q-1)^{n+1},\}
\]

\[
(q-1)N(P)N(Q) + ((q-1)^n - N(P))((q-1)^m - N(Q))\}.
\]

Since \( \delta(P*Q) = \frac{(q-1)^{n+m+1} - N(P*Q)}{(q-1)^{n+m+1}} \), we have

\[
\delta(P*Q) = \min\left\{1 - \frac{1}{q-1}, \delta(P), \delta(Q), \delta(P) + \delta(Q) - \delta(P)\delta(Q) \left( \frac{q}{q-1} \right) \right\}.
\]

**Proposition 3.6.** Let \( P \subseteq [0, q-2]^n \) and \( Q \subseteq [0, q-2]^m \), and let \( N(P) \) and \( N(Q) \) be as above. If \( N(P) \geq 2\frac{(q-1)^m}{q} \) and \( N(Q) \geq 2\frac{(q-1)^n}{q} \) then

\[
(q-1)^{n+m} \leq (q-1)N(P)N(Q) + ((q-1)^n - N(P))((q-1)^m - N(Q))
\]
Proof. Suppose \( N(P) \geq 2 \frac{(q-1)^n}{q} \) and \( N(Q) \geq 2 \frac{(q-1)^m}{q} \) and let \( F: \mathbb{R}^2 \to \mathbb{R} \) by
\[
F(x, y) = (q - 1)x + ((q - 1)y - x)((q - 1)^m - y).
\]

We wish to show that
\[
(q - 1)^n(q - 1)^m \leq F(N(P), N(Q)).
\]

By the analysis in the proof of Proposition 3.5 we know that \( F(x, y) \) is monotonically increasing with both \( x \) and \( y \) when both \( x \geq \frac{(q-1)^n}{q} \) and \( y \geq \frac{(q-1)^m}{q} \) since both partial derivatives are non-negative in this region.

In particular, we have since \( q \geq 1 \)
\[
N(P) \geq 2 \frac{(q - 1)^n}{q} \geq \frac{(q - 1)^n}{q}
\]
and similarly for \( N(Q) \). Thus,
\[
F(N(P), N(Q)) \geq F\left(2 \frac{(q - 1)^n}{q}, 2 \frac{(q - 1)^m}{q}\right)
\]
\[
= (q - 1)2 \frac{(q - 1)^n}{q} \frac{2(q - 1)^m}{q}
\]
\[
+ \left((q - 1)^n - \frac{2(q - 1)^n}{q}\right) \left((q - 1)^m - \frac{2(q - 1)^m}{q}\right)
\]
\[
= (q - 1)^{n+m}\left[\frac{(q - 1)(2/q)^2 + (1 - 2/q)^2}{q}\right]
\]
\[
= (q - 1)^{n+m}\left[4/q - 4/q^2 + 1 - 4/q + 4/q^2\right]
\]
\[
= (q - 1)^{n+m}.
\]

\[\blacksquare\]

Corollary 3.7. If both \( P \subseteq [0, q - 2]^n \) and \( Q \subseteq [0, q - 2]^m \) contain either a lattice segment of length at least 2 or a square of side length 1, then \( (q - 1)^{n+m} \leq (q - 1)N(P)N(Q) + ((q - 1)^n - N(P))((q - 1)^m - N(Q)) \).

Thus,
\[
\delta(P \ast Q) = \min \left\{ \delta(P), \delta(Q), \delta(P) + \delta(Q) - \delta(P)\delta(Q) \left(\frac{q}{q - 1}\right) \right\}.
\]

Proof. Suppose \( P \) is a polytope with a length 2 lattice segment. We can say that the three points of this segment are without loss of generality (by applying a unimodular affine transformation)
\[
(0, 0, \ldots, 0)
\]
\[
(1, 0, \ldots, 0)
\]
\[
(2, 0, \ldots, 0)
\]

Then we see that for any \( a, b \in \mathbb{F}_q^\times \)
\[
f = (x_1 - a)(x_1 - b) = ab - (a + b)x_1 + x_1^2 \in \mathcal{L}_P.
\]

We can count the zeros of \( f \) as
\[
Z(f) = 2(q - 1)^{n-1}
\]
since there are two values for \( x_1 \) and \( (q - 1) \) values for each of the \( (n - 1) \) other coordinates that result if \( f = 0 \). In particular since \( q \) is always a prime power we have that \( q \geq 2 \) and
\[
2 \frac{(q - 1)^n}{q} \leq 2 \frac{(q - 1)^n}{q - 1} = 2(q - 1)^{n-1} = Z(f) \leq N(P).
\]

Thus any polytope with a length 2 lattice segment meets the conditions for Proposition 3.6.

Similarly if \( P \) is a polytope with a length 1 lattice square then we can apply a unimodular affine transformation so that we have
\[
f = (x_1 - a)(x_2 - b) \in \mathcal{L}_P, \text{ where } a \neq b.
\]

9
We see that 
\[ Z(f) = (q - 1)^{n-1} + (q - 2)(q - 1)^{n-2} = (q - 1)^{n-2}(q - 1 + q - 2). \]

Since \( q \geq 2 \) we have 
\[ 2(q - 1)^2 = 2q^2 - 4q + 2 \leq 2q^2 - 3q = q(q - 1 + q - 2). \]

In particular, 
\[ 2 \frac{(q - 1)^n}{q} \leq \frac{(q - 1)^{n-2}}{q}q(q - 1 + q - 2) = (q - 1)^{n-2}(q - 1 + q - 2) = Z(f) \leq N(P). \]

Thus any polytope with a length 1 lattice square meets the conditions for Proposition 3.6 and we have proved the first statement.

We then note that in our case, Proposition 3.5 gives

\[ N(P + Q) = \max \{ N(P)(q - 1)^{n+1}, N(Q)(q - 1)^{n+1}, (q - 1)N(P)N(Q) + ((q - 1)^n - N(P))(q - 1)^m - N(Q) \} \]

and so

\[ \delta(P + Q) = \min \{ \delta(P), \delta(Q), \delta(P) + \delta(Q) - \delta(P)\delta(Q) \left( \frac{q}{q - 1} \right) \}. \]

\[ \square \]

3.1.2. The Direct Sum.

**Definition 3.8.** The **subdirect sum** of two integral convex polytopes \( P \subseteq \mathbb{R}^n \) and \( Q \subseteq \mathbb{R}^m \) is denoted by \( P \oplus Q \) and defined as

\[ P \oplus Q := \text{conv}(\{(p, 0) \in \mathbb{R}^{n+m} | p \in P\} \cup \{(0, q) \in \mathbb{R}^{n+m} | q \in Q\}) \]

If \( P \) and \( Q \) both contain the origin, then \( P \oplus Q \) is called the **direct sum** of \( P \) and \( Q \).

Let \( P \subseteq [0, q - 2]^n \) be an integral convex polytope that contains the origin and let \( P' = P \oplus [0, \ell] \), where \( \ell \leq q - 2 \). Our goal is to compute \( \delta(P') \). For \( 0 \leq i \leq \ell \), let \( P_i \) be the greatest integral polytope contained in the scaled polytope \( \frac{1}{\ell}P \); in other words each \( P_i \) is a slice of \( P' \), where \( x_{n+1}, \) the \( (n+1) \)st coordinate, equals \( i \) and that slice is projected to \( \mathbb{R}^n \). Note that \( P_0 = P \), \( P_{\ell} \) is the origin, and \( P_i \subseteq P_{i-1} \), so that

\[ d(P) = d(P_0) \leq d(P_1) \leq \cdots \leq d(P_\ell) = (q - 1)^n. \]

By [8, Cor. 4.3],

\[ d(C_{P'}) \geq \min_{0 \leq i \leq \ell} (q - 1 - i)d(C_{P_i}). \]

**Proposition 3.9.** In the above setup, if the minimum of \((q - 1 - i)d(C_{P_i})\), as \( i \) ranges from 0 to \( \ell \), is achieved at either \( i = 0 \) or \( i = \ell \), then \( N(P') = \max \{ N(P)(q - 1), \ell(q - 1)^n \} \) and

\[ \delta(P') = \min \left\{ \delta(P), 1 - \frac{\ell}{q - 1} \right\} = \min \{ \delta(P), \delta(0, \ell) \}. \]

**Proof.** Suppose first that the minimum occurs when \( i = 0 \). Then by [8, Cor. 4.3], \( d(C_{P'}) \geq (q - 1)d(C_{P_0}) \) so that

\[ (q - 1)^{n+1} - N(P') \geq (q - 1)((q - 1)^n - N(P)) = (q - 1)^{n+1} - (q - 1)N(P), \]

hence

\[ N(P') \leq (q - 1)N(P). \]

Consider a polynomial \( f \in L_P \) which has the maximum \( N(P) \) zeros, then \( x_{n+1}f \in L_{P'} \) and has \((q - 1)N(P)\) zeros, so \( N(P') = (q - 1)N(P) \).

If the minimum occurs at \( i = \ell \), then we have \( d(C_{P'}) \geq (q - 1 - \ell)d(C_{P_\ell}) = (q - 1 - \ell)(q - 1)^n \), so that

\[ (q - 1)^{n+1} - N(P') \geq (q - 1 - \ell)(q - 1)^n, \]

and

\[ N(P') \leq \ell(q - 1)^n. \]

Consider the polynomial \( f \in L_{P'} \), \( f := (x_{n+1} - a_1)(x_{n+1} - a_2) \cdots (x_{n+1} - a_{\ell}) \) with \( a_k \in \mathbb{F}_q^* \) all distinct. Then \( f \) has \( \ell(q - 1)^n \) zeros, so that \( N(P') = \ell(q - 1)^n \).
Therefore, \[ N(P') = \max\{N(P)(q-1), \ell(q-1)^n\}. \]

Thus, 
\[
d(C_{P'}) = \min\{(q-1)^{n+1} - N(P)(q-1), (q-1)^{n+1} - \ell(q-1)^n\} = \min\{(q-1)d(C_P), (q-1)^n(q-1-\ell)\},
\]

and
\[
\delta(P') = \min\{\delta(P), \delta([0, \ell])\}.
\]

\[\square\]

**Remark 3.10.** We conjecture that the the minimum of \((q-1-i)d(C_{P_i})\), as \(i\) ranges from 0 to \(\ell\), is always achieved at either \(i = 0\) or \(i = \ell\). In fact, numerous examples suggest that the sequence \(\{(q-1-i)d(C_{P_i})\}\) is unimodal: either it becomes (with minimum at \(i = 0\)), decreases (with minimum at \(i = \ell\)), or it increases until \(i = k\), and then decreases, hence achieving a minimum at either \(i = 0\) or \(i = \ell\).

We next prove a special case in which the condition of Proposition 3.9 holds.

**Proposition 3.11.** Let \(P \subseteq [0, q-2]^n\) be an integral convex polytope that contains the origin and \(P' = P \oplus [0, \ell]\), where \(\ell \leq q-2\). We additionally assume the following:

1. there is a polynomial \(f \in \mathcal{L}_P\) with \(Z(f) = N(P)\) that is given by a line segment through the origin of lattice length \(m\), and
2. in each \(P_i\), there is a polynomial \(f_i \in \mathcal{L}_{P_i}\) with \(Z(f_i) = N(P_i)\) that is given by that same line segment, but scaled to be in \(P_i\).

Then \((q-1-i)d(C_{P_i})\), as \(i\) ranges from 0 to \(\ell\), achieves a minimum at either \(i = 0\) or \(i = \ell\).

**Proof.** By our first assumption on \(P\), we have that 
\[
d(C_P) = (q-1)^{n-1}(q-1-m),
\]
and by the second assumption on the \(P_i\), we have that 
\[
d(C_{P_i}) = (q-1)^{n-1}(q-1-\left\lfloor \frac{\ell-i}{\ell} \cdot m \right\rfloor).
\]

We want to show that \((q-1-i)d(C_{P_i}) = (q-1-i)(q-1)^{n-1}(q-1-\left\lfloor \frac{\ell-i}{\ell} \cdot m \right\rfloor)\) attains a minimum at either \(i = 0\) or \(i = \ell\).

Consider the function \(G(x) = (q-1)^{n-1}(q-1-x)(q-1-\left(\frac{\ell-x}{\ell}\right)x)\), then \(G'(x) = (q-1)^{n-1}((q-1)(\frac{m}{\ell} - 1) + m - 2(\frac{m}{\ell})x)\) and when \(G'(x) = 0\), we have
\[
x = \frac{((q-1)(\frac{m}{\ell} - 1) + m)\ell}{2m}.
\]

Since \(G''(x) = -2\), \(x_0 := \frac{((q-1)(\frac{m}{\ell} - 1) + m)\ell}{2m}\) is a maximum of \(G(x)\). We next consider three different cases: when \(x_0 \leq 0\), \(x_0 > \ell\), and \(0 < x_0 < \ell\).

- If \(x_0 \leq 0\), then \(G(x)\) is decreasing on \([0, \ell]\), hence \((q-1-i)d(C_{P_i})\) is decreasing on \([0, \ell]\), and attains a minimum at \(i = \ell\).
- If \(x_0 > \ell\), then \(G(x)\) is increasing on \([0, \ell]\), hence \((q-1-i)d(C_{P_i})\) is increasing on \([0, \ell]\), and attains a minimum at \(i = 0\).
- If \(0 < x_0 < \ell\), then \(G(x)\) is increasing until \(x_0\) and then decreasing, this tells us that \((q-1-i)d(C_{P_i})\) is unimodal, and in particular attains a minimum at either \(i = \ell\) or \(i = 0\).

\[\square\]

**Example 3.12.** We can use Propositions 3.9 and 3.11 to find the relative minimum distance of infinite families of toric codes. For example, consider the family of simplices where \(P_0 = [0, \ell_0]\) and 
\[
P_i = P_{i-1} \oplus [0, \ell_i]
\]
where \(0 \leq \ell_i \leq q-2\) for all \(i\). The minimum distance of these general simplices was previously computed by Little and Schwarz using Vandermonde matrices [5, Theorem 2]. By Theorem 2.12, the minimum distance of each \(P_i \subseteq [0, q-2]^{n_i}\) is given by \((q-1)^{n_i} - \ell(q-1)^{n_i-1}\), where \(\ell = \max\{\ell_j\}\). Then setting \(P_{i+1} = P_i \oplus [0, \ell_{i+1}]\),
we have that each cross-section, $P_{i_j}$, satisfies assumption (2) of Proposition 3.11. Thus each $P_i$ satisfies the criteria for $P$ given in Proposition 3.11. Therefore, if $\ell = \max_{0 \leq j \leq i} \{\ell_j\}$ then

$$\delta(P_i) = \min\{\delta(P_{i-1}), \delta([0, \ell_i])\} = \min_{0 \leq j \leq i} \left\{ 1 - \frac{\ell_j}{q-1} \right\} = 1 - \frac{\ell}{q-1}.$$ 

We next work through a set of concrete examples in which Propositions 3.9 and 3.11 apply, and we illustrate how the behavior of $(q-1-i)d(C_{P_i})$ can change when the original polytope $P$ is fixed and $\ell$ varies.

**Example 3.13.** Consider the polytope $P := \text{conv}((0,0),(2,3),(4,2)) \subseteq \mathbb{R}^2$ which has seven lattice points:

![Figure 4. The polytope $P$.](image)

Note that $N(P) = 2(q-1)$, and $f = (xy - a)(xy - b) = x^2y^2 - (b + a)xy + ab$ for $a \neq b \in \mathbb{F}_q$, is a polynomial that has $N(P)$ zeros, corresponding to a line through the origin of lattice length 2.

We first analyze the direct sum of $P$ and $[0, \ell]$ when $\ell = 5$; let $P' = P \oplus [0, 5] \subseteq \mathbb{R}^3$, which has 18 lattice points. In this case the slices $P_i \subseteq \mathbb{R}^2$ are:

| $i$ | $P' \cap \{z = i\}$ | $P_i$ |
|-----|-----------------|------|
| 0   | $P = \text{conv}((0,0),(2,3),(4,2))$ | $P_0 = P$ |
| 1   | $\{(0,0,1),(2,1,1),(1,1,1),(2,2,1)\}$ | $P_1 = \text{conv}((0,0),(2,1),(1,1),(2,2))$ |
| 2   | $\{(0,0,2),(2,1,2),(1,1,2)\}$ | $P_2 = \text{conv}((0,0),(2,1),(1,1))$ |
| 3   | $\{((0,0,3),(1,1,3)\}$ | $P_3 = \text{conv}((0,0),(1,1))$ |
| 4   | $\{(0,0,4)\}$ | $P_4 = \{(0,0)\}$ |
| 5   | $\{(0,0,5)\}$ | $P_5 = \{(0,0)\}$ |

![Figure 5. (a) Polytope $P_1$ (b) Polytope $P_2$ (c) Polytope $P_3$ (d) Polytope $P_4 = P_5$.](image)

We then compute the minimum distances of each $C_{P_i}$:
| $i$ | $d(C_{P_i})$ | $(q - 1 - i)d(C_{P_i})$ |
|-----|-------------|----------------------|
| 0   | $(q - 1)^2 - 2(q - 1)$ | $(q - 1)^2(q - 3)$ |
| 1   | $(q - 1)^2 - 2(q - 1)$ | $(q - 1)(q - 2)(q - 3)$ |
| 2   | $(q - 1)^2 - (q - 1)$  | $(q - 1)(q - 2)(q - 3)$ |
| 3   | $(q - 1)^2 - (q - 1)$  | $(q - 1)(q - 2)(q - 4)$ |
| 4   | $(q - 1)^2$             | $(q - 1)^2(q - 5)$    |
| 5   | $(q - 1)^2$             | $(q - 1)^2(q - 6)$    |

Here we see that $(q - 1 - i)d(C_{P_i})$ is decreasing, so the minimum occurs when $i = 5$, and by Proposition 3.9

$$d(C_{P_{3^k}}) = (q - 1)^3 - 5(q - 1)^2.$$  

We next perform a similar analysis, keeping $P$ the same but varying $\ell$. Let $Q' := P \oplus [0, 3] \subset \mathbb{R}^3$, and for $0 \leq i \leq 3$ denote the slices by $Q_i$.

| $i$ | $Q_i$                      | $d(C_{Q_i})$ | $(q - 1 - i)d(C_{Q_i})$ |
|-----|----------------------------|-------------|----------------------|
| 0   | $P$                        | $(q - 1)^2 - 2(q - 1)$ | $(q - 1)^2(q - 3)$ |
| 1   | conv((0, 0), (2, 1), (1, 1)) | $(q - 1)(q - 2)$    | $(q - 1)(q - 2)^2$ |
| 2   | $\{(0, 0)\}$              | $(q - 1)^2$    | $(q - 1)^2(q - 3)$ |
| 3   | $\{(0, 0)\}$              | $(q - 1)^2$    | $(q - 1)^2(q - 4)$ |

With this value of $\ell$, we see that $(q - 1 - i)d(C_{Q_i})$ increases from $i = 0$ to $i = 1$, and then decreases, attaining a minimum at $i = 3$.

**Example 3.14.** There are certainly examples that don’t satisfy the assumptions of Proposition 3.11. For example, let $P \subset \mathbb{R}^2$ be the square $[0, 3] \times [0, 3]$, then a polynomial $f \in \mathcal{L}_P$ with maximum zeros is $f = (x - a_1)(x - a_2)(y - b_1)(y - b_2)$ for distinct $a_i \in \mathbb{F}_q^*$ and $b_i \in \mathbb{F}_q^*$, which corresponds to the full box, not just a line segment. If $P' = P \oplus [0, \ell]$, then each slice $P_i$ will also be a square of the form $[0, i] \times [0, i]$ for $0 \leq i \leq 3$, and hence each polynomial $f_i \in \mathcal{L}_{P_i}$ with maximum number of zeros is not given by a line segment. However, the conditions of Proposition 3.9 are met for any $\ell$; that is, the minimum of $(q - 1 - i)d(C_{P_i})$, as $i$ ranges from $0$ to $\ell$ is achieved at $i = 0$ or $i = \ell$. Indeed, if $\ell \leq 3$, the sequence $\{(q - 1 - i)d(C_{P_i})\}$ increases for $i = 0$ to $i = \ell$; and if $\ell > 3$ the sequence $\{(q - 1 - i)d(C_{P_i})\}$ increases and then decreases.

3.2. Examples of infinite families of toric codes. We now give examples of infinite families of toric codes constructed using the polytope operations defined above along with some other common operations. Both the family of boxes and simplices are considered in [10, Theorem 3.1] and [10, Proposition 4.1], but as one general construction. To clarify what fails about each family, we separate them.

3.2.1. Family of Boxes. Let $P_0 = [0, \ell_0]$ and $P_i = P_{i-1} \times [0, \ell_i]$ where $0 \leq \ell_i \leq q - 2$ for all $i$.

**Proposition 3.15.** The infinite family of toric codes corresponding to this sequence of boxes has either $\delta(P_i) \rightarrow 0$ or $R(P_i) \rightarrow 0$.

**Proof.** By Theorem 2.13, the infinite family of toric codes corresponding to this sequence of polytopes has

$$\delta(P_i) = \delta(P_{i-1}) \left(1 - \frac{\ell_i}{q - 1}\right)$$

and by counting lattice points

$$R(P_i) = R(P_{i-1}) \frac{\ell_i + 1}{q - 1}$$

We will show that either $\delta(P_i) \rightarrow 0$ or $R(P_i) \rightarrow 0$. First, assume that $\delta(P_i) \not\rightarrow 0$. This implies that all but finitely many $\ell_i = 0$ (if infinitely many $\ell_i \neq 0$ then $\delta(P_i)$ is comparable to a geometric sequence of rate $(q - 2)/(q - 1) < 1$ and thus converges to 0). But with this restriction on $\ell_i$, $R(P_i) \rightarrow 0$ as $i \rightarrow \infty$ (since now $R(P_i)$ is comparable to a geometric sequence of rate 1/(q - 1)). Similarly, assume $R(P_i) \not\rightarrow 0$. This implies
that all but finitely many \( \ell_i = q - 2 \), but then \( \delta \to 0 \) as \( i \to \infty \). Therefore, there is no way that neither \( \delta(P_i) \) nor \( R(P_i) \) tends to 0 as \( i \to \infty \), so this infinite family of toric codes is not a good infinite family. \( \square \)

3.2.2. Family of Simplices. Consider the family of simplices where \( P_0 = [0, \ell_0] \) and
\[
P_i = P_{i-1} \oplus [0, \ell_i]
\]
where \( 0 \leq \ell_i \leq q - 2 \) for all \( i \).

**Proposition 3.16.** The infinite family of toric codes corresponding to this sequence of simplices has \( \delta(P_i) \geq \delta > 0 \) for some \( 0 < \delta \leq 1 \).

**Proof.** From Example 3.12, we have that
\[
\delta(P_i) = \min\{\delta(P_{i-1}), \delta([0, \ell_i])\} = \min_{j \leq i} \left\{1 - \frac{\ell_j}{q-1}\right\}.
\]
Since \( \ell_i \leq q - 2 \) for all \( i \), we know that
\[
\delta(P_i) \geq 1 - \frac{q-2}{q-1} = \frac{1}{q-1} > 0.
\]
In particular, for any construction of \{\( P_i \), if \( \delta(P_i) \) converges, it will converge to some strictly positive value. \( \square \)

**Proposition 3.17.** The infinite family of toric codes corresponding to this sequence of simplices has \( R(P_i) \to 0 \).

**Proof.** We know that
\[
R(P_i) \leq \frac{1}{(q-1)^i} \left( \frac{i + \ell}{\ell} \right) \leq \frac{1}{(q-1)^i} \left( \frac{i + \ell}{\ell} \right)^\ell
\]
where \( \ell = \max \ell_i \). Note
\[
\lim_{i \to \infty} \frac{1}{(q-1)^i} \left( \frac{i + \ell}{\ell} \right)^\ell \to 0
\]
since its numerator is a polynomial in \( i \) and its denominator is an exponential in \( i \). Thus \( R(P_i) \to 0 \) as well.

Since \( R(P_i) \to 0 \) for any construction of \{\( P_i \), this infinite family of simplices also fails to be a good infinite family of toric codes.

3.2.3. Iteratively Taking the Join of a Polytope with Itself. Let \( P \subseteq [0, q-2]^n \) be an integral convex polytope which contains a length two lattice segment or a unit square, and let \( P := P \). For \( k \geq 0 \), define \( P_{k+1} := P_k \ast P_k \). For brevity let \( \delta_k = \delta(P_k) \). By Corollary 3.7,
\[
\delta_{k+1} = \min \left\{ \delta_k, 2\delta_k - \delta_k^2 \left( \frac{q}{q-1} \right) \right\}.
\]

**Proposition 3.18.** For the sequence given above and for \( k \geq 2 \),
\[
\delta_k = \delta_1.
\]

**Proof.** Consider the function of a real variable for \( q > 2 \)
\[
f(x) = 2x - x^2 \frac{q}{q-1}.
\]
Using calculus, we can find the maximum value of this function by considering the first and second derivatives:
\[
f'(x) = 2 - 2x \frac{q}{q-1}, \quad f''(x) = -2 \frac{q}{q-1}.
\]
We see that \( f'(x) = 0 \) exactly when \( x = \frac{q-1}{q} \), and since \( q \geq 1 \), we have \( f''(x) < 0 \) for all \( x \in \mathbb{R} \). Thus, by the second derivative test, \( f \) attains a local maximum at \( x = \frac{q-1}{q} \), whose value is
\[
f \left( \frac{q-1}{q} \right) = 2 \frac{q-1}{q} - \left( \frac{q-1}{q} \right)^2 \frac{q}{q-1} = \frac{q-1}{q}.
\]
We see that this is in fact the global maximum of $f$ since $f$ is a parabola.

Further, we see that $f(x) \geq 0$ whenever $0 \leq x \leq 1$. This follows from the fact that $f$ is concave down with $f(0) = f \left( \frac{2q-1}{q} \right) = 0$ and $0 \leq 1 \leq 2\frac{q-1}{q}$.

Finally, notice for $0 \leq x \leq \frac{q-1}{q}$ we have $x \frac{q}{q-1} \leq 1$ and so

$$f(x) - x = x - x^2 \frac{q}{q-1} = x \left( 1 - x \frac{q}{q-1} \right) \geq x(1-1) = 0.$$ 

That is, whenever $0 \leq x \leq \frac{q-1}{q}$ we have, $f(x) \geq x$.

Thus our sequence, in terms of the function $f$, is given by

$$\delta_{k+1} = \min\{\delta_k, f(\delta_k)\}.$$ 

Recall that by the properties of $\delta$ we have $0 \leq \delta_0 \leq 1$. Thus, since $0 \leq f(\delta_0) \leq \frac{q-1}{q}$, we have

$$0 \leq \delta_1 \leq \min\{\delta_0, \frac{q-1}{q}\} \leq \frac{q-1}{q}.$$ 

It follows from the above argument that $f(\delta_1) \geq \delta_1$ so that

$$0 \leq \delta_2 = \delta_1 \leq \frac{q-1}{q}.$$ 

Continuing by induction we have for all $k \geq 2$

$$\delta_k = \delta_1.$$ 

\[ \square \]

Similar to the previous infinite family of codes, we see that by appropriately setting $\delta_0$, we can get $\delta_k \to \delta > 0$. But again, recall that the number of lattice points of the join of two polytopes is the sum of the number of lattice points of those two polytopes. Then for this family of codes, we have that

$$R(P_{k+1}) = \frac{2(\dim(C_{P_j}))}{(q-1)^{2n_k+1}} \leq \frac{2(q-1)^{n_k}}{(q-1)^{2n_k+1}} = \frac{2}{(q-1)^{n_k+1}} \to 0.$$ 

Therefore, it fails to be a good family since $R(P_k) = 0$.

4. No Good Family

4.1. Unit Hypercubes. Given the examples above, we have a strong reason to believe that there is no good infinite family of toric codes as Soprunov and Soprunova describe. In order to formalize this reasoning, we introduce the following definition.

**Definition 4.1.** Let $P$ be an integral convex polytope. We define $M(P)$ to be the largest integer $m$ such that an $m$-dimensional unit hypercube is a subset of $P$, up to a unimodular affine transformation. That is

$$M(P) = \max\{ m \mid \exists \text{ a unimodular affine transformation, } A, \text{ such that } A([0,1]^m) \subseteq P \}.$$ 

**Proposition 4.2.** Let $\{P_j\}$ be an infinite family of toric codes, and suppose the sequence $\{M(P_j)\}$ is unbounded. Then if $\delta(P_j)$ converges, it converges to zero.

**Proof.** The proof of this follows from Proposition 2.11 and the value of $\delta([0,1]^k)$. As usual, let us work over $\mathbb{F}_q$.

Let $\epsilon > 0$ be arbitrary and consider a subsequence $\{P_{j}\}$ such that $M(P_{j}) \to \infty$ as $j \to \infty$. Since $M(P_j) \to \infty$, there exists an $N$ such that whenever $j \geq N$

$$\left(\frac{q-2}{q-1}\right)^{M(P_j)} < \epsilon$$

(this exists because $0 \leq \frac{q-2}{q-1} < 1$). By the definition of $M$, there exists a unimodular affine transformation $A$ such that $A([0,1]^{M(P_j)}) \subseteq P_j$. By Theorem 2.13, we have that for all $j \geq N$

$$\delta([0,1]^{M(P_j)}) = \left(\frac{q-2}{q-1}\right)^{M(P_j)}$$
Since unimodular affine transformations preserve $\delta$, we have

$$0 \leq \delta(P_j) \leq \delta(A[0, 1]^M(P_j)) = \delta([0, 1]^M(P_j)) = \left(\frac{q - 2}{q - 1}\right)^M < \epsilon.$$  

That is $\delta(P_j) \to 0$ as $j \to \infty$. Since this subsequence $\delta(P_j)$ converges to zero, and $\delta(P_i)$ converges by assumption, we have that $\delta(P_i)$ converges to zero. □

Note that this means that any family with unbounded $M(P_i)$ cannot be good, since $\delta(P_i)$ cannot converge to a strictly positive value.

Thus, we have a simple characterization of some families whose $\delta = 0$. We conjecture the following:

**Conjecture 4.3.** Let $\{P_i\}$ be an infinite family of toric codes, and suppose the sequence $\{M(P_i)\}$ is bounded. Then $R(P_i) \to 0$ as $i \to \infty$.

The intuition behind this conjecture is as follows: suppose for some infinite family $\{P_i\}$, $\{M(P_i)\}$ is bounded by $M$. By definition this means that no $P_i$ contains a lattice equivalent copy of $[0, 1]^{M+1}$. Intuitively, this means that each $P_i$ “does not take up any $(M + 1)$-dimensional volume.” We might expect that since we know each $P_i \subseteq [0, q - 2]^n$, that $R(P_i) \sim (q - 1)^M$. However, we could not find a way to verify this intuition.

Instead, we’ll relate our above characterization to the Minkowski length.

### 4.2. Minkowski Length

Recall that if $P$ and $Q$ are integral convex polytopes, we can construct a new integral convex polytope using the Minkowski sum.

**Definition 4.4.** Let $P$ and $Q$ be convex polytopes in $\mathbb{R}^n$. Their **Minkowski sum** is

$$P + Q = \{p + q \in \mathbb{R}^n \mid p \in P, q \in Q\}$$

which is again a convex polytope.

Figure 6 shows the Minkowski sum of two unit line segments in $\mathbb{R}^2$:

![Figure 6. The Minkowski Sum of Lattice Segments.](image)

Let $P$ be an integral convex lattice polytope in $\mathbb{R}^n$. Consider a Minkowski decomposition

$$P = P_1 + \ldots + P_\ell$$

into lattice polytopes $P_i$ of positive dimension (that is, none of the $P_i$ are single points). Clearly, there are only finitely many such decompositions. We let $\ell(P)$ be the largest number of summands in such decompositions of $P$, and call it the **Minkowski length** of $P$.

**Definition 4.5.** [9, Def. 1.1] The **full Minkowski length** of $P$ is the maximum of the Minkowski lengths of all subpolytopes $Q$ in $P$,

$$L(P) := \max\{\ell(Q) \mid Q \subseteq P\}.$$  

**Example 4.6.** Figure 7 illustrates the Minkowski lengths of various polytopes.
Figure 7. Polytopes with full Minkowski Lengths (a) 1 (b) 2 (c) 3. The interior polytope represents a subpolytope of the largest Minkowski sum.

Proposition 4.7. Let \( \{P_i\} \) be an infinite family of toric codes. Then \( \{M(P_i)\} \) is unbounded if and only if \( \{L(P_i)\} \) is unbounded.

Proof. We first assume that \( \{M(P_i)\} \) is unbounded and let \( N \in \mathbb{N} \) be arbitrary. Since \( \{M(P_i)\} \) is unbounded, there exists an \( i \) such that \( M(P_i) > N \). In other words, there is a lattice equivalent copy of \([0,1]^{M(P_i)}\) as a subpolytope of \( P_i \). We know that the full Minkowski length is invariant under lattice equivalence [9, Prop 1.2] and thus, since \( [0, e_1] + \ldots + [0, e_{M(P_i)}] = [0,1]^{M(P_i)} \subseteq P_i \),

we have

\[
L(P_i) \geq \ell([0,1]^{M(P_i)}) \geq M(P_i) > N.
\]

Since \( N \) was arbitrary, we see that \( \{L(P_i)\} \) is unbounded.

For the reverse implication, suppose \( \{L(P_i)\} \) is unbounded, and let \( N \in \mathbb{N} \) be arbitrary. Since \( \{L(P_i)\} \) is unbounded there exists an \( i \) such that \( L(P_i) > (q - 2)N \). That is \( P_i \) has a subpolytope \( Q \) that can be written as

\[
Q = Q_1 + \ldots + Q_{(q-2)N}
\]

where the dimension of each \( Q_\ell \) is at least one. Note that this means each \( Q_\ell \) has at least two points in it, and thus has a lattice equivalent copy of \([0,1]^1\) as a subpolytope.

For each \( Q_\ell \), let \( v_\ell \) be a direction in which \( Q_\ell \) has span. That is, two distinct points \( p, q \in Q_\ell \) differ by exactly \( v_\ell \); or \( p + v_\ell = q \).

Consider the multi-set \( V \) given by the sum over \( \{v_\ell\} \) when considered as multi-sets:

\[
V = \sum_{\ell=1}^{N(q-2)} \{v_\ell\} / \sim,
\]

where \( v \sim w \) if there exists a scaling \( \lambda \in \mathbb{R} \) such that \( \lambda v = w \). We see that since we are summing over singletons the cardinality of this multi-set is exactly \( |V| = N(q-2) \).

Further, notice that if \( q - 1 \) of the \( Q_\ell \) share the same \( v_\ell \) (or a scaling thereof), then their Minkowski sum \( Q \) is wider than \( q - 2 \) in one of the component directions of the shared \( v \). However, this is explicitly forbidden since each \( P_j \) lives in \([0, q-2]|^v\). That is, the multiplicity of \( v/ \sim \) in \( V \), \( m(v) \), is at most \( (q-2) \).

Putting these together, we see that

\[
N(q-2) = |V| = \sum_{v \in \text{Supp}(V)} m(v) \leq \sum_{v \in \text{Supp}(V)} (q-2) \leq (q-2)|\text{Supp}(V)|
\]

That is

\[
N \leq |\text{Supp}(V)|.
\]
In other words, at least \( N \) distinct direction vectors appear in \( V \). Thus, since the \( \{ Q_i \} \) mutually contain \( N \) different direction vectors, their Minkowski sum will contain a lattice equivalent copy of \([0,1]^N\). That is \( M(P_i) \geq N \), which completes the proof.

\[ \square \]

**Corollary 4.8.** Let \( \{ P_i \} \) be an infinite family of toric codes. Then if \( \{ L(P_i) \} \) is unbounded and \( \delta(P_i) \) converges, \( \delta(P_i) \to 0 \) as \( i \to \infty \).

**Proof.** This follows directly from Propositions 4.2 and 4.7. \[ \square \]

Now we can reformulate Conjecture 4.3 in terms of the Minkowski length:

**Conjecture 4.9.** Let \( \{ P_i \} \) be an infinite family of toric codes, and suppose the sequence \( \{ L(P_i) \} \) is bounded. Then \( R(P_i) \to 0 \) as \( i \to \infty \).

We next examine some special cases under which the conjecture holds:

**Proposition 4.10.** Let \( \{ P_i \} \) be an infinite family of toric codes over \( \mathbb{F}_q \). If a tail of \( \{ L(P_i) \} \) is bounded above by \( (q-3) \) then \( R(P_i) \to 0 \) as \( i \to \infty \).

**Proof.** This result follows from [6] which states a simple bound for the number of lattice points of \( P \subseteq \mathbb{R}^n \) with full Minkowski length \( L \) as

\[
|P \cap \mathbb{Z}^n| \leq (L+1)^n.
\]

Since a tail of \( \{ L(P_i) \} \) is bounded by \( (q-3) \) there exists an integer \( N \), such that whenever \( i \geq N \), \( L(P_i) \leq q-3 \). We have for all \( i \geq N \)

\[
R(P_i) = \frac{|P_i \cap \mathbb{Z}^{n_i}|}{(q-1)^{n_i}} \leq \frac{(L(P_i)+1)^{n_i}}{(q-1)^{n_i}} \leq \left( \frac{q-2}{q-1} \right)^{n_i}.
\]

Thus, we see a tail of \( \{ R(P_i) \} \) converges to 0 and so \( R(P_i) \to 0 \) as \( i \to \infty \). \[ \square \]

Note that the proof of Proposition 4.10 does not take into account the restriction that each \( P_i \subseteq [0,q-2]^{n_i} \).

**Proposition 4.11.** Let \( \{ P_i \} \) be an infinite family of toric codes. Suppose each \( P_i \) is exactly the Minkowski sum of some number of unit lattice simplices (of any dimension not greater than \( n_i \)). If \( \{ L(P_i) \} \) is bounded then \( R(P_i) \to 0 \) as \( i \to \infty \).

**Proof.** Given two integral convex polytopes \( P \) and \( Q \) in \( \mathbb{R}^n \), the Minkowski sum of \( P \) and \( Q \) has at most \( ([P \cap \mathbb{Z}^n] + [Q \cap \mathbb{Z}^n]) \) points. Since a unit simplex of dimension \( m \) has exactly \( (m+1) \) points, it follows that \( |P_i \cap \mathbb{Z}^{n_i}| \leq (n_i+1)^{L(P_i)} \), given that \( P_i \) is exactly the Minkowski sum of unit simplices. Let \( L \) be such that for all \( i \), \( L(P_i) \leq L \), then

\[
0 \leq R(P_i) \leq \frac{(n_i+1)^{L(P_i)}}{(q-1)^{n_i}} \leq \frac{(n_i+1)^L}{(q-1)^{n_i}}.
\]

Since \( n_i \to \infty \) as \( i \to \infty \) we see that \( R(P_i) \to 0 \) by the Squeeze Theorem. \[ \square \]

5. **Future Directions**

We end by posing a few open problems.

**Problem 5.1.** Prove or disprove Conjecture 4.9 (which is equivalent to Conjecture 4.3).

If Conjecture 4.9 is true, then we may consider the following: define an \( \epsilon \) good code \( C_R \) to be a toric code such that \( \delta(P) \geq \epsilon \) and \( R(P) \geq \epsilon \). Conjecture 4.9 claims that as the dimension of \( P \) goes to infinity there are no \( \epsilon \) good codes for any \( \epsilon > 0 \). This means that for every \( \epsilon > 0 \) there exists a largest dimension \( N \) for which there exists an \( \epsilon \) good code.

**Problem 5.2.** For each \( \epsilon > 0 \), what is the \( N \) after which no code of dimension \( n \geq N \) is \( \epsilon \) good?

The answer to this question also answers “If I want an optimal code (so that \( R + \delta = 1 \)) where \( \delta \) (or \( R \)) is no worse than \( \epsilon \), what is the largest dimension I can use?”
ACKNOWLEDGEMENTS

This research was completed at the REU Site: Mathematical Analysis and Applications at the University of Michigan-Dearborn. We would like to thank the National Science Foundation (DMS-1950102); the National Security Agency (H98230-21); the College of Arts, Sciences, and Letters; and the Department of Mathematics and Statistics for their support. We also would like to thank the anonymous referee for their constructive feedback.

REFERENCES

[1] Johan P. Hansen, Toric surfaces and error-correcting codes, Coding theory, cryptography and related areas, 2000, pp. 132–142.
[2] Toric varieties Hirzebruch surfaces and error-correcting codes, Appl. Algebra Engrg. Comm. Comput. 13 (2002), no. 4, 289–300. MR1953195
[3] David Joyner, Toric codes over finite fields, Appl. Algebra Engrg. Comm. Comput. 15 (2004), no. 1, 63–79. MR2142431
[4] John Little and Hal Schenck, Toric surface codes and Minkowski sums, SIAM J. Discrete Math. 20 (2006), no. 4, 999–1014. MR2272243
[5] John Little and Ryan Schwarz, On toric codes and multivariate Vandermonde matrices, Appl. Algebra Engrg. Comm. Comput. 18 (2007), no. 4, 349–367. MR2322944
[6] Kyle Meyer, Ivan Soprunov, and Jenya Soprunova, On the number of $F_q$-zeros of families of sparse trivariate polynomials, 2021.
[7] Diego Ruano, On the parameters of $r$-dimensional toric codes, Finite Fields Appl. 13 (2007), no. 4, 962–976. MR2366532
[8] Ivan Soprunov, Lattice polytopes in coding theory, J. Algebra Comb. Discrete Struct. Appl. 2 (2015), no. 2, 85–94. MR3345995
[9] Ivan Soprunov and Jenya Soprunova, Toric surface codes and Minkowski length of polygons, SIAM J. Discrete Math. 23 (2008/09), no. 1, 384–400. MR2476837
[10] Bringing toric codes to the next dimension, SIAM J. Discrete Math. 24 (2010), no. 2, 655–665. MR2661429

Mallory Dolorfino, Kalamazoo College, Kalamazoo, Michigan, USA, mallory.dolorfino19@kzoo.edu
Cordelia Horch, Occidental College, Los Angeles, California, USA, chorchoxy.edu
Kelly Jabbusch, Department of Mathematics & Statistics, University of Michigan–Dearborn, Dearborn, Michigan, USA, jabbusch@umich.edu
Ryan Martinez, Harvey Mudd College, Claremont, California, USA, rmmartinez@hmc.edu