A nonlinear model for long memory conditional heteroscedasticity

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Abstract. We discuss a class of conditionally heteroscedastic time series models satisfying the equation \( r_t = \zeta_t \sigma_t \), where \( \zeta_t \) are standardized i.i.d. r.v.’s and the conditional standard deviation \( \sigma_t \) is a nonlinear function \( Q \) of inhomogeneous linear combination of past values \( r_s, s < t \) with coefficients \( b_j \). The existence of stationary solution \( r_t \) with finite \( p \)th moment, \( 0 < p < \infty \) is obtained under some conditions on \( Q, b_j \) and \( p \)th moment of \( \zeta_0 \). Weak dependence properties of \( r_t \) are studied, including the invariance principle for partial sums of Lipschitz functions of \( r_t \). In the case of quadratic \( Q^2 \), we prove that \( r_t \) can exhibit a leverage effect and long memory, in the sense that the squared process \( r_t^2 \) has long memory autocorrelation and its normalized partial sums process converges to a fractional Brownian motion.

Keywords: ARCH model, leverage, long memory, Donsker’s invariance principle

1 Introduction

A stationary time series \( \{r_t, t \in \mathbb{Z}\} \) is said \textit{conditionally heteroscedastic} if its conditional variance \( \sigma^2_t = \text{Var}[r_t | r_s, s < t] \) is a non-constant random process. A class of conditionally heteroscedastic ARCH-type processes is defined from a standardized i.i.d. sequence \( \{\zeta_t, t \in \mathbb{Z}\} \) as solutions of stochastic equation

\begin{equation}
    r_t = \zeta_t \sigma_t, \quad \sigma_t = V(r_s, s < t),
\end{equation}

where \( V(x_1, x_2, \ldots) \) is some function of \( x_1, x_2, \ldots \).

The ARCH(\( \infty \)) model corresponds to \( V(x_1, x_2, \ldots) = \sqrt{a + \sum_{j=1}^{\infty} b_j x_j^2} \), or

\begin{equation}
    \sigma^2_t = a + \sum_{j=1}^{\infty} b_j r_{t-j}^2,
\end{equation}

or
where \( a \geq 0, b_j \geq 0 \) are coefficients.

ARCH(\( \infty \)) models include both the well-known ARCH(\( p \)) and GARCH(\( p, q \)) models of Engle [12] and Bollerslev [5]. However, despite their tremendous success, the GARCH models are not able to capture some empirical features of asset returns, in particular, the asymmetric or leverage effect discovered by Black [4], and the long memory decay in autocorrelation of squares \( \{ r_t^2 \} \). Giraitis and Surgailis [15] proved that the squared stationary solution of the ARCH(\( \infty \)) model in (1.2) with \( a > 0 \) always has short memory, in the sense that \( \sum_{j=0}^{\infty} \text{Cov}(r_0^2, r_j^2) < \infty \). (However, in the case of integrated ARCH(\( \infty \)) models with \( \sum_{j=1}^{\infty} b_j = 1 \) and \( a = 0 \) the situation is different; see [13].)

The above shortcomings of the ARCH(\( \infty \)) model motivated numerous studies proposing alternative forms of the conditional variance and the function \( V(x_1, x_2, \ldots) \) in (1.1). In particular, stochastic volatility models can display both long memory and leverage except that in their case, the conditional variance is not a function of \( r_s, s < t \) alone and therefore it is more difficult to estimate from real data in comparison with the ARCH models; see Shephard and Andersen [27]. Sentana [26] discussed a class of Quadratic ARCH (QARCH) models with \( \sigma_t^2 \) being a general quadratic form in lagged variables \( r_{t-1}, \ldots, r_{t-p} \). Sentana’s specification of \( \sigma_t^2 \) encompasses a variety of ARCH models including the asymmetric ARCH model of Engle [13] and the linear ‘standard deviation’ model of Robinson [24]. The limiting case (when \( p = \infty \)) of the last model is the LARCH model discussed in [14] (see also [15], [3], [16], [29]) and corresponding to \( V(x_1, x_2, \ldots) = a + \sum_{j=1}^{\infty} b_j x_j \), or

\[
\sigma_t = a + \sum_{j=1}^{\infty} b_j r_{t-j},
\]

(1.3)

where \( a \in \mathbb{R}, b_j \in \mathbb{R} \) are real-valued coefficients. [14] proved that the squared stationary solution \( \{ r_t^2 \} \) of the LARCH model with \( b_j \) decaying as \( j^{d-1}, 0 < d < 1/2 \) may have long memory autocorrelations. The leverage effect in the LARCH model was discussed in detail in [16]. On the other hand, volatility \( \sigma_t \) [13] of the LARCH model may assume negative values, lacking some of the usual volatility interpretation.

The present paper discusses a class of conditionally heteroscedastic models [14] with \( V \) of the form

\[
V(x_1, x_2, \ldots) = Q\left(a + \sum_{j=1}^{\infty} b_j x_j \right)
\]

(1.4)

where \( Q(x), x \in \mathbb{R} \) is a (nonlinear) function of a single real variable \( x \in \mathbb{R} \) which may be separated from below by a positive constant. Linear \( Q(x) = x \) corresponds to the LARCH model [13]. Probably, the most interesting nonlinear case of \( Q \) in (1.4) is

\[
Q(x) = \sqrt{c^2 + x^2},
\]

where \( c \geq 0 \) is a parameter. In the latter case, the model is described by equations

\[
\begin{align*}
r_t &= \zeta_t \sigma_t, \\
\sigma_t &= \sqrt{c^2 + \left( a + \sum_{s<t} b_{t-s} r_s \right)^2}.
\end{align*}
\]

(1.5)
Note \( \sigma_t \geq c \geq 0 \) in (1.5) is nonnegative and separated from 0 if \( c > 0 \). Particular cases of volatility forms in (1.5) are:

\[
\begin{align*}
\sigma_t &= \sqrt{c^2 + (a + b r_{t-1})^2} \quad \text{(Engle’s [13] asymmetric ARCH(1))}, \\
\sigma_t &= \sqrt{c^2 + \left(a + \frac{b}{p} \sum_{j=1}^{p} r_{t-j}\right)^2}, \\
\sigma_t &= |a + \sum_{j=1}^{\infty} b_j r_{t-j}| \quad (Q(x) = |x|), \\
\sigma_t &= \sqrt{c^2 + (a + b(1 - L)^{-d} - 1)r_t^2}.
\end{align*}
\]

In (1.6)-(1.9), \( a, b, c \) are real parameters, \( p \geq 1 \) an integer, \( L x_t = x_{t-1} \) is the backward shift, and \( (1 - L)^{-d} x_t = \sum_{j=0}^{\infty} \varphi_j x_{t-j}, \varphi_j = \Gamma(d + j)/\Gamma(d) \Gamma(j + 1), \varphi_0 = 1 \) is the fractional integration operator, \( 0 < d < 1/2 \). The squared volatility (conditional variance) \( \sigma_t^2 \) in (1.5) and (1.6)-(1.9) is a quadratic form in lagged returns \( r_{t-1}, r_{t-2}, \ldots \) and hence represent a particular case of Sentana’s [26] QARCH model with \( p = \infty \) studied in [28]. It should be noted, however, that the first two conditional moments do not determine the unconditional distribution in general. Therefore, (1.1) with (1.5) is a different process from the QARCH model since the latter process satisfies a linear random-coefficient equation for \( \{r_t\} \), see [26], in contrast to the nonlinear equation in (1.1).

Let us describe the main results of this paper. Section 5.2 obtains sufficient conditions on \( Q, b_j \) and \( |\mu_p| := E[\zeta_0]^p \) for the existence of stationary solution of (1.1)-(1.4) with finite moment \( E[r_t]^p < \infty \), \( p > 0 \). We use the fact that the above equations can be reduced to a ‘nonlinear moving-average’ equation

\[ X_t = \sum_{s < t} b_{t-s} \zeta_s Q(a + X_s) \]

for the linear form \( X_t = \sum_{s < t} b_{t-s} r_s \) in (1.4), and vice-versa. Section 5.3 aims at providing weak dependence properties of (1.1) with \( V \) in (1.4), in particular, the invariance principle for processes \( \{h(r_t)\}, \{g(X_t)\} \) for Lipschitz functions \( h, g \) under the assumption that \( b_j \) are summable and decay as \( j^{-\gamma} \) with \( \gamma > 1 \). Section 5.4 discusses long memory property of the ‘square root’ model in (1.5). For \( b_j \sim \beta_j j^{-d-1}, j \to \infty, 0 < d < 1/2 \) as in (1.9), we prove that the squared process \( \{r_t^2\} \) has long memory autocorrelations and its normalized partial sums process tend to a fractional Brownian motion with Hurst parameter \( H = d + 1/2 \) (Theorem 10). Finally 5.5 establishes the leverage effect in spirit of [15] consisting in the fact that the ‘leverage function’ \( h_j := \text{Cov}(\sigma_t^2, r_{t-j}) \) of the model (1.5) takes negative values for \( j = 1, \ldots, k \) if provided the coefficients \( a \) and \( b_j, 1 \leq j \leq p \) have opposite signs: \( ab_1 < 0, ab_{j+1} \leq 0, j = 2, \ldots, k \) (Proposition 11). All proofs are collected in 6 (Appendix).

Notation. In what follows, \( C, C(\cdot) \) denote generic constants, possibly dependent on the variables in brackets, which may be different at different locations. \( a_t \sim b_t \ (t \to \infty) \) is equivalent to \( \lim_{t \to \infty} a_t/b_t = 1 \).
2 Stationary solution

This section discusses the existence of a stationary solution of (1.1) with $V$ of (1.4), viz.,

$$r_t = \zeta_t Q(a + \sum_{s<t} b_{t-s} r_s), \quad t \in \mathbb{Z}. \quad (2.10)$$

Denote

$$X_t := \sum_{s<t} b_{t-s} r_s. \quad (2.11)$$

Then $r_t$ in (2.10) can be written as $r_t = \zeta_t Q(a + X_t)$ while (2.11) can be written as

$$X_t = \sum_{s<t} b_{t-s} \zeta_s Q(a + X_s). \quad (2.12)$$

In other words, stationary solution of (2.10) can be defined via stationary solution of (2.12), and vice versa.

In this section we consider a general case of (2.10)-(2.12) when the innovations may have infinite variance.

More precisely, we assume that \{\zeta_t, t \in \mathbb{Z}\} are i.i.d. r.v.’s with finite moment $|\mu|_p := E|\zeta_t|^p < \infty, p > 0$. In this paper we often use the following moment inequality.

**Proposition 1** Let \{Y_j, j \geq 1\} be a sequence of r.v.’s such that $E|Y_j|^p < \infty$ for some $p > 0$. If $p > 1$ we additionally assume that \{Y_j\} is a martingale difference sequence: $E[Y_j|Y_1, \ldots, Y_{j-1}] = 0, j = 2, 3, \ldots$. Then there exists a constant $K_p \geq 1$ depending only on $p$ and such that

$$E\left|\sum_{j=1}^{\infty} Y_j\right|^p \leq K_p \left\{\sum_{j=1}^{\infty} E|Y_j|^p, \quad 0 < p \leq 2, \quad \left(\sum_{j=1}^{\infty} (E|Y_j|^p)^{2/p}\right)^{p/2}, \quad p > 2. \quad (2.13)\right.$$  

**Remark 1** For $0 < p \leq 1$ and $p = 2$, inequality (2.13) holds with $K_p = 1$, and for $1 < p < 2$, it is known as von Bahr and Essén inequality, see \[30\], which holds with $K_p = 2$. For $p > 2$, inequality (2.13) is a consequence of the Burkholder and Rosenthal inequality (see \[6\], \[25\]). Osękowski \[23\] proved that $K_p^{1/p} \leq 4(1 + 1)^{1/p}(1 + \frac{p}{\log(p/2)})$, in particular, $K_{1/4}^{1/4} \leq 32.207$. See also \[20\].

In Proposition 4 below, we assume that $Q$ in (2.10) is a Lipschitz function, i.e., there exists $c_Q > 0$ such that

$$|Q(x) - Q(y)| \leq c_Q |x - y|, \quad x, y \in \mathbb{R}. \quad (2.14)$$

Note (2.14) implies the bound

$$Q^2(x) \leq c_1^2 + c_2^2 x^2, \quad x \in \mathbb{R}, \quad (2.15)$$

where $c_1 \geq 0, c_2 \geq c_Q$ and $c_2$ can be chosen arbitrarily close to $c_Q$; in particular, (2.15) holds with $c_2^2 = (1 + \epsilon^2)c_Q^2, c_1^2 = Q^2(0)(1 + \epsilon^{-2})$, where $\epsilon > 0$ is arbitrarily small.
Let us give some formal definitions. Let $F = \sigma(\zeta, s \leq t), t \in \mathbb{Z}$ be the sigma-field generated by $\zeta, s \leq t$. A random process $\{u_t, t \in \mathbb{Z}\}$ is called adapted (respectively, predictable) if $u_t$ is $F_t$-measurable for each $t \in \mathbb{Z}$ (respectively, $u_t$ is $F_{t-1}$-measurable for each $t \in \mathbb{Z}$). Define

$$B_p := \begin{cases} \sum_{j=1}^{\infty} |b_j|^p, & 0 < p < 2, \\ \left(\sum_{j=1}^{\infty} b_j^2\right)^{p/2}, & p \geq 2. \end{cases}$$  \tag{2.16}$$

**Definition 2** Let $p > 0$ be arbitrary.

(i) By $L^p$-solution of (2.10) we mean an adapted process $\{r_t, t \in \mathbb{Z}\}$ with $E|r_t|^p < \infty$ such that for any $t \in \mathbb{Z}$ the series $\sum_{s<t} b_{t-s}r_s$ converges in $L^p$ and (2.10) holds.

(ii) By $L^p$-solution of (2.12) we mean an predictable process $\{X_t, t \in \mathbb{Z}\}$ with $E|X_t|^p < \infty$ such that for any $t \in \mathbb{Z}$ the series $\sum_{s<t} b_{t-s}\zeta Q(a+X_s)$ converges in $L^p$ and (2.12) holds.

Proposition 3 says that equations (2.10) and (2.12) are equivalent in the sense that by solving one the these equations one readily obtains a solution to the other one.

**Proposition 3** Let $Q$ be a measurable function satisfying (2.15) with some $c_1, c_2 \geq 0$ and $\{\zeta_t\}$ be an i.i.d. sequence with $|\mu|_p = E|\zeta_0|^p < \infty$ and satisfying $E\zeta_0 = 0$ for $p > 1$. In addition, assume $B_p < \infty$.

(i) Let $\{X_t\}$ be a stationary $L^p$-solution of (2.12). Then $\{r_t := \zeta_t Q(a+X_t)\}$ is a stationary $L^p$-solution of (2.10) and

$$E|r_t|^p \leq C(1 + E|X_t|^p).$$  \tag{2.17}$$

Moreover, for $p > 1$, $\{r_t, F_t, t \in \mathbb{Z}\}$ is a martingale difference sequence with

$$E[r_t|F_{t-1}] = 0, \quad E[|r_t|^p|F_{t-1}] = |\mu|_p|Q(a + \sum_{s<t} b_{t-s}r_s)|^p. \tag{2.18}$$

(ii) Let $\{r_t\}$ be a stationary $L^p$-solution of (2.10). Then $\{X_t\}$ in (2.11) is a stationary $L^p$-solution of (2.12) such that

$$E|X_t|^p \leq CE|r_t|^p. \tag{2.19}$$

Moreover, for $p \geq 2$

$$E[X_tX_0] = Er_0^2 \sum_{s=1}^{\infty} b_{t+s}b_s, \quad t = 0, 1, \ldots. \tag{2.20}$$

The following proposition obtains a sufficient condition in (2.21) for the existence of a stationary $L^p$-solution of equations (2.12) and (2.10). Condition (2.21) involves the $p$th moment of innovations, the Lipschitz constant $c_Q$, the sum $B_p$ in (2.16) and the Rosenthal constant $K_p$ in (2.13). Part (ii) of Proposition 3 shows that for $p = 2$, condition (2.21) is close to optimal, being necessary in the case of quadratic function $Q^2$. For $p > 2$ condition (2.21) becomes more restrictive and less sharp since the optimal bound on $K_p^{1/p}$ is unknown.
Proposition 4 Let the conditions of Proposition 3 be satisfied, \( p > 0 \) is arbitrary. In addition, assume that \( Q \) satisfies the Lipschitz condition in (2.14).

(i) Let
\[
K_p^{1/p} |\mu|^{1/p} c_Q B_p^{1/p} < 1,
\]
where \( K_p \) is the absolute constant from the moment inequality in (2.13). Then there exists a unique stationary \( L^p \)-solution \( \{X_t\} \) of (2.12) and
\[
E|X_t|^p \leq \frac{C(p,Q)|\mu|B_p}{1 - K_p|\mu|^p c_Q B_p},
\]
where \( C(p,Q) < \infty \) depends only on \( p \) and \( c_1, c_2 \) in (2.15).

(ii) Assume, in addition, that \( Q^2(x) = c_1^2 + c_2^2 x^2 \), where \( c_i \geq 0, i = 1, 2 \), and \( \mu_2 = E\xi_0^2 = 1 \). Then \( c_2^2 B_2 < 1 \) is a necessary and sufficient condition for the existence of a stationary \( L^2 \)-solution \( \{X_t\} \) of (2.12) with \( a \neq 0 \).

Example 1 (The LARCH model) Let \( Q(x) = x \) and \( \{\xi_t\} \) be a standardized i.i.d. sequence with zero mean and unit variance. Then (2.12) becomes the bilinear equation
\[
X_t = \sum_{s<t} b_{t-s} \xi_s (a + X_s).
\]

The corresponding conditionally heteroscedastic process \( \{r_t = \xi_t (a + X_t)\} \) in Proposition 3(i) is the LARCH model discussed in [14], [16] and elsewhere. As shown in ([14], Thm.2.1), equation (2.23) admits a covariance stationary predictable solution if and only if \( B_2 = \sum_{j=1}^{\infty} b_j^2 < 1 \). Note that the last result agrees with Proposition 4(ii). The crucial role in the study of the LARCH model is played by the fact that its solution can be written in terms of the convergent orthogonal Volterra series
\[
X_t = a \sum_{k=1}^{\infty} \sum_{s_k < \ldots < s_1 < t} b_{t-s_1} \ldots b_{s_k-1-s_k} \xi_{s_1} \ldots \xi_{s_k}.
\]

Except for \( Q(x) = x \), in other cases of (2.12) including the ‘root model’ in (1.5), Volterra series expansions are unknown and their usefulness is doubtful.

Example 2 (Asymmetric ARCH(1)) Consider the model (1.1) with \( \sigma_t^2 \) in (1.6), viz.
\[
r_t = \xi_t (c^2 + (a + b r_{t-1})^2)^{1/2},
\]
where \( \{\xi_t\} \) are standardized i.i.d. r.v.’s. By Proposition 3(ii), equation (2.24) has a unique stationary solution with finite variance \( E r_t^2 = (a^2 + c^2)/(1 - b^2) \) if and only if \( b^2 < 1 \). As shown in Sentana [26] (see also Surgailis [28]), the random-coefficient AR(1) equation resulting in (1.1) is
\[
\tilde{r}_t = \kappa \xi_t + b \eta \tilde{r}_{t-1},
\]
where \(\{\varepsilon_t, \eta_t\}\) are i.i.d. random vectors with \(E\varepsilon_t = E\eta_t = 0\), \(E[\varepsilon_t^2] = E[\eta_t^2] = 1\), \(\rho = E[\varepsilon_t\eta_t]\) and \(\kappa > 0, \rho \in [-1, 1]\) in (2.24) are related to \(a, c\) in (2.21) by

\[
\kappa \rho = a, \quad \kappa^2 = a^2 + c^2.
\]

However, (stationary) solutions \(\{r_t\}\) and \(\{\tilde{r}_t\}\) of (2.24) and (2.25) have generally different finite-dimensional distributions (a notable exception is the case when \(\zeta_t\) and \(\{(\varepsilon_t, \eta_t)\}\) are Gaussian sequences, see [28], Corollary 2.1). This can be seen by considering the 3rd conditional moment of (2.24)

\[
E[r_t^3 | r_{t-1}] = \mu_3 (c^2 + (a + br_{t-1})^2)^{3/2}
\]

which is an irrational function of \(r_{t-1}\) (unless \(\mu_3 = E\zeta_0^3 = 0\) or \(b = 0\)), while a similar moment of (2.25)

\[
E[r_t^3 | r_{t-1}] = \kappa^3 \nu_{3,0} + 3br^2 \nu_{2,1} \tilde{r}_{t-1} + 3b^2 \kappa \nu_{1,2} \tilde{r}_{t-1}^2 + b^3 \nu_{0,3} \tilde{r}_{t-1}^3
\]

is a cubic polynomial in \(\tilde{r}_{t-1}\), where \(\nu_{i,j} := E[\varepsilon_0^i \eta_0^j]\).

For models (2.24) and (2.25), we can explicitly compute covariances \(\rho(t) = \text{Cov}(r_t^2, r_0^2)\), \(\tilde{\rho}(t) = \text{Cov}(\tilde{r}_t^2, \tilde{r}_0^2)\) and some other joint moment functions, as follows.

Let \(\mu_3 = E\zeta_0^3 = 0\), \(\mu_4 = E\zeta_0^4 < \infty\) and \(m_2 := E r_0^2\), \(m_3(t) := E r_t^2 r_0^2\), and \(m_4(t) := E r_t^2 r_0^4\), \(t \geq 0\). Then

\[
m_2 = (a^2 + c^2)/(1 - b^2), \quad m_3(0) = 0,
\]

\[
m_3(1) = E[((a^2 + c^2) + 2abr_0 + b^2r_0^2)r_0] = 2abm_2 + b^2m_3(0) = 2abm_2,
\]

\[
m_3(t) = E[((a^2 + c^2) + 2abr_{t-1} + b^2r_{t-1}^2)r_0] = b^2m_3(t - 1) = \cdots = b^2(t-1)m_3(1)
\]

\[
= \frac{2ab(a^2 + c^2)}{1 - b^2} b^{2(t-1)}, \quad t \geq 1.
\]

Similarly,

\[
m_4(0) = \mu_4 E[((a^2 + c^2) + 2abr_{t-1} + b^2r_{t-1}^2)^2]
\]

\[
= \mu_4 \{(a^2 + c^2)^2 + (2ab)^2 m_2 + b^4 m_4(0) + 2b^2(a^2 + c^2)m_2\},
\]

\[
m_4(t) = E[((a^2 + c^2) + 2abr_{t-1} + b^2r_{t-1}^2)r_0^2] = (a^2 + c^2)m_2 + b^2m_4(t - 1), \quad t \geq 1
\]

resulting in

\[
m_4(0) = \frac{\mu_4((a^2 + c^2)^2 + (2ab)^2 + 2(a^2 + c^2)b^2)m_2)}{1 - \mu_4 b^4},
\]

\[
m_4(t) = m_2(a^2 + c^2) \cdot \frac{1 - b^2t}{1 - b^2} + b^2m_4(0), \quad t \geq 1,
\]

and

\[
\rho(t) = (m_4(0) - m_2^2)b^{2t}, \quad t \geq 0.
\]

In a similar way, when the distribution of \(\zeta_0\) is symmetric one can write recursive linear equations for joint even moments \(E[r_{2p}(0)r_{2p}(t)]\) of arbitrary order \(p = 1, 2, \ldots\) involving
$E[r_{2l}(0)r_{2p}(t)], 1 \leq l \leq p - 1$ and $m_{2k}(0) = E[r_{2k}(0)], 1 \leq k \leq 2p$. These equations can be explicitly solved in terms of $a, b, c$ and $\mu_{2k}, 1 \leq k \leq 2p$.

A similar approach can be applied to find joint moments of the random-coefficient AR(1) process in (2.25), with the difference that symmetry of $(\varepsilon_0, \eta_0)$ is not needed. Let $\tilde{m}_2 := E\tilde{r}_t^2$, $\tilde{m}_3(t) := E[\tilde{r}_t^2\tilde{r}_0]$, $\tilde{m}_4(t) := E[\tilde{r}_t^2\tilde{r}_0^2]$ and $\tilde{\rho}(t) := \text{Cov}(\tilde{r}_t^2, \tilde{r}_0^2)$, $\nu_i := E[\varepsilon_0^{i}\eta_0^{j}]$. Then

\[
\tilde{m}_2 = \kappa^2/(1 - b^2),
\]

\[
\tilde{m}_3(0) = E[(\kappa\varepsilon_0 + b\eta_0\tilde{r}_{-1})^3] = \kappa^3\nu_{3,0} + 3\kappa b^2\nu_{1,2}\tilde{m}_2 + b^3\nu_{0,3}\tilde{m}_3(0),
\]

\[
\tilde{m}_3(1) = E[(\kappa + 2\kappa b\tilde{r}_0 + b^2\tilde{r}_0^2)\tilde{r}_0] = 2\kappa pb\tilde{m}_2 + b^2\tilde{m}_3(0),
\]

\[
\tilde{m}_3(t) = E[(\kappa^2 + 2\kappa pb\tilde{r}_{t-1} + b^2\tilde{r}_{t-1}^2)\tilde{r}_0] = b^2\tilde{m}_3(t-1) = \cdots = b^{2(t-1)}\tilde{m}_3(1), \quad t \geq 2,
\]

and

\[
\tilde{m}_4(0) = E[(\kappa\varepsilon_0 + b\eta_0\tilde{r}_{-1})^4] = \kappa^4\nu_{4,0} + 6\kappa b^2\nu_{2,2}\tilde{m}_2 + 4\kappa b^3\nu_{1,3}\tilde{m}_3(0) + b^4\nu_{0,4}\tilde{m}_4(0),
\]

\[
\tilde{m}_4(1) = E[(\kappa\varepsilon_1 + b\eta_0\tilde{r}_0)^2\tilde{r}_0^2] = \kappa^2\tilde{m}_2 + 2\kappa pb\tilde{m}_3(0) + b^2\tilde{m}_4(0)
\]

\[
\tilde{m}_4(t) = E[(\kappa\varepsilon_t + b\eta_0\tilde{r}_{t-1})^2\tilde{r}_0^2] = \kappa^2\tilde{m}_2 + b^2\tilde{m}_4(t-1), \quad t \geq 2,
\]

leading to

\[
\tilde{m}_3(0) = \frac{\kappa^3\nu_{3,0} + 3\kappa b^2\nu_{1,2}\tilde{m}_2}{1 - \nu_{0,3}b^3},
\]

\[
\tilde{m}_3(t) = b^{2(t-1)}(2\kappa pb\tilde{m}_2 + b^2\tilde{m}_3(0)), \quad t \geq 1.
\]

\[
\tilde{m}_4(0) = \frac{\kappa^4\nu_{4,0} + 6\kappa b^2\nu_{2,2}\tilde{m}_2 + 4\kappa b^3\nu_{1,3}\tilde{m}_3(0)}{1 - \nu_{0,4}b^4},
\]

\[
\tilde{m}_4(t) = \tilde{m}_2\kappa^2\left(1 - \frac{b^{2t}}{1 - b^2}\right) + b^{2t}(\tilde{m}_4(0) + 2\kappa pb\tilde{m}_3(0)/b), \quad t \geq 1,
\]

and

\[
\tilde{\rho}_4(t) = b^{2(t-1)}\tilde{\rho}_4(1), \quad t \geq 1,
\]

\[
\tilde{\rho}_4(1) = 2\kappa pb\tilde{m}_3(0) + b^2(\tilde{m}_4(0) - \tilde{m}_2^2).
\]

Then if $\nu_{3,0} = \nu_{1,2} = 0$ we have $\tilde{m}_3(0) = 0$ and $\tilde{\rho}_4(t) = (\tilde{m}_4(0) - \tilde{m}_2^2)b^{2t}$; moreover, $\tilde{m}_2 = m_2$ in view of (2.26). Then $\tilde{\rho}_4(t) = \rho_4(t)$ is equivalent to $\tilde{m}_4(0) = m_4(0)$, which follows from

\[
\mu_4 = \nu_{0,4} = \nu_{4,0} \quad \text{and} \quad 6\nu_{2,2} = \mu_4(4\nu_{1,1}^2 + 2),
\]

see (2.28), (2.27), (2.29). Note that (2.30) hold for centered Gaussian distribution $(\varepsilon_0, \eta_0)$ with unit variances $E\varepsilon_0^2 = E\eta_0^2 = 1$. 

8
3 Weak dependence

Various measures of weak dependence for stationary processes \( \{y_t\} = \{y, t \in \mathbb{Z}\} \) have been introduced in the literature, see e.g. [8]. Usually, the dependence between the present \( (t \geq 0) \) and the past \( (t \leq -n) \) values of \( \{y_t\} \) is measured by some dependence coefficients decaying to 0 as \( n \to \infty \). The decay rate of these coefficients plays a crucial role in establishing many asymptotic results. The classical problem is Donsker’s invariance principle:

\[
\frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} (y_t - Ey_t) \to_{D[0,1]} \sigma B(\tau), \quad \text{in the Skorohod space } D[0,1], \tag{3.31}
\]

where \( B = \{B(\tau), \tau \in [0,1]\} \) is a standard Brownian motion. The above result is useful in change-point analysis (Csörgő and Horváth [7]), financial mathematics and many other areas. Further applications of weak dependence coefficients include empirical processes [10] and the asymptotic behavior of various statistics, including the maximum likelihood estimators. See Ibragimov and Linnik [21] and the application to GARCH estimation in [22].

The present sec. discusses two measures of weak dependence - the projective weak dependence coefficients of Wu [31] and the \( \tau \)-dependence coefficients introduced in Dedecker and Prieur [9, 10] - for stationary solutions \( \{r_t\}, \{X_t\} \) of equations (2.10), (2.12). We show that the decay rate of the above weak dependence coefficients is determined by the decay rate of the moving average coefficients \( b_j \).

3.1 Projective weak dependence coefficients

Let us introduce some notation. For any r.v. \( \xi \), write \( \|\xi\|_p := E[|\xi|^p]^{1/p}, p \geq 1 \). Let \( \{y_t, t \in \mathbb{Z}\} \) be a stationary causal Bernoulli shift in i.i.d. sequence \( \{\xi_t\} \), in other words,

\[
y_t = f(\xi_s, s \leq t), \quad t \in \mathbb{Z},
\]

where \( f : \mathbb{R}^N \to \mathbb{R} \) is a measurable function. We also assume \( Ey_0 = 0, \|y_0\|_2^2 = Ey_0^2 < \infty \). Introduce the projective weak dependence coefficients

\[
\omega_p(i; \{y_t\}) := \|f_i(\xi_0) - f_i(\xi'_0)\|_p, \quad \delta_p(i; \{y_t\}) := \|f(\xi_i) - f(\xi'_i)\|_p, \tag{3.32}
\]

where \( \xi_i := (\ldots, \xi_{-1}, \xi_0, \xi_1, \ldots, \xi_t), \xi'_i := (\ldots, \xi_{-1}, \xi'_0, \xi_1, \ldots, \xi_t) \) are i.i.d. r.v.s and \( f_i(\xi_0) := E[f(\xi)|\xi_0] = E[y_t|\mathcal{F}_0] \) is the conditional expectation. Note the sequences \( \xi_i \) and \( \xi'_i \) coincide except for a single entry. Then \( \omega_p(i; \{y_t\}) \leq \delta_p(i; \{y_t\}) \) and condition

\[
\sum_{k=0}^{\infty} \omega_2(k; \{y_t\}) < \infty \tag{3.33}
\]

guarantees the weak invariance principle in (3.31), see Wu [31]. Below, we verify Wu’s condition (3.33) for \( \{X_t\}, \{r_t\} \) in (2.12), (2.10). We assume that the coefficients \( b_j \) decay as \( j^{-\gamma} \) with some \( \gamma > 0 \), viz.,

\[
\exists \gamma > 0, \ c > 0 : \ |b_j| < cj^{-\gamma}, \ \forall \ j \geq 1. \tag{3.34}
\]
Proposition 5 Let $Q$ satisfy the Lipschitz condition in (2.14), $p \geq 1$, $\mu_p = E c_p^p K_{pf_p c_p^p} B_p < 1$, and $\{X_t\}, \{r_t\}$ be stationary $L^p$-solutions of (2.12), (2.10), respectively. In addition, assume that $b_j$ satisfy (3.34) with $\gamma > \max\{1/2, 1/p\}$. Then

$$\delta_p(k; \{X_t\}) = O(k^{-\gamma}) \quad \text{and} \quad \delta_p(k; \{r_t\}) = O(k^{-\gamma}). \quad (3.35)$$

The following corollary follows by from Wu’s result in (3.33), relations $\delta_2(k,\{y_t\}) \leq C\delta_2(k,\{r_t\})$, and $\delta_2(k,\{z_t\}) \leq C\delta_2(k,\{X_t\})$ and the bounds in (3.35).

Corollary 6 Let $\{y_t := h(r_t)\}, \{z_t := h(X_t)\}$, where $\{X_t\}, \{r_t\}$ are as in Proposition $\mathfrak{g}$, $\gamma > 1$ and $h : R \to R$ is a Lipschitz function. Then

$$n^{-1/2} \sum_{t=1}^{[nr]} (y_t - E y_t) \to D[0,1] c_y B(\tau) \quad \text{and} \quad n^{-1/2} \sum_{t=1}^{[nr]} (z_t - E z_t) \to D[0,1] c_y B(\tau),$$

where $B$ is a standard Brownian motion and

$$c_y^2 := \sum_{t \in Z} E[y_0 y_t] < \infty, \quad c_z^2 := \sum_{t \in Z} E[z_0 z_t] < \infty.$$

3.2 $\tau$-weak dependence coefficients

Let $\{y_t, t \in Z\}$ be a stationary process with $\|y_0\|_p < \infty, p \in [1, \infty]$. Following Dedecker and Prieur [9, 10], we define the $\tau$-dependence coefficients

$$\tau_p(\{y_{j,i}\}_{1 \leq i \leq k}) := \sup_{f \in \Lambda_1(\mathbb{R}^k)} \| E[f(y_{j,1}, \ldots, y_{j,k})|y_t, t \leq 0] - E[f(y_{j,1}, \ldots, y_{j,k})]\|_p,$$

measuring the dependence between $y_t, t \leq 0$ and $\{y_{j,i}\}_{1 \leq i \leq k}, 0 < j_1 < \cdots < j_k$, and

$$\tau_p(n, \{y_{j}\}) := \sup_{k \geq 1} \sup_{n \leq j_1 < \cdots < j_k} \tau_p(\{y_{j,i}\}_{1 \leq i \leq k}).$$

Here, $\Lambda_1(\mathbb{R}^k)$ denotes the class of all Lipschitz functions $f : \mathbb{R}^k \to \mathbb{R}$ with

$$|f(x_1, \ldots, x_k) - f(y_1, \ldots, y_k)| \leq \sum_{i=1}^k |x_i - y_i| \quad \text{for any} \quad (x_1, \ldots, x_k), (y_1, \ldots, y_k) \in \mathbb{R}^k.$$

Proposition 7 Let the conditions of Proposition $\mathfrak{g}$ be satisfied, $p \geq 1$, $\{X_t\}, \{r_t\}$ be stationary $L^p$-solutions of (2.12), (2.10), respectively. In addition, assume that $b_j$ satisfy (3.34) with $\gamma > 1$. Then

$$\tau_p(n; \{X_t\}) = O(n^{-\gamma+1}), \quad \tau_p(n; \{r_t\}) = O(n^{-\gamma+1}). \quad (3.36)$$

The results on $\tau$-weak dependence in (11, Thm.1) together with Proposition $\mathfrak{g}$ imply the following CLT for the empirical distribution functions $F_n^X(u) := n^{-1} \sum_{t=1}^n 1(X_t \leq u), F_n^r(u) := n^{-1} \sum_{t=1}^n 1(r_t \leq u), u \in \mathbb{R}$ of stationary solutions $\{X_t\}, \{r_t\}$ of (2.12), (2.10). Let $F_n^X(u) = P(X_0 \leq u), F_n^r(u) = P(r_0 \leq u)$ denote the corresponding distribution functions. See [11] for the definition of weak convergence in the space $\ell^\infty(\mathbb{R})$ of all bounded functions on $\mathbb{R}$.
Corollary 8 Let the conditions of Proposition [7] hold with \( p = 1 \) and \( \gamma > 5 \). Moreover, assume that \( F^X, F^r \) have bounded densities. Then \( \sqrt{n}(F^X(u) - F^X(u)), u \in \mathbb{R} \) and \( \sqrt{n}(F^r(u) - F^r(u)), u \in \mathbb{R} \) converge weakly in \( \ell^\infty(\mathbb{R}) \) as \( n \to \infty \) towards Gaussian processes on \( \mathbb{R} \) with zero mean and respective covariance functions \( \sum_{k \in \mathbb{Z}} \text{Cov}(1_{X_0 \leq u}, 1_{X_k \leq v}) \) and \( \sum_{k \in \mathbb{Z}} \text{Cov}(1_{r_0 \leq u}, 1_{r_k \leq v}), \quad u, v \in \mathbb{R} \).

4 Strong dependence

The term strong dependence or long memory usually refers to stationary process \( \{y_t, t \in \mathbb{Z}\} \) whose covariance decays slowly with the lag so that its absolute series diverges, viz., \( \sum_{k=1}^\infty |\text{Cov}(y_0, y_k)| = \infty \). Since the variance of \( \sum_{t=1}^n y_t \) usually grows faster than \( n \) under long memory, Donsker’s invariance principle in (3.31) is no more valid and the limit of the partial sums process, if exists, might be quite complicated. Probably, the most important model of long memory processes is the linear, or moving average, process \( y_t = \sum_{s \leq t} b_{t-s} \zeta_s \) where \( \{\zeta_s\} \) is an i.i.d. sequence with zero mean and finite variance, and the moving average coefficients \( b_j \) decay as in (4.37) below. Various generalizations of the linear model were studied in [2], [19] and other works. See the monograph [17] for a discussion and applications of long memory processes.

It is natural to expect that the ‘long memory’ asymptotics of \( b_j \) in (4.37) induces some kind of long memory of solutions \( \{r_t\}, \{X_t\} \) of (1.1), (2.12), under general assumptions on \( Q \). Concerning the latter process, this is indeed true as shown in the following proposition.

Proposition 9 Let \( \{X_t\} \) be a stationary \( L^2 \)-solution of (2.12), where

\[
    b_j \sim \beta j^{-d-1} \quad (\exists \ 0 < d < 1/2, \ \beta > 0), \quad (4.37)
\]

and \( Q \) satisfies the Lipschitz condition in (2.14) with \( c_Q^2 B^2 = c_Q^2 \sum_{j=1}^\infty b_j^2 < 1 \). Then

\[
    \text{Cov}(X_0, X_t) \sim \lambda_2 t^{2d-1}, \quad t \to \infty \quad \text{and} \quad (4.38)
\]

\[
    n^{-d-(1/2)} \sum_{t=1}^{[nr]} X_t \to_{D[0,1]} \lambda_2 B_{d+(1/2)}(\tau),
\]

where \( B_{d+(1/2)} \) is a fractional Brownian motion with \( \text{Var}(B_{d+(1/2)}(\tau)) = \tau^{2d+1} \) and \( \lambda_2^2 := \beta^2 B(d, 1 - 2d)EQ^2(a + X_0), \lambda_2^2 := \lambda_2^2/d(1 + 2d) \).

Clearly, properties as in (4.38) do not hold for \( \{r_t = \zeta_t Q(a + X_t)\} \) which is an uncorrelated martingale difference process. Here, long memory should appear in the behavior of the volatility \( \sigma_t = Q(a + X_t) \), being ‘hidden’ inside of nonlinear kernel \( Q \). The last fact makes it much harder to prove it rigorously. In the rest of the paper we restrict ourselves to the ‘root’ model with \( Q(x) = \sqrt{c^2 + x^2} \) of (1.5), or

\[
    r_t = \zeta_t \sqrt{c^2 + \left(a + \sum_{s < t} b_{t-s} r_s\right)^2}, \quad t \in \mathbb{Z}, \quad (4.39)
\]
where (recall) \(\{\zeta_t\}\) are standardized i.i.d. r.v.s, with zero mean and unit variance, and \(b_j, j \geq 1\) are real numbers satisfying (4.37).

The following theorem shows that under some additional conditions the squared process \(\{r_t^2\}\) of (4.39) has similar long memory properties as \(\{X_t\}\) in Proposition 9. For the LARCH model (see Example 1 above), similar results were obtained in (14), Thm. 2.2, 2.3). In Theorem 10 and below, \(K_4\) is the Rosenthal’s constant from (2.13), whose numeric value is provided in Remark 1.

**Theorem 10** Let \(\{r_t\}\) be a stationary \(L^2\)-solution of (4.39)-(4.37). Assume in addition that 
\[ \mu_4 = E[\zeta_0^4] < \infty, \]
and
\[ B\mu_4^{1/4}K_4^{1/4} < 1. \]  
(4.40)

Then \(E[r_t^4] < \infty\) and
\[ \text{Cov}(r_0^2, r_t^2) \sim \kappa_1^2 2d-1, \quad t \to \infty \]  
(4.41)

where \(\kappa_1^2 := \left(\frac{2a\beta}{1-2}\right)^2 B(d, 1 - 2d)E r_0^2\). Moreover,
\[ n^{-d-1/2} \sum_{t=1}^{[nr]} (r_t^2 - Er_t^2) \to_{D[0,1]} \kappa_2 B_{d+(1/2)}(\tau), \quad n \to \infty, \]  
(4.42)

where \(B_{d+(1/2)}\) is a fractional Brownian motion with Hurst parameter \(H = d + (1/2) \in (1/2, 1)\) and \(\kappa_2^2 := \kappa_1^2/(d(1 + 2d))\).

### 5 Leverage

Given a stationary conditionally heteroscedastic time series \(\{r_t\}\) with \(E|r_t|^3 < \infty\) and conditional variance \(\sigma_t^2 = \text{Var}(r_t^2|s, s < t)\), leverage (a tendency of \(\sigma_t^2\) to move into the opposite direction as \(r_s\) for \(s < t\)) is usually measured by the covariance \(h_{t-s} = \text{Cov}(\sigma_t^2, r_s)\). Following [16], we say that \(\{r_t\}\) has leverage of order \(k\) \((1 \leq k < \infty)\) (denoted by \(\{r_t\} \in \ell(k)\)) whenever
\[ h_j < 0, \quad 1 \leq j \leq k. \]

Note for \(\{r_t\}\) in (1.1),
\[ h_j = E[r_j^2 r_0], \quad j = 0, 1, \ldots \]  
(5.43)

is the mixed moment function. Below, we show that in the case of the quadratic \(\sigma_t^2\) in (1.5), viz.,
\[ r_t = \zeta_t \sigma_t, \quad \sigma_t = \left(c^2 + \left(a + \sum_{j=1}^{\infty} b_j r_{t-j}\right)^2\right)^{1/2} \]  
(5.44)

and \(\mu_3 = E[\zeta_0^3] = 0\), the function \(h_j\) in (5.43) satisfies a linear equation in (6.72), below, which can be analyzed and the leverage effect for \(\{r_t\}\) in (5.44) can be established in spirit of [16].

Let \(L^2(Z_+)\) be the Hilbert space of all real sequences \(\psi = (\psi_j, j \in Z_+)\), with \(Z_+ = \{1, 2, \ldots \}\) with finite norm \(\|\psi\| := (\sum_{j=1}^{\infty} \psi_j^2)^{1/2} < \infty\).
As in the previous sections, let $B = \left(\sum_{j=1}^{\infty} b_j^2\right)^{1/2}$ and assume that $\{\zeta_t\}$ is an i.i.d. sequence with zero mean and unit variance; $\mu_t = E\zeta_t$, for $i = 1, 2, \ldots$.

The following proposition establishes a criterion for the presence or absence of leverage in model \[5.44\], analogous to the Thm 2.4 in \[16\].

We also note that the proof of Proposition \[11\] is much simpler than that of the above mentioned theorem, partly because of the assumption $\mu_3 = 0$ used in the derivation of equation \[6.72\].

**Proposition 11** Let $\{r_t\}$ be a stationary $L^2$-solution of \[4.39\]. Assume in addition that $B^2 < 1/5$, $\mu_4 < \infty$, $\mu_3 = 0$ and condition \[4.40\] of Theorem \[10\] guaranteeing that $E r_t^4 < \infty$ is satisfied. Then for any fixed $k$ such that $1 \leq k \leq \infty$:

(i) if $ab_1 < 0$, $ab_2 \leq 0$, $j = 2, \ldots, k$, then $\{r_t\} \in \ell(k)$

(ii) if $ab_1 > 0$, $ab_2 \geq 0$, $j = 2, \ldots, k$, then $h_j > 0$, for $j = 1, \ldots, k$.

6 Appendix: proofs

**Proof of Proposition 3**. (i) Since $\{X_t\}$ is predictable and $Q$ satisfies \[2.15\] so

$$E|r_t|^p = |\mu|^p E|Q(a + X_t)|^p$$

$$\leq |\mu|^p c_1^2 + c_2^2(a + X_t)^2$$

$$\leq C(1 + E|X_t|^p) < C < \infty,$$

proving \[2.17\]. Moreover, if $p > 1$ then $E|r_t|F_{t-1} = 0$ is a stationary martingale difference sequence. Hence by Proposition \[11\] the series in \[2.11\] converges in $L^p$ and satisfies

$$E|X_t|^p \leq C \left\{ \sum_{j=1}^{\infty} b_j^p, \begin{array}{l} 0 < p \leq 2 \\ (\sum_{j=1}^{\infty} b_j^{p/2})^p, \quad p > 2 \end{array} \right\} = CB_p < \infty.$$ 

In particular, $\zeta_t Q(a + \sum_{s \leq t} b_{s-r_s}) = \zeta_t Q(a + X_t) = r_t$ by the definition of $r_t$. Hence, $\{r_t\}$ is a $L^p$-solution of \[2.10\]. Stationarity of $\{r_t\}$ follows from stationarity of $\{X_t\}$.

Relations \[2.18\] follow from $E[\zeta_t|F_{t-1}] = 0$, $E[|\zeta_t|^p|F_{t-1}] = |\mu|^p$, $p > 1$, and the fact that $X_t$ is $F_{t-1}$-measurable.

(ii) Since $\{r_t\}$ is a $L^p$-solution of \[2.10\], so $r_t = \zeta_t Q(a + X_t)$ with $X_t$ defined in \[2.11\] and \{X_t\} satisfy \[2.12\], where the series converges in $L^p$. Also note that $\{X_t\}$ is predictable. Hence, $\{X_t\}$ is a $L^p$-solution of \[2.12\]. By \[2.13\],

$$E|\zeta_t|^p = |\mu|^p E|Q(a + X_t)|^p \leq |\mu|^p c_1^2 + c_2^2(a + X_t)^2$$

$$\leq C(1 + E|X_t|^p) < C.$$

It also easily follows that, for $p > 1$, $\{r_t, F_t, t \in \mathbb{Z}\}$ is a martingale difference sequence. Hence, by the moment inequality in \[2.13\],

$$E|X_t|^p \leq K_p \left\{ \begin{array}{l} \sum_{j=1}^{\infty} b_j^p E|\zeta_t|^p, \quad 0 < p \leq 2 \\ (\sum_{j=1}^{\infty} b_j^{p/2})^p, \quad p > 2 \end{array} \right\} = CB_p E|r_t|^p, \quad (6.45)$$

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proving (2.19). Stationarity of \( \{X_t\} \) and (2.20) are easy consequences of the above facts and stationarity of \( \{r_t\} \).

**Proof of Proposition 4** (i) For \( n \in \mathbb{N} \) define a solution of (2.12) with zero initial condition at \( t \leq -n \) as

\[
X_t^{(n)} := \begin{cases} 
0, & t \leq -n, \\
\sum_{s=-n}^{t-1} b_{t-s} \zeta_s Q(a + X_s^{(n)}), & t > -n, \quad t \in \mathbb{Z}.
\end{cases}
\tag{6.46}
\]

Let us show that \( \{X_t^{(n)}\} \) converges in \( L^p \) to a stationary solution \( \{X_t\} \) (in \( L^p \)) as \( n \to \infty \).

First, let \( 0 < p \leq 2 \). Let \( m > n \geq 0 \). Then by inequality (2.13) for any \( t > -m \) we have that

\[
E|X_t^{(m)} - X_t^{(n)}|^p = K_p|\mu|_p \left\{ \sum_{-m \leq s < -n} |b_{t-s}|^p E|Q(a + X_s^{(m)})|^p \right. \\
+ \left. \sum_{-n \leq s < t} |b_{t-s}|^p E \left| Q(a + X_s^{(n)}) - Q(a + X_s^{(m)}) \right|^p \right\} \\
= K_p|\mu|_p \{ S'_{m,n} + S''_{m,n} \}.
\]

Let \( \chi_p(n) := \sum_{j=n}^{\infty} |b_j|^p \).

From the bound \( |a + x|^2 \leq (2a^2/\epsilon) + (1 + \epsilon)x^2 \), valid for \( 0 < \epsilon < 1/2 \) \( x \in \mathbb{R} \) and \( a \geq 0 \), it follows that

\[
|c_1^2 + c_2^2(a + X_s^{(m)})^2|^{p/2} \leq c_1^p + c_2^p |(a + X_s^{(m)})^2|^{p/2} \leq C(c_1, c_2) + c_3^p (1 + \epsilon)^{p/2} |X_s^{(m)}|^p \leq C(c_1, c_2) + c_3^p |X_s^{(m)}|^p,
\]

with \( c_3 > c_2 > c_Q \) arbitrarily close to \( c_Q \). Then using (2.15) we obtain

\[
S'_{m,n} \leq \sum_{-m \leq s < -n} |b_{t-s}|^p E|c_1^2 + c_2^2(a + X_s^{(m)})^2|^p \\
\leq C(Q)K_p|\mu|_p \chi_p(t + n) + c_3^p \sum_{-m \leq s < -n} |b_{t-s}|^p E|X_s^{(m)} - X_s^{(n)}|^p,
\]

\[
S''_{m,n} \leq c_3^p \sum_{-n \leq s < t} |b_{t-s}|^p E|X_s^{(n)} - X_s^{(m)}|^p.
\]

Consequently,

\[
E|X_t^{(m)} - X_t^{(n)}|^p \leq C(Q)K_p|\mu|_p \chi_p(t + n) + K_p|\mu|_p c_3^p \sum_{-m \leq s < t} |b_{t-s}|^p E|X_s^{(n)} - X_s^{(m)}|^p.
\]

Iterating the above inequality, we obtain

\[
E|X_t^{(m)} - X_t^{(n)}|^p \leq C(Q)K_p|\mu|_p \left\{ \chi_p(t + n) + \sum_{k=1}^{\infty} (K_p|\mu|_p c_3^p)^k \right. \\
\times \left. \sum_{-m \leq s_k < \cdots < s_1 < t} |b_{t-s_1}|^p |b_{s_1-s_2}|^p \cdots |b_{s_{k-1}-s_k}|^p \chi_p(s_k + n) \right\}.
\tag{6.47}
\]

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Since $K_p |\mu|_p c_3^p B_p < 1$ by (2.21) and $\sup_{s \geq 1} \chi_p(s) \leq B_p < \infty$, the series on the r.h.s. of (3.47) is bounded uniformly in $m, n$ and tends to zero as $m, n \to \infty$ by the dominated convergence theorem. Hence, there exist $X_t, t \in \mathbb{Z}$ such that

$$\lim_{n \to \infty} E|X_t - X_t^{(n)}|^p = 0, \quad \forall \ t \in \mathbb{Z}.$$ 

Note that $\{X_t\}$ is predictable and

$$E|X_t|^p = \lim_{n \to \infty} E|X_t^{(n)}|^p \leq \frac{C(Q)K_p |\mu|_p B_p}{1 - K_p |\mu|_p c_3^p B_p} \leq \frac{C(p, Q) |\mu|_p B_p}{1 - K_p |\mu|_p c_Q^p B_p},$$

where the last inequality follows by taking $c_3 > c_Q$ sufficiently close to $c_Q$.

We also have by (2.22) and (2.14) that

$$E\left| \sum_{s < t} b_{t-s} \zeta_s Q(a + X_s) - \sum_{s=-n}^{t-1} b_{t-s} \zeta_s Q(a + X_s^{(n)}) \right|^p$$

$$= E\left| \sum_{s < -n} b_{t-s} \zeta_s Q(a + X_s) + \sum_{s=-n}^{t-1} b_{t-s} \zeta_s (Q(a + X_s) - Q(a + X_s^{(n)})) \right|^p$$

$$\leq K_p |\mu|_p \left\{ \sum_{s < -n} |b_{t-s}|^p E|Q(a + X_s)|^p + \sum_{-n \leq s < t} |b_{t-s}|^p E|Q(a + X_s) - Q(a + X_s^{(n)})|^p \right\}$$

$$\leq C \left( \sum_{s < -n} |b_{t-s}|^p + \sum_{s < t} |b_{t-s}|^p E|X_s - X_s^{(n)}|^p \right) \to 0$$

as $n \to \infty$. Whence and from (6.46) it follows that $\{X_t\}$ is a stationary $L^p$-solution of (2.12) satisfying (2.22).

To show the uniqueness of stationary $L^p$-solution of (2.12), let $\{X_t', \{X_t''\}$ be two such solutions of (2.12), and $m_p(t) := E|X_t' - X_t''|^p$.

Then $\sup_{t \in \mathbb{Z}} m_p(t) \leq M < \infty$ and $m_p(t) \leq K_p |\mu|_p c_Q^p |b_{t-s}|^p m_p(s)$ follows by (2.14).

Iterating the last equation we obtain that $m_p(t) \leq (K_p |\mu|_p c_Q^p B_p)^k M$ holds for all $k \geq 1$, where $K_p |\mu|_p c_Q^p B_p < 1$.

Hence, $m_p(t) = 0$. This proves part (i) for $0 < p \leq 2$.

The proof of part (i) for $p > 2$ is analogous. Particularly, using (2.13) as in (6.45), we obtain

$$E|X_t|^p \leq K_p |\mu|_p \left( \sum_{s < t} b_{t-s}^2 E^{2/p} |Q(a + X_s)|^p \right)^{p/2}$$

$$\leq K_p |\mu|_p \left( \sum_{s < t} b_{t-s}^2 (C(Q) + c_3^p E|X_s|^p)^{2/p} \right)^{p/2}$$

$$\leq K_p |\mu|_p B_p (C(p, Q) + c_3^p \sup_{s \in \mathbb{Z}} E|X_s|^p)$$

implying $(1 - K_p |\mu|_p c_3^p B_p) \sup_{t \in \mathbb{Z}} E|X_t|^p \leq C(p, Q) |\mu|_p B_p$ and hence the bound in (2.22) for $p > 2$, by taking $c_3$ sufficiently close to $c_Q$. This proves part (i).
(ii) Note that $Q(x) = \sqrt{c_1^2 + c_2^2x^2}$ is a Lipschitz function and satisfies (2.14) with $c_Q = c_2$. Hence by $K_2 = 1$ and part (i), a unique $L^2$-solution $\{X_t\}$ of (2.12) under the condition $c_2^2B_2 < 1$ exists. To show the necessity of the last condition, let $\{X_t\}$ be a stationary $L^2$-solution of (2.12). Then

$$
\begin{align*}
E X_t^2 & = \sum_{s<t} b_{t-s}^2 E Q^2(a + X_s) \\
& = \sum_{s<t} b_{t-s}^2 E (c_1^2 + c_2^2(a + X_s)^2) \\
& = B_2(c_1^2 + c_2^2(a^2 + EX_t^2)) > c_2^2B_2 EX_t^2
\end{align*}
$$

since $a \neq 0$. Hence, $c_2^2B_2 < 1$ unless $EX_t^2 = 0$, or $\{X_t = 0\}$ is a trivial process. Clearly, (2.12) admits a trivial solution if and only if $0 = Q(a) = \sqrt{c_1^2 + c_2^2a^2} = 0$, or $c_1 = c_2 = 0$. This proves part (ii) and the proposition. \hfill \square

The proofs of Proposition 5 and Theorem 10 use the following general lemma.

**Lemma 12** For $\alpha_j \geq 0$, $j = 1, 2, \ldots$, denote

$$
A_k := \alpha_k + \sum_{0<p<k} \sum_{0<i_1<\cdots<i_p<k} \alpha_{i_1} \alpha_{i_2-i_1} \cdots \alpha_{i_p-i_{p-1}} \alpha_{k-i_p}, \quad k = 1, 2, \ldots. \quad (6.48)
$$

Assume that $A := \sum_{j=1}^\infty \alpha_j < 1$ and

$$
\alpha_j \leq c j^{-\gamma}, \quad (\exists \ c > 0, \ \gamma > 1). \quad (6.49)
$$

Then there exists $C > 0$ such that for any $k \geq 1$

$$
A_k \leq C k^{-\gamma}. \quad (6.50)
$$

**Proof.** We have $A_k = \sum_{0\leq p<k} A_{k,p}$, where

$$
A_{k,p} := \sum_{0<i_1<\cdots<i_p<k} \alpha_{i_1} \alpha_{i_2-i_1} \cdots \alpha_{i_p-i_{p-1}} \alpha_{k-i_p}, \quad (p \geq 1), \quad A_{k,0} := \alpha_k
$$

is the inner sum in (6.48). W.l.g., assume $c \geq 1$ in (6.49). Let us prove that there exists $\lambda > 0$ such that

$$
A_{k,p} \leq c(p + 2)^\lambda A^{p+1} k^{-\gamma}, \quad \forall 0 \leq p < k < \infty. \quad (6.51)
$$

Since $A < 1$, so (6.51) and $\sum_{p>0} (p + 2)^\lambda A^{p+1} < \infty$ together imply (6.50).

By dividing both sides of (6.51) by $A^{p+1}$, it suffices to show (6.51) for $A = 1$. The proof uses induction on $p$. Clearly, (6.51) holds for $p = 0$. To prove the induction step $p - 1 \to p \geq 1$, note

$$
A_{k,p} = \sum_{0<i<k} \alpha_i A_{k-i,p-1} = \sum_{k-p+1 \leq i < k} \alpha_i A_{k-p-1} + \sum_{k-p+1 \leq i < k} \alpha_i A_{k-i,p-1}. \quad (6.52)
$$

Here, $\alpha_i (i > \frac{k}{p+1}) \leq c i^{-\gamma} (i > \frac{k}{p+1}) \leq c(p+1)^\gamma k^{-\gamma}$ and, similarly, by the inductive assumption

$$
A_{k-i,p-1} 1(k - i \geq k - \frac{k}{p+1}) \leq c(p+1)^\lambda (k - \frac{k}{p+1})^{-\gamma} = c(p+1)^\lambda (\frac{p+1}{p})^\gamma k^{-\gamma}.
$$

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Assumption $A = 1$ implies $\sum_{k>0} A_{k,p} = 1$ for any $p \geq 0$. Using the above facts from $(6.52)$ we obtain 
\[ A_{k,p} = \frac{c(p+1)^\gamma}{k^\gamma} \sum_{k/(p+1)<i<k} A_{k-i,p-1} + \frac{c(p+1)^\lambda}{k^\gamma} \left( \frac{p+1}{p} \right)^\gamma \sum_{k-k/(p+1)\leq k-i<k} \alpha_i \]
\[ \leq c(p+1)^\gamma + (p+1)^\lambda \left( \frac{p+1}{p} \right)^\gamma k^{-\gamma}. \]
Hence the proof of the induction step $p-1 \rightarrow p \geq 1$ amounts to verifying the inequality $(p+1)^\gamma + (p+1)^\lambda \left( \frac{p+1}{p} \right)^\gamma \leq (p+2)^\lambda$, or
\[ n^\gamma + n^\lambda \left( \frac{n}{n-1} \right)^\gamma \leq (n+1)^\lambda, \quad n = 2, 3, \ldots \] (6.53)
The above inequality holds with $\lambda = 3\gamma$. Indeed,
\[ n^\gamma + n^\lambda \left( \frac{n}{n-1} \right)^\gamma = n^\lambda (n^{-2\gamma} + \left( \frac{n}{n-1} \right)^\gamma) \leq n^\lambda (n^{-2} + \left( \frac{n}{n-1} \right)^\gamma) \]
\[ \leq n^\lambda (1 + \frac{1}{n-1} + \frac{1}{n^2}) \gamma \leq n^\lambda (1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}) \gamma = (n+1)^\lambda, \]
proving $(6.53)$ and the lemma, too. \hfill \square

**Proof of Proposition 4** We will give the proof for $p \geq 2$ only as the proof for $p \in [1, 2]$ is similar.

Following the notation in $(6.32)$, let $\{X'_t\}, \{r'_t\}$ be the corresponding processes (Bernoulli shifts) of the i.i.d. sequence $\xi' := (\ldots, \zeta_{-1}, \zeta'_0, \zeta_1, \zeta_2, \ldots)$ with $\zeta_0$ replaced by its independent copy $\zeta'_0$. Note that $X'_t = X_t$ ($t \leq 0$), $r'_t = r_t$ ($t < 0$). We have $\delta_p^2 (k; \{X_t\}) = (E|X_k - X'_{k|p})^{2/p} = ||X_k - X'_{k|p}||_{L_{\alpha_k}}^2$, where
\[ X_k - X'_k = \sum_{0<s<k} b_{k-s}(r_s - r'_s) + b_k(\zeta_0 - \zeta'_0)Q(a + X_0). \]
Then with $\sigma_p^2 := ||Q(a + X_0)||_{L_{\alpha_k}}^2$ using Rosenthal’s inequality $(2.13)$ similarly as in the proof of Proposition 4 we obtain
\[ ||X_k - X'_k||_{L_{\alpha_k}}^2 = K_{p}^{2/p} \left( \sum_{0<s<k} b_{k-s}^2 ||r_s - r'_s||^2 + ||\zeta_0 - \zeta'_0||_{p}^2 \right) \]
\[ \leq K_{p}^{2/p} \left( \sum_{0<s<k} b_{k-s}^2 \mu_{p}^{2/p} ||Q(a + X_s) - Q(a + X'_s)||^2 + 4\mu_{p}^{2/p} b_k^2 \sigma_p^2 \right) \]
\[ \leq K_{p}^{2/p} \left( \mu_{p}^{2/p} c_{Q} \sum_{0<s<k} b_{k-s}^2 ||X_s - X'_s||_{p}^2 + 4\mu_{p}^{2/p} b_k^2 \sigma_p^2 \right). \]
Let $\alpha_k := K_{p}^{2/p} \mu_{p}^{2/p} c_{Q}^2 b_k^2$. Iterating the last inequality we obtain
\[ \delta_p^2 (k; \{X_t\}) \leq \frac{4\sigma_p^2}{c_{Q}^{2}} \left( \alpha_k + \sum_{0<s<k} \alpha_s \alpha_{k-s} + \ldots \right) = \frac{4\sigma_p^2}{c_{Q}^{2}} \cdot A_k, \]
where $A_k$ is defined in $(6.48)$ in Lemma 12. Since $A = \sum_{k>0} \alpha_k = (K_{p}\mu_{p} c_{Q}^2 B_{p})^{2/p} < 1$ and $\alpha_k \leq Ck^{-2\gamma}$, by the above lemma we obtain $\delta_p (k; \{X_t\}) \leq Ck^{-\gamma}$, proving the
second inequality in (3.35). The proof of the first inequality in (3.35) follows similarly using \( \delta_p^2(k;\{r_t\}) = \| \delta_p(k;\{r_t\}) \|^2 \leq c_Q^2 \mu_p^{2/p} \| X_k - X_k' \|^2_p = c_Q^2 \mu_p^{2/p} \delta_p^2(k;\{X_t\}) \). \hfill \Box

Proof of Proposition 9. We use the coupling inequality in [10], providing a simple upper bound for \( \tau \)-coefficients. Let \( \{y_t^\tau\} \) be distributed as \( \{y_t\} \) and independent of \( y_s, s \leq 0 \). Then

\[
\tau_p(\{y_{t,i}\}_{1 \leq i \leq k}) \leq \sum_{i=1}^k \| y_{t,i} - y_{t,i}^\tau \|_p \quad \text{and} \quad \tau_p(n, \{y_t\}) \leq \sup_{j \geq n} \| y_j - y_j^\tau \|_p.
\]

To construct the coupling for \( \{X_t\} \), let \( \{X_t^*\} \) be the corresponding process (Bernoulli shift) of the i.i.d. sequence \( \xi^* := (\ldots, \zeta_{-2}^*, \zeta_{-1}^*, \zeta_0, \zeta_1, \ldots) \) with \( (\zeta_s, s < 0) \) an independent copy \( (\zeta_s, s < 0) \). Clearly, \( \{X_t^*\} \) is distributed as \( \{X_t\} \) and independent of \( \{X_s, s \leq 0\} \), the latter being measurable w.r.t. \( (\zeta_s, s < 0) \). Hence, the first relation in (3.36) follows from

\[
\| X_n - X_n^* \|_p = O(n^{-\gamma+1}).
\]

Towards this end, consider ‘intermediate’ i.i.d. sequence \( \xi_i^* := (\ldots, \zeta_{i-1}^*, \zeta_i^*, \zeta_{i+1}, \ldots, \zeta_0, \zeta_1, \ldots) \), \( i = 1, 2, \ldots \), \( \xi_1^* = \xi^* \). Note sequences \( \xi_i^* \) and \( \xi_i^{*-1} \) agree up to a single entry. Let \( \{X_t^*\} \) be the corresponding Bernoulli shift of the i.i.d. sequence \( \xi_t^* \). By triangle inequality, \( \| X_n - X_n^* \|_p \leq \sum_{i \geq 1} \| X_{i,n}^* - X_{i+1,n}^* \|_p \). By stationarity and Proposition 5,

\[
\| X_{i,n}^* - X_{i+1,n}^* \|_p = \delta_p(n+i, \{X_t\}) \leq C(n+i)^{-\gamma},
\]

where \( \delta_p \) is defined in (3.32). Clearly, (6.55) implies (6.54), proving the first relation in (3.36). Since \( \tau_p(n, \{r_t\}) \leq C_p \tau_p(n, \{X_t\}) \), the second relation in (3.36) follows. Proposition 7 is proved. \hfill \Box

Proof of Proposition 9. The first relation in (4.38) follows from (2.20) and (4.37). The second relation in (4.38) is follows from a general result in Abadir et al. [11], Prop.3.1, using the fact that \( \{r_s\} \) in (2.11) is a stationary ergodic martingale difference sequence. \hfill \Box

Proof of Theorem 10. The proof of Theorem 10 heavily relies on the following decomposition:

\[
(r_t^2 - Er_t^2) - \sum_{s \leq t} b_{t-s}(r_s^2 - Er_s^2) = 2aX_t + Z_t,
\]

where \( \{Z_t\} \) on the r.h.s. of (6.56) is negligible so as its memory intensity is less than the memory intensity of the main term, \( \{X_t\} \). Accordingly, \( r_t^2 - Er_t^2 = (1 - \sum_{j=1}^{\infty} b_j^2 L_j)^{-1} \xi_t \) behaves like an AR(\( \infty \)) process with long memory innovations \( \xi_t = 2aX_t + Z_t \approx 2aX_t \). A rigorous meaning to the above heuristic explanation is provided below.

By the definition of \( r_t \) in (4.39),

\[
Z_t := U_t + V_t, \quad \text{where} \quad U_t := (\zeta_t^2 - 1)Q^2(a + X_t), \quad V_t := X_t^2 - EX_t^2 - \sum_{s \leq t} b_{t-s}(r_s^2 - Er_s^2) = 2 \sum_{s_2 < s_1 < t} b_{t-s_1}b_{t-s_2}r_{s_1}r_{s_2}.
\]
Let us first check that the double series in (6.57) converges in mean square and (6.57) holds. Let
\[ X_{t,N} := \sum_{-N<s<t} b_{t-s} s, \quad V_{t,N} := 2 \sum_{-N<s<s<t} b_{t-s_1} b_{t-s_2} s_1 s_2, \]
then \( V_{t,N} = X_{t,N}^2 - E X_{t,N}^2 - \sum_{-N<s<t} b_{t-s}^2 (s^2 - E s^2) \) and, for \( M > N \),
\[ E(X_{t,N}^2 - X_{t,M}^2)^2 = E(X_{t,N} - X_{t,M})^2 (X_{t,N} + X_{t,M})^2 \leq \|X_{t,N} - X_{t,M}\|^2 \|X_{t,N} + X_{t,M}\|^2. \]
By Rosenthal’s inequality in (2.13),
\[
\|X_{t,N} + X_{t,M}\|^4 \leq C \sum_{-M<s<t} b_{t-s}^2 \leq C, \quad \text{and} \quad \|X_{t,N} - X_{t,M}\|^4 \leq C \sum_{-M<s<-N} b_{t-s}^2 \to 0 \quad (N, M \to \infty).
\]
Therefore, \( \lim_{N,M \to \infty} E(X_{t,N}^2 - X_{t,M}^2)^2 = 0. \)
The convergence in \( L^2 \) of \( EX_{t,N}^2 \) and \( \sum_{-N<s<t} b_{t-s}^2 (s^2 - E s^2) \) as \( N \to \infty \) is easy. Hence, \( V_{t,N}, N \geq 1 \) is a Cauchy sequence in \( L_2 \) and the double series in (6.57) converges as claimed above, proving (6.57).

Let us prove that \( \{Z_t\} \) in (6.58) is negligible in the sense that its (cross)covariances decays faster than the covariance of the main term, \( \{X_t\} \), viz.,
\[ E[Z_t Z_0] = o(t^{2d-1}), \quad E[X_t Z_0] = o(t^{2d-1}), \quad E[Z_t X_0] = o(t^{2d-1}) \quad (6.58) \]
as \( t \to \infty \). Note, for \( t \geq 1 \), \( E[U_t U_0] = E[V_0 U_t] = 0 \) and \( E[V_t U_0] = b_t E[\zeta_0 (\zeta^2_0 - 1) Q^2 (a + X_0) \sum_{s \leq 0} b_{t-s} s] = O(b_t) = o(t^{2d-1}) \). Hence, the first relation in (6.58) follows from
\[ E[V_t V_0] = o(t^{2d-1}), \quad t \to \infty, \quad (6.59) \]
which is proved below. Since \( E[V_t^2] < \infty, E[V_t] = 0 \) we can write the orthogonal expansion
\[ V_t = \sum_{s \leq t} P_s V_t, \]
where \( P_s V_t := E[V_t | F_s] - E[V_t | F_{s-1}] \) is the projection operator.

By orthogonality of \( P_s \),
\[ E[V_0 V_t] = \sum_{s<0} E[(P_s V_0)(P_s V_t)] \leq \sum_{s<0} \|P_s V_0\|_2 \|P_s V_t\|_2. \]
Relation (6.59) follows from
\[ \|P_s V_0\|_2^2 = o(b_{-s}^2) = o((-s)^{2(d-1)}), \quad s \to -\infty. \quad (6.60) \]
Indeed, if (6.60) is true then
\[ EV_0 V_t = o(\sum_{s<0} (-s)^{d-1}(t-s)^{d-1}) = o(t^{2d-1}), \quad t \to \infty, \]
\[ EV_0 V_t = o(\sum_{s<0} (-s)^{d-1}(t-s)^{d-1}) = o(t^{2d-1}), \quad t \to \infty, \]
proving (6.59).
Consider (6.60). We have by (6.57) and the martingale difference property of \( \{ r_s \} \) that
\[
P_s V_0 = 2 r_s b_{-s} \sum_{u<s} b_{-u} r_u
\]
and
\[
\| P_s V_0 \|_2^2 = 4 b_{-s}^2 E \left[ r_s^2 \left( \sum_{u<s} b_{-u} r_u \right)^2 \right] \leq 4 b_{-s}^2 \| r_s \|_4^2 \left( \sum_{u<s} b_{-u} r_u \right)^2.
\]
By Rosenthal’s inequality in (2.13),
\[
E \left| \sum_{u<s} b_{-u} r_u \right|^4 \leq C_4 \left( \sum_{u>s} b_{-u}^2 (E r^4_{u})^{1/2} \right)^2 \leq C \left( \sum_{u>|s|} u^{2(d-1)} \right)^2 = O(|s|^{2(2d-1)}) = o(1).
\]
Therefore,
\[
\| P_s V_0 \|_2^2 \leq C |s|^{2(d-1)+2d-1} = o(|s|^{2(d-1)}),
\]
proving (6.60), (6.59), and the first relation in (6.58). The remaining two relations in (6.58) follow easily, e.g.,
\[
E[X_t Z_0] = b_t E[r_0 (\zeta_0^2 - 1) Q^2 (a + X_0)] + 2 \sum_{s_1<0} b_{t-s_1} b_{-s_1} L_{s_1},
\]
where
\[
L_{s_1} := E[r^2_{s_1} \sum_{s_2<s_1} b_{-s_2} r_{s_2}]
\leq E^{1/2} \left| r^2_{s_1} \right| E^{1/2} \left[ \left( \sum_{s_2<s_1} b_{-s_2} r_{s_2} \right)^2 \right]
= O\left( \left( \sum_{s_2<s_1} b_{-s_2}^2 \right)^{1/2} \right) = O(\left| s_1 \right|^{d-(1/2)}), \quad s_1 \to -\infty.
\]
Therefore
\[
E[X_t Z_0] = O(t^{d-1}) + \sum_{s_1<0} (t-s_1)^{d-1} (-s_1)^{2d-(3/2)} = o(t^{2d-1}).
\]
This proves (6.58).

Next, let us prove (4.41). Recall the decomposition (6.56). Denote \( \xi_t := 2 a X_t + Z_t \), then (6.56) can be rewritten as \( r^2_t - E r^2_t - \sum_{s<t} b^2_{t-s} (r^2_s - E r^2_s) = \xi_t \), or
\[
r^2_t - E r^2_t = \sum_{i=0}^{\infty} \varphi_i \xi_{t-i}, \quad t \in \mathbb{Z}, \quad (6.61)
\]
where \( \varphi_j \geq 0, j \geq 0 \) are the coefficients of the power series
\[
\Phi(z) := \sum_{j=0}^{\infty} \varphi_j z^j = (1 - \sum_{j=1}^{\infty} b_j^2 z^j)^{-1}, \quad z \in \mathbb{C}, \quad |z| < 1
\]

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given by \( \varphi_0 := 1 \),

\[
\varphi_j := b_j^2 + \sum_{0 < k < j} \sum_{0 < s_1 < \cdots < s_{k-1} < j} b_{s_1}^2 b_{s_2 - s_1}^2 \cdots b_{s_k - s_{k-1}}^2 b_{j - s_k}^2, \quad j \geq 1.
\]

From (4.37) and Lemma 12 we infer that

\[
\varphi_t = O(t^{2d-2}), \quad t \to \infty, \quad (6.62)
\]

in particular, \( \Phi(1) = \sum_{t=0}^{\infty} \varphi_t = 1/(1 - B^2) < \infty \) and the r.h.s. of (6.61) is well-defined. Relation (5.58) implies that that

\[
\gamma_t := \text{Cov}(\xi_0, \xi_t) \sim 4a^2 \text{Cov}(X_0, X_t) \sim 4a^2 \kappa_3^2 t^{2d-1}, \quad t \to \infty \quad (6.63)
\]

with \( \kappa_3^2 = \beta^2 B(d, 1 - 2d)Er_0^2 \). Let us show that

\[
\text{Cov}(r_t^2, r_0^2) = \sum_{i,j=0}^{\infty} \varphi_i \varphi_j \gamma_{t-i+j} \sim \Phi^2(1) \gamma_t, \quad t \to \infty. \quad (6.64)
\]

With (6.63) in mind, (6.64) is equivalent to

\[
J_t := \sum_{i,j=0}^{\infty} \varphi_i \varphi_j (\gamma_{t-i+j} - \gamma_t) = o(t^{2d-1}). \quad (6.65)
\]

For a large \( L > 0 \), split \( J_t = J'_{t,L} + J''_{t,L} \), where

\[
J'_{t,L} := \sum_{i,j > 0: i + j \leq L} \varphi_i \varphi_j (\gamma_{t-i+j} - \gamma_t), \quad J''_{t,L} := \sum_{i,j > 0: i + j > L} \varphi_i \varphi_j (\gamma_{t-i+j} - \gamma_t).
\]

Clearly, (6.65) follows from

\[
t^{-2d}J'_{t,L} = o(1) \quad \forall \ L > 0, \quad \text{and} \quad \lim_{L \to \infty} \limsup_{t \to \infty} t^{-2d}J''_{t,L} = 0. \quad (6.66)
\]

The first relation in (6.66) is immediate from (6.63) since it implies \( \gamma_{t+k} - \gamma_t = o(t^{2d-1}) \) for any \( k \) fixed.

With (6.62) and (6.63), the second relation in (6.66) follows from

\[
\lim_{L \to \infty} \limsup_{t \to \infty} t^{-2d}J_{t,L} = 0.
\]

where \( J_{t,L} := \sum_{i,j > 0: i + j > L} t^{2d-2} j^{2d-2} (1 + |t + j - i|)^{2d-1} \).

Split the last sum according to whether \( |t + j - i| \geq t/2 \), or \( |t + j - i| < t/2 \). Then

\[
J'_{t,L} := \sum_{i,j > 0: i + j > L, |t + j - i| \geq t/2} t^{2d-2} j^{2d-2}(1 + |t + j - i|)^{2d-1} \leq C t^{2d-1} \sum_{i > L/2 \text{ or } j > L/2} t^{2d-2} j^{2d-2} \leq C t^{2d-1} L^{2d-1}.
\]
implying $\lim_{L \to \infty} \limsup_{t \to \infty} t^{1-2d} J''_{t,L} = 0$.

Next, since $|t + j - i| < t/2$ is equivalent to $(t/2) + j < i < (3t/2) + j$, so

$$J''_{t,L} := \left\{ \sum_{|t+j-i|<t/2,j>t} + \sum_{|t+j-i|<t/2,j<t} \right\} t^{2d-2} j^{2d-2} (1 + |t+j-i|)^{2d-1}$$

$$\leq t^{2d-2} \sum_{i>0} t^{2d-2} \sum_{j>0;|t+j-i|<t/2} (1 + |t+j-i|)^{2d-1}$$

$$+ \frac{(t/2)^{2d-2}}{t^{2d-2}} \sum_{j>0} \sum_{i>0;|t+j-i|<t/2} (1 + |t+j-i|)^{2d-1}$$

$$\leq Ct^{2d-1} t^{2d} = O(t^{4d-2}),$$

implying $\limsup_{t \to \infty} t^{1-2d} J''_{t,L} = 0$ for any $L > 0$. This proves (6.66), (6.65), and (6.64). Clearly, (4.41) follows from (6.64) and (6.63).

It remains to show the invariance principle in (4.42). With (6.64) in mind, decompose $S_n(\tau) := \sum_{t=1}^{[n\tau]}(r_t^2 - Er_t^2) = \sum_{i=1}^{3} S_{ni}(\tau)$, where

$$S_{n1}(\tau) := 2a\Phi(1) \sum_{t=1}^{[n\tau]} X_t,$$

$$S_{n2}(\tau) := \Phi(1) \sum_{t=1}^{[n\tau]} Z_t,$$

$$S_{n3}(\tau) := \sum_{t=1}^{[n\tau]} \sum_{i=0}^{\infty} \varphi_i(\xi_{t-i} - \xi_t).$$

Here, $ES_{n2}^2(\tau) = o(n^{2d+1})$ follows from (6.58). Consider

$$ES_{n3}^2(\tau) := \sum_{t,s=1}^{[n\tau]} \sum_{i,j=0}^{\infty} \varphi_i \varphi_j E(\xi_{t-i} - \xi_t)(\xi_{s-j} - \xi_s) = \sum_{t,s=1}^{[n\tau]} \rho_{t-s},$$

where $\rho_t := \sum_{i,j=0}^{\infty} \varphi_i \varphi_j (\gamma_{t+j-i} - \gamma_{t+j} - \gamma_{t-i} + \gamma_t) = o(t^{2d-1})$ follows similarly to (6.66).

Hence, $S_{ni}(\tau) = o_p(n^{d+1/2})$, $i = 2, 3$. The convergence $n^{-d-1/2} S_{n1}(\tau) \to D_{[0,1]} \kappa_2 B_{d+1/2}(\tau)$ follows from Proposition [9].

This completes the proof of Theorem [10].

**Proof of Proposition [11]** Consider equation (5.44) with zero initial condition at $t \leq -n$:

$$r_t^{(n)} := 0 \quad (t \leq -n), \quad r_t^{(n)} = \zeta_t \left( \varepsilon^2 + (a + \sum_{-n<s<t} b_{t-s} r_s^{(n)})^2 \right)^{1/2} \quad (t \geq -n).$$

Let $h_j^{(n)}(t) := E[(r_t^{(n)})^2 r_{t-j}^{(n)}], j > 0$, $m_j^{(n)}(t) := E[(r_t^{(n)})^2]$. Similarly as in the proof of Proposition [4] condition (4.40) guarantees that $\sup_{t \in \mathbb{Z}} m_j^{(n)}(t) < \infty$, $\sup_{t \in \mathbb{Z}} E[(r_t^{(n)})^4] < \infty$ and $\sup_{t \in \mathbb{Z}, j \in \mathbb{Z}, t} |h_j^{(n)}(t)| < \infty$ and $|h_j^{(n)}(t) - h_j| \to 0$ ($n \to \infty$) for $t, j > 0$ fixed.
Using \( \mathbb{E}[r_i^{(n)}] = \mathbb{E}[(r_i^{(n)})^3] = \mathbb{E}[r_i^{(n)} R_s^{(n)}] = 0, s < t \), for \( 0 \leq t - j < t + n \) we obtain

\[
\begin{align*}
    h_j^{(n)}(t) &= \mathbb{E} \left[ \left( c^2 + a^2 + 2a \sum_{-n < s < t} b_{t-s} r_s^{(n)} + \sum_{-n < s < t} b_{t-s}^2 (r_s^{(n)})^2 \right. \right. \\
    & \left. \left. + 2 \sum_{-n < s < t} b_{t-s} r_s^{(n)} r_{t-j}^{(n)} \right) \right] \\
    &= 2ab_j m_2^{(n)} (t-j) + \sum_{t-j < s < t} b_{t-s}^2 (r_{s+j}^{(n)}) + 2b_j \sum_{-n < s < t-j} b_{t-s} r_{t-j-s}^{(n)} (t-j) \\
    &= 2ab_j m_2^{(n)} (t-j) + 2b_j \sum_{0 < i < t-j+n} b_i h_i^{(n)} (t-j) + \sum_{0 < i < j} b_i^2 h_{j-i}^{(n)} (t-i). \quad (6.67)
\end{align*}
\]

Let \( H_j^{(n)}(t) := \sum_{0 < i < j} b_i^2 h_{j-i}^{(n)} (t-i) \),

\[
G_j^{(n)}(t) := 2ab_j m_2^{(n)} (t-j) + 2b_j \sum_{0 < i < t-j+n} b_i h_i^{(n)} (t-j),
\]

\( H^{(n)}(t) = (H_j^{(n)}(t), j \in \mathbb{Z}_+) \), and \( G^{(n)}(t) = (G_j^{(n)}(t), j \in \mathbb{Z}_+) \).

Then \( |G_j^{(n)}(t)| \leq |b_j|(C + 2\|h^{(n)}(t-j)\|) \parallel B \) implying

\[
\|G^{(n)}(t)\| \leq CB + 2B \left( \sum_{j > 0} b_j^2 \sup_{t > 0} \|h^{(n)}(t-j)\|^2 \right)^{1/2} \leq CB + 2B^2 \sup_{t > 0} \|h^{(n)}(t)\|. \quad (6.68)
\]

Next, by Minkowski’s inequality,

\[
\left( \sum_{j > 0} \left( \sum_{0 < i < j} b_i^2 h_{j-i}^{(n)} (t-i) \right)^2 \right)^{1/2} \leq \sum_{i > 0} b_i^2 \left( \sum_{j > i} (h_{j-i}^{(n)} (t-i))^2 \right)^{1/2}
\]

implying

\[
\sup_{t > 0} \|H^{(n)}(t)\| \leq B^2 \sup_{t > 0} \|h^{(n)}(t)\|. \quad (6.69)
\]

From (6.68), (6.69) and (6.67) we obtain that \( \sup_{t > 0} \|h^{(n)}(t)\| \leq CB + 3B^2 \sup_{t > 0} \|h^{(n)}(t)\| \), or

\[
\sup_{t > 0} \|h^{(n)}(t)\| \leq \frac{CB}{1 - 3B^2} < \infty \quad (6.70)
\]

provided that \( B^2 < 1/3 \), where the r.h.s. of (6.70) does not depend on \( n \geq 1 \).

Since \( h_j = \lim_{n \to \infty} h_j^{(n)}(t) \) for any \( t, j \) so \( \|h\| \leq \lim \sup_{n \to \infty} \|h^{(n)}(t)\| \) by Fatou’s lemma and from (6.70) we conclude that

\[
\|h\| \leq \frac{CB}{1 - 3B^2} < \infty. \quad (6.71)
\]

From (6.71) similarly to (6.67) we obtain that \( h = (h_j, j \in \mathbb{Z}_+) \in L^2(\mathbb{Z}_+) \) is a solution of the linear equation

\[
h_j = 2ab_j m_2 + \sum_{0 < i < j} b_i^2 h_{j-i} + 2b_j \sum_{i > 0} b_i h_i. \quad (6.72)
\]
Let us prove the statements (i) and (ii) of Proposition 11 for \( k = 1 \). From (6.72) it follows that

\[
h_1 = 2am_2b_1 + 2b_1 \sum_{u=1}^{\infty} h_u b_{1+u} = 2b_1 (am_2 + \sum_{u=1}^{\infty} h_u b_{1+u})
\]

Since \( |\sum_{u=1}^{\infty} h_u b_{1+u}| \leq \|h\|B \), we have \( \text{sgn}(h_1) = \text{sgn}(b_1a) \) provided \( \|h\|B < |a|m_2 \) holds. The last relation follows from (6.71) and \( B^2 < 1/5 \); indeed,

\[
\|h\|B \leq \frac{2|a|m_2B^2}{1-3B^2} \leq |a|m_2.
\]

This proves (i) and (ii) for \( k = 1 \).

The general case \( k \geq 1 \) follows similarly by induction on \( k \). Indeed, from (6.72) we have that

\[
h_k = 2b_k \left( am_2 + \sum_{u=1}^{\infty} h_u b_{k+u} \right) + \sum_{j=1}^{k-1} b_{k-j}^2 h_j.
\]

Assume \( h_1 < 0, \ldots, h_{k-1} < 0 \), then the second term \( \sum_{j=1}^{k-1} b_{k-j}^2 h_j \leq 0 \). Moreover,

\[
\left| \sum_{u=1}^{\infty} h_u b_{k+u} \right| \leq \|h\|B < |a|m_2
\]

implying that the sign of the first term is the same as \( \text{sgn}(ab_k) \).

Proposition 11 is proved. \( \square \)

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