Abstract We show that certain functional inequalities, e.g. Nash-type and Poincaré-type inequalities, for infinitesimal generators of $C_0$ semigroups are preserved under subordination in the sense of Bochner. Our result improves earlier results by Bendikov and Maheux (Trans Am Math Soc 359:3085–3097, 2007, Theorem 1.3) for fractional powers, and it also holds for non-symmetric settings. As an application, we will derive hypercontractivity, supercontractivity and ultracontractivity of subordinate semigroups.

Keywords Subordination · Bernstein function · Nash-type inequality · Super-Poincaré inequality · Weak Poincaré inequality

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1 Introduction

In this note we show that certain functional inequalities are preserved under subordination in the sense of Bochner.

Bochner’s subordination is a method to get new semigroups from a given one. Let us briefly summarize the main facts about subordination; our main reference is the monograph [12], in particular Chapter 12. Let $(T_t)_{t \geq 0}$ be a strongly continuous $(C_0)$ contraction semigroup on a Banach space $(\mathcal{B}, \| \cdot \|)$. The infinitesimal generator is the operator...
\[ Au := \lim_{t \to 0} \frac{u - T_t u}{t}, \]
\[ D(A) := \left\{ u \in B : \lim_{t \to 0} \frac{u - T_t u}{t} \text{ exists in the strong sense} \right\}. \]

A subordinator is a vaguely continuous convolution semigroup of sub-probability measures \((\mu_t)_{t \geq 0}\) on \([0, \infty)\). Subordinators are uniquely characterized by the Laplace transform:
\[ \mathcal{L} \mu_t(\lambda) = \int_{[0, \infty)} e^{-s \lambda} \mu_t(ds) = e^{-tf(\lambda)} \quad \text{for all } t \geq 0 \text{ and } \lambda \geq 0. \]

The characteristic exponent \(f : (0, \infty) \to (0, \infty)\) is a Bernstein function, i.e. a function of the form
\[ f(\lambda) = a + b\lambda + \int_{(0, \infty)} (1 - e^{-t\lambda}) \nu(dt), \]
where \(a, b \geq 0\) are nonnegative constants and \(\nu\) is a nonnegative measure on \((0, \infty)\) satisfying \(\int_{(0, \infty)} (1 \wedge t) \nu(dt) < \infty\). There are one-to-one relations between the triplet \((a, b, \nu)\), the Bernstein function \(f\) and the subordinator \((\mu_t)_{t \geq 0}\). Among the most prominent examples of Bernstein functions are the fractional powers \(f_\alpha(\lambda) = \lambda^\alpha, 0 < \alpha \leq 1\). The Bochner integral
\[ T_t^f u := \int_{[0, \infty)} T_s u \mu_t(ds), \quad t \geq 0, \quad u \in B, \]
defines a strongly continuous contraction semigroup on \(B\). We call \((T_t^f)_{t \geq 0}\) subordinate to \((T_t)_{t \geq 0}\) (with respect to the subordinator \((\mu_t)_{t \geq 0}\) or the Bernstein function \(f\)). Subordination preserves many additional properties of the original semigroup. For example, on a Hilbert space, \((T_t^f)_{t \geq 0}\) inherits symmetry from \((T_t)_{t \geq 0}\) and on an ordered Banach space \((T_t^f)_{t \geq 0}\) is sub-Markovian whenever \((T_t)_{t \geq 0}\) is. Let us write \((A^f, D(A^f))\) for the generator of \((T_t^f)_{t \geq 0}\); it is known that \(D(A)\) is an operator core of \(A^f\) and that \(A^f\) is given by Phillips’ formula
\[ A^f u = au + b u + \int_{(0, \infty)} (u - T_s u) \nu(ds), \quad u \in D(A). \]

Here \((a, b, \nu)\) is the defining triplet for \(f\) as in (1).

Bochner’s subordination gives rise to a functional calculus for generators of \(C_0\) contraction semigroups. In many situations this functional calculus coincides with classical functional calculi, e.g. the spectral calculus in Hilbert space or the Dunford–Taylor spectral calculus in Banach space, cf. [4, 12]. It is, therefore, natural to write \(f(A)\) instead of \(A^f\).

From now on we will use \(B = L^2(X, m)\) where \((X, m)\) is a measure space with a \(\sigma\)-finite measure \(m\). We write \((\cdot, \cdot)\) and \(\| \cdot \|_2\) for the scalar product and norm in \(L^2\), respectively; \(\| \cdot \|_1\) denotes the norm in \(L^1(X, m)\). To compare our result with [2, Theorem 1.3], we start with Nash-type inequalities. For the study of Nash-type inequalities and ultracontractivity of the associated operator semigroups we refer to the paper [1] and the references therein. Our main contribution to this type of functional inequalities are the following two results.

**Theorem 1** (symmetric case) Let \((T_t)_{t \geq 0}\) be a strongly continuous contraction semigroup of symmetric operators on \(L^2(X, m)\) and assume that for each \(t \geq 0\), \(T_t|_{L^2(X, m) \cap L^1(X, m)}\) has an extension which is a contraction on the space \(L^1(X, m)\), i.e. we have \(\|T_t u\|_1 \leq \|u\|_1\) for all
$u \in L^1(X, m) \cap L^2(X, m)$. Suppose that the generator $(A, D(A))$ satisfies the following Nash-type inequality:

$$\|u\|_2^2 B(\|u\|_2^2) \leq \langle A u, u \rangle, \quad u \in D(A), \quad \|u\|_1 = 1,$$

where $B : (0, \infty) \to (0, \infty)$ is any increasing function. Then, for any Bernstein function $f$, the generator $f(A)$ of the subordinate semigroup satisfies

$$\|u\|_2^2 f \left( B \left( \frac{\|u\|_2^2}{2} \right) \right) \leq \langle f(A) u, u \rangle, \quad u \in D(f(A)), \quad \|u\|_1 = 1.$$

### Remark 2
For fractional powers $A^\alpha$, $0 < \alpha < 1$, the result of Theorem 1 is due to Bendikov and Maheux [2, Theorem 1.3]; this corresponds to the Bernstein functions $f(\lambda) = \lambda^\alpha$. Our result is valid for all Bernstein functions, hence, for all subordinate generators $f(A)$. Note that [2, Theorem 1.3] claims that

$$c_1 \|u\|_2^2 (B(c_2 \|u\|_2^2))^\alpha \leq \langle A^\alpha u, u \rangle, \quad u \in D(A^\alpha), \quad \|u\|_1 = 1,$$

holds for all $0 < \alpha < 1$ with $c_1 = c_2 = 1$, but a close inspection of the proof in [2] reveals that one has to assume, in general, $c_1, c_2 \in (0, 1)$. Note that Theorem 1 yields $c_1 = c_2 = 1/2$.

If $(T_t)_{t \geq 0}$ is not symmetric, we still have the following result.

### Theorem 3
(non-symmetric case) Let $(T_t)_{t \geq 0}$ be a strongly continuous contraction semigroup on $L^2(X, m)$ and assume that for each $t \geq 0$, $T_t|_{L^2(X, m) \cap L^1(X, m)}$ has an extension which is a contraction on $L^1(X, m)$. Suppose that the generator $(A, D(A))$ satisfies the following Nash-type inequality:

$$\|u\|_2^2 B(\|u\|_2^2) \leq \text{Re} \langle A u, u \rangle, \quad u \in D(A), \quad \|u\|_1 = 1,$$

where $B : (0, \infty) \to (0, \infty)$ is any increasing function. Then, for any Bernstein function $f$, the generator $f(A)$ of the subordinate semigroup satisfies

$$\frac{\|u\|_2^2}{4} f \left( 2B \left( \frac{\|u\|_2^2}{2} \right) \right) \leq \text{Re} \langle f(A) u, u \rangle, \quad u \in D(f(A)), \quad \|u\|_1 = 1.$$

### Remark 4
(i) The assumption that $T_t$ is a contraction both in $L^2(X, m)$ and $L^1(X, m)$ is often satisfied in concrete situations. Assume that $(T_t)_{t \geq 0}$ is a strongly continuous contraction semigroup on $L^2(X, m)$ such that the operators $T_t$ are symmetric and sub-Markovian—i.e. $0 \leq T_t v \leq 1$ a.e. for all $0 \leq v \leq 1$ m-a.e. Then the following argument shows that $T_t|_{L^2(X, m) \cap L^1(X, m)}$ is a contraction on $L^1(X, m)$:

$$\langle T_t u, v \rangle = \langle u, T_t v \rangle \leq \|u\| \|v\|_{\infty} = \|v\|_{\infty} \|u\|_1 \quad u \in L^2 \cap L^1, \quad v \in L^1 \cap L^\infty.$$

In general, a sub-Markovian $L^2$-contraction operator $T_t$ is also an $L^1$-contraction if, and only if, the $L^2$-adjoint $T_t^*$ is a sub-Markovian operator, cf. [11, Lemma 2].
(ii) From (1) it follows that Bernstein functions are subadditive, thus

$$\frac{1}{2} f(2x) \leq f(x).$$

This shows that, for symmetric semigroups, (4) implies (6).
The remaining part of this paper is organized as follows. Section 2 contains some preparations needed for the proof of Theorems 1 and 3, in particular a one-to-one relation between Nash-type inequalities and estimates for the decay of the semigroups. These estimates are needed for the proof of Theorems 1 and 3 in Sect. 3. Section 4 contains several applications of our main result, e.g. the super-Poincaré and weak Poincaré inequality for subordinate semigroups and the hyper-, super- and ultracontractivity of subordinate semigroups.

2 Preliminaries

In this section we collect a few auxiliary results for the proof of Theorems 1 and 3. We begin with a differential and integral inequality, which is a consequence of [6, Appendix A, Lemma A.1, p. 193]. Note that the right hand side of the inequality (7) below is negative. This is different from the usual Gronwall–Bellman–Bihari inequality, see e.g. [5, Section 3], [3, Chapter 4 §§ 4,5] and [13, I.1.VI, I.6.IX], but it is essential for our purposes. For the sake of completeness, we include the short proof whose idea follows [6, Appendix A, Remark A.3, p. 194].

Recall that for an increasing function $G : [0, \infty) \rightarrow \mathbb{R}$ the generalized (right continuous) inverse $G^{-1} : \mathbb{R} \rightarrow [0, +\infty]$ is defined as

$$G^{-1}(y) := \inf\{t > 0 : G(t) > y\}, \quad \inf \emptyset := \infty.$$ 

If $G$ is strictly increasing, then $G^{-1}$ coincides with the usual inverse.

**Lemma 5** Let $h : [0, \infty) \rightarrow [0, \infty)$ be a differentiable function. Suppose that there exists an increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(t) > 0$ for $t > 0$, $\int_0^1 1/\varphi(t) \, dt = \infty$ and

$$h'(t) \leq -\varphi(h(t)) \quad \text{for all } t \geq 0. \tag{7}$$

Then, we have

$$h(t) \leq G^{-1}(G(h(0)) - t) \quad \text{for all } t \geq 0,$$

where $G^{-1}$ is the (generalized right continuous) inverse of

$$G(t) = \begin{cases} \int_1^t \frac{du}{\varphi(u)}, & \text{if } t \geq 1, \\ \int_1^t \frac{du}{\varphi(u)}, & \text{if } t \leq 1. \end{cases}$$

**Proof** Since $h'(t) \leq -\varphi(h(t)) \leq 0$, the set $I = \{t : h(t) > 0\}$ is a (bounded or unbounded) interval; for $t \in I$, the function $h(t)$ is strictly decreasing. With the convention $\int_b^a = -\int_a^b$, we see for all $t \in I$ that

$$G(h(t)) = \int_1^{h(t)} \frac{1}{\varphi(u)} \, du = G(h(0)) + \int_{h(0)}^{h(t)} \frac{1}{\varphi(u)} \, du$$
\[= G(h(0)) + \int_0^t \frac{h'(u)}{\varphi(h(u))} \, du \leq G(h(0)) - t.\]

If \( t \notin I \), \( G(h(t)) = G(0) = -\infty \), and the above inequality is trivial. The claim follows from the definition of the generalized inverse \( G^{-1} \).

Let \((T_t)_{t \geq 0}\) be a strongly continuous contraction semigroup of (not necessarily symmetric) operators on \( L^2 = L^2(X, m) \). Denote by \((A, D(A))\) the infinitesimal generator. Since \( \frac{d}{dt} T_t u = -T_t Au \) for all \( u \in D(A) \), we have

\[
\frac{d}{dt} \|T_t u\|^2_2 = -2 \Re \langle A T_t u, T_t u \rangle, \quad u \in D(A).
\]

**Proposition 6** Let \((T_t)_{t \geq 0}\) be a \( C_0 \) contraction semigroup on \( L^2(X, m) \) and assume that each \( T_t \mid_{L^2(X, m) \cap L^1(X, m)} \), \( t \geq 0 \), has an extension which is a contraction on \( L^1(X, m) \), i.e. \( \|T_t u\|_1 \leq \|u\|_1 \) for all \( u \in L^1(X, m) \cap L^2(X, m) \). Then the following Nash-type inequality

\[
\|u\|^2_2 B\left(\|u\|^2_2 \|u\|_1^2\right) \leq \Re \langle A u, u \rangle, \quad u \in D(A), \quad \|u\|_1 = 1
\]

with some increasing function \( B : (0, \infty) \to (0, \infty) \) holds if, and only if,

\[
\|T_t u\|^2_2 \leq G^{-1}(G(\|u\|^2_2) - t) \text{ for all } t \geq 0 \text{ and } u \in D(A), \quad \|u\|_1 = 1
\]

where

\[
G(t) = \begin{cases} 
\int_1^t \frac{ds}{2s B(s)}, & \text{if } t \geq 1, \\
-\int_t^1 \frac{ds}{2s B(s)}, & \text{if } t \leq 1.
\end{cases}
\]

**Proof** Assume that (8) holds. Then,

\[
\|u\|^2_2 B\left(\|u\|^2_2 \|u\|_1^2\right) \leq \Re \langle A u, u \rangle, \quad u \in D(A).
\]

For all \( u \in D(A) \) with \( \|u\|_1 = 1 \) we have

\[
\frac{d}{dt} \|T_t u\|^2_2 = -2 \Re \langle A T_t u, T_t u \rangle \leq -2 \|T_t u\|^2_2 B\left(\|T_t u\|^2_2 \right),
\]

Since the function \( B \) is increasing and \( \|T_t u\|_1 \leq \|u\|_1 = 1 \), we have

\[
\frac{d}{dt} \|T_t u\|^2_2 \leq -2 \|T_t u\|^2_2 B(\|T_t u\|^2_2).
\]

This, together with Lemma 5, proves (9).
For the converse we assume that (9) holds. Then, for all \( u \in D(A) \) with \( \|u\|_1 = 1 \),

\[
\text{Re}(Au, u) = -\frac{1}{2} \frac{d}{dt} \|T_t u\|_2^2 \big|_{t=0} = \frac{1}{2} \lim_{t \to 0} \frac{\|u\|_2^2 - \|T_t u\|_2^2}{t} \geq \frac{1}{2} \lim_{t \to 0} \frac{\|u\|_2^2 - G^{-1}(G(\|u\|_2^2) - t)}{t}
\]

\[
= -\frac{1}{2} \frac{d}{dt} G^{-1}(G(\|u\|_2^2) - t) \big|_{t=0}
\]

\[
= [G^{-1}(G(\|u\|_2^2) - t) \cdot B(G^{-1}(G(\|u\|_2^2) - t))] \big|_{t=0}
\]

\[
= \|u\|_2^2 B(\|u\|_2^2),
\]

which is just the Nash-type inequality (8). \( \square \)

Finally we need some elementary estimate for Bernstein functions.

**Lemma 7** Let \( f \) be a Bernstein function given by (1) where \( a = b = 0 \) and with representing measure \( \nu \). Set

\[
v_1(x) := \int_0^x v(s, \infty) \, ds.
\]

Then for \( x > 0 \),

\[
\frac{e - 1}{e} x v_1 \left( \frac{1}{x} \right) \leq f(x) \leq x v_1 \left( \frac{1}{x} \right).
\]

**Proof** By Fubini’s theorem we find

\[
x v_1 \left( \frac{1}{x} \right) = x \int_0^{1/x} v(s, \infty) \, ds = \int_0^1 v \left( \frac{t}{x}, \infty \right) \, dt
\]

\[
= \int_0^1 \int_0^{\infty} v(dy) \, dt = \int_0^\infty (xy \wedge 1) \, v(dy),
\]

see also Ōkura [9, (1.5)]. Using the following elementary inequalities

\[
\frac{e - 1}{e} (1 \wedge r) \leq 1 - e^{-r} \leq 1 \wedge r \quad \text{for} \quad r \geq 0,
\]

we conclude

\[
\frac{e - 1}{e} x v_1 \left( \frac{1}{x} \right) = \int_0^\infty \frac{e - 1}{e} (xy \wedge 1) \, v(dy) \leq \int_0^\infty (1 - e^{-xy}) \, v(dy) = f(x).
\]

The upper bound follows similarly. \( \square \)
3 Proof of the main theorems

Proof of Theorem 1. Since \( D(A) \) is an operator core for \((f(A), D(f(A)))\), it is enough to prove (4) for \( u \in D(A) \). Using Phillips’ formula (2) we find for all \( u \in D(A) \)

\[
\langle f(A)u, u \rangle = a \|u\|^2_2 + b \langle Au, u \rangle + \int_{(0, \infty)} \langle u - T_su, u \rangle v(ds).
\]

This formula and the representation (1) for \( f \) show that we may, without loss of generality, assume that \( a = b = 0 \).

Assume that (3) holds. Proposition 6 shows for \( t \geq 0 \) and \( u \in D(A) \) with \( \|u\|_1 = 1 \),

\[
\frac{\langle T_tu, u \rangle}{\|u\|^2_2} = \frac{\|T_{t/2}u\|^2_2}{\|u\|^2_2} \leq \frac{G^{-1}(G(\|u\|^2_2) - t/2)}{\|u\|^2_2}.
\]

Then,

\[
\langle f(A)u, u \rangle = \int_{(0, \infty)} \langle u - T_su, u \rangle v(ds)
\]

\[
= \|u\|^2_2 \int_{(0, \infty)} \left( 1 - \frac{\langle T_su, u \rangle}{\|u\|^2_2} \right) v(ds)
\]

\[
\geq \int_{(0, \infty)} \left( \|u\|^2_2 - G^{-1} \left( G(\|u\|^2_2) - \frac{s}{2} \right) \right) v(ds)
\]

\[
= g(\|u\|^2_2),
\]

where

\[
g(r) = \int_{(0, \infty)} \left( r - G^{-1} \left( G(r) - \frac{s}{2} \right) \right) v(ds).
\]

Furthermore, for all \( r > 0 \),

\[
g(r) = \int_{(0, \infty)} \left( r - G^{-1} \left( G(r) - \frac{s}{2} \right) \right) v(ds)
\]

\[
= \int_{(0, \infty)} \left( \int_{G(r) - s/2}^{G(r)} dG^{-1}(u) \right) v(ds)
\]

\[
= \int_{-\infty}^{r} v(2(G(r) - u), \infty) dG^{-1}(u)
\]

\[
= \int_{0}^{r} v(2(G(r) - G(u)), \infty) du.
\]
For the last equality we used that $B$ is increasing, $G(x) > -\infty$ for all $x > 0$ and $G(0) = -\infty$; this follows from

$$G(0) = - \int_0^1 \frac{du}{uB(u)} \leq \frac{-1}{B(1)} \int_0^1 \frac{du}{u} = -\infty.$$  

Using again the monotonicity of $B$, we find from the mean value theorem

$$\frac{1}{2uB(u)} \geq \frac{G(r) - G(u)}{r - u} \geq \frac{1}{2rB(r)} \text{ for all } 0 < u < r. \quad (10)$$  

Therefore,

$$g(r) \geq \int_0^r \left( \frac{1}{uB(u)} (r - u), \infty \right) du$$

$$\geq \int_{r/2}^r \left( \frac{1}{uB(u)} (r - u), \infty \right) du$$

$$\geq \int_0^{r/2} \left( \frac{2v}{rB(r/2)}, \infty \right) dv$$

$$= \frac{1}{2} r B(r/2) \int_0^{1/B(r/2)} v(s, \infty) ds.$$

A similar calculation, now using the lower bound in (10), yields

$$g(r) \leq r B(r) \int_0^{1/B(r)} v(s, \infty) ds.$$

Now we can use Lemma 7 to deduce that

$$\frac{e}{e - 1} rf(B(r)) \geq g(r) \geq \frac{r}{2} f \left( B \left( \frac{r}{2} \right) \right) \text{ for all } r > 0,$$

and the proof is complete. \qed

**Remark 8** (i) In the proof of Theorem 1, at the line (11), we can replace $r/2$ by $\varepsilon r$ for any $\varepsilon \in (0, 1)$. Then we get

$$g(r) \geq \sup_{\varepsilon \in (0, 1)} \left[ (1 - \varepsilon) r f \left( \frac{\varepsilon B(\varepsilon r)}{1 - \varepsilon} \right) \right],$$

which shows that we can improve (4) by

$$\sup_{\varepsilon \in (0, 1)} \left[ (1 - \varepsilon) \|u\|_1^2 f \left( \frac{\varepsilon B(\varepsilon \|u\|_1^2)}{1 - \varepsilon} \right) \right] \leq \langle f(A)u, u \rangle, \quad u \in D(f(A)), \quad \|u\|_1 = 1.$$

(ii) A close inspection of our proof shows that Theorem 1 remains valid if we replace the norming condition $\|u\|_1 = 1$ in (3) and (5) by the more general condition $\Phi(u) = 1$. Here $\Phi : L^2(X, m) \to [0, \infty]$ is a measurable functional satisfying $\Phi(cu) = c^2 \Phi(u)$ and $\Phi(T_t u) \leq \Phi(u)$ for all $t \geq 0$ and $\Phi(u) = 0$ if, and only if, $u = 0$. \hfill \(\blacksquare\) Springer
Proof of Theorem 3 The proof of Theorem 3 is similar to the proof of Theorem 1. Therefore we only outline the differences in the arguments. As before we can assume that the function \( f(\lambda) = \int_{(0, \infty)} (1 - e^{-\lambda t}) v(dt) \). Moreover, it is enough to verify (6) for all \( u \in D(A) \). Since \((T_t)_{t \geq 0}\) is a contraction on \( L^1(X, m) \cap L^2(X, m) \), we see from (5) and Proposition 6 that for all \( t \geq 0 \) and \( u \in D(A) \) with \( \|u\|_1 = 1 \),

\[
\|T_t u\|_2^2 \leq G^{-1}(\|u\|_2^2) - t.
\]

By the Cauchy–Schwarz inequality,

\[
\frac{\text{Re} \langle T_t u, u \rangle}{\|u\|_2^2} \leq \frac{|\langle T_t u, u \rangle|}{\|u\|_2^2} \leq \frac{\sqrt{G^{-1}(\|u\|_2^2) - t}}{\|u\|_2}.
\]

Using (2) yields that for any \( u \in D(A) \) with \( \|u\|_1 = 1 \),

\[
\text{Re} \langle f(A) u, u \rangle = \int_{(0, \infty)} \text{Re} \langle u - T_s u, u \rangle v(ds)
\]

\[
= \|u\|_2^2 \int_{(0, \infty)} \left(1 - \frac{\text{Re} \langle T_s u, u \rangle}{\|u\|_2^2}\right) v(ds)
\]

\[
\geq \|u\|_2^2 \int_{(0, \infty)} \left(1 - \frac{\sqrt{G^{-1}(\|u\|_2^2) - s}}{\|u\|_2}\right) v(ds)
\]

\[
= \|u\|_2^2 \int_{(0, \infty)} \left(1 - \frac{G^{-1}(\|u\|_2^2) - s}{\|u\|_2^2}\right) v(ds)
\]

\[
\geq \|u\|_2^2 \int_{(0, \infty)} \left(1 - \frac{G^{-1}(\|u\|_2^2) - s}{\|u\|_2^2}\right) v(ds)
\]

\[
= g(\|u\|_2^2),
\]

where

\[
g(r) = \frac{r}{2} \int_{(0, \infty)} \left(1 - \frac{G^{-1}(r - s)}{r}\right) v(ds).
\]

A similar calculation as in the proof of Theorem 1 shows

\[
g(r) = \frac{1}{2} \int_0^r v(G(r) - G(u), \infty) du \geq \frac{r}{4} f \left(2 B \left(\frac{r}{2}\right)\right),
\]

which is exactly (6).
4 Applications

We will now give some applications of our results. Throughout this section we retain the notation introduced in the previous sections. In particular, \((T_t)_{t \geq 0}\) will be a strongly continuous contraction semigroup on \(L^2(X, m)\) with generator \((A, D(A))\). We assume that \(\|T_t u\|_1 \leq \|u\|_1\) for all \(u \in L^2(X, m) \cap L^1(X, m)\) and, for simplicity, that the operators \(T_t\), \(t \geq 0\), are symmetric. By \(\Phi : L^2(X, m) \to [0, \infty]\) we denote a functional on \(L^2(X, m)\) such that for all \(c, t > 0\) and \(u \in L^2(X, m)\)

\[
\Phi(u) = 0 \Rightarrow u = 0, \Phi(cu) = c^2 \Phi(u) \quad \text{and} \quad \Phi(T_t u) \leq \Phi(u);
\]

by \(f\) we always denote a Bernstein function given by (1).

4.1 Subordinate super-Poincaré inequalities

In this section, we study the analogue of Theorem 1 for super-Poincaré inequalities. For details on super-Poincaré inequalities and their applications we refer to [14–16] or [17, Chapter 3].

**Proposition 9** Assume that \((A, D(A))\) satisfies the following super-Poincaré inequality:

\[
\|u\|_2^2 \leq r \langle A u, u \rangle + \beta(r) \Phi(u), \quad r > 0, \ u \in D(A),
\]

where \(\beta : (0, \infty) \to (0, \infty)\) is a decreasing function such that \(\lim_{r \to 0} \beta(r) = \infty\) and \(\lim_{r \to \infty} \beta(r) = 0\); moreover, we set \(\beta(0) := \infty\). Then the generator \(f(A)\) of the subordinate semigroup also satisfies a super-Poincaré inequality

\[
\|u\|_2^2 \leq r \langle f(A) u, u \rangle + \beta_f(r) \Phi(u), \quad r > 0, \ u \in D(f(A)),
\]

where

\[
\beta_f(r) = 4\beta \left( \frac{1}{2f^{-1}(2/r)} \right).
\]

**Proof** We can rewrite (12) for any \(u \in D(A)\) with \(\Phi(u) = 1\) in the following form:

\[
\|u\|_2^2 B(\|u\|_2^2) \leq \langle A u, u \rangle,
\]

where

\[
B(x) = \sup_{s > 0} \frac{1 - \beta(s)/x}{s}.
\]

Clearly, \(B(x)\) is an increasing function on \((0, \infty)\). Since \(\beta^{-1} : (0, \infty) \to (0, \infty)\), we see from

\[
\frac{1}{2\beta^{-1}(x/2)} = \frac{1 - \beta(\beta^{-1}(x/2))/x}{\beta^{-1}(x/2)} \leq B(x) = \sup_{s \geq \beta^{-1}(x)} \frac{1 - \beta(s)/x}{s} \leq \frac{1}{\beta^{-1}(x)}
\]

that \(B : (0, \infty) \to (0, \infty)\).

Using Theorem 1 and the Remark 8 (ii) yields for any \(u \in D(f(A))\) with \(\Phi(u) = 1\),

\[
\Theta(\|u\|_2^2) \leq \langle f(A) u, u \rangle,
\]

where

\[
\Theta(x) = \frac{x}{2} f \left( B \left( \frac{x}{2} \right) \right) = \frac{x}{2} \sup_{s > 0} f \left( \frac{1 - 2\beta(s)/x}{s} \right).
\]
For $r > 0$, define

$$\tilde{\beta}(r) = \sup_{s > 0} \{ \Theta^{-1}(s) - rs \}.$$

Then,

$$\|u\|_2^2 \leq r \langle f(A)u, u \rangle + \tilde{\beta}(r) \Phi(u), \quad r > 0, \ u \in D(f(A)). \quad (15)$$

Next, we will estimate $\tilde{\beta}(r)$. By (14),

$$\Theta(x) \geq \frac{x}{2} f \left( \frac{1}{2 \beta^{-1}(x/4)} \right) := \Theta_0(x),$$

which in turn implies that

$$\Theta^{-1}(x) \leq \Theta_0^{-1}(x).$$

By the definition of $\Theta_0(x)$, $\Theta_0 : (0, \infty) \rightarrow (0, \infty)$ is a strictly increasing function such that $\lim_{x \to 0} \Theta_0(x) = 0$ and $\lim_{x \to \infty} \Theta_0(x) = \infty$, and so

$$\Theta_0^{-1}(x) = 2x \left[ f \left( \frac{1}{2 \beta^{-1}(\Theta_0^{-1}(x)/4)} \right) \right]^{-1}. \quad (16)$$

On the other hand,

$$\tilde{\beta}(r) \leq \sup_{s > 0} \{ \Theta_0^{-1}(s) - rs \} = \sup_{s > 0, \Theta_0^{-1}(s) \geq rs} \Theta_0^{-1}(s).$$

From (16) we see that $\Theta_0^{-1}(s) \geq rs$ is equivalent to

$$\frac{1}{2 f^{-1}(2/r)} \leq \beta^{-1} \left( \frac{\Theta_0^{-1}(s)}{4} \right).$$

Since $\beta$ is decreasing, we can rewrite this as

$$\Theta_0^{-1}(s) \leq 4 \beta \left( \frac{1}{2 f^{-1}(2/r)} \right),$$

and so

$$\tilde{\beta}(r) \leq \sup_{s > 0, \Theta_0^{-1}(s) \leq 4 \beta \left( \frac{1}{2 f^{-1}(2/r)} \right)} \Theta_0^{-1}(s) \leq 4 \beta \left( \frac{1}{2 f^{-1}(2/r)} \right). \quad (17)$$

The proof is complete if we combine (15) and (17).

4.2 Subordinate weak Poincaré inequalities

We can also consider the subordination for weak Poincaré inequalities; for details we refer to [10] or [17, Chapter 4].

**Proposition 10** Assume that $(A, D(A))$ satisfies the following weak Poincaré inequality:

$$\|u\|_2^2 \leq \alpha(r) \langle Au, u \rangle + r \Phi(u), \quad r > 0, \ u \in D(A), \quad (18)$$

where $\alpha : (0, \infty) \rightarrow (0, \infty)$ is a decreasing function. Then the generator $f(A)$ of the subordinate semigroup also satisfies a weak Poincaré inequality

$$\|u\|_2^2 \leq \alpha_f(r) \langle f(A)u, u \rangle + r \Phi(u), \quad r > 0, \ u \in D(f(A)), \quad (19)$$

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where

\[ \alpha_f(r) = \frac{2}{\left[f\left(\frac{1}{2\alpha(r/4)}\right)\right]} . \]

**Proof** Suppose that (18) holds. As in the proof of Proposition 9 we find that

\[ \|u\|_2^2 \leq \tilde{\alpha}(r) \langle f(A)u, u \rangle + r \Phi(u), \quad r > 0, \quad u \in D(f(A)), \]

where

\[ \tilde{\alpha}(r) = \sup_{s > 0} \left\{ \frac{\Theta^{-1}(s) - r}{s} \right\} \quad \text{and} \quad \Theta(x) = \frac{x}{2} \sup_{s > 0} f\left(\frac{1 - 2s/x}{\alpha(s)}\right) . \]

If we set \( s = x/4 \),

\[ \Theta(x) \geq \frac{x}{2} f\left(\frac{1}{2\alpha(x/4)}\right) =: \Theta_0(x) , \]

and this gives us

\[ \tilde{\alpha}(r) = \sup_{s > 0} \left\{ \frac{\Theta^{-1}(s) - r}{s} \right\} \leq \sup_{s > 0, \Theta_0^{-1}(s) \geq r} \frac{\Theta^{-1}(s)}{s} . \]

According to the definition of \( \Theta_0(x) \), we have

\[ \frac{\Theta_0^{-1}(x)}{x} = 2 \left[f\left(\frac{1}{2\alpha(\Theta_0^{-1}(x)/4)}\right)\right]^{-1} . \]

Since \( \alpha \) is decreasing,

\[ \sup_{s > 0, \Theta_0^{-1}(s) \geq r} \frac{\Theta_0^{-1}(s)}{s} = \sup_{s > 0, \Theta_0^{-1}(s) \geq r} 2 \left[f\left(\frac{1}{2\alpha(\Theta_0^{-1}(s)/4)}\right)\right]^{-1} \leq 2 \left[f\left(\frac{1}{2\alpha(r/4)}\right)\right]^{-1} . \]

The required inequality (19) follows from (20), (21) and (22).

\[ \Box \]

4.3 The converses of Theorem 1 and Propositions 9 and 10

If \( A \) is a nonnegative self-adjoint operator, then it is possible to show a converse to the assertions of Theorem 1 and Propositions 9 and 10.

**Proposition 11** Let \( A \) be a nonnegative self-adjoint operator on \( L^2(X, m) \), and \( f \) be some non-degenerate Bernstein function. Let \( \Phi : L^2(X, m) \to [0, \infty) \) be a measurable functional satisfying \( \Phi(cu) = c^2 \Phi(u) \) and \( \Phi(T_t u) \leq \Phi(u) \) for all \( t \geq 0 \) and \( \Phi(u) = 0 \) if, and only if, \( u = 0 \), where \( (T_t)_{t \geq 0} \) is the semigroup generated by \( A \). If the following Nash-type inequality

\[ \|u\|_2^2 f\left(\frac{2}{f(\|u\|_2^2)}\right) \leq \langle f(A)u, u \rangle, \quad u \in D(f(A)), \quad \Phi(u) = 1 \]

holds for some increasing function \( B : (0, \infty) \to (0, \infty) \), then

\[ \|u\|_2^2 B\left(\frac{\|u\|_2^2}{f(\|u\|_2^2)}\right) \leq \langle Au, u \rangle, \quad u \in D(A), \quad \Phi(u) = 1 . \]
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Proof Every non-degenerate (i.e. non-constant) Bernstein function $f$ is strictly increasing and concave. Thus $f^{-1}$ is strictly increasing and convex. Let $(E_\lambda)_{\lambda \geq 0}$ be the spectral resolution of the self-adjoint operator $A$. Using Jensen’s inequality we get for all $u \in D(A)$ with $\|u\|_1 = 1$

$$B(\|u\|_2^2) = f^{-1} \circ f(B(\|u\|_2^2))$$

$$\leq f^{-1}\left(\frac{\langle f(A)u, u \rangle}{\|u\|_2^2}\right)$$

$$= f^{-1}\left(\int_{[0, \infty)} f(\lambda) \frac{dE_\lambda(u, u)}{\|u\|_2^2}\right)$$

$$\leq \int_{(0, \infty)} f^{-1} \circ f(\lambda) \frac{dE_\lambda(u, u)}{\|u\|_2^2}$$

$$= \frac{\langle Au, u \rangle}{\|u\|_2^2},$$

cf. also [2, Proposition 2.3]. □

Using Proposition 11 we can get the converses of Propositions 9 and 10. For example, if the following super-Poincaré inequality

$$\|u\|_2^2 \leq r \langle f(A)u, u \rangle + \beta_f(r) \Phi(u), \quad r > 0, \ u \in D(f(A))$$

holds for some decreasing function $\beta_f : (0, \infty) \to (0, \infty)$, then

$$\|u\|_2^2 \leq r \langle Au, u \rangle + \beta(r) \Phi(u), \quad r > 0, \ u \in D(A),$$

where

$$\beta(r) = 2 \beta_f \left(\frac{1}{2 f(1/r)}\right).$$

4.4 On-diagonal estimates for subordinate semigroups: Nash type inequalities

In this section $X$ is the $n$-dimensional Euclidean space $\mathbb{R}^n$ equipped with Lebesgue measure $m(dx) = dx$.

Proposition 12 Assume that $(A, D(A))$ satisfies the following Nash-type inequality

$$\|u\|_2^2 B(\|u\|_2^2) \leq \langle Au, u \rangle, \quad u \in D(A), \quad \|u\|_1 = 1,$$

where $B : (0, \infty) \to (0, \infty)$ is some increasing function. Then, if for any $t > 0$,

$$\eta(t) := \int_t^\infty \frac{du}{u f(B(u))} < \infty,$$

the subordinate semigroup $(T_t^f)_{t \geq 0}$ has a bounded kernel $p_t^f(x, y)$ with respect to Lebesgue measure, and the following on-diagonal estimate holds:

$$\text{ess sup}_{x, y \in \mathbb{R}^d} p_t^f(x, y) = \|T_t^f\|_{1 \to \infty} \leq 2 \eta^{-1}\left(\frac{t}{2}\right).$$
Proof By Theorem 1 we know that the generator \( f(A) \) of the subordinate semigroup \( (T_t^f)_{t \geq 0} \) satisfies
\[
\frac{\|u\|^2}{2} f \circ B \left( \frac{\|u\|^2}{2} \right) \leq \langle f(A) u, u \rangle, \quad u \in D(f(A)), \quad \|u\|_1 = 1.
\] (23)

Therefore the required assertion follows from [7, Proposition II.2] or [17, Theorem 3.3.17 (1), p. 158].

4.5 Contractivity of subordinate semigroups: Super- and Weak Poincaré inequalities

Let \((X, m)\) be a measure space with a σ-finite measure \(m\). Let \((T_t)_{t \geq 0}\) be a semigroup on \(L^2(X, m)\) which is bounded on \(L^p(X, m)\) for all \(p \in [1, \infty]\). This is, e.g., always the case for symmetric sub-Markovian contraction semigroups on \(L^2(X, m)\).

Recall that a semigroup \((T_t)_{t \geq 0}\) is said to be hypercontractive if \(\|T_t\|_{2 \rightarrow 4} < \infty\) for some \(t > 0\), supercontractive if \(\|T_t\|_{2 \rightarrow 4} < \infty\) for all \(t > 0\), and ultracontractive if \(\|T_t\|_{1 \rightarrow \infty} < \infty\) for all \(t > 0\). The example below improves [2, Theorem 3.1].

Proposition 13 Let \(f\) be a Bernstein function and \((T_t)_{t \geq 0}\) be an ultracontractive symmetric sub-Markovian semigroup on \(L^2(X, m)\) such that for all \(t > 0\),
\[
\|T_t\|_{1 \rightarrow \infty} \leq \exp(\lambda t^{-1/(\delta-1)})
\]
for some \(\lambda > 0\) and \(\delta > 1\). Then, we have the following statements for the subordinate semigroup \((T_t^f)_{t \geq 0}\):

(i) If \(\int_1^\infty \frac{dr}{f(r^2)} < \infty\), then \((T_t^f)_{t \geq 0}\) is ultracontractive.

(ii) If \(\lim_{r \rightarrow \infty} \frac{f^{-1}(\lambda)}{\lambda^2} = 0\), then \((T_t^f)_{t \geq 0}\) is supercontractive.

(iii) If \(\lim_{r \rightarrow \infty} \frac{f^{-1}(\lambda)}{\lambda^2} \in (0, \infty)\), then \((T_t^f)_{t \geq 0}\) is hypercontractive.

(iv) If \(\lim_{r \rightarrow \infty} \frac{f^{-1}(\lambda)}{\lambda^2} = \infty\), then \((T_t^f)_{t \geq 0}\) is not hypercontractive.

Proof Denote by \(A\) and \(f(A)\) the generators of the semigroups \((T_t)_{t \geq 0}\) and \((T_t^f)_{t \geq 0}\), respectively. By [7, Proposition II. 4] and [17, Proposition 3.3.16, p. 157], we know that the following super-Poincaré inequality holds:
\[
\|u\|^2 \leq r \langle Au, u \rangle + \beta(r) \|u\|^2, \quad r > 0, \quad u \in D(A),
\]
where
\[
\beta(r) = c_1[\exp(c_2 r^{-1/\delta}) - 1]
\]
for some \(c_1, c_2 > 0\). By Proposition 9,
\[
\|u\|^2 \leq r \langle f(A) u, u \rangle + \beta_f(r) \|u\|^2, \quad r > 0, \quad u \in D(f(A)),
\]
where
\[
\beta_f(r) = 4c_1[\exp[c_3(f^{-1}(2/r))^{1/\delta}] - 1]
\]
for some constant \(c_3 > 0\). Therefore, the required assertions follow from [17, Theorem 3.3.14, p. 156 and Theorem 3.3.13, p. 155] and the comment after Proposition 11.
We close this section with a result that shows how decay properties are inherited under subordination.

**Proposition 14** Let \((T_t)_{t \geq 0}\) be a symmetric sub-Markovian semigroup on \(L^2(X, m)\). Assume that there exist two constants \(\delta, c_0 > 0\) such that

\[
\|T_t u\|_2^2 \leq \frac{c_0 \Phi(u)}{t^\delta} \quad \text{for all } t > 0, \ u \in L^2(X, m),
\]

where \(\Phi : L^2(X, m) \to [0, \infty)\) is a functional with \(\Phi(cu) = c^2 \Phi(u)\) and \(\Phi(T_t u) \leq \Phi(u)\) for all \(c \in \mathbb{R}\) and \(t \geq 0\) and \(\Phi(u) = 0\) if, and only if, \(u = 0\). If

\[
\eta(t) := \int_t^\infty \frac{ds}{sf(s)} < \infty \quad \text{for all } t > 0,
\]

then there are constants \(c_1, c_2 > 0\) such that

\[
\|T_t f u\|_2^2 \leq c_1 [\eta^{-1}(c_2 t)]^\delta \Phi(u).
\]

**Proof** Denote by \(A\) and \(f(A)\) the generators of \((T_t)_{t \geq 0}\) and \((T_t^f)_{t \geq 0}\), respectively. From [17, Corollary 4.1.8 (1), p. 189; and Corollary 4.1.5 (2), p. 186] we know that the following weak Poincaré inequality holds:

\[
\|u\|_2^2 \leq \alpha(r) \langle A u, u \rangle + r \Phi(u), \quad r > 0, \ u \in D(A),
\]

where

\[
\alpha(r) = c_3 r^{-1/\delta}
\]

for some \(c_3 > 0\). Proposition 10 shows that

\[
\|u\|_2^2 \leq \alpha_f(r) \langle f(A) u, u \rangle + r \Phi(u), \quad r > 0, \ u \in D(f(A)),
\]

where

\[
\alpha_f(r) = 2[f(c_4 r^{1/\delta})]^{-1}
\]

for some constant \(c_4 > 0\). Therefore, the assertion follows from [17, Theorem 4.1.7, p. 188].

**Note added in Proof** After we have finished this paper, Patrick Maheux informed us that he and Ivan Gentil have, independently, obtained similar results in their (at that point still forthcoming) preprint [8]; although our findings partially overlap, the methods used here and in [8] are essentially different.

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