Variational discretization of one-dimensional elliptic optimal control problems with BV functions based on the mixed formulation

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Abstract. We consider optimal control of an elliptic two-point boundary value problem governed by functions of bounded variation (BV). The cost functional is composed of a tracking term for the state and the BV-seminorm of the control. We use the mixed formulation for the state equation together with the variational discretization approach, where we use the classical lowest order Raviart-Thomas finite elements for the state equation. Consequently the variational discrete control is a piecewise constant function over the finite element grid. We prove error estimates for the variational discretization approach in combination with the mixed formulation of the state equation and confirm our analytical findings with numerical experiments.

1 Problem formulation

We consider the optimal control problem

\[
\begin{align*}
\min_{u \in BV(\Omega)} \ J(u) := \frac{1}{2} \| y - y_d \|_{L^2(\Omega)}^2 + \alpha \| u' \|_{M(\Omega)},
\end{align*}
\]

\((P)\)

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where $y$ satisfies the one-dimensional elliptic equation

$$
\begin{cases}
-(ay')' + dy = u & \text{in } \Omega, \\
y = 0 & \text{on } \Gamma.
\end{cases}
$$

(1)

Here $\Omega = (0, 1)$ with boundary $\Gamma = \{0, 1\}$, and $\alpha > 0$ is a given parameter. We assume $a \in W^{1,\infty}(\Omega), a \geq a_0 > 0$ a.e. in $\Omega$, where $a_0$ is a constant, and $d \in L^{\infty}(\Omega), d \geq 0$ a.e. in $\Omega$. We denote the control by $u$ which we seek in $BV(\Omega)$, the state by $y \in H^1_0(\Omega)$, and the desired state by $y_d \in L^{\infty}(\Omega)$.

Our work is motivated by [21], where a similar optimal control problem is considered. There, variational discretization (from [24]) combined with the classical piecewise linear and continuous finite element approximation of the state is investigated, and also a fully discrete approach with piecewise constant control approximations. The variational discrete approach leads to the approximation order of $h^2$ for both, control in $L^1$ and state in $L^2$, whereas the fully discrete approach only gives the optimal approximation order $h$. We here propose a variational discrete approach, which automatically delivers piecewise constant control approximations. Although this approach also only delivers the optimal approximation order of $h$ for the state in $L^2$ and for the control in $L^1$, it allows a more elegant and more natural numerical analysis than the fully discrete approach presented in [21]. This is achieved by formulating the elliptic partial differential equation in its mixed form. Variational discretization based on the classical Raviart-Thomas discretization of the state equation then delivers piecewise constant control approximations, while keeping the corresponding variationally discrete, reduced optimization problem infinite-dimensional. This in turn then simplifies the numerical analysis, since e.g. the optimal control of the continuous problem in this approach can be used as comparison function in the variationally discretized optimization problem.

We give a brief overview of related literature. An early result in optimization with BV functions and regularization by BV-seminorms is [13]. Further studies involving BV functions are [4, 5, 6]. There exist studies of elliptic optimal control with total variation regularization and control in $L^{\infty}(\Omega)$, see [15, 25]. Controls from the space $BV(\Omega) \cap L^{\infty}(\Omega)$ are considered in [8]. Optimal control governed by a semilinear parabolic equation and control cost in a total bounded variation seminorm is discussed in [9], a convergence result is shown and numerical experiments are presented. A similar problem is analyzed in [10], but with semilinear elliptic equation. Numerical results for problems with BV-control are derived in [30]. In [26] the BV source in an
elliptic system is recovered. Furthermore, we remark that the inherent sparsity structure of the problem is closely related to the sparsity structure observed in optimal control problems with measures control, see e.g. [11, 12, 22, 23].

We structure this work as follows: In Section 2 we introduce the mixed formulation of the state equation, prove existence of a unique solution to the elliptic optimal control problem and derive its optimality conditions and sparsity structure. We apply variational discretization to the problem in Section 3 and discuss the resulting structure of the non-discretized controls. Then, we proceed analogously to the analysis of the continuous problem by proving existence of a solution, deriving optimality conditions and sparsity structure. We also examine error estimates. Finally, in Section 4 we explain the numerical implementation and present computational results for two different examples. We compare our findings to the experiments from [21].

2 Continuous optimality system

We begin by examining the state equation. We set $z = ay'$. Then the state $y$ is supposed to solve (1) in the following weak sense: Find $(z, y) \in H^1(\Omega) \times L^2(\Omega)$, such that

\begin{align}
\int_{\Omega} \left( \frac{1}{a} z v + v' y \right) \mathrm{d}x &= 0 \quad \forall v \in H^1(\Omega), \quad (2a) \\
\int_{\Omega} (-z' w + dy w) \mathrm{d}x &= \int_{\Omega} u w \mathrm{d}x \quad \forall w \in L^2(\Omega). \quad (2b)
\end{align}

Here we write $(z, y) = (z(u), y(u))$ for the solution of (2). We know by [28, Theorem 1] that $(z(u), y(u))$ admits a unique solution $(z, y) \in H^1(\Omega) \times H^1_0(\Omega)$, where $y$ solves (1) and $z = ay'$. Furthermore, we define the forms $a(z, v) := \int_{\Omega} \frac{1}{a} z v, \quad b(v, y) := \int_{\Omega} v' y, \quad c(y, w) := \int_{\Omega} dy w$ for all $(v, w) \in H^1(\Omega) \times L^2(\Omega)$. Then, for given $u \in L^2(\Omega)$, the pair $(z, y) = (z(u), y(u))$ solves (2), iff

\[ a(z, v) + b(v, y) - b(z, w) + c(y, w) = (u, w)_{L^2(\Omega)} = \begin{pmatrix} 0 \\ u \end{pmatrix} \left( \begin{pmatrix} v \\ w \end{pmatrix} \right)_{\Omega} \quad \forall (v, w) \in H^1(\Omega) \times L^2(\Omega). \]

Analogously, we for $s \in L^2(\Omega)$ define the pair $(q, p) = (q(s), p(s))$ as the unique solution to

\[ a(v, q) + b(q, w) - b(v, p) + c(w, p) = (s, w)_{L^2(\Omega)} = \begin{pmatrix} 0 \\ s \end{pmatrix} \left( \begin{pmatrix} v \\ w \end{pmatrix} \right)_{\Omega} \quad \forall (v, w) \in H^1(\Omega) \times L^2(\Omega). \]
Remark 1. We note that we also may allow data in the first component of the vectors \((0, u)^\top, (0, s)^\top\). However, we in the present work only consider control problems which directly affect the state \(y\) by the control \(u\), and only observations of the state \(y\), not of the derivative \(z\) of \(y\). Introducing the mixed formulation offers the opportunity of including quantities containing \(z = y'\) in the target functional. Furthermore, \([3]\) constitutes the weak form of the adjoint equation associated to \([1]\).

Let us note that problem \((2)\) for \(u \in L^2(\Omega)\) admits a unique solution \((z(u), y(u)) \in H^1_0(\Omega) \times H^1(\Omega)\), which satisfies

\[
||y(u)||_{H^2(\Omega)} + ||z(u)||_{H^1(\Omega)} \leq C ||u||_{L^2(\Omega)},
\]

with some \(C > 0\), compare \([20, \text{Lemma 2.2.}]\). Also, we with \([21, \text{Theorem 2.2.}]\) directly have

**Theorem 2.** Problem \((P)\) admits a unique solution \(\bar{u} \in BV(\Omega)\) with associated optimal state \(\bar{y} \in H^1_0(\Omega) \cap H^2(\Omega)\) and associated \(\bar{z} \in H^1(\Omega)\).

Similar to \([21, \text{Theorem 2.3.}]\), but adapted to the mixed formulation of the state equation, we provide the following optimality conditions.

**Theorem 3.** The control \(\bar{u} \in BV(\Omega)\) with associated \((\bar{z}, \bar{y}) \in H^1(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)\) is optimal for the problem \((P)\) if and only if there exists a unique pair \((\bar{q}, \bar{p}) \in H^1(\Omega) \times H^1_0(\Omega) \cap H^2(\Omega)\), such that \((\bar{u}, \bar{z}, \bar{y}, \bar{q}, \bar{p})\) and the \(H^3(\Omega)\) function \(\tilde{\Phi}(\cdot) := \int_0^\cdot \bar{p}(s)ds\) satisfy \(\tilde{\Phi}(1) = 0\) as well as

\[
\int_{\Omega} \tilde{\Phi} \, d\bar{u}' = a ||\bar{u}'||_{L(\Omega)},
\]

\[
||\tilde{\Phi}||_{C(\Omega)} \leq \sigma,
\]

\[
\int_{\Omega} \left( \frac{1}{a} \bar{z} v + v' \bar{y} \right) \, dx = 0 \quad \forall v \in H^1(\Omega),
\]

\[
\int_{\Omega} (\bar{z}' w + d \bar{y} w) \, dx = \int_{\Omega} w \bar{u} \, dx \quad \forall w \in L^2(\Omega),
\]

\[
\int_{\Omega} \left( \frac{1}{a} \bar{q} v + v' \bar{p} \right) \, dx = 0 \quad \forall v \in H^1(\Omega),
\]

\[
\int_{\Omega} (\bar{q}' w + d \bar{p} w) \, dx = \int_{\Omega} w (\bar{y} - \gamma_d) \, dx \quad \forall w \in L^2(\Omega),
\]

\[
-(\bar{p}, u - \bar{u})_{L^2(\Omega)} \leq a \left( ||u'||_{L(\Omega)} - ||\bar{u}'||_{L(\Omega)} \right) \quad \forall u \in BV(\Omega).
\]

We note that here \((\bar{q}, \bar{p}) = (q(\bar{y} - \gamma_d), p(\bar{y} - \gamma_d))\).

The problem inherits a sparsity structure, where the structure delivers information about the support of \(\bar{u}'\), not about the support of the optimal control itself. The support of \(\bar{u}'\) indicates the
location of the jumping points of the optimal control $\bar{u} \in BV(\Omega)$. For the convenience of the reader we recall [21 Corollary 1]:

**Lemma 4.** If $\bar{u}$ is optimal for $(P)$, then there hold

$$\text{supp}((\bar{u}')^+) \subset \{ x \in \Omega : \bar{\Phi}(x) = \alpha \},$$

(12)

$$\text{supp}((\bar{u}')^-) \subset \{ x \in \Omega : \bar{\Phi}(x) = -\alpha \},$$

(13)

where $\bar{u}' = (\bar{u}')^+ - (\bar{u}')^-$ is the Jordan decomposition. Moreover, we have

$$\text{supp}(\bar{u}') \subset \{ x \in \Omega : |\bar{\Phi}(x)| = \alpha \} \subset \{ x \in \Omega : \bar{p}(x) = 0 \}. $$

(14)

### 3 Variational discretization

Our aim is to introduce a piecewise constant control approximation, which is fully aligned with the discretization of our state equation. We achieve this by employing variational discretization for our optimal control problem $(P)$ combined with the classical lowest order Raviart-Thomas discretization of the mixed form of the state equation (2). Along with this discrete approach come the facts that our discrete counterpart of $(P)$ still remains infinite-dimensional and that the optimality conditions (7)-(10) remain valid with continuous variables replaced by their discrete analogues. Then the piecewise constant discretization of the adjoint state $p$ in combination with the optimality conditions for the variational discrete problem induce the piecewise constant structure of the control $u_{vd}$ through the fact that under a natural structural assumption $u_{vd}'$ is a sum of Dirac measures. This then immediately delivers that the variational discrete control $u_{vd}$ is piecewise constant.

Let $0 = x_0 < x_1 < \ldots < x_N = 1$ be a partition of $\bar{\Omega} = [0, 1]$. Then for $i = 1, \ldots, N$ we define the subintervals $I_i := (x_{i-1}, x_i)$ of size $h_i := x_i - x_{i-1}$ and set $h := \max_{1 \leq i \leq N} h_i$. Let $\chi_i$ for $i = 1, \ldots, N$ be the indicator function of interval $I_i$, i.e.

$$\chi_i(x) = \begin{cases} 
1, & x \in I_i, \\
0, & \text{else}.
\end{cases}$$

Let $e_j$ for $j = 0, \ldots, N$ denote the hat functions, i.e. those functions which are piecewise linear and continuous on the partition, satisfying $e_j(x_i) = \delta_{ij}$ for $i, j = 0, \ldots, N$. 

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We introduce the discrete spaces
\[ P_0 := \text{span}\{ \chi_i : 1 \leq i \leq N \}, \quad P_1 := \text{span}\{ e_j : 0 \leq j \leq N \}. \]

Using these spaces we get the discrete formulation of (2): Find \( y_h = \sum_{i=1}^N y_i \chi_i \in P_0 \), and \( z_h = \sum_{j=0}^N v_j e_j \in P_1 \), such that
\[
\int_\Omega \left( \frac{1}{2} z_h v_h + v'_h y_h \right) \, dx = 0 \quad \forall v_h \in P_1, \tag{15a}
\]
\[
\int_\Omega \left( -z'_h w_h + d y_h w_h \right) \, dx = \int_\Omega w_h u \, dx \quad \forall w_h \in P_0. \tag{15b}
\]

For given \( u \) we write \((z_h, y_h) = (z_h(u), y_h(u))\) for the unique solution of the discrete state equation. In the present case where \( \Omega \subset \mathbb{R} \), the space of Raviart-Thomas elements of lowest order (see e.g. [3, 28]) coincides with the chosen pair \((P_1, P_0)\). Furthermore, we stress that the control space remains \( BV(\Omega) \), so the variational discrete control \( u_{vd} \) is not discretized. The variational discrete counterpart of (P) then reads
\[
\min_{u_{vd} \in BV(\Omega)} J_h(u_{vd}) := \frac{1}{2} \| y_h(u_{vd}) - y_d \|^2_{L^2(\Omega)} + \alpha \| u'_{vd} \|_{L^\infty(\Omega)}. \tag{P_vd}
\]

As in [21, Definition 3.9., Lemma 3.10.] we define the following projection operator \( \Upsilon_h \) and collect its properties.

**Lemma 5.** For \( i = 1, \ldots, N \) let the operator \( \Upsilon_h \) be defined as:
\[
\Upsilon_h : BV(\Omega) \to P_0, \quad \Upsilon_h u_{|I_i} := \frac{1}{h} \int_{I_i} u(s) \, ds.
\]

For any \( u \in BV(\Omega) \) and \( w_h \in P_0 \) it holds
\[
(u, w_h)_{L^2(\Omega)} = (\Upsilon_h u, w_h)_{L^2(\Omega)}, \tag{16}
\]
\[
\| u - \Upsilon_h u \|_{L^1(\Omega)} \leq h \| u' \|_{L^\infty(\Omega)}, \tag{17}
\]
\[
\| (\Upsilon_h u)' \|_{L^\infty(\Omega)} \leq h \| u' \|_{L^\infty(\Omega)} \tag{18}
\]
\[
\| u - \Upsilon_h u \|_{L^\infty(\Omega)} \leq h \| u' \|_{L^\infty(\Omega)}, \quad \text{provided that } u \in W^{1,\infty}(\Omega). \tag{19}
\]
The proof can be collected from [9, Proposition 16] and [21, Lemma 3.10].

We next give the discrete counterpart of Theorem 2.

**Theorem 6.** Problem \( P_{vd} \) admits an optimal solution \( \bar{u}_{vd} \in \text{BV}(\Omega) \) and associated unique \( (\bar{z}_h, \bar{y}_h) \in P_1 \times P_0 \). For every solution \( \bar{u}_{vd} \), \( \Upsilon_h \bar{u}_{vd} \) also solves \( P_{vd} \) and \( \Upsilon_h \bar{u}_{vd} \in P_0 \) is unique. Furthermore, there exist \( C, h_0 > 0 \), such that for all \( h \in (0, h_0] \) we have

\[
\|\bar{u}_{vd}\|_{\text{BV}(\Omega)} \leq C \tag{20}
\]

for any optimal control \( \bar{u}_{vd} \).

**Proof.** Since the control \( u_{vd} \) remains continuous, i.e. is not discretized, and \( |y_h(1)| \geq \frac{1}{2}|y(1)| \) by Lemma [11] for \( h \) small enough, we can use the proof for existence of solutions from Theorem 2 verbatim, with \( y(1) \) replaced by \( y_h(1) \).

We stress that uniqueness of the control is not given in this setting, since the control is not discretized, so the control-to-state operator is in general not injective. However by definition of \( \Upsilon_h \) we know that \( \Upsilon_h \bar{u}_{vd} \) is also a solution, since \( J_h(\Upsilon_h \bar{u}_{vd}) \leq J_h(\bar{u}_{vd}) \). It is easy to see that the restriction of the mapping \( u_{vd} \mapsto y_h \) to \( P_0 \) is injective, so that the quadratic term in \( J_h \) now delivers strict convexity of \( J_h \) on \( P_0 \). Therefore, uniqueness of solution in the discrete space \( P_0 \) is evident. Due to uniqueness of the discrete solution, all projections of solutions \( \bar{u}_{vd} \in \text{BV}(\Omega) \) must coincide.

For the proof of the boundedness in the \( \text{BV} \)-norm we may proceed along the lines of the respective proof in [21, Theorem 3.5], First, we recall \( \|\bar{u}_{vd}\|_{\text{BV}(\Omega)} = \|\bar{u}_{vd}\|_{L^1(\Omega)} + \|\bar{u}_{vd}'\|_{M(\Omega)} \). Since, by optimality of \( \bar{u}_{vd} \) it holds that \( J_h(\bar{u}_{vd}) \leq J_h(0) \), we have

\[
\|\bar{u}_{vd}'\|_{M(\Omega)} \leq \frac{J_h(0)}{\alpha} = \frac{\|y_h(0) - y_d\|_{L^2(\Omega)}^2}{2\alpha} = \frac{\|y_d\|_{L^2(\Omega)}^2}{2\alpha}.
\]

Introducing \( \hat{u} := \frac{1}{|\Omega|} \int_\Omega \bar{u}_{vd} \, dx \), we from [11, Remark 3.50] have

\[
\|\bar{u}_{vd} - \hat{u}\|_{L^2(\Omega)} \leq C\|\bar{u}_{vd}'\|_{M(\Omega)},
\]

with \( C \) independent of \( h \). This implies, exploiting \( |\Omega| = 1 \)

\[
\|\bar{u}_{vd}\|_{L^1(\Omega)} \leq \|\bar{u}_{vd}\|_{L^2(\Omega)} \leq C \frac{\|y_d\|_{L^2(\Omega)}^2}{2\alpha} + |\hat{u}|.
\]

To show the boundedness of \( \bar{u}_{vd} \) in \( \text{BV}(\Omega) \) it remains to show the boundedness of \( |\hat{u}| \). Since the
mapping $u \mapsto y_h(u)$ is linear and $\tilde{u} \in \mathbb{R}$, we have $y_h(\tilde{u}) = \tilde{u}y_h(1)$, and thus

$$||\tilde{u}||_{y_h(1)}||_{L^2(\Omega)} = ||y_h(\tilde{u})||_{L^2(\Omega)} \leq ||y_h(\tilde{u} - \bar{u}_{vd})||_{L^2(\Omega)} + ||y_h(\bar{u}_{vd}) - y_{vd}||_{L^2(\Omega)} + ||y_{vd}||_{L^2(\Omega)}$$

We employ (32) to obtain

$$||y_h(\tilde{u} - \bar{u}_{vd}) - y(\tilde{u} - \bar{u}_{vd})||_{L^2(\Omega)} \leq C h \left( ||\tilde{u} - \bar{u}_{vd}||_{L^2(\Omega)} + ||y(\tilde{u} - \bar{u}_{vd})||_{H^2(\Omega)} \right)$$

Altogether, we have, using the continuity of $u \mapsto y(u) \in H^1_0(\Omega) \cap H^2(\Omega)$:

$$||\tilde{u}||_{y_h(1)}||_{L^2(\Omega)} \leq C h ||\tilde{u} - \bar{u}_{vd}||_{L^2(\Omega)} + C h ||y(\tilde{u} - \bar{u}_{vd})||_{H^2(\Omega)} + ||y(\tilde{u} - \bar{u}_{vd})||_{L^2(\Omega)} + 2 ||y_{vd}||_{L^2(\Omega)}$$

Since $|y_h(1)| \geq \frac{1}{2} |y(1)|$ for $h$ small enough we conclude $||\tilde{u}|| \leq C$, which finally delivers the boundedness $||\tilde{u}_{vd}||_{BV(\Omega)} \leq C$. \hfill $\square$

Analogously to Theorem 3 from the continuous setting we derive the optimality conditions for $(P_{vd})$.

**Theorem 7.** The control $\bar{u}_{vd} \in BV(\Omega)$ with associated $(\bar{z}_h, \bar{y}_h) = (z_h(\bar{u}_{vd}), y_h(\bar{u}_{vd})) \in P_1 \times P_0$ is optimal for the problem $(P_{vd})$ if and only if there exists a unique pair $(\bar{q}_h, \bar{p}_h) \in P_1 \times P_0$, such that $(\bar{u}_{vd}, \bar{z}_h, \bar{y}_h, \bar{q}_h, \bar{p}_h)$ and the $P_1$-function $\Phi_h(x) := \int_0^1 p_h(s) \, ds$ satisfy $\Phi_h(1) = 0$ as well as
\[ \int \tilde{\Phi}_h \, d(\mathcal{T}_h \tilde{u}_{vd})' = \alpha \| \tilde{u}_{vd}' \|_{\mathcal{L}(\Omega)}, \quad (21) \]

\[ \| \tilde{\Phi}_h \|_{C(\Omega)} \leq \alpha, \quad (22) \]

\[ \int_\Omega \left( \frac{1}{\alpha} \tilde{z}_h v_h + \tilde{v}_h' \tilde{y}_h \right) \, dx = 0 \quad \forall v_h \in P_1, \quad (23) \]

\[ \int_\Omega \left( -\tilde{z}_h' w_h + d \tilde{y}_h w_h \right) \, dx = \int_\Omega w_h \tilde{u}_{vd} \, dx \quad \forall w_h \in P_0, \quad (24) \]

\[ \int_\Omega \left( \frac{1}{\alpha} \tilde{q}_h v_h + \tilde{v}_h' \tilde{p}_h \right) \, dx = 0 \quad \forall v_h \in P_1, \quad (25) \]

\[ \int_\Omega \left( -\tilde{q}_h' w_h + d \tilde{p}_h w_h \right) \, dx = \int_\Omega w_h (\tilde{y}_h - y_d) \, dx \quad \forall w_h \in P_0, \quad (26) \]

\[ -(\tilde{p}_h, u - \tilde{u}_{vd})_{L^2(\Omega)} \leq \alpha \left( \| u' \|_{\mathcal{L}(\Omega)} - \| \tilde{u}_{vd}' \|_{\mathcal{L}(\Omega)} \right) \quad \forall u \in BV(\Omega). \quad (27) \]

**Proof.** The optimality of \( \tilde{u}_{vd} \in BV(\Omega) \) is equivalent to \( 0 \in \partial J_h(\tilde{u}_{vd}) \). By applying the chain rule and the sum rule, we then deduce

\[ -(\tilde{p}_h, \tilde{y}_h - y_d) \in \partial \left( \alpha \| \tilde{u}_{vd}' \|_{\mathcal{L}(\Omega)} \right). \quad (28) \]

We recall \((\tilde{z}_h, \tilde{y}_h) = (z_h(\tilde{u}_{vd}), y_h(\tilde{u}_{vd}))\), which gives (23) and (24). With the definition of \((\tilde{q}_h, \tilde{p}_h) = (q_h(\tilde{y}_h - y_d), \tilde{p}_h(\tilde{y}_h - y_d))\) we directly have (25) and (26), and also that (27) is an equivalent reformulation of (28). Inserting \( u = 2\tilde{u}_{vd}, u = 0, u = \tilde{u}_{vd} + \tilde{u}, \) and \( u = \tilde{u}_{vd} - \tilde{u} \) for arbitrary \( \tilde{u} \in BV(\Omega) \) in (27) delivers

\[ -(\tilde{p}_h, \tilde{u}_{vd})_{L^2(\Omega)} = \alpha \| \tilde{u}_{vd}' \|_{\mathcal{L}(\Omega)}, \quad (29) \]

\[ |(\tilde{p}_h, u)_{L^2(\Omega)}| \leq \alpha \| u' \|_{\mathcal{L}(\Omega)} \quad \forall u \in BV(\Omega). \quad (30) \]

Using (30) we conclude

\[ \tilde{\Phi}_h(1) = \int_0^1 \tilde{p}_h(s) \, ds = (\tilde{p}_h, 1)_{L^2(\Omega)} = 0, \]

and

\[ |\tilde{\Phi}_h(x)| = \left| \int_0^x \tilde{p}_h(s) \, ds \right| = \left| \int_\Omega \tilde{p}_h 1_{(0,x)} \, dx \right| = |(\tilde{p}_h, 1_{(0,x)})_{L^2(\Omega)}| \leq \alpha. \]
So, we can deduce (22). We recall the given structure of $\bar{p}_h, \bar{\tau}_h \bar{\nu}_{vd} \in P_0$, and write

$$\bar{p}_h = \sum_{i=1}^{N} \bar{p}_i \chi_i \quad \text{and} \quad \bar{\tau}_h \bar{\nu}_{vd} = \sum_{i=1}^{N} \bar{\nu}_i \chi_i.$$ 

Also, we collect the following equality for every gridpoint $x_i \in \Omega$

$$\bar{\Phi}_h(x_i) = \int_{0}^{x_i} \bar{p}_h(s) \, ds = \sum_{j=1}^{i} \bar{p}_j h_j.$$ 

Now, we calculate

$$\int_{\Omega} \Phi_h d(\bar{\gamma}_h \bar{\nu}_{vd})' = \sum_{i=1}^{N-1} (\bar{u}_{i+1} - \bar{u}_i) \Phi_h(x_i)$$

$$= \sum_{i=1}^{N-1} \bar{u}_{i+1} \Phi_h(x_i) - \bar{u}_i \Phi_h(x_{i-1}) - \bar{u}_i \bar{p}_i h_i$$

$$= \bar{u}_N \Phi_h(x_{N-1}) - \bar{u}_1 \Phi_h(0) - \sum_{i=1}^{N-1} \bar{u}_i \bar{p}_i h_i$$

$$= \Phi_h(1) - \sum_{i=1}^{N} \bar{u}_i \bar{p}_i h_i$$

$$= -(\bar{p}_h, \bar{\tau}_h \bar{\nu}_{vd})_{L^2(\Omega)}$$

$$= -(\bar{p}_h, \bar{\nu}_{vd})_{L^2(\Omega)}$$

$$\alpha \| \bar{\omega}_{vd} \|_{M(\Omega)},$$

where we use $\Phi_h(0) = \Phi_h(1) = 0$. This shows (21) and completes the proof. \(\square\)

Furthermore, we deduce a similar sparsity structure as in Lemma 4 for the continuous problem.

**Lemma 8.** If $\bar{\omega}_{vd}$ is optimal for $(P_{vd})$, then there hold

$$\text{supp}((\bar{\gamma}_h \bar{\omega}_{vd})') \subset \left\{ x \in \Omega : \Phi_h(x) = \alpha \right\},$$

$$\text{supp}((\bar{\gamma}_h \bar{\omega}_{vd})') \subset \left\{ x \in \Omega : \Phi_h(x) = -\alpha \right\},$$

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Furthermore, we have \( \bar{\Upsilon}_{h} \bar{\alpha}_v \)’ = \( \bar{\Upsilon}_{h} \bar{\alpha}_v \)' is the Jordan decomposition. Moreover, we have

\[
\text{supp}(\bar{\Upsilon}_{h} \bar{\alpha}_v) \subseteq \left\{ x \in \Omega : |\bar{\Phi}_h(x)| = \alpha \right\}.
\]

The sparsity result can be proven as in the continuous case.

Even though the control is not discretized, we can deduce information about the structure of the control from the optimality conditions and the sparsity structure, especially properties (22) and (31). Let us make the following structural assumption:

**Assumption 9.** \( \bar{p}_h(x) \neq 0 \) for all \( x \in \Omega \).

**Remark 10.** This assumption can be easily verified in a numerical setting, hence it is highly practical. Also, later on, we will formulate Assumption [17] for the continuous adjoint state \( \bar{p} \).

We then show that Assumption 9 always holds when Assumption [17] is satisfied for small enough grid sizes.

From \( \bar{p}_h(x) \neq 0 \) for all \( x \in \Omega \) we deduce \( p_i \neq 0 \) for all \( i = 1, \ldots, N \). Since \( \bar{\Phi}_h(x_i) = \bar{\Phi}_h(x_{i-1}) + \bar{p}_i h_i \) holds, we get \( \bar{\Phi}_h(x_i) \neq \bar{\Phi}_h(x_{i-1}) \) for \( i = 1, \ldots, N \). Combining this property with the facts that \( \bar{\Phi}_h \in P_1 \) and \( ||\bar{\Phi}_h||_{C(\Omega)} \leq \alpha \) we deduce

\[
\left\{ x \in \Omega : |\bar{\Phi}_h(x)| = \alpha \right\} \subseteq \{ x_i \}_{i=1}^N.
\]

Furthermore, we have \( \bar{\Phi}_h(x_{i-1}) = \bar{\Phi}_h(x_i) - \bar{p}_i h_i \) and \( \bar{\Phi}_h(x_{i+1}) = \bar{\Phi}_h(x_i) + \bar{p}_{i+1} h_{i+1} \) for any \( i = 1, \ldots, N \). Now fix some \( i \in \{1, \ldots, N\} \) and let \( \bar{\Phi}_h(x_i) = \alpha \). We then use that \( p_i \neq 0, p_{i+1} \neq 0 \) and \( ||\bar{\Phi}_h||_{C(\Omega)} \leq \alpha \) to deduce \( \bar{p}_i > 0 \) and \( \bar{p}_{i+1} < 0 \). Similarly, for \( \bar{\Phi}_h(x_i) = -\alpha \) we see \( \bar{p}_i < 0 \) and \( \bar{p}_{i+1} > 0 \). Altogether, under Assumption 9, we get

\[
\text{supp}(\bar{\Upsilon}_{h} \bar{\alpha}_v) \subseteq \left\{ x \in \Omega : |\bar{\Phi}_h(x)| = \alpha \right\} \subseteq \{ x \in \Omega : \text{sign}(\bar{p}_h(x_-)) \neq \text{sign}(\bar{p}_h(x_+)) \},
\]

where \( \nu(x_-) := \lim_{x \to x_-} \nu(s) \) and \( \nu(x_+) := \lim_{x \to x_+} \nu(s) \).

Following [21], we can express the optimal discrete control \( \bar{\Upsilon}_{h} \bar{\alpha}_v \) and its derivative as

\[
\bar{\Upsilon}_{h} \bar{\alpha}_v = \bar{a}_h + \sum_{i=1}^{N} \bar{c}_h^i \delta_{x_i}, \quad (\bar{\Upsilon}_{h} \bar{\alpha}_v)' = \sum_{i=1}^{N} \bar{c}_h^i \delta_{x_i},
\]

where \( \bar{a}_h \in \mathbb{R}, \bar{c}_h = (\bar{c}_h^1, \ldots, \bar{c}_h^N)' \in \mathbb{R}^N \). We can determine the coefficients \( \bar{a}_h \) and \( \bar{c}_h \) by solving
the finite-dimensional, convex optimization problem

\[
\min_{a_h \in \mathbb{R}, c_h \in \mathbb{R}^N} J_h(a_h, c_h) := \frac{1}{2} \|y_h - y_d\|^2_{L^2(\Omega)} + \alpha \sum_{i=1}^N |c_i^h|.
\]  

\((P_h)\)

3.1 Error estimates

For our numerical analysis we rely on finite element error estimates for mixed approximation of our elliptic two point boundary value problem. Error estimates for mixed finite elements applied to elliptic partial differential equations have been proven e.g. in [7, 18, 19, 20, 28]. For the convenience of the reader we in the following cite the respective results, adapted to our 1D situation, which we need for our numerical analysis.

Before we start, let us recall that \(BV(\Omega) \hookrightarrow L^\infty(\Omega)\) continuously for \(\Omega = (0,1)\), so that for given \(u \in BV(\Omega)\), the unique solution \((z, y)\) of (2) admits the regularity \(z \in W^{1,\infty}(\Omega)\) and \(y \in W^{2,\infty}(\Omega)\). For the solution \((q, p)\) of the adjoint equation (3) with right hand side \(y - y_d\) we thus may expect solutions \(q \in H^1(\Omega), p \in H^2(\Omega)\) for \(y_d \in L^2(\Omega)\), but also higher regularity up to \(q \in W^{3,\infty}(\Omega), p \in W^{4,\infty}(\Omega)\), if e.g. \(y_d \in W^{2,\infty}(\Omega)\) and \(a, d\) are smooth enough, e.g. \(a \in W^{3,\infty}, d \in W^{2,\infty}\). The regularity of \(y_d\) thus will have an influence on the quality of uniform error estimates for \(q\) and \(p\), as shall be seen below.

As in [16] we define the standard \(L^2(\Omega)\)-orthogonal projection \(P_h : L^2(\Omega) \rightarrow P_0\), which for \(w \in L^2(\Omega)\) is defined by

\[(w - P_h w, w_h)_{L^2(\Omega)} = 0 \quad \forall w_h \in P_0.\]

Furthermore, we shall use the Fortin projection (see [7, 16]), defined as \(\Pi_h : H^1(\Omega) \rightarrow P_1\), which for \(v \in H^1(\Omega)\) is defined by

\[((v - \Pi_h v)', w_h)_{L^2(\Omega)} = 0 \quad \forall w_h \in P_0.\]

For later use we collect the following approximation properties, e.g. from [18] (A3) and [20], Section 3] with \(2 \leq p \leq \infty\):

\[
\|w - P_h w\|_{L^p(\Omega)} \leq Ch\|w\|_{L^p(\Omega)} \quad \text{for } w \in W^{1,p}(\Omega),
\]
\[
\|v - \Pi_h v\|_{L^p(\Omega)} \leq Ch\|v\|_{L^p(\Omega)} \quad \text{for } v \in W^{1,p}(\Omega),
\]
\[
\|(v - \Pi_h v)'\|_{L^2(\Omega)} \leq Ch\|v''\|_{L^2(\Omega)} \quad \text{for } v' \in H^1(\Omega).
\]
We repeat some useful a priori error estimates for the mixed finite element approximation.

**Lemma 11.** Let \( u \in L^\infty(\Omega) \) and let \((z,y) = (z(u), y(u))\) denote the unique solution to (2). Let \((z_h(u), y_h(u))\) denote the unique corresponding mixed finite element approximation to \((z,y)\). Then, we have

\[
\|z - z_h(u)\|_{L^1(\Omega)} + \|y - y_h(u)\|_{L^1(\Omega)} \leq Ch \left( \|u\|_{L^1(\Omega)} + \|y\|_{W^2(\Omega)} \right)
\]

for all \( r \in \mathbb{N} \),

\[\forall 2 \leq r < \infty,\] (32)

with \( C \geq 0 \) only depending on \( r \) and on \( \Omega \), and

\[
\|z - z_h(u)\|_{L^r(\Omega)} + \|y - y_h(u)\|_{L^r(\Omega)} \leq Ch \log h \|u\|_{L^\infty(\Omega)}.
\]

(33)

With \( P_h \) and \( \Pi_h \) as introduced above, it holds

\[
\|z - z_h(u)\|_{L^\infty(\Omega)} \leq C \left( \|z - \Pi_h z\|_{L^\infty(\Omega)} + h \|u - P_h u\|_{L^\infty(\Omega)} \right),
\]

(34)

\[
\|P_h y - y_h(u)\|_{L^\infty(\Omega)} \leq Ch \log h \left( \|z - z_h(u)\|_{L^\infty(\Omega)} + \|u - P_h u\|_{L^\infty(\Omega)} \right).
\]

(35)

Let \( y \in W^{3,\infty}(\Omega) \), i.e. \( u \in W^{1,\infty}(\Omega) \), then

\[
h \|y - y_h(u)\|_{L^\infty(\Omega)} + |\log h|^{-1} \|z - z_h(u)\|_{L^\infty(\Omega)} \leq Ch^2 \|u\|_{W^{1,\infty}(\Omega)}.
\]

(36)

For the proof of (32) we refer to the case \( k = 0 \) in [17, Theorem 4.2.]. The results (33), (34), (35), and (36) are [18] Corollary 5.5., Lemma 4.2., Lemma 4.4., and Corollary 5.2., respectively.

We are now prepared to prove the following a priori error estimate for the state.

**Theorem 12.** Let \( \bar{u} \in BV(\Omega) \) denote the unique solution to (2) with associated unique \((\bar{z}, \bar{y})\) from (2), and let \( \bar{u}_{vd} \in BV(\Omega) \) denote a solution of (37) with uniquely determined discrete optimal \((\bar{z}_h, \bar{y}_h)\) from (15). Suppose \( y_{vd} \in W^{1,\infty}(\Omega) \). Then we have

\[
\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}^2 \leq C \left( h^2 + h \|\bar{u} - \bar{u}_{vd}\|_{L^2(\Omega)} \right),
\]

(37)

with a constant \( C > 0 \) independent of \( h \).

**Proof.** From (11) with \( u = \bar{u}_{vd} \) and (27) with \( u = \bar{u} \) we get

\[
- (\bar{p}, \bar{u}_{vd} - \bar{u})_{L^2(\Omega)} \leq \alpha \left( \|\bar{u}_{vd}\|_{M(\Omega)} - \|\bar{u}'\|_{M(\Omega)} \right);
\]

\[
- (\bar{p}_h, \bar{u} - \bar{u}_{vd})_{L^2(\Omega)} \leq \alpha \left( \|\bar{u}\|_{M(\Omega)} - \|\bar{u}_{vd}'\|_{M(\Omega)} \right);
\]
where we recall that \( \bar{\rho} = p(\bar{y} - y_d) \), \( \bar{p}_h = p(\bar{y}_h - y_d) \). Adding these inequalities delivers

\[
0 \leq (\bar{\rho} - \bar{p}_h, \bar{u}_{vd} - \bar{u})_{L^2(\Omega)} = (\bar{\rho} - p_h(\bar{y} - y_d), \bar{u}_{vd} - \bar{u}) + (p_h(\bar{y} - y_d) - \bar{p}_h, \bar{u}_{vd} - \bar{u}) =: (I) + (II).
\]

Since \( \bar{y} - y_d \in W^{1,\infty}(\Omega) \) we with (36) obtain

\[
(I) \leq \| \bar{\rho} - p_h(\bar{y} - y_d) \|_{L^\infty(\Omega)} \| \bar{u}_{vd} - \bar{u} \|_{L^1(\Omega)} \leq C h \| \bar{u}_{vd} - \bar{u} \|_{L^1(\Omega)}.
\]

Using (23)-(26), we get

\[
(II) = \int_\Omega - (z'_h - z_h(\bar{u}))(p_h(\bar{y} - y_d) - \bar{p}_h) + d(\bar{y}_h - y_h(\bar{u}))(p_h(\bar{y} - y_d) - \bar{p}_h)
= \int_\Omega - (q_h(\bar{y} - y_d) - \bar{q}_h(\bar{z}_h - z_h(\bar{u}))) + d(\bar{y}_h - y_h(\bar{u}))(p_h(\bar{y} - y_d) - \bar{p}_h)
= \int_\Omega (\bar{y}_h - y_h(\bar{u}))(\bar{y} - y_d) - (\bar{y}_h - y_h(\bar{u}))(\bar{y}_h - y_d)
= \int_\Omega (\bar{y}_h - y_h(\bar{u}))(\bar{y} - y_d)
= -\| \bar{y} - \bar{y}_h \|_{L^2(\Omega)}^2 + \int_\Omega (\bar{y} - y_h(\bar{u}))(\bar{y} - \bar{y}_h)
\leq -\frac{1}{2}\| \bar{y} - \bar{y}_h \|_{L^2(\Omega)}^2 + \frac{1}{2}\| \bar{y} - y_h(\bar{u}) \|_{L^2(\Omega)}^2.
\]

Combining the estimates for (I) and (II) we with (32) and (36) obtain

\[
\| \bar{y} - \bar{y}_h \|_{L^2(\Omega)}^2 \leq C\left( h^2 + h \| \bar{u}_{vd} - \bar{u} \|_{L^1(\Omega)} \right).
\]

\( \square \)

Let us note that with requiring only \( y_d \in L^\infty(\Omega) \) we with (33) would have obtained

\[
(I) \leq C h \log h \| \bar{u}_{vd} - \bar{u} \|_{L^1(\Omega)}.
\]

Since the variational discrete controls \( \bar{u}_{vd} \) are bounded in \( BV(\Omega) \) w.r.t. \( h \), we have

**Corollary 13.** With the suppositions of Theorem 12 there holds

\[
\| \bar{y} - \bar{y}_h \|_{L^2(\Omega)} \leq C h^2.
\]
We move on to establish an error estimate for the adjoint state.

**Theorem 14.** Let the suppositions of Theorem 12 be satisfied. Let \((\bar{q}, \bar{p}) = (q(\bar{y} - y_d), p(\bar{y} - y_d))\) and \((\bar{q}_h, \bar{p}_h) = (q_h(\bar{y}_h - y_d), p_h(\bar{y}_h - y_d))\). Then, we have

\[
\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} \leq C(h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}),
\]
\[
\|\bar{q} - \bar{q}_h\|_{L^\infty(\Omega)} \leq C(h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}).
\]

**Proof.** We have

\[
\|\bar{p} - \bar{p}_h\|_{L^\infty(\Omega)} = \|p(\bar{y} - y_d) - p_h(\bar{y}_h - y_d)\|_{L^\infty(\Omega)} \\
\leq \|p(\bar{y} - y_d) - p_h(\bar{y}_h - y_d)\|_{L^\infty(\Omega)} + \|p_h(\bar{y}_h - y_d) - p_h(\bar{y}_h - y_d)\|_{L^\infty(\Omega)}.
\]

With (36) we see

\[(I) \leq Ch\|\bar{y} - y_d\|_{W^{1,\infty}(\Omega)} \leq Ch.
\]

The second term can further be estimated by

\[(II) \leq \|p_h(\bar{y} - y_d) - P_h p(\bar{y} - y_d)\|_{L^\infty(\Omega)} + \|P_h p(\bar{y}_h - y_d) - P_h p(\bar{y}_h - y_d)\|_{L^\infty(\Omega)}
\]

\[= (IIa)
\]

\[= (IIb)
\]

\[+ \|P_h p(\bar{y}_h - y_d) - P_h p(\bar{y}_h - y_d)\|_{L^\infty(\Omega)}.
\]

With (35), the properties of the \(L^2\)-projection \(P_h\) and (33) we deduce

\[(IIa) \leq Ch \log h \left[\|q(\bar{y} - y_d) - q_h(\bar{y} - y_d)\|_{L^\infty(\Omega)} + \|\bar{y} - y_d - P_h(\bar{y} - y_d)\|_{L^\infty(\Omega)}\right]
\]

\[\leq Ch \log h \left[\|q(\bar{y} - y_d) - q_h(\bar{y} - y_d)\|_{L^\infty(\Omega)} + \|\bar{y} - y_d\|_{W^{1,\infty}(\Omega)}\right].
\]

Employing the stability of \(P_h\) we get

\[(IIb) \leq \|p(\bar{y} - y_d) - p(\bar{y}_h - y_d)\|_{L^\infty(\Omega)} \leq C\|p(\bar{y} - y_d) - p(\bar{y}_h - y_d)\|_{H^1(\Omega)} \leq C\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}.
\]
We know that $\tilde{y}_h = P_h \tilde{y}_h$ and $\|y_d - P_h y_d\|_{L^\infty(\Omega)} \leq Ch\|y_d\|_{W^{1,\infty}(\Omega)}$. Combining this with (35) delivers

$$(IIc) \leq Ch\log h\left[\|q(\tilde{y}_h - y_d) - q_h(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)} + \|\tilde{y}_h - y_d - P_h(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)}\right]$$

$$\leq Ch\log h\left[\|\tilde{y}_h - y_d\|_{L^\infty(\Omega)} + h\|y_d\|_{W^{1,\infty}(\Omega)}\right],$$

where we also have used (33) in the final estimate. Altogether, we for $h\log h^2 \leq 1$ see

$$\|\tilde{p} - \tilde{p}_h\|_{L^\infty(\Omega)} \leq C(h + \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega)}).$$

Next, we estimate

$$\|\tilde{q} - \tilde{q}_h\|_{L^\infty(\Omega)} = \|q(\tilde{y} - y_d) - q_h(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)}$$

$$\leq \|q(\tilde{y} - y_d) - q(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)} + \|q(\tilde{y}_h - y_d) - q_h(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)},$$

$$=: (III)$$

Since $\tilde{q} = \tilde{p}'$ and $q = p'$ we have

$$(III) \leq \|q(\tilde{y} - y_d) - q(\tilde{y}_h - y_d)\|_{H^1(\Omega)} \leq C\|\tilde{p}(\tilde{y} - y_d) - p(\tilde{y}_h - y_d)\|_{H^1(\Omega)} \leq C\|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega)}.$$

With (34) we get

$$(IV) \leq C\left[\|q(\tilde{y}_h - y_d) - \Pi_h q(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)} + h\log h\|\tilde{y}_h - y_d - P_h(\tilde{y}_h - y_d)\|_{L^\infty(\Omega)}\right]$$

$$=: (IVA)$$

Due to $|\Omega| = 1$ it holds

$$(IVA) \leq Ch\|q(\tilde{y}_h - y_d)\|_{W^{1,\infty}(\Omega)} \leq Ch\|\tilde{p}(\tilde{y} - y_d)\|_{W^{2,\infty}(\Omega)} \leq Ch\|\tilde{y}_h - y_d\|_{L^\infty(\Omega)} \leq Ch.$$

Also, with $\tilde{y}_h = P_h \tilde{y}_h$ we deduce

$$(IVb) \leq Ch\|y_d\|_{W^{1,\infty}(\Omega)}.$$

Consequently, for $h$ small enough we have

$$\|\tilde{q} - \tilde{q}_h\|_{L^\infty(\Omega)} \leq C(h + \|\tilde{y} - \tilde{y}_h\|_{L^2(\Omega)}).$$
Corollary 15. In particular, for \( h \) small enough we have with \( \| \bar{y} - \bar{y}_h \|_{L^2(\Omega)} \leq Ch^\frac{1}{2} \)
that
\[
\| \bar{p} - \bar{p}_h \|_{L^\infty(\Omega)} + \| \bar{q} - \bar{q}_h \|_{L^\infty(\Omega)} \leq Ch^\frac{1}{2}.
\]

It is now easy to see the following error estimate for \( \bar{\Phi} \).

Lemma 16. Let the suppositions of Theorem 14 hold. Then, we have
\[
\| \bar{\Phi} - \bar{\Phi}_h \|_{L^\infty(\Omega)} \leq C(h + \| \bar{y} - \bar{y}_h \|_{L^2(\Omega)}).
\]

Proof. By inserting the definitions \( \bar{\Phi}(x) = \int_0^x \bar{p}(s) \, ds \) and \( \bar{\Phi}_h = \int_0^x \bar{p}_h(s) \, ds \) it follows directly
that
\[
\| \bar{\Phi} - \bar{\Phi}_h \|_{L^\infty(\Omega)} \leq \| \bar{p} - \bar{p}_h \|_{L^1(\Omega)},
\]
and due to \( |\Omega| = 1 \), using (32), we also have
\[
\| \bar{p} - \bar{p}_h \|_{L^1(\Omega)} \leq \| \bar{p} - \bar{p}_h \|_{L^2(\Omega)}
\]
\[
\leq \| p(\bar{y} - y_d) - p(\bar{y}_h - y_d) \|_{L^2(\Omega)} + \| p(\bar{y}_h - y_d) - p_h(\bar{y}_h - y_d) \|_{L^2(\Omega)}
\]
\[
\leq C(\| \bar{y} - \bar{y}_h \|_{L^2(\Omega)} + h \| \bar{y}_h - y_d \|_{L^2(\Omega)})
\]
\[
\leq C(\| \bar{y} - \bar{y}_h \|_{L^2(\Omega)} + h).
\]

\[\Box\]

Note that in Lemma 16 it is sufficient to require \( y_d \in L^2(\Omega) \). Finally, we prove an error estimate for the control under the structural assumption 17 which we formulate next.

Assumption 17. Let \( \bar{p} \) denote the optimal adjoint state. The set \( \{ x \in \Omega : \bar{p}(x) = 0 \} \) is finite and all roots are simple roots, i.e. if \( \bar{p}(x) = 0 \), then \( \bar{p}'(x) \neq 0 \).

We have \( \bar{\Phi}' = \bar{p} \), so that the inclusion
\[
\left\{ x \in \Omega : |\bar{\Phi}(x)| = \alpha \right\} \subset \{ x \in \Omega : \bar{p}(x) = 0 \}
\]
holds and \( |\bar{\Phi}| \) attains the value \( \alpha \) only at finitely many points. For notation purposes we set \( \{ x \in \Omega : \bar{p}(x) = 0 \} = \{ \hat{x}_1, \ldots, \hat{x}_m \} \), with \( m = 0 \) indicating that this set is empty. From Lemma 4
we then deduce that the support of $\bar{u}'$ is finite and we can express $\bar{u}$ as

$$\bar{u} = \tilde{a} + \sum_{i=1}^{m} \tilde{c}^i 1_{(\hat{x}_i,1]},$$

where $\tilde{a} \in \mathbb{R}$ and $\tilde{c} = (\tilde{c}^1, \ldots, \tilde{c}^m)^T \in \mathbb{R}^m$.

To obtain a convergence result, we in our analysis need to estimate the difference in the jump points of the optimal control and the corresponding coefficients. We begin by analyzing the jump points across zero of the discrete adjoint state $\bar{p}_h$, which will deliver information about the support of the finite-dimensional representation $\tilde{u}_h := T_h \bar{u}_{vd}$ of the variational discrete optimal control.

**Lemma 18.** Let Assumption [17] hold. Then there exists $h_0 > 0$, such that for all $h \in (0, h_0]$ Assumption [9] is fulfilled.

**Proof.** Without loss of generality we assume $m \leq N$. For $i = 1, \ldots, m$ we have $\hat{x}_i \in (0,1)$, since $\tilde{\Phi}(x) = 0$ for $x \in \{0,1\}$ and $|\tilde{\Phi}(\hat{x}_i)| = \alpha > 0$. Then, there exists $R > 0$, such that $B_R(\hat{x}_i) \subset \Omega$ and all $B_R(\hat{x}_i)$ are pairwise disjoint. Outside of $\bigcup_i B_R(\hat{x}_i)$ it holds $|\bar{p}| \geq \epsilon$ for some $\epsilon > 0$, since $\bar{p}$ is continuous. Furthermore, for an arbitrary but fixed $\hat{x} \in \bigcup_i B_R(\hat{x}_i)$ with $\hat{x} \in (\hat{x}_{i1}, \hat{x}_{i2})$, where $i1, i2 \in \{1, \ldots, m\}$, the sign of $\bar{p}$ does not change in $(\hat{x}_{i1}, \hat{x}_{i2})$, because all roots are simple roots.

Also, $\hat{x} \in \bigcup_i B_R(\hat{x}_i)$ holds for exactly one $i \in \{1, \ldots, m\}$, since $B_R(\hat{x}_i) \cap B_R(\hat{x}_j) = \emptyset$ for $j \neq k$.

From Theorem [14] we can deduce

$$\bar{p}_h \to \bar{p} \quad \text{and} \quad \bar{q}_h \to \bar{q} \quad \text{uniformly in } h.$$

So, we may choose $h_0 > 0$, such that $|\bar{p}_h(x)| \geq \frac{\epsilon}{2}$ for all $h \in (0, h_0]$ and for all $x$ in the complement of $\bigcup_i B_R(\hat{x}_i)$. Consequently, $\bar{p}_h = 0$ can not hold outside of $\bigcup_i B_R(\hat{x}_i)$. In order to prove that Assumption [9] holds, it remains to show that $\bar{p}_h(x) = 0$ can also not hold for $x \in \bigcup_i B_R(\hat{x}_i)$.

Now let us assume that $h_0 < \frac{R}{2}$, and choose $i \in \{1, \ldots, m\}$ arbitrary but fixed. By Assumption [17] we have $\tilde{p}(\hat{x}_i) = 0$ and $\tilde{q}(\hat{x}_i) = \tilde{p}'(\hat{x}_i) \neq 0$. Since $\hat{x}_i$ is the only root of $\tilde{p}$ in $B_R(\hat{x}_i)$ we have $|\tilde{q}(x)| \geq \delta$ for all $x \in B_R(\hat{x}_i)$ and some $\delta > 0$. Without loss of generality we assume $\tilde{q}(x) \geq \delta$ for all $x \in B_R(\hat{x}_i)$. So, after a possible further reduction of $h_0$ we have $\tilde{q}_h(x) \geq \frac{\delta}{2}$ for all $x \in B_R(\hat{x}_i)$ for all $h \in (0, h_0]$ by uniform convergence of $\tilde{q}_h$ to $\tilde{q}$. Now we fix $x_i^-, x_i^+ \in B_R(\hat{x}_i)$ with

$$\bar{p}(x_i^-) < 0 \quad \text{and} \quad \bar{p}(x_i^+) > 0.$$
Due to the uniform convergence we also have \( \tilde{p}_h(x^-) < 0 \) and \( \tilde{p}_h(x^+) > 0 \) for \( h \in (0, h_0) \), after a possible reduction of \( h_0 \). Thus, a grid point \( x_{j(i)} \in B_R(\hat{x}_i) \) exists together with points \( x_{j(i)}^{\pm} \in (x_{j(i)} \pm h) \), i.e. \( x_{j(i)}^- < x_{j(i)} < x_{j(i)}^+ \), such that
\[
\tilde{p}_h(x_{j(i)}^-) < 0 \quad \text{and} \quad \tilde{p}_h(x_{j(i)}^+) > 0,
\]
holds, i.e. \( \tilde{p}_h \) at \( x_{j(i)} \) jumps from minus to plus. Next, we show that this grid point is the unique jumping point across zero of \( \tilde{p}_h \) in \( B_R(\hat{x}_i) \). Assume there exists at least one other jumping point across zero \( x_{k(i)} \neq x_{j(i)} \) in \( B_R(\hat{x}_i) \). Then we without loss of generality may assume that \( \tilde{p}_h \) at \( x_{k(i)} \) jumps from plus to minus (or is identically zero on \( (x_{k(i)} \pm h) \)). We thus find \( x_{k(i)}^- \) and \( x_{k(i)}^+ \) in \( (x_{k(i)} \pm h) \) with \( x_{k(i)}^- < x_{k(i)} < x_{k(i)}^+ \), and
\[
\tilde{p}_h(x_{k(i)}^-) \geq 0 \quad \text{and} \quad \tilde{p}_h(x_{k(i)}^+) \leq 0.
\]
We may assume this sign constellation, since if there exists exactly one additional jumping point \( x_{k(i)} \) across zero in \( B_R(\hat{x}_i) \), this relation has to hold, and if there exist several additional jumping points, at least one of them satisfies this relation. We choose the hat function \( b_{x_{k(i)}} \) with \( b_{x_{k(i)}}(x_{k(i)}) = 1 \) and \( b_{x_{k(i)}}(x_l) = 0 \) for all \( l = 1, \ldots, N \) with \( l \neq k(i) \). Then we have using (25)
\[
\frac{\delta}{2a} h \leq \int_{\Omega} \frac{1}{a} \tilde{p}_h b_{x_{k(i)}} \, dx = -\int_{\Omega} \tilde{p}_h b'_{x_{k(i)}} \, dx = -\tilde{p}_h(x_{k(i)}^-) + \tilde{p}_h(x_{k(i)}^+) \leq 0
\]
This contradicts \( \delta > 0, a > 0, h > 0 \), so we deduce that \( x_{j(i)} \) is unique. Furthermore, by the arguments given this also contradicts that \( \tilde{p}_h(x) = 0 \) holds for all \( x \) on a whole interval of the partition, which is contained in \( B_R(\hat{x}_i) \). Altogether, this implies \( \tilde{p}_h(x) \neq 0 \) for all \( x \in \Omega \), i.e. Assumption 9 holds for all \( h \in (0, h_0) \) with some \( h_0 > 0 \). Moreover, if \( \tilde{p} \) admits \( m \) simple roots \( \hat{x}_1, \ldots, \hat{x}_m \), \( \tilde{p}_h \) admits \( m \) jump points \( x_{j(1)}, \ldots, x_{j(m)} \). \( \square \)
Finally, we want to estimate the distance \( |\hat{x}_i - x_{j(i)}| \). Since \( \tilde{p} \) is continuously differentiable there
Next, we estimate the di
Lemma 19. Let Assumption 17 hold. Then there exists $h \in B_R(\hat{x})$, such that

$$\hat{p}'(\xi)(\hat{x}_i - x_{j(0)}) = \frac{\hat{p}(\hat{x}_i) - \hat{p}(x_{j(0)})}{\hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-})} \hat{p}(x_{j(0)^+}) + \frac{-\hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-})}{\hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-})} \hat{p}(x_{j(0)^-})$$

$$= \hat{a} \left( \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right) + \hat{b} \left( \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right).$$

Here, $x_{j(0)^+} \in (x_{j(0)} \pm h)$ as chosen above. Employing $\hat{p}' = \hat{a}$ and $\hat{b} = 1 - \hat{a}$, we get

$$\delta|\hat{x}_i - x_{j(0)}| \leq \hat{a}(\xi)|\hat{x}_i - x_{j(0)}|$$

$$= \hat{a} \left( \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right) + (1 - \hat{a}) \left( \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right)$$

$$\leq \hat{a} \left( \left| \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right| + \left| \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right| \right)$$

$$+ (1 - \hat{a}) \left( \left| \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right| + \left| \hat{p}(x_{j(0)^+}) - \hat{p}(x_{j(0)^-}) \right| \right)$$

$$\leq \hat{a} \left( \| \hat{p} - \hat{p}_h \|_{L^\infty(\Omega)} + \| \hat{p} \|_{W^{1,\infty}(\Omega)} |x_{j(0)^+} - x_{j(0)^-}| \right)$$

$$+ (1 - \hat{a}) \left( \| \hat{p} - \hat{p}_h \|_{L^\infty(\Omega)} + \| \hat{p} \|_{W^{1,\infty}(\Omega)} |x_{j(0)^+} - x_{j(0)^-}| \right)$$

$$\leq \| \hat{p} - \hat{p}_h \|_{L^\infty(\Omega)} + C h.$$

Finally, Theorem 14 gives

$$|\hat{x}_i - x_{j(0)}| \leq C(h + \| \bar{y} - \bar{y}_h \|_{L^2(\Omega)}).$$

(38)

Consequently, $\bar{u}_h = \Upsilon_h \bar{u}_{vd}$ can be represented with $\bar{a}_h \in \mathbb{R}$ and $\bar{c}_h = (\bar{c}_{j(1)}^h, \ldots, \bar{c}_{j(m)}^h)^T \in \mathbb{R}^m$ as follows:

$$\bar{a}_h = \bar{a}_h + \sum_{i=1}^{m} \bar{c}_{j(i)}^h 1_{(x_{j(0)}, 1)}.$$

Since $m \leq N$ and $x_{j(0)} \subset \{x_i\}_{i=1}^N$, we can add zeros in the sum to recover the representation $\bar{u}_h = \bar{a}_h + \sum_{i=1}^{N} \bar{c}_{j(i)}^h 1_{(x_{j(0)}, 1)}$. For now it is more convenient to work with the first representation.

Next, we estimate the differences in the jump heights and the constant coefficient.

Lemma 19. Let Assumption 17 hold. Then there exists $h_0 > 0$, such that for all $h \in (0, h_0]$ the
coefficients of the optimal controls $\bar{u} = \bar{u} + \sum_{i=1}^{m} \bar{e}^j I_{(\xi_i, \bar{\xi}_i]}$ and $\bar{u}_h = \bar{u}_h + \sum_{i=1}^{m} \bar{e}^j I_{(x_{i,0}, 1)}$ satisfy

$$\sum_{i=1}^{m} |\bar{e}^i - \bar{e}^j_h| \leq C \left( h + ||\bar{y} - \bar{y}_h||_{L^2(\Omega)} \right),$$  

(39)

$$|\bar{u} - \bar{u}_h| \leq C \left( h + ||\bar{y} - \bar{y}_h||_{L^2(\Omega)} \right),$$  

(40)

where $C > 0$ denotes a constant independent of $h_0$.

**Proof.** We know that there exists a $R > 0$, such that the balls $B_R(\hat{x}_i)$ are contained in $\Omega$ and are pairwise disjoint for $i = 1, \ldots, m$. For every $i = 1, \ldots, m$ we proceed as follows: Consider a function $g \in C^\infty(\Omega)$, such that $g = 1$ on $B_R(\hat{x}_i)$ and $g = 0$ on $\bar{\Omega} \setminus \bigcup_{i=1}^{m} B_{R/2}(\hat{x}_i)$. For $h$ small enough we from (38) and Corollary 13 also have $x_{j(i)} \in B_R(\hat{x}_i)$ for every $i = 1, \ldots, m$, where $x_{j(i)}$ denotes the unique jump point of $p_h$ in $B_R(\hat{x}_i)$. We have

$$\bar{u}' = \sum_{i=1}^{m} e^j \delta_{\hat{x}_i}, \quad \text{and} \quad \bar{u}'_h = \sum_{i=1}^{m} e^j_h \delta_{x_{i,0}},$$

so that by the construction of $g$, the definition of the distributional derivative, and the definition of the state equation we get for all $h \in (0, h_0]$ obtain

$$|\bar{e}^i - \bar{e}^j_h| = \left| \langle \bar{u}' - \bar{u}'_h, g \rangle_{M^1(\Omega), C(\Omega)} \right|$$

$$= \left| - (\bar{u} - \bar{u}_h, g')_{L^2(\Omega)} \right|$$

$$\leq \left| (\bar{u} - \bar{u}_h, P_h(g'))_{L^2(\Omega)} \right| + \left| (\bar{u} - \bar{u}_h, g' - P_h(g'))_{L^2(\Omega)} \right|.$$  

The second term can be estimated as follows:

$$\left| (\bar{u} - \bar{u}_h, g' - P_h(g'))_{L^2(\Omega)} \right| \leq ||\bar{u} - \bar{u}_h||_{L^2(\Omega)} ||g' - P_h(g')||_{L^2(\Omega)}$$

$$\leq \left( ||\bar{u}||_{L^2(\Omega)} + ||\bar{u}_h||_{L^2(\Omega)} \right) C h ||g'||_{L^2(\Omega)}$$

$$\leq C h,$$

where we use the definition and the properties of $P_h$ together with the bounds $||\bar{u}||_{BV(\Omega)} \leq C$ and $||\bar{u}_h||_{BV(\Omega)} = ||\bar{u}_h||_{BV(\Omega)} \leq ||\bar{u}_h||_{BV(\Omega)} \leq C$ from Theorem 6.

For the first term we use the definition of $a$ and the definition of $P_h$ to obtain
\[
\|\bar{u} - \bar{u}_h, P_h(g')\|_{L^2(\Omega)} = | - b(\bar{z} - z(\bar{u}_h), P_h(g')) + c(\bar{y} - y(\bar{u}_h), P_h(g'))| \\
\leq | - b(\bar{z}, P_h(g') - g') + c(\bar{y}, P_h(g') - g')| \\
+ | - b(\bar{z} - z(\bar{u}_h), g')| + |c(\bar{y} - y(\bar{u}_h), g')| \\
\leq \|\bar{u}\|_{L^2(\Omega)}\|P_h(g') - g\|_{L^2(\Omega)} + |b(\bar{z} - z(\bar{u}_h), g')| \\
+ |d(\bar{y} - y(\bar{u}_h))|_{L^2(\Omega)}|g' - g\|_{L^2(\Omega)} \\
\leq Ch + \int_{\Omega} a(\bar{y} - y(\bar{u}_h))g''| + C|\bar{y} - y(\bar{u}_h)||_{L^2(\Omega)} \\
\leq C(h + \|\bar{y} - y\|_{L^2(\Omega)}) \\
\leq C(h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} + \|\bar{y}_h - y(\bar{u}_h)\|_{L^2(\Omega)}) \\
\leq C(h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}).
\]

Summarizing, we see \(|\bar{e}^i - \bar{e}^i_h| \leq C(h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)})\) for every \(i = 1, \ldots, m\), which delivers (39). To see (40), we can adapt the proof of [21, Lemma 4.9,] to our setting. In particular, we have

\[
p - \bar{p}_h = p(y(\bar{u}) - y_d) - p_h(y_h(\bar{u}_h) - y_d) \\
= p\left\{\bar{a} + \sum_{i=1}^m \bar{e}^i \mathbf{1}_{(\bar{a},1)}\right\} - p_h\left\{\bar{a}_h + \sum_{i=1}^m \bar{e}^i_h \mathbf{1}_{(\bar{a}_h,1)}\right\} - (p - p_h)(y_d) \\
= (\bar{a} - \bar{a}_h)p(y(1)) + \bar{a}_h(p(y) - p_h(y_h))(1) + \sum_{i=1}^m (\bar{e}^i - \bar{e}^i_h)p(y(1,1)) \\
+ \sum_{i=1}^m \bar{e}^i_h (p(y) - p_h(y_h))(1,1) + \sum_{i=1}^m \bar{e}^i_h (p_h(y_h(1,1) - 1_{(\bar{a}_h,1)})) - (p - p_h)(y_d).
\]
By Theorem 14, the means of $\bar{p}$ and $\bar{p}_h$ vanish, so by integration we get

$$0 = (\bar{a} - \bar{a}_h) \int_{\Omega} p(y(1))dx + \bar{a}_h \int_{\Omega} (p(y) - p_h(y_h))(1)dx + \sum_{i=1}^{m} (\bar{c}_i - \bar{c}_h^{(i)}) \int_{\Omega} p(y(1_{(i,1)}))d\xi$$

$$+ \sum_{i=1}^{m} \bar{c}_h^{(i)} \int_{\Omega} (p(y) - p_h(y_h))(1_{(i,1)})dx + \sum_{i=1}^{m} \bar{c}_h^{(i)} \int_{\Omega} p_h(y_h(1_{(i,1)} - 1_{(\hat{y}_h,1)}))d\xi$$

$$- \int_{\Omega} (p - p_h)(y_d)dx.$$ 

Let us show a useful equality for general $u_1, u_2 \in L^2(\Omega)$:

$$\int_{\Omega} p(y(u_1))u_2 dx \equiv \int_{\Omega} -u_2'(y(u_1))dy + dy(u_2)p(y(u_1))dx$$

$$\int_{\Omega} 1 \int_{\Omega} q(y(u_1))c(y(u_2))dy + dy(u_2)p(y(u_1))dx$$

$$\int_{\Omega} -q'(y(u_1))y(u_2) + dy(u_2)p(y(u_1))dx$$

$$\int_{\Omega} y(u_1)y(u_2)dx.$$ 

The same equation holds true for the discrete setting by making use of (23)-(26). With $u_1 = u_2 = 1$ we see $\int_{\Omega} p(y(1))dx = \|y(1)\|_{L^1(\Omega)}^2$ and knowing that $y(1) \neq 0$ we get

$$|\bar{a} - \bar{a}_h| \leq \|y(1)\|_{L^2(\Omega)}^2 \left( |\bar{a}_h| \|(p(y) - p_h(y_h))(1)\|_{L^1(\Omega)} + \sum_{i=1}^{m} |\bar{c}_i - \bar{c}_h^{(i)}| \|p(y(1_{(i,1)}))\|_{L^1(\Omega)} \right)$$

$$+ \sum_{i=1}^{m} \bar{c}_h^{(i)} \||p(y) - p_h(y_h))(1_{(i,1)})\|_{L^1(\Omega)} + \sum_{i=1}^{m} \bar{c}_h^{(i)} \left| \int_{\Omega} p_h(y_h(1_{(i,1)} - 1_{(\hat{y}_h,1)}))d\xi \right|$$

$$+ \| (p - p_h)(y_d)\|_{L^1(\Omega)}.$$ 

Using continuity of $\hat{y}_h \mapsto \bar{p}_h, |\Omega| = 1,$ and (32) with $r = 2$, we obtain

$$\|(p(y) - p_h(y_h))(1)\|_{L^1(\Omega)} \leq \|(p(y) - p_h(y))(1)\|_{L^1(\Omega)} + \|(p_h(y) - p_h(y_h))(1)\|_{L^1(\Omega)}$$

$$\leq \|(p - p_h)(y(1))\|_{L^2(\Omega)} + C \|y(1) - y_h(1)\|_{L^2(\Omega)}$$

$$\leq Ch \|y(1)\|_{L^2(\Omega)} + C \|y(1) - y_h(1)\|_{L^2(\Omega)}$$

$$\leq Ch.$$
Since \(1_{(\hat{x},1)} \in L^{\infty}(\Omega)\), an analogous estimation can be done using (32) with \(r = 2\) for every \(i\) to see that \(\| (p(y) - p_h(\hat{y})) (1_{(\hat{x},1)}) \|_{L^1(\Omega)} \leq Ch\). Furthermore, with the helpful equality from above in the discrete setting, we have for every \(i\)

\[
\left| \int_{\Omega} p_h(\hat{y})(1_{(\hat{x},1)} - 1_{(x(j),1)}) dx \right| = \left| \int_{\Omega} \hat{y}_h(1_{(\hat{x},1)} - 1_{(x(j),1)}) y_h(1) dx \right|
\]

\[
\leq \| p_h(\hat{y}(1)) \|_{L^\infty(\Omega)} |\hat{x}_i - x_{j(i)}|
\]

\[
\leq 2 \| p(1) \|_{L^\infty(\Omega)} |\hat{x}_i - x_{j(i)}|
\]

\[
\leq Ch|\hat{x}_i - x_{j(i)}|
\]

where we used that \(h\) is chosen small enough and \(p(y(1)) \neq 0\). Also, with \(|\Omega| = 1\) and (32) with \(r = 2\) we have

\[
\|(p - p_h)(y_d)\|_{L^1(\Omega)} \leq \|(p - p_h)(y_d)\|_{L^2(\Omega)} \leq Ch \| y_d \|_{L^2(\Omega)} \leq Ch.
\]

Plugging all of this into (41), and employing \(|\bar{c}_h|_1 = \sum_{i=1}^{m} |\bar{c}_i^{(j)}|\), delivers

\[
|\bar{a} - \bar{a}_h| \leq Ch (|\bar{a}_h| + |\bar{c}_h|_1 + 1) + C \left( \sum_{i=1}^{m} |\bar{c}^{(j)} - \bar{c}_i^{(j)}| + |\bar{c}_h|_1 \sum_{i=1}^{m} |\hat{x}_i - x_{j(i)}| \right).
\]

From the definition of \(\bar{a}_h\) we obtain

\[
\frac{1}{2} \| \bar{a}_h y_h(1) + \sum_{i=1}^{m} \bar{c}_i^{(j)} y_h(1_{(x(j),1)}) - y_d \|^2_{L^2(\Omega)} + \alpha |\bar{c}_h|_1 = J_h(\bar{a}_h, \bar{c}_h) \leq J_h(0) = J(0).
\]
This delivers $|\bar{e}_h|_1 \leq \frac{J(0)}{\alpha}$. Also, for $h$ small enough

$$
\frac{1}{2} |\bar{a}_h| \|y(1)\|_{L^1(\Omega)} \leq |\bar{a}_h| \|y_h(1)\|_{L^1(\Omega)} = \|y_h(\bar{a}_h)\|_{L^1(\Omega)} \\
\leq \|y_h(\bar{a}_h) - y_d\|_{L^1(\Omega)} + \|y_d - y_h\left(\sum_{i=1}^m \bar{c}_h^{(i)} 1_{(x_j,1)}\right)\|_{L^1(\Omega)} \\
\leq C + \|y_d\|_{L^1(\Omega)} + \|\sum_{i=1}^m \bar{c}_h^{(i)} y_h(1_{(x_j,1)})\|_{L^1(\Omega)} \\
\leq C + \sum_{i=1}^m |\bar{c}_h^{(i)}| \max_{1 \leq i \leq m} \|y_h(1_{(x_j,1)})\|_{L^1(\Omega)} \\
\leq C(1 + \frac{2J(0)}{\alpha} \max_{1 \leq i \leq m} \|y(1_{(x_j,1)})\|_{L^1(\Omega)}) \\
\leq C.
$$

This implies $|\bar{a}_h| \leq C$ uniformly in $h$. Using (38) and (39), we finally obtain

$$
|\bar{a} - \bar{a}_h| \leq C(h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}).
$$

□

With the previous results we now have everything available to prove the convergence order $O(h)$ for the optimal control.

**Theorem 20.** Let Assumption [7] hold. Then there exists $h_0 > 0$, such that for all $h \in (0, h_0]$ we have

$$
\|\bar{a} - \bar{a}_h\|_{L^1(\Omega)} \leq C \left( h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \right),
$$

$$
\|\bar{a}' - \bar{a}_h'\|_{W^{1,\infty}(\Omega)} \leq C \left( h + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \right).
$$
Proof. We combine \(|Ω| = 1\), (38), (39), and (40) to get

\[
\|\bar{u} - \bar{u}_h\|_{L^1(Ω)} = \int_Ω \left| \bar{a} - \bar{a}_h + \sum_{i=1}^m (\bar{c}' \mathbb{1}_{(\hat{c}_i, 1)} - \bar{c}'_h \mathbb{1}_{(\hat{c}_i, 1)} ) \right| \, dx \\
\leq |\bar{a} - \bar{a}_h|_{Ω} + \int_Ω \sum_{i=1}^m |c'(1_{(\hat{c}_i, 1)} - 1_{(x_0, 1)})| \, dx + \int_Ω \sum_{i=1}^m |c' - c'_h| \mathbb{1}_{(x_0, 1)} \, dx \\
\leq |\bar{a} - \bar{a}_h|_{Ω} + \sum_{i=1}^m |c'| \|1_{(\hat{c}_i, 1)} - 1_{(x_0, 1)}\|_{L^1(Ω)} + \sum_{i=1}^m |c' - c'_h| \|1_{(x_0, 1)}\|_{L^1(Ω)} \\
\leq C (h + \|\bar{y} - \bar{y}_h\|_{L^2(Ω)}),
\]

Also, we have

\[
\|\bar{u}' - \bar{u}_h'\|_{W^{1,∞}(Ω)} = \sup_{\|f\|_{W^{1,∞}(Ω)} \leq 1} \int_Ω f \left( \bar{u}' - \bar{u}_h' \right) \\
= \sup_{\|f\|_{W^{1,∞}(Ω)} \leq 1} \int_Ω f \left( \sum_{i=1}^m \bar{c}' \delta_{\hat{c}_i} - \sum_{i=1}^m \bar{c}'_h \delta_{x_0} \right) \\
= \sup_{\|f\|_{W^{1,∞}(Ω)} \leq 1} \sum_{i=1}^m \bar{c}' \left( f(\hat{c}_i) - f(x_0) \right) + \left( \bar{c}' - \bar{c}'_h \right) f(x_0) \\
\leq \sup_{\|f\|_{W^{1,∞}(Ω)} \leq 1} \left\{ \max_{1 \leq i \leq m} |f(\hat{c}_i) - f(x_0)| \|\bar{c}'\|_{L^1(Ω)} + \|\bar{c}' - \bar{c}'_h\|_{L^1(Ω)} \max_{1 \leq i \leq m} |f(x_0)| \right\} \\
\leq \sup_{\|f\|_{W^{1,∞}(Ω)} \leq 1} L_f \max_{1 \leq i \leq m} |\hat{c}_i - x_0| \|\bar{c}'\|_{L^1(Ω)} + \|\bar{c}' - \bar{c}'_h\|_{L^1(Ω)} \\
\leq C (h + \|\bar{y} - \bar{y}_h\|_{L^2(Ω)}),
\]

where \(L_f \leq 1\) denotes the Lipschitz constant of \(f\).

\[\square\]

Combining this result with Theorem 12, we under the structural Assumption 17 deduce improved error estimates.

**Lemma 21.** Let Assumption 17 hold. Then there exists \(h_0 > 0\), such that for all \(h \in (0, h_0]\) we
have the following error estimates, where \( C > 0 \) denotes a constant independent of \( h \).

\[
\begin{align*}
\| \bar{y} - \tilde{y}_h \|_{L^2(\Omega)} &\leq Ch, \\
\| \bar{p} - \tilde{p}_h \|_{L^\infty(\Omega)} &\leq Ch, \\
\| \tilde{\Phi} - \tilde{\Phi}_h \|_{L^\infty(\Omega)} &\leq Ch, \\
|\tilde{a}_i - \tilde{a}_{i,h}| &\leq Ch, \\
\| \bar{u} - \tilde{u}_h \|_{L^1(\Omega)} &\leq Ch, \\
\| \bar{u}' - \tilde{u}'_h \|_{W^{1,\infty}(\Omega)} &\leq Ch.
\end{align*}
\]

**Proof.** Since \( \tilde{u}_h \) is a solution of \((P_{vd})\) and Theorem 12 holds for all solutions \( \tilde{u}_{vd} \) of \((P_{vd})\), we can plug the result from Theorem 20 into the error estimate for the state from Theorem 12 and see with Young’s inequality

\[
\| \bar{y} - \tilde{y}_h \|_{L^2(\Omega)}^2 \leq C(h^2 + h \| \bar{u} - \tilde{u}_{vd} \|_{L^1(\Omega)}) \leq C(h^2 + h \| \bar{v} - \tilde{y}_h \|_{L^2(\Omega)}) \leq Ch^2 + \frac{1}{2} \| \bar{y} - \tilde{y}_h \|_{L^2(\Omega)}^2.
\]

This delivers

\[
\| \bar{y} - \tilde{y}_h \|_{L^2(\Omega)} \leq Ch.
\]

Now we can use the above inequality in Theorem 14, Lemma 16, (38), (39), (40) and Theorem 20 to derive the remaining error estimates. \(\square\)

### 4 Computational results

We can represent the mixed formulation of the discrete state equation (15) by the following matrix equation:

\[
\begin{pmatrix}
A & B \\
B^\top & D
\end{pmatrix}
\begin{pmatrix}
z \\
y
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
-u
\end{pmatrix},
\]

(42)

where for the given grid \(0 = x_0 < x_1 < \ldots < x_N = 1\) with associated spaces \(P_0 = \text{span}\{\chi_i : 1 \leq i \leq N\}\) and \(P_1 = \text{span}\{e_j : 0 \leq j \leq N\}\) the matrix entries are given by

\[
\begin{align*}
A &= (a_{i,j})_{i,j=0}^N, & a_{i,j} &= a(e_i, e_j), \\
B &= (b_{i,j})_{i=0,j=1}^N, & b_{i,j} &= b(e_i, \chi_j), \\
D &= (d_{i,j})_{i,j=1}^N, & d_{i,j} &= c(\chi_i, \chi_j).
\end{align*}
\]
Then $A$ is symmetric positive definite, $D$ is symmetric positive semi definite and $B$ has rank $N$ (given ker $B = \{1\}$). $A \in \mathbb{R}^{(N+1)\times(N+1)}$ and $B \in \mathbb{R}^{N+1\times N}$ have the entries

$$A = \begin{pmatrix} \frac{1}{3}h_1 & \frac{1}{6}h_1 & 0 & \ldots & 0 \\ \frac{1}{6}h_1 & \frac{1}{3}(h_1 + h_2) & \frac{1}{6}h_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \frac{1}{6}h_{N-1} & \frac{1}{6}(h_{N-1} + h_N) \\ 0 & \ldots & 0 & \frac{1}{3}h_N & \frac{1}{3}h_N \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ 1 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & -1 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

The vectors contain the coefficients $z = (z_0, \ldots, z_N)^T \in \mathbb{R}^{N+1}$, $y = (y_1, \ldots, y_N)^T \in \mathbb{R}^N$, and the evaluation of the BV-function $u = (u_1, \ldots, u_N)^T \in \mathbb{R}^N$, where $u_j := \int_{x_{j-1}}^{x_j} u$ for $j = 1, \ldots, N$.

With our knowledge about the structure of $u$ we obtain $u_j = (a_h + \sum_{i=1}^{j-1} c_h^i) h_j$ for $j = 1, \ldots, N$, where $h_j = x_j - x_{j-1}$.

We use (4.2) to get $z = -A^{-1}B y$ and $(B^T A^{-1} B + D) y = u$, so that $y = (B^T A^{-1} B + D)^{-1} u$. Then we insert this into $(\hat{P}_h)$ and obtain:

$$\min_{a_h \in \mathbb{R}, c_h \in \mathbb{R}^{N-1}} f(a_h, c_h) := \frac{1}{2} \|(B^T A^{-1} B + D)^{-1} u - y\|_{L^2(\Omega)}^2 + \alpha \sum_{i=1}^{N-1} |c_h^i|,$$  

(4.1) Optimization algorithm

It follows from our variationally discrete approach that the support of $\tilde{u}_h'$ is a subset of the grid points $\{x_i\}_{i=1}^N$, so we don’t need to approximate the support like e.g. needed to be done in the classical fully discrete approach with piecewise constant controls in [21]. We start the algorithm with an empty support set and then update the set of support points in each outer iteration, where we will determine the grid points, at which the control is actually supported.

We define $m_k$ as the cardinality of support points in iteration $k$ and $t_k$ the sorted vector of all support points in iteration $k$. The outer iteration should be terminated if the support points satisfy

$$m_k = m_{k-1} \quad \text{and} \quad \|t_k - t_{k-1}\|_2 \leq \epsilon.$$  

(4.1) Optimization algorithm

Here, the second condition only needs to be checked if the first condition is fulfilled, to ensure that the support points are identical in both iterations. In [21] cycling of the outer iteration is
reported. We also observe this in our numerical experiments. We note that in [30] a solution strategy is proposed which seems to avoid cycling similar minimization problems. To detect cycling we insert a second set of termination conditions:

\[ m_k = m_{k-1} = m_{k-2} \quad \text{and} \quad \| t_k - t_{k-2} \|_2 \leq \epsilon \quad \text{and} \quad f^k < f^{k-1}, \quad (T_2) \]

where \( f^k := f(a_h^k, c_h^k, p_h^k, s^k) \). This leads to the following algorithm for the numerical solution of \( \bar{P}_h \):

\[ \text{Algorithm 22:} \]
\[
\text{input: } m_0 \in \mathbb{R}, t_0 \in \mathbb{R}^{m_0}, \epsilon > 0
\]
\[
\text{for } k = 0, 1, \ldots \text{ do}
\]
\[
\begin{align*}
\text{if } (T_1) \text{ or } (T_2) \text{ holds then} & \quad m := m_k, \bar{x}_h := t_k, \\
& \quad \text{extract } (\bar{a}_h, \bar{c}_h) \text{ from } u_h \\
& \quad \text{STOP}
\end{align*}
\]
\[
\text{Obtain } (u_h^k, y_h^k, p_h^k) \text{ by solving } \bar{P}_h \text{ on } t_k.
\]
\[
\text{Compute } t_{k+1} \in \mathbb{R}^{m_{k+1}} \text{ from } p_h^k.
\]
\[
\text{output: } \bar{x}_h \in \mathbb{R}^{m_0}, (\bar{a}_h, \bar{c}_h) \in \mathbb{R}^{m+1}
\]

We initialize our algorithm with \( \bar{a}_h = 0, \bar{c}_h = \{ \}, \epsilon = 10^{-10} \) and solve \( \bar{P}_h \) using the MATLAB routine 'fmincon' with the following choices: Algorithm: 'active-set'; MaxFunctionEvaluations: 10^5; MaxIterations: 10^4; FunctionTolerance: 10^{-12}.

4.2 Numerical Examples

As our first example, we consider the setting of [21] 5.3. Example 1 with known solution, i.e. we set \( a = 1, d = 0 \). The following choices satisfy the optimality conditions as stated in Theorem[7]

- \( c := 12 - 4 \sqrt{8}; \ x_c := \frac{1}{2\pi} \arccos(\frac{\pi}{4}); \ \alpha := 10^{-5}; \)
- \( \bar{u} := 0.5 + 1_{(x,1)} - 2 \cdot 1_{(0.5,1)} + 1.5 \cdot 1_{(1-x,1)}; \)
- \( \bar{y} := \bar{y}(\bar{u}); \)
- \( \bar{\Phi}(x) := \frac{6}{2\pi} [(1 - \cos(4\pi x)) - c(1 - \cos(2\pi x))]; \)
• \( \bar{p} := \bar{\Phi}' \);

• \( y_d := \bar{y} + \bar{p}'' \).

In Figure 1 the approximated solutions on a grid with \( h = \frac{1}{2048} \) are depicted.

![Figure 1: The variationally discrete solution to the data from Example 1 for \( h = \frac{1}{2048} \). The inclusions in (31) are clearly visible.](image)

In Figure 2 the errors between the known solutions and the solutions to the variationally discretized problem are displayed. We observe that the order of convergence is approximately \( h \), except for \( ||\bar{u} - \bar{u}_h||_{L^2(Q)} \), which as expected converges only with the half rate. In addition to plotting the errors, we also calculate the convergence order \( h^\alpha \) for the refinement from some grid size \( h_1 \) to some other grid size \( h_2 \), see Table 1, by

\[
\alpha = \frac{\log(e_{h_1})}{\log(h_2/h_1)},
\]

where \( e_{h_1} \) and \( e_{h_2} \) act as placeholders for the different errors we are examining, in particular: \( ||\bar{u} - \bar{u}_h||_{L^2(Q)} \), \( ||\bar{u} - \bar{u}_h||_{L^2(Q)} \), \( ||\bar{y} - \bar{y}_h||_{L^2(Q)} \), \( ||\bar{p} - \bar{p}_h||_{L^\infty(Q)} \), and \( ||\bar{\Phi} - \bar{\Phi}_h||_{L^\infty(Q)} \).

As a second example we use [21, 5.4. Example 2] with unknown solution, \( \alpha = 10^{-5} \) and \( y_d(x) := 0.5 \pi^2 (1 - \cos(2 \pi x)) \).

Since the solution is not known, we calculate a reference solution on the finest grid with \( h = \frac{1}{1024} \).

The results displayed in Figure 3 are then used to approximate \( \bar{u}, \bar{y}, \bar{p}, \bar{\Phi} \) for the calculation of the errors.
Figure 2: Example 1: Convergence plots of the errors of the solutions to the variationally discrete problem compared to the known exact solution.

Figure 3: The variationally discrete solution to the data from Example 2 for $h = \frac{1}{10^2}$. The inclusions in (31) are clearly visible.
Table 1: Example 1: Convergence order (potency of gridsize $h$) of the respective errors when the grid is refined from gridsize $h_1$ to gridsize $h_2$. We remark that the gridsizes are rounded.

| $h_1$ | $h_2$ | $||\bar{u} - \bar{u}_h||_{L^1}$ | $||\bar{u} - \bar{u}_h||_{L^2}$ | $||\bar{y} - \bar{y}_h||_{L^2}$ | $||\bar{p} - \bar{p}_h||_{L^\infty}$ | $||\bar{\Phi} - \Phi_h||_{L^\infty}$ |
|-------|-------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 0.2500 | 0.1250 | 0.1943                         | -0.0171                       | 0.2403                        | -0.2224                       | -0.4324                       |
| 0.1250 | 0.0625 | 1.3436                         | 0.7759                        | 1.4444                        | 1.1389                        | 2.6278                        |
| 0.0625 | 0.0313 | 1.1471                         | 0.9368                        | 1.7284                        | 0.8966                        | 1.5183                        |
| 0.0313 | 0.0156 | 0.9982                         | 0.4874                        | 1.0286                        | 1.0597                        | 0.7761                        |
| 0.0156 | 0.0078 | 1.4732                         | 2.7648                        | -0.1603                       | 0.4420                        | -2.0774                       |
| 0.0078 | 0.0039 | 0.0178                         | -2.3127                       | 1.6393                        | 1.4590                        | 3.8838                        |
| 0.0039 | 0.0020 | 0.9832                         | 0.4948                        | 0.9920                        | 0.9936                        | 1.3328                        |
| 0.0020 | 0.0010 | 0.9975                         | 0.4975                        | 0.9957                        | 1.0184                        | 0.3887                        |
| 0.0010 | 0.0005 | 0.9353                         | 0.4984                        | 0.9025                        | 0.9738                        | -0.6235                       |

Table 1: Example 1: Convergence order (potency of gridsize $h$) of the respective errors when the grid is refined from gridsize $h_1$ to gridsize $h_2$. We remark that the gridsizes are rounded.

In Figure 4 the errors between the known solutions and the solutions to the variationally discretized problem are depicted. Again, we observe that the order of convergence is approximately $h$, except for $||\bar{u} - \bar{u}_h||_{L^2(Q)}$, which converges with the half rate.

Furthermore, we calculate the convergence order $h^\alpha$ for the refinement from some gridsize $h_1$ to some other gridsize $h_2$ as explained before. The results are displayed in Table 2.

| $h_1$ | $h_2$ | $||\bar{u} - \bar{u}_h||_{L^1}$ | $||\bar{u} - \bar{u}_h||_{L^2}$ | $||\bar{y} - \bar{y}_h||_{L^2}$ | $||\bar{p} - \bar{p}_h||_{L^\infty}$ | $||\bar{\Phi} - \Phi_h||_{L^\infty}$ |
|-------|-------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 0.2500 | 0.1250 | 0.3110                         | 0.2454                        | 1.0464                        | 0.7448                        | 1.5684                        |
| 0.1250 | 0.0625 | 0.9990                         | 0.5319                        | 1.0788                        | 0.8119                        | 1.6061                        |
| 0.0625 | 0.0313 | 0.9763                         | 0.4961                        | 1.0147                        | 1.0266                        | -0.2077                       |
| 0.0313 | 0.0156 | 0.9348                         | 0.4682                        | 0.9376                        | 0.9737                        | 1.3400                        |
| 0.0156 | 0.0078 | 1.1204                         | 0.5630                        | 1.0757                        | 1.1106                        | 1.0238                        |
| 0.0078 | 0.0039 | 0.7379                         | 0.3679                        | 0.6267                        | 0.8702                        | 0.3120                        |

Table 2: Example 2: Convergence order (potency of gridsize $h$) of the respective errors when the grid is refined from gridsize $h_1$ to gridsize $h_2$. We remark that the gridsizes are rounded.

Altogether, we are able to verify the results we show in Section 3, i.e. the inclusions from (31), the sparsity structure of the control, and the error estimates for control, state, adjoint state and
multiplier.

In [21] Section 5] the same examples have been analyzed, but without employing a mixed formulation for the state equation. Under almost the same structural assumptions they get the following results: For a variational discretization approach with piecewise linear and continuous state and test functions they observe errors of the order $O(h^2)$. Additionally, for a full discretization with piecewise constant control and piecewise linear and continuous state and test functions they see errors of the order $O(h)$.

In comparison, we consider a variational discretization approach combined with a mixed formulation of the state equation discretized with lowest order Raviart Thomas elements, which corresponds to $(z_h, y_h) \in P_1 \times P_0$. We see that under the given structural assumption this leads to piecewise constant controls without discretizing the control. This clearly demonstrates that the discrete structure of the control in the variational discretization strategy can be controlled through the scheme used for the discretization of the state equation.

We note that we obtain the same approximation order for the variationally discrete, piecewise constant controls as [21] for the full discretization with piecewise constant controls. However, the numerical analysis for the variationally discrete approach in our opinion is much simpler and more natural. We also note that the mixed finite element requires more degrees of freedom than the classical finite element approximation with piecewise linear, continuous elements. This then
pays off with a more accurate approximation of the derivative of the state, which might be advantageous in situations where the accurate numerical approximation e.g. of stresses is required.

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