Study of stochastic estimates of quark loops with unbiased subtraction

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Stochastic noise estimator method is a powerful tool to calculate the disconnected insertion involving quark loops. We study the variance reduction technique with unbiased subtraction. We use the complex $Z_2$ noise to calculate the quark loops on a $16^3 \times 24$ lattice with $\beta = 6.0$ and $\kappa = 0.154$. Unbiased subtraction method is performed by using hoping parameter expansion. We report on the variance reduction for the point-split vector current as a function of the number of subtraction terms and the number of noise used.

1. INTRODUCTION

Disconnected quark loops originate from the vacuum polarization of sea quarks of the interacting operator. Whether we do lattice calculation for quenched or dynamical QCD, these quark loops contribute significantly to various hadronic matrix elements. For example, for nucleonic scalar ($\pi NN-\sigma$ term) \cite{1}, vector (strange magnetic moment) \cite{2,3}, flavor-singlet axial (quark spin) \cite{4}, and tensor (quark orbital angular momentum) \cite{5} channels, it is absolutely essential to consider disconnected quark loops. In particular, the strangeness content of the nucleon comes exclusively from these disconnected loops.

However, disconnected quark loops are difficult to simulate as they contain both diagonal and off-diagonal elements of the large inverse fermion matrix ($M$). Even for a moderate size lattice (e.g. $16^3 \times 24$) one needs to invert a $10^6 \times 10^6$ matrix which requires an enormous amount of computation time. Instead, one can use noise method to estimate required traces \cite{6,7,8} to obtain disconnected loop contribution. This study focuses on the stochastic estimation of the strangeness magnetic form factor ($G_M^s(0)$) of the nucleon and reports a systematic analysis about the number of noises required to extract its signal.

Experimentally we are still uncertain about the value (even the sign) of the strangeness magnetic moment of the nucleon. SAMPLE \cite{9} and HAPPEX \cite{10} expts. reported $G_M^s(0) = 0.01 \pm 0.29 \pm 0.31 \pm 0.07$ and $G_E^s + 0.39G_M^s(at Q^2 = 0.477 GeV^2) = 0.025 \pm 0.020 \pm 0.014$, respectively. Prediction from theoretical models vary in a wide range ($-0.75$ to $+0.30 \mu_N$) \cite{11}. Lattice QCD results also differ in conclusion. Our previous studies \cite{2} suggest $G_M^s(0) = -0.28 \pm 0.10$, while ref. \cite{3} reported very tiny signal for $G_M^s$. This work is aimed at studying why there is such a difference.

2. Noise method and unbiased subtraction

Disconnected quark loop calculation by stochastic noise method have been detailed in refs. \cite{6,1,2,3,4,5}. They used random noises to estimate various traces involving fermion matrix. Using a set of random noise vectors $\eta$, one can estimate the trace of a $N \times N$ matrix $A$ as \cite{6,8}

$$\text{Tr} (A) \equiv E \left[ < \eta^\dagger A \eta > \right].$$

(1)

Of course, this trace will be an approximation for finite number of noises and the variance of the estimator depends on the choice of the noise. It has been demonstrated that $Z_2$ noise \cite{6} is the optimal noise with minimum variance \cite{7}. For a given $L$ number of $Z_2$ noises, variance of this estimation is given by \cite{6}

$$\sigma^2_A \equiv \text{Var} \left[ < \eta^\dagger A \eta > \right] = \frac{1}{L} \sum_{m \neq n}^N |A_{mn}|^2.$$  

(2)

This variance can further be reduced by the method of unbiased subtraction \cite{8}, where a set
of $P$ traceless matrices ($Q$) are subtracted from the matrix $A$ as

$$\text{Tr}(A) = E \left[ \left\langle \eta \left( A - \sum_{p=1}^{P} \lambda_p Q^{(p)} \right) \eta \right\rangle \right],$$

(3)

where $\lambda$s are some variational coefficients. Corresponding reduced variance will be

$$\sigma^2_A(\lambda) = \frac{1}{L} \sum_{m \neq n} \left| A_{m,n} - \sum_{p=1}^{P} \lambda_p Q_{m,n}^{(p)} \right|^2.$$

(4)

This subtraction is unbiased in the sense that it does not change the expectation value of $\text{Tr}(A)$. Choice of these traceless subtraction matrices ($Q$) should be such that they match the off-diagonal behavior of the matrix $A$. For disconnected loop calculation $A$ will be replaced by the inverse fermion matrix $M^{-1}$. Previously it was shown that the above variance for $M^{-1}$ can be reduced substantially by using a set of traceless matrices obtained from the hoping parameter expansion of the fermion matrix $M$ as

$$M^{-1} = I + \kappa D + \kappa^2 D^2 + \kappa^3 D^3 + \kappa^4 D^4 + \cdots$$

(5)

For the point split conserved current, disconnected quark loop can be written as

$$\text{Loop} = \sum_x e^{-i\vec{q} \cdot \vec{x}} \text{Tr} \left[ M^{-1}(x, x + \mu)(1 + \gamma_\mu)U_{\mu}^\dagger(x) \right. - \left. M^{-1}(x + \mu, x)(1 - \gamma_\mu)U_{\mu}(x) \right].$$

(6)

Before subtracting each matrix ($I$, $\kappa D$, $\kappa^2 D^2$ etc. of Eq.(5)) from $M^{-1}$ in Eq.(6), one should make sure that it does not change the loop expectation value. In fact, all matrices with $M^{-1}$ substituted with even order of $D$ are traceless in Eq.(6). First and second terms ($I$ and $\kappa D$) are also traceless. However, starting from $\kappa^3 D^3$, all odd orders in $D$ are not traceless. So, to subtract an odd order term another matrix is need to be subtracted from it so that the resulting matrix is traceless. For example, for $\kappa^3 D^3$ term, one needs to subtract following plaquette terms from the loop:

$$-8\kappa^3 \sum_x e^{i\vec{q} \cdot \vec{x}} \sum_{\nu} \text{Tr} \square_{\nu\mu}(x) + \text{Tr} \square_{\mu\nu}^*(x - \nu),$$

$$-8\kappa^3 \sum_x e^{i\vec{q} \cdot \vec{x}} \sum_{\nu} \text{Tr} \square_{\mu\nu}(x) + \text{Tr} \square_{\nu\mu}^*(x - \nu),$$

corresponding to current in the $(1 + \gamma_\mu)$ and $(1 - \gamma_\mu)$ respectively. One should notice that in each term plaquettes are at position $x$ and $x - \nu$, and so, at $[\vec{q}] \neq 0$ one cannot use translational invariance due to the Fourier transformation factor. Similarly, one needs to subtract some chair diagrams from $\kappa^3 D^3$ terms to make it traceless.

![Figure 1](image)

**Figure 1.** Plaquette term associated with current corresponding to $1 + \gamma_\mu$. This needs to be subtracted from $\kappa^3 D^3$ term to make it traceless.

### 3. Results

Numerical simulation was done on a $16^3 \times 24$ lattice at $\kappa = 0.154$ with 60 configurations where each configuration is separated by 20,000 sweeps. Unbiased subtraction is done with terms up to $\kappa^4 D^4$. We systematically study the signal for the strange quark form factor as a function of the number of complex $Z_2$ noises used per configuration. In Fig. 2 we plot the summed ratio of three to two point functions as a function of the time slice, from which one can obtain the magnetic form factor (see ref. for notations). Valence and sea quark mass is kept fixed at $\kappa = 0.154$. First sub-figure is with 300 noises without any unbiased subtraction. Next 4 sub-figures are results with unbiased subtraction with different number of noises (30, 100, 200 and 300, respectively). It is clear from these sub-figures that we do not find any signal up to 200 noises and the signal becomes prominent at around 300 noises. The fitted slopes for 100, 200, and 300 noises are $-0.052 \pm 0.09$, $-0.060 \pm 0.048$ and $-0.092 \pm 0.040$, respectively. This implies that the signal can only be extracted out at around 300 $Z_2$ noises. Slope for the 300 noise case agrees well to our previous calculation where we used subtraction terms up to $\kappa^2 D^2$. Since this result and previous result agree at one $\kappa$, we do not carry out calculation for other $\kappa$. 

values. In our previous calculation we obtained \( G_M(0) = -0.28 \pm 0.10 \) and this systemic study of noise versus signal supports that result.

4. SUMMARY

We use noise method to extract the disconnected quark loops. As an example, we choose the strangeness magnetic form factor of the nucleon. An unbiased subtraction method is employed to reduce the variance in trace estimation. This study suggests that certain minimum number of \( Z_2 \) noises are required to extract the signal. In the case of the strangeness magnetic form factor we need around 300 complex \( Z_2 \) noises. Results of this study is consistent with our previous results [2]. We believe, this also explains why with 60 real \( Z_2 \) noises, the work of [3] did not see a signal even with a larger number of gauge configurations. In future we hope to carry out this strangeness calculation with the overlap fermion.

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Figure 2. Summed ratio of three to two point functions for different number of noises from 30 to 300. \( m \) is the fitted slope which is related to the magnetic form factor [2].