INTEGRATION OF SYSTEMS OF TWO SECOND-ORDER
ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL
PARAMETER THAT ADMIT FOUR ESSENTIAL OPERATORS

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Abstract. We discuss an algorithm for integrating systems of two
second-order ordinary differential equations (ODE) with a small parameter
that admit approximate Lie algebras with four essential generators. The
algorithm is a modification of the method of consecutive order reduction
and is based on using operators of invariant differentiation. A special
attention is given to the peculiarities of its application in dependence of
the structural properties of Lie algebras of approximate symmetries.

Keywords: system of two second-order ordinary differential equations
with a small parameter, approximate Lie algebra of generators, operator
of invariant differentiation, invariant representation, differential invar-
iant, integration of equations.

1. Introduction

Integrating rth-order ordinary differential equations (ODE) with r point symme-
tries is a classical problem in group analysis. It was solved for scalar second-order
ODE (see, e.g., [1]), third-order ODE (see [2]), systems of two second-order ODE
(see, e.g., [3, 4]).

Recently, an approach for integrating systems of p-order ODE of the form
\begin{equation}
    w^{j,(p)} = f^j\left(t, u^1, \ldots, u^n, u^{1,(1)}, \ldots, u^{n,(p-1)}\right), \quad j = 1, \ldots, n,
\end{equation}

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that admit $r$-dimensional ($r = np$) Lie algebras $L_r$ of operators

\[(2) \quad X_\alpha = \xi_\alpha (t, u^1, \ldots, u^n) \frac{\partial}{\partial t} + \sum_{j=1}^{n} \eta^j_\alpha (t, u^1, \ldots, u^n) \frac{\partial}{\partial u^j}, \quad \alpha = 1, \ldots, r,\]

was suggested in [5]. This approach is based on using an operator of invariant differentiation (OID) (see, e.g., [6]) of the admitted Lie algebra $L_r$. In [7], the first author introduced the OID for the approximate Lie algebra, proposed an algorithm for its constructing, and considered the application of this OID to order reduction of systems of ODE of the form

\[(3) \quad u^{j(p)}(t) = f_{j}^{(0)}(t, u, u', \ldots, u^{(p-1)}), \quad j = 1, \ldots, n, \quad \varepsilon \ll 1,\]

that admit approximate Lie algebras of operators (up to $o(\varepsilon^1)$)

\[X_{\alpha_0} = X_{\alpha_0, (0)} + \varepsilon X_{\alpha_0, (1)}, \quad \alpha_0 = 1, \ldots, r_0, \]
\[X_{\alpha_1} = \varepsilon X_{\alpha_1, (0)}, \quad \alpha_1 = r_0 + 1, \ldots, r.\]

In the present article, we study the applicability of the algorithm proposed in [7] to systems of the form

\[(4) \quad \begin{cases} \ddot{x} = f_0 (t, x, y, \dot{x}, \dot{y}) + \varepsilon f_1 (t, x, y, \dot{x}, \dot{y}), \\ \ddot{y} = g_0 (t, x, y, \dot{x}, \dot{y}) + \varepsilon g_1 (t, x, y, \dot{x}, \dot{y}) \end{cases}\]

that admit approximate Lie algebras $\tilde{L}$ with four essential operators. The peculiarities of the consecutive order reduction and integration of systems of ODE in dependence of the structural properties of the admitted approximate Lie algebras are described; the symmetries of reduced systems of ODE are considered and the possibilities of their applying are discussed.

2. An Invariant Representation of Systems of Two Second-Order ODE with a Small Parameter

Suppose that system (4) admits four approximate operators

\[(5) \quad X_{\alpha_0} = X_{\alpha_0, (0)} + \varepsilon X_{\alpha_0, (1)}, \quad \alpha_0 = 1, \ldots, r_0, \]
\[X_{\alpha_1} = \varepsilon X_{\alpha_1, (0)}, \quad \alpha_1 = r_0 + 1, \ldots, 4,\]

where

\[X_{\alpha, (p)} = \xi_{\alpha, (p)} (t, x, y) \frac{\partial}{\partial t} + \eta^1_{\alpha, (p)} (t, x, y) \frac{\partial}{\partial x} + \eta^2_{\alpha, (p)} (t, x, y) \frac{\partial}{\partial y}, \quad \alpha = 1, \ldots, 4, \ p = 0, 1.\]

Assume that these operators are essential for some approximate Lie algebra $\tilde{L}$, i.e., the set of the operators $\{X_{\alpha_0, \varepsilon X_{\alpha_0}, X_{\alpha_1}\}$ obtained by neglecting the terms of order $o(\varepsilon)$ provides a basis of the approximate Lie algebra $\tilde{L}$. In this case, the operators $X_{\alpha, (0)}$, $\alpha = 1, \ldots, 4$, generate an “exact” Lie algebra $L_4$ which is admitted by corresponding “unperturbed” system of two second-order ODE

\[(6) \quad \begin{cases} \ddot{x} = f_0 (t, x, y, \dot{x}, \dot{y}), \\ \ddot{y} = g_0 (t, x, y, \dot{x}, \dot{y}) \end{cases}\]

We use the following notations: the expression $f(z, \varepsilon) = o(\varepsilon)$ means $\lim_{\varepsilon \to 0} \frac{f(z, \varepsilon)}{\varepsilon} = 0$; the expression $f \approx g$ means $f(z, \varepsilon) - g(z, \varepsilon) = o(\varepsilon)$. 

The operators \( X_{\alpha_0(0)} \), \( \alpha_0 = 1, \ldots, r_0 \), generate a subalgebra \( L_{r_0} \) of the Lie algebra \( L_4 \) (see [8]). The dimension of the approximate Lie algebra \( \tilde{L} \) varies from five (if \( r_0 = 1 \)) to eight (if \( r_0 = 4 \). The operators of the form \( X_{\alpha_0} \) are called “zeroth-order” operators (with respect to \( \varepsilon \)), and operators \( X_{\alpha_1} \) are “first-order”. The corresponding operators \( X_{\alpha_0(0)} \) are called stable, and \( X_{\alpha_1(0)} \) are unstable with respect to the perturbation under consideration.

To represent system (4) in terms of the differential invariants of the admitted approximate Lie algebra, introduce the matrices \( \Delta^{(2)}_{\alpha_0(0)} \), \( \Delta^{(2)}_{\alpha_1(1)} \), \( \Lambda^{(2)}_{k(0)} \) consisting of the extended operators \( \hat{X}^{(2)}_{\alpha_0(0)} \) \( \hat{X}^{(2)}_{\alpha_1(1)} \), and \( \hat{X}^{(2)}_{\alpha_1(0)} \), \( \alpha_0 = 1, \ldots, r_0, \alpha = 1, \ldots, 4 \). For example, the matrix \( \Delta^{(2)}_{\alpha_0(0)} \) has the form

\[
\begin{pmatrix}
\xi_{1,(0)} & \eta^{1}_{1,(0)} & \eta^{2}_{1,(0)} & \zeta_{1,(0)}^{1} & \zeta_{1,(0)}^{2} & \zeta_{1,(0)}^{3} & \zeta_{1,(0)}^{4} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\xi_{r_0,(0)} & \eta^{1}_{r_0,(0)} & \eta^{2}_{r_0,(0)} & \zeta_{1,(0)}^{1} & \zeta_{1,(0)}^{2} & \zeta_{1,(0)}^{3} & \zeta_{1,(0)}^{4}
\end{pmatrix},
\]

where \( \zeta^{i,(q)}_{\alpha,(p)} \) is the coordinate of the extension of the operator \( X_{\alpha,(p)} \) to the \( q \)-th order derivative of \( x^{(q)} \) if \( i = 1 \) or \( y^{(q)} \) if \( i = 2 \) (see, e.g., [6]). In particular,

\[
\zeta^{1,(q)}_{\alpha,(p)} = D^{(q)}_t \left( \eta^{1}_{\alpha,(p)} - \dot{\xi}_{\alpha,(p)} \right) + \xi^{(q)}_{\alpha,(p)} x^{(q+1)}.
\]

Constructing differential invariants of the approximate Lie algebra \( \tilde{L} \) is reduced to solving systems of first-order partial differential equations (see [9])

\[
\begin{align*}
\Omega_0: \quad & \hat{X}^{(2)}_{\alpha_0(0)} \left( I_{(0)} \right) = 0, \quad \alpha = 1, \ldots, 4, \\
\Omega_1: \quad & \hat{X}^{(2)}_{\alpha_0(0)} \left( I_{(1)} \right) + \hat{X}^{(2)}_{\alpha_1(0)} \left( I_{(0)} \right) \approx 0, \quad \alpha_0 = 1, \ldots, r_0.
\end{align*}
\]

This system defines “zeroth-order” differential invariants (with respect to \( \varepsilon \)) of the form \( I_k = I_{k(0)} + \varepsilon I_{k(1)} \), and “first-order” differential invariants (with respect to \( \varepsilon \)) of the form \( J_k = \varepsilon J_{k(0)} \). Furthermore, the functions \( I_{k(0)} \) are invariants of the Lie algebra \( L_4 \), the functions \( I_{k(1)} \) are particular solutions to system \( \Omega_1 \), and the functions \( J_{k(0)} \) are particular solutions to the corresponding homogeneous system of equations

\[
\hat{X}^{(2)}_{\alpha_0(0)} \left( J_{s(0)} \right) = 0, \quad \alpha_0 = 1, \ldots, r_0,
\]

and \( J_{k(0)} \) are functionally independent with \( I_{k(0)} \).

It is shown in [4] that if system (6) that corresponds to system (4) defines a regular manifold (see [6]) and is representable by differential invariants of \( L_4 \) then the conditions \( \text{rg } \Delta^{(1)}_{\alpha_0(0)} = \text{rg } \Lambda^{(2)}_{\alpha_1(0)} = 4 \) are fulfilled. In this case, the approximate Lie algebra \( \tilde{L} \) has three “zeroth-order” differential invariants (with respect to \( \varepsilon \)). Two of them, \( I_1 = I_{1(0)} + \varepsilon I_{1(1)} \) and \( I_2 = I_{2(0)} + \varepsilon I_{2(1)} \), are second-order, and one of them, \( I = I_{(0)} + \varepsilon I_{(1)} \), is a first-order differential invariant (if \( \text{rg } \Lambda^{(0)}_{\alpha_1(0)} = 3 \) or an algebraic invariant (if \( \text{rg } \Lambda^{(0)}_{\alpha_1(0)} = 2 \)).

If the conditions \( \text{rg } \Delta^{(1)}_{\alpha_0(0)} = r_0, \text{rg } \Delta^{(2)}_{\alpha_0(0)} = r_0 \) are also fulfilled then the approximate Lie algebra \( \tilde{L} \) has \( 4 - r_0 \) “first-order” differential invariants (with respect to \( \varepsilon \)). Obviously, the number of “first-order” invariants depends on the number of
stable operators and varies from zero for \( r_0 = 4 \) to three for \( r_0 = 1 \). Then, according to [9], system (4) is representable as

\[
\begin{align*}
I_1^2 &= F_0(I) + \varepsilon F_1(I_{(0)}, J_{1,(0)}, \ldots, J_{4-r_0,(0)}), \\
I_2^2 &= G_0(I) + \varepsilon G_1(I_{(0)}, J_{1,(0)}, \ldots, J_{4-r_0,(0)}).
\end{align*}
\]

Note that if the condition \( \text{rg} \Delta^{(k)}_{\alpha_0,(0)} = r_0 \) does not hold then \( \Omega_1 \) gives additional equations that should be added to the system \( \Omega_0 \). However, such cases do not appear for the approximate Lie algebras under consideration.

3. AN OPERATOR OF INVARIANT DIFFERENTIATION FOR AN APPROXIMATE LIE ALGEBRA

An operator of invariant differentiation (OID) \( \lambda D_t \) for an approximate Lie algebra \( \tilde{L} \) that is admitted by system (16) is introduced by analogy with the case of an exact Lie algebra. This operator transforms one differential invariant of the Lie algebra into another one. Assume that the OID is linear with respect to \( \varepsilon \).

It is shown in [7] that an OID for an approximate Lie algebra can be obtained from the condition

\[
[\lambda D_t, X_\alpha] \approx 0.
\]

Furthermore, if the matrix \( \left( c^\gamma_{\alpha\beta} \right) \) composed of the structure constants \( c^\gamma_{\alpha\beta} \) of the exact Lie algebra \( L_4 \) (where \( \alpha, \beta \) give the row number and \( \gamma \) is the column number), satisfies the condition

\[
\text{rg} \left( c^\gamma_{\alpha\beta} \right) < 4
\]

then the factor \( \lambda \) of the OID can be found in the form

\[
\lambda = (D_t(\Phi_0 + \varepsilon \Phi_1))^{-1} \approx \frac{D_t \Phi_0 - \varepsilon D_t \Phi_1}{(D_t \Phi_0)^2}.
\]

The functions \( \Phi_0, \Phi_1 \) are obtained as particular solutions to the corresponding systems of inhomogeneous first-order partial differential equations

\[
\begin{align*}
\Pi_0: & \quad \tilde{X}_{\alpha_0,(0)}^1 \Phi_0 = K_{\alpha_0,(0)}, \quad \alpha = 1, \ldots, 4, \\
\Pi_1: & \quad \tilde{X}_{\alpha_0,(1)}^1 \Phi_1 + \tilde{X}_{\alpha_0,(1)}^1 \Phi_0 = K_{\alpha_0,(1)}, \quad \alpha_0 = 1, \ldots, r_0.
\end{align*}
\]

These systems are obtained from (8) after splitting with respect to \( \varepsilon \) and a single integration. Methods for solving such systems require studying their completeness and consistency (see, e.g., [10]). Since \( \text{rg} \Lambda^{(1)}_{4,(0)} = 4 \), the subsystem \( \Pi_0 \) is consistent for all \( K_{\alpha,(0)} \). The subsystem \( \Pi_0 \) is complete if the constants \( K_{\alpha,(0)} \) satisfy the system of algebraic equations

\[
\sum_{\alpha=1}^4 c^\alpha_{\beta\gamma} K_{\alpha,(0)} = 0, \quad \beta, \gamma = 1, \ldots, 4,
\]

where \( c^\alpha_{\beta\gamma} \) (\( \alpha, \beta, \gamma = 1, \ldots, 4 \)) are the structure constants of the Lie algebra \( L_4 \). The subsystem \( \Pi_1 \) is consistent at the solutions of \( \Pi_0 \) if the rank of \( \Pi_1 \) is equal
to \( r_0 \) (since \( \text{rg} \Delta(1)_{r_0} = r_0 \)) and complete if the constants \( K_{\alpha_0}(1) \) satisfy the system of inhomogeneous equations

\[
\sum_{\alpha_0=1}^{r_0} c^\alpha_{\beta_0 \gamma_0} K_{\alpha_0}(1) = -\sum_{\sigma=r_0+1}^{r_0+r} c^\sigma_{\beta_0 \gamma_0} K_{\sigma- r_0}(0),
\]

where \( c^\alpha_{\beta_0 \gamma_0} \), \( c^\sigma_{\beta_0 \gamma_0} \), \( \alpha_0, \beta_0, \gamma_0 = 1, \ldots, r_0, \sigma = r_0 + 1, \ldots, r_0 + 4 \), are the structure constants of the subalgebra \( L_{r_0} \) and the structure constants of the approximate Lie algebra \( \bar{L} \) (that do not coincide with the structure constants of \( L_r \) and \( L_{r_0} \)) respectively.

Consider the sum \( \Phi_0 + \varepsilon \Phi_1 \), \( \Phi_0 \neq \text{const} \) of the particular solutions \( \Phi_0 \) and \( \Phi_1 \). By [7], this sum is an invariant of the approximate Lie algebra with three essential operators which are linear combinations of operators (5) with the coefficients defined by (12) and (13).

The obtained OID is further used for constructing an approximate first integral of system (4) that is representable as (7). The main step of the proposed order reduction algorithm of a system of ODE with a small parameter is constructing an auxiliary system of differential relations connecting the differential invariants of the admitted Lie algebra, their full derivatives, and also the full derivatives of the constructed functions \( \Phi_0 \) and \( \Phi_1 \). This system can be regarded as a system of equations in full differentials, i.e. a system of first-order ODE, and its general solution gives an approximate first integral of the original system.

**Remark.** In the case of an unperturbed system of ODE, such an auxiliary system consists only of one first-order ODE. Such an equation is constructed for system (6) with four symmetries (see [4]) and for a system of \( r \)-th order ODE admitting an \( r \)-dimensional Lie algebra (see [5]).

Let us now construct a system of first-order ODE connected with system (4).

All nonsolvable four-dimensional Lie algebras are decomposable, i.e. they are representable as the direct sum of a three-dimensional simple Lie algebra and a one-dimensional subalgebra (see [11]). Moreover, condition (9) holds. Hence, for an approximate Lie algebra with four essential operators, there is always at least one OID with coefficient (10). Consider the result of the action of this OID at the invariants \( I, J_1, \ldots, J_{4-r_0} \) of (7).

Applying the OID to \( I \), we obtain the differential relation

\[
\frac{D_t \Phi_0 - \varepsilon D_t \Phi}{(D_t \Phi_0)^2} D_t \left( I(0) + \varepsilon I(1) \right) |_{(7)} = H_0 \left( I(0) \right) + \varepsilon H_0' \left( I(0) \right) I(1) + \varepsilon H_1 \left( I(0), J_1(0), \ldots, J_{4-r_0}(0) \right).
\]

Rewrite it in the form

\[
\frac{D_t I(0)}{D_t \Phi_0} - H_0 \left( I(0) \right)
+ \varepsilon \left( \frac{D_t I(1)}{D_t \Phi_0} - \frac{D_t \Phi_1 D_t I(0)}{(D_t \Phi_0)^2} - H_0' \left( I(0) \right) I(1) - H_1 \left( I(0), J_1(0), \ldots, J_{4-r_0}(0) \right) \right) = 0.
\]

One can show that this equation admits all the approximate operators \( X_\alpha \), \( \alpha = 1, \ldots, r \). The proof of this assertion follows from the systems \( \Omega_0, \Omega_1, \Pi_0, \Pi_1 \), and the identities (see [6])

\[
D_t \left( XF \right) - X \left( D_t F \right) = D_t \xi D_t F.
\]
Applying the OID to $J_k$, we obtain
\[
\frac{D_t \Phi_0 - \varepsilon D_t \Phi_1}{(D_t \Phi_0)^2} D_t \left( \varepsilon J_{k,(0)} \right) \approx \varepsilon Q_k \left( I_0, J_{1,(0)}, \ldots, J_{4-r_0,(0)} \right), \quad k = 1, \ldots, 4 - r_0.
\]

These approximate equalities are equivalent to the system of differential relations

\[
\frac{D_t J_{k,(0)}}{D_t \Phi_0} \bigg|_{(7)} = Q_k \left( I_0, J_{1,(0)}, \ldots, J_{4-r_0,(0)} \right), \quad k = 1, \ldots, 4 - r_0.
\]

It is easy to see that system (15) admits the operators $X_{\alpha_0,(0)}$ generating the subalgebra $L_{r_0}$.

In the next section, we will consider the solutions to the obtained system of differential relations (14) and (15) and the symmetries of these solutions.

4. ORDER REDUCTION FOR SYSTEMS OF TWO SECOND-ORDER ODE WITH A SMALL PARAMETER

The general algorithm of order reduction for systems of ODE with a small parameter is described in [7] and consists of the following steps:

1. calculating the approximate symmetries admitted by the system of ODE;
2. representing the system of ODE in terms of the differential invariants of an admitted approximate Lie algebra;
3. constructing an OID of the special form $(D_t(\Phi_0 + \varepsilon \Phi_1))^{-1} D_t$;
4. applying the obtained OID to the invariants of the lower orders and getting a system of first-order ODE;
5. solving the obtained equations and constructing an approximate first integral of the initial system;
6. adding the so-obtained relation to the equations of the initial system and excluding its differential consequences;
7. repeating steps 3–6.

The fulfillment of step 5 depends on the structure of the approximate Lie algebra of the initial system, and passage from step 7 to steps 3–6 requires an additional study of the symmetry properties of the reduced system. Let us examine the applicability of this algorithm to second-order systems of ODE with a small parameter depending on the structure of an admitted approximate Lie algebra $\tilde{L}$, i.e., on the structure of the corresponding exact Lie algebra $L_4$ and the dimension and structure of the subalgebra $L_{r_0}$.

4.1. Case 1. Suppose that system (4) admits an approximate Lie algebra $\tilde{L}$ with four essential operators, and that the operators $X_{\alpha_0,(0)}$, $\alpha = 1, \ldots, 4$, are stable with respect to the perturbation under consideration. Then the invariant representation (7) of system (4) does not contain “first-order” invariants, i.e., system (4) is representable as

\[
\begin{align*}
I_1^2 &= F_0 (I) + \varepsilon F_1 (I_0), \\
I_2^2 &= G_0 (I) + \varepsilon G_1 (I_0),
\end{align*}
\]

where $I$ is a first-order differential invariant if $\text{rg} L_4^{(0)} = 3$ or an algebraic invariant if $\text{rg} L_4^{(0)} = 2$. 

If the admitted approximate Lie algebra has an OID with factor (10) then an auxiliary system consists of one first-order ODE

\[ \frac{dI}{(H_0(I) + \varepsilon H_1(I))} \approx d(\Phi_0 + \varepsilon \Phi_1). \]

The general solution to this equation is a first integral of the initial system. Adding it to the initial system of ODE and excluding its differential consequences, we obtain a third-order system admitting an approximate Lie algebra with three essential operators which are the “zeroth-order” with respect to \( \varepsilon \).

If the exact Lie algebra \( L_4 \) is nonsolvable then the reduced third-order system admits an approximate Lie algebra with three essential operators, where \( L_3 \) is a simple Lie algebra, i.e., it does not contain any ideals except trivial ones. The above-described procedure does not apply to this system, and no further reduction is possible.

If the exact Lie algebra \( L_4 \) is solvable then \( L_3 \) is its solvable ideal, and we can apply the above-described algorithm to the reduced system of equations and obtain a second-order system with two stable essential symmetries. After repeating the order reduction procedure, we obtain a first-order system with one stable symmetry. Repeating the procedure once again, we obtain an approximate solution to the initial system of ODE with a small parameter.

### 4.2. Case 2.

Suppose that part of the essential operators of the approximate Lie algebra \( \hat{L} \) admitted by (4) are “first-order” with respect to \( \varepsilon \). In this case, of importance is the representability of exact Lie algebra \( L_4 \) as a sequence of embedded subalgebras

\[ L_r \supset L_{r-1} \supset \ldots \supset L_{r_0}, \quad \dim L_{r-s}/L_{r-s-1} = 1, \tag{17} \]

In [15], the authors described all subalgebras for each four-dimensional Lie algebras. An analysis of these results showed that, for any solvable Lie algebra \( L_4 \) and any its subalgebra \( L_{r_0} \), one can construct a sequence (17) containing the subalgebra \( L_{r_0} \). For nonsolvable Lie algebras \( L_4 \) such sequences do not always exist. In the present work, we consider only systems of ODE where the exact Lie algebra \( L_4 \) corresponding to the admitted approximate Lie algebra \( \hat{L} \) is solvable, and we use the results of [15] for them.

Then system (4) is represented by differential invariants as (7), where \( J_{1,(0)} \) is the invariant of \( L_{r_0} \). We can choose any other invariant \( J_{s,(0)} \) as an invariant of corresponding subalgebra \( L_{r_0+s-1} \) from (17).

Then system (15) can be reduced to the “triangular” form

\[ \frac{D_t J_{s,(0)}}{D_t \Phi_0} \bigg|_{(7)} = Q_k \left( I_{(0)}, J_{1,(0)}, \ldots, J_{s,(0)} \right), \quad s = 1, \ldots, 4 - r_0, \tag{18} \]

which is invariant with respect to subalgebra \( L_{r_0+s-1} \).

For order reduction of system (4), study the relationship between the derived algebra \( L'_4 \) and its subalgebras from (17). If an ideal \( N_3 \) containing \( L'_4 \) coincides with the subalgebra \( L_3 \) in sequence (17) then reduced system of ODE also has \( r_0 \) stable symmetries. If the ideal \( N_3 \) does not coincide with \( L_3 \) in (17) then the reduced system of ODE has \( r_0 - 1 \) stable symmetries. Then the procedure is repeated, the derived algebra \( L'_3 \) is considered. Thus, the algorithm enables us to reduce the order of the original system \( r_0 \) times or more provided that the derived algebra (or the ideal containing it) is included in the sequence (17).
In the present article, as an example, we consider the case where system (4) admits an approximate Lie algebra with four essential operators, where three of the four essential operators of the corresponding exact Lie algebra $L_4$ are stable and one is unstable, and $L_4$. The remaining cases are examined similarly.

In this case, system (4) has the invariant representation

$$
\begin{align*}
I_2 & \approx F(0) (I) + \varepsilon F(1) (I(0), J_{1,0}), \\
I_2 & \approx G(0) (I) + \varepsilon G(1) (I(0), J_{1,0}).
\end{align*}
$$

Applying the OID to the invariants $I$ and $J_1$ and expanding the results in a series in $\varepsilon$, we obtain the system

$$
\begin{align*}
D_t I(0) & = H_0 (I(0)), \\
D_t I(1) - D_t I(0) D_t \Phi_0 & = H'_0 (I(0)) I(1) + H_1 (I(0), J_{1,0}), \\
D_t J_{1,0} & = Q (I(0), J_{1,0}).
\end{align*}
$$

Here $I(0)$ is a differential invariant of $L_4$, $J_{1,0}$ is a differential invariant of the subalgebra $L_3$ generated by stable operators, and $\Phi_0$ is a differential invariant of a three-dimensional ideal $N_3$ of $L_4$.

Consider two cases separately: $N_3$ coincides with $L_3$ and it does not.

4.2.1. The Unstable Operator Is Not in $L'_4$. In this case $N_3$, coincides with $L_3$. Then we can construct an OID such that $J_{1,0} = \psi(\Phi_0)$. Then system (20) has the form

$$
\begin{align*}
\frac{dI(0)}{d\Phi_0} & = H_0 (I(0)), \\
\frac{dS}{d\Phi_0} & = H'_0 (I(0)) S + H_1 (I(0), \psi(\Phi_0)), \\
J_{1,0} & = \psi(\Phi_0).
\end{align*}
$$

The solution to this system of differential relations

$$
W(I(0) + \varepsilon I(1), \Phi_0 + \varepsilon \Phi_1) = C_1
$$

is a first integral of the initial system, and $C_1 = C_{1,0} + \varepsilon C_{1,1}$.

Adding the obtained first integral to the equations of the initial system of ODE and excluding its differential consequences, we obtain a third-order system. It admits the operators $X_{\alpha,0} + \varepsilon X_{\alpha,1}$, $\alpha = 1, 2, 3$, or their linear combinations. In other words, the third-order system admits a Lie algebra with three stable essential operators, and the corresponding exact Lie algebra $L_3$ is solvable.

The described algorithm can be applied to the reduced system of equations to obtain a second-order system with two stable essential symmetries.

After repeating the order reduction procedure, we obtain a first-order system with one stable symmetry.

Repeating the procedure once again, we obtain an approximate solution to the original system of ODE with a small parameter.
4.2.2. **The Unstable Operator Is in $L_4'$.** In this case, system (20) has the form

\[
\begin{align*}
\frac{dI(0)}{d\Phi_0} &= H_0(I(0)), \\
\frac{dS}{d\Phi_0} &= H'_0(I(0))S + H_1(I(0), J_{1,(0)}) , \\
\frac{dJ_{1,(0)}}{dI(0)} &= \frac{Q(I(0), J_{1,(0)})}{H_0(I(0))}.
\end{align*}
\]

The last equation of this system admits the operator

\[
K_{1,(0)}X_{1,(0)} + K_{2,(0)}X_{2,(0)} + K_{3,(0)}X_{3,(0)},
\]

and hence it can be integrated by quadratures. After that, we insert the solution in the second equation and reduce this system to a system of two equations. The solution to the resulting system of equations gives a first integral of the initial system. Adding it to the equations of the system and removing the differential consequences of this approximate first integral, we obtain the third-order system of the form

\[
\begin{align*}
I(0) + \varepsilon I(1) &= \varphi_0(\Phi_0 + \Phi_1) + \varepsilon\varphi_1(\Phi_0), \\
I_{2,(0)} + \varepsilon I_{2,(1)} &= \psi_0(\Phi_0 + \Phi_1) + \varepsilon\psi_1(\Phi_0).
\end{align*}
\]

This system admits two “zeroth-order” operators $Y_{1,(0)}$, $Y_{2,(0)}$, $Y_{3,(0)}$, one “first-order” operator $\varepsilon Y_{3,(0)}$. The operators $Y_{1,(0)}$, $Y_{2,(0)}$, $Y_{3,(0)}$ constitute a basis in the ideal $N_3$.

Repeating the above arguments, consider a two-dimensional ideal $N_2$ containing the derived algebra of the Lie algebra $N_3$.

If $N_2 = \langle Y_{1,(0)}, Y_{2,(0)} \rangle$, as a result of applying the algorithm, we obtain a second-order system with two stable essential symmetries. After repeating the order reduction procedure, we obtain a first-order system with one stable symmetry. The next application of the procedure gives an approximate solution to the initial system of ODE with a small parameter.

If $N_2 = \langle Y_{1,(0)}, Y_{3,(0)} \rangle$, as a result of applying the algorithm we obtain a second-order system that admits one stable operator $Z_{1,(1)}$ and one unstable operator $\varepsilon Z_{2,(0)}$. If the derived algebra of the Lie algebra $N_2$ contains no unstable operator, then, after the order reduction procedure, we obtain a first-order system. This system has one stable symmetry and hence, after repeating the procedure, we can obtain an approximate solution to the original system of ODE with a small parameter. If the derived algebra of $N_2$ contains an unstable operator then the first-order system obtained after the order reduction procedure has one unstable symmetry. In this case, it is possible to obtain the solution only for the unperturbed part of the equation.

**Example.** Consider the system of equations

\[
\begin{align*}
\ddot{x} &= \dot{x}^2 \dot{y} \frac{1}{x} + \varepsilon \left( \frac{\dot{x}^2(t\dot{y} - y)}{x} + \dot{x}^2 \dot{y} - \frac{2}{x} \right), \\
\ddot{y} &= \frac{\dot{x} e^y}{x} - \frac{\varepsilon \dot{x} e^y}{x} \left( t e^y - (t\dot{y} - y) e^y - \frac{\dot{y}}{x} \right).
\end{align*}
\]
This system admits the operators

\[ X_1 = (1 + \varepsilon t) \frac{\partial}{\partial t}, \quad X_2 = \varepsilon x \frac{\partial}{\partial x}, \quad X_3 = (1 + \varepsilon t) \frac{\partial}{\partial y}, \quad X_4 = (t + \varepsilon t^2) \frac{\partial}{\partial t} + (y + \varepsilon ty) \frac{\partial}{\partial y}. \]

These operators are essential for a six-dimensional approximate Lie algebra.

For the corresponding exact Lie algebra \( L_4 \), there is a sequence of embedded subalgebras:

\[ L_4 \supset L_{1(0)} < X_1; (0), X_3; (0), X_4; (0) \supset L_{1(0)}. \]

For an invariant representation of system (22), we use the invariants

\[ I_1 = \dot{y} + \varepsilon (t\dot{y} - y), \quad I_2 = \frac{x \ddot{x} + 2x \dot{x}^2}{x^2}, \quad I_3 = \frac{x \ddot{y} + \varepsilon t \dot{x} \dot{y}}{x}, \quad I_0 = \varepsilon x. \]

Rewrite system (22) by means of an admitted approximate Lie algebra:

\[ \begin{cases} I_2 \approx I_1 + \varepsilon I_{1(0)} I_{0(0)}, \\ I_3 \approx e^{I_1} + \varepsilon I_{1(0)} I_{0(0)}. \end{cases} \]

The function \( \Phi \) for constructing an OID is obtained from the system

\[ \begin{cases} \Phi_{0t} = 0, \\ x \Phi_{0x} + \dot{x} \Phi_{0z} = K_2, \\ \Phi_{0y} = 0, \\ t \Phi_{0t} + y \Phi_{0y} - \dot{x} \Phi_{0z} = K_4, \\ \Phi_{1t} = 0, \\ \Phi_{1y} = 0, \\ t \Phi_{1t} + y \Phi_{1y} - \dot{x} \Phi_{1z} + t^2 \Phi_{0z} + ty \Phi_{0y} - 2t \dot{x} \Phi_{0z} + (y + t \dot{y}) \Phi_{0y} = 1. \end{cases} \]

Let, for example, \( \Phi_0 = \ln x, \quad \Phi_1 = \ln t. \) Note that \( \Phi_0 = \ln I_{0(0)}. \)

Applying the operator \( \frac{x(t \ddot{x} - \varepsilon \dot{x})}{t x^2} D_t \) to the invariant \( I_1 \), we obtain

\[ \begin{cases} \frac{D_t (\dot{y})}{D_t \ln x} = e^\Phi, \\ \frac{D_t (t \dot{y} - y)}{D_t \ln x} = e^\Phi (t \dot{y} - y) + \dot{y} e^{-\ln x}. \end{cases} \]

Adding the solution to this system to the equations of the initial system (22) and excluding its differential consequences, we obtain the reduced system

\[ \begin{cases} I_1 \approx - \ln (C_1 - \Phi_0) - \varepsilon H(\Phi_0), \\ H(\Phi_0) = \frac{1}{C_1 - \Phi_0} \int (C_1 - \Phi_0) e^{-\Phi_0} \ln (C_1 - \Phi_0) d\Phi_0, \\ I_3 \approx I_1 - \varepsilon e^{-\Phi_0} \ln (C_1 - \Phi_0), \end{cases} \]

which admits the operators \( X_1, X_3, \) and \( X_4. \) This procedure can be repeated and a solution to the initial system of ODE can be obtained. □
5. Conclusion

In this article, we have shown that the application of the order reduction algorithm to systems of two second-order ODE with a small parameter that admit approximate Lie algebras with four essential operators gives an approximate solution to these systems in the case when the corresponding exact four-dimensional Lie algebra is solvable.

Applying the algorithm to higher-order systems requires an additional investigation of the structure of an admitted approximate Lie algebra of operator $\tilde{L}$, i.e., of the structure of the corresponding exact Lie algebra $L_4$ and also the dimension and structure of the subalgebra $L_{r_0}$.

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