Abstract

The value of the light quark masses combination $m_u + m_d$ is analyzed using QCD-Hadron Duality. A detailed analysis of both the perturbative QCD [to four-loops] and the hadronic parametrization needed is done. The result we get is $[m_u + m_d](1 \text{GeV}^2) = (12.8 \pm 2.5) \text{MeV}$ ($[m_u + m_d](4 \text{GeV}^2) = (9.8 \pm 1.9) \text{MeV}$) in the $\overline{\text{MS}}$ scheme.

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We are interested in the value of the light quark masses that appear in the Standard Model. These are the QCD Lagrangian quark masses or Current Algebra quark masses. This work is an update of [1] putting some emphasis in the QCD-Hadron Duality and including the recently calculated $\alpha_s^3$ QCD corrections [2]. A collection of the relevant references to earlier work can be found in [1].

The QCD Sum Rule (SR) technique has been discussed several times in this conference. (See the talks by Narison and Jamin for instance.) The relevant two-point function for calculating $m_u + m_d$ is the correlator of two divergences of the axial-vector current with the quantum numbers of the pion field $[I^G(J^P) = 1^-(0^-)]$. (See also the talk by de Rafael.)

$$\Psi_5(q^2) \equiv \int d^4x e^{iq\cdot x} \langle 0| T \{ \partial^\mu A^{1-i2}_\mu(x), \partial^\nu A^{1+i2}_\nu(0) \} |0 \rangle,$$  
(1)

with $\partial^\mu A^{1-i2}_\mu(x) = [\partial^\mu A^{1+i2}_\mu(x)]^\dagger = [m_u + m_d] [\bar{n}\gamma_5 d]$. The QCD Sum Rule technique exploits the analytic structure of $\Psi_5(q^2)$ combined with Cauchy’s theorem. The specific type of SR we shall use is the so-called Finite Energy Sum Rule (FESR). For the discussion of quark masses using QCD-Hadron Duality there are two relevant FESRs. These are the so-called 0th-moment

$$\int_0^s dt \frac{1}{\pi} \text{Im}\Psi_5(t) = \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 s^2 \left[ \frac{1}{2} + R_1(s) + \frac{C_4(O_4)}{s^2} \right]$$  
(2)

and 1st-moment

$$\int_0^s dt \frac{1}{\pi} \text{Im}\Psi_5(t) = \frac{N_c}{8\pi^2} [m_u(s) + m_d(s)]^2 s^3 \left[ 1 + R_2(s) - \frac{3}{2} \frac{C_6(O_6)}{s^3} \right],$$  
(3)

respectively. The upper limit of the integral, $s$, is the onset of the QCD continuum. The hypothesis of QCD-Hadron Duality that we have used to obtain (2) and (3) can be expressed as follows. One expects that there exists some intermediate region $[s$ belongs to that region$]$, such that above this region the description of inclusive enough quantities like $\Psi_5(q^2)$ are well approximated by perturbative QCD, i.e. local duality is approximately satisfied. In the intermediate region one can just expect to have duality with perturbative QCD for suitable averages of inclusive quantities like the moments in (2) and (3), i.e. global duality. Below that region one doesn’t expect any kind of duality.

Let me discuss the inputs needed in these FESRs. The rhs in (2) and (3) are given by

$$-\frac{1}{2\pi i} \int_s dt \Psi_5^{(2)}(t) \left( 1 - \frac{t}{s} \right)^2 \sum_{k=1}^n \frac{n + 1 - k}{n} \left( \frac{t}{s} \right)^{k-1},$$  
(4)
with $n = 1$ and $n = 2$ respectively, and $\Psi_5^{(2)}(t) \equiv \frac{d^2}{dt} \Psi_5(t)$. For large enough $Q^2 \equiv -q^2 = -t$, $\Psi_5^{(2)}(Q^2)$ is known in QCD up to four-loops [order $\alpha_s^3$] from [2, 3]. Once the global light quark masses factor scaled at $s$ together with the energy dependence and normalization factors have been pulled out, what remain from (4) is the QCD perturbative series $1 + R(s)$. This is supplemented by power corrections à la SVZ [4] parametrized by Wilson coefficients $C_i$ and condensates $\langle O_i \rangle$. The values of the power corrections coefficients are $C_4 \langle O_4 \rangle = (0.08 \pm 0.04) \text{ GeV}^4$ and $C_6 \langle O_6 \rangle = (0.04 \pm 0.03) \text{ GeV}^6$ [1].

To study the convergence of the truncated perturbative series, we have used three different ways of resumming the QCD series. The first one, that we shall call Perturbative, is obtained resumming the renormalization group logs after the Cauchy’s integral in (4) is done. In the second one, which we shall call Improved, the large running of the coupling constant around the Cauchy’s circuit (from 0 to $2\pi$) is taken into account by resumming the renormalization group logs before the Cauchy’s integral in (4) is done [5]. The third way of resumming is obtained by applying the Principle of Minimal Sensitivity (PMS) [6] to the QCD expression for $\Psi^{(2)}(Q^2)$ before the Cauchy’s integral is done. We have applied the renormalization group running to $\alpha_s(m_\tau) = 0.35 \pm 0.02$ [7] at three-loops to obtain $\alpha_s(s)$ around the Cauchy’s circuit. Below we show the QCD perturbative series $1 + R_2(s)$ for $s = 2 \text{ GeV}^2$ obtained using the three ways of resumming.

\begin{align*}
\text{Perturbative} & : 1 + 0.60 + 0.38 + 0.19 + \cdots \\
\text{Improved} & : 1 + 0.68 + 0.20 + 0.07 + \cdots \\
PMS & : 1 + 0.85 + 0.12 - 0.01 + \cdots .
\end{align*}

The second term in each line is the order $\alpha_s$ correction, the third is the order $\alpha_s^2$ and the fourth the order $\alpha_s^3$ correction. The comparison of the series in (5) shows that PMS is the best behaved but also shows how the three series show good convergence if one considers corrections up to order $\alpha_s^3$. The numerical difference between them is less than 10% at that order.

Let me now discuss the inputs needed for the lhs of (2) and (3). The integrand $\text{Im} \Psi_5(t)$ is an inclusive cross-section which, ideally, should be determined in terms of the hadronic states that couple to the vacuum through the divergence of the axial-vector current $\partial^\mu A_\mu^{1-12}(x)$, i.e. the $\pi$-pole, $3\pi$-continuum, $5\pi$-continuum, $\pi^\prime$-resonances, $N\overline{N}$, \cdots. Apart of the pion pole, we have not complete access to this spectral function yet. We do however have several pieces of information on it. In the low energy end, Chiral Perturbation Theory (CHPT) predicts

\begin{align*}
\frac{1}{\pi} \text{Im} \Psi_5(t) & = 2f_\pi^2 m_\pi^4 \delta(t - m_\pi^2) \\
& + \Theta(t - 9m_\pi^2) \frac{2f_\pi^2 m_\pi^4}{[16\pi^2 f_\pi^2]^2} \frac{t}{18} \rho^3_\chi(t) + \cdots
\end{align*}

(6)
where $f_\pi = 92.4$ MeV in this normalization and the dots stand for higher thresholds contributions. The first term is due to the pion pole and is completely fixed by the pion mass and its decay coupling constant $f_\pi$, this will become important afterwards. When $t \to 0$, the spectral function $\rho_3^{3\pi}(t)$ can be obtained at lowest order in CHPT. At lowest order in Generalized CHPT, one gets the CHPT result times some factor, see Stern’s talk. In fact, no symmetry allows to predict $\rho_3^{3\pi}(t = 0)$ and it receives corrections to all orders in CHPT proportional to pion and kaon masses. We shall let this normalization factor free and require QCD-Hadron Duality to fix it, see below.

For values of $t$ above a few hundred MeVs, CHPT is not anymore at work and we enter the resonance region. Here we have some experimental information to help us, for instance we know that there are resonance states with the quantum numbers of the pion which couple to the $3\pi$-continuum, $5\pi$-continuum, · · · . There are also two-body thresholds like $\rho_\pi, \rho'\pi, \rho_\pi', N\bar{N}$, and so on. The contributions of the resonances are in fact leading in the $1/N_c$ expansion [4, 8] and have to be included if we want to obtain QCD-Hadron Duality. In the Review of Particle Physics [9], we can find two resonances with the pion quantum numbers. Their masses and widths are: $M_1 = [1300 \pm 100]$ MeV, $\Gamma_1 = [400 \pm 200]$ MeV, $M_2 = [1770 \pm 30]$ MeV, $\Gamma_2 = [310 \pm 50]$ MeV. The most recent experimental data where both $\pi'$-resonances are seen is in [10]. One can find there the energy dependence of the $I^G(J^P) = 1^-(0^-)$ amplitude in the three-pion channel. This amplitude is proportional [in the three-pion channel] to the spectral function with the quantum numbers of the divergence of the axial-vector current we are studying. Unfortunately, the normalization of the coupling of this current to $N\bar{N}$ is unknown. That data allowed to obtain the masses and widths of the two $\pi'$-resonances since the non-resonant background in the data was found to be very small [10]. The values obtained there agree quite well with the ones reported in the Review of Particle Physics [9]. Therefore, we shall take the $I^G(J^P) = 1^-(0^-)$ data in [10] as a good estimate of the resonant shape allowing a free relative strength between both $\pi'$ resonances.

Another piece of information to add to our study of the spectral function is the success of the resonance dominance in the vector channel (VMD). One can obtain the successful VMD predictions using a spectral function with Breit-Wigner shapes for the resonances normalized to lowest order CHPT prediction [4, 8]. The difference in this case with respect to the divergence of the axial-vector channel one is that the normalization at $t = 0$ of the vector form factor is fixed since the vector current couples to the electromagnetic current. The proposal we make is to use scalar meson dominance for the $I^G(J^P) = 1^-(0^-)$ channel, so that the spectral function for the three-pion channel is the one in (4) with the substitution

$$\hat{\rho}_3^{3\pi}(t) \to \rho_3^{3\pi}(t) \equiv A \hat{\rho}_3^{3\pi}(t) |\mathcal{F}(M_1, \Gamma_1, M_2, \Gamma_2; t)|^2 \quad (7)$$

where $A$ is a free normalization factor, $\hat{\rho}_3^{3\pi}(t)$ is the lowest order CHPT result.
and

\[ F(M_1, \Gamma_1, M_2, \Gamma_2; t) \equiv \frac{1}{t - M_1^2 + i\Gamma_1 M_1} + \xi \frac{1}{t - M_2^2 + i\Gamma_2 M_2} \]

\[ - M_1^2 + i\Gamma_1 M_1 + \xi (- M_2^2 + i\Gamma_2 M_2) \]  

(8)

Here \( \xi \) is a complex parameter which fixes the relative strength between the two \( \pi' \) resonances and a possible different absorption by the nuclear target. We also included the corresponding Breit-Wigner resonant shape for the \( \rho \)-meson in the intermediate \( \rho \pi \) two-body subchannel which contribution is leading in \( 1/N_c \) and phase space \([1, 8]\). The \( \xi \) parameter can be obtained from a fit of the \( I^G(J^P) = 1^- (0^-) \) data in \([10]\). We get a good fit to the shape of the \( I^G(J^P) = 1^- (0^-) \) data by fixing \( \xi = 0.234 + i 0.1 \).

Let me now apply the technique to obtain \( m_u + m_d \) using all the information above. We only use the three-pion channel. The onset of the QCD continuum \( s \) in (2) and (3) is fixed by demanding a good QCD-Hadron Duality between

\[ R_{\text{had}}(s) = \frac{3}{2s} \int_0^s dt \frac{1}{\pi} \Im \Psi_5(t) \quad \text{and} \quad R_{\text{QCD}}(s) = \frac{1 + R_2(s) - \frac{3}{2} \frac{C_6(O_6)}{s^3}}{1 + R_1(s) + 2 \frac{C_4(O_4)}{s^2}}. \]

(9)

The plots of the ratios in (9) are shown in Figure 1. The curves labeled as

![Figure 1: Duality Ratios in (9). See text for explanation of the curves.](image)

\textit{Perturbative, Improved, and PMS correspond to} \( R_{\text{QCD}}(s) \) obtained using the
different ways of resumming the QCD series explained above. Observe the small difference between the three curves. The effect of varying $\alpha_s$ and the condensates in the range quoted above is less than 5%. Other non-perturbative contributions like for instance the possible large contribution of instantons mostly cancel in $R_{QCD}(s)$. In particular, the result in [12] produces less than 5% change in $R_{QCD}(s)$. After the good control we have on the QCD counterpart, a violation of global QCD-Hadron Duality larger than say 10% is unexpected and certainly would have to be understood.

Notice that duality cannot differentiate between shapes of spectral functions which have the area below. On the contrary, the interference of the pion pole contribution in (3) and the three-pion contribution makes the ratio $R_{had}(s)$ sensitive to changing the global normalization factor $\mathcal{A}$. The normalization factor $\mathcal{A}$ should be around 1.5 if QCD-Hadron Duality is wanted (We have used the PMS resummed series as the best behaved.). In Figure 1 we show three different normalizations: namely, the lower curve is for $\mathcal{A} = 1$ in analogy with the normalization of the conserved vector current; the curve with normalization factor $\mathcal{A} = 1.5$ is the one that has the best duality with the PMS resummed QCD series, and the higher curve is for $\mathcal{A} = 5$ where more than 20% violation of duality is obtained. Notice also that one cannot expect duality for $s$ larger than $[3 \sim 3.5]$ GeV$^2$ since from these energies on there are two-body thresholds like $NN$, $\rho\pi'$ [which couples dominantly to $5\pi$-continuum], · · · . None of these intermediate states have been included in our parametrization or in the one in [11]. These contributions are leading in the $1/N_c$ counting and dominant by phase space. This was overlooked in [11] where duality in the three-$\pi$ channel was required at energies much higher than 4 GeV$^2$. More on this issue in [8].

Once we have fixed the duality region from Figure 1 to be around 2 GeV$^2$ we can resolve for the light quark masses in that region. Using the 0-th moment (2), we get the masses in Figure 2. There we plot the $\overline{\text{MS}} m_u + m_d$ masses scaled to 1 GeV obtained using the hadronic parametrization with $\mathcal{A} = 1.5$ and the PMS resummed QCD series. The value of $m_u + m_d$ in the duality region can be read off from Figure 2. The result we get is

$$[m_u + m_d](1\text{GeV}^2) = (12.8 \pm 2.5) \text{ MeV}$$

i.e.

$$[m_u + m_d](4\text{GeV}^2) = (9.8 \pm 1.9) \text{ MeV}$$

in the $\overline{\text{MS}}$ scheme at order $\alpha_s^3$. This result updates the one in [1]. The central value is for $\mathcal{A} = 1.5$ and using the PMS resummed QCD series. The error includes the hadronic uncertainty allowing for a violation of duality of more than 10% at $s = 2$ GeV$^2$ and by varying $\alpha_s$ and the coefficients of the power corrections in the ranges given. All these uncertainties have been added in quadrature. Notice also that the results one gets using the Improved QCD resummed series or the
$m_u + m_d \overline{\text{MS}}$ masses scaled at 1 GeV.

$PMS$ one only differ by 0.1 MeV. This shows the good control we have on the QCD series. Even though our spectral function is obviously not complete [see comments above], one can see from Figure 2 that the value of the masses is quite stable between $s = 2 \text{ GeV}^2$ and $s = 4 \text{ GeV}^2$. Instanton contributions to the 0-th moment for $s$ around $3 \text{ GeV}^2$ are negligible [12] and have been included in the quoted error. Laplace Sum Rules give compatible results. The results in (10) and (11) are in perfect agreement with the bounds obtained recently in [13].

Given the good behaviour shown by the QCD counterpart, further improvements reducing the error bars in the determination of $m_u + m_d$ have to come from the very difficult measurement of the $I^G(J^P) = 1^-(0^-)$ spectral function normalization. As we tried to show, our present understanding already gives tight bounds on the values of $m_u + m_d$.

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