The heat distribution in a logarithm potential

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Abstract. All statistical information about heat can be obtained with the probability distribution of the heat functional. This paper derives analytically the expression for the distribution of the heat, through path integral, for a diffusive system in a logarithm potential. We apply the found distribution to the first passage problem and find unexpected results for the reversibility of the distribution, giving a fluctuation theorem under specific conditions of the strength parameters.

Keywords: fluctuation theorems, Brownian motion, stochastic particle dynamics

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1. Introduction

With the fast development of stochastic thermodynamics in the past two decades, fluctuations in thermodynamic quantities, such as heat, work, and entropy, for diffusive systems, have been extensively investigated [1–7]. These fluctuations can be studied through the functional distribution [8, 9] or the characteristic function [10, 11]. Exact and analytical results for heat distributions were first presented only through the generic use of path integrals, whereas in [9] the distribution of heat, for a diffusive system under the action of a harmonic potential, is obtained for the first time. Since then, several results have been obtained for the distribution of heat in cases where the force is of the linear type [9, 12–14]. As far as we know, analytical results for heat are only possible for free particles and linear forces. In the present work, we try to extend this rather short list by studying a diffusive system where the force is strongly non-linear [15], derived from a logarithmic potential.

Diffusive systems with logarithmic potential are found in the literature, and describe quite distinct phenomena such as 2D XY model with Kosterlitz–Thouless phase transition [15], DNA bubble denaturation [16], cold atom diffusion [17, 18], sleep-walk dynamics [19], resetting phenomena [20], ergodicity breaking [21], charged Brownian particle movement [22], and interaction models between poly-electrolyte polymers [24]. These two last examples have the same origin of the interaction, which is the Coulomb electrical potential of a charged line and serves to illustrate the present work. Moreover, it has been shown that a generalized Wiener–Khinchin theorem can be used to find the power spectrum for a Brownian particle diffusing in a logarithm potential [23].

Recently, when including external protocols in the system, significant attention has been given to the work distributions, where analytical and numerical results were obtained [25–27], providing a broad characterization of the distribution under different protocols. Here, we try to investigate in another direction, studying the heat, which is an important thermodynamic quantity if we are interested in Brownian machines. Another recent study is the first passage problem with protocols in [28]. The first passage problem [29] consists of finding the distribution of the first time that the particles hit an absorbing boundary condition [15, 29]. Having a time probability distribution for this first time, we are able to find the average time behavior of functionals. We use this to see the averaged behavior of the heat through this process.

Having the distribution of a functional, one may wonder whether it obeys any fluctuation theorem [30–33]. Fluctuation theorem is a link between the distribution and its reverse process. Today, we already know a unified fluctuation theorem for entropy [2, 34, 35]. For the heat, a theorem is only possible for specific conditions [36–41]. Inquisitively,
we use the derived heat distribution to look for fluctuation theorems, finding unexpected results due to the non-trivial form of the distribution.

In the present work, we use path integral formalism [42–44] to obtain the heat distribution for a logarithmic potential exactly, and analytically. We apply it to the first pass problem by taking a time average over the distribution, and we investigate the validity of fluctuation theorems for the distribution found.

In section 2, we present the stochastic thermodynamics of the model, defining the heat functional with the Langevin equation. In section 3, we solve for the heat distribution using path integrals. In section 4, we work out a simple application of the heat distribution in the first passage problem. In section 5, we investigate the role of fluctuation theorem for the heat distribution. We finish in section 6 with a discussion. In the appendix, we perform the calculation of the path integral.

2. Stochastic thermodynamics in logarithm potential

We start with the overdamped Langevin equation,

\[ \gamma \dot{x}(\tau) = -\frac{k'}{x(\tau)} + \sqrt{2D} W(\tau) \]  

(1)

where \( \gamma \) is the friction constant of the environment, \( k' \) is the constant of the force, and \( D', W(\tau) \) are the diffusion constant and the Wiener process. We normalize this equation to find a more convenient expression,

\[ \dot{x}(\tau) = -\frac{k}{x(\tau)} + \zeta(\tau), \]  

(2)

which describes an overdamped Brownian particle suffering the action of an attractive force

\[ f(x) = -\frac{k}{x} = -\partial_x U(x), \]

with \( k > 0 \), which is derived from a logarithm potential \( U(x) = k \ln x \), and a thermal force, which is a Gaussian white noise of zero average, obeying

\[ \langle \zeta(\tau)\zeta(\tau') \rangle = 2D \delta(\tau - \tau'), \]  

(3)

where \( k \) and \( D \) are the strength parameters of the problem representing, respectively, the force of the potential and the force of the reservoir. The dynamics of the particle occur in a time interval \( \tau \in [0, t] \), and are restricted for \( x(\tau) > 0 \) with boundary conditions: \( x(0) = x_0 \) and \( x(t) = x_t \). We assume a perfect absorbing boundary condition for the particle distribution in \( x(t') = 0 \); it is \( P(x = 0, t') = 0 \). The Langevin equation (2) is also known in the literature as the Bessel Process [45].

The physical character of the force in equation (2) can be understood as the electrostatic force between a uniformly charged line and an ion [47]. Instead of a force of the type \( \propto 1/r^2 \), a charged line creates a Coulomb force \( \propto 1/r \). Here, the strength of the potential \( k \) is associated with the charges of the particle and the line, and since the
force is attractive, the charges must be opposite. A realistic example, where thermal fluctuations take place, is the interaction of a polyelectrolyte and an ion, with opposite charges [24]. While the ion is still diffusing, it has not yet been captured by the attractive force of the polymer. When the ion hits the polymer, it gets stuck; hence the first passage occurs at that instant. This first passage time has a distribution [15] that fully characterizes the process, contrasting with the survival probability, which is the probability for the time that the particle still has not hit the polymer. In section 5, we apply these concepts to the heat distribution.

According to Sekimoto [1], the total heat exchange between the particle and the reservoir in the interval \( \tau \in [0, t] \) is defined as

\[
\mathcal{Q}[x] = -\int_0^t (\dot{x}(\tau) - \zeta(\tau)) \frac{dx}{d\tau} d\tau = \int_0^t k \frac{dx}{x} d\tau = k \ln \left( \frac{x_t}{x_0} \right)
\]

where we use the Langevin equation (2). This functional of the trajectory has to be defined according to the Stratonovich prescription [46]. The above equation expresses the balance of energy between the particle and the reservoir, and can be seen as the first law of stochastic thermodynamics, since \( \Delta U(x) = k \ln \left( \frac{x_t}{x_0} \right) \), and then the first law reads

\[
\mathcal{Q}[x] = \Delta U.
\]

Notice that no work is done in the system since we have not assigned any work protocol. Nevertheless, we shall see in the next section that this case still gives a non-trivial result for this functional.

### 3. Distribution for the heat functional

In stochastic thermodynamics [1, 2], heat is a functional of the particle’s trajectory \( x(t) \) [48]. Due to the dependence on the stochastic trajectory, heat is a stochastic variable, having an associated distribution \( P(Q, t) \) that contains all information about the statistical properties of the heat. To find an exact expression for \( P(Q, t) \) we will use path integral techniques [8, 43], where the distribution is given by the average

\[
P(Q, t) = \langle \delta(Q - \mathcal{Q}[x]) \rangle = \int dx_t \int dx_0 P_0(x_0) \int_{x_t,x_0} Dx \ e^{\mathcal{A}[x(t)]} \delta(Q - \mathcal{Q}[x]).
\]

In principle, we have to solve the path integral, defined in the above equation, by taking into account the constraint imposed by the Dirac delta. For our system, the heat \( \mathcal{Q}[x] \) given by equation (4) only depends on the boundary conditions \( x_0 \rightarrow x_t \), then we just take the Dirac delta out of the path integral and use it to solve one of the remaining integrals. Nevertheless, we still need to solve the path integral

\[
\int_{x_t,x_0} Dx \ e^{\mathcal{A}[x(t)]} = P[x_t, t|x_0, 0],
\]

which is the conditional probability for the Langevin equation given in equation (2), and \( \mathcal{A}[x] \) is the stochastic action [43] defined in equation (A.1). The solution of the path
The heat distribution in a logarithm potential integral in equation (6) is already known for a more general case in [49] and we give a complete derivation in appendix A.

With the simplification given by the model, we have a more simple formula for the distribution \( P(Q, t) \), which can be further simplified to

\[
P(Q, t) = \int dx_t \delta \left( Q - k \ln \frac{x_t}{x_0} \right) \int dx_0 P_0(x_0) P[x_t, t|x_0, 0],
\]

where the Dirac delta can be rewritten in a more convenient way by

\[
\delta \left( Q - k \ln \frac{x_t}{x_0} \right) = \frac{x_0 e^{Q/k}}{k} \delta \left( x_0 e^{Q/k} - x_t \right) .
\]

Integrating over \( x_t \), we find

\[
P(Q, t) = e^{Q/k} k \int dx_0 P_0(x_0) P[x_0 e^{Q/k}, t|x_0, 0].
\]

We reduce the formula to just one integral over \( x_0 \) with the initial distribution \( P_0(x_0) \). For the delta distribution as the initial distribution \( P_0(x_0) = \delta(x_0 - x_i) \), we can find an exact and analytical result for the heat distribution. Using the Dirac delta property, we solve the integral and find

\[
P(Q, t) = \frac{x_i e^{Q/k}}{k} P \left( x_i e^{Q/k}, t|x_i, 0 \right). \tag{10}
\]

Using the conditional probability given in equation (A.23), we have the explicit formula

\[
P(Q, t) = \frac{1}{4 D k t} I_{\kappa} \left( \frac{e^{Q/k} x_i^2}{2 D t} \right) \left( e^{Q/k} x_i \right)^{\kappa+2} x_i^\kappa \exp \left( -\frac{1}{4 D t} \left( e^{2Q/k} + 1 \right) x_i^2 \right), \tag{11}
\]

which is valid for all times, and \( x_i > 0 \). \( I_{\kappa} \) is the modified Bessel function of the first kind, and \( \kappa = 1/2 (k/D - 1) \) is a parameter defined in the appendix A. The exact expression for the heat distribution in equation (11) is the main result of this work. The plot of equation (11) is given in figure 1. We can see that the most probable values are in the positive \( Q > 0 \) region. As time passes, only positive values for heat are allowed. The particle is predominantly gaining energy from the reservoir.

For the asymptotic behavior, \( t \to \infty \), the argument of the Bessel function in equation (11) is small, and then the expansion for small values of the Bessel function yields

\[
I_{\kappa}(z) \propto \frac{z^\kappa}{2^\kappa \Gamma(\kappa + 1)},
\]

showing a crescent exponential tail for \( Q \). The normalization for this distribution can be checked numerically, and is given by \( \int_\infty^\infty dQ P(Q) = 1 \) as expected.

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Figure 1. The heat distribution with initial delta probability for $k = 1, D = 1, x_i = 1$ with three different times $t = 1, t = 2, t = 5$.

Figure 2. (a) Distribution for the first time heat $Q_{\text{first}}$ for $k = 1, D = 1, x_i = 1$. (b) Pictorial representation of an ion with positive charge (purple circle) and a polyelectrolyte with negative charge (blue line).

4. Application to the first passage problem

To illustrate the phenomena, we consider the case of an interaction of a polyelectrolyte, which is a charged polymer, and an ion [24]. In a system with a polyelectrolyte and an ion with opposite charge, see figure 2(b); the Coulomb interaction between them is attractive through a logarithm potential [28]. Therefore, the Langevin equation (2) of our model can be used to describe this interaction.
This kind of system exhibits the first passage problem, which corresponds to the first time, $t^*_1$, that the ion hits the polymer. The probability for the first time, $P(t^*_1)$, is given by [15],

$$P_1(t^*_1) = \frac{1}{\Gamma(\kappa + 1)} \frac{4D}{x_i^2} \left( \frac{x_i^2}{4Dt^*_1} \right)^{\kappa+2} \exp \left( -\frac{x_i^2}{4Dt^*_1} \right).$$

(12)

This distribution gives the mean first time $\langle t^*_1 \rangle_{P_1} = \int_0^\infty t^*_1 P_1(t^*_1) dt^*_1$ and can be used to find the mean behavior of functions of time in the first passage event. The heat distribution found in the previous section is valid for all times $t$. With $P(Q, t)$ we might ask: what is the statistical behavior of the heat at the first passage event? For instance, we can find the mean heat for the heat distribution 11, that is $\langle Q \rangle_Q = \int dQQP(Q, t)$, and then use the probability $P_1(t^*_1)$ to take the average again. This will be the mean value of the heat, averaged in time, giving a typical value for the heat when the first passage event occurs. We can call this mean heat the mean first time heat. Instead of doing that, we can also average the heat distribution in equation (11) in time with the first time probability in equation (12), that is

$$P_{\text{first}}(Q) = \langle P(Q, t^*_1) \rangle_{P_1} = \int_0^\infty P_1(t^*_1) P(Q, t^*_1) dt^*_1,$$

(13)

where $P_{\text{first}}(Q)$ can give us the mean first time heat, or any other moments of the first time heat. However, we cannot solve it analytically; we need to solve it numerically. Using Wolfram Mathematica [52] we can plot the distribution for the first passage heat, which is shown in figure 2(a).

When the ion moves toward the polyelectrolyte, heat is exchanged between the ion and the reservoir with distribution given by $P_{\text{first}}(Q)$, which is shown in figure 2(a), giving the typical heat for the first passage problem.

5. Fluctuation theorem

To study the path reversibility hidden in a probability distribution, we can calculate the ratio between the distribution and the inverse-path distribution. In some cases, this ratio yields a fluctuation theorem [2, 30, 33, 53]. A fluctuation theorem can be regarded as a mathematical constraint in the probabilities of functionals such as work, heat, or entropy. It is well known that for the total entropy of a stochastic system, $\Delta S_{\text{tot}}$, an integral fluctuation theorem is always satisfied $\langle e^{-\Delta S_{\text{tot}}} \rangle = 1$ [33]. For the heat, we have a direct connection with the entropy production by the system (particle) in contact with the bath through $Q[x] = TS[x]$, giving us the relation with the stochastic and total entropy

$$Q = T(\Delta S_{\text{tot}} - \Delta S_{\text{stc}}).$$

(14)

For steady states, the entropy production obeys a detailed fluctuation theorem [33], being a consequence of $\Delta S_{\text{stc}} = 0$. This gives $Q = T(\Delta S_{\text{tot}})$.

Herein, we have derived the exact expression for the heat distribution. It is then desirable to see if this distribution satisfies some kind of fluctuation theorem.
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Figure 3. Log ratio probability in the asymptotic limit (dashed line) and finite values of time $t = 1, 3, 5, 10$ (solid lines) for $k = 3, D = 6$ satisfying $k = \frac{D - 1}{2}$.

Observe that, as time increases, the log-ratio shall favor less and less the high negative values over their opposite positive ones. Actually, the picture shows only their probability ratio (which are, in fact, both small).

Experimentally, it is well known that the heat obeys a different fluctuation theorem \[50, 51\] even in an asymptotic time regime. Moreover, it is known that the fluctuation theorem for the heat is not always satisfied, even for $t \to \infty$ [41].

The distribution of $P(Q)$ is not trivial. In addition to exponentials, we also have a Bessel function $I_\kappa$ that depends on the exponential of $Q$; this lets us think that, in principle, there is no standard fluctuation theorem, as is shown below:

$$
P(Q) = \frac{I_\kappa \left( \frac{e^{Q/k} x^2}{2Dt} \right)}{I_\kappa \left( \frac{e^{-Q/k} x^2}{2Dt} \right)} \exp \left( -\frac{x^2}{2Dt} \sinh \left( \frac{2Q}{D} \right) + \frac{Q(D + k)}{Dk} + \frac{2Q}{k} \right), \quad (15)$$

It is straightforward to see that the distribution for the heat does not exhibit a standard fluctuation theorem form. However, in the asymptotic limit, when $t \to \infty$, the argument in the Bessel function becomes small, as noted in the previous section, so we can derive a power law from the Bessel function $I_\kappa$. Therefore, we find the measure of the reversibility of $Q$

$$
\ln \left( \frac{P(Q, t)}{P(-Q, t)} \right) \asymp \frac{4}{k} (1 + \kappa)Q, \quad \kappa = \frac{1}{2} \left( \frac{k}{D} - 1 \right) \quad (16)
$$

where $\asymp$ means that the equality holds only for $t \to \infty$. Even though we observe a linear dependence in $Q$, we do not have a proper fluctuation theorem. One important thing to notice is that the stochastic entropy does not vanish in the asymptotic limit (see appendix B), preventing the equality between the heat and the total entropy. It is interesting that, for a particular combination between the parameters, we can recover a

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fluctuation theorem in equation (16); this combination is

\[ k = \frac{D}{-1 + D/2} \Rightarrow \ln \left( \frac{P(Q, t)}{P(-Q, t)} \right) \approx Q. \]  

(17)

If the above condition between the strength of the potential and the strength of the bath is satisfied, a stationary state fluctuation theorem (SSFT) [53] is satisfied for the heat. Thus, we found that the fluctuation theorem for the heat is not simply satisfied, even in the steady state. Apparently, this is not unexpected, as stated in [41]. However, what is unexpected here is that we can recover the fluctuation theorem in a steady state if the strength parameter condition, equation (17), is satisfied.

The plot of the two versions of the ratio, equations (15) and (16), satisfying the condition in equation (17) are shown in figure 3, which shows the agreement between the two equations in a finite region.

6. Discussion and conclusion

In the present work, we studied the distribution of heat for a diffusive system in a logarithmic potential. We find analytical and exact results for the heat distribution through the use of path integrals, enlarging the list of few exact results in this field. Here, we limit ourselves to the case where the interaction is of the attractive type, where the first pass problem occurs. We then use the distribution found to obtain the average heat behavior during this process. To illustrate, we use the example of an ion and a charged polymer, which is carried out experimentally. Finally, we investigate the validity of the fluctuation theorem for heat through the log-ratio probability, where we find a fluctuation theorem for heat only for long times and specific values of the strength parameters.

The distribution of the heat in equation (11), obtained analytically through path integral technique, is the main result of this work. The expression obtained is a non-trivial distribution, with an exponential dependence in the argument of a modified Bessel function of the first kind, and an exponential in the argument of an exponential. These non-trivial dependencies come from the conditional probability of the Bessel process and the logarithm dependence in the boundary conditions of the heat functional in equation (4). The plot of the distribution gives us more physical insight. For fixed values of the strength parameters, we can see the time evolution of the distribution; as time passes, negative values of the heat become less probable while positive values become most probable. Then while the particle survives, the gaining of energy from the reservoir is predominant. This can be understood noting that the thermal force has to overcome the attractive force, making it more probable to the survival particle to receive energy from the reservoir.

One of the applications of the work is in the first passage problem of a charged polymer and an ion with opposite charges. Using the probability distribution for the first time that the ion hits the polymer we average the heat distribution, finding the average heat for this process, shown in figure 2. We called \( P_{\text{first}}(Q) \) the averaged distribution, which gives the average statistical behavior of the heat for the process. \( P_{\text{first}}(Q) \) has
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similar statistical properties of $P(Q,t)$, where the most probable values are for positive heat. However, the shape of the distribution becomes smoother, having a broader area.

Beyond the exact expression of the heat distribution and the application in the first passage problem, we investigate the role of fluctuation theorems for the heat distribution. Having a limited domain for the position of the particle $x(t) > 0$, and the delta distribution as the initial distribution, a fluctuation theorem is not expected. However, the log ratio of the heat distribution, equation (16), gives a fluctuation theorem for long times and a specific constraint, equation (17), between the strength parameters. As far we know, there is no physical consequence if the strength constraint is satisfied; we interpret this result as a mathematical aspect of the distribution. The fluctuation theorem for the heat is only found if a condition on the strength parameter is satisfied. In that case, we have an SSFT [53] for the heat. There is a connection between this theorem and a fluctuation theorem for the entropy [33], since the entropy production in the reservoir has a direct relation with the heat functional. The log-ratio plotted in figure 3 shows the evolution toward the fluctuation theorem in the asymptotic limit $t \to \infty$, when the strength parameter is satisfied. We can understand the necessity of a strength constraint based on the strange behavior of the heat fluctuation theorem, and the use of non-thermalized initial distributions. To be sure of this justification, an investigation using the initial thermalized condition, i.e., the Boltzmann distribution, it would be interesting to see if the strength constraint appears in such a case, is necessary and may be the subject of future work. Although the fluctuation theorem was not the initial motivation for the present work, the investigation in this direction found unexpected results.

Some questions remain and can serve as an extension of the present work; what is the effect of having an initial thermal condition? And how does heat behave when the interaction is strongly repulsive where the first pass problem does not occur?

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Appendix A. Path integral for the conditional probability

The stochastic action $A[x]$ that appears in equation (6) is defined by [42, 43, 54]

$$A[x] \to A[\dot{x}, x, t] = -\frac{1}{4D} \int_{0}^{t} \left[ \dot{x}^2 + (k^2 - 2Dk) \frac{1}{x(s)^2} \right] ds + \frac{k}{2D} \ln \left( \frac{x_t}{x_0} \right)$$

(A.1)

where we use the Stratonovich convention and the last term can be taken off of the path integral, so we define the propagator $K[x_t, t|x_0, 0]$ as

$$P[x_t, t|x_0, 0] = e^{\frac{i}{\hbar} \int_{0}^{t} \ln \left( \frac{\dot{x}}{\hbar} \right)} K[x_t, t|x_0, 0]$$

(A.2)

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where $K(x_t, t| x_0, 0)$ is now just the path integral
\[
K[x_t, t| x_0, 0] = \int_{x(0)=x_0}^{x(t)=x} D x \exp \left[ -\frac{1}{4D} \int_0^t \left( \dot{x}^2 + \left( k^2 - 2D \kappa \right) \frac{1}{x(s)^2} \right) ds \right]. \tag{A.3}
\]

In order to solve the path integral in equation (A.3), we use the approximation by piecewise linear functions described in [42]; this is essentially the same approach as in [43, 49]. We discretize the path $x(s)$ in piecewise linear functions $x_i = x(s_i)$ and the time $s_i = i \epsilon$ in $N$ pieces $i = 1, \ldots, N$, with $N\epsilon = t$ where $s_N = t$ is the final time. At the end of the calculation, we take the limit $N \to \infty$. The action $A[x]$ in equation (6) in the discretized form is
\[
A_N = -\frac{1}{4D} \sum_{i=0}^{N-1} \left( \frac{x_{i+1}^2 + x_i^2}{\epsilon} \right) + \sum_{i=0}^{N-1} \left[ \frac{x_i x_{i+1}}{2D \epsilon} - \frac{1}{2} \left( \frac{\kappa^2 - \frac{1}{4}}{x_i x_{i+1}} \right) 2D \epsilon \right] \tag{A.4}
\]
where we define the constant $(\kappa^2 - \frac{1}{4}) = \frac{k^2 - 2D \kappa}{4D^2} \to \kappa = \frac{k}{D} - 1$ for later convenience because defining $\kappa$ allows us to use the following expansion of the exponential of the last term in equation (A.4) in terms of modified Bessel functions $I_{\kappa}(x)$ [55].
\[
\exp \left( \frac{x_i x_{i+1}}{\epsilon} - \frac{1}{2} \left( \kappa^2 - \frac{1}{4} \right) \frac{\epsilon}{x_i x_{i+1}} + O(\epsilon^2) \right) = \sqrt{\frac{2\pi x_i x_{i+1}}{\epsilon}} I_{\kappa} \left( \frac{x_i x_{i+1}}{\epsilon} \right), \tag{A.5}
\]
We can simplify our path integral. This approximation works, because we want to take the limit $N \to \infty$, or equivalent, $\epsilon \to 0$, and then, terms of order $O(\epsilon^2)$ can be ignored. This is the crucial step to evaluate the path integral in equation (6). Since the beginning, we do not have a Gaussian integral; an exact solution is not available unless we transform equation (A.5).

The next step is to consider the path integral as multiple integrals in the $x$-coordinate. Since we are dealing only with the cases where $x > 0$, the integrals over $x$’s are in the range $0 \to \infty$, different from usual. Defining the $N$-integral $K_N$ as
\[
K_N = \left( \frac{1}{4\pi D \epsilon} \right)^{N/2} \int_0^\infty dx_0 \ldots \int_0^\infty dx_N \exp A_N \tag{A.6}
\]
where the first term comes from the integration measure $D x$, we can use the definition in equation (A.4) and the identity in equation (A.5)
\[
K_N = \left( \frac{1}{4\pi D \epsilon} \right)^{N/2} \int_0^\infty dx_0 \ldots \int_0^\infty dx_N \exp \left( -\frac{1}{4D} \sum_{i=0}^{N-1} \left( \frac{x_{i+1}^2 + x_i^2}{\epsilon} \right) \right)
\]
\[
\times \prod_{i=0}^{N-1} \sqrt{\frac{2\pi x_i x_{i+1}}{2D \epsilon}} I_{\kappa} \left( \frac{x_i x_{i+1}}{2D \epsilon} \right). \tag{A.7}
\]
The first term in the integral of equation (A.7) can be decomposed in
\[
-\frac{1}{4D} \sum_{i=0}^{N-1} \left( \frac{x_{i+1}^2 + x_i^2}{\epsilon} \right) = -\frac{x_0^2 + x_N^2}{4D \epsilon} - \sum_{i=1}^{N-1} \frac{x_i^2}{2D \epsilon}. \tag{A.8}
\]
and the term in the square root
\[
\prod_{i=0}^{N-1} \int_0^{2\pi x_i x_{i+1}} \frac{2\pi x_i x_{i+1}}{2D\epsilon} \rightarrow \int_{x_0 x_N}^{\prod_{i=1}^{N-1} \sqrt{2\pi x_i x_{i+1}} (\frac{1}{2D\epsilon})^{N/2} \sqrt{2\pi} \prod_{i=1}^{N-1} x_i}
\] (A.9)

Then, the integral in equation (A.7) can be rewritten as
\[
K_N = N \sqrt{x_0 x_N} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{N-1} dx_i e^{-\frac{1}{2\pi} \gamma_i^2} I_{\kappa} \left( \frac{x_0 x_1}{2D\epsilon} \right) I_{\kappa} \left( \frac{x_i x_{i+1}}{2D\epsilon} \right) x_i,
\] (A.10)

where \( N \) is a normalization factor defined by \( N = \left( \frac{1}{2\pi} \right)^N e^{-\frac{3\delta_i^2 + \delta_j^2}{4\alpha^2}} \). Note that the multiplicand now runs from \( i = 1 \) to \( i = N - 1 \), instead of starting in \( i = 0 \). We still need to integrate over all \( x_i \)'s, and to do this we use the property of Bessel functions \([55]\)
\[
\int_0^\infty e^{-a \gamma y} I_{\kappa}(a y_1) I_{\kappa}(a_2 y_2) y dy = \frac{1}{2\alpha} \exp \left( \frac{a_1^2 + a_2^2}{4\alpha^2} \right) I_{\kappa} \left( \frac{a_1 a_2}{2\alpha} \right)
\] (A.11)

This integral is valid for \( \alpha > 0 \) and \( \kappa > -1 \), which is our case.

The first integral in \( x_1 \) can be computed as
\[
\int_0^\infty e^{-\frac{a_1^2 + a_2^2}{2\alpha}} I_{\kappa} \left( \frac{x_0 x_1}{2D\epsilon} \right) I_{\kappa} \left( \frac{x_1 x_2}{2D\epsilon} \right) x_1 dx_1 = \frac{1}{2\alpha} e^{(a_1^2 + a_2^2)x_1^2} I_{\kappa} \left( \frac{x_0 x_2}{2D\epsilon} \right)
\] (A.12)

where we define \( c \equiv \frac{1}{2\alpha} \) for later convenience. The next integral in \( x_2 \) will give
\[
\frac{1}{2\alpha} e^{x_2^2} \int_0^\infty e^{-c(1+\frac{1}{4})x_2^2} I_{\kappa} \left( \frac{x_0 x_2}{2D\epsilon} \right) I_{\kappa} \left( \frac{x_3 x_2 c}{2D\epsilon} \right) x_2 dx_2 = \frac{1}{2\alpha} \cdot \frac{1}{2\alpha} \cdot e^{\frac{c^2}{4\alpha^2}} \left( 1 + \frac{1}{4} \right) I_{\kappa} \left( \frac{x_0 x_2 c}{2D\epsilon} \right)
\] (A.13)

and then, after \( N - 1 \) integrations, we arrive at
\[
\frac{1}{2} \prod_{j=1}^{N-1} \frac{1}{2\alpha} I_{\kappa} \left( x_0 x_N \right) \left( c \prod_{j=1}^{N-1} \frac{1}{2\alpha} \right) \exp \left( P_N x_N^2 + x_2^2 \frac{c}{4\gamma_{N-1}} \right)
\] (A.14)

where
\[
\gamma_0 = 1, \quad \gamma_j = \left( 1 - \frac{1}{4\gamma_{j-1}} \right), \quad P_N = \frac{c}{2} \sum_{j=1}^{N-1} \frac{1}{4\gamma_j} \left( \frac{1}{\gamma_{j-1}} \right)^2
\]

Putting all the pieces together, we have for \( K_N \) (including the normalization constant)
\[
K_N = \sqrt{x_0 x_N} \left( c \prod_{j=1}^{N-1} \frac{1}{2\alpha} \right) \left( x_0 x_N \right) \left( c \prod_{j=1}^{N-1} \frac{1}{2\alpha} \right) \exp \left( x_2^2 \left( P_N - \frac{c}{2} \right) + x_2^2 \left( c \frac{C}{4\gamma_{N-1}} - \frac{c}{2} \right) \right)
\] (A.15)
To evaluate the path probability, we need to take the limit \( N \to \infty \). The first multiplier that appears in the right side of equation (A.15) and in the argument of the Bessel \( I_\kappa \) can be calculated as

\[
\frac{1}{2} \prod_{j=1}^{N-1} \frac{1}{2\gamma_j} = \frac{1}{2^N \prod_{j=1}^{N-1} \gamma_j} = \frac{1}{2^N} \Rightarrow \frac{c}{N} = \frac{1}{2Dt},
\]

(A.16)

where this equation gives \( \prod_{j=1}^{N-1} \frac{1}{\gamma_j} = 2^N/N \), and verification of this equation is just made by induction.

The term \( P_N \), which comes multiplied with \( x_0 \), needs a more careful approach, because it has some divergences. To deal with it, we define \( \gamma_j = \frac{Q_j+1}{2Q_j} \) such that

\[
\gamma_j = 1 - \frac{1}{4\gamma_j} \to Q_{j+1} - 2Q_j + Q_{j-1} = 0 \to \dot{Q} = 0,
\]

(A.17)

where the last equation is the continuum limit of the discrete equation. The solution is just \( Q(s) = s \) for \( Q(0) = 0 \) as the initial condition. Backing to \( P_N \), we can rewrite the term which comes multiplied with \( x_0 \)

\[
P_N - \frac{1}{2} = -\frac{c}{2} + \frac{c}{2} \prod_{j=1}^{N-1} \frac{1}{2\gamma_j} \prod_{k=1}^{j-1} \frac{1}{2\gamma_j} = -\frac{c}{2} + \frac{c}{2} \sum_{j=1}^{N-1} \frac{Q_j^2}{Q_{j+1}Q_j}.
\]

(A.18)

The last term in the continuum limit \( (N \to \infty) \) is

\[
\frac{c}{2} \sum_{j=1}^{N-1} \frac{Q_j^2}{Q_{j+1}Q_j} \to \lim_{\epsilon \to 0} \frac{1}{4D} \frac{Q_1}{\epsilon^2} \int_\epsilon^t \frac{1}{Q(t')^2} dt';
\]

(A.19)

We can rewrite the term outside the integral as \( \frac{Q_1-\dot{Q}_0}{\epsilon} = \dot{Q}(0) = 1 \), because \( Q_0 = 0 \) is our initial condition. With equation (A.19), the continuum limit of equation (A.18) is

\[
\lim_{\epsilon \to \infty} P_N - \frac{c}{2} = -\frac{1}{4D} \left( \frac{1}{\epsilon} - \int_\epsilon^t \frac{1}{Q(t')^2} dt' \right) = -\frac{1}{4D} \left( \frac{1}{\epsilon} - \int_\epsilon^t \frac{1}{\epsilon^2} dt' \right)
\]

\[
= -\frac{1}{4Dt}.
\]

(A.20)

For the term multiplied with \( x_N \), \( \frac{c}{4N-1} - \frac{c}{2} \), we just note that \( \frac{1}{4N-1} = \frac{1}{2} \frac{Q_N}{Q_N} \), then the continuum limit is

\[
\lim_{\epsilon \to 0} \left( \frac{c}{4N-1} - \frac{c}{4} \right) = \lim_{\epsilon \to 0} -\frac{1}{4D} \frac{Q_N+1}{\epsilon} \frac{Q_N}{\epsilon} = -\frac{1}{2D} \frac{\dot{Q}(t)}{Q(t)} = -\frac{1}{4Dt}.
\]

(A.21)

Finally, using equations (A.16), (A.20) and (A.21) the propagator is

\[
K[x, t|x_0, 0] = \lim_{N \to \infty} K_N = \sqrt{x_0 x_t} \left( \frac{1}{2Dt} I_\kappa \left( x_0 x_t \right) \right) e^{-\frac{1}{4Dt} \left( x_t^2 + x_0^2 \right)},
\]

(A.22)
and then the conditional probability Equation (6) is

\[ P[x_t, t | x_0, 0] = \frac{x_t^{k+1}}{x_0^k} \frac{1}{2Dt} I_k \left( \frac{x_0 x_t}{2Dt} \right) e^{-\frac{1}{4Dt} (x_t^2 + x_0^2)}. \]  

(A.23)

Appendix B. Stochastic entropy

The stochastic entropy is the log of the distribution of the position of the particle [33], defined as

\[ S_{stc} = -\ln P(x, t) \]  

(B.1)

For the model studied here, the initial distribution is the Dirac delta \( \delta(x_i - x_0) \), giving, together with the transition probability equation (A.23),

\[ P(x, t) = \int dx_0 \, \delta(x_i - x_0) P[x_t, t, x_0, 0] = \frac{x_t^{k+1}}{x_i^k} \frac{1}{2Dt} I_k \left( \frac{x_i x_t}{2Dt} \right) e^{-\frac{1}{4Dt} (x_t^2 + x_i^2)}. \]  

(B.2)

Here we are interested in the asymptotic behavior in \( t \to \infty \); in this case we find

\[ P(x, t) \propto \frac{2^{-\frac{k}{2}} (Dt)^{-\frac{D+k}{2D}} (x_i x_t)^{k/D}}{x_i \Gamma \left( \frac{D+k}{2D} \right)} \sim t^{-\frac{D+k}{2D}}, \]  

(B.3)

which is a power law in \( t \). In the limit \( t \to \infty \) this function goes to zero, telling us that the stochastic entropy diverges. Moreover the initial distribution \( P(x_0) = \delta(x_i - x_0) \) is constant in time.

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