RIGHT ANGLED ARTIN GROUPS AND PARTIAL COMMUTATION, OLD AND NEW

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Abstract. We compute the $p$-central and exponent-$p$ series of all right angled Artin groups, and compute the dimensions of their subquotients. We also describe their associated Lie algebras, and relate them to the cohomology ring of the group as well as to a partially commuting polynomial ring and power series ring. We finally show how the growth series of these various objects are related to each other.

1. Introduction

Let $\Gamma$ be an undirected graph, with vertex set $V$ and edge set $E$ (consisting of 2-element subsets of $V$). A number of algebraic objects may be associated with $\Gamma$, the prominent one for us being the corresponding right angled Artin group (RAAG) $A_\Gamma$. It is the group defined in terms of generators and relations

$$A_\Gamma = \langle V \mid vw = wv \text{ whenever } \{v, w\} \in E \rangle.$$

The purpose of this note is to describe classical subgroup series in $A_\Gamma$ such as the lower-central and $p$-lower-central series, and relate them to other algebraic objects defined in terms of $\Gamma$. Many of the results quoted below appear already in other sources, though the computation of the exponent-$p$ and lower $p$-central series is new.

1.1. The actors. Let $k$ be a commutative ring. We define associative $k$-algebras

$$R_\Gamma = \langle V \mid vw = wv \text{ whenever } \{v, w\} \in E \rangle,$$

$$S_\Gamma = \langle V \mid v^2 = 0, vw = -wv \forall v, w \in V, \text{ and } vw = 0 \text{ whenever } \{v, w\} \not\in E \rangle.$$

Note that $R_\Gamma$ and $S_\Gamma$ are graded algebras with $\deg(v) = 1$ for all $v \in V$. Therefore, they admit a natural topology, in which basic neighbourhoods of 0 (say in $R_\Gamma$) are $\{x \in R_\Gamma \mid \deg(x) \geq n\}$. We denote by $\overline{R_\Gamma}$ the completion of $R_\Gamma$ in this topology. Just as $R_\Gamma$ is an algebra of partially-commuting polynomials, $\overline{R_\Gamma}$ is an algebra of partially-commuting power series.

We also define a Lie algebra over $k$

$$L_\Gamma = \langle V \mid [v, w] = 0 \text{ whenever } \{v, w\} \in E \rangle,$$

and, if $k$ is an algebra over $F_p$, a restricted Lie algebra $L_{\Gamma,p}$ with same presentation as $L_\Gamma$, see Section 5 for a review.

In particular, if $\Gamma$ is the complete graph on $d$ vertices then $A_\Gamma \cong \mathbb{Z}^d$, $R_\Gamma$ is the polynomial algebra in $d$ variables $k[X_1, \ldots, X_d]$, $S_\Gamma$ is the Grassmann algebra $\bigwedge^\ast(k^d)$, and $L_\Gamma \cong k^d$ with trivial bracket. If $\Gamma$ is the empty graph on $d$ vertices then $A_\Gamma$ is the free group $F_d$, $R_\Gamma$ is the free associative algebra on $d$ generators, $S_\Gamma \cong k \cdot 1 \oplus k^d$ with trivial multiplication except $1 \cdot x = x$, and $L_\Gamma$ is the free Lie algebra on $d$ generators.
1.2. **Subgroup series.** Let $G$ be a group, and let $\lambda: G \to R^\times$ be a representation of $G$ into an associative augmented $k$-algebra $R$ with augmentation ideal $\varpi$ (namely, an algebra equipped with an epimorphism to $k$ with kernel $\varpi$). With this representation is associated a natural sequence of subgroups, called **generalized dimension subgroups**,

$$\delta_{n,\lambda} := \lambda^{-1}(1 + \varpi^n) = \ker(G \to (R/\varpi^n)^\times).$$

In case $R = kG$ and $\lambda$ is the regular representation, we write $\delta_{n,kG}$ for $\delta_{n,\lambda}$.

There are also classical subgroup series, defined intrinsically within $G$:

- **the lower central series** $(\gamma_n)$ given by $\gamma_1 = G$ and $\gamma_n = [\gamma_{n-1}, G]$;
- **the rational lower central series** $\gamma_{n,0} = \{g \in G \mid g^k \in \gamma_n \text{ for some } k \neq 0\}$;
- for a prime $p$ fixed throughout the discussion, the **exponent-$p$ central series** $\lambda_{n,p}$ given by $\lambda_{1,p} = G$ and $\lambda_{n,p} = [\lambda_{n-1,p}, G]\lambda_{n-1,p}$, or more directly $\lambda_{n,p} = \prod_{i \geq n+1} \gamma_{n,i}^p$;
- again for a prime $p$ fixed throughout the discussion, the **Brauer-Jennings-Lazard-Zassenhaus series** $[13, 17, 33]$, also called $p$-dimension or $p$-central series, given by $\gamma_{n,p} = G$ and $\gamma_{n,p} = ([\gamma_{n-1,p}, G]\gamma_{n-1,p})^p$ or more directly $\gamma_{n,p} = \prod_{m+p \geq n} \gamma_{p,m}$.

All these series are central, meaning that $\gamma_n/\gamma_{n+1}$ belongs to the center of $G/\gamma_{n+1}$, etc. We moreover have $[\gamma_m, \gamma_n] \subseteq \gamma_{m+n}$, etc. A classical consequence [19, Section 5.3] is that $\bigoplus_{n \geq 1} \gamma_n/\gamma_{n+1}$ is a graded Lie algebra over $\mathbb{Z}$.

The groups $\gamma_{n,0}$ enjoys the extra property that $\gamma_{n,0}/\gamma_{n+1,0}$ is torsion-free, so $\bigoplus_{n \geq 1} \gamma_{n,0}/\gamma_{n+1,0}$ is $\mathbb{Z}$-free, and it is the fastest descending central series with this property. In particular, if $\gamma_n/\gamma_{n+1}$ is torsion free for each $n$, then $\gamma_{n,0} = \gamma_n$ for each $n$.

We have $\lambda_{n,p} \subseteq \lambda_{n+1,p}$ so $\bigoplus_{n \geq 1} \lambda_{n,p}/\lambda_{n+1,p}$ is an elementary abelian $p$-group. Similarly, $\gamma_{n,p} \subseteq \gamma_{n+1,p}$, and then it is classical [33] that $\bigoplus_{n \geq 1} \gamma_{n,p}/\gamma_{n+1,p}$ is a restricted Lie algebra over $F_p$. Furthermore, these series are fastest descending under these requirements.

Classical results identify $\delta_{n,kG}$ with some of the above series in case $k$ is a field: we have $\delta_{n,kG} = \gamma_{n,p}$ where $p \geq 0$ is the characteristic of $k$. However, for general $G$, the identification of $\delta_{n,ZG}$ is a fundamental open problem of group theory.

1.3. **Results.** We consider the series defined above for the group $A\Gamma$. The main purpose of this text is to exhibit numerous relations between these algebraic objects; detailed definitions and proofs will be given in subsequent sections. The main tool is an extension of Magnus’s work on the free group [19].

Recall that a commutative ring $k$ is fixed. Denote by $\varpi$ the augmentation ideal of $R\Gamma$, and by $\varpi(A\Gamma)$ the augmentation ideal of $kA\Gamma$.

**Theorem 1.1** (Augmentation ideals). For all $n$ we have

$$\varpi(A\Gamma)^n/\varpi(A\Gamma)^{n+1} \cong \varpi^n/\varpi^{n+1}.$$  

We recall that Koszul algebras are a particular kind of associative algebras (see [23] or Section 3) for which a “small” projective resolution may easily be computed. We have the following results, which for $k = \mathbb{Q}$ already appear in [24].

**Theorem 1.2** (Cohomology). We have $H^*(A\Gamma, k) = S\Gamma$.

The rings $R\Gamma$ and $S\Gamma$ are Koszul algebras, and Koszul duals to each other: $(S\Gamma)! = R\Gamma$.

**Theorem 1.3** (Central series and dimension subgroups). We have $\gamma_n = \gamma_{n,0}$ and $\cap_{n \in \mathbb{Z}} \gamma_n = \cap_{n \in \mathbb{Z}} \gamma_{n,p} = \{1\}$. In particular, $A\Gamma$ is residually torsion-free nilpotent, so by [13] Exercise 17.2.6 $A\Gamma$ is a residually finite $p$-group for every $p$. 
There is a representation \( \mu : A_\Gamma \to R_\Gamma^\infty \) given by \( v \mapsto 1 + v \) for \( v \in V \). The corresponding generalized dimension subgroups satisfy
\[
\delta_{n,\mu} = \begin{cases} 
\gamma_{n,0} & \text{if } \mathbb{Z} \subseteq k, \\
\gamma_{n,p} & \text{if } \mathbb{F}_p \subseteq k.
\end{cases}
\]
Together with Theorem 1.1 we obtain an isomorphism of filtered associative \( k \)-algebras
\[
\mathbb{k}A_\Gamma := \lim (\mathbb{k}A_{\Gamma}/\pi(G)^n) \cong \lim (R_{\Gamma}/\pi^n) = R_{\Gamma}.
\]
In particular, the classical dimension subgroups \( \delta_{n,\mathbb{k}A_{\Gamma}} \) coincide with the \( \delta_{n,\mu} \).

The Lie algebra associated with the lower central series was already determined in \([7]\) as \( L_\Gamma \). We extend this result as follows:

**Theorem 1.4 (Lie algebras).** The algebra \( R_\Gamma \) is a Hopf algebra, and we have
\[
L_\Gamma \cong \text{Primitives}(R_\Gamma) \quad \text{and} \quad R_\Gamma \cong U_p(L_\Gamma), \quad \text{the universal enveloping algebra.}
\]
Furthermore, if \( k \) is an algebra over \( \mathbb{F}_p \), then
\[
R_\Gamma \cong U_p(L_\Gamma,p), \quad \text{the } p \text{-universal enveloping algebra.}
\]
For any ring \( k \), we have \( L_\Gamma \cong \bigoplus_{n \geq 0} (\gamma_n/\gamma_{n+1}) \otimes \mathbb{Z} k \).
If \( \mathbb{F}_p \subseteq k \) then \( L_{\Gamma,p} \cong \bigoplus_{n \geq 1} (\gamma_n/p\gamma_{n+1}) \otimes \mathbb{Z} k \).
If \( \mathbb{F}_p \subseteq k \) and \( p \geq 3 \) then \( L_{\Gamma} \otimes_k [\pi] \cong \bigoplus_{n \geq 1} (\lambda_n/p\lambda_{n+1}) \otimes \mathbb{Z} k \), where \( [\pi] \) is the polynomial ring in one variable \( \pi \) of degree 1. Under the isomorphism, multiplication by \( \pi \) corresponds to the map induced by \( \lambda_n \mapsto g^p \in \lambda_{n+1,p} \).

All the above isomorphisms are natural, in the sense that they are induced by the identity map \( V \to V \), and therefore compatible with homomorphisms induced by a map of graphs \( V \to V' \).

For a graded algebra \( R = \bigoplus_{n \geq 0} R_n \) over \( k \) such that each \( R_n \) is a finitely generated free \( k \)-module, recall that its Poincaré series is the power series
\[
\Phi_R(t) = \sum_{n \geq 0} \text{rank}(R_n)t^n.
\]
For a group \( G = \langle X \rangle \), its growth series is \( \Phi_G(t) = \sum_{g \in G} t^{|g|} \), with \( |g| \) denoting the word length of \( g \in G \). The first two claims of the following result appear in \([3]\):

**Theorem 1.5 (Poincaré and growth series).** The Poincaré series of \( S_\Gamma \) is the polynomial \( \Phi_{S_\Gamma}(t) = \sum_{n \geq 0} c_n(\Gamma)t^n \), where \( c_n(\Gamma) \) denotes the number of size-\( n \) cliques (complete subgraphs) of \( \Gamma \) with \( n \) vertices.

The Poincaré series of \( R_\Gamma \) and \( S_\Gamma \) are connected by the relation
\[
\Phi_{R_\Gamma}(t) \cdot \Phi_{S_\Gamma}(-t) = 1,
\]
and the growth series of \( A_\Gamma \) is
\[
\Phi_{A_\Gamma}(t) = \Phi_{R_\Gamma} \left( \frac{2t}{1+t} \right).
\]

In our next result we determine the Malcev completion of \( A_\Gamma \). We refer to \([20,24]\) and the more recent \([23]\) for a review of this construction.

**Theorem 1.6 (Malcev completions).** Assume \( k = \mathbb{Q} \). Then there is an isomorphism \( \mu_{\text{exp}} : R_{\Gamma}/\mathbb{Q}A_{\Gamma} = \mathbb{Q}A_{\Gamma} \) of filtered, complete Hopf algebras; and \( \overline{L}_{\Gamma} \) is the Malcev Lie algebra of \( A_{\Gamma} \); and the Malcev completion of \( A_{\Gamma} \) is given on generators by
\[
A_{\Gamma} \to \exp(L_{\Gamma}) \subset \overline{R}_{\Gamma} \quad \text{via the classical power series} \quad v \mapsto \sum_{n \geq 0} \frac{v^n}{n!} \quad \forall v \in V.
\]
2. The Magnus map

We first note that, since the relations of $R_T$ and $S_T$ are homogeneous, these rings are naturally graded by setting $\text{deg}(v) = 1$ for all $v \in V$. We view $R_T$ as a ring of partially-commuting polynomials in variables $v \in V$.

Let us consider the augmentation ideal $\mathcal{I} = \langle V \rangle$ in $R_T$. It consists of all polynomials without constant term. Note that $\mathcal{I}^n$ then consists of all polynomials with vanishing degree-$(n-1)$ part. We define a topology on $R_T$ by declaring the sets $\mathcal{I}^n$ to form a basis of neighbourhoods of 0, and let $\overline{R_T}$ be the completion of $R_T$ in this topology. We thus have

$$R_T \cong \bigoplus_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}, \quad \overline{R_T} \cong \prod_{n \geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

We write $\mathcal{I}$ for the closure of $\mathcal{I}$ in $\overline{R_T}$. It consists of all power series with vanishing constant term, and similarly $\mathcal{I}^n$ consists of the power series with vanishing degree-$(n-1)$ part.

For comparison, consider the group ring $\mathbb{k}A_T$, and let $\mathcal{I}(A_T)$ denote the augmentation ideal of $\mathbb{k}A_T$; it is the ideal

$$(g-1 \mid g \in A_T) = \langle v-1 \mid v \in V \rangle.$$

We topologize $\mathbb{k}A_T$ by declaring the $\mathcal{I}(A_T)^n$ to form a basis of neighbourhoods of the identity, and let $\overline{\mathbb{k}A_T}$ denote the corresponding completion. Moreover, let $\text{gr}(\mathbb{k}A_T) := \bigoplus_{n \geq 0} \mathcal{I}(A_T)^n/\mathcal{I}(A_T)^{n+1}$ be the associated graded algebra. We are ready to prove Theorem 1.1.

Lemma 2.1. We have $R_T \cong \text{gr}(\mathbb{k}A_T)$ via the natural map

$$\alpha: R_T \ni v_1 \cdots v_k \mapsto [(v_1 - 1) \cdots (v_k - 1)] \in \text{gr}(\mathbb{k}A_T) \text{ for } v_j \in V.$$

Proof. The isomorphism between $\mathcal{I}(A_T)^n/\mathcal{I}(A_T)^{n+1}$ and the degree-$n$ subspace of $R_T$ can be proven directly, since $\mathcal{I}(A_T)^n/\mathcal{I}(A_T)^{n+1}$ is generated by expressions $(v_1 - 1) \cdots (v_n - 1)$. However, here is a shortcut: $\mathbb{k}A_T$, $\text{gr}(\mathbb{k}A_T)$, and $R_T$ are cocommutative Hopf algebras [20][29], with coproduct induced respectively by $\Delta(g) = g \otimes g$, by $\Delta((g - 1)) = [(g - 1) \otimes 1 + 1 \otimes (g - 1)]$ for $g \in A_T$ and by $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in V$. Furthermore, there is a natural Hopf algebra map $\alpha: R_T \to \text{gr}(\mathbb{k}A_T)$ induced by $v \mapsto [v - 1]$ for $v \in V$, and both of these algebras are generated in degree 1. Finally, $\alpha$ is an isomorphism in degree 1, since $\mathcal{I}(A_T)/\mathcal{I}(A_T)^2 \cong (\mathcal{I}(A_T)/\mathcal{I}(A_T)) \otimes \mathbb{k}V \cong \mathcal{I}/\mathcal{I}^2$. Here, the first isomorphism is the standard isomorphism of the first group homology as $\mathcal{I}(A_T)/\mathcal{I}(A_T)^2$. As in [26], it follows from [21] Theorems 5.18 and 6.11 that $\alpha$ is an isomorphism. \qed

We turn to the fundamental tool we use in relating the group $A_T$ with the algebra $R_T$: it is the “Magnus map”

$$\mu: \left\{ \begin{array}{c}
A_T \to 1 + \mathcal{I} \subseteq \overline{R_T}^\times \subseteq \overline{R_T}, \\
v \mapsto 1 + v \text{ for } v \in V.
\end{array} \right.$$

Here, $\overline{R_T}^\times$ is the group of multiplicative units of $\overline{R_T}$. We have to map to the completion because $\mu(v)^{-1} = 1 - v + v^2 - v^3 + \cdots$ is an infinite sum. It is immediate that the commutation relations between the $v \in V$ defining $A_T$ also hold between the $\mu(v)$, therefore $\mu$ is well defined.

It is possible to describe quite explicitly a basis of the degree-$n$ part of $R_T$. This comes hand-in-hand with a kind of normal form for elements of $A_T$.

Definition 2.2. A word $v_1^{e_1} \cdots v_k^{e_k}$ with $v_i \in V$ and $e_i \in \mathbb{Z}$ is reduced if the number $n$ of factors $v_i^{e_i}$ cannot be reduced by application of any sequence of moves which are either

(M1) remove $v_i^0$,

(M2) replace the piece $v_i^{e_i}v_{i+1}^{e_{i+1}}$ by $v_{i+1}^{e_{i+1}}v_i^{e_i}$ (if $\{v_i, v_{i+1}\} \in E$), or

(M3) replace $v_i^{e_i}v_{i+1}^{e_{i+1}}$ by $v_i^{e_{i+1}+e_{i+1}}$ (if $v_i = v_{i+1}$).
Note that none of these moves increases the number of factors.

It is known that the Magnus map $\mu$ is injective, in the case $k = \mathbb{Z}$. From this we may conclude that a reduced representative of an element of $A_\Gamma$ is essentially unique (compare [32, Section 4]):

**Proposition 2.3.** If $v_1^{e_1} \cdots v_m^{e_m}$ and $w_1^{f_1} \cdots w_n^{f_n}$ are two reduced words representing the same element $\gamma \in A_\Gamma$, then one can be obtained from the other by a finite number of applications of (M2). In particular, $n = m$ which we call the number of blocks in $\gamma$.

**Proposition 2.4.** For arbitrary $k$, the Magnus map $\mu : A_\Gamma \to \overline{A_\Gamma}$ is injective.

It maps $\gamma_n(A_\Gamma)$ into the subgroup $1 + \overline{w}^n$ of $1 + \overline{w}$. We get an induced map of graded Lie algebras

$$\mu : \bigoplus_{n \geq 1} \gamma_n(A_\Gamma)/\gamma_{n+1}(A_\Gamma) \to \bigoplus_{n \geq 1} (1 + \overline{w}^n)/(1 + \overline{w}^{n+1}) \cong \bigoplus_{n \geq 1} \overline{w}^n/\overline{w}^{n+1} = R_\Gamma,$$

where the latter Lie algebra structure is the one induced from the algebra structure.

The map induced by $\mu$ on group rings gives rise to an isomorphism of filtered associative $k$-algebras $\overline{kA_\Gamma} \rightarrow \overline{R_\Gamma}$.

In particular, $kA_\Gamma/\overline{w}^n(A_\Gamma) \cong \text{gr}_{< n}(kA_\Gamma) \cong (R_\Gamma)_{< n}$ using Lemma 2.4.

**Proof.** Let $k'$ be the image of $Z$ in $k$; it is either $Z$ or $Z/N$ for some integer $N$. The case $Z$ is already covered; if $k' = Z/N$, let $p$ be a prime number dividing $N$. We will prove the stronger statement that the map $A_\Gamma \to \overline{R_\Gamma} \to \overline{R_\Gamma} \otimes_{\mathbb{Z}} \mathbb{F}_p$ is injective, i.e., we assume without loss of generality that $k = \mathbb{F}_p$.

Let $v_1^{e_1} \cdots v_m^{e_m}$ be reduced and not trivial. By definition, $\mu(v_1^{e_1} \cdots v_m^{e_m}) = (1 + v_1)^{e_1} \cdots (1 + v_m)^{e_m}$ which is a possibly infinite (if one of the $e_i < 0$) $\mathbb{F}_p$-linear combination of words over $V$. Write $e_j = p^{\ell_j} \cdot s_j$ so that $p$ does not divide $\ell_j$.

Because we are in characteristic $p$, we have $(1 + v_j)^{e_j} = (1 + v_j^{p^{\ell_j}})$. Now, a variant of the standard argument for injectivity of the Magnus map in characteristic 0 can be applied: multiplying out (using the power series for the inverse) and using the normal form property, Proposition 2.3, we obtain a multiple of $v_1^{e_1} \cdots v_m^{e_m} \in M_p$ precisely once, with coefficient $\ell_1 \cdots \ell_n \neq 0 \in \mathbb{F}_p$. Other terms either have fewer blocks or have higher degree. Therefore, there is no complete cancellation in $\overline{R_\Gamma}$ and $\mu(v_1^{e_1} \cdots v_m^{e_m}) \neq 1$.

It is an elementary calculation in non-commutative power series that the $1 + \overline{w}^n$ form a central series of subgroups of $1 + \overline{w}$. Consequently, $\gamma_n(1 + \overline{w}) \subseteq 1 + \overline{w}^n$ and $\mu(\gamma_n(A_\Gamma)) \subseteq 1 + \overline{w}^n$. Elementary calculations in the non-commutative power series ring also show immediately that the associated graded Lie algebra $\bigoplus (1 + \overline{w}^n)/(1 + \overline{w}^{n+1})$ via the obvious map coincides with $\bigoplus \overline{w}^n/\overline{w}^{n+1}$, the graded Lie algebra induced from the associated graded algebra of $R_\Gamma$. As $R_\Gamma$ is already a graded algebra, its associated graded coincides with $R_\Gamma$. For details of these computations, compare e.g. [32, Lemma 4.10].

Finally, by definition, the induced algebra map $\overline{kA_\Gamma} \to \overline{R_\Gamma}$ preserves the filtrations by powers of the augmentation ideals and induces on the associated graded algebra the inverse of the map $\alpha$ of Lemma 2.4. As the latter is an isomorphism, the same holds for the map $\overline{\alpha}$ of completed algebras. □

3. Cohomology

To compute the structure of the cohomology ring $H^*(A_\Gamma, k)$, we first exhibit a particularly nice classifying space for $A_\Gamma$. We write $S^1 = [0, 1]/(0 \sim 1)$, and consider the subspace

$$X_\Gamma = \bigcup_{C \subseteq V \text{ a clique}} (S^1)^C \times \{0\}^{V \setminus C}$$

of the torus $(S^1)^V$. It is clear that we have $\pi_1(X_\Gamma, 0^V) = A_\Gamma$: the generator $v \in V$ is realized as a loop along the circle $(S^1)^{(v)} \times \{0\}^{V \setminus \{v\}}$, and the 2-tori in $X_\Gamma$ give
the commutation relations. Furthermore, the associated cell complex has a single vertex $0^V$, and its link is a flag complex, since every subset of a clique is a clique. Therefore, $X_\Gamma$ is a cube complex whose link is a flag complex, so $X_\Gamma$ is a locally CAT(0) space \cite{15}, see \cite[Theorem 5.18]{5}, so its universal cover is contractible.

Furthermore, the cells given in the expression of $X_\Gamma$ above form a basis of the homology of $X_\Gamma$, and dually of its cohomology: the differentials vanish identically. In cohomology, the class dual to $(S^1)^C$ is naturally the product of the classes $e^c$ dual to $(S^1)^{\{c\}}$ for all $c \in C$; in other words, the product $e^* w^*$ vanishes precisely when there is no edge $\{v, w\}$ in $E$. We have proven $H^*(A_\Gamma, \mathbb{k}) = H^*(X_\Gamma, \mathbb{k}) = S_\Gamma$, the first part of Theorem 1.2.

Consider a graded associative algebra $R$ presented as $T(W)/I$ for a $k$-module $W$ and an ideal $I \leq T(W)$. In case $I$ is generated by a subspace $I_2$ of $W^{\otimes 2}$, the algebra is called quadratic; and it then admits a Koszul dual $R^! := T(W^*)/(I_2^*)$; here by $I_2^*$ we mean the subset of $(W^*)^\otimes 2 \cong (W^{\otimes 2})^*$ annihilating $I_2$. Clearly $R^! \cong R$. Furthermore, $R$ is called Koszul if $k$ admits a minimal resolution by free $R$-modules in which all differentials increase the $R$-degree by 1. This implies that the Yoneda algebra $\text{Ext}_R(k, k)$ is isomorphic to $R^!$.

The work of Fröberg \cite{9} in particular Section 3 implies that $R_\Gamma$ and $S_\Gamma$ are Koszul, and duals of each other. His proof runs essentially as follows: consider the right $R_\Gamma$-module $P_\ast = \text{Hom}_R(S_\Gamma, R_\Gamma)$. Qua $k$-module, $S_\Gamma$ is finitely generated free with basis indexed by the cliques $C$ in $\Gamma$ (and a degree-$k$ basis element corresponding to a clique $C = \{v_0, \ldots, v_{k-1}\}$ is given by the product $v_C := v_{k-1} \cdots v_0$ — to make this definite, we pick a total ordering of the vertices and write the factors in decreasing order). Consequently, $P_\ast$ is canonically isomorphic to $\bigoplus_C v_C R_\Gamma$ where the sum is over the cliques in $\Gamma$. It is graded by $S_\Gamma$- and $R_\Gamma$-degree.

Consider the map $d(f)(p) = \sum_{v \in V} v f(vp)$ for $f \in P_\ast, p \in S_\Gamma$. In our basis, $d((v_{k-1} \cdots v_0) \cdot r) = \sum(-1)^j (v_{k-1} \cdots \hat{v}_j \cdots v_0) \cdot v_j r$. A direct computation shows that $d^2 = 0$. Note that $d$ increases the $R_\Gamma$-degree by 1, and decreases the $S_\Gamma$-degree by 1, so $R_\ast$ becomes a chain complex of finitely generated free $R_\Gamma$-modules, graded by $S_\Gamma$-degree.

We define a map $s: P_\ast \to P_{\ast+1}$ of $k$-modules on the $k$-basis element $v_C \cdot w$ for a clique $C$ of $\Gamma$ and a basis element $w$ of $R_\Gamma$ (namely, a monomial over $V$) as follows: if we can write $w = v w'$ with $v \in V, w' \in V^*$ in such a manner that $v < \min C$ (for the total ordering on $V$ picked above) and such that $C \cup \{v\}$ is a clique of $\Gamma$, then we do this, choosing $v$ minimal with this property, and we set $s(v_C \cdot v w') := v_C \cup \{v\} \cdot w'$. Otherwise, we set $s(v_C \cdot w) := 0$.

It is an elementary calculation to see that $s$ is a chain contraction, meaning $sd + ds = 1 - \epsilon$, where $\epsilon: P_\ast \to k$ is the augmentation map, projecting onto the summand of bidegree $(0, 0)$: consider $x = v_C \cdot w$. The calculation splits into three cases: if $C = \emptyset$ and $w = 1$, then $(sd + ds)(x) = 0 = (1 - \epsilon)(x)$. In the second case, $C = \{v_0, \ldots, v_k\} \neq \emptyset$ and $w$ cannot be written in the form $w = vh'$ as above; then $ds(x) = 0$ while $sd(x) = \sum(-1)^j s((v_{k-1} \cdots \hat{v}_j \cdots v_0) \cdot v_j w)$. By hypothesis, no letter in $w$ can be swapped with $v_j$ and added to $C \setminus \{v_j\}$, so all summands vanish except the 0th which is $x$. In the last case, $C = \{v_0, \ldots, v_k\}$ and $w$ can be written in the form $v_{-1} w'$ such that $C \cup \{v_{-1}\}$ is a clique in $\Gamma$, with $v_{-1} < \min C$, chosen minimal among all such possibilities. Then $v_{-1}$ commutes with all $v_j$, so

$$sd(x) = \sum_{j=0}^{k-1} (-1)^j s(v_{C \setminus \{v_j\}} \cdot v_j v_{-1} w') = \sum_{j=0}^{k-1} (-1)^j v_{C \setminus \{v_j\} \cup \{v_{-1}\}} \cdot v_j w',$$

$$ds(x) = d(v_{C \cup \{v\}} \cdot w') = \sum_{j=1}^{k-1} (-1)^{j+1} v_{C \setminus \{v_j\}} \cup \{v_{-1}\} \cdot v_j w',$$

and the terms cancel pairwise except the one with $j = -1$, giving again $(ds + sd)(x) = x$. It follows that $P_\ast$ is a free $R_\Gamma$-resolution of $k$. 

\[ \text{Reference:} \]
To show that $P_r$ is a Koszul resolution we only have to argue why it is a minimal free resolution. The image of the differential is contained in the submodule of $R_T$-degree $\geq 1$. Note that we have a surjective right $R_T$-homomorphism $\mu: \bigoplus_{v \in V} v \cdot R_T \to (R_T)_{\geq 1}; v \cdot r \mapsto vr$. Tensoring both sides over $R_T$ with $k$ gives an isomorphism

$$\mu \otimes \text{id}_k: \bigoplus_{v \in V} vk \xrightarrow{\cong} (R_T)_{\geq 1} \otimes_{R_T} k.$$  

Now consider the composition of $d_k : \text{Hom}_k((S_T)_k, R_T) \to \text{Im}(d_k)$ with the inclusion $i : \text{Im}(d_k) \to \text{Hom}_k((S_T)_{k-1}, (R_T)_{\geq 1})$ and tensor with the identity on $k$. Write $C_k$ for the set of cliques of size $k$ in $\Gamma$. We obtain, using the $v_C$ for $C \subseteq C_k$ as basis of $(S_T)_k$ and the isomorphism $(\mu \otimes \text{id}_k)^{-1}$,

$$(i \circ d_k) \otimes_{R_T} \text{id}_k: \bigoplus_{C \subseteq C_k} v_C k \to \bigoplus_{D \subseteq C_{k-1}, v \in V} v_D \cdot vk; \quad v_C \mapsto \sum \langle -1 \rangle^j v_{C \setminus \{v_j\}} \cdot v_j.$$  

This map is clearly injective, which implies inductively on $k$ that the domain of $d_k$ has minimal rank qua free $R_T$-module. For more details on minimal resolutions see [30, Appendix I].

Note now that, with $kv$ the free $k$-module with basis $V$, we have

$$R_T = T(kV)/(v \otimes w - w \otimes v | \{v, w\} \in E),$$

$$S_T = T(kV)/(u \otimes t + t \otimes u, v \otimes w | t, u, v, w \in V, \{v, w\} \notin E).$$

Let us identify $kV^{\otimes 2}$ with $(kV^{\otimes 2})^*$ by sending the basis element $v \otimes w$ to its dual element $\delta_{v\otimes w}$ given by $\delta_{v\otimes w}(v' \otimes w') = \delta_{v, v'} \delta_{w, w'}$ for $v, w, v', w' \in V$. Then immediately see that the subspace of $kV^{\otimes 2}$ generated by $G_S$ is the annihilator of the one generated by $G_T$, and therefore indeed $R_T$ and $S_T$ are Koszul dual to each other. This concludes the proof of Theorem 1.2.

We note that the usual definition of Koszul algebras is done over fields of characteristic 0; however, in our case, we need not impose any restriction on the commutative ring $k$ (other than interpreting $(kV)^*$ as naturally isomorphic to $kV$), since the rings $R_T, S_T$ are $k$-free.

### 4. Central series

Labute gave in [16] a general criterion under which a presentation $(V | R)$ of a group $G$ determines a presentation of the associated Lie algebra $L(G) = \bigoplus_{n \geq 1} \gamma_n(G)/\gamma_{n+1}(G)$. Such a group presentation is now called “mild”, and Anick gave in [2] a valuable criterion for this to happen: view all $r \in R$ as elements of the free associative algebra $T(ZV)$, under the Magnus embedding $F_V \to T(ZV)$. Let $n$ be such that $r - 1 \in \omega^n \setminus \omega^{n+1}$, and let $r'$ denote the image of $r$ in the quotient $\omega^n/\omega^{n+1}$. Then $(V | R)$ is mild if and only if $r' \in R$ is “inert”. There are powerful sufficient conditions guaranteeing that a set is inert in the free associative algebra, one of them being that it forms a Gröbner basis. It follows then quite generally that the Lie algebra $L(G)$ admits as presentation $(V | r' \forall r \in R)$, see [16 Theorem 1]; and a similar statement holds for the restricted Lie algebra $\bigoplus_{n \geq 0} \Lambda_{n,p}(G)/\Lambda_{n+1,p}(G)$, see [16 Theorem 3]. Labute’s conditions are not trivial to check, so we shall in fact recover his results rather than use them.

Now, by Proposition 2.2, the rings $R_T/\omega^n$ and $kA_T/\pi(G)^n$ are isomorphic, so the dimension subgroups $\delta_{n,A}$ and $\delta_{n,kA}$ are equal. Furthermore, since the Magnus map $\mu$ has image in the subring of $R_T$ generated by 1 and $V$, the groups $\delta_{n,A}$ depend on $k$ only via the image $k'$ of $Z$ in $k$.

We consider two cases: if $Z \subseteq k$, then the dimension subgroups associated with the rings $k$ and $Q$ agree. If, on the other hand, $F_p \subseteq k$, then the dimension subgroups associated with the rings $k$ and $F_p$ agree. In all cases, we reduce to the case $k \in \{F_p, Z\}$.

We now apply the classical results of Jennings and Hall. For $k = \mathbb{Q}$ we have $\gamma_{n,0} = \delta_{n,kA}$; compare [11,12] which treat the case of torsion-free nilpotent groups to which the general case easily reduces. For $k = F_p$ we have $\gamma_{n,p} = \delta_{n,kA}$;
compare [13] which treats the case of finite $p$-groups to which the general case easily reduces. We have proven the second part of Theorem [13]. The first part will require considerations on Lie algebras, in the next section.

5. Lie algebras

We first recall from [12] that a restricted Lie algebra, in characteristic $p$, is a Lie algebra equipped with an extra operation, written $x \mapsto x^{[p]}$, called the $p$-mapping and subject to the following axioms: for all $x, y$ in the Lie algebra and $\alpha \in \mathbb{k}$, $[y, x^{[p]}] = [y, x, \ldots, x]$ with $p$ factors `$x$'; $(\alpha x)^{[p]} = \alpha^p x^{[p]}$; and $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y)$ for the Lie expressions $s_i(X, Y)$ defined by $d/dt[X, tX + Y, \ldots, tX + Y] = \sum s_i(X, Y)t^i$ with $p-1$ factors `$tX + Y$'. For example, if $p = 2$ then $s_1(X, Y) = [X, Y]$, and if $p = 3$ then $s_1(X, Y) = [Y, X, Y]$ and $s_2(X, Y) = [X, Y, X]$.

We use the standard multi-commutator convention $[x, y, z] = [x, [y, z]]$, etc.

We adopt the usual convention that, in characteristic 0, every Lie algebra is restricted with trivial $p$-mapping. This way, from now on we can uniformly work with restricted Lie algebras.

Recall that every restricted Lie algebra $L$ has a restricted universal enveloping algebra, a unital associative algebra $U_p(L)$ equipped with a map of restricted Lie algebras $L \to U_p(L)$, universal with respect to this property. The Lie bracket in $L$ is identified with the commutator $[x, y] = xy - yx$, and the $p$-mapping in $L$ is identified with the $p$-power operation in $U_p(L)$. It is a classical fact that $U$ is a Hopf algebra, with coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$.

In a Hopf algebra $H$, call $x \in H$ primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$; the primitive elements of $H$ form a Lie subalgebra $P$ of $H$. If $H$ is a cocommutative Hopf algebra over a field, it is known [21, Theorem 5.18] that the restricted universal enveloping Hopf algebra, with coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$.

We adopt the usual convention that, in characteristic 0, every Lie algebra is restricted with trivial $p$-mapping. This way, from now on we can uniformly work with restricted Lie algebras.

Recall from Section 2 that $R_\Gamma$ is a k-free Hopf algebra. It follows that its Lie subalgebra of primitive elements $P$ admits as presentation

$$P = \langle V \mid [v, w] \rangle$$

for all $\{v, w\} \in E$, hence canonically $P \cong L_\Gamma$ with isomorphism given by the identity on $V$.

Following Magnus’ method [19, Theorem 5.12], consider $x \in P_n$, i.e. homogeneous of degree $n$. Then $x$ is a linear combination (with coefficients in $\mathbb{k}$) of a collection of bracket arrangements $\beta_i = \beta_i(v_1, \ldots, v_n)$. The assignment $P_n \ni \beta_i \mapsto \beta_i(v_1, \ldots, v_n) \in \gamma_n \subseteq A_\Gamma$ is well defined on the subset of bracket arrangements, since $\{v, w\} = 1 \in A_\Gamma$ for each $\{v, w\} \in E$. It extends $k$-linearly to a map $\nu: P_n \to \gamma_n/\gamma_{n+1} \otimes \mathbb{k}$ of $k$-modules. This map is clearly surjective, since $\gamma_n/\gamma_{n+1}$ is spanned by $n$-fold bracket arrangements, for an arbitrary group. Furthermore, the composition $\mu_L \circ \nu: P \to R_\Gamma$ with $\mu_L$ given in Proposition 2.4 is a Lie algebra map sending $v$ to $v$. Therefore this composition is the inclusion of $P$ into $R_\Gamma$ and is in particular injective. This implies that $\nu$ is an isomorphism with inverse the Magnus map $\mu_L$. This establishes the statement about $L_\Gamma$ in Theorem 1.4.

For $k = \mathbb{Z}$, as $L_\Gamma$ is $\mathbb{Z}$-free, it follows in particular that $\gamma_n(A_\Gamma)/\gamma_{n+1}(A_\Gamma)$ is torsion-free for each $n$, and therefore $\gamma_{n,0}(A_\Gamma) = \gamma_n(A_\Gamma)$ for all $n$. This finishes the proof of Theorem 1.3.

Set now $k = \mathbb{Z}$, and consider the ideal $\varpi_p = \langle p, V \rangle$ of $R_\Gamma$.

**Lemma 5.1.** The associated graded ring $\bigoplus_{n \geq 0} \varpi_p^n/\varpi_p^{n+1}$ is isomorphic to $R_\Gamma \otimes_{\mathbb{Z}} F_p[\pi]$, with $\pi$ of degree 1 mapped to $[p] \in \varpi_p/\varpi_p^2$ under the isomorphism.

**Proof.** Define a new filtration on $R_\Gamma$, with $v \in V$ of degree 1, but now in addition $p$ of degree 1. The ring $R_\Gamma$ is $\mathbb{Z}$-free. When passing to the associated graded ring for the new grading, we get on the one hand $\bigoplus \varpi_p^n/\varpi_p^{n+1}$. On the other hand,
it is obtained from the old associated graded (which is the graded algebra $R_\Gamma$) by replacing each copy of $\mathbb{Z}$ by $F_p[\pi]$, which amounts to tensoring with $F_p[\pi]$.  

In the case $p \geq 3$, we are now ready to identify the non-restricted Lie algebra $\bigoplus_{n \geq 1} \lambda_{n\cdot p}/\lambda_{n+1\cdot p}$ with $L_\Gamma \otimes_{\mathbb{Z}} F_p[\pi]$. Let us temporarily write $\nu_n := \mu^{-1}(1 + \pi_p)$. We claim:

**Lemma 5.2.** For $p \geq 3$ prime, the Magnus map $\mu$ induces a composition of (non-restricted) Lie algebra isomorphisms over $F_p[\pi]$, still written $\mu_L$,

$$\mu_L : \bigoplus_{n \geq 1} \lambda_{n\cdot p}/\lambda_{n+1\cdot p} \to \bigoplus_{n \geq 1} \nu_n/\nu_{n+1} \to L_\Gamma \otimes_{\mathbb{Z}} F_p[\pi],$$

with the first map induced by inclusion $\lambda_{n\cdot p} \leq \lambda_n$ and the second map induced by $\nu_n/\nu_{n+1} \ni [1+a] \mapsto a \in \pi_p/\pi_p^{n+1}$.

In particular, we have $\nu_n = \lambda_{n\cdot p}$.

**Proof.** To check that the first map is well-defined, it suffices to show $\lambda_{n\cdot p} \leq \lambda_n$. We have $\pi_p^n = \sum_{m=1}^{n} \lambda_p^m$. Consider $x \in \gamma_m$, so by definition $\mu(g) = 1 + x$ for some $x \in \pi_p$. We then have $\mu(\gamma_m) = (1 + x)^{\lambda_p} = 1 + \pi_p + \cdots \in 1 + \pi_p^{m+1}$, so $\mu(\gamma_m) \leq 1 + \pi_p^{m+1}$. Since $\lambda_{n\cdot p} = \prod_{m=1}^{n} \gamma_p^m$, we have shown $\lambda_{n\cdot p} \leq \lambda_n$.

Because the Magnus map $\mu : A_\Gamma \to 1 + \pi \subset R_\Gamma$ is injective by Proposition 2.3 so is the induced map $\nu_n/\nu_{n+1} \to (1 + \pi_p)/(1 + \pi_p^{n+1}) = \pi_p/\pi_p^{n+1}$, which is our second map.

Since $p \geq 3$, the assignment $\pi : g = [g^p]$ for $g \in \lambda_{n\cdot p}$ (with $g^p \in \lambda_{n+1\cdot p}$) gives $\bigoplus_{n \geq 1} \lambda_{n\cdot p}/\lambda_{n+1\cdot p}$ the structure of an $F_p[\pi]$-module (however, beware that if $p = 2$ and $n = 1$ then $(xy)^2 = x^2y^2x$ with $x \in \lambda_{2,2} \setminus \lambda_{1,2}$ for some $x, y \in \lambda_{1,2}$ and therefore the 2-power operation is not linear). We see that $\mu_L$ maps this $p$-power operation to multiplication by $\pi$ on $L_\Gamma \otimes_{\mathbb{Z}} F_p[\pi]$. It follows that $\mu_L$ is an $F_p[\pi]$-Lie algebra homomorphism. Its image contains $V$ which generates $L_\Gamma$, so $\mu_L$ is surjective. Finally, $L_\Gamma \otimes_{\mathbb{Z}} F_p[\pi]$ is the free Lie algebra over $F_p[\pi]$ modulo the relations $[v, w] = 0$ for $\{v, w\} \in E$. Those relations are clearly satisfied in the $F_p[\pi]$-Lie algebra $\bigoplus_{n \geq 1} \lambda_{n\cdot p}/\lambda_{n+1\cdot p}$, so the map $\mu_L$ is an isomorphism.

It then follows that the second map is surjective and therefore an isomorphism, so the first is also bijective, from which we deduce $\nu_n = \lambda_{n\cdot p}$.

This concludes the proof of Theorem 1.4.

Assume finally that $k$ is an algebra over $F_p$. By 26, $\bigoplus_{n \geq 1} (\gamma_{n\cdot p}/\gamma_{n+1\cdot p}) \otimes_{\mathbb{Z}} k$ is isomorphic to the primitive subalgebra of $\bigoplus_{n \geq 0} \pi(A_\Gamma)^{n}/\pi(A_\Gamma)^{n+1}$, namely to $R_\Gamma$. We have finished the proof of Theorem 1.6.

### 6. Growth series

We derive now some relations between the Poincaré series of $S_\Gamma$, $R_\Gamma$, $L_\Gamma$, and $L_{\Gamma, p}$ from general considerations. We recall that, for a graded algebra $R = \bigoplus_{n \geq 0} R_n$, its Poincaré series is $\Phi_R(t) = \sum_{n \geq 0} \text{rank}(R_n)t^n$.

First, we use Koszul duality between $R_\Gamma$ and $S_\Gamma$ to deduce $\Phi_{R_\Gamma}(t) \cdot \Phi_{S_\Gamma}(-t) = 1$. This relationship between the Poincaré series of $R_\Gamma$ and $S_\Gamma$ was already noted in 6.3.

We have $\Phi_{S_\Gamma}(t) = \sum_{n \geq 0} \text{rank} H^n(A_\Gamma, k)t^n = \sum_{n \geq 0} c_n(\Gamma)t^n$, with $c_n(\Gamma)$ the number of $n$-cliques in $\Gamma$, as follows from the description of the classifying space $X_\Gamma$ given in Section 3.

The relation between $\Phi_{R_\Gamma}$ and $\Phi_{L_\Gamma}$ is given by the Poincaré-Birkhoff-Witt theorem, namely the fact that $R_\Gamma$ and the symmetric algebra over $L_\Gamma$, respectively the degree-$p$ truncated symmetric algebra over $L_{\Gamma, p}$, are isomorphic as graded $k$-modules. It is expressed by the relation

$$\sum_{n \geq 0} a_nt^n = \prod_{n \geq 1} \left(\frac{1}{1-t^n}\right)^{b_n} = \prod_{n \geq 1} \left(\frac{1 - t^{pn}}{1 - t^n}\right)^{c_n}.$$
if $\Phi_{R_1}(t) = \sum_{n \geq 0} a_n t^n$, $\Phi_{L_1}(t) = \sum_{n \geq 1} b_n t^n$, and $\Phi_{L_{1,p}}(t) = \sum_{n \geq 1} c_n t^n$.

Finally, we consider the growth series of the group $A_{\Gamma}$. It is the function $\Phi_{A_{\Gamma}}(t) = \sum_{g \in A_{\Gamma}} t^{\lVert g \rVert}$, with $\lVert g \rVert$ the minimal number of terms of $V \cup V^{-1}$ required to write $g$ as a product. We cite [3]:

$$\Phi_{A_{\Gamma}}(t) = \Phi_{R_{C}} \left( \frac{2t}{1+t} \right).$$

Indeed, as we saw in Proposition 2.3 every element $g \in A_{\Gamma}$ can be written in the form $g = v_1^{e_1} \cdots v_n^{e_n}$ for some $e_i \in \mathbb{Z} \setminus \{0\}$ as a word of minimal length; and this expression is unique up to permuting some terms according to rule (M2). For any $(M2)$-equivalence class $(v_1, \ldots, v_n)$ of minimal-length sequences, the collection of all such terms contributes $(t + t^2 + t^3 + \cdots)^n = (t/(1-t))^n$ to the growth of $R_{\Gamma}$, because each $e_i$ can be an arbitrary positive natural number; and it contributes $(2t/(1-t))^n$ to the growth of $A_{\Gamma}$, taking into account the signs of the $e_i$. Since we obtain all elements of $A_{\Gamma}$ and all basis elements of $R_{\Gamma}$ that way, and since we have $2t/(1-t) = u/(1-u)$ for $u = 2t/(1+t)$, the result follows. We have finished the proof of Theorem 1.5.

7. Malcev completions

Recall from [24] that a Malcev Lie algebra is a Lie algebra $L$ over $\mathbb{Q}$, given with a descending filtration $(L_n)_{n \geq 1}$ of ideals such that $L$ is complete with respect to the associated topology, and satisfying $L_1 = L$ and $[L_m, L_n] \subseteq L_{m+n}$ and such that $\bigoplus_{n \geq 1} L_n/L_{n+1}$ is generated in degree 1. Every Malcev Lie algebra admits an associated exponential group $\exp(L)$, which is $L$ as a set, with product given by the Baker-Campbell-Hausdorff formula $x \cdot y = x + y + [x, y]/2 + \cdots$.

Lazard proved in [17] that every group homomorphism $\rho: G \to \exp(L)$ induces a morphism of graded Lie algebras $\bigoplus_{n \geq 1} \mathfrak{g}/\mathfrak{g}_{n+1} \otimes \mathbb{Q} \to \bigoplus_{n \geq 1} L_n/L_{n+1}$.

A Malcev completion of a group $G$ is a homomorphism $\rho: G \to \exp(L)$ for a Malcev Lie algebra $L$, universal in the sense that every representation $G/\mathfrak{g} \to \exp(L')$ for a (nilpotent) Malcev Lie algebra $L'$ factors uniquely through $\exp(L/L_n)$; see [24] Definition 2.3.

Quillen gave a direct construction of the Malcev completion of a group in [26,27]: let $\overline{G}$ be the projection of $G/\mathfrak{g}$ by $\mathfrak{g}^n$ be the completion of the group ring; then $\overline{G}$ is a complete Hopf algebra. Let $L$ be its Lie subalgebra of primitive elements; it is a Malcev Lie algebra for the filtration $L_n = L \cap \mathfrak{g}^n$. Let $\exp: L \to \overline{G}$ be the usual power series map $\exp(x) = 1 + x + x^2/2 + \cdots$ which makes sense in $\overline{G}$. Then its image $\mathfrak{g} := \exp(L)$ is a subgroup of the group of multiplicative units. It identifies with the Lie group associated to the Malcev Lie algebra $L$, and it consists precisely of the grouplike elements in $\overline{G}$, namely the $g \in 1 + \mathfrak{g}$ satisfying $\Delta(g) = g \otimes g$.

The representation $\rho: G \to \exp(L); g \mapsto g$ is the Malcev completion of $G$.

The Magnus map $\mu: A_{\Gamma} \to \overline{R_{\Gamma}}$ yields an isomorphism of associative algebras $\overline{Q A_{\Gamma}} \cong \overline{R_{\Gamma}}$. Both algebras are actually complete Hopf algebras, but the Magnus isomorphism does not preserve the Hopf algebra structure: $v \in V \subset \overline{Q A_{\Gamma}}$ is group-like, meaning $\Delta(v) = v \otimes v$ while $v \in V \subset \overline{R_{\Gamma}}$ is primitive, meaning $\Delta(v) = v \otimes 1 + 1 \otimes v$; so $\Delta(\mu(v)) = 1 \otimes v + v \otimes 1 + 1 \otimes v$ while $\mu(\mu(v)) = (1+v)\otimes (1+v)$.

The Magnus map $\mu$ is, in fact, the truncation to order 1 of a Hopf algebra isomorphism $\mu: \overline{Q A_{\Gamma}} \to \overline{R_{\Gamma}}$, given on $v \in V$ by the classical exponential series

$$\mu_{\exp}(v) = \sum_{n \geq 0} \frac{v^n}{n!} = 1 + v + O(v^2).$$

The proof that $\mu_{\exp}$ is an isomorphism of filtered associative algebras is exactly the same as that of Theorem 1.3 and will not be repeated. On the other hand, the fact that $\mu_{\exp}$ is a coalgebra map follows formally from the fact that the power series
exp maps primitive elements to group-like elements:
\[
\Delta(\mu_{\exp}(v)) = \Delta\left(\sum_{n \geq 0} v^n / n!\right) = \sum_{n \geq 0} \Delta(v^n / n!)
\]
\[
= \sum_{n \geq 0} \frac{(v \otimes 1 + 1 \otimes v)^n}{n!} = \sum_{\ell, m \geq 0} \frac{(v \otimes 1)^\ell (1 \otimes v)^m}{\ell! m!}
\]
\[
= (\exp v \otimes 1)(1 \otimes \exp v) = (\mu_{\exp} \otimes \mu_{\exp})(\Delta(v)).
\]

We have proven the first part of Theorem 1.6.

It now suffices to use this isomorphism \(\mu_{\exp}\) to make even more concrete the construction of Quillen sketched above: in \(\mathbb{Q}A_G\), the space of primitive elements is slightly mysterious (for example, it contains \(\log(g) = \log(1 - (1 - g)) = -\sum_{n \geq 1} (1 - g)^n / n\) for every \(g \in A_G\)) while its exponential is the Malcev completion naturally containing \(A_G\); and in \(\mathbb{R}A_G\), the opposite holds: the space of primitive elements is the Lie subalgebra \(L_G\) while its exponential cannot be better defined than as the exponential of \(L_G\).

In all cases, the Hopf algebra isomorphism \(\mu_{\exp}\) directly yields the remaining claims of Theorem 1.6.

8. Outlook

Baik, Petri, and Raimbault determined the subgroup growth of \(A_G\) in terms of the graph \(\Gamma\). Define \(s_n(A_G)\) as the number of subgroups of \(A_G\) of index precisely \(n\). Then \cite[Theorem A]{4} establishes
\[
\lim_{n \to \infty} \frac{\log(s_n(A_G))}{n \log(n)} = \alpha(\Gamma) - 1,
\]
i.e. \(s_n(A_G)\) grows like \((n!)^{\alpha(\Gamma) - 1}\). Here, \(\alpha(\Gamma)\) is the independence number of \(\Gamma\), the largest number of vertices such that the full subgraph of \(\Gamma\) spanned by them is discrete.

We do not discuss the rather complicated proof here. We leave it as an open question to find a corresponding result for the growth of the number of finite index Lie subalgebras of \(L_G\). Indeed, we expect that these two series are closely related and that the latter is slightly easier to control than \((s_n(A_G))\).

We have identified \(\gamma_{n, p}(A_G)\) with \(\delta_{n, p}A_G\) in Theorem 1.3. For a group \(G\), we could define \(\gamma_{n, p'}\) as the subgroup generated by \(\gamma_n\) and all \(\gamma_{i'}^{p'}\) with \(ip' \geq np'^{-1}\). When \(G\) is free, it was shown by Lazard that \(\gamma_{n, p}(G)\) coincides with the dimension subgroup \(\delta_{n, \mathbb{Z}/p^e\mathbb{Z}[G]}\) while this does not hold for general \(G\), see \cite{22}.

We leave it as an exercise to extend Lazard’s result to \(A_G\).

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