Nonlinear Responds of Lamb Waves in Plate Structure with Micro-Crack Using Frequency-Mixing Technique

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Abstract. Lamb waves have huge potential in non-destructive testing. Its nonlinear responses provide a promising method to detect micro-cracks. This work mainly analyses nonlinear responses of two Lamb waves with different frequencies in plate structure with micro-cracks. The interaction between ultrasound and two types of cracks with different lengths, widths, angles and number was performed using finite element simulations. From the results, it is shown that frequency-mixing technique is sensitive to the micro-cracks. Also, different parameters of cracks have different relationships with sum-frequency harmonic amplitudes. These relationships provide basic support for quantitative evaluation of cracks.

1. Introduction
Plate structures are common in civil life and mechanical manufacture field. If cracks appear, they will develop quickly and the whole structure is about to fail. Ref [1] indicates that appearance of cracks means the rest life of the structure is only about 10%-20% of the whole life span. Therefore, it is of significance to detect cracks at early stage to avoid more negative consequences\cite{2,3,4,5,6}.

Traditional ultrasonic testing measures some parameters to detect cracks, such as reflection and transmission coefficients, acoustic velocity and attenuation, which means we are only concerned time-domain information of the received signals. Traditional methods are sensitive to gross defects and open cracks because the cracks are big enough to be viewed as a barrier. When waves go through a gross crack, for example, the reflection wave is powerful and detectable. Nevertheless, when it come across a micro-crack, the effect caused by micro-crack is negligible. Reflection wave is perfectly small and can only be detected by precious equipment. In the same time, the noise should be reduced to particular low level to make sure the small wave is accurate. Therefore, detecting micro-cracks needs an alternative technique.

Recently, nonlinear ultrasonics attracts more and more attention because it is sensitive to micro-cracks\cite{7,8}. Although the linear responses of ultrasound cannot provide much information of micro structure of a material, the nonlinear responses of ultrasound provide a new character: frequency. Based on frequency, there are some divided methods, such as harmonic\cite{9}, resonance\cite{10}, and frequency-mixing\cite{11} method. Harmonic method is used commonly. When the harmonics increase, it can be concluded that materials degrades. However, it is still very difficult to get rid of noise from ultrasonic generation and testing system so far. Frequency mixing technique has potential to solve this issue\cite{12,13}. When two ultrasound interact with a micro-crack, two more waves will be generated.
with different frequency. If choosing two incident waves with specific frequency, the new waves can avoid confusion with harmonics generated by the system.

The ultrasonic guided wave provides a promising method to detect cracks because of long distance propagation and high efficiency compared to traditional methods[14][15][16][17]. Undoubtedly, Lamb wave, as one of the most common guided waves, has been widely adopted in the area of non-destructive testing[18][19] and material characterization. When it propagates though the plate-like structure, signals from Lamb wave will contain whole information of the propagation path. People can use the information of ultrasonic signals to evaluate the life span of materials and even to locate the position of cracks. However, due to dispersion effect which means the characteristics of Lamb waves vary along with its frequency and thickness of plate, using Lamb waves becomes more complex. The higher the frequency, the more modes generated.

Jingpin Jiao et al[20] and Shengbo Shan et al[21] have done the research about using two guided waves to detect micro-cracks in plate-like structures but their theories were wrongly based on material nonlinear constitutive relation. Moreover, their simulations only focus on one type of cracks and cannot be applied to other practical situations. In this paper, we applied frequency-mixing technique in plate structure using two Lamb waves for more full analysis of nonlinear responses.

2. Lamb wave and Nonlinear theory

There are two main theories for ultrasonic nonlinearity. The most popular one is classical nonlinearity theory which explains nonlinear resource by introducing nonlinear constitutive relation of materials. The nonlinear effect can be shown by nonlinear coefficient but the classical nonlinearity theory cannot explain the role of cracks. It can only explain the degradation of material properties. The second theory is the contact acoustic nonlinearity(CAN). CAN theory indicates that the nonlinear phenomenon was caused by asymmetry stiffness near crack area. In this section, both the classical theory and CAN theory will be briefly introduced because the classical nonlinearity theory can derive nonlinear coefficient to quantify nonlinearity and the CAN theory can explain well the crack effect. This paper is using Lamb wave to detect cracks and the Lamb wave is more complex than bulk waves. Therefore, the equations of Lamb waves are presented first.

2.1. Lamb wave

Lamb waves are based on Navier equation:

$$\mu \nabla^2 \ddot{u} + (\lambda + \mu) \nabla \nabla \cdot \ddot{u} = \rho \frac{\partial^2 \ddot{u}}{\partial t^2}$$

where $\lambda$ and $\mu$ are the Lamé constants, $\rho$ is the density, $\ddot{u}$ is the displacement vector, $\nabla^2$ is the Laplace operator, $\nabla$ is the gradient, $\nabla \cdot$ is the divergence.

According to Snell–Descartes laws, waves reflect in boundary. Also, mode transformation occurs when waves encounter a boundary. Therefore, when waves propagate in plate-like structure, waves will go through continue reflection and mode transformation at upper and lower surface and form guide waves. The dispersion equation can be derived:

$$\tan \frac{qh}{2} = \left( \frac{4k^2 pq}{q^2 - k^2} \right)^{\pm 1}$$

where exponent term $\pm 1$ specifies the symmetric/antisymmetric mode, $h$ is the plate thickness, $k$ the wave number, and $p$ and $q$ are defined as

$$p = \sqrt{k_i^2 - k^2}, \quad q = \sqrt{k_i^2 - k^2}$$

where $k_i$ and $k_j$ are the wavenumber amplitudes for the longitudinal and transversal waves, respectively. The dispersion equation can be calculated and shown in figure 1.
Figure 1. Dispersion curve of Lamb wave from numerical solution of Rayleigh-Lamb equation.

2.2. Classical nonlinearity theory and nonlinear coefficient
Consider the nonlinear constitutive relation of materials (i.e. nonlinear Hooke law):
\[ \sigma = E\varepsilon + \beta\varepsilon^2 + \cdots \]
where \( E \) is Young's modulus and \( \beta \) the second-order nonlinear elastic coefficient. Perturbation analysis is adopted here and then we can get the approximate solution[20]:

\[ u = u_0 + u_1 \]
\[ u_1 = A_1 \cos(f_1 t - k_1 x) + A_2 \cos(f_2 t - k_2 x) + x\beta \left( -\frac{A_2^2 k_2^2}{8} \cos(2f_1 t - 2k_1 x) - \frac{A_2^2 k_2^2}{8} \cos(2f_2 t - 2k_2 x) \right) \]
\[ + \frac{A_1 A_2 k_1 k_2}{4} \left\{ \cos\left[ (f_1 - f_2)t - (k_1 - k_2)x \right] - \cos\left[ (f_1 + f_2)t - (k_1 + k_2)x \right] \right\} \]

From the solution, the nonlinear coefficient can be derived:
\[ \beta' = \frac{A_1 + A_2}{A_1 A_2} \text{ or } \frac{A_1 - A_2}{A_1 A_2} \]

2.3. Contact acoustic nonlinearity theory
Contact acoustic nonlinearity theory is proposed by Solodov[22]. Near the crack area, the stiffness is asymmetric, which can be modelled by following equation:
\[ \sigma = C^0 \left[ 1 - H(\varepsilon - \varepsilon^0) \left( \frac{\Delta C}{C^0} \right) \right] \varepsilon \]
where \( C^0 \) is the stiffness when the material is intact and \( H(\varepsilon) \) is the Heaviside function which means when the strain \( \varepsilon \) is smaller than static strain \( \varepsilon^0 \), which closes the cracks, the material remains as intact. Therefore, in simulations, we need to set special constraints near the cracks. When it comes across compressional waves, the cracks will close and the whole structure will remain intact. On the other hand, if the waves force it open, then the material is no longer linear.

3. Numerical model
Finite element method (FEM) is one of the cost-effective method to simulate the wave propagation and to analyse the responses of mechanical system. Explicit solver in Abaqus software is excellent in solving complicated mechanical problem in short time and has good convergence. Therefore, we adopt Abaqus software to do the calculation.
The frequency-mixing method is applied numerically to two dimensional, homogeneous and isotropic solid plate, assuming plane strain. Material properties: density 7850 kg/m³, Young's modulus 208.42 GPa, Poisson's ratio 0.2959. The plate is 1000 × 1.5 mm² and two types of cracks are placed in the plate as shown in figure 2. Inner cracks represent the fatigue damage and the outer cracks are designed for corrosion damage. The micro-crack was placed in 200 mm from the left end of the plate. The shape of outer cracks was designed as wedge-shape. The shape of inner cracks was modelled as an ellipse. Both wedge-shape cracks and elliptic cracks are modelled with a hard-contact frictionless surface, which satisfies the asymmetric stiffness.

![Ultrasonic signal extraction]

**Figure 2.** Two types of cracks are applied to a solid plate.

As shown in figure 2, the excitation is also important because the goal is to get the frequency information. Therefore, the time domain signal of the load (figure 3a) contains two Hanning-windowed 30-cycle tone burst signals with central frequencies of 450 kHz and 600 kHz. (This excitation signal is meant to be consistent with the Ref [20]):

\[
y(t) = \left( \sin(2\pi f_1 t) + \sin(2\pi f_2 t) \right) \left( 0.08 + 0.46 \cos(2\pi f_1 t / 30) \right)
\]

In figure 3a, the excitation signal spectrum shows that the excitation signal only has two centre frequencies, which can guarantee the generated harmonics of received signals are due to nonlinear cracks.

![Excitation](image1)
![Result](image2)

**Figure 3.** Comparison between excitation signal and the extracted ultrasonic signal from result.
Mesh generation is important in FEM. The mesh element size should be much smaller than the wavelength, otherwise there is no wave propagation phenomenon. The accuracy requires that the smallest wavelength \( \lambda_{\text{min}} \) must be correctly sampled to describe the propagating mode in the frequency range for analysis. The denser the mesh, the more accurate the result. However, more density means more computing resource usage. The mesh element size should balance the accuracy of results with computing resources usage. Therefore, the spatial discretization of \( \delta x_1 \) and \( \delta x_2 \) of each element of the mesh satisfies the condition[20][23]:

\[
\frac{\lambda_{\text{min}}}{\max(\delta x_1, \delta x_2)} > 10
\]

In crack zone, the mesh must be denser to guarantee the accuracy as shown in figure 4.

Another stability condition is that the time step must be chosen so that no wave can propagate across one mesh spacing in less than one time step. Typically, the time discretization must satisfy the condition[20][23]:

\[
\Delta t < \frac{\min(\delta x_1, \delta x_2)}{c_i}
\]

4. Results and discussion

In figure 3a and 3b, by comparison, it is clear that if we add a crack, the Lamb waves will be distorted because when Lamb waves pass through a crack, the crack will open and close, which will modulate the waves which pass. We can assume one of two incident waves as force resource which forces the crack open or close, and the other one wave is being modulated when the status of cracks changes.

From figure 3b, frequency-mixing technique shows that new waves of different frequency appear, including sum-frequency wave and difference-frequency wave. Both sum-frequency and difference-frequency waves can reveal the nonlinear phenomenon. In this paper, we choose sum-frequency and its corresponding nonlinear coefficient \( \beta' = \frac{A_{f_1+f_2}}{A_1A_2} \) to analysis its potential of evaluation.

The comparison results of inner cracks and outer cracks are shown in figure 4. Four groups of simulations were performed to investigate the relationship between nonlinear coefficient \( \beta' \) and the length and width of the micro-crack. In figure 5a and 5c, the width of crack is set to 10um and in Fig, 5b and 5d, the length of crack is 1.3 mm. The results show that the sum-frequency amplitude from
extracted ultrasonic signals increase monotonically with increasing length of crack when the width of crack is fixed. The more longer the cracks, the more part of waves pass through the cracks. Also, when length of cracks is fixed, the sum-frequency harmonics decreases monotonically as the increase of cracks’ width. The more wider the cracks, the more difficult the wave propagate though.

![Comparison results of inner crack and outer crack.](image)

**Figure 5.** Comparison results of inner crack and outer crack.

![Results of more angles and crack number in plate with inner cracks.](image)

**Figure 6.** Results of more angles and crack number in plate with inner cracks.
For inner crack (fatigue crack), we investigate more about influence of crack angle and crack number in figure 6. Generally, with the angle of cracks increasing, the nonlinear coefficient decreases. However, when the crack is 30 degrees, the tendency is different. Figure 6b shows that with more cracks, the nonlinear coefficient increases initially and then decrease.

5. Conclusion
This paper presents a comprehensive analysis of application of frequency-mixing technique in plate structure. Theoretically, we introduce two main theories and compare their difference: only CAN theory explains the nonlinear responses of crack. In simulation, parameters of simulation experiment are set according to the theory. Simulation results prove the crack detection ability of frequency-mixing method. Then, it is found that the nonlinear phenomenon has strong dependence on length and width of cracks. Moreover, the results of crack angle and crack number indicate that in the future, researchers need more precious model to analyse the frequency-mixing technique.

6. References
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