Locally Cartesian Closed Categories

Huang Xu*

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Abstract

This note explains how dependent sums and products are interpreted by adjoints of the base change functor in a locally cartesian closed category. An effort is made to unpack all the definitions so as to make the concepts more transparent to new learners.

Notational conventions:

• Categories in general use the calligraphic font: \( \mathcal{C} \);
• Special categories use sans-serif: \( \text{Cat}, \text{Set} \);
• \( X \in \mathcal{C} \) means “\( X \) is an object in \( \mathcal{C} \)”;
• Compositions are in the “function order”, i.e. if \( X \xrightarrow{f} Y \xrightarrow{g} Z \), then the composite arrow is \( g \circ f \);
• \( \text{Hom}(X, Y) \) denotes the morphisms from \( X \) to \( Y \); If necessary, subscripts indicate the category in discussion: \( \text{Hom}_\mathcal{C}(X, Y) \).

1 Slices

Given a category \( \mathcal{C} \) and an object \( X \), let’s consider all the arrows into \( X \). This forms a collection of arrows

\[
\bigcup_{Y \in \mathcal{C}} \text{Hom}(Y, X).
\]

We shall take this collection of arrows as the objects of a new category, named \( \mathcal{C}/X \).

What should the morphisms be? Consider any commutative diagram of the form

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By saying that the diagram “commutes”, I mean \( g \circ u = f \). It would be natural to take \( u \) as a morphism from \( f \) to \( g \). This defines the slice category over \( X \).

**Example.** Here are some simple examples of slice categories.

- If \( C \) has a terminal object, then \( C/1 \cong C \).
- Take 2 to be the set \{blue, red\}. Set/2 is the category of **two-colored sets**. In other words, its objects are sets where each element is assigned either the color blue or red. Morphisms are set-theoretic functions that maps blue elements to blue ones, and vice versa.
- Set/\( \emptyset \) contains only one object and one morphism.
- **Exercise**: Come up with one more example. Make it as interesting as you can.

Notice that given _any_ object in a category, we can make a slice category out of it. So suppose we have two objects and a morphism \( X \xrightarrow{f} Y \). What can we say of the two slice categories?

Here, \( u \in C/X \) and \( f \circ u \in C/Y \). Therefore, there is a map from the objects of \( C/X \) to the objects of \( C/Y \). The next question to ask, is whether the map is **functorial**. Here’s the relevant diagram. The verification is left as an exercise.

We give this functor a name: \( f_! : C/X \to C/Y \).
## 2 Pullbacks

The next thing we do requires more structure in the category $\mathcal{C}$. Let’s take three objects $B \xrightarrow{f} A \xleftarrow{g} C$. If there happens to exist $X$ together with arrows $B \xleftarrow{p'} X \xrightarrow{q'} C$ such that the square commutes, and additionally...

... For every given $Y$ with morphisms $p', q'$, there is a unique arrow $Y \rightarrow X$ such that the diagram commutes. In this case, we call $X$ a **pullback**.

What are pullbacks like? We need to find arrows $p, q$ that “reconcile” $f$ and $g$. In $\textbf{Set}$, the pullback is given by the set

$$\{(b, c) \mid f(b) = g(c)\},$$

equipped with the obvious projections $p, q$.

But there’s another way to look at it. Each point $a \in A$ determines a set $f^{-1}(a) = \{b \mid f(b) = a\}$, and similarly $g^{-1}(a)$. This is called the **preimage**. In this way, $B$ can be rewritten as a union of preimages:

$$B = \bigcup_{a \in A} f^{-1}(a).$$

**EXERCISE:** In this union, each set is disjoint from each other. Can you see why? Since they are disjoint, we can use $\coprod$ instead of $\bigcup$ to emphasize this (these two symbols have the same meaning except $\coprod$ implies disjointness).

Therefore, we may regard $B$ as a space composed of “fibers” $f^{-1}(a)$. For example, if $B = \mathbb{R}^2$, and $A = \mathbb{R}$, take

$$f(x, y) = x^2 + y^2.$$  

Then $B$ is divided into concentric circles $f^{-1}(r^2)$ of radius $r$ about the origin. Note that $f^{-1}(-1)$ is empty, meaning that the fiber that lies over $-1$ is $\emptyset$.

What does this have to do with pullbacks? Well, we can rewrite $X$ in this way:

$$X = \prod_{a \in A} f^{-1}(a) \times g^{-1}(a).$$

It is another fibered space, where each fiber is the **product** of the corresponding fibers in $B$ and $C$. From this perspective, we may call the pullback as **fibered product**, denoted $B \times_A C$. **EXERCISE:** Prove that $B \times_1 C \cong B \times C$ holds in any category with a terminal object.
The reader should be familiar with the fact that $A \times (-)$ is a functor. This is in accordance with the Haskell typeclass instance `Functor ((,) a)`. In fact, pullbacks, being called the fibered product, is also a functor. To verify this, we need a diagram:

![Diagram](image)

Here the little right-angle marks say that there are two pullback squares. We need to prove that there is an arrow $(\text{fmap } h) : Y \to X$. This follows directly from the universal property of pullbacks. Next, we need the functor law.

![Diagram](image)

The reader shall complete the argument using the given diagram.

Before we move on, let’s pause for a moment and ponder what we just proved. Note that to use $(\text{fmap } h)$, the large square $Y, C, D, A$ cannot be arbitrary: The lower edge has to be $D \xrightarrow{f \circ h} A$. So what is the “functor” that we’ve just found? What is its source and target categories? It turns out that $(-) \times_A C$ is actually a functor $C/A \to C/C$! The choice of these categories are important. Note that although in the notation $B \times_A C$, the two arrows $f, g$ doesn’t appear, they are the essential ingredients. **EXERCISE:** Give an example of two pullbacks $B \times_A C$ with different $g$, such that the results are not isomorphic.

Saying that the functor is in $C/A \to C/C$ instead of $C \to C$ adds the important information of the respective arrows into $A$. And this ensures that a morphism in $C/A$ always commutes with these arrows.

To emphasize the importance of the morphisms, we write $g^* : C/A \to C/C$ for the functor. Note that the functor goes in the opposite direction of $g : C \to A$. But this does not make $g^*$ a contravariant functor. As you have proved in the previous section, $g^*$ turns $h : M \to N$ into $(\text{fmap } g^*)h : M \times_A C \to N \times_A C$, which means it is covariant.

### 3 Adjoint Yoga

Anyway, we now have a functor $g_! : C/C \to C/A$ from the first section, and $g^* : C/A \to C/C$ from the second section. In category theory, whenever you
encounter this, make a bet that they are adjoint.

What is adjunction? There are two equivalent definitions that I find the most natural. The first one describes an adjoint pair as an almost inverse pair of functors.

**Definition 1.** Two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ are called adjoint if the following holds.

For each object $X \in \mathcal{D}$, there is a morphism $\epsilon_X : FGX \to X$, and similarly for each $Y \in \mathcal{C}$ a $\eta_Y : Y \to GFY$, satisfying the following conditions:

- The assignment of morphisms $\epsilon_X$ is natural. In other words, for a morphism $f : X_1 \to X_2$, we have $\text{fmap}_F \epsilon_G f : FGX_1 \to FGX_2$, this forms a square

$$
\begin{array}{ccc}
FGX_1 & \xrightarrow{\text{fmap}_F \epsilon_G} & FGX_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{f} & X_2
\end{array}
$$

The naturality condition states that all these squares commute. Similar conditions hold for $\eta$.

- $\epsilon, \eta$ settles the situation for composing two functors. In the case of three functors, we have two maps

$$
FGFX \xrightarrow{\epsilon_{FX}} FX.
$$

These should compose to get the identity on $FX$. Similar conditions hold for $GY$.

In this case, $F$ is called the left adjoint, and $G$ the right adjoint, denoted as $F \dashv G$.

I won’t linger too much on the concept of adjunction. But here’re two quick examples.

- $U : \text{Mon} \to \text{Set}$ is a functor that maps a monoid to its underlying set. And $F : \text{Set} \to \text{Mon}$ maps a set $X$ to the collection of lists $\text{List}(X)$, with list concatenation as monoid multiplication, and the empty list $[]$ as the neutral element. $F$ is left adjoint to $U$.

- Let $\Delta : \text{Set} \to \text{Set} \times \text{Set}$ be the diagonal functor, sending $X$ to $(X, X)$. The product functor $(-) \times (-) : \text{Set} \times \text{Set} \to \text{Set}$ is the right adjoint of $\Delta$.

The second definition is more catchy:

**Definition 2.** Two functors $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ are adjoint iff

$$
\text{Hom}(FX,Y) \cong \text{Hom}(X,GY)
$$

such that the isomorphism is natural in $X$ and $Y$. 

The reader shall verify that these two definitions are equivalent, and that the two examples given are indeed adjoints (using both definitions).

Now let’s turn back to our two functors $f_!, f^*$. We draw a diagram to compose them and see what happens. First look at $f_! f^* x$.

The lower half is in $C/A$, and the upper half in $C/C$. The two dashed arrows are $x$ and $g \circ x$ under the functor $f_! f^*$. They lie in $C/A$. Now notice the red arrows generated from the pullback. Composing them with each $x \in C/A$ gives a transformation from $x$ to $f_! f^* x$. This gives $\eta_x : x \to f_! f^* x$.

What about the naturality condition? EXERCISE: Argue that the square $\bullet, \bullet, N, M$ commutes, and explain why this proves the naturality condition for $\eta$.

Next, the reverse composition $f^* f_!$. It is slightly trickier:

Here we have $x \in C/B$. Therefore, there is a well-hidden commutative square:

... Which creates the unique morphism $!$, such that $p \circ ! = \text{id}$ and $q \circ ! = x$. Now recall that $q = f^* f_! x$. Therefore, composing with $!$ gives a natural transformation $\varepsilon_x : f^* f_! x \to x$. 
The naturality condition amounts to proving that the two dashed arrows form a commutative square. This follows immediately from the universal property of pullbacks.

If you find this dizzying, why not try the other definition?

You need to find a natural isomorphism between \( \{ g \mid y = f^* x \circ g \} \) and \( \{ g \mid g \circ x = f \circ y \} \). One direction is given by composition, and the other is given by the universal property of pullbacks.

## 4 Dependent Sum

It’s time to reveal the meaning of these constructions. Recall how we can regard a morphism \( p : E \to B \) as a fibered space

\[
E = \coprod_{x:B} p^{-1}(x).
\]

So in the slice category \( \mathcal{C}/B \), everything is fibered along \( B \). If we take the map \( ! : B \to 1 \), then it induces the functor \( \mathcal{C}/B \to \mathcal{C}/1 \), which takes a fibered space \( p : E \to B \) to \( E \to 1 \).

Although this looks trivial, looking from the perspective of fibered spaces, we get something different: \( p : E \to B \) describes \( E \) with fibers over \( B \). And the functor turns it into \( ! : E \to 1 \), where all the fibers are merged into one big component. This corresponds to the **dependent sum**:

\[
\sum_{x:B} p^{-1}(x).
\]

We can generalize this by replacing the terminal object with an arbitrary object \( A \), and the morphism \( ! : B \to 1 \) with an arbitrary morphism \( f : B \to A \),
whose induced functor $f_!$ takes a “fiberwise dependent sum”, i.e. for each $a \in A$, the fiber over $a$ is
\[ \sum_{x : B_a} p^{-1}(x), \]
where $B_a$ is the fiber of $B$ over $a$.

What, then, is the functor $f^*$? Similarly we first take $A = 1$, and let $f$ be the unique morphism $! : B \to 1$. The pullback functor takes $p' : E \to 1$ to $\pi_1 : B \times E \to B$ projecting to the first component.\(^1\) In the fibered space language, it creates a trivial fibered space where each fiber looks identical to $E$.

Now generalizing to arbitrary $f : B \to A$, the pullback functor takes $p' : E \to A$ to a morphism $E \times_A B \to B$. In the category $\text{Set}$, the fibers of the new space looks like
\[ p'^{-1}(f(b)) \]
for each $b \in B$. In effect, it changes the base space from $A$ to $B$. And thus it is named the base change functor.

## 5 Towards Dependent Product

The next goal is to characterize dependent products. Following our previous experiences, it should be a functor $f_* : \mathcal{C}/B \to \mathcal{C}/A$ for $f : B \to A$. Similar to the dependent sum functor, it should take a “fiberwise dependent product”:
\[ \prod_{x : B_a} p^{-1}(x), \]
where $p : E \to B$ is regarded as a fibered space over $B$. As usual, we should consider the easy case where $A = 1$, and we only need to construct
\[ \prod_{x : B} p^{-1}(x). \]

How should it be defined? $\prod_{x : B} M$, where $M$ does not depend on $x$, is exactly the function space $M^B$. This suggests that we can define the dependent product set $\prod_{x : B} p^{-1}(x)$ as a subset of the functions $B \to \prod_{x : B} p^{-1}(x)$. Of course, to be type-correct, it needs to map $b \in B$ to an element of $p^{-1}(b)$. This can be expressed as it being a right inverse of $p$. So to sum up, our quest is now to find right inverses $? \circ p = \text{id of } p$.

### Interlude: Exponentials

Actually, we not only need to find the right inverses. In $\text{Set}$, we need a set of right inverses, which means instead of a collection of morphisms we need a single

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\(^1\)Note that now $p' \in \mathcal{C}/A$ (and we are studying the special case $A = 1$), where in the last paragraph $p \in \mathcal{C}/B$. This is because the functor $f^*$ goes in the opposite direction of $f_!$, and we need $p'$ to be in the source category of the functor we are discussing.
object that stands for the set of right inverses. Before we tackle that, we shall look at how we can create a single object that stands for the set of functions — the exponential object.

How should a set of functions behave? Given sets $X, Y$, if we have a set of functions $E = Y^X$, then we should be able to evaluate the functions at a given point $x \in X$. This is called the evaluation functional

$$ev(-, -) : E \times X \to Y.$$ 

So we already have the first parts of the definition:

**Definition 3.** Given objects $X, Y$, an exponential object is defined as an object $E$ equipped with a morphism $ev : E \times X \to Y$, such that ...

Then, as accustomed with category theory, we need some universal property. Since $ev$ already describes how to form morphisms out of $E$, our universal property describes how to create morphisms into $E$:

**Definition (Continued).** ... if there is an object $S$ with a morphism $u : S \times X \to Y$, then there is a unique morphism $v : S \to E$

such that, if the dashed arrow in the triangle is filled with $v \times id$ (which is the Haskell **first** $v = v \ *** \ id$), then the diagram commutes.

This is basically describing lambda abstraction. Given a function $u$, we have $u(s, x) \in Y$, so we can form the function $v(s) = \lambda x. u(s, x)$. 3

The exponential construction creates a functor $(-)^X$. Also, in Haskell language, the `map f` instance of $(-)^X$ is exactly `(f .)`, the left compositions.

A brilliant insight of exponentials is that they are completely characterized by currying:

**Theorem 1.** There is a natural isomorphism

$$\text{Hom}(X \times Y, Z) \cong \text{Hom}(X, Z^Y).$$

In other words,

$$(-) \times Y \vdash (-)^Y.$$ 

2The “-al” part of the word “functional” is just something that stuck with mathematicians. It doesn’t really mean anything special.

3Note how we use “pointful” notation — notation involving elements $x \in X$ etc. — to give intuition of “point-free” definitions. In this article it is only a convenient device to describe rough feelings of certain definitions. But in fact, it can be made rigorous as the internal language of a topos, where we can freely write expressions like this, and be confident that they can be translated back into the category language.
The interested reader shall complete the proof. Next, we continue on our quest of right inverses. We of course want to express the identity morphism first:

\[
\begin{array}{c}
1 \times X \\
\downarrow \\
X^X \times X \xrightarrow{\text{ev}} X
\end{array}
\]

Here the dashed line is the unique morphism \(\text{id} \times \text{id}\), where \(\text{id} : 1 \to X^X\) picks out the identity function in the object \(X^X\).

Now that we have \(\text{id}\) as our equipment, consider this pullback, where \(f : Y \to X\):

\[
\begin{array}{c}
Z \xrightarrow{f} Y^X \\
\downarrow \\
1 \xrightarrow{\text{id}} X^X
\end{array}
\]

Returning to where we tangented off, the pullback \(Z\) is, in the category \(\text{Set}\), the set \(\{g \in Y^X \mid f \circ g = \text{id}\}\). (Recall that \(g \in Y^X\) means \(g\) is a function \(X \to Y\).) This captures exactly the right inverses of \(f\).

**Fiberwise juggling**

Putting the solution in use, since a fibered space \(E = \sum_{x:B} p^{-1}(x)\) is defined by a morphism \(p : E \to B\), we need to find the space of right inverses of \(p\), which should give the space of dependent products.

\[
\begin{array}{c}
Z \xrightarrow{f} E^B \\
\downarrow \\
1 \xrightarrow{\text{id}} B^B
\end{array}
\]

This \(Z\) (considered as a fibered space \(Z \to 1\)) is then what we sought for.

Now we can generalize from \(\mathcal{C}/1\) to arbitrary slice categories \(\mathcal{C}/A\). We are now given a morphism \(f : B \to A\), and we are supposed to construct a functor \(f_* : \mathcal{C}/B \to \mathcal{C}/A\). As before, let \(p : E \to B\) be an object of \(\mathcal{C}/B\). Thinking in \(\text{Set}\)-language, we should have a “fiberwise right inverse” \(p_a^{-1}(x)\), whose domain is the fiber \(B_a = f^{-1}(a)\) of \(B\) over \(a \in A\). Its codomain would naturally be \(E_a\), which is a fiber of \(E\) when considered as a fibered space \((f \circ p) : E \to A\). Each
fiber of the dependent product object \( f \cdot p \) should look like

\[
\prod_{x \in \mathbf{B}_a} p_a^{-1}(x).
\]

The fiberwise right inverse is easy enough to construct (note that we are still working in \( \mathbf{Set} \)). We just replace everything in the previous construction.

We have the fiberwise constructions ready. How can we “collect the fibers” to create a definition that does not refer to the “points” \( a \in A \)? It looks like we are stuck. Maybe it’s time to take a retrospect of what we’ve achieved.

6 The True Nature of Slice Categories

Concepts in category theory are like elephants. You may, through analogies, theorems, or practical applications, grasp a feeling of what those concepts are like. But in truth, these feelings are only describing a part of the elephant. So let me reveal yet another part, yet another blind man’s description of elephants:

Slice categories describe local, fiberwise constructs.

Let’s return again to the definition of a fibered product.

I have added another morphism \( h \), which does not change the definition since everything commutes in this diagram. But it brings an interesting change of perspective: \( h : Z \to A \), regarded as an object in \( \mathbf{C}/A \), is exactly the usual product of the objects \( f \) and \( g \)!

On second thought this is very natural: Everything in slice categories needs to respect fibers, i.e. given two fibered spaces \( B \to A \) and \( C \to A \), any morphisms between them must map anything in the fiber \( B_a \) over \( a \) to the fiber \( C_a \). Therefore, the categorical product of two fibered spaces should also be the fiberwise product. This immediately generalizes to any construction.
EXERCISE: Define the notion of fibered coproducts, and explain why it is the coproduct in the slice category. Also, explain why the “fiberwise terminal object” is exactly id : A → A.

One thing to keep in mind: When we are talking about the category \( \mathcal{C}/A \), the fibers are considered to be over A. So when we switch to a different category \( \mathcal{C}/B \), the spaces are now considered fibered over B. That’s essentially the content of the base change functor: it changes the base space of the fiber spaces.

Armed with new weapons, we can finally write down the definition of dependent products:

\[
\begin{array}{c}
(Z \to A) \quad \quad \quad \quad (E \overset{f \circ p}{\to} A)(B \overset{f}{\to} A) \\
\downarrow \quad \quad \quad \quad \downarrow p \\
(A \overset{id}{\to} A) \quad \quad \quad \quad (B \overset{f}{\to} A)(B \overset{f}{\to} A)
\end{array}
\]

Note that this commutative diagram is entirely in the slice category \( \mathcal{C}/A \), where each object are arrows in \( \mathcal{C} \). The exponential objects are also inside the slice category. This pullback gives a space Z → A.

According to our guess at the beginning of this section, we should denote Z → A as \( f^* \). But of course we need to verify the functoriality of this construction. But it should be clear, since everything used (exponentials, products and pullbacks) is functorial.

But there is an even more succinct description of all these: the dependent product functor is exactly the right adjoint of the base change functor \( f^* \). The proof is not hard, although the diagram involved is a bit messy if you insist on drawing everything in \( \mathcal{C} \) instead of the slice categories.

7 Locally Cartesian Closed

A cartesian closed category is a category where the terminal object, all binary products and all exponentials exist. A locally cartesian closed category is a category whose slice categories are all cartesian closed. Let’s unpack the definition and see what this means.

The terminal object in a slice category \( \mathcal{C}/A \) is exactly id : A → A. So it always exists in slice categories. A binary product in a slice category, as we have discussed, is exactly the fibered product, or pullback. Therefore, a locally cartesian closed category should have all pullbacks.

What about local exponentials? If there are two objects \( p : Y \to A \) and \( q : X \to A \), then the local exponential object \( p^q : E \to A \) should be defined by the following diagram:
... Well, this looks messy. Let’s try the adjoint functor definition of exponentials: The exponential functor \((-)^Y\) is the right adjoint of the product functor \((-) \times Y\). So in other words we should find a right adjoint to the pullback functor \((-) \times_A Y\). But hey! That looks like the dependent product functor in the last section. However, the acute reader may have noticed a discrepancy: Our dependent product functor is defined as a pullback of an exponential object. It can’t exactly be the exponential functor, can it? In fact they have different codomains: Given \(f : C \to A\), the dependent product functor \(f_* : \mathcal{C}/C \to \mathcal{C}/A\) is the adjoint of the base change functor \(f^* : \mathcal{C}/A \to \mathcal{C}/C\). But when we are looking for the exponential functor, the pullback functor we want is \((-) \times_A C : \mathcal{C}/A \to \mathcal{C}/A\).

Looking at the diagram for pullbacks we see why:

The functor \((-) \times_A C : \mathcal{C}/A \to \mathcal{C}/A\) sends \(g\) to \(h\), while the functor \(f^*\) sends \(g\) to \(u\). Since \(h = f \circ u\), you can see that the functor \((-) \times_A C\) is the composition of two functors \(f \circ\).

Now we can save a tremendous amount of work with this theorem:

**Theorem 2.** Given two adjoint pairs:

\[
\begin{array}{cccc}
\mathcal{C} & \overset{F_1}{\underset{G_1}{\leftrightarrow}} & \mathcal{D} & \overset{F_2}{\underset{G_2}{\leftrightarrow}} & \mathcal{E}
\end{array}
\]

The composition also forms an adjunction

\[F_2F_1 \dashv G_1G_2.\]

**Proof.**

\[\text{Hom}_\mathcal{E}(F_2F_1X,Y) \cong \text{Hom}_\mathcal{D}(F_1X,G_2Y) \cong \text{Hom}_\mathcal{C}(X,G_1G_2Y).\]
With this diagram it is crystal clear that the fibered exponential fits exactly in the position of the question mark.

In fact, the condition that the dependent sum functor \( f_! \) (which exists in every category) has a chain of three adjoints

\[
f_! \dashv f^* \dashv f_*,
\]

is equivalent to the condition that the category is locally cartesian closed. The backward implication is precisely what we proved in the last section. As for the forward implication, it is proved by our discussion in the previous few paragraphs.

8 Prospects

This introduction has gotten way too lengthy. But I shall point out several direction to proceed before I end.

Cartesian closed category, as can be seen in the definition, serves as the semantics of simply typed lambda calculus. You might not be able to figure out the details at once, but you should see that there is a probable connection here. On the other hand, locally cartesian closed categories are central to the semantic interpretation of dependent types. Type dependency is, fundamentally, expressing fiber spaces; working with dependent types amounts to making fiberwise constructions. The classical reference for this is [1].

Although I did not mention any topology in the text, fiber spaces ultimately came from topology. And it is the fact that there is a notion of “neighbourhoodness” between fibers that makes them important — otherwise they are just random sets.

Going further in this direction, the adjunction \( f^* \dashv f_* \) is called a geometric morphism in the language of topos. If it has further adjoints, it becomes “smoother” in the geometric sense. This plays the central role in topos theory. More can be read at [2].

References

[1] Seely, R. (1984). Locally cartesian closed categories and type theory. Mathematical Proceedings of the Cambridge Philosophical Society, 95(1), 33-48. doi:10.1017/S0305004100061284

[2] Johnstone, P.T. (2002). Sketches of an Elephant: A Topos Theory Compendium. Clarendon Press.