Self-gravitating spheres of anisotropic fluid in geodesic flow

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Abstract

The fluid models mentioned in the title are classified. All characteristics of the fluid are expressed through a master potential, satisfying an ordinary second order differential equation. Different constraints are imposed on this core of relations, finding new solutions and deriving the classical results for perfect fluids and dust as particular cases. Many uncharged and charged anisotropic solutions, all conformally flat and some uniform density solutions are found. A number of solutions with linear equation among the two pressures are derived, including the case of vanishing tangential pressure.

1 Introduction

The description of gravitational collapse and evolution of compact objects under various conditions remain among the important problems of general relativity. They are described by spherically symmetric relativistic fluid models where the metric depends on time and radius. In general, they possess shear, expansion and acceleration, which makes them rather hard to study. In some physical situations, however, perfect fluid (PF) models without acceleration or dust are a good approximation. The general dust solution, known also as the LTB solution [1], [2], [3], [4] is one of the most exploited examples.
It has three branches, which are characteristic for all other solutions. Much later the PF solution for the parabolic branch was found [5], [6]. It was shown [7] that the other two branches lead to the Emden-Fowler equation [8] and some exact solutions were proposed [9]. Solutions which obey an equation of state have constant pressure and density, which is equivalent to the presence of cosmological constant [10]. The case of charged dust has also attracted attention [11], [12], [13]. Such solutions cannot be shear-free [11]. Finally, the general solution was found in mass-area coordinates [14]. Recently, a charged geodesic PF solution with constant pressure was studied [15]. The LTB model has been generalized to include dissipative fluxes [16].

On the other hand, different mechanisms have been identified through the years which create pressure anisotropy in stellar models and make the fluid imperfect [17]. Such are the exotic phase transitions during gravitational collapse, the existence of a solid core or the presence of a type P superfluid. Viscosity may also be a source of local anisotropy as well as the slow rotation of a fluid. It has been shown that the sum of two PFs, two null fluids or a perfect and a null fluid may be represented by effective anisotropic fluid models [18]. Recently it was pointed out that the same is true for PF with charge, bulk and shear viscosity [19]. These arguments have stimulated the study of anisotropic fluids. Geodesic anisotropic spheres were discussed in non-comoving coordinates and a particular solution was given [20]. Lately the notion of Euclidean stars was introduced and their properties were investigated with or without radiation [21], [22]. In the first case the fluid motion is geodesic. The evolution of the shear in geodesic fluids was studied too [23]. Unfortunately, the extensive classification of Einstein solutions [4] includes only uncharged PF. Static charged PF solutions have been described elsewhere [24].

The aim of the present paper is to generalize the above results and give a classification of the geodesic anisotropic fluid solutions, similar to that of shear-free anisotropic fluid solutions [25]. For this purpose we use the reformulation of the Einstein equations in terms of the mass function [1], [26]. In this approach, besides the new solutions, the classical results reviewed above follow as particular cases.

In Sec.2 the field equations based on the mass function are given for non-radiating spheres in geodesic flow. The fundamental potential and the main equation that it satisfies are introduced. Expressions for the other characteristics of the metric are given. Sec.3 discusses the general and some concrete solutions of the geodesic anisotropic model. In Sec.4 the particular
cases of PF and dust are investigated, making contact with the existing literature. In Sec.5 the general charged solution and its subcases of charged PF and charged dust are discussed. Sec.6 studies the case when the radial and the tangential pressures are proportional. In Sec.7 the general conformally flat solution for geodesic anisotropic fluids is found. The uniform density solution is studied in Sec.8. In Sec.9 a comparison is made between shear-free and geodesic fluids. The final section contains some discussion.

2 Field equations for geodesic fluids

Spherically symmetric relativistic fluid models are described by the metric
\[ ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2), \] (1)
where \( \nu, \lambda \) and \( R \) are functions of \( t \) and \( r \) only. The spherical coordinates are numbered as \( x^0 = t, x^1 = r, x^2 = \theta, x^3 = \varphi \). The Einstein equations read
\[ R_{ij} - \frac{1}{2}g_{ij}R_{kk} = 8\pi T_{ij}. \] (2)
Here \( R_{ij} \) is the Ricci tensor, \( T_{ij} \) is the energy-momentum tensor and we have set \( G = c = 1 \). For an anisotropic fluid model without radiation one has \[ T_{ik} = (\rho + p_t)u_iu_k - p_tg_{ik} + (p_r - p_t)\chi_i\chi_k. \] (3)
Here \( \rho \) is the energy density, \( p_r \) is the radial pressure, \( p_t \) is the tangential pressure, \( u^i \) is the four-velocity of the fluid, \( \chi^i \) is a unit spacelike vector along the radial direction. These vectors satisfy the relations
\[ u^iu_i = -1, \quad \chi^i\chi_i = 1, \quad u^i\chi_i = 0. \] (4)
The coordinates are assumed to be comoving, hence
\[ u^i = e^{-\nu}\delta^i_0, \quad \chi^i = e^{-\lambda}\delta^i_1. \] (5)
The usual field equations (2) are rather cumbersome except for the (01) component
\[ \dot{R} - \dot{R}\nu' - R'\lambda = 0. \] (6)
The dot above means time derivative, while the prime denotes a radial one. It is easier to work in the mass function formalism \[ 1, 26 \], namely
\[ m' = 4\pi \rho R^2 R', \] (7)
\[ \dot{m} = -4\pi p_r R^2 \dot{R}, \quad (8) \]

\[ \dot{\lambda} (\rho + p_r) = -\dot{\rho} - \frac{2\dot{R}}{R} (\rho + p_r), \quad (9) \]

\[ \nu' (\rho + p_r) = -p'_r + \frac{2R'}{R} (p_t - p_r). \quad (10) \]

The mass function is given by

\[ m = \frac{1}{2} R \left( e^{-2\nu} \dot{R}^2 + k \right), \quad (11) \]

\[ k = 1 - e^{-2\lambda} R^2, \quad (12) \]

where \( k \) is the binding energy \cite{28}. Bound configurations have \( 0 < k < 1 \), unbound ones have \( k < 0 \) and the marginally bound case is \( k = 0 \). These three cases are called also elliptic, hyperbolic and parabolic respectively. The second Weyl invariant reads \cite{25}

\[ \Psi_2 = \left( \frac{m}{R^3} \right)' \frac{R}{3R'} + \frac{4\pi}{3} \triangle p, \quad \triangle p = p_t - p_r. \quad (13) \]

The expansion of the fluid is

\[ \Theta = e^{-\nu} \left( \dot{\lambda} + \frac{2\dot{R}}{R} \right), \quad (14) \]

while the components of the shear tensor are proportional to

\[ \sigma = \frac{1}{3} e^{-\nu} \left( \frac{\dot{R}}{R} - \ddot{\lambda} \right). \quad (15) \]

The four acceleration is \( \ddot{u}_i = (0, \nu', 0, 0) \).

Geodesic fluids have vanishing acceleration, hence \( \nu' = 0 \) and \( \nu \) is a function of time only. It can be set to zero by a coordinate transformation. If \( k = 0 \) Eqs (6) and (12) lead to geodesic flow. The contrary is not true. When \( \nu = 0 \) and \( R' \neq 0 \) the same equations yield \( k = k(r) \) and

\[ e^{2\lambda} = \frac{R'^2}{1 - k}. \quad (16) \]
It is not possible to make $k$ vanish by a transformation of $r$. The so called Euclidean star models \[21\] have $k = 0$. Eq (8) becomes with the help of Eq (11)

$$
\left( R \dot{R}^2 \right) = -8 \pi p_r R^2 \dot{R} - k \dot{R}.
$$

(17)

This is the main equation for geodesic fluids. It is second order and contains only time derivatives. The dependence on $r$ is viewed upon as parametric. One gets rid of the first derivatives by the substitution

$$
R = Z^{2/3},
$$

(18)

which results in

$$
\dot{Z} = -6 \pi p_r Z - \frac{3}{4} k Z^{-1/3}.
$$

(19)

We can find $Z$ when $p_r$ is given or vice versa. The second case is much simpler but it is not guaranteed that the resulting $p_r$ will be physically realistic. Thus the general solution depends on one arbitrary function of $t$ and $r$.

Eq (10) becomes

$$
p_t' = \frac{2 R'}{R} (p_t - p_r)
$$

(20)

and gives an expression for the tangential pressure

$$
p_t = \frac{\left( R^2 p_r \right)'}{\left( R^2 \right)'}.
$$

(21)

When $p_r = 0$, $p_t$ vanishes too and we get a dust solution, but the opposite is not true. The mass is found from Eq (11), which becomes

$$
m = \frac{1}{2} R \left( \dot{R}^2 + k \right) = \frac{2}{9} Z^2 + \frac{k}{2} Z^{2/3}.
$$

(22)

Then the energy density follows from Eq (7)

$$
24 \pi \rho = \frac{4 \left( Z^2 \right)'}{\left( Z^2 \right)'} + 9 \left( k Z^{2/3} \right)'.
$$

(23)

The expansion and the shear scalar are given by

$$
\Theta = \frac{(Z^2)'}{(Z^2)}',
$$

(24)
\[ \sigma = -\frac{(\ln Z)'^3}{3(\ln Z)^7} \]  

(25)

A geodesic solution is also shear-free when \( Z \) and consequently \( R \) is separable. It has no expansion when \( R \) is of the form

\[ R(t,r) = [R_1(t) + R_2(r)]^{1/3}. \]  

(26)

### 3 General and particular solutions

The main equation (19) may be simplified further by the transformation

\[ Z = \mu W, \quad \frac{dt}{d\tau} = \mu^2, \]  

(27)

where \( \mu(t,r) \) determines the new time variable \( \tau \). It becomes

\[ W_{\tau\tau} = - (\ddot{\mu} + 6\pi p_r \mu) \mu^3 W - \frac{3}{4} k\mu^{8/3}W^{-1/3}. \]  

(28)

We choose \( \mu \) so that the bracket vanishes and the above equation breaks into two

\[ 6\pi p_r = -\frac{\ddot{\mu}}{\mu} = \frac{2\mu_r^2 - \mu\mu_{\tau\tau}}{\mu^5}, \]  

(29)

\[ W_{\tau\tau} = -\frac{3}{4} k\mu^{8/3}W^{-1/3}. \]  

(30)

The general solution follows when one chooses arbitrary \( k(r) \), \( \mu(t,r) \) or \( p_r(t,r) \), and finds \( W(\tau,r) \) from the above equation. This determines consecutively \( Z(t,r) \), \( R(t,r) \) and all other characteristics of the fluid. Eq (30) for \( k \neq 0 \) resembles the Emden-Fowler (EF) equation

\[ W_{\tau\tau} = A\tau^n W^l \]  

(31)

with \( A = -\frac{3}{4} k \) and \( l = -1/3 \). In order to obtain analytical solutions we set \( \mu = \tau^{3n/8} \). Eq (27) yields for \( n \neq -4/3 \) and \( n = -4/3 \) respectively

\[ \tau^{\frac{3n+4}{4}} = \frac{3n+4}{4} (t-a), \quad \tau = e^{t-a}, \]  

(32)

where \( a(r) \) is an arbitrary function of integration. Eq (29) gives for \( n \neq -4/3 \) and \( n = -4/3 \) respectively

\[ 6\pi p_r = \frac{3n(3n+8)}{4(3n+4)^2(t-a)^2}, \quad 6\pi p_r = -\frac{1}{4}. \]  

(33)
There are general solutions of Eq (30) when \( n = 0 (\mu = 1) \) or \( n = -8/3 \ (\mu = 1/\tau) \) [8], but according to Eq (29) they are particular dust solutions, which are discussed in the next section. The case \( n = -4/3 (\mu = \tau^{-1/2}) \) is also soluble, but leads to constant pressures, representing effectively a cosmological constant.

There is, however, a particular two-parameter solution of Eq (31) for any \( n, l \neq 1 \)

\[ W = \alpha \tau^{\frac{n+2}{n+1}}, \quad \alpha = \left[ \frac{(n+2)(n+l+1)}{A(l-1)^2} \right]^{\frac{1}{l+2}}. \]  

\hspace{1cm} (34)

In our case \( l \) is fixed, while \( n \) may be a function of \( r \). We obtain

\[ R = \alpha_1 (t - a), \quad \alpha_1^2 = -\frac{(3n+4)^2 k}{4(n+2)(3n+2)}. \]  

\hspace{1cm} (35)

The solution depends on three arbitrary functions of the radial coordinate \( k, a, n \). When \( a = 0 \), \( R \) is separable, hence, the solution is shear-free in addition to being geodesic.

No other explicit solutions exist when \( \mu \) is a power of \( \tau \). However, there is one with \( \mu = (ar^2 + br + c)^{-1/2} \) where \( a, b, c \) are functions of \( r \). It leads to time-independent radial pressure \( p_r(r) \). The pressure profile is given by the initial conditions and may be arbitrary. This case is better approached from Eq (19), which is integrated simply by multiplication with \( \dot{Z} \). We get the integral

\[ \int \frac{dZ}{\sqrt{-6\pi p_R Z^2 - \frac{9}{4}kZ^{2/3} + h}} = t + s, \]  

\hspace{1cm} (36)

where \( h(r), s(r) \) are arbitrary integration functions. The first is related to the mass functions through Eq (22)

\[ m = -\frac{4}{3\pi p_r R^2} + \frac{2}{9}h. \]  

\hspace{1cm} (37)

The integral in Eq (36) has an analytical expression when any of the functions \( p_r, k, h \) vanishes. Thus, when \( h = 0 \) we resort back to \( R \) to find

\[ R = \left( \frac{-3k}{8\pi p_r} \right)^{1/2} \sin \frac{2}{3} \sqrt{6\pi p_r} (t + s), \]  

\hspace{1cm} (38)
provided \( p_r > 0 \) and \( k < 0 \). When \( k = 0 \) we obtain in a similar way

\[
R^{3/2} = \sqrt{\frac{h}{6\pi p_r}} \sin \sqrt{6\pi p_r} (t + s). \tag{39}
\]

This is a concrete example of an Euclidean star. Finally, when \( p_r = 0 \) a dust solution follows, to be discussed in the next section.

Up to now we have assumed that \( k \neq 0 \) in Eq (30). The case \( k = 0 \) gives immediately

\[
W = a\tau + b, \quad \tau = \int \mu^{-2} dt, \tag{40}
\]

\[
R = \mu^{2/3} \left( a \int \mu^{-2} dt + b \right)^{2/3}, \tag{41}
\]

where \( a(r), b(r) \) and \( \mu(t, r) \) are arbitrary. The radial pressure is found from Eq (29). When \( b = 0 \), \( R \) becomes separable and the solution is shear-free too.

\section{Perfect fluid and dust}

Perfect fluid may be considered as anisotropic fluid with the equation of state (isotropy condition) \( p_r = p_t \equiv p \). In this case Eq (20) shows that \( p = p(t) \). It is enough to take a \( \mu(t) \) in Eq (29). In fact, the transformation given by Eq (27) was applied first to PF \[7\], \[9\]. The solution (36) was also proposed \[9\], without noticing that it holds for \( p(r) \) and contradicts the PF condition.

The concrete solution for \( k \neq 0 \) described by Eqs (33, 35) holds for PF too, provided \( a, n \) are constant. The general solution in the case \( k = 0 \) is given again by Eq (41), but with \( \mu \) uniform in space \[5\], \[6\]. Szafron \[29\] has found the particular solution \( \mu = t^q \) with \( q \) any real number.

When the pressures vanish, one has collapsing dust. Eq (10) shows that its motion is geodesic (without acceleration). According to Eq (8) the mass function is time-independent. Its profile characterizes the solution. The main equation (19) may be integrated once and becomes Eq (22). A second integration leads to Eq (36) with \( p_r = 0 \) and \( h = 9m/2 \), in accord with Eq (37). This is the well known LTB solution \[11\], \[2\], \[3\]. Eq (22) coincides with Eq (73) from Ref. \[16\], where one can find the analytic expressions for \( R \).
in the parabolic, hyperbolic and elliptic case. For example, when \( k = 0 \) one easily finds
\[
R = \left(3 \sqrt[2/3]{\frac{m}{2} t + b} \right),
\]
(42)
where \( m(r), b(r) \) being arbitrary functions. The solution is also shear-free when \( b = 0 \).

5 Charged fluid

Spherical symmetry allows the appearance of only an electric field \( E \) in the radial direction. The energy-momentum tensor of this field can be described as addition of the following effective energy density and pressures to the fluid
\[
\rho^E = p_t^E = -p_r^E = \frac{E^2}{8\pi}.
\]
(43)
The Maxwell equations give
\[
\frac{E^2}{8\pi} = \frac{q(r)}{R^4}, \quad 4\pi C = E e^{-\lambda},
\]
(44)
where \( C \) is the charge function of the fluid and \( q(r) \) is an arbitrary function.

The main equation in its two forms (19) and (30) becomes
\[
\ddot{Z} = -6\pi p_r Z - \frac{3}{4} k Z^{-1/3} + 6\pi q Z^{-5/3},
\]
(45)
\[
W_{\tau\tau} = -\frac{3}{4} k \mu^{8/3} W^{-1/3} + 6\pi q \mu^{4/3} W^{-5/3},
\]
(46)
while Eqs (27) and (29) do not change. Eq (21) transforms into
\[
p_t = \left(\frac{R^2 p_r}{R^2}' \right) - \frac{q'}{2R^3 R'}
\]
(47)
Now when \( p_r \) vanishes \( p_t \) is not obliged to vanish, giving a class of non-dust solutions.

For \( k \neq 0 \) Eq (46) becomes the modified EF equation
\[
W_{\tau\tau} = A_1 \tau^{n_1} W^{m_1} + A_2 \tau^{n_2} W^{m_2}
\]
(48)
when $\mu = t^{3n_1/8}$. Then

$$m_1 = -\frac{1}{3}, \quad m_2 = -\frac{5}{3}, \quad n_1 = 2n_2, \quad A_1 = -\frac{3}{4}k, \quad A_2 = 6\pi q. \quad (49)$$

There is a table of 108 explicit solutions of Eq (48) [8]. Those which satisfy the conditions above have $n_1 = 0; -\frac{8}{3}; -\frac{4}{3}$. The first two lead to vanishing $p_r$, the third has it constant. They are non-trivial and non-dust. They can be approached also as particular cases of the class of solutions with $p_r(r)$. Like for uncharged fluid one integrates directly Eq (45) to find

$$\frac{3}{2} \int \frac{RdR}{\sqrt{-6\pi p_r R^4 - \frac{2}{3}kR^2 + hR - 16\pi q}} = t + s(r). \quad (50)$$

Eq (50) coincides with Eq (36) if $q = 0$. When $h$ or $p_r$ vanishes, the integral is expressed in terms of elementary functions in five different ways depending on the relations between $k, q$ and $p_r$ or $h$.

For $k = 0$, $A_1 = 0$ and the first term in Eqs (46, 48) disappears. We get once more the usual EF equation like Eqs (30, 31) in the uncharged case, but the power of $W$ is different. This change brings a host of explicit solutions with $n_2 = 0; -\frac{4}{3}; -\frac{2}{3}; -\frac{10}{3}; -\frac{7}{3}; -\frac{5}{3}; -\frac{3}{3}; 1; 2$. The first two have vanishing radial pressure. The particular solution (34) also holds provided $l = -5/3$.

Charged PF has $p_r = p_t \equiv p$. Eq (47) becomes

$$p' = \frac{q'}{R^4}. \quad (51)$$

This is another relation between $p$ and $R$ in addition to Eq (45). A possible solution is $p(t)$ (like in the uncharged case) and $q = const$. Then $\mu = \mu(t)$ and $a, n$ should be constant in the above solutions. The isotropic pressure is given by Eq (33).

When $p' \neq 0$ one should replace $R$ from Eq (51) into Eq (46) having in mind that

$$W = \mu^{-1} R^{3/2}. \quad (52)$$

Even in the simplest case $k = 0$ this leads to contradiction, because the result is $n = -4/3$ and hence, $p = const$.

Charged dust may be considered as a subclass of the PF solutions with $p = 0$. The general solution is known, though in another coordinate system [14]. However, Eq (10) does not lead now to $\nu' = 0$ because of the electromagnetic...
additions to the energy density and the pressures given by Eq (43). We have to impose this condition in order to make the fluid flow geodesic and the solutions found here form a subclass of the general Ori solution. When \( p = 0 \) Eq (51) yields \( q (r) = q_0 = \text{const} \). It is not necessary to introduce \( W \) and \( \mu \).

Eq (45) becomes

\[
\ddot{Z} = -\frac{3}{4} kZ^{-1/3} + 6\pi q_0 Z^{-5/3}.
\]

(53)

This is the modified EF equation (48, 49) with \( n_1 = n_2 = 0 \) and \( A_2 = 6\pi q_0 \). Its solution is given by example 2.4.2.56 from [8] and has three branches.

When \( k < 0 \)

\[
t = C_1 e^{\omega \lambda} + C_2 e^{-\omega \lambda} + C_3 \lambda, \quad R = \omega \left( C_1 e^{\omega \lambda} - C_2 e^{-\omega \lambda} \right) + C_3,
\]

(54)

where \( \lambda \) is a parameter, \( \omega = \sqrt{|k|} \) and the integration functions \( C_i (r) \) satisfy the relation

\[
3 \left( A_1 C_2^2 + A_2 \right) + 16 A_1^2 C_1 C_2 = 0.
\]

(55)

When \( k > 0 \)

\[
t = C_1 \sin \omega \lambda + C_2 \cos \omega \lambda + C_3 \lambda, \quad R = \omega \left( C_1 \cos \omega \lambda - C_2 \sin \omega \lambda \right) + C_3,
\]

(56)

\[
3 \left( A_1 C_3^2 + A_2 \right) + 4 A_1^2 \left( C_1^2 + C_2^2 \right) = 0.
\]

(57)

Finally, when \( k = 0 \) Eq (53) may be integrated by multiplying with \( \dot{Z} \). The result is

\[
\int \frac{RdR}{Y} = \frac{2}{3} t + C_2,
\]

(58)

where

\[
Y = (C_1 R - 18\pi q_0)^{1/2}.
\]

(59)

Eq (58) becomes a cubic equation for \( Y \)

\[
Y^3 + 54\pi q_0 Y = C_1^2 \left( t + \frac{3}{2} C_2 \right).
\]

(60)

It may be solved for \( Y \) and \( R \). The solutions are not separable and therefore are not shear-free in accord with Ref. [11].
6 Solutions with $p_t = \gamma p_r$

This equation of state among the pressures includes the cases of perfect fluid and $p_t = 0$. Eq (21), which in principle determines the tangential pressure, now gives an expression for $p_r$ in terms of $Z$

$$p_r = h(t) Z^{\frac{4}{3} (\gamma - 1)},$$

(61)

where $h$ is an arbitrary positive function. The main equation (19) becomes

$$\ddot{Z} = -\frac{3}{4} k(r) Z^{-1/3} - 6\pi h(t) Z^{\frac{4\gamma - 1}{3}}. \tag{62}$$

This represents the modified EF equation (48) with

$$m_1 = -1/3, \quad m_2 = \frac{4\gamma - 1}{3}, \quad n_1 = 0, \quad A_1 = -\frac{3}{4} k, \quad A_2 = -6\pi, \tag{63}$$

provided $h = t^{m_2}$. The following explicit solutions exist:

When $k \neq 0$ we may take $n_2 = 0, m_2 = -5/3$ ($h = h_0, \gamma = -1$); then Eq (62) coincides in form with Eq (53) and the solution (54-57) holds. Other possibilities with the same $m_2$ are $n_2 = 1$ and $n_2 = 2$ which lead to examples 2.4.2.58 and 2.4.2.54 respectively [8]. The case $p_t = 0$ ($\gamma = 0$) reduces Eq (62) to

$$\ddot{Z} = -\left(\frac{3}{4} k + 6\pi h\right) Z^{-1/3}. \tag{64}$$

It is easily integrable for constant $h = h_0$ when it coincides in form with the LTB solution.

When $k = 0$ Eq (62) becomes the usual EF equation with $m_1 = \frac{4\gamma - 1}{3}$ provided $h = t^{m_1}$. One can choose from the 28 analytic solutions (5 one-parameter families and 23 isolated points) [8] and the particular solution (34).

Eqs (21, 61) show that when $\gamma < 1$ (including the case of vanishing tangential pressure) the radial pressure becomes infinite at $R = 0$, adding another singularity to the curvature singularity of the LTB solution. This is true provided there is no bounce of the collapsing fluid and the point $R = 0$ is covered by it after some time. A similar situation is described in Ref. [28].
7 Conformally flat solutions

These solutions have $\Psi_2 = 0$. Combining Eqs (13, 20) yields for geodesic fluids

$$\Psi_2 = \left(\frac{m}{R^3} + 2\pi p_r\right) R \frac{\dot{R}}{3R'} \quad (65)$$

The vanishing of the second Weyl invariant requires

$$\frac{m}{R^3} + 2\pi p_r = f(t), \quad (66)$$

where $f$ is an arbitrary positive function since $m$ and $p_r$ are positive for realistic star models. Eq (8) transforms into

$$\dot{X} = -\frac{\dot{R}}{R} X \equiv f - 2\pi p_r. \quad (67)$$

Inserting Eq (22) into Eq (66) results in an expression for $p_r$

$$2\pi p_r = f - \frac{\dot{R}^2 + k}{2R^2}. \quad (68)$$

We combine this equation with Eq (66) to find

$$2R\ddot{R} - \dot{R}^2 = -4f R^2 + k, \quad (69)$$

which is analogous to Eq (17). The first derivative is eliminated by the substitution $R = Y^2$

$$\ddot{Y} = -fY + \frac{k}{4}Y^{-3}. \quad (70)$$

This is the main equation in the conformally flat case, replacing Eq (19). It has no reference to the pressure, although $f$ enters it in a similar way. Once again we use the transformation $Y = \mu W$ which splits Eq (70) into two parts

$$\ddot{\mu} + f \mu = 0, \quad W_{\tau\tau} = \frac{k}{4}W^{-3}, \quad (71)$$

where $\mu = \mu(t)$. Once again we obtain a EF equation. This time it is easily integrated by multiplying with $W_{\tau}$. We get

$$R = b\mu^2 \left(\frac{1}{4}V^2 + \frac{k}{b^2}\right), \quad V \equiv \int \mu^{-2}dt + c. \quad (72)$$
Here \( b(r), c(r) \) are arbitrary integration functions and \( \mu^2 \) is an arbitrary positive function of time. This formula is valid for any \( k \). In the marginal case \( f = 0 \) (which requires some radial tension) it simplifies to

\[
R = \frac{b}{4} (t + c)^2 + \frac{k}{b}. \tag{73}
\]

The general conformally flat geodesic solution is also shear-free provided \( c = 0 = k \).

Let us find now the conformally flat PF solution. Eq (20) shows that \( p = p(t) \). Eq (68) becomes

\[
\frac{\dot{R}^2 + k}{2R^2} = X(t). \tag{74}
\]

Eq (22) then gives for the mass function

\[
m = \frac{1}{2} X(t) R^3 \tag{75}
\]

and Eq (7) shows that the energy density is uniform, \( \rho = \rho(t) \). Putting the general solution (72) into Eq (74) yields

\[
\frac{1 + 2\mu \dot{\mu} V}{\frac{1}{4} V^2 + \frac{k}{b^2}} = X_1(t). \tag{76}
\]

The l.h.s. should depend only on time, which is ensured by setting to constants \( c = c_0 \) and \( k/b^2 = c_1 \). \( R \) becomes

\[
R = b \mu^2 \left[ \frac{1}{4} \left( \int \mu^{-2} dt + c_0 \right)^2 + c_1 \right]. \tag{77}
\]

The solution is shear-free because \( R \) is separable.

The conformally flat dust solution is a subcase of the PF solution with vanishing pressure. Eqs (67, 68) become

\[
\frac{j}{3f} = -\frac{\dot{R}}{R}, \quad \dot{R}^2 + k = 2fR^2. \tag{78}
\]

They lead to

\[
R = g(r) f(t)^{-1/3}, \tag{79}
\]
where $g(r)$ is arbitrary, but positive. Obviously $k/g^2 = c_2 = \text{const.}$ The equation is integrated by the substitution $y = f^{1/3}$ and the final result is

$$R = \left( \frac{3}{\sqrt{2}} \right)^{2/3} g(r) (t + c_3)^{2/3},$$

where $c_3$ is some constant. This formula resembles the $k = 0$ case of the LTB solution given by Eq (42). Like the PF solution (77), the dust solution is shear-free.

### 8 Uniform density

The energy density in this case is a function of time, $\rho(t)$. Eq (7) gives upon integration

$$m = \frac{4\pi}{3} \rho R^3,$$

where an integration function has been set to zero[30]. Plugging this formula in Eq (22) yields

$$\frac{8\pi}{3} \rho = \frac{\dot{R}^2 + k}{R^2}. \quad (83)$$

With the help of Eq (8) it is transformed into

$$3k = R^2 \left[ 8\pi \rho - \frac{\dot{\rho}^2}{3 (\rho + p_r)^2} \right]. \quad (84)$$

If we put Eq (82) into Eq (8) an expression for the radial pressure is obtained

$$-4\pi p_r = \frac{\ddot{R}}{R} + \frac{4\pi}{3} \rho. \quad (85)$$

Let us discuss first the case $k = 0$. Eq (84) defines $p_r$ in terms of the arbitrary $\rho(t)$

$$p_r = \pm \frac{\dot{\rho}}{\sqrt{24\pi \rho}} - \rho. \quad (86)$$
It shows that $p_r = p_r(t)$ and the fluid is perfect. Eq (83) defines $R$

$$R = a(r) \exp \left( \pm \sqrt{\frac{8\pi}{3}} \int \sqrt{\rho} dt \right)$$

(87)

with integration function $a(r)$. The solution is shear-free.

In the dust subcase Eq (86) becomes an equation for $\rho$. Its solution is

$$\rho = \frac{1}{6\pi (t + t_0)^2},$$

(88)

where $t_0$ is an integration constant. Then $R$ simplifies

$$R = a(r) (t + t_0)^{2/3}.$$  

(89)

It coincides in form with the conformally flat dust solution given by Eq (81). However the function $g(r)$ there depends on $k$ and it holds for any $k$.

Let us consider now the case $k \neq 0$. We set

$$R(t, r) = \sqrt{|k(r)|} P(t, r).$$

(90)

Eq (83) becomes

$$\dot{P}^2 = \frac{8\pi}{3} \rho P^2 - \varepsilon, \quad \varepsilon \equiv \text{sign} k.$$  

(91)

This equation is not integrable in general and separable solutions are not possible. For PF, however, Eqs (84, 90) show that $P = P(t)$ and we may take it as an arbitrary function which determines $\rho$ and $R$. The pressure is found from Eq (85). In the dust subcase $m = m(r)$ and Eq (82) becomes

$$\rho P^3 = \frac{3m}{4\pi |k|^{3/2}}.$$  

(92)

The l.h.s. depends on $t$, while the r.h.s. depends on $r$, hence, both of them are constant, $3c_0/8\pi$. Then Eq (91) reads

$$\dot{P}^2 = \frac{c_0 - \varepsilon P}{P}$$  

(93)

and may be integrated, giving a rather complicated inexplicit expression for $P$

$$-\sqrt{P(c_0 - \varepsilon P)} + \frac{1}{2} \ln \left| \frac{P}{P - \varepsilon c_0} \right| = \varepsilon (t + t_0).$$

(94)

It is not possible to recover the $k = 0$ case by setting $\varepsilon = 0$. 

16
9 Comparison with shear-free anisotropic fluid

The structure of the present classification of geodesic anisotropic fluid spheres is analogous to that of shear-free anisotropic fluid spheres \[25\], denoted further as I. In both cases the (01) component of the Einstein equations may be integrated. As a result all metric components depend on \(R\) plus arbitrary functions of one variable like \(\Theta(t)\) in I and \(k(r)\) here. The latter function divides any solution into two or three branches, namely \(k = 0\) and \(k \neq 0\) \((k > 0\) and \(k < 0\) in some cases).

In both cases the fundamental ingredient is a second order differential equation for \(R\) in only one of the variables \(t\) or \(r\). It is based on the expression for the mass function, which luckily contains derivatives of only one kind. For shear-free fluids this is Eq (I 34) with only first order derivatives in \(r\). It leads to the main Eq (I 36), which is of second order. For geodesic fluids the mass function is given by Eq (22), containing only a time derivative. It leads to the main Eq (19), which is of second order in time derivatives. All other characteristics of the fluid are expressed through the solutions of the main equation.

Shear, expansion and acceleration are usually expressed through the metric. An essential role in the formalism plays a second formula for the vanishing characteristic, which holds for any fluid sphere. It elevates its dependence from metric components to sources (pressures and energy density). In I it is given by Eq (I 29) for the shear, whose derivation required the mass decomposition formula (I 15). Here this is Eq (10) for the four-acceleration, which enters the usual set of Einstein equations, based on the mass function. It is not necessary to use the mass decomposition formula except for the definition of \(\Psi_2\) given by Eq (13).

The main equation contains no first derivatives and just one non-derivative term, when one passes from \(R\) to \(L = r/R\) in I or to \(Z = R^{3/2}\) here. For geodesic fluids a second step is necessary, namely the transformation in Eq (27). The general solution of the main solution is based on one arbitrary function. In I this may be \(R, L, Z, \Delta p\) or \(\Psi_2\). Here these are \(R, Z\) or \(p_r\). When \(R\) is chosen, solutions are found quite easily, however, they may be unphysical and the passage to PF is rather involved. Therefore we take \(Z\) or \(\Delta p\) as arbitrary functions in I or \(p_r\) here and solve the main equation. Analytic solutions of it are found by reduction to an EF equation. This happens when an arbitrary function is taken as a power of \(r^2\) in I or \(\tau\) in the present paper.
The passage to PF is controlled by $\Delta p$ in I (it enters directly Eq (I 36)) and by $p'_r = 0$ here, which makes the radial pressure dependent on time only. For geodesic fluids the additional case of dust exists. It is treated as a subcase of PF, when the pressure vanishes.

Charged fluid has additions to the pressures and density which bring a second term to the main equation. It becomes the modified EF equation and possesses a number of one-parameter or isolated solutions.

There is one easily integrable charged case when some arbitrary functions are set to constants for shear-free fluids or $p_r = p_r (r)$ for geodesic fluids. The corresponding integral (I 52) leads to the Weierstrass elliptic function in I and to Eq (50) here, which has analytic expression in some cases. In I an equation of state for PF emerges from this case. Here the necessary condition for this is constant pressure, which represents a cosmological constant in accord with Ref. [10] and is rather trivial. The case of charged dust is completely integrable.

The general conformally flat solution is found in both papers and is given by Eq (I 64) and Eq (72) respectively. The solution for PF follows when an arbitrary function depending on $r$ is put to zero in I, or two such functions are made constant for geodesic fluids. Conformally flat dust is obtained by further simplification. Conformally flat PF solutions intertwine with uniform density solutions.

The general uniform density solution is found in I, while here this is done in the $k = 0$ case for anisotropic fluid and in all cases for PF and dust.

An important case is to impose a linear equation of state $p_t = \gamma p_r$ between the two pressures, which is a generalization of the isotropy condition $\gamma = 1$, leading to PF. In both cases there are analytic solutions for $R$ at some discrete values of $\gamma$, including the case of vanishing tangential pressure $\gamma = 0$.

A geodesic solution may be shear-free too. This happens when $R$ is separable. Many of the above solutions become separable when the integration functions are chosen appropriately. However, we were unable to find solutions that are both geodesic and expansion-free, i.e. satisfy Eq (26), except for the trivial case $R (r)$.

As a final remark, the EF equation or its modified version appear frequently in both the shear-free and the acceleration-free case. Integrals which lead to the Weierstrass function are as frequent in I but here they do not appear at all.
10 Discussion

Recently we have stressed the importance of anisotropic fluid models for astrophysics [19]. The fluid flow has three important characteristics - shear, acceleration and expansion. It is hard to obtain general solutions when all three are non-trivial. We have classified in another paper the shear-free anisotropic spheres [25]. In the present paper the same is done for geodesic flows, where the acceleration vanishes.

Taking the fluid anisotropic means accepting the maximum freedom allowed by spherical symmetry. The general solution depends on an arbitrary function and a core of simple relations is obtained. Only after that we start to impose different constraints aiming the system at particular cases.

Thus, PF is this core with imposed isotropy condition \( p_r = p_t \), charged fluid is a neutral fluid with a special kind of anisotropy, conformal flatness means the constraint \( \Psi_2 = 0 \), uniform density constrains the energy density to a function of time, anisotropy may be studied in more detail by a linear equation among the pressures, which includes the cases where one of them vanishes. In this way the relations among the particular cases become much clearer due to the relations of all of them to the core. Time and again the second order Emden-Fowler equations appear, which sometimes may be integrated to first order and even to implicit integral formulas for the solution, like Eqs (36, 50, 58). In many cases they are expressed through elementary functions.

Imposing one constraint fixes the arbitrary potential of the general model but up to functions of time or radius. Imposing a second constraint makes the system overdetermined, yet special solutions still exist when the arbitrary functions become constants. In total, we find many uncharged and charged anisotropic solutions, all conformally flat solutions, a large class of solutions with proportional or vanishing pressures and some uniform density solutions. The general geodesic dust solutions are found explicitly in all cases, building upon the LTB solution.

The present classification is based on the mass function formalism, which is applicable only to metrics varying with time. Therefore the static case should be studied on the base of the usual Einstein equations (2) which simplify a lot in this case.

Much work remains to be done. One should classify as a next step the expansion-free anisotropic spheres. Physically realistic solutions for collapsing star models should be distinguished from the above classes, cases and
subcases with numerous arbitrary functions and constants of integration. Junction conditions give additional constraints. It seems possible to generalize the above constructions also to radiating anisotropic spheres, which have attracted a lot of interest recently.

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