PRESENTATIONS OF BRAID GROUPS OF TYPE A
ARISING FROM \((m + 2)\)-ANGULATIONS OF REGULAR POLYGONS

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Abstract. We describe presentations of braid groups of type A arising from coloured quivers of mutation type A. We show that these can be interpreted geometrically as generalised triangulations of regular polygons.

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1. Introduction

The concept of \((m + 2)\)-angulation of a regular \((nm + 2)\)-sided polygon was studied in [1] by Baur and Marsh in 2006 to give a geometric description of the \(m\)-cluster category of type \(A_{n-1}\). A few years later, Buan and Thomas introduced \(m\)-coloured quivers and \(m\)-coloured quiver mutations in [2], adapting the classical concept of quiver mutation given by Fomin and Zelevinsky in [3] to the setting of higher cluster categories.

Classically, the concepts of quiver of mutation type \(A_{n-1}\) and triangulation of a regular \((n + 2)\) polygon coincide (see [4]). In Section 2 of [7], the authors associate a group to any quiver of mutation type \(A_{n-1}\), that they show being isomorphic to the braid group of type \(A_{n-1}\). They give an interpretation of such presentations, that already appeared in [8], in terms of triangulations of a regular \((n + 2)\)-gon. This construction allows them to interpret the aforementioned presentations from a cluster algebra point of view.

This paper generalises the result of [7] just described, using the concepts introduced in [1] and [2]. We propose a description of some presentations given in [8] from an higher cluster category perspective. In particular, we describe presentations of the braid group of type \(A_{n-1}\) arising from coloured quiver mutation. The main theorem we will prove is the following.

**Theorem 1.1.** Fix two integers \(m, n \geq 1\). Let \(Q\) be an \(m\)-coloured quiver of mutation type \(A_{n-1}\), with vertices \(1, \ldots, n - 1\). If \(i \rightarrow j \in Q_1\), let \(c_{ij}\) be its colour. Define the group \(B_Q\) to be generated by \(s_1, \ldots, s_{n-1}\) subject to the following relations.

1. \(s_i s_j = s_j s_i\) if there is no arrow between \(i\) and \(j\) (in either direction);
2. \(s_i s_j s_i = s_j s_i s_j\) if there is a pair of arrows \(i \rightarrow j \leftarrow i\).
is a subquiver of $Q$, and $c_{i_1,i_2} + c_{i_2,i_3} + c_{i_3,i_1} = 2m + 1$.

Then $B_Q$ is isomorphic to the braid group of type $\tilde{A}_{n-1}$.

We prove this result by giving explicit group isomorphisms between $B_Q$ and $B_{\mu_k(Q)}$, the group associated to the $m$-coloured quiver mutation of $Q$ at some vertex $k$. The proof relies on the geometric description of $m$-coloured quivers of mutation type $\tilde{A}_{n-1}$ as $(m+2)$-angulations of a regular $(nm+2)$-sided polygon which was proved in [9]. However, an explicit proof that the quiver associated to an $(m+2)$-angulation of a regular polygon is always of mutation type $\tilde{A}_{n-1}$ is not given in [9]. Therefore, we will go through the proof of this fact, and then tackle Theorem 1.1.

The relations in Theorem 1.1 are the same given by Grant and Marsh in [7] for uncoloured quivers of mutation type $A$. However, in the uncoloured setting, the only possible full subquivers of a quiver $Q$ of mutation type $A$ are linear quivers and 3-cycles. In the coloured setting, if we fix an integer $m \geq 2$ and choose $n \in \mathbb{N}$ big enough, we can have as full subquivers of an $m$-coloured quiver $Q$ of mutation type $\tilde{A}_{n-1}$ as many $k$-cycles as we want, for $k = 2, \ldots, m+2$. We will show that relations involving $k$-cycles also hold for all $k$, and follow from the ones in Theorem 1.1.

The pictures in this paper have been done using the interactive mathematics software Geogebra [5].

2. Coloured quivers

Throughout the paper, $m, n \geq 1$ will be fixed integers. The following definitions can be found in [2].

Definition 2.1. An $m$-coloured quiver $Q$ consists of vertices $1, \ldots, n$ and coloured arrows $i \xrightarrow{c} j$, where $c \in \{0, \ldots, m\}$.

Let $q_{ij}^{(c)}$ denote the number of arrows from $i$ to $j$ of colour $c$. We consider $m$-coloured quivers satisfying the following conditions.

(I) No loops: $q_{ii}^{(c)} = 0$ for all $c = 0, \ldots, m$.

(II) Monochromaticity: $q_{ij}^{(c)} \neq 0$ implies $q_{ij}^{(c')} = 0$ for all $c' \neq c$.

(III) Skew-symmetry: $q_{ij}^{(c)} = q_{ji}^{(m-c)}$ for all $c = 0, \ldots, m$.

Remark 2.2. A standard quiver (i.e., a quiver without colours on the arrows) shall be interpreted as a 1-coloured quiver whose arrows have colour 1. For example, a quiver of type $A_3$ with the usual orientation of the arrows
will be interpreted as the following 1-coloured quiver.

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow 3 & \sim & & 1 & \stackrel{(1)}{\rightarrow} 2 & \stackrel{(1)}{\rightarrow} 3
\end{align*}
\]

**Remark 2.3.** Notice that, because of the skew symmetry property, if an \(m\)-coloured quiver \(Q\) has an arrow \(i \xrightarrow{(c)} j\), \(c \in \{0, \ldots, m\}\), then it also has an arrow \(j \xrightarrow{(m-c)} i\).

Thus, we may sometimes draw

\[
\begin{align*}
i & \xrightarrow{(c)} j
\end{align*}
\]

instead of

\[
\begin{align*}
i & \xrightarrow{(0)} \leftarrow \xleftarrow{(1)} j
\end{align*}
\]

**Example 2.4.** Let \(m = 3\), \(n = 5\). Then an example of 3-coloured quiver on vertices 1, \ldots, 5 is the following.

\[
\begin{align*}
1 & \xrightarrow{(1)} 3 & \xleftarrow{(0)} 2 & \xrightarrow{(1)} 4 & 5
\end{align*}
\]

We might draw this as

\[
\begin{align*}
1 & \xrightarrow{(1)} 3 & \xleftarrow{(0)} 2 & \xrightarrow{(1)} 4
\end{align*}
\]

We now introduce the concept of mutation of an \(m\)-coloured quiver at a vertex \(k\).

**Definition 2.5.** Let \(Q\) be an \(m\)-coloured quiver, and \(k \in \{1, \ldots, n\}\) one of its vertices. The \textbf{\(m\)-coloured quiver mutation} of \(Q\) at vertex \(k\) is the \(m\)-coloured quiver \(\tilde{Q} = \mu_k(Q)\) defined by

\[
\tilde{q}_{ij}^{(c)} = \begin{cases} 
q_{ij}^{(c+1)} & \text{if } k = i \\
q_{ij}^{(c-1)} & \text{if } k = j \\
\max \{0, q_{ij}^{(c)} - \sum_{t \neq c} (q_{ik}^{(t)} - q_{ik}^{(c-1)})q_{kj}^{(0)} + q_{ik}^{(m)}(q_{kj}^{(c)} - q_{kj}^{(c+1)})\} & \text{if } i \neq k \neq j
\end{cases}
\]

where we set \(q_{ij}^{(m+1)} = q_{ij}^{(0)}\) and \(q_{ij}^{(-1)} = q_{ij}^{(m)}\).

**Remark 2.6.** We will later associate to \(m\)-coloured quivers of some type a generalised triangulation of a regular polygon. In this interpretation, the quiver mutation defined above corresponds to the counterclockwise rotation of a diagonal in the polygon.
If we replace all the occurrences of $c + 1$ by $c - 1$ (and vice versa) in the definition of $\tilde{q}_{ij}^{(c)}$ given above, then we would get a definition of $m$-coloured quiver mutation that is inverse to Definition 2.5. With this alternative definition, mutations should be interpreted as clockwise rotations of a diagonal in the associated generalised triangulation of a regular polygon.

**Remark 2.7.** Let $Q$ be a standard (uncoloured) quiver. If we interpret $Q$ as a 1-coloured quiver as in Remark 2.2, then it is easy to check that the definition of 1-coloured quiver mutation given above agrees with the standard definition of quiver mutation given in literature (see [3], §4).

The above definition is slightly different to the one given in [2]. Indeed, in [2] the authors swap the conditions $k = i$ and $k = j$. Our choice will be justified later (see Remark 2.9 and Section 4).

One can give an equivalent definition of coloured quiver mutation, that is in general easier to work with.

**Proposition 2.8.** Let $Q$ be an $m$-coloured quiver, and $k \in \{1, \ldots, n\}$ one of its vertices. Then the following algorithm correctly computes the $m$-coloured quiver mutation $\mu_k(Q)$ of $Q$ at vertex $k$. The colours $-1$ and $m+1$ that could arise shall be interpreted as $m$ and $0$, respectively.

**Step 1.** Add 1 to the colour of the arrows going into $k$, and subtract 1 to the colour of the arrows going out of $k$.

**Step 2.** For each of the following type of arrows

\[
\begin{array}{ccc}
  (c) & (0) \\[6pt]
  i & k & j
\end{array}
\quad \text{with } i \neq j \text{ and } c \neq m, \text{ add the pair of arrows}
\]

\[
\begin{array}{ccc}
  (c) \\[6pt]
  i & (m-c) & j
\end{array}
\]

**Step 3.** If the graph obtained violates the monochromaticity property (II) of Definition 2.1 because for some pair of vertices $i$ and $j$ there are arrows from $i$ to $j$ which have more than one different colour, cancel the same number of arrows of each colour, until property (II) is satisfied.

**Remark 2.9.** The above Proposition can be found in Section 10 of [2]. However, the authors don’t impose the condition $c \neq m$ in Step 2. This is likely a mistake for the following two reasons:

- Let $Q = i \xrightarrow{(m)} k \xleftarrow{(0)} j$, $i \neq j$. 


If we mutate at vertex \( k \), Definition 2.5 yields \( q_{ij}^{(m)} = 0 \) and hence there should be no pair of arrows \( i \rightarrow j \) in \( \mu_k(Q) \) while, according to Section 10 of [2], there should be a pair of arrows \( i \rightarrow j \).

- The second reason will be fully explained in Section 4. There, we will associate to some \( m \)-coloured quivers a combinatorial object, called \( (m+2) \)-angulation of an \( mn+2 \)-gon. On such \( (m+2) \)-angulations we will define an operation, that we want to be compatible with the quiver mutation defined above. One can check that, in order for this to happen, the condition \( c \neq m \) is crucial.

**Example 2.10.** Let \( m = 2 \) and let \( Q \) be the following quiver.

```
2
(1) \rightarrow (0)
1 \rightarrow (2)
```

Applying Steps 1. and 2. of Proposition 2.8 for \( k = 2 \) yields the quiver

```
2
(1) \rightarrow (0)
1 \rightarrow (2)
```

However, the pair of vertices 1 and 3 violate the monochromaticity property of coloured quivers. Thus, if we apply Step 3. of Proposition 2.8 we get that the mutation of \( Q \) at vertex 2 is

\[
\mu_2(Q) = 1 \rightarrow (2) \rightarrow (2) 3.
\]

**Definition 2.11.** We call the following \( m \)-coloured quiver \( \overrightarrow{A_{n-1}} \).

\[
1 \rightarrow (m) \rightarrow 2 \rightarrow (m) \rightarrow \cdots \rightarrow (m) \rightarrow (n-2) \rightarrow (m) \rightarrow (n-1)
\]

**Definition 2.12.** An \( m \)-coloured quiver \( Q \) is called of mutation type \( \overrightarrow{A_{n-1}} \) if \( Q = \mu_{i_1} \cdots \mu_{i_\ell} (\overrightarrow{A_{n-1}}) \) for some \( i_1, \ldots, i_\ell \in \{1, \ldots, n-1\} \).

3. \((m+2)\)-ANGULATIONS OF A REGULAR POLYGON

Recall \( n, m \geq 1 \) are fixed integers.

In the following, \( \Pi \) will denote a regular \( nm+2 \) sided polygon, with vertices numbered clockwise from 1 to \( nm+2 \).

We introduce the concept of \((m+2)\)-angulation of \( \Pi \). In the following we will see that this is closely related to \( m \)-coloured quivers of mutation type \( \overrightarrow{A_{n-1}} \).
Definition 3.1. • A diagonal of $\Pi$ is a pair $(i, j)$ with $i \neq j$, $i, j \in \{1, \ldots, nm + 2\}$. The diagonal $(i, j)$ will be interpreted the same as the diagonal $(j, i)$.

• Two diagonals $(i_1, j_1), (i_2, j_2)$ with $i_1 < j_1$ and $i_2 < j_2$ of $\Pi$ are called intersecting if $i_1 < i_2 < j_1 < j_2$ or $i_2 < i_1 < j_2 < j_1$.

• An $m$-diagonal of $\Pi$ is a diagonal of $\Pi$ of the form $(i, i + jm + 1)$ for some $i \in \{1, \ldots, nm + 2\}$ and $j \in \{1, \ldots, n - 1\}$.

• An $(m + 2)$-angulation of $\Pi$ is a maximal collection of non intersecting $m$-diagonals.

Remark 3.2. One can give a geometrical interpretation the first three items of Definition 3.1.

• A diagonal of $\Pi$ is a segment connecting two distinct vertices of $\Pi$.

• Two diagonals of $\Pi$ are called intersecting if they have a common point inside $\Pi$.

• An $m$-diagonal of $\Pi$ is a diagonal of $\Pi$ which divides $\Pi$ into an $(mj + 2)$-gon and an $(m(n - j) + 2)$-gon for some $j = 1, \ldots, n - 1$.

We will mainly use the geometric description of the concepts defined above, since it allows us to draw pictures. However, one might use a purely combinatorial approach as well.

The following results follow directly from the definition of $(m + 2)$-angulation.

Proposition 3.3. Let $\Delta$ be an $(m + 2)$-angulation of $\Pi$. Then the following hold.

1. The number of $m$-diagonals of $\Delta$ is $n - 1$.
2. $\Delta$ defines $n$ distinct $(m + 2)$-gons $P_1, \ldots, P_n$ whose union is $\Pi$.
3. If $n \geq 2$ then, for each $\gamma \in \Delta$ there are exactly two $(m + 2)$-gons $P_\gamma^{(1)}, P_\gamma^{(2)}$ determined by $\Delta$ that have $\gamma$ as an edge. We call $P_\gamma$ the $(2m + 2)$-gon given by the union of $P_\gamma^{(1)}$ and $P_\gamma^{(2)}$.

Proof. The first two statements can be proved by induction on $n$.

1. If $n = 1$, then $\Pi$ is an $(m + 2)$-gon. Hence $\Delta$ is empty, that is, the number of $m$-diagonals of $\Delta$ is 0, and $\Pi$ is trivially union of one $(m + 2)$-gon, that is $\Pi$ itself.
2. If $n > 1$, let $\gamma \in \Delta$ be an $m$-diagonal. Then $\gamma$ divides $\Pi$ into an $(mj + 2)$-gon $P$ and a $m(n - j) + 2$-gon $P'$ for some $j = 1, \ldots, n - 1$.
   - By induction hypothesis, we have that $\Delta$ induces an $(m + 2)$-angulation on $P$ (resp. $P'$) consisting of $j - 1$ $m$-diagonals (resp. $n - j - 1$ $m$-diagonals), say $\delta_1, \ldots, \delta_{j-1}$ (resp. $\delta_{j+1}, \ldots, \delta_{n-1}$). Hence $\Delta = \{\delta_1, \ldots, \delta_{j-1}, \gamma, \delta_{j+1}, \ldots, \delta_{n-1}\}$, so the first statement follows.
   - By induction hypothesis, $P$ can be written as disjoint union of $j$ $(m + 2)$-gons, say $P_1, \ldots, P_j$, while $P'$ can be written as union of $n - j$ $(m + 2)$-gons, say $P_{j+1}, \ldots, P_n$. Hence $\Delta$ can be written as
union of the \((m+2)\)-gons \(P_1, \ldots, P_n\), and so we get the second statement.

As for (3), we know that \(\gamma\) divides \(\Pi\) into \((mj+2)\)-gon \(\Pi_1\) and an \((m(n-j)+2)\)-gon \(\Pi_2\) for some \(j \in \{1, \ldots, n-1\}\). Let \(\Delta_i\) be the restriction of \(\Delta\) to \(\Pi_i\), for \(i=1,2\). By (2), \(\Delta_1\) (resp. \(\Delta_2\)) defines \(j\) (resp. \(n-j\)) \((m+2)\)-gons whose union is \(\Pi_1\) (resp. \(\Pi_2\)). Therefore \(P^{(1)}\) and \(P^{(2)}\) are the \((m+2)\)-gons that have \(\gamma\) as an edge in \(\Pi_1\) and \(\Pi_2\), respectively. \(\square\)

**Definition 3.4.** Let \(\Delta\) be an \((m+2)\)-angulation of \(\Pi\), and \(\gamma \in \Delta\) be an \(m\)-diagonal. Let \(P_\gamma\) be the \((2m+2)\)-gon introduced in Proposition 3.3. Let \(\{a_0, \ldots, a_{2m+1}\}\) be the vertices of \(P_\gamma\), with \(a_0 < a_1 < \ldots < a_{2m+1}\). Then the **mutation** of \(\Delta\) at \(\gamma\) is the following \((m+2)\)-angulation of \(\Pi\)

\[
r_\gamma(\Delta) = (\Delta \setminus \gamma) \cup \{\sigma(\gamma)\},
\]

where, if \(\gamma = (a_i, a_{i+m+1})\) for some \(i = 0, \ldots, m\), then

\[
\sigma(\gamma) = (a_{i-1 \mod 2m+2}, a_{i+m \mod 2m+2}).
\]

**Remark 3.5.** The \((m+2)\)-angulation \(r_\gamma(\Delta)\) can be geometrically interpreted as the \((m+2)\)-angulation of \(\Pi\) obtained by fixing all the diagonals of \(\Delta\) except from \(\gamma\), that is rotated counterclockwise inside the \((2m+2)\)-gon \(P_\gamma\).

**Proposition 3.6.** Let \(\Delta\) be an \((m+2)\)-angulation of \(\Pi\), \(\gamma \in \Delta\). Then \(r_\gamma\) has order \(m+1\).

**Proof.** Let \(\{a_0, \ldots, a_{2m+1}\}\) be the vertices of \(P_\gamma\), with \(a_0 < a_1 < \ldots < a_{2m+1}\), and let \(i \in \{0, \ldots, m\}\) be such that \(\gamma = (a_i, a_{i+m+1})\). Definition 3.4 implies that

\[
\sigma^j(\gamma) = (a_{i-j \mod 2m+2}, a_{i+m+1-j \mod 2m+2})
\]

for all \(j \geq 0\). Hence

\[
\sigma^{m+1}(\gamma) = (a_i- (m+1) \mod 2m+2, a_i \mod 2m+2) = (a_i+m+1, a_i)
\]

is the \(m\)-diagonal \(\gamma\) by Definition 3.1. It is also straightforward to check that \(\sigma^j(\gamma) \neq \gamma\) for all \(1 \leq j < m+1\). Thus \(\sigma\) has order \(m+1\). The definition of \(r_\gamma\) given in Definition 3.4 implies that

\[
r^j_\gamma(\Delta) = (\Delta \setminus \gamma) \cup \{\sigma^j(\gamma)\}
\]

for all \(j \geq 1\). Therefore, also \(r_\gamma\) has order \(m+1\). \(\square\)

**Example 3.7.** Let \(m = 2, n = 5\). Consider the diagonal \(\gamma\) in the 4-angulation \(\Delta\) of the regular dodecagon given below. The hexagon \(P_\gamma\) is coloured in picture, and the result of applying \(r_\gamma\) is displayed.
The application of $r_\gamma$ might hence be thought as a counterclockwise rotation of $\gamma$ inside the hexagon $P_\gamma$.

4. CONNECTION BETWEEN $(m + 2)$-ANGULATIONS AND $m$-COLOURED QUIVERS

In this section we introduce a map $\Psi$, that associates an $m$-coloured quiver to an $(m + 2)$-angulation of $\Pi$. This construction will be done in a way such that the two concepts of mutation we introduced (at a vertex for an $m$-coloured quiver, and at a diagonal for an $(m + 2)$-angulation of $\Pi$) commute. This result, that we will prove in detail, is used in [9] to show that $\Psi$ induces a bijection between $(m + 2)$-angulations of $\Pi$ and $m$-coloured quivers of mutation type $\xrightarrow{\sim} A_{n-1}$.

**Definition 4.1.** Let $\Delta$ be an $(m + 2)$-angulation of $\Pi$. We associate to $\Delta$ an $m$-coloured quiver $\Psi(\Delta)$ as follows.

- The vertices of $\Psi(\Delta)$ are the $m$-diagonals of $\Delta$.
- If $\gamma, \delta$ are $m$-diagonals of $\Delta$ which are edges of some $(m + 2)$-gon in the $(m + 2)$-angulation $\Delta$, then $\Psi(\Delta)$ has an arrow from $\gamma$ to $\delta$. In this case, the colour of the arrow is the number of edges forming the segment of the boundary of the $(m + 2)$-gon which lie between $\gamma$ and $\delta$, counterclockwise from $\gamma$ and clockwise from $\delta$.

**Example 4.2.** The 2-coloured quiver $\Psi(r_\gamma(\Delta))$ associated to the 4-angulation $r_\gamma(\Delta)$ in Example 3.7 is given by

$$
\sigma(\gamma) \xrightarrow{(2)} \delta \\
\varepsilon \xrightarrow{(0)} \gamma \\
\zeta \xrightarrow{(2)} \eta
$$
In the following, we will identify $\Pi$ with a circle with $mn + 2$ marked points, numbered clockwise from 1 to $mn + 2$.

**Definition 4.3.** We call $\hat{A}_{n-1}$ the following $(m + 2)$-angulation of $\Pi$.

![Diagram of a circle with marked points and angles]

**Remark 4.4.** Consider the $m$-coloured quiver $\overrightarrow{A_{n-1}}$ introduced in Definition 2.11. Then
\[ \overrightarrow{A_{n-1}} = \Psi(\hat{A}_{n-1}) \]

We want to prove that every $(m + 2)$-angulation of $\Pi$ can be obtained from $\hat{A}_{n-1}$ by iteratively mutating it at some $m$-diagonals. In order to do this, we introduce the concept of distance between a diagonal of an $(m + 2)$-angulation $\Delta$ of $\Pi$ and the vertex 1.

**Definition 4.5.** Let $\Delta$ be an $(m + 2)$-angulation of $\Pi$, $\gamma \in \Delta$ one of its $m$-diagonals. We define the distance $d_\Delta(\gamma)$ between $\gamma$ and the vertex 1 of $\Pi$ as follows:

- $d_\Delta(\gamma) = 0$ if $\gamma = (1, i)$ for some $i$;
- $d_\Delta(\gamma) = k$ if $\gamma$ is an edge of a $(2m + 2)$-gon $P_\delta$ defined in Proposition 3.3, for some $\delta \in \Delta$ with $d_\Delta(\delta) = k - 1$.

**Example 4.6.** Consider the 4-angulation $r_\gamma(\Delta)$ of $\Pi$ given in Example 3.7. Then $d_{r_\gamma(\Delta)}(\delta) = 0 = d_{r_\gamma(\Delta)}(\sigma(\gamma))$, $d_{r_\gamma(\Delta)}(\varepsilon) = 1$, $d_{r_\gamma(\Delta)}(\zeta) = 2$.

**Lemma 4.7.** Let $\Delta$ be an $(m + 2)$-angulation of $\Pi$. Then there exist $m$-diagonals $\gamma_1, \ldots, \gamma_\ell$ such that
\[ \Delta = r_{\gamma_1} \cdots r_{\gamma_\ell}(\hat{A}_{n-1}). \]

**Proof.** We prove the result by induction on $k = \max\{d_\Delta(\gamma) | \gamma \in \Delta\}$.

- If $k = 0$, then $\Delta = \hat{A}_{n-1}$.
- Let $k > 0$. Order the $m$-diagonals of $\Delta$ as
  \[ \Delta = \{\gamma_{0,1}, \ldots, \gamma_{0,t_0}, \gamma_{1,1}, \ldots, \gamma_{1,t_1}, \ldots, \gamma_{k,1}, \ldots, \gamma_{k,t_k}\}, \]
  where $d_\Delta(\gamma_{i,j}) = i$ for all $i, j$. Fix $j \in \{1, \ldots, t_1\}$. Since $d_\Delta(\gamma_{1,j}) = 1$, then we know that the $(2m + 2)$-gon $P_{\gamma_{1,j}}$ contains the vertex...
1. Hence we can mutate the $m$-diagonal $\gamma_{1,j}$ a certain number of times, say $s_j$, so that $\sigma^{s_j}(\gamma_{1,j})$ has 1 as endpoint. Now, consider the following $(m + 2)$-angulation of $\Pi$.

$$\Delta' = r_{\gamma_{1,t_1}}^{s_1} \cdots r_{\gamma_{1,t_k}}^{s_k}(\Delta) = \{\gamma_{0,1}, \ldots, \gamma_{0,t_0}, \sigma^{s_1}(\gamma_{1,1}), \ldots, \sigma^{s_k}(\gamma_{1,t_1}), \ldots, \gamma_{k,1}, \ldots, \gamma_{k,t_k}\}.$$  

By assumption we have $d_\Delta(\gamma_{0,j}) = 0$ for all $j$.

Also, $d_\Delta'(\gamma_{i,j}') = i - 1$ for $i = 1, \ldots, k$ and all $j$. This can be shown by induction on $i \geq 1$.

- If $i = 1$, then by construction we know that all the $m$-diagonals $\gamma_{1,j}' = \sigma^{s_i}(\gamma_{1,j})$ have 1 as endpoint. This means that $d_\Delta(\gamma_{1,j}') = 0$ for all $j$.

- Let $i > 1$, and fix $j \in \{1, \ldots, t_i\}$. Then, by construction, we can find an $m$-diagonal $\gamma_{i-1,t_i}'$ that is an edge of $P_{i-1,t}$. But $d_\Delta(\gamma_{i-1,t_i}') = i - 2$ by induction hypothesis, and thus $d_\Delta'(\gamma_{i,j}') = i - 1$ by definition of distance $d_\Delta'$.

Hence $\max\{d_\Delta(\gamma') | \gamma' \in \Delta'\} = k - 1$. Thus, by induction we get

$$\Delta_{n-1} = r_{\tilde{\gamma}_1} \cdots r_{\tilde{\gamma}_s}(\Delta') = r_{\tilde{\gamma}_1} \cdots r_{\tilde{\gamma}_s} r_{\gamma_{1,t_1}}^{s_1} \cdots r_{\gamma_{1,1}}^{s_1}(\Delta)$$

for some $m$-diagonals $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_s$.

But all mutations $r_\gamma$ have finite order $m + 1$ by Proposition 3.6 and hence we can invert all mutations in $[\Pi]$. Therefore we get the statement.

\[\square\]

**Definition 4.8.** Let $Q$ be an $m$-coloured quiver, $k \in Q_0$ one of its vertices. Define

- $\mathcal{N}_{Q,k} = \{i \in Q_0 | \text{there are arrows } i \to k\}$ to be the neighbour-
hood of $k$ in $Q$.

- $F_{Q,k}$ as the full subquiver of $Q$ with vertex set $\mathcal{N}_{Q,k} \cup \{k\}$.

**Definition 4.9.**

- We call an $m$-coloured quiver $Q$ **complete** if, for every pair of vertices $i,j \in Q_0$, there is exactly one pair of arrows $i \to j$.

- Let $Q, Q'$ quivers with vertex sets $\{k, u_1, \ldots, u_j\}$ and $\{k, v_1, \ldots, v_j\}$ respectively, for some $i, j \geq 0$. Then the **glueing of $Q$ and $Q'$ at vertex $k$** is the quiver with vertex set $\{k, u_1, \ldots, u_i, v_1, \ldots, v_j\}$ and arrows $Q_1 \cup Q'_1$.

The following Lemma will help us understand how the $m$-coloured quiver associated to an $(m + 2)$-angulation of $\Pi$ mutates at a vertex. Its proof follows by a straightforward application of Definition 4.1.
Lemma 4.10. Let $Q$ be an $m$-coloured quiver, $k \in Q_0$. Suppose that $F_{Q,k}$ has the following shape

where the arrows can have any colour.

In words, suppose that $F_{Q,k}$ is the quiver obtained by gluing the complete quiver on $\{k, u_1, \ldots, u_a\}$ and the complete quiver on $\{k, v_1, \ldots, v_b\}$ at vertex $k$, for some $a, b \geq 0$.

For each pair of vertices $h, \ell \in N_{Q,k}$ connected by an arrow, let $c_{h\ell}$ be the colour of the arrow $h \to \ell$. Suppose that $c_{k,u_a} < c_{k,u_a-1} < \ldots < c_{k,u_1}$ and $c_{k,v_b} < c_{k,v_b-1} < \ldots < c_{k,v_1}$.

Then the mutation $\mu_k(Q)$ of $Q$ at vertex $k$ fixes the arrows (together with their colours) of $Q_1 \setminus (F_{Q,k})_0$, and acts as follows on $F_{Q,k}$ (the blue vertices will denote those vertices that have moved, and red arrows will denote the new arrows):

a) If $c_{k,u_a} \neq 0 \neq c_{k,v_b}$, then $\mu_k(F_{Q,k})$ has the same shape as $F_{Q,k}$:

and the colours of the arrows change as follows.

\[ \begin{align*}
\tilde{c}_{u_i,u_j} &= c_{u_i,u_j}, & i, j &\in \{1, \ldots, a\} \\
\tilde{c}_{v_i,v_j} &= c_{v_i,v_j}, & i, j &\in \{1, \ldots, b\} \\
\tilde{c}_{k,u_i} &= c_{k,u_i} - 1, & i &\in \{1, \ldots, a\} \\
\tilde{c}_{k,v_i} &= c_{k,v_i} - 1, & i &\in \{1, \ldots, b\}
\end{align*} \]

b) If $c_{k,u_a} = 0$, $c_{k,v_b} \neq 0$, then $\mu_k(F_{Q,k})$ is given by
and the colours of the arrows change as follows.

\[ \tilde{c}_{u_i, u_j} = c_{u_i, u_j}, \quad i, j \in \{1, \ldots, a - 1\} \]
\[ \tilde{c}_{v_i, v_j} = c_{v_i, v_j}, \quad i, j \in \{1, \ldots, b - 1\} \]
\[ \tilde{c}_{k, u_i} = c_{k, u_i} - 1, \quad i \in \{1, \ldots, a\} \]
\[ \tilde{c}_{k, v_i} = c_{k, v_i} - 1, \quad i \in \{1, \ldots, b\} \]
\[ \tilde{c}_{v_i, k} = c_{v_i, k}, \quad i \in \{1, \ldots, b\} \]

\( \tilde{c}_{u_i, v} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)
\( \tilde{c}_{v_i, u} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)

\( \tilde{c}_{u_i, v} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)
\( \tilde{c}_{v_i, u} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)

\( \tilde{c}_{u_i, v} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)
\( \tilde{c}_{v_i, u} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)

\( \tilde{c}_{v_i, v} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)
\( \tilde{c}_{u_i, u} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)

\( \tilde{c}_{u_i, v} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)
\( \tilde{c}_{v_i, u} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)

\( \tilde{c}_{u_i, v} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)
\( \tilde{c}_{v_i, u} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)

\( \tilde{c}_{v_i, v} = c_{v_i, k} \), \( i \in \{1, \ldots, b - 1\} \)
\( \tilde{c}_{u_i, u} = c_{u_i, k} \), \( i \in \{1, \ldots, a - 1\} \)
The following proposition is stated in [2], after Proposition 11.1. We give an explicit proof of this fact that relies on the action of $\mu_k$ on $m$-coloured quivers described in Lemma 4.10.

**Proposition 4.11.** Let $\Delta$ be an $(m+2)$-angulation of $\Pi$, and consider an $m$-diagonal $\gamma \in \Delta$. Then

$$\Psi(r_\gamma(\Delta)) = \mu_\gamma(\Psi(\Delta)).$$

**Proof.** Let $P_\gamma$ be the regular $(2m+2)$-gon defined in Proposition 3.3. Let $P^{(1)}_\gamma$, $P^{(2)}_\gamma$ be the two $(m+2)$-gons induced by $\Delta$ having $\gamma$ as an edge, so that $P_\gamma = P^{(1)}_\gamma \cup P^{(2)}_\gamma$.

Let $\delta_1^{(1)}, \ldots, \delta_a^{(1)}, \delta_{a+1}^{(1)} = \gamma$ (resp. $\delta_1^{(2)}, \ldots, \delta_b^{(2)}, \delta_{b+1}^{(2)} = \gamma$) be the edges of $P^{(1)}_\gamma$ (resp. of $P^{(2)}_\gamma$) that are $m$-diagonals of $\Delta$, ordered clockwise, for some $a, b \geq 0$.

Let $\ell_i^{(1)}$ (resp. $\ell_i^{(2)}$) be the number of edges forming the segment of the boundary of $P^{(1)}_\gamma$ (resp. $P^{(2)}_\gamma$) which lies between $\delta_i^{(1)}$ and $\delta_{i-1}^{(1)}$ (resp. between $\delta_i^{(2)}$ and $\delta_{i-1}^{(2)}$), counterclockwise from $\delta_i^{(1)}$ (resp. $\delta_i^{(2)}$) for $i = 1, \ldots, a+1$ (resp. $i = 1, \ldots, b+1$). By convention, we set $\delta_0^{(1)} = \delta_0^{(2)} = \gamma$. See Figure 1 for a geometric description.

![Figure 1. Local $(m+2)$-angulation $\Delta$ around $\gamma$](image)
Using Definition 4.1 we get that $F_{\Psi(\Delta), \gamma}$ is the following quiver,

where the colour of the arrows $\delta_i^{(1)} \to \delta_j^{(1)}$ is $c_i^{(1)} = \ell_i^{(1)} + \ell_{i-1}^{(1)} + \ldots + \ell_j^{(1)} + (i-j-1)$ for all $1 \leq j < i \leq a+1$, and the colour of the arrows $\delta_i^{(2)} \to \delta_j^{(2)}$ is $c_i^{(2)} = \ell_i^{(2)} + \ell_{i-1}^{(2)} + \ldots + \ell_j^{(2)} + (i-j+1)$ for all $1 \leq j < i \leq b+1$.

Notice that $c_{a+1,a}^{(1)} < c_{a+1,a-1}^{(1)} < \ldots < c_{a+1,1}^{(1)}$, $c_{b+1,b}^{(2)} < c_{b+1,b-1}^{(2)} < \ldots < c_{b+1,1}^{(2)}$.

Therefore we can apply Lemma 4.10 that gives us all the possible mutations of $\Psi(\Delta)$ at vertex $\gamma$. In particular, $\mu_\gamma$ fixes the arrows (together with their colours) of $\Psi(\Delta) \setminus F_{\Psi(\Delta), \gamma}$, and it acts on $F_{\Psi(\Delta), \gamma}$ as explained in Lemma 4.10. This computes $\mu_\gamma(\Psi(\Delta))$.

We now compute $r_\gamma(\Delta)$. First of all notice that $r_\gamma$ fixes all the $m$-diagonals of $\Delta$ apart from $\gamma$. Hence we may as well just represent how $r_\gamma$ acts on $\Delta$ locally around $\gamma$, that is, on $P^{(1)}_\gamma$ and $P^{(2)}_\gamma$. We have four possible cases, depending on the values of $\ell_{a+1}^{(1)}$ and $\ell_{b+1}^{(2)}$ (see Figure 2 in the next page).

Now, applying the definition of $\Psi$ (see Definition 4.1), one gets that the quiver $\Psi(r_\gamma(\Delta))$ associated to $r_\gamma(\Delta)$ is exactly the one given in Lemma 4.10.

For example, if $\ell_{a+1}^{(1)} = 0, \ell_{b+1}^{(2)} \neq 0$, one gets the commutative diagram in Figure 3.

Hence we get the statement.  □
Presentations of braid groups of type $A$ from $(m + 2)$-angulations of regular polygons

Figure 2. Mutations $r_\gamma(\Delta)$ at $\gamma$
Figure 3. Commutative diagram for $\ell^{(1)}_{i+1} = 0$, $\ell^{(2)}_{j+1} \neq 0$
Remark 4.12. The above Proposition is the second reason why we defined \( m \)-coloured quiver mutation in a slightly different way compared to [2]. Consider the following example:

\[
Q = 1 \xleftarrow{(0)} 2 \xrightarrow{(2)} 3 \quad m = 2.
\]

Then, if we mutate at vertex 2 using Proposition 10.1 in [2], we get that \( \mu_2(Q) \) is the following quiver.

\[
\begin{array}{c}
1 \\
\downarrow \Psi \\
\end{array}
\xleftarrow{(0)} \xrightarrow{(2)}
\begin{array}{c}
2 \\
\downarrow \Psi \\
\end{array}
\xrightarrow{(0)} \xleftarrow{(2)}
\begin{array}{c}
3 \\
\downarrow \\
\end{array}
\]

However, the diagram in Figure 4 does not commute.

\[
\begin{array}{c}
1 \\
\downarrow \Psi \\
\end{array}
\xleftarrow{(0)} \xrightarrow{(2)}
\begin{array}{c}
2 \\
\downarrow \Psi \\
\end{array}
\xrightarrow{(0)} \xleftarrow{(2)}
\begin{array}{c}
3 \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
\downarrow \\
\end{array}
\xleftarrow{(0)} \xrightarrow{(2)}
\begin{array}{c}
2 \\
\downarrow \Psi \\
\end{array}
\xrightarrow{(0)} \xleftarrow{(2)}
\begin{array}{c}
3 \\
\downarrow \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
\downarrow \\
\end{array}
\xleftarrow{(0)} \xrightarrow{(2)}
\begin{array}{c}
2 \\
\downarrow \Psi \\
\end{array}
\xrightarrow{(0)} \xleftarrow{(2)}
\begin{array}{c}
3 \\
\downarrow \\
\end{array}
\]

**Figure 4.** Diagram obtained from Proposition 10.1 in [2]

This suggests that the correct definition of \( m \)-coloured quiver mutation is the one in Proposition 2.8 that in this example correctly gives

\[
\mu_2(Q) = 1 \xleftarrow{(0)} 2 \xrightarrow{(2)} 3.
\]

Proposition 4.11 is used in the following Theorem, whose proof can be found in Section 4 of [9].

**Theorem 4.13.** Let \( \mathcal{M}_{n-1,m} \) be the set of \( m \)-coloured quivers of mutation type \( A_{n-1} \), and \( \mathcal{T}_{m,n} \) the set of \( (m+2) \)-angulations of \( \Pi \). Then the map \( \Psi \) introduced in Definition 4.11 induces a bijection

\[
\Psi : (\mathcal{T}_{m,n}/\sim) \rightarrow \mathcal{M}_{n-1,m}/\mathfrak{S}_{n-1},
\]
where $\Delta \sim \Delta'$ if $\Delta'$ can be obtained rotating $\Delta$, and the symmetric group $S_{n-1}$ on $n-1$ letters acts on an $m$-coloured quiver by permuting its vertices.

**Corollary 4.14.** Let $Q$ be an $m$-coloured quiver of mutation type $A_n^{-1}$ and $k \in Q_0$. Then $\mu_k$, the mutation of $Q$ at vertex $k$, has order $m + 1$.

**Proof.** This follows directly from Proposition 3.6 and Theorem 4.13. $\blacksquare$

**Remark 4.15.** Corollary 4.14 holds for arbitrary quivers in the case $m = 1$. Equivalently, for all (uncoloured) quivers $Q$ and $k \in Q_0$, then $\mu_k$ has order $m + 1 = 2$.

This is not in general true for $m > 1$. For example, let $m = 2$ and consider the following 2-coloured quiver $Q$ and its repeated mutations at vertex $k = 2$ obtained applying repeatedly Proposition 2.8:

```
\[ Q \rightarrow \mu_2(Q) \rightarrow \mu_2^2(Q) \rightarrow \mu_2^3(Q) \]
```

Hence we have $\mu_2^3(Q) \neq Q$.

Therefore, for an arbitrary $m$-coloured quiver $Q$ and $k \in Q_0$ we have $\mu_k^{m+1}(Q) \neq Q$, and the condition that $Q$ is of mutation type $A_n^{-1}$ is crucial.

**Corollary 4.16.** Let $Q$ be an $m$-coloured quiver of mutation type $A_n^{-1}$, and let $i, j \in Q_0$. Then there is at most one arrow $i \rightarrow j$.

**Proof.** Thanks to Theorem 4.13 this is equivalent to saying that, for each pair of $m$-diagonals $\gamma, \gamma'$ in an $(m+2)$-angulation of $\Pi$, there is at most one $(m+2)$-gon inside $\Pi$ having both $\gamma$ and $\gamma'$ as edges, that is trivially true. $\blacksquare$

Let $\Delta$ be an $(m+2)$-angulation of $\Pi$, and $\gamma_1, \ldots, \gamma_a \in \Delta$ be $m$-diagonals ordered clockwise that are edges of an $(m+2)$-gon $P$ inside $\Pi$ determined by $\Delta$, for some $i \geq 2$. Let $\ell^k$ be the number of edges of $P$ lying between $\gamma_{k+1}$ and $\gamma_k$, $k = 1, \ldots, a$, where we set $\gamma_{a+1} = \gamma_1$.

Let $Q$ be an $m$-coloured quiver associated to $\Delta$ by $\Psi$, and let $\ell_k$ be the colour of the arrow $\gamma_k \rightarrow \gamma_{k+1}$, for $k = 1, \ldots, a$.

Notice that, by the skew symmetry property of $m$-coloured quivers, we have

\[ \ell_k = m - \ell^k, \]

for all $k = 1, \ldots, a$.

**Proposition 4.17.** With the same notation as above, the complete quiver on vertices $\gamma_1, \ldots, \gamma_a$ is a subquiver of $Q$. 
Furthermore, the following formulas hold.
\[ \ell_1 + \ldots + \ell_a = (a - 1)(m + 1) - 1, \quad \ell_1^1 + \ldots + \ell_a^a = m - a + 2 \]

Proof. We can represent the \((m+2)\)-gon \(P\) and the \(m\)-diagonals \(\gamma_1, \ldots, \gamma_a \in \Delta\) as follows.

Using the definition of \(\Psi\), we can see that the subquiver of \(Q\) associated to \(\gamma_1, \ldots, \gamma_a\) by \(\Psi\) is the following complete quiver on vertices \(\gamma_1, \ldots, \gamma_a\),

with arrows
\[ \gamma_k \xrightarrow{\ell_k} \gamma_{k+1}, \quad k = 1, \ldots, a. \]

Now, since \(P\) is an \((m+2)\)-gon, we have \(\ell_1 + \ldots + \ell_a = m + 2\), that is,
\[ \ell_1^1 + \ldots + \ell_a^a = m - a + 2. \]

Also, since \(\ell_k = m - \ell^k\) for all \(k = 1, \ldots, a\), we get:
\[ \ell_1 + \ldots + \ell_a = (m - \ell_1) + \ldots + (m - \ell_a) \]
\[ = am - (\ell_1^1 + \ldots + \ell_a^a) = am - (m - a + 2) \]
\[ = (a - 1)(m + 1) - 1. \]

Example 4.18. Let \(m = 2, n = 4\), and consider the following 12-angulation of a regular dodecagon \(P\).
The associated 2-coloured quiver $Q$ is given by

Let $c_{i,j}$ be the colour of the arrow $i \to j$ in $Q$. Then the two formulas from Proposition 4.17 hold.

For example, if we consider the vertices 1,2,3,4, we have

$$c_{1,4} + c_{4,3} + c_{3,2} + c_{2,1} = 0 + 0 + 0 + 0 = 0 = 2 - 4 + 2$$
$$c_{1,2} + c_{2,3} + c_{3,4} + c_{4,1} = 2 + 2 + 2 + 2 = 9 = 3 \cdot 3 - 1.$$

Also, if we consider vertices 1,2,4, we have

$$c_{1,4} + c_{4,2} + c_{2,1} = 0 + 1 + 0 = 1 = 2 - 3 + 2$$
$$c_{1,2} + c_{2,4} + c_{4,1} = 2 + 1 + 2 = 5 = 2 \cdot 3 - 1.$$

We can now define in a natural way a group arising from an $m$-coloured quiver of mutation type $\overrightarrow{A_{n-1}}$.

5. **Presentations of braid groups of type $\overrightarrow{A_{n-1}}$**

**Definition 5.1.** Let $Q$ be an $m$-coloured quiver of mutation type $\overrightarrow{A_{n-1}}$, with vertices 1,\ldots,n−1. For an arrow $i \to j$, let $c_{ij}$ be its colour. Define the group $B_Q$ to be generated by $s_1,\ldots,s_{n-1}$ subject to the relations:

1. $s_is_j = s_js_i$ if there is no arrow between $i$ and $j$ (in either direction);
2. $s_is_js_i = s_js_is_j$ if there is a pair of arrows $i \leftarrow \leftrightarrow \rightarrow j$;
(3) \[ s_{i_1} s_{i_2} s_{i_3} s_{i_1} = s_{i_2} s_{i_3} s_{i_1} s_{i_2} = s_{i_3} s_{i_1} s_{i_2} s_{i_3} \] if

is a subquiver of \( Q \) and \( c_{i_1,i_2} + c_{i_2,i_3} + c_{i_3,i_1} = 2m + 1. \)

**Remark 5.2.** Let \( Q \) be an \( m \)-coloured quiver of mutation type \( \overrightarrow{A_{n-1}} \). By Theorem 4.13 we can find an \( (m+2) \)-angulation \( \Delta \) of \( \Pi \) such that \( \Psi(\Delta) = Q \). Therefore any subquiver of \( Q \) of the form

\[
\begin{array}{ccc}
& i_1 & \\
\downarrow & & \downarrow \\
& i_2 & \rightarrow \rightarrow i_3
\end{array}
\]

corresponds to three \( m \)-diagonals \( i_1, i_2, i_3 \in \Delta \) that, by Definition of \( \Psi \), are edges of the same \( (m+2) \)-gon identified by \( \Delta \) in \( \Pi \). Thus, Proposition 4.17 applied in the case \( i = 3 \) tells us that exactly one of the following holds true.

- The sum of the colours of the arrows going clockwise is \( 2m + 1 \);
- The sum of the colours of the arrows going counterclockwise is \( 2m+1 \).

Therefore any complete full subquiver of \( Q \) on three vertices gives a relation in \( B_Q \) of type (3) in Definition 5.1.

**Remark 5.3.** Consider the \( m \)-coloured quiver \( \overrightarrow{A_{n-1}} \).

\[
\begin{array}{c}
1 \overrightarrow{(0)} \rightarrow \overrightarrow{(m)} 2 \overrightarrow{(0)} \rightarrow \overrightarrow{(m)} \cdots \overrightarrow{(0)} (n-2) \overrightarrow{(m)} (n-1).
\end{array}
\]

Then its associated group \( B_{\overrightarrow{A_{n-1}}} \) is the braid group of type \( A_{n-1} \), given in terms of its standard presentation. More precisely, it is generated by \( s_1, \ldots, s_{n-1} \) subject to the relations

\[
\begin{align*}
&s_i s_j = s_j s_i, \quad \text{if } |i - j| > 1 \\
&s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}
\end{align*}
\]

**Example 5.4.** Let \( m = 2 \), and consider the following 2-coloured quiver \( Q \) of mutation type \( \overrightarrow{A_4} \).

\[
\begin{array}{c}
1 \overrightarrow{(2)} \rightarrow \overrightarrow{(1)} 2 \overrightarrow{(0)} \rightarrow \overrightarrow{(0)} 3 \overrightarrow{(1)} \rightarrow \overrightarrow{(2)} 4
\end{array}
\]
Then the group $B_Q$ associated to $Q$ is generated by $s_1, s_2, s_3, s_4$ subject to the following relations:

\[
\begin{align*}
    s_1 s_4 &= s_4 s_1 \\
    s_2 s_4 &= s_4 s_2 \\
    s_1 s_2 s_1 &= s_2 s_1 s_2 \\
    s_1 s_3 s_1 &= s_3 s_1 s_3 \\
    s_2 s_3 s_2 &= s_3 s_2 s_3 \\
    s_3 s_4 s_3 &= s_4 s_3 s_4 \\
    s_2 s_3 s_2 s_1 &= s_3 s_2 s_1 s_3 = s_2 s_1 s_3 s_2.
\end{align*}
\]

Notice that the quiver $Q$ and the associated group $B_Q$ can also be obtained in the uncoloured case (see [7]). In Example 5.11 we will give an example of a 2-coloured quiver $Q$ of mutation type $\overrightarrow{A_4}$ that does not arise from the uncoloured theory.

Let $Q$ be an $m$-coloured quiver of mutation type $\overrightarrow{A_{n-1}}$, $k \in Q_0$. Let $Q' = \mu_k(Q)$ be its coloured quiver mutation at $k$. Let $\{s_i| i \in Q\}$ (resp. $\{t_i| i \in Q'_0\}$) be the generators of $B_Q$ (resp. $B_{Q'}$) introduced in Definition 5.1. Let $F_Q$ be the free group with generators $s_i, i \in Q_0$.

**Definition 5.5.** Let $\varphi_k : F_Q \rightarrow B_{Q'}$ be the group homomorphism defined by

\[
\varphi_k(s_i) = \begin{cases} 
    t_k t_i t_k^{-1}, & \text{if } k \rightarrow i \\
    t_i, & \text{otherwise}
\end{cases}
\]

**Proposition 5.6.** The group homomorphism defined above induces a group homomorphism $\varphi_k : B_Q \rightarrow B_{Q'}$.

**Proof.** We need to show that all the relations in $B_Q$ are preserved by $\varphi_k$.

By Theorem 4.13 we can find an $(m+2)$-angulation $\Delta$ of $\Pi$ such that $\Psi(\Delta) = Q$.

Consider the $m$-diagonal $k \in \Delta$. By Proposition 3.3(3), there are exactly two $(m+2)$-gons $P_k^{(1)}, P_k^{(2)}$ in $\Pi$ identified by $\Delta$ that have $k$ as an edge. Let $u_0, u_1, \ldots, u_a$ (resp. $v_0, v_1, \ldots, v_b$) be the $m$-diagonals of $\Delta$ that are edges of $P_k^{(1)}$ (resp. of $P_k^{(2)}$), ordered clockwise, for some $a, b \geq 0$, where we set $u_0 = v_0 = k$. By Proposition 3.3(3), we also know that, if $a > 0$ (resp. $b > 0$), then $u_a$ (resp. $v_b$) will be edge of another $(m+2)$-gon $P_{u_a}$ (resp. $P_{v_b}$) in $\Pi$ identified by $\Delta$. Let $u_a, w_1, \ldots, w_\ell$ (resp. $v_b, z_1, \ldots, z_h$) be the $m$-diagonals of $\Delta$ that are edges of $P_{u_a}$ (resp. $P_{v_b}$), ordered clockwise, for some $\ell, h \geq 0$.

Applying the definition of $\Psi$, we get that the union of the subquivers $F_{Q,k}, F_{Q,u_a}$ and $F_{Q,v_b}$ of $Q$ is the following glueing of complete subquivers...
Presentations of braid groups of type $A$ from $(m+2)$-angulations of regular polygons of $Q$.

Notice that the relations in $B_Q$ involving $s_{u_a}$ are:

1. $s_{u_a} s_p = s_p s_{u_a}$ for all $p \notin \mathcal{N}_{Q,u_a}$;
2. $s_{u_a} s_p s_{u_a} = s_p s_{u_a} s_p$ for all $p \in \mathcal{N}_{Q,u_a}$;
3. $s_{u_a} s_{u_i} s_{u_j} s_{u_a} = s_{u_i} s_{u_j} s_{u_a} s_{u_i}$ for all $i, j \in \{0, \ldots, a-1\}$, with $i < j$;
4. $s_{u_a} s_{w_i} s_{w_j} s_{u_a} = s_{w_i} s_{w_j} s_{u_a} s_{w_i} = s_{w_j} s_{u_a} s_{w_i} s_{w_j}$ for all $i, j \in \{1, \ldots, \ell\}$, with $i < j$.

The relations in $B_Q$ involving $s_{v_b}$ are analogous.

Now, Lemma 4.10 thus tells us the result of the mutation of $Q$ at $k$.

We need to check that the relations defining $B_Q$ are preserved by $\varphi_k$. We split the cases as in Lemma 4.10. For convenience, let $c$ (resp. $d$) be the colour of the arrow $k \to u_a$ (resp. of the arrow $k \to v_b$). Let $\tilde{s}_p := \varphi_k(s_p)$ for all $p \in Q_0$.

a) If $c \neq 0 \neq d$, then $\tilde{s}_p = s_p$ for all $p \in Q_0$, and the result follows trivially.

b) If $c = 0$ and $d \neq 0$ then, if we apply $\mu_k$, the shape of the subquiver of $Q$ previously drawn changes as follows.
By definition of $\varphi_k$, we have $\tilde{s}_{u_a} = \varphi_k(s_{u_a}) = t_{k} t_{u_a}^{-1}$ and $\tilde{s}_p = \varphi_k(s_p) = t_p$ for all $p \neq u_a$.

It is enough to check that the relations involving $s_{u_a}$ are preserved.

(1) Suppose that $p \notin N_{Q,u_a}$. Then:
- if $p \notin N_{Q,k}$ then, using $t_k t_p = t_p t_k$ and $t_{u_a} t_p = t_p t_{u_a}$ we get
  $$\tilde{s}_{u_a} \tilde{s}_p = t_{k} t_{u_a} t_k^{-1} t_p = t_p t_{k} t_{u_a} t_k^{-1} = \tilde{s}_p \tilde{s}_{u_a}.$$
- if $p \in N_{Q,k}$, then $p = v_i$ for some $i \in \{1, \ldots, b\}$. Thus, using $t_k t_{v_i} t_k = t_{v_i} t_k t_u t_{v_i}$ and $t_k t_{v_i} t_k = t_{v_i} t_k t_{v_i}$, we get
  $$\tilde{s}_{u_a} \tilde{s}_{v_i} = t_{k} t_{u_a} t_k^{-1} t_{v_i} = t_{v_i} t_k t_{u_a} t_k^{-1} t_{v_i} t_k^{-1} t_{v_i}$$
  $$= t_{v_i} t_k t_{u_a} t_{v_i} t_k^{-1} t_{v_i} t_k^{-1} t_{v_i} = t_{v_i} t_k t_{u_a} t_k^{-1} = \tilde{s}_{v_i} \tilde{s}_{u_a}.$$

(2) Suppose $p \in N_{Q,u_a}$. Then:
- if $p \notin N_{Q,k}$, then $p = u_i$ for some $i \in \{0, \ldots, a-1\}$. Then, using the relations $t_k t_{u_a} t_k = t_{u_a} t_k t_{u_a}$, $t_{k} t_{u_i} t_k = t_{u_i} t_k t_{u_i}$ and $t_{u_a} t_{u_i} = t_{u_i} t_{u_a}$ we get
  $$\tilde{s}_{u_a} \tilde{s}_{u_i} \tilde{s}_{u_a} = t_{k} t_{u_a} t_k^{-1} t_{u_i} t_k t_{u_a} t_k^{-1} = t_{u_i}^{-1} t_k t_{u_a} t_{u_i}^{-1} t_{k} t_{u_a}$$
  $$= t_{u_i}^{-1} t_k t_{u_a} t_{u_i} = t_{u_i}^{-1} t_k t_{u_a} t_{u_i}$$
  $$= t_{u_i}^{-1} t_k t_{u_a} t_{u_i} = \tilde{s}_{u_i} \tilde{s}_{u_a} \tilde{s}_{u_i}.$$
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d) If $\tilde{s}$ relations involving $\tilde{s}_u \tilde{s}_a \tilde{s}_w = \tilde{s}_t k_i u w i = \tilde{s}_t t w u a t_k^{-1} t w_i$
when using $t_k t w_i = t_w t_k$ and $t_u a t_w t_u a = t_w t_u a t_k^{-1}$ we get

\[
\tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1} = t_u a t_w t_k u a t_k^{-1} = t_w t_k u a t_k^{-1} t w_i.
\]

(3) If $i, j \in \{0, \ldots, a - 1\}$ and $i < j$ then, using $t_k t u a t_k = t_u a t_k u a$,

\[
t_u a t_u a = t_u a t_u a, t_u a t_j a = t_a t_u a, t_k t u a t_j a = t_a t_k u a t_j a \text{ instead of}
\]

\[
t_k t u a t_j a t_k = t_u a t_k u a t_j a, \text{ one can also show that}
\]

\[
\tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1} = \tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1} \text{ for some}
\]

\[
\tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1} = \tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1}.
\]

(4) If $i, j \in \{0, \ldots, \ell\}$ and $i < j$ then, using $t_k t w_j = t_w t_k$,

\[
t_k t w_j = t_w t_k \text{ and } t_u a t_w t_k t_u a = t_u a t_w t_k t_u a \text{ we get}
\]

\[
\tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_j t_k u a t_k^{-1} = t_k t w_j t_k u a t_k^{-1} t w_i t_u a t_k^{-1} = t_k t w_j t_k u a t_k^{-1} t w_i t_u a t_k^{-1} \text{ instead of}
\]

\[
t_k t w_j t_k u a t_k^{-1} t w_i t_u a t_k^{-1} = \tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1} \text{ for all}
\]

\[
\tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1} = \tilde{s}_u \tilde{s}_w \tilde{s}_u a t_k^{-1} t w_i t_u a t_k^{-1}.
\]

c) The case $c \neq 0, d = 0$ is analogous to b).

d) If $c = d = 0$, then $\tilde{s}_u a = t_k t u a t_k^{-1}$, $\tilde{s}_v b = t_k t v b t_k^{-1}$ and $\tilde{s}_p = t_p$
for all $p \neq u a, v a$. Again, we just need to check that the relations involving $s_u a$ and $s_v b$ are preserved. We will just check that the relations involving $s_u a$ are preserved, since one can apply use the same methods for $s_v b$.

(1) Suppose $p \notin N Q, u a$. Then:

- if $p \neq v b$, then one can show that

\[
\tilde{s}_u a \tilde{s}_p = \tilde{s}_p \tilde{s}_u a.
\]

exactly as in b(1) and b(2).

- if $p = v b$ then, using the relation $t_u a t_v b = t_v b t_u a$ we get

\[
\tilde{s}_u a \tilde{s}_v b = t_k t u a t_k^{-1} t_k t v b t_k^{-1} = \tilde{s}_v b t_k t u a t_k^{-1} t_k t v b t_k^{-1} = \tilde{s}_v b \tilde{s}_u a.
\]
(2) If \( p \in \mathcal{N}_{Q,u_a}, p \neq u_a \), then one can show that
\[
\tilde{s}_{u_a} \tilde{s}_p \tilde{s}_{u_a} = \tilde{s}_p \tilde{s}_{u_a} \tilde{s}_{u_a}
\]
exactly as in b(2).

(3) If \( i, j \in \{0, \ldots, a - 1\} \) with \( i < j \), then one can show that
\[
\tilde{s}_{u_a} \tilde{s}_{u_i} \tilde{s}_{u_a} \tilde{s}_{u_i} = \tilde{s}_{u_i} \tilde{s}_{u_j} \tilde{s}_{u_a} \tilde{s}_{u_j}
\]
exactly as in b(3).

(4) If \( i, j \in \{1, \ldots, \ell\} \) with \( i < j \), then one can show that
\[
\tilde{s}_{u_a} \tilde{s}_{w_i} \tilde{s}_{u_j} \tilde{s}_{u_a} = \tilde{s}_{w_i} \tilde{s}_{w_j} \tilde{s}_{u_a} \tilde{s}_{w_j}
\]
exactly as in b(4).

\[\square\]

**Proposition 5.7.** Let \( Q \) be an \( m \)-coloured quiver of mutation type \( A_{n-1} \rightarrow \), \( k \in Q_0 \), and \( k^{(c)} \rightarrow \ell \) an arrow of colour \( c \) in \( Q \). Let \( \{t_i^{(j)}|i \in Q_0\} \) be the generators of \( B \mu_k(Q) \) for \( j = 0, \ldots, m + 1 \). Then
\[
\varphi_k^j(t^{(0)}_\ell) = \begin{cases} t^{(j)}_\ell, & \text{for } j = 0, \ldots, c \\ t^{(j)}_k t^{(j)}_\ell (t^{(j)}_k)^{-1}, & \text{for } j = c + 1, \ldots, m + 1 \end{cases}
\]
where we set \( t^{(m+1)}_w = t^{(0)}_w \) for all \( w \in Q_0 \).

**Proof.** By the equivalent definition of \( m \)-coloured quiver mutation given in Proposition 2.8, we know that, for each application of \( \mu_k \), the colour of the arrow \( k \rightarrow \ell \) decreases by one. Hence, the colour of the arrow \( k \rightarrow \ell \) changes as follows.

\[
\begin{array}{cccc}
Q & \rightarrow & \mu_k(Q) & \rightarrow \cdots \rightarrow \mu_c^j(Q) \\
(k^{(c)} \rightarrow \ell) & \mapsto & (k^{(c-1)} \rightarrow \ell) & \mapsto \cdots \mapsto (k^{(0)} \rightarrow \ell)
\end{array}
\]

Therefore, if we apply \( \varphi_k \) on \( t^{(0)}_\ell \) multiple times, we get
\[
B_Q \rightarrow B_{\mu_k(Q)} \rightarrow \cdots \rightarrow B_{\mu_c^j(Q)}
\]
\[
t^{(0)}_\ell \mapsto t^{(1)}_\ell \mapsto \cdots \mapsto t^{(c)}_\ell
\]

Thus \( \varphi_k^j(t^{(0)}_\ell) = t^{(j)}_\ell \) for \( j = 0, \ldots, c \).

Now, the arrow \( k \rightarrow \ell \) has colour 0 in \( \mu_c^j(Q) \). Hence the colour of the arrow \( k \rightarrow \ell \) changes as follows.

\[
\begin{array}{cccc}
\mu_c^j(Q) & \rightarrow & \mu_c^{j+1}(Q) & \rightarrow \cdots \rightarrow \mu_c^{m+1}(Q) = Q \\
(k^{(0)} \rightarrow \ell) & \mapsto & (k^{(m)} \rightarrow \ell) & \mapsto \cdots \mapsto (k^{(c)} \rightarrow \ell)
\end{array}
\]

Therefore, if we apply \( \varphi_k \) on \( t^{(c)}_\ell \) multiple times, we get
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Thus $\varphi_j^i(t_\ell^{(0)}) = t_\ell^{(j)}i^{(j)}(t_\ell^{(i)})^{-1}$ for all $j = c+1, \ldots, m+1$. □

**Theorem 5.8.** The group homomorphism $\varphi_k : B_{Q} \to B_{Q'}$ is an isomorphism.

**Proof.** Let $\{t_p | p \in Q_0\}$ be the set of generators for $B_{Q}$ given in Definition 5.1. Consider the following composition of group homomorphisms.

$$\phi : B_{Q} \xrightarrow{\varphi_k} B_{\mu_k(Q)} \xrightarrow{\varphi_k^m} B_{\mu_k^{m+1}(Q)} = B_{Q}$$

Let $\ell \in N_{Q,k}$. Then, using Proposition 5.7, we can say that $\phi(t_\ell) = t_k t_\ell t_k^{-1}$. Furthermore, if $w \notin N_{Q,k}$, then $t_w$ commutes with $t_k$. Thus $\phi(t_w) = t_w = t_k t_w t_k^{-1}$. Hence $\phi(t_p) = t_k t_p t_k^{-1}$ for all $p \in Q_0$.

So $\phi$ is conjugation by $t_k$, that is an isomorphism. Hence $\varphi_k$ is injective.

We can use the same argument to show that the composition

$$\tilde{\phi} : B_{\mu_k(Q)} \xrightarrow{\varphi_k^m} B_{\mu_k^{m+1}(Q)} = B_{Q} \xrightarrow{\varphi_k} B_{\mu_k(Q)}$$

is conjugation by $t_k$, and hence it is an isomorphism. This proves that $\varphi_k$ is surjective, and hence it is an isomorphism. □

**Corollary 5.9.** Let $Q$ be an $m$-coloured quiver of mutation type $\overrightarrow{A_{n-1}}$, and $k \in Q_0$. Let $\{s_i | i \in Q_0\}$ (resp. $\{t_i | i \in Q_0\}$) be a generating set for $B_{Q}$ (resp. $B_{\mu_k(Q)}$). Then $\varphi_k^{-1} : B_{\mu_k(Q)} \to B_{Q}$ is computed by

$$\varphi_k^{-1}(t_i) = \begin{cases} s_k^{-1} s_i s_k, & \text{if } k \xrightarrow{(m)} i \\ s_i, & \text{otherwise.} \end{cases}$$

**Corollary 5.10.** Let $Q$ be an $m$-coloured quiver of mutation type $\overrightarrow{A_{n-1}}$. Then its associated group $B_{Q}$ is isomorphic to the braid group of type $A_{n-1}$.

**Proof.** By Remark 5.3, we know that $B_{\overrightarrow{A_{n-1}}}$ is the braid group of type $A_{n-1}$. Hence Theorem 5.8 implies the statement. □

**Example 5.11.** Consider the 4-angulation of a regular dodecagon given in Example 4.18.
The associated 2-coloured quiver $\Psi(\Delta)$ is the following.

Therefore $B_{\Psi(\Delta)}$ is the group generated by $s_1, s_2, s_3, s_4$, subject to relations

\begin{align*}
  s_1 s_2 s_1 &= s_2 s_1 s_2 \\
  s_2 s_3 s_2 &= s_3 s_2 s_3 \\
  s_3 s_4 s_3 &= s_4 s_3 s_4 \\
  s_4 s_1 s_4 &= s_1 s_4 s_1 \\
  s_1 s_2 s_3 s_1 &= s_2 s_3 s_1 s_2 = s_3 s_1 s_2 s_3 \\
  s_1 s_2 s_3 s_4 s_1 &= s_2 s_3 s_4 s_1 s_2 = s_3 s_4 s_1 s_2 s_3 = s_4 s_1 s_2 s_3 s_4.
\end{align*}

Notice that also cycle-type relations involving all the four vertices 1, 2, 3, 4 hold. Specifically

\begin{align*}
  s_1 s_2 s_3 s_4 s_1 &= s_2 s_3 s_4 s_1 s_2 = s_3 s_4 s_1 s_2 s_3.
\end{align*}
For example, we can get the first relation using (2), (3) and (4) as follows:

\[
\begin{align*}
    s_1 s_2 s_3 s_4 s_1 &\rightarrow s_2 s_3 s_1 s_2 s_1^{-1} s_4 s_1 \rightarrow s_2 s_3 s_1 s_2 s_4 s_1 s_4^{-1} \\
    \rightarrow s_2 s_3 s_4 s_1 s_2 s_4 s_1^{-1} &\rightarrow s_2 s_3 s_4 s_2 s_2.
\end{align*}
\]

The phenomenon described in Example 5.11 can be generalised to subquivers of \(Q\) that are complete graphs on an arbitrary number of vertices. This result can be found in Remark 3.2 of [8], but we include it together with its proof for completeness.

**Remark 5.12.** Let \(j \geq 3\), and \(i_1, \ldots, i_j \in \{1, \ldots, n-1\}\), with \(i_1 < i_2 < \ldots < i_j\). Let \(Q\) be an \(m\)-coloured quiver of mutation type \(A_{n-1}\), and \(\{s_i \mid i \in Q_0\}\) the set of generators of \(B_Q\) introduced in Definition 5.1.

Suppose that the complete quiver on \(i_1, \ldots, i_j\) is a subquiver of \(Q\). Let \(\ell_{p,q}\) be the colour of the arrow \(i_p \to i_q\), for all \(p, q \in \{1, \ldots, j\}\), \(p \neq q\). Then, if

\[\ell_{a,b} + \ell_{b,c} + \ell_{c,a} = 2m + 1\]

for all \(a, b, c \in \{1, \ldots, j\}\) with \(a < b < c\), the following relation holds in \(B_Q\).

\[s_{i_1} s_{i_2} \cdots s_{i_j} s_{i_1} = s_{i_2} s_{i_3} \cdots s_{i_j} s_{i_1} s_{i_2}.\]

**Proof.** We prove the result by induction on \(j \geq 3\).

- If \(j = 3\), then the result follows directly from the definition of \(B_Q\).
- If \(j > 3\), then by induction hypothesis we know that

\[s_{i_1} s_{i_2} \cdots s_{i_{j-1}} s_{i_j} = s_{i_2} s_{i_3} \cdots s_{i_{j-1}} s_{i_1} s_{i_2}.\]

Using this relation together with \(s_{i_1} s_{i_2} s_{i_1} = s_{i_j} s_{i_1} s_{i_j}\) and

\[s_{i_1} s_{i_2} s_{i_3} s_{i_1} = s_{i_j} s_{i_1} s_{i_2} s_{i_j},\]

we get

\[s_{i_1} \cdots s_{i_{j-1}} s_{i_j} s_{i_1} = s_{i_2} \cdots s_{i_{j-1}} s_{i_1} s_{i_2} s_{i_1} s_{i_2} = s_{i_2} \cdots s_{i_{j-1}} s_{i_1} s_{i_2} s_{i_1} s_{i_2} s_{i_1} s_{i_2} = s_{i_2} \cdots s_{i_{j-1}} s_{i_1} s_{i_2} s_{i_1} s_{i_2}.\]

\[\square\]

**Remark 5.13.** In Section 4.3 of [6] the author defines a group \(G^d_s\) that acts on an higher zigzag algebra of type \(A\), for fixed \(s, d \geq 1\).

The construction of the group \(G^d_s\) for \(s = 2\) is the following. Consider the quiver \(Q^d_2\) given by

\[
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
& d + 1 & \leftarrow \ldots
\end{array}
\]

Then the group \(G^d_2\) is generated by \(s_1, \ldots, s_{d+1}\) subject to relations

\[s_{i_1} s_{i_2} \cdots s_{i_j} s_{i_1} = s_{i_2} s_{i_1} s_{i_2} \cdots s_{i_j} s_{i_1} s_{i_2} = \cdots = s_{i_j} s_{i_1} \cdots s_{i_{j-1}} s_{i_j}\]

for all \(1 \leq i_1 < i_2 < \ldots < i_j \leq d + 1\) and \(j \geq 2\).
Quivers of shape $Q_2^d$ arise in my work. Indeed, if we take $m = d - 1$ and $n = d + 2$, then the $m$-coloured quiver associated via the map $\Psi$ to the $(m + 2)$-angulation

$$\Delta^{(d)} = \{(1, d + 1), (d + 1, 2d + 1), (2d + 1, 3d + 1), \ldots, (d^2 + 1, 1)\}$$

of a regular $(mn + 2)$-gon is a complete quiver on $n - 1 = d + 1$ vertices, and contains $Q_2^d$ as a subquiver.

For example, for $d = 4$, the 5-angulation $\Delta^{(4)}$ of the 20-gon described above is given by the following diagram:

![Diagram of a 5-angulation of a 20-gon](image)

and the quiver associated via $\Psi$ to $\Delta^{(4)}$, this contains $Q_2^4$ as subquiver.

Therefore, Remark 5.12 implies that not all the relations from (5) are necessary. The only relations one actually needs to define $G_2^d$ are

$$s_{i_1} s_{i_2} s_{i_3} = s_{i_2} s_{i_1} s_{i_3}, \quad \text{for } 1 \leq i_1 < i_2 < i_3 \leq d + 1 \quad \text{and} \quad s_{i_1} s_{i_2} = s_{i_3} s_{i_1} s_{i_2}, \quad \text{for } 1 \leq i_1 < i_2 \leq d + 1.$$
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