Canonical path integral quantization of the finite dimensional systems with constraints*

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Abstract

The path integral formulation of constrained systems leads to obtain the equations of motion as total differential equations in many variables. If these equations are integrable then one can construct a valid and a canonical phase space coordinates. The path integral is obtained as an integration over the canonical phase space coordinates. This approach is applied to obtain the path integral for three singular systems and it shown that in our formulation there is no need to distinguish between first and second- class constraints, no need for fixing any gauge, as will as no need to enlarge the phase space.

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1 Introduction

The canonical formulation [1-3] gives the set of Hamilton-Jacobi partial differential equations (HJPDE) as

\[
H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_a}) = 0,
\]

\[\alpha, \beta = 0, n - r + 1, \ldots, n, a = 1, \ldots, n - r,\]  

(1)

where

\[
H'_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha,
\]

(2)

and \(H_0\) is defined as

\[
H_0 = p_a w_a + p_\mu q_\mu|_{p_\nu = -H_\nu} - L(t, q_i, \dot{q}_i, \dot{q}_a = w_a),
\]

\[\mu, \nu = n - r + 1, \ldots, n.\]  

(3)

The equations of motion are obtained as total differential equations in many variables as follows:

\[
dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha;
\]

(4)

\[
dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha;
\]

(5)

\[
dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha;
\]

(6)

\[
dZ = (-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha;
\]

(7)

where \(Z = S(t_\alpha; q_a)\). The set of equations (4-7) is integrable [3] if

\[
dH'_0 = 0, \quad (8)
\]

\[
dH'_\mu = 0, \mu = n - p + 1, \ldots, n, \quad (9)
\]

or in equivalent form

\[
[H'_\alpha, \ H'_\beta] = 0 \ \forall \ \alpha, \beta. \quad (10)
\]
Equations of motion reveal the fact that the Hamiltonians $H'_\alpha$ are considered as the infinitesimal generators of canonical transformations given by parameters $t_\alpha$ and the set of canonical phase-space coordinates $q_a$ and $p_a$ is obtained as functions of $t_\alpha$, besides the canonical action integral is obtained in terms of the canonical coordinates. In this case, the path integral representation may be written as \([4-7]\)

$$
D(q'_a, t'_\alpha; q_a, t_\alpha) = \int_{q_a}^D q^a \, Dp^a \times 
\exp i\{ \int_{t_\alpha}^{t'_\alpha} [-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}] dt_\alpha \},
$$

$$
a = 1, \ldots, n - r, \alpha = 0, n - r + 1, \ldots, n. \quad (11)$$

Now we will study the path integral quantization of the finite dimensional systems considering systems with first and second class constraints.

2 Examples

2.1 A system with first class constraints

As a first example let us consider the following Lagrangian \([1]\)

$$
L = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j + b \dot{q}_2 - c, \quad i, j = 1, 2, 3. \quad (12)
$$

The generalized momenta read as

$$
p_1 = a_1 \dot{q}_1, \quad p_2 = a_2 (\dot{q}_3 - \dot{q}_2) + b, \quad p_3 = a_2 (\dot{q}_3 - \dot{q}_2). \quad (13)
$$

The canonical method \([1-3]\) leads us to obtain the set of the Hamilton- Jacobi partial differential equations as follows:

$$
H'_0 = p_0 + \frac{1}{2} (\frac{p_1^2}{a_1} - \frac{p_3^2}{a_2}) + c = 0, \quad (14)
$$

$$
H'_2 = p_2 + p_3 - b = 0, \quad (15)
$$

where $a$, $b$ and $c$ are constants.
The equations of motion are obtained as total differential equations in many variables as follows:

\[
dq_1 = \frac{\partial H'_0}{\partial q_1} dt + \frac{\partial H'_2}{\partial p_1} dq_2 = p_1 dt,
\]

\[
dq_3 = \frac{\partial H'_0}{\partial p_3} dt + \frac{\partial H'_2}{\partial p_3} dq_2 = -\frac{p_1}{a_2} dt + dq_2,
\]

\[
dp_1 = -\frac{\partial H'_0}{\partial q_1} dt - \frac{\partial H'_2}{\partial q_1} dq_2 = 0,
\]

\[
dp_2 = -\frac{\partial H'_0}{\partial q_2} dt - \frac{\partial H'_2}{\partial q_2} dq_2 = 0,
\]

\[
dp_3 = -\frac{\partial H'_0}{\partial q_3} dt - \frac{\partial H'_2}{\partial q_2} dq_2 = 0,
\]

\[
dp_0 = -\frac{\partial H'_0}{\partial t} dt - \frac{\partial H'_2}{\partial t} dq_2 = 0.
\]

From the integrability conditions (8,9) since the total variations \(dH'_0 = 0, dH'_2 = 0\) are satisfied identically, the set of equations are integrable and the canonical phase space coordinates are obtained in terms of parameters \(t\) and \(q_2\) as follows:

\[
q_1 \equiv q_1(t, q_2), \quad p_1 \equiv p_1(t, q_2),
\]

\[
q_3 \equiv q_3(t, q_2), \quad p_3 \equiv p_3(t, q_2).
\]

Making use of (7) and (14, 15), the canonical action integral is obtained as follows

\[
z = \int \{-c + \frac{p_1^2}{2a_1} - \frac{p_3^2}{2a_2}\} dt + bdq_2.
\]

Now the path integral for the given system is obtained as an integration over the canonical phase space coordinates \((q_1, p_1; q_3, p_3)\) as follows:

\[
D(q'_1, q'_3, t', q'_2; q_1, q_3, t, q_2) = \int_{q_1, q_3}^{q_1', q_3'} Dq_1 \ Dq_3 \ Dp_1 \ Dp_3 \times
\exp i \left[ \int_{t, q_2}^{t', q'_2} \left( -c + \frac{p_1^2}{2a_1} - \frac{p_3^2}{2a_2}\right) dt + bdq_2 \right].
\]
2.2 A system with second class constraints

Let us consider the singular Lagrangian

\[ L = \frac{1}{2} q_1^2 - \frac{1}{4} (q_2^2 - 2q_2q_3 + q_3^2) + (q_1 + q_3)q_2 - q_1 - q_2 - q_3^2. \]  

(26)

The generalized momenta read as

\[ p_1 = q_1, \quad p_2 = \frac{1}{2} (q_3 - q_2) + q_1 + q_3, \quad p_3 = \frac{1}{2} (q_2 - q_3). \]  

(27)

Since the rank of the Hessian matrix is two one of the momenta is depending on the others. Thus we have

\[ \dot{q}_1 = p_1 = w_1, \quad \dot{q}_3 = \dot{q}_2 - 2p_3, \]  

(28)

\[ p_2 = -p_3 + q_1 + q_3 = -H_2. \]  

(29)

The Hamiltonian \( H_0 \) is defined as

\[ H_0 = -L(q_1, q_2, \dot{q}_2, w_1, w_3) + p_1w_1 + p_3w_3 + (-p_3 + q_1 + q_3)\dot{q}_2, \]  

(30)

or

\[ H_0 = \frac{1}{2} (p_1^2 - 2p_3^2) + q_1 + q_2 + q_3^2. \]  

(31)

Thus the Hamiltonians \( H'_0 \) and \( H'_2 \) are obtained as

\[ H'_0 = p_0 + \frac{1}{2} (p_1^2 - 2p_3^2) + q_1 + q_2 + q_3^2 = 0, \]  

(32)

\[ H'_2 = p_2 + p_3 - q_1 - q_3 = 0. \]  

(33)

Making use of (32) and (33) the equations of motion of this system are obtained as

\[ dq_1 = \frac{\partial H'_0}{\partial p_1} dt + \frac{\partial H'_2}{\partial p_1} dq_2 = p_1 dt, \]  

(34)

\[ dq_3 = \frac{\partial H'_0}{\partial p_3} dt + \frac{\partial H'_2}{\partial p_3} dq_2 = -2p_3 dt + dq_2, \]  

(35)

\[ dp_1 = -\frac{\partial H'_0}{\partial q_1} dt - \frac{\partial H'_2}{\partial q_1} dq_2 = -dt + dq_2, \]  

(36)

\[ dp_2 = -\frac{\partial H'_0}{\partial q_2} dt - \frac{\partial H'_2}{\partial q_2} dq_2 = -dt, \]  

(37)

\[ dp_3 = -\frac{\partial H'_0}{\partial q_3} dt - \frac{\partial H'_2}{\partial q_3} dq_2 = -2q_3 dt + dq_2, \]  

(38)

\[ dp_0 = -\frac{\partial H'_0}{\partial t} dt - \frac{\partial H'_2}{\partial t} dq_2 = 0. \]  

(39)
To check whether this system is integrable or not, let us consider the variation of $H'_0$, where

$$dH'_0 = dp_0 + dq_1 + dq_2 + p_1 dp_1 + 2q_3 dq_3 - 2p_3 dp_3.$$  (40)

Making use of eqs. of motion (34-39) one obtains

$$dH'_0 = H'_3 dq_2,$$  (41)

where

$$H'_3 = 2p_3 - 2q_3 - p_1 - 1.$$  (42)

Since $H'_3$ is not identically zero, we consider it as a new constraint. Thus for a valid theory, variation of $H'_3$ should be zero. Thus one gets

$$dH'_3 = (1 - 4q_3 + 4p_3) dt - dq_2 = 0,$$  (43)

which can be expressed as

$$\dot{q}_2 - (1 - 4q_3 + 4p_3) = 0.$$  (44)

Again considering the variation of $H'_2$ one obtains

$$\ddot{q}_2 - 8(p_3 - q_3) = 0.$$  (45)

Combining Eqs. (44) and (45), differential equation for $q_2$ is determined as

$$\ddot{q}_2 - 2\dot{q}_2 + 2 = 0,$$  (46)

which has the following solution

$$q_2 = 2A \exp(2t) + t + C,$$  (47)

where $A$ and $C$ are arbitrary constants. Besides the variation of $H'_2$ is

$$dH'_2 = dp_2 + dp_3 - dq_1 - dq_3,$$  (48)

or

$$dH'_2 = H'_3 dt.$$  (49)

The set of equations (34-39) is integrable, hence the canonical action integral is calculated as

$$z = \int [(-H_0 + p_1^2 - 2p_3^2) dt + (-H_2 + p_3) dq_2],$$  (50)
or
\[ z = \int (-H_0 + p_1^2 - 2p_3^2 + q_1 + q_3)dt. \] (51)

Making use of (11) and (51), the path integral for this system is obtained as
\[
D(q'_1, q'_3, t', q'_2; q_1, q_3, t, q_2) = \int_{q_1, q_3}^{q'_1, q'_3} Dq_1 Dq_3 Dp_1 Dp_3 \times \exp i \left[ \int_t^{t'} (-H_0 + p_1^2 - 2p_3^2 + q_1 + q_3)dt \right]. \] (52)

3 Path integral for canonically transformed systems

In this section we will consider the path integral quantization of singular systems after performing some canonical transformations.

Let us perform the canonical transformations
\[
t_\alpha = (t, q_\mu) \rightarrow T_\alpha = (t, Q_\mu), \quad p_0 \rightarrow p_0, \quad q_a \rightarrow Q_a, \quad p_a \rightarrow P_a, \quad a = 1, \ldots, n-r, \quad \alpha = 0, n-r+1, \ldots, n.
\] (53)

In this case the Hamiltonians \( H'_\alpha \) are transformed as follows
\[ H'_\alpha \rightarrow K'_\alpha = P_\alpha + K_\alpha = 0. \] (54)

The transformations (53) and (54) lead us to obtain the path integral representation for this system as
\[
D(Q'_a, T'_\alpha; Q_a, T_\alpha) = \int_{Q_a}^{Q'_a} DQ^a DP^a \times \exp i \left\{ \int_{T_\alpha}^{T'_\alpha} [-K_\alpha + P_\alpha \frac{\partial K'_\alpha}{\partial P_\alpha}]dT_\alpha \right\},
\] (55)

\[ a = 1, \ldots, n-r, \quad \alpha = 0, n-r+1, \ldots, n. \]

The procedure described above will be demonstrated by the following example.

Let us consider the Lagrangian on the three-dimensional configuration space \( R^3 = (x, y, z) \) [8]:
\[
L = \frac{1}{2r^2} (x \ddot{x} + y \ddot{y} + z \ddot{z})^2 - V(x^2 + y^2 + z^2),
\] (56)
where \( r^2 = x^2 + y^2 + z^2 \).

The canonical momenta are obtained as

\[
p_x = \frac{x}{2r^2} (x\dot{x} + y\dot{y} + z\dot{z}), \quad n_y = \frac{y}{2r^2} (x\dot{x} + y\dot{y} + z\dot{z}), \quad n_z = \frac{z}{2r^2} (x\dot{x} + y\dot{y} + z\dot{z}).
\]

(57)

Since the rank of the Hessian matrix is one, the canonical method leads us to obtain the set of Hamilton-Jacobi partial differential equations as follows

\[
H'_0 = p_0 + V(x^2 + y^2 + z^2) + \frac{p_x^2}{2x^2} (x^2 + y^2 + z^2) = 0,
\]

(58)

\[
H'_1 = p_y - y \frac{p_x}{x}, \quad H'_2 = p_z - z \frac{p_x}{x}.
\]

(59)

The equations of motion are obtained as a set of total differential equations as follows

\[
dx = \frac{p_x}{x^2} (x^2 + y^2 + z^2) dt - \frac{y}{x} dy - \frac{z}{x} dz,
\]

(60)

\[
dp_x = -\frac{\partial V}{\partial x} dt + \frac{p_x^2}{x^2} (y^2 + z^2) dt - \frac{zp_x}{x^2} dy - \frac{yp_x}{x^2} dz,
\]

(61)

\[
dp_y = \left( -\frac{\partial V}{\partial y} - \frac{yp_x^2}{x^2} \right) dt + \frac{p_x}{x} dy,
\]

(62)

\[
dp_z = \left( -\frac{\partial V}{\partial z} - \frac{zp_x^2}{x^2} \right) dt + \frac{p_x}{x} dz,
\]

(63)

\[
dp_0 = 0.
\]

(64)

To have a consistent theory one should consider the variations of \( H'_0, H'_1 \) and \( H'_2 \). In fact, one can show that the total variations for each one of them is identically zero. Hence, this system is integrable and the canonical phase space coordinates \((x, p_x)\) are obtained in terms of parameters \((t, y, z)\) and one can use the procedure described in section (1) to obtain the path integral for this system as an integration over the canonical phase space coordinates \(x, p_x\).

Now let us perform the canonical transformations

\[
y \to y, \quad z \to z, \quad x \to \sqrt{R^2 - y^2 - z^2},
\]

(65)

\[
p_x \to \sqrt{R^2 - y^2 - z^2} \frac{Pr}{R}, \quad p_y \to P_y + \frac{yPr}{R}, \quad p_z \to P_z + \frac{zPr}{R}.
\]

(66)

The corresponding set of Hamilton-Jacobi partial differential equations read as

\[
K'_0 = p_0 + \frac{P^2_x}{2} + V(R^2) = 0,
\]

(67)
\[ K'_1 = P_y = 0, \quad \text{(68)} \]
\[ K'_2 = P_z = 0. \quad \text{(69)} \]

This set leads us to obtain the total differential equations
\[ dR = P_R dt, \quad dP_R = -\frac{\partial V}{\partial R} dt, \quad dP_y = 0, \quad dP_z = 0, \quad dp_0 = 0. \quad \text{(70)} \]

Integrability conditions require the total variations of \( K'_0, K'_1 \) and \( K'_2 \) vanish. In fact
\[ dK'_0 = dK'_1 = dK'_2 = 0. \quad \text{(71)} \]

The set of equations (70) is integrable and the canonical phase space coordinates \((R, P_R)\) are obtained in terms of parameters \((t, y, z)\). Besides the canonical action integral is calculated as
\[ dz = \left( \frac{P^2_R}{2} - V(R^2) \right) dt. \quad \text{(72)} \]

Making use of (55) and (72) the path integral representation for this system is obtained as
\[ D(R, y, z, t; R', y', z', t') = \int_R^{R'} DR DP_R \times \exp i \left\{ \int_t^{t'} \left[ \frac{P^2_R}{2} - V(R^2) \right] dt \right\}. \quad \text{(73)} \]

4 Conclusion

The Path integral formulation of constrained systems is obtained using the canonical path integral method introduced by Muslih [4-7]. The staring point of this method is variational principle. The Hamiltonian treatment of constrained systems leads to a set of Hamilton -Jacobi partial differential equations, which leads to obtain the equations of motion as total differential equations in many variables. The equations are integrable if the corresponding system of partial differential equations is a Jacobi system. In this case one can construct a valid and canonical phase space coordinates \( q_a \) and \( p_a \) in terms of parameters \( t_\alpha \) and the path integral is obtained directly as an integration over the canonical phase space coordinates \( q_a \) and \( p_a \).

In the first example since this system is integrable, then the canonical phase space coordinates \((q_1, q_3, p_1, p_3)\) are obtained as an integration over
these canonical phase space coordinates directly without using any gauge fixing conditions [9, 10]. The second example is integrable and the path integral is obtained as an integration over the canonical phase space coordinates \((q_1, q_3, p_1, p_3)\). In the usual formulation [11] one has to integrate over the extended phase space \((q_1, q_2, q_3, p_1, p_2, p_3)\) and after integration over the redundant variables \((q_2, p_2)\) one can arrive at the result (52).

The path integral for singular systems is obtained when a suitable canonical transformation is used. For the system (56), after performing the canonical transformations (53), we obtain the equations of motion as total differential equations which are integrable. In this case canonical phase space coordinates \((R, P_R)\) are obtained in terms of parameters \((t, y, z)\) and the path integral (73) is obtained as an integration over the canonical phase space coordinates \((R, P_R)\).

As a conclusion it is obvious that the Muslih [4-7] method is a direct method to obtain the path integral for constrained systems as an integration over the canonical phase space coordinates \((q_a, p_a)\). It is obvious from the given examples that when applying this method their is no need to distinguish between first and second class constraints, no need to use gauge fixing conditions, no need to enlarge the phase space, as well as no need to add auxiliary dynamical variables expanding the phase space beyond its original classical formulation, including no ghosts. All is needed the set of the Hamilton Jacobi partial differential equations and the set of the equations of motion. Then one should test whether these equations are integrable or not. If the integrability conditions are not satisfied identically, then the total variation of them should be introduced as new constraints of the theory. Repeating this procedure as many times as needed one may obtain a set of conditions. The number of independent parameters of the theory is determined directly, without imposing any gauge fixing conditions by this set.

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