Rotors in triangles and tethrahedra

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1 Introduction

A polytope $P$ is circumscribed about a convex body $\phi \subset \mathbb{R}^n$ if $\phi \subset P$ and each facet of $P$ is contained in a support hyperplane of $\phi$. We say that a convex body $\phi \subset \mathbb{R}^n$ is a rotor of a polytope $P$ if for each rotation $\rho$ of $\mathbb{R}^n$ there exist a translation $\tau$ so that $P$ is circumscribed about $\tau \rho \phi$.

If $Q^n$ is the $n$-dimensional cube then a convex body $\Phi$ is a rotor of $Q^n$ if and only if $\Phi$ has constant width. However, there are convex polytopes that have rotors which are not of constant width.

A survey of results in this area has been given by Golberg [4]. See also the book Convex Figures of Blotyanskii and Yaglom [3].

It is well known that if $\Phi$ is a convex plane figure which is a rotor in the polygon $P$, then every support line of $\Phi$ intersects its boundary in exactly one point, and if $\Phi$ intersects each side of $P$ at the points $\{x_1, \ldots, x_n\}$, then the normals of $\Phi$ at these points are concurrent.

In this paper we shall prove that if $P$ is a triangle, then there is a baricentric formula that describes the curvature of bd$\Phi$ at the contact points. We prove also that if $\Phi \subset \mathbb{R}^3$ is a convex body which is a rotor in a tetrahedron $T$ then the normal lines of $\Phi$ at the contact points with $T$ generically belong to one ruling of a quadric surface.

2 Rotors in the triangle

Consider $\Phi$ a smooth rotor in the triangle $T$ and suppose that the three sides of $T$ intersect the boundary of $\Phi$ at the points $x_1, x_2, x_3$, respectively. As in the case of constant width bodies in which the radii of curvature of the boundary at the ends of a binormal sum to $h$, we are interested in a formula that involves the curvatures of the boundary of $\Phi$ at $x_1, x_2, x_3$.

A $C^m$ framed curve $(\alpha, \lambda)$ is a curve of class $C^m$ given by a parametrization of the following form: there is a support function $P : (-\delta, \delta) \to \mathbb{R}$ of class $C^m$, $m \geq 2$, such that $\alpha(\theta) = P(\theta)u(\theta_0 + \theta) + P'(\theta)u'(\theta_0 + \theta)$ and $\lambda$ is the tangent line through $\alpha(0) = x$, in the direction $x^\perp$. Therefore, $P'(0) = 0$ and $\alpha(0) = P(0)u(\theta_0)$ is the closest point of the line $\lambda$ to the origin and the normal
line of $\alpha$ at $\alpha(0)$ passes through the origin. Where $u(\theta) = (\cos \theta, \sin \theta)$ and $u'(\theta) = (-\sin \theta, \cos \theta)$, for every $\theta \in \mathbb{R}$.

A sliding along two given $C^n$ framed curves $(\alpha_1, \lambda_1)$ and $(\alpha_2, \lambda_2)$ is a one parameter family of Euclidean isometries $L_\theta, \theta \in (-\epsilon, \epsilon), \epsilon > 0$, satisfying

- $L_0$ is the identity map,
- $L_\theta$ rotates the plane by an angle of $\theta$,
- $L_\theta(\lambda_i)$ is a tangent line of the curve $\alpha_i$, for each $\theta \in (-\epsilon, \epsilon)$ and $i = 1, 2$.

**Lemma 1.** Let $(\alpha_1, \lambda_1)$ and $(\alpha_2, \lambda_2)$ be two $C^n$ framed curves. Suppose that their normal lines at $\alpha_1(0) = x_1$ and $\alpha_2(0) = x_2$ are not parallel and are concurrent at the origin. Then

1. there is a unique sliding $L_\theta, \theta \in (-\epsilon, \epsilon), \epsilon > 0$, along them,
2. there is a $C^n$ map $f : (-\epsilon, \epsilon) \to \mathbb{R}^2$ such that $L_\theta(x) = R_\theta(x) + f(\theta)$, for every $x \in \mathbb{R}^2, f(0) = f'(0) = 0$, where $R_\theta$ is the rotation of the plane about the origin by an angle of $\theta$.
3. If the origin does not lie in the line $\lambda_3$, then the envelope of $\{L_\theta(\lambda_3)\}_{\theta \in (-\epsilon, \epsilon)}$ is a $C^n$ framed curve $(\alpha_3, \lambda_3)$, such that the tangent line at $\alpha_3(0)$ is $\lambda_3$ and the normal line at $\alpha_3(0)$ passes through the origin.

**Proof.** Let $E$ be the Lie Group of orientation-preserving isometries of the Euclidean space $\mathbb{R}^2$. Let $R_\theta$ denote the rotation about the origin by an angle of $\theta$. Since every $g \in E$ takes the form $g(x) = R_\theta(x) + f$ for some $\theta$ and a fixed $f \in \mathbb{R}^2$, we will identify a neighborhood of the identity in $E$ with $(-\gamma, \gamma) \times \mathbb{R}^2 \subset \mathbb{R}^3$, via the mapping $(\theta, f) \to R_\theta + f$. Observe that the identity in $E$ is identified with the origin in $\mathbb{R}^3$.

Given a $C^n$ framed curve $(\alpha, \lambda)$ with support function $\mathcal{P}(\theta)$, consider the set

$$\mathcal{S} = \{g \in E \mid g(\lambda) \text{ is a tangent line to } \alpha\}$$

defined in the neighborhood of the identity in $E$ (or of the origin in $\mathbb{R}^3$). We shall prove that $\mathcal{S}$ is a surface of class $C^n$. Indeed, we have the following explicit parametrization: consider the map $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\psi(\theta, t) = (\theta, h(\theta, t))$, where $h(\theta, t) = (\mathcal{P}(\theta) - \mathcal{P}(0))u(\theta_0 + \theta) + tu'(\theta_0 + \theta)$. It is not difficult to verify that for every $-\delta \leq \theta \leq \delta$ and $t \in R$, the isometry $L_\theta + h(\theta, t)$ sends the line $\lambda$ to a tangent line of $\alpha$. Furthermore,

$$\frac{d\psi}{d\theta}(0) = (1, \mathcal{P}'(0)u(\theta_0)) = (1, 0, 0)$$

and

$$\frac{d\psi}{dt}(0) = (0, u'(\theta_0))$$

Moreover, it follows that the normal vector to $\mathcal{S}$ at the origin is $(0, -u(\theta_0))$. 


Now, given two $C^m$ framed curves, $(\alpha_1, \lambda_1)$ and $(\alpha_2, \lambda_2)$, let $S_1$ and $S_2$ be their corresponding surfaces. If $\alpha_i(0) = P_i(0)u(\theta_i)$, then the normal vector to $S_i$ at the origin is $(0, -u(\theta_i))$, $i = 1, 2$, and since $\theta_1 \neq \theta_2$, we have that in a neighborhood of the origin $S_1$ and $S_2$ intersect transversally in a curve of the form $(\theta, f(\theta))$ and hence the sliding can be written as 

$$L_\theta = R_\theta + f(\theta)$$

where $f : (-\epsilon, \epsilon) \to \mathbb{R}^2$ is of class $C^m$.

Thus, for $i = 1, 2$ the support function of $\alpha_i$ is given by 

$$P_i(\theta) = P_i(0) + \langle f(\theta), u(\theta_i + \theta) \rangle.$$ 

where $\langle \cdot, \cdot \rangle$ denotes the interior product.

This implies that $f(0) = 0$ and furthermore, $0 = P_i'(0) = \langle f'(0), u(\theta_i) \rangle$.

Since $\theta_1 \neq \theta_2$, then $f'(0) = 0$.

Finally, let $\theta_3$ be such that $u(\theta_3)$ is orthogonal to the line $\lambda_3$ and let $r_3$ be the distance from $\lambda_3$ to the origin. Then the support function of $\alpha_3$ is given by 

$$P_3(\theta) = r_3 + \langle f(\theta), u(\theta_3 + \theta) \rangle$$

and $P_3'(0) = 0$ as we wished. \qed

For curves of constant width $h$, the sum of the radii of curvature at extreme points of every diameter is $h$. For rotors in a triangle, the analogous result is the following baricentric formula.

**Theorem 1.** Let $\Phi$ be a rotor in the triangle $T$ with vertices $\{A_1, A_2, A_3\}$. Suppose the boundary of $\Phi$ is twice continuous differentiable and let $x_3 = \Phi \cap A_1A_2$, $x_1 = \Phi \cap A_2A_3$ and $x_2 = \Phi \cap A_3A_1$. Let $\{a_1, a_2, a_3\}$ be the baricentric coordinates of the point $O$ with respect to the triangle $T$, where $O$ is the point at which the normal lines to $T$ at the points $x_1$, $x_2$ and $x_3$ concur. If $r_i$ is the distance from $O$ to $x_i$ and $\kappa_i$ the curvature of the boundary of $\Phi$ at $x_i$, $i = 1, 2, 3$, then

$$\frac{a_1}{\kappa_1 r_1} + \frac{a_2}{\kappa_2 r_2} + \frac{a_3}{\kappa_3 r_3} = 1.$$
Proof. Let \( \alpha_i : (-\epsilon, \epsilon) \to \mathbb{R}^2 \) be a \( C^2 \)-parametrization of a neighborhood of the boundary of \( \Phi \) around \( x_i \), with \( \alpha_i(0) = x_i \) and let \( \lambda_i \) be the line through \( A_{i+1}A_{i+2} \mod 3 \), so that \( (\alpha_i, \lambda_i) \) are \( C^2 \) framed curves, whose corresponding normal lines at \( x_i \) are concurrent at \( O \). Suppose without loss of generality that \( O \) is the origin. By Lemma 1, there is a sliding along the three framed curves. That is, there is a one parameter family of Euclidean isometries \( L_\theta \), \( \theta \in (-\epsilon, \epsilon), \epsilon > 0 \), satisfying

- \( L_0 \) is the identity map,
- \( L_\theta \) rotates the plane by an angle of \( \theta \),
- \( L_\theta(\lambda_i) \) is a tangent line of the curve \( \alpha_i \), for each \( \theta \in (-\epsilon, \epsilon) \) and \( i = 1, 2, 3 \).

Furthermore, there is a \( C^2 \) map \( f : (-\epsilon, \epsilon) \to \mathbb{R}^2 \) such that

\[
L_\theta(x) = R_\theta(x) + f(\theta),
\]

for every \( x \in \mathbb{R}^2 \), \( f(0) = f'(0) = 0 \), where \( R_\theta \) is the rotation of the plane through the origin by an angle of \( \theta \).

Let \( \mathcal{P}_i(\theta) \) be the pedal function of the framed curve \( \alpha_i \), with \( \mathcal{P}_i(0) = r_i = |x_i|, i = 1, 2, 3 \). Hence, \( \mathcal{P}_i'(0) = 0 \) and the radius of curvature of the boundary of \( \Phi \) at \( x_i \) is

\[
\frac{1}{\kappa_i} = \mathcal{P}_i(0) + \mathcal{P}_i''(0).
\]

On the other hand, \( \mathcal{P}_i(\theta) = |L_\theta(x_i)| = |R_\theta(x_i) + f(\theta)| \). Hence,

\[
\mathcal{P}_i(\theta)^2 = \langle R_\theta(x_i) + f(\theta), R_\theta(x_i) + f(\theta) \rangle.
\]

So,

\[
\mathcal{P}_i(\theta)\mathcal{P}_i'(\theta) = \langle R_\theta(x_i) + f(\theta), R_\theta(x_i) + f'(\theta) \rangle.
\]

Let \( h_i(\theta) = \langle R_\theta(x_i), f'(\theta) \rangle + \langle R_\theta(x_i), f'(\theta) \rangle + \langle f'(\theta), f'(\theta) \rangle \) in such a way that

\[
\mathcal{P}_i'(\theta) = \frac{h_i(\theta)}{\mathcal{P}_i(\theta)}
\]

and

\[
\mathcal{P}_i''(\theta) = \frac{h_i'(\theta)\mathcal{P}_i(\theta)^2 - h_i(\theta)^2}{\mathcal{P}_i(\theta)^3}.
\]

Note that \( h_i(0) = 0 \) and \( h_i'(0) = \langle f''(0), x_i \rangle \).

Since the radius of curvature of \( \partial \Phi \) at \( x_i \) is given by \( \mathcal{P}_i(0) + \mathcal{P}_i''(0) \), we have that for \( i = 1, 2, 3 \)

\[
\frac{1}{\kappa_i} = r_i + \frac{\langle f''(0), x_i \rangle}{r_i}.
\]
Let \( \{b_1, b_2, b_3\} \) be the baricentric coordinates of the origin \( O \) with respect to the triangle with vertices \( \{x_1, x_2, x_3\} \). That is: \( b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \), with \( b_1 + b_2 + b_3 = 1 \). Hence, for \( i = 1, 2, 3 \),

\[
\frac{b_i r_i^2}{\kappa_i r_i} = b_i r_i^2 + \langle f''(0), b_i x_i \rangle,
\]

and therefore,

\[
\sum b_i r_i^2 = \sum b_i r_i^2 + 0.
\]

To conclude the proof of the theorem, it will be enough to prove that

\[
a_i = \frac{b_i r_i^2}{b_1 r_1^2 + b_2 r_2^2 + b_3 r_3^2}.
\]

The basic property that defines \( A_i \) is

\[
\langle A_i, x_j \rangle = \langle x_j, x_j \rangle = r_j^2 \quad \text{for} \quad i \neq j.
\]

Using it, one easily obtains that

\[
\langle b_1 r_1^2 A_1 + b_2 r_2^2 A_2 + b_3 r_3^2 A_3, x_j \rangle = \langle r_j^2 A_j, b_1 x_1 + b_2 x_2 + b_3 x_3 \rangle = 0,
\]

for \( j = 1, 2, 3 \). This implies that \( b_1 r_1^2 A_1 + b_2 r_2^2 A_2 + b_3 r_3^2 A_3 = 0 \) because the \( x_j \) generate \( \mathbb{R}^2 \), and from here

\[
\sum b_i r_i^2 A_1 + \sum b_i r_i^2 A_2 + \sum b_i r_i^2 A_3 = 0.
\]

It follows that

\[
\frac{a_1}{\kappa_1 r_1} + \frac{a_2}{\kappa_2 r_2} + \frac{a_3}{\kappa_3 r_3} = 1,
\]

as we wished.

\[\square\]

3 The relation with immobilization problems

Immobilization problems were introduced by Kuperberg \[5\] and also appeared in \[8\]. They were motivated by grasping problems in robotics (\[6\] and \[7\]).

Let \( \Phi \subset \mathbb{R}^n \) be a convex body. A collection of points \( X \) on the boundary of \( \Phi \) is said to immobilize \( \Phi \) if any small rigid movement of \( \Phi \) causes one point in \( X \) to penetrate the interior of \( \Phi \). In the plane, for the case in which three points \( X = \{x_1, x_2, x_3\} \) lie in the boundary \( \Phi \), there is a baricentric formula involving the curvature of \( \partial \Phi \) at \( x_i \) that allows us to know if \( X \) immobilizes \( \Phi \). See \[1\].

**Theorem 2.** Let \( \Phi \) be a twice continuous differentiable convex figure and let \( X = \{x_1, x_2, x_3\} \) be three points in the boundary of \( \Phi \), whose normals are concurrent at the point \( O \). Let \( \{a_1, a_2, a_3\} \) be the baricentric coordinates of the point \( O \) with respect to the vertices of the triangle formed be the three support lines...
of $\Phi$ at $x_1$, $x_2$ and $x_3$. Also, let $r_i$ be the distance from $O$ to $x_i$, let $\kappa_i$ be the curvature of the boundary of $\Phi$ at $x_i$, $i = 1, 2, 3$, and let

$$\omega = a_1 \kappa_1 r_1 + a_2 \kappa_2 r_2 + a_3 \kappa_3 r_3.$$  

Then, if $\omega < 1$, \{x_1, x_2, x_3\} immobilize $\Phi$, and if $\omega > 1$, they do not.

There is a duality between Theorem 2 and Theorem 1. While in Theorem 2, we have a rigid segment sliding along the boundary of the convex figure $\Phi$, in Theorem 1, we have a rigid angle (formed by two lines) sliding along the boundary of $\Phi$.

In dimension three, immobilization results are much more complicated. See [2]. To characterize when four points in the faces of a tetrahedron $T$ immobilize $T$ we require the following definition.

Let $\{L_1, L_2, L_3, L_4\}$ be four directionally independent lines in $\mathbb{R}^3$. We say that they belong generically to one ruling of a quadric surface if

- they are concurrent,
- they belong to one ruling of a quadric surface, or
- they meet in pairs and the planes these pairs generate meet in the line through the intersecting points.

**Theorem 3.** A necessary and sufficient condition for four points $\{x_1, x_2, x_3, x_4\}$, in the corresponding four faces of a tetrahedron $T$, to immobilize it, is that the normal lines to $T$ at $x_1, x_2, x_3$ and $x_4$ belong generically to one ruling of a quadratic surface.

The “duality” mentioned above, gives us the following theorem for rotors in a tetrahedron.

**Theorem 4.** Let $\Phi$ a twice continuous differentiable rotor in the tetrahedron $T$, and let $\{x_1, x_2, x_3, x_4\}$ be the points of the boundary of $\Phi$ that intersect the four faces of $T$. Then, the normal lines to $T$ at $x_1, x_2, x_3$ and $x_4$ belong generically to one ruling of a quadratic surface.

**Proof.** Consider a tetrahedron $T$ that circumscribes $\Phi$. For every $\rho \in SO(3)$, let $T(\rho)$ be the tetrahedron directly homothehtic to $\rho T$ circumscribing $\Phi$ and let $V_\Phi(\rho)$ be the volume of of $T(\rho)$. It is not difficult to see that $V_\Phi(\rho)$ depends continuously on $\rho$. 

We will prove that if $\rho_0$ is a local maximum of $V_\Phi(\rho)$, then the four normal lines to the boundary of $\Phi$ at the points that touch the four faces of $T(\rho_0)$, belong generically to one ruling of a quadratic surface. If this is so, then the proof the theorem is complete because $\Phi$ is a rotor in $T$ if and only if $V_\Phi(\rho)$ is constant. For the proof of the above statement, it will be sufficient to consider the case in which $\Phi$ is a tetrahedron. The reason is that if $a, b, c$ and $d$ are the points in which the sides of $T(\rho_0)$ touch the boundary of $\Phi$, then $\rho_0$ is also a local maximum of $V_K(\rho)$, where $K$ is the tetrahedron with vertices $\{a, b, c, d\}$. 

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Let $H_a, H_b, H_c$ and $H_d$ be four planes containing the faces of the tetrahedron $T(\rho_0)$, in such a way that $a \in H_a, b \in H_b, c \in H_c$ and $d \in H_d$, respectively. Assume now that $a T(\rho_0)$ is a rigid tetrahedron sliding along $a, b, c$. That is, $T(\rho_0)$ is sliding rigidly in such a way that the points $a, b, c$ remain fixed but inside the planes $H_a, H_b$ and $H_c$, and during the rigid sliding movement of $T(\rho_0)$, the fixed point $d$ is always inside $T(\rho_0)$.

The proof of Theorem 4 now follows straightforward from the proof or Theorem 3 in [2], but this time we consider, instead of a rigid triangle sliding along three fixed planes, the dual situation of a 3-dimensional rigid sector (the angle between three planes $H_a, H_b$ and $H_c$) sliding along three fixed points $a, b, c$.

\[ \square \]

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