Big Rip in SO(1, 1) phantom universe

Yi-Huan Wei

Department of Physics, Bohai University, Jinzhou 121000, Liaoning, China

Abstract

For the inverse linear potential, the $SO(1, 1)$ field behaves as phantom for late time and the Big Rip will occur. The field approaches zero as time approaches the Big Rip, here. For this potential the phantom equation of state takes the late-time minimum $w_\Phi = -3$. We give some discussions that the Big Rip in the $SO(1, 1)$ model may be treated as either the transition point of universe from expansion to extract phase or the final state. In the latter picture of the universe, the field has the $T$ symmetry and the scale factor possesses the $CT$ symmetry, for which the $SO(1, 1)$ charge $\bar{Q}$ plays a crucial role.

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I. INTRODUCTION

The recent type Ia supernova (SN Ia) observations [1] and the cosmic microwave background radiation (CMBR) measurements [2] indicates that the universe is expanding acceleratedly and thus there exists the dark energy in it [3, 4, 5, 6, 7, 8, 9]. Dark energy may be very likely to possess a super-negative pressure, i.e., the phantom energy [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Phantom violates the weak energy condition and may be unstable, so that the phantom universe may evolve to the Big Rip singularity [7, 8]. In the scalar phantom model with $L = -\frac{1}{2}\dot{\phi}^2 - V(\phi)$, for a constant equation of state $w_X$, there are the scale factor $a \sim (t_{br} - t)^{2/3(1+w_X)}$ and the field $\phi \sim \ln(t - t_{br})$ with $t_{br}$ a Big Rip time [7, 8]. By an appropriate choice of the equation of state $w_X$, there can always be $a(t_{br} - t) = a(t - t_{br})$, but the field isn’t well-defined for $t > t_{br}$. Clearly, the above phantom solution $(a, \phi)$ only describes the phantom universe before the Big Rip. For it, we can have the two possible conclusions: Either such a solution $(a, \phi)$ describes an actual Big Rip which indicates a final state of the universe, or it should be incomplete since the field can not well-defined for $t > t_{br}$ providing that the Big Rip isn’t a final state.

What a Big Rip really means is still puzzling. Taking into account the effects of the quantum gravity, the correction to the behavior of phantom can become important near the singularity [11, 12, 13], so that the Big Rip of the phantom universe may be evaded or moderates at least. The latter case implies that the quantum effects should only give a suppression on the Big Rip derived in a classical gravitational theorem. In other words, a Big Rip may actually indicate a critical state of phantom universe between expansion and extract phase if a Big Rip means an epoch that the quantum effects dominate, or it should still be the final state of the universe providing that the quantum effects only is subordinate and can not stop the phantom universe evolving to the future sudden singularity.

The $SO(1, 1)$ model shows some distinctive features [23]. For the exponential potential, the phantom universe may evade the future sudden singularity and eventually settle into the de Sitter phase [22]. In this paper, we will find a new class of the Big Rip in the $SO(1, 1)$ phantom model, for which the scale factor blows up but the field is regular at the Big Rip. In Sec. II, we analyze the late-time behaviors of the scale factor and the field for the inverse linear potential by using some approximation conditions, and derive the late-time field, the
late-time scale factor and the late-time energy densities. In Sec. III, we first show the late-time phantom equation of state, and then give some discussions of the Big Rip. Here, the Big Rip may be considered as a final state or treated as a critical point from expansion to contract phase. The former situation is conjectured to describe a pair of the universes having the common Big Rip.

II. LATE-TIME UNIVERSE WITH INVERSE POWER-LAW POTENTIAL

The dark energy model with the Lagrangian $\mathcal{L} = \frac{1}{2}(\dot{\Phi}^2 - \dot{\theta}^2\Phi^2) - V$ is derived from the $SO(1,1)$ model by defining $\Phi = \sqrt{\phi_1^2 - \phi_2^2}$ and $\tanh \Theta = \frac{\phi_2}{\phi_1}$, where $\phi_1$ and $\phi_2$ are the two components of the $SO(1,1)$ field, $\Phi$ and $\theta$ can be called the norm and the rotation angle of the $SO(1,1)$ vector (field). The $SO(1,1)$ model may also be considered as a generalization to the quintessence [22, 23], in other words, the quintessence corresponds to the case of the constant $\theta$ in the $SO(1,1)$ model. In this model, the field behaves like phantom for $\dot{\Phi}^2 < \dot{\theta}^2\Phi^2$ and quintessence for $\dot{\Phi}^2 > \dot{\theta}^2\Phi^2$. Here, we will focus on the phantom case and discuss the late-time behavior of the field which is sufficient to analyze the problem of the Big Rip. For a spatially flat, isotropic and homogeneous universe, the late time Einstein equation and equations of motion for $\Phi$ and $\theta$ read [23]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_\Phi,$$

(1)

$$\ddot{\Phi} + 3H\dot{\Phi} + \dot{\theta}^2\Phi + V'(\Phi) = 0,$$

(2)

$$\dot{\theta} = \frac{c}{a^3\Phi^2},$$

(3)

and the energy density and pressure are given by

$$\rho_\Phi = \rho_k + \rho_c + V, \quad p_\Phi = \rho_k + \rho_c - V,$$

(4)

where

$$\rho_k = \frac{1}{2}\dot{\Phi}^2, \quad \rho_c = -\frac{1}{2}\Phi^2\dot{\theta}^2,$$

(5)

a dot and a prime denote $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \Phi}$, respectively. The constant $c$ resembles the $U(1)$ charge $Q$ in the spintessence model [6], which is the $SO(1,1)$ charge and we henceforth mark as $\bar{Q}$. 
From Eq. (4), one can see \( w_\Phi = \frac{\rho_k}{\rho_\Phi} < -1 \) for \( \rho_k + \rho_c < 0 \) and thus for this case the model behaves like phantom. For the exponential potential the late-time universe appears to be stable and will finally be in the de Sitter expansion phase [22]. Another potential studied greatly in quintessence, phantom, tachyon and other models [3, 24, 25, 26, 27] is the inverse power potential. Here, we will consider the following potential of the form
\[
V = V(\Phi) = \frac{V_0}{\Phi},
\]
with the constant \( V_0 > 0 \) having the unit \( M^5 \) and \( M \) energy unit.

Eq. (2) has an extra term \( \dot{\theta}^2 \Phi \) in contrast to the equation of motion of quintessence, which proves to play a crucial role for the phantom case. For convenience, introduce \( \eta_1 \) by
\[
\ddot{\Phi} + 3H \dot{\Phi} = \eta_1 \dot{\theta}^2 \Phi, \tag{7}
\]
where \( \eta_1 \) is assumed to be a small quantity for late time. Putting Eq. (7) and \( V' = -V_0 \Phi^{-2} \) in Eq. (2) leads to the following equation
\[
(1 + \eta_1) \dot{Q}^2 a^{-6} - V_0 \Phi = 0, \tag{8}
\]
which yields the scale factor
\[
a = \left[ \frac{V_0 \Phi}{Q^2(1 + \eta_1)} \right]^{-\frac{1}{6}}. \tag{9}
\]
From (9), it follows the Hubble parameter
\[
H = \frac{1}{6} \left[ -\dot{\Phi} \Phi^{-1} + \dot{\eta}_1 (1 + \eta_1)^{-1} \right], \tag{10}
\]
the first term on the left-hand of which takes a dominant place for late time, as will be seen.

Defining another parameter \( \eta_2 \) by
\[
\eta_2 = -\frac{\rho_k}{\rho_c}, \tag{11}
\]
with \( \rho_c = -\frac{V}{2(1 + \eta_1)} \), which is also assumed to be a small quantity for late time, and substituting the total energy density \( \rho_\Phi = \rho_k + \rho_c + V \) in Eq. (11), then we have
\[
H = \pm \mu_P^{-1} \delta \sqrt{V}, \tag{12}
\]
with \( \delta = \sqrt{1 - \frac{\rho_k}{2(1 + \eta_1)}} \approx \frac{1}{\sqrt{2}} \left[ 1 + \frac{1}{2} (\eta_1 + \eta_2) \right] \) for late time, \( \mu_P = \sqrt{3} M_P \) and \( M_P = 1/\sqrt{8\pi G} \) the reduced Planck mass, where the sign + and − correspond to the cases of expansion and contract universe, respectively. Combining Eqs. (10) and (12) yields the following equation
\[
- \frac{1}{6} [\dot{\Phi} \Phi^{-1} + \xi] = \pm \delta H_P \Phi^{-\frac{1}{2}}, \tag{13}
\]
with \( H_P = \frac{\sqrt{V_0}}{\mu_P} \), where \( \xi = \dot{\eta}_1(1 + \eta_1)^{-1} \), which takes the form \( \xi \simeq \dot{\eta}_1(1 - \eta_1) \), and \( \delta \) can vary very slowly for late time, noticing that \( \eta_1 \) and \( \eta_2 \) have been assumed to be two small quantities for late time.

It is difficult to find exact solution of (13). The direct and useful way to determine whether the universe has a future singularity is to analyze the late-time behavior of the solution. In the following, we will only discuss the solution of (13) in the late-time situation. Treating \( \delta \) as a constant, then from Eq. (13) we obtain

\[
\Phi = \left[ \mp 3\delta H_P(t - t_{br}) + \zeta \right]^2,
\]

where \( t_{br} \) is an integration constant and has the time unit, \( \zeta \) is given by

\[
\zeta = \frac{1}{2} \int \xi \Phi \frac{1}{2} dt.
\]

Substituting (14) in Eq. (9), we obtain the late-time scale factor

\[
a = \left[ \frac{V_0}{Q^2(1 + \eta_1)} \right]^{-\frac{2}{3}} \left[ \mp 3\delta H_P(t - t_{br}) + \zeta \right]^{-\frac{2}{3}}.
\]

Noting that \( \delta \simeq \frac{1}{\sqrt{2}} \) for late time and neglecting \( \zeta \), then (14) and (16) reduce to

\[
\Phi \simeq \frac{3V_0}{2M_P^2} (t - t_{br})^2, \quad a \simeq \left( \frac{\sqrt{3}V_0}{\sqrt{2}M_P} \right)^{-\frac{1}{3}} \left[ \mp Q^{-1}(t - t_{br}) \right]^{-\frac{2}{3}},
\]

where the charge \( Q \) is assumed to be positive, the sign \( - \) is taken for \( t < t_{br} \) and \( + \) for \( t > t_{br} \), which imply the Hubble parameter \( H > 0 \) for \( t < t_{br} \) and \( H < 0 \) for \( t > t_{br} \). The variations of the late-time scale factor and field with time are shown in Fig.1.

The approximation expression (17) for \( \Phi \) and \( a \) is sufficient for deriving the leading terms of the late-time \( \rho_k \), \( \rho_c \) and \( V \). Putting \( a \) and \( \Phi \) given in (17) into Eqs. (5) and (6), then we obtain

\[
\rho_k \simeq \frac{9}{2} V_0^2 M_P^{-4} (t - t_{br})^2,
\]

\[
\rho_c \simeq -\frac{1}{3} M_P^2 (t - t_{br})^{-2},
\]

\[
V \simeq \frac{2}{3} M_P^2 (t - t_{br})^{-2},
\]

which show that \( \rho_c \) and \( V \) become infinite and \( \rho_k \) diminishes to zero as \( t \to t_{br} \), and give the late-time energy density of the universe \( \rho_\Phi = \rho_k + \rho_c + V \simeq \frac{1}{3} M_P^2 (t - t_{br})^{-2} \). Noting
FIG. 1: In this figure, $\tau$ is defined as $\tau = t - t_{br}$, the scale factor $a$ and the field $\Phi$ are given in the units $\frac{3V_0}{2M_P^2}$ and $(\frac{\sqrt{3V_0}}{\sqrt{2qM_P}})^{\frac{1}{3}}$, respectively.

that $\dot{\theta}^2\Phi = -2\rho_c\Phi^{-1}$ and the late-time Hubble parameter $H \simeq -\frac{1}{3}(t - t_{br})^{-1}$, then from (17) there are

$$\dot{\theta}^2\Phi = \frac{4}{9}V_0^{-1}M_P^4(t - t_{br})^{-4}, \quad \ddot{\Phi} = 3V_0M_P^{-2}, \quad 3H\dot{\Phi} = -3V_0M_P^{-2}. \quad (21)$$

Clearly, $\eta_1 = \frac{\dot{\Phi} + 3H\dot{\Phi}}{\dot{\theta}^2\Phi} \simeq 0$ and $\eta_2 = -\frac{\rho_k}{\rho_c} \sim O[(t - t_{br})^4]$ are indeed the two small quantities for late time, as anticipated above. From $\xi \simeq \eta_1(1 - \eta_1)$ and $\Phi$ in (17), it follows that $\zeta \sim 0$ for late time. Thus, the scale factor $a$ and the field $\Phi$ in Eq. (17) and the energy densities $\rho_k$, $\rho_c$ and $V$ given in Eqs. (18)-(20) become more and more precise as time approaches the Big Rip.

III. CONCLUSIONS AND DISCUSSIONS

In the previous section, it has been shown that the late-time $\rho_c$ and $V$ are quadratically divergent and $\rho_k$ approaches zero as $t \to t_{br}$. From Eqs. (18)-(20), the late-time equation of state is given by $w_\Phi = \frac{\rho_k + \rho_c - V}{\rho_k + \rho_c + V} \simeq -3 + 54V_0^2M_P^{-6}(t - t_{br})^4$, which reaches its minimum $-3$ at the Big Rip. Eqs. (18)-(20) show that near the Big Rip the total energy density of
the universe grows according to the law \((t - t_{br})^{-2}\) same as that in given in [7, 8]. Eq. (17) also appears the late-time scale factor obeys the same law \(a \sim (t - t_{br})^{2/3(w_{k})}\) given in terms of \(w_{k}\) for a constant equation of state. However, the scale factor in (17) can contain an additional information from the charge \(\bar{Q}\), as will seen in the following. Besides, the descriptions for the dynamics fields are also quite different, in the scale phantom model the field has a logarithmatic divergence at the Big Rip, while in the \(SO(1, 1)\) phantom model the field can be regular at it, as shown in Fig.1.

The Big Rip may imply the nonanalyticity of the scale factor resulted from the model itself, which may be removed in the quantum gravity theorem [13]. If so, then the Big Rip can be specified as a critical state of the universe. For this treatment, a phantom universe will undergo the two different stages: the expansion and the contract phase. For us, such a Big Rip should be viewed as the end of our universe, but it can look like an initial state for the living who uses \(\tau = 0\), instead of \(t = 0\), as the beginning of the time. Alternatively, the Big Rip may be considered as the final state [7, 8]. For \(t > t_{br}\), from Eq. (17), we have

\[
\Phi \simeq \frac{3V_0}{2M_P^2} \tau^2, \quad a \simeq \left(\frac{\sqrt{3V_0}}{\sqrt{2}M_P}\right)^{1/2}(\bar{Q}^{-1})^{-\frac{1}{2}},
\]

(22)

with \(\tau = t - t_{br}\). Eq. (22) shows clearly the symmetry transformation \(\tau \rightarrow -\tau\) (the time-reversal transformation for the redefined time \(\tau\), which is still called \(T\), here) and \(\bar{Q} \rightarrow -\bar{Q}\) (looking like charge conjugate transformation in particle physics, which is also called \(C\)) about the Big Rip. As a result, the field is \(T\)-invariant and the scale factor is \(CT\)-invariant. This symmetry can induce us to imagine the Big Rip as an infinite plane mirror, to which the scale factor has the imaginary image. This can suggest a pair of the universes characterized by the charge \(\bar{Q}\) and having the common Big Rip, one with \(\bar{Q} > 0\) is the universe on which we live and for us the other with \(\bar{Q} < 0\) is the image universe which may be an actual universe for the human being who live there.

In summary, there evidently are some inviting characteristics for the \(SO(1, 1)\) Big Rip. As a mathematic interest, the field and the scale factor are well-defined both in the range \(t > t_{br}\) and \(t < t_{br}\); The field becomes zero at the Big Rip, instead of infinity, this indicates a fundamental difference from the one given in the single field phantom model for which both the scale factor and the field blow up; The field possesses \(T\) symmetry and the scale factor has \(CT\) symmetry, which can suggest a pair of the universes having the same final state.
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