Quintessence dynamics with two scalar fields and mixed kinetic terms

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The dynamical properties of a model of dark energy in which two scalar fields are coupled by a noncanonical kinetic term are studied. We show that overall the addition of the coupling has only minor effects on the dynamics of the two-field system for both potentials studied, even preserving many of the features of the assisted quintessence scenario. The coupling of the kinetic terms enlarges the regions of stability of the critical points. When the potential is of an additive form, we find the kinetic coupling has an interesting effect on the dynamics of the fields as they approach the inflationary attractor, with the result that the combined equation of state of the scalar fields can approach $-1$ during the transition from a matter dominated universe to the recent period of acceleration.

I. INTRODUCTION

One goal of cosmology is to understand the origin of the observed accelerated expansion of the universe (see [1] and references therein). To date, there are several suggestions, including the cosmological constant, slowly evolving scalar fields and modifications to Einstein’s theory of general relativity. Since scalar fields are predicted by many particle physics theories, scalar field models of dark energy, such as quintessence [2, 3, 4] or k-essence [5, 6, 7] have been studied in considerable depth in the past. One requirement a satisfactory model of dark energy must fulfill is that it leads to an equation of state (EOS) close to $w = -1$, in order to agree with current observational data. For single scalar fields, exponential potentials with slope $\lambda$ lead to a scalar field dominated universe with late-time accelerated expansion if $\lambda < \sqrt{3}$. In this case, the duration of the matter dominated epoch depends on the initial conditions for the scalar field. Thus, the situation is not better than that with a cosmological constant. If, on the other hand, $\lambda > \sqrt{3(1+w)}$, the scalar field scales with the dominant fluid (with EOS $w$ [8, 9]). This would help the initial condition problem, but unfortunately, the scaling solution and the accelerating solution are mutually exclusive. The situation with inverse power-law potentials is better in the sense that a wide range of initial conditions end up with the same cosmology at late times, but in order for the theory to be consistent with observational data, the exponent has to be small.

On the other hand, if quintessence is indeed the scenario realized in nature, the quintessence sector might turn out to be rather nontrivial. There might, for example, be several scalar fields interacting [10, 11, 12, 13] and/or the scalar fields might interact with matter in the universe [14, 15, 16, 17]. In this paper we will study interacting scalar fields as a model for dark energy and study whether this can shed some light on the issues in quintessence model building. Models with multiple scalar fields have been considered in the past: partly to study isocurvature ( entropy) perturbations (e.g., [18, 19, 20, 21]) and/or non-Gaussianity generated during inflation (see e.g., [22, 23, 24, 25] and references therein). It was also found that the cumulative effect of many scalar fields could relax the constraints on the inflationary potential [26, 27, 28]. These ideas have also been used in models of dark energy (assisted quintessence [13, 29, 30]). Like the case with inflation in the very early universe, there is no reason why dark energy is not driven by several interacting scalar fields. In a scenario with several scalar fields one might hope to find an explanation for the coincidence problem, i.e. why dark energy dominates today and not earlier during the cosmic history. In single scalar field models this is rather difficult to achieve (see e.g., [1]).

We consider a simple extension of the standard case with canonical normalized fields by allowing for a cross-term in the kinetic energy of the scalar field, which, in the case of a homogeneous and isotropic universe, is proportional to $\dot{\phi} \chi$. This is the simplest extension possible in the two-field case without changing the properties of the potential energy. Kinetic interactions between multiple scalar fields have been previously studied in the context of models in which the dark energy equation of state can become less than $-1$ [31, 32].

The paper is organized as follows: in the next section we present the model and discuss some aspects of the dynamics of the system, such as effective exponents. In Secs. III and IV we perform a critical point analysis for two types of exponential potentials and in Sec. V we present the results of a numerical analysis. In Sec. VI we consider the case of a varying coupling function. Our conclusions can be found in Sec. VII.

II. THE MODEL

The general action for theories which describe the dynamics of two scalar fields with noncanonical kinetic
terms can be written as
\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{2} + P(X_{mn}, \phi_m) \right) \]  
(1)
where \(m, n = 1, 2\) and
\[ X_{mn} = -\frac{1}{2} (\partial_{\mu} \phi_m \partial^\mu \phi_n), \]
(2)
in units where \(M_{Pl} = 1/\sqrt{8\pi G} = 1\). In this paper, we will focus on a model in which a modification to the standard two-field system, determined by the coupling \(a(\phi, \chi)\), is introduced. The scalar field Lagrangian becomes
\[ P = X_{11} + X_{22} + a(\phi, \chi) X_{12} - V(\phi, \chi), \]
(3)
where, for simplicity, we have introduced the notation \(\phi_1 = \phi\) and \(\phi_2 = \chi\). In the following we specialize to a flat, homogeneous and isotropic universe, where the expansion is described by the scale factor \(R(t)\).

If \(a\) is constant, one can perform a field redefinition and diagonalise the kinetic terms so the action takes a canonical form. This is not possible in the general case, which is considered in Sec. [VI]. We will work in the field basis defined above, which facilitates comparison with previous work [13]. In a similar way to the analysis presented in [33, 34], solutions with constant \(a\) can be considered instantaneous critical points, valid for a particular value of \(a\).

In order to study scaling solutions in this unusual system we consider the scalar field system in conjunction with a perfect fluid of energy density \(\rho\), pressure \(p\) and EOS \(p = (\gamma - 1)\rho\). The Friedmann equations are
\[ 3H^2 = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\chi}^2 + \frac{2}{3} \phi \dot{\chi} + V + \rho, \]
(4)
\[ \dot{H} = -\frac{1}{2} (\dot{\phi}^2 + \dot{\chi}^2 + a\dot{\phi} \dot{\chi} + \gamma \rho), \]
(5)
and the variation of the action with respect to the fields yields
\[ \ddot{\phi} + 3H \dot{\phi} \left(1 - a^2 \right) + 3H \dot{\chi} \left(1 - a^2 \right) + V_\phi = 0, \]
(6)
\[ \ddot{\chi} + 3H \dot{\chi} \left(1 - a^2 \right) + V_\chi = 0, \]
(7)
where \(V_\phi = \partial V/\partial \phi\), \(V_\chi = \partial V/\partial \chi\) and \(H = \dot{R}/R\). The dot indicates a derivative with respect to time \(t\). The appearance of second time derivatives of both scalar fields in each equation is characteristic of systems of this type. However, in the case at hand the equations can be separated for \(a^2 \neq 4\), yielding
\[ \ddot{\phi} \left(1 - \frac{a^2}{4}\right) + 3H \dot{\phi} \left(1 - \frac{a^2}{4}\right) + V_\phi - \frac{a}{2} V_\chi = 0, \]
(8)
\[ \ddot{\chi} \left(1 - \frac{a^2}{4}\right) + 3H \dot{\chi} \left(1 - \frac{a^2}{4}\right) + V_\chi - \frac{a}{2} V_\phi = 0. \]
(9)
To understand the behaviour when \(a = \pm 2\), consider the reparametrization \(\phi = \frac{1}{2} (\psi_1 + \psi_2)\), \(\chi = \frac{1}{2} (\psi_1 - \psi_2)\). Substituting these expressions into Eqs. (8) and (9) gives
\[ (1 + \frac{a}{2}) (\ddot{\psi}_1 + 3H \dot{\psi}_1) + 2 \frac{\partial V}{\partial \psi_1} = 0, \]
(10)
\[ (1 - \frac{a}{2}) (\ddot{\psi}_2 + 3H \dot{\psi}_2) + 2 \frac{\partial V}{\partial \psi_2} = 0. \]
(11)
It can be seen from these equations that when \(a = 2\), \(\psi_2\) does not play a dynamical role and its value is determined by \(\psi_1\). Similarly, when \(a = -2\), \(\psi_1\) is not independent of

![FIG. 1: Assisted case parameter space showing the stability of the critical points for (a) \(a = -1\) and (b) \(a = 1\) with \(\gamma = 1\). The shaded area indicates the region with \(\Omega_{\text{fields}} = 1\). When \(a\) is negative there are six stable critical points, defined by the lines \(2\lambda + a\mu = 0\), \(2\mu + a\lambda = 0\) and \(\chi^2 = 3\gamma\). As \(a \to 0\), points E and F dominate the parameter space and when \(a\) is positive, only these points are stable.](Image)
the effective exponent for the assisted potential in our method to that used in [35] to solve for the system near the critical points. We will perform a dynamical systems analysis, similar to that used in [9], to convert the equations of motion into an autonomous system and investigate the properties of the system near the critical points.

### A. Potentials

We will study two forms of the scalar field potential, the first an additive exponential potential similar to that used in assisted inflation [26, 27, 28] and assisted quintessence [13, 29, 30] scenarios,

\[ V_a(\phi, \chi) = e^{-\lambda \phi} + e^{-\mu \chi}. \]  

(12)

This shall be referred to henceforth as the assisted potential. The attractive feature of this potential is that in the case where no matter is present, one can find a solution where the two-field system behaves like a single field with a potential less steep than either field. This allows scalar fields that are incapable of driving acceleration by themselves to co-operate with each other to sustain cosmic acceleration. It is well known that the scale factor \( R(t) \) in a single field system with potential \( V(\phi) = e^{-\lambda \phi} \) evolves as \( R \propto t^p \), where \( p = 2/\lambda^2 \). Applying a similar method to that used in [35] to solve for \( p = 2/\lambda^2 \) yields the effective exponent for the assisted potential in our system,

\[ \frac{1}{\lambda^2_{\text{eff}}} = \frac{1}{\mu^2} + \frac{1}{\lambda^2} + \frac{a}{\mu \lambda}. \]  

(13)

In addition, we shall also consider a multiplicative exponential potential, previously studied in the context of inflation in [25, 36, 37],

\[ V_a(\phi, \chi) = e^{-\lambda \phi} \mu \chi. \]  

(14)

This will be referred to as the soft potential, as its behaviour is similar to that of soft inflation models [38] in which the presence of additional scalar fields inhibits the ability of the model to produce acceleration. Defining the effective exponent for the soft potential by \( p = 2/\mu^2_{\text{eff}} \) and repeating the analysis above, we find

\[ \frac{1}{\mu^2_{\text{eff}}} = \frac{1}{\lambda^2 + \mu^2 - a \lambda \mu}. \]  

(15)

The effective exponents \( \lambda^2_{\text{eff}} \) and \( \mu^2_{\text{eff}} \) can be used to simplify the untidy expressions one finds when matter is included.

### III. ASSISTED POTENTIAL

After defining new variables,

\[ x_1 = \frac{\dot{\phi}}{\sqrt{6} H}, \quad x_2 = \frac{\dot{\chi}}{\sqrt{3} H}, \quad y_1 = e^{-\lambda \phi/2}, \quad y_2 = e^{-\mu \chi/2}, \]  

(16)

the autonomous system is

\[ x_1' = -3x_1 + \sqrt{\frac{3}{2}} \left( 1 - \frac{a^2}{4} \right)^{-1} (\lambda y_2 - \frac{a}{2} \mu y_2) + x_1 \Xi_a, \]  

(17)

\[ x_2' = -3x_2 + \sqrt{\frac{3}{2}} \left( 1 - \frac{a^2}{4} \right)^{-1} (\mu y_2 - \frac{a}{2} \lambda y_1) + x_2 \Xi_a, \]  

(18)

\[ y_1' = -\sqrt{\frac{3}{2}} \lambda x_1 y_1 + y_1 \Xi_a, \]  

(19)

\[ y_2' = -\sqrt{\frac{3}{2}} \mu x_2 y_2 + y_2 \Xi_a, \]  

(20)

where ‘ indicates a derivative with respect to \( N = \ln R \) and

\[ \Xi_a = \frac{3}{2} [2x_1^2 + 2x_2^2 + 2ax_1 x_2 \]  

\[ + \gamma (1 - x_1^2 - x_2^2 - ax_1 x_2 - y_1^2 - y_2^2)]. \]  

(21)
We define the contribution of the fields to the energy density of the universe by

\[ \Omega_{\text{fields}} = x_1^2 + x_2^2 + ax_1x_2 + y_1^2 + y_2^2, \]  

(22)

so that the constraint \( 0 \leq \Omega_{\text{fields}} \leq 1 \) arises from [11].

The critical points corresponding to this system are shown in Table II. It is useful to compare these results to those in Table I in [13], where the authors examine a system of two canonical scalar fields with the same potential (originally studied with nonzero curvature in [11]). The points with \( y_1, y_2 < 0 \) have been excluded as unphysical. It is also possible to place further constraints on the parameter space by excluding the solutions with negative values of \( x_1 \) and \( x_2 \) i.e. where the fields roll up the potential. However, as there are no stable solutions of this kind, the unadulterated results are shown. The results of the analysis are summarised in Table II and the stability regions of the points are plotted in Fig. 1.

| Point   | Existence               | Conditions for Stability | \( \Omega_{\text{fields}} \) | \( \gamma_{\text{fields}} \) |
|---------|-------------------------|--------------------------|-----------------------------|------------------------------|
| A       | All \( \mu, \lambda, a \) | Unstable; \( \lambda > 4 \) | 0                           | Undefined                   |
| B       | \( x_2^2 < (1 - a^2/4)^{-1} \) | Unstable; \( \lambda > 4 \) | 1                           | 2                            |
| C1      | \( \mu \geq \sqrt{3a} (4 - a^2) \) | \( \mu < -a\lambda/2 \) | \( \frac{3a^2}{4(4 - a^2)} \) | 1 \( \gamma \)               |
| D1      | \( \mu \leq \sqrt{2a} (4 - a^2) \) | \( \mu < \min \left[ -a\lambda/2, \sqrt{3a} (4 - a^2) \right] \) | 1                           | \( \frac{4\mu^2}{3(4 - a^2)} \) |
| C2      | \( \lambda \geq \sqrt{3a/4 - a^2} \) | \( \lambda < -a\mu/2 \) | \( \frac{3a^2}{4(4 - a^2)} \) | 1 \( \gamma \)               |
| D2      | \( \lambda \leq \sqrt{2a} (4 - a^2) \) | \( \lambda < \min \left[ -a\mu/2, \sqrt{3a} (4 - a^2) \right] \) | 1                           | \( \frac{4\mu^2}{3(4 - a^2)} \) |
| E       | \( 2\mu + a\lambda > 0 : 2\mu + a\lambda > 0 : \lambda_{\text{eff}}^2 > 3\gamma \) | Stable; \( \lambda_{\text{eff}}^2 < 3\gamma \) | \( 3\gamma / \lambda_{\text{eff}} \) | 1 \( \lambda_{\text{eff}}^2 / 3 \) |
| F       | \( 2\lambda + a\mu > 0 : 2\mu + a\lambda > 0 : 0 < \lambda_{\text{eff}}^2 < 6 \) | Stable; \( \lambda_{\text{eff}}^2 < 3\gamma \) | \( 3\gamma / \lambda_{\text{eff}} \) | 1 \( \lambda_{\text{eff}}^2 / 3 \) |

**TABLE II: Stability of points for the assisted potential**

with \( x_2 \) constrained only by the requirement that \( x_1, x_2 \in \mathbb{R} \). The variables are real if the term under the square root is positive, giving the condition \( x_2^2 < (1 - a^2/4)^{-1} \). Together, the + and − solutions form a continuous locus of points, describing an ellipse in the \( x_1, x_2 \) plane (it is a circle when \( a = 0 \)). As \( a^2 \rightarrow 4 \), the eccentricity increases and the shape rotates, anticlockwise for positive \( a \) and clockwise for negative \( a \). The eigenvalues are

\[ e_1 = 3(2 - \gamma) > 0 \]

\[ e_2 = 0 \]

\[ e_3 = 3 - \sqrt{2a} \]

\[ e_4 = 3 - \sqrt{\frac{2a}{\lambda}} \left( -\frac{1}{2} x_2a \pm \frac{1}{2} \sqrt{x_2^2(a^2 - 4) + 4} \right) \]

(25)

The ± in \( e_4 \) corresponds to that in (22); \( e_1 \) is positive, so the point is either an unstable node or saddle depending on the values of \( x_2, a, \mu \) and \( \lambda \).

**Point A**

This is a trivial solution in which the scalar fields have no role to play. There are no constraints on the existence of the point so it is valid for all parameter values. As \( \Omega_{\text{fields}} = 0 \) and the \( y_1 \) and \( y_2 \) values are 0 the EOS is undefined. The eigenvalues are

\[ e_1 = e_2 = \frac{3}{2} \gamma > 0 \]

\[ e_3 = e_4 = -\frac{1}{2} (2 - \gamma) < 0, \]

(23)

so this point is a saddle.

**Point B**

The point B is a kinetically driven solution with \( \Omega_{\text{fields}} = 1 \), given by,

\[ x_1 = -\frac{1}{2} x_2a \pm \frac{1}{2} \sqrt{x_2^2(a^2 - 4) + 4} \]

\[ x_2 = x_2, \quad y_1 = y_2 = 0, \]

(24)
a plotted against large values of \( \lambda \).

(b) \( \gamma \) and \( e \) responding to this point are

\[
e_1 = -\frac{3}{4}(2 - \gamma) < 0, \\
e_2 = \frac{4}{9}(2\mu + a\lambda), \\
e_{3,4} = \frac{3(2 - \gamma)}{4} \\
\times \left[ 1 \pm \sqrt{\frac{6\gamma^2(4 - a^2) + \mu^2(2 - 9\gamma)}{\mu^2(2 - \gamma)}} \right].
\] (26)

Using the condition for existence, one can show

\[6\gamma^2(4 - a^2) + \mu^2(2 - 9\gamma) \leq \mu^2(2 - \gamma),\]

so \( e_{3,4} \leq -\frac{3}{4}(2 - \gamma)(1 \pm 1) \leq 0 \). Thus, C1 is stable when \( e_2 \) is negative i.e. \( \mu < -a\lambda/2 \).

- **Point D1**

D1 is a scalar field-dominant solution (\( \Omega_{\text{fields}} = 1 \)) analogous to 5 in [13], so the only existence condition comes from \( y_2 \). Therefore, the point exists when \( \mu \leq \sqrt{\frac{3}{2}(4 - a^2)} \). The eigenvalues for this point are

\[
e_1 = \frac{\mu(2\mu + a\lambda)}{(4 - a^2)}, \\
e_2 = \frac{4\mu^2}{(4 - a^2)} - 3\gamma \\
e_{3,4} = \frac{2\mu^2}{(4 - a^2)} - 3 \leq 0
\] (27)

where the existence condition ensures that \( e_{3,4} \) are negative. If the point is to be stable both \( e_1 \) and \( e_2 \) must be negative, so the point is a stable node if

\[
\mu < \min \left[ -\frac{a\lambda}{2}, \sqrt{\frac{3\gamma}{4}(4 - a^2)} \right].
\]

- **Points C2 and D2**

These points are related to C1 and D2 by the transformation \( x_1 \leftrightarrow x_2 \), \( y_1 \leftrightarrow y_2 \), \( \lambda \leftrightarrow \mu \). Thus, the existence and stability conditions are in terms of \( \lambda \) instead of \( \mu \).

- **Point E**

The point E is a scaling solution analogous to 8 in [13]. \( y_1 \) and \( y_2 \) are square roots and \( \Omega_{\text{fields}} \neq 1 \) so there are three conditions that have to be satisfied for the point to exist. To ensure that \( y_1 \) and \( y_2 \) are real and nonzero, one has to impose \( 2\mu + a\lambda > 0 \) and \( 2\lambda + a\mu > 0 \). The third condition is given in terms of \( \lambda_{\text{eff}} \) by \( \lambda_{\text{eff}}^3 \geq 3\gamma \). The eigenvalues for this point are

\[
e_{1,2} = -\frac{3}{4}(2 - \gamma) \\
\times \left[ 1 \pm \sqrt{\frac{24\gamma^2(2\lambda_{\text{eff}}^2 + 2 - 9\gamma)}{(2 - \gamma)}} \right],
\] (28)

\[
e_{3,4} = -\frac{3}{4}(2 - \gamma) \\
\times \left[ 1 \pm \sqrt{\frac{8\gamma(2\mu + a\lambda)(2\lambda + a\mu)}{\lambda\mu(2 - \gamma)(4 - a^2)}} \right].
\] (29)

Using the existence condition, one finds \( \frac{24\gamma^2}{\lambda_{\text{eff}}^3} + 2 - 9\gamma \leq (2 - \gamma) \). Substituting this into (28) gives \( e_{1,2} \leq -\frac{3}{4}(2 - \gamma)(1 \pm 1) \leq 0 \). It is clear that the term under the square root in (29) must be less than 1 if both eigenvalues are to be negative. This means the point is always stable when the conditions for existence are satisfied.

- **Point F**

F is a scalar field-dominant solution analogous to 7...
in [13]. The $y_1$ and $y_2$ values are

\[
\begin{align*}
y_1 &= \frac{1}{\mu \lambda} \sqrt{\frac{\lambda^2_{\text{eff}} \mu (\mu + a \lambda)}{6} \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right)} \\
y_2 &= \frac{1}{\mu \lambda} \sqrt{\frac{\lambda^2_{\text{eff}} \lambda (\lambda + a \mu)}{6} \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right)},
\end{align*}
\]

so for the point to be real we require

\[
\begin{align*}
\lambda^2_{\text{eff}} (2 \mu + a \lambda) \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right) &> 0 \\
\lambda^2_{\text{eff}} (2 \lambda + a \mu) \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right) &> 0.
\end{align*}
\]

Assuming \(1 - \frac{\lambda^2_{\text{eff}}}{6}< 0\) and using [13] leads to a contradiction, so the conditions for the point to exist are,

\[
0 < \lambda^2_{\text{eff}} < 6, \quad (2 \mu + a \lambda) > 0, \quad (2 \lambda + a \mu) > 0. \tag{31}
\]

The eigenvalues corresponding to this point are

\[
\begin{align*}
e_1 &= -3 \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right) < 0 \\
e_2 &= \lambda^2_{\text{eff}} - 3 \gamma \\
e_{3,4} &= \frac{3}{2} \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right) \\
&\times \left[1 \pm \sqrt{\frac{4 \lambda^2_{\text{eff}} (2 \mu + a \lambda)(2 \lambda + a \mu)}{3 \lambda \mu (4 - a^2) \left(1 - \frac{\lambda^2_{\text{eff}}}{6}\right)}}\right]. \tag{32}
\end{align*}
\]

$e_{3,4}$ are negative if the term under the square root is less than $1$, which is always true as the factor in the second term under the square root is positive. Therefore, the point is stable when $\lambda^2_{\text{eff}} < 3 \gamma$.

In the case where $a = 0$, $\lambda_{\text{eff}}$ is smaller than $\lambda$ and $\mu$. This means that the points E and F are similar to the scaling and scalar field-dominant solutions of a single scalar field [1] with a flatter potential, which naturally makes an accelerating solution more favourable (cf. assisted inflationary scenarios [20]). The fact that $a$ can be both positive and negative complicates this slightly but the general principle is the same. When $a \geq 0$ the conditions for existence and stability in terms of $\lambda_{\text{eff}}$ are the same as in [13] although it must be noted that the additive factor of $a/\lambda \mu$ means that $\lambda_{\text{eff}}$ is generally smaller for given values of $\lambda$ and $\mu$. As $a \rightarrow -2$, the stability region for the scalar field dominated points shrinks. When $a < -1$ one can find combinations of fields that each have a small exponent (so considered individually they would dominate the universe) that are out of range of the $\Omega_{\text{fields}} = 1$ attractors.

The observed value of the EOS of dark energy is very close to $w = -1$. A problem identified by the authors
### IV. SOFT POTENTIAL

We define new variables for the three-dimensional soft system,

$$x_1 = \frac{\dot{\phi}}{\sqrt{6H}}, \quad x_2 = \frac{\dot{\chi}}{\sqrt{6H}}, \quad y = \frac{\sqrt{\Omega_{\text{fields}}}}{\sqrt{3H}}$$  \tag{33}$$

Using these, the autonomous system is

$$x'_1 = -3x_1 + \sqrt{\frac{3}{2}} \left(1 - \frac{q^2}{4}\right)^{-1} \left(\lambda - \frac{q}{\mu}\right) y^2 + x_1 \Xi_s, \quad \tag{34}$$
$$x'_2 = -3x_2 + \sqrt{\frac{3}{2}} \left(1 - \frac{q^2}{4}\right)^{-1} \left(\mu - \frac{q}{\lambda}\right) y^2 + x_2 \Xi_s, \quad \tag{35}$$
$$y' = -\sqrt{\frac{3}{2}} (\lambda x_1 + \mu x_2) y + y \Xi_s, \quad \tag{36}$$

where

$$\Xi_s = \frac{3}{2} [2x_1^2 + 2x_2^2 + 2a x_1 x_2 + \gamma (1 - x_1^2 - x_2^2 - ax_1 x_2 - y^2)]. \quad \tag{37}$$

We again have the constraint $0 \leq \Omega_{\text{fields}} \leq 1$ from $\Omega$, but in this case the contribution of the fields to the energy density of the universe is given by

$$\Omega_{\text{fields}} = x_1^2 + x_2^2 + ax_1 x_2 + y^2. \quad \tag{38}$$

The critical points for this system are shown in Table III again excluding unphysical values of $y$. The stability and existence conditions are investigated below, with a summary of the results in Table IV. The stability regions of the points are plotted in Fig. 2.

- **Points A and B**
  These points are kinetically driven and are equivalent to points A and B in the assisted case. As the potential terms are zero in each case, the stability analysis is identical to that in the previous section.

- **Point C**
  C is actually two points, one with the + sign in $x_1$ and $x_2$ and the other with the − sign. These points

| Existence | Stability | $\Omega_{\text{fields}}$ | $\gamma_{\text{fields}}$ |
|-----------|-----------|-----------------|-----------------|
| A         | All $\mu, \lambda, a$ | Unstable | 0 | Undefined |
| D         | $x_2^2 < (1 - a^2/4)^{-1}$ | Unstable | 1 | 2 |
| C         | $(\mu_{\text{eff}}/6 - 1) > 0$ | Unstable | 1 | 2 |
| D         | $\mu_{\text{eff}} < 6$ | $\mu_{\text{eff}} < 3\gamma$ | 1 | $\mu_{\text{eff}}/3$ |
| E         | $\mu_{\text{eff}} \geq 3\gamma$ | Stable | $3\gamma/\mu_{\text{eff}}$ | $\gamma$ |

**TABLE IV: Stability of points for the soft potential**

| $\Omega_{\text{fields}}$ | $\gamma_{\text{fields}}$ |
|-----------------|-----------------|
| A               | 0 | Undefined |
| D               | 1 | 2 |
| C               | 1 | 2 |
| D               | 1 | $\mu_{\text{eff}}/3$ |
| E               | $3\gamma/\mu_{\text{eff}}$ | $\gamma$ |

**TABLE III: Critical points for the soft potential**

$\Omega_{\text{fields}} = 1$ [the grey shaded region in Fig. 1(a)] has a similar shape to that when $a$ is zero or positive but is covered by the three points D1, D2 and F. $w_{\text{fields}}$ and the $x_1$, $x_2$, $y_1$ and $y_2$ values are continuous on the boundary. In the region governed by D1, only the $\chi$ field has a nonzero potential and, regardless of the second field, its exponent $\mu$ solely determines the character of the critical point. D2 is similarly dependent on only one field. These points account for more and more of the scalar field-dominant ($\Omega_{\text{fields}} = 1$) region of the parameter as $a \rightarrow -2$. The effect of this is that whenever the value of $\lambda$ is much larger than that of $\mu$ or vice versa the values of $x_1$, $y_2$, $w_{\text{fields}}$ etc. are sensitive only to changes in the smaller of the two exponents.

When $a$ is negative, the region of the $(\mu, \lambda)$-parameter space for which $\Omega_{\text{fields}} = 1$ has a similar shape to that when $a$ is zero or positive but is covered by the three points D1, D2 and F. $w_{\text{fields}}$ and the $x_1$, $x_2$, $y_1$ and $y_2$ values are continuous on the boundary. In the region governed by D1, only the $\chi$ field has a nonzero potential and, regardless of the second field, its exponent $\mu$ solely determines the character of the critical point. D2 is similarly dependent on only one field. These points account for more and more of the scalar field-dominant ($\Omega_{\text{fields}} = 1$) region of the parameter as $a \rightarrow -2$. The effect of this is that whenever the value of $\lambda$ is much larger than that of $\mu$ or vice versa the values of $x_1$, $y_2$, $w_{\text{fields}}$ etc. are sensitive only to changes in the smaller of the two exponents.
are also kinetically driven, with surprisingly complicated expressions for $x_1$ and $x_2$. The condition for existence is determined by the condition that the terms under the square roots must be positive. In terms of $\mu_{\text{eff}}$, this means that

$$(\mu_{\text{eff}}^2/6 - 1) > 0,$$

to ensure that $x_1$ and $x_2$ are real. The only nonzero eigenvalue for these points is

$$e_1 = 3(2 - \gamma) > 0,$$

so the points are always unstable nodes.

**Point D**

D is a scalar field dominated solution somewhat similar to point F in the assisted case, as it is stable for values of the effective exponent less than $3\gamma$. To ensure the $y$ value is real and nonzero, one requires $\mu_{\text{eff}} < 6$. The eigenvalues are

$$e_{1,2} = \frac{1}{2}\left(\mu_{\text{eff}}^2 - 6\right) < 0 \quad e_3 = \mu_{\text{eff}}^2 - 3\gamma,$$

so the point is a stable node if one applies the stronger constraint $\mu_{\text{eff}} < 3\gamma$.

**Point E**

This is a scaling solution similar to point E in the assisted case. The square root in $y$ means that $\mu_{\text{eff}}^2 > 0$. The density parameter of the fields has the value $3\gamma/\mu_{\text{eff}}^2$ so the condition $\Omega_{\text{fields}} \leq 1$ means that $\mu_{\text{eff}}^2 \geq 3\gamma$ must be satisfied if the point is to exist. The eigenvalues are

$$
e_1 = -\frac{3}{2} < 0$$

$$
e_{2,3} = -\frac{3(2 - \gamma)}{4}$$

$$\times \left[ 1 \pm \sqrt{\frac{24\gamma^2/\mu_{\text{eff}}^2 + 2 - 9\gamma}{2 - \gamma}} \right].$$

(V. NUMERICAL ANALYSIS)

Ideally, one would like the energy density of the scalar fields to scale with the dominant matter component before causing a period of acceleration. Although, as we have seen in the previous section, there exist scaling solutions for this system, they are of limited phenomenological interest. This is due to the fact that the regions for which scaling occurs and those that give rise to acceleration are mutually exclusive and the boundaries do not evolve in time. Depending on the values of the constant parameters $\lambda$, $\mu$ and $a$, the system either does not exhibit acceleration or the scalar fields quickly dominate the universe. There is an interesting scenario mentioned.

FIG. 4: The EOS for the soft case. (a) is a contour plot showing how $w_{\text{fields}}$ varies with $\mu$ and $\lambda$ for $a = 0$ and $\gamma = 1$. For large values of $\lambda$ and $\mu$, point E is stable so $w_{\text{fields}} = 0$. (b) plots $w_{\text{fields}}$ along the line $\mu = \lambda$ for $a = -1$ (dot dashed), $a = 0$ (solid) and $a = 1$ (dashed). In (c) $w_{\text{fields}}$ is plotted against $a$ for $\lambda = \mu = 0.4$ (solid), $\lambda = \mu = 0.8$ (dot-dash) and $\lambda = \mu = 1.2$ (dashed).
FIG. 5: Numerical integration for the case $a = 0$, $\lambda = 3$, $\mu = 1$ and $\gamma = 1$. As in [13], the initial conditions ($\phi = 1$, $\chi = 30$, $\dot{\chi} = \dot{\phi} = 0$) are chosen so that $\chi$ is initially negligible. The upper left panel shows $\Omega_{\text{fields}}$ (solid) and $\Omega_m$ (dashed). The lower left panel shows $w_{\text{fields}}$; the upper and lower dotted lines show the expected values at critical points E and F respectively. The upper right panel shows the evolution of $x_1$ (black) and $y_1$ (grey) and the lower right panel shows the evolution of $x_2$ (black) and $y_2$ (grey). In both of these the dotted lines are the expected values at point E and the dot-dashed lines are the expected values at point F. It will be seen that the dynamical variables approach the critical point values with very little oscillation.

This behaviour can be understood by examining the evolution of $x_1$, $x_2$, $y_1$ and $y_2$. In the $a = 0$ case and with the $\chi$ field initially negligible, the kinetic ($x_2$) and potential ($y_2$) terms of the $\chi$ field do not play a role until eventually $y_2$ increases. In the $a = 1.99$ case, $\exp(-\mu\chi_{\text{ini}})$ is small so $V_\chi \ll V_\phi$. In this case, the equations of mo-
As in the $a = 0$ case, the system first approaches the scaling point $E$, which means $x_1 \to \sqrt{\frac{2}{3} \gamma / \lambda}$. Thus, the evolution of $y_1$ is quickly halted and, until the $\chi$-field becomes significant, the fields only occupy a fraction of the energy density of the universe. Effects of this type are not seen in the case of the soft potential as the condition $V_\chi \ll V_\phi$ corresponds to $\mu \gg \lambda$, which is not realised in the region where the $\Omega\text{fields} = 1$ attractor is stable.

This feedback effect occurs whenever $a$ is close to 2 and $V_\chi \ll V_\phi$ (or $V_\phi \ll V_\chi$) so is affected both by the initial conditions and the relative sizes of $\lambda$ and $\mu$. It is noteworthy that one does not actually require the initial conditions of the two fields to be different to see the scaling effect and the decrease in $\omega\text{fields}$ (see Fig. 7). If the initial contribution of the scalar fields to energy density is small compared to that of matter, a small difference in the values of $\mu$ and $\lambda$ will yield the condition $x_1 \approx -x_2$ that pushes the EOS to $-1$. When the system is initially close to the critical point there is no scaling regime, but as long as $V_\chi \ll V_\phi$ and $a \approx 2$, one still observes the effect on $\omega\text{fields}$. However, although the scalar field dominated
solution is an attractor, the question of when the scalar fields become cosmologically important is dependent on the initial conditions.

We can gain further insight into this behaviour (in the case of constant $a$) by performing a field redefinition to diagonalise the kinetic terms,

$$
\varphi_1 \equiv \sqrt{\frac{1}{2} \left(1 - \frac{a}{2}\right)} (\phi - \chi),
$$

$$
\varphi_2 \equiv \sqrt{\frac{1}{2} \left(1 + \frac{a}{2}\right)} (\phi + \chi),
$$

so that the Lagrangian for the scalar field system with the assisted potential becomes,

$$
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \varphi_1 \partial^{\mu} \varphi_1) - \frac{1}{2} (\partial_{\mu} \varphi_2 \partial^{\mu} \varphi_2) - e^{-\lambda (A \varphi_1 + B \varphi_2)} - e^{-\mu (B \varphi_2 + A \varphi_1)},
$$

with $V_1 = e^{-\lambda A \varphi_1}$ and $V_2 = e^{\mu A \varphi_1}$ constant. This is similar to the case in [10], where a double exponential potential is used to give a scaling regime followed by acceleration driven by the scalar field.

VI. FIELD DEPENDENT COUPLING

In the previous analysis, we assumed that the kinetic coupling, $a$, is constant in time and independent of the scalar fields. In this section we relax this assumption and extend the analysis to include $a = a(\phi, \chi)$ for the assisted potential. The Friedmann equations (4) and (5) are unaffected but the equations of motion (8) and (9) become

$$
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \varphi_2 \partial^{\mu} \varphi_2) - V_1 e^{\lambda \varphi_2/2} - V_2 e^{-\mu \varphi_2/2},
$$

where $V_1 = e^{-\lambda A \varphi_1}$ and $V_2 = e^{\mu A \varphi_1}$ constant.
have additional terms on the right-hand side,

\[ \ddot{\phi} + 3H\dot{\phi} = \left(1 - a^2 \right)^{-1} \left( V_{\phi} - \frac{a^2}{2} V_{\chi} \right) \]

\[ \ddot{\chi} + 3H\dot{\chi} = \left(1 - a^2 \right)^{-1} \left( V_{\chi} - \frac{a^2}{2} V_{\phi} \right) \]

where \( a_{\phi} \) and \( a_{\chi} \) are the partial derivatives of \( a \) with respect to the fields \( \phi \) and \( \chi \) respectively. Our previous analysis is valid for two cases of particular interest: a slowly varying coupling and a rapid change between approximately constant values of \( a \). In the former case, the effect of the extra terms on the dynamics of the fields is very slight and we can consider the asymptotic solutions discussed in Sec. III as instantaneous critical points (cf. 33, 34), valid for particular values of \( a(\phi, \chi) \). The system will evolve toward one of the stable critical points as before; as \( a \) varies only slowly, the solution will track the variation in \( a \). This can be seen in Fig. 8 where the solution tracks the point \( F \) as \( a \) slowly increases to larger values.

When the rate of change of \( a \) is faster than the typical response time of the system, in order to understand the behaviour one would typically have to study the effect of the terms in \( a_{\phi} \) and \( a_{\chi} \) in (47) and (48) for a particular model. An interesting exception to this is when the timescale during which \( a \) changes is short. An example of this is illustrated in Fig. 9 where we have chosen \( a(\phi, \chi) \propto \tanh(8(\phi + \chi) + 25) \), with \( \phi_{\text{ini}} = \chi_{\text{ini}} = 2 \), \( \mu = 2.1 \) and \( \gamma = 1 \). The quantities plotted are the same as those in Fig. 8 except that when point \( F \) is unstable, the dotted line in the second panel shows \( w_{\text{fields}} = 0 \). Initially the point \( E \) is stable and the system exhibits scaling. After \( a \) changes rapidly the existence conditions for \( E \) are no longer satisfied and the system evolves toward the field dominant point \( F \).

FIG. 8: A slowly varying coupling function, modelled by \( a(\phi, \chi) = \log_{10} \left( \frac{1}{2} (\phi + \chi) + 5 - \frac{a^2}{2} \right) \), with \( \phi_{\text{ini}} = \chi_{\text{ini}} = 2 \), \( \mu = 2.1 \) and \( \gamma = 1 \). The upper panel shows \( \Omega_{\text{fields}} \) (solid) and \( \Omega_{\text{ini}} \) (dashed) and the middle panel shows the EOS (solid) and the value of \( a \) at the critical point, \( (w_{\text{fields}})_F \) (dotted). The lower panel shows the evolution of \( a \). The system initially heads toward the field-dominant point \( F \). As \( a \) increases, \( w_{\text{fields}} \) tracks the change in \( (w_{\text{fields}})_F \).

FIG. 9: A rapid shift in \( a \), modelled by \( a(\phi, \chi) = 1.9 \times \tanh(8(\phi + \chi) + 25) \), with \( \phi_{\text{ini}} = \chi_{\text{ini}} = 2 \), \( \mu = 2.1 \) and \( \gamma = 1 \). The quantities plotted are the same as those in Fig. 8 except that when point \( F \) is unstable, the dotted line in the second panel shows \( (w_{\text{fields}})_E = 0 \). Initially the point \( E \) is stable and the system exhibits scaling. After \( a \) changes rapidly the existence conditions for \( E \) are no longer satisfied and the system evolves toward the field dominant point \( F \).
VII. DISCUSSION

Although the use of scalar fields to model the acceleration of the universe could well be described as standard practice in cosmology, the fundamental physics that underpins these phenomenological models is by no means clear. It is wise, therefore, to consider the possibility that the observed acceleration of the universe may have more exotic origins. The recent focus on scalar fields with non-canonical kinetic terms derived from string-inspired models has shown that models that deviate from the standard lore are capable of producing fruitful results that display qualitatively different behaviour. In this work we have considered a model of dark energy in which the kinetic terms of two scalar fields are coupled by a term in the Lagrangian of the form $a(\partial_{\alpha} \phi \partial^\alpha \chi)$. This allowed us to study the simplest extension of the standard case without modifying the potential energy. After analysing the phase plane structure of the model for two different potentials, we found that the basic features of the uncoupled case were preserved and the most important effect of the extra degree of freedom was to change the range of parameters that lead to the different types of solutions. In particular, there exist stable points corresponding to scalar field-dominant or scaling solutions for every value of $\lambda$, $\mu$ and $a$ considered.

In the case of the assisted potential we found that for negative $a$, critical points C1, D1, C2 and D2 that existed but were unstable in the case without the kinetic coupling became stable. This means that when $a$ goes from positive to negative values, there are three different regions in the $\lambda$, $\mu$-parameter space that give rise to accelerated solutions, each with a different value of the EOS, $w_{\text{fields}}$. This is less interesting from a phenomenological point of view, however, as except when the values of the potential exponents are very small, negative $a$ values have $w_{\text{fields}}$ larger than the observed dark energy EOS. In the case of positive $a$, one finds the EOS to be closer to $-1$ than that in the model with $a = 0$, though in the region of interest ($w_{\text{fields}} \approx -1$) the effect of the kinetic coupling is extremely slight. The soft potential evinced a similar dependence on $a$, though it is of less interest as a cosmological model as accelerated behaviour only occurs with small values of the exponents in all cases.

Although models with exponential potentials exhibit both scaling and scalar field-dominant solutions, it is difficult to solve the coincidence problem without fine-tuning the initial conditions or allowing the parameters to vary in time. In Sec. IV we discuss how the kinetic coupling affects the dynamics of the fields as they evolve towards the critical points. When $a$ is close to 2 one finds that the value of $w_{\text{fields}}$ becomes very close to $-1$ for a brief period before the scalar fields dominate. This is a partial resolution of the perennial problem of the EOS being too large.

This analysis is revealing as the addition of a cross-kinetic term has only a relatively minor effect on the dynamics of the two-field system for both cases studied, even preserving many of the features of the assisted quintessence scenario. In the case of the assisted potential, the kinetic coupling has an interesting effect on the dynamics of the fields as they approach the stable solution, with the result that the EOS of the scalar fields can approach $-1$ during the transition from a matter dominated universe to the recent period of acceleration, something that does not occur in assisted quintessence or the single field case. A natural extension to this work is to consider the case when the scalar fields are coupled directly to the matter fluid by a similar mechanism to that described in [13]. In this case, one would expect the value and stability ranges of the critical points to change dramatically due to the extra degree(s) of freedom, but it would be interesting to see whether the additive potential still allows large values of the potential exponents. The work could also be extended by treating $\lambda$ and $\mu$ as dynamical variables in a manner similar to [34], in which case it may be possible to obtain solutions with a large range of initial conditions that track the background solution and later dominate.

As we have discussed, our results are also applicable in the case of a slowly varying coupling function $a(\phi, \chi)$ as well as the case in which the coupling undergoes a rapid change but is constant (or evolving very slowly) after the transition.

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