CLOSED MAGNETIC GEODESICS ON $S^2$

MATTHIAS SCHNEIDER

Abstract. We give existence results for simple closed curves with prescribed geodesic curvature on $S^2$, which correspond to periodic orbits of a charge in a magnetic field.

1. Introduction

The trajectory of a charged particle on an orientable Riemannian surface $(N, g)$ in a magnetic field given by the magnetic field form $\Omega = k \, dA$, where $k : N \to \mathbb{R}$ is the magnitude of the magnetic field and $dA$ is the area form on $N$, corresponds to a curve $\gamma$ on $N$ that solves

$$D_{t,g} \dot{\gamma} = k(\gamma) J_g(\gamma) \gamma$$

where $D_{t,g}$ is the covariant derivative with respect to $g$, and $J_g(x)$ is the rotation by $\pi/2$ in $T_x N$ measured with $g$ and the orientation chosen on $N$. A curve $\gamma$ in $N$ that solves (1.1) will be called a ($k$-)magnetic geodesic. We refer to [3, 5, 11] for the Hamiltonian description of the motion of a charge in a magnetic field. Taking the scalar product of (1.1) with $\dot{\gamma}$ we see that if $\gamma$ is a magnetic geodesic then $(\gamma, \dot{\gamma})$ lies on the energy level

$$E_c := \{ (x, V) \in TN : |V|_g = c \}.$$ 

The geodesic curvature $k_g(\gamma, t)$ of an immersed curve $\gamma$ at $t$ is defined by

$$k_g(\gamma, t) := |\dot{\gamma}(t)|^{-2}_g \langle (D_{t,g} \dot{\gamma})(t), N_g(\gamma(t)) \rangle_g,$$

where $N_g(\gamma(t))$ denotes the unit normal of $\gamma$ at $t$ given by

$$N_g(\gamma(t)) := |\dot{\gamma}(t)|^{-1}_g J_g(\gamma(t)) \dot{\gamma}(t).$$

By (1.1), a nonconstant curve $\gamma$ on $E_c$ is a $k$-magnetic geodesic if and only if its geodesic curvature $k_g(\gamma, t)$ is given by $k(\gamma(t))/c$. We will take advantage of the latter description and consider the equation

$$D_{t,g} \dot{\gamma} = |\dot{\gamma}| g k(\gamma) J_g(\gamma) \gamma.$$ 

We call equation (1.2) the prescribed geodesic curvature equation, as its solutions $\gamma$ are constant speed curves with geodesic curvature $k_g(\gamma, t)$ given by $k(\gamma(t))$. For fixed $k$ and $c > 0$ the equations (1.1) and (1.2) are equivalent in the following sense: If $\gamma$ is a nonconstant solution of (1.2) with $k$ replaced by $k(\gamma(t))$ then $\gamma$ is a solution of (1.1) with $c = k(\gamma(t))/c$.
by \( k/c \), then the curve \( t \mapsto \gamma(ct/|\gamma|_g) \) is a \( k \)-magnetic geodesic on \( E_c \) and a \( k \)-magnetic geodesic on \( E_k \) solves \((1.2)\) with \( k \) replaced by \( k/c \).

We study the existence of closed curves with prescribed geodesic curvature or equivalently the existence of periodic magnetic geodesics on prescribed energy levels \( E_k \).

There are different approaches to this problem, the Morse-Novikov theory for (possibly multi-valued) variational functionals (see [20, 24, 25]), the theory of dynamical systems using methods from symplectic geometry (see [3, 10–14, 22]) and Aubry-Mather’s theory (see [5]).

We suggest a new approach, instead of looking for critical points of the (possibly multi-valued) action functional we consider solutions to (1.2) as zeros of the vector field \( X_{k,g} \) defined on the Sobolev space \( H^{2,2}(S^1, N) \) as follows: For \( \gamma \in H^{2,2}(S^1, N) \) we let \( X_{k,g}(\gamma) \) be the unique weak solution of

\[
( -D_{t,g}^2 + 1)X_{k,g}(\gamma) = -D_{t,g}\dot{\gamma} + |\gamma|_g k(\gamma)J_g(\gamma)\dot{\gamma}
\]

in \( T_\gamma H^{2,2}(S^1, N) \). The uniqueness implies that any zero of \( X_{k,g} \) is a weak solution of \((1.2)\) which is a classical solution in \( C^2(S^1, N) \) applying standard regularity theory. The vector field \( X_{k,g} \) as well as the set of solutions to \((1.2)\) is invariant under a circle action: For \( \theta \in S^1 = \mathbb{R}/\mathbb{Z} \) and \( \gamma \in H^{2,2}(S^1, N) \) we define \( \theta \ast \gamma \in H^{2,2}(S^1, N) \) by

\[
\theta \ast \gamma(t) = \gamma(t + \theta).
\]

Moreover, for \( V \in T_\gamma H^{2,2}(S^1, N) \) we let

\[
\theta \ast V := V(\cdot + \theta) \in T_{\theta \ast \gamma} H^{2,2}(S^1, N).
\]

Then \( X_{k,g}(\theta \ast \gamma) = \theta \ast X_{k,g}(\gamma) \) for any \( \gamma \in H^{2,2}(S^1, N) \) and \( \theta \in S^1 \). Thus, any zero gives rise to a \( S^1 \)-orbit of zeros and we say that two solutions \( \gamma_1 \) and \( \gamma_2 \) of \((1.2)\) are (geometrically) distinct, if \( S^1 \ast \gamma_1 \neq S^1 \ast \gamma_2 \).

We will apply this approach to the case \( N = S^2 \), equipped with a smooth metric \( g \), and \( k \) a positive smooth function on \( S^2 \). We shall prove

**Theorem 1.1.** Let \( g \) be a smooth metric and \( k \) a positive smooth function on \( S^2 \). Suppose that one of the following three assumptions is satisfied,

\[
4\inf(k) \geq (\text{inj}(g))^{-1} (2\pi + (\sup K_g^-) \text{vol}(S^2, g)), \quad (1.4)
\]

\[
K_g > 0 \text{ and } 2\inf(k) \geq \sup(K_g) \frac{1}{2}, \quad (1.5)
\]

\[
\sup(K_g) < 4\inf(K_g), \quad (1.6)
\]

where \( K_g \) denotes the Gauss curvature, \( K_g^- := -\min(K_g, 0) \), and \( \text{inj}(g) \) the injectivity radius of \((S^2, g)\). Then there is a simple curve \( \gamma \in C^2(S^1, S^2) \) that solves \((1.2)\) and the number of simple solutions of \((1.2)\) is even provided they are all nondegenerate.

Concerning the existence of closed \( k \)-magnetic geodesics for a positive smooth function \( k \) on \((S^2, g)\) the following is known (see [10, 11])
(i) if $c > 0$ is sufficiently small, then $E_c$ contains two simple closed magnetic geodesics,
(ii) if $g$ is sufficiently close to the round metric $g_0$ and $k$ is sufficiently close to a positive constant, then there is a closed magnetic geodesic in every energy level $E_c$,
(iii) if $c > 0$ is sufficiently large, then $E_c$ contains a closed magnetic geodesic.

Using the equivalence between (1.1) and (1.2) we obtain from Theorem 1.1 Corollary 1.2. Let $g$ be a smooth metric, $k$ a positive smooth function on $S^2$, and $c > 0$. Suppose that one of the following three assumptions is satisfied,

$$c \leq 4\left(\inf(k)\right)\left(\text{inj}(g)\right)\left(2\pi + \left(\sup K_g\right)\text{vol}(S^2, g)\right)^{-1},$$  \hspace{1cm} (1.7)

$$K_g > 0 \text{ and } c \leq 2\inf(k)\left(\sup(K_g)\right)^{-\frac{1}{2}},$$  \hspace{1cm} (1.8)

$$\sup(k) < 4\inf(K_g).$$

Then $E_c$ contains a simple magnetic geodesic and the number of simple magnetic geodesics in $E_c$ is even provided they are all nondegenerate.

Condition (1.7) should be compared to the existence results in (i) and gives bounds on the required smallness of $c$ in terms of geometric quantities. To show that (1.7) is useful despite the implicit definition of $\text{inj}(g)$, we apply an estimate of $\text{inj}(g)$ in [17] and obtain (1.8) as a special case. The pinching condition (1.6) extends the existence result in (ii) and shows for instance that on the round sphere there is a simple curve of prescribed geodesic curvature $k$ for any positive function $k$, which gives a partial solution to a problem posed by Arnold in [4, 1994-35, 1996-18] concerning the existence of magnetic geodesics on $S^2$ on every energy level $E_c$.

By the famous Lusternik-Schnirelmann theorem there are at least three simple closed geodesics on every Riemannian two sphere $(S^2, g)$. As a byproduct of our analysis we show that in general, even if $k$ is very close to 0, there are no more than two simple closed magnetic geodesics on $S^2$ (see also [13, Sec. 7]).

**Theorem 1.3.** Let $g_0$ be the round metric on $S^2$. For any positive constant $k_0 > 0$ there is a smooth function $k$ on $S^2$, which can be chosen arbitrarily close to $k_0$, such that there are exactly two simple solutions of (1.2).

The proof of our existence results is organized as follows. After setting up notation in Section 2 and introducing the classes of maps and spaces needed for our analysis we define in Section 3 a $S^1$-equivariant Poincaré-Hopf index or $S^1$-degree,

$$\chi_{S^1}(X, M) \in \mathbb{Z},$$

where $M$ is a $S^1$-invariant subset of prime curves in $H^2(S^1, S^2)$ and $X$ belongs to a class of $S^1$-invariant vector fields. The index $\chi_{S^1}(X, M)$ is related to the extension of the Leray-Schauder degree to intrinsic nonlinear
Equivariant degree theories have been defined and applied to differential equations by many authors, we refer to [6, 7, 9, 15, 16] and the references therein. However, we do not see how to apply these results directly to (1.2).

The vector field $X$ therein. Section 4 is devoted to the computation of $\chi$ to count simple periodic solutions of (1.2). We remark that the standard degree $\chi(X, M)$, that does not take the $S^1$ invariance into account, vanishes as it detects only fixed points under the $S^1$-action, i.e. constant solutions. Equivariant degree theories have been defined and applied to differential equations by many authors, we refer to [6, 7, 9, 15, 16] and the references therein. However, we do not see how to apply these results directly to (1.2).

The vector field $X_{k,g}$ corresponding to the prescribed geodesic curvature problem falls into the class of vector fields, where the $S^1$-degree is defined. Section 4 is devoted to the computation of $\chi_{S^1}(X_{k_0,g_0}, M)$, where $k_0$ is a positive constant, $g_0$ is the round metric of $S^2$, and $M$ is the set of simple regular curves in $H^{2,2}(S^1, S^2)$. We call equation (1.2) with $k \equiv k_0$ and $g = g_0$ the unperturbed problem, which is analyzed in detail. The family of simple solutions to the unperturbed problem corresponds to parallels of radius $(1+\frac{k_0^2}{2})^{-1/2}$ with respect to any fixed north pole and is thus isomorphic to $S^2$. In order to compute the $S^1$-degree we slightly perturb the constant function $k_0$ and end up with exactly two nondegenerate solutions of degree $-1$. This implies that $\chi_{S^1}(X_{k_0,g_0}, M) = -2$. Section 5 contains the a priori estimates showing that the set of simple solutions to (1.2) is compact in $M$ under each of the assumptions (1.4)-(1.6). This yields together with the perturbative analysis in Section 4 the proof of Theorem 1.3 and allows to construct an admissible homotopy of vector fields between $X_{k_0,g_0}$ and $X_{k,g}$ whenever $k$ and $g$ satisfy the assumptions of Theorem 1.1. The homotopy invariance of the $S^1$-equivariant Poincaré-Hopf index then shows

$$\chi_{S^1}(X_{k,g}, M) = \chi_{S^1}(X_{k_0,g_0}, M) = -2.$$  

The existence result is given in Section 6.

2. Preliminaries

Let $S^2 = \partial B_1(0) \subset \mathbb{R}^3$ be the standard round sphere with induced metric $g_0$ and orientation such that the rotation $J_{g_0}(y)$ is given for $y \in S^2$ by

$$J_{g_0}(y)(v) := y \times v$$

for all $v \in T_y S^2$,

where $\times$ denotes the cross product in $\mathbb{R}^3$. If we equip $S^2$ with a general Riemannian metric $g$, then the rotation by $\pi/2$ measured with $g$ is given by

$$J_g(y)v = (G(y))^{-1}J_{g_0}(y)(G(y))v \quad \forall v \in T_y S^2,$$

where $G(y)$ denotes the positive symmetric map $G(y) \in \mathcal{L}(T_y S^2)$ satisfying

$$\langle v, w \rangle_{T_y S^2, g} = \langle G(y)v, G(y)w \rangle_{T_y S^2, g_0} \quad \forall v, w \in T_y S^2.$$

We consider for $m \in \mathbb{N}_0$ the set of Sobolev functions

$$H^{m,2}(S^1, S^2) := \{ \gamma \in H^{m,2}(S^1, \mathbb{R}^3) : \gamma(t) \in \partial B_1(0) \text{ for a.e. } t \in S^1. \}$$

For $m \geq 1$ the set $H^{m,2}(S^1, S^2)$ is a sub-manifold of the Hilbert space $H^{m,2}(S^1, \mathbb{R}^3)$ and is contained in $C^{m-1}(S^1, \mathbb{R}^3)$. Hence, if $m \geq 1$ then
\( \gamma \in H^{m,2}(S^1, S^2) \) satisfies \( \gamma(t) \in \partial B_1(0) \) for all \( t \in S^1 \). In this case the tangent space \( T_\gamma H^{m,2}(S^1, S^2) \) of \( H^{m,2}(S^1, S^2) \) at \( \gamma \in H^{m,2}(S^1, S^2) \) is given by

\[
T_\gamma H^{m,2}(S^1, S^2) := \{ V \in H^{m,2}(S^1, \mathbb{R}^3) : V(t) \in T_\gamma(t)S^2 \text{ for all } t \in S^1 \}.
\]

For \( m = 0 \) the set \( H^{0,2}(S^1, S^2) = L^2(S^1, S^2) \) fails to be a manifold. In this case we define for \( \gamma \in H^{1,2}(S^1, S^2) \) the space \( T_\gamma L^2(S^1, S^2) \) by

\[
T_\gamma L^2(S^1, S^2) := \{ V \in L^2(S^1, \mathbb{R}^3) : V(t) \in T_\gamma(t)S^2 \text{ for almost every } t \in S^1 \}.
\]

A metric \( g \) on \( S^2 \) induces a metric on \( H^{m,2}(S^1, S^2) \) for \( m \geq 1 \) by setting for \( \gamma \in H^{m,2}(S^1, S^2) \) and \( V, W \in T_\gamma H^{m,2}(S^1, S^2) \)

\[
\langle W, V \rangle_{T_\gamma H^{m,2}(S^1, S^2), g} := \int_{S^1} \langle (-1)^m \frac{1}{T} (D_{t,g})^m + 1 \rangle V(t), (-1)^m \frac{1}{T} (D_{t,g})^m + 1 \rangle W(t) \rangle_{\gamma(t), g} dt,
\]

where \( \lfloor m/2 \rfloor \) denotes the largest integer that does not exceed \( m/2 \).

Let \( X \) be a differentiable vector field on \( H^{2,2}(S^1, S^2) \). Then the covariant (Frechet) derivative \( D_g X \),

\[
D_g X : TH^{2,2}(S^1, S^2) \to TH^{2,2}(S^1, S^2),
\]

of the vector field \( X \) with respect to the metric induced by \( g \) is defined as follows: For \( \gamma \in H^{2,2}(S^1, S^2) \) and \( V \in T_\gamma H^{2,2}(S^1, S^2) \) we consider a \( C^1 \)-curve

\[
(-\varepsilon, \varepsilon) \ni s \mapsto \gamma_s \in H^{2,2}(S^1, S^2)
\]

satisfying

\[
\gamma_0 = \gamma \text{ and } \frac{d}{ds} \gamma_s|_{s=0} = V,
\]

and define

\[
D_g X|_{\gamma}[V](t) := D_{g,s} \left( X(\gamma_s(t)) \right)|_{s=0}.
\]

For the vector field theory on infinite dimensional manifolds it is convenient to work with Rothe maps instead of compact perturbations of the identity, because the class of Rothe maps is open in the space of linear continuous maps. We recall the definition and properties of Rothe maps given in [27] for the sake of the readers convenience. For a Banach space \( E \) we denote by \( GL(E) \) the set of invertible maps in \( L(E) \) and by \( S(E) \) the set

\[
S(E) = \{ T \in GL(E) : (tT + (1 - t)I) \in GL(E) \text{ for all } t \in [0,1] \}.
\]

Then the set of Rothe maps \( R(E) \) is defined by

\[
R(E) := \{ A \in L(E) : A = T + C, T \in S(E) \text{ and } C \text{ compact} \}.
\]
The set $\mathcal{R}(E)$ is open in $\mathcal{L}(E)$ and consists of Fredholm operators of index 0. Moreover, $\mathcal{GR}(E) := \mathcal{R}(E) \cap \mathcal{GL}(E)$ has two components, $\mathcal{GR}^\pm(E)$, with $I \in \mathcal{GR}^+(E)$. For $A \in \mathcal{GR}(E)$ we let

$$\text{sgn}A = \begin{cases} +1 & \text{if } A \in \mathcal{GR}^+(E), \\ -1 & \text{if } A \in \mathcal{GR}^-(E). \end{cases}$$

If $A = I + C \in \mathcal{GL}(E)$, where $C$ is compact, then $A \in \mathcal{GR}(E)$ and $\text{sgn}A$ is given by the the usual Leray-Schauder degree of $A$.

Since $g$ and $k$ are smooth, $X_{k,g}$ is a smooth vector field (see [26, Sec. 6]) on the set $H^{2,2}_{reg}(S^1, S^2)$ of regular curves,

$$H^{2,2}_{reg}(S^1, S^2) := \{ \gamma \in H^{2,2}(S^1, S^2) : \dot{\gamma}(t) \neq 0 \text{ for all } t \in S^1 \}.$$

To compute $D_gX_{k,g}|_\gamma(V)$ we observe

$$D_{s,g}( - D^2_{t,g} + 1)X_{k,g}(\gamma(s)) = D_{s,g}( - D_{t,g}\dot{\gamma}s + |\gamma_s|_{g}k(\gamma_s)J_g(\gamma_s)\dot{\gamma}_s)$$

$$= -D^2_{t,g} \frac{d\gamma_s}{ds} - R_g \left( \frac{d\gamma_s}{ds}, \dot{\gamma}_s \right)\dot{\gamma}_s + D_{s,g}( |\gamma_s|_{g}k(\gamma_s)J_g(\gamma_s)\dot{\gamma}_s).$$

Evaluating at $s = 0$ we obtain

$$D_{s,g}( - D^2_{t,g} + 1)X_{k,g}(\gamma(s))|_{s=0}$$

$$= -D^2_{t,g}V - R_g(V, \dot{\gamma})\dot{\gamma} + |\gamma|_{g}^{-1}(D_{t,g}V, \dot{\gamma})_gk(\gamma)J_g(\gamma)\dot{\gamma}$$

$$+ |\gamma|_{g}(k'(\gamma)V)J_g(\gamma)\dot{\gamma} + |\gamma|_{g}k(\gamma)(D_gJ_g|_\gamma V)\dot{\gamma} + |\gamma|_{g}k(\gamma)J_g(\gamma)D_{t,g}V. \tag{2.1}$$

Moreover, we have

$$D_{s,g}( - D^2_{t,g} + 1)X_{k,g}(\gamma(s))|_{s=0}$$

$$= -D_{s,g}D^2_{t,g}X_{k,g}(\gamma(s))|_{s=0} + D_{s,g}X_{k,g}(\gamma(s))|_{s=0}$$

$$= ( - D^2_{t,g} + 1)D_gX_{k,g}|_\gamma(V) - D_{t,g} \left( R_g(V, \dot{\gamma})X_{k,g}(\gamma) \right)$$

$$- R_g(V, \dot{\gamma})D_{t,g}X_{k,g}(\gamma). \tag{2.2}$$

Equating (2.1) and (2.2) at a critical point $\gamma$ of $X_{k,g}$ leads to

$$(- D^2_{t,g} + 1)D_gX_{k,g}|_\gamma(V)$$

$$= -D^2_{t,g}V - R_g(V, \dot{\gamma})\dot{\gamma} + |\gamma|_{g}^{-1}(D_{t,g}V, \dot{\gamma})_gk(\gamma)J_g(\gamma)\dot{\gamma}$$

$$+ |\gamma|_{g}(k'(\gamma)V)J_g(\gamma)\dot{\gamma} + |\gamma|_{g}k(\gamma)(D_gJ_g|_\gamma V)\dot{\gamma} + |\gamma|_{g}k(\gamma)J_g(\gamma)D_{t,g}V. \tag{2.3}$$

We note that (see also [27, Thm. 6.1])

$$(- D^2_{t,g} + 1)D_gX_{k,g}|_\gamma(V) = (- D^2_{t,g} + 1)V + T(V),$$

where $T$ is a linear map from $T_\gamma H^{2,2}(S^1, S^2)$ to $T_\gamma L^2(S^1, S^2)$ that depends only on the first derivatives of $V$ and is therefore compact. Taking the inverse $(- D^2_{t,g} + 1)^{-1}$ we deduce that $D_gX_{k,g}|_\gamma$ is the form identity + compact and thus a Rothe map.
For $m \geq 1$ the exponential map $\text{Exp}_g : TH^{m,2}(S^1, S^2) \to H^{m,2}(S^1, S^2)$ is defined for $\gamma \in H^{m,2}(S^1, S^2)$ and $V \in T_\gamma H^{m,2}(S^1, S^2)$ by

$$\text{Exp}_{\gamma,g}(V)(t) := \text{Exp}_{\gamma(t),g}(V(t)),$$

where $\text{Exp}_{\gamma,g}$ denotes the exponential map on $(S^2, g)$ at $z \in S^2$. Due to its pointwise definition

$$\theta \ast \text{Exp}_{\gamma,g}(V)(t) = \text{Exp}_{\theta \ast \gamma,g}(\theta \ast V)(t).$$

### 3. The $S^1$-Poincaré-Hopf Index

For $\gamma \in H^{2,2}(S^1, S^2)$ we define the form $\omega_g(\gamma) \in (T_\gamma H^{2,2}(S^1, S^2))^\ast$ by

$$\omega_g(\gamma)(V) := \int_0^1 \langle \dot{\gamma}(t), (-(D_{t,g})^2 + 1)V(t) \rangle_{\gamma(t),g} dt = \langle \dot{\gamma}, V \rangle_{T_\gamma H^{1,2}(S^1, S^2),g}.$$ 

Approximating $\dot{\gamma}$ by vector fields contained in $T_\gamma H^{2,2}(S^1, S^2)$, it is easy to see that $\omega_g(\gamma) \neq 0$, if $\gamma \neq \text{const}$. If $\gamma \in H^{3,2}(S^1, S^2)$, then $\omega_g(\gamma)$ extends to a linear form in $(T_\gamma L^2(S^1, S^2))^\ast$ by

$$\omega_g(\gamma)(V) := \langle (-(D_{t,g})^2 + 1)\dot{\gamma}, V \rangle_{T_\gamma L^2(S^1, S^2),g}.$$ 

From Riesz’ representation theorem there is $W_g(\gamma) \in T_\gamma H^{2,2}(S^1, S^2)$ such that

$$\omega_g(\gamma)(V) = \langle V, W_g(\gamma) \rangle_{T_\gamma H^{2,2}(S^1, S^2),g} \forall V \in T_\gamma H^{2,2}(S^1, S^2),$$

and

$$\langle W_g(\gamma) \rangle^\perp = \langle \dot{\gamma} \rangle^\perp \cap T_\gamma H^{2,2}(S^1, S^2).$$

Hence

$$W_g(\gamma) = \langle (-(D_{t,g})^2 + 1)\dot{\gamma} \rangle^\perp$$

and $W_g$ is a $C^2$ vector field on $H^{2,2}(S^1, S^2)$.

The form $\omega_g(\gamma)$ and the vector field $W_g(\gamma)$ are equivariant under the $S^1$-action in the sense that for all $\theta \in S^1$ and $V \in T_\gamma H^{2,2}(S^1, S^2)$ we have

$$w_{\theta \ast \gamma,g}(\theta \ast V) = \omega_g(\gamma)(V) \text{ and } W_{\theta \ast \gamma,g} = \theta \ast W_g(\gamma).$$

Using the vector field $W_g$ we define a vector bundle $SH^{2,2}(S^1, S^2)$ by

$$SH^{2,2}(S^1, S^2) := \{ (\gamma,V) \in \text{TH}^{2,2}(S^1, S^2) : \gamma \neq \text{const} \text{ and } V \in \langle W_g(\gamma) \rangle^\perp \}.$$

Note that $SH^{2,2}(S^1, S^2)$ is $S^1$-invariant, as

$$\langle V, W_g(\gamma) \rangle_{T_\gamma H^{2,2}(S^1, S^2),g} \Rightarrow (\theta \ast \gamma, \theta \ast V) \in SH^{2,2}(S^1, S^2) \forall \theta \in S^1.$$

For $\gamma \in H^{2,2}(S^1, S^2) \setminus \{ \text{const} \}$ we consider the map

$$\psi_{\gamma,g} : T_\gamma H^{2,2}(S^1, S^2) \times T_\gamma H^{2,2}(S^1, S^2) \to SH^{2,2}(S^1, S^2)$$

defined by

$$\psi_{\gamma,g}(V,U) := \left( \text{Exp}_{\gamma,g} V, \text{Proj}_{\langle W_g(\text{Exp}_{\gamma,g} V) \rangle^\perp} (D\text{Exp}_{\gamma,g} |V,U) \right).$$

(3.2)
The differential of $\psi_{\gamma,g}$ at $(0,0)$ is given by
\[
D\psi_{\gamma,g}|_{(0,0)}(V,U) = (V,U - \|W_g(\gamma)\|^{-2}(U,W_g(\gamma)))_{T_\gamma H^{2,2}(S^1, S^2), g W_g(\gamma)}.
\]
Consequently, there is $\delta = \delta(\gamma, g) > 0$ such that $\psi_{\gamma,g}$ restricted to
\[
B_\delta(0) \times B_\delta(0) \cap (W_g(\gamma))^\perp \subset T_\gamma H^{2,2}(S^1, S^2) \times T_\gamma H^{2,2}(S^1, S^2)
\]
is a chart for the manifold $SH^{2,2}(S^1, S^2)$ at $(\gamma, 0)$. The construction is
$S^1$-equivariant, for
\[
\psi_{\theta \ast \gamma,g}(\theta \ast V, \theta \ast U) = \theta \ast \psi_{\gamma,g}(V, U) \forall \theta \in S^1
\]
and we may choose $\delta(\gamma, g) = \delta(\theta \ast \gamma, g)$ for all $\theta \in S^1$. Shrinking $\delta(\gamma, g)$ we may assume, as $\text{Exp}_{\gamma,g}$ is also a chart for $H^{k,2}(S^1, S^2)$ with $1 \leq k \leq 4$ and by (3.1),
\[
T_{\text{Exp}_{\gamma,g}(V)} H^{1,2}(S^1, S^2) = \langle D_1 \text{Exp}_{\gamma,g}(V) \rangle \oplus D\text{Exp}_{\gamma,g}|_{V}(\langle \gamma \rangle)^{1 + 2}, (3.3)
\]
\[
T_{\text{Exp}_{\gamma,g}(V)} H^{2,2}(S^1, S^2) = \langle W_g(\text{Exp}_{\gamma,g}(V)) \rangle \oplus D\text{Exp}_{\gamma,g}|_{V}(\langle W_g(\gamma) \rangle)^{1}, (3.4)
\]
\[
\text{Proj}_{(W_g(\text{Exp}_{\gamma,g}(V)))} \circ D\text{Exp}_{\gamma,g}|_{V} : \langle W_g(\gamma) \rangle \rightarrow \sim (W_g(\text{Exp}_{\gamma,g}(V))^1, (3.5)
\]
and the norm of the projections corresponding to the decompositions in (3.3) and (3.4) as well as the norm of the map in (3.5) and its inverse are uniformly bounded with respect to $V$.

The circle action is only continuous and not differentiable on $H^{2,2}(S^1, S^2)$ as for instance the candidate for the differential of the map $\theta \rightarrow \theta \ast \gamma$ at $\theta = 0$, $\gamma$, is in general only in $T_\gamma H^{1,2}(S^1, S^2)$. We prove the existence of a slice of the $S^1$-action (see [18, Lem. 2.2.8] and the references therein) at a curve $\gamma$ with higher regularity and obtain additional differentiability of the slice map.

**Lemma 3.1** (Slice lemma). Let $\gamma \in H^{3,2}(S^1, S^2)$ be a prime curve, i.e. a curve with trivial isotropy group $\{ \theta \in S^1 : \theta \ast \gamma = \gamma \}$. Then there is an open neighborhood $U$ of $0$ in $T_\gamma H^{2,2}(S^1, S^2)$, such that the map
\[
\Sigma_{\gamma,g} : S^1 \times U \cap (W_g(\gamma))^\perp \rightarrow H^{2,2}(S^1, S^2),
\]
defined by
\[
\Sigma_{\gamma,g}(\theta, V) := \theta \ast \text{Exp}_{\gamma,g}(V),
\]
is a homeomorphism onto its range, which is open in $H^{2,2}(S^1, S^2)$. Moreover, the inverse $(\Sigma_{\gamma,g})^{-1}$ satisfies
\[
\text{Proj}_{S^1} \circ (\Sigma_{\gamma,g})^{-1} \in C^2 \left( \Sigma_{\gamma,g}(S^1 \times U \cap (W_g(\gamma))^\perp), S^1 \right).
\]

**Proof.** Fix a prime curve $\gamma \in H^{3,2}(S^1, S^2)$. We consider for $\delta_0 > 0$ the map
\[
F_{\gamma,g} : B_{\delta_0}(0) \times B_{\delta_0}(0) \subset \mathbb{R} / \mathbb{Z} \times T_\gamma H^{2,2}(S^1, S^2) \rightarrow \mathbb{R}
\]
defined by
\[
F_{\gamma,g}(\theta, V) := \omega_g(\gamma) \left( \text{Exp}_{\gamma,g}^{-1}(\theta \ast \text{Exp}_{\gamma,g}(V)) \right).
\]
Note that, as $S^1$ acts continuously on $H^{2,2}(S^1, S^2)$ and $Exp_{\gamma,g}$ is a local diffeomorphism, after shrinking $\delta_0 > 0$ the map $F_{\gamma,g}$ is well defined. $Exp_{\gamma,g}$ is a smooth map, such that for fixed $\theta$ the map $V \mapsto F_{\gamma,g}(\theta, V)$ is also smooth. Moreover, since $Exp_{\gamma,g}(V)$ is in $H^{2,2}(S^1, S^2)$ and $DExp_{\gamma,g}|_V$ maps $L^2$ vector fields along $\gamma$ into $L^2$ vector fields along $Exp_{\gamma,g}(V)$, the map

$$\theta \mapsto Exp^{-1}_{\gamma,g}(\theta \ast Exp_{\gamma,g}(V))$$

is $C^2$ from $B_{\delta_0}(0) \subset \mathbb{R}/\mathbb{Z}$ to $T_\gamma L^2(S^1, S^2)$, the space of $L^2$ vector fields along $\gamma$. For $\gamma \in H^{3,2}(S^1, S^2)$ the form $\omega_2(\gamma)$ is in $(T_\gamma L^2(S^1, S^2))^*$. Thus, $\theta \mapsto F_{\gamma,g}(\theta, V)$ is $C^2$ as well as $F_{\gamma,g}$. Fix $V \in T_\gamma H^{2,2}(S^1, S^2)$. Since

$$D_\theta F_{\gamma,g}|_{(0,0)} = \omega_2(\gamma)(\dot{\gamma}), \neq 0.$$ 

by the implicit function theorem and after shrinking $\delta_0 > 0$ we get a unique $C^2$-map

$$\sigma_{\gamma,g}: B_{\delta_0}(0) \subset T_\gamma H^{2,2}(S^1, S^2) \to B_{\delta_0}(0) \subset \mathbb{R}/\mathbb{Z}$$

such that

$$F_{\gamma,g}(\sigma_{\gamma,g}(V), V) \equiv 0 \text{ in } B_{\delta_0}(0) \subset T_\gamma H^{2,2}(S^1, S^2).$$

Hence, we may define locally around $\gamma$

$$V_{\gamma,g}(\alpha) := Exp^{-1}_{\gamma,g}(\sigma_{\gamma,g}(Exp^{-1}_{\gamma,g}(\alpha)) \ast \alpha) \in (W_g(\gamma))^\perp.$$ 

Using the uniqueness of $\sigma_{\gamma,g}$ and the fact that $\gamma$ is prime it is standard to see that $\Sigma_{\gamma,g}$ is injective and that the inverse is given locally around $\theta_0 \ast \gamma$ for fixed $\theta_0 \in S^1$ by

$$\Sigma_{\gamma,g}^{-1}(\theta_0, 0) + (-\sigma_{\gamma,g} \circ Exp^{-1}_{\gamma,g} \circ (-\theta_0 \ast), V_{\gamma,g} \circ (-\theta_0 \ast)).$$

This finishes the proof. \(\square\)

We will compute the Poincaré-Hopf index for the following class of vector fields.

**Definition 3.2.** Let $M$ be an open $S^1$-invariant subset of prime curves in $H^{2,2}(S^1, S^2)$. A $C^2$ vector field $X$ on $M$ is called $(M, g, S^1)$-admissible, if

1. $X$ is $S^1$-equivariant, i.e. $X(\theta \ast \gamma) = \theta \ast X(\gamma)$ for all $(\theta, \gamma) \in S^1 \times M$.
2. $X$ is proper in $M$, i.e. the set $\{\gamma \in M : X(\gamma) = 0\}$ is compact,
3. $X$ is orthogonal to $W_g$, i.e. $w_g(\gamma)(X(\gamma)) = 0$ for all $\gamma \in M$.
4. $X$ is a Rothe field, i.e. if $X(S^1 \ast \gamma) = 0$ then $D_g X|_\gamma \in \mathcal{R}(T_\gamma H^{2,2}(S^1, S^2))$ and $Proj_{[W_g(\gamma)]^\perp} \circ D_g X|_\gamma \in \mathcal{R}([W_g(\gamma)]^\perp),$ (5) $X$ is elliptic, i.e. there is $\varepsilon > 0$ such that for all finite sets of charts

$$\{(Exp_{\gamma_i, g}, B_{2\delta_0}(0)) : \gamma_i \in H^{1,2}(S^1, S^2) \text{ for } 1 \leq i \leq n\},$$

and finite sets

$$\{W_i \in T_{\gamma_i} H^{1,2}(S^1, S^2) : \|W_i\|_{T_{\gamma_i} H^{1,2}(S^1, S^2)} < \varepsilon \text{ for } 1 \leq i \leq n\},$$
there holds: If \( \alpha \in \cap_{i=1}^{n} \text{Exp}_{\gamma_i,g}(B_{\delta_i}(0)) \subset H^{2,2}(S^1, S^2) \) satisfies
\[
X(\alpha) = \sum_{i=1}^{n} \text{Proj}_{(W_g(\alpha))} \circ D\text{Exp}_{\gamma_i,g}|_{\text{Exp}_{\gamma_i,g}^{-1}(\alpha)}(W_i)
\]
then \( \alpha \) is in \( H^{4,2}(S^1, S^2) \).

Property (4) does not depend on the particular element \( \gamma \) of the critical orbit \( S^1 \ast \gamma \), because from \( \theta \ast X(\gamma) = X(\theta \ast \gamma) \) we get
\[
D_gX|_\gamma = (-\theta \ast) \circ D_gX|_{\theta \ast \gamma} \circ (\theta \ast).
\]
and Rothe maps are invariant under conjugacy. Concerning the regularity property (5), taking \( W_i = 0 \), we deduce that if \( X(\gamma) = 0 \) then \( \gamma \in H^{4,2}(S^1, S^2) \). Furthermore, if \( \gamma \in H^{4,2}(S^1, S^2) \) then the map \( \theta \mapsto \theta \ast \gamma \) is \( C^2 \) from \( S^1 \) to \( H^{2,2}(S^1, S^2) \). Hence, if \( X(\gamma) = 0 \) then
\[
0 = D_\theta(X(\theta \ast \gamma))|_{\theta = 0} = D_gX(\gamma),
\]
such that the kernel of \( D_gX|_\gamma \) at a critical orbit \( S^1 \ast \gamma \) is nontrivial. The parameter \( \varepsilon > 0 \) ensures that (5) remains stable under small perturbations used in the Sard-Smale lemma below. If \( X \) is a vector field orthogonal to \( W_g \) and \( X(\gamma) = 0 \), then
\[
0 = D((X(\alpha), W_g(\alpha))|_{T_{\gamma}H^{2,2}(S^1, S^2),g})|_{\gamma} = (D_gX|_\gamma, W_g(\gamma))|_{T_{\gamma}H^{2,2}(S^1, S^2),g}
\]
where the various curvature terms and terms containing derivatives of \( W_g \) vanish as \( X(\gamma) = 0 \). Thus, \( X(\gamma) = 0 \) implies
\[
D_gX|_\gamma : T_{\gamma}H^{2,2}(S^1, S^2) \to (W_g(\gamma))^\perp,
\]
and the projection \( \text{Proj}_{(W_g(\gamma))^\perp} \) in (4) is unnecessary.

**Lemma 3.3.** The vector field \( X_{k,g} \), defined in [1, 3], is \( S^1 \)-equivariant, orthogonal to \( W_g \), elliptic, and a \( C^2 \)-Rothe field with respect to the set \( H^{2,2}_{reg}(S^1, S^2) \) of regular curves.

**Proof.** From Section [1] and Section [2] the vector field \( X_{k,g} \) is \( S^1 \)-equivariant and a \( C^2 \)-Rothe field. Furthermore, we obtain for \( \alpha \in H^{2,2}(S^1, S^2) \)
\[
(X_{k,g}(\alpha), W_g(\alpha))|_{T_{\gamma}H^{2,2}(S^1, S^2),g} = \int_{S^1} \langle \dot{\alpha}(t), (-D_{\gamma}^2 + 1)X_{k,g}(\alpha)(t) \rangle_g dt
\]
\[
= \int_{S^1} \langle \dot{\alpha}(t), -D_t\dot{\alpha}(t) + |\dot{\alpha}(t)|_g k(\alpha(t))J_g(\alpha(t))\dot{\alpha}(t) \rangle_g dt
\]
\[
= -\int_{S^1} \langle \dot{\alpha}(t), D_t\dot{\alpha}(t) \rangle_g dt = -\int_{S^1} \frac{1}{2} \frac{d}{dt} \langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle_g dt = 0.
\]
To show that \( X_{k,g} \) is elliptic, we fix
\[
\{ (\gamma_i, W_i) \in TH^{4,2}(S^1, S^2) : W_i \in B_{\delta_i}(0), 1 \leq i \leq n \},
\]
where \((\exp_{\gamma_i, g}, B_{2\delta}(0))\) is a chart around \(\gamma_i\), and \(\alpha \in \bigcap_{i=1}^{n} \exp_{\gamma_i, g}(B_{\delta}(0))\) satisfying

\[
X_{k,g}(\alpha) = \sum_{i=1}^{n} \text{Proj}_{(W_g(\alpha))^\perp} \circ D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i).
\]

Then

\[
D_{t,g}\dot{\alpha} - |\dot{\alpha}|_g k(\alpha)J_g(\alpha)\dot{\alpha} = (-D_{t,g}^2 + 1) \sum_{i=1}^{n} \text{Proj}_{(W_g(\alpha))^\perp} \circ D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i).
\]

We fix \(1 \leq i \leq n\) and get

\[
D_{t,g}^2 \text{Proj}_{(W_g(\alpha))^\perp} \circ D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i) = D_{t,g}^2 \left(D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i)\right) - \langle D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i), W_g(\alpha)\rangle D_{t,g}W_g(\alpha),
\]

as well as

\[
D_{t,g}^2 (D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i))(t) = D^2\exp_{\gamma_i(0), g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)(t)} D_{t,g}^2 \exp_{\gamma_i, g}^{-1}(\alpha)(t)(W_i(t)) + R_{1,i}(t)
\]

\[
= D^2\exp_{\gamma_i(0), g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)(t)} D(\exp_{\gamma_i(0), g})^{-1}\alpha(t) D_{t,g}\dot{\alpha}(t)(W_i(t)) + R_{2,i}(t),
\]

where \(R_{1,i}\) and \(R_{2,i}\) consist of lower order terms containing only derivatives of \(\alpha\) up to order 1 and derivatives of \(\gamma_i\) and \(W_i\) up to order 2. Thus \(\alpha\) is a solution of

\[
(1 - A(t))D_{t,g}\dot{\alpha} = |\dot{\alpha}|_g k(\alpha)J_g(\alpha)\dot{\alpha} + R(t)
\]

\[
- \sum_{i=1}^{n} \langle D\exp_{\gamma_i, g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)}(W_i), W_g(\alpha)\rangle (-D_{t,g}^2 + 1)W_g(\alpha), \quad (3.9)
\]

where \(R\) contains only derivatives of \(\alpha\) up to order 1 and derivatives of \(\gamma_i\) and \(W_i\) up to order 2 and \(A(t) \in L(T_{\gamma_i(t)}S^2)\) is given by

\[
V \mapsto \sum_{i=1}^{n} D^2\exp_{\gamma_i(0), g} |_{\exp_{\gamma_i, g}^{-1}(\alpha)(t)} D(\exp_{\gamma_i(0), g})^{-1}\alpha(t) V(W_i(t)).
\]

Since \(H^{2,2}\)-bounds yield \(L^\infty\)-bounds, choosing max \(\|W_i\|\) small enough independently of \(\{\gamma_i\}\) and \(\alpha\) we may assume \(\|A(t)\| < \frac{1}{2}\) and \(A\) is of class \(H^{2,2}\) with respect to \(t\). Since \(\gamma_i\) and \(W_i\) are in \(H^{4,2}\) and \((-D_{t,g}^2 + 1)W_g(\alpha) = \dot{\alpha}\), the right hand side of \((3.9)\) is in \(H^{1,2}\). By standard regularity results \(\alpha\) is in \(H^{3,2}\), such that the right hand side of \((3.9)\) is in \(H^{2,2}\), which yields \(\alpha \in H^{4,2}\). Consequently, \(X_{k,g}\) is elliptic.

\[\square\]

**Definition 3.4.** Let \(M\) be an open \(S^1\)-invariant subset of prime curves in \(H^{2,2}(S^1, S^2)\), \(S^1 \ast \gamma \subset M\), and \(X\) a \((M, g, S^1)\)-admissible vector field on \(M\). The orbit \(S^1 \ast \gamma\) is called a critical orbit of \(X\), if \(X(\gamma) = 0\).
The orbit $S^1 \ast \gamma$ is called a nondegenerate critical orbit of $X$, if $X(\gamma) = 0$ and
\[ D_gX|_\gamma : \langle W_g(\gamma) \rangle^\perp \to \langle W_g(\gamma) \rangle^\perp \] (3.10)
is an isomorphism.
If $S^1 \ast \gamma$ is critical, then using the chart $\psi_{\gamma,g}$ given in (3.2) we define after possibly shrinking $\delta > 0$ a map $X^\gamma \in C^2(B_\delta(0) \cap \langle W_g(\gamma) \rangle^\perp, \langle W_g(\gamma) \rangle^\perp)$ by
\[ X^\gamma(V) := \text{Proj}_2 \circ \psi_{\gamma,g}^{-1}(\text{Exp}_{\gamma,g}(V), X(\text{Exp}_{\gamma,g}(V))), \] (3.11)
where Proj denotes the projection on the second component.

The nondegeneracy of a critical orbit does not depend on the choice of $\gamma$ in $S^1 \ast \gamma$.

**Lemma 3.5.** Under the assumptions of Definition 3.4 a tangent vector $V \in B_\delta(0) \cap \langle W_g(\gamma) \rangle^\perp$ is a (nondegenerate) zero of $X^\gamma$ if and only if $S^1 \ast \text{Exp}_{\gamma,g}(V)$ is a (nondegenerate) critical orbit of $X$.

**Proof.** From the fact that $X(\text{Exp}_{\gamma,g}(V)) \perp W_g(\text{Exp}_{\gamma,g}(V))$ we get
\[ X^\gamma(V) = 0 \iff X(\text{Exp}_{\gamma,g}(V)) = 0. \]
Moreover, if $X^\gamma(V) = 0$, then
\[
D^\gamma|_V = \text{Proj}_2 \circ D\psi_{\gamma,g}^{-1}(\text{Exp}_{\gamma,g}(V),0) \\
\circ (D\text{Exp}_{\gamma,g}|_V, D_gX|_{\text{Exp}_{\gamma,g}(V)} \circ D\text{Exp}_{\gamma,g}|_V) \\
= A^{-1} \circ D_gX|_{\text{Exp}_{\gamma,g}(V)} \circ D\text{Exp}_{\gamma,g}|_V,
\]
where $A : \langle W_g(\gamma) \rangle^\perp \to \langle W_g(\text{Exp}_{\gamma,g}(V)) \rangle^\perp$ is given by
\[
A := \text{Proj}_{\langle W_g(\text{Exp}_{\gamma,g}(V)) \rangle^\perp} \circ D\text{Exp}_{\gamma,g}|_V.
\]
By (3.5) the map $A$ is an isomorphism. Consequently, the map $D^\gamma|_V$ is invertible, if and only if
\[
D_gX|_{\text{Exp}_{\gamma,g}(V)} \circ D\text{Exp}_{\gamma,g}|_V : \langle W_g(\text{Exp}_{\gamma,g}(V)) \rangle^\perp \xrightarrow{\cong} \langle W_g(\text{Exp}_{\gamma,g}(V)) \rangle^\perp
\] (3.12)
is an isomorphism. The injectivity in (3.12), (3.3), and (3.7) implies that the kernel of $D_gX|_{\text{Exp}_{\gamma,g}(V)}$ is one dimensional and given by $\langle D_t\text{Exp}_{\gamma,g}(V) \rangle$. As $D_gX|_{\text{Exp}_{\gamma,g}(V)}$ is a Rothe map and thus a Fredholm operator of index 0, we deduce that (3.12) implies the nondegeneracy of $\text{Exp}_{\gamma,g}(V)$. If (3.10) holds with $\gamma$ replaced by $\text{Exp}_{\gamma,g}(V)$, then the kernel of $D_gX|_{\text{Exp}_{\gamma,g}(V)}$ is one dimensional, and from (3.3) we infer that (3.12) holds, which finishes the proof. 

**Definition 3.6.** Let $g_t$ for $t \in [0,1]$ be a family of smooth metrics on $S^2$, which induces a corresponding family of metrics on $H^2.2(S^1, S^2)$, still denoted by $g_t$. Let $M$ be an open $S^1$-invariant subset of prime curves in
Lemma 3.8. Under the assumptions of Definition 3.7 the tuple $(t, V) \in B_3(t_0) \times B_3(0) \cap (W_{g_{t_0}}(\gamma))^\perp_{g_{t_0}}$ is a (nondegenerate) zero of $X^{t_0,0}$ if and only if the orbit $(t, S^1 \ast \{\gamma\}_{g_{t_0}}) \circ \{\gamma\}_{g_{t_0}}$ is a (nondegenerate) zero of $X$.

We give a $S^1$ equivariant version of the Sard-Smale lemma [21, 23].
Lemma 3.9. Let $M$ be an open $S^1$-invariant subset of prime curves in $H^{2,\alpha}(S^1, S^2)$ and $X$ a $(M, g, S^1)$-admissible vector field on $M$. Let $\mathcal{U}$ be an open neighborhood of the zeros of $X$. Then there exists a $(M, g, S^1)$-admissible vector field $Y$ such that $Y$ has only finitely many isolated, non-degenerate zeros, $Y$ equals $X$ outside $\mathcal{U}$ and there is a $(M, g, S^1)$-homotopy connecting $X$ and $Y$.

Proof. As $X$ is proper and $X^{-1}(0) \subset H^{4,\alpha}(S^1, S^2)$ using Lemma 3.1 we may cover $X^{-1}(0)$ with finitely many open sets

$$X^{-1}(0) \subset \bigcup_{i=1}^{n} S^{1} \ast \exp_{\gamma_{i}, g}(B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}),$$

where $\delta_{i} > 0$, the slice $\Sigma_{\gamma_{i}, g}$ is defined in $S^{1} \times B_{\delta_{i}}(0)$, and $X^{\gamma_{i}}$ is defined in $B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}$ for $i = 1, \ldots, n$. Then $DX^{\gamma_{i}}|_{0}$ is in $\mathcal{R}(\langle W_{g}(\gamma_{i}) \rangle^{\perp})$, which is open in $\mathcal{L}(\langle W_{g}(\gamma_{i}) \rangle^{\perp})$. Thus $DX^{\gamma_{i}}|_{V}$ remains a Rothe map for $V$ close to 0 and consequently a Fredholm operator of index 0. As Fredholm maps are locally proper, we may assume for all $1 \leq i \leq n$ that the map $X^{\gamma_{i}}$ is proper and Rothe on $\overline{B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}}$, i.e.

$$DX^{\gamma_{i}}|_{V} \in \mathcal{R}(\langle W_{g}(\gamma_{i}) \rangle^{\perp}) \forall V \in \overline{B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}},$$

$\overline{B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}} \cap (X^{\gamma_{i}})^{-1}(K)$ is compact $\forall K \subset \langle W_{g}(\gamma_{i}) \rangle^{\perp}$ compact.

To construct $Y$ we proceed step by step and construct $Y_{j}$ such that

(i) $Y_{j}$ equals $X$ outside $\bigcup_{i=1}^{j-1} S^{1} \ast \exp_{\gamma_{i}, g}(B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp})$,

(ii) $Y_{j}^{-1}(0)$ is a subset of

$$\bigcup_{i=1}^{j} S^{1} \ast \exp_{\gamma_{i}, g}(B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}),$$

(iii) the critical orbits of $Y_{j}$ in

$$\bigcup_{i=1}^{j} S^{1} \ast \exp_{\gamma_{i}, g}(B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp})$$

are isolated and nondegenerate.

Since each $X^{\gamma_{i}}$ is proper, $\|X(\cdot)\|$ is bounded below by a positive constant in

$$\bigcup_{i=1}^{n} S^{1} \ast \exp_{\gamma_{i}, g}(B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}) \setminus \bigcup_{i=1}^{n} S^{1} \ast \exp_{\gamma_{i}, g}(B_{\delta_{i}}(0)).$$

Consequently, (ii) remains valid for all small perturbations of $X$.

We start with $Y_{0} := X$. In the 1st step we consider $Y_{1}^{\gamma_{1}}$. By the Sard-Smale lemma there is $V_{j} \in \langle W_{g}(\gamma_{j}) \rangle^{\perp} \cap T_{\gamma_{j}} H^{4,\alpha}(S^1, S^2)$ arbitrarily close to zero, such that $Y_{j}^{\gamma_{1}} + V_{j}$ has only nondegenerate zeros in $\overline{B_{\delta_{i}}(0) \cap \langle W_{g}(\gamma_{i}) \rangle^{\perp}}$.

Since $\gamma_{j} \in H^{4,\alpha}(S^1, S^2)$, the map $\theta \mapsto \theta * \gamma_{j}$ is in $C^{2}(S^1, H^{2,\alpha}(S^1, S^2))$ and $S^{1} * \gamma_{j}$ is a $C^{2}$ sub-manifold of $H^{2,\alpha}(S^1, S^2)$. Shrinking $\delta_{j} > 0$ we may assume
the distance function \( d_g(\cdot, S^1 \star \gamma_j) \) in the Riemannian manifold \( H^{2,2}(S^1, S^2) \) satisfies
\[
d_g(\cdot, S^1 \star \gamma_j)^2 \in C^2(S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp), \mathbb{R}),
\]
and there are \( \varepsilon_{j,1}, \varepsilon_{j,2} > 0 \) such that the set
\[
\{ \gamma \in S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp) : \varepsilon_{j,1} \leq d_g(\gamma, S^1 \star \gamma_j) \leq \varepsilon_{j,2} \}
\]
is contained in
\[
S^1 \star \text{Exp}_{\gamma_j, g}(\overline{(B_{2\delta_j}(0) \setminus B_{\delta_j}(0))} \cap \langle W_g(\gamma_j) \rangle^\perp).
\]
We take a cut-off function \( \eta \in C_c^\infty([\mathbb{R}, [0, 1]] \) such that \( \eta \equiv 1 \) in \( [0, \varepsilon_{j,1}] \) and \( \eta(x) = 0 \) for \( x \geq \varepsilon_{j,2} \). Using Lemma 3.1 we define
\[
\theta_j \in C^2(S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp), S^1)
\]
by \( \theta_j := \text{Proj}_{S^1} \circ (\Sigma_{\gamma_j, g})^{-1} \) and the vector field \( Y_j \) on \( M \) by
\[
Y_j(\gamma) := Y_{j-1}(\gamma),
\]
if \( \gamma \not\in S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp) \) and
\[
Y_j(\gamma) := Y_{j-1}(\gamma) + \eta(d_g(\gamma, S^1 \star \gamma_j)) - \psi_{\theta_j(\gamma)^* \gamma_j, g}(\text{Exp}_{\theta_j(\gamma)^* \gamma_j, g}(\gamma), \theta_j(\gamma) \star V_j),
\]
if \( \gamma \in S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp) \).

Note that the map \( \theta \mapsto (\theta \star \gamma_j, \theta \star V_j) \) is in \( C^2(S^1, TH^{2,2}(S^1, S^2)) \) as \( (\gamma_j, V_j) \in TH^{4,2}(S^1, S^2) \). It is easy to see that \( Y_j \) is a \( S^1 \)-equivariant \( C^2 \) vector field, which is orthogonal to \( W_g \) by construction. If \( \|V_j\| \) is small enough, then (i)-(iii) continue to hold for \( Y_j \) as well as the Rothe property, because Rothe maps and nondegenerate critical orbits are stable under small perturbations. Moreover, \( \cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j \) is proper for any \( t \in [0, \pi/2] \), because \( \cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j \) equals \( Y_{j-1} \) outside \( S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp) \), which is proper, and the zeros of \( \cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j \) inside \( S^1 \star \text{Exp}_{\gamma_j, g}(B_{2\delta_j}(0) \cap \langle W_g(\gamma_j) \rangle^\perp) \) are contained in the compact set \( S^1 \star \text{Exp}_{\gamma_j, g}((V_{j-1}^\gamma)^{-1}([0,1] V_j)) \). If \( Y_{j-1} \) is elliptic with constant \( \varepsilon_{j-1} > 0 \), then taking \( \|V_j\| \) small enough \( \cos(t)^2 Y_{j-1} + \sin(t)^2 Y_j \) remains elliptic with constant \( \varepsilon_j = \varepsilon_{j-1}/2 \), because \( Y_j(\gamma) \) and \( Y_{j-1}(\gamma) \) differ only by
\[
\lambda \text{Proj}_{\langle W_g(\gamma) \rangle^\perp} \circ D\text{Exp}_{\theta_j(\gamma)^* \gamma_j, g}|_{\text{Exp}_{\theta_j(\gamma)^* \gamma_j, g}(\gamma)}^\perp (\theta_j(\gamma) \star V_j),
\]
where \( \lambda \in [0, 1] \) and \( \theta_j(\gamma) \star \gamma_j \) and \( \theta_j(\gamma) \star V_j \) are in \( H^{4,2} \).

For \( j = n \) we arrive at the desired vector-field \( Y \).

Essentially the same arguments lead to the following lemma.

**Lemma 3.10.** Let \( M \) be an open \( S^1 \)-invariant subset of prime curves in \( H^{2,2}(S^1, S^2) \), \( g_t \) for \( t \in [0, 1] \) a smooth family of metrics on \( S^2 \), and \( X \) a \( (M, g_t, S^1) \)-homotopy between two vector-fields \( X_0 \) and \( X_1 \) on \( M \), which have only finitely many critical orbits in \( M \), that are all nondegenerate. Let \( U \)
be an open neighborhood of the zeros of \(X\). Then there exists a \((M,g_t, S^1)\)-homotopy \(Y\) and \(\varepsilon > 0\) such that \(Y_t(\gamma) = X_t(\gamma)\) for
\[(t, \gamma) \in ([0, \varepsilon] \cup [1 - \varepsilon, 1]) \times M \cup ([0, 1] \times M) \setminus U,
\]
and
\[DY|_{t, \gamma} : \mathbb{R} \times \langle W_{g_t}(\gamma) \rangle^{\perp \cdot g_t} \to \langle W_{g_t}(\gamma) \rangle^{\perp \cdot g_t}\]
is surjective for all zeros \((t, \gamma)\) of \(Y\).

For the rest of this section we let \(M\) be an open \(S^1\)-invariant subset of prime curves in \(H^{2,2}(S^1, S^2)\) and \(X\) a \((M,g, S^1)\)-admissible vector field on \(M\). We shall define the \(S^1\)-equivariant Poincaré-Hopf index \(\chi_{S^1}(X,M)\) of the vector-field \(X\) with respect to the set \(M\). We begin with the definition of the local degree of an isolated, nondegenerate critical orbit of \(X\).

We fix a nondegenerate critical orbit \(S^1 \ast \gamma_0\) of \(X\) in \(M\). As \(X\) is \((M,g, S^1)\)-admissible, \(DX|_{\gamma_0} \in \mathcal{GR}((W_g(\gamma_0))^\perp)\) and we define the local degree of \(X\) at \(S^1 \ast \gamma_0\) by
\[\deg_{\text{loc},S^1}(X, S^1 \ast \gamma_0) := \text{sgn}DgX|_{\gamma_0}.
\]
From (3.6) the local degree does not depend on the choice of \(\gamma_0\) in \(S^1 \ast \gamma_0\).

**Definition 3.11** (\(S^1\)-degree). Let \(X\) be \((M,g, S^1)\)-admissible. From Lemma 3.9 there is a vector field \(Y\), which is \((M,g, S^1)\)-homotopic to \(X\), with only finitely many zeros, that are all nondegenerate. The \(S^1\)-equivariant Poincaré-Hopf index (or \(S^1\)-degree) of \(X\) in \(M\) is defined by
\[\chi_{S^1}(X,M) := \sum_{\{S^1 \ast \gamma \subset M : Y(S^1 \ast \gamma) = 0\}} \deg_{\text{loc},S^1}(Y, S^1 \ast \gamma).
\]

To show that the definition does not depend on the particular choice of \(Y\), and that the \(S^1\)-degree does not change under homotopies in the class of \((M,g, S^1)\)-admissible vector-fields we prove

**Lemma 3.12.** Let \(g_t\) be a continuous family of metrics on \(H^{2,2}(S^1, S^2)\) for \(t \in [0,1]\). Suppose \(X\) is a \((M,g_t, S^1)\)-homotopy between \(X_0\) and \(X_1\), such that the zeros of \(X_0\) and \(X_1\) are isolated and nondegenerate. Then
\[\chi_{S^1}(X_0, M) = \chi_{S^1}(X_1, M).
\]

**Proof.** By Lemma 3.10 we may assume the the homotopy \(X\) is nondegenerate, i.e. \(DX_{t,\gamma}\) is surjective whenever \(X(t, S^1 \ast \gamma) = 0\).

Fix \((t_0, \gamma_0) \in X^{-1}(0)\). From the implicit function theorem, Lemma 3.1 and Lemma 3.5 there is a regular \(C^1\) curve \(c = (c_t, c_\gamma) \in C^1(I, \mathbb{R} \times M)\) with \(I = (-1,1)\) for \(t_0 \in (0,1)\) and \(I = [0,1)\) for \(t_0 \in \{0,1\}\), such that \(X(c(s)) \equiv 0, c(0) = (t_0, \gamma_0),\) and the map
\[S^1 \times I \ni (\theta, s) \mapsto (c_t(s), \theta \ast c_\gamma(s)) = \theta \ast c(s)
\]
parametrizes the zero set \(X^{-1}(0)\) locally around \((t_0, \gamma_0)\), where we define the action of \(S^1\) on tuples \((t, \gamma)\) by \(\theta \ast (t, \gamma) := (t, \theta \ast \gamma).\)
The ellipticity of $X_t$ shows that $c_t(s) \in H^{4,2}(S^1, S^2)$, thus $\dot{c}_t(s)$ is in $T_{c_t(s)}H^{2,2}(S^1, S^2)$ and from (3.1) we deduce that

$$\dot{c}_t(s) \text{ is transversal to } (W_{g_t(s)}(c_t(s)))^\perp g_t(s).$$

Since $0 \neq c'(0) \in \mathbb{R} \times \langle W_{g_0(\gamma_0)}(\gamma_0) \rangle^\perp g_0$ we see from the construction of $c$ that we may assume for all $s \in I$

$$c'(s) \text{ is transversal to } (0, \dot{c}_t(s)). \quad (3.16)$$

By the $S^1$-equivariance of $X$, (3.16), and the fact that $D_{g_\gamma(s)}X|_{c_t(s), c_t(s)}$ is a Fredholm operator of index 1 with image $\langle W_{g_\gamma(s)}(c_t(s)) \rangle^\perp g_\gamma(s)$ of codimension 1 we find

$$\ker D_{g_\gamma(s)}X|_{c_t(s), c_t(s)} = \langle c'(s), (0, \dot{c}_t(s)) \rangle. \quad (3.17)$$

Fix $(c_1, I_1)$ and $(c_2, I_2)$ such that $S^1 * c_1(s_1) = S^1 * c_2(s_2)$ for some $s_1 \in I_1$ and $s_2 \in I_2$. Then from the uniqueness part in the construction of $c_2$ we get $\theta_2 \in S^1$ such that $\theta_2 * c_2(s_2) = c_1(s_1)$. From its construction $\theta_2 * c_2'(s_2)$ is contained in the kernel of $DX|_{c_1(s_1)}$ spanned by $\langle c_1'(s_1), (0, (c_1)_\gamma(s_1)) \rangle$. Since $c_1'(s_1)$ and $\theta_2 * c_2'(s_2)$ are both transversal to $(0, (c_1)_\gamma(s_1))$ there is $0 \neq \lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}$ such that

$$\theta_2 * c_2'(s_2) = \lambda_1 c_1'(s_1) + \lambda_2 (0, (c_1)_\gamma(s_1)).$$

We choose a function $\theta_2 \in C^1(I, \mathbb{R}/\mathbb{Z})$ satisfying $\theta_2(s_2) = \theta_2$ and $\theta_2'(s_2) = -\lambda_2$, define $\tilde{c}_2 \in C^1(I, M)$ by $\tilde{c}_2(s) := \theta_2(s) * c_2(s)$, and get

$$\tilde{c}_2(s_2) = \theta_2 * c_1'(s_1) + (0, \theta_2 * (c_2)_\gamma(s_2))\theta_2'(s_2) = \lambda_1 c_1'(s_1).$$

With an additional change in the $s$ parameter we may easily arrive at $\tilde{c}_2'(s_2) = c_1'(s_1)$ in such a way that the map $(\theta, s) \mapsto \theta * \tilde{c}_2(s)$ still parametrizes $S^1 * c_2(I_2)$. This gives a recipe how to obtain from two overlapping local parameterizations $(c_1, I_1)$ and $(c_2, I_2)$ of $X^{-1}(0)$ a parametrization of the union $S^1 * c_1(I_1) \cup S^1 * c_2(I_2)$. As in the classification of one dimensional manifolds [19] we deduce that $X^{-1}(0)$ is a two dimensional manifold with components diffeomorphic to $S^1 \times S^1$ or $S^1 \times [0, 1]$. Let $P$ be a component of $X^{-1}(0)$ with boundary, i.e. of the type $S^1 \times [0, 1]$, such that a parametrization of $P$ is given by

$$(\theta, s) \in S^1 \times [0, 1] \mapsto \theta * c(s),$$

where $c \in C^1([0, 1], [0, 1] \times M)$. First we change $c$ to arrive at

$$c'(s) \in \mathbb{R} \times \langle W_{g_\gamma(s)}(c_\gamma(s)) \rangle^\perp g_\gamma(s) \subset \mathbb{R} \times T_{c(s)}H^{2,2}(S^1, S^2). \quad (3.18)$$

To this end we note that from the definition of $W_g$ we have

$$\mathbb{R} \times T_{c(s)}H^{2,2}(S^1, S^2) = \mathbb{R} \times \langle W_{g(s)}(c_\gamma(s)) \rangle^\perp g(s) \oplus \langle (0, \dot{c}_\gamma(s)) \rangle$$

and denote by Proj$_1$ the projection onto $\mathbb{R} \times \langle W_{g(s)}(c_\gamma(s)) \rangle^\perp g(s)$ with respect to this decomposition. There holds

$$c'(s) = \text{Proj}_1(c'(s)) + \lambda(s)(0, \dot{c}_\gamma(s)).$$
We take $\theta \in C^1([0,1],\mathbb{R})$ such that $\theta'(s) = -\lambda(s)$ and define $\tilde{c}(s) := \theta(s) \cdot c(s)$. Then
\[
\tilde{c}'(s) = \left( c'_t(s), \theta(s) \cdot (c'_t(s) - \lambda(s)c_t(s)) \right)
\]
\[
\in \mathbb{R} \times \theta(s) \cdot \langle W_{g_{\tilde{c}(s)}}(c_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}} = \mathbb{R} \times \langle W_{g_{\tilde{c}(s)}}(\tilde{c}_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}}.
\]
Thus, replacing $c$ with $\tilde{c}$ we may assume \((3.18)\) holds.

Consider for $s \in [0,1]$ the family of operators
\[
F_{s} : \mathbb{R} \times \langle W_{g_{\tilde{c}(s)}}(c_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}} \rightarrow \mathbb{R} \times \langle W_{g_{\tilde{c}(s)}}(c_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}}
\]
defined by
\[
F_{s}(\tau,V) := \left( (c'_t(s), (\tau,V)) \right)_{\mathbb{R} \times T_{c_t(s)}H^{2,2}(S^1,S^2)} D_{g_{\tilde{c}(s)}}X|_{c(s)}(\tau,V).
\]
Since
\[
\ker(D_{g_{\tilde{c}(s)}}X|_{c(s)}) \cap \mathbb{R} \times \langle W_{g_{\tilde{c}(s)}}(c_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}} = \langle c'_t(s) \rangle,
\]
\[
D_{g}X|_{c(s)}(\mathbb{R} \times \langle W_{g_{\tilde{c}(s)}}(c_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}}) = \langle W_{g_{\tilde{c}(s)}}(c_t(s)) \rangle_{\tilde{g}_{\tilde{c}(s)}},
\]
each $F_s$ is an isomorphism. Moreover, the Rothe property of $X$ implies that each $F_s$ is a Rothe map, because $F_s$ is obtained from $DX|_{c(s)}$ through a change in finite dimensions. Consequently, $\text{sgn}(F_s)$ is well defined and by its homotopy invariance independent of $s \in [0,1]$. If $c'_t(s) \neq 0$ we have again by the homotopy invariance $\text{sgn}(F_s) = \text{sgn}(F_0)$, where
\[
\tilde{F}_{s}(\tau,V) := F_{s}(\tau,V + (c'_t(s))^{-1}\tau c'_t(s)).
\]
We have
\[
\tilde{F}_s = \left( \begin{array}{cc} (c'_t(s))^{-1}\|c'(s)\|^2 & \langle c'_t(s), \cdot \rangle_{D_{g}X|c(s)} \end{array} \right) \sim \left( \begin{array}{cc} (c'_t(s))^{-1}\|c'(s)\|^2 & 0 \\ 0 & D_{g}X|c(s) \end{array} \right).
\]
Hence, for all $s \in [0,1]$ such that $c'_t(s) \neq 0$ there holds
\[
\text{sgn}(F_s) = \text{sgn}(\tilde{F}_s) = \text{sgn}(c'_t(s))\text{sgn}(D_{g}X|c(s)). \tag{3.19}
\]
Let $S^1 \ast \alpha_1, \ldots, S^1 \ast \alpha_{k_0}$ be the critical orbits of $X_0$ and $S^1 \ast \beta_1, \ldots, S^1 \ast \beta_{k_1}$ be the critical orbits of $X_1$. The critical orbits of $X_0$ and $X_1$ are boundary points of $X^{-1}(0)$. From \((3.19)\) we get
\begin{itemize}
  \item $\text{sgn}DX_{0|\alpha_i} = -\text{sgn}DX_{0|\alpha_j}$, if $S^1 \ast \alpha_i$ and $S^1 \ast \alpha_j$ are boundary orbits of the same component of $X^{-1}(0)$,
  \item $\text{sgn}DX_{1|\beta_i} = -\text{sgn}DX_{1|\beta_j}$, if $S^1 \ast \beta_i$ and $S^1 \ast \beta_j$ are boundary orbits of the same component of $X^{-1}(0)$,
  \item $\text{sgn}DX_{0|\alpha_i} = \text{sgn}DX_{1|\beta_i}$, if $S^1 \ast \alpha_i$ and $S^1 \ast \beta_j$ are boundary orbits of the same component of $X^{-1}(0)$.
\end{itemize}
Putting the above facts together, we see that
\[
\chi_{S^1}(X_0,M) = \chi_{S^1}(X_1,M).
\]
4. The Unperturbed Problem

Let $S^2 = \partial B_1(0) \subset \mathbb{R}^3$ be the standard round sphere with induced metric $g_0$. Then the prescribed geodesic curvature equation with $k \equiv k_0$ on $(S^2, g_0)$ is given by

$$\text{Proj}_\gamma \tilde{\gamma} = |\gamma| k_0 \gamma \times \dot{\gamma},$$  \hspace{1cm} (4.1)

where $\gamma \in H^{2,2}(S^1, S^2)$, $\dot{\gamma}$ and $\tilde{\gamma}$ are the usual derivatives of $\gamma$ considered as a curve in $\mathbb{R}^3$, $|\dot{\gamma}|$ is the euclidean norm of $\dot{\gamma}$ in $\mathbb{R}^3$. Differentiating twice the identity $|\gamma|^2 = 1$ we find $\langle \tilde{\gamma}, \gamma \rangle + |\gamma|^2 \equiv 0$ and (4.1) is equivalent to

$$\tilde{\gamma} = |\gamma| k_0 \gamma \times \dot{\gamma} - |\gamma|^2 \gamma.$$  \hspace{1cm} (4.2)

In order to solve the ordinary differential (4.2) we fix initial conditions

$$\gamma(0) = \gamma_0 \in S^2 \text{ and } \dot{\gamma}(0) = \tilde{v}_0 \in T_{\gamma_0} S^2.$$

If $\tilde{v}_0 = 0$ then $\gamma$ is given by the constant curve $\gamma \equiv \gamma_0$. We may assume in the sequel

$$\lambda := |\tilde{v}_0| > 0.$$

If $k_0 \neq 0$ then there is a unique $r = r(k_0) \in (-1, 1) \setminus \{0\}$ such that

$$k_0 = \frac{\sqrt{1 - r^2}}{r}.$$

For $k_0 = 0$, the case of geodesics, we may take $r = \pm 1$.

We define for $\lambda > 0$ and a positive oriented orthonormal system $\{v_0, v_1, w\}$ the function $\alpha \in C^\infty(\mathbb{R}, S^2)$ by

$$\alpha(t, \lambda, v_0, v_1, w) := \sqrt{1 - r^2} w + r \cos(\lambda r^{-1} t)v_1 + r \sin(\lambda r^{-1} t)v_0$$

A direct calculation shows that $\alpha(\cdot, \lambda, v_0, v_1, w)$ solves (4.2). Moreover, if we take for given $(\gamma_0, \tilde{v}_0)$ the positive oriented orthonormal system $(v_0, v_1, w)$ defined by

$$v_0 := \lambda^{-1} \tilde{v}_0, \hspace{0.5cm} v_1 := r \gamma_0 + \sqrt{1 - r^2}(v_0 \times \gamma_0), \hspace{0.5cm} w := (v_1 \times v_0)$$

and $\lambda > 0$ as above, then $\alpha(\cdot, \lambda, v_0, v_1, w)$ satisfies the initial conditions

$$\alpha(0, \lambda, v_0, v_1, w) = \gamma_0, \hspace{0.5cm} \dot{\alpha}(0, \lambda, v_0, v_1, w) = \tilde{v}_0.$$

Since we are only interested in solutions in $H^{2,2}(S^1, S^2)$ we get an extra condition on $\lambda$, i.e. the 1-periodicity leads to

$$\lambda \in 2\pi \mathbb{Z}.$$

Hence the simple solutions in $H^{2,2}(S^1, S^2)$ of equation (4.1) are given by

$$\{\alpha(\cdot, 2\pi |r|, v_0, v_1, w) : \{v_0, v_1, w\} \text{ is a positive orthonormal system in } \mathbb{R}^3\}.$$

$SO(3)$ acts on solutions: if $\gamma$ solves (4.1) so does $A \circ \gamma$ for any $A \in SO(3)$. We have

$$A \circ \alpha(\cdot, 2\pi |r|, v_0, v_1, w) = \alpha(\cdot, 2\pi |r|, A(v_0), A(v_1), A(w)),$$
and the set of solutions is parametrized by $A \in SO(3)$. It is easy to see, that

$$\alpha(\cdot, 2\pi |r|, v_0, v_1, w) = \theta * \alpha(\cdot, 2\pi |r|, v_0', v_1', w')$$

for some $\theta \in S^1$ if and only if $w = w'$. Consequently the set of critical orbits $\mathcal{Z}$ is parametrized by $w \in S^2$.

We need to compute the kernel of $D_{g_0}X_{k_0,g_0}|_\alpha$ at a solution $\alpha = \alpha(\cdot, v_0, v_1, w)$ for some fixed system $(v_0, v_1, w)$. We note that for $V \in T_\alpha H^{2,2}(S^1, S^2)$

$$R_{g_0}(V, \dot{\alpha}) \dot{\alpha} = V |\dot{\alpha}|^2 - \langle V, \dot{\alpha} \rangle \dot{\alpha}$$

and hence by (2.3)

$$D_{g_0}X_{k_0,g_0}|_\alpha(V) = (-D_{g_0,t}^2 + 1)^{-1} (-D_{t,g_0}^2 V - V |\dot{\alpha}|^2 + \langle V, \dot{\alpha} \rangle \dot{\alpha} + |\dot{\alpha}|^{-1} (D_{t,g_0} V, \dot{\alpha}) k_0 (\alpha \times \dot{\alpha}) + |\dot{\alpha}| k_0 (\alpha \times D_{t,g_0} V). \quad (4.3)$$

Due to the geometric origin of equation (1.1) we deduce that

$$W_1(t, v_0, v_1, w) := \dot{\alpha} = 2\pi r(-\sin(2\pi t)v_1 + \cos(2\pi t)v_0),$$

$$W_1(0, v_0, v_1, w) = 2\pi r v_0, \quad D_{t,g_0} W_1(0, v_0, v_1, w) = -4\pi^2 r^3 k_0 (k_0 v_1 - w),$$

$$W_0(t, v_0, v_1, w) := \dot{t} \dot{\alpha}, \quad W_0(0, v_0, v_1, w) = 0, \quad D_{t,g_0} W_0(0, v_0, v_1, w) = 2\pi r v_0,$$

solve $D_{g_0}X_{k_0,g_0}|_\alpha(W) = 0$. The vector-field $W_1$ corresponds to invariance with respect to the $S^1$-action, $\theta \mapsto \alpha(\cdot + \theta)$, and $W_0$ stems from the change of parameter $s \mapsto \alpha(\cdot s)$. The $SO(3)$ invariance leads to two additional vector-fields in the kernel of $D_{g_0}X_{k_0,g_0}|_\alpha$, i.e. we let

$$w_{1,s} := \cos(s) w + \sin(s) v_1, \quad v_0 = v_0, \quad w_{1,s} = w_{1,s} v_0 = \cos(s) v_1 - \sin(s) w,$$

$$w_{2,s} := \cos(s) w + \sin(s) v_0, \quad v_1 = v_1, \quad w_{2,s} v_0 = w_{2,s} v_1 = \cos(s) v_0 - \sin(s) w$$

and get

$$W_2(t, v_0, v_1, w) := \frac{d}{ds} (\alpha(\cdot, 2\pi r, v_0, v_1, w_1,s)|_{s=0} = r k_0 v_1 - r \cos(2\pi t) w,$$

$$W_2(0, v_0, v_1, w) = \sqrt{1 - r^2} v_1 - r w, \quad D_{t,g_0} W_2(0, v_0, v_1, w) = 0,$$

$$W_3(t, v_0, v_1, w) := \frac{d}{ds} (\alpha(\cdot, 2\pi r, v_0, s, v_1, w_2,s)|_{s=0} = r k_0 v_0 - r \sin(2\pi t) w,$$

$$W_3(0, v_0, v_1, w) = r k_0 v_0, \quad D_{t,g_0} W_3(0, v_0, v_1, w) = 2\pi r^3 (k_0 v_0 - w). \quad (4.4)$$

We will omit the dependence of $W_i$ on $(v_0, v_1, w)$, if there is no possibility of confusion. Since the initial values of $W_0, \ldots, W_3$ are linearly independent in $(T_\alpha H^{2,2}(S^1, S^2))^2$, the vector-fields are a basis of the kernel of $D_{g_0}X_{k_0,g_0}|_\alpha$. As only $W_1, \ldots, W_3$ are periodic, we obtain

$$\text{kernel}(D_{g_0}X_{k_0,g_0}|_\alpha) = \langle W_1, W_2, W_3 \rangle. \quad (4.5)$$

To find the range of $D_{g_0}X_{k_0,g_0}|_\alpha$ we note that the moving frame $\{\dot{\alpha}, \alpha \times \dot{\alpha}\}$ is an orthogonal system in $T_\alpha S^2$ for any $t \in S^1$. Thus any $V \in T_\alpha H^{2,2}(S^1, S^2)$
may be written as
\[ V = \lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha}) \]
for some functions \( \lambda_1, \lambda_2 \in H^{2,2}(S^1, \mathbb{R}) \). Using the fact that
\[ D_{t,g_0} \dot{\alpha} = |\dot{\alpha}| k_0 (\alpha \times \dot{\alpha}) \text{ and } D_{t,g_0} (\alpha \times \dot{\alpha}) = -|\dot{\alpha}| k_0 \dot{\alpha}, \]
we may express \( D_{t,g_0} V \) and \((D_{t,g_0})^2 V\) in terms of \( \lambda_1 \) and \( \lambda_2 \). This leads to
\[ D_{g_0} X_{k_0,g_0}|_\alpha (V) = (-D_{t,g_0}^2 + 1)^{-1} (-\lambda''_1 + 2\pi \sqrt{1 - r^2 \lambda'_2}) \dot{\alpha} \]
\[ + (-\lambda''_2 - (2\pi)^2 \lambda_2) (\alpha \times \dot{\alpha}). \] (4.6)

Concerning \( W_1, \ldots, W_3 \) and \( W_g \) we find
\[ W_1(t) = \dot{\alpha}(t), \]
\[ W_2(t) = -\frac{1}{2\pi r} \left( \sqrt{1 - r^2} \sin(2\pi t) \dot{\alpha}(t) + \cos(2\pi t) (\alpha \times \dot{\alpha}) \right), \]
\[ W_3(t) = -\frac{1}{2\pi r} \left( -\sqrt{1 - r^2} \cos(2\pi t) \dot{\alpha}(t) + \sin(2\pi t) (\alpha \times \dot{\alpha}) \right) \]
\[ W_{g_0}(\alpha) = (1 + |\dot{\alpha}|^2 k_0^2)^{-1} \dot{\alpha} = (1 + |\dot{\alpha}|^2 k_0^2)^{-1} W_1(\alpha). \] (4.7)

**Lemma 4.1.** For any solution \( \alpha \) of the unperturbed problem there holds
\[ \{0\} = \langle W_1(\alpha), W_2(\alpha), W_3(\alpha) \rangle \cap R(D_{g_0} X_{k_0,g_0}|_\alpha), \]
\[ \langle W_1(\alpha) \rangle^\perp = \langle W_2(\alpha), W_3(\alpha) \rangle \oplus R(D_{g_0} X_{k_0,g_0}|_\alpha) \]

**Proof.** We omit the dependence of \( W_i \) on \( \alpha \). For \( \lambda_1, \lambda_2 \in H^{2,2}(S^1, \mathbb{R}) \) we have
\[ (-D_{t,g_0}^2 + 1)(\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha})) \]
\[ = (-\lambda''_1 + 4\pi \sqrt{1 - r^2 \lambda'_2} + (4\pi^2 (1 - r^2) + 1) \lambda_1) \dot{\alpha} \]
\[ + (-\lambda''_2 - 4\pi \sqrt{1 - r^2 \lambda'_2} + (4\pi^2 (1 - r^2) + 1) \lambda_1) (\alpha \times \dot{\alpha}). \]

Hence we get by direct calculations
\[ (-D_{t,g_0}^2 + 1)(W_1) = (4\pi^2 (1 + r^2 + 1) \dot{\alpha}, \]
\[ (-D_{t,g_0}^2 + 1)(-2\pi r W_2) = \sqrt{1 - r^2} (-4\pi^2 r^2 + 1) \sin(2\pi t) \dot{\alpha} \]
\[ + (4\pi^2 r^2 + 1) \cos(2\pi t) (\alpha \times \dot{\alpha}), \] (4.8)
\[ (-D_{t,g_0}^2 + 1)(-2\pi r W_3) = -\sqrt{1 - r^2} (-4\pi^2 r^2 + 1) \cos(2\pi t) \dot{\alpha} \]
\[ + (4\pi^2 r^2 + 1) \sin(2\pi t) (\alpha \times \dot{\alpha}). \] (4.9)

Consequently, by (3.8) and (4.7) the vector \( W_1 \) is orthogonal to \( \langle W_2, W_3 \rangle \) and to \( R(D_{g_0} X_{k_0,g_0}|_\alpha) \) in \( T_\alpha H^{2,2}(S^1, S^2) \). As in \( L^2(S^1, \mathbb{R}) \)
\[ \lambda''_2 + (2\pi)^2 \lambda_2 \perp_{L^2} \langle \cos(2\pi t), \sin(2\pi t) \rangle, \]
\[ \langle \lambda''_1, \lambda'_2 \rangle \perp_{L^2} \text{ const}, \]
where we get
\[
\{0\} = (-D^2_{t, g_0} + 1)(W_1, W_2, W_3) \cap (-D^2_{t, g_0} + 1)D_{g_0}X_{k_0, g_0}|_\alpha(T\alpha H^{2, 2}(S^1, S^2))
\]
and the claim follows for $D_{g_0}X_{k_0, g_0}|_\alpha$ is a Fredholm operator of index 0. □

To analyze the range of $D_{g_0}X_{k_0, g_0}$ we see for $\alpha \in \mathcal{Z}$
\[
R(DX_{k_0, g_0}|_\alpha) = \{( -D^2_{t, g_0} + 1)^{-1}( -\lambda'' + 2\pi \sqrt{1 - r^2}\lambda')\dot{\alpha} \\
- (\lambda'' + (2\pi)^2 \lambda_\alpha)(\alpha \times \dot{\alpha}) : \lambda_1, \lambda_2 \in H^{2, 2}(S^1, \mathbb{R})\}
\]
\[
= \{(-D^2_{t, g_0} + 1)^{-1}(\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha})) : \lambda_1, \lambda_2 \in L^2(S^1, \mathbb{R}), \lambda_1 \perp L^2 1, \lambda_2 \perp L^2 \langle \cos(2\pi t), \sin(2\pi t) \rangle \}
\]
\[
= \langle (\alpha \times \dot{\alpha}) \rangle \oplus E_+,
\]
where $E_+$ is given by
\[
E_+ = \{( -D^2_{t, g_0} + 1)^{-1}(\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha})) : \lambda_1, \lambda_2 \in L^2(S^1, \mathbb{R}), \lambda_1 \perp L^2 1, \lambda_2 \perp L^2 \langle \cos(2\pi t), \sin(2\pi t) \rangle \}
\]

We have for $V = \lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha})$ in $T\alpha H^{2, 2}(S^1, S^2)$
\[
DX_{k_0, g_0}|_\alpha(V) \in E_+ \iff \lambda_2 \perp L^2 1 \iff V \perp L^2 (\alpha \times \dot{\alpha}).
\]

We fix $V = (-D^2_{t, g_0} + 1)^{-1}(\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha})) \in E_+$. Then
\[
\int_{S^1} V(\alpha \times \dot{\alpha}) = \int_{S^1} (-D^2_{t, g_0} + 1)^{-1}(\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha}))(\alpha \times \dot{\alpha})
\]
\[
= \int_{S^1} (\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha}))(-D^2_{t, g_0} + 1)^{-1}(\alpha \times \dot{\alpha})
\]
\[
= (4\pi^2(1 - r^2) + 1)^{-1} \int_{S^1} (\lambda_1 \dot{\alpha} + \lambda_2 (\alpha \times \dot{\alpha}))(\alpha \times \dot{\alpha}) = 0.
\]

Consequently, $DX_{k_0, g_0}|_\alpha(E_+) = E_+$. Moreover,
\[
\langle (-D^2_{t, g_0} + 1)DX_{k_0, g_0}|_\alpha(V), V \rangle_{L^2_2}
\]
\[
= \int_{S^2} (\lambda_1')^2 - 2\pi \sqrt{1 - r^2}\lambda_1' \lambda_2 + (\lambda_2')^2 - 4\pi^2(\lambda_2)^2
\]
\[
\ge \int_{S^2} (\lambda_1')^2 - \frac{1}{4}(\lambda_1')^2 - 4\pi^2(1 - r^2)(\lambda_2)^2 + (\lambda_2')^2 - 4\pi^2(\lambda_2)^2
\]
\[
\ge \frac{3}{4}(\lambda_1')^2 + 12\pi^2(\lambda_2)^2,
\]
where we used the fact that for $\lambda_2 \perp (1, \cos(2\pi t), \sin(2\pi t))$
\[
\int_{S^1}(\lambda_2')^2 - 4\pi^2(\lambda_2)^2 \ge \int_{S^1} 16\pi^2(\lambda_2)^2.
\]
where the vector field $K$ is given by

$$\frac{\varepsilon K}{\varepsilon} \in \mathbb{R}$$

we see that $(DX_{k,0}\alpha)_{R(DX_{k,0}\alpha)}$ with respect to the decomposition

$$R(DX_{k,0}\alpha) = (\langle \alpha \times \dot{\alpha} \rangle \oplus E_+)$$

is given by

$$(DX_{k,0}\alpha)_{R(DX_{k,0}\alpha)} = \begin{pmatrix} -\text{const} & 0 \\ 0 & (DX_{k,0}\alpha)_{|E_+} \end{pmatrix} \sim \begin{pmatrix} -1 & 0 \\ 0 & \text{id}_{E_+} \end{pmatrix},$$

such that

$$\text{sgn}(DX_{k,0}\alpha)_{R(DX_{k,0}\alpha)} = -1. \quad (4.10)$$

To compute the $S^1$-degree of the unperturbed vector field $X_{k,0}$ for $k \equiv k_0 > 0$ we consider for $k_1 \in C^2(S^2, \mathbb{R})$, which will be chosen later, and $\varepsilon \in \mathbb{R}$, which is assumed to be very small, the perturbed vector field $X_{0,\varepsilon}$ defined by

$$X_{0,\varepsilon}(\gamma) := (-D^2_{t,0} + 1)^{-1}(-D_{t,0}\dot{\gamma} + |\gamma|_{k_0 + \varepsilon k_1(\gamma)} \gamma \times \dot{\gamma})$$

$$= X_{k_0,0}(\gamma) + \varepsilon K_1(\gamma),$$

where the vector field $K_1$ is given by

$$K_1(\gamma) := (-D^2_{t,0} + 1)^{-1}|\gamma|_{k_0 + \varepsilon k_1(\gamma)} \gamma \times \dot{\gamma}.$$ We fix $\alpha_0 \in \mathcal{Z}$ and a parametrization $\varphi$ of $\mathcal{Z}$, which maps an open neighborhood of 0 in $\langle W_1(\alpha_0), W_2(\alpha_0), W_3(\alpha_0) \rangle$ into $\mathcal{Z}$, such that

$$\varphi(0) = \alpha_0 \text{ and } D\varphi|_0 = \text{id}.$$ As $\mathcal{Z}$ consists of smooth functions, $\mathcal{Z}$ is a sub-manifold of $H^{m,2}(S^1, S^2)$ for $1 \leq m < \infty$. We define a map $\Phi$ from an open neighborhood $\mathcal{U}$ of 0 in

$$T_{\alpha_0}H^{2,2}(S^1, S^2) = \langle W_1(\alpha_0), W_2(\alpha_0), W_3(\alpha_0) \rangle \oplus \text{Range}(DX_{0,0}|_{\alpha_0})$$
to $H^{2,2}(S^1, S^2)$ by

$$\Phi(W, U) := \exp_{0,90}(\exp_{0,90}^{-1}(\varphi(W)) + U).$$

Then $(\Phi, \mathcal{U})$ is a chart of $H^{2,2}(S^1, S^2)$ around $\alpha_0$ such that $\mathcal{U}$ is an open neighborhood of 0 in $T_{\alpha_0}H^{2,2}(S^1, S^2)$, and

$$\Phi(0) = \alpha_0, \ D\Phi|_0 = \text{id}, \ \Phi^{-1}(\mathcal{Z} \cap \Phi(\mathcal{U})) = \mathcal{U} \cap \langle W_1(\alpha_0), W_2(\alpha_0), W_3(\alpha_0) \rangle.$$
From the properties of $\text{Exp}_{\alpha_0,g_0}$ the map $\Phi$ is a chart of of $H^{k,2}(S^1, S^2)$ around $\alpha_0$ for any $1 \leq k \leq 4$ and shrinking $U$ we may assume that (3.3) continue to hold with $\text{Exp}_{\gamma,g}$ replaced by $\Phi$, i.e.

\begin{align*}
T_{\Phi(V)} H^{1,2}(S^1, S^2) &= \left( \frac{d}{dt} \Phi(V) \right) \oplus D\Phi|_{V} (\langle \alpha_0 \rangle_{\perp}, H^{1,2}), \\
T_{\Phi(V)} H^{2,2}(S^1, S^2) &= \langle W_{g_0}(\Phi(V)) \rangle \oplus D\Phi|_{V} (\langle W_{g_0}(\alpha_0) \rangle_{\perp}),
\end{align*}
(4.11)

where the isomorphism \(A\) from the properties of $\text{Proj}_{W_{g_0}(\Phi(V))_{\perp}} \circ D\Phi|_{V} : \langle W_{g_0}(\alpha_0) \rangle_{\perp} \cong \langle W_{g_0}(\Phi(V)) \rangle_{\perp},
(4.13)

and the norm of the projections in (4.11) and (4.12) as well as the norm of the map in (4.13) and its inverse are uniformly bounded with respect to $V$. For $\alpha_0 \in \mathcal{Z}$ the vectors $W_1(\alpha_0)$ and $W_{g_0}(\alpha_0)$ are collinear and we use $\langle W_1(\alpha_0) \rangle$ instead of $\langle W_{g_0}(\alpha_0) \rangle$ in the analysis of the unperturbed problem below.

As in (3.2) we get a chart $\Psi$ for the bundle $SH^{2,2}(S^1, S^2)$ around $(\alpha_0, 0)$,

\[ \Psi: U \times U \cap \langle W_1(\alpha_0) \rangle_{\perp} \rightarrow SH^{2,2}(S^1, S^2), \]

\[ \Psi(V, U) := (\Phi(V), \text{Proj}_{W_{g_0}(\Phi(V))_{\perp}} \circ D\Phi|_{V}(U)). \]

Analogous to (3.11) we define

\[ X_{\Phi,g_0,\varepsilon}^{\Phi} : U \cap \langle W_1(\alpha_0) \rangle_{\perp} \rightarrow \langle W_1(\alpha_0) \rangle_{\perp}, \]

by

\[ X_{\Phi,g_0,\varepsilon}^{\Phi}(V) := \text{Proj}_2 \circ \Psi^{-1}(\Phi(V), X_{\Phi,g_0,\varepsilon}(\Phi(V))). \]

Replacing $\text{Exp}_{\gamma,g}$ by $\Phi$ it is easy to see that Lemma 3.3 carries over to $X_{\Phi,g_0,\varepsilon}^{\Phi}$, i.e.

\[ V \in U \cap \langle W_1(\alpha_0) \rangle_{\perp} \text{ is a (nondegenerate) zero of } X_{\Phi,g_0,\varepsilon}^{\Phi} \text{ if and only if } \]

\[ S^1 \ast \Phi(V) \text{ is a (nondegenerate) critical orbit of } X_{\Phi,g_0,\varepsilon}^{\Phi}, \]

(4.14)

and if $X_{\Phi,g_0,\varepsilon}^{\Phi}(V) = 0$, then after shrinking $U$

\[ DX_{\Phi,g_0,\varepsilon}^{\Phi}|_{V} = A_{\Phi}^{-1} \circ DX_{\Phi,g_0,\varepsilon}|_{\Phi(V)} \circ D\Phi|_{V}, \]

(4.15)

where the isomorphism $A_{\Phi} : \langle W_1(\alpha_0) \rangle_{\perp} \rightarrow \langle W_{g_0}(\Phi(V)) \rangle_{\perp}$ is given by

\[ A_{\Phi} = \text{Proj}_{W_{g_0}(\Phi(V))_{\perp}} \circ D\Phi|_{V}. \]

From Lemma 4.1 we may assume

\[ U \cap \langle W_1(\alpha_0) \rangle_{\perp} = U_1 \times U_2, \]

where $U_1$ and $U_2$ are open neighborhoods of 0 in $\langle W_2(\alpha_0), W_3(\alpha_0) \rangle$ and $R(D_{g_0}X_{k_0,\varepsilon}|_{\alpha_0})$. We denote for $\alpha \in \mathcal{Z}$ by $P_2(\alpha)$ the projection onto $R(D_{g_0}X_{k_0,\varepsilon}|_{\alpha})$ with respect to the decomposition

\[ \langle W_1(\alpha) \rangle_{\perp} = \langle W_2(\alpha), W_3(\alpha) \rangle \oplus R(D_{g_0}X_{k_0,\varepsilon}|_{\alpha}), \]
and by \( P_i(\alpha) \) the projection onto \( \langle W_2(\alpha), W_3(\alpha) \rangle \). Moreover, for \( W \in U_1 \) we define for \( i = 1, 2 \)

\[
P_i^\Phi(W) := (A_W)^{-1} \circ P_i(\Phi(W)) \circ A_W.
\]

The projections \( P_1^\Phi(W) \) and \( P_2^\Phi(W) \) correspond to the decomposition

\[
\langle W_1(\alpha_0) \rangle^\perp = \langle W_2(\alpha_0), W_3(\alpha_0) \rangle \oplus R(D_{g_0}X_{g_0,0}\mid W),
\]

(4.16)
as we have for \( W \in U_1 \)

\[
DX_{g_0,0}\mid W = A_W^{-1} \circ DX_{g_0,0}\mid \Phi(W) \circ A_W.
\]

**Lemma 4.2.** For \( \alpha_0 \in Z \) after possibly shrinking \( U \) there are \( \epsilon_0 > 0 \) and

\[
U \in C^2([-\epsilon_0, \epsilon_0] \times U_1, \langle W_1(\alpha_0) \rangle^\perp),
\]

\[
R \in C^2([-\epsilon_0, \epsilon_0] \times U_1, \langle W_2(\alpha_0), W_2(\alpha_0) \rangle),
\]
such that for all \( (\epsilon, W) \in [-\epsilon_0, \epsilon_0] \times U_1 \)

\[
R(\epsilon, W) = X_{g_0,\epsilon}(W + U(\epsilon, W)),
\]

\[
0 = P_1^\Phi(W) \circ U(\epsilon, W),
\]

\[
O(\epsilon)_{\epsilon \to 0} = \|U(\epsilon, W)\| + \|D_W U(\epsilon, W)\| + \|R(\epsilon, W)\| + \|D_W R(\epsilon, W)\|,
\]

\[
R(\epsilon, W) = \epsilon P_1^\Phi(W) \circ K_1^\Phi(W) + o(\epsilon)_{\epsilon \to 0},
\]

\[
U(\epsilon, W) = -\epsilon(DX_{g_0,\epsilon}\mid W)^{-1} \circ P_2^\Phi(W) \circ K_1^\Phi(W) + o(\epsilon)_{\epsilon \to 0}.
\]

Moreover, \( U(\epsilon, W) \) and \( R(\epsilon, W) \) are unique, in the sense that, if \( (\epsilon, W, U, R) \) in \( [-\epsilon_0, \epsilon_0] \times U_1 \times U \cap \langle W_1(\alpha_0) \rangle^\perp \times U_1 \) satisfies

\[
X_{g_0,\epsilon}(W + U) = R \text{ and } P_1^\Phi(W)(U) = 0,
\]

then \( U = U(\epsilon, W) \) and \( R = R(\epsilon, W) \).

**Proof.** We define a \( C^2 \)-function \( H \)

\[
H : \mathbb{R} \times U_1 \times U \cap \langle W_1(\alpha_0) \rangle^\perp \times \langle W_2(\alpha_0), W_3(\alpha_0) \rangle
\]

\[
\to \langle W_1(\alpha_0) \rangle^\perp \times \langle W_2(\alpha_0), W_3(\alpha_0) \rangle,
\]

by

\[
H(\epsilon, W, U, R) := \left( X_{g_0,\epsilon}(W + U) - R, P_1^\Phi(W)(U) \right).
\]

We have in \( \mathcal{L}(\langle W_1(\alpha_0) \rangle^\perp \times \langle W_2(\alpha_0), W_3(\alpha_0) \rangle) \)

\[
D_{(U,R)}H|_{(0,0,0,0)} = \begin{pmatrix}
DX_{g_0,0}\mid W & -id
\end{pmatrix}
\]

\[
0
\]

where we used the fact that \( X_{g_0,0}(W) = 0 \) and (4.15). From (4.5) and Lemma 4.1, we see that \( D_{(U,R)}H|_{(0,0,0,0)} \) is an isomorphism. By the implicit function theorem, after possibly shrinking \( U \), we get \( \epsilon_0 > 0 \) and unique functions \( U = U(\epsilon, W) \) and \( R = R(\epsilon, W) \) such that \( H(\epsilon, W, U(\epsilon, W), R(\epsilon, W)) = 0 \) for all \( (\epsilon, W) \in [-\epsilon_0, \epsilon_0] \times U_1 \), and \( D_{(U,R)}H|_{\epsilon,W,U,R} \) is uniformly invertible.
for \((\varepsilon, W, U, R) \in [-\varepsilon_0, \varepsilon_0] \times U_1 \times U_2\). This yields the existence and uniqueness part of the claim.

The uniqueness implies \(U(0, W) = 0\) and \(R(0, W) = 0\) for all \(W \in U_1\). As \(U\) and \(R\) are differentiable we find \(U(\varepsilon, W) = O(\varepsilon)\) and \(R(\varepsilon, W) = O(\varepsilon)\) as \(\varepsilon \to 0\). Moreover, taking the derivative with respect to \(W\) we see

\[
0 = D_W H|_{0, W, 0, 0} + D_{(U, R)} H|_{0, W, 0, 0} \left(D_W U(0, W), D_W R(0, W)\right)^T
\]

Since \(H(0, W, 0, 0) \equiv 0\) we have \(D_W H|_{0, W, 0, 0} = 0\), which implies

\[
(D_W U(0, W), D_W R(0, W)) = (0, 0),
\]

because \(D_{(U, R)} H|_{0, W, 0, 0}\) is invertible. This gives the desired estimate for \(D_W U\) and \(D_W R\).

Moreover, taking the derivative with respect to \(\varepsilon\) at \((0, W, 0, 0)\) we see as above

\[
0 = D_{\varepsilon} H|_{0, W, 0, 0} + D_{(U, R)} H|_{0, W, 0, 0} \left(D_{\varepsilon} U(0, W), D_{\varepsilon} R(0, W)\right)^T
\]

Consequently,

\[
D_{\varepsilon} R(0, W) = P_1^\Phi(W) \circ K_1^\Phi(W),
\]

\[
D_{\varepsilon} U(0, W) = -(DX_{\Phi, 0}|_W)^{-1} \circ P_2^\Phi(W) \circ K_1^\Phi(W)
\]

This yields the claim. \(\square\)

**Lemma 4.3.** Under the assumptions of Lemma 4.2 we have as \(\varepsilon \to 0\)

\[
X_{\Phi, \varepsilon}^\Phi(W + U(\varepsilon, W)) = \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2)\varepsilon \to 0,
\]

where \(K_1^\Phi\) is the vector-field \(K_1\) in the coordinates \(\Phi\), i.e.

\[
K_1^\Phi = X_{\Phi, 1} - X_{\Phi, 0}.
\]

**Proof.** Since \(U(\varepsilon, W) = O(\varepsilon)\) we find

\[
X_{\Phi, \varepsilon}^\Phi(W + U(\varepsilon, W))
\]

\[
= P_1^\Phi(W) \circ X_{\Phi, \varepsilon}(W + U(\varepsilon, W))
\]

\[
= P_1^\Phi(W) \circ X_{\Phi, 0}(W + U(\varepsilon, W)) + \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W + U(\varepsilon, W))
\]

\[
= P_1^\Phi(W) \circ DX_{\Phi, 0}|_W U(\varepsilon, W) + \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2)
\]

\[
= \varepsilon P_1^\Phi(W) \circ K_1^\Phi(W) + O(\varepsilon^2)\varepsilon \to 0.
\]

\(\square\)

**Lemma 4.4.** Under the assumptions of Lemma 4.2 suppose 0 is a nondegenerate zero of the vector-field \(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot)\), in the sense that \(P_1^\Phi(0) \circ K_1^\Phi(0) = 0\) and

\[
D_W (P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_0 \in \mathcal{L}((W_2(\alpha_0), W_3(\alpha_0)))
\]
is an isomorphism. Then, after possibly shrinking $\varepsilon_0$ and $\mathcal{U}$, for any $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ there is a unique $W(\varepsilon) \in \mathcal{U}_1$ such that

$$X_{g_0,\varepsilon}^\Phi(W(\varepsilon) + U(\varepsilon, W(\varepsilon))) = 0,$$

$$W(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Moreover, $V(\varepsilon) := W(\varepsilon) + U(\varepsilon, W(\varepsilon))$ is the only zero of $X_{g_0,\varepsilon}^\Phi$ in $\mathcal{U} \cap \langle W_1(\alpha_0) \rangle_\perp$ and is nondegenerate with

$$\text{sgn}(DX_{g_0,\varepsilon}^\Phi|_{V(\varepsilon)}) = -\det(D_W(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot)))_{\mathcal{U}_1}.$$

**Proof.** Using Lemma [4.2](#) and the estimates for $U$ and $D_W U$ we find

$$D_W \left(X_{g_0,\varepsilon}^\Phi(\cdot + U(\varepsilon, \cdot))\right)|_W = 0$$

with respect to $W$ we obtain

$$0 = (D_W P_1^\Phi|_W) \circ DX_{g_0,0}^\Phi|_W U(\varepsilon, W)$$

$$+ P_1^\Phi(W) \circ \left(D^2 X_{g_0,0}^\Phi|_W U(\varepsilon, W) + DX_{g_0,0}^\Phi|_W D_W U(\varepsilon, W)\right).$$

Since $P_1^\Phi(W) \circ DX_{g_0,0}^\Phi|_W = 0$, combining (4.17) and (4.18) leads to

$$D_W \left(X_{g_0,\varepsilon}^\Phi(\cdot + U(\varepsilon, \cdot))\right)|_W = \varepsilon D_W \left(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot)\right)|_W + O(\varepsilon^2).$$

We define $F : [-\varepsilon_0, \varepsilon_0] \times \mathcal{U}_1 \to \langle W_2(\alpha_0), W_3(\alpha_0) \rangle$ by

$$F(\varepsilon, W) := \varepsilon^{-1} P_1^\Phi(W) \circ X_{g_0,\varepsilon}(W + U(\varepsilon, W)).$$

Note that by Lemma [4.3](#) the function $F$ extends continuously to $\varepsilon = 0$. By (4.19) we have

$$D_W F|_{\varepsilon, W} = D_W \left(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot)\right)|_W + O(\varepsilon),$$

and $F$ is in $C^1$ with $D_W F|_{(0,0)}$ invertible. Consequently, by the implicit function theorem after shrinking $\varepsilon_0$ and $\mathcal{U}$ there is a unique $C^1$-function $W = W(\varepsilon)$ such that $F(\varepsilon, W(\varepsilon)) = 0$ and

$$X_{g_0,\varepsilon}^\Phi(W(\varepsilon) + U(\varepsilon, W(\varepsilon))) = 0.$$
Shrinking $\mathcal{U}$ we may assume that any $V \in \mathcal{U} \cap (W_1(\alpha_0))^\perp$ admits a unique decomposition $V = W_V + U_V$, where $U_V = P_2^\Phi(W_V)V$. From the construction in Lemma 4.2 and the analysis above we see that for $(\varepsilon, V) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{U} \cap (W_1(\alpha_0))^\perp$

$$X_{g_0,\varepsilon}^\Phi(V) = 0 \iff X_{g_0,\varepsilon}(W_V + U_V) = 0$$

$$\iff U_V = U(\varepsilon, W_V) \text{ and } X_{g_0,\varepsilon}(W_V + U(\varepsilon, W_V)) = 0$$

$$\iff V = W(\varepsilon) + U(\varepsilon, W(\varepsilon)).$$

We use the decomposition in (4.16) to compute the local degree of $X_{g_0,\varepsilon}^\Phi$ in $V(\varepsilon) := W(\varepsilon) + U(\varepsilon, W(\varepsilon))$ as $\varepsilon \to 0$. As $U(\varepsilon, W) = O(\varepsilon)$ we find

$$DX_{g_0,\varepsilon}|_{V(\varepsilon)} = DX_{g_0,0}|_{W(\varepsilon)} + D^2X_{g_0,0}|_{W(\varepsilon)} U(\varepsilon, W(\varepsilon)) + \varepsilon DK_1^\Phi|_{W(\varepsilon)} + O(\varepsilon^2)$$

(4.20)

Differentiating for fixed $\tilde{W} \in \langle W_2(\alpha_0), W_3(\alpha_0) \rangle$ the identity $DX_{g_0,0}|_W \tilde{W} \equiv 0$, we obtain $D^2X_{g_0,0}|_W \tilde{W} \equiv 0$ and thus by (4.20)

$$DX_{g_0,\varepsilon}|_{V(\varepsilon)} \tilde{W} = (\varepsilon DK_1^\Phi|_{W(\varepsilon)} + O(\varepsilon^2)) \tilde{W}.$$

For $\tilde{U} \in R(D_{g_0}X_{g_0,0}|_W)$ we get from (4.20)

$$DX_{g_0,\varepsilon}|_{V(\varepsilon)} \tilde{U} = (DX_{g_0,0}|_{W(\varepsilon)} + O(\varepsilon)) \tilde{U}.$$

Consequently, with respect to the decomposition in (4.16)

$$DX_{g_0,\varepsilon}|_{V(\varepsilon)} = \begin{pmatrix} \varepsilon P_1^\Phi(W(\varepsilon)) \circ DK_1^\Phi|_{W(\varepsilon)} & 0 \\ 0 & DX_{g_0,0}|_{W(\varepsilon)} \end{pmatrix} + \begin{pmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(\varepsilon) \end{pmatrix}.$$ 

This shows that shrinking $\varepsilon_0 > 0$ we may assume that $V(\varepsilon)$ is a nondegenerate zero of $X_{g_0,\varepsilon}^\Phi$ for all $|\varepsilon| \leq \varepsilon_0$ and by (4.10)

$$\text{sgn}(DX_{g_0,\varepsilon}|_{V(\varepsilon)}) = \det(D(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_{\varepsilon}) \text{sgn}(DX_{g_0,0}|_{W(\varepsilon)})$$

$$= -\det(D(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_{\varepsilon}).$$

This finishes the proof. \hfill $\square$

**Lemma 4.5.** Under the assumptions of Lemma 4.2 suppose $\alpha_0$ is a nondegenerate zero of the vector-field $P_1(\cdot) \circ K_1(\cdot)$ on $\mathcal{Z}$, in the sense that $P_1(\alpha_0) \circ K_1(\alpha_0) = 0$ and

$$D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0} \in \mathcal{L}\langle W_2(\alpha_0), W_3(\alpha_0) \rangle$$

is an isomorphism. Then for any $0 < \varepsilon < \varepsilon_0$ there is $\gamma(\varepsilon) \in \Phi(\mathcal{U})$ satisfying

$$X_{g_0,\varepsilon}(\gamma(\varepsilon)) = 0 \text{ and } \gamma(\varepsilon) \to \alpha_0 \text{ as } \varepsilon \to 0.$$

Moreover, $S_1^\ast \gamma(\varepsilon)$ is the unique critical orbit of $X_{g_0,\varepsilon}$ in $\mathcal{U}$ and is nondegenerate with

$$\deg_{\text{loc}, S_1}(X_{g_0,\varepsilon}, S_1^\ast \gamma(\varepsilon)) = -\det(D_{\mathcal{Z}}(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}).$$
Proof. We note that as $P_1(\alpha_0) \circ K_1(\alpha_0) = 0$

$$D_Z(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0} = D(P_1^\Phi(\cdot) \circ K_1^\Phi(\cdot))|_{\alpha_0}. $$

Consequently, the assumptions of Lemma 4.4 are satisfied and we may define for $0 < \varepsilon < \varepsilon_0$ the curve $\gamma(\varepsilon)$ by

$$\gamma(\varepsilon) := \Phi(V(\varepsilon)) \in H^{2,2}(S^1, S^2).$$

From (4.14) we infer that $\gamma(\varepsilon)$ is the unique zero of $X_{g_0,\varepsilon}$ in $\Phi((W_1)^\perp \cap U)$ and $S^1 \ast \gamma(\varepsilon)$ is a nondegenerate critical orbit. It is easy to see that the existence of a slice in Lemma 3.1 remains valid if we replace $\Exp$ by $\Phi$. Consequently, $S^1 \ast \gamma(\varepsilon)$ is the unique critical orbit of $X_{g_0,\varepsilon}$ in $S^1 \ast \Phi((W_1)^\perp \cap U)$, which is an open neighborhood of $S^1 \ast \alpha_0$ in $H^{2,2}(S^1, S^2)$.

We fix $0 < \varepsilon < \varepsilon_0$ and consider for $s \in [0, 1]$ the family of maps

$$Y_s := A^{-1}_{V(\varepsilon)} \circ DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ ((1 - s) + s \Proj_{(W_1(\gamma(\varepsilon)))^\perp}) D\Phi|_{V(\varepsilon)}.$$

Since $DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}$ restricted to $(W_1(\gamma(\varepsilon)))^\perp$ is of the form $id - compact$, writing

$$DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} = DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ \Proj_{(W_1(\gamma(\varepsilon)))^\perp} + DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ \Proj_{(W_1(\gamma(\varepsilon)))^\perp},$$

we deduce that $Y_s = id - compact$ for all $s \in [0, 1]$. From Lemma 4.4 we have that $Y_0$ is invertible and satisfies

$$Y_0 = DX_{g_0,\varepsilon}|_{V(\varepsilon)} \circ \Phi$$

and

$$\text{sgn}(Y_0) = -\det(D_Z(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}).$$

As $DX_{g_0,\varepsilon}|_{V(\varepsilon)}$ is invertible, the kernel of $DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}$ is given by $\langle \gamma(\varepsilon) \rangle$. Since $\gamma(\varepsilon)$ converges to $\alpha_0$ as $\varepsilon \to 0$ and $\dot{\alpha}_0 = W_1(\alpha_0)$ we get

$$\dot{\gamma}(\varepsilon) = W_1(\gamma(\varepsilon)) + o(1)_{\varepsilon \to 0},$$

which implies together with (4.14) that $\langle \dot{\gamma}(\varepsilon) \rangle$ is transversal to the range of

$$((1 - s) + s \Proj_{(W_1(\gamma(\varepsilon)))^\perp}) \circ D\Phi|_{V(\varepsilon)}$$

for all $s \in [0, 1]$. Consequently, $Y_s$ remains invertible when $s$ moves from $0$ to $1$. Due to the homotopy invariance we finally obtain

$$\text{sgn}(Y_0) = -\text{sgn}(\det(D_Z(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_0}))$$

$$= \text{sgn}(Y_1) = \text{sgn}(A^{-1}_{V(\varepsilon)} \circ DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)} \circ A_{V(\varepsilon)})$$

$$= \text{sgn}(DX_{g_0,\varepsilon}|_{\gamma(\varepsilon)}) = \deg_{\loc,S^1}(X_{g_0,\varepsilon}, S^1 \ast \gamma(\varepsilon)).$$

This finishes the proof. \hfill $\square$

In order to compute the $S^1$-degree of $X_{g_0,\varepsilon}$ we define the function $k_1$ by

$$k_1(x) := \langle x, e_3 \rangle \text{ for } x \in S^2 = \partial B_1(0) \subset \mathbb{R}^3,$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{R}^3$. The corresponding vector-field $K_1$ on $H^{2,2}(S^1, S^2)$ is given by

$$K_1(\alpha) = (-D^2_{t,g_0} + 1)^{-1}(|\dot{\alpha}|(\alpha, e_3)(\alpha \times \dot{\alpha})).$$
We note that for \( \alpha = \alpha(\cdot, 2\pi |r|, v_0, v_1, w) \in \mathcal{Z} \) we have
\[
(-D^2_{t,0|} + 1)K_1(\alpha)
= (\sqrt{1 - r^2}(w,e_3) + r \cos(2\pi \cdot)(v_1, e_3) + r \sin(2\pi \cdot)(v_0, e_3))(\alpha \times \dot{\alpha})
= -\frac{2\pi r^2}{4\pi^2 r^2 + 1}(\alpha^2 + 1)((v_1, e_3)W_2(\alpha) + (v_0, e_3)W_3(\alpha)) + h(\alpha),
\]
where \((-D^2_{t,0|} + 1)^{-1}h(\alpha)\) is in the range of \(DX_{k_0,0}\alpha\) by (4.6)-(4.9). Hence,
\[
P_1(\alpha) \circ K_1(\alpha) = -\frac{2\pi r^2}{4\pi^2 r^2 + 1}(v_1, e_3)W_2(\alpha) + \frac{2\pi r^2}{4\pi^2 r^2 + 1}(v_0, e_3)W_3(\alpha),
\]
and there are exactly two critical orbits of \(P_1(\alpha) \circ K_1(\alpha)\) on \(\mathcal{Z}\) given by
\[
\{\alpha = \alpha(\cdot, 2\pi |r|, v_0, v_1, w) \in \mathcal{Z} : w = \pm e_3\} = S^1 \ast \alpha_+ \cup S^1 \ast \alpha_-,\]
where
\[
\alpha_+ = \alpha(\cdot, 2\pi |r|, e_1, e_2, e_3)\quad \text{and} \quad \alpha_- = \alpha(\cdot, 2\pi |r|, -e_1, e_2, -e_3).
\]
The curves \(\alpha_\pm\) correspond to two parallels with respect to the north pole \(e_3\) and curvature \(k_0\). Using the formulas for \(W_2\) and \(W_3\) in (4.4) we find with respect to the basis \(\{W_2(\alpha_\pm), W_3(\alpha_\pm)\}\)
\[
D(P_1(\cdot) \circ K_1(\cdot))|_{\alpha_\pm} = \frac{2\pi r^2}{4\pi^2 r^2 + 1} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.
\]
Thus, we may apply Lemma 4.5 and get two critical orbits \(\alpha_\pm(\varepsilon)\) for \(X_{g_0,\varepsilon}\) converging to \(\alpha_\pm\) as \(\varepsilon \to 0\).

**Lemma 4.6.** Let \(M\) be the set of simple, regular curves in \(H^{2,2}(S^1, S^2)\). Then \(\chi_{S^1}(X_{k_0,0|} M) = -2\).

**Proof.** We choose \(k_1 = \langle \cdot, e_3 \rangle\) as above. From Lemmas 4.2 and 4.3 there is an open neighborhood \(U\) of \(\mathcal{Z}\) such that for all \(0 < \varepsilon < \varepsilon_0\) the critical orbits of \(X_{g_0,\varepsilon}\) in \(U\) are given exactly by \(\alpha_\pm(\varepsilon)\). Indeed, suppose a sequence \((\alpha_n)\) of zeros of \(X_{g_0,\varepsilon_n}\) converges to \(\mathcal{Z}\). Then necessarily
\[
\varepsilon_n \to 0 \quad \text{and} \quad \alpha_n \to \alpha_0 \in \mathcal{Z}
\]
as \(n \to \infty\). For large \(n\) we use the chart \(\Phi\) around \(\alpha_0\) as in Lemma 4.2.

From the existence of a slice in Lemma 3.1 we get a sequence \(\theta_n \in \mathbb{R}/\mathbb{Z}\) converging to 0 such that
\[
\theta_n \ast \alpha_n = \Phi(V_n) \quad \text{for some} \quad V_n \in \langle W_1(\alpha_0) \rangle^\perp.
\]
As in the proof of Lemma 4.3 we may decompose
\[
V_n = \Phi^{-1}(\theta_n \ast \alpha_n) = W_n + U_n,
\]
where \(W_n \in \langle W_1(\alpha_0) \rangle^\perp\) and \(U_n \in R(DX_{k_0,0}|_{W_n})\). From the uniqueness part of Lemma 4.2 as \(X_{g_0,\varepsilon_n}(W_n + U_n) = 0\), we get \(U_n = U(\varepsilon_n, W_n)\). By Lemma 4.3 we see that necessarily \(P_1(\alpha_0) \circ K_1(\alpha_0) = 0\), such that \(S^1 \ast \alpha_0 \in \{S^1 \ast \alpha_\pm\}\). From Lemma 4.5 we finally deduce that \(S^1 \ast \alpha_n \in \{S^1 \ast \alpha_\pm(\varepsilon_n)\}\).
From the definition of the \( S^1 \)-equivariant Poincaré-Hopf index and the classification of the simple zeros of \( X_{k_0, g_0} \) there holds for small \( \varepsilon > 0 \)
\[
\chi_{S^1}(X_{k_0, g_0}, M) = \chi_{S^1}(X_{k_0, g_0}, \mathcal{U}) = \chi_{S^1}(X_{g_0, \varepsilon}, \mathcal{U}) = -2.
\]
\[
\square
\]

5. A PRIORI ESTIMATES

We fix a continuous family of metrics \( \{ g_t : t \in [0, 1] \} \) on \( S^2 \) and a continuous family of positive continuous function \( \{ k_t : t \in [0, 1] \} \) on \( S^2 \). We let \( X_t \) be the vector field on \( H^{2,2}(S^1, S^2) \) defined by
\[
X_t := X_{k_t, g_t}.
\]
We denote by \( M \subset H^{2,2}(S^1, S^2) \) the set
\[
M := \{ \gamma \in H^{2,2}(S^1, S^2) : \gamma \text{ is simple and regular.} \}.
\]
We shall give sufficient conditions assuring that the set
\[
X^{-1}(0) := \{ (\gamma, t) \in M \times [0, 1] : X_t(\gamma) = 0 \}
\]
is compact in \( M \times [0, 1] \). Fix \( (\gamma, t) \in X^{-1}(0) \). The Gauss-Bonnet formula yields
\[
\int_{\gamma} k_t \, ds + \int_{\Omega_{\gamma}} K_{g_t} \, dg_t = 2\pi,
\]
where \( \Omega_{\gamma} \) denotes the interior of \( \gamma \) with respect to the normal \( N_{g_t} \) and \( K_{g_t} \) is the Gauss curvature of \( (S^2, g_t) \). To obtain a contradiction assume that there is \( (\gamma_n, t_n) \) in \( X^{-1}(0) \) such that \( L(\gamma_n) \to 0 \) as \( n \to \infty \). Then the left hand side in the Gauss-Bonnet formula, as \( k_t \) and \( K_{g_t} \) are uniformly bounded, tends to 0, which is impossible. Consequently, the length \( L(\gamma) \) of \( \gamma \) satisfies
\[
c \leq L(\gamma) \leq \left( \inf \{ k_t(x) \} \right)^{-1} (2\pi + \sup_{t \in [0, 1]} \{ (\sup K_{g_t}^-) vol(S^2, g_t) \}), \quad (5.1)
\]
for some positive constant \( c = c(\{ k_t \}, \{ g_t \}) \) and \( K_{g_t}^- := -\min(K_{g_t}, 0) \).

Suppose \( (\gamma_n, t_n) \) in \( X^{-1}(0) \) converges to \( (\gamma_0, t_0) \) in \( H^{2,2}(S^1, S^2) \), such that
\[
\gamma_0 \not\in M.
\]
Then by (5.1) the curve \( \gamma_0 \) is non-constant and regular, hence there is \( s_1 \neq s_2 \) in \( \mathbb{R}/\mathbb{Z} \) such that \( \gamma_0(s_1) = \gamma_0(s_2) \). As \( \gamma_n \) are simple curves, parametrized proportional to arc-length we see that \( \dot{\gamma}_0(s_1) = \pm \dot{\gamma}_0(s_2) \). If \( \dot{\gamma}_0(s_1) = \dot{\gamma}_0(s_2) \) then by the unique solvability of the initial value problem
\[
\gamma_0(\cdot + (s_1 - s_2)) = \gamma_0(\cdot).
\]
If \( \dot{\gamma}_0(s_1) = -\dot{\gamma}_0(s_2) \) then we write \( \gamma \) close to \( s_1 \) and \( s_2 \) as a graph over the tangent direction \( \dot{\gamma}_0(s_1) \) in normal coordinates \( \text{Exp}_{\gamma_0(s_1)} \). By the maximum
principle we find
\[ \gamma_0(s_1 + t) = \text{Exp}_{\gamma_0(s_1),g} \left( t\dot{\gamma}_0(s_1) + a(t)N_g(\gamma_0(s_1)) \right), \]
\[ \gamma_0(s_2 + t) = \text{Exp}_{\gamma_0(s_1),g} \left( -t\dot{\gamma}_0(s_1) - b(t)N_g(\gamma_0(s_1)) \right), \]
where \( a(t) \) and \( b(t) \) are positive for \( t \neq 0 \). Consequently, if \( \dot{\gamma}_0(s_1) = -\dot{\gamma}_0(s_2) \) then \( \gamma_0 \) touches itself at \( \gamma_0(s_1) \), locally separated by the geodesic through \( \gamma_0(s_1) \) with velocity \( \dot{\gamma}_0(s_1) \). Thus, \( \gamma_0 \) is a \( m \)-fold covering for some \( m \in \mathbb{N} \) of a curve \( \alpha \), which is almost simple in the sense that \( \alpha \) can only touch itself as described above. Using stereographic coordinates \( S \) there is a point \( p_0 \) close to the curve \( \gamma_0 \), such that the winding number of \( S(\gamma_0) \) around \( S(p_0) \) is \( \pm m \). Since \( \gamma_0 \) is a limit of simple curves, by the stability of the winding number, we deduce \( m = 1 \).

We denote by \((\Omega_0, g)\) the interior of \( \gamma_0 \) considered as a Riemannian surface with boundary of positive geodesic curvature. Fix a touching point \( \gamma_0(s_1) = \gamma_0(s_2) \). The point \( \gamma_0(s_1) = \gamma_0(s_2) \) corresponds to two different boundary points of \( \Omega_0 \). Denote by \( \beta \) the curve of minimal length in \( \Omega_0 \) connecting the two boundary points. From a regularity result for variational problems with constraints (see [1, 2]) the minimizer \( \beta \) is a \( C^1 \)-curve. By the maximum principle \( \beta \) cannot touch the boundary of \( \Omega_0 \) and is therefore a \( C^2 \) geodesic in the interior of \( \Omega_0 \). Moreover, as a minimizer, \( \beta \) is stable and going back to \( S^2 \) the curve \( \beta \) is a geodesic loop which is stable with respect to variations with fixed end-points. Thus
\[ \text{inj}(g_\alpha) \leq \frac{1}{2} L(\beta) < \frac{1}{4} L(\gamma_0). \]  
\[ (5.2) \]

This leads to

Lemma 5.1. \( X^{-1}(0) \) is compact in \( M \times [0, 1] \) under each of the following assumptions

\[ \inf_{(t,x) \in [0,1] \times S^2} \left\{ k_t \right\} \geq \frac{1}{4} \sup_{t \in [0,1]} \left( \left( \text{inj}(g_t) \right)^{-1} \left( 2\pi + (\sup K_{g_t}) \text{vol}(S^2, g_t) \right) \right), \]  
\[ (5.3) \]

\[ K_{g_t} > 0 \ \forall t \in [0,1] \text{ and } \inf_{(t,x) \in [0,1] \times S^2} \left\{ k_t \right\} \geq \frac{1}{2} \sup_{t \in [0,1]} \left( \left( \sup K_{g_t} \right)^{\frac{1}{2}} \right), \]  
\[ (5.4) \]

\[ K_{g_t} > 0 \ \forall t \in [0,1] \text{ and } (\sup K_{g_t}) < 4 \left( \inf K_{g_t} \right) \text{ for all } t \in [0,1], \]  
\[ (5.5) \]

where \( \text{inj}(g_t) \) denotes the injectivity radius of \((S^2, g_t)\).

Proof. We first show that \( X^{-1}(0) \) is closed under each of the above assumptions. Suppose \((\gamma_n, t_n) \in X^{-1}(0) \) converges to some \((\gamma_0, t_0) \) in \( H^{2,2}(S^1, S^2) \). To obtain a contradiction assume \((\gamma_0, t_0) \notin X^{-1}(0) \), i.e. \( \gamma_0 \) is not simple. Then by the above analysis \( \gamma_0 \) touches itself at some point \( \gamma_0(s_1) = \gamma_0(s_2) \) and there is a stable, nontrivial geodesic loop \( \beta \), which yields a bound from above on the injectivity radius in \((5.2)\) by the length of \( \gamma_0 \). If \( \gamma_0 \) is too short this is impossible. The estimate on the length of \( \gamma_0 \) in \((5.1)\) leads to
the contradiction under the assumption (5.3). If $K_{t_0} > 0$ then by [17, Thm 2.6.9]

$$inj(g_{t_0}) \geq \pi \left( \sup K_{t_0} \right)^{-\frac{1}{2}},$$

and (5.4) is a special case of (5.3).

Moreover, by Bonnet-Meyer’s theorem, as $\beta$ is a stable geodesic loop, its length is bounded by

$$L(\beta) \leq \frac{\pi}{\sqrt{\inf K_{t_0}}},$$

which yields together with (5.6) the contradiction assuming (5.5).

To deduce the compactness of $X^{-1}(0)$ we fix a sequence $(\gamma_n, t_n) \in X^{-1}(0).$ By (5.1) the length $L_{g_{t_n}}(\gamma_n)$ is uniformly bounded. Since each $\gamma_n$ is parametrized proportional to arc-length, $(|\dot{\gamma}_n|_{g_{t_n}})$ is uniformly bounded. Using the equation (1.2) and standard elliptic regularity $(\gamma_n)$ is bounded in $H^{2,2}(S_1, S_2).$ Hence we may choose a subsequence, which converges in $H^{2,2}(S_1, S_2)$ and by the first part of the proof in $X^{-1}(0)$ under each of the above assumptions. This yields the claim.

\[\square\]

**Corollary 5.2.** Let $k_0 > 0$ and $k_1 \in C^\infty(S^2, \mathbb{R})$ be given by (4.21). Then there is $\varepsilon_0 > 0$ such that for any $|\varepsilon| \leq \varepsilon_0$ the vector field $X_{g_0, \varepsilon}$ has exactly two critical orbits $S^1 \star \alpha_{\pm}(\varepsilon)$ in $M,$ which are nondegenerate and converge to the orbits of the parallels $\alpha(\cdot, 2\pi |r|, \pm e_1, e_2, e_3)$ as $\varepsilon \to 0.$

**Proof.** Consider the metrics $g_t \equiv g_0,$ the functions $k_t := k_0 + tk_1,$ and the corresponding vector fields $X_t := X_{g_0, t}.$ The zeros of $X_0$ in $M$ are given by $Z,$ the manifold of solutions to the unperturbed problem. The compactness of $X^{-1}(0)$ implies that the zeros of $X_t$ converge to $Z$ as $t \to 0.$ From the proof of Lemma 4.6 there are exactly two critical orbits for $|t|$ small enough close to $Z$ with the claimed behavior. \[\square\]

6. **Existence results**

We give the proof of our main existence result.

**Proof of Theorem 1.1.** We consider the family of metrics $\{g_t : t \in [0, 1]\}$ defined by

$$g_t := (1 - t)g_0 + tg.$$ 

Since $\{g_t\}$ is a compact family of metrics, there is a constant $k_0 > 0$ such that

$$k_0 > \frac{1}{4} \sup_{t \in [0,1]} \left( \left( inj(g_t) \right)^{-1} \left( 2\pi + (\sup K_{g_t}^-) \text{vol}(S^2, g_t) \right) \right).$$

We denote by $M$ the set of simple regular curves in $H^{2,2}(S^1, S^2).$ From condition (5.3) in Lemma 5.1 the homotopy

$$[0, 1] \ni t \mapsto X_{k_0, g_t}$$
is \((M,g,S^1)\)-admissible and hence from Lemma 3.12 and Lemma 4.6
\[ -2 = \chi_{S^1}(X_{k_0,g}, M) = \chi_{S^1}(X_{k_0,0}, M). \]
We define the family of functions \(\{k_t : t \in [0,1]\}\) by
\[ k_t := (1-t)k_0 + tk \]
and consider the homotopy
\[ [0,1] \ni t \mapsto X_{k_t,g}. \]
Under each of the above assumptions we may apply Lemma 5.1 to deduce that the homotopy is \((M,g,S^1)\)-admissible, and thus
\[ -2 = \chi_{S^1}(X_{k_0,g}, M) = \chi_{S^1}(X_{k,0}, M). \]

\[ \square \]

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RUPRECHT-KARLS-UNIVERSITÄT, IM NEUENHEIMER FELD 288, 69120 HEIDELBERG, GERMANY,

E-mail address: mschneid@mathi.uni-heidelberg.de