Multi-dimensional Avikainen’s estimates

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Abstract

Avikainen proved in [4] the estimate $E[|f(X) - f(\hat{X})|^q] \leq C(p, q)E[|X - \hat{X}|^p]^{\frac{q}{p}}$ for $p, q \in [1, \infty)$, one-dimensional random variables $X$ with the bounded density function and $\hat{X}$, and a function $f$ of bounded variation in $\mathbb{R}$. In this article, we will provide multi-dimensional analogues of this estimate for functions of bounded variation in $\mathbb{R}^d$, Orlicz–Sobolev spaces, Sobolev spaces with variable exponents and fractional Sobolev spaces. The main idea of our arguments is to use Hardy–Littlewood maximal estimates and pointwise characterizations of these function spaces. We will apply main statements to numerical analysis on irregular functionals of a solution to stochastic differential equations based on the Euler–Maruyama scheme and the multilevel Monte Carlo method, and to estimates of the $L^2$-time regularity of decoupled forward–backward stochastic differential equations with irregular terminal conditions.

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1 Introduction

Numerical analysis for stochastic differential equations (SDEs) is one of significant research subjects in the field of stochastic calculus, and it has been studied from both sides of the theory and application. As a numerical scheme for solutions to SDE $dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$, $t \in [0, T]$ driven by a Brownian motion $B$, one often use the Euler–Maruyama scheme $X^{(h)}$ with time step $h \in (0, T)$ (see, [51]), and the weak and strong rates of its convergence has been widely studied, e.g., [32, 33, 54, 63, 68, 81] for weak rate, [4, 7, 13, 35, 63, 65, 67] for strong rate, [2, 12, 67] for backward schemes and [65, 74] for tamed schemes. In particular, Bally and Talay [6] provided the expansion $E[f(X(T))] - E[f(X^{(h)}(T))] = -Ch + Q_h h^2$ for any bounded

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measurable function \( f \) and times step \( h = T/n \), which shows the weak rate of convergence is one (see, also \([32, 54, 53, 52]\)). On the other hand, in this article, for an irregular function \( f \) (e.g., bounded variation or Sobolev differentiable) and \( q \in [1, \infty) \), we are interested in the strong rate of convergence for

\[
E \left[ \left| f(X(T)) - f(X^{(h)}(T)) \right|^q \right] \tag{1}
\]

(see, Corollary 3.6), in order to apply the multilevel Monte Carlo (MLMC) method. Heinrich \([41]\) firstly introduced the MLMC method for parametric integrations, and Giles \([28]\) proposed the method for SDEs based on the Euler–Maruyama scheme \( X^{(h)} \) as a generalization of the statistical Romberg method by Kebaier \([49]\). The MLMC method can reduce the computational complexity for the mean squared error \( E[|\hat{Y} - E[f(X(T)])|^2] \), where \( \hat{Y} \) is an estimator for \( f(X(T)) \), and requires the rate of convergence for \( \mathbb{H} \) with \( q = 2 \). If the function \( f \) is \( \alpha \)-Hölder continuous with \( \alpha \in (0, 1] \), then (1) can be estimated above by \( \|f\|_\alpha^2 E[|X(T) - X^{(h)}(T)|^{2\alpha}] \), where \( \|f\|_\alpha := \sup_{x \neq y} |f(x) - f(y)|^\alpha \), and thus its strong rate of convergence can be obtained under suitable assumptions on the coefficients of SDE (e.g., \([7, 13, 35, 51, 63, 65, 68, 69]\)). However, if the function \( f \) is irregular, then such an estimate is not trivial.

Motivated by such a problem, Avikainen \([4]\) proved that for \( p, q \in [1, \infty) \), one-dimensional random variables \( X \) with the bounded density function \( p_X \) and \( \hat{X} \), and a function \( f \) of bounded variation in \( \mathbb{R} \) (e.g., payoff of the binary options in mathematical finance), it holds that

\[
E \left[ \left| f(X) - f(\hat{X}) \right|^q \right] \leq 3^{q+1} V(f)^q \left( \sup_{x \in \mathbb{R}} p_X(x) \right)^{\frac{q-1}{q}} E \left[ \left| X - \hat{X} \right|^p \right]^{\frac{q}{p}}, \tag{2}
\]

where \( V(f) \) is the total variation of \( f \). Note that this estimate is optimal, that is, the equality holds for some \( X, \hat{X} \) and \( f \). The proof of this estimate is based on the following three ideas. The first idea is to use the Lipschitz continuity of the distribution function of \( X \), which is equivalent to the existence of the bounded density function of \( X \). If the distribution function of \( X \) is Hölder continuous instead of the Lipschitz continuity, then a generalized Avikainen’s estimate holds, and then it can be applied to error estimates for numerical schemes of stochastic processes (see, \([31]\)). The second idea is to use Skorokhod’s “explicit” representation of \( X \) for embedding to the probability space \(([0, 1], \mathcal{B}([0, 1]), \text{Leb}) \) (e.g., Section 3 in \([35]\)). For multi-dimensional random variables, this representation is known as Skorokhod’s embedding theorem (e.g., Theorem 2.7 in \([46]\)). However, it might be difficult to apply it for extending the estimate \( (2) \) in the multi-dimensional case since its representation is not explicit. The third idea is that every function \( f \) of (normalized) bounded variation in \( \mathbb{R} \) can be expressed as the integral of the indicator function of the form \( 1_{(1, 1]}(x) \) with respect to the signed measure \( \nu(dx) \) of bounded variation. Then the estimate \( (2) \) can be obtained by considering the estimate \( (2) \) for \( f(x) = 1_{(1, 1]}(x) \) (see, Lemma 3.4 in \([3]\) and Proposition 5.3 in \([30]\) for simpler proof).

In this article, inspired by \([4]\), we will provide multi-dimensional analogues of Avikainen’s estimate \( (2) \). Unfortunately, as mentioned above, it might be difficult to use the ideas of the one-dimensional case, and to the best of our knowledge, there is no study of multi-dimensional Avikainen’s estimates. Therefore instead of them, as a completely different approach, we will use the Hardy–Littlewood maximal operator \( M \) and its various estimates for locally finite vector valued
measures $\nu$, which is defined by

$$M\nu(x) := \sup_{s > 0} \int_{B(x; s)} |\nu|(z) \, dz, \quad f(x) := \int_B |\nu|(z) \, dz := \frac{|\nu|(B(x; s))}{\text{Leb}(B(x; s))}, \quad x \in \mathbb{R}^d,$$

where $|\nu|$ is the total variation of $\nu$ and $B(x; r)$ is a closed ball of $\mathbb{R}^d$ with center $x$ and radius $r$. This operator is significant in the fields of real analysis and harmonic analysis, and satisfies the following Hardy–Littlewood maximal weak type estimate

$$\text{Leb}(\{ x \in \mathbb{R}^d ; \ M\nu(x) > \lambda \}) \leq A_1 |\nu|(\mathbb{R}^d)^{\lambda^{-1}}, \quad \lambda > 0.$$  

Using this estimate, we will prove that for any random variables $X, \tilde{X} : \Omega \to \mathbb{R}^d$ with the density functions $p_X$ and $p_{\tilde{X}}$ with respect to Lebesgue measure, respectively, $f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $p \in (0, \infty)$ and $q \in [1, \infty)$, if either $p_X$ or $p_{\tilde{X}}$ is bounded, then it holds that

$$E \left[ |f(X) - f(\tilde{X})|^q \right] \leq C(p, q) E \left[ |X - \tilde{X}|^p \right]^{\frac{q}{p}} \tag{3}$$

for some constant $C(p, q)$ (for more details, see, Theorem 2.11). Here, $BV(\mathbb{R}^d)$ is the class of functions $f$ of bounded variation in $\mathbb{R}^d$, which is a subset of $L^1(\mathbb{R}^d)$ such that the total variation $|Df|(\mathbb{R}^d) = \int_{\mathbb{R}^d} |Df|$ of the Radon measure $Df$ defined by

$$\int_{\mathbb{R}^d} |Df| := \sup \left\{ \int_{\mathbb{R}^d} f(x) \text{div}g(x) \, dx ; \ g \in C^1_c(\mathbb{R}^d; \mathbb{R}^d) \text{ and } \sup_{x \in \mathbb{R}^d} |g(x)| \leq 1 \right\}$$

is finite, where the Radon measure $Df$ is defined as the generalized derivative formulated as integration by parts for functions of bounded variations (for more details, see, Section 2.1). The most important property of $f \in BV(\mathbb{R}^d)$ which we use in this article is the following pointwise estimate (see, Lemma 2.13 and Remark 2.14)

$$|f(x) - f(y)| \leq K_1 |x - y| \{ M_{2|x-y|}(Df)(x) + M_{2|x-y|}(Df)(y) \}, \quad \text{a.e. } x, y \in \mathbb{R}^d, \tag{4}$$

where for $R > 0$, $M_R\nu$ is the restricted Hardy–Littlewood maximal function defined by

$$M_R\nu(x) := \sup_{0 < s \leq R} \int_{B(x; s)} |\nu|(z), \quad x \in \mathbb{R}^d.$$  

It is worth noting that Hajłasz [36, 37] characterized Sobolev spaces $W^{1,p}(\mathbb{R}^d)$, $1 \leq p < \infty$ by using a pointwise estimate similar to (3), and defined Sobolev spaces on metric spaces using its pointwise estimate. On the other hand, Lahti and Tuominen [61], and Tuominen [84] generalized this characterization to $BV(\mathbb{R}^d)$ and the Orlicz–Sobolev space $W^{1,\Phi}(\mathbb{R}^d)$ with a Young function $\Phi$ such that both $\Phi$ and its complementary function $\Psi$ satisfy the $\Delta_2$-condition (or are doubling). Moreover, the Sobolev space $W^{1,p}(\cdot)(\mathbb{R}^d)$ with a variable exponent $p : \mathbb{R}^d \to [1, \infty]$ and the fractional Sobolev space $W^{s,p}(\mathbb{R}^d)$ for $(s, p) \in (0, 1) \times [1, \infty)$ also satisfy some pointwise estimates similar to (3). Inspired by these facts, we will also provide estimates similar to (3) for $f \in \{ W^{1,\Phi}(\mathbb{R}^d) \cup W^{1,p(\cdot)}(\mathbb{R}^d) \cup W^{s,p}(\mathbb{R}^d) \} \cap L^\infty(\mathbb{R}^d)$. Note that our approach to use Hardy–Littlewood maximal estimates and pointwise estimates is also applicable to the one-dimensional case.
Finally, we will present the other application of Avikanen’s estimate. Recently several numerical schemes based on machine learning for high-dimensional forward–backward stochastic differential equations (FBSDEs) have been studied (e.g., [19] [39] [43]). In particular, Hure, Pham and Warin [43] proposed numerical schemes based on a backward dynamic programming equation, and they provided an upper bound of their squared error. Applying the estimate [3] to its upper bound, we can ensure the convergence of their numerical schemes for solutions to high-dimensional FBSDEs with irregular terminal conditions (see, Theorem 3.11 for the $L^2$-time regularity of decoupled FBSDEs with irregular terminal conditions.

This article is structured as follows. In section 2, we first recall the definitions of functions of bounded variation in $\mathbb{R}^d$, Orlicz–Sobolev spaces, Sobolev spaces with variable exponents, fractional Sobolev spaces and the Hardy–Littlewood maximal function, and recall its estimates on their function spaces. Then we will provide multi-dimensional analogues of Avikainen’s estimate (see, Theorem 2.11, 2.17, 2.20, 2.22) for their function spaces. In section 3, we will apply main statements to numerical analysis on irregular functionals of a solution to stochastic differential equations based on the Euler–Maruyama scheme and the multilevel Monte Carlo method, and to estimates of the $L^2$-time regularity of decoupled FBSDEs with irregular terminal conditions.

Notations

We give some basic notations and definitions used throughout this article. We consider that elements of $\mathbb{R}^d$ are column vectors, and for $x \in \mathbb{R}^d$, we denote $x = (x_1, \ldots, x_d)^\top$. Let $C^r_c(U; \mathbb{R}^r)$ be the space of $\mathbb{R}^r$-valued functions on an open set $U$ of $\mathbb{R}^d$ with compact support such that the first continuous partial derivatives on $U$ exist. For smooth functions $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}$, we define the divergence of $f$ by $\text{div} f := \sum_{i=1}^d \frac{\partial f_i}{\partial x_i}$, and the gradient of $g$ by $\nabla g = (\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_d})^\top$.

For a bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$, the supremum norm of $f$ is defined by $\|f\|_\infty := \text{ess sup}_{x \in \mathbb{R}^d} |f(x)|$. For measurable functions $f, g : \mathbb{R}^d \to \mathbb{R}$, we denote $\|f\|_{L^r(U; \mathbb{R}^d, g)} := \left\{ \left( \int_{\mathbb{R}^d} |f(x)|^r g(x) \mathrm{d}x \right)^{1/r} \right\}$, if $r \in [1, \infty)$, $\|f\|_\infty$, if $r = \infty$

and the class of all functions with $\|f\|_{L^r(U; \mathbb{R}^d, g)} < \infty$ by $L^r(U; \mathbb{R}^d, g)$. In particular, if $g \equiv 1$, then we use the notation $L^r(U; \mathbb{R}^d)$ as usual $L^r$ space in $\mathbb{R}^d$. For $s > 0$ and $x \in \mathbb{R}^d$, we denote open and closed balls by $U(x; s) := \{ y \in \mathbb{R}^d : |y - x| < s \}$ and $B(x; s) := \{ y \in \mathbb{R}^d : |y - x| \leq s \}$, respectively. For an invertible $d \times d$-matrix $A = (A_{i,j})_{1 \leq i,j \leq d}$, we denote $|A|^2 := \sum_{i,j=1}^d A_{i,j}^2$ and $g_A(x, y) = \frac{\exp(-\frac{1}{2} (A^{-1}(y-x),(y-x)))}{(2\pi)^{d/2} |\det A|^{1/2}}$, and $g_c(x, y) = g_{cI}(x, y)$ for $c > 0$, where the matrix $I$ is the identity matrix. We denote the gamma function by $\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$ for $x \in (0, \infty)$.

2 Multi-dimensional Avikainen’s estimates

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $d \geq 1$. In this section, we provide multi-dimensional analogues of Avikainen’s estimate for any random variables $X$ with the bounded density function, $\hat{X}$ with the density function and for functions of bounded variation in $\mathbb{R}^d$, Orlicz–Sobolev spaces, Sobolev spaces with variable exponents and fractional Sobolev spaces.
2.1 Function spaces

In this subsection, we provide the definitions of functions of bounded variation in $\mathbb{R}^d$, Orlicz–Sobolev spaces, Sobolev spaces with variable exponents and fractional Sobolev spaces.

**Bounded variation in $\mathbb{R}^d$**

We first recall the definition of functions of bounded variation in an open subset $U$ of $\mathbb{R}^d$. For more detail, we refer to [21, 31]. A function $f \in L^1(U)$ has bounded variation in $U$, denoted by $f \in BV(U)$, if

$$
\sup \left\{ \int_U f(x) \text{div} g(x) \, dx \ : \ g \in C^1_c(U; \mathbb{R}^d), \ \sup_{x \in U} |g(x)| \leq 1 \right\} < \infty.
$$

It follows from the structure theorem (e.g., Section 5.1, Theorem 1 in [21]) that for $f \in BV(U)$, there exists a vector valued Radon measure $Df$ on $(U, \mathcal{B}(U))$ such that the following integration by parts formula holds:

$$
\int_U f(x) \text{div} g(x) \, dx = - \int_U \langle g(x), Df(dx) \rangle_{\mathbb{R}^d}, \ \text{for all} \ g \in C^1_c(U; \mathbb{R}^d).
$$

Then we denote

$$
\int_U |Df| := \sup \left\{ \int_U f(x) \text{div} g(x) \, dx \ : \ g \in C^1_c(U; \mathbb{R}^d) \ \text{and} \ \sup_{x \in U} |g(x)| \leq 1 \right\}
$$

and we call it the total variation of $f$ in $U$. A function $f \in L^1_{\text{loc}}(U)$ has locally bounded variation in $U$, denoted by $f \in BV_{\text{loc}}(U)$, if $\int_V |Df| < \infty$ for any open set $V \subset U$ such that its closer $\overline{V}$ is compact and $\overline{V} \subset U$.

We say a Borel subset $E$ of $\mathbb{R}^d$ has locally finite perimeter in $U$ if $1_E \in BV_{\text{loc}}(U)$, and we say $E$ is a Caccioppoli set if it has locally finite perimeter in every bounded open subset $U$ of $\mathbb{R}^d$.

**Remark 2.1.**

(i) $BV(U)$ is a Banach space with the norm $\|f\|_{BV(U)} := \|f\|_{L^1(U)} + \int_U |Df|$ (see, Remark 1.12 in [31]).

(ii) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $BV(U)$ which converges to $f$ in $L^1_{\text{loc}}(U)$. Then it holds that $\int_U |Df| \leq \liminf_{n \to \infty} \int_U |Df_n|$ (semi-continuity, e.g., Section 5.2, Theorem 1 in [21] or Theorem 1.9 in [31]).

(iii) Sobolev’s inequality holds on $BV(\mathbb{R}^d)$, that is, if $f \in BV(\mathbb{R}^d)$ and $d \geq 2$, then there exists $C > 0$ such that $\|f\|_{L^d(\mathbb{R}^d)} \leq C \int_{\mathbb{R}^d} |Df|$, (see, Section 5.6, Theorem 1 (i) in [21] or Theorem 1.28 (A) in [31]). And if $d = 1$, then $\|f\|_{L^\infty} \leq \int_{\mathbb{R}} |Df|$. Indeed, in the same way as the proof of Theorem 1 in Section 5.6 of [21], we choose $f_k \in C^1_c(\mathbb{R}; \mathbb{R})$, $k \in \mathbb{N}$ such that $f_k \to f$ a.e. and $\int_{\mathbb{R}} |f'_k(z)| \, dz \to \int_{\mathbb{R}} |Df|$ as $k \to \infty$. Then by the fundamental theorem of calculus, $|f_k(x)| \leq \int_x^\infty f'_k(z) \, dz \leq \int_{\mathbb{R}} |f'_k(z)| \, dz$, which implies $\|f\|_{L^\infty} \leq \int_{\mathbb{R}} |Df|$.

(iv) In the theory of stochastic calculus, Caccioppoli sets are related to reflected Brownian motions (see, [14, 25, 26]).
Example 2.2. (i) Let \( W^{1,1}(U) \) be the Sobolev space. Then \( W^{1,1}(U) \subset BV(U) \) (see, e.g., Section 5.1, Example 1 in [21] or Example 1.2 in [31]).

(ii) Let \( E \) be a bounded subset of \( \mathbb{R}^d \) with \( C^2 \) boundary. Then \( 1_E \in BV(U) \setminus W^{1,1}(U) \). In this case, it holds that the Gauss–Green formula

\[
\int_E \nabla u(x) \cdot \mathbf{v}(x) \, dx = \int_{\partial E} u(x) \mathbf{v}(x) \cdot \mathbf{n}(x) \, d\mathbf{S}(x),
\]

for all \( u \in C^1(U) \). We now recall the definition of the Orlicz–Sobolev space

\[
W^{1,\Phi}(\mathbb{R}^d) = \{ u \in L^1(\mathbb{R}^d) : \nabla u \in L^\Phi(\mathbb{R}^d) \},
\]

where \( \Phi : [0, \infty) \to [0, \infty) \) is a Young function.

Orlicz–Sobolev spaces

We now recall the definition of the Orlicz–Sobolev space \( W^{1,\Phi}(\mathbb{R}^d) \) for given a Young function \( \Phi \). For more detail, we refer to [40] [44]. We first recall the definitions of Young functions and N-functions. A convex function \( \Phi : [0, \infty) \to [0, \infty) \) is called a Young function if it satisfies the conditions: \( \Phi(0) = 0 \) and \( \lim_{x \to \infty} \Phi(x) = \infty \). A Young function \( \Phi \) has the following integral form

\[
\Phi(x) = \int_0^x \varphi(y) \, dy, \quad x \in [0, \infty),
\]

where \( \varphi : [0, \infty) \to [0, \infty) \) is non-decreasing and left continuous such that \( \varphi(0) = 0 \), and if \( \varphi(x) = \infty \) for \( x \geq a \geq 0 \), then \( \Phi(x) = \infty \) for \( x \geq a \) (see, e.g., Section 1.3, Corollary 2 in [73]). For a Young function \( \Phi \), the complementary function \( \Psi : [0, \infty) \to [0, \infty) \) of \( \Phi \) and the generalized inverse \( \Phi^{-1} : [0, \infty) \to [0, \infty) \) of \( \Phi \) are defined by

\[
\Psi(x) := \sup_{y \geq 0} \{ y \in [0, \infty) : \Phi(y) \leq x \}, \quad \Phi^{-1}(x) := \inf_{y \geq 0} \{ y \in [0, \infty) : \Phi(y) \geq x \},
\]

where \( \varphi^{-1}(x) := \inf \{ y \geq 0 : \varphi(y) > x \}, \) \( x \geq 0 \). A Young function \( \Phi \) is called an N-function if it is continuous such that \( \Phi(x) = 0 \) if \( x = 0 \), and \( \lim_{x \to 0} \Phi(x)/x = 0 \), \( \lim_{x \to \infty} \Phi(x)/x = \infty \). Here, continuity in the topology of \( C([0, \infty); [0, \infty]) \) means that \( \lim_{y \to x} \Phi(y) = \Phi(x) \) for every point \( x \in [0, \infty) \) regardless of whether \( \Phi(x) \) is finite or infinite (e.g., page 14 on [40]). For a given Young function \( \Phi \), the Orlicz space \( L^\Phi(\mathbb{R}^d) \) is defined by

\[
L^\Phi(\mathbb{R}^d) := \bigcup_{\alpha > 0} \{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} \Phi(\alpha |f(x)|) \, dx < \infty \}.
\]

This function space is a Banach space with the Luxemburg norm

\[
\|f\|_{L^\Phi(\mathbb{R}^d)} := \inf \{ \lambda > 0 : \int_{\mathbb{R}^d} \Phi(\lambda |f(x)|) \, dx \leq 1 \}.
\]
(e.g., Section 3.3, Theorem 10 in [74]), and if Ψ is the complementary function of Φ, then the generalized Hölder’s inequality

$$
\int_{\mathbb{R}^d} |f(x)g(x)|dx \leq 2\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}
$$

holds for any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ (e.g., Section 3.3, Proposition 1 in [74]).

A Young function Φ satisfies the $\Delta_2$-condition (or is doubling) if there exists $C > 0$ such that for each $x > 0$, $\Phi(2x) \leq C\Phi(x)$. More specifically, various characterizations (equivalent conditions) of the $\Delta_2$-condition are described in Section 2.3 of [74].

For a Young function Φ, the Orlicz–Sobolev space $W^{1,\Phi}(\mathbb{R}^d)$ is defined by

$$
W^{1,\Phi}(\mathbb{R}^d) := \{ f \in L^\Phi(\mathbb{R}^d) : |Df| \in L^\Phi(\mathbb{R}^d) \},
$$

where $Df := (D_1f, \ldots, D_df)^T$ is the vector of the first order weak partial derivatives $D_if$ of $f$ for $i = 1, \ldots, d$ (e.g., Section 9.3, Definition 1 in [74]).

**Remark 2.3.**

(i) The complementary function Ψ to a Young function Φ is also a Young function (e.g., page 10 on [74], or page 14 on [40] and Lemma 2.4.2 in [40]).

(ii) The complementary function Ψ to an N-function Φ is also an N-function. Indeed, Ψ is continuous since Ψ is convex and $\Psi(0) = 0$ by Remark 2.3 (i) (e.g., page 14 on [40] and Lemma 2.4.2 in [40]). We next assume $\Psi(x) = 0$. Then for $y > 0$, we obtain $x \leq \Psi(y)/y$. Thus we have $x = 0$ by $\lim_{y \to 0} \Psi(y)/y = 0$. On the other hand, since $\varphi$ is non-decreasing, we obtain $\Phi(x)/x \leq \varphi(x) \leq \Phi(2x)/x$, $x > 0$. Thus Φ satisfies $\lim_{x \to 0} \Phi(x)/x = 0$ and $\lim_{x \to \infty} \Phi(x)/x = \infty$ if and only if $\varphi$ satisfies $\lim_{x \to 0} \varphi(x) = 0$ and $\lim_{x \to \infty} \varphi(x) = \infty$. This implies that $\lim_{x \to 0} \Psi(x)/x = 0$ and $\lim_{x \to \infty} \Psi(x)/x = \infty$.

(iii) In general, the complementary function Ψ to a Young function Φ does not always satisfy the $\Delta_2$-condition even if Φ satisfies the $\Delta_2$-condition (e.g., $\Phi(x) = \int_0^x \log(1 + y)dy = (1 + x) \log(1 + x) - x$, $x \in [0, \infty)$).

(iv) The following inclusion relations hold $W^{1,\Phi}(\mathbb{R}^d) \subset W^{1,1}_{loc}(\mathbb{R}^d) \subset BV_{loc}(\mathbb{R}^d)$. Indeed, the first relation is shown by using Jensen’s inequality for the convex function Φ or using the generalized Hölder’s inequality [5].

(v) If a Young function Φ satisfies the $\Delta_2$-condition, then the Orlicz space $L^\Phi(\mathbb{R}^d)$ coincides with the set of all functions $f$ which satisfy $\int_{\mathbb{R}^d} \Phi(|f(x)|)dx < \infty$.

**Example 2.4.**

(i) Let $1/p + 1/q = 1$ with $1 < p < \infty$ and $\Phi(x) := x^p/p$, $x \in [0, \infty)$. Then $\Phi$ and its complementary $\Psi(x) = x^q/q$, $x \in [0, \infty)$ are N-functions and satisfy the $\Delta_2$-condition. Moreover, the Orlicz spaces $L^\Phi(\mathbb{R}^d)$ and $L^\Psi(\mathbb{R}^d)$ are equivalent to the classical Lebesgue spaces $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, respectively.

(ii) Let $p > 1$ and $\alpha > 0$, or $p > 1 - \alpha$ and $-1 \leq \alpha < 0$. Then the function $\Phi(x) := x^p(\log(e + x))^\alpha$, $x \in [0, \infty)$ and its complementary function $\Psi$ are N-functions and satisfy the $\Delta_2$-condition. Note that the Orlicz–Sobolev spaces with such $\Phi$ are used and studied in [3] [47]. We only check that $\Phi$ and $\Psi$ satisfy the $\Delta_2$-condition. Let $x > 0$. Since $\log(e + x) \leq \log(e + 2x) \leq 2\log(e + x)$,
we obtain $\Phi(2x) \leq 2^p \max\{1, 2\alpha\} \Phi(x)$. On the other hand, if $p > 1$ and $\alpha > 0$, we obtain for any $y \geq 0$,

$$y(2x) - \Phi(y) \leq 2yx - y^p \left\{ \log \left( e + \frac{y}{2^\pi} \right) \right\}^\alpha = 2^{\pi+\alpha} \left\{ \frac{y}{2^\pi} x - \Phi \left( \frac{y}{2^\pi} \right) \right\}.
$$

Thus $\Psi(2x) \leq 2^{\pi+\alpha} \Psi(x)$. If $p > 1 - \alpha$ and $-1 \leq \alpha < 0$, since the function $y \mapsto \log(e + y)$ is concave, we obtain for any $y \geq 0$,

$$y(2x) - \Phi(y) \leq 2yx - 2^{\pi+\alpha} y^p \left\{ \log \left( e + \frac{y}{2^\pi} \right) \right\}^\alpha = 2^{\pi+\alpha} \left\{ \frac{y}{2^\pi} x - \Phi \left( \frac{y}{2^\pi} \right) \right\}.
$$

Thus $\Psi(2x) \leq 2^{\pi+\alpha+1} \Psi(x)$.

(iii) Let $\Phi$ be an N-function and let $f \in L^1(\mathbb{R}^d)$ be a continuous and strictly positive function with $\lim_{|x| \to \infty} f(x) = 0$, then $f \in L^\Phi(\mathbb{R}^d)$. Indeed, for any $\alpha > 0$, there exists $K > 0$ such that $C_\alpha := \sup_{|x| > K} \frac{\Phi(\alpha f(x))}{\alpha f(x)} < \infty$. Thus since $\Phi$ is non-decreasing and $f$ is bounded on $B(0; K)$, we obtain

$$\int_{\mathbb{R}^d} \Phi(\alpha |f(x)|) \, dx \leq \text{Leb}(B(0; K)) \Phi(\alpha \sup_{x \in B(0; K)} |f(x)|) + \alpha C_\alpha \| f \|_{L^1(\mathbb{R}^d)} < \infty.
$$

**Sobolev spaces with variable exponents**

We next recall the definition of the Sobolev space $W^{1,p}((\mathbb{R}^d))$ with a variable exponent $p$. This function space is defined as one of generalized Orlicz spaces (also known as Musielak-Orlicz spaces) by the modular $f \mapsto \int_{\mathbb{R}^d} |f(x)|^p(x) \, dx$. For more detail, we refer to [18].

A measurable function $p : \mathbb{R}^d \to [1, \infty]$ is called a variable exponent on $\mathbb{R}^d$, and denoted by $p \in \mathcal{P}(\mathbb{R}^d)$. We define

$$p^- := \text{ess inf}_{x \in \mathbb{R}^d} p(x) \quad \text{and} \quad p^+ := \text{ess sup}_{x \in \mathbb{R}^d} p(x).
$$

The Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$ with a variable exponent $p \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^d) := \bigcup_{\alpha > 0} \left\{ f : \mathbb{R}^d \to \mathbb{R} : \int_{\mathbb{R}^d} |\alpha f(x)|^{p(\alpha f(x))} \, dx < \infty \right\}
$$

This function space is a Banach space with respect to the norm

$$\| f \|_{L^{p(\cdot)}(\mathbb{R}^d)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \left| \frac{f(x)}{\lambda} \right|^{p(\lambda)} \, dx \leq 1 \right\}
$$

(e.g., Theorem 3.2.7 in [18]). Let $p, q, s \in \mathcal{P}(\mathbb{R}^d)$ and assume that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \ a.e. \ x \in \mathbb{R}^d
$$

Then for any $f \in L^{p(\cdot)}(\mathbb{R}^d)$ and $g \in L^{q(\cdot)}(\mathbb{R}^d)$, the generalized Hölder’s inequality

$$\| fg \|_{L^{s(\cdot)}(\mathbb{R}^d)} \leq 2 \| f \|_{L^{p(\cdot)}(\mathbb{R}^d)} \| g \|_{L^{q(\cdot)}(\mathbb{R}^d)}
$$

(6)
holds (see, Lemma 3.2.20 in [18]). In the case $s = p = q = \infty$, we use the convention $s/p = s/q = 1$.

We say a function $f : \mathbb{R}^d \to \mathbb{R}$ is locally log-Hölder continuous on $\mathbb{R}^d$ if there exists $C > 0$ such that for any $x, y \in \mathbb{R}^d$,

$$|f(x) - f(y)| \leq \frac{C}{\log(e + 1/|x - y|)}.$$  

More specifically, various characterizations (equivalent conditions) of the locally log-Hölder continuity is described in Lemma 4.1.6 of [18]. We say that $f : \mathbb{R}^d \to \mathbb{R}$ satisfies the log-Hölder decay condition if there exist $f_\infty \in \mathbb{R}$ and $C > 0$ such that for any $x \in \mathbb{R}^d$,

$$|f(x) - f_\infty| \leq \frac{C}{\log(e + |x|)}.$$  \hspace{1cm} (7)

We say that $f : \mathbb{R}^d \to \mathbb{R}$ is globally log-Hölder continuous on $\mathbb{R}^d$ if it is locally log-Hölder continuous on $\mathbb{R}^d$ and satisfies the log-Hölder decay condition. If $f : \mathbb{R}^d \to \mathbb{R}$ is globally log-Hölder continuous on $\mathbb{R}^d$, then the constant $f_\infty$ in (7) is unique and $f$ is bounded (e.g., page 100 on [18]). We define

$$\mathcal{P}^{\log}(\mathbb{R}^d) := \{ p \in \mathcal{P}(\mathbb{R}^d) : 1/p \text{ is globally log-Hölder continuous} \}$$

and define $p_\infty$ by $1/p_\infty := \lim_{|x| \to \infty} 1/p(x)$. As usual we use the convention $1/\infty := 0$.

The Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^d)$ with a variable exponent $p \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$W^{1,p(\cdot)}(\mathbb{R}^d) := \left\{ f \in L^{p(\cdot)}(\mathbb{R}^d) : |Df| \in L^{p(\cdot)}(\mathbb{R}^d) \right\},$$

where $Df := (D_1 f, \ldots, D_d f)^T$ is the vector of the first order weak partial derivatives $D_i f$ of $f$ for $i = 1, \ldots, d$ (see, e.g., Definition 8.1.2 in [18]).

**Remark 2.5.**

(i) For $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$, although $1/p$ is bounded, $p$ is not always bounded (e.g., page 101 on [18]).

(ii) $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ if and only if $p^* := p/(p - 1) \in \mathcal{P}^{\log}(\mathbb{R}^d)$, and then $(p_\infty)^* = (p^*)_\infty$ (e.g., page 101 on [18]).

(iii) For $p \in \mathcal{P}(\mathbb{R}^d)$ with $p^* < \infty$, $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$ if and only if $p$ is globally log-Hölder continuous. This is due to the fact that $p \mapsto 1/p$ is a bilipschitz mapping from $[p^-, p^+]$ to $[1/p^+, 1/p^-]$ (e.g., Remark 4.1.5 in [18]).

(iv) Note that the following inclusion relations hold $W^{1,p(\cdot)}(\mathbb{R}^d) \subset W^{1,p^*}_{\text{loc}}(\mathbb{R}^d) \subset W^{1,1}_{\text{loc}}(\mathbb{R}^d)$. Indeed, the first relation is shown by using the generalized Hölder’s inequality [10].

**Example 2.6.** Let $d = 1$ and

$$p(x) := \max \left\{ 1 - e^{3|x|}, \min \left\{ \frac{6}{5}, \max \left\{ \frac{1}{2}, \frac{3}{2} - x^2 \right\} \right\} \right\} + 1, \quad x \in \mathbb{R}.$$  

Then $p \in \mathcal{P}^{\log}(\mathbb{R})$ and $1 < p^- < p^+ < \infty$ (e.g., Example 1.3 in [60], or Example 9.1.15 and Example 9.1.16 in [87]).
Fractional Sobolev spaces

We finally recall the definition of the fractional Sobolev space $W^{s,p}(\mathbb{R}^d)$. For more detail, we refer to [17].

Let $s \in (0, 1)$ be a fractional exponent and $p \in [1, \infty)$. The fractional Sobolev space $W^{s,p}(\mathbb{R}^d)$ is defined by

$$W^{s,p}(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) : \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \in L^p(\mathbb{R}^d \times \mathbb{R}^d) \right\}.$$ 

In the literature, the fractional Sobolev space is also called the Aronszajn, Gagliardo or Slobodeckij space.

For $s \in (0, 1)$ and $p \in [1, \infty)$ and $f \in W^{s,p}(\mathbb{R}^d)$, we denote the operator $G_{s,p}$ by

$$G_{s,p}f(x) := \left( \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} dy \right)^{1/p}, \quad x \in \mathbb{R}^d.$$ 

Then $G_{s,p}f \in L^p(\mathbb{R}^d)$.

2.2 Hardy–Littlewood maximal function and estimates

In this subsection, we provide the definition of the Hardy–Littlewood maximal function and recall its estimates on function spaces defined in Section 2.1, which are well-known in the fields of real analysis and harmonic analysis.

Let $\nu$ be a locally finite vector valued measure on $\mathbb{R}^d$. The Hardy–Littlewood maximal operator $M$ for $\nu$ is defined by

$$M\nu(x) := \sup_{s>0} \frac{1}{|B(x,s)|} \int_{B(x,s)} \nu(z) dz,$$

where $|\nu|$ is the total variation of $\nu$, and if $\nu(x) = f(x)dx$, then we denote $Mf(x)$.

The following lemma is well-known as the Hardy–Littlewood maximal weak and strong type estimates.

**Lemma 2.7.**

(i) Weak type estimate (e.g., Chapter III, Section 4.1, (a) in [79]). There exists $A_1 > 0$ such that for any finite Borel measure $\nu$ on $\mathbb{R}^d$ and $\lambda > 0$,

$$\text{Leb}\left( \{ x \in \mathbb{R}^d : M\nu(x) > \lambda \} \right) \leq A_1 |\nu|(\mathbb{R}^d)\lambda^{-1}.$$ 

(ii) Strong type estimate (e.g., Chapter I, Section 1.3, Theorem 1 (c) in [79]). For any $p \in (1, \infty)$, there exists $A_p > 0$ such that for any $f \in L^p(\mathbb{R}^d)$,

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}.$$ 

**Remark 2.8.** (i) The estimate in Lemma 2.7 (i) can be shown by the same way as the proof of Theorem 1 (b) in Chapter I, Section 4.1 of [79] as an application of Vitali’s covering lemma (e.g., Chapter I, Section 1.6, Lemma in [79]), and the constant $A_1$ can be chosen as $A_1 = 5^d$. 


(ii) The Hardy–Littlewood maximal operator is used to prove the flow property of ordinary differential equations (ODEs) and stochastic differential equations (SDEs) with Sobolev coefficients. In particular, by using this maximal operator, Crippa and De Lellis [13] proved the existence of a unique regular Lagrangian flow for ODEs with a local Sobolev coefficient, and Zhang [39] (also see, [40]) studied the stochastic homeomorphism flows property for SDEs with local Sobolev coefficients.

The following lemma shows that the $\Delta_2$-condition is equivalent to the Hardy–Littlewood maximal strong type estimate on the Orlicz space.

**Lemma 2.9** (Theorem 2.1 in [27]). Let $\Phi$ be an $N$-function and $\Psi$ be its complementary function. Then $\Psi$ satisfies the $\Delta_2$-condition if and only if there exists $A_\Phi > 0$ such that for any $f \in L^\Phi(\mathbb{R}^d)$,

$$
\|Mf\|_{L^\Phi(\mathbb{R}^d)} \leq A_\Phi \|f\|_{L^\Phi(\mathbb{R}^d)}.
$$

The following lemma shows that the Hardy–Littlewood maximal strong type estimate holds on the Sobolev space $W^{1,p}([0,T] \times \mathbb{R}^d)$ with a variable exponents $p \in P_{\text{loc}}(\mathbb{R}^d)$.

**Lemma 2.10** (e.g., Theorem 4.3.8 in [13]). Let $p \in P_{\text{loc}}(\mathbb{R}^d)$ with $1 < p^-$. Then there exists $A_{p(\cdot)} > 0$ such that for any $f \in L^{p(\cdot)}(\mathbb{R}^d)$,

$$
\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^d)} \leq A_{p(\cdot)} \|f\|_{L^{p(\cdot)}(\mathbb{R}^d)}.
$$

### 2.3 Main statements

In this subsection, we state our main statements of this article as multi-dimensional analogues of Avikainen’s estimate. We use notations $1/\infty := 0$ and $1/0 := \infty$ for convenience.

We first consider the case of bounded variation in $\mathbb{R}^d$.

**Theorem 2.11.** Let $X, \hat{X} : \Omega \to \mathbb{R}^d$ be random variables which admit density functions $p_X$ and $\hat{p}_X$ with respect to Lebesgue measure, respectively, and let $r \in (1, \infty)$. Suppose that $p_X \in L^\infty(\mathbb{R}^d)$. Then for any $f \in BV(\mathbb{R}^d) \cap L^r(\mathbb{R}^d, p_X) \cap L^r(\mathbb{R}^d, \hat{p}_X)$, $p \in (0, \infty)$ and $q \in [1, r)$, it holds that

$$
\mathbb{E} \left[ \left| f(X) - f(\hat{X}) \right|^q \right] \leq C_{BV}(p, q, r) \mathbb{E} \left[ |X - \hat{X}|^p \right]^{\frac{q}{p(r-1)}},
$$

where the constant $C_{BV}(p, q, r)$ is defined by

$$
C_{BV}(p, q, r) :=
\begin{cases}
2K_0^p \|f\|_\infty + A_1 \|p_X\|_\infty \int_{\mathbb{R}^d} |Df| & \text{if } r = \infty, \\
2^{q-1} \left( 2K_0^p + A_1 \|p_X\|_\infty \int_{\mathbb{R}^d} |Df| \right) \left( \|f\|_{L^{r}(\mathbb{R}^d, p_X)} + \|f\|_{L^{r}(\mathbb{R}^d, \hat{p}_X)} \right)^{r-1} & \text{if } r \in (1, \infty), \ \mathbb{E}[|X - \hat{X}|^p] < 1, \\
\left( \|f\|_{L^{r}(\mathbb{R}^d, p_X)} + \|f\|_{L^{r}(\mathbb{R}^d, \hat{p}_X)} \right)^q & \text{if } r \in (1, \infty), \ \mathbb{E}[|X - \hat{X}|^p] \geq 1.
\end{cases}
$$

Here, $K_0$ and $A_1$ are the constants of the pointwise estimate [9] in Lemma 2.13 and of the Hardy–Littlewood maximal weak type estimate in Lemma 2.7 (i), respectively.
Lemma 2.13. (i) In Theorem 2.11 we need the existence of density functions for both $X$ and $\hat{X}$ in order to use the pointwise estimate (9) in Lemma 2.13. However, we assume the boundedness only for one of them.

(ii) Recall that if $d = 1$, then $BV(\mathbb{R}) \subset L^\infty(\mathbb{R})$ (see, Remark 2.12 (iii)).

(iii) In the estimate (8) for $r = \infty$, the power $1/(p + 1)$ is optimal, that is, there exist $f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and random variables $X$ with the bounded density and $\hat{X}$ such that both sides in (8) coincide for some constant $C_{BV}(p, q, \infty)$ (see, Theorem 2.3 (ii) in [4] for one-dimensional case). Indeed, let $X = (X_1, \ldots, X_d)^\top$ and $X_1, \ldots, X_d$ be independent uniformly distributed random variables on $[0, 1]$. For given $\varepsilon \in (0, 1)$, we define $\hat{X} = (\hat{X}_1, \ldots, \hat{X}_d)^\top$ by

$$\hat{X}_1 := \begin{cases} X_1, & \text{if } X_1 \in \left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right], \\ X_1 + \frac{\varepsilon}{2}, & \text{if } X_1 \in \left[\frac{1-\varepsilon}{2}, \frac{1}{2}\right] \end{cases}$$

and $\hat{X}_i := X_i$ for $i = 2, \ldots, d$. Then for any $p \in (0, \infty)$, we obtain

$$\mathbb{E} \left[ \left| X - \hat{X} \right|^p \right] = \left( \frac{\varepsilon}{2} \right)^p \mathbb{P} \left( X_1 \in \left[\frac{1-\varepsilon}{2}, \frac{1}{2}\right] \right) = \left( \frac{\varepsilon}{2} \right)^{p+1}.$$

Hence for $f = 1_{[1/2, 1]} \times F \in BV(\mathbb{R}^d)$, $F \in \mathcal{B}(\mathbb{R}^{d-1})$ and $q \in [1, \infty)$, we have

$$\mathbb{E} \left[ 1_{[1/2, 1]} \times F(X) - 1_{[1/2, 1]} \times F(\hat{X}) \right]^q = \mathbb{E} \left[ 1_F(X_2, \ldots, X_d) 1_{[1/2, 1]}(X_1) - 1_{[1/2, 1]}(\hat{X}_1) \right]$$

$$= \mathbb{E} \left[ 1_F(X_2, \ldots, X_d) 1_{[1/2, 1]}(X_1) \right] - \mathbb{E} \left[ 1_F(\hat{X}_1) \right] = \mathbb{E} \left[ 1_F(\hat{X}_1) \right] = \mathbb{E} \left[ 1_{\mathbb{R}^{d-1}}(F \cap [0, 1]^{d-1}) \right] \mathbb{P} \left( X_1 \in \left[\frac{1-\varepsilon}{2}, \frac{1}{2}\right] \right)$$

$$= \mathbb{E} \left[ 1_{\mathbb{R}^{d-1}}(F \cap [0, 1]^{d-1}) \right] \frac{\varepsilon}{2} = \mathbb{E} \left[ 1_{\mathbb{R}^{d-1}}(F \cap [0, 1]^{d-1}) \right] \left( \frac{\varepsilon}{2} \right)^{p+1}.$$

Thus both sides in (8) coincide.

(iv) In the case of $r \in (1, \infty)$ and $\mathbb{E}[|X - \hat{X}|^p] \geq 1$, the power of the right hand side of the estimate (8) does not necessarily have to be $\frac{r+1}{p}$ and can be chosen arbitrarily. Indeed, for any $\alpha \geq 0$, since $\mathbb{E}[|X - \hat{X}|^p]^\alpha \geq 1$, we obtain

$$\mathbb{E} \left[ \left| f(X) - f(\hat{X}) \right|^q \right] \leq \left( \mathbb{E}[|X - \hat{X}|^p]^\alpha \right)^q \leq \left( \mathbb{E}[|X - \hat{X}|^p]^\alpha \right)^q \mathbb{E} \left[ \left| X - \hat{X} \right|^q \right]^\alpha.$$

Before proving Theorem 2.11, we give a pointwise estimate for functions of locally bounded variation in $\mathbb{R}^d$, which plays a crucial role in our arguments.

Lemma 2.13. Let $f \in BV_{loc}(\mathbb{R}^d)$. Then there exist a constant $K_0 > 0$ and a Lebesgue null set $N \in \mathcal{B}(\mathbb{R}^d)$ such that for all $x, y \in \mathbb{R}^d \setminus N$,

$$|f(x) - f(y)| \leq K_0|x - y| M_{2|x-y|}(Df)(x). \quad (9)$$
Remark 2.14. By the same way as the proof of Lemma 2.13, the following pointwise estimate holds:

$$|f(x) - f(y)| \leq K_1|x - y| \{M_2|x-y|(Df)(x) + M_2|x-y|(Df)(y)\}, \text{ a.e. } x, y \in \mathbb{R}^d$$  \hspace{1cm} (10)

for some $K_1 > 0$. Note that Theorem 3 in [61] shows that functions in $BV(\mathbb{R}^d)$ can be characterized by this estimate. If both $X$ and $\tilde{X}$ admit bounded density functions, we can use the pointwise estimate (10) for proving Theorem 2.11 and Lemma 2.15, but we assume boundedness only for one of them. Thus we need to modify the pointwise estimate from (10) to (9).

Proof of Lemma 2.13. The proof is based on Theorem 3.2 in [38]. We first note that if $d \geq 2$, by using Jensen’s inequality and Poincaré’s inequality for functions of locally bounded variation (see, e.g., Section 5.5, Theorem 1 (ii) in [21]), there exists a constant $C_0 > 0$ such that for any $x \in \mathbb{R}^d$ and $r > 0$,

$$\int_{B(x,r)} |f(z) - (f)_{x,r}| \, dz \leq \left(\int_{B(x,r)} |f(z) - (f)_{x,r}|^{d+1} \, dz \right)^{\frac{1}{d+1}} \leq \frac{C_0}{\text{Leb}(U(x;r))^{\frac{d+1}{d}}} \int_{U(x;r)} |Df| \leq C_d r M_r(Df)(x),$$  \hspace{1cm} (11)

where $(f)_{x,r} := \int_{B(x,r)} f(z) \, dz$ and $C_d := C_0 \frac{\Gamma(d/2 + 1)^{-1/d}}{\sqrt{\pi}}$. If $d = 1$, there exists $\{f_k\}_{k \in \mathbb{N}} \subset C^1(U(x;r);\mathbb{R})$ such that $f_k \to f$ in $L^1(U(x;r))$ and $\int_{U(x;r)} |f_k'(z)| \, dz \to \int_{U(x;r)} |Df| \, dz$ as $k \to \infty$ (e.g., Section 5.2, Theorem 2 in [21] or Theorem 1.17 in [31]). Then by using Fatou’s lemma and Lemma 1 in Section 4.5 of [21] with $p = 1$, there exists $C_1 > 0$ such that

$$\int_{B(x,r)} |f(z) - (f)_{x,r}| \, dz \leq C_1 r \liminf_{k \to \infty} \int_{U(x;r)} \int_{U(x;r)} |f(z) - f_k(y)| \, dy \, dz \leq C_1 r \liminf_{k \to \infty} \int_{U(x;r)} \int_{U(x;r)} |f_k'(y)| \, dy \, dz \leq C_1 r M_r(Df)(x).$$  \hspace{1cm} (12)

Moreover, the Lebesgue differentiation theorem (e.g., Section 1.7, Theorem 1 in [21]) shows that there exists a Lebesgue null set $N \in \mathcal{B}(\mathbb{R}^d)$ such that for any $x \in \mathbb{R}^d \setminus N$,

$$\lim_{r \to 0} (f)_{x,r} = f(x).$$  \hspace{1cm} (13)

Let $x, y \in \mathbb{R}^d \setminus N$ be fixed and denote $r_i := 2^{-i}|x - y|$ for $i \in \mathbb{N} \cup \{0\}$. Then by using (13), we obtain

$$|f(x) - (f)_{x,r_0}| \leq \sum_{i=0}^{\infty} |(f)_{x,r_{i+1}} - (f)_{x,r_i}| \leq \sum_{i=0}^{\infty} \int_{B(x;r_{i+1})} |f(z) - (f)_{x,r_i}| \, dz$$

13
By using this trick, we obtain
\[ |f(x) - (f)_{x,r_0}| \leq 2^{d+1}C_d|x - y|M_{|x-y|}(Df)(x). \]  
(14)

By the same way, since \( B(y;r_0) \subset B(x;2r_0) \), we have
\[ |f(y) - (f)_{y,r_0}| \leq 2^{d+1}C_d|x - y|M_{|x-y|}(Df)(y) \leq 2^{2d+1}C_d|x - y|M_{2|x-y|}(Df)(x). \]  
(15)

By combining (14), (15) and (16), we conclude the proof.

By using the Hardy–Littlewood maximal weak type estimate in Lemma 2.7 (i) and the pointwise estimate in Lemma 2.13, we first prove the estimate for indicator functions \( 1_E \in BV(\mathbb{R}^d) \), which is a multi-dimensional analogue of Lemma 3.4 in [4] and Proposition 5.3 in [30].

**Lemma 2.15.** Let \( X, \tilde{X} : \Omega \to \mathbb{R}^d \) be random variables which admit density functions \( p_X \) and \( p_{\tilde{X}} \) with respect to Lebesgue measure, respectively. Suppose that \( p_X \in L^\infty(\mathbb{R}^d) \). Then for any \( E \in \mathcal{B}(\mathbb{R}^d) \) with \( 1_E \in BV(\mathbb{R}^d) \) and \( p, q \in (0, \infty) \), it holds that
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\tilde{X}) \right|^q \right] \leq \left( K_0^q + A_1\|p_X\|_\infty \int_{\mathbb{R}^d} |D1_E| \right) \mathbb{E} \left[ |X - \tilde{X}|^p \right]^{\frac{q}{p}}. \]  
(17)

**Proof.** If \( \mathbb{E}[|X - \tilde{X}|^p] = 0 \) then \( X = \tilde{X} \) almost surely, and thus the statement is obvious.

We assume \( \mathbb{E}[|X - \tilde{X}|^p] > 0 \). For \( \lambda > 0 \), we define the event \( \Omega(D1_E, \lambda) \in \mathcal{F} \) by
\[
\Omega(D1_E, \lambda) := \{M(D1_E)(X) > \lambda\}. \]

We first remark that for any \( x, y \in \mathbb{R}^d \), it holds that
\[
|1_E(x) - 1_E(y)|^q = |1_E(x) - 1_E(y)|^p. \]  
(18)

By using this trick, we obtain
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\tilde{X}) \right|^q \right] = \mathbb{E} \left[ \left| 1_E(X) - 1_E(\tilde{X}) \right|^p 1_{\Omega(D1_E, \lambda)} \right] + \mathbb{E} \left[ \left| 1_E(X) - 1_E(\tilde{X}) \right|^p 1_{\Omega(D1_E, \lambda)^c} \right]. \]  
(19)
On the event $\Omega(D1_{E}, \lambda)$, since $X$ has a bounded density function, by using Lemma 2.13 (i), we have
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\hat{X}) \right|^p \mathbb{1}_{\Omega(D1_{E}, \lambda)} \right] \leq \mathbb{P}(M(D1_{E})(X) > \lambda) \leq A_1 \|pX\|_\infty \int_{R^d} |D1_{E}| \lambda^{-1}.
\] (19)

Let $N \in \mathcal{B}(R^d)$ be the Lebesgue null set defined on Lemma 2.13. On the event $\Omega(D1_{E}, \lambda)^c$, since $X$ and $\hat{X}$ have density functions, by Lemma 2.13 we obtain
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\hat{X}) \right|^p \mathbb{1}_{\Omega(D1_{E}, \lambda)^c} \right] = \mathbb{E} \left[ \left| 1_E(X) - 1_E(\hat{X}) \right|^p \mathbb{1}_{\Omega(D1_{E}, \lambda)^c} 1_{R^d \setminus N}(X) 1_{R^d \setminus N}(\hat{X}) \right]
\leq K_0^p \mathbb{E} \left[ |X - \hat{X}|^p M(D1_{E})(X)^p \mathbb{1}_{\Omega(D1_{E}, \lambda)^c} \right]
\leq K_0^p \lambda^p \mathbb{E} \left[ |X - \hat{X}|^p \right].
\] (20)

Hence, by (19) and (20), we have
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\hat{X}) \right|^q \right] \leq A_1 \|pX\|_\infty \int_{R^d} |D1_{E}| \lambda^{-1} + K_0^p \lambda^p \mathbb{E} \left[ |X - \hat{X}|^p \right].
\]

Now we choose $\lambda := \mathbb{E}[|X - \hat{X}|^p]^{-R} > 0$ for some $R > 0$. Then we obtain
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\hat{X}) \right|^q \right] \leq A_1 \|pX\|_\infty \int_{R^d} |D1_{E}| \mathbb{E} \left[ |X - \hat{X}|^p \right]^R + K_0^p \mathbb{E} \left[ |X - \hat{X}|^p \right]^{1-pR}.
\]

By choosing $R$ as $R = 1 - pR$, that is, $R = \frac{1}{p + 1}$, then we have
\[
\mathbb{E} \left[ \left| 1_E(X) - 1_E(\hat{X}) \right|^q \right] \leq \left( A_1 \|pX\|_\infty \int_{R^d} |D1_{E}| + K_0^p \right) \mathbb{E} \left[ |X - \hat{X}|^p \right]^{\frac{1}{p+1}},
\]
which concludes the statement. \(\Box\)

**Remark 2.16.** Note that the equation (18) is the key trick for replacing the power $q$ in the left hand side of Avikainen’s estimates (8) and (17) by $p$ in the right hand side.

By using Lemma 2.13 with the coarea formula for functions of bounded variation, we now prove Theorem 2.11 for general functions $f \in BV(R^d)$.

**Proof of Theorem 2.11.** For $\lambda > 0$, we define the event $\Omega(f, \lambda) \in \mathcal{F}$ by
\[
\Omega(f, \lambda) := \{|f(X)| > \lambda\} \cup \{|f(\hat{X})| > \lambda\}.
\]

For $t \in \mathbb{R}$, we define $E_t := \{x \in R^d : f(x) > t\}$. Then for any $x, y \in R^d$, it holds that
\[
|f(x) - f(y)| = \int_{f(x) \wedge f(y)}^{f(x) \vee f(y)} |1_{E_t}(x) - 1_{E_t}(y)| dt.
\]

Hence, since $q \in [1, \infty)$, by using Jensen’s inequality, it holds that
\[
\mathbb{E} \left[ \left| f(X) - f(\hat{X}) \right|^q \mathbb{1}_{\Omega(f, \lambda)^c} \right] \leq (2\lambda)^{q-1} \int_{-\lambda}^{\lambda} \mathbb{E} \left[ \left| 1_{E_t}(X) - 1_{E_t}(\hat{X}) \right|^q \right] dt.
\]
It follows from Theorem 1 (i) in Section 5.5 of [21] that \( \mathbf{1}_{E_t} \in BV(\mathbb{R}^d) \) almost every \( t \in \mathbb{R} \). Note that by the coarea formula for functions of bounded variation (see, e.g., Theorem 1 (ii) in Section 5.5 of [21]), it holds that
\[
\int_{\mathbb{R}} dt \int_{\mathbb{R}^d} |D\mathbf{1}_{E_t}| = \int_{\mathbb{R}^d} |Df|.
\]
Therefore, by using Lemma 2.15 with \( E = E_t \), we obtain
\[
\mathbb{E} \left[ \left( f(X) - f(\hat{X}) \right)^q \mathbf{1}_{\Omega(f, \lambda)} \right] \leq (2\lambda)^{q-1} \int_{\lambda}^\infty \left( K_0^p + A_1 \|p_X\|_\infty \right) dt \mathbb{E} \left[ |X - \hat{X}|^p \right]^{(q-1)/r}
\leq (2\lambda)^{q-1} \left( 2K_0^p + A_1 \|p_X\|_\infty \right) \mathbb{E} \left[ |X - \hat{X}|^p \right]^{(q-1)/r}. \tag{21}
\]
If \( r = \infty \) (i.e., \( f \) is bounded), then by choosing \( \lambda := \|f\|_\infty \), it holds that \( \mathbb{P}(\Omega(f, \lambda)^c) = 1 \). Thus the estimate (21) implies that the estimate (8) in the case of \( r = \infty \) holds.

We next show the estimate (8) in the case of \( r \in (1, \infty) \) and \( \mathbb{E}[|X - \hat{X}|^p] < 1 \). On the event \( \Omega(f, \lambda) \), by using Hölder’s inequality with \( \frac{1}{r/q} + \frac{1}{r/(r-q)} = 1 \), we obtain
\[
\mathbb{E} \left[ \left( f(X) - f(\hat{X}) \right)^q \mathbf{1}_{\Omega(f, \lambda)} \right] \leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^q \mathbb{P}(\Omega(f, \lambda))^{1-\frac{q}{r}} \leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^r \lambda^{-(r-q)}. \tag{22}
\]
We choose \( \lambda := (\mathbb{E}[|X - \hat{X}|^p])^{1/r} > 1 \). Then by (21) and (22), we have
\[
\mathbb{E} \left[ \left( f(X) - f(\hat{X}) \right)^q \right] \leq 2^{q-1} \lambda^q \left( 2K_0^p + A_1 \|p_X\|_\infty \right) \mathbb{E} \left[ |X - \hat{X}|^p \right]^{1/r} \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^r \lambda^{-(r-q)} = C_{BV}(p, q, r) \mathbb{E} \left[ |X - \hat{X}|^p \right]^{1/r} \lambda^{-(r-q)},
\]
which concludes the estimate (8) in the case of \( r \in (1, \infty) \) and \( \mathbb{E}[|X - \hat{X}|^p] < 1 \). In the case of \( r \in (1, \infty) \) and \( \mathbb{E}[|X - \hat{X}|^p] \geq 1 \), it is already shown in Remark 2.12 (iv).

Orlicz–Sobolev spaces and Sobolev spaces with variable exponents

For a function \( f \) in \( BV_{\text{loc}}(\mathbb{R}^d) \) or \( W^{1,1}_{\text{loc}}(\mathbb{R}^d) \), \( \int_{\mathbb{R}^d} |Df| \) might not be finite, and thus it is difficult to estimate the probability \( \mathbb{P}(M(Df)(X) > \lambda) \). Therefore, from here we will consider Avikainen’s estimates for several subspaces of \( W^{1,1}_{\text{loc}}(\mathbb{R}^d) \).

We first consider the case of the Orlicz–Sobolev space.

**Theorem 2.17.** Let \( \Phi \) be an \( N \)-function and \( \Psi \) be its complementary function. Suppose that \( \Psi \) satisfies the \( \Delta_2 \)-condition. Let \( X, \hat{X} : \Omega \to \mathbb{R}^d \) be random variables which admit density functions \( p_X \) and \( p_{\hat{X}} \) with respect to Lebesgue measure, respectively, and let \( r \in (1, \infty) \). Suppose that \( p_X \in \)
$L^\infty(\mathbb{R}^d) \cup L^\Psi(\mathbb{R}^d)$. Then for any $f \in W^{1,\Phi}(\mathbb{R}^d) \cap L^r(\mathbb{R}^d, p_X) \cap L^r(\mathbb{R}^d, p_{\hat{X}})$ and $q \in (0, r)$, it holds that

$$
\mathbb{E} \left[ |f(X) - f(\hat{X})|^q \right] 
\leq \begin{cases} 
C_{W^{1,\Phi}}(q, r, \infty) \inf_{\lambda > 0} \left\{ \lambda^{-(1-\frac{q}{r})} + (\Phi^{-1}(\lambda))^q \mathbb{E} \left[ |X - \hat{X}|^q \right] \right\}, & \text{if } p_X \in L^\infty(\mathbb{R}^d), \\
C_{W^{1,\Phi}}(q, r, \Psi) \mathbb{E} \left[ |X - \hat{X}|^q \right]^{\frac{1-q/r}{q}}, & \text{if } p_X \in L^\Psi(\mathbb{R}^d),
\end{cases}
$$

(23)

where the constants $C_{W^{1,\Phi}}(q, r, \infty)$ and $C_{W^{1,\Phi}}(q, r, \Psi)$ are defined by

$$
C_{W^{1,\Phi}}(q, r, \infty) := \max \left\{ \left( \frac{K_0}{\alpha} \right)^q, \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|Df\|_{L^r(\mathbb{R}^d, p_{\hat{X}})} \right)^q \right\} \left( A_1 \|p_X\|_\infty \|\Phi(\alpha|Df|)\|_{L^1(\mathbb{R}^d)} \right)^{1 - \frac{q}{r}}.
$$

$$
C_{W^{1,\Phi}}(q, r, \Psi) := K_0^q + \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|Df\|_{L^r(\mathbb{R}^d, p_{\hat{X}})} \right)^q \left( 2A_1 \|p_X\|_{L^\Psi(\mathbb{R}^d)} \|Df\|_{L^\Psi(\mathbb{R}^d)} \right)^{1 - \frac{q}{r}}.
$$

Here, $K_0$, $A_1$, and $A_1$ are the constants of the pointwise estimate (9) in Lemma 2.13 of the Hardy-Littlewood maximal weak and strong type estimates in Lemma 2.7 (i) and Lemma 2.9 respectively, and $\alpha$ is a positive constant such that $\|\Phi(\alpha|Df|)\|_{L^1(\mathbb{R}^d)} < \infty$.

**Remark 2.18.**

(i) In the right hand side of the estimates (23), the power inside of the expectation is $q$ not $p$ in $(0, \infty)$ unlike the case of $BV(\mathbb{R}^d)$ (see, Theorem 2.11). The reason is that we do not know the indicator function $1_{E_t}, E_t := \{x \in \mathbb{R}^d : f(x) > t\}$ belongs to $BV(\mathbb{R}^d)$ or $W^{1,\Phi}(\mathbb{R}^d)$, and thus we cannot apply the trick (18) for replacing the power $q$ by $p$.

(ii) Let $\Phi$ be a Young function. Since $W^{1,\Phi}(\mathbb{R}^d) \subset BV_{\text{loc}}(\mathbb{R}^d)$ (see, Remark 2.13 (iv)), the pointwise estimates in Lemma 2.13 and Remark 2.14 hold for $f \in W^{1,\Phi}(\mathbb{R}^d)$. Moreover, by using Jensen’s inequality for the convex function $\Phi$, for almost every $x, y, \hat{X} \in \mathbb{R}^d$,

$$
|f(x) - f(y)| \leq K_0 |x - y| \Phi^{-1}(M_{2|x-y|}(\Phi(|Df|))(x))
$$

and

$$
|f(x) - f(y)| \leq K_1 |x - y| \left\{ \Phi^{-1}(M_{2|x-y|}(\Phi(|Df|))(x)) + \Phi^{-1}(M_{2|x-y|}(\Phi(|Df|))(y)) \right\}.
$$

(25)

Theorem 1.2 in [34] shows that functions $f \in W^{1,\Phi}(\mathbb{R}^d)$ can be characterized by the estimate (25).

As a conclusion of Theorem 2.11 and Theorem 2.17 noting Example 2.4 (i), we obtain the following estimates for the Sobolev space $W^{1,p}(\mathbb{R}^d)$ for $p \in [1, \infty)$.

**Corollary 2.19.** Let $X, \hat{X} : \Omega \to \mathbb{R}^d$ be random variables which admit density functions $p_X$ and $p_{\hat{X}}$ with respect to Lebesgue measure, respectively, and let $r \in (1, \infty), p \in [1, \infty)$ and $p^* := p/(p-1)$. Suppose that $p_X \in L^\infty(\mathbb{R}^d) \cup L^{p^*}(\mathbb{R}^d)$. Then for any $f \in W^{1,p}(\mathbb{R}^d) \cap L^r(\mathbb{R}^d, p_X) \cap L^r(\mathbb{R}^d, p_{\hat{X}})$ and $q \in (0, r)$, there exist $C_{W^{1,p}}(q, r, \infty) > 0$ and $C_{W^{1,p}}(q, r, p^*) > 0$ such that

$$
\mathbb{E} \left[ |f(X) - f(\hat{X})|^q \right] \leq \begin{cases} 
C_{W^{1,p}}(q, r, \infty) \mathbb{E} \left[ |X - \hat{X}|^q \right]^{\frac{1-q/p}{1-q/r}}, & \text{if } p_X \in L^\infty(\mathbb{R}^d), \\
C_{W^{1,p}}(q, r, p^*) \mathbb{E} \left[ |X - \hat{X}|^q \right]^{\frac{1-q/p}{1-q/r}}, & \text{if } p_X \in L^{p^*}(\mathbb{R}^d).
\end{cases}
$$

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**Proof of Theorem 2.17.** We first assume that \( p_X \in L^\infty(\mathbb{R}^d) \). Since \(|Df| \in L^\Psi(\mathbb{R}^d)\), there exists \( \alpha > 0 \) such that \( \|\Phi(\alpha |Df|)\|_{L^1(\mathbb{R}^d)} < \infty \). Then for \( \lambda > 0 \), we define the event \( \Omega(\Phi(\alpha |Df|), \lambda) \in \mathcal{F} \) by

\[
\Omega(\Phi(\alpha |Df|), \lambda) := \{ M(\Phi(\alpha |Df|))(X) > \lambda \}.
\]

Since \( X \) has a bounded density, by using Lemma 2.7 (i), we obtain

\[
P(\Omega(\Phi(\alpha |Df|), \lambda)) \leq A_1 \|p_X\|_\infty \|\Phi(\alpha |Df|)\|_{L^1(\mathbb{R}^d)} \lambda^{-1}.
\]

Hence by using Hölder’s inequality with \( \frac{1}{r} + \frac{1}{r/(r-q)} = 1 \) in the case of \( r \in (1, \infty) \) and by using the boundedness of \( f \) in the case of \( r = \infty \), we have

\[
\mathbb{E} \left[ |f(X) - f(\hat{X})|^q \mathbf{1}_{\Omega(\Phi(\alpha |Df|), \lambda)} \right] \\
\leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^q \mathbb{P}(\Omega(\Phi(\alpha |Df|), \lambda))^{1-\frac{q}{r}} \\
\leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^q \left( A_1 \|p_X\|_\infty \|\Phi(\alpha |Df|)\|_{L^1(\mathbb{R}^d)} \right)^{1-\frac{q}{r}} \lambda^{-(1-\frac{q}{r})}.
\]  

(26)

Let \( N \in \mathcal{B}(\mathbb{R}^d) \) be the Lebesgue null set defined on Lemma 2.13. On the event \( \Omega(\Phi(\alpha |Df|), \lambda)^c \), density functions and \( \Phi^{-1} \) are non-decreasing, by similar way as (24) in Remark 2.18 (ii), we obtain

\[
\mathbb{E} \left[ |f(X) - f(\hat{X})|^q \mathbf{1}_{\Omega(\Phi(\alpha |Df|), \lambda)^c} \right] \\
= \mathbb{E} \left[ |f(X) - f(\hat{X})|^q \mathbf{1}_{\Omega(\Phi(\alpha |Df|), \lambda)^c} \mathbf{1}_{\mathbb{R}^d \setminus N}(X) \mathbf{1}_{\mathbb{R}^d \setminus N}(\hat{X}) \right] \\
\leq \left( \frac{K_0}{\alpha} \right)^q \mathbb{E} \left[ |X - \hat{X}|^q \|\Phi^{-1}(M(\Phi(\alpha |Df|))(X))\|^q \mathbf{1}_{\Omega(\Phi(\alpha |Df|), \lambda)^c} \right] \\
\leq \left( \frac{K_0}{\alpha} \right)^q \|\Phi^{-1}(\lambda)\|^q \mathbb{E} \left[ |X - \hat{X}|^q \right].
\]

(27)

Hence, by (26) and (27), we have

\[
\mathbb{E} \left[ |f(X) - f(\hat{X})|^q \right] \leq C_{W^{1,*}}(q, r, \infty) \left( \lambda^{-1+\frac{q}{r}} + \|\Phi^{-1}(\lambda)\|^q \mathbb{E} \left[ |X - \hat{X}|^q \right] \right),
\]

which concludes the statement for \( p_X \in L^\infty(\mathbb{R}^d) \).

Now we suppose \( p_X \in L^\Psi(\mathbb{R}^d) \). For \( \lambda > 0 \), we define the event \( \Omega(Df, \lambda) \in \mathcal{F} \) by

\[
\Omega(Df, \lambda) := \{ M(Df)(X) > \lambda \}.
\]

Since \( \Psi \) satisfies the \( \Delta_2 \)-condition, it follows from Lemma 2.9 that

\[
\|M(Df)\|_{L^\Psi(\mathbb{R}^d)} \leq A \|Df\|_{L^\Psi(\mathbb{R}^d)}.
\]

Hence by using the Markov inequality and the generalized Hölder’s inequality [10], we obtain

\[
P(\Omega(Df, \lambda)) \leq \int_{\mathbb{R}^d} M(Df)(x)p_X(x)dx \lambda^{-1}
\]
We choose $\lambda$ where the constant $C$ of H"older's inequality (6) for by the same way as the proof of Theorem 2.17, we can prove the statement by using the generalized Theorem 2.20.

be random variables which admit density functions $Df, \lambda$

On the event $\Omega(Df, \lambda)$

We can use the Hardy–Littlewood maximal strong type estimate in Lemma 2.10 since

Proof. We can use the Hardy–Littlewood maximal strong type estimate in Lemma 2.14 since $p \in \mathcal{P}^{\log}(\mathbb{R}^d)$. Moreover, it holds that $W^{1,p}(\mathbb{R}^d) \subset BV_{loc}(\mathbb{R}^d)$ (see, Remark 2.3 (iv)). Therefore, by the same way as the proof of Theorem 2.17 we can prove the statement by using the generalized H"older’s inequality (6) for $M(Df) \in L^{p}(\mathbb{R}^d)$ and $p_X \in L^{p}(\mathbb{R}^d)$, and thus it will be omitted.

Remark 2.21. For the Sobolev space with a variable exponent $p$, it is difficult to obtain Avikainen’s estimate in the case of $p_X \in L^{\infty}(\mathbb{R}^d)$. The reason is that since the variable exponent $p$ is not constant, we cannot use Jensen’s inequality in the same way as the estimate 27.
Fractional Sobolev spaces

We finally consider Avikainen’s estimates for fractional Sobolev spaces.

**Theorem 2.22.** Let $s \in (0, 1)$, $p \in [1, \infty)$ and $p^* := p/(p - 1)$. Let $X, \tilde{X} : \Omega \to \mathbb{R}^d$ be random variables which admit the density functions $p_X$ and $p_{\tilde{X}}$ with respect to Lebesgue measure, respectively, and let $r \in (1, \infty]$. Suppose that $p_X \in L^\infty(\mathbb{R}^d) \cup L^{p^*}(\mathbb{R}^d)$. Then for any $f \in W^{s,p}(\mathbb{R}^d) \cap L^r(\mathbb{R}^d, p_X) \cap L^{r^*}(\mathbb{R}^d, p_{\tilde{X}})$ and $q \in (0, r)$, it holds that

$$
\mathbb{E} \left[ |f(X) - f(\tilde{X})|^q \right] \leq \begin{cases} 
C_{W^{s,p}}(q, r, \infty) \mathbb{E} \left[ |X - \tilde{X}|^{qs} \right] \frac{p(1 - q/r)}{s + p(1 - q/r)}, & \text{if } p_X \in L^\infty(\mathbb{R}^d), \\
C_{W^{s,p}}(q, r, p^*) \mathbb{E} \left[ |X - \tilde{X}|^{qs} \right] \frac{1 - q/r}{s + q/r}, & \text{if } p_X \in L^{p^*}(\mathbb{R}^d),
\end{cases}
$$

where

$$C_{W^{s,p}}(q, r, \infty) := K_0(s, p)^q + \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^{r^*}(\mathbb{R}^d, p_{\tilde{X}})} \right)^q (A_1 \|p_X\|_{\infty} \|G_{s,p,f}\|_{L^1(\mathbb{R}^d)})^{1 - \frac{q}{r}},$$

$$C_{W^{s,p}}(q, r, p^*) := K_0(s, p)^q + \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^{r^*}(\mathbb{R}^d, p_{\tilde{X}})} \right)^q (A_p \|p_X\|_{L^{p^*}(\mathbb{R}^d)} \|G_{s,p,f}\|_{L^p(\mathbb{R}^d)})^{1 - \frac{q}{r}}.$$

Here, $K_0(s, p)$, $A_1$ and $A_p$ are the constants of pointwise estimate (30) in Lemma 2.23 and of the Hardy–Littlewood maximal weak and strong type estimates in Lemma 2.22 (i) and (ii), respectively.

Before proving Theorem 2.22, we give a pointwise estimate for functions in $W^{s,p}(\mathbb{R}^d)$, which plays a crucial role in our argument.

**Lemma 2.23.** Let $s \in (0, 1)$, $p \in [1, \infty)$ and $f \in W^{s,p}(\mathbb{R}^d)$. Then there exist a constant $K_0(s, p) > 0$ and a Lebesgue null set $N \in \mathcal{B}(\mathbb{R}^d)$ such that for all $x, y \in \mathbb{R}^d \setminus N$,

$$|f(x) - f(y)| \leq K_0(s, p) |x - y|^s M_{2|x-y|}(G_{s,p,f})(x). \quad (30)$$

**Remark 2.24.** By the same way as the proof of Lemma 2.23, the following pointwise estimate holds:

$$|f(x) - f(y)| \leq K_1(s, p) |x - y|^s \left\{ M_{2|x-y|}(G_{s,p,f})(x) + M_{2|x-y|}(G_{s,p,f})(y) \right\}, \text{ a.e. } x, y \in \mathbb{R}^d \quad (31)$$

for some $K_1(s, p) > 0$. Note that Yang [86] introduced Hajlasz–Sobolev space $W^{s,p}(X)$ on a metric measure space $X$ of homogeneous type by using the pointwise estimate similar to (31) (see, Definition 1.4 in [86]).

**Proof of Lemma 2.23.** The proof is similar to Lemma 2.13. By using Jensen’s inequality, for any $x \in \mathbb{R}^d$ and $r > 0$,

$$\int_{B(x;r)} |f(z) - (f)_{x,r}| dz \leq \int_{B(x;r)} \left( \int_{B(x;r)} |f(z) - f(y)|^p dy \right)^{1/p} dz \leq (2r)^{(d+sp)/p} \int_{B(x;r)} \left( \int_{B(x;r)} |f(z) - f(y)|^p dy \right)^{1/p} dz$$
where \( C_0(s, p) := 2^{(d + sp)/p} \left( \frac{\Gamma(d/2 + 1)}{\pi^{d/2}} \right)^{1/p} \). Let \( N \in \mathcal{B}(\mathbb{R}^d) \) be the Lebesgue null set defined on \( \mathbb{R}^d \).

Then, by the same way as the proof of Lemma 2.13 for fixed \( x, y \in \mathbb{R}^d \setminus N \) and \( r_i := 2^{-i}|x - y| \), for \( i \in \mathbb{N} \cup \{0\} \), we obtain

\[
|f(x) - (f)_{x,r_0}| \leq 2^d \sum_{i=0}^{\infty} \left| f(z) - (f)_{x,r_i} \right| \, dz
\]

\[
\leq 2^d C_0(s, p) M_{|x-y|}(G_{s,p,f})(x) \sum_{i=0}^{\infty} r_i^s
\]

\[
= \frac{2^{s+d}}{2^s - 1} C_0(s, p) M_{|x-y|}(G_{s,p,f})(x)|x-y|^s. \quad (32)
\]

By the same way, since \( B(y; r_0) \subset B(x; 2r_0) \), we have

\[
|f(y) - (f)_{y,r_0}| \leq 2^{s+d} C_0(s, p) M_{|x-y|}(G_{s,p,f})(y)|x-y|^s
\]

\[
\leq 2^{s+d+1} C_0(s, p) M_{|x-y|}(G_{s,p,f})(x)|x-y|^s. \quad (33)
\]

On the other hand, by the same way as proof of Lemma 2.13 it holds from (32) that

\[
|f(x)_{x,r_0} - (f)_{y,r_0}| \leq 2^{d+1} \int_{B(x; 2r_0)} \left| f(z) - (f)_{x,r_0} \right| \, dz
\]

\[
\leq 2^{s+d+1} C_0(s, p)|x-y|^s M_{|x-y|}(G_{s,p,f})(x). \quad (35)
\]

By combining (33), (34) and (35), we conclude the proof. \( \square \)

**Proof of Theorem 2.22**. We first assume \( p_X \in L^\infty(\mathbb{R}^d) \). For \( \lambda > 0 \), we define the event \( \Omega(|G_{s,p,f}|^p, \lambda) \in \mathcal{F} \) by

\[
\Omega(|G_{s,p,f}|^p, \lambda) := \{ M(|G_{s,p,f}|^p)(X) > \lambda \}.
\]

Since \( |G_{s,p,f}|^p \in L^1(\mathbb{R}^d) \), by using Lemma 2.7 (i), we obtain

\[
\mathbb{P}(\Omega(|G_{s,p,f}|^p, \lambda)) \leq A_1 \|p_X\|_\infty \|G_{s,p,f}|^p\|_{L^1(\mathbb{R}^d)} \lambda^{-1}.
\]

Hence by using Hölder’s inequality with \( \frac{1}{r/q} + \frac{1}{r/(r-q)} = 1 \) in the case of \( r \in (1, \infty) \) and by using the boundedness of \( f \) in the case of \( r = \infty \), we have

\[
\mathbb{E} \left[ \left| f(X) - f(\mathbb{X}) \right|^q \mathbf{1}_{\Omega(|G_{s,p,f}|^p, \lambda)} \right]
\]

\[
\leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_\mathbb{X})} \right)^q \mathbb{P}(\Omega(|G_{s,p,f}|^p, \lambda))^{1-\frac{q}{r}}.
\]
Hence by using Hölder’s inequality with

\[ \left( \|f\|_{L^r(\mathbb{R}^d)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^q (A_1 \|p_X\|_{\infty} \|G_{s,p,f}\|^p_{L^1(\mathbb{R}^d)})^{1-\frac{q}{r}} \lambda^{-(1-\frac{q}{r})}. \]  

(36)

Let \( N \in \mathcal{B}(\mathbb{R}^d) \) be the Lebesgue null set defined on Lemma 2.23. On the event \( \Omega(|G_{s,p,f}|^p, \lambda)^c \), since \( X \) and \( \tilde{X} \) have density functions, by Lemma 2.23 and using Jensen’s inequality, we obtain

\[
E \left[ |f(X) - f(\tilde{X})|^q \mathbf{1}_{\Omega(|G_{s,p,f}|^p, \lambda)^c} \right] = E \left[ |f(X) - f(\tilde{X})|^q \mathbf{1}_{\Omega(|G_{s,p,f}|^p, \lambda)^c} \mathbf{1}_{\Omega(\mathbb{R}^d \setminus N)} \mathbf{1}_{\Omega(\mathbb{R}^d \setminus N)(\tilde{X})} \right]
\leq K_0(s, p)^q E \left[ |X - \tilde{X}|^{qs} M(G_{s,p,f})(X)^q \mathbf{1}_{\Omega(|G_{s,p,f}|^p, \lambda)^c} \right]
\leq K_0(s, p)^q E \left[ |X - \tilde{X}|^{qs} M(G_{s,p,f})(X)^q \mathbf{1}_{\Omega(|G_{s,p,f}|^p, \lambda)^c} \right]
\leq K_0(s, p)^q \lambda^\frac{q}{r} E \left[ |X - \tilde{X}|^{qs} \right].
\]  

(37)

By choosing \( \lambda := E|X - \tilde{X}|^{qs} \frac{q}{r} \), we conclude the statement for \( p_X \in L^\infty(\mathbb{R}^d) \) from (36) and (37).

Now we suppose \( p_X \in L^{p^*}(\mathbb{R}^d) \). For \( \lambda > 0 \), we define the event \( \Omega(G_{s,p,f}, \lambda) \in \mathcal{F} \) by

\[ \Omega(G_{s,p,f}, \lambda) := \{ M(G_{s,p,f})(X) > \lambda \}. \]

By using the Markov inequality, Hölder’s inequality and Lemma 2.4 (ii), we obtain

\[
P(\Omega(G_{s,p,f}, \lambda)) \leq \int_{\mathbb{R}^d} M(G_{s,p,f})(x)p_X(x)dx \lambda^{-1}
\leq \|p_X\|_{L^{p^*}(\mathbb{R}^d)} \|M(G_{s,p,f})\|_{L^p(\mathbb{R}^d)} \lambda^{-1}
\leq A_1 \|p_X\|_{L^{p^*}(\mathbb{R}^d)} \|G_{s,p,f}\|_{L^p(\mathbb{R}^d)} \lambda^{-1}.
\]

Hence by using Hölder’s inequality with \( \frac{q}{r/q + \frac{1}{r}} = 1 \) in the case of \( r \in (1, \infty) \) and by using the boundedness of \( f \) in the case of \( r = \infty \), we have

\[
E \left[ |f(X) - f(\tilde{X})|^q \mathbf{1}_{\Omega(G_{s,p,f}, \lambda)} \right]
\leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^q P(\Omega(G_{s,p,f}, \lambda))^{1-\frac{q}{r}}
\leq \left( \|f\|_{L^r(\mathbb{R}^d, p_X)} + \|f\|_{L^r(\mathbb{R}^d, p_X)} \right)^q (A_1 \|p_X\|_{L^{p^*}(\mathbb{R}^d)} \|G_{s,p,f}\|_{L^p(\mathbb{R}^d)})^{1-\frac{q}{r}} \lambda^{-(1-\frac{q}{r})}.
\]  

(38)

On the event \( \Omega(G_{s,p,f}, \lambda)^c \), since \( X \) and \( \tilde{X} \) have density functions, by Lemma 2.23 we obtain

\[
E \left[ |f(X) - f(\tilde{X})|^q \mathbf{1}_{\Omega(G_{s,p,f}, \lambda)^c} \right] = E \left[ |f(X) - f(\tilde{X})|^q \mathbf{1}_{\Omega(G_{s,p,f}, \lambda)^c} \mathbf{1}_{\Omega(\mathbb{R}^d \setminus N)} \mathbf{1}_{\Omega(\mathbb{R}^d \setminus N)(\tilde{X})} \right]
\leq K_0(s, p)^q E \left[ |X - \tilde{X}|^{qs} M(G_{s,p,f})(X)^q \mathbf{1}_{\Omega(G_{s,p,f}, \lambda)^c} \right]
\leq K_0(s, p)^q \lambda^\frac{q}{r} E \left[ |X - \tilde{X}|^{qs} \right].
\]  

(39)

By choosing \( \lambda := E|X - \tilde{X}|^{qs} \frac{q}{r} \), we conclude the statement for \( p_X \in L^{p^*}(\mathbb{R}^d) \) from (38) and (39).
3 Applications

In this section, we apply Avikanen’s estimates proved in Section 2 to numerical analysis on irregular functionals of a solution to stochastic differential equations (SDEs) based on the multilevel Monte Carlo method, and to estimates of the $L^2$-time regularity of decoupled forward–backward stochastic differential equations (FBSDEs) with irregular terminal conditions.

3.1 Upper bound and integrability of density functions

In order to apply Avikanen’s estimates proved in Section 2, we need an appropriate upper bound or integrability of density functions. In this subsection, we give some examples of random variables with a bounded or integrable density function, which are studied in various ways.

We first give the well-known fact as a conclusion of Lévy’s inversion formula.

**Example 3.1.** Let $X : \Omega \rightarrow \mathbb{R}^d$ be a random variable. If the characteristic function $\varphi_X(\xi) := \mathbb{E}[e^{i\sqrt{-1}\langle \xi, X \rangle}]$ belongs to $L^1(\mathbb{R}^d)$, then by using Lévy’s inversion formula, $X$ admits a continuous density function $p_X$ of the form

$$p_X(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle x, \xi \rangle} \varphi_X(\xi) d\xi$$

(see, e.g., Proposition 2.5 in [76] or Theorem 16.6 in [85]), and thus $X$ has a bounded density function.

Next, we recall the Gaussian two-sided bound for density functions of solutions to SDEs driven by a Brownian motion.

**Example 3.2.** Let $B = (B(t))_{t \in [0, T]}$ be a $d$-dimensional standard Brownian motion, and let $X = (X(t))_{t \in [0, T]}$ be a solution to the following $d$-dimensional Markovian SDE of the form

$$dX(t) = b(t, X(t)) dt + \sigma(t, X(t)) dB(t), \quad X(0) = x \in \mathbb{R}^d, \quad t \in [0, T],$$

(40)

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and the diffusion matrix $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ are measurable functions. Suppose that $b$ is bounded and $\sigma$ satisfies the following two conditions.

(i) $a := \sigma \sigma^T$ is $\alpha$-Hölder continuous in space and $\alpha/2$-Hölder continuous in time for some $\alpha \in (0, 1]$, that is,

$$\|a\|_\alpha := \sup_{t \in [0, T], x \neq y} \frac{|a(t, x) - a(t, y)|}{|x - y|^\alpha} + \sup_{x \in \mathbb{R}^d, t \neq s} \frac{|a(t, x) - a(s, x)|}{|t - s|^\alpha/2} < \infty.$$

(ii) The diffusion coefficient $\sigma$ is bounded and uniformly elliptic, that is, there exist $\underline{\sigma}, \overline{\sigma} > 0$ such that for any $(t, x, \xi) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, $\underline{\sigma} |\xi|^2 \leq \langle a(t, x) \xi, \xi \rangle \leq \overline{\sigma} |\xi|^2$.

Then it is well-known that for all $t \in (0, T]$, $X(t)$ admits a density function $p_t(\cdot, \cdot)$ with respect to Lebesgue measure which has the Gaussian two sided bound, that is, there exist $C_- > 0$ and $c_\pm > 0$ such that for any $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$C_- g_{c_- t} (x, y) \leq p_t(x, y) \leq C_+ g_{c_+ t} (x, y)$$

(41)
of solutions to SDEs driven by a fractional Brownian motions (see, [8, 9]).

Moreover, the same or similar bounds [11] hold for SDEs with a path–dependent or an unbounded drift (see, Theorem 2.5. in [60] and Theorem 3.4 in [80]), and the Gaussian two sided bound [11] holds for Brownian motions with a signed measure valued drift belonging to the Kato class $K_{4,1}$ (see, Theorem 3.14 in [50]). In particular, for Brownian motions with a bounded drift (that is, $\sigma \equiv I$ in [11]), it holds that a sharp two–sided bound

$$q_t^{b,\alpha (\cdot, \cdot) \leq C \leq q_t^{y,\alpha (\cdot, \cdot)} (x, y),$$

where for $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R},$

$$q_t^{a,\alpha (x, \alpha)} = \prod_{i=1}^d \frac{2}{\sqrt{2\pi t}} \int_{|x+\alpha_i|/\sqrt{t}}^{\infty} z_i \exp \left( -\frac{(z_i - \beta \sqrt{t})^2}{2} \right) \, dz_i$$

(see [73, 72, 80]).

It is also well-known that the Gaussian two–sided bound holds for the fundamental solution $\Gamma(s, x; t, y)$ of parabolic equations in the divergence form $(\partial_{ss} + \frac{1}{2} \sum_{i,j=1}^d \partial_{x_i x_j} a_{i,j}(x) \partial_{x_j}) u(s, x) = 0$ (see, [3]), and there exists a Hunt process with the transition density function $\Gamma$ (see, Example 4.5.2, Theorem A.2.2 in [24] and Theorem I 9.4 in [12]).

On the other hand, Malliavin calculus can be used to study the regularity and upper bounds of density functions. Indeed, it is known that under Hörmander’s and the smoothness conditions on the coefficients, $X(t)$ admits a bounded and smooth density function (see, e.g., Theorem 6.16 in [78] and see, also [16, 52] for the Gaussian type estimates for density functions of solutions to degenerate SDEs). We also note that the Gaussian type two sided bound holds for density functions of solutions to SDEs driven by a fractional Brownian motions (see, [13, 24]).

The next example shows density estimates for path-dependent stochastic differential equations.

**Example 3.3.** Let $B = (B(t))_{t \in [0,T]}$ be a $d$-dimensional standard Brownian motion, and let $X = (X(t))_{t \in [0,T]}$ be a solution to the following $d$-dimensional path–dependent SDE of the form

$$dX(t) = b(t, X) \, dt + \sigma(t, X) \, dB(t), \quad X(0) = x \in \mathbb{R}^d, \quad t \in [0,T],$$

where the drift coefficient $b : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ and the diffusion matrix $\sigma : [0, T] \times C([0, T]; \mathbb{R}^d) \rightarrow \mathbb{R}^{d \times d}$ are measurable functions.

(i) Suppose that the coefficients $b$ and $\sigma$ are continuous in time and bounded continuously Gâteaux differentiable up to order $n+2$ in space, and $\sigma$ is uniformly elliptic. Then, by using Malliavin calculus, it is shown that for all $t \in (0, T]$, $X(t)$ admits a density function with respect to Lebesgue measure which belongs to $C^0_b(\mathbb{R}^d, \mathbb{R})$ (see, [59]).

(ii) Suppose that the coefficients $b$ and $\sigma$ are bounded, $\sigma$ is uniformly elliptic and there exist $\varepsilon > 0$ and $C > 0$ such that for any $(s, t, \omega) \in [0, T] \times [0, T] \times C([0, T]; \mathbb{R}^d)$ with $s < t,$

$$\sup_{1 \leq j \leq d} |\sigma_j(t, \omega) - \sigma_j(s, \omega)| \leq C \left\{ \log \left( \frac{1}{\sup_{u \leq \omega \leq \tau} |\omega_u - \omega_s|} \right) \right\}^{-(2+\varepsilon)},$$

The next example shows density estimates for path-dependent stochastic differential equations.
where \( \sigma_j := (\sigma_{1,j}, \ldots, \sigma_{d,j})^\top \). Then, by using an interpolation method, it is shown that for all \( t \in (0, T] \), \( X(t) \) admits a density function \( p_t(x, \cdot) \) with respect to Lebesgue measure which belongs to \( L_{\text{log}} (\mathbb{R}^d) \), where \( e_{\log}(x) := (1 + |x|) \log(1 + |x|) \) (see, Theorem 3.1 in [5]).

Finally, we give examples for a two-sided bound of density functions of solutions to SDEs driven by a rotation invariant \( \alpha \)-stable process.

**Example 3.4.** Let \( Z = (Z(t))_{t \in [0,T]} \) be a rotation invariant \( \alpha \)-stable process in \( \mathbb{R}^d \) with \( \alpha \in (0, 2) \) and \( \mathbb{E}[e^{\sqrt{\xi} Z(t)}] = e^{-t|\xi|^\alpha} \), \( \xi \in \mathbb{R}^d \) (see Theorem 14.14 in [76]), and let \( X = (X(t))_{t \in [0,T]} \) be a solution to the following \( d \)-dimensional SDE of the form

\[
\frac{dX(t)}{dt} = b(X(t))dt + \sigma(X(t-))dZ(t), \quad X(0) = x \in \mathbb{R}^d, \quad t \in [0,T],
\]

where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \) are bounded measurable functions. Suppose that the drift coefficient \( b \) is \( \gamma \)-Hölder continuous with \( \gamma \in (0, 1] \), and the jump intensity coefficient \( \alpha := |\sigma|^\alpha \) is \( \eta \)-Hölder continuous with \( \eta \in (0, 1] \) and uniformly positive, that is, there exists \( a > 0 \) such that for any \( x \in \mathbb{R}^d \), \( a(x) \geq a \). Under the balance condition \( \alpha + \gamma > 1 \), Kulik [55] proved that the existence of a unique weak solution to the equation [112]. Moreover, by using the parametrix method, he showed that for all \( t \in (0, T] \), \( X(t) \) admits a density function \( p_t(x, \cdot) \) with respect to Lebesgue measure and gave its two-sided bound, that is, there exist \( C_{\pm} > 0 \) such that for any \( (t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
C_- \bar{p}_t(x, y) \leq p_t(x, y) \leq C_+ \bar{p}_t(x, y),
\]

where

\[
\bar{p}_t(x, y) := \frac{1}{t^{d/\gamma \alpha(x)/\alpha}} \int_{1}^{\infty} \frac{g(y - v_t(x))}{t^{1/\alpha \gamma(x)^{1/\alpha}}} \, dv_t(x),
\]

\( g^{(\gamma)} \) is the density function of \( Z(1) \) and \( \{v_t(x)\}_{t \in [0, T]} \) is a solution to ODE \( dv_t(x) = b(v_t(x))dt \) with \( v_0(x) = x \) (see Theorem 2.1 and Theorem 2.2 in [58]). Note that if \( \gamma < 1 \), then such a solution of ODE may fail to be uniqueness, and if \( \alpha + \gamma < 1 \), then a solution of SDE [112] may fail to be uniqueness in law (see, Theorem 3.2 (ii) in [59]).

Moreover, by the asymptotic behaviour of \( g^{(\gamma)} \) (see, e.g., Theorem 2.1 in [77]), we have \( g^{(\gamma)}(x) \leq C \min\{1, |x|^{-d-\gamma} \} \) for some \( C > 0 \), which implies that \( p_t(x, \cdot) \in L^\infty(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d) \cap L^{r^+}(\mathbb{R}^d) \) for the complementary function \( \Psi \) of an \( N \)-function \( \Phi \) (see, Example 2.4 (iii)) and \( p^*(\cdot) := p(\cdot)/(p(\cdot) - 1) \) for \( p \in \mathcal{P}(\mathbb{R}^d) \) with \( 1 < p^- \leq p^+ < \infty \), (see, also [48, 52, 76] for upper bounds of density functions of Lévy processes).

### 3.2 Multilevel Monte Carlo method

In this subsection, we apply Avikainen’s estimates proved in Section 2 to the multilevel Monte Carlo method for solutions to SDE [10]. We first define the union of function spaces \( F(\mathbb{R}^d) \) by

\[
F(\mathbb{R}^d) := \left\{ BV(\mathbb{R}^d) \cup W^{1,\Phi}(\mathbb{R}^d) \cup W^{1,p(\cdot)}(\mathbb{R}^d) \cup W^{s,p}(\mathbb{R}^d) \right\} \cap L^\infty(\mathbb{R}^d)
\]

for an \( N \)-function \( \Phi \) with its complementary function \( \Psi \) which satisfies the \( \Delta_2 \)-condition, a variable exponent \( p(\cdot) \in \mathcal{P}^\infty(\mathbb{R}^d) \) with \( 1 < p^- \leq p^+ < \infty \) and \( (s, p) \in (0, 1] \times [1, \infty) \).
We consider the computational complexity of the mean squared error (MSE) to estimate the expectation of \( P := f(X(T)) \) for some measurable function \( f : \mathbb{R}^d \to \mathbb{R} \) with \( \mathbb{E}[|f(X(T))|] < \infty \), by using the standard and multilevel Monte Carlo method.

We first recall the standard Monte Carlo method. Let \( X^{(h)} = (X^{(h)}(t))_{t \in [0,T]} \) be the Euler–Maruyama scheme for SDE (40) with time step \( h \in (0, T) \), which is defined by
\[
dX^{(h)}(t) = b(\eta_h(t), X^{(h)}(\eta_h(t)))dt + \sigma(\eta_h(t), X^{(h)}(\eta_h(t)))dB(t), \quad X^{(h)}(0) = X(0), \quad t \in [0, T],
\]
where \( \eta_h(s) := kh \) and \( k \) is the natural number such that \( s/h - 1 < k \leq s/h \). We denote \( \hat{P}^{(h)} := f(X^{(h)}(T)) \). Let \( \hat{Y}^{(h)} \) be an estimator for \( \hat{P}^{(h)} \). For example, one may use \( \hat{Y}^{(h)} \) as the arithmetic mean, that is,
\[
\hat{Y}^{(h)} := \frac{1}{N} \sum_{i=1}^{N} \hat{P}^{(h,i)},
\]
where \( \hat{P}^{(h,1)}, \ldots, \hat{P}^{(h,N)} \) are i.i.d. random variables which have the same distribution as \( \hat{P}^{(h)} \). We suppose that the weak rate of convergence for \( X^{(h)} \) is \( \alpha > 0 \), that is, there exists \( c_0 > 0 \) such that
\[
|\mathbb{E}[f(X(T))] - \mathbb{E}[f(X^{(h)}(T))]| \leq c_0 h^\alpha.
\]
Then the mean squared error is estimated as follows:
\[
\text{MSE} := \mathbb{E}[|\hat{Y}^{(h)} - \hat{P}[P]|^2] = \mathbb{E}[|\hat{Y}^{(h)} - \mathbb{E}[\hat{Y}^{(h)}]|^2] + \mathbb{E}[\mathbb{E}[\hat{P}^{(h)}] - \mathbb{E}[\hat{P}[P]]]^2 \leq \text{Var}[\hat{P}^{(h)}]N^{-1} + c_0^2 h^{2\alpha}.
\]
Therefore, if we would like to make \( \text{MSE} \leq \varepsilon^2 \) for a given \( \varepsilon > 0 \), we choose \( N \) and \( h \) to satisfy \( \text{Var}[\hat{P}^{(h)}]N^{-1} \leq \varepsilon^2 / 2 \) and \( c_0^2 h^{2\alpha} \leq \varepsilon^2 / 2 \). Then the computational complexity for \( \hat{Y}^{(h)} \) is estimated above by \( c_1 \varepsilon^{-(2+1/\alpha)} \) for some constant \( c_1 > 0 \).

Now we recall the multilevel Monte Carlo method. Let \( M \in \mathbb{N} \) and \( X_\ell = (X_\ell(t))_{t \in [0,T]}, \ell = 0, \ldots, L \) be numerical approximations to \( X \) with each time step \( h_\ell := T/M^\ell \) and define \( \hat{P}_\ell := f(X_\ell(T)) \). Then it holds that
\[
\mathbb{E}[\hat{P}_\ell] = \sum_{\ell=0}^{L} \mathbb{E}[\hat{P}_\ell - \hat{P}_{\ell-1}],
\]
where \( \hat{P}_{-1} := 0 \). Let \( \hat{Y}_\ell, \ell = 0, \ldots, L \) be independent estimators for each \( \mathbb{E}[\hat{P}_\ell - \hat{P}_{\ell-1}] \) and \( C_\ell, \ell = 0, \ldots, L \) be their corresponding computational complexities. For example, one may use \( \hat{Y}_\ell \) as the arithmetic mean, that is,
\[
\hat{Y}_\ell = \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \left( \hat{P}^{(i)}_\ell - \hat{P}^{(i)}_{\ell-1} \right),
\]
where \( \hat{P}^{(1)}_{\ell} - \hat{P}^{(1)}_{\ell-1}, \ldots, \hat{P}^{(N_\ell)}_{\ell} - \hat{P}^{(N_\ell)}_{\ell-1} \) are i.i.d random variables which have the same distribution as \( \hat{P}_\ell - \hat{P}_{\ell-1} \). Note that the random variable \( \hat{P}_\ell - \hat{P}_{\ell-1} \) is the deference between two discrete approximations with different time steps \( h_\ell \) and \( h_{\ell-1} \), and the key point is that they are defined by the same Brownian motion. We define the estimator \( \hat{Y} := \sum_{\ell=0}^{L} \hat{Y}_\ell \) and its computational complexity \( C_{\text{MLMC}} := \sum_{\ell=0}^{L} C_\ell \). Then the following complexity theorem holds for the MLMC method.
Theorem 3.5 (Complexity theorem, Theorem 3.1 in [28], Theorem 2.1 [29]). Let $X$ be a solution to SDE (10), and denote $P := f(X(T))$. We assume that $\{\hat{P}_t\}_{t=0,\ldots,L}$, estimators $\{\hat{Y}_t\}_{t=0,\ldots,L}$ and their corresponding computational complexities $\{C_r\}_{r=0,\ldots,L}$ satisfy the following conditions: there exist positive constants $c_1, c_2, c_3$ and $\alpha, \beta$ such that $\alpha > \beta/2$ and (i) $E[P] - E[\hat{P}_t] \leq c_1 h^2_t$; (ii) $E[\hat{Y}_t] = E[\hat{P}_t - \hat{P}_{t-1}]$; (iii) $\text{Var}[\hat{Y}_t] \leq c_2 N^{-\frac{1}{2}}_t h^2_t$; (iv) $C_t \leq c_3 N_t h^{-1}_t$. Then for any $\varepsilon \in (0,1/e)$, the mean squared error is estimated by

$$MSE := E\left[ (\hat{Y} - E[P])^2 \right] \leq \varepsilon^2$$

with the computational complexity $C_{MLMC}$ for $\hat{Y}$ bounded by

$$C_{MLMC} \leq \begin{cases} 
    c_4 \varepsilon^{-2}, & \text{if } \beta \in (1, \infty), \\
    c_4 \varepsilon^{-2} (\log \varepsilon)^2, & \text{if } \beta = 1, \\
    c_4 \varepsilon^{-2 + (1-\beta)/\alpha}, & \text{if } \beta \in (0,1)
\end{cases}$$

for some constants $c_4 > 0$.

Before applying Theorem 3.5 to the Euler–Maruyama scheme, we provide the strong rate of convergence for $f$ for irregular functions $f \in F(\mathbb{R}^d)$.

Corollary 3.6. Suppose that the coefficients $b$ and $\sigma$ of SDE (10) are bounded and Lipschitz continuous in space, and $1/2$-Hölder continuous in time, and $\sigma$ is uniformly elliptic. Then for any $f \in F(\mathbb{R}^d)$, $q \in [1, \infty)$ and $\delta \in (0,1)$, there exist $C_{EM}(q, \delta) > 0$ and $C_{EM}(q) > 0$ such that

$$E\left[ |f(X(T)) - f(X^{(h)}(T))|^q \right] \leq C_{EM}(q, \delta) h^\frac{\delta}{2},$$

$$C_{EM}(q) h^{-\frac{d+\delta}{2}},$$

$$C_{EM}(q) h^{-\frac{d+\delta}{p^*}},$$

for $f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $f \in \{W^{1,\beta}(\mathbb{R}^d) \cup W^{1,p(q)}(\mathbb{R}^d) \} \cap L^\infty(\mathbb{R}^d)$, $f \in W^{s,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, respectively.

Proof. We first note that under the assumptions on the coefficients, the strong rate of convergence for the Euler–Maruyama scheme is $1/2$, that is, for any $p > 0$, there exists a constant $C_p > 0$ such that $E[|X(T) - X^{(h)}(T)|^p] \leq C_p h^{1/2}$ (see, e.g., [21]). It follows from Example 3.2 that $X(T)$ and $X^{(h)}(T)$ admit density functions with respect to Lebesgue measure and the density function $p_T(x, \cdot)$ of $X(T)$ has the Gaussian upper bound (11), and thus $p_T(x, \cdot) \in L^\infty(\mathbb{R}^d) \cap L^\Psi(\mathbb{R}^d) \cap L^{p(r)}(\mathbb{R}^d)$. Therefore, by using Theorem 2.11, 2.17, 2.20, and Corollary 2.19 with $r = \infty$, we conclude the statement.

Remark 3.7. Recently under non-Lipschitz coefficients, the strong rate of convergence for the Euler–Maruyama scheme are widely studied (see, e.g., [7, 13, 28, 35, 57, 63, 65, 68, 69]).

As applications of Theorem 3.5 and Corollary 3.6, we have the following two examples for irregular functions $f \in F(\mathbb{R}^d)$.

Example 3.8. Let $X_t$ be the Euler–Maruyama scheme with time step $h_t = T/M$ and $\hat{Y}_t$ be the estimator defined by (13). Suppose the coefficients $b$ and $\sigma$ of SDE (10) satisfy $b \in C_0^3([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in C_0^3([0,T] \times \mathbb{R}^d, \mathbb{R}^{d \times d})$ and $\partial_t \sigma \in C_0^3([0,T] \times \mathbb{R}^d, \mathbb{R}^{d \times d})$, and $\sigma$ is uniformly elliptic. Then it follows from Theorem 2.5 in [32] that for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$,
the weak rate of convergence $\alpha$ in Theorem 3.3 is one (see also, Theorem 3.5 in [6], Corollary 22 in [34], Theorem 1.1 in [54] and Theorem 1 in [82]). Note that

$$\text{Var}[\hat{Y}_t] = \mathbb{E}[|\hat{Y}_t - \mathbb{E}[\hat{Y}_t]|^2] = N^{-1}_c \text{Var}[\hat{P}_t - \hat{P}_{t-1}]$$

for each $\ell = 0,\ldots,L$. Hence by using Corollary 3.7 for any $f \in F(\mathbb{R}^d)$, we have

$$\text{Var}[\hat{P}_t - \hat{P}_{t-1}] \leq \left| \mathbb{E}[\hat{P}_t] - \mathbb{E}[\hat{P}_{t-1}] \right|^2 + 2\mathbb{E}[|P - \hat{P}_t|^2] + 2\mathbb{E}[|P - \hat{P}_{t-1}|^2]$$

$$\leq c_0^2 \{ h_\ell + h_{\ell-1} \}^2 + \left\{ \begin{array}{ll}
2C_{EM}(2, \delta) \{ h^\frac{1}{2}_\ell + h^\frac{1}{2}_{\ell-1} \}, & \text{if } f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \\
2C_{EM}(2) \{ h^\frac{1}{2}_\ell + h^\frac{1}{2}_{\ell-1} \}, & \text{if } f \in \{ W^{1,\Phi}(\mathbb{R}^d) \cup W^{1,p}(\mathbb{R}^d) \} \cap L^\infty(\mathbb{R}^d), \\
2C_{EM}(2) \{ h^\frac{1}{2}_\ell + h^\frac{1}{2}_{\ell-1} \}, & \text{if } f \in W^{s,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).
\end{array} \right.$$ 

Hence, for $f \in F(\mathbb{R}^d)$, the computational complexity $C_{MLMC}$ for $\hat{Y}$ is bounded above by

$$C_{MLMC} \leq \left\{ \begin{array}{ll}
c_4 \varepsilon^{-\frac{1-\delta}{2}}, & \text{if } f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \\
c_4 \varepsilon^{-2-\delta}, & \text{if } f \in \{ W^{1,\Phi}(\mathbb{R}^d) \cup W^{1,p}(\mathbb{R}^d) \} \cap L^\infty(\mathbb{R}^d), \\
c_4 \varepsilon^{-2-\frac{\delta}{2}+\frac{2}{p-1}} & \text{if } f \in W^{s,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).
\end{array} \right.$$ 

**Example 3.9.** Let $X_t$ be the Euler–Maruyama scheme with time step $h_t = T/M^\ell$ and $\hat{Y}_t$ be the estimator defined by (43). Suppose the coefficients $b$ and $\sigma$ of SDE [10] are bounded and Lipschitz continuous in space, and $1/2$-Hölder continuous in time, and $\sigma$ is uniformly elliptic. Then for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$ and $\delta \in (0,1)$, the weak rate of convergence $\alpha$ in Theorem 3.3 is $\delta/2$, (see Theorem 1.1 in [34]). Therefore, for $f \in F(\mathbb{R}^d)$, the computational complexity $C_{MLMC}$ for $\hat{Y}$ is bounded above by

$$C_{MLMC} \leq \left\{ \begin{array}{ll}
c_4 \varepsilon^{-1-\frac{\delta}{2}}, & \text{if } f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \\
c_4 \varepsilon^{-2-\delta}, & \text{if } f \in \{ W^{1,\Phi}(\mathbb{R}^d) \cup W^{1,p}(\mathbb{R}^d) \} \cap L^\infty(\mathbb{R}^d), \\
c_4 \varepsilon^{-2-\frac{\delta}{2}+\frac{2}{p-1}} & \text{if } f \in W^{s,p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).
\end{array} \right.$$ 

### 3.3 $L^2$-time regularity of BSDEs

In this subsection, inspired by [19] [33] [39] [43] [88], we apply Avikainen’s estimates proved in Section 2 to numerical analysis for BSDEs with irregular terminal functions $g \in F(\mathbb{R}^d)$. Let $(X,Y,Z)$ be a solution of the following (Markovian) decoupled forward-backward stochastic differential equation (FBSDE):

$$(44) \quad X(t) = x + \int_0^t b(s,X(s))ds + \int_0^t \sigma(s,X(s))dB(s), \quad x \in \mathbb{R}^d, \ t \in [0,T],$$

$$Y(t) = g(X(T)) + \int_t^T f(s,X(s),Y(s),Z(s))ds - \int_t^T Z(s)^\top dB(s), \ t \in [0,T],$$

where $B$ is a $d$-dimensional $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion, and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are measurable functions. For details of the theory and applications of BSDEs, we refer to [10] [19] [20] [33] [39] [43] [71] [88].

We need the following assumptions on the coefficients of FBSDE (44).
Assumption 3.10. (i) The drift coefficient $b$ and the diffusion coefficient $\sigma$ are bounded and twice continuously differentiable in space, and their partial derivatives are uniformly bounded and $\gamma$-Hölder continuous with $\gamma \in (0, 1]$. Moreover, $b$ and $\sigma$ are $1/2$-Hölder continuous in time, and $\sigma$ is uniformly elliptic.

(ii) The driver $f$ is continuous and continuously differentiable in space, and its partial derivative is uniformly bounded. Moreover, $\int_0^T |f(s, 0, 0, 0)| ds < \infty$.

It is known (see, [43, 88]) that the rate of convergence for numerical schemes of (43) is derived by the $L^2$-time regularity $\varepsilon(Z, \pi)$, that is, for a given time mesh $\pi : 0 = t_0 < \cdots < t_n = T$,

$$
\varepsilon(Z, \pi) := \mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z(t) - Z(t_i)|^2 dt \right], \quad Z(t_i) := \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} Z(t) dt \mid \mathcal{F}_t.
$$

(45)

For providing the error estimate of $\varepsilon(Z, \pi)$, we consider the following space:

$$
\mathcal{L}_{2, \alpha} := \left\{ g : \mathbb{R}^d \rightarrow \mathbb{R} ; \; \mathbb{E} \left[ |g(X(T))|^2 \right] + \sup_{0 \leq t < T} \left( \mathbb{E} \left[ |g(X(T)) - \mathbb{E}[g(X(T)) \mid \mathcal{F}_t]|^2 \right] \right)^{\alpha} < \infty \right\}.
$$

Under Assumption 3.10 Gobet and Makhlouf [33] provided an upper bound of the $L^2$-time regularity $\varepsilon(Z, \pi)$ for $g \in \mathcal{L}_{2, \alpha}$ (see, Theorem 3.1 and Theorem 3.2 in [33]). By using these theorems and Avikainen’s estimate proved in Section 2, we can provide the order of the $L^2$-time regularity $\varepsilon(Z, \pi)$ for $g \in F(\mathbb{R}^d)$.

Theorem 3.11. Assumption 3.10 hold. Let $g \in F(\mathbb{R}^d)$ and $(X, Y, Z)$ be a solution of decoupled FBSDE (43), and let $\pi$ be the equidistant time mesh, that is, $\pi = \{t_i = iT/n : i = 0, \ldots, n\}$, and let $\delta \in (0, 1)$. Then, there exist constants $C_\delta > 0$ and $C > 0$ which do not depend on $n$ such that

$$
\varepsilon(Z, \pi) \leq \begin{cases} 
C_\delta n^{-\frac{\delta}{2}}, & \text{if } g \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \\
C_n^{-\frac{\delta}{2}}, & \text{if } g \in \{W^{1,\infty}(\mathbb{R}^d) \cup W^{1, p(\cdot)}(\mathbb{R}^d)\} \cap L^\infty(\mathbb{R}^d), \\
C_n^{-\frac{\delta}{2}}, & \text{if } g \in W^{s, p}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d). 
\end{cases}
$$

Proof. We only prove the statement for $g \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The proofs for the other function spaces are similar. We first note that since $g$ is bounded, $\mathbb{E}[|g(X(T))|^2] < \infty$. From Theorem 3.2 (a) in [33], it suffices to prove that for any $\delta \in (0, 1)$,

$$
\sup_{0 \leq t < T} \frac{\mathbb{E} \left[ |g(X(T)) - \mathbb{E}[g(X(T)) \mid \mathcal{F}_t]|^2 \right]}{(T - t)^{\frac{\delta}{2}}} < \infty.
$$

Under Assumption 3.10 (i), the stochastic process $X$ admits a transition probability density $p(s, x; t, \cdot)$ with respect to Lebesgue measure, and it has the Gaussian upper bound, that is, there exist $C_+ > 0$ and $c_+ > 0$ such that for any $x, y \in \mathbb{R}^d$ and $0 \leq s < t \leq T$,

$$
p(s, x; t, y) \leq C_+ g_{c_+(t-s)}(x, y)
$$

29
for some constant $C$. The squared error of their schemes has the sum of the mean squared error. Schemes based on a dynamic programming equation are proposed, and the upper bound of the error is upper bound for machine learning have been studied (e.g., [19, 39, 43]). In particular, in [43], several backward schemes based on a dynamic programming equation are proposed, and the upper bound of their schemes has been derived.

Remark 3.12. Recently, numerical schemes for high-dimensional forward–backward SDEs based on machine learning have been studied (e.g., Theorem 6.4.5 in [22]). Therefore, by using the Markov property of $X$, Jensen’s inequality and change of variables, for any $0 < t < T$, we have

$$\mathbb{E} \left[ |g(X(T)) - \mathbb{E}[g(X(T)) | \mathcal{F}_t]|^2 \right] = \mathbb{E} \left[ \int_{\mathbb{R}^d} \{g(X(T)) - g(y)\} p(t, X(t); T, y) dy \right]^2$$

$$\leq \mathbb{E} \left[ \int_{\mathbb{R}^d} (g(X(T)) - g(y))^2 p(t, X(t); T, y) dy \right]$$

$$\leq C_t \int_{\mathbb{R}^d} \mathbb{E} \left[ |g(X(T)) - g(y + X(t))|^2 \right] g_{c_+}^{(T-t)}(0, y) dy.$$

Note that $X(T)$ and $y + X(t)$ admit density functions with respect to Lebesgue measure, and the density function of $X(T)$ has the Gaussian upper bound (see, (41)). Thus by using Theorem 2.11 (i), we obtain

$$\mathbb{E} \left[ |g(X(T)) - \mathbb{E}[g(X(T)) | \mathcal{F}_t]|^2 \right]$$

$$\leq C_t C_{BV}(p, 2, \infty) \int_{\mathbb{R}^d} \mathbb{E} \left[ |X(T) - (y + X(t))|^p \right] \frac{1}{\sqrt{t}} g_{c_+}^{(T-t)}(0, y) dy$$

$$\leq 2^{p+1} C_t C_{BV}(p, 2, \infty) \left( \mathbb{E} \left[ |X(T) - X(t)|^p \right] \frac{1}{\sqrt{t}} + \int_{\mathbb{R}^d} |y| \frac{1}{\sqrt{t}} g_{c_+}^{(T-t)}(0, y) dy \right)$$

$$\leq C(p)|T - t|^{\frac{p}{2}}$$

for some constant $C(p) > 0$. Hence by choosing $p := \frac{4}{3} \delta$, we conclude the proof. \qed

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