Levinson’s theorem and reflectionless one-dimensional potentials

D. E. Zambrano*
Departamento de Física
Universidad Nacional de Colombia, Sede Bogotá
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We studied two possible approaches to one-dimensional Levinson’s theorem for Schödinger equation, by using the reflectionless potential problem. In the first one, we restrict the 3-dimensional theorem in a direct way. In the other one, we use the theorem proposed by Dong, Ma and Klauss [1]. We find failures in both of two approaches. In order to see this, we explicitly evaluate the phase shift using Schrödinger equation. A solution of equation is obtain by means a procedure proposed by Jaffe [2].

Keywords: Levinson’s theorem, reflectionless potentials, bounded and semi-bounded states, phase shift.

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I. INTRODUCTION

If we consider a potential with the form[2]:

\[ v_\ell = -\ell(\ell + 1)\text{sech}^2 x, \]

where \( \ell \geq 0 \) is an integer, transmission probability across this potential is 1. These kind of potential are known as reflectionless potential. Schrödinger’s equation with this kind of potentials has an exact solution (cf. [2]). In this paper we compare the result of Levinson’s theorem with the solution of Schrödinger equation. In section II we find the solution of this system and the phase shift for semi-bounded states, as in [2]. In section III restrict the 3-dimensional theorem and apply its result to the system. We find that yields a wrong result. In section IV we test the theorem proposed by Ma, Dong and Klauss[1], finding the same inconsistence. Finally we discuss the results in section V.

*Electronic address: dezambranor@unal.edu.co
II. PHASE SHIFT FOR ZERO MOMENTUM

Following the method proposed by Jaffe [2], we find the phase shift for bounded state with zero momentum, so-called *semi-bounded* states. Beginning from Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - V_0 \text{sech}^2(br) \right] \Psi(r) = E \Psi(r),$$

we write this equation for dimensionless parameters by means

$$x = br, \quad v_0 = \frac{2mV_0b^2}{\hbar^2}, \quad k^2 = \frac{2mb^2E}{\hbar^2}.$$  \hspace{1cm} (3)

taking $v_0 = \ell(\ell + 1)$ where $l \geq 0$ is an integer, we obtain a potential shape as in Eq. (1),

$$\hat{H}_\ell \Psi(x) = \left[ \hat{p}^2 - \ell(\ell + 1) \text{sech}^2x \right] \Psi = k^2 \Psi(x).$$  \hspace{1cm} (4)

So we have

- For $k^2 > 0$ a scattering state,
- For $k^2 \geq 0$ and $||\Psi|| < \infty$ a bounded state,

where $\Psi$ is the wave-function. In accordance with [2], we define the operators

$$\hat{a}_\ell = \hat{p} - i\ell \tanh x, \quad \hat{a}_\ell^\dagger = \hat{p} + i\ell \tanh x.$$  \hspace{1cm} (5)

Since $[\hat{x}, \hat{p}] = i$, is clear that

$$\hat{A}_\ell \equiv \hat{a}_\ell^\dagger \hat{a}_\ell = \hat{H}_\ell + \hat{1} \ell^2, \quad \hat{B}_\ell \equiv \hat{a}_\ell \hat{a}_\ell^\dagger = \hat{H}_{\ell-1} + \hat{1} \ell^2.$$  \hspace{1cm} (6)

Now, we define the state $|0\rangle_\ell$ by $\hat{a}_\ell |0\rangle_\ell = |\circ\rangle$, where $|\circ\rangle$ is null ket. Denoting $\Psi_\ell^{(0)}(x) = \langle x |0\rangle_\ell$, we obtain

$$\left[ -i \frac{d}{dx} - i\ell \tanh x \right] \Psi_\ell^{(0)}(x) = 0.$$  \hspace{1cm} (7)

The solution for this equation is

$$\Psi_\ell^{(0)}(x) = N_\ell \text{sech}^\ell(x), \quad \ell \geq 0,$$  \hspace{1cm} (8)

where $N_\ell$ is a normalisation constant. Due to linearity of the operator $\hat{a}_\ell^\dagger$, $\hat{A}_\ell |0\rangle_\ell = 0 |0\rangle_\ell$, then $|0\rangle_\ell$ is an eigenstate of $\hat{A}_\ell$ associated to the zero eigenvalue, and is the ground state because does not have any node. In the other hand, if $|\Psi\rangle$ is an eigenstate of $\hat{A}_\ell$ associated to the eigenvalue $\psi \neq 0$ we have

$$\hat{a}_\ell \left( \hat{A}_\ell |\Psi\rangle \right) = \left( \hat{a}_\ell \hat{a}_\ell^\dagger \right) \hat{a}_\ell |\Psi\rangle = \hat{B}_\ell \left( \hat{a}_\ell |\Psi\rangle \right) = \psi \left( \hat{a}_\ell |\Psi\rangle \right),$$  \hspace{1cm} (9)
then if $\psi \neq 0$, $|\Psi'\rangle = \hat{a}_\ell |\Psi\rangle$ is an eigenvalue of $\hat{\mathcal{B}}_\ell$ associated to the same eigenvalue $\psi$. Explicitly
\begin{align}
\hat{\mathcal{B}}_\ell |\Psi\rangle &= \hat{H}_\ell |\Psi\rangle + \ell^2 |\Psi\rangle = \psi |\Psi\rangle, \quad (10a) \\
\hat{\mathcal{B}}_\ell |\Psi'\rangle &= \hat{H}_{\ell-1} |\Psi'\rangle + \ell^2 |\Psi'\rangle = \psi |\Psi'\rangle, \quad \psi \neq 0. \quad (10b)
\end{align}

Therefore
\begin{align}
\hat{H}_\ell |\Psi\rangle &= \left(\psi - \ell^2\right) |\Psi\rangle, \quad (11a) \\
\hat{H}_{\ell-1} |\Psi'\rangle &= \left(\psi - \ell^2\right) |\Psi'\rangle, \quad \psi \neq 0 \quad (11b)
\end{align}

thus $|\Psi\rangle$ and $\hat{a}_\ell^\dagger |\Psi\rangle$ form the set of eigenstates of $\hat{H}_\ell \neq \hat{H}_{\ell-1}$, respectively, and these Hamiltonians have the same spectrum, except by the eigenvalue associated to the ground state of $\hat{H}_\ell$. This is a very important result obtained by Jaffe [2].

For $\ell = 0$, we have that $\hat{H}_0 = \hat{p}^2$, i.e. free particle case, a well-known system. Energy eigenstates are
\begin{equation}
\Psi_0(k, x) \equiv \langle x | k \rangle_0 = \exp(ikx), \quad (12)
\end{equation}

where subindex zero indicates $\ell = 0$. Energy eigenvalues are $E(k) = k^2$, since we are working with dimensionless quantities.

For $\ell = 1$, we construct the eigenfunctions of $\hat{H}_1$ for a given $k^2$. We define $|k\rangle_1 = \hat{a}_1^\dagger |k\rangle_0$. If we apply $\hat{H}_1$ to $|k\rangle_1$ we obtain
\begin{equation}
\hat{H}_1 |k\rangle_1 = (\hat{\mathcal{B}}_1 - 1)\hat{a}_1^\dagger |k\rangle_0 = \hat{a}_1^\dagger \mathcal{B}_1 |k\rangle_0 - \hat{a}_1^\dagger |k\rangle_0 = (k^2 + 1)\hat{a}_1^\dagger |k\rangle_0 - \hat{a}_1^\dagger |k\rangle_0 = k^2 \hat{a}_1^\dagger |k\rangle_0 = k^2 |k\rangle_1. \quad (13)
\end{equation}

therefore $|k\rangle_1$ is an eigenstate of $\hat{H}_1$ associated to the eigenvalue $k^2$, as we hope, and so the wave function is
\begin{equation}
\Psi_1(k, x) = \langle x | k \rangle_1 = \langle x | \hat{a}_1^\dagger |k\rangle_0 = (k + i \tanh x) \exp(ikx). \quad (14)
\end{equation}

In the region before to potential well, i.e. for $x \to -\infty$, we may express the wave-function as an asymptotic incident function $I_1(k)e^{ikx}$ and a reflected wave function $R_1(k)e^{-ikx}$. In the same manner, in the region after to the potential well, i.e. $x \to \infty$, we may write the wave-function as $T_1(k)e^{ikx}$, the transmitted part,
\begin{equation}
\lim_{x \to -\infty} \Psi_1(k, x) = I_1(k)e^{ikx} + R_1(k)e^{-ikx}, \quad \lim_{x \to \infty} \Psi_1(k, x) = T_1(k)e^{ikx} \quad (15)
\end{equation}
From Eq. (14) we can see that
\[ R_1(k) = 0, \quad T_1(k) = k + i, \quad I_1(k) = k - i. \] (16)

At this moment we can observe several aspects. First, since the reflection coefficient \( R(k) \) is null, there not exist reflected wave function, i.e. reflectionless. Second, the ratio between incident and transmitted waves is
\[
\frac{T(k)}{I(k)} = \frac{k + i}{k - i} = \exp \left( 2i \arctan \left( \frac{1}{k} \right) \right).
\] (17)

This indicates that the interaction with the potential only yields a phase shift, therefore the transmission probability is 1. Phase shift is defined as [3]
\[
\delta_\ell = \frac{1}{2} \arg \left( \frac{T(k)}{I(k)} \right),
\] (18)
so, for \( \ell = 1 \) we have,
\[
\delta_1 = \arctan \left( \frac{1}{k} \right). \] (19)

Repeating the same procedure \( \ell \) times, eigenfunctions can be constructed by means
\[
\Psi_\ell(k, x) = \hat{a}_j^\dagger \cdots \hat{a}_1^\dagger \Psi_0(k, x) = \prod_{j=1}^\ell (k + ij \tanh x) \exp(ikx),
\] (20)

Last expression allows us to obtain
\[
R_\ell(k) = 0, \quad T_\ell(k) = \prod_{j=1}^\ell (k + ij), \quad I_\ell(k) = \prod_{j=1}^\ell (k - ij).
\] (21)

Hence, for any \( \ell \) there not exist reflected wave and
\[
\frac{T_\ell(k)}{I_\ell(k)} = \prod_{j=1}^\ell \frac{(k + ij)}{(k - ij)} = \exp \left( 2i \sum_{j=1}^\ell \arctan \left( \frac{j}{k} \right) \right)
\] (22)
indicating only a phase shift of \( \delta_\ell(k) \) given by
\[
\delta_\ell(k) = \sum_{j=1}^\ell \arctan \left( \frac{j}{k} \right) \] (23)

Taking the limit \( k \to 0 \), i.e. for zero momentum, we have
\[
\delta_\ell(0) = \lim_{k \to 0} \sum_{j=1}^\ell \arctan \left( \frac{j}{k} \right) = \frac{\pi}{2} \ell
\] (24)
III. LEVINSON’S THEOREM

Levinson’s theorem [3] establishes that the phase shift $\delta_\ell(0)$ for states of zero momentum $y$ and angular momentum $\ell$ for 3-dimensional Schrödinger equation with a spherical symmetric potential is given by

$$\delta_\ell(0) = \begin{cases} n_\ell \pi + \pi/2 & \text{critical case} \\ n_\ell \pi & \text{no critical case} \end{cases} \tag{25}$$

where $n_\ell$ is the number of bounded states with angular momentum $\ell$. Critical case means existence of semi-bounded state (also energy zero or half-bound state), i.e. $k^2 = 0$, but wave function is finite and not necessarily square integrable. Potential $V(r)$ must fulfills the asymptotic condition:

$$r^2|V(r)| \to 0, \quad r \to \infty. \tag{26}$$

We would restrict this theorem to one-dimensional case, supposing an even potential (symmetric in one dimension) which fulfills:

$$x^2|V(x)| \to 0, \quad x \to \infty. \tag{27}$$

For the potential (1), the conditions of restricted theorem are satisfied, since

$$\lim_{x \to \infty} x^2 v_\ell(x^2) = -4\ell(\ell + 1) \lim_{x \to \infty} \frac{x^2}{(e^x + e^{-x})^2} = 0 \tag{28}$$

In conclusion the restricted theorem is applicable to reflectionless potentials. For $\ell = 0$, in addition to the plane waves, i.e. the eigenfunctions for free particle, it must exist a the ground state $|0\rangle_0$ determined by

$$d\Psi_0^{(0)}(0, x) dx = 0 \tag{29}$$

hence $\Psi_0(0, x) = N_0$, is a constant and is associated to eigenvalues $E_0^{(0)} = 0$, then it is a semi-bounded state and since there not exist bounded states[1], this state represents the critical case for (25), then the restricted theorem states

$$\delta_0 = \frac{\pi}{2}. \tag{30}$$

This result contradicts (24). Therefore is not valid restrict the theorem (25) to the one-dimensional case.
IV. LEVINSON’S THEOREM FOR WAVE FUNCTION WITH PARITY

In the last section we show that restrict the 3-dimensional theorem is not valid. In 3-dimensional case we have an additional ingredient: the angular momentum, which is lost if we restrict to a one-dimensional case in a direct way. Ma, Dong and Klauss[1] found a restriction, in principle valid, for the Levinson theorem. This restriction affirms for the one-dimensional case:

For one-dimensional Schrödinger equation with an even potential $V(x)$ which fulfills (27) the phase shift for a state with definite parity at zero momentum is

$$
\begin{align*}
\delta_e(0) &= n_e \pi + \pi/2, \\
\delta_o(0) &= n_o \pi \\
\delta_e(0) &= n_e \pi \\
\delta_o(0) &= n_o \pi + \pi/2,
\end{align*}
$$

(31)

where $n_e$ and $n_o$ are the number of bounded states for the even and odd wave-function, respectively and $\delta_e(0)$ and $\delta_o(0)$ are the phase shift at zero momentum for even and odd case. Critical case occurs if there exist a semi-bounded state.

For $\ell = 0$, the wave function at zero momentum is $\Psi_0^{(0)}(x)$, which is an even function. In the other hand, we have a semi-bounded state and not bounded ones, then (31) affirms $\delta_0 = 0$ which is true.

Now, we analyse the $\ell = 1$ case. The spectrum of the Hamiltonian is the same than $\ell = 0$ except for an eigenvalue associated to the state $|0\rangle_1$, which is evaluated using (11a) for $\psi = 0$,

$$
\hat{H}_1 |0\rangle_1 = -|0\rangle_1.
$$

(32)

then $E_1^{(0)} = -1$, a bounded state. In order to obtain the wave function $\Psi_1^{(0)}(x)$ we replace $\ell = 1$ in Eq. (8),

$$
\Psi_1^{(0)}(x) = \langle x|0\rangle_1 = N_1 \text{sech}x.
$$

(33)

This function is even. Replacing $\Psi_0^{(0)}$ in (20) we obtain the zero energy (semi-bounded) wave function

$$
\Psi_1^{0}(x) = iN_0 \text{tanh} x
$$

(34)

whose parity is odd. There are not any odd bounded states, thus the phase shift determined by (31) is $\delta_1(0) = \pi/2$, yielding the same phase shift than (24) for $\ell = 1$. Note that the bounded
state \( |0\rangle_1 \) does not contribute to the phase shift, since have a distinct parity than the half-bounded state.

In the case \( \ell = 2 \), we have a bounded state for \( E = -1 \), and its corresponding eigenfunction is

\[
\Psi_2^{(1)} = \langle x | \hat{a}_2^\dagger | 0 \rangle_1 \propto (k + 2i\tanh x) \text{sech} x \propto \frac{\tanh x}{\cosh x},
\]

which is an odd function. Additionally, we have a bounded state for \( E_2^{(0)} = -4 \) given by

\[
\Psi_2^{(0)} = N_2 \text{sech}^2 x
\]

which is even. Also we obtain an even semi-bounded state for \( k^2 = 0 \) given by

\[
\Psi_2^0(x) = -N_0 \tanh^2 x
\]

Hence, there are two bounded states, one odd and other even, and an even semi-bounded state associated to \( k^2 = -1 \), then (31) establishes that \( \delta_{\ell}(0) \) is a semi-integer times \( \pi \), contradicting the phase shift given by the Schrödinger equation (see Eq. (24)), which value is \( \pi \).

V. DISCUSSION

Although the potential (1) fullfils the conditions of the restricted theorem, in his two versions, the direct restriction fails from \( \ell = 0 \), instead the approach of Ma, Dong and Klauss fails from \( \ell = 2 \).

Is clear that Schrödinger equation provides the actual result for phase shift, it must exist a hide or ignored condition in the demonstration, in the case of [1]. In addition, is clear that in one-dimensional case the angular momentum is not considered, since have not one-dimensional sense. The idea proposed in [1] is well method to include the lack of angular behaviour, but is not enough, some class of information is lost in the process.

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