Order isomorphisms on convex functions in windows

S. Artstein-Avidan, D. Florentin and V. Milman

October 14, 2015

Abstract

In this paper we give a characterization of all order isomorphisms on some classes of convex functions. We deal with the class $C_{\text{cvx}}(K)$ consisting of lower-semi-continuous convex functions defined on a convex set $K$, and its subclass $C_{\text{cvx}_0}(K)$ of non negative functions attaining the value zero at the origin. We show that any order isomorphism on these classes must be induced by a point map on the epi-graphs of the functions, and determine the exact form of this map. To this end we study convexity preserving maps on subsets of $\mathbb{R}^n$, and also in this area we have some new interpretations, and proofs.

1 Introduction

In recent years, a big research project initiated by the first and third named authors has been carried out, in which a characterization of various transforms by their most simple and basic properties has been found. Among these are the Fourier transform (see [2] and [1]), the Legendre transform (see [5]), polarity for convex sets (see [10], [6], [16]), the derivative (see [4]) and various other transforms. This paper is part of this effort.

One example for such characterization is the understanding of bijective order isomorphisms for certain partially ordered sets (see Section 3 for definitions and details). In this category one includes the Legendre transform, which is, up to linear terms, the unique bijective order reversing map on the set of all convex (lower-semi-continuous) functions on $\mathbb{R}^n$, with respect to the pointwise order. In this paper we discuss analogous results where the convex functions are defined on a “window”, that is on a convex set $K \subseteq \mathbb{R}^n$, and several other variants as well.

One main tool in the proof of such results is the fundamental theorem of affine geometry, which states that an injective mapping on $\mathbb{R}^n$ which sends lines to lines, and whose image is not contained in a line, must be affine linear. When working with “windows”, one immediately encounters a need for a similar theorem for maps defined on a subset of $\mathbb{R}^n$. Such theorems exist in the literature, and a mapping which maps intervals to intervals on a subset of $\mathbb{R}^n$ must be of a very specific form, which we call here “fractional linear”, discussed in Section 2. A remark about this name is in need: the mappings are of the form: $\frac{Ax + b}{(c,x) + d}$ for $A \in L_n(\mathbb{R}), b,c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, with some extra restriction. In the literature, the name “fractional linear maps” sometimes refers to Möbius transformations, which is not the case here (note that a Möbius transformation on a subset of $\mathbb{C} = \mathbb{R}^2$ does not preserve intervals, but these notions are indeed connected - see Example 2.7). One could name them “permissable projective transformation” but we prefer to think about them exclusively in $\mathbb{R}^n$ and not on the projective space. Another option was to call them “convexity preserving maps”, which describes their action rather than their functional form, but this hides the fact that they are of a very simple form.

*The research was supported in part by Israel Science Foundation: first and second named authors by grant No. 865/07, third named author by grant No. 491/04. The authors were also supported in part by BSF grant No. 2006079.
The classification of interval preserving transforms of a convex subset of $\mathbb{R}^n$ is known (see [15]). However, since this is an essential part for our study of the order isomorphisms on functions in windows, we dedicate the whole of Section 2 to this topic. We also provide some new insights and results, and give a seemingly new geometric proof of the main fact which is that such maps are fractional linear. Some parts of this section are elementary and may be known to the reader, but we include them as they too serve as intuition for the way these maps behave.

In Sections 3-5 we turn to the main topic of this paper, namely characterization of order preserving (and reversing) isomorphisms on classes of convex functions defined on windows. Let $T \subseteq K$ be two closed convex sets. The class of all lower-semi-continuous convex functions $\{f : K \to \mathbb{R} \cup \{\infty\}\}$ is denoted $Cvx(K)$, and its subclass of non negative functions satisfying $f(T) = 0$ is denoted $Cvx_T(K)$. In Section 3 we give background on general order isomorphisms. In Sections 4, 5 we deal with characterization of such transforms on $Cvx(K)$ and $Cvx_0(K)$ respectively. In both cases, the proof is based on finding a subset of convex functions which are extremal, in some sense. The extremal elements are relatively simple functions, which can be described by a point in $\mathbb{R}^{n+1}$. We show that an order isomorphism is determined by its action on the extremal family, and that its restriction to this family must be a bijection. Therefore the transform induces a bijective point map on a subset of $\mathbb{R}^{n+1}$. By applying our uniqueness theorem, we show that this point map must be fractional linear. We then discuss some generalizations of these theorems to other classes of non-negative convex functions.

2 Interval Preserving Maps

We start this section with a simple but curious fact which is stated again and proved as Theorem 2.27 below. This fact demonstrates the idea that fractional linear maps should be a key ingredient in convexity theory. Consider a convex body $K$ (actually any closed convex set will do) which includes the origin and is included in the half-space $H_1 = \{x_1 < 1\} \subseteq \mathbb{R}^n$. One may take its polar, defined by

$$K^\circ = \{y : \sup_{x \in K} \langle x, y \rangle \leq 1\}.$$  

Since polarity reverses the partial order of inclusion on closed convex sets including the origin, $K^\circ$ includes the set $[0, e_1]$ which is the polar of $H_1$. Translate it by $-e_1$ (so now it includes $[-e_1, 0]$, and in particular includes the origin) and then take its polar again. In other words, we constructed a map mapping certain convex sets (which include 0 and are included in the half-space $H_1$) to convex sets, given by

$$F(K) = (K^\circ - e_1)^\circ.$$  

Clearly this mapping is order preserving.

While polarity is a “global” operation, it turns out that this mapping is actually induced by a point map on $H_1$, $\tilde{F} : H_1 \to \mathbb{R}^n$, which preserves intervals, and can be explicitly written as $\tilde{F}(x) = \frac{x}{x_1}$. (This is a simple calculation, and for completeness we provide it in the proof of Theorem 2.27 below.)

This map is a special case of the so called “fractional linear maps” which are the main topic of this section.

2.1 Definition and simple observations

**Definition 2.1.** Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^n$ is called an interval preserving map, if $f$ maps every interval $[x, y] \subseteq D$ to an interval $[z, w]$.

**Lemma 2.2.** Let $D \subseteq \mathbb{R}^n$, and let $f : D \to \mathbb{R}^n$ be an injective interval preserving map. Then for every $x, y \in D$ with $[x, y] \subseteq D$ we have that $f([x, y]) = [f(x), f(y)]$.  

The interval where the last inclusion is due to convexity of $f$.

Proof. Indeed, assume that $f([x,y]) = [w,z]$, and that, say, $f(y) \in (w,z)$. Pick a point $b \in (w,f(y))$, and a point $b' \in (f(y),z)$. Then for some $a,a' \in (x,y)$, $b = f(a)$ and $b' = f(a')$. Consider $f([a,a'])$. It is an interval that includes the points $b$ and $b'$, and therefore it includes $f(y)$, whereas $y \notin [a,a']$ - in contradiction to the injectivity of $f$.

Lemma 2.3. Let $D \subseteq \mathbb{R}^n$, and let $f : D \to \mathbb{R}^n$ be an injective interval preserving map. Then the inverse $f^{-1} : f(D) \to D$ is interval preserving.

Proof. Let $I = [f(a),f(b)]$ be an interval in the image, and $f(c) \in I$. From Lemma 2.2 $f([a,b]) = I$, so by injectivity, $c \in [a,b]$.

Remark 2.4. Clearly, an interval preserving map $f$ must be convexity preserving, i.e. $f$ must map every convex set $K$ to a convex set $f(K)$. We will actually need the opposite direction, given in the following lemma.

Lemma 2.5. Let $D \subseteq \mathbb{R}^n$ be a convex set, and let $f : D \to \mathbb{R}^n$ be an injective interval preserving map. Then the inverse image of a convex set in $f(D)$ is convex.

Proof. Let $K \subseteq f(D)$ be a convex set, and let $x,y \in f^{-1}(K)$. We wish to show that $[x,y] \subseteq f^{-1}(K)$. The interval $[x,y]$ is contained in the convex domain $D$, and by Lemma 2.2 we know that

$$f([x,y]) = [f(x),f(y)] \subseteq K,$$

where the last inclusion is due to convexity of $K$. This implies $[x,y] \subseteq f^{-1}(K)$.

Lemma 2.6. Let $D \subseteq \mathbb{R}^n$ be an open domain and $f : D \to \mathbb{R}^n$ an injective interval preserving map, then $f$ is continuous.

Proof. Let us prove that $f$ is continuous at a point $x \in D$. We may assume that $D$ is convex (restrict $f$ to an open convex neighborhood of $x$). Let $y = f(x) \in f(D)$ and $B_y$ an open ball containing $y$. If $x \in int(f^{-1}(B_y))$ we are finished (we have a neighborhood of $x$ that is mapped into $B_y$). We claim this must be the case. Indeed, assume otherwise, then $x$ is on the boundary of the set $f^{-1}(B_y)$, which by Lemma 2.5, is convex. Let $x_{out} \in D$ such that $[x,x_{out}] \cap f^{-1}(B_y) = \{x\}$. Then $f([x,x_{out}])$ is an interval $I \subseteq f(D)$ such that $I \cap B_y = \{y\}$, but no such interval exists, since $B_y$ is open.

2.2 Fractional linear maps

Clearly, linear maps are interval preserving. It turns out that when the domain of the map is contained in a half-space of $\mathbb{R}^n$, there is a larger family of (injective) interval preserving maps. Indeed, fix a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$, let $A \in L_n(\mathbb{R})$ be a linear map, $b,c \in \mathbb{R}^n$ two vectors and $d \in \mathbb{R}$ some constant, then the map

$$v \mapsto \frac{1}{\langle c,v \rangle + d} (Av + b)$$

is defined on the open half space $\langle c,v \rangle < -d$ and is interval preserving. One can check interval preservation directly, or deduce it from the projective description in Section 2.2.1, as well as an injectivity argument. A necessary and sufficient condition for this map to be injective is that the associated matrix $\hat{A}$ (defined below) is invertible. The matrix $A$ itself need not be invertible, for an example see Remark 2.9. We call these maps “fractional linear maps”, see the introduction for a remark about this name.
2.2.1 Projective description

Consider the projective space $\mathbb{RP}^n = P(\mathbb{R}^{n+1})$, the set of 1-dimensional subspaces of $\mathbb{R}^{n+1}$. It is easily seen that

$$\mathbb{RP}^n = \mathbb{R}^n \cup \mathbb{RP}^{n-1},$$

where one can geometrically think of $\mathbb{R}^n$ as $\mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1}$, so that each line which is not on the hyperplane $e_{n+1}^\perp \approx \mathbb{R}^n$, intersects the shifted copy of $\mathbb{R}^n$ at exactly one point. The lines which lie on $e_{n+1}^\perp$ are thus lines in an $n$-dimensional subspace, and are identified with $\mathbb{RP}^{n-1}$.

Any linear transformation on $\mathbb{R}^{n+1}$ induces a transformation on $\mathbb{RP}^n$, mapping lines to lines. Thus it induces in particular a map $F : \mathbb{R}^n \to \mathbb{R}^n \cup \mathbb{RP}^{n-1}$. It is easily checked that the part mapped to $\mathbb{RP}^{n-1}$ is either empty - in which case the induced transformation on $\mathbb{R}^n$ is linear, or an affine hyperplane $H$ - in which case the induced transformation $F : \mathbb{R}^n \setminus H \to \mathbb{R}^n$ is fractional linear.

Indeed, if the matrix associated with the original transformation in $L(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ is $\hat{A} \in GL_{n+1}$, then the hyperplane in $\mathbb{R}^n$ mapped to $e_{n+1}^\perp$ is simply $\{x \in \mathbb{R}^n : (\hat{A}(x,1))_{n+1} = 0\}$. If $\hat{A}$ is given by

$$\hat{A} = \begin{pmatrix} A & b \\ c^T & d \end{pmatrix}$$

with $A \in M_{n \times n}, b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, then the set $\{x : (c,x) + d \neq 0\} \subset \mathbb{R}^n$ is exactly the pre image of $\mathbb{RP}^{n-1} \setminus \{x : \langle c,x \rangle + d = 0\}$ under $F$. It is an $n-1$ dimensional subspace if $c \neq 0$, and empty if $c = 0$ ($\hat{A} \in GL_{n+1}$ implies $d \neq 0$ in that case).

Pick any vector $x \in \mathbb{R}^n$ which is not in this hyperplane, then it is mapped to $y = \hat{A}(x,1) \in \mathbb{R}^{n+1}$ which has a non-vanishing $(n+1)^{th}$ coordinate, $y_{n+1} = \langle c,x \rangle + d$. Normalize $y \to y/y_{n+1}$, so that the last coordinate is 1, and consider only the first $n$ coordinates of this vector (we denote the projection to the first $n$ coordinates by $P_n$). Thus, under the above map, $x$ is mapped to

$$F(x) = P_n \left( \hat{A}(x,1)/(\hat{A}(x,1))_{n+1} \right) = \frac{Ax + b}{\langle c,x \rangle + d}.$$  -----------

(1)

Denote the domain of the map by $D \subseteq \mathbb{R}^n$. Clearly, if $D = \mathbb{R}^n$, the condition $\phi(x) = (\hat{A}(x,1))_{n+1} \neq 0$ for all $x \in D$ implies that this (affine linear) function $\phi(x)$ is constant, which means that our induced map is affine linear. However, when $D$ is contained in a half space (for example, if $D$ is a convex set strictly contained in $\mathbb{R}^n$), there are many choices of $\hat{A}$ which satisfy this condition. Indeed, $(\hat{A}(x,1))_{n+1} = \langle c,x \rangle + d$ for some $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ ($(c,d)$ is the $(n+1)^{th}$ row of $\hat{A}$), and the condition is that

$$\forall x \in D \quad \langle c,x \rangle \neq -d,$$

which can be satisfied for appropriate chosen $c$ and $d$; for every direction $c$ in which $D$ is bounded, there exists a critical $d$ such that from this value onwards the condition is satisfied (the critical $d$ may or may not be chosen, depending on the boundary of $D$). Other ways of describing these maps will be given in Section 2.4.

Notation. For future reference we denote the map $F$ associated with a matrix $\hat{A}$ by $F_{\hat{A}}$ and the matrix $\hat{A}$ associated with a map $F$ by $A_F$. Note that $\hat{A}$ is defined uniquely up to a multiplicative constant. We say that $A_F$ induces $F$, and may also write $F \sim A_F$.

Example 2.7. We are, in fact, very much familiar with one class of projective transformations: Möbius transformations of the extended complex plane. These are just projective transformations of the complex projective line $\mathbb{P}^1(\mathbb{C})$ to itself. We describe points in $\mathbb{P}^1(\mathbb{C})$ by homogeneous coordinates $[z_0, z_1]$, and then a projective transformation $\tau$ is given by $\tau([z_0, z_1]) = [az_0 + bz_1, cz_0 + dz_1]$ where $ad - bc \neq 0$. This corresponds to the invertible linear transformation

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
It is convenient to write $P^1(\mathbb{C}) = \mathbb{C} \cup \{+\infty\}$ where the point $\{+\infty\}$ is now the 1-dimensional space $z_1 = 0$. Then if $z_1 \neq 0$, $[z_0, z_1] = [z, 1]$ and $\tau([z, 1]) = [az + b, cz + d]$ and if $cz + d \neq 0$ we can write $\tau([z, 1]) = [(az + b)/(cz + d), 1]$ which is the usual form of a Möbius transformation, i.e.

$$z \mapsto \frac{az + b}{cz + d}.$$ 

The advantage of projective geometry is that the point $1 = [1, 0]$ plays no special role. If $cz + d = 0$ we can still write $\tau([z, 1]) = [az + b, cz + d] = [az + b, 0] = [1, 0]$ and if $z = 1$ (i.e. $[z_0, z_1] = [1, 0]$) then we have $\tau([1, 0]) = [a, c]$. In this note, however, we work over $\mathbb{R}$ and these transformations when considered as acting over $\mathbb{R}^2$ do not preserve intervals.

**Example 2.8.** If we view the real projective plane $P^2(\mathbb{R})$ in the same way, we get some less familiar transformations. Write $P^2(\mathbb{R}) = \mathbb{R}^2 \cup P^1(\mathbb{R})$ where the projective line at infinity is $z = 0$. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ can then be written as the matrix

$$T = \begin{pmatrix} a_{11} & a_{12} & b_1 \\
 a_{21} & a_{22} & b_2 \\
 c_1 & c_2 & d \end{pmatrix},$$

and its action on $[x, y, 1]$ can be expressed, with $v = (x, y) \in \mathbb{R}^2$, as

$$v \to \frac{1}{\langle c, v \rangle + d} (Av + b)$$

where $A$ is the matrix $\{a_{ij}\}$ and $b, c$ are the vectors $(b_1, b_2), (c_1, c_2)$. Each such transformation can be considered as a composition of an invertible linear transformation, a translation and an inversion $v \to v/\langle c, v \rangle + d)$. Clearly it is easier here to consider projective transformations defined by $3 \times 3$ matrices.

**Remark 2.9.** Consider the matrix

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 \\
 0 & 0 & 1 \\
 0 & 1 & 0 \end{pmatrix}.$$ 

It gives rise to the transformation

$$(x, y) \mapsto \begin{pmatrix} x \\
 y \\
 1 \end{pmatrix},$$

which is injective (where it is defined). The upper $2 \times 2$ block (or $n \times n$, in the general case) is not invertible, though. We will get back to this transformation in later sections.

### 2.2.2 Basic properties

1. **Preservation of intervals.** It is very easy to check that the map $F$ defined above in (1) preserves intervals. Indeed, an interval in $\mathbb{R}^n$ is a subset of a line, which corresponds to a two dimensional plane in $\mathbb{R}^{n+1}$. The latter is mapped by $A_F$ to a two dimensional plane, and after the radial projection to the level $x_{n+1} = 1$ we again get a line.

2. **Maximal domain.** A non affine fractional linear map $F$ can be extended to a half space. The only restriction is that for $x \in D$ one has $\langle c, x \rangle \neq -d$, that is, $D$ cannot intersect some given affine hyperplane $H$. Since we are interested in a convex domain, we must choose one side, which means the domain can be extended to a half space. It is not immediately clear why it cannot be extended further. To see why it cannot be extended further **while preserving intervals**, consider a point $x_0 \in H$. We shall see that there is no way to define $F$ on $x_0$. Indeed, take two rays emanating from $x_0$ into the domain of $F$, say $H^+$ (and not on $H$); $\{x_0 + \lambda y : \lambda > 0\}, \{x_0 + \lambda y' : \lambda > 0\}$. The fact
that the rays are not on \( H \) means that \( \langle c, y \rangle \neq 0 \), likewise for \( y' \). Moreover, \( \langle c, y \rangle \) and \( \langle c, y' \rangle \) have the 
same sign (say, positive, if we are in \( H^+ \)). Assume \( F(x) = \frac{Ax + b}{\langle c, x \rangle + d} \). Remember \( \langle c, x_0 \rangle = -d \). Then 
\[ F(x_0 + \lambda y) = \frac{A(x_0 + \lambda y) + b}{\langle c, x_0 + \lambda y \rangle + d} = \frac{Ax_0 + b + \lambda \cdot \left( \frac{Ax_0 + b}{\langle c, y \rangle} \right)}{\langle c, x_0 + \lambda y \rangle + d}, \]
and similarly \( F(x_0 + \lambda y') = \frac{Ay' + 1}{\langle c, y' \rangle + 1} \cdot \left( \frac{Ax_0 + b}{\langle c, y \rangle} \right) \). We see the two rays are mapped to two parallel half lines, which by injectivity of \( F \) are not identical, and therefore \( F(x_0) \) cannot be chosen so that it lies on both these lines. This means \( F \) cannot be extended to a domain which intersects \( H \), and still preserve intervals.

3. The image. The image of a (non-affine) fractional linear map \( F \), whose domain is maximal (meaning it is an open half space) is an open half space. Indeed, let \( \hat{A} = A_F \) be the associated matrix, and let \( \hat{A}(\{(x, 1) : x \in \mathbb{R}^n\}) = E \subseteq \mathbb{R}^{n+1} \). Then the image of \( F \) is the radial projection into \( \{(x, 1) : x \in \mathbb{R}^n\} \) of the part of \( E \) with positive \((n + 1)\)th coordinate. It is easily checked that this is a half space in \( \{(x, 1) : x \in \mathbb{R}^n\} \sim \mathbb{R}^n \), whose boundary is the hyperplane

\[ \partial(\text{Im}(F)) = \{Ax : \langle c, x \rangle = 1\}, \quad \text{where} \quad \hat{A} = \begin{pmatrix} A & b \\ c^T & d \end{pmatrix}. \]

Note that it does not depend on \( b \) and \( d \).

4. Composition. It is easily checked that \( A_{F \circ G} = A_F \cdot A_G \). In particular, the composition of two fractional linear maps is again a fractional linear map. As for the domains: The maximal domain of each of the maps is a half space, and so is the image, thus the map is formally defined only on \( G^{-1}(\text{Im}(G) \cap \text{dom}(F)) \), and by the previous remarks it can be extended to be defined on some half space.

5. The inverse map. It is easily checked that \( A_{F^{-1}} = A_F^{-1} \) and in particular, every fractional linear map has an inverse, which is also fractional linear. The domain of \( F^{-1} \) is the image of \( F \), which by previous remarks is exactly the radial projection into \( \{(x, 1) : x \in \mathbb{R}^n\} \) of the part of \( E = A(\{(x, 1) : x \in \mathbb{R}^n\}) \) with, say, positive \((n + 1)\)th coordinate.

2.2.3 More properties

To continue we first need two properties of fractional linear maps, given in Lemma 2.10 and Lemma 2.15. The first is a transitivity result.

**Lemma 2.10.** Fix a point \( p \) in the interior of the simplex \( \Delta = \{z = \sum z_i e_i : 0 \leq z_i, \sum z_i \leq 1\} \), where \( \{e_i\}_{i=1}^n \) is the standard basis of \( \mathbb{R}^n \). Given \( n + 2 \) points, \( x_0, x_1, \ldots, x_n \) in \( \mathbb{R}^n \), such that \( y \) is in the interior of \( \text{conv}(\{x_i\}) \), there exists an open convex domain \( D \) which contains the points and a fractional linear map \( F : D \rightarrow \mathbb{R}^n \) such that for \( 1 \leq i \leq n \), \( F(x_i) = e_i \), \( F(x_0) = 0 \) and \( F(y) = p \).

**Remark 2.11.** By invertibility (see Item 5. above) an equivalent formulation is as follows: there exists a fractional linear map \( F : \Delta \rightarrow \mathbb{R}^n \) such that for \( 1 \leq i \leq n \), \( F(e_i) = x_i \), \( F(0) = x_0 \) and \( F(p) = y \).

**Remark 2.12.** From Lemma 2.14, it will follow that the map in Lemma 2.10 is unique.

**Remark 2.13.** Let us compare Lemma 2.10 with the more standard transitivity result of projective geometry, which can be found for example in [13] (Theorem 2, page 59):

Let \( A_1, \ldots, A_{n+2} \) and \( B_1, \ldots, B_{n+2} \) be two sets of points in general position in \( \mathbb{R}P^n \). Then there exists a unique projective transformation \( f : \mathbb{R}P^n \rightarrow \mathbb{R}P^n \) such that \( f(A_i) = B_i \) for \( i = 1, \ldots, n+2 \). Indeed, they have the same flavor, however we demand more (in both sets, one point is in the convex hull of all the others) and get more; the whole convex hull is in the domain of the fractional linear map (i.e. it is mapped, within \( \mathbb{R}P^n = \mathbb{R}^n \cup \mathbb{R}P^{n-1} \), to the part not in \( \mathbb{R}P^{n-1} \)).

**Proof of Lemma 2.10.** First let us build an affine linear map which maps \( x_i \) to \( e_i \) for \( i = 1, \ldots, n \) and \( x_0 \) to 0. This is clearly possible by linear algebra. So we are left with the following task: given
z in the interior of the simplex, build a fractional linear map \( F \) whose domain contains the simplex, such that \( F(e_i) = e_i, F(0) = 0 \) and \( F(z) = p. \)

To describe this map \( F \), consider its associated matrix \( \hat{A} \) in \( GL_{n+1} \). Let us give the matrix elements which produce the desired map. Let the matrix be given by

\[
\hat{A} = \begin{pmatrix}
0 & \vdots \\
A & 0 \\
c^T & d
\end{pmatrix},
\]

where \( A \) is an \( n \times n \) matrix, \( c \in \mathbb{R}^n \), and \( d \in \mathbb{R}^+ \). Let \( A \) be the diagonal matrix with diagonal entries \( A_{i,i} = \frac{p_i}{\tau} \), let \( d = \frac{i-\sum \tau_i}{\sum \tau_i} \), and let the vector \( c \) be given by \( c_i = \frac{\tau_i}{\tau} - d \). The matrix induces a fractional linear map on the domain \( \{ x : \langle c, x \rangle > -d \} \). We must verify that the points \( 0, \{ e_i \}_{i=1}^n \) are in this domain. Indeed, \( d > 0 \) since the points are in the simplex, and also \( c_i > -d \). Finally, it is easily checked that the associated fractional linear map satisfies the desired conditions.

Once we know that the map from Lemma 2.10 exists, it follows that it is unique. Indeed, by the Theorem quoted in Remark 2.13, there exists only one fractional linear map which maps \( n + 2 \) given points to another \( n + 2 \) given points. We formulate it below, and for completeness provide the proof.

**Lemma 2.14.** Let \( F_1 : D_1 \rightarrow \mathbb{R}^n \) and \( F_2 : D_2 \rightarrow \mathbb{R}^n \) be two fractional linear maps, where \( D_i \subseteq \mathbb{R}^n \). Let \( \{ x_i \}_{i=0}^{n+1} \) be \( (n + 2) \) points in \( D_1 \cap D_2 \) such that one is in the interior of the convex hull of the others. If \( F_1(x_i) = F_2(x_i) \) for every \( 0 \leq i \leq n + 1 \), then the two maps coincide on all of \( D_1 \cap D_2 \) and moreover, are induced by the same matrix in \( GL_{n+1} \) (up to multiplication by a non-zero scalar).

**Proof of Lemma 2.14.** Without loss of generality, by Lemma 2.10, we can assume that \( x_0 = 0, x_i = e_i \) for \( i = 1, \ldots, n \), and that \( x_{n+1} = p \) is any point we desire in the interior of the convex hull of \( \{ x_i \}_{i=1}^n \). Furthermore, by the same lemma, we may assume that \( F_1(x_i) = F_2(x_i) = x_i \) for all \( i \), and therefore we may simply compare, say, \( F_1 \) to \( Id \) (and then also \( F_2 \)). Consider the matrix which induces \( F_1 \), given by some

\[
\begin{pmatrix}
A & b \\
c^T & d
\end{pmatrix},
\]

where \( A \) is an \( n \times n \) matrix, \( b \) and \( c \) are vectors in \( \mathbb{R}^n \) and \( d \in \mathbb{R} \). In fact, \( d \neq 0 \) since 0 is in the domain of \( F_1 \), which is \( \{ x : \langle c, x \rangle > -d \} \). Without loss of generality we let \( d = 1 \). From the condition \( F_1(0) = 0 \) we see that \( b = 0 \). From \( F_1(e_i) = e_i \) we see that \( A \) is diagonal, let us denote \( A_{i,i} = a_i \), so that \( c_i = a_i - 1 \). Finally, for \( F_1(p) = p \) we see that

\[
p_i = \frac{a_i p_i}{1 + \sum (a_j - 1)p_j}.
\]

This implies that for all \( i, a_i = 1 + \sum (a_j - 1)p_j \), and in particular \( a_1 = \ldots = a_n \). This means that \( (a_1 - 1)(1 - \sum p_j) = 0 \), and since \( p \) is not on the hyperplane passing through \( \{ e_i \}_{i=1}^n \), it implies \( a_i = 1 \) for all \( i \), that is, \( F_1 \) is the identity mapping. The same holds for \( F_2 \).

As a consequence we get the following useful fact:

**Corollary 2.15.** Let \( F_1 : D_1 \rightarrow \mathbb{R}^n \) and \( F_2 : D_2 \rightarrow \mathbb{R}^n \) be two fractional linear maps, where \( D_i \subseteq \mathbb{R}^n \). Let \( D \subseteq D_1 \cap D_2 \) be some open domain in \( \mathbb{R}^n \) such that \( F_1|_D = F_2|_D \). Then the two maps coincide, i.e. they are induced by the same matrix, and their maximal extension is the same function, with the same maximal domain.
2.3 Uniqueness

When the domain of an interval preserving map is assumed to be all of $\mathbb{R}^n$, it is a well known classical theorem that the map must be affine linear, as stated in the fundamental theorem of affine geometry, quoted below as Theorem 2.16. As a reference see, for example, [13], or [3] for the projective counterpart. More generally, interval preservation can be replaced by “collineation”. More far reaching generalizations also exist, and we refer the reader to the forthcoming [9] where an elaborate account of these is given.

**Theorem 2.16.** [The Fundamental Theorem of Affine Geometry] Let $m \geq 2$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a bijective interval preserving map. Then $f$ must be an affine transformation.

In this section we discuss the fact that when the domain is a convex set (or, more generally, a connected open domain), the only interval preserving maps are fractional linear. Then concludes that it must be projective linear. We work in a more elementary way, never leaving $\mathbb{R}^n$. However, Shiffman’s result is in a more general setting where not all intervals are assumed to be mapped to intervals, but only a subfamily which is large enough. This is important in some applications, in particular in the proof of Theorem 5.23.

The main theorem discussed in this section is the following.

**Theorem 2.17.** Let $n \geq 2$ and let $K \subseteq \mathbb{R}^n$ be a convex set with non empty interior. If $F : K \rightarrow \mathbb{R}^n$ is an injective interval preserving map, then $F$ is a fractional linear map.

The proof of the theorem relies on the following lemma:

**Lemma 2.18.** Assume $n \geq 2$. Let $\Delta \subseteq U \subseteq \mathbb{R}^n$, where $U$ is an open set, and $\Delta$ is a non-degenerate simplex with vertices $x_0, \ldots, x_n$. Let $p$ belong to the interior of $\Delta$. If $F : U \rightarrow \mathbb{R}^n$ is an injective interval preserving map that fixes all $(n+1)$ vertices of $\Delta$ and the interior point, that is $F(x_i) = x_i$ for every $0 \leq i \leq n$, and $F(p) = p$, then $F|_{\Delta} = Id|_{\Delta}$.

**Proof of Lemma 2.18.** The proof goes by induction on the dimension $n$. Begin with $n = 2$. Consider a two dimensional simplex $\Delta$, that is, a triangle in $\mathbb{R}^2$, with vertices $a, b, c$, and a point $p \in int(\Delta)$. Since $F$ is injective and interval preserving, by Lemma 2.6 it is continuous, which implies that the set $D = \{x \in \Delta : F(x) = x\}$ is closed.

Let us check that all the edges are contained in $D$. Assume the contrary, namely that there is a point $e \in [a, b]$, $e \not\in D$. Since $D$ is closed, there exists an interval $[a', b'] \subseteq [a, b]$, such that $a', b' \in D$, but $(a', b') \cap D = \emptyset$. Now we will find a point $e' \in (a', b') \cap D$ in contradiction, thus concluding that no such $e$ exists. Let us find two points $a'' \in [a', c]$ and $b'' \in [b', c]$, such that $a'', b'' \in D$. To this end, consider the intervals $[a', c]$, $[b', c]$. They are both mapped to themselves by $F$, and both intersect the line $L$ containing $a$ and $p$, for which we have $F(L) \subseteq L$. Let $a'' \in [a', c]$ and $b'' \in [b', c]$ be the points of intersection with $L$. Then $F(a'') = a''$ since this is the only point in $[a', c]$ and in $L$, and similarly $F(b'') = b''$.

Now we look at the intersection of $[a'', b']$ with $[b', a']$. This is a point $p'$ in the interior of the triangle $a'b'c$. The line between $c$ and $p'$ intersects with $[a', b']$ at some point $e' \in (a', b')$, and by the same argument as before, $e' \in D$. We get a contradiction which proves that $F$ is the identity map on the edges of $\Delta$.

Next, for every point $y$ in the interior, we draw two intervals containing $y$ - each connecting a vertex with an edge, and get that the two intervals must be mapped to themselves (since the end points are on the edges and are thus mapped to themselves). This implies, as before, $F(y) = y$, which completes the proof for $n = 2$. 

8
For the inductive step, we assume that the proposition is true for dimension \( n - 1 \), and prove it for dimension \( n \). Let \( \Delta \) be an \( n \) dimensional simplex. Denote by \( \Delta_i := \text{Conv}\{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\} \) the face of \( \Delta \) opposite to \( x_i \). First we claim that \( F(\Delta_i) = \Delta_i \). Indeed, this is due to interval preservation, together with the fact that the vertices are mapped to themselves. Denote by \( y \in \text{relint}(\Delta_i) \) the unique point in the intersection of \( \Delta_i \), with the line connecting \( x_i \) and \( p \in \text{int}(\Delta) \). Interval preservation implies that \( F(y) \) remains on this line, and since it must remain on the face, we get \( F(y) = y \). By applying the claim to the \((n-1)\) dimensional simplex \( \Delta_i \), we conclude that \( F|_{\Delta_i} = \text{Id}|_{\Delta_i} \). The fact that the restriction of \( F \) to each of the faces is the identity, combined with interval preservation, implies that \( F|_{\Delta} = \text{Id}|_{\Delta} \) simply by representing a point in the interior as the intersection of two intervals with endpoints on faces.

By the transitivity result from Lemma 2.10, we may state a corollary of the above lemma for general maps on the simplex.

**Corollary 2.19.** Assume \( n \geq 2 \). Let \( \Delta \subseteq U \subseteq \mathbb{R}^n \), where \( U \) is an open set, and \( \Delta \) is a non-degenerate simplex with vertices \( x_0, \ldots, x_n \). If \( F : U \to \mathbb{R}^n \) is an injective interval preserving map then there exists a fractional linear map \( F_A \) such that \( F|_{\Delta} = F_A|_{\Delta} \).

**Proof of Corollary 2.19.** Let \( p \) belong to the interior of \( \Delta \). The main step is to show that the mapping \( F \) maps the point \( p \) to a point in the interior of \( \text{Conv}\{F(x_i)\}_{i=0}^n \), so that we may invoke transitivity and Lemma 2.18. To this end we shall use induction and prove the following claim: an injective interval preserving map must map simplices of dimension \( k \), for any \( k \geq 1 \), to simplices of the same dimension, whose vertices are the images of the original vertices. Once this is done, an interior point must be mapped to an interior point by injectivity of \( F \). The case \( k = 1 \) is almost by definition (see Lemma 2.2). Assume this is the case for simplices of dimension \( \leq k \) and let \( y_0, \ldots, y_{k+1} \), the vertices of some \((k+1)\) dimensional simplex, in general position, be given. By induction, the relative boundary of the convex hull is mapped to the relative boundary of the simplex \( \{F(y_j)\}_{j=0}^{k+1} \).

Since a point in the interior can be written as the intersection of two intervals with endpoints on the boundary, we get that the interior of the simplex \( \text{Conv}\{y_j\}_{j=0}^{k+1} \) is mapped to the interior of \( \text{Conv}\{F(y_j)\}_{j=0}^{k+1} \), as needed. Applying this, we have that the points \( \{x_i\}_{i=0}^n \) are mapped to points \( \{F(x_i)\}_{i=0}^n \) which are the vertices of a non degenerate simplex, \( F(\Delta) = \text{Conv}\{F(x_i)\}_{i=0}^n =: \Delta' \), and for any point \( p \in \text{int}(\Delta) \) we have that \( F(p) \in \text{int}(\Delta') \). To prove the corollary, chose any \( p \in \Delta \), and compose \( F \) with some fractional linear \( G \) so that \( (G \circ F)(x_i) = x_i \) for \( i = 0, \ldots, n \) and \( (G \circ F)(p) = p \). Using Lemma 2.18 we have that \( G \circ F = \text{Id} \) on \( \Delta \), and therefore \( F|_{\Delta} = G^{-1}|_{\Delta} \), which is fractional linear, as claimed.

**Proof of Theorem 2.17.** First we prove the theorem under the assumption that \( K \) is open and convex, and at the end of the proof we remark on the extension to general convex \( K \) (with non empty interior).

First we note that for every simplex \( \Delta \) inside \( K \) the statement holds: consider \( n + 2 \) points \( x_0, \ldots, x_n, p \in \mathbb{R}^n \), arranged as a simplex \( \Delta \) and a point in its interior, as in Corollary 2.19. Since \( F \) is injective and interval preserving, by Corollary 2.19 \( F|_{\Delta} \) is fractional linear.

Next, consider the union of two simplices \( \Delta_1 \) and \( \Delta_2 \) such that the intersection has a non empty
interior. \( F|_{\Delta_i} \) is fractional linear on each simplex \( \Delta_1 \) and \( \Delta_2 \), and these mappings coincide on the intersection, so they must be induced by the same matrix, by Corollary 2.15.

Finally, by covering the domain \( K \) with simplices so that each two are connected by a chain of simplices \( \{\Delta_i\}_{i=0}^{N} \), with the property that the intersection of \( \Delta_i \) and \( \Delta_{i+1} \) has a non empty interior, we get that there is one map which induces all of the maps \( F|_{\Delta_i} \) for all these simplices, meaning that \( F \) itself is a fractional linear map. Such a covering exists, for example an infinite family \( \{\Delta_{x,y} : x, y \in K\} \), where \( \Delta_{x,y} \) is some simplex which contains \( x \) and \( y \) in the interior will do (such a simplex exists for every \( x \) and \( y \)). This completes the proof in the case where \( K \) is open.

For a general convex \( K \) with non empty interior we must deal with the boundary of \( K \). We know there exists a fractional linear map \( G : U \to \mathbb{R}^n \) s.t. \( F|_{\text{int}(K)} = G|_{\text{int}(K)} \), where \( U \) is the maximal domain of \( G \) (an open half space), and of course \( \text{int}(K) \subseteq U \). We wish to show that \( K \subseteq U \), and that \( F = G \) also on \( K \cap \partial K \). We first claim that \( x \in U \), for which we need only show that \( x \not\in H = \partial U \). However, we have shown in item 2 of Section 2.2.2 that \( G \) cannot be extended to be defined on any point of \( H \) so that it is still interval preserving, from which we conclude \( K \subseteq U \). Indeed, this was shown by considering two points \( a, b \) in the interior of \( K \), to which correspond intervals \( [a, x] \) and \( [b, x] \) which are mapped to intervals, by \( G \). Were \( x \) on the boundary, these intervals would have been parallel, and no way to define \( F(x) \) would have existed. When \( x \not\in H \), the intervals \( [G(a), G(x)] \) and \( [G(b), G(x)] \) have a unique point of intersection \( G(x) \), and we conclude that \( F(x) = G(x) \).

\[ \square \]

**Remark 2.20.** Theorem 2.17 can be proved for a general open connected set \( K \); we only used convexity of \( K \) when arguing that \( K \) can be covered by simplices to get the wanted chains. This argument holds also whenever \( K \) is open and connected. Indeed, to get this covering we took between every two points \( x, y \in K \) a simplex \( \Delta_{x,y} \). This simplex is now replaced by a chain of simplices connecting \( x \) and \( y \), constructed using an \( \epsilon \) neighborhood of the path between \( x \) and \( y \).

To complete the picture let us also attend to the case \( n = 1 \), although this will not be used in the sequel. Obviously, a similar theorem cannot be proved in \( \mathbb{R} \), since, for example, all continuous functions are interval preserving. The next theorem, Theorem 2.23, gives a characterization of one dimensional fractional linear maps. The theorem is a local version of the more well-known fact from projective geometry, stating that maps preserving cross ratio are linear when the domain and range are lines, and projective when the domain and range are extended lines.

We recall that the **cross ratio** of four numbers (thought of as coordinates of points on a line) is defined to be

\[
[a, b, c, d] := \left(\frac{c-a}{c-b}\right) \div \left(\frac{d-a}{d-b}\right).
\]

For details and discussion see, for example, [13].

**Remark 2.21.** Note that \([a, b, c, x] = [a', b', c', x']\) implies \(x' = \frac{\alpha x + \beta y}{\gamma x + \delta y}\), where \(\alpha, \beta, \gamma, \delta\) are some function of \(a, b, c, a', b', c'\). Conversely, every fractional linear map on \( \mathbb{R} \) preserves the cross ratio of any four points in its domain.

**Remark 2.22.** Regarding permutations of \(a, b, c, d\), we have the following:

\[
[A, B, C, D] = [B, A, C, D]^{-1},
\]

\[
[a, b, c, d] = [a, b, D, C]^{-1},
\]

\[
[a, B, C, d] = 1 - [a, C, B, d],
\]

and using the rule for these three transpositions, the cross ratio of any permutation of \(a, b, c, d\) can be derived from \([a, b, c, d]\). Moreover, as a consequence, we see that if we have \([a, b, c, d] = [x, y, z, w]\), then for every permutation \(\sigma\) we also have that \([\sigma(a), \sigma(b), \sigma(c), \sigma(d)] = [\sigma(x), \sigma(y), \sigma(z), \sigma(w)]\).
A basic notion when dealing with one dimensional fractional linear maps is the projection of one line to another line, through a so-called “focus point” situated outside the two lines. See [13] for more details on the relation between fractional linear maps, preservation of cross ratio, and projection.

**Theorem 2.23.** Let $I \subseteq \mathbb{R}$ be a convex set, either bounded or not, and $f : I \to \mathbb{R}$. Assume further that $f$ preserves cross ratio on $I$, so for every four distinct points $a < b < c < d \in I$

$$[f(a), f(b), f(c), f(d)] = [a, b, c, d].$$

Then $f$ is fractional linear on $I$. In fact, it is true also if $a, b, c \in I$ are three (distinct) fixed points, and we assume only that $f$ preserves cross ratio of $a, b, c, d$ for any $d \in I \setminus \{a, b, c\}$.

**Proof.** Let $a, b, c \in I$ such that $a < b < c$, and $f$ preserves cross ratio of $a, b, c, x$ for any $x \in I \setminus \{a, b, c\}$. Let $x \in I$. We consider four cases: $x < a$, $a < x < b$, $b < x < c$, and $c < x$. For each case, the preservation of cross ratio yields a different equation:

- $x < a \Rightarrow [f(x), f(a), f(b), f(c)] = [x, a, b, c],$
- $a < x < b \Rightarrow [f(a), f(x), f(b), f(c)] = [a, x, b, c],$
- $b < x < c \Rightarrow [f(a), f(b), f(x), f(c)] = [a, b, x, c],$
- $c < x \Rightarrow [f(a), f(b), f(c), f(x)] = [a, b, c, x].$

By Remark 2.22, each of these equations implies $[f(a), f(b), f(c), f(x)] = [a, b, c, x]$, and thus by Remark 2.21, we get $f(x) = \alpha x + \beta$ for some $\alpha, \beta, \gamma, \delta$ which depend only on $a, b, c, f(a), f(b), f(c)$. Therefore $f$ is a fractional linear map on $I$. \qed

### 2.4 Other representations and properties

#### 2.4.1 Canonical form

In what follows, we denote by $x = (x_1, \ldots, x_n)$ the coordinates of a point $x$ with respect to the standard basis $\{e_i\}$.

**Definition 2.24.** Let $H^+$ be the half space $\{x_1 > 1\}$. The mapping $F_0 : H^+ \to H^+$ given by

$$F_0(x) = \frac{x}{x_1 - 1}$$

will be called the *canonical* fractional linear map.

It is useful to note that the group of fractional linear maps is generated by its subgroup of affine linear maps, and the above map.

**Theorem 2.25.** Let $F$ be an injective non-affine fractional linear map with $F(x_0) = y_0$. Then there exist $B, C \in GL_n$ such that $B(F(Cx + x_0) - y_0) = F_0(x)$.

**Proof of Theorem 2.25.** Define $G(x) := F(x + x_0) - y_0$, then $G(0) = 0$. $G$ is an injective non-affine fractional linear map, with an inducing matrix of the form:

$$
\begin{pmatrix}
A' & b \\
ct & d
\end{pmatrix}.
$$

From $0 \in \text{Dom}(G)$ it follows $d \neq 0$, so (using the multiplicative degree of freedom) we let $d = -1$. Also, $G(0) = 0$ implies $b = 0$. Since $G$ is injective, the inducing matrix is invertible, and by $b = 0$
this implies that $A' \in GL_n$. Non-linearity of $G$ implies $c \neq 0$. Therefore we can write for some $A' \in GL_n$, $0 \neq c \in \mathbb{R}^n$, that
\[
G(x) = \frac{A'x}{\langle c, x \rangle - 1}.
\]
Pick $C \in GL_n$ such that $C'c = e_1$. We get $\langle c, Cx \rangle = \langle e_1, x \rangle = x_1$. Therefore
\[
G(Cx) = \frac{A'Cx}{x_1 - 1}.
\]
Finally, by letting $B = (A' C)^{-1}$, we get $(B \circ G \circ C)(x) = \frac{x}{x_1 - 1}$, and so
\[
B(F(Cx + x_0) - y_0) = \frac{x}{x_1 - 1},
\]
as required.

\[\square\]

**Remark 2.26.** For simplicity, assume below $x_0 = y_0 = 0$. The representation in Theorem 2.25 is clearly not unique, as $C$ can be chosen in any way satisfying just one linear condition, and $B$ depends on $C$. Another form which can be given is:
\[
C^{-1}A'^{-1}FC = \frac{x}{x_1 - 1},
\]
where $A'$ is uniquely determined, and $C$ as before. Yet a third way to view this representation is:
\[
F(x) = \frac{A'x}{\langle c, x \rangle - 1},
\]
as was shown in the proof. This form has the advantage of emphasizing the degrees of freedom of a fractional linear map, since both the point $c$ and the matrix $A'$ are determined uniquely.

### 2.4.2 Geometric structure

The mapping $F_0(x) = \frac{x}{x_1 - 1}$ is defined on $H^+ = \{x_1 > 1\}$, and satisfies $F_0(H^+) = H^+$. It is an involution on $H^+$ (and on $H^- = \{x_1 < 1\}$ as well). Denote the boundary of $H^+$ by $H$.

For every affine hyperplane parallel to $H$, namely $H_t = \{x : x_1 = t\}$ (for $t \neq 1$), we have $F_0(H_t) = H_{f(t)}$, where $f(t) = \frac{1}{t - 1}$. The restriction $F_0 : H_t \to H_{f(t)}$, thought of as a map on $\mathbb{R}^{n-1}$, is a linear map - in fact, it is simply a scalar map: $x \mapsto \frac{1}{t - 1}x$. In particular we see that in this family of parallel hyperplanes (shifts of $H$), parallel hyperplanes are mapped to parallel hyperplanes. This behavior is unique to shifts of $H$. Indeed, take $v \in \mathbb{R}^n$, then ($F_0^{-1} = F_0$):
\[
F_0(\{x : \langle x, v \rangle = c\}) = F_0(\{x : \langle x, v \rangle = c\}) = \{x : \langle F_0(x), v \rangle = c\} = \{x : \langle x, v \rangle = c(x_1 - 1)\} = \{x : \langle x, v \rangle = c(x, e_1) - c\} = \{x : \langle x, v - ce_1 \rangle = -c\}.
\]

And so we see that if $v \neq \lambda e_1$, hyperplanes parallel to $v^\perp$ are mapped to hyperplanes which are not parallel; $(v - ce_1)^\perp \neq (v - c'e_1)^\perp$ for $c \neq c'$.

These considerations, by Theorem 2.25, may be applied to a general fractional linear mapping $F$. There are two hyperplanes, the first of which, say $H_1$, is the boundary of the maximal domain of $F$, and the second, $H_2$, is the boundary of the image of $F$, such that any translate of $H_1$ (which is in the domain) is mapped to a translate of $H_2$, and moreover, the map $F$ restricted to each translate of $H_1$ is linear. In any other direction, however, two parallel hyperplanes are mapped to two hyperplanes which are not parallel.

As for a linear subspace $V$ of $\mathbb{R}^n$ of dimension $0 \leq k \leq n$, we have $F_0(V) = V$ (by this we mean $F_0(V \cap H^-) = V \cap H^-$ and $F_0(V \cap H^+) = V \cap H^+$, since $F_0$ is not defined on the intersection with
For \( n - 1 \) dimensional subspaces, we have seen it in the formula given above for the image of hyperplanes under \( F_0 \); substituting \( c = 0 \) yields \( F_0(v^+) = v^+ \) for every \( v \in \mathbb{R}^n \). But in fact it is true trivially for subspaces of any dimension; simply note that \( F_0(x) \) is in the direction of \( x \). In fact, this is a particular case of the more general phenomenon; lines (more precisely: their intersection with the domain) through a fixed point in the domain of \( F \), \( x_0 \in \text{dom}(F) \), are mapped into lines through \( F(x_0) \). This is due to interval preservation of \( F \). Since \( F \) is smooth, this mapping of lines (but not of points along the lines) is the linear map given by the differential of \( F \), \( dF(x_0) \).

We can say even more about the geometric structure of \( F \). For a point \( y_0 \) on the boundary of the maximal domain of \( F \), the family of all the rays emanating from the point \( y_0 \) (into the domain) is mapped to the family of all half lines in the image of \( F \) which are parallel to some vector \( y'_0 \), and vice versa. Again, by Theorem 2.25 it is enough to show this for the specific map \( F_0(x) = \frac{x}{x_1-1} \). Consider a point \( \hat{y} = (1, y) \) on \( H \); a ray emanating from \( \hat{y} \) into the domain can be written, for some \((1, u) \in H\) as
\[
R = \{(1, y) + t(1, u) : t \in \mathbb{R}^+\}.
\]
It is mapped to the half line
\[
l' = \{F((1, y) + t(1, u)) : t \in \mathbb{R}^+\} = \{(1, u) + \frac{1}{t}(1, y) : t \in \mathbb{R}^+\} = \{(1, u) + s(1, y) : s \in \mathbb{R}^+\}.
\]
So we have seen that for \( a \) and \( b \) on \( H \), the ray \( a + b\mathbb{R}^+ \) is mapped under \( F_0 \) to \( b + a\mathbb{R}^+ \) and vice versa. For example all rays emanating from the point \( e_1 \in H \) are mapped to all lines perpendicular to \( H \). Note that the part of \( l \) which is close to the point \( \hat{y} \) (small \( t \)) is mapped to the part of \( l' \) which is far from the hyperplane \( H \) (large \( s \)). In a sense, the point \( \hat{y} \) is mapped to “infinity” in direction opposite to \( H \).

This also shows that fractional linear maps act as a lens on straight lines intersecting the defining hyperplane. Indeed, a cone of rays with base \( B \), emanating from the point \( a \) in \( H \), is mapped to a half infinite cylinder with base \( B \), in the direction \( a \). If \( a \in B \), the corresponding line is the only one in the cone which is mapped to itself. When considering a general non-affine fractional linear map, we get that an infinite cone with base \( B \) is mapped to a half infinite cylinder with base \( T(B) \) for some linear \( T \), and vice versa. Of course, if the fractional linear map is affine it also does this, but by mapping cones to themselves and cylinders to themselves.

2.5 Additional results

2.5.1 Fractional linear maps and polarity

For a closed convex set \( T \) containing 0, denote its polar set as before by \( T^c \). We claim that in a sense, the “root” of a fractional linear map is the polar map. The following theorem states that the so called “distortion” of fractional linear maps corresponds to two actions of polarity, each with respect to a different point of origin.
Theorem 2.27. Let $0 \in K \subseteq \{x_1 < 1\} \subseteq \mathbb{R}^n$ be a closed convex set. Then for the canonical form of a fractional linear map, $F_0(x) = \frac{x}{x_1 - 1}$, the following holds:

$$F_0(K) = (e_1 - K^o)^o.$$

In [10], [16] the authors prove uniqueness theorems for order isomorphisms on various families of convex sets. Here we see new such maps, on the family of closed convex bodies which are contained in a half space. Uniqueness of these maps in some weak sense (among point maps) follows immediately from the uniqueness theorem 2.17. Applying techniques from those papers one can get uniqueness of these maps among all order isomorphisms on this class of convex bodies.

Proof of Theorem 2.27. Let $T$ be a closed convex set. Clearly

$$[0, e_1] \subseteq T \iff [-e_1, 0] \subseteq T - e_1 \iff (T - e_1)^o \subseteq \{x_1 > -1\},$$

and therefore under our assumptions for every $x \in (T - e_1)^o$ we have $0 < 1 + x_1$. We define $G(-x) = -F_0(x)$, or explicitly $G(x) = \frac{-x}{1 + x_1}$. Note that $F_0$ is an involution on $\{x_1 \neq 1\}$, and hence $G$ is an involution on $\{x_1 \neq -1\}$. Compute

$$(T - e_1)^o = \{x \in \mathbb{R}^n : \langle x, y - e_1 \rangle \leq 1 \ \forall y \in T\}
= \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 + x_1 \ \forall y \in T\}
= \left\{ x \in \mathbb{R}^n : \left(\frac{-x}{1 + x_1}, -y\right) \leq 1 \ \forall y \in T \right\}
= \{x \in \mathbb{R}^n : G(x), -y \leq 1 \ \forall y \in T\}
= \{G^{-1}(x) \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall y \in (-T)\}
= G^{-1}(\{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall y \in (-T)\})
= G^{-1}((-T)^o) = G(-T^o) = -F_0(T^o),$$

which in turn implies

$$F_0(T^o) = (e_1 - T)^o,$$

for sets $T$ which contain the interval $[0, e_1]$, or conversely, such that $T^o \subseteq \{x_1 < 1\}$. Therefore we can formulate it in the following way, for a closed convex $K \subseteq \{x_1 < 1\}$ such that $0 \in K$ we have

$$F_0(K) = (e_1 - K^o)^o.$$

□

Remark 2.28. Recall that $\{x_1 = 1\}$ is the defining hyperplane of $F_0$, so we cannot hope to get that result for $K$ which intersects this hyperplane. In the other side of this hyperplane, however, we do not have 0, and again cannot work with $K^o$.

Remark 2.29. By Theorem 2.25, once we understand the action of $F_0$ on convex bodies, we understand the action of all (non-affine) fractional linear maps on convex bodies, and the only difference is in some linear maps and translations.

2.5.2 Sets that can be preserved

The fractional linear maps clearly have a non-linear “distortion” of the image. As we saw above, when approaching the defining hyperplane, the map diverges. However, fractional linear maps preserve some structure, for example, they preserve combinatorial structure of polytopes (number of vertices, faces of every dimension, intersection between faces, etc). We will investigate which sets can be preserved by fractional linear maps.

We present some examples of simple convex sets $K$ for which there exist fractional linear maps $F$ with $F(K) = K$. This will also shed some light on the question: “given sets $K_1$, $K_2$, does there
exist a fractional linear map $F$ such that $F(K_1) = K_2$. This question will have consequences in the next section, where we deal with classes of functions supported on convex sets (“windows”), and see that the existence of any order isomorphism between two such classes depends on the existence of a fractional linear map between the corresponding windows (more precisely; between the corresponding cylinders, either $K_1 \times \mathbb{R}^+$ or $K_1 \times \mathbb{R}$).

Let us start with an explicit two dimensional example: A non-affine fractional linear map which preserves the Euclidean disk.

**Example 2.30. Euclidean ball, 2 dimensions.** Define $T : D \to \mathbb{R}^2$, where $D = \{(x, y) \in \mathbb{R}^2 : x < 2\}$, in the following way:

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix} \mapsto \begin{pmatrix}
  T_1(x) \\
  T_2(x, y)
\end{pmatrix} = \begin{pmatrix}
  \frac{2x - 1}{\sqrt{3}y} \\
  \frac{\sqrt{3}y}{x - 2}
\end{pmatrix}.
$$

Note that $x^2 + y^2 = 1$ implies $T_1(x)^2 + T_2(x, y)^2 = 1$, that is, $S^1$ is mapped to itself by $T$. It is easy to check that $T$ maps $S^1$ onto itself. By the interval preservation property of $T$, this implies that the unit ball is mapped to itself. Note that $T(0) \neq 0$, with correspondence to Theorem 2.37.

**Example 2.31. Ellipsoids in $n$ dimensions.** The above explicit example can be extended easily to the Euclidean ball in $\mathbb{R}^n$. However, let us discuss this case, or more generally, the case of an ellipsoid in $\mathbb{R}^n$, in a slightly more abstract way. Note that a conic section is always mapped by a fractional linear map (so, it is bounded away from the defining hyperplane), it is mapped to a conic section, but since $F$ is continuous, this must be a compact conic section, and in particular a bounded one. Thus, $F(\mathcal{E})$ is an ellipsoid $\mathcal{E}'$. Finally, since any two ellipsoids can be mapped to one another via an affine linear map, we can find an invertible affine transformation $A$ such that $AF(\mathcal{E}) = \mathcal{E}$, and $AF$ is a non-affine linear fractional map.

Before moving on to the next convex set, we mention that for Euclidean balls (and hence ellipsoids) we also have a transitivity result, in the flavor of Lemma 2.10 for simplices. It is given in the following proposition.

**Proposition 2.32.** Let $B_n$ denote the open unit ball in $\mathbb{R}^n$, and $\mathcal{E}$ be some open ellipsoid, with $p \in \mathcal{E}$. Then there exists a bijective fractional linear map $F : \mathcal{E} \to B_n$ with $F(p) = 0$.

**Proof.** There exists an affine linear map that maps $\mathcal{E}$ to $B_n$ and $p$ to $p'$, and an orthogonal transformation which maps $p'$ to $\lambda e_1$ for $0 < \lambda < 1$. If $\lambda = 0$ we are done, with $F$ being an affine map. Assume otherwise; then by invertibility of f.l. maps, our task is to find a bijective fractional linear map $G : B_n \to B_n$ such that $G(0) = \lambda e_1$, for a given $0 < \lambda < 1$. Let $a := 1/\lambda$, $c := \sqrt{a^2 - 1}$ (so, $1 < a, 0 < c$). One possible choice of $G$ is induced by the $(n + 1) \times (n + 1)$ matrix

$$
A_G = \begin{pmatrix}
  D & e_1 \\
  e_1^T & a
\end{pmatrix},
$$

where $D$ is diagonal with eigenvalues $\{a, c, \ldots, c\}$. The direct formula corresponding to that choice of $G$ is:

$$
G \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_n
\end{pmatrix} = \frac{1}{x_1 + a} \begin{pmatrix}
  ax_1 + 1 \\
  cx_2 \\
  \vdots \\
  cx_n
\end{pmatrix}.
$$
We turn to the second example which is again in $\mathbb{R}^2$, a trapezoid.

**Example 2.33. Trapezoid.** Let $\alpha > 0$, and $D = \{(x, y) \in \mathbb{R}^2 : x < 1 + \alpha^{-1}\}$. Define $T : D \to \mathbb{R}^2$ and $A : \mathbb{R}^2 \to \mathbb{R}^2$ in the following way:

$$
T \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} T_1(x) \\ T_2(x, y) \end{array} \right) = \left( \frac{x}{1 + \alpha - \alpha x}, \frac{y}{1 + \alpha} \right), \quad A \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \frac{1 - y}{x} \right).
$$

The affine linear map $A$ is the $\pi/2$ rotation around $(1/2, 1/2)$, and so it maps the four points $(0, 0), (1, 0), (0, 1), (1, 1)$ to themselves in a cyclic manner i.e. to $(1, 0), (0, 1), (1, 1), (0, 0)$ respectively. The fractional linear map $T$ fixes the three points $(0, 0), (1, 0), (0, 1)$, and maps $(1, 1)$ to $(1, 1 + \alpha)$.

Denote by $K$ the trapezoid with vertices $(0, 0), (1, 0), (0, 1), (1, 1 + \alpha)$, and consider $F : K \to K$ defined by $F := T \circ A \circ T^{-1}$. It is obvious that $F$ is not affine, and that it maps the four vertices of $K$ to themselves cyclically, thus by interval preservation, $F(K) = K$. These two facts can also be verified from the direct formula of $F$:

$$
\left( \begin{array}{c} x \\ y \end{array} \right) \mapsto \left( \frac{\alpha y + 1}{\alpha x + \alpha x + 1}, \frac{\alpha x y}{\alpha^2 + 1} \right).
$$

Note, had we chosen $\alpha = 0$, our trapezoid $K$ would be a square, and we would get that $T$, therefore $F$, are both affine maps, and thus we see that at least with this construction, we did not get a non-affine fractional linear map that preserves the cube in $\mathbb{R}^2$. This is, in fact, a general result in $\mathbb{R}^n$.

We denote by $Q^n$ the unit ball of the $l_\infty$ norm in $\mathbb{R}^n$, and by $B^n_1$ the unit ball of the $l_1$ norm in $\mathbb{R}^n$:

$$
Q^n := \{ x \in \mathbb{R}^n : -1 \leq x_i \leq 1, \quad i = 1, \ldots, n \},
$$

$$
B^n_1 := \{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1, \quad i = 1, \ldots, n \}.
$$

**Theorem 2.34.** Any bijective fractional linear map $F : Q^n \to Q^n$ is affine.

**Theorem 2.35.** Any bijective fractional linear map $F : B^n_1 \to B^n_1$ is affine.

We use the following lemma:

**Lemma 2.36.** Let $K \subset \mathbb{R}^n$ be a non-degenerate closed polytope, and $f : K \to \mathbb{R}^n$ a fractional linear map. If two pairs of opposite and parallel facets are mapped to such pairs, the map must be affine.

**Proof.** By Section 2.4.2, if $f$ is not affine, there is only one direction in which $f$ maps parallel hyperplanes to parallel hyperplanes. Therefore, if two $n - 1$ dimensional subsets are parallel (but are not contained in the same hyperplane), and mapped to parallel sets, they must lie on a translate of the defining hyperplane of $f$. Assume that $F_1, F_2$ are two parallel facets of $K$, and likewise $F_3, F_4$. There is no hyperplane whose shifts contain all four facets, since $K$ is a polytope of full dimension (there are no more than two parallel facets). Therefore, the fact that the pair $F_1, F_2$ is mapped to a similar pair, and likewise $F_3, F_4$, implies that $f$ is affine.

**Proof of Theorems 2.34, 2.35.** Both the facets of $Q^n$ and of $B^n_1$ have the property that every two non-opposite facets intersect. Therefore, every pair of opposite facets is mapped to such a pair. In particular, we have two such pairs, and by the previous lemma this implies that $f$ is affine.

Next, we prove that if $K$ is a centrally symmetric convex body, the only fractional linear maps which may preserve both $K$ and $\{0\}$ are affine.
Theorem 2.37. Let $K \subseteq \mathbb{R}^n$ be a closed, convex, centrally symmetric body, and let $F : K \to K$ be a bijective fractional linear map. If $F(0) = 0$, then $F$ is linear.

Proof of Theorem 2.37. As usual, since $F(0) = 0$ we assume that the inducing matrix of $F$ has the form:

$$F \sim \begin{pmatrix} A & 0 \\ v^T & -1 \end{pmatrix},$$

where $A \in GL_n$, and $0, v \in \mathbb{R}^n$. Therefore $F(x) = \frac{Ax + v}{\langle v, x \rangle - 1}$.

We need to show that $v = 0$. Otherwise, let $x \in \partial K$ be such that $\langle v, x \rangle \neq 0$ (for example, take $x$ in the direction of $v$). The interval $[x, -x]$ is mapped by $F$ to the interval $[F(x), F(-x)]$. Since $F$ is surjective, $F(x)$ and $F(-x)$ are also on the boundary of $K$, and by the formula they are in opposite direction, which means that $F(-x) = -F(x)$, by symmetry of $K$. By $\|F(x)\| = \|F(-x)\|$ we get $|\langle v, x \rangle + 1| = |\langle v, x \rangle - 1|$, meaning $\langle v, x \rangle = 0$, in contradiction to our choice of $x$. Thus we conclude $v = 0$, which means that $F$ is linear. \hfill $\square$

Remark 2.38. The theorem remains correct also when the condition “closed” is omitted. If the closure of $K$ is contained in the maximal domain of $F$ (the half space parallel to the defining hyperplane), then by continuity of $F$ we get that the same conditions hold for the closure of $K$, apply the theorem, and conclude that $F$ is linear. In the other case, i.e. when the closure of $K$ intersects the defining hyperplane, one must be more careful, and we omit the details completing the proof.

Remark 2.39. The condition $F(0) = 0$ cannot be omitted. Indeed, we have seen examples of symmetric bodies preserved by non-affine fractional linear maps, for instance in Example 2.30.

Theorem 2.40. Let $\Delta \subseteq \mathbb{R}^n$ be a closed, non-degenerate simplex, and $p \in \Delta$ its center of mass. If $F : \Delta \to \Delta$ is a bijective fractional linear map with $F(p) = p$, then $F$ is affine linear.

Proof of Theorem 2.40. Denote by $x_0, ..., x_n$ the vertices of $\Delta$. Let $A : \Delta \to \Delta$ be the affine map defined by the conditions $A(x_i) = F(x_i)$, $i = 0, ..., n$. Such a map obviously exists, moreover it is unique, and it is invertible. Note that $A(\Delta) = \Delta$ implies $A(p) = p$, since the center of mass is a linear invariant. By Lemma 2.14, this implies $F = A$, meaning that $F$ is affine linear. \hfill $\square$

Remark 2.41. As in the case of symmetric bodies, the condition $F(p) = p$ cannot be omitted. In fact we have seen in Lemma 2.10 a transitivity result, stating that fractional linear maps can map any simplex to itself, with an arbitrary permutation on the vertices, and in addition map a given point inside - say, the center of mass - to an arbitrary point inside. In the last theorem we have seen that among these maps, the affine maps are the only ones which map the center of mass to itself.

However, the choice of a different point inside will not give the same result. Meaning, for any point $p'$ in the interior of $\Delta$ which is not the center of mass, there exists a non-affine fractional linear map $F$ such that $F(\Delta) = \Delta$ and $F(p') = p'$. The construction is quite simple - find a linear map $A : \Delta \to \Delta$ which permutes the vertices and does not fix the point $p'$ (such a map is easily seen to exist), and then compose it with a fractional linear which fixes the vertices but “restores” $A(p')$ to $p'$ (that map will be non-affine, since the only affine map which fixes all the vertices is the identity map). This composition is the wanted map.

3 Background on order isomorphisms

Our main interest in what follows is order preserving and order reversing transforms on convex functions, when the functions are restricted to being defined on a convex body in $\mathbb{R}^n$ rather than the whole space. It turns out that this restriction changes the picture entirely, and a new family of transformations appears. These transformations are based on fractional linear maps, which we studied in detail in Section 2.
3.1 General order isomorphisms

Definition 3.1. If $S_1, S_2$ are partially ordered sets, and $T : S_1 \to S_2$ is a bijective transform, such that for every $f, g \in S_1$: $f \leq g$ $\iff$ $Tf \leq Tg$, we say that $T$ is an order preserving isomorphism.

Definition 3.2. If $S_1, S_2$ are partially ordered sets, and $T : S_1 \to S_2$ is a bijective transform, such that for every $f, g \in S_1$: $f \leq g$ $\iff$ $Tf \geq Tg$, we say that $T$ is an order reversing isomorphism.

Definition 3.3. A partially ordered set $S$ is said to be closed under supremum, if for every $\{f_\alpha\} \subseteq S$, there exists a unique element in $S$, denoted $\text{sup}\{f_\alpha\}$, with the following two properties:

1. For every $\alpha$, $f_\alpha \leq \text{sup}\{f_\alpha\}$ (bounding from above).
2. If $g \in S$ also bounds $\{f_\alpha\}$ from above, then $\text{sup}\{f_\alpha\} \leq g$ (minimality).

Definition 3.4. A partially ordered set $S$ is said to be closed under infimum, if for every $\{f_\alpha\} \subseteq S$, there exists a unique element $f \in S$, with the following two properties:

1. For every $\alpha$, $f \leq f_\alpha$ (bounding from below).
2. If $g \in S$ also bounds $\{f_\alpha\}$ from below, then $g \leq f$ (maximality).

Consider the case where $S$ is a partially ordered set which contains a minimal element, and is closed under supremum. When $S$ is one of the classes of convex functions we deal with, $\text{sup}\{f_\alpha\}$ may be given by the pointwise supremum. However, the corresponding pointwise $\text{inf}\{f_\alpha\}$ operation may not give a convex function. To obtain an infimum operation (denoted $\hat{\text{inf}}$), we use the supremum operation in the following way:

$$\hat{\text{inf}}\{f_\alpha\} := \text{sup}\{g \in S : \forall \alpha \in A \ g \leq f_\alpha\}.$$

That is, $\hat{\text{inf}}\{f_\alpha\}$ is the largest element which is below the family $\{f_\alpha\}$. Using $\hat{\text{inf}}$, we see that these classes are also closed under infimum; the first property is due to the minimality of sup, and the second holds since sup is a bound from above. Dealing with convex functions, we have:

1. $\hat{\text{inf}}\{f_\alpha\} \leq \text{inf}\{f_\alpha\}$.
2. When $\text{inf}\{f_\alpha\}$ is already a convex function, $\text{inf}\{f_\alpha\} = \hat{\text{inf}}\{f_\alpha\}$.

For example, if $f$ is a convex function, then $f = \text{inf}\{\delta_{x,f(x)}\}$ (recall that $\delta_{x,c}(y) = +\infty$ for $y \neq x$, and $\delta_{x,c}(x) = c$). Thus $\text{inf}\{\delta_{x,f(x)}\} = \hat{\text{inf}}\{\delta_{x,f(x)}\} = f$.

Next we follow Proposition 2.2 from [7], which states that an order preserving isomorphism $T$ must satisfy $T(\text{sup}\{f_\alpha\}) = \text{sup}\{Tf_\alpha\}$ and $T(\hat{\text{inf}}\{f_\alpha\}) = \hat{\text{inf}}\{Tf_\alpha\}$, that is, sup and $\text{inf}$ are preserved by $T$. Similarly, an order reversing isomorphism satisfies $T(\text{sup}\{f_\alpha\}) = \text{inf}\{Tf_\alpha\}$ and $T(\hat{\text{inf}}\{f_\alpha\}) = \text{sup}\{Tf_\alpha\}$, that is, sup and $\text{inf}$ are interchanged by $T$. We will prove this lemma for the case of order isomorphisms and order reversing isomorphisms between two possibly different partially ordered sets.

Proposition 3.5. Let $S_1, S_2$ be partially ordered sets closed under supremum and infimum, and let $T : S_1 \to S_2$ be an order preserving isomorphism. Then for any family $f_\alpha \in S_1$ we have

- $T(\text{sup}\{f_\alpha\}) = \text{sup}\{Tf_\alpha\}$
- $T(\hat{\text{inf}}\{f_\alpha\}) = \hat{\text{inf}}\{Tf_\alpha\}$

Proposition 3.6. Let $S_1, S_2$ be partially ordered sets closed under supremum and infimum, and let $T : S_1 \to S_2$ be an order reversing isomorphism. Then for any family $f_\alpha \in S_1$ we have

- $T(\text{inf}\{f_\alpha\}) = \text{sup}\{Tf_\alpha\}$
- $T(\hat{\text{inf}}\{f_\alpha\}) = \text{inf}\{Tf_\alpha\}$
Both proofs are almost identical to the proof of Proposition 2.2 in \cite{7}, but we cannot apply it directly, since here the domain and image of $T$ may be different sets. Therefore we prove below only Proposition 3.5 (the proof of Proposition 3.6 follows the exact same lines).

**Proof of Proposition 3.5.** Let $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{S}_1$. Denote $f = \sup \{f_\alpha\}$, and $g$ such that $Tg = \sup \{Tf_\alpha\}$ - such $g$ exists due to surjectivity of $T$. We wish to show that $Tf = Tg$, i.e. $f = g$. Since $f \geq f_\alpha$ for all $\alpha$, we get $Tf \geq Tf_\alpha$ for all $\alpha$, thus $Tf \geq \sup \{Tf_\alpha\} = Tg$, which implies $f \geq g$. On the other hand, since $Tg \geq Tf_\alpha$ for all $\alpha$, we have $g \geq f_\alpha$ for all $\alpha$, thus $g \geq \sup \{f_\alpha\} = f$. We have seen $f \geq g$ and $g \geq f$; therefore $f = g$.

For $\inf$, denote $f = \inf \{f_\alpha\}$, and $g$ such that $Tg = \inf \{Tf_\alpha\}$. We wish to show that $Tf = Tg$, i.e. $f = g$. Since $f \leq f_\alpha$ for all $\alpha$, we get $Tf \leq Tf_\alpha$ for all $\alpha$, thus $Tf \leq \inf \{Tf_\alpha\} = Tg$, which implies $f \leq g$. On the other hand, since $Tg \leq Tf_\alpha$ for all $\alpha$, we get $g \leq f_\alpha$ for all $\alpha$, thus $g \leq \inf \{f_\alpha\} = f$. We have seen $f \geq g$ and $g \geq f$, therefore $f = g$. \qed

### 3.2 Order isomorphisms of convex functions

In a recent series of papers, the first and third named authors have crystallized the concept of duality and investigated order reversing isomorphisms (called there “abstract duality”) for various classes of objects and functions, see \cite{5}, \cite{6}. The main theorem in \cite{5} can be stated in two equivalent forms which we quote here for future reference.

Recall the Legendre transform $L$ for a function $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$; one first fixes a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ (that is, a pairing between the space and the dual space). The Legendre transform $L$ is then defined by

$$
(2) \quad (L\phi)(x) = \sup_y \{\langle x, y \rangle - \phi(y)\}.
$$

It is an involution on the class of all lower-semi-continuous convex functions on $\mathbb{R}^n$, denoted $Cvx(\mathbb{R}^n)$. More precisely, $Cvx(\mathbb{R}^n)$ consists of all convex l.s.c. functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, together with the constant $-\infty$ function.

**Theorem 3.7.** Let $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ be an order reversing involution. Then there exist $C_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}^n$ and a symmetric transformation $B \in GL_n$, such that

$$(\mathcal{T}\phi)(x) = (L\phi)(Bx + v_0) + \langle x, v_0 \rangle + C_0.$$ 

We call these two properties “abstract duality”, and so we say that on the class $Cvx(\mathbb{R}^n)$ there is, up to linear terms, only one duality transform, $L$. More generally we have:

**Theorem 3.8.** Let $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ be an order reversing isomorphism. Then, there exist $C_0 \in \mathbb{R}, C_1 \in \mathbb{R}^+, v_0, v_1 \in \mathbb{R}^n$ and $B \in GL_n$, such that

$$(\mathcal{T}\phi)(x) = C_0 + \langle v_1, x \rangle + C_1 (L\phi)(Bx + v_0)).$$

As usual, this is equivalent to the following

**Theorem 3.9.** Let $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ be an order preserving isomorphism. Then there exist $C_0 \in \mathbb{R}, C_1 \in \mathbb{R}^+, v_0, v_1 \in \mathbb{R}^n$ and $B \in GL_n$, such that

$$(\mathcal{T}\phi)(x) = C_1 \phi(Bx + v_0) + \langle v_1, x \rangle + C_0.$$ 

### 3.3 Order isomorphisms of geometric convex functions

The subclass of $Cvx(\mathbb{R}^n)$ consisting of non negative functions with $f(0) = 0$ is denoted by $Cvx_0(\mathbb{R}^n)$. Next we follow \cite{8} to define two transforms $\mathcal{J}$ and $\mathcal{A}$ on this class. Consider the following transform,
defined on $\text{Cvx}_0(\mathbb{R}^n)$:

$$(Af)(x) = \begin{cases} \sup_{y \in \mathbb{R}^n : f(y) > 0} \frac{(x,y) - 1}{f(y)} & \text{if } x \in \{f^{-1}(0)\}^\circ \\ +\infty & \text{if } x \not\in \{f^{-1}(0)\}^\circ \end{cases}.$$  

(with the convention $\sup \emptyset = 0$). One may check that it is order reversing. This transform (with its counterpart $J$ defined below) first appeared in the classical monograph [14], but remained practically unnoticed until recently. For details, a geometric description, and more, see [8]. Next define:

$$J = LA = AL.$$  

Clearly, as a composition of two order reversing isomorphisms, it is an order preserving isomorphism. The formula for $J$ can be computed (again, see [8] for details), and has the form:

$$(Jf)(x) = \inf\{r > 0 : f(x/r) \leq 1/r\},$$

with the convention $\inf \emptyset = +\infty$. It turns out that, apart from the identity transform, up to linear variants, this is the only order preserving transform on the class $\text{Cvx}_0(\mathbb{R}^n)$. It was shown in [8] that the following uniqueness theorems for $J$ hold.

**Theorem 3.10.** If $T : \text{Cvx}_0(\mathbb{R}^+) \to \text{Cvx}_0(\mathbb{R}^+)$ is an order isomorphism, then there exist two constants $\alpha > 0$ and $\beta > 0$ such that either (a-la-$I$) for every $\phi \in \text{Cvx}_0(\mathbb{R}^+)$,

$$(T\phi)(x) = \beta \phi(x/\alpha),$$

or (a-la-$J$), for every $\phi \in \text{Cvx}_0(\mathbb{R}^+)$,

$$(T\phi)(x) = \beta (J\phi)(x/\alpha).$$

In higher dimensions, it was shown that

**Theorem 3.11.** Let $n \geq 2$. Any order isomorphism $T : \text{Cvx}_0(\mathbb{R}^n) \to \text{Cvx}_0(\mathbb{R}^n)$ is either of the form $Tf = C_0 f \circ B$ or of the form $Tf = C_0 (Jf) \circ B$ for some $B \in \text{GL}_n$ and $C_0 > 0$.

It is interesting to notice, and will be quite important in the sequel, that the map (on functions) $J$ is actually induced by a point map on the epi-graphs of those functions. Indeed, one can check that for every $f \in \text{Cvx}_0(\mathbb{R}^n)$, the bijective map $F : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \times \mathbb{R}^+$ given by

$$F(x,y) = \left(\begin{array}{c} x \\ y, \frac{1}{y} \end{array}\right),$$

satisfies

$$\text{epi}(Jf) = F(\text{epi}(f)),$$

where

$$\text{epi}(f) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^+ : f(x) < y\}.$$  

See [8] for details. Moreover, we see that $F$ is actually a fractional linear map. We will get back to this issue frequently in the next two sections.

Clearly, if we have a point map which preserves the set “epi-graphs of (a certain subset of) convex functions” then it induces an order preserving transform on this subset. It is not clear that, in some cases, any order preserving transform is induced by such a point map. However, this turns out to be the case both in the theorems described above, and in all theorems in the next two sections. Let us emphasize that this is also, usually, the idea behind the proof. First one shows that the transform must be induced by some point map, and moreover, one which preserves intervals. Next one uses some theorem which classifies all interval preserving maps (for example, the fundamental theorem of affine geometry, or Theorem 2.17), and finally one checks which of these maps really induces a transform on the right class, by this getting a full classification of order preserving transforms.
3.4 Order reversing isomorphisms

Considering order reversing transforms, the situation is slightly different, since there are two different cases. The first case is when one is given a set on which there is a known order reversing transform, such as $\mathcal{L}$ on $Cvx(\mathbb{R}^n)$ or on $Cvx_0(\mathbb{R}^n)$, for example. In that case the classification of order reversing transforms is completely equivalent to the classification of order preserving ones, by composing each of them with the known transform. For example, the theorems above give the following:

**Theorem 3.12.** Let $n \geq 2$. If $T : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ is an order reversing involution, then $T$ is of the form $Tf = (\mathcal{L}f) \circ B + C_0$, for some symmetric $B \in GL_n$ and $C_0 \in \mathbb{R}$.

**Theorem 3.13.** Let $n \geq 2$. If $T : Cvx_0(\mathbb{R}^n) \to Cvx_0(\mathbb{R}^n)$ is an order reversing involution, then $T$ is either of the form $Tf = (\mathcal{L}f) \circ B$, or of the form $Tf = C_0(Af) \circ B$, for some symmetric $B \in GL_n$ and $C_0 > 0$.

However, there exists a second case in which there is no order reversing transform and this requires a different treatment, since one cannot use the above mentioned strategy, and is forced to find the real obstruction for the existence of such a transform (see [7] for examples). In Section 4.5 we deal with order reversing isomorphisms on $Cvx(K)$, and show that when $K \neq \mathbb{R}^n$, there are no such transforms.

4 The cone of convex functions on a window

4.1 Introduction

We investigate the question of characterizing order isomorphisms on convex functions, when the domain of the functions is not the whole of $\mathbb{R}^n$ but a convex subset. One such example which has already been studied (see [8]) is the case of geometric convex functions on $\mathbb{R}^+$. Since this example is central also for our setting, we describe it in detail below. First, let us recall the following definition:

**Definition 4.1.** The class of all lower-semi-continuous convex functions $f : K \to \mathbb{R} \cup \{\infty\}$ together with the constant $-\infty$ function on $K$ will be denoted $Cvx(K)$. It can be naturally embedded into $Cvx(\mathbb{R}^n)$ by assigning to $f$ the value $+\infty$ outside $K$.

We often call $K$ a window, on which we observe the functions of $Cvx(\mathbb{R}^n)$. Our first results regard a description of order isomorphisms on the class of convex functions defined on a window. We state two versions, one of which does not assume surjectivity, but in which the order preservation condition is replaced by a slightly stronger condition of preservation of supremum and generalized infimum.

**Theorem 4.2.** Let $n \geq 1$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior. If $T : Cvx(K_1) \to Cvx(K_2)$ is an order preserving isomorphism, then there exists a bijective fractional linear map $F : K_1 \times \mathbb{R} \to K_2 \times \mathbb{R}$, such that $T$ is given by

$$epi(Tf) = F(epi(f)).$$

In particular, $K_2$ is a fractional linear image of $K_1$.

**Theorem 4.3.** Let $n \geq 1$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior. If $T : Cvx(K_1) \to Cvx(K_2)$ is an injective transform satisfying:

1. $T(sup_\alpha f_\alpha) = sup_\alpha Tf_\alpha$.
2. $T(\hat{inf}_\alpha f_\alpha) = \hat{inf}_\alpha Tf_\alpha$.

21
for any family \( \{f_a\} \subseteq Cvx(K_1) \), then there exist \( K'_2 \subseteq K_2 \), and a bijective fractional linear map \( F : K_1 \times \mathbb{R} \rightarrow K'_2 \times \mathbb{R} \), such that \( T \) is given by

\[
\text{epi}(Tf) = F(\text{epi}(f)).
\]

Note that for \( x \not\in K'_2 \) we get \( (Tf)(x) = +\infty \).

Remark 4.4. Let us elaborate on the meaning of the equation \( epi(Tf) = F(epi(f)) \). When \( F \) induces a transform on \( Cvx(K) \), it is shown in Section 4.4 that up to some affine linear functional \( L_1 \), \( F \) is of the form

\[
F(x, y) = \left( \frac{Ax + u}{v, x} + d \right) \left( \frac{y}{v, x} + d \right),
\]

where \( A \in L_n(\mathbb{R}) \), \( u, v \in \mathbb{R}^n \), and \( d \in \mathbb{R} \). Denoting \( L_0 = \langle v, \cdot \rangle + d \) for the affine linear functional in the denominator, and \( F_b(x) = \frac{Ax + u}{v, x} \) for the base-map (projection of \( F \) to the first \( n \) coordinates), we conclude that

\[
(Tf) = \left( \frac{f}{L_0} \right) \circ F_b^{-1} + L_1,
\]

where \( L_1 \) is some affine linear functional and \( F_b^{-1} : K_2 \rightarrow K_1 \) is bijective. Note that \( L_0 \) and \( F_b \) are not independent, since \( L_0 \) must vanish on the defining hyperplane of \( F_b \) (where \( F_b \) is not defined). Moreover, note that for a general \( f \), the function \( F_b^{-1} \) may not be convex, but the composition with \( F_b^{-1} \) exactly compensates this problem, and the result is again a convex function. In the special case of \( A = I, u = 0, L_0(x) = x_1 + 1, L_1(x) \equiv 0 \) we get \( F(x, y) = (\frac{x}{x_1+1}, \frac{y}{x_1+1}) \), and \( (Tf)(x) = (1- x_1)f(\frac{x}{x_1+1}) \). This simpler form of the transform is not general, but if one allows linear actions on the epi-graphs, before and after \( F \) acts on them, it suffices to consider this form. There is another important, different, instance of the equation \( epi(Tf) = F(epi(f)) \), which may occur when the transform is defined on the subset of \( Cvx(K) \) consisting of non-negative functions vanishing at the origin. We state it now for comparison and elaborate below (Theorem 5.2). A transform of this second, essentially different, type (a-la-\( \mathcal{J} \), see [8]), corresponds to the inducing fractional linear map:

\[
F_{\mathcal{J}}(x, y) = (\frac{x}{y}, \frac{1}{y}),
\]

and to the explicit formula:

\[
(\mathcal{J}f)(x) = \inf\{r > 0 : rf(\frac{x}{r}) \leq 1\}.
\]

4.2 The bijective case

Proof of Theorem 4.2. The proof is composed of several steps.

Extremality of delta functions. As in [8], we define the following family \( P \) of extremal functions: \( f \in P \) if every two functions above \( f \) are comparable, that is:

\[
f \leq g, h \quad \Rightarrow \quad g \leq h \quad \text{or} \quad h \leq g.
\]

This implies that the support of \( f \) (the set on which \( f \) is finite) consists of only one point. We call these functions \textbf{delta functions}, and denote by \( \delta_{x,c} \) the function which equals \( c \) at the point \( x \), and \( +\infty \) elsewhere.

\( T \) is a bijection between the family \( P \) in \( Cvx(K_1) \) and the family \( P \) in \( Cvx(K_2) \), since this property is defined only using the “\( \leq \)” relation, which \( T \) preserves in both directions. Thus \( T(\delta_{x,c}) = \delta_{x',c} \).
δ_{y,d}, and this map between delta functions is bijective. This allows us to define a bijection $F : K_1 \times \mathbb{R} \to K_2 \times \mathbb{R}$; $F(x,c) = (y(x,c), d(x,c))$, such that $T(\delta_{x,c}) = \delta_{F(x,c)}$. In fact, we get that $y = y(x)$ and $d = d(x,c)$ because two functions $\delta_{x,c}$ and $\delta_{x,c'}$ are comparable, and so must be mapped to comparable functions. Note that also $y(x)$ is bijective. Indeed, it is injective since the images of two functions are comparable if, and only if, the original functions are comparable, and it is surjective since all delta functions are in the image of $T$.

**Preservation of intervals.** The “projection” of $F$ to the first $n$ coordinates, i.e. the mapping $x \mapsto y(x)$, is a bijective interval preserving map. Indeed, assume $y(x_1) = y_1$, $y(x_2) = y_2$, and $x_1 \in [x_1, x_2]$. Since $\delta_{x_3,0} \geq \inf\{\delta_{x_1,0}, \delta_{x_2,0}\}$, the function $\delta_{x_3,0}$ must be mapped to a function $\delta_{y_3,d_3}$ which is above $\inf\{\delta_{y_1,d_1}, \delta_{y_2,d_2}\}$. Since $\inf\{\delta_{y_1,d_1}, \delta_{y_2,d_2}\}$ is $+\infty$ outside $[y_1, y_2]$, this implies $y_3 \in [y_1, y_2]$. For $n \geq 2$, it implies that $y(x)$ is fractional linear, by Theorem 2.17. In fact this is true also when $n = 1$, but for $n = 1$ it follows from interval preservation of $F$ itself. To see that $F$ is interval preserving, consider $(x_3, c_3)$ on the interval between $(x_1, c_1)$ and $(x_2, c_2)$. We know it is mapped to $(y_3, d_3)$ with $y_3 \in [y_1, y_2]$ and moreover, letting $y_3 = \lambda y_1 + (1 - \lambda)y_2$, we know $d_3 \geq \lambda d_1 + (1 - \lambda)d_2$. Using surjectivity, we deduce that $F(x_3, c_3) = \delta_{y_3, c_3}$ is above the function $\inf\{\delta_{y_1, c_1}, \delta_{y_2, c_2}\}$. Therefore we get that $y(x)$ is fractional linear, by Theorem 2.17. In fact this is true also when $n = 1$, but for $n = 1$ it follows from interval preservation of $F$ itself. To see that $F$ is interval preserving, consider $(x_3, c_3)$ on the interval between $(x_1, c_1)$ and $(x_2, c_2)$. We know it is mapped to $(y_3, d_3)$ with $y_3 \in [y_1, y_2]$ and moreover, letting $y_3 = \lambda y_1 + (1 - \lambda)y_2$, we know $d_3 \geq \lambda d_1 + (1 - \lambda)d_2$. Using surjectivity, we deduce that $F(x_3, c_3) = \delta_{y_3, c_3}$ is above the function $\inf\{\delta_{y_1, c_1}, \delta_{y_2, c_2}\}$.

Since $F$ is an injective interval preserving map, we may apply Theorem 2.17, to conclude that $F$ is a fractional linear map.

To complete the proof of Theorem 4.2, let $f \in Cvx(K_1)$, and write it as

$$f = \inf\{\delta_{x,y} : (x, y) \in \text{epi}(f)\}.$$ 

Then

$$\Rightarrow T f = \hat{\text{inf}}\{T(\delta_{x,y}) : (x, y) \in \text{epi}(f)\}$$

$$= \hat{\text{inf}}\{\delta_{F(x,y)} : (x, y) \in \text{epi}(f)\}$$

$$= \hat{\text{inf}}\{\delta_{x,y} : (x, y) \in \text{F(epi(f))}\}.$$ 

On the other hand:

$$T f = \hat{\text{inf}}\{\delta_{x,y} : (x, y) \in \text{epi}(T f)\}.$$ 

Therefore we get

$$\text{epi}(T f) = F(\text{epi}(f)),$$

as desired. This completes the proof.

Of course, there are restrictions on the structure of $F$ for it to induce such a transform. This is elaborated in Section 4.4.

### 4.3 The injective case

We next move to the case of injective transforms. Let us first remark why in Theorem 4.3 we had to change the conditions from mere order preservation to preservation of sup and inf.

**Remark 4.5.** In the bijective case, order preservation (in both directions) is equivalent to preservation of sup and inf. One direction is given in Proposition 3.5, and the other is given here:

$$f \preceq g \quad \Rightarrow \quad \mathcal{T}(g) = \mathcal{T}(\text{sup}\{f, g\}) = \text{sup}\{\mathcal{T}(f), \mathcal{T}(g)\} \quad \Rightarrow \quad \mathcal{T}(f) \preceq \mathcal{T}(g),$$

$$\mathcal{T}(f) \preceq \mathcal{T}(g) \Rightarrow \mathcal{T}(g) = \text{sup}\{\mathcal{T}(f), \mathcal{T}(g)\} = \mathcal{T}(\text{sup}\{f, g\}) \Rightarrow g = \text{sup}\{f, g\} \Rightarrow f \preceq g.$$

This direction is true also in the injective case (preservation of sup and $\hat{\text{inf}}$ implies order preservation), but the opposite (order preservation in both directions implies preservation of sup and inf) is not, as shown in the following example. The following $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ is injective and $f \leq g$ if and only if $\mathcal{T} f \leq \mathcal{T} g$:

$$(\mathcal{T} f)(x) = f(x) + x_1^2.$$
But $\mathcal{T}$ does \textit{not} map \text{\inf} to \text{\inf}. The reason behind this fact is that $\mathcal{T}$ is not surjective. Moreover, there exist $f, g$, such that $\text{\inf}\{\mathcal{T}(f), \mathcal{T}(g)\}$ is not in the image of $\mathcal{T}$, and in particular it is not equal to $\mathcal{T}(\text{\inf}\{f, g\})$; for example take $f(x) = x_1, g(x) = -x_1$.

For the proof of the more general Theorem 4.3, we need the following known geometric lemma. The dimension of a set $K$ denotes the minimal dimension of an affine subspace which contains the set.

\textbf{Lemma 4.6.} In an $m$-dimensional affine space, let $M$ be a closed convex set. Let $\mathcal{F}$ be a family of $m$-dimensional closed convex sets such that $K \neq M$ for all $K \in \mathcal{F}$, and $K_1 \cap K_2 = M$ whenever $K_1 \neq K_2$ and $K_1, K_2 \in \mathcal{F}$. Then $\mathcal{F}$ is at most countable.

We reformulate it, to better suit our need:

\textbf{Lemma 4.7.} Let $M \subseteq \mathbb{R}^n$ be a fixed closed convex set of dimension $m$. Let $\mathcal{F}$ be an uncountable family of closed convex sets such that $K \neq M$ for all $K \in \mathcal{F}$, and $K_1 \cap K_2 = M$ whenever $K_1 \neq K_2$ and $K_1, K_2 \in \mathcal{F}$. Then for at least one set $K \in \mathcal{F}$, $\dim(K) \geq m + 1$. In particular, $m \leq n - 1$.

Lemma 4.7 follows from Lemma 4.6, where the minimal subspace which contains $M$ is taken to be the $m$-dimensional affine space of Lemma 4.6. Our application of this lemma requires a little more, so we prove:

\textbf{Lemma 4.8.} Let $M \subseteq \mathbb{R}^n$ be a fixed closed convex set of dimension $m$. Let $\mathcal{F}$ be an uncountable family of closed convex sets such that $K \neq M$ for all $K \in \mathcal{F}$, and $K_1 \cap K_2 = M$ whenever $K_1 \neq K_2$ and $K_1, K_2 \in \mathcal{F}$. Then for at least one set $K \in \mathcal{F}$, $\dim(K) \geq m + 1$. Moreover, $m \leq n - 2$.

\textbf{Proof.} We wish to prove that $m \neq n - 1$; the rest follows from Lemma 4.7. Assume otherwise, then let $H = \{(x, u) = c\}$ be the affine subspace of dimension $n - 1$ which contains $M$. Our assumption is that the relative interior of $M$ in $H$ is not empty. The set $\{K \in \mathcal{F} : K \subseteq H\}$ is at most countable, by Lemma 4.6. Since $\mathcal{F}$ is not countable, there are at least three sets which are not contained in $H$, and therefore (without loss of generality) we have $A, B \in \mathcal{F}$ such that $A \cap H^+ \neq \emptyset$, $B \cap H^+ \neq \emptyset$, where $H^+ := \{(x, u) > c\}$. Let $a \in A, b \in B$ such that $a, b \in H^+$, and let $x \in M$ be a point in the relative interior of $M$. Since $\text{conv}(M, a) \subseteq A$, we conclude that there is some open half ball of the form $B(x, r) \cap H^+$ contained in $A$, and likewise for $B$. The two half balls have non empty intersection, in contradiction to $A \cap B = M$. \hfill \Box

We will use this lemma for epi-graphs of functions. Noting that $\text{epi}(\text{\max}\{f, g\}) = \text{epi}(f) \cap \text{epi}(g)$, we get the following lemma for convex functions:

\textbf{Lemma 4.9.} Let $M : \mathbb{R}^n \to \mathbb{R}$ be a fixed convex function, such that $\text{epi}(M) \subseteq \mathbb{R}^{n+1}$ is of dimension $m$. Let $\mathcal{F}$ be an uncountable family of convex functions such that $f \prec M$ for all $f \in \mathcal{F}$, and $\text{\max}\{f_1, f_2\} = M$ whenever $f_1, f_2 \in \mathcal{F}$ and $f_1 \neq f_2$. Then for at least one function $f \in \mathcal{F}$, $\dim(\text{epi}(f)) \geq m + 1$. Moreover, $m \leq n - 1$.

\textbf{Proof of Theorem 4.3.} We start by checking where the constant function $+\infty$ is mapped to. Let us call its image $f_{\infty}$. Consider the family $\{\delta_{x}\}_{x \in K_1}$, and its image $\{\mathcal{T} \delta_{x}\}_{x \in K_1}$. It is uncountable, and every two functions in the second family satisfy $\text{\max}\{g_1, g_2\} = f_{\infty}$.

This means, by Lemma 4.9, that there exists $x_1 \in K_1$ such that the dimension of the epi-graph of $\mathcal{T} \delta_{x_1}$ must be higher by at least 1 than the dimension of the epi-graph of $f_{\infty}$. Similarly, for $x_1$ we construct an uncountable family of functions $\{\delta_{x_1, y}\}_{y \in K_1}$ such that the maximum of every two is $\delta_{x_1}$, and by applying Lemma 4.9 again we get that there exists at least one such function, the image of which has an epi-graph with dimension higher by at least 1 than the dimension of the epi-graph.
of $\mathcal{T}\delta_{x_1}$. After repeating this construction an overall of $n - 1$ times, we conclude that there exist $x_1, \ldots, x_{n-1} \in K_1$ such that the epi-graph of the function $\mathcal{T}\delta_{\text{conv}}(x_1, \ldots, x_{n-1})$ is of dimension higher by at least $n - 1$ than the dimension of the epi-graph of $f_\infty$. Applying Lemma 4.9 one last time, we get that the dimension of the epi-graph of $\mathcal{T}\delta_{\text{conv}}(x_1, \ldots, x_{n-1})$ is at most $(n + 1) - 2 = n - 1$. This means that the epi-graph of $f_\infty$ is of dimension 0, that is, $f_\infty = +\infty$.

This also shows that $\mathcal{T}(\delta_{x,c}) = \delta_{y,d}$. Indeed, since the only epi-graph with dimension 0 has already been designated to $f_\infty$, the dimension of the epi-graph of $\mathcal{T}(\delta_{x,c})$ is at least 1; but we may construct a chain as above which implies that it is also at most 1. We define the injective map $F : K_1 \times \mathbb{R} \to K_2 \times \mathbb{R}$ by the relation $\mathcal{T}(\delta_{x,c}) = \delta_{F(x,c)}$, and denote $F(x,c) = (y(x,c), d(x,c))$.

In fact, we get that $y = y(x)$ and $d = d(x,c)$ because the two functions $\delta_{x,c}$ and $\delta_{x,c'}$ are comparable, and so must be mapped to comparable functions (by Remark 4.5). Note that $y(x)$ is injective because the images of two functions are comparable if and only if the original functions are comparable. In addition, $y(x)$ is interval preserving. Indeed, assume $y(x_1) = y_1$, $y(x_2) = y_2$, and $x_3 \in [x_1, x_2]$. Since $\delta_{x_3} \geq \inf\{\delta_{x_1}, \delta_{x_2}\}$, the function $\delta_{x_3}$ must be mapped to a function $\delta_{y_3}$ which is above $\inf\{\delta_{y_1, c_1}, \delta_{y_2, c_2}\}$, which implies $y_3 \in [y_1, y_2]$. For $n \geq 2$, the fact that $y(x)$ is an injective interval preserving map implies that it is fractional linear, by Theorem 2.17. Actually this is true also for $n = 1$, but it only follows from the fact that $(x, c) \mapsto (y, d)$ is also interval preserving, which we will next show.

**Remark.** We note that until this point in the proof (for $n \geq 2$) we only use the max/min condition, and not the stronger assumed condition for sup/inf; we already get that the map $F$ is very restricted: it is a fractional linear map on the base, and some one dimensional map $d_s(c)$ on each fiber, and all these maps $d_s$ must join together to preserve convexity of epi-graphs. This seems to restrict $d(x,c)$ enough to determine its form, but we chose to continue using a different argument, which works also for $n = 1$, but requires the preservation of sup/inf.

To see that $F$ is interval preserving consider the function $\min\{\delta_{x_1, c_1}, \delta_{x_2, c_2}\}$, which is $+\infty$ outside the interval $[x_1, x_2]$ and linear in it, with $f(x_1) = c_1$ and $f(x_2) = c_2$. By assumption, it is mapped to $\min\{\delta_{y_1, d_1}, \delta_{y_2, d_2}\}$. Taking $(x_3, c_3) \in [(x_1, c_1), (x_2, c_2)]$ we have that $\delta_{x_3, c_3} \geq \min\{\delta_{y_1, d_1}, \delta_{y_2, d_2}\}$ and so the point $(y_3, d_3)$ lies above or on the segment $[(y_1, d_1), (y_2, d_2)]$.

On the other hand, look at $x_3 = \lambda x_1 + (1 - \lambda)x_2$ and $c_3 < \lambda c_1 + (1 - \lambda)c_2$. That is, we take a point $(x_3, c'_3)$ which is under the segment $[(x_1, c_1), (x_2, c_2)]$. From the “only if” condition, we have that $\mathcal{T}(\delta_{x_3, c'_3}) \geq \min\{\delta_{x_1, c_1}, \delta_{x_2, c_2}\}$. So $(y_3, d'_3)$ is under the segment $[(y_1, d_1), (y_2, d_2)]$, since $y_3 \in [y_1, y_2]$ and it cannot be above or on it. Since $\delta_{y_3, c_3} = \sup\{\delta_{y_1, c_1}, \delta_{y_2, c_2}\}$, we may use the condition of supremum to get $d_3 = \sup\{d'_3\}$, and thus $(y_3, d_3)$ is below or on the segment $[(y_1, d_1), (y_2, d_2)]$. Together with what we saw before, this implies $(y_3, d_3) \in [(y_1, d_1), (y_2, d_2)]$.

So, we have shown that $F : K_1 \times \mathbb{R} \to K_2 \times \mathbb{R}$ is an injective interval preserving map, and we may apply Theorem 2.17 to conclude that it is fractional linear.

To complete the proof of Theorem 4.3, we proceed in exactly the same way as in the proof of Theorem 4.2, to conclude that

\[
\mathcal{T}f = \hat{\inf}\{\delta_{x,y} : (x,y) \in epi(f)\}
= \inf\{\delta_{x,y} : (x,y) \in epi(\mathcal{T}f)\},
\]

and thus

\[epi(\mathcal{T}f) = F(epi(f)),\]

which completes the proof. \qed

Both proofs generalize without any complication to various other settings in which one considers different classes, such as the class of all non negative functions in $Cvx(\mathbb{R}^n)$, or in $Cvx(K)$, or more generally:

\[S_{f_0} = Cvx(\mathbb{R}^n) \cap \{f : f_0 \leq f\},\]

for some fixed $f_0 \in Cvx(\mathbb{R}^n)$. We get:
Theorem 4.10. Let \( n \geq 1 \), and let \( f_1, f_2 \in \text{Cvx}(\mathbb{R}^n) \) be convex functions with support of full dimension. If \( T : S_{f_1} \to S_{f_2} \) is an order isomorphism, then there exists a bijective fractional linear map \( F : \text{epi}(f_1) \to \text{epi}(f_2) \), such that \( T \) is given by

\[
\text{epi}(Tf) = F(\text{epi}(f)).
\]

Theorem 4.11. Let \( n \geq 1 \), and let \( f_1, f_2 \in \text{Cvx}(\mathbb{R}^n) \) be convex functions with support of full dimension. If \( T : S_{f_1} \to S_{f_2} \) is an injective transform satisfying:

1. \( T(\sup_\alpha f_\alpha) = \sup_\alpha T f_\alpha \).
2. \( T(\hat{\inf}_\alpha f_\alpha) = \hat{\inf}_\alpha T f_\alpha \).

for any family \( \{f_\alpha\} \subseteq S_{f_1} \), then there exist \( f'_2 \in S_{f_2} \), and a bijective fractional linear map \( F : \text{epi}(f_1) \to \text{epi}(f'_2) \), such that \( T \) is given by

\[
\text{epi}(Tf) = F(\text{epi}(f)).
\]

It is tempting to consider Theorems 4.2 and 4.3 as manifestations of Theorems 4.10 and 4.11, where \( f_i \) is the function which attains only the values \(-\infty\) on \( K_i \) and \(+\infty\) outside \( K_i \). The only problem is that these functions are not elements of \( \text{Cvx}(\mathbb{R}^n) \), but in fact Theorems 4.10 and 4.11 can be further generalized without any effort. Instead of considering only classes of the form \( S_{f_0} = \{ f \in \text{Cvx}(\mathbb{R}^n) : \text{epi}(f) \subseteq \text{epi}(f_0) \} \), consider also \( \{ f \in \text{Cvx}(\mathbb{R}^n) : \text{epi}(f) \subseteq K \} \), where \( K \) is some convex set (in the case of Theorems 4.2 and 4.3, \( K \) is the infinite cylinder \( K_1 \times \mathbb{R} \)).

4.4 Classification of admissible fractional linear maps

Since fractional linear maps send intervals to intervals, it is clear (a-posteriori, once we know the transform is induced by a fractional linear map) that a delta function \( \delta_{y,d} \) is mapped to a delta function \( \delta_{y,d'} \); since these are the only functions with epi-graphs that are half-lines. Moreover, by order preservation, we see that \( y \) is a function only of \( x \). Observations of this kind allow us to classify the type of fractional linear maps that induce transforms as in Theorem 4.2.

Let the inducing matrix \( A_F \in GL_{n+2} \) be given by

\[
A_F = \begin{pmatrix}
  u_1' & u_1 \\
  A & \vdots & \vdots \\
  u_n' & u_n \\
  v_1' & \cdots & v_n' & a & b \\
  v_1 & \cdots & v_n & c & d
\end{pmatrix},
\]

where \( A \) is an \( n \times n \) matrix, \( v, v', u, u' \in \mathbb{R}^n \), and \( a, b, c, d \in \mathbb{R} \).

The infinite cylinder \( K_1 \times \mathbb{R} \) is contained in the domain of \( F \), so it must not intersect the defining hyperplane \( H = \{ \langle x, x \rangle + cy = -d \} \), which implies \( c = 0 \). In particular, \( K_1 \subseteq \{ \langle x, x \rangle > -d \} \) (the sign of the denominator is constant on \( \text{dom}(F) \), and we choose it to be positive; we may do so due to the multiplicative degree of freedom in the choice of \( A_F \)).

Since \( F \) must map fibers \( \{ (x, y) : y \in \mathbb{R} \} \) to fibers, we see that for \( i = 1, \ldots, n \), \( F((x, y))_i = \left(\frac{A x + y u' + u}{(v, x) + d} \right)_i \) does not depend on \( y \), which implies \( u' = 0 \).

Let \( f \in \text{Cvx}(K_1) \). The image of \( \text{epi}(f) \) must be the epi-graph of some \( g \in \text{Cvx}(K_2) \). Since we have chosen a positive sign for the denominator, this simply means that \( a > 0 \), and we choose \( a = 1 \), thus exhausting the multiplicative degree of freedom in the choice of \( A_F \).
Finally, let $F'$ be the map corresponding to the following $(n+1) \times (n+1)$ matrix, having removed the next to last row and column from $A_F$:

$$A_F' = \begin{pmatrix} A & u \\ v^T & d \end{pmatrix}. $$

The map $F' : K_1 \rightarrow K_2$ is fractional linear, and corresponds to the action of $F$ on fibers (the “projection” of $F$ to $\mathbb{R}^n$). Thus $A_F'$ must be invertible. We note that this condition always holds; we have $A_F \in \text{GL}_{n+2}$, and since the $(n+1)^{th}$ column of $A_F$ is $e_{n+1}$, $\det(A_F') = \pm \det(A_F) \neq 0$.

We claim that these restrictions are not only necessary but also sufficient:

**Proposition 4.12.** Let $K_1 \subseteq \mathbb{R}^n$ be a convex set with interior, for $n \geq 1$. Let $A$ be an $n \times n$ matrix, $u, v, v' \in \mathbb{R}^n$, $b, d \in \mathbb{R}$, and let $F, F'$ be the fractional linear maps defined by the following matrices:

$$A_F = \begin{pmatrix} 0 \\ A \\ \vdots \\ u \\ 0 \\ v^T \end{pmatrix}, \quad A_F' = \begin{pmatrix} A \\ u \\ \vdots \\ v^T \end{pmatrix}. $$

If the following two conditions are satisfied:

1. $K_1 \subseteq \{ \langle v, x \rangle > -d \}$.
2. $A_F' \in \text{GL}_{n+1}$, or equivalently $A_F \in \text{GL}_{n+2}$.

then $F$ induces an order isomorphism from $Cvx(K_1)$ to $Cvx(K_2)$ by its action on epi-graphs, where $K_2 = F'(K_1)$.

**Proof.** The following four conditions must be checked: that epi-graphs are mapped to epi-graphs, that convexity of the functions is preserved under the transform, that it is bijective, and that it is order preserving. Bijectivity and convexity preservation follow easily by the bijectivity and interval preservation properties of fractional linear maps, and order preservation is immediate for transforms induced by a point map. The fact that epi-graphs are mapped to epi-graphs follows from the zeros in the $(n+1)^{th}$ (next to last) column of $A_F$.

Denote the map from the fiber above $x_1$ to the fiber above $F'(x_1) = x_2$ by $F_{x_1} : \mathbb{R} \rightarrow \mathbb{R}$. It is an affine linear map, given by

$$F_{x_1}(y) = \frac{\langle v', x_1 \rangle + y + b}{\langle v, x_1 \rangle + d}. $$

**Remark 4.13.** Letting $x_2 = F'(x_1)$ we get

$$(Tf)(x_2) = F_{x_1}(f(x_1)). $$

Note that there is a sort of coupling between the “projected” map $F'$, which determines the $x \in \mathbb{R}^n$ dependency, and $F_{x_1}$, which determines the $y$ dependency. More precisely: given $F'$, the transform induced by $F$ is determined, up to multiplication by a positive scalar, and addition of an affine linear function. We next show that the linear part is determined by $v'$ and $b$. Consider a transform $T$ induced by a map $F$, where

$$A_F = \begin{pmatrix} 0 \\ A \\ \vdots \\ u \\ 0 \\ v^T \end{pmatrix}. $$

27
Next, consider the transform: $(\tilde{T}f)(x) = (Tf)(x) + \langle x, w \rangle + \epsilon$, induced by a map $\tilde{F}$, where $w \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}$. As before, denote

$$A_{\tilde{F}} = \begin{pmatrix} 0 & \tilde{A} & \tilde{u} \\ \tilde{v}^T & 0 & \tilde{b} \\ \tilde{v}^T & 0 & \tilde{d} \end{pmatrix}.$$ 

Then $A = \tilde{A}, u = \tilde{u}, v = \tilde{v}$, and $d = \tilde{d}$. The only difference is in the next to last row, namely $\nu'$ and $b$, and a simple calculation shows that

$$\begin{pmatrix} \tilde{v} \\ \tilde{b} \end{pmatrix} = \begin{pmatrix} A^T & v \\ u^T & d \end{pmatrix} \begin{pmatrix} w \\ \epsilon \end{pmatrix}.$$ 

The matrix appearing above is exactly $A_{\tilde{F}}^T$, so it is invertible, and therefore, the set of all $v,b$ corresponds exactly to the set of all affine linear additions to $T$ (clearly these affine additions do not harm the properties of order preservation, bijectivity, etc.).

### 4.5 Order reversing isomorphisms

The Legendre transform $\mathcal{L} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$, is the unique order reversing isomorphism on $Cvx(\mathbb{R}^n)$. The corresponding question for windows is, given $K_1, K_2 \subseteq \mathbb{R}^n$, what are all the possible order reversing isomorphisms between $Cvx(K_1)$ and $Cvx(K_2)$? It turns out that there are no such order reversing isomorphisms, except in the aforementioned case where $K_1 = K_2 = \mathbb{R}^n$. This is due to the fact that the delta functions “have nowhere to be mapped to”. We formulate this simple observation in the following Proposition 4.17. To this end we use the following two definitions.

**Definition 4.14.** Let $P_K \subset Cvx(K)$ denote the following subset of extremal functions:

$$P_K := \{ f \in Cvx(K) : g, h \geq f \Rightarrow g, h \text{ are comparable} \}.$$ 

**Definition 4.15.** Let $Q_K \subset Cvx(K)$ denote the following subset of extremal functions (dual to $P$):

$$Q_K := \{ f \in Cvx(K) : g, h \leq f \Rightarrow g, h \text{ are comparable} \}.$$ 

Recall that in this new notation, for any closed convex $K$ (actually, for any $K \subseteq \mathbb{R}^n$), $P_K$ consists exactly of the delta functions. In $Cvx(\mathbb{R}^n)$, it is clear that $Q_{\mathbb{R}^n}$ consists of linear functions; it follows from the fact that the only functions below $f = \langle c, x \rangle + d$ are of the form $g(x) = \langle c, x \rangle + d'$, for $d' < d$. In the next lemma we see that when $K \neq \mathbb{R}^n$ is a convex set with non empty interior, $Q_K = \emptyset$.

**Lemma 4.16.** If $K \subseteq \mathbb{R}^n$ is a convex set with non empty interior, then $Q_K = \emptyset$.

**Proof.** Clearly, if $f$ is a non linear convex function, $f \notin Q$ (take two hyperplanes supporting $epi(f)$ in different directions). For a linear function $f$, one may easily construct two non-parallel linear functions below it, which are not comparable (they will satisfy $g(x), h(x) \leq f(x)$ for every $x \in K$, not for every $x \in \mathbb{R}^n$). Note that the fact that $K$ has non empty interior is essential, otherwise there is no guarantee that the functions will differ on $K$, as demonstrated by the example of $K$ being a subspace.

We have shown in the proof of Theorem 4.2 that an order preserving isomorphism $T : Cvx(K_1) \to Cvx(K_2)$ defines a bijection from $P_{K_1}$ to $P_{K_2}$. Similarly, an order reversing isomorphism defines a bijection from $P_{K_1}$ to $Q_{K_2}$ (and from $Q_{K_1}$ to $P_{K_2}$, of course), which is why we say $Q$ is “dual” to $P$.
**Proposition 4.17.** Let $n \geq 1$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior, such that either $K_1 \neq \mathbb{R}^n$ or $K_2 \neq \mathbb{R}^n$. Then there does not exist any order reversing isomorphism $\mathcal{T} : \text{Cvx}(K_1) \to \text{Cvx}(K_2)$.

**Proof of Proposition 4.17.** Without loss of generality, assume $K_2 \neq \mathbb{R}^n$ (otherwise consider $\mathcal{T}^{-1}$). Let $x \in K_1$, then $\delta_{x,0} \in \mathcal{P}_{K_1}$. Therefore $\mathcal{T}(\delta_{x,0}) \in \mathcal{Q}_{K_2}$, which contradicts the conclusion of Lemma 4.16. \hfill \square

5 Geometric convex functions on a window

Recall the definition of geometric convex functions on a window:

**Definition 5.1.** For a convex set $K \subseteq \mathbb{R}^n$ with $0 \in K$, the subclass of $\text{Cvx}(K)$ containing non negative functions satisfying $f(0) = 0$ is called the class of geometric convex functions, and denoted by $\text{Cvx}_0(K)$, i.e.

$$\text{Cvx}_0(K) = \{ f \in \text{Cvx}(K) : f \geq 0, f(0) = 0 \}.$$  

It is naturally embedded in $\text{Cvx}_0(\mathbb{R}^n)$ by assigning to $f$ the value $+\infty$ outside $K$. Therefore an equivalent definition is

$$\text{Cvx}_0(K) = \{ f \in \text{Cvx}(\mathbb{R}^n) : 1_K \leq f \leq 1_{\{0\}} \}$$

where $1_K$ denotes the convex indicator function of $K$, which is zero on $K$ and $+\infty$ elsewhere, and similarly $1_{\{0\}}$. Note that these functions are usually denoted by $1_K^\circ$, however, we never use in this paper the standard characteristic functions, so this notation can not lead to a misunderstanding.

In this section we deal with order isomorphisms from $\text{Cvx}_0(K_1)$ to $\text{Cvx}_0(K_2)$, where $K_i$ are convex sets (containing 0, of course), and some generalizations of these classes.

As the example of $J$ in $\text{Cvx}_0(\mathbb{R}^n)$ (which was discussed in Section 3.3) shows us, the case of $\text{Cvx}_0(K)$ is more involved than $\text{Cvx}(K)$, and a transform can be more complicated than a mere fractional linear change in the domain with the corresponding change in the fiber. Indeed, here we know already of an example where an indicator function is not mapped to such.

However, for the cases of $K = \mathbb{R}^+$ and $K = \mathbb{R}^n$ we do have theorems of the sort, see Theorem 3.10 and Theorem 3.11. There, the transform is given by a fractional linear point map on the epi-graphs. In each of these cases we observe two different types of behavior; one where fibers are mapped to fibers (a-la-$I$), and one when they are not (a-la-$J$).

In this section we generalize these theorems to apply to an order isomorphism $\mathcal{T} : \text{Cvx}_0(K_1) \to \text{Cvx}_0(K_2)$, for convex domains $K_1, K_2$.

**Theorem 5.2.** Let $n \geq 2$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior. If $\mathcal{T} : \text{Cvx}_0(K_1) \to \text{Cvx}_0(K_2)$ is an order preserving isomorphism, then there exists a bijective fractional linear map $F : K_1 \times \mathbb{R}^+ \to K_2 \times \mathbb{R}^+$, such that $\mathcal{T}$ is given by

$$\text{epi}(\mathcal{T}f) = F(\text{epi}(f)).$$

The case $n = 1$ is slightly different since the two domains $\mathbb{R}^+$ and $\mathbb{R}^-$ do not interact. Other than that, the result is the same, for example see Theorem 5.7.

**Remark 5.3.** Of course, it is not true that every fractional linear map on $K_1 \times \mathbb{R}^+$ induces such a transform. A discussion of which fractional linear maps do induce such a transform (similar to that in Section 4.4) is given in Section 5.3.2.

**Remark 5.4.** In Section 5.3.2 we will also see that there is a difference between the cases $0 \in \partial K$ and $0 \in \text{int}(K)$, where in the former a “$J$-type” transform does exist, and in the latter it does not (except in the case $K_1 = K_2 = \mathbb{R}^n$).
First, we will prove the one-dimensional theorem. We will do this in two ways. The first (in Section 5.1) is by using the known uniqueness Theorem 3.10 for $\mathcal{J}$ and $\mathcal{I}$. The second is a direct proof, which we postpone to Section 5.3.1. We add this second proof for two purposes; to make the paper self contained, and also to clarify the case of a transform is fractional linear. The first family of transforms (a-la-$\mathcal{F}$) for each of these two families of transforms, the transform is induced by a point map on the epi-graphs, that is when the domain of all functions is bounded.

Second, we will prove the multi-dimension theorem, in the following stages: we show that the transform must act “ray-wise”. Then, on each ray, we could already apply the one-dimensional conclusion, but in fact we need much less - thus we continue directly and show that two extremal families of functions, namely linear functions and indicator functions, determine the full shape of $\mathcal{J}$. The extremality property forces the transform to act bijectively on these two families, and in a monotone way. Here, we do not need to discover the exact rule of this monotone mapping (even though we have it, since we’ve solved the one dimensional case). Instead, we prove that there is some point map on the epi-graphs, controlling the rule of the transform for a third family, namely triangle functions. We show that this point map is interval preserving, and then apply Theorem 2.17 to show that it is fractional linear. Finally, we show that the rule of the transform for triangles determines the whole transform, thus completing the proof. This plan follows the proof from [8] of the case $K_1 = K_2 = \mathbb{R}^n$.

5.1 Dimension one

In [8], the first and third named authors showed that essentially, any order isomorphism $\mathcal{T}: \text{Cvx}_0(\mathbb{R}^+ \to \text{Cvx}_0(\mathbb{R}^+)$ is either $\mathcal{I}$ or $\mathcal{J}$, see Theorem 3.10. We note that in this case, indeed, for each of these two families of transforms, the transform is induced by a point map on the epi-graphs which is fractional linear. The first family of transforms (a-la-$\mathcal{I}$) is given by

$$(\mathcal{T}\phi)(x) = \beta \phi(x/\alpha),$$

for positive $\alpha$ and $\beta$, and the inducing maps are $F^\mathcal{I}_{\alpha,\beta}(x,y) = (\alpha x, \beta y)$. The second family of transforms (a-la-$\mathcal{J}$) is given by

$$(\mathcal{T}\phi)(x) = \beta (\mathcal{J}\phi)(x/\alpha),$$

for positive $\alpha$ and $\beta$, and the inducing maps are $F^\mathcal{J}_{\alpha,\beta}(x,y) = \left(\frac{\alpha x}{y}, \frac{\beta}{y}\right)$.

We introduce a third transform, with a parameter $z > 0$, to be able to switch between the bounded and non bounded cases;

**Definition 5.5.** Let $z > 0$, and $F_z: [0, z) \times \mathbb{R}^+ \to \mathbb{R}^+ \times \mathbb{R}^+$ be the bijective fractional linear map defined by $F_z(x,y) = \left(\frac{x}{z-x}, \frac{y}{z-x}\right)$.

**Lemma 5.6.** $F_z$ induces an order isomorphism $\mathcal{T}_z: \text{Cvx}_0([0,z)) \to \text{Cvx}_0(\mathbb{R}^+)$ by its action on epi-graphs, that is $\text{epi}(\mathcal{T}_z(f)) = F_z(\text{epi}(f))$.

**Proof.** To see that a transform defined using a point map on the epi-graphs, is an order isomorphism, three things need to be checked; that it is well defined, that it is bijective, and that it preserves order in both directions. For $\mathcal{T}_z$ to be well defined, $F_z$ must map epi-graphs of geometric convex functions to epi-graphs of geometric convex functions. Since $F_z$ is fractional linear, it is interval preserving, thus a convex epi-graph is mapped to some convex set. Among all convex sets, epi-graphs of geometric convex functions are characterized by two inclusions;

$$\{(0,y): y > 0\} = \text{epi}(1_{\{0\}}) \subseteq \text{epi}(f) \subseteq \text{epi}(1_K) = \{(x,y): x \in K, y > 0\}.$$

Note that $F_z$ maps the half line $\{(0,y): y > 0\}$ onto itself, and the entire domain $[0,z) \times \mathbb{R}^+$ onto the image $\mathbb{R}^+ \times \mathbb{R}^+$. Therefore also $F_z(\text{epi}(f))$ is between these two sets, which means it
is the epi-graph of some geometric convex function. Bijectivity of $F_z$ implies bijectivity of $T_z$. Since $f \leq g \Leftrightarrow epi(g) \subseteq epi(f)$, a transform induced by a bijective point map on the epi-graphs, automatically preserves order in both directions.

We are ready to prove the one dimensional theorem, dealing with $I_1, I_2 \subseteq \mathbb{R}$ which may be either bounded intervals or half lines.

**Theorem 5.7.** Let $I_1 \subseteq \mathbb{R}$ be either of the form $I_1 = [0, x_1)$ for some positive $x_1$, or $I_1 = [0, \infty)$, and likewise $I_2$. If $T : Cvx_0(I_1) \to Cvx_0(I_2)$ is an order isomorphism, then there exists a bijective fractional linear map $F : I_1 \times \mathbb{R} \to I_2 \times \mathbb{R}^+$, such that $T$ is given by

$$epi(T f) = F(epi(f)).$$

**Proof of Theorem 5.7.** Define $\tilde{T} : Cvx_0(\mathbb{R}^+) \to Cvx_0(\mathbb{R}^+)$ in the following way:

If $I_1 = [0, x_1)$ and $I_2 = [0, \infty)$, then: $\tilde{T} := T \circ T_x^{-1}$

If $I_1 = [0, x_1)$ and $I_2 = [0, x_2)$, then: $\tilde{T} := T_{x_2} \circ T \circ T_{x_1}^{-1}$

If $I_1 = [0, \infty)$ and $I_2 = [0, x_2)$, then: $\tilde{T} := T$

If $I_1 = [0, \infty)$ and $I_2 = [0, \infty)$, then: $\tilde{T} := T$

$\tilde{T}$ is clearly an order isomorphism. Next, by simply applying Theorem 3.10, we get that our original $T$ is some composition of the transforms $I, J, T_z$, and $T_z^{-1}$, which are all induced by fractional linear point maps on the epi-graphs. Thus we conclude that $T$ is also induced by such a map.

**Remark 5.8.** For transforms on (or to) $Cvx_0([0, z])$ simply note that all elements of $Cvx_0([0, z])$ are non decreasing and lower-semi-continuous functions, and thus have a unique extension to $[0, z]$, which preserves order in both directions. Therefore, by embedding $Cvx_0([0, z]) = Cvx_0([0, z])$ (where $f$ is mapped to its unique extension) we get an order isomorphism of the form described in Theorem 5.7, and thus have the same result for closed intervals $[0, z]$, where epi-graphs are taken without the point $z$. In particular we see that there exist order isomorphisms between $Cvx_0([0, z])$ and $Cvx_0(\mathbb{R}^+)$. 

### 5.1.1 Table of one dimension transforms

Straightforward computation of the transform in each of the cases gives, in each of the four scenarios, two types of transforms; a-la-identity and a-la-$J$. We list them here, indicated by the fractional linear maps which induce them, namely $F_{a,b} : I_1 \times \mathbb{R} \to I_2 \times \mathbb{R}^+$. Each family is two-parametric, for convenience we choose the parameters $a, b$ such that $a, b > 0$ gives exactly all the functions in the family:

| $I_1$ | $I_2$ | a-la-$\mathcal{I}$: $F_{a,b}(x,y)$ | a-la-$\mathcal{J}$: $F_{a,b}(x,y)$ |
|-------|-------|---------------------------------|---------------------------------|
| $[0, x_1)$ | $[0, x_2)$ | $x \over x(1-a) + x_1 a$ · $(x \over by)$ | $x \over by$ · $a(x_1 - x)$ |
| $[0, x_1)$ | $[0, \infty)$ | $x \over x_1 - x$ · $(x \over by)$ | $x \over by$ · $a(x_1 - x)$ |
| $[0, \infty)$ | $[0, x_2)$ | $x \over ax + x_1$ · $(x \over by)$ | $x \over by$ · $a(x_1 - x)$ |
| $[0, \infty)$ | $[0, \infty)$ | $a$ · $(x \over by)$ | $b \over y$ · $(x \over a)$ |

There is an essential difference between the $\mathcal{I}$-type and $\mathcal{J}$-type transforms; they handle differently the extremal elements of $Cvx_0(I)$, which are indicators and linear functions (see Section 5.2.2 for exact definitions). The $\mathcal{I}$-type transforms map indicators to themselves (bijectionally), and likewise linear functions. The $\mathcal{J}$-type transforms, however, interchange between the two sub-families, mapping indicators to linear functions (bijeccitively) and vice versa. In the inducing maps, we also have a natural distinction between the $\mathcal{I}$-type and $\mathcal{J}$-type maps. In both cases the determinant of the Jacobian of the inducing map never vanishes; it is positive for $\mathcal{I}$-type maps, and negative for $\mathcal{J}$-type maps.
5.2 Multi dimension

5.2.1 Acting on rays

We next prove that in the $n$-dimensional case, one merely deals with many copies of the one dimensional problem (in fact, the case of functions on $\mathbb{R}^+$).

The next lemma states that an order isomorphism basically works in the following way: first, there is a permutation on the rays, and then on each ray, the transform acts independently of the functions’ values on other rays.

There are two nuances here; first, if $K \neq \mathbb{R}^n$, then in some directions it does not contain a full ray. Since this does not affect the argumentation in any way, we don’t distinguish between a full ray ($\mathbb{R}^+z$) and a restricted ray ($\mathbb{R}^+z \cap K$), which may be a bounded interval, and use “ray” to describe both. Second, if $0 \in \text{int}(K)$, then the set of all relevant rays can be described by $S^{n-1}$, but if $0 \in \partial K$, then there are less relevant rays (in some directions $z$, $\mathbb{R}^+z \cap K = \{0\}$). Therefore we are again forced to add another definition, for the set of all relevant rays - $S(K) \subseteq S^{n-1}$. $S(K) := \{z \in S^{n-1} : \mathbb{R}^+z \cap K \neq \{0\}\}$. In what follows, the support of a function is defined to be the (closure of) the set on which it is finite; $\{x : f(x) < \infty\}$

Lemma 5.9. Let $n \geq 2$, and let $K_1, K_2 \subseteq \mathbb{R}^n$ be convex sets with non empty interior. If $T : \text{Cvx}_0(K_1) \rightarrow \text{Cvx}_0(K_2)$ is an order preserving isomorphism, then there exists a bijection $\Phi : S(K_1) \rightarrow S(K_2)$, such that any function supported on $\mathbb{R}^+y$ is mapped to a function supported on $\mathbb{R}^+z$, for $z = \Phi(y)$. Moreover, $T$ acts ray-wise, namely $(Tf)|_{\mathbb{R}^+z}$ depends only on $f|_{\mathbb{R}^+y}$, for $z = \Phi(y)$.

We remark that if we were to prove the theorem directly for order reversing transformations then we would not encounter this ray-wise behavior, and get a transform $\mathcal{A}$ (or $\mathcal{L}$) which, miraculously, when combined with $\mathcal{L}$ acts ray-wise. Later on, it will follow that $\Phi$ must be induced by a linear map.

The proof uses the following simple observation: if $x, y \in S(K_1)$ are two different points, and $f_x, f_y \in \text{Cvx}_0(K_1)$ are two functions supported on $\mathbb{R}^+x$ and $\mathbb{R}^+y$ respectively, then $\max\{f_x, f_y\} = 1_{\{0\}}$, and thus also $\max\{Tf_x, Tf_y\} = 1_{\{0\}}$, which means that $Tf_x$ and $Tf_y$ are supported on different sets.

Proof of Lemma 5.9. For two functions $f, g$ to have $\max\{f, g\} = 1_{\{0\}}$ they must be supported on two sets whose intersection equals $\{0\}$. A function with support in a line cannot be mapped to one whose support includes two positively-linearly-independent points because then $T^{-1}$ would map two functions whose support intersects at $\{0\}$ only, to functions supported on the same ray - impossible. Thus functions supported on a given ray are all mapped to functions supported on another fixed ray. By invertibility, we get that this defines a mapping $\Phi : S(K_1) \rightarrow S(K_2)$ which is bijective.

As for the ray-wise action of $T$, the values of $Tf$ on $\mathbb{R}^+z$ are the same as the values of $\max\{Tf, R_z\}$, where $R_z$ denotes the function which is 0 on $\mathbb{R}^+z \cap K_2$ and $+\infty$ elsewhere. This maximum is the image of the function $\max\{f, R_y\}$, because $TR_y = R_z$ (each being the smallest function supported on the corresponding ray). Since $\max\{f, R_y\}$ does not depend on the values $f$ attains outside $\mathbb{R}^+y$, our claim follows.

5.2.2 Extremal elements and monotonicity

Restricted to a ray $I$, we consider two families of extremal functions in $\text{Cvx}_0(I)$: indicator functions, and linear functions.

a) $1_{[0,z]}$ which equals to 0 on $[0, z]$ and $+\infty$ elsewhere (indicator).

b) $L_e(t) = \max\{ct, 1_I(t)\}$ (linear).
Formally, the function $l_c$ is defined on the whole of $\mathbb{R}^n$, therefore it is not really linear, but we will use this name in short. All the $J$-type transforms switch (a) and (b) - bijectively, and all the $I$-type transforms fix (a) and fix (b) - again, bijectively. We will show that this is no coincidence - a general order isomorphism $\mathcal{T}$ must act in one of these two ways. We derive this from two properties of these families - the extremality property, and the non-comparability relation between these two families.

**Definition 5.10.** A function $f \in Cvx_0(I)$ is called extremal if there exist no two functions $g, h \in Cvx_0(I)$ such that $g \gtrless f$ and $h \gtrless f$ but $\max\{g, h\} \geq f$.

In the language of epi-graphs, this means that for $\text{epi}(f)$ to contain $A \cap B$, it must contain either $A$ or $B$ - whenever $A, B$ are also epi-graphs of geometric convex functions.

We claim that extremality characterizes indicator and linear functions in $Cvx_0(I)$:

**Lemma 5.11.** The only extremal functions in $Cvx_0(I)$ are either of the form $1_{[0,z]}$ for some $z \in I$ or of the form $l_c$ for some $c \in \mathbb{R}^+$. 

**Proof of Lemma 5.11.** It is easy to check that both families are extremal. To show that any extremal function $f \in Cvx_0(I)$, must be of one of the two forms, we first show that if it assumes some value $0 < c \neq \infty$, it must be linear. Indeed, let $f(x) = c$. Without loss of generality we may assume $x \in \text{int}(I)$, since $f$ is lower-semi-continuous. Consider the function $1_{[0,x]}$ assuming 0 in the interval $[0,x]$ and $+\infty$ elsewhere: $f \neq 1_{[0,x]}$, since $1_{[0,x]}(x) = 0 < f(x)$. Consider the function $L_x(y) = cy$. By convexity of $f$, on the interval $[0,x]$, $f \leq L_x$. Since outside $[0,x]$ we have $f \leq 1_{[0,x]}$, this implies $f \leq \max\{1_{[0,x]}, L_x\}$, and so by extremality it must be that $f \leq L_x$. Since $x$ is in the interior of $I$, this means that $f = L_x$, and therefore $f$ is linear. The only other option is that $f$ assumes only the values 0 and $+\infty$, which implies it is an indicator function, by convexity.

**Lemma 5.12.** If $\mathcal{T} : Cvx_0(I_1) \to Cvx_0(I_2)$ is an order isomorphism then either:

$\mathcal{T}$ is a bijection from linear functions to indicators, and a bijection from indicators to linear functions, or:

$\mathcal{T}$ is a bijection from linear functions to themselves, and a bijection from indicator functions to themselves.

**Proof.** Extremality is preserved under $\mathcal{T}$. Indeed, if there exist two functions $g, h \in Cvx_0(I_2)$ such that $g \gtrless \mathcal{T}f$ and $h \gtrless \mathcal{T}f$ but $\max\{g, h\} \geq \mathcal{T}f$, then the functions $\mathcal{T}^{-1}g$ and $\mathcal{T}^{-1}h$ contradict extremality for $f$. So, we see that the family of all extremal functions is mapped to itself, and by Lemma 5.11 this family is exactly the union of linear and indicator functions. Since $\mathcal{T}^{-1}$ shares the same properties as $\mathcal{T}$, we see that the map is surjective.

Secondly, all linear functions are comparable to one another and all indicator functions are comparable to one another (by $f$ and $g$ comparable we mean that either $f \leq g$ or $g \leq f$). However, no indicator function is comparable to a linear function - except for the trivial examples of $1_{[0]}$ and 0, whose behavior is obvious - since in $Cvx_0(I)$, these are the maximal and minimal elements (they are also the only mutual elements in both families). Hence, once we know that one linear function is mapped to a linear function then all of them must be, and then all indicator functions are mapped to indicators. The alternative is of course that all linear functions are mapped to indicators, and then all indicators are mapped to linear functions.

In this last Lemma, a dichotomy, not apparent at first sight, appears. We have two very different possibilities, one corresponding to $I$, the identity transform (which clearly maps linear functions to themselves, likewise for indicator functions), and the other possibility corresponds to the transform $J$, which - as can be checked - maps linear functions to indicator functions and vice-versa. Despite this dichotomy, in the statement of the next lemma we do not need to separate the two cases.

Next we claim that $\mathcal{T}$ is a monotone bijection on each of the extremal families. Monotonicity has a meaning here since both families are fully ordered subsets of $Cvx_0(I)$ - “chains” - bounded together by the minimal and maximal elements $f_0 = 0$ and $f_\infty = 1_{[0]}$. 

33
If $\mathcal{T}$ maps linear functions to themselves (and likewise indicator functions), we define $S : I_1 \to I_2$ to be the function for which $\mathcal{T}_I[0,x] = 1_{[0,S(x)]}$, and $A : \mathbb{R}^+ \to \mathbb{R}^+$, for which $\mathcal{T}(l_c) = l_{A(c)}$. If $\mathcal{T}$ interchanges between the two families, we define $S : I_1 \to \mathbb{R}^+$ to be the function for which $\mathcal{T}_I[0,x] = l_{S(x)}$, and $A : \mathbb{R}^+ \to I_2$, for which $\mathcal{T}(l_c) = 1_{[0,A(c)]}$. In this next simple lemma we formulate the monotonicity property:

**Lemma 5.13.** Assume $\mathcal{T} : Cvx_0(I_1) \to Cvx_0(I_2)$ is an order isomorphism.

If $\mathcal{T}$ maps linear functions to themselves, then $S$ and $A$ are increasing bijections.

If $\mathcal{T}$ interchanges between the two families, then $S$ and $A$ are decreasing bijections.

**Proof.** $S$ and $A$ are bijections, since $\mathcal{T}$ is a bijection. Note that $1_{[0,x]} \leq 1_{[0,y]} \iff x \geq y$ and $l_c \leq l_d \iff c \leq d$. Therefore, if $\mathcal{T}$ fixes each of the families, $S$ and $A$ are increasing, and if $\mathcal{T}$ switches between the families, $S$ and $A$ are decreasing. \hfill $\square$

### 5.2.3 Triangles functions - completing the proof

Next, we handle another family of functions, “triangle” functions. We show it is preserved under $\mathcal{T}$, and that the rule of the transform for it is monotone. We show that when leaving the one-dimensional perspective, the rule of the transform for triangles is controlled by an interval preserving bijection; and thus we apply our uniqueness theorem for such maps, Theorem 2.17. Finally we show that the transform is determined by its behavior on triangles, which proves Theorem 5.2.

For $z \in K$ and $c \in \mathbb{R}^+$, we introduce the “triangle” functions, denoted $\langle z, c \rangle \in Cvx_0(K)$:

$$
\langle z, c \rangle(x) = \begin{cases} 
  c|x|, & \text{if } x \in [0, z] \\
  +\infty, & \text{otherwise.}
\end{cases}
$$

Note that they are one-dimensional (i.e. supported on a ray), so they can be thought of as elements of $Cvx_0(I)$ where $I$ is a ray, and then $\langle z, c \rangle = \max\{1_{[0,z]}, l_c\}$.

**Lemma 5.14.** If $\mathcal{T} : Cvx_0(I_1) \to Cvx_0(I_2)$ is an order isomorphism then a triangle function $\langle z, c \rangle$ is mapped under $\mathcal{T}$ to a triangle function $\langle z', c' \rangle$, where $(z', c')$ is a function of $(z, c)$.

**Proof of Lemma 5.14.** A triangle is the maximum of an indicator and a linear function. By Proposition 3.5 $\mathcal{T}$ respects sup and inf, and thus in both cases of Lemma 5.12, a triangle is mapped to the maximum of an indicator and a linear function; that is, to a triangle. \hfill $\square$

**Remark.** Since in Lemma 5.13 we showed that $\mathcal{T}$ maps indicator and linear functions in a monotone way, it is obvious that this is the case also for triangles, meaning either $\mathcal{T}(\langle z, c \rangle) = \langle S(z), A(c) \rangle$, or $\mathcal{T}(\langle z, c \rangle) = \langle A(c), S(z) \rangle$, and in both cases, fixing any of the parameters $z$ or $c$ and changing the other monotonously, changes also the triangle in the image monotonously. Since we already know the exact shape of 1D transforms, we could have concluded this immediately. However, in what follows, we only use the fact that $\mathcal{T}(\langle z, c \rangle) = \langle y, d \rangle$, and that this map is monotone, meaning that on a fixed ray, either $y = y(x)$, $d = d(c)$, and both functions are bijective and increasing, or $y = y(c)$, $d = d(x)$, and both functions are bijective and decreasing.

We return to the $n$-dimensional picture, the first time since we reduced the discussion to ray-wise action. We wish to see how the different mappings of triangles on different rays all fit together. To this end, we replace the “parametrization” of triangles, from the point $z$ (indicating the support of the function) and the slope $c$, to the point $z$ and the value of the function at that point $h = c|z|$. To avoid abuse of notation, for $h = c|z|$ we will denote $\langle z, c \rangle$ by $\langle z, h \rangle$. With this notation, we denote by $F : (K_1 \setminus \{0\}) \times \mathbb{R}^+ \to (K_2 \setminus \{0\}) \times \mathbb{R}^+$ the bijective map for which $\mathcal{T}\langle z, h \rangle = \langle F(z, h) \rangle$.

**Proposition 5.15.** Let $n \geq 2$, $K_1, K_2 \subseteq \mathbb{R}^n$ convex sets with non empty interior, and $\mathcal{T} : Cvx_0(K_1) \to Cvx_0(K_2)$ an order preserving isomorphism. Assume $F : (K_1 \setminus \{0\}) \times \mathbb{R}^+ \to (K_2 \setminus \{0\}) \times \mathbb{R}^+$
Proof of Theorem 5.2. For every following simple equality epigraph $F(x,h)$ for every $(x,h) \in (K_1 \setminus \{0\}) \times \mathbb{R}^+$. Then $F$ is a fractional linear map.

**Proof of Proposition 5.15.** First we show that the restriction of $F$ to any domain for which $(0,0)$ is an extreme point, is fractional linear. Let $(x_1,h_1),(x_2,h_2) \in K_1 \times \mathbb{R}^+$ such that $0 \not\in [x_1,x_2]$. This merely means that our argument does not hold if $x_1$ and $x_2$ are on opposite rays. Letting $(x_3,h_3) \in [(x_1,h_1),(x_2,h_2)]$, and denoting $F(x_i,h_i) = (y_i,l_i)$, we need to prove that $(y_3,l_3) \in [(y_1,l_1),(y_2,l_2)]$. If $x_i$ are on the same ray, then it follows from the one dimensional case, handled in Section 5.3.1, that the restriction of $F$ to this line is fractional linear, and in particular it maps intervals to intervals, that is $F([(x_1,h_1),(x_2,h_2)]) = [(y_1,l_1),(y_2,l_2)]$. Assume otherwise, that $x_i$ are linearly independent. Note that $\psi^{x_3,h_3} \geq \inf\{\psi^{x_1,h_1},\psi^{x_2,h_2}\}$, and that in this inequality $x_3$ is maximal, and $h_3$ is minimal. Therefore $\psi^{y_3,l_3} \geq \inf\{\psi^{y_1,l_1},\psi^{y_2,l_2}\}$, and in this inequality - due to the monotonicity of $T$ on triangles - again $y_3$ is maximal, and $l_3$ is minimal (recall that in Lemma 5.13 we saw that if indicators and linear functions are exchanged, $S$ and $A$ are decreasing, and if they are preserved, $S$ and $A$ are increasing - thus in any case maximality of $x_3$ and minimality of $h_3$ coincides with maximality of $y_3$ and minimality of $l_3$). Therefore $y_3$, which lies on a different ray than those of $y_1,y_2$ ($\Phi$ is bijective), is in the triangle with vertices $0,y_1,y_2$, and due to its maximality - $y_3 \in [y_1,y_2]$. Moreover, the point $(y_3,l_3)$ is above or on the interval $[(y_1,l_1),(y_2,l_2)]$, and due to its minimality, it is on this line. Therefore $(y_3,l_3) \in [(y_1,l_1),(y_2,l_2)]$, which means that $F$ preserves intervals which do not intersect the positive $h$-axis; $\{(0,h) : h \geq 0\}$. In other words, the restriction of $F$ to any domain for which $(0,0)$ is an extreme point, is interval preserving. By applying Theorem 2.17, we conclude that $F$ is fractional linear on each such domain, and thus, since $n \geq 2$, we may use Corollary 2.15 to conclude that $F$ is a fractional linear map on the whole of $(K_1 \setminus \{0\}) \times \mathbb{R}^+$. □

**Remark.** The proof of Proposition 5.15 does not work in one dimension, since the only two rays: $\mathbb{R}^+,\mathbb{R}^-$ cannot interact - they have 0 in their convex hull, and therefore a direct proof is needed in this case, to show that the transform is given by a fractional linear map on the epi-graphs. In fact, while it is true for transforms on a ray, it is indeed not the case for transforms on $Cvx_0(\mathbb{R})$, or on $Cvx_0(I)$ where $I$ is an interval containing 0 in the interior.

**Remark.** The function $F$ which is defined formally only for $(x,h) \in (K \setminus \{0\}) \times \mathbb{R}^+$, can in fact be extended to $K \times \mathbb{R}^+$, since the defining hyperplane of $F$ does not intersect $epi(1_{\{0\}}) = \{(0,h) : h > 0\}$. Indeed, it is obvious that if it intersects this ray in one point it must contain the whole ray. In such a case, it follows from the properties of fractional linear maps, that rays emanating from a point in the hyperplane are mapped to parallel rays emanating from the hyperplane. Such a point map does not induce a transform on $Cvx_0(K)$. Therefore $F$ can be defined on the whole of $K \times \mathbb{R}^+$. Moreover, using the fact that the supremum of all triangles is $1_{\{0\}}$, we get that $F(epi(1_{\{0\}})) = epi(1_{\{0\}})$.

Finally, knowing that the transform rule for triangle functions is controlled by a fractional linear map $F$, we turn to see that this is also the case for the epi-graph of any function. We use the following simple equality $epi(f) = \{(x,h) \in (K \setminus \{0\}) \times \mathbb{R}^+ : \psi_{f} > f\} \cup epi(1_{\{0\}})$ which holds for every $f \in Cvx_0(K)$.

**Proof of Theorem 5.2.** By the previous proposition, there exists a bijective fractional linear map $F : (K_1 \setminus \{0\}) \times \mathbb{R}^+ \to (K_2 \setminus \{0\}) \times \mathbb{R}^+$, and we need to show that $F(epi(f)) = epi(Tf)$.

$$
epi(Tf) = \{(y,l) \in (K_2 \setminus \{0\}) \times \mathbb{R}^+ : \psi_{y,l} > Tf\} \cup epi(1_{\{0\}})$$

$$= F(\{(x,h) \in (K_1 \setminus \{0\}) \times \mathbb{R}^+ : \psi_{F(x,h)} > Tf\}) \cup F(epi(1_{\{0\}}))$$

$$= F(\{(x,h) \in (K_1 \setminus \{0\}) \times \mathbb{R}^+ : T\psi_{x,h} > Tf\}) \cup F(epi(1_{\{0\}}))$$

$$= F(\{(x,h) \in (K_1 \setminus \{0\}) \times \mathbb{R}^+ : \psi_{x,h} > f\}) \cup F(epi(1_{\{0\}}))$$

$$= F(epi(f)) □$$
5.3 Additional results

5.3.1 Direct uniqueness proof in the one dimensional bounded case

We focus on the possible transforms in the case where linear functions are mapped to themselves, likewise indicator functions. Clearly, the function \( S : I_1 \to I_2 \) for which we have that \( T \) is bijective and increasing (so it is continuous as well). Similarly \( A : \mathbb{R}^+ \to \mathbb{R}^+ \), for which \( \mathcal{T}(c) = I_{A(c)} \), is bijective, increasing, and continuous. Note that we deal now only with \( I_1 \) and \( I_2 \) which are bounded, which means that \( S \) maps an interval to an interval, and \( A \) maps a full ray to a full ray.

**Lemma 5.16.** Let \( I_1 = [0, x_1) \), \( I_2 = [0, x_2) \), where \( x_i \in \mathbb{R} \) are two positive numbers. Let \( T : C_{x_0}(I_1) \to C_{x_0}(I_2) \) be an order preserving isomorphism. Assume further, that for some increasing bijective function \( S : I_1 \to I_2 \), we have \( T \) is bijective, increasing, and continuous. Note that we deal now only with \( I_1 \) and \( I_2 \) which are bounded, which means that \( S \) maps an interval to an interval, and \( A \) maps a full ray to a full ray.

**Proof of Lemma 5.16.** Denote as before \( s_{x,c} = \max \{ 1_{[x,x]} \} \), and similarly \( g_{x,c} = \inf \{ 1_{[x,x]} \} \). We get (on \( I_2 \) replace \( x_1 \) by \( x_2 \)):

\[
\begin{align*}
g_{x,c}(z) = \begin{cases} 0 & \text{if } z \in [0, x] \\ c(z-x) \frac{x_1}{x_1-x} & \text{if } z \in [x, x_1] \\ +\infty & \text{otherwise} \end{cases}
\end{align*}
\]

By Proposition 3.5 we get \( T(s_{x,c}) = s_{S(x),A(c)} \). Let \( \alpha \) be an order preserving isomorphism. Assume further, that for some increasing bijective function \( A : \mathbb{R}^+ \to \mathbb{R}^+ \), we have that \( T \) is bijective, increasing, and continuous. Note that we deal now only with \( I_1 \) and \( I_2 \) which are bounded, which means that \( S \) maps an interval to an interval, and \( A \) maps a full ray to a full ray.

\[
\begin{align*}
g_{S(x),A(c)}(z) &= \begin{cases} 0 & \text{if } z \in [0, x] \\ c(z-x) \frac{x_1}{x_1-x} & \text{if } z \in [x, x_1] \\ +\infty & \text{otherwise} \end{cases}
\end{align*}
\]

with equality between the two functions at the point \( S(x) \), meaning:

\[
A \left( c \frac{x_1-tx}{x_1-x_1} \right) \cdot (S(x) - S(tx)) \cdot \left( \frac{x_2}{S(x) - S(tx)} \right) = A(c)S(x)
\]

for every \( 0 < t < 1 \), every \( 0 < x < x_1 \), and every \( 0 < c \). By defining \( u = \frac{x_1-tx}{x_1-tx_1} \) and rearranging the equation, we get:

\[
A(cu) = \frac{A(c)A(u)}{A(1)}
\]

In particular, the ratio \( \frac{A(cu)}{A(c)} \) does not depend on \( c \) - thus it is equal to \( \frac{A(u)}{A(1)} \), and we may write

\[
A(cu) = \frac{A(c)A(u)}{A(1)}
\]

which holds for all \( 0 < c \) and \( 1 < u \) (see the definition of \( u \)). For \( u = 1 \) it is true trivially. For \( 0 < u < 1 \) we denote \( u' := 1/u > 1 \). Noticing the symmetry between \( u \) and \( c \), we interchange their roles to see that \( A(1) = \frac{A(u)A(u')}{A(1)} \), and write:

\[
A(cu) = \frac{A(1)}{A(cu)} \cdot \frac{1}{\frac{A(u')}{A(1)}} = \frac{A(cu)}{A(1)} \cdot \frac{1}{\frac{A(u)}{A(1)}} = \frac{A(cu)}{A(1)} \cdot \frac{1}{\frac{A(u)}{A(1)}}.
\]

36
for some fixed $\alpha > 0$ and $\gamma$.

Therefore, $\frac{A(cu)}{A(u)} = u^\gamma$. Returning to equation (5) with this new information, and substituting $u = \frac{x_1-tx}{x_1-tx_1}$, we get

$$
\left(\frac{x_1-tx}{x_1-tx_1}\right)^\gamma = \frac{S(x)}{S(x) - S(tx)} \cdot \left(\frac{x_2 - S(tx)}{x_2}\right).
$$

This can be written also as

$$
S(tx) = S(x) \cdot \frac{x_2 \left(\frac{x_1-tx}{x_1-tx_1}\right)^\gamma - x_2}{x_2 \left(\frac{x_1-tx}{x_1-tx_1}\right) - S(x)},
$$

to show that for a given $0 < x < x_1$, $f(t) := S(tx)$ is differentiable as a function of $t$, for all $0 < t < 1$. This means $S$ is differentiable in $(0, x_1)$ (the interior of $I_1$).

Denote $D_{a,b} = \frac{S(b) - S(a)}{b-a}$ for $a, b \in [0, x_1]$, and similarly $D_{a,a} = S'(a)$ for $a \in (0, x_1)$, so that $D_{a,b} \to D_{a,a}$ when $b \to a$. Note that for $a \neq b$, $0 < D_{a,b} < \infty$. Rearranging equation (7) yields:

$$
\left(\frac{x_1-tx}{x_1-tx_1}\right)^{\gamma-1} = \frac{D_{0,x} \cdot D_{tx,x_1}}{D_{0,x_1} \cdot D_{tx,x}}.
$$

Choose $x < x_1$ such that $S'(x) \neq 0$ and let $t \to 1^-$, then the right hand side of the equation tends to a finite, strictly positive number, and since $\frac{x_1-tx}{x_1-tx_1} \to \infty$ when $t \to 1$, this implies $\gamma = 1$. Therefore for every $0 < t < 1, 0 < x < x_1$ we have:

$$
D_{0,x_1} \cdot D_{tx,x_1} = D_{0,x} \cdot D_{tx,x},
$$
or alternatively:

$$
[0, tx, x, x_1] = [S(0), S(tx), S(x), S(x_1)]
$$

which by Theorem 2.23 implies that $S$ is fractional linear. Combined with $S(0) = 0, S(x_1) = x_2$, $S'(x) > 0$, and $\gamma = 1$, this implies that $S$ and $A$ each belongs to a one-parametric family of maps of the form

$$
S(x) = x_2 \cdot \frac{x/x_1}{d(x/x_1 - 1) + 1} \quad A(c) = \alpha c
$$

where $d < 1$ and $\alpha < 0$. 

\( \square \)

### 5.3.2 Classification of admissible fractional linear maps

We wish to fully classify the type of fractional linear maps that induce transforms as in Theorem 5.2. (The one dimensional case was fully described in Section 5.1). Denote by $A_\infty = \{(y, y) : y > 0\}$ the epi-graph of $\delta_{0,0} = 1_{(0)}$: the maximal function in $Cvx_0(K)$, and by $A_0^1 = \{(x, y) : x \in K_1, y > 0\}$ the epi-graph of $1_{K_1}$: the minimal function in $Cvx_0(K_1)$ (similarly $A_0^n = \{(x, y) : x \in K_2, y > 0\}$ for $Cvx_0(K_2)$). Since $Cvx_0(K) = \{f \in Cvx(\mathbb{R}^n) : 1_K \leq f \leq 1_{(0)}\}$, it turns out that a necessary and sufficient condition for a bijection $F : K_1 \times \mathbb{R}^+ \to K_2 \times \mathbb{R}^+$ to induce an order isomorphism is that it maps the minimal and maximal elements in $Cvx_0(K_1)$ to the minimal and maximal elements in $Cvx_0(K_2)$, namely:

$$
F(A_\infty) = A_\infty.
$$
Indeed, since $F$ is a bijection from the cylinder $K_1 \times \mathbb{R}^+$ to the cylinder $K_2 \times \mathbb{R}^+$, we see that the transform is bijective. Order preservation (in both directions) is automatic for point-map-induced transforms. One must check that $F(\text{epi}(f))$ is an epi-graph of some convex function, which follows from it being a convex set containing the fiber $A_{\infty}$. Since $A_{\infty} \subseteq \text{epi}(f) \subseteq A_0^1$, we get $A_{\infty} \subseteq F(\text{epi}(f)) \subseteq A_0^2$, meaning that $F(\text{epi}(f))$ is an epi-graph of a function in $Cvx_0(K_2)$. Therefore, we give the description of a general fractional linear map $F$ which satisfies (8) and (9). Let the matrix $A_F$ be given by

$$A_F = \begin{pmatrix} v_1' & u_1' \\ A & \vdots & \vdots \\ v_n' & u_n' \\ v_1 & \cdots & v_n & a & b \\ u_1 & \cdots & u_n & c & d \end{pmatrix},$$

for $A \in L_n(\mathbb{R})$, $v, v', u, u' \in \mathbb{R}^n$, and $a, b, c, d \in \mathbb{R}$. Thus $F$ is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} (Ax + yv' + u')/(v,x) + cy + d \\ (v,x) + ay + b/(u,x) + cy + d \end{pmatrix}$$

Condition (8) means that $\begin{pmatrix} 0 \\ y \end{pmatrix}$ is mapped to $\begin{pmatrix} 0 \\ g(y) \end{pmatrix}$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is some bijection, and therefore

$$\frac{yv' + u'}{cy + d} = 0 \quad \text{for all } y > 0,$$

which implies $v' = u' = 0$. For $g(y) = \frac{ay + b}{cy + d}$ to be a bijection there exist only two options, corresponding to the two types of transforms on $Cvx_0(K)$: either $g$ is increasing, and then $g(y) = \frac{y}{d}$ for some $d > 0$, which is associated with the $I$-type transforms, or $g$ is decreasing, and then $g(y) = \frac{b}{y}$ for some $b > 0$, which is associated with the $J$-type transforms. We denote these two different cases by $F^I$ and $F^J$, and (using the multiplicative degree of freedom in $A_F$) get:

$$A_{F^I} = \begin{pmatrix} 0 & 0 \\ \vdots \\ 0 & 0 \\ v_1 & \cdots & v_n & 1 & 0 \\ u_1 & \cdots & u_n & 0 & d \end{pmatrix}, \quad A_{F^J} = \begin{pmatrix} 0 & 0 \\ \vdots \\ A & \vdots \\ v_1 & \cdots & v_n & 0 & b \\ u_1 & \cdots & u_n & 1 & 0 \end{pmatrix}$$

Note that in both cases, $A_F \in GL_{n+2} \iff A \in GL_n$. Turning to condition (9), we separate the two cases, dealing first with the $I$-type.

This case is very similar to the $Cvx(K)$ case (see discussion in Section 4.4), where the preservation of infinite cylinders is replaced with preservation of a part of those cylinders. Since

$$F^I \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax/(u,x)+d \\ y/(u,x)+d \end{pmatrix}$$

we have that $\langle v,x \rangle + y/(u,x)+d > 0$ for all $x \in K_1$ and $y > 0$, which implies $K_1 \subseteq \{\langle u,x \rangle + d > 0\}$. Since $\langle v,x \rangle + y$ maps $\mathbb{R}^+$ to $\mathbb{R}^+$ (as a function of $y$), $\langle v, x \rangle = 0$ for all $x \in K_1$, which implies $v = 0$ (recall that $K_1$ has interior). The general form of an $I$-type inducing map, is thus given, for $A \in GL_n$, $u \in \mathbb{R}^n$, and $d > 0$, such that $K_1 \subseteq \{\langle u,x \rangle + d > 0\}$ by

$$F^I \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax/(u,x)+d \\ y/(u,x)+d \end{pmatrix}.$$
For the $J$-type case, we know that

$$F^J \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \frac{Ax}{(u,x)+y} \\ \frac{(v,x)+b}{(u,x)+y} \end{array} \right).$$

Therefore \(\frac{(v,x)+b}{(u,x)+y} > 0\) for all \(x \in K_1\) and \(y > 0\), which implies \(K_1 \subseteq \{(u,x) \geq 0\}\), and also \(K_1 \subseteq \{-v,x \leq b\}\).

In the image, we know that each fiber \(\{(x_2, y)\}\) above a point \(x_2 \in K_2\) must contain all positive \(y\). The fiber above \(\frac{Ax_0}{(u,x_0)+y_0}\) is given by \(\frac{t(v,x_0)+b}{t(u,x_0)+y_0}\), and is the image of the ray \((tx_0, ty_0)\) in \(K_1 \times \mathbb{R}^+\), which may be bounded or not. If it is bounded, say of the form \([0,0),(x_0, y_0]\), we must have \(\langle v, x_0 \rangle = -b\). If it is not bounded, we must have \(\langle v, x_0 \rangle = 0\). Therefore we handle the following cases separately:

**A cone** \(K_1\): In this case, all rays \((tx_0, ty_0)\) in \(K_1 \times \mathbb{R}^+\) are not bounded, therefore all directions \(x_0\) in \(K_1\) satisfy \(\langle v, x_0 \rangle = 0\), and therefore \(v = 0\), since \(K_1\) has interior.

**Bounded** \(K_1\): In this case, all rays \((tx_0, ty_0)\) in \(K_1 \times \mathbb{R}^+\) are bounded, therefore all rays in \(K_1\) emanating from the origin have end points in the hyperplane \(\{(v, \cdot) = -b\}\) (in particular, \(v \neq 0\)). This means that \(K_1\) is a truncated cone, i.e. \(K_1 = K_1 \cap S_1\), where \(K_1\) is the minimal cone containing \(K_1\) and \(S_1\) is the slab \(\{0 \leq \langle -v, \cdot \rangle \leq b\}\).

**General** \(K_1\): In this case, some rays \((tx_0, ty_0)\) in \(K_1 \times \mathbb{R}^+\) are bounded, which implies \(v \neq 0\). All non bounded directions \(x_0\) must satisfy as before \(\langle v, x_0 \rangle = 0\), which implies that \(K_1\) is bounded in directions \(x_0 \notin v^\perp\), and \(K_1 \cap v^\perp\) is a (degenerate) cone. As in the bounded case, we get \(K_1 = K_1 \cap S_1\).

We can sum up the above three options as follows. The set \(K_1\) is the intersection of some cone, with the (possibly degenerate; if \(v = 0\)) slab \(\{0 \leq \langle -v, \cdot \rangle \leq b\}\).

Similarly, since in every direction, \(K_2\) is given by \(\{tAx : 0 \leq t \leq \langle u, x \rangle^{-1}\}\) for some \(x \in K_1\) (if \(\langle u, x \rangle = 0\) we let \(\langle u, x \rangle^{-1} = \infty\)), it contains full rays in all directions \(A(u^\perp)\), and in other directions it contains intervals with end points \(x_0\) which satisfy \(\langle A^{-T} u, x_0 \rangle = 1\). This implies, as before, that \(K_2\) is some cone, intersected with the (possibly degenerate) slab \(\{0 \leq \langle A^{-T} u, \cdot \rangle \leq 1\}\).

One can further investigate the possible restrictions on \(v, u, A\) with respect to the bodies \(K_i\), but it involves considering different cases for \(K_1\) and \(K_2\). We do not go into this in detail but instead give a few examples.

**Remark 5.17.** Under the condition \(0 \in \text{int}(K_1)\), the only \(J\)-type order isomorphism \(T : Cvx_0(K_1) \rightarrow Cvx_0(K_2)\) is possible when \(K_1 = K_2 = \mathbb{R}^n\). Indeed, in that case \(K_1 \subseteq \{(u,x) \geq 0\}\) implies \(u = 0\), which means that the projection of \(F^J\) to the first \(n\) coordinates is \(P_n F^J \left( \begin{array}{c} x \\ y \end{array} \right) = \frac{1}{y} Ax\). Clearly, since \(A\) is invertible, and \(K_1\) contains \(0\) in the interior, this means \(K_2 = \mathbb{R}^n\), and by the exact same argument also \(K_1 = \mathbb{R}^n\). In addition, \(K_1 \subseteq \{(v,x) + b \geq 0\}\) implies \(v = 0\), thus the general form of \(F^J\) which induces a transform on \(Cvx_0(\mathbb{R}^n)\) is

$$F^J \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \frac{Ax}{y} \\ \frac{b}{y} \end{array} \right)$$

for \(A \in GL_n\) and \(b > 0\), as stated in Theorem 3.11.

**Example 5.18.** When \(u \neq 0\), the defining hyperplane of \(F^J\) intersects the cylinder \(K_1 \times \mathbb{R}\), and the defining hyperplane of the image intersects the cylinder \(K_2 \times \mathbb{R}\). This is a restriction on the bodies \(K_i\); since \(K_2\) is the intersection of a cylinder (which has its base on the defining hyperplane of the image) with a half space \((y > 0)\), we have (see Section 2.4.2) that \(K_1 \times \mathbb{R}^+\) is the intersection of a cone (emanating from the origin) with a half space. This is true also for \(K_2 \times \mathbb{R}^+\), and so both our
bodies are simultaneously the intersection of a half space with a cylinder and the intersection of a half space with a cone. For example, let $K_1 = (\mathbb{R}^+)^n$ and $K_2 = \text{conv} \{0,e_1,\ldots,e_n\}$, and let $F^J$ be given by

$$A_{F^J} = \begin{pmatrix}
0 & 0 & \vdots & \vdots \\
I_n & \vdots & \vdots & \vdots \\
0 & 0 & \vdots \\
0 & \cdots & 0 & b \\
1 & \cdots & 1 & 1 & 0
\end{pmatrix}. $$

**Example 5.19.** Let $K = K_1 = K_2$ be the slab $\{0 \leq x_1 \leq 1\}$. The following matrix induces a mapping $F^J : K \times \mathbb{R}^+ \to K \times \mathbb{R}^+$:

$$A_{F^J} = \begin{pmatrix}
0 & 0 \\
I_n & \vdots \\
0 & 0 \\
-\varepsilon_1^T & 0 & 1 \\
\varepsilon_1^T & 1 & 0
\end{pmatrix},$$

which induces an order isomorphism on $Cvx_0(K)$.

### 5.4 Generalized geometric convex functions

#### 5.4.1 Introduction

**Definition 5.20.** Let $n \geq 2$, and let $T \subset K$ be two closed convex sets. The subclass of $Cvx(\mathbb{R}^n)$ consisting of functions above $1_K$ and below $1_T$ will be denoted $Cvx_T(K)$, that is

$$Cvx_T(K) := \{f \in Cvx(\mathbb{R}^n) : 1_K \leq f \leq 1_T\}. $$

**Remarks**

1. In the case $n = 1$ this definition would still make sense, but it does not really generalize the case of $T = \{0\}$. Indeed, $Cvx_T(K)$ is isomorphic to $Cvx_0([0,1])$ if $K \setminus T$ is connected, and to $Cvx_0([-1,1])$ otherwise.

2. When $T = \emptyset$, this is the case of convex functions on a window, $Cvx(K)$.

3. When $T = \{0\}$, this is the case of geometric convex functions on a window, $Cvx_0(K)$.

4. Throughout this section we will assume that $K$ is of dimension $n$, and that the interior of $K \setminus T$ is connected.

**Definition 5.21.** Let $\mathcal{T} : Cvx_{T_1}(K_1) \to Cvx_{T_2}(K_2)$ be an order preserving isomorphism, and $F : K_1 \times \mathbb{R}^+ \to K_2 \times \mathbb{R}^+$ a fractional linear map such that $\text{epi}(Tf) = F(\text{epi}(f))$ for every $f \in Cvx_{T_1}(K_1)$. The transform $\mathcal{T}$ and the map $F$ are said to be of $I$-type in two cases: the first, if $F$ is linear map, and the second, if $F$ is a non affine fractional linear map with its defining hyperplane containing a ray in the $\mathbb{R}^+$ direction. Otherwise, $\mathcal{T}$ and $F$ are said to be of $J$-type.

Note that this definition coincides with that of the particular case $Cvx_0(K)$, given in the previous section, 5.3.2.

**Definition 5.22.** Let $\mathcal{T} : Cvx_{T_1}(K_1) \to Cvx_{T_2}(K_2)$ be an order reversing isomorphism. We say that $\mathcal{T}$ is of $A$-type if the composition $\mathcal{T} \circ A$ is an order preserving isomorphism of $I$-type, otherwise we say $\mathcal{T}$ is of $L$-type.

We deal with order isomorphisms from $Cvx_{T_1}(K_1)$ to $Cvx_{T_2}(K_2)$. We show that order preserving isomorphisms are induced by fractional linear point maps on $K_1 \times \mathbb{R}^+$, which are always of $I$-type.
We show that up to a composition with such transforms, the only order reversing isomorphism is the geometric duality $\mathcal{A}$. It may be formulated for order preserving or for order reversing transforms:

**Theorem 5.23.** Let $n \geq 2$, and let $T_1 \subset K_1 \subset \mathbb{R}^n, T_2 \subset K_2 \subset \mathbb{R}^n$ be four non empty, convex, compact sets, and assume that $\text{int}(K_1) \neq \emptyset$, and that $\text{int}(K_1 \setminus T_1)$ is connected. If $T : \text{Cvx}_T(K_1) \rightarrow \text{Cvx}_T(K_2)$ is an order preserving isomorphism, then there exists a fractional linear map $F : K_1 \times \mathbb{R}^+ \rightarrow K_2 \times \mathbb{R}^+$ such that for every $f \in \text{Cvx}_T(K_1)$, we have

$$\text{epi}(Tf) = F(\text{epi}(f)).$$

Moreover, $F$ is of $T$-type, and in particular $T_2$ is a fractional linear image of $T_1$, and $K_2$ is a fractional linear image of $K_1$.

**Theorem 5.24.** Let $n \geq 2$, let $K \subset T \subset \mathbb{R}^n$ be two convex sets such that $0 \in \text{int}(K)$, and assume that $K$ does not contain a full line, and that $\text{int}(T \setminus K)$ is connected. Let $T' \subset K' \subset \mathbb{R}^n$ be two non empty, convex, compact sets, and assume that $\text{int}(K') \neq \emptyset$. If $T : \text{Cvx}_K(T) \rightarrow \text{Cvx}_T(K')$ is an order reversing isomorphism, then $T$ is of $\mathcal{A}$-type. In particular, $T', K'$ are fractional linear images of $T, K$ respectively.

**Proof of Theorem 5.24.** The composition $\tilde{T} := T \circ A : \text{Cvx}_T(K) \rightarrow \text{Cvx}_T(K')$ is an order preserving isomorphism, and the assumptions on $T$ and $K$ imply that $T, K, T', K'$ satisfy the conditions of Theorem 5.23. Indeed, $T, K$ are non empty convex sets, and $0 \in \text{int}(K)$ implies that they are compact. Since $K$ does not contain a full line, $K'$ is not contained in any hyperplane, thus it has non empty interior. It is easy to check that for two convex sets $A \subset B$, $\text{int}(B \setminus A)$ is connected if and only if $\text{int}(A \setminus B)$ is connected, thus we may apply Theorem 5.23. We get that $\tilde{T}$ is induced by some fractional linear map $F : T \times \mathbb{R}^+ \rightarrow T' \times \mathbb{R}^+$, which is of $T$-type, thus $T$ is of $\mathcal{A}$-type, as desired.

For the proof of Theorem 5.23, we first need to define and characterize extremal elements in the class $\text{Cvx}_T(K)$. Then we show that extremal elements are mapped to such, which will imply that the transform induces a point map on a subset of $\mathbb{R}^{n+1}$. We show that this point map is interval preserving for a sufficiently large set of intervals, in order to use a theorem of Shiffman [15], which states that the map is fractional linear. Finally we show that under our assumptions, the transform is of $T$-type, thus completing the proof of Theorem 5.23. We will need the following notations throughout this section.

- Let $n \geq 2$, and let $A \subset B \subset \mathbb{R}^n$ be two closed convex sets. We denote

$$\mathcal{K}^n(A, B) = \{K \subset \mathbb{R}^n : K \text{ is closed, convex, and } A \subset K \subset B\}.$$ 

For $\mathcal{K}^n(\emptyset, \mathbb{R}^n)$ we simply write $\mathcal{K}^n$. Note that if $T \neq \emptyset$, any element in $\mathcal{K}^{n+1}(\text{epi}(1_T), \text{epi}(1_K))$ is an epi-graph of some function $f \in \text{Cvx}_T(K)$.

- For the convex hull of two sets $A$ and $B$ we write

$$A \vee B = \bigcap_{K \in \mathcal{K}^n(A \cup B) \subset K} K.$$ 

### 5.4.2 Extremal elements

**Definition 5.25.** A set $K \in \mathcal{K}^n(A, B)$ is called extremal if $\forall T, P \in \mathcal{K}^n(A, B):$

$$K = T \vee P \quad \implies \quad K = T, \quad \text{or} \quad K = P.$$ 

**Definition 5.26.** A function $f \in \text{Cvx}_T(K)$ is called extremal if $\forall g, h \in \text{Cvx}_T(K):$

$$f = \inf\{h, g\} \quad \implies \quad f = h, \quad \text{or} \quad f = g.$$
Another formulation of which is:
\[ epi(f) = epi(h) \lor epi(g) \quad \implies \quad f = h, \quad \text{or} \quad f = g, \]
which (in the case \( T \neq \emptyset \)), means that \( epi(f) \) is extremal in \( K^{n+1}(epi(1_T), epi(1_K)) \).

Recall that for bijective transforms, order-preservation in both directions is equivalent to preservation of the lattice operations inf and sup (see Proposition 3.5 and Remark 4.5). Since the extremality property is defined by the inf operation, all extremal elements in the domain are mapped to all extremal elements in the range.

In the next few lemmas we investigate extremal elements of \( Conv(K) \). We need the following simple observation.

**Lemma 5.27.** Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be an affine linear functional and \( K \subset \mathbb{R}^n \) a closed, convex set that does not contain a ray on which \( \varphi \) is constant. If \( \varphi(K) > 0 \), then there exists some \( c \in \mathbb{R} \) such that \( \varphi(K) \geq c > 0 \).

**Proof.** Consider the slab \( S = \varphi^{-1}([0,1]) \). If the intersection \( K \cap S \) is empty then we may take \( c = 1 \). Assume otherwise, then \( K \cap S \) is a closed convex set, and moreover, it is bounded. Indeed, the slab \( S \) contains only rays on which \( \varphi \) is constant, and \( K \) contains no such rays, therefore \( K \cap S \) contains no rays, and one can easily verify that for a convex set this is equivalent to boundedness. Since \( K \) is compact and \( \varphi \) is continuous, there exists \( x_0 \in K \) such that \( \varphi(K) \geq \varphi(x_0) \equiv c > 0 \).

**Lemma 5.28.** Let \( n \geq 2 \), and let \( T \subset K \subset \mathbb{R}^n \) be two non empty, compact, convex sets. Consider the subsets \( A = T \times \mathbb{R}^+, B = K \times \mathbb{R}^+ \) of \( \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \). If \( K \in K^{n+1}(A, B) \) is extremal, then \( K = A \lor \{x\} \), for some \( x \in B \).

**Proof of Lemma 5.28.** Let \( K \in K^{n+1}(A, B) \) be extremal. By a Krein-Milman type theorem for non compact sets, see [11], \( K \) is the convex hull of its extreme points and extreme rays. Since the only rays in \( K \) are translates of \( \{0\} \times \mathbb{R}^+ \), and any extreme ray emanate from an extreme point, \( K \) is the convex hull of \( A \) and its extreme points. Finally, since the set of exposed points is dense in the set of extreme points, see [17], if we denote by \( E \) the set of exposed extreme points of \( K \) which are not in \( A \), we have \( K = A \lor E \) (actually in [17] this is proved for compact convex sets, but the non compact case follows as an immediate consequence, and also appears in a more general setting of normed spaces in [12], as Theorem 2.3).

Let \( x_1 \in E \), and let \( \varphi_1 \) be an affine functional such that \( \varphi_1(K \setminus \{x_1\}) > 0 \) and \( \varphi_1(x_1) = 0 \). Note that \( \varphi_1 \) cannot be constant on translates of \( \{0\} \times \mathbb{R}^+ \), since then it would be constant 0 on the translate of \( \{0\} \times \mathbb{R}^+ \) emanating from \( x_1 \), contradicting strict positivity on \( K \setminus \{x_1\} \). If \( E \subseteq A \lor \{x_1\} \), the proof is complete. Assume otherwise; that there exists \( x_2 \in E \setminus \{A \lor \{x_1\}\} \). We may separate \( x_2 \) from the closed set \( A \lor \{x_1\} \) by an affine functional \( \varphi_2 \) such that \( \varphi_2(A \lor \{x_1\}) > 0 \) and \( \varphi_2(x_2) < 0 \). Denote by \( H_2^+ \) the (closed) half space on which \( \varphi_2 \leq 0 \) and by \( H_2^- \) the (closed) half space on which \( \varphi_2 \geq 0 \). Consider the sets \( K^+ = A \lor (E \cap H_2^+) \), \( K^- = A \lor (E \cap H_2^-) \). Clearly \( K^+ \in K^{n+1}(A, B) \) and \( K = K^+ \lor K^- \), thus by extremality of \( K \) we must have either \( K = K^+ \) or \( K = K^- \). Since both \( A \) and \( E \cap H_2^- \) are contained in \( H_2^- \), so is \( K^+ \), thus \( x_2 \notin K^+ \). This implies \( K \neq K^+ \), i.e. \( K = K^- \). We next show that \( x_1 \notin K^- \), which leads to the wanted contradiction. To this end we claim that \( \varphi_1(K^-) > 0 = \varphi_1(x_1) \). Indeed, \( \varphi_1(A) > 0 \), and the only rays contained in \( A \) are translates of \( \{0\} \times \mathbb{R}^+ \), on which \( \varphi_1 \) is not constant. Thus, by Lemma 5.27, there exists some constant \( c \) such that \( \varphi_1(A) \geq c > 0 \). Similarly \( \varphi_1(K \cap H_2^-) \geq c' > 0 \). For the convex hull we get \( \varphi_1(K^-) \geq \min\{c, c'\} > 0 \), so \( x_1 \notin K^- \).

The following is a simpler version of Lemma 5.28, which we do not use in this paper but add it to complete the picture.

**Lemma 5.29.** Let \( n \geq 2 \), and let \( A \subset B \subset \mathbb{R}^n \) be two compact convex sets. If \( K \in K^n(A, B) \) is extremal, then \( K = A \lor \{x\} \), for some \( x \in B \).
We omit the proof, as it is contained in the proof of the previous lemma (the use of Lemma 5.27 is replaced by a straightforward compactness argument).

A reformulation of Lemma 5.28 is:

**Lemma 5.30.** Let \( n \geq 2 \), and let \( T \subset K \subset \mathbb{R}^n \) be two non empty, compact, convex sets. If \( f \in \text{Cvx}_T(K) \) is extremal, then either:

- \( f = 1_T \), or:
- \( f = \inf \{1_T, \delta_{k,h}\} \) for some \( k \in K \setminus T \) and \( h \geq 0 \).

**Proof of Lemma 5.30.** By Lemma 5.28, \( \text{epi}(f) = \text{epi}(1_T) \lor \{x\} \) for some \( x \in \text{epi}(1_K) \). If \( x \in \text{epi}(1_T) \), then \( f = 1_T \). If \( x \not\in \text{epi}(1_T) \), then \( f = \inf \{1_T, \delta_{k,h}\} \) for some \( k, h \) as stated above. \( \square \)

### 5.4.3 The point map

So far we have seen that an order isomorphism \( \mathcal{T} : \text{Cvx}_{T_1}(K_1) \to \text{Cvx}_{T_2}(K_2) \) is in particular a bijection between the extremal families. Clearly \( \mathcal{T}(1_{T_1}) = 1_{T_2} \). Aside of the maximal element \( 1_{T_1} \), each extremal function in \( \text{Cvx}_{T_1}(K_1) \) corresponds to a point in \( \mathbb{R}^{n+1} \), thus \( \mathcal{T} \) induces a bijective point map \( F : (K_1 \setminus T_1) \times \mathbb{R}^+ \to (K_2 \setminus T_2) \times \mathbb{R}^+ \). Denote by \( E_1 \) the interior of the set \( (K_1 \setminus T_1) \times \mathbb{R}^+ \) (by our assumption, it is connected). The sets \( (K_1 \setminus T_1) \times \mathbb{R}^+ \) inherit the partial order structure of \( \text{Cvx}_{T_1}(K_1) \), after restriction to the set of extremal elements, and the bijective map \( F : (K_1 \setminus T_1) \times \mathbb{R}^+ \to (K_2 \setminus T_2) \times \mathbb{R}^+ \) is an order isomorphism. In the following lemmas we will use the fact that the injective map \( F|_{E_1} : E_1 \to (K_2 \setminus T_2) \times \mathbb{R}^+ \) is an order isomorphism on its image, to prove that for some intervals \( [a,b] \subset E_1 \), \( F([a,b]) \) is again an interval (these can be characterized as the ones that, extended to a full line, do not intersect \( \text{epi}(1_{T_1}) \)). Since the use of the uniqueness Theorem 2.17 requires the preservation of all intervals, we apply a result by Shiffman from [15], which roughly states that if a set of points is covered by an open set of intervals which are all mapped to intervals, then the inducing map is fractional linear. More precisely, denote by \( \mathcal{L}(\mathbb{R}^n) \) the set of all lines in \( \mathbb{R}^n \), not necessarily intersecting the origin. It may be seen as a subset of the Grassmannian \( G_{n+1,2} \), therefore it is equipped with the usual inherited metric topology (for some details see Remark 4 below). Denoting by \( \mathcal{L}(U) \subseteq \mathcal{L}(\mathbb{R}^n) \) the set of all such lines intersecting a given set \( U \subseteq \mathbb{R}^n \), we have

**Theorem 5.31.** [15] Let \( n \geq 2 \), let \( U \) be an open connected set in \( \mathbb{R}^n \), and let \( \mathcal{L}_0 \) be an open subset of \( \mathcal{L}(U) \), which covers \( U \), i.e. \( U \subseteq \cup_{l \in \mathcal{L}_0} l \). Assume that \( F : U \to \mathbb{R}^n \) is a continuous injective map, and that \( F(l \cap U) \) is contained in a line for all \( l \in \mathcal{L}_0 \). Then \( F \) is fractional linear.

**Remarks**

1. Theorem 5.31 is adjusted to the real, linear, setting (i.e. when \( U \) is a subset of \( \mathbb{R}^n \), which is embedded in \( \mathbb{R}^{P^n} \)), and is a particular case of the more general statement Shiffman proves in [15]. The general result applies for subsets of \( \mathbb{R}^{P^n} \) or \( \mathbb{C}^{P^n} \), and states that the map \( F \) is projective linear.

2. In [15], Theorem 5.31 is proved for \( \mathbb{R}^{P^n} \) and \( \mathbb{C}^{P^n} \) simultaneously. However, considering only the case of \( \mathbb{R}^{P^n} \), one may check (by following the proof in [15]), that in this case continuity is actually not required, and may be replaced by the following weaker condition: if \( I \subset U \) is an interval and \( I \subset l \in \mathcal{L}_0 \), then \( F(I) \) is again an interval. We will use this stronger version of Theorem 5.31.

3. In our setting, we have epi-graphs of functions in \( \text{Cvx}_T(K) \), therefore we apply Theorem 5.31 to the function \( F \) defined on the set \( U = E_1 \subset \mathbb{R}^{n+1} \).

4. A line in \( \mathcal{L}(\mathbb{R}^n) \) is determined by its closest point to the origin and its direction. That is, for every \( l \in \mathcal{L}(\mathbb{R}^n) \) let \( x_l \in l \) be the unique point satisfying \( |x_l| = \min \{|x| : x \in l\} \), and let \( u_l \in \{x_l\}^{\perp} \) be one of the two points satisfying \( l = \{x_l + tu_l, t \in \mathbb{R}\}, |u_l| = 1 \) (the other being \(-u_l\)). Note that directions in \( \{x_l\}^{\perp} \) correspond to \( S^{n-2} \) if \( x_l \neq 0 \), and to \( S^{n-1} \) if \( x_l = 0 \). Denoting the line \( l \) by the pair \((x_l, u_l)\), we
get a correspondence between \( \mathcal{L}(\mathbb{R}^n) \subset G_{n+1,2} \) and \( (\mathbb{R}^n \setminus \{0\}) \times S^{n-2} \cup \{0\} \times S^{n-1} \), which is \( 1-1 \), modulo the \( \pm \) choice in the direction \( u \). The metric \( d \) on \( \mathcal{L}(\mathbb{R}^n) \) is inherited from that on \( G_{n+1,2} \), and it follows that \( d((x,u_1),(x,u_2)) = |u_1 - u_2| \), and that \( d((x,u),(x',u')) = d((x,1),(x',1)) \), where \( \hat{x} = \frac{x}{|x|} \).

A neighborhood of \((x,u)\) is therefore constructed by perturbing simultaneously \( x \) and \( u \). It can be checked that such a perturbation contains the following “cylinder” of lines; fix a point \( z \in (x,u) \), let \( M > 0 \), and let \( a, b \in (x,u) \) satisfy \( |a - z| = |b - z| = M \), and let \( A, B \) be open balls of radius \( 1/M \) and centers \( a, b \) respectively. We take our “cylinder” of lines to be \( \mathcal{L}_{l,z,M} := \mathcal{L}(A) \cap \mathcal{L}(B) \). For every \( z \in l \) and every \( M > 0 \), there exists a small perturbation of \( l = (x,u) \) which is contained in \( \mathcal{L}_{l,z,M} \).

More precisely, there exists \( \varepsilon > 0 \) such that \( G_{l,\varepsilon} = \{(y,v) : |y-x| + |v-u| < \varepsilon\} \subset \mathcal{L}_{l,z,M} \). This fact is useful in the proof of Lemma 5.32.

Let \( \tilde{\mathcal{L}}_0 := \mathcal{L}(E_1) \setminus \mathcal{L}(T_1 \times \mathbb{R}^+) \), that is, \( \tilde{\mathcal{L}}_0 \) is the set of lines through \( E_1 \) (the domain of \( F \)), which do not intersect the inner half cylinder \( T_1 \times \mathbb{R}^+ \). In Lemma 5.32 we prove that the interior of \( \tilde{\mathcal{L}}_0 \), denoted \( \mathcal{L}_0 \), is an open subset of \( \mathcal{L}(E_1) \) which covers \( E_1 \), and in Lemma 5.34 we prove that \( F(l \cap E_1) \) is contained in a line for all \( l \in \mathcal{L}_0 \), and that intervals which are segments of lines in \( \mathcal{L}_0 \) are mapped to intervals.

**Lemma 5.32.** The open set \( \mathcal{L}_0 = \text{int}(\tilde{\mathcal{L}}_0) \) described above, covers \( E_1 \). That is,

\[
E_1 \subseteq \bigcup_{l \in \mathcal{L}_0} l.
\]

**Proof of Lemma 5.32.** Let \( x \in E_1 \). We may separate \( x \) from the closed set \( T_1 \times \mathbb{R}^+ \) by a hyperplane. Denote by \( H \) the translate of this hyperplane containing \( x \). We claim that if \( l \subset H \) is a line containing \( x \), which is not parallel to \( \{0\} \times \mathbb{R}^+ \), then \( l \in \mathcal{L}_0 \). Indeed, it is clear that \( l \in \mathcal{L}_0 \). Consider the set of lines \( \mathcal{L}_{l,z,M} \), for some \( M > 0 \) (see the last remark). It is an open neighborhood of \( l \), and since \( T_1 \) is compact and \( l \) is not parallel to \( \{0\} \times \mathbb{R}^+ \), we have (for large enough \( M \)) that \( \mathcal{L}_M \subset \tilde{\mathcal{L}}_0 \), thus \( l \in \mathcal{L}_0 \). This implies \( x \in \bigcup_{l \in \mathcal{L}_0} l \), and hence \( \mathcal{L}_0 \) covers \( E_1 \).

Next we prove that the set \( \mathcal{L}_0 \) consists exactly of all the lines in \( \mathcal{L}(E_1) \), with the property that points along these lines are not comparable.

**Lemma 5.33.** Let \( a, b \in E_1 \) be two different points, and let \( l_{a,b} \) be the line containing \( a \) and \( b \). Then \( l_{a,b} \notin \mathcal{L}_0 \) if and only if \( a \) and \( b \) are comparable.

**Proof of Lemma 5.33.** The point \( a \) is “greater” than the point \( b \), if and only if \( a \in \text{epi}(1_{T_1}) \cup \{b\} \), therefore \( a \) and \( b \) are comparable if and only if \( l_{a,b} \) is in the closure of \( \mathcal{L}(\{b\}) \cap \mathcal{L}(T_1 \times \mathbb{R}^+) \subset \mathcal{L}(E_1) \setminus \mathcal{L}_0 \) (in fact, the closure is only necessary if \( a \) and \( b \) are on the same translate of \( \{0\} \times \mathbb{R}^+ \)). This closure does not intersect \( \mathcal{L}_0 \), the interior of \( \tilde{\mathcal{L}}_0 \), therefore we have shown:

\[
a, b \text{ are comparable} \quad \Rightarrow \quad l_{a,b} \notin \mathcal{L}_0.
\]

If \( l_{a,b} \notin \mathcal{L}_0 \) there are two cases. First assume \( l_{a,b} \notin \tilde{\mathcal{L}}_0 \) (that is, \( l_{a,b} \) intersects \( T_1 \times \mathbb{R}^+ \)). Thus \( a \) and \( b \) are comparable (one is in the convex hull of the other and \( \text{epi}(1_{T_1}) \)). Otherwise, assume \( l_{a,b} \in \mathcal{L}_0 \). Since it is not in the interior \( \mathcal{L}_0 \), we get \( l_{a,b} \in \partial \mathcal{L}_0 \). Since \( \mathcal{L}(E_1) \setminus \bigcup_{x \in K_1 \setminus T_1} \{x\} \times \mathbb{R} \) is open, and \( \mathcal{L}(T_1 \times \mathbb{R}^+) \) is closed, \( \mathcal{L}_0 = \text{int}(\mathcal{L}_0) \) must contain \( \left( \mathcal{L}(E_1) \setminus \bigcup_{x \in K_1 \setminus T_1} \{x\} \times \mathbb{R} \right) \setminus \mathcal{L}(T_1 \times \mathbb{R}^+) \), and therefore \( \mathcal{L}_0 = \tilde{\mathcal{L}}_0 \setminus \bigcup_{x \in K_1 \setminus T_1} \{x\} \times \mathbb{R} \). Thus \( l_{a,b} \in \partial \mathcal{L}_0 \) implies that \( l_{a,b} \) is parallel to the ray \( \{0\} \times \mathbb{R} \), and that \( a \) and \( b \) are comparable. Thus we have shown:

\[
l_{a,b} \notin \mathcal{L}_0 \quad \Rightarrow \quad a, b \text{ are comparable}.
\]

**Lemma 5.34.** If \( l \in \mathcal{L}_0 \), then \( F(l \cap E_1) \) is contained in a line. Moreover, if \( I \subset E_1 \) is an interval and \( I \subset l \in \mathcal{L}_0 \), then \( F(I) \) is again an interval.
Proof of Lemma 5.34. The intersection of every \( l \in \mathcal{L}_0 \) with the convex set \( K_1 \times \mathbb{R}^+ \) is either a ray or an interval. Since \( l \in \mathcal{L}_0 \), it does not intersect \( T_1 \times \mathbb{R}^+ \), and therefore also \( l \cap E_1 \) is either a ray or an interval. Thus, by Lemma 5.33, it is enough to show that for every two non comparable points \( a, b \in E_1 \), we have \( F([a,b]) = [F(a), F(b)] \). Denote for every \( x \in E_1 \) by \( \delta_x \) the function with epi-graph \( (T_1 \times \mathbb{R}^+) \cup \{x\} \). Of all the extremal functions \( \delta_x \), only those corresponding to \( x \in [a,b] \) have the following minimality property: \( \delta_x \geq \inf\{\delta_a, \delta_b\} \), and for every \( y \) with \( \delta_y \geq \inf\{\delta_a, \delta_b\} \), we have \( \delta_y \not< \delta_x \). This property is preserved by \( F \), therefore the interval \( [a,b] \) is mapped to the interval \( [F(a), F(b)] \). \( \square \)

Proof of Theorem 5.23. The set \( E_1 \) is open and connected. Therefore, by Lemmas 5.32 and 5.34, we may apply Theorem 5.31 (see Remark 2 after Theorem 5.31) to the map \( F|_{E_1} \), and conclude it is fractional linear. To see that \( F : (K_1 \setminus T_1) \times \mathbb{R}^+ \to (K_2 \setminus T_2) \times \mathbb{R}^+ \) is fractional linear, note that a point in the boundary of \( (K_1 \setminus T_1) \times \mathbb{R}^+ \) is the infimum of all the points below it which are in \( E_1 \). To see that \( F \) induces the transform \( T : Cvx_{T_1}(K_1) \to Cvx_{T_2}(K_2) \), note that the epi-graph of a function \( f \in Cvx_{T_1}(K_1) \) corresponds to the set of extremal functions above it, and that \( f \) is given as the infimum of those extremal functions. Finally we need to show that \( F \) is of \( \mathcal{I} \)-type, that is, assuming \( F \) is a non affine fractional linear map, we need to show that the defining hyperplane is parallel to the \( \mathbb{R}^+ \) direction. If it is not, then by Section 2.4.2, the half cylinder \( K_1 \times \mathbb{R}^+ \) is mapped to some cone, which must be \( K_2 \times \mathbb{R}^+ \). But since \( K_2 \) is compact, \( K_2 \times \mathbb{R}^+ \) is not a cone. Therefore the map \( F \) is either affine, or it is non affine, but with a defining hyperplane containing the direction of the epi-graphs (the ray \( \{0\} \times \mathbb{R}^+ \)). \( \square \)

References

[1] Alesker, S., Artstein-Avidan, S., Faifman, D. and Milman, V., A characterization of product preserving maps with applications to a characterization of the Fourier transform. To appear in Illinois Journal of Mathematics.

[2] Alesker, S., Artstein-Avidan, S. and Milman, V., A characterization of the Fourier transform and related topics. Linear and complex analysis, 11–26, Amer. Math. Soc. Transl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

[3] Artin E., Geometric Algebra. Wiley-Interscience (1988) ISBN: 0471608394.

[4] Artstein-Avidan, S. König, H. and Milman, V., The chain rule as a functional equation. J. Funct. Anal., to appear.

[5] Artstein-Avidan, S. and Milman, V., The concept of duality in convex analysis, and the characterization of the Legendre transform. Ann. of Math. (2) 169 (2009), no. 2, 661–674.

[6] Artstein-Avidan, S. and Milman, V., The concept of duality for measure projections of convex bodies. J. Funct. Anal 254 (2008), 2648-2666.

[7] Artstein-Avidan, S. and Milman, V., A characterization of the concept of duality. Electronic Research Announcements in Mathematical Sciences, AIMS (2007), Vol. 14, Pages 48-65.

[8] Artstein-Avidan, S. and Milman, V., Hidden structures in the class of convex functions and a new duality transform. J. Eur. Math. Soc. 13, 975-1004.

[9] Artstein-Avidan, S. and Slomka, B.A., A new fundamental theorem of affine geometry and applications. Preprint.

[10] Böröczky, K. and Schneider, R., A characterization of the duality mapping for convex bodies. Geom. and Funct. Anal. 18 (2008), 657-667.

[11] Klee, V. L., Extremal structure of convex sets. Arch. Math. 8 (1957), 234-240.
[12] Klee, V. L., *Extremal structure of convex sets II*. Math. Zeitschr. 69 (1958), 90-104.

[13] Prasolov, V. V. and Tikhomirov, V. M. *Geometry (English summary)*. Translated from the 1997 Russian original by O. V. Sipacheva. Translations of Mathematical Monographs, 200. American Mathematical Society, Providence, RI, 2001. xii+257 pp. ISBN: 0-8218-2038-9.

[14] Rockafellar, R.T., Convex Analysis, Princeton University Press, 1970.

[15] Shiffman, B., *Synthetic projective geometry and Poincaré’s theorem on automorphisms of the ball*. Enseign. Math. (2) 41 (1995), no. 3-4, 201–215.

[16] Slomka, B.A., *On duality and endomorphisms of lattices of closed convex sets*. Adv. Geom. (2011) Vol. 11, Issue 2, Pages 225-239.

[17] Straszewicz, S., *Über exponierte Punkte abgeschlossener Punktmengen*, Fund. Math. 24 (1935) 139-143.