A Direct Ultrametric Approach to Additive Complexity and the Shub-Smale Tau Conjecture

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Abstract

The Shub-Smale Tau Conjecture is a hypothesis relating the number of integral roots of a polynomial $f$ in one variable and the Straight-Line Program (SLP) complexity of $f$. A consequence of the truth of this conjecture is that, for the Blum-Shub-Smale model over the complex numbers, P differs from NP. We prove two weak versions of the Tau Conjecture and in so doing show that the Tau Conjecture follows from an even more plausible hypothesis.

Our results follow from a new $p$-adic analogue of earlier work relating real algebraic geometry to additive complexity. For instance, we can show that a non-zero univariate polynomial of additive complexity $s$ can have no more than $1 + s^3(s + 1)(7.5)^s = O(e^s \log s)$ roots in the 2-adic rational numbers $\mathbb{Q}_2$, thus dramatically improving an earlier result of the author. This immediately implies the same bound on the number of ordinary rational roots, whereas the best previous upper bound via earlier techniques from real algebraic geometry was a quantity in $\Omega((22.6)^s)$.

This paper presents another step in the author’s program of establishing an algorithmic arithmetic version of fewnomial theory.

1 Introduction

We show how ultrametric geometry can be used to give new sharper lower bounds for the additive complexity of polynomials. As a consequence, we advance further a number-theoretic approach toward an analogue of the $P \neq \text{NP}$ conjecture for the BSS model over $\mathbb{C}$. Let us now clarify all these notions. (Simple explicit examples of our constructions appear in Section 1.1 below.)

The Blum-Shub-Smale (BSS) model over $\mathbb{C}$ can be thought of roughly as a Turing machine augmented with additional registers that perform arithmetic operations on complex numbers at unit cost. (See [BCSS98] for a much more complete description.) This model of computation possesses a natural analogue of the classical $P \neq \text{NP}$ question: the $P^C \neq \text{NP}^C$ question. This new analogue was introduced with the hope of (a) enabling researchers to enlarge their usual bag of tricks with the arsenal of number theory and analysis and (b) gaining new information on the original $P \neq \text{NP}$ question.

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Indeed, it is now known (see, e.g., [Shu93 SS95 BCSS98]) that $P_C = NP_C \implies NP \subseteq BPP$. Since the latter containment is widely disbelieved, this makes it plausible that $P_C \neq NP_C$. The implications of $P_C \neq NP_C$ for classical disbeliever, still not clear. However, via an examination of the Boolean parts of $P_C$ and $NP_C$ one can see that $P_C \neq NP_C$ would provide some evidence that $BPP \neq AM$ [Kol96 Roj03b]. Better still, Mike Shub and Steve Smale were able to obtain an elegant number-theoretic statement that implies $P_C \neq NP_C$ (see [SS95] or [BCSS98] Thm. 3, Pg. 127). Recall that $\mathbb{Z}$ denotes the integers.

**Definition 1** Suppose a polynomial in one variable $f \in \mathbb{Z}[x_1]$ is expressed as a sequence of the form $(1, x_1, f_2, \ldots, f_N)$, where $f_N = f(x_1)$, $f_0 := 1$, $f_1 := x_1$, and for all $i \geq 2$ we have that $f_i$ is a sum, difference, or product of some pair of elements $(f_j, f_k)$ with $j, k < i$. Such a computational sequence is a straight-line program (SLP). We then let $\tau(f)$ denote the smallest possible value of $N - 1$, i.e., the smallest length for such a computation of $f$. 

**The Shub-Smale $\tau$-Conjecture** There is an absolute constant $\kappa \geq 1$ such that for all $f \in \mathbb{Z}[x_1] \setminus \{0\}$, the number of distinct roots of $f$ in $\mathbb{Z}$ is no more than $(\tau(f) + 1)^\kappa$.

It is easily checked that $\deg f \leq 2^{\tau(f)}$ so we at least know that $f$ has no more than $2^{\tau(f)}$ integral roots. However, little else is known about the relation of $\tau(f)$ to the number of integral roots of $f$: as of mid-2003, the $\tau$-conjecture remains open, even in the special case $\kappa = 1$. The polynomial $(x - 2^1)(x - 2^2) \cdots (x - 2^9)$ easily shows that the $\tau$-conjecture fails if we allow $\kappa < 1$.

To state our first main result, let us introduce a family of weaker hypotheses depending on an integer parameter $p$.

**The $p$-adic Digit Conjecture** There is an absolute constant $c_p > 0$ such that for all $f \in \mathbb{Z}[x_1] \setminus \{0\}$, the number of distinct roots $x \in \mathbb{Z}$ of $f$ with $x \equiv 1 \mod p$ is no more than $(\tau(f) + 1)^{c_p}$.

Clearly, the $\tau$-conjecture implies the $p$-adic Digit Conjecture for any integer $p$. However, a strong converse holds as well.

**Theorem 1** Suppose the $p$-adic Digit Conjecture is true for some fixed prime $p$. Then $P_C \neq NP_C$. In particular, if the $p$-adic Digit Conjecture is true for some fixed prime $p$, then the $\tau$-conjecture is true.

The last implication follows easily from our next main theorem.

**Definition 2** Letting $R$ be any ring, we say that $f \in R[x_1]$ has additive complexity $\leq s$ (over $R$) iff there exist constants $c_1, d_1, \ldots, c_s$, $d_s, c_{s+1} \in R$ and arrays of nonnegative integers $\left[m_{i,j}\right]$ and $\left[m'_{i,j}\right]$ with $f(x) = c_{s+1} \prod_{i=0}^{s} X_i^{m_{i,s+1}}$, where $X_0 = x$, $X_1 = c_1 X_0^{m_{0,1}} + d_1 X_0^{m_{0,1}}$, and $X_j = c_j \left( \prod_{i=0}^{j-1} X_i^{m_{i,j}} \right) + d_j \left( \prod_{i=0}^{j-1} X_i^{m'_{i,j}} \right)$ for all $j \in \{2, \ldots, s\}$. We then define the additive complexity (over $R$) of $f$, $\sigma_R(f)$, to be the least $s$ in such a presentation of $f$ as an algebraic expression. Finally, additive complexity without appellation, and $\sigma(f)$ will be understood to mean $\sigma_{\mathbb{Z}}(f)$. 

Note in particular that repeated additions or subtractions in different sub-expressions are thus not counted, e.g., $\sigma(9(x - 7)^{99}(2x + 1)^{43} - 11(x - 7)^{999}(2x + 1)^3) \leq 3$. Recall that for any prime number $p \in \mathbb{Z}$, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ respectively denote the $p$-adic integers, the $p$-adic rational numbers, and the $p$-adic complex numbers (cf. Section 2).
Definition 3  Given any $p$-adic integer $x \in \mathbb{Z}_p$ define its $p$-adic valuation, $\text{ord}_p x$, to be the largest $s$ such that $p^s$ divides $x$ (so $\text{ord}_p 0 := +\infty$). We then extend $\text{ord}_p$ to $\mathbb{Q}_p$ by setting $\text{ord}_p \frac{m}{n} := \text{ord}_p m - \text{ord}_p n$, for any nonzero $m, n \in \mathbb{Z}_p$. Finally, we define the $p$-adic norm of any $p$-adic rational number $x$, $|x|_p$, to be $p^{-\text{ord}_p x}$. ∙

Theorem 2  Let $p$ be any prime number and let $\#$ denote the operation of taking set cardinality. Also, following the notation above, let $N_p(s) := \max_{f \in \mathbb{Z}[x]} \# \{|x|_p \mid x \in \mathbb{C}_p \setminus \{0\}$ is a root of $f$. Then

$$s \leq N_p(s) \leq s(s + 1)/2.$$ 

Furthermore, the lower bound can be realized even if one restricts to roots in $\mathbb{Z}$. In particular, the number of possible locations for the first nonzero base-$p$ digit of an integral root of $f$ is no more than $s(s + 1)/2$.

Remark 1  Alternatively, the quantity $N_p(s)$ can be thought of as the $p$-adic analogue of the maximum number of distinct absolute values that can occur for the roots of a fixed polynomial of bounded additive complexity. Note in particular that $(x + 1)^d - 1$ has additive complexity $\leq 2$ but exactly $\lfloor \frac{d}{2} \rfloor$ distinct absolute values for its complex roots. So, differing radically from the analogous situation over $\mathbb{C}$, the maximum number of distinct absolute values over $\mathbb{C}_p$ is not only a finite function of $s$ and $p$, but sub-quadratic and independent of $p$. ∙

Since $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{Q}_p \subset \mathbb{C}_p$, we can thus think of Theorem 2 as a weakening of the $\tau$-conjecture where we count $p$-adic absolute values of roots instead of roots. The author believes that a lower bound quadratic in $s$ is also possible.

Theorem 2 in turn implies the truth of another weak version of the $\tau$-conjecture: Theorem 3 below, which may also be of independent interest in arithmetic circuit complexity.

Theorem 3  Any polynomial $f \in \mathbb{Z}[x] \setminus \{0\}$ of additive complexity $\leq s$ has no more than $O \left( pe^s \log s \right)$ roots in $\mathbb{Q}_p$. More precisely, $1 + (p - 1)s^2(7.5)^s N_p(s)$ suffices as an upper bound. In particular, the number of rational roots of an $f \in \mathbb{Z}[x] \setminus \{0\}$ with additive complexity $\leq s$ over $\mathbb{Z}$ is no more than $1 + s^3(s + 1)(7.5)^s s!$.

Remark 2  Note that $\tau(f)$ is already unbounded for polynomials of the form $a + bx^d$, which clearly have additive complexity 1. So $\sigma(f)$ can be tremendously smaller than $\tau(f)$. Our theorem above can thus be interpreted as a weak, but definitely non-trivial version of the $\tau$-conjecture. ∙

Theorem 3 can be sharpened further (see Section 6 for the details) and even the coarse bound on rational roots above is the best to date in terms of additive complexity.

Theorem 3 follows easily from a weakening of the $p$-adic Digit Conjecture that we actually can prove: Theorem 4 below gives an upper bound on the number of roots of $f$ with “first $p$-adic digit 1” while, not polynomial in $\tau(f)$, is exponential in $\sigma(f)$, explicit, and applies to $p$-adic complex roots as well.

Theorem 4  For any $r > 0$, let $C_p'(s, r)$ denote the maximum, over all $f \in \mathbb{Z}[x]$ with $\sigma(f) \leq s$, of

$$\# \left\{ \text{Roots of } f \text{ in the closed complex } p\text{-adic disk of radius } \frac{1}{p^r} \text{ about } 1 \right\}.$$ 

Then $C_p'(s, r) \leq s^2 s! \left( 3 + \frac{2}{r} \log_p \left( \frac{2}{\tau \log p} \right) \right)^s$. 

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Theorem 4 can also be sharpened even further and the details are in Section 5. The author suspects that the bound above can even be improved to \((1 - \log x)^{O(s)}\).

The proofs of Theorems 1, 2, 3 and 4 appear respectively in Sections 3, 4, 5 and 6. In closing this introduction we note that all our preceding results admit natural extensions to \(p\)-adic fields (i.e., finite algebraic extensions of the usual \(p\)-adic numbers), roots of bounded degree over number fields (finite algebraic extensions of the ordinary rational numbers), and even systems of multivariate polynomial equations. These extensions will be detailed in the full version of this paper.

Let us now further detail earlier work on these problems as well as some background on \(p\)-adic numbers.

### 1.1 Related Work and More Examples

Let us first point out that there are examples showing that we must avoid \(\mathbb{R}\) if we are to use field arithmetic tricks to solve the \(\tau\)-conjecture.

**Example 1** Consider the recurrence \(g_{j+1} := 4g_j (1-g_j)\) with \(g_1 := 4x(1-x)\). It is then easily checked\(^1\) that \(g_j(x) - x\) has exactly \(2^j\) roots in the open interval \((0,1)\), but \(\tau(g_j(x) - x) = O(j)\). \(\diamond\)

The existence of an analogous example over the \(p\)-adic rationals is still an open question.

That the \(\tau\)-conjecture is still open is a testament to the fact that we know far less about the complexity measures \(\sigma\) and \(\tau\) than we should. For example, there is still no more elegant method known to compute \(\tau\) for a fixed polynomial than brute force enumeration. Also, the computability of additive complexity is still an open question, although a more efficient variant (allowing radicals as well) can be computed in triply exponential time [GK98].

As for earlier earlier approaches to the \(\tau\)-conjecture, some investigations into \(\tau\) were initiated by de Melo and Svaiter [dMS96] (in the special case of constant polynomials) and Moreira [Mor97]. Unfortunately, not much more is known than (a) \(\tau(f)\) is “usually” bounded below by \(h(f)/\log h(f)\) (where \(h(f)\) is \((1 + \deg f) \max \log |c|\) and the maximum ranges over all coefficients \(c\) of \(f\)) and (b) \(\tau \leq 1.1 h(f)/\log h(f)\) for \(h(f)\) sufficiently large [Mor97, Thm. 3].

So information relating the integral roots of \(f\) with \(\tau(f)\) was essentially non-existent, at least until independent work of Dima Yu. Grigoriev [Gri82] and Jean-Jacques Risler [Ris85] that related additive complexity with the number of real roots. (This was also preceded by important seminal work of Allan Borodin and Stephen A. Cook [BC76] first proving the surprising fact that one could indeed bound the number of real roots in terms of additive complexity.) In particular, they showed that an \(f \in \mathbb{R}[x_1]\) with \(\sigma_{\mathbb{R}}(f) \leq s\) could have no more than \((s+2)^{3s+12}(9s^2+5s+2)/2\) (or \(\Omega((22.6)^{s^2})\) real roots [Ris85, Pg. 181, Line 6].

It was then discovered around 2001 that one could derive even sharper bounds by working \(p\)-adically. In particular, [Roj02] gives an upper bound of \(1 + s^2(22.5)^s!\) for the number of \(2\)-adic rational roots of an \(f \in \mathbb{Q}_2[x_1]\) with \(\sigma_{\mathbb{Q}_2}(f) \leq s\), and states explicit generalizations to \(\mathbb{Q}_p\), algebraic extensions, roots of bounded degree over number fields, and systems of multivariate polynomial equations. Our results thus improve the last bound by a factor close to \(3^s\) and isolates the portion where the (necessary?) exponentiality is coming from.

Historically, the results of [Roj02] were derived as a consequence of a higher-dimensional result, also due to the author, relying on a less efficient encoding: monomial term expansions. For completeness, we paraphrase the most relevant special case of the result below.

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\(^1\)This example is well-known in dynamical systems, and the author thanks Gregorio Malajovich for pointing it out.
Arithmetic Multivariate Descartes’ Rule (Special Case) \cite[Cor. 1 of Sec. 3]{Roj03a}

Let \( p \) be any prime and let \( \mathbb{Q}_p^* := \mathbb{Q}_p \setminus \{0\} \). Suppose \( f_1, \ldots, f_k \in \mathbb{Q}_p[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\} \), \( F := (f_1, \ldots, f_k) \), \( m_i \) is the total number of distinct exponent vectors appearing in \( f_i \) (assuming all polynomials are written as sums of monomials), and \( N_i \) is the number of variables occurring in \( f_i \).

Finally, let \( m := (m_1, \ldots, m_n) \), \( N := (N_1, \ldots, N_n) \), and define \( B(L, m, N) \) to be the maximum number of isolated roots in \((\mathbb{Q}_p)^n\) of such an \( F \), counting multiplicities.\(^2\) Then

\[
B(L, m, N) \leq \left( \prod_{i=1}^n (p-1)m_i(m_i - 1)/2 \right) \left[ \prod_{i=1}^n \left\{ c(m_i - 1)N_i \left[ 1 + \log_p \left( \frac{(m_i-1)\#N_i}{\log^\#N_ip} \right) \right] \right\} \right],
\]

where \( c := \frac{e}{e-1} \leq 1.582 \).\(^3\)

This result generalized earlier work of Lenstra for the case of sparse polynomials in one variable \cite{Len99}, and was derived via an analogous result counting \( p \)-adic complex roots close to the point \((1, \ldots, 1)\).

The latter result, which provided the first effective version of earlier model-theoretic work of Denef, Lipshitz, and van den Dries \cite{DvdD88, Lip88}, can also be viewed as an arithmetic extension \cite{Roj03a} of Khovanski’s Theorem on Complex Fewnomials \cite{Kho91}.

\( p \)-adic Complex Fewnomial Theorem \cite[Thm. 2 of Sec. 1.1]{Roj03a}

Following the notation above, suppose instead now that \( k = n \) and that the coefficients of \( F \) can lie in \( \mathbb{C}_p \). Let \( r_1, \ldots, r_n > 0 \), \( r := (r_1, \ldots, r_n) \), and let \( C_p(m, N, r) \) denote the maximum number of geometrically isolated roots \((x_1, \ldots, x_n) \in \mathbb{C}_p^n\) of \( F \) with \( \text{ord}_p(x_i - 1) \geq r_i \) for all \( i \), counting multiplicities. Then \( C_p(m, N, r) = 0 \) (if \( m_i \leq 1 \) for some \( i \)) and

\[
C_p(m, N, r) \leq \left| e^n \prod_{i=1}^n \left\{ (m_i - 1) + \log_p \left( \frac{(m_i - 1)\#N_i}{\prod_{j \in N_i} r_j \log^\#N_ip} \right) \right\} \right| / r_i \]

(if \( m_1, \ldots, m_n \geq 2 \)), where \# denotes the operation of taking set cardinality.\(^\blacksquare\)

In essence, our approach replaces the use of Arithmetic Multivariate Descartes’ Rule in \cite{Roj03a} by simple and direct combinatorial argument. Furthermore, we need the \( p \)-adic Complex Fewnomial Theorem only for Theorem \( \square \)

\section{Background and Useful Tools}

Let \( p \) be any prime number in the ring of ordinary integers \( \mathbb{Z} \) and recall that \( \mathbb{Z}_p \), the ring of \( p \)-adic integers, can be identified with the set of all “leftwardly infinite” sequences of the form \( \cdots d_2d_1d_0 \), where \( d_i \in \{0, \ldots, p-1\} \) for all \( i \geq 0 \). (In particular, \( \mathbb{Z} \) embeds naturally in \( \mathbb{Z}_p \) as the set of all \( p \)-adic integers having a finite \( p \)-adic expansion.) Addition and multiplication are then performed just as with base \( p \) integers, noting that the carries may propagate infinitely. Put another way, the last \( n \) digits of any \( p \)-adic integer calculation can be determined simply by working in the ring \( \mathbb{Z}/p^{n+1}\mathbb{Z} \).

The \( p \)-adic rational numbers are then obtained by allowing finitely many \( (\text{base } p) \) digits “after the decimal point”. For instance, the ordinary rational number \( \frac{12345}{49} \) can be considered a \( 7 \)-adic rational by expressing it as the sequence of \( 7 \)-adic digits 506.64, which in turn should be interpreted as \( 5 \cdot 7^2 + 6 + \frac{6}{7} + \frac{4}{49} \).

Following the notation of Definition \( \boxplus \) of the last section, it is then easy to show that \( \|x - y\|_p \) defines a metric on \( \mathbb{Q}_p \) and that \( \mathbb{Q}_p \) is \textbf{complete} with respect to this metric.\(^3\) So the fields \( \mathbb{Q}_p \) form

\(^2\)For the sake of simplicity, one can safely assume that the multiplicity of any isolated root is always a positive integer.

\(^3\)This means that all Cauchy sequences converge, i.e., if \( \{a_i\} \) satisfies \( \max_{i,j \geq N} \|a_i - a_j\|_p \to 0 \) as \( N \to \infty \) then \( \{a_i\} \) has a well-defined limit. \cite{Kob84}
an alternative collection of metric completions of \( \mathbb{Q} \); the real numbers \( \mathbb{R} \) being the completion of \( \mathbb{Q} \) we usually see first in school.

It is also convenient to define the \( p \)-adic analogue of the complex numbers: We let \( \mathbb{C}_p \) denote the completion, with respect to the obvious extensions of \( | \cdot |_p \), of the algebraic closure of \( \mathbb{Q}_p \). It turns out that \( \mathbb{C}_p \) is itself algebraically closed, so no further closures or completions need be taken. In particular, \( \text{ord}_p \mathbb{C}_p = \mathbb{Q} \) [Kob81].

One of the most amusing facts about the \( p \)-adic numbers is that their behavior as roots of polynomials can be described quite elegantly in terms of polyhedral geometry.

**Definition 4** Given any polynomial \( f(x_1) = \sum_{a \in A} c_a x_1^a \in \mathbb{C}_p[x_1] \), its \( p \)-adic Newton polygon, \( \text{Newt}_p(f) \), is the convex hull of \( \{(a, \text{ord}_p c_a) \mid a \in A\} \) in \( \mathbb{R}^2 \). Also, given any polygon \( Q \subseteq \mathbb{R}^2 \) and \( w = (w_1, w_2) \in \mathbb{R}^2 \), we let its **face with inner normal** \( Q^w \), \( Q^w \), be the set \( \{(a_1, a_2) \in Q \mid a_1 w_1 + a_2 w_2 \text{ minimal}\} \). In particular, we call \( Q^w \) a vertex (resp. improper, an edge) iff \( Q^w \) is a point (resp. \( Q \), not \( Q \) or a point). Finally, the **lower hull** of \( Q \) is the union of all faces of \( Q \) with an inner normal of the form \( (v, 1) \).

By exploiting the definition of a vertex one can easily prove the following useful facts on Newton polygons.

**Proposition 1** (See, e.g., [Kob87].) Following the notation above, the number of roots \( \zeta \in \mathbb{C}_p \) of \( f \) with \( \text{ord}_p \zeta = v \) is exactly the length of the orthogonal projection of \( \text{Newt}_p(f)^{(v, 1)} \) onto the \( x \)-axis.

We are now ready to prove our main theorems.

### 3 The Proof of Theorem 1

That the truth of the \( \tau \)-conjecture implies \( \text{P}_C \neq \text{NP}_C \) was proved by Shub and Smale in [SS95]. So we need only prove the last implication of our theorem.

Toward this end, suppose that the \( p \)-adic Digit Conjecture were true for some fixed prime \( p \). Note that any nonzero integral root \( x \) of \( f \) must have its first nonzero base-\( p \) digit in \( \{1, \ldots, p-1\} \). Furthermore, by Theorem 2 there are clearly no more than \( N_p(\sigma(f)) \leq N_p(\tau(f)) \leq \tau(f)(\tau(f)+1)/2 \) possible locations for this first digit of \( x \). Therefore, there are at most \( pN_p(\tau(f))(1 + \tau(f))^{c_p} \leq (1 + \tau(f))^{c_p+1+\log_2 p} \) possibilities for \( x \in \mathbb{Z} \setminus \{0\} \). Since 0 is the only remaining possibility for a root of \( f \), we are done.

### 4 The Proof of Theorem 2

The first fundamental lemma for our approach to additive complexity is the following.

**Lemma 1** Following the notation of Theorem 2 \( N_p(s) \leq s(s+1)/2 \).

\(^{4}\text{i.e., smallest convex set containing...}\)
Proof: Recalling the notation of Definition 2, note that we can define \( f(x) \) via a sequence \((X_0, \ldots, X_s)\) where \( X_{i+1} \) is a binomial or monomial in \( X_0, \ldots, X_i \) for all \( i \in \{0, \ldots, s\} \), with \( X_0 = x \), \( X_{s+1} = f(x) \), and \( s = \sigma(f) \). In particular, letting \( L_i \) denote the number of edges on the lower hull of Newt\(_p\)(\( X_i \)), Proposition 2 immediately yields a recursive inequality for \( L_i \): \( L_{i+1} \leq 2 \sum_{j=1}^{i} L_j \) for all \( i \geq 1 \). (That \( L_0 = 0 \) and \( L_1 = 1 \) is easily verified directly.) This recurrence easily yields \( L_s = O(3^s) \) and thus the same upper bound on \( N_p(s) \) by Proposition 1. However, with a little extra effort, we can do far better.

In particular, let \( F_i := E_+ (X_i) \) for all \( i \geq 0 \). Then

\[
F_{i+1} = E_+ \left( \text{Newt}_p \left( c_{i+1} \left( \prod_{j=0}^{i} X_j^{m_{j,i+1}} \right) + d_{i+1} \left( \prod_{j=0}^{i} X_j^{m_{j,i+1}} \right) \right) \right),
\]

which by Proposition 2 is \( E_+ (\text{Conv}(P \cup Q)) \) where \( P := \text{Newt}_p \left( c_{i+1} \left( \prod_{j=0}^{i} X_j^{m_{j,i+1}} \right) \right) \) and \( Q := \text{Newt}_p \left( d_{i+1} \left( \prod_{j=0}^{i} X_j^{m_{j,i+1}} \right) \right) \). In particular, note that any edge of the lower hull of \( \text{Conv}(P \cup Q) \) must be either (a) an edge of the lower hull of \( P \), (b) an edge of the lower hull of \( Q \), or (c) a line segment connecting a vertex of the lower hull of \( P \) with a vertex of the lower hull of \( Q \). Moreover, since we can order the vertices of \( P \) and \( Q \) in, say, counter-clockwise order, no more than \( 1 + \#(F_0 \cup \cdots \cup F_i) \) edges can occur in case (c).

Put another way, this means that \( \#(F_{i+1} \setminus (F_0 \cup \cdots \cup F_i)) \leq 1 + \#(F_0 \cup \cdots \cup F_i) \). So by induction, we easily obtain that \( L_{i+1} \leq L_i + i + 1 \). So we are done.

We are now ready to prove Theorem 2.

Proof of Theorem 2 Thanks to Lemma 1 we need only prove that \( N_p(s) \geq s \). In particular, we need only exhibit a polynomial \( f \in \mathbb{Z}[x_1] \) of additive complexity \( \leq s \) with at least \( s \) different valuations for its roots in \( \mathbb{Z} \). Considering \((x - 1)(x - p) \cdots (x - p^{s-1})\), we are done.

Remark 3 Note that we could have instead worked over any ring \( R \subseteq \mathbb{C}_p \) containing \( \mathbb{Z} \) by replacing \( \sigma(f) \) with \( \sigma_R(f) \). The obvious generalization of Theorem 2 then clearly holds.

5 The Proof of Theorem 4

First note that by the definition of additive complexity, \( x \in \mathbb{C}_p \) is a root of \( f \implies (X_0, \ldots, X_s) \in \mathbb{C}_p^{s+1} \) is a geometrically isolated root of the polynomial system \( G = 0 \), where the corresponding equations are exactly

\[
c_{s+2} \prod_{i=1}^{s+1} X_i^{m_{i,s+2}} = 0
\]

\[
X_2 = c_2 X_1^{m_{1,2}} + d_2 X_1^{m_{2,2}}
\]

\[\vdots\]

\[
X_{s+1} = c_{s+1} \left( \prod_{j=1}^{s} X_j^{m_{j,s+1}} \right) + d_{s+1} \left( \prod_{j=1}^{s} X_j^{m_{j,s+1}} \right),
\]

where \( s := \sigma(f) \), \( X_1 = x \), \( f(x) = c_{s+1} \prod_{i=1}^{s+1} X_i^{m_{i,s+2}} \), and the \( c_i \), \( d_i \), \( c_{i}^{(j)} \), \( m_{i,j} \), and \( m_{i,j}' \) are suitable constants. This follows easily from the fact that corresponding quotient rings \( \mathbb{C}_p[x]/(f) \) and
\[ \mathbb{C}_p[X_1, \ldots, X_{s+1}] / \langle G \rangle \] are isomorphic, thus making \( \mathbb{C}_p[x]/\langle f \rangle \) and \( \mathbb{C}_p[X_1, \ldots, X_{s+1}] / \langle G \rangle \) isomorphic.

So we now need only count the geometrically isolated roots \((X_1, \ldots, X_{s+1}) \in \mathbb{C}_p^{s+1} \) of \( G \), with \(|X_i - 1|_p \leq \frac{1}{p} \), precisely enough to conclude. Toward this end, note that by a simple rescaling (and since the \( p \)-adic Complex Fewnomial Theorem is independent of the underlying coefficients) that it suffices to count roots of \((X_1, \ldots, X_{s+1}) \in \mathbb{C}_p^{s+1} \) of \( G \) with \(|X_i - 1|_p \leq \frac{1}{p} \) for all \( i \).

Note then that the first equation of \( G = O \) imply that at least one \( X_i \) must be 0. Note also that if one of the variables \( X_1, \ldots, X_{\ell+1} \) is 0, then the first \( 1 + \ell \) equations of \( G \) completely determine \((X_1, \ldots, X_{\ell+1})\). Furthermore, by virtue of the last \( s - \ell \) equations of \( G \), the value of \((X_1, \ldots, X_{\ell+1})\) uniquely determines the value of \((X_{\ell+2}, \ldots, X_{s+1})\). So it in fact suffices to find the total number of geometrically isolated roots (with all coordinates nonzero) of all systems of the form \( G' = O \), where the equations of \( G' \) are exactly \((0 = 0)\) or

\[
X_2 = c_2 X_i^{m_{1,2}} + d_2 X_i^{m_{1,2}} \\
X_{\ell} = c_\ell \left( \prod_{i=1}^{\ell-1} X_i^{m_{i,\ell}} \right) + d_\ell \left( \prod_{i=1}^{\ell-1} X_i^{m_{i,\ell}} \right) \\
0 = c_{\ell+1} \left( \prod_{i=1}^{\ell} X_i^{m_{i,\ell+1}} \right) + d_{\ell+1} \left( \prod_{i=1}^{\ell} X_i^{m_{i,\ell+1}} \right),
\]

where \( \ell \) ranges over \( \{1, \ldots, n\} \). Note in particular that the \( j^{th} \) equation above involves no more than \( j + 1 \) variables for all \( j \in \{1, \ldots, \ell - 1\} \), and that the \( \ell^{th} \) equation involves no more than \( \ell \) variables.

Recalling the notation of the \( p \)-adic Complex Fewnomial Theorem, we then see that \( G \) has no more than

\[
1, 1 + C_p(2, 1, r) \quad, \quad \rho := 1 + C_p(2, 1, r) + (C_p(2, 1, r)C_p(3, 1, r)) \quad, \quad \text{or}
\]

\[
\rho + \sum_{\ell=3}^s C_p((2, 3, \ldots, 3), (2, 3, \ldots, \ell, \ell), \{r\}^\ell) \quad \text{geometrically isolated roots in} \quad \mathbb{C}_p^{s+1} \quad \text{satisfying} \quad |x_i - 1|_p \leq \frac{1}{p} \quad \text{for all} \quad i, \quad \text{according as} \quad s = 0, 1, 2, \text{or } \geq 3.
\]

So, by our earlier observations, these quantities are upper bounds for \( C'_p(0, r), C'_p(1, r), C'_p(2, r), \) and \( C'_p(s, r) \) respectively. So by an elementary calculation, we are done.

\section{The Proof of Theorem \ref{thm:complexity}}

Note that any nonzero \( p \)-adic rational root \( x \) of \( f \) must have its first nonzero digit in \( \{1, \ldots, p-1\} \). Furthermore, by Theorem \ref{thm:complexity} there are clearly no more than \( N_p(s) \leq s(s + 1)/2 \) possible locations for this first digit of \( x \). Therefore, by Theorem \ref{thm:complexity} there are at most \( 1 + pN_p(s)C'_p(s, 1) \) possibilities for \( x \), so we are done.

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