The discriminant of symmetric matrices as a sum of squares and the orthogonal group

Mátyás Domokos*

Rényi Institute of Mathematics, Hungarian Academy of Sciences,
P.O. Box 127, 1364 Budapest, Hungary, E-mail: domokos@renyi.hu

Abstract

It is proved that the discriminant of $n \times n$ real symmetric matrices can be written as a sum of squares, where the number of summands equals the dimension of the space of $n$-variable spherical harmonics of degree $n$. The discriminant of three by three real symmetric matrices is explicitly presented as a sum of five squares, and it is shown that the discriminant of four by four real symmetric matrices can be written as a sum of seven squares. These results improve theorems of Kummer from 1843 and Borchardt from 1846.

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1 Introduction

The discriminant of a degree $n$ monic polynomial $p$ with complex roots $\lambda_1, \ldots, \lambda_n$ equals

$$\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2.$$ 

Recall that it can be written as a polynomial function of the coefficients of $p$. By the discriminant $\delta(A)$ of an $n \times n$ real symmetric matrix $A$ we mean the discriminant of the characteristic polynomial of $A$. Recall that $\delta(A)$ is a homogeneous polynomial function in the entries of $A$, of degree $n(n - 1)$. Moreover, $\delta(A) = 0$ if and only if the matrix $A$ is degenerate (i.e. has multiple eigenvalues).

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Denote by $\mathcal{M}$ the space of $n \times n$ real symmetric matrices ($n \geq 2$). It is a vector space of dimension $n(n+1)/2$ over $\mathbb{R}$. It contains the subset $\mathcal{E}$ of degenerate real symmetric matrices, a real algebraic subvariety. Although $\mathcal{E}$ is the zero locus of the single polynomial $\delta \in \mathbb{R}[\mathcal{M}]$, it has codimension two in $\mathcal{M}$ by a result of Neumann and Wigner (cf. [8]). An algebraic explanation of the fact that the codimension is greater than 1 is that $\delta$ can be written as a sum of squares in $\mathbb{R}[\mathcal{M}]$. An explicit presentation of $\delta$ as a sum of seven squares was given in the nineteenth century by Kummer [6] for $n = 3$ (see Remark 7.4 for details), and by Borchardt [1] for arbitrary $n$. More recent approaches to the problem can be found in Ilyushechkin [7], Lax [9], Parlett [11]. Denote by $\mu(n)$ the minimal number of summands in a representation of $\delta$ as a sum of squares. The exact value of $\mu(n)$ is known only for $n \leq 3$: a straightforward calculation yields $\mu(2) = 2$, and the equality $\mu(3) = 5$ will be proved here.

The approach of Lax [9] to this problem is substantially different from the other works mentioned above, as it makes a crucial use of the conjugation action of the orthogonal group on the space of symmetric matrices. The present paper develops further the key ideas from [9], by exploiting deeper the representation theory of the orthogonal group. Our main results are the following. Theorem 6.2 asserts that the degree $n(n-1)/2$ homogeneous component of the vanishing ideal of the variety of degenerate real symmetric $n \times n$ matrices contains an $SO_n$-submodule isomorphic to the space $H^n(\mathbb{R}^n)$ of $n$-variable spherical harmonics of degree $n$. Consequently, the discriminant can be written as the sum of $\dim(H^n(\mathbb{R}^n)) = (2n-1) - (2n-3)$ squares. Note that the number $\dim(H^3(\mathbb{R}^3)) = 7$ agrees with the bound for $\mu(3)$ obtained by Kummer (or recently by Parrilo [12] using an algorithm based on semidefinite programming). However, for the case of $3 \times 3$ symmetric matrices we give an explicit presentation of the discriminant as the sum of five squares, and prove that $\mu(3) = 5$, see Theorem 7.3; this shows that $\mu(n)$ may be strictly smaller than the minimal dimension of an $SO_n$-invariant subspace in the degree $n(n-1)/2$ homogeneous component of the vanishing ideal of degenerate symmetric matrices. In Theorem 8.1 we locate some further irreducible $SO_4$-module summands in the degree six homogeneous component of the vanishing ideal of degenerate symmetric $4 \times 4$ matrices, and conclude that the discriminant in this case can be written as the sum of seven squares.

The paper is organized as follows. First in Section 2 we place the problem into a more general context, relating degeneracy loci of linear actions of compact Lie groups, and point out the existence of some generalized "discriminants" that are sums of squares by basic principles of representation theory; this is done mainly for sake of completeness of the picture, only Lemma 2.1 is logically necessary for the rest of the paper. Using an observation of Lax...
(see Lemma 3.3), we conclude in Section 3 that $\mu(n)$ is bounded by the minimal dimension of an $SO_n$-invariant subspace in the degree $n(n-1)/2$ homogeneous component of the vanishing ideal of degenerate symmetric $n \times n$ matrices, see Lemma 3.2. Section 4 contains a crucial new ingredient in our work: an explicit construction of an $O_n$-module homomorphism $T^*$ from the $(n-1)$th exterior power of the space of trace zero symmetric $n \times n$ matrices into the degree $n(n-1)/2$ homogeneous component of the vanishing ideal of degenerate matrices. In Section 5 we recall the facts from the representation theory of the orthogonal group that we shall use. Our best general upper bound for $\mu(n)$ (cf. Theorem 6.2) is discussed in Section 6. The results on the $3 \times 3$ and $4 \times 4$ cases are contained in Sections 7 and 8. In Section 9 for general $n$ we locate some irreducible $SO_n$-module summands in the kernel of $T^*$; in the special cases $n = 3, 4$ these results are used in the previous two sections.

2 General discriminants

Let $G$ be a compact Lie group (over $\mathbb{R}$) with a (smooth) representation on the finite dimensional real vector space $V$. Denote by $\mathbb{R}[V]$ the algebra of real valued polynomial functions on $V$ (the coordinate ring of $V$ in the terminology of algebraic geometry). There is an induced representation of $G$ on $\mathbb{R}[V]$ given by the formula $(g \cdot f)(v) := f(g^{-1}v)$ ($g \in G, v \in V, f \in \mathbb{R}[V]$). Note that the homogeneous components of $\mathbb{R}[V]$ are $G$-stable. By a $G$-invariant we mean a polynomial $f \in \mathbb{R}[V]$ with $g \cdot f = f$ for all $g \in G$.

Lemma 2.1 Any finite dimensional $G$-stable subspace $W$ in $\mathbb{R}[V]$ has an $\mathbb{R}$-basis $h_1, \ldots, h_n$ such that $H := \sum_{i=1}^n h_i^2$ is $G$-invariant. Moreover, the zero locus of the polynomial $H$ in $V$ coincides with the common zero locus of the elements of $W$.

Proof. This is just a reformulation of the well known fact that since $G$ is compact, for any finite dimensional representation of $G$ there exists a $G$-invariant scalar product (i.e. a symmetric, positive definite bilinear form) on the underlying vector space. Fix a $G$-invariant scalar product on $W$, choose an orthonormal basis $h_1, \ldots, h_n$ in $W$, and set $H := \sum_{i=1}^n h_i^2$. The orthonormality of the $h_i$ means that for each $g \in G$, we have $g \cdot h_j = \sum_{i=1}^n A_{ij}(g)h_i$, where $A_{ij}(g)$ are the matrix coefficients of the $G$-invariant scalar product.
where \((A_{ij}(g))_{n \times n}\) is a real orthogonal matrix. Consequently,

\[
g \cdot H = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} A_{ij}(g) h_i \right)^2 = \sum_{j=1}^{n} \sum_{i,k=1}^{n} A_{ij}(g) A_{kj}(g) h_i h_k = \sum_{i=1}^{n} h_i h_k \sum_{j=1}^{n} A_{ij}(g) A_{kj}(g) = \sum_{i=1}^{n} h_i^2 = H,
\]

so \(H\) is \(G\)-invariant. Moreover, suppose that \(0 = H(v) = \sum_{i=1}^{n} h_i(v)^2\) for some \(v \in V\). By non-negativity of the summands we have \(h_i(v) = 0\) for \(i = 1, \ldots, n\). Since any \(f \in W\) is a linear combinations of the \(h_i\), we conclude \(f(v) = 0\). □

For a positive integer \(r\) set

\[V^{<r} := \{ v \in V \mid \dim(G \cdot v) < r \}.\]

Note that the dimension of any orbit \(G \cdot v\) is less than or equal to \(\dim(G)\).

**Proposition 2.2** Let \(d\) denote the maximal dimension of an orbit in \(V\). For \(r = 1, \ldots, d\), there exists a non-zero \(G\)-invariant polynomial function \(D_r\) on \(V\), homogeneous of degree \(2r\), such that \(D_r(v) = 0\) if and only if \(v \in V^{<r}\). Moreover, \(D_r\) is the sum of squares of homogeneous polynomials of degree \(r\), and the number of summands is less than or equal to \((\binom{\dim(G)}{r} \cdot \binom{\dim(V)}{r})\) (product of binomial coefficients).

**Proof.** Fix \(r \in \{1, \ldots, d\}\). One can find polynomial equations on \(V\) whose common zero locus is \(V^{<r}\) as follows. The dimension of the orbit of \(v \in V\) is the difference of the dimension of \(G\) and the dimension of the stabilizer subgroup \(G_v\) of \(v\) in \(G\). One can pass to the tangent representation of the Lie algebra \(\text{Lie}(G)\) on \(V\), this is a Lie algebra homomorphism \(\tau : \text{Lie}(G) \rightarrow \text{gl}(V)\) (where \(\text{gl}(V)\) is the Lie algebra of all linear transformations of \(V\)). The dimension of \(G_v\) is the same as the dimension of its Lie algebra. Now

\[\text{Lie}(G_v) = \{ A \in \text{Lie}(G) \mid \tau(A)v = 0 \}\]

For each \(v \in V\) denote by \(L_v\) the \(\mathbb{R}\)-linear map \(\text{Lie}(G) \rightarrow V\) given by \(L_v(A) := \tau(A)v\) for \(A \in \text{Lie}(G)\). Note that the map \(V \rightarrow \text{hom}_\mathbb{R}(\text{Lie}(G), V), v \mapsto L_v\) is \(\mathbb{R}\)-linear. Moreover, \(\ker(L_v) = \text{Lie}(G_v)\), hence the dimension of the orbit of \(v\) equals the dimension of the image of \(L_v\). Thus \(v\) belongs to \(V^{<r}\) if and only if the rank of \(L_v\) is less than \(r\). Fixing a basis in \(V\) and in \(\text{Lie}(G), L_v\) is identified with a matrix of size \(\dim(V) \times \dim(G)\), whose \((i, j)\)-entry equals
\( \xi_{ij}(v) \), where \( \xi_{ij} \) are linear forms on \( V \). Consequently, \( V^{<r} \) is the common zero locus of the determinants of the \( r \times r \)-minors of the matrix \( (\xi_{ij}) \). The determinant of an \( r \times r \) minor is a degree \( r \) homogeneous element in \( \mathbb{R}[V] \) (unless it is the zero polynomial). The assumption \( r \leq d \) implies that not all of these determinants are identically zero.

Next we show that these degree \( r \) homogeneous polynomials span a \( G \)-stable subspace in \( \mathbb{R}[V] \). Indeed, it is easy to see that

\[
L_{g^{-1}v} = T(g^{-1}) \circ L_v \circ \text{ad}(g)
\]

where \( \text{ad} : G \rightarrow GL(\text{Lie}(G)) \) denotes the adjoint representation of \( G \) on its Lie algebra, and \( T : G \rightarrow GL(V) \) is the given representation of \( G \) on \( V \). Consequently, the image under \( g \in G \) of the set of determinants of the \( r \times r \) minors of \( (\xi_{ij})_{i=1,\ldots,\dim(G)} \) is the set of determinants of the \( r \times r \) minors of \( P(\xi_{ij})Q \), where \( P, Q \) are the matrices of \( T(g^{-1}) \), \( \text{ad}(g) \) with respect to the chosen bases of \( V, \text{Lie}(G) \). By the Binet-Cauchy formula we conclude that the determinants of the \( r \times r \) minors of \( (\xi_{ij}) \) span a \( G \)-stable subspace in \( \mathbb{R}[V] \).

Now the Proposition follows from Lemma 2.1.

An example of the general setup discussed above is the case of the orthogonal group \( G := O_n \) acting on the space \( V := \mathcal{M} \) of real symmetric \( n \times n \) matrices by conjugation: for \( g \in O_n \) and \( A \in \mathcal{M} \) we have \( g \cdot A := gAg^T \) (matrix multiplication). Then the stabilizer of a matrix with distinct eigenvalues is zero-dimensional, so the maximal dimension of an orbit is \( \dim(O_n) = n(n-1)/2 \), and \( \mathcal{E} := V^{<n(n-1)/2} \) is the set of degenerate matrices. Moreover, the Lie algebra of \( O_n \) is the space of skew-symmetric matrices (with the commutator as the Lie bracket). So in this case we get back exactly the polynomials vanishing on \( \mathcal{E} \) that are constructed in [9]. By Proposition 2.2 we conclude the existence of a degree \( n(n-1) \) homogeneous \( O_n \)-invariant \( D \), which is a sum of squares, and whose zero locus is \( \mathcal{E} \). As it is pointed out in [9], up to non-zero scalars the discriminant is the only degree \( n(n-1) \) homogeneous \( O_n \)-invariant vanishing on \( \mathcal{E} \). So \( D \) and \( \delta \) coincide up to scalar, and since both take non-negative values only, the scalar is positive. Thus \( \delta \) is a sum of squares, with \( \mu(n) \leq \binom{n(n+1)/2}{n(n-1)/2} = \binom{n(n+1)/2}{n} \) summands.

Exploiting some special features of this example, the bound for \( \mu(n) \) will be drastically improved in the following sections. The above considerations motivate the following general question:

**Question.** Do the polynomials constructed in the proof of Proposition 2.2 generate the vanishing ideal of \( V^{<r} \)?
3 Bounding $\mu(n)$ with the dimension of some irreducible representation

As a representation of $O_n$, the space $\mathcal{M}$ decomposes as

$$\mathcal{M} = \mathbb{R}I \oplus \mathcal{N} \quad (1)$$

where $\mathcal{N}$ stands for the codimension one subspace of trace zero matrices, and $I$ is the $n \times n$ identity matrix. The projection onto the second direct summand in (1) identifies $\mathbb{R}[\mathcal{N}]$ with an $O_n$-stable subalgebra of $\mathbb{R}[\mathcal{M}]$. Moreover, $\mathbb{R}[\mathcal{M}] = \mathbb{R}[\mathcal{N}][\text{Tr}]$ is a polynomial ring over $\mathbb{R}[\mathcal{N}]$ generated by the trace function $\text{Tr} : \mathcal{M} \to \mathbb{R}$ (which is $O_n$-invariant). Write $\mathcal{F} := \mathcal{E} \cap \mathcal{N}$ for the set of degenerate trace zero symmetric matrices. Denote by $I(\mathcal{E})$ the ideal in $\mathbb{R}[\mathcal{M}]$ consisting of the polynomials that vanish on $\mathcal{E}$. Similarly, $I(\mathcal{F})$ stands for the vanishing ideal of $\mathcal{F}$ in $\mathbb{R}[\mathcal{N}]$. Obviously we have $\mathcal{E} = \mathbb{R}I \oplus \mathcal{F}$, implying

$$I(\mathcal{E}) = I(\mathcal{F})[\text{Tr}] \quad (2)$$

Therefore the study of $I(\mathcal{E})$ is essentially equivalent to the study of $I(\mathcal{F})$.

The definition of the discriminant in terms of the eigenvalues implies that $\delta$ belongs to the subalgebra $\mathbb{R}[\mathcal{N}]$ of $\mathbb{R}[\mathcal{M}]$. From now on we shall focus on the algebra $\mathbb{R}[\mathcal{N}]$ and the ideal $I(\mathcal{F})$.

**Remark 3.1** Since $\dim(\mathcal{N})$ is one less than $\dim(\mathcal{M})$, the argument in the end of Section 2 yields the improved bound $\mu(n) \leq \left(\frac{n(n+1)/2-1}{n-1}\right)$. This is already better than the bound $\left(\frac{n(n+1)}{n}\right) - (n-1)\left(\frac{n}{n-1}\right)$ obtained in Section 4.3 of [11]. For $n = 4$ we get $\mu(4) \leq 84$, a result stated by Borchardt [1]. For $n = 3$ we get $\mu(3) \leq 10$. However, the better bound $\mu(3) \leq 7$ is obtained in [6]. This improvement will be generalized for arbitrary $n$ in Section 6.

Restricting to diagonal matrices one easily shows that no polynomial of degree less than $n(n-1)/2$ vanishes on $\mathcal{F}$. By Proposition 2.2 (and the explanation afterwards), the degree $n(n-1)/2$ homogeneous component $I(\mathcal{F})_{n(n-1)/2}$ of the ideal $I(\mathcal{F})$ is non-zero; even more, the common zero locus of $I(\mathcal{F})_{n(n-1)/2}$ is $\mathcal{F}$. Moreover, $I(\mathcal{F})_{n(n-1)/2}$ is an $O_n$-submodule in $\mathbb{R}[\mathcal{N}]$. Indeed, the subset $\mathcal{F}$ of $\mathcal{N}$ is $O_n$-stable, hence the ideal $I(\mathcal{F})$ is an $O_n$-submodule of $\mathbb{R}[\mathcal{N}]$. Since the action of $O_n$ preserves the grading on $\mathbb{R}[\mathcal{N}]$, the homogeneous component $I(\mathcal{F})_{n(n-1)/2}$ is an $O_n$-submodule. To get the best bounds on $\mu(n)$, we shall switch from $O_n$ to its subgroup $SO_n$ consisting of the orthogonal matrices with determinant one ($SO_n$ is called the *special orthogonal group*).
Lemma 3.2 Any non-zero $SO_n$-submodule of $I(F)_{n(n-1)/2}$ has a basis $\{f_i\}$ such that $\delta = \sum f_i^2$. Consequently, $\mu(n)$ is less than or equal to the minimal dimension of an irreducible $SO_n$-submodule contained in $I(F)_{n(n-1)/2}$.

Proof. Let $W$ be a non-zero $SO_n$-invariant subspace of the degree $n(n-1)/2$ homogeneous component of $I(F)$. By Lemma 2.1, $W$ has a basis $\{f_i\}$ such that $D := \sum f_i^2$ is $SO_n$-invariant (non-zero by positivity of the summands). Moreover, $\deg(D) = n(n-1)$ and $D$ vanishes on $F$. By Lemma 3.3 below, $D = c\delta$ for some positive scalar $c \in \mathbb{R}$, so $\delta = \sum_i (c^{-1/2} f_i)^2$. □

The following lemma is due to Lax; it is stated in [9] for $\mathcal{M}$ and $\mathcal{E}$, but obviously holds in the form below by (2):

Lemma 3.3 Up to scalar multiples, $\delta$ is the only degree $n(n-1)$ homogeneous $SO_n$-invariant polynomial function on $\mathcal{N}$ that vanishes on $F$.

4 An $O_n$-submodule in $I(F)$

Next we turn to a crucial step in the present paper, and provide a simple construction of a non-zero $O_n$-submodule in $I(F)_{n(n-1)/2}$. (This construction seems to suit better for computations than the construction in the proof of Proposition 2.2.)

One has the $O_n$-equivariant polynomial maps

$$H_i : \mathcal{N} \rightarrow \mathcal{N}, \quad A \mapsto A^i - \frac{1}{n} \text{Tr}(A^i) I$$

for $i = 1, 2, \ldots$. Using them one defines a map from $\mathcal{N}$ to the degree $n-1$ exterior power of $\mathcal{N}$:

$$T : \mathcal{N} \rightarrow \bigwedge^{n-1} \mathcal{N}, \quad A \mapsto A \wedge H_2(A) \wedge \cdots \wedge H_{n-1}(A)$$

Proposition 4.1 For $A \in \mathcal{N}$ we have $T(A) = 0$ if and only if $A$ belongs to $\mathcal{F}$.

Proof. Denote by $\mathcal{D}$ the space of trace zero diagonal matrices, $\mathcal{D}_1$ the subspace of $\mathcal{D}$ consisting of the matrices whose first two diagonal entries coincide, and $\mathcal{D}_0$ the subset of matrices with distinct diagonal entries. Clearly the $H_j$ map $\mathcal{D}_1$ into itself, so $T(\mathcal{D}_1) \subseteq \bigwedge^{n-1} \mathcal{D}_1 = 0$, since $\dim_{\mathbb{R}}(\mathcal{D}_1) < n-1$. On the other hand, we claim that for $A \in \mathcal{D}_0$ the $H_j(A)$ ($j = 1, \ldots, n-1$) are linearly independent, and therefore span $\mathcal{D}$. Indeed, $A \in \mathcal{D}_0$ has distinct diagonal
Consider the Vandermonde matrix $V := (a_i^{j-1})_{i,j=1}^n$, its columns are linearly independent. Denote by $V'$ the matrix obtained from $V$ by subtracting from the $j$th column of $V$ the first column of $V$ multiplied by $1/n \sum_{i=1}^n a_i^{j-1}$, for $j = 2, \ldots, n$. Clearly, the columns of $V'$ are linearly independent. Since $H_{j-1}(A)$ can be identified with the $j$th column of $V'$ for $j = 2, \ldots, n$, our claim follows. Consequently, $\mathcal{T}(A) \in \bigwedge^{n-1}D$ is non-zero.

If $A \in \mathcal{F}$, then the $O_n$-orbit of $A$ intersects $D_1$, therefore by $O_n$-equivariance of $\mathcal{T}$ we conclude that $\mathcal{T}(A) = 0$. Similarly, if $A \in \mathcal{N} \setminus \mathcal{F}$, then the $O_n$-orbit of $A$ intersects $D_0$, consequently $\mathcal{T}(A) \neq 0$. □

Now $\mathcal{T}$ induces a non-zero $O_n$-equivariant linear map $\mathcal{T}^*$ from the dual space of $\bigwedge^{n-1}\mathcal{N}$ defined as follows:

$$\mathcal{T}^* : (\bigwedge^{n-1}\mathcal{N})^* \to \mathcal{I}(\mathcal{F})_{n(n-1)/2}$$

$$(\mathcal{T}^*(\xi))(A) := \xi(\mathcal{T}(A)) \text{ for } \xi \in (\bigwedge^{n-1}\mathcal{N})^*, A \in \mathcal{N}$$

Indeed, $\mathcal{T}$ is a polynomial map, and in the terminology of algebraic geometry, $\mathcal{T}^*$ is the restriction to $(\bigwedge^{n-1}\mathcal{N})^* \subset \mathbb{R}[\bigwedge^{n-1}\mathcal{N}]$ of the comorphism of the morphism $\mathcal{T}$ of affine algebraic varieties. Since the polynomial map $\mathcal{T}$ is homogeneous of degree $1 + 2 + \cdots + (n-1) = n(n-1)/2$, the image of $\mathcal{T}^*$ is contained in the degree $n(n-1)/2$ homogeneous component of $\mathbb{R}[\mathcal{N}]$. Since $\mathcal{T}$ is $O_n$-equivariant, the same holds for $\mathcal{T}^*$. By Proposition 4.1, the image of $\mathcal{T}^*$ is a subspace of $\mathcal{I}(\mathcal{F})$, furthermore, the common zero locus in $\mathcal{N}$ of the polynomials from the image of $\mathcal{T}^*$ is $\mathcal{F}$. In particular, $\mathcal{T}^*$ is non-zero.

5 Representations of $O_n$

A classical reference for the material in this section is [14]; see also [13], [4], [3] for more modern treatments. By a representation of $O_n$ (resp. $SO_n$) we mean a Lie group homomorphism from $O_n$ (or $SO_n$) into the real Lie group of all linear transformations of a finite dimensional vector space over the field of real numbers. Since these groups are compact, all representations decompose as a sum of irreducibles, and all representations are self-dual. The irreducible representations of $O_n$ and $SO_n$ all appear as summands in the tensor powers of the defining representation of $O_n$ on $\mathbb{R}^n$ (see [14]), and the isomorphism classes of irreducible representations of $O_n$ are traditionally labeled by partitions. By a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ we mean a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ of non-negative integers. For $j = 1, 2, \ldots, n$, set $h_i(\lambda) := \{j \mid \lambda_j \geq i\}$ (the length of the $i$th column of the Young diagram
of $\lambda$). The isomorphism classes of irreducible representations of $O_n$ are in bijection with partitions $\lambda$ satisfying $h_1(\lambda) + h_2(\lambda) \leq n$ (see for example Section 6.5 in [13]). Denote by $V_\lambda$ the irreducible $O_n$-module corresponding to $\lambda$. For the partition $(d) = (d, 0, \ldots, 0)$ we have that $V_d \cong \mathcal{H}^d(\mathbb{R}^n)$, the space of spherical harmonics of degree $d$ in $n$ variables. It can be constructed as follows: consider the natural representation of $O_n$ on the coordinate ring \( \mathbb{R}[x_1, \ldots, x_n] \) of $\mathbb{R}^n$, restrict to the degree $d$ homogeneous component, and take its factor space by the degree $d$ homogeneous multiples of $x_1^2 + \cdots + x_n^2$.

The restriction of an irreducible $O_n$-module to $SO_n$ either stays irreducible, or is the sum of two non-isomorphic irreducibles (having the same dimension). The details are as follows (they can be found for example on page 164 in [14]): If $h_1(\lambda) < n/2$, then the restriction $W_\lambda := \text{Res}_{SO_n}^O \lambda \big V_\lambda \big \big |_{SO_n}$ remains irreducible over the special orthogonal group $SO_n$. Moreover, denoting by $\lambda^0$ the partition with $h_1(\lambda^0) = n - h_1(\lambda)$ and $h_i(\lambda^0) = h_i(\lambda)$ for $i > 1$, we have that $V_{\lambda^0}$ is isomorphic to the tensor product of $V_\lambda$ and the determinant representation of $O_n$, hence the restriction to $SO_n$ of $V_\lambda$ is also isomorphic to $W_\lambda$. When $n = 2l + 1$ is odd, then $\{W_\lambda \mid h_1(\lambda) \leq n/2\}$ is a complete list of isomorphism classes of irreducible $SO_n$-modules. When $n = 4m$ is divisible by four and $h_1(\lambda) = n/2$, then $\text{Res}_{SO_n}^O \lambda \big V_\lambda \big \big |_{SO_n}$ decomposes as the direct sum $W_\lambda \oplus W(\lambda_1, \ldots, \lambda_{2m-1}, -\lambda_2)$ of two non-isomorphic irreducible $SO_n$-modules having the same dimension, and $\{W(\lambda_1, \ldots, \lambda_{2m}) \mid \lambda_1 \geq \cdots \geq \lambda_{2m-1} \geq |\lambda_2|\}$ is a complete list of isomorphism classes of irreducible representations of $SO_n$. When $n = 4m + 2$ and $h_1(\lambda) = n/2$, then $\text{Res}_{SO_n}^O \lambda \big V_\lambda \big \big |_{SO_n}$ remains irreducible over $SO_n$, and $\{W_\lambda \mid h_1(\lambda) < n/2\} \cup \{\text{Res}_{SO_n}^O \lambda \mid h_1(\lambda) = n/2\}$ is a complete list of isomorphism classes of irreducible $SO_n$-modules.

Although in our problem we are dealing with real representations of real Lie groups, in order to study concrete representations we shall apply the so-called highest weight theory, and therefore we shall need to change to representations of the complex orthogonal groups $O_n(\mathbb{C}) := \{A \in \mathbb{C}^{n \times n} \mid A^T A = I\}$ and $SO_n(\mathbb{C}) := \{A \in O_n(\mathbb{C}) \mid \det(A) = 1\}$. The passage is as follows: First recall that a complex representation of a real Lie group $G$ is a (real) Lie group homomorphism from $G$ into the group of invertible linear transformations of some finite dimensional complex vector space. For any representation of a real Lie group $G$ on some finite dimensional real vector space $V$ there is an associated complex representation of $G$ (called its complexification): namely, consider the induced $\mathbb{C}$-linear action of $G$ on $\mathbb{C} \otimes_{\mathbb{R}} V$. The complexification of an irreducible $O_n$-module or $SO_n$-module stays irreducible, with the exception that when $n = 4m + 2$ and $h_1(\lambda) = n/2$, then the restriction to $SO_n$ of the complexification of the irreducible $O_n$-module $V_\lambda$ splits as the sum $W_\lambda + W(\lambda_1, \ldots, \lambda_{2m}, -\lambda_{n/2})$ of two non-isomorphic equidimensional irreducible
complex representations of $SO_n$ (just like as it happens already over the reals when $n = 4m$). Next recall that representations of $O_n$ or $SO_n$ are polynomial, that is, the matrix elements of a representation are polynomials in the matrix entries of the elements of our group. Therefore the complexification of a representation on $V$ extends to a polynomial representation of $O_n(\mathbb{C})$ or $SO_n(\mathbb{C})$ on the complexified vector space $\mathbb{C} \otimes_{\mathbb{R}} V$. This extension is unique (since the equations defining our groups inside the space of $n \times n$ matrices are the same in the complex and the real cases). Given an irreducible $O_n$-module (or $SO_n$-module) $V_\lambda$ or $W_\lambda$, we keep the same symbol to denote the corresponding irreducible polynomial representations of the corresponding complex group $O_n(\mathbb{C})$ or $SO_n(\mathbb{C})$. It is clear from the discussion above that given a representation of $O_n$ or $SO_n$ on $V$, the multiplicity of an irreducible representation $V_\lambda$ or $W_\lambda$ as a summand in $V$ is the same as the multiplicity of the corresponding irreducible representation of $O_n(\mathbb{C})$ or $SO_n(\mathbb{C})$ as a summand in $\mathbb{C} \otimes_{\mathbb{R}} V$.

The so-called highest weight theory is a standard tool to decompose a given polynomial $SO_n(\mathbb{C})$-module as a sum of irreducibles. To apply highest weight theory it is convenient to perform a linear change of variables and work with the orthogonal group

$$O_n(\mathbb{C}, J) := \{ A \in \mathbb{C}^{n \times n} \mid A^T J A = J \}$$

$$SO_n(\mathbb{C}, J) := \{ A \in O_n(\mathbb{C}, J) \mid \det(A) = 1 \}$$

preserving the symmetric bilinear form on $\mathbb{C}^n$ with matrix $J$, where for $n = 2l$ even we have $J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, a $2 \times 2$ block matrix, with the $l \times l$ identity matrix $I$ in the off-diagonal positions, and the zero matrix in the diagonal positions, and for $n = 2l+1$ odd we have $J = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Denote by $T$ the subgroup of $SO_n(\mathbb{C}, J)$ consisting of the diagonal matrices

$$\{ t = \text{diag}(t_1, \ldots, t_l, t_1^{-1}, \ldots, t_l^{-1}) \mid t_1, \ldots, t_l \in \mathbb{C}^\times \}$$

when $n = 2l$ and

$$\{ t = \text{diag}(t_1, \ldots, t_l, t_1^{-1}, \ldots, t_l^{-1}, 1) \mid t_1, \ldots, t_l \in \mathbb{C}^\times \}$$

when $n = 2l+1$. Then $T$ is a maximal torus in $SO_n(\mathbb{C}, J)$ (in the terminology of algebraic groups). Characters of $T$ are identified with $l$-tuples of integers: given $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{Z}^l$ and $t \in T$ as above we write $\alpha(t) := \prod_{l-1}^{l} t_i^{\alpha_i}$. An element $v$ in an $SO_n(\mathbb{C}, J)$-module $V$ is called a weight vector if for some
character $\alpha$ of $\mathbb{T}$ we have $t \cdot v = \alpha(t)v$ ($t \in \mathbb{T}$); in this case we call $\alpha$ the weight of $v$. Denote by $u^+$ the unipotent radical given for example in section 10.4.1 in [13] of the positive Borel subalgebra of the Lie algebra $so_n(\mathbb{C}, J)$ of $SO_n(\mathbb{C}, J)$. A non-zero element $w$ in an $SO_n(\mathbb{C}, J)$-module $W$ is called a highest weight vector of weight $\lambda = (\lambda_1, \ldots, \lambda_l)$ if $w$ is annihilated by $u^+$ (the Lie algebra $so_n(\mathbb{C})$ acts on $V$ via the tangent representation of the given representation of $SO_n(\mathbb{C})$), and $t \cdot w = \lambda(t)w$ for all $t \in \mathbb{T}$. Such a vector generates an irreducible $SO_n(\mathbb{C}, J)$-submodule in $W$ isomorphic to $W\lambda$. We recall that there is a standard partial ordering of weights in representation theory: the weight $\alpha$ is greater than the weight $\beta$ if $\alpha - \beta$ is a sum of positive roots of the Lie algebra $so_n(\mathbb{C}, J)$. Now $\lambda$ is the unique maximal element (with respect to this partial ordering) among the weights of $\mathbb{T}$ that occur in $W\lambda$.

6 Spherical harmonics in the vanishing ideal of degenerate matrices

Denote by $\mathcal{M}_C$ the space of $n \times n$ complex symmetric matrices. As a module over $O_n(\mathbb{C})$ it decomposes as $\mathcal{M}_C = N_C \oplus \mathbb{C}I$, where $N_C$ is the subspace of trace zero $n \times n$ complex symmetric matrices. View $N_C$ as a complex affine variety with coordinate ring $\mathbb{C}[N_C]$. The same formulae as in [3] and [4] give an $O_n(\mathbb{C})$-equivariant polynomial map $T_C : N_C \to \bigwedge_{n-1} N_C$, and we want to decompose the image of the dual of $\bigwedge_{n-1} N_C$ under the comorphism $T^*_C : \mathbb{C}[\bigwedge_{n-1} N_C] \to \mathbb{C}[N_C]$ of $T$. Clearly $N_C$ contains the subsets $\mathcal{N} \supset \mathcal{F}$. Denote by $\mathcal{F}_C$ the closure of $\mathcal{F}$ in the Zariski topology of the complex affine space $N_C$, and denote by $\mathcal{I}(\mathcal{F}_C)$ the vanishing ideal in $\mathbb{C}[N_C]$ of $\mathcal{F}_C$. Note that $\mathcal{I}(\mathcal{F}_C)$ is spanned over $\mathbb{C}$ by its real subspace $\mathcal{I}(\mathcal{F})$, and $\mathcal{I}(\mathcal{F}_C) \supset \mathcal{T}_C^*((\bigwedge^n N_C)^*)$ spanned over $\mathbb{C}$ by $T^*(((\bigwedge^n N_C)^*)$. As explained in Section 5 the $O_n$-module structure of $T^*(((\bigwedge^n N_C)^*)$ and $\mathcal{I}(\mathcal{F})$ can be read off from the $O_n(\mathbb{C})$-module structure of $T_C^*(((\bigwedge^n N_C)^*)$ and $\mathcal{I}(\mathcal{F}_C)$, so from now on we shall focus on the complex objects. (Let us stress explicitly that $\mathcal{F}_C$ is properly contained in the set of all complex trace zero symmetric matrices with multiple eigenvalues; the latter is an irreducible complex hypersurface in $N_C$, namely the set of all complex zeros of the discriminant, whereas $\mathcal{F}_C$ is a codimension two complex algebraic subvariety of $N_C$.)

As we indicated in Section 5 we change to the groups $O_n(\mathbb{C}, J)$ and $SO_n(\mathbb{C}, J)$ preserving the symmetric bilinear form on $\mathbb{C}^n$ with matrix $J$. Accordingly, $\mathcal{M}_C$ has to be replaced by the space

$$\mathcal{M}_{C,J} := \{ A \in \mathbb{C}^{n \times n} \mid A^T = JAJ^{-1} \}$$
of self-adjoint linear operators on \((\mathbb{C}^n, J)\), on which \(O_n(\mathbb{C}, J)\) acts by conjugation: for \(g \in O_n(\mathbb{C}, J)\) and \(A \in M_{C,J}\) we have \(g \cdot A = gAg^{-1}\) (matrix multiplication on the right hand side). The space \(N_C\) of trace zero symmetric matrices has to be replaced by
\[
N_{C,J} := \{A \in M_{C,J} \mid \text{Tr}(A) = 0\}.
\]

Note that if \(K\) is an \(n \times n\) matrix with \(J = K^T \overline{K}\), then conjugation by \(K^{-1}\) gives isomorphisms \(g \mapsto K^{-1}gK\), \(O_n(\mathbb{C}) \to O_n(\mathbb{C}, J)\) and \(A \mapsto K^{-1}AK\), \(M_C \to M_{C,J}\) that intertwines the actions of the orthogonal groups. Taking this into account it is easy to see that the \(O_n(\mathbb{C}, J)\)-equivariant polynomial map \(T_{C,J} : N_{C,J} \to \bigwedge^{n-1} N_{C,J}\) corresponding to \(T_{C}\) is given by the same formulae as in [3] and [4]. (For sake of completeness we mention that \(N^{\ast}_{C,J} \cong W_2(\mathbb{C}_n, J)\)-modules. Furthermore, the symmetric bilinear map \(\mathbb{C}^n \times \mathbb{C}^n \to M_{C,J}\), \((v, w) \mapsto \frac{1}{2}(v \cdot w^T + w \cdot v^T) \cdot J\) induces an \(O_n(\mathbb{C}, J)\)-module isomorphism between the symmetric tensor square \(S^2(\mathbb{C}^n)\) of the defining \(O_n(\mathbb{C}, J)\)-module \(\mathbb{C}^n\) and \(M_{C,J}\). Morever, \((A, B) \mapsto \text{Tr}(AB)\) is an \(O_n(\mathbb{C}, J)\)-invariant bilinear form on \(M_{C,J}\) that can be used to fix \(O_n(\mathbb{C}, J)\)-module isomorphisms \(M^{\ast}_{C,J} \cong M_{C,J}\) and \(N^{\ast}_{C,J} \cong N_{C,J}\).)

**Proposition 6.1** The \(SO_n(\mathbb{C}, J)\)-module \(T^{\ast}_{C,J}((\bigwedge^{n-1} N_{C,J})^\ast)\) contains a summand isomorphic to \(W_{(n)}\).

**Proof.** Denote \(x_{ij}\) the function on \(N_{C,J}\) mapping an \(n \times n\) matrix in \(N_{C,J}\) to its \((i, j)\)-entry. For a diagonal matrix \(t \in \mathbb{T}\) (cf. Section 5) we have that \(t \cdot x_{ij}\) is the \((i, j)\)-entry of \(t^{-1}(x_{ij})_{ij=1}^n t\) (matrix multiplication). In particular, all the \(x_{ij}\) are weight vectors in \(N^{\ast}_{C,J}\) (and \(x_{i+1,1}\) is the unique highest weight vector in \(N^{\ast}_{C,J}\), it has weight \((2)\)).

Consequently,
\[
x := x_{2,1} \wedge x_{3,1} \wedge \cdots \wedge x_{n,1} \in \bigwedge^{n-1} N^{\ast}_{C,J}
\]
is a weight vector, and one computes easily that its weight is \((n)\). Moreover, it is a highest weight vector, since its weight is maximal with respect to the lexicographic ordering among the weights that occur in \(\bigwedge^{n-1} N^{\ast}_{C,J}\) (one can easily see that for all other weights \(\alpha = (\alpha_1, \ldots, \alpha_l)\) in \(\bigwedge^{n-1} N^{\ast}_{C,J}\) we have \(\alpha_1 < n\), hence it is also maximal with respect to the standard partial ordering of weights mentioned in the end of Section 5.

Next use the usual identification \(\bigwedge^{n-1}(N^{\ast}_{C,J}) \cong (\bigwedge^{n-1} N_{C,J})^\ast\), so for \(x_1, \ldots, x_{n-1} \in N^{\ast}_{C,J}\) and \(A_1, \ldots, A_{n-1} \in N_{C,J}\), the value of \(x_1 \wedge \cdots \wedge x_{n-1}\)
(viewed as a linear form on $\bigwedge^{n-1} \mathcal{N}_{C,j}$) at $A_1 \wedge \cdots \wedge A_{n-1}$ equals

$$ \sum_{\pi \in \text{Sym}(n-1)} \text{sign}(\pi)x_1(A_{\pi(1)}) \cdots x_{n-1}(A_{\pi(n-1)})$$

(where the summation ranges over the full symmetric group $\text{Sym}(n-1)$ of degree $n-1$). In particular, this means that the value of $T^*_C,J(x)$ on $A \in \mathcal{N}_{C,j}$ equals the determinant of the $(n-1) \times (n-1)$ matrix, whose $i$th column is the first column (with the $(1,1)$-entry removed) of $A^i$. Now take for $A$ the matrix of the linear transformation permuting the standard basis vectors $e_1, \ldots, e_n \in \mathbb{C}^n$ cyclically as follows:

$$ e_1 \mapsto e_{i+1} \mapsto e_{i+2} \mapsto \cdots \mapsto e_n \mapsto e_i \mapsto e_{i-1} \mapsto \cdots \mapsto e_2 \mapsto e_1 $$

(for $n = 4$ the matrix $A$ is displayed in the proof of Theorem 8.1). It is easy to see that $A$ belongs to $\mathcal{N}_{C,j}$. The first columns (with the first entry removed) of the first $n-1$ powers of $A$ exhaust the set of standard basis vectors in $\mathbb{C}^{n-1}$, showing that $T^*_C,J(x)(A) \neq 0$. Consequently, $T^*_C,J(x)$ is non-zero, and so it is a highest weight vector of weight $(n)$, generating an $\text{SO}_n(\mathbb{C}, J)$-submodule isomorphic to $W(n)$.

**Theorem 6.2** The degree $n(n-1)/2$ homogeneous component of the vanishing ideal $\mathcal{I}(\mathcal{F})$ of degenerate trace zero symmetric $n \times n$ real matrices contains an $\text{SO}_n$-submodule isomorphic to $\mathcal{H}^n(\mathbb{R}^n)$, the space of $n$-variable spherical harmonics of degree $n$. Consequently, the discriminant of $n \times n$ symmetric matrices can be written as the sum of $(2n-1) - (2n-3)$ squares.

**Proof.** As explained in Section 5 and in the beginning of Section 6, Proposition 6.1 implies the first statement. The second statement follows by Lemma 3.2.

Based on Kummer’s result $\mu(3) \leq 7 = \dim(\mathcal{H}^3(\mathbb{R}^3))$, Peter Lax surmized the inequality $\mu(n) \leq (\binom{2n}{n-1}) - (\binom{2n-3}{n-1})$ in his letter [10] to the author. This is a drastic improvement compared to the general upper bounds for $\mu(n)$ appearing in prior work known to us (cf. Remark 3.1). On the other hand, as we shall see in Sections 7 and 8, this inequality is not always sharp.

## 7 The case $n = 3$

**Proposition 7.1** For $n = 3$, the degree three homogeneous component $\mathcal{I}(\mathcal{F})_3$ of the vanishing ideal of degenerate trace zero symmetric matrices is isomorphic to the seven dimensional irreducible $\text{SO}_3$-module $\mathcal{H}^3(\mathbb{R}^3)$, and coincides with the image under the map $T^*$ of $(\bigwedge^2 \mathcal{N})^*$.

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Proof. Set \( K := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Then we have \( K^T K = J \). As explained in Sections 5 and 6, it is sufficient to prove that the degree three homogeneous component of \( I(F_{\mathbb{C},J}) \) is isomorphic as an \( SO_3(\mathbb{C}, J) \)-module to \( W(3) \), and coincides with \( T_C^*((\wedge^2 N_{\mathbb{C},J})^*) \), where

\[
F_{\mathbb{C},J} := \{ K^{-1} A K \mid A \in F \}
\]

(so \( F_{\mathbb{C},J} \) is the Zariski closure in \( N_{\mathbb{C},J} \) of the subset \( K^{-1} \cdot F \cdot K \)). The character of the \( SO_3(\mathbb{C}, J) \)-module \( N_{\mathbb{C},J} \) (i.e. the trace of the group element diag\((t, t^{-1}, 1)\) as a linear operator on \( N_{\mathbb{C},J} \)) equals \( t^3 + t^2 + 2t + 2t^{-1} + t^{-2} + t^{-3} \), hence the character of \( \wedge^2 N_{\mathbb{C},J} \) is

\[
t^3 + t^2 + 2t + 2t^{-1} + t^{-2} + t^{-3}.
\]

Since the character of \( W(d) \) is \( \sum_{j=-d}^{d} t^j \), we conclude that

\[
(\wedge^2 N_{\mathbb{C},J})^* \cong \wedge^2 N_{\mathbb{C},J} \cong W(1) + W(3).
\]

The first summand is the defining representation of \( SO_3(\mathbb{C}, J) \) on \( \mathbb{C}^3 \), and it is isomorphic to the adjoint representation on \( so_3(\mathbb{C}, J) \). It follows from the considerations in Section 9 that the kernel of the map \( \kappa \) (defined in Section 9) is isomorphic to the irreducible \( SO_3(\mathbb{C}, J) \)-module \( W(3) \), hence \( T_C^*((\wedge^2 N_{\mathbb{C},J})^*) \cong W(3) \) (say by the special case \( n = 3 \) of Theorem 6.2).

The degree three homogeneous component of \( \mathbb{C}[N_{\mathbb{C},J}] \) is isomorphic as an \( SO_3(\mathbb{C}, J) \)-module to the third symmetric tensor power \( S^3(W(2)) \), so its character is

\[
t^6 + t^5 + 2t^4 + 3t^3 + 4t^2 + 4t + 5 + 4t^{-1} + 4t^{-2} + 3t^{-3} + 2t^{-4} + t^{-5} + t^{-6},
\]

implying

\[
\mathbb{C}[N_{\mathbb{C},J}]_3 \cong W(6) + W(4) + W(3) + W(2) + W(0).
\]

Next we determine the highest weight vectors of the irreducible summands in the above decomposition. The unipotent radical of the positive Borel subalgebra of the Lie algebra \( so_3(\mathbb{C}, J) \) is one-dimensional spanned by \( E := E_{13} - E_{32} \), where \( E_{ij} \) stands for the \( 3 \times 3 \) matrix unit whose only non-zero entry is a 1 in the \((i, j)\)-position (see for example Section 10.4.1 in [13]). The following table gives the effect of \( E \) on the basis elements \( x_{ij} \in N^*_{\mathbb{C},J} \) (the coordinate function mapping a matrix in \( N_{\mathbb{C},J} \) to its \((i, j)\)-entry), as well as the weights of the \( x_{ij} \). To compute it note that the action of the Lie algebra
so\(_3(\mathbb{C}, J)\) is the following: \(A \in so_3(\mathbb{C}, J)\) maps \(B \in \mathcal{N}_{\mathbb{C}, J}\) to \([A, B] := AB - BA\) (matrix multiplication on the right hand side). Consequently, \(E\) sends \(x_{ij} \in \mathcal{N}_{\mathbb{C}, J}\) to the \((i, j)\)-entry of \([[x_{ij}]]_{j=1}^3, E\). The matrix \(H := E_{11} - E_{22}\) spans the Cartan subalgebra of \(so_3(\mathbb{C}, J)\), and the weight of \(x_{ij}\) is \(k \in \mathbb{Z}\) if \(H\) maps \(x_{ij}\) to \(kx_{ij}\). Note that we have the following linear relations in \(\mathcal{N}_{\mathbb{C}, J}\):

\[
x_{22} = x_{11}, \quad x_{33} = -2x_{11}, \quad x_{13} = x_{32}, \quad x_{23} = x_{31}.
\]

From this information one easily works out the following table:

| \(x \in \mathcal{N}_{\mathbb{C}, J}^*\) | \(x_{21}\) | \(x_{11}\) | \(x_{31}\) | \(x_{12}\) | \(x_{32}\) |
|---|---|---|---|---|---|
| weight of \(x\) | 2 | 0 | 1 | -2 | -1 |
| \(E(x)\) | 0 | -x_{31} | x_{21} | -2x_{32} | 3x_{11} |

The coefficient of \(t^2\) in the character of \(S^3(\mathcal{N}_{\mathbb{C}, J}^*)\) is 4, hence the weight subspace of weight 2 in \(S^3(\mathcal{N}_{\mathbb{C}, J}^*)\) is 4-dimensional. From the above table one easily sees that \(x_{21}x_{11}, x_{21}x_{31}, x_{32}, x_{21}^2x_{12}, x_{11}x_{21}^2\) is a basis of this weight space, and computes the images under \(E\) of these weight vectors. For example,

\[
E(x_{21}x_{11}) = E(x_{21})x_{11} + x_{21}E(x_{11})x_{11} + x_{21}x_{11}E(x_{11}) = -2x_{21}x_{11}x_{31}.
\]

Solving a system of linear equations one gets explicitly the elements in this weight space annihilated by \(E \in so_3(\mathbb{C}, J)\). It turns out that (up to non-zero scalar multiples) there is one highest weight vector of weight 2 in \(S^3(\mathcal{N}_{\mathbb{J}, \mathbb{C}}^*)\). One finds similarly all the highest weight vectors in the degree three homogeneous component of the coordinate ring of \(\mathcal{N}_{\mathbb{C}, J}\), they are listed in the following table (the highest weight vector of weight 0 is a scalar multiple of the function \(A \mapsto \text{Tr}(A^3)\)).

| weight | highest weight vector in \(S^3(\mathcal{N}_{\mathbb{C}, J}^*)\) |
|---|---|
| 6 | \(x_{21}^3\) |
| 4 | \(2x_{21}^2x_{11} + x_{21}x_{31}^2\) |
| 3 | \(x_{31}^3 + 3x_{21}x_{31}x_{11} - x_{21}^2x_{32}\) |
| 2 | \(3x_{21}x_{11}^2 + 2x_{21}x_{31}x_{32} + x_{21}^2x_{12}\) |
| 0 | \(-2x_{11}^3 + 2x_{11}x_{12}x_{21} - 2x_{11}x_{32}x_{31} + x_{12}x_{31}^2 + x_{32}x_{21}\) |

The diagonal matrix \(\text{diag}(2, -4, 2)\) belongs to \(\mathcal{F}\), hence

\[
K^{-1} \cdot \text{diag}(2, -4, 2) \cdot K = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

belongs to \(\mathcal{F}_{\mathbb{C}, J}\). Direct computation shows that there is only one polynomial in the second table vanishing on this matrix, namely the highest weight vector with weight 3. Consequently, \(\mathcal{I}(\mathcal{F}_{\mathbb{C}, J})_3 \cong W_3\) as \(SO_3(\mathbb{C}, J)\)-modules. \(\square\)
Proposition 7.2 Let $L$ denote the linear map from the symmetric tensor square of $\mathcal{I}(\mathcal{F})_3$ into $\mathbb{R}[N]_6$ induced by the multiplication map $\mathcal{I}(\mathcal{F})_3 \times \mathcal{I}(\mathcal{F})_3 \to \mathbb{R}[N]_6$, $(f_1, f_2) \mapsto f_1 f_2$. Then the kernel of $L$ is 5-dimensional, and is isomorphic to the $SO_3$-module $W(2) \cong \mathcal{H}^2(\mathbb{R}^3) \cong N$. 

Proof. We shall compute explicit highest weight vectors in the symmetric tensor square $S^2(\mathcal{I}(\mathcal{F},J)_3)$, and select those that are mapped to zero under $L_{\mathcal{C},J} : S^2(\mathcal{I}(\mathcal{F},J)_3) \to \mathbb{C}[N_{\mathcal{C},J}]_6$, the complexified version of $L$.

Fix a highest weight vector $y_3$ in the irreducible $SO_3(\mathbb{C},J)$-module $W(3)$. So $E(y_3) = 0$, and $y_3$ is uniquely determined up to a non-zero scalar multiple. Then there is a unique basis $\{y_k \mid k = 3, 2, 1, 0, -1, -2, -3\}$ in $W(3)$ such that $E(y_k) = y_{k+1}$ for $k = -3, -2, \ldots, 2$. Set $F := E_{31} - E_{23}$ and $H := E_{11} - E_{22}$. Then $F$ spans the unipotent radical of the negative Borel subalgebra of the Lie algebra $so_3(\mathbb{C},J)$, and $H$ spans the Cartan subalgebra of $so_3(\mathbb{C},J)$.

Moreover, $H(y_k) = ky_k$, i.e. $y_k$ is a weight vector with weight $k$ for $k = -3, \ldots, 3$. The relation $H = [E,F] = EF - FE$ shows that $y_2 = \frac{1}{4}F(y_3)$, $y_1 = \frac{1}{6}F(y_2)$, $y_0 = \frac{1}{6}F(y_1)$, $y_1 = \frac{1}{6}F(y_2)$, $y_2 = \frac{1}{6}F(y_1)$, $y_2 = \frac{1}{6}F(y_1)$, $y_3 = \frac{1}{6}F(y_2)$. Furthermore, $F(y_{-3}) = 0$. An easy character calculation yields

$$S^2(W(3)) \cong W(6) + W(4) + W(2) + W(0).$$

The highest weight vectors of the first three summands are

$$w(6) := y_3^2, \quad w(4) := 2y_3y_1 - y_2^2, \quad w(2) := 2y_3y_{-1} - 2y_2y_0 + y_1^2.$$ 

Denote by $\iota$ the unique $SO_3(\mathbb{C},J)$-module isomorphism $W(3) \to \mathcal{I}(\mathcal{F},J)_3$ mapping $y_3$ to the highest weight vector $x_3^3 + 3x_{21}x_{31}x_{11} - x_{32}^2x_{32} - x_{21}^2x_{12}$ of $\mathcal{I}(\mathcal{F},J)_3$ computed in the proof of Proposition 7.1. Keep the notation $\iota$ also for the induced isomorphism $S^2(W(3)) \to S^2(\mathcal{I}(\mathcal{F},J)_3)$. The effect of $F$ on the variables $x_{ij}$ can be computed similarly to the first table in the proof of Proposition 7.1.

\[
\begin{array}{c|c|c|c|c|c}
 x & x_{21} & x_{31} & x_{11} & x_{32} & x_{12} \\
 F(x) & 2x_{31} & -3x_{11} & x_{32} & -x_{12} & 0 \\
\end{array}
\]

Consequently,

$$\iota(y_2) = \frac{1}{3}F(\iota(y_3)) = \frac{1}{3}(-3x_{31}^2x_{11} - 9x_{21}x_{11}^2 - x_{31}x_{21}x_{32} + x_{21}^2x_{12})$$

$$\iota(y_1) = \frac{1}{5}F(\iota(y_2)) = \frac{1}{3}(-x_{31}^2x_{32} - 3x_{21}x_{11}x_{32} + x_{31}x_{21}x_{12})$$

$$\iota(y_0) = \frac{1}{6}F(\iota(y_1)) = \frac{1}{6}(x_{31}^2x_{12} - x_{21}x_{32})$$

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\[ \iota(y_{-1}) = \frac{1}{6} F \iota(y_0) = \frac{1}{18} (-3x_{31}x_{11}x_{12} - x_{31}x_{32}^2 + x_{21}x_{32}x_{12}) \]

Now one gets by direct computation that \( L_{C,J} \circ \iota(w(2)) = 2 \iota(y_3) \iota(y_{-1}) - 2 \iota(y_2) \iota(y_0) + \iota(y_1)^2 = 0 \), whereas \( w(6), w(4) \) do not belong to the kernel of \( L_{C,J} \circ \iota \). Obviously \( L_{C,J} \circ \iota(w(0)) \) is a non-zero scalar multiple of the discriminant (by Lemma 3.2), hence is non-zero. \( \square \)

It is well known that the discriminant of the trace zero symmetric \( 3 \times 3 \) matrix

\[
\begin{pmatrix}
  a & d & e \\
  d & b & f \\
  e & f & c
\end{pmatrix}
\]

where \( c = -a - b \)

is

\[ \delta = -4p^3 - 27q^2 \]

where

\[
\begin{align*}
p &= ab + ac + bc - d^2 - e^2 - f^2 \\
  &= -a^2 - ab - b^2 - d^2 - e^2 - f^2
\end{align*}
\]

and

\[
\begin{align*}
q &= -abc + af^2 + be^2 + cd^2 - 2df \\
  &= a^2b + ab^2 - ad^2 + af^2 - bd^2 + be^2 - 2def.
\end{align*}
\]

**Theorem 7.3** The discriminant \( \delta = -4p^3 - 27q^2 \) (where \( p, q \) are given above) of \( 3 \times 3 \) trace zero symmetric matrices can be written as

\[
\begin{align*}
\delta &= 27(af - be - df)^2 \\
  &+ (2a^2 + 3a^2b - 3ab^2 - ad^2 + 2ae^2 - af^2 - 2b^3 + bd^2 + be^2 - 2bf^2)^2 \\
  &+ (4a^2d + 10abd + 3ae + 4b^2d + 3bef - 2d^3 + de^2 + df^2)^2 \\
  &+ 4(a^2e + abe + 3adf - 2b^2e + 3bdf - 2d^2e + e^3 + ef)^2 \\
  &+ 4(2a^2f - abf - 3ade - b^2f - 3bde + 2d^2f - e^2f - f^3)^2.
\end{align*}
\]

Moreover, \( \delta = -4p^3 - 27q^2 \) can not be written as the sum of four (or less) squares in \( \mathbb{R}[\mathcal{N}] = \mathbb{R}[a, b, d, e, f] \), so \( \mu(3) = 5 \).

**Proof.** The formula can be checked by direct computation. We provide a representation theoretic proof, that shows the way we found it. Keep the
notation introduced in the proof of Proposition 7.2. Applying $F$ successively to $w_{(2)}$ we obtain the following basis in the summand $W_{(2)}$ of $S^2(W_{(3)})$:

\[
\begin{align*}
    k_2 &= 2y_3y_{-1} - 2y_2y_0 + y_1^2, \\
    k_1 &= 5y_3y_{-2} - 3y_2y_{-1} + y_1y_0, \\
    k_0 &= 5y_3y_{-3} - 3y_1y_{-1} + 2y_0^2, \\
    k_{-1} &= 5y_2y_{-3} - 5y_1y_{-2} + 2y_0y_{-1}, \\
    k_{-2} &= 5y_1y_{-3} - 10y_0y_{-2} + 6y_0^2.
\end{align*}
\]

It is easy to see that the highest weight vector in the trivial summand $W_{(0)}$ of $S^2(W_{(2)})$ is

\[
w_{(0)} = 2y_3y_{-3} - 2y_2y_{-2} + 2y_1y_{-1} - y_0^2.
\]

Identify $S^2(W_{(3)}) \cong S^2(\mathcal{I}(\mathcal{F}_{C,J}))$ via the isomorphism $\iota$ (see the proof of Proposition 7.2). Now $L_{C,J} \circ \iota(w_{(0)})$ is a non-zero scalar multiple of the discriminant $\delta \in \mathbb{C}[N_{C,J}]$, since $\delta$ is the only degree six $SO_3(\mathbb{C},J)$-invariant in $\mathcal{I}(\mathcal{F}_{C,J})$. Moreover, $L_{C,J} \circ \iota(k_i) = 0$ for $i = -2, -1, 0, 1, 2$ by the proof of Proposition 7.2. Consequently, there is a non-zero $c \in \mathbb{C}$ with

\[
c\delta = L_{C,J} \circ \iota(5w_{(0)} - 2k_0) = L_{C,J} \circ \iota(-10y_2y_{-2} + 16y_1y_{-1} - 9y_0^2).
\]

Complete the computation of the effect of $\iota$ on the basis of weight vectors of $W_{(3)}$ started in the proof of Proposition 7.2

\[
\iota(y_{-2}) = \frac{1}{5} F(\iota(y_{-1})) = \frac{1}{90} (9x_{11}^2x_{12} + x_{31}x_{32}x_{12} + 3x_{11}x_{32}^2 - x_{21}x_{12}^2)
\]

\[
\iota(y_{-3}) = \frac{1}{3} F(\iota(y_{-2})) = \frac{1}{90} (3x_{11}x_{12}x_{32} - x_{31}x_{12}^2 + x_{32}^3)
\]

Denote by $\sigma : \mathbb{C}[N_{C,J}] \rightarrow \mathbb{C}[N_{C}] = \mathbb{C}[a, b, d, e, f]$ the $\mathbb{C}$-algebra isomorphism given by

\[
\begin{pmatrix}
    \sigma(x_{11}) & \sigma(x_{12}) & \sigma(x_{13}) \\
    \sigma(x_{21}) & \sigma(x_{22}) & \sigma(x_{23}) \\
    \sigma(x_{31}) & \sigma(x_{32}) & \sigma(x_{33})
\end{pmatrix} = K^{-1} Y K
\]

where

\[
Y := \begin{pmatrix}
    a & d & e \\
    d & b & f \\
    e & f & -a - b
\end{pmatrix}
\]

and $K$ is the base change matrix given at the beginning of the proof of Proposition 7.1 so

\[
\begin{align*}
    \sigma(x_{11}) &= \frac{1}{2} (a + b), \\
    \sigma(x_{21}) &= \frac{1}{2} (a - b) + i \cdot d, \\
    \sigma(x_{31}) &= \frac{1}{\sqrt{2}} (e + i \cdot f), \\
    \sigma(x_{12}) &= \frac{1}{2} (a - b) - i \cdot d, \\
    \sigma(x_{32}) &= \frac{1}{\sqrt{2}} (e - i \cdot f).
\end{align*}
\]
Note that since the characteristic polynomial of a matrix is conjugation invariant, it follows from (6) that the coefficients of the characteristic polynomial of \((x_{ij})_{3 \times 3}\) are mapped by \(\sigma\) to the characteristic coefficients of the characteristic polynomial of \(Y\). Consequently, \(\sigma\) maps \(\delta \in \mathbb{C}[N_{C,J}]\) to \(\delta \in \mathbb{C}[N_{C}]\).

Introduce the following operation on \(\mathbb{C}[N_{C}] = \mathbb{C}[a, b, d, e, f]\): a polynomial \(p \in \mathbb{C}[N_{C}]\) can be uniquely written as \(\text{Re}(p) + i \cdot \text{Im}(p)\), where \(\text{Re}(p), \text{Im}(p)\) belong to the \(\mathbb{R}\)-subalgebra \(\mathbb{R}[N] = \mathbb{R}[a, b, d, e, f]\) of \(\mathbb{C}[N_{C}]\). Then set \(\overline{p} := \text{Re}(p) - i \cdot \text{Im}(p)\). Note that \(p \cdot \overline{p} = \text{Re}(p)^2 + \text{Im}(p)^2\), and \(p - \overline{p} = 2i \cdot \text{Im}(p)\).

Now observe that

\[
\sigma(x_{12}) = \overline{x_{21}}, \quad \sigma(x_{32}) = \overline{x_{31}}, \quad \text{and} \quad \sigma(x_{11}) = \sigma(x_{11}).
\]

Setting \(z_j := \sigma(\iota(y_j))\) for \(j = \pm 1, \pm 2\), and \(z_0 := \frac{1}{6}\sigma(x_{31}^2 x_{12})\), we have

\[
z_{-2} = -\frac{1}{30} z_{-2}, \quad z_{-1} = \frac{1}{6} z_{-1}, \\
\sigma(\iota(y_0)) = z_0 - \overline{z_0}.
\]

Therefore by (5) we get the following equality in \(\mathbb{C}[N_{C}]\):

\[
c \delta = \frac{1}{3}(\text{Re}(z_2)^2 + \text{Im}(z_2)^2) + \frac{8}{3}(\text{Re}(z_1)^2 + \text{Im}(z_1)^2) + 36 \cdot \text{Im}(z_0)^2.
\]

The right hand side is the sum of squares of five real polynomials; the constant turns out to be \(\frac{1}{27}\). Multiplying by \(4 \cdot 27\) the above equality one gets the formula in our theorem.

Next we show that the discriminant can not be written as the sum of four (or less) squares. Suppose to the contrary that \(\delta = f_1^2 + f_2^2 + f_3^2 + f_4^2\) for some \(f_i \in \mathbb{R}[N]\). Then all the \(f_i\) are homogeneous of degree 3, and all vanish on \(\mathcal{F}\), hence \(f_i \in \mathcal{I}(\mathcal{F})_3\). Denote by \(h_i\) the elements in the \(SO_8(\mathbb{C}, J)\)-module \(W_3\) with \(\sigma \circ \iota(h_i) = f_i\). Since \(\sigma\) maps \(\delta \in \mathbb{C}[N_{C,J}]\) to \(\delta \in \mathbb{C}[N_{C}]\), we have the equality \(\sum_{i=1}^4 \iota(h_i)^2 = \delta \in \mathbb{C}[N_{C,J}]\), i.e. \(L_{C,J} \circ \iota(\sum_{i=1}^4 h_i^2) = \delta \in \mathbb{C}[N_{C,J}]\). Consequently, \(c \sum_{i=1}^4 h_i^2 - w_{(0)}\) belongs to the kernel of \(L_{C,J} \circ \iota\) for some non-zero \(c \in \mathbb{C}\). By Proposition (7.2) there exists scalars \(a_2, a_1, a_0, a_{-1}, a_{-2} \in \mathbb{C}\) such that we have the following equality in \(S^2(W_3)\):

\[
c(h_1^2 + h_2^2 + h_3^2 + h_4^2) = w_{(0)} + 2 \sum_{j=-2}^{2} a_j k_j. \quad (7)
\]

The choice of the basis \(y_3, y_2, y_1, y_0, y_{-1}, y_{-2}, y_{-3}\) induces an identification between \(S^2(W_3)\) and the space of \(7 \times 7\) complex matrices. The element on
the right hand side of (7) corresponds to
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 2a_2 & 5a_1 & 1 + 5a_0 \\
0 & 0 & 0 & -2a_2 & -3a_1 & -1 & 5a_{-1} \\
0 & 2a_2 & a_1 & 1 - 3a_0 & -5a_{-1} & 5a_{-2} \\
0 & -2a_2 & a_1 & 1 - 4a_0 & 2a_{-1} & -10a_{-2} & 0 \\
2a_2 & -3a_1 & 1 - 3a_0 & 2a_{-1} & 12a_{-2} & 0 & 0 \\
5a_1 & -1 & -5a_{-1} & -10a_{-2} & 0 & 0 & 0 \\
1 + 5a_0 & 5a_{-1} & 5a_{-2} & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The left hand side of (7) implies that the rank of the above matrix is at most four. Therefore the determinant of the left upper 5 × 5 minor shows successively that \(0 = a_1 = a_{-2} = a_{-1}\). Then the rank of our matrix is five if \(1 + 5a_0\) or \(1 - 3a_0\) equals zero, the rank is six if \(-1 + 4a_0 = 0\), and the rank is seven otherwise. This is a contradiction, hence \(\delta\) can not be written as the sum of four (or less) squares. □

Remark 7.4 For comparison we give the expression for \(\delta = -4p^3 - 27q^2\) as a sum of seven squares that is obtained from the formula of Kummer \([6]\) after restriction to \(N\):

\[
\delta = 15(ade + 2bde - d^2 f + e^2 f)^2 + 15(-2adf - bdf + d^2 e - e f^2)^2 + 15(af - be f - d e^2 + df^2)^2 + (-4a^2 f + 2abf + 3ade + 2b^2 f - d^2 f - e^2 f + f^3)^2 + (2a^2 e + 2abe - 4b^2 e + 3bdf - d^2 e + 2e^3 - e f^2)^2 + (-4a^2 d - 10abd - 3ae f - 4b^2 d - 3be f + 2d^3 - de^2 - df^2)^2 + (-2a^3 - 3a^2 b + 3ab^2 + ad^2 - 2ae^2 + af^2 + 2b^3 - bd^2 - be^2 + 2bf^2)^2.
\]

We mention that an alternative way to arrive at Kummer’s formula was given by Jacobi \([5]\). Computational aspects of the problem of writing a form as a sum of squares are discussed by Parrilo in \([12]\); in particular, using a method based on semidefinite programming Parrilo finds the same presentation for the 3 × 3 discriminant as Kummer! Observe that the last two summands on the right hand side above appear also as the second and third summands on the right hand side of our formula in Theorem 7.3. Specializing \(a \rightarrow 0\), \(b \rightarrow 0\) in the above equality one recovers the expression for the discriminant \(4(d^2 + e^2 + f^2)^3 - 108d^2 e^2 f^2\) of

\[
\begin{pmatrix}
0 & d & e \\
d & 0 & f \\
e & f & 0
\end{pmatrix}
\]
as the sum of six squares found in [9]. The specialization $a \mapsto 0$, $b \mapsto 0$ of the formula in Theorem 7.3 yields
\[
4(d^2 + e^2 + f^2)^3 - 108de^2f^2 = 27(-de^2 + df^2)^2 + (-2d^3 + de^2 + df^2)^2 \\
+ 4(-2d^2e + e^3 + ef^2)^2 + 4(2d^2f - e^2f - f^3)^2
\]
(a sum of four squares on the right hand side).

**Remark 7.5** Comparing Proposition 7.1 and Theorem 7.3 we see that $\mu(n)$ can be strictly smaller than the minimal dimension of an irreducible $SO_n$-submodule in the degree $n(n-1)/2$ homogeneous component of the vanishing ideal of $\mathcal{F}$.

8 The case $n = 4$

**Theorem 8.1** When $n = 4$, the image of $(\bigwedge^3 N)^*$ under $T^*$ in the degree 6 homogeneous component of $I(\mathcal{F})$ is isomorphic as an $SO_4$-module to
\[
W_{(3,3)} + W_{(3,-3)} + W_{(4)}
\]
(the dimensions of the summands are 7, 7, 25). Consequently, the discriminant of $4 \times 4$ symmetric matrices can be written as the sum of seven squares.

**Proof.** The second statement follows from the first by Lemma 3.2.

To prove the first statement we may turn to the analogous statement for $SO_4(\mathbb{C}, J)$ and $N_{\mathbb{C}, J}$ (see the explanation in Sections 5 and 6). By a standard character calculation (the character of $W_{\lambda}$ is given for example in Section 24.2 of [3]) one obtains
\[
\bigwedge^3 N_{\mathbb{C}, J} \cong W_{(3,3)} + W_{(3,-3)} + W_{(4)} + W_{(3,1)} + W_{(3,-1)} + W_{(2)} + W_{(1,1)} + W_{(1,-1)}
\]
as $SO_4(\mathbb{C}, J)$-modules. Since $\dim(W_{\lambda}) = (\lambda_1 + 1)^2 - \lambda_2^2$ (see for example page 410 in [3]), the dimensions of the summands above are 7, 7, 25, 15, 15, 9, 3. (We note that the considerations below will show that all the above irreducible $SO_4$-modules occur as summands in $\bigwedge^3 N_{\mathbb{C}, J}^*$. Since their dimensions sum up to 84 = dim($\bigwedge^3 N_{\mathbb{C}, J}^*$), the above $SO_4$-module isomorphism follows without the character calculation.) Since $W_{(1,1)} + W_{(1,-1)}$ is isomorphic to the adjoint representation of $SO_4(\mathbb{C}, J)$ on its Lie algebra $so_4(\mathbb{C}, J)$, these two summands are annihilated by $T_{\mathbb{C}, J}^*$ by the considerations about the map $\kappa$ in Section 9. The summand $W_{(2)} \cong N_{\mathbb{C}, J}$ is also annihilated by $T_{\mathbb{C}, J}^*$, see
Section [9] The result of Section [5] in the special case $n = 4$ says that $W_{(4)}$ occurs as a summand in the image of $\mathcal{T}_{\mathcal{C}, J}^*$. Now we shall find the highest weight vectors of weight $(3, 3)$ respectively $(3, 1)$ in $\Lambda^3 \mathcal{N}_{\mathcal{C}, J}^*$. The unipotent radical of the positive Borel subalgebra of $\mathfrak{so}_4(\mathbb{C}, J)$ is spanned by $E_1 := E_{12} - E_{43}$ and $E_2 := E_{14} - E_{23}$, and the Cartan subalgebra of $\mathfrak{so}_4(\mathbb{C}, J)$ is spanned by $H_1 := E_{11} - E_{33}, H_2 := E_{22} - E_{44}$ (see for example Section 10.4.1 in [13]). Denote by $x_{ij}$ $(1 \leq i, j \leq 4)$ the usual coordinate functions on $\mathcal{N}_{\mathbb{C}, J}$. Then $x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{31}, x_{41}, x_{42}$ is a basis in $\mathcal{N}_{\mathbb{C}, J}$, and we have the relations $-x_{22} = x_{33} = -x_{44} = x_{11}, x_{23} = x_{14}, x_{32} = x_{41}, x_{34} = x_{21}, x_{43} = x_{12}$. Recall that $SO_4(\mathbb{C}, J)$ acts on $\mathcal{N}_{\mathbb{C}, J}$ by conjugation. The tangent representation of the dual representation of $SO_4(\mathbb{C}, J)$ on $\mathcal{N}_{\mathbb{C}, J}^*$ is the following representation of the Lie algebra $\mathfrak{so}_4(\mathbb{C}, J)$: for a Lie algebra element $A \in \mathfrak{so}_4(\mathbb{C}, J)$ and $x_{ij} \in \mathcal{N}_{\mathbb{C}, J}^*$ we have that $A(x_{ij})$ is the $(i, j)$ entry of the matrix commutator $XA - AX$, where $X = (x_{ij})^n_{i,j=1}$. Recall that given a representation of $\mathfrak{so}_4(\mathbb{C}, J)$ on some vector space $V$, we say that $v \in V$ is a weight vector of weight $(\alpha_1, \alpha_2) \in \mathbb{Z}^2$ if $H_i(v) = \alpha_i v$ holds for $i = 1, 2$. In our case the $x_{ij}$ are all weight vectors. One gets the following table:

| $x$ | $x_{11}$ | $x_{21}$ | $x_{31}$ | $x_{41}$ | $x_{42}$ | $x_{12}$ | $x_{13}$ | $x_{14}$ | $x_{24}$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| weight | 0, 0 | 1, -1 | 2, 0 | 1, 1 | 0, 2 | -1, 1 | -2, 0 | -1, -1 | 0, -2 |
| $E_1(x)$ | $-x_{21}$ | 0 | 0 | $x_{31}$ | $2x_{41}$ | $2x_{11}$ | $-2x_{14}$ | $-x_{24}$ | 0 |
| $E_2(x)$ | $-x_{41}$ | $x_{31}$ | 0 | 0 | 0 | $-x_{42}$ | $-2x_{12}$ | $2x_{11}$ | $2x_{21}$ |

The $(3, 3)$ weight space in $\Lambda^3 \mathcal{N}_{\mathbb{C}, J}^*$ is spanned by $x_{31} \wedge x_{41} \wedge x_{42}$, and this element is annihilated both by $E_1$ and $E_2$, as one can easily check using the table above. So this is a highest weight vector of weight $(3, 3)$. Set

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{N}_{\mathbb{C}, J}, A^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, A^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We have

$$\mathcal{T}_{\mathbb{C}, J}^*(x_{31} \wedge x_{41} \wedge x_{42})(A) = \det \begin{pmatrix} x_{31}(A) & x_{31}(A^2) & x_{31}(A^3) \\ x_{41}(A) & x_{41}(A^2) & x_{41}(A^3) \\ x_{42}(A) & x_{42}(A^2) & x_{42}(A^3) \end{pmatrix} = 1$$

so $x_{31} \wedge x_{41} \wedge x_{42}$ is not annihilated by $\mathcal{T}_{\mathbb{C}, J}^*$, hence $W_{(3,3)}$ occurs as a summand in the image of $T^*$. Since this image is an $O_4(\mathbb{C}, J)$-module, $W_{(3,-3)}$ is also a summand in the image.

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The \((3, 1)\) weight space is spanned by \(x_{21} \wedge x_{31} \wedge x_{42}\) and \(x_{11} \wedge x_{31} \wedge x_{41}\). Both are annihilated by \(E_2\), whereas

\[
E_1(x_{21} \wedge x_{31} \wedge x_{42}) = 2x_{21} \wedge x_{31} \wedge x_{41}, \quad E_1(x_{11} \wedge x_{31} \wedge x_{41}) = -x_{21} \wedge x_{31} \wedge x_{41}.
\]

Therefore there is one (up to non-zero scalar multiples) highest weight vector of weight \((3, 1)\), namely \(x_{21} \wedge x_{31} \wedge x_{42} + 2x_{11} \wedge x_{31} \wedge x_{41}\). We checked using CoCoA[2] that \(\mathcal{T}_{C, J}^*\) maps this highest weight vector to zero. Consequently, \(W_{(3, 1)}\) is not a summand in the image of \(\mathcal{T}_{C, J}^*\). Since the image is an \(O_4(C, J)\)-submodule, \(W_{(3, -1)}\) is not a summand in the image either.

\[\square\]

**Remark 8.2** Borchardt [1] notes at the end of his paper that his general process for writing the discriminant as a sum of squares yields in the case \(n = 4\) an expression with 135 summands, and states that the number of summands can be taken down to 84. Moreover, he adds that the number 84 may be further decreased, but does not specify the numbers.

**Remark 8.3** Combining Theorems 7.3 and 8.1 one gets \(5 \leq \mu(4) \leq 7\), since by Theorem 3 in [9], \(\mu(n)\) is an increasing function of \(n\).

## 9 About the kernel of \(\mathcal{T}^*\)

We do not have a general formula for the multiplicities of the irreducible \(SO_n\)-summands in \((\wedge^{n-1} \mathcal{N})^* \cong \wedge^{n-1} \mathcal{N}\). In Section 6 we found an irreducible \(SO_n\)-submodule in \((\wedge^{n-1} \mathcal{N})^*\) that is not mapped to zero under \(\mathcal{T}^*\). Here we present two constructions of some irreducible \(O_n\)-submodules (resp. \(SO_n\)-submodules) in the kernel of \(\mathcal{T}^*\).

Denote by \(so_n\) the Lie algebra of \(SO_n\); it can be identified with the space of \(n \times n\) skew-symmetric matrices, and the adjoint representation of \(O_n\) is identified with the conjugation action. Moreover, the conjugation representation of \(O_n\) on the space \(\mathbb{R}^{n \times n}\) of \(n \times n\) matrices decomposes as \(\mathbb{R}^{n \times n} = I \oplus \mathcal{N} \oplus so_n\).

We have an alternating multilinear \(O_n\)-equivariant map

\[
\mathcal{N} \oplus \cdots \oplus \mathcal{N} \rightarrow \mathbb{R}^{n \times n}, \quad (A_1, \ldots, A_{n-1}) \mapsto \sum_{\pi \in \text{Sym}(n-1)} \text{sign}(\pi) A_{\pi(1)} \cdots A_{\pi(n-1)}
\]

where the summation above is over the full symmetric group \(\text{Sym}(n - 1)\) of degree \(n - 1\). It induces an \(O_n\)-module morphism

\[
\kappa : \bigwedge^{n-1} \mathcal{N} \rightarrow \mathbb{R}^{n \times n}.
\]

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Denote by $\rho \in \text{Sym}(n-1)$ the permutation $\rho(i) = n - i$ ($i = 1, \ldots, n-1$). Then
\[
\text{sign}(\rho) = \begin{cases} 
-1, & \text{if } n \text{ or } n-3 \text{ is divisible by 4} \\
1, & \text{if } n-1 \text{ or } n-2 \text{ is divisible by 4}
\end{cases}
\]
The transpose of $\kappa(A_1 \wedge \cdots \wedge A_{n-1})$ (where the $A_i$ are symmetric) is
\[
\sum_{\pi \in \text{Sym}(n-1)} \text{sign}(\pi)A_{\pi(n-1)} \cdots A_{\pi(1)} = \sum_{\pi \in \text{Sym}(n-1)} \text{sign}(\pi)A_{\pi(1)} \cdots A_{\pi(n-1)}
= \text{sign}(\rho)\kappa(A_1 \wedge \cdots \wedge A_{n-1})
\]
so $\text{im}(\kappa) \subseteq \mathcal{M}$ when $n$ is congruent to 1 or 2 modulo 4, whereas $\text{im}(\kappa) \subseteq \text{so}_n$ otherwise. Denoting by $E_{ij}$ the $n \times n$ matrix unit with the entry 1 in the $(i, j)$ position and zeros everywhere else, we have
\[
\kappa((E_{12} + E_{21}) \wedge (E_{23} + E_{32}) \wedge \cdots \wedge (E_{n-1,n} + E_{n,n-1})) = E_{1n} + \text{sign}(\rho)E_{n1}
\]
hence $\kappa$ is non-zero. By irreducibility of $\text{so}_n$ it follows that $\text{im}(\kappa) = \text{so}_n$ if $n$ or $n-3$ is divisible by 4, so $\text{so}_n$ is an $\text{O}_n$-module direct summand in $\bigwedge^{n-1}\mathcal{N}$. When $n-1$ or $n-2$ is divisible by 4, the equality
\[
\kappa((E_{12} + E_{21}) \wedge (E_{23} + E_{32}) \wedge \cdots \wedge (E_{n-1,1} + E_{1,n-1})) = 2\sum_{j=1}^{n-1} (-1)^{n(j-1)}E_{jj}
\]
shows that neither $\mathcal{N}$ nor $\mathbb{R}I$ contains the image of $\kappa$, hence $\text{im}(\kappa) = \mathcal{M}$, and $\mathcal{M}$ is an $\text{O}_n$-module direct summand in $\bigwedge^{n-1}\mathcal{N}$.

Since the matrices $H_i(A)$ pairwise commute for all $A \in \mathcal{N}$ (see (3)), it follows that $\kappa \circ \mathcal{T} = 0$, so the image of $\mathcal{T}$ is contained in $\ker(\kappa)$, implying that the $\text{O}_n$-module map $\mathcal{T}^*$ factors through the natural surjection $(\bigwedge^{n-1}\mathcal{N})^* \rightarrow \ker(\kappa)^*$ (induced by the inclusion of $\ker(\kappa)$ into $\bigwedge^{n-1}\mathcal{N}$). Consequently, $\mathcal{T}^*((\bigwedge^{n-1}\mathcal{N})^*)$ is a non-zero homomorphic image of the $\text{O}_n$-module $\ker(\kappa)^* \cong \ker(\kappa)$.

Next for $n = 2l \geq 4$ even we construct an $\text{SO}_n$-module surjection $\gamma : \bigwedge^{n-1}\mathcal{N} \rightarrow \mathcal{N}$; note that $\gamma$ is not $\text{O}_n$-equivariant. For an $n \times n$ skew-symmetric matrix $C$ denote by $\text{Pf}(C)$ the pfaffian (see for example section 5.3.6 in [3]). Note that for $A, B \in \mathcal{N}$ their commutator $[A, B] = AB - BA$ is skew symmetric. Define the functions $F_\alpha$ on $\mathcal{N}^n = \mathcal{N} \oplus \cdots \oplus \mathcal{N}$ by
\[
\text{Pf}(t_1[A_1, A_2] + \cdots + t_l[A_{n-1}, A_n]) = \sum_{\alpha_1 + \cdots + \alpha_l = l} t_1^{\alpha_1} \cdots t_l^{\alpha_l} F_\alpha(A_1, \ldots, A_n)
\]
where $t_1, \ldots, t_l$ are commuting indeterminates. Set $F := F_{(1, \ldots, 1)}$, so $F$ is an $n$-variable multilinear $\text{SO}_n$-invariant function on $\mathcal{N}$. Define
\[
G(A_1, \ldots, A_n) := \frac{1}{n(n-2)\cdots 2} \sum_{\pi \in \text{Sym}(n)} \text{sign}(\pi)F(A_{\pi(1)}, \ldots, A_{\pi(n)})
\]
an alternating $n$-variable multilinear $SO_n$-invariant on $\mathcal{N}$. The function $G$ is non-zero, since $G(B_1, C_1, \ldots, B_l, C_l) = 1$ for the substitution

$$B_i := E_{2i-1,2i-1} - E_{2i,2i}, \quad C_i := \frac{1}{2}(E_{2i-1,2i} + E_{2i,2i-1}), \quad i = 1, \ldots, l.$$  

(8)

Since $G$ is multilinear, it is naturally identified with

\[ \tilde{G} \in (\mathcal{N} \otimes \cdots \otimes \mathcal{N})^* \cong \mathcal{N}^* \otimes \cdots \otimes \mathcal{N}^*. \]

Identify the last tensor factor $\mathcal{N}^*$ on the right hand side with $\mathcal{N}$ (using the trace form on $\mathcal{N}$), so view $\tilde{G}$ as an element of

\[ \tilde{G} \in ((\mathcal{N}^* \otimes \cdots \otimes \mathcal{N}^*) \otimes \mathcal{N})^{SO_n} \cong \text{hom}_{SO_n}(\mathcal{N} \otimes \cdots \otimes \mathcal{N}, \mathcal{N}). \]

Moreover, since $G$ is alternating, $\tilde{G}$ factors through the natural surjection

\[ \mathcal{N} \otimes \cdots \otimes \mathcal{N} \to \wedge^{n-1} \mathcal{N} \] and yields the desired non-zero element $\gamma \in \text{hom}_{SO_n}(\wedge^{n-1} \mathcal{N}, \mathcal{N})$. It is easy to see that $\gamma \circ T = 0$: indeed, the commutator of any two of $H_i(A), \ i = 1, \ldots, n - 1$ (see (3)) is zero, hence $F(1, \ldots, 1)$ becomes zero under a substitution of the arguments in any order by $A, H_2(A), \ldots, H_{n-1}(A), B$ (where $A, B \in \mathcal{N}$ are arbitrary). So $\gamma^*$ embeds $\mathcal{N}^* \cong \mathcal{N}$ as an $SO_n$-module direct summand in the kernel of $T^*$.

When $n-2$ is divisible by 4, denote by $\kappa_1$ the composition of the projection $\mathcal{M} = \mathcal{N} \oplus \mathbb{R}I \to \mathcal{N}$. Then $\gamma$ and $\kappa_1$ are both $SO_n$-module surjections from $\wedge^{n-1} \mathcal{N}$ to $\mathcal{N}$. However, they are not scalar multiples of each other, since $\kappa_1$ is $O_n$-equivariant, whereas $\gamma$ is not. (Alternatively, $\kappa(B_1 \wedge C_1 \wedge \cdots \wedge B_l) = 0$, where $B_i, C_j$ were defined in (8), whereas $\gamma(B_1 \wedge C_1 \wedge \cdots \wedge B_l) \neq 0$ as we pointed out above.) Consequently, the irreducible $SO_n$-module $\mathcal{N}$ appears with multiplicity $\geq 2$ as a summand in $\wedge^{n-1} \mathcal{N}$ when $n - 2$ is divisible by 4.

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