Research article

Quantum Hermite-Hadamard type integral inequalities for convex stochastic processes

Thanin Sitthiwirattham¹, Muhammad Aamir Ali²*, Hüseyin Budak³ and Saowaluck Chasreechai⁴

¹ Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, 10300, Thailand
² Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China
³ Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey
⁴ Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand

* Correspondence: Email: mahr.muhammad.aamir@gmail.com.

Abstract: In this paper, we introduce the notions of $q$-mean square integral for stochastic processes and co-ordinated stochastic processes. Furthermore, we establish some new quantum Hermite-Hadamard type inequalities for convex stochastic processes and co-ordinated stochastic processes via newly defined integrals. It is also revealed that the results presented in this research transformed into some already proved results by considering the limits as $q, q_1, q_2 \to 1^-$ in the newly obtained results.

Keywords: Hermite-Hadamard inequality; Jensen inequality; convex stochastic processes

Mathematics Subject Classification: 26D10, 26D15, 26A51

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention with an interest in elementary mathematics. A number of mathematicians have devoted their efforts to generalizing, refining and extending it for different classes of functions, such as convex mappings.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see [26, 49]). These inequalities state that if $F : I \to \mathbb{R}$ is a convex function
on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then
\[
F\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} F(x)dx \leq \frac{F(a) + F(b)}{2}. \tag{1.1}
\]
Both inequalities hold in the reversed direction if $F$ is concave. For more recent developments of inequality (1.1), one can consult [2, 3, 24, 25, 27, 48, 50, 60, 61].

The stochastic processes may be described in a general sense, and it has piqued the interest of many researchers due to its numerous applications in disciplines, such as physics, mathematics, economics, and engineering, therefore in 1980, K. Nikodem [44] introduced the notion of convex stochastic processes and discussed their regularity properties. In [54], A. Skowroński discussed some more results for convex stochastic processes which generalize to some results about the classical convex functions. After that, D. Kotrys established Hermite-Hadamard inequality for the convex stochastic processes in [38]. The inequality states that if a stochastic processes $X : I \times \Omega \to \mathbb{R}$ is a Jensen-convex, mean-square continuous in the interval $I$. Then for any $u, v \in I$ we have
\[
X\left(\frac{u+v}{2}, \cdot \right) \leq \frac{1}{v-u} \int_{a}^{b} X(t, \cdot) dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (\text{a.e.}). \tag{1.2}
\]

For more results regarding the inequality (1.2), one can read [14, 18, 19, 30, 36, 39, 42, 43, 51–53, 57, 58].

On the other side, in the domain of $q$-analysis, many works are being carried out initiating from Euler in order to attain adeptness in mathematics that constructs quantum computing $q$-calculus considered as a relationship between physics and mathematics. In different areas of mathematics, it has numerous applications, such as combinatorics, number theory, basic hypergeometric functions, orthogonal polynomials, and other sciences, mechanics, the theory of relativity, and quantum theory [10, 29, 35]. Quantum calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [15, 17]. Apparently, Euler invented this important mathematics branch. He used the $q$ parameter in Newton’s work on infinite series. Later, in a methodical manner, the $q$-calculus that knew without limits calculus was firstly given by F. H. Jackson [28, 33]. In 1966, W. Al-Salam [13] introduced a $q$-analogue of the $q$-fractional integral and $q$-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, J. Tariboon and S. K. Ntouyas introduced $aD_q$-difference operator and $q_a$-integral in [56]. In 2020, S. Bermudo et al. introduced the notion of $bD_q$ derivative and $q^b$-integral in [16].

Many integral inequalities have been studied using quantum and post-quantum integrals for various types of functions. For example, in [5, 8, 11, 12, 16, 21, 22, 34, 41, 45], the authors used $aD_q, bD_q$-derivatives and $q_a, q^b$-integrals to prove Hermite-Hadamard integral inequalities and their left-right estimates for convex and coordinated convex functions. In [46], M. A. Noor et al. presented a generalized version of quantum integral inequalities. For generalized quasi-convex functions, E. R. Nwaeze et al. proved certain parameterized quantum integral inequalities in [47]. M. A. Khan et al. proved quantum Hermite-Hadamard inequality using the green function in [37]. H. Budak et al. [20], M. A. Ali et al. [4, 6] and M. Vivas-Cortez et al. [59] developed new quantum Simpson’s and quantum Newton’s type inequalities for convex and coordinated convex functions. For quantum Ostrowski’s inequalities for convex and co-ordinated convex functions on can consult [7, 9, 23].
Recently, in [32], W. U. Haq introduced the notions about the \( q \)-mean square integral and gave the following quantum version of the inequality (1.2).

**Theorem 1.1.** If a stochastic process \( X : I \times \Omega \to \mathbb{R} \) is Jensen-convex and mean-square continuous in the interval \( I \), then for any \( u, v \in I \) we have

\[
X\left(\frac{qu + v}{2}, \cdot \right) \leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot) \, \alpha_{q} \, d_{q}t \leq \frac{qX(u, \cdot) + X(v, \cdot)}{[2]_{q}} \quad (a.e.).
\]

Despite the fact that stochastic processes theory and applications have advanced significantly in recent years, there are still many new and challenging problems in the areas of theory, analysis, and application, which include fields such as stochastic control, Markov chains, renewal processes, actuarial science, and so on.

Inspired by these ongoing studies, we introduce the notions of \( q \)-mean square integrals with respect to \( b \) and prove some new Hermite-Hadamard type inequality for convex stochastic processes. Moreover, we introduce four different variants of \( q_{1}q_{2} \)-mean square integrals for co-ordinated stochastic processes and prove some new Hermite-Hadamard type inequalities for co-ordinated stochastic processes.

This paper’s organization is as follows: We summarize the convex stochastic processes in Section 2, and some related work is given in this setup. In Section 3, we review the notions of \( q \)-calculus and some related research in this direction. In Sections 4 and 5, we prove Hermite-Hadamard type inequalities for the convex stochastic and co-ordinated convex stochastic processes. The relationship between the findings obtained and the comparable outcomes in the current literature is also discussed. Some findings and further directions for future study are found in Section 6. We assume that the analysis initiated in this paper could provide researchers working on integral inequalities and their applications with a strong source of inspiration.

### 2. Convex stochastic processes

Let \((\Omega, \mathcal{A}, P)\) be an arbitrary probability space. A function \( X : \Omega \to \mathbb{R} \) is called a random variable if it is \( \mathcal{A} \)-measurable. A function \( X : I \times \Omega \to \mathbb{R} \), where \( I \subset \mathbb{R} \) is an interval, is called a stochastic process if for every \( t \in I \) the function \( X(t, \cdot) \) is a random variable.

Recall that the stochastic process \( X : I \times \Omega \to \mathbb{R} \) is called

(i) **continuous in probability** in interval \( I \), if for all \( t_{0} \in I \) we have

\[
P \lim_{t \to t_{0}} X(t, \cdot) = X(t_{0}, \cdot),
\]

where \( P \lim \) denotes the limit in probability.

(ii) **mean-square continuous** in the interval \( I \), if for all \( t_{0} \in I \)

\[
\lim_{t \to t_{0}} E\left[(X(t) - X(t_{0}))^{2}\right] = 0,
\]

where \( E[X(t)] \) denotes the expectation value of the random variable \( X(t, \cdot) \).

Indeed, mean-square continuity implies continuity in probability, but the converse implication is not true.
Definition 2.1. Suppose we are given a sequence \( \{ \Delta^m \} \) of partitions, \( \Delta^m = \{ a_{m,0}, \ldots, a_{m,n_m} \} \). We say that the sequence \( \{ \Delta^m \} \) is a normal sequence of partitions if the length of the greatest interval in the \( n \)-th partition tends to zero, i.e.,

\[
\lim_{m \to \infty} \sup_{1 \leq i \leq n_m} |a_{m,i} - a_{m,i-1}| = 0.
\]

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties (see [55]).

Let \( X : I \times \Omega \to \mathbb{R} \) be a stochastic process with \( E[X(t)^2] < \infty \) for all \( t \in I \). Let \( [a, b] \subset I \), \( a = t_0 < t_1 < t_2 < \ldots < t_n = b \) be a partition of \( [a, b] \) and \( \Theta_k \in [t_{k-1}, t_k] \) for all \( k = 1, \ldots, n \). A random variable \( Y : \Omega \to \mathbb{R} \) is called the mean-square integral of the process \( X \) on \( [a, b] \), if we have

\[
\lim_{n \to \infty} E \left[ \left( \sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y \right)^2 \right] = 0
\]

for all normal sequence of partitions of the interval \([a, b]\) and for all \( \Theta_k \in [t_{k-1}, t_k] \), \( k = 1, \ldots, n \). Then, we write

\[
Y(\cdot) = \int_{a}^{b} X(s, \cdot) ds \text{ (a.e.).}
\]

For the existence of the mean-square integral, it is enough to assume the mean-square continuity of the stochastic process \( X \).

Throughout the paper, we will frequently use the monotonicity of the mean-square integral. If \( X(t, \cdot) \leq Y(t, \cdot) \) (a.e.) in some interval \([a, b]\), then

\[
\int_{a}^{b} X(t, \cdot) dt \leq \int_{a}^{b} Y(t, \cdot) dt \text{ (a.e.).}
\]

Of course, this inequality is an immediate consequence of the definition of the mean-square integral.

Definition 2.2. We say that a stochastic process \( X : I \times \Omega \to \mathbb{R} \) is convex, if for all \( \lambda \in [0, 1] \) and \( u, v \in I \) the inequality

\[
X(\lambda u + (1 - \lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot) \text{ (a.e.)}
\] (2.1)

is satisfied. If the above inequality is assumed only for \( \lambda = \frac{1}{2} \), then the process \( X \) is Jensen-convex or \( \frac{1}{2} \)-convex. A stochastic process \( X \) is concave if \((-X)\) is convex. Some interesting properties of convex and Jensen-convex processes are presented in [44, 55].

Now, we present some results proved by D. Kotrys [38] about Hermite-Hadamard inequality for convex stochastic processes.

Lemma 2.1. If \( X : I \times \Omega \to \mathbb{R} \) is a stochastic process of the form \( X(t, \cdot) = A(\cdot) t + B(\cdot) \), where \( A, B : \Omega \to \mathbb{R} \) are random variables, such that \( E[A^2] < \infty, E[B^2] < \infty \) and \([a, b] \subset I \), then

\[
\int_{a}^{b} X(t, \cdot) dt = A(\cdot) \frac{b^2 - a^2}{2} + B(\cdot) (b - a) \text{ (a.e.).}
\]
Proposition 2.1. Let \( X : I \times \Omega \to \mathbb{R} \) be a convex stochastic process and \( t_0 \in \text{int}I \). Then there exists a random variable \( A : \Omega \to \mathbb{R} \) such that \( X \) is supported at \( t_0 \) by the process \( A(\cdot)(t-t_0) + X(t_0, \cdot) \). That is
\[
X(t, \cdot) \geq A(\cdot)(t-t_0) + X(t_0, \cdot) \quad (a.e.)
\]
for all \( t \in I \).

Theorem 2.1. If a stochastic process \( X : I \times \Omega \to \mathbb{R} \) is Jensen-convex, mean-square continuous in the interval \( I \). Then for any \( u, v \in I \) we have
\[
X\left(\frac{u+v}{2}, \cdot\right) = \frac{1}{v-u} \int_u^v X(t, \cdot) \, dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (a.e.). \tag{2.2}
\]

In [39], D. Kotrys introduced the concept of strongly convex stochastic processes and investigated their properties.

Definition 2.3. Let \( C : \Omega \to \mathbb{R} \) denote a positive random variable. A stochastic process \( X : I \times \Omega \to \mathbb{R} \) is called strongly convex with modulus \( C(\cdot) > 0 \), if for all \( \lambda \in [0, 1] \) and \( u, v \in I \) the inequality
\[
X(\lambda u + (1-\lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) - C(\cdot)\lambda(1-\lambda)(u-v)^2 \quad \text{a.e.}
\]
is satisfied. If the above inequality is assumed only for \( \lambda = \frac{1}{2} \), then the process \( X \) is strongly Jensen-convex with modulus \( C(\cdot) \).

In [31], F. M. Hafiz gave the following definition of stochastic mean-square fractional integrals:

Definition 2.4. For the stochastic process \( X : I \times \Omega \to \mathbb{R} \), the concept of stochastic mean-square fractional integrals \( I^\alpha_{u+} \) and \( I^\alpha_{v-} \) of \( X \) of order \( \alpha > 0 \) is defined by
\[
I^\alpha_{u+}[X](t) = \frac{1}{\Gamma(\alpha)} \int_u^t (t-s)^{\alpha-1} X(x, s) \, ds \quad (a.e.), \quad t > u
\]
and
\[
I^\alpha_{v-}[X](t) = \frac{1}{\Gamma(\alpha)} \int_t^v (s-t)^{\alpha-1} X(x, s) \, ds \quad (a.e.), \quad t < v.
\]

Using this concept of stochastic mean-square fractional integrals \( I^\alpha_{u+} \) and \( I^\alpha_{v-} \), H. Agahi and A. Babakhani proved the following Hermite-Hadamard type inequality for convex stochastic processes:

Theorem 2.2. [1] Let \( X : I \times \Omega \to \mathbb{R} \) be a Jensen-convex stochastic process that is mean-square continuous in the interval \( I \). Then for any \( u, v \in I \), the following Hermite-Hadamard inequality
\[
X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} \left[ I^\alpha_{u+}[X](v) + I^\alpha_{v-}[X](u) \right] \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (a.e.) \tag{2.3}
\]
holds, where \( \alpha > 0 \).
3. Quantum calculus and related inequalities

In this section, we present some required definitions. Additionally, here and further we use the following notation (see [35]):

\[ [n]_q = \frac{1 - q^n}{1 - q} = \sum_{n=0}^{q^n-1} q^n, \quad q \in (0, 1). \]

In [33], F. H. Jackson gave the \(q\)-Jackson integral from 0 to \(b\) for \(0 < q < 1\) as follows:

\[
\int_0^b F(x) \, d_q x = (1 - q) \sum_{n=0}^{\infty} q^n F(bq^n) \tag{3.1}
\]
promdied the sums converges absolutely. Furthermore, he gave the \(q\)-Jackson integral in a generic interval \([a, b]\) as:

\[
\int_a^b F(x) \, d_q x = \int_0^b F(x) \, d_q x - \int_0^a F(x) \, d_q x.
\]

**Definition 3.1.** [56] For a continuous function \(F : [a, b] \to \mathbb{R}\), the \(q\)-derivative of \(F\) at \(x \in [a, b]\) is characterized by the expression:

\[
aD_q F(x) = \frac{F(x) - F(qx + (1 - q)a)}{(1 - q)(x - a)}, \quad x \neq a.
\]  

If \(x = a\), we define \(aD_q F(a) = \lim_{x \to a} aD_q F(x)\) if it exists and it is finite.

**Definition 3.2.** [56] Let \(F : [a, b] \to \mathbb{R}\) be a continuous function. Then, the \(q\)-definite integral on \([a, b]\) is defined as:

\[
\int_a^b F(x) \, d_q x = (b - a) \sum_{n=0}^{\infty} q^n F(q^n b + (1 - q^n)a) = (b - a) \int_0^1 F((1 - t)a + tb) \, d_q t.
\]

In [11], N. Alp et al. proved the following \(q\)-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

**Theorem 3.1.** If \(F : [a, b] \to \mathbb{R}\) is a convex differentiable function on \([a, b]\) and \(0 < q < 1\). Then, the \(q\)-Hermite-Hadamard inequalities are expressed as:

\[
F\left(\frac{qa + b}{[2]_q}\right) \leq \frac{1}{b - a} \int_a^b F(x) \, d_q x \leq \frac{qF(a) + F(b)}{[2]_q}.
\]  

On the other hand, S. Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:
The q-Hermite-Hadamard inequalities are expressed as:

**Theorem 3.2.** [16] If \( F \) is a convex function on \( [a, b] \), then for any convex function \( F \) on \( [a, b] \),

\[
\int_a^b F(x) \, dq_x = (b-a) \sum_{n=0}^{\infty} q^n F(q^n a + (1-q^n) b) = (b-a) \int_0^1 F(ta + (1-t) b) \, dt.
\]

If \( x = b \), we define \( \int_a^b F(x) \, dq_x = \lim_{x \to b} \int_a^b F(x) \, dq_x \) if it exists and it is finite.

**Definition 3.3.** [16] For a continuous function \( F : [a, b] \to \mathbb{R} \), the \( q \)-derivative of \( F \) at \( x \in [a, b] \) is characterized by the expression:

\[
\frac{b \, D_q F(x)}{b} = \frac{F(qx + (1-q)b) - F(x)}{(1-q)(b-x)}, \quad x \neq b.
\]

**Corollary 3.1.** [16] For any convex function \( F \) on \( [a, b] \), the \( q \)-Hermite-Hadamard inequalities are expressed as:

\[
F \left( \frac{a + qb}{2} \right) \leq \frac{1}{b-a} \int_a^b F(x) \, dq_x \leq \frac{F(a) + qF(b)}{2}.
\] (3.4)

From Theorem 3.1 and Theorem 3.2, one can obtain the following inequalities:

**Theorem 3.2.** [16] If \( F : [a, b] \to \mathbb{R} \) is a convex differentiable function on \( [a, b] \) and \( 0 < q < 1 \). Then, the \( q \)-Hermite-Hadamard inequalities are expressed as:

\[
F \left( \frac{a + qb}{2} \right) \leq \frac{1}{b-a} \int_a^b F(x) \, dq_x \leq \frac{F(a) + qF(b)}{2},
\] (3.4)

**Corollary 3.1.** [16] For any convex function \( F : [a, b] \to \mathbb{R} \) and \( 0 < q < 1 \), we have

\[
F \left( \frac{qa + b}{2} \right) + F \left( \frac{a + qb}{2} \right) \leq \frac{1}{b-a} \left\{ \int_a^b F(x) \, dq_x + \int_a^b F(x) \, dq_x \right\} \leq F(a) + F(b)
\] (3.5)

and

\[
F \left( \frac{a + b}{2} \right) \leq \frac{1}{2(b-a)} \left\{ \int_a^b F(x) \, dq_x + \int_a^b F(x) \, dq_x \right\} \leq \frac{F(a) + F(b)}{2}.
\] (3.6)

In [40], M. A. Latif defined \( q_{ac} \)-integral and partial \( q \)-derivatives for two variables functions as follows:

**Definition 3.5.** Suppose that \( F : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R} \) is continuous function. The definite \( q_{ac} \)-integral on \( [a, b] \times [c, d] \) is defined by

\[
\int_a^x \int_c^y F(t, s) \, dq_{2}s \, dq_{1}t = (1-q_1)(1-q_2)(x-a)(y-c)
\]

\[
\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^n F(q_1^n a, q_2^n c, q_1^n y + (1-q_1^n) a, q_2^n y + (1-q_2^n) c)
\]

for \((x, y) \in [a, b] \times [c, d]\).
On the other hand, H. Budak et al. gave the following definitions of $q^d_a$, $q^b$ and $q^{bd}$ integrals:

**Definition 3.6.** [22] Suppose that $F : [a, b] \times [c, d] \subset \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Then the following $q^d_a$, $q^b$ and $q^{bd}$ integrals on $[a, b] \times [c, d]$ are defined by

\[
\int_a^b \int_y^d F(t, s) \, dq^d_{s,t} = (1 - q_1)(1 - q_2)(x - a)(d - y) \\
\times \sum_{n=0}^\infty \sum_{m=0}^\infty q^n_1 q^n_2 F(q^n_1 x + (1 - q^n_1) a, q^n_2 y + (1 - q^n_2) d)
\]

\[
\int_x^b \int_c^y F(t, s) \, dq^b_{s,t} = (1 - q_1)(1 - q_2)(b - x)(y - c) \\
\times \sum_{n=0}^\infty \sum_{m=0}^\infty q^n_1 q^n_2 F(q^n_1 x + (1 - q^n_1) b, q^n_2 y + (1 - q^n_2) c)
\]

and

\[
\int_x^b \int_y^d F(t, s) \, dq^{bd}_{s,t} = (1 - q_1)(1 - q_2)(b - x)(d - y) \\
\times \sum_{n=0}^\infty \sum_{m=0}^\infty q^n_1 q^n_2 F(q^n_1 x + (1 - q^n_1) b, q^n_2 y + (1 - q^n_2) d)
\]

respectively, for $(x, y) \in [a, b] \times [c, d]$.

Recently, W. U. Haq gave the following definition of $q_a$-mean square integral and related results for quantum stochastic process:

**Definition 3.7.** [32] For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the $q_a$-mean integral of $X$ on $[a, b] \subset I$ with $E[(X(t))^2] < \infty$ for all $t \in I$, is a random variable $\Lambda : \Omega \to \mathbb{R}$ which satisfying the following equality:

\[
E\left[\left((1 - q)(b - a) \sum_{n=0}^\infty q^n X(q^n b + (1 - q^n) a) - \Lambda\right)^2\right] = 0.
\]

Hence, we can state

\[
\Lambda(.) = \int_a^b X(t, .) \, dq_a t \quad (a.e.),
\]

for the existence of the $q_a$-mean integral, the stochastic process must be mean square continuous.

**Lemma 3.1.** [32] If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t, .) = A(\cdot) t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E[A^2] < \infty$, $E[B^2] < \infty$ and $[a, b] \subset I$, then

\[
\int_a^b X(t, .) \, dq_a t = A(\cdot) \frac{(b - a)(b + qa)}{[2]_q} + B(\cdot)(b - a) \quad (a.e.).
\]
Theorem 3.3. [32] If a stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $I$, then for any $u, v \in I$ we have

$$X \left( \frac{qu + v}{2q}, \cdot \right) \leq \frac{1}{v - u} \int_u^v X(t, \cdot) \, q \, dt \leq \frac{qX(u, \cdot) + X(v, \cdot)}{2q} \quad (a.e.).$$  \hspace{1cm} (3.10)

4. Quantum Hermite-Hadamard inequality for convex stochastic processes

In this section, we introduce the notion of $q^b$-mean square integral for stochastic process and establish some new inequalities of Hermite-Hadamard type for convex stochastic process using the $q^b$-mean square integral.

Definition 4.1. For the stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$, the $q^b$-mean integral of $X$ on $[a, b] \subset I$ with $E \left[ (X(t))^2 \right] < \infty$ for all $t \in I$, is a random variable $\Lambda : \Omega \rightarrow \mathbb{R}$ which satisfying the following equality:

$$E \left[ \left( (1 - q)(b - a) \sum_{n=0}^{\infty} q^n X(q^n a + (1 - q^n) b) - \Lambda \right)^2 \right] = 0.$$

Hence, we can state

$$\Lambda (\cdot) = \int_a^b X(t, \cdot) \, b \, dq_t \quad (a.e.),$$

for the existence of the $q^b$-mean integral, the stochastic process must be a mean square continuous.

Lemma 4.1. If $X : I \times \Omega \rightarrow \mathbb{R}$ is a stochastic process of the form $X(t, \cdot) = A(\cdot) t + B(\cdot)$, where $A, B : \Omega \rightarrow \mathbb{R}$ are random variables, such that $E \left[ A^2 \right] < \infty$, $E \left[ B^2 \right] < \infty$ and $[a, b] \subset I$, then

$$\int_a^b X(t, \cdot) \, b \, dq_t = A(\cdot) \frac{(b - a)(qb + a)}{2q} + B(\cdot) (b - a) \quad (a.e.).$$

Theorem 4.1. If a stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $I$, then for any $u, v \in I$ we have

$$X \left( \frac{u + qv}{2q}, \cdot \right) \leq \frac{1}{v - u} \int_u^v X(t, \cdot) \, v \, dt \leq \frac{X(u, \cdot) + qX(v, \cdot)}{2q} \quad (a.e.).$$  \hspace{1cm} (4.1)

Proof. Since the process $X$ is mean-square continuous, it is continuous in probability. Nikodem in [44] proved that every Jensen-convex and continuous in probability stochastic process is convex. Since $X$ is convex, then by Proposition 2.1 it has a supporting process at any point $t_0 \in int I$. Let’s take a support at $t_0 = \frac{u + qv}{2q}$, then we have

$$X(t, \cdot) \geq A(\cdot) \left( t - \frac{u + qv}{2q} \right) + X \left( \frac{u + qv}{2q}, \cdot \right) \quad (a.e.).$$
By $q^v$-integral, we have
\[
\int_u^v X(t, \cdot)^v d_q t \geq A(\cdot) \frac{(v - u)(qv + u)}{[2]_q} - A(\cdot) \frac{u + qv}{[2]_q} (v - u) + X \left( \frac{u + qv}{[2]_q}, \cdot \right) (v - u).
\]
That is
\[
\frac{1}{v - u} \int_u^v X(t, \cdot)^v d_q t \geq X \left( \frac{u + qv}{[2]_q}, \cdot \right)
\]
which proves the first inequality in (4.1).

Since stochastic processes $X$ is convex, we have
\[
X(t, \cdot) = X \left( \frac{t - u}{v - u} v + \frac{v - t}{v - u} u, \cdot \right)
\leq \frac{t - u}{v - u} X(v, \cdot) + \frac{v - t}{v - u} X(u, \cdot)
= \frac{X(v, \cdot) - X(u, \cdot)}{v - u} t + \frac{vX(u, \cdot) - uX(v, \cdot)}{v - u}.
\]
By using $q^v$-integral, we have
\[
\int_u^v X(t, \cdot)^v d_q t \leq \frac{X(v, \cdot) - X(u, \cdot)}{v - u} \frac{u + qv}{[2]_q} (v - u) + \frac{vX(u, \cdot) - uX(v, \cdot)}{v - u} (v - u)
= \frac{X(u, \cdot) + qX(v, \cdot)}{[2]_q} (v - u).
\]
This completes the proof. \(\square\)

**Remark 4.1.** If we set $q \to 1^{-}$ in Theorem 4.1, then Theorem 4.1 reduces to Theorem 2.1.

Adding the results in Theorems 3.1 and 4.1 yields the next corollary.

**Corollary 4.1.** If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $I$, then for any $u, v \in I$ we have
\[
\frac{1}{2} \left[ X \left( \frac{qu + v}{[2]_q}, \cdot \right) + X \left( \frac{u + qv}{[2]_q}, \cdot \right) \right]
\leq \frac{1}{2(v - u)} \left[ \int_u^v X(t, \cdot)^v d_q t + \int_u^v X(t, \cdot)^v d_q t \right]
\leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (a.e.).
\]
Corollary 4.2. If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is Jensen-convex and mean-square continuous in the interval $I$, then for any $u, v \in I$ we have

$$X\left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{2(v - u)} \left[ \int_u^v X(t, \cdot) \, dq_t + \int_u^v X(t, \cdot) \, dq_t \right] \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (a.e.).$$

**Proof.** Since $X$ is convex stochastic process, we have

$$X\left(\frac{u + v}{2}, \cdot\right) = X\left(\frac{1}{2} [2]_q u + \frac{1}{2} [2]_q v, \cdot\right) \leq \frac{1}{2} \left[ X\left(\frac{qu + v}{[2]_q}, \cdot\right) + X\left(\frac{u + qv}{[2]_q}, \cdot\right) \right].$$

This completes the proof. □

Theorem 4.2. If a stochastic process $X : I \times \Omega \to \mathbb{R}$ is strongly Jensen-convex with modulus $C(\cdot)$ and mean-square continuous in the interval $I$ such that $E[C^2]$, then for any $u, v \in I$ we have

$$X\left(\frac{u + qv}{[2]_q}, \cdot\right) + C(\cdot) \left[ \frac{q(u + qv)^2 + u^2 + qv^2}{[2]_q [3]_q} - \left(\frac{u + qv}{[2]_q}\right)^2 \right] \leq \frac{1}{v - u} \int_u^v X(t, \cdot) \, dq_t \leq \frac{X(u, \cdot) + X(v, \cdot)}{[2]_q} \quad (a.e.).$$

**Proof.** If $X$ is strongly convex with modulus $C(\cdot)$, then the process $Y(t, \cdot) = X(t, \cdot) - C(\cdot)t^2$ is convex. Therefore, if we apply the inequality (4.1) to process $Y(t, \cdot)$, then we have

$$Y\left(\frac{u + qv}{[2]_q}, \cdot\right) \leq \frac{1}{v - u} \int_u^v Y(t, \cdot) \, dq_t \leq \frac{Y(u, \cdot) + qY(v, \cdot)}{[2]_q} \quad (a.e.).$$

That is,

$$X\left(\frac{u + qv}{[2]_q}, \cdot\right) - C(\cdot) \left(\frac{u + qv}{[2]_q}\right)^2 \leq \frac{1}{v - u} \int_u^v \left[ X(t, \cdot) - C(\cdot)t^2 \right] \, dq_t$$

Aims Mathematics
\[ \frac{X(u, \cdot) - C(\cdot)u^2 + qX(v, \cdot) - C(\cdot)v^2}{[2]_q} \] (a.e.).

By calculating the integrals, we have
\[
\begin{align*}
X\left(\frac{u + qv}{[2]_q}, \cdot\right) + C(\cdot)\left[ q\frac{(u + qv)^2 + u^2 + qv^2}{[2]_q[3]_q} - \left(\frac{u + qv}{[2]_q}\right)^2 \right] \\
\leq \frac{1}{v - u} \int_v^u X(t, \cdot) \, ^v d_q t \\
\leq \frac{X(u, \cdot) + qX(v, \cdot)}{[2]_q} + C(\cdot)\left[ q\frac{(u + qv)^2 + u^2 + qv^2}{[2]_q[3]_q} - \frac{u^2 + qv^2}{[2]_q} \right] (a.e.)
\end{align*}
\]
which gives the required result. \( \Box \)

**Remark 4.2.** If we set \( q \to 1^- \) in Theorem 4.2, then Theorem 4.2 reduces to [39, Theorem 11].

5. Quantum Hermite-Hadamard inequality for co-ordinated convex stochastic processes

In this section, we review the definition of co-ordinated convex stochastic process and introduce the notion of \( q_1q_2 \)-mean square integrals for co-ordinated stochastic processes. Moreover, we prove Hermite-Hadamard inequalities for co-ordinated convex stochastic process using the \( q_1q_2 \)-mean square integrals which is the main motivation of this section.

**Definition 5.1.** [53] Let \( \Lambda := T_1 \times T_2, T_1, T_2 \subset \mathbb{R} \). A stochastic process \( X : \Lambda \times \Omega \to \mathbb{R} \) is called co-ordinated convex on \( \Lambda \) if the following inequality holds for all \( \alpha, \beta \in [0, 1] \) and \( (t_1, s_1), (t_2, s_2) \in \Lambda \):
\[
\begin{align*}
X((\alpha t_1 + (1 - \alpha) t_2, \beta s_1 + (1 - \beta) s_2), \cdot) & \leq \alpha \beta X((t_1, s_1), \cdot) + \alpha (1 - \beta) X((t_1, s_2), \cdot) \\
+ (1 - \alpha) \beta X((t_2, s_1), \cdot) + (1 - \alpha) (1 - \beta) X((t_2, s_2), \cdot).
\end{align*}
\]

We can give the definitions of \( q_1q_2 \)-mean square integrals as follows:

**Definition 5.2.** Let \( X : \Lambda \times \Omega \to \mathbb{R} \) be a stochastic process with \( E\left[X(t, s)^2\right] < \infty \) for all \( (t, s) \in \Lambda \).

i) A random variable \( \Pi_1 : \Omega \to \mathbb{R} \) is called the \( q_{ac} \)-mean-square integral of the process \( X \) on \( [a, b] \times [c, d] \subset \Lambda \), if we have
\[
E\left[\left(1 - q_1\right)(1 - q_2)(b - a)(d - c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m X(q_1^n b + (1 - q_1^n) a, q_2^m d + (1 - q_2^m) c) - \Pi_1\right)^2\right] = 0.
\]

Then, we can state
\[
\Pi_1(\cdot) = \int_a^b \int_c^d X((t, s), \cdot) \, _c d_{q_2} s \, _a d_{q_1} t \quad (a.e.)
\]
ii) A random variable $\Pi_2 : \Omega \to \mathbb{R}$ is called the $q^2_d$-mean-square integral of the process $X$ on $[a, b] \times [c, d] \subset \Lambda$, if we have

$$E\left[ \left( (1 - q_1) (1 - q_2) (b - a) (d - c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \sum_{a}^{b} \sum_{c}^{d} q^n_d q^m_d X (q^n_d a + (1 - q^n_d) b, q^m_d c + (1 - q^m_d) d) - \Pi_2 \right)^2 \right] = 0.$$ 

Then, we can define

$$\Pi_2 (\cdot) = \int_{a}^{b} \int_{c}^{d} X ((t, s), \cdot) \ d q_2^s \ d q_1^t \ (a.e.).$$

iii) A random variable $\Pi_3 : \Omega \to \mathbb{R}$ is called the $q^3_d$-mean-square integral of the process $X$ on $[a, b] \times [c, d] \subset \Lambda$, if we have

$$E\left[ \left( (1 - q_1) (1 - q_2) (b - a) (d - c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \sum_{a}^{b} \sum_{c}^{d} q^n_d q^m_d X (q^n_d a + (1 - q^n_d) b, q^m_d c + (1 - q^m_d) d) - \Pi_3 \right)^2 \right] = 0.$$ 

Then, we can define

$$\Pi_3 (\cdot) = \int_{a}^{b} \int_{c}^{d} X ((t, s), \cdot) \ c d q_2^s \ b d q_1^t \ (a.e.).$$

iv) A random variable $\Pi_4 : \Omega \to \mathbb{R}$ is called the $q^{bd}$-mean-square integral of the process $X$ on $[a, b] \times [c, d] \subset \Lambda$, if we have

$$E\left[ \left( (1 - q_1) (1 - q_2) (b - a) (d - c) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \sum_{a}^{b} \sum_{c}^{d} q^n_d q^m_d X (q^n_d a + (1 - q^n_d) b, q^m_d c + (1 - q^m_d) d) - \Pi_4 \right)^2 \right] = 0.$$ 

Then, we can state

$$\Pi_4 (\cdot) = \int_{a}^{b} \int_{c}^{d} X ((t, s), \cdot) \ d d q_2^s \ b d q_1^t \ (a.e.).$$

For the existence of the $q_1 q_2$-mean-square integrals, it is enough to assume that the stochastic process $X$ is the mean-square continuous.

**Theorem 5.1.** If $X : \Lambda \times \Omega \to \mathbb{R}$ is a co-ordinated convex stochastic process on $\Lambda$, then we have

$$X \left( \begin{bmatrix} u_1 q_1 + v_1 \\ [2] q_1 \\ u_2 q_2 + v_2 \\ [2] q_2 \end{bmatrix}, \cdot \right)$$

\begin{align*}
\leq & \frac{1}{2} \left[ \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X \left( t, \frac{u_2 q_2 + v_2}{[2] q_2}, \cdot \right) \ u_1 d q_1^t \\
+ & \frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X \left( \frac{u_1 q_1 + v_1}{[2] q_1}, \cdot \right) \ u_2 d q_2^s \right] 
\end{align*}
By similar argument applied for mapping Z : T₁ × Ω → ℝ, Z(t, ·) = X((t, s), ·), we get

\[
\frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X\left(\left(\frac{q_1u_1 + v_1}{[2]^2_{q_1}}, s\right), \cdot \right) u_2 d_{q_2} s = \frac{q_1}{[2]_{q_1}} \frac{1}{2} \int_{0}^{1} X((v_1, s), \cdot) u_2 d_{q_2} s
\]
By adding the inequalities (5.3) and (5.4), we obtain the second and third inequalities in (5.1).

By the first inequality in (3.10), we get

\[
X \left( \left( \frac{u_1 q_1 + v_1}{[2]_{q_1}}, \frac{u_2 q_2 + v_2}{[2]_{q_2}} \right), s \right) \leq \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X \left( \left( t, \frac{u_2 q_2 + v_2}{[2]_{q_2}} \right), s \right) u_1 d q_1 t
\]

and

\[
X \left( \left( \frac{u_1 q_1 + v_1}{[2]_{q_1}}, \frac{u_2 q_2 + v_2}{[2]_{q_2}} \right), s \right) \leq \frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X \left( \left( \frac{u_1 q_1 + v_1}{[2]_{q_1}}, t \right), u_2 d q_2 s \right)
\]

This gives, by addition the inequalities (5.3) and (5.4), the first inequality in (5.1).

Finally, by second inequality in (3.10), we have

\[
\frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X \left( \left( u_1, s \right), \frac{u_2 q_2}{[2]_{q_2}} \right) u_2 d q_2 s \leq \frac{q_2 X \left( \left( u_1, u_2 \right), \frac{u_2 q_2}{[2]_{q_2}} \right) + X \left( \left( u_1, v_2 \right), \frac{u_2 q_2}{[2]_{q_2}} \right)}{[2]_{q_2}}
\]

and

\[
\frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X \left( \left( t, u_2 \right), \frac{u_2 q_2}{[2]_{q_2}} \right) u_1 d q_1 t \leq \frac{q_1 X \left( \left( u_1, u_2 \right), \frac{u_2 q_2}{[2]_{q_2}} \right) + X \left( \left( v_1, u_2 \right), \frac{u_2 q_2}{[2]_{q_2}} \right)}{[2]_{q_1}}
\]

This completes the proof.

**Theorem 5.2.** If \( X : \Lambda \times \Omega \to \mathbb{R} \) is a co-ordinated convex stochastic process on \( \Lambda \), then we have

\[
X \left( \left( \frac{u_1 + v_1 q_1}{[2]_{q_1}}, \frac{u_2 q_2 + v_2}{[2]_{q_2}} \right), s \right) \leq \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X \left( \left( t, \frac{q_2 u_2 + v_2}{[2]_{q_2}} \right), s \right) u_1 d q_1 t
\]
Then it follows that

$$\frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X \left( \left( \frac{u_1 + q_1 v_1}{[2]_{q_1}}, s \right), \cdot \right) w_2 d_q s$$

\[ \leq \frac{1}{(v_1 - u_1)(v_2 - u_2)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} X \left( (t, s), \cdot \right) u_2 d_q s \ w_1 d_q t \]

\[ \leq \frac{1}{2[2]_{q_1} (v_2 - u_2)} \int_{u_2}^{v_2} X \left( (u_1, s), \cdot \right) u_2 d_q s + \frac{q_1}{2[2]_{q_1} (v_2 - u_2)} \int_{u_2}^{v_2} X \left( (v_1, s), \cdot \right) u_2 d_q s \]

\[ + \frac{q_2}{2[2]_{q_2} (v_1 - u_1)} \int_{u_1}^{v_1} X \left( (t, u_2), \cdot \right) v_2 d_q t + \frac{1}{2[2]_{q_2} (v_1 - u_1)} \int_{u_1}^{v_1} X \left( (t, v_2), \cdot \right) v_1 d_q t \]

\[ \leq q_2 X \left( (u_1, u_2), \cdot \right) + X \left( (u_1, v_2), \cdot \right) + q_1 q_2 X \left( (v_1, u_2), \cdot \right) + q_1 X \left( (v_1, v_2), \cdot \right) \]

\[ \frac{1}{[2]_{q_1} [2]_{q_2}} \]

for \(0 < q_1, q_2 < 1\).

**Proof.** By using \(q^\nu\)-integration in (5.2), we have

$$\frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X \left( \left( t, \frac{q_2 u_2 + v_2}{[2]_{q_2}}, \right), \cdot \right) v_1 d_q t$$

(5.12)

$$\leq \frac{1}{(v_1 - u_1)(v_2 - u_2)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} X \left( (t, s), \cdot \right) u_2 d_q s \ v_1 d_q t$$

$$\leq \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} q_2 X \left( (t, u_2), \cdot \right) + X \left( (t, v_2), \cdot \right) \frac{1}{[2]_{q_2}} v_1 d_q t \quad (\text{a.e.})$$

By applying Theorem 4.1 to mapping \(Z : T_1 \times \Omega \rightarrow \mathbb{R}, Z(t, \cdot) = X ((t, s), \cdot)\), we have

$$Z \left( \frac{u_1 + q_1 v_1}{[2]_{q_1}}, \cdot \right) \leq \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} Z \left( t, \cdot \right) v_1 d_q t \leq \frac{Z (u_1, \cdot) + q_1 Z (v_1, \cdot)}{[2]_{q_1}}$$

(a.e.)

That is,

$$X \left( \left( \frac{u_1 + q_1 v_1}{[2]_{q_1}}, s \right), \cdot \right) \leq \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X \left( (t, s), \cdot \right) v_1 d_q t \leq \frac{X \left( (u_1, s), \cdot \right) + q_1 X \left( (v_1, s), \cdot \right)}{[2]_{q_1}}$$

(a.e.)

Then it follows that

$$\frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X \left( \left( \frac{u_1 + q_1 v_1}{[2]_{q_1}}, s \right), \cdot \right) w_2 d_q s$$

(5.13)
\begin{align*}
\leq & \frac{1}{(v_1 - u_1)(v_2 - u_2)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} X((t, s), \cdot) \ u_2 d_q s \ v_1 t d_q t \\
\leq & \frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X((u_1, s), \cdot) + q_1 X((v_1, s), \cdot) \ [2]_{q_1} u_2 d_q s \ (a.e.).
\end{align*}

Adding the inequalities (5.12) and (5.13), then we obtain the second and third inequalities in (5.11).

By the first inequality in (3.10), we get
\begin{equation}
X\left(\left(\frac{u_1 + v_1 q_1}{[2]_{q_1}}, \frac{u_2 q_2 + v_2}{[2]_{q_2}}\right), \cdot\right) \leq \frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X\left(\left(\frac{u_1 q_1 + v_1}{[2]_{q_1}}, s\right), \cdot\right) u_2 d_q s \ (5.14)
\end{equation}
and by the first inequality in (4.1), we have
\begin{equation}
X\left(\left(\frac{u_1 + v_1 q_1}{[2]_{q_1}}, \frac{u_2 q_2 + v_2}{[2]_{q_2}}\right), \cdot\right) \leq \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X\left(\left(t, \frac{u_2 q_2 + v_2}{[2]_{q_2}}\right), \cdot\right) v_1 d_q t \ . \ (5.15)
\end{equation}

Then, by adding the inequalities (5.14) and (5.15), we have the first inequality in (5.11).

Finally, by using second inequality in (4.1), we get
\begin{equation}
\frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X((t, u_2), \cdot) \ v_1 d_q t \leq \frac{X((u_1, u_2), \cdot) + q_1 X((v_1, u_2), \cdot)}{[2]_{q_1}} \ (5.16)
\end{equation}
and
\begin{equation}
\frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X((t, v_2), \cdot) \ v_1 d_q t \leq \frac{X((u_1, v_2), \cdot) + q_1 X((v_1, v_2), \cdot)}{[2]_{q_1}} \ . \ (5.17)
\end{equation}

By the inequalities (5.7), (5.8), (5.16) and (5.17), then one can obtain the last inequality in (5.11). Thus, the proof is completed. \hfill \Box

**Theorem 5.3.** If $X : \Lambda \times \Omega \rightarrow \mathbb{R}$ is a co-ordinated convex stochastic process on $\Lambda$, then we have
\begin{align*}
X\left(\left(\frac{u_1 q_1 + v_1}{[2]_{q_1}}, \frac{u_2 + v_2 q_2}{[2]_{q_2}}\right), \cdot\right) & \leq \frac{1}{2} \left[ \frac{1}{v_1 - u_1} \int_{u_1}^{v_1} X\left(\left(t, \frac{u_2 + v_2 q_2}{[2]_{q_2}}\right), \cdot\right) u_1 d_q t \right. \\
& \left. + \frac{1}{v_2 - u_2} \int_{u_2}^{v_2} X\left(\left(\frac{u_1 q_1 + v_1}{[2]_{q_1}}, s\right), \cdot\right) u_2 d_q s \right]
\end{align*}
and

\[
X \left( \left[ \frac{u_1 + v_1 q_1}{2 q_1}, \frac{u_2 + v_2 q_2}{2 q_2} \right] \right)
\leq \frac{1}{2} \left\{ \int_{v_1}^{v_2} X \left( \frac{t - u_2 + v_2 q_2}{2 q_2} \right) dt \right. \\
+ \left. \int_{u_2}^{v_2} X \left( \frac{u_1 + v_1 q_1}{2 q_1} \right) dt \right\} \leq \frac{1}{(v_1 - u_1)(v_2 - u_2)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} X((t,s),\cdot) v_2 dq_2 s \quad v_1 dq_1 t
\]

for \(0 < q_1, q_2 < 1\).

**Proof.** The proof is similar to the proof of Theorems 5.1 and 5.2 by using Theorems 4.1 and 5.1. \(\square\)

**Remark 5.1.** By taking the limit \(q \to 1^-\), then Theorem 5.1, Theorem 5.2 and Theorem 5.3 reduce to [53, Theorem 2.5].

6. Conclusions

In this investigation, we have introduced the notions for \(q\)-mean square stochastic processes. We have derived some new quantum inequalities of Hermite-Hadamard type for convex stochastic process and co-ordinated stochastic processes using the newly defined integrals. Moreover, we have proved that the results offered in this research are the strong generalization of several known results inside the literature. It is an interesting and new problem that the upcoming mathematicians can offer the similar inequalities for different kinds of stochastic convexities in their future research.

AIMS Mathematics
Acknowledgments

This research was funded by King Mongkut’s University of Technology North Bangkok. Contract no. KMUTNB-63-KNOW-18. This work was partially supported by National Natural Science Foundation of China (Grant No. 11971241).

Conflict of interest

The authors declare no conflict of interest.

References

1. H. Agahi, A. Babakhani, On fractional stochastic inequalities related to Hermite-Hadamard and Jensen types for convex stochastic processes, *Aequationes Math.*, **90** (2016), 1035–1043.
2. P. Agarwal, M. Jleli, M. Tomar, Certain Hermite-Hadamard type inequalities via generalized $k$-fractional integrals, *J. Inequal. Appl.*, **2017** (2017), 1–10.
3. P. Agarwal, Some inequalities involving Hadamard-type $k$-fractional integral operators, *Math. Meth. Appl. Sci.*, **40** (2017), 3882–3891.
4. M. A. Ali, H. Budak, Z. Zhang, H. Yildirim, Some new Simpson’s type inequalities for co-ordinated convex functions in quantum calculus, *Math. Meth. Appl. Sci.*, **44** (2021), 4515–4540.
5. M. A. Ali, H. Budak, M. Abbas, Y. M. Chu, Quantum Hermite-Hadamard-type inequalities for functions with convex absolute values of second $q^{2}$-derivatives, *Adv. Differ. Equ.*, **2021** (2021), 1–12.
6. M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, Y. M. Chu, New quantum boundaries for quantum Simpson’s and quantum Newton’s type inequalities for preinvex functions, *Adv. Differ. Equ.*, **2021** (2021), 1–21.
7. M. A. Ali, Y. M. Chu, H. Budak, A. Akkurt, H. Yildirim, Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables, *Adv. Differ. Equ.*, **2021** (2021), 1–26.
8. M. A. Ali, N. Alp, H. Budak, Y. M. Chu, Z. Zhang, On some new quantum midpoint type inequalities for twice quantum differentiable convex functions, *Open Math.*, **19** (2021), 427–439.
9. M. A. Ali, H. Budak, A. Akkurt, Y. M. Chu, Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus, *Open Math.*, **19** (2021), 440–449.
10. M. Alomari, $q$-Bernoulli inequality, *Turk. J. Sci.*, **3** (2018), 32–39.
11. N. Alp, M. Z. Sarıkaya, M. Kunt İ. İşcan, $q$-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions, *J. King Saud Univ. Sci.*, **30** (2018), 193–203.
12. N. Alp, M. Z. Sarıkaya, Hermite Hadamard’s type inequalities for co-ordinated convex functions on quantum integral, *Appl. Math. E-Notes*, **20** (2020), 341–356.
13. W. Al-Salam, Some fractional $q$-integrals and $q$-derivatives, *Proc. Edinburgh Math. Soc.*, **15** (1966), 135–140.
14. D. Barraez, L. Gonzalez, N. Merentes, A. M. Moros, On $h$–convex stochastic process, *Mathematica Aeterna*, 5 (2015), 571–581.

15. F. Benatti, M. Fannes, R. Floreanini, D. Petritis, *Quantum Information, Computation and Cryptography: An Introductory Survey of Theory, Technology and Experiments*, Springer Science and Business Media, 2010.

16. S. Bermudo, P. Kórus, J. N. Valdés, On $q$-Hermite-Hadamard inequalities for general convex functions, *Acta Math. Hung.*, 162 (2020), 364–374.

17. A. Bokulich, G. Jaeger, *Philosophy of Quantum Information Theory and Entanglement*, Cambridge University Press, 2010.

18. H. Budak, M. Z. Sarikaya, A new Hermite-Hadamard inequality for $h$-convex stochastic processes, *Ntmsci.*, 7 (2019), 356–363.

19. H. Budak, Sarikaya, On generalized stochastic fractional integrals and related inequalities, *Mod. Stoch-Theory App.*, 5 (2018), 471–481.

20. H. Budak, S. Erden, M. A. Ali, Simpson and Newton type inequalities for convex functions via newly defined quantum integrals, *Math. Meth. Appl. Sci.*, 44 (2020), 378–390.

21. H. Budak, Some trapezoid and midpoint type inequalities for newly defined quantum integrals, *Proyecciones*, 40 (2021), 199–215.

22. H. Budak, M. A. Ali, M. Tarhanaci, Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions, *J. Optim. Theory Appl.*, 186 (2020), 899–910.

23. H. Budak, M. A. Ali, N. Alp, Y. M. Chu, Quantum Ostrowski type integral inequalities, *J. Math. Inequal.*, 2021.

24. S. I. Butt, S. Yousuf, A. O. Akdemir, M. A. Dokuyucu, New Hadamard-type integral inequalities via a general form of fractional integral operators, *Chaos Solition Fract.*, 148 (2021), 111025.

25. S. I. Butt, A. O. Akdemir, N. Nadeem, N. Malaiki, İ. İşcan, T. Abdeljawad, $(m,n)$-Harmonically polynomial convex functions and some Hadamard type inequalities on the coordinates, *AIMS Math.*, 6 (2021), 4677–4690.

26. S. S. Dragomir, C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

27. S. S. Dragomir, On Hadamard’s inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese J. Math.*, 4 (2001), 775–788.

28. T. Ernst, *The History of $Q$-Calculus And New Method*, Sweden: Department of Mathematics, Uppsala University, 2000.

29. T. Ernst, A *Comprehensive Treatment of $q$-Calculus*, Springer Basel, 2012.

30. L. Gonzalez, N. Merentes, M. Valera-Lopez, Some estimates on the Hermite-Hadamard inequality through convex and quasi-convex stochastic processes, *Mathematica Aeterna*, 5 (2015), 745–767.

31. F. M. Hafiz, The fractional calculus for some stochastic processes, *Stoch. Anal. Appl.*, 22 (2004), 507–523.

32. W. U. Haq, Certain inequalities of Hermite-Hadamard-Fejer type involving $q$-stochastic integral, *Iran. J. Sci. Technol.*, 44 (2020), 487–491.
33. F. H. Jackson, On a \(q\)-definite integrals, *Quarterly J. Pure Appl. Math.*, 41 (1910), 193–203.
34. S. Jhanthanam, T. Jessada, S. K. Ntouyas, N. Kamsing, On \(q\)-Hermite-Hadamard inequalities for differentiable convex functions, *Mathematics*, 7 (2019), 632.
35. V. Kac, P. Cheung, *Quantum Calculus*, Springer, 2001.
36. V. Karahan, O. K. U. R. Nrgül, İ. İşcan, Hermite-Hadamard type inequalities for convex stochastic processes on \(n\)-coordinates, *Turk. J. Math. Comput. Sci.*, 10 (2018), 256–262.
37. M. A. Khan, M. Noor, E. R. Nwaeze, Y. M. Chu, Quantum Hermite-Hadamard inequality by means of a green function, *Adv. Differ. Equ.*, 2020 (2020), 1–20.
38. D. Kotrys, Hermite-Hadamard inequality for convex stochastic processes, *Aequationes Math.*, 83 (2012), 143–151.
39. D. Kotrys, Remarks on strongly convex stochastic processes, *Aequationes Math.*, 86 (2013), 91–98.
40. M. A. Latif, S. S. Dragomir, E. Momoniat, Some \(q\)-analogues of Hermite-Hadamard inequality of functions of two variables on finite rectangles in the plane, *J. King Saud University-Science*, 29 (2017), 263–273.
41. W. Liu, Z. Hefeng, Some quantum estimates of Hermite-Hadamard inequalities for convex functions, *J. Appl. Anal. Comput.*, 7 (2016), 501–522.
42. S. Maden, M. Tomar, E. Set, Hermite-Hadamard type inequalities for \(s\)-convex stochastic processes in first sense, *Pure Appl. Math Lett.*, 2015 (2015), 1–7.
43. J. Materano, N. Merentes, M. Valera-Lopez, Some estimates on the Simpson’s type inequalities through \(s\)-convex and quasi-convex stochastic processes, *Mathematica Aeterna*, 5 (2015), 673–705.
44. K. Nikodem, On convex stochastic processes, *Aequationes Math.*, 20 (1980), 184–197.
45. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum estimates for Hermite-Hadamard inequalities, *Appl. Math. Comput.*, 251 (2015), 675–679.
46. M. A. Noor, K. I. Noor, M. U. Awan, Some quantum integral inequalities via preinvex functions, *Appl. Math. Comput.*, 269 (2015), 242–251.
47. E. R. Nwaeze, A. M. Tameru, New parameterized quantum integral inequalities via \(\eta\)-quasiconvexity, *Adv. Differ. Equ.*, 2019 (2019), 1–12.
48. M. E. Özdemir, M. A. Latif, A. O. Akdemir, On some Hadamard-type inequalities for product of two \(h\)-convex functions on the co-ordinates, *Turk. J. Sci.*, 1 (2016), 41–58.
49. J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
50. S. Rashid, S. I. Butt, S. Kanwal, H. Ahmed, M. K. Wang, Quantum integral inequalities with respect to Raina’s function via coordinated generalized \(-\)-convex functions with applications, *J. Funct. Spaces*, 2021 (2021).
51. M. Z. Sarıkaya, H. Yalız, H. Budak, Some integral inequalities for convex stochastic process, *Acta Math. Univ. Comenian.*, 85 (2016), 155–164.
52. E. Set, M. Tomar, S. Maden, Hermite-Hadamard type inequalities for s–convex stochastic processes in the second sense, *Turk. J. Anal. Number Theory*, 2 (2014), 202–207.

53. E. Set, M. Z. Sarikaya, M. Tomar, Hermite-Hadamard type inequalities for coordinated convex stochastic processes, *Mathematica Aeterna*, 5 (2015), 363–382.

54. A. Skowroński, On some properties of J-convex stochastic processes, *Aequationes Math.*, 44 (1992), 249–258.

55. K. Sobczyk, *Stochastic Differential Equations with Applications to Physics and Engineering*, Kluwer, Dordrecht, 1991.

56. J. Tariboon, S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations, *Adv. Differ. Equ.*, 2013 (2013), 1–19.

57. M. Tomar, E. Set, S. Maden, Hermite-Hadamard type inequalities for log-convex stochastic processes, *J. New Theory*, 2 (2015), 23–32.

58. M. Tomar, E. Set, N. O. Bekar, On Hermite-Hadamard type inequalities for strongly-log convex stochastic processes, *J. Global Eng. Stud.*, 1 (2014), 53–61.

59. M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, Z. Zhang, Some new Newton’s type integral inequalities for co-ordinated convex functions in quantum calculus, *Symmetry*, 12 (2020), 1476.

60. S. Zhao, S. I. Butt, W. Nazir, J. Nasir, M. Umar, Y. Liu, Some Hermite-Jensen-Mercer type inequalities for k-Caputo-fractional derivatives and related results, *Adv. Differ. Equ.*, 2020 (2020), 1–17.

61. J. Zhao, S. I. Butt, J. Nasir, Z. Wang, I. Tilli, Hermite-Jensen-Mercer type inequalities for Caputo fractional derivatives, *J. Funct. Spaces*, 2020 (2020).