POLYMERS AND TOPOLOGICAL FIELD THEORY: A 2 LOOP COMPUTATION

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Abstract

Within the Quantum Action Principle framework we show the perturbative renormalizability of previously proposed topological lagrangian à la Witten-Fujikawa describing polymers, then we perform a 2 loop computation. The theory turns out to have the same predictive power of De Gennes theory, even though its running coupling constants exhibit a very peculiar behaviour. Moreover we argue that the theory presents two phases, a topological and a non topological one.

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Introduction.

In a previous work [1] we proposed a topological theory à la Fujikawa-Witten [4, 3, 2] describing the self–avoiding walks (hereafter SAW), i.e. the polymers in the De Gennes model [5, 6].

The aim of this approach was the exact computation of the critical exponents of SAW; this would have been achieved by an exact computation of the theory β-function(s) relying on the ”topologicity” of the theory.

In this article we want to prove that the theory we proposed is actually perturbatively renormalizable and to perform the calculation of the interesting quantities up to the second loop. The explicit computation reveals that , in spite of the topologicity of the theory, the hope for an exact computation of β-function is not fulfilled, nonetheless the theory has some interesting features such as the doubling of the coupling constants, which exhibit a very peculiar behaviour under the RG flow, and the possibility for a spontaneous breaking of the topological phase.

The article is divided as follows: in section one we review the relation among our model and those of De Gennes ([5, 6]) and of Parisi-Sourlas-Mc Kane ([7, 8]); in section two we prove of the perturbative renormalizability in the formalism of Quantum Action Principle [9]-[10]); in section three we explain the two loop computation in the framework of background field method ([11]); in section four we discuss the renormalization group flow, finally we draw our conclusions.

1 The topological theory of polymers.

We start discussing briefly the relation between our model and the MPS one. The renormalization requires a slight generalization of the previous lagrangian [1] that can easily understood as the necessity of including all the BRST invariant terms with the same dimension. The lagrangian is now given by:

\[
\mathcal{L} = \rho b^\dagger b + ib^\dagger (-\Delta + m^2) \phi + i\phi^\dagger (-\Delta + m^2) b - i\xi^\dagger (-\Delta + m^2) \eta + i\eta^\dagger (-\Delta + m^2) \xi \\
+ \lambda \left( i b^\dagger \phi - i \phi^\dagger b - i \xi^\dagger \eta - i \eta^\dagger \xi \right)^2 + \nu \left( b^\dagger \phi + \phi^\dagger b - \xi^\dagger \eta + \eta^\dagger \xi \right)^2
\]  

(1.1)

where \( b = (b_1, \ldots, b_N)^T \), \( \phi = (\phi_1, \ldots, \phi_N)^T \) are two vectors of N complex Lorentz scalars (N being an arbitrary natural number), \( \xi = (\xi_1, \ldots, \xi_N)^T \),
\( \eta = (\eta_1, \ldots, \eta_N)^T \) are two vectors of \( N \) complex Lorentz scalar ghosts, and \( \rho, \lambda \) and \( \nu \) are arbitrary positive numbers satisfying the condition \( \lambda > \nu \). The dimension and the BRST charge of the fields is given in the following table:

|     | \( \phi \) | \( b \) | \( \xi \) | \( \eta \) |
|-----|------------|--------|--------|--------|
| dim | 0          | 2      | 1      | 1      |
| \( \Phi \Pi \) | 0        | 0      | -1     | 1      |

and discussed in the next section. The two adimensional coupling constants \( \lambda \) and \( \nu \) are related to the \( O(n \to 0) \) coupling constant \( g \) by

\[
g = \lambda - \nu \tag{1.2}
\]

this implies that \( \lambda = \nu \) is a complicate way of describing a free theory. The proof of this relation and of the formal equivalence between De Gennes theory and the present one is based upon the equality of the two points Green function of the two theories; this can easily be achieved by rewriting (1.1) using two auxiliary fields \( \alpha, \beta \) as

\[
L = \rho b^\dagger b + i b^\dagger O \phi + i \phi^\dagger O^\dagger b - i \xi^\dagger O \eta + i \eta^\dagger O^\dagger \xi + \alpha^2 + \beta^2 \tag{1.3}
\]

where

\[
O = (-\Delta + m^2 + 2i\sqrt{\lambda}+ 2\sqrt{\nu})
\]

and by using the McKane-Parisi-Sourlas trick [7, 8] on De Gennes theory [5, 6] along with (1.2), in such a way that De Gennes theory can be rewritten as:

\[
L_{O(n \to 0)} = \phi^\dagger O \phi + \psi^\dagger O \psi + \alpha^2 + \beta^2
\]

We want to stress that this equivalence is true only if \( \lambda, \nu > 0 \) because otherwise, after integrating over \( b \) in (1.3), the remaining effective action would not be bounded from below.

For computational purpose it is better to rewrite (1.1) as

\[
L = \rho \varphi^\dagger U \varphi + i \varphi^\dagger Y (-\Delta + m^2) \varphi - \psi^\dagger X (-\Delta + m^2) \psi + \lambda (\varphi^\dagger X \varphi - i \psi^\dagger Y \psi)^2 + \nu (\varphi^\dagger Y \varphi + i \psi^\dagger X \psi)^2 \tag{1.4}
\]

where we introduced the following matrices

\[
X = \begin{pmatrix} 0 & i \mathbb{I}_N \\ -i \mathbb{I}_N & 0 \end{pmatrix} = \sigma_1 \otimes \mathbb{I}_N \quad Y = \begin{pmatrix} 0 & \mathbb{I}_N \\ \mathbb{I}_N & 0 \end{pmatrix} = \sigma_2 \otimes \mathbb{I}_N
\]
\[ U = \begin{pmatrix} 1_N & 0 \\ 0 & 0 \end{pmatrix} \]

and we defined
\[ \varphi = \begin{pmatrix} b \\ \phi \end{pmatrix} \quad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \]

Notice the symmetry in the four-field terms
\[(\lambda, X, Y) \leftrightarrow (\nu, Y, -X)\]

that turns out to be useful in performing the actual computation.

\section{Symmetries and renormalization}

In the previous section we gave the topological lagrangian along with the dimension of the fields and their BRST charge, without motivating these choices and the terminology.

Now we proceed in explaining the field dimensions, they are deduced looking at the explicit form of the free theory propagators:

\[
\begin{align*}
< \phi_j^*(x)\phi_k(0) > &= \delta_{jk} \int d^4k \ e^{ikx} \frac{\rho}{(k^2 + m^2)^2} \\
< \phi_j^*(x)b_k(0) > &= \delta_{jk} \int d^4k \ e^{ikx} \frac{-i}{(k^2 + m^2)} \\
< \xi_j^*(x)\eta_k(0) > &= \delta_{jk} \int d^4k \ e^{ikx} \frac{-i}{(k^2 + m^2)} \quad (2.1)
\end{align*}
\]

One could wonder about the dimensionality of the parameter \(\rho\) and, consequently, of the other fields, but what justifies setting to zero the dimensionality of \(\rho\) is just the power counting in which the propagator \(\phi\phi\) has dimension -4 (in unit of mass). Just because of this noncanonical dimension of the \(\phi\phi\) propagator we performed the usual power counting both with \(\rho \neq 0\) and \(\rho = 0\). It is easy to show that the critical dimension is four (as it should be to reproduce De Gennes theory) and that if we indicate with \(E_\phi, E_b, E_\xi, E_\eta\) the number of external legs of the fields \(\phi, b, \xi, \eta\) in a truncated diagram, at the critical dimensionality \(D_{cr} = 4\), there are only the following superficially
We are ready to discuss both the symmetries and the broken symmetries of the action:

1. **the discrete symmetry:**

\[ \phi \rightarrow \phi^*, b \rightarrow b^*, \xi \rightarrow \xi^*, \eta \rightarrow \eta^* \]  

(2.3)

that is responsible for the non appearance in the lagrangian of a term

\[
\left(i b^\dagger \phi - i \phi^\dagger b - i \xi^\dagger \eta - i \eta^\dagger \xi\right) \left(b^\dagger \phi + \phi^\dagger b - \xi^\dagger \eta + \eta^\dagger \xi\right)
\]

which is odd under such a transformation; this symmetry allows to construct the action only from the real part of functionals (possibly with complex coefficients) and from even power of the imaginary part of functionals (possibly with complex coefficients);

2. **\(GL(N, \mathbb{C})_{\text{bos}}\)** broken to **\(U(N, \mathbb{R})\)** by the term proportional to \(b^\dagger b\):

\[
\begin{align*}
\delta^b(\theta^a) \phi & = i G_\alpha \phi \theta^a, & \delta^b(\theta^a) \phi^\dagger & = -i \phi^\dagger G_\alpha \phi \theta^{a*} \\
\delta^b(\theta^a) b^\dagger & = -i b^\dagger G_\alpha \theta^a, & \delta^b(\theta^a) b & = i G_\alpha \theta^{a*}
\end{align*}
\]

(2.4)

where \(G_\alpha\) are the generators of \(gl(N, \mathbb{C})\) and \(\theta_\alpha\) is the complex parameter of the transformation.

We use the following explicit representation for the generators

\[ (G_\alpha)_{pq} \equiv (G_{(ab)})_{pq} = -i \delta_{ap} \delta_{bq} \quad (G_{(ab)} = -G_{(ba)}) \]

Considering \(gl(N)\) over the complex field allows to vary independently \((\phi, b^*)\) from \((\phi^*, b)\), in fact we can build the following generators of the
decomplexified algebra:

\[ \delta^{(+)}_\alpha = \frac{1}{2}(\delta_\alpha(\theta_\alpha = 1) - i(\delta_\alpha(\theta_\alpha = i)) ) \]
\[ \delta^{(-)}_\alpha = \frac{1}{2}(\delta_\alpha(\theta_\alpha = 1) + i(\delta_\alpha(\theta_\alpha = i)) ) \]  

whose action on the bosonic fields is given by

\[ \delta^{(+)}_{(ab)} \phi_p = \delta_{ap} \phi_b, \quad \delta^{(+)}_{(ab)} b_p^* = -\delta_{bp} b_a^*, \quad \delta^{(+)}_{(ab)} a_p^* = \delta^{(+)}_{(ab)} b_p = 0 \]
\[ \delta^{(-)}_{(ab)} \phi_p = \delta_{ap} \phi_b^*, \quad \delta^{(-)}_{(ab)} b_p = -\delta_{bp} b_a, \quad \delta^{(-)}_{(ab)} a_p^* = \delta^{(-)}_{(ab)} b_p^* = 0 \]  

This symmetry is however broken, in fact we find immediately the breaking under the transformation by a complex parameter \( \theta_\alpha \):

\[ \delta L = i \rho \ b^\dagger (G^\dagger_\alpha - G_\alpha) b \ \text{Re} \theta^\alpha + \rho \ b^\dagger (G^\dagger_\alpha + G_\alpha) b \ \text{Im} \theta^\alpha \]

the breaking vanishes when restricting the symmetry to the \( u(N, \mathbb{R}) \) generated by \( T_A \) whose explicit representation is given by:

\[ T_{(aa)} = iG_{(aa)} \]
\[ T_{(ab)} = G_{(ab)} + G^\dagger_{(ab)} \quad T_{2(ab)} = i(G_{(ab)} - G^\dagger_{(ab)}) \quad \text{with} \quad a < b \]

3. \( GL(N, \mathbb{C}) \) ferm with complex parameter \( \theta_\alpha \):

\[ \delta^f(\theta_\alpha)^{\eta} = iG_\alpha \eta \ \theta^\alpha \quad \delta^f(\theta_\alpha)^{\eta^\dagger} = -i\eta^\dagger G^\dagger_\alpha \ \theta^{\alpha^*} \]
\[ \delta^f(\theta_\alpha)^{\xi^\dagger} = -i \ \xi^\dagger G_\alpha \ \theta^\alpha \quad \delta^f(\theta_\alpha)^{\xi} = i \ G^\dagger_\alpha \xi \ \theta^{\alpha^*} \]  

As in the previous case of the broken bosonic \( GL(N, \mathbb{C}) \), it is possible to build \( \delta^f(\pm) \) and vary independently the couple \( (\eta, \xi^\dagger) \) from \( (\eta^\dagger, \xi) \).

4. BRST-like transformations with complex parameter \( \theta^\alpha \):

\[ \hat{\delta}(\theta^\alpha)^{\phi} = iG_\alpha \eta \ \theta^\alpha, \quad \hat{\delta}(\theta^\alpha)^{\eta} = 0 \]
\[ \hat{\delta}(\theta^\alpha)^{\xi^\dagger} = i\xi^\dagger G_\alpha \ \theta^\alpha, \quad \hat{\delta}(\theta^\alpha)^{b} = 0 \]  

5
In particular the generator of the "canonical" BRST is 

\[ s = \sum_{a=1}^{N} \hat{\delta}_{(aa)} \]

in such a way we can rewrite the lagrangian (1.1) as

\[
L = s [ \rho \ b^\dagger \xi + i \xi^\dagger (-\Delta + m^2) \phi + i \phi^\dagger (-\Delta + m^2) \xi \\
+ \lambda \left( i b^\dagger \phi - i \phi^\dagger b - i \xi \phi - i \phi \xi \right) \\
+ \nu \left( b^\dagger \phi + \phi^\dagger b - \xi \phi + \phi \xi \right) ]
\]

Notice that the explicit form and existence of \( s \) justifies the dimensions and the charges of the fields we gave. Exactly as before, we can build \( \hat{\delta}^{(\pm)} \) and vary independently \((\phi, \xi^*)\) from \((\phi^*, \xi)\).

5. antiBRST-like transformations broken by \( b^\dagger b \):

\[
\hat{\delta}_{\alpha} \eta = i G_{\alpha}^\dagger \phi, \quad \hat{\delta}_{\alpha} \phi = 0
\]

\[
\hat{\delta}_{\alpha} b^\dagger = -i \xi^\dagger G_{\alpha}^\dagger, \quad \hat{\delta}_{\alpha} \xi = 0
\]

The breaking is:

\[
\hat{\delta}(\theta^\alpha) \mathcal{L} = i \ \rho \ (b^\dagger G_{\alpha}^\dagger \xi - \xi^\dagger G_{\alpha} b) \ \text{Re}(\theta^\alpha) + \ \rho \ (b^\dagger G_{\alpha}^\dagger \xi + \xi^\dagger G_{\alpha} b) \ \text{Im}(\theta^\alpha)
\]

Since the broken symmetries are broken by a term of dimension four, it would be very difficult to keep them under control, so we prefer to give up these symmetries and to consider only the unbroken ones.

In order to implement the Ward-Takahashi identities (WTI) we introduce the following functional operators:

1.

\[
u(N, \mathbb{R}) \implies W^b_A = \int \ i (T_A \phi)_p \frac{\delta}{\delta \phi_p} - i (b^\dagger T_A)_p \frac{\delta}{\delta b^\dagger_p} + (c.c.) \tag{2.10}
\]

2.

\[
gl(N, \mathbb{C}) \implies \begin{cases} 
W^{f(+)}_\alpha = \int \ i (G_{\alpha} \eta)_p \frac{\delta}{\delta \eta_p} - i (\xi^\dagger G_{\alpha})_p \frac{\delta}{\delta \xi^\dagger_p} \\
W^{f(-)}_\alpha = \int \ i (G_{\alpha}^\dagger \xi)_p \frac{\delta}{\delta \xi_p} - i (\eta^\dagger G_{\alpha}^\dagger)_p \frac{\delta}{\delta \eta^\dagger_p}
\end{cases} \tag{2.11}
\]

3.

\[
\text{BRST-like} \implies \begin{cases} 
S^{(+)}_\alpha = \int \ i (G_{\alpha} \eta)_p \frac{\delta}{\delta \eta_p} + i (b^\dagger G_{\alpha})_p \frac{\delta}{\delta b^\dagger_p} \\
S^{(-)}_\alpha = \int \ i (G_{\alpha}^\dagger b)_p \frac{\delta}{\delta b_p} + i (\eta^\dagger G_{\alpha}^\dagger)_p \frac{\delta}{\delta \eta^\dagger_p}
\end{cases} \tag{2.12}
\]
Notice that since the fields are doublets under the BRST it is not necessary
to introduce external sources for different kind of BRST multiplets.
All the symmetries are contained in the following WTI:

\[ [W^b_A, W^b_B] \Gamma = f^C_{AB} W^b_C \Gamma \]  
(2.13)

\[ [W^b_A, W^{f(\pm)}_\beta] \Gamma = 0 \]  
(2.14)

\[ [W^b_A, S^{(\pm)}_\beta] \Gamma = g_{\alpha \beta}^{\gamma(\pm)} S^{(\pm)}_{\gamma} \Gamma \]  
(2.15)

\[ [W^{f(\pm)}_\alpha, W^{f(\pm)}_\beta] \Gamma = f_{\alpha \beta}^{\gamma(\pm)} W^f_{\gamma} \Gamma \]  
(2.16)

\[ [W^{f(\pm)}_\alpha, W^{f(\mp)}_\beta] \Gamma = 0 \]  
(2.17)

\[ [W^{f(\pm)}_\alpha, S^{(\pm)}_\beta] \Gamma = g_{\alpha|\beta}^{\gamma(\pm)} W^f_{\gamma} \Gamma \]  
(2.18)

\[ [W^{f(\pm)}_\alpha, S^{(\mp)}_\beta] \Gamma = 0 \]  
(2.19)

\[ [S^{(\pm)}_\alpha, S^{(\pm)}_\beta] \Gamma = 0 \]  
(2.20)

where

\[ [T_A, T_B] = i f^C_{AB} T_C \]

\[ [G_\alpha, G_\beta] = i f_{\alpha \beta}^{\gamma(\pm)} G_\gamma \]

\[ T_A G_\alpha = i g_{\alpha \alpha}^{\beta(\pm)} G_\beta = i g_{\alpha \alpha}^{\beta(+) \star} G_\beta \]

\[ G_\alpha G_\beta = -i g_{\alpha \beta}^{\gamma(\pm)} G_\gamma = i g_{\alpha \beta}^{\gamma(+) \star} G_\gamma \]

It can be shown (with a big amount of algebra) that these symmetries are
not anomalous.

After the discussion of the WTI we can discuss the stability of the clas-
sical action \( \Gamma_{cl} \). This amount to impose the conditions (2.13-2.20) to the
perturbed action \( \Gamma' = \Gamma_{cl} + \Delta \).

The \( U(1)^N_f \) symmetry ( \( W^{f(\pm)}_{aa} \Delta = 0 \) ) implies that every term of \( \Delta \) has to
be built using an equal number of conjugate ghost fields and ghost fields\(^1\); in
the mean time the ghost charge implies that every term should contain
an equal number of \( \xi \) and \( \eta \). Taking also in account the discrete symmetry,

\(^1\) This assertion is intuitively obvious, however a rigorous proof is based on the ob-
servation that \( W^{f(\pm)}_{aa} = N(\eta_a) - N(\eta^*_a) + N(\xi^*_a) - N(\xi_a) \) where \( N(\eta_a) = \int \eta_a \frac{\delta}{\delta \eta_a} \) can be
interpreted as an occupation number operator.
the dimension of the fields and the fact that we are looking at an integrated functional, the explicit most general form of $\Delta$ is

$$\Delta = \int \xi^a \eta_b \ f_{ab}[\phi, \phi^*, b, b^*] + \xi_a \eta_b \ f_{ab}[\phi^*, \phi, b^*, b]$$

$$+ \partial^a \xi^b \eta_b \ g_{ab}[\phi, \phi^*] + \partial^a \xi^b \eta_b \ g_{ab}[\phi^*, \phi]$$

$$+ \partial^a \xi_a \partial^b \eta_b \ h_{\mu \nu ab}(\phi, \phi^*) + \partial^a \xi_a \partial^b \eta_b \ h_{\mu \nu ab}(\phi^*, \phi)$$

$$+ \xi_a \xi^b \eta_c \eta_d \ l_{1 \ a|b|cd}(\phi, \phi^* + \xi_b \xi^b \eta_c \eta_d \ l_{1 \ a|b|cd}(\phi^*, \phi)$$

$$+ \xi_a \xi^b \eta_c \eta_d \ l_{2 \ a|b|cd}(\phi, \phi^*) + n[\phi, \phi^*, b, b^*]$$

(2.21)

From Lorentz invariance and applying $W_{\langle ab \rangle}^{f(\pm)} (a \neq b)$ to this expression, it reduces to:

$$\Delta = \int \xi^\dagger \eta \ f[\phi, \phi^*, b, b^*] - \eta^\dagger \xi \ f[\phi^*, \phi, b^*, b]$$

$$+ \partial^a \xi^b \eta_b \ (\partial_\mu \phi g^{(1)}(\phi, \phi^*) + \partial_\mu \phi^* g^{(2)}(\phi, \phi^*))$$

$$- \eta^\dagger \partial^a \xi \ (\partial_\mu \phi^* g^{(1)}(\phi^*, \phi) + \partial_\mu \phi g^{(2)}(\phi, \phi))$$

$$+ \partial^a \xi^\dagger \partial_\mu \eta \ h(\phi, \phi^*) - \partial^a \eta^\dagger \partial_\mu \xi \ h(\phi^*, \phi) + \xi^\dagger \eta^\dagger \xi \ l_{2}(\phi, \phi^*)$$

$$+ (\xi^\dagger \eta)^2 \ l_{1}(\phi, \phi^*) + (\eta^\dagger \xi)^2 \ l_{1}(\phi^*, \phi) + n[\phi, \phi^*, b, b^*]$$

(2.22)

From $U(1)^N_b \otimes U(1)^7_f$ and $S^{(\pm)} \Delta|_{b^0} = 0$ in the sector without derivatives, we get $l_1, l_2$ constants and:

$$\Delta|_{b^0} = \int -\frac{1}{4} (2l_1 + l_2) \left( b^\dagger \phi - \phi^\dagger b - \xi^\dagger \eta - \eta^\dagger \xi \right)^2$$

$$+ \frac{1}{4} (2l_1 - l_2) \left( b^\dagger \phi + \phi^\dagger b - \xi^\dagger \eta + \eta^\dagger \xi \right)^2$$

$$+ n_0 b^\dagger b - f_0 (b^\dagger \phi + \phi^\dagger b - \xi^\dagger \eta + \eta^\dagger \xi)$$

(2.23)

where $f_0$ and $n_0$ are constants. Examining the sector with two derivatives, it is easy to realize that terms proportional to $m^2$, i.e. with the structure $m^2 \partial^2 \phi$, are absent; then from $S^{(\pm)} \Delta|_{b^2} = S^{(\pm)} \Delta|_{\phi^2} = 0$, it is not difficult to prove that

$$\Delta|_{b^2} = n \int b^\dagger \phi + \phi^\dagger b - \xi^\dagger \Delta \phi + \eta^\dagger \Delta \eta$$

Finally we can set immediately to zero the four derivatives part of $\Delta$ because it is impossible to have diagrams with $p^4$ behaviour (2.22).
3 The quantum corrections.

In the following we will use the background field method (11). To this aim
we split the fields as follows:

\[
\varphi \to \Phi + \varphi_{\text{quant}} \quad \psi \to \Psi + \psi_{\text{quant}} \quad (3.1)
\]

where \(\Phi, \Psi\) are the classical background fields. Performing this splitting (and
dropping the specification quant) the lagrangian (1.4) becomes:

\[
\mathcal{L}_{\text{quant}} = +i \varphi^\dagger M_1 \varphi + \varphi^\dagger M_2 \varphi + \varphi^\dagger M_3 \varphi^* + \psi^\dagger N_1 \psi + \psi^\dagger N_2 \psi + \psi^\dagger N_3 \psi^*
\]
\[+ \varphi^\dagger \tilde{\Omega}_1 \psi + \varphi^\dagger \tilde{\Omega}_2 \psi + \psi^\dagger \Omega_1 \varphi + \psi^\dagger \Omega_2 \varphi^*
\]
\[+ A_{2ijk} \varphi_i \varphi_j \varphi_k^* + A_{3ijk} \varphi_i \varphi_j \varphi_k^* + A_{2ijk} \psi_i \psi_j \psi_k^* + A_{3ijk} \psi_i \psi_j \psi_k^*
\]
\[+ B_{3ijk} \varphi_i \psi_j \psi_k^* + B_{4ijk} \varphi_i \psi_j \varphi_k^* + B_{4ijk} \psi_i \psi_j \varphi_k^* + B_{4ijk} \psi_i \psi_j \psi_k^*
\]
\[+ C_{ijkm} \varphi_i \varphi_j \varphi_k \varphi_m^* + D_{ijkm} \psi_i \psi_j \psi_k \psi_m^* + E_{ijkm} \varphi_i \psi_j \psi_k \psi_m^* \quad (3.2)
\]

where all the coefficients can be obtained easily from (1.4) using (3.1); ex-
plicitly we get

\[
\begin{align*}
\varphi^\dagger M_1 \varphi &= \lambda \Phi^\dagger X \varphi \Phi^\dagger X \varphi + \nu \Phi^\dagger Y \varphi \Phi^\dagger Y \varphi \\
M_2 &= a U + 2\lambda (\Phi^\dagger X \Phi - i \Psi^\dagger Y \Psi)X \\
&\quad + [i \ m^2 + 2\nu (\Phi^\dagger Y \Phi + i \Psi^\dagger X \Psi)]Y \\
&\quad + 2\lambda X \Phi \Phi^\dagger X + 2\nu Y \Phi \Phi^\dagger Y \\
\varphi^\dagger M_3 \varphi^* &= \lambda \phi^\dagger X \Phi \phi^\dagger X \Phi + \nu \phi^\dagger Y \Phi \phi^\dagger Y \Phi \\
\psi^\dagger N_1 \psi &= -\lambda \Psi^\dagger Y \psi \Psi^\dagger Y \psi - \nu \Psi^\dagger X \psi \Psi^\dagger X \psi \\
N_2 &= -2i\lambda (\Phi^\dagger X \Phi - i \Psi^\dagger Y \Psi)Y \\
&\quad + [-m^2 + 2\nu (\Phi^\dagger Y \Phi + i \Psi^\dagger X \Psi)]X \\
&\quad - 2\lambda Y \Psi \Psi^\dagger Y - 2\nu X \Psi \Psi^\dagger X \\
\psi^\dagger N_3 \psi^* &= -\lambda \psi^\dagger Y \Psi \psi^\dagger Y \Psi - \nu \psi^\dagger X \Psi \psi^\dagger X \Psi \\
\varphi^\dagger \tilde{\Omega}_1 \psi &= 2i\lambda \Phi^\dagger X \varphi \Phi^\dagger Y \psi - 2i\nu \Phi^\dagger Y \varphi \Phi^\dagger X \psi \\
\varphi^\dagger \tilde{\Omega}_2 \psi &= 2i\lambda \phi^\dagger X \Phi \Phi^\dagger Y \psi - 2i\nu \phi^\dagger Y \Phi \Phi^\dagger X \psi \\
\psi^\dagger \tilde{\Omega}_1 \varphi &= 2i\lambda \Phi^\dagger X \varphi \psi^\dagger Y \Psi - 2i\nu \Phi^\dagger Y \varphi \psi^\dagger X \Psi
\end{align*}
\]
We performed the computation of Feynman graphs in $D = 4 - 2\epsilon$ and within the $\overline{\text{MS}}$ scheme. In figs. 1, 2, 3, 4 two representatives for each of four different kinds of graphs involved in the computation are given. Here we want only to point out that the diagrams like those of of Fig. 2 give $Z_\varphi$, those similar to those of Fig. 3 (i.e. those containing at least either one factor $M_2$ or one $N_2$) generate and cancel the overlapping divergences proportional to currents similar to $\Phi^\dagger X\Phi(x)$, $\Phi^\dagger(y)X\Phi(y)$ while the graphs (like those) of Fig. 4 cancel the overlapping divergences containing currents of the kind $\Phi^\dagger(x)X\Phi(y)\Phi^\dagger(y)X\Phi(x)$.

The one loop computation (graphs of Fig. 1) yields:

$$
\begin{align*}
\delta_1\lambda &= \frac{1}{(4\pi)^2\epsilon}4\lambda(\lambda - 3\nu) \\
\delta_1\nu &= \frac{1}{(4\pi)^2\epsilon}4(\lambda^2 - \lambda\nu + 2\nu^2) \\
\delta_1m^2 &= \frac{1}{(4\pi)^2\epsilon}2m^2(\lambda - \nu) \\
\delta_1\rho &= \frac{1}{(4\pi)^2\epsilon}2\rho(\lambda + \nu)
\end{align*}
$$

The notation $2\rangle$ Dashed lines are $\bar{\psi}\psi$ propagators, continuous lines are $\varphi^*\varphi$ propagators.
The two loops computation yields:

\[
\delta_{(2)} \lambda = \frac{1}{(4\pi)^4} \left[ \frac{4\lambda}{\epsilon^2} (-5\lambda^2 + 18\lambda\nu - 21\nu^2) \right] \\
\delta_{(2)} \nu = -\frac{1}{(4\pi)^4} \left[ \frac{4}{\epsilon^2} (6\lambda^3 - 24\lambda^2\nu + 18\lambda\nu^2 - 16\nu^3) \right] \\
\delta_{(2)} m^2 = \frac{1}{(4\pi)^4} m^2 \left[ \frac{10}{\epsilon^2} - \frac{6}{\epsilon} \right] (\lambda - \nu)^2 \\
\delta_{(2)} \rho = -\frac{1}{(4\pi)^4} 2\rho \left[ -\frac{1}{\epsilon^2} (\lambda^2 + 6\lambda\nu + 5\nu^2) + \frac{1}{\epsilon} (-\lambda^2 + 2\lambda\nu + 3\nu^2) \right] \\
Z_{\phi}^{(2)} = Z_{b}^{(2)} = Z_{\xi}^{(2)} = Z_{\eta}^{(2)} = -\frac{1}{(4\pi)^4} \epsilon (\lambda - \nu)^2
\] (3.5)

where \(Z_{\phi}\) is the wave function renormalization (\(\varphi_{bare} = Z_{\phi}^{1/2}\varphi_{ren}\)).

One could wonder why setting \(Z_{\phi} = Z_{b} = Z_{\xi} = Z_{\eta}\) when \(b\) and \(\varphi\) have a different dimension; the answer lies in the fact that renormalization fixes \(Z_{\phi}Z_{b}\), \(Z_{b}Z_{\varphi} = Z_{\xi}Z_{\eta}\), \(Z_{\lambda}(Z_{b}Z_{\varphi})^{1/2}\) and \(Z_{\nu}(Z_{b}Z_{\varphi})^{1/2}\), while leaving two free parameters (\(Z_{b}\) and \(Z_{\eta}\), for instance). This arbitrariness is however easily understood as the possibility of redefining \(b\) and \(\eta\) inside the path integral; because of this interpretation, this arbitrariness does not affect the physics.

Notice that there is also another natural choice for the free parameters: \(Z_{\rho} = 1, Z_{b} = Z_{\xi}\), so that \(\rho\) becomes a free constant and not a coupling constant; we want to stress that even in the delta gauge (\(\rho = 0\), \(\delta g = 0\)) the theory does not become finite and the quantum corrections to \(\lambda, \nu\) do not change.

4 The RG flow.

As it is easy to see \(Z_{\phi}, \delta m^2\) and \(\delta \lambda - \delta \nu\) are expressible as a function of \(g = \lambda - \nu\), in fact from (3.4, 3.3) we get:

\[
\begin{align*}
Z_{\phi} &= -\frac{1}{(4\pi)^4} \epsilon^2 g^2 \\
\delta m^2 &= \frac{1}{(4\pi)^2} 2m^2 g + \frac{1}{(4\pi)^4} m^2 \epsilon^2 \frac{10}{\epsilon^2} - \frac{6}{\epsilon} \\
\delta g &= \delta \lambda - \delta \nu = \frac{1}{(4\pi)^2} 8g^2 + \frac{1}{(4\pi)^4} g^3 \frac{64}{\epsilon^2} - \frac{44}{\epsilon}
\end{align*}
\] (4.6)
These are exactly the quantum corrections obtainable in the theory \(O(n \to 0)\) and this strongly suggests, even if it does not prove, that the perturbative expansion of \(Z_\phi, \delta m^2\) and \(\delta \lambda - \delta \nu\) in our theory is equal to that of the corresponding quantities in \(O(n \to 0)\) theory \(^3\) . Nevertheless the two coupling constants exhibit a very peculiar behaviour under the RG flow, moreover there are not acceptable fixed point beside the trivial one \((\lambda = \nu = 0)\). In order to show this explicitly, let us compute the \(\beta\) and \(\gamma\) functions, we get in \(D = 4 - 2\epsilon\):

\[
\begin{align*}
\beta_\lambda &= -2\epsilon \lambda + \frac{1}{(4\pi)^2} 8(\lambda^2 - 3\nu) + \frac{1}{(4\pi)^4} 16(-5\lambda^3 + 18\lambda^2\nu - 21\lambda\nu^2) \\
\beta_\nu &= -2\epsilon \nu - \frac{1}{(4\pi)^2} 8(\lambda^2 - \lambda\nu + 2\nu^2) + \frac{1}{(4\pi)^4} 16(6\lambda^3 - 15\lambda^2\nu + 12\lambda\nu^2 - 11\nu^3) \\
\beta_\rho &= -\frac{1}{(4\pi)^2} 4\rho(\lambda + \nu) + \frac{1}{(4\pi)^4} 8\rho(\lambda^2 - 2\lambda\nu - 3\nu^2) \\
\gamma_m &= \frac{1}{(4\pi)^2} 4m^2(\lambda - \nu) - \frac{1}{(4\pi)^4} 24m^2(\lambda - \nu)^2 \quad (4.7)
\end{align*}
\]

Integrating the \(\beta\) differential equations at one loop we get easily (integrating firstly \(g(\mu)\), then \(\lambda(\mu)\) and finally getting \(\nu(\mu)\) as the difference of the previous two functions):

\[
\begin{align*}
\lambda(\mu) &= \frac{g_0}{2} \frac{1}{1 - \frac{2g_0}{\pi^2}x} \frac{1}{1 - \left(1 - \frac{g_0}{\pi^2}x\right)^{1/2} \left(1 - \frac{g_0}{2\lambda_0}\right)} \\
\nu(\mu) &= \frac{g_0}{2} \frac{1}{1 - \frac{2g_0}{\pi^2}x} \left(\frac{1}{2} \frac{1}{1 - \left(1 - \frac{g_0}{\pi^2}x\right)^{1/2} \left(1 - \frac{g_0}{2\lambda_0}\right)} - 1\right)
\end{align*}
\]

(4.8)

where \(\lambda_0 = \lambda(\mu_0), \nu_0 = \nu(\mu_0)\) and \(g_0 = g(\mu_0) = \lambda_0 - \nu_0\) and \(x = \log\left(\frac{\mu}{\mu_0}\right)\).

It is easy to see that \(\lambda(\mu)\) has two singularities (fig. 5): one is the usual

---

\(^3\) Would we have chosen \(Z'_\rho = 1, Z'_b = Z'_\xi\), this would not have been completely true, nevertheless what really matters, the physical quantities, would have behaved exactly as \(O(n \to 0)\) theory: for instance, the two points function \(\langle \phi^* b \rangle\) depends only on \(Z'_\phi Z'_b = Z_\phi Z_b = Z^2_\phi(g)\)

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Landau pole of $\phi^4$ at $x_L = \log\left(\frac{\mu}{\mu_0}\right) = \frac{\pi^2}{g_0}$ and the other is at $x_P = \log\left(\frac{\mu}{\mu_0}\right) = \frac{\pi^2}{g_0} \left(1 - \frac{1}{(1 - \frac{m}{2\lambda_0})^2}\right)$

This latter singularity is shared also by $\nu(\mu)$ because it takes place at a finite value of $g(\mu)$. What is the meaning of this singularity? There are two possibilities; it could be either a breakdown of the perturbative expansion (and in this case it is probably related to the specific formulation of the theory) or a problem intrinsic to the theory (12). What makes more reliable the first possibility is that this pole is present even when the theory is free, i.e. setting $\lambda = \nu$, in this case $\beta_\lambda = \beta_\nu = -\frac{1}{\pi^2} \lambda^2$ would lead to a singularity in $x_P = \log\left(\frac{\mu}{\mu_0}\right) = -\frac{\pi^2}{\lambda_0}$.

There is also an other singular point of the perturbative expansion of the theory: it happens when $\nu(\mu)$ crosses the zero and then it becomes negative, in that case the theory is not bounded from below anymore as can easily seen from (1.3); this happens for $\log\left(\frac{\mu}{\mu_0}\right) > x_Z = \frac{\pi^2}{g_0} \left(1 - \frac{1}{(2 - \frac{m}{2\lambda_0})^2}\right)$

The singular points of $\rho$ are at $x_L$ where it diverges and at $x_P$ where it vanishes, but differently from the previous singular behaviours, these can be eliminated setting $\rho = 0$, that, as shown by (3.4,3.5), does not change the physics.

5 Conclusion.

In this paper we have demonstrated that the topological theory we proposed is renormalizable and we have explicitly computed its two loop perturbative expansion, however the main aim of our approach, the exact computation of the critical indexes of SAW, has revealed unreachable, nevertheless this topological theory reveals interesting features:

1. even in the delta gauge it is not finite;
2. it has two phases, one of which has an explicit breaking of the topological character.
There is an heuristic way to see immediately the existence of two phases. It consists of a mean field approximation in which all the fields are constants, whence the action can be written as

\[ S = (\rho|b|^2 + 4\lambda w^2 - 2i m^2 z + 4\nu z^2) V \]

where \( V \) is the volume, \( w = Im(b^*\phi) + iRe(\xi^*\eta) \) and \( z = Re(b^*\phi) - iIm(\xi^*\eta) \). If we try to minimize this action and we consider that it is limited from below, we get immediately that \( b = w = 0 \) and \( z = -\frac{im^2}{4\nu} \), that implies

\[ <b^*\phi + \phi^*b> = 0 \neq <\xi^*\eta - \eta^*\xi> = \frac{im^2}{2\nu} \]

while they should be equal in order not to break the BRST symmetry.

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