Asymptotic Analysis of MAP Estimation via the Replica Method and Applications to Compressed Sensing

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Abstract—The replica method is a non-rigorous but widely-accepted technique from statistical physics used in the asymptotic analysis of large, random, nonlinear problems. This paper applies the replica method to non-Gaussian maximum a posteriori (MAP) estimation. It is shown that with random linear measurements and Gaussian noise, the asymptotic behavior of the MAP estimate of an n-dimensional vector “decouples” as n scalar MAP estimators. The result is a counterpart to Guo and Verdú’s replica analysis of minimum mean-squared error estimation.

The replica MAP analysis can be readily applied to many estimators used in compressed sensing, including basis pursuit, lasso, linear estimation with thresholding, and zero norm-regularized estimation. In the case of lasso estimation the scalar estimator reduces to a soft-thresholding operator, and for zero norm-regularized estimation it reduces to a hard-threshold. Among other benefits, the replica method provides a computationally-tractable method for exactly computing various performance metrics including mean-squared error and sparsity pattern recovery probability.

Index Terms—Compressed sensing, Laplace’s method, large deviations, least absolute shrinkage and selection operator (lasso), nonlinear estimation, non-Gaussian estimation, random matrices, sparsity, spin glasses, statistical mechanics, thresholding

I. INTRODUCTION

Estimating a vector \( x \in \mathbb{R}^n \) from measurements of the form

\[
y = \Phi x + w,
\]

where \( \Phi \in \mathbb{R}^{m \times n} \) represents a known measurement matrix and \( w \in \mathbb{R}^m \) represents measurement errors or noise, is a generic problem that arises in a range of circumstances. One of the most basic estimators for \( x \) is the maximum a posteriori (MAP) estimate

\[
\hat{x}_{\text{map}}(y) = \arg \max_{x \in \mathbb{R}^n} p_{x|y}(x|y),
\]

which is defined assuming some prior on \( x \). For most priors, the MAP estimate is nonlinear and its behavior is not easily characterizable. Even if the priors for \( x \) and \( w \) are separable, the analysis of the MAP estimate may be difficult since the matrix \( \Phi \) couples the \( n \) unknown components of \( x \) with the \( m \) measurements in the vector \( y \).

The primary contribution of this paper is to show that with certain large random \( \Phi \) and Gaussian \( w \), there is an asymptotic decoupling of (1) into \( n \) scalar MAP estimation problems. Each equivalent scalar problem has an appropriate scalar prior and Gaussian noise with an effective noise level. The analysis yields the asymptotic joint distribution of each component \( x_j \) of \( x \) and its corresponding estimate \( \hat{x}_j \) in the MAP estimate vector \( \hat{x}_{\text{map}}(y) \). From the joint distribution, various further computations can be made, such as the mean-squared error (MSE) of the MAP estimate or the error probability of a hypothesis test computed from the MAP estimate.

The analysis can quantify the effect of using a postulated prior different from the true prior. This has two important consequences: First, for many priors, the exact MAP estimate is computationally intractable; one can use our method to determine the asymptotic performance when using an approximate prior that simplifies computations. Second, when MAP is not the criterion of interest, many popular estimation algorithms can be seen as MAP estimators with respect to a postulated prior. This is the case for the basis pursuit and lasso estimators used in compressed sensing.

A. Replica Method

Our analysis is based on a powerful but non-rigorous technique from statistical physics known as the replica method. The replica method was originally developed by Edwards and Anderson [1] to study the statistical mechanics of spin glasses. Although not fully rigorous from the perspective of probability theory, the technique was able to provide explicit solutions for a range of complex problems where many other methods had previously failed. Indeed, the replica method and related ideas from statistical mechanics have found success in a number of classic NP-hard problems including the traveling salesman problem [2], graph partitioning [3], k-SAT [4] and others [5]. Statistical physics methods have also been applied to the study of error correcting codes [6], [7].

The replica method was first applied to the study of nonlinear MAP estimation problems by Tanaka [8]. He considered multiuser detection for large CDMA systems with random spreading sequences. Müller [9] considered a mathematically-similar problem for MIMO communication systems. In the context of the estimation problem considered here, Tanaka’s and Müller’s papers essentially characterized the behavior of
the MAP estimator of a vector $x$ with i.i.d. binary components observed through linear measurements of the form $\Phi$ with a large random $\Phi$ and Gaussian $w$.

Tanaka’s results were then generalized in a remarkable paper by Guo and Verdú [10] to vectors $x$ with arbitrary distributions. Guo and Verdú’s result was also able to incorporate a large class of minimum postulated MSE estimators, where the estimator may assume a prior that is different from the actual prior. The result in this paper is the corresponding MAP statement to Guo and Verdú’s result. In fact, our result is derived from Guo and Verdú’s by taking appropriate limits with large deviations arguments.

The non-rigorous aspect of the replica method involves a set of assumptions that include a self-averaging property, the validity of a “replica trick,” and the ability to exchange certain limits. Some progress has been made in formally proving these assumptions; a survey of this work can be found in [11]. Also, some of the predictions of the replica method have been validated rigorously by other means. For example, Montanari and Tse [12] have confirmed Tanaka’s formula in certain regimes using density evolution and belief propagation.

To emphasize our dependence on these unproven assumptions, we will refer to Guo and Verdú’s result as the Replica MMSE Claim. Our main result, which depends on Guo and Verdú’s analysis, will be called the Replica MAP Claim.

### B. Applications to Compressed Sensing

As an application of our main result, we will develop a few analyses of estimation problems that arise in compressed sensing [13]–[15]. In compressed sensing, one estimates a sparse vector $x$ from random linear measurements. A vector $x$ is sparse when its number of nonzero entries $k$ is smaller than its length $n$. Generically, optimal estimation of $x$ with a sparse prior is NP-hard [16]. Thus, most attention has focused on greedy heuristics such as matching pursuit [17]–[20] and convex relaxations such as basis pursuit [21] or lasso [22]. While successful in practice, these algorithms are difficult to analyze precisely.

Compressed sensing of sparse $x$ through $\Phi$ (using inner products with rows of $\Phi$) is mathematically identical to sparse approximation of $y$ with respect to columns of $\Phi$. An important set of results for both sparse approximation and compressed sensing are the deterministic conditions on the coherence of $\Phi$ that are sufficient to guarantee good performance of the suboptimal methods mentioned above [23]–[25]. These conditions can be satisfied with high probability for certain large random measurement matrices. Compressed sensing has provided many sufficient conditions that are easier to satisfy than the initial coherence-based conditions. However, despite this progress, the exact performance of most sparse estimators is still not known precisely, even in the asymptotic case of large random measurement matrices. Most results describe the estimation performance via bounds, and the tightness of these bounds is generally not known.

There are, of course, notable exceptions including [26] and [27] which provide matching necessary and sufficient conditions for recovery of strictly sparse vectors with basis pursuit and lasso. However, even these results only consider exact recovery and are limited to measurements that are noise-free or measurements with a signal-to-noise ratio (SNR) that scales to infinity.

Many common sparse estimators can be seen as MAP estimators with certain postulated priors. Most importantly, lasso and basis pursuit are MAP estimators assuming a Laplacian prior. Other commonly-used sparse estimation algorithms, including linear estimation with and without thresholding and zero norm-regularized estimators, can also be seen as MAP-based estimators. For these MAP-based sparse estimation algorithms, we can apply the replica method to provide a novel analysis with a number of important features:

- **Asymptotic exactness:** Most importantly, the replica method provides—under the assumption of the replica hypotheses—not just bounds, but the exact asymptotic behavior of MAP-based sparse estimators. This in turns permits exact expressions for the various performance metrics such as MSE or fraction of support recovery. The expressions apply for arbitrary ratios $k/n$, $n/m$, and SNR.

- **Connections to thresholding:** The scalar model provided by the Replica MAP Claim is appealing in that it reduces the analysis of a complicated vector-valued estimation problem to a simple equivalent scalar model. This model is particularly simple for lasso estimation. In this case, the replica analysis shows that the asymptotic behavior of the lasso estimate of any component of $x$ is equivalent to that component being corrupted by Gaussian noise and soft-thresholded. Similarly, zero norm-regularized estimation is equivalent to hard thresholding.

- **Application to arbitrary distributions:** The replica analysis can incorporate arbitrary distributions on $x$ including several sparsity models, such as Laplacian, generalized Gaussian and Gaussian mixture priors. Discrete distributions can also be studied.

It should be pointed out that this work is not the first to use ideas from statistical physics for the study of sparse estimation. Merhav, Guo and Shamai [28] consider, among other applications, the estimation of a sparse vector $x$, through measurements of the form $y = x + w$. In their model, there is no measurement matrix such as $\Phi$ in (1), but the components of $x$ are possibly correlated. Their work derives explicit expressions for the MMSE as a function of the probability distribution on the number of nonzero components. The analysis does not rely on replica assumptions and is fully rigorous. More recently, Kabashima, Wadayama and Tanaka [29] have used the replica method to derive precise conditions on which sparse signals can be recovered with $l_p$-based relaxations such as lasso. Their analysis does not consider noise, but can find conditions on recovery on the entire vector $x$, not just individual components.

### C. Outline

The remainder of the paper is organized as follows. The precise estimation problem is described in Section II. We review the Replica MMSE Claim of Guo and Verdú in Section III. We then present our main result, the Replica
MAP Claim, in Section \[ \text{V} \] The results are applied to the analysis of compressed sensing algorithms in Section \[ \text{VI} \] which is followed by numerical simulations in Section \[ \text{VII} \]. Future work is given in Section \[ \text{VIII} \]. The proof of the main result is somewhat long and given in a set of appendices; Appendix \[ \text{II} \] provides an overview of the proof and a guide through the appendices with detailed arguments.

II. ESTIMATION PROBLEM AND ASSUMPTIONS

Consider the estimation of a random vector \( x \in \mathbb{R}^n \) from linear measurements of the form
\[
y = \Phi x + w = AS^{1/2}x + w,
\] where \( y \in \mathbb{R}^m \) is a vector of observations, \( \Phi = AS^{1/2} \in \mathbb{R}^{m \times n} \) is a measurement matrix, \( S \) is a diagonal matrix of positive scale factors,
\[
S = \text{diag}(s_1, \ldots, s_n), \quad s_j > 0,
\] and \( w \in \mathbb{R}^m \) is zero-mean, white Gaussian noise. We consider a sequence of such problems indexed by \( n \), with \( n \to \infty \). For each \( n \), the problem is to determine an estimate \( \hat{x} \) of \( x \) from the observations \( y \) knowing the measurement matrix \( A \) and scale factor matrix \( S \).

The components \( x_j \) of \( x \) are modeled as zero mean and i.i.d. with some prior probability distribution \( p_0(x_j) \). The per-component variance of the Gaussian noise is \( \mathbb{E}|w_j|^2 = \sigma_0^2 \). We use the subscript “0” on the prior and noise level to omit for brevity.

In (3), we have factored \( \Phi = AS^{1/2} \) so that even with the i.i.d. assumption on \( x_j \)'s above and an i.i.d. assumption on entries of \( A \), the model can capture variations in powers of the components of \( x \) that are known a priori at the estimator. Specifically, multiplication by \( S^{1/2} \) scales the variance of the \( j \)th component of \( x \) by a factor \( s_j \). Variations in the power of \( x \) that are not known to the estimator should be captured in the distribution of \( x \).

We summarize the situation and make additional assumptions to specify the problem precisely as follows:
(a) The number of measurements \( m = m(n) \) is a deterministic quantity that varies with \( n \) and satisfies
\[
\lim_{n \to \infty} n/m(n) = \beta
\] for some \( \beta \geq 0 \). (The dependence of \( m \) on \( n \) is usually omitted for brevity.)
(b) The components \( x_j \) of \( x \) are i.i.d. with probability distribution \( p_0(x_j) \).
(c) The noise \( w \) is Gaussian with \( w \sim \mathcal{N}(0, \sigma_0^2 I_m) \).
(d) The components of the matrix \( A \) are i.i.d. zero mean with variance \( 1/m \).
(e) The scale factors \( s_j \) are i.i.d. and satisfy \( s_j > 0 \) almost surely.
(f) The scale factor matrix \( S \), measurement matrix \( A \), vector \( x \) and noise \( w \) are all independent.

III. REVIEW OF THE REPLICA MMSE CLAIM

We begin by reviewing the Replica MMSE Claim of Guo and Verdú [10].

A. Minimum Postulated MSE Estimators

The Replica MMSE Claim concerns the asymptotic behavior of estimators that minimize MSE under certain postulated prior distributions. To define the concept, suppose one is given a “postulated” prior distribution \( p_{\text{post}} \) and a postulated noise level \( \sigma_{\text{post}}^2 \) that may be different from the true values \( p_0 \) and \( \sigma_0^2 \). We define the minimum postulated MSE (MPMSE) estimate of \( x \) as
\[
\hat{x}_{\text{mpmse}}(y) = \mathbb{E}(x \mid y; p_{\text{post}}, \sigma_{\text{post}}^2)
= \int x p_{x\mid y}(x \mid y; p_{\text{post}}, \sigma_{\text{post}}^2) \, dx,
\] where the normalization constant is
\[
C = \int \exp\left(-\frac{1}{2\sigma_0^2}||y - AS^{1/2}x||^2\right) q(x) \, dx
\] and \( q(x) \) is the joint p.d.f.
\[
q(x) = \prod_{j=1}^{n} q(x_j).
\]

In the case when \( p_{\text{post}} = p_0 \) and \( \sigma_{\text{post}}^2 = \sigma_0^2 \), so that the postulated and true values agree, the MPMSE estimator reduces to the true MMSE estimate.

B. Replica MMSE Claim

The essence of the Replica MMSE Claim is that the asymptotic behavior of the MPMSE estimator is described by an equivalent scalar estimator. Let \( q(x) \) be a probability distribution defined on some set \( \mathcal{X} \subseteq \mathbb{R} \). Given \( \mu > 0 \), let \( p_{x\mid z}(x \mid z; q, \mu) \) be the conditional distribution
\[
p_{x\mid z}(x \mid z; q, \mu)
= \left[ \int_{\mathcal{X}} \phi(z - x; \mu) q(x) \, dx \right]^{-1} \phi(z - x; \mu) q(x)
\] where \( \phi(\cdot) \) is the Gaussian distribution
\[
\phi(v; \mu) = \frac{1}{\sqrt{2\pi \mu}} e^{-|v|^2/(2\mu)}.
\]
The distribution \( p_{x\mid z}(x \mid z; q, \mu) \) is the conditional distribution of the scalar random variable \( x \sim q(x) \) from an observation of the form
\[
z = x + \sqrt{\mu} v,
\]
where $v \sim \mathcal{N}(0, 1)$. Using this distribution, we can define the scalar conditional MMSE estimate,

$\hat{x}^{\text{mmse}}(z; q, \mu) = \int_{x \in \mathcal{X}} xp_x | z | x \hat{z} \, dx$. \hfill (10)

Also, given two distributions, $p_0(x)$ and $p_1(x)$, and two noise levels, $\mu_0 > 0$ and $\mu_1 > 0$, define

\[ \text{mse}(p_1, p_0, \mu_1, \mu_0, z) = \int_{x \in \mathcal{X}} |x - \hat{x}^{\text{mmse}}(z; p_1, \mu_1)|^2 p_{x | z}(x | z; p_0, \mu_0) \, dx, \] \hfill (11)

which is the MSE in estimating the scalar $x$ from the variable $z$ in (9) when $x$ has a true distribution $x \sim p_0(x)$ and the noise level is $\mu = \mu_0$, but the estimator assumes a distribution $x \sim p_1(x)$ and noise level $\mu = \mu_1$.

**Replica MMSE Claim [10]:** Consider the estimation problem in Section II. Let $\hat{x}^{\text{mmse}}(y)$ be the MP MSE estimator based on a postulated prior $p_{\text{post}}$ and postulated noise level $\sigma_{\text{post}}$. For each $n$, let $j = j(n)$ be some deterministic component index with $j(n) \in \{1, \ldots, n\}$. Then there exist effective noise levels $\sigma_{\text{eff}}^2$ and $\sigma_{\text{p-eff}}^2$ such that:

(a) As $n \to \infty$, the random vectors $(x_j, s_j, \hat{x}_j^{\text{mmse}})$ converge in distribution to the random vector $(x, s, \hat{x})$ shown in Fig. 1. Here, $x$, $s$, and $v$ are independent with $x \sim p_0(x)$, $s \sim p_S(s)$, $v \sim \mathcal{N}(0, 1)$, and

\[ \hat{x} = \hat{x}^{\text{mmse}}(z; p_{\text{post}}, \mu_p) \] \hfill (12a)

\[ z = x + \sqrt{\mu v} \] \hfill (12b)

where $\mu = \sigma_{\text{eff}}^2 / s$ and $\mu_p = \sigma_{\text{p-eff}}^2 / s$.

(b) The effective noise levels satisfy the equations

\[ \sigma_{\text{eff}}^2 = \sigma_0^2 + \beta E[s \text{mse}(p_{\text{post}}, p_0, \mu_p, \mu, z)] \] \hfill (13a)

\[ \sigma_{\text{p-eff}}^2 = \sigma_{\text{post}}^2 + \beta E[s \text{mse}(p_{\text{post}}, p_0, \mu_p, \mu, z)] \] \hfill (13b)

where the expectations are taken over $s \sim p_S(s)$ and $z$ generated by (12b).

The Replica MMSE Claim asserts that the asymptotic behavior of the joint estimation of the $n$-dimensional vector $x$ can be described by $n$ equivalent scalar estimators. In the scalar estimation problem, a component $x_j \sim p_0(x)$ is corrupted by additive Gaussian noise yielding a noisy measurement $z$. The additive noise variance is $\mu = \sigma_{\text{eff}}^2 / s$, which is the effective noise divided by the scale factor $s$. The estimate of that component is then described by the (generally nonlinear) scalar estimator $\hat{x}(z; p_{\text{post}}, \mu_p)$.

The effective noise levels $\sigma_{\text{eff}}^2$ and $\sigma_{\text{p-eff}}^2$ are described by the solutions to fixed-point equations (13). Note that $\sigma_{\text{eff}}^2$ and $\sigma_{\text{p-eff}}^2$ appear implicitly on the left- and right-hand sides of these equations via the terms $\mu$ and $\mu_p$. In general, there is no closed form solution to these equations. However, the expectations can be evaluated via numerical integration.

It is important to point out that there may, in general, be multiple solutions to the fixed-point equations (13). In this case, it turns out that the true solution is the minimizer of a certain Gibbs’ function described in [10].

**C. Effective Noise and Multiuser Efficiency**

To understand the significance of the effective noise level $\sigma_{\text{eff}}^2$, it is useful to consider the following estimation problem with side information. Suppose that when estimating the component $x_j$ an estimator is given as side information the values of all the other components $x_{\ell}, \ell \neq j$. Then, this hypothetical estimator with side information can “subtract out” the effect of all the known components and compute

\[ z_j = \frac{1}{|a_j|^2} a_j^* \left( y - \sum_{\ell \neq j} \sqrt{\eta} a_{\ell} x_{\ell} \right), \]

where $a_{\ell}$ is the $\ell$th column of the measurement matrix $A$. It is easily checked that

\[ z_j = \frac{1}{|a_j|^2} a_j^* \left( \sqrt{\eta} a_j x_j + w \right) \]

\[ = x_j + \sqrt{\mu_0 v_j}, \] \hfill (14)

where

\[ v_j = \frac{1}{\sigma_0 |a_j|^2} a_j^* w, \quad \mu_0 = \frac{\sigma_0^2}{s_j}. \]

Thus, (14) shows that with side information, estimation of $x_j$ reduces to a scalar estimation problem where $x_j$ is corrupted by additive noise $v_j$. Since $w$ is Gaussian with mean zero and per-component variance $\sigma_0^2$, $v_j$ is Gaussian with mean zero and variance $1/|a_j|^2$. Also, since $a_j$ is an $m$-dimensional vector whose components are i.i.d. with variance $1/m$, $|a_j|^2 \to 1$ as $m \to \infty$. Therefore, for large $m$, $v_j$ will approach $\mathcal{N}(0, 1)$.

Comparing (14) with (12b), we see that the equivalent scalar model predicted by the Replica MMSE Claim in (12b) is identical to the estimation with perfect side information (14), except that the noise level is increased by a factor

\[ 1/\eta = \mu / \mu_0 = \sigma_{\text{eff}}^2 / \sigma_0^2. \] \hfill (15)

In multiuser detection, the factor $\eta$ is called the multiuser efficiency [30], [31].

The multiuser efficiency can be interpreted as degradation in the effective signal-to-noise ratio (SNR): With perfect side information, an estimator using $z_j$ in (14) can estimate $x_j$ with an effective SNR of

\[ \text{SNR}_0(s) = \frac{1}{\mu_0} E[x_j^2] = \frac{s}{\sigma_0^2} E[x_j^2]. \] \hfill (16)
In CDMA multiuser detection, the factor SNR$_0(s)$ is called the post-despreading SNR with no multiple access interference. The Replica MMSE Claim shows that without side information, the effective SNR is given by

$$\text{SNR}(s) = \frac{1}{h} E[x_j^2] = \frac{s}{\sigma^2_{\text{eff}}} E[x_j^2].$$  \hspace{1cm} (17)

Therefore, the multiuser efficiency $\eta$ is the ratio of the effective SNR with and without perfect side information.

### D. Replica Assumptions

As described in Section I-A, the Replica MMSE Claim is not formally proven. We introduce the following definition to call attention to when we are explicitly assuming that the Replica MMSE Claim holds.

**Definition 1:** Consider the estimation problem in Section II. A postulated prior $p_{\text{post}}$ and noise level $\sigma^2_{\text{post}}$ are said to satisfy the Replica MMSE Claim when the corresponding postulated MMSE estimator $\hat{x}_{\text{mmse}}$ satisfies properties (a) and (b) of the Replica MMSE Claim.

### IV. Replica MAP Claim

We now turn to MAP estimation. Let $X \subseteq \mathbb{R}$ be some (measurable) set and consider an estimator of the form

$$\hat{x}_{\text{map}}(y) = \arg \min_{x \in X} \frac{1}{2\gamma} \| y - Ax^{1/2} x \|^2 + \sum_{j=1}^{n} f(x_j),$$  \hspace{1cm} (18)

where $\gamma > 0$ is an algorithm parameter and $f : X \rightarrow \mathbb{R}$ is some scalar-valued, non-negative cost function. We will assume that the objective function in (18) has a unique essential minimizer for almost all $y$.

The estimator (18) can be interpreted as a MAP estimator. To see this, suppose that for $u$ sufficiently large,

$$\int_{x \in X} e^{-uf(x)} dx < \infty,$$  \hspace{1cm} (19)

where we have overloaded the notation $f(\cdot)$ such that

$$f(x) = \sum_{j=1}^{n} f(x_j).$$

When (19) is satisfied, we can define the prior probability distribution

$$p_u(x) = \left[ \int_{x \in X} \exp(-uf(x)) dx \right]^{-1} \exp(-uf(x)).$$  \hspace{1cm} (20)

Also, let

$$\sigma^2_{\text{u}} = \gamma / u.$$  \hspace{1cm} (21)

Substituting (20) and (21) into (6), we see that

$$p_{x|y}(x \mid y ; p_u, \sigma^2_{\text{u}}) = C_u \exp \left[ -u \left( \frac{1}{2\gamma} \| y - Ax^{1/2} x \|^2 + f(x) \right) \right]$$  \hspace{1cm} (22)

for some constant $C_u$ that does not depend on $x$. (The scaling of the noise variance along with $p_u$ enables the factorization in the exponent of (22).) Comparing to (18), we see that

$$\hat{x}_{\text{map}}(y) = \arg \max_{x \in X} p_{x|y}(x \mid y ; p_u, \sigma^2_{\text{u}}).$$

Thus for all sufficiently large $u$, we indeed have a MAP estimate—assuming the prior $p_u$ and noise level $\sigma^2_{\text{u}}$.

To analyze this MAP estimator, we consider a sequence of MMSE estimators. For each $u$, let

$$\hat{x}_{\text{u}}^u(y) = E \left( x \mid y ; p_u, \sigma^2_{\text{u}} \right),$$  \hspace{1cm} (23)

which is the MMSE estimator of $x$ under the postulated prior $p_u$ in (20) and noise level $\sigma^2_{\text{u}}$ in (21). Using a standard large deviations argument, one can show that under suitable conditions

$$\lim_{u \to \infty} \hat{x}_{\text{u}}^u(y) = \hat{x}_{\text{map}}(y)$$

for all $y$. A formal proof is given in Appendix I (see Lemma 4). Under the assumption that the behaviors of the MMSE estimators are described by the Replica MMSE Claim, we can then extrapolate the behavior of the MAP estimator. This will yield our main result, the Replica MAP Claim.

To state the claim, define the scalar MAP estimator

$$\hat{x}_{\text{scalar}}(z ; \lambda) = \arg \min_{x \in \mathcal{X}} F(x, z, \lambda)$$  \hspace{1cm} (24)

where

$$F(x, z, \lambda) = \frac{1}{2\lambda} \| z - x \|^2 + f(x).$$  \hspace{1cm} (25)

The estimator (24) plays a similar role as the scalar MMSE estimator (10).

The Replica MAP Claim pertains to the estimator (18) applied to the sequence of estimation problems defined in Section II. Our assumptions are as follows:

**Assumption 1:** For all $u > 0$ sufficiently large, assume the postulated prior $p_u$ in (20) and noise level $\sigma^2_{\text{u}}$ in (21) satisfy the Replica MMSE Claim (see Definition 1).

**Assumption 2:** Let $\sigma^2_{\text{eff}}(u)$ and $\sigma^2_{\text{p-eff}}(u)$ be the effective noise levels when using the prior $p_u$ and noise level $\sigma^2_{\text{u}}$.

Assume the following limits exist:

$$\sigma^2_{\text{eff, map}} = \lim_{u \to \infty} \sigma^2_{\text{eff}}(u),$$

$$\gamma_p = \lim_{u \to \infty} u \sigma^2_{\text{p-eff}}(u).$$

**Assumption 3:** For each $n$, $\hat{x}_{\text{y}}^u(n)$ is the MMSE estimate of the component $x_j$ for some index $j \in \{1, \ldots, n\}$ based on the postulated prior $p_u$ and noise level $\sigma^2_{\text{u}}$. Then, assume that limits can be interchanged to give the following equality:

$$\lim_{u \to \infty} \lim_{n \to \infty} \hat{x}_{\text{y}}^u(n) = \lim_{n \to \infty} \lim_{u \to \infty} \hat{x}_{\text{y}}^u(n),$$

where the limits are in distribution.

**Assumption 4:** For every $n$, $A$, and $S$, assume that for almost all $y$, the minimization in (18) achieves a unique essential minimum. Here, essential should be understood in the standard measure theoretic sense in that the minimum and essential infimum agree.

**Assumption 5:** Assume that $f(x)$ is non-negative and satisfies

$$\lim_{|x| \to \infty} f(x) = \infty,$$

where the limit must hold over all sequences $x \in \mathcal{X}$ with $|x| \to \infty$. If $\mathcal{X}$ is compact, this limit is automatically satisfied (since there are no sequences in $\mathcal{X}$ with $|x| \to \infty$).
with $f$ of interest. In fact, we will explicitly calculate

tions 1–4, we will verify Assumptions 5 and 6 for the problems

is non-rigorous.

Assumptions 1–4 reflect the main limitations of the replica

section and uniqueness of the MAP estimate. Unlike Assump-
tions 5 and 6, the Replica MMSE Claim has not been formally
proven. We make the additional Assumptions 2–4, which are

in Assumption 2. Here, $x$ and $v$ are independent with

A REPLICA MAP CLAIM AND APPLICATIONS TO COMPRESSED SENSING

V. ANALYSIS OF COMPRESSED SENSING

Our results thus far hold for any separable distribution for $x$
(see Section II) and under mild conditions on the cost function $f$
(see especially Assumption 5, but other assumptions also
implicitly constrain $f$). In this section, we provide additional
details on replica analysis for choices of $f$ that yield MAP
estimators relevant to compressed sensing. Since the role of $f$
is to determine the estimator, this is not the same as choosing
spars prior for $x$. Numerical evaluations of asymptotic
performance with sparse priors for $x$ are given in Section VI.

A. Linear Estimation

We first apply the Replica MAP Claim to the simple case of
linear estimation. Linear estimators only use second-order
statistics and generally do not directly exploit sparsity or other
aspects of the distribution of the unknown vector $x$. Nonetheless,
for sparse estimation problems, linear estimators can be
used as a first step in estimation, followed by thresholding or
other nonlinear operations [32], [33]. It is therefore worthwhile
to analyze the behavior of linear estimators even in the context
of sparse priors.

The asymptotic behavior of linear estimators with large ran-
dom measurement matrices is well known. For example, using
the Marčenko-Pastur theorem [34], Verdú and Shamai [35]
characterized the behavior of linear estimators with large
i.i.d. matrices $A$ and constant scale factors $S = I$. Tse and Hanly [36] extended the analysis to general $S$. Guo and
Verdú [10] showed that both of these results can be recovered
as special cases of the general Replica MMSE Claim. We
show here that the Replica MAP Claim can also recover
these results. Although this analysis will not provide any new
results, walking through the computations will illustrate how
the Replica MAP Claim is used.

To simplify the notation, suppose that the true prior on $x$
is such that each component has zero mean and unit variance.
Choose the cost function

$$f(x) = \frac{1}{2} |x|^2,$$

which corresponds to the negative log of a Gaussian prior also
with zero mean and unit variance. With this cost function, the
MAP estimator (18) reduces to the linear estimator

$$\hat{x}(y) = S^{1/2} A' (A S A' + \gamma I)^{-1} y.$$ (30)

When $\gamma = s^2_0$, the true noise variance, the estimator (30) is the linear MMSE estimate.

Now, let us compute the effective noise levels from the
Replica MAP Claim. First note that $F(x, z, \lambda)$ in (25) is given by

$$F(x, z, \lambda) = \frac{1}{2\lambda} |z - x|^2 + \frac{1}{2} |x|^2,$$
and therefore the scalar MAP estimator in (24) is given by
\[ \hat{x}^\text{map}_\text{scalar}(z; \lambda) = \frac{1}{1 + \lambda} z. \] (31)

A simple calculation also shows that \( \sigma^2(z, \lambda) \) in (27) is given by
\[ \sigma^2(z, \lambda) = \frac{\lambda}{1 + \lambda}. \] (32)

Now, as part (a) of the Replica MAP Claim, let \( \mu = \sigma^2_{\text{eff, map}}/s \) and \( \lambda_p = \gamma_p/s \). Observe that
\[
E \left[ s|x - \hat{x}^\text{map}_\text{scalar}(z; \lambda_p)|^2 \right] \\
= E \left[ s|x - \frac{1}{1 + \lambda_p} z|^2 \right] \\
= \frac{s}{1 + \lambda_p} \left( \lambda + \beta \right) E \left[ s|\lambda p|/1 + \lambda_p \right] \\
= \frac{s(\lambda^2 + \mu)}{(1 + \lambda_p)^2},
\]
where (a) follows from (31); (b) follows from (28b); and (c) follows from the fact that \( x \) and \( v \) are uncorrelated with zero mean and unit variance. Substituting (32) and (33) into the fixed-point equations (29), we see that the limiting noise levels \( \sigma^2_{\text{eff, map}} \) and \( \gamma_p \) must satisfy
\[
\sigma^2_{\text{eff, map}} = \sigma^2_0 + \beta E \left[ s(\lambda^2 + \mu)/\left(1 + \lambda_p\right)^2 \right], \\
\gamma_p = \gamma + \beta E \left[ s\lambda p/1 + \lambda_p \right],
\]
where the expectation is over \( s \sim ps(s) \). In the case when \( \gamma = \sigma^2_0 \), it can be verified that a solution to these fixed-point equations is \( \sigma^2_{\text{eff, map}} = \gamma_p \), which results in \( \mu = \lambda_p \) and
\[
\sigma^2_{\text{eff, map}} = \sigma^2_0 + \beta E \left[ s\lambda p/1 + \lambda_p \right] \\
= \sigma^2_0 + \beta E \left[ s\sigma^2_{\text{eff, map}}/s + \sigma^2_{\text{eff, map}} \right].
\] (34)

The expression (34) is precisely the Tse-Hanly formula [36] for the effective interference. Given a distribution on \( s \), this expression can be solved numerically for \( \sigma^2_{\text{eff, map}} \). In the special case of constant \( s \), (34) reduces to Verdú and Shamai’s result in [37] and can be solved via a quadratic equation.

The Replica MAP Claim now states that for any component index \( j \), the asymptotic joint distribution of \( (x_j, sj, \hat{x}_j) \) is described by \( x_j \) corrupted by additive Gaussian noise with variance \( \sigma^2_{\text{eff, map}}/s \) followed by a scalar linear estimator.

As described in [10], the above analysis can also be applied to other linear estimators including the matched filter (where \( \gamma \to \infty \)) or the decorrelating receiver (\( \gamma \to 0 \)).

**B. Lasso Estimation**

We next consider lasso estimation, which is widely used for estimation of sparse vectors. The lasso estimate [22] (sometimes referred to as basis pursuit denoising [21]) is given by
\[
\chi^\text{lasso}(y) = \arg \min_{x \in \mathbb{R}^n} \frac{1}{2\gamma} ||y - A^\dagger x||_2^2 + ||x||_1,
\] (35)

where \( \gamma > 0 \) is an algorithm parameter. The estimator is essentially a least-squares estimator with an additional \( \|x\|_1 \) regularization term to encourage sparsity in the solution. The parameter \( \gamma \) is selected to trade off the sparsity of the estimate with the prediction error. An appealing feature of lasso estimation is that the minimization in (35) is convex; lasso thus enables computationally-tractable algorithms for finding sparse estimates.

The lasso estimator (35) is identical to the MAP estimator (18) with the cost function
\[
f(x) = |x|.
\]

With this cost function, \( F(x, z, \lambda) \) in (25) is given by
\[
F(x, z, \lambda) = \frac{1}{2\lambda} |z - x|^2 + |x|,
\]

and therefore the scalar MAP estimator in (24) is given by
\[
\hat{x}^\text{map}_\text{scalar}(z; \lambda) = T^\text{soft}_{\text{map}}(z),
\] (36)

where \( T^\text{soft}_{\text{map}}(z) \) is the soft thresholding operator
\[
T^\text{soft}_{\text{map}}(z) = \begin{cases} 
  z - \lambda, & \text{if } z > \lambda; \\
  0, & \text{if } |z| \leq \lambda; \\
  z + \lambda, & \text{if } z < -\lambda. 
\end{cases}
\] (37)

The Replica MAP Claim now states that there exists effective noise levels \( \sigma^2_{\text{eff, map}} \) and \( \gamma_p \) such that for any component index \( j \), the random vector \( (x_j, sj, \hat{x}_j) \) converges in distribution to the vector \((x, s, \hat{x})\) where \( x \sim p_0(x) \), \( s \sim ps(s) \), and \( \hat{x} \) is given by
\[
\hat{x} = T^\text{soft}_{\text{map}}(z), \quad z = x + \sqrt{\lambda v},
\] (38)

where \( v \sim \mathcal{N}(0, 1) \), \( \lambda_p = \gamma_p/s \), and \( \mu = \sigma^2_{\text{eff, map}}/s \). Hence, the asymptotic behavior of lasso has a remarkably simple description: the asymptotic distribution of the lasso estimate \( \hat{x}_j \) of the component \( x_j \) is identical to \( x_j \) being corrupted by Gaussian noise and then soft-thresholded to yield the estimate \( \hat{x}_j \).

This soft-threshold description has an appealing interpretation. Consider the case when the measurement matrix \( A = I \). In this case, the lasso estimator (35) reduces to \( n \) scalar estimates,
\[
\hat{x}_j = T^\text{soft}_{\text{map}}(x_j + \sqrt{\mu_0 v_j}), \quad j = 1, 2, \ldots, n,
\] (39)

where \( v_j \sim \mathcal{N}(0, 1) \), \( \lambda = \gamma/s \), and \( \mu_0 = \sigma^2_0/s \). Comparing (38) and (39), we see that the asymptotic distribution of \((x_j, sj, \hat{x}_j)\) with large random \( A \) is identical to the distribution in the trivial case where \( A = I \), except that the noise levels \( \gamma \) and \( \sigma^2_0 \) are replaced by effective noise levels \( \gamma_p \) and \( \sigma^2_{\text{eff, map}} \).

To calculate the effective noise levels, one can perform a simple calculation to show that \( \sigma^2(z, \lambda) \) in (27) is given by
\[
\sigma^2(z, \lambda) = \begin{cases} 
  \lambda, & \text{if } |z| > \lambda; \\
  0, & \text{if } |z| \leq \lambda. 
\end{cases}
\] (40)

Hence,
\[
E \left[ s\sigma^2(z, \lambda_p) \right] = E \left[ s\lambda p \right] \Pr(|z| > \lambda_p) \\
= \gamma_p \Pr(|z| > \gamma_p/s),
\] (41)
where we have used the fact that $\lambda_p = \gamma_p/s$. Substituting (36) and (41) into (29), we obtain the fixed-point equations

$$\sigma_{\text{eff, map}}^2 = \sigma_0^2 + \beta \mathbb{E} \left[ s|x - T_{\lambda_p}^{\text{soft}}(z) |^2 \right]$$

$$\gamma_p = \gamma + \beta \gamma_p \Pr(|z| > \gamma_p/s),$$

where the expectations are taken with respect to $x \sim p_0(x)$, $s \sim p_S(s)$, and $z$ in (38). Again, while these fixed-point equations do not have a closed-form solution, they can be relatively easily solved numerically given distributions of $x$ and $s$.

C. Zero Norm-Regularized Estimation

Lasso can be regarded as a convex relaxation of zero norm-regularized estimation

$$\hat{x}^\text{zero}(y) = \arg \min_{x \in \mathbb{R}^n} \frac{1}{2\gamma}\|y - AS^{1/2}x\|_2^2 + \|x\|_0,$$

where $\|x\|_0$ is the number of nonzero components of $x$. For certain strictly sparse priors, zero norm-regularized estimation may provide better performance than lasso. While computing the zero norm-regularized estimate is generally very difficult, we can use the replica analysis to provide a simple characterization of its performance. This analysis can provide a bound on the achievable performance by practical algorithms.

To apply the Replica MAP Claim to the zero norm-regularized estimator (43), we observe that the zero norm-regularized estimator is identical to the MAP estimator (18) with the cost function

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } x \neq 0. \end{cases}$$

(44)

Technically, this cost function does not satisfy the conditions of the Replica MAP Claim. For one thing, without bounding the range of $x$, the bound (19) is not satisfied. Also, the minimum of (24) does not agree with the essential infimum. To avoid this problem, we can consider an approximation of (44),

$$f_{\delta,M}(x) = \begin{cases} 0, & \text{if } |x| < \delta; \\ 1, & \text{if } |x| \in [\delta, M], \end{cases}$$

which is defined on the set $\mathcal{X'} = \{x : |x| \leq M\}$. We can then take the limits $\delta \to 0$ and $M \to \infty$. For space considerations and to simplify the presentation, we will just apply the Replica MAP Claim with $f(x)$ in (44) and omit the details in taking the appropriate limits.

With $f(x)$ given by (44), the scalar MAP estimator in (24) is given by

$$\hat{x}_{\text{scalar}}^\text{map}(z; \lambda) = T_{\lambda_p}^{\text{hard}}(z), \quad t = \sqrt{2\lambda},$$

(45)

where $T_{\lambda_p}^{\text{hard}}$ is the hard thresholding operator,

$$T_{\lambda_p}^{\text{hard}}(z) = \begin{cases} z, & \text{if } |z| > \lambda_p; \\ 0, & \text{if } |z| \leq \lambda_p. \end{cases}$$

(46)

Now, similar to the case of lasso estimation, the Replica MAP Claim states there exists effective noise levels $\sigma_{\text{eff, map}}^2$ and $\gamma_p$ such that for any component index $j$, the random vector $(x_j, s_j, \hat{x}_j)$ converges in distribution to the vector $(x, s, \hat{x})$ where $x \sim p_0(x)$, $s \sim p_S(s)$, and $\hat{x}$ is given by

$$\hat{x} = T_t^{\text{hard}}(z), \quad z = x + \sqrt{\mu}v,$$

(47)

where $v \sim \mathcal{N}(0, 1)$, $\lambda_p = \gamma_p/s$, $\mu = \sigma_{\text{eff, map}}^2/s$, and

$$t = \sqrt{2\gamma_p} = \sqrt{2\lambda_p/s}.$$

(48)

Thus, the zero norm-regularized estimation of a vector $x$ is equivalent to $n$ scalar components corrupted by some effective noise level $\sigma_{\text{eff, map}}^2$ and hard-thresholded based on an effective noise level $\gamma_p$.

The fixed-point equations for the effective noise levels $\sigma_{\text{eff, map}}^2$ and $\gamma_p$ can be computed similarly to the case of lasso. Specifically, one can verify that (40) and (41) are both satisfied for the hard thresholding operator as well. Substituting (41) and (43) into (29), we obtain the fixed-point equations

$$\sigma_{\text{eff, map}}^2 = \sigma_0^2 + \beta \mathbb{E} \left[ s|x - T_{\lambda_p}^{\text{hard}}(z) |^2 \right],$$

$$\gamma_p = \gamma + \beta \gamma_p \Pr(|z| > \gamma_p),$$

(49a)

(49b)

where the expectations are taken with respect to $x \sim p_0(x)$, $s \sim p_S(s)$, $z$ in (47), and $t$ given by (48). These fixed-point equations can be solved numerically.

D. Optimal Regularization

The lasso estimator (35) and zero norm-regularized estimator (43) require the setting of a regularization parameter $\gamma$. Qualitatively, the parameter provides a mechanism to trade off the sparsity level of the estimate with the prediction error. One of the benefits of the replica analysis is that it provides a simple mechanism for optimizing the parameter level given the problem statistics.

Consider first the lasso estimator (35) with some $\beta > 0$ and distributions $x \sim p_0(x)$ and $s \sim p_S(s)$. Observe that there exists a solution to (42b) with $\gamma > 0$ if and only if

$$\Pr(|z| > \gamma_p/s) < 1/\beta.$$ 

(50)

This leads to a natural optimization: we consider an optimization over two variables $\sigma_{\text{eff, map}}^2$ and $\gamma_p$, where we minimize $\sigma_{\text{eff, map}}^2$ subject to (42a) and (50).

One simple procedure for performing this minimization is as follows: We start with $t = 0$ and some initial value of $\sigma_{\text{eff, map}}^2(0)$. For any iteration $t \geq 0$, we update $\sigma_{\text{eff, map}}^2(t)$ with the minimization

$$\sigma_{\text{eff, map}}^2(t + 1) = \sigma_0^2 + \beta \min_{\gamma_p} \mathbb{E} \left[ s|x - T_{\lambda_p}^{\text{soft}}(z) |^2 \right],$$

(51)

where, on the right-hand side, the expectation is taken over $x \sim p_0(x)$, $s \sim p_S(s)$, $z$ in (38), $\mu = \sigma_{\text{eff, map}}^2(t)/s$, and $\lambda_p = \gamma_p/s$. The minimization in (51) is over $\gamma_p > 0$ subject to (50).

One can show that with a sufficiently high initial condition, the sequence $\sigma_{\text{eff, map}}^2(t)$ monotonically decreases to a local minimum of the objective function. Given the final value for $\gamma_p$, one can then recover $\gamma$ from (42b). A similar procedure can be used for the zero norm-regularized estimator.
We simulated various estimators and compared their performances against the asymptotic values predicted by the replica analysis. For each value of $\beta$, we performed 1000 Monte Carlo trials of each estimator. For each trial, we measured the normalized squared error (SE) in dB

$$10 \log_{10} \left( \frac{\|\hat{x} - x\|^2}{\|x\|^2} \right),$$

where $\hat{x}$ is the estimate of $x$. The results are shown in Fig. 3 with each set of 1000 trials represented by the median normalized SE in dB.

The top curve shows the performance of the linear MMSE estimator (30). As discussed in Section V-A the Replica MAP Claim applied to the case of a constant scale matrix $S = I$ reduces to Verdú and Shamai’s result in [37]. As can be seen in Fig. 3, the result predicts the simulated performance of the linear estimator extremely well.

The next curve shows the lasso estimator (35) with the factor $\gamma$ selected to minimize the MSE as described in Section V-D. To compute the predicted value of the MSE from the Replica MAP Claim, we numerically solve the fixed-point equations (42) to obtain the effective noise levels $\sigma_{\text{eff, map}}^2$ and $\gamma_p$. We then use the scalar MAP model with the estimator (36) to predict the MSE. We see from Fig. 3 that the predicted MSE matches the median SE within 0.3 dB over a range of $\beta$ values. We believe that this level of accuracy in predicting lasso’s performance is not achievable with any other method.

Fig. 3 also shows the theoretical minimum MSE (computed by the Replica MMSE Claim) and the theoretical MSE from the zero norm-regularized estimator as computed in Section V-C. For these two cases, the estimators cannot be simulated since they involve NP-hard computations. But we have depicted the curve to show that the replica method can be used to calculate the gap between practical and impractical algorithms. Interestingly, we see that there is about a 2 to 2.5 dB gap between lasso and zero norm-regularized estimation, and another 1 to 2 dB gap between zero norm-regularized estimation and optimal MMSE.

It is, of course, not surprising that zero norm-regularized estimation performs better than lasso for the strictly sparse prior considered in this simulation, and that optimal MMSE performs better yet. However, what is valuable is that replica analysis can quantify the precise performance differences.

In Fig. 3, we plotted the median SE since there is actually considerable variation in the SE over the random realizations of the problem parameters. To illustrate the degree of variability, Fig. 4 shows the CDF of the SE values over the 1000 Monte Carlo trials. We see that there is large variation in the SE, especially at the smaller dimension $n = 100$. While the median value agrees well with the theoretical replica limit, any particular instance of the problem can vary considerably from that limit. This is perhaps the most significant drawback of the replica method: at lower dimensions, the replica method may provide accurate predictions of the median behavior, but does not bound the variations from the median.

As one might expect, at the higher dimension of $n = 500$, the level of variability is reduced and the observed SE begins to concentrate around the replica limit. In his original paper [8], Tanaka assumes that concentration of the SE will occur; he calls this the self-averaging assumption. Fig. 4 provides some empirical evidence that self-averaging does indeed occur. However, even at $n = 500$, the variation is not insignificant. As a result, caution should be exercised in using the replica predictions on particular low-dimensional instances.
valued random variable with probability mass function can analyze two different scenarios for the lasso estimator:

where \( s \) is a random power level and \( u \) is the post-power variation unknown. The parameter \( \rho \) represents the sparsity ratio and we chose a value of \( \rho = 0.1 \). The measurements were generated by

\[
y = Ax + w = AS^{1/2} u + w,
\]

where \( A \) is an i.i.d. Gaussian measurement matrix and \( w \) is Gaussian noise. As in the previous section, the post-despreading SNR with side-information was normalized to 10 dB.

The factor \( s_j \) in (52) accounts for power variations in \( x_j \). We considered two random distributions for \( s_j \): (a) \( s_j = 1 \), so that the power level is constant; and (b) \( s_j \) is uniform (in dB scale) over a 10 dB range with average unit power.

In case (b), when there is variation in the power levels, we can analyze two different scenarios for the lasso estimator:

- **Power variations unknown**: If the power level \( s_j \) in (52) is unknown to the estimator, then we can apply the standard lasso estimator:

\[
\hat{x}(y) = \arg\min_{x \in \mathbb{R}^n} \frac{1}{2\gamma} \|y - Ax\|_2^2 + \|x\|_1,
\]

which does not need knowledge of the power levels \( s_j \).

To analyze the behavior of this estimator with the replica method, we simply incorporate variations of both \( u_j \) and \( s_j \) into the prior of \( x_j \) and assume a constant scale factor \( s \) in the replica equations,

- **Power variations known**: If the power levels \( s_j \) are known, the estimator can compute

\[
\hat{u}(y) = \arg\min_{u \in \mathbb{R}^n} \frac{1}{2\gamma} \|y - AS^{1/2} u\|_2^2 + \|u\|_1
\]

and then \( \hat{x} = S^{1/2} \hat{u} \). This can be analyzed with the replica method by incorporating the distribution of \( s_j \) into the scale factors.

Fig. 5 shows the performance of the lasso estimator for the different power range scenarios. As before, for each \( \beta \), the figure plots the median SE over 1000 Monte Carlo simulation trials. Fig. 5 also shows the theoretical asymptotic performance as predicted by the Replica MAP Claim. Simulated values are based on a vector dimension of \( n = 100 \) and optimal selection of \( \gamma \) as described in Section V.D.

We see that in all three cases (constant power and power variations unknown and known to the estimator), the replica prediction is in excellent agreement with the simulated performance. With one exception, the replica method matches the simulated performance within 0.2 dB. The one exception is for \( \beta = 2.5 \) with constant power, where the replica method underpredicts the median MSE by about 1 dB. A simulation at a higher dimension of \( n = 500 \) (not shown here) reduced this discrepancy to 0.2 dB, suggesting that the replica method is still asymptotically correct.

We can also observe two interesting phenomena in Fig. 5.
First, the lasso method’s performance with constant power is almost identical to the performance with unknown power variations for values of \( \beta < 2 \). However, at higher values of \( \beta \), the power variations actually improve the performance of the lasso method, even though the average power is the same in both cases. Wainwright’s analysis [26] demonstrated the significance of the minimum component power in dictating lasso’s performance. The above simulation and the corresponding replica predictions suggest that dynamic range may also play a role in the performance of lasso. That increased dynamic range can improve the performance of sparse estimation has been observed for other estimators [38], [39].

A second phenomena we see in Fig. 5 is that knowing the power variations and incorporating them into the measurement matrix can actually degrade the performance of lasso. Indeed, knowing the power variations appears to result in a 1 to 2 dB loss in MSE performance.

Of course, one cannot conclude from this one simulation that these effects of dynamic range hold more generally. The study of the effect of dynamic range is interesting and beyond the scope of this work. The point is that the replica method provides a simple analytic method for quantifying the effect of dynamic range that appears to match actual performance well.

C. Support Recovery with Thresholding

In estimating vectors with strictly sparse priors, one important problem is to detect the locations of the nonzero components in the vector \( x \). This problem, sometimes called support recovery, arises for example in subset selection in linear regression [40], where finding the support of the vector \( x \) corresponds to determining a subset of features with strong linear influence on some observed data \( y \). Several works have attempted to find conditions under which the support of a sparse vector \( x \) can be fully detected [26], [33], [41] or partially detected [42]–[45]. Unfortunately, with the exception of [26], the only available results are bounds that are not tight.

One of the uses of the Replica MAP claim is to exactly predict the fraction of support that can be detected correctly. To see how to predict the support recovery performance, observe that the Replica MAP Claim provides the asymptotic joint distribution for the vector \((x_j, s_j, \hat{x}_j)\), where \( x_j \) is the component of the unknown vector, \( s_j \) is the corresponding scale factor and \( \hat{x}_j \) is the component estimate. Now, in support recovery, we want to estimate \( \theta_j \), the indicator function that \( x_j \) is non-zero

\[
\theta_j = \begin{cases} 
1, & \text{if } x_j \neq 0; \\
0, & \text{if } x_j \neq 0.
\end{cases}
\]

One natural estimate for \( \theta_j \) is to compare the magnitude of the component estimate \( \hat{x}_j \) to some scale-dependent threshold \( t(s_j) \).

\[
\hat{\theta}_j = \begin{cases} 
1, & \text{if } |\hat{x}_j| > t(s_j); \\
0, & \text{if } |\hat{x}_j| \leq t(s_j).
\end{cases}
\]

This idea of using thresholding for sparsity detection has been proposed in [32] and [46]. Using the joint distribution \((x_j, s_j, \hat{x}_j)\), one can then compute the probability of sparsity misdetection

\[
perr = Pr(\hat{\theta}_j \neq \theta_j).
\]

The probability of error can be minimized over the threshold levels \( t(s) \).

To verify this calculation, we generated random vectors \( x \) with \( n = 100 \) i.i.d. components given by (52) and (53). We used a constant power \( (s_j = 1) \) and a sparsity fraction of \( \rho = 0.2 \). As before, the observations \( y \) were generated with an i.i.d. Gaussian matrix with \( \text{SNR}_0 = 10 \text{ dB} \).

Fig. 6 compares the theoretical probability of sparsity misdetection predicted by the replica method against the actual probability of misdetection based on the average of 1000 Monte Carlo trials. We tested two algorithms: linear MMSE estimation and lasso estimation. For lasso, the regularization parameter was selected for minimum MMSE as described in Section V.D. The results show a good match.

VII. CONCLUSIONS AND FUTURE WORK

We have applied the replica method from statistical physics for computing the asymptotic performance of MAP estimation of non-Gaussian vectors with large random linear measurements. The method can be readily applied to problems in compressed sensing. While the method is not theoretically rigorous, simulations show an excellent ability to predict the performance for a range of algorithms, performance metrics, and input distributions. Indeed, we believe that the replica method provides the only method to date for asymptotically-exact prediction of performance of compressed sensing algorithms that can apply in a large range of circumstances.

Moreover, we believe that the availability of a simple scalar model that exactly characterizes certain sparse estimators opens up numerous avenues for analysis. For one thing, it
would be useful to see if the replica analysis of lasso can be used to recover the scaling laws of Wainwright [26] and Donoho and Tanner [27] for support recovery and to extend the latter to the noisy setting. Also, the best known bounds for MSE performance in sparse estimation are given by Haupt and Nowak [47] and Candès and Tao [48]. Since the replica analysis is asymptotically exact, we may be able to obtain much tighter analytic expressions. In a similar vein, several researchers have attempted to find information-theoretic lower bounds with optimal estimation [33, 41, 49]. Using the replica analysis of optimal estimators, one may be able to improve these scaling laws as well.

Finally, there is a well-understood connection between statistical mechanics and belief propagation-based decoding of error correcting codes [6], [7]. These connections may suggest improved iterative algorithms for sparse estimation as well.

**APPENDIX I**

**PROOF OVERVIEW**

Fix a deterministic sequence of indices \( j = j(n) \) with \( j(n) \in \{1, \ldots, n\} \). For each \( n \), define the random vector triples

\[
\theta^u(n) = (x_j(n), s_j(n), \hat{x}^u_j(n)), \quad \theta^\text{map}(n) = (x_j(n), s_j(n), \hat{x}^\text{map}_j(n)),
\]

where \( x_j(n), \hat{x}^u_j(n), \) and \( \hat{x}^\text{map}_j(n) \) are the \( j \)-th components of the random vectors \( \hat{x}^u(y) \), and \( \hat{x}^\text{map}(y) \), and \( s_j(n) \) is the \( j \)-th diagonal entry of the matrix \( S \).

For each \( u \), we will use the notation

\[
\hat{x}^\text{scalar}(z; \lambda) = \hat{x}^\text{mmse}_\text{scalar}(z; p_u, \lambda/u),
\]

where \( p_u \) is defined in (20) and \( \hat{x}^\text{mmse}(z; \cdot) \) is defined in (10). Also, for every \( \sigma \) and \( \gamma > 0 \), define the random vectors

\[
\theta^u_\text{scalar}(\sigma^2, \gamma) = (x, s, \hat{x}^\text{scalar}(z; \gamma/s)), \quad \theta^\text{map}_\text{scalar}(\sigma^2, \gamma) = (x, s, \hat{x}^\text{map}(z; \gamma/s)),
\]

where \( x \) and \( s \) are independent with \( x \sim p_0(x), s \sim p_S(s) \), and

\[
z = x + \frac{\sigma}{\sqrt{s}} u,
\]

with \( v \sim \mathcal{N}(0, 1) \).

Now, to prove the Replica MAP Claim, we need to show that (under the stated assumptions)

\[
\lim_{n \to \infty} \theta^\text{map}(n) = \theta^\text{map}_\text{scalar}(\sigma^2_{\text{eff, map}}, \gamma_p),
\]

where the limit is in distribution and the noise levels \( \sigma^2_{\text{eff, map}} \) and \( \gamma_p \) satisfy part (b) of the claim. This desired equivalence is depicted in the right column of Fig. 7.

To show this limit we first observe that under Assumption II for \( u \) sufficiently large, the postulated prior distribution \( p_u(x) \) in (20) and noise level \( \sigma^2_{\text{p}} \) in (21) are assumed to satisfy the Replica MMSE Claim. Satisfying the Replica MMSE Claim implies that

\[
\lim_{n \to \infty} (x_j(n), s_j(n), \hat{x}^u_j(n)) = (x, s, \hat{x}^\text{mmse}_\text{scalar}(z; p_u, \sigma^2_{\text{p-eff}}/u)/s),
\]

where the limit is in distribution, \( x \sim p_0(x), s \sim p_S(s), \) and

\[
z = x + \frac{\sigma_{\text{eff}}(u)}{\sqrt{s}} v, \quad v \sim \mathcal{N}(0, 1).
\]

Using the notation above, we can rewrite this limit as

\[
\lim_{n \to \infty} \theta^u(n) = \begin{align*}
\theta^\text{map}(n) &= (x_j(n), s_j(n), \hat{x}^u_j(n)) \\
\theta^\text{map}_\text{scalar}(\sigma^2_{\text{p-eff}}(u)/s) &= (x, s, \hat{x}^\text{scalar}(z; p_u, \sigma^2_{\text{p-eff}}(u)/s)) \\
\theta^\text{map}_\text{scalar}(\sigma^2_{\text{p-eff}}(u)/s) &= (x, s, \hat{x}^\text{map}_\text{scalar}(z; \gamma_p/u))
\end{align*}
\]

where all the limits are in distribution and (a) follows from the definition of \( \theta^u(n) \) in (56a); (b) follows from (61); (c) follows from (57) and (d) follows from (58a). This equivalence is depicted in the left column of Fig. 7.

The key part of the proof is to use a large deviations argument to show that for almost all \( y \)

\[
\lim_{u \to \infty} \hat{x}^u(y) = \hat{x}^\text{map}(y).
\]

This limit in turn shows (see Lemma 3 of Appendix III) that for every \( n \),

\[
\lim_{u \to \infty} \theta^u(n) = \theta^\text{map}(n)
\]

almost surely and in distribution. A large deviation argument is also used to show that for every \( \lambda \) and almost all \( z \),

\[
\lim_{u \to \infty} \hat{x}^\text{scalar}(z; \lambda) = \hat{x}^\text{scalar}_\text{scalar}(z; \lambda).
\]

Combining this with the limits in Assumption II we will see (see Lemma 7 of Appendix IV) that

\[
\lim_{u \to \infty} \theta^\text{scalar}(\sigma^2_{\text{eff}}(u), u\sigma^2_{\text{p-eff}}(u)) = \theta^\text{map}_\text{scalar}(\sigma^2_{\text{eff, map}}, \gamma_p)
\]

almost surely and in distribution.

The equivalences (63) and (64) are shown as rows in Fig. 7. As shown, they combine with the Replica MMSE Claim to prove the Replica MAP Claim. In equations instead of
diagrammatic form, the combination of limits is
\[
\lim_{n \to \infty} \theta_{\text{map}}(n) = \lim_{n \to \infty} \lim_{u \to \infty} \theta(u(n)) = \lim_{u \to \infty} \lim_{n \to \infty} \theta(u(n)) = \lim_{u \to \infty} \theta_{\text{map}}(u) = \theta_{\text{map}}(\sigma_{\text{eff}}^2(u), u\sigma_{\text{p-eff}}^2(u))
\]
where all the limits are in distribution (and (a) follows from (62); (b) follows from Assumption [3]; (c) follows from (62); and (d) follows from (64). This proves (60) and part (a) of the claim.

Therefore, to prove the claim we prove the limit (63) in Appendix [II] and the limit (64) in Appendix [V] and show that the limiting noise levels satisfy the fixed-point equations in part (b) of the claim in Appendix [V]. Before these results are given, we review in Appendix [II] some requisite results from large deviations theory.

**APPENDIX II**

**LARGE DEVIATIONS RESULTS**

We begin by reviewing some standard results from large deviations theory. The basic result we need is Laplace’s principle as described in [50].

**Lemma 1 (Laplace’s Principle):** Let \( \varphi(x) \) be any measurable function defined on some measurable subset \( D \subseteq \mathbb{R}^n \) such that
\[
\int_{x \in D} \exp(-\varphi(x)) \, dx < \infty.
\]
Then
\[
\lim_{u \to \infty} \frac{1}{u} \log \int_{x \in D} \exp(-u\varphi(x)) \, dx = -\inf_{x \in D} \varphi(x).
\]
Given \( \varphi(x) \) as in Lemma 1, define the probability distribution
\[
q_u(x) = \left[ \int_{x \in D} \exp(-u\varphi(x)) \, dx \right]^{-1} \exp(-u\varphi(x)).
\]
We want to evaluate expectations of the form
\[
\lim_{u \to \infty} \int_{x \in D} g(u,x)q_u(x) \, dx
\]
for some real-valued measurable function \( g(u,x) \). The following lemma shows that this integral is described by the behavior of \( g(u,x) \) in a neighborhood of the minimizer of \( \varphi(x) \).

**Lemma 2:** Suppose that \( \varphi(x) \) and \( g(u,x) \) are real-valued measurable functions satisfying:

(a) The function \( \varphi(x) \) satisfies (65) and has a unique essential minimizer \( \hat{x} \in \mathbb{R}^n \) such that for every open neighborhood \( U \) of \( \hat{x} \),
\[
\inf_{x \in U} \varphi(x) > \varphi(\hat{x}).
\]
(b) The function \( g(u,x) > 0 \) and satisfies
\[
\limsup_{u \to \infty} \sup_{x \in U} \frac{\log g(u,x)}{u(\varphi(x) - \varphi(\hat{x}))} \leq 0
\]
for every open neighborhood \( U \) of \( \hat{x} \).

(c) There exists a constant \( g_\infty \) such that for every \( \epsilon > 0 \), there exists a neighborhood \( U \) of \( \hat{x} \) such that
\[
\limsup_{u \to \infty} \left| \int_{u} g(u,x)q_u(x) \, dx - g_\infty \right| \leq \epsilon.
\]
Then,
\[
\lim_{u \to \infty} \int_{x \in U} g(u,x)q_u(x) \, dx = g_\infty.
\]

**Proof:** Due to item (c), we simply have to show that for any open neighborhood \( U \) of \( \hat{x} \),
\[
\limsup_{u \to \infty} \int_{x \in U} g(u,x)q_u(x) \, dx = 0.
\]
To this end, let
\[
Z(u) = \log \int_{x \in U} g(u,x)q_u(x) \, dx.
\]
We need to show that \( Z(u) \to -\infty \) as \( u \to \infty \). Using the definition of \( q_u(x) \) in (66), it is easy to check that
\[
Z(u) = Z_1(u) - Z_2(u),
\]
where
\[
Z_1(u) = \log \int_{x \in U} g(u,x) \exp\left(-u(\varphi(x) - \varphi(\hat{x}))\right) \, dx,
\]
\[
Z_2(u) = \log \int_{x \in \hat{x}} \exp\left(-u(\varphi(\hat{x}) - \varphi(x))\right) \, dx.
\]
Now,
\[
M = \inf_{x \in U} \varphi(x) - \varphi(\hat{x}).
\]
By item (a), \( M > 0 \). Therefore, we can find a \( \delta > 0 \) such that
\[
-M(1-\delta) + 3\delta < 0.
\]
Now, from item (b), there exists a \( u_0 \) such that for all \( u > u_0 \),
\[
Z_1(u) \leq \log \int_{x \in U} \exp\left(-u(1-\delta)(\varphi(x) - \varphi(\hat{x}))\right) \, dx.
\]
By Laplace’s principle, we can find a \( u_1 \) such that for all \( u > u_1 \),
\[
Z_1(u) \leq u \left[ \delta - \inf_{x \in U} (1-\delta)(\varphi(x) - \varphi(\hat{x})) \right] = u(M(1-\delta) + \delta).
\]
Also, since \( \hat{x} \) is an essential minimizer of \( \varphi(x) \),
\[
\inf_{x \in U} \varphi(x) = \varphi(\hat{x}).
\]
Therefore, by Laplace’s principle, there exists a \( u_2 \) such that for \( u > u_2 \),
\[
Z_2(u) \geq u \left[ -\delta - \inf_{x \in U} (\varphi(x) - \varphi(\hat{x})) \right] = -u\delta.
\]
Substituting (69) and (70) into (67) we see that for \( u \) sufficiently large,
\[
Z(u) \leq u(M(1-\delta) + \delta) + u\delta < -u\delta,
\]
where the last inequality follows from (68). This shows \( Z(u) \to -\infty \) as \( u \to \infty \) and the proof is complete.

One simple application of this lemma is as follows:
**Lemma 3**: Let $\varphi(x)$ and $h(x)$ be real-valued measurable functions such that the distribution $q_u(x)$ satisfies the following:

(a) The function $\varphi(x)$ has a unique essential minimizer $\hat{x}$ such that for every open neighborhood $U$ of $\hat{x}$,

$$\inf_{x \in U} \varphi(x) > \varphi(\hat{x}).$$

(b) The function $h(x)$ is continuous at $\hat{x}$.

(c) There exists a $c > 0$ such that for all $x \neq \hat{x}$,

$$\varphi(x) - \varphi(\hat{x}) \geq c \log |h(x) - h(\hat{x})|.$$

Then,

$$\lim_{u \to \infty} \int h(x) q_u(x) \, dx = h(\hat{x}).$$

**Proof**: We will apply Lemma 2 with $g(u, x) = |h(x) - h(\hat{x})|$ and $g_u = 0$. Item (a) of this lemma shows that $\varphi(x)$ satisfies item (a) in Lemma 2.

To verify that item (b) of Lemma 2 holds, observe that item (a) of Lemma 2 satisfies item (a) in Lemma 2.

Theorem 2 shows that item (b) holds. Let $\epsilon > 0$. Since $h(x)$ is continuous at $\hat{x}$, there exists an open neighborhood $U$ of $x$ such that $g(u, x) < \epsilon$ for all $x \in U$ and $u$. This implies that for all $u$,

$$\int_U g(u, x) q_u(x) \, dx < \epsilon \int_U q_u(x) \, dx \leq \epsilon,$$

which shows that $g(u, x)$ satisfies item (c) of Lemma 2. Thus,

$$\left| \int h(x) q_u(x) \, dx - h(\hat{x}) \right| = \left| \int (h(x) - h(\hat{x})) q_u(x) \, dx \right| \leq \int |h(x) - h(\hat{x})| q_u(x) \, dx \leq \int g(u, x) q_u(x) \, dx \to 0,$$

where the last limit is as $u \to \infty$ and follows from Lemma 2.

**Appendix III**

**Evaluation of $\lim_{u \to \infty} \hat{x}^u(y)$**

We can now apply Laplace’s principle in the previous section to prove (63). We begin by examining the pointwise convergence of the MMSE estimator $\hat{x}^u(y)$.

**Lemma 4**: For every $n$, $A$, and $S$ and almost all $y$,

$$\lim_{u \to \infty} \hat{x}^u(y) = \hat{x}_{\text{map}}(y),$$

where $\hat{x}^u(y)$ is the MMSE estimator in (23) and $\hat{x}_{\text{map}}(y)$ is the MAP estimator in (18).

**Proof**: The lemma is a direct application of Lemma 5. Fix $n$, $y$, $A$, and $S$ and let

$$\varphi(x) = \frac{1}{2\lambda} \|y - AS^{1/2}x\|^2 + f(x).$$

The definition of $\hat{x}_{\text{map}}(y)$ in (13) shows that

$$\hat{x}_{\text{map}}(y) = \arg \min_{x \in X^n} \varphi(x).$$

Assumption 4 shows that this minimizer is unique for almost all $y$. Also (23) shows that

$$p_{x|y}(x | y; p_u, \sigma_u^2) = \left[ \int_{x \in X^n} \exp(-u \varphi(x)) \, dx \right]^{-1} \exp(-u \varphi(x)) = q_u(x),$$

where $q_u(x)$ is given in (66) with $D = X^n$. Therefore, using (23),

$$\hat{x}^u(y) = E(x | y; p_u, \sigma_u^2) = \int_{x \in X^n} x q_u(x) \, dx. \quad (72)$$

Now, to prove the lemma, we need to show that

$$\lim_{u \to \infty} \hat{x}^u_j(y) = \hat{x}_{j, \text{map}}^u(y)$$

for every component $j = 1, \ldots, n$. To this end, fix a component index $j$. Using (72), we can write the $j$th component of $\hat{x}^u(y)$ as

$$\hat{x}^u_j(y) = \int_{x \in X^n} h(x) q_u(x) \, dx,$$

where $h(x) = x_j$. The function $h(x)$ is continuous. Also, using Assumption 5 it is straightforward to show that item (c) of Lemma 5 is satisfied for some $c > 0$. Thus, the hypotheses of Lemma 5 are satisfied and we have the limit

$$\lim_{u \to \infty} \hat{x}^u_j(y) = h(\hat{x}_{\text{map}}(y)) = \hat{x}_{j, \text{map}}(y).$$

This proves the lemma.

**Lemma 5**: Consider the random vectors $\theta^n(n)$ and $\theta_{\text{map}}^u(n)$ be defined in (56a) and (66), respectively. Then, for all $n$,

$$\lim_{u \to \infty} \theta^n(n) = \theta_{\text{map}}^u(n) \quad (73)$$

almost surely and in distribution.

**Proof**: The vectors $\theta^n(n)$ and $\theta_{\text{map}}^u(n)$ are deterministic functions of $x(n)$, $A(n)$, $S(n)$, and $y$. Lemma 4 shows that the limit (73) holds for any values of $x(n)$, $A(n)$, and $S(n)$, and almost all $y$. Since $y$ has a continuous probability distribution (due to the additive noise $w$ in (3)), the set of values where this limit does not hold must have probability zero. Thus, the limit (73) holds almost surely, and therefore, also in distribution.
**Lemma 6:** Consider the scalar estimators $\hat{x}_{\text{scalar}}^u(z ; \lambda)$ defined in (57) and $\hat{x}_{\text{map}}(z ; \lambda)$ defined in (24). For all $\lambda > 0$ and almost all $z$, we have the deterministic limit
\[ \lim_{u \to \infty} \hat{x}_{\text{scalar}}^u(z ; \lambda) = \hat{x}_{\text{map}}(z ; \lambda). \]

**Proof:** The proof is similar to that of Lemma 4. Fix $z$ and $\lambda$ and consider the conditional distribution $p_{x|z}(x | z ; p_u, \lambda/u)$. Using (7) along with the definition of $p_u(x)$ in (20) and an argument similar to the proof of Lemma 4, it is easily checked that
\[ p_{x|z}(x | z ; p_u, \lambda/u) = q_u(x), \]
where $q_u(x)$ is given by (66) with $\mathcal{D} = \mathcal{X}$ and
\[ \varphi(x) = F(x, z, \lambda), \]
where $F(x, z, \lambda)$ is defined in (25). Using (57) and (10),
\[ \hat{x}_{\text{scalar}}^u(z ; \lambda) = \hat{x}_{\text{scalar}}^{\text{mmse}}(z ; p_u, \lambda/u) = \int_{x \in \mathcal{X}} x p_{x|z}(x | z ; p_u, \lambda/u) \, dx = \int_{x \in \mathcal{X}} h(x) q_u(x) \, dx, \]
with $h(x) = x$.

We can now apply Lemma 3. The definition of $\hat{x}_{\text{map}}(z ; \lambda)$ in (24) shows that
\[ \hat{x}_{\text{map}}(z ; \lambda) = \arg \min_{x \in \mathcal{X}} \varphi(x). \]  
(76)

Assumption 6 shows that for all $\lambda > 0$ and almost all $z$, this minimization is unique so
\[ \varphi(x) > \varphi(\hat{x}_{\text{map}}(z ; \lambda)), \]
for all $x \neq \hat{x}_{\text{map}}(z ; \lambda)$. Also, using (25),
\[ \lim_{|x| \to \infty} \varphi(x) \stackrel{(a)}{=} \lim_{|x| \to \infty} F(x, z, \lambda) \]
\[ \stackrel{(b)}{=} \lim_{|x| \to \infty} f(x) \stackrel{(c)}{=} \infty \]  
(77)
where (a) follows from (75); (b) follows from (25); and (c) follows from Assumption 5. Equations (76) and (77) show that item (a) of Lemma 3 is satisfied. Item (b) of Lemma 3 is also clearly satisfied since $h(x) = x$ is continuous.

Also, using Assumption 5, it is straightforward to show that item (c) of Lemma 3 is satisfied for some $c > 0$. Thus, all the hypotheses of Lemma 3 are satisfied and we have the limit
\[ \lim_{u \to \infty} \hat{x}_{\text{scalar}}^u(z ; \lambda) = h(\hat{x}_{\text{map}}(z ; \lambda)) = \hat{x}_{\text{map}}(z ; \lambda). \]

This proves the lemma.

We now turn to convergence of the random variable $\theta^u_{\text{scalar}}(\sigma_{\text{eff}}^2(u), u \sigma_{\text{eff}}^2(u))$.

**Lemma 7:** Consider the random vectors $\theta_{\text{scalar}}^u(\sigma^2, \gamma)$ defined in (58a) and $\theta_{\text{map}}(\sigma^2, \gamma)$ in (58b). Let $\sigma_{\text{eff}}^2(u)$, $\sigma_{\text{eff}}^2(u)$, $\sigma_{\text{eff, map}}^2$, and $\gamma_p$ be as defined in Assumption 2. Then the following limit holds:
\[ \lim_{u \to \infty} \theta_{\text{scalar}}^u(\sigma_{\text{eff}}^2(u), u \sigma_{\text{eff}}^2(u)) = \theta_{\text{map}}(\sigma_{\text{eff, map}}^2, \gamma_p). \]  
(78)

almost surely and in distribution.

**Proof:** The proof is similar to that of Lemma 5. For any $\sigma^2$ and $\gamma > 0$, the vectors $\theta_{\text{scalar}}^u(\sigma^2, \gamma)$ and $\theta_{\text{map}}(\sigma^2, \gamma)$ are deterministic functions of the random variables $x \sim p_0(x)$, $s \sim p_S(s)$, and $z$ given (59) with $v \sim N(0, 1)$. Lemma 6 shows that the limit
\[ \lim_{u \to \infty} \theta_{\text{scalar}}^u(\sigma^2, \gamma) = \theta_{\text{map}}(\sigma^2, \gamma) \]  
(79)
holds for any values of $\sigma^2$, $\gamma$, $x$, and $s$ and almost all $z$. Also, if we fix $x$, $s$, and $v$, by Assumption 6 the function
\[ \hat{x}_{\text{scalar}}^u(z ; \gamma/s) = \hat{x}_{\text{map}}(x + \sigma \sqrt{s}, \gamma/s) \]
is continuous in $\gamma$ and $\sigma^2$ for almost all values of $v$. Therefore, we can combine (79) with the limits in Assumption 2 to show that
\[ \lim_{u \to \infty} \theta_{\text{scalar}}^u(\sigma_{\text{eff}}^2(u), u \sigma_{\text{eff}}^2(u)) = \theta_{\text{map}}(\sigma_{\text{eff, map}}^2, \gamma_p) \]  
for almost all $x$ and $s$ and almost all $z$. Since $z$ has a continuous probability distribution (due to the additive noise $v$ in (59)), the set of values where this limit does not hold must have probability zero. Thus, the limit (78) holds almost surely, and therefore, also in distribution.

**Appendix V

Proof of the Fixed-Point Equations**

For the final part of the proof, we need to show that the limits $\sigma_{\text{eff, map}}^2$ and $\gamma_p$ in Assumption 2 satisfy the fixed-point equations (29). The proof is straightforward, but we just need to keep track of the notation properly. We begin with the following limit.

**Lemma 8:** The following limit holds:
\[ \lim_{u \to \infty} E[s \text{mse}(p_u, p_0, \mu_p, \mu^u, z^u)] \]
\[ = \ E[s|x - \hat{x}_{\text{map}}^u(z ; \lambda)|^2], \]
where the expectations are taken over $x \sim p_0(x)$ and $s \sim p_S(s)$, and $z$ and $z^u$ are the random variables
\[ z^u = x + \sqrt{\mu^u} v, \]
\[ z = x + \sqrt{\mu} v, \]
with $v \sim N(0, 1)$ and $\mu^u = \sigma_{\text{eff}}^2(u)/s$, $\mu_p = \sigma_{\text{eff}}^2(u)/s$, $\mu = \sigma_{\text{eff, map}}^2/s$, and $\lambda = \gamma_p/s$.

**Proof:** Using the definitions of mse in (11) and $\hat{x}_{\text{scalar}}^u(z ; \lambda)$ in (57),
\[ \text{mse}(p_u, p_0, \mu_p, \mu^u, z^u) \]
\[ = \int_{x \in \mathcal{X}} |x - \hat{x}_{\text{scalar}}^u(z^u ; p_u, \mu^u)|^2 p_{x|z}(x | z^u ; p_0, \mu^u) \, dx \]
\[ = \int_{x \in \mathcal{X}} |x - \hat{x}_{\text{scalar}}^u(z^u ; \mu_p/u)|^2 p_{x|z}(x | z^u ; p_0, \mu^u) \, dx. \]
Therefore, fixing $s$ (and hence $\mu_p^u$ and $\mu^u$), we obtain the conditional expectation
\[ E[s \text{mse}(p_u, p_0, \mu_p, \mu^u, z^u) | s] \]
\[ = E[s|x - \hat{x}_{\text{scalar}}^u(z^u ; \mu_p/u)|^2 | s], \]  
(81)
where the expectation on the right is over $x \sim p_0(x)$ and $z^u$ given by (80a).

Also, observe that the definitions $\mu_0 = \sigma^2_{\text{eff}}(u)/s$ and $\mu = \sigma^2_{\text{eff, map}}/s$ and along with the limit in Assumption 2 show that

$$\lim_{u \to \infty} \mu_0 = \mu. \quad (82)$$

Similarly, since $\mu_0^u = \sigma^2_{\text{eff}}(u)/s$ and $\lambda = \gamma_0/s$, Assumption 2 shows that

$$\lim_{u \to \infty} \mu_0^u = \lambda. \quad (83)$$

Taking the limit as $u \to \infty$,

$$\lim_{u \to \infty} \mathbb{E} \left[ s \text{mse}(p_0, p_0, \mu_0^u, \mu_0^u, z^u) \right]$$

where (a) follows from (81); (b) follows from (83); (c) follows from Lemma 6.

The previous lemma enables us to evaluate the limit of the MSE in (29a). To evaluate the limit of the MSE in (29a), we need the following lemma.

**Lemma 9:** Fix $z$ and $\lambda$, and let

$$g(u, x) = |x|/x|\hat{x}^2|, \quad \hat{x} = \hat{z}\text{scalar}(z; \lambda). \quad (84)$$

Also, let $\varphi(x)$ be given by (75) and $q_0(x)$ be given by (66) with $D = \mathcal{X}$. Then, for any $\epsilon > 0$, there exists an open neighborhood $U \subseteq \mathcal{X}$ of $\hat{x}$ such that

$$\lim_{u \to \infty} \sup_{x \in U} \left| \int_{x \in U} g(u, x)q_0(x) \, dx - \delta(z, \lambda) \right| < \epsilon,$$

where $\delta(z, \lambda)$ is given in Assumption 4.

**Proof:** The proof is straightforward but somewhat tedious. We will just outline the main steps. Let $\delta > 0$. Using Assumption 5, one can find an open neighborhood $U \subseteq \mathcal{X}$ of $\hat{x}$ such that for all $x \in U$ and $u > 0$,

$$\phi(x, \delta_2(u)) \leq \exp(-u(\varphi(x) - \varphi(\hat{x}))) \leq \phi(x, \delta_2(u)),$$

where $\phi(x, \delta_2)$ is the unnormalized Gaussian distribution

$$\phi(x, \delta_2) = \exp\left(-\frac{1}{2\delta_2^2}|x - \hat{x}|^2\right)$$

and

$$\delta_2^2(u) = (1 - \delta)^2(z, \lambda)/u, \quad \delta_2^2(u) = (1 - \delta)^2(z, \lambda)/u.$$ COMBINING THE BOUNDS IN (85) WITH THE DEFINITION OF $q_0$ IN (66) AND THE FACT THAT $U \subseteq \mathcal{X}$ SHOWS THAT FOR ALL $x \in U$ AND $u > 0$, $q_0(x) = \left[ \int_{x \in \mathcal{X}} e^{-u\varphi(x)} \, dx \right]^{-1} e^{-u\varphi(x)} \leq \left[ \int_{x \in \mathcal{X}} \phi(x, \delta_2(u)) \, dx \right]^{-1} \phi(x, \delta_2^2(u)).$

Therefore,

$$\int_{x \in U} g(u, x)q_0(x) \, dx = \int_{x \in U} |x|/x|\hat{x}|^2q_0(x) \, dx$$

$$\leq \left[ \int_{x \in U} \phi(x, \delta_2^2(u)) \, dx \right]^{-1} \phi(x, \delta_2^2(u)). \quad (86)$$

Now, it can be verified that

$$\lim_{u \to \infty} \int_{x \in U} u^{1/2}\phi(x, \delta_2^2(u)) \, dx = \sqrt{2\pi(1 - \delta)}\sigma(z, \lambda) \quad (87)$$

and

$$\lim_{u \to \infty} \int_{x \in U} u^{3/2}|x - \hat{x}|^2\phi(x, \delta_2^2(u)) \, dx = \sqrt{2\pi(1 - \delta)}\sigma(z, \lambda). \quad (88)$$

Substituting (87) and (88) into (86) shows that

$$\lim_{u \to \infty} \sup_{x \in U} \int_{x \in U} u^{1/2}\phi(x, \delta_2^2(u)) \, dx \leq \left( \frac{1 + \delta)^{3/2}}{1 - \delta - \delta^2(z, \lambda)} \quad (89)$$

Therefore, with appropriate selection of $\delta$, one can find a neighborhood $U$ of $\hat{x}$ such that

$$\limsup_{u \to \infty} \left| \int_{x \in \mathcal{X}} g(u, x)q_0(x) \, dx - \delta(z, \lambda) \right| < \epsilon,$$

and this proves the lemma.

**Using the above result, we can evaluate the scalar MSE.**

**Lemma 10:** Using the notation of Lemma 8,

$$\lim_{u \to \infty} \mathbb{E} \left[ \mu \text{mse}(p_0, p_0, \mu_0^u, \mu_0^u, z) \right] = \mathbb{E} \left[ \sigma^2(z, \gamma_0/s) \right].$$

**Proof:** This is an application of Lemma 2. Fix $z$ and $\lambda$ and define $g(u, x)$ as in (34). As in the proof of Lemma 6, the conditional distribution $p_{\text{ex}}(x | z; p_0, \lambda/s)$ is given by (74) with $\varphi(x)$ given by (75). The definition of $\hat{z}\text{scalar}(z; \lambda)$ in (24) shows that $\hat{z}\text{scalar}(z; \lambda)$ minimizes $\varphi(x)$. Similar to the proof of Lemma 6, one can show that items (a) and (b) of Lemma 2 are satisfied. Also, Lemma 9 shows that item (c) of Lemma 2 holds with $g_0 = \sigma^2(z, \lambda)$. Therefore, all the hypotheses of Lemma 2 are satisfied and

$$\lim_{u \to \infty} \int_{x \in \mathcal{X}} u|x - \hat{z}\text{scalar}(z; \lambda)|^2q_0(x) \, dx = \sigma^2(z, \lambda), \quad (90)$$

for all $\lambda$ and almost all $z$.

Now

$$\mu \text{mse}(p_0, p_0, \lambda/s, \lambda/s, u)$$

$$\leq \int_{x \in \mathcal{X}} |x - \hat{z}\text{mse}(z; p_0, \lambda/s)|^2q_0(x) \, dx$$

$$\leq \int_{x \in \mathcal{X}} |x - \hat{z}\text{mse}(z; p_0, \lambda/s)|^2q_0(x) \, dx$$

$$\leq \int_{x \in \mathcal{X}} |x - \hat{z}\text{mse}(z; \lambda/s)|^2q_0(x) \, dx,$$
where (a) is the definition of mse in (11); (b) follows from (73); and (c) follows from (57). Taking the limit of this expression

\[
\begin{align*}
\lim_{u \to \infty} u \text{mse}(p_u, p_u, \lambda/u, \lambda/u, z) &\overset{(a)}{=} \lim_{u \to \infty} \int_{x \in \mathcal{X}} u|x - \hat{z}^{\text{scalar}}(z; \lambda)|^2 q_u(x) \, dx \\
&\overset{(b)}{=} \lim_{u \to \infty} \int_{x \in \mathcal{X}} u|x - \hat{z}^{\text{map}}(z; \lambda)|^2 q_u(x) \, dx \\
&\overset{(c)}{=} \sigma^2(z, \lambda),
\end{align*}
\]

(91)

where (a) follows from (90); (b) follows from Lemma 6 and (c) follows from (89).

The variables \( z^u \) and \( z \) in (80a) and (80b) as well as \( \mu^u \) and \( \mu_p^u \) are deterministic functions of \( x, v, s, \) and \( u \). Fixing \( x, v, \) and \( s \) and taking the limit with respect to \( u \) we obtain the deterministic limit

\[
\begin{align*}
\lim_{u \to \infty} u \text{mse}(p_u, p_u, \mu_p^u, \mu_p^u, z^u) &\overset{(a)}{=} \lim_{u \to \infty} u \text{mse}(p_u, p_u, \mu_p^u, \mu_p^u, z^u) \\
&\overset{(b)}{=} \lim_{u \to \infty} \sigma^2(z^u, u\sigma^2_p-\text{eff}(u)/s, z^u) \\
&\overset{(c)}{=} \lim_{u \to \infty} \sigma^2(z, u\sigma^2_p-\text{eff}(u)/s) \\
&\overset{(d)}{=} \sigma^2(z, \gamma_p/s),
\end{align*}
\]

(92)

where (a) follows from the definitions of \( \mu^u \) and \( \mu_p^u \) in Lemma 8; (b) follows from (91); (c) follows from the limit (proved in Lemma 8) that \( z^u \to z \) as \( u \to \infty \); and (d) follows from the limit in Assumption 2.

Finally, observe that for any prior \( p \) and noise level \( \mu \),

\[
\text{mse}(p, p, \mu, \mu, z) \leq \mu,
\]

since the MSE error must be smaller than the additive noise level \( \mu \). Therefore, for any \( u \) and \( z \),

\[
\begin{align*}
\text{mse}(p_u, p_u, \mu^u_p, \mu^u_p, z^u) &\leq u\mu^u_p = u\sigma^2_p(u),
\end{align*}
\]

where we have used the definition \( \mu^u_p = \sigma^2_p(u)/s \). Since \( u\sigma^2_p(u) \) converges, there must exists a constant \( M > 0 \) such that

\[
\begin{align*}
\text{mse}(p_u, p_u, \mu^u_p, \mu^u_p, z^u) &\leq u\mu^u_p \leq M,
\end{align*}
\]

for all \( u, s \) and \( z^u \). The lemma now follows from applying the Dominated Convergence Theorem and taking expectations of both sides of (92).

We can now show that the limiting noise values satisfy the fixed-point equations.

**Lemma 11:** The limiting effective noise levels \( \sigma^2_{\text{eff, map}} \) and \( \gamma_p \) in Assumption 2 satisfy the fixed-point equations (29a) and (29b).

**Proof:** The noise levels \( \sigma^2_{\text{eff}}(u) \) and \( \sigma^2_{p-\text{eff}}(u) \) satisfy the fixed-point equations (13a) and (13b) of the Replica MMSE Claim with the postulated prior \( p_{\text{post}} = p_u \) and noise level \( \sigma^2_{\text{post}} = \gamma/u \). Therefore, using the notation in Lemma 8

\[
\begin{align*}
\sigma^2_{\text{eff}}(u) &\overset{(a)}{=} \sigma^2_0 + \beta \mathbb{E} \left[ s \text{mse}(p_u, p_0, \mu^u_p, \mu^u_p, z^u) \right] \text{(93a)} \\
\sigma^2_{p-\text{eff}}(u) &\overset{(b)}{=} \gamma + \beta \mathbb{E} \left[ u \text{mse}(p_u, p_u, \mu^u_p, \mu^u_p, z^u) \right] \text{(93b)}
\end{align*}
\]

where (as defined in Lemma 8), \( \mu^u = \sigma^2_{\text{eff}}(u)/s \) and \( \mu^u_p = \sigma^2_{p-\text{eff}}(u)/s \) and the expectations are taken over \( s \sim p_0(s) \), \( x \sim p_0(x) \), and \( z^u \) in (80a).

Therefore,

\[
\begin{align*}
\sigma^2_{\text{eff, map}} &\overset{(a)}{=} \lim_{u \to \infty} \sigma^2_{\text{eff}}(u) \\
&\overset{(b)}{=} \sigma^2_0 + \beta \mathbb{E} \left[ s \text{mse}(p_u, p_0, \mu^u_p, \mu^u_p, z^u) \right] \\
&\overset{(c)}{=} \sigma^2_0 + \beta \mathbb{E} \left[ s|x - \hat{z}^{\text{map}}(z; \lambda)|^2 \right]
\end{align*}
\]

where (a) follows from the limit in Assumption 2; (b) follows from (93a); and (c) follows from Lemma 8. This shows that (29a) is satisfied.

Similarly,

\[
\begin{align*}
\gamma_p &\overset{(a)}{=} \lim_{u \to \infty} u \sigma^2_{p-\text{eff}}(u) \\
&\overset{(b)}{=} \gamma + \beta \mathbb{E} \left[ s \text{mse}(p_u, p_u, \mu^u_p, \mu^u_p, z^u) \right] \\
&\overset{(c)}{=} \gamma + \beta \mathbb{E} \left[ s \sigma^2(z, \lambda_p) \right]
\end{align*}
\]

where (a) follows from the limit in Assumption 2; (b) follows from (93b); and (c) follows from Lemma 10. This shows that (29b) is satisfied.

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