Renormalization of the Majorana bound state
decay length in a perpendicular magnetic field

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Orbital effects of a magnetic field in a proximitized semiconductor nanowire are studied in the
context of the spatial extent of Majorana bound states. We develop analytical model that explains
the impact of concurring effects of paramagnetic coupling of the nanowire bands via the kinetic
energy operator and spin-orbit interaction on the Majorana modes. We find, that the perpendicular
field, so far considered as to be detrimental to the Majorana fermion formation, is in fact helpful in
establishing the topological zero-energy-modes in a finite system due to significant decrease in the
Majorana decay length.

I. INTRODUCTION

Topological superconductivity created by a combination
of the spin-orbit (SO) coupling and the Zeeman effect
in a low-dimensional semiconductor-superconductor
heterostructure is foreseen as prospective in the quantu-
computation that exploits manipulation of Major-
ana bound states (MBSs). The elementary braiding
operation of Majoranas requires at least a three terminal
junction. In the pursuit of topological quantum gates
creation multiterminal networks of MBSs are real-
ized experimentally in a form of nanowire crosses and
hashtags. In those structures orientation of the mag-
tic field perpendicular to the substrate is preferable
as it allows to induce the topological phase in all the
nanowire branches owing to perpendicular orientation of
the field with respect to the SO coupling direction.

In previous experimental research, large in-plane, type-II
superconducting electrodes were sputtered after the
nanowire deposition on the substrate preventing
application of the perpendicular magnetic field due to
the vortex formation in the superconductor slab. This
obstacle has been recently overcome by the realization of
2DEG and nanowire hybrid devices in a bottom-up
synthesis with a thin Aluminum shell. This guarantees
a pristine interface between superconductor and semi-
conductor materials ensuring the hard superconducting
gap and allows for observation of a long-predicted
current doubling in a proximitized quantum point
contact or $2e^2/h$ quantization of Majorana modes.

Most importantly, the exploitation of the thin supercon-
ducting shell enables arbitrary alignment of the magnetic
field without destroying the superconductivity and by
that opens a wide perspective for the topological quan-
tum gate creation.

Studies that took into account effects of the magnetic
field beyond sole-Zeeman-interaction focused on the im-
pact of the field orientation on the phase boundary and
painted somewhat discouraging picture of topologi-
cal phase instability due to the gap closing in tilted
magnetic field. It has been shown however that this is
rather an effect of numerical treatment of the vector
potential, and in fact the topological phase can still ex-
ist in perpendicular orientation of the field. In this work,
exploiting analytical treatment of the orbital effects of
the perpendicular magnetic field we address problem of
the real-space extent of Majorana modes, crucial in the
light of topological protection of MBSs in a finite-system
quantum gates.

Very recently O. Dmytruk and J. Klinovaja pointed
out that the diamagnetic effect of the magnetic field or-
iented along the radially symmetric nanowire acts as an
effective chemical potential that reduces the energy os-
cillations of the overlapping MBSs. This is true only for
the parallel magnetic field orientation when the param-
agnetic effect does not couple the orbital and spin de-
grees of freedom. As we show, in general case when the
magnetic field has a component perpendicular to the
nanowire axis or, as in our case, it is simply perpendicu-
lar to the nanowire, the paramagnetic coupling of modes
with different orbital excitation and opposite spin renor-
malizes not only the chemical potential but also the effec-
tive mass and the SO coupling constant. As a result both
the diamagnetic and paramagnetic terms in the kinetic
energy operator contribute to the decrease of the spatial
extent of the Majorana modes. On the other hand the
magnetic field entering through momentum operator in
the SO coupling Hamiltonian has distinct and detrimen-
tal effects on the reduction of the decay length, which
is crucial for current generation of devices where SO in-
teraction is particularly strong. We find surprising
equalization of the above mentioned effects that results
in the decay length comparable, but still less than in the
case without the orbital effects of the magnetic field.
II. THEORETICAL MODEL

We start by writing down Bogoliubov-de Gennes Hamiltonian for a proximitized semiconductor nanowire, 

\[ H = \left( \frac{\hbar^2 k^2}{2m^*} - \mu \right) \sigma_0 \tau_z + \Delta \sigma_0 \tau_x + \alpha (\sigma_x k_y - \sigma_y k_z) \sigma_0 \tau_y + \frac{eB}{\hbar} \sigma_z \tau_0. \]  

(1)

where \( \alpha \) is the Rashba SO coupling constant, \( E_z = \frac{g}{2} \mu_B B \) is the Zeeman term, \( B \) is the external magnetic field oriented in the \( z \)-direction, perpendicular to the nanowire plane and \( \Delta \) is the effective induced pair-potential in Lorentz gauge acting on spin- and particle-hole degrees of freedom, respectively. The uniform \( \Delta \) in Eq. (1) corresponds to a system in the weak-coupling regime, where the superconductor-semiconductor interface is non-transparent or to the presence of low-energy modes with minimum in the interface transparency. The orbital effects of the magnetic field are included through the canonical momentum, \( k = -i \nabla + eA/\hbar \cdot \tau_z \) with the vector-potential in Lorentz gauge \( A = [-yB, 0, 0] \).

Figure 1. a) Two dimensional crosssection of the proximitized semiconductor nanowire subjected to the magnetic field \( B \) perpendicular to the nanowire axis. The colormap shows an exemplary density distribution of the Majorana bound state. b) Dispersion relations \( E(k) \) obtained in the model of Eq. (1) without the orbital effects (solid black), with the kinetic-paramagnetic term (green dashed) and with the diamagnetic term (red dashed). Their joint effect is presented in panel c) with the violet curve. In c) the blue dashed curve corresponds to the results with the canonical momentum included also in the SO coupling Hamiltonian (SO-paramagnetic term). The black crosses are the results obtained in numerical calculations. Results obtained for \( \mu = 3 \) meV, \( \alpha = 50 \) meVnm, and \( B = 0.7 \) T.

It has been demonstrated that the most convenient situation for realization of MBSs is when a single band of transverse quantization is occupied. We therefore focus on the low electron density regime where the system is tuned such the phase transition occurs due to a single band crossing the Fermi level. Since we are specifically interested in the orbital effects of the magnetic field, for simplicity and without loss of generality, we decouple the spatial degree of freedom collinear with the magnetic field \( z \)-direction and consider the Hamiltonian \( \hat{H} \) in the \( x-y \) plane. Figure 1a) shows the schematic illustration of the considered nanowire oriented along \( x \) direction with the width \( W \) along \( y \) (transverse) direction.

In contrary to the diamagnetic term \( \sim y^2 B^2 \) which solely shifts energies of the electronic states, the paramagnetic effect \( \sim k_x y B \) originating from both the kinetic energy (kinetic-paramagnetic term) and the SO interaction (SO-paramagnetic term), couples the orbital degrees of freedom in the \( y \) direction via the wave vector \( k_x \). For Majorana fermions in the ground state of transverse quantization, the most significant effect comes from the coupling with the first excited state. For this purpose we write the Hamiltonian \( \hat{H} \), in the basis of two lowest eigenstates of infinite quantum well of width \( W \), centered at \( y = 0 \), \( \Psi_n(y) = \sqrt{2/W} \sin[n\pi(y + W/2)/W] \). We obtain

\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \]

(2)

with

\[ H_{11(22)} = H_{1D} + (E_{1(2)} + E_{\text{dia}}) \sigma_0 \tau_z, \]

(3)

\[ H_{12} = H_{21} = \varepsilon_p k_x \sigma_0 \tau_0 + E_p^{SO} \sigma_y \tau_0 + E_{\tau_z}^{SO} \sigma_z \tau_z, \]

(4)

where \( E_n = n^2 \pi^2 \hbar^2 / 2mW^2 \) is the energy of orbital excitation in the \( y \)-direction, \( E_{\text{dia}} = \langle \Psi_n | y^2 | \Psi_n \rangle \epsilon B^2 / 2m \) is the diamagnetic term in the \( n \)-th subband \( \langle \Psi_1 | y^2 | \Psi_1 \rangle = (\pi^2 - 6)W^2 / 12\pi^2 \) and \( \langle \Psi_2 | y^2 | \Psi_2 \rangle = (2\pi^2 - 3)W^2 / 24\pi^2 \) that acts as the extra chemical potential being different for each band. The parameter \( \varepsilon_p = -\langle \Psi_1 | y | \Psi_2 \rangle \epsilon B^2 / m = 16W \epsilon B^2 / 9m \pi^2 \) results from the kinetic-paramagnetic effect that due to parity of the transverse modes is non-zero only in the off-diagonal submatrices \( (H_{12}, H_{21}) \). It mixes the transverse modes with magnitude proportional to the wave-vector \( k_x \). In Eq. (4), \( E_p^{SO} = \langle \Psi_1 | y | \Psi_2 \rangle \alpha eB / \hbar = -16W \alpha eB / 9m \pi^2 \) corresponds to the substitution of the canonical momentum in the SO Hamiltonian that accounts for mixing bands with different transverse excitation and spin along the \( z \)-direction. \( E_{\tau_z}^{SO} = i \alpha \langle \Psi_1 | \partial / \partial y | \Psi_2 \rangle / 3W \) is a correction due to the transverse part of the Rashba Hamiltonian. In Eq. (4) \( H_{1D} \) is the one-dimensional Hamiltonian along the wire axis

\[ H_{1D} = \left( \frac{\hbar^2 k_x^2}{2m^*} - \mu \right) \sigma_0 \tau_z + \Delta \sigma_0 \tau_x \]

(5)

\[ -\alpha k_x \sigma_y \tau_0 + E_{\tau_z}^{SO} \sigma_z \tau_0. \]

In order to solve the problem analytically we use the fact that \( E_2 \) is the largest energy in the system. Then, the Hamiltonian \( H_{22} \) can be further simplified to diagonal form neglecting \( \Delta, \alpha, E_z \) dependencies [for details see the Appendix A]. As we will see further, this approximation works well and the calculated dispersions \( E(k) \) remain
in a good agreement with results obtained from the full numerical diagonalization of the Hamiltonian \(\hat{H}_{1D}\). Using the folding-down transformation,

\[
H(E) = H_{11} - H_{12}(H_{22} - E)^{-1} H_{21},
\]

the 8 \times 8 Hamiltonian \(\hat{H}_{1D}\) can be reduced into the 4 \times 4 effective Hamiltonian. As we derived, it has the form of \(H_{1D}\). Eq. \(\hat{H}_{1D}\) with the renormalized effective mass \(m^*\), chemical potential \(\mu\) and SO coupling constant \(\tilde{\alpha}\) given by the formulas:

\[
\frac{1}{m^*} = \frac{1}{m^*} - \frac{2e^2}{\hbar^2 E_2},
\]

\[
\tilde{\mu} = \mu - E_1 - E_1^{dia} + \frac{(E_p^{SO} - E_1^{SO})^2}{E_2},
\]

\[
\tilde{\alpha} = \alpha + 2 \frac{E_p^{SO} \xi_p}{E_2}.
\]

By diagonalization of the renormalized Hamiltonian \(\hat{H}_{1D}\) we find the energies

\[
E(k_x) = \pm \sqrt{\hat{\xi}^2 + E_k^2 + \Delta_{SO}^2 \pm 2\sqrt{E_k^2 \hat{\xi}^2_1 + \Delta_{SO}^2 \hat{\xi}^2_1 + \Delta^2 E_k^2}},
\]

where \(\hat{\xi}_1 = \hbar^2 k^2/2m^* - \mu\) and \(\Delta_{SO} = \tilde{\alpha}k_x\).

### III. RESULTS

#### A. Low-energy analysis

The real space extent of MBSs with the wave-function \(\sim e^{-z/\xi}\) is determined by the decay length \(\xi\). For wires with length \(L > \xi\), the modes overlap and their energies are shifted away from zero impairing the topological protection. The localization length \(\xi\) can be determined from the dispersion \(E(k)\) of the translation invariant system by considering the evanescent modes at zero energy. Away from the topological transition, \(\xi\) is characterized by properties of the gapped Dirac cones at \(k \neq 0\) with \(\xi = h\nu/\Delta^*\), in analogy to standard superconducting coherence length, where \(\nu = dE/dk\) is the Fermi velocity and \(\Delta^*\) is the induced gap [see Fig. 1b].

Let us consider the band-structure given by Eq. (10), taking the parameters for the state-of-the-art structures made on InSb nanowires covered by the Al shell. We adopt the effective electron mass \(m^* = 0.014\), induced gap \(\Delta = 250\ \mu\text{eV}\) and considerable g-factor of \(g = -5\). The SO interaction strength has been recently probed in various types of measurements and proved to be quite diverse, ranging from \(\alpha \approx 20\ \text{meVnm}\) in the spin-qubit manipulation experiments \(\alpha = 50 - 100\ \text{meVnm}\) for the weak antilocalization measurement up to extremely high values \(\alpha = 120\ \text{meVnm}\) or \(\alpha = 266\ \text{meVnm}\) in the transport experiments probing the helical gap. Therefore, in the following we analyze results for a range of \(\alpha\) values. We take the width of the wire \(W = 104\ \text{nm}\) [see Appendix B for the analysis of the width dependence].

In Fig. 1b we plot the positive-energy bands with the orbital effects of the magnetic field neglected (black solid curve) and with the kinetic-paramagnetic term (green dashed curve) and diamagnetic term (red dashed curve) included. By the analysis of these curves together with the formulas (7), (8) we conclude that the diamagnetic effect lowers the chemical potential, while the paramagnetic effect widens the dispersion relation \(E(k_x)\) due to rescaling of the effective mass. In turn, both these effects decrease the slope of the cones at \(k \neq 0\) decreasing the decay length \(\xi\). Their joint outcome is depicted in Fig. 1c with the violet curve. Inspecting Eqs. (8), (9) we note that the paramagnetic term through the SO interaction Hamiltonian counteracts the rescaling of the chemical potential and decreases the induced gap due the reduction of the SO coupling constant [see the blue dashed curve in Fig. 1c].

In Fig. 2a we plot map of the decay length \(\xi\) in the topological phase \(E_2^z > \Delta^2 + \tilde{\mu}^2\) determined from the imaginary part of the wave-vector of the zero-energy solutions of the renormalized Hamiltonian Eq. (3) for a translational invariant system. Due to rescaling of the chemical potential the contour of the topological regime in the diagram clearly deviates from the hyperbolic shape [see the inset to Fig. 2a] for the phase diagram without the orbital effects] as recently measured for parallel orientation of the field \(B\). With the white curve we depict the topological transition without the SO-paramagnetic effects, which shows narrower range of \(B\) values for which the system is in the topological regime. The minimal decay length in the topological regime on the maps of Fig. 2a is significantly decreased from 41 nm for the case of neglected orbital effects, to 17 nm for the orbital effects included.

When the energy scale set by the SO coupling times the gap \(\Delta\) is smaller than both the Zeeman energy \(E_z\) and the chemical potential \(\mu\) squared one can quantitiy the Majorana decay length posterior the topological transition as,

\[
\xi \simeq \frac{1}{\tilde{\alpha} \Delta} \sqrt{\left(\frac{\hbar^2}{m^*} \tilde{\mu} + \tilde{\alpha}^2\right)^2 + \left(\frac{\hbar^2}{m^*}\right)^2 (E_2^z - \Delta^2 - \tilde{\mu}^2)}.
\]
solid curves that approach the dashed ones obtained with the SO-paramagnetic effect which is manifested by the detrimental effect on the decay length reduction through resulting in overall increase of the same time, the SO coupling constant $\tilde{\alpha}$ counteracting the diamagnetic term contribution. On ary $E$ in the renormalized model of Eq. (5) for $\alpha$ Figure 2. a) Majorana bound state decay length obtained effective chemical potential $\tilde{\mu}$ increases luted. The increase of the magnetic field effects of SO-paramagnetic contribution are more convo-
front the red dot-dashed curve with the dashed one]. The $\alpha$ suppression of the decay length, up to a factor of three for and 1 $\xi$ and by that reduce $\xi$, leading to the strong suppression of the decay length, up to a factor of three for the case of the strong SO coupling $\alpha = 50$ meVnm [confront the red dot-dashed curve with the dashed one]. The effects of SO-paramagnetic contribution are more convoluted. The increase of the magnetic field increases the effective chemical potential $\tilde{\mu}$ through $E^p_{SO}$ term in Eq. $\tilde{\alpha}$ counteracting the diamagnetic term contribution. On the same time, the SO coupling constant $\tilde{\alpha}$ is decreased resulting in overall increase of $\xi \sim 1/\tilde{\alpha}$. This leads to a detrimental effect on the decay length reduction through the SO-paramagnetic effect which is manifested by the solid curves that approach the dashed ones obtained with the sole Zeeman splitting. Note however that for the both decay length obtained from Eq. (11) reproduces the lengths inferred qualitatively from the band structure. The orbital effects through the kinetic energy operator decrease $\tilde{\mu}$ and $1/\tilde{m}^*$ and by that reduce $\xi$, leading to the strong suppression of the decay length, up to a factor of three for the case of the strong SO coupling $\alpha = 50$ meVnm [confront the red dot-dashed curve with the dashed one]. The effects of SO-paramagnetic contribution are more convoluted. The increase of the magnetic field increases the effective chemical potential $\tilde{\mu}$ through $E^p_{SO}$ term in Eq. $\tilde{\alpha}$ counteracting the diamagnetic term contribution. On the same time, the SO coupling constant $\tilde{\alpha}$ is decreased resulting in overall increase of $\xi \sim 1/\tilde{\alpha}$. This leads to a detrimental effect on the decay length reduction through the SO-paramagnetic effect which is manifested by the solid curves that approach the dashed ones obtained with the sole Zeeman splitting. Note however that for the both considered Rashba strengths, $\xi$ still remains lower with the orbital magnetic effects included as compared to the case of the Zeeman splitting only.

The crosses in Fig. 2 b) corresponds to numerically obtained decay lengths from the renormalized Hamiltonian Eq. (5). We observe excellent agreement of the two approaches. The only difference is the divergence of the decay length just after the phase transition, where the decay length is dictated by the gapped Dirac cone at $k = 0$ and which is not captured by the formula Eq. (11).

B. Beyond low-energy approximation.

The above consideration focused on the band mixing limited only to the two lowest-energy transverse subbands. For stronger magnetic field or SO coupling this approach must unavoidably break. Furthermore the validity of decay lengths obtained through Eq. (11) is limited to small $\Delta$ and $\tilde{\phi}$. Now we turn our attention to exact solution of the Hamiltonian (1) to test the validity of the developed theory and to extend our study beyond the above mentioned limits. For this purpose, we diagonalize numerically the Hamiltonian (1) on a square mesh with $\Delta x = \Delta y = 4$ nm. The orbital effects of the magnetic field are incorporated using Peierls substitution of the hopping elements $t_{nm} \rightarrow t_{nm} \exp [-ie \int A dz/\hbar]$. In Fig. 1 (c) with the black crosses we depict the dispersion relation obtained from the numerical calculations. As we see the agreement between the analytical and the numerical results is exceptionally good.

Finally, we complement the study with numerical assessment of the decay lengths. The spatial extent of MBSs is obtained numerically as the largest decay length $\xi = \max Re[\kappa]^{-1}$ of the evanescent waves $\Psi \sim e^{-\kappa x}$ at zero energy, with $\kappa$ being the eigenvalue of the translational operator (10). We perform eigendecomposition of the translation operator for a infinite wire described by the Hamiltonian (1). The topological transition is determined from the change of the topological invariant calculated as the determinant of the reflection matrix $\text{sgn}(\det[R])$ of a finite, translation invariant slice of the wire contacted with infinite normal and superconducting electrodes (20). All calculations are performed in Kwant package (20).

Figure 3 a) with open circles presents numerical results for the decay lengths. We observe great agreement with the developed theory that persist up to the topological phase closing, where the increasing magnetic field and wave-vector breaks the applicability of both the two-band model and validity of the formula given by Eq. (11). Interestingly, in this regime, in the exact calculation we observe further beneficial reduction of the decay length which leads to decreasing amplitude of the energy oscillation of the overlapping MBSs [see Appendix C].

Inspecting the decay length as a function of the chemical potential we observe that the tunability of the $\xi$ is
much more pronounced with the orbital effects included – see the circles in Fig. 3b). This is understandable since now the magnetic field modifies $\xi$ not only through $E_z$ but also by all the renormalized parameters $1/\tilde{m}^*$, $\tilde{\mu}$, $\tilde{\alpha}$ in Eq. (11).

IV. CONCLUSIONS

Summarizing, we studied impact of the orbital effects of the perpendicular magnetic field on the decay length of the Majorana bound states in the proximitized semiconductor wire. We provided quasi-one-dimensional analytical model that allows to quantify the energies and decay lengths of Majorana modes in the low-density limit, which we validated by comparison with exact numerical calculations. We found that the reduction of the decay length via diamagnetic rescaling of the chemical potential is assisted by the change of the effective mass due to subband mixing by the paramagnetic term in the kinetic energy operator. On the other hand, the vector potential entering through the SO coupling Hamiltonian has rudimentary effect on the decay length reduction by the enhancement of the chemical potential and the decrease of the Rashba coupling constant. We found however, that in total, the spatial extent of the Majorana modes is still less than without the orbital magnetic effects in favor of the topological robustness of Majorana states in finite size quantum gate devices on composite nanowires$^{7,8}$.

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Appendix A: Derivation of the analytic formula for $E(k)$ with the orbital effects

In this subsection we demonstrate in details the folding-down procedure which leads to a single-band model with effective mass, chemical potential and spin-orbit coupling renormalized due to the orbital effects. In the presence of a magnetic field $B$ the orbital effects couple spatial degrees of freedom in the direction perpendicular to $B$. Bearing in mind that Majorana bound states are formed in the ground state of transverse quantization, in the low-energy approximation the system can be described in the framework of the two-band Hamiltonian. For this purpose we write the Hamiltonian (1) from the main paper, in a basis of the two lowest eigenstates of infinite quantum well in the $y$-direction, $\Psi_n(y) = \sqrt{2/W} \sin(n\pi(y + W/2)/W)$. We obtain

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

with the diagonal elements given by

\[
H_{11(22)} = \begin{pmatrix} \frac{k^2}{2m} - \mu_{1(2)} + \frac{1}{2}g\mu_B B & \Delta & i\alpha k_x & 0 \\ -\Delta & -\frac{k^2}{2m} + \mu_{1(2)} + \frac{1}{2}g\mu_B B & 0 & -i\alpha k_x \\ -i\alpha k_x & 0 & \frac{k^2}{2m} - \mu_{1(2)} - \frac{1}{2}g\mu_B B & \Delta \\ 0 & -i\alpha k_x & \Delta & -\frac{k^2}{2m} + \mu_{1(2)} - \frac{1}{2}g\mu_B B \end{pmatrix},
\]
where $\alpha$ is the Rahsba spin-orbit coupling constant, the chemical potential $\mu_n = \mu - E_n - H_n^{\text{dia}}$ with $n = 1, 2$, where $E_n = n^2 \pi^2 \hbar^2 / 2mW^2$ is the energy excitation in the $y$-direction. $H_n^{\text{dia}} = \langle \Psi_n | y^2 | \Psi_n \rangle e^2 B^2 / 2m$ is the diagonal term in the $n$'th subband with $\langle \Psi_1 | y^2 | \Psi_1 \rangle = (\pi^2 - 6)W^2 / 12\pi^2$ and $\langle \Psi_2 | y^2 | \Psi_2 \rangle = (2\pi^2 - 3)W^2 / 24\pi^2$, respectively, where $\langle \Psi_n \rangle = (\psi_n^\uparrow, \psi_n^\downarrow, \psi_n^\uparrow, -\psi_n^\downarrow)$.

The off-diagonal elements of $\langle A1 \rangle$ have the form

\[
\begin{pmatrix}
\varepsilon_p k_x & 0 & -i E_p^{\text{SO}} + E_\perp^{\text{SO}} & 0 \\
0 & \varepsilon_p k_x & 0 & -i E_p^{\text{SO}} - E_\perp^{\text{SO}} \\
i E_p^{\text{SO}} + E_\perp^{\text{SO}} & 0 & \varepsilon_p k_x & 0 \\
i E_p^{\text{SO}} - E_\perp^{\text{SO}} & 0 & 0 & \varepsilon_p k_x
\end{pmatrix},
\]

which $\varepsilon_p = -\langle \Psi_1 | y^2 | \Psi_2 \rangle eBh / m = 16W eBh / 9m\pi^2$, $E_p^{\text{SO}} = \hbar^2 / 2m\alpha / 3W$.

Using the folding-down transformation,

\[
H(E) = H_{11} - H_{12}(H_{22} - E)^{-1} H_{21}
\]

the $8 \times 8$ Hamiltonian $\langle A1 \rangle$ can be reduced into the $4 \times 4$ effective Hamiltonian. Based on the fact that $E_2$ is the largest energy in the system $E_2 \ll (\alpha^2 m^* / 2\hbar^2, \Delta, 1 / 2g\mu_B B)$, we can neglect the spin-orbit coupling, superconducting pairing and the Zeeman splitting in the first excited state. Then, $(H_{22} - E)^{-1}$ takes the diagonal form

\[
(H_{22} - E)^{-1} = \left[ \left( \frac{\hbar^2 k_x^2}{2m} - \mu + E_2 + H_2^{\text{dia}} \right) \sigma_0 \tau_z - E \sigma_0 \tau_0 \right]^{-1}
\]

where $\sigma_0 \tau_z = \sigma_0 \tau_z - \sigma_0 \tau_z - \sigma_0 \tau_z - \sigma_0 \tau_z$, $\tau_z = 1$ or $\tau_z = 0$.

\[
H_{12}(H_{22} - E)^{-1} H_{21} = \begin{pmatrix}
\varepsilon_p k_x & 0 & 0 & 0 \\
0 & \varepsilon_p k_x & 0 & 0 \\
0 & 0 & \varepsilon_p k_x & 0 \\
0 & 0 & 0 & \varepsilon_p k_x
\end{pmatrix}
\]

the folding-down procedure leads to

\[
H = \left( \hbar^2 k_x^2 / 2m^* - \tilde{\mu} \right) \sigma_0 \tau_z + \Delta \sigma_0 \tau_x - \tilde{\alpha} k_x \sigma_0 \tau_z + E_2 \sigma_z \tau_0,
\]

where $\tilde{m}^*$, $\tilde{\mu}$, $\tilde{\Delta}$ are the effective mass, chemical potential and SO coupling energy normalized due to the presence of the orbital effects

\[
\frac{1}{\tilde{m}} = \frac{1}{m^*} - \frac{2 \varepsilon_p}{\hbar^2 E_2},
\]

\[
\tilde{\mu} = \mu - E_1 - E_1^{\text{dia}} + \frac{(E_p^{\text{SO}} - E_2^{\text{SO}})^2}{E_2},
\]

\[
\tilde{\alpha} = \alpha + \frac{2 E_p^{\text{SO}} \varepsilon_p}{E_2},
\]

\[
\frac{\hbar^2 k_x^2}{2m^*} + \Delta^{\text{dia}} + \Delta^2 \pm 2 \sqrt{E_2^2 \xi_1^2 + \Delta^2 \xi_1^2 + \Delta^2 E_2^2}.
\]

where $\xi_1 = \hbar^2 k_x^2 / 2m^* - \tilde{\mu}$ and $\tilde{\Delta} = \tilde{\alpha} k_x$.

**Appendix B: Impact of the nanowire width**

We inspect the impact of the wire width on the MBSs decay length. Without the orbital effects the nanowire width does not affect $\xi$ as can be observed in Fig. [4] provided that we tune the chemical potential such the
phase transition occurs for the minimal $B$ in the phase diagram. Inclusion of the orbital effects significantly decreases the decay length at the cost of reduction of the topological phase size in $B$.

Figure 4. Decay length $\xi$ versus the magnetic field $B$ for three values of the nanowire width for the orbital effects included (neglected) with circles (crosses). Results obtained in numerical calculation for $\alpha = 50$ meVnm and $W = 72$ nm – $\mu = 5$ meV, $W = 104$ nm – $\mu = 2.3$ meV, $W = 136$ nm – $\mu = 1.3$ meV.

Figure 5. Numerically obtained energy spectra without a) and with b) orbital effects included. Results for $\mu = 2.3$ meV, $W = 104$ nm, $\alpha = 50$ meVnm calculated for a finite system with the length $L = 1000$ nm.

Without the orbital magnetic effects the energy of overlapping MBSs in a finite system deviates from zero, oscillating with a growing amplitude when $B$ increases accordingly to Eq. (11) of the main text. Inclusion of the orbital effects decreases $\xi$ near the phase transition at high magnetic fields and through that limits the energy oscillations of MBSs, as can be observed in Fig. 4.

Appendix C: Energy spectra

1 Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
2 J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma, Phys. Rev. Lett. 104, 040502 (2010).
3 A. Y. Kitaev, Annals of Physics 303, 2 (2003).
4 J. Alicea, Y. Oreg, G. Refael, F. v. Oppen, and M. P. A. Fisher, Nat. Phys. 7, 412 (2011).
5 B. v. Heck, A. R. Akhmerov, F. Hassler, M. Burrello, and C. W. J. Beenakker, New J. Phys. 14, 035019 (2012).
6 T. Hyart, B. van Heck, I. C. Fulga, M. Burrello, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 88, 035121 (2013).
7 S. R. Plissard, L. van Weperen, D. Car, M. A. Verheijen, G. W. G. Immink, J. Kamnhuber, L. J. Cornelissen, D. B. Szombati, A. Geresdi, S. M. Frolov, L. P. Kouwenhoven, and E. P. A. M. Bakkers, Nat. Nano. 8, 859 (2013).
8 E. M. T. Fadaly, H. Zhang, S. Conesa-Boj, D. Car, O. Gül, S. R. Plissard, R. L. M. Op het Veld, S. Kölling, L. P. Kouwenhoven, and E. P. A. M. Bakkers, Nano Lett. 17, 6511 (2017).
9 S. Gazibegovic, D. Car, H. Zhang, S. C. Balk, J. A. Logan, M. W. A. de Moor, M. C. Cassidy, R. Schmits, D. Xu, G. Wang, P. Krogstrup, R. L. M. Op het Veld, K. Zuo, Y. Vos, J. Shen, D. Bouman, B. Shojaei, D. Pennachio, J. S. Lee, P. J. van Veldhoven, S. Koelling, M. A. Verheijen, L. P. Kouwenhoven, C. J. Palmstrøm, and E. P. A. M. Bakkers, Nature 548, 434 (2017).
10 V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. a. M. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
11 M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. 12, 6414 (2012).
12 A. D. K. Finck, D. J. Van Harlingen, P. K. Mohseni, K. Jung, and X. Li, Phys. Rev. Lett. 110, 126406 (2013).
13 H. O. H. Churchill, V. Fatemi, K. Grove-Rasmussen, M. T. Deng, P. Caroff, H. Q. Xu, and C. M. Marcus, Phys. Rev. B 87, 241401 (2013).
14 J. Chen, P. Yu, J. Stenger, M. Hocevar, D. Car, S. R.
15 J. Shabani, M. Kjaergaard, H. J. Suominen, Y. Kim, F. Nichele, K. Pakrouskis, T. Stankevic, R. M. Lutchyn, P. Kroghstrup, R. Feidenhans’l, S. Kraemer, C. Nayak, M. Troyer, C. M. Marcus, and C. J. Palmstrøm, Phys. Rev. B 93, 155402 (2016).

16 P. Kroghstrup, N. L. B. Ziino, W. Chang, S. M. Albrecht, M. H. Madsen, E. Johnson, J. Nygård, C. M. Marcus, and T. S. Jespersen, Nat. Mater. 14, 400 (2015).

17 M. Kjaergaard, H. J. Suominen, M. P. Nowak, A. R. Akhmerov, J. Shabani, C. J. Palmstrøm, F. Nichele, and C. M. Marcus, Phys. Rev. Applied 7, 034029 (2017).

18 W. Chang, S. M. Albrecht, T. S. Jespersen, F. Kuemmeth, P. Kroghstrup, J. Nygård, and C. M. Marcus, Nat. Nano. 10, 232 (2015).

19 M. Kjaergaard, F. Nichele, H. J. Suominen, M. P. Nowak, M. Wimmer, A. R. Akhmerov, J. A. Folk, K. Flensberg, J. Shabani, C. J. Palmstrøm, and C. M. Marcus, Nat. Commun. 7, 12841 (2016).

20 H. Zhang, C.-X. Liu, S. Gazibegovic, D. Xu, J. A. Logan, G. Wang, N. van Loo, J. D. S. Bommer, M. W. A. de Moor, D. Car, R. L. M. O. h. Veld, P. J. van Veldhoven, S. Koelling, M. A. Verheijen, M. Pendharkar, D. J. Pennachio, B. Shojaei, J. S. Lee, C. J. Palmstrøm, E. P. A. M. Bakkers, S. D. Sarma, and L. P. Kouwenhoven, arXiv:1710.10701 (2017).

21 S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Kroghstrup, and C. M. Marcus, Nature 531, 206 (2016).

22 J. S. Lim, L. Serra, R. López, and R. Aguado, Phys. Rev. B 86, 121103 (2012).

23 J. Osca and L. Serra, Phys. Rev. B 91, 235417 (2015).

24 B. Nijholt and A. R. Akhmerov, Phys. Rev. B 93, 235434 (2016).

25 B. Kiczek and A. Ptok, J. Phys.: Condens. Matter 29, 495301 (2017).

26 O. Dmytruk and J. Klinovaja, arXiv:1710.01671 (2017).

27 I. van Weperen, B. Tarasinski, D. Eeltink, V. S. Pribiag, S. R. Plissard, E. P. A. M. Bakkers, L. P. Kouwenhoven, and M. Wimmer, Phys. Rev. B 91, 201413 (2015).

28 S. Heedt, N. T. Ziani, F. Crépin, W. Prost, S. Tellenkamp, J. Schubert, D. Grützmacher, B. Trauzettel, and T. Schäpers, Nat. Phys. 13, 563 (2017).

29 J. Kammhuber, M. C. Cassidy, F. Pei, M. P. Nowak, A. Vuik, O. Gül, D. Car, S. R. Plissard, E. P. A. M. Bakkers, M. Wimmer, and L. P. Kouwenhoven, Nat. Commun. 8, 478 (2017).

30 Y. Peng, F. Pientka, L. I. Glazman, and F. von Oppen, Phys. Rev. Lett. 114, 106801 (2015).

31 D. Sticlet, B. Nijholt, and A. Akhmerov, Phys. Rev. B 95, 115421 (2017).

32 S. D. Sarma, M. Freedman, and C. Nayak, npj Quantum Information 1, 15001 (2015).

33 G. W. Winkler, D. Varjas, R. Skolasinski, A. A. Soluyanov, M. Troyer, and M. Wimmer, Phys. Rev. Lett. 119, 037701 (2017).

34 S. Nadj-Perge, V. S. Pribiag, J. W. G. van den Berg, K. Zuo, S. R. Plissard, E. P. A. M. Bakkers, S. M. Frolov, and L. P. Kouwenhoven, Phys. Rev. Lett. 108, 166801 (2012).

35 S. Das Sarma, J. D. Sau, and T. D. Stanescu, Phys. Rev. B 86, 220506 (2012).

36 I. C. Fulga, F. Hassler, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 83, 155429 (2011).

37 C. W. Groth, M. Wimmer, A. R. Akhmerov, and X. Waintal, New J. Phys. 16, 063065 (2014).