REALIZATION OF COHERENT STATE LIE ALGEBRAS BY DIFFERENTIAL OPERATORS

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Abstract. A realization of coherent state Lie algebras by first-order differential operators with holomorphic polynomial coefficients on Kähler coherent state orbits is presented. Explicit formulas involving the Bernoulli numbers and the structure constants for the semisimple Lie groups are proved.

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1. Introduction

The starting point of this investigation is the standard Segal-Bargmann-Fock realization $a \mapsto \frac{\partial}{\partial z}; \quad a^+ \mapsto z$ of the canonical commutation relations $[a, a^+] = 1$ on the symmetric Fock space $\mathcal{F} := \Gamma^\text{hol}(\mathbb{C}, \frac{1}{2\pi} \exp(-|z|^2)dz \wedge d\bar{z})$ attached to the Hilbert space $\mathcal{H} := L^2(\mathbb{R}, dx)$. The Segal-Bargmann-Fock realization can be considered as a representation by differential operators of the real 3-dimensional Heisenberg algebra $\mathfrak{g}_{HW} = < is1 + za^+ - \bar{z}a >_{s \in \mathbb{R}; z \in \mathbb{C}}$ of the Heisenberg-Weyl group $\text{HW}$. We can look at this construction from group-theoretic point of view, considering the complex number

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z as local coordinate on the homogeneous manifold $M := HW/\mathbb{R} \cong \mathbb{C}$. Glauber \cite{Glauber25} has attached field coherent states (CS) to the points of the manifold $M$.

We shall consider instead Glauber’s \cite{Glauber25} field CS generalized CS in the sense of Perelomov \cite{Perelomov86} based on homogeneous manifolds $M = G/H$. We restrict ourself to Kähler homogeneous spaces $M = G/H$ associated to the so called CS-groups $G$, see \cite{Neeb97, Neeb98, Perelomov86, Perelomov96, Perelomov98} and several works of Neeb quoted in \cite{Neeb97}. The CS-groups are groups whose quotients with stationary groups are manifolds which admit a holomorphic embedding in a projective Hilbert space. This class of groups contains all compact groups, all simple hermitian groups, certain solvable groups and also mixed groups as the semidirect product of the Heisenberg group and the symplectic group \cite{Neeb97}. We are interested in the realization of the CS-Lie algebras by first order holomorphic differential operators with polynomial coefficients.

The present work extends our previous results \cite{Berceanu05, Berceanu06}. The differential action of the generators of the groups on coherent state manifolds which have the structure of hermitian symmetric spaces can be written down as a sum of two terms, one a polynomial $P$, and the second one a sum of partial derivatives times some polynomials $Q$-s, the degree of polynomials being less than 3 \cite{Berceanu05, Berceanu06}. It is interesting to investigate the same problem as in \cite{Berceanu05, Berceanu06} on flag manifolds \cite{Neeb97}. Some results are available \cite{Berceanu21}, but they are not easily handled. We give explicit formulas of the polynomials $P$ and $Q$-s in the case of semisimple Lie groups and also the simplest example of the compact nonsymmetric space $SU(3)/S(U(1) \times U(1) \times U(1))$, where the degree of the polynomials is already 3.

The paper is laid out as follows. §2 collects some more of less known facts about CS-groups and CS-representations. We follow \cite{Perelomov86, Perelomov96, Perelomov98} and \cite{Neeb97}. Many of the facts summarized in §2 have been already detailed in \cite{Berceanu05} and \cite{Berceanu06}. The definition of CS-groups is contained in §2.1. §2.2 defines the so called Perelomov’s generalized coherent state vectors in the context of the CS-groups. In §2.3 we recall the construction of the symmetric Fock space of functions $\mathcal{F}_\gamma$ on which the differential operators act. In §3 we construct the representations of Lie algebras of CS-groups by differential operators. §3.1 recall some known facts about multipliers in the context of coherent states. Data on hermitian representations and differential operators are summarized in §3.2. Simple examples are presented in §3.3 the Heisenberg-Weyl group and $\mathfrak{sl}(2, \mathbb{C})$. §4 is dedicated to the semisimple case. The construction of Perelomov’s coherent state vectors in this case is contained in §4.1. Our main new results are contained in Theorem 1 in §4.2. The simplest compact and non-compact nonsymmetric examples are contained in §4.3 $SU(3)/S(U(1) \times U(1) \times U(1))$ and $Sp(3, \mathbb{R})/S(U(1) \times U(1) \times U(1))$.

In this paper we give the proof of the formulas referring to the semisimple case. These formulas contain the Bernoulli numbers and the structure constants. Let us recall that the Bernoulli numbers appear \cite{Kontsevich97} also in connection with Kontsevich’s universal formula for deformation quantization \cite{Kontsevich97}, in the context of the Duflo-Kirillov isomorphism \cite{Duflo82} and Kashiwara-Vergne conjecture \cite{Kashiwara09}. Applications of the formulas here proved to explicit boson expansions for collective models on Kähler CS-orbits have been given in \cite{Berceanu07}. Using the same formulas of the differential action of the generators of semisimple Lie groups, we have written down the equations of motion generated by linear generators of the groups on CS-orbits \cite{Berceanu08}. We have not included in this text the
holomorphic representation of CS-groups of semidirect product type presented at the Conference Operator algebras and Mathematical Physics in Sinaia, see [11].

We have underlined the deep relationship between coherent states and geometry [9]. Here we are interested in the algebraic aspect. Our approach is closely related with those of reference [2], where are considered differential operators acting on coherent states constructed on Lie algebras.

Let us also mention the “reflection symmetry” approach [27] to study simultaneously the representations of the HW group and semisimple Lie groups. Applying the “restriction principle” as a particular case it is obtained the Segal-Bargmann-Hall transform, a generalization of the standard Segal-Bargmann transform for compact groups, see references in [28].

We want to underline that in this paper we do not give an explicit construction of the symmetric Fock space of functions $F_M$ on which the differential operators act. For hermitian symmetric spaces a case by case investigation was started by Hua [31] and developed by many authors, see e.g. [55, 14, 58, 59, 26, 57, 24, 48].

We use for the scalar product the convention: $(\lambda x, y) = \lambda(x, y), x, y \in \mathcal{H}, \lambda \in \mathbb{C}$.

2. CS-representations, CS-vectors, reproducing kernel

Perelomov’s paper [51] generated the interest of theoretical physicists (e.g. [50]) and mathematicians (e.g. [11, 42]) in understanding the mathematical aspects of CS-representations. Here we follow the formulation of Lisiecki [38, 39, 40] and Neeb [47]. The whole section just fixes the definition and collects some more or less known facts about CS-representations. More details and proofs can be found in [8, 10].

2.1. Coherent state representations. Let us consider the triplet $(G, \pi, \mathcal{H})$, where $\pi$ is a continuous, unitary representation of the Lie group $G$ on the separable complex Hilbert space $\mathcal{H}$. Let us denote by $\mathcal{H}^\infty$ the smooth vectors. Let us pick up $e_0 \in \mathcal{H}^\infty$ and let the notation: $e_{g,0} := \pi(g) e_0, g \in G$. We have an action $G \times \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty, g.e_0 := e_{g,0}$. When there is no possibility of confusion, we write just $e_g$ for $e_{g,0}$.

Let us denote by $[\cdot] : \mathcal{H}^\times := \mathcal{H} \setminus \{0\} \rightarrow \mathbb{P} (\mathcal{H}) = \mathcal{H}^\times / \sim$ the projection with respect to the equivalence relation $[\lambda x] \sim [x], \lambda \in \mathbb{C}^\times, x \in \mathcal{H}^\times$. So, $[\cdot] : \mathcal{H}^\times \rightarrow \mathbb{P}(\mathcal{H}), [v] = \mathbb{C}v$.

The action $G \times \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ extends to the action $G \times \mathbb{P}(\mathcal{H}^\infty) \rightarrow \mathbb{P}(\mathcal{H}^\infty), g.[v] := [g.v]$.

Let us now denote by $H$ the isotropy group $H := G_{[e_0]} := \{g \in G | g.e_0 \in \mathcal{C}e_0\}$.

We shall consider (generalized) coherent states on complex homogeneous manifolds $M \cong G/H$, imposing the restriction that $M$ be a complex submanifold of $\mathbb{P}(\mathcal{H}^\infty)$.

Definition 1. a) The orbit $M$ is called a CS-orbit if there exists a holomorphic embedding $\iota : M \hookrightarrow \mathbb{P}(\mathcal{H}^\infty)$. In such a case $M$ is also called CS-manifold.

b) $(\pi, \mathcal{H})$ is called a CS-representation if there exists a cyclic vector $0 \neq e_0 \in \mathcal{H}^\infty$ such that $M$ is a CS-orbit.

c) The groups $G$ which admit CS-representations are called CS-groups, and their Lie algebras $\mathfrak{g}$ are called CS-Lie algebras.

For $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, let us define the unbounded operator $d\pi(X)$ on $\mathcal{H}$ by $d\pi(X).v := d/dt|_{t=0} \pi(\exp tX).v$, whenever the limit on the right hand side exists. The operator $d\pi(X)$ is the closure of its restriction to $\mathcal{H}^\infty$. In
particular, \( id\pi(X)|_{\mathcal{H}^\infty} \) is essentially selfadjoint (cf. Proposition X.1.5. p. 391 in [47]). We obtain a representation of the Lie algebra \( \mathfrak{g} \) on \( \mathcal{H}^\infty \), the derived representation, and we denote \( X.v := d\pi(X).v \) for \( X \in \mathfrak{g} \), \( v \in \mathcal{H}^\infty \). Extending \( d\pi \) by complex linearity, we get a representation of the universal enveloping algebra of the complex Lie algebra \( \mathfrak{g}_C \) on the complex vector space \( \mathcal{H}^\infty \)

\[
(2.1) \quad d\pi : \mathfrak{u}(\mathfrak{g}_C) \to B_0(\mathcal{H}^\infty), \text{ with } d\pi(X).v := \frac{d}{dt} \bigg|_{t=0} \pi(\exp tX).v, \ X \in \mathfrak{g}.
\]

\( B_0(\mathcal{H}^0) \subset \mathcal{L}(\mathcal{H}) \), where \( \mathcal{H}^0 := \mathcal{H}^\infty \), denotes the set of linear operators \( A : \mathcal{H}^0 \to \mathcal{H}^0 \) which have a formal adjoint \( A^* : \mathcal{H}^0 \to \mathcal{H}^0 \), i.e. \( (x, Ay) = (A^*x, y) \) for all \( x, y \in \mathcal{H}^0 \). Note that if \( B_0(\mathcal{H}^0) \) is the set of unbounded operators on \( \mathcal{H} \), then the domain \( \mathcal{D}(A^*) \) contains \( \mathcal{H}^0 \) and \( A^*\mathcal{H}^0 \subset \mathcal{H}^0 \), and it makes sense to refer to the closure of \( A \in B_0(\mathcal{H}^0) \) (cf. [47] p. 29; here \( A^* \) is the adjoint of \( A \)).

We denote by \( B := \langle \exp \mathfrak{g}_C, \mathfrak{b} \rangle \) the Lie group corresponding to the Lie algebra \( \mathfrak{b} \), with \( \mathfrak{b}(v) := \{ X \in \mathfrak{g}_C : X.v \in \mathbb{C}v \} = (\mathfrak{g}_C)[v] \). The group \( B \) is closed in the complexification \( G_C \) of \( G \), cf. Lemma XII.1.2. p. 495 in [47]. The complex structure on \( M \) is induced by an embedding in a complex manifold, \( i_1 : M \cong G/H \to G_C/B \). We consider such manifolds which admit a holomorphic embedding \( i_2 : G_C/B \to \mathbb{P}(\mathcal{H}^\infty) \). Then the embedding \( \iota = i_2 \circ i_1, \ i : M \to \mathbb{P}(\mathcal{H}^\infty) \) is a holomorphic embedding, and the complex structure comes as in Theorem XV.1.1 and Proposition XV.1.2 p. 646 in [47].

We conclude this paragraph recalling some known facts about CS-orbits and CS-representations (cf. [47]). Firstly, note that: If \( G[v] \) is a CS-orbit, then \( v \) is an analytic vector (cf. Prop. XV.2.2 p. 651 in [47]). Now, let \( G \) be a connected Lie group such that \( \mathfrak{g} \) contains a compactly embedded Cartan algebra \( \mathfrak{t} \). Choosing \( e_0 := v_{\lambda} \), where \( \lambda \in \mathfrak{t}^* \) is a primitive element of a unitary highest weight representation \( (\pi, \mathcal{H}_\lambda) \), then:

\( G.[v_{\lambda}] \) is a complex orbit in bijection via the momentum map with the coadjoint orbit \( \mathfrak{t} \) and every unitary highest weight representation is a CS-representation (cf. Proposition XV.2.6 p. 652 in [47]). The CS-representations of connected Lie groups are irreducible (cf. Proposition XV.2.7 p. 652 in [47]). Conversely, if \( \mathfrak{g} \) is an admissible Lie algebra, \( G \) a connected Lie group with Lie algebra \( \mathfrak{g} \), and \( (\pi, \mathcal{H}) \) a CS-representation of \( G \) with discrete kernel, then \( \pi \) is a unitary highest weight representation. If \( G[e_0] \) is a CS-orbit, then \( e_0 \) is a primitive element for an appropriate positive system of roots with respect to a compactly embedded Cartan subalgebra and the orbit is the unique complex orbit in \( \mathbb{P}(\mathcal{H}^\infty) \) (cf. Theorems XV.2.10 - 11 p. 655 in [47]).

2.2. Coherent state vectors. Now we construct what we call Perelomov’s generalized coherent state vectors, or simply CS-vectors, based on the CS-homogeneous manifolds \( M \cong G/H \).

We denote also by \( \pi \) the holomorphic extension of the representation \( \pi \) of \( G \) to the complexification \( G_C \) of \( G \), whenever this holomorphic extension exists. In fact, it can be shown that in the situations under interest in this paper, this holomorphic extension exists [43][46]. Then there exists a homomorphism \( \chi_0(\lambda) : H \to \mathbb{T}, (\chi : B \to \mathbb{C}^\times) \), such that \( H = \{ g \in G | e_g = \chi_0(g)e_0 \} \) (respectively, \( B = \{ g \in G_C | e_g = \chi(g)e_0 \} \)), where \( \mathbb{T} \) denotes the torus \( \mathbb{T} := \{ z \in \mathbb{C} | |z| = 1 \} \).
For the homogeneous space \( M = G/H \) of cosets \( \{gH\} \), let \( \lambda : G \to G/H \) be the natural projection \( g \mapsto gH \), and let \( o := \lambda(1) \), where \( 1 \) is the unit element of \( G \). Choosing a section \( \sigma : G/H \to G \) such that \( \sigma(o) = 1 \), every element \( g \in G \) can be written down as \( g = \tilde{g}(g) h(g) \), where \( \tilde{g}(g) \in G/H \) and \( h(g) \in H \). Then we have

\[
\tag{2.2}
e_g = e^{i\alpha(h(g))} e_{\tilde{g}(g)}, \quad e^{i\alpha(h(g))} := \chi_0(h).
\]

Now we take into account that \( M \) also admits an embedding in \( G_C/B \). We choose a local system of coordinates parametrized by \( z_g \) (denoted also simply \( z \), where there is no possibility of confusion) on \( G_C/B \). Choosing a section \( G_C/B \to G_C \) such that every element \( g \in G_C \) can be written down as \( g = \tilde{g}_b(g) \), where \( \tilde{g}_b \in G_C/B \), and \( b(g) \in B \), we have

\[
\tag{2.3}
e_g = \Lambda(g) e_{z_g}, \quad \Lambda(g) := \chi(b(g)) = e^{i\alpha(b(g))} (e_{z_g}, e_{z_g})^{-\frac{1}{2}}.
\]

Let us denote by \( m \) the vector subspace of the Lie algebra \( g \) orthogonal to \( h \), i.e. the vector space decomposition \( g = h + m \). It can be shown that for CS-groups the vector space decomposition \( g = h + m \) is \( \text{Ad} \) \( H \)-invariant. The homogeneous spaces \( M \cong G/H \) with this decomposition are called reductive spaces (cf. [49]) and the CS-manifolds are reductive spaces (cf. [10]). So, the tangent space to \( M \) at \( o \) can be identified with \( m \).

Let \( \tilde{g}(g) = \exp X, \tilde{g}(g) \in G/H, X \in m, e_{\tilde{g}(g)} = \exp(X)e_0 \). Note that \( T_o(G/H) \cong g/h \cong g_c/b \cong (b + b)/b \cong b/h_c \), where we have a linear isomorphism \( \alpha : g/\hbar \cong g_c/b \), \( \alpha(X + \hbar) = X + b \) (cf. [45]). We can take instead of \( m \subset g \) the subspace \( m' \subset g_c \) complementary to \( b \), or the subspace of \( b \) complementary to \( h_c \). If we choose a local canonical system of coordinates \( \{z_m\} \) with respect to the basis \( \{X_\alpha\} \) in \( m' \), then we can introduce the vectors

\[
\tag{2.4}
e_z = \exp \left( \sum_{X_\alpha \in m'} z_\alpha X_\alpha \right). e_0 \in \mathcal{H}.
\]

We get

\[
\tag{2.5}
e_{\sigma(z)} = \pi(\sigma(z)). e_0, \quad z \in M,
\]

and we choose local coordinates in a neighborhood \( V_0 \subset M \) of \( z = 0 \) corresponding to \( \sigma(o) = e \in G \) such that

\[
\tag{2.6}
e_{\sigma(z)} = N(z) e_z, \quad N(z) = (e_z, e_z)^{-1/2}.
\]

Equations (2.4), (2.5), and (2.6) define locally the coherent vector mapping

\[
\tag{2.7}
\varphi : M \to \tilde{\mathcal{H}}, \quad \varphi(z) = e_z,
\]

where \( \tilde{\mathcal{H}} \) denotes the Hilbert space conjugate to \( \mathcal{H} \). We call the vectors \( e_z \in \tilde{\mathcal{H}} \) indexed by the points \( z \in M \) Perelomov’s coherent state vectors.

2.3. Reproducing kernel. Let us introduce the function \( f'_\psi : G_C \to \mathbb{C}, f'_\psi(g) := (e_g, \psi), g \in G, \psi \in \mathcal{H} \). Then \( f'_\psi(gb) = \chi(b)^{-1} f'_\psi(g), g \in G_C, b \in B \), where \( \chi \) is the continuous homomorphism of the isotropy subgroup \( B \) of \( G_C \) in \( \mathbb{C}^\times \). The coherent states realize the space of holomorphic global sections \( \Gamma^{\text{hol}}(M, L_\chi) = H^0(M, L_\chi) \) on the \( G_C \)-homogeneous line bundle \( L_\chi \) associated by means of the character \( \chi \) to the principal
The holomorphic line bundle is $L_\chi := M \times_\chi \mathbb{C}$, also denoted $L := M \times_B C$ (cf. [17, 50]).

The local trivialization of the line bundle $L_\chi$ associates to every $\psi \in \mathcal{H}$ a holomorphic function $f_\psi$ on a open set in $M \hookrightarrow G_C/B$. Let the notation $G_S := G_C \setminus S$, where $S$ is the set $S := \{ g \in G_C | \alpha_g = 0 \}$, and $\alpha_g := (e_g, e_0)$. $G_S$ is a dense subset of $G_C$. We introduce the function $f_\psi : G_S \to \mathbb{C}$, $f_\psi(g) = \frac{f_\psi(g)}{\alpha_g}, \psi \in \mathcal{H}, g \in G_S$. The function $f_\psi(g)$ on $G_S$ is actually a function of the natural projection $\lambda(g)$, $\lambda : G \to G/H$, holomorphic in $M_S := \lambda(G_S)$.

Supposing that the line bundle $L_\chi$ is already very ample, the symmetric Fock space $\mathcal{F}_\mathcal{H}$ is defined as the set of functions corresponding to sections such that $\{ f \in L^2(M, L) \cap \mathcal{O}(M, L) | (f, f)_{\mathcal{F}_\mathcal{H}} < \infty \}$ with respect to the scalar product

$$
(f, g)_{\mathcal{F}_\mathcal{H}} = \int_M \bar{f}(z)g(z)d\nu_M(z, \bar{z}),
$$

where $d\nu_M(z, \bar{z})$ is the quasi-invariant measure on $M$

$$
d\nu_M(z, \bar{z}) = \frac{\Omega_M(z, \bar{z})}{(e_z, e_{\bar{z}})}.
$$

Here $\Omega_M$ is the $G$-invariant volume form

$$
\Omega_M := (-1)^{\binom{n}{2}} \frac{1}{n!} \omega \wedge \cdots \wedge \omega, \quad n \text{ times}
$$

and the Kähler two-form $\omega$ on $M$ is given by

$$
\omega(z) = i \sum_{\alpha, \beta \in \Delta_{m'}} g_{\alpha, \beta} dz_\alpha \wedge d\bar{z}_\beta, \quad g_{\alpha, \beta}(z) = \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log(e_z, e_{\bar{z}}).
$$

It can be shown (cf. [43]) that the space of functions $\mathcal{F}_\mathcal{H}$ identified with $L^{2,\text{hol}}(M, L_\chi)$ is a closed subspace of $L^2(M, L_\chi)$ with continuous point evaluation and eq. (2.8) is nothing else than the Parseval overcompleteness identity [13]

$$
(f_1, f_2) = \int_{M/G/H} (f_1, e_z)(e_{\bar{z}}, f_2)d\nu_M(z, \bar{z}), \quad (f_1, f_2)_{\mathcal{F}_\mathcal{H}}.
$$

It can be seen that the relation (2.8) (or eq. (2.12)) on homogeneous manifolds fits into Rawnsley’s global realization [53] of Berezin’s coherent states on quantizable Kähler manifolds [12], modulo Rawnsley’s “epsilon” function [53, 20], a constant for homogeneous quantization. If $(M, \omega)$ is a Kähler manifold and $(L, h, \nabla)$ is a (quantum) holomorphic line bundle $L$ on $M$, where $h$ is the hermitian metric and $\nabla$ is the connection compatible with the metric and the complex structure, then $h(z, \bar{z}) = (e_z, e_{\bar{z}})^{-1}$ and the Kähler potential is $-\log h(z)$.

Let us now introduce the map

$$
\Phi : \mathcal{H}^* \to \mathcal{F}_\mathcal{H}, \Phi(\psi) := f_\psi, \quad f_\psi(z) = \Phi(\psi)(z) = (\varphi(z), \psi)_{\mathcal{H}} = (e_z, \psi)_{\mathcal{H}}, \quad z \in \mathcal{V}_0,
$$

where we have identified the space $\mathcal{F}_\mathcal{H}$ complex conjugate to $\mathcal{H}$ with the dual space $\mathcal{H}^*$ of $\mathcal{H}$. Our supposition that $L_\chi$ is already a very ample line bundle implies the validity of
Parseval overcompletness identity \(^{(2.12)}\) (cf. Theorem XII.5.6 p. 542 in [47], Remark VIII.5 in [43], and Theorem XII.5.14 p. 552 in [47]).

It can be defined a function \(K, K : M \times \overline{M} \to \mathbb{C}\), which on \(\mathcal{V}_0 \times \overline{\mathcal{V}}_0\) reads
\begin{equation}
K(z, \overline{w}) := K_w(z) = (e_z, \overline{e_w})_\mathcal{K}.
\end{equation}

Taking into account \(^{(2.13)}\), it follows (see Proposition 1 in [8]) that if the line bundle \(L\) is very ample, then the function \(K\) \(^{(2.14)}\) is a reproducing kernel, the symmetric Fock space \(\mathcal{F}_\mathcal{K}\) is the reproducing kernel Hilbert space \(\mathcal{H}_K \subset \mathbb{C}^M\) associated to the kernel \(K\), and the evaluation map \(\Phi\) defined in eqs. \(^{(2.13)}\) extends to an isometry
\begin{equation}
(\psi_1, \psi_2)_{\mathcal{F}_\mathcal{K}} = (\Phi(\psi_1), \Phi(\psi_2))_{\mathcal{F}_\mathcal{K}} = (f_{\psi_1}, f_{\psi_2})_{\mathcal{F}_\mathcal{K}} = \int_M \overline{T}_{\psi_1}(z) f_{\psi_2}(z) d\nu_M(z).
\end{equation}

3. REPRESENTATIONS OF COHESIVE STATE LIE ALGEBRAS BY DIFFERENTIAL OPERATORS

3.1. Multipliers and coherent states. Recalling the definition of the function \(f_\psi\) given in \(^{(2.3)}\) we have
\begin{equation}
f_\psi(z) = (e_z, \psi) = (\pi(\overline{g})e_0, \psi) = (\pi(\overline{g})e_0, e_0), \quad z \in M, \psi \in \mathcal{H}.
\end{equation}

We get
\begin{equation}
f_{\pi(\overline{g}), \psi}(z) = \mu(g', z) f_\psi(g^{-1}.z),
\end{equation}

where
\begin{equation}
\mu(g', z) = \frac{(\pi(g^{-1})e_0, e_0)}{(\pi(\overline{g})e_0, e_0)} = \frac{\Lambda(g^{-1}g)}{\Lambda(g)}.
\end{equation}

or
\begin{equation}
\mu(g', z) = \Lambda(\overline{g}')(e_z, e_z) = e^{i\alpha(\overline{g}')} \left(\frac{e_z, e_z}{e_z, e_z}\right)^{1/2}.
\end{equation}

The following assertion is easily checked up using successively eq. \(^{(2.13)}\):

Remark 1. Let us consider the relation \(^{(3.1)}\). Then we have \(^{(3.2)}\), where \(\mu\) can be written down as in equations \(^{(3.3)}\), \(^{(3.4)}\). We have the relation \(\mu(g, z) = J(g^{-1}, z)^{-1}\), i.e. the multiplier \(\mu\) is the cocycle in the unitary representation \((\pi_K, \mathcal{H}_K)\) attached to the positive definite holomorphic kernel \(K\) defined by equation \(^{(2.14)}\),
\begin{equation}
(\pi_K(g).f)(x) := J(g^{-1}, x)^{-1} f(g^{-1}.x),
\end{equation}

and the cocycle verifies the relation \(J(g_1, g_2, z) = J(g_1, g_2z)J(g_2, z)\).

Note that the prescription \(^{(3.3)}\) defines a continuous action of \(G\) on \(\text{Hol}(M, \mathbb{C})\) with respect to the compact open topology on the space \(\text{Hol}(M, \mathbb{C})\). If \(K : M \times \overline{M} \to \mathbb{C}\) is a continuous positive definite kernel holomorphic in the first argument satisfying \(K(g.x, \overline{g.y}) = J(g, x) K(x, \overline{y}) J(g, y)^*\), \(g \in G, x, y \in M\), then the action of \(G\) leaves the reproducing kernel Hilbert space \(\mathcal{H}_K \subseteq \text{Hol}(M, \mathbb{C})\) invariant and defines a continuous unitary representation \((\pi_K, \mathcal{H}_K)\) on this space (cf. Prop. IV.1.9 p. 104 in Ref. [47]).
3.2. Hermitian representations and differential operators. Let us consider again the triplet \((G, \pi, \mathcal{H})\). Let \(\mathfrak{g}\) be the Lie algebra of \(G\) and let us denote by \(\mathfrak{s} := \mathcal{U}(\mathfrak{g}_C)\) the semigroup associated with the universal enveloping algebra equipped with the anti-linear involution extending the antiautomorphism \(X \mapsto X^* := -X\) of \(\mathfrak{g}_C\). The derived representation \(d\pi\) defined by eq. (2.11) is a hermitian representation of \(\mathfrak{s}\) on \(\mathcal{H}^0 := \mathcal{H}_0^\infty\) (cf. Neeb [47], p. 30). As we have already noted, the unitarity and the continuity of the representation \(\pi\) implies that \(id\pi(X)|_{\mathcal{H}^\infty}\) is essentially selfadjoint. Let us denote his image in \(B_0(\mathcal{H}^0)\) with \(A_M := d\pi(\mathfrak{s})\). If \(\Phi : \mathcal{H}^* \to \mathcal{F}_{\mathfrak{g}_C}\) is the isometry (2.13), we are interested in the study of the image of \(A_M\) via \(\Phi\) as subset in the algebra of holomorphic, linear differential operators, \(\Phi A_M \Phi^{-1} := A_M \subset \mathcal{D}_M\).

The sheaf \(\mathcal{D}_M\) (or simply \(\mathfrak{D}\)) of holomorphic, finite order, linear differential operators on \(M\) is a subalgebra of homomorphisms \(\mathcal{Hom}_C(\mathcal{O}_M, \mathcal{O}_M)\) generated by the sheaf \(\mathcal{O}_M\) of germs of holomorphic functions of \(M\) and the vector fields. We consider also the subalgebra \(\mathfrak{A}_M\) of \(\mathfrak{A}_M\) of differential operators with holomorphic polynomial coefficients.

Let \(U := \mathcal{V}_0\) in \(M\), endowed with the coordinates \((z_1, z_2, \cdots, z_n)\). We set \(\partial_i := \frac{\partial}{\partial z_i}\) and \(\partial^* := \partial_1^* \partial_2^* \cdots \partial_n^*\), \(\alpha := (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n\). The sections of \(\mathcal{D}_M\) on \(U\) are \(A : f \mapsto \sum_\alpha a_\alpha \partial^\alpha f\), \(a_\alpha \in \Gamma(U, \mathcal{O})\), the \(a_\alpha\)-s being zero except a finite number.

For \(k \in \mathbb{N}\), let us denote by \(\mathcal{D}_k\) the subsheaf of differential operators of degree \(\leq k\) and by \(\mathcal{D}'_k\) the subsheaf of elements of \(\mathcal{D}_k\) without constant terms. \(\mathcal{D}_0\) is identified with \(\mathcal{O}\) and \(\mathcal{D}'_1\) with the sheaf of vector fields. The filtration of \(\mathcal{D}_M\) induces a filtration on \(\mathfrak{A}_M\).

Summarizing, we have a correspondence between the following three objects:

\[
\mathfrak{g} \ni X \mapsto X \in \mathfrak{A}_M \mapsto X \in A_M \subset \mathcal{D}_M, \text{ differential operator on } \mathcal{F}_{\mathfrak{g}_C}.
\]

Using eq. (2.12) and the reproducing kernel properties, it is easy to emphasize the correspondence between the operators \(L \in B_0(\mathcal{H}_0)\) and their images \(\mathbb{L} = \Phi L \Phi^{-1}\) in \(A_M\) defined on \(M_S\).

Remark 2. Let us consider \(\phi, \psi \in \mathcal{H}\), and \(L \in B_0(\mathcal{H}_0)\) related by

\[
\phi = L\psi.
\]

Then their images \(f_\phi, f_\psi \in \mathcal{F}_{\mathfrak{g}_C}\) are related by

\[
f_\phi(z) = \mathbb{L}(z)f_\psi(z),
\]

where the operator \(\mathbb{L} = \Phi L \Phi^{-1}\in \mathfrak{A}_M\) is determined by its symbol \(K^L\), expressed locally as

\[
K^L(z, \bar{w}) := (e_z, L e_w) = \mathbb{L}(z)(e_z, e_w).
\]

Now we can see that

Proposition 1. If \(\Phi\) is the isometry (2.13), then \(\Phi d\pi(\mathfrak{g}_C) \Phi^{-1} \subset \mathcal{D}_1\).

Proof. Let us consider an element in \(\mathfrak{g}_C\) and his image in \(\mathcal{D}_M\), via the correspondence (3.6), i.e.:

\[
\mathfrak{g}_C \ni X \mapsto X \in \mathcal{D}_M; \ X_z(f_\psi(z)) = X_z(e_z, \psi) = (e_z, X \psi).
\]
The action \( G \times M \to M \) is a holomorphic one and we have successively:

\[
X_z(f_\psi(z)) = (e_z, d\pi(X)\psi) = \frac{d}{dt}|_{t=0} (e_z, \pi(\exp(tX))\psi) = \frac{d}{dt}|_{t=0} \mu(\exp(tX), z) f_\psi(\exp(-tX).z)
\]

We have finally

\[
(3.10) \quad X_z(f_\psi(z)) = \left( P_X(z) + \sum Q^i_X(z) \frac{\partial}{\partial z_i} \right) f_\psi(z),
\]

where

\[
P_X(z) := \frac{d}{dt}|_{t=0} \mu(\exp(tX), z), \quad Q^i_X(z) := \frac{d}{dt}|_{t=0} (\exp(-tX).z)_i.
\]

Now we formulate the following assertion:

**Remark 3.** If \((G, \pi)\) is a CS-representation, then \(A_M\) is a subalgebra of holomorphic differential operators with polynomial coefficients, i.e. \(A_M \subset A_M \subset D_M\).

More exactly, for \(X \in \mathfrak{g}\) and \(X := d\pi(X) \in A_M\), let us consider his image \(X \in A_M\) as in relation (3.4), acting on the space of functions \(F_M^G\). Then, for CS-representations, we have that \(X \in A_1 = A_0 \oplus A'_1\).

Explicitly, if \(\lambda \in \Delta\) is a root and \(X_\lambda\) is in a base of the Lie algebra \(\mathfrak{g}_C\) of \(G_C\), then his image \(X_\lambda \in D_M\) acts as a first order differential operator on the symmetric Fock space \(F_M^G\)

\[
(3.11) \quad X_\lambda = P_\lambda + \sum_{\beta \in \Delta_m'} Q_{\lambda,\beta} \partial_\beta, \quad \lambda \in \Delta,
\]

where \(P_\lambda\) and \(Q_{\lambda,\beta}\) are polynomials in \(z\) and \(m'\) is the subset of \(\mathfrak{g}_C\) which appears in the definition (2.4) of the coherent vectors.

Actually, we don’t have a proof of this assertion for the general case of CS-groups. For the compact case, there exists the calculation of Dobaczewski [21], which in fact can be extended also to real semisimple Lie algebras. For compact hermitian symmetric spaces it was shown [3] that the degrees of the polynomials \(P\) and \(Q\)-s are \(\leq 2\) and similarly for the non-compact hermitian symmetric case [6]. Neeb [47] gives a proof of this Remark for CS-representations for the (unimodular) Harish-Chandra type groups.

Let us also remember that: If \(G\) is an admissible Lie group such that the universal complexification \(G \to G_C\) is injective and \(G_C\) is simply connected, then \(G\) is of Harish-Chandra type (cf. Proposition V.3 in [43]). Differentiating eq. (3.5) in order to obtain the derived representation (2.8), we get two terms, one in \(D_0\) and the other one in \(D'_1\), as was shown in Proposition [11]. A proof that the two parts are in fact \(A_0\) and respectively \(A'_1\) is contained in Prop. XII.2.1 p. 515 in [47] for the groups of Harish-Chandra type in the particular situation where the space \(p^+\) in Lemma VII.2.16 p. 241 in [47] is abelian. We present below explicit formulas for semisimple Lie groups and also the
simplest example where the maximum degree of $P$ and $Q$ is 3.

3.3. Simple examples.

3.3.1. Canonical commutation relations, Glauber’s coherent states and the Heisenberg-Weyl Group. The example of the HW group is sketched here only to check up that the formalization in previous sections leads in particular to the standard realization of the canonical commutation relations (CCR) on $\mathcal{F}_{\mathcal{H}^1}$, i.e. $a \mapsto \frac{\partial}{\partial z}$, $a^+ \mapsto z$.

The HW group here is the group with the 3-dimensional real Lie algebra isomorphic to the Heisenberg algebra $h_1 \equiv g_{HW} = \langle is1 + za^+ - \bar{z}a >_{s \in \mathbb{R}, z \in \mathbb{C}}$, where the bosonic creation (annihilation) operators $a^+$ (respectively $a$) verify the CCR relations $[a, a^+] = 1$, and the action of the annihilation operator on the vacuum is $ae_0 = 0$.

Let $\mathcal{H} := L^2(\mathbb{R}, dx)$. Then $\mathcal{F}_{\mathcal{H}} := \Gamma^\text{hol}(\mathbb{C}, \frac{1}{2\pi} \exp(-|z|^2)dz \land d\bar{z})$. The infinite-dimensional irreducible unitary Schrödinger representations $\pi_\lambda$ of the HW group are indexed by $\lambda \in \mathbb{R}$, where the infinitesimal character of the representation is $\chi'_\lambda(z) = 2\pi i \lambda$, $z \in \mathbb{C}$, the center of the Lie algebra of the group, and we take the standard representation ($\lambda = 1$). The CS-manifold $M$ for the HW group is the quotient $HW/\mathbb{R} \approx \mathbb{C}$.

Let us choose the section $\sigma : M \approx \mathbb{C} \rightarrow HW, \sigma(z) = (0, z)$. The CS-vectors (2.5) for the HW-group (Glauber’s CS field [25]) are given by the unitary displacement operator acting on the ground state

$$e_{\sigma(z)} := \exp(\bar{z}a^+ - za)e_0 = e^{-|z|^2}e_z,$$

where the Perelomov’s CS-vectors are

$$e_z := \exp(za^+)e_0,$$

and the constant $N$ of eq. (2.6) here has the value given in (3.12) because

$$(e_{z'}, e_z) = \exp(\bar{z}'z).$$

The coherent vectors are eigenvectors of the annihilation operator $ae_z = ze_z$.

It is easy to see that

$$\langle e_{z'}, a^+e_z \rangle = \bar{z}'(e_{z'}, e_z),$$

which is compatible with the formal equation

$$a^+e_z = \frac{\partial}{\partial z} e_z,$$

a formula also noted by Glauber [25].

Equation (3.14) leads to the known expression of the reproducing kernel for $M \approx \mathbb{C}$

$$K(z, \bar{w}) := (e_z, e_{\bar{w}}) := fe_{\bar{w}}(z) = \exp(z\bar{w}).$$

$K_{\bar{w}} : z \mapsto e^{z\bar{w}}$ are contained in $\mathcal{F}_{\mathcal{H}}$ and $K$ is a positive kernel on $\mathbb{C}$. We find

$$K^{\cdot\cdot\cdot}(z, \bar{w}) = (e_z, a^+e_{\bar{w}}) = \frac{\partial}{\partial \bar{w}}(e_z, e_{\bar{w}}) = \bar{z}(e_z, e_{\bar{w}}),$$

e.i.

$$\Phi a^+ \Phi^{-1}(z) = z \in \mathfrak{a}_0.$$
Also
\[ K^a(z, w) = (e_z, ae_{\bar{w}}) = \bar{w}(e_z, e_{\bar{w}}) = \frac{\partial}{\partial z}(e_z, e_{\bar{w}}), \]
i.e.
\[ \Phi(a\Phi^{-1}(z) = \frac{\partial}{\partial z} \in \mathfrak{a}^t. \]

The operator \( \Phi a \Phi^{-1} (\Phi a^+ \Phi^{-1}) \) corresponding to \( a \) (respectively, \( a^+ \)) is acting on the pre-Hilbert space \( \mathcal{H}_K^0 \subset \mathcal{F}_K \) corresponding to the reproducing kernel \( K(z, \bar{w}) \) \((3.17)\), \( w \) fixed. \( a \) and \( a^+ \) are formal adjoint on the pre-Hilbert space \( \mathcal{H}_K^0 \), \( (av, w) = (v, a^+ w) \), \( v, w \in \mathcal{H}_K^0 \), and \( a^+ \) is \( a^\dagger \) in the notation of \( \mathfrak{sl}(n, \mathbb{C}) \).

Note that the principal vectors \( e_{\bar{w}} \in \mathcal{F}_K \) in Bargmann’s terminology (see eq. (1.10) in \[ \mathfrak{sl}(n, \mathbb{C}) \]), \( e_{\bar{w}}(z) = e^{\bar{w}z} \), correspond to the coherent vectors (3.13) parametrized with \( \bar{w}, \bar{v}, \bar{w} \) in Bargmann’s terminology (see eq. (1.10) in \( \mathfrak{sl}(n, \mathbb{C}) \)).

### 3.3.2. \( \mathfrak{sl}(2, \mathbb{C}) \)

Let us now consider the generators of \( \mathfrak{sl}(2, \mathbb{C}) \)
\[ J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
which verifies the commutation relations:
\[ [J_0, J_\pm] = \pm J_\pm; [J_-, J_+] = -2J_0. \]

Then we can see that

**Remark 4.** Proposition 7 for \( \mathfrak{sl}(2, \mathbb{C}) \) is realized as
\[ J_+ = -\frac{\partial}{\partial z}, \quad J_- = -2jz + z^2 \frac{\partial}{\partial z}, \quad J_0 = j - z \frac{\partial}{\partial z}. \]

**Proof.** Indeed, let
\[ g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}). \]

Then eq. \((3.2)\) becomes
\[ f_{\pi(c')}\psi(z) = (a' - c'z)^2f_{\psi}(\{d'z - b'(a' - c'z)^{-1}\}]. \]

For example, the calculation for \( X = J_+ \) corresponds to
\[ g'(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \quad \mu(g', z) = 1; \quad f_{\pi(c')}\psi(z) = f_{\psi}(z - t), \]
and by taking the derivatives at \( t = 0 \) we get the first relation \((3.23)\). The other ones are obtained similarly.

If we use the CS-vectors \( e_z = e^{z J_+ e_{j,-j}} \) for the minimal weight, i.e.
\[ J_+ e_{j,-j} \neq 0; \quad J_- e_{j,-j} = 0; \quad J_0 e_{j,-j} = -j e_{j,-j}, \]
then we get formally
\[ J_+ e_z = \partial e_z; \quad J_- e_z = (2jz - z^2 \partial)e_z; \quad J_0 e_z = (-j + z \partial)e_z. \]
Equations (3.23) and (3.24) differs by an overall “-” sign. See also Remark 5.

4. The semisimple case

4.1. Perelomov’s coherent vectors for semisimple Lie groups. All representations of compact Lie groups are CS-representations because these representations are highest weight representations. Kostant and Sternberg [36] showed that for any representation of a compact group $G$ the orbit to a projectivized highest weight vector is the only Kähler coherent state orbit. Harish-Chandra [29] has defined highest weight representations for non-compact semisimple (or even reductive) Lie groups. He has classified square integrable highest weight representations. This classification has been fully realized by Enright, Howe and Wallach, and independently by Jakobsen [23]. Lisiecki has emphasized (cf. [38] and Theorem 6.1 in [40]) that: a non-compact semisimple Lie group is a CS-group if and only if it is hermitian. If this is the case, the CS-representations of $G$ are precisely the highest weight representations. Each of them has a unique CS-orbit, which is the orbit through highest line. The starting point of the proof of Lisiecki is the paper of Borel [16], where it is proved: a noncompact semisimple Lie group $G$ admits a homogeneous Kähler manifold if and only if it is of hermitian type, and such a manifold is of the form $G/Z_{G(S)}$, where $Z_{G(S)}$ is the centralizer of a torus $S \subset G$; moreover, it is a holomorphic fiber bundle over the Hermitian symmetric space $G/K$, where $K$ is a maximal compact subgroup of $G$, with (compact) flag manifolds $K/Z_{G(S)}$ as fibers.

Let us consider again the triplet $(G, \pi, H)$ where $(G, \pi)$ is a CS-representation. Then this representation can be realized as an extreme weight representation. For linear connected reductive groups with $Z_K(\mathfrak{z}) = k$, where $\mathfrak{z}$ denotes the center of the Lie algebra $k$ of $K$, the effective representation is furnished by the Harish-Chandra theorem (cf. e.g. [34], p. 158). The theorem furnishes the holomorphic discrete series for the non-compact case, and for the compact case it is equivalent with the Borel-Weil theorem ([54]; also cf. [34], p. 143).

We use standard notation referring to Lie algebras of a complex semisimple Lie group $G$ [60]. In this case $\Delta \equiv \Delta_+, i.e. \Delta_+ = \{0\}$, i.e. all roots are semisimple.

$\mathfrak{g}$ – complex semisimple Lie algebra
$\mathfrak{t} \subset \mathfrak{g}$ – Cartan subalgebra
$\mathfrak{b} = \mathfrak{t} + \mathfrak{b}^u$ – Borel subalgebra
$\mathfrak{b}^u = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ – the nilradical of $\mathfrak{b}$
$\Sigma$ – root system for $(\mathfrak{g}, \mathfrak{t})$
$\Sigma^+$ – a positive root system
$\Psi$ – a simple root system for $\Sigma$
$\sum \alpha \in \Psi n_\mu(\alpha)\mu$ – unique, $n_\mu \in \mathbb{N}, n_\mu(\alpha) \geq 0 \text{ if } \alpha \in \Sigma^+; n_\mu(\alpha) \leq 0 \text{ if } \alpha \in \Sigma^-$
$\Psi \supset \Phi \rightarrow \Phi^r = \{\alpha \in \Sigma; n_\mu(\alpha) = 0 \text{ whenever } \mu \notin \Phi\}$
$\Phi^u = \{\alpha \in \Sigma; n_\mu(\alpha) > 0 \text{ for some } \mu \notin \Phi\} = \Sigma^+ \setminus \{\Sigma^+ \cap \Phi^r\}$
$p_\Phi = p_\phi^r + p_\phi^u$ – parabolic subalgebras of $\mathfrak{g}$ corresponding to $\Phi \subset \Psi$
$p_\phi = \mathfrak{t} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ – the reductive part of $p_\Phi$
$p_\phi^u = \sum_{\alpha \in \Phi^u} \mathfrak{g}_\alpha$ – the unipotent part of $p_\Phi$
$\Delta_0 = \Phi^r; \Delta_- = -\Phi^u; \Delta_+ = \Phi^u$
$B = \{g \in G; \text{Ad}(g)\mathfrak{b} = \mathfrak{b}\}$ – Borel subgroup (maximal solvable)
\[ P = \{ g \in G; \text{Ad}(g) p = p \} \]  – parabolic subgroup (contains a Borel subgroup).

In the notation of Definitions VII.2.4 p. 234, VII.2.6 p. 236 and VII.2.22, p. 244 in [47] we have \( \Phi^\prime \equiv \Delta^\prime \) and \( \Phi^\prime \equiv \Delta_k \).

We also need the commutation relations in the Cartan-Weyl basis [30]

\[
\begin{align*}
[H_i, H_j] &= 0, & i = 1, \ldots, r, H_i \in \mathfrak{t}, \\
[H_i, E_\alpha] &= \alpha_i E_\alpha, & \alpha_i = \alpha(H_i), \\
[E_\alpha, E_\beta] &= n_{\alpha,\beta} E_{\alpha+\beta}, & \alpha + \beta \in \Delta \setminus \{0\}, \\
[E_\alpha, E_{-\alpha}] &= H_\alpha = \sum \alpha_i H_i.
\end{align*}
\]

(4.1)

As a consequence, we have also the commutation relations:

\[
\begin{align*}
[H_{-\gamma}, E_\gamma] &= -\gamma H; & \gamma H := (\gamma, H) = \sum_{j=1}^r \gamma_j H_j; \\
[H, E_\alpha] &= \alpha(H) E_\alpha.
\end{align*}
\]

(4.2)

If the extreme weight \( j \) (here minimal) of the representation has the components \( j = (j_1, \ldots, j_r) \), where \( r \) is the rank of the Cartan algebra, then

\[
\begin{align*}
H_k e_j = j_k e_j, & \; k = 1, \ldots, r; \\
E_{\alpha} e_j &= 0, & \alpha \in \Delta_- \cup \Delta_0.
\end{align*}
\]

(4.3)

Now we take into account that \((\pi, \mathfrak{H})\) is a unitary representation of the group \( G \) on the Hilbert space \( \mathfrak{H} \). Recall that \( i\pi(X)|_{\mathfrak{g}_{\text{cov}}} \) is essentially selfadjoint. If \( \{H_k, E_\alpha\} \) is the Cartan-Weyl base (4.1) of complex Lie algebra \( \mathfrak{g} \), a base of the compact real form of \( \mathfrak{g} \) is \( iH_k, i(E_\alpha + E_{-\alpha}), E_\alpha - E_{-\alpha}, \alpha \in \Delta_+ \). The essentially selfadjointness condition implies that \( H_k^* = H_k \) and \( E_\alpha^* = E_{-\alpha}, \alpha \in \Delta_+ \). A base of a real (noncompact) form of \( \mathfrak{g} \) is \( iH_k, E_\alpha + E_{-\alpha}, -i(E_\alpha - E_{-\alpha}) \). If we denote \( K_\alpha := iE_\alpha \), then we have also \( K_\alpha^* = K_{-\alpha} \). So, it is convenient to introduce the notation

\[
F_\alpha := \begin{cases} E_\alpha, & \text{for the compact case} \\
K_\alpha, & \text{for the noncompact case} \end{cases}
\]

For any element \( X \in \mathfrak{g} \) the corresponding \( X \in \mathfrak{A}_M \) is a linear combination

\[
X = \sum c_k^i H_k + \sum_{\alpha \in \Delta_+} b_\alpha F_\alpha - \bar{b}_\alpha F_{-\alpha},
\]

where \( c_k^i \in \mathbb{R}, \; b_\alpha, \bar{b}_\alpha \in \mathbb{C} \). So, for \( G \ni g = e^X, \; X \in \mathfrak{g} \) we have the following realization of equation (2.2)

\[
e_{g,j} = \exp(X) e_j = e^{i\alpha(g)} e_{b,j}, \; e_{b,j} := \exp(\sum_{\alpha \in \Delta_+} b_\alpha F_\alpha - \bar{b}_\alpha F_\alpha) e_j.
\]

In accord to (2.4), the Perelomov’s CS-vectors are

\[
e_{b,j} = N(z) e_{z,j}, \; e_{z,j} = \exp(\sum_{\alpha \in \Delta_+} z_\alpha F_\alpha) e_j,
\]

(4.4)

where \( z_\alpha \) are local coordinates for the coordinate neighborhood \( \mathcal{V}_0 \subset M \).
4.2. Differential operators on semisimple Lie group orbits. We start introducing the notation
\[ Z := \sum_{\alpha \in \Delta_+} z_\alpha E_\alpha, \quad \partial_\alpha(Z) = E_\alpha, \quad \partial_\alpha = \frac{\partial}{\partial z_\alpha}, \quad \alpha \in \Delta_. \]
With this notation, the Perelomov’s coherent state vectors are
\[ e_{z,j} = \exp Z e_j, \]
but when not necessarily, the subindex \( j \) will be omitted.

In this paragraph we use a formal method to get the holomorphic differential action (3.11) of a generator \( X \) of the Lie algebra \( g \) of the group \( G \) on the homogeneous space \( M = G/H \). This method was developed in [5] (see also [4]) and applied in [6].

Let us consider Perelomov’s coherent state vectors (4.5). We associate to every generator \( X \in g \) a formal operator \( D_X \) on \( A_M \), where \( X := d\pi(X) \). Then we make the following

**Remark 5.** Let us suppose that we have the relation
\[ X . e_z = D_X(z) . e_z, \]
where \( e_z \) is the Perelomov’s state vector (2.4), in particular (4.5), belonging to the Hilbert space \( \mathcal{H} \) of the unitary continuous representation \( \pi \), and \( z \) are local coordinates on the homogeneous manifold \( M = G/H \). We suppose that \( D_X \) is a first order differential operator with polynomial coefficients of the form (3.11). Then the differential action \( X \) on the symmetric Fock space \( \mathcal{F}_H \)
\[ X_z(e_z, e_{\bar{w}}) := (e_z, X . e_{\bar{w}}) \]
is given by
\[ X_z = D_+(z). \]
We can also write down the relation
\[ X_{\bar{w}}(e_z, e_{\bar{w}}) = (X^+)_{\bar{w}}(e_z, e_{\bar{w}}). \]

**Proof.** Indeed, let
\[ f_{e_{\bar{w}}}(z) = (e_z, e_{\bar{w}}) = K(z, \bar{w}). \]
Then
\[ K^X(z, \bar{w}) = (e_z, X . e_{\bar{w}}) = (X^+. e_z, e_{\bar{w}}) = D_+(z)(e_z, e_{\bar{w}}). \]

If \( G \) is a Lie group and \( g \) is its Lie algebra, we shall use the formula (cf. [19] III, §6.4, Corollary 3, p. 313 )
\[ \text{Ad}(\exp Z) = \exp \text{ad}_Z, \quad Z \in g, \]
i.e. (cf. [19], II, §6.5, eq. (22)):
\[ e^{\bar{z}}Xe^{-Z} = \sum_{n \geq 0} \frac{1}{n!} \text{ad}_Z^n X, \quad X, Z \in g, \]
where
\[ \text{ad}_Y X = [Y, X], \quad \text{ad}^m_Y X = [Y, \text{ad}^{m-1}_Y X], \quad m > 1, \quad \text{ad}^0_Y X = X. \]

We also use the relation
\begin{equation}
(4.11) \quad e^Z \partial_\alpha (e^{-Z}) = - \left[ \partial_\alpha (Z) + \sum_{n \geq 1} \frac{1}{(n + 1)!} \text{ad}_Z^n \partial_\alpha (Z) \right],
\end{equation}
\[ \partial_\alpha (Y) = \partial_\alpha Y - Y \partial_\alpha = -\text{ad}_Y (\partial_\alpha). \]

We recall the definition of the Bernoulli numbers \( B_i \):\(^1\)
\begin{align*}
(4.12) \quad & \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2} x + \sum_{k \geq 1} (-1)^{k-1} \frac{B_k x^{2k}}{(2k)!} = \sum_{n \geq 0} c_n x^n, \\
(4.13) \quad & c_0 = 1; \ c_1 = \frac{1}{2}; \ c_{2k+1} = 0; \ c_{2k} = \frac{(-1)^{k-1}}{(2k)!} B_k, \\
(4.14) \quad & B_1 = \frac{1}{6}; \ B_2 = \frac{1}{30}; \ B_3 = \frac{1}{42}; \ B_4 = \frac{1}{300} \ldots
\end{align*}
We need:

**Lemma 1.** Let the relation:
\begin{equation}
(4.15) \quad \frac{1}{n!} = \sum_{k=0}^{n} c_k \frac{1}{(n-k+1)!}.
\end{equation}
Then the constants \( c_k \) of eq. (4.15) verifies the definition (4.12).

**Proof.** We have successively:
\begin{align*}
\sum_{n \geq 0} \frac{1}{n!} x^n &= \sum_{k,m,n \geq k} c_k \frac{x^n}{(n-k+1)!} \\
&= \sum_{k,m \geq 0} c_k \frac{1}{(m+1)!} x^{m+k} \\
&= \sum_{k \geq 0} c_k x^k \sum_{m \geq 0} \frac{x^m}{(m+1)!}.
\end{align*}
We obtain
\[ xe^x = \sum_{k \geq 0} c_k x^k [e^x - 1]. \]

We need also another formula similar to (4.15).
Lemma 2. Let the constants $d_k$ be defined by the relation:

\[(4.16) \quad \frac{1}{(n+2)!} = \sum_{k=0}^{n} d_k \frac{1}{(n-k+1)!}.\]

Then the constants $c$ and $d$ are related by

\[(4.17) \quad d_k = (-1)^k c_{k+1}.\]

Proof.

\[
\sum_{n \geq 0} \frac{1}{(n+2)!} x^{n+2} = \sum_{n \geq 0} \sum_{k=0}^{n} d_k \frac{x^{n+2}}{(n-k+1)!} = \sum_{k \geq 0} d_k x^{k+1} \sum_{m \geq 0} \frac{x^{m+1}}{(m+1)!}.
\]

So

\[
(e^x - 1) \sum_{k \geq 0} d_k x^{k+1} = e^x - x - 1.
\]

\[
\sum_{k \geq 0} d_k x^k = \frac{e^x - 1 - x}{x(e^x - 1)} = \frac{1}{x} - \frac{1}{e^x - 1} = -\frac{1}{x} \sum_{n \geq 1} c_n (-x)^n = \sum_{k \geq 0} c_{k+1} (-1)^k x^k.
\]

Eq. (4.12) was used. Equation (4.17) is proved. $\square$

Now we formulate the main result of the present paper:

Theorem 1. Let $G$ be a semisimple Lie group admitting a CS-representation $\pi$. If $X_\lambda \in \mathfrak{g}$ is a generator of the group $G$, then the corresponding holomorphic first-order differential operator $X_\lambda$ associated to the derived representation $d\pi$, $X_\lambda \in \mathcal{D}_1 = \mathcal{D}_0 \oplus \mathcal{D}'_1$, has polynomial coefficients, $X_\lambda \in \mathfrak{A}_1$. More exactly,

\[(4.18) \quad X_\lambda = P_\lambda + \sum_{\beta \in \Delta_+} Q_{\lambda,\beta} \partial_\beta, \lambda \in \Delta,
\]

where $P_\lambda$ and $Q_{\lambda,\beta}$ are polynomials in $z$ on the $G$-homogeneous CS-manifold $M$.

Explicitly, the differential operators $\{E_\alpha, H_i\}$ corresponding to the Cartan-Weyl base $\{E_\alpha, H_i\}$ (4.7) are as follows:
a) For $\alpha \in \Delta_+$,

\[(4.19)\]

$$E_\alpha = \nu \sum_{k \geq 0} c_k \sum_{\beta \in \Delta_+} p_{k\alpha\beta}(z) \partial_{\alpha+\beta},$$

where the coefficients $c_k$, related to the Bernoulli numbers by eq. (4.13), are given by eq. (4.15). The polynomials $p_{k\alpha\beta}, k \in \mathbb{N}, \alpha \in \Delta_+$ are given by the equation:

\[(4.20)\]

$$p_{k\alpha\beta}(z) = \sum_{\alpha_1, \ldots, \alpha_k} n_{\alpha_1, \ldots, \alpha_k} z_{\alpha_1} \cdots z_{\alpha_k}, \quad k \geq 1,$$

where

\[(4.21)\]

$$n_{\alpha_1, \ldots, \alpha_k} = n_{\alpha_1, \alpha} n_{\alpha_2, \alpha_1 + \alpha} \cdots n_{\alpha_k, \alpha_1 + \cdots + \alpha_{k-1}}, \quad (k \geq 1, \alpha_0 = 0),$$

and $n_{\alpha\beta}, \alpha, \beta \in \Delta_+$ are the structure constants of eq. (4.1), and for $k = 0$ the sum (4.19) is just $\partial_\alpha$.

The expression (4.19) can be put also into a form in which the Bernoulli numbers are explicit:

\[(4.22)\]

$$E_\alpha = \partial_\alpha + \frac{1}{2} \sum_{\beta \in \Delta_+} z_\beta n_{\beta, \alpha} \partial_{\alpha+\beta} + \sum_{k \geq 1} \frac{(-1)^{k-1}}{(2k)!} B_k \sum_{\beta \in \Delta_+} p_{k\alpha\beta} \partial_{\alpha+\beta}.$$

The degree of the polynomial $p$ has the property: degree $p_{k\alpha\beta} \leq \nu$; $p_{k\alpha\beta}$ as a function of $z$ contains only even powers. The table below contains the values of $\nu$.

| Degree $\nu$ for simple Lie algebras |
|------------------------------------|
| $A_l : \nu = l - 1 \quad l \geq 1$ |
| $B_l : \nu = 2l - 2 \quad l \geq 1$ |
| $C_l : \nu = 2l - 2 \quad l \geq 2$ |
| $D_l : \nu = 2l - 4 \quad l \geq 3$ |
| $E_6 : \nu = 10$ |
| $E_7 : \nu = 16$ |
| $E_8 : \nu = 28$ |
| $F_4 : \nu = 10$ |
| $G_2 : \nu = 4$ |

b) The differential action of the generators of the Cartan algebra is:

\[(4.23)\]

$$H = j + \sum_{\beta \in \Delta_+} \beta z_\beta \partial_\beta.$$

c) If $(\alpha, j) = 0$, then

\[(4.24)\]

$$E_\alpha = - \sum_{\beta \in \Delta_+} n_{\beta, -\alpha} z_\beta \partial_\beta.$$

d) If $\gamma \in \Delta_-$ is a simple root, then

\[(4.25)\]

$$E_\gamma = j \gamma z_{-\gamma} + \sum_{k \geq 0} d_k \sum_{\delta, \beta \in \Delta_+} q_{\gamma\delta}(z) p_{k\delta\beta}(z) \partial_{\beta+\delta},$$

where the coefficients $d$ are expressed through the coefficients $c$ by eq. (4.17).
The expression of the polynomials \( q_\gamma \delta, \gamma \in \Delta_-, \delta \in \Delta_+ \) is

\[
q_\gamma \delta = -\gamma z_\gamma \delta + \sum_{\mu \in \Delta_k} \zeta_{\delta - \gamma - \mu} n_{\delta - \mu} n_{\gamma - \mu}.
\]

In the case of a Hermitian symmetric space eq. \((4.19)\) becomes just:

\[
E_\alpha = \partial_\alpha,
\]
while eq. \((4.25)\) becomes

\[
-\sum_{\mu \in \Delta_+} \zeta_{\delta - \gamma - \mu} n_{\delta - \mu} n_{\gamma - \mu}.
\]

Proof. a) Let \( \alpha \in \Delta_+ \).

We apply the formula \((4.10)\):

\[
e^Z E_\alpha e^{-Z} = \sum_{n \geq 0} \frac{1}{n!} \text{ad}_Z^n E_\alpha
= \sum_{k,n \geq 0} \frac{c_k}{(n-k+1)!} \text{ad}_Z^n E_\alpha
= \sum_{k \geq 0} \sum_{m \geq 0} \frac{1}{(m+1)!} \text{ad}_Z^m (\text{ad}_Z^k E_\alpha).
\]

But

\[
\text{ad}_Z^k E_\alpha = \sum_{\alpha_1, \ldots, \alpha_k} \zeta_{\alpha_1} \cdots \zeta_{\alpha_k} n_{\alpha_1 + \cdots + \alpha_k} E_{\alpha + \alpha_1 + \cdots + \alpha_k},
\]

and the expression \((4.19)\) is obtained by successive application of the third commutation relation \((4.11)\). The sum \( \alpha + \alpha_1 + \cdots + \alpha_k \) goes until a \( k = \nu \) corresponding to the largest root (cf. \[18\], Chapter VI, Tables pp. 250-273). So the expression \((4.20)\) follows.

The relation

\[
\text{ad}_Z^k E_\alpha = \sum_{\beta} p_{k\alpha \beta}(z) E_{\beta + \alpha}
\]
leads to

\[
e^Z E_\alpha e^{-Z} = -\sum_{k \geq 0} c_k \sum_{\beta} p_{k\alpha \beta}(z) e^Z \partial_{\alpha + \beta} (e^{-Z}).
\]

The relations \((4.19)\), \((4.22)\) are proved.

For example, for the \( A_\ell \)-series \[18\]:

\[
A_\ell : \circ \circ \circ \cdots \cdots \circ \circ
\]

The maximal root is: \( \alpha_1 + \cdots + \alpha_\ell \). This implies the degree \( \nu \) for simple Lie algebra \( A_\ell \). Similarly for the other cases.
b), c) The differential actions corresponding to the generators of the Cartan algebra (eq. (4.23) and eq. (4.24)) were calculated in [5, 6] using the formula (4.10) and the commutation relations (4.1).

d) Let $\gamma \in \Delta_-$ be simple root. Then $[E_\alpha, E_\gamma] = n_{\alpha \gamma} E_{\alpha + \gamma}$. It is observed that $\alpha \in \Delta_+$, $\alpha + \gamma \in \Delta$ implies $\alpha + \gamma \in \Delta_-$.

Indeed, if: $\alpha + \gamma \in \Delta_-$ then $\gamma = -\alpha + \delta, \delta \in \Delta_-, -\alpha \in \Delta_-$, i.e. $\gamma$ is not simple. But this is not true!

So:

$$\alpha + \gamma \begin{cases} \in \Delta_0, & \text{or} \\ \in \Delta_+, & \text{or} \\ = 0. & \end{cases}$$

Now we do some preliminary calculation:

$$[Z, E_\gamma] = \sum_{\alpha \in \Delta, \alpha + \gamma \in \Delta_+} z_{\alpha} n_{\alpha \gamma} E_{\alpha + \gamma} + \sum_{\alpha \in \Delta_+, \alpha + \gamma \in \Delta_0} z_{\alpha} n_{\alpha \gamma} E_{\alpha + \gamma} - z_{\gamma \gamma} H.$$ 

Next

$$[Z, H] = \sum_{\alpha \in \Delta_+} z_{\alpha} [E_\alpha, H] = - \sum_{\alpha \in \Delta_+} \alpha z_{\alpha} E_\alpha.$$ 

Also:

$$[Z, E_\beta] = \sum_{\mu \in \Delta_+, \mu + \beta \in \Delta_+} z_{\mu} n_{\mu \beta} E_{\mu + \beta}, \beta \in \Delta_0, \beta = \alpha + \gamma.$$ 

We have used the relations (4.11), (4.12).

We apply again the formula (4.10):

$$e^Z E_\gamma e^{-Z} = \sum_{n \geq 0} \frac{1}{n!} \text{ad}_Z^n E_\gamma =$$

$$= E_\gamma + \sum_{m \geq 0} \frac{1}{(m + 1)!} \text{ad}_Z^m [Z, E_\gamma] =$$

$$= E_\gamma - \sum_{\alpha \in \Delta_+, \alpha + \gamma \in \Delta_+} z_{\alpha} n_{\alpha \gamma} e^Z \partial_{\gamma + \alpha} (e^{-Z}) +$$

$$+ \sum_{\alpha \in \Delta_+, \alpha + \gamma \in \Delta_0} z_{\alpha} n_{\alpha \gamma} E_{\alpha + \gamma} - \gamma z, H + R,$$

where

(4.28) $R := \sum_{m \geq 1} \frac{1}{(m + 1)!} \text{ad}_Z^{m-1} [q],$

and

(4.29) $q := -\gamma z - \gamma \sum_{\alpha \in \Delta_+} \alpha z_{\alpha} E_\alpha + \sum_{\alpha, \mu, \alpha + \gamma \in \Delta_+, \alpha + \gamma + \mu \in \Delta_+} z_{\alpha} n_{\alpha \gamma} z_{\mu} n_{\mu, \alpha + \gamma} E_{\mu + \alpha + \gamma}.$
Changing the summation variable \( \alpha \to \delta \) in the first sum in the expression (4.29) and denoting \( \mu + \alpha + \gamma \to \delta \) in the second sum of the same expression, we get finally for \( q \) the formula

\[
q = \sum_{\delta \in \Delta_+} q_\gamma E_\delta ,
\]

and the formula (4.26) is proved.

We continue to calculate \( R \):

\[
R = \sum_{m \geq 0} \frac{1}{(m + 2)!} \text{ad}_Z^m \sum_{\delta \in \Delta_+} q_\gamma E_\delta .
\]

Now we use eq. (4.16). Then

\[
(4.31) \quad R = - \sum_{k \geq 0} d_k \sum_{\beta, \delta \in \Delta_+} q_{\gamma, \beta} p_{k\delta\beta}(z) e^Z \partial_{\delta+\beta}(e^{-Z}) .
\]

So we get finally:

\[
e^Z E_\gamma e^{-Z} = E_\gamma - \sum_{\alpha \in \Delta_+} \sum_{\alpha+\gamma \in \Delta_+} z_\alpha n_{\alpha\gamma} E_{\alpha+\gamma} - \gamma z H + \sum_{\alpha \in \Delta_+} \sum_{\alpha+\gamma \in \Delta_+} z_\alpha n_{\alpha\gamma} E_{\alpha+\gamma} + R ,
\]

and eq. (4.25) is proved. \( \square \)

4.3. Examples.

4.3.1. \( M = SU(3)/S(U(1) \times U(1) \times U(1)) \). In this section we follow closely [37].

The commutation relations of the generators are:

\[
[C_{ij}, C_{kl}] = \delta_{jk} C_{il} - \delta_{il} C_{kj}, \quad 1 \leq i, j \leq 3 .
\]

Let us consider the following parametrizations useful for the Gauss decomposition and also in the definition of the coherent states for the manifold \( M \):

\[
V_+(\zeta) := \exp(\zeta_{12} C_{12} + \zeta_{13} C_{13} + \zeta_{23} C_{23}),
\]

\[
(4.34) \quad V'_+(z) := \exp(z_{23} C_{23}) \exp(z_{12} C_{12} + z_{13} C_{13}).
\]

Let us denote by the same letter \( C_{ij} \) the \( n \times n \)-matrix having all elements 0 except at the intersection of the line \( i \) with the column \( j \), that is \( C_{ij} = (\delta_{ai} \delta_{bj})_{1 \leq a, b \leq n} \). Here \( n = 3 \). Then:

\[
V_+(\zeta) = \begin{pmatrix}
1 & \zeta_{12} & \zeta_{13} + \frac{1}{2}\zeta_{12}\zeta_{23} \\
0 & 1 & \zeta_{23} \\
0 & 0 & 1
\end{pmatrix},
\]
\[ V'_+(z) = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}. \]

Now observing that for
\[ z_{12} = \zeta_{12}; \quad z_{13} = \zeta_{13} + \frac{1}{2}\zeta_{12}\zeta_{23}; \quad z_{23} = \zeta_{23}, \]
we get
\[ V_+(\zeta) = V'_+(z). \]

So we have two parametrizations of the compact non-symmetric flag manifold
\[ M = SU(3)/S(U(1)) \times U(1) \times U(1): \]
one in \( \zeta \), given by eq. (4.35) and the other one in \( z \),
given by (4.36), which are identified using the relations (4.37).

Let us consider also the vectors
\[ \phi_z = [V'_+(z)]^+ \phi_w = \exp(\bar{z}_{12}C_{21} + \bar{z}_{13}C_{31}) \exp(\bar{z}_{23}C_{32})\phi_w. \]

\( \phi_w \) is chosen as maximal weight vector corresponding to the weight \( w = (w_1, w_2, w_3) \) such that \( j_1 = \omega_1 - \omega_2 \geq 0, j_2 = \omega_2 - \omega_3 \geq 0 \) and the lowering operators are \( C_{ij}, i > j \),
while \( C_{ii} \) corresponds to the Cartan algebra, i.e.
\[ \begin{cases} 
C_{ij}\phi_w &\neq 0, \ i > j; \\
C_{ij}\phi_w & = 0, \ i < j; \\
C_{ii}\phi_w & = w_i\phi_w. 
\end{cases} \]

The coherent vectors corresponding to the representation \( \pi_w \) determined by eqs. (4.40)
are introduced as
\[ e_z = \pi_w((V'_+(z))\phi_w). \]

Denoting by \( Z \) the matrix
\[ Z = \begin{pmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{pmatrix}, \]
the reproducing kernel which determines the scalar product \( (e_z, e_{\bar{z}}) \) has the expression:
\[ K(ZZ^+) = \Delta_1^2(ZZ^+)\Delta_2^2(ZZ^+); \]
\[ \Delta_1 = 1 + |z_{12}|^2 + |z_{13}|^2; \]
\[ \Delta_2 = (1 + |z_{12}|^2 + |z_{13}|^2)(1 + |z_{23}|^2) - |z_{12} + z_{13}z_{23}|^2. \]

In particular, it is observed that \( z_{23} = 0 \) corresponds to the manifold \( SU(3)/S(U(2) \times U(1)) = G_1(C^3) = CP^2. \)

In order to compare with the scalar product for coherent states on \( M \approx CP^2 \approx G_1(C^3) \) we remember that in the case of the Grassmannian we have used in [5, 6] a weight which here corresponds to \( w_1 = 1, w_2 = w_3 = 0 \) and then on \( CP^2 \) the reproducing kernel is just
\[ K(ZZ^+) = \Delta_1. \]
We underline that the calculation given below, which will proof Lemma 4, is algebraic, and we do not use the value of the reproducing kernel.

Let us introduce the simplifying notation

\[ E(z) := V'_+(z). \]

We shall find the operators \( \tilde{C}_{ij} \) such that

\[ EC_{ij} = \tilde{C}_{ij}E. \]

Then

\[ (e_z, C_{ij}e_z) = (\phi_w, EC_{ij}E^+\phi_w) = (\phi_w, \tilde{C}_{ij}EE^+\phi_w) = \tilde{C}_{ij}(e_z, e_z). \]

In the coherent state representation (4.41) \( C \) is the differential operator associated to the operator \( \pi_w(C) \).

**Lemma 3.** The operators \( \tilde{C}_{ij} \) associated to the operators \( C_{ij} \) as in (4.44) are given by the formulas:

\[
\begin{align*}
\tilde{C}_{11} & = C_{11} - z_{12} \frac{\partial}{\partial z_{12}} - z_{13} \frac{\partial}{\partial z_{13}}, \\
\tilde{C}_{12} & = \frac{\partial}{\partial z_{12}}, \\
\tilde{C}_{13} & = \frac{\partial}{\partial z_{13}}, \\
\tilde{C}_{21} & = C_{21} + z_{12}(C_{11} - C_{22}) - z_{12}^2 \frac{\partial}{\partial z_{12}} - (z_{13} - z_{12}z_{23}) \frac{\partial}{\partial z_{23}} - z_{12}z_{13} \frac{\partial}{\partial z_{13}}, \\
\tilde{C}_{22} & = C_{22} + z_{12} \frac{\partial}{\partial z_{12}} - z_{23} \frac{\partial}{\partial z_{23}}, \\
\tilde{C}_{23} & = \frac{\partial}{\partial z_{23}} + z_{12} \frac{\partial}{\partial z_{13}}, \\
\tilde{C}_{31} & = C_{31} + z_{23}C_{21} - z_{12}C_{32} + z_{13}(C_{11} - C_{33}) - z_{12}z_{23}(C_{22} - C_{33}) - z_{13}^2 \frac{\partial}{\partial z_{13}} - z_{23}(z_{13} - z_{12}z_{23}) \frac{\partial}{\partial z_{23}} - z_{12}z_{13} \frac{\partial}{\partial z_{12}}, \\
\tilde{C}_{32} & = C_{32} + z_{23}(C_{22} - C_{33}) - z_{23}^2 \frac{\partial}{\partial z_{23}} + z_{13} \frac{\partial}{\partial z_{12}}, \\
\tilde{C}_{33} & = C_{33} + z_{23} \frac{\partial}{\partial z_{23}} + z_{13} \frac{\partial}{\partial z_{13}}.
\end{align*}
\]

**Proof.** First, it is observed that

\[ \frac{\partial E}{\partial z_{23}} = C_{23}E; \quad \frac{\partial E}{\partial z_{13}} = C_{13}E; \quad \frac{\partial E}{\partial z_{12}} = (C_{12} - z_{23}C_{13})E. \]

Then formula (4.10) is applied, taking into account the commutation relations (4.32). One important observation is that in the relation (4.34) the generators in the second exponential commutes and in fact this equation is expressed in one-parameter subgroups.
Another useful relation is
\[
\exp(z_{23}C_{23}) \exp(z_{12}C_{12} + z_{13}C_{13}) \exp(-z_{23}C_{23}) = \exp(z_{12}C_{12} + (-z_{12}z_{23} + z_{13})C_{13})
\]
\[\square\]

**Lemma 4.** The differential operators \( C_{ij} \) associated to the generators \( C_{ij} \) are given by the formulas:

\[
\begin{align*}
C_{11} &= -z_{12}\partial_{12} - z_{13}\partial_{13} + w_1, \\
C_{12} &= \partial_{12}, \\
C_{13} &= \partial_{13}, \\
C_{21} &= -z_{12}^2\partial_{12} - z_{12}z_{13}\partial_{13} + (z_{12}z_{23} - z_{13})\partial_{23} + (w_1 - w_2)z_{12}, \\
C_{22} &= z_{12}\partial_{12} - z_{23}\partial_{23} + w_2, \\
C_{23} &= z_{12}\partial_{13} + \partial_{23}, \\
C_{31} &= -z_{12}z_{13}\partial_{12} - z_{13}^2\partial_{13} + (z_{12}z_{23} - z_{13})z_{23}\partial_{23} + (w_1 - w_3)z_{13} - (w_2 - w_3)z_{23}z_{23}, \\
C_{32} &= z_{13}\partial_{12} - z_{23}^2\partial_{23} + (w_2 - w_3)z_{23}, \\
C_{33} &= z_{13}\partial_{13} + z_{23}\partial_{23} + w_3.
\end{align*}
\]

**Proof.** The operators determined in Lemma 3 are used taking into account eqs. (4.40).
\[\square\]

We have underlined the apparition of a third-degree polynomial multiplying the partial derivative of \( C_{31} \). Note also the relation \( C_{11} + C_{22} + C_{33} = w_1 + w_2 + w_3 \).

4.3.2. \( M = Sp(3, \mathbb{R})/S(U(1) \times U(1) \times U(1)) \). This is an example of a non-symmetric, non-compact manifold. Other simple examples can be constructed taking quotients of the groups \( SO^*(6) \) or \( SU(2, 1) \).

Firstly, note that:
\[
Sp(3, \mathbb{R})/S(U(1) \times U(1) \times U(1)) = Sp(3, \mathbb{R})/SU(3) \times SU(3)/S(U(1) \times U(1) \times U(1)).
\]

The expression of the reproducing kernel is:
\[
K(\zeta \zeta^+) = \Delta_1^{j_1}(\zeta \zeta^+)\Delta_2^{j_2}(\zeta \zeta^+)\Delta_3^{j_3}(\zeta \zeta^+),
\]
where
\[
\zeta = ZW; \quad WW^+ = (1 - SS^+)^{-1}; \quad S = S^t,
\]

\( W \) is a \( 3 \times 3 \) triangular matrix and
\[
\zeta \zeta^+ = Z(1 - SS^+)^{-1}Z^+.
\]
The differential action of the generators is given by the formulas \((i, j = 1 - 3)\):

\[
\bar{C}_{ij} = C_{ij} + \sum_{r=1}^{3} s_{ir} \left( \frac{\partial}{\partial s_{jr}} + \frac{\partial}{\partial s_{rj}} \right), \\
\bar{X}_{ij} = \frac{1}{2} \left( \frac{\partial}{\partial s_{ij}} + \frac{\partial}{\partial s_{ji}} \right), \\
\bar{Y}_{ij} = \frac{1}{2} \sum_{r,r'=1}^{3} s_{ir} s_{jr'} \left( \frac{\partial}{\partial s_{rr'}} + \frac{\partial}{\partial s_{r'r}} \right) + \frac{1}{2} \sum_{r=1}^{3} (s_{ir} C_{jr} + s_{jr} C_{ir}).
\]

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