CUTTING SETS OF CONTINUOUS FUNCTIONS ON THE UNIT INTERVAL

MAREK BALCERZAK, PIOTR NOWAKOWSKI, AND MICHAŁ POPLAWSKI

Abstract. For a function $f : [0,1] \to \mathbb{R}$, we consider the set $E(f)$ of points at which $f$ cuts the real axis. Given $f : [0,1] \to \mathbb{R}$ and a Cantor set $D \subset [0,1]$ with $\{0,1\} \subset D$, we obtain conditions equivalent to the conjunction $f \in C[0,1]$ (or $f \in C^\infty[0,1]$) and $D \subset E(f)$. This generalizes some ideas of Zabeti. We observe that, if $f$ is continuous, then $E(f)$ is a closed nowhere dense subset of $f^{-1}([0])$ where each $x \in \{0,1\} \cap E(f)$ is an accumulation point of $E(f)$. Our main result states that, for a closed nowhere dense set $F \subset [0,1]$ with each $x \in \{0,1\} \cap E(f)$ being an accumulation point of $F$, there exists $f \in C^\infty[0,1]$ such that $F = E(f)$.

1. Introduction

Our research has been inspired by the studies by Zabeti [6]. According to [6], we say that a function $f : [0,1] \to \mathbb{R}$ cuts the real axis at a point $x \in [0,1]$ provided that $f(x) = 0$, and in every neighbourhood $U$ of $x$ there exist $y, z \in U$ such that $f(y) < 0 < f(z)$. Zabeti in [6] showed two examples of continuous functions cutting the real axis at every point of a measurable set of positive measure. In the first example, a continuous function is of infinite variation, and in the second example, a function has derivatives of all orders.

Given a function $f : [0,1] \to \mathbb{R}$, the set of all points $x \in [0,1]$ at which $f$ cuts the real axis will be called the cutting set for $f$ and denoted by $E(f)$.

Proposition 1.1. If $f : [0,1] \to \mathbb{R}$ is a continuous function then $E(f)$ is a closed nowhere dense subset of $f^{-1}([0])$. Additionally, each $x \in \{0,1\} \cap E(f)$ is an accumulation point of $E(f)$.

Proof. Assume that a sequence $(x_n)$ of points in $E(f)$ is convergent to $x \in [0,1]$. Then $f(x) = 0$ by the continuity of $f$. Fix a neighbourhood $U$ of $x$ and find $x_n \in U$. Pick a neighbourhood $V \subset U$ of $x_n$. Then there are points $y, z \in V$ such that $f(y) < 0 < f(z)$. Hence $x \in E(f)$. Thus $E(f)$ is closed.

Suppose that $E(f)$ is dense in some nondegenerate interval $I$. Since $f$ is continuous, it should be constantly equal to zero on $I$ which contradicts the fact that it cuts the real axis at every point of $E(f)$. Consequently, $E(f)$ is nowhere dense. The final assertion follows from the definition of $E(f)$.

As it is shown in [6, Theorem 1], there exists a continuous function $f : [0,1] \to \mathbb{R}$ with $E(f)$ containing a Cantor-type set of positive measure. This result will be discussed and generalized in Section 2. Observe that, by Proposition 1.1, the set $E(f)$ cannot be of full measure in any nondegenerate interval.

If we do not assume that $f$ is continuous, a situation is different.

2010 Mathematics Subject Classification. 54C30, 54E52, 26A30, 26A24.

Key words and phrases. Cutting set, Baire category, Cantor-type sets, continuous functions, functions of class $C^\infty$, variation of a real function.

Piotr Nowakowski was supported by the GA ČR project 20-22230L and RVO: 67985840.
Example 1. Consider (see [2]) a differentiable function \(g: [0, 1] \to \mathbb{R}\) such that for any interval \((a, b) \subset [0, 1]\) there exist points \(x_1, x_2 \in (a, b)\) such that \(g'(x_1) < 0\) and \(g'(x_2) > 0\). This shows that the function \(g\) is nowhere monotone in \([0, 1]\). Observe that \(E(g') = \{x: g'(x) = 0\}\). Also, the set \(E(g')\) is dense. Indeed, take any open interval \((a, b) \subset [0, 1]\) and pick \(x, y \in (a, b)\) such that \(g'(x) < 0 < g'(y)\). Since the derivative \(g'\) has the Darboux property (see [5]), there exists a point \(z\) between \(x\) and \(y\) such that \(g'(z) = 0\). It is known that the derivative \(g'\) is Baire 1 (we have \(g'(x) = \lim_{n \to \infty} n(g(x + 1/n) - g(x))\) for \(x \in [0, 1]\)). Hence \(\{x: g'(x) = 0\}\) is of type \(G_\delta\) (see [1, Theorem 5.14]). Thus \(E(g')\) is a comeager set.

The article is organized as follows. In Sections 2 and 3, we discuss conditions under which a Cantor set is contained in \(E(f)\) for \(f \in C[0, 1]\) and \(f \in C^\infty[0, 1]\), respectively. This refers to Theorems 1 and 2 of Zabeti from his paper [6]. Section 4 contains the main result where we reverse the situation described in Proposition 5.14. Namely, for a closed set \(F \subset [0, 1]\) with the respective properties, we construct a function \(f \in C^\infty[0, 1]\) such that \(E(f) = F\). In Section 5, we formulate an open problem concerning the measure of the cutting set of a typical function in \(C[0, 1]\). The Appendix presents a proof of the result, being a byproduct of our considerations, which states that the complement of the set of functions in \(C[0, 1]\) with infinite variation is a \(\sigma\)-porous set.

2. Cantor sets included in the cutting sets of continuous functions

By the Brouwer theorem [3, Theorem 7.4], a topological space is homeomorphic to the classical ternary Cantor set in \([0, 1]\) if and only if, it is perfect, nonempty compact metrizable and zero-dimensional. Among subspaces of \(\mathbb{R}\) those spaces are exactly nonempty perfect, compact and nowhere dense sets. Such sets will be called Cantor sets in \(\mathbb{R}\).

Zabeti in [6] considered a subclass of Cantor subsets of \([0, 1]\). Because of a construction, they are usually called central Cantor sets. Every central Cantor set is generated by a sequence \((\xi_n)_{n \geq 0}\) with positive real terms where \(\xi_0 := 1\), and \(0 < \xi_{n+1} < (1/2)\xi_n\) for all \(n \geq 0\). In the first step, we remove from \([0, 1]\) the open middle interval of length \(\xi_0 - 2\xi_1\). Then we obtain a set \(D_1\) as the union of two closed intervals \([0, \xi_1], [1 - \xi_1, 1]\). So, the Lebesgue measure \(\lambda(D_1)\) is \(2\xi_1\). Having constructed \(D_n\) as the union of \(2^n\) constituent intervals of length \(\xi_n\), we remove from each of them the open middle interval of length \(\xi_n - 2\xi_{n+1}\). Then we obtain a set \(D_{n+1}\) that is the union of \(2^{n+1}\) pairwise disjoint closed intervals of length \(\xi_{n+1}\). The intersection \(D := \bigcap_n D_n\) is the central Cantor set generated by \((\xi_n)_{n \geq 0}\). Clearly, the measure \(\lambda(D)\) equals \(\lim_{n \to \infty} 2^n \xi_n\).

Given a function \(f: [0, 1] \to \mathbb{R}\) and a closed interval \(I \subset [0, 1]\), we denote by \(\text{Var}(f, I)\), the variation of \(f\) restricted to \(I\). In particular, \(\text{Var}(f, [0, 1])\) is the total variation of \(f\) on \([0, 1]\). A result of [6] shows, for a given central Cantor set \(D\) of positive measure \(\alpha \in (0, 1)\), an example of a continuous function \(f: [0, 1] \to \mathbb{R}\) with infinite variation on \([0, 1]\), that cuts the real axis at every point of \(D\). One of our aims is to obtain a strengthened description of this phenomenon. Our approach is more general. Firstly, we consider all Cantor sets in \([0, 1]\), not necessarily central Cantor sets of positive measure as it was done in [6]. Secondly, we will observe that the main idea of the Zabeti example characterizes all continuous functions \(f\) cutting the real axis at any point of a Cantor set, and in fact, we can make \(f\) of finite or infinite variation as we want.
Note that, in the respective interpretation, continuous functions of infinite variation on $[0,1]$ are typical since they form a comeager set in the Banach space $C[0,1]$ with the supremum norm. This fact is known since a function of finite variation on $[0,1]$ is differentiable almost everywhere, and continuous nowhere differentiable functions on $[0,1]$ form a comeager set in $C[0,1]$; see [4, Section 11]. However, we propose a stronger version of this fact.

Let us recall the notions of porous and $\sigma$-porous sets in a metric space (see [7], [8]). Let $X$ be a metric space. By $B(x,r)$ we will denote the open ball with center $x \in X$ and radius $r > 0$. Given $E \subset X$, $x \in X$ and $R > 0$, we write $\gamma(x,R,E) := \sup\{r > 0 : \exists z \in X \ B(z,r) \subset B(x,R) \setminus E\}$. We define the porosity of $E$ at $x$ as

$$p(E,x) := 2 \limsup_{\varepsilon \rightarrow 0^+} \frac{\gamma(x,\varepsilon,E)}{\varepsilon}.$$  

We say that a set $E$ is porous if its porosity is positive at each $x \in E$. We say that $E$ is $\sigma$-porous if it is a countable union of porous sets. Observe that every porous set is nowhere dense, so every $\sigma$-porous set is meager.

**Theorem 2.1.** The subset $U$ of $C[0,1]$ that consists of continuous functions with infinite variation on $[0,1]$ is a $G_\delta$ comeager set in this space. Moreover, its complement is $\sigma$-porous.

We postpone a proof of this result to the Appendix.

Now, let us present the promised characterization extending the first example of Zabeti [6].

**Theorem 2.2.** Fix $f : [0,1] \rightarrow \mathbb{R}$. Let $D \subset [0,1]$ be a Cantor set with $\{0,1\} \subset D$, and let $W_n$, $n \in \mathbb{N}$, be a one-to-one sequence of all connected components of $[0,1] \setminus D$. Then $f \in C[0,1]$ with $D \subset E(f)$ if and only if the following conditions hold simultaneously

(i) $f|_D = 0$, functions $f|_{\text{cl}(W_n)}$, $n \in \mathbb{N}$, are continuous, and the series $\sum_n \chi_{\text{cl}(W_n)} f$ is uniformly convergent on $[0,1]$;

(ii) $D \subset \text{cl} \left( \bigcup_{n \in M^-} W_n \right) \cap \text{cl} \left( \bigcup_{n \in M^+} W_n \right)$ where $M^- := \{n : W_n \cap f^{-1}((-\infty,0]) \neq \emptyset\}$ and $M^+ := \{n : W_n \cap f^{-1}([0,\infty]) \neq \emptyset\}$.

**Proof.** **Necessity.** To show (i) it suffices to prove that the series $\sum_n \chi_{\text{cl}(W_n)} f$ is uniformly convergent on $[0,1]$. Suppose it is not the case. So, the uniform Cauchy condition fails. Hence there exist $\varepsilon > 0$, infinitely many indices $n_1 < n_2 < \ldots$ and points $x_j \in W_{n_j}$ such that $|f(x_j)| \geq \varepsilon$ for all $j \in \mathbb{N}$. Pick a convergent subsequence $x_{k_j} \rightarrow x$. Since $\lambda(W_{n_j}) \rightarrow 0$, we have $x \in D$. But $f(x) = 0$ which together with $|f(x_{k_j})| \geq \varepsilon$ for $j \in \mathbb{N}$ contradicts the continuity of $f$ at $x$.

To show (ii) note that, since $D \subset E(f)$, around every point $x \in D$ one can find points $y$ and $z$ with $f(y) < 0 < f(z)$. Since $f|_D = 0$, we have $y \in \bigcup_{n \in M^-} W_n$ and $z \in \bigcup_{n \in M^+} W_n$.

**Sufficiency.** From (i) it follows that $f$ is continuous on $[0,1]$. Let $x \in D$. Then $f(x) = 0$ and by (ii), in every neighbourhood of $x$ we can pick points $y, z \in \bigcup_{n \in \mathbb{N}} W_n$ such that $f(y) < 0 < f(z)$ as desired. □

A variant of the above theorem, where one requires the infinite variation of $f \in C[0,1]$, can be stated as follows.

**Theorem 2.3.** Fix $f : [0,1] \rightarrow \mathbb{R}$. Let $D \subset [0,1]$ be a Cantor set with $\{0,1\} \subset D$, and let $W_n$, $n \in \mathbb{N}$, be a one-to-one sequence of all connected components of $[0,1] \setminus D$. Then $f \in C[0,1]$ with
Var\( (f, [0, 1]) = \infty \) and \( D \subset E(f) \) if and only if conditions (i), (ii) stated in Theorem 2.2 hold together with

(iii) \( \sum_{n \in \mathbb{N}} \text{Var}(f, \text{cl}(W_n)) = \infty. \)

Proof. Since \( f|_D = 0 \), it is enough to observe that \( \text{Var}(f, [0, 1]) = \sum_{n \in \mathbb{N}} \text{Var}(f, \text{cl}(W_n)) \). The rest is a consequence of the previous theorem. \( \square \)

Remark 2. The example given in [6] Theorem 1] follows the scheme of sufficiency in Theorem 2.3. Namely, the author considers a central Cantor set \( D \) generated by a sequence \((\xi_n)_{n \geq 0}\) with \( \lambda(D) > 0 \). If \( W = (a, b) \) is any open interval that has been removed from \([0, 1]\) in the \( n \)-th step of the construction of \( D \), the value \( f(x) \) for \( x \in W \) is simply defined as \( c_n \sin \left( \frac{2\pi}{b-a}x \right) \) where \( c_n > 0 \) with

(2) \[ \sum_n c_n < \infty \quad \text{and} \quad \lim_{n \to \infty} 2^n c_n = \infty. \]

Also, let \( f(x) := 0 \) for each \( x \in D \). The first part of (2) guarantees that the series \( \sum_k f|_{\text{cl}(W_k)} \) is uniformly convergent on \([0, 1]\) (cf. condition (i)), and the second part of (2) forces that the total variation of \( f \) on \([0, 1]\) is infinite. Indeed, note that \( \text{Var}(f, \text{cl}(W)) = 2c_n \) and then it suffices to use (iii). The definition of \( f \) on \( W \) clearly implies that condition (ii) holds.

In the case of an arbitrary Cantor set \( D \subset [0, 1] \) with \( \{0, 1\} \subset D \), one can mimic this construction. Firstly, we order the sequence of all open components \( W_n, n \in \mathbb{N} \), of \([0, 1]\) \( \cap D \) with respect to decreasing lengths (if a few intervals have the same length, they can be ordered arbitrarily). Using this ordering, we define \( f|_{\text{cl}(W_n)}, n \in \mathbb{N} \), step by step taking care to obtain (i) and (ii). We let \( f(x) := 0 \) for \( x \in D \). Additionally, if we want to control \( V(f, [0, 1]) \), we should use condition (iii).

3. The case of functions of class \( C^\infty[0, 1] \)

We denote by \( C^\infty[0, 1] \) the vector space of functions \( f : [0, 1] \to \mathbb{R} \) that have derivatives of all orders on \([0, 1]\). This is a complete space when we use the metric \( d \) given by

\[
d(f, g) = \sum_{n=0}^{\infty} 2^{-n} \min\{1, \|f^{(n)}(x) - g^{(n)}(x)\|\}
\]

where \( \| \cdot \| \) denotes as before, the supremum norm. In [6] Theorem 2], the author constructed a function \( f \in C^\infty[0, 1] \) and a measurable set \( D \subset [0, 1] \) (of type \( G_\delta \) of positive measure (that can be arbitrarily close to 1) such that \( D \subset E(f) \). In this case, the choice of a set \( D \) is different from that proposed previously in [6] Theorem 1] since \( D \) is not a Cantor set. However, we have observed that the previous technique still works after the respective modification. Namely, the counterpart of Theorem 2.2 holds in the following version.

**Theorem 3.1.** Fix \( f : [0, 1] \to \mathbb{R} \). Let \( D \) be a Cantor set in \([0, 1] \) with \( \{0, 1\} \subset D \), and let \( W_n, n \in \mathbb{N} \), be a one-to-one sequence of all connected components of \([0, 1] \) \( \cap D \). Then \( f \) is in \( C^\infty[0, 1] \) with \( D \subset E(f) \) if and only if the following conditions hold simultaneously

(i) \( f|_D = 0 \), functions \( f|_{\text{cl}(W_n)}, n \in \mathbb{N} \), are in \( C^\infty(\text{cl}(W_n)) \), and the series \( \sum_n \chi_{\text{cl}(W_n)} f^{(p)} \), for \( p \geq 0 \), are uniformly convergent on \([0, 1]\);

(ii) \( D \subset \text{cl}(\bigcup_{n \in M^-} W_n) \cap \text{cl}(\bigcup_{n \in M^+} W_n) \) where \( M^- := \{n : W_n \cap f^{-1}((-\infty, 0]) \neq \emptyset\} \) and \( M^+ := \{n : W_n \cap f^{-1}([0, \infty]) \neq \emptyset\} \).
The proof is analogous to that for Theorem 2.2.

The following example modifies the idea of [3, Theorem 2]. We repeat the method constructing the respective function $f \in C^\infty[0, 1]$. Using the sufficiency part of Theorem 3.1, we will ensure that, for a given Cantor set $D \subset [0, 1]$, $f$ can be chosen so that $D \subset E(f)$. Therefore, we connect somehow the ideas of the two results stated in [6].

**Example 3.** Consider the following function $h : \mathbb{R} \to \mathbb{R}$ which has derivatives of all orders on $\mathbb{R}$ and obtains non-zero (positive) values exactly on $(0, 1)$:

$$h(x) := \begin{cases} e^{-x^{-2}} e^{-(x-1)^{-2}}, & \text{if } x \in (0, 1); \\ 0, & \text{otherwise.} \end{cases}$$

Fix a Cantor set $D$ in $[0, 1]$ with $\{0, 1\} \subset D$, and let $W_n$, $n \in \mathbb{N}$, be a one-to-one sequence of all connected components of $[0, 1] \setminus D$. With any $W_n := (a_n, b_n)$, we associate the function $f_n : [0, 1] \to \mathbb{R}$ given by

$$f_n(x) := c_n h \left( \frac{x - a_n}{b_n - a_n} \right) \sin \left( \frac{2\pi(x - a_n)}{b_n - a_n} \right)$$

where a sequence $(c_k)$ will be chosen later. Then $f_n|_{\text{cl}(W_n)} \in C^\infty(\text{cl}(W_n))$ and, thanks to the function sinus, it attains negative and positive values on $W_n$. Denote $\epsilon_n := b_n - a_n$.

Fix an integer $p \geq 0$. By the Leibniz rule, we can calculate the $p$-th derivative of $f_n$ at a point $x \in W_n$ as follows

$$f_n^{(p)}(x) = c_n \epsilon_n^{-p} (2\pi)^p \sum_{k=0}^{p} \binom{p}{k} h^{(k)} \left( \frac{x - a_n}{\epsilon_n} \right) (2\pi)^{-k} \sin^{(p-k)} \left( \frac{2\pi(x - a_n)}{\epsilon_n} \right).$$

Note that

$$(4) \quad ||f_n^{(p)}|| \leq c_n \epsilon_n^{-p} M_p \quad \text{for all } n \in \mathbb{N}$$

where $M_p := (2\pi)^p \sum_{k=0}^{p} \binom{p}{k} (2\pi)^{-k} ||h^{(k)}||$. Hence every series $\sum_{n=1}^{\infty} f_n^{(p)}$, $p \geq 0$, is uniformly convergent on $[0, 1]$ whenever $(\epsilon_n)$ is chosen so that $\sum_{n=1}^{\infty} c_n \epsilon_n^{-p} < \infty$ since then the Weierstrass test works. For instance, the choice $c_n := \frac{1}{n^p} \exp \left( \frac{1}{\epsilon_n} \right)$ is good since $\lim_{x \to 0^+} x^{-p} \exp \left( \frac{1}{x} \right) = 0$.

Therefore, we have checked that conditions (i) and (ii) of Theorem 3.1 are satisfied if $f := \sum_{n=1}^{\infty} f_n$, and so, $f \in C^\infty[0, 1]$ with $D \subset E(f)$.

4. **Main result**

An interesting question is connected with the converse to the situation described in Proposition 3.1. Namely, given a nonempty closed nowhere dense set $F \subset [0, 1]$ with $\{0, 1\} \cap F$ consisting of accumulation points of $F$, we need to find a function $f \in C[0, 1]$ (or even, $f \in C^\infty[0, 1]$) such that $E(f) = F$.

Note that Example 3 does not give a solution to this problem in the case if $F$ is a Cantor set with $\{0, 1\} \subset F$ since functions $f_n$, defined in Example 3, produce some points in $E(f)$ which are not in $F$.

Our main theorem will give a positive answer to the above question with a function of class $C^\infty[0, 1]$.

Let $I \subset \mathbb{R}$ be a bounded interval. By $l(I), r(I)$ and $|I|$ we will denote the left endpoint of $I$, the right endpoint of $I$ and the length of $I$, respectively. We say that an interval $I$ is a neighbour of an interval $J$ if $I$ and $J$ have a common endpoint. We denote $\{0, 1\}^{<\mathbb{N}} := \bigcup_{n=0}^{\infty} \{0, 1\}^n$. 
Theorem 4.1. Let $F \subset [0, 1]$ be a closed nowhere dense set with $\{0, 1\} \cap F$ (if nonempty) consisting of accumulation points of $F$. Then there exists a function $f \in C^\infty[0, 1]$ with $E(f) = F$.

Proof. Fix $F$ as in the assumption of the theorem. Since $F$ is closed, (by the classical Cantor-Bendixson theorem) it can be expressed as a union of a perfect set $D$ and a countable set $Q$. Assume that $D \neq \emptyset$ since otherwise, the proof is simpler. The set $D$ is a Cantor set as a perfect and nowhere dense subset of $[0, 1]$. We will express $D$ in the fashion similar to that used for the Cantor ternary set. Let $I$ be a minimal closed interval containing $D$. Let $J$ denote the leftmost longest connected component of $[0, 1] \setminus D$. This is called a gap of level 0. Denote by $I_0$ and $I_1$ (respectively) the left and the right components of $I \setminus J$; they will be called basic intervals of level 1. Inside the intervals $I_k$, for $k = 0, 1$, we pick the leftmost longest components of $I_k \setminus D$; we denote them $J_k$ and call gaps of level 1. The further construction goes inductively. For any $n \in \mathbb{N}$, we obtain the family of basic intervals $I_s$, $s \in \{0, 1\}^n$ of level $n$ and gaps $J_s$ of level $n$ inside of them. Note that $D = \bigcap_{n \in \mathbb{N}} D_n$ where $D_n$ is the union of basic intervals of level $n$. For every $x \in D$ there is a unique sequence $(x_1, x_2, \ldots) \in \{0, 1\}^\mathbb{N}$ such that $\{x\} = \bigcap_{n \in \mathbb{N}} I_{x^n} [x_n]$ where $x_n := (x_1, \ldots, x_n)$ for $n \in \mathbb{N}$.

To simplify the further reasoning suppose that $l(I) = 0$ and $r(I) = 1$. If it is not the case, then we just consider two additional intervals $(0, l(I))$ and $(r(I), 1)$ that can play a role of gaps.

We can present the countable set $Q$ as $\{x_n\}_n \cup \{y_n\}_n$ where $x_n$’s are isolated points in $F$, and $y_n$’s are accumulation points of the set $\{x_n\}_n$. (Symbols $\{x_n\}_n$ and $\{y_n\}_n$ mean that these sets are countable; they may be finite or even empty.) Observe that each element $x_n$ belongs to some interval $J_s$. For a fixed $s$, the set $J_s \setminus F = J_s \setminus Q$ is open, so it can be presented as the union of a countable (possibly finite) disjoint family of open components numbered as $J_{s,(i)}$ with $i$’s belonging either to the whole $\mathbb{N}$ or to an initial segment of $\mathbb{N}$.

Consider the function $h \in C^\infty[0, 1]$ as in Example 4. Given an interval $P = [a, b]$, define $h_P : [0, 1] \to \mathbb{R}$ by $h_P(x) := h \left( \frac{x-a}{b-a} \right)$.

Fix $k \in \mathbb{N} \cup \{0\}$ and $s \in \{0, 1\}^k$ (for $k = 0$, $s$ is equal to the empty sequence, so $J_s = J$). Consider the interval $J_s$. To any interval $J_{s,(i)}$ we assign a function $f_{s,(i)} := \pm c_{s,(i)} h_{cl} J_{s,(i)}$ where $c_{s,(i)} := \frac{1}{n \geq 2} \exp \left( \frac{1}{|J_{s,(i)}|} \right)$. The key point is the choice of the signs $\pm$. If $J_{s,(1)} = J_s$ (that is, if there are no $x_n$’s and $y_n$’s in $J_s$), then let $f_{s,(1)} := -c_{s,(1)} h_{cl} J_{s,(1)}$ if the length of $s$ is odd, and $f_{s,(1)} := c_{s,(1)} h_{cl} J_{s,(1)}$ if the length of $s$ is even.

In the remaining cases we proceed as follows. The set $J_s \setminus \{y_n\}_n$ is open, so it is the union of a countable disjoint family $\{V_m : m \in M\}$ of open components. Note that every interval $J_{s,(i)}$ is contained in exactly one interval $V_m$. So, it is enough to fix $V_m$ and show the choice of sign of functions $f_{s,(i)}$ for all intervals $J_{s,(i)}$ contained in $V_m$. Denote by $V_m$ the family of all such intervals. Observe that $V_m = (V_m \cap \{x_n\}_n) \cup \bigcup V_m$. Intervals $J_{s,(i)}$ in $V_m$ can be ordered with respect to the following ordering: $K \preceq L$ if $K = L$ or there are $K_1, \ldots, K_l \in V_m$ with $K = K_1, L = K_l$ and $r(K_i) = l(K_{i+1})$ for $i = 1, \ldots, l - 1$. Then $\preceq$ on $V_m$ is isomorphic to the usual ordering on exactly one of the following sets:

1. $\{1, \ldots, j\}$ for some $j \in \mathbb{N}$, if the family $V_m$ is finite;
2. $-\mathbb{N}$, if $l(V_m)$ is a right-hand point of accumulation of $x_n$’s in $V_m$, and $r(V_m)$ is not a left-hand point of accumulation of $x_n$’s in $V_m$;
3. $\mathbb{N}$, if $l(V_m)$ is not a right-hand point of accumulation of $x_n$’s in $V_m$, and $r(V_m)$ is a left-hand point.
of accumulation of \(x_n\)’s in \(V_m\);

\(4^0 \mathbb{Z}\), if \(l(V_m)\) is a right-hand point of accumulation of \(x_n\)’s in \(V_m\), and \(r(V_m)\) is a left-hand point of accumulation of \(x_n\)’s in \(V_m\).

In each of these cases, by simple induction, we alternate the signs of functions \(f_{s, (i)}\) for neighbouring intervals \(J_{s, (i)}\) in \(V_m\). For instance, in case \(4^0\), we renumber intervals \(J_{s, (i)}\) in \(V_m\) (according to the respective isomorphism) as \(J_k^s, k \in \mathbb{Z}\), and choose the sign of the respective \(f_{s, (i)}\) for \(J_0^s\) as \(+\). Then choose the sign \(−\) for \(J_1^1, J_{−1}^1\), next we choose the sign \(+\) for \(J_2^1, J_{−2}^1\) and so on.

By the above method, we have defined all the functions \(f_{s, (i)}\) for the intervals \(J_{s, (i)}\) in such a way that they have different signs on any two neighbours.

Now, fix an interval \(J_{s, (i)}\) with \(s \in \{0, 1\}^n\), and fix an integer \(p \geq 0\). We can calculate the \(p\)-th derivative of \(f_{s, (i)}\) at a point \(x \in J_{s, (i)}\) as follows

\[
f_{s, (i)}^{(p)}(x) = \pm c_{s, (i)} \frac{1}{|J_{s, (i)}|^p} h_{s, (i)}^{(p)} \left( \frac{x - l(J_{s, (i)})}{|J_{s, (i)}|} \right).
\]

Note that

\[
\|f_{s, (i)}^{(p)}\| \leq c_{s, (i)} \frac{\|h_{s, (i)}^{(p)}\|}{|J_{s, (i)}|^p} = \frac{1}{n^2 2^t} \exp \left( \frac{-1}{|J_{s, (i)}|} \right) \cdot \frac{\|h_{s, (i)}^{(p)}\|}{|J_{s, (i)}|^p} \leq \frac{M_p}{n^2 2^t} \quad \text{for all } i,
\]

where \(M_p = \|h_{s, (i)}^{(p)}\| \cdot \sup_{x \in [0, 1]} \frac{1}{x^p} < \infty\). Hence every series \(\sum_{i} f_{s, (i)}^{(p)}\), \(p \geq 0\), is uniformly convergent on \([0, 1]\) by the Weierstrass test. Therefore, the function \(f_s = \sum_{i} f_{s, (i)}\) is in \(C^\infty[0, 1]\) because every function \(f_{s, (i)}\) is in \(C^\infty[0, 1]\). Also \(g_n = \sum_{s \in \{0, 1\}^n} f_s\) is in \(C^\infty[0, 1]\). From \([\text{6}]\) it follows that, for any \(p \geq 0\) and \(n \in \mathbb{N}\), we have

\[
\|g_n^{(p)}\| \leq \frac{M_p}{n^2}.
\]

Hence every series \(\sum_{n=1}^{\infty} g_n^{(p)}\), \(p \geq 0\), is uniformly convergent on \([0, 1]\) since the Weierstrass test works. Therefore, the function \(f = \sum_{n=0}^{\infty} g_n\) is in \(C^\infty[0, 1]\).

We will show that \(F = E(f)\). First, fix \(x \in F\) and a neighbourhood \(U\) of \(x\). If \(x \in D\), then, by the construction, \(f(x) = 0\). Also we can find two gaps \(J'\) and \(J''\) obtained in the construction of \(D\) such that \(J' \cup J'' \subseteq U\) where \(J'\) is of odd level and \(J''\) is of even level. But there are \(y' \in J'\) and \(y'' \in J''\) such that \(f(y') < 0\) and \(f(y'') > 0\), so \(x \in E(f)\).

If \(x = x_n\) for some \(n\), then also \(f(x) = 0\). Moreover, \(x_n = r(J_{s, (i)}) = l(J_{s, (j)})\) for some \(s \in \{0, 1\}^n\) and \(i, j \in \mathbb{N}\). The intervals \(J_{s, (i)}\) and \(J_{s, (j)}\) are neighbours, thus \(f\) attains positive values on one of them and negative values on the other. Hence \(x \in E(f)\).

If \(x = y_m\) for some \(m\), then \(f(x) = 0\) and there is an increasing sequence of indices \((k_n)\) such that \(x_{k_n} \to y_m\). Without loss of generality we can assume that all of the terms \(x_{k_n}\) belong to the interval \(J_s\) for some \(s \in \{0, 1\}^n\). Since \(f\) is continuous, then \(f(y_m) = \lim_{n \to \infty} f(x_{k_n}) = 0\). Moreover, since \(U\) contains infinitely many points \(x_{k_n}\), there are some neighbouring intervals \(J_{s, (i)}, J_{s, (j)}\) contained in \(U\). Thus \(f\) attains positive values on one of them and negative on the other. Hence \(x \in E(f)\).

On the other hand, if \(x \notin F\), then \(x \notin J_s \setminus \bigcup_n \{x_n\}_n \cup \bigcup_n \{y_n\}_n\) for some \(s \in \{0, 1\}^n\). Hence \(x \notin J_{s, (j)}\) for some \(j \in \mathbb{N}\). But then \(f(x) > 0\) or \(f(x) < 0\), so \(x \notin E(f)\). This ends the proof of equality \(E(f) = F\). □
5. An open problem

It would be interesting to decide which situation is more common (in the Baire category sense) for functions \( f \in C[0,1] \): when \( \lambda(E(f)) = 0 \) or when \( \lambda(E(f)) > 0 \). Of course, we should restrict this question to functions \( f \in C[0,1] \) for which \( E(f) \neq \emptyset \). Firstly, we need to assume \( f^{-1}([0]) \neq \emptyset \). Secondly, since \( E(f) \) is nowhere dense, it is natural to assume that the interior \( \text{Int}(f^{-1}([0])) \) is empty. Also, we should admit a situation where \( E(f) \) is of positive measure. To this aim, fix \( \alpha \in (0,1) \) and assume that \( \lambda(f^{-1}([0])) \geq \alpha \). Thus it is reasonable to consider the following set

\[
ZC_\alpha[0,1] := \{ f \in C[0,1] : f^{-1}([0]) \neq \emptyset \text{ and } \text{Int}(f^{-1}([0])) = \emptyset \text{ and } \lambda(f^{-1}([0])) \geq \alpha \}.
\]

Proposition 5.1. \( ZC_\alpha[0,1] \) is a Polish subspace of \( C[0,1] \).

Proof. By the Alexandrov theorem, it suffices to show that \( ZC_\alpha[0,1] \) is of type \( G_\delta \). Firstly, note that the set \( \{ f \in C[0,1] : f^{-1}([0]) \neq \emptyset \} \) is open since its complement

\[
\{ f \in C[0,1] : \exists x \in [0,1], f(x) = 0 \}
\]

is closed which follows from the compactness of \([0,1]\).

Next, given a base \( (U_n)_{n \in \mathbb{N}} \) of open sets in \([0,1]\), observe that

\[
\{ f \in C[0,1] : \text{Int}(f^{-1}([0])) = \emptyset \} = \bigcap_{n \in \mathbb{N}} \bigcup_{x \in U_n} \{ f \in C[0,1] : f(x) \neq 0 \}
\]

and this set is of type \( G_\delta \).

Finally, note that the set \( \{ f \in C[0,1] : \lambda(f^{-1}([0])) \geq \alpha \} \) is closed since its complement can be expressed in the form

\[
\{ f \in C[0,1] : (\exists U \text{ - open, } \lambda(U) < \alpha) f^{-1}([0]) \subset U \}.
\]

Here the set \( \{ f \in C[0,1] : f^{-1}([0]) \subset U \} \) is open since \( \{ f \in C[0,1] : (\exists x \in [0,1] \setminus U) f(x) = 0 \} \) is closed. Consequently, by definition \((G)\), the set \( ZC_\alpha[0,1] \) is of type \( G_\delta \). \( \square \)

Problem 4. Given \( \alpha \in (0,1) \), establish the Baire category of the following complementary sets in the Polish space \( ZC_\alpha[0,1] \)

\[
\{ f \in ZC_\alpha[0,1] : \lambda(E(f)) = 0 \}, \{ f \in ZC_\alpha[0,1] : \lambda(E(f)) > 0 \}.
\]

6. Appendix: the proof of Theorem 2.1

Proof. Observe that \( U = \bigcap_{n \in \mathbb{N}} U_n \) where

\[
U_n := \bigcup_{0 = x_0 < x_1 < \cdots < x_k = 1} \left\{ f \in C[0,1] : \sum_{i=1}^k |f(x_i) - f(x_{i-1})| > n \right\}.
\]

The sets \( U_n \) are open, so \( U \) is of type \( G_\delta \).

We will prove that each of the sets \( C[0,1] \setminus U_n \) is porous. Fix \( f \in C[0,1] \) and \( \varepsilon > 0 \). For any \( n \in \mathbb{N} \), we can find a polygonal function \( g \in C[0,1] \) such that \( ||g|| = \frac{\varepsilon}{2} \) and there is a partition \( 0 = x_0 < x_1 < \cdots < x_k = 1 \) of \([0,1]\) satisfying

\[
\text{Var}(g, [0,1]) = \sum_{i=1}^k |g(x_i) - g(x_{i-1})| > \text{Var}(f, [0,1]) + n.
\]
Indeed, it suffices to put $g(x) := \varepsilon \operatorname{dist}(\frac{x}{\varepsilon}, \mathbb{Z})$ for $x \in [0, 1]$ with $k \in 2\mathbb{N}$ such that $\frac{k\varepsilon}{4} > \operatorname{Var}(f, [0, 1]) + n$.

In the role of $x_i$’s, we take the first coordinates of vertices of the polygonal function $g$, that is, points of the form $\frac{i}{k}$ for $i = 0, \ldots, k$. Note that

$$
\sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| = \frac{k\varepsilon}{2}.
$$

Now, we will show that

$$(8) \quad B\left(f + g, \frac{\varepsilon}{8}\right) \subset B(f, \varepsilon) \cap U_n.$$ 

We have $B(f + g, \frac{\varepsilon}{8}) = f + B(g, \frac{\varepsilon}{8})$. Fix any $h \in B(g, \frac{\varepsilon}{8})$. Then

$$||f + h - f|| \leq ||h - g|| + ||g|| < \frac{\varepsilon}{8} + \frac{\varepsilon}{2} = \frac{5\varepsilon}{8} < \varepsilon.$$ 

This yields $B(f + g, \frac{\varepsilon}{8}) \subset B(f, \varepsilon)$. For the taken $h$, put $\delta_i := h(x_i) - g(x_i)$ if $i = 0, 1, \ldots, k$. Clearly, $\delta_i \in (-\frac{\varepsilon}{8}, \frac{\varepsilon}{8})$. Thus

$$\sum_{i=1}^{k} |h(x_i) - h(x_{i-1})| = \sum_{i=1}^{k} |g(x_i) + \delta_i - g(x_{i-1}) - \delta_{i-1}|$$

$$\geq \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| - \sum_{i=1}^{k} |\delta_{i-1} - \delta_i| > \frac{k\varepsilon}{2} - k\frac{2\varepsilon}{8} = \frac{k\varepsilon}{4} > \operatorname{Var}(f, [0, 1]) + n.$$ 

Consequently,

$$\sum_{i=1}^{k} |(f + h)(x_i) - (f + h)(x_{i-1})| \geq \sum_{i=1}^{k} |h(x_i) - h(x_{i-1})| - \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

$$> \operatorname{Var}(f, [0, 1]) + n - \operatorname{Var}(f, [0, 1]) = n.$$ 

Hence $B(f + g, \frac{\varepsilon}{8}) \subset U_n$.

Now, by [3], it follows that $\gamma(f, \varepsilon, C[0, 1] \setminus U_n) \geq \frac{\varepsilon}{8}$. Thus, for any $f \in C[0, 1]$ (in particular, for all $f \in C[0, 1] \setminus U_n$) we have

$$p(C[0, 1] \setminus U_n, f) = 2 \liminf_{\varepsilon \to 0^+} \frac{\gamma(f, \varepsilon, C[0, 1] \setminus U_n)}{\varepsilon} \geq 2 \frac{\varepsilon}{8\varepsilon} = 1 > 0.$$ 

Therefore, the set $C[0, 1] \setminus U$ is porous for all $n \in \mathbb{N}$. So, the set $C[0, 1] \setminus U = \bigcup_{n \in \mathbb{N}} (C[0, 1] \setminus U_n)$ is $\sigma$-porous as desired. \[\square\]

From the above proof it follows that the set $C[0, 1] \setminus U$ has even a stronger property. Namely, each set $C[0, 1] \setminus U_n$, $n \in \mathbb{N}$, has positive lower porosity at every point where the lower porosity is defined as in formula (11) with lim sup replaced by lim inf. This kind of porosity was considered in [3].

REFERENCES

[1] R. A. Gordon, The Integrals of Lebesgue, Denjoy, Perron and Henstock, AMS, Providence 1994.
[2] Y. Katznelson, K. Stromberg, Every differentiable, nowhere monotone, functions, Amer. Math. Monthly 81 (1974), 349–354.
[3] A. S. Kechris, Classical Descriptive Set Theory, Springer 1995.
[4] J. C. Oxtoby, Measure and Category, Springer 1980.
[5] W. Rudin, Principles of Mathematical Analysis, McGraff–Hill, New York 1976.
[6] O. Zabeti, Continuous functions that cut the real axis very often, Expo. Math. 34 (2016), 101–105.
[7] L. Zajiček, *Porosity and $\sigma$-porosity*, Real Anal. Exchange 13 (1987-1988), 314–350.

[8] L. Zajiček, *On $\sigma$-porous sets in abstract spaces*, Abstr. Appl. Anal. 5 (2005), 509–534.

Institute of Mathematics, Lodz University of Technology, ul. Wólczańska 215, 93-005 Łódź, Poland

*Email address*: marek.balcerzak@p.lodz.pl

Institute of Mathematics, Lodz University of Technology, ul. Wólczańska 215, 93-005 Łódź, Poland

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

*Email address*: piotr.nowakowski@edu.p.lodz.pl

Department of Exact and Natural Sciences Jan Kochanowski University in Kielce, ul. Uniwersytecka 7, 25-406 Kielce, Poland

*Email address*: michal.poplawski.m@gmail.com