AN INTEGRAL REPRESENTATION OF OPERATOR MEANS VIA
THE POWER MEANS AND AN APPLICATION TO THE
ANDO-HIAI INEQUALITY

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Abstract. An integral representation of an operator mean via the power means is
obtained. As an application, we shall give explicit condition of operator means that
the Ando-Hiai inequality holds.

1. Introduction

The theory of operator means have been started in [7]. In this paper, the operator
gometric mean has been defined. Then the axiom of operator means was given in [3].
In this discussion, Kubo and Ando obtained that there is a one-to-one correspondence
between operator means and operator monotone functions. This operator monotone
function is called a representing function of an operator mean. Applying the Loewner’s
theorem, each representing function of operator means has an integral representation
via the harmonic means. Using such an integral representation, we can obtain a
lot of properties of operator means by using the properties of the harmonic means.
However it is difficult to obtain individual properties of operator means by using the
integral representation. In this paper, we shall give an integral representation of a
representing function of an operator mean via the power mean. Because of the power
mean interpolates the arithmetic, the geometric and the harmonic means, this integral
representation can be considered as a more precise result than the known result. In
fact, we shall give a property of operator means which are greater than the geometric
means, and give an explicit condition of operator means that the Ando-Hiai inequality
holds.

In what follows let \( \mathcal{H} \) be a Hilbert space with an inner product \( \langle \cdot, \cdot \rangle \), and \( \mathcal{B}(\mathcal{H}) \)
be a set of all bounded linear operators on \( \mathcal{H} \). An operator \( A \in \mathcal{B}(\mathcal{H}) \) is positive
definite (resp. positive semi-definite) if \( \langle Ax, x \rangle > 0 \) (resp. \( \langle Ax, x \rangle \geq 0 \)) holds for all
non-zero \( x \in \mathcal{H} \). If \( A \) is positive semi-definite, we denote \( A \geq 0 \). Let \( \mathcal{PS}, \mathcal{P} \subset \mathcal{B}(\mathcal{H}) \)
be the sets of all positive semi-definite and positive definite operators, respectively.
For self-adjoint operators \( A \) and \( B \), \( A \geq B \) is defined by \( A - B \geq 0 \). A real-valued
function \( f \) defined on an interval \( I \) satisfying
\[
B \leq A \implies f(B) \leq f(A)
\]
for all self-adjoint operators \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( \sigma(A), \sigma(B) \subset I \) is called an
operator monotone function, where \( \sigma(X) \) means the spectrum of \( X \in \mathcal{B}(\mathcal{H}) \).

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1.1. Operator mean.

**Definition 1** (Operator mean, [5]). Let $\sigma : \mathcal{PS}^2 \to \mathcal{P} \mathcal{S}$ be a binary operation. If $\sigma$ satisfies the following four conditions, then $\sigma$ is called an operator mean.

1. If $A \leq C$ and $B \leq D$, then $\sigma(A, B) \leq \sigma(C, D)$,
2. $X^* \sigma(A, B) X \leq \sigma(X^* A X, X^* B X)$ for all $X \in \mathcal{B}(\mathcal{H})$,
3. $\sigma$ is upper semi-continuous on $\mathcal{PS}^2$,
4. $\sigma(I, I) = I$, where $I$ means the identity operator in $\mathcal{B}(\mathcal{H})$.

We notice that if $X$ is invertible in (2), then equality holds.

**Theorem A** ([5]). Let $\sigma$ be an operator mean. Then there exists an operator monotone function $f$ on $(0, \infty)$ such that $f(1) = 1$ and

$$\sigma(A, B) = A^{\frac{t}{2}}f(A^{\frac{-t}{2}}BA^{\frac{-t}{2}})A^{\frac{t}{2}}$$

for all $A \in \mathcal{P}$ and $B \in \mathcal{PS}$. A function $f$ is called a representing function of an operator mean $\sigma$.

Let $\varepsilon$ be a positive real number. Then we have $A_\varepsilon = A + \varepsilon I, B_\varepsilon = B + \varepsilon I \in \mathcal{P}$ for $A, B \in \mathcal{PS}$, and we can compute an operator mean $\sigma(A, B)$ by $\sigma(A, B) = \lim_{\varepsilon \to 0} \sigma(A_\varepsilon, B_\varepsilon)$. We note that for an operator mean $\sigma$ with a representing function $f$, $f'(1) = \lambda \in [0, 1]$ (cf. [3]), and we call $\sigma$ a $\lambda$-weighted operator mean. Typical examples of operator means are the $\lambda$-weighted geometric and $\lambda$-weighted power means. These representing functions are $f(x) = x^\lambda$ and $f(x) = (1 - \lambda + \lambda x^t)^{\frac{1}{t}}$, respectively, where $\lambda \in [0, 1]$ and $t \in [-1, 1]$ (in the case $t = 0$, we consider $t \to 0$). The weighted power mean interpolates the arithmetic, the geometric and the harmonic means by putting $t = 1, 0, -1$, respectively. In what follows, the $\lambda$-weighted geometric and $\lambda$-weighted power means of $A, B \in \mathcal{P}$ are denoted by $A_\lambda^\natural B$ and $P_t(\lambda; A, B)$, respectively, i.e.,

$$A_\lambda^\natural B = A^{\frac{t}{2}}(A^{\frac{-t}{2}}BA^{\frac{-t}{2}})^\lambda A^{\frac{t}{2}},$$

$$P_t(\lambda; A, B) = A^{\frac{t}{2}}\left[1 - \lambda + \lambda(A^{\frac{-t}{2}}BA^{\frac{-t}{2}})^t\right]^\frac{1}{t}A^{\frac{t}{2}}.$$

In what follows $p_t$ denotes the representing function of the power mean $P_t$, i.e.,

$$p_t(\lambda; x) = (1 - \lambda + \lambda x^t)^{\frac{1}{t}}.$$

$p_t$ is monotone increasing on a parameter $t \in [-1, 1]$. The Loewner’s theorem gives us an integral representation of an operator monotone function on $(0, \infty)$ such that $f(1) = 1$ as follows.

**Theorem B** (cf. [3]). Let $\sigma$ be an operator mean, and $f_\sigma$ be a representing function of $\sigma$. Then there exists a provability measure $d\mu$ on $[0, 1]$ such that

$$f_\sigma(x) = \int_0^1 p_{-1}(\lambda; x)d\mu(\lambda) = \int_0^1 [(1 - \lambda) + \lambda x^{-1}]^{-1}d\mu(\lambda).$$

It is a very useful result, and can give us a lot of properties of operator means. However it is difficult to give us individual properties of operator means. According
to Theorem [B], we have $f'(1) = \int_0^1 \lambda d\mu(\lambda)$. Using the harmonic -arithmetic mean inequality, we have

$$f(x) = \int_0^1 [(1 - \lambda) + \lambda x^{-1}]^{1} d\mu(\lambda)$$

$$\leq \int_0^1 [(1 - \lambda) + \lambda x] d\mu(\lambda) = 1 - f'(1) + f'(1)x$$

holds for all $x \in (0, \infty)$,

The first aim of this paper is to generalize Theorem [B] into more precise form by using the power means.

1.2. The Ando-Hiai inequality. The Ando-Hiai inequality is one of the most important operator inequalities.

**Theorem C** (The Ando-Hiai inequality, [1]). Let $A, B \in \mathcal{PS}$. If for each $t \in [0, 1]$, $A^t B \leq I$, then $A^r B^r \leq I$ holds for all $r \geq 1$.

This result only discusses the operator geometric means, and it has been generalized in [9, 10], for example. Especially, in [9], Wada obtained an equivalent condition of operator means that Theorem C holds.

**Theorem D** ([9]). Let $\sigma$ be an operator mean with a representing function $f_\sigma$. Then the following conditions are equivalent:

1. $f_\sigma(x)^r \leq f_\sigma(x'^r)$ for all $r \geq 1$ and $x \in (0, \infty)$,
2. $\sigma(A, B) \leq I$ implies $\sigma(A^r, B^r) \leq I$ for all $r \geq 1$ and $A, B \in \mathcal{PS}$.

Wada focused on the condition (1) in [9], and he defined two sets of operator monotone functions as follows. Let $\mathcal{M}$ be a set of all operator monotone functions on $(0, \infty)$ such that $f(1) = 1$ for $f \in \mathcal{M}$, i.e., $\mathcal{M}$ is a set of all representing functions of all operator means. Let

$$PMI = \{f \in \mathcal{M} | f(x)^r \leq f(x'^r) \text{ for all } r \geq 1 \text{ and } x \in (0, \infty)\},$$

$$PMD = \{f \in \mathcal{M} | f(x)^r \geq f(x'^r) \text{ for all } r \geq 1 \text{ and } x \in (0, \infty)\}.$$

If $f \in PMI$, then the correspondence operator mean satisfies (2) in Theorem [D] and if $f \in PMD$, then the correspondence operator mean satisfies the opposite inequality of (2) in Theorem [D]. See [9] for more information. The second aim of this paper is to give a explicit condition of operator monotone functions in $PMI$ or $PMD$.

This paper is organized as follows: In Section 2, we shall give an integral representation of a function in $\mathcal{M}$ via representing functions of the power means. In Section 3, we shall give an explicit condition of operator means that Theorem [D] (2) holds.

## 2. Integral Representation

In this section, we shall give an integral representation of a function in $\mathcal{M}$ via the power means. For $t \in [-1, 1]$, let

$$C_t = \{f \in \mathcal{M} | p_t(f'(1); x) \leq f(x) \text{ for all } x \in (0, \infty)\}.$$
Theorem 1. For each \( t \in [-1, 1] \) and \( f \in C_t \), there exists a probability measure \( d\mu \) on \([0, 1]\) such that
\[
f(x) = \int_0^1 p_t(\lambda; x)d\mu(\lambda).
\]

Proof. First of all, the case of \( t = 1 \) is obvious because \( C_1 = \{ p_1(\lambda; x) | \lambda \in [0, 1] \} \) by (1.1). Hence we only consider \( t \in [-1, 1) \). We notice that \( C_t \) is a closed set in the topology of point-wise convergence of functions. \( M = C_{-1} \) is compact (cf. 3, 4). Hence \( C_t \) is closed and compact since \( C_t \subseteq C_{-1} \).

We shall prove that \( C_t \) is a convex set. Let \( f, g \in C_t \). Then for each \( \alpha \in [0, 1] \),
\[
p_t((1 - \alpha)f'(1) + \alpha g'(1); x) \leq (1 - \alpha)p_t(f'(1); x) + \alpha p_t(g'(1); x)
\]
holds for all \( x \in (0, \infty) \) since \( p_t(\lambda; x) \) is convex on \( \lambda \in [0, 1] \). Hence \((1 - \alpha)f(x) + \alpha g(x) \in C_t \), and \( C_t \) is a convex set. Therefore \( C_t \) is a convex, closed and compact set.

By the Krein-Milman’s theorem, we have \( C_t = \operatorname{conv} \left( \operatorname{Ext}(C_t) \right) \), where \( \operatorname{conv}(X) \) and \( \operatorname{Ext}(X) \) are convex hull and the set of all extreme points of a set \( X \), respectively.

Next, we shall show that every \( p_t \) is an extreme point of \( C_t \). Assume that
\[
(2.1) \quad \alpha f(x) + (1 - \alpha)g(x) = p_t(\mu; x)
\]
holds for some \( f, g \in C_t, \mu \in [0, 1] \) and all \( x \in (0, \infty) \). Then
\[
p_t(\mu; x) = \alpha f(x) + (1 - \alpha)g(x)
\]
(2.2)
\[
\geq \alpha p_t(f'(1); x) + (1 - \alpha) p_t(g'(1); x)
\]
\[
\geq p_t(\alpha f'(1) + (1 - \alpha)g'(1); x).
\]
Since \( \mu = p_t(\mu; 1) \) and (2.1), we have \( \mu = \alpha f'(1) + (1 - \alpha)g'(1) \), and by (2.2), we have
\[
\alpha p_t(f'(1); x) + (1 - \alpha)p_t(g'(1); x) = p_t(\mu; x).
\]
Since \( p_t(\mu; x) \) is strictly convex on \( \mu \in [0, 1] \), we have \( \mu = f'(1) = g'(1) \), and \( p_t(\mu; x) = f(x) = g(x) \). Hence \( p_t(\mu; x) \) is an extreme point of \( C_t \).

Therefore each element in \( C_t \) can be representing by a convex combination of \( p_t \). Hence Theorem 1 is completed. \( \square \)

Especially, if \( t \rightarrow 0 \), then we have the following corollary.

Corollary 2. For each \( f \in C_0 \), there exists a probability measure \( d\mu \) on \([0, 1]\) such that
\[
f(x) = \int_0^1 x^\lambda d\mu(\lambda).
\]

We remark that Theorem 1 can be extended from \( p_t \) into any representing function of an operator mean such that it is strict convex on the weight parameter. The weighted operator mean is obtained by using the algorithms in 2 and 3. However the author does not know any operator mean which is strict convex on the weight parameter except the power means.
3. An application to the Ando-Hiai inequality

In this section, we shall discuss the Ando-Hiai inequality. Especially, we shall give concrete examples some operator means which satisfy the Ando-Hiai inequality.

**Theorem 3.** Let \( \sigma \) be an operator mean with a representing function \( f_\sigma \). Then the following conditions are equivalent:

1. \( x^{f_\sigma'(1)} \leq f_\sigma(x) \) for all \( x \in (0, \infty) \).
2. \( \sigma(A, B) \leq I \) implies \( \sigma(A^r, B^r) \leq I \) for all \( r \geq 1 \) and \( A, B \in \mathcal{PS} \).

Combined with Theorems \( \text{D} \) and \( \text{B} \) we can get

\[ \text{PMI} = \{ f \in \mathcal{M} \mid x^{f'(1)} \leq f(x) \text{ for all } x \in (0, \infty) \} \]

To prove Theorem 3, we shall give the following lemma.

**Lemma 4.** Let \( f \in \mathcal{M} \). Then the following conditions are equivalent.

1. \( f(x)^p \leq f(x^p) \) holds for all \( r \geq 1 \) and \( x \in (0, \infty) \).
2. \( x^{f'(1)} \leq f(x) \) holds for all \( x \in (0, \infty) \).

**Proof.** Proof of (1) \( \Rightarrow \) (2). (1) is equivalent to \( f(x^p)^{\frac{1}{p}} \leq f(x) \) for all \( p \in (0, 1] \) and \( x \in (0, \infty) \). Then we have \( \lim_{p \to 0} f(x^p)^{\frac{1}{p}} \leq f(x) \). Here by the H’lospital’s theorem, we have

\[
\lim_{p \to 0} \log f(x^p)^{\frac{1}{p}} = \lim_{p \to 0} \frac{\log f(x^p)}{p} = \frac{f'(x^p)x^p \log x}{f(x^p)} = \log x^{f'(1)}.
\]

Hence we have (2).

Proof of (2) \( \Rightarrow \) (1). If \( f \) satisfies (2), then \( f \in C_0 \). By Theorem \( \text{H} \) there exists a probability measure \( d\mu \) on \( [0, 1] \) such that

\[
f(x) = \int_0^1 x^\lambda d\mu(\lambda).
\]

Since \( t^{\frac{1}{p}} \) is a convex function for \( p \in (0, 1) \), we have

\[
f(x^p)^{\frac{1}{p}} = \left( \int_0^1 x^{p\lambda} dE_\lambda \right)^{\frac{1}{p}} \leq \int_0^1 x^\lambda dE_\lambda = f(x).
\]

Therefore the proof is completed. \( \square \)

**Proof of Theorem B** By Lemma \( \text{H} \) (1) of Theorems \( \text{D} \) and \( \text{B} \) are equivalent. Hence the proof is completed. \( \square \)

Using Theorem \( \text{B} \) we can obtain examples of operator means such that Theorem \( \text{B} \) (2) holds.

**Examples.**

1. The Logarithmic mean: The representing function is \( f(x) = \frac{x-1}{\log x} \), and it satisfies \( \sqrt{x} \leq f(x) \leq \frac{1+x}{2} \) for all \( x \in (0, \infty) \). Hence the operator logarithmic mean \( L(A, B) \) satisfies that \( L(A, B) \leq I \) implies \( L(A^r, B^r) \leq I \) for all \( r \geq 1 \) and \( A, B \in \mathcal{PS} \).
(2) The identric mean: The representing function is \( f(x) = \exp\left(\frac{x \log x}{x-1} - 1\right) \). It can be obtained as \( F_{1,0}(x) \), where
\[
F_{p,q}(x) = \left( \int_0^1 [1 - \lambda + \lambda x^p]^{\frac{q}{p}} d\lambda \right)^{\frac{1}{q}} = \left( \frac{p}{p+q} \frac{x^{p+q} - 1}{x^p - 1} \right)^{\frac{1}{q}}.
\]
\( F_{p,q}(x) \) is a representing function of the extension of the power difference mean. In [3], \( F_{p,q} \) is monotone increasing on each parameter \( p, q \in [-1, 1] \) (the cases \( p = 0 \) and \( q = 0 \) are defined by taking limits). Then we have
\[
\frac{1+x}{2} = F_{1,1}(x) \geq F_{1,0}(x) = \exp\left(\frac{x \log x}{x-1} - 1\right) \geq F_{0,0}(x) = \sqrt{x}.
\]
Hence the operator identric mean \( \mathcal{I}(A, B) \) satisfies that \( \mathcal{I}(A, B) \leq I \) implies \( \mathcal{I}(A^r, B^r) \leq I \) for all \( r \geq 1 \) and \( A, B \in \mathcal{PS} \).

(3) The Heinz mean: For \( t \in [0, 1] \), the Heinz mean \( M_t \) is defined as follows:
\[
M_t(A, B) = \frac{A^{\frac{1-t}{t}}B + A^{\frac{t}{1-t}}B}{2}.
\]
It is easy that
\[
A^{\frac{t}{1-t}}B \leq M_t(A, B) \leq \frac{A+B}{2}.
\]
Hence \( M_t(A, B) \leq I \) implies \( M_t(A^r, B^r) \leq I \) for all \( r \geq 1 \) and \( A, B \in \mathcal{PS} \).

In [9], it is shown that \( f(x) \in PMI \) if and only if \( f^+(x) := f(x^{-1})^{-1}, f^-(x) := x f(x)^{-1} \in PMD \). Hence we have the following.
\[
PMD = \{ f \in M | x f^{(1)}(x) \geq f(x) \text{ for all } x \in (0, \infty) \}.
\]

**Theorem 3.** Let \( \sigma \) be an operator mean with a representing function \( f_\sigma \). Then the following conditions are equivalent:
\begin{enumerate}
  \item \( x f^{(1)}_\sigma(x) \geq f_\sigma(x) \text{ for all } x \in (0, \infty) \),
  \item \( \sigma(A, B) \geq I \text{ implies } \sigma(A^r, B^r) \geq I \text{ for all } r \geq 1 \) and \( A, B \in \mathcal{PS} \).
\end{enumerate}

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