VALUATIONS FROM REPRESENTATION THEORY AND TROPICAL GEOMETRY

CHRISTOPHER MANON

Abstract. We recall the space of seminorms discussed by Payne in [P] and define a slight modification, the space of graded valuations. After explaining how these spaces relate to tropical geometry, we describe examples of valuations which come from the representation theory of reductive groups.

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1. INTRODUCTION

This note is written with intention of creating a context for recent results of the author [M1], [M2], involving a set of valuations of certain commutative algebras related to the representation theory of a reductive group. These valuations are intrinsic to the algebra, but were shown to define a point in the tropical variety of the associated ideal of any presentation. We explain this by viewing these objects as generalizations of logarithms of the multiplicative seminorms of Berkovich, also studied recently by Payne, [P], we will call a logarithm of a seminorm a generalized valuation. Roughly, the set of generalized valuations $F_A$ ($(Spec(A)_{an}$ in [P]) for an algebra $A$ has the property that for any presentation of a subalgebra

$$0 \longrightarrow I \longrightarrow K[X] \longrightarrow \phi \longrightarrow A$$

we obtain a map $\hat{\phi} : F_A \rightarrow tr(I)$ to the tropical variety of the ideal $I$, this map is continuous with respect to a natural topology. We view the space $F_A$ as a step in the direction of creating a theory of tropical geometry which is intrinsic to an algebra or scheme, and is not dependent upon embedding information. We will also define a slight generalization of the space of generalized valuations, the space of graded valuations $F_A^s$ for a grading $A = \bigoplus A_s$ of the underlying vector space of the algebra $A$. This set up appears naturally in several contexts, we will use

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it to study algebras related to the representation theory of reductive groups, but any context where algebras are multigraded or multifiltered by combinatorial data ought to see an application. The second part of this note is devoted to showing how to construct valuations from representation theoretic data attached to a ring $A$. Graded valuations are not always necessary in our examples, but we will see that it is much easier to prove that a given function is a graded valuation, and that this distinction is almost as good for applications. We should also mention that rings with other types of extra data have natural graded valuations, see for example [LM] and [KKH] for constructions for rings of global sections of line bundles on projective varieties.

2. Definitions

In this section we will define the space of generalized valuations and the space of graded valuations.

**Definition 2.1.** Let $A$ be an algebra. We call $v : A \rightarrow \mathbb{R}$ a valuation if the following conditions are satisfied.

1. $v(0) = -\infty$

2. For any two elements $a, b \in A$ we have $v(a + b) \leq \max\{v(a), v(b)\}$.

3. For any two elements $a, b \in A$ we have $v(ab) = v(a) + v(b)$

This is the logarithm of the definition of a multiplicative seminorm. Using the fact that $\mathbb{R} \cup \{-\infty\}$ is a tropical algebra with $\max = \oplus$ and $+ = \otimes$, we can reformulate these conditions as follows.

1. $v(0)$ is the additive identity

2. $v(a + b) \oplus v(a) \oplus v(b) = v(a) \oplus v(b)$

3. $v(ab) = v(a) \otimes v(b)$

Throughout this note we will move back and forth between the tropical notation and the classical notation. The set of all such valuations is referred to as $\mathcal{F}_A$ from now on. The requirement that $v(a + b)$ only be less than or equal to $v(a) \oplus v(b)$ may look a little awkward, this is because highest terms can cancel in a sum, we express this with the following proposition.

**Proposition 2.2.** Let $v$ be a valuation for $A$, and suppose $v(a + b) \neq v(a) \oplus v(b)$ then $v(a) = v(b)$.

**Proof.** Suppose $v(a) = v(a) \oplus v(b)$ then we have

1. $v(a) = v(a + b - b) \leq v(a + b) \oplus v(b)$

If $v(a + b) \oplus v(b) = v(a + b)$, then we must have

2. $v(a + b) < v(a) \leq v(a + b)$

which is a contradiction. Therefore $v(b) = v(a) \oplus v(b) = v(a)$. \qed
2.1. Cone structures on subsets of $\mathcal{F}_A$.

**Definition 2.3.** For $v, w \in \mathcal{F}_A$ we write $v \Rightarrow w$ if $v(a) \leq v(b)$ implies $w(a) \leq w(b)$. Tropically this is $v(a) + v(b) = v(b)$ implies $w(a) + w(b) = w(b)$.

We say $v$ and $w$ are on the same facet if $v \Rightarrow w$ and $w \Rightarrow v$, in which case we write $v \iff w$. Note that for any positive real number $R \in \mathbb{R}$, $Rv$ is valuation and we have $v \iff Rv$. The relation $\Rightarrow$ is transitive, reflexive, and $\iff$ defines an equivalence relation.

**Proposition 2.4.** If $v \Rightarrow w_1, w_2$ in $\mathcal{F}_A$ then the function $w_1 + w_2 : A \rightarrow \mathbb{R}_{\geq 0}$ defines a valuation with $v \Rightarrow w_1 + w_2$.

**Proof.** First we will check the three conditions defining valuations. The first and third conditions follow trivially. For the second condition, note that we have

\[(3) \quad w_1 + w_2(a + b) = w_1(a + b) + w_2(a + b) \leq \max\{w_1(a), w_1(b)\} + \max\{w_2(a), w_2(b)\}\]

Suppose that $w_1(a) < w_1(b)$ then $v(a) < v(b)$ and we must have $w_2(a) \leq w_2(b)$, in which case $\max\{w_1(a), w_1(b)\} + \max\{w_2(a), w_2(b)\} = w_1(b) + w_2(b)$. Similarly if $w_1(a) = w_1(b)$ and $w_2(a) < w_2(b)$ we still have $\max\{w_1(a), w_1(b)\} + \max\{w_2(a), w_2(b)\} = w_1(b) + w_2(b)$. Either way we can write

\[(4) \quad w_1 + w_2(a + b) \leq \max\{w_1(a), w_1(b)\} + \max\{w_2(a), w_2(b)\}\]

\[(5) \quad \max\{w_1(a) + w_2(a), w_1(b) + w_2(b)\} = \max\{w_1 + w_2(a), w_1 + w_2(b)\},\]

this proves that $w_1 + w_2$ is a valuation. Now if $v(a) = v(b)$ then $w_1(a) = w_1(b)$ and $w_2(a) = w_2(b)$. Similarly if $v(a) < v(b)$ then $w_1(a) \leq w_1(b)$ and $w_2(a) \leq w_2(b)$. This allows us to conclude that $v \Rightarrow w_1 + w_2$.

For a given $v \in \mathcal{F}_A$ the set of all $w$ such that $v \Rightarrow w$ forms a cone in this way. It is of course possible for $v_1 \Rightarrow w$ and $v_2 \Rightarrow w$ for incomparable $v_1$ and $v_2$, in this way we view $\mathcal{F}_A$ as a complex of cones glued along common boundary cones. We can even give these cones an integral structure by looking at the distinguished valuations $v : A \rightarrow \mathbb{R}$ such that $v(a) \in \mathbb{Z}$.

**Definition 2.5.** For $v, w \in \mathcal{F}_A$ we say $v \rightarrow w$ if for any $\{x_1, \ldots, x_n\} = X \subset A$ with associated ideal $I \subset \mathbb{Z}[X]$ in the symbols $X$, we have $\text{in}_v(\text{in}_w(I)) = \text{in}_w(I)$.

**Lemma 2.6.** If $v \Rightarrow w$ then $v \rightarrow w$.

**Proof.** This follows directly from the definitions, in particular for any two elements $a$ and $b$ we must have $v(a) \leq v(b)$ implies $w(a) \leq w(b)$. Hence, if

\[(6) \quad v(\phi(C_i x^{i(i)})) \leq v(C_j \phi(x^{j(j)}))\]

we must have

\[(7) \quad w(\phi(C_i x^{i(i)})) \leq w(C_j \phi(x^{j(j)}))\]

$\square$
In particular, if we have \( v \equiv w \) then \( \text{Max}_v(F) = \text{Max}_w(F) \) for all \( F \in I \). Clearly for any \( R \in \mathbb{R}^{\geq 0} \) we have \( Rv \equiv v \).

**Proposition 2.7.** If \( v \to w_1, w_2 \) then \( w_1 + w_2 \) is a generalized valuation and \( v \to w_1 + w_2 \).

**Proof.** Clearly \( w_1 + w_2(0) = -\infty \), and \( w_1 + w_2(ab) = w_1(ab) + w_2(ab) = w_1(a) + w_1(b) + w_2(a) + w_2(b) = w_1 + w_2(a) + w_1 + w_2(b) \). Now consider the elements \([a + b], [a], [b] \in A\) (brackets for emphasis), which satisfy the relation

\[
[a + b] = [a] + [b]
\]

So by definition we must have containment of maximum terms \( \text{Max}_v(a + b, a, b) \subset \text{Max}_w(a + b, a, b) \). Hence if \( w_1(a) < w_1(b) \) then \( w_1(b) = w_1(a + b) \) and \( v(b) = v(a + b) \) and \( w_2(a) \leq w_2(b) \). In this case we have the following,

\[
w_1 + w_2(a + b) = w_1(a + b) + w_2(a + b) \leq \max\{w_1(a), w_1(b)\} + \max\{w_1(a), w_1(b)\} = w_1(b) + w_2(b) = w_1 + w_2(b)
\]

A similar statement holds if \( w_1(a) > w_1(b) \) and if \( w_1(a) = w_1(b) \). This proves the proposition. \( \square \)

The relation \( \to \) is also reflexive and transitive, and \( \equiv \) defines an equivalence relation. This implies that both \( \Rightarrow \) and \( \to \) divide \( \mathcal{F}_A \) up into cones, and \( \Rightarrow \) refines \( \to \).

Also, fixing a particular set \( X \subset A \) defines a relation \( \to_X \) on \( A \), and each of these is refined by \( \Rightarrow \), but these relations do not capture enough information for the above proof to work (not every set contains \( a + b, a, \) and \( b \)), so they do not necessarily form cones.

### 2.2. Valuations over a fixed field.

**Definition 2.8.** Fix a field \( K \subset A \) and a valuation \( v : K \to \mathbb{R} \). Let \( \mathcal{F}_A(K) \subset \mathcal{F}_A \) be the set of generalized valuations such that \( v(k) = -v(k) \) for all \( k \in K \).

Note that this set is not always complex of cones, as \( v + w \) is never in \( \mathcal{F}_A(K) \) if the valuation \( v : K \to \mathbb{R} \) is nontrivial. The subsets defined by taking a trivial valuation, or the subset of all \( v : A \to \mathbb{R} \) such that \( v(k) = -v(k) \) for all \( k \in K \) for some fixed \( R \in \mathbb{R}^{\geq 0} \) is closed under scaling and addition within the cones, and so inherits the same type of structures, \( \Rightarrow \), found on \( \mathcal{F}_A \). In [F] it is shown that field extensions \( L \subset K \) which preserve the chosen valuation give the same tropical varieties when the ideal is generated by polynomials with coefficients in \( L \). Note that this does not imply that \( \mathcal{F}_A(L) = \mathcal{F}_A(K) \), as there are valuations of \( A \) which agree with the valuation on \( L \), but not \( K \).

**Remark 2.9.** This extension of scalars result for a field inclusion \( L \subset K \) is amusing when \( L \) has the trivial valuation. It can be used to prove the theorem in [SpSt] asserting that the image of the "tropicalization map" for a field with valuation matches the tropical variety over a subfield with trivial valuation.
Theorem 2.10. For any map from a polynomial ring

\[ 0 \longrightarrow I \longrightarrow K[X] \longrightarrow A \phi \rightarrow A \]

where \( X = \{x_1, \ldots, x_n\} \) we get a map \( \hat{\phi} : \mathcal{F}_A(K) \rightarrow \text{tr}(I) \) defined by

\[ v \mapsto (v(\phi(x_1)), \ldots, v(\phi(x_n))). \]

Furthermore, this map respects the cone structure on the tropical variety \( \text{tr}(I) \).

The last part of the theorem was discussed in the previous section for both the relations \( \Rightarrow \) and \( \rightarrow \). Before we launch into a proof of the other part of this theorem let’s consider a monomial \( x_1^{a_1} \cdots x_n^{a_n} = \tilde{x}^\alpha \in K[X] \). For \( v \in \mathcal{F}_A \), we must have \( v(\tilde{x}^\alpha) = \sum v(x_i)a_i \), by the definition of a valuation. Now we fix a form in the \( x_i \),

\[ F(x_1, \ldots, x_n) = \sum C_i \tilde{x}^\alpha(i) \]

which is contained in \( I \). First we prove that \( \hat{\phi}(v) \) is a tropical point.

Lemma 2.11. \( \hat{\phi}(v) \) weights at least two monomials in \( F(x_1, \ldots, x_n) \) with the highest weight.

Proof. This essentially follows from proposition 2.2 but we will write out the proof. First note that \( v(F(\tilde{x})) = -\infty \). Select the monomial term with the highest weight \( C_1 \tilde{x}^{\alpha(1)} \),

\[ \bigoplus v(C_i \tilde{x}^\alpha(i)) \leq v(C_1 \tilde{x}^{\alpha(1)}). \]

Now we can perform the same trick used in the proof of proposition 2.2

\[ v(C_1 \tilde{x}^{\alpha(1)}) = v(C_1 \tilde{x}^{\alpha(1)}) + \sum_{i \geq 2} C_i \tilde{x}^\alpha(i) - \sum_{i \geq 2} C_i \tilde{x}^\alpha(i) \leq \]

\[ v(F(\tilde{x})) \oplus v(\sum_{i \geq 2} C_i \tilde{x}^\alpha(i)) = v(\sum_{i \geq 2} C_i \tilde{x}^\alpha(i)) = \bigoplus v(C_i \tilde{x}^\alpha(i)). \]

This proves that \( v(C_1 \tilde{x}^{\alpha(1)}) = \bigoplus v(C_i \tilde{x}^\alpha(i)) \), so at least two terms must be given the highest weight by \( v \).

Since \( F(x_1, \ldots, x_n) \) was arbitrary, we have shown the initial ideal of \( I \) with respect to \( \hat{\phi}(v) \) is monomial free, and therefore this defines a point in \( \text{tr}(I) \). In the case with trivial valuation, \( \hat{\phi}(v) \) is a point in the Gröbner fan by construction. This completes the proof of the theorem. In [P], theorem 1.1 Payne uses the maps \( \hat{\phi} : \mathcal{F}_A(K) \rightarrow \text{tr}(I) \) to define a homeomorphism,

\[ \mathcal{F}_A(K) \cong \lim_{\leftarrow} \text{tr}(I), \]

where the limit is over all ideals \( I \) which present \( A \) as a \( K \) algebra. We have seen how the map works in one direction, starting with a point \( y_i \in \text{Lim}_{\rightarrow} \text{tr}(I) \) we can define a valuation on \( A \) by choosing a generating set of \( A \) which includes \( f \in A \), and letting \( v_{y_i}(f) \) be the value of the component of \( y_i \) on \( f \) on this generating set, this value is well-defined because of properties of the inverse limit. To prove that
this actually defines a valuation, one looks at generating sets which contain two chosen elements \( f \) and \( g \), and their product \( fg \) and sum \( f + g \). Indeed, suppose we choose a generating set which includes \( f \), \( g \), and \( f + g \), then the relation

\[
[f + g] = [f] + [g]
\]

(brackets added for emphasis) holds in the ideal defining the presentation. Any weighting \( w \) in the tropical variety must satisfy \( w(f + g) = w(f) \geq w(g) \), \( w(f) = w(g) \), or \( w(f) = w(g) \), so we can always write \( w(f + g) \leq \max\{w(f), w(g)\} \). If the generating set of the ideal contains \( f \), \( g \), and \( fg \) then any term weighting must satisfy \( w(fg) = w(f) + w(g) \). In \([12]\) it is also shown that \( \mathcal{F}_A(K) \) surjects onto any given tropical variety associated to a presentation of \( A \) by a \( K \)-polynomial algebra, so any given tropical point comes from a valuation in some sense. However, this does not imply that valuations are the only way to get tropical points.

2.3. Generalization to graded valuations. We define graded valuations. These objects are useful for capturing intrinsic tropical properties of a ring with extra grading structure.

**Definition 2.12.** We call \( v : A \to \mathbb{R} \) a graded valuation with respect to a grading \( A = \bigoplus A_s \) if the following are satisfied.

1. \( v(0) = -\infty \)
2. \( v(ab) = v(a) + v(b) \) for \( a \) and \( b \) homogeneous.
3. \( v(a + b) \leq v(a) + v(b) \)

We denote the set of all graded valuations with respect to \( s \) by \( \mathcal{F}^s_A \).

Note that it is easier to prove something is a graded valuation because less equalities need be checked from condition 2. Graded valuations behave much like valuations, in particular they have a similar universality property for tropical varieties of embeddings given by homogenous parameters.

**Proposition 2.13.** Let \( v \in \mathcal{F}^s_A(K) \subset \mathcal{F}_A^s \) be a graded valuation which agrees with \( -v : K \to \mathbb{R} \) on \( K \subset A \), and let

\[
0 \longrightarrow I \longrightarrow K[X] \overset{\phi}{\longrightarrow} A
\]

be a presentation of a subalgebra by \( s \)-homogeneous elements. This defines a map \( \phi : \mathcal{F}_A \to tr(I) \).

**Proof.** This follows from the same argument used in theorem 2.10.

As a consequence of this proposition there is a natural map,

\[
\mathcal{F}^s_A(K) \to \lim_{\leftarrow} [tr(I), X \subset S]
\]

where the limit is over all presentations by homogeneous generating sets. There is also a natural map \( \mathcal{F}_A \to \mathcal{F}^s_A \) given by weighting homogeneous terms in \( A \) with
respects to \( s \) with generalized valuations. In fact, if \( s' \) is a grading on \( A \) which refines \( s \) then we have inclusions,

\[
\mathcal{F}_A \subset \mathcal{F}_A^s \subset \mathcal{F}_A^{s'}.
\]

In this way every \( \mathcal{F}_A^s \) is contained in \( \mathcal{F}_A^B \) for the grading given by a basis \( B \subset A \). The relation \( \Rightarrow \) makes sense for \( \mathcal{F}_A^s \) as well, and the cone structure it induces is respected by the above inclusions. The \( \rightarrow \) relation also makes sense.

**Proposition 2.14.** If \( v \in \mathcal{F}_A \subset \mathcal{F}_A^s \) and \( v \Rightarrow w \) then \( w \in \mathcal{F}_A \).

**Proof.** This follows from the definitions. \(\square\)

The following example shows how one can have a graded valuation which is not a valuation.

**Example 2.15.** We will look at the polynomial algebra \( \mathbb{C}[x, y, z] \). Consider the following assignment of weights,

\[
v(x) = v(y) = v(z) = 1 \\
v(C) = 0
\]

We will select the grading given by the one dimensional subspaces defined by each monomial. It is obvious that for any two homogeneous elements, the weights of a product add. Now, suppose we choose to weight \( xy + xz \) less than 2. Then we get the following.

\[
v(x(y + z)) = v(xy + xz) < 2 = v(x) + v(y + z)
\]

So this weighting fails as a valuation, but is a graded valuation.

The theory of generalized valuations leaves a lot of room for variations which can take into account the variety of structures one can put on a ring. We have seen the graded variant above, but we could also look at the equivariant case, where we restrict attention to valuations which are preserved by a group action. In the examples we will see the use of the trivial valuation on \( \mathbb{C} \). We can define the space of (graded) valuations \( v \) for a \( \mathbb{C} \)-algebra by simple stipulating that \( v(C) = 0 \) for all non-zero complex numbers \( C \). We also point out that \( \mathbb{R} \) can be replaced by other tropical algebras which satisfy similar properties, namely any totally ordered group or semigroup. In the examples we will see the use of \( \mathbb{R}_{\geq 0} \) where the role of \( -\infty \) is filled by the lower bound 0. Indeed, any well-ordered tropical algebra will do, with the lower bound as the additive identity.

### 3. The Functor of Valuations

We now turn our attention to the properties of \( \mathcal{F}_A \) and \( \mathcal{F}_A^s \) as the algebra \( A \) is allowed to vary. If \( f : A \to B \) is a ring homomorphism the pullback of a valuation may not continue to be a valuation. This is because the map \( f \) may have a nontrivial kernel, which must map to the lower bound of the tropical algebra being used. Take this algebra to be \( \mathbb{R}_{\geq 0} \), with lower bound 0. If \( a \) is any element then \( v(a) = v(ai) - v(i) \) for any element \( i \) in the kernel of the map, this is 0. For this reason we focus on injections of algebras, where kernels cannot cause problems.
Proposition 3.1. If $f : A \rightarrow B$ is an injection of algebras, then for any valuation $v \in \mathcal{F}_B$, the pullback $f^*(v)$ is a valuation. Furthermore, if $v \Rightarrow w$ then $f^*(v) \Rightarrow f^*(w)$, and if $v \rightarrow w$ then $f^*(v) \rightarrow f^*(w)$.

Proof. We need only note that by definition, any product of elements $a \times b$ in $A$ must have a highest weight term with weight equal to $v(a) + v(b) = f^*(v)(a) + f^*(v)(b)$. Also by definition if $v(a) \leq v(b)$ then $w(a) \leq w(b)$, so if $f^*(v)(a) \leq f^*(v)(b)$ then $f^*(w)(a) \leq f^*(w)(b)$. □

It is also easy to check that $(f \circ g)^* = g^* \circ f^*$ for injections $f$ and $g$. These facts have a graded form as well.

Proposition 3.2. Let $f : A \rightarrow B$ be an injection of algebras which respects underlying gradings on $A$ and $B$.

1. $A = \bigoplus A_s$, $B = \bigoplus B_t$

2. If $a \in A$ is homogeneous then $f(b) \in B$ is homogeneous.

Then for a graded valuation $v$ on $B$, the function $f^*(v)$ is a graded valuation.

The inclusion $\mathcal{F}^s_A \subset \mathcal{F}_A$ for a grading $s'$ which refines $s$ is a special case of this proposition.

4. Examples

In this section we will go through two examples and an amusing generalization of the objects we have been discussing.

Example 4.1. Consider the injection of rings defined by inverting an element.

$$A \xrightarrow{i_f} \frac{1}{f}A$$

Note that we must have $0 = v(1) = v(f + 1) = v(f) + v(1)$. Suppose $v$ and $w$ are two distinct valuations on $\frac{1}{f}A$, then for some element we must have

$$v(a_0 + \frac{1}{f}a_1 + \ldots + \frac{1}{f^n}a_n) \neq w(a_0 + \frac{1}{f}a_1 + \ldots + \frac{1}{f^n}a_n)$$

If $v(f) \neq w(f)$ then these elements define distinct elements in $\mathcal{F}_A$, suppose then that $v(f) = w(f)$. We also must have

$$v(a_0 + \frac{1}{f}a_1 + \ldots + \frac{1}{f^n}a_n) + nv(f) = v(f^n(a_0 + \frac{1}{f}a_1 + \ldots + \frac{1}{f^n}a_n)) = v(f^n a_0 + f^{n-1}a_1 + \ldots + a_n).$$

Since $v(f) = w(f)$, this implies that $v$ and $w$ must still differ on an element of $A$. This implies that

$$i_f^* : \mathcal{F}_{\frac{1}{f}A} \rightarrow \mathcal{F}_A$$

is an injection of complexes of cones. This generalizes to an inversion of any set of elements.
Example 4.2. We’ll consider the polynomial algebra $\mathbb{C}[t]$. By definition, the value $v(t)$ determines $v(t^n) = nv(t)$. For a general polynomial $p(t) = a_n t^n + \ldots + a_0$ we have the following,

\[(22) \quad v(t^n) = v(a_n t^n) = v(p(t) - (p(t) - a_n t^n)) \leq v(p(t)) \oplus v(p(t) - a_n t^n)\]

\[(23) \quad v(p(t)) \leq v(a_n t^n) \oplus v(p(t) - a_n t^n)\]

\[(24) \quad v(p(t) - a_n t^n) < v(a_n t^n)\]

this implies that

\[(25) \quad v(p(t)) \leq v(t^n) \leq v(p(t))\]

so $v(p(t)) = v(t) \deg(p(t))$. This proves that $\mathcal{F}_{\mathbb{C}[t]} = \mathbb{R}_{\geq 0}$.

We could have also proved this using the fact that any polynomial over $\mathbb{C}$ factors, but the content of this example does not depend on the field being algebraically closed.

Example 4.3. We have already mentioned that the tropical algebra $\mathbb{R} \cup \{-\infty\}$ is not necessary to the formulation of generalized valuations. Suppose for example we took a noncommutative tropical algebra, a general totally ordered semigroup $S$, with a lower bound $O$, and define tropical operations $s_1 \otimes s_2 = s_1 s_2$ and $s_1 \oplus s_2 = \sup\{s_1, s_2\}$. We could look at functions $v : A \to S$ such that the following are satisfied.

\[(1) \quad v(0) = O\]

\[(2) \quad v(ab) = v(a) \otimes v(b)\]

\[(3) \quad v(a + b) \leq v(a) \oplus v(b)\]

We call this the set of generalized valuations of $A$ with coefficients in $S$, $\mathcal{F}_{A,S}$. We could then look at the ideal defined by a subset $X \subset A$ by a map from the noncommutative polynomial algebra on $X$.

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
& \longrightarrow & \mathbb{C} < X > \\
& \phi & \longrightarrow A
\end{array}
\]

Each such $v$ defines a partial ordering on the free noncommutative monoid on $X$ symbols.

\[
\begin{array}{ccc}
< X > & \longrightarrow & \mathbb{C} < X > \\
& \longrightarrow & A \\
& \phi & \longrightarrow S
\end{array}
\]

Much of what we’ve formulated above carries over formally to this case. From the conditions above one can prove that for each form $F(X) \in I$ there are at least two monomials in $F$ which are given the maximum value by $v$. One could take the set of all monoidal maps $\text{Hom}(< X >, S)$ and the subset $\text{tr}_S(I) \subset \text{Hom}(< X >, S)$ of maps $v$ such that $\text{in}_v(I)$ is monomial free, there is a map

\[(26) \quad \hat{\phi} : \mathcal{F}_{A,S} \to \text{tr}_S(I) \subset \text{Hom}(< X >, S)\]
5. Examples from representation theory

In this section we will discuss how to obtain graded valuations from some familiar rings related to the representation theory of reductive groups. The examples we will discuss will all involve an algebra $A$, with a grading of the underlying vector space given by a monoid $S$,

$$A = \bigoplus_{s \in S} A_s.$$ 

Orderings $v : S \to \mathbb{R}$ will be given such that multiplication in $A$ is lower-triangular with respect to $v$. So the weight of a sum of two elements from different graded pieces can then be defined as the max, $v(a + b) = v(a) \oplus v(b)$. The following is a consequence of these conventions.

**Proposition 5.1.** If $A$, $v$ and $S$ are as above then $v$ defines a graded valuation on $A$.

This situation is made more interesting if the valuation and the monoid behave well together. For instance, if for any two elements $a \in A_{s_1}$ and $b \in A_{s_2}$ we have $ab \in A_{s_1 + s_2} \oplus \ldots$, with $(ab)_{s_1 + s_2} \neq 0$ and $v(s_1 + s_2) > v(s)$ for the other $s$ appearing in the decomposition of $ab$. If these conditions are satisfied for $A$ then it is also true for graded subalgebras of $A$. If the grading $S$ has only 1-dimensional components, then graded valuations that possess this property define filtrations on $A$ which have monoidal associated graded algebras, as do subalgebras which respect the grading. With the next theorem we see that with some assumptions we can guarantee the existence of generalized valuations (that need not be graded), this is satisfied by special variants of the valuations constructed in our examples and the valuations from algebraic geometry presented in [LM] and [KKh].

**Theorem 5.2.** Suppose $A$ has a direct sum decomposition $\bigoplus_S A_s$ by the elements of a monoid $S$, and the following are satisfied.

1. the expansion of $a_{s_1} b_{s_2}$ contains a component of weight $s_1 + s_2$

2. there is a total order weighting $w$ of $S$

Then any $v : A \to \mathbb{R}$ for which $w \Rightarrow v$ defines a generalized valuation of $A$.

**Proof.** It suffices by proposition 2.14 to prove this for $w$. Suppose that for inhomogeneous elements $A = a_{s_1} + \ldots + a_{s_n}$ and $B = b_{t_1} + \ldots + b_{t_m}$ that $w(AB) < w(A) + w(B)$ (note that this is all that could go wrong). For each pair let $a_{s_i} b_{t_j} = \sum c^k_{s_i, t_j}$ be the expansion into homogeneous elements, with $c^\text{top}_{s_i, t_j}$ the highest component by $w$, which must have a grading of $s_i + s_j$. Let $s_1$ and $t_1$ have the highest grades with respect to $w$. We must have that the sum of all elements of grade $s_1 + t_1$ is 0. We can assume without loss of generality that all of these components are of the form $c^k_{s_i, t_j}$. To see this note that if the grade of $c^k_{s_i, t_j}$ is $s_1 + t_1$ then we must have $w(s_1 + t_1) < w(s_i + t_j)$ unless $k = \text{top}$. So we have

$$\sum c^\text{top}_{s_i, t_j} = 0$$

for some collection $\{(i, j)\} \subset [1, n] \times [1, m]$, with $s_1 + t_1 = s_i + t_j$, and these are all the components of weight $s_1 + t_1$. This implies that $w(s_1) = w(s_i)$ and $w(t_1) = w(t_j)$
for all $i, j$, because if any of these were inequalities, $s_1 + t_1$ would not be highest by $w$, or $s_1 + t_1 \neq s_i + t_j$. But $w$ is a total ordering, so we must have $s_1 = s_i$ and $t_1 = t_j$, and we must have $v(A'B') \neq v(A') + v(B')$ for pure elements $A'$ and $B'$, a contradiction. So, the highest component of $AB$ by $w$ (therefore with unique $S$ weight) cannot vanish.

□

From this theorem we can conclude that weakenings of the total orderings of the dual canonical basis and gradings of branching algebras to be discussed below are generalized valuations. For dual canonical bases, the total orderings themselves give toric degenerations of the associated algebra. We also mention the following corollary of the proof of theorem 5.2.

**Corollary 5.3.** If the associated graded algebra of an algebra $A = \bigoplus S A_s$ with respect to $w : S \rightarrow \mathbb{R}$ is a domain, and $w$ well-orders the elements in the expansion of any product, then $w$ is a generalized valuation.

**Proof.** Suppose $w$ is not a generalized valuation, then some product of inhomogeneous elements $AB = (\sum a_s)(\sum b_t) = \sum a_s b_t = \sum c_{s_t}$ must have all of its top weighted terms cancel out. By the same reasoning as in the proof of theorem 5.2 we have that the product of the sums of the subsets of the top $w$-weighted homogeneous terms of $A$ and $B$, $A' = \sum_{top} a_s, B' = \sum b_t$ must have $v(A'B') < v(A) + v(B)$. Therefore this product vanishes in the associated graded algebra. (it may be instructive to see why the product $AB$ does not necessarily vanish in the associated graded)

□

This corollary can be used to show that special graded valuations on branching algebras are actually generalized valuations, we will mention more below.

### 5.1. Rings with a reductive group action.

In this example we will follow the books by Grosshans and Dolgachev, [Gr] and [D]. Let $A$ be any commutative algebra over $\mathbb{C}$ with an action of a reductive group $G$. Then $A$ has a direct sum decomposition into weighted components from the category of finite dimensional representations of $G$.

\[
A = \bigoplus_{\lambda \in S_G} A^{U+}(\lambda) \otimes V(\lambda^*)
\]

Here, $A^{U+}(\lambda)$ is the $\lambda$ weight component of the invariants of $A$ with respect to the action the maximal unipotent $U_+ \subset G$. Let $\mathfrak{h}$ be the Cartan subalgebra corresponding to a chosen maximal torus of $G$. Multiplication in this algebra is lower triangular in the sense that if $a$ and $b$ are from the $\lambda$ and $\gamma$ components of $A$ then $ab$ decomposes into homogeneous elements from components of weights less than or equal to $\lambda + \gamma$ as dominant weights. Any functional $\mathfrak{h}^* \rightarrow \mathbb{R}_{\geq 0}$ which respects the ordering on dominant weights then defines graded valuation of $A$ for the $G$-decomposition.

More generally, one could also start with a graded valuation on $A^{U+}$ with respect to the grading $A^{U+} = \bigoplus_{\lambda \in S_G} A^{U+}(\lambda)$ and define a graded valuation on $A$ by ignoring the $V(\lambda^*)$ components of elements in the $A^{U+}(\lambda) \otimes V(\lambda^*)$ summand. Note that graded valuations constructed this way are always $G$-invariant.
5.2. Dual canonical bases. In this example we use the same set up as the last example, with $A$ a $\mathbb{C}$-algebra which carries the action of a reductive group $G$, we follow the paper of Alexeev and Brion, [AB]. Each representation $V(\lambda)$ of $G$ has a distinguished basis $B(\lambda)$, called the dual canonical basis. This basis was defined by Lusztig and has been studied by many authors, see for example [AB] and [BZ1]. We can refine the grading of the previous example to the one given by the components $A^{U_+}(\lambda)$ tensored with members of $B(\lambda)$.

$$A = \bigoplus_{\lambda, \phi \in S_G, B} A^{U_+}(\lambda) \otimes \mathbb{C}b_{\lambda, \phi}$$

Multiplication for the dual canonical basis is lower triangular with respect an ordering of weights called string parameters which index the basis, these depend on a chosen decomposition of the longest element of the Weyl group, see [AB] or [BZ2]. Any functional $h : B \to \mathbb{R}_{\geq 0}$ which respects this ordering then gives a graded valuation. Note that this example specializes to the previous example for any functional which only sees the dominant weight information.

5.3. Branching algebras. Branching algebras are discussed in both [M1] and [M2], they are multigraded commutative algebras over $\mathbb{C}$, where the dimensions of the graded components give the branching multiplicities for a morphism of reductive groups. For a reductive group $G$, let $C_G$ denote the cone of dominant $G$-weights. Let $U_+ \subset G$ be a maximal unipotent subgroup, we let $R(G)$ denote the commutative algebra defined as follows.

$$R(G) = \mathbb{C}[G/U_+] = \mathbb{C}[G]^{U_+} = \bigoplus_{\lambda \in C_G} V(\lambda^*) \otimes V(\lambda)$$

For a map of reductive groups $\phi : H \to G$ we define the full branching algebra for $\phi$ as follows.

$$\mathfrak{A}(\phi) = [R(H) \otimes R(G)]^H = R(G)^{U(H)}$$

As vector spaces, these algebras decompose into meaningful graded components.

$$\mathfrak{A}(\phi) = \bigoplus_{\mu, \lambda \in C_H \times C_G} Hom_H(W(\mu), V(\lambda))$$

We have called these full branching algebras because multigraded subalgebras of $\mathfrak{A}(\phi)$ are also meaningful for representation theory, see [M2]. The reader’s favorite subcones of $C_H \times C_G$ define subalgebras of $\mathfrak{A}(\phi)$ which share many useful features. In particular the subcone $\{0\} \times C_G$ defines the subalgebra of $H$-invariants in $R(G)$ with respect the action through $\phi$. Now take $\phi = \pi \circ \psi$ to be any factorization of $\phi$ in the category of reductive groups. $H \xrightarrow{\psi} K \xrightarrow{\pi} G$ On the level of vector spaces this refines the direct sum decomposition with respect to intermediate branching over $K$.

$$Hom_H(W(\mu), V(\lambda)) = \bigoplus_{\eta \in C_K} Hom_H(W(\mu), Y(\eta)) \otimes Hom(Y(\eta), V(\lambda))$$
Multiplication with respect to this refinement is "lower triangular" in the sense that the multiplication of two elements,
\[
W(\mu_1) \xrightarrow{f_1} Y(\eta_1) \xrightarrow{g_1} V(\lambda_1)
\]
\[
W(\mu_2) \xrightarrow{f_2} Y(\eta_2) \xrightarrow{g_2} V(\lambda_2)
\]
Yields a sum of homogeneous terms of the form
\[
W(\mu_1 + \mu_2) \xrightarrow{f \circ f_2} Y(\eta_1 + \eta_2) \xrightarrow{g \circ g_2} V(\lambda_1 + \lambda_2)
\]
with \(\eta \leq \eta_1 + \eta_2\) as dominant weights. Also, there is always a nonzero highest weight term with \(\eta = \eta_1 + \eta_2\), of the form
\[
W(\mu_1 + \mu_2) \xrightarrow{f_1 \circ f_2} Y(\eta_1 + \eta_2) \xrightarrow{g_1 \circ g_2} V(\lambda_1 + \lambda_2)
\]
where \(f_1 \circ f_2\) and \(g_1 \circ g_2\) are the multiplications in the branching algebras \(A(\psi)\) and \(A(\pi)\) respectively. One can now choose functionals \(\bar{h}\) on the monoid \(C_H \times C_K \times C_G\) to produce graded filtrations of \(A(\phi)\). It is shown in [M1] and [M2] that if the functional \(\bar{h}\) is non-negative on all positive roots, it defines a graded valuation. Furthermore, if the functional is strictly positive on positive roots then it defines a filtration with a remarkable associated graded.

\[(34)\] \[
gr_{\bar{h}}[A(\psi \circ \pi)] = [A(\psi) \otimes A(\pi)]^T
\]

Here \(T\) is a maximal torus of \(K\), which picks out the correct tensors of graded components. These are the generalized valuations mentioned previously in the discussion of corollary [5.3]. The same reasoning can be used to show that graded valuations defined similarly for the algebra of conformal blocks and studied in [M1] are actually generalized valuations. For each factorization of a morphism \(\phi = \pi_1 \circ \ldots \circ \pi_k\) in the category of reductive groups we obtain a cone of suitable functionals. Given a functional \(\bar{h} = (h_1, \ldots, h_{k+1})\) for a factorization \((\pi_1, \ldots, \pi_k)\) we can set a component \(h_i\) to 0 and obtain a new functional \(h_i\) for the factorization \((\pi_1, \ldots, \pi_{i+1} \circ \pi_i, \ldots, \pi_k)\). In this way some of the structure of category of reductive groups is realized in a polyhedral complex of graded valuations on branching algebras. Also, multigraded subalgebras of the full branching algebra inherit these valuations by the functor properties, in particular this holds for the algebra of invariants \(R(G)^H\).

An amusing special case of a branching algebra filtration as above is given by observing that any representation of a reductive group \(V\) defines a map \(G \rightarrow GL_n\) where \(n = \text{dim}(V)\). This map factors the map from the trivial group \(1 \rightarrow GL_n\), and so defines a multilfiltration of \(A(1 \rightarrow GL_n) = R(GL_n)\). In this way any representation of a reductive group can be associated with a cone of graded valuations of \(R(GL_n)\) for some \(n\).

5.4. More valuations from branching structures. Let \(A\) be an algebra over \(\mathbb{C}\) with an action of a reductive group \(G\), and let \(\phi : H \rightarrow G\) be a morphism of reductive groups. We consider the algebra \([R(H) \otimes A]^H\). From the direct sum decomposition of \(A\) we get the following direct sum decomposition of \([R(H) \otimes A]^H\).

\[(35)\] \[
[R(H) \otimes A]^H = \bigoplus_{\mu, \lambda \in C_H \times C_G} A^{U+}(\lambda^*) \otimes \text{Hom}_H(W(\mu), V(\lambda))
\]
One can prove that multiplication $[R(H) \otimes A]^H$ has a similar "lower triangular" property with respect to the dominant weight data in this decomposition. In this way $[R(H) \otimes A]^H$ inherits the valuations on $\mathcal{A}(\phi)$ discussed in the previous example. The same holds for multigraded graded subalgebras of $[R(H) \otimes A]^H$, such as the algebra of invariants, $A^H$.

(36) $$A^H = \bigoplus \, A^{U_+}(\lambda^*) \otimes \text{Hom}_H(\mathbb{C}, V(\lambda))$$

Incidentally, both of these algebras also inherit any graded valuation of $A^{U_+}$ as in the first example above.

5.5. More valuations from dual canonical bases. The functor properties of graded valuations are useful for passing combinatorial structures from a ring to a subring. We return to the dual canonical bases, discussed in a previous example. Recall that these define a direct sum decomposition of an irreducible representation $V(\lambda) = \bigoplus \mathbb{C}b_\phi$, and therefore define a grading on the underlying vector space of $R(G)$ by the monoid of string parameters for a chosen decomposition of the longest element of the Weyl group. Recall also that for suitably assigned orders on these parameters, the multiplication in $R(G)$ is "lower triangular." Both of these properties therefore pass to subrings which respect this multigrading. Let $G' \subset G$ be a levi subgroup of $G$, which corresponds to a subset of the positive roots $\mathcal{R}_{G'} \subset \mathcal{R}_G$. The subspaces

(37) $$\text{Hom}_H(W(\mu), V(\lambda)) = \{v | e_i(v) = 0, h_i(v) = \mu(h_i)v, \alpha_i \in \mathcal{R}_{G'} \} \subset V(\lambda),$$

inherit a basis from the dual canonical basis of $V(\lambda)$, where $e_i$ is the raising operator and $h_i$ is the generator of the Cartan subalgebra of $\mathfrak{g}$ corresponding to the root $\alpha_i$. Therefore, for a Levi subgroup $G' \subset G$, we can construct graded valuations acting on the string parameters of the dual canonical basis, and the associated graded algebras of these multifiltrations are monoidal, as in [AB]. For a flag of Levi subgroups $G_0 \subset G_1 \subset \ldots \subset G$, this can be used to construct graded valuations which refine those constructed in the previous example on the branching algebra.

By a similar argument, the same is true for other rings that inherit the dual canonical basis. Consider the algebra $R(G) \otimes \mathbb{C}[T]$, for a maximal torus $T \subset G$. We will let $T$ act on $R(G)$ on the left and on $T$ by $t^{-1}$, and consider the subring of invariants.

(38) $$[R(G) \otimes \mathbb{C}[T]]^T = \bigoplus_{\mu, \lambda} V_\mu(\lambda) \otimes \mathbb{C}t^{-\mu}$$

Here $V_\mu(\lambda)$ is the subspace of $V(\lambda)$ with left $T$–character equal to $\mu$. It is known that this subspace inherits the dual canonical basis. This ring is particularly interesting because it contains the projective coordinate rings of weight varieties of $G/B$ as multigraded subalgebras. By the functor property, these rings inherit graded valuations from $R(G) \otimes \mathbb{C}[T]]^T$.

Now we consider opposite maximal unipotents $U_+$ and $U_-$ in $G$, and the map of varieties,
where a calculation on Lie algebras shows that the bottom row is dense in the top row. An invariant function \( f \in \mathbb{C}[G/U_+ \times G/U_+] \) satisfies

\[
(39) \quad f(u_1 t_1, u_2 t_2) = f(t_1, u_1^{-1} u_2 t_2),
\]

so we can realize \( \mathbb{C}[G/U_+ \times G/U_+] \) as a subalgebra of \( R(G) \otimes \mathbb{C}[T] \). The algebra \( \mathbb{C}[G/U_+ \times G/U_+] \) is multigraded by components which classify the multiplicities of invariants in triple tensor products of irreducible representations of \( G \).

On the level of vector spaces we have

\[
(40) \quad \text{Hom}_G(V(\alpha), V(\beta) \otimes V(\gamma)) = \text{Hom}_{U_+}(Cv_\alpha, V(\beta) \otimes V(\gamma)) = V_{\alpha - \beta, \beta} \otimes Cv_\beta
\]

Here \( V_{\alpha - \beta, \beta} \) is the subspace of vectors \( v \in V(\gamma) \) such that \( h_i(v) = \alpha - \beta(h_i)v \) and \( e_i^{\beta(h_i)+1} v = 0 \) for \( h_i \) the basis member of the Cartan subalgebra of \( g \) and \( e_i \) the raising operator in \( g \) corresponding to the positive root \( \alpha_i \). For this see chapter XVIII, page 383 of [Zh]. It is also known that this subspace inherits the dual canonical basis. The algebra \( \mathbb{C}[G/U_+ \times G/U_+] \) is isomorphic to the branching algebra \( A(\Delta_2) \), where \( \Delta_2 : G \rightarrow G \times G \) is the diagonal morphism. The "lower triangular" multiplication property of dual canonical basis therefore also defines monoidal associated graded algebras of this algebra in the fashion of [AB]. To our knowledge these have not been studied.

All three subspaces discussed in this example inherit a basis from the dual canonical basis because it satisfies the so-called good basis property, for this see [Ma]. We believe that the space of graded valuations of \( R(G) \) with respect to the dual canonical basis grading should be a useful combinatorial object for the representation theory of \( G \).

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