POINCARÉ INEQUALITIES AND UNIFORM RECTIFIABILITY

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ABSTRACT. We show that any $d$-Ahlfors regular subset of $\mathbb{R}^n$ supporting a weak $(1,d)$-Poincaré inequality with respect to surface measure is uniformly rectifiable.

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1. INTRODUCTION

For $p \geq 1$, a metric measure space $(X, \mu)$ admits a weak $(1,p)$-Poincaré inequality for all measurable functions $u$ with constants $C, \lambda \geq 1$ if $\mu$ is locally finite and

$$\int_{B} |u - u_B| d\mu \leq C \text{diam } B \left( \int_{\lambda B} \rho^p d\mu \right)^{\frac{1}{p}}$$

where $\rho$ is any upper gradient for $u$, meaning for every $x, y \in X$,

$$|u(x) - u(y)| \leq \int_{\gamma} \rho$$

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for any rectifiable curve $\gamma$ connecting $x$ to $y$ in $X$.

This condition was introduced by Heinonen and Koskela in [HK98], as it is a property shared by a large class of metric spaces (such as the Heisenberg group and Ahlfors regular Riemannian manifolds of non-negative Ricci curvature) and imposes certain geometric properties on the metric space $X$. For example, $X$ must be quasiconvex and in fact there are quantitatively many curves running through $X$ (we will be more specific about this later). Also, a classical result of Cheeger says that if $\mu$ is a doubling measure and $X$ is complete, then the metric space admits a differentiable structure akin to the way smooth manifolds do, that is, it admits a partition into sets upon which there are maps into some Euclidean space playing the role of chart maps, and in particular one has a Rademacher-type theorem saying that Lipschitz functions may be differentiated a.e. with respect to these chart maps [Che99] (see also [Kei04] for an improvement on this result, [KM16] for a compact primer to these results, and [BKO19] for a shorter proof using Guth’s multilinear Kakeya inequality for neighbourhoods of Lipschitz graphs). Having a differentiable structure does not imply having a Poincaré inequality, since such a set can be totally disconnected. We should also mention that Bate has characterized Lipschitz differentiability spaces in terms of having Alberti representations, which is weaker than having a Poincaré inequality [Bat15].

Additionally, in Cheeger’s original setting, if $\mu \ll \mathcal{H}^d$ (where $\mathcal{H}^d$ is Hausdorff measure, see [Mat95]) and $X$ can be bi-Lipschitz embedded into Euclidean space, Cheeger showed $X$ is $d$-rectifiable, meaning it can be covered up to $\mathcal{H}^d$-measure zero by countably many Lipschitz images of subsets of $\mathbb{R}^d$ [Che99, Theorem 14.3], and in fact later it was shown that $\mu \ll \mathcal{H}^d$ is not necessary as it is implied by having a differentiable structure [DPMR17].

On the other hand, there are well-known examples of metric spaces satisfying these properties (apart from being subsets of Euclidean space) that are not rectifiable, let alone uniformly, such as the Heisenberg group and Laakso spaces [Laa00]. However, recently Bate and Li have characterized $d$-dimensional rectifiable metric spaces as those metric spaces with positive and finite $d$-dimensional densities (with respect to Hausdorff measure) and for which there is a differentiable structure such that the chart maps are $d$-dimensional [BL17].

The purpose of this note is to develop a quantitative version of Cheeger’s original result, or in other words, to determine what better rectifiable structure we can attain if we know more about the differentiable structure of a set. We do so for Ahlfors $d$-regular subsets of Euclidean space, which are
sets $E \subseteq \mathbb{R}^n$ for which there is a constant $A$ so that

$$\frac{r^d}{A} \leq \mathcal{H}^d(E \cap B(x, r)) \leq Ar^d \quad \text{for all } x \in E \text{ and } 0 < r < \text{diam } E.$$ 

Our main result is as follows:

**Main Theorem.** Let $n > d \geq 2$ be integers and $X \subseteq \mathbb{R}^n$ be an Ahlfors $d$-regular set with constant $A \geq 1$ supporting a weak $(1, d)$-Poincaré inequality with respect to $\mathcal{H}^d|X$ with constants $C, \lambda \geq 1$. Then $X$ is uniformly $d$-rectifiable (UR), meaning there are constants $L, c > 0$ so that for every $x \in X$ and $0 < r < \text{diam } X$, there is an $L$-bi-Lipschitz image of a subset of $\mathbb{R}^d$ contained in $X \cap B(x, r)$ of $\mathcal{H}^d$-measure at least $cr^d$. The constants $L$ and $c$ depend on $n, C, \lambda$, and $A$.

Uniformly rectifiable sets were introduced by David and Semmes in [DS91], and are a sort of quantitative version of a rectifiable set, in the sense that $X$ is UR if it is rectifiable by the same amount and Lipschitz constant in every ball. They feature in various results that characterize when a certain quantitative property holds on an Ahlfors regular set. For example, certain classes of singular integral operators are bounded on an Ahlfors regular set if and only if that set is UR [DS91].

One previous result similar to our Main Theorem is due to Merhej [Mer17], who showed that if additionally $d = n - 1$ and the unit normal vectors to the set have small BMO norm, and in this case locally $X$ is contained in a bi-Lipschitz image of $\mathbb{R}^{n-1}$ (rather than just containing big pieces of $\mathbb{R}^{n-1}$ as in the definition of UR). She also has a higher codimensional version of this result [Mer16], which again requires some small oscillation of the tangents in the set $X$.

There are other similar results for sets that inherit a Poincaré condition from some stronger topological assumptions: David (not the aforementioned one) showed that any compact Ahlfors $d$-regular locally linearly contractible complete oriented topological $d$-manifold is UR [Dav16, Theorem 1.13], and such spaces support a weak $(1, d)$-Poincaré inequality by [Sem96] (see also [HK98, Theorem 6.11]). This is more general than our result in that it holds for non-Euclidean metric spaces, although the topological condition is more restrictive than being Loewner.

The proof of the Main Theorem goes roughly as follows: the Poincaré inequality implies that there are quantitatively many curves passing through the set by a result of Heinonen and Koskela. Using Dorronsoro’s theorem, we can show that, for many $x \in X$ and $r > 0$, and for any $(d - 1)$-dimensional plane $V$, we can find parts of $X$ that lie close to a line segment passing through $x$ in $B(x, r)$ and have large angle from $V$. Inductively, this
means we can actually find parts of $X$ close to $d$ many line segments passing through $x$ that have large angle from each other. We then use similar arguments to show that, for most balls on $X$, $X$ is approximately contained in a $d$-dimensional plane in those balls (otherwise, we could also find parts of $X$ close to a $(d + 1)$st-line passing through each $x$, but we know $X$ is $d$-rectifiable and so it must be approximately $d$-flat somewhere, violating the existence of this extra line). These two geometric properties imply that in fact $X$ is close to a $d$-dimensional plane in the Hausdorff metric, and this implies uniform rectifiability by a result of David and Semmes. We point out that this aspect of finding approximate line segments in many directions is in a way reminiscent of how Bate finds Alberti representations in differentiability spaces [Bat15].

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2. Preliminaries

2.1. Notation. We will write $a \lesssim b$ if there is a constant $C > 0$ so that $a \leq Cb$, $a \lesssim_t b$ if the constant depends on the parameter $t$, and $a \sim b$ and $a \sim_t b$ to mean $a \lesssim b \lesssim a$ and $a \lesssim_t b \lesssim_t a$ respectively. We will assume all implied constants depend on $n$, $d$, and also on the Poincaré and Ahlfors regularity constants for $X$, and hence write $\sim$ instead of $\sim_{d,n,A,C,\lambda}$.

Whenever $A, B \subset \mathbb{R}^n$ we define

$$\text{dist}(A, B) = \inf \{|x - y|; x \in A, y \in B\},$$

and $\text{dist}(x, A) = \text{dist}(\{x\}, A)$.

Let $\text{diam} A$ denote the diameter of $A$ defined as

$$\text{diam} A = \sup \{|x - y|; x, y \in A\}.$$

We let $B(x, r)$ denote the closed ball centered at $x$ of radius $r$ in $\mathbb{R}^n$, and we will often write $B = B(0, 1)$. If $B$ is a generic ball, we will write $x_B$ for its center and $r_B$ for its radius, so $B = B(x_B, r_B)$. We let $\mathcal{G}(d, n)$ denote the Grassmannian, that is, the set of $d$-dimensional subspaces of $\mathbb{R}^n$ (that is, the $d$-dimensional planes containing the origin), and $\mathcal{A}(d, n)$ denote the affine Grassmannian, which is the set of all $d$-dimensional planes in $\mathbb{R}^n$ (not necessarily containing the origin).

Given a plane $V \in \mathcal{A}(d, n)$, we let $\pi_V : \mathbb{R}^n \to V$ denote the projection into $V$, $V' \in \mathcal{G}(d, n)$ the $d$-dimensional plane parallel to $V$ and containing the origin, and $V^\perp \in \mathcal{G}(n - d, n)$ the orthogonal complement of $V'$. Given
two planes $V, W$ with $\dim V \leq \dim W$, we let
\[
\angle(V, W) = |\pi|_{W^\perp} = \sup_{x \in V \cap B} \dist(x, W).
\]
that is, $\angle(V, W)$ is the norm of the linear operator $\pi_W : V \to W^\perp$. Note
that if $L$ is a line, then $\angle(L, W)$ is comparable to the usual angle between $L$ and $W$. If $V, W \in \mathscr{A}(d, n)$, we let $\angle(V, W) := \angle(V', W')$. Note that from
the above definition, if $\dim U \leq \dim V \leq \dim W$, then
\[
\angle(U, W) \leq \angle(U, V) + \angle(V, W).
\]

2.2. Curves and Modulus. In this section we introduce the notion of modulus
of curve families. For a more in depth treatment, see [Hei01] or [Vuo88].

By a curve $\gamma$, we will mean any continuous image of a closed interval
$I \subseteq X$. Given $\gamma$, we will denote this function also as $\gamma : I \to X$. We define
the length of $\gamma$ as
\[
\ell(\gamma) = \sup_{t_1 < \cdots < t_k} \sum |\gamma(t_i) - \gamma(t_{i+1})|
\]
where the supremum is over all sequences $a = t_1 < \cdots < t_k = b$ if the endpoints of $I$ are $a$ and $b$. If $I$ is not closed, we define the length of $\gamma$
to be the supremum over the lengths of all subcurves with closed domain.
If $\gamma$ is of finite length, we say $\gamma$ rectifiable then $\gamma$ factors as $\gamma = \gamma_s \circ s_\gamma$
where $s_\gamma : I \to [0, \ell(\gamma)]$ is so that $s_\gamma(t) = \ell(\gamma|_{[0,t]})$ and $\gamma_s$ is the arclength parametrization, that is, a $1$-Lipschitz function $\gamma_s : [0, \ell(\gamma)] \to X$
with $\gamma(I) = \gamma$. We will assume all rectifiable curves below are arclength
parametrized. If all closed subcurves are rectifiable, we say $\gamma$ is locally rectifiable.

Given a metric space $X$, a Borel measure $\mu$, a family of curves $\Gamma$ in $X$,
and a Borel function $\rho$, we say $\rho$ is admissible for $\Gamma$ if for each rectifiable
curve $\gamma \in \Gamma$,
\[
\int_\gamma \rho := \int_0^{\ell(\gamma)} \rho \circ \gamma \geq 1 \text{ for all } \gamma \in \Gamma.
\]
Note that this notation means we are integrating $\rho$ composed with the function $\gamma$ and not $\rho$ on the image of $\gamma$. However, the former is at least the latter: since the arclength parametrization is $1$-Lipschitz, $\mathcal{H}^1(\gamma(A)) \leq |A|$.
for any \( A \subseteq [0, \ell(\gamma)] \) (see [Mat95, Theorem 7.5]), and so

\[
\int_\gamma \rho = \int_0^\infty |\{ t \in [0, \ell(\gamma)] : \rho \circ \gamma(t) > \lambda \}| d\lambda
\]

\[
= \int_0^\infty |\gamma^{-1}(\{ x \in \gamma : \rho(x) > \lambda \})| d\lambda
\]

\[
\geq \int_0^\infty \mathcal{H}^1(\{ x \in \gamma : \rho(x) > \lambda \}) d\lambda
\]

\[
= \int_\gamma \rho d\mathcal{H}^1
\]

although these two integrals may not equal, for example if \( \gamma \) doubles back on itself. If \( \gamma \) is only locally rectifiable, we define \( \int_\gamma \rho \) to be the supremum of \( \int_{\gamma'} \rho \) over all rectifiable subcurves \( \gamma' \).

We define the \( p \)-modulus of \( \Gamma \) to be

\[
\inf \left\{ \int \rho^p d\mu : \rho \text{ admissible for } \Gamma \right\}.
\]

We say \( (X, \mu) \) is a \( p \)-Loewner space if, whenever \( E, F \subseteq X \) are two disjoint continua, and \( \Gamma(E, F) \) is the collection of curves in \( X \) starting in \( E \) and ending in \( F \), then

\[
\text{Mod}_p(\Gamma(E, F)) \geq t \quad \text{whenever } \Delta(E, F) := \frac{\text{dist}(E, F)}{\min\{\text{diam } E, \text{diam } F\}} \leq t.
\]

The following lemma is quite standard, but we include a proof for completeness.

**Lemma 2.1.** Let \( B \) be a ball in \( X \) and \( E, F \subseteq Q \) two disjoint continua so that \( \Delta(E, F) \leq t \). Let \( \Gamma_{C, B}(E, F) \) be those curves in \( \Gamma(E, F) \) of length at most \( Ct_B \). Then for \( C \) large enough (depending on \( t \) and the Loewner constants),

\[
\text{Mod}_d(\Gamma_{C, B}(E, F)) \geq C, t \quad \text{1.}
\]

**Proof.** Recall that there is a constant \( C_0 \) depending on \( t \) and the Loewner constants so that

\[
\text{Mod}_d(\Gamma(E, F)) \geq C_0.
\]

First let \( \Gamma_1 \) be those curves in \( \Gamma(E, F) \) that contain a point outside of \( AB \) for some \( A \geq 2 \) to be chosen shortly. Let

\[
\rho_1(x) = \frac{1}{r_B + |x - x_B|} \frac{1}{\log A} 1_{AB}
\]
Then it is not hard to show that $\int_\gamma \rho_1 \gtrsim 1$ for all $\gamma \in \Gamma_1$. Thus,

$$\text{Mod}_d(\Gamma_1) \leq \int \rho^p d\mu \lesssim (\log A)^{1-d}. \tag{2.13}$$

See [Hei01, Theorem 7.18] for a proof of a similar estimate. Choose $A$ large enough (depending on $d$ and $C_0$) so that

$$\text{Mod}_d(\Gamma_1) < \frac{C_0}{4}.$$ 

Now let $\Gamma_2$ be those curves in $\Gamma(E,F)$ contained in $AB$ but so that their length is at least $C r_B$. Let $\rho = \frac{1}{C r_B} I_{AB}$. Then $\rho$ is admissible for $\Gamma_2$, and so

$$\text{Mod}_d(\Gamma_2) \leq \int \rho^p \lesssim (Ar_B)^d (Cr_B)^{-d} = \frac{A^d}{C^d}.$$ 

Hence, we can pick $C$ depending on $A$ and $C_0$ (and so just really on $C_0$) so that

$$\text{Mod}_d(\Gamma_2) < \frac{C_0}{4}.$$ 

Thus, by the subadditivity of the modulus (see [Hei01, Equation (7.7)]),

$$\text{Mod}_d(\Gamma(E,F)) \geq \text{Mod}_d(\Gamma_{C,B}(E,F)) - \text{Mod}_d(\Gamma_1) - \text{Mod}_d(\Gamma_2) > \frac{C_0}{2}. \tag{2.14}$$

The connection between the Poincaré inequality and Loewner spaces is via the following result.

**Theorem 2.2.** An Ahlfors $d$-regular metric measure space $(X, \mu)$ admits a weak $(1,d)$-Poincaré inequality if and only if it is a $d$-Loewner space.

This follows from [HK98, Theorems 5.7 and 5.12]. Note that the first of these theorems (the forward implication) requires $X$ to be $\phi$-convex; we won’t define this, but it is satisfied when $X$ is quasiconvex, which holds when $X$ has a weak $(1,d)$-Poincaré inequality by a theorem of Semmes (see the appendices of [Che99, KM16]).

### 2.3. Compactness Lemma.

Recall that a sequence of compact sets $X_j$ converge to another compact set $X$ in the Hausdorff metric in $\mathbb{R}^n$ if

$$\lim_{j \to \infty} \max \left\{ \sup_{x \in X} \text{dist}(x, X_j), \sup_{x \in X_j} \text{dist}(x, X) \right\} = 0.$$ 

Given closed nonempty but possibly unbounded sets $X_j$ and $X$ in $\mathbb{R}^n$, we will say $X_j \to X$ in the Hausdorff metric if for each $R > 0$ there is $\varepsilon_j \downarrow 0$
so that \( X_j \cap B(0, R + \varepsilon_j) \) converges to \( X \cap B(0, R) \) in the Hausdorff metric, or equivalently, if

\[
\lim_{j \to \infty} \max \left\{ \sup_{x \in X \cap B(0, R)} \text{dist}(x, X_j), \sup_{x \in X_j \cap B(0, R)} \text{dist}(x, X) \right\} = 0 \quad \forall \ R > 0.
\]

We will also use the notion of convergence of pointed metric measure spaces. There is a lot of notation to unpack here, so we instead refer the reader to [Kei03, Section 2] for the terminology. His definition of measured (Gromov-Hausdorff) convergence is different than that used in [Che99], but after passing to a subsequence they are equivalent (see [Kei03, Section 1.3]).

Here we recall a well-known compactness lemma. We couldn’t find an exact statement of this result, but give a sketch of the proof.

**Lemma 2.3.** Let \( X_j \subseteq \mathbb{R}^n \) be a sequence of Ahlfors \( d \)-regular sets admitting a weak \((1, p)\)-Poincaré inequality for some \( p > 1 \) with the same constants, and suppose \( 0 \in X_j \). Then there is a subsequence that converges in the Hausdorff distance to an Ahlfors \( d \)-regular set also satisfying a weak \((1, p)\)-Poincaré inequality.

**Proof.** By the main result of [KZ08], there is \( \varepsilon > 0 \) (independent of \( j \)) so that each \( X_j \) also satisfies a weak \((1, p - \varepsilon)\)-Poincaré inequality quantitatively, and with the same constants for each \( j \). Let \( p_j \) be the path-metric on \( X_j \). As mentioned above, the \( X_j \) are uniformly quasiconvex, and thus \((X_j, \rho_j)\) is bi-Lipschitz equivalent to \((X_j, |\cdot|)\), where \(|\cdot|\) is the usual Euclidean metric.

By [Che99, Theorems 9.1 and 9.6], we can pass to a subsequence so that the pointed metric spaces \((X_j, 0, \rho_j, \mu_j)\) converge in the Gromov-Hausdorff sense to a pointed metric space \((X_0, p_0, \rho, \mu_0)\) also satisfying a weak \((1, p')\)-Poincaré inequality for each \( p' > p - \varepsilon \), and so in particular when \( p' = p \).

It is not hard to show that \( \mu_0 \) is Ahlfors regular. Note that as \((X_j, \rho_j)\) is bi-Lipschitz equivalent to \((X_j, |\cdot|)\) via some maps \( g_j : (X_j, \rho_j) \to (X_j, |\cdot|)\) so that \( g_j(0) = 0 \), we may pass to a subsequence via Arzela-Ascoli to find a bi-Lipschitz map \( g : (X_0, \rho_0) \to (X, |\cdot|)\). Thus, \( X \) also satisfies a Poincaré inequality with respect to \( g[\mu_0] \). Since \( \mu_0 \) is Ahlfors regular and \( g \) is bi-Lipschitz, this measure is comparable to surface measure on \( X \), and so \( X \) satisfies a weak \((1, p - \varepsilon)\)-Poincaré inequality. By Jensen’s inequality, it also satisfies a \((1, p)\)-Poincaré inequality. \(\square\)

2.4. \( \beta \)-numbers. For \( V \in \mathcal{A}(d, n) \), let

\[
\beta_X(x, r, V) = \sup_{y \in B(x, r) \cap X} \frac{\text{dist}(y, V)}{r}, \quad \beta_X(x, r) = \inf_{V \in \mathcal{A}(d, n)} \beta(x, r, V).
\]
Given a ball $B(x,r)$ centered on $X$, we will also sometimes write $\beta_X(B(x,r))$ for $\beta_X(x,r)$. It is not hard to show that, if $B(x,r) \subseteq B(y,s)$ are centered on $X$, then

$$\beta_X(x,r,V) \leq \frac{s}{r}\beta_X(x,s,V).$$

Furthermore, for $r > 0$ and $x, y \in X$,

$$\beta_X(x,r,V) \leq \beta_X(y,r,V) + \frac{|x - y|}{r}.$$  

The compactness result above gives us an easy first quantitative version of Cheeger’s rectifiability theorem that we will need later to prove the full version we seek:

**Lemma 2.4.** Let $X \subseteq \mathbb{R}^n$ be a $d$-Loewner space. For all $\varepsilon \in (0, 1/2)$, $x \in X$, and $r > 0$, there is $r' \geq \varepsilon r$ and $x' \in B(x,r/2) \cap X$ so that

$$\beta_X(x',r') < \varepsilon.$$  

**Proof.** It suffices to prove the lemma in the case that $x = 0$ and $r = 1$. Suppose there was $\varepsilon > 0$ and a sequence of $d$-Loewner spaces $X_j \subseteq \mathbb{R}^n$ with the same constants so that for all $x' \in B(0, 1/2) \cap X_j$ and $r' \geq 1/j$,

$$\beta_{X_j}(x', r') \geq \varepsilon.$$  

These spaces satisfy a weak $(1,d)$-Poincaré inequality with the same constants for all $j$. We can pass to a subsequence so that they converge in the Hausdorff metric to another $d$-regular set satisfying a weak $(1,d)$-Poincaré inequality. By [Che99, Theorem 14.2], $X$ is $d$-rectifiable, and since $X$ is Ahlfors $d$-regular, $X$ has a tangent at some point $x \in X \cap B(0, 1/2)$ (see the discussion after [Vil17, Definition 1.7]), so there is a plane $P$ passing through $x$ and $r > 0$ small so that

$$\beta_X(x,r, P) < \varepsilon/4.$$  

There is $\varepsilon_j$ so that $X_j \cap B(0, 1 + \varepsilon_j)$ converges to $X$ in the Hausdorff metric, so for $j$ large enough,

$$\sup_{x' \in X_j \cap B(x,r)} \text{dist}(x', X) < \frac{\varepsilon r}{4}. $$

In particular, for $j$ large enough we can find $x_j \in X_j \cap B(x,r/2)$ and so that for each $y' \in B(x_j,r/2) \cap X_j$, there is $y \in X$ with $|y - y'| < \frac{\varepsilon r}{4}$. Thus,

$$|y - x| \leq |y - y'| + |y' - x_j| + |x_j - x| \leq \frac{\varepsilon r}{4} + \frac{r}{2} + \frac{\varepsilon r}{4} < r$$

so $y \in B \cap X$. Thus,

$$\text{dist}(y', P) \leq |y' - y| + \text{dist}(y, P) \overset{\text{(2.4)}}{<} \frac{\varepsilon r}{4} + \frac{\varepsilon r}{4} = \frac{\varepsilon r}{2}.$$
If we take the supremum over all $y' \in B(x_j, r/2) \cap X_j$, then for $1/j < r/2$, by how we chose the $X_j$, 

$$\varepsilon \leq \beta_{X_j}(x_j, r/2) < \varepsilon,$$

which is a contradiction. □

2.5. **Christ-David Cubes.** We recall the following version of “dyadic cubes” for metric spaces, first introduced by David [Dav88] for Ahlfors regular sets, but generalized in [Chr90] and [HM12].

**Theorem 2.5.** Let $X$ be a doubling metric space. Let $X_k$ be a nested sequence of maximal $\rho^k$-nets for $X$ where $\rho < 1/1000$ and let $c_0 = 1/500$. For each $n \in \mathbb{Z}$ there is a collection $\mathcal{D}_k$ of “cubes,” which are Borel subsets of $X$ such that the following hold.

1. For every integer $k$, $X = \bigcup_{Q \in \mathcal{D}_k} Q$.
2. If $Q, Q' \in \mathcal{D} = \bigcup \mathcal{D}_k$ and $Q \cap Q' \neq \emptyset$, then $Q \subseteq Q'$ or $Q' \subseteq Q$.
3. For $Q \in \mathcal{D}$, let $k(Q)$ be the unique integer so that $Q \in \mathcal{D}_k$ and set $\ell(Q) = 5\rho^k(Q)$. Then there is $\zeta_Q \in X$ so that

$$\begin{align*}
B_{X}(\zeta_Q, c_0\ell(Q)) \subseteq Q \subseteq B_{X}(\zeta_Q, \ell(Q))
\end{align*}$$

and $X_k = \{\zeta_Q : Q \in \mathcal{D}_k\}$.

3. **Proof of the Main Theorem**

3.1. **The bilateral weak geometric lemma.** For $x \in X$, $r > 0$ and $V \in \mathcal{A}(d, n)$, we define the $d$-dimensional bilateral $\beta$-number with respect to $V$ to be

$$b\beta_X(x, r, V) = r^{-1}\left(\sup_{y \in X \cap B(x, r)} \text{dist}(y, V) + \sup_{y \in V \cap B(x, r)} \text{dist}(y, X)\right)$$

and then define

$$b\beta_X(x, r) = \inf_{V \in \mathcal{A}(d, n)} b\beta(x, r, V).$$

The main black-box theorem we will use is the following characterization of uniform rectifiability due to David and Semmes [DS93, Theorem I.2.4]. Here, we write $|A| = \mathcal{H}^d(A)$ for $A \subseteq X$.

**Lemma 3.1 (The Bilateral Weak Geometric Lemma (BWGL)).** An Ahlfors $d$-regular set $X \subseteq \mathbb{R}^n$ is UR if and only if, for each $\varepsilon > 0$ and $R \in \mathcal{D}$,

$$\sum_{Q \subseteq R, b\beta(2B_R) \geq \varepsilon} |Q| \lesssim_{\varepsilon} |R|.$$

We will establish that $X$ in the Main Theorem is UR by proving it satisfies the BWGL over the course of the following sections.
3.2. Finding approximate line segments in \( X \). For \( x \in X, r > 0, V \) a plane of dimension between 1 and \( n - 1 \), and \( \theta > 0 \), define
\[
\eta^V_X(x, r) = \inf_L \sup_{y \in B(x, r) \cap L} \frac{\text{dist}(y, X)}{r}
\]
where the infimum is over all lines \( L \) passing through \( x \) so that if \( e_L \) is the vector parallel to \( L \), then
\[
\angle(L, V) = \text{dist}(e_L, V') \geq \theta.
\]
We record a few basic properties of the \( \eta \)-numbers. Firstly, since all lines pass through \( x \) in this definition, we immediately have
\[
0 \leq \eta^V_X(x, r) \leq 1.
\]

**Lemma 3.2.** Let \( x \in X \). Then
\[
(3.1) \quad \eta^V_X(x, r) \leq \frac{s}{r} \eta^V_X(x, s) \quad \text{for all } 0 < r \leq s.
\]
Also, if \( x, y \in X \), then
\[
(3.2) \quad \eta^V_X(x, r) \leq \eta^V_X(y, r) + \frac{|x - y|}{r}.
\]

**Proof.** For the first of these equations, let \( L \) be any line passing through \( x \). Then
\[
r \eta^V_X(x, r) \leq \sup_{y \in B(x, r) \cap L} \text{dist}(y, X) \leq \sup_{y \in B(x, s) \cap L} \text{dist}(y, X)
\]
and infimizing over all \( L \), we obtain
\[
r \eta^V_X(x, r) \leq s \eta^V_X(x, s).
\]
For the second equation, let \( L \) be the line that infimizes \( \eta^V_X(x, r) \). Let \( L' = L + y - x \). Then \( L' \) passes through \( y \) and also has angle at least \( \theta \) with \( V \). If \( z' \in L' \cap B(y, r) \), then \( z := z' - y + x \in B(x, r) \cap L \), and so there is \( z'' \in X \) with \( |z - z''| \leq \eta^V_X(x, r)r \). Thus,
\[
\text{dist}(z', X) \leq |z' - z''| \leq |z' - z| + |z - z''| \leq |x - y| + \eta^V_X(x, r)r.
\]
Dividing both sides by \( r \) and taking the supremum over all \( z' \in B(y, r) \cap L' \) gives (3.2).

The main objective of this section is the following lemma.

**Lemma 3.3.** Let \( V \in G(d - 1, n) \). For \( Q \in \mathcal{D} \), let
\[
\eta_V(Q) = \sup_{x \in Q} \eta^V_X(x, \ell(Q)).
\]
There is $\theta > 0$ so that, for $\delta > 0$,
\[ \sum_{Q \subseteq R, \eta(Q) \geq \delta} |Q| \lesssim |R| \quad \text{for all } R \in \mathcal{D}. \]

We will require a lemma:

**Lemma 3.4.** Let $C$ be as in Lemma 2.1. There is $\theta > 0$ so that the following holds. Let $V$ be a plane through the origin of dimension between 1 and $n - 1$. Let $B$ be a ball centered on $X$ with $0 < r_B < \text{diam } X$ and suppose there is $x_0 \in \frac{1}{2}B \cap X$ with $\text{dist}(x_0, V + x_B) \geq cr_B$. Then there is $E_B^V \subseteq CB \cap X$ so that $|E_B^V| \gtrsim c$ and

\[ (3.3) \quad \int_0^{r_B} \eta_B^V(x, r)^2 \frac{dr}{r} \lesssim_{c, X} 1 \quad \text{for all } x \in E_B. \]

Some of the ideas for this proof come from [Jon88], [Dav91, Section III.4], and [JKV97].

**Proof.** Fix $V$, let $\theta > 0$ to be decided later, and write $\eta = \eta_B^V$.

Let $A_1 \subseteq \frac{c}{4}B \cap X$ and $A_2 \subseteq B(x_0, \frac{c}{4}r_B) \cap X$ be two continua of diameter at lest $\frac{c}{4}r_B$ (which exist since $X$ is connected).

For technical reasons, it will be more convenient to work with loops rather than curves. For a family of rectifiable curves $\Gamma_0$ and $\gamma \in \Gamma_0$, let $\dot{\gamma} = \gamma$ on $[0, \ell(\gamma)]$ and $\dot{\gamma}(t) = \gamma(2\ell(\gamma) - t)$ for $t \in [\ell(\gamma), 2\ell(\gamma)]$. Then $\dot{\gamma}$ is a curve of length $2\ell(\gamma)$. Let $\dot{\Gamma}_0 = \{ \dot{\gamma} : \gamma \in \Gamma_0 \}$. Then one can show

\[ \text{Mod}_d(\dot{\Gamma}_0) = 2^{-d}\text{Mod}_d(\Gamma_0). \]

Let $\Gamma = \dot{\Gamma}_{C,B}(A_1, A_2)$. By Lemma 2.1 and the above observations,

\[ \text{Mod}_d(\Gamma) \gtrsim 1. \]

Note that since the curves in $\Gamma_{C,B}(A_1, A_2)$ start and end in $\frac{1}{2}B$, they must be contained in $CB$ (otherwise their lengths would be at least $2(C - 1)r_B$, and we can assume $C$ is large enough so that $2(C - 1)r_B > Cr_B$, and thus their length would be too big). Let

\[ \rho(x) = r_B^{-1} \exp \left( - \int_0^{r_B} \eta(x, r)^2 \frac{dr}{r} \right) 1_{CB}. \]

Our aim is to show that some multiple of $\rho$ is admissible for $\Gamma$. If this is the case, then by Ahlfors regularity

\[ 1 \lesssim \text{Mod}_d(\Gamma) \lesssim \int \rho^d \lesssim \int_{CB} r_B^{-d} \lesssim 1, \]

and so by Chebychev’s inequality, there must be a set $E_B^V \subseteq CB$ with $|E_B^V| \sim |Q|$ and where $\rho \lesssim 1$ on $E_B^V$.

Let $\gamma \in \Gamma$ (see Figure 1 for reference).
Recall that $\gamma \subseteq CB$ by the definition of $\Gamma$. Our goal now is to show

\[ \int_{\gamma} \rho \gtrsim 1. \tag{3.4} \]

Let $\gamma : [0, \ell(\gamma)] \to \mathbb{R}^n$ denote its $1$-Lipschitz arclength parametrization. Let $\pi$ be the orthogonal projection into $V^\perp$. Then

\[ |\pi(\gamma)| \geq \frac{c}{2} r_B. \tag{3.5} \]

Without loss of generality, we’ll scale things so that $[0, \ell(\gamma)] = [0, 1]$ (recall that since $\gamma \in \Gamma$, $1 = \ell(\gamma) \sim r_B$) and translate so that $\gamma(0) = 0$. We recall the following theorem.

**Theorem 3.5** (Dorronsoro, [Dor85], Theorem 6). Let $1 \leq p < p(d)$ where
For $x \in \mathbb{R}^d$, $r > 0$, and $f \in W^{1,2}(\mathbb{R}^d)$, define

$$
\Omega_{f,p}(x,r) = \inf_A \left( \int_{B(x,r)} \left( \frac{|f - A|}{r} \right)^p \right)^{\frac{1}{p}}
$$

where the infimum is over all affine maps $A : \mathbb{R}^d \to \mathbb{R}$. Set

$$
\Omega_p(f) := \int_{\mathbb{R}^d} \int_0^\infty \Omega_{f,p}(x,r)^2 \frac{dr}{r} \, dx.
$$

Then

$$
\Omega_p(f) \lesssim_{d,p} ||\nabla f||_2^2.
$$

We can extend $\gamma$ to the whole real line by setting $\gamma(t) = 0$ for $t \notin [0, \ell(\gamma)]$ (recall that by our definition of $\Gamma$, $\gamma(0) = \gamma(\ell(\gamma))$, so our extension is $1$-Lipschitz on all of $\mathbb{R}$). For an interval $I$ in the real line, let

$$
\Omega(I) = |I|^{-1} \inf_A ||\gamma - A||_{L^\infty(I)}.
$$

where the infimum is over all affine functions $A : \mathbb{R} \to \mathbb{R}^n$. Then it is not hard to show (see [Azz16b, Lemma 2.5]) that,

$$
\sum_I \Omega(3I)^2 |I| \lesssim \Omega_{\infty}(\gamma) \lesssim 1
$$

where the sum is over all dyadic intervals in $\mathbb{R}$.

Let $A_I$ be the affine map that achieves the infimum in $\Omega(3I)$. Let $\delta > 0$ and $\mathscr{I}_1$ be those maximal intervals $I \subseteq [0,1]$ for which

$$
\text{diam}(\pi \circ \gamma(I)) < \delta |I|.
$$

Let $A > 0$ and let $\mathscr{I}_2$ be those maximal intervals $I$ for which

$$
\sum_{I \subseteq J \subseteq [0,1]} \Omega(3J)^2 > A.
$$

Then by Chebychev’s inequality,

$$
\left| \bigcup_{\mathscr{I}_2} I \right| \lesssim \frac{1}{A}.
$$
Let \( \{I_j\} \) be a subcollection of the intervals \( \mathcal{I} \) so that no point in \( \bigcup_{I \in \mathcal{I}} I \) is contained in more than two of the \( I_j \). Then
\[
\left| \pi \circ \gamma \left( \bigcup_{I \in \mathcal{I}} I \right) \right| = \left| \pi \circ \gamma \left( \bigcup_{I \in \mathcal{I}} I_j \right) \right| \leq \sum \left| \pi \circ \gamma (I_j) \right| < \sum \delta |I_j|
\]
(3.8)
\[
\leq 2\delta \left| \bigcup_{I \in \mathcal{I}} I \right| \leq 2\delta.
\]
Since \( |\pi(\gamma)| \gtrsim 1 \), this means that for \( \delta \) small enough and \( A \) large enough (depending on \( M \)), if we set
\[
E_\gamma = [0, 1] \setminus \bigcup_{I \in \mathcal{I}_{1} \cup \mathcal{I}_{2}} I, \quad F_\gamma = \gamma(E_\gamma),
\]
then since \( \gamma \) and \( \pi \) are 1-Lipschitz,
\[
|E_\gamma| \geq |F_\gamma| \geq |\pi(F_\gamma)| \geq |\pi \circ \gamma ([0, 1])| - 2\delta - \frac{C'}{A} \geq \frac{c}{2} - 2\delta - \frac{C'}{A}
\]
where \( C' \) is the constant implicit in (3.7), and so for \( A \) large and \( \delta \) small,
\[
|E_\gamma| \geq |F_\gamma| \geq \frac{c}{4}.
\]
Let \( L_I = A_I(\mathbb{R}) \). We claim that if \( I \cap E_\gamma \neq \emptyset \) and \( \Omega(3I) < \varepsilon \), then
\[
\angle (L_I, V) = |\pi_{|L_I}| \gtrsim \delta.
\]
We then pick \( \theta \ll \delta \) so that \( \angle (L_I, V) \geq \theta \). Indeed, under these assumptions, \( I \) is not contained in an interval from \( \mathcal{F} \), and so \( \text{diam} (\pi \circ \gamma (I)) \geq \delta |I| \). Since \( \Omega(3I) < \varepsilon \), for \( \varepsilon \ll \delta \) we then have that
\[
|E_\gamma| \geq |F_\gamma| \geq \frac{c}{4}.
\]
which implies
\[
|A_I(a) - A_I(b)| \geq \frac{\delta}{2} |a - b| \text{ for all } a, b \in \mathbb{R}
\]
(3.10)
Let \( X \in F_\gamma \) and \( x' \in E_\gamma \) be so that \( \gamma(x') = x \). Let \( I \) be a dyadic interval containing \( x' \) and let \( r \) be so that
\[
|I|/2 \leq 8r/\delta < |I|.
\]
(3.11)
Then
\[
|x - A_I(x')| = |\gamma(x') - A_I(x')| \leq |3I|\Omega(3I).
\]
Thus, recalling that \( \Omega(3I) < \varepsilon \ll 1 \),
\[
A_I^{-1}(B(x, 2r)) \subseteq A_I^{-1}(B(A(x'), 2r + |x - A(x')|)) \subseteq B(x', 2(2r + \varepsilon|3I|)/\delta) \subseteq B(x', 8r/\delta) \subseteq 3I
\]
(3.12)
Then
\[ \sup_{z \in L_I \cap B(x, 2r)} \text{dist}(z, X) \leq \sup_{z \in L_I \cap B(x, 2r)} \text{dist}(z, \gamma) = \sup_{y \in A_I^{-1}(B(x, r))} |A_I(y) - \gamma(y)| \]
\[ \leq \sup_{y \in \mathbb{M}} |A_I(y) - \gamma(y)| = |3I|\Omega(3I) \leq \frac{48}{\delta} r \Omega(3I) \]  \hspace{1cm} (3.11)

Now, the line \( L_I \) doesn’t pass through \( x \) necessarily, so let \( L_{x, r} \) be the line parallel to \( L_I \) passing through \( x \). Then for \( \varepsilon < \frac{\delta}{48} \),
\[ \text{dist}(L_{x, r}, L_I) \leq \frac{3I}{\Omega(3I)} \leq \frac{48}{\delta} \varepsilon < r \]  \hspace{1cm} (3.12)
and so for each \( z \in L_{x, r} \cap B(x, r) \), the closest point to \( z \) in \( L_I \) is contained in \( B(x, 2r) \). Thus,
\[ \sup_{z \in L_{x, r} \cap B(x, r)} \text{dist}(z, X) \leq |3I|\Omega(3I) + \sup_{z \in L_I \cap B(x, 2r)} \text{dist}(z, X) \leq \frac{96}{\delta} r \Omega(3I). \]

Hence,
\[ \int_0^{r_B} \eta(x, r)^2 \frac{dr}{r} \leq \int_{\delta/8}^1 \frac{dr}{r} + \sum_{x \in I \subseteq [0, 1]} \int_{|I|^{\delta/16}} |I|^{\delta/8} \eta(x, r)^2 \frac{dr}{r} \]
\[ \leq \delta + \sum_{x \in I \subseteq [0, 1]} \Omega(3I)^2 \frac{\delta}{\delta/16} r \Omega(3I) + \sum_{x \in I \subseteq [0, 1]} 1 \frac{dr}{r} \]
\[ \leq 1 + \Omega(3I)^2 + \varepsilon^{-2} \sum_{x \in I \subseteq [0, 1]} \Omega(3I)^2 \lesssim \frac{\varepsilon^2}{\varepsilon^2}. \]

In particular, (recall \( r_B \sim 1 \))
\[ \int_{\gamma} \rho \geq r_B^{-1} e^{-CA/\varepsilon^2} |F_{\gamma}| \gtrsim e^{-CA/\varepsilon^2}. \]

This proves (3.4).

\( \square \)

**Proof of Lemma 3.3.** First notice that since \( X \) is Ahlfors \( d \)-regular, for any \( V \in \mathcal{G}(d-1, n) \) and any cube \( Q \in \mathcal{D} \), the ball \( B'_Q := c_0 B_Q \) satisfies the conditions of Lemma 3.4. Let
\[ E_Q = E_{B'_Q} \subseteq c_0 B_Q \cap X. \]

We will use the following lemma.

**Lemma 3.6.** [DS93, Lemma VI.1.12] Let \( \alpha : \mathcal{C} \to [0, \infty) \) be given and suppose there is \( N > 0 \) and \( \eta > 0 \) so that for all \( R \in \mathcal{D} \),
\[ \left| \left\{ x \in R : \sum_{x \in Q \subseteq R} \alpha(Q) \leq N \right\} \right| \geq \eta |R|. \]
Then
\[ \sum_{Q \subseteq R} \alpha(Q) \lesssim_{N, \eta} |R| \text{ for all } R \in \mathcal{D}. \]

Thus, our lemma will follow once we show the following claim: if for \( \delta > 0 \) and \( Q \in \mathcal{D} \) we set
\[ \eta_\delta(Q) = \begin{cases} 1 & \eta_V(Q) \geq \delta \\ 0 & \eta_V(Q) < \delta \end{cases} \]
then for each \( R \in \mathcal{D} \) there is \( G_R \subseteq R \) so that \( |G_R| \gtrsim |R| \) and
\[ \sum_{x \in Q \subseteq R} \eta_\delta(Q) \lesssim_{\delta} 1 \text{ for all } x \in G_R. \]

Recall that \( |E_R| \geq c|R| \) for some constant \( c \). Let \( Q_j \) be the maximal cubes in \( R \) for which \( |Q_j \cap E_R| < \frac{c}{2}|Q_j| \). Then
\[ \sum |Q_j \cap E_R| \leq \frac{c}{2} \sum |Q_j| \leq \frac{c}{2} |R| \leq \frac{1}{2} |E_R| \]
so if we set \( F_R = E_R \setminus \bigcup Q_j \), then
\[ |F_R| \geq \frac{1}{2} |E_R| \geq \frac{c}{2} |R| \]
and \( F_R \) has the property that
\[ |E_R \cap Q| \geq \frac{c}{2} |Q| \text{ whenever } Q \cap F_R \neq \emptyset \text{ and } Q \subseteq R. \]

We require the following lemma.

**Lemma 3.7.** [Azz16a, Section 3] Let \( \mu \) be a \( C_\mu \)-doubling measure and let \( \mathcal{D} \) the cubes from Theorem 2.5 for \( X = \text{supp} \mu \) with admissible constants \( c_0 \) and \( \rho \). Let \( E \subseteq Q_0 \in \mathcal{D} \), \( M > 1 \), \( \delta > 0 \), and set
\[ \mathcal{P} = \{ Q \subseteq Q_0 : Q \cap E \neq \emptyset, \exists \xi \in B(\zeta_Q, M\ell(Q)) \text{ such that } \text{dist}(\xi, E) \geq \delta \ell(Q) \}. \]
Then there is \( C_1 = C_1(M, \delta, C_\mu) > 0 \) so that, for all \( Q' \subseteq Q_0 \),
\[ \sum_{Q \subseteq Q'} \mu(Q) \leq C_1 \mu(Q'). \]

This is slightly different from the original statement of this lemma, but it is a consequence (see for example [AM16]).

Let \( \mathcal{P}_R \) be the cubes from Lemma 3.7 with \( \mu = \mathcal{H}^d|_X \), \( E = F_R \), and \( \delta/4 \) in place of \( \delta \), and set
\[ \mathcal{C}_R = \{ Q \subseteq R : Q \cap F_R \neq \emptyset \text{ and } Q \notin \mathcal{P}_R \}. \]
Let \( \varepsilon > 0 \) and \( Q \in \mathcal{C}_R \) be so that

\[
\int_{E_R \cap Q} \int_{\ell(Q)/\rho} \eta(x, r)^2 \frac{dr}{r} < \varepsilon
\]

where again we set \( \eta = \eta_{X}^{V, \theta} \). We claim that, for \( \varepsilon \) small enough this implies \( \eta(Q) < \delta \).

By (3.1), (3.15) implies

\[
\int_{E_R \cap Q} \eta(x, \ell(Q))^2 \lesssim \varepsilon.
\]

By Chebychev, this implies that if

\[
S_Q := \{ x \in E_R \cap Q : \eta(x, \ell(Q)) < \varepsilon^{\frac{1}{4}} \}
\]

then

\[
|Q \cap E_R \setminus S_Q| \leq \varepsilon^{-1/2} \int_{E_R \cap Q} \eta(x, \ell(Q))^2 < \varepsilon^{\frac{3}{2}} |Q|
\]

Let \( x \in F_R \cap Q \) and let \( Q(x) \) be the largest cube in \( Q \setminus S_Q \) containing \( x \) (if it exists). Then

\[
|Q(x)| \leq \frac{2}{c} |E_R \cap Q(x)| \leq \frac{2}{c} |E_R \cap Q \setminus S_Q| < \frac{2}{c} \varepsilon^{\frac{1}{4}} |Q|.
\]

Thus, since \( Q(x) \) intersects \( F_R \) and the parent of \( Q(x) \) (which has diameter \( \rho^{-1} \text{diam} \ Q(x) \)) intersects \( S_Q \), for \( \varepsilon > 0 \) small (depending on \( \delta \) and \( c \)) and since \( X \) is Ahlfors \( d \)-regular,

\[
\text{dist}(x, S_Q) \leq \frac{\text{diam}(Q(x))}{\rho} \lesssim \frac{|Q(x)|^{1/d}}{\rho} \leq \left( \frac{2}{c} \right)^{1/d} \frac{\varepsilon^{\frac{1}{4}}}{\rho} |Q|^{\frac{1}{d}} < \frac{\delta}{4} \ell(Q).
\]

Thus, if \( Q \in \mathcal{C}_R \), then every \( x \in Q \) is at most \( \frac{\delta}{4} \ell(Q) \) from a point in \( F_R \), and so in fact by definition of \( \mathcal{C}_R \),

\[
\text{dist}(x, S_Q) < \frac{\delta}{2} \ell(Q) \quad \text{for all } x \in Q, \ Q \in \mathcal{C}_R.
\]

Thus, by (3.2), for \( \varepsilon^{\frac{1}{4}} < \frac{\delta}{2} \),

\[
\eta(x, \ell(Q)) \leq \varepsilon^{\frac{1}{4}} + \frac{\delta}{2} \frac{\ell(Q)}{\ell(Q)} \ell(Q) < \delta \quad \text{for all } x \in Q.
\]
This proves the claim. Thus,

\[
\sum_{Q \in \mathcal{C}_R} \eta(Q)|Q| \leq \sum_{Q \in \mathcal{C}_R} |Q| \leq \frac{2}{c} \sum_{Q \in \mathcal{C}_R} |E_R \cap Q|
\]

\[
(3.15) \leq \frac{2}{c} \varepsilon \sum_{Q \subseteq F_R, Q \neq \emptyset} \int_{E_R \cap Q} \frac{\ell(Q) \rho}{\eta(Q)} \frac{d\eta}{d\rho} \leq \frac{2}{c} \int_{E_R} \int_{0}^{\ell(R)/\rho} \eta(x, r)^2 \frac{d\eta}{d\rho} \leq \frac{2}{c} \varepsilon |R|.
\]

Thus,

\[
\sum_{Q \subseteq \mathcal{C}_R} \eta(Q)|Q| \leq \sum_{Q \subseteq F_R} |Q| + \sum_{Q \in \mathcal{C}_R} \eta(Q)|Q| \lesssim \varepsilon |R|.
\]

Let \( N \) be a large integer and

\[
G_R = \left\{ x \in F_R : \sum_{x \in Q \subseteq R} \eta(Q) \leq N \right\}.
\]

Then

\[
|F_R \setminus G_R| \leq \frac{1}{N} \int_{F_R \setminus G_R} \sum_{x \in Q \subseteq R} \eta(Q)
\]

\[
= \frac{1}{N} \sum_{Q \subseteq F_R, Q \neq \emptyset} \eta(Q) |F_R \cap Q|
\]

\[
\leq \frac{1}{N} \sum_{Q \subseteq F_R, Q \neq \emptyset} \eta(Q) |Q| \lesssim \frac{|R|}{N}.
\]

So for \( N \) large enough,

\[
|G_R| \geq \frac{|F_R|}{2} \geq \frac{c}{4} |R|.
\]

This proves the lemma. \( \square \)

3.3. Finding \( d \)-approximate line segments though each point.

**Lemma 3.8.** For \( Q \in \mathcal{D} \), let

\[
\eta_X(x, r) = \inf_{L_1, \ldots, L_d} \sup_{y \in (L_1 \cup \cdots \cup L_d) \cap B(x, r)} \frac{\text{dist}(y, X)}{r}
\]
where the infimum is over all lines \( L_1, \ldots, L_d \) passing through \( x \) so that 
\( \angle(L_i, L_j) \geq \theta/2 \) and set
\[
\eta(Q) = \sup_{x \in Q} \eta_X(x, \ell(Q)).
\]

Then for \( \delta > 0 \),
\[
\sum_{Q \subseteq R} |Q| \lesssim |R| \quad \text{for all } R \in \mathcal{D}.
\]

**Proof.** Let \( V_1, \ldots, V_N \) be a maximally \( \theta/2 \)-separated set in \( G(d-1, n) \) with respect to the metric
\[
d(V, U) := \angle(V, U).
\]

Let
\[
\mathcal{B}_\delta = \{ Q \in \mathcal{D} : \eta_{V_1}(Q) \geq \delta \}, \quad \mathcal{B} = \bigcup_i \mathcal{B}_\delta,
\]
\( \mathcal{G} = \mathcal{D} \setminus \mathcal{B} \).

Let \( x \in Q \in \mathcal{G} \). Without loss of generality, we will assume \( x = 0 \). Then \( \eta_{V_1}(Q) < \delta \), and so there is a line \( L_1(x) \) passing through \( x \) so that
\[
\sup_{y \in L_1(x) \cap B(x, \ell(Q))} \frac{\text{dist}(y, X)}{\ell(Q)} < \delta.
\]

Pick a \((d-1)\)-plane \( V_1 \) so that \( \angle(L_1(x), V_1) < \theta/2 \). Then \( \eta_{V_1}(Q) < \delta \), and so there is a line \( L_2(x) \) passing through \( x \) so that \( \angle(L_2(x), V_1) \geq \theta \), hence
\[
\angle(L_2(x), L_1(x)) \overset{(2.1)}{=} \angle(L_2(x), V_1) - \angle(L_1(x), V_1) \geq \frac{\theta}{2} \ell(Q).
\]

and
\[
\sup_{y \in L_2(x) \cap B(x, \ell(Q))} \frac{\text{dist}(y, X)}{\ell(Q)} < \delta.
\]

Inductively, for \( 2 \leq k \leq d-1 \), if \( U_k \) is the plane spanned by the lines \( L_1(x), \ldots, L_k(x) \), then we can find \( V_k \) a \((d-1)\)-plane so that \( \angle(U_k, V_k) < \theta/2 \). Then \( \eta_{V_{k+1}}(Q) < \delta \), and so there is \( L_{k+1}(x) \) passing through \( x \) so that \( \angle(L_{k+1}(x), V_k) \geq \theta \), and so
\[
\angle(L_{k+1}(x), U_k) \geq \angle(L_{k+1}(x), V_k) - \angle(U_k, V_k) \geq \frac{\theta}{2} \ell(Q)
\]

and
\[
\sup_{y \in L_{k+1}(x) \cap B(x, \ell(Q))} \frac{\text{dist}(y, X)}{\ell(Q)} < \delta.
\]
The final lines we obtain $L_1(x), \ldots, L_d(x)$ have angle at least $\theta/2$ between each other. Since we can find such lines for each $x \in Q$, this implies $\eta(Q) < \delta$ for all $Q \in \mathcal{G}$, and

$$\sum_{Q \in R \atop \eta(Q) \geq \delta} |Q| \leq \sum_{Q \in \mathcal{G}} \sum_{Q \leq R} |Q| \lesssim |R|.$$

\[ \square \]

3.4. The weak geometric lemma. We now use very similar arguments to get control on how flat $X$ is.

Lemma 3.9. For $R \in \mathcal{D}$ and $\delta > 0$,

$$\sum_{Q \in R \atop \beta(Q) \geq \delta} \eta(Q) \lesssim |R|.$$

This is the so-called weak geometric lemma (WGL) in the argot of David and Semmes [DS91, Chapter 5]. The name is a bit misleading since it is a property and not a theorem or implication: it really means that the above inequality holds, but this alone does not imply UR. It is still an important step toward our objective.

The WGL will follow from a few lemmas.

Lemma 3.10. For $V \in \mathcal{G}(d, n)$. For $\beta > 0$ small enough, there is $\theta' > 0$ depending on $\beta$ so that if

$$\xi_{V, \theta', \beta}(x, r) = \begin{cases} 0 & \beta_X(x, r, V) < \beta \\ \eta_{X, \theta'}(x, r) & \beta_X(x, r, V) \geq \beta \end{cases}.$$ 

Let $Q \in \mathcal{D}$. For $\beta > 0$ small enough, there is $E_{Q, V}' \subseteq Q$ with $|E_{Q, V}'| \geq |Q|$ and

$$\int_0^{\ell(Q)} \xi_V(x, r)^2 \frac{dr}{r} \lesssim 1 \text{ for all } x \in E_{Q, V}'.$$

Proof. Let $\xi = \xi_{V, \theta', \beta}$. For $Q \in \mathcal{D}$, let

$$\beta(Q, V) = \sup_{y \in 2Q \cap X} \frac{\text{dist}(y, V)}{\ell(Q)} \quad \text{and} \quad \beta(Q) = \inf_{V \in \mathcal{G}(d, n)} \beta(Q, V).$$

Let $Q \in \mathcal{D}$, let $\mathcal{Q}$ be the maximal cube in $Q$ containing the same center as $Q$ so that $\ell(\mathcal{Q}) < \frac{\ell(Q)}{4}$, and let $Q_j$ be those maximal cubes in $\mathcal{Q}$ for which $\beta(Q_j, V) \geq \beta^2$ and set $S = \bigcup Q_j$.

Case 1: If $|S| < \frac{1}{2} |\mathcal{Q}|$, then set $E_{Q, V}' = Q \setminus S$, so then

$$|E_{Q, V}'| \geq \frac{1}{2} |\mathcal{Q}| \gtrsim |Q|.$$
For \( x \in E'_{Q,V} \) and \( 0 < r < \ell(Q) \), we can find \( Q' \subseteq Q \) containing \( x \) so that \( \rho \ell(Q') < r \leq \ell(Q') \), and since \( x \in Q' \) and \( x \in E'_{Q,V} \),

\[
\beta_X(x, r, V) \lesssim \beta_X(x, 2\ell(Q'), V) \leq \beta(Q', V) < \beta^2
\]

so for \( \beta > 0 \) small enough, \( \beta_X(x, r, V) < \beta \) and so \( \xi(x, r) = 0 \). In particular,

\[
\int_0^{\ell(Q)} \xi(x, r)^2 \frac{dr}{r} = 0 \quad \text{for all } x \in E'_{Q,V}.
\]

**Case 2:** Alternatively, suppose \( |S| \geq \frac{|Q|}{2} \). Let \( j_k \) be such that \( \{4CB_{Q_{jk}}\} \) is a \( 5r \)-subcovering of \( \{4CB_{Q_j}\} \). Since \( \beta(2B_{Q_j}, V) \geq \varepsilon \), Lemma 3.4 implies there is a set

\[
E^V_{2B_{Q_j}} \subseteq 4CB_{Q_j} \cap X \subseteq Q
\]

upon which

\[
\int_0^{2\ell(Q_j)} \eta_X^{V,\theta'}(x, r)^2 \frac{dr}{r} \lesssim_{\beta, X} 1 \quad \text{for all } x \in E^V_{2B_{Q_j}}
\]

for some \( \theta' \) small enough depending on \( \beta \). Since the balls \( \{4CB_{Q_{jk}}\} \) are disjoint, if we set

\[
E'_{Q,V} = \bigcup E^V_{2B_{Q_j}};
\]

Then \( E'_{Q,V} \subseteq Q \) and

\[
|E'_{Q,V}| = \sum |E_{2B_{Q_{jk}}}| \gtrsim \sum |Q_{jk}| \sim \sum |10B_{Q_{jk}} \cap X| \geq \left| \bigcup 10B_{Q_{jk}} \cap X \right| \geq \frac{1}{2}|Q| \gtrsim |Q|.
\]

Now let \( x \in E'_{Q,V} \), then there is \( Q_j \) containing \( x \). For \( r > \ell(Q_j) \), there is \( Q' \subseteq Q \) properly containing \( Q_j \) so that \( \rho \ell(Q') < r \leq \ell(Q') \). Thus, since \( Q_j \) was maximal and \( B(x, r) \subseteq 2B_{Q'} \),

\[
\beta(x, r, V) \lesssim \beta(Q', V) < \beta^2
\]

so for \( \varepsilon \) small enough \( \beta(x, r, V) < \beta \), and so

\[
\xi(x, r) = 0 \quad \text{for all } \ell(Q_j) < r \leq \ell(Q).
\]

Thus,

\[
\int_0^{\ell(Q)} \xi(x, r)^2 \frac{dr}{r} = \int_0^{\ell(Q_j)} \xi(x, r)^2 \frac{dr}{r} \leq \int_0^{\ell(Q_j)} \eta_X^{V,\theta'}(x, r)^2 \frac{dr}{r} \lesssim_{\beta, X} 1.
\]

This proves the lemma. \( \square \)
Lemma 3.11. Choose $\beta$ and $\theta'$ so that the conclusions of the previous lemma hold. For $Q \in \mathcal{D}$, let

$$\xi_V(Q) = \sup_{x \in Q} \xi_V^{\theta', \beta/\rho}(x, 2\ell(Q)).$$

Then for $\delta > 0$,

$$\sum_{Q \in R, \xi_V(Q) \geq \delta} |Q| \lesssim |R|. \leqno{(3.17)}$$

Proof. This is shown in much the same way as Lemma 3.3. Let $\delta > 0$. Let $E_R = E_{R,V}'$, so again $|E_R| \geq c|R|$ for some constant $c$ depending on $\beta$. Let $Q_j$ be the maximal cubes in $R$ for which $|Q_j \cap E_R| < c^2 \ell(Q_j)$ and set $F_R = E_R \setminus \bigcup Q_j$, so again we have $|F_R| \geq c^2 |R|$ and

$$|E_R \cap Q| \geq \frac{c}{2} |Q| \quad \text{whenever} \quad Q \cap F_R \neq \emptyset \quad \text{and} \quad Q \subseteq R. \leqno{(3.18)}$$

Define $F_R$ just as before, let $\mathcal{P}_R$ be the cubes from Lemma 3.7 with $\beta/2$ in place of $\delta$. Define $\mathcal{C}_R$ just as before. Let $\varepsilon > 0$ and $Q \in \mathcal{C}_R$ be so that

$$\sum_{Q \in R, \xi_V(Q) \geq \delta} |Q| \lesssim |R|. \leqno{(3.19)}$$

Note that

$$\xi_V^{\theta', \beta s/r}(x, r) \leq \frac{s}{r} \xi_V^{\theta', \beta}(x, s) \quad \text{for} \quad r \leq s. \leqno{(3.20)}$$

Indeed, if $\beta_X(x, s, V) < \beta$, then (2.2) implies $\beta_X(x, r, V) < \beta s/r$, and so

$$\xi_V^{\theta', \beta s/r}(x, r) = 0 \leq \frac{s}{r} \xi_V^{\theta', \beta}(x, s). \leqno{(3.21)}$$

Otherwise, if $\beta_X(x, s, V) \geq \beta$, then

$$\frac{s}{r} \xi_V^{\theta', \beta}(x, s) = \frac{s}{r} \eta_X^{\theta', \beta}(x, s) \overset{(3.1)}{=} \eta_X^{\theta', \beta}(x, r) \geq \xi_V^{\theta', \beta s/r}(x, r). \leqno{(3.22)}$$

This proves (3.19). Thus, (3.18) implies

$$\sum_{Q \in R, \xi_V(Q) \geq \delta} |Q| \lesssim |R|. \leqno{(3.23)}$$

We claim that, for $\varepsilon$ small enough this implies

$$\xi(Q) < \delta. \leqno{(3.24)}$$

Again set

$$S_Q = \{ x \in E_R \cap Q : \xi_V^{\theta', \beta/\rho}(x, 2\ell(Q)) < \varepsilon^{1/4} \}. \leqno{(3.25)}$$

Hence, for $\varepsilon$ small enough, we again have by Chebychev

$$|Q \cap E_R \setminus S_Q| < \varepsilon^{3/2} |Q|. \leqno{(3.26)}$$
and with the same proof as before, for $\varepsilon > 0$ small enough (depending on $\delta$ and $\beta$),

$$\text{dist}(x, S_Q) < \min \left\{ \frac{\delta}{2}, \frac{\beta}{\rho} \right\} \ell(Q) \quad \text{for all } x \in Q, \ Q \in \mathcal{C}_R.$$  

Hence, for $x \in Q \in \mathcal{C}_R$, there is $x' \in S_Q$ with $|x - x'| < \min \left\{ \frac{\delta}{2}, \frac{\beta}{\rho} \right\} \ell(Q)$.

**Case 1:** If $\beta(x', 2\ell(Q), V) < \frac{\beta}{2\rho}$, then

$$\beta_X(x, 2\ell(Q), V) \leq \frac{|x - x'|}{2\ell(Q)} + \beta_X(x', 2\ell(Q), V) < \frac{\beta}{2\rho} + \frac{\beta}{2\rho} < \frac{\beta}{\rho}.$$  

and so $\xi^{\theta, \beta/\rho}_V(x, 2\ell(Q), V) = 0 < \delta$.

**Case 2:** Alternatively, if $\beta_X(x', 2\ell(Q), V) \geq \beta/\rho$, then for $\varepsilon > 0$ small enough (depending on $\delta$), because $x' \in S_Q$,

$$\xi^{\theta, \beta/\rho}_V(x, 2\ell(Q), V) \leq \frac{|x - x'|}{2\ell(Q)} + \eta^{\theta'}_X(x', 2\ell(Q), V)$$

$$\leq \frac{\delta}{2} + \xi^{\theta, \beta/\rho}_V(x', 2\ell(Q)) < \frac{\delta}{2} + \varepsilon + \delta < \delta.$$  

This proves (3.20). In particular,

(3.21) if $\xi_V(Q) \geq \delta$, then $\int_{E_R \cap Q} \int_{\ell(Q)}^{2\ell(Q)/\rho} \xi^{\theta, \beta}_V(x, r)^2 \frac{dr}{r} \geq \varepsilon$.

Let

$$\xi_\delta(Q) = \begin{cases} 1 & \xi_V(Q) \geq \delta \\ 0 & \xi_V(Q) < \delta \end{cases}$$

Then

$$\sum_{Q \in \mathcal{C}_R} \xi_\delta(Q)|Q| \leq \sum_{Q \in \mathcal{C}_R} |Q| \leq \frac{2}{c} \sum_{Q \in \mathcal{C}_R} |E_R \cap Q|$$

$$\leq \frac{2}{c\varepsilon} \sum_{Q \in \mathcal{C}_R} \int_{E_R \cap Q} \int_{\ell(Q)}^{2\ell(Q)/\rho} \xi^{\theta, \beta}_V(x, r)^2 \frac{dr}{r}$$

$$\leq \frac{2}{c\varepsilon} \int_{E_R} \int_0^{\ell(R)/\rho} \xi^{\theta, \beta}_V(x, r) \frac{dr}{r} \lesssim |R|.$$
Just as before, for each $R \in \mathcal{D}$ we can now find $G_R \subseteq R$ so that $|G_R| \gtrsim |R|$ and
\[
\sum_{x \in Q \subseteq R} \xi_\delta(Q) \lesssim_\delta 1 \quad \text{for all } x \in G_R.
\]
This completes the proof. \hfill \Box

Lemma 3.12. For $Q \in \mathcal{D}$, let
\[
\xi(Q) = \inf_V \xi_V(Q).
\]
For all $\beta > 0$ small enough so that Lemma 3.10 holds, there is $\varepsilon > 0$ so that if $\eta(Q) < \varepsilon$ and $\xi(Q) < \varepsilon$, then $\beta(B_Q) \lesssim \beta$.

Proof. Suppose $\eta(Q) < \varepsilon$ and $\xi(Q) < \varepsilon$ but $\beta(B_Q) \geq A\beta$ for some large constant $A > 0$.

Let $\theta'' > 0$, which will be determined shortly and will depend on $\theta$ and $\theta'$ (and so ultimately on $X$ and $\beta$, but not on $\varepsilon$). By Lemma 2.4, there is a ball $B' \subseteq c_0 B_Q$ centered on $X$ so that $\beta(B') < \theta''$ and $r_{B'} \geq c_{\theta''} \ell(Q)$ for some $c_{\theta''} > 0$. Let $x$ be the center of this ball, so by Theorem 2.5 $x \in Q$.

By Lemma 3.8, since $\eta(Q) < \varepsilon$ and $x \in Q$ there are lines $L_1(x), \ldots, L_d(x)$ passing through $x$ so that
\[
\sup_{y \in B(x, 2\ell(Q)) \cap (L_1(x) \cup \cdots \cup L_d(x))} \frac{\text{dist}(y, X)}{\ell(Q)} < \varepsilon.
\]
Since $\beta(B_Q) \geq A\beta$,
\[
\beta(x, 2\ell(Q)) \underset{(2.2)}{\gtrsim} \beta(B_Q) \geq A\beta.
\]
Thus, for $A$ large enough, this implies that for all $V \in \mathcal{G}(d, n)$,
\[
\beta_X(x, 2\ell(Q), V) \geq \beta_X(x, 2\ell(Q)) \geq \beta / \rho
\]
and so
\[
\xi_{\theta', \beta / \rho}^V(x, 2\ell(Q)) = \eta_X^{V, \theta'}(x, 2\ell(Q)) < \varepsilon
\]
where the last inequality follows from the assumption that $\eta(Q) < \varepsilon$. Let $V$ be the $d$-plane containing the lines $L_i(x)$. Since $\xi_V(Q) < \varepsilon$, there is a line $L_{d+1}(x)$ passing through $x$ so that
\[
\angle(L_{d+1}(x), V) \geq \theta'
\]
and
\[
\sup_{y \in B(x, 2\ell(Q)) \cap L_{d+1}(x)} \frac{\text{dist}(y, X)}{\ell(Q)} < \varepsilon.
\]
But for $\varepsilon$ small enough (depending on $\theta$ and $\theta'$, and $c_{\theta''}$), this implies that $\beta(B') \gtrsim \min\{\theta, \theta'\}$. Since $\beta(B') < \theta''$, this is impossible for $\theta' \ll \min\{\theta, \theta'\}$, which gives a contradiction. \hfill \Box
Proof of Lemma 3.9. By Lemmas 3.8, Lemma 3.11, and 3.12, for $\beta > 0$ small enough,

$$\sum_{Q \subseteq R, \beta(B_Q) \geq A\beta} |Q| \leq \sum_{Q \subseteq R, \beta(B_Q) \geq 4\beta, \eta(Q) \geq \varepsilon} |Q| + \sum_{Q \subseteq R, \beta(B_Q) \geq A\beta, \eta(Q) < \varepsilon} |Q| \lesssim |R| + \sum_{Q \subseteq R, \xi(Q) \geq \varepsilon} |Q| \lesssim |R|$$

\[\square\]

3.5. The end of the proof. In this section we finally establish the BWGL and finish the proof of the Main Theorem.

Lemma 3.13. For all $\delta > 0$ there are $M > 1$ and $\varepsilon > 0$ so that if $B$ is a ball centered on $X$ with $r_B < M^{-1} \text{diam} \ X$, $\beta(MB) < \varepsilon$, and $\sup_{x \in B \cap X} \eta(x, Mr_B) < \varepsilon$, then $b\beta(B) < \delta$.

Proof. Without loss of generality, we can assume $B = \mathbb{B}$. Suppose instead that for all $j$ we could find $d$-Loewner spaces $X_j \subseteq \mathbb{R}^n$ (with the same constants) containing 0 so that $\beta(j\mathbb{B}) < \frac{1}{j^2}$ and $\sup_{x \in \mathbb{B} \cap X_j} \eta(x, j) < \frac{1}{j^2}$, but $b\beta_{X_j}(\mathbb{B}) \geq \delta$ for some $\delta > 0$. We can pass to a subsequence so that this converges in the Hausdorff metric to an Ahlfors $d$-regular set $X$ containing 0 and with the property that for all $x \in X$ there are $d$ lines $L_1(x), \ldots, L_d(x) \subseteq X$ containing $x$ with angles at least $\theta > 0$ apart, and so that $\beta_X(r\mathbb{B}) = 0$ for all $r > 0$. In particular, $X$ is contained in a $d$-dimensional plane, which we can assume without loss of generality to be $\mathbb{R}^d$. Moreover, $b\beta_X(\mathbb{B}) \geq \delta$.

Since $X \subseteq \mathbb{R}^d$, this implies there is $z \in \mathbb{B} \cap \mathbb{R}^d$ with $\text{dist}(z, X) \geq \delta$. Let $x \in \partial B(z, \delta) \cap X$. If $V$ is the $(d - 1)$-dimensional plane in $\mathbb{R}^d$ tangent to $B(z, \delta) \cap \mathbb{R}^d$ at $x$, then there is at least one $i$ so that $L_i(x)$ is not parallel with $V$, so in particular, $L_i(x) \cap (\mathbb{R}^d \setminus B(z, r))^o \neq \emptyset$ (where we are taking the interior with respect to $\mathbb{R}^d$), but then $L_i(x) \cap \mathbb{R}^d \setminus X \neq \emptyset$, whereas $L_i(x) \subseteq X$, so we get a contradiction. \[\square\]

Corollary 3.14. For $\delta > 0$ there is $N \in \mathbb{N}$ and $\varepsilon > 0$ so that if $Q \in \mathcal{D}$, $Q^N$ is the $N$-th generation ancestor of $Q$, $\beta_X(B_{Q^N}) < \varepsilon$, and $\eta_X(Q^N) < \varepsilon$, then $b\beta(B_Q) < \delta$.

This follows from the previous lemma, and we leave the details to the reader.
We now finish the proof of the main theorem. Observe that the map sending $Q \to Q^N$ is at most $C(N)$-to-1, and so
\[
\sum_{Q \subseteq R, \beta(\partial Q) \geq \varepsilon} |Q| \leq \sum_{Q \subseteq R, \beta(\partial Q) \geq \varepsilon} |Q| + \sum_{Q \subseteq R, \beta(\partial Q) \geq \varepsilon} |Q| \lesssim \sum_{Q \subseteq R, \beta(\partial Q) \geq \varepsilon} |Q| + \sum_{Q \subseteq R, \beta(\partial Q) \geq \varepsilon} |Q| \lesssim |R^N| \lesssim |R|.
\]

Now $X$ is UR by the bilateral weak geometric lemma. This completes the proof of the Main Theorem.

REFERENCES

[Azz16] J. Azzam. Bi-Lipschitz parts of quasisymmetric mappings. Rev. Mat. Iberoam., 32(2):589–648, 2016. 17

[Azz16b] J. Azzam. Sets of absolute continuity for harmonic measure in NTA domains. Potential Anal., 45(3):403–433, 2016. 14

[AM16] J. Azzam and M. Mourgoglou. A characterization of 1-rectifiable doubling measures with connected supports. Anal. PDE, 9(1):99–109, 2016. 17

[Bat15] D. Bate. Structure of measures in Lipschitz differentiability spaces. J. Amer. Math. Soc., 28(2):421–482, 2015. 2, 4

[BKO19] D. Bate, I. Kangasniemi, and T. Orponen. Cheeger’s differentiation theorem via the multilinear kakeya inequality. arXiv preprint arXiv:1904.00808, 2019. 2

[BL17] D. Bate and S. Li. Characterizations of rectifiable metric measure spaces. Ann. Sci. Éc. Norm. Supér. (4), 50(1):1–37, 2017. 2

[Che99] J. Cheeger. Differentiability of lipschitz functions on metric measure spaces. Geometric and functional analysis, 9(3):428–517, 1999. 2, 7, 8, 9

[Chr90] M. Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math., 60/61(2):601–628, 1990. 10

[Dav88] G. David. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. Rev. Mat. Iberoamericana, 4(1):73–114, 1988. 10

[Dav91] G. David. Wavelets and singular integrals on curves and surfaces, volume 1465 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1991. 12

[DS91] G. David and S. W. Semmes. Singular integrals and rectifiable sets in $\mathbb{R}^n$: Beyond Lipschitz graphs. Astérisque, (193):152, 1991. 3, 21

[DS93] G. David and S. W. Semmes. Analysis of and on uniformly rectifiable sets, volume 38 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1993. 10, 16

[Dav16] G. C. David. Bi-Lipschitz pieces between manifolds. Rev. Mat. Iberoam., 32(1):175–218, 2016. 3

[DPMR17] G. De Philippis, A. Marchese, and F. Rindler. On a conjecture of Cheeger. In Measure theory in non-smooth spaces, Partial Differ. Equ. Meas. Theory, pages 145–155. De Gruyter Open, Warsaw, 2017. 2

[Dor85] J. R. Dorronsoro. A characterization of potential spaces. Proc. Amer. Math. Soc., 95(1):21–31, 1985. 13

[Hei01] J. Heinonen. Lectures on analysis on metric spaces. Universitext. Springer-Verlag, New York, 2001. 5, 7

[HK98] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. Acta Math., 181(1):1–61, 1998. 2, 3, 7
[HM12] Hytönen, T.; Martikainen, H. Non-homogeneous $T^b$ theorem and random dyadic cubes on metric measure spaces. *J. Geom. Anal.* 22 (2012), no. 4, 1071–1107.

[Jon88] P. W. Jones. Lipschitz and bi-Lipschitz functions. *Rev. Mat. Iberoamericana*, 4(1):115–121, 1988.

[Kei03] S. Keith. Modulus and the Poincaré inequality on metric measure spaces. *Math. Z.*, 245(2):255–292, 2003.

[Kei04] S. Keith. A differentiable structure for metric measure spaces. *Adv. Math.*, 183(2):271–315, 2004.

[KZ08] S. Keith and X. Zhong. The poincaré inequality is an open ended condition. *Ann. of Math.*(2), 167(2):575–599, 2008.

[KM16] B. Kleiner and J. M. Mackay. Differentiable structures on metric measure spaces: a primer. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), 16(1):41–64, 2016.

[Laa00] T. J. Laakso. Ahlfors $Q$-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, 10(1):111–123, 2000.

[Mat95] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[Mer16] J. Merhej. Poincaré-type inequalities and finding good parameterizations. *Mathematische Zeitschrift*, pages 1–33, 2016.

[Mer17] J. Merhej. On the geometry of rectifiable sets with Carleson and Poincaré-type conditions. *Indiana Univ. Math. J.*, 66(5):1659–1706, 2017.

[JKV97] A.M. Vargas Rey, N.H. Katz, and P. W. Jones. Checkerboards, lipschitz functions and uniform rectifiability. *Revista matemática iberoamericana*, 13(1):189–210, 1997.

[Sem96] S. W. Semmes. Finding curves on general spaces through quantitative topology, with applications to Sobolev and Poincaré inequalities. *Selecta Math. (N.S.*), 2(2):155–295, 1996.

[Vil17] M. Villa. Tangent points of $d$-lower content regular sets and $\beta$-numbers. *arXiv preprint arXiv:1712.02823, to appear in J. London. Math. Soc.*, 2017.

[Vuo88] M. Vuorinen. *Conformal geometry and quasiregular mappings*, volume 1319 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.