On rich points and incidences with restricted sets of lines in 3-space

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Abstract

Let $L$ be a set of $n$ lines in $\mathbb{R}^3$ that is contained, when represented as points in the four-dimensional Plücker space of lines in $\mathbb{R}^3$, in an irreducible variety $T$ of constant degree which is non-degenerate with respect to $L$ (see below). We show:

(1) If $T$ is two-dimensional, the number of $r$-rich points (points incident to at least $r$ lines of $L$) is $O\left(\frac{n^{4/3} + \varepsilon}{r^2}\right)$, for $r \geq 3$ and for any $\varepsilon > 0$, and, if at most $n^{1/3}$ lines of $L$ lie on any common regulus, there are at most $O\left(\frac{n^{4/3} + \varepsilon}{r}\right)$ 2-rich points. For $r$ larger than some sufficiently large constant, the number of $r$-rich points is also $O\left(\frac{n}{r}\right)$.

As an application, we deduce (with an $\varepsilon$-loss in the exponent) the bound obtained by Pach and de Zeeuw [17] on the number of distinct distances determined by $n$ points on an irreducible algebraic curve of constant degree in the plane that is not a line nor a circle.

(2) If $T$ is two-dimensional, the number of incidences between $L$ and a set of $m$ points in $\mathbb{R}^3$ is $O(m + n)$.

(3) If $T$ is three-dimensional and nonlinear, the number of incidences between $L$ and a set of $m$ points in $\mathbb{R}^3$ is $O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right)$, provided that no plane contains more than $s$ of the points. When $s = O(\min\{n^{3/5}/m^{2/5}, m^{1/2}\})$, the bound becomes $O(m^{3/5}n^{3/5} + m + n)$.

As an application, we prove that the number of incidences between $m$ points and $n$ lines in $\mathbb{R}^4$ contained in a quadratic hypersurface (which does not contain a hyperplane) is $O(m^{3/5}n^{3/5} + m + n)$.

The proofs use, in addition to various tools from algebraic geometry, recent bounds on the number of incidences between points and algebraic curves in the plane.

1 Introduction

The setup: Incidences between a set of points and a restricted set of lines in $\mathbb{R}^3$. Let $P$ be a set of $m$ points and $L$ a set of $n$ lines in $\mathbb{R}^3$. We consider the problem

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of obtaining sharp incidence bounds between the points of $P$ and the lines of $L$, when the lines of $L$, considered as points in the four-dimensional Plücker space of lines in $\mathbb{R}^3$, are restricted to lie on a two- or three-dimensional constant-degree algebraic variety $T$. The topic of incidences between points and lines is a fundamental topic in incidence geometry, significantly boosted since Guth and Katz’s seminal work [11] on point-line incidences in $\mathbb{R}^3$. Instead of asking for a bound on the number of incidences between points and lines, we can ask, for each $r \geq 3$, for a bound on the number of $r$-rich points in a set of lines, which are the points that are incident to at least $r$ of the lines. As it turns out, the two questions are equivalent. The related, and finer problem of bounding the number of 2-rich points, determined by a set of $n$ lines in $\mathbb{R}^3$, studied in [11], is also discussed in this paper, under the restricted setup considered here. Building on recent works of Sharir and Zahl [30] and Zahl [34], we are able to improve Guth and Katz’s point-line incidence bounds when the lines in $L$ are restricted to lie on a two- or three-dimensional variety $T$ in the Plücker space.

Background: Points and curves, the planar case. The study of incidences between points and curves has a rich history, starting with the simplest instance of points and lines in the plane, where we have (see also [4, 32]):

**Theorem 1.1 (Szemerédi and Trotter [33])** The maximum number of incidences between $m$ points and $n$ lines in the plane is $\Theta\left(m^{2/3}n^{2/3} + m + n\right)$.

In fact, an equivalent formulation of Szemerédi-Trotter theorem asserts that, given $n$ lines in the plane, the number of points that are incident to at least $r$ of the lines, for any parameter $2 \leq r \leq n$, which we call $r$-rich points and denote the set of these points by $P_{\geq r}(L)$, satisfies

$$|P_{\geq r}(L)| = O\left(\frac{n^2}{r^3} + \frac{n}{r}\right).$$

(1)

Still in the plane, Pach and Sharir [18] extended this bound to incidence bounds between points and curves with $k$ degrees of freedom, namely, for each set of $k$ distinct points, there are only $\mu = O(1)$ curves that pass through all of them, and each pair of curves intersect in at most $\mu$ points; $\mu$ is called the multiplicity (of the degrees of freedom). Here is their result, tailored to the case of algebraic curves.

**Theorem 1.2 (Pach and Sharir [18])** Let $P$ be a set of $m$ points in $\mathbb{R}^2$ and let $C$ be a set of $n$ bounded-degree algebraic curves in $\mathbb{R}^2$ with $k$ degrees of freedom and with multiplicity $\mu$. Then (where the constant of proportionality depends on $k$ and $\mu$)

$$I(P, C) = O\left(\frac{m^{k}{k-2}}{2{k-1}} n^{2k-2} + m + n\right).$$

Except for the case $k = 2$ (lines have two degrees of freedom), the upper bound is not known, and is strongly suspected to be too large (see [1, 2, 16] for an improvement for the case of circles and similar curves).

Recently, Sharir and Zahl [30] have considered general families of constant-degree algebraic curves in the plane that belong to an $s$-dimensional family of curves. This means that each curve in such a family can be represented by a constant number of real parameters, so that, in this parametric space, the points representing the curves lie in an $s$-dimensional
algebraic variety $F$ of some constant degree (the so-called “complexity” of $F$). See [30] for details.

**Theorem 1.3 (Sharir and Zahl [30])** Let $C$ be a set of $n$ algebraic plane curves that belong to an $s$-dimensional family $F$ of curves of maximum constant degree $E$, no two of which share a common irreducible component, and let $P$ be a set of $m$ points in the plane. Then, for any $\varepsilon > 0$, the number $I(P,C)$ of incidences between the points of $P$ and the curves of $C$ satisfies

$$I(P,C) = O\left(m^{2/3} n^{5/3 - 4\varepsilon} + m^{2/3} n^{2/3} + m + n\right),$$

where the constant of proportionality depends on $\varepsilon$, $s$, $E$, and the complexity of the family $F$.

Except for the factor $O(n^\varepsilon)$, this is a significant improvement over the bound in Theorem 1.2 (for $s \geq 3$), in cases where the assumptions in Theorem 1.3 imply (as they often do) that $C$ has $k = s$ degrees of freedom. Examples where $k = s$ are abundant in the plane. For example, lines have $k = s = 2$ (two points determine a line, and two real parameters specify a line. For circles we have $k = s = 3$, for unit circles we have $k = s = 2$, and for general conic sections we have $k = s = 5$.

**Incidences with lines in three dimensions.** The groundbreaking work of Guth and Katz [11] implies the sharper bound $O(m^{1/2} n^{3/4} + m^{2/3} n^{1/3} q^{1/3} + m + n)$ on the number of incidences between $m$ points and $n$ lines in $\mathbb{R}^3$, provided that no plane contains more than $q$ of the given lines. We use the following variant of this result.

**Theorem 1.4** Let $P$ be a set of $m$ points in $\mathbb{R}^3$, and let $L$ be a set of $n$ lines in $\mathbb{R}^3$, so that no 2-flat contains more than $s$ points of $P$. Then

$$I(P,L) = O(m^{1/2} n^{3/4} + m^{1/3} n^{2/3} s^{1/3} + m + n).$$

Note that this symmetric assumption (no more than $s$ points, instead of no more than $q$ lines, on a plane) causes the roles of $m$ and $n$ to switch in the second term, from their roles in the Guth–Katz bound.

**Sketch of proof.** The proof is very similar to that in [11]. Briefly, it is based on constructing a *partitioning polynomial* $f$ of some suitable degree. It then bounds the number of incidences within the cells of the partition (connected components of $\mathbb{R}^3 \setminus Z(f)$, where $Z(f)$ is the zero set of $f$), and the number of incidences on $Z(f)$. The latter task is performed separately for each irreducible component of $Z(f)$. We follow the proof in [11] verbatim, except for the analysis of incidences on the planar irreducible components of $Z(f)$. For that step, both arguments use the Szemerédi-Trotter bound for point-line incidences in the plane (Theorem 1.1), for each of these planes, except that the analysis in [11] exploits the assumption that no plane contains more than $q$ of the lines, whereas we exploit the symmetric assumption that no plane contains more than $s$ of the points. Reasoning as in [11], but swapping the roles of points and lines, we obtain the modified second term in the bound in

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1. This bound is not explicitly stated in [11], but it readily follows from the analysis given there, and by now it is generally attributed to that work.
the theorem. See Section 5 for a similar reasoning for the restricted cases considered here.

Plugging the bound of Theorem 1.4 into the proof of [26, Theorem 1.3(a)], we get

**Theorem 1.5** Let $P$ be a set of $m$ points in $\mathbb{R}^3$, and let $L$ be a set of $n$ lines in $\mathbb{R}^d$, for $d \geq 3$, so that all the points and lines lie in a common two-dimensional algebraic variety $V$ of degree $D$ that does not contain any 2-flat, and so that no 2-flat contains more than $s$ points of $P$. Then

$$I(P, L) = O(m^{1/2}n^{1/2}D^{1/2} + m^{1/3}D^{4/3}s^{1/3} + m + n).$$

Guth and Katz’s work has lead to many recent works on incidences between points and lines or other curves in three and higher dimensions; see [3, 12, 21, 22, 24, 26] for a sample.

Of particular significance is the recent work of Guth and Zahl [12] on the number of 2-rich points in a collection of algebraic curves of constant degree, namely, points incident to at least two of the given curves. For the case of lines, Guth and Katz [11] have shown that the number of such points is $O(n^{3/2})$, when no plane or regulus contains more than $O(n^{1/2})$ lines. Guth and Zahl obtain the same asymptotic bound for general algebraic curves, under analogous (but stricter) restrictive assumptions (concerning surfaces that are doubly ruled by the given family of curves).

The new bounds that we will derive require the extension to three dimensions of the notions of having $k$ degrees of freedom and of being an $s$-dimensional family of curves. The definitions of these concepts, as given above for the planar case, extend, more or less verbatim, to three (or higher) dimensions, but, even in typical situations, these two concepts do not coincide anymore. For example, lines in three dimensions have two degrees of freedom and of being an

**Our results.** We obtain improved incidence bounds when the lines of $L$, as points in Plücker space, lie on a two- or three-dimensional variety $T$. When $T$ is two-dimensional and non-planar, the number of $r$-rich points is $O(n^{4/3+\varepsilon}/r^2)$, for $r \geq 3$ and for any $\varepsilon > 0$, and, if at most $n^{1/3}$ lines of $L$ lie on any common regulus, there are at most $O(n^{1/3+\varepsilon})$ 2-rich points. For $r$ larger than some sufficiently large constant, the number of $r$-rich points is also $O(n/r)$, which is a better bound when $r = O(n^{1/3})$. These bounds improve significantly, for the restricted context at hand, the bound $O(n^{3/2}/r^2)$ due to Guth and Katz [11] (which holds when no plane or regulus contains more than $O(n^{1/2})$ lines). Moreover, the number of incidences between $L$ and a set of $m$ points in $\mathbb{R}^3$ is $O(m + n)$, again a significant improvement, in our context, over the previous bound in [11].

As an application, we show that the number of distinct distances determined by $n$ points on an irreducible algebraic curve of constant degree in the plane that is not a line nor a circle, is $\Omega(n^{4/3-\varepsilon})$, for any $\varepsilon > 0$, which is (with an $\varepsilon$-loss in the exponent) the bound obtained by Pach and de Zeeuw [17].

If $T$ is three-dimensional and nonlinear, the number of incidences between $L$ and a set of $m$ points in $\mathbb{R}^3$ is $O\left(m^{3/5}n^{3/5} + (m^{1/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right)$, provided that no plane contains more than $s$ of the points. When $s = O(\min\{n^{3/5}/m^{2/5}, m^{1/2}\})$, the bound becomes $O(m^{3/5}n^{3/5} + m + n)$. 

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An interesting novel feature of our result is that, like Theorem 1.4, it is obtained under an assumption that restricts the number of points that can lie on a common plane (instead of restricting the number of coplanar lines in the previous studies). Very few earlier works have used this kind of restriction; see Elekes et al. [5] for one of the few exceptions.

Similar bounds have recently been obtained by the authors for other special cases of the incidence problem [29, 30], using related but different approaches.

As an application, we prove that the number of incidences between \(m\) points and \(n\) lines in \(\mathbb{R}^4\) contained in a quadratic hypersurface (which does not contain a hyperplane) is \(O(m^{3/5}n^{3/5} + m + n)\).

All our bounds are significant improvements, under the restricted scenarios assumed in this work, over the standard incidence bounds in three dimensions, and shed, as we believe, new light on the structure of point-line incidences in three dimensions.

As is standard in the ‘modern’ study of incidence geometry, the analysis is based on the polynomial partitioning technique (see [10, 11] for details), combined with a variety of tools from algebraic geometry.

We remark that, wherever needed in the analysis, we switch to the (projective 3-space over) the complex field, which simplifies it and lets us use numerous tools from algebraic geometry, available in this setting. The passage from the complex projective setup back to the real affine one is easy—the former is a generalization of the latter. The real affine setup is needed only for constructing a polynomial partitioning, which is meaningless over \(\mathbb{C}\). Once we are, say, within the zero set \(Z(f)\) of the partitioning polynomial \(f\), we can switch to the complex projective setup, and reap the benefits just noted.

2 Rich points determined by a two-dimensional family of lines

As already said, we parameterize lines in three dimensions by their Plücker coordinates, as follows (see, e.g., Griffiths and Harris [9, Section 1.5]). For a pair of distinct points \(x, y \in \mathbb{P}^3\), given in projective coordinates as \(x = (x_0, x_1, x_2, x_3)\) and \(y = (y_0, y_1, y_2, y_3)\), let \(\ell_{x,y}\) denote the (unique) line in \(\mathbb{P}^3\) incident to both \(x\) and \(y\). The Plücker coordinates of \(\ell_{x,y}\) are given in projective coordinates in \(\mathbb{P}^5\) as \((\pi_{0,1}, \pi_{0,2}, \pi_{0,3}, \pi_{2,3}, \pi_{3,1}, \pi_{1,2})\), where \(\pi_{i,j} = x_i y_j - x_j y_i\). Under this parameterization, the set of lines in \(\mathbb{P}^3\) corresponds bijectively to the set of points in \(\mathbb{P}^5\) lying on the Klein quadric \(Q\) given by the quadratic equation

\[
\pi_{0,1}\pi_{2,3} + \pi_{0,2}\pi_{3,1} + \pi_{0,3}\pi_{1,2} = 0
\]

(which is indeed always satisfied by the Plücker coordinates of a line).

As we will be working both in the primal three-dimensional space and the dual five-dimensional projective Plücker space, or rather in the Klein quadric \(Q\) contained in that space, we will denote the Plücker point image of a line \(\ell\) by \(\hat{\ell}\) throughout the paper. For a set \(L\) of lines in \(\mathbb{R}^3\) or in \(\mathbb{C}^3\), we denote\(^2\) \(\hat{L} = \{\hat{\ell} \mid \ell \in L\}\).

\(^2\)As mentioned in the introduction, one can freely pass between a real algebraic variety, or other constructs, and its complexification, so, for convenience, we will mostly argue over the complex domain.
Given a surface $V$ in $\mathbb{P}^3$, the set of lines fully contained in $V$, represented by their Plücker coordinates in $\mathbb{P}^5$, is a subvariety of the Klein quadric $Q$, which is denoted by $F(V)$, and is called the Fano variety of $V$; see Harris [13, Lecture 6, page 63] for details, and [13, Example 6.19] for an illustration, and for a proof that $F(V)$ is indeed a variety.

Let $H$ denote a plane in $\mathbb{C}^3$, and let $H^* = \{\tilde{\ell} \mid \ell \subset H\}$ be the 2-flat in Plücker space consisting of the points that represent the lines fully contained in $H$ (see Rudnev [19] for why $H^*$ is indeed a 2-flat and for more details).

For a set $L$ of lines, we put
\[ H(L) = \{H_{\ell,\ell'}^* \mid \ell, \ell' \in L \text{ and are coplanar}\}, \]
where for coplanar lines $\ell, \ell'$, $H_{\ell,\ell'}^*$ is the (unique) 2-flat containing $\ell$ and $\ell'$.

In this paper, we study incidences between a set of points $P \subset \mathbb{R}^3$, and a set of lines $L$ in $\mathbb{R}^3$, whose Plücker images lie on some irreducible algebraic subvariety $T$ of the Klein quadric $Q$, which is of constant degree, and which has dimension either 2 or 3.

In this section we restrict ourselves to the case where $\dim(T) = 2$. For a set $L$ of lines, we say that the (two-dimensional) variety $T$ is non-degenerate with respect to $L$ if

(i) $T$ is irreducible of constant degree,

(ii) $T$ is not a 2-flat, and

(iii) the intersection of $T$ with each 2-flat $H^* \in H(L)$ consists of $O(1)$ points (representing lines).

Note that condition (iii) is what one would expect to hold in a generic situation in a four-dimensional space. The simpler case where $T$ is a 2-flat will not be considered in this work. The reason is that when many of the input lines lie in a plane $h$ in the primal 3-space, the dual of $h$ in the Plücker space is a 2-flat, which could be $T$ itself. In this case the best upper bound that one could get for the number of $r$-rich points would be the worst-case bound $O(n^2/r^3 + n/r)$ of Szemerédi and Trotter [33].

**Theorem 2.1** Let $L$ be a set of $n$ lines in $\mathbb{R}^3$, such that, in Plücker space, $L$ is a subset of some two-dimensional variety $T$ that is non-degenerate with respect to $L$.

(a) The number of $r$-rich points determined by $L$ is $|P_{\geq r}(L)| = O(n^{4/3+\varepsilon}/r^2)$, for any $\varepsilon > 0$ and $r \geq 3$.

(b) If, in addition, the number of lines of $L$ contained in any common regulus is at most $n^{1/3}$ then the number of 2-rich points determined by $L$ is $|P_{\geq 2}(L)| = O(n^{4/3+\varepsilon})$, for any $\varepsilon > 0$.

**Proof.** First here is a high-level overview of the proof. After a pruning step, we may assume that the set $\gamma_{\ell}$, for a line $\ell \in L$, of the Plücker images of the lines $\ell$ that are coplanar with $\ell$ and $\tilde{\ell} \in T$, is a one-dimensional curve in $T$. An $r$-rich point generates $\Omega(r^2)$ incidences between the Plücker points in $\tilde{L}$ and the curves $\gamma_{\ell}$, so it suffices to bound the number of such incidences. There are two kinds of curves, those that represent the lines in one ruling
of some regulus, and those that do not. For \( r \)-rich points, with \( r \geq 3 \), only the latter kind of curves matter, and a suitable application of Theorem 1.3 allows us to obtain an upper bound for the number of such incidences. For 2-rich points (part (b) of the theorem), the regulus-curves also play a part, and the analysis is complicated because these curves do not have to be distinct. Still, the assumptions in (b) allow us to handle this case and get the desired bound.

The following notation will be useful later on in the paper. For each line \( \ell \in \mathcal{Q} \), define the variety \( S_\ell \) (in Plücker space) to be

\[
S_\ell = \{ \tilde{\ell} \in \mathcal{Q} \mid \ell, \ell' \text{ are coplanar} \}.
\]

If the Plücker coordinates of \( \tilde{\ell} \) are \((\pi_{0,1}, \pi_{0,2}, \pi_{0,3}, \pi_{2,3}, \pi_{3,1}, \pi_{1,2})\), then

\[
S_\ell = \{ (\pi'_{0,1}, \pi'_{0,2}, \pi'_{0,3}, \pi'_{2,3}, \pi'_{3,1}, \pi'_{1,2}) \in \mathcal{Q} \mid \pi_{0,1}\pi'_{2,3} + \pi_{0,2}\pi'_{3,1} + \pi_{0,3}\pi'_{1,2} + \pi'_{0,1}\pi_{2,3} + \pi'_{0,2}\pi_{3,1} + \pi'_{0,3}\pi_{1,2} = 0 \}.
\]

In particular, Equation (2) implies that \( \tilde{\ell} \in S_\ell \). We see that, for every line \( \ell \), the variety \( S_\ell \) is the intersection of \( \mathcal{Q} \) with a hyperplane, so it is a three-dimensional quadratic surface contained in \( \mathcal{Q} \) and containing \( \tilde{\ell} \). We say that a line \( \ell \) is exceptional with respect to \( T \) if \( T \subset S_\ell \). We say that a point \( p \in \mathbb{R}^3 \) is exceptional with respect to \( T \) if the set of the Plücker images of the lines incident to \( p \) in 3-space, which we denote by \( S_p \) and which is known to be a 2-plane in \( \mathcal{Q} \) (see the proof of Lemma 2.2 below), is equal to \( T \).

**Lemma 2.2** (i) If \( T \) is non-degenerate, there are no exceptional points with respect to \( T \). (ii) Even when \( T \) is degenerate, there can be at most one exceptional point.

**Proof.** We recall, e.g., from Rudnev [19], that a 2-flat contained in \( \mathcal{Q} \) parameterizes either the set of lines in \( \mathbb{C}^3 \) that are incident to some point, or the set of lines contained in a plane in \( \mathbb{C}^3 \). The set of lines that are incident to an exceptional point is thus a 2-flat, which, as \( T \) is an irreducible two-dimensional surface, must be equal to \( T \), contradicting the assumption that \( T \) is non-degenerate, so (i) follows. For the proof of (ii), we observe that if \( p, p' \) are two exceptional points, then \( S_p, S_{p'} \subset T \) and \( S_p \neq S_{p'} \), contradicting the assumption that \( T \) is two-dimensional and irreducible. \( \square \)

**Lemma 2.3** There are at most two exceptional lines with respect to \( T \).

**Proof.** Assume to the contrary that there are three exceptional lines. Assume first that two of these lines are coplanar, and denote them by \( \ell_1 \) and \( \ell_2 \). Then \( T \subset S_{\ell_1} \cap S_{\ell_2} \), i.e., \( T \) is contained in the set of the Plücker images of the lines intersecting both \( \ell_1 \) and \( \ell_2 \). If \( \ell_1 \) and \( \ell_2 \) do not intersect one another, then \( S_{\ell_1} \cap S_{\ell_2} = H^*_{\ell_1, \ell_2} \). Otherwise, letting \( p = \ell_1 \cap \ell_2 \), we have \( S_{\ell_1} \cap S_{\ell_2} = H^*_{\ell_1, \ell_2} \cup S_p \), i.e., it is a union of two 2-flats. In both cases, \( T \) is a 2-flat, contradicting our assumption.

We may thus assume there are (at least) three lines \( \ell_1, \ell_2 \) and \( \ell_3 \) that are pairwise skew, such that \( T \subset S_{\ell_1} \cap S_{\ell_2} \cap S_{\ell_3} \). As is well known (see, e.g., [7] Theorem 16.4 and [26] Lemma 2.2), the Plücker images of lines that intersect \( r \geq 3 \) pairwise-skew lines \( \ell_1, \ldots, \ell_r \) belong to one ruling of a regulus, and the Plücker images of \( \ell_1, \ldots, \ell_r \) belong to the other ruling
of this regulus. That is, $S_{\ell_1} \cap S_{\ell_2} \cap S_{\ell_3}$ is one ruling of the regulus generated by the lines intersecting $\ell_1, \ell_2$ and $\ell_3$, which is a quadratic curve in the Plücker space, contradicting the fact that $T$ is two-dimensional. It is a curve because the ruling defines a one-parameter family of lines, which yields a curve in Plücker space; the fact that this curve is quadratic, an easy consequence of the fact that the regulus spans a quadratic surface, is not important for the argument. □

We prune away, as we may, the exceptional point, if such a point exists (in the degenerate situation), and the (at most) two exceptional lines, thereby losing at most $2(n-1)+1 < 2n$ 2-rich points.

For each of the (remaining) lines $\ell \in L$, the intersection $S_{\ell} \cap T$ is a curve contained in $T$ (possibly also containing a discrete finite subset), which we denote by $\gamma_{\ell}$. Define

$$C = \{ \gamma_{\ell} \mid \ell \text{ is not exceptional} \}.$$  \hspace{1cm} (3)

We have the following simple observation, whose trivial proof is omitted.

**Lemma 2.4** Let $p$ be an $r$-rich point, with $r \geq 2$, and denote the lines of $L$ incident to $p$ as $\ell_1, \ldots, \ell_s$, for some $s \geq r$. Then, for each pair of indices $1 \leq i \neq j \leq s$, $\tilde{\ell}_i$ is incident to $\gamma_{\ell_j}$, and for every incidence $\ell' \in \gamma_{\ell}$, for a pair of lines $\ell, \ell'$, there is at most one point $p \in C^3$ that induces it, in the sense that both $\ell$ and $\ell'$ are incident to $p$.

The lemma asserts that each $r$-rich point contributes at least $r(r-1)$ incidences between the Plücker points of the lines of $L$ and the curves $\gamma_{\ell}$ of $C$ (as curves in $Q$). Hence, to bound the number of $r$-rich points, it suffices to bound the number of incidences between the Plücker images of the lines in $L$ and the curves of $C$ (and then divide the bound by $r(r-1)$). For any curve $\gamma_{\ell}$, its corresponding discrete subset of $O(1)$ points contributes only $O(1)$ incidences, for a total of $O(n)$ incidences. We may therefore ignore all these discrete subsets.

The notion of *dimensionality* for families of curves (see the definition preceding Theorem 1.3) easily extends in a natural way to collections $C$ of higher-dimensional algebraic varieties.

**Lemma 2.5** The family $C$ defined in (3) is two-dimensional.

**Proof.** Each curve $\gamma_{\ell}$ of $C$ can be parameterized by the parameters of the corresponding line $\ell$, and the Plücker images of these lines lie in the two-dimensional variety $T$, so it takes only two real parameters to specify such a line. □

When analyzing incidences between the Plücker points in $\tilde{L}$ and the curves $\gamma_{\ell_j}$, as in Lemma 2.4, some care has to be exercised, to handle situations in which many of the curves $\gamma_{\ell}$ share a common irreducible component (or even coincide).

(≥ 3)-rich points. Assume that $\ell_1, \ldots, \ell_\xi \in L$ are such that $\gamma_{\ell_1}, \ldots, \gamma_{\ell_\xi}$ all share a common curve, for some $\xi \geq 3$. If some pair of lines $\ell_i, \ell_j$ are coplanar, we write $H_{i,j}$ for the (unique) plane $H_{\ell_i, \ell_j}$ containing both $\ell_i$ and $\ell_j$. As in the proof of Lemma 2.3, (i) if $\ell_i$ and $\ell_j$ are parallel then $S_{\ell_i} \cap S_{\ell_j} = H_{i,j}$, and (ii) if $\ell_i$ and $\ell_j$ intersect in a point $p$
then \( S_{\ell_i} \cap S_{\ell_j} = H_{i,j}^* \cup S_p \), so \( S_{\ell_i} \cap S_{\ell_j} \) is either a 2-flat or the union of two 2-flats (in \( Q \)). Therefore,

\[
\gamma_{\ell_i} \cap \gamma_{\ell_j} = S_{\ell_i} \cap S_{\ell_j} \cap T = H_{i,j}^* \cap T \quad \text{or} \quad (H_{i,j}^* \cup S_p) \cap T,
\]

and the right hand sides of these equations are the intersection of one or two 2-flats with \( T \). Since \( T \) is assumed to be non-degenerate, it follows that \( \gamma_{\ell_i} \cap \gamma_{\ell_j} \) is a finite set of points, and thus \( \gamma_{\ell_i} \) and \( \gamma_{\ell_j} \) cannot intersect in a common curve. We can thus assume that \( \ell_1, \ldots, \ell_\xi \) are pairwise skew (and that \( \gamma_{\ell_1}, \ldots, \gamma_{\ell_\xi} \) intersect in a common curve).

**Lemma 2.6** Assume that the arc (in Plücker space) \( \gamma := \bigcap_{i=1}^\xi \gamma_{\ell_i} \) is nonempty (and is not a finite set), where \( \ell_1, \ldots, \ell_\xi \) are \( \xi \) pairwise-skew lines, with \( \xi \geq 3 \). Then \( \gamma \) parameterizes one ruling of a regulus, and, for each line \( \ell \in T \) such that \( \ell \in \gamma \), \( \ell \) intersects \( \ell_1, \ldots, \ell_\xi \), and thus \( \ell \) lies in the curve that represents the other ruling of the same regulus.

**Proof.** The proof is similar to the proof of Lemma 2.3. The intersection \( \bigcap_{i=1}^\xi S_{\ell_i} \) consists of the Plücker points that represent the lines that intersect the \( \xi \geq 3 \) pairwise-skew lines \( \ell_1, \ldots, \ell_\xi \). Thus, as already noted (see [7]), all these lines belong to one ruling of a regulus, and the Plücker points of \( \ell_1, \ldots, \ell_\xi \) belong to the other ruling of this regulus. Therefore, \( \gamma \) parameterizes one ruling of a regulus, and \( \ell_1, \ldots, \ell_\xi \) belong to the other ruling of this regulus, as asserted. \( \square \)

Partition the set of irreducible components of the curves \( \gamma_{\ell_i} \), over all lines \( \ell \in L \) that are not exceptional, into two subsets \( \mathcal{C}_0 \) and \( \mathcal{C}_1 \), where \( \mathcal{C}_0 \) (resp., \( \mathcal{C}_1 \)) contains all the components that do not (resp., do) parameterize one ruling of some regulus. Since \( \deg(\gamma_{\ell_i}) \leq \deg(\gamma_{T}) = O(1) \), for each \( \ell \in L \), it follows that \( |\mathcal{C}_0| = |\mathcal{C}_1| = O(n) \). We partition the set of incidences into incidences between the Plücker points in \( \tilde{L} \) and the curves in \( \mathcal{C}_0 \), and incidences with the curves in \( \mathcal{C}_1 \). We remind the reader that at this stage we are only concerned with incidences induced by a concurrence of at least \( r \geq 3 \) lines of \( L \) at some \((r\text{-rich})\) point \( p \).

By Lemma 2.4 any \( r\text{-rich} \) point \( p \), for \( r \geq 3 \), corresponds to incidences between the Plücker points \( \ell \) that represent lines \( \ell \) of \( L \) that are incident to \( p \) and the (at least three) curves \( \gamma_{\ell_i} \) that are associated with these lines, and any such incidence can arise for at most one point \( p \). One possibility is that the Plücker point \( \ell \), for a line \( \ell \) incident to \( p \), is incident to a common component of at least three of these curves, call them \( \gamma_{\ell_1}, \gamma_{\ell_2}, \gamma_{\ell_3} \). However, the analysis preceding Lemma 2.6 implies that \( \ell_1^p, \ell_2^p, \ell_3^p \) are pairwise skew, which is impossible as they are all incident to \( p \). Hence an incidence between the Plücker image of a line and a common component of at least three curves \( \gamma_{\ell_i} \) does not generate any \( r\text{-rich} \) points, for \( r \geq 3 \), and, by construction, curves in \( \mathcal{C}_1 \) also do not generate any \( r\text{-rich} \) points, for \( r \geq 3 \). We may therefore assume that every curve in \( \mathcal{C}_0 \) is an irreducible component of at most two curves in \( \mathcal{C} \).

Summarizing, the number of \( r\text{-rich} \) points, with \( r \geq 3 \), is proportional to the number of incidences between the Plücker points in \( \tilde{L} \) and the distinct curves in \( \mathcal{C}_0 \), divided by \( \binom{\xi}{2} \), and there is no contribution by the curves in \( \mathcal{C}_1 \).

**2-rich points.** The situation is different for 2-rich points, which may arise also as incidences between the Plücker points of lines in \( L \) and curves in \( \mathcal{C}_1 \). Handling them requires more care, and is done as follows. A proper 2-rich point \( p \), namely a point that is incident
to precisely two lines $\ell_p$ and $\ell'_p$ of $L$, corresponds to an incidence between the Plücker point of $\ell_p$ and the curve $\gamma_{\ell_p}$ (and also between the Plücker point of $\ell'_p$ and the curve $\gamma_{\ell_p}$). We count this incidence at most $\deg(\gamma_{\ell'}) = O(1)$ times, once for each irreducible component of the curve $\gamma_{\ell_p}$. It therefore suffices to count incidences between the Plücker points in $\tilde{L}$ and the curves in $C_0$ (as we have just argued, this is relevant only for curves of multiplicity at most 2) and in $C_1$ (which may have an arbitrary multiplicity).

Consider first incidences with curves of $C_0$. By projecting $T$ onto some generic plane, the number of incidences between the $n$ points of $\tilde{L}$ and the curves of $C_0$ is the same as the number of incidences between the projected points and the projected curves. Since $C$ is a two-dimensional family of curves (Lemma 2.5), so is $C_0$. It therefore follows, by Theorem 1.3 (with $s = 2$), that the number of these incidences is $O(n^{4/3+\varepsilon})$, for any $\varepsilon > 0$. As argued above, this gives us the bound $O(n^{4/3+\varepsilon}/\varepsilon^2)$ on the number of $r$-rich points, for $r \geq 3$, thereby establishing part (a) of Theorem 2.1.

This also gives us the bound $O(n^{4/3+\varepsilon})$ for the number of 2-rich points that correspond to incidences formed with the curves of $C_0$. For the remaining 2-rich points, which correspond to incidences between lines in $L$ (points of $L^*$) with curves of $C_1$ (which may appear with arbitrarily large multiplicity), we recall that each of the curves in $C_1$ represents one ruling of some regulus, and that we have assumed that no regulus contains more than $n^{1/3}$ lines of $L$. Hence the each curve in $C_1$ is incident to the Plücker points of at most $n^{1/3}$ lines in $L$, which implies that the number of incidences with these curves, counted with multiplicity, is at most $O(n^{4/3})$. Hence part (b) of the theorem also follows, and the proof is thus completed. 

3 Application: Distinct distances between points on an algebraic curve in the plane

Let $P$ be a set of $n$ points on an irreducible algebraic curve $\gamma$ of constant degree in the plane, which is not a line or a circle. We apply the Elekes-Sharir-Guth-Katz framework [11], and define a set $L$ of $n(n-1)$ lines in the parametric 3-space of rotations (rigid motions) in the plane, as $L = \{h_{a,b} \mid a \neq b \in P\}$, where $h_{a,b}$ is the locus of all rotations that map $a$ to $b$, which is indeed a line with a suitable parameterization; see [6, 11] for details. Then $L$ is contained in the two-dimensional family of lines $C = \{h_{x,y} \mid x,y \in \gamma\}$. It is easy to verify that $C$ is irreducible and is not a 2-flat. The property that $C$ is not a 2-flat will follow from the arguments following Lemma 3.1. To see that $C$ is irreducible, define a smooth morphism $\Phi: \gamma \times \gamma \to C$ by $\Phi(x,y) = h_{x,y}$. If $C$ were reducible, we could write $C = C_1 \cup C_2$, and then $\gamma \times \gamma = \Phi^{-1}(C_1) \cup \Phi^{-1}(C_2)$, implying that $\gamma \times \gamma$ is reducible, from which it follows that $\gamma$ too is reducible (see, e.g., Hartshorne [14, Exercise 3.15(a)]), contrary to assumption.

We will show below that, after pruning away some lines of $L$ (which will not affect the asymptotic bounds derived in the analysis), the number of remaining lines in $L$ that are contained in a common plane or regulus in 3-space is $O(1)$, and thus $C$ is non-degenerate with respect to $L$. Indeed, having $O(1)$ points in the dual space $H^*_{\ell,\ell'}$ follows by having, in primal space, $O(1)$ lines contained in the the plane $H_{\ell,\ell'}$.

In more detail, let $\Delta$ denote the number of distinct distances determined by $P$. We
count the number of quadruples
\[ \{(a, b, a', b') \in P^4 \mid |ab| = |a'b'|\}, \]
in two different ways. First, let \( N_k \) (resp., \( N_{\geq k} \)) denote the number of rotations of multiplicity exactly (resp., at least) \( k \); that is, rotations that map exactly (resp., at least) \( k \) points of \( P \) to \( k \) other points of \( P \). By construction, a rotation of multiplicity at least \( k \) is mapped to a \( k \)-rich point with respect to the lines of \( L \). By Theorem 2.1, replacing \( \varepsilon \) by \( \varepsilon/2 \), we have
\[ N_{\geq k} = O((n^2)^{4/3+\varepsilon/2}/k^2) = O(n^{8/3+\varepsilon}/k^2), \]
provided that the number of lines in \( L \) contained in a common regulus is \( O(|L|^{1/3}) \). We will shortly argue that this is indeed the case, and will also show that no plane, except for a constant number of exceptional planes, contains more than \( O(1) \) lines of \( L \). Consider first the case of coplanar lines. A standard observation is that one can parameterize the set of lines of the form \( h_{a,b} \), with \( a, b \in \mathbb{C}^2 \), that are contained in some fixed plane, by rigid motions with negative determinants (i.e., motions that involve reflections), as follows.

Let \( \tau \) be any rigid motion of the plane with a negative determinant. Simple calculations show that \( \tau \) must be a reflection around some line \( \ell \) in the plane, followed by a translation by some vector \( t \in \mathbb{C} \) in the direction of \( \ell \). Such transformations are also known as glide reflections.

We have the following lemma.

**Lemma 3.1** (i) The lines of the form \( h_{\xi, \tau(\xi)} \), with \( \xi \in \mathbb{C}^2 \), are all distinct and contained in a common plane, and every line in that plane is of this form, for a suitable point \( \xi \).

(ii) This gives a bijection between the set of nonvertical planes in \( \mathbb{R}^3 \) and the set of rigid motions with negative determinants (i.e., glide reflections), in the sense that the plane determines the rotation \( \tau \) and vice versa.

**Sketch of proof.** The plane \( \pi \) in part (i) is obtained as follows. Its intersection with the \( xy \)-plane is the line \( \ell \), and the angle \( \beta \) that it forms with the \( xy \)-plane is given by \( \tan \beta = \frac{1}{5} \). See Figure 1 for an illustration of this claim. This also gives a recipe to reconstruct \( \tau \) from \( \pi \).

Assume that the lines \( h_{a_i,b_i} \), for \( i = 1, \ldots, r \), are contained in some non-vertical plane. Then, by Lemma 3.1 there is a glide reflection \( \tau \), such that \( h_{a_i,b_i} = h_{a_i,\tau(a_i)} \) (this latter property follows because \( h_{a,b} \neq h_{a',b'} \) when \( (a, b) \neq (a', b') \)), and thus \( b_i = \tau(a_i) \), for \( i = 1, \ldots, r \). However, this implies that \( a_1, \ldots, a_r \in \gamma \cap \tau^{-1}(\gamma) \), which implies that \( \gamma = \tau(\gamma) \) (assuming that \( r \) is larger than the degree of \( \gamma \)). By Pach and de Zeeuw [17, Lemma 2.5], this can happen for at most \( 4 \deg(\gamma) = O(1) \) rotations, unless \( \gamma \) is a line orthogonal to or coinciding with \( \ell \) (assuming \( \gamma \) to be irreducible), or \( \gamma \) is a circle and \( \ell \) goes through its center. Since both these possibilities have been excluded, we are left with at most \( O(1) \) glide reflections \( \tau \) (that is, planes in 3-space) that satisfy \( \gamma = \tau(\gamma) \). Fix one of these planes \( \pi \), with a corresponding glide reflection \( \tau \), and note that the number of lines \( h_{a,\tau(a)} \) is at most \( n \). Hence there are at most \( O(n^2) \) rich points (that is, 2-rich points) formed solely by lines in \( \pi \). Any other rich point occurs at an intersection of a line of \( L \) (not contained in \( \pi \)) with \( \pi \), and the number of these points is \( O(n^2) \). To recap, the \( O(1) \) exceptional rotations contribute at most \( O(n^2) \) \( r \)-rich points, for any \( r \geq 2 \), well below our bound. We remove
the lines lying in these $O(1)$ planes, and, among the remaining lines, no plane contains more than $O(1)$ lines.

This argument also implies that $C$ cannot be a 2-flat. Indeed, 2-flats within the Klein quadric $Q$ come either from lines in $\mathbb{R}^3$ that are contained in a common plane or from lines that pass through a common point (see Rudnev and Selig [20] for details). The former situation is impossible because, as we have argued, it is impossible for all (or just for too many of) the lines $h_{x,y}$, for $x,y \in \gamma$, to be contained in a common plane in the primal 3-space. The latter situation is also easy, as the point in $\mathbb{R}^3$ common to the lines $h_{x,y}$ is a rotation that maps every point $x \in \gamma$ to every other point $y \in \gamma$, which is clearly impossible.

The situation is simpler for lines in a common regulus. Assume that the lines $h_{a_i,b_i}$, for $i = 1, \ldots, r$, are contained in some regulus $\sigma$. In general, any regulus in $\mathbb{C}^3$ is equivalent to the hyperbolic paraboloid $z = xy$ by a suitable change of coordinates, so we may assume that $\sigma$ is $z = xy$. The lines contained in $\sigma$ are either of the form $x = a$, $z = ay$ or of the form $y = a$, $z = ax$, for $a \in \mathbb{C}$. We may assume, without loss of generality, that our lines are of the first kind, and then it is easy to verify that these lines can be expressed as $h_{(a - \frac{1}{a},0),(a + \frac{1}{a},0)}$, for $a \in \mathbb{C}$. Hence, if there are at least $k$ values $a$, where $k$ is the degree of $\gamma$, such that both points $(a - \frac{1}{a},0)$ and $(a + \frac{1}{a},0)$ belong to $\gamma$, then, since $\gamma$ is irreducible and of constant degree, it must coincide with the $x$-axis, contradicting our assumption that $\gamma$ is not a line.

We have thus shown that no plane or regulus contains more than a constant number of lines of $L$, except for at most $O(1)$ special planes, whose effect on the asserted bound is negligible, and which we ignore by removing all the lines contained in these planes. Hence, arguing as in [6, 11] and using (4), the number of quadruples is at most

$$\sum_{k=2}^{n} \binom{k}{2} N_k \leq \sum_{k=2}^{n} k N_{\geq k} = O \left( \sum_{k=2}^{n} \frac{n^{8/3+\varepsilon}}{k} \right) = O(n^{8/3+\varepsilon} \log n) = O(n^{8/3+2\varepsilon}). \quad (5)$$

On the other hand, by Elekes’s analysis, which is based on the Cauchy-Schwarz inequality
(see, e.g., Guth and Katz [6 [11]], the number of quadruples is also $\Omega(n^4/\Delta)$, implying that the number of distinct distances satisfies $\Delta = \Omega(n^{4/3-2\varepsilon})$. This result was obtained earlier in [17], without the $\varepsilon$-loss in the exponent, but the proof here is much simpler, and we hope that it will find similar applications of this kind.

4 Incidences between points and lines in a two-dimensional family of lines

The main result of this section is the following theorem.

**Theorem 4.1** Let $P$ be a set of $m$ points in $\mathbb{R}^3$, and let $L$ be a set of $n$ lines in $\mathbb{R}^3$, such that the set $\tilde{L}$ of the Plücker images of the lines of $L$ is contained in some two-dimensional, non-planar, irreducible variety $T$ of constant degree, as in Section 2. Then $I(P, L) = O(m + n)$.

**Proof.** As observed above, for a point $p \in \mathbb{C}^3$, the set $S_p$ of the Plücker images of lines that are incident to $p$ form a 2-flat in Plücker space, and that 2-flat is contained in $Q$. We first observe that even if $T$ were a 2-flat in $Q$, and, for some $p \in \mathbb{R}^3$, we had $T = S_p$, then all the lines with Plücker images in $T$ would be incident to $p$, and thus for any point $q \neq p$, the number of incidences between $q$ and the lines with Plücker images in $T$ would be at most one (there is only one line in $\mathbb{C}^3$ that is incident to $p$ and $q$), so the number of incidences would be $O(m + n)$ in this case too. Thus, we may assume that for every $p \in \mathbb{C}^3$, $T \neq S_p$.

As $T$ is two-dimensional, Bézout’s theorem [8] implies that for every $p \in \mathbb{C}^3$, the intersection $S_p \cap T$ is a union of a constant number of curves of constant degree and a discrete set of a constant number of points, in Plücker space.

We next argue that $V$ cannot be two-dimensional (recall that $V \subset \mathbb{C}^3$). Assume to the contrary that $V$ is two-dimensional. By definition, for each $p \in V$, we have that $T$ contains a one-dimensional set of the Plücker images of lines that are incident to $p$ (namely, those images forming the curve $S_p \cap T$ in the Plücker space).

**Lemma 4.2** If $V$ is a 2-flat in the primal $\mathbb{C}^3$, then $T$ is a 2-flat in $Q$. 

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Proof. For a point \( p \in \mathbb{C}^3 \), let \( X_p := \{ \ell \in S_p \cap T \mid \ell \subset V \} = S_p \cap T \cap V^* \), where \( V^* = \{ \ell \mid \ell \subset V \} \). By Bézout’s Theorem, \( S_p \cap T \) is a union of \( O(1) \) constant-degree curves contained in \( S_p \), and \( O(1) \) Plücker images of lines (note that \( S_p \cap T \) can also be a discrete finite set). Let 
\[
U := \{ p \in V \mid X_p \text{ is a curve in Plücker space} \}.
\]
We claim that \( U \) is a subvariety of \( V \). To see this, define \( Y = \{(p, \ell) \in V \times (T \cap V^*) \mid \ell \in S_p \} \). Clearly \( Y \) is an algebraic variety whose fiber over point \( p \in V \) of the projection onto \( V \) is \( Y_p = T \cap V^* \cap S_p \), and by the Theorem of the Fibers [13 Corollary 11.13] (see also Theorem 6.2 in [26, Theorem 6.2]), we deduce that \( \{ p \in V \mid \dim(Y_p) \geq 1 \} \) is an algebraic variety, and this is precisely \( U \).

If \( U \) is two-dimensional, then \( V = U \), implying that for every point \( p \) in the 2-flat \( V \), there are infinitely many lines \( \ell \) that are incident to \( p \) and contained in \( V \) and that \( \ell^* \in T \). This implies that \( T = V^* \), and that \( T \) is a 2-flat.

Otherwise, \( U \) is at most one-dimensional, and for every \( p \in V \setminus U \), there is a set \( K \), consisting of \( O(1) \) Plücker images \( \tilde{\ell} \) in \( S_p \cap T \) of lines that are contained in \( V \) (in the primal 3-space). For a pair \( p \neq q \in V \setminus U \), \( S_p \) and \( S_q \) are both 1-dimensional (in \( Q \)), and, outside the set \( K \) of constantly many Plücker images of lines, \( S_p \) and \( S_q \) are disjoint. Thus, \( \dim(T) = 3 \), as it is parameterized by three parameters, two for the point \( q \in V \), and one for the 1-dimensional curve \( S_p \), contradicting the assumption that \( \dim(T) = 2 \). □

We formalize the argument from Lemma 4.2 and generalize it to the general two-dimensional case, as follows. For a point \( p \in \mathbb{C}^3 \), the set \( X_p := \{ \ell \in S_p \cap T \mid \ell \subset V \} = S_p \cap T \cap V^* \) is a union of \( O(1) \) constant-degree curves contained in \( S_p \), and \( O(1) \) additional Plücker images of lines.

As observed in [26 Appendix], a surface is ruled by lines if every point in a Zariski-open dense subset on the surface is incident to a line that is fully contained in it. The same is true for infinitely ruled surfaces, namely, it suffices to have a Zariski-open dense subset on the surface, each of whose points is incident to infinitely many lines that are fully contained in the surface. Therefore, if the set \( U \), defined in the proof of Lemma 4.2 is two-dimensional, it follows that \( V \) is infinitely ruled by lines, and by [24, Theorem 3.11], it follows that \( V \) must be a 2-flat (in the primal 3-space), and Lemma 4.2 implies that \( T \) is a 2-flat in \( Q \), contradicting our assumption. Otherwise, \( U \) is at most one-dimensional, and for every \( p \in V \setminus U \), there are only \( O(1) \) Plücker images \( \tilde{\ell} \) in \( S_p \cap T \) of lines \( \ell \) that are contained in \( V \). Hence, for any pair \( p \neq q \in V \setminus U \), \( S_p \) and \( S_q \) are both 1-dimensional and, outside a possible set \( K \) of \( O(1) \) Plücker images of lines that are contained in \( V \), \( S_p \) and \( S_q \) are disjoint. It thus follows that \( \dim(T) = 3 \), as it is parameterized by three parameters, two for the point \( q \in V \), and one for the 1-dimensional curve \( S_p \), contradicting the assumption that \( \dim(T) = 2 \). Thus \( V \) is a curve, contradicting our assumption that \( V \) is two-dimensional.

We have thus argued that \( V \) cannot be two-dimensional, so \( V \) must be one-dimensional. By definition of \( V \), every point \( p \in P \setminus V \) is incident to at most \( O(1) \) lines of \( L \), for a total of \( O(m) \) incidences. Thus, we may assume that all the points of \( P \) are contained in the curve \( V \). For each line \( \ell \in L \), if \( \ell \) is not contained in \( V \) it contributes at most \( O(\deg V) = O(1) \) incidences with \( P \). Thus, we get a total of \( O(n) \) incidences, except for at most \( O(\deg V) = O(1) \) lines that are contained in \( V \), for a total of \( O(m) \) additional incidences.
This completes the proof of the theorem. □

The following corollary is an immediate consequence of the theorem.

**Corollary 4.3** Let $T$ be a two-dimensional, non-planar, irreducible subvariety, of constant degree, of the Klein quadric $Q$. Then, there exists a constant $r_0 = r_0(\deg(T))$ so that, if $L$ is a set of $n$ lines in $\mathbb{R}^3$ whose Plücker images are points in $T$ then, for $r \geq r_0$, the set $P_{\geq r}(L)$ of $r$-rich points determined by $L$ satisfies $|P_{\geq r}(L)| = O(n/r)$.

## 5 Incidences between points and lines in a three-dimensional family of lines

In this section we prove the following result.

**Theorem 5.1** The number of incidences between $m$ points in $\mathbb{R}^3$ and $n$ lines in $\mathbb{R}^3$ whose Plücker images are contained in an irreducible nonlinear constant-degree three-dimensional variety $T$ is

$$O\left(m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n\right),$$

provided that no plane contains more than $s$ of the points. If $s = O(\min\{n^{3/5}/m^{2/5}, m^{1/2}\})$, the bound becomes $O(m^{3/5}n^{3/5} + m + n)$.

**Proof.** Recall that $S_p$ is the 2-flat in Plücker space that consists of the images of all lines passing through a point $p \in \mathbb{R}^3$, and define

$$W := \{p \in \mathbb{C}^3 \mid S_p \cap T \text{ is two-dimensional}\}.$$

**Lemma 5.2** $W$ is an algebraic variety of dimension at most 2 and of constant degree.

**Proof.** Since $S_p$ is a 2-flat, $S_p \cap T$ is two-dimensional if and only if $S_p \subset T$. Similarly to the Fano variety of lines, the Grassmannian manifold of 2-flats contained in a constant-degree variety is an algebraic variety of constant degree [9], so $W$ is algebraic of constant degree. To bound the dimension of $W$, we repeat the proof of [26 Theorem 2.3(a)], which proceeds by counting the dimensions of the fibers that arise in the problem. Here we omit the details and give the high-level idea. Assume to the contrary that $W$ is three-dimensional, i.e., $W = \mathbb{C}^3$, so for every point $p \in \mathbb{C}^3$, the 2-flat $S_p$ is contained in $T$. Omitting details, we note that each $p \in \mathbb{C}^3$ contributes a two-dimensional set ($\dim(S_p) = 2$), but then every line is counted by the infinitely many points incident to it. A standard dimension counting argument then implies that $\dim(F(T)) \geq 4$, where $F(T)$ is the Fano variety of lines contained in $T$. By [24 Theorem 3.11], this implies that $T$ has to be a 3-flat, contradicting our assumption. □

We first treat incidences with points $p \in P \cap W$. We decompose $W$ into its $O(1)$ irreducible components, and treat each component separately. If a component $W_0$ of $W$ is not a 2-flat then, by [26 Corollary 1.4], the number of incidences between points contained in $W_0$ and lines in $L$ is $O(m + n)$. If $W_0$ is a 2-flat, we invoke the Szemerédi-Trotter
bound in Theorem 1.1 and get the bound $O(s^{2/3}n^{2/3} + s + n)$, using our assumption that no 2-flat contains more than $s$ points of $P$. This in turn can be upper bounded by $O(s^{1/3}m^{1/3}n^{4/3} + m + n)$, which is subsumed by the bound asserted by the theorem.

Next, we treat incidences involving points in $\mathbb{R}^3 \setminus W$, i.e., points that are incident to a one-dimensional family of lines with Plücker images in $T$. In this case we use duality, replacing each point $p$ in $\mathbb{C}^3$ with the one-dimensional curve $\gamma_p$ of the Plücker images $\ell$ of lines $\ell$ that are incident to $p$ and have Plücker images in $T$. This yields a family of $m$ constant-degree curves that is a family of pseudo-lines. (Two such curves $\gamma_p$ and $\gamma_q$ intersect in at most one point, representing the (unique) line connecting $p$ and $q$, if its Plücker image lies in $T$.) We replace each of the $n$ lines in $L$ by its Plücker image, and obtain an incidence problem between $n$ points and $m$ pseudo-lines within the variety $T$, a three-dimensional subset of the four-dimensional Klein quadric $Q$. Using a generic projection of $T$ onto $\mathbb{R}^3$ (in which all projected points are distinct and no pair of projected curves overlap), the analysis then proceeds by invoking Zahl [34, Lemma 4.1], which extends the Guth–Katz incidence bound from incidences with lines to incidences with pseudo-lines. Specifically, Zahl shows that the number of incidences between $n$ points and $m$ pseudo-lines in $\mathbb{R}^3$, assuming that these pseudo-lines are constant-degree algebraic curves, is $O(n^{1/2}m^{3/4} + n^{2/3}m^{1/3} + m + n)$, where $\xi$ is an upper bound on the number of pseudo-lines that are contained in any common two-dimensional surface contained in $T$ that is infinitely ruled by curves from the infinite family from which our pseudo-lines are taken.

As argued in Guth and Zahl [12], any such surface must be of degree at most $100E^2$, where $E$ is the degree of the pseudo-lines $\gamma_p$, so it is sometimes convenient, especially when no simple characterization of such infinitely ruled surfaces is known, to impose the stronger assumption that no surface of degree at most $100E^2$ contains more than $\xi$ pseudo-lines. In general, without having a good characterization of the infinitely-ruled surfaces, this assumption is too restrictive, and difficult to verify. One of the main technical contribution of the analysis in this section is to exploit the dual nature of the present setup, and replace this assumption by the simpler and more natural assumption that, in the original primal 3-space, no plane contains more than $s$ points of $P$, allowing us to replace $\xi$ by $s$.

To see this we argue as follows. Let $S$ be a surface contained in $Q$ that is infinitely ruled by curves from $C_0$, and define the variety\footnote{It is easy to verify that $V_S$ is defined by polynomial equations and is thus an algebraic variety.} $V_S := \{p \in \mathbb{C}^3 \mid \text{an irreducible component of } c_p \subseteq S\}$. We have

**Lemma 5.3** $V_S$ is two-dimensional.

**Proof.** We assume, as we may, that $S \subset T$, and note that then $V_S = \{p \in \mathbb{C}^3 \mid \text{dim}(S_p \cap S) = 1\}$, and that $V_S$ parameterizes curves $c_p$ that are contained in $S$. By using resultant theory (see, e.g., [24, Section 2.4]), it is also easy to verify that $V_S$ is an algebraic variety.

Since $S$ is infinitely ruled by curves in $C_0$, it follows that for every $\ell \in S$, there exists an infinite family of curves in $C_0$ that are contained in $S$ and incident to $\ell$. Since every curve in $C_0$ is of the form $S_p \cap S$ with $p \in \ell$, it follows that each point $\ell \in S$ is incident to a 1-dimensional family of curves of $C_0$ (which is the family corresponding to points in $\ell$). As $S_p \cap S_q$ is a point, it follows that pairs of curves in $C_0$ intersect in at most one point. By dimension counting, it follows that $V_S$ is two-dimensional. \(\square\)
By definition of $S$, for every point $\ell \in S$, there are infinitely many curves of the form $c_p$, for $p \in V_S$, that are contained in $S$ and incident to $\ell$. For each such curve $c_p$, in the primal space $\mathbb{C}^3$, the corresponding point $p$ is incident to the line $\ell$. Therefore, there are infinitely many points of $\ell$ that are contained in $V_S$, and as $V_S$ is an algebraic variety, it follows that $\ell \subset V_S$.

Therefore, we have a two-dimensional variety $V_S \subset \mathbb{C}^3$ containing lines that are parameterized by a two-dimensional variety $S \subset T$, implying that $V_S$ is a plane. (Indeed, the Fano variety of lines contained in a two-dimensional irreducible variety that is not a plane is at most one-dimensional.)

In summary, the lines in $\mathbb{C}^3$ whose Plücker coordinates lie in $S$ are contained in a plane $V_S$. As observed above (and argued in Rudnev [19]), the set of lines that are contained in a plane in $\mathbb{C}^3$ are dual to a 2-flat in the Klein quadric $Q$, implying that $S$ is a plane.

For a point $p \in V_S$, we have $S_p \cap T = c_p \subset S$, implying that $S_p \cap T = S_p \cap S$, and the right-hand side is an intersection of two 2-flats. Except for at most one point $p_0$, we get that for every (other) $p \in V_S$, the curve $S_p \cap S$ is a line contained in $S$.

Since the set of lines fully contained in $S$ is two-dimensional, it follows that all the lines fully contained in $S$ are of the form $S_p \cap S$. By assumption, the number of points in $P$ that are contained in a common plane is at most $s$, and duality implies that the number of lines of the form $S_p \cap S$ fully contained in $S$ is at most $s$.

We thus obtain the incidence bound

$$I(P, L) = O(n^{1/2}m^{3/4} + n^{2/3}m^{1/3}s^{1/3} + m + n).$$

Since $n^{1/2}m^{3/4} \leq m^{3/2}$ when $n \leq m^{3/2}$, and $n^{1/2}m^{3/4} \leq n$ otherwise, we get the following bootstrapping bound

$$I(P, L) = O(m^{3/2} + n^{2/3}m^{1/3}s^{1/3} + n).$$  \hspace{1cm} (6)

The analysis then proceeds by “starting over” in primal space, i.e., by constructing a partitioning polynomial $g$ of degree $O(D)$, for a suitable value of $D$, to be fixed shortly, using the techniques in [10, 11], so that each connected component (cell) $\tau$ of $\mathbb{R}^3 \setminus Z(g)$ contains at most $m/D^3$ points of $P$ and is crossed by at most $n/D^2$ lines of $L$ (but any number of points and lines can be contained in the zero set $Z(g)$).

**Incidences within the cells.** We first bound the number of incidences within the partition cells. We apply the bootstrapping bound in (6) to each cell $\tau$ and sum the bound over all components, to obtain the bound

$$O\left( D^3((m/D^3)^{3/2} + (n/D^2)^{2/3}(m/D^3)^{1/3}s^{1/3} + n/D^2) \right)$$

$$= O\left( \frac{m^{3/2}}{D^{3/2}} + n^{2/3}m^{1/3}D^{2/3}s^{1/3} + nD \right).$$

---

4The set of lines of the form $S_p \cap S$ is a two-dimensional variety contained in the set of lines contained in $S$, which is an irreducible two-dimensional variety. Moreover, for two distinct points $p, q \in V_S$, the intersection $S_p \cap S_q$ is equal to the point in Plücker space that represent the unique line that is incident to $p$ and $q$. Therefore, the lines $S_p \cap S$, for $p \in V_S$, are distinct. Thus the two varieties must be equal.
To balance the first and last terms, we choose $D = m^{3/5}/n^{2/5}$. For this to make sense, we require that $1 \leq D \leq \min\{m^{1/3}, n^{1/2}\}$, or, equivalently, that $n \leq m^{3/2}$ and $m \leq n^{3/2}$. When the first inequality does not hold, we do not use any partitioning and just apply [1] to obtain the bound $I(P, L) = O(n^{2/5}m^{1/3}s^{1/3} + n)$. When the second inequality does not hold, we choose $D = an^{1/2}$, for a suitable constant $a$, which satisfies the inequalities $1 \leq D \leq \min\{m^{1/3}, n^{1/2}\}$. In fact, we can construct a polynomial $g$ of this degree so that all the lines of $L$ are fully contained in $Z(g)$ (see, e.g., [15]), and we may therefore assume that all the points of $P$ are also contained in $Z(g)$, as the other points contribute no incidences. That is, in this case there are no incidences within the cells.

In the middle range, our choice of $D$ yields the bound $O(m^{3/5}n^{3/5} + m^{11/15}n^{2/5}s^{1/3})$. Combining all the bounds, the number of incidences within the partition cells is

$$O \left( m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + n \right). \quad (7)$$

**Incidences on the zero set.** Consider next incidences involving points that lie on $Z(g)$. A line $\ell$ that is not fully contained in $Z(g)$ crosses it in at most $O(D)$ points, for an overall $O(nD)$ bound, which is subsumed by the bound (7) for incidences within the cells. It therefore remains to bound the number of incidences between the points of $P$ on $Z(g)$ and the lines that are fully contained in $Z(g)$.

We handle each irreducible component of $Z(g)$ separately. For non-planar components, Theorem [1,5], combined with Hölder’s inequality (for summing up the bounds over the irreducible components) implies that the number of incidences between points and lines contained in $Z(g)$, but not in any planar component of $Z(g)$, is

$$O(m^{1/2}n^{1/2}D^{1/2} + m^{1/3}D^{4/3}s^{1/3} + m + n).$$

In the middle range $n^{2/3} \leq m \leq n^{3/2}$, the choice of $D = m^{3/5}/n^{2/5}$ is easily seen to yield the desired bound $O(m^{3/5}n^{3/5} + m + n)$ (as the second term in the bound is dominated by the first term for this range of $m$). The case $m < n^{2/3}$ has already been handled, by a single application of [1], which yields the bound $O(n^{2/3}m^{1/3}s^{1/3} + n)$. When $m > n^{3/2}$, the choice of $D = an^{1/2}$, as made above, yields the bound $O(n^{2/3}m^{1/3}s^{1/3} + m)$.

For the planar components, we use the standard technique of assigning each point and line to the first planar component that contains it (according to some arbitrary enumeration of the components). The number of incidences between points and lines assigned to different components is $O(nD) = O(m^{3/5}n^{3/5} + m + n)$ (the right-hand side does indeed bound the left-hand side for each of the sub-ranges). For incidences between points and lines assigned to the same planar component, we apply the Szemerédi-Trotter bound (Theorem [1,1]) to each component and sum the resulting bounds over the components. The assumption that each plane contains at most $s$ points, combined with Hölder’s inequality, yields the bound $O(m^{1/3}n^{2/3}s^{1/3} + m + n)$.

That is, the number of incidences with points on $Z(g)$ is bounded by

$$O \left( m^{3/5}n^{3/5} + m^{1/3}n^{2/3}s^{1/3} + m + n \right). \quad (8)$$

Combining with the bound (7) for incidences within the cells, we get the overall bound

$$I(P, L) = O \left( m^{3/5}n^{3/5} + (m^{11/15}n^{2/5} + m^{1/3}n^{2/3})s^{1/3} + m + n \right),$$
thereby completing the proof of the theorem. □

**Remark.** An interesting open challenge in incidence geometry is to sharpen the Guth-Katz bound [11] when the number of lines in any common plane is at most some constant. When the lines in $L$ are contained, as points in Plücker space, in an irreducible nonlinear constant-degree three-dimensional variety $T$ then, while we cannot deduce that the number of lines contained in a common plane is constant, we can nevertheless deduce the following useful property. For any plane $\Pi \subset \mathbb{C}^3$, $T \cap \Pi^\ast$ (recall that $\Pi^\ast$ is the 2-flat dual to $\Pi$, consisting of all the points dual to lines that are contained in $\Pi$) is a constant-degree curve, and thus, except for $O(1)$ points, every point in $\Pi$ is incident to $O(1)$ lines in $T$, implying that the number of incidences in a common plane is $O(m_\Pi + n_\Pi)$, where $m_\Pi$ ($n_\Pi$) is the number of points (lines) contained in $\Pi$. Such a linear bound on the number of incidences within a plane is a key property for deriving improved incidence bounds, as demonstrated in this work. For Theorem 5.1, we also added the condition that $m_\Pi \leq s$, for every plane $\Pi$, to further improve the bound.

6 Application: Incidences between points and lines on a quadric in four dimensions

Solomon and Zhang [31] give a configuration of $m$ points and $n$ lines in a quadratic hypersurface in $\mathbb{R}^4$, having $\Omega(m^{2/3}n^{1/2} + m + n)$ incidences. The following theorem follows as a corollary from the previous section.

**Theorem 6.1** Let $P$ be a set of $m$ points and $L$ a set of $n$ lines contained in a quadratic hypersurface $S \subset \mathbb{C}^4$ such that no 2-flat contains more than $s = O(n^{3/5}/m^{2/5})$ of the points of $P$. Then $I(P, L) = O(m^{3/5}n^{3/5} + m + n)$.

**Remark.** When $m = O(n^{3/2})$, the lower bound $\Omega(m^{2/5}n^{1/2} + m + n)$ obtained in [31] is (asymptotically) smaller than the upper bound $O(m^{3/5}n^{3/5} + m + n)$ asserted in Theorem 6.1. Closing this gap remains a challenging open problem.

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