ON THE LOCUS OF CURVES WITH AN ODD SUBCANONICAL MARKED POINT

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ABSTRACT. We present an explicit construction of a compactification of the locus of smooth curves whose symmetric Weierstrass semigroup at a marked point is odd. The construction is an extension of Stoehr’s techniques using Pinkham’s equivariant deformation of monomial curves by exploring syzygies. As an application we prove the rationality of the locus for genus at most six.

1. Introduction

Let \( \mathcal{H}_{2g-2} \) be the locus of compact Riemann surfaces (smooth projective algebraic curves) of genus \( g > 1 \) with a fixed abelian differential vanishing at a point to order \( 2g - 2 \). In a remarkable work Kontsevich–Zorich [16, Thm. 1] showed that \( \mathcal{H}_{2g-2} \) has exactly three irreducible components, namely the locus \( \mathcal{H}_{2g-2}^{\text{hyp}} \) of hyperelliptic points, the even \( \mathcal{H}_{2g-2}^{\text{even}} \) and the odd \( \mathcal{H}_{2g-2}^{\text{odd}} \) points. Ten years later Bullock [5, Thm. 2.1] characterized a general point of each component, a general point of \( \mathcal{H}_{2g-2}^{\text{hyp}} \) has Weierstrass gaps \( \{1, 3, 5, \ldots, 2g - 3, 2g - 1\} \), a general point of \( \mathcal{H}_{2g-2}^{\text{odd}} \) has Weierstrass semigroup \( \{1, 2, 3, \ldots, g - 1, 2g - 1\} \) and finally a general point of \( \mathcal{H}_{2g-2}^{\text{even}} \) has Weierstrass gaps \( \{1, 2, 3, \ldots, g - 2, g, 2g - 1\} \). Say that an abelian differential with a zero at a point of order \( 2g - 2 \) is equivalent to required that this point is subcanonical, [5, Def. 1], i.e. the associated Weierstrass semigroup of this point is symmetric.

Let \( \mathcal{M}^S_{g,1} \) be the moduli space of smooth pointed curves of genus \( g > 1 \) with a fixed symmetric Weierstrass semigroup \( S \) at the marked point. There are two powerful tools to investigate the moduli spaces \( \mathcal{M}^S_{g,1} \), both based on deformation of suitable curves. On the one hand Eisenbud–Harris [11] deformed stable curves and uses limit linear series to study properties of \( \mathcal{M}^S_{g,1} \) as a locally closed subset of \( \mathcal{M}_{g,1} \). On the other hand, Pinkham [22] studied the moduli \( \mathcal{M}^S_{g,1} \) by using equivariant deformation theory, deforming monomial curves. Following a proposal given by Mumford [13] on Petri’s analysis of the canonical ideal, Stoehr [24] constructed a compactification of \( \mathcal{M}^S_{g,1} \) when \( S \) is symmetric by allowing Gorenstein curves at its bordering. Stoehr’s techniques avoid suitable classes of symmetric semigroups, more precisely, it is assumed that the multiplicity \( n_1 \) of \( S \) satisfies \( 3 < n_1 < g \), and that \( S \neq \langle 4, 5 \rangle \), avoiding the general points of \( \mathcal{H}_{2g-2}^{\text{hyp}} \) and of \( \mathcal{H}_{2g-2}^{\text{odd}} \) of the Kontsevich–Zorich space \( \mathcal{H}_{2g-2} \). Another successful approach to study families of Weierstrass points can be done by considering (generalized) Wronskians and its derivatives, we refer to [12, 13].

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In this paper we extend Stoehr’s techniques in order to construct in a rather explicit way a compactification \( \overline{\mathcal{M}}_{g,1}^S \) of the moduli space \( \mathcal{M}_{g,1}^S \) when \( S \) is a symmetric numerical semigroup different from the hyperelliptic \( (2, 2g + 1) \). Numerical semigroups of odd type tends to be realized as Weierstrass semigroups of possibly singular Gorenstein curves which are a triple recovering of the projective line \( \mathbb{P}^1 \), i.e. 3-gonal singular curves, see Lemma [3.1] below. Hence the canonical ideal of the monomial Gorenstein curve associated to a numerical odd semigroup cannot be generated by only quadratic forms as required, cf. Lemma [3.3] of the present work.

Given a non-hyperelliptic symmetric semigroup \( S \neq (2, 2g + 1) \), we deform the ideal (which is generate by quadratic and cubic forms) of the associated canonically embedded monomial Gorenstein curve. By analyzing suitable syzygies of canonical ideals, see Lemma [3.1], we get a compactification of \( \mathcal{M}_{g,1}^S \) by allowing Gorenstein singularities at its bordering, cf. Theorem [3.9]. The compactification is (by construction) a closed subset of the weighted projective space \( \mathbb{P}(k[T_1, \ldots, k[S]]) \), where \( T_1 \) stands for the negatively graded part of the first module of cotangent complex associated to a suitable monomial curve. Since our construction is completely explicit, we are able to produce non-trivial examples and investigate the global geometry of the moduli spaces \( \mathcal{M}_{g,1}^S \). In the last section of this paper we illustrate our techniques computing the equations of \( \overline{\mathcal{M}}_{g,1}^S \) when \( S \) is odd of genus 5, \( S = \{5, 6, 7, 8\} \) and of genus 6, \( S = \{6, 7, 8, 9, 10\} \).

2. Gorenstein subcanonical curves and Weierstrass Points

Let \( C \) be a complete integral Gorenstein curve of arithmetical genus \( g > 1 \) defined over an algebraically field \( k \). Throughout this section we assume that \( C \) is subcanonical, i.e. there is a rational function on \( C \) with pole divisor \( (2g - 2)P \), where \( P \) is a nonsingular point of \( C \). The dualizing sheaf \( \omega \) of \( C \) is an \( O_C((2g - 2)P) \), and the vector space of its global sections is

\[
H^0(C, \omega) = k \cdot x_0 \oplus k \cdot x_1 \oplus \cdots \oplus k \cdot x_{n_g - 1}
\]

where \( x_n \) is a rational function on \( C \) whose pole divisor is \( n_iP \), for \( i \geq 1 \), with \( n_0 := 0 \) and \( n_{g-1} = 2g - 2 \). Equivalently, the base point \( P \in C \) is a Weierstrass point with gap sequence \( 1 = \ell_1 < \ell_2 < \cdots < \ell_g = 2g - 1 \), whose symmetric Weierstrass semigroup \( S \) of genus \( g \) is canonically generated by its first \( g \) non-gaps, \( \langle n_0, n_1, \ldots, n_{g-1} \rangle = S \). We recall that a semigroup \( S \) of genus \( g \) is symmetric if its Frobenius number \( \ell_g \) is the largest possible, namely \( \ell_g = 2g - 1 \). Equivalently, \( S \) is symmetric if and only if \( \ell_i = \ell_g - n_{g-i} \), for all \( i = 1, \ldots, g \).

Let us assume that \( C \) is also non-hyperelliptic, thus its dualizing sheaf \( \omega \) induces an embedding in the \((g - 1)\)-dimensional projective space \( \mathbb{P}^{g-1} \) defined over \( k \),

\[
(x_{n_0} : \cdots : x_{n_g-1}) : C \overset{\omega}{\longrightarrow} \mathbb{P}^{g-1} = \mathbb{P}(H^0(C, \omega)).
\]

Therefore, \( C \) can be identified with its image under the canonical embedding. Hence \( C \subset \mathbb{P}^{g-1} \) is a projective curve of genus \( g \) and degree \( 2g - 2 \).

Reciprocally, every nonhyperelliptic symmetric numerical semigroup \( S \) of genus \( g > 1 \) can be realized as a Weierstrass semigroup of a canonical Gorenstein curve. We just consider the canonical generators \( 0 = n_0 < n_1, \ldots, n_{g-1} = 2g - 2 \) of \( S \) and take the induced (canonical) monomial curve

\[
C^{(0)} := \{ (s^{n_0}t^{\ell_g-1} : s^{n_1}t^{\ell_{g-1}-1} : \cdots : s^{n_g-2}t^{\ell_2-1} : s^{n_{g-1}}t^{\ell_1-1}) | (s : t) \in \mathbb{P}^1 \} \subset \mathbb{P}^{g-1}.
\]
It can be checked that it has a unique singular point, namely \((1 : 0 : \cdots : 0)\), which is unibranch and has singularity degree \(g\). Since the semigroup \(\mathcal{S}\) is symmetric, \(C^{(0)}\) is a Gorenstein curve. The contact orders with hyperplanes at its unique point \(P = (0 : \cdots : 1)\) at the infinity are exactly \(\ell_i - 1, \ i = 1, \ldots, g\) (the vanishing sequence). Thus \(C^{(0)}\) has degree \(2g - 2\) and its Weierstrass semigroup at \(P\) is \(\mathcal{S}\).

According to Enriques–Babbage’s theorem for smooth curves, cf. \([1]\), if we assume that \(C\) is not isomorphic to a plane quintic, then its ideal can be generated by quadratic forms, when it is non-trigonal, and by quadratic and cubic forms when it is trigonal.

An extended version of Max Noether’s theorem for complete integral non-hyperelliptic curves, cf. \([8, 17]\), states that there is a surjective homomorphism

\[
\text{Sym}^r(H^0(C, \omega)) \longrightarrow H^0(C, \omega^r)
\]

for all \(r \geq 1\). In the following, we recall a suitable proof of Max-Noether’s theorem for subcanonical curves given by Stöhr in \([24]\).

Let \(C\) be a complete non-hyperelliptic Gorenstein curve with a subcanonical point \(P\). Since \(C\) is non-hyperelliptic, we must to assume that the symmetric semigroup \(\mathcal{S}\) is not hyperelliptic, i.e. \(2 \notin \mathcal{S}\), equivalently \(\mathcal{S} \neq \langle 2, 2g + 1 \rangle\). Now, for each nongap \(s \leq 4g - 4\), we consider the partitions of \(s\) as sums of two nongaps,

\[
s = a_s + b_s, \ a_s \leq b_s \leq 2g - 2,
\]

with \(a_s\) the smallest possible nongap. From Oliveira’s paper \([19\text{, Thm. 1.3}]\) the following \(3g - 3\) rational functions \(x_{a_s}x_{b_s}\) of \(C\) form a \(P\)-hermitian basis for the space of the global sections of the bicanonical divisor \(\omega^2 = \mathcal{O}_C((4g - 4)P)\). Now, for each integer \(r \geq 3\) a \(P\)-hermitian basis for the space \(H^0(C, \omega^r)\) is given by the \(r\)-monomials expressions

\[
x_{n_0}^{r-1}x_{n_1}, \ x_{n_0}^{r-2}x_{a_s}x_{b_s}x_{n_{a_s-1}}, \ x_{n_0}^{r-3}x_{a_s}x_{2a_{s-1}}x_{n_{a_{s-1}}}
\]

\((i = 0, \ldots, g - 1)\),

\[
x_{n_0}^{r-1}x_{n_1}x_{2g-n_1}x_{n_{g-2}}x_{n_{g-1}}
\]

\((i = 0, \ldots, r - 3)\).

Note that the pole orders of the above \((2r - 1)(g - 1)\) rational functions are pairwise different, so they form a linearly independent set in \(H^0(C, \omega^r)\).

Let \(I(C) = \oplus_{r=2}^{\infty}I_r(C)\) be the homogeneous canonical ideal of \(C \subset \mathbb{P}^{g-1}\). As an immediate consequence of the existence of the above \(P\)-hermitian basis of \(r\)-monomials for \(H^0(C, \omega^r)\), the homomorphism

\[
k[X_{n_0}, \ldots, X_{n_{g-1}}]_r \longrightarrow H^0(C, \omega^r)
\]

induced by the substitutions \(X_{n_i} \mapsto x_{n_i}\) is surjective for each \(r \geq 1\). Thus we get a proof of Max-Noether’s theorem for non-hyperelliptic Gorenstein curves with a subcanonical point.

By virtue of Riemann’s theorem, for each \(r \geq 2\), the codimension of \(I_r(C)\) in the \((r + g - 1)\)-dimensional vector space \(k[X_{n_0}, \ldots, X_{n_{g-1}}]_r\) of homogeneous \(r\)-monomials is equal to \((2r - 1)(g - 1)\). So the vector space of quadratic and cubic relations have dimensions

\[
\dim I_2(C) = \frac{(g - 2)(g - 3)}{2} \quad \text{and} \quad \dim I_3(C) = \binom{g+2}{3} - (5g - 5),
\]

respectively.

For each \(r \geq 2\), we define the vector subspace \(\Lambda_r\) of \(k[X_{n_0}, \ldots, X_{n_{g-1}}]_r\) spanned by the lifting of the above \(P\)-hermitian \(r\)-monomial basis of \(H^0(C, \omega^r)\). It is spanned
by the $r$-monomials in $X_{n_0}, \ldots, X_{n_{g-1}}$ whose weights are pairwise different between all the nongaps $n \leq r(2g-2)$. Since $\Lambda_r \cap I_r(\mathcal{C}) = 0$ and 
\[ \dim \Lambda_r = \dim H^0(\mathcal{C}, \omega^r) = \text{codim} I_r(\mathcal{C}), \]
we obtain
\[ k[X_{n_0}, \ldots, X_{n_{g-1}}]_r = I_r(\mathcal{C}) \oplus \Lambda_r, \text{ for each } r \geq 2. \]
Let $r\mathcal{S}$ be the set of all sums of $r$ nongaps not bigger than $2g-2$. Oliveira showed, cf. [19] theorem 1.5, that each nongap smaller than or equal to $r(2g-2)$ belongs to $r\mathcal{S}$. Moreover, each sum of $r$ nongaps $\leq 2g-2$ is a nongap $\leq r(2g-2)$. Consequently, $\#r\mathcal{S} = (2r-1)(g-1)$ and therefore
\[ \#r\mathcal{S} = \dim H^0(\mathcal{C}, \omega^r). \]
In particular, for each nongap $s \leq 4g-4$ we list all the partitions $s = a_{si} + b_{si} \in 2\mathcal{S}$, where
\[ a_{si} \leq b_{si} \leq 2g-2 \quad (i = 0, \ldots, \nu_s) \quad \text{and} \quad a_s := a_{s0} < a_{s1} < a_{s2} < \ldots < a_{s\nu_s}. \]
Since $x_{a_{si}}x_{b_{si}} \in H^0(\mathcal{C}, sP)$ and $\{x_{a_{si}}x_{b_{si}}\}$ is the above fixed basis, we can write
\[ x_{a_{si}}x_{b_{si}} = \sum_{n=0}^{s} c_{sin}x_{a_{si}}x_{b_{si}}, \]
for each $i = 0, \ldots, \nu_s$, where the coefficients $c_{si}$ are uniquely determined constants and the summation index only varies through nongaps. In the same way, for each nongap $\sigma \leq 6g-6$ we consider the partitions $\sigma = a_{\sigma j} + b_{\sigma j} + c_{\sigma j} \in 3\mathcal{S}$ where $a_{\sigma j} \leq b_{\sigma j} \leq c_{\sigma j} \leq 2g-2 \quad (j = 0, \ldots, \nu_\sigma)$ with $a_{\sigma} := a_{\sigma 0} < a_{\sigma 1} < \ldots < a_{\sigma \nu_\sigma}$ and $b_{\sigma} := b_{\sigma 0} > b_{\sigma 1} > \ldots > b_{\sigma \nu_\sigma}$. Analogously, we can write
\[ x_{a_{\sigma j}}x_{b_{\sigma j}}x_{c_{\sigma j}} = \sum_{n=0}^{s} d_{\sigma jn}x_{a_{\sigma j}}x_{b_{\sigma j}}x_{c_{\sigma j}}, \]
for each integer $j = 0, \ldots, \nu_\sigma$, where the coefficients $d_{\sigma jn}$ are uniquely determined constants and the summation index only varies through nongaps.

Multiplying the functions $x_{n_0}, \ldots, x_{n_{g-1}}$ by constants we do not change the $P$-hermitian property of the above basis, thus we can normalize the coefficients $c_{si}$ and $d_{\sigma jn}$. Therefore, the $(\binom{g+1}{2}) - (3g-3) = \frac{1}{2}(g-3)(g-2)$ quadratic forms
\[ (2) \quad F_{si} = X_{a_{si}}X_{b_{si}} - X_{a_{si}}X_{b_{si}} - \sum_{n=0}^{s-1} c_{sin}X_{a_{si}}X_{b_{si}}, \]
and the $(\binom{g+2}{3}) - (5g-5)$ cubic forms
\[ (3) \quad G_{\sigma j} = X_{a_{\sigma j}}X_{b_{\sigma j}}X_{c_{\sigma j}} - X_{a_{\sigma j}}X_{b_{\sigma j}}X_{c_{\sigma j}} - \sum_{n=0}^{\sigma-1} d_{\sigma jn}X_{a_{\sigma j}}X_{b_{\sigma j}}X_{c_{\sigma j}}, \]
vanish identically on $\mathcal{C}$. We attach to the variable $X_n$, the weight $n$, to the coefficient $c_{sin}$, the weight $s-n$ and to $d_{\sigma jn}$, the weight $\sigma - n$. Thus the above quadric and cubic forms are also isobaric forms.

In the view of Henriques–Babbage’s theorem we want to assure that the canonical ideal of $\mathcal{C}$ is generated by the above quadratic and cubic forms. We assume that the non-hyperelliptic symmetric semigroup $\mathcal{S}$ is a non-trivial semigroup of genus $g > 1$, which is equivalent to assume that the multiplicity $n_1$ of $\mathcal{S}$ satisfies $2 < n_1 \leq g$. By
a theorem of Oliveira [19, Thm. 1.7], if we consider \( 3 < n_1 < g \), then there is at least one quadratic form, i.e. \( \nu_i \geq 1 \), whenever \( s = n_i + 2g - 2 \) for \( i = 0, \ldots, g - 3 \).

In this case Contiero–Stoehr [7] gave an algorithmic proof that the canonical ideal of a Gorenstein curve \( C \subset \mathbb{P}^{g-1} \) with Weierstrass semigroup \( S \) at the base point is generated by only quadratic relations. If we assume that \( 3 \in S \) then its genus has residue 1 or 0 module 3, hence \( S := (3, g + 1) \). In this case we already know that \( \overline{\mathcal{M}}^{c, 1}_{g, 1} = \mathcal{P}(\mathcal{L}^1_{k[S][k]}). \) If \( S = (4, 5) \), then \( C \) is isomorphic to a plane quintic where the quadric hypersurfaces containing \( C \) is the Veronese surface.

In the excluded case \( S = \mathbb{N}\setminus\{1, 2, \ldots, g - 1, 2g - 1\} \), the curve \( C \) is possibly trigonal, so its canonical ideal can not be generated by only quadratic relations. In the next section we investigate the Weierstrass semigroup of trigonal complete curves and then, we will give an algorithmic proof that the canonical ideal of a complete Gorenstein curve with symmetric Weierstrass semigroup

\[
S := \mathbb{N}\setminus\{1, 2, \ldots, g - 1, 2g - 1\} = \langle 0, g, g + 1, \ldots, 2g - 2 \rangle
\]

at a smooth non-ramified point is generated by quadric and cubics forms.

### 3. Curves with odd subcanonical points

Let \( C \) be a complete integral curve of arithmetic genus \( g \) defined over an algebraically closed field \( k \). A linear system of dimension \( r \) on \( C \) is a set of the form

\[
\mathcal{L} = \mathcal{L}(\mathcal{F}, V) := \{x^{-1}\mathcal{F} \mid x \in V \setminus 0\}
\]

where \( \mathcal{F} \) is a coherent fractional ideal sheaf on \( C \) and \( V \) is a vector subspace of \( H^0(C, \mathcal{F}) \) of dimension \( r + 1 \).

The notion of linear systems on curves presented here is characterized by interchanging bundles by torsion free sheaves of rank 1. This is a meaningful approach since they may possess non-removable base points, see Coppens [9].

The degree of the linear system \( \mathcal{L} \) is the integer \( \deg \mathcal{F} := \chi(\mathcal{F}) - \chi(\mathcal{O}_C) \), where \( \chi \) denotes the Euler characteristic. Note, in particular, that if \( \mathcal{O}_C \subset \mathcal{F} \) then

\[
\deg \mathcal{F} = \sum_{P \in C} \dim(\mathcal{F}_P/\mathcal{O}_C, P).
\]

The notation \( g^d_d \) stands for a linear system of degree \( d \) and dimension \( r \). The linear system is said to be complete if \( V = H^0(C, \mathcal{F}) \), in this case one simply writes \( \mathcal{L} = |\mathcal{F}|. \) According to E. Ballico’s [2] p. 363, Dfn. 2.1 (3)], the gonality of \( C \) is the smallest \( d \) for which there exists a \( g^d_d \) on \( C \), or equivalently, a torsion free sheaf \( \mathcal{F} \) of rank 1 on \( C \) with degree \( d \) and \( h^0(C, \mathcal{F}) \geq 2 \).

The following lemma is a straightforward generalization of a Kim’s result [15, Thm. 2.6] characterizing the Weierstrass semigroup associated to a non-ramification point of a trigonal curve.

**Lemma 3.1.** Let \( C \) be a complete integral trigonal curve of arithmetical genus \( g \geq 5 \) and \( P \in C \) a Weierstrass non-ramification point. The Weierstrass semigroup \( S \) of \( C \) at \( P \) is of the form

\[
\{0, m, m + 1, m + 2, \ldots, m + (s - g), s + 2, s + 3, s + 4, \ldots\},
\]

for some \( s \) and \( m \) such that \( g \geq m \geq \left\lfloor \frac{s + 1}{2} \right\rfloor + 1 \). In particular, in the symmetric case we get the odd semigroup

\[
S = \{0, g, g + 1, \ldots, 2g - 2, 2g, 2g + 1, 2g + 2, \ldots\}.
\]
Proof. Let $\ell_g$ be the Frobenius number of the Weierstrass semigroup $\mathcal{S}$ associated to $P \in \mathcal{C}$. Equivalently, the integer $s := \ell_g - 1$ is the largest such that the divisor $D_0 = sP$ is special. Since $P$ is a Weierstrass point, it is immediate that $g \leq \ell_g - 1 \leq 2g - 2$. By the maximality of $s$

$$\dim \lvert \mathcal{O}(D_0) \rvert = s - g + 1.$$ 

Since $D_0$ is a special divisor, let be

$$\omega_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}(D_0 + P_1 + P_2 + \ldots + P_{2g-2-s})$$

the dualizing sheaf of $\mathcal{C}$ where $P_i \in \mathcal{C}, P_i \neq P$, with $i = 1, \ldots, 2g - 2 - s$. As $P$ is not a ramification point, the first nongap $m$ is greater than 3, and so $\lvert mP \rvert$ is not compounded of $g_3^1$. By considering the divisor

$$D := (s - m)P + P_1 + P_2 + \ldots + P_{2g-2-s}$$

we see that $\lvert D \rvert$ is compounded of $g_3^1$ because $\omega_{\mathcal{C}} = \mathcal{O}_{\mathcal{C}}(mP) \otimes \mathcal{O}_{\mathcal{C}}(D)$.

Applying the Riemann-Roch theorem, $\dim \lvert D \rvert = g - m$, hence we can write $\lvert D \rvert = (g - m)g_3^1 + B$, where $B$ is the base locus of $\lvert D \rvert$. For each element $R$ of $g_3^1$ with $R \geq P$, we have $R = P + Q_1 + Q_2$, with $P \neq Q_1$ and $P \neq Q_2$ because $P$ is not a ramification point of $\mathcal{C}$, thus

$$D = (g - m)(P + Q_1 + Q_2) + B = (s - m)P + P_1 + P_2 + \ldots + P_{2g-2-s},$$

and by the maximality of $s$

$$P_1 + P_2 + \ldots + P_{2g-2-s} \geq (g - m)Q_1 + (g - m)Q_2,$$

implying $2(g - m) \leq 2g - 2 - s$. Therefore, $m \geq \lceil \frac{s+1}{2} \rceil + 1$.

On the other hand,

$$B \geq (s - g)P,$$

which means that $(s - g)P$ is contained in the base locus of $\lvert D \rvert$. Consequently each divisor $iP$ is not in the base locus of $\lvert mP + iP \rvert$, $i = 0, \ldots, s - g$, and thus $m, m + 1, \ldots, m + s - g$ are nongaps of $\mathcal{S}$.

Now by definition of $s$ and by Riemann-Roch theorem, $\dim \lvert (s + 1)P \rvert = \dim \lvert sP \rvert$, which implies that $s + 1$ is a gap of $\mathcal{S}$. Since the divisor $(r - 1)P$ is nonspecial for each integer $r \geq s + 2$, it follows that

$$\dim \lvert rP \rvert = r - g = \dim \lvert (r - 1)P \rvert + 1,$$

so each $r \geq s + 2$ is a nongap. In this way the set

$$S = \{0, m, m + 1, \ldots, m + (s - g), s + 2, \ldots\}$$

is contained in $\mathcal{S}$ and the cardinality of $\mathbb{N} - S$ is $g$. \hfill \Box

Let us consider the odd numerical semigroup $\mathcal{S} := \langle 0, g, g+1, \ldots, 2g-2 \rangle$ of genus $g \geq 5$. We now fix $\frac{1}{2}(g-3)(g-2)$ initial quadratic forms like in \((2)\)

$$F_{si}^{(0)} := X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s}$$

and $(g+2) - (5g - 5)$ initial cubic forms like in \((3)\)

$$G_{cj}^{(0)} := X_{a_{cj}}X_{b_{cj}}X_{c_{cj}} - X_{a_s}X_{b_s}X_{c_s}.$$ 

It is clear that a considerable amount of cubic forms are just multiplies of quadratic ones, the next result explicitly find them.
Proposition 3.2. Let $\mathcal{S} := \langle 0, g, g + 1, \ldots, 2g - 2 \rangle$. There are exactly $\varphi = \binom{g+3}{3} - (5g - 5) - \eta$, with

$$
\eta = (g - 3)(g - 2) + (g - 2) \left\lfloor \frac{g - 2}{2} \right\rfloor + \left\lfloor \frac{g - 3}{2} \right\rfloor + \sum_{j=1}^{g-4} \left\lfloor \frac{g - 2 - j}{2} \right\rfloor
$$

initial cubic forms which are not multiples of the quadratic ones.

Proof. Since the fixed basis for $\Lambda_2$ is $\{X_{02}^2, X_0X_g, X_gX_g, X_gX_{g+1}X_{2g-2}\}$ with $i = 0, \ldots, g - 2$ and $j = 1, \ldots, g - 2$, the initial quadratic forms are

$$
F_{si}^{(0)} = X_{a,i}X_{b,i} - X_gX_{g+i} \quad \text{and} \quad F_{sl}^{(0)} = X_{a,i}X_{b,i} - X_{g+j}X_{g+2},
$$

where the 2-monomials nonbasis elements of $\Lambda_2$ are the products $X_{g+i}X_{g+j}$ where $1 \leq i \leq j = 1, \ldots, g - 3$. While the fixed basis for $\Delta_3$ is

$$
\{X_0^2X_i, X_0X_aX_b, X_aX_bX_{2g-2}, X_gX_{2g-3}\},
$$

with $i = 0, g, g + 1, \ldots, 2g - 2$ and $\{X_aX_b\}$ the above fixed basis for $\Delta_2$. Set $F := F_{si}^{(0)}$ for a initial quadratic form. It is clear that the $(g - 3)(g - 2)$ products $X_0F$ and $X_{2g-2}F$ are cubic forms for every $F$. Since the monomials $X_{g+k}X_{g+i}X_{g+j}$ are in $\Lambda_3$ just for $i = g - 2, k = 0, \ldots, g - 3$ and for $i = g - 3, k = 0$. Hence we get the following $(g - 2) \left\lfloor \frac{g - 2}{2} \right\rfloor + \left\lfloor \frac{g - 3}{2} \right\rfloor$ cubic forms

$$
X_{g+k}(X_{a,i}X_{b,i} - X_gX_{2g-2}), \quad k = 0, \ldots, g - 3
$$

and

$$
X_g(X_{a,i}X_{b,i} - X_gX_{2g-3}).
$$

In the remaining case, $X_{g+k}X_{g+j}X_{2g-2} \in \Lambda_3$ just for $k = 0, j = 1, \ldots, g - 2$. So we get the following $\sum_{j=1}^{g-4} \left\lfloor \frac{g - 2 - j}{2} \right\rfloor$ initial cubic forms

$$
X_g(X_{a,i}X_{b,i} - X_{g+j}X_{2g-2}), \quad j = 1, \ldots, g - 4.
$$

It is straightforward that the quadratic $F_{si}^{(0)}$ and cubic forms $G_{a\sigma}^{(0)}$ vanish identically on the monomial curve $C^{(0)}$. The next lemma show that they generate the ideal of $C^{(0)}$.

Lemma 3.3. The canonical ideal $I(C^{(0)})$ is generated by the $\frac{1}{2}(g - 2)(g - 3)$ quadratic forms $F_{si}^{(0)}$ and by the $\varphi$ cubic forms $G_{a\sigma}^{(0)}$.

Proof. Since the $I(C^{(0)})$ is generated by homogeneous and isobaric forms, all we have to do is to show that for a homogeneous and isobaric form belongs to $I(C^{(0)})$ if and only if belongs to the ideal $\mathcal{J}$ generated by the binomials $F_{si}^{(0)}$ and $G_{a\sigma}^{(0)}$. It is just obvious that $\mathcal{J} \subseteq I(C^{(0)})$. For the opposite inclusion we order the monomials $\prod_{k=0}^{g-1} X_{b_k}$ according to the lexicographic ordering of the vectors

$$
\left( \sum_{i_k} i_k, \sum_{n_k} n_k i_k, -i_0, -i_{g-1}, \ldots, -i_1 \right).
$$
In this way, the binomials \( F_{s_i}^{(0)} \) and \( G_{\sigma j}^{(0)} \) form a Groebner basis for \( \mathcal{J} \). Now, for each homogeneous form \( F \) of degree \( r \) which is also isobaric of weight \( \omega \) we divide it by the Groebner basis getting a decomposition

\[
F = \sum H_{s_i}F_{s_i}^{(0)} + T_{\sigma j}G_{\sigma j}^{(0)} + R
\]

where \( R \in \Lambda_r \) and \( H_{s_i} \) and \( T_{\sigma j} \) are homogeneous of degree \( r-2 \) and \( r-3 \) respectively, and weight \( \omega - s \) and \( \omega - \sigma \), respectively. The remainder \( R \) is the only monomial in \( \Lambda_r \) of weight \( \omega \) whose coefficient is equal to the sum of the coefficients of \( F \). Since \( F \in I(C^{(0)}) \) the sum of its coefficients is equal to zero, then \( R = 0 \). \( \square \)

A different proof of the above lemma can be found in \([14, \text{Thm. 1.1}]\) by noting that the symmetric semigroup \( S = \{0, g, g+1, \ldots, 2g-2\} \) is generated by a generalized arithmetic sequence. So the ideal \( I(C^{(0)}) \) of the monomial curve \( C^{(0)} \) is also generated by the \( 2 \times 2 \) minors of suitable two matrices. It can be seen immediately that the ideal given by this \( 2 \times 2 \) minors is equal to the ideal generated by the binomials \( F_{s_i}^{(0)} \) and \( G_{\sigma j}^{(0)} \).

The following lemma is a generalization of result in \([7, \text{Lemma 2.3}]\), where due to the assumptions the authors just deal with the first syzygies of quadratic forms. Here we also deal with syzygies of cubic forms, which will induce nonlinear syzygies, see the equations \([7, \text{pg. 14, and 8}]\), pg. 17.

**Syzygy Lemma 3.4.** For each of the \( \frac{1}{2}(g-3)(g-4) \) quadratic forms \( F_{s'i'}^{(0)} \) different from \( F_{n_i+2g-2,1}^{(0)} (i = 1, \ldots, g-3) \) there is a syzygy of the form

\[
X_{2g-2}F_{s'i'}^{(0)} + \sum_{n=1}^{(s'i')} X_n F_{s}^{(0)} = 0
\]

and for each cubic forms \( G_{\sigma j}^{(0)} \) different from \( G_{4g-4,1}^{(0)} \), there is a syzygy of the form

\[
X_{2g-2}G_{\sigma j}^{(0)} + \sum_{q=1}^{(\sigma j') \neq (\sigma j)} X_q G_{\sigma j}^{(0)} = 0,
\]

where the coefficients \( t_{n_{si}}^{(s'i')} \), \( t_{q_{\sigma j}}^{(\sigma j')} \) are integers equal to 1, -1 or 0, and where the sum is taken over the nongaps \( n, q < 2g-2 \), the double indexes \( si \) with \( s + n = 2g-2 + s' \) and \( \sigma j \) with \( q + \sigma = 2g-2 + \sigma' \).

**Proof.** Given a quadratic form \( F = F_{s'i'}^{(0)} \) or \( F = -F_{s'i'}^{(0)} \), we can write

\[
F = X_m X_n - X_q X_r,
\]

where \( m, n, q, r \) are nongaps satisfying \( m + n = q + r \) and \( q < m \leq n < r < 2g-2 \). If \( r + 1 \) is a gap then, by symmetry, \( k := 2g-2 - r + n \) is a nongap and we find the syzygy

\[
X_{2g-2}(X_m X_n - X_q X_r) + X_r(X_q X_{2g-2} - X_m X_k) - X_m(X_n X_{2g-2} - X_r X_k) = 0,
\]

The binomials in the brackets can be written as \( F_{s_j}^{(0)} - F_{s_j}^{(0)} \), \( F_{s_j}^{(0)} \) or \( -F_{s_j}^{(0)} \). Analogously if \( m + 1 \) is a gap then we take the nongap \( k := 2g-2 - m + r \) and we obtain a syzygy as above. Now we can assume that \( r + 1 \) and \( m + 1 \) are nongaps, hence we have the syzygy

\[
X_{2g-2}(X_m X_n - X_q X_r) + X_q(X_{2g-2} X_r - X_{2g-3} X_{r+1}) -
X_{2g-3}(X_{m+1} X_n - X_q X_{r+1}) - X_n(X_m X_{2g-2} - X_{2g-3} X_{m+1}) = 0.
\]
For a cubic form, if we put $G = G^{(0)}_{\sigma_j}$ or $G = -G^{(0)}_{\sigma_j}$ then we can write
\[ G = X_mX_nX_p - X_qX_rX_t, \]
where $m, n, p, q, r, s$ are nongaps satisfying $m + n + p = q + r + t$ and $q < m \leq n \leq r \leq p < t \leq 2g - 2$.

If $p + 1$ is a gap then, by symmetry, the integer $k := 2g - 2 - p + q$ is a nongap smaller than $2g - 2$, hence we have the syzygy
\[ X_{2g-2}(X_{m}X_{n}X_{p} - X_{q}X_{r}X_{t}) + X_{r}(X_{2g-2}X_{t}X_{q} - X_{t}X_{p}X_{k}) - X_{p}(X_{2g-2}X_{m}X_{n} - X_{r}X_{t}X_{k}) = 0, \]
where the binomials in the brackets can be written as $G^{(0)}_{\sigma_j} - G^{(0)}_{\sigma_i}$, $G^{(0)}_{\sigma_j}$ or $-G^{(0)}_{\sigma_i}$. Analogously, if $r + 1$ is a gap then $k := 2g - 2 - r + p$ is a nongap, and therefore we obtain the syzygy
\[ X_{2g-2}(X_{m}X_{n}X_{p} - X_{q}X_{r}X_{t}) + X_{m}(X_{k}X_{r}X_{n} - X_{2g-2}X_{p}X_{n}) - X_{r}(X_{k}X_{m}X_{n} - X_{2g-2}X_{r}X_{q}) = 0. \]

Now we can assume that $p + 1$ and $r + 1$ are the nongaps. We just take the syzygy
\[ X_{2g-2}(X_{m}X_{n}X_{p} - X_{q}X_{r}X_{t}) + X_{2g-3}(X_{r+1}X_{q}X_{t} - X_{p+1}X_{n}X_{m}) - X_{m}(X_{p}X_{2g-2}X_{n} - X_{p+1}X_{2g-3}X_{n}) - X_{q}(X_{2g-3}X_{r+1}X_{t} - X_{2g-2}X_{r}X_{t}) = 0. \]

**Remark 3.5.** The $\eta$ syzygies corresponding to the cubic forms which are multiples of the quadratics are trivial, therefore we just to consider syzygies for the $\varphi - 1$ cubic forms, however, these $\varphi - 1$ syzygies are not necessarily linear.

**Lemma 3.6.** Let $I$ be the ideal generated by the $\frac{1}{2}(g - 2)(g - 3)$ quadratic forms $F_{si}$ and by the $\varphi$ cubic forms $G_{\sigma_j}$. Then,
\[ k[X_{n_0}, \ldots, X_{n_{g-1}}]_r = I_r + \Lambda_r \text{ for each } r \geq 2. \]

**Proof.** Let $F$ be a homogeneous polynomial of degree $r$ and weight $w$. Let $S$ be its quasi-homogeneous component of weight $w$ and $R$ the unique monomial in $\Lambda_r$ of weight $w$ whose coefficient is the sum of the coefficients of $S$. Thus, $S - R \in I(\mathcal{C}^{(0)})$ and by Lemma 3.4 we can write the expression
\[ S - R = \sum_{s_i} S_{s_i}F_{s_i}^{(0)} + \sum_{\sigma_j} H_{\sigma_j}G_{\sigma_j}^{(0)}. \]
Replacing each polynomial $S_{s_i}$ and $H_{\sigma_j}$ with its homogeneous component of degree $r - 2$ and $r - 3$, respectively, we can take $S_{s_i}$ and $H_{\sigma_j}$ homogeneous of degree $r - 2$ and $r - 3$, respectively. Likewise, we can assume that $S_{s_i}$ and $H_{\sigma_j}$ are quasi-homogeneous of weight $w - s$ and $w - \sigma$, respectively. Then the polynomial
\[ F - R - \sum_{s_i} S_{s_i}F_{s_i}^{(0)} - \sum_{\sigma_j} H_{\sigma_j}G_{\sigma_j}^{(0)} \]
is homogeneous of degree $r$ and weight smaller than $w$. Now, the proof follows by induction on $w$. \[ \square \]

**Remark 3.7.** We see that if the curve $\mathcal{C}$ is not trigonal, then the last summand in \[ (6) \] does not appear because the ideal $I(\mathcal{C}^{(0)})$ is generated by the $\frac{1}{2}(g - 2)(g - 3)$ quadratic forms $F_{s_i}^{(0)}$. 


Let us now invert the considerations of the previous section. Instead of take a pointed canonical Gorenstein curve whose Weierstrass semigroup at the marked point is $S := (g, g+1, \ldots, 2g-2)$, we consider the semigroup $S$ and the associated monomial curve $C^{(0)}$. We want to deform it in order to get a Gorenstein curve with a marked point whose Weierstrass semigroup is also $S$. By Lemma 3.3 the ideal of the monomial curve $C^{(0)}$ is generated by the $(g-2)(g-3)$ quadratic forms $F_{si}$ and by the $\varphi$ cubic forms $G_{s}^{(0)}$. So, let us consider a pre-deformation of the ideal of $C^{(0)}$ which is

$$F_{si} = X_{a_{si}}X_{b_{si}} - X_{a_{s}}X_{b_{s}} - \sum_{n=0}^{s-1} c_{sin}X_{a_{n}}X_{b_{n}},$$

and

$$G_{s} = X_{a_{s_{j}}}X_{b_{s_{j}}}X_{c_{s_{j}}} - X_{a_{s}}X_{b_{s}}X_{c_{s}} - \sum_{n=0}^{\sigma_{s_{j}}-1} d_{s_{j}n}X_{a_{n_{j}}}X_{b_{n_{j}}}X_{c_{n}},$$

where the coefficients $c_{sin}$ and $d_{s_{j}n}$ belongs to the ground field $k$. It is clear that we are looking for conditions on this coefficients such that this pre-deformation is a deformation: a curve of degree $2g-2$ and genus $g$ with a marked point whose Weierstrass semigroup is $S$. The main idea is to apply the Syzygy Lemma and erase the superscript zeros of the quadratic and cubic forms and then, by means of (i), get the conditions on the coefficients.

Replacing the left-hand side of the equation (4) of the Syzygy Lemma the binomials $F_{s_{i_{i}}}^{(0)}$ and $F_{s_{i}}^{(0)}$ with the quadratic forms $F_{s_{i}}^{(0)}$ and $F_{si}$ we obtain for each of the $rac{1}{2}(g-3)(g-4)$ double indexes $s_{i}^{i_{i}}$ a linear combination of cubic monomials of weight less than $s_{j}^{i_{i}} + 2g - 2$, which by Lemma 3.3 admits the decomposition

$$X_{2g-2}F_{s_{i}}^{(0)} + \sum_{n=si}^{\epsilon_{\eta_{ni}}} X_{n}F_{si} = \sum_{n=si}^{\eta_{ni}} X_{n}F_{si} + R_{s_{i}},$$

where the sum on the right-hand side is taken over all the nongaps $n \leq 2g - 2$, the double indexes $si$ with $n + s < s_{j}^{i_{i}} + 2g - 2$, the coefficients $\epsilon_{\eta_{ni}}$ are constants and where $R_{s_{i}}$ is a linear combination of cubic monomials of pairwise different weights less than $s_{j}^{i_{i}} + 2g - 2$.

Repeating the above procedure for the equation (4) on the Syzygy Lemma, we obtain a decomposition

$$X_{2g-2}G_{s_{j}} = \sum_{\rho_{\eta_{qr}}q_{\eta_{qr}}} X_{q}G_{s_{j}} = \sum_{m=q_{\eta_{qr}}}^{\mu_{\eta_{qr}}} X_{m}^{(\sigma_{j})}X_{q}F_{s_{j}} + \sum_{q_{\eta_{qr}}}^{\nu_{\eta_{qr}}} X_{q}^{(\sigma_{j})}X_{q}G_{s_{j}} + R_{s_{j}},$$

where the sum on the right-hand side is taken over the nongaps $m, q \leq 2g - 2$, the indexes $mq\sigma$ and $q\sigma$ with $m + q + \sigma < 2g - 2 + \sigma'$ and $q + \sigma < 2g - 2 + \sigma'$, the coefficients $\mu_{\eta_{qr}}$ and $\nu_{\eta_{qr}}$ are constants and where $R_{s_{j}}$ is a linear combination of quartic monomials of pairwise different weights less than $2g - 2 + \sigma'$.

For each nongap $m < s_{j}^{i_{i}} + 2g + 2$ let $q_{s_{i}^{i_{i}}m}$ be the unique coefficient of $R_{s_{i}^{i_{i}}}$ (resp. $R_{s_{j}^{i_{i}}}^{(0)}$) of weight $m$ (resp. $r$). We do not lost information about the coefficients $R_{s_{i}^{i_{i}}}$ and $R_{s_{j}^{i_{i}}}^{(0)}$, replacing the variables $X_{n}$ by powers $t^{n}$ of an indeterminate $t$. Hence it is convenient to consider the polynomials

$$R_{s_{i}^{i_{i}}}(t^{n_{0}}, \ldots, t^{n_{s_{i}^{i_{i}}}}) = \sum_{m=0}^{s_{j}^{i_{i}} + 2g - 2} q_{s_{i}^{i_{i}}m}t^{m}$$
and

\[ R_{s',j'}(t^{n_0}, \ldots, t^{n_{g-1}}) = \sum_{r=0}^{\sigma' + 2g - 2} \vartheta_{s',j'} t^r. \]

We can assume that the coefficients \( \vartheta_{s',j'} \) are quasi-homogeneous polynomial expressions of weight \( s' + 2g - 2 - m \) in the constants \( c_{s'j'} \) while the coefficients \( \vartheta_{s,j'} \) are quasi-homogeneous polynomial expressions of weight \( \sigma' + 2g - 2 - r \) in the constants \( d_{s'j} \).

**Theorem 3.8.** Let \( S \) be a nonhyperelliptic and non-ordinary numerical symmetric semigroup of genus \( g \). The respective \( \frac{1}{2}(g - 2)(g - 3) \) quadratic forms, \( F_{s'i} = F_{s'i}^{(0)} - \sum_{n=0}^{s'-1} c_{s'a} x_{a} x_{b} \) and \( G_{s'j} = G_{s'j}^{(0)} - \sum_{n=0}^{s'} d_{s'j} x_{a} x_{b} x_{c} \) cut out a canonical integral Gorenstein curve on \( \mathbb{P}^{g-1} \) if and only if the coefficients \( c_{s'j}, d_{s'j} \) satisfy the quasi-homogeneous equations \( \vartheta_{s'i} = 0 \) and \( \vartheta_{s,j'} = 0 \). In this case, the point \( P = (0 : \ldots : 0 : 1) \) is a smooth point of the canonical curve with Weierstrass semigroup \( S \).

**Proof.** We first assume that the \( \frac{1}{2}(g - 2)(g - 3) \) quadratic forms \( F_{s'i} \) and the \( \varphi \) cubic forms \( G_{s'j} \) cut out a canonical curve \( C \subset \mathbb{P}^{g-1} \). Since each \( R_{s'i} \) and \( R_{s'j'} \) belongs to the ideal \( I \), follows that \( R_{s'i}(x_{n_0}, \ldots, x_{n_{g-1}}) = R_{s'j'}(x_{n_0}, \ldots, x_{n_{g-1}}) = 0 \) for each pair of index \( s'i \) and \( s'j' \). We can write

\[ R_{s'i}(x_{n_0}, \ldots, x_{n_{g-1}}) = \sum_{m=0}^{s' + 2g - 2} \vartheta_{s'i} m z_{s'i} m \]

and

\[ R_{s'j'}(x_{n_0}, \ldots, x_{n_{g-1}}) = \sum_{r=0}^{\sigma' + 2g - 2} \vartheta_{s'j'} r z_{s'j'} r, \]

where the \( z_{s'i} m \), \( z_{s'j'} r \) are monomial expressions of weights \( m \) and \( r \) respectively in the projective coordinates functions \( x_{n_0}, \ldots, x_{n_{g-1}} \), and hence \( z_{s'i} m \) has pole divisor \( mP \) while \( z_{s'j'} r \) has pole divisor \( rP \). Then we conclude that \( \vartheta_{s'i} m = \vartheta_{s'j'} r = 0 \).

On the opposite, let us assume that the coefficients \( c_{s'j}, d_{s'j} \) satisfy the equations \( \vartheta_{s'i} m = 0 \) and \( \vartheta_{s'} j' r = 0 \). Since the \( g - 3 \) quadric hypersurfaces \( V(F_{n_i+2g-2,1}) \subset \mathbb{P}^{g-1}(i = 1, \ldots, g - 3) \) and the cubic hypersurface \( V(G_{4g-4,1}) \) intersect transversally at \( P \), in an open neighborhood of \( P \) their intersection has an unique irreducible component which contains \( P \), and so this component is a projective integral algebraic curve, say \( C \), which is smooth at \( P \) and whose tangent line is the intersection of their tangent hyperplanes \( V(X_i), i = 0, \ldots, g - 3 \).

Let \( y_{n_0}, \ldots, y_{n_{g-1}} \) be the projective coordinate functions of \( C \) and we look for the affine open \( y_{n_{g-1}} = 1 \). Since the local coordinate ring of \( C \) at \( P \) is a discrete valuation ring and \( n_{g-1} - n_{g-2} = l_2 - l_1 = 1 \), we have that \( t := y_{n_{g-2}} \) is a local parameter of \( C \) at \( P \), and \( y_{n_0}, \ldots, y_{n_{g-3}} \) are the power series in \( t \) of order greater than 1. More precisely, comparing coefficients in the \( g - 3 \) equations

\[ F_{n_i+2g-2}(y_{n_0}, \ldots, y_{n_{g-2}}, y_{n_{g-1}}) = 0, \quad i = 1, \ldots, g - 3 \]

and

\[ G_{4g-4,1}(y_{n_0}, \ldots, y_{n_{g-2}}, y_{n_{g-1}}) = 0 \]
one sees that

\[ y_{ni} = t^{n_i-1} + (\text{sum of higher orders terms}) \]

\[ = t^{g-1} + (\text{sum of higher orders terms}), \]

for each integer \( i = 0, \ldots, g - 1 \). This means that the \( g \) integers \( l_i - 1 \) \((i = 1, \ldots, g)\) are the contact orders of the curve \( C \subset \mathbb{P}^{g-1} \) with the hyperplanes at \( P \). In particular, the curve \( C \) is not contained in any hyperplane.

By assumption, \( \varrho_{s'i'm} = 0 \) and \( \vartheta_{\sigma'j'r} = 0 \) for each pair of double indexes \( s'i' \) and \( \sigma'j' \), respectively. Hence, we obtain the syzygies

\[ X_{2g-2}F_{s'i'} + \sum_{nsi} \epsilon_{nssi} X_n F_{si} - \sum_{nsi} \eta_{nssi} X_n F_{si} = 0 \]

and

\[ X_{2g-2}G_{\sigma'j'} + \sum_{\rho \sigma j} \rho_{\rho \sigma j} X_q G_{\sigma j} - \sum_{mq \sigma j} \mu_{mq \sigma j} X_m X_q G_{\sigma j} - \sum_{qs \sigma j} \nu_{qs \sigma j} X_q G_{\sigma j} = 0. \]

Replacing the variables \( X_{n_0}, \ldots, X_{n_{g-1}} \) by the projective coordinates functions \( y_{n_0}, \ldots, y_{n_{g-1}} \) we get two systems: a system with \( \frac{1}{2}(g-3)(g-4) \) linear homogeneous equations in the \( \frac{1}{2}(g-3)(g-4) \) functions \( F_{s'i'}(y_{n_0}, \ldots, y_{n_{g-1}}) \) with the coefficients in the domain \( k[[t]] \) of formal power series; the second system is composed by \( \varrho - 1 \) linear homogeneous equations in the \( \varrho - 1 \) functions \( G_{\sigma'j'}(y_{n_0}, \ldots, y_{n_{g-1}}) \) with the coefficients in the domain \( k[[t]] \) of formal power series. Since the triple indexes \( nsi \) of the coefficients \( \epsilon_{nssi}^{(s'i')} \), respectively, \( \eta_{nssi}^{(s'i')} \), satisfy the inequalities \( n < 2g - 2 \) and \( n + s = 2g - 2 + s' \), respectively, \( n \leq 2g - 2 \) and \( n + s < 2g - 2 + s' \), the diagonal entries of the matrix of the system have constant terms 1, while the remaining entries have positive orders. Therefore, the matrix is invertible, and so the equation \( F_{si}(y_{n_0}, \ldots, y_{n_{g-1}}) = 0 \) holds for each double index \( si \). In the second system, the indexes \( qs \sigma j, mq \sigma j \) and \( n \sigma j \) of the coefficients \( \rho_{qs \sigma j}^{(s'i')} \), \( \mu_{mq \sigma j}^{(s'i')} \) and \( \nu_{nsi \sigma j}^{(s'i')} \), respectively, are such that satisfy the inequalities \( q < 2g - 2 \) and \( q + \sigma = 2g - 2 + s' \), respectively, \( m, q < 2g - 2 \) and \( m + q + \sigma < 2g - 2 + s' \). So the diagonal entries of the matrix of the system have constant terms 1, while the remaining entries have positive orders, hence the matrix is also invertible. This means that the equation \( G_{\sigma j}(y_{n_0}, \ldots, y_{n_{g-1}}) = 0 \) holds for each double index \( \sigma j \). Therefore, we shown that \( I \subset I(C) \), where \( I \) is the ideal generated by the \( \frac{1}{2}(g-2)(g-3) \) quadratic forms \( F_{si} \) and by the \( \varrho \) cubic forms \( G_{\sigma j} \).

By virtue of Lemma 3.6 codim \( I_r \leq \dim \Lambda_r \) for each \( r \geq 2 \). Since \( I_r(C) \cap \Lambda_r = 0 \), we deduce \( \dim \Lambda_r \leq \text{codim } I_r(C) \) and we obtain

\[ \text{codim } I_r(C) = \text{codim } I_r = \dim \Lambda_r = (2g - 2)r + 1 - g. \]

Thus \( I(C) = I \) and the curve \( C \subset \mathbb{P}^{g-1} \) has Hilbert polynomial \( (2g - 2)r + 1 - g \), hence \( C \) has degree \( 2g - 2 \) and arithmetic genus \( g \).

Intersecting the curve \( C \) with the hyperplane \( V(X_{2g-2}) \) we obtain the divisor \( D := (2g - 2)P \) of degree \( 2g - 2 \), whose complete linear system \( |D| \) has dimension at least \( g - 1 \), and so by Riemann–Roch theorem for complete integral curves the Cartier divisor \( D \) is canonical, and \( C \) is a canonical Gorenstein curve. \( \square \)
ON THE LOCUS OF CURVES WITH AN ODD SUBCANONICAL MARKED POINT

Note that the fixed $P$-hermitian basis $x_{n_0}, x_{n_1}, \ldots, x_{n_g-1}$ of $H^0(C, (2g - 2)P)$ is uniquely determined up to a linear transformation $x_{n_i} \mapsto \sum_{j=i}^{g-1} c_{ij} x_{n_j}$, with $(c_{ij}) \in \text{GL}_g(k)$ an upper triangular matrix whose diagonal entries are of the form $c_{ii} = c^{n_i}, i = 0, \ldots, g - 1$, for some non-zero constant $c$, due the normalizations $c_{s_{i0}} = 1$. We assume that the characteristic of the field of constants $k$ is zero (or a prime not dividing any of the differences $m - n$ with $m, n$ nongaps such that $m < n \leq 2g - 2$).

If the symmetric semigroup is non-odd we can normalize $\frac{1}{2}g(g - 1)$ coefficients $c_{s_{in}}$ of the quadratic forms to be zero, for each $i = 1, \ldots, g - 1$ we just transform

$$X_{n_i} \mapsto X_{n_i} + \sum_{j=1}^{i} c_{n_i,n_i-j} X_{n_i-j}$$

and proceed by induction on the weight of the coefficients, as in [7, pg. 587]. In the odd case, we can normalize $g - 3$ coefficients of the cubic form $G_{4g-4.1}$ by transforming

$$X_{2g-4} \mapsto X_{2g-4} + \sum_{i=1}^{g-3} d_{2g-4,n_{g-3-i}} X_{n_{g-3-i}},$$

and by transforming

$$X_{n_i} \mapsto X_{n_i} + \sum_{j=1}^{i} c_{n_i,n_i-j} X_{n_i-j}$$

with $n_i \neq n_{g-3} = 2g - 4$ we can normalize the remaining $\frac{1}{2}g(g - 1) - (g - 3)$ coefficients of the quadratic forms $F_{n_i+2g-2.1}$.

Due all the normalizations the only freedom left to us is to transform $x_{n_i} \mapsto c^{n_i} x_{n_i}, i = 0, \ldots, g - 1$ for some non-zero constant $c \in k$. Therefore, we have showed:

**Theorem 3.9.** Let $S$ be a nonhyperelliptic and non-ordinary symmetric semigroup of genus $g \geq 5$ The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $S$ correspond bijectively to the orbits of the $G_m(k)$-action

$$(c, \ldots, c_{s_{in}}, \ldots) \mapsto (\ldots, c^{n-n}c_{s_{in}}, \ldots)$$
on the affine quasi-cone of the vectors whose coordinates are the coefficients $c_{s_{in}}, d_{s_{jhn}}$ of the normalized quadratic and cubic forms $F_{sl}$ and $G_{s_{j}}$ satisfying the quasi-homogeneous equations $\partial_{s_{l}^iv_{nm}} = \partial_{s_{l}^iv_{ir}} = 0$.

The dimension of the moduli spaces $\mathcal{M}_{g,1}^S$ for any $S$ is known for a few special cases. A great lower bound was obtained by Pflueger in [20], where the effective weight is an upper bound for codimension of $\mathcal{M}_{g,1}^S$ in $\mathcal{M}_{g,1}$. On the other hand, an upper bound follows from a formula obtained by Deligne [10]. Both bounds are sharp but there are examples where the strict inequalities hold, see [21] and [7]. In the case of odd symmetric semigroups $S = \langle g, g + 1, \ldots, 2g - 2 \rangle$ Rim–Vitulli [24] showed that $S$ is negatively graduated, hence Pflueger’s lower bound and Deligne’s upper bound are equal to $2g - 1 = \dim \mathcal{M}_{g,1}^S = \dim \mathcal{M}_{g,1}^S$. 

4. Odd numerical semigroups of genus at most six

We start this section with the following observation on the rationality of $\overline{M^g_{5,1}}$ for $S$ symmetric and generated by less than five elements, which was also noted in [7]. If the symmetric semigroup $S$ is generated by 4 elements, using Pinkham’s equivariant deformation theory [22], Buchsbaum-Eisenbud’s structure theorem for Gorenstein ideals of codimension 3 (see [3, p. 466]), one can deduce that the affine monomial curve $C^{(0)}$ can be negatively smoothed without any obstructions (see [4, 25, 26 Satz 7.1]), hence

$$\overline{M^g_{5,1}} = \mathbb{P}(T_{k[5]}[k]).$$

Although the above observation assures that $\overline{M^g_{5,1}} = \mathbb{P}^9$ for $S := \langle 5, 6, 7, 8 \rangle$, we believe that it is relevant to illustrate our techniques in an example not so involved with large computations.

4.1. Odd of genus five. Let $C^{(0)}$ be the canonical monomial Gorenstein curve of genus 5 associated to the odd symmetric semigroup of genus also 5. Up to change of coordinates can we write:

$$C^{(0)} := \{(a^8 : a^3b^5 : a^2b^6 : a^1b^7 : b^8) \mid (a : b) \in \mathbb{P}^1 \} \subseteq \mathbb{P}^4.$$

The symmetric Weierstrass semigroup of the smooth point $P = (0 : 0 : 0 : 0 : 1)$ is $S := \langle 5, 6, 7, 8 \rangle$. Following Lemma [3,3] the ideal of $C^{(0)}$ can be generated by the following seven isobaric and homogeneous forms

$$F^{(0)}_{12} = X_6^2 - X_5X_7, \quad F^{(0)}_{13} = X_6X_7 - X_5X_8,$$

$$F^{(0)}_{14} = X_6^2 - X_6X_8, \quad G^{(0)}_{15} = X_5^3 - X_6X_7X_8,$$

$$G^{(0)}_{16} = X_5^2X_6 - X_6X_7^2, \quad G^{(0)}_{17} = X_6^3 - X_5X_7X_8,$$

$$G^{(0)}_{21} = X_7^3 - X_5X_8^2.$$ 

For each nongap $n \in S$ we take the rational function $x_n$ with pole divisor $nP$. Writing each one of the seven rational functions $x_6^2, x_6x_7, x_6^2, x_5^3, x_5^2x_6, x_5^2$ and $x_7^3$ as linear combination of the basis elements of the vector spaces $H^0(C, 2g - 2)$ and $H^0(C, 3(2g - 2))$, respectively, we obtain in the variables $X_0, X_5, X_6, X_7, X_8$, the polynomials

$$F_i = F^{(0)}_i - \sum_{j=1}^i c_{ij}Z_{i-j}, \quad (i = 12, 13, 14),$$

and

$$G_i = G^{(0)}_i - \sum_{j=1}^i d_{ij}Z_{i-j}, \quad (i = 15, 16, 18, 21),$$

where $Z_{i-j}$ stands for the basis monomial of weight $i-j$, and the summation index $j$ varies only through the integers such that $i - j \in S$.

By using the transformations $X_i \mapsto X_i + \sum_{j=1}^{i-1} \lambda_jX_{i-j}$, we can normalize the following ten coefficients

$$c_{12,1} = c_{12,2} = c_{12,7} = c_{13,1} = c_{13,2} = c_{13,3} = c_{13,8} = d_{16,1} = d_{16,6} = d_{21,5} = 0.$$
By applying the Syzygy Lemma \[\text{Lemma 5.4}\] we obtain the following four syzygies of the canonical monomial curve $C^{(0)}$

\[
\begin{align*}
X_8F_{12}^{(0)} - X_7F_{13}^{(0)} + X_6F_{14}^{(0)} &= 0, \\
X_6G_{13}^{(0)} - X_5X_4F_{12}^{(0)} + X_5G_{18}^{(0)} - X_7G_{10}^{(0)} &= 0, \\
X_8G_{18}^{(0)} - X_5G_{21}^{(0)} + X_5X_7F_{14}^{(0)} - X_6X_8F_{12}^{(0)} &= 0, \\
X_8G_{21} - X_7X_8F_{14}^{(0)} + X_6G_{0}^{(0)} &= 0.
\end{align*}
\]

(7)

Replacing each left-hand side of the above syzygies the binomials $F_{s,i}^{(0)}, F_{s,i'}^{(0)}, G_{s,j}^{(0)}$ and $G_{s,j'}^{(0)}$, by the quadratic and cubic forms $F_{s,i}, F_{s,i'}, G_{s,j}$ and $G_{s,j'},$ respectively, and applying the division algorithm recursively until all the monomials of these new equations belong to the basis $A_3$ or $A_4$, we get the following four polynomial equations

\[
\begin{align*}
X_8F_{12} - X_7F_{13} + X_6F_{14} &= -F_{12}(c_{14,3}X_5 + c_{14,8}X_0) + F_{14}c_{13,6}X_0 - G_{16}c_{14,4} \\
&+ F_{13}(c_{13,7}X_0 - c_{14,2}X_5 - c_{14,7}X_0), \\
X_8G_{15} - X_6G_{17} + X_5G_{18} - X_7G_{16} &= (c_{12,6}X_0X_5 - d_{18,1}X_0X_8)F_{12} \\
&- (c_{14,3}d_{16,4} + c_{14,3}d_{15,3}d_{18,1} + d_{18,7})X_0G_{16} + (d_{16,5}X_5 + c_{12,5}X_5)X_0F_{13} \\
&+ (d_{16,9}X_0 - d_{18,1}X_5 + d_{15,8}d_{18,1}X_0 + d_{15,3}d_{18,1}X_5 + d_{16,4}X_5)X_0F_{14} \\
&+ (d_{16,10}X_0 + d_{15,9}d_{18,1}X_9 + d_{15,1}d_{18,1}X_8 + d_{15,4}d_{18,1}X_5 + d_{16,2}X_8)X_0F_{13} \\
&+ (c_{14,4}d_{16,4}X_0 - c_{14,4}d_{15,3}d_{18,1}X_0 - d_{18,1}X_7 - d_{18,8}X_0)G_{15}.
\end{align*}
\]

We now determine the weighted vector space $T_{k[S]}^1$, which is (up to an isomorphism) the locus of the linearizations of the above 4 equations, all we have to do is substituting by zero the right hand side of each equation. These four equations give rise to other 20 linear equations obtained by replacing $X_m \mapsto t^m$. We can solve this linear system as follows:

\[
\begin{align*}
d_{16,10} &= d_{15,10}, \quad d_{16,9} = d_{15,9}, \quad d_{16,8} = d_{15,8}, \quad c_{14,7} = c_{13,7}, \quad d_{18,7} = c_{13,7}, \quad d_{15,7} = -c_{13,7}, \\
d_{21,7} &= 2c_{13,7}, \quad c_{14,6} = -c_{12,6}, \quad d_{21,6} = -c_{12,6}, \quad d_{18,6} = c_{12,6}, \quad d_{16,5} = d_{15,5}, \\
c_{14,4} &= -c_{12,4}, \quad d_{16,4} = d_{15,4}, \quad d_{21,4} = -c_{12,4}, \quad d_{18,4} = c_{12,4}, \quad d_{16,3} = d_{15,3}, \quad d_{16,2} = d_{15,2}.
\end{align*}
\]
We can verify that the weighted vector space $T_{k[S]}^1$ depends only on the ten coefficients $d_{15,2}, d_{15,3}, c_{12,4}, d_{15,4}, d_{15,5}, c_{13,7}, d_{15,8}, d_{15,9}, d_{15,10}$, which implies

$$\dim T_{k[S]}^1 = 10.$$ 

More precisely, counting the coefficients of weight $s$, we obtain the dimension of the graded component of $T_{k[S]}^1$ of negative weight $-s$:

$$\dim T_{-s}^1 = 1, \quad (s = -10, -9, -8, -7, -6, -5, -3, -2) \quad \text{and} \quad \dim T_{-4}^1 = 2.$$ 

For the remainder integers, the dimension of $T_{-s}^1$ is zero. In particular, the compactified moduli space $\overline{M}_{k[S]}^{1,1}$ can be realized as closed subspace of the 9-dimensional weighted projective space $\mathbb{P}(T_{k[S]}^1)$. 

Finally, we solve the four polynomial equations of the previous page to obtain the equations of the moduli variety $\overline{M}_{k[S]}^{1,1}$. By replacing $X_n \mapsto t_n^s$ the compactified moduli space $\overline{M}_{k[S]}^{1,1}$ is cut out by 70 equations which depend on 64 variables, we can solve them in the following way:

- 18 coefficients which are identically zero, namely:
  \[
  c_{12,5} = c_{13,5} = c_{13,6} = c_{14,1} = c_{14,2} = c_{14,3} = d_{15,1} = d_{16,11} = d_{18,1} = 0, \\
  d_{18,2} = d_{18,3} = d_{18,5} = d_{18,8} = d_{18,11} = d_{21,1} = d_{21,2} = d_{21,3} = d_{21,10} = 0.
  \]

- 11 linear equations:
  \[
  c_{14,4} = -c_{12,4}, \quad d_{15,7} = -c_{13,7}, \quad d_{16,2} = d_{15,2}, \quad d_{16,4} = d_{15,4}, \\
  d_{16,5} = d_{15,5}, \quad d_{16,9} = d_{15,9}, \quad d_{18,4} = c_{12,4}, \quad d_{18,6} = c_{12,6}, \\
  d_{18,7} = c_{13,7}, \quad d_{21,4} = -c_{12,4}, \quad d_{16,3} = d_{15,3}.
  \]

- 17 quadratic polynomials and isobarics:
  \[
  c_{12,12} = -c_{12,4}d_{15,8}, \quad c_{12,12} = -c_{12,4}d_{15,8}, \quad c_{13,13} = c_{12,4}d_{15,9}, \\
  c_{14,6} = -c_{12,4}d_{15,2} - c_{12,6}, \quad c_{14,6} = -c_{12,4}d_{15,2} - c_{12,6}, \quad c_{14,7} = -c_{12,4}d_{15,3} + c_{13,7}, \\
  c_{14,8} = -c_{12,4}d_{15,4}, \quad c_{14,8} = -c_{12,4}d_{15,4}, \quad c_{14,9} = -c_{12,4}d_{15,5}, \\
  c_{14,14} = -c_{12,4}d_{15,10}, \quad c_{14,14} = -c_{12,4}d_{15,10}, \quad d_{15,15} = c_{12,6}d_{15,9} + c_{13,7}d_{15,8}, \\
  d_{16,8} = -c_{12,4}d_{15,4} + d_{15,8}, \quad d_{16,8} = -c_{12,4}d_{15,4} + d_{15,8}, \quad d_{16,10} = -c_{12,6}d_{15,4} - c_{13,7}d_{15,3} + d_{15,10}, \\
  d_{18,12} = -c_{12,4}d_{15,8} - c_{12,6}, \quad d_{18,12} = -c_{12,4}d_{15,8} - c_{12,6}, \quad d_{21,6} = -c_{12,4}d_{15,2} - c_{12,6}, \\
  d_{18,13} = c_{12,4}d_{15,9}, \quad d_{18,13} = c_{12,4}d_{15,9}, \quad d_{21,7} = -c_{12,4}d_{15,3} + 2c_{13,7}, \\
  d_{21,8} = -c_{12,4}d_{15,4}, \quad d_{21,8} = -c_{12,4}d_{15,4}, \quad d_{21,9} = -c_{12,4}d_{15,5}, \\
  d_{18,10} = -c_{12,4}d_{15,6}, \quad d_{18,10} = -c_{12,4}d_{15,6}.
  \]

- and the following 8:
  \[
  d_{16,16} = -c_{12,4}d_{15,3}d_{15,9} + c_{12,4}d_{15,4}d_{15,8}, \quad d_{18,18} = c_{12,4}c_{12,6}d_{15,8}, \\
  d_{21,13} = -c_{12,4}d_{15,2}d_{15,3} - c_{12,4}c_{12,6}d_{15,3} + c_{12,4}c_{13,7}d_{15,2} + c_{12,4}d_{15,9} + c_{12,6}c_{13,7}, \\
  d_{21,14} = -c_{12,4}d_{15,3}^2 + 2c_{12,4}c_{13,7}d_{15,3} - c_{12,4}d_{15,10} - c_{13,7}^2, \\
  d_{21,15} = -c_{12,4}d_{15,3}d_{15,4} + 2c_{12,4}c_{13,7}d_{15,4}, \quad d_{21,11} = -c_{12,4}^2d_{15,3} + c_{12,4}c_{13,7}, \\
  d_{21,16} = -c_{12,4}d_{15,3}d_{15,5} + c_{12,4}c_{13,7}d_{15,5}, \quad d_{21,21} = -c_{12,4}^2d_{15,3}d_{15,10} + c_{12,4}^2d_{15,4}d_{15,9} + c_{12,4}c_{13,7}d_{15,10}.
  \]

We note that there are 16 missing equations from the 70 announced, but each one of these 16 is redundant. We also note that no one condition on the 10 coefficients.
of the ambient space $T_{k|S||k}^{1,-}$ appears, which means
\[ \mathcal{M}_{5,1}^{S} = \mathbb{P}(T_{k|S||k}^{1,-}) \cong \mathbb{P}^{9}, \quad \text{with } \alpha = (2, 3, 4, 5, 6, 7, 8, 9, 10). \]

4.2. Odd of genus six. Let $C^{(0)}$ be the canonical monomial Gorenstein curve of genus 6 associated to the odd symmetric semigroup $S := < 6, 7, 8, 9, 10 >$. Take $P = (0 : 0 : 0 : 0 : 0 : 1)$ a smooth point in $C^{(0)}$ whose Weierstrass semigroup is $S$. Applying Lemma (3.3), the generators of the ideal of $C^{(0)}$ are the following 6 quadratic and 8 cubic forms:

\[
\begin{align*}
F_{14}^{(0)} &= X_7^2 - X_6 X_9, \\
F_{16,1}^{(0)} &= X_7 X_9 - X_6 X_{10}, \\
F_{17}^{(0)} &= X_8 X_9 - X_7 X_{10}, \\
G_{18}^{(0)} &= X_8^3 - X_6 X_8 X_{10}, \\
G_{19}^{(0)} &= X_8^2 X_7 - X_6 X_9 X_{10}, \\
G_{20}^{(0)} &= X_8^2 X_9 - X_6 X_{10}, \\
G_{21}^{(0)} &= X_6 X_9 - X_6 X_{10}, \\
G_{22}^{(0)} &= X_6^3 - X_6 X_9, \\
G_{23}^{(0)} &= X_6^2 - X_6^2 X_{10}, \\
G_{24}^{(0)} &= X_6^2 - X_6 X_{10}.
\end{align*}
\]

We consider a pre-deformation of the ideal of $C^{(0)}$ as follows:

\[ F_i = F_i^{(0)} - \sum_{j=1}^{i} c_{ij} Z_{i-j}, \quad (i = 14, \ldots, 18 \text{ and } i = 16, 1) \]

and

\[ G_i = G_i^{(0)} - \sum_{j=1}^{i} d_{ij} Z_{i-j}, \quad (i = 18, \ldots, 22, 26, 27 \text{ and } i = 20, 1). \]

where $Z_{i-j}$ is a polynomial of weight $i - j$, whenever $i - j$ is a nongap of $S$. By suitable transformations of the variables $X_0, X_6, X_7, X_8, X_9, X_{10}$, we are able to normalize the following 15 coefficients:

\[
\begin{align*}
c_{14,1} &= c_{15,1} = c_{16,1,1} = d_{18,1} = d_{18,2} = c_{15,2} = c_{16,1,2} = c_{15,3} = 0, \\
c_{16,1,3} &= c_{16,1,4} = c_{15,6} = c_{14,7} = c_{14,8} = c_{15,9} = c_{16,1,10} = 0.
\end{align*}
\]

We also consider the ten syzygies of the monomial curve $C^{(0)}$, which are induced by the Syzygy Lemma (3.3).

\[
\begin{align*}
X_{10} F_{14}^{(0)} - X_8 F_{16,1}^{(0)} + X_7 F_{17}^{(0)} &= 0, \\
X_{10} F_{15}^{(0)} - X_9 F_{16,1}^{(0)} + X_7 F_{18}^{(0)} &= 0, \\
X_{10} F_{16}^{(0)} - X_{10} F_{16,1}^{(0)} - X_9 F_{17}^{(0)} + X_8 F_{18}^{(0)} &= 0, \\
X_{10} G_{18}^{(0)} - X_8 G_{20}^{(0)} + X_6^2 F_{16}^{(0)} &= 0, \\
X_{10} G_{19}^{(0)} - X_9 G_{20,1}^{(0)} + X_6 X_7 F_{16}^{(0)} &= 0, \\
X_{10} G_{20}^{(0)} - X_{10} G_{20,1}^{(0)} + X_6 X_{10} F_{14}^{(0)} &= 0, \\
X_{10} G_{21}^{(0)} - X_7 X_{10} F_{14}^{(0)} - X_6 X_{10} F_{15}^{(0)} &= 0, \\
X_{10} G_{22}^{(0)} - X_6 X_{10} F_{16}^{(0)} - X_8 X_{10} F_{14}^{(0)} &= 0, \\
X_{10} G_{23}^{(0)} - X_{10}^2 F_{16,1}^{(0)} - X_9 X_{10} F_{17}^{(0)} &= 0, \\
X_{10} G_{24}^{(0)} - X_{10}^2 F_{17}^{(0)} - X_9 X_{10} F_{18}^{(0)} &= 0.
\end{align*}
\]
The 10 above syzygies of the monomial curve give rise to 10 polynomial equations between the 14 polynomials $F_i$'s and $G_j$'s.

\[
\begin{align*}
X_{10}F_{14} &- X_8F_{16,1} + X_7F_{17}, \\
X_{10}F_{15} &- X_9F_{16,1} + X_7F_{18}, \\
X_{10}F_{16} &- X_{10}F_{16,1} - X_9F_{17} + X_8F_{18}, \\
X_{10}G_{18} &- X_8G_{20} + X_7^2F_{16}, \\
X_{10}G_{19} &- X_9G_{20,1} + X_8X_7F_{16,1}, \\
X_{10}G_{20} &- X_{10}G_{20,1} + X_6X_10F_{14}, \\
X_{10}G_{21} &- X_7X_10F_{14} - X_6X_{10}F_{15}, \\
X_{10}G_{22} &- X_6X_10F_{16} - X_8X_{10}F_{14}, \\
X_{10}G_{26} &- X_{10}^2F_{16,1} - X_9X_{10}F_{17}, \\
X_{10}G_{27} &- X_{10}^2F_{17} - X_9X_{10}F_{18}.
\end{align*}
\]

Again, we compute the linearization of the above ten polynomials, which is isomorphic to the weighted vector space $T_{k[S]}^{1,-}$. To do this, we make the substitutions $X_i \mapsto t^i$ and solve a homogeneous linear system with 60 equations. We can solve it in way that the solution depends only on the 15 coefficients:

\[
\begin{align*}
d_{18,12}, & \quad d_{18,11}, \quad c_{15,8}, \quad c_{16,1,9}, \quad c_{16,1,8}, \quad c_{15,7}, \quad c_{14,6}, \quad d_{18,6}, \\
d_{18,10}, & \quad c_{14,5}, \quad d_{18,5}, \quad c_{14,4}, \quad d_{18,4}, \quad d_{18,3}, \quad c_{14,2}.
\end{align*}
\]

Therefore the compactified moduli space $\overline{M}_{6,1}^S$ can be realized as a closed subset of the 14-dimensional weighted projective space $\mathbb{P}(T_{k[S]}^{1,-}) \cong \mathbb{P}_{\alpha}^{14}$, where $\alpha = (2,3,4,4,5,5,6,6,7,8,8,9,10,11,12)$. Since the odd symmetric semigroup $S$ is negatively graded, cf. [23], the moduli space $\overline{M}_{6,1}^S$ has codimension three in $\mathcal{M}_{6,1}$, cf. [20, 10]. Hence $\overline{M}_{6,1}^S$ has dimension 11.

Now we have to take each polynomial in (9) and make successive divisions in order that all its monomials belongs to the basis $\Lambda_3$ or $\Lambda_4$, it is possible by virtue of Lemma 3.6. This procedure is completely computational and we can make it by using a suitable software on computer algebra, like Singular or Maple. Here we do not display the resulting polynomials, just because they have a huge number of monomials. Then, we make the substitutions $X_i \mapsto t^i$, with $i = 6,7,8,9,10$, on the 10 polynomials whose monomials are in $\Lambda_3$ and $\Lambda_4$ and solve 188 polynomial equations. This system can be solved by increasing weights whose solution depends only on the 15 coefficients of the linearization that here we rename them:

\[
\begin{align*}
d_{18,4} := b_i & \quad (i = 3,4,5,6,10,11,12), \\
c_{14,3} := a_j & \quad (j = 2,4,5,6), \\
c_{16,1,8} := b_8, & \quad c_{15,7} := a_7, \quad c_{15,8} := a_8, \quad c_{16,1,9} := a_9.
\end{align*}
\]

By Theorem 3.9 we can conclude that the moduli space $\overline{M}_{6,1}^S$ is given by the zero locus of following 5 isobaric polynomials.

\[
\begin{align*}
\vartheta_{15} := 4a_5a_6 & - a_2a_5b_8 + a_4a_5b_6 - a_4b_3b_8 + a_5^3 + a_5^2b_5 + a_4b_{11} + a_5b_{10} + 2a_7b_8, \\
\vartheta_{13} := 2a_2a_5a_6 + a_4^2a_5 + a_4^2b_5 & + a_4a_5b_4 + a_4a_6b_3 + a_5^2b_3 - a_4a_9 + a_5a_8 - a_5b_8 - 2a_6a_7, \\
\vartheta_{17} := a_5b_{12} & - a_2a_5^3 - 2a_2a_5^2b_5 - a_4a_5b_8 - a_4b_3b_8 + 2a_5^2a_7 + a_5a_7b_5 - a_5a_8b_4 - a_5a_9b_3 - a_6b_{11} + a_9b_8.
\end{align*}
\]
and the homogeneous graded part of degree nonzero. To see this, we use the description of $S_{\text{dim} 11}$. We also note that Bullock [6, Thm. 1] proved that the moduli spaces are stably rationals when $2g \leq 6$, with the possible exceptions $< 6, 7, 8, 9, 10 >$ and $< 5, 7, 8, 9, 11 >$, the last one is not subcanonical.

For a given monomial curve $C$ associated to a semigroup $S$, its obstruction space lies in the second cohomological module of cotangent complex $T^2 := T^2[k[S]/k]$. As noted in the beginning of the last section of this work, if $S$ is symmetric and generated by less than five elements, the monomial curve $C$ can be smoothed without any obstructions, which implies that $\overline{M_{6,1}^S}$ is the weighted projective space $\mathbb{P}(T^2-(k[S]/k))$. The obstructions spaces of the two examples of this section are nonzero. To see this, we use the description of $T^2$ given by Buchweitz in [3, Thm 2.3.1], and we can conclude that for genus five, $S = < 5, 6, 7, 8 >$, the homogeneous graded part of degree $-9$ of $T^2$ has dimension 1, for genus 6, $S = < 6, 7, 8, 9, 10 >$, the homogeneous graded part of degree $-13$ has dimension 1, and in both cases $T^1$ and $T^2$ are negatively graded.

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