High-order nonreciprocal add-drop filter

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Topological photonics have led to the robust optical behavior of the device, which has solved the problem of the influence of manufacturing defects and perturbations on the device performance. Meanwhile, temporal coupled-mode theory (t-CMT) has been developed and applied widely. However, the t-CMT of cascaded coupling cavities (CCC) system and its corresponding high-order filter has yet to be established. Here the t-CMT of CCC system is established based on the existing t-CMT. By combining the CCC with the topological waveguides, a versatile design scheme of the high-order nonreciprocal add-drop filter (HONAF) is proposed. The relationship between coupling effect of cavities and transmission and filtering performance of HONAF is analyzed quantitatively, then a method to improve the transmission efficiency and quality factor of the filter is given. Based on the combination of gyromagnetic photonic crystals and decagonal Penrose-type photonic quasicrystals, a HONAF is proposed. The transmission and filtering performance of the HONAF are numerically analyzed, which verifies the consistency between the theoretical prediction and the numerical simulation. The t-CMT of CCC system established can be widely used in the coupled resonator optical waveguides and their related systems. The designed HONAF can also be widely used in communication systems.

I. INTRODUCTION

Optical integrated circuits with large density, high speed transmission and low energy consumption have developed rapidly [1]. With the continuous development of micronano lithography and subwavelength structure manufacturing technologies [2–4], novel micronano photonics devices have been proposed, such as fibers [5–8], sensors [6], lenses [9, 10], couplers [11], absorbers [12–15], resonators [16–21], and other photonic crystal (PC) devices [22]. Optical resonator is the core part of many micronano photonics devices due to its excellent natural frequency selection characteristics, such as coupled resonator optical waveguides (CROWS) [16] filters [23], sensors [24], lasers [25], etc. The frequency-division/recombination of signals in optical filters plays an irreplaceable role in optical integrated circuits. Therefore, the design and characteristics of optical filters have been researched for a long time [18, 23, 26–32].

At present, the filters constructed by a single cavity have the characteristics of multiple drop channels, non-reciprocity, high transmission efficiency (T) and quality factor (Q), and the radiation loss inherent in the PC cavities has been reduced to a low level [17–20, 26, 32]. However, these filters are unable to immune to the defects in waveguides or cavities, and the transmission efficiency is sensitive to the coupling phase shift between waveguides and cavities, which brings great troubles to device fabrication and application [16, 20, 27]. Furthermore, the performance of the single cavity filters for the out-of-band rejection ratios and flat tops bandwidth tunable property is poor. In fact, the practical function also needs to be further expanded [17, 18, 20, 26, 27]. In order to overcome these problems, a topological nontrivial waveguide and cavity was introduced into PC filters, which can be immune to the defects in structure, bringing the transmission characteristic of reciprocity, and the problem that the transmission efficiency is sensitive to the coupling phase shift of optical waveguides and cavities is solved [19, 26]. Therefore, PCs with topological properties are widely used, such as isolators [33], one-way waveguides [34–43], resonators [19, 21, 44, 45], lasers [46] and logic gates [47], etc. In addition, the structure of the filters cascaded with microring resonators can improve the out-of-band rejection ratios, possess flat tops bandwidth tunable property and realize multiple filtering, but its structure is more complex and the composition of the material is not common [28, 29]. Moreover, the existing temporal coupled-mode theory (t-CMT) can only accurately predict the transmission efficiency and quality factor of a single cavity or a small number of cavities coupling systems [22, 26]. In order to make it more universal, that is, to predict the transmission efficiency and quality factor of any cascaded coupling cavities (CCC) system, and to improve its structure to optimize performance parameters, it is indispensable to explore and derive the applicable t-CMT equations for infinite CCC system.

In Section II, based on the existing t-CMT, the t-CMT of CCC system is derived. The transmission and filtering characteristics of four-channel high-order nonreciprocal add-drop filter (HONAF) coupled by topological PC waveguides and the CCC are deeply researched to find a multifunctional and high-performance filter and a universal design scheme. In Section III, in order to verify the strictness and practicability of theoretical analysis, combining the gyromagnetic photonic crystals (GPCs) [38] and photonic quasicrystals (PQCs) [48, 49] (excel-
lent frequency selectivity) with abundant defect mode properties, a composite HONAF is proposed, whose performance index is analyzed when the number of cascaded cavities varies. Point defects (size and material of several scatterers changed) and line defects are introduced to verify the properties of the waveguide channels protected by topology. The main work of this paper is briefly summarized in the Section IV.

II. THE t-CMT OF CCC SYSTEM

Weak coupling is an indispensable condition of the t-CMT, which guarantees the evanescent wave coupling relationship between waveguides and cavities, and the favourable frequency selection behavior of cavities. Weak coupling can be achieved by placing enough scatterers around the cavities [22]. Therefore, it is assumed that the relationship between waveguide channels and the cascaded cavities in the HONAF is weakly coupled. The operating principle of the HONAF system is shown in Fig. 1. Two nonreciprocal waveguides constitute the bus waveguide W1 and the add/drop waveguide W2 respectively, which is coupled with cascaded cavities to form a four-port system. The signal in the bus waveguide passes through the frequency selection behavior of the cavities and drops the signal of the specific frequency to the P3 or P4 port determined by the nonreciprocal direction, i.e. the signal drop channel of the HONAF system can be transformed by nonreciprocal direction. Moreover, if the light wave signal is input from the add/drop waveguide channel, the function of add signal can be realized.

For convenience, two nonreciprocal waveguides and cascaded cavities are both structurally and functionally symmetric, which supports only one mode within the desired frequency range. When the field source is placed on port P1 and the drop port is set as P4 [Fig. 1], then the transmission efficiencies of all the ports of the HONAF (derivation process see S.I within the Supplemental Materials) are

$$T_{2(1)} = \left| \frac{S_2}{S_1} \right|^2 = 1 - \frac{2/(\tau_1\tau_2)^{1/2}}{j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2}^2,$$

$$T_{2(N)} = \left| \frac{S_2}{S_1} \right|^2, (N \geq 2)$$

$$T_{3(N)} = \left| \frac{S_3}{S_1} \right|^2 = 0, (N \geq 1),$$

$$T_{4(1)} = \left| \frac{S_4}{S_1} \right|^2 = \frac{2/(\tau_1\tau_2)^{1/2}}{j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2}^2,$$

$$T_{4(N)} = \left| \frac{S_4}{S_1} \right|^2, (N \geq 2)$$

$$= \frac{2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_a)^{N-1}}{dH_{N-1} + cF_{N-1}}^2,$$

where,

$$dH_{N-1} + cF_{N-1} =$$

$$\frac{a b d + 2a c + 2c d - b c + (a d + c)(b^2 + 4c)^{1/2}}{2^{N-1}(b^2 + 4c)^{1/2}(b + (b^2 + 4c)^{1/2})^{2-N}}$$

$$- \frac{a b d + 2a c + 2c d - b c - (a d + c)(b^2 + 4c)^{1/2}}{2^{N-1}(b^2 + 4c)^{1/2}(b + (b^2 + 4c)^{1/2})^{2-N}},$$

$$a = j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_a,$$

$$b = j2\pi(f - f_0) + 1/\tau_0 + 2/\tau_a,$$

$$c = j\mu(-j\mu + 2/\tau_a),$$

$$d = j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_a,$$

where, the physical significances of parameters a, b, c and d are complex frequencies. j is an imaginary unit, and f is the mixed frequency of the input time harmonic field. $f_0$ is the resonance frequency of a single isolated cavity, and N represents the number of cascaded cavities (i.e.
the order of the HONAF). According to the definitions of $\tau_a$ and $\mu$, the change in $\mu$ will lead to the change in $\tau_a$, assuming that

$$\tau_a = \hat{A} \mu,$$  \hspace{1cm} (11)

where $\hat{A}$ is the operator factor, representing the relationship between $\tau_a$ and $\mu$. Generally, $\tau_a$ increases with the increase of $\mu$. The specific functional relationship between them needs to be obtained according to the specific structure.

Then, the transmission efficiency of drop port $P_4$ is specifically analyzed, i.e. (5). If $a$, $b$, $c$ and $d$ are substituted into (5), the higher-order function of $f - f_0$ can be obtained, and the highest-order is $N$. Specifically, the numerator of the $T_{4(N)}$ expression is a constant, while the denominator is a high-order polynomial in terms of $f - f_0$. Therefore, within a large frequency range, the $T_{4(N)}$ transmission spectrums will possess multiple formants. According to the definition of transmission efficiency of filter [22], $T_{\max} = 1$. Therefore, if the maximum value of transmission efficiency is set to $T_{4(N)} = 1$, the $N$-order equation of $f - f_0$ is obtained by simplification. There are at most $N$ real solutions to this equation, i.e. there are at most $N$ formants in the transmission spectrums of $T_{4(N)}$, whose positions are determined by $\tau_0$, $\tau_1$, $\tau_2$, $\tau_a$, and $\mu$.

Next, the laws and reasons of producing multiple formants are explored. According to the $T_{4(N)}$ expression, when $N \geq 2$, the influence term of interaction factor $\mu$ between the nearest-neighbor cavities appears in (5). Therefore, it is assumed that the coupling distance is far and there are enough transmission media between the nearest-neighbor cavities. In this case, the coupling form between the nearest-neighbor cavities is evanescent wave, so that $\mu \to 0$, then the (5) changes to

$$T_{4(N)} = \left| \frac{2^N / (\tau_1^{1/2} \tau_2^{1/2} \tau_a^{N-1})}{j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_a} \times \frac{1}{j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_2 + 1/\tau_a} \times \frac{1}{j2\pi(f - f_0) + 1/\tau_0 + 2/\tau_a} \right|^{N-2} \right|,$$

\hspace{1cm} (12)

where $N \geq 2$. According to (12), the numerator order is the same as the denominator order in $T_{4(N)}$ expression. Hence, the transmission spectrums with $T_{4(N)}$ expression only possess a single formant, which is located at the frequency point $f_0$, and the peak value of transmission efficiency is related to $\tau_0$, $\tau_1$, $\tau_2$ and $\tau_a$. Furthermore, as can be seen from the denominator of (12), the larger the value of $N$ is, the sharper the formant at $f_0$ (when $f$ approaches $f_0$ from a distance, $T$ can quickly reach the maximum value) is. Therefore, as $N$ increases, the bandwidth of the formant decreases. According to the definition of $Q$ [22], this means that $Q$ will increase as $N$ increases.

In the case of ignoring the influence of $\mu$, the influence of the size relation between $\tau_1$, $\tau_2$ and $\tau_a$ on the transmission efficiency of the drop port $P_4$ is discussed. If the designed HONAF satisfies the condition $\tau_1 \neq \tau_2 \neq \tau_a$, the maximum value of $T_{4(N)}$ can be obtained from (4) and (12), that is

$$T_{4(1)\max} = \frac{4\tau_1 \tau_2}{(\tau_1 \tau_2 / \tau_0 + \tau_1 + \tau_2)^2};$$  \hspace{1cm} (13)

$$T_{4(N)\max} = \frac{4^N \tau_1 \tau_2 \tau_a^2 (\tau_a / \tau_0 + 2)^{4-2N}}{(\tau_1 \tau_2 / \tau_0 + \tau_1 + \tau_2)^2(\tau_2 \tau_a / \tau_0 + \tau_2 + \tau_a)^2} (N \geq 2).$$  \hspace{1cm} (14)

In the general structure, the values of $\tau_1$, $\tau_2$ and $\tau_a$ are diminutive, i.e. the relation $1 > T_{4(1)\max} > T_{4(N)\max}, (N \geq 2)$ is valid, indicating that the transmission efficiency of the drop port $P_4$ will decrease with the increase of $N$ without ignoring the influence of $\tau_0$. If the influence of $\tau_0$ is ignored in cascaded cavities, it can be obtained

$$T_{4(1)\max} = \frac{4\tau_1 \tau_2}{(\tau_1 + \tau_2)^2};$$  \hspace{1cm} (15)

$$T_{4(N)\max} = \frac{16\tau_1 \tau_2 \tau_a^2}{(\tau_1 + \tau_a)^2(\tau_2 + \tau_a)^2} (N \geq 2).$$  \hspace{1cm} (16)

Obviously, there is $1 > T_{4(1)\max} > T_{4(2)\max} = T_{4(N)\max}, (N \geq 2)$, i.e. in the case of $N \geq 2$, $T_{4(N)\max}$ is a constant. If the designed HONAF satisfies the condition (which can be satisfied if the leaky components [17] of cavities are completely symmetrical),

$$\tau_1 = \tau_2 = \tau_a,$$  \hspace{1cm} (17)

and if the influence of $\tau_0$ is ignored, then, it can be seen from (15) and (16) that $T_{4(N)\max} = 1$ is valid obviously. Specific structure parameters are substituted into (4) and (12), then the transmission spectrums of the HONAF can be obtained (see S. II within the Supplemental Materials), that intuitively shows the correctness of the theory mentioned above. Therefore, a filter with high $Q$ can be obtained by cascaded cavities without changing the resonance frequency of the single isolated cavity. Moreover, the above studies found that only when $\tau_1 = \tau_2 = \tau_a$, the maximum drop efficiency of the HONAF could reach 100 percent. Next, the quality factor expression ($Q_N$) of the $N^{th}$-order HONAF, in the case of $\tau_1 = \tau_2 = \tau_a$, $\tau_0 = 0$ and $\mu = 0$, is derived (see S. III within the Supplemental Materials)

$$Q_N = Q_0 (2^{1/N} - 1)^{-1/2},$$  \hspace{1cm} (18)

where $Q_0$ is the quality factor of the single isolated cavity. Obviously, $Q_N$ is a function of $N$, when $N \to \infty$, $Q_N \to \infty$. However, in the actual situation (with energy loss), as $N$ increases, the loss of the whole CCC system will inevitably increase, hence $Q_N$ cannot approach infinity. The first derivative of $Q_N$ with respect to $N$ is $Q_N' >$
0, i.e. $Q_N$ increases monotonically with the increase of $N$. The second derivative $Q''_N < 0$, i.e. the line spectrum of $Q_N$ is upward convex type. Therefore, as $N$ increases, the growth rate of $Q_N$ becomes smaller and smaller. The cascade of more cavities will no longer play a leading role. However, in this case, $N$ is a large value, and the self-loss of the CCC system will play a leading role. The $Q_0$ of a single cavity is substituted into (18) to obtain the spectrum of $Q_N$ changing with $N$ in the CCC system, which is consistent with the above analysis (see S. III within the Supplemental Materials).

It’s worth noting that the $\mu \to 0$ condition is harder to satisfy. The reason is that the interaction between cascaded cavities always exists, even if the coupling distance between the nearest-neighbor cavities is very long, there is still a weak interaction, and it can be seen from (5) that this kind of interaction increases with the increase of $N$ (In the denominator of the $T_{4(N)}$ expression, the influence of $\mu a$ is strengthened with the increase of $N$). Hence, for the higher-order HONAF, the influence of interaction between cascaded cavities cannot be overcome theoretically. If the coupling distance between the cavities is too large, new problems will arise, such as the coupling efficiency between the waveguide channels and the cavities will decrease, thus reducing the drop efficiencies.

As the coupling distance between the cascaded cavities is reduced, the coupling form between the nearest-neighbor cavities will gradually change from evanescent wave coupling to near-field coupling. However, the discussion in this paper is limited to the category of evanescent wave coupling, and it is assumed that the HONAF still satisfies the weak coupling condition. The interaction factor $\mu$ will play a leading role in this process, so that the CCC system will show more excellent characteristics. According to (11), the change of $\mu$ leads to the change of $\tau_a$ (the specific change depends on the specific structure). From (5), it can be seen that the $\mu$ will have an impact on the transmission efficiency of the drop port, which will be intensified with the increase of $N$ (In the denominator of (5), the influence of $\mu$ polynomial will increase). Once $\mu \neq 0$, $\mu$ polynomial will cause transmission spectrums of the $T_{4(N)}$ to split from a single formant into multiple formants. When $\mu$ is slightly greater than zero, peak spikes appear in the single formant in the $T_{4(N)}$ transmission spectrums. When $\mu$ is large enough, the single formant will completely split into multiple formants in the $T_{4(N)}$ transmission spectrums. As long as $\mu \neq 0$, the splitting effect will persist. It can also be seen from (5) that the number of formants generated by the influence $\mu$ is $N$, and the positions of the formants after splitting are symmetrical about $f_0$. According to (5), the obtain the transmission spectrums of the HONAF with specific structural parameters shown that the conclusion is consistent with the above. Under the same parameters, when $N$ is large, single formant is more likely to split into multiple formants (see S. IV within the Supplemental Materials). This means that in the actual HONAF, even if the coupling distance between the nearest-neighbor cavities is large, as $N$ increases, its transmission spectrums will inevitably split from single formant to multiple formants.

It is important to note that when the coupling distance between the cascaded cavities is too close, the strong coupling will make the interaction between the cascaded cavities becoming more complex, even the cascaded cavities have merged into a whole, the property of the isolated cavity will gradually disappear, to create a new cavity, the emergence of new properties, such as different resonance frequency from that of isolated cavity, the number of formants is no longer equal to $N$, and the resonance frequency is not symmetrical about $f_0$. The above theory is no longer universally applicable in this case.

Furthermore, the transmission efficiency equations ((1) to (5)) of the HONAF discussed above is also generally applicable to CROWs. Simply remove the two waveguide channels (the bus waveguide and the add/drop waveguide) or change the transmission direction of the two waveguides to coincide with that of the cascaded cavities (i.e. to be couplers for input and output ports). Equation (5) can be used to research the transmission characteristics of the CROWs (see S. V within the Supplemental Materials).

### III. A HONAF VERIFYING THE t-CMT

In this Section, a HONAF is designed to verify the correctness of the theory above, which can also be applied to the actual communication system. Two topological waveguide channels are constructed by using two-dimensional (2D) square lattice GPCs and decagonal Penrose-type PQC, and the decagonal Penrose-type PQC is used as the cavity to obtain the four-port system of waveguide - CCC - waveguide.

The decagonal Penrose-type PQC is shown in Fig. 2(a) [49]. The yellow scatterers in Fig. 2(a) are selected as the basic cascaded unit (isolated cavity) to construct the HONAF with cascaded cavities, as shown in Fig. 2(b) (In order to prove the correctness of the theoretical prediction in Section II and its generality with the simulation results, the basic cascaded unit $S$ can be arbitrarily taken to construct HONAF $S$. The model and results can be referred to S. VI within the Supplemental Materials). In order to make the coupling between waveguides and cavities better meet the weak coupling condition and ensure favourable frequency selectivity, the coupling distance between waveguides and cavities is increased, i.e. more scatterers (white scatterers in Fig. 2), are taken in the coupling cavity unit adjacent to the waveguide.

Ports $P_1$ and $P_2$ are provided to constitute the bus waveguide, while ports $P_3$ and $P_4$ constitute the add/drop waveguide. If the signal source is placed on port $P_1$, and external dc magnetic field in the direction of $-z$ and $+z$ is applied to the left and right GPCs respectively (the nonreciprocal directions are counterclockwise
FIG. 2. (a) Decagonal Penrose-type PQCs; (b) 2nd-order HONAF model.

and clockwise respectively), as shown in Fig. 2(b), so that the port P4 is the drop channel. If P3 is taken as the input port, and the direction of the external dc magnetic field in the GPCs is not changed, the HONAF will realize the function of adding the signal and input the signal into the bus.

In the GPCs (the blue medium columns in Fig. 2(b)), the scatterer is yttrium-iron-garnet (YIG) with permittivity $\varepsilon_1 = 15\varepsilon_0$, and permeability $\mu_0$ in the absence of external dc magnetic field. The initial structure parameters are set as that, lattice constant $a_1 = 14$ mm, scatterer radius $r_1 = 0.11a_1$, and the background material is air. In the PQCs (the yellow and white medium columns in Fig. 2(b)), the scatterer is alumina with permittivity $\varepsilon_2 = 15\varepsilon_0$, and permeability $\mu_0$. The lattice constant and the scatterer radius are $a_2 = 10$ mm and $r_2 = 0.15a_2$ respectively, and the background material is also air. The coupling channel width between the GPCs and the PQCs is $d = 0.875a_1$. When the external dc magnetic field is 1600 G and the frequency is 4.28 GHz [38], the permeability of the GPCs is the matrix tensor in the form of

$$\mu = \begin{bmatrix}
14\mu_0 & \pm 12.4i\mu_0 & 0 \\
\mp 12.4i\mu_0 & 14\mu_0 & 0 \\
0 & 0 & \mu_0
\end{bmatrix},$$

(19)

since the magnetic field breaks the time-reversal symmetry, the two straight waveguides possess nonreciprocal characteristics [50].

Next, the transmission performance of the HONAF is analyzed. With initial structure parameters (whose value obtained after structural parameters optimization) and the order $N = 1$ to 6, the transmission spectrum of the HONAF can be obtained [Fig. 3]. As can be seen from Fig. 3, when the HONAF is in each resonance state, $T_{2(N)} \approx 0$, $T_{4(N)} \approx 1$ and $T_{3(N)} \equiv 0$, the drop efficiency reaches almost 100 percent. Moreover, with the increase of $N$, more and more formants in the HONAF are generated, i.e. $\mu$ causes the resonance frequency of the isolated cavity to split, producing multiple resonance frequencies, and the number of resonance frequencies is equal to the number $N$ of cascaded cavities. In other words, the function of multiple filtering can be realized through cascaded cavities. The function of multiband transmission can be realized if it is applied to CROWs. Theoretically, the $N$ can be increased continuously to obtain the HONAF with more channels and realize the filtering function with any number of channels. It can be seen from Fig. 3 that when $N$ increases, the bandwidth of formants decreases, and the formants bandwidth at both ends is much narrower, while the formants bandwidth in the middle is much wider. Furthermore, with the increase of $N$, the out-of-band rejection ratios of the transmission spectrums of port P4 also increases. The above numerical simulation results are basically consistent with the theoretical prediction. It is worth noting that the cascaded cavities produce multiple resonance frequencies that are not symmetric with respect to $f_0$, but are randomly dispersed in the frequency interval around $f_0$, which is slightly different from the theoretical prediction. The reason is that the coupling distance between cascaded cavities is diminutive, resulting the interaction between the cascaded cavities is strong, and the interaction is not only from the near-est-neighbor cavities, but also from the next-nearest-neighbor cavities. In this case, the interaction coefficient cannot be simply described by $\mu$, more complex interaction factors need to be introduced, which is different from the original hypothesis of the theory above. Moreover, when the coupling distance between cascaded cavities is too diminutive, the t-CMT will no longer be accurate [22]. The explanation of this situation by the theory derived above is not reasonable, therefore, the numerical simulation is slightly inconsistent with the theory. In addition, by analyzing the electric field distribution laws in the cascaded cavities, it is also proved that the reason why the HONAF possesses multiple filtering function lies in the interaction between cascaded cavities from the perspective of photon local-

FIG. 3. Transmission spectrum of the HONAF
FIG. 4. The $Q$ of HONAF in each resonance state

Next, the quality factor distribution laws of the HONAF are analyzed. With the initial structure parameters and the order $N = 1$ to 24, the $Q$ in each resonance state are obtained by numerical calculation [Fig. 4]. As can be seen from Fig. 4, with the increase of $N$, the average quality factor $Q_{ave} = 1/N \sum_{i=1}^{N} Q_{Ni}$ also increases, indicating that performance of the HONAF becomes better with the increase of $N$. For the HONAF with particular order, the distribution of $Q$ at each resonance state is U-shaped. The $Q$ is the largest one at the resonance state of both left and right ends, while it is the smallest one at the intermediate resonance state (e.g. for the 3rd-order HONAF, $f_{032}$ is called the intermediate resonance state, while $f_{031}$ and $f_{033}$ are respectively called the left and right resonance states, and the other order HONAFs are similar). Moreover, the distribution laws of $Q$ at each resonant state is also consistent with the photon localization laws in the cascaded cavities mentioned above (see S. VII within the Supplemental Materials). At the intermediate resonance state, the photons are concentrated in the cavities at the left and right ends, which are easy to leak into the waveguide channels, hence the $Q$ is much smaller. When the cascaded cavities are at the resonance state of both left and right ends, the photons are concentrated in the intermediate cavities, which are not easy to leak, and the $Q$ is naturally much larger. This is also the reason why the $Q$ increases rapidly with the increase of $N$ (it is more and more difficult for photons to leak out) at the resonance state of both left and right ends of the HONAF, while the $Q$ increases slowly at the intermediate resonance state.

It is worth noting that the drop channels of the HONAF can be regulated by external dc magnetic field, i.e. the direction of the external dc magnetic field can be controlled to adjust the nonreciprocal direction of the add/drop waveguide, which determines the drop channels. For the HONAF designed above, change the external dc magnetic field direction of GPCs in the right half to $-z$, then the drop channel becomes $P_3$. With the original structural parameters, the transmission spectrums and the distribution diagrams of electric field modulus value at resonance state in 5th-order HONAF with defects or not are shown in Fig. 5. When the external dc magnetic field direction of the right half part of GPCs is $-z$, $T_{3(1)} \approx 1$ and $T_{2(1)} \approx 0$, the drop efficiency is also almost 100 percent. Meanwhile, $T_{4(1)} \equiv 0$ due to the nonreciprocal waveguide. By comparing Fig. 5(a) and 5(b), it can be seen that $P_3$ or $P_4$ is used as the drop channel, transmission efficiencies of the HONAF are basically the same. Therefore, changing the drop port will not affect the performance of the HONAF. Moreover, the nonreciprocal waveguide channels designed possess a topological nontrivial state, i.e. when there are defects in the HONAF waveguides, the light wave in the waveguides can bypass the defects and achieve 100 percent forward transmission. It can be found from Figs. 5(c), 5(d) and 5(e) that, light wave can bypass the defects and transmit forward without loss, indicating that waveguide channels in the HONAF is topologically protected.

FIG. 5. The distribution diagrams of electric field modulus value at resonance state $f_{052}$ in 5th-order HONAF in the different cases: (a) and (b), the external dc magnetic field direction of GPCs in the right half part is $+z$ and $-z$, respectively; (c), (d) and (e), reduce and increase the radius of the scatterer as indicated by the green dotted circle, replace the quasicrystal material alumina with silicon marked by green ellipse dotted box and an ideal electrical conductor marked by the white line, respectively. The green double arrow represents the light source and the green thick line represents the monitors.

IV. CONCLUSIONS

In this paper, the t-CMT is used to derive the transmission efficiency equations of the CCC system, which is applicable to the general situation. A design scheme of universal HONAF is proposed, which can improve the $Q$ and $T$ through cascaded cavities and realize the function of multi-channel filtering. It is concluded that
the interaction between cascaded cavities leads to the splitting of transmission spectrums from a single formant to multiple formants. In the case of $\mu = 0$, $\tau_0 = 0$ and $\tau_1 = \tau_2 = \tau_\alpha$, the change equation of the $Q_N$ of the single formant HONAF increases with the increase of $N$ is derived. A HONAF composed of the GPCs and the decagonal Penrose-type PQC is designed. With the change of $N$, the transmission characteristics and the change laws of $Q$ of the HONAF are obtained, which verifies the consistency between theoretical prediction and numerical simulation. By adjusting the direction of the external dc magnetic field, the convertibility of drop channel of the HONAF is verified, and the topological protection of the waveguide channels is also verified by introducing point defects (size and material of two scatterers changed) and line defects.

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Supplementary materials for
High-order nonreciprocal add-drop filter
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I. TRANSMISSION EFFICIENCY EQUATIONS OF THE HONAF

The following equations are derived to predict the transmission efficiency of HONAF with cascaded cavities. According to the t-CMT [SF1, SF2], in the cavities and waveguides, the evolution equations of the normalized amplitude of the fields over time are

\[
\frac{dA_{N1}}{dt} = (j2\pi f_0 - 1/\tau_0 - 1/\tau_1 - 1/\tau_0)A_{N1} - j\mu A_{N2} + \sqrt{2/\tau_1}S_1,
\]

(S1)

\[
\frac{dA_{N_i}}{dt} = (j2\pi f_0 - 1/\tau_0 - 2/\tau_0)A_{N_i} - j\mu A_{N(i-1)} - j\mu A_{N(i+1)} + \sqrt{2/\tau_a}S_{N(i-1)}, \quad (i = 2, 3, ..., N - 1),
\]

(S2)

\[
\frac{dA_{NN}}{dt} = (j2\pi f_0 - 1/\tau_0 - 1/\tau_2 - 1/\tau_0)A_{NN} - j\mu A_{N(N-1)} + \sqrt{2/\tau_a}S_{NN},
\]

(S3)

\[
S_{N_i} = \sqrt{2/\tau_a}A_{N_i}, \quad (i = 1, 2, ..., N - 1),
\]

(S4)

\[
S_{NN} = \sqrt{2/\tau_a}A_{NN}.
\]

(S5)

Suppose port \( P_4 \) is the drop channel, according to the nonreciprocity of waveguides, \( S_3 = 0 \), then,

\[
S_4 = S_{NN}.
\]

(S6)

According to the law of conservation of energy,

\[
S_2 = S_1 - S_4.
\]

(S7)

The physical significances of all the parameters are consistent with the text. In addition, the interaction between the next-nearest neighbor cavities has been ignored by the (S1) to (S3). When the coupling distance between the next-nearest neighbor cavities is far, and after the isolation of the intermediate cavities, this kind of neglect is allowed.

The input time harmonic fields, the HONAF system is a linear system, whose frequency is conserved. Therefore, if the mixed frequency of the fields input at port \( P_1 \) is \( f \), the fields oscillate in the system in the form of \( e^{j2\pi ft} \), and the result can be obtained

\[
\frac{dA_{Ni}}{dt} = j2\pi fA_{Ni}, \quad (i = 1, 2, ..., N).
\]

(S8)

Substituting (S4), (S5) and (S8) into the (S1) to (S3), and simplify the terms to get

\[
[j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_a]A_{N1} = -j\mu A_{N2} + \sqrt{2/\tau_1}S_1,
\]

(S9)

\[
[j2\pi(f - f_0) + 1/\tau_0 + 2/\tau_0]A_{N_i} = (-j\mu + 2/\tau_a)A_{N(i-1)} - j\mu A_{N(i+1)}, \quad (i = 2, 3, ..., N - 1),
\]

(S10)

\[
[j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_2 + 1/\tau_a]A_{NN} = (-j\mu + 2/\tau_a)A_{NN}.
\]

(S11)

Introduce parameters \( a, b, c, d \), and set as

\[
a = j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_a,
\]

(S12)

\[
b = j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_a,
\]

(S13)

\[
c = j\mu(-j\mu + 2/\tau_a),
\]

(S14)

\[
d = j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_2 + 1/\tau_a.
\]

(S15)

Substituting (S12) to (S15) into (S9) to (S11), the expression of \( A_{N1} \) can be solved from (S9). In (S10), set \( i = 2 \) and substitute the expression of \( A_{N1} \) into (S10) to solve the expression of \( A_{N2} \). Then set \( i = 3 \) in (S10), and substitute the expression of \( A_{N2} \) into the (S10) to solve the expression of \( A_{N3} \). Repeat this process to get intermediate equations,

\[
aA_{N1} = -j\mu A_{N2} + \sqrt{2/\tau_1}S_1,
\]

(S16)

\[
(ab + c)A_{N2} = -j\mu a A_{N3} + \sqrt{2/\tau_1}(-j\mu + 2/\tau_a)S_1,
\]

(S17)

\[
[(ab + c)b + ac]A_{N3} = -j\mu(ab + c)A_{N4} + \sqrt{2/\tau_1}(-j\mu + 2/\tau_a)^2S_1,
\]

(S18)

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\[
\{(ab + c)b + ac\}A_{N4} = -j\mu \\
\times[(ab + c)b + ac]A_{N5} + \sqrt{2/\tau_1(-j\mu + 2/\tau_a)^3}S_1,
\]  
(S19)

... 

The coefficients of \(A_{Ni}\) in (S16) to (S19) possess an iterative relation, which is actually determined by the relation between (S1) to (S5), and the intermediate equations are set as

\[
H_nA_{Nn} = -j\mu F_nA_{N(n+1)} + \sqrt{2/\tau_1} \times (-j\mu + 2/\tau_a)^{n-1}S_1, \quad (n = 1, 2, ..., N - 1),
\]  
(S20)

\(H_n\) and \(F_n\) are defined as sequence of functions of \(n\), where \(n\) is a positive integer, and \(n_{max} = N - 1\). From the (S16) to (S19), can get

\[
H_1 = a,
\]  
(S21)

\[
H_2 = bH_1 + c,
\]  
(S22)

\[
H_3 = bH_2 + cH_1,
\]  
(S23)

\[
H_4 = bH_3 + cH_2,
\]  
(S24)

\[
F_1 = 1,
\]  
(S25)

\[
F_2 = a,
\]  
(S26)

\[
F_3 = bF_2 + c,
\]  
(S27)

\[
F_4 = bF_3 + cF_2.
\]  
(S28)

The general term formulas of \(H_n\) and \(F_n\) can be obtained from (S21) to (S28). The correctness of this recursive method is proved by the original (S1) to (S5). Therefore,

\[
H_{n+2} = bH_{n+1} + cH_n, \quad (n = 1toN - 1),
\]  
(S29)

\[
F_{n+2} = bF_{n+1} + cF_n, \quad (n = 1toN - 1).
\]  
(S30)

Obviously, the recursive expressions of \(H_n\) and \(F_n\) are consistent. The general term formulas of \(H_n\) and \(F_n\) is obtained by using the method of characteristic equation. Hence, let the general term formulas of \(H_n\) and \(F_n\) be respectively,

\[
H_n = w_1x_1^n + w_2x_2^n,
\]  
(S31)

\[
F_n = v_1x_1^n + v_2x_2^n,
\]  
(S32)

where, \(w_1, w_2, v_1\) and \(v_2\) are the coefficients, \(x_{1,2}\) is the solution of the characteristic equation, and

\[
x_{1,2} = \frac{b \pm \sqrt{b^2 + 4c}}{2},
\]  
(S33)

\[
w_1 = \frac{ab + c - ax_2}{x_1^2 - x_1x_2}, \quad w_2 = \frac{ab + c - ax_1}{x_2^2 - x_1x_2},
\]  
(S34)

\[
v_1 = \frac{a - x_2}{x_1^2 - x_1x_2}, \quad v_2 = \frac{a - x_1}{x_2^2 - x_1x_2}.
\]  
(S35)

By substituting (S34) and (S35) into (S31) and (S32) respectively, the general term formulas of \(H_n\) and \(F_n\) are obtained as

\[
H_n = \frac{(ab + c - ax_2)x_1^{n-1}}{x_1 - x_2} + \frac{(ab + c - ax_1)x_2^{n-1}}{x_2 - x_1},
\]  
(S36)

\[
F_n = \frac{(a - x_2)x_1^{n-1}}{x_1 - x_2} + \frac{(a - x_1)x_2^{n-1}}{x_2 - x_1}.
\]  
(S37)

Substitute (S36) and (S37) back into (S20), and let \(n = N - 1, (n \geq 1 \rightarrow N \geq 2)\), the expression of \(A_{N(N-1)}\) can be solved. Substitute this expression into (S11), and the expression of \(A_{NN}\) can be obtained as

\[
A_{NN} = \frac{\sqrt{2/\tau_1}(-j\mu + 2/\tau_a)^{N-1}S_1}{dH_{N-1} + cF_{N-1}}.
\]  
(S38)

Then substitute (S38) into (S5), and utilize (S6) and (S7), then get

\[
S_4 = S_{NN} = \frac{2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_a)^{N-1}S_1}{dH_{N-1} + cF_{N-1}},
\]  
(S39)

\[
S_2 = S_1 - S_4 = (1 - \frac{2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_a)^{N-1}}{dH_{N-1} + cF_{N-1}})S_1.
\]  
(S40)

Hence, the transmission efficiencies of HONAF system is

\[
T_{2(N)} = \left| \frac{S_2}{S_1} \right|^2 = \left| 1 - \frac{2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_a)^{N-1}}{dH_{N-1} + cF_{N-1}} \right|^2, \quad (N \geq 2),
\]  
(S41)

\[
T_{3(N)} = \left| \frac{S_3}{S_1} \right|^2 = 0, \quad (N \geq 1),
\]  
(S42)

\[
T_{4(N)} = \left| \frac{S_4}{S_1} \right|^2 = \left| \frac{2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_a)^{N-1}}{dH_{N-1} + cF_{N-1}} \right|^2, \quad (N \geq 2).
\]  
(S43)
In the case of $N = 1$, it’s easy to get
\[
T_{2(1)} = |1 - \frac{2}{j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2}|^2, \tag{S44}
\]
\[
T_{4(1)} = \left| \frac{2}{j2\pi(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2} \right|^2. \tag{S45}
\]

Now, consider the polynomial $dH_{N-1} + cF_{N-1}$. Substitute (S33) into (S36) and (S37), set $n = N - 1$, ($N \geq 2$), and then substitute $dH_{N-1} + cF_{N-1}$, which is combined and simplified to get
\[
dH_{N-1} + cF_{N-1} = \frac{abd + 2ac + 2cd - bc + (ad + c)(b^2 + 4c)^{1/2}}{2^{N-1}(b^2 + 4c)^{1/2}[b + (b^2 + 4c)^{1/2}]^{2-N}}, \tag{S46}
\]
\[
= \frac{1}{2^{N-1}(b^2 + 4c)^{1/2}[b - (b^2 + 4c)^{1/2}]^{2-N}}.
\]

The expressions for $T_{4(2)}$, $T_{4(3)}$, and $T_{4(4)}$ are listed below,
\[
T_{4(2)} = \left| 2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_0)/\{\sqrt{2\pi}(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2 + 1/\tau_3\} \right|^2, \tag{S47}
\]
\[
T_{4(3)} = \left| 2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_0)^2/\{\sqrt{2\pi}(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2 + 1/\tau_3\} \right|^2, \tag{S48}
\]
\[
T_{4(4)} = \left| 2/(\tau_1\tau_2)^{1/2}(-j\mu + 2/\tau_0)^3/\{\sqrt{2\pi}(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_2 + 1/\tau_3\} \right|^2. \tag{S49}
\]

\section*{II. IN THE CASE OF $\mu \to 0$, TRANSMISSION SPECTRUMS OF THE HONAF SYSTEM}

When order $N = 1$ to 3, according to (S45) and (S53), the transmission spectrums of all the ports of HONAF under a specific $f_0$, $\tau_1$, $\tau_2$ and $\tau_a$ (ignoring the influence of $\tau_0$) are obtained, as shown in Fig. S1.

It can be intuitively seen from Fig. S1(a) to S1(c) that when $\tau_1$, $\tau_2$ and $\tau_a$ are not the same, $1 > T_{4(1)\text{max}} > T_{4(2)\text{max}} = T_{4(3)\text{max}}$. According to Fig. S1(d) to S1(f), when $\tau_1 = \tau_2 = \tau_a$, $T_{4(N)\text{max}} = 1$. It can also be intuitively seen from Fig. S1 that the bandwidth of transmission spectrums decreases with the increase of $N$, i.e. the $Q$ of the HONAF will increase with the increase of $N$.

\section*{III. IN THE CASE OF $\tau_1 = \tau_2 = \tau_a, \tau_0 = 0$ AND $\mu = 0$, QUALITY FACTOR EQUATIONS OF THE HONAF SYSTEM}

The quality factor is generally defined as $Q = \pi f_0 \tau$, ($Q = f_0/\Delta f$ can also be defined from the transmission spectrums, $\Delta f$ is the full width at half maximum (FWHM) of the passband) [SF3], and $1/\tau = \sum_{i=1}^{\infty} 1/\tau_i$, where $\tau$ represents the total lifetimes, $\tau_i$ represents the lifetimes of each leaky components, and $f_0$ represents the resonance frequency of cavity [SF1]. Assume that the quality factor of a single isolated cavity is $Q_0$, for the model in this paper (the self-loss of the cavities is ignored), it can be obtained
\[
Q_0 = \pi f_0 \tau, \tag{S50}
\]
\[
1/\tau = 1/\tau_1 + 1/\tau_2, (N = 1), \tag{S51}
\]
\[
1/\tau = 1/\tau_1 + 1/\tau_a = 1/\tau_2 + 1/\tau_a, (N \geq 1), \tag{S52}
\]
\[
(12) \text{ and } (17) \text{ in the text are also utilized, that is}
\]
\[
T_{4(N)} = \left| \frac{2^{2N}/(\tau_1^{1/2}\tau_2^{1/2}\tau_a^{N-1})}{\sqrt{2\pi}(f - f_0) + 1/\tau_0 + 1/\tau_1 + 1/\tau_a} \right|^2
\times \left| \frac{1}{\sqrt{2\pi}(f - f_0) + 1/\tau_0 + 1/\tau_2 + 1/\tau_a} \right|^2,
\]
\[
\tau_1 = \tau_2 = \tau_a. \tag{S54}
\]

Substituting (S50), (S51) and (S54) into (S45); (S50), (S52) and (S54) into (S53), and then combine and simplify to get
\[
T_{4(N)} = \left[ \frac{1/2Q^2}{(f - f_0)^2/f_0^2 + 1/(2Q_0 f_0)^2} \right]^N, (N \geq 1). \tag{S55}
\]

Assume that the above equation is $T_{4(N)}(f = f_1) = 1/2$, $f_1$ is the position of the FWHM frequency points from the transmission spectrums, and utilize
\[
Q_N = f_0/\{2f_1 - 2f_0\}. \tag{S56}
\]

Simplify to get
\[
Q_N = Q_0(2^{1/N} - 1)^{-1/2}, (N \geq 1). \tag{S57}
\]
IV. TRANSMISSION SPECTRUMS OF THE HONAF SYSTEM WITH μ PLAYING A LEADING ROLE

According to (S43), under some parameter values of $N$, $\tau_1$, $\tau_2$, $\tau_a$, and $\mu$, the transmission spectrums of all the ports of the HONAF (Specific expressions refer (S47) to (S49)) is shown in Fig. S3.

As can be seen from Fig. S3(a) to S3(c), with the increase of $\mu$, $T_{4(N)}$ is divided from a single formant ($\mu = 0$) into multiple formants ($\mu \neq 0$, and the number of formants is equal to $N$). When $\mu$ is diminutive, the formants appear peak spikes and split to both ends, as shown in Fig. S3(b) black arrow. When $\mu$ is large enough, the formants split completely, as shown in Fig. S3(c). By comparing the $N = 2$ and $N = 3$ transmission spectrums of port $P_4$ in Figs. S3(b) and S3(c), it is found that when $N$ value is large, it is more likely to split into multiple formants. This means that in the actual HONAF structure, even if the coupling distance between the nearest neighbor cavities is large, as $N$ increases, its transmission spectrums will inevitably split from single formant to multiple formants. By comparing the transmission spectrums of $P_4$ port in Figs. S3(c) and S3(d), it is found that when $\mu$ continues to increase ($\tau_a$ changes correspondingly; $\tau_1$ and $\tau_2$ remain unchanged, and $\tau_0 = 0$), the distance between the formants will increase, which the bandwidth will decrease. It can be seen from Figs. S3(e) and S3(f) that the formants on both
FIG. S3. The transmission spectrums of all the ports of the HONAF. Assuming $f_0 = 0.56365 \times c/a$, $\tau_1 = \tau_2 = 8.5765 \times 10^{-7}$ s, ignoring the influence of $\tau_0$, (a)-(c) is the transmission spectrums of port $P_4$ under different $\tau_a$ and $\mu$ when $N = 2, 3$. (d)-(f) respectively represent the transmission spectrums of all the ports when $N = 2, 3$ and 4, and when $\tau_a$ and $\mu$ are determined. Parameter values (a) $\tau_a = 8.5765 \times 10^{-7}$ s, $\mu = 0$; (b) $\tau_a = 8.5765 \times 10^{-4}$ s, $\mu = 3.0670 \times 10^6$ s$^{-1}$; (c) $\tau_a = 1.5765 \times 10^{-3}$ s, $\mu = 1.6670 \times 10^7$ s$^{-1}$; (d)-(f) $\tau_a = 2.8677 \times 10^{-1}$ s, $\mu = 1.3674 \times 10^8$ s$^{-1}$.

sides possess a diminutive bandwidth, while the formants in the middle possess a large bandwidth. Furthermore, the central frequency points of each formants after splitting are symmetric about $f_0$.

V. TRANSMISSION EFFICIENCY EQUATIONS OF THE CROWS SYSTEM

The CROWS are composed of multiple cavities coupled side by side in a cascade [SF3, SF4]. Therefore, the two waveguide channels of the above four-port system are deleted, or the waveguide channels are changed into input and output couplers (i.e. the waveguide is the input and output port), which constitute the basic model of the universal CROWS system.

It is assumed that the input and output ends are fully coupled to the cascaded cavities and that the waveguide and each single cavity support only one mode in the same frequency band. The normalized amplitudes of the input and output fields of the waveguide are $S_x$ and $S_y$, respectively. According to the equations above, the output end $S_y$ of the CROWS is

$$S_y = S_{NN} = S_4,$$

and $S_x = S_1$, then the transmission efficiency of the output is

$$T = \left| \frac{S_y}{S_x} \right|^2 = \left| \frac{S_4}{S_1} \right|^2. \tag{S59}$$

Obviously, this equation is consistent with (S43). This equation can be used for the analysis of any CROWS system, is not limited to the input and output is similar the coupler of reciprocity waveguide structure CROWS. Moreover, this equation can be used to analyze the variation of the transmission efficiency of the CROWS with the coupling intensity of the cascade cavities (limited to the weakly coupled system, and the input and output ends are fully coupled to the cascaded cavities).

VI. MODEL AND PERFORMANCE ANALYSIS OF THE HONAF S

The theoretical model established in this paper can be widely used in various cascaded cavities systems. In order to prove the correctness of the theoretical prediction and the consistency with the simulation results, the basic cascaded unit S (yellow scatterers in Fig. S4(a)) is arbitrarily taken, which constructs the HONAF S, as shown in Fig. S4(b). The topological waveguide channels are constructed by using GPCs, whose structure parameters and related settings are consistent with the text HONAF. As can be seen from the structure of HONAF S in Fig. S4(b), the coupling distance between the nearest-neighbor cavities is enough large, so that $\mu$ is enough diminutive, and it can be considered that $\mu = 0$. According to Section II of the text, the formants should be degenerate with different $N$ values, with only one single
The transmission performance of the HONAF S is analyzed below. With initial structure parameters and when order $N = 1$ to 6, the transmission spectrums of all the ports of HONAF S are obtained, as shown in Fig. S5. The numerical simulation results of the HONAF S in Fig. S5 are basically consistent with the theoretical prediction. When the HONAF S is in resonance state and $N = 1$, $T_2(1) \approx 0$, $T_4(1) \approx 1$, the drop efficiency reaches almost 100 percent, while $T_3(1) \equiv 0$ due to the nonreciprocal waveguide. When $N = 2$ to 6, $T_2(N) > 0$, $T_4(N) < 1$, the reason is $\tau_1 \neq \tau_a$ ($\tau_1 = \tau_2$ is known from the symmetry of the structure) as mentioned in Section II of the text. In addition, when $N$ increases, peak spikes appear in the $P_4$ port transmission spectrum. This is because, with the increase of $N$, the interaction between cascaded cavities is enhanced, so that single formant tends to split into multiple formants. Moreover, with the increase of $N$, the bandwidth of a single formant decreases, which also conforms to the theoretical results. When $N = 1$ to 6, the transmission spectrums of port $P_4$ is shown in Fig. S6.

As can be seen from Fig. S6, with the increase of $N$, the out-of-band rejection ratios of the HONAF S also increases. When $N = 6$, the out-of-band rejection ratios reaches 85 dB. With the increase of $N$, the formants in the $T_4(N)$ transmission spectrums gradually changes from a peak to a rectangular peak (or flat peak), whose position moves toward the high frequency (i.e. blueshifts), as shown by the black arrow in Fig. S6. According to Fig. S5, it is found that the increase of the out-of-band rejection ratios, the appearance of the rectangular peak and the blueshifts of the formants are all caused by the enhancement of the interaction between cascaded cavities with the increase of $N$.

The distribution laws of electric fields $E_z$ in the cascaded cavities resonance of the HONAF S are analyzed below. Through the studies, it is found that when $N$ increases, there are $N$ distribution forms of cascaded cavities electric field, i.e. there are $N$ resonance states of the HONAF S. When the order $N = 1$ to 3, the electric fields distribution diagrams of cascaded cavities resonance are obtained, as shown in Fig. S7.

Obviously, the resonance form of cascaded cavities is whispering gallery modes. In this kind of structure, the photons are concentrated in a single cavity and do not

FIG. S4. (a) Decagonal Penrose PQCs structure; (b) 2nd-order HONAF model.

FIG. S5. Transmission spectrums of all the ports of the HONAF S.

FIG. S6. When $N = 1$ to 6, transmission spectrums of port $P_4$ of the HONAF S.

FIG. S7. $N = 1$ to 3, the cascaded cavities of HONAF S is in resonance, the electric fields $E_z$ distribution diagrams. $f_0$ represents the resonance frequency, and the subscript $Ni$ represents the $i$-th resonance state of the cascaded $N$ cavities, the same as below.
remain between adjacent cavities, indicating that the interaction between cascaded cavities is weak. The electric fields in the cascaded cavities are distributed in positive and negative phases, and for a certain value of $N \geq 2$, the electric fields directions of each pair of adjacent cavities in the cascaded cavities are both symmetric and antisymmetric. Because the interaction between the cascaded cavities is diminutive, the difference between the resonance frequencies of different resonance states under a certain $N$ value is very diminutive. Therefore, from the perspective of the transmission spectrums, each resonance frequency point is degenerate. In fact, as long as $\mu \neq 0$ is satisfied, the frequency splitting situation must exist. With the increase of $N$ value, the number of resonant states of the cascaded cavities increases, and the influence of $\mu$ also increases, so that the frequency splitting trend becomes obvious. As shown in Fig. S5, the transmission spectrums of the HONAF S shows peak spikes.

According to each resonant state of the cascaded cavities, a quality factor $Q$ can be obtained by numerical calculation to measure the filtering performance of this resonant state. Therefore, the quality factor of each resonant state in Fig. S7 is calculated: $Q_{011} = 60792$, $Q_{021} = 121338$, $Q_{022} = 121326$, $Q_{031} = 216732$, $Q_{032} = 121368$, and $Q_{033} = 272423$. However, according to (S54), $Q_{N=1} = 61910$, $Q_{N=2} = 118419$, and $Q_{N=3} = 118440$ can be calculated from the transmission spectrums. When $N \neq 1$, the $Q$ value obtained by the above two methods describes a completely different state. The former represents the frequency selection performance of a single resonant frequency point in the cascaded cavities, and the corresponding frequency point is completely split. This calculation method is not applicable to models such as the HONAF S (degeneracy of each resonant frequency point). The latter describes the frequency selection performance of cascaded cavities resonance frequency degeneracy, which should be selected in practical application for the models such as the HONAF S. The difference superposition of the resonance frequencies points (blueshifts of the resonance frequencies points, without complete degeneracy) increases the bandwidth of the formants after superposition. Hence, the exception of $Q_{N=1} < Q_{N=2}$, in the case of $N \geq 2$, the actual quality factor $Q$ is going to decrease as $N$ goes up. Therefore, in this case, more cavities through cascades have little contribution to the increase of quality factor of the HONAF S, the number of cascaded cavities $N_{max} = 2$. Increasing the coupling distance between the cascaded cavities is an attempt to improve the quality factor (i.e. reducing the interaction factor $\mu$ between the cascaded cavities). However, as mentioned in Section II of the text, excessive coupling distance will lead to a decrease in the coupling rate between the waveguide channels and the cavities and a decrease in drop efficiency. However, for CROWs, the bandwidth increases and the transmission spectrums changes from a peak to a rectangular peak are excellent [SF3].

VII. DISTRIBUTION LAWS OF THE ELECTRIC FIELDS $E_z$ IN THE HONAF

The distribution laws of electric fields $E_z$ in the cascaded cavities resonance of the HONAF are analyzed. When the order $N = 1$ to 6, the electric fields distribution diagrams of the cascaded cavities in resonance can be calculated, as shown in Fig. S8. The resonance mode of cascaded cavities is also whispering gallery modes. The electric fields in the cascaded cavities are distributed in positive and negative phases, and for a certain value of $N \geq 2$, the electric fields directions of each pair of cavities are both symmetric and antisymmetric. As can be seen from Fig. S8, when the HONAF is in the intermediate resonance state the photons are concentrated in the cavities at the left and right ends, while in the resonance state at the left and right ends, the photons are concentrated in the cavities in the middle, and the trend of this law is more obvious with the increase of $N$. In addition, because the coupling distance between the cascaded cavities is close, the cascaded cavities almost form a whole structure. The photons are not only localized in the cavities, but also localized between adjacent cavities. Therefore, the cascaded cavities can be regarded as a whole cavity structure, which possesses multiple resonance frequencies. The positions of these resonance frequencies are determined by the structure of the whole cavity itself. In this case, the distribution of multiple resonance frequencies in the whole cavity is more complex, which may possess symmetrical properties or be dispersed randomly. The results are consistent with those obtained in Section II of text.

FIG. S8. When $N = 1$ to 6, the cascaded cavities of the HONAF is in resonance, the electric fields $E_z$ distribution diagrams.
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