LEFT 3-ENGEL ELEMENTS IN GROUPS

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Abstract. In this paper we study left 3-Engel elements in groups. In particular, we prove that for any prime $p$ and any left 3-Engel element $x$ of finite $p$-power order in a group $G$, $x^p$ is in the Baer radical of $G$. Also it is proved that $\langle x, y \rangle$ is nilpotent of class 4 for every two left 3-Engel elements in a group $G$.

1. Introduction and results

Let $G$ be any group and $n$ be a non-negative integer. For any two elements $a$ and $b$ of $G$ we define inductively $[a, n b]$ the $n$-Engel commutator of the pair $(a, b)$, as follows:

$$[a, 0] := a, \quad [a, b] := a^{-1}b^{-1}ab, \quad \text{and} \quad [a, n b] := [[a, n-1 b], b] \quad \text{for all} \quad n > 0.$$ 

An element $x$ of $G$ is called a left $n$-Engel element if $[g, n x] = 1$ for all $g \in G$. We denote by $L_n(G)$, the set of all left $n$-Engel elements of $G$. So $L_0(G) = 1$, $L_1(G) = Z(G)$ the centre of $G$, and it can be easily seen that

$$L_2(G) = \{ x \in G \mid \langle x \rangle^G \text{ is abelian } \},$$

where $\langle x \rangle^G$ denotes the normal closure of $x$ in $G$. Therefore $L_2(G)$ is contained in $B(G)$ the Baer radical of $G$, and in particular it is contained in $HP(G)$ the Hirsch-Plotkin radical of $G$. In general for an arbitrary group $K$ it is not necessary that $L_n(K) \subseteq HP(K)$. For suppose, for a contradiction, that $L_n(K) \subseteq HP(K)$ for all $n$ and all groups $K$. By a deep result of Ivanov [1], there is a finitely generated infinite group $M$ of exponent $2^k$ for some positive integer $k$. Suppose that $k$ is the least integer with this property, so every finitely generated group of exponent dividing $2^{k-1}$ is finite. Let $a$ be any element of $M$ of order 2 and $x$ an arbitrary element of $M$. It is easy to see that $[x, m a] = [x, a](-2)^{m-1}$ for all positive integers $m$. Thus every element of order 2 of $M$ is in $L_{k+1}(M)$. So by hypothesis, $M/HP(M)$ is of exponent dividing $2^{k-1}$ and so it is finite. Since $M$ is finitely generated, $HP(G)$ is also. But this yields that $HP(G)$ is a periodic finitely generated nilpotent group and so it is finite. It follows that $M$ is finite, a contradiction.

Hence the question which naturally arises, is that: What is the least positive integer $n$ such that $L_n(G) \not\subseteq HP(G)$? To study this question it should first study the case $n = 3$, since

$$L_0(G) \subseteq L_1(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots.$$ 

The main object of this paper is to study $L_3(G)$. As far as we know there is no example of a group $G$ for which $L_3(G) \not\subseteq HP(G)$. The corresponding subset to
Lemma 2.1. Suppose that $L_0(G)$ which can be similarly defined, is $R_n(G)$ the set of all right $n$-Engel elements of $G$: an element $x$ of $G$ is called a right $n$-Engel element if $[x, y] = 1$ for all $y \in G$. There is a well-known relation between these two subsets of $G$ due to Heineken [4] Theorem 7.11: $(R_n(G))^{-1} \subseteq L_{n+1}(G)$ for all integers $n > 0$. It is clear that $R_0(G) = 1$, $R_1(G) = Z(G)$ and by a result of Kappe [4] Corollary 1 Theorem 7.13, $R_2(G)$ is a characteristic subgroup of $G$. It is known also that $R_2(G) \subseteq L_2(G)$ [4] Theorem 7.13 (i)]. Recently Newell [3] has shown that the normal closure of every element of $R_3(G)$ is nilpotent of class at most 3. This shows, of course, that $R_3(G) \subseteq B(G) \subseteq HP(G)$. An early example was given by Macdonald [2] shows that the inverse and square of a right 3-Engel element need not be right 3-Engel, thus $R_3(G)$ is not necessarily a subgroup. But as we show in Corollary 2.2 $L_3(G)$ is closed under the taking all powers. It is interesting to note that $(R_3(G))^2 \subseteq L_3(G)$ and $(R_3(G))^4 \subseteq R_3(G)$ [3] Remark to Theorem 1].

The main results of this paper are the followings.

Theorem 1.1. Let $G$ be a group and $p$ be a prime number. If $x \in L_3(G)$ and $x^p = 1$ for some integer $n > 1$, then $(x^p)^G$ is soluble of derived length at most $n - 1$ and $x^p$ belongs to $B(G)$, the Baer radical of $G$. In particular, $x^p$ belongs to $H^P(G)$, the Hirsch-Plotkin radical of $G$.

Theorem 1.2. Let $G$ be any group and $a, b \in L_3(G)$. Then $(a, b)$ is nilpotent of class at most 4.

2. PROOFS

Lemma 2.1. Suppose that $G$ is an arbitrary group and $x, y \in G$. Then $[y, x] = [y^{-1}, x] = 1$ if and only if $(x, x^y)$ is nilpotent of class at most 2.

Proof. Since $[y, x] = [y^{-1}, x^{-1}, [x^{-1}, y]]^{-1}$

where $\epsilon \in \{-1, 1\}$, we have

$[y, x] = [y^{-1}, x] = 1$

$\iff [x^{-1}, [x^{-1}, [x^{-1}, y]]] = [x^{-1}, [x^{-1}, [x^{-1}, y^{-1}]]] = 1$

$\iff [x^{-x^{-y}}, x^{-1}] = [x^{-x^{-y}}, x^{-1}] = 1$

$\iff [x^{-y}, x^{-1}] = [x^{-y}, x^{-1}] = 1$

$\iff [x^{-y}, x^{-1}] = [x^{-y}, x^{-1}] = 1$

$\iff (x^{-1}, x^{-y})$ is nilpotent of class at most 2.

So we have the following characterization of left 3-Engel elements in a group. We denote by $\mathcal{N}_2$ the class of nilpotent groups of class at most 2.

Corollary 2.2. For an arbitrary group $G$,

$L_3(G) = \{x \in G \mid (x, x^y) \in \mathcal{N}_2 \text{ for all } y \in G\}$.

In particular every power of a left 3-Engel element is also a left 3-Engel element.
Proposition 2.3. A group generated by a set of left 3-Engel elements of finite orders such that their orders are pairwise coprime is abelian.

Proof. It is enough to show that \([a, b] = 1\) for any two left 3-Engel elements \(a\) and \(b\) such that \(\gcd(|a|, |b|) = 1\). By Corollary 2.2, \(K = \langle a, a^b \rangle\) and \(H = \langle b, b^a \rangle\) are both nilpotent. Thus \([a, b] = a^{-1}a^b = (b^{-1})^a b \in K \cap H\). Since \(\gcd(|a|, |b|) = 1\) and \(H\) and \(K\) are both nilpotent, \(\gcd(|H|, |K|) = 1\). It follows that \([a, b] = 1\), as required.

Lemma 2.4. Let \(p\) be a prime number, \(G\) be a group and \(x \in L_3(G)\). If \(x^{p^n} = 1\) for some integer \(n \geq 2\), then \(x^{p^{n-1}} \in L_2(G)\).

Proof. Let \(y\) be an arbitrary element of \(G\). By Corollary 2.2, \(\langle x, x^y \rangle\) is nilpotent of class at most 2. Thus

\[
\begin{align*}
[(x^{-y})^{p^{n-1}}, x^{p^{n-1}}] &= \\
[(x^{-y})^{p^{n-2}}, x^{p^{n-1}}]^{p} &= \text{since } [x^{-y}, x] \in Z(\langle x, x^y \rangle) \\
[(x^{-y})^{p^{n-2}}, x^{p^n}] &= 1
\end{align*}
\]

But \([y, x^{p^{n-1}}, x^{p^n}] = [(x^{-y})^{p^{n-1}}, x^{p^n}] = 1\). This completes the proof.

Proof of Theorem 1.1. By Lemma 2.4 and Corollary 2.2,

\[
1 \leq \langle x^{p^{n-1}} \rangle^G \leq \langle x^{p^{n-2}} \rangle^G \leq \cdots \leq \langle x^p \rangle^G
\]

is a series of normal subgroups of \(G\) with abelian factors. This implies that \(K = \langle x^p \rangle^G\) is soluble of derived length at most \(n - 1\). By Corollary 2.2, \(x^p\) and so all its conjugates in \(G\) belong to \(L_3(G)\) and in particular they are in \(L_3(K)\). Now a result of Gruenberg [3] Theorem 7.35 implies that \(B(K) = K\). But \(K \trianglelefteq G\) which yields that \(x^p \in K \leq B(G) \leq HP(G)\).

In the following calculations, one must be careful with notation. As usual \(u^{g_1+g_2}\) is shorthand notation for \(u^{g_1}u^{g_2}\). This means that

\[
u^{(g_1+g_2)(h_1+h_2)} = u^{(g_1+g_2)h_1}u^{(g_1+g_2)h_2}
\]

which does not have to be equal to \(u^{g_1(h_1+h_2)}u^{g_2(h_1+h_2)}\). We also have that

\[
u^{(g_1+g_2)(-h)} = (u^{g_1}u^{g_2})^{-1}h
\]

which is equal to \(u^{-g_2h-g_1h}\). This does not have to be the same as \(u^{-g_1h-g_2h}\). The following remark easily follows from Corollary 2.2. We use in sequel this remark, sometimes without any reference.

Remark 1. Let \(G\) be any group and \(a \in L_3(G)\). Then

\[
[a, x]^{a^2} = [a, x]^{2a-1} \text{ and } [a, x]^{a^{-1}} = [a, x]^{-a+2}\text{ for all } x \in G.
\]

Lemma 2.5. Let \(G\) be any group and \(a, b \in L_3(G)\). Then

\[
\langle [a, b], [a, b]^a, [a, b]^b, [a, b]^{ab} \rangle
\]
Lemma 2.6. Let $\langle a, b \rangle$ be a group of order 3. Then both of the commutators $[a, b]_2$ and $[a, b]_3$ belong to $C_2([a, b])$.

Proof. By Theorem 1.2, $[a, b]_2$ and $[a, b]_3$ are normal subgroups of $[a, b]$. Thus $[a, b]_2$ and $[a, b]_3$ are normal subgroups of $[a, b]$.

Proof of Theorem 1.2. We first prove that the derived subgroup of $\langle a, b \rangle$ is abelian. Since $\langle a, b \rangle'$ is the normal closure of $[a, b]$ in $\langle a, b \rangle$, it is enough to show that $[a, b]$ is in the centre of $\langle a, b \rangle'$ and by Lemma 2.1 it suffices to prove that $[a, b]_2, [a, b]_3= 1$.

Let $a^2 = [a, b]$ and $c = [a, b]$. Then $\langle a, a_1 \rangle$, $\langle a_1, a_2 \rangle$ and $\langle a_1, a_2 \rangle$ are each nilpotent of class at most 2, by Lemma 2.1. Now $c = a^{-1}a_1$ and $c^b = a_1^{-1}a_2$ and these commute by Lemma 2.1. But

$$1 = [c, b] = [a_1^{-1}a_2, a^{-1}a_1] = [a_1^{-1}a_2, a_1][a_1^{-1}a_2, a^{-1}]a_1.$$

Hence $[a_1, a_2] = 1$. Now $[a_1, a] = [a, b]$ and $a_2 = a^{-2}[c, b]$. Since $[c, a]$ commutes with $a$ and $c$, it follows that $[c, a, [a, b]] = 1$. Now it follows from Lemma 2.1 that $[c, a, b] = 1$. Now by Remark 1 we have $c^{-1} = c^{-b}c^2$. It follows that $[c, a, b] = 1$.

Thus $\langle a, b \rangle$ is metabelian. Now for completing the proof, it is enough to show that $[a, b, x_1, x_2, x_3] = 1$ for all $x_1, x_2, x_3 \in \{a, b\}$. Since $\langle a, b \rangle$ is metabelian we have $[a, b, x_1, x_2, x_3] = [a, b, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ for all permutations $\sigma$ on the set $\{1, 2, 3\}$. Now $a, b \in L_3(G)$, it is easy to see that the equality (*) is satisfied by all $x_1, x_2, x_3 \in \{a, b\}$. This completes the proof.
The author could not prove a similar result to Theorem 1.2 for groups generated by 3 left 3-Engel elements. We finish this paper by the following question.

**Question.** Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that every nilpotent group generated by $d$ left 3-Engel elements is nilpotent of class at most $f(d)$?

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