Global Weak Solutions to the Density-Dependent Hall-Magnetohydrodynamics System

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Abstract. We are concerned with the global existence of finite energy weak solutions to 3D density-dependent magnetohydrodynamics system with Hall-effect set in a general smooth bounded domain. The perfectly conducting wall boundary condition is imposed on the magnetic field. Due to the degeneracy of Hall-effect term (a higher-order trilinear term) in vacuum, we assumed initial density lies in the bounded function space and having a positive lower bound. Particularly, these bounds are preserved by the density transport equation which helps to yield a satisfactory compactness argument of the magnetic field.

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1. Introduction and Main Results

1.1. Introduction

In this paper, we consider the following three dimensional density-dependent or inhomogeneous incompressible magnetohydrodynamics system that includes the Hall-effect (Hall-MHD):

\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho) d(u)) + \nabla P &= \text{curl}B \times B, \\
\text{div} u &= 0, \\
\partial_t B + \text{curl} \left( B \times u + h \frac{\text{curl}B \times B}{\rho} \right) &= -\text{curl} \left( \frac{\text{curl}B}{\sigma(\rho)} \right).
\end{align*}

The unknowns are the density of the fluid $\rho$, the fluid velocity $u \in \mathbb{R}^3$, the magnetic field $B \in \mathbb{R}^3$ and the scalar pressure $P$. We denote by $d(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the stress tensor, $\mu = \mu(\rho)$ the fluid viscosity and $\sigma = \sigma(\rho)$ its electrical conductivity (this dependence enables us to consider the motion of several immiscible fluids with various viscosities and conductivities), both conductivity and viscosity being positive continuous functions on $[0, \infty)$. The number $h$ measures the magnitude of the Hall-effect term $\text{curl}((\text{curl}B \times B)/\rho)$ compared to the typical length scale of the fluid.

The above system is used to model the evolution of electrically conducting fluids such as plasmas or electrolytes (then, $u$ represents the ion velocity) and takes into account the fact that in a moving conductive fluid, the magnetic field can induce currents which, in turn, polarize the fluid and change the magnetic field. In the work of Acheritogaray, Degond, Frouvelle and Liu [1], they derived the following generalized Ohm’s law from the two-fluids Navier–Stokes–Maxwell model under suitable scaling hypotheses

$$j = \sigma(E + u \times B + \nabla(\ln \rho) - h(j \times B)/\rho),$$
where \( j = \text{curl} \mathbf{B} \) is the current density and \( \mathbf{E} \) the electric field. Jang and Masmoudi in [23] also gave a derivation of Hall-effect. The Maxwell–Faraday equations:

\[
\partial_t \mathbf{B} + \text{curl} \mathbf{E} = 0
\]

with above generalized Ohm’s law then gives rise to (1.4). Compared with the classical inhomogeneous incompressible MHD equations, the density-dependent Hall-MHD system have an additional Hall-effect term which is believed to be the key to understanding the problem of magnetic reconnection, as observed in space plasmas, star formation, neutron stars and geo-dynamo (see for example [3,17,21,22]). Meanwhile, since Hall-effect term is a tri-linear term and degenerate in vacuum, it makes the mathematical analysis of the density-dependent Hall-MHD system much more complicated.

The primary aim of this paper is to establish the existence of global weak solutions that could be called “solutions à la Leray” to the density-dependent Hall-MHD system by analogy with the classical global existence results for the incompressible homogeneous Navier–Stokes equations obtained by Leray [24] and the density-dependent Navier–Stokes equations by Lions [26]. Without taking into consideration of Hall-effect term (i.e. \( h = 0 \)), many works have already been addressed. For instance, with constant density, global existence for standard viscous resistive incompressible MHD system has been previously proved by Duvaut and Lions [13] and Serangane and Temam [28], while with inhomogeneous density, Gerbeau and Le Bris [18] proved existence of a global weak solution. Consider incompressible homogeneous Hall-MHD system, the global existence of Leray-Hopf weak solutions to the periodic case was first studied in [1], later, Chae et al. [4] treated the \( \mathbb{R}^3 \) case as well as the local-in-time existence of classical solutions with initial data in regular Sobolev spaces. Weak–strong uniqueness and energy identity have been investigated by Dumas and Sueur in [12]. Partial regularity has been studied by Chae and Wolf in a series works [6–8] and later by Zeng and Zhang [31]. Very recently, Dai [9] shows a non-uniqueness result for weak solutions having Leray-Hopf type regularity, Danchin and the author in [10,11] have first established existence results for initial data with critical regularity in Besov spaces and Sobolev spaces.

As far as we know, there are few results for the Hall-MHD system in a general bounded domain. Unlike MHD system in a bounded domain, the appearance of the Hall-effect term gives us additional difficulties in non-linear analysis and on boundary condition for magnetic field. Let \( \Omega \) be a smooth, bounded, fixed connected open subset of \( \mathbb{R}^3 \). We shall denote by \( \mathbf{n} \) the outward-pointing normal to \( \Omega \).

For the boundary conditions, we consider the no-slip boundary condition for the fluid velocity \( \mathbf{u} \):

\[
\mathbf{u} = 0 \quad \text{on} \quad \partial \Omega, \quad (1.5)
\]

the perfectly conducting wall boundary condition for the magnetic field (see [1] for the homogeneous case), that is assuming zero normal component of the magnetic field and zero tangential component of the electric field:

\[
\begin{cases}
\mathbf{B} \cdot \mathbf{n} = 0 & \text{on} \quad \partial \Omega, \\
\left( h \frac{\text{curl} \mathbf{B} \times \mathbf{B}}{\rho} + \frac{\text{curl} \mathbf{B}}{\sigma} \right) \times \mathbf{n} = 0 & \text{on} \quad \partial \Omega. 
\end{cases} \quad (1.6)
\]

Above nonlinear boundary condition has been emphasized by Chae et al. in [1,5] for the homogeneous case. With our best knowledge, the only well-posedness result of strong solutions with perfectly conducting wall boundary conditions is due to Mulone and Solonnikov [25]. Recently, Han et al. [19] considered slip boundary condition for \( \mathbf{u} \) and no-slip boundary condition for \( \mathbf{B} \) and established global weak solutions to the Hall-MHD system with the ion-slip effect in a bounded domain. Later, Han and Hwang [20] imposed a new boundary condition and proved local well-posedness of strong solutions with a regularity criteria.

Due to the degeneracy of Hall-effect term in vacuum, we shall suppose in this paper that the initial density does not vanish. Thanks to the nature of (1.1) with (1.3), this property will be preserved along the time and it is crucial for our later analysis. If there exists vacuum, we have no idea how to make sure the Hall-effect term is well-defined. Let us look at the compressible Hall-magnetohydrodynamics system in e.g. [15,30] for a while,

\[
\partial_t \rho + \text{div} (\rho \mathbf{u}) = 0,
\]
\[ \partial_t (\rho u) + \text{div} (\rho u \otimes u) - \mu \Delta u + \nabla P(\rho) = \text{curl} B \times B, \]
\[ \partial_t B + \text{curl} \left( B \times u + h \frac{\text{curl} B \times B}{\rho} \right) = \nu \Delta B. \]

The celebrated results of Lions [27] and later by Fereisl [16] on the weak solutions for compressible Navier–Stokes equations may not be successfully applied to the above model, since without the incompressibility condition (1.3) on the velocity field we are not able to control vacuum regions even if we assume there is no vacuum at beginning. Still, global low-energy weak solutions of 3D compressible MHD equations with density positive and essentially bounded were recently established in Suen and Hoff [29].

### 1.2. A Priori Estimate and Functional Spaces

Different from the work of Gerbeau and Le Bris, where they only assumed positive initial density, we need to assume initial density having a positive lower bound for technical reason (see Remark 1). Given that \( \inf \rho_0 > 0 \), it is reasonable to make an initial hypothesis on the velocity. We thus impose the following initial conditions:

\[ \rho|_{t=0} = \rho_0 \geq 0 \quad \text{in} \ \Omega, \quad (1.7) \]
\[ u|_{t=0} = u_0 \quad \text{in} \ \Omega, \quad (1.8) \]
\[ B|_{t=0} = B_0 \quad \text{in} \ \Omega. \quad (1.9) \]

And we shall suppose in the sequel that for \( \xi \in [\bar{\rho}, \infty) \),

\[ 0 < \mu \leq \mu(\xi) \leq \bar{\mu}, \quad (1.10) \]
\[ 0 < \sigma \leq \sigma(\xi) \leq \bar{\sigma}. \quad (1.11) \]

Now, we formally derive an a priori energy estimate. We first remark that (1.1) and the incompressibility condition (1.3) immediately imply that

\[ \rho \leq \rho(t,x) \leq \|\rho_0\|_{L^\infty, \ a.e. t, x}. \]

Next, recall that for all velocity fields \( u \) and density \( \rho \)

\[ \text{div} (\rho u \otimes u) = u \text{div} (\rho u) + \rho u \cdot \nabla u, \quad (1.12) \]

in the sense of distributions on \( \Omega \). Moreover, we shall make frequent use of the following formula of vector analysis: for all vector fields \( v \) and \( w \) in \( \mathbb{R}^3 \) we have

\[ \int_{\Omega} \text{curl} v \cdot w \, dx = \int_{\partial \Omega} (n \times v) \cdot w \, dS + \int_{\Omega} v \cdot \text{curl} w \, dx, \quad (1.13) \]

whenever these integrals make sense. Here \( dS \) is the standard surface measure of \( \partial \Omega \).

Thanks to (1.1) and (1.12), we multiply (1.2) by \( u \), integrate over \( \Omega \) and use boundary condition (1.5) to get

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u|^2 \, dx + \frac{1}{2} \int_{\Omega} \mu(\rho) |\nabla u + \nabla u^T|^2 \, dx = \int_{\Omega} (\text{curl} B \times B) \cdot u \, dx. \quad (1.14) \]

Multiplying (1.4) by \( B \), integrating over \( \Omega \) and using (1.13), we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |B|^2 \, dx + \int_{\Omega} \left( \frac{\text{curl} B \times B}{\rho} + \frac{\text{curl} B}{\sigma(\rho)} \right) \cdot \text{curl} B \, dx \]
\[ = - \int_{\partial \Omega} \left( n \times \left( \frac{\text{curl} B \times B}{\rho} + \frac{\text{curl} B}{\sigma(\rho)} \right) \right) \cdot B \, dS. \]

Thus, using the facts that

\[ (\text{curl} B \times B) \cdot \text{curl} B = 0, \]
\[ (B \times u) \cdot \text{curl} B = (\text{curl} B \times B) \cdot u, \]
and boundary conditions (1.5), (1.6) we further get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathbf{B}|^2 \, dx + \int_{\Omega} \frac{\left| \text{curl} \mathbf{B} \right|^2}{\sigma} \, dx = - \int_{\Omega} (\text{curl} \mathbf{B} \times \mathbf{B}) \cdot \mathbf{u} \, dx.
\] 
(1.15)

Adding up (1.14) and (1.15), one has the following energy equality
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho|\mathbf{u}|^2 + |\mathbf{B}|^2) \, dx + \frac{1}{2} \int_{\Omega} \mu(\rho)|\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 \, dx + \int_{\Omega} \frac{|\text{curl} \mathbf{B}|^2}{\sigma(\rho)} \, dx = 0.
\] 
(1.16)

It is well-known and wide open in [24,26], when dealing with global weak solutions of nonlinear partial differential equations, the global weak solutions we obtained usually satisfy the energy inequality instead of energy equality
\[
\frac{1}{2} \int_{\Omega} (\rho|\mathbf{u}|^2 + |\mathbf{B}|^2) \, dx + \frac{1}{2} \int_{0}^{T} \int_{\Omega} \mu(\rho)|\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 \, dx \, dt + \int_{0}^{T} \int_{\Omega} \frac{|\text{curl} \mathbf{B}|^2}{\sigma(\rho)} \, dx \, dt \leq \frac{1}{2} \int_{\Omega} (\rho_0|\mathbf{u}_0|^2 + |\mathbf{B}_0|^2) \, dx.
\] 
(1.17)

Let us notice from Inequality (1.10) and condition $\text{div} \mathbf{u} = 0$ that
\[
\int_{\Omega} \mu(\rho)|\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 \, dx \geq 2\mu \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx.
\]

Then thanks to Inequality (1.11) we finally have
\[
\frac{1}{2} \int_{\Omega} (\rho|\mathbf{u}|^2 + |\mathbf{B}|^2) \, dx + \mu \int_{0}^{T} \int_{\Omega} |\nabla \mathbf{u}|^2 \, dx \, dt + \frac{1}{\sigma} \int_{0}^{T} \int_{\Omega} |\text{curl} \mathbf{B}|^2 \, dx \, dt \leq \frac{1}{2} \int_{\Omega} (\rho_0|\mathbf{u}_0|^2 + |\mathbf{B}_0|^2) \, dx.
\] 
(1.18)

In order to formulate our problem and the main results, let us recall the definition of some functional spaces that we shall use throughout this paper. The space $\mathcal{D}(\Omega)$ is defined as the space of smooth functions compactly supported in the domain $\Omega$, and $\mathcal{D}'(\Omega)$ as the space of distributions on $\Omega$. For $X$ a Banach space, $p \in [1, \infty]$ and $T > 0$, the notation $L^p(0, T; X)$ or $L^p_f(0, T; X)$ designates the set of measurable functions $f : [0, T] \rightarrow X$ with $t \mapsto \|f(t)\|_X$ in $L^p(0, T)$, endowed with the norm $\| \cdot \|_{L^p_f(X)} := \| \cdot \|_{L^p(0, T; X)}$, and agree that $C([0, T]; X)$ denotes the set of continuous functions from $[0, T]$ to $X$. Slightly abusively, we will keep the same notations for multi-component functions. The space $H^1(\Omega)$ denotes the space of $L^2$ functions of $f$ on $\Omega$ such that $\nabla f$ also belongs to $L^2(\Omega)$. The Hilbertian norm is defined by
\[
\|f\|^2_{H^1(\Omega)} := \|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)}.
\]

The space $H^1_0(\Omega)$ is defined as the closure of $\mathcal{D}(\Omega)$ for the $H^1(\Omega)$ norm, and the space $H^{-1}(\Omega)$ as the dual space of $H^1_0(\Omega)$ for the $\mathcal{D}' \times \mathcal{D}$ duality. Then we introduce the vector-valued spaces:

$\mathcal{V} = \{ \mathbf{u} \in \mathcal{D}(\Omega), \text{ div } \mathbf{u} = 0 \}$,

$\mathcal{H} = \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega)$,

$= \{ \mathbf{u} \in L^2(\Omega), \text{ div } \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0 \}$,

$V_1 = \text{the closure of } \mathcal{V} \text{ in } H^1(\Omega)$,

$= \{ \mathbf{u} \in H^1_0(\Omega), \text{ div } \mathbf{u} = 0 \}$,

$\mathcal{W} = \{ \mathbf{B} \in C^\infty(\Omega), \text{ div } \mathbf{B} = 0, \mathbf{B} \cdot \mathbf{n}|_{\partial \Omega} = 0 \}$,

$W_1 = \text{the closure of } \mathcal{W} \text{ in } H^1(\Omega)$.
\[ \{ \mathbf{B} \in H^1(\Omega), \; \text{div} \mathbf{B} = 0, \; \mathbf{B} \cdot \mathbf{n}|_{\partial \Omega} = 0 \} , \]

and their norms

\[
\| \mathbf{u} \|_{V_1} = \| \nabla \mathbf{u} \|_{L^2(\Omega)}, \\
\| \mathbf{B} \|_{W_1} = \| \text{curl} \mathbf{B} \|_{L^2(\Omega)}. 
\]

We equip \( \mathcal{H} \) with the following inner product

\[
(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx, \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{H}. 
\]

Just remark that, one can establish that \( \| \cdot \|_{V_1} \) and \( \| \cdot \|_{W_1} \) define two norms which are equivalent to that introduced by \( H^1(\Omega) \) on \( V_1 \) and \( W_1 \), respectively (cf. Duvaut and Lions [13], Sermange and Temam [28]).

In accordance with (1.3), we assume that \( \text{div} \mathbf{u}_0 = 0 \) and, for physical consistency, since a magnetic field has to be divergence free, we suppose that \( \text{div} \mathbf{B}_0 = 0 \), too, a property that is conserved through the evolution. With above notations and Inequality (1.18), we propose the following assumptions on the initial data:

\[
\rho_0 \in L^\infty(\Omega), \quad \rho_0 \geq \rho > 0, \quad \rho \in C([0,T]; L^p(\Omega)), \quad 1 \leq p < \infty, \\
\mathbf{u}_0 \in \mathcal{H}, \quad \mathbf{B}_0 \in \mathcal{H}. 
\]

In a same fashion with [18,26], we define our weak solutions as follows.

**Definition 1.1.** We say that \((\rho, \mathbf{u}, \mathbf{B})\) is a global weak solution of the problem (1.1)–(1.9) with the initial assumptions (1.19)–(1.21), if for any \( T > 0 \), \((\rho, \mathbf{u}, \mathbf{B})\) satisfies the following properties:

\[
\rho \geq \rho, \quad \rho \in L^\infty((0,T) \times \Omega), \quad \rho \in C([0,T]; L^p(\Omega)), \quad 1 \leq p < \infty, \\
\mathbf{u} \in L^2(0,T; V_1) \quad \text{and} \quad \rho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)), \\
\mathbf{B} \in L^2(0,T; W_1) \cap L^\infty(0,T; L^2(\Omega)), 
\]

moreover, for any \( \phi \in C^1([0,T]; V_1) \) with \( \phi(T,\cdot) = 0 \),

\[
- \int_0^T \int_{\Omega} (\rho \partial_t \phi + \rho \mathbf{u} \cdot \nabla \phi) \, dxdt = \int_{\Omega} \rho_0 \phi(0, x) \, dx; \quad (1.22) 
\]

for any \( \Phi \in C^1([0,T]; V_1) \) and \( \Phi(T,\cdot) = 0 \),

\[
\int_0^T \int_{\Omega} \left( -\rho \mathbf{u} \cdot \partial_t \Phi - (\rho \mathbf{u} \otimes \mathbf{u}) \cdot \nabla \Phi + 2\mu(\rho)\mathbf{d}(\mathbf{u}) \cdot \mathbf{d}(\Phi) \right) \, dxdt \\
= \int_0^T \int_{\Omega} (\text{curl} \mathbf{B} \times \mathbf{B}) \cdot \Phi \, dxdt + \int_{\Omega} \rho_0 \mathbf{u}_0 \cdot \Phi(0, x) \, dx; \quad (1.23) 
\]

for any \( \Psi \in C^1([0,T]; W_1) \) with \( \Psi(T,\cdot) = 0 \),

\[
\int_0^T \int_{\Omega} \left( -\mathbf{B} \cdot \partial_t \Psi + \left( \mathbf{B} \times \mathbf{u} + h \frac{\text{curl} \mathbf{B} \times \mathbf{B}}{\rho} + \frac{\text{curl} \mathbf{B}}{\sigma(\rho)} \right) \cdot \text{curl} \Psi \right) \, dxdt \\
= \int_{\Omega} \mathbf{B}_0 \cdot \Psi(0, x) \, dx; \quad (1.24) 
\]

and finally, the energy inequality (1.17) holds for all \( t \in [0,T] \).
1.3. Main Results

In comparison with the models studied in [18, 26], the main difficulty of proving the existence of weak solutions lies in the Lorentz force term $\text{curl} \mathbf{B} \times \mathbf{B}$ in the Navier–Stokes Eq. (1.2) while $\mathbf{B}$ satisfies a quasi-linear parabolic equations with a tri-linear term involving density, we have to recover some compactness on $\mathbf{B}$ in order to pass to the limit in the nonlinear terms.

We now state our main theorem.

**Theorem 1.2.** Under the regularity assumptions (1.19)–(1.21) on the initial data, the initial-boundary value problem to the density-dependent Hall-MHD system (1.1)–(1.9) has a weak solution in the sense of Definition 1.1. Furthermore, we have for all $0 \leq \alpha \leq \beta < \infty$

$$\text{meas}\{x \in \mathbb{R}^3, \alpha \leq \rho(t, x) \leq \beta\} \text{ is independent of } t \geq 0,$$

and

$$\mathbf{u}, \mathbf{B} \in \mathcal{C}_w([0, T]; L^2(\Omega)).$$

We remark that the weak solutions obtained from Theorem 1.2 do not necessarily satisfy the second boundary condition in (1.6). The next theorem states a weak–strong uniqueness property of such solutions: any global weak solution coincides with a more regular solution as long as the latter exists.

**Theorem 1.3.** Assume $\mu, \sigma$ are locally Lipschitz continuous. Let $(\rho, \mathbf{u}, \mathbf{B})$ be a weak solution obtained from Theorem 1.2. If there exists another weak solution $(\hat{\rho}, \hat{\mathbf{u}}, \hat{\mathbf{B}})$ such that $(\hat{\rho}, \hat{\mathbf{u}}, \hat{\mathbf{B}}) \in \mathcal{C}^1([0, T]; H^2(\Omega))$ which satisfies boundary conditions (1.5)–(1.6) and $\nabla \hat{\mathbf{u}}, \nabla \hat{\mathbf{B}} \in L^2(0, T; L^\infty(\Omega))$, and at initial time

$$\hat{\rho}|_{t=0} = \rho_0, \quad \hat{\mathbf{u}}|_{t=0} = \mathbf{u}_0, \quad \hat{\mathbf{B}}|_{t=0} = \mathbf{B}_0 \quad \text{in } \Omega,$$

then we have $(\rho, \mathbf{u}, \mathbf{B}) \equiv (\hat{\rho}, \hat{\mathbf{u}}, \hat{\mathbf{B}})$ a.e. in $(0, T) \times \Omega$.

In the rest of this paper, we first set up our approximation scheme and establish the existence of solutions to the approximation problem in Sect. 2. Then in Sect. 3, in order to recover the original system, we deduce some compactness results and finally finish the proof of our main theorem. In the end, we prove Theorem 1.3.

Throughout this paper, we use $C$ to denote a general positive constant which may different from line to line.

2. Approximation Scheme

The essential idea to prove the existence of a weak solution to the density-dependent Hall-MHD system is to introduce an approximation problem, that allows one to define (1.1) as a classical transport equation. Now, presenting the approximation problem and showing the existence of a regular solution to this problem is our purpose for the next two steps.

2.1. First Step: A Linearized Problem

At this step, we prove a preliminary result for a linearized problem with prescribed density, which will be useful in Sect. 2.2. For this purpose, we first define two finite dimensional spaces for $n \in \mathbb{N}^*$

$$\mathcal{V}_n = \text{span}\{\Theta_i\}_{i=1}^n \quad \text{and} \quad \mathcal{W}_n = \text{span}\{\Gamma_i\}_{i=1}^n,$$

where $\{\Theta_i\}_{i=1}^\infty \subset \mathcal{V}$ and $\{\Gamma_i\}_{i=1}^\infty \subset \mathcal{W}$ are orthonormal basis of $V_1$ and $W_1$ respectively.

1The space $\mathcal{C}_w([0, T]; L^2(\Omega))$ denotes continuity on the interval $[0, T]$ with values in the weak topology of $L^2(\Omega)$. 
For $\rho > 0$, $v, H \in \mathbb{R}^3$ and $n, T > 0$ arbitrarily fixed such that
\begin{align}
\rho, \partial_t \rho &\in C([0, T], C^1(\Omega)), \quad 0 < \rho_1 \leq \rho(t, x) \leq \rho_2, \quad (2.1) \\
v &\in C([0, T]; V_n) \text{ such that } \partial_t \rho + \text{div}(\rho v) = 0, \quad (2.2) \\
H &\in C([0, T]; W_n), \quad (2.3)
\end{align}
the linearized problem is to find a couple of vector-valued functions $(u, B)$ such that for any $\Phi \in V_n$ and any $\Psi \in W_n$, one has
\begin{equation}
\int_\Omega \rho(\partial_t u + v \cdot \nabla u) \cdot \Phi \, dx + \int_\Omega 2\mu(\rho) d(u) \cdot d(\Phi) \, dx = \int_\Omega (\text{curl} B \times H) \cdot \Phi \, dx, \quad (2.4)
\end{equation}
and
\begin{equation}
\int_\Omega \partial_t B \cdot \Psi \, dx + \int_\Omega \left( H \times u + h \frac{\text{curl} B \times H}{\rho} + \frac{\text{curl} B}{\sigma(\rho)} \right) \cdot \text{curl} \Psi \, dx = 0, \quad (2.5)
\end{equation}
and $(u, B)$ satisfies initial condition $(u|_{t=0}, B|_{t=0}) = (u_0, B_0)$.

To show an existence result for the above linearized problem, we will follow the classical Galerkin method.

**Proposition 2.1.** Let $(u_0, B_0) \in V_n \times W_n$. Under the assumptions $(2.1)$–$(2.3)$, there exists a unique pair of solution $(u, B) \in C^1([0, T]; V_n) \times C^1([0, T]; W_n)$ to the problem $(2.4)$–$(2.5)$. Moreover, we have energy equality $(1.16)$.

**Proof.** We look for a solution $(u, B)$ under the form
\begin{equation}
\begin{aligned}
u &= \sum_{i=1}^n \alpha_i(t) \Theta_i, \\
B &= \sum_{i=1}^n \beta_i(t) \Gamma_i.
\end{aligned} \tag{2.6}
\end{equation}

Look at weak form $(2.4)$ and $(2.5)$, replace $\Phi$ by $\Theta_j$ and $\Psi$ by $\Gamma_j$ for $j = 1, \ldots, n$, respectively, we find that scalar functions $\alpha_i$ and $\beta_i$ ($i = 1, \ldots, n$) are solutions of the following linear ODEs
\begin{equation}
\begin{aligned}
\left( \int_\Omega \rho \Theta_j \cdot \Theta_i \, dx \right) \frac{d\alpha_i}{dt} + \left( \int_\Omega (\rho v \cdot \nabla \Theta_i) \cdot \Theta_j + 2\mu(\rho) d(\Theta_i) \cdot d(\Theta_j) \, dx \right) \alpha_i \\
- \left( \int_\Omega (\text{curl} \Gamma_i \times H) \cdot \Theta_j \, dx \right) \beta_i &= 0, \\
\left( \int_\Omega \Gamma_i \cdot \Gamma_j \, dx \right) \frac{d\beta_i}{dt} + \left( \int_\Omega \left( \frac{h\text{curl} \Gamma_i \times H}{\rho} + \frac{\text{curl} \Gamma_i}{\sigma(\rho)} \right) \cdot \text{curl} \Gamma_j \, dx \right) \beta_i \\
+ \left( \int_\Omega (H \times \Theta_i) \cdot \text{curl} \Gamma_j \, dx \right) \alpha_i &= 0,
\end{aligned} \tag{2.7}
\end{equation}
with initial data for $i = 1, \ldots, n$, defined by
\begin{equation}
\begin{aligned}
\alpha_i(0) &= \int_\Omega u_0 \cdot \Theta_i \, dx, \\
\beta_i(0) &= \int_\Omega B_0 \cdot \Gamma_i \, dx.
\end{aligned} \tag{2.8}
\end{equation}

Since the family $(\sqrt{\rho} \Theta_i)_{i=1,\ldots,n}$ (resp. $(\sqrt{\rho} \Gamma_i)_{i=1,\ldots,n}$) with $\rho \geq \rho$ is free, it follows that the matrix $(\int_\Omega \rho \Theta_i \cdot \Theta_j \, dx)_{n \times n}$ (resp. $(\int_\Omega \rho \Gamma_i \cdot \Gamma_j \, dx)_{n \times n}$) is nonsingular for any $t \in [0, T]$. We note that the coefficients lie in the above ODEs are continuous on $[0, T]$ from assumptions $(2.1)$–$(2.3)$. Thanks to the Cauchy-Lipschitz theorem, $(\alpha(t), \beta(t))$ exists uniquely and is continuous on $[0, T]$, where $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))^T \in \mathbb{R}^n$, $\beta(t) = (\beta_1(t), \ldots, \beta_n(t))^T \in \mathbb{R}^n$. Thus, we obtain with $(2.7)$ and $(2.8)$ that $(u, B) \in C^1([0, T], V_n) \times C^1([0, T], W_n)$. 


In view of this regularity, we multiply the first equations in (2.7) by \( \alpha_j \) and second equations by \( \beta_j \), then we sum them for \( i, j = 1, \ldots, n \). This yields
\[
\int_{\Omega} \rho (\partial_t \mathbf{u} + \mathbf{v} \cdot \nabla \mathbf{u}) \cdot \mathbf{u} \, dx + \int_{\Omega} 2\mu(\rho)|d(\mathbf{u})|^2 \, dx - \int_{\Omega} (\text{curl} \mathbf{B} \times \mathbf{H}) \cdot \mathbf{u} \, dx = 0,
\]
and
\[
\int_{\Omega} \partial_t \mathbf{B} \cdot \mathbf{B} \, dx + \int_{\Omega} \frac{|\text{curl} \mathbf{B}|^2}{\sigma(\rho)} \, dx + \int_{\Omega} (\mathbf{H} \times \mathbf{u}) \cdot \text{curl} \mathbf{B} \, dx = 0.
\]
By integration by parts with equation in (2.2), we get (1.16). This completes the proof of Proposition 2.1.

2.2. Second Step: A Regularized Approximation Problem

In this step, we solve a non-linear regularized approximation problem by using Schauder’s fixed-point theorem (see Theorem II.3.9 in [2]) and Proposition 2.1.

The following definition of mollifier can be found in e.g. [14] with several properties. Define
\[
\Omega_\epsilon := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \epsilon \},
\]
and \( \eta \in C^\infty(\mathbb{R}^3) \) the standard mollifier by
\[
\eta(x) := \begin{cases} 
C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\
0 & \text{if } |x| \geq 1,
\end{cases}
\]
for some normalizing constant \( C \) such that \( \int_{\mathbb{R}^3} \eta \, dx = 1 \). Let \( \epsilon \in (0, 1) \), define
\[
\eta_\epsilon(x) := \frac{1}{\epsilon^3} \eta\left(\frac{x}{\epsilon}\right).
\]
Now, we are ready to define approximate initial data. Since we assumed our initial density to be bounded below by \( \rho \), we set
\[
\tilde{\rho} = \begin{cases} 
\rho_0 & \text{in } \Omega \\
\rho & \text{in } \mathbb{R}^3 \setminus \Omega,
\end{cases}
\]
and
\[
\rho_0^\epsilon = \tilde{\rho}_0 * \eta_\epsilon.
\]
Then the initial density for the approximate system is defined by
\[
\rho_n|_{t=0} = \rho_0^\epsilon.
\]
Thanks to assumption (1.19), it is clear for some universal constant \( C_0 \) independent of \( \epsilon \), we have
\[
\rho \leq \rho_0^\epsilon \leq C_0,
\]
and \( \rho_0^\epsilon \in C^\infty(\overline{\Omega}) \)
\[
\lim_{\epsilon \to 0} \rho_0^\epsilon = \rho_0 \quad \text{in } L^p(\Omega) \quad (1 \leq p < \infty).
\]
We would also need to regularize \( \mu(\xi), \sigma(\xi) \) like in [26]. Assume \( \mu_\epsilon(\xi) \) is a \( C([0, \infty)) \) function bounded away from zero, which is constant for \( \xi \geq 0 \) large and such that \( \sup_{[0, \infty)} |\mu_\epsilon - \mu| < \epsilon \). We set
\[
\tilde{\mu_\epsilon}(\rho) = \begin{cases} 
\mu_\epsilon(\rho) & \text{in } \Omega \\
1 & \text{in } \mathbb{R}^3 \setminus \Omega,
\end{cases}
\]
and define \( \mu^\epsilon = \tilde{\mu_\epsilon}(\rho) * \eta_\epsilon|_{\Omega} \). We define \( \sigma^\epsilon \) from \( \sigma \) like \( \mu^\epsilon \) from \( \mu \).
We set the initial condition for approximate velocity field and magnetic field as
\[ u_n|_{t=0} = u_{0,n}^\epsilon = P_{\mathcal{V}_0} u_0^\epsilon \] (2.12)
and
\[ B_n|_{t=0} = B_{0,n}^\epsilon = P_{\mathcal{W}_0} B_0^\epsilon, \] (2.13)
where
\[ u_0^\epsilon = ((u_0 I_{\Omega_0}) \ast \eta_\epsilon), \]
\[ B_0^\epsilon = ((B_0 I_{\Omega_0}) \ast \eta_\epsilon) \]
and \( P_{\mathcal{V}_0} \) (resp. \( P_{\mathcal{W}_0} \)) is the orthogonal projection in \( L^2(\Omega) \) onto \( \mathcal{V}_n \) (resp. \( \mathcal{W}_n \)).

Fixed \( \epsilon \), our approximation problem is stated as follows (for simplicity, we omit the upscript \( \epsilon \) for solutions).

**Definition 2.2.** For any given \( T > 0 \), we say \((\rho_n, u_n, B_n)\) with
\[ \rho_n \in C([0, T] \times \Omega), \quad u_n \in C([0, T]; \mathcal{V}_n), \quad B_n \in C([0, T]; \mathcal{W}_n) \]
is a global weak solution of the following approximation problem
\[
\begin{aligned}
&\partial_t \rho_n + \text{div} (\rho_n u_n) = 0, \quad \text{(2.14)} \\
&\partial_t (\rho_n u_n) + \text{div} (\rho u_n \otimes u_n) - \text{div} (2\mu^\epsilon \rho_n d(u_n)) + \nabla P_n = \text{curl} B_n \times B_n, \quad \text{(2.15)} \\
&\text{div} u_n = 0, \quad \text{(2.16)} \\
&\partial_t B_n + \text{curl} \left( B_n \times u_n + h \frac{\text{curl} B_n \times B_n}{\rho_n} \right) = -\text{curl} \left( \frac{\text{curl} B_n}{\sigma^\epsilon(\rho_n)} \right), \quad \text{(2.17)} \\
&\text{div} B_n = 0, \quad \text{(2.18)}
\end{aligned}
\]
with the initial conditions (2.9), (2.12), (2.13) and boundary conditions
\[
\begin{cases}
B_n \cdot n = 0 & \text{on } \partial \Omega, \\
\left( h \frac{\text{curl} B_n \times B_n}{\rho_n} + \frac{\text{curl} B_n}{\sigma^\epsilon(\rho_n)} \right) \times n = 0 & \text{on } \partial \Omega,
\end{cases}
\] (2.19)
(2.20)
if (2.14)–(2.20) are satisfied in the weak sense of Definition 1.1 with the test function spaces in (1.23) and (1.24) replaced by the restriction on \( \mathcal{V}_n \) and \( \mathcal{W}_n \), respectively.

With the above definition, we have the following existence result.

**Theorem 2.3.** There exists a global weak solution \((\rho_n, u_n, B_n)\) to the above initial-boundary value problem.

**Proof.** In order to solve this nonlinear approximation problem by Schauder’s fixed-point theorem, we shall construct a operator \( F_n : I_n \to I_n \), where the convex set
\[ I_n := \left\{ (\bar{u}, \bar{B}) \in C([0, T]; \mathcal{V}_n) \times C([0, T]; \mathcal{W}_n) : \sup_{t \in [0,T]} \| (\bar{u}, \bar{B}) \|_{L^2(\Omega)} \leq R_0 \right\}, \]
with \( R_0 \) a constant to be determined later. We denote the input of \( F_n \) to be \((\bar{u}_n, \bar{B}_n)\), and the corresponding output \( F_n(\bar{u}_n, \bar{B}_n) \) to be \((u_n, B_n)\). Then we define our operator as follows. At first, with the input \((\bar{u}_n, \bar{B}_n)\), we consider the following linear problem
\[
\begin{cases}
\partial_t (\rho_n) + \text{div} (\rho_n \bar{u}_n) = 0, \\
\rho_n|_{t=0} = \rho_0^\epsilon.
\end{cases}
\] (2.21)
This is a classical transport equation since, \( \bar{u}_n \) is regular, \( \text{div} \bar{u}_n = 0 \) and vanishes near \( \partial \Omega \). Thus \( \rho_n \) is uniquely given by
\[
\rho_n(t, x) = \rho_0^\epsilon(X(0; x, t)), \quad \forall (t, x) \in [0, T] \times \bar{\Omega},
\]
where $X$ is the solution of the ODE
\[
\begin{align*}
\frac{dX}{ds} &= \bar{u}_n(s, X), \\
X(t; x, t) &= x.
\end{align*}
\]

Obviously, we have
\[
\rho \leq \rho_n \leq C_0. \tag{2.22}
\]

Since $\rho_n^\prime$ is smooth, so $\rho_n \in C^1([0, T] \times \bar{\Omega})$ and is bounded in this space uniformly in $(\bar{u}_n, \bar{B}_n)$.

Now, we set $\rho = \rho_n$, $v = \bar{u}_n$, $H = \bar{B}_n$ and replace $\mu$ by $\mu^\prime(\rho_n)$, $\sigma$ by $\sigma^\prime(\rho_n)$ and we invoke Proposition 2.1 to define $(\bar{u}_n, \bar{B}_n)$ as the solution of problem:
\[
\int_{\Omega} \rho_n (\partial_t \bar{u}_n + \bar{u}_n \cdot \nabla \bar{u}_n) \cdot \Phi dx + \int_{\Omega} 2\mu^\prime(\rho_n) d(\bar{u}_n) \cdot d(\Phi) dx
\]
\[
= \int_{\Omega} (\text{curl } \bar{B}_n \times \bar{B}_n) \cdot \Phi dx, \tag{2.23}
\]

for any $\Phi \in \mathcal{V}_\Omega$;
\[
\int_{\Omega} \partial_t \bar{B}_n \cdot \Psi dx + \int_{\Omega} \left( \bar{B}_n \times \bar{u}_n + \frac{h}{\rho_n} \frac{\text{curl } \bar{B}_n \times \bar{B}_n}{\sigma^\prime(\rho_n)} \right) \cdot \text{curl } \Psi dx = 0, \tag{2.24}
\]

for any $\Psi \in \mathcal{W}_\Omega$, while $\bar{u}_n$ and $\bar{B}_n$ satisfy initial condition (2.12)–(2.13).

Next, let us choose $R_0$ such that $(\bar{u}_n, \bar{B}_n) \in I_n$. Thanks to Proposition 2.1, $(\bar{u}_n, \bar{B}_n)$ belongs to $C([0, T]; \mathcal{V}_\Omega) \times C([0, T]; \mathcal{W}_\Omega)$ and satisfies
\[
\frac{d}{dt} \int_{\Omega} (\rho_n |\partial_t \bar{u}_n|^2 + |\partial_t \bar{B}_n|^2) dx + \int_{\Omega} (4\mu^\prime(\rho_n)|d(\bar{u}_n)|^2 + 2\frac{|\text{curl } \bar{B}_n|^2}{\sigma^\prime(\rho_n)}) dx = 0. \tag{2.25}
\]

This with (2.22) and div $\bar{u}_n = 0$ lead to
\[
\sup_{t \in [0, T]} \|\bar{u}_n\|_{L^2(\Omega)} + \sup_{t \in [0, T]} \|\bar{B}_n\|_{L^2(\Omega)} + \|\bar{u}_n\|_{L^2(0, T; \mathcal{V}_1)} + \|\bar{B}_n\|_{L^2(0, T; \mathcal{W}_1)} \leq C_1, \tag{2.26}
\]
where $C_1$ is a constant independent of $R_0, \bar{u}_n, \bar{B}_n$. Hence, by taking $R_0 = C_1$, we have $(\bar{u}_n, \bar{B}_n) \in I_n$.

Then we prove the compactness of mapping $F_n$. In fact, with uniform bounds (2.26) in hand, in view of the famous Aubin–Lions lemma (see Theorem II. 5.16 in [2]), we only need to show some uniform bounds for $\partial_t \bar{u}_n$ and $\partial_t \bar{B}_n$ in suitable spaces. Using Proposition 2.1 again, we know that actually $(\partial_t \bar{u}_n, \partial_t \bar{B}_n)$ belongs to $C([0, T]; \mathcal{V}_\Omega) \times C([0, T]; \mathcal{W}_\Omega)$. Hence by taking $\Phi = \partial_t \bar{u}_n$, $\Psi = \partial_t \bar{B}_n$ in weak formulations (2.23)–(2.24), we have
\[
\int_{\Omega} (\rho_n |\partial_t \bar{u}_n|^2 + |\partial_t \bar{B}_n|^2) dx
\]
\[
= - \int_{\Omega} \left( (\rho_n \bar{u}_n \cdot \nabla \bar{u}_n) \cdot \partial_t \bar{u}_n + 2\mu^\prime(\rho_n) d(\bar{u}_n) \cdot d(\partial_t \bar{u}_n) \right) dx
\]
\[
+ \int_{\Omega} \frac{h}{\rho_n} \frac{\text{curl } \bar{B}_n \times \bar{B}_n}{\sigma^\prime(\rho_n)} \cdot \text{curl } (\partial_t \bar{B}_n) dx
\]
\[
= \int_{\Omega} (\text{curl } \bar{B}_n \times \bar{B}_n) \cdot \partial_t \bar{u}_n dx.
\]

Since all norms in a finite dimensional space are equivalent and thanks to (2.26), we obtain by Hölder’s inequality that
\[
\rho \|\partial_t \bar{u}_n\|_{L^2(\Omega)}^2 + \|\partial_t \bar{B}_n\|_{L^2(\Omega)}^2
\]
\[
\leq C_0 \|\bar{u}_n\|_{L^\infty(\Omega)} \|
abla \bar{u}_n\|_{L^2(\Omega)} \|\partial_t \bar{u}_n\|_{L^2(\Omega)} + 2\mu \|\nabla \bar{u}_n\|_{L^2(\Omega)} \|\nabla \partial_t \bar{u}_n\|_{L^2(\Omega)}
\]
\[
+ (\|\bar{B}_n\|_{L^\infty(\Omega)} \|ar{u}_n\|_{L^2(\Omega)} + \frac{h}{\rho_n} |\text{curl } \bar{B}_n|_{L^2(\Omega)} \|\bar{B}_n\|_{L^\infty(\Omega)} + \frac{1}{\sigma} |\text{curl } \bar{B}_n|_{L^2(\Omega)})
\]
\[ \| \text{curl} (\partial_t \mathbf{B}_m) \|_{L^2(\Omega)} + \| \text{curl} \mathbf{B}_n \|_{L^2(\Omega)} + \| \partial_t \mathbf{u}_n \|_{L^2(\Omega)} \leq C (\| \partial_t \mathbf{u}_n \|_{L^2(\Omega)} + \| \partial_t \mathbf{B}_n \|_{L^2(\Omega)}) \]

and further

\[ \| \partial_t \mathbf{u}_n \|_{L^2(\Omega)} + \| \partial_t \mathbf{B}_n \|_{L^2(\Omega)} \leq C, \tag{2.27} \]

where \( C \) is a constant independent on \( n, \epsilon \). We thus conclude that \( F_n \) is a compact operator on \( I_n \) to itself.

In order to apply the Schauder’s theorem, we still have to check the continuity of \( F_n \). It suffices to prove that the mapping is sequentially continuous. Let \( \{ (\mathbf{u}_m^n, \mathbf{B}_m^n) \}_{m \geq 1} \subset I_n \) be a sequence which strongly converges to \( (\mathbf{u}_n, \mathbf{B}_n) \) in \( I_n \). Recall our definition of mapping \( F_n \), we denote \( \rho_m^n \) as the corresponding solution to Eq. (2.21) and let \( (\mathbf{u}_m^n, \mathbf{B}_m^n) = F_n(\mathbf{u}_n, \mathbf{B}_n) \) as the corresponding solution to problem (2.23)–(2.24). It is clear when solving Eq. (2.21), we have that \( \{\rho_m^n\}_{m \geq 1} \) is bounded in \( C^1([0,T] \times \bar{\Omega}) \) uniformly in \( \{ (\mathbf{u}_m^n, \mathbf{B}_m^n) \}_{m \geq 1} \). Thus the Aubin–Lions lemma implies that \( \{\rho_m^n\}_{m \geq 1} \) is pre-compact in \( C([0,T] \times \bar{\Omega}) \). For \( \{ (\mathbf{u}_m^n, \mathbf{B}_m^n) \}_{m \geq 1} \), we know it is a subset of \( I_n \) and the control of \( \{ (\partial_t \mathbf{u}_m^n, \partial_t \mathbf{B}_m^n) \}_{m \geq 1} \) in \( L^\infty(0,T;L^2(\Omega)) \) can also be obtained by following a same procedure as to get Inequality (2.27). So one more application of the Aubin–Lions lemma gives that \( \{ (\mathbf{u}_m^n, \mathbf{B}_m^n) \}_{m \geq 1} \) is pre-compact in \( C([0,T];L^2(\Omega)) \).

Without loss of generality, as we can always replace our original sequence by a weakly converging subsequence, we conclude that \( (\mathbf{u}_m^n, \mathbf{B}_m^n) \) converges to \( (\mathbf{u}_n, \mathbf{B}_n) \) a solution of problem (2.23)–(2.24) when \( m \) goes to \( \infty \). Since problem (2.21) and (2.23)–(2.24) are all linear problems and the solutions are shown to be unique, one has \( (\mathbf{u}_m^n, \mathbf{B}_m^n) = F_n(\mathbf{u}_n, \mathbf{B}_n) \), that is \( (\mathbf{u}_m^n, \mathbf{B}_m^n) \) converges to \( F_n(\mathbf{u}_n, \mathbf{B}_n) \) as \( m \to \infty \).

From the Schauder’s fixed-point theorem, there exists a fixed point \( (\mathbf{u}_n, \mathbf{B}_n) \) of \( F_n \) in \( I_n \). It means that with input \( (\mathbf{u}_n, \mathbf{B}_n) \) one can well-define \( \rho_n \) as the solution of Eq. (2.14). Moreover, following the definition of mapping \( F_n \), the weak formulations in the Definition 2.2. Finally, we prove (1.25) for \( \rho_n \). Let \( \gamma_m \) be a function of \( C^1(\mathbb{R};\mathbb{R}) \). Multiplying (2.14) by \( \gamma_m(\rho_n) \) and using \( \text{div} \mathbf{u}_n = 0 \) we have

\[ \partial_t \gamma_m(\rho_n) + \mathbf{u}_n \cdot \nabla \gamma_m(\rho_n) = 0. \]

We integrate this equation on \([0,T] \times \Omega\) and use again that \( \text{div} \mathbf{u}_n = 0 \), with \( \mathbf{u}_n \) vanishes near the boundary to obtain

\[ \int_{\Omega} \gamma_m(\rho_n(t,x)) dx = \int_{\Omega} \gamma_m(\rho_0(x)) dx. \tag{2.30} \]

For \( 0 \leq \alpha \leq \beta < \infty \) we choose for \( m \) large enough \( 0 \leq \gamma_m \leq 1 \) such that

\[
\begin{align*}
\gamma_m(\lambda) &= 0 \quad \text{if } \lambda \notin [\alpha, \beta], \\
\gamma_m(\lambda) &= 1 \quad \text{if } \lambda \in [\alpha + \frac{1}{m}, \beta - \frac{1}{m}].
\end{align*}
\]

Letting \( m \to \infty \) in (2.30) we deduce that (1.25) holds for \( \rho_n \),

\[ \int_{\Omega} 1_{[\alpha,\beta]}(\rho_n(t,x)) dx = \int_{\Omega} 1_{[\alpha,\beta]}(\rho_0(x)) dx \]

where \( 1_{[\alpha,\beta]}(\lambda) \) is the characteristic function on \([\alpha, \beta]\). This completes the proof of Theorem 2.3. \( \square \)
3. Convergence of the Approximation Problem

The aim of this last section is to prove our main Theorem 1.2 by passing to the limit in the regularized approximation problem stated in Definition 2.2 as \( n \to \infty \) and \( \varepsilon \to 0 \). The fundamental tool is a compactness result due to P.-L. Lions \([26]\) that we recall here for its importance.

**Theorem 3.1.** We suppose that two sequences \( \rho_n \) and \( u_n \) are given satisfying \( \rho_n \in \mathcal{C}([0,T];L^1(\Omega)), 0 \leq \rho_n \leq C \) a.e. on \( (0,T) \times \Omega \), \( u_n \in L^2(0,T;H^1_0(\Omega)) \), \( \|u_n\|_{L^2(0,T;H^1_0(\Omega))} \leq C \) and \( \text{div} \, u_n = 0 \) (\( C \) denotes various constants independent of \( n \)). We assume:

\[
\begin{aligned}
\rho_n |_{t=0} &= \rho_0, \\
\partial_t \rho_n + \text{div} \, (\rho_n u_n) &= 0 \quad \text{in} \quad \mathcal{D}'((0,T) \times \Omega) ,
\end{aligned}
\]

and

\[
\rho_0 \to \rho_0 \quad \text{in} \quad L^1(\Omega),
\]

\[
u_n \to \nu \quad \text{weakly in} \quad L^2(0,T;H^1_0(\Omega)).
\]

Then:

1. \( \rho_n \) converges in \( \mathcal{C}([0,T];L^p(\Omega)) \) for all \( 1 \leq p < \infty \) to the unique \( \rho \) belonging to \( \mathcal{C}([0,T];L^1(\Omega)) \) bounded on \( (0,T) \times \Omega \) solution of

\[
\begin{aligned}
\partial_t \rho + \text{div} \, (\rho u) &= 0 \quad \text{in} \quad \mathcal{D}'((0,T) \times \Omega), \\
\rho|_{t=0} &= \rho_0 \quad \text{a.e. in} \quad \Omega .
\end{aligned}
\]

2. We assume in addition that \( \rho_n|u_n|^2 \) is bounded in \( L^\infty(0,T;L^1(\Omega)) \) and that we have for some \( l \geq 1 \)

\[
|<\partial_t (\rho_n u_n), \Phi>| \leq C\|\Phi\|_{L^2(0,T;H^1(\Omega))}
\]

for all \( \Phi \in \mathcal{D}((0,T) \times \Omega) \) such that \( \text{div} \, \Phi = 0 \) on \( (0,T) \times \Omega \). Then

\[
\sqrt{\rho_n} u_n \to \sqrt{\rho} u \quad \text{in} \quad L^q(0,T;L^r(\Omega)) \quad \text{for} \quad 2 < q < \infty, \quad 1 \leq r < \frac{6q}{3q-4}
\]

\[
u_n \to \nu \quad \text{in} \quad L^\theta(0,T;L^{3\theta}(\Omega)) \quad \text{for} \quad 1 \leq \theta < 2 \quad \text{on the set} \quad \{(t,x)| \rho(t,x) > 0\}.
\]

3.1. Pass to the Limit as \( n \to \infty \)

For fixed \( \varepsilon \), we denote by \( (\rho_n,u_n,B_n) \) the smooth approximate solution given by Theorem 2.3. From Inequality (2.25) it is clear that for any fixed \( T > 0 \), one has

\[
\int_\Omega (\rho_n|u_n|^2 + |B_n|^2) \, dx + \int_0^T \int_\Omega \left( 4\mu^\varepsilon(\rho_n)|d(u_n)|^2 + 2\frac{|\text{curl} \, B_n|^2}{\sigma^\varepsilon(\rho_n)} \right) \, dxdt \leq \int_\Omega (\rho_0^\varepsilon|u_0^\varepsilon|^2 + |B_0^\varepsilon|^2) \, dx .
\]

(3.1)

Note that \( \rho_n \) is bounded away from 0 uniformly, then we have the following bounds independent of \( n \):

\[
\|\rho_n|u_n|^2\|_{L^\infty(0,T;L^1(\Omega))} \leq C, \quad (3.2)
\]

\[
\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad (3.3)
\]

\[
\|u_n\|_{L^2(0,T;V_1)} \leq C, \quad (3.4)
\]

\[
\|B_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad (3.5)
\]

\[
\|B_n\|_{L^2(0,T;W_1)} \leq C, \quad (3.6)
\]

which imply that as \( n \) goes to infinity, up to extraction (we will extract subsequence if necessary),

\[
u_n \to \nu^\varepsilon \quad \text{weakly in} \quad L^2(0,T;V_1), \quad \nu_n \rightharpoonup \nu^\varepsilon \quad \text{weakly}^* \quad \text{in} \quad L^\infty(0,T;V)
\]
\( \mathbf{B}_n \rightarrow \mathbf{B}^\varepsilon \) weakly in \( L^2(0,T;W_1) \), \( \mathbf{B}_n \rightarrow \mathbf{B}^\varepsilon \) weakly* in \( L^\infty(0,T;W) \).

In view of properties (2.11), (2.22) and Inequality (3.4), the first assertion of Theorem 3.1 implies that
\[
\rho_n \rightarrow \rho^\varepsilon \quad \text{in} \quad C([0,T];L^p(\Omega)) \quad \text{with} \quad 1 \leq p < \infty
\]
and
\[
\partial_t \rho^\varepsilon + \text{div} \left( \rho^\varepsilon \mathbf{u}^\varepsilon \right) = 0.
\]

Our goal is now to pass to the limit as \( n \) goes to infinity in the following weak formulations, which originally from the approximation problem that have stated in Definition 2.2,

\[
\int_0^T \int_\Omega \left( -\rho_n \mathbf{u}_n \cdot \partial_t \Phi \, dx \, dt - \rho_n \mathbf{u}_n \otimes \mathbf{u}_n \cdot \nabla \Phi + 2\mu^\varepsilon (\rho_n) d(\mathbf{u}_n) \cdot d(\Phi) \right) \, dx
\]
\[
\quad = - \int_0^T \int_\Omega \left( \text{curl} \mathbf{B}_n \times \mathbf{B}_n \right) \cdot \Phi \, dx \, dt + \int_\Omega \rho_0^\varepsilon \mathbf{u}_{0,n} \cdot \Phi(0,x) \, dx,
\]

for any \( \Phi \in C^1([0,T] \times \mathcal{V}_n) \) with \( \text{div} \Phi = 0 \) and \( \Phi(T,\cdot) = 0 \);

\[
\int_0^T \int_\Omega \left( \mathbf{u}_n - \mathbf{B}_n \cdot \partial_t \Psi + \left( \mathbf{B}_n \times \mathbf{u}_n + h \frac{\text{curl} \mathbf{B}_n \times \mathbf{B}_n}{\rho_n} \right) \cdot \text{curl} \Psi \right) \, dx \, dt
\]
\[
\quad = \int_\Omega \mathbf{B}^\varepsilon_{0,n} \cdot \Psi(0,x) \, ds,
\]
for any \( \Psi \in C^1([0,T] \times \mathcal{W}_n) \) with \( \Psi(T,\cdot) = 0 \).

It turns out that we need to apply the second part of Theorem 3.1 to pass to the limit. Since we already get Inequality (3.2), let us estimate \( \partial_t (\rho_n \mathbf{u}_n), \Phi \) from weak formulation (3.8) with \( \Phi \in \mathcal{D}((0,T) \times \Omega) \), \( \text{div} \Phi = 0 \).

First, by Sobolev embeddings : \( H^1(\Omega) \hookrightarrow L^6(\Omega) \), \( H^{\frac{1}{2}}(\Omega) \hookrightarrow L^\infty(\Omega) \), we have

\[
| \int_0^T \int_\Omega \rho_n \mathbf{u}_n \otimes \mathbf{u}_n \cdot \nabla \Phi \, dx \, dt |
\]
\[
\leq \| \rho_n \|_{L^\infty(0,T) \times \Omega} \| \mathbf{u}_n \|_{L_6^\infty(\Omega)} \| \mathbf{u}_n \|_{L_2^6(L^\infty)} \| \nabla \Phi \|_{L^2_2(L^\infty)}
\]
\[
\leq C_0 C_2 \| \Phi \|_{L_2^\infty(H^{\frac{1}{2}})}
\]

and by \( \text{div} \mathbf{u}_n = 0 \), \( \text{div} \Phi = 0 \),

\[
| \int_0^T \int_\Omega 2\mu^\varepsilon (\rho_n) d(\mathbf{u}_n) \cdot d(\Phi) \, dx \, dt |
\]
\[
\leq C \| \mu^\varepsilon \|_{L^\infty(0,T) \times \Omega} \| d(\mathbf{u}_n) \|_{L_2^\infty(L^2)} \| d(\Phi) \|_{L_2^2(L^2)}
\]
\[
\leq C_2 \| \Phi \|_{L_2^\infty(H^{1})}.
\]

Using the inequality \( \| \Phi \|_{L^\infty(\Omega)} \leq C \| \Phi \|_{H^s} \) for \( s > \frac{3}{2} \), we have

\[
| \int_0^T \int_\Omega (\text{curl} \mathbf{B}_n \times \mathbf{B}_n) \cdot \Phi \, dx \, dt |
\]
\[
\leq \| (\text{curl} \mathbf{B}_n) \|_{L_2^2(L^2)} \| \mathbf{B}_n \|_{L_2^2(L^\infty)} \| \Phi \|_{L_2^\infty(L^\infty)}
\]
\[
\leq C_2 \| \Phi \|_{L_2^\infty(H^s)}.
\]

Therefore for any \( l > \frac{3}{2} \), the second assertion of Theorem 3.1 with strong convergence of \( \rho_n \) then imply that as \( n \rightarrow \infty \)

\[
\rho_n \mathbf{u}_n \rightarrow \rho^\varepsilon \mathbf{u}^\varepsilon \quad \text{in} \quad L^q(0,T;L^r(\Omega)) \quad \text{for} \quad 2 < q < \infty, \quad 1 \leq r < \frac{6q}{3q-4}
\]
\( \mathbf{u}_n \to \mathbf{u}^\varepsilon \) in \( L^\theta(0, T; L^{3\theta}(\Omega)) \) for \( 1 \leq \theta < 2 \). \hfill (3.10)

Let us prove the strong convergence of \( \mathbf{B}_n \) to \( \mathbf{B}^\varepsilon \) in \( L^2(0, T; W_1) \) at this moment. Indeed, from formula (2.29), for any \( \Psi \in L^4(0, T; H^2(\Omega)) \), one has

\[
| < \partial_t \mathbf{B}_n \cdot \Psi > | 
\leq | \int_0^T \int_\Omega (\mathbf{B}_n \times \mathbf{u}_n + h \frac{\text{curl} \mathbf{B}_n \times \mathbf{B}_n}{\rho_n} + \frac{\text{curl} \mathbf{B}_n}{\sigma^\varepsilon(\rho_n)}) \cdot \text{curl} \mathbf{u} \, dx \, dt | 
\leq \| \mathbf{B}_n \|_{L^4_T(L^3)} \| \mathbf{u}_n \|_{L^6_T(L^6)} \| \nabla \Phi \|_{L^4_T(L^2)} 
+ \frac{h}{\rho} \| \text{curl} \mathbf{B}_n \|_{L^4_T(L^2)} \| \mathbf{B}_n \|_{L^6_T(L^6)} \| \nabla \Phi \|_{L^4_T(L^2)} 
+ \frac{1}{\sigma} \| \text{curl} \mathbf{B}_n \|_{L^4_T(L^2)} \| \nabla \Phi \|_{L^6_T(L^6)} 
\leq C \| \mathbf{B} \|_{L^4_T(L^3)} 
\text{and thus } \{ \partial_t \mathbf{B}_n \}_{n \geq 1} \text{ is bounded in } L^4_T(0, T; H^2(\Omega)).
\]

**Remark 1.** If there is no positive lower bound to initial density but only \( \rho_0 \geq 0 \), for fixed \( \varepsilon \) we can define the initial condition for approximate density as in [18, 26], where \( \rho^n \geq \varepsilon \) is constructed. However, when passing to the limit as \( \varepsilon \to 0 \) we will lose above uniform bound with respect to \( \varepsilon \) due to the appearance of Hall-effect term. This is the reason why we need to assume \( \inf \rho_0 > 0 \).

Keeping estimate (3.6) in mind, we infer from the Aubin–Lions lemma that

\[
\mathbf{B}_n \to \mathbf{B} \quad \text{in } L^2(0, T; L^6(\Omega)). \hfill (3.12)
\]

The weak and strong convergences obtained for \( \rho_n, \mathbf{u}_n \), and \( \mathbf{B}_n \) enable us to pass to the limit in the weak formulations (3.8)–(3.9) like in [18], except for the Hall-effect term. To deal with it, we write

\[
\frac{\text{curl} \mathbf{B}_n \times \mathbf{B}_n}{\rho} - \frac{\text{curl} \mathbf{B} \times \mathbf{B}}{\rho} = \frac{\rho - \rho_n}{\rho_0 \rho} (\text{curl} \mathbf{B}_n \times \mathbf{B}_n) + \frac{\text{curl} \mathbf{B}_n \times (\mathbf{B}_n - \mathbf{B})}{\rho} + \frac{(\text{curl} \mathbf{B}_n - \text{curl} \mathbf{B}) \times \mathbf{B}}{\rho}.
\]

Thanks to bounds (3.5), (3.6) and convergence property (3.7) we know that the first term strongly tends to 0 in \( L^1(0, T; L^6(\Omega)) \), while the second one strongly tends to 0 in \( L^1(0, T; L^6(\Omega)) \) due to (3.12). Finally, the third term weakly tends to 0 in \( L^1(0, T; L^6(\Omega)) \) since \( \text{curl} \mathbf{B}_n \) is weakly convergent to \( \text{curl} \mathbf{B} \) in \( L^2((0, T) \times \Omega) \) and \( \mathbf{B} \) lies in \( L^2(0, T; L^6(\Omega)) \), \( \rho \) lies in \( L^\infty((0, T) \times \Omega) \).

For the initial values, by definition of \( \mathbf{u}_{0,n}^\varepsilon \) and \( \mathbf{B}_{0,n}^\varepsilon \), we have

\[
\int_\Omega \rho_0^\varepsilon \mathbf{u}_{0,n}^\varepsilon \cdot \Phi(0, x) \, dx \to \int_\Omega \rho_0^\varepsilon \mathbf{u}_0^\varepsilon \cdot \Phi(0, x) \, dx = \int_\Omega \mathbf{m}_0^\varepsilon \cdot \Phi(0, x) \, dx,
\]

\[
\int_\Omega \mathbf{B}_{0,n}^\varepsilon \cdot \Psi(0, x) \, dx \to \int_\Omega \mathbf{B}_0^\varepsilon \cdot \Phi(0, x) \, dx.
\]

In conclusion, passing to the limit in (3.8)–(3.9) and energy inequality (3.1), we have obtained the following result:

**Proposition 3.2.** For any \( T > 0 \), there exists a solution \( (\rho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{B}^\varepsilon) \) which satisfies

\[
\partial_t \rho + \text{div} (\rho \mathbf{u}) = 0,
\]

\[
\partial_t (\rho \mathbf{u}) + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) \equiv - \text{div} (2\mu^\varepsilon(\rho) d(\mathbf{u})) + \nabla P = \text{curl} \mathbf{B} \times \mathbf{B},
\]

\[
\text{div} \mathbf{u} = 0,
\]

\[
\partial_t \mathbf{B} + \text{curl} \left( \mathbf{B} \times \mathbf{u} + h \frac{\text{curl} \mathbf{B} \times \mathbf{B}}{\rho} \right) = -\text{curl} \left( \frac{\text{curl} \mathbf{B}}{\sigma^\varepsilon(\rho)} \right).
\]
\[ \text{div } \mathbf{B} = 0, \]

in the sense of Definition 2.2 with the initial data
\[
\rho|_{t=0} = \rho_0^\epsilon, \quad u|_{t=0} = u_0^\epsilon, \quad B|_{t=0} = B_0^\epsilon.
\]

Moreover, that solution satisfies the following energy inequality:
\[
\int_\Omega (\rho^\epsilon |u^\epsilon|^2 + |B^\epsilon|^2) \, dx + \int_0^T \int_\Omega \mu^\epsilon(\rho^\epsilon) | \nabla u^\epsilon + (\nabla u^\epsilon)^T |^2 \, dx \, dt + 2 \int_0^T \int_\Omega \frac{|\text{curl } B^\epsilon|^2}{\sigma^\epsilon(\rho^\epsilon)} \, dx \, dt \leq \int_\Omega (\rho_0^\epsilon |u_0^\epsilon|^2 + |B_0^\epsilon|^2) \, dx, \tag{3.13}
\]

and
\[
\int_\Omega 1_{[\alpha, \beta]}(\rho^\epsilon(t, x)) \, dx = \int_\Omega 1_{[\alpha, \beta]}(\rho_0^\epsilon(x)) \, dx. \tag{3.14}
\]

3.2. Pass to the Limit as $\epsilon \to 0$

For this passage to the limit, there is no additional difficulty compare to the previous step since $\rho^\epsilon$ is still bounded away from zero by $\underline{\rho}$ uniformly. In particular, from energy inequality (3.13) we have as $\epsilon \to 0$,
\[
u^\epsilon \to u \quad \text{weakly in } L^2(0, T; V_1), \quad u^\epsilon \to u \quad \text{weakly}^* \text{ in } L^\infty(0, T; V)
\]
\[
B^\epsilon \to B \quad \text{weakly in } L^2(0, T; W_1), \quad B^\epsilon \to B \quad \text{weakly}^* \text{ in } L^\infty(0, T; W),
\]

and by Theorem 3.1
\[
\rho^\epsilon \to \rho \quad \text{in } C([0, T]; L^p(\Omega)) \quad \text{with} \quad 1 \leq p < \infty,
\]
\[
\rho^\epsilon u^\epsilon \to \rho u \quad \text{in } L^q(0, T; L^r(\Omega)) \quad \text{for } 2 < q < \infty, \quad 1 \leq r < \frac{6q}{3q-4},
\]
\[
u^\epsilon \to u \quad \text{in } L^\theta(0, T; L^{3\theta}(\Omega)) \quad \text{for } 1 \leq \theta < 2. \tag{3.15}
\]

Moreover, again by the Aubin–Lions lemma,
\[
B^\epsilon \to B \quad \text{in } L^2(0, T; L^2(\Omega)).
\]

Just as mentioned in Remark 1 that $\rho^\epsilon$ bounded below by $\underline{\rho}$ is essential for getting uniform bound of $\partial_t B^\epsilon$ in $L^2(0, T; H^{-2})$.

In view of the construction of $\mu^\epsilon$, $\sigma^\epsilon$, one has
\[
\mu^\epsilon \to \mu \quad \text{in } C([0, T]; L^p(\Omega)), \quad \forall 1 \leq p \leq \infty,
\]
\[
\sigma^\epsilon \to \sigma \quad \text{in } C([0, T]; L^p(\Omega)), \quad \forall 1 \leq p \leq \infty.
\]

Hence, the above convergence properties with convergences of initial values ensure the existence part of Theorem 1.2. We can check as in [26] (page 71–73) that any solution built as above satisfies the energy inequality (1.17). Look at Equality (3.14), an $\epsilon/2$ argument with convergence properties (2.11), (3.15) for $\rho_0^\epsilon$ and $\rho_0^\epsilon$ then implies property (1.25).

To prove that $B \in C_w([0, T]; L^2(\Omega))$, one first need to notice that $\partial_t B$ lies in $L^1(0, T; H^{-2})$, in particular $B$ is almost everywhere equal to a continuous function from $[0, T]$ into $H^{-2}(\Omega)$. Finally, $B \in L^\infty(0, T; L^2)$ together with the fact $H^2(\Omega)$ is dense in $L^2(\Omega)$ implies that $B$ is weakly continuous from $[0, T]$ into $L^2(\Omega)$. The continuity $u \in C_w([0, T]; L^2(\Omega))$ will based on similar argument.

This concludes the proof of Theorem 1.2. \qed
4. Weak–Strong Uniqueness

In this section, we prove a weak–strong uniqueness property for global weak solutions obtained from Theorem 1.2. We first remark that, in view of regularities \((\dot{\rho}, \dot{u}, \dot{B}) \in C^1([0, T]; H^2(\Omega))\) we actually could take \(\dot{u}\) and \(\dot{B}\) as test functions in the weak formulations (1.23), (1.24), and then get the following equalities for all \(t \in (0, T)\)

\[
\int_{\Omega} \rho u \cdot \dot{u} \, dx + 2 \int_{0}^{t} \int_{\Omega} \mu(\rho) d(u) \cdot d(\dot{u}) \, dx \, ds - \int_{0}^{t} \int_{\Omega} (\text{curl} B \times B) \cdot (\dot{u}) \, dx \, ds
\]

\[
= \int_{\Omega} \rho_0 |u_0|^2 \, dx + \int_{0}^{t} \int_{\Omega} \rho u \cdot \left( \partial_t \dot{u} + u \cdot \nabla \dot{u} \right) \, dx \, ds, \tag{4.1}
\]

\[
\int_{\Omega} B \cdot \dot{B} \, dx + \int_{0}^{t} \int_{\Omega} \left( \frac{\text{curl} B \times B}{\rho} + \frac{\text{curl} B \cdot \dot{B}}{\sigma(\rho)} \right) \cdot \text{curl} \dot{B} \, dx \, ds
\]

\[
= \int_{\Omega} |B_0|^2 \, dx + \int_{0}^{t} \int_{\Omega} B \cdot \partial_t \dot{B} \, dx \, ds - \int_{0}^{t} \int_{\Omega} (\text{curl} \dot{B} \times B) \cdot u \, dx \, ds. \tag{4.2}
\]

Above, we have used the following two vector identities

\[
(\rho u \otimes u) \cdot \nabla \dot{u} = \rho u \cdot (u \cdot \nabla \dot{u}),
\]

\[
(B \times u) \cdot \text{curl} \dot{B} = (\text{curl} \dot{B} \times B) \cdot u.
\]

Let us notice that there exists a gradient term \(\nabla \dot{P}\) that belongs to \(L^2(0, T; L^6(\Omega))\) coupled with \((\dot{\rho}, \dot{u}, \dot{B})\). Next we write

\[
\rho(\partial_t \dot{u} + u \cdot \nabla \dot{u}) = \text{div} (2\mu(\rho)d(\dot{u})) + \nabla \dot{P} - \text{curl} \dot{B} \times \dot{B}
\]

\[
= (\rho - \dot{\rho})(\partial_t \dot{u} + \dot{u} \cdot \nabla \dot{u}) + \rho(u - \dot{u}) \cdot \nabla \dot{u} - \text{div} (2(\mu(\rho) - \mu(\dot{\rho}))d(\dot{u})), \tag{4.3}
\]

\[
\partial_t \dot{B} + \text{curl} \left( \dot{B} \times \dot{u} + \frac{\text{curl} \dot{B} \times \dot{B}}{\rho} \right) + \text{curl} \left( \frac{\text{curl} \dot{B}}{\sigma(\rho)} \right) = 0, \tag{4.4}
\]

and if we multiply (4.3), (4.4) by \(u, B\), respectively, and integrate over \((0, t) \times \Omega\), then we have by integrations by parts

\[
\int_{0}^{t} \int_{\Omega} \rho u \cdot (\partial_t \dot{u} + u \cdot \nabla \dot{u}) \, dx \, ds + 2 \int_{0}^{t} \int_{\Omega} \mu(\rho) d(u) \cdot d(\dot{u}) \, dx \, ds
\]

\[
= \int_{0}^{t} \int_{\Omega} \left( \text{curl} B \times B + (\rho - \dot{\rho})(\partial_t \dot{u} + \dot{u} \cdot \nabla \dot{u}) + \rho(u - \dot{u}) \cdot \nabla \dot{u} \right) \cdot u \, dx \, ds
\]

\[
+ \int_{0}^{t} \int_{\Omega} 2(\mu(\rho) - \mu(\dot{\rho}))d(\dot{u}) \cdot d(u) \, dx \, ds, \tag{4.5}
\]

and

\[
\int_{0}^{t} \int_{\Omega} \left( \partial_t \dot{B} \cdot B + \frac{\text{curl} \dot{B} \cdot B}{\sigma(\rho)} \right) \, dx \, ds
\]

\[
= \int_{0}^{t} \int_{\Omega} \left( (\sigma(\rho) - \sigma(\dot{\rho})) \frac{\text{curl} \dot{B}}{\sigma(\rho)\sigma(\dot{\rho})} \right) \cdot \text{curl} B
\]

\[
- \left( \dot{B} \times \dot{u} + \frac{\text{curl} \dot{B} \times \dot{B}}{\dot{\rho}} \right) \cdot \text{curl} B \right) \, dx \, ds. \tag{4.6}
\]

Bringing (4.5), (4.6) into (4.1), (4.2), we obtain

\[
\int_{\Omega} (\rho u \cdot \dot{u} + B \cdot \dot{B}) \, dx + \int_{0}^{t} \int_{\Omega} \left( 4\mu(\rho)d(u) \cdot d(\dot{u}) + 2\frac{\text{curl} \dot{B} \cdot B}{\sigma(\rho)} \right) \, dx \, ds
\]

\[
= \int_{0}^{t} \int_{\Omega} \left( \text{curl} B - \text{curl} \dot{B} \right) \times (B - \dot{B}) \cdot \dot{u} - \left( \text{curl} \dot{B} \times (B - \dot{B}) \right) \cdot (u - \dot{u}) \tag{4.7}
\]
\[\begin{align*}
+ & h \left( \frac{\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}}{\hat{\rho}} \right) \times (\hat{\mathbf{B}} - \mathbf{B}) \cdot \text{curl } \hat{\mathbf{B}} + h(\rho - \hat{\rho}) \left( \frac{\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}}{\hat{\rho}} \right) \cdot \text{curl } \hat{\mathbf{B}} \\
+ & (\sigma(\hat{\rho}) - \sigma(\rho)) \frac{\text{curl } \hat{\mathbf{B}}}{\sigma(\rho)\sigma(\hat{\rho})} \cdot \text{curl } \mathbf{B} + ((\rho - \hat{\rho})(\partial_e \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}) + \rho(\mathbf{u} - \hat{\mathbf{u}}) \cdot \nabla \hat{\mathbf{u}}) \cdot \mathbf{u} \\
+ & 2(\mu(\rho) - \mu(\hat{\rho}))d(\hat{\mathbf{u}}) \cdot d(\mathbf{u}) \right) dx ds + \int_\Omega (\rho_0|\mathbf{u}_0|^2 + |\mathbf{B}_0|^2) dx.
\end{align*}\]

Then we multiply (4.3), (4.4) by \(\hat{\mathbf{u}}, \hat{\mathbf{B}}\) respectively, and integrate over \((0, t) \times \Omega\) to find
\[\begin{align*}
\frac{1}{2} & \int_\Omega \rho|\hat{\mathbf{u}}|^2 dx + \int_0^t \int_\Omega \mu(\rho)|d(\hat{\mathbf{u}})|^2 dx ds \\
= & \frac{1}{2} \int_\Omega \rho_0|\mathbf{u}_0|^2 dx + \int_0^t \int_\Omega \left( (\text{curl } \mathbf{B} \times \mathbf{B} + (\rho - \hat{\rho})(\partial_e \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}) + \rho(\mathbf{u} - \hat{\mathbf{u}}) \cdot \nabla \hat{\mathbf{u}}) \cdot \mathbf{u} \\
+ & 2(\mu(\rho) - \mu(\hat{\rho}))d(\hat{\mathbf{u}}) \cdot d(\mathbf{u}) \right)^2 dx ds
\end{align*}\]

and
\[\begin{align*}
\frac{1}{2} & \int_\Omega |\hat{\mathbf{B}}|^2 dx + \int_0^t \int_\Omega \frac{\text{curl } \hat{\mathbf{B}}^2}{\sigma(\rho)} dx ds \\
= & \frac{1}{2} \int_\Omega |\mathbf{B}_0|^2 dx - \int_0^t \int_\Omega \left( (\text{curl } \mathbf{B} \times \mathbf{B}) \cdot \hat{\mathbf{u}} - (\sigma(\rho) - \sigma(\hat{\rho})) \left[ \frac{\text{curl } \mathbf{B}}{\sigma(\rho)} \right] \right. \left. \cdot \hat{\mathbf{B}} \right)^2 dx ds.
\end{align*}\]

Finally, we add up (4.9), (4.10) and energy inequality (1.17), and subtract (4.8), thanks to (1.10), (1.11) and (1.25), we have
\[\begin{align*}
\frac{1}{2} & (\rho||\mathbf{u} - \hat{\mathbf{u}}||^2_{L^2} + ||\mathbf{B} - \hat{\mathbf{B}}||^2_{L^2}) + \int_0^t \left( \mu||\nabla \mathbf{u} - \nabla \hat{\mathbf{u}}||^2_{L^2} + \frac{1}{\sigma}||\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}||^2_{L^2} \right) ds \\
\leq & \int_0^t \int_\Omega \left( (\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}) \times (\hat{\mathbf{B}} - \mathbf{B}) \cdot \text{curl } \hat{\mathbf{B}} \\
+ & h \left( \frac{\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}}{\hat{\rho}} \right) \times (\hat{\mathbf{B}} - \mathbf{B}) \cdot \text{curl } \hat{\mathbf{B}} \\
+ & h(\rho - \hat{\rho}) \left( \frac{\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}}{\hat{\rho}} \right) \cdot \text{curl } \hat{\mathbf{B}} \\
+ & (\sigma(\hat{\rho}) - \sigma(\rho)) \frac{\text{curl } \hat{\mathbf{B}}}{\sigma(\rho)\sigma(\hat{\rho})} \cdot (\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}) \\
+ & ((\rho - \hat{\rho})(\partial_e \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}}) + \rho(\mathbf{u} - \hat{\mathbf{u}}) \cdot \nabla \hat{\mathbf{u}}) \cdot (\hat{\mathbf{u}} - \mathbf{u}) \\
+ & 2(\mu(\rho) - \mu(\hat{\rho}))d(\hat{\mathbf{u}}) \cdot d(\mathbf{u}) \right) dx ds.
\end{align*}\]

Hence, by Hölder inequality, we have
\[\begin{align*}
\frac{1}{2} & (\rho||\mathbf{u} - \hat{\mathbf{u}}||^2_{L^2} + ||\mathbf{B} - \hat{\mathbf{B}}||^2_{L^2}) + \int_0^t \left( \mu||\nabla \mathbf{u} - \nabla \hat{\mathbf{u}}||^2_{L^2} + \frac{1}{\sigma}||\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}||^2_{L^2} \right) ds \\
\leq & C \int_0^t \left( ||\mathbf{B} - \hat{\mathbf{B}}||_{L^2} ||\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}||_{L^2} + ||\mathbf{B} - \hat{\mathbf{B}}||_{L^\infty} \frac{1}{\rho} ||\mathbf{B} - \hat{\mathbf{B}}||^2_{L^2} \right) ds \\
+ & ||\mathbf{B} - \hat{\mathbf{B}}||_{L^2} ||\text{curl } \mathbf{B}||_{L^\infty} ||\text{curl } \hat{\mathbf{B}}||_{L^\infty} \frac{1}{\rho} ||\mathbf{B} - \hat{\mathbf{B}}||^2_{L^2} ||\text{curl } \mathbf{B}||_{L^\infty} ||\text{curl } \hat{\mathbf{B}}||_{L^\infty} \frac{1}{\rho} ||\mathbf{B} - \hat{\mathbf{B}}||^2_{L^2} \right) ds \\
+ & \int_0^t \frac{h(\rho - \hat{\rho})}{\rho} \left( \frac{\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}}{\hat{\rho}} \right) \times \mathbf{B} \cdot \text{curl } \mathbf{B} dx \\
= & \int_0^t \frac{h(\rho - \hat{\rho})}{\rho} \left( \frac{\text{curl } \mathbf{B} - \text{curl } \hat{\mathbf{B}}}{\hat{\rho}} \right) \times \mathbf{B} \cdot \text{curl } \mathbf{B} dx \\
+ & \int_0^t \int_\Omega (\rho_0|\mathbf{u}_0|^2 + |\mathbf{B}_0|^2) dx ds.
\end{align*}\]
\[ \leq h \frac{1}{\rho} \| \rho \|_{L^\infty} \left( \frac{1}{\rho} \| \nabla \rho \|_{L^2} \right) + \| \rho - \hat{\rho} \|_{L^2} \| B - \hat{B} \|_{L^2} \]

Now, we apply Young’s inequality and sometimes Sobolev embeddings to (4.11) and deduce from the assumptions on \((\hat{\rho}, \hat{u}, \hat{B})\) that for all \(t \in (0, T)\)

\[ \frac{1}{2} \rho \| u - \hat{u} \|_{L^2}^2 + \| B - \hat{B} \|_{L^2}^2 + \left( \mu \| \nabla u - \nabla \hat{u} \|_{L^2}^2 + \frac{1}{\sigma} \| \nabla \hat{B} \|_{L^2}^2 \right) dt \]

\[ \leq \int_0^t C(s) \left( \| u - \hat{u} \|_{L^2}^2 + \| B - \hat{B} \|_{L^2}^2 + \| \rho - \hat{\rho} \|_{L^2}^2 \right) ds, \]

where \(C(s)\) denotes a various measurable function in \(L^1(0, T)\).

It remains to estimate \(\| \rho - \hat{\rho} \|_{L^2}\) as in [26]. In fact, we have

\[ \frac{1}{2} \rho \| u - \hat{u} \|_{L^2}^2 \leq \int_0^t \| \nabla \hat{\rho} \|_{L^3} \| \rho - \hat{\rho} \|_{L^2} ds. \]

Combining above two estimates, we write

\[ \frac{1}{2} \rho \| u - \hat{u} \|_{L^2}^2 + \| B - \hat{B} \|_{L^2}^2 + \| \rho - \hat{\rho} \|_{L^2}^2 \]

\[ + \int_0^t \left( \mu \| \nabla u - \nabla \hat{u} \|_{L^2}^2 + \frac{1}{\sigma} \| \nabla \hat{B} \|_{L^2}^2 \right) ds \]

\[ \leq \int_0^t C(s) \left( \| u - \hat{u} \|_{L^2}^2 + \| B - \hat{B} \|_{L^2}^2 + \| \rho - \hat{\rho} \|_{L^2}^2 \right) ds, \]

and conclude by applying Grönwall’s lemma that \((\hat{\rho}, \hat{u}, \hat{B}) \equiv (\rho, u, B)\) a.e. in \((0, T) \times \Omega\).

\[ \square \]

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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