On the Notion of Semi-Randers Space

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Abstract

The two currents frameworks for indefinite Finslerian space-times are introduced. We discuss some of the limitations of both formalisms, in particular the notion of semi-Randers. The key point is the issue of the gauge invariance associated with gauge transformations of the 1-form appearing in the Randers function. Therefore gauge invariance spoils the space-time metric interpretation and Lagrangian interpretation of a Randers space. Instead, we argue in favor of a sheaf formulation of the classical dynamics.

1 Introduction

The origin of the notion of Randers space can be traced back at least to the work of G. Randers [1]. However, a proper discussion of the positivity and non-degeneracy of the resultant metric was not performed, specially the discussion of the gauge invariance and its physical implications, were inconclusive.

If one does not consider the issue of the gauge invariance, whereas for positive definite metrics there is an established theory [2, chapter 11], for indefinite Randers spaces the question there is a lack of a convincing and universally accepted formulation. The situation is quite different, however, when one tries to formulate a structure allowing gauge transformations of the 1-form that defines the Randers function.

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Currently, there are two formalisms for indefinite Finsler space (which we call semi-Finsler structures). We discuss their differences and limitations, in particular when one considers the corresponding definition of Randers-type spaces in these frameworks. We explain why the situation is considered unsatisfactory from a physical point of view. This motivates a new mathematical formulation for Randers spaces.

Therefore, we consider in section 3 a refinement of the usual Randers-type space which is compatible with the gauge invariance. This will imply that a Randers structure is essentially non-metric, since they will appear singularities in the domain of definition of the fundamental tensor.

2 Semi-Randers Spaces as Space-Time Structures

Let $M$ be a smooth $n$-dimensional manifold, $TM$ its tangent bundle manifold with $TM \supset N$, and $\pi : N \to M$ a fiber bundle over $M$. $(x, y)$ is a local coordinate system over a neighborhood of $TM$.

We consider the following two definitions of semi-Finsler structures being currently used in the literature:

1. Asanov’s definition [3],

   **Definition 2.1** A semi-Finsler structure $F$ defined on the $n$-dimensional manifold $M$ is a positive, real function $F : N \to [0, \infty[$ such that:

   (a) It is smooth in $N$,

   (b) It is positive homogeneous of degree 1 in $y$,

   $$F(x, \lambda y) = \lambda F(x, y), \ \forall \lambda > 0,$$

   (c) The vertical Hessian matrix

   $$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}$$  \hspace{1cm} (2.1)

   is non-degenerate over $N$.

   $g_{ij}(x, y)$ is the matrix of the fundamental tensor. The set $N_x$ is called the admissible set of tangent vectors at $x$; the disjoint union $N = \bigsqcup_{x \in M} N_x$ is the admissible set of vectors over $M$.

   In principle $N \subset TM$ is unspecified. This point will be problematic when we consider Randers-type metrics.

2. Beem’s definition [4], [5]

   **Definition 2.2** A semi-Finsler structure defined on the $n$-dimensional manifold $M$ is a real function $L : TM \to \mathbb{R}$ such that
(a) It is smooth in the slit tangent bundle \( \tilde{\mathbf{N}} := \mathbf{T}\mathbf{M} \setminus \{0\} \)

(b) It is positive homogeneous of degree 2 in \( y \),

\[
L(x, \lambda y) = \lambda^2 L(x, y), \quad \forall \lambda > 0,
\]

(c) The Hessian matrix

\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}
\]

is non-degenerate on \( \tilde{\mathbf{N}} \).

Some differences between the above definitions are highlighted below:

1. Using Beem’s definition it is possible to speak in an invariant way of light-like vectors and to derive Finslerian geodesics, including lightlike geodesics, from a variational principle [6]. By construction, in Asanov’s formalism it is not possible to do that in an invariant way, because lightlike vectors are excluded in the formalism from the beginning, since nothing is said about how to extend the function \( F^2 \) smoothly to the lightcone.

2. In Asanov’s framework, given a parameterized curve \( \sigma : I \rightarrow \mathbf{M}, I \subset \mathbf{R} \) on the semi-Finsler manifold \( (\mathbf{M}, F) \) such that \( \dot{\sigma} \in \mathbf{N} \) for all \( t \in I \), the length functional acting on \( \sigma \) is given by the line integral

\[
\sigma(t) \rightarrow \mathcal{E}_A(\sigma) := \int_{t_{\min}}^{t_{\max}} F(\sigma(t), \dot{\sigma}(t)) dt, \quad I = [t_{\min}, t_{\max}].
\]

Due to the homogeneity condition of the Finsler function \( F \), \( \mathcal{E}_A(\sigma) \) is re-parametrization invariant. On the other hand, if we consider Beem’s definition, the energy functional is given by the following expression [6]:

\[
\sigma(t) \rightarrow \mathcal{E}_B(\sigma) := \int_{t_{\min}}^{t_{\max}} L(\sigma(t), \dot{\sigma}(t)) dt, \quad I = [t_{\min}, t_{\max}].
\]

Beem’s energy functional is not re-parametrization invariant, because \( L \) is homogeneous of degree 2 in \( y \).

3. A third difference emerges when we consider the category of Randers-type spaces:

**Definition 2.3** (semi-Randers Space as semi-Finsler Space)

In Asanov’s framework, a semi-Randers space is characterized by a semi-Finsler function of the form:

\[
F(x, y) = \sqrt{\eta_{ij}(x) y^i y^j} + A(x, y),
\]

where \( \eta_{ij}(x) dx^i \otimes dx^j \) defines a (semi)-Riemannian metric and \( A(x, y) := A_i(x) y^i \) is the value of the action of the 1-form \( A(x) = A_i dx^i \) acting on \( y \in \mathbf{N}_x \).

In the positive definite case (when \( \eta \) is a Riemannian metric), the requirement that \( g_{ij} \) being non-degenerate implies that the 1-form \( (A_1, \ldots, A_n) \) is bounded by the Riemannian
metric \( \eta \):

\[ A_i A_j \eta^{ij} < 1, \quad \eta_{ik} \eta^{kj} = \delta^j_i. \]

The indefinite case is quite different. First of all, there is not a Riemannian metric from the beginning that allows for a definition of a norm. Therefore, the criterion for non-degeneracy becomes not trivial. Secondly, only the variation of the length functional (2.4) (not directly the integrand itself) is invariant under the gauge transformation \( A \rightarrow A + d\lambda \), while \( F \) is not invariant.

The notion of Randers space in Beem’s formalism is even more problematic. In this case, there is not a formulation of semi-Randers space (because eq. (2.5) is of degree 1 in \( y \)).

3 Non-Lagrangian Notion of Semi-Randers Space

Given a manifold \( M \), let us assume the existence of a smooth semi-Riemannian structure \( \eta \) on \( M \),

\[ \eta : TM \times TM \rightarrow \mathbb{R} \]

\[ (X, Y) \mapsto \eta_{ij}(x)X^i Y^j, \quad X, Y \in T_x M \]

smooth on \( x, X^i, Y^j \). Since we will use the square root \( \sqrt{\eta_{ij}(x)X^i Y^j} \), we also require that:

1. The square root function is continuous on \( TM \)
2. \( \sqrt{\eta_{ij}(x)X^i Y^j} \) is smooth on \( N := \bigcup_{x \in TM} \{ y \in T_x M, \ \eta_{ij}(x) y^i y^j > 0 \} \)
3. \( \sqrt{-\eta_{ij}(x)X^i Y^j} \) is smooth in \( N := \bigcup_{x \in TM} \{ y \in T_x M, \ \eta_{ij}(x) X^i Y^j < 0 \} \).

Let us assume a closed 2-form \( F \) on \( M \). Then, we propose a notion of semi-Randers space based on the following

**Definition 3.1** A semi-Randers space consists on a triplet \((M, \eta, F)\), where \( M \) is a space-time manifold, \( \eta \) is a semi-Riemannian metric continuous on \( M \) and smooth on \( TM \setminus CM \) with \( F \in \wedge^2 M \) such that \( dF = 0 \).

\( F \) is on the second de Rham cohomology group \( H^2(M) \). Due to Poincaré’s lemma, there exists a locally smooth 1-form \( A \) such that \( dA = F \). Any pair of locally smooth 1-form \( \tilde{A} = A + d\lambda \), with \( \lambda \) a locally smooth real function defined on the given open neighborhood, are equivalent and produce under exterior differentiation the same cohomology class \( F \in H^2(M) \), which representative is \( F \): \( d(d\lambda + A) = dA = F \). Note that we are speaking of locally smooth 1-forms \( A \) and a globally smooth 2-forms \( F \). Therefore, instead of giving \( F \), one can consider equivalence class of 1-forms:

\[ [A] := \{ \tilde{A} = A + d\lambda, \ dA = F \ \text{in the intersection of the opens sets where} \ A \ \text{and} \ \lambda \ \text{are defined} \}, \]

with \( A \) a piecewise smooth 1-form defined on an open set \( U \subset M \). Therefore, if two forms \( ^1A \) and \( ^2A \) are in \( [A] \), \( ^1A = ^2A \). This implies that \( d(^1A - ^2A) = 0 \). Therefore, for each point \( x \) of
the open neighborhood $U_{ij}$ where both potentials are defined, there is a locally smooth function defined in a open neighborhood of $x \in U(x) \subset U_{ij}$ such that $(^1A - J^A) = d\lambda$ (using Poincaré's lemma).

Therefore, we give an alternative definition of semi-Randers space:

**Definition 3.2** A semi-Randers space consists of a triplet $(M, \eta, [A])$, where $M$ is a space-time manifold, $\eta$ is a semi-Riemannian metric continuous on $M$ and smooth on $TM \setminus CM$ and the class of locally smooth 1-forms $A \in [A]$ is defined as before.

We have just proved that both definitions are equivalent. However, we adopt the second definition, because it has the advantage that allows us to discuss some local issues related with the inverse variational problem associated with the Lorentz force equation. This is why, even is the topology of $M$ is trivial, we speak of piecewise smooth forms. Note that two neighborhoods $U_1$ and $U_2$, the potentials are $^1A$ and $^2A$. Since they are in the same class $[A]$, in the intersection $U_1 \cap U_2 \neq \emptyset$, they are related by $^1A = d\lambda_{12} + ^2A$. Note in general that there is not restriction on the group $H^1M$, since the class $[A]$ is different than the class $F$.

For each of the representatives $A \in [A]$ there is defined on $M$ a function $F_A$ by the following expression:

1. $F_A(x, y) = \sqrt{\eta_{ij}(x) y^i y^j} + A_i(x) y^i$, for $\eta_{ij}(x) y^i y^j \geq 0$,
2. $F_A(x, y) = \sqrt{-\eta_{ij}(x) y^i y^j} + A_i(x) y^i$, for $\eta_{ij}(x) y^i y^j < 0$.

(3.1)

The following are direct consequences from the definition:

**Proposition 3.3** Let $(M, \eta, [A])$ be a semi-Randers space with $\eta$ a semi-Riemannian metric, $A \in [A]$ and $F_A$ given by equation (3.1). Then:

1. On the light cone $C_xM := \{ y \in T_xM | \eta_{ij}(x) y^i y^j = 0 \}$ $F_A$ is of class $C^0$, for $\eta \in C^0$ and $A \in \Lambda^1M$.
2. The subset where $\eta_{xx}(y, y) \neq 0$ is an open subset of $T_xM$ and $F$ is smooth on $T_xM \setminus C_xM$, for $\eta$ smooth and $A \in \Lambda^1M$.
3. There is no condition of non-degeneracy of the fundamental tensor $g_{ij}$. Therefore, it is not required that the representative $A$ must to be bounded by 1.
4. The function $F$ is positive homogeneous of degree 1 in $y$.

However, without the requirement that the fundamental tensor $g_{ij}$ be non-degenerate, it is not possible to define a geodesic equation from a variational principle [7].

The natural variational principle is based on the following functional. Consider a locally smooth 1-form $A$ defined in an open neighborhood of $M$. Then let us consider a curve $\sigma : I \rightarrow M$
with \( \sigma I \) included in the domain of definition of the locally smooth 1-form \( A \). Then let us consider

the functional acting on \( \sigma \) given by the action:

\[
\mathcal{E}_{F_A}(\sigma) = \int_{\sigma} F_A(\sigma(\tau), \dot{\sigma}(\tau)) \, d\tau
\]

and with \( \tau \) the proper time associated to \( \eta \) along the curve \( \sigma \). Let us restrict to the gauge invariance under functions \( \lambda \) vanishing on the boundary \( \partial M \). Then, the functional is gauge invariant, up to a constant: if we choose another gauge potential \( \tilde{A} = A + d\lambda \), then \( \mathcal{E}_{F_A}(\sigma) = \mathcal{E}_{\tilde{F}_A}(\sigma) + \text{constant} \), as we will show shortly. Therefore, the functional is well defined on a given semi-Randers space \((M, \eta, [A])\). We denote the corresponding functional acting on \( \sigma \) by \( \mathcal{E}_F(\sigma) \).

It is well known that the condition that guarantees the construction of the first variation formula and the existence and uniqueness of the corresponding geodesics is that the vertical Hessian \( g_{ij} \) must be non-degenerate [7]. Also note that one cannot guarantee that given a particular potential \( A \in [A] \) Hessian of \( F_A \) is non-degenerate. However, due to the possibility to do gauge transformations in the gauge potential \( A(x) \mapsto A(x) + d\lambda(x) \), one has the following proposition:

**Proposition 3.4** Let \((M, \eta, [A])\) be a semi-Randers space. Assume that:

1. The dynamics is such that the values of \( y^0 \to \infty \).
2. The image of any curve \( \sigma \) on the manifold \( M \) are compact.

Then,

1. There is a representative \( \tilde{A} \in [A] \) such that the hessian of the functional \( F_{\tilde{A}} \) is non-degenerate, except for a sub-set of co-dimension 1 in each unit hyperboloid \( \Sigma_x \).
2. The functional (3.2) is well defined on the Randers space \((M, \eta, [A])\), except for a constant depending on the representative \( \tilde{A} \in [A] \).

**Proof:** Using the gauge invariance of \( \mathcal{E}_F(\sigma) \) up to a constant, we can find locally an element \( \tilde{A} \in [A] \) such that \( \tilde{A}(x)i\tilde{A}(x)j \eta^{ij} < 1 \) in an open neighborhood. Consider that we start with a 1-form \( A \) which is not bounded by 1 and then consider the equivalent representative \( \tilde{A}(x) + d\lambda(x) \). Then, the requirement that the hessian of \( \tilde{A} \) is non-degenerate is, using the generalization of the condition [2, pg 289]:

\[
0 < |2 + \frac{\tilde{A}_i(x)y^i + \sqrt{\eta_{ij}(x)y^iy^j}(\eta^{ij}\tilde{A}_i\tilde{A}_j)}{\sqrt{\eta_{ij}(x)y^iy^j} + \tilde{A}_i(x)y^i}|.
\]

In the unit hyperboloid, this condition is:

\[
0 < |2 + \frac{\tilde{A}_i(x)y^i + \eta^{ij}\tilde{A}_i\tilde{A}_j}{1 + \tilde{A}_i(x)y^i}|.
\]
Let us assume that (if it is negative, the treatment is similar) that
\[
2 + \frac{\bar{A}_i(x)y^i + \eta^{ij} \bar{A}_i\bar{A}_j}{1 + \bar{A}_i(x)y^i} > 0.
\]
Therefore, there is a positive and locally smooth function \(\epsilon^2(x)\) such that:
\[
\epsilon^2(x, y) = 2 + \frac{\bar{A}_i(x)y^i + \eta^{ij} \bar{A}_i\bar{A}_j}{1 + \bar{A}_i(x)y^i}.
\]
In order to be able to right down this condition one needs that \(1 + \bar{A}_i(x)y^i \neq 0\). Therefore, the conditions where this does not hold, is equivalent to the intersection of the hyperplane
\[
P_x := \{y \in T_x M, \ | 1 + \bar{A}_i(x)y^i \} = 0
\]
with the unit hyperboloid \(\Sigma_x\). The intersection has maximal co-dimension 1. Let us write in detail the above condition of positiveness:
\[
\epsilon^2(x, y) = 2 + \frac{(A_i(x) + \partial_i\lambda(x))y^i + \eta^{ij}(A_i\partial_j\lambda(x))(A_j + \partial_j\lambda(x))}{1 + (A_i(x) + \partial_i\lambda(x))y^i}.
\]
This condition can be re-writing:
\[
\epsilon^2(x, y) \left(1 + (A_i(x) + \partial_i\lambda(x))y^i\right) = 2 \left(1 + (A_i(x) + \partial_i\lambda(x))y^i\right)(A_i(x) + \partial_i\lambda(x))y^i + \eta^{ij}(A_i + \partial_i\lambda(x))(A_j + \partial_j\lambda(x)).
\]
We can decompose this condition in two conditions and show that they can be satisfied simultaneously in the limit \(y^0 \to \infty\):
\[
\epsilon^2(x, y) \left(1 + (A_i(x) + \partial_i\lambda(x))y^i\right) = 2 \left(1 + (A_i(x) + \partial_i\lambda(x))y^i\right)(A_i(x) + \partial_i\lambda(x))y^i + A, \tag{3.3}
\]
\[
A = \eta^{ij}(A_i + \partial_i\lambda(x))(A_j + \partial_j\lambda(x)); \tag{3.4}
\]
\(|A| < 1\) a constant. We can see from eq. (3.3) that in the limit \(y^0 \to \infty\)
\[
\lim_{y^0 \to \infty} \epsilon^2(x, y) = 3.
\]
Therefore, for big enough \(y^0\), \(\epsilon^2(x, y)\) will be bigger than zero.

The partial differential equation (3.4) has a local solution. The argument is that it can be re-interpreted as a Hamilton-Jacobi equation, which has always local solutions.

Since the image of any curve \(\sigma\) on the manifold \(M\) is compact, one can find a finite open cover of \(\sigma(I)\), \(\{U_k\}\) such that over each \(U_k\) there is a gauge potential \(k\bar{A} \in [A]\) such that the hessian \(g\) is non-degenerate. By construction, on each pair of intersecting neighborhoods \(kU\) and \(lU\), there is a gauge function \(\lambda^{kl}(x)\) such that \(k\bar{A}(x) = l\bar{A}(x) + d\lambda^{kl}(x)\); this is because both \(k\bar{A}\) and \(l\bar{A}\) are in the same class \([A]\). In this way, using a finite covering of \(M\), one obtains a global representative of the class \([A]\) such that the corresponding Hessian of \(F_{\bar{A}}\) is not bounded.
When the functional $\mathcal{E}_F(\sigma)$ acting on the curve $\sigma$ which image is defined over several domains $U_k$, it is evaluated using the representatives $\tilde{A}_k$, the evaluation differs by boundary terms, which comes from the evaluation of the $d\lambda$ terms on the boundary points in the intersection $\mathcal{E} U \cap \mathcal{E} U \cap \sigma(I)$. This differences does not contribute to the first variation of the functional. Therefore, the first variation of the functional (3.2) exists and does not depend of the representative. \hfill $\Box$

**Remark 1.** It is important to notice that in principle the non-degeneracy of the metric $g_{ij}$, that is, the vertical Hessian of the function $\frac{1}{2} F^2(x, y)$ implies not necessarily that it has the same signature than the semi-Riemannian metric $\eta$.

**Remark 2.** It is always possible to find a neighborhood where the expression

$$2 + \frac{\tilde{A}_i(x)y^i + \eta^{ij}(x)\tilde{A}_i(x)\tilde{A}_j(x)}{1 + \tilde{A}_i(x)y^i}$$

is either strictly positive or strictly negative. The argument goes as follows. If one has

$$2 + \frac{A_i(x)y^i + \eta^{ij}(x)A_i(x)A_j(x)}{1 + A_i(x)y^i} = 0$$

it can be re-written (except for a set of maximum co-dimension 1) as

$$(A_i(x)y^i + \eta^{ij}(x)A_i(x)A_j(x)) = -2(1 + A_i(x)y^i)$$

If one consider the 1-form $A$ fixed, then the scaling behavior of this condition with $y \rightarrow cy$ for long values of $c$ is inconsistent:

$$(A_i(x)y^i + \eta^{ij}(x)A_i(x)A_j(x)) = -2(1 + A_i(x)y^i) \rightarrow (A_i(cy^i + \eta^{ij}(x)A_i(x)A_j(x)) = -2(1 + A_i(cx^i).$$

If the value of $c$ is very large, except for special values of $y$, the scaling of this expression is of the form

$$A_i(x)cy^i = -2A_i(x)c y^i.$$ 

This is a condition that can only be full-filled when $A_i(x)c y^i = 0$. The incompatibility of the scaling is the signal that one can find domains of large value of the components of $y$ where the expression is strictly positive or strictly negative. This domains are of interest for us.

**Remark 3.** If the curve $\sigma$ is parameterized with respect to a parameter such that the Finslerian arc-length $F(\sigma, \dot{\sigma})$ is constant along the geodesic, the geodesic equations have a more complicated form (see for instance [2, pg. 296]) and they are not invariant under arbitrary gauge transformations of the 1-form $A \rightarrow A + d\lambda$, $\lambda \in \mathcal{F}(M)$.

**Remark 4.** The fact that we are speaking of local data makes natural to consider sheaf cohomology [8, chapter 6] as the basic ingredient in the definition of Randers spaces. Sheaf theory is the theory that enable to treat local objects defined on sheafs, like locally smooth functions on a manifold or locally smooth manifolds [9, chapter 2]. In this case, the definition of the class $[A]$ is on the second class of the sheaf cohomology on the manifold $M$.

With definition (3.2), the problem of how to introduce gauge invariance in a Randers-type space is almost solved. However, as a result, the class $[A]$ and the Riemannian metric $\eta$ are
unrelated geometric objects. This is in contradiction with the spirit of Randers spaces as a space-time asymmetric structure.

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