**On phase asymmetries in oscillatory pipe flow**

Daniel Feldmann and Claus Wagner

**Abstract**  We present results from direct numerical simulations (DNS) of oscillatory pipe flow at several dimensionless frequencies $Wo \in \{6.5, 13, 26\}$ and one fixed shear Reynolds number $Re_{\tau} = 1440$. Starting from a fully-developed turbulent velocity field at that $Re_{\tau}$, the oscillatory flow either relaminarises or reaches a conditionally turbulent or strongly asymmetric state depending on $Wo$. The numerical method is validated by demonstrating excellent agreement of our DNS results with experimental data and analytical predictions from literature for the limiting cases of non-oscillating but turbulent and oscillating but laminar pipe flow. For an oscillating turbulent pipe flow we further found a very good agreement between qualitative descriptions of the characteristic flow features observed in experiments and our DNS. Here, we focus on the observation of a strongly asymmetric behaviour between the positive and the negative half-cycles of the oscillatory pipe flow at $Wo = 6.5$.

**1 Introduction**

For many industrial applications and especially for bio-fluid dynamics, e.g. respiratory airflow (oscillatory) or vascular blood flow (pulsatile), decay and amplification of turbulence play an important role. The onset of turbulence in such wall-bounded time-periodic fluid motions considerably affect the mixing and transport efficiency and can rapidly change shear forces acting on the system’s wall. Contrarily to the onset of turbulence in statistically steady pipe flows, see e.g. Avila et al. [1], turbulence in statistically unsteady pipe flows is far from understood. Some of the few...
existing theoretical, numerical and experimental studies on transition to turbulence in such flows are summarised and discussed in e.g. [4, 5]. The most important results are also collated in fig. 1. To our knowledge, the existing numerical studies only consider oscillatory flows over a flat plate or in two-dimensional channels and cover rather few and low Womersley numbers \( Wo \), or focus on pulsatile rather than oscillatory flows. To complement the findings summarised in fig. 1 we perform three-dimensional direct numerical simulations (DNS) of oscillatory pipe. In the following we report and discuss the vastly asymmetric behaviour of an oscillatory pipe flow at \( Wo = 6.5 \).

2 Numerical method

We consider an incompressible Newtonian fluid with constant properties confined by a cylindrical wall of diameter \( D \) and length \( L \). The governing equations read

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \frac{\partial \mathbf{u}}{\partial t} + \mathbf{(u \cdot \nabla)u} + \nabla p' - \frac{1}{Re_\tau} \nabla^2 \mathbf{u} = -\mathbf{P}(t),
\]

where the flow is driven in axial direction \( z \) by the prescribed mean pressure gradient

\[
P_z(t) = \frac{\partial \langle p \rangle}{\partial z} \equiv -4 \cos \left( \frac{4W_0^2 t}{Re_\tau} \right), \quad P_\phi \equiv 0, \quad P_r \equiv 0 \quad \text{with} \quad p = p' + \langle p \rangle.
\]

The time is denoted by \( t \) and \( \mathbf{u} \) is the velocity vector with its components \( u_z \), \( u_\phi \), and \( u_r \), pointing along the cylindrical coordinates in axial \( (z) \), azimuthal \( (\phi) \) and radial \( (r) \).
direction. According to the Reynolds decomposition the pressure \( p \) is separated in its fluctuating part \( p' \) and its mean part \( \langle p \rangle \), such that periodic boundary conditions can be employed for \( p' \) in \( z \) and in \( \phi \). The control parameters and the normalisation in eqs. (1) and (2) are given by the Womersley number \( \text{Wo} = \frac{D}{2} \sqrt{\frac{\omega}{\nu}} \) and the shear Reynolds number \( \text{Re}_\tau = u_\tau D/\nu \). Here, \( \omega = \frac{2\pi}{T} \) is the oscillation frequency, \( u_\tau \) is the friction velocity of a fully-developed turbulent pipe flow at \( \text{Wo} = 0 \), and \( \nu \) is the kinematic viscosity. The oscillatory pipe flow is fully characterised by \( \text{Wo} \) and the peak Reynolds number \( \text{Re} = u_p D/\nu \), where \( u_p = \max u_b(t) \) is the peak velocity within \( 0 \leq t < T \) and \( u_b \) is the instantaneous bulk velocity. Eqs. (1) and (2) are complemented by periodic boundary conditions in \( z \) and in \( \phi \) and with no-slip conditions at \( r = D/2 \). They are discretised using a fourth order accurate finite volume method on staggered grids, based on Schumann’s [9] volume balance procedure. Time advancement is done using a leapfrog-Euler time integration scheme where the time step is adapted using a von Neuman stability criterion. Further details on the numerical method are given in [8, 11, 4].

3 Fully turbulent test case (\( \text{Wo} = 0 \))

To provide adequate initial conditions for all oscillating pipe flow DNS and to demonstrate the capability of the used numerical method to properly reproduce the physics of turbulent wall-bounded shear flows, we performed DNS of a non-oscillating but fully-developed turbulent pipe flow at \( \text{Re}_\tau = 1440 \). The resulting bulk Reynolds number of the statistically-steady turbulent pipe flow is \( \text{Re} = 25960 \). The radial profile of the mean axial velocity is shown in fig. 2a). There is an ex-

![Fig. 2](image-url)
cellent agreement with the linear law within the viscous sub-layer \((r^+ < 5)\) close to the pipe wall as well as with the logarithmic law within the inertial sub-layer \((30 < r^+ < 100)\). Furthermore, there is a good agreement between our DNS results and experimental data obtained by Durst et al. [2] using laser-Doppler anemometry (LDA) in a pipe flow at a slightly lower \(Re = 18500\). Fig. 2b) shows all three components of the RMS velocity fluctuations. There is a fairly good overall agreement with the experimental results, despite the lower \(Re\) of the LDA data. The used spatial resolution expressed in wall units \((\nu/u\tau)\), denoted by \(^+\), is given in tab. 1.

As discussed in [4], the pipe domain of length \(L^+ = 1800\) is sufficiently long and the spatial resolution is sufficiently fine to capture all relevant length scales in the turbulent pipe flow. For all simulations the adaptive time step varies in the range \(6 \times 10^{-6} < \Delta t < 3.9 \times 10^{-5}\) and is thus much smaller than the Kolmogorov time scales in the fully-developed turbulent pipe flow.

### Table 1 Resulting flow state, characteristic parameters, computational domain length and spatial resolution (given in wall units denoted by \(^+\)) of the performed pipe flow simulations at \(Re_\varepsilon = 1440\).

| Flow state  | \(Wo\) | \(Re\)  | \(L^+\) | \(\Delta z^+ \times r^+ \Delta \varphi \times \Delta r^+\) |
|-------------|--------|---------|---------|-------------------------------------------------|
| Fully turb. | 0      | 25960   | 1800, 7200 | 7.0 \times (0.03 – 4.4) \times (0.5 – 11.1) |
| Fully turb. | 0      | 25960   | 1700, 5100 | 6.7 \times (0.02 – 4.4) \times (0.5 – 6.7) |
| Asymmetric | 6.5    | 21000 / 57370 | 7200 | 7.0 \times (0.03 – 4.4) \times (0.5 – 11.1) |
| Asymmetric | 6.5    | 21000 / 57370 | 1700, 5100 | 6.7 \times (0.02 – 4.4) \times (0.5 – 6.7) |
| Cond. turb. | 13     | 11510   | 1800    | 7.0 \times (0.03 – 4.4) \times (0.5 – 11.1) |
| Cond. turb. | 13     | 11510   | 1700, 5100 | 6.7 \times (0.02 – 4.4) \times (0.5 – 6.7) |
| Laminar    | 26     | 2910    | 1800, 7200 | 7.0 \times (0.03 – 4.4) \times (0.5 – 11.1) |

4 Laminar oscillating test case \((Wo = 26)\)

For a laminar, axially symmetric and fully-developed oscillatory pipe flow there exists a closed-form solution to eqs. (1) and (2) for the axial velocity component, hereinafter referred to as Sexl-Womersley (SW) solution [10, 14]. It reads

\[
\begin{align*}
   u_{c,SW}(r,t) &= \Re \left\{ -i \frac{Re_\varepsilon}{Wo^2} \left[ 1 - \frac{J_0 \left( \frac{\nu}{\nu_\tau} Wor \right)}{J_0 \left( \frac{\nu}{\nu_\tau} Wo \right)} \right] e^{i4Wo^2/Re_\varepsilon} \right\} \\
\end{align*}
\]

using our normalisation, where \(J_0\) is Bessel’s function of first kind and zeroth order and \(i\) is the imaginary unit. For \(Wo \to 0\) the SW solution takes the form of a quasi-steady version of the Hagen-Poiseuille solution for laminar pipe flow with a parabolic distribution. For increasing \(Wo\), the flow gradually becomes unsteady. The parabolic velocity profile disappears in advantage of a plateau like velocity distribution in the central region of the pipe, where inertial forces now become dominant.
The velocity maximum occurs in combination with inflection points in the vicinity of the pipe wall rather than at the pipe centre line. In this near-wall region, i.e. the Stokes layer (of thickness $\delta$), the viscous forces are still dominant. During phases of bulk flow reversal (RV) a phenomenon called coaxial counterflow takes place within the Stokes layer, when the viscous near-wall flow and the inviscid core flow are directed in opposite ways at the same time. All these features, which are typical for SW flow are exemplarily shown in fig. 3 for a moderately high Womersley number of $Wo = 26$.

We performed a DNS of oscillatory pipe flow at $Re_\tau = 1440$ and $Wo = 26$ using the fully-developed turbulent flow field described in sec. 3 as initial condition. At $t = 0$ the driving pressure gradient starts to vary between $-4$ and $4$ in a single harmonic manner with a period of $T = \pi Re_\tau/2Wo^2 = 3.3$. As a consequence the mean flow starts to decelerate; the turbulence intensity decreases likewise. We found that the flow field laminarises entirely within the first 20 oscillation cycles. However, to converge to a fully symmetric oscillatory pipe flow as described by eq. 3 it takes the flow at least another 50 periods due to inertia. Fig. 3 reflects an excellent overall agreement between radial profiles of $u_z$ extracted from our DNS results for $t > 70T$, predictions from laminar theory (SW), and experimental results obtained by Eckmann & Grotberg [3] using LDA in an oscillating laminar pipe flow at a comparable setup at $Wo = 26.5$ and $Re = 8430$.

Despite the relatively high $Re$, the well-correlated initial turbulence is not sustained in the resulting oscillatory flow and our numerical study thus supports the experimental results by Eckmann & Grotberg [3] who found oscillatory pipe flow to be laminar at this $Re_\delta = \sqrt{1/2Re/\delta} = 79$. High-frequency time series of $p$ and $u$ (not shown here) do not reflect any fluctuations over
the last 50 cycles and thus we are not able to confirm the small near-wall instabilities prognosticated by Trukenmüller [13] from quasi-steady linear stability analysis.

5 Conditionally turbulent case ($Wo = 13$)

For $Wo = 13$ the oscillation period is increased by a factor of four ($T = 13.4$) and so is the computational effort to simulate one cycle with this halved $Wo$ and the same shear Reynolds number. When compared to the case with $Wo = 26$ the flow converges within much fewer cycles to a pure oscillatory state, as reported earlier [4,5]. This faster relaxation from the non-oscillating initial condition can be mainly explained by two mechanisms. First, the smaller $Wo$ implies a longer phase of deceleration (DC); i.e. the driving force acts for a longer period of time against the initial bulk flow direction. Second, the flow does not become laminar. Turbulent mixing enhances the deceleration of the bulk flow by intensified wall-normal transport of kinetic energy from the high-momentum fluid in the core region to the near-wall Stokes layer, where viscous forces dominate. After the flow has reached an oscillatory state ($t > 27$), the final peak Reynolds number results in $Re = 11510 \pm 480$ and thus $Re_\delta = 626$. Despite the flow being intermittently turbulent, this value is surprisingly close to $Re = 11010$ predicted by the SW solution for a completely laminar flow, see fig. 1.

Fig. 4 presents time series of $u_z$ to illustrate the conditionally turbulent character of this oscillatory pipe flow. It can be seen that small velocity fluctuations are abruptly amplified near the wall during the early stage of DC, e.g. $205 < t < 207$. During the bulk flow reversal (RV), e.g. $t \approx 200$, these fluctuations are damped again and they further decay during the following acceleration phase (AC); at least close to the wall. This asymmetry between AC and DC is also reflected by the time signal of the wall-shear stress plotted in fig. 4. All these observations are in very good qualitative agreement with observations from experimental investigations by e.g. Eckmann & Grotberg [3], who report on turbulent bursts in the vicinity of the wall.
On phase asymmetries in oscillatory pipe flow during DC and a resulting asymmetry in each half-cycle for oscillatory pipe flows at $Re_\delta > 500$.

6 Asymmetric case ($Wo = 6.5$)

The temporal evolution of all three velocity components recorded at different radial locations is shown in fig. 5 for an oscillatory pipe flow at $Wo = 6.5$. There is a strong asymmetry between half-cycles of positive and negative bulk flow values, even though the prescribed driving pressure gradient is symmetric. The velocity field is uniform and of laminar nature during every other half-cycle with $u_b < 0$, e.g., $72.3 < t < 107.6$. At the end of those half-cycles there is always a sudden breakdown of the laminar state. Each subsequent half-cycle with $u_b > 0$, e.g., $107.6 < t < 125.9$, is then characterised by strong velocity fluctuations and a turbulent flow field.

The recurring of such events like the RV and the laminar breakdown is highly periodic, already after the first DC phase. For example, the length of the laminar
half-cycles is \( T_{\text{lam}} = 35.2 \pm 0.1 \) and the length of the turbulent ones is \( T_{\text{turb}} = 18.3 \pm 0.1 \). Thus, we exclude the possibility that the found asymmetric oscillation is only a transient state which will disappear in advantage of a symmetric one.

In the DC phase of the laminar half-cycle a phase lag develops in the velocity field between the near-wall and the central region due to the interplay of viscous and inertial forces. This can be seen in fig. 5b) at \( t \approx 100.4 \), where the fluid in the vicinity of the wall is moving in the positive z direction already, whereas the fluid further away from the wall is still moving in the negative direction. Such a phase lag is typical for SW flow at \( Wo > 1 \) and leads to maxima and inflection points in the velocity distribution, cf. fig. 5a) The latter also represent a source of instabilities, which are most likely triggered by small residual disturbances remaining in the secondary components \((u_\phi, u_z)\) of the velocity field since the last turbulent half-cycle. Fig. 5a) shows the structure of such disturbances in the azimuthal velocity component during the laminar AC phase. Fig. 5c) highlights the longsome decay of the kinetic energy content in \( u_\phi \) and \( u_z \) in the course of the laminar half-cycle. Nevertheless, right before the event of laminar breakdown the secondary velocity components are still \( O(10^{-2}) \). They rapidly grow by three orders of magnitude within two or three convective time units and lead to a highly turbulent flow field, e.g. \( 104.6 < t < 107.1 \).

The following positive half-cycle is characterised by chaotic velocity fluctuations during early AC and fully-developed turbulence during the rest of the half-cycle, as shown in fig. 6b). Due to the onset of turbulence the wall-shear stress increases, see fig. 5d), and thus the turbulent flow only accelerates up to \( u_b = 14.6 \pm 0.2 \). This corresponds to \( Re = 21000 \) which is of course much lower than the predicted value from laminar theory, see fig. 1. Fig. 5e) further reveals that the turbulence intensity is decreasing in accordance with the decelerating bulk flow. Nevertheless, the velocity fluctuations are still \( O(10^{-1}) \) at RV. The wall-normal momentum transport due to turbulent mixing prevents the formation of near-wall velocity maxima, inflection points

\[
\begin{align*}
\text{a) } t &= 83.4 \\
\text{b) } t &= 114.2
\end{align*}
\]

Fig. 6 Colour coded contour plots of the azimuthal velocity component \( u_\phi \), as predicted by DNS of oscillatory flow at \( Wo = 6.5 \) in a pipe domain of length \( L^+ = 7200 \).
On phase asymmetries in oscillatory pipe flow and co-axial counter flows. In fact, the velocity distribution during RV is rather uniform; i.e. $u_r \approx 0$ for all $r$. The initial condition for any laminar half-cycle is therefore very similar to the classic scenario of a start-up flow in a pipe. Missing inflection points and the stabilising effect of acceleration promote the ongoing laminarisation. The initially resting fluid and the much lower wall-shear stress allow the laminar flow to accelerate up to $u_b = 39.8 \pm 0.1$. This corresponds to $Re = 57,370$ which is much higher compared to SW theory, see fig. [1]. Therefore, the laminar as well as the turbulent half-cycle is characterised by a $Re_{\delta, crit}$ much higher than the critical value $Re_{\delta, crit} \approx 550$ for the onset of turbulence found experimentally by Hino et al. [7], Eckmann & Grotberg [3], and Zhao & Cheng [15].

7 Conclusion

We report on strong asymmetries in an oscillatory pipe flow at $Wo = 6.5$ and $Re_\tau = 1440$ investigated by means of DNS using a prescribed pressure force driving the flow. Despite the problem being symmetric in terms of its boundary conditions, the resulting flow is turbulent during half-cycles of positive bulk flow values and laminar during half-cycles of negative bulk flow values. The turbulent phase is shorter, the peak flow rate is smaller, and the following bulk flow reversal happens earlier, when compared to predictions from SW theory for laminar oscillating pipe flow. Contrarily, the laminar phase is longer, the reached peak flow rate is larger, and the following RV happens later as expected from SW theory. As a result, the oscillating pipe flow develops asymmetric transport and mixing properties.

The used numerical method and the spatio-temporal resolution was validated for the problem at hand by demonstrating excellent agreement of our DNS results with experimental data and analytical predictions from literature for the two limiting cases of a non-oscillating but turbulent pipe flow ($Wo = 0$) and an oscillating but laminar pipe flow ($Wo = 26$). For an oscillating and turbulent pipe flow ($Wo = 13$) we found very good agreement with qualitative descriptions of the flow field observed in experiments. For the non-oscillating case ($Wo = 0$) we have shown earlier [4], that $L^+ = 1800$ is a sufficiently long pipe domain and the coarser grid is sufficient to capture most of the turbulent scales. For $Wo = 13$, results from DNS employing three different pipe domains with $L^+ = 1700$ up to $L^+ = 5100$ qualitatively show the same behaviour of the oscillating flow field. For $Wo = 6.5$ we performed additional DNS in even longer domains to make sure that the asymmetry is not an artefact caused by the finite pipe length. Results from DNS using $L^+ = 1800$ and $L^+ = 7200$ show the same behaviour of the oscillating flow field. A more detailed study of the influence of the length of the computation pipe domain in oscillatory turbulent flow is the subject of our ongoing work [6].

To our knowledge, this asymmetric behaviour was never found in experiments on oscillatory pipe flow published in literature. The major difference between our DNS and all experimental set-ups is the forcing mechanism which drives the oscillatory flow through the pipe. In pipe flow experiments it is most convenient to drive
the flow by a reciprocating piston; i.e. the flow rate is prescribed and thus always follows a sine function. The pressure gradient across the pipe is a result and not necessarily symmetric. Contrarily, in our DNS the pressure gradient is prescribed to follow a sine function due to convenience. The temporal evolution of the bulk flow is therefore a result and not necessarily symmetric. The asymmetry is introduced by the reduced length of the turbulent half-cycle with $T_{\text{lam}} / T_{\text{turb}} = 1.9$. Thus the driving force is rather small at this preterm RV; but further increasing. On the other hand the length of the laminar half-cycle is enlarged and therefore the driving force has already reached its peak value at this delayed RV. For completely laminar and fully developed oscillatory pipe flows a difference in the driving mechanism only results in a phase shift. But for non-laminar oscillating pipe flows, this difference can lead to strong asymmetries either in the temporal evolution of the resulting flow or in the temporal evolution of the resulting pressure drop. Thus, the difference in boundary conditions is a crucial point and has to be considered for future numerical studies on oscillatory pipe flow.

References

1. Avila, K., Moxey, D., de Lozar, A., Avila, M., Barkley, D., Hof, B.: The onset of turbulence in pipe flow. Science 333(6039), 192–196 (2011)
2. Durst, F., Kikura, H., Lekakis, I., Jovanović, J., Ye, Q.: Wall shear stress determination from near-wall mean velocity data in turbulent pipe and channel flows. Experiments in Fluids 20, 417–428 (1996)
3. Eckmann, D.M., Grobert, J.B.: Experiments on transition to turbulence in oscillatory pipe flow. Journal of Fluid Mechanics 222, 329–350 (1991)
4. Feldmann, D., Wagner, C.: Direct numerical simulation of fully developed turbulent and oscillatory pipe flows at $Re_\tau = 1440$. Journal of Turbulence 13(32), 1–28 (2012)
5. Feldmann, D., Wagner, C.: Numerical study on the decay and amplification of turbulence in oscillatory pipe flow. In: Notes on Numerical Fluid Mechanics and Multidisciplinary Design, vol. 125, pp. 75–81 (2014)
6. Feldmann, D., Wagner, C.: On the influence of computational domain length on turbulence in oscillating pipe flow. In: Turbulence and Shear Flow Phenomena, vol. 9, pp. 1–6 (2015)
7. Hino, M., Sawamoto, M., Takasu, S.: Experiments on transition to turbulence in an oscillatory pipe flow. Journal of Fluid Mechanics 75(02), 193–207 (1976)
8. Schmitt, L., Richter, K., Friedrich, R.: Large-eddy simulation of turbulent boundary layer and channel flow at high Reynolds number. In: Direct and Large Eddy Simulation of Turbulence, pp. 161–176 (1986)
9. Schumann, U.: Ein Verfahren zur direkten numerischen Simulation turbulenter Strömungen in Platten- und Ringspaltkanälen und über seine Anwendung zur Untersuchung von Turbulenzmodellen. Dissertation, Universität Karlsruhe (TH), Karlsruhe (1973)
10. Sexl, T.: Über den von E. G. Richardson entdeckten Annulareffekt. Zeitschrift für Physik A 61, 349–362 (1930)
11. Shishkina, O., Wagner, C.: A fourth order finite volume scheme for turbulent flow simulations in cylindrical domains. Computers & Fluids 36(2), 484–497 (2007)
12. Thomas, C., Bassom, A.P., Blennerhassett, P.J.: The linear stability of oscillating pipe flow. Physics of Fluids 24, 1–11 (2012)
13. Trukenmüller, K.E.: Stabilitätstheorie für die oszillierende Rohrströmung. Dissertation, Helmut–Schmidt–Universität, Hamburg (2006)
14. Womersley, J.R.: Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known. Journal of Physiology 127, 553–563 (1955)
15. Zhao, T.S., Cheng, P.: Experimental studies on the onset of turbulence and frictional losses in an oscillatory turbulent pipe flow. International Journal of Heat and Fluid Flow 17(4), 356–362 (1996)