On the BKP hierarchy: additional symmetries, Fay identity and Adler-Shiota-van Moerbeke formula

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Abstract

We give an alternative proof of the Adler-Shiota-van Moerbeke formula for the BKP hierarchy. The proof is based on a simple expression for the generator of additional symmetries and the Fay identity of the BKP hierarchy.

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1 Introduction

There are two important symmetries associated with the Kadomtsev-Petviashvili (KP) hierarchy [4]. One is the Sato’s Bäcklund symmetry generated by vertex operator [4] and the other the additional symmetries by Orlov-Schulman operator [12, 5]. They have been realized from such several points of view as conformal algebras and string equation of matrix models in the 2-d quantum gravity etc. (see e.g. [20] and references therein) Both of them commute with the hierarchy flows but do not commute between themselves. These two symmetries looking quite different initially are in fact related to each other via the so-called Adler-Shiota-van Moerbeke (ASV) formula [1, 2] which states that additional symmetries acting on the wave function is equivalent to Sato’s Bäcklund symmetries acting on the corresponding tau function, and the associated algebra of additional symmetries is then lifted to its central extension, the algebra of Bäcklund symmetries. The proof was then simplified by Dickey [7] using the notion of resolvent operators and the Fay identity of the KP hierarchy.

In this work, we consider the KP hierarchy of B type [4] (or BKP hierarchy for short) which is a reduction of the ordinary KP hierarchy (or KP hierarchy of A type). Here B stands for the infinite dimensional Lie algebra \( \mathfrak{go}(\infty) \), in contrast to \( \mathfrak{gl}(\infty) \) for the KP. The BKP hierarchy possesses many integrable structures as the KP hierarchy, for example, Lax equations, linear system, \( \tau \)-function, Hirota bilinear form, free fermion representation, soliton and quasi-periodic solutions etc. (see [4] for a review) In [18] van de Leur obtained the corresponding ASV formula for the BKP hierarchy, which is similar to but different from that of the KP hierarchy. Motivated by the KP hierarchy, we attempt to give an alternative proof of the ASV formula for the BKP hierarchy based on Dickey’s approach [6, 7]. Our main results contain two parts. The first part is to give a simple expression of the generator of additional symmetries of the BKP hierarchy, which as a by-product provides an origin of the special form of the Lax operator for the constrained BKP (cBKP) hierarchy [3, 11]. The second part is to derive the Fay identity and its differential form from the bilinear identity which together with the first part gives a conceptually simple proof of the ASV formula for the BKP hierarchy. Let us recall some basic properties of the BKP hierarchy.
2 The BKP hierarchy

The BKP hierarchy \[4\] can be defined by the Lax equation

\[ \partial_{2n+1}L = [B_{2n+1}, L], \quad B_{2n+1} = (L^{2n+1})_+, \quad n = 0, 1, 2, \cdots \] (1)

where the Lax operator has the form

\[ L = \partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \cdots, \]

with coefficient functions \( u_i \) depending on the time variables \( t = (t_1 = x, t_3, t_5, \cdots) \) and satisfies the constraint

\[ L^* = -\partial L \partial^{-1}. \] (2)

Here we use the notations: \( \partial_{2n+1} f = \partial f / \partial t_{2n+1} \), \( (\sum_i a_i \partial^i)_+ = \sum_{i \geq 0} a_i \partial^i \), \( (\sum_i a_i \partial^i)_- = \sum_{i < 0} a_i \partial^i \), \( (\sum_i a_i \partial^i)^* = \sum_i (-\partial)^i a_i \). It can be shown \[4\] that the constraint (2) is equivalent to the condition \( (B_{2n+1})_{[0]} = 0 \).

The Lax equation (1) can be described by the compatibility condition of the linear system

\[ Lw(t, \lambda) = \lambda w(t, \lambda), \quad \partial_{2n+1} w(t, \lambda) = B_{2n+1} w(t, \lambda), \] (3)

where \( w(t, \lambda) \) is called wave function (or Baker function) of the system and \( \lambda \) is the spectral parameter. The whole hierarchy can be expressed in terms of a dressing operator, the so-called Sato operator \( W \)[4], as

\[ L = W \partial W^{-1}, \quad w(t, \lambda) = We^{\xi(t, \lambda)}, \] (4)

where \( \xi(t, \lambda) = \sum_{i=0} t_{2i+1} \lambda^{2i+1} \) and \( W = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i} \). Then the Lax equation (1) is equivalent to the Sato equation

\[ \partial_{2n+1} W = -(L^{2n+1})_- W \] (5)

where we refer \( (L^{2n+1})_- \) to the generator of inner symmetries of the hierarchy. Using the dressing form (4), the constraint (2) can be expressed as[14, 16]

\[ W^* \partial W = \partial. \] (6)

From (4) the solutions of the linear system (3) has the form

\[ w(t, \lambda) = \hat{w}(t, \lambda)e^{\xi(t, \lambda)}, \] (7)

where \( \hat{w}(t, \lambda) = 1 + w_1 / \lambda + w_2 / \lambda^2 + \cdots \).
Lemma 1. For pseudodifferential operators $P, Q$, the following formula holds.

$$\text{res}_\lambda[\lambda^{-1}(\partial^j P e^{\lambda x})(Q e^{-\lambda x})] = \text{res}_0(\partial^j P \partial^{-1} Q^*)$$

where we denote the symbols $\text{res}_z(\sum_i a_i z^i) = a_{-1}$ and $\text{res}_0(\sum_i b_i \partial^i) = b_{-1}$.

Using Lemma 1 it can be shown [4] that $w(t, \lambda)$ is a wave function of the BKP hierarchy if and only if it satisfies the bilinear identity

$$\text{res}_\lambda(\lambda^{-1} w(t, \lambda) w(t', -\lambda)) = 1.$$  \hfill (8)

In fact, from the bilinear identity (8), solutions of the BKP hierarchy can be characterized by a single function $\tau(t)$ called $\tau$-function such that [4]

$$\hat{w}(t, \lambda) = \frac{\tau(t_1 - \frac{2}{\lambda}, t_3 - \frac{2}{\lambda^3}, t_5 - \frac{2}{\lambda^5}, \cdots)}{\tau(t)} \equiv \frac{\tau(t - 2[\lambda^{-1}])}{\tau(t)}. \hfill (9)$$

3 Vertex operators and Sato’s Bäcklund transformations

From (7) and (9) the wave function $w(t, \lambda)$ can be written as

$$w(t, \lambda) = \frac{X_B(t, \lambda) \tau(t)}{\tau(t)},$$

where $X_B(t, \lambda)$ is the so-called vertex operator, defined by [4]

$$X_B(t, \lambda) = e^{\xi(t, \lambda)} e^{-2 \sum_{n=0}^{\infty} \frac{\lambda^{-2n-1}}{2n+1} \partial^{2n+1}} \equiv e^{\xi(t, \lambda)} G(\lambda).$$

Also another useful vertex operator $X_B(\lambda, \mu)$ can be defined as

$$X_B(\lambda, \mu) = e^{-\xi(t, \lambda)} e^{\xi(t, \mu)} G(-\lambda) G(\mu).$$

Proposition 2. (Sato’s Bäcklund transformations) The vertex operator $X_B(\lambda, \mu)$ provides infinitesimal transformations on the space of $\tau$-function, namely, if $\tau(t)$ is a solution then $\tau(t) + \epsilon X_B(\lambda, \mu) \tau(t)$ is a solution as well.

Introducing the symbol $\alpha(z) = \sum_{n \in \mathbb{Z}_{\text{odd}}} \alpha_n z^{-n}/n$ with

$$\alpha_n = \begin{cases} 2\partial/\partial t_n & n > 0 \\ |n|t_{|n|} & n < 0 \end{cases}$$
that satisfy the commutation relation

\[ [\alpha_n, \alpha_m] = 2n\delta_{n,-m}, \quad n \in \mathbb{Z}_{\text{odd}}. \]

Then the vertex operator \( X_B(\lambda, \mu) \) can be expressed as

\[ X_B(\lambda, \mu) = :e^{\alpha(\lambda) - \alpha(\mu)} : \]

where the normal ordering \( : \) demands that \( \alpha_n>0 \) must be placed to the right of \( \alpha_n<0 \).

One can Taylor expand \( X_B(\lambda, \mu) \) around \( \mu = \lambda \) for large \( \lambda \) as

\[ X_B(\lambda, \mu) = \sum_{m=0}^\infty \frac{(\mu - \lambda)^m}{m!} W^{(m)}(\lambda) = \sum_{m=0}^\infty \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^\infty \lambda^{-m-l} W_l^{(m)}, \]

with

\[ W^{(m)}(\lambda) = \partial_\mu^m X_B(\lambda, \mu)|_{\mu=\lambda} = :\partial_\lambda^m e^{-\alpha(\lambda)} \cdot e^{\alpha(\lambda)} : = P_m(-\partial_\lambda \alpha(\lambda)) : \]

where \( P_m(u(z)) \) is the so-called Faà di Bruno polynomials (see e.g. [8]) defined by the recurrence relations

\[ P_{m+1}(u) = (\partial_z + u)P_m(u). \]

For instance, \( P_0 = 1, P_1 = u, P_2 = u' + u^2, P_3 = u'' + 3uu' + u^3. \) Then differential operators \( W_l^{(m)} \) can be computed as

\[ W_n^{(0)} = \delta_{n,0}, \]

\[ W_n^{(1)} = \left\{ \begin{array}{ll} \alpha_n & n \in \mathbb{Z}_{\text{odd}} \\ 0 & n \in \mathbb{Z}_{\text{even}} \end{array} \right., \]

\[ W_n^{(2)} = \left\{ \begin{array}{ll} -(n+1)\alpha_n & n \in \mathbb{Z}_{\text{odd}} \\ \sum_{i+j=n} \alpha_i \alpha_j & n \in \mathbb{Z}_{\text{even}} \end{array} \right., \]

\[ W_n^{(3)} = \left\{ \begin{array}{ll} (n+1)(n+2)\alpha_n - \sum_{i+j+k=n} \alpha_i \alpha_j \alpha_k & n \in \mathbb{Z}_{\text{odd}} \\ -\frac{3}{2}(n+2) \sum_{i+j=n} \alpha_i \alpha_j & n \in \mathbb{Z}_{\text{even}} \end{array} \right., \]

etc. Moreover, through the fermion-boson correspondence, the representation of Lie algebra \( go(\infty) \) on \( \mathbb{C}[t_1, t_3, t_5, \cdots] \) is given by [11]

\[ Z_B(\lambda, \mu) = \frac{1}{2\mu - \lambda} (X_B(\lambda, \mu) - 1), \]

which after Taylor expanding around \( \mu = \lambda \) for large \( \lambda \) has the form

\[ Z_B(\lambda, \mu) = \sum_{m=0}^\infty \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^\infty \lambda^{-m-l} Z_l^{(m+1)}. \]

It is easy to show that differential operators \( Z_l^{(m)} \) are related to \( W_l^{(m)} \) as

\[ Z_l^{(1)} = W_l^{(1)}, \quad Z_l^{(m+1)} = W_l^{(m+1)} + \frac{1}{2} W_l^{(m)}, \quad m \geq 1, \]

and constitute an infinite-dimensional Lie algebra called \( W_{1+\infty}^B \)-algebra which is a subalgebra of \( W_{1+\infty} \) associated with the KP hierarchy.
4 Orlov-Schulman operator and additional symmetries

Due to the work of Orlov and Schulman [12], the Lax formulation can be extended by introducing the Orlov-Schulman operator $M$ defined by

$$M = WTW^{-1}, \quad \Gamma = \sum_{n=0}^{\infty} (2n+1)t_{2n+1} \partial^{2n},$$  \ \ \ \ (11)

which satisfies

$$\partial_{2n+1} M = [B_{2n+1}, M], \quad [L, M] = 1.$$  

Thus the linear system (3) should be extended to

$$Lw = zw, \quad Mw = \partial_z w, \quad \partial_{2n+1} w = B_{2n+1} w.$$  

Note that on the space of wave function $w(t, z)$, $(L, M)$ is anti-isomorphic to $(z, \partial_z)$ since $[z, \partial_z] = -1$. More general, one has $M^m L^l w = z^l \partial_z^m w$, $L^l M^m w = \partial_z^m z^l w$, and $\partial_{2n+1}(M^m L^l) = [B_{2n+1}, M^m L^l]$. On the other hand, one can introduce the adjoint wave function $w^*(t, z)$ and the adjoint Orlov-Schulman operator $M^*$ as

$$w^*(t, z) = (W^*)^{-1} e^{-\xi(t,z)} = -z^{-1} w_x(t, -z),$$  \ \ \ \ (12)

and

$$M^* = (WTW^{-1})^* = (L^*)^{-1} \partial M \partial^{-1} L^*,$$  \ \ \ \ (13)

where we have used (6) and $\Gamma^* = \Gamma$. Then $[L^*, M^*] = [M, L]^* = -1$, and

$$L^* w^* = zw^*, \quad M^* w^* = \partial_z w^*, \quad \partial_{2n+1} w^* = -B_{2n+1}^* w^*.$$  

Now let us introduce a new set of parameters $\hat{t}_{ml}$ to the system so that additional symmetries of the BKP hierarchy can be expressed as

$$\hat{\partial}_{ml} W = -(A_{ml}(L, M))_+ W, \quad \hat{\partial}_{ml} = \partial / \partial \hat{t}_{ml}$$  \ \ \ \ (14)

where $A_{ml}(L, M)$ are monomials in $L$ and $M$ and $A(L, M) = \sum_{ml} c_{ml} A_{ml}(L, M)$ is a generator of additional symmetries. Then through the dressing formulas (4) and (11) we have

$$\hat{\partial}_{ml} L = -[(A_{ml}(L, M))_+, L], \quad \hat{\partial}_{ml} M = -[(A_{ml}(L, M))_+, M],$$  

and

$$\hat{\partial}_{ml} A_{nk}(L, M) = -[(A_{ml}(L, M))_+, A_{nk}(L, M)].$$  \ \ \ \ (15)
Proposition 3. The additional flows commute with the hierarchy flows, i.e.

\[ [\hat{\partial}_{ml}, \partial_{2k+1}] = 0. \]

Thus they are symmetries of the BKP hierarchy.

Proof. From (5) and (14) we have

\[ [\hat{\partial}_{ml}, \partial_{2k+1}] W = -\hat{\partial}_{ml} L^{2k+1}_W + \partial_{2k+1} (A_{ml}(L, M))_W \]

Therefore one has \( \hat{\partial}_{ml} W^* = \partial \cdot \hat{\partial}_{ml} W^{-1} \cdot \partial^{-1} \) which together with (14) implies that \( \partial^{-1} (A_{ml}(L, M)^*) \cdot \partial = -(A_{ml}(L, M))_- \). However it is sufficient to consider the condition \( 16 \)

\[ \partial^{-1} A_{ml}(L, M)^* \partial = -A_{ml}(L, M), \] (16)

which is crucial to determine the algebra associated with additional symmetries. Substituting \( M^{m} L^l \) into (16) and using (13) we have \( \partial^{-1} (M^{m} L^l)^* \partial = (-1)^l l^{-1} M^{m} L \) and thus the additional flows of the BKP hierarchy are generated by

\[ A_{ml}(L, M) = M^{m} L^l - (-1)^l l^{-1} M^{m} L. \]

Proposition 4. The noncommutative vector fields \( \hat{\partial}_{ml} \) acting on Sato operator form a \( W_{B,\infty}^- \)-algebra (centerless \( W_{B,1+\infty,\infty}^- \)-algebra):

\[ [\hat{\partial}_{ml}, \hat{\partial}_{nk}] = \sum_{pq} C_{nk,ml}^{pq} \hat{\partial}_{pq}, \]

where \( C_{nk,ml}^{pq} \) are structure constants of the algebra.

Proof. Using (14) and (15) we have

\[ [\hat{\partial}_{ml}, \hat{\partial}_{nk}] W = [A_{ml}(L, M), A_{nk}(L, M)]_W W. \]
Note that \([A_{ml}(L, M), A_{nk}(L, M)]\) satisfies the constraint \([16]\), i.e.,

\[
\partial^{-1}[A_{ml}(L, M), A_{nk}(L, M)]^* \partial = -[A_{ml}(L, M), A_{nk}(L, M)],
\]

thus

\[
[A_{ml}(L, M), A_{nk}(L, M)] = \sum_{pq} C_{ml,nk}^{pq} A_{pq}(L, M).
\]

This implies that

\[
[\hat{\partial}_{ml}, \hat{\partial}_{nk}] W = -[A_{nk}(L, M), A_{ml}(L, M)]_+ W = \sum_{pq} C_{nk,m\ell}^{pq} \hat{\partial}_{pq} W. \]

Noticing that \(A_{0,2k}(L, M) = 0\) and \(A_{0,2k+1}(L, M) = 2L^{2k+1}\). Even though \(\hat{\partial}_{0,2k} L = \hat{\partial}_{0,2k} M = 0\), and \(\hat{\partial}_{0,2k+1} L = 2[\hat{L}^{2k+1}, L]\); however \(\hat{\partial}_{0,2k+1} M = 2[\hat{L}^{2k+1}, M] - 2(2k + 1)L^{2k}\). Hence \(\hat{\partial}_{0,2l+1}(l \geq 0)\) can not be identified with \(\hat{\partial}_{2l+1}\) due to the fact that these additional symmetries are explicitly time-dependent.

Let us define another generator \(Y_B(\lambda, \mu)\) of additional symmetries as \([18]\)

\[
Y_B(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}(A_{m,m+l}(L, M))_-, \quad (17)
\]

\[
= \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \lambda^{-l-m-1}(M^m L^{m+l} - (-1)^{m+l} L^{m+l-1} M^m L)_-.
\]

We mention that for \(\lambda = \mu\), the generator \(Y_B(\lambda, \lambda) = 2\sum_{l \in \mathbb{Z}_{odd}} \lambda^{-l-1} L_l^\mu\) can be viewed as the resolvent operator of the BKP hierarchy. The main result in this section is the following.

**Theorem 5.** The generator \(Y_B(\lambda, \mu)\) can be expressed as

\[
Y_B(\lambda, \mu) = \frac{1}{\lambda} [w(t, -\lambda) \partial^{-1} \cdot w_x(t, \mu) - w(t, \mu) \partial^{-1} \cdot w_x(t, -\lambda)].
\]

Proof. Using Lemma \([1]\) and the identity \((P)_- = \sum_{i=1}^{\infty} \partial^{-i} \text{res}_z(\partial^{i-1} P)\) for a pseudodifferential operator \(P = \sum_i a_i \partial^i\) (in particular, for \(P = f \partial^{-1}, f \partial^{-1} = \sum_{i=1}^{\infty} \partial^{-i} f(i-1)\)), we have

\[
(M^m L^{m+l})_- = \sum_{i=1}^{\infty} \partial^{-i} \text{res}_z[z^{-1}(\partial^{-i-1}(M^m W \partial^{m+l+1} e^z))(W^*)^{-1} e^{-z}]],
\]

\[
= \sum_{i=1}^{\infty} \text{res}_z[z^{m+l} \partial^{-i}(M^m w)^{(i-1)} \cdot w^*(t, z)],
\]

\[
= \text{res}_z[z^{m+l} \partial^{-i} w^*(t, z)].
\]
Similarly,

$$(L^{m+l-1} M^m L)_- = \text{res}_z [z(\partial_z^m z^{m+l-1} w(t, z)) \partial^{-1} \cdot w^*(t, z)].$$

Then

$$Y_B(\lambda, \mu) = \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} \sum_{l=-\infty}^{\infty} \{ \lambda^{-l-m-1} \text{res}_z [z^{m+l} (\partial_z^m w(t, z)) \partial^{-1} \cdot w^*(t, z)]

+ (-\lambda)^{-m-l-1} \text{res}_z [z (\partial_z^m z^{m+l-1} w(t, z)) \partial^{-1} \cdot w^*(t, z)] \},

= \text{res}_z \left[ \sum_{n=-\infty}^{\infty} \frac{z^n}{\lambda^{n+1}} \sum_{m=0}^{\infty} \frac{(\mu - \lambda)^m}{m!} (\partial_z^m w(t, z)) \partial^{-1} \cdot w^*(t, z) \right]

+ \frac{\mu}{\lambda} \text{res}_z \left[ \sum_{n=-\infty}^{\infty} \frac{(z + \mu - \lambda)^n}{(\lambda)^{n+1}} w(t, z + \mu - \lambda) \partial^{-1} \cdot w^*(t, z) \right],

= w(t, \mu) \partial^{-1} \cdot w^*(t, \lambda) + \frac{\mu}{\lambda} w(t, -\lambda) \partial^{-1} \cdot w^*(t, -\mu),

= \frac{1}{\lambda} w(t, -\lambda) \partial^{-1} \cdot w_z(t, \mu) - w(t, \mu) \partial^{-1} \cdot w_z(t, -\lambda),$$

where we have used the formula $\text{res}_z (\sum_{n=-\infty}^{\infty} (z^n/\lambda^{n+1}) f(z)) = f(\lambda)$ and \([12]\).  

Based on Dickey’s observation \([6, 7]\), one can define eigenfunctions $\phi_i(t)$ as

$$\phi_1(t) = \int d\lambda \rho_1(\lambda) w(t, -\lambda)/\lambda, \quad \phi_2(t) = \int d\mu \rho_2(\mu) w(t, \mu),$$

where $\rho_i(\lambda)$ are some weighting functions, so that

$$\partial_{2n+1} \phi_i(t) = B_{2n+1} \phi_i(t), \quad i = 1, 2.$$  

This enables one to introduce the so-called constrained BKP (cBKP) hierarchy defined by the Lax operator \([5, 11]\)

$$K = L^n = L^n_+ + \phi_1(t) \partial^{-1} \cdot \phi_2(t) \partial^{-1} \cdot \phi_1(t).$$  \(18\)

It is quite interesting to contrast two kind of reductions of the BKP hierarchy. For the $n$th-reduced BKP hierarchy ($n = 1, 3, 5, \ldots$) \([4, 19]\), the associated Lax operator $L^n = L^n_+$ (or $L^n_-$ = 0) which is related to inner symmetries of the hierarchy, while for the cBKP, the negative part $L^n_-$ is constructed from a particular linear combination of $Y_B(\lambda, \mu)$ and thus related to additional symmetries of the hierarchy. It is not hard to verify that the negative part of $K$ in \([18]\) is indeed compatible with the Lax equation

$$\partial_{2k+1} K = [K_{2k+1}^+, K].$$

Hence the above discussions provide an origin for the special form of the Lax operator \([18]\) from symmetry point of view.
5 Fay identity and Adler-Shiota-van Moerbeke formula

Having discussed the Bäcklund symmetry and additional symmetries of the BKP hierarchy, now we like to show that these two symmetries are essentially the same and are connected by a kind of ASV formula that was previously proved by van der Leur [18]. Our proof relies on the Fay identity and its differential form of \( \tau \)-functions.

**Proposition 6.** (Fay identity) The \( \tau \)-function of the BKP hierarchy satisfies the Fay quadrisecant identity:

\[
\sum_{(s_1, s_2, s_3)} \frac{(s_1 - s_0)(s_1 + s_2)(s_1 + s_3)}{(s_1 + s_0)(s_1 - s_2)(s_1 - s_3)} \tau(t + 2[s_0] + 2[s_1]) \tau(t + 2[s_2] + 2[s_3]) \\
+ \frac{(s_0 - s_1)(s_0 - s_2)(s_0 - s_3)}{(s_0 + s_1)(s_0 + s_2)(s_0 + s_3)} \tau(t + 2[s_0] + 2[s_1] + 2[s_2] + 2[s_3]) \tau(t) = 0 \tag{19}
\]

where \((s_1, s_2, s_3)\) stands for cyclic permutations of \(s_1, s_2, s_3\).

**Proof.** From the bilinear identity (8) we have

\[
\text{res}_z(z^{-1}G(z)\tau(t)G'(z)\tau(t')e^{\xi(t-t',z)}) = \tau(t)\tau(t').
\]

Setting \(t = x - y\) and \(t' = x + y\) to rewrite it as

\[
\text{res}_z(z^{-1}\tau(x - y - 2[z^{-1}])\tau(x + y + 2[z^{-1}])e^{-2\xi(y,z)}) = \tau(x - y)\tau(x + y).
\]

Now replace \(x\) as \(x + [s_0] + [s_1] + [s_2] + [s_3]\) and \(y\) as \([s_0] - [s_1] - [s_2] - [s_3]\) we have

\[
\text{res}_z\left(\frac{z^{-1}(1 - s_0z)(1 + s_1z)(1 + s_2z)(1 + s_3z)}{(1 + s_0z)(1 - s_1z)(1 - s_2z)(1 - s_3z)} \cdot \frac{\tau(x + 2[s_1] + 2[s_2] + 2[s_3] - 2[z^{-1}])}{\tau(x + 2[s_0] + 2[s_3])}\right) = 1.
\]

Computing the residue with poles at \(z = \infty, -s_0^{-1}, -s_1^{-1}, -s_2^{-1}, -s_3^{-1}\), we reach the Fay identity. ■

Just as the case of the KP hierarchy whose quasi-periodic solutions satisfy the Fay trisecant identity on Jacobian varieties [9, 13], the Fay quadrisecant identity for the BKP hierarchy can be viewed as an identity of theta functions on Prym varieties [14, 15, 10]. Next let us derive its differential form.
Proposition 7. (Differential Fay identity) The following equation holds.

\[
\left(\frac{1}{s_2^2} - \frac{1}{s_1^2}\right) \{\tau(t + 2[s_1])\tau(t + 2[s_2]) - \tau(t + 2[s_1] + 2[s_2])\tau(t)\} = \left(\frac{1}{s_2} + \frac{1}{s_1}\right) \{\partial\tau(t + 2[s_2])\tau(t + 2[s_1]) - \partial\tau(t + 2[s_1])\tau(t + 2[s_2])\} + \left(\frac{1}{s_2} - \frac{1}{s_1}\right) \{\tau(t + 2[s_1] + 2[s_2])\partial\tau(t) - \partial\tau(t + 2[s_1] + 2[s_2])\tau(t)\}. \tag{20}
\]

Proof. Setting \(s_0 = 0\) in the Fay identity (19) we have

\[
\tau(t + 2[s_1] + 2[s_2] + 2[s_3])\tau(t) = \sum_{(s_1, s_2, s_3)} \left(\frac{s_1 + s_2}{s_1 - s_2}\right) \left(\frac{s_1 + s_3}{s_1 - s_3}\right) \tau(t + 2[s_2] + 2[s_3])\tau(t + 2[s_1]) \tag{21}
\]

The differential Fay identity (20) can be proved by differentiating the above equation with respect to \(s_3\) and then setting \(s_3 = 0\). ■

We remark that the differential Fay identity (20) was also derived by Takasaki [17] by differentiating the bilinear identity over \(t'\) and setting \(t' = t + 2[\lambda^{-1}] + 2[\mu^{-1}]\).

Now we can give a simple proof of the ASV formula for the BKP hierarchy.

Theorem 8. (Adler-Shiota-van Moerbeke and van de Leur [18]) The following formula

\[
X_B(\lambda, \mu)w(t, z) = 2\lambda \left(\frac{\lambda - \mu}{\lambda + \mu}\right) Y_B(\lambda, \mu)w(t, z), \tag{22}
\]

holds, where it should be understood that the vertex operator \(X_B(\lambda, \mu)\) acting on \(w(t, z)\) is generated by its action on the \(\tau\) function.

Proof. The left hand side of (22) is given by

\[
X_B(\lambda, \mu)w(t, z) = e^{\xi(t, z)} \left[\frac{\tau(t)G(z)X_B(\lambda, \mu)\tau(t) - G(z)\tau(t) \cdot X_B(\lambda, \mu)\tau(t)}{\tau(t)^2}\right]
= e^{\xi(t, z) - \xi(t, \lambda) + \xi(t, \mu)} \left\{\left(\frac{z + \lambda}{z - \lambda}\right) \left(\frac{z - \mu}{z + \mu}\right) \tau(t)\tau(t + 2[\lambda^{-1}] - 2[\mu^{-1}])\right\}/\tau^2(t),
= e^{\xi(t, z) - \xi(t, \lambda) + \xi(t, \mu)} \left(\frac{\lambda - \mu}{\lambda + \mu}\right) \left\{\left(\frac{z + \lambda}{z - \lambda}\right) \tau(t - 2[\mu^{-1}])\tau(t + 2[\lambda^{-1}] - 2[z^{-1}])\right\}/\tau^2(t), \tag{23}
= e^{\xi(t, z) - \xi(t, \lambda) + \xi(t, \mu)} \left(\frac{z + \lambda}{z - \lambda}\right) \tau(t)\tau(t + 2[\lambda^{-1}] - 2[\mu^{-1}] - 2[z^{-1}]) - \left(\frac{z - \mu}{z + \mu}\right) \tau(t - 2[z^{-1}] - 2[\mu^{-1}])\tau(t + 2[\lambda^{-1}])/\tau^2(t),
\]

\[\]
where we have used (21) for \( s_1 = -z^{-1}, s_2 = \lambda^{-1}, s_3 = -\mu^{-1} \) to reach the last equality. On the other hand, the right hand side of (22) is given by

\[
2\lambda \left( \frac{\lambda - \mu}{\lambda + \mu} \right) Y_B(\lambda, \mu) w(t, z)
= 2 \left( \frac{\lambda - \mu}{\lambda + \mu} \right) \left[ w(t, -\lambda) \partial^{-1} w_x(t, \mu) w(t, z) - w(t, \mu) \partial^{-1} w_x(t, -\lambda) w(t, z) \right],
\]

\[
\left( \frac{\lambda - \mu}{\lambda + \mu} \right) \left[ w(t, -\lambda) \partial^{-1} (w_x(t, \mu) w(t, z) - w(t, \mu) w_x(t, z))
- w(t, \mu) \partial^{-1} (w_x(t, -\lambda) w(t, z) - w(t, -\lambda) w_x(t, z)) \right]
\]

(24)

where we have used integration by part to reach the second equality. Comparing (24) with (23), it is sufficient to prove the following

\[
\left( \frac{z + \lambda}{z - \lambda} \right) \left( e^{\xi(t, z) - \xi(t, \lambda)} \frac{\tau(t + 2[\lambda^{-1}] - 2[z^{-1}])}{\tau(t)} \right) x
= w(t, -\lambda) w_x(t, z) - w_x(t, -\lambda) w(t, z),
\]

(25)

and

\[
\left( \frac{z - \mu}{z + \mu} \right) \left( e^{\xi(t, z) + \xi(t, \mu)} \frac{\tau(t - 2[z^{-1}] - 2[\mu^{-1}])}{\tau(t)} \right) x
= w(t, \mu) w_x(t, z) - w_x(t, \mu) w(t, z).
\]

(26)

In terms of \( \tau \)-function, (25) can be expressed as

\[
(z^2 - \lambda^2) \{ \tau(t + 2[\lambda^{-1}] - 2[z^{-1}]) \tau(t) - \tau(t + 2[\lambda^{-1}] \tau(t - 2[z^{-1}])) \}
= (z + \lambda) \{ \tau(t + 2[\lambda^{-1}] - 2[z^{-1}]) \partial \tau(t) - \partial \tau(t + 2[\lambda^{-1}] - 2[z^{-1}]) \tau(t) \}
+ (z - \lambda) \{ \tau(t + 2[\lambda^{-1}]) \partial \tau(t - 2[z^{-1}]) - \partial \tau(t + 2[\lambda^{-1}]) \tau(t - 2[z^{-1}]) \},
\]

which, after changing of variable \( t = t' + 2[z^{-1}] \), is just the differential Fay identity (20) with \( s_1 = \lambda^{-1} \) and \( s_2 = z^{-1} \). Similarly, (26) can be verified in the same manner.

\[ \blacksquare \]

**Corollary 9.** The vector fields \( \hat{\theta}_{m,m+1} \) of additional symmetries acting on \( \tau \)-function can be expressed as

\[
\hat{\theta}_{m,m+1} \tau = Z_i^{(m+1)}(\tau).
\]

(27)

where \( Z_i^{(m)} \) are generators of the \( W_{1+\infty}^B \)-algebra defined by (10).

Proof. Observing the fact that

\[
X_B(\lambda, \mu) w(t, z) = w(t, z)(G(z) - 1) \left[ \frac{(X_B(\lambda, \mu) - 1) \tau(t)}{\tau(t)} \right]
\]
and

\[(A_{m,m+l})_+w(t, z) = -w(t, z)(G(z) - 1) \left[ \frac{\partial_{m,m+l} \tau(t)}{\tau(t)} \right].\]

Then (27) is an immediate consequence of (10), (17) and the ASV formula (22). ■

Therefore, the \(w_{\infty}^B\)-algebra of additional symmetries, acting on Sato operator (or wave function) is lifted to its central extension, the \(W_{1+\infty}^B\)-algebra of Bäcklund symmetries, acting on \(\tau\)-function.

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