Exact Sequence Analysis for Three-Dimensional HP Lattice Proteins

Reinhard Schiemann, Michael Bachmann and Wolfhard Janke
Institut für Theoretische Physik, Universität Leipzig,
Augustusplatz 10/11, D-04109 Leipzig, Germany

We have exactly enumerated all sequences and conformations of HP proteins with chains of up to 19 monomers on the simple cubic lattice. For two variants of the hydrophobic-polar (HP) model, where only two types of monomers are distinguished, we determined and statistically analyzed designing sequences, i.e., sequences that have a non-degenerate ground state. Furthermore we were interested in characteristic thermodynamic properties of HP proteins with designing sequences. In order to be able to perform these exact studies, we applied an efficient enumeration method based on contact sets.

PACS numbers: 05.10.-a, 87.15.Aa, 87.15.Cc

I. INTRODUCTION

Real proteins are build up of sequences of amino acids covalently linked by peptide bonds. Twenty different types of amino acids occurring in protein sequences are known. For a protein consisting of \(N\) amino acid residues there are thus in principle \(20^N\) possibilities to form sequences or primary protein structures. Single domain polypeptides usually possess \(N = 30 \ldots 400\) residues; proteins built up of several domains can consist of up to 4000 amino acids. Only a few of the \(20^N\) possible proteins, however, are actually realized in nature and are functional in a sense that they fulfill a specific task in a biological system. This requires the native structure of the protein to be unique and stable against moderate fluctuations of the environmental chemical and physical conditions. It is widely believed that the native state resides in a deep funnel-like minimum of the free energy landscape \(\mathcal{F}\). Since the energy of a protein depends on its sequence, it seems plausible that only such sequences of residues are favored whose associated energy landscape shows up a pronounced global minimum. On the other hand, from the conformational point of view, it can be estimated that the number of structures proteins typically fold into, is only of order 1000 — this is at least two orders of magnitude less than the number of known proteins.

Hence, exposing the nature of the relationship between sequences (primary structure) and conformations (secondary and higher structures) is one of the main aspects in protein research \(\S\). Attacking this general problem by means of computer simulations based on realistic interactions is currently impossible. There are two major reasons being responsible for this. Firstly, the precise form of the energy function in an all-atom approach containing all molecular and nuclear interactions within the polypeptide as well as the influence of the solvent is still under consideration. An important question is what “level of detail” is necessary to model proteins in general. Considering an exemplified sequence of amino acid residues, the use of different force fields usually leads to different predictions of the native state. Secondly, even if a reliable model would exist, the sequence space is too large to be completely scanned by enumeration in order to search for the small number of sequences with appropriate free energy landscape (the number of primary structures of very short peptides with, say only 10 residues, is \(20^{10} \approx 10^{13}\)).

In order to have a chance to perform such an analysis, the model must be drastically simplified. The simplest model to describe very qualitatively the folding behavior of proteins is the HP model \(\S\), where the continuous conformational space is reduced to discrete regular lattices and conformations of proteins are modeled as self-avoiding walks restricted to the lattice. In this model it is assumed that the hydrophobic interaction is the essential driving force towards a native fold. It is expected that the hydrophobic side chains are screened from the aqueous environment by hydrophilic residues. Therefore, the sequences of HP proteins consist of only two types of monomers (or classes of amino acids), amino acids with high hydrophobicity are treated as hydrophobic monomers (\(H\)), while the class of polar (or hydrophilic) residues is represented by polar monomers (\(P\)). In order to achieve the formation of a hydrophobic core surrounded by a shell of polar monomers, the interaction between hydrophobic monomers is attractive in the standard formulation of the model. All other interactions are neglected. Variants of the HP model also take into account (weaker) interactions between \(H\) and \(P\) monomers as well as between polar monomers \(\S\).

Although it is obvious that this model can describe the folding process very roughly only \(\S\), much work has been done to find lowest-energy states and their degeneracy for given sequences, or in the inverse problem, to identify all sequences of given length whose native conformation matches a given target structure. As simple as this model seems to be, it has been proven to be an NP-complete problem in two and three dimensions \(\S\). Therefore, sophisticated algorithms were

\(\S\)
applied to find lowest energy states for chains of up to 136 monomers. The methods applied are based on very different algorithms, ranging from exact enumeration in two dimensions \[ \mathbb{R}^2 \] and three dimensions on cubic (compact) lattices \[ \mathbb{Z}^2 \] and \[ \mathbb{Z}^3 \], and hydrophobic core construction methods \[ \mathbb{Z}_2 \], \[ \mathbb{Z}_3 \] over genetic algorithms \[ \mathbb{Z} \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \], Monte Carlo simulations with different types of move sets \[ \mathbb{Z} \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \], and generalized ensemble approaches \[ \mathbb{Z} \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \] \[ \mathbb{Z}_{12} \] \[ \mathbb{Z}_{13} \] over genetic algorithms \[ \mathbb{Z} \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \].

Monte Carlo simulations with different types of move sets \[ \mathbb{Z} \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \] , and generalized ensemble approaches \[ \mathbb{Z} \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \] \[ \mathbb{Z}_{12} \] \[ \mathbb{Z}_{13} \] over genetic algorithms \[ \mathbb{Z} \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \].

In this work, we apply an exact enumeration method to three-dimensional HP proteins being not necessarily compact on the simple cubic (s.c.) lattice. For efficiency, we enumerated contact sets for chains of given length instead of conformations. In order to study the interplay between sequences and conformations and to investigate peculiarities of designing sequences, we perform a statistical analysis of the complete spaces of conformations and sequences for chains of up to \( N = 19 \) monomers.

In Section \( \text{II} \) we give a review on the two variants of the HP model we use in our study, the original HP model and a variant taking into account an additional interaction between hydrophobic and polar residues. This is followed by Section \( \text{III} \) where we discuss self-avoiding conformations and contact sets. Then, in Section \( \text{IV} \) we perform an exact statistical analysis of properties of designing sequences and native conformations with lengths up to 19 monomers in comparison with the bulk of all possibilities to form sequences and to generate conformations, respectively. Since the exact data obtained with our algorithm can be rearranged in terms of the energy levels of the conformations, we are also able to determine the densities of states for all sequences. This allows for the study of the energetic thermodynamic properties of sequences whose associated ground state is unique or not, and their comparison from a thermodynamic point of view. We do just that in Section \( \text{V} \). The paper is then concluded by summarising our results in Section \( \text{VI} \).

II. HP MODELS

A monomer of a HP sequence \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N) \) is characterized by its residual type \( \sigma_i = P \) for polar and \( \sigma_i = H \) for hydrophobic residues, the place \( 1 \leq i \leq N \) within the chain of length \( N \), and the spatial position \( x \) to be measured in units of the lattice spacing. A conformation is then symbolized by the vector of the coordinates of successive monomers, \( X = (x_1, x_2, \ldots, x_N) \). We denote by \( d_{ij} = |x_i - x_j| \) the distance between the \( i \)th and the \( j \)th monomer. The bond length between adjacent monomers in the chain is identical with the spacing of the used regular lattice with coordination number \( k \). These covalent bonds are thus not stretchable. A monomer and its non-bonded nearest neighbors may form contacts. Therefore, the maximum number of contacts of a monomer within the chain is \((k-2)(k-1)\) for the monomers at the ends of the chain. To account for the excluded volume, lattice proteins are self-avoiding, i.e., two monomers cannot occupy the same lattice site. The general energy function of the non-covalent interactions reads in energy units \( \varepsilon \) (we set \( \varepsilon = 1 \) in the following)

\[
E = \varepsilon \sum_{\langle i,j \rangle \ni \sigma \neq \sigma'} C_{ij} U_{\sigma,\sigma'},
\]

where \( C_{ij} = (1 - \delta_{i+1,j}) \Delta(x_{ij} - 1) \) with

\[
\Delta(z) = \begin{cases} 
1, & z = 0, \\
0, & z \neq 0
\end{cases}
\]

is a symmetric \( N \times N \) matrix called contact map and

\[
U_{\sigma,\sigma'} = \begin{pmatrix} u_{HH} & u_{HP} \\
u_{PH} & u_{PP} \end{pmatrix}
\]

is the \( 2 \times 2 \) interaction matrix. Its elements \( u_{\sigma,\sigma'} \) correspond to the energy of \( HH \), \( HP \), and \( PP \) contacts. For labeling purposes we shall adopt the convention that \( \sigma_i = 0 \equiv P \) and \( \sigma_i = 1 \equiv H \).

In the simplest formulation \( \mathbb{R}^2 \), which we will refer to as HP model in the following, only the attractive hydrophobic interaction is nonzero, \( u_{HH}^{HP} = -1 \), while \( u_{HP}^{HP} = u_{PP}^{HP} = 0 \). Therefore, \( U_{\sigma,\sigma'}^{HP} = -\delta_{\sigma,H}\delta_{\sigma',H} \). This model has been extensively used to identify ground states of HP sequences, some of which are believed to show up qualitative properties comparable with realistic proteins whose 20-letter sequence was translated into the 2-letter code of the HP model \( \mathbb{R}^2 \) \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \]. As we will see later on, this simple form of the HP model suffers, however, from the fact that the lowest-energy states are usually highly degenerate and therefore the number of designing sequences (i.e., sequences with unique ground state – up to the usual translational, rotational, and reflection symmetries) is very small, at least on the simple cubic lattice.

For a more reliable statistical sequence analysis, we compare with another model of HP type, as proposed in Ref. \[ \mathbb{Z}_2 \]. This model was motivated by results from an analysis of inter-residue contact energies between real amino acids \[ \mathbb{Z}_3 \]. To this end, an attractive nonzero energy contribution for contacts between \( H \) and \( P \) monomers is assumed \[ \mathbb{Z}_2 \]. In what follows we call this the MHP (mixed HP) model. The elements of the interaction matrix \[ \mathbb{Z}_0 \] \[ \mathbb{Z}_1 \] \[ \mathbb{Z}_2 \] \[ \mathbb{Z}_3 \] \[ \mathbb{Z}_4 \] \[ \mathbb{Z}_5 \] \[ \mathbb{Z}_6 \] \[ \mathbb{Z}_7 \] \[ \mathbb{Z}_8 \] \[ \mathbb{Z}_9 \] \[ \mathbb{Z}_{10} \] \[ \mathbb{Z}_{12} \] \[ \mathbb{Z}_{13} \].

Additional \( H-P \) interaction breaks conformational symmetries yielding a much higher number of designing sequences on cubic lattices.

III. SELF-AVOIDING WALKS AND CONTACT MATRICES

Since lattice polymers are self-avoiding walks, the total number of conformations for a chain with \( N \) monomers
In our study are comparably demanding. Hence, the computational efforts are significant. Rather, we scan the combined space of HP sequences and conformations which contains possible combinations. Therefore, the computational efforts in our study are comparably demanding.

In Table I, the resulting numbers of contact sets \( M_n \) are summarized and, although also growing exponentially (see Figs. 1(a) and (b)), the gain of efficiency by enumerating contact sets, is documented by the ratio between \( C_n \) and \( M_n \). In models with the general form (1), where the calculation of the energy reduces to the summation over contacts (i.e., pairs of monomers being nearest neighbors on the lattice but nonadjacent along the chain) of a given conformation, the number of conformations that must necessarily be enumerated can drastically be decreased by considering only classes of conformations, so-called contact sets \[14, 44\]. A contact set is uniquely characterized by a corresponding contact map (or contact matrix), but a single conformation is not. Thus, for determining energetic quantities of different sequences, it is sufficient to carry out enumerations over contact sets. In a first step, however, the contact sets and their degeneracy, i.e., the number of conformations belonging to each set, must be determined and stored. Then, the loop over all non-redundant sequences is performed for all contact sets instead of conformations. The technical details of our implementation will be described elsewhere [45].

In Table I, the effective coordination number of the lattice, \( \gamma \), is a non-universal amplitude. In this expression, \( C_n \) is the effective coordination number of the lattice, \( \gamma \) is a universal exponent, and \( A \) is a non-universal amplitude. In Table I, we have listed the exactly enumerated number for self-avoiding conformations of chains with up to \( N = n + 1 = 19 \) monomers. Based on these data we estimated \( A \gamma_n \approx 4.68 \) and \( \gamma_n \approx 1.6 \) by extrapolating the results obtained with the ratio method \[36, 38\]. These results are in good agreement with previous enumeration results \[39, 41-41\], Monte Carlo methods \[42\] and field-theoretic estimates for \( \gamma \) \[43\]. We should note that it is not the aim of this work to extend the numbers of walks \( C_n \) in Table I which has already been enumerated up to \( n = 26 \) steps (and hence \( C_{27} \approx 5.49 \times 10^{17} \) self-avoiding conformations with \( N = 27 \) monomers) \[40\]. Rather, we scan the combined space of HP sequences and conformations which contains for chains of \( N = 19 \) monomers \( 2^{19}C_{19} \approx 1.17 \times 10^{18} \) possible combinations. Therefore, the computational efforts in our study are comparably demanding.

In models with the general form (1), where the calculation of the energy reduces to the summation over contacts (i.e., pairs of monomers being nearest neighbors on the lattice but nonadjacent along the chain) of a given conformation, the number of conformations that must necessarily be enumerated can drastically be decreased by considering only classes of conformations, so-called contact sets [14, 44]. A contact set is uniquely characterized by a corresponding contact map (or contact matrix), but a single conformation is not. Thus, for determining energetic quantities of different sequences, it is sufficient to carry out enumerations over contact sets. In a first step, however, the contact sets and their degeneracy, i.e., the number of conformations belonging to each set, must be determined and stored. Then, the loop over all non-redundant sequences is performed for all contact sets instead of conformations. The technical details of our implementation will be described elsewhere [45].

In Table I, the effective coordination number of the lattice, \( \gamma \), is a non-universal amplitude. In this expression, \( C_n \) is the effective coordination number of the lattice, \( \gamma \) is a universal exponent, and \( A \) is a non-universal amplitude. In Table I, we have listed the exactly enumerated number for self-avoiding conformations of chains with up to \( N = n + 1 = 19 \) monomers. Based on these data we estimated \( A \gamma_n \approx 4.68 \) and \( \gamma_n \approx 1.6 \) by extrapolating the results obtained with the ratio method [36, 38]. These results are in good agreement with previous enumeration results [39, 41-41], Monte Carlo methods [42] and field-theoretic estimates for \( \gamma \) [43]. We should note that it is not the aim of this work to extend the numbers of walks \( C_n \) in Table I which has already been enumerated up to \( n = 26 \) steps (and hence \( C_{27} \approx 5.49 \times 10^{17} \) self-avoiding conformations with \( N = 27 \) monomers) [40]. Rather, we scan the combined space of HP sequences and conformations which contains for chains of \( N = 19 \) monomers \( 2^{19}C_{19} \approx 1.17 \times 10^{18} \) possible combinations. Therefore, the computational efforts in our study are comparably demanding.

In models with the general form (1), where the calculation of the energy reduces to the summation over contacts (i.e., pairs of monomers being nearest neighbors on the lattice but nonadjacent along the chain) of a given conformation, the number of conformations that must necessarily be enumerated can drastically be decreased by considering only classes of conformations, so-called contact sets [14, 44]. A contact set is uniquely characterized by a corresponding contact map (or contact matrix), but a single conformation is not. Thus, for determining energetic quantities of different sequences, it is sufficient to carry out enumerations over contact sets. In a first step, however, the contact sets and their degeneracy, i.e., the number of conformations belonging to each set, must be determined and stored. Then, the loop over all non-redundant sequences is performed for all contact sets instead of conformations. The technical details of our implementation will be described elsewhere [45].

### Table I: Number of conformations \( C_n \) (with a global symmetry factor of 6 divided out) and contact matrices \( M_n \) for chains with \( N \) monomers (or, equivalently, self-avoiding walks with \( n = N - 1 \) steps).

| \( N \) | \( n \) | \( \frac{C_n}{M_n} \) | \( M_n \) | \( \frac{C_n}{M_n} \) |
|---|---|---|---|---|
| 4 | 3 | 25 | 2 | 13 |
| 5 | 4 | 121 | 3 | 40 |
| 6 | 5 | 589 | 9 | 65 |
| 7 | 6 | 2281 | 20 | 141 |
| 8 | 7 | 13565 | 66 | 206 |
| 9 | 8 | 64661 | 188 | 344 |
| 10 | 9 | 308981 | 699 | 442 |
| 11 | 10 | 1468313 | 2180 | 674 |
| 12 | 11 | 6989025 | 8738 | 800 |
| 13 | 12 | 33140457 | 29779 | 1113 |
| 14 | 13 | 157329085 | 121872 | 1290 |
| 15 | 14 | 744818613 | 434313 | 1715 |
| 16 | 15 | 3529191009 | 1806495 | 1954 |
| 17 | 16 | 16686979329 | 6601370 | 2528 |
| 18 | 17 | 78955042017 | 27510000 | 2869 |
| 19 | 18 | 372953947349 | 102111542 | 3652 |

FIG. 1: (a) Dependence of the numbers of self-avoiding walks \( C_n \) and contact matrices \( M_n \) on the number of steps \( n = N - 1 \). (b) Ratios of numbers of self-avoiding walks \( \frac{C_n}{M_n} = \frac{C_n}{C_n-1} \) and contact matrices \( \frac{M_n}{M_n-1} \). The dotted lines indicate the values the respective series converge to, \( \frac{C_n}{M_n} \approx 4.68 \) and \( \frac{M_n}{M_n} \approx 4.38 \), respectively.

\[
C_n = A \gamma_n \approx 4.68 \text{ for } n = N - 1.
\]
TABLE II: Number of designing sequences $S_N$ (only relevant sequences) in the HP and MHP model

| $N$ | $S^H_N$ | $S^{MHP}_N$ |
|-----|---------|-------------|
| 4   | 0 0 0   | 7 0 0 6     |
| 5   | 0 0 0 2 | 0 0 0 13    |
| 6   | 0 0 0 0 | 0 0 0 11    |
| 7   | 0 0 0 1 | 0 0 0 124   |
| 8   | 0 0 0 1 | 0 0 0 14 66 |
| 9   | 0 0 0 1 | 0 0 0 97 486|
| 10  | 0 0 0 1 | 0 0 0 2196 9491|
| 11  | 0 0 0 1 | 0 0 0 4885 |

IV. EXACT STATISTICAL ANALYSIS OF DESIGNING SEQUENCES

In this section, we analyze the complete sets $S_N$ of designing sequences for HP proteins of given numbers of residues $N \leq 19$. A sequence $\sigma$ is called designing, if there is only one conformation associated with the native ground state, not counting rotation, translation, and reflection symmetries that altogether contribute on a simple cubic lattice a symmetry factor 6 for linear, 24 for planar, and 48 for conformations spreading into all three spatial directions. In Table II we have listed the numbers of designing sequences $S_N$ we found for the two models. In contrast to previous investigations of HP proteins on the square lattice [10], the number of designing sequences obtained with the pure HP model is extremely small on the simple cubic lattice. This does not allow for a reasonable statistical study of general properties of designing sequences, at least for very short chains. The situation is much better using the more adequate MHP model. The first quantity under consideration is the hydrophobicity of a sequence $\sigma$, i.e., the number of hydrophobic monomers $N_H$, normalized with respect to the total number of residues:

$$m(\sigma) = \frac{N_H}{N} = \frac{1}{N} \sum_{i=1}^{N} \sigma_i.$$ (5)

The average hydrophobicity over a set of designing sequences of given length $N$ is then defined by

$$\langle m \rangle_N = \frac{1}{2^N} \sum_{\sigma \in S_N} m(\sigma).$$ (6)

In Fig. 2 we have plotted $\langle m \rangle_N$ as function of the sequence length $N$. The plots do not show up a clear tendency to what average hydrophobicity they converge for long chains. This to know would be, however, of some interest for the design of a biased algorithm of Monte Carlo type that searches the combined sequence and conformational space for candidates of designing sequences with lengths, where enumeration is no longer applicable. A distinct indication that designing sequences have in most cases hydrophobicities different from 0 could be used as a bias in order to reduce the section of the sequence space to be scanned, since the number of all possible sequences with given hydrophobicity has a peak at $m = 0.5$ (see Fig. 2(a)) which becomes the more pronounced the higher the number of residues is.

It should be noted that the hydrophobicity distribution for all these sequences is not binomial since in our analysis we have distinguished only sequences that we call relevant, i.e., two sequences that are symmetric under reversal of their residues are identified and enter only once into the statistics. Therefore we consider, for example, only 10 relevant sequences with length $N = 4$ instead of $2^4 = 16$. Taking into account all $2^N$ sequences would obviously lead to a binomial distribution for $N_H$, since there are then exactly

$$\binom{N}{N_H}$$ (7)

sequences with $N_H$ hydrophobic monomers.

In Fig. 2(a) we have plotted both, the distribution of hydrophobicity of the designing sequences with $N = 18$ monomers in the MHP model and, for comparison, the distribution of all sequences with $N = 18$. For this example, we see that the width of the hydrophobicity distribution for the designing sequences, which has its peak at $\langle m \rangle_{18}^{MHP} \approx 0.537 > 0.5$, is smaller than that of the distribution over all sequences. In order to gain more insight how the hydrophobicity distributions differ, we have compared the widths of both distributions in their dependence on the chain length $N \leq 19$. This is shown in Fig. 2(b). It seems that for $N \to \infty$ the widths of the hydrophobicity distributions for the designing sequences asymptotically approach the curve of the widths of the hydrophobicity distributions of all sequences.

The hydrophobicity profile

$$p_i = \frac{1}{2^N} \sum_{\sigma \in S_N} \left( \sigma_i + \sigma_{N-i+1} \right), \quad i = 1, 2, \ldots, N,$$ (8)

is a measure for the probability to find a hydrophobic monomer in a distance $i$ from the nearest end of a designing sequence. Thus, this quantity gives an impression of how the $H$ monomers are on average distributed along the chain. In Figs. 2(a) and (b) the profiles for...
designing sequences in the MHP model are plotted for respective chains with even \( N = 14, 16, 18 \) and odd numbers \( N = 15, 17, 19 \) of residues. As a first remarkable result, we see that for odd numbers of monomers the profile shows up periodic oscillations, i.e., if the \( n \)th monomer is preferably hydrophobic the \((n + 1)\)th residue is with lower probability. As this effect is stronger for \( N = 17 \) than for \( N = 19 \), we expect that the amplitude of these oscillations decreases with increasing number of monomers. The behavior of the chains with even number of monomers (Fig. 4(a)) is less spectacular. Here, for increasing number of monomers, the probability seems to become more and more equally distributed. Therefore, it is more interesting to study how each monomer of the designing sequences is involved in the formation of \( HH \) contacts (as well as \( HP \) contacts in the MHP model) being favored in low-energy conformations. We define the hydrophobic contact density profile by

\[
q_i = \frac{1}{2SN} \sum_{\sigma \in S_N} \sum_{j=1}^N \left[ C_{ij} \sigma_i \sigma_j + C_{N-i+1,j} \sigma_{N-i+1} \sigma_j \right],
\]

where \( C_{ij} \) is the contact map defined after Eq. 1. The higher the affinity of the \( i \)th monomer to form contacts (preferably if it is hydrophobic), the bigger the value of \( q_i \). This profile is shown in Fig. 5 for both models, where we have again separated even and odd numbers of residues. From the two profiles for the HP model \( (N = 18, 19) \), we observe that there is a strong tendency of the monomers at the ends of the chain \((i = 1, N)\) to form hydrophobic \((HH)\) contacts. The reason is that these two monomers can have 5 nearest neighbors on the s.c. lattice, i.e., there is one more possibility for them to form a favorable energetic contact than for monomers residing within the chain. In the MHP model, the behavior is less unique, since also \( HP \) contacts are attractive and the tendency that the ends are preferably hydrophobic is much weaker.

After having discussed sequential properties of designing sequences, we now analyze the properties of their unique ground-state structures, the native conformations. From Table III we read off that the number of different native conformations \( D_N \) is usually much smaller

| \( N \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( D_N^{HP} \) | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 8 | 28 | 708 | 1447 |
| \( D_N^{MHP} \) | 1 | 0 | 0 | 2 | 2 | 0 | 5 | 6 | 30 | 8 | 31 | 58 | 258 | 708 | 1447 | 1623 |
than the number of designing sequences, i.e., several designing sequences share the same ground-state conformation. The number of designing sequences that fold into a certain given target conformation $X^{(0)}$ (or conformations being trivially symmetric to this by translations, rotations, and reflections) is called designability [46]:

$$F_N(X^{(0)}) = \sum_{\sigma \in S_N} \Delta \left( X_{gs}(\sigma) - X^{(0)} \right),$$

where $X_{gs}(\sigma)$ is the native (ground-state) conformation of the designing sequence $\sigma$. The function $\Delta(Z)$ is the generalization of Eq. (2) to $3N$-dimensional vectors. It is unity for $Z = 0$ and zero otherwise.

The designability is plotted in Fig. 5 for all native conformations that HP proteins with $N = 17$, $18$, and $19$ monomers can form in the MHP model. In this figure, the abscissa is the rank of the conformations, ordered according to their designability. The conformation with the lowest rank is therefore the most designable structure and we see that a majority of the designing sequences folds into a few number of highly designable conformations, while only a small number of designing sequences possesses a native conformation with low designability (note that the plot is logarithmic). Similar results were found, for example in Ref. [47], where the designability of compact conformations on cuboid lattices was investigated in detail. The left picture in Fig. 5 shows the conformation with the lowest rank (or highest designability) with $N = 18$ monomers.

From our analysis we see that this characteristic distribution of the designing sequences is not restricted to cuboid lattices only. This result is less trivial than one may think at first sight. As we will show later on in this paper in the discussion of the radius of gyration, native conformations are very compact, but only very few conformations are maximally compact (at least for $N \leq 19$). For longer sequences similar results were found in Ref. [30]. Highly designable conformations are of great interest, since it is expected that they form a frame making them stable against mutations and thermodynamic fluctuations. Such fundamental structures are also relevant in nature, where in particular secondary structures (helices, sheets, hairpins) supply proteins with a stable backbone [47].

Conformational properties of polymers are usually studied in terms of the squared end-to-end distance

$$R_e^2 = (x_N - x_1)^2$$

FIG. 7: Structure ($N = 18$) with the highest designability of all native conformations (left) and with minimal radius of gyration (right).
The averages were obtained by calculating all possible self-avoiding walks. The same quantities for the native conformations found in the MHP model and to as measures for the compactness of a conformation. In where \( \langle R^2_g \rangle \) on a s.c. lattice. Figure 8(a) shows that, compared to widths of all self-avoiding walks (SA W). We have also inserted the native conformations in the MHP model compared with those of the distributions of the squared end-to-end distances are also very small. Even for heteropolymers with \( N = 19 \) monomers in total, there are virtually no native conformations, where the distance between the ends is larger than 3 lattice sites. We have checked this for the HP model, too, and found the same effect. Since the number of native conformations is very small in this model, we have not included these results in the figure. Depicting the average squared radius of gyration \( \langle R^2_g \rangle \) and the widths of the corresponding distribution of the radius of gyration in Fig. 8(b) for all self-avoiding conformations as well as for the native ones, we see that these results confirm the above remarks. As the average end-to-end distances of native conformations are much smaller than those for the bulk of all conformations, we observe the same trend for the mean squared radii of gyration \( \langle R^2_g \rangle_{\text{MHP}} \) and \( \langle R^2_g \rangle_{\text{SA W}} \) and the widths \( b^2_{R^2_g} \) and \( b^2_{R^2_g}^{\text{MHP}} \) as well. In particular, the width \( b^2_{R^2_g}^{\text{MHP}} \) is so small, that virtually all native conformations possess the same radius of gyration. For this reason, we have also searched for the conformations having the smallest radius of gyration \( R^2_{g_{\text{min}}} \) (these conformations are not necessarily native as we will see!) and inserted these values into this figure, too. We observe that these values differ only slightly from \( \langle R^2_g \rangle_{\text{MHP}} \). Thus we conclude that native conformations are very compact, but not necessarily maximally compact. This property has already been utilized in enumerations being performed \textit{a priori} on compact lattices \( 2 \times 11 \times 17 \), where, however, the proteins are confined by hand to live in small cuboids (e.g. of size \( 3 \times 3 \times 3 \) or \( 4 \times 3 \times 3 \)). Our results on the general s.c. lattice confirm that this assumption is justified to a great extent. Nevertheless, the slight deviation from the minimal radius of gyration native conformations exhibit is a remarkable result as it concerns about 90% of the whole set of native conformations! This can be seen in Fig. 8 where we have plotted the distribution of the squared radii of gyration for all self-avoiding conformations with \( N = 18 \) and the native states in the MHP model. All native conformations have a very small radius of gyration but only a few of them share the smallest possible value. A structure with the smallest radius of gyration is shown on the right-hand side of Fig. 8. It obviously differs from the most-designable conformation drawn on the left of the same figure.

\[
\langle R^2_{g_{\text{SA W}}} \rangle \sim n^{2\nu} \text{ with } \nu \approx 0.59 \text{ (see Ref. 48 for a recent summary of estimates for } \nu \text{), the average end-to-end distance } \langle R^2_{g_{\text{MHP}}} \rangle \text{ for the native conformations only is much smaller. For even number of monomers, the ends of a HP protein can form contacts with each other on the s.c. lattice. Accordingly, the values of } \langle R^2_{g_{\text{MHP}}} \rangle \text{ are smaller for } N \text{ being even and the even-odd oscillations are very pronounced. The widths (or standard deviations) } b^2_{R^2_g} \text{ of the distributions of the squared end-to-end distances are also very small.}
\]

\[
\langle \rangle \text{ and the squared radius of gyration }
\]

\[
R_g^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2,
\]

where \( \bar{x} = \sum_i x_i / N \) is the center of mass of the polymer.

In polymer physics both quantities are usually referred to as measures for the compactness of a conformation. In Fig. 8(a) we compare the \( N \)-dependence of the averages of the native conformations found in the MHP model and all possible self-avoiding walks. The same quantities for the squared radius of gyration are shown in Fig. 8(b). The averages were obtained by calculating

\[
\langle R^2_{g_{\text{SA W}}} \rangle = \frac{1}{C_N} \sum_{X \in C_N} R^2_{g_{\text{SA W}}}(X),
\]

\[
\langle R^2_{g_{\text{MHP}}} \rangle = \frac{1}{S_N} \sum_{\sigma \in S_N} R^2_{g_{\text{MHP}}}(X_{gs}(\sigma)),
\]

where \( C_N \) is the set of all self-avoiding conformations on a s.c. lattice. Figure 8(a) shows that, compared to...
FIG. 9: Distribution $h_{R_g^2}$ (normalized to $\sum h_{R_g^2} = 1$) of squared radii of gyration (normalized with respect to the maximal radius of gyration $R_{g,\text{max}} = (N^2 - 1)/12$ of a completely stretched conformation) of native conformations with $N = 18$ in the MHP model, compared with the histogram for all self-avoiding conformations. The vertical line refers to the minimal radius of gyration $(R_{g,\text{min}}^2/R_{g,\text{max}}^2 = 0.0579$ for $N = 18$) and an associated structure is shown on the right-hand side of Fig. 4. The inset shows the distribution up to $R_{g,\text{max}}^2/R_{g,\text{max}}^2 = 0.5$.

V. DENSITY OF STATES AND THERMODYNAMICS

In this section we systematically compare energetic thermodynamic quantities of designing and non-designing sequences. In Ref. [49] it was conjectured for exemplified sequences of comparable 14mers, one of them being designing, that designing sequences in the HP model seem to show up a much more pronounced low-temperature peak in the specific heat than the non-designing examples. This peak may be interpreted as kind of a conformational transition between structures with compact hydrophobic cores (ground states) and states where the whole conformation is highly compact (globules) [29, 30]. Another peak in the specific heat at higher temperatures, which is exhibited by all lattice proteins, is an indication for the usual globule – coil transition between compact and untangled conformations.

In order to study energetic thermodynamic quantities such as mean energy and specific heat we determined from our enumerated conformations for a given sequence the density of states $g(E)$ that conveniently allows the calculation of the partition sum $Z(T) = \sum_E g(E) \exp(-E/k_BT)$ and the moments $(E^k)_T = \sum_E E^k g(E) \exp(-E/k_BT)/Z$, where the subscript $T$ indicates the difference of calculating thermal mean values based on the Boltzmann probability from averages previously introduced in this paper. Then the specific heat is given by the fluctuation formula $C_V = ((E^2)_T - (E)_T^2)/k_BT^2$.

A. Sequences in the HP Model

In the HP model with pure hydrophobic interaction the density of states shows up a monotonic growth with increasing energy, at least for the short chains in our study (for longer chains, e.g. the 42mer investigated in Ref. [29, 30], the number of states in the high-energy region decreases, i.e., the density of states possesses a global maximum at an energy $E$ residing within the interval $E_{\text{min}} < E < E_{\text{max}} = 0$). For a reasonable comparison of the behavior of designing and non-designing sequences, we have focused on 18mers having the same hydrophobicity ($m_H = 8$) and ground-state energy $E_{\text{min}} = -9$. There are in total 527 sequences with these properties, two of which are designing. The densities of states for the two designing sequences and an example of a non-designing sequence are plotted in Fig. 10. We have already divided out a global symmetry factor 6 (number of possible directions for the link connecting the first two monomers) that all conformations on a s.c. lattice have in common. Since the ground-state conformations of the designing sequences spread into all three dimensions, an additional symmetry factor $4 \times 2 = 8$ (4 for rotations around the first bond, 2 for a remaining independent reflection) makes a number of conformations obsolete and the ground-state degeneracy of the designing sequences is indeed unity. Obviously this is not the case for the sequences we identified as non-designing. In fact, the uniqueness of the ground states of designing sequences is a remarkable property as there are not less than $\sim 10^{10}$ possible conformations of HP lattice proteins with 18 monomers. As we also see in Fig. 10, the
FIG. 11: (a) Mean energy $\langle E \rangle_T$ and (b) specific heat $C_V(T)$ for the two designing sequences with $N = 18$, $m_H = 8$, and $E_{\text{min}} = -9$ (solid lines) in the HP model, whose densities of states were plotted in Fig. 10. The curves of the same quantities for the 525 non-designing sequences are completely included within the respective areas between the dashed lines. The low-temperature peak of the specific heat (near $T = 0.14$) is most pronounced for the two designing sequences which behave similarly for low temperatures.

FIG. 12: Minimum and maximum boundaries for the densities of states of the 13 designing (filled boxes, connected by solid lines) and the 40 non-designing sequences (open circles, connected by dashed lines) with 18 monomers, hydrophobicity $m_H = 3$, and ground-state energy $E_{\text{min}} \approx -5.478$ in the MHP model. Once more, a global symmetry factor 6 has already been divided out.

The upper bound of the specific heats for non-designing sequences in Fig. 11(b) exposes two peaks. By analyzing our data for all 525 non-designing sequences we found that there are two groups: some of them experience two conformational transitions, while others do not show a characteristic low-temperature behavior. Thus, the only appearance of these two peaks is not a unique, characteristic property of designing sequences. In order to quantify this observation, we have studied all relevant 32,896 sequences with 16 monomers. Only one of these sequences is designing ($H P_2 H P_2 H P H P_2 P H P H$, with minimum energy $E_{\text{min}} = -9$), but in total there are 593 sequences, i.e., 1.8% of the relevant sequences, corresponding to curves of specific heats with two local maxima. It should be noted that the degeneracies of the ground states associated with these sequences are comparably small.

B. Properties of the MHP Model

In the MHP model, the energy levels are no longer equally spaced due to the additional non-integer $H P$ interaction. Moreover, the absolute value of the energy of the lattice heteropolymer is not necessarily identical with the number of hydrophobic contacts. The formation of a highly compact core is still desirable, but also the attractive contacts between $H$ and $P$ monomers reduce the energy of the heteropolymer. For this reason, the relatively manifest distinction between “phases” with compact $H$-core states and entirely compact conformations is expected to be much less pronounced, even for the designing sequences.

Once more, we have first enumerated the densities of states for a set of sequences that have similar proper-
ties but differ only in the ordering of the sequence. For this study we chose all 18mers sharing the same hydrophobicity $m_H = 3$ and identical ground-state energy $E_{\text{min}} = 2u_{\text{HH}}^{\text{MHP}} + 8u_{\text{HP}}^{\text{MHP}} \approx -5.478$, since there are 2 $HH$ and 8 $HP$ contacts in each of the ground-state conformations. We found 13 designing and 40 non-designing sequences that satisfy these specifications. In order not to be confused by too many curves, in Fig. 12 again only the minimum and maximum boundaries of the designing sequences (solid lines) are shown as well as the corresponding bounds for the non-designing sequences (dashed lines). We observe that the regions enclosed by the minimum and maximum boundaries of the designing sequences hardly differ, it is difficult to identify a particular thermodynamic behavior being characteristic for designing sequences only. This is indeed true as can be seen from Figs. 13(a) and (b), where we have plotted the lower and upper boundaries for the respective mean energies and specific heats of these 18mers. In contrast to the results for the HP model (cf. Figs. 11(a) and (b)), where, within a certain low-temperature interval, the regions enclosing the curves for the designing and non-designing sequences do not overlap, a separation of this kind is not apparent in the MHP model. Nevertheless, these figures also show that for very low temperatures ($0 < T < 0.1$), the general behavior is very similar for all designing sequences, but it is not for the non-designing sequences, where the temperature dependence of energy and thus specific heat can be significantly different.

VI. SUMMARY

We have exactly analyzed the combined space of sequences and conformations for proteins on the simple cubic lattice for two HP-type models that differ in the contact energy between hydrophobic and polar monomers. In the original HP model this interaction is zero, while in the more realistic MHP model there is a nonzero contribution as suggested by the Miyazawa-Jernigan matrix of contact energies between amino acids in proteins. Since there were only a few known exact results for heteropolymers in 3D, in particular on compact cuboid lattices, we generated by exact enumeration the sets of designing sequences and native conformations on non-compact simple cubic lattices. We studied, how their properties, measured, e.g., in terms of quantities like end-to-end distance, radius of gyration, designability, etc., differ from the bulk of all possible sequences and all self-avoiding conformations, respectively. We found that ground states of designing sequences, i.e. native conformations, have a much smaller mean end-to-end distance than the set of all conformations with the same length. Moreover, we confirmed that these conformations are very compact, i.e., they have a smaller mean radius of gyration than the whole set. This is valid for both models under consideration.

We have also studied energetic thermodynamic properties, in order to investigate how characteristic the low-temperature behavior of designing compared to non-designing sequences is. We determined the densities of states for respective sets of selected 18mers with comparable properties. In the HP model, where the number of designing sequences is rather small compared with the MHP model, we could observe that energetic fluctuations are different for designing and non-designing sequences within a certain low-temperature region. Designing sequences show up a more pronounced low-temperature peak in the specific heat being related to a conformational transition between low-energy states with hydrophobic core and highly compact globules. For the MHP model the situation is more diffuse, and a clear distinction between designing and non-designing sequences based on
characteristic thermodynamic properties is not uniquely possible. Nevertheless, we have also seen in this model that designing sequences behave similarly for very low temperature while non-designing sequences react quite differently on changes of the temperature, over the entire range of temperatures.

VII. ACKNOWLEDGEMENTS

This work is partially supported by the German-Israel-Foundation (GIF) under contract No. I-653-181.14/1999. One of us (R.S.) acknowledges support by the Studienstiftung des deutschen Volkes.

[1] K. A. Dill, Prot. Sci. 8, 1166 (1999).
[2] C. Tang, Physica A 288, 31 (2000).
[3] K. A. Dill, Biochemistry 24, 1501 (1985); K. F. Lau and K. A. Dill, Macromolecules 22, 3986 (1989).
[4] J. E. Shakhnovich, Phys. Rev. Lett. 72, 3907 (1994).
[5] K. Yue, K. M. Fiebig, P. D. Thomas, H. S. Chan, E. I. Shakhnovich, and K. A. Dill, Proc. Natl. Acad. Sci. U.S.A. 92, 325 (1995).
[6] M. Vendruscolo and E. Domany, J. Chem. Phys. 109, 11101 (1998).
[7] H. S. Chan and E. Bornberg-Bauer, Appl. Bioinf. 1, 121 (2002).
[8] D. Crenceni, D. Goldman, C. Papadimitriou, A. Piccolboni, and M. Yannakakis, ibid., 423 (1998).
[9] A. Irbäck and E. Sandelin, J. Phys. Chem. 108, 2245 (1998).
[10] A. Irbäck and C. Troein, J. Biol. Phys. 28, 1 (2002).
[11] H. Cejtin, J. Edler, A. Gottlieb, R. Helling, H. Li, J. Philbin, C. Tang, and N. Wingreen, J. Chem. Phys. 116, 352 (2002).
[12] K. Yue and K. A. Dill, Phys. Rev. E 48, 2267 (1993); Proc. Natl. Acad. Sci. U.S.A. 92, 146 (1995).
[13] M. C. Beutler and K. A. Dill, Prot. Sci. 5, 2037 (1996).
[14] R. Unger and J. Moult, J. Mol. Biol. 231, 75 (1993).
[15] N. Kranzloger, W. E. Hart, J. Smith, and D. A. Pelta, Proc. Genetic and Evolutionary Computation Conf. (GECCO’99), Orlando, 1999, p. 1596.
[16] Y. Cui, H. W. Wong, E. Bornberg-Bauer, and H. S. Chan, Proc. Natl. Acad. Sci. U.S.A. 99, 809 (2002).
[17] N. Lesh, M. Mitzenmacher, and S. Whitesides, Int. Conf. Research in Computational Molecular Biology (RECOMB’03), Berlin, 2003, p. 188.
[18] T. Jiang, Q. Cui, G. Shi, and S. Ma, J. Chem. Phys. 119, 4592 (2003).
[19] F. Seno, M. Vendruscolo, A. Maritan, and J. R. Banavar, Phys. Rev. Lett. 77, 1901 (1996).
[20] R. Ramakrishnan, B. Ramachandran, and J. F. Pekny, J. Chem. Phys. 106, 2418 (1997).
[21] A. Irbäck, C. Peterson, F. Potthast, and E. Sandelin, Phys. Rev. E 58, R5249 (1998).
[22] L. W. Lee and J.-S. Wang, Phys. Rev. E 64, 056112 (2001).
[23] G. Shikenji, M. Kikuchi, and Y. Iba, Phys. Rev. Lett. 83, 1886 (1999); and references therein.
[24] M. N. Rosenbluth and A. W. Rosenbluth, J. Chem. Phys. 23, 356 (1955).
[25] D. Aldous and U. Vazirani, ‘Go with the Winners’ Algorithms, 35th Annual Symposium on Foundations of Computer Science, Santa Fe, 1994, p. 492.
[26] P. Grassberger and W. Nadler, ‘Go with the Winners’ Simulations, in: Computational Statistical Physics – From Billiards to Monte Carlo, edited by K. H. Hoffmann and M. Schreiber (Springer, Berlin, 2002), p. 169, and references therein.
[27] P. Grassberger, Phys. Rev. E 56, 3682 (1997); H. Frauenkron, U. Bastolla, E. Gerstner, P. Grassberger, and W. Nadler, Phys. Rev. Lett. 80, 3149 (1998); U. Bastolla, H. Frauenkron, E. Gerstner, P. Grassberger, and W. Nadler, Proteins 32, 52 (1998).
[28] H.-P. Hsu, V. Mehta, W. Nadler, and P. Grassberger, J. Chem. Phys. 118, 444 (2003); Phys. Rev. E 68, 21113 (2003).
[29] M. Bachmann and W. Janke, Phys. Rev. Lett. 91, 208105 (2003).
[30] M. Bachmann and W. Janke, J. Chem. Phys. 120, 6779 (2004).
[31] R. J. Najmanovich, J. L. deLyra, and V. B. Henriques, Physica A 249, 374 (1998).
[32] J. M. Lattman, J. M. Fiebig, and K. A. Dill, Biochemistry 38, 6158 (1994).
[33] L. Toma and S. Toma, Prot. Sci. 5, 147 (1996).
[34] S. Miyazawa and R. L. Jermigan, J. Mol. Biol. 256, 623 (1996).
[35] In contrast to Ref. 2, we have kept $w_{uHH}^{MHP} = w_{uHH}^H = -1$ and rescaled the value of $w_{uHH}^M$ by the factor of 2.3. This number was chosen in Ref. 2 as a result of an analysis for the inter-residue energies of contacts between hydrophobic amino acids and contacts between hydrophobic and polar residues which motivated the relation $2u_{uHH}^M > u_{uPP} + u_{uHH}^P$.
[36] D. S. Gaunt and A. J. Guttmann, Asymptotic Analysis of Coefficients, in Phase Transitions and Critical Phenomena, edited by C. Domb and M. S. Green (Academic Press, London, 1974), p. 181.
[37] J. L. Guttmann and A. J. Guttmann, J. Phys. A: Math. Gen. 26, 2485 (1993).
[38] see, e.g. H. E. Stanley, Introduction to Phase Transitions and Critical Phenomena (Oxford University Press, New York, 1987); A. J. Guttmann, Asymptotic Analysis of Power-Series Expansions, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. L. Lebowitz.
[39] D. MacDonald, D. L. Hunter, K. Kelly, and N. Jan, J. Phys. A: Math. Gen. 25, 1429 (1992).
[40] D. MacDonald, S. Joseph, D. L. Hunter, L. L. Moseley, N. Jan, and A. J. Guttmann, J. Phys. A: Math. Gen. 33, 5973 (2000).
[41] M. Chen and K. Y. Lin, J. Phys. A: Math. Gen. 35, 1501 (2002).
[42] S. Caracciolo, A. S. Causo, and A. Pelissetto, Phys. Rev. E 57, R1215 (1998).
[43] R. Guida and J. Zinn-Justin, J. Phys. A: Math. Gen. 31, 8103 (1998).
[44] M. Vendruscolo and E. Domany, Folding & Design 2, 295 (1997); ibid. 3, 329 (1998).
[45] R. Schiemann, M. Bachmann, and W. Janke, to be published.
[46] E. G. Emberly, J. Miller, C. Zeng, N. S. Wingreen, and C. Tang, Proteins 47, 295 (2002).
[47] H. Li, R. Helling, C. Tang, and N. Wingreen, Science 273, 666 (1996).
[48] L. H. Wong and A. L. Owczarek, J. Phys. A: Math. Gen. 36, 9635 (2003).
[49] M. Bachmann and W. Janke, Acta Physica Polonica 34, 4689 (2003).