T-folds, doubled geometry, and the $SU(2)$ WZW model

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Abstract

The $SU(2)$ WZW model at large level $N$ can be interpreted semiclassically as string theory on $S^3$ with $N$ units of Neveu-Schwarz $H$-flux. While globally geometric, the model nevertheless exhibits an interesting doubled geometry possessing features in common with nongeometric string theory compactifications, for example, nonzero $Q$-flux. Therefore, it can serve as a fertile testing ground through which to improve our understanding of more exotic compactifications, in a context in which we have a firm understanding of the background from standard techniques. Three frameworks have been used to systematize the study of nongeometric backgrounds: the T-fold construction, Hitchin’s generalized geometry, and fully doubled geometry. All of these double the standard description in some way, in order to geometrize the combined metric and Neveu Schwarz B-field data. We present the T-fold and fully doubled descriptions of WZW models, first for $SU(2)$ and then for general group. Applying the formalism of Hull and Reid-Edwards, we indeed recover the physical metric and $H$-flux of the WZW model from the doubled description. As additional checks, we reproduce the abelian T-duality group and known semiclassical spectrum of D-branes.

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1 Introduction

Over the last two decades, a beautiful picture has emerged for the geometric encoding of low energy quantum field theories in string theory. Results from the early 1990s for Calabi-Yau compactifications related field content to Betti numbers \[7\], and interactions to intersection numbers in classical and quantum cohomology rings \[69\], with mirror symmetry relating the two \[18, 55, 8, 29, 30, 2\]. From physical reasoning, we gained early hints at the rich interrelation between complex and symplectic geometry, which despite two decades continues to bear fruit as a fertile area of mathematical investigation integrating many subdisciplines. (For an overview, see Refs. \[41, 6\].) The duality revolution of the mid 1990s brought us D-branes as the microscopic carriers of Ramond-Ramond (RR) charge \[60\]. With it, came AdS/CFT correspondence \[56\] and an understanding of the role in compactifications of microscopically consistent objects that appear to violate classical energy conditions.\[4, 66, 14, 72, 10\]. These objects evaded classical no-go theorems forbidding internal Neveu-Schwarz and Ramond-Ramond fluxes \[57\]. The fluxes generate a superpotential \[32\] which was studied systematically beginning in the early 2000s \[14, 24, 49\], and which, together with instanton effects, generically lifts all moduli in type IIB string theory \[51\].\[3\] Virtually all model building in type II string theory today takes place in orientifold models with internal flux. (See, e.g., Ref. \[15\].)

Yet, this story is not complete. The combination of flux and mirror symmetry highlighted the insufficiency of standard geometric compactifications to describe the full set of topological data describing a string theory compactification. In the context of effective field theory, the problem is easy to understand. For example, consider KKLT type compactifications of type IIB string theory \[51\]. Ignoring the subsector of the theory from localized objects (D-branes and singularities), the matter comes from \(H^2\), and the gauge fields from \(H^3\). On the other hand, the fluxes are cohomology classes in \(H^3\). We cannot achieve the most general gauge couplings in the low energy effective field theory, since the counting is only sufficient to couple the gauge fields to a single matter multiplet (from the universal hypermultiplet). Moreover, the gauge group is abelian. The question naturally arises, how to realize general gaugings (gauge group plus couplings) of the compactified supergravity theory. While well understood in the context of localized objects, this is still poorly understood in the bulk.

Duality arguments \[50, 67\] suggest that the basic problem is not a fundamental constraint of the microscopic theory, but an insufficiency of the standard hierarchical choice of topological data to describe the most general compactifications. One cannot first choose the topology of a compact manifold, and then subsequently choose the topology of various other bundle-like objects on that manifold. For the Riemannian structure, one cannot first choose the

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\[1\] For example, orientifold planes and D-branes wrapped on cycles of \(\int R \wedge R < 0\) display this property.

\[2\] The presence of moduli immediately renders a compactification unrealistic. The moduli fields generate long range interactions not observed in nature, and lead to overclosure of the universe in cosmology.
The curvature of spacetime, and then the curvature of various fields living in that spacetime. The two are intrinsically interrelated, and must be described as a whole. From the point of view of string theory, this is not so surprising. After all, everything is built from one object, strings. However, this also implies that everything is probed by strings, and the corresponding notion of spacetime geometry need not agree with our intuition from a spacetime that can be probed by point particles.

In Refs. [42, 43, 44, 13, 45] it was argued that the appropriate mathematical setting for realizing the integrated approach to string compactifications motivated in the previous paragraph is a doubled spacetime roughly thought of as that seen simultaneously by left and right moving degrees of freedom. For example, a string on a circle carries momentum and a winding number, or equivalently, independent left and right momenta. The space of states of a single such string is spanned by $|p_L, p_R, N_{osc}⟩$, where $N_{osc}$ describes the modes of oscillation. In a Fourier transformed basis, this becomes $|x_L, x_R, N_{osc}⟩$, with $x_L$ and $x_R$ independent parameters, which we would like to interpret as coordinates on a space of twice the usual dimension. While this argument invoked a circle, it is clear that the same momentum and position parameters would persist as independent integration constants in locally solving the string wave equation in an open set. In general, one expects to obtain a global description by sewing together these open sets in a suitable way.

Even in the context of the standard worldsheet sigma model, a similar doubling is apparent. For example, in conformal gauge ($\gamma^{ab} \propto \delta^{ab}$), the worldsheet Polyakov action

$$S = \int d^2 \sigma L = \frac{1}{4\pi\alpha'} \int d^2 \sigma \left( \sqrt{\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu + i \epsilon^{ab} B_{\mu\nu} \partial_a x^m \partial_b X^n \right)$$

(1.1)

gives Hamiltonian density

$$\mathcal{H} = \frac{1}{4\pi\alpha'} \left( G_{mn} X^m X^n + G^{mn} (\alpha' p_m + B_{mp} X^p) (\alpha' p_q + B_{nq} X^q) \right).$$

(1.2)

Here, $p_m = (2\pi/\sqrt{\gamma}) \partial L / \partial \dot{X}^m$ is the canonical momentum, $\dot{X}^m(\sigma) = dX^m(\sigma)/d\sigma^2$, and $X^m(\sigma) = dX^m(\sigma)/d\sigma^1$. If there is even a local description of a compactification in terms of a standard geometric one, so that we can locally apply this sigma model, with $X^m(\sigma)$ embedding the string worldsheet into an open manifold $M$, then $(X^m, p_m)$ can be thought of as valued in the tangent plus cotangent bundle $(T \oplus T^*)M$. The Hamiltonian density gives a Riemannian structure on $(T \oplus T^*)M$. There is a canonical $O(d,d)$ invariant structure as well, from the quantity $p_m X^m / (4\pi)$. Here $d = \text{dim}(M)$. The integrated sum and difference of these two quantities give the Virasoro generators $L_0$ and $\bar{L}_0$ in the left moving and right moving worldsheet sectors. Thus, the local worldsheet geometry of the compactification furnishes a Riemannian structure and a (constant) $O(d,d)$ structure on $(T \oplus T^*)M$.

The geometry of $(T + T^*)M$ was explored by Hitchin and Gualtieri in Refs. [36, 31], pioneering a branch of mathematics called generalized complex geometry. A foundational
observation is that a complex structure $\mathcal{J}^M_N$ on $(T + T^*)M$ can be constructed from either a complex structure $J^m_n$ on $M$ (from $\mathcal{J} = \text{diag}(J, -J^T)$), or a symplectic structure $\omega_{mn}$ on $M$ (putting $\omega$ and $\omega^{-1}$ on the off-diagonal). This allows one to unify the rich complex and symplectic geometric structures of mirror symmetry as well as interpolate between the two. We will refer to the geometry of $(T + T^*)M$ as generalized geometry, whether or not complex structures are introduced.

Generalized complex geometry partially rises to the task of supplying the missing supergravity gaugings of type II compactifications [26, 27]. Pure spinors on the doubled tangent bundle correspond to maximal isotropic subbundles, i.e., $d$-dimensional subbundles that are null with respect to the $O(d, d)$ metric. Thus a local pure spinor determines a local projection to a $d$ dimensional subbundle, which generalizes $TM$. Pure spinors also define almost complex structures of the generalized geometry reducing the $O(d, d)$ structure to $U(d/2, d/2)$. Via a map to differential forms, the pure spinors corresponding to the symplectic and complex structures of a Calabi-Yau manifold become $e^{-\omega}$ and the holomorphic $(d, 0)$-form $\Omega$, and the structure group is reduced to $U(d/2, d/2) \cap U(d/2, d/2) = SU(d/2) \times SU(d/2)$ [31]. Calabi-Yau 3-folds give a generalized geometry with $SU(3) \times SU(3)$ holonomy. When the spinors are instead covariantly conserved by a connection with torsion, there is $SU(3) \times SU(3)$ structure. In the latter case, the map of pure spinors to differential form need not give $\omega$ and $\Omega$ of definite degree, and the torsion need not map $p$ forms to $p + 1$ forms. Instead we obtain torsion data furnishing maps of the form

$$K_{mnp}: \Omega^0 \rightarrow \Omega^3, \quad f_{mn}^p: \Omega^2 \rightarrow \Omega^3, \quad Q_{mnp}^q: \Omega^3 \rightarrow \Omega^2, \quad R_{mnp}^q: \Omega^3 \rightarrow \Omega^6.$$  (1.3)

The same data can be used to define a Roytenberg bracket giving the generalized geometry the structure of an $\alpha$-Lie $\beta$-algebroid, where the appropriate prefixes $\alpha$ and $\beta$ depend on which of $K$, $f$, $Q$, $R$ are nonzero. Note that Eq. (1.3) is exactly the data required to furnish the full set of $\mathcal{N} = 2$ gauged supergravity couplings [26, 27]. The main deficiency of this approach is that it generically describes a string theory compactification only locally when $Q$ is nonzero, and does not appear to apply even locally when $R$ is nonzero. As discussed in connection with Eq. (1.2), generalized geometry applies locally, when compactification locally has a description in terms of a standard sigma model.

What, then, is the global structure, and what do we mean by a nongeometric compactification? In this context, there are two basic constructions, monodrofolds [34, 11, 35], and doubled geometry [44, 13, 45]. A monodrofold can be thought of a two step compactification. In the first step, we are given a compactification having a discrete gauge symmetry $\Gamma_{\text{modular}}$. For example, geometric moduli spaces take the form of a noncompact Teichmüller space quotiented by a modular group. For the common NSNS sector of a toroidal string space
compactification on $T^d$, the metric and $B$-field moduli parametrize a coset space of the form $O(d, d)/(O(d) \times O(d))$ and are identified under the action of the T-duality group $O(d, d; \mathbb{Z})$. Then, in further compactifying on a base manifold, in the second step, it is possible to specify nontrivial monodromies in the duality group, when traversing noncontractible loops in the base. In response to the question, “is the result a CFT?” we take the point of view that the monodromies are part the basic topological data in defining the worldsheet sigma model. Conformality is a dynamical question for the effective field theory, whose equations of motion are the vanishing beta function conditions of the worldsheet theory.

When the T-duality group is used in this construction, the result is called a T-fold \cite{IIB-T-folds, T-folds}. In this case, the monodromies generically mix the metric and $B$-field of the fiber theory, so that neither is globally well defined. For example, consider a $T^2$. In this case, the T-duality group is $O(2, 2; \mathbb{Z})$, which is the same as $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ up to $\mathbb{Z}_2$ identifications. The latter acts by fractional linear transformation on the complex structure modulus $\tau$ and complexified Kähler modulus $\rho = b + iv$, where $b = \int B$ on the $T^2$, and $v$ is the $T^2$ volume. Now further compactify on a circle. Monodromies $\rho \mapsto \rho + N$ or $\tau \mapsto \tau + N$ correspond to $H$-flux, and a nontrivial $T^2$-bundle, respectively. However, $-1/\rho \mapsto -1/\rho + N$ is also a perfectly good monodromy and mixes the metric and $B$-field. In this case, one finds

$$ds^2 = dz^2 + \frac{1}{1 + (Nz)^2}(dx^2 + dy^2), \quad B_{xy} = -\frac{Nz}{1 + (Nz)^2},$$

(1.4)

where $x, y$ are coordinates on the $T^2$ fiber, and $z$ is the coordinate on the $S^1$ base. Locally, this gives a perfectly well defined metric and $B$-field. Globally, neither is single valued on $S^1$. The next simplest T-fold base after $S^1$ is $\mathbb{P}^1$. Via F-theory/heterotic duality, it is also relatively straightforward to describe T-folds over $\mathbb{P}^1$ base, which generalize heterotic compactifications on K3 \cite{F-theory/K3}. While T-folds do not have a global geometry or topology in the conventional sense, they do in a doubled sense \cite{F-theory/K3}. If the fiber is doubled to include both the physical torus $T^d$

\footnote{See Ref. \cite{orbifold} for a generalization of the orbifold construction, which gives the modular invariant partition functions of flat monodrofolds over $S^1$. Here, flat means that the monodromies have fixed points so that the moduli do not have to vary over the base $S^1$.}

\footnote{There may or may not be static solutions. The data may give time dependent “runaway” solutions or domain wall solutions where the volume modulus or dilaton runs away pathologically, as is true for generic supergravity gaugings. Only certain choices of topological data will lead to physically interesting results.}

\footnote{When the base manifold is large, the monodromies require small spatial gradients of fields, and correspondingly small masses for the lifted moduli, so that the relevant effective field theory truncations are justified. However, we will not worry about this a priori. The supergravity regime is always in some sense nongeneric in a theory in which the basic unit is string scale. In any realistic compactification, one requires a small parameter so that corrections are controlled.}

\footnote{This particular example can be converted to a geometric one by a global T-duality transformation $(\rho', \tau') = (-1/r, \tau)$. However, we choose monodromy $(\rho, \tau) \mapsto (-1/\rho, -1/\tau)$ rather than $(\rho, \tau) \mapsto (-1/(-1/\rho + N), \tau)$, then the result is nongeometric in all T-duality frames.}
and T-dual torus $\tilde{T}^d$, then the $O(d, d)$ monodromies act linearly as transition functions on the fiber. The full specification of topological data consists of the topology of the doubled fibration, together with a choice of $B$-field on the base. The metric and $B$-field on the fiber are neatly packaged into a single Riemannian metric of the same form as Eq. (1.2), which is well defined on the doubled fibration. Thus, the doubled fibers carry a very similar structure to that described in the context of generalized geometry, except that the space itself is doubled, not just the tangent bundle. To locally recover the standard nondoubled description requires a choice of polarization,\(^8\) that is, a choice of dual $\tilde{T}^d$ in the $T^{2d}$ fiber, which must be null with respect to the $O(d, d)$ metric. A global polarization defines a global projection from the total space of the doubled fibration to the total space of the physical $T^2$ fibration. When a global polarization exists, we recover the standard sigma model description globally. Polarizations are thus much like the pure spinors of generalized geometry in that they define physical subbundles of doubled bundles.

Doubled geometry takes the T-fold construction one step further, by doubling all $d$ dimensions [44, 13, 28, 45]. It has been applied to generalizations of toroidal compactifications, and furnishes the gaugings of the common NSNS sector of their supergravity theories. Just as gauged analogs of a Calabi-Yau compactification have been realized via generalized geometry with $SU(3) \times SU(3)$ structure, so too in the trivial holonomy case, in the doubled geometry context, we seek a geometry with identity structure, i.e., a parallelizable manifold. For technical reasons we further restrict to “consistent absolute parallelism” [28] where the metric is constant in the frame basis. The possibilities are $S^7$ or a Lie group, and we discard the former since it is not $2d$ dimensional. Compactifications based on a fully doubled group manifold, as opposed to the torus-fiber doubling of Ref. [42] or circle-base doubling of Ref. [12], were first considered by Dall’Agata and Prezas in Ref. [13]. They were subsequently studied by Graña et al. in Sec. 5.3 of Ref. [28], and in depth by Hull and Reid-Edwards in Refs. [44, 45].

Starting from the doubled space

$$\mathcal{X}_{2d} = \Gamma \backslash G_{2d},$$

where $\Gamma$ is a discrete subgroup, Hull and Reid-Edwards build a formalism for describing compactifications that yield gauged supergravities with gauge group $G_{2d}$ [44, 15]. Given a Lie algebra frame $T_M$, the left-invariant forms $P^M$ furnish a coframe. The additional required data beyond $G_{2d}$ and $\Gamma$ is a constant, signature $(d, d)$ symmetric metric $L_{MN}$ which must be invariant under the action of $G_{2d}$.

The formulation of a worldsheet theory [15] on this doubled space is a true generalization of the Polyakov action [14]. To locally recover the standard action, when this is possible,
requires a choice of polarization on the tangent bundle of $\mathcal{X}_{2d}$. Given a choice of polarization, it is convenient to choose a frame $T_M = (Z_m, X^m)$ putting the $O(d, d)$ metric $L_{MN}$ into canonical form, with identity matrices off diagonal. One then defines a Riemannian metric, of the form (1.2). The metrics in other polarizations are locally related by $O(d, d)$ transformation. $G_{2d}$-invariance of $L_{MN}$ implies that the Lie algebra takes the form

$$[Z_m, Z_n] = K_{mnp}X^p + f_{mn}^p Z_p,$$

$$[Z_m, X^n] = f_{pm}^n X^p + Q_{mnp} Z_p,$$

$$[X^m, X^n] = Q_{mn}^p X^p + R_{mnp} Z_p.$$ 

In the case that $R_{mnp} = 0$, the $X^m$ close to form a subgroup $\tilde{G}_d \subset G_{2d}$, which can be quotiented out, to leave the physical $d$-dimensional geometry $X_d$. This is a local statement, and whether or not $R_{mnp} = 0$ depends upon the local choice of polarization. In this case, the worldsheet model of Hull and Reid-Edwards indeed agrees with the standard Polyakov action. For such polarizations, it is natural to seek a relation between the doubled geometry of Hull and Reid-Edwards, and the corresponding generalized geometry. This has been done in detail for the special cases with $f, K$ or $f, Q$ nonzero in Ref. [64]. In these cases, the Lie algebra of the doubled gauge group $G_{2d}$ indeed agrees with the Roytenberg bracket on $(T + T^*)X_d$.

In this paper we present the T-fold and doubled geometry descriptions of Wess-Zumino-Witten (WZW) models at large level $n$, emphasizing the special case of $SU(2)$. The $SU(2)$ case describes strings propagating on a 3-sphere of radius $\sqrt{n\alpha'}$ with $n$ units of Neveu-Schwarz $H$-flux. Our motivations for studying WZW models in this context are as follows. (For earlier work on WZW models in doubled and generalized geometry, see Refs. [12, 13, 33].)

1. We have outlined three approaches to nongeometric string theory compactifications above: T-folds, generalized geometry, and doubled geometry. It would be interesting to further clarify the relation between them. Additional examples are needed to elucidate the generalized geometry and doubled geometry approaches, particularly compact examples, and ideally one with a clear CFT description. A noncontroversial example yielding interesting generalized and doubled geometries is well suited to this goal, even if geometric. And in this case, the generalized geometry is applicable globally.

2. The $SU(2)$ WZW model is somewhat counterintuitive. The naive expectation is that it would have $f \neq 0$, since $f$ is conventionally associated with a space whose 1-forms close with torsion. It would also have $K \neq 0$, conventionally associated with $H$-flux. The remaining $Q$ and $R$ would naively vanish, since these are conventionally thought of as obstructions to global and local geometry, respectively. However, these structure constants do not describe an $SU(2) \times SU(2)$ gauge algebra. Instead, setting the $SU(2)$
generators equal to the sum and difference of \( Z_m, X^m \) gives nonvanishing \( K \) and \( Q \). Where did the naive intuition fail?

3. There are at least two discrete groups involved. What fixes them? The doubled geometry is purported to encompass not only the discrete abelian T-duality group that is a symmetry of string theory, but also the nonabelian and Poisson-Lie T-dualities, which generically fail beyond tree level. What characterizes the restriction of the polarization choices to those of abelian T-duality? And what fixes the discrete group \( \Gamma \) in the definition of the doubled space? For a WZW model at level \( n \), it seems natural that the integer \( n \) should show up in the defining topological data (for example, through a \( \mathbb{Z}_n \) factor in \( \Gamma \)) and not just in the choice of polarization.

4. There has been only a modest amount of work on D-branes in the context of T-folds and doubled geometry [42, 1, 53]. However, they do have a relatively straightforward description in doubled geometry. They wrap maximal isotropic submanifolds. Do the D-brane predictions of doubled geometry agree with the well-known results for WZW models?

5. Ultimately we are interested in an analogous doubled geometry for gauged analogs of Calabi-Yau compactifications. Mirror pairs of Calabi-Yau manifolds are \( T^3 \) fibrations. We know how to double the \( T^3 \) fiber, and even how to twist this fibration [71]. How do we double the base? The base is a rational homology sphere, which we can think of as analogous to an \( S^3 \). For the quintic, it is \( S^3/\mathbb{Z}_5 \). The \( SU(2) \cong S^3 \) WZW model provides a case study of the doubled geometry of one such \( S^3 \), albeit a very special one with \( H \)-flux.

An outline of the paper is as follows:

In Sec. 2, we fix notation, and describe the physical metric and \( H \)-flux of a WZW model, first for \( SU(2) \), and then for general gauge group. The reader is referred to App. A for conventions and basic facts about Lie Algebras. We allude to the worldsheet description only minimally throughout the paper. App. B contains a review of the basic worldsheet and CFT aspects of WZW models.

Sec. 3 is devoted to T-folds. We review the definition of a T-fold, including its \( O(d,d) \) and Riemannian metrics, and the procedure for recovering a physical background given a choice of polarization. The T-fold description of the WZW model for \( SU(2) \) and then for general gauge group are presented in Secs. 3.2 and 3.3. For \( SU(2) \), the T-fold is a doubled

\[ ^{10} \text{That the } SU(2) \text{ WZW model has only } K \text{ and } Q \text{ nonzero, or alternatively } f \text{ and } R \text{ nonzero, can be found in Refs. [42, 13]. See also Ref. [3] for an interesting follow-up on [13], which matches a worldsheet analysis to the gaugings of } \mathcal{N} = 4 \text{ supergravity, and describes the gauging for all cases with one or two of } K, f, Q, R \text{ nonzero.} \]
Hopf fibration quotiented by \( \mathbb{Z}_n \). The physical \( S^1 \) fibration gives a sphere, and dual \( \tilde{S}^1 \) fibration gives a Lens space, with T-duality interchanging the two. For a general group, the physical fiber is the Cartan torus \( T^r \), and half the dual coordinate is valued in \((T^r)^*/(\mathbb{Z}_n)^r\), where \((T^r)^*\) denotes the Cartan torus of the dual gauge group.\(^{11}\) T-duality again simply introduces a factor of \( \mathbb{Z}_n \) quotienting the physical space for each \( U(1) \) dualized. The total space of the T-fold is interpreted as the group manifold \((U(1)^r)_L \times G^{\text{WZW}}_R\), which suggests that for WZW models, the T-fold is embedded in the fully doubled space as a subgroup.\(^{12}\)

Sec. 4 is the heart of the paper. We refer the reader to the introduction of Sec. 4 for a more complete overview of the results of this section. The first half covers generalities. Subsecs. 4.1.1 through 4.1.5 provide a review of the formalism of Hull and Reid-Edwards. Sec. 4.1.6 describes the recovery of the physical from doubled geometry. Here, we emphasize that \( Q \)-flux alone is not an obstruction to global geometry, but rather the interplay of \( Q \) and \( \Gamma \); the condition for global geometry is \( \Gamma \)-invariant \( Q \). The double geometry of WZW models is discussed in Sec. 4.2. The doubled space takes the form

\[
X_{2d} = \Gamma \backslash (G_1 \times G_2),
\]

where \( G_1 \) and \( G_2 \) are two copies of the physical WZW group \( G^{\text{WZW}} \). Global polarizations are choices of maximal isotropic subgroup \( \tilde{G} \) conjugate to \( G_{\text{diag}} \). Quotienting the doubled space \( X_{2d} \) by \( \tilde{G} \) gives the physical space \( G^{\text{WZW}} \). In the diagonal polarization, the projection to the physical target space is \( \pi : (g_1, g_2) \mapsto g_{\text{phys}} = g_1^{-1}g_2 \). The symmetry under right multiplication in \( G_{2d} \) gives the \( g_{\text{phys}} \mapsto \Omega_1 g_{\text{phys}} \Omega_2^{-1} \) symmetry of the physical model. We show that the correct physical metric and \( H \)-flux are indeed recovered, including the condition \( r^2 = n\alpha' \). In Secs. 4.2.5 and 4.2.6 we consider general polarizations and describe abelian T-duality. The known results for semiclassical D-branes in WZW models are reproduced in Sec. 4.2.7. Finally, in Sec. 4.2.8 we describe restrictions on the discrete group \( \Gamma \), but do not fully resolve the question of what \( \Gamma \) is at level \( n \) for each choice of modular invariant. In Sec. 5 we conclude with a summary of results and discussion of open questions.

2 The 3D physical background

2.1 Target space description of the \( SU(2) \) WZW model

Consider a 3-sphere \( S^3 \) of radius \( \sqrt{n\alpha'} \) (in string frame) and \( n \) units of \( H \)-flux,

\[
d^2s_{\text{phys}}^2 = r^2d^2s_3^2, \quad r^2 = n\alpha', \quad \text{and} \quad H = 2N\alpha' \omega_{S^3}. \quad (2.1)
\]

\(^{11}\)For simply connected groups, \( G^* \) is \( G \) quotiented by its center. Therefore, \( SU(2)^* = SU(2)/\mathbb{Z}_2 \) and the two factors of 2 cancel in this case.

\(^{12}\)This contrasts to the example of a \( T^3 \) with \( H \)-flux, for which the T-fold appears to arise by partial projection from the fully doubled space to the physical base.
Here, $d\mathcal{s}_3^2$ and $\omega_3$ are the metric and volume form on a unit $S^3$,

$$d\mathcal{s}_3^2 = \frac{1}{4} \left( (d\phi^1)^2 + (d\phi^2)^2 + (d\phi^3)^2 + 2 \cos \phi^1 d\phi^2 d\phi^3 \right),$$  \hspace{1cm} (2.2)$$

$$\omega_3 = \frac{1}{8} \sin \phi^1 d\phi^1 \wedge d\phi^2 \wedge d\phi^3, \quad \int_{S^3} \omega_3 = 2\pi^2.$$  \hspace{1cm} (2.3)$$

The polar angle $\phi^1$ takes values in the interval $I_1 = [0, \pi]$ and the sum and difference of the azimuthal angles, $\phi^2 \pm \phi^3$, are periodic modulo $4\pi$. The normalization of $H$ follows from the quantization condition

$$\frac{1}{2\pi \alpha'} \int_{S^3} H = 2\pi n, \quad n \in \mathbb{Z},$$  \hspace{1cm} (2.4)$$

which ensures that the phase

$$\exp \left( \frac{i}{2\pi \alpha'} \int_{\Sigma} B \right)$$  \hspace{1cm} (2.5)$$

is single valued in the string path integral. A convenient gauge choice for $B$ is

$$B = -\frac{1}{4} n\alpha' \cos \phi^1 d\phi^2 \wedge d\phi^3.$$  \hspace{1cm} (2.6)$$

As discussed in App. B, the background (2.1) arises as the semiclassical description of the $SU(2)$ Wess-Zumino-Witten (WZW) model, at large level $n$. The group manifold $SU(2)$ has the topology of a 3-sphere, and can be parametrized as

$$g(\phi^1, \phi^2, \phi^3) = e^{-i\phi^2 \sigma_3/2} e^{-i\phi^1 \sigma_1/2} e^{-i\phi^3 \sigma_3/2},$$  \hspace{1cm} (2.7)$$

where the $\sigma_m$ are the Pauli spin matrices. Then, $g^{-1} dg = -\frac{i}{2} \sigma_m \lambda^m$, in terms of the left-invariant 1-forms,

$$\lambda^1 = \cos \phi^3 d\phi^1 + \sin \phi^3 \sin \phi^1 d\phi^2,$$

$$\lambda^2 = -\sin \phi^3 d\phi^1 + \cos \phi^3 \sin \phi^1 d\phi^2,$$

$$\lambda^3 = d\phi^3 + \cos \phi^1 d\phi^2.$$  \hspace{1cm} (2.8)$$

The $\lambda^p$ satisfy $d\lambda^p + \frac{1}{2} \epsilon_{mnp} \lambda^m \wedge \lambda^n = 0$ and their product is

$$\lambda^1 \wedge \lambda^2 \wedge \lambda^3 = \sin \phi^1 d\phi^1 \wedge d\phi^2 \wedge d\phi^3 = 8\omega_3,$$  \hspace{1cm} (2.9)$$

where $\omega_3$ is the volume form on a unit $S^3$. Aside from the factor of $n\alpha'$, the metric (2.1) is the unit $SU(2)$ metric metric $ds^2 = \frac{1}{4} \left( (\lambda^1)^2 + (\lambda^2)^2 + (\lambda^3)^2 \right)$.\[13\] Thus, a fundamental domain of the azimuthal angles is $0 \leq \phi^2 < 2\pi$ and $0 \leq \phi^3 < 4\pi$.\[11\]
2.2 Target space description of the WZW model for general group

For a general group $G^{WZW}$, with Lie algebra

$$[t_m, t_n] = c_{mn}^p t_p, \quad (2.10)$$

and left-invariant Maurer-Cartan form

$$g^{-1} dg = \lambda = \lambda^p t_p, \quad \text{where} \quad d\lambda + \frac{1}{2} c_{mn}^p \lambda^m \wedge \lambda^n = 0, \quad (2.11)$$

the target space description of the WZW model at level $n$ is as follows. The metric of Eq. (2.1) generalizes to

$$ds^2_{\text{phys}} = r^2 ds_G^2, \quad r^2 = n\alpha', \quad (2.12)$$

in terms of “unit” $G^{WZW}$ metric

$$ds_G^2 = -\frac{1}{4} \text{tr}'(\lambda\lambda) = \frac{1}{4} \psi^2 d_{mn} \lambda^m \lambda^n, \quad (2.13)$$

and the $H$-flux becomes

$$H = -\frac{n}{12} \text{Tr}'(\lambda \wedge \lambda) = -\frac{n}{12} c_{mnp} \lambda^m \wedge \lambda^n \wedge \lambda^p, \quad \text{where} \quad c_{mnp} = c_{mn}^q c_{qp}. \quad (2.14)$$

Here, $d_{mn}$ is the normalized Killing form, $\psi^2$ is the length squared of a long root, and $\hat{n} = \psi^2 n/2$. Note that $d_{mn}$ is related to the nonnormalized Killing form

$$\tilde{d}_{mn} = -c_{mp}^q c_{mq}^p \quad (2.15)$$

via $\tilde{d}_{mn} = h^\vee \psi^2 d_{mn}$, where $h^\vee$ is the dual Coxeter number of $G^{WZW}$. We refer the reader to App. A for additional Lie algebra conventions and to App. B for the worldsheet description of a WZW model.

3 The 4D T-fold description: doubled Hopf fibration

A T-fold is a generalization of a $T^n$ fibration, in which the transition functions are allowed to lie in the T-duality group $O(n, n)$ rather than its geometric subgroup. In this section, we review the definition of a T-fold and then present the T-fold description of the large level $SU(2)$ WZW model as an $S^1 \times \tilde{S}^1$ fibration over $S^2$. The physical $S^1$ fibration is the Hopf fibration of the physical space $SU(2) \cong S^3$ and the dual $\tilde{S}^1$ fibration defines the Lens space $SU(2)/\mathbb{Z}_n \cong L(n,1)$ T-dual to this background. (It is also possible to give a T-fold description as a $T^2 \times \tilde{T}^2$ fibration over the interval $I_1$. However, the latter is somewhat less natural since the fibers degenerate. See App. C) Finally, in Sec. 3.3 we generalize from $SU(2)$ to arbitrary group.
3.1 T-fold generalities

3.1.1 T-fold backgrounds vs. geometric backgrounds with $B$-field

Recall that a Riemannian manifold is a differentiable manifold $M$ endowed with a metric $G_{mn}$. A $B$-field is conventionally thought of as a $U(1)$ gerbe connection, that is, a 2-form potential for the $H$-flux, analogous to the 1-form potential $A$ of electromagnetism.\(^{14}\)

Of particular interest in string theory is the case that $M$ is a $T^n$ fibration over some base manifold $\mathcal{B}$. The simplest supersymmetry preserving string theory backgrounds ($T^n$, K3, CY$_n$ and products thereof) are generically of this type (due to the special Lagrangian $T^n$ fibration of generic CY$_n$). A manifold $M$ is a $T^n$ fibration if it admits a projection to a base $\mathcal{B}$ and is diffeomorphic to $T^n \times U$ over sufficiently small patches $U \subset \mathcal{B}$ away from singular fibers. Globally, the patches are sewn together via transition functions in $GL(n)$ acting on the fibers.\(^{15}\)

As defined by Hull \cite{42, 43}, a T-fold is a generalization of a $T^n$ fibration with $B$-field, which permits not only geometric transition functions in $GL(n)$ but also T-duality transition functions in $O(n, n)$. Since these transition functions can mix the metric and $B$-field, neither $G$ nor $B$ necessarily has a global interpretation as a metric or gerbe connection on any dim $\mathcal{B} + n$ dimensional space; nor does there necessarily exist a global dim $\mathcal{B} + n$ dimensional topology or associated gerbe topology. From a dim $\mathcal{B} + n$ dimensional point of view, a T-fold is nongeometric.

However, a T-fold does always have a global doubled geometry and topology. For a T-fold (in contrast to the construction of Sec. 4), this doubling refers to the fiber only. One simply considers the product of the physical torus $T^n$ fiber and T-dual torus fiber $\tilde{T}^n$ over each patch $U \subset \mathcal{B}$. Since $O(n, n) \subset GL(2n)$, the T-duality transition functions become ordinary transition function on the doubled $T^n$ fiber, $T^{2n}$. The topology and curvature of this $T^{2n}$ fibration characterize all topological and curvature information that one would seek in the pair $(M, B)$ of the geometric case, except for the $B$-field $B_{\mathcal{B}}$ on the base, which must be specified separately.\(^{16}\)

\(^{14}\)In electromagnetism, the local 1-form $A$ (defined in each coordinate patch of the manifold $M$) is a connection on a $U(1)$ principle bundle. The curvature of this bundle is the global 2-form $F = dA$, i.e., the field strength. Finally, the topology of the bundle is characterized by the cohomology class $[F] \in H^2(M, \mathbb{Z})$. Similarly, a gerbe of connection $B$ is characterized by curvature $H$ and topology $[H] \in (2\pi)^2 \alpha' H^3(M, \mathbb{Z})$.

\(^{15}\)To be precise, the transition functions of the tangent bundle are in $GL(n)$. For the coordinates, the homogeneous part of the transition functions lies in the same $GL(n)$, the inhomogenous part (translations in $T^n$) is a semidirect $U(1)^n$, and the full structure group is $GL(n) \rtimes U(1)^n$.

\(^{16}\)To be precise, $B_{\mathcal{B}}$ is the pullback of a gerbe connection on the base to a gerbe connection on the total space of the $T^{2n}$ fibration.
3.1.2 Metrics and polarizations

On a T-fold, we define two metrics: a constant $O(n,n)$ invariant fiber metric, and a Riemannian metric on the total space,

\[
\begin{align*}
&ds_{O(n,n)}^2 = \mathcal{L}_{IJ} \eta^I \eta^J, \\
&ds_{T\text{-fold}}^2 = ds_B^2 + \mathcal{H}_{IJ} \eta^I \eta^J.
\end{align*}
\]  

(3.1a)

(3.1b)

Here, the $\eta^I = dx^I + A^I$ are the global fiber 1-forms, where $A^I$ is the $T^{2n}$ connection. The Riemannian fiber metric $\mathcal{H}_{IJ}(y)$ is a symmetric matrix such that

\[
\mathcal{H}^T \mathcal{L}^{-1} \mathcal{H} = \mathcal{L}.
\]

(3.2)

Given a polarization or choice of null decomposition $T^{2n} = T^n \times \tilde{T}^n$ into physical and dual subspaces over each patch $U \subset \mathcal{B}$, it is convenient to choose a basis of fiber 1-forms $\eta_I = (\eta^i, \tilde{\eta}^i)$ that respects the decomposition. We similarly choose a coordinate decomposition so that the $x^i (\tilde{x}^i)$ are coordinates on the physical subspace $T^n$ (dual subspace $\tilde{T}^n$). Then,

\[
\begin{align*}
\mathcal{L}_{IJ} &= \begin{pmatrix} 0 & L_i^j \\ (L^T)^i_j & 0 \end{pmatrix}, \\
\mathcal{H}_{IJ} &= \begin{pmatrix} G + B^T G^{-1} B & B^T G^{-1} L \\ L^T G^{-1} B & L^T G^{-1} L \end{pmatrix}_{IJ}.
\end{align*}
\]

(3.3)

In a canonical basis with coordinate periodicities $x^i \cong x^i + 2\pi \nu$ and $\tilde{x}^i \cong \tilde{x}^i + 2\pi \tilde{\nu}$, the former becomes

\[
\mathcal{L}_{IJ} = \frac{1}{\nu \tilde{\nu}} \begin{pmatrix} 0 & \delta^j_i \\ \delta^i_j & 0 \end{pmatrix}, \quad \text{i.e.}, \quad L_i^j = \frac{1}{\nu \tilde{\nu}} \delta^j_i.
\]

(3.4)

With these definitions, the two metrics become

\[
\begin{align*}
&ds_{O(n,n)}^2 = 2L_i^j \eta^i \tilde{\eta}^j, \\
&ds_{T\text{-fold}}^2 = ds_B^2(y) + G_{ij} \eta^i \eta^j + G^{ij}(L^k_i \tilde{\eta}^k + B_{ik} \eta^k)(L^l_j \tilde{\eta}^l + B_{lj} \eta^l).
\end{align*}
\]

(3.5a)

(3.5b)

Here, and in all subsequent sections, we set $\alpha' = 1$ for simplicity. The field $B_{ij}$ parametrizes the off diagonal components of the $T^n \times \tilde{T}^n$ Riemannian metric. Given a choice of doubled fiber coordinates $x^i$, we can write $\eta^I = dx^i + A^i$, where the connection $A^I(y)$ depends only on the base coordinates $y^a$. Then, given a polarization, we decompose $A^I$ as $A^I = (A^i, (L^{-1})^i_j B_j)$, so that

\[
\eta^I = dx^i + A^i \quad \text{and} \quad \tilde{\eta}_i = d\tilde{x}_i + (L^{-1})^i_j B_j.
\]

(3.6)

3.1.3 Recovery of the physical background

Given a choice of gerbe connection $B_B$ on the base, and a choice of polarization over a patch $U \subset \mathcal{B}$, the physical metric and $B$-field in this patch are

\[
\begin{align*}
ds^2 &= ds_B^2(y) + G_{ij} \eta^i \eta^j, \\
B &= B_B + (dx^i + \frac{1}{2} A^i) \wedge B_i + \frac{1}{2} B_{ij} (dx^i + A^i) \wedge (dx^j + A^j).
\end{align*}
\]

(3.7a)

(3.7b)
Here, $B_B$ is a local 2-form on the base and the $B_i$ are local 1-forms on the base. Note that the polarization need not be defined globally, so that the recovery of the physical background and standard sigma model, is only patchwise. If a global polarization exists, then the compactification is geometric and described globally by a standard sigma model. In this case, the total space of the T-fold is dual torus fibration over the physical space, and it possible to recover the physical background by global projection. Otherwise, the compactification is only locally geometric, and is globally nongeometric.

From the worldsheet point of view, a T-fold background is not really so different from a globally geometric compactification with $B$-field: In each patch, we have a standard sigma model description and the usual $\beta$-function equations. Globally, for either a manifold or a T-fold, transition functions are necessary in order to relate the sigma model Lagrangian in overlaps between coordinate patches.

3.1.4 T-duality action on fiber

T-duality acts by $O(n, n; \mathbb{Z})$ transformation as

$$\eta^i \mapsto O^i{}_{j} \eta^j, \quad \mathcal{H} \mapsto O^{-1T}{}^i{}_{k} \mathcal{H} O^{-1T}{}^j{}_{l}, \quad \mathcal{O} \in O(n, n; \mathbb{Z}).$$

(3.8)

The $B$-field component $B_B$ in Eq. (3.7b) is a T-duality invariant. Here, the $O(n, n)$ condition is $O^T \mathcal{L} O = \mathcal{L}$, and the restriction to $\mathbb{Z}$ indicates that $O^i{}_{j}$ preserves the lattice defining the doubled torus. This is the “active” point of view. For the passive transformations, the 1-forms $\eta^a$ and Riemannian metric are held fixed, and we consider different choices of polarization on the same doubled fibration.

3.2 T-fold description of the $SU(2)$ WZW model

We now show that the $SU(2)$ WZW model admits a T-fold description as the $\mathbb{Z}_n$ quotient of a doubled Hopf fibration over $S^2$. In this section, we consider only the semiclassical background (2.1). In Sec. 4, we also briefly comment on the CFT interpretation.

A 3-sphere can be thought of as the Hopf fibration of $S^1$ over $S^2$ with $-1$ unit of Euler class. Following this interpretation, we write Eq. (2.2) as

$$ds^2_{S^3} = \frac{1}{4} \left( ds^2_{S^2} + (d\phi^3 + A^3)^2 \right) \quad \text{with} \quad A^3 = \cos \phi^1 d\phi^2,$$

(3.9)

where the metric and volume form on a unit 2-sphere are

$$ds^2_{S^2} = (d\phi^1)^2 + \sin^2 \phi^1 (d\phi^2)^2 \quad \text{and} \quad \omega_{S^2} = \sin \phi^1 d\phi^1 \wedge d\phi^2.$$

(3.10)

The topology of this fibration is characterized by its Euler class $dA^3 = -\omega_{S^2}$, viewed as an element of $H^2(S^2, \mathbb{Z})$. Here, the coordinate $\phi^3$ is periodic modulo $4\pi$ on each fiber, and $\phi^2$ is the usual $S^2$ azimuthal angle periodic modulo $2\pi$ on the base. (See Footnote 13.)
Now consider the background (2.11). In the decomposition of Eq. (3.7b), the $B$-field (2.6) has only a 1-form component,

$$B_3 = \frac{1}{4} n \cos \phi^1 d\phi^2 \quad \text{with} \quad dB_3 = -\frac{1}{4} n \omega_{S^2}. \quad (3.11)$$

The NS sector topological data (the spatial $S^3$ topology, and cohomology class $[H] \in (2\pi)^2 H^3(M, \mathbb{Z})$), and Riemannian data (the choice of a physical spatial metric and gerbe connection), are completely encoded in the topology and Riemannian metric of the T-fold background, once a polarization is specified.

The $O(1,1)$ metric (3.1a) and Riemannian metric (3.1b) are

$$ds^2_{O(1,1)} = \frac{1}{2} \eta^3 \tilde{\eta}_3, \quad (3.12a)$$

$$ds^2_{\text{T-fold}} = \frac{1}{4} \left( n \left( (\eta^1)^2 + (\eta^2)^2 + (\eta^3)^2 \right) + \frac{1}{n} (\tilde{\eta}_3)^2 \right), \quad (3.12b)$$

where $\eta^i = \lambda^i$ of Sec. 2.11 and

$$\tilde{\eta}_3 = d\tilde{\phi}_3 + n \cos \phi^1 d\phi^2, \quad \text{with} \quad d\tilde{\eta}_3 = -n \omega_{S^2}. \quad (3.13)$$

In contrast, the 1-form on the Hopf fiber of the physical $S^3$ is

$$\eta_3 = d\phi^3 + \cos \phi^1 d\phi^2, \quad \text{with} \quad d\eta_3 = -\omega_{S^2}. \quad (3.14)$$

Here, we are working in the convention $\phi^3 \simeq \phi^3 + 4\pi$ and $\tilde{\phi}_3 \simeq \tilde{\phi}_3 + 4\pi$, so that $\nu = \tilde{\nu} = 2$ in the notation of Sec. 3.1.2.

The physical metric is obtained by dropping the $\tilde{\eta}_3$ term from Eq. (3.12b). T-duality inversion of the physical $S^1$ fiber interchanges $\eta_3$ and $\tilde{\eta}_3$, and thus exchanges $-n$ unit of Euler class and $n$ units of $H$-flux with $-n$ units of Euler class and 1 unit of $H$-flux. The T-dual background is a Lens space $L(n,1) = S^3/\mathbb{Z}_n$ with the minimal quantum of $H$-flux. Indeed, it is known that at level $n$, the $SU(2)$ WZW model and $SU(2)/\mathbb{Z}_n$ WZW orbifold are exactly equivalent as CFTs\footnote{For $n = n_1n_2$, the equivalence of the $SU(2)/\mathbb{Z}_{n_1}$ and $SU(2)/\mathbb{Z}_{n_2}$ CFTs has also been demonstrated \cite{58}. This is the T-duality equivalence of $S^3/\mathbb{Z}_{n_1}$ with $n_2$ units of $H$-flux and $S^3/\mathbb{Z}_{n_2}$ with $n_1$ units of $H$-flux, which describe the near horizon angular geometry of a system of $n_1$ KK monopoles and $n_2$ NS5-branes, and $n_2$ KK monopoles and $n_1$ NS5-branes, respectively.} \cite{21, 58, 17}.

The $\mathbb{Z}_n$ quotient can be seen more clearly by writing

$$\tilde{\chi}^3 = d\tilde{\phi}^3 + \cos \phi^1 d\phi^2, \quad (3.15)$$

with $\tilde{\phi}^3 = \tilde{\phi}_3/n$ periodic modulo $4\pi/n$ instead of mod $4\pi$. The metrics then takes the form

$$ds^2_{O(1,1)} = \frac{n}{2} \lambda^3 \tilde{\chi}^3, \quad (3.16)$$

$$ds^2_{\text{T-fold}} = \frac{n}{4} \left( (\lambda^1)^2 + (\lambda^2)^2 + (\lambda^3)^2 + (\tilde{\chi}^3)^2 \right), \quad (3.17)$$
with
\[ \lambda^3 = d\phi^3 + \cos\phi^1 d\phi^2, \quad \text{and} \quad \tilde{\lambda}^3 = d\tilde{\phi}^3 + \cos\phi^1 d\phi^2. \] (3.18)

On the \( \mathbb{Z}_n \) covering space \((\tilde{\phi}^3 \approx \tilde{\phi}^3 + 4\pi)\), the physical and dual \( S^1 \) fibrations each define a total space \( S^3 \). The \( \mathbb{Z}_n \) acts freely by translation on the dual Hopf fiber: \( \tilde{\phi}^3 \mapsto \tilde{\phi}^3 + 4\pi/n \). \(^{18}\)

### 3.3 T-fold description of the WZW model for general group

The \( SU(2) \) discussion of the previous section readily generalizes to an arbitrary compact semisimple Lie group \( G^{WZW} \). The Cartan subalgebra generates a \( U(1)^r \) isometry, which endows \( G^{WZW} \) with the structure of a \( T^r \) Cartan torus fibration over the coset \( G^{WZW}/U(1)^r \). The \( B \)-field of the WZW model defines a formal \( \tilde{T}^r \) fibration over the same base. In Sec. 3.3.1, we describe the physical fibration of the WZW background and present the metric and \( B \)-field in the form required by Eqs. (3.7). Then, in Sec. 3.3.2, we show how the same data is encoded the doubled fibration of the T-fold.

#### 3.3.1 The physical \( T^r \) fibration

In terms of the Chevalley basis \(^{19}\) of \( g^{WZW} \) (defined in App. A.3), write
\[ t_i = -ih_i, \quad t_{1a} = -i(e_a + f_{-a}), \quad \text{and} \quad t_{2a} = -(e_a - f_{-a}), \] (3.19)

and let \( a \) denote an index that runs over \( 1i \) and \( 2i \). Then, the Lie algebra takes the form
\[ [t_i, t_j] = 0, \quad [t_a, t_i] = c_{ai}^j t_j, \quad [t_a, t_b] = c_{ab}^i t_i, \] (3.20)

and the normalized Killing form takes the block diagonal form \( \text{diag}(d_{ij}, d_{ab}) \). The \( G^{WZW} \) invariance of the Killing form implies that the lowered index structure constant is completely antisymmetric. Therefore, the two structure constants of Eq. (3.20) are related by
\[ c_{ia}^c d_{cb} = c_{ab}^j d_{ji} = c_{iab}. \] (3.21)

Since the \( t_i \) generate an abelian subalgebra, the group manifold \( G^{WZW} \) is fibered by Cartan tori \( T^r = U(1)^r \). A generic element can be parametrized by coordinates \( (x^i, y^a) \) as
\[ g(x,y) = \exp(y^a t_a) \prod_{m=1}^{r} \exp(x^i t_i). \] (3.22)

---

\(^{18}\)Note that the linear combination \( \lambda^3 - \tilde{\lambda}^3 \) is closed. A linear combination of the Hopf fibers is therefore trivially fibered, and topologically, the covering space of the T-fold factors as \( S^3 \times S^1 \). This will come up again in Sec. 3.3.

\(^{19}\)In the notation used here and in App. A.3, the \( e_a \) \( (f_{-a}) \) include both the fundamental Chevalley generators \( e_i \) \( (f_i) \) of the defining commutation relations and the descendents obtained from multiple commutators of the \( e_i \) \( (f_i) \).
The Killing isomorphism (c.f. App. A.2) maps the Cartan subalgebra \( h \) to the dual space \( h^\ast \) by trading Chevalley generators \( h_j \) for coroots \( \alpha^{\vee}_j \). Thus \( ix^jt_j \) maps to \( x \), where

\[
x = x^j\alpha^{\vee}_j = x_jw^{(j)} \quad \text{and} \quad x_j = d_{jk}x^k.
\]

(3.23)

Here the \( w^{(i)} \) are the basis of weights, dual to the \( \alpha_j \). Taking into account the periodic identifications, we have

\[
x \in h^\ast/(2\pi\Lambda^\vee) \cong T^r,
\]

(3.24)
i.e., \( x \cong x+2\pi\alpha^{\vee} \), where \( \alpha^{\vee} \) is any coroot, or in components, \( x^j \cong x^j+2\pi \) and \( x_j \cong x_j+d_{jk}N^k \), where the \( N^k \) are integers.

The metric (2.12) can be written in the fibration form

\[
d_{\text{phys}}^2 = d_{B}^2 + \frac{\hat{n}}{2}d_{ij}(dx^i + A^i)(dx^j + A^j),
\]

(3.25)

where the \( x \)-dependence drops out of the metric on the base \( B = G^{WZW}/U(1)^r \),

\[
ds_B^2(y) = \frac{\hat{n}}{2}d_{ab}\lambda^a\lambda^b,
\]

(3.26)

and the fiber 1-forms satisfy

\[
d\lambda^i = dA^i = -\frac{1}{2}c^i_{ab}\lambda^a \wedge \lambda^b.
\]

(3.27)

The \( H \)-flux (2.14) becomes

\[
H = \frac{\hat{n}}{4}c_{ab}\lambda^i \wedge \lambda^a \wedge \lambda^b,
\]

(3.28)
and can be obtained from a \( B \)-field of the form (3.7), with

\[
B_B = \frac{\hat{n}}{4}d_{ij}A^i \wedge A^j, \quad B_i = \frac{\hat{n}}{2}d_{ij}A^j \quad \text{and} \quad B_{ij} = 0.
\]

(3.29)

### 3.3.2 The doubled fibration

In the T-fold description, the torus fibration of \( G^{WZW} \) is promoted to a doubled fibration by including both the Cartan torus \( T^r \) and dual Cartan torus \( \tilde{T}^r \) fibers. The description follows straightforwardly from Secs. 3.1 and 3.3.1 once we specify the \( O(r,r) \) fiber metric (3.3).

Let \( \tilde{x}_i \) and \( \tilde{t}^i \) denote the dual coordinates and generators, respectively. Then \( i\tilde{x}_mt^m \) canonically maps to \( \tilde{x} \in h^\ast \), where

\[
\tilde{x} = \tilde{x}_m w^{(m)} = \tilde{x}^m w_m \quad \text{and} \quad \tilde{x}^m = d_{mn}^{\ast}\tilde{x}_n.
\]

(3.30)

Taking into account the periodic identifications, we have

\[
\tilde{x} \in h^\ast/(2\pi(\Lambda^\vee)^\ast) \cong \tilde{T}^r,
\]

(3.31)
i.e., $\tilde{x} \cong \tilde{x} + 2\pi w^\vee$, where $w^\vee$ is any weight, or in components, $\tilde{x}_j \cong \tilde{x}_j + 2\pi$ and $\tilde{x}^j \cong \tilde{x}^j + d^k N_k$, where the $N_k$ are integers.

Since the coordinates $(x^i, \tilde{x}_i)$ have the canonical $2\pi$ periodicities, $L^i_j$ of Eq. (3.3) takes the canonical form $\delta^i_j$ in this basis.

The doubled metrics (3.5) become

$$ds^2_{O(n,n)} = 2L^i_l \eta^i \tilde{\eta}^l,$$  \hspace{1cm} (3.32a)

$$ds^2_{T\text{-fold}} = ds^2_B(y) + \frac{\hat{n}}{2} d^i j \eta^i \eta^j + \frac{2}{\hat{n}} d^i j \tilde{\eta}^i \tilde{\eta}^j,$$  \hspace{1cm} (3.32b)

where

$$\eta^i = \lambda^i = dx^i + A^i \quad \text{and} \quad \tilde{\eta}^i = d\tilde{x}^i + \frac{\hat{n}}{2} d^i j A^i.$$  \hspace{1cm} (3.33)

Proceeding as in the $SU(2)$ case, it is natural to define new coordinates on the dual fiber, $\tilde{x}'^i = (2/\hat{n}) d^i j \tilde{x}_j$. Then,

$$ds^2_{O(n,n)} = 2L^i_l \lambda^i \tilde{\lambda}^l, \quad L^i_j = \frac{\hat{n}}{2} d^i j,$$  \hspace{1cm} (3.34a)

$$ds^2_{T\text{-fold}} = ds^2_B(y) + \frac{\hat{n}}{2} d^i j (\lambda^i \lambda^j + \tilde{\lambda}^i \tilde{\lambda}^j),$$  \hspace{1cm} (3.34b)

where

$$\lambda^i = dx^i + A^i, \quad \tilde{\lambda}^i = d\tilde{x}^i + A^i.$$  \hspace{1cm} (3.35)

This puts the T-fold Riemannian metric in the form of doubled Cartan torus fibration with the same connection for either factor, as was obtained in Sec. 3.2 for the $SU(2)$ case. However, we still need to account for the modified periodicities of the $\tilde{x}'^i$. The coordinate $\tilde{x}' = (2/\hat{n})\tilde{x}$ satisfies

$$\frac{\tilde{x}'}{2} \in \frac{\mathfrak{h}^*}{(2\pi/\hat{n})(\Lambda^\vee)^*} = \frac{\mathfrak{h}^*}{(2\pi/\hat{n})\Lambda^\vee}/C,$$  \hspace{1cm} (3.36)

where we have used the fact that the ratio of the weight lattice to the root lattice is $C$, the center of the group. For $G^{WZW}$ a simply laced group, we have $(\psi^2/2)\Lambda = \Lambda^\vee$. In this case, the two coordinate periodicities are given by

$$x \in \left( \frac{\mathfrak{h}^*}{2\pi\Lambda^\vee} \right) \quad \text{and} \quad \frac{\tilde{x}'}{2} \in \left( \frac{\mathfrak{h}^*}{(2\pi/\hat{n})\Lambda^\vee} \right)/C.$$  \hspace{1cm} (3.37)

For $SU(2)$, with $C = \mathbb{Z}_2$, this becomes

$$x \in \left( \frac{\mathfrak{h}^*}{2\pi\Lambda^\vee} \right) \quad \text{and} \quad \frac{\tilde{x}'}{2} \in \left( \frac{\mathfrak{h}^*}{(2\pi/\hat{n})\Lambda^\vee} \right) = \left( \frac{\mathfrak{h}^*}{2\pi\Lambda^\vee} \right)/\mathbb{Z}_n,$$  \hspace{1cm} (3.38)

so we indeed obtain two copies of the same Cartan torus fibration (in this case, the Hopf fibration), up to a quotient by $\mathbb{Z}_n$ on the second factor, in agreement with Sec. 3.2.
Let us interpret what we have done. The group $G^{WZW}$ represents both the physical space and the left and right action on the group manifold $g_{\text{phys}} \mapsto \Omega_L g_{\text{phys}} \Omega_R^{-1}$. Let us view the Lie algebra of the previous section as that of the right action. There is no harm in considering a slightly larger group, which also includes the left action of the Cartan subalgebra. Let us add superscripts $L$ and $R$ to Cartan generators to distinguish between left and right actions.

The Lie algebra becomes

$$[t_i^R, t_j^R] = 0, \quad [t_a, t_i^R] = c_{ai} t_j^R, \quad [t_a, t_b] = c_{ab} t_i^R, \quad [t_i^L, t_j^R] = 0, \quad [t_i^L, t_a] = 0,$$

(3.39)

where the new generators just contribute an abelian $(U(1)^r)_L$. It is convenient to define physical and dual Cartan generators $t_i$ and $\tilde{t}_i$ via

$$t_i^L = t_i - \tilde{t}_i, \quad t_i^R = t_i + \tilde{t}_i.$$

(3.40)

In this basis, the Lie algebra becomes

$$[t_i, t_j] = [\tilde{t}_i, \tilde{t}_j] = 0, \quad [t_a, t_i] = [t_a, \tilde{t}_i] = c_{ai} (t_j + \tilde{t}_j), \quad [t_a, t_b] = c_{ab} (t_i + \tilde{t}_i).$$

(3.41)

The doubled fibration described in this section is the group manifold of this Lie algebra.

The appearance of $\tilde{x}'/2$ rather than $\tilde{x}'$ in Eq. (3.37) has the following interpretation. The Killing form on the enlarged algebra, in the $(t_i^L, t_i^R, t_a)$ basis is $\text{diag}(d_{ij}, d_{ij}, d_{ab})$. In the $(t_i, \tilde{t}_i, t_a)$ basis it is $\text{diag}(\frac{1}{2}d_{ij}, \frac{1}{2}d_{ij}, d_{ab})$. Thus, the natural dual generator with upper index is $2d^i \tilde{t}_i = 2\tilde{t}_i$ with conjugate coordinate $\tilde{x}'/2$.

Define left and right fiber coordinates by $x^i = x^i_L + x^i_R$ and $\tilde{x}^i = -x^i_L + x^i_R$. Then,

$$\lambda^i t_i + \tilde{\lambda}^i t_i = \lambda^i_L t_i^L + \lambda^i_R t_i^R,$$

(3.42)

where

$$\lambda^i_L = \frac{1}{2}(\lambda^i - \tilde{\lambda}^i) = dx^i_L \quad \text{and} \quad \lambda^i_R = \frac{1}{2}(\lambda^i + \tilde{\lambda}^i) = dx^i_R + A^i.$$

(3.43)

Thus, it is no coincidence that the difference between the 1-forms in Eq. (3.35) is trivially fibered. The total space of the T-fold factorizes as the produce of a left $U(1)^r$ and a right $G^{WZW}$.

T-duality interchanges factors of $n$ between the two denominators of Eq. (3.37). A straightforward generalization of this result, starting with an orbifold of the original space by $\bigoplus_{i=1}^r \mathbb{Z}_{n_i}$, is

$$x \in \left( \frac{\mathfrak{h}^*}{2\pi A^\Lambda} \right) / \bigoplus_{i=1}^r \mathbb{Z}_{n_i} \quad \text{and} \quad \tilde{x}' = \left( \frac{\mathfrak{h}^*}{2\pi A^\Lambda} / C \right) / \bigoplus_{i=1}^r \mathbb{Z}_{\tilde{n}_i},$$

(3.44)

where $n_i \tilde{n}_i = n$ (no sum) for $i = 1, \ldots, r$. T-duality on the $i$th $U(1)$ interchanges $n_i$ and $\tilde{n}_i$.

Looking back at the $SU(2)$ example, and comparing Eqs. (2.7) with the results of Sec. 3.2 we see that the effect of T-duality at level $n$ is a right quotient, replacing $SU(2)$ by $SU(2)/\mathbb{Z}_n$. Thus, in the present context, we expect the discrete groups to quotient the $(U(1)^r)_R$ fiber of the $(G^{WZW})_R$ factor of the T-fold topology, leaving the $(U(1)^r)_L$ factor unchanged.
4 The 6D fully doubled description: $\Gamma \backslash (S^3 \times S^3)$

In this section we describe the doubled geometry of Hull and Reid-Edwards, focusing on the doubled geometry of WZW models, and then the special case of the $SU(2)$ WZW model at level $n$. The first half of the section covers generalities. Subsecs. 4.1.1 through 4.1.5 provide a careful review of the formalism of Hull and Reid-Edwards. Sec. 4.1.6 deals with the recovery of the physical from doubled geometry. The general idea was already sketched in the introduction. When $R \neq 0$, there is a closed subgroup $\tilde{G}$ with structure constants $Q$, by which we can quotient to obtain the physical geometry. This always works locally, but a potential obstruction is the discrete group $\Gamma$. The relevant condition for global geometry is $\Gamma$-invariant $Q$: conjugation by elements of $\Gamma$ should preserve $\tilde{G}$. This guarantees that there is a global polarization. Thus, we see that $Q$-flux alone is not an obstruction to global geometry, but rather the interplay of $Q$ and $\Gamma$. Finally we describe the procedure of Hull and Reid-Edwards for defining local horizontal and vertical 1-forms, and locally extracting the physical metric and $B$-field from the doubled geometry.

This brings us to the doubled geometry of WZW models in Sec. 4.2. The doubled space takes the form

$$X_{2d} = \Gamma \backslash (G_1 \times G_2).$$

Here $G_1$ and $G_2$ are two copies of the physical WZW group $G_{WZW}^\text{phys}$. Global polarizations are choices of maximal isotropic subgroup $\tilde{G}$ conjugate to $G_{\text{diag}}$. Let us focus on the choice $G_{\text{diag}}$. Then, the projection to the physical target space is $\pi: (g_1, g_2) \mapsto g_{\text{phys}} = g_1^{-1}g_2$. In the doubled sigma model, $g_1^{-1}(z, \bar{z})$ and $g_2(z, \bar{z})$ are analogs of the chiral fields $g_L(z)$ and $g_R(\bar{z})$ in the physical model. A gauging of the left action of $G_{\text{diag}}$ ensures that the correct chiral coordinate dependence is restored. The symmetry under right multiplication in $G_{2d}$ gives the $g_{\text{phys}} \mapsto \Omega_1 g_{\text{phys}} \Omega_2^{-1}$ symmetry of the physical model. By expressing the local procedure of Hull and Reid-Edwards in terms of global 1-forms, we show that the correct physical metric and $H$-flux are indeed recovered, including the condition $r^2 = na'$. When the total space is viewed as a fibration over the physical base, the horizontal and vertical 1-forms are

$$\lambda_{\text{phys}} = \lambda_2 - g_{\text{phys}}^{-1}\lambda_1 g_{\text{phys}} \quad \text{and} \quad \omega = \lambda_2 + g_{\text{phys}}^{-1}\lambda_1 g_{\text{phys}},$$

where $\lambda$ denotes a left-invariant form. The simpler linear combinations $\lambda_1 \pm \lambda_2$ define the totally antisymmetric structure constants $H, f, Q, R$, with $H, Q$ nonzero. However, they are twisted relative to the natural forms on the fiber and base. This resolves the naive confusion about why $f$ vanishes.

In Secs. 4.2.5 and 4.2.6 we consider polarizations $\tilde{G}_b = bG_{\text{diag}}b^{-1}$ and interpret ordinary abelian T-duality on the Cartan torus in terms of a restricted subgroup of $b \in G_{2d}$. The same maximal isotropic subspaces furnish possible D-brane worldvolumes in the doubled description. Using this observation, in Sec. 4.2.7 we reproduce the known results for semi-classical D-branes in WZW models. Finally, in Sec. 4.2.8 we describe restrictions on the
discrete group $\Gamma$, but do not fully resolve the question of what $\Gamma$ is at level $n$ for each choice of modular invariant.

\section{Doubled geometry generalities}

In Ref. \cite{45} (based on earlier work \cite{42, 43, 44}, see also Sec. 5.3 of Ref. \cite{28}), Hull and Reid-Edwards present a framework to describe compactifications that are analogs of torus reductions, twisted by general NSNS sector discrete data. The basic idea is a natural extension of the previous section: We would like to extend the doubled fibration of Sec. 3 to a fully doubled space. The topological choice is then that of the doubled manifold $X_{2d}$. In addition, we require a locally flat $O(d,d)$ invariant metric and a compatible Riemannian metric. Then, given a choice of polarization, we can recover the conventional sigma model description in each patch, provided there is no “R-flux” (defined below) locally obstructing the projection from the total space to a physical base.

In this paper, we will stick to the purely bosonic WZW model, however, the natural expectation is that in a supersymmetric context the $G$-structure (structure group of the frame bundle) of $X_{2d}$ determines the amount of supersymmetry preserved by the low energy action\footnote{Since the twisting (NSNS discrete data) gauges the supergravity theory, the vacua will spontaneously break some or all of the supersymmetry, and preserve less supersymmetry than the action.} compared to that in flat space\footnote{For analogs of K3 or CY, this group structure is the expectation for the Hitchin generalized geometry \cite{36, 31, 48, 26, 27}. (See also Ref. \cite{22} for an authoritative discussion of $G$-structures in string theory compactifications, preceding their application to generalized geometry.) One might question whether analogous statements should hold for a suitable supersymmetric generalization of the doubled geometry of Hull and Reid-Edwards. A piece of evidence to the affirmative, is the agreement verified in Ref. \cite{64} between Lie brackets on the doubled geometry and twisted Courant brackets on the generalized geometry, for the special case of backgrounds with $K$ and $f$ flux only.}

\begin{align*}
\text{unbroken supersymmetry} & \rightarrow \text{identity structure (i.e., parallelizable } X_{2d}), \\
\frac{1}{2}\text{ supersymmetry} & \rightarrow SU(2) \times SU(2) \text{ structure}, \\
\frac{1}{4}\text{ supersymmetry} & \rightarrow SU(3) \times SU(3) \text{ structure},
\end{align*}

and so on. The corresponding compactifications generalize purely geometric $T^d$, $K3 \times T^{d-4}$ and $CY_3 \times T^{d-6}$ compactifications with no flux, by introducing NSNS data that twists the compactification and gauges the low energy supergravity theory.

The work of Hull and Reid-Edwards focuses on the bosonic sector in the case that the action preserves the same amount of supersymmetry as flat space. The NSNS sector topological data includes the doubled space $X_{2d}$, a constant $O(d,d)$ metric, and a choice of polarization, each of which we now describe. The NSNS sector continuous moduli appear in a Riemannian metric on $X_{2d}$.
4.1.1 Doubled space $\mathcal{X}_{2d}$

In the framework of Hull and Reid-Edwards, the manifold $\mathcal{X}_{2d}$ is a twisted doubled torus, defined as the coset of a group manifold $G_{2d}$ by some discrete subgroup $\Gamma \subset G_{2d}$,

$$\mathcal{X}_{2d} = \Gamma \backslash G_{2d}. \quad (4.1)$$

As we will see, the spacetime gauge symmetry arises from the right $G_{2d}$ action on $\mathcal{X}_{2d}$. Since this isometry is preserved by the left quotient by $\Gamma$, the discrete group $\Gamma$ is part of the data that needs to be specified in order to globally define the model. Such a quotient is necessary when $G_{2d}$ is noncompact, in order to obtain a compact physical space $\mathcal{X}_d$ (or its suitable nongeometric generalization), i.e., finite 4D Planck mass. However, $\Gamma$ is part of the topological data that needs to be specified even when $G_{2d}$ is compact. Our convention is that $G_{2d}$ is simply connected. An arbitrary Lie group $G'_{2d}$ can be written as the quotient of its universal cover $\tilde{G}_{2d}$ by a discrete normal subgroup $\Gamma \in G_{2d}$.\footnote{The universal covering group $\tilde{G}_{2d}$ is simply connected. For $\Gamma$ a normal subgroup of $G_{2d}$ the coset $\Gamma \backslash G_{2d} = G_{2d}/\Gamma$ is a subgroup. For $\Gamma$ discrete, $\pi(\Gamma \backslash G_{2d}) \cong \Gamma$.}

Thus, $\Gamma$ can be nontrivial, even when the doubled target space $\mathcal{X}_{2d}$ is a group manifold.

Given a basis for the Lie algebra $g_{2d}$, the corresponding left invariant vector fields form a frame $\{T_M\}$ trivializing the tangent bundle $T\tilde{G}_{2d}$, and the dual left invariant 1-forms $P^M$ defined by $g^{-1}dg = \mathcal{P} = T_M P^M$ form a coframe trivializing the cotangent bundle $T^* \tilde{G}_{2d}$. The frame and coframe satisfy

$$[T_M, T_N] = t_{MN}^P T_P, \quad dP^P + \frac{1}{2}t_{MN}^P P^M \wedge P^N, \quad (4.2)$$

for the same structure constants $t_{MN}^P$.

4.1.2 $O(d, d)$ invariant metric

The next piece of data we require is a locally flat $O(d, d)$ invariant metric on $\mathcal{X}_{2d}$,

$$ds_{O(d, d)}^2 = \mathcal{L}_{MN} P^M \mathcal{P}^N, \quad (4.3)$$

or equivalently, an inner product

$$\langle T_M, T_N \rangle = \mathcal{L}_{MN} \quad (4.4)$$

on the Lie algebra $g_{2d}$: $[T_M, T_N] = t_{MN}^P T_P$. By locally flat, we mean that $\mathcal{L}_{MN}$ is a constant matrix of signature $(d, d)$. A restriction on the choice of $G_{2d}$ is that its action must preserve the $O(d, d)$ metric. This means that $\langle [T_P, T_M], T_N \rangle + \langle T_M, [T_P, T_N] \rangle = 0$, from which the lowered index structure constants $t_{MNP} = t_{MNQ} \mathcal{L}_{QP}$ are totally antisymmetric.\footnote{From the point of view of the low energy effective field theory, $G_{2d}$ must have a well defined action on the scalars. Therefore, it must be the semidirect product of a subgroup of $O(d, d)$ (the isometry group of the scalar manifold described in Sec. [11.1.4] and a group under which the scalars are not charged.}
4.1.3 Polarization and $K, f, Q, R$-flux

A choice of polarization is a choice of projection defining the physical subbundle of the tangent bundle (or equivalently, cotangent bundle) over each patch $U \subset \mathcal{X}_{2d}$. The projection must be null with respect to the $O(d, d)$ metric. The choice of polarization is as much part of the defining data of the string compactification as $\mathcal{X}_{2d}$, in that different polarizations can, but need not necessarily, define equivalent vacua. Given a polarization, it is natural to choose a corresponding basis $T_M = (Z_m, X^m)$ and dual basis $\mathcal{P}^N = (\mathcal{P}^m, \tilde{\mathcal{P}}_m)$, so that $\mathcal{L}_{MN}$ takes the form

$$\mathcal{L}_{MN} = \begin{pmatrix} 0 & L_m^n \\ (L^T)^m_n & 0 \end{pmatrix}. \quad (4.5)$$

In this basis, the Lie algebra $\mathfrak{g}_{2d}$ takes the form

$$\begin{align*}
[Z_m, Z_n] &= K_{mnp}X^p + f_{mn}^qZ_p, \\
[Z_m, X^n] &= f_{pm}^nX^p + Q^m_nZ_p, \\
[X^m, X^n] &= Q^{mn}_pX^p + R^{mnp}Z_p,
\end{align*} \quad (4.6)$$

where the same $f$ and $Q$ appear twice due to the antisymmetry of $t_{MNP}$. The structure constants are referred to as $K, f, Q, R$-flux, a name motivated by their appearance in the 3-form

$$\mathcal{K} = -\frac{1}{3} \mathcal{L}_{MN}\mathcal{P}^M \wedge d\mathcal{P}^N = \frac{1}{6} t_{MNP}\mathcal{P}^M \wedge \mathcal{P}^N \wedge \mathcal{P}^P. \quad (4.7)$$

The isometry group $\mathcal{G}_{2d}$ of $\mathcal{X}_{2d} = \Gamma \backslash \mathcal{G}_{2d}$ acts by right action and completely geometrizes the gauge group. (In contrast, in the non-doubled physical background, the gauge group is half due to Kaluza-Klein gauge fields $A^m\mu$ and half due to winding gauge fields $B_{m\mu}$, with generators $Z_m$ and $X^m$, respectively.) In the parallelizable case, this observation can be taken as a definition of the doubled geometry given the gauge group: the doubled geometry $\mathcal{X}_{2d}$ is a continuous representation of group $\mathcal{G}_{2d}$. The gauge transformations are just the translations on $\mathcal{X}_{2d}$.

4.1.4 Riemannian metric

Finally, we define a Riemannian metric on $\mathcal{X}_{2d}$,

$$ds^2 = \mathcal{M}_{MN}\mathcal{P}^M \mathcal{P}^N = g_{mn}\mathcal{P}^m\mathcal{P}^n + g^{mn}(L_m^p\mathcal{P}_p + b_{mp}\mathcal{P}^p)(L_n^q\mathcal{P}_q + b_{nq}\mathcal{P}^q). \quad (4.8)$$

24As discussed in Refs. [63, 64], the vacua are generically expected to be related by a Poisson-Lie or nonabelian T-duality, which is a symmetry at tree level in string theory, but not at higher loop. Inequivalent polarizations are potentially even more interesting than equivalent ones, in that they should correspond to inequivalent quantum completions of the same classical theory.
As for standard toroidal compactifications, the NSNS sector deformation space includes a coset space $\Gamma\text{modular} \backslash O(d,d)/O(d) \times O(d)$, parametrized by a symmetric $2d \times 2d$ matrix,

$$\mathcal{M}_{MN} = \begin{pmatrix} g_{mn} + b_{mp} b_{qn}^T (L^T)^m p g^{pq} b_{qn} & b_{mp} g^{pq} L_q^n \\ (L^T)^m p g^{pq} b_{qn} & (L^T)^m p g^{pq} L_q^n \end{pmatrix},$$

(4.9)

which we can also write in terms of a vielbein as

$$\mathcal{M} = \mathcal{E}^T \mathcal{E}$$

where

$$\mathcal{E}_N^A = \begin{pmatrix} e^a_n \\ (e^{-1T})_a p b_{pn} \\ (e^{-1T})_a p L_p^n \end{pmatrix},$$

(4.10)

where the lowercase $g_{mn}$ and $b_{mn}$ denote moduli with no $X_{2d}$ coordinate dependence, and where $e^a_n$ is a vielbein for $g_{mn}$. Here, $g_{mn}$ and $b_{mn}$ are not the physical metric and $B$-field, but do parametrize them, as described at the end of Sec. 4.1.6. The Riemannian metric is compatible with the $O(d,d)$ metric in the sense that $\mathcal{E}_A = \mathcal{E}_A^M \mathcal{P}^M$ suffices to put both metrics in unit form,

$$d s_{O(d,d)}^2 = \mathcal{L}_{AB} \mathcal{E}^A \mathcal{E}^B$$

and

$$d s_{\text{Riem}}^2 = \delta_{AB} \mathcal{E}^A \mathcal{E}^B$$

where

$$\mathcal{L}_{AB} = \begin{pmatrix} 0 & \delta_a^b \\ \delta_a^b & 0 \end{pmatrix}.$$

(4.11)

### 4.1.5 Scalar potential and effective field theory

For toroidal compactifications ($t_{MN}^P = 0$), $g_{mn}$ and $b_{mn}$ are exact moduli. More generally, some of these moduli are lifted by a scalar potential $V(\mathcal{M}) = \left(\frac{1}{12} \mathcal{M}^{MQ} \mathcal{M}^{NR} \mathcal{M}^{PS} - \frac{1}{4} \mathcal{M}^{MQ} L^{NR} L^{PS}\right) t_{MN} t_{QRS}.$

(4.12)

This generalizes the more familiar potential due to $H$-flux. We recognize the coefficient of $\mathcal{M}^{MQ}$ in the second term as $-1/4$ times the nonnormalized Killing form of $G_{2d}$

$$\tilde{D}_{MQ} = -t_{MN}^P t_{QP}^N.$$

(4.13)

We will treat the moduli matrix $\mathcal{M}_{MN}$ as a space of (constant) deformation parameters rather than (noncompact-coordinate dependent) low energy fields, though they are of course promoted to fields in the low energy action. In the compactified supergravity theory, the scalar potential might lift some of these moduli, or they might acquire dependence on the noncompact spacetime coordinates (as in cosmological or domain wall solutions). For compactification on the $SU(2)_n$ WZW model studied in this article, the potential lifts some of the moduli, and the linear dilaton compensates for the nonzero vacuum energy. In the supersymmetric context, this is the near horizon $\mathbb{R}^{6,1} \times S^3$ throat geometry of a stack of $n$ NS5-branes. (See App. D.)

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25Depending on context (bosonic vs. type II vs. heterotic), there is a possible additional $LLLtt$ term $40$. However, this term vanishes for the case $G_{2d} = SU(2) \times SU(2)$ of interest in this article.

26The conventionally normalized Killing form is $D_{MQ} = \tilde{D}_{MQ}/(\psi^2 h^N)$, where $\psi^2$ is the length squared of any long root, and $h^N$ is the dual Coxeter number of $G$. (See App. A.1)
4.1.6 Recovery of the physical background

Philosophy

To recover the physical background and standard sigma model in each patch, we would like to eliminate all reference to the dual coordinates $\tilde{x}$ and the dual directions in the tangent and cotangent bundles. When this is possible, the background is not really so different globally from that described by a standard sigma model (defined patchwise on a manifold with $B$-field). In this case, the background is said to be locally geometric. For globally nongeometric compactifications, there are two qualitatively different cases: the tame case, which is locally geometric, and the wild case, in which there is an obstruction to recovering the standard sigma model description even locally \cite{45}. The tame/wild distinction is a polarization-dependent statement, as is the decomposition of the structure constants $t_{MN}^P$ of $g_{2d}$ into $K$, $f$, $Q$, and $R$. As noted in Refs. \cite{47, 12, 53, 44, 28}, and elucidated in this context in Ref. \cite{45}, nonvanishing $R_{mnp}$ obstructs local geometry.

Roughly speaking, a background is locally geometric if on each patch $U \in X_{2d}$ the polarization allows us to canonically define fields $G_{mn}(x, \tilde{x})$ and $B_{mn}(x, \tilde{x})$ that are independent of the dual coordinates $\tilde{x}$. (Otherwise the background is a coherent state that includes winding modes and not just momentum modes). Intuitively, this is the statement that the doubled geometry is fibered over the physical geometry, so there exists a projection from $X_{2d}$ to a physical base $X_d$. An embedding $X_d \hookrightarrow X_{2d}$ wouldn’t do, since we would then expect $G_{mn}(x, \tilde{x})$ and $B_{mn}(x, \tilde{x})$ to depend on the transverse location $\tilde{x}_m$ of the embedding, whereas a geometric background should be independent of dual coordinates. This intuition turns out to be exactly right.

Implementation

When $R = 0$, the choice of polarization defines a projection from doubled to physical geometry

$$\pi: X_{2d} \to X_d, \quad (4.14)$$

at least patchwise. When $Q$ is $\Gamma$-invariant, the projection is globally well defined and the compactification is geometric, otherwise it is only locally geometric.

Global geometry: $R = 0$ and $\Gamma$-invariant $Q$

When $R = 0$, the generators $X^m$ form a closed subalgebra

$$[X^m, X^n] = Q_{nm}^p X^p, \quad (4.15)$$

and corresponding subgroup $\tilde{G}_d \subset \tilde{G}_{2d}$. This suggests that we can quotient $X_{2d}$ by $\tilde{G}_d$ to obtain a physical compactification manifold $X_d$.

$$X_d = \tilde{G}_d \backslash X_{2d}. \quad (4.16)$$

26
If this succeeds, the doubled space $X_{2d}$ is globally a principle $\tilde{G}_d$-bundle over the physical space $X_d$, 

$$\tilde{G}_d \mapsto X_{2d} \to X_d.$$ (4.17)

This is implemented in the doubled sigma model of Hull and Reid-Edwards [45] by gauging the left action of the group $\tilde{G}_d$.

A possible global obstruction is the left coset by $\Gamma$ in the definition $X_{2d} = \Gamma \backslash G_{2d}$. There are two natural conditions one might seek to impose to ensure the persistence of a global projection to physical base after quotienting by $\Gamma$: (i) the $\tilde{G}$-action on $\tilde{G} \backslash G_{2d}$ is well defined, or (ii) the $\tilde{G}$-action on $\Gamma \backslash G_{2d}$ is well defined [27]. Which is the appropriate condition?

Provided there is a global polarization with $R = 0$, the projection exists globally. The group $\tilde{G}_d$ defines such a polarization on $G_{2d}$, since its left action gives a null decomposition of every tangent space into the Lie algebra of $\tilde{G}_d$ and its complement. It also defines a global polarization on the quotient $X_{2d} = \Gamma \backslash G_{2d}$, provided the null tangent subspaces generated by left $\tilde{G}$ action agree at identified points $a$ and $\gamma a$ in $G_{2d}$, where $\gamma \in \Gamma$. For agreement, conjugation by an arbitrary $\gamma \in \Gamma$ must preserve the subgroup $\tilde{G}_d \subset G$. This is condition (i) (see Footnote [27]).

Condition (ii) is too strong. For a global polarization, we require only that the $\Gamma$-quotient preserve the group $\tilde{G}_d$, not the individual elements.

**Local geometry: $R = 0$ and $\Gamma$-noninvariant $Q$**

The $\Gamma$-invariance condition just described guarantees the existence of a *global* projection. When it fails, the $\tilde{G}$ actions on $\Gamma$-identified points do not agree. The Lie algebra of $\tilde{G}_d$ determines a different $d$-dimensional null subspace $T^\text{phys}_p X_{2d} \subset T_p X_{2d}$, depending on which preimage of the point $p \in X_{2d}$ we choose in $G_{2d}$. So, in each contractible open set $U \subset X_{2d}$, $\tilde{G}$ does not define a unique polarization, but up to dim $\Gamma$ of them. We are free to make an arbitrary choice to break this ambiguity. Then, we have a valid polarization in each open set, and well defined projection to physical subspace in each open set. Thus we are able to recover a $d$-dimensional geometry locally but not globally. As in the previous case, the choice of polarization defines a gauging of the doubled sigma model of Hull and Reid-Edwards [45], now patchwise in each open set of $X_{2d}$. The gauging permits patchwise recovery of the standard sigma model.

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27 As noted in Ref. [45], condition (i) requires that for every $\gamma \in \Gamma$ and $\tilde{h} \in \tilde{G}$, $\gamma \tilde{h} \gamma^{-1} = \tilde{h}'$, for some $\tilde{h}' \in \tilde{G}$. For (ii), the roles of $\tilde{G}$ and $\Gamma$ are reversed: for every $\tilde{h} \in \tilde{G}$ and $\gamma \in \Gamma$, $\tilde{h} \gamma \tilde{h}^{-1} = \gamma'$, for some $\gamma' \in \Gamma$. In the latter case, since $\Gamma$ is discrete and $\tilde{G}$ is continuously connected to the identity, deforming $\tilde{g}$ to the identity shows that $\gamma' = \gamma$; thus, $\tilde{h} \gamma \tilde{h} = \gamma$, which shows that $\tilde{h}$ and $\gamma$ commute, i.e., $\Gamma$ lies in the commutant of $\tilde{G}_d$ in $G_{2d}$.
Physical metric and $H$-flux

When $R = 0$, we can locally recover the physical background as follows. The remainder of this subsection is primarily a review of Sec. 2.6 of Ref. [45]. The reader is encouraged to consult Ref. [45] for further details. Given a choice of polarization, and corresponding left invariant frame $(Z_m, X^m)$, an arbitrary element $g \in G_{2d}$ can be written

$$g(x, \tilde{x}) = \tilde{h}(\tilde{x}) h(x), \quad \text{where} \quad \tilde{h}(\tilde{x}) = \exp(\tilde{x}_m X^m) \quad \text{and} \quad h(x) = \exp(x^m Z_m). \quad (4.18)$$

The left invariant 1-forms $P^M$ are the components of the Maurer-Cartan form $\mathcal{P} = P^M T_M = g^{-1}dg$. It is convenient to define a 1-form $\Phi = h^P h^{-1} = \tilde{h}^{-1}d(\tilde{h})h^{-1}$, which is a left invariant with respect to $\tilde{h}$ and right invariant with respect to $h$. Then

$$\mathcal{P}^M = (P^m, P_m) = \mathcal{V}^M_N(x) \Phi^N, \quad (4.19)$$

where $\mathcal{V}^M_N = (\text{Ad}_{h^{-1}(x)})^M_N$ is the adjoint action of $h^{-1}(x)$, and

$$\Phi = dh \ h^{-1} + \tilde{h}^{-1} d\tilde{h}. \quad (4.20)$$

In the $(Z_m, X^m)$ basis, we can expand the Lie valued 1-forms on the right hand side as

$$r = dh \ h^{-1} = r^m Z_m + r_m X^m, \quad (4.21)$$

$$\tilde{\ell} = \tilde{h}^{-1}d\tilde{h} = \tilde{\ell}^m Z_m + \tilde{\ell}_m X^m. \quad (4.22)$$

Thus, $\Phi^M = (p^m, q_m)$, where

$$p^m = r^m(x) + \tilde{\ell}^m(\tilde{x}), \quad (4.23)$$

$$q_m = \tilde{\ell}_m(\tilde{x}) + r_m(x). \quad (4.24)$$

From the Lie algebra (4.6), we see that

$$\tilde{\ell}^m = 0 \quad \text{if} \quad R^{mnp} = 0 \quad \text{and the} \quad X^m \quad \text{close to generate a group} \quad \tilde{G}_d \quad (4.25)$$

$$r_m = 0 \quad \text{if} \quad K_{mnp} = 0 \quad \text{and the} \quad Z_m \quad \text{close to generate a group} \quad G_d. \quad (4.26)$$

The former is the case of interest here. In this case, the physical 1-forms $p^m(x) = r^m(x)$ depend strictly on the physical coordinates $x$, while the second term in $q_m = \tilde{\ell}_m(\tilde{x}) + r_m(x)$ can be thought of as encoding the fibration of the doubled geometry over the physical base (in a coordinate patch).

The Riemannian metric on $X_{2d}$ is

$$ds^2_{\text{Riem}} = \mathcal{M}_{MN} \mathcal{P}^M \mathcal{P}^N = \mathcal{H}_{MN}(x) \Phi^M \Phi^N, \quad (4.27)$$

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28Here, our convention for $\mathcal{V}^M_N(x)$ differs by a transpose from Ref. [45] and agrees with Ref. [64].
where
\[ H_{MN}(x) = M_{PQ} V(x)^P M V(x)^Q N, \] (4.28)
with no \( \mathcal{X}_{2d} \) coordinate dependence in \( M_{PQ} \).

Like \( M_{MN} \), the Riemannian metric \( H_{MN}(x) \) also is symmetric \( O(d, d) \) matrix satisfying
\[ H^T L^{-1} H = L. \] (4.29)

Therefore, it defines a \( d \) dimensional symmetric and antisymmetric tensor fields
\[ G_{mn} = G_{mn}(x) \quad \text{and} \quad B^M_{mn} = B^M_{mn}(x), \] (4.30)
such that
\[ H(x) = \begin{pmatrix} G + (B^M)^T G^{-1} B^M & (B^M)^T G^{-1} L \\ L^T G^{-1} B^M & L^T G^{-1} L \end{pmatrix}. \] (4.31)

That is, we can locally extract from \( H_{MN}(x) \) a metric and \( B \)-field living in the physical subspace of \( (T^* \mathcal{X}_{2d})^2 \) with respect to our choice of polarization:
\[ ds^2_{\text{phys}} = G_{mn} p^m \otimes p^n, \quad B^M = \frac{1}{2} B^M_{mn} p^m \wedge p^n. \] (4.32)

When \( R \neq 0 \), we have \( p^m = p^m(x, \tilde{x}) \), so these fields still functionally depend \( \tilde{x} \), even if their tensorial components lie strictly in the physical directions. However, in the case that \( R = 0 \), the functional dependence on \( \tilde{x} \) drops out, and these fields can be thought of as pullbacks of quantities defined on the physical base \( \mathcal{X}_d = \tilde{G}_d \backslash \mathcal{X}_{2d} \), or at least the patchwise analog of this in the local geometry discussion earlier in this section.

When \( R = 0 \), the field \( G_{mn} p^m p^n = G_{mn} p^m \tau p^n \ d\tau \ dx \ dx^s \) is the physical metric in the local geometric description of the background. The field \( B^M \), while convenient in this framework, is not quite the standard \( B \)-field, but only the moduli dependent part. It determines (the pullback of) the \( H \)-flux via
\[ H = dB^M - \frac{1}{2} d(L_m p^m \wedge \tilde{q}_n) - \frac{1}{2} \mathcal{K}, \] (4.33)
where \( d \) is the exterior derivative on the doubled space \( \mathcal{X}_{2d} \), and \( \mathcal{K} \) was defined in Eq. (4.7).

This expression for \( H \) was derived entirely from the worldsheet description of Hull and Reid-Edwards, in the course of verifying the equivalence of the doubled and standard sigma model descriptions when \( R = 0 \) (cf. Sec. 6.2 of Ref. [45]). The worldsheet description of Hull and Reid-Edwards takes the form of a gauged sigma model on the doubled space, with the polarization determining the particular gauging in each patch.\[ ^{29} \]

\[ ^{29} \text{It would be interesting to rederive this equation from a spacetime point of view, and to provide intuition on why this 3-form has physical components only, i.e., can locally be viewed as a pullback of a 3-form on physical space when } R = 0. \]
4.2 Doubled description of the WZW model for general group

We now apply the formalism of the previous subsection to the doubled description of the level $n$ WZW model for arbitrary group $G_{WZW}$. As we will see, the doubled space is

$$X_{2d} = \Gamma \backslash G_{2d}, \text{ where } G_{2d} = G_1 \times G_2,$$

with $G_1$ and $G_2$ two copies of $G_{WZW}$, associated with the left and right moving worldsheet sectors. The physical space $X_d = G_{phys} \cong G_{WZW}$ is obtained by identifying elements of $G_{2d}$ under the left action of the diagonal subgroup,

$$G_{phys} \cong \tilde{G} \backslash X_{2d} \text{ where } \tilde{G} = G_{diag} \subset G_1 \times G_2.$$

Other choices of $\tilde{G}$ conjugate to $G_{diag}$ give other global polarizations. Up to subtleties involving $\Gamma$, the doubled worldsheet theory of Hull and Reid-Edwards is a gauged WZW model with group $G_1 \times G_2$ and gauge group $G_{diag}$.

At level $n$, the doubled metrics in the chiral basis are

$$ds^2_{O(d,d)} = -\frac{n}{2} \text{tr}'(\lambda_2 \lambda_2 - \lambda_1 \lambda_1) = \hat{n} \frac{d_{mn}}{2} (\lambda_2^m \lambda_2^n - \lambda_1^m \lambda_1^n),$$

(4.34)

$$ds^2_{Riem} = -\frac{n}{2} \text{tr}'(\lambda_2 \lambda_2 + \lambda_1 \lambda_1) = \frac{\hat{n}}{2} d_{mn} (\lambda_2^m \lambda_2^n + \lambda_1^m \lambda_1^n),$$

(4.35)

where $\lambda_i = g_{i}^{-1} dg_i$ is the left-invariant Maurer-Cartan form on $G_{i}^{WZW}$, and $\hat{n} = n \psi^2 / 2$. (See App. [3] for Lie Algebra conventions.) The second line assumes the WZW point in moduli space $g_{mn} = \frac{1}{2} \hat{n} d_{mn}$ and $b_{mn} = 0$. The formalism of Sec. 4 gives the metric at a general point in moduli space, including would-be moduli that are lifted by the potential. The global horizontal and vertical forms adapted to the $\tilde{G}$ fibration are given in Eq. (4.85), and in this basis,

$$ds^2_{Riem} = \frac{\hat{n}}{2} d_{mn} \left( \frac{r^2}{n} \lambda_{phys}^m \lambda_{phys}^n + \frac{n}{r^2} \omega^m \omega^n \right),$$

(4.36)

when the overall radial modulus is allowed to vary away from $n$. For generic moduli $g_{mn}$ not proportional to $d_{mn}$, the Riemannian metric in this basis is position dependent.

We show that the local procedure described in Sec. 4.1.6 indeed gives the correct physical metric (2.12) and $H$-flux (2.14) when expressed in terms of global 1-forms. Allowing only the overall volume modulus to vary, we review the worldsheet arguments for $r^2 = n$ and show that the effective potential (4.12) indeed stabilizes the radial modulus to this value. In addition to the standard polarization with $\tilde{G} = G_{diag}$, other polarizations are obtained by conjugation, $\tilde{G} = b - 1 G_{diag} b$, and we relate the abelian T-duality group to a restricted subgroup of $b$. These are closely related to the spectrum of D-branes of the theory, as we explain. Finally, in Sec. 4.2.8 we discuss the choice of discrete group $\Gamma$, tentatively identifying $\Gamma/C_{diag}$ with $\mathbb{Z}_n$ at level $n$, where $C$ is the center of $G_{WZW}$. 

30
4.2.1 Symmetries

First consider $G^{WZW} = SU(2)$. The symmetry of the physical target space $SU(2)_{\text{phys}} \cong S^3$ is $O(4)$. The subgroup $SO(4) \subset O(4)$ is realized as left and right multiplication on points $g_{\text{phys}} \in SU(2)_{\text{phys}}$:

$$g_{\text{phys}} \mapsto \Omega_1 g_{\text{phys}} \Omega_2^{-1}, \quad \text{where} \quad \Omega_1 \in SU(2)_1 \quad \text{and} \quad \Omega_2 \in SU(2)_2.$$  \hspace{1cm} (4.37)

Since the center $-1 \in SU(2)_i$ gives the same action $k \mapsto -k$ for left or right multiplication, we have $SO(4) \cong \mathbb{Z}_2^{\text{diag}} \setminus (SU(2)_1 \times SU(2)_2)$, where $\mathbb{Z}_2^{\text{diag}}$ is generated by $(-1, -1)$, and we have chosen for later convenience to write the quotient on the left.\footnote{A different $\mathbb{Z}_2$, orientation reversal $k \mapsto k^{-1}$ of $SU(2)$, promotes $SO(4)$ to $O(4)$. This $\mathbb{Z}_2$ interchanges $SU(2)_1$ and $SU(2)_2$. For the oriented string, this is not a symmetry of the theory.} This is the gauge group in the spacetime description of the $SU(2)$ WZW model at level $n$. The $SU(2)_i$ subgroups for $i = 1$ and 2 are associated with currents in the left and right moving worldsheet sectors. We use subscripts 1 and 2 rather than $L$ and $R$ to avoid confusion with the left and right $G_6$-action on the doubled space $X_6$, which does not correspond to left and right worldsheet sector.

Analogous statements hold when $SU(2)$ is generalized to an arbitrary Lie group $G^{WZW}$. In this case $\mathbb{Z}_2^{\text{diag}}$ becomes $C^{\text{diag}}$, where $C$ is the center of $G^{WZW}$. The physical gauge group from left and right multiplication of points $g_{\text{phys}} \in G^{WZW}_{\text{phys}}$ is $C^{\text{diag}} \setminus (G_1 \times G_2)$.

4.2.2 The doubled space

The fully doubled space $X_{2d}$ must form a continuous representation of the gauge group $C^{\text{diag}} \setminus (G_1 \times G_2)$, where $G_1$ and $G_2$ are two copies of $G^{WZW}$. Therefore, in the notation of the previous section, we expect that

$$X_{2d} = \Gamma \backslash G_{2d} \quad \text{where} \quad G_{2d} = G_1 \times G_2,$$ \hspace{1cm} (4.38)

where $\Gamma \subset G_{2d}$ is a discrete subgroup. For simplicity of exposition, we will assume trivial $\Gamma$, and consider the modification (and motivation) for more general $\Gamma$ in Sec. 4.2.8.

Let us write

$$g = (g_1, g_2) \quad \text{for} \quad g \in G_{2d} = G_1 \times G_2,$$ \hspace{1cm} (4.39)

and choose the standard product group composition law on $G_{2d}$,

$$g \circ g' = (g_1, g_2) \circ (g'_1, g'_2) = (g_1 g'_1, g_2 g'_2).$$ \hspace{1cm} (4.40)

Right $G_{2d}$-action on $X_{2d} = G_{2d}$

$$g \mapsto g\Omega^{-1} = (g_1 \Omega_1^{-1}, g_2 \Omega_2^{-1})$$ \hspace{1cm} (4.41)
should project to the gauge action
\[ g_{\text{phys}} \mapsto \Omega_1 g_{\text{phys}} \Omega_2^{-1}, \quad g_{\text{phys}} \in G_{\text{phys}}^{\text{WZW}}, \]
on the physical space. This implies that the projection from doubled to physical space is
\[ \pi: \mathcal{G}_{2d} \to G_{\text{phys}}^{\text{WZW}}, \quad g = (g_1, g_2) \mapsto g_{\text{phys}} = g_1^{-1} g_2. \]

To recover the physical geometry \( X_d = G_{\text{phys}}^{\text{WZW}} \) from the doubled geometry \( \mathcal{G}_{2d} \), we can quotient \( \mathcal{G}_{2d} \) by the left action of the diagonal subgroup \( \tilde{G} = G_{\text{diag}} \subset G_1 \times G_2 \), which leaves \( g_{\text{phys}} = g_1^{-1} g_2 \) invariant. Here, \( G_{\text{diag}} \) consists of elements \((\tilde{h}, \tilde{h})\) \( G_{\text{2d}} \). Thus,
\[ G_{\text{phys}}^{\text{WZW}} = G_{\text{diag}} \backslash (G_1 \times G_2). \]

As noted in App. \[\text{B} \], the general solution to the classical equations of motion of the WZW model is
\[ g_{\text{phys}}(z, \bar{z}) = g_L(z) g_R(\bar{z}) \quad \text{(on-shell)}. \]
Comparing to Eq. \[\text{(4.43)} \], we obtain the on-shell identifications
\[ g_1^{-1}(z, \bar{z}) = g_L(\bar{z}), \quad g_2(z, \bar{z}) = g_R(z), \quad \text{(on-shell)}. \]
The doubled sigma model of Ref. \[\text{[45]} \], promotes \( g_L(z) \) and \( g_R(\bar{z}) \) to fields \( g_1^{-1}(z, \bar{z}) \) and \( g_2(z, \bar{z}) \), each of which has the full \( z, \bar{z} \) dependence, thus doubling the target space from \( G_{\text{phys}}^{\text{WZW}} \) to \( \mathcal{G}_{2d} = G_1 \times G_2 \). The choice of polarization in each patch of \( \mathcal{G}_{2d} \) defines a gauging of the doubled sigma model. In the case at hand, the gauge identifications precisely implement the left quotient by \( \tilde{G} = G_{\text{diag}} \) globally, so the formalism of Hull and Reid-Edwards indeed reproduces the standard physical sigma model.

### 4.2.3 Recovery of the physical WZW background

**Embeddings and local trivialization**

Consider an arbitrary simply connected group \( G^{\text{WZW}} \), with doubled group \( \mathcal{G}_{2d} = G_1 \times G_2 \) with the group composition law \[\text{(4.40)}. \] In addition to the standard left and right embeddings, it is useful to define submanifolds \( \tilde{G}, G \subset \mathcal{G}_{2d} \) composed of diagonal and anti-diagonal diagonal elements, \((a, a)\) and \((a^{-1}, a)\), respectively. Both map bijectively to \( G^{\text{WZW}} \), via
\[ \tilde{\iota}: \quad G^{\text{WZW}} \to \tilde{G}, \quad a \mapsto (a, a), \quad \text{(4.47)} \]
\[ \iota: \quad G^{\text{WZW}} \to G, \quad a \mapsto (a^{-1}, a). \quad \text{(4.48)} \]

Under the \( G \) group composition law \[\text{(4.40)} \], the former closes and the latter does not:

On \( \tilde{G} \):
\[ (a, a) \circ (b, b) = (ab, ab), \]

On \( G \):
\[ (a^{-1}, a) \circ (b^{-1}, b) = (a^{-1} b^{-1}, ab) \neq ((ab)^{-1}, ab). \]
Thus, \( \tilde{G} = G_{\text{diag}} \subset G \) is a subgroup of \( G \) isomorphic to \( G^{WZW} \), whereas \( G \) is not as subgroup, though topologically both are embeddings \( G^{WZW} \hookrightarrow G_{2d} \).

Following Sec. 4.1.6, we write an arbitrary element of \( G_{2d} \) as

\[
\mathbf{g}(x, \tilde{x}) = \mathbf{h}(\tilde{x}) \circ \mathbf{h}(x) = (\tilde{h}h^{-1}, \tilde{h}h). \tag{4.49}
\]

Here,

\[
\tilde{h} = \iota(\tilde{h}) = (\tilde{h}, \tilde{h}) \in \tilde{G} \quad \text{and} \quad h = \iota(h) = (h^{-1}, h) \in G, \tag{4.50}
\]

where \( \tilde{h}, h \in G^{WZW} \), parametrized by coordinates \( \tilde{x} \) and \( x \), respectively.

The projection

\[
\pi : G_{2d} \rightarrow G^{WZW}_{\text{phys}}, \quad \text{mapping} \quad \mathbf{g} = (g_1, g_2) \mapsto g_{\text{phys}} = g_1^{-1}g_2 = h^2, \tag{4.51}
\]

gives \( G \) the structure of a \( G^{WZW} \) fibration over \( G^{WZW} \). This is the map from doubled group \( G_{2d} = G_1 \times G_2 \) to the physical group \( G^{WZW}_{\text{phys}} \) of the standard WZW model. The map

\[
\phi = \iota \circ \iota : G^{WZW} \times G^{WZW} \rightarrow G, \quad h, \tilde{h} \mapsto (\tilde{h}h^{-1}, \tilde{h}h), \tag{4.52}
\]

gives a local trivialization of \( G \) as a \( G^{WZW} \) fibration, where we view \( \tilde{h}(\tilde{x}) \) as the fiber coordinate and \( h(x) \) as the base coordinate. It is surjective, but not one-to-one. However, the many-to-oneness is entirely due to using \( h \) rather than \( g_{\text{phys}} = h^2 \) to parametrize the base.

### Polarization and \( O(d,d) \) metric

To recover the physical WZW background, we must first choose a polarization on \( X_{2d} \) and specify the \( O(d,d) \) metric \( \left[ 13 \right] \). The polarization has already been implicitly defined by Eq. (4.49). We now make this more explicit. In the present context, it is convenient to write

\[
T_M = (Z_m, X_m), \quad x^M = (x^m, \tilde{x}^m) \tag{4.53}
\]

without flipping the index placement for the dual generators and coordinates. A chiral basis \( \left[ 32 \right] \) for \( g_{2d} \) is \( T^1_m = (t_m, 0) \) and \( T^2_m = (0, t_m) \), where the \( t_m \) generate the Lie algebra \( (2.10) \) of \( G^{WZW} \). In terms of these generators, the polarization is specified by writing

\[
-T^1_m = Z_m - X_m \quad \text{left-moving worldsheet sector}, \tag{4.54}
\]

\[
T^2_m = Z_m + X_m \quad \text{right-moving worldsheet sector}. \tag{4.55}
\]

---

\(^{31}\) The product group of the present section presents a notational inconvenience. To avoid a excess of bold type earlier in the paper, \( h, \tilde{h} \) and \( g \) of Sec. 4.1.6 correspond to \( \mathbf{h}, \tilde{\mathbf{h}} \) and \( \mathbf{g} \) here.

\(^{32}\) Equivalently, \( \sigma : g_{\text{phys}} \mapsto (h, h) = (g_{\text{phys}}^{1/2}, g_{\text{phys}}^{1/2}) \) gives a local section, and then a generic element of \( G_{2d} \) can be written \( \mathbf{g} = \overline{\iota}(h) \circ \sigma(g_{\text{phys}}) \). The section is not global, since the function \( g_{\text{phys}}^{1/2} \) can be defined locally but not globally on \( G_{\text{phys}} \). In contrast, the left and right embeddings \( \sigma_L : g_{\text{phys}} \mapsto (g_{\text{phys}}^{-1}, 1) \) and \( \sigma_R : g_{\text{phys}} \mapsto (1, g_{\text{phys}}) \) do give global sections, in terms of which the generic element of \( G_{2d} \) can be written \( \mathbf{g} = \overline{\iota}(g_2) \circ \sigma_L(g_{\text{phys}}) \) and \( \mathbf{g} = \overline{\iota}(g_1) \circ \sigma_R(g_{\text{phys}}) \) respectively.

\(^{33}\) Recall that \( g_L = g_1^{-1} \) and \( g_R = g_2 \) from Eq. (4.49). Thus the generators in the left-moving and right-moving worldsheet sectors are \( -T_1 \) and \( T_2 \), respectively.
Then,
\[
\begin{align*}
\tilde{h}(x) &= \exp(\bar{x}^m X_m), \quad \bar{h}(\bar{x}) = \exp\left(\frac{1}{2}\bar{x}^m t_m\right), \\
h(x) &= \exp(x^m Z_m), \quad h(x) = \exp\left(\frac{1}{2}x^m t_m\right), \quad \text{and} \quad g_{\text{phys}} = h^2 = \exp(x^m t_m).
\end{align*}
\] (4.56)

With these definitions, the Lie algebra of \(G_{2d}\) in the \((T_1, T_2)\) basis is
\[
[\tilde{T}_1, \tilde{T}_1] = c_{mn} p T_p, \quad [\tilde{T}_2, \tilde{T}_2] = c_{mn} p T_p, \quad [\tilde{T}_1, \tilde{T}_2] = 0.
\] (4.58)

In the \((Z, X)\) basis, the Lie algebra is of the general form (4.6), with the simpler index placement
\[
\begin{align*}
[Z_m, Z_n] &= K_{mn} p X_p + f_{mn} p Z_p, \\
[Z_m, X_n] &= f_{mn} p X_p + Q_{mn} p Z_p, \\
[X_m, X_n] &= Q_{mn} p X_p + R_{mn} p Z_p.
\end{align*}
\] (4.59)

From Eqs. (4.54) and (4.58), we find
\[
K_{mn} p = Q_{mn} p = \frac{1}{2} c_{mn} p, \quad f_{mn} p = R_{mn} p = 0.
\] (4.60)

The root lattice of \(G\) is the vector sum of the root lattices of \(G_1\) and \(G_2\). Thus, the nonnormalized Killing form on \(g_{2d}\) in this basis is
\[
\tilde{D}_{MN} = \text{diag}(\tilde{d}_{mn}, \tilde{d}_{mn}) \quad ((T_1, T_2)\text{ basis}).
\] (4.61)

The dual Coxeter numbers of all three algebras is the same, \(h^\vee (g_{2d}) = h^\vee (g_1) = h^\vee (g_2)\), and we choose the natural convention in which the length squared of long roots \(\psi^2\) is the same as well. Then, the normalized Killing form \(D_{mn} = h^\vee \psi^2 \tilde{D}_{mn}\) similarly satisfies
\[
D_{MN} = \text{diag}(d_{mn}, d_{mn}) \quad ((T_1, T_2)\text{ basis}).
\] (4.62)

In the \((Z, X)\) basis, these last two equations become
\[
\begin{align*}
\tilde{D}_{MN} &= \frac{1}{2} \text{diag}(\tilde{d}_{mn}, \tilde{d}_{mn}) \quad ((Z, X)\text{ basis}), \\
D_{MN} &= \frac{1}{2} \text{diag}(d_{mn}, d_{mn}) \quad ((Z, X)\text{ basis}).
\end{align*}
\] (4.63) (4.64)

The \(O(d, d)\) metric in the \((Z, X)\) basis is given by Eq. (4.3), with
\[
L_{MN} = \left( \begin{array}{cc} 0 & L_{mn} \\ L_{mn}^T & 0 \end{array} \right), \quad \text{where} \quad L_{mn} = \frac{1}{4} \psi^2 d_{mn} = \frac{1}{2} \hat{n} d_{mn}.
\] (4.65)

This choice is unique, since \(\frac{1}{2} \hat{n} d_{mn}\) is the only quantity with the correct index structure that appears in the definition (B.12) of the physical WZW model. The flux (2.14) \(H_{mnp} = \frac{1}{2} \hat{n} d_{pq} c_{mn} q\) and (on-shell) metric (2.12) \(G_{mn} = \frac{1}{2} \hat{n} d_{mn}\) are determined solely by the tensors \(\frac{1}{2} \hat{n} d_{mn}\) and \(c_{mn}^p\) on the group manifold \(G^{\text{WZW}}\).
Doubled metrics and local fibration structure

We now perform the local analysis of Sec. 4.1.6 to obtain the local horizontal and vertical 1-forms $p^m$ and $\tilde{q}_m$ and the doubled Riemannian and $O(d,d)$ metrics in these coordinates. When the radial modulus is stabilized to $r = n$, we show that these give the quoted results (4.34).

The $G$ left-invariant 1-form is

$$\mathcal{P} = g^{-1} dg = (\tilde{h} h^{-1} , \tilde{h} h)^{-1} \circ (d(\tilde{h} h^{-1}) , d(\tilde{h} h))$$

(4.66)

where $\iota h = (h^{-1} , h)$, and $\lambda , \rho , \tilde{\lambda} , \tilde{\rho}$ are the invariant 1-forms constructed from $h(x) \in G^{WZW}$,

$$\lambda(x) = h^{-1} dh, \quad \rho(x) = dh h^{-1}, \quad \tilde{\lambda}(\tilde{x}) = \tilde{h}^{-1} d\tilde{h}, \quad \tilde{\rho}(\tilde{x}) = d\tilde{h} h^{-1}. \quad (4.67)$$

Thus,

$$\Phi = (\tilde{\lambda} - \lambda, \tilde{\lambda} + \rho) = (\tilde{\lambda} - \lambda)^m T^1_m + (\tilde{\lambda} + \rho)^m T^2_m \quad \text{and} \quad \mathcal{V} = \text{Ad}_{h^{-1}}, \quad (4.68)$$

in the notation of Sec. 4.1.6 In the $(Z_m, X_m)$ basis,

$$\Phi = p^m Z_m + \tilde{q}^m X_m, \quad (4.69)$$

where

$$p^m = - (\tilde{\lambda} - \lambda)^m + (\tilde{\lambda} + \rho)^m = (\lambda + \rho)^m, \quad \tilde{q}^m = (\tilde{\lambda} - \lambda)^m + (\tilde{\lambda} + \rho)^m = (2\tilde{\lambda} + \rho - \lambda)^m. \quad (4.70)$$

The doubled Riemannian metric was given in terms of the metric and $B$-field moduli $g_{mn}$ and $b_{mn}$ in Eq. (4.8). For simplicity, let us set $b_{mn} = 0$ and allow only the overall radial modulus to vary

$$b_{mn} = 0, \quad g_{mn} = \frac{1}{4} r^2 \psi^2 d_{mn}. \quad (4.71)$$

Then, Eq. (4.8) becomes

$$ds^2_{\text{Riem}} = \frac{r^2 \psi^2}{4} d_{mn} p^m p^n + \frac{4}{r^2 \psi^2} d_{mn} \left( \frac{1}{4} n \psi^2 d_{mn} \tilde{q}^p \right) \left( \frac{1}{4} n \psi^2 d_{mn} \tilde{q}^q \right)$$

$$= \frac{\hat{n}}{2} d_{mn} \left( \frac{r^2}{n} p^m p^n + \frac{n}{r^2} q^m q^n \right), \quad \hat{n} = \frac{1}{2} \psi^2 n. \quad (4.72)$$

As we will see below, the modulus $r^2$ is stabilized to $n$. Therefore,

$$ds^2_{\text{Riem}} = \frac{\hat{n}}{2} d_{mn} \left( p^m p^n + q^m q^n \right) \quad = - \frac{n}{4} \text{tr}'(pp + qq) \quad (r^2 = n), \quad (4.73)$$

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where \( p = p^m t_m \) and \( q = q^m t_m \) in terms of the generators \( t_m \) of \( G^{WZW} \). For the choice of moduli (4.71), the \( O(d,d) \) metric becomes

\[
ds_{O(d,d)}^2 = \hat{\eta} d m p^m q^n = -\frac{\hat{\eta}}{2} tr'(pq),
\]

(4.74)

Observing that \( \mathcal{P} = (g_1^{-1}dg_1, g_2^{-1}g_2) = (\lambda_1, \lambda_2) \), and comparing Eqs. (4.66) and (4.68), we see that

\[
p = -h^{-1} \lambda_1 h + h \lambda_2 h^{-1} \quad \text{and} \quad q = h^{-1} \lambda_1 h + h \lambda_2 h^{-1},
\]

(4.75)

which, by the cyclic property of the trace, gives

\[
ds_{Riem}^2 = -\frac{n}{2} tr' \left( \lambda_1 \lambda_1 + \lambda_2 \lambda_2 \right) \quad (r^2 = n),
\]

\[
ds_{O(d,d)}^2 = \frac{n}{2} tr' \left( \lambda_1 \lambda_1 - \lambda_2 \lambda_2 \right) \quad (r \text{ arbitrary}),
\]

as claimed in Eq. (4.34).

Following Hull and Reid-Edwards, we would like to interpret \( p^m \) and \( \tilde{q}_m \) as local horizontal and vertical 1-forms on \( G_{2d} \), viewed as a fibration of \( G_{\text{diag}} \) over \( G_{\text{phys}} \). Let us examine these forms. In the notation of Sec. 4.1.6, we have

\[
r = (\rho^m + \lambda^m) Z_m + (\rho^m - \lambda^m) X_m \quad \text{and} \quad \tilde{\ell} = 2\tilde{\lambda}^m X_m,
\]

(4.76)

so indeed

\[
p^m = r^m_Z = \rho^m + \lambda^m,
\]

(4.77)

\[
\tilde{q}^m = \tilde{\ell}^m_X + r^m_X = (2\lambda + \rho - \lambda)^m,
\]

(4.78)

where, in terms of the structure constants \( c_{mn}^p \) of \( G_{WZW} \),

\[
d\lambda^p + \frac{1}{2} c_{mn}^p \lambda^n \wedge \lambda^m = 0, \quad d\rho^p - \frac{1}{2} c_{mn}^p \rho^m \wedge \rho^n = 0.
\]

(4.79)

**Interpretation of \( p \)**

The projection \( \pi \) takes \( g = (g_1, g_2) \) to \( g_{\text{phys}} = g_1^{-1}g_2 = h^2 \), so the left invariant 1-form on the physical base \( G_{\text{phys}} \cong G_{WZW} \) is

\[
\lambda_{\text{phys}} = g_{\text{phys}}^{-1}dg_{\text{phys}} = h^{-2}d(h^2)
\]

\[
= h^{-2}(dhh + hdh) = h^{-1}(h^{-1}dh + dh h^{-1})h
\]

\[
= h^{-1} ph.
\]

(4.80)

Thus, \( p \) is just the left invariant global 1-form \( \lambda_{\text{phys}} \) on the physical base \( G_{\text{phys}} \), up to the adjoint action of \( h \).

---

34In the end, only \( g_{\text{phys}} = h^2 \) and not \( h \) should appear in the physical metric. The factor of \( \text{Ad}_h \) in \( p^m = (\text{Ad}_h)^n \lambda^n \) combines with similar factors in the vielbein \( V = \text{Ad}_h \) to leave a result for \( H \) that only depends on \( h^2 \) for all values of the moduli \( g_{mn} \) and \( b_{mn} \). This can also be seen from \( ds_H^2 = M_{MN} P^m P^N \) with \( P = \frac{1}{2} (g_{\text{phys}}(\omega - \lambda_{\text{phys}}) g_{\text{phys}}^{-1}, \omega + \lambda_{\text{phys}}) \).
Interpretation of $q$

Likewise the global vertical 1-form

$$\omega = h^{-1}qh = 2\lambda_2 - \lambda_{\text{phys}}$$

defines the curvature 2-form $\Omega = d\omega + \omega \wedge \omega$ of the fibration, which can be shown to be

$$\Omega = \frac{1}{2}(\omega - \lambda_{\text{phys}}) \wedge (\omega - \lambda_{\text{phys}}).$$

The forms $\omega$ and $\Omega$ on $G$ are related to the local potential $A$ and field strength $F$ on the base $G_{\text{phys}}$, via

$$A = \sigma^* \omega, \quad F = \sigma^* \Omega,$$

where $\sigma$ is the corresponding choice of section. For the local analysis of Sec. 4.1.6, the choice of local section is $\sigma: G_{\text{phys}} \rightarrow G$, $h^2 \mapsto \iota(h) = (h^{-1}, h)$.

Note that the $O(d, d)$ metric expressed in terms of $p, q$ in Eq. (4.74) takes the same form when expressed in terms of $\lambda_{\text{phys}}, \omega$,

$$ds^2_{O(d,d)} = \hat{n} d_{mn} \lambda^m_{\text{phys}} \lambda^n_{\text{phys}} = -\frac{\hat{n}}{2} \text{tr}'(\lambda_{\text{phys}} \omega).$$

The change of variables $(\lambda_{\text{phys}}, \omega) = h^{-1}(p, q)h$ is a local transformation in $G_{WZW}/C_{WZW} \subset O(d) \subset O(d, d)$, where $C_{WZW}$ is the center of $G_{WZW}$. In terms of the chiral basis,

$$\lambda_{\text{phys}} = \lambda_2 - g_{\text{phys}}^{-1} \lambda_1 g_{\text{phys}} \quad \text{and} \quad \omega = \lambda_2 + g_{\text{phys}}^{-1} \lambda_1 g_{\text{phys}}, \quad g_{\text{phys}} = g_1^{-1} g_2.$$

Global recovery of group metric and $H$-flux

We now verify that the local analysis of Sec. 4.1.6 when expressed in terms of the global horizontal and vertical 1-forms $\lambda_{\text{phys}}$ and $\omega$, gives the correct group metric and $H$-flux on the physical space $G_{WZW}$. From Eqs. (4.32) and (4.33) with appropriately modified index placement, the physical metric and $H$-flux are given by

$$ds^2_{\text{phys}} = G_{mn} p^m p^n, \quad H = dB^M - \frac{1}{2} d(L_{mn} p^m \wedge q^n) + \frac{1}{2} \mathcal{K}.$$ (4.86)

For the choice of moduli (4.71) above, this metric indeed reproduces Eq. (2.12),

$$ds^2_{\text{phys}} = \frac{1}{4} r^2 \psi^2 d_{mn} \lambda^m_{\text{phys}} \lambda^n_{\text{phys}} = \frac{1}{4} r^2 \text{tr}'(\lambda_{\text{phys}} \lambda_{\text{phys}}).$$

(Conjugation of $\lambda_{\text{phys}}$ by $h$ leaves the trace invariant and replaces $\lambda_{\text{phys}}$ by $p$ in Eq. 4.88.)

It is shown in App. E that Eq. (4.87) also reproduces the $H$-flux (2.14),

$$H = \frac{\hat{n}}{12} c_{mpn} \lambda^m_{\text{phys}} \wedge \lambda^n_{\text{phys}} \wedge \lambda^p_{\text{phys}} = -\frac{n}{12} \text{Tr}'(\lambda_{\text{phys}} \wedge \lambda_{\text{phys}} \wedge \lambda_{\text{phys}}).$$

Therefore, for the simple choice of moduli above, the doubled description of Hull and Reid-Edwards correctly reproduces the physical sigma model background for any WZW model.
4.2.4 Stabilization of radial modulus

Worldsheet sigma model description at large radius

For definiteness, this section treats the bosonic string, however, an analogous discussion can be given for the common NSNS section in the supersymmetric case. The worldsheet description of the bosonic $SU(2)$ WZW model is given in App. B. The theory is conformal provided the beta functions of the sigma model vanish. At large radius, it suffices to work to first order in $\alpha'$. Allowing only the overall volume modulus of the $S^3$ to vary, the first order beta functions are (cf. Eq. (15.4.28) of Ref. [62], with $q = 2n\alpha'$)

$$\beta^G_{mn} = 2G_{mn} \left( \frac{1}{r^2} - \frac{n^2}{r^6} \right), \quad (4.90a)$$

$$\beta^\Phi = \frac{1}{2} - \frac{n^2}{r^6}, \quad (4.90b)$$

where $H = 2n\omega_{S^3}$. Here, $n \in \mathbb{Z}$ from flux quantization,

$$\frac{1}{2\pi} \int_{S^3} H = 2\pi n \in 2\pi \mathbb{Z} \quad (2\pi^2 = \text{Volume of } S^3), \quad (4.91)$$

so solving $\beta^G_{mn} = 0$ determines the radius $r^2 = n$. The deficit in central charge from $c = 6\beta^\Phi < 3$ is compensated by a surplus in the noncompact dimensions from a linear dilaton in the radial direction. (Here, we have in mind the near horizon “throat” geometry $\mathbb{R}^{6,1} \times S^3$ of a stack of $n$ NS 5-branes. See App. D.)

What enters into the above beta functions on $S^3$ is the metric and $H$-flux only, since there is no dilaton profile on $S^3$. For the full bosonic string theory, we have, in general, to first order in $\alpha'$ (cf. Eq. (3.7.14) of Ref. [61]),

$$\beta^G_{MN} = R_{MN} + 2\nabla_M \nabla_N \Phi - \frac{1}{4} H_{MPQ} H_{N}^{PQ}, \quad (4.92a)$$

$$\beta^B_{MN} = -\frac{1}{2} e^{2\Phi} \nabla^P \left( e^{-2\Phi} H_{PMN} \right), \quad (4.92b)$$

$$\beta^\Phi = \frac{(D - 26)}{6\alpha'} - \frac{1}{2} \nabla^2 \Phi + (\partial \Phi)^2 - \frac{1}{4 \cdot 3!} H_{MNP} H^{MNP}. \quad (4.92c)$$

Setting $R_{mn} = (2/r^2)G_{mn}$, and $\frac{1}{2} H_{mpq} H_{n}^{pq} = 4n^2 G_{mn}/r^6$ on $S^3$ gives the quoted $\beta^G_{mn}$ in Eq. (4.90). Setting $\frac{1}{3} H_{mnp} H^{mnp} = 4n^2/r^6$ and including only the $S^3$ contribution of 3 to $(D - 26)$, we obtain the quoted $\beta^\Phi$.

A similar analysis can be performed when $SU(2)$ is replaced by an arbitrary group $G^{WZW}$, since the assumption of a homogeneous space allows a corresponding simplification of the $\beta$-function equations. We will not provide that analysis here, however, the result is that the first equation of (4.90) is unchanged, and the second is multiplied by $\text{dim}(G^{WZW})/3$. 

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Exact CFT description

As shown in App. B, when \( r^2 = n \) the WZW model possesses chirally conserved currents on the worldsheet. Upon quantization, these currents generate a level \( n \) affine \( SU(2) \) algebra, with respect to which the WZW model is a Sugawara model, whose Virasoro algebra is constructed entirely from the currents. This allows for many exact statements, including the all \( \alpha' \) order central charge (B.33), which agrees with the sigma model analysis above to first order in \( \alpha' \), or equivalently, first order in the \( 1/n \) expansion.

Effective field theory description

The scalar potential in Hull’s doubled formalism was given in Eq. (4.12) and gives

\[
0 = 4 \frac{\partial V(M)}{\partial M^{MQ}} = \left( M^{NR} M^{PS} - L^{NR} L^{PS} \right) t_{MNP} t_{QRS} = M^{NR} M^{PS} t_{MNP} t_{QRS} - \tilde{D}_{MQ},
\]

(4.93)

Here, \( M^{MN} \) is the inverse moduli matrix

\[
M^{MN} = \begin{pmatrix} g^{-1} & g^{-1}bL^{-1T} \\ L^{-1}b^Tg^{-1} & L^{-1}(g + b^Tg^{-1}b)L^{-1T} \end{pmatrix},
\]

(4.94)

with \( L_{mn} \) given by Eq. (4.65); \( \tilde{D}_{MQ} \) is the nonnormalized Killing form, given by (4.63) in the \((Z_m, X_m)\) basis.

We focus on the simplified case (4.71) that only the radial modulus is allowed to vary: \( g_{mn} = \frac{1}{4} r^2 \psi^2 d_{mn} \) and \( b_{mn} = 0 \). This gives

\[
M^{MN} = \frac{2}{n} \text{diag}(m^{mn}, \tilde{m}^{mn}), \quad \text{where} \quad m^{mn} = (n/r^2)d_{mn}, \quad \tilde{m}^{mn} = (r^2/n)d_{mn}.
\]

(4.95)

From the structure constants \( t^{MNP} \) of Eq. (4.60), we find the following nonzero \( t_{MNP} \):

\[
K_{mnp} = Q_{mnp} = \hat{n} c_{mnp}, \quad \text{where} \quad c_{mnp} = d_{mq} c_{np}.
\]

(4.96)

Then, using \( c_{mp} q c_{mq}^p = -\tilde{d}_{mn} \), the nontrivial components of Eq. (4.93) are the upper left block

\[
0 = m^{nr} m^{ps} K_{mnp} K_{qrs} + \tilde{m}^{nr} \tilde{m}^{ps} Q_{mnp} Q_{qrs} - \frac{1}{2} \tilde{d}_{mq} = \frac{1}{4} ((n/r^2)^2 + (r^2/n)^2 - 2) \tilde{d}_{mq} = \frac{1}{4} (n/r^2 - r^2/n)^2 \tilde{d}_{mq},
\]

(4.97)
and lower right block

\[ 0 = m^r \tilde{m}^s Q_{pmn} Q_{sqr} + \tilde{m}^r m^s Q_{pnm} Q_{sqm} - \frac{1}{2} \tilde{d}_{mq} \]

\[ = \frac{1}{4} (1 + 1 - 2) \tilde{d}_{mq} = 0, \]

with off-diagonal blocks vanishing identically. Equality holds Eq. (4.97) when

\[ r^2 = n, \]

as desired.

### 4.2.5 Other global polarizations

We noted in Sec. 4.1.6 that any maximal isotropic subgroup \( \tilde{G}_d \subset G_{2d} \), i.e., any dimension \( d \) subgroup that is null with respect to the \( O(d,d) \) metric, defines a global polarization on \( G_{2d} \). In addition to the diagonal subgroup of \( G_{2d} \), any conjugate group \( \tilde{G}_b = b G_{\text{diag}} b^{-1} \)

satisfies this criterion. Note that the same group is obtained from any \( b = (b_1, b_2) \) that differ only by right multiplication by an element of \( G_{\text{diag}} \). Therefore \( G_b \) is determined by \( \pi(b^{-1}) = b_1 b_2^{-1} \). Let us define a projection

\[ \pi_b : G_{2d} \to G_{\text{phys}}^{\text{WZW}}, \text{ mapping } g = (g_1, g_2) \mapsto g_{\text{phys}} = (b_1^{-1} g_1^{-1} b_1)(b_2^{-1} g_2 b_2). \]

In this case, we have the (on-shell) identifications

\[ g_L(z) = b_1^{-1} g_1^{-1} b_1, \quad g_R(\bar{z}) = b_2^{-1} g_2 b_2. \]

Proceeding as in Sec. 4.2.3, we have a new group isomorphism

\[ \tilde{\iota}_b = b \circ \tilde{\iota} \circ b^{-1} : G_{\text{WZW}}^{\text{WZW}} \to \tilde{G}_b, \quad a \mapsto (b_1 a b_1^{-1}, b_2 a b_2^{-1}). \]

We write an arbitrary element of \( G_{2d} \) as

\[ g(x, \bar{x}) = \tilde{h}_b(\bar{x}) \circ h(x) = (b_1 \tilde{h} b_1^{-1} h^{-1}, b_2 \tilde{h} b_2^{-1} h). \]

Here,

\[ \tilde{h}_b = \tilde{\iota}_b(h) = (b_1 \tilde{h} b_1^{-1}, b_2 \tilde{h} b_2^{-1}) \in \tilde{G}_b \quad \text{and} \quad h = \iota(h) = (h^{-1}, h) \in G, \]

where \( \tilde{h}_b, h \in G_{\text{WZW}}^{\text{WZW}}, \) parametrized by coordinates \( \bar{x} \) and \( x \), respectively.
The projection $\pi_b$ gives $\mathcal{G}$ the structure of a $G_{\text{WZW}}$ fibration over $G_{\text{WZW}}$. For each choice of $b$, it defines the corresponding map from doubled group $\mathcal{G}_{2d} = G_1 \times G_2$ to the physical group $G_{\text{phys}}$ of the standard WZW model. For $b = I$, the identity on $\mathcal{G}_{2d}$, we recover the previously defined projection and polarization.

The map
\[
\phi(h, \tilde{h}) = \iota(h) \circ \pi_b(\tilde{h}) : \quad G_{\text{WZW}} \times G_{\text{WZW}} \rightarrow \mathcal{G},
\]
gives a local trivialization of $\mathcal{G}$ as a $G_{\text{WZW}}$ fibration, where we view $\tilde{h}(x)$ as the fiber coordinate and $h(x) = g_{\text{phys}}^{-1/2}$ as the base coordinate. (See the corresponding discussion in Sec. 4.2.3, and Footnote 32.)

For trivial discrete group $\Gamma$, we again recover the physical space $G_{\text{phys}}$ with metric (2.12) and $n$ units of $H$-flux, proceeding as before for any choice of global polarization $\mathcal{G}_b$. When $\Gamma$ is not contained in the center of $\mathcal{G}_{2d}$, only special choices of polarization will give rise to a geometric compactification. This is the condition of “$\Gamma$-invariant $Q$” described in Sec. 4.1.6.

### 4.2.6 Abelian T-duality

For toroidal compactifications, the T-duality group is the subgroup of the $T^d \times \tilde{T}^d$ lattice automorphisms preserving the $O(d,d)$ metric. The $\mathcal{G}_b$ polarization choices of Sec. 4.2.5 with $b$ varying over $\mathcal{G}_{2d}$, are expected to be related to nonabelian or Poisson-Lie T-duality, which is not in general a symmetry beyond tree level in the genus expansion, but relates also inequivalent conformal field theories [45, 64]. However, for $b$ lying in a subgroup, which we now describe, the $\mathcal{G}_b$ polarization choices correspond to ordinary abelian T-duality.

A choice of Cartan subalgebra of $g_{2d}$ gives $\Gamma \backslash \mathcal{G}_{2d}$ the structure of a $T^r \times \tilde{T}^r$ fibration, where $r$ is the rank of $G_{\text{WZW}}$. The lattice of $T^r \times \tilde{T}^r$ is the coroot lattice of $\mathcal{G}_{2d}$ mod $\Gamma$. The restriction of the $O(d,d)$ metric to the fiber gives an $O(r,r)$ metric. One expects that the abelian T-duality group (i.e., T-duality relative to the Cartan torus as opposed to the full nonabelian group) will be the subgroup of inner automorphisms $\mathcal{G}_{2d} \rightarrow b^{-1}\mathcal{G}_{2d}b$ that restrict to automorphisms of the lattice, and that preserve the $O(r,r)$ metric.

The T-duality inversion of a single $U(1)$ then acts as follows. Given a choice of $U(1)_t \subset G_{\text{WZW}}$ with generator $t$, we have two groups $U(1)_Z, U(1)_X \subset \mathcal{G}_{2d}$, generated by $Z = \frac{1}{2}(-t,t)$ and $X = \frac{1}{2}(t,t)$, and consisting of elements of the form $(\omega^{-1}, \omega)$ and $(\omega, \omega)$, respectively. Here, $\omega$ is obtained by exponentiating $t$ to some power. Now, suppose that $b^{-1}tb = -t$. Then, conjugation of $\mathcal{G}_{2d}$ by $b^{-1} = (b^{-1}, 1)$ at fixed polarization $\tilde{G} = G_{\text{diag}}$ interchanges $Z$ and $X$ and hence the two $U(1)$s. This is the active point of view. The passive point of view fixes $\mathcal{G}_{2d}$ while conjugating the polarization as in Eq. (4.100).

More generally, when the lattice of $T^r \times \tilde{T}^r$ also has a discrete symmetry in the diagonal subgroup $O(r)_{\text{diag}} \subset O(r,r)$ containing an element $(b_2,b_2)$, we have $b_2^{-1}tb_2 = t$, and can follow the conjugation of the previous paragraph with a conjugation of by $(b_2^{-1},b_2^{-1})$ to obtain a general abelian T-duality transformation. This gives total conjugation of $\mathcal{G}_{2d}$ by...
\((b_1, b_2)^{-1} = (b_2^{-1}, b_1^{-1}) \circ (b_2 b_1^{-1}, 1)\), where \(b_1 b_2^{-1} = \pi(b^{-1})\) plays the role of \(b\) in the previous paragraph. Again, this is the active point of view, and the passive point of view conjugates the polarization from \(G_{\text{diag}}\) to \(\tilde{G}_b\) at fixed \(G_{2d}\).

### 4.2.7 D-branes

As noted in Ref. [42] and studied in detail in Ref. [53], classically, the submanifolds on which we can wrap D-branes in the fully doubled description are \(d\)-dimensional submanifolds that are null with respect to the \(O(d, d)\) metric, also known as maximal isotropic submanifolds. D-branes wrapped on these submanifolds project to different combinations of lower dimensional D-branes in the physical space, depending on the choice of polarization.

The submanifolds \(\tilde{G}_b\) satisfy the requisite condition. Let us make contact with the well known results for D-branes in WZW models [52, 19, 68, 58]. We refer the reader to App. [B.2.3] for a discussion of the WZW model current algebra. Writing \(g_{\text{phys}} = g_L(z)g_R(\bar{z})\) in Eq. (B.22) of that appendix, and identifying \(g_L = g_1^{-1}\) and \(g_R = g_2\) on-shell in the standard polarization, as in Eq. (4.102), we have

\[
J(z) = -\hat{n}g_1^{-1}\partial g_1, \quad \bar{J}(\bar{z}) = -\hat{n}g_2^{-1}\partial g_2. \tag{4.107}
\]

Define \(b = \pi(b^{-1}) = b_1 b_2^{-1}\) and \(b' = \pi(b) = b_1^{-1} b_2\), which may be chosen independently. Since the submanifold \(\tilde{G}_b\) of (4.100) is characterized by the condition \(g_1 = bg_2 b^{-1}\), we have \(J(z) = b\bar{J}(\bar{z})b^{-1}\), i.e.,

\[
J^m = (\text{Ad}_b)^m J^0 \quad \text{on} \quad \tilde{G}_b \subset G_{2d}. \tag{4.108}
\]

Here, we have set \(z = \bar{z}\) at the boundary of the worldsheet. This is precisely Dirichlet boundary condition characterizing D-manifolds in the doubled geometry, and will project to an identical condition on the physical D-manifold. In the standard polarization \(\tilde{G} = G_{\text{diag}}\), the physical D-manifold \(\pi(\tilde{G}_b)\) obtained in this way is the conjugacy class

\[
[b'] = \{gb'g^{-1} \mid g \in G_{\text{WZW}}^{\text{phys}}\}, \tag{4.109}
\]

right multiplied by \(b\). Both Eq. (4.108) and the identification of D-manifolds with conjugacy classes agree with the standard symmetry preserving D-branes of WZW models [52, 19, 68]. Also well understood are symmetry breaking branes satisfying twisted boundary conditions \(J(z) = b_{\text{phys}}^{-1}\bar{J}(\bar{z})f(b_{\text{phys}})\), where \(f\) is a Dynkin diagram automorphism [52, 19]. These are similarly described in the doubled description.

### 4.2.8 The discrete group \(\Gamma\)

**Conservative observations**

We have already noted in Sec. [4.2.1] that \(Z_{2d}^{\text{diag}} \subset \Gamma\) if we require the group of right isometries of \(X_{2d}\) to act faithfully as the gauge group. Let us assume the standard polarization \(\tilde{G} = G_{\text{diag}}\).
As argued in the global geometry discussion of Sec. 4.1.6, a restriction on $\Gamma$ is that conjugation by arbitrary $\gamma \in \Gamma$ must map $\tilde{G}$ to itself. Let us write $\gamma = (\gamma_1, \gamma_2)$. Under conjugation by $\gamma$, an element $(a, a) \in \tilde{G}_d$ is mapped to $(\gamma_1 a \gamma_1^{-1}, \gamma_2 a \gamma_2^{-1})$, which lies in $\tilde{G}$ iff $\gamma_1 a \gamma_1^{-1} = \gamma_2 a \gamma_2^{-1}$. That is, $\gamma_2^{-1} \gamma_1$ commutes with $a$ for arbitrary $a \in G^{WZW}$, from which $\gamma_2^{-1} \gamma_1$ lies in the center $C^{WZW}$ of $G^{WZW}$. Therefore, $\gamma = (\gamma_1, \gamma_1 c) = (\gamma_2 c^{-1}, \gamma_2)$, for some $c \in C$, i.e.,

$$\Gamma \subset \tilde{G} \times C_2 = \tilde{G} \times C_1,$$  \hspace{1cm} (4.110)

where $C_1 = (C^{WZW}, 1)$ and $C_2 = (1, C^{WZW})$. This result is also obtained in the $\tilde{G}_b$ polarizations.

**Case 1:** $\Gamma \subset \tilde{G}$. In this case, the projection $\pi$ maps $\Gamma$ to the identity, and the physical target space is $X^\text{phys}_d = G^{WZW}$.

**Case 2:** $\Gamma \not\subset \tilde{G}$. In this case, the projection $\pi$ maps $\Gamma$ to a nontrivial subgroup $\Gamma^{\text{phys}}$ of the center $C^{WZW}$ of $G^{WZW}$. The physical target space is a geometric orbifold

$$X^\text{phys}_d = \Gamma^{\text{phys}} \backslash G^{WZW}, \quad \Gamma^{\text{phys}} \subset C^{WZW}.$$  \hspace{1cm} (4.111)

The WZW model at level $n$ involves a choice of modular invariant: a specification of which combinations of representations $R, \tilde{R}$ of $G^{WZW}$ appear in the spectrum for left and right moving states, respectively, subject to constraints from $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The simplest choice is the diagonal invariant $R = \tilde{R}$, which should correspond to Case 1 above. The next simplest choices are those constructed from outer automorphisms, which are precisely the quotients by subgroups of the center of Case 2.

**Speculative observations**

We now offer more speculative observations regarding the choice of discrete group $\Gamma$, focusing on the case $G^{WZW} = SU(2)$. If the arguments leading to Eq. (4.110) are correct, then the result would similarly hold in any other $\tilde{G}_b$ polarization that leads to a geometric compactification. It is hard to see how there could then exist more than one polarization compatible with $\Gamma$, in order to obtain the T-dual physical spaces $SU(2)$ and $SU(2)/\mathbb{Z}_n$ depending on polarization. Moreover, $\Gamma^{\text{phys}} \subset C$ in Eq. (4.111), so quotients by $\mathbb{Z}_{n>2}$ would be impossible for $SU(2)$ with $C = \mathbb{Z}_2$. Therefore, let us relax the restriction that conjugation by an arbitrary $\gamma \in \Gamma$ map $\tilde{G}_b$ to itself, and proceed more heuristically.

By analogy to the T-fold discussion in Sec. 3.3, it is natural to suppose that at level $n$, the discrete group is

$$\Gamma = (\mathbb{Z}_n)^r \ltimes C^{\text{diag}} \subset G^{\text{diag}},$$  \hspace{1cm} (4.112)

Additional modular invariant follow from the methods of conformal embeddings (e.g., $\hat{su}(2)_{16} \oplus \hat{su}_3 \subset (\hat{E}_8)_1$, for the $E_7$ modular invariant of the $SU(2)$ WZW model at level 16) and Galois permutations, however a general construction is lacking. For a overview of modular invariants in WZW models, see Ch. 17 of Ref. [16].
where \( r \) is the rank of \( G^{WZW} \) and \( C \) is the center of \( G^{2d} \). Here, the semidirect product notation means that \( \Gamma \) is such that \( C^{\Gamma} \times C^{WZW} \). Indeed, for the polarization choice \( \tilde{G} = G^{\text{diag}} \), the discrete group \( \Gamma \) then acts only on the fibers, so that we have physical space

\[
X_d = G^{\text{phys}} \text{ and fibers}
\]

\[
\Gamma \backslash C^{\text{diag}} \cong \left( (\mathbb{Z}_n)^r \times C^{WZW} \right) \backslash G^{WZW}
\]

\[
\cong (\mathbb{Z}_n)^r \backslash \left( (G^{WZW})^* \right) \quad \text{for simply laced } G^{WZW},
\]

where \( * \) denotes the dual of a group, in agreement with the naive intuition that “the fibers represent the T-dual space.” However, this last piece of naive intuition is incorrect—the T-dual spaces arise from projections associated to other permissible polarization choices, not from the fibers themselves—so it is necessary to be more careful.

Suppose that \( \gamma \in \Gamma \). The identifications

\[
g \sim \gamma g
\]

on \( G^{2d} \) induce identifications

\[
g_{\text{phys}} \sim \pi_b(\gamma g)
\]

on \( \pi_b(G^{2d}) = \tilde{G}_b \backslash G^{2d} \), in the \( \tilde{G}_b \) polarizations of the previous section. This gives

\[
g_{\text{phys}} \sim \gamma_{\text{phys}} g_{\text{phys}},
\]

along with other identifications that cannot be written in terms of a \( G^L \times G^R \) action on \( g_{\text{phys}} \). Let us tentatively ignore the others, although it seems unjustified to do so. Indeed, these additional identifications are present precisely when the restriction mentioned above is violated. Here, \( g_{\text{phys}} = \pi_b(g) \) and \( \gamma_{\text{phys}} = \pi_b(\gamma) \), and the restriction is equivalent to requiring

\[
\pi_b(\gamma g) = \pi_b(\gamma)\pi_b(g),
\]

i.e., that the \( \Gamma \)-action on \( G^{2d} \) induces a group action on equivalence classes in \( \tilde{G}_b \backslash G^{2d} \).

We will instead explore the weaker condition implied by Eq. (4.116), that the \( \gamma_{\text{phys}} \) form a group,

\[
\pi_b(\gamma_1 \gamma_2) = \pi_b(\gamma_1)\pi_b(\gamma_2).
\]

\[36\] Even more desirable would be a quotient of “all directions of \( \tilde{G} \) by a factor of \( n \),” since this would allow the quotient to be defined independent of a choice of Cartan subalgebra. This does not appear to be possible. For example, viewing the physical fiber coordinate as \( \tilde{h}^{\pm} \) rather than \( \tilde{h} \) fails to do the trick, since the Maurer-Cartan form \( P \) would not be single valued on this \( 1/n \)-fold cover.

\[37\] The dual of a group is obtained by interchanging its root and weight lattices. For simply laced groups, \( G^* \cong C \backslash G \), with \( C = \mathbb{Z}_{r+1} \) for \( SU(r+1) \), \( \mathbb{Z}_4 \) \((r \text{ odd})\) and \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) \((r \text{ even})\) for \( SO(2r) \), and \( \mathbb{Z}_{3,2,1} \) for \( E_{6,7,8} \). The non simply laced groups \( F_4 \) and \( G_2 \) are self dual and the dual of \( Sp(2r) \) is \( SO(2r+1) \). See App. 13.A. of Ref. [16].
For definiteness, let us focus on $SU(2)$. Then, for the appropriate choice of $\Gamma$, and a subset of the possible polarizations on $G_6$, we expect to obtain physical space $X_3 = SU(2)$ or $SU(2)/\mathbb{Z}_n$. Under what conditions is $\pi_b: \Gamma \rightarrow \Gamma_{\text{phys}}$ a group homomorphism with $\Gamma_{\text{phys}} = \mathbb{Z}_n$?

One solution is as follows. First, suppose that $\Gamma$ is a cyclic group $\mathbb{Z}_m \subset G_{\text{diag}}$ generated by $\omega = (\omega, \omega)$. Then, \[
\omega_{\text{phys}} = (b_1^{-1}(\omega^{-1}b_1)(b_2^{-1}\omega b_2),
\]
where each factor on the right hand side is a rotation by angle $4\pi/m$ about some axis. By suitable choice of $b_1$ and $b_2$, any desired axis can be obtained for either factor. Thus, there exist $b_1$ and $b_2$ such that the two factors are equal, \[
b_1^{-1}\omega^{-1}b_1 = b_2^{-1}\omega b_2 \equiv \omega'.
\]
This equation for $\pi(b^{-1}) = b_1b_2^{-1}$ is the condition for the polarizations $G_b$ and $G_{\text{diag}}$ to be related by an abelian T-duality tranformation, as described in Sec. 4.2.6. For any solution, \[
\omega_{\text{phys}} = \omega'^2 \quad \text{where} \quad \omega'^m = 1.
\]
Choosing $m = 2n$, we obtain the desired homomorphism \[
\pi_b: \mathbb{Z}_{2n} \rightarrow \mathbb{Z}_n, \quad \text{mapping} \quad (\omega, \omega) \mapsto \omega_{\text{phys}},
\]
with kernel $\mathbb{Z}^{\text{diag}}_{2n}$.

In summary, the tentative conclusion for $G^{\text{WZW}} = SU(2)$ is as follows. If there is room for relaxing the condition (4.110) and imposing only the identifications (4.116) on the physical space, then the desired results for $G^{\text{WZW}} = SU(2)$ are obtained from the choice \[
X_6 = \Gamma \backslash G_6, \quad \text{where} \quad G_6 = SU(2)_1 \times SU(2)_2 \quad \text{and} \quad \Gamma = \mathbb{Z}_{2n} \subset SU(2)_{\text{diag}},
\]
where $(\omega, \omega)$ denotes a generator of $\mathbb{Z}_{2n}$. This is indeed of the general form (4.123). For polarizations choices $\tilde{G}_b$ with $b$ satisfying Eq. (4.120), the physical space is then the Lens space $SU(2)_{\text{phys}}/\Gamma_{\text{phys}}$, where $\Gamma_{\text{phys}} = \pi_b(\Gamma) \cong \mathbb{Z}_n$. For polarization choices such that $b_{\text{phys}}$ commutes with $\omega$, the physical space is $SU(2)_{\text{phys}}$.

\[38\]Other polarizations that are not suitably compatible with $\Gamma$, are expected to lead to nongeometric compactifications.

\[39\]In the $SU(2)$ conventions of Sec. 2.1, the generators $t_m = -\frac{i}{2}\sigma_m$ multiplied by $4\pi$ exponentiate to unity.

\[40\]Another way to state this condition is $(\omega^{-1}, \omega) \in \tilde{G}_b \cap G$, where $G = iG^{\text{WZW}}$ is the submanifold of $G_6$ of elements of the form $(h^{-1}, h)$. This means that $\tilde{G}_b \cap G$ contains the whole $U(1) \ni (\omega^{-1}, \omega)$. 

45
ADE modular invariants

At each level $n$, the $SU(2)$ WZW model gives rise to one, two, or three different CFTs distinguished by a choice of $A$, $D$, or $E$ modular invariant. The $A_{n+1}$ series exists at all levels $n$. The $D_{n/2+2}$ series exists at all even levels $n$, and is orbifold of the $A_{n+1}$ model quotiented by its $\mathbb{Z}_2$ center, with target space $SO(3)$. The $E_6$, $E_7$, and $E_8$ models exist at levels $n = 10, 16$, and $28$, respectively. Other free orbifolds exist; however, these are either equivalent to the ADE models or do not have the full $\mathbb{Z}_2^{\text{diag}}$ symmetry.

It is natural to seek to relate the choice of discrete group $\Gamma$ to the choice of ADE modular invariant. Let us focus on the $A_{n+1}$ and $D_{n/2+2}$ series, which have a large $n$ semiclassical interpretation:

**$A_{n+1}$ series.** For the $A_{n+1}$ series, the target space is $SU(2)$, therefore $\Gamma \in \tilde{G} = SU(2)^{\text{diag}}$. If the tentative choice (4.123) is correct, this corresponds to $\Gamma = \mathbb{Z}_{2n}$ and $\Gamma/\mathbb{Z}_2^{\text{diag}} = \mathbb{Z}_n$.

**$D_{n/2+1}$ series.** For the $D_{n/2+1}$ series to be a physical $\mathbb{Z}_2$ orbifold of the $A_{n+1}$ series for even $n$, with target space $SU(2)/\mathbb{Z}_2^{\text{center}}$, Eq. (4.111) implies that $\Gamma$ is the extension of that of $A_{n+1}$ by the independent $\mathbb{Z}_2$ in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ center of $SU(2) \times SU(2)$.

More generally, in the case that the physical space is $SU(2)$ rather than $SU(2)/\mathbb{Z}_2^{\text{center}}$, Eq. (4.110) implies that the discrete group $\Gamma$ is a subgroup of $SU(2)^{\text{diag}}$ containing $\mathbb{Z}_2^{\text{diag}}$, so that $\Gamma/\mathbb{Z}_2^{\text{diag}} \in SO(3)$ is a cyclic, dihedral, or polyhedral finite group. This includes many possibilities beyond the $\mathbb{Z}_n$ cyclic case of our $A_n$ description above, whose physicality will be explored in future work.

## 5 Conclusions

**Summary of results**

The two primary results of this paper are as follows:

1. A construction of the T-fold and fully doubled descriptions of WZW models in the formalism of Ref. [45], using $SU(2)$ as a guide.

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41 As noted in Sec. 3.2 and App. B.2.5 at level $n$, the $SU(2)$ and $SU(2)/\mathbb{Z}_n$ models are equivalent as CFTs.
42 At level $n = n_1 n_2$, the free orbifolds $SU(2)/\mathbb{Z}_{n_1}$ and $SU(2)/\mathbb{Z}_{n_2}$ are equivalent as CFTs [58]. See Ref. [39] for further generalizations and a discussion of discrete torsion in this context.
43 From the point of view of the doubled space, the $A_{n+1}$ model would then appear to be related to the $A_1$ model via a generalized $\mathbb{Z}_n$ orbifold, however this is not quite correct, since the $O(3, 3)$ metrics also differ by a factor of $n$. 
2. A demonstration that the procedure given by Hull and Reid-Edwards in Ref. [45] for recovering physical from doubled geometry indeed reproduces the physical WZW metric (2.12) and $H$-flux (2.14)

Along the way, we have provided several additional details and consistency checks of the formalism of Hull and Reid-Edwards:

1. For the T-folds, we have given an interpretation of the total space as the group manifold $(U(1)^r)_L \times G^{WZW}_R$, where $U(1)^r$ is the Cartan torus, and have interpreted the physical $T^r$ fibration and dual $\tilde{T}^r$ fibrations in terms of explicit lattices and connection 1-forms.

2. In the fully doubled description, the total space is

$$X_{2d} = \Gamma \backslash (G_1 \times G_2),$$

where $G_1$ and $G_2$ are two copies of $G^{WZW}$.

3. Polarization choices are determined by maximal isotropic submanifolds of $G_1 \times G_2$. A natural family is $\tilde{G}_b = b G_{\text{diag}} b^{-1}$.

4. Given a polarization choice, the doubled space projects to physical target space $\tilde{G} \backslash G_{2d}$ (up to discrete identifications), which is in one-to-one correspondence with $G^{WZW}$.

5. The map giving the projection is $\pi: g \mapsto g_{\text{phys}} = g_1^{-1} g_2$ for the diagonal polarization, with suitable modification for the other $\tilde{G}_b$ polarizations. Thus, we can interpret $g_1^{-1}(z, \bar{z})$ and $g_2(z, \bar{z})$ as off-shell analogs of $g_L(z)$ and $g_R(\bar{z})$ of the physical WZW model. The appropriate left (right) moving constraint coming from gauging $\tilde{G}$ in the worldsheet description of Hull and Reid-Edwards.

6. When expressed in terms of global horizontal and vertical 1-forms $\lambda_{\text{phys}}$ and $\omega$ defined in Sec. 42.3, the local prescription of Hull and Reid-Edwards indeed globally reproduces the metric and $H$-flux of the WZW model.

7. As additional consistency checks, we have reproduced the moduli constraint $r^2 = n\alpha'$, the abelian T-duality group, and the classical D-brane spectrum, working solely in the doubled description.

---

44The recovery is trivial in the T-fold case.

45A maximal isotropic submanifold is a $d$-dimensional submanifolds that are null with respect to the $O(d,d)$ metric.
The discrete group $\Gamma$ and recovery of WZW orbifolds

The main unresolved question within the scope of this paper is the choice of discrete group $\Gamma$. A natural expectation from Secs. 3.3 and 4.2.8 is $\Gamma = \mathbb{Z}_{2n}$ or $\mathbb{Z}_n \times \mathbb{Z}_2$ for the $SU(2)$ model with $A$ modular invariant, and more generally, that

$$\Gamma \backslash G_{\text{diag}} \cong (\mathbb{Z}_n)^r \times C_{\text{WZW}} \backslash G_{\text{WZW}}$$

$$\cong (\mathbb{Z}_n)^r \backslash (G_{\text{WZW}})^*$$

for simply laced $G_{\text{WZW}}$, where $*$ denotes the dual group. T-duality is known to relate the physical $G_{\text{WZW}}$ model and $G_{\text{WZW}}/(\mathbb{Z}_n)^r$ free orbifold at level $n$ (and many intermediate orbifolds in between). The doubled description should reproduce all T-dual descriptions, depending on the choice of polarization. Unfortunately, the T-duality analysis in Sec. 4.2.8 combined with the restriction (4.110) on $\Gamma$ suggests that $\Gamma$ containing $\mathbb{Z}_{n>2}$ is incompatible with a global polarization in the T-dual frame that is expected to give target space $SU(2)/\mathbb{Z}_n$.

Another possibility is that the $\mathbb{Z}_n$ quotient is determined dynamically. The sigma model of Hull and Reid-Edwards involves a chiral gauging of a $G_1 \times G_2$ model that starts out with $(G_1 \times G_2)_L \times (G_1 \times G_2)_R$ global symmetry. A subgroup conjugate to the diagonal (vector) subgroup of $(G_1 \times G_2)_L$ is gauged. By analogy to the story described in Ref. [58], one might expect this gauging to generate an anomaly in the global antidagonal (axial) subgroup of $(G_1 \times G_2)_L$, breaking the commutant of the field strength from a $U(1)$ to a $\mathbb{Z}_n$ at level $n$. A third possibility is that both stories are correct, with the vestiges of the global symmetry providing an interpretation and/or means of eliminating the unwanted identifications beyond (4.116) in Sec. 4.2.8. A final possibility is that it is simply not possible to describe the $G_{\text{WZW}}/(\mathbb{Z}_n)^r$ models in this formalism, in a way that makes their equivalence to the $G_{\text{WZW}}$ model manifest.

Resolving this issue should provide additional insight into framework of Ref. [45], perhaps at the level of its quantum dynamics. The following is also worth highlighting. Given a choice of polarization, the structure constants of $G_{2d}$ define the $K, f, Q, R$ flux of Ref. [67]. In that polarization, the $R$-flux is traditionally identified with the obstruction to a local geometric description (in terms of a standard sigma model), and the $Q$-flux with an obstruction to global but not local geometry. As we have argued in Sec. 4.1.6, the $Q$-flux is only an obstruction to global geometry, when $Q$ is not $\Gamma$-invariant, i.e., when conjugation by $\Gamma$ does not preserve the subgroup $\tilde{G} \subset G_{2d}$:

$$[X^m, X^n] = Q^{mn}_p X_p.$$  

It is also tempting to try to relate $\Gamma$ to ADE subgroups of $SU(2)$ for the $SU(2)$ model with ADE modular invariant.

Note that the indices 1 and 2 give the chirality in this context, not $L$ and $R$.

These statements only apply relative to the particular polarization used to decompose the structure constants into $K, f, Q, R$. The say nothing about the existence of another polarization in which a subset of the $K, f, Q, R$ might vanish.
For polarizations reproducing the physical $SU(2)$ WZW model, it is fairly clear that $\Gamma \in \tilde{G}$, so that this condition is satisfied. And indeed, the model is geometric. Unambiguous recovery of the T-dual $SU(2)/\mathbb{Z}_n$ from the doubled description will provide a good probe of the validity of this criterion.

**Broader questions for the future**

The broader goals toward which this investigation aims are:

Effective field theory goal: To generalize the notion of a string theory and supergravity compactification to accommodate generic gaugings (gauge group and gauge/matter couplings) of the low energy effective field theory.

Microscopic goal: To understand the NSNS sector topological and Riemannian choice *defining* a string theory compactification in this generalized context.

As discussed in the introduction, there are currently at least three different approaches toward these goals: the T-fold description, doubled geometry, and generalized geometry. As geometric compactifications, WZW models can be consistently described in all three formalisms. Their doubled and generalized geometries are nontrivial since the $O(d,d)$ structure is twisted relative to the fiber-base decomposition of 1-forms. Therefore, they possess features more typical of nongeometric models, such as $Q$-flux, and should provide a fruitful context for illuminating all three approaches as well as the relations among them. In this paper, we have presented the T-fold and doubled descriptions of WZW models. In companion papers, we hope to present the description of WZW models via generalized geometry, and the relation between all three descriptions, building on Ref. [64] which restricted its scope to $f, K$ or $f, Q$ nonzero.\footnote{Even for T-folds and doubled geometry, the general relation is not clear. For a $T^3$ with $H$-flux, the doubled geometry is a $U(1)^3$ fibration over $T^3$, and the T-fold seems to be obtained by partial projection. For the chiral WZW models, the T-fold seems to be embedded in the doubled geometry as a subgroup.}

Ultimately, what is most interesting is the analogous global description suitable for arbitrary gaugings of $\mathcal{N} = 2$ or $\mathcal{N} = 1$ supergravities. The doubled geometry of Hull and Reid-Edwards describes the common NSNS sector of gauged analogs of toroidal reductions. The physical times dual torus $T^d \times \tilde{T}^d$ is replaced by a (similarly parallelizable) group manifold $G_{2d}$ with simultaneous $O(d,d)$ and Riemannian structure. For $\mathcal{N} = 2$, generalized geometry provides an excellent *local* description in terms of a geometry that doubles the tangent bundle rather than the space itself.\footnote{36, 31, 48, 26, 27} However it is not valid *globally* except for geometric compactifications. Calabi-Yau $n$-folds themselves have a natural T-fold description as the fiber product of the two mirror Strominger-Yau-Zaslow fibrations\footnote{70} over the same base. But, what is the general description of the fully gauged analog of a Calabi-Yau compactification and can the metric plus $B$-field topological data be similarly geometrized? In
the $\mathcal{N} = 2$ context, it expected to be a rich structure integrating the many beautiful results of complex and symplectic geometry and replacing Hitchin’s doubled tangent bundle with a doubling of the manifold itself. While the simple parallelizable context described by Ref. [45] might seem special and highly dependent on the group structure and degree of homogeneity, it does suggest generalizations, and it must be remembered that the simplest K3 surface and Calabi-Yau manifolds are obtained as resolved or deformed orbifolds of tori. Might the doubled description of Hull and Reid-Edwards be orbifolded to yield similar generalizations of reduced symmetry? And if so, what is the doubled geometry of K3 surface?

A direction that may serve as a guide is the development of a doubled effective field theory, furnishing the equations that these doubled spaces need to satisfy, and from which their structure potentially can be deduced. This effective field theory resembles Hitchin’s generalized geometry in that the physical $B$-field explicitly enters (in contrast to the formalism of Ref. [45], c.f. Eq. (4.33)). On the other hand, it resembles the formalism of Hull and Reid-Edwards, in that it is a theory on a doubled space, not simply the doubled tangent bundle. Much headway has been made in this direction over the past two years, and applications to examples will likely further illuminate the formalism.

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\begin{footnote}{For example, $K, f, Q, R$ become torsion. The underlying integral structure could be naturally related to Leray-Hirsch spectral sequence, as in Ref. [71]. Likewise, one expects that the doubled geometry parametrizing twisted analogs of K3 or Calabi-Yau compactifications geometrizes the space of diffeomorphism and $B$-field transformations to $O(d, d)$-compatible diffeomorphisms on the doubled space.}
\end{footnote}
A Lie algebra conventions

Here, we establish the notation and conventions used for Lie algebras. The discussion closely follows Sec. 13.1 of Ref. [16].

A.1 Basic definitions

Given a basis \{t_m\} for the Lie algebra \(g\) of \(G\), we write

\[
[t_m, t_n] = c_{mn}^p t_p,
\]

in terms of structure constants \(c_{mn}^p\). For \(G\) semisimple, the Killing form

\[
d(X, Y) = -\frac{1}{h^\vee \psi^2} \text{tr} (\text{Ad} X \text{ Ad} Y) \quad \iff \quad d_{mn} = -\frac{1}{h^\vee \psi^2} c_{mp}^q c_{nq}^p
\]

\[(A.2)\]

gives a positive definite inner product on \(g\). Here, \(h^\vee\) is the dual Coxeter number of \(g\), and \(\psi^2 = d_{mn} \psi_m \psi_n\) is the length squared of any long root. A tilde denotes the Killing form without the prefactors:

\[
\tilde{d}_{mn} = -c_{mp}^q c_{nq}^p.
\]

(A.3)

Traces in all representations of \(g\) are proportional. In a representation \(R\), we define the Dynkin index \(x_R\) via

\[
\text{tr}_R(t_m t_n) = -\psi^2 x_R d_{mn}.
\]

(A.4)

The Dynkin index in the adjoint representation is the dual Coxeter number \(h^\vee\), by the definition of \(d_{mn}\).[52] It is convenient to define a representation independent trace

\[
\text{tr}'(T_m T_n) = \frac{1}{x_R} \text{tr}_R(t_m t_n) = -\psi^2 d_{mn}.
\]

(A.5)

This is the trace that appears in the WZW action as described in App. [3].

For \(SU(2)\), we have \(h^\vee = 2\), \(x_f = 1/2\), and conventionally choose a Lie algebra basis such that \(c_{mn}^p = \epsilon_{mpn}\). In the fundamental (spinor) representation, we represent \(t_m\) by \(-\frac{i}{2} \sigma_m\), where \(\sigma_1, \sigma_2, \sigma_3\) are the Pauli matrices. Then,

\[
\text{tr}_f(t_m t_n) = -\frac{1}{2} \delta_{mn}, \quad \text{tr}_{\text{Ad}}(t_m t_n) = -2 \delta_{mn}, \quad \text{tr}'(t_m t_n) = -\delta_{mn},
\]

(A.6)

from which \(\psi^2 d_{mn} = \frac{1}{2} \delta_{mn}\), \(\psi^2 \tilde{d}_{mn} = 2 \delta_{mn}\), and \(G_{mn} = \frac{1}{4} \psi^2 \delta_{mn}\) for \(SU(2)\), as in Eq. (2.2).

[52]Given the root lattice of \(g\), this definition determines \(d_{mn}\) (and the length-squared of roots) only up to an overall rescaling, which is then fixed by specifying \(\psi^2\). In the standard normalization convention, \(\psi^2 = 2\).

[53]For reference, \(h^\vee\) is \(N\) for \(SU(N)\), \(N + 1\) for \(Sp(2N)\), \(N - 2\) for \(SO(N)\), 12, 18, 248 for \(E_6, E_7, E_8\), 9 for \(F_4\), and 4 for \(G_2\); furthermore, \(x_f = 1\) in all of these cases except for \(SU(N)\) where \(x_f = 1/2\).
A.2 Cartan-Weyl basis, roots, and inner products

For any Lie algebra $g$, we can choose a maximally commuting set of generators $H_i$, $[H_i, H_j] = 0, \ m = 1, \ldots, r$, (A.7) where $r$ is the rank of $G$. This subalgebra of $g$ is called a Cartan subalgebra $h$, and exponentiates to generate a maximal torus $T^r \subset G$, for $G$ simply connected. In a Cartan-Weyl basis $g$, the remaining generators $E_\alpha$ are chosen so that they are eigenvectors of the $H_i$, $[H_i, E_\alpha] = \alpha_i E_\alpha$. (A.8)

For a unitary representation, the $H_i$ are taken to be Hermitian and $E^{\alpha}_- = E_\alpha^\dagger$. Here, we have labeled the generators by their roots $\alpha = (\alpha_1, \ldots, \alpha_r) = \alpha_i e^i$, where $e^i$ is the standard Cartesian basis. Let $\Delta$ denote the set of all roots. The roots Lie in the dual vector space $h^*$, since $\alpha$ gives a natural map from any $\beta^i T_i \in h$ to $\alpha_i \beta^i$. Using the Jacobi identity, it can be shown that $[H_i, [E_\alpha, E_\beta]] = (\alpha_i + \beta_i) E_{\alpha+\beta}$, from which $[E_\alpha, E_\beta]$ is: (i) in the Cartan subalgebra when $\alpha + \beta = 0$, (ii) proportional to $E_{\alpha+\beta}$ when $\alpha + \beta \in \Delta$, and (iii) equal to zero otherwise. The $E_\alpha$ are normalized so that

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & 0 \neq \alpha + \beta \in \Delta, \\ \frac{2}{|\alpha|^2} \alpha \cdot H & \alpha = -\beta, \\ 0 & \text{otherwise}, \end{cases} \quad \text{(A.9)}$$

where $N_{\alpha\beta} = \text{constant}$. Here $\alpha \cdot \beta = d^{ij} \alpha_i \beta_j$ and $|\alpha|^2 = \alpha \cdot \alpha$, where $d_{ij}$ is the restriction of the normalized Killing form to the Cartan subalgebra, and $d^{ij}$ is its inverse. Note that $d_{ij}$ gives an isomorphism $h \rightarrow h^*$, mapping $\alpha^i H_i \mapsto \alpha = \alpha_i e^i$, where $\alpha_i = d_{ij} \alpha^j$. (A.10)

We will refer to this map as the Killing isomorphism.

A.3 Chevalley basis, coroots, and Cartan matrix

It is possible to define a notion of positive roots $\Delta_+$ and negative roots $\Delta_-$ such that $\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$. Simple roots are positive roots that cannot be written as the sum of two positive roots. There are $r$ simple roots $\alpha^{(i)}, i = 1, \ldots, r$. Their inner products define the Cartan matrix

$$A^i_j = \alpha^{(i)} \cdot \alpha^\vee_{(j)}. \quad \text{(A.11)}$$

54 In the Cartan-Weyl basis, it can be shown that $d(X, Y)$ is nonzero only for $(X, Y)$ equal to two elements of the Cartan subalgebra or $(E_\alpha, E_{-\alpha})$. 52
whose elements are integers. Here, $\alpha^\vee = 2\alpha/|\alpha|^2$ is the coroot associated to a root $\alpha$. A Chevalley basis is defined as follows. For each simple root, define generators
\begin{equation}
  e_i = E_{\alpha(i)}, \quad f_m = E_{-\alpha(i)}, \quad \text{and} \quad h_i = \alpha^\vee_{(i)} \cdot H.
\end{equation}
Then, the Killing isomorphism maps
\begin{equation}
  h_i \mapsto \alpha^\vee_{(i)}
\end{equation}
and the commutation relations become
\begin{align}
  [h_i, h_j] &= 0, \\
  [h_i, e_j] &= +A^j_i e_j \quad \text{(no sum)}, \\
  [h_i, f_j] &= -A^j_i f_j \quad \text{(no sum)}, \\
  [e_i, f_j] &= \delta_{ij} h_j \quad \text{(no sum)}. 
\end{align}

The remaining generators with roots in $\Delta_+$ ($\Delta_-$) are obtained from multiple commutators of the $e_i$ ($f_i$) among themselves. This process terminates, due to the Serre relations
\begin{align}
  [\text{Ad}(e_i)]^{1-A^j_i} e_j &= 0, \\
  [\text{Ad}(f_i)]^{1-A^j_i} f_j &= 0,
\end{align}
where $\text{Ad}(a)b = [a, b]$.

It is convenient to let $e_\alpha$ ($f_{-\alpha}$) denote the full set of generators obtained from the $e_i$ ($f_i$) in this way, including all nonvanishing multiple commutators, with signs chosen so that $f_{-\alpha} = e_\alpha^\dagger$ for a unitary representation. Then, the Chevalley basis is \{h_i, e_\alpha, f_{-\alpha}\}, with integer structure constants.

The Cartan generators in the Chevalley basis satisfy the useful property that
\begin{equation}
  \exp(2\pi ih_j) = 1, \quad \text{for } j = 1, \ldots, r.
\end{equation}
Thus, after accounting for the periodic identifications, the Killing isomorphism gives a map between the Cartan torus $T^r$ and $\mathfrak{h}^*/(2\pi \Lambda^\vee)$,
\begin{equation}
  \exp(ix^j h_j) \mapsto x^j \alpha_{(j)}.
\end{equation}
For $SU(2)$, a Chevalley basis is $h = \sigma_3$, $e = \frac{1}{2}(\sigma_1 + i\sigma_2)$, and $f = \frac{1}{2}(\sigma_1 - i\sigma_2)$, satisfying
\begin{equation}
  [e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.
\end{equation}
A.4 Lattices

Three lattices are conventionally defined in the vector space $\mathfrak{h}$*. The root lattice $\Lambda$, coroot lattice $\Lambda^\vee$, and weight lattice $(\Lambda^\vee)^*$ are obtained by taking integer linear combinations of the simple roots $\alpha^{(i)}$, simple coroots $\alpha^{\vee(\ell)}$, and weights $w^{(i)}$, respectively. Here, the weights are defined by

$$w^{(i)} \cdot \alpha^{\vee(\ell)} = \delta_{ij},$$

so that the weight lattice is dual to the coroot lattice. For $d^{ij}$ computed in the Chevalley basis, we have $\omega^{(i)} = d^{ij} \alpha^{\vee(j)}$.

The weight lattice is a sublattice of the root lattice, with quotient $(\Lambda^\vee)^*/L = C$, the center of the group. For simply laced groups (e.g., the ADE groups), we have $(\alpha^{(i)})^2 = \psi^2$ for all roots. Then, $\Lambda^\vee = (2/\psi^2)\Lambda$, so that in the standard convention $\psi^2 = 2$, roots and coroots agree and $\Lambda = \Lambda^\vee$.

B Worldsheet description of the WZW model

This Appendix reviews the basic results for WZW models. We use a Lie algebra convention in which group elements are obtained by exponentiation of generators without additional factors of $i$. Otherwise, the discussion closely follows Ch. 15 of Ref. [16]. (See also Ref. [23] and Ch. 15 of Ref. [62].) As a preliminary, Sec. B.1 first describes the purely geometric model, with no $H$-flux. While this model exhibits a global symmetry, it is not conformal, and there is no chirally conserved current. In Sec. B.2, we introduce $H$-flux via a Wess-Zumino term to obtain the WZW model. Allowing the overall volume modulus to vary, we show obtain a conformal model with chirally conserved currents at level $n$ when $r^2 = n\alpha'$.

Finally, we describe the chiral primary states of the $SU(2)$ WZW model at level $n$, and the T-duality map that between the $SU(2)/\mathbb{Z}_p$ and $SU(2)/\mathbb{Z}_q$ models at level $n = pq$.

B.1 Geometric sigma model

B.1.1 Action and symmetries

The nonlinear sigma model action describing a string propagating on a semisimple compact group manifold $G$ of radius $r$ (relative to the “unit metric” $\frac{1}{2} \psi^2 d_{mn}$, as defined below) is

$$S_0 = - \frac{r^2}{16\pi\alpha'} \int d^2 \sigma \text{tr}'(\partial_a g^{-1} \partial^a g) = - \frac{r^2}{8\pi\alpha'} \int d^2 z \text{tr}'(\partial g^{-1} \partial g).$$

(B.1)

This action has a natural $G \times G$ global symmetry from left and right multiplication,

$$g(z, \bar{z}) \mapsto g' = \Omega_L g(z, \bar{z}) \Omega_R^{-1}.$$  

(B.2)
Here, \( g(z, \bar{z}), \Omega_L, \) and \( \Omega_R \) take values in a unitary representation of \( G \), and \( \text{tr}' \) denotes the representation independent trace, as defined in App. A.

In terms of the left-invariant Maurer Cartan form \( \lambda = \lambda^m t_m = g^{-1} dg \) and coordinate fields \( X^i \) on the group manifold, the sigma model action becomes

\[
S_0 = \frac{1}{2\pi \alpha'} \int d^2 z G_{mn} D \lambda^m \bar{D} \lambda^n, \quad \text{where} \quad G_{mn} = \frac{1}{4} r^2 \psi^2 d_{mn}. \tag{B.3}
\]
Here, \( D\theta = \lambda^m t_m \partial X^i \) and \( \bar{D} \lambda = \lambda^m t_m \bar{\partial} X^i \).

### B.1.2 Equations of motion and conserved currents

Under \( g \mapsto g + \delta g \), the variation of the action is

\[
\delta S = -\frac{r^2}{8\pi \alpha'} \int d^2 \sigma \text{tr}' \left( \delta g \ g^{-1} \partial^a (\partial_a g \ g^{-1}) \right)
= -\frac{r^2}{8\pi \alpha'} \int d^2 \sigma \text{tr}' \left( g^{-1} \delta g \ \partial^a (g^{-1} \partial_a g) \right), \tag{B.4}
\]
from which the equations of motion are

\[
\partial^a (\partial_a g \ g^{-1}) = 0, \quad \text{or equivalently,} \quad \partial^a (g^{-1} \partial_a g) = 0. \tag{B.5}
\]
Integration by parts in the first or second line of Eq. (B.4) shows that \( \delta S = 0 \) for \( \delta g \ g^{-1} \) = constant or \( g^{-1} \delta g = \) constant, respectively. This is the infinitesimal form of the global \( G \times G \) symmetry,

\[
\delta g(z, \bar{z}) = \epsilon_L g(z, \bar{z}) - g(z, \bar{z}) \epsilon_R, \tag{B.6}
\]
which agrees with Eq. (B.2) for \( \Omega_{L,R} = \exp(\epsilon_{L,R}) \) and infinitesimal \( \epsilon_{L,R} \). The corresponding Noether current

\[
J_a = \frac{r^2}{4\alpha'} \text{Tr}' \left( -\epsilon_L \partial_a g \ g^{-1} + \epsilon_R g^{-1} \partial_a g \right), \tag{B.7}
\]
is conserved, as a consequence of the equations of motion. The current conservation law combines \( J_z \) and \( J_{\bar{z}} \),

\[
\partial J_z + \bar{\partial} J_{\bar{z}} = 0. \tag{B.8}
\]
which are not separately conserved, nor is there a local symmetry.

### B.2 Wess-Zumino-Witten model

#### B.2.1 Action and symmetries

It is possible to promote the global \( G \times G \) symmetry of the previous model to a local chiral symmetry

\[
g(z, \bar{z}) \mapsto g' = \Omega_L(z) g(z, \bar{z}) \Omega_R^{-1}(\bar{z}), \quad \text{(finite)}, \tag{B.9}
\]
\[
\delta g(z, \bar{z}) = \epsilon_L(z) g(z, \bar{z}) - g(z, \bar{z}) \epsilon_R(\bar{z}), \quad \text{(infinitesimal)}, \tag{B.10}
\]
with separately conserved left and right (holomorphic and antiholomorphic) currents, through the addition of a Wess-Zumino term proportional to

$$\Gamma = \frac{1}{24\pi} \int_M \text{tr}'(\lambda^3), \quad \lambda = g^{-1}dg,$$  \hspace{1cm} (B.11)

where \( M \) is any 3-manifold bounded by the worldsheet. The complete action is

$$S = S_0 + n\Gamma = -\frac{r^2}{8\pi\alpha'} \int d^2z \text{tr}'(\partial g^{-1}\bar{\partial}g) - \frac{n}{24\pi} \int_M \text{tr}'(\lambda^3).$$ \hspace{1cm} (B.12)

The Wess-Zumino term contributes a boundary \( H \)-flux term to the action,

$$\frac{1}{2\pi\alpha'} \int_M H, \quad \text{where} \quad H = -\frac{n\alpha'}{12} \text{tr}'(\lambda^3)$$ \hspace{1cm} (B.13)

which is well defined in the path integral (independent of the choice of \( M \)), provided \( n \) is an integer. From

$$\text{tr}'(\lambda \wedge \lambda \wedge \lambda) = \frac{1}{2} \ lambda^m \wedge \lambda^n \wedge \lambda^p \text{tr}'([T_m, T_n]T_p)$$  \hspace{1cm} (B.14)

we can also write

$$H = \frac{\hat{n}\alpha'}{12} c_{mnp} \lambda^m \wedge \lambda^n \wedge \lambda^p, \quad \text{where} \quad \hat{n} = \psi^2n, \quad c_{mnp} = c_{mnq}d_{qp}.$$ \hspace{1cm} (B.15)

### B.2.2 Equations of motion and conserved currents

The equations of motion (B.5) become

$$\left(1 + \frac{n\alpha'}{r^2}\right) \partial(g^{-1}\bar{\partial}g) + \left(1 - \frac{n\alpha'}{r^2}\right) \bar{\partial}(g^{-1}\partial g) = 0,$$ \hspace{1cm} (B.16)

or, equivalently

$$\left(1 - \frac{n\alpha'}{r^2}\right) \partial(\bar{\partial}g g^{-1}) + \left(1 + \frac{n\alpha'}{r^2}\right) \bar{\partial}(\partial g g^{-1}) = 0.$$ \hspace{1cm} (B.17)

The conserved current (B.7) becomes

$$J_z = \frac{r^2}{4\alpha'} \text{Tr}'\left[-\epsilon_L \left(1 + \frac{n\alpha'}{r^2}\right) \partial g \ g^{-1} + \epsilon_R \left(1 - \frac{n\alpha'}{r^2}\right) g^{-1} \partial g\right],$$

$$J_{\bar{z}} = \frac{r^2}{4\alpha'} \text{Tr}'\left[-\epsilon_L \left(1 - \frac{n\alpha'}{r^2}\right) \bar{\partial} g \ g^{-1} + \epsilon_R \left(1 + \frac{n\alpha'}{r^2}\right) g^{-1} \bar{\partial} g\right].$$ \hspace{1cm} (B.18)

For \( r^2 = n\alpha' > 0 \), we obtain a conformal field theory with the desired chiral conservation laws \( n \geq 0 \).

$$\bar{\partial}J_L^z = 0 \quad \text{and} \quad \bar{\partial}J_R^z = 0,$$ \hspace{1cm} (B.19)

\( 55 \)For \( n < 0 \), the roles of \( z \) and \( \bar{z} \) are interchanged, and \( n \) is replaced by \(|n|\) in what follows.

56
where
\[ J^L_z(z) = -\frac{n}{2} \text{tr}'(\epsilon_L \partial g g^{-1}) \quad \text{and} \quad J^R_{\bar{z}}(\bar{z}) = \frac{n}{2} \text{tr}'(\epsilon_R g^{-1} \partial g), \] (B.20)
associated with a \( G \times G \) current algebra of central charge \( n \). This is the level \( n \) WZW model with group \( G \).

For \( r^2 = n\alpha' > 0 \), the general solution to the classical equations of motion is
\[ g(z, \bar{z}) = g_L(z) g_R(\bar{z}), \] (B.21)
for arbitrary \( g_L(z), g_R(\bar{z}) \), analogous to \( X(z, \bar{z}) = X_L(z) + X_R(z) \) for a free boson.

**B.2.3 Affine Lie algebra**

Let us write
\[ J(z) = J^m(z) T_m = \hat{n} \partial g g^{-1}, \] (B.22)
\[ \bar{J}(\bar{z}) = \bar{J}^m(\bar{z}) T_m = -\hat{n} g^{-1} \bar{\partial} g, \] (B.23)
where \( \hat{n} = \psi^2 n/2 \). Then, in terms of \( \epsilon_{L,R} = \epsilon_{L,R}^m T_m \), the currents (B.20) become
\[ J^L_{\bar{z}}(z) = d_{mn} \epsilon^m_L J^n(z), \] (B.24)
\[ J^R_{\bar{z}}(\bar{z}) = d_{mn} \epsilon^m_R \bar{J}^n(\bar{z}). \] (B.25)

The Laurent coefficients \( J^m_k \) of the \( J^m(z) \) generate an affine Lie algebra
\[ [J^m_k, J^n_l] = f^{mn}_{\; \; p} J^p_k - \hat{n} k d^{mn} \delta_{k+l,0}, \] (B.26)
with the \( \bar{J}^m_k \) satisfying the same algebra. Here, \( f^{mn}_{\; \; p} \) is obtained from \( f_{mn}^p \) by raising and lowering with \( d_{mn} \). In terms of \( \hat{n} \), the level is \( n = 2\hat{n} / \psi^2 \) and is a nonnegative integer \( \text{[56]} \).

Locally near the identity of \( G \), we can expand
\[ g(z, \bar{z}) = \exp(X^m t_m) = 1 + X^m(z, \bar{z}) t_m + O(X^2) \] (B.27)
to write
\[ S = \frac{1}{4\pi} \int d^2 z \hat{n} d_{mn} \partial X^m \partial X^n + O(X^3), \] (B.28)
\[ J^m(z) = \hat{n} \partial X^m + O(X^2), \quad \bar{J}^m(\bar{z}) = -\hat{n} \bar{\partial} X^m + O(X^2). \] (B.29)

In the neighborhood of the origin, we can treat \( X^m \) as a free boson to confirm that the central term of the \( J^m J^n \) OPE is indeed \( \hat{n} - d^{mn}/z^2 \), in agreement with Eq. (B.26).

\text{[56]} Thus, \( n = \hat{n} \) in the standard normalization convention \( \psi^2 = 2 \).
B.2.4 Sugawara description and states

The WZW model is a Sugawara model, which means that it is a CFT whose stress tensor is constructed entirely from the currents. The stress tensor is

\[ T_{zz}(z) = -\frac{1}{(n + h^\vee)} \psi^2 \tau_{mn}(z^m J^n(z), \text{ (B.30)} \]

so that the Virasoro generators are

\[ L_0 = -\frac{1}{(n + h^\vee)} \psi^2 \sum_{k=1}^{\infty} J^m J^n_{k-l}, \text{ (B.31)} \]

\[ L_k = -\frac{1}{(n + h^\vee)} \psi^2 \sum_{l=-\infty}^{\infty} J^m J^n_{k-l}, \text{ (B.32)} \]

and similarly for \( T_{z\bar{z}}(\bar{z}) \) and \( \bar{L}_k \).

The central charge of the Sugawara model for a Lie algebra \( g \) at level \( n \) is

\[ c_{\varnothing,n} = (\text{dim } g) \left( 1 - \frac{h^\vee}{n + h^\vee} \right). \text{ (B.33)} \]

At large level (large radius), where the semiclassical sigma model interpretation of the WZW model is a good approximation, \( c_{\varnothing,n} \) falls short of the classical dimension \( \text{dim } g \) of the group manifold, by a deficit \( (\text{dim } g)h^\vee/n + O(1/n^2) \). Therefore, in any critical string theory background, the CFT in the remaining spacetime dimensions will need to compensate for the deficit. The simplest possibility is a linear dilaton background (see App. D).

The states are as follows. The primaries \( |R_i \rangle \otimes |\bar{R}\rangle \) are labeled by vectors \( i \) and \( \bar{i} \) in representations \( R \) and \( \bar{R} \) of \( G \), on which \( J^m_0 \) and \( \bar{J}^m_0 \) act as

\[ J^m_0 |R_i \rangle \otimes |\bar{R}\rangle = (T^m)^{ij}_j |R_j \rangle \otimes |\bar{R}\rangle, \text{ (B.34)} \]

\[ \bar{J}^m_0 |R_i \rangle \otimes |\bar{R}\rangle = (T^m)^{ij}_j |R_i \rangle \otimes |\bar{R}_j \rangle, \text{ (B.35)} \]

where \( T^m = d^{mn}T_n \). The descendents are obtained by acting with the raising operators \( J^m_k \) and \( \bar{J}^m_k \) for \( k, l > 0 \).

Let us restrict to group \( G = SU(2) \) and set \( \psi^2 = 2 \). Then, \( h^\vee = 2 \) and \( d_{mn} = \frac{1}{2}\delta_{mn} \), so that

\[ c = 3 \left( 1 - \frac{2}{n + 2} \right), \text{ (B.36)} \]

and

\[ L_0 = -\frac{1}{4(n+2)} J^m_0 J^n_0 + N_{osc}, \quad N_{osc} = -\frac{2}{4(n+2)} \sum_{k=1}^{\infty} J^m_{-k} J^n_k, \text{ (B.37)} \]

\[ \bar{L}_0 = -\frac{1}{4(n+2)} \bar{J}^m_0 \bar{J}^n_0 + \bar{N}_{osc}, \quad \bar{N}_{osc} = -\frac{2}{4(n+2)} \sum_{k=1}^{\infty} \bar{J}^m_{-k} \bar{J}^n_k. \text{ (B.38)} \]
A basis of primary states is given by

\[ |jm⟩ \otimes |\bar{m}\rangle \quad (G = SU(2)), \quad (B.39) \]

where \( j, m \) (\( \bar{j}, \bar{m} \)) are the standard angular momentum quantum numbers in the left (right) moving sectors, with \( 2j \) and \( 2\bar{j} \) nonnegative integers. On the primaries,

\[ L_0 = \frac{1}{n+2} j(j+1), \quad \bar{L}_0 = \frac{1}{n+2} \bar{j}(\bar{j}+1) \quad \text{(on primaries)}. \quad (B.40) \]

At level \( n \), it can be shown that \( 0 \leq j, \bar{j} \leq n/2 \), so the number of primaries is finite. The only question is which pairs \( j, \bar{j} \) appear. For the standard diagonal modular invariant at level \( n \) (the \( A_{n+1} \) modular invariant), all pairs \( j = \bar{j} \leq n/2 \) arise. Aside from the maximum value of \( j \), this is exactly as expected. The primaries carry the quantum numbers of the non-oscillator zero modes. Since \( SU(2) \) is simply connected, there is no winding, and the zero modes are simply the momenta of a point particle moving on \( S^3 \). There are two commuting momentum components \( iJ^3_0 = m \) and \( i\bar{J}^3_0 = \bar{m} \), which generate left and right multiplication by \( U(1)_{\sigma_3} \) (i.e., shifts of the coordinates \( \phi^2 \) and \( \phi^3 \) of Sec. 2). For \( S^3 \) embedded in \( \mathbb{C}^2 \) as in App. C, \( m \pm \bar{m} \) are the angular momenta corresponding to rotation in each \( \mathbb{C}^1 \) (i.e., shifts of \( \xi_3 \) and \( \xi_2 \)).

For \( n \) even, it is possible to orbifold the \( A_{n+1} \) model by the \( \mathbb{Z}_2 \) symmetry \((-1)^{2j}\) to obtain the \( D_{n/2+2} \) model. This gives the theory with target space \( SU(2)/\mathbb{Z}_2^{\text{center}} = SO(3) \). From the point of view of the \( A_n \) primaries, this projects out half-integer \( j = \bar{j} \), and introduces a twisted sector with \( j = n/2 - j \). The twisted sector consists of integer \( j, \bar{j} \) for \( n \equiv 0 \mod 4 \) and half-integer \( j, \bar{j} \) for \( n \equiv 2 \mod 4 \). Finally, at the special levels \( n = 10, 16, 28 \) there are exceptional models with \( E_6, E_7, E_8 \) modular invariants (c.f. Ref. [16]).

\section*{B.2.5 Orbifolds and T-duality}

For \( G = SU(2) \), in contrast to toroidal compactifications, there are two physical momentum quantum numbers and no winding quantum numbers: the left and right momenta \( m \) and \( \bar{m} \) are eigenvalues of the state under the generators of physical motions on the group manifold generated by the \(-i\sigma_3/2 \) action by left or right multiplication. Since \( SU(2) \) is simply connected, there are no winding sectors. A particle moving on the \( SU(2) \) group manifold would have the same quantum numbers as the primaries (B.39), with the diagonal constraint \( j = \bar{j} \).

At level \( n \), T-duality maps the \( SU(2)_n \) WZW model to the freely acting \( \mathbb{Z}_n \) WZW orbifold of \( SU(2)_n \) on the left or right.\(^{57}\) For definiteness, let us assume that it is the \( \mathbb{Z}_n \) generated by \( \omega = \exp\left(-2\pi i/n\sigma_3\right) \) acting on the right, as in Secs. 2.1 and 3.2. More generally, at level \( n = pq \), T-duality maps the orbifold \( SU(2)_{pq}/\mathbb{Z}_n \) to \( SU(2)_{pq}/\mathbb{Z}_q \).\(^{21, 58, 17}\)

\(^{57}\)It is possible to perform either a left or right T-duality, corresponding to the choice of \( O(2, 2) \) elements with \( \pm 1 \) on the off-diagonal.
The untwisted sector primaries of the orbifold \( SU(2)_{pq} / \mathbb{Z}_p \) consists the subset of states \( \{\bar{m}\} \) such that \( \bar{m} \) is divisible by \( p \). For \( p = pq \), only the \( \bar{m} = 0 \) state satisfies this condition. For \( p \) a proper divisor of \( pq \) there will be other states as well. The manifold \( SU(2) / \mathbb{Z}_p \) has fundamental group \( \mathbb{Z}_p \). The \( \sigma_3 \) Hopf fiber, which is a boundary in \( SU(2) \) is now a \( \mathbb{Z}_p \) torsion cycle. The twisted sectors are labeled by winding numbers \( \bar{w} = 1, \ldots, p - 1 \), and carry shifted momenta. The right moving chiral primary states of the \( SU(2)_{pq} / \mathbb{Z}_p \) model are thus labeled as\( |\bar{j}, \bar{m}, \bar{w}\rangle \) (\( SU(2)_{pq} / \mathbb{Z}_p \) model).\( \) (B.41)

The \( SU(2)_{pq} \) and \( SU(2)_{pq} / \mathbb{Z}_p \) models are not equivalent as conformal field theories for \( q \neq 1 \). On the other hand, the \( SU(2)_{pq} / \mathbb{Z}_p \) and \( SU(2)_{pq} / \mathbb{Z}_q \) models are equivalent. The precise mapping of states can be found in Refs. [58, 17]. The duality naturally generalizes to an arbitrary group by choosing independent \( \bar{m}_i, \bar{n}_i, p_i, q_i, i = 1, \ldots, r \), for each Cartan generator, where \( r \) is the rank of the group [21].

## C \( S^3 \) as a \( T^2 \) fibration

A 3-sphere can also be viewed as a \( T^2 \) fibration over the interval \( I_1 = [0, \pi] \), with the first \( S^1 \) shrinking at one end of the interval and the second \( S^1 \) shrinking at the other end. Written in terms of the coordinates \( \phi^1, \xi^2 = \frac{1}{2}(\phi^2 - \phi^3) \) and \( \xi^3 = \frac{1}{2}(\phi^2 + \phi^3) \), the metric on the unit \( S^3 \) becomes
\[
d s^2_{S^3} = \frac{1}{4}(d\phi^1)^2 + \sin^2(\frac{\phi^1}{2})(d\xi^2)^2 + \cos^2(\frac{\phi^1}{2})(d\xi^3)^2.
\] (C.1)

The coordinates \( (\phi^1/2, \xi^2, \xi^3) \) are known as Hopf coordinates. They arise naturally from the point of view of the embedding \( S^3 \subset \mathbb{C}^2 \). The locus
\[
S^3 = \left\{ (z_1, z_2) \in \mathbb{C}^2 \left| |z_1|^2 + |z_2|^2 = 1 \right. \right\},
\] (C.2)
can be parametrized by
\[
\begin{align*}
    z_1 &= e^{i\xi^2} \sin\left(\frac{\phi^1}{2}\right) \quad \text{and} \quad z_2 &= e^{i\xi^3} \cos\left(\frac{\phi^1}{2}\right).
\end{align*}
\] (C.3)

The usual \( \mathbb{C}^2 \) metric \( ds^2 = |dz_1|^2 + |dz_2|^2 \) restricted to the \( S^3 \) gives the metric \( C.1 \) [58].

Since circles in the \( T^2 \) fiber vanish at the endpoints of the base \( I_1 \), the \( T \)-dual circles blow-up at these points, and the 5d T-fold description with \( T^2 \times \tilde{T}^2 \) fiber over \( I_1 \) is singular.

---

58 The Hopf fiber of \( S^3 \) is the boundary of the half “large 2-sphere” discussed in Footnote 59 of App. C.

59 In this description, the \( \eta^2 \) fibration at fixed \( \eta^3 \) (or vice versa) gives half of a great 2-sphere. For example, consider the great 2-sphere \( \text{Im} \ z_2 = 0 \): the hemisphere \( \text{Re} \ z_2 \geq 0 \) is obtained from \( \eta^3 = 0 \) and the hemisphere \( \text{Re} \ z_2 \leq 0 \) is obtained from \( \eta^3 = \pi \).
The near horizon geometry of an NS5-brane

The supergravity background describing an NS5-brane as a soliton in an ambient 10D flat spacetime of parallel metric $\eta_{\mu\nu} = \text{diag}(-1, +1, \ldots, +1)$ and transverse metric $G_{mn}$ is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n, \quad g_{mn} = e^{2\Phi} G_{mn}, \quad (D.1)$$

$$e^{2\Phi} = e^{2\Phi_\infty} + \frac{n}{y^2}, \quad y^2 = G_{mn} y^m y^n, \quad (D.2)$$

$$H_{mnp} = -\text{Vol}(g) q_{mnp} \partial_q (2\Phi) \quad \leftrightarrow \quad H = 2n \omega_{S^3}. \quad (D.3)$$

Here a subscript $(g)$ denotes the metric $g_{mn}$. The last equation can be written as $\ast (g) H = e^{2\Phi} d(e^{-2\Phi})$, which corresponds to a generalized calibration of 1, in the sense of Ref. 22.\textsuperscript{60}

Here our orientation convention is that the volume form in the metric $G_{mn}$ is $y^3 dy \wedge \omega_{S^3}$, where $y$ is the transverse radial coordinate. The flux through the angular $S^3$, and the Bianchi identity for $H$ are

$$\frac{1}{2\pi} \int_{S^3} H = 2\pi n, \quad d \ast (e^{-2\Phi} H) = 0. \quad (D.4)$$

Dirac quantization requires that $n \in \mathbb{Z}$. For generic constant $G_{mn}$, the $S^3$ is not round, but has metric defined by

$$G_{mn} dy^m dy^n = dy^2 + y^2 ds^2_{(S^3)}, \quad \text{i.e.,} \quad ds^2_{(S^3)} = \frac{1}{y^2} (G_{mn} dy^m dy^n - dy^2). \quad (D.5)$$

After compactifying from 10D to 7D, we can think of the NS5-brane as a domain wall in 7D,

$$ds^2 = ds^2_{7} + e^{2\Phi} y^2 ds^2_{(S^3)}, \quad \text{where} \quad ds^2_{7} = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2\Phi} dy^2. \quad (D.6)$$

In the 7D interpretation, $y \geq 0$ is the coordinate transverse to the domain wall.

Near $y = 0$, we have $e^{2\Phi} \simeq n/y^2$, so the near horizon geometry is an infinite throat, with 10D metric neatly factorizing as $\mathbb{R}^{6,1} \times S^3$:

$$ds^2 \simeq ds^2_{\mathbb{R}^{6,1}} + ds^2_{(S^3)}, \quad \text{where} \quad ds^2_{\mathbb{R}^{6,1}} = \eta_{\mu\nu} dx^\mu dx^\nu + dx^7 dx^7, \quad x^7 = \sqrt{n} \log(y/\sqrt{n}). \quad (D.7)$$

In terms of $x^7$, the near horizon dilaton is linear,

$$\Phi \simeq -x^7/\sqrt{q}. \quad (D.8)$$

The additional central charge from the varying dilaton in $\mathbb{R}^{6,1}$ exactly compensates for the deficit in central charge from $S^3$. For example, in the bosonic theory, the deficit in $c = 6\beta \Phi$ from Eq. (4.90) is $6n^2/r^6$ to leading order in $\alpha'$, and the surplus from linear dilaton (D.8) is $6/n$, so the two contributions indeed cancel for $r^2 = n$.

For $G_{mn} \propto \delta_{mn}$, the worldsheet CFT describing the near horizon background factorizes as the $SU(2)_N$ WZW model times a linear dilaton theory. Generic $G_{mn}$ are obtained by marginal deformation of the WZW CFT away from the WZW point.

\textsuperscript{60}This is as expected, since the NS5-brane “wraps” a point, with volume form 1, in the transverse $\mathbb{R}^4$.\n
61
E Derivation of $H$-flux from doubled geometry

The physical $H$-flux determined by the doubled description of the WZW model is given by Eq. (4.87),

$$H = dB^M - \frac{1}{2} d(L_{mn} p^m \wedge \bar{q}^n) + \frac{1}{2} K.$$ 

For $L_{mn}$, $p^m$ and $q^n$ as defined in Sec. 4.2.3, we have

$$\frac{1}{2} L_{mn} p^m \wedge \bar{q}^n = \hat{n} d_{mn} p^m \wedge \bar{q}^n = -\frac{n}{8} \text{tr}'(p \wedge \bar{q}). \quad (E.1)$$

It can be shown that

$$dp = \frac{1}{2} (p \wedge \bar{q} + \bar{q} \wedge p) - (p \wedge \bar{\lambda} + \bar{\lambda} \wedge p), \quad (E.2)$$

$$dq = \frac{1}{2} (p \wedge p + \bar{q} \wedge \bar{q}) - (\bar{q} \wedge \bar{\lambda} + \bar{\lambda} \wedge \bar{q}). \quad (E.3)$$

Therefore,

$$-d(p \wedge \bar{q}) = -dp \wedge \bar{q} + p \wedge \bar{q} = \frac{1}{2}(p \wedge p \wedge p - \bar{q} \wedge p \wedge \bar{q}) + \bar{\lambda} \wedge p \wedge \bar{q} - p \wedge q \wedge \bar{\lambda}, \quad (E.4)$$

and

$$-\frac{1}{2} d(L_{mn} p^m \wedge \bar{q}^n) = -\frac{n}{16} \text{tr}'(p \wedge p \wedge p - \bar{q} \wedge p \wedge \bar{q}), \quad (E.6)$$

using the cyclic property of the trace.

Next, from Eq. (4.60) for the $t_{MNP}$, we have

$$K = \frac{1}{6} t_{MNP} M^{P} N^{P} P^{P} = \frac{1}{6} t_{MNP} \Phi^{M} \Phi^{N} \Phi^{P}$$

$$= \frac{1}{6} K_{mnp} p^m \wedge p^n \wedge p^p + \frac{1}{2} Q_{mnp} q^m \wedge \bar{q}^n \wedge \bar{q}^p$$

$$= \frac{2}{3} c_{mnp} (\frac{1}{2} p^m \wedge p^n \wedge p^p + \frac{1}{2} p^m \wedge \bar{q}^n \wedge \bar{q}^p)$$

$$= -\frac{n}{8} \text{tr}(\frac{1}{2} p \wedge p \wedge p + p \wedge q \wedge \bar{q}). \quad (E.7)$$

Combining the last two results, Eq. (4.87) becomes

$$H = dB^M - \frac{n}{12} \text{tr}'(p \wedge p \wedge p) = dB^M + \frac{n}{12} c_{mnp} p^m \wedge p^n \wedge p^p. \quad (E.8)$$

Since $p = h \lambda_{\text{phys}} h^{-1}$, the cyclic property of the trace allows us to eliminate the factors of $h$, leaving the desired result

$$H = dB^M - \frac{n}{12} \text{tr}'(\lambda_{\text{phys}} \wedge \lambda_{\text{phys}} \wedge \lambda_{\text{phys}}) = dB^M + \frac{n}{12} c_{mnp} \lambda_{\text{phys}}^m \wedge \lambda_{\text{phys}}^n \wedge \lambda_{\text{phys}}^p. \quad (E.9)$$

For $G_{\text{WZW}} = SU(2)$, with $\hat{n} = n$ and $d_{mn} = \frac{1}{2} \delta_{mn}$ as in App. A.1, we have $c_{mnp} = 2 \epsilon_{mnp}$ in terms of the totally antisymmetric tensor

$$\epsilon_{mnp} \equiv \begin{cases} +1 & \text{if cyclic permutation of 123,} \\ -1 & \text{if anticyclic permutation of 123,} \\ 0 & \text{otherwise.} \end{cases}$$
The previous result becomes

\[
H = dB^M + \frac{n}{24} \epsilon_{rst} \lambda_{\text{phys}}^r \wedge \lambda_{\text{phys}}^s \wedge \lambda_{\text{phys}}^t
= dB^M + \frac{n}{4} 8 \omega_{S^3},
\]  

(E.10)

in agreement with Eq. (2.1), for a choice of moduli such that \(B^M = 0\).

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