A short proof that the Coulomb-gauge potentials yield the retarded fields

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Abstract
A short demonstration that the potentials in the Coulomb gauge yield the retarded electric and magnetic fields is presented. This demonstration is relatively simple and can be presented in an advanced undergraduate course of electromagnetic theory.

1. Introduction

After the Lorenz gauge, the most popular gauge in electrodynamics is the Coulomb gauge in which the scalar potential $\Phi_C$ displays the properties of acausality and instantaneous propagation. However, the electric field generated by the Coulomb-gauge potentials $\Phi_C$ and $A_C$ is a retarded field satisfying the properties of causality and propagation at the speed of light $c$. The proof of this well-known result seems to be, however, rather involved. Most textbooks usually do not present this proof and in the best of cases some of them quote a paper by Brill and Goodman [1] who presented in the 1960s an elaborate demonstration (restricted to sources with harmonic time dependence) that potentials in the Coulomb and Lorenz gauges yield the same retarded fields. Starting the 2000s, Jackson [2] derived a novel expression for $A_C$ and demonstrated how this expression and the usual expression for the potential $\Phi_C$ lead to the retarded fields. However, Jackson’s formula for $A_C$ is not so easy to obtain. More recently, other authors [3–7] developed alternative procedures to show that the potentials $\Phi_C$ and $A_C$ generate the well-known retarded fields.

In this paper I present a short proof that the Coulomb-gauge potentials yield the retarded electric and magnetic fields. The quantities $-\nabla \Phi_C$ and $-\partial A_C/\partial t$ are expressed in terms of retarded integrals with local and non-local sources. The combination $-\nabla \Phi_C - \partial A_C/\partial t$ identically cancels the integrals with non-local sources and identifies with the retarded electric field in SI units. The retarded magnetic field is directly obtained by taking the curl to $A_C$. This short proof involves simple vector operations on retarded quantities and may be presented in
an advanced undergraduate course of electromagnetic theory. To emphasize the simplicity of this proof, it is compared with the more elaborated proof given by Jackson [2].

2. The short proof

The electric and magnetic fields expressed in terms of the Coulomb-gauge potentials are
\[
\mathbf{E} = -\nabla \Phi_C - \frac{\partial \mathbf{A}_C}{\partial t},
\]
\[
\mathbf{B} = \nabla \times \mathbf{A}_C.
\]

The equations for the potentials \(\Phi_C\) and \(\mathbf{A}_C\) are given by
\[
\nabla^2 \Phi_C = -\frac{\rho}{\epsilon_0},
\]
\[
\Box^2 \mathbf{A}_C = -\mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\nabla^2 \Phi_C}{R},
\]
where \(\Box^2 \equiv \nabla^2 - (1/c^2) \partial^2/\partial t^2 \). The retarded solution of (4) can be written as\(^1\)
\[
\mathbf{A}_C = \frac{\mu_0}{4\pi} \int \frac{d^3 x'}{R} \left[ \frac{\nabla' \rho}{R} + \frac{1}{4\pi c^2} \int \frac{d^3 x'}{R} \left[ \frac{\partial^2 \nabla' \Phi_C}{\partial t^2} \right] \right].
\]

Consider now the identity
\[
-\Box^2 \nabla \Phi_C \equiv -\nabla^2 \nabla \Phi_C + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \nabla \Phi_C.
\]
Equations (3) and (7) and the result \(\nabla^2 \nabla \Phi_C = \nabla \nabla^2 \Phi_C\) yield the following wave equation:
\[
-\Box^2 \nabla \Phi_C = \nabla \frac{\rho}{\epsilon_0} + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \nabla \Phi_C.
\]
The retarded solution of this equation is given by\(^2\)
\[
\nabla \Phi_C = -\frac{1}{4\pi \epsilon_0} \int \frac{d^3 x'}{R} \left[ \frac{\nabla' \rho}{R} + \frac{1}{4\pi c^2} \int \frac{d^3 x'}{R} \left[ \frac{\partial \nabla' \Phi_C}{\partial t'} \right] \right].
\]
Equations (6) and (9) show that each of the terms \(-\partial \mathbf{A}_C/\partial t\) and \(-\nabla \Phi_C\) can be expressed as retarded integrals with local and non-local sources. The second integral in (9) exactly cancels the second integral in (6). Therefore, if (6) and (9) are used in (1), then one obtains the electric field expressed in its usual retarded form [8]:
\[
\mathbf{E} = -\frac{1}{4\pi \epsilon_0} \int \frac{d^3 x'}{R} \left[ \frac{\nabla' \rho}{R} + \frac{1}{4\pi c^2} \int \frac{d^3 x'}{R} \left[ \frac{\partial \nabla' \Phi_C}{\partial t'} \right] \right].
\]

Curl of (5) and an integration by parts give the magnetic field in its usual retarded form:
\[
\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{d^3 x'}{R} \left[ \frac{\nabla' \times \mathbf{J}}{R} \right] = \frac{\mu_0}{4\pi} \int \frac{d^3 x'}{R} \left[ \frac{\nabla' \times \mathbf{J}}{R} \right].
\]
The above proof that \(\Phi_C\) and \(\mathbf{A}_C\) yield the retarded fields is relatively simple and could be presented in an advanced undergraduate course on electromagnetism.

\(^1\) To verify that (5) satisfies (4), one takes \(\Box^2\) to (5), uses the general identity \(\Box^2 [\ ] / R = -4\pi [\ ] \delta(x - x')\) with \(\delta\) being the Dirac delta function, and integrates over all space. The general identity is proved in [9].

\(^2\) Strictly, (9) is not a solution for \(-\nabla \Phi_C\), but only an integral representation given in terms of its second-order time derivatives and of the charge density. One can verify that (9) satisfies (8) by taking \(\Box^2\) to (9), using \(\Box^2 [\ ] / R = -4\pi [\ ] \delta(x - x')\) and integrating over all space.
3. Jackson’s proof

Jackson’s proof that the Coulomb-gauge potentials yield the retarded electric field can be outlined as follows. The instantaneous solution of (3) reads

$$\Phi_C(x, t) = \frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{\rho(x', t)}{R}.$$  (12)

The gradient of (12) and an integration by parts allow one to express the field $E$ in (1) as

$$E = \frac{1}{4\pi \varepsilon_0} \int d^3x' \frac{\rho(x', t)}{R^2} - \frac{\partial A_C}{\partial t}.$$  (13)

The instantaneous (first) term in (13) must be a spurious quantity because the field $E$ is expected to be a retarded field. This means that the term $-\partial A_C/\partial t$ must contain a piece that identically eliminates the first term in (13). Jackson [2] has constructed an expression for $A_C$ using an indirect approach based on the gauge function that transforms the Lorenz-gauge potentials into the Coulomb-gauge potentials [2]:

$$A_C(x, t) = \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left( [\mathbf{J} - c\mathbf{\hat{R}}\rho] + \mathbf{J} \left( \frac{\partial}{\partial t} \right)\mathbf{\hat{R}} \right),$$  (14)

where $\mathbf{\hat{R}} = \mathbf{R}/R = (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|$. Time derivative of (14) and the property $\partial/\partial t = -\partial/\partial \tau$ yield the expected result [2]:

$$-\frac{\partial A_C}{\partial t} = \frac{1}{4\pi \varepsilon_0} \int d^3x' \left( \mathbf{\hat{R}}[\rho] + \mathbf{\hat{R}}[\partial \rho/\partial t'] - \left[ \frac{\partial \mathbf{J}}{\partial t'} \mathbf{\hat{R}} \right] \frac{R}{R_c^2} \right) - \frac{1}{4\pi \varepsilon_0} \int d^3x' \rho(x', t) \mathbf{\hat{R}}.$$  (15)

The last term in (15) exactly cancels the first instantaneous piece in (13). Therefore, from (13) and (15) one obtains the retarded electric field in the form given by Jefimenko [8]:

$$E = \frac{1}{4\pi \varepsilon_0} \int d^3x' \left( \mathbf{\hat{R}}[\rho] + \mathbf{\hat{R}}[\partial \rho/\partial t'] - \left[ \frac{\partial \mathbf{J}}{\partial t'} \mathbf{\hat{R}} \right] \frac{R}{R_c^2} \right).$$  (16)

The curl of (14) gives the magnetic field expressed in its usual retarded form [8] (this calculation is similar to that in (11) because of the result $\nabla \times \mathbf{R} = 0$). The practical difficulty in Jackson’s proof is that the derivation of the expression for $A_C$ in (14) is somewhat laborious [2].

4. Comparing the two proofs

The short proof that $\Phi_C$ and $A_C$ yield the retarded electric field can be drawn as follows:

$$E = -\nabla \Phi_C = -\nabla \frac{\mu_0}{4\pi} \int d^3x' \frac{[\nabla \mathbf{J}]}{R} - \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[ \frac{\partial^2 \nabla \Phi_C}{\partial t'^2} \right].$$

$$E = -\nabla \Phi_C = -\frac{\mu_0}{4\pi} \int d^3x' \frac{[\nabla \mathbf{J}]}{R} + \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{R} \left[ \frac{\partial^2 \nabla \Phi_C}{\partial t'^2} \right].$$  (17)

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3 Maxwell’s equations for sources confined in space and bounded in time admit retarded and advanced solutions (or more in general a combination of them). One chooses the retarded solutions because they satisfy the causality principle. See [10].
The second and last terms identically cancel, and therefore they represent spurious contributions which do not appear in the final expression for the retarded electric field. Jackson’s proof that $\Phi_C$ and $A_C$ yield the retarded electric field can be drawn as follows:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(x', t')\hat{R}}{R^2} \left( -\nabla \Phi_C - \frac{\partial A_C}{\partial t} \right) + \frac{1}{4\pi\epsilon_0} \int d^3x' \left( \frac{\hat{R}[\rho]}{R^2} + \frac{\hat{R}[\partial \rho/\partial t']}{Rc} - \frac{[\partial \mathbf{J}/\partial t']}{Rc^2} \right) - \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(x', t')\hat{R}}{R^2}.$$  \hspace{1cm} (18)

The first and last terms identically cancel, and therefore they represent spurious contributions which do not appear in the final expression for the retarded electric field.

Note added in proof. Note that (5), (6) and (9) are not solutions, but non-local coupled integral equations. To show that the Coulomb-gauge potentials yield retarded fields one can equivalently use solutions (as in Jackson’s approach) or coupled integral equations (as in the present paper). Note also that the ‘integral equation’ in (5) can be transformed into the ‘solution’ in (14) after some calculation. I thank Professor J D Jackson for calling my attention to this point.

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