FLAT DEFORMATIONS OF $\mathbb{P}^n$

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Abstract. In this paper we study projective flat deformations of $\mathbb{P}^n$. We prove that the singular fibers of a projective flat deformation of $\mathbb{P}^n$ appear either in codimension 1 or over singular points of the base. We also describe projective flat deformations of $\mathbb{P}^n$ with smooth total space, and discuss flatness criteria.

1. Introduction

It is a well-known theorem of Siu that $\mathbb{P}^n$ is rigid (see [Siu92, Main Theorem]). This means that, if $\pi : X \to Y$ is a smooth proper morphism between connected complex manifolds, and if the general fiber of $\pi$ is isomorphic to $\mathbb{P}^n$, then every fiber of $\pi$ is isomorphic to $\mathbb{P}^n$. The aim of these notes is to prove similar results for projective flat deformations of $\mathbb{P}^n$.

A $\mathbb{P}^n$-bundle is a smooth projective morphism between complex analytic spaces whose fibers are all isomorphic to $\mathbb{P}^n$. The simplest examples are scrolls. These are $\mathbb{P}^n$-bundles $\pi : X \to Y$ satisfying the following equivalent conditions.

1. There is a locally free sheaf $\mathcal{E}$ of rank $n + 1$ on $Y$, and an isomorphism $X \cong \mathbb{P}(\mathcal{E})$ over $Y$.
2. There is a line bundle $\mathcal{L}$ on $X$ whose restriction to every fiber $X_t$ satisfies $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^n}(1)$.
3. The morphism $\pi$ admits a section $\sigma : Y \to X$.

We call a line bundle $\mathcal{L}$ as in (2) a global $\mathcal{O}(1)$ for $\pi$. Given $\mathcal{E}$ as in (1), the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a global $\mathcal{O}(1)$ for $\pi$. Conversely, given $\mathcal{L}$ as in (2), $\mathcal{E}$ can be taken to be $\pi_*\mathcal{L}$. The equivalence with (3) can be seen by considering the associated $\mathbb{P}^n$-bundle of hyperplanes (see [Mü80, p.134]).

We recall the following characterization of scrolls, due to Fujita.

Theorem F1 ([Fuj75, Corollary 5.4]). Let $X$ and $Y$ be irreducible and reduced complex analytic spaces, and $\pi : X \to Y$ a proper flat morphism whose fibers are all irreducible and reduced. Suppose that the general fiber of $\pi$ is isomorphic to $\mathbb{P}^n$, and that there exists a $\pi$-ample line bundle $\mathcal{L}$ on $X$ such that $\mathcal{L}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^n}(1)$ for general $t \in Y$. Then $\pi$ is a $\mathbb{P}^n$-bundle and $\mathcal{L}$ is a global $\mathcal{O}(1)$ for $\pi$.

Every $\mathbb{P}^n$-bundle over a smooth curve carries a global $\mathcal{O}(1)$. In general, not every $\mathbb{P}^n$-bundle is a scroll, although this is the case locally in the étale topology. (See [Art82] for the connection between this condition and Brauer-Severi varieties.) So, it is natural to look for more general characterizations of $\mathbb{P}^n$-bundles, without requiring the existence of a global $\mathcal{O}(1)$. We start by observing that Theorem F1 does not hold if we drop the assumptions on the line bundle $\mathcal{L}$. This is illustrated in the following example.
**Example 1.** Let $\nu : \mathbb{P}^n \to \mathbb{P}^N$ be the $d$-uple embedding of $\mathbb{P}^n$, with $n, d \geq 2$. Denote by $V$ the image of $\nu$ in $\mathbb{P}^N$, and let $C(V) \subset \mathbb{P}^{N+1}$ be the cone over $V$ with vertex $P$. Let $\Gamma$ be a general pencil of hyperplane sections of $C(V)$ in $\mathbb{P}^{N+1}$. It gives rise to a projective flat morphism $\pi : X \to Y = \mathbb{P}^1$ whose fibers are precisely the members of $\Gamma$. There is a unique member of $\Gamma$ that passes through $P$. It is isomorphic to the cone over a general hyperplane sections of $V$ in $\mathbb{P}^N$. Let $o \in Y$ be the point corresponding to this singular member of $\Gamma$. Then $X_t \cong \mathbb{P}^n$ for every $t \in Y \setminus \{o\}$, while $X_o$ is a singular cone.

We call the reader’s attention to the following properties of $\pi$.

- The locus of $Y$ over which the fibers are not isomorphic to $\mathbb{P}^n$ has codimension one in $Y$.
- The total space $X$ is singular.

We will see that these properties are typical for non-smooth flat deformations of $\mathbb{P}^n$.

The following is the key result in our study of flat deformations of $\mathbb{P}^n$. We henceforth denote the unit ball of $\mathbb{C}^m$ by $\Delta^m$.

**Theorem 2.** Let $X$ be a complex analytic space, and $\pi : X \to \Delta^m$ a projective surjective flat morphism, with $m \geq 2$. Suppose that $X_t \cong \mathbb{P}^n$ for every $t \in \Delta^m \setminus \{\bar{0}\}$. Then $X$ is smooth and $\pi$ is a scroll.

As a consequence of Theorem 2, the singular fibers of a projective flat deformation of $\mathbb{P}^n$ appear either in codimension 1, or over rather singular points of the base. To state this precisely, we introduce some notation. Given a surjective morphism $\pi : X \to Y$ between algebraic varieties, we denote by $S_\pi$ the locus of points of $Y$ over which $\pi$ is not smooth. It is a closed subset of $Y$.

**Corollary 3.** Let $\pi : X \to Y$ be a projective surjective flat morphism between algebraic varieties with general fiber isomorphic to $\mathbb{P}^n$, and fix $y \in S_\pi$. Suppose that there is a surjective quasi-finite morphism from a smooth variety onto a neighborhood of $y$ in $Y$. Then $S_\pi$ has pure codimension 1 at $y$.

Next we describe projective flat deformations of $\mathbb{P}^n$ with smooth total space. In order to state our result, we introduce some more notation.

**Notation-Remark 4.** Let $\pi : X \to Y$ be a proper surjective equidimensional morphism between normal algebraic varieties. We denote by $R_\pi$ the locus of points of $Y$ over which the fibers of $\pi$ are reducible. Note that $R_\pi$ is a constructible set. Indeed, let $d$ denote the relative dimension of $\pi$. Then, for every $y \in Y$, $H^{2d}(X_y, \mathbb{Z})$ is free and its rank is the number of irreducible components of $X_y$. On the other hand, the sheaf $R^{2d}\pi_*\mathbb{Z}_X$ is constructible by the proper base change theorem on étale cohomology and Artin’s comparison theorem (see [Mil80, Theorem VI.2.1]).

**Theorem 5.** Let $X$ be a smooth complex quasi-projective variety, $Y$ a normal complex quasi-projective variety, and $\pi : X \to Y$ a proper surjective flat morphism. Suppose that the general fiber of $\pi$ is isomorphic to $\mathbb{P}^n$. Then either

1. $Y$ is smooth and $\pi : X \to Y$ is a $\mathbb{P}^n$-bundle; or
2. $S_\pi$ is of pure codimension 1, and $S_\pi = R_\pi$.

The second case described in Theorem 5 is exemplified by suitable blow-ups of $\mathbb{P}^n$-bundles.
It is also useful to have characterizations of $\mathbb{P}^n$-bundles without flatness assumptions. These can be obtained by applying our flatness criteria discussed in Section 3.

**Notation.** Our ground field is always $\mathbb{C}$.

Let $\pi : X \to Y$ be a morphism of complex analytic spaces. Given $t \in Y$, we denote by $X_t$ the scheme-theoretical fiber over $t$. We refer to the reduced scheme $(X_t)_{\text{red}}$ as the set-theoretical fiber over $t$.

Given a locally free sheaf $E$ on a complex analytic space $X$, we denote by $\mathbb{P}(E)$ the Grothendieck projectivization $\text{Proj}_X(\text{Sym}(E))$.

Varieties are always assumed to be irreducible and reduced.

### 2. Proofs

In order to characterize $\mathbb{P}^n$-bundles that are not necessarily scrolls, we will use the following lemma to construct “local $\mathcal{O}(1)$’s”. This result follows from the proof of [AM97, Lemma 3.3]. We reprove it here for the reader’s convenience.

**Lemma 6.** Let $X$ and $U$ be complex analytic spaces, with $U \setminus \{o\} \cong \Delta^m \setminus \{0\}$ for some point $o \in U$ and some $m \geq 2$. Let $\pi : X \to U$ be a surjective projective morphism whose restriction to $X \setminus \pi^{-1}(o)$ is a $\mathbb{P}^n$-bundle, and assume that $\text{codim}_X (\pi^{-1}(o)) \geq 2$. Let $\mathcal{M}$ be a $\pi$-ample line bundle on $X$, and let $d \in \mathbb{Z}_{\geq 0}$ be such that $\mathcal{M}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^n}(d)$ for every $t \in U \setminus \{o\}$. Then there exists a coherent sheaf $\mathcal{L}$ on $X$ such that $\mathcal{L}|_{X \setminus \pi^{-1}(o)}$ is invertible and $(\mathcal{L}|_{X \setminus \pi^{-1}(o)})^\otimes d \cong \mathcal{M}|_{X \setminus \pi^{-1}(o)}$. If, moreover, $X$ is smooth, then there is a line bundle $\mathcal{L}$ on $X$ such that $\mathcal{L}^\otimes d \cong \mathcal{M}$.

**Proof.** Set $U^* = U \setminus \{o\}$, $X^* = X \setminus \pi^{-1}(o)$, and $\mathcal{M}^* = \mathcal{M}|_{X^*}$. We will apply Leray spectral sequence to the morphism $\pi|_{X^*} : X^* \to U^*$ and the locally constant sheaf $\mathbb{Z}_{X^*}$. Set $E_2^{p,q} := H^p(U^*, R^q(\pi|_{X^*}), \mathbb{Z}_{X^*})$, and denote the differentials of the corresponding spectral sequence by $d_2^{p,q}$. The cohomology classes $c_1(\mathcal{M})^k$ yield the Leray-Hirsch theorem for $\mathbb{Q}_{X^*}$ (see [Vo92, Theorem 7.33]) and its proof, or [BT82, p.192-3]). Hence, $d_2^{p,q} \otimes \mathbb{Q} = 0$ for $r \geq 2$, and abutment at $E_2$ follows by recursion on $r$, using that the $E_2^{p,q}$’s are free abelian groups. So there is an isomorphism of $H^*(U^*, \mathbb{Z})$-algebras

$$H^*(X^*, \mathbb{Z}) \cong H^*(U^*, \mathbb{Z}) \otimes H^*(\mathbb{P}^n, \mathbb{Z}).$$

In particular, $H^2(X^*, \mathbb{Z}) \cong \mathbb{Z}$, and the cokernel of the composed Chern class map

$$c_1 : \text{Pic}(X^*) \to H^2(X^*, \mathbb{Z}) \cong \mathbb{Z}$$

is finite. On the other hand, this cokernel injects into $H^2(X^*, \mathcal{O}_{X^*})$, which is torsion-free. Hence $c_1$ is surjective.

Note that $c_1(\mathcal{M}^*) = d$. Since the kernel of $c_1$ is divisible, there is a line bundle $\mathcal{L}$ on $X^*$ such that $(\mathcal{L}^*)^\otimes d \cong \mathcal{M}^*$. We take $\mathcal{L}$ to be a coherent sheaf on $X$ extending $\mathcal{L}^*$ (see [Ser66, Theorem 1]). If $X$ is smooth, then, since $\text{codim}_X (X \setminus X^*) \geq 2$, there is an isomorphism $\mathcal{P}(\mathcal{L}^*) \cong \mathcal{P}(\mathcal{L})$. Let $\mathcal{L} \in \text{Pic}(X)$ correspond to $\mathcal{L}^* \in \text{Pic}(X^*)$. Then $\mathcal{L}^\otimes d \cong \mathcal{M}$.

**Proof of Theorem 5.** In order to show that $\pi$ is a $\mathbb{P}^n$-bundle, it suffices to prove that the scheme-theoretical fiber $X_0$ over $0 \in \Delta^m$ is isomorphic to $\mathbb{P}^n$. By intersecting $\Delta^m$ with a 2-plane passing through $0$, we may assume that $m = 2$.

Let $\mathcal{M}$ be a $\pi$-ample line bundle on $X$, and let $d \in \mathbb{Z}$ be such that $\mathcal{M}|_{X_t} \cong \mathcal{O}_{\mathbb{P}^n}(d)$ for every $t \in \Delta^2 \setminus \{0\}$. The restriction $\mathcal{M}|_{X_0}$ is ample. Hence, by replacing $\mathcal{M}$
with a sufficiently high tensor power if necessary, we may assume that $M|_{X_t}$ is very ample and that $H^i(X_t, M|_{X_t}) = 0$ for every $t \in \Delta^2$ and $i > 0$. Since $\pi$ is flat, $\chi(X_t, M|_{X_t})$ is constant on $t \in \Delta^2$, and hence so is $h^0(X_t, M|_{X_t})$. Set $\mathcal{F} = \pi_* M$. Then $\mathcal{F}$ is locally free and the natural map $\pi^* \mathcal{F} \to M$ is surjective (see for example [Loo75] Theorem 1.4)). This yields an embedding $i : X \to \mathbb{P}(\mathcal{F})$ over $\Delta^2$.

By Lemma 6 there is a coherent sheaf $\mathcal{E}$ on $X$ such that $\mathcal{E}|_{X \setminus \pi^{-1}(\bar{0})}$ is invertible and $(\mathcal{E}|_{X \setminus \pi^{-1}(\bar{0})})^{\otimes d} \cong M|_{X \setminus \pi^{-1}(\bar{0})}$. Set $\mathcal{E} = \pi_*(\mathcal{F})^{\vee \vee}$. Since $\Delta^2$ is smooth and two-dimensional, $\mathcal{E}$ is locally free (see for example [OSS80] II. Lemma 1.1.10)). Consider the projectivization $p : \mathbb{P}(\mathcal{E}) \to \Delta^2$, with tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. There is an isomorphism $\varphi : \mathbb{P}(\mathcal{E}) \setminus p^{-1}(\bar{0}) \cong X \setminus \pi^{-1}(\bar{0})$ such that $\varphi^*(M|_{X \setminus \pi^{-1}(\bar{0})}) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)|_{\mathbb{P}(\mathcal{E}) \setminus p^{-1}(\bar{0})}$.

Note that $p_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$ is locally free and that the natural map $p^* p_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$ is surjective. The corresponding morphism $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(p_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d))$ is an embedding over $\Delta^2$, which restricts to the $d$-uple embedding of $\mathbb{P}^n$ on each fiber of $p$. By construction, the locally free sheaves $p_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$ and $\mathcal{F}$ are isomorphic over $\bigcup \{0\}$, hence isomorphic over $\Delta^2$. Thus, there is an embedding $j : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{F})$ over $\Delta^2$ such that $j = i \circ \varphi$ on $\mathbb{P}(\mathcal{E}) \setminus p^{-1}(\bar{0})$. It follows that the closure of $i(X \setminus \pi^{-1}(\bar{0}))$ in $\mathbb{P}(\mathcal{E})$ is $j(\mathbb{P}(\mathcal{E}))$, and thus $X \cong \mathbb{P}(\mathcal{E})$.

Once we know that $\pi$ is a $\mathbb{P}^n$-bundle, we use Lemma 6 to construct a global $\mathcal{O}(1)$ for $\pi$.

**Proof of Corollary 5.** Let $U$ be a smooth variety, and $U \to V$ a surjective quasi-finite morphism onto an open subset $V \subset Y$. Theorem 2 applied to the induced flat morphism $X \times_Y U \to U$ shows that $S_\pi$ has pure codimension 1 on $V$.

**Proof of Theorem 5.** First we prove that $S_\pi$ is either empty or has pure codimension 1. Given $y \in S_\pi$, let $Z \subset X$ be a complete intersection of $n$ general very ample divisors on $X$. Then $Z$ is smooth by Bertini’s Theorem, and $\pi|_Z : Z \to Y$ is finite over a neighborhood of $y$. Our claim follows from Corollary 5.

If $S_\pi = \emptyset$, then $Y$ is smooth. Indeed, given $y \in Y$, by Bertini’s Theorem, we can take $Z$ as above such that $\pi|_Z : Z \to Y$ is unramified over $y$.

Next, we show that $R_\pi$ is Zariski dense in $S_\pi$. Let $C \subset Y$ be a smooth curve obtained as complete intersection of general very ample divisors on $Y$. Then $X_C = \pi^{-1}(C)$ is smooth by Bertini’s Theorem, and $C$ intersects every irreducible component of $S_\pi$ at general points. Set $U := C \setminus R_\pi$ and $X_U := \pi^{-1}(U)$. The relative Picard number $\rho(X_U/U)$ equals 1, since every fiber of $\pi$ over $U$ is irreducible. Let $V \subset U$ be an open subset over which $\pi$ is a $\mathbb{P}^n$-bundle. Since $\dim V = 1$, every $\mathbb{P}^n$-bundle over $V$ is a scroll. Hence there is a line bundle $\mathcal{L}_V$ on $\pi^{-1}(V)$ such that $\mathcal{L}_V|_{X_t} \cong \mathcal{O}_{\mathbb{P}^n}(1)$ for every $t \in V$. We can extend $\mathcal{L}_V$ to a line bundle $\mathcal{L}$ on $X_C$. The restriction of $\mathcal{L}$ to $X_U$ is $\pi$-ample since $\rho(X_U/U) = 1$. Since $c_1(\mathcal{L})^n \cdot X_t = 1$ for every $t \in C$, all fibers of $\pi$ over $U$ are reduced. Theorem F1 then implies that $\pi|_{X_U} : X_U \to U$ is a $\mathbb{P}^n$-bundle. This shows that $R_\pi = S_\pi$.

3. **Flatness criteria**

It is often useful to have characterizations of $\mathbb{P}^n$-bundles without flatness assumptions. In this context, we recall the following result of Fujita.

**Theorem F2** ([Fuj87] Lemma 2.12]). Let $X$ be a smooth complex projective variety, $Y$ a normal complex projective variety, and $\pi : X \to Y$ a surjective equidimensional
morphism. Let $L$ be an ample line bundle on $X$, and suppose that $(X_t, L|_{X_t}) \cong (\mathbb{P}^n, O_{\mathbb{P}^n}(1))$ for general $t \in Y$. Then $Y$ is smooth, $\pi$ is a $\mathbb{P}^n$-bundle, and $L$ is a global $O(1)$ for $\pi$.

One reduces Theorem F2 to Theorem F1 by applying the following flatness criterion.

**Criterion 7** ([Gro66, 6.1.5]). An equidimensional proper morphism $\pi : X \to Y$ of algebraic varieties is flat, provided that $Y$ is smooth and $X$ is locally Cohen-Macaulay.

**Remark 8.** Let $\pi : X \to Y$ be a finite flat morphism of algebraic varieties. Suppose that $Y$ is locally Cohen-Macaulay at a point $y = \pi(x)$. We claim that $X$ is locally Cohen-Macaulay at $x$. Indeed, by [Eis95, Corollary 18.17], this is the case if $Y$ is smooth at $y$. The general case can be reduced to this one by applying Noether normalization theorem to $Y$ and observing Criterion 7.

More refined flatness criteria can be found in [Kol95] and [Kol11]. The problem is more delicate under the presence of everywhere nonreduced fibers. The next example illustrates this situation.

**Example 9.** Let $\sigma$ be an involution of $\mathbb{P}^n$. Consider the diagonal action of $\mu_2$ on $\mathbb{P}^n \times \mathbb{A}^2$, where the action on $\mathbb{P}^n$ is given by $\sigma$, and the action on $\mathbb{A}^2$ is given by the antipodal map. Set $X := (\mathbb{P}^n \times \mathbb{A}^2)/\mu_2$, $Y := \mathbb{A}^2/\mu_2$, and denote by $o \in Y$ the unique singular point of $Y$. The actions induce a proper equidimensional morphism $\pi : X \to Y$ such that $X_t \cong \mathbb{P}^n$ for $t \in Y \setminus \{o\}$, while $X_o$ is not generically reduced. Moreover $(X_o)_{\text{red}} \cong \mathbb{P}^n/\mu_2$. Note that $\pi : X \to Y$ is not flat. This can be seen by considering the induced morphism $X \times_Y \mathbb{A}^2 \to \mathbb{A}^2$ and applying Theorem 7.

We end the paper with some flatness criteria.

**Proposition 10.** Let $\pi : X \to Y$ be a projective surjective equidimensional morphism between normal algebraic varieties. Then $\pi$ is flat at $x \in X$, provided that one of the following conditions holds.

1. $\pi$ has connected fibers, $X$ is smooth at $x$, and there is a surjective quasi-finite morphism from a smooth variety onto a neighborhood of $\pi(x)$ in $Y$ that is étale in codimension 1.
2. $X$ is locally Cohen-Macaulay at $x$, and there is a finite flat morphism from a smooth variety onto a neighborhood of $\pi(x)$ in $Y$.

**Remark 11.** The last part of condition (1) above holds for example if the fiber through $x$ has a generically reduced irreducible component, or if $\pi(x)$ is a quotient singularity of $Y$.

**Proof.** Assume (1) holds. Let $U \to V$ be a surjective quasi-finite morphism from a smooth variety onto a neighborhood of $\pi(x)$, étale in codimension 1. Set $X' := X \times_Y U$, and note that it is irreducible since $\pi$ has connected fibers and $X$ is normal (and so is the general fiber of $\pi$). The induced morphism $\varphi : X' \to X$ is quasi-finite and étale in codimension 1. By purity of the branch locus (see [AK71]), we conclude that $\varphi$ is étale, and thus $X'$ is smooth at any point $x'$ over $x \in X$. The induced morphism $\pi' : X' \to U$ is then flat at $x'$ by Criterion 7 hence $\pi$ is flat at $x$ (see [Har77, Theorem III.9.9]).

Assume (2) holds. Let $U \to V$ be a finite flat morphism from a smooth variety onto a neighborhood of $\pi(x)$. Set $X' := X \times_Y U$. By Remark 8 $X'$ is locally
Cohen-Macaualay at any point \( x' \) over \( x \in X \). The induced morphism \( \pi' : X' \rightarrow U \) is then flat at \( x' \) by Criterion\(^7\) hence \( \pi \) is flat at \( x \).

\[ \square \]

**Remark 12.** Theorem F2 can be generalized to normal varieties \( X \) (see [AD12, Proposition 4.10]). Other characterizations of scrolls in a similar vein appear in [HN12].

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