Explicit solution for a vibrating bar with viscous boundaries and internal damper

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Abstract We investigate longitudinal vibrations of a bar subjected to viscous boundary conditions at each end and an internal damper at an arbitrary point along the bar’s length. The system is described by four independent parameters and exhibits a variety of behaviors including rigid motion, super stability/instability and zero damping. The solution is obtained by applying the Laplace transform to the equation of motion and computing the Green’s function of the transformed problem. This leads to an unconventional eigenvalue-like problem with the spectral variable in the boundary conditions. The eigenmodes of the problem are necessarily complex-valued and are not orthogonal in the usual inner product. Nonetheless, in generic cases we obtain an explicit eigenmode expansion for the response of the bar to initial conditions and external force. For some special values of parameters the system of eigenmodes may become incomplete, or no non-trivial eigenmodes may exist at all. We thoroughly analyze physical and mathematical reasons for this behavior and explicitly identify the corresponding parameter values. In particular, when no eigenmodes exist, we obtain closed form solutions. Theoretical analysis is complemented by numerical simulations, and analytic solutions are compared to computations using finite elements.

Keywords Longitudinal vibrations · Modal decomposition · Vibratory response · Viscous boundary conditions

1 Introduction

In this article, we analyze longitudinal vibrations of a bar with dampers attached at each end as well as at an internal point of the bar. This type of problem occurs in modeling structures containing shock absorbers and in control of continuous structures with discrete elements. Mathematically, the problem reduces to solving the wave equation modified by a Dirac delta term with viscous boundary conditions. When the boundary conditions are classical, e.g., the ends are free or clamped, separation of variables or Laplace transforms reduce the situation to a boundary eigenvalue problem for a second-order ODE called the Sturm–Liouville problem. These problems are self-adjoint and
admit a complete system of orthogonal eigenmodes the solution can be expanded into with coefficients determined from initial values using the orthogonality.

When viscous boundaries are present the Laplace transform leads to a boundary value problem with the spectral parameter entering boundary conditions. One still gets a system of eigenmodes, but they are not orthogonal in the usual inner product, and the eigenvalues are general complex numbers reflecting the non-self-adjoint nature of the problem. For some critical values of damping parameters the system of eigenmodes may not be complete, and even when it is finding the expansion coefficients in terms of initial data is non-trivial because the eigenmodes are not orthogonal. Although, studied by mathematicians [1,2] such problems and their properties are rather sparsely treated in the engineering literature, nevertheless see [3–6] and Example 4 in [7, Chap. 4]. Hull [3] was the first to treat a bar with a viscous boundary at one end and clamped at the other, but he utilized a non-standard approach to decoupling the equations of motions and provided a response only for a harmonic driving force. Udwadia [6] appears to be the first to provide a complete closed form solution to this problem via Laplace transform.

Adding an internal damper at an arbitrary point of the bar, as we do in this article, significantly complicates a closed form solution to the problem. In particular, it is no longer possible to find analytic formulas for the eigenvalues since their determination depends on solving algebraic equations of arbitrarily high degree. However, if the eigenvalues are found numerically the solution can be written explicitly in a closed form. We obtain the analytical solution by taking the Laplace transform and finding the Green’s function for the resulting boundary eigenvalue problem. In our analysis, we are able to take advantage of general mathematical results that simplify calculations considerably. For example, although, we do find a cumbersome explicit formula for the Green’s function it is not necessary to find the expansion of its inverse Laplace transform, which depends on eigenmodes and eigenvalues only. Moreover, many qualitative traits of the solution can be gleaned from the characteristic equation for eigenvalues directly without computing the vibratory response. Internal damper in a problem with free ends was considered in [8].

Behavior of the bar is controlled by four dimensionless parameters, the damping coefficients $h_1$ and $h_2$ at the left and right ends, the internal damping coefficient $h_3$, and the ratio $a/L$ characterizing the position of the internal damper ($a$ is the distance to the damper from the left end of the bar and $L$ is its length). Since, the dimension of the parameter space is four it can not be easily visualized. As the parameters are varied, the bar exhibits a variety of behaviors including rigid motions, zero damping, super stability, and instability. Although, a four-dimensional diagram can not be drawn we give analytic conditions for all these types of behavior. Much of the unusual behavior is due to the fact that we do not restrict $h_i$’s to positive values they would take if the dampers are realized as dashpots. For the negative values, we are dealing with so-called active dashpots, or rather “pushpots”, that add energy to the bar instead of damping it. Such discrete elements are sometimes used in control problems for continuous structures [6].

Perhaps the most striking observation is the extreme sensitivity of the eigenvalue distribution to the nature of the number $a/L$. When this number is rational the eigenvalues are generically distributed along $p$ vertical lines in the complex plane, where $p$ is the denominator of $a/L$ in lowest terms. This significantly complicates expansion into eigenmodes since increasingly larger numbers of them have to be kept. When $a/L$ is irrational this distribution appears random. Vibratory response on the other hand is qualitatively insensitive to the placement of the internal damper, and in practice one may want to use ratios with small denominators like $1/2, 1/3, 2/3$, etc. A flip side of these observations is that while FEM produces good approximation for the vibratory response at least for short times it performs poorly in approximating the eigenvalues. In fact, it produces spurious eigenvalues with large real parts that do not converge to actual eigenvalues as the number of elements is increased. When the real parts of the spurious eigenvalues are positive FEM will lead to large errors in the vibratory response at large times.

We organize our presentation as follows. The next section gives the precise problem statement and describes our approach toward solving it and the main results obtained. In Sects. 3, 4, we respectively derive analytic formulas for the eigenmodes, and reduce computing the eigenvalues to solve an algebraic equation for $a/L$ rational. Section 5 discusses in more detail the case, when the damper is placed exactly in the middle of the bar, i.e., $a/L = 1/2$. We compute the eigenvalues explicitly and also give explicit conditions for the undamped behavior of the bar. The Laplace transform of the Green’s function of our problem is computed in Sect. 6 and we discuss its expansion.