Hourglass alternative and the finiteness conjecture for the spectral characteristics of sets of non-negative matrices

Victor Kozyakin

Institute for Information Transmission Problems
Russian Academy of Sciences
Bolshoj Karetny lane 19, Moscow 127994 GSP-4, Russia

Abstract

Recently Blondel, Nesterov and Protasov proved [1, 2] that the finiteness conjecture holds for the generalized and the lower spectral radii of the sets of non-negative matrices with independent row/column uncertainty. We show that this result can be obtained as a simple consequence of the so-called hourglass alternative used in [3], by the author and his companions, to analyze the minimax relations between the spectral radii of matrix products. Axiomatization of the statements that constitute the hourglass alternative makes it possible to define a new class of sets of positive matrices having the finiteness property, which includes the sets of non-negative matrices with independent row uncertainty. This class of matrices, supplemented by the zero and identity matrices, forms a semiring with the Minkowski operations of addition and multiplication of matrix sets, which gives means to construct new sets of non-negative matrices possessing the finiteness property for the generalized and the lower spectral radii.

Keywords: Matrix products, Non-negative matrices, Joint spectral radius, Lower spectral radius, Finiteness conjecture

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1. Introduction

One of the characteristics that describes the exponential growth rate of the matrix products with factors from a set of matrices, is the so-called joint or generalized spectral radius. Denote by $\mathcal{M}(N,M)$ the set of all real $(N \times M)$-matrices. This set of matrices is naturally identified with the space $\mathbb{R}^{N \times M}$ and therefore, depending on the context, it can be interpreted as topological, metric or normed space. The joint spectral radius [4] of a set of matrices $\mathcal{A} \subset \mathcal{M}(N,N)$ is defined as the value of

$$\rho(\mathcal{A}) = \lim_{n \to \infty} \sup_{A_i \in \mathcal{A}} \|A_n \cdots A_1\|^{1/n} = \inf_{n \geq 1} \sup_{A_i \in \mathcal{A}} \|A_n \cdots A_1\|^{1/n},$$

(1)

\`Funded by the Russian Science Foundation, Project No. 14-50-00150.
Email address: kozyakin@iitp.ru (Victor Kozyakin)
URL: http://www.iitp.ru/en/users/46.htm (Victor Kozyakin)

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where $\| \cdot \|$ is a matrix norm on $\mathcal{M}(N, M)$ generated by the corresponding vector norm on $\mathbb{R}^N$. The *generalized spectral radius* [5, 6] of a bounded set of matrices $\mathcal{A} \subset \mathcal{M}(N, N)$ is the quantity

$$\hat{\rho}(\mathcal{A}) = \lim_{n \to \infty} \sup_{A_i \in \mathcal{A}} \rho(A_n \cdots A_1)^{1/n} = \sup_{n \geq 1} \sup_{A_i \in \mathcal{A}} \rho(A_n \cdots A_1)^{1/n},$$

where $\rho(\cdot)$ is the spectral radius of a matrix, i.e. the maximum of modules of its eigenvalues. If the norms of matrices from the set $\mathcal{A}$ are uniformly bounded then by the Berger-Wang theorem [7] $\rho(\mathcal{A}) = \hat{\rho}(\mathcal{A})$. In the case when the set of matrices $\mathcal{A}$ is compact (closed and bounded), the suprema over $A_i \in \mathcal{A}$ in (1) and (2) may be replaced by maxima.

By replacing the suprema over $A_i \in \mathcal{A}$ in (2) with infima (or with minima, in the case of a compact set of matrices) one can obtain the so-called *joint spectral subradius* or *lower spectral radius* [8]:

$$\check{\rho}(\mathcal{A}) = \lim_{n \to \infty} \inf_{A_i \in \mathcal{A}} \|A_n \cdots A_1\|^{1/n} = \inf_{n \geq 1} \inf_{A_i \in \mathcal{A}} \|A_n \cdots A_1\|^{1/n},$$

which also (for arbitrary, not necessarily bounded set of matrices) may be expressed in terms of the spectral radii instead of norms:

$$\check{\rho}(\mathcal{A}) = \lim_{n \to \infty} \inf_{A_i \in \mathcal{A}} \rho(A_n \cdots A_1)^{1/n} = \inf_{n \geq 1} \inf_{A_i \in \mathcal{A}} \rho(A_n \cdots A_1)^{1/n},$$

as was shown in [8, Theorem B1] for finite sets $\mathcal{A}$, and in [9, Lemma 1.12] and [10, Theorem 1] for arbitrary sets $\mathcal{A}$.

The possibility of explicit calculation of the spectral characteristics of sets of matrices is conventionally associated with the validity of the *finiteness conjecture* according to which the limit in formulas (2) and (4) is attained at some finite value of $n$. For the generalized spectral radius this conjecture was set up by Lagarias and Wang [11] and subsequently disproved by Bousch and Mairesse [12]. Later on there appeared a few alternative counterexamples [13–15]. However, all these counterexamples were pure ‘non-existence’ results which provided no constructive description of the sets of matrices for which the finiteness conjecture fails. The first explicit counterexample to the finiteness conjecture was built in [16], while the general methods of constructing of such a type of counterexamples were presented recently in [17, 18]. The lower radius in a sense is more complex object for analysis than the generalized spectral radius because it generally depends on $\mathcal{A}$ not continuously [19, 20]. However, for the lower spectral radius, disproof of the finiteness conjecture was found to be even easier [12, 21] than for the generalized spectral radius.

Despite the fact that in general the finiteness conjecture is false, attempts to discover new classes of matrices for which it still occurs continues. However, it should be borne in mind that the validity of the finiteness conjecture for some class of matrices provides only a theoretical possibility to explicitly calculate the related spectral characteristics, because in practice calculation of the spectral radii $\rho(A_n \cdots A_1)$ for all possible sets of matrices $A_1, \ldots, A_n \in \mathcal{A}$ may require too much computing resources, even for relatively small values of $n$. Therefore, from a practical point of view, the most interesting are the cases when the finiteness conjecture is valid for small values of $n$. 

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The finiteness conjecture is obviously satisfied for the sets of commuting matrices as well as for sets consisting of upper or lower triangular matrices or of matrices that are isometries in some norm up to a scalar factor (that is, \(\|Ax\| \equiv \lambda_A \|x\|\) for some \(\lambda_A\)). It is less obvious that the finiteness conjecture holds for the class of ‘symmetric’ bounded sets of matrices characterizing by the property that together with each matrix the corresponding set contains also the (complex) conjugate matrix [22 Proposition 18]. In particular, this class includes all the sets of self-adjoint matrices. One of the most interesting classes of matrices for which the finiteness conjecture is valid, for both the generalized and the lower spectral radius, is the so-called class of non-negative matrices with independent row uncertainty [1]. Note that in all these cases, the generalized spectral radius coincides with the spectral radius of a single matrix from \(\mathcal{A}\) or with the spectral radius of the product of a pair of such matrices. In [23] it was demonstrated that the finiteness conjecture is valid for any pair of 2 \(\times\) 2 binary matrices, i.e. matrices with the elements \{0, 1\}, and in [24] a similar result was proved for any pair of 2 \(\times\) 2 sign-matrices, i.e. matrices with the elements \{-1, 0, 1\}. Finally, in [25-29] it was shown that the finiteness conjecture holds for any bounded family of matrices \(\mathcal{A}\), whose matrices, except perhaps one, have rank 1. There are also a number of works with less constructive sufficient conditions for attainability of the generalized spectral radius on a finite product of matrices, see, e.g., the references in [30].

So, calculating the joint and lower spectral radii is a challenging problem, and only for exceptional classes of matrices these characteristics may be found explicitly and expressed by a ‘closed formula’, see, e.g., [30, 31] and the bibliography therein.

Outline the structure of the work. In this section, we have presented a brief overview of the results related to the finiteness conjecture for the spectral characteristics of matrices. In Section 2, we remind the definition of the sets of non-negative matrices with independent row uncertainty, and then in Theorem 1 give a new proof of the related Blondel-Nesterov-Protasov results on finiteness [2]. A key point of this proof is the so-called hourglass alternative, Lemma 1, that has been proposed in [3] to analyze the minimax relations between the spectral radii of matrix products. In Section 3, assertions of Lemma 1 are taken as axioms for the definition of the sets of positive matrices, called hourglass- or \(\mathcal{H}\)-sets, satisfying the hourglass alternative. In Theorem 2 we show that the totality of all \(\mathcal{H}\)-sets of matrices, supplemented by the zero and the identity matrices, forms a semiring. This opens up the possibility of constructing new classes of matrices for which analogues of Theorem 1 are true. The main result of such a kind, Theorem 3, is proved in Section 4, and in Corollary 1 we show that all the assertions on finiteness of the spectral characteristics remain valid for the sets of matrices, obtained as a polynomial Minkowski combination of compact sets of non-negative matrices with independent row uncertainty. In Section 5 we present a general fact about the relationship between the spectral characteristics of sets of matrices and their convex hulls. Concluding remarks are given in Section 6.

2. Sets of matrices with independent row uncertainty

As was noted in Section 1, one of the most interesting classes of matrices for which the finiteness conjecture holds, both for the generalized and lower spectral radius, is the so-called class of non-negative matrices with independent row uncertainty [1]. In this
A matric will be called positive if it is a matric satisfying the inclusions of all the rows constituting the sets. We will use the notation of all the rows constituting the sets. Clearly, any singleton set of matrices is an IRU-set. An IRU-set of matrices will be called positive if so are all its matrices which is equivalent to positivity of all the rows constituting the sets.

If an IRU-set is formed by a set of rows, then the following quantities are well defined:

\[ \rho_{\text{min}}(\mathcal{A}) = \min_{A \in \mathcal{A}} \rho(A), \quad \rho_{\text{max}}(\mathcal{A}) = \max_{A \in \mathcal{A}} \rho(A). \]

We will use the notation

\[ \hat{\rho}_n(\mathcal{A}) = \sup_{A_i \in \mathcal{A}} \rho(A_n \cdots A_1)^{1/n}, \quad \tilde{\rho}_n(\mathcal{A}) = \inf_{A_i \in \mathcal{A}} \rho(A_n \cdots A_1)^{1/n}. \]

As shows the following theorem, the finiteness conjecture is valid for compact IRU-sets of positive matrices.

**Theorem 1.** Let \( \mathcal{A} \) be a compact IRU-set of positive matrices and \( \mathcal{A}^\prime \) be a compact set of matrices satisfying the inclusions \( \mathcal{A} \subseteq \mathcal{A}^\prime \subseteq \text{co}(\mathcal{A}) \), where \( \text{co}(\mathcal{A}) \) stands for the convex hull of the set \( \mathcal{A} \). Then

(i) \( \tilde{\rho}_n(\mathcal{A}^\prime) = \rho_{\text{min}}(\mathcal{A}) \) for all \( n \geq 1 \), and therefore \( \hat{\rho}(\mathcal{A}^\prime) = \rho_{\text{min}}(\mathcal{A}) = \rho_{\text{min}}(\mathcal{A}) \);

(ii) \( \hat{\rho}_n(\mathcal{A}^\prime) = \rho_{\text{max}}(\mathcal{A}) \) for all \( n \geq 1 \), and therefore \( \tilde{\rho}(\mathcal{A}^\prime) = \rho_{\text{max}}(\mathcal{A}) = \rho_{\text{max}}(\mathcal{A}) \).

For the cases \( \mathcal{A} = \mathcal{A}^\prime \) and \( \mathcal{A} = \text{co}(\mathcal{A}) \) this theorem in a somewhat different formulation is proved in [2]. The next example demonstrates that none of the equalities \( \rho_{\text{max}}(\mathcal{A}^\prime) = \rho_{\text{max}}(\mathcal{A}) \) and \( \rho_{\text{min}}(\mathcal{A}^\prime) = \rho_{\text{min}}(\mathcal{A}) \) holds for arbitrary sets of matrices.

**Example 1.** Consider the sets of matrices \( \mathcal{A} = \{A_1, A_2\} \) and \( \mathcal{B} = \{B_1, B_2\} \), where

\[
A_1 = \begin{pmatrix}
0 & 2 \\
0 & 0
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 0 \\
2 & 0
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
2 & 0 \\
0 & 0
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
0 & 0 \\
0 & 2
\end{pmatrix}.
\]

Then \( \rho_{\text{max}}(\mathcal{A}) < \rho_{\text{max}}(\text{co}(\mathcal{A})) \) and \( \rho_{\text{min}}(\mathcal{B}) > \rho_{\text{min}}(\text{co}(\mathcal{B})) \) because

\[
\rho_{\text{max}}(\mathcal{A}) = \max_{A \in \mathcal{A}} \rho(A) = 0, \quad \rho_{\text{max}}(\text{co}(\mathcal{A})) = \max_{A \in \text{co}(\mathcal{A})} \rho(A) \geq \rho\left(\frac{1}{2}(A_1 + A_2)\right) = 1,
\]

\[
\rho_{\text{min}}(\mathcal{B}) = \min_{B \in \mathcal{B}} \rho(B) = 2, \quad \rho_{\text{min}}(\text{co}(\mathcal{B})) = \min_{B \in \text{co}(\mathcal{B})} \rho(B) \leq \rho\left(\frac{1}{2}(B_1 + B_2)\right) = 1.
\]

**Remark.** If an IRU-set \( \mathcal{A} \) is formed by a set of rows \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N \), then its convex hull \( \text{co}(\mathcal{A}) \) is the IRU-set formed by the set of rows \( \text{co}(\mathcal{A}_1), \text{co}(\mathcal{A}_2), \ldots, \text{co}(\mathcal{A}_N) \).
2.1. Hourglass alternative

To prove Theorem 1 we will need some definitions and a number of supporting facts. For vectors \( x, y \in \mathbb{R}^N \), we write \( x \geq y \) or \( x > y \), if all coordinates of the vector \( x \) are not less or strictly greater, respectively, than the corresponding coordinates of the vector \( y \). Similar notation will be applied to matrices.

In the space \( \mathbb{R}^1 \) of real numbers any two elements \( x \) and \( y \) are comparable, i.e. either \( x \leq y \) or \( y \leq x \). In this case we say that the space \( \mathbb{R}^1 \) is linearly ordered. In the spaces \( \mathbb{R}^N \) with \( N > 1 \) the situation is quite different. Here there exist infinitely many pairs of non-comparable elements, and the failure of the inequality \( x \geq y \) does not imply the inverse inequality \( x \leq y \). The existence of noncomparable elements leads to the fact that if, for some \( x \), the system of linear inequalities

\[
Ax \geq v, \quad A \in \mathcal{A} \subset \mathcal{M}(N, M)
\]

has no solution, then it does not mean that for some matrix \( \tilde{A} \in \mathcal{A} \) the inverse inequality \( \tilde{A}x \leq v \) will be valid. Examples of corresponding sets of matrices \( \mathcal{A} \) can be easily constructed. However, as the following lemma shows, for the sets of matrices with independent row uncertainty all is not so bad, and for linear inequalities an analogue of the linear ordering of solutions holds.

**Lemma 1.** Let \( \mathcal{A} \subset \mathcal{M}(N, M) \) be an IRU-set of matrices and let \( \tilde{A}u = v \) for some matrix \( \tilde{A} \in \mathcal{A} \) and vectors \( u, v > 0 \).

We first prove assertion H1. If the inequality \( Au \geq v \) holds for all \( A \in \mathcal{A} \) or there exists a matrix \( A \in \mathcal{A} \) such that \( \tilde{A}u \leq v \) and \( Au \neq v \).

H1: either \( Au \geq v \) for all \( A \in \mathcal{A} \) or there exists a matrix \( A \in \mathcal{A} \) such that \( \tilde{A}u \leq v \) and \( Au \neq v \).

H2: either \( Au \leq v \) for all \( A \in \mathcal{A} \) or there exists a matrix \( A \in \mathcal{A} \) such that \( \tilde{A}u \geq v \) and \( Au \neq v \).

Assertions H1 and H2 have a simple geometrical interpretation. Imagine that the sets \( B_l = \{ x : x \leq v \} \) and \( B_u = \{ x : x \geq v \} \) form the lower and upper bulbs of an hourglass with the neck at the point \( v \). Then Lemma 1 asserts that either all the grains \( Au \) fill one of the bulbs (upper or lower), or there remains at least one grain in the other bulb (lower or upper, respectively). Such an interpretation gives reason to call Lemma 1 the hourglass alternative. This alternative will play a key role in the proof of Theorem 1 as well as in its extension to a new class of matrices. The hourglass alternative has been proposed by the author in [3] to analyze the minimax relations between the spectral radii of matrix products.

**Proof of Lemma 1.** Given an IRU-set of matrices \( \mathcal{A} \subset \mathcal{M}(N, M) \), let \( \tilde{A}u = v \) for some matrix \( \tilde{A} \in \mathcal{A} \) and vectors \( u, v > 0 \).

We first prove assertion H1. If the inequality \( Au \geq v \) holds for all \( A \in \mathcal{A} \) then there is nothing to prove. So let us suppose that for some matrix \( A = (a_{ij}) \in \mathcal{A} \) the inequality \( Au \geq v \) is not satisfied. Then representing the vectors \( u \) and \( v \) in the coordinate form

\[
u = (u_1, u_2, \ldots, u_M)^\top, \quad v = (v_1, v_2, \ldots, v_N)^\top,
\]

we obtain that

\[
a_{i1}u_1 + a_{i2}u_2 + \cdots + a_{iM}u_M < v_i
\]
for some $i \in \{1, 2, \ldots, N\}$; without loss of generality we can assume that $i = 1$. In this case, for the matrix

$$
\tilde{A} = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1M} \\
  \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2M} \\
  \cdots & \cdots & \cdots & \cdots \\
  \tilde{a}_{N1} & \tilde{a}_{N2} & \cdots & \tilde{a}_{NM}
\end{pmatrix},
$$

which is obtained from the matrix $\tilde{A} = (\tilde{a}_{ij})$ by changing the first row to the row $a_1 = (a_{11}, a_{12}, \ldots, a_{1M})$ and therefore also belongs to $\mathcal{A}$, we have the inequalities

$$
a_{11}u_1 + a_{12}u_2 + \cdots + a_{1M}u_M < v_1 \quad \text{and} \quad \tilde{a}_{i1}u_1 + \tilde{a}_{i2}u_2 + \cdots + \tilde{a}_{iM}u_M = v_i, \quad i = 2, 3, \ldots, N.
$$

Consequently, $\tilde{A}u \leq \mathbf{v}$ and $\tilde{A}u \neq \mathbf{v}$, which completes the proof of assertion H1.

Assertion H2 is proved similarly.

We now show how Lemma 1 can be used to analyze the spectral characteristics of sets of matrices. The spectral radius of an $(N \times N)$-matrix $A$ is defined as the maximal modulus of its eigenvalues and denoted by $\rho(A)$. The spectral radius depends continuously on the matrix, and in the case $A > 0$ by the Perron-Frobenius theorem [32, Theorem 8.2.2] the number $\rho(A)$ is a simple eigenvalue of the matrix $A$ whereas all the other eigenvalues of $A$ are strictly less than $\rho(A)$ by modulus. The eigenvector $v = (v_1, v_2, \ldots, v_N)^T$ corresponding to the eigenvalue $\rho(A)$ (normalized, for example, by the equality $v_1 + v_2 + \cdots + v_N = 1$) is uniquely determined and positive.

In the following lemma we consolidate some facts of the theory of positive matrices, which in general are well known, but references to which are spreaded among various publications.

**Lemma 2.** Let $A$ be a non-negative $(N \times N)$-matrix, then the following assertions hold:

(i) if $Au \leq \lambda u$ for some vector $u > 0$, then $\lambda \geq 0$ and $\rho(A) \leq \lambda$;

(ii) moreover, if in conditions of (i) $A > 0$ and $Au \neq \lambda u$, then $\rho(A) < \lambda$;

(iii) if $Au \geq \lambda u$ for some non-zero vector $u \geq 0$ and some number $\lambda \geq 0$, then $\rho(A) \geq \lambda$;

(iv) moreover, if in conditions of (iii) $A > 0$ and $Au \neq \lambda u$, then $\rho(A) > \lambda$.

**Proof.** As stated in [32, Corollary 8.1.29], for any nonnegative matrix $A$ and numbers $\alpha, \beta \geq 0$, the inequalities

$$
\alpha \leq \rho(A) \leq \beta
$$

are valid provided that $\alpha u \leq Au \leq \beta u$ for some $u > 0$, from which assertion (i) immediately follows. Let us prove three remaining assertions.

(iii) Let $Au \leq \lambda u$ for $u > 0$, where $A > 0$ and $Au \neq \lambda u$. Then at least one coordinate of the vector $Au - \lambda u \leq 0$ is strictly negative. Therefore, the condition $A > 0$ implies...
strict negativity of all the coordinates of the vector $A(Au - \lambda u)$. Then there exists $\varepsilon > 0$ such that $A(Au - \lambda u) \leq -\varepsilon u$ and therefore $A^2u = A(Au - \lambda u) + \lambda Au \leq (\lambda^2 - \varepsilon)u$. Then, by (5), we get $\rho(A^2) \leq \lambda^2 - \varepsilon$, and thus $\rho(A) \leq \sqrt{\lambda^2 - \varepsilon} < \lambda$.

(iii) The condition $Au \geq \lambda u$ with a non-zero $u \geq 0$ implies $A^n u \geq \lambda^n u$ for any $n \geq 1$. Then $\|A^n\| \cdot \|u\| \geq \|A^n u\| \geq \lambda^n \|u\|$, where $\| \cdot \|$ is any norm monotone with respect to coordinates of a non-negative vector, e.g. the Euclidean norm or the max-norm. Therefore, $\|A^n\| \geq \lambda^n$, and by Gelfand’s formula [32, Corollary 5.6.14] $\rho(A) = \lim_{n \to \infty} \|A^n\|^{1/n} \geq \lambda$.

(iv) Now let $A > 0$ and $Au \neq \lambda u$. Then at least one coordinate of the vector $Au - \lambda u \geq 0$ is strictly positive. Therefore, the condition $A > 0$ implies strict positivity of all coordinates of the vector $A(Au - \lambda u)$. Then there exists $\varepsilon > 0$ such that $A(Au - \lambda u) \geq \varepsilon u$ and therefore $A^2u = A(Au - \lambda u) + \lambda Au \geq (\lambda^2 + \varepsilon)u$. This, by assertion (iii) applied to the matrix $A^2$, implies $\rho(A^2) \geq \lambda^2 + \varepsilon$, and thus $\rho(A) \geq \sqrt{\lambda^2 + \varepsilon} > \lambda$.

The lemma is proved.

The proofs of Lemmas 1 and 2 are borrowed from [3] and presented here only for the sake of completeness of presentation. Lemma 2 resembles Lemma 1 from [2]. The next lemma shows that for the IRU-sets of positive matrices there are valid assertions in a certain sense inverse to Lemma 2.

**Lemma 3.** Let $\mathcal{A} \subset \mathcal{M}(N, N)$ be a compact IRU-set of positive matrices, then the following assertions hold:

(i) if $\bar{A} \in \mathcal{A}$ is a matrix satisfying $\rho(\bar{A}) = \rho_{\min}(\mathcal{A})$ and $\bar{v}$ is its positive eigenvector corresponding to the eigenvalue $\rho(\bar{A})$, then $\bar{A} \bar{v} \geq \rho_{\min}(\mathcal{A}) \bar{v}$ for all $A \in \mathcal{A}$;

(ii) if $\bar{A} \in \mathcal{A}$ is a matrix satisfying $\rho(\bar{A}) = \rho_{\max}(\mathcal{A})$ and $\bar{v}$ is its positive eigenvector corresponding to the eigenvalue $\rho(\bar{A})$, then $\bar{A} \bar{v} \leq \rho_{\max}(\mathcal{A}) \bar{v}$ for all $A \in \mathcal{A}$.

**Proof.** To prove assertion (i) let us note that $\bar{A} \bar{v} = \rho_{\min}(\mathcal{A}) \bar{v}$. Then by assertion (i) of Lemma 1 either $\bar{A} \bar{v} \geq \rho_{\min}(\mathcal{A}) \bar{v}$ for all $A \in \mathcal{A}$ or there exists a matrix $\bar{A} \in \mathcal{A}$ such that $\bar{A} \bar{v} \leq \rho_{\min}(\mathcal{A}) \bar{v}$ and $\bar{A} \bar{v} \neq \rho_{\min}(\mathcal{A}) \bar{v}$. In the latter case, by Lemma 2 there would be valid the inequality $\rho(\bar{A}) < \rho_{\min}(\mathcal{A})$ which contradicts to the definition of $\rho_{\min}(\mathcal{A})$. Hence, the inequality $\bar{A} \bar{v} \geq \rho_{\min}(\mathcal{A}) \bar{v}$ holds for all $A \in \mathcal{A}$.

Assertion (ii) is proved similarly.

Now all is ready to prove Theorem 1.

### 2.2. Proof of Theorem 1

To prove assertion (i) choose a matrix $\bar{A} \in \mathcal{A}$ for which $\rho(\bar{A}) = \rho_{\min}(\mathcal{A})$ and denote by $\bar{v} = (\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_N)^T$ a positive eigenvector of $\bar{A}$ corresponding to the eigenvalue $\rho(\bar{A})$. Then by assertion (i) of Lemma 3 $\bar{A} \bar{v} \geq \rho_{\min}(\mathcal{A}) \bar{v}$ for all $A \in \mathcal{A}$ and therefore for all $A \in \mathcal{A}$. Hence $A_{i_1} \cdots A_{i_k} \bar{v} \geq \rho_{\min}(\mathcal{A}^k) \bar{v}$ for all $A_{i_j} \in \mathcal{A}$. Consequently, by Lemma 2 $\rho(A_{i_1} \cdots A_{i_k}) \geq \rho_{\min}(\mathcal{A})$ for all $A_{i_j} \in \mathcal{A}$ and therefore

$$\rho_{\min}(\mathcal{A}^k) \geq \rho_{\min}(\mathcal{A}) \rho_{\min}(\mathcal{A}^{k-1}).$$

On the other hand, since $\bar{A} \in \mathcal{A} \subset \mathcal{A}^k$ then clearly

$$\rho_{\min}(\mathcal{A}) \leq \rho(\mathcal{A}^k)^{1/n} \leq \rho_{\min}(\mathcal{A}),$$
which implies \( \hat{\rho}_n(\mathcal{A}) = \rho_{\min}(\mathcal{A}) \) for all \( n \geq 1 \). Observing now that \( \hat{\rho}_1(\mathcal{A}) = \rho_{\min}(\mathcal{A}) \), we complete the proof of assertion (i).

The proof of assertion (ii) is carried out by verbatim repetition of the proof of assertion (i) by taking instead of \( \tilde{A} \) a matrix maximizing the spectral radii of matrices from \( \mathcal{A} \) and instead of the estimates for \( \rho_{\min}(\mathcal{A}) \) the corresponding estimates for \( \rho_{\max}(\mathcal{A}) \), and then using assertion (ii) of Lemma 3 instead of assertion (i).

3. \( \mathcal{H} \)-sets of matrices

Apart from general properties of positive matrices given in Lemma 2, the proof of Theorem 1 relies only on those properties of IRU-sets of matrices which were formulated in Lemma 1 as statements H1 and H2 of the hourglass alternative. Therefore, it is natural to axiomatize the class of matrices satisfying the statements H1 and H2, and to study its properties.

3.1. Main definitions

A set of positive matrices \( \mathcal{A} \subset \mathcal{M}(N, M) \) will be called \( \mathcal{H} \)-set or hourglass set if every time the equality \( \tilde{A}u = v \) is true for some matrix \( \tilde{A} \in \mathcal{A} \) and vectors \( u, v > 0 \) there are also true assertions H1 and H2 of Lemma 1.

A trivial example of \( \mathcal{H} \)-sets are linearly ordered sets \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \) composed of positive matrices \( A_i \) satisfying the inequalities \( 0 < A_1 < A_2 < \cdots < A_n \). In this case, for each \( u > 0 \), the vectors \( A_1u, A_2u, \ldots, A_nu \) are strictly positive and linearly ordered, which yields the validity of assertions H1 and H2 for \( \mathcal{A} \). A less trivial and more interesting example of \( \mathcal{H} \)-sets, as follows from Lemma 1, is the class of sets of positive matrices with independent row uncertainty.

Not every set of positive matrices is an \( \mathcal{H} \)-set. A relevant example could easily be built for the set \( \mathcal{A} = \{A, B\} \) consisting of two \((2 \times 2)\)-matrices. In this case, for \( \mathcal{A} \) to be an \( \mathcal{H} \)-set, it is necessary that the vectors \( Au \) and \( Bu \) were comparable for any vector \( u > 0 \), that is, either \( Au \leq Bu \) or \( Bu \leq Au \). But this is not fulfilled, for example, in the case when \( AB = P \), where \( P \) is any projection on the linear hull of the vector \((-1, 1)\).

Let us describe some general properties of the class of \( \mathcal{H} \)-sets of matrices. Introduce the operations of Minkowski addition and multiplication for sets of matrices:

\[
\mathcal{A} + \mathcal{B} = \{A + B : A \in \mathcal{A}, B \in \mathcal{B}\}, \quad \mathcal{A} \mathcal{B} = \{AB : A \in \mathcal{A}, B \in \mathcal{B}\},
\]

and also the operation of multiplication of a set of matrices by a scalar:

\[
t\mathcal{A} = \{tA : t \in \mathbb{R}, A \in \mathcal{A}\}.
\]

Naturally, the operation of addition is admissible if and only if the matrices from the sets \( \mathcal{A} \) and \( \mathcal{B} \) are of the same size, while the operation of multiplication is admissible if and only if the sizes of the matrices from sets \( \mathcal{A} \) and \( \mathcal{B} \) are matched: dimension of the rows of the matrices from \( \mathcal{A} \) is the same as dimension of the columns of the matrices from \( \mathcal{B} \). Problems with matching of sizes do not arise when one considers sets of square matrices of the same size.

In what follows, we will need to make various kinds of limiting transitions with the matrices from the sets under consideration as well as with the sets of matrices themselves.
Theorem 2. The following is true:

(i) \( \mathcal{A} + \mathcal{B} \in \mathcal{H}(N, M) \) if \( \mathcal{A} \in \mathcal{H}(N, M) \) and \( \mathcal{B} \in \mathcal{H}(N, M) \);

(ii) \( \mathcal{A} \mathcal{B} \in \mathcal{H}(N, Q) \) if \( \mathcal{A} \in \mathcal{H}(N, M) \) and \( \mathcal{B} \in \mathcal{H}(M, Q) \);

(iii) \( t \mathcal{A} = \mathcal{A} t \in \mathcal{H}(N, M) \) if \( t > 0 \) and \( \mathcal{A} \in \mathcal{H}(N, M) \).

Proof. First prove (i). Show the validity of assertion H1 for the sum \( \mathcal{A} + \mathcal{B} \). Let, for some matrix \( C \in \mathcal{A} + \mathcal{B} \) and vectors \( u, v > 0 \), the equality \( Cu = v \) holds. Then, by definition of the set \( \mathcal{A} + \mathcal{B} \), there exist matrices \( \hat{A} \in \mathcal{A} \) and \( \hat{B} \in \mathcal{B} \) such that \( C = \hat{A} + \hat{B} \), and hence \( (\hat{A} + \hat{B})u = v \). Denote \( \hat{A}u = v_1 \) and \( \hat{B}u = v_2 \) then \( v_1 + v_2 = v \), where \( v_1, v_2 > 0 \) due to the positivity of the matrices \( \hat{A} \) and \( \hat{B} \). If

\[
Au \geq v_1, \quad Bu \geq v_2 \quad \text{for all} \quad A \in \mathcal{A}, \quad B \in \mathcal{B},
\]

then, for all \( A + B \in \mathcal{A} + \mathcal{B} \), there will be valid also the inequality \( (A + B)u \geq v_1 + v_2 = v \). Thus, in this case assertion H1 is proven.

Now, let (6) fail, and let, to be specific, the inequality \( Au \geq v_1 \) be not valid for at least one matrix \( A \in \mathcal{A} \). Then, since \( \mathcal{A} \in \mathcal{H}(N, M) \), assertion H1 for the set of matrices \( \mathcal{A} \) implies the existence of a matrix \( \hat{A} \in \mathcal{A} \) such that \( Au \leq v_1 \) and \( \hat{A}u \neq v_1 \). In this case the matrix \( \hat{A} + \hat{B} \in \mathcal{A} + \mathcal{B} \) will satisfy the inequality \( (\hat{A} + \hat{B})u \leq v_1 + v_2 = v \), and moreover, \( (\hat{A} + \hat{B})u \neq v \) since \( \hat{B}u = v_2 \) while \( \hat{A}u \neq v_1 \). Thus, assertion H1 for the set \( \mathcal{A} + \mathcal{B} \) is also valid in the case when (6) fails.

The proof of assertion H2 for the set \( \mathcal{A} + \mathcal{B} \) is similar. Compactness of the set \( \mathcal{A} + \mathcal{B} \) in the case when the sets \( \mathcal{A} \) and \( \mathcal{B} \) are compact is evident.

We now prove (ii). Show the validity of assertion H1 for the product \( \mathcal{A} \mathcal{B} \). Suppose that \( Cu = v \) for some matrix \( C \in \mathcal{A} \mathcal{B} \) and vectors \( u, v > 0 \). Then, by definition of the set \( \mathcal{A} \mathcal{B} \), there exist matrices \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that \( \hat{A} \hat{B}u = v \). By denoting \( \hat{B}u = w \) we obtain, due to the positivity of the matrix \( \hat{B} \) and the vector \( u \), that \( w > 0 \) and \( \hat{A}w = u \). If

\[
Aw \geq v, \quad Bu \geq w \quad \text{for all} \quad A \in \mathcal{A}, \quad B \in \mathcal{B},
\]

then, due to the positivity of the matrices \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \), for all \( AB \in \mathcal{A} \mathcal{B} \) there will be valid also the inequalities \( ABu \geq Aw \geq v \). Thus in this case assertion H1 is proved.

Now, let (7) fail, and let, to be specific, the inequality \( Bu \geq w \) be not valid for at least one matrix \( B \in \mathcal{B} \). Then, since \( \mathcal{B} \in \mathcal{H}(N, M) \), assertion H1 for the set of matrices \( \mathcal{B} \) implies the existence of a matrix \( \hat{B} \in \mathcal{B} \) such that \( \hat{B}u \leq w \) and \( \hat{B}u \neq w \). But in this case the matrix \( \hat{A} \hat{B} \in \mathcal{A} \mathcal{B} \), due to the positivity of the matrix \( \hat{A} \), will satisfy the

\[\text{(i)}\]
inequality \( \bar{A} Bu \leq \bar{A} w = v \), and then \( \bar{A} Bu \neq v \) since \( Bu \leq w \), \( Bu \neq w \) and the matrix \( \bar{A} \) is positive. Thus, assertion H1 for the set \( \mathcal{A} \mathcal{B} \) is also valid in the case when (7) fails.

The proof of assertion H2 for the set \( \mathcal{A} \mathcal{B} \) is similar. Compactness of the set \( \mathcal{A} \mathcal{B} \) in the case when the sets \( \mathcal{A} \) and \( \mathcal{B} \) are compact is evident.

The proof of (iii) is also evident.

Remark. The requirement of positivity for the matrices and the vectors \( u,v \) in the definition of \( \mathcal{H} \)-sets was introduced to ensure the inclusion \( \mathcal{A} \mathcal{B} \in \mathcal{H}(N,Q) \) in Theorem 2, as well as to provide an opportunity to further use of Lemma 2 for the analysis of the spectral properties of the sets of matrices from \( \mathcal{H}(N,Q) \).

By Theorem 2 the totality of sets of square matrices \( \mathcal{H}(N,N) \) is enabled by additive and multiplicative group operations, but itself is not a group, neither additive nor multiplicative. However, after adding the zero additive element \( \{0\} \) and the multiplicative identity element \( \{I\} \) to \( \mathcal{H}(N,N) \), the resulting totality \( \mathcal{H}(N,N) \cup \{0\} \cup \{I\} \) becomes a semiring [33].

Theorem 2 implies that any finite sum of any finite products of sets of matrices from \( \mathcal{H}(N,N) \) is again a set of matrices from \( \mathcal{H}(N,N) \). Moreover, for any integers \( n,d \geq 1 \), all the polynomial sets of matrices

\[
P(\mathcal{A}_1, \mathcal{A}_1, \ldots, \mathcal{A}_n) = \sum_{k=1}^{d} \sum_{i_1, i_2, \ldots, i_k \in \{1,2,\ldots,n\}} p_{i_1, i_2, \ldots, i_k} \mathcal{A}_{i_1} \mathcal{A}_{i_2} \cdots \mathcal{A}_{i_k},
\]  

(8)

where \( \mathcal{A}_1, \mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathcal{H}(N,N) \) and the scalar coefficients \( p_{i_1, i_2, \ldots, i_k} \) are positive, belong to the set \( \mathcal{H}(N,N) \).

The polynomials (8) allow to construct not only the elements \( P(\mathcal{A}_1, \mathcal{A}_1, \ldots, \mathcal{A}_n) \) of the set \( \mathcal{H}(N,N) \) but also the elements of arbitrary sets \( \mathcal{H}(N,M) \), by taking the arguments \( \mathcal{A}_1, \mathcal{A}_1, \ldots, \mathcal{A}_n \) from the sets \( \mathcal{H}(N_i, M_i) \) with arbitrary matrix sizes \( N_i \times M_i \). One must only ensure that the products \( \mathcal{A}_{i_1} \mathcal{A}_{i_2} \cdots \mathcal{A}_{i_k} \) would be admissible and determine the sets of matrices of dimension \( N \times M \).

We have presented above two types of non-trivial \( \mathcal{H} \)-sets of matrices, the sets of matrices with independent row uncertainty and the linearly ordered sets of positive matrices. In this connection, let us denote by \( \mathcal{H}_*(N,M) \) the totality of all sets of \( (N \times M) \)-matrices which can be obtained as admissible finite sums of finite products of the sets of positive matrices with independent rows uncertainty or the sets of linearly ordered positive matrices. In other words, \( \mathcal{H}_*(N,M) \) is the totality of all sets of matrices that can be represented as the values of polynomials (8) with the arguments taken from the sets of the matrices belonging to \( \mathcal{U}(N_i, M_i) \cup \mathcal{L}(N_i, M_i) \).

Question. Does equality \( \mathcal{H}_*(N,M) = \mathcal{H}(N,M) \) hold?

The answer to this question is probably negative, but we do not aware of any counterexamples.

3.2. Closure of the set \( \mathcal{H}(N,M) \)

When considering various types of problems related to the sets of matrices, it is desirable to be able to perform limit transitions. In fact, for further goals we would like
to be able to extend some facts relevant to $\mathcal{H}$-sets of positive matrices to the same kind of sets of matrices, but with non-negative elements. To achieve this, without going too deep into the variety of all topologies on spaces of subsets, we confine ourselves to the description of only one of them, the topology specified by the Hausdorff metric.

Given some matrix norm $\| \cdot \|$ on $\mathcal{M}(N,M)$, denote by $\mathcal{K}(N,M)$ the totality of all compact subsets of $\mathcal{M}(N,M)$. Then for any two sets of matrices $\mathcal{A}, \mathcal{B} \in \mathcal{K}(N,M)$ there is defined the Hausdorff metric

$$H(\mathcal{A}, \mathcal{B}) = \max \left\{ \sup_{A \in \mathcal{A}} \inf_{B \in \mathcal{B}} \| A - B \|, \sup_{B \in \mathcal{B}} \inf_{A \in \mathcal{A}} \| A - B \| \right\},$$

in which $\mathcal{K}(N,M)$ becomes a full metric space. Then $\mathcal{H}(N,M) \subset \mathcal{K}(N,M)$, equipped with the Hausdorff metric, also becomes a metric space.

As is known, see, e.g., [34, Chapter E, Proposition 5], any mapping $F(\mathcal{A})$ acting from $\mathcal{K}(N,M)$ into itself is continuous in the Hausdorff metric at some point $\mathcal{A}_0$ if and only if it is both upper and lower semicontinuous. It is also known [35, Section 1.3] that the mappings $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} + \mathcal{B}$, $(\mathcal{A}, \mathcal{B}) \mapsto \mathcal{A} \cdot \mathcal{B}$, $\mathcal{A} \mapsto \mathcal{A} \times \cdots \times \mathcal{A}$, $\mathcal{A} \mapsto \text{co}(\mathcal{A})$, where $\mathcal{A}$ and $\mathcal{B}$ are compact sets, are both upper and lower semicontinuous. Then these mappings are continuous in the Hausdorff metric, and any polynomial mapping (8) possesses the same continuity properties.

Denote by $\overline{\mathcal{H}}(N,M)$ the closure of the set $\mathcal{H}(N,M)$ in the Hausdorff metric. It is obvious that $\{0\}, \{I\} \in \overline{\mathcal{H}}(N,M)$, and since the Minkowski addition and multiplication of matrix sets are continuous in the Hausdorff metric, then by Theorem 3 the set $\overline{\mathcal{H}}(N,N)$ is a semiring. However, the answer to the question when, for some $\mathcal{A}$, the inclusion $\mathcal{A} \in \overline{\mathcal{H}}(N,M)$ holds, requires further analysis. We restrict ourselves to the description of only one case where the answer to this question can be given explicitly.

Let $\mathbf{1}$ stand for the matrix (of appropriate size) with all elements equal to $1$. First note that each set of linearly ordered non-negative matrices $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, that is a set whose matrices $A_i$ satisfy the inequalities $0 \leq A_1 \leq A_2 \leq \cdots \leq A_n$, is a limiting point in the Hausdorff metric of the family of linearly ordered sets of positive matrices

$$\mathcal{A}(\varepsilon) = \{A_1 + \varepsilon \mathbf{1}, A_2 + 2\varepsilon \mathbf{1}, \ldots, A_n + n\varepsilon \mathbf{1}\}, \quad \varepsilon > 0.$$

Further, let $\mathcal{A}$ be an IRU-set of non-negative matrices. Then any set of matrices

$$\mathcal{A}(\varepsilon) = \mathcal{A} + \varepsilon \mathbf{1} = \{A + \varepsilon \mathbf{1} : A \in \mathcal{A}\}, \quad \varepsilon > 0,$$

is also an IRU-set, this time consisting of positive matrices. To verify this, it suffices to note that if a set of matrices $\mathcal{A}$ is defined by sets of rows $\mathcal{A}_i$, then the set $\mathcal{A} + \varepsilon \mathbf{1}$ will be defined by the sets of rows $\mathcal{A}_i + \varepsilon \mathbf{1} = \{a + \varepsilon \mathbf{1} : a \in \mathcal{A}_i\}$, where $\mathbf{1}$ is the unit row of appropriate size. Moreover, this IRU-set $\mathcal{A}$ of non-negative matrices, as well as in the previous case, will be a limiting point in the Hausdorff metric for the positive family of IRU-sets $\mathcal{A}(\varepsilon)$, $\varepsilon > 0$.

These observations imply the following lemma.

**Lemma 4.** The values of any polynomial mapping (8) with the arguments from finite linearly ordered sets of non-negative matrices or from IRU-sets of non-negative matrices belong to the closure in the Hausdorff metric of the totality of positive $\mathcal{H}$-sets of matrices.
4. Main results

In this section we present a generalization of Theorem 1 to the case of \( \mathcal{H} \)-sets of matrices.

**Theorem 3.** Let \( \mathcal{A} \in \overline{\mathcal{H}}(N,N) \), and let \( \tilde{\mathcal{A}} \) be a set of matrices satisfying the inclusions \( \mathcal{A} \subseteq \tilde{\mathcal{A}} \subseteq \text{co}(\mathcal{A}) \). Then

(i) \( \tilde{\rho}_n(\mathcal{A}) = \rho_{\text{min}}(\mathcal{A}) \) for all \( n \geq 1 \), and therefore \( \tilde{\rho}(\mathcal{A}) = \rho_{\text{min}}(\mathcal{A}) = \rho_{\text{min}}(\mathcal{A}) \);

(ii) \( \tilde{\rho}_n(\mathcal{A}) = \rho_{\text{max}}(\mathcal{A}) \) for all \( n \geq 1 \), and therefore \( \tilde{\rho}(\mathcal{A}) = \rho_{\text{max}}(\mathcal{A}) = \rho_{\text{max}}(\mathcal{A}) \).

**Proof.** If \( \mathcal{A} \in \mathcal{H}(N,N) \) then, by definition, the set \( \mathcal{A} \) consists of positive matrices. Therefore, for \( \mathcal{A} \in \mathcal{H}(N,N) \) assertions H1 and H2 of Lemma 1 hold, which implies that Lemma 3 is valid, too. Then the proof of the theorem word for word repeats the proof of Theorem 1. Thus, we need only consider the case when \( \mathcal{A} \in \overline{\mathcal{H}}(N,N) \) but \( \mathcal{A} \not\in \mathcal{H}(N,N) \).

First prove that, for every \( n \geq 1 \), there are valid the equalities

\[
\tilde{\rho}_n(\mathcal{A}) = \rho_{\text{min}}(\mathcal{A}), \quad \tilde{\rho}_n(\text{co}(\mathcal{A})) = \rho_{\text{min}}(\mathcal{A})
\]

(9)

Since \( \mathcal{A} \in \overline{\mathcal{H}}(N,N) \) then there exists a sequence of sets of matrices \( \mathcal{A}_k \in \mathcal{H}(N,N) \), \( k = 1, 2, \ldots \), converging to \( \mathcal{A} \) in the Hausdorff metric. Then, as it has been already proved, for each \( n, k \geq 1 \) we have the equalities

\[
\tilde{\rho}_n(\mathcal{A}_k) = \rho_{\text{min}}(\mathcal{A}_k), \quad \tilde{\rho}_n(\text{co}(\mathcal{A}_k)) = \rho_{\text{min}}(\mathcal{A}_k),
\]

(10)

and therefore it is natural to try to get (9) by limiting transition from (10). To do this, we recall the following simplified version of Berge’s Maximum Theorem [34, Chapter E, Section 3].

**Lemma 5.** Let \( X \) and \( Y \) be metric spaces, \( \Gamma : X \to Y \) be a multivalued mapping with compact values, and \( \varphi \) be a continuous real function on \( X \times Y \). If the mapping \( \Gamma \) is continuous, that is both upper and lower semicontinuous, at a point \( x_0 \in X \) then both functions \( M(x) = \max_{y \in \Gamma(x)} \varphi(x,y) \) and \( m(x) = \min_{y \in \Gamma(x)} \varphi(x,y) \) are also continuous at the point \( x_0 \).

To use this lemma we will treat \( \mathcal{M}(N,N) \) as a metric space, and take the following notation:

\[
X = \mathcal{K}(N,N), \quad Y = \mathcal{M}(N,N) \times \cdots \times \mathcal{M}(N,N),
\]

\[
x = \mathcal{A} \in X, \quad y = (A_1, \ldots, A_n) \in Y, \quad \Gamma(x) = \mathcal{A} \times \cdots \times \mathcal{A}, \quad \varphi(x,y) = \varphi(y) = \rho(A_n \cdots A_1),
\]

Here, the function \( \varphi(x,y) \), which in fact depends on a single argument \( y \), is continuous. The multivalued mapping \( \Gamma(x) \), for each \( x = \mathcal{A} \in \mathcal{K}(N,N) \), takes compact values and is also continuous in the Hausdorff metric, see, e.g., [35, Section 1.3]. Therefore, \( \min_{y \in \Gamma(x)} \varphi(x,y) = \rho_{\text{min}}(\mathcal{A}) \), and by Lemma 5 the function \( \rho_{\text{min}}(\mathcal{A}) \) is continuous in \( \mathcal{A} \in \mathcal{K}(N,N) \). Similarly, choosing as \( \varphi(x,y) \) the functions of the form \( \varphi(x,y) \equiv \varphi(y) = \rho(A_n \cdots A_1)^{1/n} \) for various \( n \geq 1 \), we obtain from Lemma 5 continuity of the functions \( \tilde{\rho}_n(\mathcal{A}) \) in \( \mathcal{A} \in \mathcal{K}(N,N) \) for all \( n \geq 1 \). And choosing as \( \Gamma(x) \) the multivalued
mapping \( \Gamma(x) = \text{co}(\mathcal{A}) \times \cdots \times \text{co}(\mathcal{A}) \), which also takes compact values and is continuous in the Hausdorff metric because in the Hausdorff metric it is continuous the mapping \( \mathcal{A} \mapsto \text{co}(\mathcal{A}) \) [35, Section 1.3], we obtain similarly that the functions \( \rho_n(\text{co}(\mathcal{A})) \) are continuous in \( \mathcal{A} \in \mathcal{K}(N, N) \) for every \( n \geq 1 \).

Thus, we have shown that all the functions in equalities (10) are continuous in \( \mathcal{A} \in \mathcal{K}(N, N) \) from which, taking the limit as \( \mathcal{A}_k \to \mathcal{A} \in \mathcal{H}(N, N) \), we obtain (9).

Let now \( \mathcal{A} \) be a compact set of matrices satisfying the inclusions \( \mathcal{A} \subseteq \tilde{\mathcal{A}} \subseteq \text{co}(\mathcal{A}) \).

Then, since the quantities \( \tilde{\rho}_n(\cdot) \) are defined as infima over the corresponding sets, we have the inequalities

\[
\tilde{\rho}_n(\text{co}(\mathcal{A})) \leq \tilde{\rho}_n(\tilde{\mathcal{A}}) \leq \tilde{\rho}_n(\mathcal{A}).
\]

Therefore, by virtue of the already proven equalities (9), for each \( n \geq 1 \), there holds the equality

\[
\tilde{\rho}_n(\tilde{\mathcal{A}}) = \rho_{\min}(\mathcal{A}),
\]

and then, due to (4), \( \tilde{\rho}(\mathcal{A}) = \rho_{\min}(\mathcal{A})\) and then \( \rho_{\min}(\mathcal{A}) = \rho_{\min}(\mathcal{A}) \). Assertion (i) of Theorem 3 is completely proved.

The proof of assertion (ii) is similar.

On application of Theorem 3, among the first there arises the question about verification of the inclusion \( \mathcal{A} \in \mathcal{H}(N, N) \), for given sets of matrices \( \mathcal{A} \). One such case has been described in Lemma 4, which implies the following corollary.

**Corollary 1.** Let \( \mathcal{A} \) be a set of matrices obtained as the value of a polynomial mapping (8), whose arguments are finite linearly ordered sets of non-negative matrices or compact IRU-sets of non-negative matrices. Then for any compact set of matrices \( \mathcal{A} \) satisfying the inclusions \( \mathcal{A} \subseteq \mathcal{A} \subseteq \text{co}(\mathcal{A}) \) assertions of Theorem 3 hold.

5. Spectral characteristics of convex hulls of matrix sets

Theorem 3 implies that

\[
\hat{\rho}(\mathcal{A}) = \hat{\rho}(\text{co}(\mathcal{A})), \quad \hat{\rho}(\mathcal{A}) = \hat{\rho}(\text{co}(\mathcal{A})),
\]

for any set \( \mathcal{A} \in \mathcal{H}(N, M) \). In fact, it is known [9, 31] that the first of equalities (11) holds for arbitrary (not necessarily non-negative) sets of matrices \( \mathcal{A} \subset \mathcal{M}(N, N) \), which follows from the obvious observation that

\[
\sup_{A_n \in \mathcal{A}} \|A_n \cdots A_1\| = \sup_{A_n \in \text{co}(\mathcal{A})} \|A_n \cdots A_1\|
\]

for any norm. The second equality in (11) for general sets of matrices is not true, as is seen from the example of the set \( \mathcal{A} = \{I, -I\} \), for which \( \hat{\rho}(\mathcal{A}) = 1 \) while \( \hat{\rho}(\text{co}(\mathcal{A})) = 0 \). In this regard, we note the following general assertion.

**Theorem 4.** For any bounded set of non-negative matrices \( \mathcal{A} \subset \mathcal{M}(N, N) \) the second of equalities (11) holds.
Proof. We will need some auxiliary facts. Let us take in the definition (3) the norm \( \| x \| = \sum_i |x_i| \) and notice that in this case \( \| x \| = \sum_i x_i \) for any \( x = (x_1, x_2, \ldots, x_N)^T \geq 0 \), which implies that
\[
\left\| \sum u_i \right\| = \sum_i \| u_i \| \tag{12}
\]
for any finite set of vectors \( u_i \geq 0 \). Notice also that
\[
\| Ae \| \geq \rho(A), \quad \text{where} \quad e = (1, 1, \ldots, 1)^T, \tag{13}
\]
for any matrix \( A \geq 0 \). Indeed, if inequality (13) is not true then \( \| Ae \| < \rho(A) \), which means that all coordinates of \( Ae \) are less than \( \rho(A) \), i.e. \( Ae < \rho(A)e \). This leads, by assertion (i) of Lemma 2, to the self-contradictory inequality \( \rho(A) < \rho(A) \).

To prove the equality \( \tilde{\rho}(co(\mathcal{A})) = \tilde{\rho}(\mathcal{A}) \) let us observe first that
\[
\tilde{\rho}(co(\mathcal{A})) \leq \tilde{\rho}(\mathcal{A}), \tag{14}
\]
since \( \mathcal{A} \subseteq co(\mathcal{A}) \) while due to the definition (4) both sides of this inequality are infima of the same expression over \( co(\mathcal{A}) \) and \( \mathcal{A} \) respectively.

Now, given \( n \geq 1 \), let for each \( i = 1, 2, \ldots, n \) a matrix \( A_i \) be a finite convex combinations of matrices \( A_j^{(i)} \in \mathcal{A}_j \), that is \( A_i = \sum_j \mu_j^{(i)} A_j^{(i)} \in co(\mathcal{A}) \) with some \( \mu_j^{(i)} \geq 0 \) and \( \sum_j \mu_j^{(i)} = 1 \). Then in view of (12)
\[
\| A_n \cdots A_1 \| \cdot \| e \| \geq \| A_n \cdots A_1 e \| = \left\| \left( \sum_{j_n} \mu_{j_n}^{(n)} A_{j_n}^{(n)} \right) \cdots \left( \sum_{j_1} \mu_{j_1}^{(1)} A_{j_1}^{(1)} \right) e \right\| = \sum_{j_n} \cdots \sum_{j_1} \mu_{j_n}^{(n)} \cdots \mu_{j_1}^{(1)} \| A_{j_n}^{(n)} \cdots A_{j_1}^{(1)} e \|. \tag{15}
\]
Here \( \| e \| = N \), and by (13) \( \| A_{j_n}^{(n)} \cdots A_{j_1}^{(1)} e \| \geq \rho(A_{j_n}^{(n)} \cdots A_{j_1}^{(1)}) \geq \tilde{\rho}_n(\mathcal{A}) \). Therefore,
\[
N \| A_n \cdots A_1 \| \geq \left( \sum_{j_n} \cdots \sum_{j_1} \mu_{j_n}^{(n)} \cdots \mu_{j_1}^{(1)} \right) \tilde{\rho}_n(\mathcal{A}). \tag{16}
\]
Moreover, since \( \sum_{j_n} \cdots \sum_{j_1} \mu_{j_n}^{(n)} \cdots \mu_{j_1}^{(1)} = 1 \) then
\[
\| A_n \cdots A_1 \| \geq \frac{1}{N} \tilde{\rho}_n(\mathcal{A}). \tag{17}
\]
Taking in this last inequality the infimum over all \( A_1, \ldots, A_n \in co(\mathcal{A}) \) we obtain the inequalities
\[
\inf_{A_i \in co(\mathcal{A})} \| A_n \cdots A_1 \| \geq \frac{1}{N} \tilde{\rho}_n(\mathcal{A}), \quad n \geq 1, \tag{18}
\]
which we substitute in (3):
\[
\tilde{\rho}(co(\mathcal{A})) = \lim_{n \to \infty} \inf_{A_i \in co(\mathcal{A})} \| A_n \cdots A_1 \|^\frac{1}{n} \geq \lim_{n \to \infty} \left( \frac{1}{N} \right)^\frac{1}{n} \tilde{\rho}_n(\mathcal{A})^\frac{1}{n} = \tilde{\rho}(\mathcal{A}),
\]
and together with (14) this yields the required equality. \( \square \)
6. Concluding remarks

6.1. Sets of matrices with independent column uncertainty

Since the spectral radius does not change during the transposition of a matrix, then all the assertions of Theorem 3 remain to be valid for the sets of matrices taken from the totality of $H'$-sets of matrices which is obtained by transposition of matrices from $H$-sets. In particular, in the course of transposing, the sets of matrices with independent row uncertainty turn into the so-called sets of matrices with independent column uncertainty [1].

Note that for the $H'$-sets of matrices the hourglass alternative, generally speaking, is not valid. In this connection the question naturally arises about further expansion of classes of matrices for which the theorems proven in the article hold.

6.2. Terminology

We have borrowed the term ‘set of matrices with independent row (or column) uncertainty’ from the recent work [1], although such a kind of sets of matrices in fact have been used for a long time in the theory of parallel computing and the theory of asynchronous systems. In [36] the same sets of matrices got the name product families.

In the special case when each of the rows of the matrix $A'$ coincides with the corresponding row of either a predetermined matrix $A$ or the identity matrix $I$, this type of matrices is sometimes called [37] mixtures of the matrices $A$ and $I$, see also a brief survey in [38]. An example, in which the mixtures of matrices arise, is the so-called linear asynchronous system $x_{n+1} = A_n x_n$, wherein at each time one or more components of the state vector are updated independently of each other, i.e. each coordinate of the vector $x_{n+1}$ can take the value of the corresponding coordinates of $A x_n$ or $x_n$.

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