Distinction for Unipotent $p$-Adic Groups

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Abstract
Let $F$ be a $p$-adic field and $U$ be a unipotent group defined over $F$, and set $U = U(F)$. Let $\sigma$ be an involution of $U$ defined over $F$. Adapting the arguments of Yves Benoist (J Funct Anal 59(2):211–253, 1984; Mem Soc Math France 15:1–37, 1984) in the real case, we prove the following result: an irreducible representation $\pi$ of $U$ is $U^{\sigma}$-distinguished if and only if it is $\sigma$-self-dual and in this case $\text{Hom}_{U^{\sigma}}(\pi, \mathbb{C})$ has dimension one. When $\sigma$ is a Galois involution, these results imply a bijective correspondence between the set $\text{Irr}(U^{\sigma})$ of isomorphism classes of irreducible representations of $U^{\sigma}$ and the set $\text{Irr}_{U^{\sigma} - \text{dist}}(U)$ of isomorphism classes of distinguished irreducible representations of $U$.

Keywords Distinguished representations · Unipotent $p$-adic groups · Kirillov parametrization

Mathematics Subject Classification 22E50

1 Introduction
Let $G$ be a connected algebraic group defined over a field $F$, and $\sigma$ be an $F$-rational involution of $G$. One says that a complex representation $\pi$ of $G = G(F)$ is distinguished if $\text{Hom}_{G^{\sigma}}(\pi, \mathbb{C}) \neq 0$. One is in general interested in computing the dimension of $\text{Hom}_{G^{\sigma}}(\pi, \mathbb{C})$ when $\pi$ is irreducible, as well as understanding the relation between irreducible distinction and conjugate self-duality.

One extensively studied situation is that of distinction by a Galois involution. Let $E/F$ be a separable extension of quadratic field, and take $G = \text{Res}_{E/F}(H)$ for $H$ to be a connected algebraic group defined over $F$. Then, $\sigma$ is taken to be the corresponding
Galois involution. A case of interest is that of finite fields, in which case it has been shown in [10, Theorem 2] that an irreducible representation $\pi$ of $G$ which is stable is distinguished if and only if it is conjugate self-dual: $\pi^\vee \simeq \pi^\sigma$.

The question of the relation between distinction and conjugate self-duality as well as that of the dimension of $\text{Hom}_H(\pi, \mathbb{C})$ remains interesting for smooth representations when $F$ is $p$-adic, and it has attracted a lot of attention when $G$ is reductive. The answer is not known in general, but a conjectural and very precise answer in terms of Langlands parameters is provided by [11]. It in particular roughly says that if $\pi$ is an irreducible distinguished (by a certain quadratic character) representation of $G$, then $\pi^\vee$ and $\pi^\sigma$ should be in the same L-packet, and moreover there should be a correspondence between irreducible distinguished representations of $G$ and irreducible representations of $H^{\text{op}}(F)$, where the opposition group $H^{\text{op}}$ is a certain reductive group defined over $F$ and isomorphic to $H$ over $E$.

Going back to a general involution, still with $F$ a $p$-adic field, it seems that such questions have not attracted as much attention when $G$ is unipotent. It turns out that the different answers, provided by this paper, are simple as well as their proofs. In fact they were completely solved when $F = \mathbb{R}$ by Benoist [1,2], where moreover a Plancherel formula for the corresponding symmetric space was established. Our results are the same and the proofs are very close, though sometimes the arguments have to be different. Let us quickly describe the content of this note.

If $G = U$ is unipotent, then a smooth irreducible representation of $U = U(F)$ is distinguished if and only if it is conjugate self-dual, in which case $\text{Hom}_{U^\sigma}(\pi, \mathbb{C})$ has dimension one (Proposition 5.1 and Theorem 5.2). Moreover when $\sigma$ is a Galois involution, there is a bijective correspondence between distinguished irreducible representations of $U$ and representations of $U^\sigma$ (Corollary 5.3). Hence, setting $H = U$, in a certain sense $U^{\text{op}} = U$ when $U$ is unipotent.

As in [2] and [1], all proofs are based on the Kirillov construction and parametrization [8,13] of irreducible representations of $U$. In fact, as the Kirillov construction in the case of smooth irreducible representations of $p$-adic fields seems not to be fully written in detail in the literature, we do this work in Sect. 3 for the convenience of the reader. Note that the classification for continuous irreducible unitary representations of $U$ on Hilbert spaces is available in several papers (see [5] and the references there), so that the Kirillov classification of smooth irreducible unitary representations of $U$ can certainly be deduced from it by considering the injection of this category into that of irreducible unitary representations on Hilbert spaces by taking smooth vectors (though we could not find a proof of this result in the case at hand), but in any case we give a direct proof here, for which we claim no originality other than that we did not find it written as such in the literature. We make use of a result of [13], which is very well suited to obtain Kirillov’s classification in a quick manner.

2 Notations

In this paper, $F$ is a $p$-adic field, i.e., a finite extension of $\mathbb{Q}_p$, with ring of integers $O_F$. We consider $U$ a (necessarily connected) unipotent group defined over $F$. We denote by $U$ a connected unipotent group defined over $F$ with Lie algebra $\mathcal{N}$ so that...
is an isomorphism of algebraic $F$-varieties with reciprocal map $\ln$ [4, Proposition 4.1].

We set $U = U(F)$ and $\mathcal{N} = \mathcal{N}(F)$; the map $\exp$ restricts as a homeomorphism from $\mathcal{N}$ to $U$. We will say that $U'$ is an $F$-subgroup of $U$ if it is the $F$-points of a closed algebraic subgroup $U'$ of $U$ defined over $F$. The map $\exp$ induces a bijection between $\mathcal{N}$ and $U$. We will say that $U'$ is an $F$-subgroup of $U$ if it is the $F$-points of a closed algebraic subgroup $U'$ of $U$ defined over $F$. Moreover if $U'$ is an $F$-subgroup of $U$, then $U/U' \cong (U/U')(F)$ by [12, 14.2.6], and this bijection becomes a group isomorphism if $U'$ is normal in $U$ in which case both quotients identify to $\mathcal{N}/\mathcal{N}' = (\mathcal{N}/\mathcal{N}')(F)$ via $\exp$.

We denote by $Z$ the center of $\mathcal{N}$, and by $Z$ the center $\exp(Z)$ of $U$.

As a convention if $U_i$ or $U'$ is an $F$-subgroup of $U$, we will denote by $\mathcal{N}_i$ or $\mathcal{N}'$ its Lie algebra.

A fundamental example of unipotent group is the Heisenberg group:

$$U = H_3 = \left\{ h(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \; x, \; y, \; z \in F \right\}.$$ 

We will denote by

$$L = \{ h(0, y, z), \; y, \; z \in F \}$$

its normal Lagrangian subgroup $H_3$.

We denote by $\text{Irr}(U)$ the set of isomorphism classes of (always smooth) irreducible representations of $U$ and by $\text{Irr}_{\sigma - \text{dist}}(U)$ the subset of isomorphism classes of distinguished irreducible representations of $U$. For $\pi \in \text{Irr}(U)$, we will write $c_\pi$ its central character. We will say that a representation is unitary if it preserves a positive definite hermitian form. We write $\text{ind}$ for compact induction and $\text{Ind}$ for induction (in our situation normalized induction will coincide with non-normalized induction). We recall that if $\pi'$ is a smooth representation of a closed subgroup $U'$ of $U$, then if $\text{ind}_{U'}^U(\pi)$ is admissible we have $\text{ind}_{U'}^U(\pi) = \text{Ind}_{U'}^U(\pi)$.

### 3 The Kirillov Classification

#### 3.1 Definitions

In this section, we fix $\psi : F \to \mathbb{C}_u$ a non trivial character. Take $\phi \in \text{Hom}_F(N, F)$ and let $\mathcal{N}'$ be a Lie sub-algebra of $\mathcal{N}$. We will say that the pair $(\phi, \mathcal{N}')$ is polarized for $\mathcal{N}'$ if $\mathcal{N}'$ is a totally isotropic sub-space of maximal dimension for the $F$-bilinear form

$$B_\phi : \mathcal{N} \times \mathcal{N} \to F,$$
defined by

\[ B_\phi(X, Y) = \phi([X, Y]). \]

We denote by \( \mathcal{P}(N) \) the set of polarized pairs for \( N \). The group \( U \) acts on \( \mathcal{P}(N) \) by the formula

\[ u.(\phi, N') = (\phi \circ \text{Ad}(u)^{-1}, \text{Ad}(u)(N')). \]

More generally, it acts by the same formula on the set of pairs \( (\phi, N') \), where \( \phi \) is a linear form on \( N \) and \( N' \) is a sub-algebra (or even a vector subspace) of \( N \).

Whether \( (\phi, N') \) is polarized or not, as soon as \( N' \) is totally isotropic for \( B_\phi \), the linear form \( \phi \) defines a character \( \psi_\phi \) of \( U' := \exp(N') \) given by

\[ \psi_\phi(u') = \phi(\ln(u')). \]

We set

\[ \pi(U', U, \psi_\phi) = \text{ind}_{U'}^U(\psi_\phi). \]

Note that if \( (\phi, N') \) and \( (\phi', N'') \) are in the same \( U \)-orbit, then the inducing data \( (\psi_\phi, U') \) and \( (\psi_{\phi'}, U'') \) are conjugate and

\[ \pi(U', U, \psi_\phi) \simeq \pi(U'', U, \psi_{\phi'}). \]

The author of [13] notices in [13, Sect. 6] that the results of [8] on unitary representations of real unipotent groups apply with the same proofs to unitary representations (acting on Hilbert spaces) of unipotent \( p \)-adic groups. They also apply to smooth representations of unipotent \( p \)-adic groups with the same proofs. For the sake of completeness, we will recall the proofs, using handy results from [13, Proof of Theorem 4].

### 3.2 Preparation

In this paragraph, we suppose that \( Z \) is of dimension 1. By Kirillov’s lemma [8, Lemma 4.1], there is a “canonical” decomposition

\[ N = F \cdot X \oplus F \cdot Y \oplus F \cdot Z \oplus W, \]

which means that the vectors \( X, Y, Z \) and the \( F \)-vector space \( W \) have the following properties:

(i) \( Z = F \cdot Z \).
(ii) \( [X, Y] = Z \).
(iii) \( [Y, W] = \{0\} \).
The Lie sub-algebra

\[\mathcal{N}_0 := F \cdot Y \oplus F \cdot Z \oplus W\]

is automatically a codimension 1 ideal of \(\mathcal{N}\) and we set

\[U_0 = \exp(\mathcal{N}_0).\]

Note that \(\text{Vect}_F(X, Y, Z)\) is a Lie algebra isomorphic to that of \(H_3\), hence \(\exp(\text{Vect}_F(X, Y, Z))\) is a closed subgroup of \(U\) isomorphic to \(H_3\). We set

\[h(x, y, z) = \exp(y \cdot Y) \exp(x \cdot X) \exp(z \cdot Z)\]

and use \(h\) to consider \(H_3\) as a subgroup of \(U\) which satisfies

\[H_3 \cap U_0 = L.\]

We note that \(Y\) and \(Z\) are central in \(\mathcal{N}_0\); hence, they belong to \(\mathcal{N}'\) whenever \((\phi, \mathcal{N}') \in \mathcal{P}(\mathcal{N}_0)\).

By [13, Proof of Theorem 4] we have:

**Proposition 3.1** Let \(\pi\) be an irreducible representation of \(U\) with non-trivial central character \(c_{\pi}\), then there is \(\pi_0 \in \text{Irr}(U_0)\) such that

\[\pi = \text{ind}_{U_0}^U(\pi_0).\]

In fact, one can choose \(\pi_0\) such that if we identify the space of \(\pi\) with \(C_\infty^c(F, V_{\pi_0})\) via the map \(f \mapsto [x \mapsto f(h(x, 0, 0))]\), setting \(\chi(z) = c_{\pi}(h(0, 0, z))\), we have

\[(\pi(u_0)f)(0) = \pi_0(u_0)f(0)\]

for any \(u_0 \in U_0\) and

\[(\pi(h(x, y, z))f)(x') = \chi(z + x'y)f(x' + x).\]

Note that Eq. (3.1) is automatically satisfied when \(\pi = \text{ind}_{U_0}^U(\pi_0)\). On the other hand, Eq. (3.2) is not. One can in fact characterize the representations \(\pi_0\) of \(U_0\) in the above proposition:

**Lemma 3.2** The irreducible representation \(\pi_0\) is such that Eq. (3.2) is satisfied if and only if \(c_{\pi_0}\) is trivial on \(h(0, F, 0)\).
Proof Suppose that Eq. (3.2) is satisfied. Then by Eqs. (3.1) and (3.2) evaluated at \( x' = x = z = 0 \), we see that the group \( h(0, E, 0) \) acts trivially on \( V_{\pi_0} \). Conversely, suppose that \( h(0, E, 0) \) acts trivially on \( V_{\pi_0} \). Then,

\[
(\pi(h(x, y, z))f)(x') = \pi(h(x', 0, 0)h(x, y, z))f(0) = \pi(h(x + x', y, z + x'y))f(0) \\
= \pi(h(0, y, z + yx')h(x + x', 0, 0))f(0) \\
= \pi_0(h(0, y, z + x'y)(\pi(h(x + x', 0, 0))f)(0) \\
= \chi(z + x'y)(\pi(h(x + x', 0, 0))f)(0) = \chi(z + x'y)f(x + x').
\]

We will say that \( \pi_0 \in \text{Irr}(U_0) \) as in Lemma 3.2 is good.

3.3 Classification

An immediate corollary of Proposition 3.1 proved in [13] is:

Corollary 3.3 Any \( \pi \in \text{Irr}(U) \) is admissible and unitary.

Proof By induction on \( \text{dim}(U) \). If \( \text{dim}(U) = 1 \), it is clear. If not, if either \( \text{dim}(Z) \geq 2 \) or if \( c_\pi \) is trivial, then setting \( K = \text{Ker}(c_\pi) \), the group \( \overline{U} = U/\text{Ker}(c_\pi) \) has dimension smaller than that of \( U \) and we conclude by induction because \( \pi \) is a representation of \( \overline{U} \). If \( \text{dim}(Z) = 1 \) and \( c_\pi \) is nontrivial, we can write \( \pi = \text{ind}_{U_0}^U(\pi_0) \), thanks to Proposition 3.1. In this case, \( \pi_0 \) must be irreducible so by induction it is unitary and admissible, from which we already conclude that \( \pi \) is unitary. Moreover, take a function \( f \in \text{ind}_{U_0}^U(\pi_0) \simeq C_c^\infty(F, V_{\pi_0}) \) which is fixed by \( U(O_F) \). Then by Eq. (3.2), \( f \) is an \( O_F \)-invariant function on \( F \) which must vanish outside the orthogonal of \( O_F \) with respect to \( \chi \), so it is determined by its values on a finite set \( A \), and moreover its image is a subset of the finite dimensional space \( V_{\pi_0}^K \) where \( K = \cap_{a \in A} a^{-1}U_0(O_F)a \). This means that \( \text{ind}_{U_0}^U(\pi_0) \) has finite dimension so that \( \pi \) is admissible. ⊓⊔

Because irreducible representations are unitary, the following can be proved.

Corollary 3.4 Suppose that \( Z \) has dimension 1, and let \( \pi_0 \in \text{Irr}(U_0) \) be a good representation, then \( \pi = \text{ind}_{U_0}^U(\pi_0) \) is irreducible. Moreover if \( \pi'_0 \in \text{Irr}(U_0) \) is another good representation such that \( \pi = \text{ind}_{U_0}^U(\pi'_0) \), then \( \pi'_0 \simeq \pi_0 \).

Proof Because \( \pi_0 \) is unitary, so is \( \pi \); hence, \( \pi \) is semi-simple, and it is thus sufficient to prove that \( \text{Hom}_U(\pi, \pi) \) is one dimensional. Now, Eq. (3.2) is satisfied for \( \pi_0 \) and \( \pi \) thanks to our hypothesis, and the proof of Corollary 3.2 shows that \( \pi \) is in fact admissible, so

\[
\pi = \text{Ind}_{U_0}^U(\pi_0) \simeq \text{Ind}_{U_0}^U(\pi'_0).
\]

Hence, one has

\[
\text{Hom}_U(\pi, \pi) \simeq \text{Hom}_U(\pi, \pi'_0),
\]
and it remains to show that this latter space is one dimensional when $\pi'_0 \simeq \pi_0$ and $\{0\}$ otherwise. Take $L \in \text{Hom}_{U_0}(\pi, \pi'_0)$. We identify $\pi$ with $C^\infty_0(F, V_{\pi_0})$. For $\phi \in C^\infty_0(F)$ and $f \in C^\infty_0(F, V_{\pi_0})$, we set

$$\pi(\phi) f = \int_F \phi(y) \pi(0, y, 0) f dy.$$ 

Note that

$$(\pi(\phi) f)(x) = \hat{\phi}(x) f(x),$$

where the Fourier transform is taken with respect to $\chi$ and the fixed Haar measure on $F$. On the other hand because $c_{\pi_0}(h(0, F, 0)) = \{1\}$, there is $c > 0$ such that

$$L(\pi(\phi) f) = \pi'_0(\phi) L(f) = c\hat{\phi}(0) L(f),$$

giving the equality

$$L(\hat{\phi} f) = c\hat{\phi}(0) L(f)$$

for all $\phi \in C^\infty_c(F)$ and $f \in C^\infty_c(F, V_{\pi_0})$. In particular if $f(0) = 0$, taking $\hat{\phi}$ the characteristic function of a small enough compact open subgroup of $F$, we see that $L(f) = 0$. This implies that there exists $L_0 \in \text{Hom}_{U_0}(V_{\pi_0}, V_{\pi'_0})$ such that

$$L = [\phi \mapsto \phi(0)] \otimes L_0.$$ 

We thus just exhibited a linear injection $L \mapsto L_0$ of $\text{Hom}_{U_0}(\pi, \pi_0)$ into $\text{Hom}_{U_0}(V_{\pi_0}, V_{\pi'_0})$ which is zero if $\pi'_0 \not\simeq \pi_0$ and one dimensional by Schur’s lemma otherwise. This concludes the proof.

Before we state Kirillov’s classification, let us state another lemma.

**Lemma 3.5** Let $(\phi, N')$ be a pair where $\phi$ is a linear form on $N$ and $N'$ is a subalgebra of $N$, such that $B_\phi$ is isotropic on $N'$, but which is not polarized, then $\pi(U, U, \psi_\phi)$ is reducible.

**Proof** By transitivity of induction and because reducible representations induce to reducible ones, it is enough to show this when $(\phi, N)$ is polarized. In this case $\psi_\phi$ defines a character of the whole group $U$. Suppose that $\text{ind}_U^U(\psi_\phi)$ was irreducible, in particular we would have $\text{ind}_U^U(\psi_\phi) = \text{Ind}_U^U(\psi_\phi)$ by admissibility of irreducible representations. But then

$$\text{Hom}_U(\psi_\phi, \text{Ind}_U^U(\psi_\phi)) \simeq \text{Hom}_U(\psi_\phi, \psi_\phi) \neq 0$$

which is absurd as it would imply that $\text{Ind}_U^U(\psi_\phi)$ is a character, which it is not by assumption. \qed
We can now obtain Kirillov’s classification.

**Theorem 3.6** (1) Let \((\mathcal{N}, \phi)\) be a pair consisting of a sub-algebra of \(\mathcal{N}\) and a linear form \(\phi\) on \(\mathcal{N}\) such that \(\mathcal{N}'\) is isotropic for \(B_\phi\). The representation \(\pi(U', U, \psi_\phi)\) is irreducible if and only if \((\phi, \mathcal{N}')\) is polarized for \(\mathcal{N}\).

(2) Any irreducible representation of \(U\) is of the form \(\pi(U', U, \psi_\phi)\) with \((\phi, \mathcal{N}')\) polarized for \(\mathcal{N}\).

(3) Two irreducible representations \(\pi(U', U, \psi_\phi)\) and \(\pi(U'', U, \psi_\psi)\) are isomorphic if and only if \((\phi, \mathcal{N}')\) and \((\phi', \mathcal{N}'')\) are in the same \(U\)-orbit.

**Proof** Thanks to Lemma 3.5, the first point will be proved if we show that \(\pi(U', U, \psi_\phi)\) is irreducible when \((\mathcal{N}, \phi)\) is polarized for \(\mathcal{N}\). We do an induction on \(\dim(U)\). If it is 1, there is nothing to prove. If not, we take \(\pi = \pi(U', U, \psi_\phi)\) with \((\phi, \mathcal{N}') \in \mathcal{P}(\mathcal{N})\). If \(\dim(U) > 1\) and \(c_\pi\) is trivial or if \(\dim(Z) > 1\), take \(H \in Z\) such that \(\phi(H) = 0\), then \(\pi\) is in fact a representation of \(U/\exp(F.H)\), and \((\mathcal{N}'/F.H, \tilde{\phi}) \in \mathcal{P}(\mathcal{N}/F.H)\), so we conclude by induction. If \(\dim(Z) = 1\) and \(c_\pi\) is non-trivial, then according to [8, Lemma 5.1] we can suppose that \(\mathcal{N}'\) is a sub-algebra of \(\mathcal{N}_0\) and that \(\phi(Y) = 0\). Then the pair \((\phi|_{\mathcal{N}_0}, \mathcal{N}')\) is polarized for \(\mathcal{N}_0\) and by induction the representation \(\pi_0 = \pi(U', U_0, \psi_{\phi|_{\mathcal{N}_0}})\) is irreducible; it is moreover good because \(\phi(Y) = 0\). But then, \(\pi(U'', U, \psi_\psi) = \text{ind}_U^{U''} (\pi_0)\) is irreducible thanks to Corollary 3.4.

For point 2), we do again an induction on \(\dim(U)\), the one-dimension case being obvious. Then if \(c_\pi\) is trivial or if \(\dim(Z) > 1\), we conclude by induction. If not, \(\pi = \text{ind}_{U_0}^{U}(\pi_0)\) with \(\pi_0\) being good. By induction, \(\pi_0\) is in \(\mathcal{P}(\mathcal{N}_0)\). Then extending \(\phi_0\) to a linear form \(\phi\) on \(\mathcal{N} = F \cdot X \oplus \mathcal{N}_0\), we claim that the pair \((\phi, \mathcal{N}')\) remains polarized for \(\mathcal{N}\). Indeed if it was not, then one would have \(\phi([X', \mathcal{N}']) = 0\) for \(X' \notin \mathcal{N}_0\). Writing \(X' = aX + N_0\) with \(N_0 \in \mathcal{N}_0\), then in particular one would have \(\phi([aX + N_0, Y]) = 0\), but \([aX + N_0, Y] = aZ + 0 = aZ\), so this would mean that \(\phi(Z) = 0\), i.e., that \(c_\pi\) is trivial, which it is not.

Point 3) is proved by induction on \(\dim(U)\) as well, and we only focus on the case \(\dim(Z) = 1\) and \(c_\pi \neq 1\). By [8, Lemma 5.1], we can suppose that both \(\mathcal{N}''\) and \(\mathcal{N}'\) are sub-algebras of \(\mathcal{N}_0\) and that \(\phi(Y) = \phi'(Y) = 0\). In particular, \(\pi_0 = \pi(U', U_0, \psi_{\phi|_{\mathcal{N}_0}})\) and \(\pi_0 = \pi(U'', U_0, \psi_{\phi'|_{\mathcal{N}_0}})\) are both good, and both induce to \(\pi\) so they are isomorphic by Corollary 3.4. By induction, this means that \((\phi|_{\mathcal{N}_0}, \mathcal{N}')\) and \((\phi'|_{\mathcal{N}_0}, \mathcal{N}'')\) are \(U_0\)-conjugate. It has been explained before [8, Lemma 5.2] at the end of the proof of [8, Theorem 5.2] that this implies that \((\phi, \mathcal{N}')\) and \((\phi, \mathcal{N}'')\) are indeed \(U\)-conjugate. \(\square\)

**4 Unipotent Symmetric Spaces**

We recall that the map \(x \mapsto x^2\) is bijective from \(U\) to itself. We set \(U^\sigma := \text{ker}(\sigma)\). We have a polar decomposition on \(U\).

**Lemma 4.1** The multiplication map \(m : U^\sigma \times U^\sigma \to U\) given by \(m(u^+, u^-) = u^+ u^-\) is a homeomorphism.

**Proof** This is is just [2, Proposition 2.1, 3)], the proof of which is valid in our setting. \(\square\)
We will use the following fixed point result in replacement of that used in [2, Proof of Lemma 4.3.1]. It could be used in ibid. as well.

**Lemma 4.2** Let $X$ be the $F$-points of an $F$-algebraic variety on which $U$ acts in an $F$-rational manner, and $\sigma$ be an $F$-rational involution of $X$ (i.e., we have two involutions on different sets which we denote by the same letter) such that $\sigma(u \cdot x) = \sigma(u) \cdot \sigma(x)$ for all $u \in U$ and $x \in X$. Then, a $U$-orbit in $X$ is $\sigma$-stable if and only if it contains a fixed point of $\sigma$.

**Proof** Take $O = U \cdot x$ a $U$-orbit in $X$. If it contains a $\sigma$-fixed point $y$, then $y = u \cdot x$ and $\sigma(x) = \sigma(u) \cdot y = \sigma(u)u^{-1} \cdot x$ so $O$ is $\sigma$-stable. Conversely, suppose that $\sigma(O) = O$. We denote by $K$ the stabilizer of $x$ (it is an $F$-subgroup of $U$). If $K = U$, there is nothing to prove. If not because $U$ is unipotent, there is a sequence $K \triangleleft V \triangleleft U$ with $V$ a normal $F$-subgroup of $U$ such that $U/V$ is commutative of dimension 1 (this property can be proved by induction on $(\dim(U), \dim(U) − \dim(K))$ with lexicographic ordering). Now, $\sigma(x) = u \cdot x$ for $u \in U$ by assumption. This implies that $\sigma(u)u$ belongs to $K$ hence to $V$, so $\sigma(u) = uu^{-1} \in U/V$, i.e.,

$$\frac{u^+}{u^-} = \frac{u^-}{u^+} = \frac{u^+}{u^-} ⇔ \frac{u^+}{u^-} = \bar{1}$$

in $U/V$, which implies that $u^+ = \bar{1} \in U/V$, so that $u^+ \in V$. However because $u^+$ is fixed by $\sigma$, it implies that

$$u^+ \in V \cap \sigma(V).$$

Note that because $V$ is normal in $U$, so is $\sigma(V)$ and hence

$$V \cap \sigma(V) \triangleleft U.$$

We set

$$u_1 = (u^-)^{1/2}$$

so that

$$\sigma(u_1) = u_1^{-1}$$

(because this relation is true when squared) and

$$v = u_1^{-1}uu_1^{-1} = u_1^{-1}u^+u_1,$$

hence

$$v \in V \cap \sigma(V).$$
So setting

\[ y = u_1 \cdot x, \]

this implies that

\[ \sigma(y) = \sigma(u_1) \cdot \sigma(x) = u_1^{-1} u \cdot x = v \cdot y. \]

Hence, \( \sigma(y) \) and \( y \) are in the same \( V \cap \sigma(V) \)-orbit \( O' \) inside \( O \). Now because \( V \cap \sigma(V) \) is \( \sigma \)-stable and has dimension smaller than that of \( U \), we conclude by induction that \( \sigma \) fixes a point of \( O' \), and hence a point of \( O \).

We make \( \sigma \) act on the set of pairs \((\phi, V)\) with \( \phi \) a linear form on \( N \) and \( V \) an \( F \)-subspace of \( N \) (it is a disjoint union of varieties of the form \( N^* \times G_k(N) \) where \( G_k(N) \) is the Grassmanian variety of \( k \)-dimensional subspaces of \( N \)) by the formula:

\[ \sigma(\phi, V) = (-\phi^\sigma, V^\sigma). \]

Then Lemma 4.2 implies:

**Lemma 4.3** Let \((\phi, N')\) be a polarized pair for \( N \). Then, \( \sigma(\phi, N') \) and \((\phi, N')\) are in the same \( U \)-orbit if and only if there is a \( \sigma \)-fixed polarized pair in the \( U \)-orbit of \((\phi, N')\).

Finally, we have:

**Lemma 4.4** Let \((\phi, N')\) and \((\phi', N'')\) be two \( \sigma \)-fixed \( U \)-conjugate polarized pairs for \( N \), then they are \( U^\sigma \)-conjugate.

**Proof** It follows from the polar decomposition as in [2, Lemma 4.3.1, b)]. Note that in the proof of ibid. it is enough to argue that if \( u^2 \) is in the stabilizer of \( \phi \), then clearly \( u \) is because the stabilizer in question is unipotent as well (so that \( u \mapsto u^2 \) is a bijection of it).

\[ \square \]

**5 Distinction, Conjugate Self-Duality and Multiplicity One**

We now recover the results we are interested in from [2] and [1], with the same proofs. Multiplicity one and conjugate self-duality for distinguished representations of \( U \) follow from the Gelfand–Kazhdan argument, or more precisely its simplification by Bernstein–Zelevinsky ([3]). We indeed notice that the space of double cosets

\[ U^\sigma \backslash U / U^\sigma \]

is fixed by the anti-involution

\[ \theta(g) \rightarrow \sigma(g)^{-1}, \]
thanks to Lemma 4.1. In particular, any bi-$U^\sigma$-invariant distribution on $U$ is fixed by $\theta$ thanks to [3, Theorems 6.13 and 6.15]. This implies as in [6], or more precisely as in [9, Lemma 4.2], that for any irreducible representation $\pi \in \text{Irr}(U)$ one has

$$\dim(\text{Hom}_{U^\sigma}(\pi, \mathbb{C})) \cdot \dim(\text{Hom}_{U^\sigma}(\pi^\vee, \mathbb{C})) \leq 1.$$ 

**Proposition 5.1** For $\pi \in \text{Irr}_{U^\sigma}(U)$, one has $\dim(\text{Hom}_{U^\sigma}(\pi, \mathbb{C})) \leq 1$ and $\pi^\vee \simeq \pi_\sigma$.

**Proof** Suppose that $\pi$ is distinguished and take $L \in \text{Hom}_{U^\sigma}(\pi, \mathbb{C}) - \{0\}$. Because $\pi$ is unitary, its contragredient $\pi^\vee$ is isomorphic to $\overline{\pi}$, where $\overline{\pi} = c \circ \pi \circ c^{-1}$ with $c$ the complex conjugation on the space of $\pi$ obtained by the choice of a basis of this space. In particular, $\overline{L} = L \circ c^{-1} \in \text{Hom}_{U^\sigma}(\pi, \mathbb{C})$. Then the map $T_{\overline{L}, \overline{L}} : f \in C^\infty_c(U) \mapsto \overline{L}(\pi(f)L)$ is a bi-$U^\sigma$-invariant and hence fixed by $\theta$. We conclude by applying [7, Lemma 3] (where we take $H_1 = H_2 = U^\sigma$ and $\chi_2(zu^+) = \chi_1(zu^+) = c_\pi(z)$ for $u^+ \in U^\sigma$ and $z \in \mathbb{Z}$, remembering that $c_\pi$ is necessarily trivial on $Z^\sigma$).

Denote by $\mathcal{P}^\sigma(\mathcal{N})$ the set of $\sigma$-fixed polarized pairs for $\mathcal{N}$. It is a set acted upon by $U^\sigma$.

**Theorem 5.2** A representation $\pi \in \text{Irr}_{U^\sigma}(U)$ is distinguished if and only if $\pi^\vee = \pi_\sigma$. Moreover, the map $U^\sigma.(\phi, \mathcal{N}') \mapsto \pi(U, U', \psi_\phi)$ is a bijection from $U^\sigma \setminus \mathcal{P}^\sigma(\mathcal{N})$ to $\text{Irr}_{U^\sigma}(U)$.

**Proof** Suppose that $\pi = \pi(U', U, \psi_\phi) \in \text{Irr}(U)$ is conjugate self-dual, then $\sigma(\phi, \mathcal{N}')$ and $(\phi', \mathcal{N})$ are in the same $U$ orbit, which must contain a $\sigma$-fixed polar pair for $\mathcal{N}$ thanks to Lemma 4.3. So we can in fact suppose that $(\phi, \mathcal{N}')$ is fixed by $\sigma$. Because $\phi' = -\phi$, the character $\psi_\phi$ is conjugate self-dual. However because $u \mapsto u^2$ is a bijection of $U^\sigma$, the map $u \in U' \mapsto \sigma(u)u$ is surjective on $U'^\sigma$, so the character $\psi_\phi$ is $U'^\sigma$-distinguished. Then $\pi$ is distinguished, with explicit linear nonzero $U^\sigma$-invariant linear form given on $\pi$ by

$$\lambda : f \mapsto \int_{U'^\sigma \setminus U^\sigma} f(u)du.$$

To finish the proof, it remains to prove the injectivity of the map $U^\sigma.(\phi, \mathcal{N}') \mapsto \pi(U, U', \psi_\phi)$, which is Lemma 4.4.

In particular in the case of the Galois involution, one gets a bijective correspondence between $\text{Irr}(U^\sigma)$ and $\text{Irr}_{U^\sigma}(U)$. Indeed, $U = \text{Res}_{E/F}(U^\sigma)$ for $E$, a quadratic extension of $F$. Write $\delta$ for an element of $E - F$ with square in $F$. One can identify $\mathcal{P}^\sigma(\mathcal{N})$ to the set $\mathcal{P}(\mathcal{N}^\sigma)$ by the map

$$C : (\phi_\sigma, \mathcal{N}'_\sigma) \rightarrow (\phi, \mathcal{N})$$
where
\[ N' = N' \otimes E \]
and
\[ \phi(N + \delta N') = \phi(N'). \]

This yields:

**Corollary 5.3** When \( E/F \) is a Galois involution, the map \( \pi(U'_\sigma, U^\sigma, \psi_{\phi}) \to \pi(U', U, \psi_{\phi}) \) is a bijective correspondence from \( \text{Irr}(U^\sigma) \) to \( \text{Irr}_{U^\sigma - \text{dist}}(U) \).

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