Trace anomaly of the conformal gauge field

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Abstract

The proposed by Bastianelli and van Nieuwenhuizen new method of calculations of trace anomalies is applied in the conformal gauge field case. The result is then reproduced by the heat equation method. An error in previous calculation is corrected. It is pointed out that the introducing gauge symmetries into a given system by a field-enlarging transformation can result in unexpected quantum effects even for trivial configurations.

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1 Introduction

The trace anomaly in a scalar theory with possible additional fields, although less mundane than its chiral counterpart [1, 2], has been intensively studied for a long time. This is because in quantum field theory on curved spacetime, the conformal symmetry is an important issue. One can obtain a conformally invariant Lagrangian from the Lagrangian

$$L = \frac{1}{2} \int d^n x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

by adding the term $-\zeta R \phi^2$, where $R$ denotes the scalar curvature and $\zeta = \frac{n-2}{4(n-1)}$. The resulting Lagrangian is then invariant with respect to:

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{2\alpha} g_{\mu\nu} ; \quad \phi \rightarrow \phi' = e^{(2-n)\alpha} \phi,$$

where $\alpha$ is an arbitrary real function. It is also possible to obtain a Lagrangian invariant with respect to (2) by introducing an additional vector field to (1) [3]. The appropriate Lagrangian

$$L = \frac{1}{2} \int d^n x \sqrt{-g} g^{\mu\nu} (\partial_\mu + A_\mu) \phi (\partial_\nu + A_\nu) \phi$$

is then invariant with respect to (2) provided the vector field transforms as

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha, \quad n > 2$$
that is as a gauge field (conformal gauge field). For $n = 2$ the gauge field $A_\mu$ is invariant (and redundant). The gauge group is the multiplicative group $R - \{0\}$, where $R$ denotes the real numbers. What more interesting is one can combine both methods of achieving conformal invariance in (1). The result is

$$L = \frac{1}{2} \int d^n x \sqrt{-g} \left\{ g^{\mu\nu} (\partial_\mu + A_\mu) \phi (\partial_\nu + A_\nu) \phi - a \phi^2 (\zeta R + A_\mu A^\mu + \nabla_\mu A^\mu) \right\},$$

(5)

where an additional parameter $a$ has been introduced. The case of $a = 1$ corresponds to the ordinary conformally invariant scalar field Lagrangian.

### 2 The trace anomaly

The conformal gauge field is not a gauge field in the usual sense. An ordinary gauge field is conformally invariant. Note that for $n \neq 4$ one cannot add the conformal gauge field kinetic term $F_{\mu\nu} F^{\mu\nu}$ to (5) because it spoils the conformal invariance. Further we will restrict ourselves to the four dimensional case. It seems to be interesting to apply the proposed recently by Bastianelli and Nieuwenhuizen method of calculating the trace anomaly [4]. This is a generalization of the ideas of Alvarez-Gaume’ and Witten [5] to the conformal case. The method consists in replacing n-dimensional operators like $x^\mu$, $\partial_\mu$, $\gamma^\mu g_{\mu\nu}$, $A_\mu$, and $T_a$ (generator of the gauge group) by the their
quantum mechanical analog. All you have to worry about is to keep the appropriate (anti)commutation rules. One then gets the following formula for the conformal anomaly (the reader is referred to [4] for details):

\[ I^n = \lim_{\beta \to 0} \frac{1}{(2\pi \beta)^\frac{n}{2}} \int d^n x \sqrt{g} < 0 | T e^{-\frac{1}{\beta} S_{\text{int}}} | 0 > , \]  

(6)

where the prime over the lim symbol denotes that we should take only the \( \beta \) independent (finite) part. One can directly follow the route proposed in [4].

The Hamiltonian that corresponds to (5) have the form:

\[ H = -\frac{1}{2} (\nabla^\mu + A^\mu) (\partial_\mu + A_\mu) - \frac{1}{8} R + V , \]  

(7)

where the "scalar potential" \( V \) is given by

\[ V = \frac{a}{2} \left( \frac{1}{6} R + A^\mu A_\mu + \nabla^\mu A_\mu \right) + \frac{1}{8} R \]

. By repeating the calculation and using the proposed in [4] time ordering prescriptions one gets

\[ I^4 = \frac{-1}{28800\pi^2} \left\{ 10 R_{\mu \nu \sigma \tau} R^{\mu \nu \sigma \tau} - 10 R_{\mu \nu} R^{\mu \nu} - 60 \Box R + 25 R^2 + 150 F_{\mu \nu} F^{\mu \nu} 
\]

\[ + 300 a (\Box - R) \left( \frac{1}{6} R + A^\mu A_\mu + \nabla^\mu A_\mu \right) + 900 a^2 \left( \frac{1}{6} R + A^\mu A_\mu + \nabla^\mu A_\mu \right)^2 \} . \]

(8)

This result is in apparent disagreement with the one presented in [6, 7]. To check our result we will repeat the calculation of the trace anomaly by the
heat kernel method [8]. The trace anomaly in four dimensional spacetime is given by [9]

\[ I^4 = -\frac{a_2}{16\pi^2}, \tag{9} \]

where \(a_2\) is calculated for the operator

\[ D = -g^{\mu\nu} (\nabla^\mu + A^\mu) (\partial^\mu + A_\mu) + V, \tag{10} \]

where

\[ V = a \left( \frac{1}{6} R + A^\mu A_\mu + \nabla^\mu A_\mu \right). \tag{11} \]

The calculation is straightforward and leads to

\[ I^4 = \frac{1}{28800\pi} \left\{ 10R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau} - 10R_{\mu\nu} R^{\mu\nu} - 60\Box R + 25R^2 + 150F_{\mu\nu} F^{\mu\nu} \\
+ 300 (\Box - R) V + 900 V^2 \right\}. \tag{12} \]

By inserting the \(V\) given by (11) we reproduce the formula (8). So the the method of [4] gives the correct result. One can also calculate the trace in a more general case, e.g. when fermions are present. You have only to worry about the appropriate form of the Hamiltonian and the time ordering prescriptions.
3 Concluding remarks

We would like to conclude by several remarks. First, the use of the "Feynman approach" (perturbative expansion) gives another interpretation in terms interaction vortices of various terms that appear in the trace formula. The reader is encouraged to repeat at least part of the calculation presented in [4]. Secondly, the perturbative expansion (6) can be compared with the formula [9]

\[ I^n = -\frac{a_2^n}{(4\pi)^2} \]  \hspace{1cm} (13)

and used to analyse subtleties of the \(\zeta\)-function regularization method [10-12]. For example it can be used to determine the coefficients \(a_n(x, x')\) in the limit \(x \to x'\) for \(n\) even:

\[ a_n = \text{finite part of } \lim_{\beta \to 0} \frac{-2^n}{\beta^n} < 0|Te^{-\frac{1}{\beta}S_{\text{int}}}|0> \]  \hspace{1cm} (14)

Finally, let us notice that the Lagrangian (3) can be also obtained by the field-enlarging transformation [13, 14]:

\[ g_{\mu\nu} \to g_{\mu\nu}e^{2\alpha} \]  \hspace{1cm} (15)

and

\[ \phi \to \phi e^{-\alpha} \]  \hspace{1cm} (16)
Then, if one introduces the conformal gauge field $A_\mu$ via $A_\mu = \partial_\mu \alpha$, one gets (3). Of course, in this case $F_{\mu\nu} = 0$ so that the kinetic term is absent from (3) and (5). Even such a trivial conformal gauge configuration can produce a non-vanishing anomaly. This points out that the introducing gauge symmetries into a given model by a field-enlarging transformations can result in unexpected quantum effects because it may not be possible to find a local interaction term that cancel the anomaly contribution. One can use a local scale transformation to remove the scalar field (set $\phi = 1$). Then one gets an Einstein gravity coupled to an auxilliary vector field. Such a model is not scale-invariant. We can say that it is a conformal gauge fixed form of (5).

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