1D Ising model using the Kronecker sum and Kronecker product

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Abstract
Calculations in the Ising model can be cumbersome and non-intuitive. Here we provide a formulation that addresses these issues for 1D scenarios. We represent the microstates of spin interactions as a diagonal matrix. This is done using two operations: the Kronecker sum and Kronecker product. The calculations thus become a simple matter of manipulating diagonal matrices. We address the following problems in this work: spins in the magnetic field, open-chain 1D Ising model, closed-chain 1D Ising model and the 1D Ising model in an external magnetic field. We believe that this representation will help provide students and experts with a simple yet powerful technique to carry out calculations in this model.

Keywords: Ising model, Kronecker sum, Kronecker product

1. Introduction
It is remarkable that many complicated statistical systems can be studied using relatively simple mathematical models involving lattice arrangements of molecules and considering the nearest-neighbour interactions [1]. One such model that has seen a wide range of applications is the Ising model proposed in 1925 by Wilhelm Lenz and solved for the 1D spin lattice by Ernst Ising as a part of his doctoral thesis [1, 2].

It is ironic, that after showing that there could be no phase transitions in 1D systems at $T \neq 0$ and erroneously concluding that this was true for higher dimensional systems as well, Ising gave up study of this model and realized much later that his name had become immortal because of it [3, 4]. In today’s world, the Ising model has found wide ranging applications in

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various fields. It is one of the simplest models that shows phase transitions of statistical systems in higher dimensions. Building on the works of Kramers and Wannier [5], Lars Onsager gave an exact solution for the 2D Ising model which showed phase transitions and is considered one of the landmarks in theoretical physics [6]. This model has also played a crucial role in studying alloys [7, 8], spin glasses [9, 10], in neuroscience [11, 12], in studies of financial markets and social sciences [13–16] and studying epidemics and pandemics with reference to the recent COVID-19 outbreak [17, 18]. Heisenberg’s model, which was inspired by the Ising model [1] is finding wide scale applications in quantum information and quantum computing [19–22].

Even though the model was proposed almost a century ago, it is clear that applications are still being found in many important areas. At its heart, calculations in the Ising model involve counting various microstates of the system. This procedure then helps us to calculate the partition function which embeds information of the macroscopic properties of the system. The most widely taught method to solve the Ising model exactly is the transfer matrix method [23]. Mathematically, solving the Ising model is a combinatorial problem and people have given purely combinatorial techniques. For e.g. Kac and Ward’s work [24] using combinatorics to yield the partition function of 2D Ising model and Feynman’s contribution toward this work [25]. Numerical techniques have also been applied to study higher dimensional Ising models [26–30] and studying long range interactions in Ising chains; see [31–33] and references therein.

Our aim here is to provide a method that is physically more intuitive and less cumbersome for the 1D scenario. Numerous approaches exist to analytically solve the Ising model (for some recent works see [34–36]) for various geometries, configurations and models [37–41]. However, we believe that the method presented in this article provides a simple and yet powerful approach using the operations of the Kronecker sum and Kronecker product.

The paper is organized as follows: in section 2, we review the definition and properties of the Kronecker sum and Kronecker product and discuss them in the context of diagonal matrices. In section 3, we provide a detailed prescription to obtain partition function for spin interaction Hamiltonians using spin-1/2 particles in the absence of external magnetic field employing the Kronecker product and sum operations. In section 4 we develop our approach for non-interacting spins in the presence of an external magnetic field. Section 5 is dedicated to solving the 1D Ising model for open/close chains in the absence and presence of external magnetic fields.

2. Preliminaries

In this section, we review the definitions and properties of the Kronecker sum and Kronecker product. The Kronecker product (⊗) operation (also known as tensor product) is defined as following [42]. If $S$ is a $m \times n$ matrix and $T$ is a $p \times q$ matrix then the Kronecker product is a $pm \times nq$ matrix:

$$S \otimes T = \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{m1} & \cdots & s_{mn} \end{bmatrix} \otimes \begin{bmatrix} s_{11T} & \cdots & s_{1nT} \\ \vdots & \ddots & \vdots \\ s_{m1T} & \cdots & s_{mnT} \end{bmatrix}.$$  \hspace{1cm} (1)

The Kronecker product is an operation on matrices of arbitrary sizes. It is important to note that the Kronecker product is associative however, non-commutative. Moreover, it is distributive over the usual addition, i.e. $(A + B) \otimes C = A \otimes C + B \otimes C$. In the following sections, we will
mostly deal with square diagonal matrices, therefore, it is convenient to use the following notation. For diagonal matrices $S_{2\times2}$ and $T_{m\times m}$, only taking into account the diagonal entries, one can write $S = \text{diag}(s_{11}, s_{22})$ and $T = \text{diag}(t_{11}, \ldots, t_{mm})$, with the Kronecker product as,

$$S \otimes T = \text{diag}(s_{11}t_{11}, \ldots, s_{11}t_{mm}, s_{22}t_{11}, \ldots, s_{22}t_{mm}).$$

(2)

Consider two square matrix $S$ and $T$ of order $m$ and $n$ respectively, the Kronecker sum ($\oplus$) operation is defined as

$$S \oplus T = S \otimes I_m + I_n \otimes T,$$

(3)

where, $I_x$ is an identity matrix of order $x$. As evident, the dimension of $S \oplus T$ is $mn$, therefore similar to the Kronecker product, the Kronecker sum also increases the dimension. It is also non-commutative and associative operator, however under the product, the Kronecker sum is not distributive over the usual addition i.e. $(A + B) \oplus C \neq A \oplus C + B \oplus C$. For diagonal matrices $S = \text{diag}(s_{11}, s_{22})$ and $T = \text{diag}(t_{11}, \ldots, t_{mm})$, it can be compactly written as

$$S \oplus T = \text{diag}((s_{11} + t_{11}), \ldots, (s_{11} + t_{mm}), (s_{22} + t_{11}), \ldots, (s_{22} + t_{mm})).$$

(4)

In the following sections, we make use of these two operations to represent the Hamiltonian of spin chain systems in various scenarios, leading to an efficient and simple procedure to obtain partition functions without explicit consideration of the involved microstates. For more explicit examples of the Kronecker product and Kronecker sum, see the appendix.

3. Representing spin interactions

In this work, we restrict our discussion to spin-half particles. There are two eigenstates corresponding to spin-half particles, spin-up ($\uparrow$) and spin-down ($\downarrow$). Therefore, the microstates of a spin-half particle are given by the set $\{\uparrow, \downarrow\}$. The matrix representation of the microstates of a spin-half particle is given by,

$$S = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix},$$

(5)

where the values $+1$ and $-1$ corresponds to spin-up and spin-down states, respectively. In this section we use ($\uparrow$, $\downarrow$) to represent the states which illustrate the counting of microstates using the Kronecker sum and Kronecker product for a given interaction. In the following, we consider the Hamiltonian involving product of spins, sum of spins and finally a combination of both.

Case 1. Product of spins

To begin with, consider a system of two spins with the Hamiltonian given by,

$$H = S_1S_2.$$

(6)

Since each spin can independently be in spin-up and spin-down state, microstates of this interaction is given by the set $\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$. The partition function is obtained as,

$$Z = \sum_{\text{microstates}} \exp(-\beta H) = \sum_{S_1, S_2} \exp(-\beta S_1S_2)$$

$$= e^{-\beta\uparrow\uparrow} + e^{-\beta\uparrow\downarrow} + e^{-\beta\downarrow\uparrow} + e^{-\beta\downarrow\downarrow}.$$

(7)

It is to be noted that in the above we have explicitly considered all the microstates involving configuration of individual spins. The above procedure can be modeled by using a Kronecker
product operation. We can write the matrix representation of both spins as, $S_1 = \text{diag}(\uparrow, \downarrow)$ and $S_2 = \text{diag}(\uparrow, \downarrow)$ and observing that the Kronecker product between them yields $S_1 \otimes S_2 = \text{diag}(\uparrow \uparrow, \uparrow \downarrow, \downarrow \uparrow, \downarrow \downarrow)$. The partition function is then given by,

$$ Z = \text{Tr} \left( \exp \left( -\beta S_1 \otimes S_2 \right) \right) $$

$$ = e^{-\beta \uparrow \uparrow} + e^{-\beta \uparrow \downarrow} + e^{-\beta \downarrow \uparrow} + e^{-\beta \downarrow \downarrow}. \quad (8) $$

Therefore, one can represent all the microstates of the Hamiltonian, $H = S_1 S_2$ as a diagonal matrix $\hat{H} = S_1 \otimes S_2$. In general, the interaction of $n$-spins of type $H = S_1 S_2 S_3 \ldots S_n$ having $2^n$ microstates can be represented by,

$$ \hat{H} = S_1 \otimes S_2 \otimes S_3 \otimes \cdots \otimes S_n. \quad (9) $$

The partition function is given by,

$$ Z = \text{Tr} \left( e^{-\beta \hat{H}} \right). $$

Therefore, the Kronecker product operation represents the microstates of the product of the independent spins.

Case 2. Sum of spins

Consider a two spin system with the Hamiltonian given by, $H = S_1 + S_2$. Through explicit counting one obtains the microstates of this interaction as \{$(\uparrow + \uparrow)$, $(\uparrow + \downarrow)$, $(\downarrow + \uparrow)$, $(\downarrow + \downarrow)$\}. Interestingly, these microstates can be modeled using Kronecker sum as,

$$ \hat{H} = S_1 \oplus S_2 = \text{diag}(\uparrow, \downarrow) \oplus \text{diag}(\uparrow, \downarrow) \quad (10) $$

$$ = \text{diag}(\uparrow \uparrow, \uparrow \downarrow, \downarrow \uparrow, \downarrow \downarrow) \oplus \text{diag}(\uparrow, \downarrow) $$

We can generalise the above for $n$-spins with the Hamiltonian $H = S_1 + S_2 + \cdots + S_n$ having $2^n$ microstates through,

$$ \hat{H} = S_1 \oplus S_2 \oplus \cdots \oplus S_n. \quad (11) $$

Therefore, the Kronecker sum ($\oplus$) operation represents the microstates of the sum of the independent spins.

Case 3. Sum of product of spins

For a more general case where we have both product and sum in a Hamiltonian, for example, interaction of type $H = S_1 S_2 + S_3$ for a three spin system which have the $2^3 = 8$ microstates, through explicit counting one obtain the microstates as following,

$$ \{ (\uparrow \uparrow + \uparrow), (\uparrow \uparrow + \downarrow), (\uparrow \downarrow + \uparrow), (\downarrow \uparrow + \uparrow), (\downarrow \downarrow + \downarrow), (\downarrow \downarrow + \uparrow), (\uparrow \uparrow + \uparrow), (\uparrow \downarrow + \downarrow) \}. \quad (12) $$

It is now straightforward to obtain the above through the Kronecker sum and product by simply using the previous cases one after the other. First, there is a product between $S_1$ and $S_2$ followed by a Kronecker sum with $S_3$. The matrix representing the microstates is then given by,

$$ \hat{H} = (S_1 \otimes S_2) \oplus S_3 $$

$$ = \text{diag}(\uparrow \uparrow, \uparrow \downarrow, \downarrow \uparrow, \downarrow \downarrow) \oplus \text{diag}(\uparrow, \downarrow) $$

$$ = \text{diag}(\uparrow \uparrow \uparrow, \uparrow \uparrow \downarrow, \uparrow \downarrow \uparrow, \uparrow \downarrow \downarrow, \downarrow \uparrow \uparrow, \downarrow \uparrow \downarrow, \downarrow \downarrow \uparrow, \downarrow \downarrow \downarrow) $$
In the above cases, we are explicitly counting the microstates but in an organized way. Thus, we can represent the microstates of an interaction of independent spins by replacing product with tensor product and sum with the Kronecker sum. The resultant diagonal matrix gives all microstates corresponding to the interaction under consideration.

4. Spins in the presence of magnetic field

Here, we make use of the approach described in previous section to obtain explicit expressions for partition functions for system of mutually non-interacting spins present in an external magnetic field. The Hamiltonian for a spin $S_1$ in the external magnetic field is given by,

$$H = -kS_1, \quad (13)$$

where, $k$ is a positive constant [23].

For the spin-half case, $S_1$ can take two configurations $\{\uparrow, \downarrow\}$. Adding one more spin $S_2$ in this system and not considering the mutual interaction between spins, the Hamiltonian for this system in the presence of external magnetic field is given by,

$$H = -k (S_1 + S_2). \quad (14)$$

As discussed in the previous section and using matrix representation of equation (5), the microstates of this system are obtained through,

$$\hat{H} = -k (S_1 \oplus S_2)$$

$$= -k \; \text{diag} (1, -1) \oplus \text{diag} (1, 1)$$

$$= -k \; \text{diag} (2, 0, 0, -2). \quad (15)$$

It is now straightforward to evaluate the partition function. One obtains,

$$Z = \text{Tr} \left( \exp \left( -\beta \hat{H} \right) \right)$$

$$= \text{Tr} \left( \exp \left( \beta \; \text{diag} (2k, 0, 0, -2k) \right) \right).$$

From the above diagonal matrix, one observes that the degeneracy of each energy level are as follows: energy level $2k, 0, -2k$ have the degeneracy 1, 2, and 1, respectively. Therefore, the expression for the partition function evaluates to,

$$Z = 2 + e^{2k\beta} + e^{-2k\beta}$$

$$= 2(1 + \cosh (2k\beta)). \quad (16)$$

Further adding one more spin into the system, the partition function becomes,

$$Z = \text{Tr} \left( \exp (k\beta \; \text{diag} (2, 0, 0, -2) \oplus \text{diag} (1, 1)) \right)$$

$$= \text{Tr} \left( \exp (k\beta \; \text{diag} (3, 1, 1, -1, -1, -1, 1, -3)) \right),$$

where degeneracy corresponding to the energies $3k, 1k, -1k, \text{ and } -3k$ are given by 1, 3, 3 and 1 respectively. Therefore, we observe that using this Kronecker sum and product structure simplifies the counting of degeneracy corresponding to various energy levels. Since it does
not explicitly refer to the specific configuration, counting of degeneracy reduces to identifying equivalent elements in a diagonal matrix. With this, the partition function evaluates to

\[ Z = e^{3k\beta} + 3e^{k\beta} + 3e^{-k\beta} + e^{-3k\beta} \]

\[ = 2(\cosh 3k\beta + 3 \cosh k\beta). \quad (17) \]

For \( n \) mutually non-interacting spins in an external magnetic field, the Hamiltonian is written as,

\[ \hat{H} = k(S_1 \circledast S_2 \circledast S_3 \circledast \cdots \circledast S_n). \quad (18) \]

5. 1D Ising model

In the Ising interaction, we consider the spin–lattice, and each spin interacts only with its neighbouring spin by a product type interaction. The Hamiltonian of the Ising model is given by \( H = -j \sum_{\langle ik \rangle} S_i S_k \), where \( \langle ik \rangle \) represents sum over neighbouring spins and \( j \) is a coupling constant [23]. In the following, we consider various configurations and provide a prescription to find the partition function employing the Kronecker sum and product.

Case 1. 1D open-chain

Consider a three spins chain as shown in the figure 1. The Hamiltonian for this system is given by

\[ H = -j(S_1 S_2 + S_2 S_3). \quad (19) \]

Here, the spin \( S_2 \) is a common term between \( S_1 \) and \( S_3 \). It is not immediately obvious how to write this interaction in terms of the Kronecker product and Kronecker sum. One observes the microstates of this system as

\[ \{ (\uparrow \uparrow + \uparrow \downarrow), (\uparrow \uparrow + \downarrow \uparrow), (\uparrow \downarrow + \downarrow \uparrow), (\uparrow \downarrow + \downarrow \downarrow), \}

\[ (\downarrow \uparrow + \uparrow \uparrow), (\downarrow \uparrow + \downarrow \downarrow), (\downarrow \downarrow + \downarrow \uparrow), (\downarrow \downarrow + \uparrow \downarrow) \}. \quad (20) \]

It is evident that for the Hamiltonian given by (19), the two terms, \( S_1 S_2 \) and \( S_2 S_3 \) can only take values \(+1\) and \(-1\). Moreover, it is the value of \( S_3 \) which determines the product term \( S_2 S_3 \), for specific choices of the values of \( S_1 \) and \( S_2 \). Therefore, the microstates in (20) are equivalent to the following,

\[ \{ (\uparrow \uparrow + \uparrow \downarrow), (\uparrow \uparrow + \downarrow \downarrow), (\downarrow \uparrow + \downarrow \uparrow), (\downarrow \downarrow + \downarrow \uparrow), (\downarrow \downarrow + \uparrow \downarrow), (\downarrow \downarrow + \uparrow \downarrow) \}. \quad (21) \]

Therefore, the value of \( S_3 \) can be independently added to the first term \( S_1 S_2 \). In terms of the values corresponding to the microstates, the Hamiltonians \( H = -j(S_1 S_2 + S_2 S_3) \) and

\[ Z = e^{3k\beta} + 3e^{k\beta} + 3e^{-k\beta} + e^{-3k\beta} \]

\[ = 2(\cosh 3k\beta + 3 \cosh k\beta). \]

For \( n \) mutually non-interacting spins in an external magnetic field, the Hamiltonian is written as,

\[ \hat{H} = k(S_1 \circledast S_2 \circledast S_3 \circledast \cdots \circledast S_n). \]
\( H = -j(S_1S_2 + S_3) \) are equivalent. In term of the Kronecker sum and Kronecker product, the Hamiltonian (19) can be written as,

\[
\hat{H} = -j ((S_1 \otimes S_2) \oplus S_3).
\]  

(22)

If we add one more spin to this linear chain, we get the following Hamiltonian,

\[
H = -j (S_1S_2 + S_2S_3 + S_3S_4).
\]  

(23)

Again we can do the same thing, this time we first fix \( S_1 \) and \( S_2 \), we can see that the second term only depends on the value of \( S_3 \), and then fixing that value of \( S_3 \), we permute over \( S_4 \). It is represented by,

\[
H = -j (S_1S_2 + S_3 + S_4).
\]  

(24)

In term of Kronecker sum and Kronecker product, the above Hamiltonian can be written as,

\[
\hat{H} = -j ((S_1 \otimes S_2) \oplus S_3 \oplus S_4).
\]  

(25)

Carrying on in the same way, we can write the 1D linear open-chain Ising model of \( n \)-spins in the Kronecker product and Kronecker sum as:

\[
\hat{H} = -j \{(S_1 \otimes S_2) \oplus S_3 \oplus \cdots \oplus S_n \}.
\]  

(26)

Case 2. 1D closed-chain

For a 1D closed-chain Ising model of three spins, the Hamiltonian is given by

\[
H = -j (S_1S_2 + S_2S_3 + S_3S_1).
\]  

(27)

There are three independent spins: \( S_1, S_2 \) and \( S_3 \). Therefore, the total number of microstates are, \( 2^3 = 8 \). The microstates of this system are:

\[ [3, -1, -1, -1, -1, -1, 1, 3]. \]

As shown earlier, for three spins as open-chain, one can represent the microstates by,

\[
\hat{H} = -j ((S_1 \otimes S_2) \oplus S_3).
\]  

(28)

To describe three spins closed-chain system, we cannot directly use the Kronecker sum and Kronecker product, like

\[
\hat{H} = -j ((S_1 \otimes S_2) \oplus S_3 \oplus S_1),
\]

or

\[
\hat{H} = -j ((S_1 \otimes S_2) \oplus S_3 \otimes S_1).
\]

We can do this because these expressions will yield the number of microstates as \( 2^4 \). It will not describe a 1D closed-loop problem of three spins, which require eight microstates.
The only way we can proceed from an open-chain to closed-chain is by adding an $8 \times 8$ diagonal matrix ($D_3$) to the open-chain expression as following,

$$\hat{H} = - j ((S_1 \otimes S_2) \oplus S_3 + D_3).$$  \hfill (29)

In a similar manner, for the case of four spins, the Hamiltonian is given by

$$H = - j (S_1 S_2 + S_3 S_4 + S_4 S_1).$$  \hfill (30)

With four independent spins, the total number of microstates are $2^4 = 16$. To describe the closed-chain of four spins, we have to add $16 \times 16$ diagonal matrix ($D_4$) to the open-chain expression as,

$$\hat{H} = - j ((S_1 \otimes S_2) \oplus S_3 \oplus S_4 + D_4).$$  \hfill (31)

And for $n$ spins closed-chain, we have to add an $2^n \times 2^n$ diagonal matrix ($D_n$) to the open-chain term,

$$\hat{H} = - j ((S_1 \otimes S_2) \oplus \cdots \oplus S_n + D_n).$$  \hfill (32)

It turns out that the only consistent way of generalizing the expression of closed-chain from open-chain, is to choose

$$D_n = S_1 \otimes S_2 \otimes \cdots \otimes S_n.$$  \hfill (33)

Therefore, for closed-chain three spins, the microstates are represented by the expression:

$$\hat{H} = - j \left\{ (S_1 \otimes S_2) \oplus S_3 + D_3 \right\}$$
$$= - j \left\{ (S_1 \otimes S_2) \oplus S_3 + S_1 \otimes S_2 \otimes S_3 \right\}$$
$$= \text{diag}(3, -1, 1, 1, 1, 1, 1, -1).$$  \hfill (34)

To describe the closed-chain Ising model of $n$-spins, one has to add a diagonal matrix ($D_n$) to the open-chain expression. The correct diagonal matrix is the one which generalizes closed-chain expression from open-chain for $n$-spins. It turns out that there is only one such expression given by,

$$D_n = S_1 \otimes S_2 \otimes \cdots \otimes S_n.$$  \hfill (33)

Case 3. Open-chain in the presence of magnetic field

Considering three spins, the Hamiltonian for the Ising model in the presence of external magnetic field has the following expression [23],

$$H = - j (S_1 S_2 + S_3) - k (S_1 + S_2 + S_3),$$  \hfill (35)

where the first term is the Ising terms and the second term is due to external magnetic field, with $j$ and $k$ representing the intraparticle correlation coefficient and the magnetic field strength, respectively. Through explicit configurations of spins, one obtains the microstates of this system as,

$$[(-2 j + 3 k), (k), (2 j - k), (+ k), (- k), (2 j + k), (+ k), (-2 j + 3 k)].$$  \hfill (36)

Yet we have dealt with the Ising term and magnetic field term separately and obtained the corresponding Hamiltonians in terms of the Kronecker sum and Kronecker product. To write
the open-chain Ising model in the presence of an external magnetic field, the obvious way to proceed is to use the previous methods directly:

\[
\hat{H} = -j (S_1 \otimes S_2) \otimes S_3 - k (S_1 \oplus S_2) \oplus S_3
\]

\[
= -j \ \text{diag} (2, 0, -2, 0, -2, 0) - k \ \text{diag} (3, 1, 1, -1, 1, -1, -3). \tag{37}
\]

Here, we observe that the Ising term and the magnetic term individually yield the correct energies and degeneracy, namely, \((S_1 \otimes S_2) \otimes S_3\) agrees with the explicit counting for microstates of \(S_1 + S_2 + S_3\) and \(S_1 \oplus S_2 \oplus S_3\) agrees with the explicit counting for microstates of \(S_1 + S_2 + S_3\). However, as evident from the calculation in (37), which yields the following microstates,

\[
\hat{H} = \text{diag} ((-2j - 3k), -k, -k, (2j + k), -k, (2j + k), (-2j + k), 3k), \tag{38}
\]

which clearly does not agrees with the microstates obtained through explicit counting (36) for the Hamiltonian in (35). Therefore, the straightforward generalisation, namely, adding up the individual representation for the Ising term and magnetic field term does not yield the correct result. This occurs because of the disparity in the correspondence of the respective diagonal terms to the spin configuration. For instance, the last term in the expression \((S_1 \otimes S_2) \otimes S_3\) is zero, which correspond to the configuration where two of the spins are oriented opposite to the third, whereas the last term in the expression \(S_1 \oplus S_2 \oplus S_3\) is minus three, corresponding to all of them aligned in the same direction (opposite to the magnetic field in this case). Naturally, combining them as in (37) does not yield the expected outcome. A simple similarity transformation on the Ising term resolves this issue, and can be generalised for \(n\)-spins. The correct order of the diagonal elements in the three-spin chain case are, \(\text{diag}(2, 0, -2, 0, 0, -2, 0, 2)\), which is obtained after performing the following similarity transformation, \(M_3^{-1}[(S_1 \otimes S_2) \oplus S_3]M_3\), where \(M_3\) is a block diagonal matrix given by,

\[
M_3 = \text{diag} (I_2, \sigma_x, I_2, \sigma_x), \tag{39}
\]

where,

\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{40}
\]

It is straightforward to check that \(M_3^2 = I_8\). So \(M_3^{-1} = M_3\), where \(I_8\) is identity matrix of order 8. Therefore, the correct representation of microstates for the Hamiltonian in (35) is given by

\[
\hat{H} = -j \{M_3 [(S_1 \otimes S_2) \oplus S_3] M_3\} - k (S_1 \oplus S_2 \oplus S_3). \tag{41}
\]

We will call the modified Ising term \(G_3\):

\[
G_3 = M_3 [(S_1 \otimes S_2) \oplus S_3] M_3. \tag{42}
\]

This evaluates to,

\[
\hat{H} = -jG_3 - k (S_1 \oplus S_2 \oplus S_3)
\]

\[
= \text{diag} ((-2j - 3k), (-k), (2j + k), (+k),
\]

\[
(-k), (2j + k), (+k), (-2j + 3k)). \tag{43}
\]
If we add one more spin in the system, then the Hamiltonian can be written in terms of the Kronecker sum and Kronecker product by repeated use of similarity transformation as following,

\[
\hat{H} = -j \{M_4(G_3 \oplus S_3)M_4\} - k(S_1 \oplus S_2 \oplus S_3 \oplus S_4),
\] (44)

where \(M_4 = \text{diag}(I_2, \sigma_x, I_2, \sigma_x, I_2, \sigma_x, I_2, \sigma_x)\)

Generalising to a system containing \(n\) spins, one obtains the following,

\[
\hat{H} = -jG_n - k(S_1 \oplus S_2 \oplus S_3 \oplus \cdots \oplus S_n),
\] (45)

where \(G_n = M_n(G_{n-1} \oplus S_n)M_n\). The transformation matrix \(M_n\) only contains a sequence of repeated block matrices, namely \(I_2\) and \(\sigma_x\), and can be written in a compact form,

\[M_2 = \text{diag}(I_2, \sigma_x)\]
\[M_3 = \text{diag}(I_2, \sigma_x, I_2, \sigma_x)
\]
\[M_4 = \text{diag}(I_2, \sigma_x, I_2, \sigma_x, I_2, \sigma_x, I_2, \sigma_x)
\]

Case 4. Closed-chain in the presence of magnetic field

For the closed-chain Ising model in an external magnetic field, we follow the same procedure we followed to go from 1D open-chain to 1D closed-chain in the absence of magnetic field. We add a \(2^n \times 2^n\) diagonal matrix \((D_n)\) to the Ising term (figures 2, 3 and 4).

\[
\hat{H} = -j(G_n + D_n) - k(S_1 \oplus S_2 \oplus S_3 \oplus \cdots \oplus S_n).
\] (46)

Here again it turns out that the only consistent way of generalizing the expression to \(n\)-spins for closed-chain in external magnetic field is to choose

\[D_n = S_1 \oplus I_{2^{n-2}} \otimes S_n,
\]

where \(I_{2^{n-2}}\) is an identity matrix of order \(2^{n-2}\). For instance, for the three spins, the microstates are given by the diagonal entries of

\[
\hat{H} = -j(G_3 + D_3) - k(S_1 \oplus S_2 \oplus S_3)
\]
\[= -j(G_3 + S_1 \otimes I_2 \otimes S_3) - k(S_1 \oplus S_2 \oplus S_3),
\] (47)

and for four spins, the microstates are given by the diagonal entries of

\[
\hat{H} = -j(G_4 + D_4) - k(S_1 \oplus S_2 \oplus S_3 \oplus S_4)
\]
\[= -j(G_4 + S_1 \otimes I_4 \otimes S_4) - k(S_1 \oplus S_2 \oplus S_3 \oplus S_4).\] (48)

This concludes our discussion on representing various interactions using the Kronecker sum and product. We note in passing that the presented prescription can be simply generalised for higher spin eigenvalues (such as spin \(-1\)) systems for open chains as long as all the spins are identical. Chains with non-identical spins require a bit more care, and will be dealt with elsewhere.
6. Conclusions

In this paper we have provided a method of representing microstates of various spin interactions in the 1D Ising model using Kronecker sum ($\oplus$) and Kronecker product ($\otimes$) operations on matrices. The partition function, which gives all the relevant information about the system being studied, is found by taking the trace of the exponential of the resultant matrix.

This method was applied to open and closed 1D chains and we readily obtained the correct values. We also applied this approach for spin interactions with external magnetic field and were able to find the correct count for the Ising and magnetic term separately. Using a similarity transformation we corrected for the order of terms in the diagonal matrix and obtained results which match with the literature.

Solutions in 1D Ising models involve counting of microstates which can become difficult to keep track of when not done systematically. Our system provides a systematic way of doing such calculations while also giving an intuitive grasp of the underlying mechanism and we believe that this method can find useful applications in undergraduate classrooms as well as practicing researchers in the field because of its computational friendly formalism.
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Appendix

Kronecker product: \cite{42} given a $m \times n$ matrix $S$ and a $p \times q$ matrix $T$, then their kronecker product or tensor product $W = S \otimes T$ is an $(mp) \times (nq)$ matrix with elements given by $W_{\alpha\beta} = S_{ij} T_{kl}$, where $\alpha = p(i-1) + k$ and $\beta = q(j-1) + l$. For example, the tensor product of a $2 \times 2$ matrix $S$ and a $3 \times 2$ matrix $T$ is given by

$$S \otimes T = \begin{bmatrix} s_{11}T & s_{12}T \\ s_{21}T & s_{22}T \end{bmatrix}.\,$$

For two diagonal matrices,

$$S_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \text{diag}(a, b); \quad S_2 = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix} = \text{diag}(p, q, r),$$

using the above definition,

$$S_1 \otimes S_2 = \text{diag}(a, b) \otimes \text{diag}(p, q, r) = \begin{bmatrix} a \ 0 \\ 0 \ b \end{bmatrix} \otimes \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix} = \begin{bmatrix} ap & 0 & 0 & 0 & 0 \\ 0 & aq & 0 & 0 & 0 \\ 0 & 0 & ar & 0 & 0 \\ 0 & 0 & 0 & bp & 0 \\ 0 & 0 & 0 & 0 & bq \end{bmatrix}.$$  

Kronecker sum: \cite{42} it is defined over diagonal matrices as follows. Given a $m \times m$ matrix $A$ and a $n \times n$ matrix $B$, the Kronecker sum of $A$ and $B$ is defined as,

$$A \oplus B = A \otimes I_n + I_m \otimes B,$$

where $I_n$ and $I_m$ represents $n \times n$ and $m \times m$ identity matrices, respectively. For example, consider the following matrices

$$A_1 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \text{diag}(a, b); \quad A_2 = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \text{diag}(p, q),$$

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the Kronecker sum of $A_1$ and $A_2$ will be,

$$A_1 \oplus A_2 = A_1 \otimes I_2 + I_2 \otimes A_2 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

$$= \begin{bmatrix} a + p & 0 & 0 & 0 \\ 0 & a + q & 0 & 0 \\ 0 & 0 & b + p & 0 \\ 0 & 0 & 0 & b + q \end{bmatrix}$$

$$= \text{diag}(a + p, a + q, b + p, b + q).$$

We have made use of similar calculations in the main text. Following are few important properties of Kronecker sum and product,

1. $(A + B) \otimes C = A \otimes C + B \otimes C$
2. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
3. $\alpha(A \otimes B) = (\alpha A) \otimes B = A \otimes (\alpha B)$
4. $A \otimes B \neq B \otimes A$
5. $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
6. $(A + B) \oplus C \neq A \oplus C + B \oplus C$
7. $(\alpha X) \oplus (\alpha Y) = \alpha(X \oplus Y)$,

where, $\alpha$ is a scalar, $X, Y$ are diagonal matrices and $A, B$ are arbitrary matrices.

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