Smallest number of incident directions for topological derivative imaging: a numerical study

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Abstract. Various studies have confirmed that only a small number of incident field directions are needed to apply the topological derivative in inverse scattering problem. In this paper, we consider a topological derivative-based technique with a small number of such directions for imaging thin, curve-like dielectric inhomogeneities embedded in a homogeneous domain, and explore a suitable condition via various numerical simulation results.

1. Introduction
Identifying the shape of unknown crack-like thin defects in homogeneous domains is an interesting and remarkable inverse scattering problem. For this purpose, various iterative and non-iterative algorithms have been developed. From various studies [1, 2, 3], it has emerged that iterative-based methods require the derivation of a complex Fréchet derivative, a priori information of the unknown target, considerable computational resources, and a good initial guess. In contrast, non-iterative techniques have no such requirements, although a large number of incident fields with various directions and corresponding scattered field data are essential [4, 5, 6, 7].

The topological derivative is a non-iterative imaging technique for which, unlike various other non-iterative techniques, good results can be achieved even with a small number of incident field directions [8, 9]. However, the fewest such directions that give acceptable results are not known. Motivated by this, we aim to identify the smallest number through numerical simulations.

This paper is organized as follows. In Section 2, we introduce the direct scattering problem and the topological derivative. In Section 3, we present various numerical results and explore the smallest number of incident directions. Our conclusions are given in Section 4.

2. Direct scattering problem and topological derivative
Let \( \Omega \subset \mathbb{R}^2 \) be a homogeneous domain with a smooth boundary \( \partial \Omega \). In this paper, we assume that a homogeneous thin dielectric inhomogeneity \( \Gamma \) of small thickness \( 2h \) is completely hidden in \( \Omega \). For this, we set \( \Gamma \) to be of the form:

\[
\Gamma = \{ x + \eta n(x) : x \in \sigma, -h \leq \eta \leq h \},
\]

where \( \sigma \) is a simple smooth curve that describes the supporting curve of \( \Gamma \) and \( n(x) \) is the unit normal to \( \sigma \) at \( x \). In this paper, \( \omega = 2\pi/\lambda \) is a fixed angular frequency, where \( \lambda \) is a given wavelength such that \( h \ll \lambda \), and all materials are characterized by their dielectric permittivity...
at ω. For simplicity, let ε₀ and ε⋆ be the permittivities of Ω and Γ, respectively. We can then define the piecewise constant permittivity ε(x) as

\[ ε(x) := \begin{cases} 
ε₀ & \text{for } x \in Ω \setminus Γ \\
ε⋆ & \text{for } x \in Γ 
\end{cases} \]

Let \( u^{(n)}(x; ω) \) be the time-harmonic total field that satisfies the Helmholtz equation

\[
\begin{align*}
\Delta u^{(n)}(x; ω) + ω^2 ε(x)u^{(n)}(x; ω) &= 0 \quad \text{for } x \in Ω, \\
∇u^{(n)}(x; ω) \cdot ν(x) &= ∇e^{iωθ_n \cdot x} \cdot ν(x) \quad \text{for } x \in ∂Ω,
\end{align*}
\]

(1)

with transmission conditions on the boundary of Γ. Here, ν(x) denotes the unit normal vector at x ∈ ∂Ω, and θ_n denotes a vector on the unit circle \( S^1 \) such that

\[ \theta_n = [\cos(θ_n), \sin(θ_n)]^T = \left[ \cos \left( \frac{2π(n − 1)}{N − 1} \right), \sin \left( \frac{2π(n − 1)}{N − 1} \right) \right]^T. \]

In the same manner, let \( u_B^{(n)}(x; ω) = e^{iωθ_n \cdot x} \) be the solution of (1) without Γ.

To introduce the topological derivative, let us consider the following energy functional:

\[ E(Ω; ω) := \frac{1}{2} \sum_{n=1}^{N} \int_{Ω} |u^{(n)}(x; ω) − u_B^{(n)}(x; ω)|^2 dS(x). \]

(2)

The topological derivative measures the influence of creating a small inclusion at a certain point inside Ω. In this paper, we create a dielectric inhomogeneity, say Σ, of small diameter r at z ∈ Ω \( ∖ ∂Ω \), and denote the corresponding domain as Ω|Σ. Then, based on [9], the topological derivative \( d_T E(z; ω) \) satisfies the following asymptotic formula:

\[ E(Ω|Σ; ω) = E(Ω; ω) + φ(r; ω)d_T E(z; ω) + o(φ(r; ω)), \]

and can be represented as

\[ d_T E(z; ω) = \text{Re} \sum_{n=1}^{N} \left( \frac{u_A^{(n)}(z; ω)u_B^{(n)}(z; ω)}{u_A^{(n)}(z; ω)} \right), \]

where \( u_A^{(n)}(x; ω) \) satisfies the following adjoint problem:

\[
\begin{align*}
\Delta u_A^{(n)}(x; ω) + ω^2 u_A^{(n)}(x; ω) &= 0 \quad \text{for } x \in Ω, \\
∇u_A^{(n)}(x; ω) \cdot ν(x) &= u^{(n)}(x; ω) − u_B^{(n)}(x; ω) \quad \text{for } x \in ∂Ω.
\end{align*}
\]

(3)

The points at which \( d_T E(z; ω) \) attains its maximum positive value are expected to be on σ. Hence, the shape of Γ can be identified via the map of \( d_T E(z; ω) \) [10]. The shape of Γ is obtained in only one iteration. Hence, this is faster than the traditional iterative method.
3. Simulation results and smallest number of incident directions
In this section, the results of numerical simulations with small \( N \) are presented. Before starting, it is worth mentioning that the application of multi-frequencies guarantees better imaging performance than the application of a single frequency (see [8] for a detailed discussion). Hence, we consider the following normalized multi-frequency topological derivative:

\[
I_{TD}(z; K) := \frac{1}{K} \sum_{k=1}^{K} \frac{d_T \mathcal{E}(z; \omega_j)}{\max [d_T \mathcal{E}(z; \omega_j)]}, \quad k = 1, 2, \ldots, K. \tag{4}
\]

To perform the numerical simulations, the homogeneous domain \( \Omega \) is chosen as a unit circle centered at the origin, and two \( \sigma_j \) specifying the thin inclusions \( \Gamma_j \) are chosen as

\[
\sigma_1 = \left\{ [s - 0.2, -0.5s^2 + 0.5] : -0.5 \leq s \leq 0.5 \right\}
\]
\[
\sigma_2 = \left\{ [s + 0.2, s^3 + s^2 - 0.6] : -0.5 \leq s \leq 0.5 \right\}.
\]

The thickness \( h \) of every \( \Gamma_j \) is set to 0.02, \( \varepsilon_0 \) is chosen as 1, and \( K = 10 \) different frequencies with \( \omega_1 = 2\pi/0.7 \) and \( \omega_{10} = 2\pi/0.3 \) are applied. Let \( \varepsilon_j \) denote the permittivity of \( \Gamma_j \), \( j = 1, 2 \), and set \( \varepsilon_j = 5 \). In all of our results, a white Gaussian noise with 15 dB signal-to-noise ratio (SNR) has been added to the unperturbed data \( u^{(n)}(x; \omega_k) \).

Figure 1 exhibits the maps of \( I_{TD}(z; 10) \) with \( N = 1, 2, \ldots, 6 \) when the inhomogeneity is \( \Gamma_1 \). Based on the results, we can clearly observe that, when \( N \) is a small odd number, it is very hard to identify the shape of \( \Gamma_1 \). Note that it is still impossible to recognize the shape of \( \Gamma_1 \) when \( N = 2 \), but very good results can be obtained when \( N \) is an even number greater than 4. A similar phenomenon can be seen in Figure 2 when the inhomogeneity is \( \Gamma_2 \).

Figure 3 shows the maps of \( I_{TD}(z; 10) \) for imaging multiple thin inclusions \( \Gamma_1 \cup \Gamma_2 \) with \( \varepsilon_1 = \varepsilon_2 = 5 \). Unlike the single inhomogeneity case, although the existence of two inhomogeneities
can be recognized, it is hard to identify their shape. However, when \( N \) is an even number greater than 4, an outline of the inhomogeneities is retrieved.

For the final example, let us consider the imaging of \( \Gamma_1 \cup \Gamma_2 \) under the same configuration as the previous example except that \( \varepsilon_1 = 10 \) and \( \varepsilon_2 = 5 \). It is interesting to observe that, although
the shape of $\Gamma_2$ cannot be retrieved, the shape of $\Gamma_1$ retrieved by our method is close to the true one. Similar to the previous example, we can observe that the shape of $\Gamma_2$ can be obtained when $N$ is an even number greater than 4.

![Maps of $I_{\text{TD}}(z; 10)$ for $\Gamma_1 \cup \Gamma_2$ with different permittivities.](image)

### Figure 4.

4. Concluding remark
We considered the applicability of the topological derivative with a small number of fields of incident directions for the one-step iterative imaging of thin dielectric inhomogeneities in a homogeneous domain. From various results for single and multiple inhomogeneities, we can conclude that, unlike many non-iterative techniques, the topological derivative requires a small number of incident directions, and this must be an even number greater than 4. As this finding is somewhat heuristic, we intend to develop the underlying mathematical theory in future work.

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