Sub-Heisenberg estimation strategies are ineffective

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In interferometry, sub-Heisenberg strategies claim to achieve a phase estimation error smaller than the inverse of the mean number of photons employed (Heisenberg bound). Here we show that one can achieve a comparable precision without performing any measurement, just using the large prior information that sub-Heisenberg strategies require. For uniform prior (i.e. no prior information), we prove that these strategies cannot achieve more than a fixed gain of about 1.73 over Heisenberg-limited interferometry. Analogous results hold for arbitrary single-mode prior distributions. These results extend also beyond interferometry: the effective error in estimating any parameter is lower bounded by a quantity proportional to the inverse expectation value (above a ground state) of the generator of translations of the parameter.

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It has been known for a long time that, under very general conditions\textsuperscript{1}\textsuperscript{13}, the precision $\Delta \phi$ in determining the phase in interferometry is bounded by the inverse of the mean total number of photons $N$ used in the estimation as $\Delta \phi \geq 1/N$: the Heisenberg bound to optical interferometry. Recent work\textsuperscript{1}\textsuperscript{2} has challenged the validity of such bound in regimes not covered by prior proofs, raising the question of whether it is possible to implement sub-Heisenberg strategies where the error has a scaling smaller than $1/N$. A negative answer was given in\textsuperscript{13} for those estimation strategies that are capable of reducing significantly the initial uncertainty on the phase. This result was obtained in the theoretical framework of quantum estimation theory\textsuperscript{14}\textsuperscript{13}, invoking the quantum speed limit theorem\textsuperscript{16}. We note that there have been prior debates on sub-Heisenberg scaling\textsuperscript{19}\textsuperscript{22}, and proposals to use nonlinear schemes\textsuperscript{22}\textsuperscript{23}: a detailed review is found in\textsuperscript{13}.

Namely, in this regime we prove that no sub-Heisenberg scaling can be attained, independently of the probability distribution that characterizes the prior information on the parameter. In the opposite regime (High Prior Information (HPI) regime) instead, we show that, even though sub-Heisenberg scalings are not prohibited by the ZZ bound, the resulting accuracy is of the same order of the one obtainable by guessing a random value distributed according to the prior distribution (without performing any measurement). This last result is derived under specific, but reasonable, assumptions on the prior distribution. In particular, we show that for uniform prior the ZZ bound forces any sub-Heisenberg scheme to accuracies which are only a meager 1.73 times larger than the one obtainable via a random guess.

The final part of the paper contains material which is not directly related with the Heisenberg bound issue, but which makes use of the same technique developed in the previous sections. Specifically, generalizing previous results of\textsuperscript{18}, we show how the quantum ZZ bound can be used to derive a weighted uncertainty relation weighted by generic prior distributions.

\textit{Lower bound on the weighted precision:—} Let $x$ be the parameter to be measured, say the relative optical phase acquired by an optical pulse when traveling through the two arms of an interferometer. We allow $x$ to take any possible real value and assign a prior probability distribution $p(x)$ characterized by the quantity $W$ which measures the initial uncertainty of the problem – e.g. for $p$ Gaussian, $W$ can be identified with the standard deviation\textsuperscript{24}. As is customary in quantum metrology\textsuperscript{13}, we assume $x$ to be encoded into a state $\rho_x$ of a quantum probe system (say the optical pulse emerging from the Mach-Zehnder interferometer) which we measure obtaining the random outcome $y$. We can quantify the accuracy of the estimation through the weighted Root Mean
Square Error (RMSE) [13]:
\[
\Delta Y := \left[ \int dx \int dy \ p(y|x) \ |X(y) - x|^2 \right]^{1/2},
\]
where \(X(y)\) is the estimation of \(x\) we construct from the outcome \(y\), and where \(p(y|x)\) is the conditional probability distribution of getting a certain \(y\) given \(x\). A lower bound for \(\Delta Y\) follows from the quantum ZZ bound [13],
\[
\Delta Y \geq \left\{ \frac{1}{2} \int_0^\infty dz \int_{-\infty}^{+\infty} dx \min[p(x), p(x+z)] \right. \\
\times \left[ 1 - \sqrt{1 - F(\rho_x, \rho_{x+z})} \right\} \right\}^{1/2},
\]
where \(F\) is the fidelity between the states \(\rho_x\) and \(\rho_{x+z}\) that correspond to a true value of the parameter of \(x\) and \(x+z\) respectively (an alternative, slightly stronger inequality is presented in the Appendix). The parameter \(x\) is mapped onto the probe state \(\rho_x\) by a unitary
\[
x \rightarrow \rho_x = e^{-ixH} \rho_0 e^{ixH},
\]
with \(H\) the generator of translations of \(x\) (an effective Hamiltonian). Then, a bound for the fidelity \(F\) is [13] [10]
\[
F(\rho_x, \rho_{x+z}) \geq \alpha^{-1}(2H / \pi) ,
\]
where \(H := \text{Tr} [H \rho_0] - H_0\) is the expectation value of the generator \(H\) above the ground level \(H_0\) of \(H\), and the function \(\alpha^{-1}\) is the inverse of the function \(\alpha(\epsilon)\) defined in Ref. [13] which vanishes for \(\epsilon \geq 1\) and is approximated by \(4 \arccos^2(\sqrt{\epsilon}) / 2 \pi^2\) for \(\epsilon \in [0, 1]\). In interferometry, \(H\) is the number operator of the optical mode [13] and \(H\) is with the mean photon number \(N\) employed in the experiment. Replacing (3) into (2) we find [25]
\[
\Delta Y \geq \Delta Y_{\text{LB}}, \quad \text{where}
\]
\[
\Delta Y_{\text{LB}} := \left\{ \frac{2}{\pi} \int_0^1 dt \ E(x_0t) \left[ 1 - \sqrt{1 - \alpha^{-1}(t)} \right] \right\}^{1/2}
\]
with \(x_0 := \pi / (2H)\) and
\[
E(z) := \int_{-\infty}^{+\infty} dx \min[p(x), p(x+z)]
\]
being a function which satisfies the constraint \(0 \leq E \leq 1\), with \(E(0) = 1\) and \(E(z) \simeq 0\) for \(z \gg W\) [24].

Asymptotic regimes:— The inequality (3) is an extension of the bound given in Ref. [13], as it also contains the prior information in the function \(E\). There is no guarantee that the accuracy \(\Delta Y\) can reach \(\Delta Y_{\text{LB}}\): indeed most likely the achievable optimal threshold for \(\Delta Y\) is larger, see also [13]. Still the inequality (3) is useful to analyze whether sub-Heisenberg scalings are possible or not. In particular, when the integral in (3) can be approximated by a constant, the factor \(x_0^2\) implies a \(1/H\) scaling for \(\Delta Y\) as requested by the Heisenberg bound. This happens for instance in the LPI regime considered in [13] where the Heisenberg scaling \(1/H\) guarantees a large improvement over the prior uncertainty measured by \(W\), i.e. when the ratio \(t_0 := W / x_0 = 2H W / \pi\) diverges. In this case in fact, independently from the specific form of the prior distribution \(p(x)\), one has \(E(x_0t) = E(W/t_0)\) for all values of the integral in (3) and hence
\[
\Delta Y_{\text{LB}}^{(\text{LPI})} := \lim_{t_0 \to \infty} \Delta Y_{\text{LB}} = x_0 \sqrt{\frac{A}{2}} = \sqrt{\frac{A}{2}} \frac{\pi}{2H},
\]
with \(A := \int_0^1 dt [1 - \sqrt{1 - \alpha^{-1}(t)}] \simeq 0.042\). (9)

A similar result was derived in [18] under the assumption of a uniform prior distribution \(p(x)\), and by bounding \(\alpha^{-1}\) of Eq. (4) with a linear function. Since the asymptotic value \(\Delta Y_{\text{LB}}^{(\text{LPI})}\) does not depend on the specific choice of the prior and has a Heisenberg scaling, we will use it as a benchmark to evaluate the performances of sub-Heisenberg strategies [20].

Consider now the HPI regime in which the prior uncertainty is already much better than the one we could possibly get from a Heisenberg-like scaling. Indicatively this can be identified by the condition \(t_0 \ll 1\). Since for all \(t \neq 0\), \(E(x_0t) = E(W/t_0)\) vanishes when \(t_0 \to 0\) [24] one expects that \(\Delta Y_{\text{LB}}\) of Eq. (3) reduces drastically. As a consequence, the bound (3) no longer prevents \(\Delta Y\) from dropping below the \(1/H^2\) scaling defined by the benchmark value of Eq. (5). However, this is hardly surprising: in the HPI regime the real question is whether or not one can get a significant improvement with respect to the initial uncertainty. Drawing general conclusions on this issue is difficult due to the complex dependence of (3) from the prior probability \(p(x)\). Interestingly, in the case of single-mode distributions \(p(x)\) (i.e. with a single maximum) we can show that the improvement on \(\Delta Y\) vanishes for \(t_0 \to 0\). To prove this we note that for single-mode distributions Eq. (3) yields
\[
E(x_0t) = 1 - \int_{y_m - x_0t/2}^{y_m + x_0t/2} dx \ p(x),
\]
with \(y_m := y_0 + x_0t/2\), where \(y_0\) is the point where \(p(y_0) = p(y_0 + x_0t)\) (such point is unique for single-mode distributions). Remembering that the probability distribution \(p(x)\) has a width \(W\), from (3) it is clear that for \(t_0 \to 0\) the function \(E(x_0t) = E(W/t_0)\) is non-null only when \(t \simeq 0\). Hence, to gauge the lower bound (3) in the HPI regime we can remove the term in square brackets in (3), since \(\alpha^{-1}(t) \approx 1\) for \(t \approx 0\). Accordingly, (3) becomes
\[
\Delta^2 Y_{\text{LB}} = \frac{2}{\pi^2} \int_0^1 dt \ E(x_0t) = \Gamma + \frac{2}{4} \int_0^1 dt \sqrt{P(y_m + \frac{x_0t}{2})} + p(y_m - \frac{x_0t}{2}) = \Gamma + \int_{y_m - \frac{x_0}{2}}^{y_m + \frac{x_0}{2}} dx \ (x - y_m)^2 p(x),
\]
where the first equality follows from an integration by parts, whence \( \Gamma := x_0^2 E(x_0)/4 \), and the second equality is a change of integration variables. If the prior \( p(x) \) has a variance \( \Delta^2 X \), for \( t_0 \to 0 \) the above expression is

\[
\Delta Y_{\text{LB}}^{(\text{HPI})} := \lim_{t_0 \to 0} \Delta Y_{\text{LB}} = \sqrt{\Delta^2 X + (y_0 - \mu)^2} \geq \Delta X(11)
\]

where \( \mu \) is the prior distribution mean, and where we used the fact that \( \lim_{t_0 \to \infty} \Gamma = 0 \) \([24]\). The asymptotic inequality \([1]\) shows that the estimation error is lower bounded by the standard deviation \( \Delta X \) of the prior distribution: the same precision of a strategy that just takes a random guess (according to the prior distribution) for the parameter to estimate. Therefore, sub-Heisenberg schemes are useless in this regime: one can attain the same precision just by “guessing” the parameter.

Intermediate regimes:— Having discussed the general behavior of the bound \([3]\) for large and small \( t_0 \), here we analyze some examples of prior distributions for which the function \( \Delta Y_{\text{LB}} \) can be explicitly evaluated. This allows us to study regimes which are intermediate between HPI and LPI, showing that sub-Heisenberg strategies can deliver advantages which, at most, are limited to a (small) constant factor with respect to the prior accuracy.

Probably the most interesting case is when one has no prior information about the true value of the parameter, which corresponds to a uniform prior \( p(x) \) of width \( W \) with equal weight for all possible values \([18]\). In this case, we have \( E(z) = 1 - z/W \) for \( z \leq W \), and \( E(z) = 0 \) otherwise. Thus, Eq. \([1]\) becomes

\[
\Delta Y_{\text{LB}} = x_0 \sqrt{A(t_0) - B(t_0)/t_0^2}, \quad \text{with} \quad A(t_0) := \int_0^{\min(t_0,1)} dt \ t \left[ 1 - \sqrt{1 - \alpha^{-1}(t)} \right], \quad B(t_0) := \int_0^{\min(t_0,1)} dt \ t^2 \left[ 1 - \sqrt{1 - \alpha^{-1}(t)} \right],
\]

which is plotted in Fig. 2. In agreement with the analysis of the previous section, for large \( t_0 \to \infty \) Eq. \([12]\) yields the universal limit \([3]\) — indeed \( A(\infty) = A \) and \( B(\infty)/t_0 \to 0 \). Moreover, the trivial estimation strategy of choosing a random value according to the prior distribution gives an error \( \Delta X = W/\sqrt{12} = x_0 t_0/\sqrt{12} \).

In the HPI regime (\( t_0 \ll 1 \)) the term in square brackets in \([13]\) and \([14]\) can again be removed since \( \alpha^{-1}(t) \sim 1 \) for \( t \sim 0 \). Hence, the lower bound \([12]\) gives \( \Delta Y_{\text{LB}} = x_0 \sqrt{t_0^2/2 - t_0^2/3} = W/\sqrt{12} = \Delta X \), matching the trivial random estimation procedure in agreement with \([1]\). Furthermore, the largest gap between the lower bound \([3]\) and the prior uncertainty is reached for \( t_0 \simeq 0.5 \) (vertical line in Fig. 2): even assuming that a sub-Heisenberg strategy could reach \([3]\), this will only provide a relative gain of about 1.73 with respect to the initial uncertainty.

Similar results have been obtained for other prior distributions. In particular, in Fig. 2 we report the results obtained for Gaussian and for a bimodal step-like prior distribution. In all cases we record maximum gains of order 2 over the prior uncertainties.

Conclusions:— Using the quantum ZZ bound recently introduced in \([18]\) we have shown that, independently of the prior information available at the beginning of the protocol, the resulting accuracy cannot beat the Heisenberg scaling \( 1/\mathcal{H} \) in the LPI regime (which is arguably the most relevant for practical implementations). Moreover, we have shown that in the other cases where the prior is sufficiently large to allow for a sub-Heisenberg scaling, the resulting accuracy cannot attain a significant enhancement over the one available before starting the estimation procedure.

A similar analysis can be done by replacing \( \mathcal{H} \) with the standard deviation \( \Delta \mathcal{H} \) of the generator \( H \). This yields a generalized uncertainty relation \([25]\) weighted by the prior: it is sufficient to replace the inequality \([3]\) with the Bhattacharyya-like inequality \([29]\) of Ref. \([4]\), i.e.
\[ F(\rho_x, \rho_{x+\frac{\pi}{2}}) \geq \cos^2(\Delta H z) \] for \( \Delta H z \leq \frac{\pi}{2} \). Replacing this into (4), we obtain

\[ \Delta Y \geq \left\{ \frac{\delta_0^2}{2} \int_0^1 dt \int_{-\infty}^{+\infty} dx \left[ p(x) + p(x + x_0 t) \right] \right\}^{\frac{1}{2}}, \tag{15} \]

where \( \delta_0 := \pi/(2\Delta H) \). From this it follows again that in the asymptotic regime \( t_0 \to \infty \) one has \( \Delta Y \geq \delta_0 \sqrt{A'/2} \) where \( A' = 1/2 - 4/\pi^2 \approx 0.095 \) is a constant that does not depend on the prior. In contrast, for \( t_0 \to 0 \), and assuming a single-mode prior, the bound (13) becomes \( \Delta Y \geq \Delta X \), showing that the initial uncertainty again determines the scaling of the accuracy.

**Appendix:** An inequality slightly stronger than (2) can be obtained by using the quantum ZZ-bound [18]

\[ \Delta Y \geq \left\{ \frac{1}{2} \int_0^{+\infty} dz \int_{-\infty}^{+\infty} dx \left[ p(x) + p(x + x + z) \right] \right\}^{\frac{1}{2}}, \tag{16} \]

where \( P_{r_e}(x, x + z) \) is the minimum error probability of a hypothesis-testing problem which aims to discriminate between the states \( R_0 := \rho_x \) and \( R_1 := \rho_{x+\frac{\pi}{2}} \) assuming that they are produced with probabilities \( P_0 = p(x)/[p(x) + p(x + x + z)] \) and \( P_1 = 1 - P_0 \), respectively. Introducing two purifications \( \Psi_0 \) and \( \Psi_1 \) of \( R_0 \) and \( R_1 \) respectively, which verify the identity \( F(\rho_{01}, \rho_{1}) = F(\Psi_0, \Psi_1) \), the quantity \( P_{r_e}(x, x + z) \) can be bounded as follows

\[
P_{r_e}(x, x + z) \geq \frac{1}{2} \left[ 1 - \| P_0 R_0 - P_1 R_1 \|_1 \right] \geq \frac{1}{2} \left[ 1 - \| P_0 \Psi_0 - P_1 \Psi_1 \|_1 \right] = \frac{1}{2} \left[ 1 - \sqrt{1 - 4 P_0 P_1 F(\Psi_0, \Psi_1)} \right] = \frac{1}{2} \left[ 1 - \sqrt{1 - 4 P_0 P_1 F(R_0, R_1)} \right],
\]

where the first inequality derives from [16] (see also Ref. [18]) and the second from the fact that the trace distance is decreasing under the action of partial trace. Replacing this into (16) and using (14) we then find

\[ \Delta Y \geq \left\{ \frac{x_0^2}{4} \int_0^1 dt \int_{-\infty}^{+\infty} dx \left[ p(x) + p(x + x_0 t) \right] \right\}^{\frac{1}{2}} \tag{17} \times \left[ 1 - \sqrt{1 - 4 \frac{x_0^2}{4} \left( \frac{p(x + x_0 t)}{p(x) + p(x + x_0 t)} \right) \alpha^{-1}(t)} \right]^{\frac{1}{2}}. \]

By direct evaluation it results that at least for values of \( x_0 \) and for the prior distributions analyzed in the text, (17) provides a stronger bound than (3), see Fig. 2. However, in the LPI regime (17) and (3) give the same asymptotic constraint. Indeed, since \( p(x + x_0 t) = p(x + W t)/t_0 \), it follows that for \( t_0 \to \infty, p(x + x_0 t) \to p(x) \) and thus the lhs of (17) converges to the quantity \( \Delta Y_{\text{LPI}} \) of Eq. (3).

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[24] Operationally \( W \) is defined by the requirement that \( p(x) \) vanishes for \( |x| \gg W \). In particular, if the distribution \( p(x) \) possesses a variance this implies that the function \( E(z) \) defined in Eq. (4) nullifies when \( z \gg W \).
[25] In writing Eq. (6) we used the fact that \( \alpha^{-1}(t) = 0 \) for \( t \to 1 \) to choose the upper integration limit.
[26] Since \( E \) is always smaller than 1, the lower bound in (6) can never be larger than the asymptotic value \( x_0 \sqrt{A/2} \).
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