Hidden symmetries, trivial conservation laws and Casimir invariants in geophysical fluid dynamics

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Abstract

From a manifestly invariant Lagrangian density based on Clebsch fields and suitable for geophysical fluid dynamics, non-trivial conservation laws and their associated symmetries are described in arbitrary coordinates via Noether’s first theorem. Potential vorticity conservation is however shown to be a trivial law of the second kind with no relevance to Noether’s first theorem. A canonical Hamiltonian formulation is obtained in which Dirac constraints explicitly include the possibly time-dependent metric tensor. It is shown that all Dirac constraints are primary and of the second class, which implies that no infinite-dimensional symmetry transformations of Clebsch fields exist and that Noether’s second theorem does not apply to the governing equations. Therefore, the considered Lagrangian density does not admit a symmetry associated with potential vorticity conservation via Noether’s two theorems. Finally, the corresponding non-canonical Hamiltonian structure with time-dependent strong constraints is derived using tensor components for arbitrary coordinates. The existence of Casimir invariants is linked to trivial conservation laws of the second kind and to symmetries that become hidden after a transformation away from canonical dynamical fields.

1. Introduction

Hamilton’s least action principle provides an elegant way of describing the dynamics of physical systems. Equations of motion arise from a variational problem for the action functional, which on its turn is defined by a single fundamental object called Lagrangian density. In this case, preservation of physical properties—e.g. covariance, the adequate approximate limits, the underlying symmetries and corresponding conservation laws, and so on—translates into constraints on the form of the acceptable Lagrangian densities.

Covariance becomes manifest in Newtonian physics when the Lagrangian density is written in terms of 4-dimensional tensor components suitable for any non-inertial coordinate system in which time intervals are absolute. The corresponding Hamiltonian formulation may also be obtained using Dirac’s theory for constrained systems (Dirac 1950, 1964).

In this paper, symmetries and their corresponding non-trivial conservation laws are studied and Noether’s first theorem is reviewed in the context of a covariant formulation of fluid mechanics, implying that results presented here are valid in all admissible coordinate systems. Noether’s second theorem is also discussed due to its relation to infinite-dimensional symmetries in under-determined dynamical systems. The Hamiltonian formalism with constraints is used since it provides a powerful tool to verify whether Noether’s second theorem is relevant in the study of fluid dynamics.

The formalism and the Lagrangian density used in this paper are outlined in section 2. In section 3, a demonstration of Noether’s first theorem is presented. The demonstration considers field transformations as well as coordinate transformations, and accounts for the presence of the metric tensor and external forcing. Symmetries of the equations of motion leading to mass, entropy, momentum, and energy conservation are studied in arbitrary coordinates. In section 4, potential vorticity conservation—a trivial law of the second...
kind—is obtained independently of Noether’s first theorem (see also Charron and Zadra 2018). In section 5, canonical and non-canonical Hamiltonian formulations are presented in arbitrary coordinates. One objective of this paper is to point out that the constraints on momenta and fields, and the non-vanishing determinant of their corresponding Poisson brackets, imply that the equations of motion for inviscid fluid dynamics (unapproximated and geometrically approximated) described by Clebsch fields are not under-determined, nor are they invariant under infinite-dimensional symmetry transformations. Noether’s second theorem is therefore of no relevance in that case. This implies that potential vorticity conservation cannot be associated with a symmetry via Noether’s two theorems. Another objective of this paper is to verify that a trivial conservation law of the second kind in fluid mechanics—that is, Ertel’s theorem—translates into a Casimir invariant in a non-canonical Hamiltonian formulation. The link between, on the one hand, symmetries that become hidden after a transformation from canonical dynamical fields to non-canonical ones and, on the other hand, Casimir invariants is also discussed in section 5. Conclusions are drawn in section 6.

2. A manifestly invariant Lagrangian density

This section reviews and summarizes the main results from Zadra and Charron (2015) and serves to establish definitions and the notation. A Lagrangian density $\mathcal{L}$ is a scalar that can be used to describe the evolution of a dynamical system. Hamilton’s least action principle states that solutions of the equations of motion are those whose action functional

$$ S[\psi_{(1)}, \psi_{(2)}, \ldots, \psi_{(P)}] = \int d^4x \sqrt{g} \mathcal{L} $$

is stationary, i.e. when

$$ \delta S = \int d^4x \sqrt{g} \sum_{p=1}^{P} \Lambda_{(\psi_{(p)})} \delta \psi_{(p)} = 0, $$

where $P$ is the total number of dynamical fields $\psi_{(p)}$ and $\delta \psi_{(p)}$ indicates arbitrary perturbations of those fields with given and fixed initial, final, and boundary values. The quantity $d^4x \sqrt{g} \equiv dx^0 dx^1 dx^2 dx^3 \sqrt{g}$, where $x^0$ is time and $x^1, x^2, x^3$ are spatial curvilinear coordinates, is an invariant space-time volume element, and $g$ is the determinant of the symmetric covariant metric tensor $g_{\mu\nu}$. This condition on the action leads to the Euler–Lagrange equations. In the case where the Lagrangian density $\mathcal{L}$ may be solely written in terms of scalar fields and their first derivatives, and assuming the metric tensor is prescribed (i.e. not a dynamical field, as is the case in geophysical fluid dynamics), the Euler–Lagrange equations of motion take the form

$$ \Lambda_{(\psi_{(p)})} = 0, $$

where

$$ \Lambda_{(\psi_{(p)})} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_{(p)}} - \left( \frac{\partial \mathcal{L}}{\partial (\psi_{(p)})} \right)_{,\mu} $$

for all the scalar fields on which the Lagrangian density depends (Zadra and Charron 2015). The formalism and notation used in this paper are those of Charron et al (2014), Charron and Zadra (2014, 2015), Zadra and Charron (2015). In particular, tensors are defined with respect to synchronous coordinate transformations (i.e. time intervals are absolute). The admissible spatial coordinate transformations are otherwise general, and may involve the time. A comma followed by an index (here $\mu$) indicates an ordinary partial derivative with respect to $x^\mu$, where $\mu$ takes on the values 0, 1, 2, or 3. A colon followed by an index indicates a covariant derivative. Repeated Greek indices are summed from 0 to 3, and Latin indices from 1 to 3, unless otherwise indicated.

Zadra and Charron (2015) have shown that the manifestly invariant Lagrangian density

$$ \mathcal{L} = -\rho \left( \frac{1}{2} h^{\mu\nu} \dot{v}_\mu v_\nu + \Phi + I + g^{00} v_\mu \right) $$

suitably describes the Newtonian dynamics of an inviscid fluid in arbitrary coordinates under the influence of an external scalar gravitational potential $\Phi$. This Lagrangian density corresponds to $\mathcal{L}_{(IB)}$ in Zadra and Charron (2015). The scalar specific internal energy of the fluid is described by $I = K(\rho, s)$, where $\rho$ is the fluid density, and $s$ its specific entropy. The covariant 4-vector field $v_\mu$ is given by a Clebsch decomposition.

— Manifestly invariant’ means that the Lagrangian density is written using tensors only with balancing indices and null contravariant indices.
\[ v_\mu = \alpha_{\mu} + \beta s_{\mu} + \sum_{r=1}^{N} \gamma_{(r)} \lambda_{(r) \mu}, \]  

(2.6)

where the scalar fields \( \alpha, \beta, \gamma_{(r)}, \) and \( \lambda_{(r)} \) are Clebsch potentials (see e.g. Lin 1963), with \( N \geq 2 \), suitable for any flow (Bretherton 1970). Note that the specific entropy scalar field \( s \) is also interpreted as a Clebsch potential. Taking into account the density and the ensemble of Clebsch potentials, there are \( P = 4 + 2N \) dynamical scalar fields.

The symmetric contravariant tensor \( h^{\alpha \nu} = g^{\alpha \nu} - g^{\alpha \mu} g_{\mu \nu} \), written in terms of the symmetric contravariant space-time metric tensor \( g^{\alpha \nu} \), describes the geometry of space (instead of space-time), within the admissible class of synchronous coordinate transformations. The term \( \mu h^{\alpha \nu} v_\mu v_\nu / 2 \) is the scalar kinetic energy density of the flow as calculated in an inertial coordinate system. Because this kinetic energy density is a manifestly invariant field, its value at any given point in space-time is the same in all admissible (inertial as well as non-inertial) coordinate systems. The scalar \( g^{\alpha \nu} v_\mu \) includes the contribution of non-inertial acceleration terms (such as the Coriolis effect) to the Lagrangian density. The absolute nature of time intervals in Newtonian mechanics imposes a constraint on the space-time metric tensor: the component \( g^{00} \) must be a non-zero constant (taken here as unity). Therefore, the contravariant 4-vector \( u^\mu = h^{0\mu} = h^{00} \) identically vanishes. The absolute nature of time intervals implies that \( \Gamma^0_{\alpha \nu} = 0 \), where \( \Gamma^0_{\alpha \nu} \) are Christoffel symbols of the second kind. Moreover, it also implies that a contravariant tensor of rank \( n \) becomes a tensor of rank \( n - m (0 \leq m \leq n) \) when \( m \) of its indices are set to zero. In general, this rule does not apply to covariant tensors. The contravariant 4-velocity vector field \( u^\mu = dx^\mu / dt \) is taken to have component \( u^0 = dx^0 / dt = 1 \). It is related to the covariant 4-vector field \( v_\mu \) by

\[ u^\mu = h^{0\nu} v_\nu + g^{0\mu}. \]  

(2.7)

The above Lagrangian density and Euler–Lagrange equations lead to the equations governing the evolution of the Clebsch fields and fluid density, which are provided by (65)–(70) in Charron and Zadra (2015). The momentum equations are also derived from these equations in their appendix E.

The above description of fluid dynamics involves a Clebsch decomposition of the 4-velocity field. In practice, given a vector field, Clebsch potentials may be difficult to obtain. A particle-like Lagrangian formulation without Clebsch potentials exists at the expense of imposing a priori constraints on mass and entropy conservation. See Salmon (1998), Zadra and Charron (2015) and references therein. A field formulation in arbitrary coordinates is arguably more satisfying, as mass and entropy conservation equations emerge from Hamilton’s least action principle as dynamical equations.

3. Noether’s first theorem and non-trivially conserved 4-currents

Since Noether (1918) and Bessel-Hagen (1921), it is known that continuous symmetries of the equations of motion lead to conserved currents. For covariant theories defined in 4-dimensional space-time, these conservation laws may be written as 4-dimensional divergent-free equations, e.g. \( j^\mu = 0 \) or equivalently \( (\sqrt{g} \ J^\mu)\mu = 0 \). In this section, Noether’s first theorem is re-derived in the specific context of the formalism outlined in section 2: synchronous coordinate transformations in a metric space-time (not necessarily flat, thus allowing for geometric approximations, see Charron and Zadra 2014) with a Lagrangian density depending on scalar fields and their first derivatives, the metric tensor, and an external scalar potential.

The symmetry transformations here considered are generated by: 1) transformations of the dynamical fields themselves at a given point in space-time, called active field transformations; and 2) transformations induced by changes of the coordinates—which may be interpreted as passive transformations at a given point in space-time, as will be seen below.

3.1. Active transformations

First, consider infinitesimal active field transformations only. Perturbations to the scalar fields \( \psi_\lambda(p) = \psi_\lambda(x^0, x^1, x^2, x^3) \) are written as

\[ \delta \psi_\lambda \equiv \tilde{\psi}_\lambda (x) - \psi_\lambda (x), \]  

(3.1)

where \( \tilde{\psi}_\lambda (p) \) is a perturbed scalar field. The perturbations \( \delta \psi_\lambda (p) \) may depend on all the fields but they take place at fixed space-time points. They leave the metric tensor, its determinant, and all external fields unchanged. It follows that active transformations and gradients are operations that commute, i.e.

\[ \delta (\psi_\lambda (p)_{,\mu}) = (\delta \psi_\lambda (p))_{,\mu}. \]  

(3.2)

Any active transformation \( \delta \psi_\lambda(p) \) induces a perturbation to the integrand of the action functional, which may be written as
Therefore, one may write

\[ \delta \mathcal{L} = \sqrt{g} \delta \mathcal{L}, \]

\[ = \sqrt{g} \sum_{p=1}^{P} \left( \frac{\partial \mathcal{L}}{\partial \psi(p)} \delta \psi(p) + \frac{\partial \mathcal{L}}{\partial (\psi(p),\mu)} \delta (\psi(p),\mu) \right), \]

\[ = \sqrt{g} \sum_{p=1}^{P} \Lambda(\psi(p)) \delta \psi(p) + \sqrt{g} \sum_{p=1}^{P} \frac{\partial \mathcal{L}}{\partial (\psi(p),\mu)} \delta (\psi(p),\mu) \]  

(3.3)

from (2.4) and (3.2). The relation above is true, whether the transformation is a symmetry or not. If in addition one finds that a particular transformation \( \delta \psi(p) \), identically satisfies the condition

\[ \delta \mathcal{L} \equiv J^\mu,\mu \]  

(3.4)

off-shell (i.e. without assuming that the equations of motion (2.3) are satisfied) for some 4-vector \( J^\mu \), then that transformation is a symmetry. In that case, it follows that

\[ \left( J^\mu - \sum_{p=1}^{P} \frac{\partial \mathcal{L}}{\partial (\psi(p),\mu)} \delta \psi(p) \right) \equiv \sum_{p=1}^{P} \Lambda(\psi(p)) \delta \psi(p), \]

(3.5)

This is an identity valid off-shell. In particular, the right-hand side vanishes on-shell (i.e. when the equations of motion (2.3) are satisfied), and (3.5) becomes the conservation of the 4-current

\[ k^\mu = J^\mu - \sum_{p=1}^{P} \frac{\partial \mathcal{L}}{\partial (\psi(p),\mu)} \delta \psi(p). \]

This is Noether’s first theorem for active transformations of fields. Note however that Noether (1918) considered only Lagrangian densities that are strictly invariant under symmetry transformations (i.e. the particular case where \( J^\mu = 0 \)). The generalization to invariant Lagrangian densities up to a divergence is made in Bessel-Hagen (1921), who states that he owes this generalization to an oral communication with E Noether. Kosmann-Schwarzbach (2011) provides mathematical, physical, and historical perspectives on Noether’s two theorems.

3.2. Passive transformations

Consider now an infinitesimal synchronous transformation of coordinates parametrized as \( x^\rho \rightarrow \tilde{x}^\rho = x^\rho + \epsilon^\rho \), where \( \epsilon^\rho \) is an infinitesimal 4-vector. The spatial components \( \epsilon^\rho \) may be functions of the space-time coordinates; in other words, the infinitesimal coordinate transformation of the spatial components is assumed to be local. In contrast, the time component \( \epsilon^0 \) must be at most an infinitesimally small constant due to the absolute nature of time intervals in Newtonian mechanics; only time shifting by a constant is admissible here.

By definition, a scalar field preserves its values at all space-time points under a coordinate transformation. Therefore, one may write \( \tilde{\psi}(\tilde{x}) = \tilde{\psi}(x) \). Here, \( \tilde{x} \) and \( x \) represent the same space-time point expressed in two different coordinate systems. In the coordinate system \( \tilde{x}^\rho \), the functional form of the scalar is different from the functional form of the same scalar in the coordinate system \( x^\rho \). This difference is symbolized by writing the scalar field as \( \tilde{\psi} \) in the coordinate system \( \tilde{x}^\rho \). To first order, one may therefore write

\[ \tilde{\psi}(\tilde{x}) = \tilde{\psi}(x) + \tilde{\psi}_{x^\rho}(x) \epsilon^\rho, \]

(3.6)

\[ = \psi(x). \]

(3.7)

At a point \( x \) of the original coordinate system, define the perturbations

\[ \delta \psi(x) = \psi(x) = -\tilde{\psi}_{x^\rho}(x) \epsilon^\rho = -\psi_{x^\rho}(x) \epsilon^\rho. \]

(3.8)

The last equality stems from the fact that only first order perturbations are considered.

This shows that an infinitesimal coordinate transformation may be interpreted as a particular field transformation at a fixed point in space-time. This type of transformation is called a passive field transformation. The fields are transformed at fixed points within the original coordinate system as the result of an infinitesimal coordinate transformation.

The Lagrangian density depends on the scalar dynamical fields but also on their first derivatives. The first derivatives of the scalar dynamical fields are covariant 4-vectors. Therefore, one must define how an infinitesimal coordinate transformation on a covariant 4-vector is written in terms of passive transformations. By definition, a covariant 4-vector \( A_\mu(x) \) transforms as

\[ A_\mu(x) = \frac{\partial \tilde{x}^\mu}{\partial x^\rho} \tilde{A}_\rho(\tilde{x}). \]

(3.9)
For the infinitesimal transformation \( \tilde{x}^\mu = x^\mu + \epsilon^\mu \), one writes to first order
\[
A_\nu(x) = (\delta^\nu_\mu + \epsilon^\nu_\mu) \tilde{A}_\mu(\tilde{x}),
\]
\[
= \tilde{A}_\mu(\tilde{x}) + \tilde{A}_\mu(\tilde{x}) \epsilon^\nu_\mu,
\]
\[
= \tilde{A}_\mu(x) + \tilde{A}_{\nu;\mu}(x) \epsilon^\nu_\mu + \tilde{A}_\mu(x) \epsilon^\nu_\mu,
\]
\[
= \tilde{A}_\nu(x) + A_{\nu;\mu}(x) \epsilon^\nu_\mu + A_{\mu}(x) \epsilon^\nu_\mu.
\]
One may therefore define
\[
\delta A_\mu \equiv \tilde{A}_\mu(x) - A_\mu(x),
\]
\[
= -A_{\mu;\nu}(x) \epsilon^\nu_\mu - A_{\nu;\mu}(x) \epsilon^\nu_\mu\]
\[
= -A_{\nu;\mu}(x) \epsilon^\nu_\mu - A_{\mu}(x) \epsilon^\nu_\mu,
\]
In the special case where \( A_\mu = \psi_{(p)\mu} \), the term \( A_{\mu;\nu} = A_{\nu;\mu} \) vanishes because ordinary derivatives commute. This yields
\[
\delta(\psi_{(p)\mu}) = -\psi_{(p)\nu}(x) \epsilon^\nu_\mu = (\delta\psi_{(p)})_\mu.
\]
As was the case for active field transformations, the passive perturbation of the gradient of a scalar field is equal to the gradient of the passive perturbation.

An infinitesimal coordinate transformations interpreted as a passive transformation induces changes not only to the dynamical fields but also to the metric tensor, its determinant, and to the external fields such as the gravitational potential. For passive transformations of the action functional, all perturbations occur within the original coordinate system \( x^\mu \), and the integration limits are consequently kept fixed. Therefore,
\[
\delta S = \delta \int d^4x \sqrt{g} \mathcal{L} = \int d^4x \, \delta(\sqrt{g} \mathcal{L}).
\]
The variation \( \delta(\sqrt{g} \mathcal{L}) \) is written
\[
\delta(\sqrt{g} \mathcal{L}) = \sqrt{g} \sum_{p=1}^p \left( \frac{\partial \mathcal{L}}{\partial \psi_{(p)}} \delta\psi_{(p)} + \frac{\partial \mathcal{L}}{\partial (\psi_{(p)\mu})} \delta(\psi_{(p)\mu}) \right) + \frac{\partial(\sqrt{g} \mathcal{L})}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \sqrt{g} \frac{\partial \mathcal{L}}{\partial \Phi} \delta \Phi,
\]
where \( \delta g^{\mu\nu} \) is given by (A.20), and \( \delta \Phi = -\Phi_{;\mu} \epsilon^\mu_\mu \) as in (3.8) since \( \Phi \) is a scalar. It is convenient to define the symmetric tensor \( t_{\mu\nu} \) as
\[
t_{\mu\nu} \equiv -2 \frac{\partial(\sqrt{g} \mathcal{L})}{\partial g^{\mu\nu}},
\]
which is the covariant mass-momentum tensor \( T_{\mu\nu} = \rho u_{\mu} u_{\nu} + h_{\mu\nu} p \) (where \( p \) is pressure) plus a term that vanishes on-shell (see appendix A.1). One may then rewrite the variation
\[
\delta(\sqrt{g} \mathcal{L}) = \sqrt{g} \sum_{p=1}^p \left( \lambda(\psi_{(p)}) \delta\psi_{(p)} - \frac{1}{2} t_{\mu\nu} \delta g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial \Phi} \delta \Phi + \left( \sum_{p=1}^p \frac{\partial \mathcal{L}}{\partial (\psi_{(p)\mu})} \delta(\psi_{(p)\mu}) \right)_{\mu} \right),
\]
from (2.4) and (3.17).

In addition, the quantity \( \delta(\sqrt{g} \mathcal{L}) \) may always be rewritten as a divergence under a passive transformation. To see this, consider the perturbations of \( \sqrt{g} \) and \( \mathcal{L} \) separately. The former is expressed as
\[
\delta(\sqrt{g}) = -\sqrt{g} \epsilon^\mu_\mu\]
(see appendix A.2), and the latter, being a scalar, as
\[
\delta \mathcal{L} = -\mathcal{L}_{;\mu} \epsilon^\mu,
\]
similarly to (3.8). Therefore,
\[
\delta(\sqrt{g} \mathcal{L}) = -(\sqrt{g} \mathcal{L}_{;\mu})_\mu = -\sqrt{g}(\mathcal{L}_{;\mu})_\mu.
\]
This expression is nothing but the statement that equations of motion are covariant: a coordinate transformation does not change the form of the equations of motion since the integrand defining the action functional changes by a total divergence under any infinitesimal coordinate transformation. In itself, this does not automatically translate into an actual symmetry. From (3.8), (3.21) and (3.24), one writes
\[
\left( \sum_{p=1}^p \frac{\partial \mathcal{L}}{\partial (\psi_{(p)\mu})} \delta(\psi_{(p)}) \right)_{\mu} = \sum_{p=1}^p \lambda(\psi_{(p)}) \delta\psi_{(p)} - \frac{1}{2} t_{\mu\nu} \delta g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial \Phi} \delta \Phi.
\]
On-shell, (3.25) represents the conservation of a 4-current
\[
j^{\mu} = \left( \sum_{\rho=1}^{p} \frac{\partial \mathcal{L}}{\partial (\psi_{\rho}^{*})} \psi_{\rho}^{*} - \delta^{\mu}_{\nu} \mathcal{L} \right) e^{\nu}
\] (3.26)
if the two symmetry conditions \( \delta g^{\mu\nu} = 0 \) and \( \delta \Phi = 0 \) are satisfied. These symmetry conditions on the metric tensor and external forcing are equivalent to
\[
e^{\nu\mu} = -e^{\nu\mu},
\] (3.27)
\[
\Phi_{\mu} e^{\mu} = 0
\] (3.28)
(see appendix A.3 and (3.8)). Notice that more general symmetry conditions may be found if one establishes that \( t_{\mu\nu} \delta g^{\mu\nu} \equiv A^{\mu}_{\rho} \) and \( \rho \delta \Phi \equiv B^{\mu}_{\rho} \) off-shell for some 4-vectors \( A^{\mu} \) and \( B^{\mu} \). However, these more general symmetry conditions will not be explored below. Notice also that adding (3.5) to (3.25) yields a more general expression of Noether’s first theorem than what is presented in Webb (2018, his equation 4.25), insofar as the variation of an external forcing \( \Phi \) and metric tensor \( g^{\mu\nu} \) are here considered in its derivation.

The conserved 4-current is associated with the symmetry conditions (3.27) and (3.28). This is Noether’s first theorem for infinitesimal coordinate transformations interpreted here as passive transformations on dynamical fields, the metric tensor, and the external forcing.

### 3.3. Symmetries of the equations of motion, not necessarily of the action functional

In general, active and passive transformations are not special cases of the arbitrary variations defined in the context of Hamilton’s least action principle: initial, final, and boundary field values are not necessarily kept fixed under active and passive transformations.

Two Lagrangian densities that differ by a covariant divergence lead to the same equations of motion from Hamilton’s least action principle. The transformation of \( \sqrt{g} \mathcal{L} \) into
\[
\sqrt{g} \mathcal{L} + (\delta + \bar{\delta})(\sqrt{g} \mathcal{L}) = \sqrt{g} (\mathcal{L} + (J^{\mu} - \mathcal{L} e^{\mu})_{,\mu}),
\] (3.29)
following which the equations of motion are automatically unchanged\(^3\), does not imply that the action functional \( S \) itself is necessarily unchanged. In the 4-volume integration that leads to a transformed action functional \( S' = S + (\delta + \bar{\delta}) S \), possible remaining boundary terms induced by the transformation mean that \( (\delta + \bar{\delta}) S \) does not vanish in general. However, \( (\delta + \bar{\delta}) S \) does not need to vanish for conserved currents to exist. The covariance of the equations of motion under the transformations and the realization of the symmetry conditions suffice.

In sum, active transformations are symmetry transformations if a 4-vector \( f^{\mu} \) exists such that \( \delta (\sqrt{g} \mathcal{L}) = (\sqrt{g} f^{\mu})_{,\mu} \) off-shell. For passive transformations, \( \delta (\sqrt{g} \mathcal{L}) \) can always be expressed as a total divergence because, in physics, acceptable equations of motion are covariant. Passive transformations are symmetry transformations if specific conditions on the external forcing and metric tensor are satisfied. The conservation laws associated with these active and passive transformations are called *non-trivial*; they exist off-shell only, and they arise from internal or space-time symmetries of the equations of motion. These non-trivial conservation laws may be contrasted with *trivial* conservation laws of the second kind, which exist off-shell and cannot be obtained via Noether’s first theorem (Olver 1993, Charron and Zadra 2018).

### 3.4. Symmetries from active transformations of fields: mass and entropy conservation

As an example of symmetries from active field transformations that lead to conserved currents, consider the following transformations of the dynamical fields:
\[
\delta \rho = 0,
\] (3.30)
\[
\delta \alpha = \epsilon \left( \mathcal{F} - \sum_{i=1}^{N} \gamma_{(r)} \frac{\partial \mathcal{F}}{\partial \gamma_{(r)}} \right),
\] (3.31)
\[
\delta \beta = -\epsilon \frac{\partial \mathcal{F}}{\partial \beta},
\] (3.32)
\[
\delta s = 0,
\] (3.33)
\(^3\) However, the new Lagrangian density \( \mathcal{L}' = \mathcal{L} + (J^{\mu} - \mathcal{L} e^{\mu})_{,\mu} \) now also depends on the second derivatives of the scalar dynamical fields. In this case, the arbitrary variations defined in the context of Hamilton’s least action principle \( \delta (\psi_{\rho}^{*}) \) and \( \delta (\psi_{\rho}) \) both vanish at the 4-volume integration limits, and the Euler–Lagrange equations take the form
\[
\frac{\partial \mathcal{L}'}{\partial \psi_{\rho}^{*}} - \left( \frac{\partial \mathcal{L}'}{\partial (\psi_{\rho}^{*}; \psi_{\rho})} \right)_{,\rho} + \left( \frac{\partial \mathcal{L}'}{\partial (\psi_{\rho}^{*}; \psi_{\rho})} \right)_{,\rho} = 0,
\]
\[ \delta \gamma_{(r)} = -\epsilon \frac{\partial F}{\partial \lambda_{(r)}}, \tag{3.34} \]

\[ \delta \lambda_{(r)} = \epsilon \frac{\partial F}{\partial \gamma_{(r)}}, \tag{3.35} \]

where \( \epsilon \) is a small constant parameter and \( F = F(s, \gamma_{(1)}^1, \lambda_{(1)}, \ldots, \gamma_{(N)}^N, \lambda_{(N)}) \) any differentiable function that depends locally on \( s, \gamma_{(r)}^1 \) and \( \lambda_{(r)} \) but not on their space-time derivatives. It may be verified that these transformations leave the vector field \( v_\gamma \) and the Lagrangian density unchanged (\( \delta v_\gamma = 0 \) and \( \delta L = 0 \)). One may then choose \( J^\mu = 0 \) from (3.4). From (2.5) and (3.5), one finds the associated on-shell conserved 4-current

\[ k^\mu = -\sum_{p=1}^{p} \frac{\partial L}{\partial (\psi_{(p),\mu})} \delta \psi_{(p),\nu} = \rho \mu^\mu \left( \delta \alpha + \beta \delta s + \sum_{r=1}^{N} \gamma_{(r)} \delta \lambda_{(r)} \right) = \epsilon \rho \mu^\mu F. \tag{3.36} \]

In particular, the choice \( F = s \) leads to the conservation of entropy, and \( F = 1 \) to the conservation of mass. The transformations (3.30)–(3.35) are called a gauge symmetry by e.g. Henyey (1982) and Webb and Anco (2017). Notice however that they represent a global, not a local, gauge symmetry as \( \epsilon \) is a constant. The existence of local gauge (infinite-dimensional) symmetries would imply that the associated transformations involve arbitrary functions of space-time and that the dynamical system is under-determined. As will be shown in subsection 5.1.1, local gauge freedom does not exist for the dynamical equations describing density and the Clebsch functions of space-time and that the dynamical system is under-determined. As will be shown in Webb and Anco (2017).

3.5. Space-time symmetries: momentum and energy conservation

Continuous space-time transformations are not necessarily symmetries in the sense of Noether. For continuous space-time transformations to be symmetry transformations, specific and concurrent conditions on the metric tensor and external forcing must be satisfied. In this case, these conditions are given by (3.27)–(3.28).

Conserved 4-currents arising from purely space-time symmetries are obtained from (3.25) for passive transformations. From the Lagrangian density (2.5) and the definitions (2.6) and (2.7), one obtains

\[ \sum_{p=1}^{p} \frac{\partial L}{\partial (\psi_{(p),\mu})} \psi_{(p),\nu} = -\rho \mu^\mu v_\nu. \tag{3.37} \]

Define an energy density 4-vector similar to (45) in Charron et al (2014) as

\[ E^\mu \equiv \rho \mu^\mu \left( \frac{1}{2} g_{\alpha\beta} u^\alpha u^\beta + I + \Phi' \right) + u_\nu h_{\nu\mu} p. \tag{3.38} \]

(\( \Phi' \equiv \Phi + 1 \) and \( \Phi \) obviously represent the same gravitational field). The 4-current is written, after some algebra and from (65) in Zadra and Charron (2015), as

\[ j^\mu \equiv \sum_{p=1}^{p} \frac{\partial L}{\partial (\psi_{(p),\mu})} \psi_{(p),\nu} e^\nu - L e^\mu, \tag{3.39} \]

\[ = -\rho \mu^\mu v_\nu e^\nu - p e^\mu - \rho \lambda_{(0)} e^\mu, \tag{3.40} \]

\[ = e^0 E^\mu - e^\nu T^\nu_{\mu} + \rho \lambda_{(0)} (e^0 u^\mu - e^\mu). \tag{3.41} \]

On-shell, (3.25) becomes

\[ (e^0 E^\mu - e^\nu T^\nu_{\mu})_{;\mu} = -T_{\mu\nu} e^{\mu\nu} + \rho \Phi_{\mu} e^{\mu}, \tag{3.42} \]

where \( T_{\mu\nu} \) is the covariant mass-momentum tensor. If the symmetry conditions (3.27) and (3.28) are satisfied, the right-hand side vanishes (recall that \( T_{\mu\nu} = T_{\nu\mu} \) would then be contracted with an antisymmetric tensor \( e^\nu_{\mu\nu} = -e^\nu_{\mu\nu} \)) and the 4-current \( e^0 E^\mu - e^\nu T^\nu_{\mu} \) becomes covariant divergent-free.

Suppose that one chooses \( e^i = 0 \). It has been shown by Charron and Zadra (2014) that the symmetry conditions (3.27) and (3.28) are satisfied when having \( g_{\mu\nu,0} = 0, \Phi_{0} = 0, e^i = \text{constant (} i = 1 \text{ or } 2 \text{ or } 3 \), and \( e^k = 0 \) (for any \( k \neq i \)) concurrently in at least one coordinate system. In that case, (3.42) represents the conservation of momentum (linear, angular, or other) in any admissible coordinate system.

Similarly, if one chooses \( e^0 = 0 \), the symmetry conditions (3.27) and (3.28) are satisfied when having \( g_{\mu\nu,0} = 0, \Phi_{0} = 0, \) and \( e^k = 0 \) concurrently in at least one coordinate system (see Charron and Zadra 2015). In that case, (3.42) represents the conservation of energy in any admissible coordinate system.

In sum, in a given coordinate system conservation equations for momentum or energy exist if both the gravitational potential and the metric tensor exhibit the appropriate symmetries.
4. Triviality of potential vorticity conservation

In the preceding section, mass and entropy conservation has been associated with a one-parameter global internal symmetry of the equations of motion, while momentum and energy conservation has been associated with space-time symmetries of the metric tensor and external forcing. Another fundamental conservation law in geophysical fluid dynamics is that of Ertel’s potential vorticity. As will be shown below, the equation describing potential vorticity conservation is an off-shell identity and a trivial law (see also Charron and Zadra 2018). Therefore, Noether’s first theorem is inadequate to associate potential vorticity conservation with a symmetry.

Consider a 4-current \( c_\mu \equiv F^\mu\nu_{\nu,\mu} \), where \( F^\mu\nu \) is an antisymmetric tensor. The covariant divergence of \( c_\mu \) identically vanishes:

\[
\rho^{\mu\nu} c_\mu = F^\mu\nu_{\nu,\mu} = \frac{1}{\sqrt{\gamma}} (\sqrt{\gamma} F^\mu\nu)_{\nu,\mu} = 0. \tag{4.1}
\]

Following the terminology of Olver (1993), (4.1) is a trivial conservation law of the second kind and is obtained off-shell.

Potential vorticity \( q \) is written

\[
q = \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} u_{\alpha,\gamma,\beta,\mu} = \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} w_{\alpha,\gamma,\beta,\mu} = \sum_{\tau=1}^N \frac{\partial_\nu s_{\nu,\tau,\mu}}{\gamma} \tau_{\tau,\nu,\mu,\xi,\nu}. \tag{4.2}
\]

The fields defining \( q \) are considered off-shell. Define also

\[
q^{(\tau)} = \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} \tau_{\tau,\nu,\mu,\xi,\nu} \tag{4.3}
\]

(no sum over \( \tau \)), a 4-vector \( A^{\mu}_{(\tau)} \) as

\[
A^{\mu}_{(\tau)} = \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} \tau_{\tau,\nu,\mu,\xi,\nu} \tag{4.4}
\]

(again, no sum over \( \tau \)), and the antisymmetric tensor \( F^{\mu\nu}_{(\tau)} \) as

\[
F^{\mu\nu}_{(\tau)} = u^\beta A^{\mu \beta}_{(\tau)} - u^{\mu} A^{\beta}_{(\tau)}. \tag{4.5}
\]

The 4-current \( \rho u^\beta q^{(\tau)} \) may be written as

\[
\rho u^\beta q^{(\tau)} = \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} u^\beta s_{\mu,\nu,\tau,\mu,\xi,\nu} = (u^\beta A^{\mu \beta}_{(\tau)})_{\mu} - u^{\beta \mu} A^{\mu}_{(\tau)}. \tag{4.6}
\]

By virtue of (A.21) and (A.23),

\[
u^\mu A^{\mu}_{(\tau)} = \frac{1}{\sqrt{\gamma}} (s_{\tau,\nu,\mu,\xi,\nu} (\partial_\nu s_{\nu,\mu}) - \partial_\nu s_{\nu,\mu} u^{\mu} - \partial_\nu s_{\nu,\mu} u^{\mu}),
\]

\[
= (u^\beta A^{\beta}_{(\tau)})_{\mu} - \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} (s_{\tau,\nu,\mu,\xi,\nu} + s_{\tau,\nu,\mu,\xi,\nu} + s_{\tau,\nu,\mu,\xi,\nu}). \tag{4.7}
\]

One may therefore define a 4-current \( c^{(\tau)}_\mu \) as

\[
c^{(\tau)}_\mu \equiv \rho u^\beta q^{(\tau)} = \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} (s_{\tau,\nu,\mu,\xi,\nu} + s_{\tau,\nu,\mu,\xi,\nu} + s_{\tau,\nu,\mu,\xi,\nu}),
\]

\[
= \rho u^\beta q^{(\tau)} + \frac{\partial_\nu s_{\nu,\mu}}{\sqrt{\gamma}} (\rho^{-1} s_{\tau,\nu,\mu,\xi,\nu} + s_{\tau,\nu,\mu,\xi,\nu} + s_{\tau,\nu,\mu,\xi,\nu}),
\]

\[
= F^{\mu\nu}_{(\tau)} \gamma_{\nu,\mu}. \tag{4.8}
\]

This 4-current is therefore identically divergent-free \( (c^{(\tau)}_{\mu,\nu} = 0) \) by virtue of (4.1). On-shell, it becomes \( \rho u^\beta q^{(\tau)} \).

In the case of the current described by (4.8), the charge density \( c^{(\tau)}_\mu \) is \( \rho q^{(\tau)} \) off-shell.
Because the conservation equation

\[ c^\mu : \mu = \left( \sum_{i=1}^{N} c^\mu (y_i, \tilde{y}) \right)_{, \mu} = 0 \]  
\[(4.9)\]

is a trivial conservation law of the second kind, potential vorticity conservation cannot be associated with a particular symmetry of the equations of motion via Noether’s first theorem. The conservation law (4.9) is an identity and is demonstrated without assuming that the equations of motion are satisfied. The triviality of potential vorticity conservation was also demonstrated using the fields \( u^i, \rho, \) and \( s \)—instead of Clebsch potentials—by Charron and Zadra (2018). When the equations of motion are under-determined, trivial conservation laws of the second kind may be associated with infinite-dimensional symmetries via Noether’s second theorem. In the following section, the applicability of Noether’s second theorem to fluid dynamics is investigated using Dirac’s theory of constrained Hamiltonian systems.

5. Hamiltonian formulations

The manifestly covariant Lagrangian field theory in arbitrary coordinates presented above uses Clebsch potentials as dynamical fields. In practice, it is often difficult to determine these potentials. The Hamiltonian formulation allows to choose more traditional dynamical fields, such as \( u^i \). Moreover, if one is able to write the dynamical evolution of a physical system within a Hamiltonian formalism, one may take advantage of a vast body of general techniques to tackle specific problems, for instance non-linear stability theorems (see e.g. Shepherd 1990, and references therein). In addition and perhaps more importantly in the context of this paper, the theory of constrained Hamiltonians developed by Dirac (1950, 1964) and its classification of phase-space constraints provide a systematic method to determine the existence of degrees of freedom associated with infinite-dimensional symmetries which are related to Noether’s second theorem.

Hamiltonian formulations applied to fluid flows are reviewed for example in Shepherd (1990), Salmon (1998), Morrison (1998) and references therein. The use of Dirac’s theory of constrained Hamiltonians is not new in fluid mechanics. For instance, Salmon (1988) used Dirac constraints to study semi-geostrophic flows, Vanneste and Bokhove (2002) to study nearly geostrophic balanced models, Chandre et al (2012) to study incompressible flows. Chandre et al (2012) provide a list of references in which Dirac’s theory is used in (magneto)hydrodynamics. Here, Dirac’s theory of constrained Hamiltonian systems is employed not to describe approximated fluid equations or because it would be necessary for transitioning from a Lagrangian to a Hamiltonian description of fluid dynamics but to characterize the class of phase-space constraints that the degeneracy of the Lagrangian density (2.5) induces. Knowing the class of phase-space constraints provides information on the types of symmetries a dynamical system possesses.

In this section, canonical and non-canonical Hamiltonian formulations are developed with time-dependent constraints in arbitrary coordinates admissible in Newtonian fluid mechanics. One objective is to clarify the relation between Casimir invariants and the types of associated conservation laws. As for the previous sections, results are valid under any geometric approximations, i.e. in curved (Riemannian) space as long as time intervals remain absolute. Dynamical approximations are not explicitly considered but the methods described below could be applied under such approximations.

5.1. Canonical Hamiltonian formulation with time-dependent constraints

In a canonical Hamiltonian formulation, the ‘velocities’ \( \psi_{(p,0)} \) of the dynamical fields \( \psi_{(p)} \) are replaced by ‘momenta’ defined by

\[ \pi_{(\psi_{(p,0)})} = \frac{\partial (\sqrt{\mathcal{L}})}{\partial (\dot{\psi}_{(p,0)})} = \sqrt{\mathcal{L}} \frac{\partial \mathcal{L}}{\partial (\dot{\psi}_{(p,0)})}. \]  
\[(5.1)\]

The phase space of the dynamical system consists of the \( P \) dynamical fields \( \psi_{(p)} \) and the \( P \) conjugate momenta \( \pi_{(\psi_{(p)})} \). In particular for the Lagrangian density (2.5), the momenta of the \( 4 + 2N \) scalar fields are \( \pi_{(\rho)} = 0 \), \( \pi_{(\tilde{\rho})} = -\sqrt{\mathcal{L}} \rho \), \( \pi_{(\beta)} = 0 \), \( \pi_{(\tilde{\beta})} = -\sqrt{\mathcal{L}} \rho_{\beta} \), \( \pi_{(\chi)} = 0 \), and \( \pi_{(\tilde{\chi})} = -\sqrt{\mathcal{L}} \rho_{\chi} \). In principle, these relations are used to express the ‘velocities’ in terms of \( \psi_{(p)} \) and the corresponding momenta. However, when a Lagrangian density depends linearly on the ‘velocities’, this is not possible and one finds constraints in phase space. This is the case of the Lagrangian density for geophysical fluid dynamics, where one therefore finds \( P \) constraints \( \phi_{(\psi_{(p)})} \) written as

\[ \phi_{(\rho)} = \pi_{(\rho)} \approx 0, \]  
\[(5.2)\]
\[ \phi_{(\beta)} = \pi_{(\beta)} + \sqrt{\mathcal{L}} \rho \approx 0, \]  
\[(5.3)\]
\[ \phi_{(\tilde{\beta})} = \pi_{(\tilde{\beta})} \approx 0, \]  
\[(5.4)\]
where the notation $\approx 0$ is explained later on. The Hamiltonian $H_C$ is

$$H_C = \int d^3x \mathcal{H}_C,$$  

(5.8)

where its density $\mathcal{H}_C$ is

$$\mathcal{H}_C = \sum_{p=1}^{P} (\pi_{(p)} \dot{\psi}_{(p)} + c_{(p)}') - \sqrt{g} \mathcal{L},$$

(5.9)

The $c_{(p)}'$s, where $c_{(p)}' = c_{(p)}' + \psi_{(p)}$, are at this stage unknown functions to be determined. This Hamiltonian density incorporates the null constraints. Although it obeys the principle of Newtonian relativity, it is not manifestly invariant because time is singled out in the Hamiltonian formalism.

The equations of motion are obtained from the Poisson brackets$^4$:

$$\{ \psi_{(p)}(x), \pi_{(p)}(y) \} = \{ \psi_{(p)}(x), \mathcal{H}_C(y) \},$$

(5.10)

and

$$\pi_{(p)}(x) = \int d^3y \{ \pi_{(p)}(x), \mathcal{H}_C(y) \}.$$  

(5.11)

The fundamental (canonical) Poisson brackets are

$$\{ \psi_{(p)}(x), \pi_{(p)}(y) \} = -\{ \pi_{(p)}(x), \psi_{(q)}(y) \} = \delta_{pq} \delta^{(3)}(x - y),$$

(5.12)

and zero otherwise.

Following the terminology and notation of Dirac (1964), the $P$ conditions $\phi_{(p)} \approx 0$ are called primary constraints. The symbol $\approx 0$ indicates that they are weak equations in the sense that Poisson brackets must be calculated before actually applying the constraints.

The primary constraints $\phi_{(p)} \approx 0$ may not be the only constraints of the dynamical system. The conditions $\phi_{(p)} \approx 0$ should remain valid at all times, which is equivalent to imposing that

$$\phi_{(p)}(x) \equiv \{ \phi_{(p)}(x), H_C \} + \left( \frac{\partial \phi_{(p)}(x)}{\partial t} \right)_{\psi, \pi} \approx 0.$$  

(5.14)

This relation means that the time derivative of a null constraint at a given point in space-time remains weakly zero. The last term before $\approx 0$ exists when a constraint has an explicit time dependence, for example involving the metric terms. The symbol $\approx 0$ means that the time derivative is taken keeping the dynamical fields $\psi_{(p)}$ and their conjugate momenta $\pi_{(p)}$ unchanged. If these $\approx 0$ equations are not equivalent to the already known primary constraints, they imply either new constraints (called secondary constraints) independent of the $c_{(p)}'$s, or they determine the $c_{(p)}'$s. When (5.14) is applied successively on the constraints $\phi_{(p)}'$, $\phi_{(a)}'$, $\phi_{(b)}'$, $\phi_{(c)}'$, and $\phi_{(d)}'$, one may determine all the $c_{(p)}'$s:

$$c_{(p)} = -\frac{1}{\sqrt{g}} \left(\sqrt{g} \rho \dot{\psi}_{(p)}\right)_0 + \rho \left(\sqrt{g}\right)_0,$$  

(5.15)

$$c_{(a)} = -(B + g^{\alpha \beta} u_\beta) + \beta \dot{u}_\alpha + \sum_{r=1}^{N} \gamma_{(r)} u^r \lambda_{(r)},$$

(5.16)

$$c_{(b)} = -u^r \lambda_{(r)} + T,$$  

(5.17)

$$c_{(c)} = -u^i \gamma_{(i)},$$

(5.18)

$$c_{(d)} = -u^i \gamma_{(i)},$$  

(5.19)

$^4$ In this section, the fields and metric terms are considered at the same time $t$, therefore $x$ and $y$ are meant to represent two spatial—not space-time—points in the same coordinate system.
where

\[ \mathcal{B} = \frac{1}{2} h^{ij} u_i u_j + \Phi + I + \frac{P}{\rho} \]  

(5.21)

is the Bernoulli function and \( T \) is temperature. Consequently, there are no secondary constraints. The Hamiltonian \( H_C \) is now fully determined and the equations of motion (5.10) are

\[ \rho_0 = -\frac{1}{\sqrt{8}}((\sqrt{8} \rho u^i)_i + \rho(\sqrt{8})_0), \]  

(5.22)

\[ \alpha_{0,i} = -(B + \sqrt{8} \rho^i u_i) + \beta u^i s_i + \sum_{r=1}^{N} \gamma_{(r)} u^i \lambda_{(r),i} \]  

\[ = -u^i \alpha_{i} + \frac{1}{2} h^{ij} u_i u_j - I - \Phi + \frac{\rho}{\rho}, \]  

(5.23)

\[ \beta_0 = -u^i \beta_i + T, \]  

(5.24)

\[ s_0 = -u^i s_i, \]  

(5.25)

\[ \gamma_{(r),0} = -u^i \gamma_{(r),i}, \]  

(5.26)

\[ \lambda_{(r),0} = -u^i \lambda_{(r),i}. \]  

(5.27)

These are equivalent to (71)–(76) in Zadra and Charron (2015).

5.1.1. Absence of infinite-dimensional symmetries

Dirac (1964) makes the distinction between first-class and second-class constraints. First-class constraints exist if and only if the determinant of the matrix \( M \) vanishes, where its matrix elements are

\[ M_{pq}(x, y) = \{ \phi_{(q)}(x), \phi_{(p)}(y) \}, \]  

(5.28)

including all (primary and secondary) constraints. In this case, when dynamical fields are ordered as \( (\rho, \alpha, \beta, \gamma_{(1)}, \gamma_{(2)}, \ldots, \gamma_{(N)} \) and for \( N = 2 \), this matrix reads

\[
\begin{pmatrix}
0 & -1 & 0 & -\beta & 0 & -\gamma_{(1)} & 0 & -\gamma_{(2)} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\gamma_{(2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]  

(5.29)

From the definition \( M_{pq}(x, y) \equiv \sqrt{8} \delta^{(3)}(x - y) \mathcal{M}(x) \), one may verify that

\[ \det(\mathcal{M}) = \rho^{2(N+1)} \neq 0, \]  

(5.30)

which holds for any positive integer \( N \). There are consequently no first-class constraints. All constraints are of the second class. Dirac (1964) shows that infinite-dimensional symmetries—that depend on arbitrary functions of space-time—exist only when there are first-class constraints. Therefore, there are no infinite-dimensional symmetries and the dynamical system presented in this paper is not under-determined.

In this specific case, Noether’s second theorem becomes irrelevant because it only applies to under-determined systems. Noether’s second theorem and the associated differential identities among the equations of motion for under-determined dynamical systems, which imply the existence of trivial conservation laws of the second kind, cannot be invoked to explain potential vorticity conservation. Potential vorticity conservation is therefore not associated with a symmetry via Noether’s two theorems under the Lagrangian density (2.5).

If potential vorticity conservation is to be associated with an infinite-dimensional symmetry via Noether’s second theorem, one must first find a Lagrangian density that leads to a set of under-determined equations of motion for classical fluids. However, it may be shown that not only (2.5) but all three Lagrangian densities known to describe classical fluids in arbitrary coordinates (Zadra and Charron 2015, and references therein) do not lead to under-determined equations of motion and therefore do not admit infinite-dimensional symmetries. The Lagrangian density \( \mathcal{L}_{(2)} \) in Zadra and Charron (2015) leads to 16 Dirac constraints (13 primary, 3 secondary) but none are of the first class. In addition, the Lagrangian density \( \mathcal{L}_{(1)} \) in Zadra and Charron (2015)—the particle-like formulation used, for instance, by Padhye and Morrison (1996a, 1996b),
Charron and Zadra (2018)—leads to three equations of motion \( \Lambda_{(\rho)} \) that are linear combinations of the Euler–Lagrange equations (2.4) (and the first-order derivative of one of them) obtained from the Lagrangian density (2.5):

\[
\Lambda_{(\rho)} = \rho \Lambda_{(\rho)} - (\alpha_{i} - u_{i}) \Lambda_{(\alpha)} - \beta_{i} \Lambda_{(\beta)} - s_{i} \Lambda_{(s)} - \sum_{r=1}^{N} (\gamma_{r}(\tau_{i}) \Lambda_{(\tau_{i})} + \lambda_{(r)} \Lambda_{(\lambda_{r})}),
\]

(5.31)

to which two external constraints on mass and entropy conservation \( \Lambda_{(\rho)} = 0 \) and \( \Lambda_{(s)} = 0 \), respectively) are imposed. Therefore, the \( \Lambda_{(\rho)} \)'s and the two externally imposed constraints cannot be under-determined and cannot lead to infinite-dimensional symmetries via Noether’s second theorem because the right-hand side of (5.31) is made of equations of motion that are not under-determined.

Then why do some authors still claim that potential vorticity conservation in compressible fluids can be associated with the particle-relabeling transformation? As shown in Charron and Zadra (2018), Noether’s first theorem was (mis)used to infer this association (Newcomb 1967, Padhye and Morrison 1996b). However, Noether’s first theorem is irrelevant to associate a trivial conservation law of the second kind with a symmetry: trivial conservation laws of the second kind exist off-shell while conservation laws obtained from Noether’s first theorem exist on-shell only. Noether’s second theorem, and ‘generalizations’ thereof, are also sometimes invoked (Padhye and Morrison 1996a, 1996b), however, the fact that the equations of motion are not under-determined prevents the use of Noether’s second theorem with the three Lagrangian densities presented in Zadra and Charron (2015): there are no infinite-dimensional symmetries in the system described by these three Lagrangian densities.

The source of this mistaken association between potential vorticity conservation and particle-relabeling seems to boil down to two factors.

Firstly, it appears that the concepts of covariance—i.e. invariance in the form of the governing equations—and symmetry are sometimes treated as being equivalent. Because the equations of motion are covariant, an arbitrary particle-relabeling—i.e. a coordinate transformation within the sub-class of comoving frames, see Charron and Zadra (2018)—leaves the tensorial form of the equations of motion unchanged. This does not imply that a dynamically relevant symmetry exists. In Newtonian mechanics, an ‘invariance of form’ should not be mistaken for a dynamically relevant symmetry: covariance and symmetry are distinct concepts. The situation is different in general relativity where the metric tensor is a dynamical field. The Bianchi relations for the Ricci tensor represent differential identities among Einstein’s field equations. In Newtonian mechanics however, the metric tensor is not a dynamical field and the arbitrariness in the choice of coordinates does not imply the existence of differential identities among the Euler–Lagrange equations.

Secondly, Noether’s second theorem is based on the existence of homogeneous differential identities among the Euler–Lagrange equations for under-determined dynamical systems (see Olver 1993, equation 5.97). The under-determination leads to the existence of trivial conservation laws of the second kind associated with infinite-dimensional symmetries. However, Padhye and Morrison (1996b) seem to suggest that inhomogeneous differential identities (their equation (31)) imply under-determination. This is however not the case: inhomogeneous differential identities such as (4.9) may exist in dynamical systems that are not under-determined. To associate a trivial conservation law of the second kind such as potential vorticity conservation with an infinite-dimensional symmetry, one must first establish the under-determination of the dynamical system. One systematic method is to verify whether first-class Dirac constraints exist—see for instance Kosmann-Schwarzbach (2011), section 6.2. It turns out that, as far as the three Lagrangian densities \( L_{(i)} \), \( L_{(ii)} \), and \( L_{(iii)} \) described in Zadra and Charron (2015) are concerned, there are no such first-class constraints in their corresponding Hamiltonian descriptions.

5.2. Non-canonical Hamiltonian structure and symplectic form

In Dirac’s theory on generalized Hamiltonian systems, weak constraints become strong constraints, and Poisson brackets are replaced by Dirac brackets.

In this case, once constraints become strong, the Hamiltonian density \( \mathcal{H}_{C} \) given by (5.9) becomes \( \sqrt{\mathcal{F}}(E^{0} - T_{0}^{0}) \). It is not in general the total energy density in all admissible coordinate systems. It reduces to the total energy density only in particular coordinate systems in which the components \( T_{0}^{0} \) do not contribute to the total energy, as follows from (3.42).

A non-canonical Hamiltonian formulation may be obtained by constructing the Dirac brackets. One must first find the inverse of \( M \)—which must be built with second-class constraints only—from the following definition:

\[
\sum_{r=1}^{P} \int d^{3}z \; M^{-1}(pr)(x, z) M_{(pq)}(z, y) = \delta_{(pq)} \delta^{(3)}(x - y). \quad (5.32)
\]
For $N = 2$, one obtains

$$M^{-1}(x, y) = \frac{\delta^{(3)}(x - y)}{\sqrt{g} \rho} \begin{pmatrix}
0 & \rho & 0 & 0 & 0 & 0 & 0 \\
-\rho & 0 & \beta & 0 & \gamma_{(1)} & 0 & \gamma_{(2)} \\
0 & -\beta & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (5.33)$$

This is easily generalized to any positive integer $N$.

A Dirac bracket between $A$ and $B$ is defined by

$$[A(x), B(y)] \equiv \{A(x), B(y)\} - \sum_{p=1}^{P} \sum_{q=1}^{P} \int d^3z \int d^3w \{A(x), \phi_{(\psi_p)}(w)\} M^{-1}_{(\psi_p)}(w, z) \phi_{(\psi_q)}(z), B(y)\} \quad (5.34)$$

(Dirac 1964). This definition implies that the Dirac bracket between a second-class constraint and an arbitrary field $B(y)$, written $[\phi_{(\psi_p)}(x), B(y)]$, vanishes by construction. Therefore, when using Dirac brackets instead of Poisson brackets, second-class constraints become strong constraints; one may apply the second-class constraints from the start and there is no need to first calculate the brackets involving the second-class constraints.

The time evolution of a field $A(x)$ at a point $x$ is provided by

$$A_{,\beta}(x) = [A(x), H_C] + \left(\frac{\partial A(x)}{\partial t}\right)_{\psi, \pi}. \quad (5.35)$$

The last term is non-vanishing only when an explicit time dependence of $A$ exists.

A way to take into account the explicit time dependence of a field is to follow Gitman and Tyutin (1990, section 7.2) and formally expand the phase space of dynamical variables to include the time itself and its conjugate momentum density $\pi_{(0)}(y)$. In this extended phase space, the Poisson bracket of any function $A(x)$ with $\pi_{(0)}(y)$ is

$$[A(x), \pi_{(0)}(y)] = \left(\frac{\partial A(x)}{\partial t}\right)_{\psi, \pi} \delta^{(3)}(x - y). \quad (5.36)$$

Consequently, the Poisson bracket of a function $A(x)$ with $\pi_{(0)}(y)$ is non-zero only if $A(x)$ has an explicit time dependence. After defining a new Hamiltonian $H'$ as

$$H' = H_C + \int d^3x \pi_{(0)}(x), \quad (5.37)$$

(5.35) becomes

$$A_{,\beta}(x) = [A(x), H'], \quad [A(x), H'] \quad \sum_{p=1}^{P} \sum_{q=1}^{P} \int d^3z \int d^3w \{A(x), \phi_{(\psi_p)}(w)\} M^{-1}_{(\psi_p)}(w, z) \phi_{(\psi_q)}(z), H'], \quad (5.38)$$

from (5.14) and (5.34). Moreover because the Dirac bracket between a second-class constraint and any field vanishes by construction, one may explicitly set $\phi_{(\psi_p)}$ to zero in (5.9) when using the non-canonical formalism. One therefore obtains the Hamiltonian $H$, where

$$H = \int d^3x \left[\sqrt{g} \rho \frac{1}{2} h^{ij} u_i u_j + \Phi + I + g^{ij} u_i + \pi_{(0)}\right]. \quad (5.39)$$

The use of the non-canonical formalism (i.e. Dirac brackets instead of canonical Poisson brackets) allows to replace $H'$ by $H$ in all equations. Therefore,

$$A_{,\beta}(x) = [A(x), H]. \quad (5.40)$$
One must calculate the Dirac brackets among all the dynamical fields of the extended phase space. Note that the momenta $\pi_{(\psi)}$ do not appear in the Hamiltonian (5.39), as the constraints are now strong constraints. From the definition (5.34) and $[\psi_\rho(x), \psi_\eta(y)] = 0$, one may find among the $4 + 2N$ dynamical fields $\psi_\rho$

$[\rho(x), \alpha(y)] = -[\alpha(x), \rho(y)] = (\sqrt{g})^{-1}\delta^{(3)}(x - y), \quad (5.41)$

$[\alpha(x), \beta(y)] = -[\beta(x), \alpha(y)] = (\sqrt{g})^{-1}\delta^{(3)}(x - y), \quad (5.42)$

$[\alpha(x), \gamma_{(\eta)}(y)] = -[\gamma_{(\eta)}(x), \alpha(y)] = (\sqrt{g})^{-1}\gamma_{(\eta)}\delta^{(3)}(x - y), \quad (5.43)$

$[\beta(x), s(y)] = -[s(x), \beta(y)] = (\sqrt{g})^{-1}\delta^{(3)}(x - y), \quad (5.44)$

$[\gamma_{(\eta)}(x), \lambda_{(\eta)}(y)] = -[\lambda_{(\eta)}(x), \gamma_{(\eta)}(y)] = (\sqrt{g})^{-1}\delta^{(3)}(x - y), \quad (5.45)$

and zero otherwise. The only non-vanishing Dirac bracket between the dynamical fields $\psi_\rho$ and $\pi_{(\eta)}$ is

$[\rho(x), \pi_{(\eta)}(y)] = -[\pi_{(\eta)}(x), \rho(y)] = -\rho(\sqrt{g})^{-1}(\sqrt{g})_{\alpha\beta}\delta^{(3)}(x - y). \quad (5.46)$

For a generic external field $\Psi$ (e.g. $g_{\mu\nu} \sqrt{g}$, $\Phi$), one has

$[\Psi(x), \pi_{(\eta)}(y)] = -[\pi_{(\eta)}(x), \Psi(y)] = \Psi_{,\alpha}\delta^{(3)}(x - y). \quad (5.47)$

The above equations imply

$[\sqrt{g}(x)\rho(x), \pi_{(\eta)}(y)] = 0. \quad (5.48)$

The equations of motion with a possibly time-varying metric tensor are written as

$\psi_{\rho,0} = [\psi_{\rho}, H]. \quad (5.49)$

It may be verified that (5.49) reproduces (5.22)–(5.27). The term $\pi_{(\eta)}$ in (5.39) may be omitted if one chooses a time-independent metric tensor.

The form of $H$ suggests that dynamical fields may be transformed from Clebsch potentials to tensor wind components. The Hamiltonian $H$ may be rewritten as

$H = \int d^3x \left[ \frac{1}{2} g_{\mu\nu} u^\mu u^\nu + \Phi \right] + \pi_{(\eta)} \quad (5.50)$

after some algebra and after defining an effective gravitational potential $\Phi \equiv \Phi - h_{00}/2$. From the relation

$u^i = \hbar^{\eta} \left[ \alpha_{,j} + \beta s_{,j} + \sum_{r=1}^{N} \gamma_{(\eta)}\lambda_{(\eta)} \right] + g^{\eta i}, \quad (5.51)$

one calculates the following Dirac brackets:

$[\rho(x), u^i(y)] = (\sqrt{g}(x))^{-1}\hbar^{\eta}(y) \frac{\partial}{\partial y^i} \delta^{(3)}(x - y), \quad (5.52)$

$[u^i(x), \rho(y)] = -(\sqrt{g}(y))^{-1}\hbar^{\eta}(x) \frac{\partial}{\partial x^i} \delta^{(3)}(x - y), \quad (5.53)$

$[s(x), u^i(y)] = -(\sqrt{g})^{-1}\hbar^{\eta}(x) s_{,i} \delta^{(3)}(x - y), \quad (5.54)$

$[u^i(x), s(y)] = (\sqrt{g})^{-1}\hbar^{\eta}s_{,i} \delta^{(3)}(x - y), \quad (5.55)$

$[u^i(x), u^j(y)] = (\sqrt{g})^{-1}(h^{i}u^j_\alpha - h^{j}u^i_\alpha) \delta^{(3)}(x - y) \quad (5.56)$

((5.56) is demonstrated in appendix B). These are the only non-vanishing Dirac brackets among the dynamical fields $\rho$, $s$ and $u^i$ on which the Hamiltonian (5.50) depends. In addition to these, one must also include (5.46) as well as

$[u^i(x), \pi_{(\eta)}(y)] = (\hbar^{\eta}_{,\alpha}(g_{\alpha\beta} + g_{\beta\alpha}) + g^{\eta i}) \delta^{(3)}(x - y), \quad (5.57)$

and

$[t^i(x), \pi_{(\eta)}(y)] = \delta^{(3)}(x - y) \quad (5.58)$

in the list of fundamental brackets. The expression $t^i(x)$ is meant to represent the same time at each spatial point, not that time depends on space. The terms $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols of the second kind:
\[ \Gamma_{\mu
u} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}). \] 

(5.59)

This non-canonical Hamiltonian structure is a generalization to arbitrary coordinates of what is presented in, for example, Shepherd (1990) and Morrison (1998).

The non-canonical Hamiltonian structure is therefore established in terms of observable and measurable fields (except \( q_i, i \)), and any admissible coordinate system may be used. The equations of motion are

\[
\begin{align*}
\rho_0 & = [\rho, H], \\
\sigma_{0} & = [s, H], \\
u_{\alpha,0} & = [u_{\alpha}, H].
\end{align*}
\]

(5.60)

Equations (5.60) and (5.61) lead to (5.22) and (5.25) respectively, while (5.62) to

\[
u_{i,0} = -u_{i} u_{,j} - \Gamma_{00}^{i} - 2 \Gamma_{j0}^{i} u_{j} - \Gamma_{ij}^{l} u_{k} - h^{i} (\Phi_{ij} + \frac{1}{\rho} \eta_{ij}).
\]

(see appendix C). Equation (5.63) is equivalent to (40) in Charron et al (2014) provided that viscosity is neglected.

Define a state vector \( \eta \), for instance \( \eta = (\rho, \alpha, \beta, s, \gamma_{ij}, \lambda_{ij}, t, \pi_{ij}) \) or \( \eta = (\rho, s, u_{i}, u^{2}, u^{3}, t, \pi_{ij}) \) with \( \eta_{ij} \) representing each dynamical field. Define also an operator \( J(x, y) \) with component \((a, b)\) as

\[ J_{ab}(x, y) = [\eta_{a}(x), \eta_{b}(y)]. \]

(5.64)

The non-zero components of this antisymmetric operator are provided by (5.41)–(5.45), (5.46), (5.58), or by (5.46), (5.52)–(5.58)—depending on the choice of dynamical variables.

The Dirac bracket of two fields \( A(x) \) and \( B(y) \) that depend on the dynamical fields is written

\[
\begin{align*}
[A(x), B(y)] & = \sum_{a=1}^{p} \int d^{3}w \frac{\delta A(x)}{\delta \eta_{a}(w)} [\eta_{a}(w), B(y)], \\
& = \sum_{a=1}^{p} \sum_{b=1}^{p} \int d^{3}w \int d^{3}z \frac{\delta A(x)}{\delta \eta_{a}(w)} \eta_{b}(w, \eta_{b}(z)) \frac{\delta B(y)}{\delta \eta_{b}(z)}, \\
& = \sum_{a=1}^{p} \sum_{b=1}^{p} \int d^{3}w \int d^{3}z \frac{\delta A(x)}{\delta \eta_{a}(w)} J_{ab}(w, z) \frac{\delta B(y)}{\delta \eta_{b}(z)},
\end{align*}
\]

(5.65)

where \( P \) is now the total number of dynamical fields in the extended phase space \((6 + 2N) \text{ or } 7, \text{ depending on the choice of dynamical fields})\). This is the symplectic form. The terms like \( \delta F(x)/\delta G(y) \) are meant to represent functional derivatives, for example \( \delta \eta_{a}(x)/\delta \eta_{b}(y) = \delta_{ab}\delta^{(3)}(x - y) \).

The time evolution of a field \( A(x) \) that depends on the dynamical fields is therefore written

\[ A_{,\alpha}(x) = [A(x), H] = \sum_{a=1}^{p} \sum_{b=1}^{p} \int d^{3}w \int d^{3}z \int d^{3}y \frac{\delta A(x)}{\delta \eta_{a}(w)} J_{ab}(w, z) \frac{\delta H(y)}{\delta \eta_{b}(z)}. \]

(5.66)

The equations of motion take the form

\[ \eta_{a,0}(x) = \sum_{b=1}^{p} \int d^{3}z \int d^{3}y J_{ab}(x, z) \frac{\delta H(y)}{\delta \eta_{b}(z)}. \]

(5.67)

### 5.3. Casimir invariants

By definition, a Casimir invariant \( C \) (sometimes called a distinguished functional) exists if its Dirac brackets with all the dynamical fields vanish:

\[
[C, \eta_{a}(x)] = 0 = [\delta \eta_{a}(x), C],
\]

\[
\sum_{b=1}^{p} \int d^{3}z [\delta \eta_{a}(x), \eta_{b}(z)] \frac{\delta C}{\delta \eta_{b}(z)},
\]

\[
\sum_{b=1}^{p} \int d^{3}z \int d^{3}y J_{ab}(x, z) \frac{\delta C}{\delta \eta_{b}(z)}.
\]

(5.68)

for all \( a \) and for all points \( x \) within the interior domain, where \( C \) is the density associated with the Casimir functional \( C \), i.e. \( C = \int d^{3}y C(y) \). Equation (5.68) means that Casimir invariants exist if the kernel of the operator \( J(x, y) \) is not an empty set. They are not absolute; their existence depends on the choice of dynamical variables. Obviously, because \( C \) has vanishing Dirac brackets with all the dynamical fields, its Dirac bracket with the Hamiltonian \( H \) also vanishes, and \( C \) is a constant of the motion given suitable boundary conditions.
Suppose one chooses \((\rho, \alpha, \beta, \gamma, \lambda, \pi)\) as the dynamical fields in the extended phase space. One may verify that the quantity

\[
F = \int_D d^3x \sqrt{g} \rho \mathcal{F}(s, \gamma, \lambda, \pi) ,
\]

where \(\mathcal{F}\) is any local function of \(s, \gamma, \lambda, \pi\), is not a Casimir invariant—in particular, this includes total mass \((\mathcal{F} = 1)\) and total entropy \((\mathcal{F} = s)\). Recall that the conserved charge \(F\) is obtained from a non-trivial conservation law and is associated with an internal symmetry, as shown in sub-section 3.4. However, one may verify that the quantity

\[
Q = \int_D d^3x \sqrt{g} \rho q,
\]

where \(q\) is Ertel’s potential vorticity, has vanishing Dirac brackets with all the dynamical fields, and is therefore a Casimir invariant. As previously shown in section 4 and in Charron and Zadra (2018), the conserved charge \(Q\) is obtained from a trivial conservation law. It cannot be associated with a symmetry via Noether’s two theorems. This is the only known Casimir invariant (up to a trivial multiplicative constant) when using \((\rho, \alpha, \beta, \gamma, \lambda, \pi)\) as dynamical fields. Notice that although

\[
R = \int_D d^3x \sqrt{g} \rho Q(q) ,
\]

where \(Q(q)\) is any local function of \(q\), is a constant of the motion, it is not in general a Casimir invariant. In this case,

\[
[R, \alpha(y)] = Q - q \frac{dQ}{dq} + \frac{d^2Q}{dq^2} \sum_{r=1}^N \frac{\partial \alpha_i}{\partial q^r} \lambda^{(r)} (y)
\]

for any point \(y\) located in the interior domain \(D\). This expression vanishes only for \(Q(q) = bq\) \((b\) a constant). Although \(R\) is a constant of the motion on-shell, it may be verified that it is not in general explicitly obtained from an off-shell trivial law of the second kind, as opposed to \(Q\). When canonical variables (i.e. Clebsch potentials) are used as dynamical fields, the existence of the only Casimir invariant is directly linked to the existence of the only known trivial conservation law of the second kind, i.e. an off-shell conservation law.

Suppose now that one chooses \((\rho, s, u^1, u^2, u^3, t, \pi)\) instead as the dynamical fields. It may be verified that the quantity

\[
C = \int_D d^3x \sqrt{g} \rho C(q, s),
\]

where \(C(q, s)\) is any local function of Ertel’s potential vorticity \(q\) and specific entropy \(s\), has vanishing Dirac brackets with the seven dynamical fields and is a Casimir invariant. In this case, no explicit or visible symmetries are associated with the time invariance of \(C\). In particular, total mass \((C(q, s) = 1)\) and total entropy \((C(q, s) = s)\) become Casimir invariants. Notice that in contrast with the previous choice of dynamical variables (Clebsch potentials), the function \(C(q, s) = Q(q)\) now leads to a Casimir invariant. This may be explained from the fact that the Clebsch potentials whose Dirac brackets with \(R\) did not vanish are no longer dynamical fields.

A trivial conservation law of the second kind—that is, Ertel’s theorem—as well as the non-trivial conservation laws that are associated with hidden symmetries translate into Casimir invariants in the non-canonical Hamiltonian formulation. Notice that it is not because the transformations (3.30)–(3.35) are a global gauge symmetry that the associated conservation laws translate into Casimir invariants. Using Noether’s first theorem, it was shown that mass and entropy conservation laws are associated with internal symmetries under an active transformation of Clebsch fields. In addition, it was confirmed that momentum and energy conservation laws are associated with space-time symmetries of the external forcing and metric tensor. These results were presented as examples of applications of Noether’s first

6. Summary and conclusions

Starting from a manifestly invariant Lagrangian density for geophysical fluids—expressed as a function of fluid density, specific entropy, other Clebsch potentials, the metric tensor and an external gravitational potential—some symmetry properties of the equations of motion were investigated in the context of arbitrary coordinates. With that objective in mind, a review of Noether’s first theorem for field transformations as well as coordinate transformations was presented. Using Noether’s first theorem, it was shown that mass and entropy conservation laws are associated with internal symmetries under an active transformation of Clebsch fields. In addition, it was confirmed that momentum and energy conservation laws are associated with space-time symmetries of the external forcing and metric tensor. These results were presented as examples of applications of Noether’s first
theorem leading to non-trivial conservation laws. A non-trivial conservation law obtained via Noether’s first theorem requires the existence of a symmetry and is valid on-shell only.

Potential vorticity conservation was shown to be a trivial law of the second kind. Noether’s first theorem is therefore irrelevant to explain potential vorticity conservation.

Non-canonical Hamiltonian structures were derived in arbitrary coordinates from a canonical Hamiltonian formalism with weak constraints. In particular, a time-dependent metric tensor is accounted for by formally extending the phase space of dynamical fields to include time itself and its conjugate momentum. Using Dirac’s theory for constrained Hamiltonian systems, it was shown that the governing equations for Clebsch potentials and density are not invariant under any infinite-dimensional symmetry transformation, therefore they are not under-determined and Noether’s second theorem does not apply. This remains true for geometrically approximated equations—for instance, when the thin-shell approximation is used. From the Lagrangian density considered in this paper, Noether’s first and second theorems cannot be invoked to associate a symmetry with potential vorticity conservation. The same can be said of the equations of motion obtained from the Lagrangian densities $L_{(1)}$ and $L_{(II)}$ in Zadra and Charron (2015), the former being used in the context of a particle-like formulation of fluid motion with externally imposed constraints on mass and entropy conservation.

The conservation of Ertel’s potential vorticity—a trivial law of the second kind—translates into a Casimir invariant. The other Casimir invariants are related to hidden global symmetries associated with mass and entropy conservation when using the non-canonical velocity components as dynamical fields instead of canonical Clebsch potentials.

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Appendix A. Useful identities

A.1. Mass-momentum tensor

The mass-momentum tensor is related to the derivative of the Lagrangian density with respect to the metric tensor:

$$t_{\mu \nu} \equiv -\frac{2}{\sqrt{g}} \frac{\partial (\sqrt{g} L)}{\partial g^{\mu \nu}} = -2 \left( \frac{L}{\sqrt{g}} \frac{\partial (\sqrt{g})}{\partial g^{\mu \nu}} + \frac{\partial L}{\partial g^{\mu \nu}} \right)$$  \hspace{1cm} (A.1)

The derivative of $\sqrt{g}$ with respect to $x^\alpha$ is written

$$(\sqrt{g})_{,\alpha} = \frac{\partial (\sqrt{g})}{\partial g^{\mu \nu}} g^{\mu \nu, \alpha} = \sqrt{g} \Gamma^\mu {_{\alpha \beta}} = \frac{1}{2} \sqrt{g} g^{\mu \nu} g_{\mu, \alpha},$$

$$= -\frac{1}{2} \sqrt{g} g_{\mu \nu} g^{\mu \nu, \alpha},$$

$$= -\frac{1}{2} \sqrt{g} (h_{\mu \nu} + \delta^\alpha_\mu \delta^\nu_\alpha) g^{\mu \nu, \alpha},$$

$$= -\frac{1}{2} \sqrt{g} h_{\mu \nu} g^{\mu \nu, \alpha}$$  \hspace{1cm} (A.2)

from (22) in Charron et al (2014) and $g^{00} = 1$. The term $\Gamma^\mu {_{\alpha \beta}} = g^{\mu \nu} (g_{\alpha \beta, \nu} + g_{\nu \beta, \alpha} - g_{\alpha \beta, \nu})/2$ is a Christoffel symbol of the second kind. This implies that

$$\frac{\partial (\sqrt{g})}{\partial g^{\mu \nu}} = -\frac{1}{2} \sqrt{g} h_{\mu \nu}$$  \hspace{1cm} (A.3)

and that

$$-2 \frac{L}{\sqrt{g}} \frac{\partial (\sqrt{g})}{\partial g^{\mu \nu}} = h_{\mu \nu} p + h_{\mu \nu} \rho \Lambda_{(p)}$$  \hspace{1cm} (A.4)
from (65) in Zadra and Charron (2015). Note however that this relation does not hold for \( \mu = \nu = 0 \) because \( g^{00} \) is a constant.

The term \( \partial L / \partial g^{\mu \nu} \) is most directly obtained from the Lagrangian density (2.5) when symmetrically rearranged, while making use of

\[
\begin{align*}
{u^0} = 1 &= g^{00} \\
v_i = u_i \\
h^{\mu \nu} v_\nu &= (g^{ij} - g^{0i} g^{0j}) u_i u_j.
\end{align*}
\]

The Lagrangian density is written

\[
L = -\frac{\rho}{2} (g^{ij} u_i u_j - g^{0i} g^{0j} u_i u_j + g^{0i} u_i + g^{0j} u_j + 2I + 2\Phi + 2\nu_0).
\]

It is seen that

\[
-\frac{\partial L}{\partial g^{ij}} = \rho u_i u_j,
\]

and that

\[
-2 \frac{\partial L}{\partial g^{00}} = \rho (1 - g^{00} u_i) u_i
\]

\[
= \rho (1 - g^{00} u_i + g^{00} u_0) u_i
\]

\[
= \rho (1 - u^0 + u_0) u_i,
\]

\[
= \rho u_0 u_i.
\]

After defining the covariant mass-momentum tensor as \( T_{\mu \nu} = \rho u_\mu u_\nu + h_{\mu \nu} p \), the tensor \( t_{\mu \nu} \) is therefore written

\[
t_{\mu \nu} = T_{\mu \nu} + h_{\mu \nu} \rho \Lambda_{(\nu)}
\]

except when \( \mu = \nu = 0 \). However, as a consequence of \( \tilde{g}^{00} \) being zero, the tensor component \( t_{00} \) never appears in (3.21). One may therefore define it arbitrarily. Equation (A.11) may then be used without restrictions even when \( \mu = \nu = 0 \).

A.2. Passive variation of \( \sqrt{g} \)

From the transformation

\[
g_{\mu \nu} (x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} g_{\alpha \beta}(x),
\]

one deduces

\[
\sqrt{\tilde{g}} (\tilde{x}) = J^{-1} \sqrt{g} (x),
\]

where \( g (x) \) and \( \tilde{g} (\tilde{x}) \) are the determinant of the covariant metric tensor in the original and transformed coordinate systems, respectively. The quantity \( J \) is the Jacobian of the transformation. For an infinitesimal transformation \( \tilde{x}^\mu = x^\mu + \epsilon^\mu \),

\[
J = \begin{vmatrix}
1 + \epsilon^0,_{0} & \epsilon^1,_{0} & \epsilon^2,_{0} & \epsilon^3,_{0} \\
\epsilon^0,_{1} & 1 + \epsilon^1,_{1} & \epsilon^2,_{1} & \epsilon^3,_{1} \\
\epsilon^0,_{2} & \epsilon^1,_{2} & 1 + \epsilon^2,_{2} & \epsilon^3,_{2} \\
\epsilon^0,_{3} & \epsilon^1,_{3} & \epsilon^2,_{3} & 1 + \epsilon^3,_{3}
\end{vmatrix}
\]

\[
= 1 + \epsilon_{\mu,\mu}
\]

to first order. Therefore, \( J^{-1} = 1 - \epsilon_{\mu,\mu} \) to first order. After writing

\[
\sqrt{\tilde{g}} (\tilde{x}) = \sqrt{\tilde{g}} (x) + (\sqrt{\tilde{g}} (x))_{\mu} \epsilon^\mu,
\]

\[
= \sqrt{\tilde{g}} (x) + (\sqrt{\tilde{g}} (x))_{\mu} \epsilon^\mu,
\]

\[
= \sqrt{\tilde{g}} (x) - \sqrt{g} (x) \epsilon^\mu_{\mu}
\]

to first order, and defining

\[
\tilde{\delta} (\sqrt{g}) \equiv \sqrt{\tilde{g}} (x) - \sqrt{g} (x),
\]

(16)
one obtains

$$\hat{\delta}(\sqrt{g}) = -(\sqrt{g} \epsilon^\mu)_\mu = -\sqrt{g} \epsilon^\mu, \quad \text{(A.17)}$$

### A.3. Passive variation of $g^{\mu \nu}$

From the transformation $g^{\mu \nu}(x) = \partial \tilde{g}^{\mu \nu} / \partial x^{\alpha} g^{\alpha \beta}(x)$, one may write to first order

$$\tilde{g}^{\mu \nu}(\tilde{x}) = g^{\mu \nu}(x) + (g^{\mu \nu}(x))_{,\beta} \epsilon^\beta,$$

$$\tilde{g}^{\mu \nu}(x) = g^{\mu \nu}(x) + g^{\mu \nu}(x)^{,\beta} \epsilon^\beta + g^{\mu \nu}(x) \epsilon^\nu \epsilon^\mu. \quad \text{(A.19)}$$

After defining $\hat{\delta} g^{\mu \nu} \equiv \tilde{g}^{\mu \nu}(x) - g^{\mu \nu}(x)$, one obtains

$$\hat{\delta} g^{\mu \nu} = g^{\alpha \mu} \epsilon^\alpha + g^{\nu \alpha} \epsilon^\nu - g^{\mu \nu} \epsilon^\beta,$$

$$\hat{\delta} g^{\mu \nu} = g^{\alpha \mu} \epsilon^\alpha + g^{\nu \alpha} \epsilon^\nu - g^{\alpha \beta} \Gamma^\alpha_{\beta \alpha} \epsilon^\beta + g^{\alpha \beta} \Gamma^\alpha_{\beta \alpha} \epsilon^\beta + g^{\alpha \beta} \Gamma^\alpha_{\beta \alpha} \epsilon^\beta,$$

$$\epsilon^\mu + \epsilon^\nu \quad \text{(A.20)}$$

from the definition of covariant derivatives for first and second rank tensors, and the fact that $g^{\mu \nu} \epsilon^\beta$ vanishes. Relation (A.20) is compatible with $\hat{\delta} g^{\mu 0} = 0$ because $\epsilon^0$ is a constant.

If a symmetry exists such that $\hat{\delta} g^{\mu \nu} = 0$, then $\epsilon^{\mu \nu} = -\epsilon^{\nu \mu}$, or equivalently $\epsilon_{\mu \nu} = -\epsilon_{\nu \mu}$ after contractions with the covariant metric tensor.

### A.4. Two tensor identities

1. The intrinsic derivative of the gradient of a scalar $f$ is written

$$\frac{D}{Dt} (f_{\mu}) \equiv u^\nu f_{\mu \nu} = \hat{f}_\mu - f_\mu u^\nu, \quad \text{(A.21)}$$

where $\hat{f} \equiv u^\nu f_\nu$.

2. One may verify the identity

$$\epsilon^{\alpha \mu \sigma} A^\beta_\alpha + \epsilon^{\beta \mu \sigma} A^\alpha_\mu + \epsilon^{\beta \mu \sigma} A^\nu_\sigma = \epsilon^{\beta \mu \sigma} A^\alpha_\alpha, \quad \text{(22.2)}$$

contract it with $\delta^\alpha_\beta$, and replace $A^\mu_\alpha$ by $u^\mu_\alpha$ to obtain

$$\epsilon^{\beta \mu \sigma} u^\mu_\alpha + \epsilon^{0 \mu \sigma} u^\sigma_\alpha + \epsilon^{0 \mu \alpha} u^\mu_\alpha = \epsilon^{\beta \mu \sigma} u^\sigma_\alpha. \quad \text{(A.23)}$$

This holds because $u^\sigma_\alpha = 0$ (recall that $u^0 = 1$ and $\Gamma^0_{\mu \nu} = 0$). This is a tensor identity when multiplied by $(\sqrt{g})^{-1}$.

### Appendix B. Demonstration of (5.56)

First, the Dirac brackets of the Clebsch potentials with $u_j$ are calculated from (5.42)–(5.45) and from $u_j = \alpha_j + \beta_{s_j} + \sum_{r=1}^N \gamma_{(r_1)} \lambda_{(r_2)}$:

$$[\alpha(x), u_j(y)] = (\rho(y) \sqrt{g}(y))^{-1} \left( \beta(y) \frac{\partial s(y)}{\partial y^j} + \sum_{r=1}^N \gamma_{(r)}(y) \frac{\partial \lambda_{(r)}(y)}{\partial y^j} \delta^{(3)}(x - y), \right)$$

$$= (\rho(y) \sqrt{g}(y))^{-1} \left( \beta(x) \frac{\partial s(x)}{\partial y^j} + \sum_{r=1}^N \gamma_{(r)}(x) \frac{\partial \lambda_{(r)}(x)}{\partial y^j}. \right) \delta^{(3)}(x - y), \quad \text{(B.1)}$$
\[ [\beta(x), u_j(y)] = - \frac{\partial}{\partial y^j}((\rho(y) \sqrt{g}(y))^{-1} \beta(y) \delta^{(3)}(x - y)) \]
\[ + \beta(y) \frac{\partial}{\partial y^j}((\rho(y) \sqrt{g}(y))^{-1} \delta^{(3)}(x - y)) \]
\[ = -(\rho(y) \sqrt{g}(y))^{-1} \frac{\partial \beta(y)}{\partial y^j} \delta^{(3)}(x - y), \quad (B.2) \]
\[ [s(x), u_j(y)] = -(\rho(y) \sqrt{g}(y))^{-1} \frac{\partial s(y)}{\partial y^j} \delta^{(3)}(x - y), \quad (B.3) \]
\[ [\gamma_{0i}(x), u_j(y)] = -(\rho(y) \sqrt{g}(y))^{-1} \frac{\partial \gamma_{0i}(y)}{\partial y^j} \delta^{(3)}(x - y), \quad (B.4) \]
\[ [\lambda_{0i}(x), u_j(y)] = -(\rho(y) \sqrt{g}(y))^{-1} \frac{\partial \lambda_{0i}(y)}{\partial y^j} \delta^{(3)}(x - y). \quad (B.5) \]

From
\[ [u_j(x), u_j(y)] = \frac{\partial}{\partial x^i} [\alpha(x), u_j(y)] + \beta(x) \frac{\partial}{\partial x^j} [s(x), u_j(y)] + \frac{\partial s(x)}{\partial x^j} [\beta(x), u_j(y)] \]
\[ + \sum_{r=1}^{N} \left[ \gamma_{0i}(x) \frac{\partial}{\partial x^i} [\gamma_{0i}(x), u_j(y)] + \gamma_{0i}(x) \frac{\partial}{\partial x^i} [\gamma_{0i}(x), u_j(y)] \right] \]
\[ = (\rho(y) \sqrt{g}(y))^{-1} \left( \frac{\partial \beta(x)}{\partial y^j} \delta^{(3)}(x - y) \right. \]
\[ - \left. (\rho(y) \sqrt{g}(y))^{-1} \frac{\partial \beta(y)}{\partial y^j} \delta^{(3)}(x - y) \right) \]
\[ = (\rho(x) \sqrt{g}(x))^{-1} \left( \frac{\partial \beta(x)}{\partial y^j} \delta^{(3)}(x - y) \right. \]
\[ - \left. (\rho(x) \sqrt{g}(x))^{-1} \frac{\partial \beta(y)}{\partial y^j} \delta^{(3)}(x - y) \right) \]
\[ = (\sqrt{g} \rho)^{-1} [(u_{j,i} - u_{i,j}) \delta^{(3)}(x - y)] \]
\[ = (\sqrt{g} \rho)^{-1} [(u_{j,i} - u_{i,j}) \delta^{(3)}(x - y)]. \quad (B.6) \]

one obtains
\[ [u^i(x), u^j(y)] = h^k(x) h^l(y) [u_k(x), u_l(y)], \]
\[ = (\sqrt{g} \rho)^{-1} h^k h^l (u_{k,l} - u_{l,k}) \delta^{(3)}(x - y), \]
\[ = (\sqrt{g} \rho)^{-1} (h^k u_{k,l} - h^l u_{k,l}) \delta^{(3)}(x - y). \quad (B.7) \]

Appendix C. Demonstration of (5.63)

From (5.50) and (5.57), one writes
\[ [u^i(x), H] = \int d^3y \ [u^i(x), \mathcal{H}_D(y)] - \Gamma_{0i}^i - \frac{1}{2} h^{ij} h_{\alpha j} - h^{ij} g_{\alpha 0} u^i, \quad (C.1) \]

where
\[ \mathcal{H}_D = \sqrt{g} \rho \left( \frac{1}{2} g_{ik} u^i u^j + I + \Phi \right). \]

However,
\[ [u^i(x), \mathcal{H}_D(y)] = \sqrt{g} [u^i(x), \rho(y)] \left( \frac{1}{2} g_{ik} u^i u^j + I + \Phi \right) \]
\[ + \sqrt{g} \rho [g_{ik} u^i [u^i(x), u^j(y)] + [u^i(x), I(y)]] \]
\[ = \sqrt{g} [u^i(x), \rho(y)] \left( \frac{1}{2} g_{ik} u^i u^j + I + \Phi \right) + \sqrt{g} \rho [g_{ik} u^i [u^i(x), u^j(y)]] \]
\[ + \frac{p}{\rho^2} [u^i(x), \rho(y)] + T [u^i(x), s(y)]. \quad (C.2) \]
This means that

\[
\int d^3y \left[ u^i(x), H_{\nu\lambda}(y) \right] = -h^i \left( \frac{1}{2} g_{k\ell} u^\ell u^i + I + \Phi_i \right)_j + g_{k\ell} u^k (h^{i\ell} u^i_j - h^{i\ell} u^i_j)
\]

\[
- h^i \left( \frac{1}{\rho} p^j \right)_j + h^i T_{s,j},
\]

\[
- h^i \left( \frac{1}{2} g_{k\ell} u^\ell u^i + \Phi_k \right)_j - u^{i\ell} u^i_j + h^i \left( \frac{1}{2} g_{k\ell} u^\ell u^i \right)_j - \frac{1}{\rho} h^i p^j, \]

\[
- h^i \left( \frac{1}{2} g_{k\ell} u^\ell u^i + \Phi_k \right)_j - u^{i\ell} u^i_j + h^i \left( \frac{1}{2} g_{k\ell} u^\ell u^i \right)_j - h^i u_{0;j}
\]

\[
- \frac{1}{\rho} h^i p^j + h^i h_{00,j} + h^i (g_{\ell\ell} u^\ell)_j,
\]

\[
- u^{i\ell} u^i_j - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right)_j + h^i u_{0;j} + h^i u_{0;j},
\]

\[
- u^{i\ell} u^i_j - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right)_j + h^i \Gamma_{k0} u^k + h^i \Gamma_{k0},
\]

\[
- u^{i\ell} u^i_j - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right)_j + h^i \Gamma_{k0} u^k + h^i \Gamma_{k0} u^k + h^i \Gamma_{k0} u^k - h^i \Gamma_{j0} u^k,
\]

\[
+ \frac{1}{2} h^i h_{00,j},
\]

\[
- u^{i\ell} u^i_j - \Gamma_{j0} u^i - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right)_j + h^i g_{k0,0} u^k + \frac{1}{2} h^i h_{00,j},
\]

\[
- u^{i\ell} u^i_j - u^i \Gamma_{j0} u^\ell - \Gamma_{j0} u^i - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right)_j + h^i g_{k0,0} u^k + \frac{1}{2} h^i h_{00,j},
\]

\[
+ h^i g_{k,0} u^k + \frac{1}{2} h^i h_{00,j},
\]

\[
- u^{i\ell} u^i_j - 2 \Gamma_{j0} u^i - \Gamma_{j0} u^i - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right)_j + h^i g_{k,0} u^k + \frac{1}{2} h^i h_{00,j},
\]

\[
(C.3)
\]

Therefore from (C.1),

\[
[u^i(x), H] = -u^{i\ell} u^i_j - \Gamma_{j0} u^i - 2 \Gamma_{j0} u^i - \Gamma_{j0} u^i - h^i \left( \frac{1}{\rho} p^j + \Phi_j \right). \quad (C.4)
\]

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