Integrability and Identification in Multinomial Choice Models

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Abstract

McFadden’s random-utility model of multinomial choice has long been the workhorse of applied research. We establish shape-restrictions under which multinomial choice-probability functions can be rationalized via random-utility models with nonparametric unobserved heterogeneity and general income-effects. When combined with an additional restriction, the above conditions are equivalent to the canonical Additive Random Utility Model. The sufficiency-proof is constructive, and facilitates nonparametric identification of preference-distributions without requiring identification-at-infinity type arguments. A corollary shows that Slutsky-symmetry, a key condition for previous rationalizability results, is equivalent to absence of income-effects. Our results imply theory-consistent nonparametric bounds for choice-probabilities on counterfactual budget-sets. They also apply to widely used random-coefficient models, upon conditioning on observable choice characteristics. The theory of partial differential equations plays a key role in our analysis.

Keywords: Multinomial Choice, Unobserved Heterogeneity, Random Utility, Integrability, Slutsky-Symmetry, Income Effects, Partial Differential Equations, Nonparametric Identification, Random Coefficient Models, Bounds on Counterfactuals.

JEL Codes: C14, C25, D11.

1 Introduction

The random utility model of multinomial choice (McFadden, 1973) has gained enormous popularity among applied economists. However, there has been limited research on the micro-theoretic underpinning of such models, and in particular, on the question of ‘integrability’, i.e. which choice

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probability functions are logically consistent with a random utility model. Apart from obvious theoretical interest, this question has practical implications for empirical modelling of individual demand as well as predicting aggregate demand and welfare on counterfactual budget-sets that arise from a new tax or subsidy or changes in choice-sets due to addition or elimination of choice-options. In particular, any utility distribution that rationalizes a given demand dataset can be used, in addition to shape restrictions implied by economic theory, to construct nonparametric, theory-consistent bounds on such counterfactuals.

There has been comparatively more work on integrability in empirical demand models with continuous goods, c.f. Lewbel, 2001. More recently, Dette, Hoderlein and Neumayer 2016 and Hausman and Newey 2016 have derived integrability conditions for choice of a single continuous good and Bhattacharya 2020 has obtained them for binary choice settings under general (i.e. not necessarily additive) heterogeneity. The multinomial discrete choice case differs fundamentally from the single continuous good setting because the price of different alternatives are generically distinct, unlike continuous choice where the per unit price is constant across choices.

In the present paper, we first show that in multinomial choice settings that allow for non-parametric unobserved heterogeneity and income effects, there is a set of shape restrictions on conditional choice probability functions which together are sufficient for integrability. The proof of this result is constructive, and the rationalizing utility functions are obtained by inverting solutions of certain partial differential equations (PDEs). The way in which PDEs arise here is unrelated to Roy’s Identity (c.f. Mas-Colell et al, 1995, Proposition 3.G.4); the partial derivatives appearing in the PDE are of the average demand function, not the indirect utility function. Together with an additional restriction, the above conditions are then shown to be both necessary and sufficient for the canonical additive random utility model (ARUM) of McFadden. In our analysis of integrability, we leave the joint distribution of unobserved heterogeneity terms nonparametric. Unlike the computationally intensive algorithmic approach of McFadden and Richter 1990, further investigated in Kitamura and Stoye 2018, our conditions are closed-form and analytic, and can therefore be imposed on choice probability functions during estimation; they are also global, in the sense that their forms do not depend on how many and which budget sets happen to be observed in a specific dataset. On the other hand, MR and KS’s approach work under unrestricted heterogeneity, whereas our set-up is the canonical model with additive heterogeneity but also covers more flexible models like the widely-used random coefficient setting (e.g. mixed logit) which, conditional on observed covariates, have an additive structure.
For discrete choice, Daly and Zachary 1978 provided a set of closed-form, global conditions under which closed-form choice-probability functions can be justified as having arisen from preference maximization by a heterogeneous population. These conditions were independently derived in Armstrong and Vickers, 2015, who improved upon the Daly-Zachary results by including an outside option in the choice set. In all of these results, a key condition for integrability is Slutsky symmetry, analogous to the classic textbook case for demand systems with continuous goods. As a corollary of our main theorem, we show that in the multinomial setting, Daly-Zachary’s Slutsky symmetry is equivalent to the absence of income effects, i.e. that conditional choice probabilities do not depend on the decision-makers’ income. The “necessity” part is easy to show, but showing “sufficiency”, i.e. that Slutsky symmetry implies absence of income effects is non-trivial.

Next, we show how our integrability results can be used to nonparametrically identify the underlying preference distributions from empirical choice-probabilities. A key restriction delivering this identification result – viz. invertibility of sub-utilities in the numeraire due to non-satiation – is based on economic theory, as opposed to statistical assumptions. This is in contrast to existing results on identification of multinomial choice models, which either rely on statistical/mathematical assumptions, e.g. utilities being linearly separable in a covariate with large support, c.f. Matzkin 1993 (see also Allen and Rehbeck 2019 for related results). An important distinguishing feature of our set-up is that the arguments of choice-probability functions, viz. price and income, arise from budget constraints and they play very specific roles in the proof of integrability and the identification strategy. In that sense, our approach utilizes the basic economic theory of utility maximization subject to budget constraints, in contrast to the approach of Matzkin or Allen and Rehbeck that treat the arguments of choice-probabilities in a more abstract, statistical way. An important empirical consequence of this is that our results lead to nonparametric, theory-consistent bounds for choice probabilities on counterfactual budget sets. No such bounds on counterfactuals are possible in the set-up of Matzkin or Allen and Rehbeck unless utility indices and the heterogeneity distribution are assumed to have a known parametric form. Furthermore, from a purely methodological standpoint, achieving nonparametric identification by solving PDEs appears to be novel in the discrete choice literature.

1This is distinct from Slutsky negativity c.f. Bhattacharya 2021 for the general (i.e. not necessarily additive) heterogeneity case. Dagsvik and Karlstrom 2005 provide some related results for the setting where unobserved heterogeneity is both additive and is assumed to have known distribution. See also Fosgerau et al 2013 and Delle Site 2014.
Next, we discuss the empirical usefulness of our results by showing how they can be used (a) to analyze random coefficient models that are popular in applied work, e.g. McFadden-Train’s mixed logit or the BLP model, and (b) to calculate theory-consistent bounds for demand and welfare on counterfactual budget sets, e.g. those resulting from prospective introduction of new taxes and subsidies, price-changes due to mergers and potential changes in choice sets e.g. due to removal of alternatives.

The plan for the rest of the paper is as follows. Section 2 discusses integrability for multinomial choice in presence of income effects, and presents Lemma 1 and Theorem 1, the two key results of this paper, followed by a discussion of Daly-Zachary’s Slutsky symmetry condition and its connection with lack of income effects. Section 3 discusses four further points, viz. the implication of the integrability result for nonparametric identification of preference distributions, incorporation of covariates into the analysis, the applicability of these results to random coefficient models and using these results to calculate bounds on counterfactual choice probabilities. Section 4 concludes. A short appendix at the end presents two mathematical results on partial and ordinary differential equations that are intensively used in this paper, as well as proofs of the two main results.

Throughout the paper, we will assume continuous differentiability of the choice probability function in prices and income to sufficient orders and, to avoid repetitions, not include this separately each time among the conditions for our results.

2 Set-up and Key Results

Consider a setting of multinomial choice, where the discrete alternatives are indexed by $j = 0, 1, \ldots, J$, individual income is $y$, price of alternative $j$ is $p_j$; if alternative 0 refers to the outside option, i.e. not buying any of the alternatives, then $p_0 \equiv 0$. Let the utility from consuming the $j$th alternative and a quantity $z$ of the numeraire be given by $U(j, z)$, where $U(j, \cdot)$ is not necessarily linear. The consumer’s problem is $\max_{j \in \{0, 1, \ldots, J\}} [U(j, a_j) + \epsilon_j]$, subject to the budget constraint $z \leq y - p_j$, where $y$ is the consumer’s income, $p_j$ is the price of alternative $j$ faced by the consumer, and $\epsilon_j$ is unobserved heterogeneity in the consumer’s preferences. If $U(j, \cdot)$ is strictly increasing (i.e. non-satiation in the numeraire), then we can rewrite the consumer problem as $\max_{j \in \{0, 1, \ldots, J\}} [U(j, a_j) + \epsilon_j]$, where $a_j \equiv y - p_j$, $a_0 = y$. Denote the structural probability of choosing alternative $j \in \{0, \ldots, J\}$ at $a \equiv (a_0, \ldots, a_J)$ by $q_j(a)$. In words, if we randomly sample individuals from the population, and offer the vector $a$ to each sampled individual, then a fraction
$q_j(a)$ will choose alternative $j$, in expectation. It is easy to incorporate other attributes of the alternatives and characteristics of consumers in our analysis, and we outline how to that in Section 3. For now, we suppress other covariates for clarity of exposition. Note that the above structure covers models for bundles, c.f. Gentzkow 2007. For example, if the choice set is \(\{0\}, \{1\}, \{2\}, \{1, 2\}\), then that model is equivalent to a multinomial model with 4 alternatives where the price of option \(\{1, 2\}\) is \(p_1 + p_2\).

The key question of this paper is whether utility maximization in the above setting of multinomial choice that allows for income effects (corresponding to \(U(j, \cdot)\) being nonlinear) impose any restriction on choice-probabilities. To answer this question, we first introduce a condition that we call ‘Slutsky invariance’.

\[(A):\text{ For any }a, \text{ and any pair of alternatives }k \neq l, \text{ the ratio } \frac{\partial}{\partial a} q_l(a) / \frac{\partial}{\partial a} q_k(a) \text{ depends only on } a_k \text{ and } a_l.\]

**Motivation:** To see where this restriction comes from, consider the above setting of multinomial choice, and let the utility from consuming the \(j\)th alternative and a quantity \(z\) of the numeraire be given by \(U(j, z) + \varepsilon_j\). The \(\{\varepsilon_j\}\), which represent unobserved heterogeneity in preferences, are allowed to have any arbitrary and unspecified joint distribution in the population (subject to the resulting choice probability functions being smooth). If \(U(j, \cdot)\) is strictly increasing, i.e., preferences are non-satiated in the numeraire, then we can replace \(z = y - p_j \equiv a_j\), and rewrite the consumer problem as

\[
\max_{j \in \{0, 1, ..., J\}} [U(j, a_j) + \varepsilon_j]. \tag{1}
\]

To allow for income effects, we let \(U(j, a_j) \equiv h_j(a_j)\), where \(h_j(\cdot)\) are smooth, possibly nonlinear, strictly increasing, *unspecified* functions of the \(a_j\)’s. When \(h_j(\cdot)\) are nonlinear, the conditional choice-probabilities will depend on income, i.e., there are non-zero income effects. This structure is also observationally equivalent to a utility structure where unobserved heterogeneity is not additively separable from the \(a_j\)’s (see below) in the utility function.

Now, for the above set-up, the choice probability for the 0th alternative is given by

\[
q_0(a) = \Pr(\cap_{j \neq 0} \{h_0(a_0) + \varepsilon_0 > h_j(a_j) + \varepsilon_j\})
\]

\[
= \Pr(\cap_{j \neq 0} \{h_0(a_0) - h_j(a_j) > \varepsilon_j - \varepsilon_0\})
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(\varepsilon) \, d\varepsilon \, ... \, d\varepsilon_1 \, d\varepsilon_0. \tag{2}
\]
Therefore, by the first fundamental theorem of calculus,

\[
\frac{\partial}{\partial a_1} q_0 (a) = -h_1' (a_1) \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g \left( \frac{\varepsilon_0}{h_0 (a_0) + \varepsilon_0} \right) (h_0 (a_0) - h_1 (a_1)) + \varepsilon_0, \varepsilon_2, \ldots, \varepsilon_J \right] d\varepsilon_1 \ldots d\varepsilon_J d\varepsilon_0. \tag{3}
\]

Similarly,

\[
q_1 (a) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g (\varepsilon) d\varepsilon_1 \ldots d\varepsilon_J d\varepsilon_0 d\varepsilon_1,
\]

implying by the first fundamental theorem and chain-rule that

\[
\frac{\partial}{\partial a_0} q_1 (a) = -h_0' (a_0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g \left( \frac{h_1 (a_1) - h_0 (a_0) + \varepsilon_1}{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_J} \right) d\varepsilon_1 \ldots d\varepsilon_J d\varepsilon_0 \tag{4}
\]

\[
\overset{(1)}{=} -h_0' (a_0) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g \left( \frac{s_0 - h_1 (a_1) + h_0 (a_0)}{\varepsilon_2, \ldots, \varepsilon_J} \right) d\varepsilon_1 \ldots d\varepsilon_J d\varepsilon_0
\]

\[
= \frac{h_0' (a_0)}{h_1' (a_1)} \frac{\partial}{\partial a_1} q_0 (a), \text{ using } (3),
\]

where the second equality \(\overset{(1)}{=}\) follows by substituting \(s_0 = h_1 (a_1) - h_0 (a_0) + \varepsilon_1\) in \(\overset{(4)}{=}\).

The same argument can be repeated for any other pair of alternatives \(l \neq k\), to obtain

\[
\frac{\partial}{\partial a_l} q_l (a) = \frac{h_0' (a_0)}{h_1' (a_1)} \frac{\partial}{\partial a_1} q_k (a), \tag{5}
\]

for all \(a\), and it is clear that the RHS of \(\overset{(5)}{=}\) depends only on \(a_k\) and \(a_l\), and thus satisfies condition (A) above.

**Remark 1** Condition \(\overset{(5)}{=}\) has no relation with the Independence of Irrelevant Alternatives (IIA) property. Indeed, the model above will **not** have the IIA property if the \(\varepsilon_j\)'s are correlated across alternatives (i.e. across \(j\)), but it will continue to satisfy \(\overset{(5)}{=}\), since uncorrelatedness of \(\varepsilon\) was not used to derive \(\overset{(5)}{=}\).
Main Results: We now state and prove our main results. The first result is that the Slutsky invariance condition stated above, plus two shape-restrictions on \( q_j (\cdot) \)'s are jointly sufficient for integrability, i.e., under those restrictions on \( q_j (\cdot) \)'s, we can find a set of utility functions and a joint distribution of unobserved preference heterogeneity, such that individual maximization of these utilities will indeed produce the conditional choice-probabilities \( \{ q_j (\cdot) \}, \ j = 0, 1, ..., J \).

To state and prove our first result, we will use the following additional notation: let \( a_{-j} \) denote the vector \((a_0, a_1, ... a_{j-1}, a_{j+1}, ... a_J)\) and let for each \( j = 0, 1, ... J \), \( \lim_{a_{-j} \downarrow c^{(j)}(a_{j})} \) denote that each \( k \)th component of \( a_{-j} \) goes to a constant \( c_k^{(j)}(a_j) \) with \( c^{(j)}(a_j) = (c_0^{(j)}(a_j), ... , c_{j-1}^{(j)}(a_j), c_{j+1}^{(j)}(a_j), ... , c_J^{(j)}(a_j)) \). Similarly, \( \lim_{a_{j} \downarrow d^{(j)}(a_{-j})} \) denotes that for fixed \( a_{-j}, a_j \) decreases to a constant \( d^{(j)}(a_{-j}) \) (whose value depends on \( a_{-j} \)).

Lemma 1 Suppose that the following three conditions are satisfied by the choice-probabilities \( \{ q_j (a) \} \):

(i) For each \( j = 0, 1, ..., J \), and each \( a \), \( q_j (a) \) is strictly increasing in \( a_j \) and strictly decreasing in \( a_k \) for \( k \neq j \), continuously differentiable in each argument, and for all \( j \), there exist a joint density function \( f \) and \( m \) exist, are continuous, and satisfy \((-1)^r \sum_{\alpha_0 \alpha_1 ... \alpha_{r-1} \alpha_{r+1} ... \alpha_j} q_r(a) \geq 0 \).

Then there exist random variables \( V = (V_0, V_1, ..., V_{m-1}, V_{m+1}, ..., V_J) \) with support \( \{ Y \subseteq \mathbb{R}^J \) and a joint density function \( f (\cdot) \), and ‘utility’ functions \( w_j (a, v_j) : \mathbb{R} \times V_j \rightarrow \mathbb{R} \), such that \( w_j (\cdot, v_j) \) are strictly increasing and continuous, \( w_m (a_m, v_m) = q_j \), and

\[
q_j (a_0, a_1, ..., a_J) = \int V \cap_{k \neq j} \{ w_j (a_j, v_j) \geq w_k (a_k, v_k) \} f (v) dv
\]

for each \( j = 0, 1, ... J \). Thus the utility functions \( \{ w_j (a, v_j) \} \) and heterogeneity distribution \( f (\cdot) \) rationalize the choice probabilities \( \{ q_j (a) \} \). (Proof in Appendix)

Condition (i) is intuitive, and corresponds to preferences being non-satiated in the quantity of numeraire. Indeed, if choice probabilities are generated by the structure

\[
q_j (a) = \int V \{ W_j (a_j, \eta) \geq \max_{r \in \{0, 1, ..., J\} \setminus \{j\}} W_r (a_r, \eta) \} f (\eta) d\eta
\]
where \( W_j(\eta) \) are strictly increasing and continuous, and their distributions sufficiently smooth, then condition (i) must hold. The limiting condition \( \lim_{a_j \rightarrow c_j} q_j(a_j) = 1 \) means that holding \( a_j \) fixed, if we lower \( \{a_k, k \neq j\} \) sufficiently, then the probability of choosing \( j \) rises to 1. For example, if the price of each alternative \( k \neq j \) becomes sufficiently high, then eventually everyone will choose \( j \).

Similarly, \( \lim_{a_j \rightarrow d_j} q_j(a_j) = 0 \) means that holding income and prices of other alternatives fixed, if the price of the \( j \)th alternative increases sufficiently, then its aggregate demand will become zero.

Condition (iii) is related to the existence of a density function for unobserved heterogeneity. For models with parametrically specified heterogeneity distributions, condition (iii) was previously used to recover underlying utility functions (c.f. McFadden, 1978 just above Eqn. 12, and McFadden 1981). The motivation for condition (ii) was discussed right before Lemma 1. The proof of this lemma, detailed in the appendix, is based on differentiating the identity \( \sum_{j=0}^{J} q_j(a_j) = 1 \), applying condition (ii) and solving the resulting partial differential equation.

Note that by using the utility functions and heterogeneity distribution obtained via Lemma 1, one can simulate choice probabilities at the observed \( \mathbf{a}'s \). To do this, for any pair of alternatives \( j \neq m \) a least squares projection of \( \frac{\partial}{\partial a_m} q_j(a_j) / \frac{\partial}{\partial a_m} q_m(a_m) \) on a polynomial sieve in \( a_j, a_m \) would be used to generate the coefficient functions of the PDEs, which are then solved to obtain the utility functions and the heterogeneity distribution (see the section "Identification" below for further details), as in Lemma 1. One can then test whether these simulated choice probabilities equal the observed choice-probabilities. Passing this test would imply that the observed choice probabilities can be rationalized.

**Remark 2** The utility function for each alternative \( j \), viz. \( w_j(a_j, v_j) \), constructed in the proof of Lemma 1, consists of a scalar heterogeneity \( v_j \). However, the individual demand function for alternative \( j \) has \( J \) separate sources of heterogeneity, i.e.

\[
Q_j(a, v) = 1 \left\{ w_j(a_j, v_j) \geq \max_{r \in \{0, 1, ..., J\} \setminus \{j\}} w_r(a_r, v_r) \right\}
\]

\[
= Q_j \left( a_0, a_1, ...a_J, \underbrace{v_1, v_2, ..., v_J}_{J \text{ dimensional heterogeneity}} \right)
\]

Thus, we have rationalized a \((J + 1)\) dimensional choice probability function via a \(J\)-dimensional heterogeneity distribution.

The above result establishes a set of conditions for a choice probability function to be rationalized via a random utility model. The constructed model, however, is not linear in unobserved
heterogeneity. The next result shows that when combined with an additional requirement, the three conditions above are necessary and sufficient for integrability via an additive random utility model.

**Theorem 1** Assume the same set-up as in Lemma 1, and assume that Conditions (i) and (iii) of Lemma 1 hold. Additionally it holds that for all $j \neq m$, (ii') \[ \frac{\partial}{\partial a_m} q_j (a) / \frac{\partial}{\partial a_j} q_m (a) \] depends only on $a_j, a_m$, and is of the form

\[ \frac{\partial}{\partial a_m} q_j (a) / \frac{\partial}{\partial a_j} q_m (a) = G_m (a_m) / G_j (a_j), \]

where $G_j (\cdot), G_0 (\cdot) > 0$, for all $j \neq m$. Then there exist strictly increasing utility functions $U (j, \cdot) : \mathbb{R} \to \mathbb{R}$, and $J$ dimensional unobserved heterogeneity $(v_1, ..., v_J) \equiv (\varepsilon_1 - \varepsilon_0, ..., \varepsilon_J - \varepsilon_0)$ with continuous density such that for all $j = 0, 1, ..., J$.

\[ q_j (a_0, a_1, ..., a_J) = \text{Pr} \left[ \bigcap_{k \neq j} \{ U (j, a_j) + \varepsilon_j \geq U (k, a_k) + \varepsilon_k \} \right]. \quad (6) \]

Conditions (i), (ii'), (iii) are also necessary for (6) to hold (proof in Appendix).

**Conditions in standard form:** We have expressed choice probabilities as functions of the $a_j$s, as opposed to $p_j$s and $y$, since it is more natural to impose monotonicity of a function in its arguments, rather than on combination of derivatives with respect to arguments. If choice probabilities are instead expressed in the standard form with income and prices as arguments, one has

\[ q_j (a_0, a_1, ..., a_J) = \bar{q}_j (a_0, a_0 - a_1, ..., a_0 - a_J) \]

\[ = \bar{q}_j (y, p_1, ..., p_J) \equiv \bar{q}_j (y, p). \]

Then the shape restrictions, i.e. condition (i) become: for each $j = 1, ..., J$, $\partial \bar{q}_j (y, p) / \partial p_j \leq 0$, $\partial \bar{q}_j (p, y) / \partial p_k \geq 0$ for all $k \neq j$, and $\sum_{k=1}^J \partial \bar{q}_j (y, p) / \partial p_k + \partial \bar{q}_j (y, p) / \partial y \leq 0$ for all $j = 1, ..., J$. The forms of these expressions bear similarity to Slutsky inequality conditions in standard, deterministic

\[ \frac{\partial}{\partial a_m} q_j (a) / \frac{\partial}{\partial a_j} q_m (a) = H_m (a_m) \times H_j (a_j), \]

or equivalently,

\[ \ln \left( \frac{\partial}{\partial a_m} q_j (a) / \frac{\partial}{\partial a_j} q_m (a) \right) = h_m (a_m) + h_j (a_j), \]

where $H_m (\cdot)$ and $H_j (\cdot)$ are positive functions, and $h_m (\cdot), h_j (\cdot)$ are real-valued functions.
demand analysis for continuous goods. An important difference with the standard continuous case is that our condition is
\[
\sum_{k=1}^{J} \frac{\partial \tilde{q}_j(y, p)}{\partial p_k} + \frac{\partial \tilde{q}_j(y, p)}{\partial y} \leq 0,
\]
(7)
in contrast to the standard continuous case where the Slutsky condition is
\[
\sum_{k=1}^{J} \frac{\partial q_j(y, p)}{\partial p_k} + \frac{\partial q_j(y, p)}{\partial y} \leq 0.
\]
(8)

Condition (ii) becomes: for all \( j = 1, 2, ..., J \),
\[
\sum_{k=1}^{J} \frac{\partial \tilde{q}_j(y, p)}{\partial p_k} + \frac{\partial \tilde{q}_j(y, p)}{\partial y} \frac{\partial \tilde{q}_0(y, p)}{\partial p_j} \leq 0.
\]

2.1 Daly-Zachary’s Slutsky-Symmetry

In the above set-up, Daly-Zachary’s Slutsky symmetry conditions are that for any two alternatives \( k, l \in \{0, 1, ..., J\}, k \neq l \),
\[
\frac{\partial}{\partial a_l} q_k(a) = \frac{\partial}{\partial a_k} q_l(a)
\]
(9)

We first show that the classic random utility model with no income effects implies (9). We then show that Slutsky symmetry (9) implies absence of income effects.

**Necessity:** The canonical random utility model of multinomial choice assumes that the systematic part of the utility from consuming the \( j \)th alternative at income \( y \) and price \( p_j \) is given

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\(^4\)Daly-Zachary defines choice probabilities as functions of price and income, \( \tilde{q}_j(p_0, p_1, ..., p_J, y) \). This is equivalent to our notation of \( q_j(a_0, a_1, ..., a_J) \) with \( a_0 = y, a_1 = y - p_1, ..., a_J = y - p_J \), in that one can move back and forth between the two notations, since

\[
q_j(a_0, a_1, ..., a_J) = \tilde{q}_j(a_0 - a_1, a_0 - a_2, ..., a_0 - a_J), \quad \text{and}
\]
\[
\tilde{q}_j(p_1, p_2, ..., p_J, y) = q_j(y, y - p_1, y - p_2, ..., y - p_J).
\]

"Slutsky symmetry" in Daly-Zachary’s notation is that \( \partial \tilde{q}_k/\partial p_j = \partial \tilde{q}_j/\partial p_k \) for all \( j \neq k \) (if alternative 0 is the outside option, then the corresponding condition is \( \partial \tilde{q}_0/\partial p_j = \partial \tilde{q}_j/\partial y \)). This is identical to (9) in our notation.
by

\[ U(j, a_j) \equiv a_j, \quad (10) \]

where \( a_j = y - p_j \) as above. Income effects are zero since demand depends on the \( a \)'s via the differences \( a_j - a_k = (y - p_j) - (y - p_k) = p_k - p_j \). Then (5) with \( h_j(a_j) = a_j \), i.e. \( h'_j(a_j) = 1 \) implies

\[
\frac{\partial}{\partial a_k} q_l(a_l) = 1,
\]

for all \( a \). This shows that in the canonical random utility model with no income effects, Daly-Zachary’s Slutsky symmetry condition holds.

**Proposition 1 (Sufficiency):** In the above set-up, Daly-Zachary’s Slutsky symmetry implies absence of income effects.

**Proof.** First note that because \( \sum_{k=0}^{J} q_k(a) = 1 \), differentiating both sides w.r.t. \( a_l \) gives

\[
\frac{\partial}{\partial a_l} q_l(a) + \sum_{k=0, k \neq l}^{J} \frac{\partial}{\partial a_l} q_k(a) = 0.
\]

(12)

Substituting (9) in (12), we get:

\[
\frac{\partial}{\partial a_l} q_l(a) + \sum_{k=0, k \neq l}^{J} \frac{\partial}{\partial a_k} q_l(a) = 0.
\]

(13)

This is a linear, homogeneous partial differential equation in \( q_l(\cdot) \), and can be solved via the method of characteristics (c.f. Courant, 1962, Chapter I.5 and II.2, summarized briefly in the Appendix). The characteristic curve, i.e. the \( J \)-dimensional subspace on which \( q_l(a) \) remains constant, can be obtained by solving the so-called “characteristic” Ordinary Differential Equations (see appendix): \( \frac{da_k}{da_l} = 1, k = 0, ...l - 1, l + 1, ..., J \),

(14)

with generic solutions \( a_k - a_l = c_k, k = 0, ...l - 1, l + 1, ..., J \). This means that general solutions to (13) are of the form

\[
q_l(a) = H_l(a_0 - a_l, a_1 - a_l, ..., a_{l-1} - a_l, a_{l+1} - a_l, ..., a_J - a_l),
\]

(15)

where \( H_l(\cdot) \) is any arbitrary continuously differentiable function. Thus \( q_l(a) \) depends on the \( (J + 1) \)-dimensional argument \( (a_0, a_1, a_2, ...a_J) \) through a \( J \)-dimensional vector

\[
(a_0 - a_l, a_1 - a_l, a_2 - a_l, ..., a_{l-1} - a_l, a_{l+1} - a_l, ..., a_J - a_l).
\]
That (15) is a solution to (13) can also be verified directly by partially differentiating the RHS of (15), and verifying that it satisfies (13). Finally, note that

\[(a_0 - a_1, a_1 - a_2, ..., a_{l-1} - a_l, a_{l+1} - a_l, ..., a_J - a_l)\]

and so (15) implies that \(q_l(a)\) does not depend on income. Since \(l\) is arbitrary, we have shown that Slutsky symmetry implies that income effects are absent.

3 Further Points

3.1 Identification

Lemma 1 can be used to identify utilities and the heterogeneity distributions nonparametrically from choice-probabilities observed in a dataset. Nonparametric identification of multinomial choice models (without any discussion of integrability) has been studied previously in the econometric literature, c.f. Matzkin, 1993, 2007 and Allen and Rehbeck, 2019. Since our proof of integrability presented in Lemma 1 is constructive, it provides an alternative and novel way to obtain identification by solving PDEs. Unlike Matzkin 1993, our identification strategy does not rely on identification-at-infinity type arguments nor on linear separability in a regressor with large support (c.f. Matzkin 2007), but does require smoothness.

Specifically, our identification approach is as follows. Suppose that the choice-probabilities are generated by maximization of the utilities \(u_j \equiv \{h_j(a_j) + \varepsilon_j\}, \ j = 0, ..., J\), where the utility functions \(h_j(\cdot)\) are strictly increasing and continuous and hence invertible, but otherwise unknown. Observe that an observationally equivalent utility structure is where utility for the 0th alternative is \(a_0\) and that for the \(j\)th alternative is \(h_0^{-1}\left(h_j(a_j) + \varepsilon_j - \varepsilon_0\right) \equiv w_j(a_j, v_j)\), in that these utilities will produce exactly the same choice probabilities as the \(\{u_j\}\)s. We work under this normalization from now on. We also note in passing that the \(w_j(a_j, v_j)\) are not necessarily additive in the unobserved heterogeneity \(v_j\).

Let \(a\) and \(q_j(a)\) be as above. We can use the proof of Lemma 1 to identify the \(w_j(a_j, v_j)\) functions and the joint distribution of \((v_1, ..., v_J)\) from the \(\{q_j(a)\}\), as follows. First, note that

\[q_0(a) = \Pr(\cap_{j \neq 0} \{a_0 > w_j(a_j, v_j)\}) = \Pr[\cap_{j \neq 0} \{v_j < \omega_j(a_j, a_0)\}],\]
If this implication is not rejected, denote the RHS of (18) as
\[ q_j(a) = \Pr[v_j(a_j, v_j) > w_j(a_j, v_j) > w_1(a_j, v_j), \ldots, w_J(a_j, v_J)] \]
and therefore, by the chain-rule, the first fundamental theorem of calculus, and using \( w_j(a_j, \omega_j(a_j, a_0)) = a_0 \), we have that
\[
\frac{\partial}{\partial a_0} q_j(a) = -\frac{\partial \omega_j(a_j, a_0)}{\partial a_0} \times \int_{-\infty}^{w_1(a_1, a_0)} \cdots \int_{-\infty}^{w_J(a_j, a_0)} f(v_1, \ldots, v_J) dv_J \ldots dv_1
\]
and thus from (16) and (17), we have that
\[
\frac{\partial \omega_j(a_j, a_0)}{\partial a_0} = \frac{\partial}{\partial a_0} q_j(a) / \frac{\partial}{\partial a_j} q_j(a),
\]
which is the same as (4). The RHS of (18) is nonparametrically identifiable from the data, and under the hypothesis of the model, is solely a function of \( a_0 \) and \( a_j \), which is a testable implication. If this implication is not rejected, denote the RHS of (18) as \( t_j(a_j, a_0) \) (this \( t_j(\cdot, \cdot) \) can be estimated by, say a least squares projection of \( \frac{\partial}{\partial a_0} q_j(a) / \frac{\partial}{\partial a_j} q_j(a) \) on a polynomial sieve in \( a_j, a_0 \)). Then solve the PDE
\[
\frac{\partial \omega_j(a_j, a_0)}{\partial a_0} + \frac{\partial \omega_j(a_j, a_0)}{\partial a_j} t_j(a_j, a_0) = 0,
\]
for the \( \omega_j(\cdot, \cdot) \)'s as outlined in the proof of Lemma 1 below (see (33) and (34)), where \( \omega_j(a_j, a_0) \) is strictly increasing in \( a_0 \) and strictly decreasing in \( a_j \), and obtain the \( w_j(a_j, v_j) \) by inverting the solution \( \omega_j(a_j, a_0) \)'s w.r.t. \( a_0 \), and the joint density of \( v \) using (12).

### 3.2 Incorporating Covariates

In our discussion above, choice probabilities \( q_j(\cdot) \) defined in Section 2, correspond to so-called “average structural function”, c.f. Blundell and Powell 2003, 2004. Estimating these from a non-experimental dataset might be non-trivial when observed budget sets (i.e. price and/or income) are...
correlated with unobserved individual preferences across the cross-section of consumers. A common empirical assumption is that budget sets and preferences are independent, conditional on a set of observed covariates. Hence it is useful to see how to adapt the above results to the presence of covariates.

Suppose in addition to price and income, we also observe a vector of characteristics \( z_j \) for each alternative \( j = 1, \ldots, J \). Assume that the choice-probabilities are generated by maximization of the utilities

\[
\begin{align*}
u_0 &\equiv \{ h_0 (a_0) + \varepsilon_0 \}, \quad u_j \equiv \{ h_j (a_j, z_j) + \varepsilon_j \}, \\
&j = 1, \ldots, J,
\end{align*}
\]

where \( h_0 (a) \) and each \( h_j (a, z) \) are strictly increasing and continuous in \( a \), and hence invertible. Then an observationally equivalent utility structure is where utility for the 0th alternative is \( a_0 \) and that for the \( j \)th alternative is

\[

\begin{align*}
&h_0^{-1} \left( h_j (a_j, z_j) + \varepsilon_j - \varepsilon_0 \right) \equiv w_j (a_j, z_j, v_j),
\end{align*}
\]

which is in general not linear or separable in \( v_j \). Working off this normalization, and essentially repeating the same steps as above holding \( z_j \) fixed, lead to the conclusion that for each \( z_j \),

\[

\frac{\partial \omega_j (a_j, a_0, z_j)}{\partial a_0} + \frac{\partial \omega_j (a_j, a_0, z_j)}{\partial a_j} t_j (a_j, a_0, z_j) = 0,
\]

The RHS of (21) is observable from the data, and for each fixed \( z_j \), is solely a function of \( a_0 \), \( a_j \), which is a testable implication. If this implication is not rejected, denote the RHS of (21) as \( t_j (a_j, a_0, z_j) \), just as above. Then for each each fixed \( z_j \), solve the PDE
to obtain the \( \omega_j (a_j, a_0, z_j) \), invert w.r.t. \( a_0 \) to obtain the utilities \( w_j (a_j, v_j, z_j) \) and the joint density of \( v \) using the analog of (39), where we utilize the inverse of \( \omega_j (a_j, a_0, z_j) \) w.r.t. \( a_j \), analogous to (38).

\[\text{4}\]

If even conditional on covariates, independence of preferences and budget sets is suspect, then one needs to employ a “control function” type strategy (c.f. Blundell and Powell, 2004) to estimate the structural choice-probabilities. Indeed, our results above explore the connection between random utility models and “structural” choice probabilities. So, given the extensive econometric literature on estimating structural parameters under endogeneity, we refrain from discussing the consistent estimation of \( q_j (\cdot) \) any further.

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3.3 Empirical Implications: Bounds on Counterfactuals

A key empirical implication of our results is that they can be used to obtain bounds for predicted demand on counterfactual budget sets. We demonstrate how to construct such bounds in the two leading cases of interest, viz. price changes and elimination/addition of alternatives.

**Price Changes:** Denote the support of observed price and income by \( \mathcal{A} \) and suppose we have to predict demand for alternative 1 at a counterfactual \( a' = (a'_0, a'_1, ..., a'_J) \notin \mathcal{A} \). Such counterfactual budget sets may arise due to potential price changes, e.g. those caused by taxes and subsidies or firm-mergers (c.f. Berry and Pakes 1993). To predict this counterfactual demand, let \( \mathcal{A}_j \) denote the set of values of \( a_j \)'s that appear in \( \mathcal{A} \), and \( \mathcal{A}_{jk} \) denote the collection of values taken by the pairs \( \{a_j, a_k\} \), \( j \neq k \) that appear in \( \mathcal{A} \). Now, using Lemma 1, we obtain the utility functions \( w_j (a_j, v_j), j = 1, 2, ..., J \) for \( a_j \in \mathcal{A}_j \), and the joint distribution \( f(\cdot) \) of the unobserved heterogeneity \( \{v_1, v_2, ..., v_J\} \). Recall that our parameter of interest is

\[
q_1 (a') = \int 1 \{w_1 (a'_1, v_1) \geq w_0 (a'_0, v_0), w_1 (a'_1, v_1) \geq w_2 (a'_2, v_2), ..., w_1 (a'_1, v_1) \geq w_J (a'_J, v_J)\} f(v) dv.
\]

Now, for the pair \( (a_1, a_2) \in \mathcal{A}_{12} \), we have that \( w_1 (a_1, v_1) \geq w_2 (a_2, v_2) \Rightarrow w_1 (a'_1, v_1) \geq w_2 (a'_2, v_2) \) whenever \( a'_1 = a_1 \), \( a'_2 = a_2 \) for any pair \( (a_1, a_2) \). Accordingly, define \( w_0 (a'_0, v_0) \equiv a'_0 \), and for each \( j = 0, 2, ..., J \) and the upper and lower bound for \( 1 \{w_1 (a'_1, v_1) \geq w_j (a'_j, v_j)\} \) by

\[
l (a'_1, a'_j, v_1, v_j) = \sup_{(a_1, a_j) \in \mathcal{A}_{1j}} \inf_{a'_1 \leq a_1, a'_j \geq a_j} 1 \{w_1 (a_1, v_1) \geq w_j (a_j, v_j)\}
\]

\[
u (a'_1, a'_j, v_1, v_j) = \inf_{(a_1, a_j) \in \mathcal{A}_{1j}} \sup_{a'_1 \geq a_1, a'_j \geq a_j} 1 \{w_1 (a_1, v_1) \geq w_j (a_j, v_j)\}.
\]

(22)

Therefore, lower and upper bounds on \( q_1 (a') \) are given by

\[
LB_1 (a') = \int \prod_{j=0}^{J} \left[ l (a'_1, a'_j, v_1, v_j) \right] f(v) dv
\]

\[
UB_1 (a') = \int \prod_{j=0}^{J} \left[ u (a'_1, a'_j, v_1, v_j) \right] f(v) dv.
\]

(23)

Since the utility functions \( w_j (a_j, v_j), j = 1, 2, ..., J \) for \( a_j \in \mathcal{A}_j \), and the joint distribution \( f(\cdot) \) of the unobserved heterogeneity \( \{v_1, v_2, ..., v_J\} \) are identified using Lemma 1, so are \( LB_1 (a') \) and \( UB_1 (a') \).
To get simultaneous bounds on \( \{q_j (a')\}, j = 0, ..., J \), we have to impose the constraint that the sum of lower bounds and the sum of upper bounds over \( j = 0, 1, ... J \) must equal 1. This amounts to finding the set of \( \tilde{q}_j (a') \), \( j = 0, ..., J \) such that

\[
LB_j (a') \leq \tilde{q}_j (a') \leq UB_j (a'), \quad \sum_{j=0}^{J} \tilde{q}_j (a') = 1,
\]

(24)

where \( LB_j (a') \) and \( UB_j (a') \), defined in (23), are point-identified and satisfy the shape restrictions of Lemma 1 (i). Note that (24) is a set of linear equality/inequality constraints in \( \tilde{q}_j (a') \) and can be computed using simplex methods. Molinari 2020 discusses several substantive econometric problems that have such linear structure. The bounds (24) on demand in turn provide bounds for welfare calculations corresponding to changes in prices or quality of the products, or addition and elimination of options, since welfare expressions for such cases are known functionals of choice probabilities, c.f. Bhattacharya 2018. The bounds in (24) are sharp because the choice probabilities \( \{q_j (a) \cup \tilde{q}_j (a')\}_{j=0,...,J} \) on \( A \cup \{a'\} \) where \( \tilde{q}_j (a') \) satisfies (24), satisfy all conditions of Lemma 1 and can therefore be rationalized by the same utility functions and heterogeneity distribution as those that rationalize \( \{q_j (a)\}_{j=0,...,J} \) on \( A \).

Allen and Rehbeck 2019 derive bounds for \( \{q_j (a')\}_{j=0,...,J} \) when \( a' \notin A \) by assuming the additive structure \( w_j (a_j, v_j) = w_j (a_j) + v_j \) and that \( w_j (a'_j) \) is known even if \( a' \notin A \). This is possible if \( w_j (\cdot) \) and the joint distribution of unobserved heterogeneity are parametrically specified, and the values of these parameters are known from the observed choice probabilities. In contrast, the bounds in (24) do not require such arbitrary parametric restrictions on the utility indices \( w_j (\cdot, \cdot) \).

**Change in Choice Sets:** From an initial situation described by the set-up, suppose alternative \( J \) is eliminated from the choice-set. Then the choice probability \( q_j (a \setminus \{J\}) \) of alternative \( j \in \{0, 1, 2, ..., J - 1\} \) can be obtained as follows. First the utilities \( w_j (a_j, v_j) \) and the joint density \( f_{\mathbf{v}} (v_1, v_{J-1}, v_J) \) are obtained by applying Lemma 1 to the original choice probabilities when the entire choice set was available. Then the joint density \( f_{\mathbf{v},-J} (v_1, ... v_{J-1}) \) is obtained as

\[
\int_{-\infty}^{\infty} f_{\mathbf{v},-J} (v_1, v_2, ..., v_{J-1}, v_J) dv_J
\]

Finally, the choice probability \( q_j (a \setminus \{J\}) \) of alternative \( j \in \{0, 1, 2, ..., J - 1\} \) is obtained as

\[
\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \prod_{k=0,k\neq j}^{J-1} 1 \{w_j (a_j, v_j) \geq w_k (a_k, v_k)\} f_{\mathbf{v},-J} (v_1, ... v_{J-1}) dv_{J-1} ... dv_1,
\]

(25)

which is point-identified.
3.4 Random Coefficient Models

A popular specification of choice probabilities in applied work is the ‘random coefficient’ model such as the mixed logit or BLP (c.f. Berry 1994. McFadden and Train 2000, Gautier and Kitamura 2013). McFadden and Train 2000 show that essentially all choice probability functions generated via utility maximization can be approximated arbitrarily well by an appropriately defined mixed multinomial logit model. In a random coefficient setting, the utility of the $i$th individual from choosing the $j$th alternative is specified as

$$U_{ij} = \eta_{ij0} + \sum_{k=1}^{K} z_{jk} \eta_{ik} + \eta_{ip} U \left( y_i - p_j \right),$$

where $z_j = \{z_{j1}, ..., z_{JK}\}_{j=1,...,J}$ represents a vector of $K$ observed characteristics of alternative $j$, and $(\eta_{ij0}, \eta_{i1}, ..., \eta_{iK}, \eta_{ip})$ is a random coefficient vector where $\eta_{ip} > 0$ with probability 1 (reflecting non-satiation in the quantity of numeraire), and $U (\cdot)$ is a potentially nonlinear, unknown sub-utility function. Then

$$U_{ij} > U_{il} \iff \eta_{ij0} + \sum_{k=1}^{K} z_{jk} \eta_{ik} + \eta_{ip} U \left( y_i - p_j \right) > \eta_{il0} + \sum_{k=1}^{K} z_{lk} \eta_{ik} + \eta_{ip} U \left( y_i - p_l \right)$$

$$\iff U \left( y_i - p_j \right) + \frac{\eta_{ij0}}{\eta_{ip}} \sum_{k=1}^{K} z_{jk} \frac{\eta_{ik}}{\eta_{ip}} > U \left( y_i - p_l \right) + \frac{\eta_{il0}}{\eta_{ip}} \sum_{k=1}^{K} z_{lk} \frac{\eta_{ik}}{\eta_{ip}}$$

which amounts to choice based on the utility functions $U \left( y_i - p_j \right) + \varepsilon_{ji} \left( z_j \right)$. Therefore, for each realization of $z = \{z_j\}_{j=1,...,J}$, the conditions and thus conclusions of Theorem 1 hold; the only difference is that the structural choice probabilities appearing in the statement of the theorem will have to be defined conditional on $z = \{z_j\}_{j=1,...,J}$. Similarly, the identification argument of Sec 4.1 will work conditional on $z$, implying that the joint distribution of $\{\varepsilon_{ji} (z_j)\}$, $j = 1, ..., J$ is exactly identified while $U (\cdot)$ may be over-identified. For example, one would expect the characteristics of alternatives viz. $z$ to remain identical across consumers in a single market (e.g. the frequency of various modes of public transport are likely to be identical across individuals in the same locality). Then the $q_j (\cdot; z)$’s and their partial derivatives are identified via the variation in income $y$, and hence in $a_0 = y$ and $a_j \equiv y - p_j$ for $j = 1, ..., J$, across individuals in the same market and, additionally, any variation in price within and across markets with the same observed $z$’s. Applying

---

5 If $U (\cdot)$ is linear, then income drops out of choice probabilities, which is a strong and testable restriction.
the identification argument outlined in Section 3.1, one obtains the distribution of \( \{ \varepsilon_{ji}(z_j) \} \), \( j = 1, \ldots, J \) conditional on each realization of \( z \) and the utility indices. These objects will yield bounds on choice probabilities when the budget set takes counterfactual values due to potential policy interventions, by applying (23) or (24) conditional on the \( z \)'s.

Note further that knowledge of the distribution of the (suitably normalized) \( \eta \)'s will allow one to bound choice probabilities when not only the budget set but also covariates take counterfactual values. If the number of markets is large, the distribution of random coefficients is identical in each market, and there is sufficient independent variation of the \( z \)'s across markets, then one can identify the distribution of the normalized \( \eta \)'s from the distribution of the \( \varepsilon_j(z_j) \)'s by using the Cramer-Wold theorem (c.f. Billingsley 1995, Theorem 29.4, Beran and Hall 1992). To see this, let the value of \( z_j \) in market \( m \) be denoted by \( z_j^m \), and denote \( \varepsilon_{ji}(z_j) = \gamma_{i}z_j^m \), where \( z_j^m \) is observed, and the object of interest is the distribution of the unobserved random coefficients \( \gamma \) which are the normalized values of the \( \eta \)'s. Then, using Lemma 1, we obtain the joint distribution of \( (\gamma'z_i^m, \ldots, \gamma'z_J^m) \) in market \( m \), and therefore the marginal of \( \gamma'z_i^m \). Doing this in each market gives us the marginal distribution of each of the projections \( \{ \gamma'z_i^m \} \), \( m = 1, \ldots, M \). Now applying the approach of Beran and Hall 1992 as \( M \to \infty \) identifies the distribution of \( \gamma \) under appropriate regularity conditions. The precision of the corresponding estimator can be increased by using information on all \( J \) alternatives, i.e. \( \{ \gamma'z_i^m \} \), \( m = 1, \ldots, M \), \( j = 1, 2, \ldots, J \).

We conclude this subsection with the observation that Lemma 1 also applies to more general models e.g. where utilities are given by

\[
U_{ij} = \eta_{ip}U(y_i - p_j, z_j) + \varepsilon_{ji}(z_j),
\]

where \( \eta_{ip} > 0 \) with probability 1, and the unobserved \( \varepsilon_{ji}(z_j) \) is not necessarily linear in \( z_j \). Condition (ii) of Lemma 1, conditional on observed covariates, is therefore a testable implication of all such models.

4 Conclusion

This paper provides a unified analysis of integrability and identification in multinomial discrete choice models. It establishes closed-form shape-restrictions on choice-probability functions, under which multinomial choice probabilities can be rationalized via random utility models. These conditions are shown to be necessary and sufficient for the additive random utility model of McFadden. Our results apply equally to random coefficient models like mixed logit – widely used in
IO applications – because conditional on observed characteristics, these are observationally equivalent to models with additive heterogeneity. Our theoretical results are obtained via application of the classical theory of partial differential equations, whose use in economics and econometrics is relatively novel. The key empirical implications of our results are that they lead to (a) non-parametric identification of random utility models using economic theory as opposed to statistical assumptions, (b) specification of multinomial choice models in applied work that is consistent with economic theory while allowing for fully nonparametric utility functions, unobserved heterogeneity and income-effects, and (c) calculation of theory-consistent nonparametric bounds for demand and welfare on counterfactual budget sets, e.g. those arising from price change due to a tax or subsidy, firm-mergers and changes in the number of available alternatives.
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5 Appendix

Two basic results from the theory of partial and ordinary differential equations are used to prove Lemma 1; here we state those results. We will use the notation $C^1$ to indicate a function that is once continuously differentiable.

**Result 1 (Method of Characteristics):** Consider the linear homogeneous PDE

$$\frac{\partial \sigma(x, y, z)}{\partial x} + g_2(x, y) \frac{\partial \sigma(x, y, z)}{\partial y} + g_3(x, z) \frac{\partial \sigma(x, y, z)}{\partial z} = 0. \quad (27)$$

Suppose $g_2$ and $g_3$ are $C^1$ and do not vanish simultaneously. Then a general solution to this equation is given by

$$\sigma(x, y, z) = \phi(h_2(x, y), h_3(x, z)), \quad (28)$$

where $\phi(\cdot)$ is any arbitrary $C^1$ function, and $h_2(x, y) = c_2$ and $h_3(x, z) = c_3$ are general solutions to the ordinary differential equations

$$\frac{dx}{1} = \frac{dy}{g_2(x, y)} = \frac{dz}{g_3(x, z)}, \quad (29)$$

i.e. $\frac{dy}{dx} = g_2(x, y), \frac{dz}{dx} = g_3(x, z)$. The ODE (29) are known as the ”characteristic equations” of the linear PDE (27), and existence of a solution to the PDE (27) amounts to existence of a solution of the ODE (29), c.f. Courant, 1962, Chapter I.5, II.2. The intuitive reason for this is that (27) means that the vector $(1, g_2(x, y), g_3(x, z))$ is a tangent to any level curve $\sigma(x, y, z) = c$. Therefore, for any parametrization $(x(t), y(t), z(t))$ defining the level curve $\sigma(x(t), y(t), z(t)) = c$, the corresponding tangent vector $\left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$ equals the vector $(1, g_2(x(t), y(t)), g_3(x(t), z(t)))$. The formal statement of this result, c.f. Zachmanoglou and Thoe 1986 Theorem 4.1, is that (a) if $S$ is a level set of the solution $\sigma(x, y, z)$ of (27), then for every point of $S$, the solution curve of (29) passing through that point lies entirely on $S$; conversely, (b) if at every point $(x_0, y_0, z_0)$, the solution curve of (29) passing through $(x_0, y_0, z_0)$ lies entirely on the level surface of the function $\sigma(x, y, z)$ passing through $(x_0, y_0, z_0)$, then $\sigma(x, y, z)$ is a solution to (27). Sub-statement (a) is proved by showing that for any solution curve of (29) given by the parametrization $(x(t), y(t), z(t))$, we must have that $\frac{dx}{dt} = 1, \frac{dy}{dt} = g_2(x, y), \frac{dz}{dt} = g_3(x, z)$; therefore,

$$\frac{d}{dt} \sigma(x(t), y(t), z(t)) = \frac{\partial \sigma(x, y, z)}{\partial x} \frac{dx}{dt} + \frac{\partial \sigma(x, y, z)}{\partial y} \frac{dy}{dt} + \frac{\partial \sigma(x, y, z)}{\partial z} \frac{dz}{dt} = \frac{\partial \sigma(x, y, z)}{\partial x} + g_2(x, y) \frac{\partial \sigma(x, y, z)}{\partial y} + g_3(x, z) \frac{\partial \sigma(x, y, z)}{\partial z} = 0.$$
Sub-statement (b) is proved by noting that if the solution curve of (29) is described by the parametrization \((x(t), y(t), z(t))\), then the vector \(\left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)\) is tangent to that curve; therefore, the vector \((1, g_2(x(t), y(t)), g_3(x(t), z(t)))\) is tangent to the curve \((x(t), y(t), z(t))\) and hence to the level surface \(S\) of \(\sigma(x, y, z)\) because \((x(t), y(t), z(t))\) lies on \(S\); therefore, we must have that the gradient of \(\sigma(x, y, z)\) is orthogonal to \((1, g_2(x, y), g_3(x, z)))\), i.e. (27) holds.

In (28), \(\phi(\cdot, \cdot)\) can be chosen to be strictly increasing in both arguments. A unique choice of \(\phi(\cdot, \cdot)\) is pinned down by boundary conditions; in our application, these amount to equating \(\phi(h_2(x, y), h_3(x, z))\) to observed choice probability functions.

**Result 2 (Solution of the Characteristic ODE):** The second result restates a global version of the Picard-Lindelöf theorem that establishes conditions for existence of a solution to a first-order ODE.

**Picard-Lindelöf Theorem:** Suppose that a function \(g : R \times R \rightarrow R\) is continuous, and on each strip \(S_a = \{(x, y) : |x| \leq a, \ |y| < \infty\}\), \(g(x, y)\) is Lipschitz in \(y\). Then the ordinary differential equation \(n'(x) = g(x, n(x))\), has a general solution \(n(\cdot) : R \rightarrow R\) with \(n(\cdot)\) being \(C^1\). (See, for instance, Coddington, 1961, Theorem 9 and corollary).

This result is proved by showing that under the assumptions of the lemma, the map \(n(\cdot) : \int_{x_0}^x g(s, n(s)) \, ds\) for any arbitrary \(x_0\) is a contraction, thereby ensuring, via the Banach fixed point theorem, the existence of \(n(\cdot)\) satisfying

\[
n(x) = n(x_0) + \int_{x_0}^x g(s, n(s)) \, ds.
\]

**Proof of Lemma 1**

**Proof.** WLOG take \(m = 0\), and use condition (ii) of the Lemma to define

\[
t_{j0}(a_j, a_0) \equiv \frac{\partial}{\partial a_0} q_j(a) / \frac{\partial}{\partial a_j} q_0(a) \geq 0.
\] (30)

Now, because \(\sum_{j=0}^J q_j(a) = 1\), differentiating both sides w.r.t. \(a_0\) gives

\[
\frac{\partial}{\partial a_0} q_0(a) + \sum_{j=1}^J \frac{\partial}{\partial a_0} q_j(a) = 0.
\] (31)

Substituting (30) in (31), we get the linear, homogeneous, partial differential equation in \(q_0(\cdot)\):

\[
\frac{\partial}{\partial a_0} q_0(a) + \sum_{j=1}^J \frac{\partial}{\partial a_j} q_0(a) \times t_{j0}(a_j, a_0) = 0.
\] (32)
This PDE can be solved via the method of characteristics (see Result 1 above), giving the characteristic ordinary differential equations:

\[
\frac{da_j}{da_0} = t_{j0}(a_j, a_0),
\]

for \( j = 1, ..., J \). Using the Picard-Lindelöf theorem (Result 2 above) and the principle of solving linear homogeneous PDEs, we obtain the general solutions of (33) given by \( \omega_j(a_j, a_0) = \text{cons} \), where \( \omega_j(a_j, a_0) \) is differentiable, strictly increasing in \( a_0 \) and strictly decreasing in \( a_j \), and satisfies

\[
\frac{\partial \omega_j(a_j, a_0)}{\partial a_0} + \frac{\partial \omega_j(a_j, a_0)}{\partial a_j} t_{j0}(a_j, a_0) = 0,
\]

and also, using (30)

\[
- \frac{\partial \omega_j(a_j, a_0)}{\partial a_0} \frac{\partial \omega_j(a_j, a_0)}{\partial a_j} \equiv \frac{\partial}{\partial a_0} q_j(a) / \frac{\partial}{\partial a_j} q_0(a).
\]

A general solution \( q_0(a) \) is therefore of the form

\[
q_0(a) = H_0(\omega_1(a_1, a_0), \omega_2(a_2, a_0), ..., \omega_J(a_J, a_0)),
\]

where \( H_0(\cdot) \) can be chosen to be strictly increasing and \( C^1 \) in each argument, and with continuous \( J \)th order cross partial derivatives. Since \( q_0(a) \) is observed, the exact functional form of \( H_0(\cdot) \) is pinned down by (36), for any set of solutions \( \omega_j(\cdot, \cdot) \) to the ODEs (33). This corresponds to the so-called "initial condition" in the PDE nomenclature. In particular, given any \( a_0 \), the value of \( H_0(x_1, x_2, ..., x_J) \) at any vector \( (x_1, x_2, ..., x_J) \) is given by

\[
H_0(x_1, x_2, ..., x_J) = q_0(a_0, b_1(x_1, a_0), ...b_J(x_J, a_0)),
\]

where \( b_j(x_j, a_0) \) is defined by the solution \( b \) to

\[
\omega_j(b, a_0) = x_j
\]

In this construction, the choice of \( a_0 \) is immaterial. That is, for two choices \( a_0 \neq a'_0 \),

\[
q_0(a_0, b_1(x_1, a_0), ...b_J(x_J, a_0)) = H_0(\omega_1(b_1(x_1, a_0), a_0), \omega_2(b_2(x_2, a_0), a_0), ..., \omega_J(b_J(x_J, a_0), a_0)) \text{ from (33)}
\]

\[
= H_0(x_1, x_2, ..., x_J) \text{ from (37)}
\]

\[
= H_0(\omega_1(b_1(x_1, a'_0), a'_0), \omega_2(b_2(x_2, a'_0), a'_0), ..., \omega_J(b_J(x_J, a'_0), a'_0))
\]

\[
= q_0(a'_0, b_1(x_1, a'_0), ...b_J(x_J, a'_0)).
\]
Having obtained the $\omega_j (\cdot , \cdot)$’s from $\ref{eq:omega}$ and $\ref{eq:omega2}$, for each $j = 1, \ldots, J$, define the function $w_j (a_j, v)$ by inversion, i.e.

$$w_j (a_j, v) = \{a_0 : \omega_j (a_j, a_0) = v\}. \quad (40)$$

Note that by construction, $w_j (a_j, v)$ is strictly increasing and continuous in $a_j$ for each $v$. The $w_j (\cdot , \cdot)$’s will play the role of ‘utilities’ in our proof of integrability. Set $w_0 (a_0, v_0) \equiv a_0$.

We now show how to construct the distribution of heterogeneity. Let $\mathcal{V}_j$ denote the co-domain of $\omega_j (\cdot , \cdot)$, and let

$$\mathcal{V}_j = \mathcal{V}_j \cap \left\{ \omega_j (a_j, a_0) : \prod_{j=1}^J \left( \frac{\partial}{\partial a_0} \omega_j (a_j, a_0) \times \frac{\partial}{\partial a_j} \omega_j (a_j, a_0) \right) \neq 0 \right\},$$

and let $\mathcal{V} \equiv \times_{j=1}^J \mathcal{V}_j$. Now, given any vector $\mathbf{v} \equiv (v_1, \ldots, v_J) \in \mathcal{V}$, define the cumulative distribution function at $\mathbf{v}$ as

$$F (v_1, \ldots, v_J) = q_0 (a_0, a_1, \ldots, a_J),$$

where the vector $(a_0, a_1, \ldots, a_J)$ satisfies $v_j = \omega_j (a_j, a_0)$, for each $j = 1, \ldots, J$. It follows from $\ref{eq:omega}$ and $\ref{eq:omega2}$ that this function is well-defined. The above CDF implies the density function $f : \mathcal{V} \to \mathbb{R}^+$:

$$f (v_1, \ldots, v_J) = \frac{\partial^J}{\partial a_1 \cdots \partial a_J} q_0 (a_0, a_1, \ldots, a_J) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J} \prod_{j=1}^J \frac{\partial}{\partial a_j} \omega_j (a_j, a_0) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J} \prod_{j=1}^J \omega_j (a_j, a_0) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J}$$

$$= \prod_{j=1}^J \left( \frac{\partial}{\partial a_j} \omega_j (a_j, a_0) \right) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J} \times \frac{\partial}{\partial a_0} q_k (a_0, a_1, \ldots, a_J) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J},$$

from $\ref{eq:omega2}$

$$= - \prod_{j=1}^J \left( \frac{\partial}{\partial a_j} \omega_j (a_j, a_0) \right) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J} \frac{\partial}{\partial a_0} q_k (a_0, a_1, \ldots, a_J) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J}$$

$$= - \prod_{j=1}^J \left( \frac{\partial}{\partial a_j} \omega_j (a_j, a_0) \right) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J} \frac{\partial}{\partial a_0} q_k (a_0, a_1, \ldots, a_J) \bigg|_{v_j = \omega_j (a_j, a_0), \; j = 1, \ldots, J}, \quad \text{from } \ref{eq:omega2}$$

Since $\frac{\partial}{\partial a_0 q_k (a_0, a_1, \ldots, a_J)}$ has sign $(-1)^J$ and $\frac{\partial}{\partial a_j} \omega_j (a_j, a_0) < 0$, and $\frac{\partial}{\partial a_0} \omega_j (a_j, a_0) > 0$.
0 on $\mathcal{V}$, each of the above expressions has numerator and denominator of the same sign, and is thus non-negative. We verify below that this joint density integrates to 1.

We now show that the above construction of $w_j (\cdot, \cdot)$ (c.f. (40)) and the joint density of heterogeneity (39) and (42) will indeed produce the original choice probabilities. To see this for alternative 1, consider the integral

$$
\int_{\mathcal{V}} 1 \left\{ w_1 (a_1, v_1) \geq \max_{k \in \{0, 2, \ldots, J\}} w_k (a_k, v_k) \right\} f (v_1, v_2, \ldots, v_J) dv_1 \ldots dv_J
$$

$$= \int_{\mathcal{V}} 1 \left\{ v_1 \geq \omega_1 (a_1, a_0), k \in \{2, \ldots, J\} \right\} \{ v_k \leq \omega_k (a_k, w_1 (a_1, v_1)) \} f (v_1, v_2, \ldots, v_J) dv_1 \ldots dv_J
$$

Consider the substitution $(v_1, v_2, \ldots, v_J) \rightarrow (x_1, x_2, \ldots, x_J)$ given by $v_1 = \omega_1 (a_1, x_1)$ (so that $x_1 = w_1 (a_1, v_1)$), and for $k = 2, \ldots, J$, $v_k = \omega_k (x_k, x_1)$, which transforms the above integral to

$$
\int_{a_0}^{\infty} \int_{a_2}^{\infty} \ldots \int_{a_J}^{\infty} \left[ f (\omega_1 (a_1, x_1), \omega_2 (x_2, x_1), \ldots, \omega_J (x_J, x_1)) \times \frac{\partial \omega_1 (a_1, x_1)}{\partial x_1} \times \prod_{k=2}^{J} \frac{\partial \omega_k (x_k, x_1)}{\partial x_k} \right] dx_1 \ldots dx_J
$$

$$= \int_{a_0}^{\infty} \int_{a_2}^{\infty} \ldots \int_{a_J}^{\infty} \left\{ -\frac{\partial^J}{\partial x_1 \partial x_2 \ldots \partial x_J} q_1 (x_1, a_1, x_2, \ldots, x_J) \right\} dx_1 \ldots dx_J
$$

$$= q_1 (a_0, a_1, a_2, \ldots, a_J). \quad (43)
$$

Exactly analogous steps for $j = 2, \ldots, J$, and using (42), lead to the conclusion that for all $j \geq 1$,

$$
\int 1 \left\{ w_j (a_j, v_j) \geq \max_{k \in \{0, 1, 2, \ldots, J\} \setminus \{j\}} w_k (a_k, v_k) \right\} f (v_1, v_2, \ldots, v_J) dv_1 \ldots dv_J
$$

$$= q_j (a_0, a_1, a_2, \ldots, a_J).$$

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Also, note that
\[
\int 1 \left\{ a_0 \geq \max_{k \in \{1, 2, ..., J\}} w_k (a_k, v_k) \right\} f (v_1, v_2, ..., v_J) dv_1...dv_J
\]
substitute \( v_j \to x_j \) satisfying \( v_j = \omega_j (x_j, a_0) \)
\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left[ f (\omega_1 (a_1, x_1), \omega_2 (x_2, x_1), ..., \omega_J (x_J, x_1)) \right] dx_2...dx_Jdx_1
\]
\[
= (-1)^J \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \left\{ \prod_{k=2}^{J} \frac{\partial^{J}}{\partial x_1 \partial x_2...\partial x_J} q_1 (x_1, a_1, x_2, ...x_J) \right\} dx_2...dx_Jdx_1
\]
\[
= q_1 (-\infty, a_1, -\infty, ..., -\infty)
\]
\[
= 1, \text{ by condition (i) of Lemma 1.}
\]

Finally, to show that the joint density (11) integrates to 1, use exactly the same substitution as the one leading to (13), and observe that
\[
\int f (v_1, v_2, ..., v_J) dv_1...dv_J
\]
\[
= \int_{\omega_1(a_1,a_0)}^{\infty} ... \int_{\omega_J(a_J,a_0)}^{\infty} f (v_1, v_2, ..., v_J) dv_J...dv_1
\]

Thus we have shown that a population endowed with our constructed \( w_j (\cdot, v_j) \) as utilities, together with the joint density of heterogeneity given by (39) would indeed produce the choice probabilities \{q_j (\cdot,...)\} for each \( j = 0, 1, ..., J \). ■

**Proof of Theorem 1**

**Proof.** Necessity is obvious. In particular, condition (ii') is a direct consequence of equation (3).

To prove sufficiency, WLOG take \( m = 0 \), and let \( \tilde{G}_j (a_j) \) and \( \tilde{G}_0 (a_0) \) be the primitive integrals of \( G_j (a_j) \) and \( G_0 (a_0) \), i.e. \( \frac{d}{da_j} \tilde{G}_j (a_j) = G_j (a_j) \) and \( \frac{d}{da_0} \tilde{G}_0 (a_0) = G_0 (a_0) \); note that \( \tilde{G}_j (a_j) \) and \( \tilde{G}_0 (a_0) \) are strictly increasing and continuous since they have strictly positive derivatives. Then, by exactly analogous steps that led to (36), we have that condition (ii') of Theorem 1, viz. 28
\[
\frac{\partial}{\partial a_m} q_j(a) = G_m(a_m) / G_j(a_j)
\]
has a general solution of the form
\[
q_0(a) = H(\bar{G}_1(a_1) - \bar{G}_0(a_0), ..., \bar{G}_J(a_J) - \bar{G}_0(a_0)),
\]
where \(H(\cdot)\) is an arbitrary smooth function mapping \(\mathbb{R}^J \to [0, 1]\). In particular, we can take \(H(\cdot)\) to be nondecreasing in each argument, and we have that \(\bar{G}_j(\cdot), j = 0, ..., J\) are strictly increasing and continuous. Following exactly analogous steps to the proof of Lemma 1, we get that \(q_j(a)\) is rationalized by the utility functions
\[
w_0(a_0, \eta) = a_0, \quad w_j(a_j, \eta) = \bar{G}_j^{-1}(\bar{G}_j(a_j) - v_j),
\]
with the CDF for the joint distribution of the unobserved heterogeneity \(\eta \equiv (v_1, ..., v_J)\) given by
\[
F_{\eta}(v_1, v_2, ..., v_J) = q_0(a_0, \bar{G}_1^{-1}(\bar{G}_0(a_0) + v_1), ..., \bar{G}_J^{-1}(\bar{G}_0(a_0) + v_J)) = H(v_1, ..., v_J)
\]
(44) (45)

Just as in (39), the choice of \(a_0\) is immaterial here. Note further that the above model is observationally equivalent to one where utilities are given by
\[
W_0(a_0, \eta) = \bar{G}_0(a_0), \quad W_j(a_j, \eta) = \bar{G}_j(a_j) - v_j, \quad j = 1, ..., J,
\]
with the joint CDF of \(\eta \equiv (v_1, ..., v_J)\) still given by (45). This is precisely the ARUM model. That these distribution implies
\[
q_j(a) = \Pr[\cap_{k \neq j} W_j(a_j, \eta) \geq W_k(a_k, \eta)]
\]
for all \(j = 0, ..., J\) can be established following the exact same steps as in the proof of Lemma 1 above, with \(\omega_j(a_j, a_0)\) replaced by \(\bar{G}_j(a_j) - \bar{G}_0(a_0)\) everywhere. ■

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