Construction of a periodic standing wave for $n$ corotating vortex filaments arising from a central configuration.

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Abstract

The model for $n$ almost-parallel vortex filaments in a three dimensional fluid takes in consideration the interaction between different filaments and an approximation for the self-induction. The main technique to construct the standing wave is the Nash-Moser scheme that leads to a Lyapunov-Schmidt reduction in a Cantor set. The methods are a combination of the analysis of Hamiltonian dynamical systems and the linear theory related to Anderson Localization.

Introduction

Many PDE’s exhibit the structure of an infinite dimensional Hamiltonian systems. In Hamiltonian PDE’s, the proof of periodic solutions have a small divisors problem, caused by the spectral properties of the linearized operators. This problem is closely analogous to the analytic challenge of KAM theory.

To construct periodic solutions, Craig and Wayne developed a Lyapunov–Schmidt reduction using a Nash-Moser method in [4]. This technique has the advantage of not requiring the second Melnikov’s condition present in KAM methods. This procedure was generalized by Bourgain to construct quasiperiodic solutions, for instance see [2], and later on by Berti and Bolle, see [3].
The local induction approximation for a vortex filament is an equation for the movement of a self-interacting infinitesimal vortex tube in a three-dimensional fluid. Later on, a system of equations was proposed for the interaction of \( n \) almost-parallel vortex filaments in \([1]\).

In this paper, we proof bifurcation of periodic standing waves for \( n \) parallel vortex filaments that originally revolve around an axis, in positions determined by an arbitrary central configuration. In this case, the amplitude of the perturbation is limited to a Cantor set.

The proof is based in a Lyapunov-Schmidt reduction to solve the range equation by a Nash-Moser procedure. This is a Newton method with smoothing operators via finite dimensional Fourier truncations. To prove the convergence of the procedure, the inverse of a sequence of finite dimensional matrices of increasing dimension has to be investigated.

Fröhlich-Spencer like estimates are used to control the inverse of the projected linearization, assuming that the spectrum of the linear map in singular regions has some properties. Thus, restrictions in the kernel and frequency have to be imposed in order to satisfy the hypothesis. In this way, the fast convergence of the Newton method overcomes the growing rate of the norm of the projected inverse. However, at the end of the process, the range equation has been solved only in a Cantor set.

The bifurcation equation is analyzed with the use of the symmetries. In this case there are two bifurcation branches, one of standing waves and one of traveling waves. The standing wave is a breaking of symmetries from a one-dimensional orbit to a three-dimensional orbit. Estimates on the measure of the intersection of the branch of standing waves with the Cantor set finish the proof.

The Nash-Moser procedure has been previously applied to the nonlinear wave equations \([4]\), and the NLS equation \([5]\). The implementation of the Nash-Moser method in this problem is relevant to understand fluid dynamics. Moreover, this result is intended as a first step in the analysis of the quasiperiodic case, which is of great mathematical complexity.

We present here a simplified proof of the Fröhlich-Spencer like estimates. These estimates integrate clusters of singular sites to the project inverse, which are the key to prove the convergence of the Nash-Moser procedure. Moreover, the estimates can be used to solve any small divisor problem from periodic solutions in one-dimensional Hamiltonian PDE’s.

In section 1, we set the existence of standing waves as a problem of bifurcation in a space with symmetries. In section 2, we discuss the Lyapunov-
Schmidt reduction with the use of the Nash-Moser procedure. In section 3, we present the estimate in the projected inverse, assuming hypothesis of separation of singular sites and estimates of their spectrums. This step is the key to prove the convergence of the Nash-Moser procedure.

In section 4, we estimate the measure of good parameter where the hypothesis are satisfied. In section 5, we prove that the intersection of the bifurcation branch with good parameters have measure. Finally, in the Appendix, we prove a theorem on the global bifurcation of periodic traveling waves, as well, we discuss the symmetries of the standing waves.

1 Setting the problem

Let \( u_j(t, s) \in \mathbb{C} \) be the positions of \( n \) filaments with circulations equal to 1. From [11], the equations for the \( n \) almost-parallel vortex filaments are

\[
\partial_t u_j = i \left( \partial_{ss} u_j + \sum_{i=1}^{n} \frac{u_j - u_i}{|u_j - u_i|^2} \right),
\]

(1)

Homothetic solutions for the vortex filament problem are solutions with \( u_j(t, s) = a_j u(t, s) \), where \( a_j \)'s are complex numbers. In this kind of solutions the shape of the intersections of the filaments with a complex plane is homothetic with the shape of any other plane at any time.

Homothetic solutions of the form \( a_j u(s, t) \) must satisfy the equations

\[
\partial_t u = i \left( \partial_{ss} u + |u|^{-2} u \right),
\]

(2)

and

\[
a_j = \sum_{i=1}^{n} \frac{a_j - a_i}{|a_j - a_i|^2}.
\]

(3)

Solutions to the equation (3) are central configurations in the \( n \)-vortex problem, and there are plenty of this solutions in the literature. For instance, in [10] and [8], a polygonal central configurations with a central filament is discussed.

Remark 1 The solutions to equation (2) also generate homothetic solutions when there is a central filament with different circulation that remains at the central axe, see [10]. The set of solutions \( a_j u(s, t) \) foliate an invariant manifold. For two filaments, the complement of this subspace is foliated by solutions of the linear Schrödinger equation [10].
Remark 2  Equation (2) is a Hamiltonian system given by \( u_t = i \nabla H(u) \) with
\[
H = \int_0^{2\pi} \left( -\frac{1}{2} |\partial_s u|^2 + \ln |u| \right) ds.
\]

Since the Hamiltonian \( H(u) \) is autonomous, phase invariant, and invariant by translations, then the energy \( H \), the angular momentum \( I = \frac{1}{2} \int_0^{2\pi} |u|^2 ds \), and the momentum \( W = \int_0^{2\pi} (iu, \partial_s u) ds \) are conserved quantities.

The simplest solutions to equation (2) are relative equilibria of the form \( u(t, s) = u(s) e^{i\omega t} \). In these solutions the filaments turn around one axis at the same constant speed. For this class of solutions equation (2) becomes
\[
\omega u = \partial_{ss} u + |u|^{-2} u. \tag{4}
\]

One trivial set of solutions of the previous equation is given by \( u(s) = ae^{i\sigma s} \) with \( \omega = -\sigma^2 + a^{-2} \). Thus
\[
u(t, s) = ae^{i(\omega t + \sigma s)} \text{ with } \omega = -\sigma^2 + a^{-2} \tag{5}
\]
is a continuum of solutions of equation (2) parametrized by \( a \). This continuum originates the one dimensional orbit \( ae^{i\theta} e^{i(\omega t + \sigma s)} \) for \( \theta \in S^1 \).

Remark 3  The equation (2) is invariant under the Gauss transformation
\[
e^{-i\alpha^2 t} e^{i\alpha s} u(t, s - 2\alpha t). \tag{6}
\]

Using the Gauss transformation, the solution (5) generates the parametrized family
\[
ae^{i((\omega - 2\alpha\sigma - \alpha^2)t + (\sigma + \alpha)s)}.
\]

Since \( \omega = -\sigma^2 + a^{-2} \), choosing \( \alpha = -\sigma \) the solution becomes \( ae^{i\omega t} \) with \( \omega = a^{-2} \). Hence, using the Gauss symmetry, different branches \( ae^{i(\omega t + \sigma s)} \) are transform in the branch \( \sigma = 0 \).

By the previous remark, without loss of generalization, we may reduce the study to bifurcation of periodic solutions close to \( ae^{i(\omega t + \sigma s)} \) for \( \sigma = 0 \). Moreover, using the Gauss transform, any periodic solution near \( ae^{i\omega t} \) is reproduced as a solution near \( ae^{i(\omega t + \sigma s)} \) for an arbitrary \( \sigma \in \mathbb{R} \).
Remark 4  Since equation (2) is invariant under the scaling transform

$$\tau^{-1}u(\tau^2 t, \tau s),$$

then any $T$-periodic boundary condition in $s$ may be fix to $T = 2\pi$. However, once the period in $s$ has been fixed, the amplitude $a$ cannot be scaled further.

By the previous remark, only bifurcation of $2\pi$-periodic solutions in space is considered. However, periodic solutions close to $ae^{i\omega t}$ have different qualitative behaviors depending on the amplitude $a$, then the amplitude has to be considered as a free parameter.

To look for bifurcation of $2\pi/\Omega$-periodic in time solutions close to $e^{i\omega t}a$, we do the change of coordinates

$$u(t, s) = ae^{\omega t}v(\Omega t, s).$$

(7)

With this change of coordinates, the $2\pi/\Omega$-periodic solutions of the equation (2) are solutions of

$$i\Omega \partial_t v = -\partial_{ss} v + \omega(1 - |v|^2)v.$$

(8)

As the equation depends on the amplitude $a$ only through $\omega$, a better choice of the free parameter is the angular frequency $\omega$.

Remark 5  Equation (4) for relative equilibriums may be solved by the method of quadratures, as in the Kepler problem. In polar coordinates with $u = re^{i\theta}$, the equation becomes $\partial_s \theta = c/r^2$ where $c$ is the angular momentum, and

$$\partial_s^2 r - c^2 r^{-3} + r^{-1} - \omega r = 0.$$

(9)

If $c = 0$, then $\theta = \theta_0$ is constant. In this case, the filaments are in a plane that pass through the central axis. Since $\omega$ is positive, there is a unique equilibrium at $r = \omega^{-1/2}$, that corresponds to the $2\pi$-periodic solution (2) with $\sigma = 0$. There are more bounded solutions where $r(s) \to 0$ when $s \to \pm\infty$, but the model is not valid when the filaments get close. There are also unbounded solutions and these solutions may have physical relevance.

If $c \neq 0$, then the dependency of equations in $s$ can be eliminated, as in the Kepler problem. Changing variables as $\rho = r^{-1}$ (see (7)), then $\partial_s r = -c\partial_\theta \rho$ and $\partial_{ss} r = -c^2 \rho^2 \partial_{\theta\theta} \rho$. Thus, the equation becomes

$$c^2 \partial_{\theta\theta} \rho + c^2 \rho - \rho^{-1} + \omega \rho^{-3} = 0.$$

(10)
This equation may be integrated from \(c^2 (\partial_t \rho)^2 + V(\rho) = E\) with the potential

\[
V(\rho) = c^2 \rho^2 - \ln \rho^2 - \omega \rho^2. 
\]

Equation (11) have two equilibriums for \(\omega \in (0, 1/4c^2)\) given by

\[
\rho_{\pm}^2 = \frac{1}{2c^2} \left(1 \pm \sqrt{1 - 4c^2 \omega}\right).
\]

Since \(\omega = -c^2 \rho_{\pm}^4 + \rho_{\pm}^2\) and \(\theta(s) = c\rho_{\pm}^2 s\), then the equilibria \(\rho_{\pm}\) correspond to solutions \(u = \rho_{\pm}^{-1} e^{i(\omega t + \omega \rho_{\pm}^2 s)}\), which are the helix solutions (??).

Moreover, the equilibrium \(\rho_+\) is a minimum of \(V\), and there are periodic solutions close to \(\rho_+\). The continuum of periodic solutions near \(\rho_+\) is of like elliptic helices that live in a bounded ring. Actually, it may be proven that the projection in the plane of these periodic solutions are asymptotically ellipses. These like elliptic helices may generate more complex solutions using the Gauss transform.

2 The Nash-Moser method

By generic arguments of bifurcation, see the appendix, there are at least one bifurcating branch of traveling waves and one of standing waves. Without resonances, these bifurcation branches are the only ones. Given that the traveling wave do not present a small divisor problem, we shall give a rigorous proof of the existence of the standing wave first.

The proof the standing wave, we do a Nash-Moser method in a subspace of symmetries (21), to simplify the proof in many aspects: singular regions consist of isolated sites, the linearization is not degenerated and the kernel is one dimensional.

Remark 6 In a work in preparation, we are working in the cuasiperiodic case. In this case, we cannot evade the problem of big clusters of small eigenvalues. Thus, an analysis without simplifications is being prepared in this case.

Using the change of variables \(v = 1 + u\) in equation (8), for a small perturbation \(u\), the bifurcation of periodic solutions correspond to zeros of the map

\[
f(u; \Omega) = -i\Omega u_t - u_{ss} + \omega(u + \bar{u}) + \omega g(u, \bar{u}).
\]
The nonlinearity is
\[ g(u, \bar{u}) = \sum_{n=2}^{\infty} (-1)^n \bar{u}^n = \frac{\bar{u}^2}{1 + \bar{u}}. \]

The next theorem is the main result of the present work.

**Theorem 7** For \( \omega \) Diophantine with \( |q \omega - p| \geq \gamma/|q|^T \), there exists \( r_0 > 0 \) and a cantor subset \( C \) of \([0, r_0]\), such that for all \( r \) in \( C \), the map \( f \) has analytic \( 2\pi \)-periodic solutions parametrized by \( r \), where solutions \( (u(s, t; r), \Omega(r)) \) satisfy

\[
\begin{align*}
u(t, s; r) &= r \cos s (\cos t - i \Omega_0 \sin t) + O(r^2), \\
\Omega(r) &= \Omega_0 + \Omega_2 r^2 + O(r^3) \text{ with } \Omega_2 \neq 0.
\end{align*}
\]

The cantor set \( C \) has measure \( r_0(1 - \varepsilon) \). And the solutions \( u(s, t) \) have the symmetries

\[ u(t, s) = u(t, -s) = \bar{u}(-t, s). \quad (11) \]

**Remark 8** Actually, solutions of the previous theorem satisfies also the symmetry \( u \)
\[
(s, t) = u(s + \pi, t + \pi).
\]

We omit the proof order to simplify the procedure. Notice that \( e^{i\varphi} u_j(t + \varphi_t, s + \varphi_s) \) is also a solution for any \( \varphi_s \in [0, 2\pi) \), and that the solution has an orbit of a 3-torus.

As a consequence, the following result for the \( n \) vortex filament problem holds.

**Theorem 9** Let \( a_j \) be a central configuration with

\[
\omega a_j = \sum_{i=1(i \neq j)}^{n} \frac{a_j - a_i}{|a_j - a_i|^2},
\]

then the solution \( u \) of the previous theorem generates the solution for the \( n \) vortex filament problem \((J)\) given by

\[ u_j(t, s) = a_j e^{i\omega} (1 + u(\Omega(r)t, s; r)), \]

where \( \omega \) is a Diophantine frequency of rotation around the central axis, and \( r \) is a small amplitude of the perturbed solution in a Cantor set.
Let us define the Hilbert space

\[ L^2_{\text{sym}}(T^2; \mathbb{C}) = \{ u \in L^2(T^2; \mathbb{C}) : u(t, s) = u(t, -s) = \bar{u}(-t, s) \}, \]

with the inner product

\[ \langle u, w \rangle = \frac{1}{(2\pi)^2} \int_{T^2} u \bar{w} dt ds. \]

Thus, any function \( u \in L^2_{\text{sym}} \) may be written as

\[ u = \sum_{k \in \mathbb{N}} a_{0,k} \cos ks + \sum_{j \in \mathbb{N}^+, k \in \mathbb{N}} (a_{j,k} \cos jt + ib_{j,k} \sin jt) \cos ks \]

with \( a_{j,k}, b_{j,k} \in \mathbb{R} \), where \( \mathbb{N} = \{0, 1, \ldots\} \) and \( \mathbb{N}^+ = \{1, 2, \ldots\} \).

The linearization of the map \( f \) is

\[ L(\Omega)u = -i\Omega u_t - u_{ss} + \omega(u + \bar{u}). \]

In the Fourier component \( j = 0 \), the linear map is

\[ L(\Omega)(a_{0,k} \cos ks) = (k^2 + 2\omega)a_{0,k} \cos ks. \]

In the Fourier component \( j \in \mathbb{N}^+ \), the linear map is

\[ L(\Omega) \left( \begin{pmatrix} a_{j,k} \\ b_{j,k} \end{pmatrix} \cdot \begin{pmatrix} \cos jt \\ \sin jt \end{pmatrix} \cos ks \right) = M_{j,k} \begin{pmatrix} a_{j,k} \\ b_{j,k} \end{pmatrix} \cdot \begin{pmatrix} \cos jt \\ i\sin jt \end{pmatrix} \cos ks, \]

where \( M_{j,k} \) is the matrix

\[ M_{j,k}(\Omega) = \begin{pmatrix} k^2 + 2\omega & -\Omega j \\ -\Omega j & k^2 \end{pmatrix}. \]

The matrix \( M_{j,k} \) has eigenvalues

\[ \lambda_{j,k,\pm 1} = k^2 + \omega \pm \sqrt{j^2\Omega^2 + \omega^2}, \]

and normalized eigenvectors

\[ v_{j,k,\pm 1} = \frac{1}{c_{j,\pm 1}} \begin{pmatrix} \omega \pm \sqrt{j^2\Omega^2 + \omega^2} \\ -j\Omega \end{pmatrix}, \]

where

\[ c_{j,\pm 1} = \sqrt{2} \left( \omega^2 + j^2\Omega^2 \pm \omega \sqrt{j^2\Omega^2 + \omega^2} \right)^{1/2}. \]
Definition 10 For $(j,k,l) \in \mathbb{N}^+ \times \mathbb{N} \times \{1, -1\}$, we define the functions

$$e_{j,k,l} = c_k v_{j,k,l} \cdot \begin{pmatrix} \cos jt \\ i \sin jt \end{pmatrix} \cos ks,$$

where $c_0 = \sqrt{2}$ and $c_k = 2$ for $k \neq 0$. For $(j,k,l) \in \{0\} \times \mathbb{N} \times \{1\}$, we define $e_{0,0,1} = 1$ and $e_{0,k,1} = \sqrt{2} \cos ks$.

Since $L(\Omega)e_x = \lambda_x e_x$ with

$$x \in \Lambda = \mathbb{N}^+ \times \mathbb{N} \times \{1, -1\} \cup \{0\} \times \mathbb{N} \times \{1\},$$

then every function $u \in L^2_{\text{sym}}$ may be written as

$$u(t,s) = \sum_{x \in \Lambda} \langle u, e_x \rangle e_x.$$

Thus, the eigenfunctions $\{e_x\}_{x \in \Lambda}$ form a complete orthonormal set for the linear operator $L$, this is

$$L(\Omega)u = \sum_{x \in \Lambda} \lambda_x \langle u, e_x \rangle e_x.$$

For $u \in L^2_{\text{sym}}$, we define the norm

$$|u|_\sigma = \sum_{(j,k,l) \in \Lambda} \langle u, e_{j,k,l} \rangle^2 e^{2(j,k)}|\sigma \langle (j,k) \rangle|^{2s},$$

where $\langle (j,k) \rangle = \sqrt{1 + j^2 + k^2}$.

Remark 11 Even though the eigenfunction $e_x$ depends of $\Omega$, and the norm $|u|_\sigma$ is defined trough this system of eigenfunctions, the definition of the norm does not depend of $\Omega$ because the system $\{e_x\}_{x \in \Lambda}$ is orthonormal.

Lemma 12 Define the spaces $h_\sigma$ of analytic functions as

$$h_\sigma = \{u \in L^2_{\text{sym}}(T^2; \mathbb{C}) : |u|_\sigma < \infty\}.$$

The space $h_\sigma$ is an algebra for $s > n/2$,

$$|uv|_\sigma \leq c_{\sigma,s} |u|_\sigma |v|_\sigma.$$
Proof. Let \( u \in L^2(T^2, \mathbb{C}) \), then the norm
\[
\|u\|_\sigma = \sum_{(j,k) \in \mathbb{Z}^2} \left| \langle u, e^{i(jt+ks)} \rangle \right|^2 e^{2j(j,k)|\sigma} e^2| (j,k)|^2 s
\]
has the algebra property under the convolution for \( s > n/2 \), see [5]. Since \( \|uv\|_\sigma < c'_\sigma,s \|u\|_\sigma \|v\|_\sigma \), the result follows from the fact that the two norms \(|\cdot|_\sigma\) and \(\|\cdot\|_\sigma\) are equivalent in \( L^2_{sym} \),
\[
\frac{1}{16} \|u\|_\sigma < |u|_\sigma < \frac{1}{4} \|u\|_\sigma.
\]

Lemma 13 The nonlinear operator \( g(u, \bar{u}) \) satisfy
\[
|g(u, \bar{u})|_\sigma < c_\sigma |u|_\sigma.
\]
Thus, the map \( f \) is well defined in \( h_\sigma \) and continuous for \( |u|_\sigma < c^{-1}_\sigma \).

Proof. That the map is well defined follows from the equivariant property and the first statement. To prove first inequality, we use the algebraic property
\[
|g(u, \bar{u})|_\sigma = \left| \sum_{n=2}^\infty (-1)^n \bar{u}^n \right|_\sigma \leq \sum_{n=2}^\infty c_{\sigma,s}^{n-1} |u|^n_\sigma = \frac{c_{\sigma,s} |u|^2_\sigma}{1 - c_{\sigma,s} |u|_\sigma} < c_\sigma |u|_\sigma.
\]

The eigenvalue \( \lambda_{j,k,-1}(\Omega) \) is zero when the frequency is equal to
\[
\Omega_{j,k} = j^{-1} \sqrt{k^4 + 2k^2 \omega}.
\]
Without loss of generalization, see appendix, the bifurcation of periodic solutions can be analyzed only when the Fourier mode \( \lambda_{1,1,-1}(\Omega_0) = 0 \). This happens when
\[
\Omega_0 = \Omega_{11} = \sqrt{1 + 2 \omega}.
\]
In the next proposition we prove that there is no-resonances when \( \omega \) is irrational.

Proposition 14 For \( \omega \) irrational, the kernel of the map \( L(\Omega_0) \) has dimension one corresponding to \( e_{1,1,-1} \).
Proof. The eigenvalues of $M_{j,k}(\Omega_0)$ are $\lambda_{j,k,l}(\Omega_0)$ for $l = \pm 1$. The determinant of the matrix $M_{j,k}(\Omega_0)$ is

$$d_{j,k}(\Omega_0) = \lambda_{j,k,1}\lambda_{j,k,-1} = 2(k^2 - j^2)\omega + (k^4 - j^2).$$

Since the frequency $\Omega_0$ is chosen such that $M_{1,1}$ is not invertible, then $d_{1,1}(\Omega_0) = 0$. Since $\omega$ is an irrational number, then the determinant $d_{j,k}$ is zero only if both numbers $k^2 - j^2$ and $k^4 - j^2$ are zero. For $(j,k,l) \in \Lambda$, the condition happens only when $(j,k) = (1,1)$ or $(j,k) = (0,0)$. But the eigenvalue $\lambda_{0,0,1} = 2\omega$ is always positive, then $\lambda_{1,1,-1}(\Omega_0)$ is the only zero eigenvalue in $\Lambda$.

Remark 15 Notice that $\lambda_{0,0,-1}$ is always zero. Thus, using the subspace of symmetries, the problem of resonances does not exist in the lattice $\Lambda$.

Let us define $P_A$ as the projection in the Fourier components $x \in A \subset \Lambda$,

$$P_Au = \sum_{x \in A} \langle u, e_x \rangle e_x.$$ 

Let $N$ be the set of lattice point where the linear map $L(\Omega_0)$ is zero, then $N = \{(1,1,-1) \in \Lambda\}$. Thus, $P_{\Lambda\setminus N}$ is the projection on the one dimensional kernel, and $P_{\Lambda\setminus N}$ is the projection on the complement.

Define $u = v + w$ with

$$v = P_Nu \quad \text{and} \quad w = P_{\Lambda \setminus N}u,$$

then the zeros of $f$ are the solutions of

$$P_Nf(v + w; \Omega) = 0 \quad \text{and} \quad P_{\Lambda \setminus N}f(v + w; \Omega) = 0.$$

Let $r \in \mathbb{R}$ be a parametrization of the kernel of $f'(0; \Omega_0)$ given by

$$v(r) = re_{1,1,-1} \in L^2_{\text{sym}}(T^2; \mathbb{C}).$$

The plan is about using a Nash-Moser method to solve $w(r, \Omega)$ from the range equation $P_{\Lambda \setminus N}f(v(r) + w; \Omega) = 0$. Here, the function $w(r, \Omega)$ can be construct only in a cantor set of parameters close to $(0, \Omega_0)$.

For the first step, we need the following proposition
Lemma 16 For a Diophantine number $\omega$, if $\Omega$ is such that $|\Omega - \Omega_0| < c\gamma/L_0^{2r+1}$, then

$$|\lambda_{j,k,-1}(\Omega)| > c\gamma/L_0^{2r+2}$$

for all $(j, k)$ with $|(j, k)| < L_0$.

Proof. From the definition of the determinant of $M_{j,k}(\Omega_0)$, and $|q\omega - p| \geq \gamma/|q|^r$, then

$$|d_{j,k}(\Omega_0)| \geq \gamma/\left|2 (k^2 - j^2)\right|^r \geq c\gamma/L_0^{2r}.$$

Since $|\lambda_{j,k,1}| \leq cL_0^2$, then

$$|\lambda_{j,k,-1}(\Omega_0)| \geq c\gamma/L_0^{2r+2}.$$

The result follows from the inequalities

$$|\lambda_{j,k,-1}(\Omega)| \geq |\lambda_{j,k,-1}(\Omega_0)| - |\lambda_{j,k,-1}(\Omega) - \lambda_{j,k,-1}(\Omega_0)|$$

$$\gtrsim \gamma/L_0^{2r+2} - |j||\Omega - \Omega_0| \gtrsim \gamma/L_0^{2r+2}$$

A first approximation to the solution is constructed projecting in a big ball $B_0$ of radius $L_0$ in $\Lambda \setminus N$, where

$$B_0 = \{(j, k) \in \Lambda \setminus N : |j| + |k| < L_0\}.$$

Thus, by the implicit function theorem and the lemma above, there is a solution $w_0(r, \Omega)$ of $P_{B_0}f(v(r) + w_0, \Omega) = 0$.

Let $P_{B_n}$ be the projection in the ball

$$B_n = \{(j, k) \in \Lambda \setminus N : |j| + |k| < L_n\}$$

with $L_n = 2^n L_0$. Then the Nash-Moser method is about proving that $w_n$ is a better approximation at each step, where

$$w_n(r, \Omega) = w_{n-1}(r, \Omega) + \delta w_n,$$

and $\delta w_n$ is the correction given by

$$\delta w_n = -G_{B_n}P_{B_n}f(w_{n-1}(r, \Omega); r, \Omega) \quad \text{with}$$

$$G_{B_n} = (P_{B_n} \partial_w f(w_{n-1}(r, \Omega); r, \Omega) P_{B_n})^{-1}.$$  

To prove the convergence of the method, we need to find the estimate

$$\|G_{B_n}\|_{\sigma_n} < 1/d_n,$$  

(12)

where $\sigma_n = \sigma_{n-1} - \gamma_n$ and $(r, \Omega) \in N_n$.  

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Theorem 17 If $d_n = L_n^{-\beta}$ for $\beta > 3/2$ and $\sum_{n=1}^{\infty} \gamma_n \to \sigma_0/2$, from the estimate (12), see the result in [6], then $w_n(r, \Omega)$ converges to a solution $w(r, \Omega) \in h_{\sigma_0/2}$ of

$$P_{\Lambda \setminus \mathcal{N}} f(w(r, \Omega) + v(r), \Omega) = 0,$$

for $(r, \Omega)$ in $\mathcal{N} = \cap_{n=1}^{\infty} \mathcal{N}_n$.

3 Control of the inverse

The estimates presented here can be used to solve any small divisor problems when at each step of the iteration, the singular regions remain bounded. This is the case when one looks for periodic solutions in Hamiltonian PDE’s of dimension one. For the particular case of the $n$ vortex filament problem, the hypotheses are verified in the next section.

The linear map is given by

$$\partial_w f(v(r) + w; \Omega) = P_{\Lambda \setminus \mathcal{N}} (L(\Omega) + \omega \partial_w g(v + w)) P_{\Lambda \setminus \mathcal{N}},$$

and it is equivalent to the matrix

$$H = D(\Omega) + T(w; r, \Omega),$$

where $D$ is the diagonal matrix

$$D(\Omega) = \text{diag}_{x \in \Lambda \setminus \mathcal{N}} (\lambda_x(\Omega)),$$

and $T$ is the Toeplitz matrix

$$T(w; r, \Omega)(x, y) = \langle \omega \partial_w g(v(r) + w) e_y, e_x \rangle$$

for $x, y \in \Lambda \setminus \mathcal{N}$.

The estimate of the matrix

$$G_{B_n} = (P_{B_n} H P_{B_n})^{-1},$$

is done in the operator norm

$$\|G\|_\sigma := \max \left\{ \sup_x \sum_y |G(x, y)| e^{\sigma|x-y|} \langle x - y \rangle^s \right\}, \sup_y \sum_x |G(x, y)| e^{\sigma|x-y|} \langle x - y \rangle^s \right\}. $$
Definition 18 For a matrix $H$, we define

$$H^B_A = P_A HP_B.$$  

Notice that with this notation $(G_E)^B_A = P_A(P_E HP_E)^{-1}P_B$.

Definition 19 A site $x \in \Lambda$ is regular if $|\lambda_x| > d_0$. The subset of regular sites is denoted by $A \subset \Lambda$. A site is singular if $|\lambda_x| \leq d_0$. We define $S_n \subset \Lambda$ as the subset of all singular sites in the ring $B_n \setminus B_{n-1}$. Then, we have the disjoint union

$$\mathbb{Z}^2 = A \cup \bigcup_{n=1}^{\infty} S_n$$

These following hypothesis are necessary to control the norm of $G_{B_n}$.

(\textbf{h1}) The non-diagonal Toeplitz matrix $T$ has the exponential decay property

$$\|T(u)\|_\sigma \leq Cr_0$$

for $|u|_\sigma < r_0$.

(\textbf{h2}) The set of singular points in $B_n\setminus B_{n-1}$ is the union of bounded and separated singular regions, thus $S_n = \bigcup_j S_j$ with $\text{rad}(S_j) < c_0$ and for any region $S_j, S_i \subset S_n \cup S_{n-1}$, it holds that

$$\text{dist}(S_i, S_j) > 4l_n.$$  

(\textbf{h3}) Let $C(S_j)$ be a neighborhood of radius $l_n$ around $S_j$, then the spectrum of $H_{S_j}$ and $H_{C(S_j)}$ are bounded away from zero by $d_n$, 

$$\text{spec}(H_{S_j}), \text{ spec}(H_{C(S_j)}) \subset \mathbb{R}\setminus [-d_n, d_n].$$

Definition 20 A singular site $x \in S_n$ is tame if there exists a neighborhood $C_n(x)$ such that $\text{dist}(x, \Lambda \setminus C_n(x)) > l_n$, $\text{rad}(C_n(x)) \leq 2l_n$, and

$$\||G_{C_n(x)}||_\sigma < \frac{1}{d_n}.$$  

All singular sites which satisfy hypothesis (\textbf{h2}) and (\textbf{h3}) are tame with

$$C_n(S_j) = \{x \in \Lambda : \text{dist}(x, S_j) < l_n\}.$$  

We omit the indices $n$ and $j$ in the proof of this fact.
Lemma 21  From hypotheses (h1)-(h3), for \( cr_0/d_0 < 1/2 \) we have

\[
\|G_E\|_\sigma \leq \frac{2}{d_0}, \|G_S\|_\sigma < \frac{C}{d_n} \quad \text{and} \quad \|(G_{C(S)})^S_S\|_\sigma \leq \frac{C}{d_n},
\]

for any regular subset \( E \subset A \).

Proof. Since \( E \) is regular, \( \left\| (D_E^E)^{-1} T_E^E \right\|_\sigma \leq cr_0/d_0 < 1/2 \), then

\[
\|G_E\|_\sigma = \left\| (I + (D_E^E)^{-1} T_E^E)^{-1} \right\|_\sigma \leq \frac{2}{d_0}.
\]

Since \( H_{C_S} \) is self-adjoint, and hypothesis (h3) holds, then the \( l^2 \)-norm of the inverse of \( H_{C_S} \) is bounded by \( 1/d_n \), \( \|G_{C(S)}\|_0 \leq 1/d_n \). Since the set \( S \) has radius bounded by \( c_0 \), then

\[
\|(G_{C(S)})^S_S\|_\sigma \leq e^{\sigma c_0 c_S} \|(G_{C(S)})^S_S\|_0 < e^{\sigma c_0 c_S}/d_n
\]

The proof is similar for \( G_S \). □

Proposition 22  By hypothesis (h1) and the above lemma, then

\[
\|G_{C(S)}\|_\sigma < \frac{C}{d_n}.
\]

Proof. Since singular regions are separated, the set \( E = C(S) \setminus S \) is regular. From the self-adjoint property only two cases need to be verified, \( \|(G_{C(S)})^E_E\|_\sigma < C/d_n \) and \( \|(G_{C(S)})^E_E\|_\sigma < C/d_n \).

Let us define the connection matrix as

\[
\Gamma := H_{C(S)} - H_E \oplus H_S = T_E^S + T_S^E.
\]

From resolvent identities

\[
G_{C(S)} = G_E \oplus G_S - G_E \oplus G_S \Gamma G_{C(S)},
\]

then

\[
(G_{C(S)})^S_E = G_E T_E^S (G_{C(S)})^S_S.
\]

(13)

Thus, for the first inequality we have

\[
\|(G_{C(S)})^S_E\|_\sigma \leq C \frac{r_0}{d_n} \frac{1}{d_n}.
\]
From resolvent identities,

\[ G_{C(S)} = G_E \oplus G_S - G_E \oplus G_S \Gamma G_E \oplus G_S + G_E \oplus G_S \Gamma G_{C(S)} \Gamma G_E \oplus G_S, \]

then

\[ (G_{C(S)})_E^E = G_E + G_E T_E^S (G_{C(S)})_S^T S G_E. \] (14)

Thus, for the second inequality we have

\[ \| (G_{C(S)})_E^E \|_\sigma \leq 2 d_0 + C r^2 \frac{1}{d_0^2 d_n} < C \frac{1}{d_n}. \]

Using hypotheses (h1)-(h3), we have proven that all singular sites in \( S_n \) are tame with \( C_n(x) = C_n(S_j) \) for all \( x \in S_j \). To prove the main theorem of this section we need the following lemma.

**Lemma 23** Let \( P\{x\}G \) be the projection on the \( x \)-row of \( G \), then

\[ \| G \|_{\sigma - \gamma} \leq c_\gamma \sup_x \| P\{x\}G \|_\sigma, \]

where

\[ c_\gamma = c \sum_{x \in \mathbb{Z}_2} e^{-\gamma |x|} \lesssim \gamma^{-2}. \]

**Proof.** The result follows from the inequality

\[ \sup_x \sum_y |G(x, y)| e^{(\sigma - \gamma) |x-y|} \langle x - y \rangle^s \leq \sup_x \sum_y \| P\{x\}G \|_\sigma e^{-\gamma |x-y|} \leq c_\gamma \sup_x \| P\{x\}G \|_\sigma, \]

and from the analogous for the supremum over \( y \). ■

We defined recursively \( \sigma_n = \sigma_{n-1} - \gamma_n \). From conditions (h1)-(h3) we have the following estimate for the inverse.

**Theorem 24** If \( c r_0/d_0 < 1/4 \), and \( c_{\gamma_n} r_0 e^{-\gamma_n l_n}/d_n < 1/4 \) for all \( n \), then

\[ \| G_{E_n} \|_{\sigma_n} \lesssim c_{\gamma_n}/d_n \]

for any set \( E_n = B_n \cup A \) with \( A \) a regular subset.
Proof. We define $A_n$ as the set of regular sites

$$A_n = E_n \setminus (B_{n-1} \cup S_n).$$

Thus, at each step, the set $E_n$ is the union of the ball $B_{n-1}$, the regular subset $A_n$, and the set of tame singular sites $S_n$ in $B_n \setminus B_{n-1}$.

Let $C_n(B_{n-1}) = \{x : \text{dist}(x, B_{n-1}) \leq l_n\}$, let us define the matrix $L_n$ as

$$L_n = G_{A_n} + P_{B_{n-1}} G_{C_n(B_{n-1})} + \sum_{x \in S_n} P_{\{x\}} G_{C_n(x)}.$$

Thus, we have that

$$L_n (H_{E_n}^{E_n}) = G_{A_n} (H_{A_n}^{A_n} + H_{E_n \setminus A_n}^{E_n \setminus A_n}) + P_{B_{n-1}} G_{C_n(B_{n-1})} (H_{C_n(B_{n-1})}^{C_n(B_{n-1})} + H_{E_n \setminus C_n(B_{n-1})}^{E_n \setminus C_n(B_{n-1})})$$

$$+ \sum_{x \in S_n} P_{\{x\}} G_{C_n(x)} (H_{C_n(x)}^{C_n(x)} + H_{E_n \setminus C_n(x)}^{E_n \setminus C_n(x)})$$

$$= I_{B_n} + \Gamma_n,$$

where

$$\Gamma_n = G_{A_n} H_{A_n}^{E_n \setminus A_n} + P_{B_{n-1}} G_{C_n(B_{n-1})} H_{C_n(B_{n-1})}^{E_n \setminus C_n(B_{n-1})} + \sum_{x \in S_n} P_{\{x\}} G_{C_n(x)} H_{C_n(x)}^{E_n \setminus C_n(x)}.$$

We conclude that

$$G_{E_n} = (H_{E_n}^{E_n})^{-1} = (I_{E_n} + \Gamma_n)^{-1} L_n.$$

Since $C_n(B_{n-1}) \subset B_{n-1} \cup A$ by the separation property (h2), then from the next lemma we have that

$$\|G_{C_n(B_{n-1})}\|_{\sigma_{n-1}} \lesssim c_{\gamma_n} / d_{n-1}. \tag{15}$$

Thus, we have that

$$\|L_n\|_{\sigma_n} \leq 2/d_0 + c_{\gamma_n} / d_{n-1} + c_\gamma \sup_{x \in S_n} \|G_{C_n(x)}\|_{\sigma_{n-1}} \lesssim c_\gamma / d_n.$$

Thus, if we prove the inequality $\|\Gamma_n\|_{\sigma_n} \leq 3/4$, we conclude that $\|G_{E_n}\|_{\sigma_n} \lesssim c_\gamma / d_n$.  

17
Since \( A_n \) is regular, then \( \left\| G_{A_n} H_{A_n}^{E_n \setminus A_n} \right\|_{\sigma_n} \leq c r_0 / d_0 < 1/4 \). We have for \( x \in S_n \) that \( \left\| G_{C_n(x)} \right\|_{\sigma_n} \lesssim 1/d_n \), then

\[
\left\| P_x G_{C_n(x)} H_{C_n(x)}^{E_n \setminus C_n(x)} \right\|_{\sigma_n} \lesssim \left\| G_{C_n(x)} H_{C_n(x)}^{E_n \setminus C_n(x)} \right\|_{\sigma_{n-1}} e^{-\gamma n l_n} \lesssim \frac{1}{d_n} r_0 e^{-\gamma n l_n}.
\]

In a similar way, from (15) we have that

\[
\left\| P_{B_{n-1}} G_{C_n(B_{n-1})} H_{C_n(B_{n-1})}^{E_n \setminus C_n(B_{n-1})} \right\|_{\sigma_n} \lesssim \left\| G_{C_n(B_{n-1})} H_{C_n(B_{n-1})}^{E_n \setminus C_n(B_{n-1})} \right\|_{\sigma_{n-1}} e^{-\gamma n l_n} \lesssim \frac{c_{\gamma_{n-1}}}{d_{n-1}} r_0 e^{-\gamma n l_n}.
\]

Thus, we conclude that

\[
\left\| \Gamma_n \right\|_{\sigma_{n-1} - 2\gamma_n} \leq \frac{1}{4} + \frac{c_{\gamma_{n-1}}}{d_{n-1}} r_0 e^{-\gamma n l_n} + \frac{c_{\gamma_n}}{d_n} r_0 e^{-\gamma n l_n} \leq \frac{3}{4}.
\]

Lemma 25 For any for any set \( E_{n-1} = B_{n-1} \cup A \) with \( A \) regular we have that

\[
\left\| G_{E_{n-1}}(w_n) \right\|_{\sigma_{n-1}} \lesssim \frac{c_{\gamma_{n-1}}}{d_{n-1}}.
\]

Proof. We assume from previous step that

\[
\left\| G_{E_{n-1}}(w_{n-1}) \right\|_{\sigma_{n-1}} \lesssim \frac{c_{\gamma_{n-1}}}{d_{n-1}}.
\]

The difference of the Hamiltonians is

\[
R_{n-1}(v) = H_{E_{n-1}}(w_n) - H_{E_{n-1}}(w_{n-1}) = T_{E_{n-1}}(w_n) - T_{E_{n-1}}(w_{n-1}).
\]

From Taylor’s theorem, since \( \delta w_n = w_n - w_{n-1} \), we have

\[
R_{n-1}(v) = T'_{E_{n-1}}(w_{n-1}; v)[\theta(\delta w_n)].
\]

Since \( \left\| T'(w_{n-1}; v) \right\|_{\sigma_{n-1}} \lesssim \left\| w_{n-1} \right\|_{\sigma_{n-1}} \left| v \right|^{2} \) with \( w_{n-1} \) and \( \left| v \right| \) bounded for all \( n \), then

\[
\left\| R_{n-1}(v) \right\|_{\sigma_{n-1}} \lesssim \left\| \delta w_n \right\|_{\sigma_{n-1}} \lesssim \varepsilon_{n-1}.
\]

By inductive hypothesis, we have that

\[
\left\| G_{E_{n-1}}(w_{n-1}; v) R_{n-1} \right\|_{\sigma_{n-1}} \lesssim \frac{\varepsilon_{n-1}}{d_{n-1}} \leq 1/2,
\]

the result follows from

\[
G_{E_{n-1}}(w_n; v) = G_{E_{n-1}}(w_{n-1}; v)(I_{E_{n-1}} + G_{E_{n-1}} R_{n-1})^{-1}.
\]
4 Verification of hypothesis

In this section we prove the exponential decay of the Toeplitz matrix $T$. Then, the separation property of the singular regions is discussed. Finally, the estimate of the spectrum of the Hamiltonians in the singular regions is proven for parameters $(r, \Omega)$ in a subset $N_n$.

4.1 (h1) Exponential decay

Lemma 26 If $|u|_\sigma < r_0$, then $|T(u)|_\sigma < Cr_0$.

Proof. Let $u$ be a function with $|u|_\sigma < r_0$, by the algebra property of the norm we have

$$h(t, s) = \omega \sum_{n=2}^{\infty} n (-1)^n u^{n-1}. $$

Since the function satisfy $|h|_\sigma < Cr_0$, then by definition

$$|\langle h, e_{j,k,l} \rangle| < Cr_0 e^{-\sigma |(j,k)|} (j,k)^{-s}. $$

Let $x_n = (j_n, k_n, l_n) \in \Lambda$, then

$$T(x_1, x_2) = \langle he_{x_2}, e_{x_1} \rangle = \sum_{x_3 \in \Lambda} \langle h, e_{x_3} \rangle \langle e_{x_2} e_{x_3}, e_{x_1} \rangle$$

Since $\langle e_{x_2} e_{x_3}, e_{x_1} \rangle = 0$ when $j_3 \notin \{ \pm j_1 \pm j_2 \}$ or $k_3 \notin \{ \pm k_1 \pm k_2 \}$, and since $|\langle e_{x_2} e_{x_3}, e_{x_1} \rangle| \leq 1$, then

$$|T(x_1, x_2)| \leq \sum_{l_3=\pm 1, j_3 \in \{ \pm j_1 \pm j_2 \}, k_3 \in \{ \pm k_1 \pm k_2 \}} |\langle h, e_{j_3,k_3,l_3} \rangle| \leq 2 \sum_{j_3 \in \{ \pm j_1 \pm j_2 \}, k_3 \in \{ \pm k_1 \pm k_2 \}} Cr_0 e^{-\sigma |(j_3,k_3)|} (j_3,k_3)^{-s}. $$

There are only four elements in the sum, and they satisfy $j_3 \geq |j_1 - j_2|$ and $k_3 \geq |k_1 - k_2|$, then

$$|T(x_1, x_2)| \leq 8Cr_0 e^{-\sigma |(j_1-j_2)| (j_1-k_2)|} (|j_1 - j_2|, |k_1 - k_2|)^{-s}. $$
4.2 (h2) Separation property

We say that \((j, k, l) \in \Lambda\) is a singular site if \(|\lambda_{j,k,l}(\Omega)| \leq d_0\).

**Lemma 27** Let \((j_1, k_1)\) and \((j_2, k_2)\) be two different singular sites, then for sufficiently small \(d_0\), we have that

\[ |j_1 - j_2| \geq C |k_1 + k_2|, \]

where \(C\) is a constant that depends on \(\Omega\) and \(\omega\). Furthermore, the constant is absolute for \((\Omega, \omega)\) in neighborhood of \((\Omega_0, \omega_0)\).

**Proof.** The sites of the form \((j, k, 1)\) are never singular if \(d_0 \ll 1\). Given that \(\lambda_{j,k,-1} = k^2 + \omega - \sqrt{(j\Omega)^2 + \omega^2}\), then

\[ |k_1^2 - k_2^2| - C\Omega |j_1 - j_2| \leq |\lambda_{j_1,k_1,-1} - \lambda_{j_2,k_2,-1}| \leq 2d_0. \]

If \(k_1 = k_2\), taking \(d_0\) small enough such that \(d_0 \leq C\Omega/2\), then \(j_1 = j_2\).

Finally, if \(k_1 \neq k_2\), there exists a constant \(c\) such that

\[ |j_1 - j_2| \geq \frac{1}{C\Omega} \left( |k_1^2 - k_2^2| - 2d_0 \right) \geq C |k_1 + k_2|. \]

Let \(S = \{(j_0, k_0, -1)\}\) be a singular site in the ball \(B_{n+1}/B_n\), then by the previous lemma, the neighborhood

\[ C_n(S) = \{(j, k, l) : |(j, k) - (j_0, k_0)| < l_n\} \]

contains only one singular site for \(l_n = CL_n^{1/2}\). ■

4.3 (h3) Good parameters

In this section the spectrum of the local Hamiltonians are analyzed

\[ H_C(w_n; r, \Omega) = P_C(D(\Omega) + T(v(r) + w_n(r, \Omega); \Omega))P_C \]

for \(C = S\) and \(C = C(S)\).

For \(|r| \leq r_0\), there is only one eigenvalue of \(H_C(w_n, \Omega)\) with norm less than \(d_0/2\) if \(r_0 \ll d_0/2\). Let \(e(r, \Omega)\) be that eigenvalue of \(H_{C(S)}\) with

\[ |e(r, \Omega)| < d_n \ll d_0/2, \]

then \(e(r, \Omega)\) is isolated from the other eigenvalues, and then analytic.
Lemma 28 For $|r| < r_0$, there exists a constant $C > 0$ such that
\[ \partial_\Omega e(r, \Omega) \leq -CL_n. \]

Proof. Since $e(r, \Omega)$ is analytic, then
\[ \partial_\Omega e(r, \Omega) = \partial_\Omega e(0, \Omega) + O(r) = \partial_\Omega \lambda_{j,k,-1} + O(r). \]

By an explicitly calculation
\[ \partial_\Omega \lambda_{j,k,-1} = \frac{-j^2 \Omega}{\sqrt{j^2 \Omega^2 + \omega^2}} \geq -C\Omega L_n, \]
because the site $(j, k)$ is singular with the estimate $|j| \geq C\Omega L_n$.

From the above lemma, for a fix $r < r_0$, the eigenvalue $e(r, \Omega)$ is a monotone decreasing function of $\Omega$. Since $e(r, \Omega)$ is analytic, then there is an unique analytic function $\Omega_z(r)$ such that $e(r, \Omega_z(r)) = 0$. For $r = 0$, $e(0, \Omega) = \lambda_{j,k,-1}(\Omega)$, thus
\[ \Omega_z(0) = \Omega_{j,k} = \frac{1}{j} \sqrt{k^4 + 2k^2 \omega} \]
and $(j, k, -1) \in \mathbb{S}$. ■

Lemma 29 We have that
\[ \Omega_z(r) = \Omega_{j,k} + \frac{1}{L_n} O(r^2). \]

Proof. Since $e(r, \Omega)$ is analytic then
\[ e(r, \Omega) = e(0, \Omega_{j,k}) + \partial_\Omega e(0, \Omega_{j,k})(\Omega - \Omega_{j,k}) + \partial_r e(0, \Omega_{j,k})r + h.o.t \]

By the Feynman-Hellman formula,
\[ \partial_r e(0, \Omega_{j,k}) = \langle T'(0)[\partial_r v, \psi_0], \psi_0 \rangle. \]

In the space of functions $L^2_{sym}$, the functions are $\psi_0 = e_{j,k,-1}$, $\partial_r v = e_{1,1,-1}$, and
\[ d^2 g(0)[w_1, w_2] = \partial_{w_1}^2 g(0) \bar{w}_1 \bar{w}_2 = 2 \bar{w}_1 \bar{w}_2. \]
Thus, for any \( j, k \neq 1/2 \), we have that
\[
\langle \psi_0, T'(0)[\partial_r v, \psi_0] \rangle = \langle 2\omega \bar{e}_{1,1,-1} \bar{e}_{j,k,-1}, e_{j,k,-1} \rangle = 0.
\]

Since \( \partial_2 c(0, \Omega_{j,k}) < -CL_n \), then
\[
\partial_2 c(0, \Omega_{j,k}) (\Omega_z - \Omega_{j,k}) + h.o.t. = 0
\]
Using the implicit function theorem, we have that \( \Omega_z - \Omega_{j,k} \) is a function of \( r \), and
\[
\Omega_z - \Omega_{j,k} = \frac{1}{L_n} O(r^2).
\]

Let \( N_{j,k} \) be the neighborhood of the curve \( \Omega_z(r) \) given by
\[
N_{j,k} = \{(r, \Omega) : |\Omega_z(r) - \Omega| < C \frac{d_n}{L_n} \},
\]
by previous lemma and the mean value theorem, the eigenvalue satisfy \(|e(r, \Omega)| > d_n\) if \((r, \Omega) \notin N_{j,k}\). Thus, the hypothesis (h3) is true for the parameters in
\[
\mathcal{N}_n = \bigcup_{(j,k) \in S_n} N_{j,k},
\]
where the union is over all singular sites in \( B_n \backslash B_{n-1} \).

5 Intersection property

In this section we prove that the bifurcation curve of parameter corresponding to the standing waves is given by
\[
\mathcal{C} = \{(r, \Omega(r)) : \Omega(r) = \Omega_0 + \Omega_2 r^2 + O(r^3) \},
\]
where \( \Omega_2 \neq 0 \). First, we prove that the intersection of \( \mathcal{C} \) with the Cantor set \( \mathcal{N} = \cap_{n=1}^{\infty} \mathcal{N}_n \) has measure when \( \Omega_2 \neq 0 \). Later, the non-degeneracy of the curve \( \mathcal{C} \) is proven.
5.1 Measure of good parameters

Lemma 30 Let \( r_- \) and \( r_+ \) bet the minimum and the maximum of \( \{ r : (r, \Omega) \in C \cap N_{j,k} \} \), we have that

\[
|r_-^2 - r_+^2| \leq \frac{C}{\Omega_2 L_n} d_n.
\]

Proof. Let \( r_0 \) be such that \( \Omega_z(r_0) = \Omega(r_0) \), then the point \( (r_0, \Omega(r_0)) \) is the intersection of \( \Omega_z(r) \) and \( \Omega(r) \). Since \( \Omega(r) = \Omega_0 + \Omega_2 r^2 + O(r^3) \), then

\[
|\Omega(r_-) - \Omega(r_0)| \geq \frac{\Omega_2}{2} |r_-^2 - r_0^2|.
\]

By the previous lemma

\[
|\Omega_z(r_0) - \Omega_z(r_-)| \leq \frac{C}{L_n} |r_-^2 - r_0^2|.
\]

Since \( \Omega(r_-), \Omega_z(r_-) \in N_{j,k} \), then

\[
\frac{Cd_n}{L_n} \geq |\Omega(r_-) - \Omega_z(r_-)| \geq |\Omega(r_-) - \Omega_z(r_0)| - |\Omega_z(r_0) - \Omega_z(r_-)|
\]

\[
\geq \frac{\Omega_2}{2} |r_-^2 - r_0^2| - \frac{C}{L_n} |r_-^2 - r_0^2| \geq \frac{\Omega_2}{4} |r_-^2 - r_0^2|.
\]

Analogously, for the estimate of \( r_+ \) we have that \( |r_+^2 - r_0^2| \leq \frac{Cd_n}{\Omega_2 L_n} \). The lemma follows from the triangle inequality.

Lemma 31 If \( \Omega_2 \neq 0 \), the measure of the set \( \{ r : (r, \Omega) \in C \cap N_{j,k} \} \) is bounded by

\[
|\{ r : (r, \Omega) \in C \cap N_{j,k} \}| < \frac{C}{\sqrt{\Omega_2} \sqrt{L_n}} d_n.
\]

Proof. For a singular site \( \Omega_0 \) \( |j| \sim k^2 \), then

\[
\Omega_z(0) = \Omega_{j,k} = |k/j| \sqrt{k^2 + 2\omega} \sim \Omega_0 \sqrt{1 + 2\omega k^{-2}}.
\]

Thus,

\[
|\Omega_0 - \Omega_z(0)| = \Omega_0 \left| \frac{-2\omega k^{-2}}{1 + \sqrt{1 + 2\omega k^{-2}}} \right| \geq \Omega_0 \left| 2\omega k^{-2} \right| \geq \frac{C}{L_n}.
\]
By the definition of \(r_-, |\Omega_z(r_-) - \Omega(r_-)| < Cd_n/L_n\), and by the properties of \(\Omega_z\), \(|\Omega_z(0) - \Omega_z(r_-)| < Cr^2/L_n\), then

\[
\frac{C}{L_n} r_-^2 + \frac{C d_n}{L_n} > |\Omega(r_-) - \Omega_z(0)|. 
\]

Since the curve \(C\) is of the form \(\Omega(r) = \Omega_0 + \Omega_2 r^2 + O(r^3)\), then

\[
|\Omega(r_-) - \Omega_z(0)| > |\Omega_0 - \Omega_z(0)| - 2\Omega_2 r_-^2.
\]

Thus,

\[
\left(2\Omega_2 + \frac{C}{L_n}\right) r_-^2 > |\Omega_0 - \Omega_z(0)| - \frac{C d_n}{L_n} > \frac{C}{L_n}(1 - Cd_n) > \frac{C}{2L_n}.
\]

For \(\Omega_2 \neq 0\), then \(r_+ > r_- > \frac{C}{\sqrt{\Omega_2 L_n}}\). By the above lemma,

\[
r_+ - r_- \leq \frac{C d_n}{\Omega_2 L_n} \frac{C \sqrt{\Omega_2 L_n}}{L_n} \leq \frac{C d_n}{\sqrt{\Omega_2 L_n}}.
\]

**Proposition 32** Let \(d_n = r_0 \gamma L_n^{-\beta}\), then the measure of good parameters is

\[
|\{r \in [0, r_0) : (r, \Omega) \in \mathcal{N} \cap \mathcal{C}\}| > r_0(1 - \gamma C_\beta),
\]

where

\[
C_\beta = \frac{C}{\sqrt{\Omega_2}} \sum_{n=1}^{\infty} L_n^{3/2-\beta} < \infty
\]

for \(\Omega_2 \neq 0\) and \(\beta > 3/2\).

**Proof.** Since there are at most \(cL_n^2\) singular sites at the \(n\) step, then by the previous lemma

\[
|\{r \in [0, r_0) : (r, \Omega(r)) \notin \mathcal{N}_n \cap \mathcal{C}\}| \leq r_0 L_n^2 \frac{C}{\sqrt{\Omega_2}} \frac{d_n}{\sqrt{L_n}}.
\]

Thus

\[
|\{r \in [0, r_0) : (r, \Omega) \in \mathcal{N} \cap \mathcal{C}\}| = r_0 - r_0 \gamma C \sum_{n=1}^{\infty} L_n^{3/2} d_n \leq r_0(1 - C_\beta \gamma).
\]

\[\Box\]
5.2 Curvature of the bifurcation branch

In this section we want to prove that $\Omega_2 \neq 0$, using the expansion

$$u = ru_1 + r^2u_2 + h.o.t. \text{ and }$$

$$\Omega(r) = \Omega_0 + r\Omega_1 + r^2\Omega_2 + h.o.t.,$$

in

$$f(w; \Omega) = Lu + i(\Omega(r) - \Omega_0)u_t + \omega\bar{u}^2 - \omega\bar{u}^3 + h.o.t. = 0,$$

where $L = L(\Omega_0)$ is the linear map.

At order $r$, we have that $Lu_1 = 0$, and then $u_1 = e_{1,1,-1}(\Omega_0)$. At order $r^2$ we have

$$Lu_2 - i\Omega_1 \partial_t u_1 + \omega\bar{u}_1^2 = 0.$$

Multiplying by $u_1$, integrating by parts, and using that $L$ is self-adjoint with $Lu_1 = 0$, we have that

$$\Omega_1 \langle i\partial_t u_1, u_1 \rangle = \omega \langle \bar{u}_1^2, u_1 \rangle.$$

The basis $e_{j,k,l}$ at $\Omega = \Omega_0$ is given by $e_{0,0,1} = 1$,

$$e_{0,2,1} = \sqrt{2}\cos 2s,$$
$$e_{1,1,-1} = 2(a\cos t + ib\sin t)\cos s,$$
$$e_{2,0,\pm1} = \sqrt{2}(a_{\pm}\cos 2t + ib_{\pm}\sin 2t),$$
$$e_{2,2,\pm1} = 2(a_{\pm}\cos 2t + ib_{\pm}\sin 2t)\cos 2s,$$

where $(a, b)^T = v_{1,1,-1}$ and $(a_{\pm}, b_{\pm})^T = v_{2,0,\pm1}$. Since

$$\langle i\partial_t u_1, u_1 \rangle = \frac{1}{4\pi^2} \int_{T^2} i\partial_t u_1 \bar{u}_1 dt ds = -2ab,$$

then

$$\langle \bar{u}_1^2, u_1 \rangle = \frac{1}{4\pi^2} \int_{T^2} e_{1,1,-1}^3 dt ds = 0. \quad (17)$$

Since $\Omega_1 = 0$ and $u_2 = -\omega L^{-1}\bar{u}_1^2$, then

$$\bar{u}_2 = -\omega L^{-1}u_1^2.$$
At order $r^3$, we have that
\[ Lu_3 - i\Omega_2 \partial_t u_1 + 2\omega \bar{u}_1 u_2 - \omega \bar{u}_1^3 = 0. \]

Multiplying by $u_1$ and integrating by parts, then
\[ -\Omega_2 \langle i\partial_t u, u_1 \rangle = \omega \langle 2\omega \bar{u}_1 L^{-1}(u_1^2) + \bar{u}_1^3, u_1 \rangle. \]

Thus, we have that
\[ \Omega_2 = \frac{\omega}{2ab} \left( \langle \bar{u}_1^3, u_1 \rangle + 2\omega \langle L^{-1}u_1^2, u_1^2 \rangle \right). \]

To calculate the first product use that $\int_0^{2\pi} \cos^4 \theta d\theta = 3\pi/4$ and $\int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta = \pi/4$, then
\[ \langle \bar{u}_1^3, u_1 \rangle = \frac{4}{\pi^2} \left( a^4 \frac{3\pi}{4} - 6a^2b^2 \frac{3\pi}{4} + b^4 \frac{3\pi}{4} \right) \frac{3\pi}{4} \]
\[ = \frac{9}{4} (a^4 - 2a^2b^2 + b^4) = \frac{9}{4} (a^2 - b^2)^2 \]

Since
\[ \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2} \sqrt{1 + \omega}} \begin{pmatrix} 1 \\ \sqrt{1 + 2\omega} \end{pmatrix}, \]
then
\[ (a^2 - b^2)^2 = \frac{\omega^2}{(1 + \omega)^2} \quad \text{and} \quad 2ab = \frac{\sqrt{1 + 2\omega}}{1 + \omega}. \]  \hspace{1cm} (18)

We conclude from the next proposition that
\[ \Omega_2 = \frac{1}{6} \frac{\omega^2}{(\omega + 1)(\omega + 2) \sqrt{2\omega + 1}} (4\omega^3 + 29\omega^2 + 33\omega - 6). \]

Since $\Omega_2 = 0$ at only one point $\omega_0 > 0$, excluding $\omega_0$ the curve has the property for the intersection with the Cantor set $\mathcal{N}$.

**Proposition 33** We have that
\[ \langle L^{-1}u_1^2, u_1^2 \rangle = \frac{1}{24 (\omega + 1)^2 (\omega + 2)} (8\omega^3 + 31\omega^2 + 12\omega - 12). \]
Proof. To calculate \( \langle L^{-1}u_1^2, u_1^2 \rangle \), we use the expression for \( u_1^2 \) given by
\[
u_1^2 = e_{1,1,-1}^2 = (a^2 - b^2 + \cos 2t + i2ab \sin 2t)(1 + \cos 2s).
\]

Projecting the vector \((1, 2ab)\) in the orthonormal components \((a_+, b_+)\) and \((a_-, b_-)\), we have that
\[
u_1^2 = (a_2^2 - b_2^2) \left( e_{0,0,1}^2 + \frac{1}{\sqrt{2}} e_{0,2,1} \right)
\]
\[
+ (a_+ + 2abb_+) \left( \frac{1}{\sqrt{2}} e_{2,0,1} + \frac{1}{2} e_{2,2,1} \right)
\]
\[
+ (a_- + 2abb_-) \left( \frac{1}{\sqrt{2}} e_{2,0,-1} + \frac{1}{2} e_{2,2,-1} \right).
\]

Thus, we have that
\[
\langle L^{-1}u_1^2, u_1^2 \rangle = (a^2 - b^2)^2 \left( \lambda_{0,0,1}^{-1} + \frac{1}{2} \lambda_{0,2,1}^{-1} \right) + P
\]
(19)
where \( P \) is the polynomial
\[
P = (a_+ + 2abb_+)^2 \left( \frac{1}{2} \lambda_{2,0,1}^{-1} + \frac{1}{4} \lambda_{2,2,1}^{-1} \right)
\]
\[
+ (a_- + 2abb_-)^2 \left( \frac{1}{2} \lambda_{2,0,-1}^{-1} + \frac{1}{4} \lambda_{2,2,-1}^{-1} \right).
\]

For the first term we have that
\[
\left( \lambda_{0,0,1}^{-1} + \frac{1}{2} \lambda_{0,2,1}^{-1} \right) (a^2 - b^2)^2 = \frac{1}{4} \frac{\omega}{(\omega + 1)^2 (\omega + 2)} (3\omega + 4).
\]

To calculate \( P \), define the polynomial
\[
Q = \sqrt{\omega^2 + 8\omega + 4},
\]
then \( a_\pm \) and \( b_\pm \) are given by
\[
\left( \begin{array}{c}
  a_+ \\
  b_+
\end{array} \right) = \frac{1}{\sqrt{2} \sqrt{Q^2 + \omega Q}} \left( \begin{array}{c}
  \omega \pm Q \\
  -2\sqrt{1 + 2\omega}
\end{array} \right),
\]
and the eigenvalues are \( \lambda_{0,0,1} = 2\omega \), \( \lambda_{0,2,1} = 2(\omega + 2) \), \( \lambda_{2,0,\pm} = \omega \pm Q \) and \( \lambda_{2,2,\pm} = \omega \pm Q + 4 \).
Thus, we have that
\[
\frac{1}{2} \lambda_{2,0}^{-1} + \frac{1}{4} \lambda_{2,1}^{-1} = \frac{1}{48} \frac{1}{2\omega + 1} (2\omega^2 + 3\omega + 4 \pm (5 - 2\omega)Q),
\]
and
\[
(a_{\pm} + 2\omega b_{\pm})^2 = \frac{1}{2(Q^2 \pm \omega Q)} \left( \omega \pm Q - 2 \frac{1 + 2\omega}{1 + \omega} \right)^2.
\]
After computations with maple, we have that
\[
P = \frac{1}{24} \left( \frac{1}{\omega + 1} \right)^2 (8\omega^2 - 3\omega - 6).
\]

6 Appendix: Symmetries

Periodic solutions of equation (8) are zeros of the map
\[
f(v) = -i\Omega \partial_t v - \partial_{ss} v + \omega (1 - |v|^{-2}) v.
\]
The map \( f \) is \( T^3 \)-equivariant with the action of \((\varphi_\theta, \varphi_s, \varphi_t) \in [0, 2\pi)^3 \) given by
\[
\rho(\varphi_\theta, \varphi_s, \varphi_t)v = e^{i\varphi_\theta} v(t + \varphi_t, s + \varphi_s).
\]
In addition, the map is equivariant by the actions
\[
\rho(\kappa_s)v(t, s) = v(t, -s) \quad \text{and} \quad \rho(\kappa_t)v(t, s) = \bar{v}(-t, s).
\]
Since
\[
\rho(\varphi_\theta, \varphi_s, \varphi_t)\rho(\kappa_s) = \rho(\kappa_s)\rho(\varphi_\theta, -\varphi_s, \varphi_t),
\]
\[
\rho(\varphi_\theta, \varphi_s, \varphi_t)\rho(\kappa_t) = \rho(\kappa_t)\rho(-\varphi_\theta, \varphi_s, -\varphi_t),
\]
then the map \( f \) is equivariant by the action of the non-abelian group
\[
\Gamma = O(2) \times (T^2 \cup \kappa_t T^2).
\]
The equilibrium \( v_0 = 1 \) is fixed by the actions of \( \kappa_s, \kappa_t \) and \((0, \varphi_s, \varphi_t) \in T^3 \), then the isotropy group of \( v_0 \) is
\[
\Gamma_{v_0} = O(2) \times (S^1 \cup \kappa_t S^1).
\]
The orbit of \( v_0 \) has dimension one, with
\[
\Gamma v_0 = \{ e^{i\varphi} : \varphi \in S_1 \}.
\]
Isotropy groups

The linear map \( f'(1) \) in coordinates \( u = (v, \bar{v}) \in \mathbb{C}^2 \) is given by

\[
L(\Omega)u = \begin{pmatrix} -i\Omega \partial_t - \partial_{ss} + \omega & \omega \\ \omega & -i\Omega \partial_t - \partial_{ss} + \omega \end{pmatrix} u.
\]

The linear map \( L \) in Fourier components is

\[
L(\Omega)u = \sum_{(j,k) \in \mathbb{Z}^2} M_{j,k}(\Omega) u_{j,k} e^{i(jt + ks)},
\]

where the matrix

\[
M_{j,k}(\Omega) = \begin{pmatrix} \Omega j + k^2 + \omega & \omega \\ \omega & -\Omega j + k^2 + \omega \end{pmatrix}
\]

has eigenvalues \( \lambda_{j,k,\pm1}(\Omega) \) as before.

For \( j = 1 \), the eigenvalue \( \lambda_{1,k,-1}(\Omega) \) is zero at

\[
\Omega_0 = |k| \sqrt{k^2 + 2\omega}.
\]

Let \((a, b)\) be the eigenvector of \( M_{1,k}(\Omega_0) \) corresponding to the eigenvalue \( \lambda_{1,k,-1} \). Thus, the function

\[
v(z_1, z_2) = z_1ae^{i(ks + t)} + \bar{z}_1be^{-i(k(s + t)} + z_2ae^{i(-ks + t)} + \bar{z}_2be^{-i(-ks + t)}
\]

is in the kernel of \( f'(1; \Omega_k) \) for \((z_1, z_2) \in \mathbb{C}^2\).

The action of \( \Gamma_{u_0} \) in the parametrization of the kernel \((z_1, z_2)\) is given by

\[
\rho(\varphi_s, \varphi_t)(z_1, z_2) = e^{i\varphi_t}(e^{ik\varphi_s}z_1, e^{-ik\varphi_s}z_2)
\]

\[
\rho(\kappa_s)(z_1, z_2) = (z_2, z_1), \quad \text{and} \quad \rho(\kappa_t)(z_1, z_2) = (\bar{z}_2, \bar{z}_1).
\]

The isotropy groups of the action is inherited in the bifurcating solutions. Using the elements \( \kappa_s \) and \( \varphi_t \in S^1 \), we may assume without loss of generalization that the first coordinate is real and positive \( z_1 = a \in \mathbb{R} \), unless the two coordinates are zero.

\((a, 0)\) If \( z_2 = 0 \), then the isotropy group is generated by

\((0, -\varphi/k, \varphi)\) and \( \kappa_s\kappa_t \).

This isotropy group \( \Gamma_{(a,0)} \) is isomorphic to \( \tilde{S}_1 \cup (\kappa_s\kappa_t)\tilde{S}_1 \), and the orbit of \((a, 0)\) contains a 2-torus.
If \( z_2 \neq 0 \), using the action of \((0, -\varphi/k, \varphi)\) that fixes the first coordinate and acts multiplying by \( e^{2\varphi} \) in the second, we may assume that \( z_2 \) is real.

(a,a) If \( z_2 = z_1 \), then the isotropy group of \((a, a)\) is generated by
\[
(0, -\pi/k, \pi), \kappa_s \text{ and } \kappa_t.
\]
This isotropic group \( \Gamma_{(a,a)} \) is finite, and their orbit contains a 3-torus.
In other cases, the isotropy group is generated by \((0, -\pi/k, \pi)\).

(a,0) Traveling waves

Functions in the fixed point space of \( \Gamma_{(a,0)} \) satisfy
\[
v(t, s) = \rho(0, -\varphi/k, \varphi)v(t, s) = v(t + \varphi, s - \varphi/k),
\]
thus they are of the form
\[
v(t, s) = \sum_{l \in \mathbb{Z}} v_{l, lk} e^{i(lt + lks)}.
\]
In this case, all the non-zero Fourier components lie in a line on the lattice. Moreover, from \( v(t, s) = \bar{v}(-t, -s) \), we have that \( uv_{l, lk} \in \mathbb{R} \).
Since \( f \) is \( \Gamma \times S^1 \)-equivariant, the map \( f \) sends the fixed point space of \( \Gamma_{(a,0)} \) into itself. The restriction of \( L \) to this subspace is given by
\[
L(u) = \sum_{l \in \mathbb{Z}} M_{l, kl} u_{l, lk} e^{i(lt + slk)}.
\]
Moreover, for \( l = 1 \), if \( \omega \) is irrational, the kernel has dimension one at
\[
\Omega_0 = |kk| \sqrt{k^2 + 2\omega}.
\]
Since \( \lambda_{1,k,-1}(\Omega) \) changes sign at \( \Omega_0 \), by arguments of topological bifurcation \cite{9}, there is a global bifurcation of periodic solutions near \((1, \Omega_0)\).

**Theorem 34** The map \( f(v; \Omega) \) has a global bifurcation of periodic traveling solutions from \((1, \Omega_0)\). For the filament problem these are solutions of the form
\[
u_j(t, s) = a_j e^{i\omega t} v(\Omega t + ks),
\]
where \( v \) is a 2\( \pi \)-periodic that starts from 1.

**Remark 35** In \cite{11} there is a proof of traveling waves that decay to zero in infinite. Also, in \cite{8} there is a proof of global bifurcations of periodic traveling waves for the a polygonal relative equilibrium. The proof in \cite{8} uses equivariant degree to manage the symmetries.
(a,a) Standing waves

The group $\Gamma_{(a,a)}$ is generated by the elements $(0, -\pi/k, \pi), \kappa_s$ and $\kappa_t$. Thus, functions with isotropy group $\Gamma_{(a,a)}$ satisfy

$$v(t, s) = v(t + \pi, s - \pi/k) = v(t, -s) = \bar{v}(-t, s). \quad (21)$$

These solutions are the standing waves. Since $v_0 = 1$ is the only solution of the orbit $e^{i\theta}v_0$ that satisfy the symmetries (21), then $f'(v_0)$ is not degenerate when restricted to the subspace of functions (21). We have taken advantage of this fact to prove the existence of the standing wave.

**Remark 36** In order to solve the small divisor problem, the frequency $\omega$ has been chosen Diophantine. In this case there is no resonant Fourier components, and all branches from different $k$’s are given by the same branch $k = 1$ and the transformation $\tau^{-1}u(\tau^2t, \tau s)$ for some $\tau$. Thus, the case we have analyzed in the paper for $k = 1$ is the general one.

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