A quantum statistical approach to simplified stock markets

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Abstract
We use standard perturbation techniques originally formulated in quantum (statistical) mechanics in the analysis of a toy model of a stock market which is given in terms of bosonic operators. In particular we discuss the probability of transition from a given value of the portfolio of a certain trader to a different one. This computation can also be carried out using some kind of Feynman graphs adapted to the present context.
I Introduction and motivations

In some recent papers, [1, 2, 3], we have discussed why and how a quantum mechanical framework, and in particular operator algebras and the number representation, can be used in the analysis of some simplified models of stock markets. These models are just prototypes of real stock markets because, among the other simplifications, we are not considering financial derivatives. For this reason our interest looks different from that widely discussed in [4], even if the general settings appear to be very close (and very close also to the framework used in [5]). The main reason for using operator algebras in the analysis of these simplified closed stock markets comes from the following considerations: in the closed market we have in mind the total amount of cash stays constant. Also, the total number of shares does not change with time. Moreover, when a trader $\tau$ interacts with a second trader $\sigma$, they change money and shares in a discrete fashion: for instance, $\tau$ increments his number of shares of 1 unit while his cash decrements of a certain number of monetary units (which is the minimum amount of cash existing in the market: 1 cent of dollar, for example), which is exactly the price of the share. Of course, for the trader $\sigma$ the situation is just reversed. So we have at least two quantities, the cash and the number of shares, which change discontinuously as multiples of two fixed quantities. In [1, 2, 3] we also have two other quantities defining our simplified market: the price of the share (in the cited papers the traders can exchange just a single kind of shares!) and the market supply, i.e. the overall tendency of the market to sell a share. It is clear that also the price of the share must change discontinuously, and that’s why we have assumed that also the market supply is labeled by a discrete quantity.

Operator algebras and quantum statistical mechanics provide a very natural settings for discussing such a system. Indeed they produce a natural way for: (a) describing quantities which change with discrete steps; (b) obtaining the differential equations for the relevant variables of the system under consideration, the so-called observables of the system; (c) finding conserved quantities; (d) compute transition probabilities.

For these reasons we have suggested in [1, 2, 3] an operator-valued scheme for the description of such a simplified market. Let us see why, neglecting here all the many mathematical complications arising mainly from the fact that our operators are unbounded, and limiting our introduction to few important facts in quantum mechanics and second
quantization which will be used in the next sections. More details can be found, for instance, in [6, 7] and [8], as well as in [1, 2, 3].

Let \( H \) be an Hilbert space and \( B(H) \) the set of all the bounded operators on \( H \). Let \( S \) be our (closed) physical system and \( \mathfrak{A} \) the set of all the operators, which may be unbounded, useful for a complete description of \( S \), which includes the observables of \( S \). The description of the time evolution of \( S \) is driven by a self-adjoint operator \( H = H^\dagger \) which is called the \textit{hamiltonian} of \( S \) and which in standard quantum mechanics represents the energy of \( S \). In the \textit{Heisenberg} picture the time evolution of an observable \( X \in \mathfrak{A} \) is given by

\[
X(t) = e^{iHt}Xe^{-iHt}
\]

or, equivalently, by the solution of the differential equation

\[
\frac{dX(t)}{dt} = ie^{iHt}[H, X]e^{-iHt} = i[H, X(t)],
\]

where \([A, B] := AB - BA\) is the \textit{commutator} between \( A \) and \( B \). The time evolution defined in this way is usually a one parameter group of automorphisms of \( \mathfrak{A} \). The wave function \( \Psi \) of \( S \) is constant in time.

In the Scrödinger picture the situation is just reversed: an observable \( X \in \mathfrak{A} \) does not evolve in time (but if it has some explicit dependence on \( t \)) while the wave function \( \Psi \) of \( S \) satisfies the Scrödinger equation

\[
i\frac{\partial \Psi(t)}{\partial t} = H \Psi(t),
\]

whose formal solution is, if \( H \) does not depend on time, \( \Psi(t) = e^{-iHt}\Psi \).

In our paper a special role is played by the so called \textit{canonical commutation relations} (CCR): we say that a set of operators \( \{a_l, a_l^\dagger, l = 1, 2, \ldots, L\} \) satisfy the CCR if the following hold:

\[
[a_l, a_n] = [a_l^\dagger, a_n^\dagger] = 0,
\]

\[
[a_l, a_n^\dagger] = \delta_{ln} \mathbb{1},
\]

for all \( l, n = 1, 2, \ldots, L \). Here \( \mathbb{1} \) is the identity operator of \( B(H) \). These operators, which are widely analyzed in any textbook in quantum mechanics, see [6] for instance, are those which are used to describe \( L \) different \textit{modes} of bosons. From these operators we can construct \( \hat{n}_l = a_l^\dagger a_l \) and \( \hat{N} = \sum_{l=1}^L \hat{n}_l \) which are both self-adjoint. In particular \( \hat{n}_l \) is the \textit{number operator} for the \( l \)-th mode, while \( \hat{N} \) is the \textit{number operator} of \( S \).

The Hilbert space of our system is constructed as follows: we introduce the \textit{vacuum} of the theory, that is a vector \( \varphi_0 \) which is annihilated by all the operators \( a_l \): \( a_l \varphi_0 = 0 \).
for all $l = 1, 2, \ldots, L$. Then we act on $\varphi_0$ with the operators $a_l^\dagger$ and their powers:

$$\varphi_{n_1,n_2,\ldots,n_L} := \frac{1}{\sqrt{n_1!n_2!\cdots n_L!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_L^\dagger)^{n_L}\varphi_0, \quad (1.4)$$

$n_l = 0, 1, 2, \ldots$ for all $l$. These vectors form an orthonormal set and are eigenstates of both $\hat{n}_l$ and $\hat{N}$: $\hat{n}_l\varphi_{n_1,n_2,\ldots,n_L} = n_l\varphi_{n_1,n_2,\ldots,n_L}$ and $\hat{N}\varphi_{n_1,n_2,\ldots,n_L} = N\varphi_{n_1,n_2,\ldots,n_L}$, where $N = \sum_{l=1}^{L} n_l$. Moreover, using the CCR we deduce that $\hat{n}_l (a_l\varphi_{n_1,n_2,\ldots,n_L}) = (n_l - 1)(a_l\varphi_{n_1,n_2,\ldots,n_L})$ and $\hat{n}_l (a_l^\dagger\varphi_{n_1,n_2,\ldots,n_L}) = (n_l + 1)(a_l^\dagger\varphi_{n_1,n_2,\ldots,n_L})$, for all $l$. For these reasons the following interpretation is given: if the $L$ different modes of bosons of $S$ are described by the vector $\varphi_{n_1,n_2,\ldots,n_L}$, this implies that $n_1$ bosons are in the first mode, $n_2$ in the second mode, and so on. The operator $\hat{n}_l$ acts on $\varphi_{n_1,n_2,\ldots,n_L}$ and returns $n_l$, which is exactly the number of bosons in the $l$-th mode. The operator $\hat{N}$ counts the total number of bosons. Moreover, the operator $a_l$ destroys a boson in the $l$-th mode, while $a_l^\dagger$ creates a boson in the same mode. This is why $a_l$ and $a_l^\dagger$ are usually called the annihilation and the creation operators.

The Hilbert space $\mathcal{H}$ is obtained by taking the closure of the linear span of all the vectors in (1.4). An operator $Z \in \mathfrak{A}$ is a constant of motion if it commutes with $H$. Indeed in this case equation (1.2) implies that $\dot{Z}(t) = 0$, so that $Z(t) = Z$ for all $t$.

The vector $\varphi_{n_1,n_2,\ldots,n_L}$ in (1.4) defines a vector (or number) state over the algebra $\mathfrak{A}$ as

$$\omega_{n_1,n_2,\ldots,n_L}(X) = \langle \varphi_{n_1,n_2,\ldots,n_L}, X\varphi_{n_1,n_2,\ldots,n_L} \rangle, \quad (1.5)$$

where $\langle , \rangle$ is the scalar product in $\mathcal{H}$. As we have discussed in [1, 2], these states may be used to project from quantum to classical dynamics and to fix the initial conditions of the market.

The paper is organized as follows:

In Section II we introduce a new model, slightly different from the one proposed in [2], and we deduce some of its features and the related equations of motion, working in the Heisenberg picture. One of the main improvements with respect to [2] is that several kind of shares (and not just one!) will be considered here.

In Section III we adopt a different point of view, using the Schrödinger picture to deduce the transition probability from a given initial situation to a final state, corresponding to
two different values of the portfolios of the various traders. Since the reader might not be familiar with the tools adopted, we will be rather explicit in the derivation of our results.

Section IV is devoted to the conclusions and to our plans for the future.

II The model and first considerations

Let us consider $N$ different traders $\tau_1, \tau_2, \ldots, \tau_N$, exchanging $L$ different kind of shares $\sigma_1, \sigma_2, \ldots, \sigma_L$. Each trader has a starting amount of cash, which is used during the trading procedure: the cash of the trader who sells a share increases while the cash of the trader who buys that share consequently decreases. The absolute value of these variations is the price of the share at the time in which the transaction takes place. Following our previous results we start introducing a set of bosonic operators which are listed, together with their economical meaning, in the following table. We adopt here latin indexes to label the traders and greek indexes for the shares: $j = 1, 2, \ldots, N$ and $\alpha = 1, 2, \ldots, L$.

| the operator and.. | ...its economical meaning |
|--------------------|---------------------------|
| $a_{j,\alpha}$     | annihilates a share $\sigma_\alpha$ in the portfolio of $\tau_j$ |
| $a_{j,\alpha}^\dagger$ | creates a share $\sigma_\alpha$ in the portfolio of $\tau_j$ |
| $\hat{n}_{j,\alpha} = a_{j,\alpha}^\dagger a_{j,\alpha}$ | counts the number of share $\sigma_\alpha$ in the portfolio of $\tau_j$ |
| $c_j$               | annihilates a monetary unit in the portfolio of $\tau_j$ |
| $c_j^\dagger$       | creates a monetary unit in the portfolio of $\tau_j$ |
| $\hat{k}_j = c_j^\dagger c_j$ | counts the number of monetary units in the portfolio of $\tau_j$ |
| $p_\alpha$          | lowers the price of the share $\sigma_\alpha$ of one unit of cash |
| $p_\alpha^\dagger$  | increases the price of the share $\sigma_\alpha$ of one unit of cash |
| $\hat{P}_\alpha = p_\alpha^\dagger p_\alpha$ | gives the value of the share $\sigma_\alpha$ |

Table 1.- List of operators and of their economical meaning.

These operators are bosonic in the sense that they satisfy the following commutation rules

$$[c_j, c_k^\dagger] = \mathbb{1} \delta_{j,k}, \quad [p_\alpha, p_\beta^\dagger] = \mathbb{1} \delta_{\alpha,\beta} \quad [a_{j,\alpha}, a_{k,\beta}^\dagger] = \mathbb{1} \delta_{j,k} \delta_{\alpha,\beta},$$  

(2.1)
while all the other commutators are zero.

As discussed in the Introduction, in \[2, 3\] we have also introduced another set of operators related to the market supply which was used to deduce the dynamics of the price of the single kind of share considered there. However, the mechanism proposed in those papers, thought being reasonable, is too naive and gives no insight on the nature of the market itself. For this reason in \[3, 9\] we have also considered a different point of view, leaving open the problem of finding the dynamics of the price and focusing the attention on the time evolution of the portfolio of a fixed trader. This is the same point of view which we briefly consider in this section, while we will comment on other possibilities in the rest of the paper. More precisely, as we have discussed in the Introduction, the dynamical behavior of our market is driven by a certain Hamiltonian \( \hat{H} \). We assume that \( \hat{H} \) can be written as

\[
\hat{H} = \hat{H}_0 + \lambda \hat{H}_I,
\]

with

\[
\begin{align*}
\hat{H}_0 &= \sum_{j,\alpha} \omega_{j,\alpha} \hat{n}_{j,\alpha} + \sum_j \omega_j \hat{k}_j, \\
\hat{H}_I &= \sum_{i,j,\alpha} \hat{p}_{i,j}^{(\alpha)} \left( a_{i,\alpha}^\dagger a_{j,\alpha} c_{i}^\dagger c_{j}^{\alpha} + h.c. \right).
\end{align*}
\]

(2.2)

Here h.c. stands for hermitian conjugate, \( c_{i}^\dagger \) and \( c_{j}^{\alpha} \) are defined as in \(2\), and \( \omega_{j,\alpha}, \omega_j \) and \( p_{i,j}^{(\alpha)} \) are positive real numbers. In particular these last coefficients assume different values depending on the possibility of \( \tau_i \) to interact with \( \tau_j \) and exchanging a share \( \sigma_\alpha \): for instance \( p_{2,5}^{(1)} = 0 \) if there is no way for \( \tau_2 \) and \( \tau_5 \) to exchange a share \( \sigma_1 \). Notice that this does not exclude that, for instance, they could exchange a share \( \sigma_2 \), so that \( p_{2,5}^{(2)} \neq 0 \). With this in mind it is natural to put \( p_{i,i}^{(\alpha)} = 0 \) and \( p_{i,j}^{(\alpha)} = p_{j,i}^{(\alpha)} \).

Going back to (2.2), we observe that \( \hat{H}_0 \) is nothing but the standard free hamiltonian which is used for many-body systems like the ones we are considering here (where the bodies are nothing but the traders, and the cash). More interesting is the meaning of the interaction hamiltonian \( \hat{H}_I \). To understand \( \hat{H}_I \) we consider its action on a vector like

\[
\varphi_{\{n_{j,\alpha};\{k_j\};\{P_\alpha\}}} := \frac{a_{1,1}^{n_{1,1}} \cdots a_{N,L}^{n_{N,L}} c_1^{k_1} \cdots c_N^{k_N} \prod_{P_1} \prod_{P_L} P_1 \cdots P_L}{\sqrt{n_{1,1}! \cdots n_{N,L}! k_1! \cdots k_L! P_1! \cdots P_L!}} \varphi_0,
\]

(2.3)

where \( \varphi_0 \) is the vacuum of all the annihilation operators involved here, see Section I.
Because of the CCR we deduce that the action of a single contribution of $H_I, a_{i,\alpha}^\dagger a_{j,\alpha} c_j^\dagger \hat{P}_\alpha$, on $\varphi(n_j,\alpha);(k_j);(P_\alpha)$ is proportional to another vector $\varphi(n'_{j,\alpha});(k'_j);(P'_{\alpha})$ with just 4 different quantum numbers. In particular $n_{j,\alpha}, n_{i,\alpha}, k_j$ and $k_i$ are replaced respectively by $n_{j,\alpha}-1, n_{i,\alpha}+1, k_j+P_\alpha$ and $k_i-P_\alpha$ (if this is larger or equal than zero, otherwise the vector is annihilated). This means that $\tau_j$ is selling a share $\sigma_{\alpha}$ to $\tau_i$ and earning money from this operation. For this reason it is convenient to introduce the following selling and buying operators:

$$x_{j,\alpha} := a_{j,\alpha} c_j^\dagger \hat{P}_\alpha, \quad x_{j,\alpha}^\dagger := a_{j,\alpha}^\dagger c_j \hat{P}_\alpha$$

With these definitions and using the properties of the coefficients $p_{i,j}^{(\alpha)}$ we can rewrite $H_I$ as

$$H_I = 2 \sum_{i,j,\alpha} p_{i,j}^{(\alpha)} x_{i,\alpha}^\dagger x_{j,\alpha} \Rightarrow H = \sum_{j,\alpha} \omega_{j,\alpha} \hat{n}_{j,\alpha} + \sum_j \omega_j \hat{k}_j + 2\lambda \sum_{i,j,\alpha} p_{i,j}^{(\alpha)} x_{i,\alpha}^\dagger x_{j,\alpha}$$

In [9] we have discussed the role of $H_{\text{price}}$ which should be used to deduce the time evolution of the operators $\hat{P}_\alpha, \alpha = 1, 2, \ldots, L$, and which will not be fixed in these notes. This will be justified below, after getting the differential equations of motion for our system. Again in [9] we have also shown that $\hat{H}$ corresponds to a closed market where the money and the total number of shares of each type are conserved. Indeed, calling $\hat{N}_\alpha := \sum_{l=1}^N \hat{n}_{l,\alpha}$ and $\hat{K} := \sum_{l=1}^N \hat{k}_l$ we have seen that, for all $\alpha$, $[\hat{H}, \hat{N}_\alpha] = [\hat{H}, \hat{K}] = 0$. Hence $\hat{N}_\alpha$ and $\hat{K}$ are integrals of motion, as expected. Of course, something different may happen in realistic markets. For instance, a given company could decide to issue more stocks in the market or to split existing stocks. The related $\hat{N}_\alpha$, say $\hat{N}_{\alpha_0}$, is no longer a constant of motion. Hence $\hat{H}$ should be modified in order to have $[\hat{H}, \hat{N}_{\alpha_0}] \neq 0$. For that it is enough to add in $\hat{H}$ some source or sink contribution, trying to preserve the self-adjointness of $\hat{H}$. However, this is not always the more natural choice. For instance, if we want to include in the model interactions with the environment (economical, political, social inputs), it could be more convenient to use non-hermitean operators, like the generators appearing in the analysis of quantum dynamical semigroups used to describe open systems, [10]. This aspect will not be considered here.
Let us now define the *portfolio operator* of the trader \( \tau_l \) as

\[
\hat{\Pi}_l(t) = \sum_{\alpha=1}^{L} \hat{P}_{\alpha}(t) \hat{n}_{l,\alpha}(t) + \hat{k}_l(t).
\]  

(2.6)

This is a natural definition, since it is just the sum of the cash and of the total value of the shares that \( \tau_l \) possesses at time \( t \). Once again, we stress that in our simplified model there is no room for the financial derivatives.

Using (1.2) and the commutation rules assumed so far we derive the following system of equations:

\[
\begin{align*}
\frac{d\hat{n}_{l,\alpha}(t)}{dt} &= 2i\lambda \sum_{j=1}^{N} p_{l,j}^{(\alpha)} \left( x_{j,\alpha}^\dagger(t) x_{l,\alpha}(t) - x_{l,\alpha}^\dagger(t) x_{j,\alpha}(t) \right), \\
\frac{d\hat{k}_l(t)}{dt} &= -2i\lambda \sum_{j=1}^{N} \sum_{\alpha=1}^{L} p_{l,j}^{(\alpha)} \hat{P}_{\alpha}(t) \left( x_{j,\alpha}^\dagger(t) x_{l,\alpha}(t) - x_{l,\alpha}^\dagger(t) x_{j,\alpha}(t) \right), \\
\frac{dx_{l,\alpha}(t)}{dt} &= i x_{l,\alpha}(t) (\omega_l \hat{P}_{\alpha}(t) - \omega_{l,\alpha}) + \\
&\quad + 2i\lambda \sum_{j=1}^{N} \sum_{\beta=1}^{L} p_{l,j}^{(\beta)} [x_{l,\alpha}(t), x_{l,\alpha}(t)] x_{j,\beta}(t),
\end{align*}
\]

(2.7)

which, together with their adjoints, produce a closed system of differential equation. Notice that these equations imply that \( \sum_{\alpha=1}^{L} \hat{P}_{\alpha}(t) \frac{d}{dt}\hat{n}_{l,\alpha}(t) + \frac{d}{dt}\hat{k}_l(t) = 0 \). This system is now replaced by a semi-classical approximation which is obtained replacing the time dependent operators \( \hat{P}_{\alpha}(t) \) with \( L \) classical fields \( P_{\alpha}(t) \) which are deduced by empirical data. This is the reason why we were not interested in fixing \( H_{prices} \) in \( \hat{H} \): \( \hat{P}_{\alpha}(t) \) is replaced by \( P_{\alpha}(t) \) which are no longer internal degrees of freedom of the model but, rather than this, simple external classical fields. Neglecting all the details, which can be found in [9], we observe that the first non trivial contribution in \( \lambda \) is

\[
\begin{align*}
\hat{n}_{l,\alpha}(t) &:= \omega_{\{n_{j,\alpha}\};\{k_{l}\};\{P_{\alpha}\}}(\hat{n}_{l,\alpha}(t)) = \\
&= n_{l,\alpha} - 8\lambda^2 \sum_{j=1}^{N} \left( p_{l,j}^{(\alpha)} \right)^2 \tilde{M}_{j,l,\alpha} \Re(\Theta_{j,l;\alpha}^{(2)}(t)) =: n_{l,\alpha} + \delta n_{l,\alpha}(t), \\
n_{l,\alpha} &:= 8\lambda^2 \sum_{j=1}^{N} \sum_{\alpha=1}^{L} \left( p_{l,j}^{(\alpha)} \right)^2 \tilde{M}_{j,l,\alpha} \Re(\Theta_{j,l;\alpha}^{(3)}(t)) =: k_l + \delta k_l(t),
\end{align*}
\]

(2.8)

where we have defined:

\[
\begin{align*}
\tilde{M}_{j,l,\alpha} &= M_{j,l,\alpha} - M_{j,l;\alpha}, \\
M_{j,l;\alpha} &= n_{j,\alpha} n_{l,\alpha} \frac{(k_{j} + P_{\alpha})!}{k_{l}!} - n_{j,\alpha} (1 + n_{l,\alpha}) \frac{(k_{j} + P_{\alpha})!}{k_{l}!} \frac{k_{l}!}{(k_{j} - P_{\alpha})!},
\end{align*}
\]

(2.9)

8
and

\[
\begin{align*}
\Theta_{j,l}\alpha^{(3)}(t) &= \int_0^t P\alpha(t') \Theta_{j,l}\alpha^{(1)}(t') e^{-i\Theta_{j,l}\alpha^{(0)}(t')} dt', \\
\Theta_{j,l}\alpha^{(2)}(t) &= \int_0^t P\alpha(t') e^{-i\Theta_{j,l}\alpha^{(0)}(t')} dt', \\
\Theta_{j,l}\alpha^{(1)}(t) &= \int_0^t e^{-i\Theta_{j,l}\alpha^{(0)}(t')} dt', \\
\Theta_{j,l}\alpha^{(0)}(t) &= (\omega_j - \omega_l) \int_0^t P\alpha(t') dt' - (\omega_j,\alpha - \omega_l,\alpha)t.
\end{align*}
\]

The time dependence of the portfolio can now be written as

\[
\Pi_l(t) := \omega\{n_{j,\alpha};\{k_j;\{\Pi_l(t)\}} = \Pi_l(0) + \delta \Pi_l(t),
\]

with

\[
\delta \Pi_l(t) = \sum_{\alpha=1}^L n_{l,\alpha}(P\alpha(t) - P\alpha(0)) + \sum_{\alpha=1}^L P\alpha(t) \delta n_{l,\alpha}(t) + \delta k_l(t).
\]

It should be emphasized that, even under the approximations we are considering here, we still find $\sum_{\alpha=1}^L P\alpha(t) \dot{n}_{l,\alpha}(t) + \dot{k}_l(t) = 0$. However, we are loosing the conservation laws we have discussed before: both $\dot{N}_l$ and $\dot{K}$ do not commute with the effective hamiltonian which produces the semi-classical version of (2.7) anymore, and therefore they are not constant in time. This imposes some strict constraint on the validity of our expansion, as we have discussed already in [3] and suggests the different approach to the problem which we will discuss in the next section. Again we refer to [9] for more comments on these results.

### III A time dependent point of view

In this section we will consider a slightly different point of view. Our hamiltonian has no $H_{price}$ contribution at all, since the price operators $\hat{P}\alpha$, $\alpha = 1, \ldots, L$, are now replaced from the very beginning by external classical fields $P\alpha(t)$, whose time dependence describes, as an input of the model, the variation of the prices of the shares. Incidentally, this implies that possible fast changes of the prices are automatically included in the model through the analytic expressions of the functions $P\alpha(t)$. Hence the interaction hamiltonian $H_I$ in (2.5) turns out to be a time dependent operator, $H_I(t)$. More in details, the hamiltonian
\[ H = H_0 + \lambda H_I(t) \] of the model looks like the one in (2.5) but with the following time-dependent selling and buying operators:

\[ x_{j,\alpha}(t) := a_{j,\alpha} c_j^{\dagger} P_\alpha(t), \quad x_{j,\alpha}^\dagger(t) := a_{j,\alpha}^{\dagger} c_j P_\alpha(t) \quad (3.1) \]

The \( L \) price functions \( P_\alpha(t) \) will be taken piecewise constant, since the price of a share changes discontinuously: it has a certain value before the transaction and (in general) a different value after the transaction. This new value does not change until the next transaction takes place. More in details, we introduce a time step \( h \) which we call the time of transaction, and we divide the interval \([0, t]\) in subintervals of duration \( h \): \([0, t[ = [t_0, t_1] \cup [t_1, t_2] \cdots [t_{M-1}, t_M] \), where \( t_0 = 0, \ t_1 = h, \ldots, \ t_{M-1} = (M-1)h = t - h, \ t_M = Mh = t \). Hence \( h = t/M \). As for the prices, for \( \alpha = 1, \ldots, L \) we put

\[ P_\alpha(t) = \begin{cases} P_{\alpha,0}, & t \in [t_0, t_1[, \\ P_{\alpha,1}, & t \in [t_1, t_2[, \\ \ldots, \\ P_{\alpha,M-1}, & t \in [t_{M-1}, t_M[ \end{cases} \quad (3.2) \]

An orthonormal basis in the Hilbert space of the model \( \mathcal{H} \) is now the set of vectors defined as

\[ \varphi_{\{n_{j,\alpha}\};\{k_j\}} := a_{1,1}^{\dagger} \cdots a_{N,L}^{\dagger} c_1^{\dagger} \cdots c_N^{\dagger} k_1! \cdots k_N! \varphi_0, \quad (3.3) \]

where \( \varphi_0 \) is the vacuum of all the annihilation operators involved here. They differ from the ones in (2.3) since the price operators disappear, of course. To simplify the notation we introduce a set \( \mathcal{F} = \{\{n_{j,\alpha}\};\{k_j\}\} \) so that the vectors of the basis will be simply written as \( \varphi_{\mathcal{F}} \).

The main problem we want to discuss here is the following: suppose that at \( t = 0 \) the market is described by a vector \( \varphi_{\mathcal{F}_0} \). This means that, since \( \mathcal{F}_0 = \{\{n_{j,\alpha}^0\};\{k_j^0\}\} \), at \( t = 0 \) the trader \( \tau_1 \) has \( n_{11}^0 \) shares of \( \sigma_1 \), \( n_{12}^0 \) shares of \( \sigma_2 \), \ldots, and \( k_1^0 \) units of cash. Analogously, the trader \( \tau_2 \) has \( n_{21}^0 \) shares of \( \sigma_1 \), \( n_{22}^0 \) shares of \( \sigma_2 \), \ldots, and \( k_2^0 \) units of cash. And so on. We want to compute the probability that at time \( t \) the market has moved to the configuration \( \mathcal{F}_f = \{\{n_{j,\alpha}^f\};\{k_j^f\}\} \). This means that, for example, \( \tau_1 \) has now \( n_{11}^f \) shares of \( \sigma_1 \), \( n_{12}^f \) shares of \( \sigma_2 \), \ldots, and \( k_1^f \) units of cash.
Similar problems are very well known in ordinary quantum mechanics: we need to compute a probability transition from the original state $\varphi_{F_0}$ to a final state $\varphi_{F_f}$, and therefore we will use here the standard time-dependent perturbation scheme for which we refer to [11]. The main difference with respect to what we have done in the previous section is the use of the Schrödinger rather than the Heisenberg picture. Hence the market is described by a time-dependent wave function $\Psi(t)$ which, for $t = 0$, reduces to $\varphi_{F_0}$: $\Psi(0) = \varphi_{F_0}$. The transition probability we are looking for is

$$P_{F_0 \rightarrow F_f}(t) := |\langle \varphi_{F_f}, \Psi(t) \rangle|^2 \quad (3.4)$$

The computation of $P_{F_0 \rightarrow F_f}(t)$ is a standard exercise, [11]. In order to make the paper accessible also to those people who are not familiar with quantum mechanics, we give here the main steps of its derivation.

Since the set of the vectors $\varphi_{F}$ is an orthonormal basis in $\mathcal{H}$ the wave function $\Psi(t)$ can be written as

$$\Psi(t) = \sum_{F} c_F(t) e^{-iE_{F}t} \varphi_{F}, \quad (3.5)$$

where $E_{F}$ is the eigenvalue of $H_0$ defined as

$$H_0 \varphi_{F} = E_{F} \varphi_{F}, \quad \Rightarrow \quad E_{F} = \sum_{j,\alpha} \omega_{j,\alpha} n_{j,\alpha} + \sum_{j} \omega_{j} k_{j}. \quad (3.6)$$

This is a consequence of the fact that $\varphi_{F}$ in (3.3) is an eigenstate of $H_0$ in (2.2). Using the quantum mechanical terminology, we sometimes call $E_{F}$ the free energy of $\varphi_{F}$. Putting (3.5) in (3.4), and recalling that $\langle \varphi_{F}, \varphi_{G} \rangle = \delta_{F,G}$, we have

$$P_{F_0 \rightarrow F_f}(t) := |c_{F_f}(t)|^2 \quad (3.7)$$

The answer to our original question is therefore given if we are able to compute $c_{F_f}(t)$ in (3.5). Due to the analytic form of our hamiltonian, this cannot be done exactly. However, several possible perturbation schemes exist in the literature. We will adopt here a simple perturbation expansion in the interaction parameter $\lambda$ appearing in the hamiltonian (2.5).

In other words, we look for the coefficients in (3.5) having the form

$$c_{F}(t) = c_{F}^{(0)}(t) + \lambda c_{F}^{(1)}(t) + \lambda^2 c_{F}^{(2)}(t) + \cdots \quad (3.8)$$
Each $c_{F}^{(j)}(t)$ satisfies a differential equation which can be deduced as follows: first we recall that $\Psi(t)$ satisfies the Schrödinger equation $i\frac{\partial\Psi(t)}{\partial t} = H(t)\Psi(t)$. Replacing (3.5) in this equation and using the orthonormality of the vectors $\varphi_F$’s, we find that

$$\dot{c}_{F}(t) = -i\lambda \sum_{F} c_{F}(t) e^{i(E_{F'}-E_{F})t} < \varphi_{F'}, H_{1}(t)\varphi_{F}>$$  \hspace{1cm} (3.9)$$

Replacing now (3.8) in (3.9) we find the following infinite set of differential equations, which we can solve, in principle, up to the desired order in $\lambda$:

$$\begin{align*}
\dot{c}_{F}^{(0)}(t) &= 0, \\
\dot{c}_{F}^{(1)}(t) &= -i \sum_{F} c_{F}^{(0)}(t) e^{i(E_{F'}-E_{F})t} < \varphi_{F'}, H_{1}(t)\varphi_{F}> , \\
\dot{c}_{F}^{(2)}(t) &= -i \sum_{F} c_{F}^{(1)}(t) e^{i(E_{F'}-E_{F})t} < \varphi_{F'}, H_{1}(t)\varphi_{F}> , \\
&\ldots. \\
\end{align*}$$  \hspace{1cm} (3.10)$$

The first equation, together with the assumed initial condition, gives $c_{F}^{(0)}(t) = c_{F}^{(0)}(0) = \delta_{F,F_0}$. When we replace this solution in the differential equation for $c_{F}^{(1)}(t)$ we get, recalling again that $\Psi(0) = \varphi_{F_0}$,

$$c_{F}^{(1)}(t) = -i \int_{0}^{t} e^{i(E_{F'}-E_{F_0})t_1} < \varphi_{F'}, H_{1}(t_1)\varphi_{F_0}> dt_1$$  \hspace{1cm} (3.11)$$

Using this in (3.10) we further get

$$c_{F}^{(2)}(t) = (-i)^2 \sum_{F} \int_{0}^{t} \left( \int_{0}^{t_2} e^{i(E_{F}-E_{F_0})t_1} h_{F,F_0}(t_1) dt_1 \right) e^{i(E_{F'}-E_{F})t_2} h_{F',F}(t_2) dt_2, \hspace{1cm} (3.12)$$

where we have introduced the shorthand notation

$$h_{F,G}(t) := < \varphi_{F}, H_{1}(t)\varphi_{G}>$$  \hspace{1cm} (3.13)$$

### III.1 First order corrections

We continue our analysis computing $P_{F_0\to F}(t)$ in (3.7) up to the first order corrections in $\lambda$ and assuming that $F_f$ is different from $F_0$. Hence we have

$$P_{F_0\to F_f}(t) = \left| c_{F_f}^{(1)}(t) \right|^2 = \lambda^2 \left| \int_{0}^{t} e^{i(E_{F_f}-E_{F_0})t_1} h_{F_f,F_0}(t_1) dt_1 \right|^2$$  \hspace{1cm} (3.14)$$
Using (3.2) and introducing \( \delta E = E_{F_f} - E_{F_0} \), after some algebra we get

\[
P_{\mathcal{F}_0 \rightarrow \mathcal{F}_f}(t) = \lambda^2 \left( \frac{\delta E h/2}{\delta E/2} \right)^2 \left| \sum_{k=0}^{M-1} h_{\mathcal{F}_f, \mathcal{F}_0}(t_k) e^{it_k \delta E} \right|^2
\] (3.15)

The computation of the matrix elements \( h_{\mathcal{F}_f, \mathcal{F}_0}(t_k) \) is easily performed. Indeed, because of some standard properties of the bosonic operators, we find that

\[
a_{i, \alpha}^{\dagger} a_{j, \alpha} c_{i} P_{\alpha, k} \varphi_{0, k} = \Gamma_{i, j; \alpha}^{(k)} \varphi_{0, k}^{(i, j; \alpha)}
\]

where

\[
\Gamma_{i, j; \alpha}^{(k)} := \sqrt{\frac{(k^o_j + P_{\alpha, k})!}{k^o_j!} \frac{k^o_i!}{(k^o_i - P_{\alpha, k})!} n^o_{j, \alpha} (1 + n^o_{i, \alpha})}
\] (3.16)

and \( \varphi_{0, k}^{(i, j; \alpha)} \) differs from \( \varphi_{0, k} \) only for the following replacements: \( n^o_{j, \alpha} \rightarrow n^o_{j, \alpha} - 1, n^o_{i, \alpha} \rightarrow n^o_{i, \alpha} + 1, k^o_j \rightarrow k^o_j + P_{\alpha, k}, k^o_i \rightarrow k^o_i - P_{\alpha, k} \). Notice that in our computations we are implicitly assuming that \( k^o_i \geq P_{\alpha, k} \), for all \( i, k \) and \( \alpha \). This is because otherwise the trader \( \tau_i \) would have not enough money to buy a share \( \sigma_\alpha \).

We find that

\[
h_{\mathcal{F}_f, \mathcal{F}_0}(t_k) = 2 \sum_{i, j; \alpha} p_{i, j; \alpha}^{(k)} \Gamma_{i, j; \alpha}^{(k)} < \varphi_{\mathcal{F}_f}, \varphi_{\mathcal{F}_0, k}^{(i, j, \alpha)} >
\] (3.17)

Of course, due to the orthogonality of the vectors \( \varphi_{\mathcal{F}} \)'s, the scalar product \( < \varphi_{\mathcal{F}_f}, \varphi_{\mathcal{F}_0, k}^{(i, j, \alpha)} > \) is different from zero (and equal to one) if and only if \( n^f_{j, \alpha} = n^o_{j, \alpha} - 1, n^f_{i, \alpha} = n^o_{i, \alpha} + 1, k^f_j = k^o_j - P_{\alpha, k} \) and \( k^f_i = k^o_i + P_{\alpha, k} \), and all the other new and old quantum numbers coincide.

For concreteness sake we now consider two simple situations: in the first example below we just assume that the prices of the various shares do not change with \( t \). In the second example we consider the case in which only few changes occur, and we take \( M = 3 \).

**Example 1: constant prices**

Let us assume that, for all \( k \) and for all \( \alpha \), \( P_{\alpha, k} = P_{\alpha}(t_k) = P_{\alpha} \). This means that \( \Gamma_{i, j; \alpha}^{(k)}, \varphi_{\mathcal{F}_0, k}^{(i, j, \alpha)} \) and the related vectors \( \varphi_{\mathcal{F}_0, k}^{(i, j, \alpha)} \) do not depend on \( k \). Hence \( h_{\mathcal{F}_f, \mathcal{F}_0}(t_k) \) is also independent of \( k \). After few computation we get

\[
P_{\mathcal{F}_0 \rightarrow \mathcal{F}_f}(t) = \lambda^2 \left( \frac{\sin(\delta E t/2)}{\delta E/2} \right)^2 \left| h_{\mathcal{F}_f, \mathcal{F}_0}(0) \right|^2
\] (3.18)
to which corresponds a transition probability per unit of time

\[ p_{\mathcal{F}_0 \to \mathcal{F}_f} = \lim_{t, \infty} \frac{1}{t} P_{\mathcal{F}_0 \to \mathcal{F}_f}(t) = 2\pi \lambda^2 \delta(E_{\mathcal{F}_f} - E_{\mathcal{F}_0}) |h_{\mathcal{F}_f, \mathcal{F}_0}(0)|^2, \tag{3.19} \]

which shows that, in this limit, a transition between two states is possible only if the two states have the same free energy. The presence of \( h_{\mathcal{F}_f, \mathcal{F}_0}(0) \) in the final result shows, using our previous remark, that at the order we are considering here a transition is possible only if \( \varphi_{\mathcal{F}_0} \) does not differ from \( \varphi_{\mathcal{F}_f} \) for more than one share in two of the \( n_{j,\alpha} \)'s and for more than \( P_\alpha \) in two of the \( k_j \)'s. All the other transitions are forbidden.

**Example 2: few changes in the price**

Let us now fix \( M = 3 \). Formula (3.15) can be rewritten as

\[
P_{\mathcal{F}_0 \to \mathcal{F}_f}(t) = 4\lambda^2 \left( \frac{\sin(\delta Eh/2)}{\delta E/2} \right)^2 \left| \sum_{i,j,\alpha} P_{i,j}^{(\alpha)} (\Gamma_{i,j,\alpha}^{(0)} < \varphi_{\mathcal{F}_f}, \varphi_{\mathcal{F}_0}^{(i,j,\alpha)} >) \right| + \\
+ \Gamma_{i,j,\alpha}^{(1)} < \varphi_{\mathcal{F}_f}, \varphi_{\mathcal{F}_0}^{(i,j,\alpha)} > e^{ih\delta E} + \Gamma_{i,j,\alpha}^{(2)} < \varphi_{\mathcal{F}_f}, \varphi_{\mathcal{F}_0}^{(i,j,\alpha)} > e^{2ih\delta E} \right|^2 \tag{3.20} \]

The meaning of this formula is not very different from the one discussed in the previous example: if we restrict to this order of approximation, the only possibilities for a transition \( \mathcal{F}_0 \to \mathcal{F}_f \) to occur are those already discussed in Example 1 above. We will see that, in order to get something different, we need to go to higher orders in \( \lambda \). In other words, even if the prices depend on time, not new relevant features appear in the transition probabilities.

As for the validity of the approximation, let us consider the easiest situation: we have constant prices (Example 1) and, moreover, in the summation in (3.17) only one contribution survives, the one with \( i_0, j_0 \) and \( \alpha_0 \). Then we have \( h_{\mathcal{F}_f, \mathcal{F}_0}(t_0) = 2p_{i_0,j_0}^{(\alpha_0)} \Gamma_{i_0,j_0,\alpha_0} < \varphi_{\mathcal{F}_f}, \varphi_{\mathcal{F}_0}^{(i_0,j_0,\alpha_0)} > \). Because of (3.18), and since our approximation becomes meaningless if \( P_{\mathcal{F}_0 \to \mathcal{F}_f}(t) \) exceeds one, it is necessary to have small \( \lambda \), small \( p_{i,j}^{(\alpha)} \) and large \( \delta E \) (if this is possible). However, due to the analytic expression for \( \Gamma_{i_0,j_0,\alpha_0} \), see (3.16), we must pay attention to the values of the \( n_{j,\alpha} \)'s and of the \( k_j \)'s, since, if they are large, the approximation may likely break down very soon in \( t \).

Let us now finally see what can be said about the portfolio of the trader \( \tau_1 \). Since we know the initial state of the system, then we know the value of its portfolio at time
zero: extending our original definition in (2.6) we have $\hat{\Pi}_l(0) = \sum_{\alpha=1}^{L} P_{\alpha}(0) \hat{n}_{l,\alpha}(0) + \hat{k}_l(0)$. As a matter of fact, the knowledge of $\varphi_{F_0}$ implies that we know the time zero value of the portfolios of all the traders, clearly! Formula (3.15) gives the transition probability from $\varphi_{F_0}$ to $\varphi_{F_f}$. This probability is just a single contribution in the computation of the transition probability from a given $\hat{\Pi}_l(0)$ to a certain $\hat{\Pi}_l(t)$, since the same value of the portfolio can be recovered at time $t$ for very many different states $\varphi_{F_f}$: all the sets $G$ with the same $n_{l,\alpha}^f$ and $k_{l}^f$ give rise to the same portfolio for $\tau_l$. Hence, if we call $\tilde{F}$ the set of all these sets, we just have to sum up over all these different contributions:

$$P_{\hat{\Pi}_l \rightarrow \hat{\Pi}_l}(t) = \sum_{G \in \tilde{F}} P_{\varphi_{F_0} \rightarrow G}(t)$$

### III.2 Second order corrections

We start considering the easiest situation, i.e. the case of a time independent perturbation $H_I$: the prices are constant in time. Hence the integrals in formula (3.12) can be easily computed and the result is the following:

$$c^{(2)}_{F_f}(t) = \sum_{F} h_{F_f,F}(0) h_{F,F_0}(0) \mathcal{E}_{F,F_0,F_f}(t),$$

where

$$\mathcal{E}_{F,F_0,F_f}(t) = \frac{1}{E_F - E_{F_0}} \left( \frac{e^{i(E_{F_f} - E_{F_0})t} - 1}{E_{F_f} - E_{F_0}} - \frac{e^{i(E_{F_f} - E_{F_0})t} - 1}{E_{F_f} - E_{F}} \right).$$

Recalling definition (3.13), we rewrite equation (3.21) as $c^{(2)}_{F_f}(t) = \sum_{F} \varphi_{F_f}, H_I \varphi_{F} > \mathcal{E}_{F,F_0,F_f}(t)$ which explicitly shows that up to this order in our perturbation expansion transitions between states which differ, e.g., for 2 shares are allowed: it is enough that some intermediate state $\varphi_{F}$ differs for (plus) one share from $\varphi_{F_0}$ and for (minus) one share from $\varphi_{F_f}$.

If the $P_{\alpha}(t)$'s depend on time the situation is a bit more complicated but not very different. Going back to Example 2 above, and considering then a simple (but not trivial) situation in which the prices of the shares really change, we can perform the computation and we find

$$c^{(2)}_{F_f}(t) = (-i)^2 \sum_{F} \{h_{F_f,F}(t_0) h_{F,F_0}(t_0) J_0(F,F_0,F_f; t_1) +$$

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\[ +h_{\mathcal{F}_f, \mathcal{F}}(t_1) (h_{\mathcal{F}_f, \mathcal{F}_0}(t_0) I_0(\mathcal{F}, \mathcal{F}_0; t_1) I_1(\mathcal{F}_f, \mathcal{F}; t_2) + h_{\mathcal{F}_f, \mathcal{F}_0}(t_1) I_1(\mathcal{F}, \mathcal{F}_0, \mathcal{F}_f; t_2)) + \\
+ h_{\mathcal{F}_f, \mathcal{F}}(t_2) [h_{\mathcal{F}_f, \mathcal{F}_0}(t_0) I_0(\mathcal{F}, \mathcal{F}_0; t_1) + h_{\mathcal{F}_f, \mathcal{F}_0}(t_1) I_1(\mathcal{F}, \mathcal{F}_0; t_2)] I_2(\mathcal{F}_f, \mathcal{F}; t_3) + \\
+ h_{\mathcal{F}_f, \mathcal{F}}(t_2) J_2(\mathcal{F}, \mathcal{F}_0, \mathcal{F}_f; t_3)] \]

where we have introduced the functions

\[ I_j(\mathcal{F}, \mathcal{G}; t) := \int_{t_j}^{t} e^{i(E_{\mathcal{F}_f} - E_{\mathcal{G}}) t'} dt' = \frac{1}{i(E_{\mathcal{F}_f} - E_{\mathcal{G}})} (e^{i(E_{\mathcal{F}_f} - E_{\mathcal{G}}) t} - e^{i(E_{\mathcal{F}_f} - E_{\mathcal{G}}) t_j}) \]

and

\[ J_j(\mathcal{F}, \mathcal{G}, \mathcal{L}; t) := \int_{t_j}^{t} I_j(\mathcal{F}, \mathcal{G}; t') e^{i(E_{\mathcal{L}_f} - E_{\mathcal{F}}) t'} dt'. \]

 Needless to say, this last integral could be explicitly computed but we will not show here the explicit result, since it will not be used.

The same comments as above about the possibility of having a non zero transition probability can be repeated also for equation (3.22): it is enough that the time-depending perturbation connect \( \varphi_{\mathcal{F}_0} \) to \( \varphi_{\mathcal{F}_f} \) via some intermediate state \( \varphi_{\mathcal{F}} \) in a single time sub-interval in order to permit a transition. If this never happens in \([0, t]\), then the transition probability is zero. We can see the problem from a different point of view: if some transition takes place in the interval \([0, t]\), there must be another state, \( \varphi_{\mathcal{F}_f}' \), different from \( \varphi_{\mathcal{F}_f} \), such that the transition probability \( P_{\mathcal{F}_0 \rightarrow \mathcal{F}_f'}(t) \) is non zero.

### III.3 Feynman graphs

Following [11] we now try to connect the analytic expression of a given approximation of \( c_{\mathcal{F}_f}(t) \) with some kind of Feynman graph in such a way that the higher orders could be easily written considering a certain set of rules which we will obviously call Feynman rules.

The starting point is given by the expressions (3.11) and (3.12) for \( c_{\mathcal{F}_f}^{(1)}(t) \) and \( c_{\mathcal{F}_f}^{(2)}(t) \), which is convenient to rewrite in the following form:

\[ c_{\mathcal{F}_f}^{(1)}(t) = -i \int_{0}^{t} e^{iE_{\mathcal{F}_f} t_1} < \varphi_{\mathcal{F}_f}, H_I(t_1) \varphi_{\mathcal{F}_0} > e^{-iE_{\mathcal{F}_0} t_1} dt_1 \]
and

\[ c^{(2)}_{\mathcal{F}_f}(t) = (-i)^2 \sum_{\mathcal{F}} \int_0^t dt_2 \int_0^{t_2} dt_1 e^{iE_{\mathcal{F}_f}t_2} \langle \varphi_{\mathcal{F}_f}, H_1(t_2)\varphi_{\mathcal{F}} \rangle e^{-iE_{\mathcal{F}_f}t_2} \times \]
\[ \times e^{iE_{\mathcal{F}_f}t_1} \langle \varphi_{\mathcal{F}}, H(t_1)\varphi_{\mathcal{F}_0} \rangle e^{-iE_{\mathcal{F}_f}t_1} \quad (3.24) \]

A graphical way to describe \( c^{(1)}_{\mathcal{F}_f}(t) \) is given in the figure below: at \( t = t_0 \) the state of the system is \( \varphi_{\mathcal{F}_0} \), which evolves freely (and therefore \( e^{-iE_{\mathcal{F}_0}t_1}\varphi_{\mathcal{F}_0} \) appears) until the interaction occurs, at \( t = t_1 \). After the interaction the system is moved to the state \( \varphi_{\mathcal{F}_f} \), which evolves freely (and therefore \( e^{-iE_{\mathcal{F}_f}t_1}\varphi_{\mathcal{F}_f} \) appears, and the different sign in \( (3.23) \) is due to the anti-linearity of the scalar product in the first variable.). The free evolutions are the upward arrows, while the interaction between the initial and the final states, \( \langle \varphi_{\mathcal{F}_f}, H_1(t_1)\varphi_{\mathcal{F}_0} \rangle \), is described by an horizontal wavy line. Obviously, since the interaction may occur at any time between 0 and \( t \), we have to integrate on all these possible \( t_1 \)'s and multiply the result for \(-i\).

![Figure 1: graphical expression for \( c^{(1)}_{\mathcal{F}_f}(t) \)](image)

In a similar way we can construct the Feynman graph for \( c^{(2)}_{\mathcal{F}_f}(t) \), \( c^{(3)}_{\mathcal{F}_f}(t) \) and so on. For example \( c^{(2)}_{\mathcal{F}_f}(t) \) can be deduced by a graph like the one in Figure 2, where two interactions occur, the first at \( t = t_1 \) and the second at \( t = t_2 \):
Because of the double interaction we have to integrate twice the result, since \( t_1 \in (0, t_2) \) and \( t_2 \in (0, t) \). For the same reason we have to sum over all the possible intermediate states, \( \varphi_F \). The free time evolution for the various free fields also appear, as well as a \((−i)^2\). Following these same rules we could also give at least a formal expression for the other coefficients, as \( c^{(3)}_{\varphi_f}(t) \), \( c^{(4)}_{\varphi_f}(t) \) and so on: the third order correction \( c^{(3)}_{\varphi_f}(t) \) contains, for instance, a double sum on the intermediate states, allowing in this way a transition from a state with, say, \( n^o_{i,\alpha} \) shares to a state with \( n^f_{i,\alpha} = n^o_{i,\alpha} + 3 \) shares, a triple time integral and a factor \((−i)^3\).

**IV Conclusions**

We have shown how quantum statistical dynamics can be adopted to construct and analyze simplified models of closed stock markets, where no derivatives are considered. In particular we have shown that both the Schrödinger and the Heisenberg pictures can be successfully used in the perturbative analysis of the time evolution of the market: however, the approximations considered in the Heisenberg picture are not completely under control because of the many assumptions adopted as it happens using the Schrödinger wave function of the market and looking for transition probabilities.

We have also shown that the Feynman graphs technique can be adopted for the pertur-
bative analysis of our market, and some simple rules to write down the integral analytic expression for the transition probabilities have been deduced.

Of course a more detailed analysis of the model should be performed, in particular looking for those adjustments which can make more realistic the market we have described so far. These should include for instance source and sink effects to mimicate non conservation of the number of shares, short terms exchanges, financial derivatives and so on. Incidentally, we believe that the hamiltonian $H$ in (2.5) and the related differential equations in (2.7), could be interesting by themselves, hopefully in the description of some (realistic?) many-body system. We hope to consider this aspect in the next future.

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