PHASE TRANSITION IN FIREFLY CELLULAR AUTOMATA ON FINITE TREES

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Abstract

We study a one-parameter family of discrete dynamical systems called the $\kappa$-color firefly cellular automata (FCAs), which were introduced recently by the author. At each discrete time $t$, each vertex in a graph has a state in $\{0, \ldots, \kappa - 1\}$, and a special state $b(\kappa) = \lfloor \frac{\kappa - 1}{2} \rfloor$ is designated as the ‘blinking’ state. At step $t$, simultaneously for all vertices, the state of a vertex increments from $k$ to $k + 1 \mod \kappa$ unless $k > b(\kappa)$ and at least one of its neighbors is in the state $b(\kappa)$. A central question about this system is that on what class of network topologies synchrony is guaranteed to emerge. In a previous work, we have shown that for $\kappa \in \{3, 4, 5\}$, every $\kappa$-coloring on a finite tree synchronizes if the maximum degree is less than $\kappa$, and asked whether this behavior holds for all $\kappa$. In this paper, we answer the question positively for $\kappa = 6$ and negatively for all $\kappa \geq 7$ by constructing counterexamples on trees with maximum degree at most $\kappa/2 + 1$.

Keywords:
synchronization, coupled oscillators, cellular automaton, finite trees, phase transition

1. Introduction

Many biological complex systems consist of levels of hierarchies of locally interacting dynamic units, whose internal dynamics are induced by non-linear aggregation of local interactions between units at lower levels. Top levels are forced to have a certain macro-behavior suitable for survival, which is miraculously supplied by the right micro-level local interactions, forged by the evolutionary process. This chain of emergent dynamics is at the heart of the challenge we are facing in understanding not only biological systems, but also many other complex systems in our society as well as in designing cooperative control protocol of large networked systems [15], [19].

Consisting of only two levels of hierarchies with simple internal dynamics for units at the bottom level, system of coupled oscillators has been a central subject in non-linear dynamical systems literature for decades [18]. As populations of blinking fireflies [4] and circadian pacemaker cells [7] do, two neighboring oscillators are coupled so that they tend to synchronize their phase or frequency, and the question is that whether such local tendency to synchrony does lead to global synchronization in the entire network. Despite their simplicity they exhibit many fundamental difficulties which repel our traditional reductionist approach based on linear methods, and yet our enhanced knowledge on such systems is finding fruitful applications, ranging from robotic vehicle networks [15] to electric power networks [6], and more recently, to distributed control of wireless sensor networks [11], [17], [21], [20].

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Formulated in a discrete setting, understanding the tight interplay between non-linear local interaction of coupled oscillators and underlying network topology gives a fascinating combinatorial problem. A combinatorial framework on modeling complex systems is called a generalized cellular automaton (GCA), which we describe here. Given a simple connected graph $G = (V, E)$ and a fixed integer $κ ≥ 2$, the microstate of the system at a given discrete time $t ≥ 0$ is given by a $κ$-coloring of vertices $X_t : V → Z_κ = Z/κZ$. A given initial coloring $X_0$ evolves in discrete time via iterating a fixed deterministic transition map (or coupling) $τ : X_t ↦ X_{t+1}$, which depends only on local information at each time step. This generates a trajectory $(X_t)_{t≥0}$, and its limiting behavior in relation to the topology of $G$ and the parameter $κ$ is of our interest.

The problem of designing a GCA model for coupled oscillators which has the capacity to synchronize arbitrary $κ$-coloring on a class of finite graphs has been known as the digital clock synchronization problem in distributed algorithms literature. If one allows $κ$ to grow with the size of $G$, then there is such a solution which works on arbitrary finite graphs (e.g., see Dolev [5] or Arora et al. [1]). Roughly speaking, the idea is that if $κ$ is large enough, then one can let every vertex to adapt the locally maximum color within distance 1 at each time step in parallel; then the globally maximum color would propagate and “eat up” all vertices. In fact, this idea of “tuning toward maximum” dates back to a famous consensus algorithm by Lamport [12]. One can readily see that such algorithm relies on some notion of global total ordering among colors of vertices, which is not supplied for fixed $κ$ due to a cyclic nature of the color space. In fact, this issue arising from the cyclic hierarchy between colors is fundamental to our problem, and in fact is a key source which generates interesting emergent behavior in the system. Hence we may restrict ourselves on GCA models with $κ$ independent of $G$.

Dolev [5] showed that no GCA model using a fixed $κ$ is able to synchronizes arbitrary $κ$-coloring on all connected finite graphs. Roughly speaking, for any such given GCA model, one can construct a symmetric configuration on a cycle of some length so that the vertices have no way to break the symmetry by blindly following a homogeneous local rule. On trees, however, such a construction is topologically prohibited so one may hope that there exists a $κ$-color GCA model which synchronizes all initial $κ$-colorings on any finite trees. Indeed, a 3-color GCA model was studied by Herman and Ghosh [10], and odd $κ ≥ 3$ models by Boulinier, Petit, and Villain [2]. When $κ = 3$, the latter model coincides with another well-known GCA model called the cyclic cellular automaton, which was introduced by Bramson and Griffiths [4] as a discrete time analogue of the cyclic particle systems. In a recent work with Gravner and Sivakoff [8], we studied the limiting behavior of 3-color cyclic cellular automaton together with the 3-color Greenberg-Hastings model [9] on infinite trees using probabilistic methods.

The model we are interested in the present work is a one-parameter family of GCAs which we call the $κ$-color finitely cellular automata (FCAs), proposed by the author in a recent work [13] as a discrete model for pulse-coupled inhibitory oscillators. The model is defined for each integer $κ ≥ 3$. Among $κ$ possible colors for each vertices, a special state $b(κ) = \left\lfloor \frac{κ - 1}{2} \right\rfloor$ is designated as the ‘blinking’ color. In a network of $κ$-state identical oscillators, each oscillator updates from state $i$ to $i + 1 (\text{mod } κ)$ unless it sees a neighbor and notices that its phase is ahead of the blinking neighbor, in which case it waits for one iteration without update. More precisely, the transition map $τ : X_t ↦ X_{t+1}$ for the $κ$-color FCA is given as follows:

\[
(\text{FCA}) \quad X_{t+1}(v) = \begin{cases} 
X_t(v) & \text{if } X_t(v) > b(κ) \text{ and } |\{u ∈ N(v) : X_t(u) = b(κ)\}| \geq 1 \\
X_t(v) + 1 (\text{mod } κ) & \text{otherwise}
\end{cases}
\]

where $N(v)$ denotes the set of all neighbors of $v$ in $G$. We call a unit of time a “second”. We say a vertex $v$ blinks at time $t$ if $X_t(v) = b(κ)$, is pulled at time $t$ if $X_{t+1}(v) = X_t(v)$, and pulls its neighbor $u$ at time $t$ if $u$ is pulled at time $t$ and $v$ blinks at time $t$. Given a $κ$-color FCA
trajectory \((X_t)_{t \geq 0}\) on a graph \(G = (V, E)\), we say \(X_t\) (or \(X_0\)) synchronizes if there exists \(N \geq 0\) such that \(X_t \equiv \text{Const.}\) for all \(t > N\).

Being a deterministic dynamical system with finite state space for each vertex, any \(\kappa\)-color FCA trajectory \((X_t)_{t \geq 0}\) on any finite graph \(G = (V, E)\) must converge to a periodic limit cycle. Limit cycles can be either a synchronous or asynchronous periodic orbit, as illustrated in the examples of 6-color FCA trajectories in Figures[1]. Note that \(b(6) = 2\) is the blinking state in this case, so every vertex of state 3, 4, or 5 with a state 2 neighbor stops evolving for 1 second and all the other vertices evolve to the next state.

![FCA Trajectories](image)

Figure 1: Two examples of synchronizing 6-color FCA trajectories are shown in (a) and (b). In (c) and (d), the last configurations are symmetric to the initial ones, so the networks do not synchronize.

In [13], we have shown that for any \(\kappa \geq 3\), arbitrary \(\kappa\)-coloring on finite paths synchronizes in finite time (Theorem 2). This result is pushed further on infinite paths in a joint work with David Sivakoff [14], in the sense that if the initial \(\kappa\)-coloring on the integer lattice \(\mathbb{Z}\) is given at random according to the uniform product measure, then the probability that there is only one color on a fixed finite interval at time \(t\) converges to 1.

On finite trees, however, one does not have such a universal synchronization behavior for all \(\kappa \geq 3\). As illustrated by example (c) in Figure[1] there exists a tree with a non-synchronizing 6-configuration. The obstruction there is that the center of a star with many leaves could be delayed by the leaves constantly. In general, let \(v\) be a vertex in finite tree \(T\) with degree \(\geq \kappa\), and let \(T_1, \ldots, T_m\) be the connected components of \(T - v\), the graph obtained from \(T\) by deleting \(v\) together with edges incident to it. Note that \(m \geq \kappa\). Assign state \(i \mod \kappa\) to every vertex of \(T_i\), and assign any state \(> \kappa/2\) to vertex \(v\). Then \(v\) never blinks and each component \(T_i\) never get pulled by \(v\), which is essentially the counterexample in Figure[1] (c). Therefore if every \(n\)-configuration on \(T\) synchronizes, then necessarily \(T\) has maximum degree \(< \kappa\). In [13], we showed that such a necessary local condition to synchronize arbitrary \(\kappa\)-coloring on a tree is also sufficient for \(\kappa \in \{3,4,5\}\), but not necessarily for \(\kappa = 7\). In the present work, we characterize this behavior for all \(\kappa \geq 3\) and obtain the following result:

**Theorem 1.**

(i) *If \(\kappa \in \{3,4,5,6\}\) and \(T = (V, E)\) is any finite tree, then every \(\kappa\)-coloring on \(T\) synchronizes iff \(T\) has maximum degree \(< \kappa\).*

(ii) *If \(\kappa \geq 7\), then there exists a finite tree \(T = (V, E)\) with maximum degree \(\leq \kappa/2 + 1\) and a non-synchronizing \(\kappa\)-coloring on \(T\).*
In this context, Theorem 1 tells us that there is a critical number of colors “between” 6 and 7; with fewer colors, maximum degree $< \kappa$ implies synchronization of arbitrary $\kappa$-coloring, and with more colors, there are non-synchronizing examples on trees with maximum degree $\leq \kappa/2 + 1$. This is analogous to the clustering-fixation phase transition of the $\kappa$-color cyclic cellular automaton on $\mathbb{Z}$ (see Fisch [7].)

The most substantial result in the above theorem is the $\kappa = 6$ case of part (i). To highlight some of the difficulties in the $\kappa = 6$ case, we briefly recap our strategy for $\kappa \in \{3,4,5\}$ cases in [13]. Let $(T, X_0)$ be a minimal counterexample to Theorem 4.1. Viewing $T$ as a rooted tree, for each $v \in V(T)$, denote by $T_v$ the subtree consisting of $v$ and all of its descendants. For $\kappa \in \{3,4,5\}$, the minimality enforces very low entropy on possible local dynamics on $T_v$, and considering possible fluctuation on $v$ from the complement $T - T_v$, we rule out each of such local dynamics and obtain contradiction. For $\kappa = 3$ and 5, it is enough to take $T_v$ to be of depth 1; for $\kappa = 4$, it is sufficient to analyze depth 2 subtrees.

But ‘near the criticality’ when $\kappa = 6$, the entropy of induced dynamics gets substantially high so that local dynamics analysis up to any fixed depth is not enough. To overcome this difficulty, we first observe that the induced local dynamics on $T_v$ give constraints on the inter-blinking times of the root of $v$, which we denote by $v^{-1}$. By combining all possible constraints on $v^{-1}$ from its descendant subtrees, we can then deduce the constraints on the root of $v^{-1}$. Proceeding recursively, which involves long and technical analysis, we arrive at the contradiction that the root of entire tree $T$ must have its parent.

This paper is organized as follows. We summarize some basic facts and lemmas about FCA we have established in [13] in Section 2. A quick proof of Theorem 1(ii) is given in Section 3. An outline of the Proof of Theorem 4.1 for $\kappa = 6$ will be given in in Section 4, together with its proof assuming two key lemmas. Section 5 is devoted to a preliminary analysis on induced local dynamics from a minimal counterexample to the $\kappa = 6$ assertion. In subsequent sections, Section 6 and 7, we prove the two key lemmas and complete the proof of Theorem 4.1 for $\kappa = 6$.

2. Generalities and the branch width lemma

It will be convenient to introduce a geometric representation of the FCA dynamics. Let $(X_t)_{t \geq 0}$ be a $\kappa$-color FCA dynamics on a graph $G = (V,E)$. The idea is to consider the induced dynamics $(Y_t)_{t \geq 0}$, where $Y_t : V \cup \{\alpha\} = Z_\kappa$ is the relative configuration given by

$$Y_t(x) = \begin{cases} b(\kappa) - t \mod \kappa & \text{if } x = \alpha \\ X_t(x) - t + b(\kappa) \mod \kappa & \text{otherwise.} \end{cases}$$

We refer to the value $Y_t(v)$ the phase of $v$ at time $t$. Note that in the original dynamics, a node blinks whenever $X_t = b(\kappa)$, so in the relative dynamics $(Y_t)_{t \geq 0}$, a node blinks whenever it has phase $-t \mod \kappa$. In this relative dynamics vertices keep the same phase until they get pulled, in which case they decrease their phase by 1. A comparison between the original dynamics and the relative dynamics in case $\kappa = 6$ is illustrated by example in Figure 2. The geometric representation of the relative FCA dynamics in the second row in Figure 2 is what we call the relative circular representation. The hexagon represents the phase space, which is the original color space $Z_6$ modulo rotation, increasing in clockwise orientation. The open circle inside it, called the activator, revolves around the phase space counterclockwise at unit speed, whose location at time $t$ is $-t \mod 6$. Hence whenever a vertex has the same phase as the activator, the node blinks.
Let \( u, v \) be two vertices in \( G \). The *counterclockwise displacement of \( v \) from \( u \) at time \( t \) is defined by
\[
\delta_t(u, v) := X_t(u) - X_t(v) \pmod{k}.
\]
We say \( v \) is *counterclockwise to \( u \) and \( u \) is *clockwise to \( v \) at \( t \) if \( \delta_t(u, v) < n/2 \), and \( u \) is *opposite to \( v \) if \( \delta_t(u, v) = n/2 \), which can happen only if \( n \) is even. Suppose \( u \) and \( v \) are adjacent in \( G \).
We say \( v \) is a *clockwise neighbor of \( u \) at \( t \) if \( v \) is clockwise to \( u \) at \( t \), and *counterclockwise neighbor at \( t \) otherwise. The *width of a \( X_t \) (or \( Y_t \)) is defined to be the quantity
\[
w(X_t) := \min_{v \in V} \max_{u \in V} \delta_t(u, v),
\]
which is the length of the shortest path on the color space \( \mathbb{Z}_n \) (viewed as a cycle of length \( n \)) that covers all states of the vertices in the configuration. For instance, the first configuration in Figure 2 has width 4, whereas the last one has width 3. For any subgraph \( B \subset G \), we denote by \( w_B(X) \) the width of the restricted configuration \( X|_{V(B)} \) on \( B \).

A classic observation in the theory of pulse-coupled oscillators is that the width \( w(X_t) \) at time \( t \) converges to 0 monotonically if \( w(X_0) < \kappa/2 \). Roughly speaking, the intuition is that if at some point the width at time \( t = s \) is strictly less then half of the perimeter of color space \( \mathbb{Z}_x \), then one can define a global total ordering on all occupied phases at time \( t = s \) from the most lagging to the most advancing. Under a very mild condition on the coupling, this total ordering is respected by the dynamics and the farthest displacement monotonically decreases. For more details see Lemma 2.2 and following discussions in [13]. Its key mechanism is illustrated in the example in Figure 3.

A natural extension of the above observation to a proper subgraph is our starting point to understand FCA dynamics on finite trees, which we shall introduce now. A connected subgraph \( S \subset G \) is called a *k-star* if it has a vertex \( v \), called the *center*, such that all the other vertices of \( S \) are leaves in \( G \). A k-star \( S \) is called a *k-branch* if the center of \( S \) has only one neighbor in \( G - S \), which we may call the *root* of \( S \). We may denote a k-branch by \( B \) rather than by \( S \). Note that branches are smallest induced subgraphs of trees with a single vertex.
adjacent to its complement. Hence it is the smallest subgraph which gets minimal perturbation from outside and yet it should have a simple internal dynamics. Our observation is that if \( w_B(X_0) < \kappa/2 - 1 \), then such small branch width is maintained in the dynamics and the global dynamics restricts on the complement \( G - B \).

We say a dynamic \((X_t)_{t \geq 0}\) on \( G \) restricts on \( H \subset G \) if the restriction \( X_t \mapsto X_t|_H \) and transition map \( r \) commute, i.e., the induced restricted dynamic \((X_t|_H)_{t \geq 0}\) follows the same transition map on \( H \). We say the dynamic \((X_t)_{t \geq 1}\) on \( G \) restricts on \( H \) eventually if there exists \( r \geq 0 \) such that \((X_t)_{t \geq r} \) restricts on \( H \).

**Lemma 2.1** (branch width lemma). Let \( G = (V, E) \) be a graph with a \( k \)-branch \( B \) rooted at some vertex \( w \in V \). Let \( u \) be the center of \( B \) and \( l_1, \ldots, l_k, k \geq 1 \) be its leaves. Let \( H \) be the graph obtained from \( G \) by deleting the leaves of this branch. Let \( X_0 \) be a \( \kappa \)-coloring on \( G \) for any \( \kappa \geq 3 \) with \( w_B(X_0) < \kappa/2 - 1 \). Then we have the followings:

(i) if \( u \) is clockwise to all leaves of \( B \) at some time \( r \leq n(w_B(X_0) + 1) \) and \( w_B(X_r) \leq w_B(X_0) \);
(ii) \( u \) is clockwise to all leaves of \( B \) for all \( t \geq r \), and \( w_B(X_t) \leq w_B(X_0) + 1 \) for all \( t \geq 0 \);
(iii) if \( u \) is clockwise to all leaves at \( t = r \), then the dynamic \((X_t)_{t \geq r} \) restricts on \( H \);
(iv) if every \( \kappa \)-coloring on \( H \) synchronizes, then \( X_0 \) synchronizes.

A detailed proof can be found in [13], and here we give a brief sketch through an example. Suppose \( \kappa = 8 \) and \( k = 3 \). Since the coupling is inhibitory, the leaves of \( B \) and the root \( w \) only pulls \( u \) until it becomes the most lagging one in \( B \). So eventually, we will have a situation as in the first diagram in Figure 4, where the branch width \( w_B \) is still strictly less than \( \kappa / 2 - 1 \) and the center \( u \) is at most lagging in \( B \). Now the root \( w \) pulls \( v \) at most once in every \( \kappa \) seconds, increasing the branch width by 1. But since \( w \) is closer to the outside of the branch, the increased branch width is still small (< \( \kappa / 2 \)) and the leaves do not pull \( u \) until its next blink. Then the center \( u \) blinks and pulls all leaves, decreasing the branch width by 1. Hence the original branch width is recovered, and this scenario repeats over and over again. In this cycle the leaves never pull the center, so the dynamics restricts on \( H \).

![Figure 4: An illustration of branch width recovery for \( n = 8 \) and \( k = 3 \). Once \( u \) is the most lagging, \( w \) can pull \( u \) to increase the branch width by 1 but \( u \) pulls the most advancing leaves and decrease the branch width by 1, before \( w \) blinks again. Note that \( w \) could get external pulls from its neighbors different from \( u \) but it doesn't affect our argument.](image)

3. FCA on finite trees when \( \kappa \geq 7 \).

In this section, we prove the following result, which implies Theorem 1.1 for \( \kappa \geq 7 \):

**Theorem 2.1.** Let \( \kappa \geq 7 \) be an integer.

(i) If \( \kappa = 2m - 1 \geq 3 \) is odd, there exists a finite tree \( T = (V, E) \) with maximum degree \( m \) and a \( \kappa \)-coloring \( X_0 : V \to \mathbb{Z}_\kappa \) such that \( X_0 \) is non-synchronizing whose period divides \( 3\kappa^2 + \kappa \).

(ii) There exists a tree of maximum degree 4 with a non-synchronizing 8-coloring which is 60 periodic.

(iii) If \( \kappa = 2m \geq 10 \) is even, there exists a finite tree \( T = (V, E) \) with maximum degree \( m + 1 \) and a \( \kappa \)-coloring \( X_0 : V \to \mathbb{Z}_\kappa \) such that \( X_0 \) is non-synchronizing whose period divides \( 5\kappa^2 + \kappa \).
Proof. (i) Note that \( b(\kappa) = m - 1 \geq 3 \). Let \( p, q \) be integers such that \( p, q \geq 2 \) and \( p + q = m \). Let \( T = (V, E) \) be a star with \( m \) leaves \( v_1, \ldots, v_p, u_1, \ldots, u_q \) and center \( w \). Define a relative \( \kappa \)-coloring \( Y_0 : V \cup \{a\} \to Z_\kappa \) on \( T \) by

\[
Y_0(x) = \begin{cases} 
  m - 1 & \text{if } x \in \{w, a\} \\
  i & \text{if } x = v_i \text{ for some } 1 \leq i \leq p \\
  m + j & \text{if } x = u_j \text{ for some } 1 \leq j \leq q.
\end{cases}
\]

To show the assertion for \( Y_j \), we claim that

\[
Y_{3k+1}(x) = Y_0(x) - 1 \quad \forall x \in V.
\]

We begin with observing that in the first \( \kappa \) iterations, \( w \) blinks once at time 0 to pull each of \( u_j \)'s, and is pulled by each of \( v_i \)'s followed by each of \( v_j \)'s. Hence we have

\[
Y_k(x) = \begin{cases} 
  -1 & \text{if } x = w \\
  i & \text{if } x = v_i \text{ for some } 1 \leq i \leq p \\
  m + j - 1 & \text{if } x = u_j \text{ for some } 1 \leq j \leq q.
\end{cases}
\]

Then since \( 0 \leq \delta_\kappa(v_i, w) \leq m \) for each \( 1 \leq i \leq p \), \( w \) is not pulled by any of the \( u_j \)'s during \([\kappa, \kappa + m] \). Then it pulls all of \( u_j \)'s at time \( t = \kappa + m \), and is pulled by each of \( u_j \)'s during \([\kappa + m, 2\kappa] \). This yields

\[
Y_{2k}(x) = \begin{cases} 
  -1 - q & \text{if } x = w \\
  i - 1 & \text{if } x = v_i \text{ for some } 1 \leq i \leq p \\
  m + j - 1 & \text{if } x = u_j \text{ for some } 1 \leq j \leq q.
\end{cases}
\]

Finally, \( \delta_{2k}(w, v_p) = m \), so \( w \) is pulled by each of \( v_j \)'s during \([2\kappa, 2\kappa + m - 1]\) so that \( Y_{2k+m-1}(w) = -1 - q - p = m - 2 \mod \kappa \). Then \( \delta_{2k+m-1}(w, u_q) = m + a - 2 \in [m, \kappa - 1] \), so \( w \) is not pulled by any of \( u_j \)'s during \([\kappa + m - 1, 3\kappa] \). Moreover, \( Y_{3k+1}(w) = Y_{3k+1}(a) = m - 2 \). This shows the claim.

(ii) Let \( T = (V, E) \) be a tree where \( V = \{v_1, \ldots, v_8\} \) and edges are determined by \( N(v_4) = \{v_1, v_2, v_3, v_5\} \) and \( N(v_5) = \{v_4, v_6, v_7, v_8\} \). Note that \( T \) has maximum degree 4. Let \( X_0 : V \to Z_8 \) be the initial 8-coloring on \( T \) defined by

\[
(X_0(v_1), \ldots, X_0(v_8)) = (1, 5, 7, 5, 6, 0, 3, 6).
\]

Then it is straightforward to check that \( X_t \) satisfies the assertion.

(iii) Note that \( b(\kappa) = m - 1 \). Let \( T = (V, E) \) be a star with center \( w \) and \( L = V \setminus \{w\} \) the set of all leaves. For each integers \( p, q, r \geq 2 \) such that \( p + q + r = m + 1 \), let \( Y_{p,q,r} \) be the set of all relative \( \kappa \)-colorings \( Y^{p,q,r} \) on \( T \) with the following properties:

(a) \( Y^{p,q,r}(w) = Y^{p,q,r}(a) = m - 1 \)

(b) \( Y^{p,q,r}[L] = [1, p] \cup [m - q, m - 1] \cup [m + p, m + p + r - 1] \).

Note that \( Y^{p,q,r} \) uses exactly \( p + q + r = m + 1 \) colors for the leaves. Fix an initial \( \kappa \)-coloring \( Y_0 \in Y_{p,q,r} \). By a similar reasoning as in (i), it is easy to check that \( Y_{3m+q-m+q} = \kappa \). By iterating this observation, we get \( Y_{6m+q+r-q-r} \in Y_{p,q,r} \) and \( Y_{6m+q+p+r-m+p+q+r} = Y_{5k+1} + 1 \in Y_{p,q,r} \). In fact, it is easy to see that \( Y_{5k+1} + 1 = Y_0 \) on \( V \). As in the proof of (i), this shows that \( X_t \) does not synchronize and its period divides \( 5k^2 + \kappa \).

\( \square \)

For (ii), we remark that all 8-colorings on any star synchronizes.
4. Outline of the proof of Theorem 1 for $\kappa = 6$.

In the rest of this paper, we devote ourselves to prove the following statement:

**Theorem 4.1.** Let $T$ be a finite tree and let $X_0$ be a 6-coloring on $T$. Then $X_0$ synchronizes if and only if every vertex of $T$ blinks infinitely often in the orbit $(X_t)_{t \geq 0}$.

Note that this implies the $\kappa = 6$ case of Theorem 1 by the following lemma (Lemma 5 in [13]):

**Lemma 4.2.** Let $G = (V, E)$ be a graph and let $u$ be a vertex. Suppose $\deg_G(u) < \kappa$. Let $X_0$ be any $\kappa$-coloring on $G$. Then $u$ blinks infinitely often in the orbit $(X_t)_{t \geq 0}$.

Trees have nice recursive property when viewed as rooted trees. A rooted tree $T = (V, E)$ is a tree with a designated vertex $r$ called the root. For each $v \in V$, let $P_v$ be the unique path from $r$ to $v$, and let $v^-$ be the unique neighbor of $v$ in $P_v$. For distinct vertices $u, v$ in $T$, we say $u$ is the parent of $v$ and $v$ is a child of $u$ if $u = v^-$. Write $u \leq v$ if $u \in V(P_v)$. We say $u$ is an ancestor of $v$ and $v$ is a descendant of $u$ if $u \leq v$. For each $v \in V$, we define the descendant subtree $T_v$ to be the subtree of $T$ consisting of all descendants of $v$. The depth of $T_v$, denoted by $\dep(v)$, is the maximum level of vertices in $T_v$. We say a descendant subtree $T_v$ is a terminal branch if it is a branch and either $T_v = T$, or $v^- \in V$ and for all children $u$ of $v^-$, $T_u$ is either a singleton or a branch. Note that any rooted tree with depth $\geq 2$ has at least one terminal branch.

Now for each $\kappa \geq 3$, call a pair $(T, X_0)$ of a finite tree and a $\kappa$-coloring on it a minimal counterexample if (1) $X_0$ does not synchronize, (2) every vertex in $T$ blinks infinitely often in the dynamic $(X_t)_{t \geq 0}$, and (3) $|V(T)| \leq |V(T')|$ if $(T', X_0')$ is another pair with satisfying (1) and (2). We may assume without loss of generality that $(X_t)_{t \geq 0}$ is periodic, by choosing $X_0$ from the periodic limit cycle. Note that by the minimality of $T$ and Lemma 2.1 every branch in a minimal counterexample must have branch width $\geq \kappa/2 - 1$ for all times. This enforces a very specific local dynamics on branches which easily led to contradiction in case of $\kappa \in \{3, 4, 5\}$. Namely, let $(T, X_0)$ be a minimal counterexample for $\kappa \in \{3, 4, 5\}$. In [13], we have proceeded as follows:

- $\kappa = 3$. Every branch in $T$ eventually has small branch width, a contradiction;
- $\kappa = 5$. If $B$ is a branch in a $T$, then the dynamics restricts eventually restricts on $T$ less the leaves of $B$, a contradiction;
- $\kappa = 4$. If $T_v$ is a terminal branch in $T$, then the dynamics restricts on $T - T_v$ eventually, a contradiction.

The argument for $\kappa = 4$ is notable. Suppose $w$ is a vertex in $T$ such that each descendant tree $T_v$ at its children is either a leaf or a branch (e.g., take $w$ to be the parent of the center of a terminal branch). Each subtree $T_v$ must have one of a few local dynamics enforced from the minimality, which gives some constraint on the local dynamic on $w$. Roughly speaking, the contradiction is obtained by showing that such constraints from multiple components are not compatible. Unfortunately, however, for $\kappa = 6$, the enforced dynamics on branches still has large entropy and one should consider the ensemble of all possible local dynamics joining at the common root. Moreover, analyzing enforced dynamics on depth 2 descendant trees is not enough; in fact, we have to go all the way down to the root to get a contradiction.

Let $(T, X_0)$ be a minimal counterexample for $\kappa = 6$, and we fix this notation hereafter throughout later sections. Let $T_v$ be a proper descendant subtree in $T$. We say $T_v$ is open if the induced dynamics on $T_v$ requires $v$ to be pulled by its parent $v^-$. In particular, the whole tree $T = T_r$ cannot be open. If $T_v$ is open, then the minimality will force particular induced local
dynamics on its root $v^-$. To represent induced local dynamics on a single vertex concisely, we introduce the following notion. For each vertex $v$ in $T$, let $b_{t,i}(v)$ be the time of $i$th blink of $v$ in the orbit, for each $i \geq 1$. The $i$th blinking gap is given by $g_i(v) = b_{t,i+1}(v) - b_{t,i}(v)$. The blinking sequence of $v$ is defined by the sequence $(g_i(v))_{i \geq 1}$ of blinking gaps of $v$. Note that since we are in a periodic orbit, the blinking sequence of any $v \in V$ repeats a finite sequence of blinking gaps. There will be mainly four different types of entorced dynamics on $T_v$ as we define below:

**Definition 4.3.** Let $(T, X_0)$ as before and let $T_v$ an open descendant subtree of $T$.

(i) We say $T_v$ is of type (a) iff $g_i(v) \equiv 12 \text{ and } 2 \in \{X_{t+4}(v^-), X_{t+5}(v^-)\}$ whenever $X_i(v) = 2$;

(ii) We say $T_v$ is of type (b) iff $g_i(v)$ alternates 9 and 7, and 2 $\in \{X_{t+2}(v^-), X_{t+4}(v^-)\}$ whenever $X_i(v) = X_{t+9}(v) = 2$;

(iii) We say $T_v$ is fractal of type 10/9 iff $g_i(v)$ alternates 10 and 9, and 2 $\in \{X_{t+1}(v^-), X_{t+4}(v^-)\}$ whenever $X_i(v) = X_{t+10}(v) = 2$;

(iv) We say $T_v$ is fractal of type 11/8 iff $g_i(v)$ alternates 11 and 8, and 2 $\in \{X_{t+2}(v^-), X_{t+4}(v^-)\}$ whenever $X_i(v) = X_{t+11}(v) = 2$.

We say $T_v$ is fractal if its fractal of either types.

Below we give a more direct characterization of type (a), (b), and fractal subtrees in terms of the induced dynamics on $v$ and its parent $v^-$. Suppose a descendant subtree $T_v$ is of type (a). From the definition, it is easy to see that the local dynamics on $v$ and $v^-$ are given by concatenating the following four sequences:

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\
v^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\
v^- & 4 & 4 & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\
v^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\
v^- & 5 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 \\
\end{array}
\]

where time goes from left to right and none of $-$’s are 2. Each of the above sequences describe dynamics on $v$ and $v^-$ for 12 iterations, and the sequence $(P)(P)$ obtained by concatenating the sequence (P) twice describes 24 iterations, for instance. The four possibilities came from considering possible instances of $v^-$ blinking after its first blink during each blinking gap 12 of $v$.

Similarly, if $T_v$ is of type (b), then the local dynamics on $v$ and $v^-$ are given by concatenating the following six sequences:

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\
v^- & 0 & 1 & 2 & 3 & - & - & - & - & - & - & - & - & - & - & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\
v^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - & - & - & - & - & - & - \\
\end{array}
\]

\[
\begin{array}{c}
v & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\
v^- & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - \\
\end{array}
\]
being rooted at \( r \)

Proper descendant subtree that dep

A terminal branch, which is open by Proposition 4.4, a contradiction. Hence we may assume

Let \( \kappa \)

This would yield that \( T \)

Next, the induction step is based on the recursive property of fractal branches stated in the following lemma:

\[
\begin{align*}
\nu & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_7 \quad a_8 \quad - \quad - \quad - \quad - \\
\nu & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad b_3 \quad b_4 \quad b_5 \quad b_7 \quad b_8 \quad - \quad - \quad - \quad - \\
\nu & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 5 \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad d_3 \quad d_4 \quad d_5 \quad d_6 \quad d_8 \quad - \quad - \quad - \quad - \\
\end{align*}
\]

where none of \( \sim \)'s are 2, as before.

Finally, the same holds for the following two sequences

\[
\begin{align*}
\nu & \quad 2 \quad 3 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad - \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_7 \quad a_8 \quad - \quad - \quad - \\
\nu & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 5 \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad b_3 \quad b_4 \quad b_5 \quad - \quad - \quad - \\
\nu & \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 5 \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad d_3 \quad d_4 \quad d_5 \quad d_6 \quad d_8 \quad - \quad - \quad - \\
\end{align*}
\]

When \( T_v \) is fractal of type 10/9, and with the following two sequences for \( T_w \) fractal of type 11/8:

\[
\begin{align*}
\nu & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad c_6 \quad c_8 \quad - \quad - \quad - \\
\nu & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
\nu^- & \quad 5 \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad d_3 \quad d_4 \quad d_5 \quad - \quad - \quad - \\
\end{align*}
\]

As before, none of \( \sim \)'s are 2 in any of the sequences above, but other instances \( a_i \)'s, \( b_i \)'s, \( c_i \)'s, and \( d_i \)'s could be 2. We could specify all sequences when \( \nu^- \) could blink among those instances, but there would be too many cases in doing so.

Now we outline the proof of Theorem 4.1 for \( \kappa = 6 \). In a nutshell, we show that every proper descendant subtree \( T_v \) of depth \( \geq 1 \) is fractal. In particular, every component in \( T_v - r \) will be either a singleton or fractal. A recursive property of fractal subtrees would then yield that the whole tree is fractal, and in particular, open. This contradiction shows that minimal counterexample for \( \kappa = 6 \) does not exist. To give more detail, we first show by using Lemma 2.4 that every branch must be open and of type (a) or (b), or fractal of type 10/9. Furthermore, we will show that if \( T_v \) is a terminal branch, then it cannot be of type (a) or (b), as stated in the following lemma:

**Lemma 4.4.** Let \((T, X_0)\) be as before. Then every terminal branch of \( T \) is fractal.

Next, the induction step is based on the recursive property of fractal branches stated in the following lemma:

**Lemma 4.5.** Let \((T, X_0)\) be as before. Let \( w \in V \) and suppose that each connected component of \( T_w - w \) is either a singleton or a fractal branch. Then \( T_w \) is open and fractal. In particular, \( w^- \in V(T) \).

These two lemmas easily imply the main theorem.

**Proof of Theorem 4.1** for \( \kappa = 6 \). It suffices to show the "if" part. For the contrary, suppose there exists a minimal counterexample \((T, X_0)\) for \( \kappa = 6 \). Choose a vertex \( r \) and view \( T \) as being rooted at \( r \). By minimality, the depth of \( T = T_r \) is at least 1. If \( \text{deg}(r) = 1 \), then \( T = T_r \) is a terminal branch, which is open by Proposition 4.4, a contradiction. Hence we may assume that \( \text{deg}(r) \geq 2 \).

It suffices to show that for every non-leaf and non-root vertex \( v \), \( T_v \) is fractal. Indeed, this would yield that \( T_r - r \) is a disjoint union of leaves and fractal subtrees, but by Lemma
$T_v$ must be fractal, a contradiction. We proceed by an induction on $\text{dep}(v^-) \geq 2$. For the base step, note that $\text{dep}(v^-) = 2$ means that $T_v - v^-$ is a disjoint union of leaves and terminal branches. Since by Lemma 4.4 terminal branches are fractal, Lemma 4.5 gives that $T_v$ is fractal. The induction step follows similarly. If $\text{dep}(v^-) = d \geq 3$, then $T_v - v^-$ is a disjoint union of leaves and depth $< d - 1$ descendant subtrees. By induction hypothesis and Lemma 4.5, $T_v$ must be fractal. This shows the assertion. ■

5. Analysis of enforced local orbits on branches

Throughout this section, $(T, X_0)$ is a minimal counterexample for $\kappa = 6$. For each descendant subtree $T_v$, the minimality forces a particular local dynamic on $T_v$ and see how they restrict the dynamics on $v^-$ when $T_v$ is either a leaf, branch, or a fractal branch. Furthermore, we investigate possible ensemble of such constraint on the local dynamics of $v^-$ when it has multiple descendant subtrees rooted at itself. A conceptual background is a classic technique in dynamical systems literature called the Poincaré return map, which is to look at transitions between snapshots of system configuration where a particular vertex takes a particular state. We adapt this concept in a local setting: we consider all possible local configurations on a descendant subtree $T_v$ in which $v$ blinks. Since we are assuming that $v$ blinks infinitely often in the dynamic, the global periodic orbit $(X_t)_{t \geq 0}$ must induce a periodic orbit on such special local configurations, together with constraints on the local dynamics on $v^-$. We will rely heavily on diagrammatic analysis to study possible blinking sequences of $v^-$ and their ensemble. We shall represent local dynamics on $T_v$ often as a weighted digraph, in which edge weights represent blinking gaps of $v$ and nodes could be snapshots of local configurations or a finite sequence of local dynamics. Let us first introduce some terminologies. Let $D = (V, E)$ be a digraph with vertex and edge weights $\omega : V \cup E \rightarrow \mathbb{N} \cup \{0\}$. We say a sequence $(a_n)$ of positive integers is generated by $D$ if there exists a directed walk $P = v_1 e_1 v_2 e_2, \cdots$ in $D$ such that $(a_n)$ can be obtained from the sequence $\omega(v_1), \omega(e_1), \omega(v_2), \cdots$ by dropping the zero terms. For example, consider a digraph $D$ with vertex set $V = \{X, Y\}$ and edge set $E = \{(XX), (XY), (YX), (YY)\}$ with weights given as in Figure 19.

![Figure 5: A digraph that can generate finite sequence 3,11,6,7,6,5,3.](image)

Whenever a vertex has weight 0, we shall omit the weight in the diagram. Notice that the directed walk $P = X, (XX), X, (XY), Y, (YY), Y, (YX), X, (XX)$ gives the sequence of weights 0,3,11,6,7,6,5,0,3. By dropping out the zero terms, we see that the given digraph can generate the sequence 3,11,6,7,6,5,3.

A first example comes from analyzing local dynamics on a vertex with a leaf neighbor. We begin with a simple example.

**Proposition 4.1.** Let $(T, X_0)$ as before. Let $v$ be a vertex in $T$ with a leaf neighbor $u$. Then $u$ is never opposite to $v$ whenever $v$ blinks. That is, $X_t(u) \neq 5$ if $X_t(v) = 2$.

**Proof.** Back-tracking three iterations from such local configuration leads to contradiction, as
below:

\[
\begin{array}{c|c}
\text{v} & \text{u} \\
2|5 & 1|4 & 0|3 & 5|2
\end{array}
\]

as blinking \(u\) must have been pulled \(v\) at color 5.

**Proposition 4.2.** Let \((T, X_0)\) as before. Let \(v\) be a vertex in \(T\) with a leaf neighbor \(u\). Then the blinking sequence of \(v\) is given by 

\[
(a_i + 6k_i)_{i \geq 1}
\]

where \((a_i)_{i \geq 1}\) is generated by the digraph in Figure 6 and \((k_i)_{i \geq 1}\) is some sequence of non-negative integers which depend on the dynamics.

![Figure 6: A weighted digraph on five possible non-opposite local configurations on the 1-star \(v + u\).](image)

The blinking gap of \(v\) corresponding each transition is given by \(a + 6k\) where \(a\) the edge weight and \(k\) is some nonnegative integer depending on the structure and dynamics on \(G\).

**Proof.** There are five local configurations on the 1-star \(v + u\) with center \(v\) where \(v\) blinks such that \(u\) is not opposite to \(v\). The above digraph shows every possible transition between such five non-opposite local configurations. For the edge weights, note that since \(v\) is the only neighbor of the leaf \(u\), once \(v\) blinks, \(u\) maintains its phase until the next blink of \(v\). This determines the blinking gap of \(v\) during each transition in the above digraph modulo 6.

For example, consider the transition \(d \rightarrow e\) in Figure 6 which is shown in Figure 7.

![Figure 7: The transition \(d \rightarrow e\) in Figure 6. The blinking gap must be \(9 + 6k\) for some nonnegative integer \(k\).](image)

Since the center pulls the leaf initially, the phase of leaf moves one step clockwise after the first iteration. Now the leaf does not move until the next blink of the center, so to get the bottom left local configuration in Figure 16, the center must be at the top of the hexagon by the time it blinks again for the first time. Hence looking that the initial and terminal phase of the activator, we conclude that the blinking gap of \(v\) during this transition is 9 modulo 6. Other edge weights are determined in similar way. This shows the assertion.

Next, we analyze forced local dynamics on branches. In [13] Lemma 3.2, we showed that 1-branches eventually gets small branch width and contradicts the minimality by Lemma 2.1. Hence \(T\) does not have a 1-branch. Moreover, if \(B\) is any \(k\)-branch in \(T\), then all the \(k\) leaves
there should maintain distinct colors for all times, since otherwise we can delete some leaves
and restrict the dynamics on $T$ on a proper subtree, which contradicts the minimality. The
following proposition gives how the blinking sequence of a vertex $v$ is restricted if it has mul-
tiple leaves, which includes the case when $v$ is a center of a branch in $T$. Its proof is given at
the end of this section.

**Proposition 4.3.** Let $(T, X_0)$ be as before. Suppose $T$ has a $k$-star $S$ for $k \geq 2$ with center $v$. Then
we have the followings:

(i) The induced local dynamics on $S$ is given by one of the four digraphs in Figure 8. In partic-
ular, the blinking sequence of $v$ is given by $(a_i + 6k_i)_{i \geq 1}$ where $(a_i)_{i \geq 1}$ is generated by the
digraph in Figure 8 and $(k_i)_{i \geq 1}$ is some sequence of non-negative integers which depend on the
dynamics.

(ii) If $S = T_v$ is a branch, then the induced local dynamics on $T_v$ only uses the five shaded local
configuration in Figure 8.

(iii) If $S = T_v$ has local dynamics given by Figure 8 (a), (b), or (c), then it is open and of type (a),
(b), or fractal of type 10/9, respectively.

Next, we investigate how the three types of closed orbits on a branch restricts the blinking
sequence of its root. Let $(a_n), (b_n)$ be two sequences of real numbers. We say the sequence
$(b_n)$ refines $(a_n)$ and $(b_n)$ is a coarsening of $(a_n)$ if there exists an increasing sequence
$(d_n)$ of natural numbers such that

$$a_n = \sum_{d_n \leq k < d_{n+1}} b_k.$$ 

For instance, the sequence 1, 2, 3, 4, · · · refines 3, 7, 11, 15, · · · since $3 = 1 + 2, 7 = 3 + 4, 11 = 5 + 6,$
and so on.

**Proposition 4.4.** Let $(T, X_0)$ be as before. Let $T_v$ be a branch in $T$ with $v^- \in T$. Then we have
the followings:

(i) If $T_v$ is of type (a), then the blinking sequence of $v^-$ is generated by the digraph (A) in Figure 9.

(ii) If $T_v$ is of type (b) then the blinking sequence of $v^-$ refines a sequence generated by di-
graph(B1) in Figure 9.

(iii) In case of (ii), the blinking sequence $(g_i)$ of $w$ is refined by some sequence $(b_m)_{i \geq 1}$ generated
by diagram (B2) in Figure 9. Furthermore, $(g_i)_{i \geq 1}$ can be obtained from $(b_m)$ by merging
some consecutive terms $b_m, b_{m+1}$ into $b_m + b_{m+1} + b_{m+1}$, where $b_m$ is a vertex weight and $b_{m+1}$ is
following edge weight.
(iv) In case of (iii), the sequence \((b_m)\) cannot be generated by a closed walk on (B2) which only uses nodes Y or Z.

\[
(A) \quad (b_6) = (6, 11, 7, 12, 13)
\]

\[
(B1) \quad (b_1) = (1, 18, 16)
\]

\[
(B2) \quad (b_{12}) = (12, 7, 9, 0, 10, 7, 12, 7, 9, 0, 10, 7, 9)
\]

Figure 9: (A) a digraph generating the blinking sequence of the root \(v^-\) of type (a) branches; (B1) a digraph generating a coarsening of blinking sequence of \(v^-\) of type (b) branches; (B2) a digraph generating a refinement of blinking sequence of \(v^-\) of type (b) branches.

**Proof.** The proof follows mostly from definitions. Let \(T_v\) is of type (a). Then concatenating sequences \((P)\cdot(S)\) gives a complete description of the blinking sequence of \(v^-\). For instance, if string \((P)(P)(Q)\) is used in the local dynamics, then \(v^-\) blinks exactly once in sequence \((P)\), and blinks after 12 iterations again in \((P)\), and then its next blink in \((Q)\) takes 13 iterations. In digraph \((A)\) in Figure 9 this is represented as going through the loop at node \(P\) twice and then using the edge \((PQ)\). Note that the diagram \((A)\) lacks loop at node \(S\) and edges from \(S\) to \(P\) or \(Q\), since those sequences cannot be concatenated in such order; the color of \(v^-\) at the end and beginning does not match. To explain the use of node weight on \(S\) in diagram \((A)\) in Figure 9, consider the string of sequences \((P)\cdot(S)\cdot(Q)\). After the blink within sequence \((P)\), \(v^-\) blinks for the first time in sequence \((S)\) after 12 iterations, and then again for the second time after 6 iterations within sequence \((S)\). Then it takes 7 iterations to blink again within sequence \((Q)\). In terms of diagrams, we walk though the edge weight 12 of \((PS)\), and then node weight 6 of \(S\), and then edge weight 7 of \((SQ)\). This shows (i).

For type (b) branches, observe that if \(v^-\) blinks as sparse as possible in the dynamics, then it would only use the “long periodic” sequences \((I)\) and \((J)\), in which case its blinking sequence is generated by diagram \((B1)\) in Figure 9. On the other hand, if \(v^-\) blinks as often as possible, only those four “short periodic” sequences \((X)\cdot(W)\) would be used and its blinking sequence is generated by Figure 9. In general, the actual local dynamics on \(v\) and \(v^-\) could use all combinations, which means that \(v^-\) could blink within long periodic sequences \((I)\) and \((J)\) or could skip the second blinks in short periodic sequences \((X)\cdot(W)\). Thus the actual blinking sequence of \(v^-\) refines a sequence generated by diagram \((B1)\), but could be coarser than a sequence generated by diagram \((B2)\); skipping second blinks within short periodic sequences corresponds to merging node weights with the following edge weights in diagram \((B2)\). For example, the string \((X)(J)(Z)\) is represented on diagram \((B2)\) by the directed walk \(X(XW), W + (WZ), Z\), which generates the sequence 6, 12, \((7+7)\), 9. This shows (ii) and (iii).

Lastly, suppose \(v^-\) only uses sequences \((Y)\) and \((Z)\). Note that the center \(v\) does not pull \(v^-\) in those sequences, since \(v^-\) has colors \(\leq 2\) whenever the center has color 2. Hence if the induced local orbit on \(v\) and \(v^-\) is given by an infinite subsequences of \((Y)\) and \((Z)\) only, then the dynamics restricts on \(T - T_v\), a contradiction. This shows (iv).

**Proposition 4.5.** Let \((T, X_0)\) be as before. Let \(T_w\) be a fractal branch in \(T\) with \(w^- \in V(T)\). Then the blinking sequence of \(w^-\) refines a sequence generated by digraph \((F10-9)\) or \((F11-8)\).
corresponding to the type of \( T_w \).

![Diagrams](image)

Figure 10: The blinking sequence of \( w^- \) refines a sequence generated by (F10/9) or (F11/8) depending on the type of \( T_w \).

**Proof.** Let \( T_w \) be of type 10/9. According to the definition, the dynamic on \( w \) and \( w^- \) during consecutive blinking gaps of 10 and 9 of \( w \) is given by concatenations of the four sequences (F1)-(F4) in Section 3. Note that \( w^- \) may or may not blink at some of the \( a_i \)'s, \( b_i \)'s, \( c_i \)'s, or \( d_i \)'s. But if one ignores such blinks within each sequence of 19 iterations (Fi)'s, the blinking gap of \( w^- \) must be generated by the digraphs in Figure 10. For instance, if sequence (F1) is followed by (F2), then it takes 22 iterations for \( w^- \) to blink at the beginning of each (Fi)'s. Thus the actual blinking sequence of \( w^- \) must refine a sequence generated by digraphs in Figure 10 depending on the type of \( T_w \).

**Lemma 4.7.** Let \( (T, X_0) \) as before. Suppose \( w \in V(T) \) such that each component of \( T_w - w \) is either a singleton, branch, or fractal. Then branches of type (a) or (b) or fractal of either types rooted at \( w \) are mutually exclusive.

**Proof.** Suppose there are both type (a) and (b) branches rooted at \( w \). Then by Proposition 4.4 the blinking sequence of \( w \) is generated by the diagram (A) and must refine a sequence generated by diagram (B1) in Figure 9. It is easy to see that the sum of the edge and vertex weights in any directed walk in diagram (A) cannot be 14 or 16. This means that any sequence generated by (A) cannot refine a sequence which contains a term of 14 or 16. But any sequence generated by (a directed closed walk in) diagram (b1) must contain a term of 14 or 16. Hence this is impossible.

Next, suppose there are one branch \( B \) and a fractal branch \( F \) rooted at \( w \). Suppose \( B \) is of type (a). Then the blinking sequence of \( w \) must be generated by diagram (A) in Figure 9 and refine a sequence generated by (F10/9) or (F11/8) in Figure 10. First note that there is no way to refine 16 and 21 using the blinking gaps in diagram (A). Hence the blinking sequence of \( w \) must refine the constant sequence of gap 19. It remains to show that sequence 19, 19, 19 cannot be refine by any sequence generated by diagram (A). Note that there are 3 ways to refine 19 using weights in diagram (A): \( (YX)(XY) \), \( (WY)(YY) \), and \( (YX)(XX) \)(here we may take \( Y = Z \)). Notice that none of them uses gap 6 inside sequence (W) or begins with an edge emanating from node X in diagram (A). Hence the first 19 must be refined by \( (WY)(YY) \), and the second 19 must be refined by \( (YX)(XX) \), but then following 19 cannot be refined. Thus shows that branch of type (a) is exclusive.

Now suppose the branch \( B \) is of type (b). A blinking sequence generated by diagram (B2) in Figure 8 must refine a sequence generated by (F10/9) or (F11/8) in Figure 10. To this end, we claim the following: among all directed walks in diagram (B2),

(a) \( Z, (ZX) \) is the only walk which generates a sequence (9, 7, 6) that refines 22;
(b) \( W, (WY) \), \( Y \) is the only walk which generates a sequence (7, 7, 7) that refines 21;
(c) \( (XY) \), \( Y \) is the only walk which generates a sequence (10, 7) that refines 17;
(d) \((XZ), Z\) and \((XW), W\) are the only walks which generate sequences \((10, 9\) and \(12, 7, \text{ respectively})\) refining \(19\).

To see this, for instance, consider possible ways to refine \(22\) using diagram \((B2)\). If gap \(12\) is used, then it must be \(22 = 10 + 12\), but \(12\) cannot be preceded or be followed by \(10\); if \(11\) is used, it must be \(22 = 11 + 11\), but this is also impossible; if \(10\) is used, then \(12\) must be properly refined, but this is impossible; if \(9\) is used, then \(13 = 6 + 7\) is the only way to refine \(13\), and \(Z(ZX)X\) is the only way to generate \(6, 7\), and \(9\) consecutively. This shows \((a)\), and the other claims can be shown similarly.

Now we show that any sequence generated by diagram \((B2)\) refines no sequence generated by \((F11/8)\). By \((c)\) and \((d)\), no refinement of \(17\) can be followed or preceded by any refinement of \(19\). Since in diagram \((F11/8)\) \(19\) always follows \(17\), we see that \(17\) cannot be refined. This yields that the blinking sequence of \(w\) may only refine the constant sequence of \(19\), but by \((d)\) any refinement of \(19\) begins with node \(X\) and ends with nodes \(Z\) or \(W\), so \(19\) cannot be refined repeatedly.

It remains to show that any sequence generated by diagram \((B2)\) refines no sequence generated by \((F10/9)\). We have seen in the previous paragraph that the constant sequence \(19\) cannot be refined. So if ever \(19\) is refined, then at some point a refinement of \(22\) or \(16\) should follow. But by \((a)\) and \((d)\), the refinement of \(22\) cannot follow any refinement of \(19\). This makes that the directed walk in diagram \((F10/9)\) which generates a sequence refined by some sequence generated from diagram \((B2)\) cannot use the loop at node \(F1\), and consequently, also the right-left edge of weight \(16\); this implies that only a constant sequence \(19\) from \((F10/9)\) can be refined, which contradicts our earlier observation. This shows the assertion.

Proof of Proposition \(4.3\). By the minimality, we may assume that the number of distinct phases occupied by the leaves in \(S\) is at least \(2\) and constant in time. At each time \(t\), by a component we mean the set of consecutive states on the leaves on the hexagon \(Z_6\); the size of a component is the number of distinct phases in it. Notice that by Proposition \(4.1\), whenever \(v\) blinks, every component must lie entirely clockwise or counterclockwise without any leaf opposite to \(v\). Hence the number of components is a non-increasing functions in time, which must be constant in time since we are in a periodic orbit. Let us call any local configuration in such closed orbit stable.

Figure 8 (a) is the only closed local orbit with a single component of size \(3\); Figure 8 (d) shows the closed orbit with a single component of size \(2\); Figure 8 (c) shows the closed orbit with 2 components of size \(1\) and \(2\); Figure 8 (b) for 2 components of size \(1\) for both. Notice that any configuration of a component of size \(\geq 4\) is unstable, likewise any one with two components with one component of size \(\geq 3\), any one with two components with both has size \(\geq 2\). So the nine configurations in Figure 8 gives all stable local configurations. By the time-invariance types, transitions between local configurations in different types are impossible. Possible transitions within each type and their minimal transition times are investigated similarly in Figure 7. For instance, Figure 11 illustrates possible transitions from Figure 8(b) to \(b_2\). This shows \((i)\).
To see (ii), suppose $S = T_v$ is a branch. Note that $v^-$ is the only external neighbor of $v$, so $v$ can get at most one external pull from $v^-$ in every 6 seconds. This makes the five unshaded local configurations in Figure 8 impossible to appear on branches. For instance, consider possible transitions from the Figure 8 $b_4$, which is shown in Figure 12. During the second transition of length 5 from the second to third column, either $v^-$ pulls $v$ as in the dotted bottom transition or not as in the solid upper transition. When $v$ blinks for the next time, none of the resulting configuration in the last column is stable. This shows the bottom left local configuration in Figure 8 is impossible on branches. Similar argument applies for other four unshaded local configurations in Figure 8. Thus there are exactly three possible types of closed orbit for this branch as asserted in (ii).

Now we show (iii). First suppose the local dynamic on $T_v$ is given by Figure 8 (a). In such local orbit, in terms of standard representation, the leaves must have colors 0, 1 and 2 whenever $v$ blinks. The following sequence shows the first 8 iterations starting from such local configuration (Figure 8 $a_1$): 

\[
\begin{array}{cccccccccccc}
\text{leaves} & 0 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 9 & 10 \\
\nu & 2 & 3 & 3 & a_1 & a_2 & a_3 & a_4 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\
\nu^- & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\
\end{array}
\]

Clearly $a_3 \neq 2$, and it is easy to check that $a_3 \neq 5$ leads to a different local configuration at the next blink of $v$: hence we must have $a_3 = 5$. This requires $2 \in \{b_4, b_5, b_6\}$, which in particular yields that $T_v$ is open. But $b_2 = 2$ leads to a contradiction since it would yield $b_1 = 5$ and $b_2 = 0$; so $2 \in \{b_5, b_6\}$. We extend sequence (1) as follows:

\[
\begin{array}{cccccccccccc}
\text{leaves} & 0 & 1 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 9 & 10 \\
\nu & 2 & 3 & 3 & a_1 & a_2 & 5 & 5 & 5 & 5 & 5 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\
\nu^- & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\
\end{array}
\]

Note that $2 \in \{b_5, b_6\}$ yields $b_9 \neq 2$, so $x_1 = 0$, $x_2 = 1$, and $X_3 = 2$. This shows a single transition from Figure 8 $a_1$ to itself takes exactly 12 seconds, and since $2 \in \{b_5, b_6\}$, $T_v$ is of type (a) definition.
Next, suppose the local dynamic on $T_v$ is given by Figure 8 (b). The argument is similar for type (a). We will show that the transition $b_2 \rightarrow b_1$ and $b_1 \rightarrow b_2$ in Figure 8 takes 9 and 7 seconds, respectively. We look at the first 9 iterations starting from Figure 8 $b_2$:

leaves: \[
\begin{array}{cccccccc}
1 & 2 & 3 & 3 & 3 & 4 & 5 & 8 \\
\nu & 41 & 42 & 53 & 04 & 15 & 20 & 31 & 42 \\
\nu^- & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8
\end{array}
\]

We need to have $2 \in \{a_3, a_4, a_5\}$ since otherwise $x_5 = 2$ and the resulting local configuration is not Figure 8 $b_1$. This makes $x_3 = 5$ and we may extend the sequence further:

leaves: \[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 3 & 3 & 4 & 5 & 5 & 0 & 1 & 2 & 3 \\
\nu & 41 & 42 & 53 & 04 & 15 & 20 & 31 & 42 & 53 & 04 & 14 & 25 & 30 & 41 & 52 & 03 & 14 \\
\nu^- & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17}
\end{array}
\]

This in particular shows that the transition $b_2 \rightarrow b_1$ in Figure 8 takes 9 seconds. Furthermore, $a_4 \neq 2$ since it leads to a contradiction by back-tracking in time, so we have $2 \in \{a_3, a_5\}$. Hence by definition, $T_v$ would be of type (b) if the transition $b_1 \rightarrow b_1$ in Figure 8 takes 7 seconds, i.e., $b_3 = 2$. To this end, it is enough to show that $2 \notin \{a_{11}, a_{13}, a_{14}\}$. Indeed, $a_{11} \neq 2$ since otherwise $b_3 = 0$ so the local configuration ‘$v$|leaves’ after two more iterations from the end of the above sequence would be $2|30$, which is not what we should have as in Figure 8 $b_2$. Similarly, $2 \in \{a_{13}, a_{14}\}$ leads to a wrong local configuration $2|30$, so $2 \notin \{a_{11}, a_{13}, a_{14}\}$. Thus $T_v$ is of type (b).

Finally, suppose the local dynamic on $T_v$ is given by Figure 8 (c). First five iterations from Figure 8 $c_1$ is as follows:

leaves: \[
\begin{array}{cccccccc}
3 & 4 & 13 & 245 & 350 & 501 & 502 \\
\nu & 03 & 14 & 304 & 245 & 350 & 401 & 502 \\
\nu^- & w_1 & w_2 & w_3 & w_4 & w_5 & w_6
\end{array}
\]

In order for this local dynamics lead to Figure 8 $c_2$, we need to have $2 \in \{w_2, w_4, w_5\}$. However, $w_4 = 22$ would lead to a contradiction by back-tracking up to $w_1$, so $2 \notin \{w_2, w_5\}$. An entirely similar argument for previous cases shows that the transitions $c_1 \rightarrow c_2$ and $c_2 \rightarrow c_1$ in Figure 8 take exactly 10 and 9 seconds, respectively. Thus $T_v$ is fractal of type 10/9. This shows the assertion. $
$

6. Proof of Lemma 4.4

By Proposition 4.3 (iii) we know that type (c) terminal branches are fractal, so in order to show Lemma 4.4 it suffices to show that no terminal branches can be of type (a) or (b). We begin by ruling out type (a) terminal branches.

**Proposition 5.1.** Let $(T, X_0)$ be as before. If there are two type (a) terminal branches $B$ and $B'$ rooted at the same vertex $w$, then one of the two branches must only use the sequence (P), and the other must only use (Q), which are given in the proof of Proposition 4.4.

**Proof.** First note that if the blinking sequence of $w$ ever uses the term 11, then because there is only one weight of 11 in Figure 8(a), both branches undergo the sequence (S) in synchrony. Since $w$ fluctuates the centers of $B$ and $B'$ in the same way, the two branches will be in synchrony thereafter, contradicting the minimality. Thus we may assume that $w$ never have a
blinking gap 11. Similarly, we may assume that blinking gap 13 never appears for \( w \). In general, the same argument applies to any unique sequence generated by diagram (A) in Figure 9 such as 7-7, 12-6, and 12-7. Once we exclude such segments, the only possible directed closed walk in Figure 9(A) is the one that uses loops on nodes (P), (Q) or (R). Since the induced dynamics on \( w \) must coincide, this is possible only if on of the two branches constantly use sequence (P) and the other (R), as asserted.

\[ \square \]

**Proposition 5.2.** Let \((T, X_0)\) be as before. Then there is no terminal branch of type (a).

**Proof.** Suppose there is a terminal branch \( T_v \) of type (a). Then each component in \( T_v - v^- \) is either a leaf or a branch. First suppose there is another branch, say \( T_u \), rooted at \( v^- \). By Lemma 4.7, it must be of type (a) as well. By Proposition 5.1, we may assume that \( T_v \) only uses sequence (P) and \( T_u \) only (R). Any more branch rooted at \( w \) will be redundant. So we assume these two are the only branches rooted at \( w \). Consider the following sequence, which is obtained by overlapping (P) and (R) by matching dynamics on \( v^- = u^- \):

\[
\begin{align*}
\text{(PR)} & \quad u \quad 2 \quad 3 \quad 3 \quad 4 \quad 5 \quad 5 \quad 5 \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
\text{(PRI)} & \quad v \quad 1 \quad 2 \quad 3 \quad 3 \quad 4 \quad 4 \quad 5 \quad 5 \quad 5 \quad 5 \quad 0 \quad 1 \quad 2 \\
\text{leaves} & \quad 234 \quad 345 \quad 450 \quad 501 \quad 012 \quad 123 \quad 234 \quad 345 \quad 450 \quad 501 \quad 012 \quad 123 \quad 234
\end{align*}
\]

Note that since \(* \neq 2\), this sequence requires \( v^- \) to be pulled four times in 6 iterations when it goes through the \(-\)'s. Since any vertex blinks at most once in 6 seconds, this means that \( v^- \) must have at least 4 external neighbors except \( v \) and \( u \). Thus except its own parent \( v^- \), it must have at least three leaves. By Proposition 4.3, the local dynamics on this 3-star centered at \( v^- \) the local dynamic should be given by Figure 8(a) or (c). However, the latter is not possible since in our circumstance the blinking sequence of \( v^- \) is the constant sequence 12, 12, \cdots, which is not the form of 10 + 6k1, 9 + 6k2, 10 + 6k3, \cdots. Thus the 3-star centered at \( v^- \) must go through type (a) closed orbit. In particular, whenever \( v^- \) blinks, its three leaves must have colors 0, 1, 2. Adding this to \((P + Q)\), we see that the local dynamics on \( T_v^- \) must be of the concatenation of the following sequence.

There are multiple contradictions at this point: \( v^- \) still needs to be pulled twice from external neighbors when it goes through \(-\)'s in the above sequence (PRI) but \( v^- \) is the only remaining external neighbor; whenever \( u \) or \( v \) pulls \( v^- \), some leaf pulls \( v^- \) together, so the branches \( T_v \) and \( T_u \) are not contributing anything to the dynamics on \( v^- \), contradicting the minimality.

Hence we may assume that there is no other branch rooted at \( v^- \). Observe that since \( v^- \) has color 4 at the end of sequence (S), it must be concatenated with the other three, so sequences (P)-(R) must be used at least once in the periodic local dynamics. Note that \( v^- \) must have color 4 or 5 at the end of sequences (P)+(R) in order to be concatenated by a following one. Since \( v^- \) does not blink and \( v \) does not pull \( v^- \) within those three sequences, it means that \( v^- \) must be pulled either by its own parent \( v^- \) or by its leaves, at least four times during the last six iterations in the three sequences. Since every vertex blinks at most once in every six iterations, this yields that \( v^- \) needs to have at least three leaves except its own parent. On this 3-star centered at \( v^- \), Proposition 4.3 again enforces the local dynamics given by Figure 8(a) (cf. (c) is not the case as before). Thus every blinking gap of \( v^- \) should be of the form 12 + 6k. Since sequence (S) contains a blinking gap 6, it cannot be used and only the other three can
be. Now the same sequence \((PRI)\) shows that whenever \(T_v\) goes through sequences (P) or (R), \(v^-\) is pulled by some leaf whenever it is pulled by \(v\). Since \(v\) blinks at exactly same time in sequences (Q) and (R), this means \(T_v\) is redundant to the dynamics of \(v^-\), contradicting the minimality. This shows the assertion.

Now we rule out type (b) terminal branches.

**Proposition 5.3.** Let \((T, X_0)\) as before. Suppose \(T\) has a terminal branch \(T_u\) of type (b). Then the local dynamic on \(T_v\) does not use the long periodic sequences (I) and (J). In particular, the exact blinking sequence of \(v^-\) is generated by digraph (B2) in Figure 9.

**Proof.** Suppose for contrary that sequences (I) and (J) do appear. This means \(v^-\) have blinking gap from one of the four edge weights in Figure 9(B1). We are going to show that \(T_v\) is the only branch rooted at \(v^-\) and \(v^-\) has at most two leaves. The assertion then easily follows. Indeed, in the last 6 iterations in both sequences (I) and (J), \(v^-\) must be pulled at least four times. Since \(v\) does not pull \(v^-\) during this period, its parent \(v^-\) and two other leaves cannot provide this.

We first show that \(v^-\) has at most two leaves. Suppose not. By Proposition 4.3, the 3-star centered at \(v^-\) has local dynamics given by Figure 8(a) or (c). Suppose the former. Then the blinking sequence of \(v^-\) is of the form \(12 + 6k_1, 12 + 6k_2, \cdots\). Among the weights in Figure 9(B1), the edge weight 18 of (I) is the only one of that form, and the following blinking gap of \(v^-\) should be either 16 of (J), 14 of (II), or their refinements. The first two are not of the prescribed form, so it must be their refinements. In Figure 9(B2), the edge (II) in the coarsened diagram (B1) corresponds to the node \(Y\) and edge (YW) combined. Thus any refined blinking gap of \(v^-\) must use the node weight 7 at \(W\) in diagram (B2), which also conflicts with the prescribed form. Hence \(v^-\) and its leaves cannot have local dynamics given by Figure 8(a).

Assuming local dynamics on \(T_v\) given by Figure 9(c), the blinking sequence of \(v^-\) must be of the form \(10 + 6k_1, 9 + 6k_2, 10 + 6k_3, \cdots\). Notice that there is no weight of \(9 + 6k\) for \(k \geq 1\) in diagram (B1) and (B2) in Figure 9, so the blinking sequence of \(v^-\) must be of the form \(10 + 6k_1, 9 + 6k_2, \cdots\). The only weights of the from \(10 + 6k\) in (B1) and (B2) in Figure 9 are 10 and 16. This yields that the blinking sequence \(v^-\) must consist of three terms 9, 10, and 16, where 10 and 16 is followed by 9 and must be followed by 10 or 16. We shall see this is impossible. Note that the sequence 10-9 is uniquely generated by the walk \((XX, Z)\), \(Z\) in Figure 9(B2), but no edge emanating from node \(Z\) in that digraph has weight 10 or 16. Thus 10 is not a blinking gap of \(v^-\), so the blinking sequence must alternate 9 and 16. But such a sequence cannot refine any sequence generated by Figure 9(B1), a contradiction. This shows that \(v^-\) has at most two leaves.

It remains to show that there is no other branch rooted at \(v^-\). Suppose for contrary that another branch \(T_u\) is rooted at \(v^-\). By Lemma 4.7, \(T_u\) is of type (b). By the minimality, branches \(T_v\) and \(T_u\) must have distinct dynamics. Now if \(v^-\) has blinking gap 14 or 18, then since those gaps are uniquely generated by Figure 9(B1), the two branches must be synchronized thereafter, a contradiction. Thus \(v^-\) never have blinking gaps 14 or 18, but does use gap 16, which are given by the loops at node \(I\) or \(J\) in Figure 9(B1). We may assume that when \(v^-\) has blinking gap 16, \(T_v\) and \(T_u\) undergo loops (II) and (JJ) in Figure 9(B1), respectively. Note that the loop (II) is represented by the node \(X\) and its loop \((XX)\) combined in the refining digraph Figure 9(B2), so the blinking gap of \(v^-\) that follows 16 should be coming from the four edges emanating from node \(X\) in the same digraph, which only give 10 or 12. By the parallel reasoning, loop (JJ) must be followed by an edge emanating from nodes \(Z\) or \(W\) in Figure 9, which yields the next blinking gap should be either 7 or 9, a contradiction. Hence \(T_v\) is the unique branch rooted at \(v^-\). This shows the assertion.
Proposition 5.4. Let \((T, X_0)\) as before. Suppose \(T\) has a terminal branch \(T_v\) of type (b). Then \(T_v\) is the only branch rooted at \(v^–\).

Proof. Suppose for contrary there is another branch \(T_u\) rooted at \(v^–\). By Lemma 4.7, we know that \(T_u\) must be of type (b), and by Proposition 5.3, they never use long periodic sequences (I) and (J) so that the blinking sequence of \(v^–\) is generated by digraph (B2) in Figure 9. By minimality, these two branches must not be synchronized. This means that we must be able to find two distinct closed walks in digraph (B2) which generate the same sequence. Since the weights 6, 11, and 12 are unique in the diagram, any such blinking sequence cannot use those numbers. Thus we may delete the node \(X\) together with all the indecent edges, and also the edge \((Y W)\) of weight 11 from the digraph. The resulting digraph, which generates the blinking sequence of \(v^–\) in our current situation, is provided below:

![Diagram](image)

Figure 13: If \(T_v\) is a terminal branch of type (b), then the blinking sequence of \(v^–\) is generated by this digraph.

Note that by Proposition 4.4 (iv), both branches must use (W) at least once. We claim that the blinking sequence of \(v^–\) never repeat 9 twice. This would yield the assertion as follows. Under this assumption, it would be impossible to use the edge \((ZW)\); thus no edge heading toward (W) is available, so after a branch uses the node (W), then it must be confined there. Thus both branches use node W only (recall that we are in a periodic orbit), and since they should generate the same blinking sequence for \(v^–\), they must have synchronized dynamics, a contradiction.

Thus it suffices to show that the blinking sequence of \(w\) cannot repeat 9 twice. Suppose for contrary that \(g_1(v^–) = g_2(v^–) = 9\). Observe that there are only two ways to generate 9-9 from Figure 9 (B2) with node (X) deleted: \((YZ)\), \(Z\) and \(Z\), \((ZW)\). Thus we may assume \(T_v\) goes through \((YZ)\) and \(T_u\) goes through \((ZW)\) simultaneously. Since the string 7-7-7 is uniquely generated by \(W, (WY), Y\) in the above digraph, it never appears in the blinking sequence of \(v^–\). This forces \(T_u\) to be confined at node \(W\) after the third blink, forcing \((g_i)_{i \geq 3}\) to alternate 7 and 9. This contradicts the periodicity of the blinking sequence, so string 9-9 never appears in the blinking sequence of \(v^–\). This shows the assertion.

Proposition 5.5. Let \((T, X_0)\) as before. Then \(T\) has no terminal branch of type (b).

Proof. Suppose for contrary that \(T_v\) is a terminal branch of type (b). By Proposition 5.4, we know that there is no other branch rooted at \(v^–\). Thus all neighbors of \(v^–\) except \(v\) and its own parent \(v^–\) are leaves. Furthermore, by Proposition 5.3, the local dynamics on \(T_v\) uses only those short periodic sequences (X)-(W) and the blinking sequence of \(v^–\) is generated by digraph (B2) in Figure 9.

We first show that sequence (X) is never used. To see this, notice that in sequence (X), \(v^–\) to be pulled at least four times during the last six iterations. Since \(v\) does not pull \(v^–\) during this period, \(v^–\) must have at least three leaves. By Proposition 4.3, \(v^–\) has exactly three leaves.
and the 3-star centered at $v^-$ has local dynamics given by Figure 8 (a) or (c). This implies that the blinking gap of $v^-$ must always be the form of $12 + 6k_1$, $10 + 6k_2$, $9 + 6k_3$ for $k_1, k_2, k_3 \geq 0$. But sequence (X) forces $v^-$ to have blinking gap 6, which is not of the prescribed form, a contradiction. Thus sequence (X) is never used by the local dynamics on $T_{v^-}$. In particular, $g_i(v^-) \in \{7, 9, 11\}$ for all $i \geq 1$.

Next, we show that sequences (Y) and (Z) also are not used by the local dynamics on $T_{v^-}$. Suppose not. Proposition 4.4 (iv), we know that sequence (W) is used (periodically). In particular, $v^-$ has blinking gap 7 periodically. Observe that sequence (Y) itself requires $v^-$ to be pulled four times in the last six iterations, and sequence $(Z)(W)$ concatenated also requires the same. Thus $v^-$ needs to have at least two leaves. Combining Proposition 4.3 together with the conclusion of the previous paragraph and the fact that $v^-$ does have blinking gap 7, we see that the local dynamics on the star centered at $v^-$ must only use local configurations Figure 8 $b_1$ or $b_2$, and blinking sequence of $v^-$ alternates 7 and 9. But then once $T_{v^-}$ uses the node W in digraph (B2) in Figure 9, it must confined on node W, contradicting to the periodicity of local dynamics. Thus $T_{v^-}$ only uses sequence (W), and the blinking sequence of $v^-$ alternates 7 and 9.

Note that the concatenated sequence (W)(W) requires $v^-$ to be pulled at least three times at the end of the first (W). Since $v$ does not pull $v^-$ during this period, $v^-$ needs to have at least one leaf. If it has at least two leaves, then combining sequence (W) with the local dynamics on $v^-$ together with its leaves given by Figure 8 (b), the local dynamics on $T_{v^-}$ is given by repeating the following sequence

\[
\begin{array}{cccccccccccc}
\nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\
\nu^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{leaves} & 20 & 31 & 4 & 53 & 04 & 14 & 25 & 30 & 41 & 52 & 03 & 14 & 24 & 35 & 40 & 51 & 02 \\
\end{array}
\]

(WI)

Note that in the above sequence, whenever $v$ blinks, one of the two leaves of $v^-$ blinks as well. Hence the branch $T_{v^-}$ is redundant to the dynamics of $v^-$, which contradicts minimality. So we may assume $v^-$ has exactly one leaf. The local dynamics on this 1-star centered at $v^-$ is given by the digraph in Figure 6. The only compatible closed walk there which generates a sequence that alternates 7 and 9 is $e, (ea), a, (ad), d, (dd), d, (de), e$ (up to choice of starting node). Notice that during the loop $(ee)$, the leaf of $v^-$ has color 4 when $w$ is blinking. If we plug the leaf in (W), we get the following sequence:

\[
\begin{array}{cccccccccccc}
\nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\
\nu^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{leaf} & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 5 & 0 & 1 & 2 & 3 & 4 & 4 & 5 & 0 & 1 & 2 \\
\end{array}
\]

(WI1)

which should appear in the local dynamics on $T_{v^-}$ periodically. Now the last four iteration is conflicting, since the leaf of $v^-$ does not contribute to extra pull on $v^-$. This shows the assertion.

\[
\Box
\]

7. Proof of Lemma 4.5

In this section we show Lemma 4.5. Let $(T, X_0)$ as before and let $T_{v^-}$ as stated in Lemma 4.5. We say a neighbor of $v^-$ external if it is either a leaf or its own parent $v^-$. By Proposition 4.3 $v^-$ has at most three leaves. Hence $v^-$ can have at most four external pulls during every six iterations. Since large blinking gaps of $v^-$ generally require lots of external pulls, it would be not likely under our hypothesis. In fact, blinking gaps of $v^-$ can be at most 11, as stated in the following proposition:
Proposition 6.1. Let \((T, X_0)\) be as before. Suppose that each connected component of \(T_{v^—} - v^—\) is either a singleton or fractal. Further assume that at least one such component which is fractal. Then the blinking gaps of \(v^—\) are bounded above by 11.

Our strategy for showing the above statement is the following: we collect all possible subsequences arising from the two sequences \((F_1), (F_2)\) and their eight concatenations \((F_i)(F_j)\) for \((i, j) \in \{1, 2\}^2 \cup \{2, 3\}^2\) with respect to the induced blinking gap of \(v^—\), and count the number of required external pulls. A detailed proof of this statement is given at the end of this section.

For further discussions, we give a full list of possible subsequences of \((F_i)(F_j)\) generating a fixed blinking gap of \(v^—\) generating blinking gaps \(\leq 11\) for \(v^—\) below:

(11) Blinking gap 11:

\[
\begin{align*}
(F1)(a_5) & \quad 3 \quad 3 \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(F2)(b_8) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
(F3)(c_6) & \quad - \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(F4)(d_9) & \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \\
(a_2)(F1) & \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(a_5 \text{ or } b_5)(F2) & \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
(c_3 \text{ or } d_3)(F3) & \quad 1 \quad 2 \quad 3 \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(c_5 \text{ or } d_5)(F4) & \quad 3 \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
\end{align*}
\]

\[
\begin{align*}
v^— & \quad 2 \quad 3 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad 5 \quad 0 \quad 1 \quad 2 \\
\end{align*}
\]

(10) Blinking gap 10:

\[
\begin{align*}
(F1)(a_4) & \quad 3 \quad 3 \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(F2)(b_7) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
(F3)(c_5) & \quad - \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(a_5 \text{ or } b_5)(F1) & \quad 2 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(c_3 \text{ or } d_3)(F3) & \quad 2 \quad 3 \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad * \\
(c_5 \text{ or } d_5)(F4) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
\end{align*}
\]

\[
\begin{align*}
v^— & \quad 2 \quad 3 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad 5 \quad 0 \quad 1 \quad 2 \\
\end{align*}
\]

(9) Blinking gap 9:

\[
\begin{align*}
(F1)(a_3) & \quad 3 \quad 3 \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
(F3)(c_4) & \quad - \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
(F4)(d_6) & \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(a_4 \text{ or } b_4)(F1) & \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(a_7 \text{ or } b_7)(F2) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
(c_5 \text{ or } d_5)(F3) & \quad 3 \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
\end{align*}
\]

\[
\begin{align*}
v^— & \quad 2 \quad 3 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad 5 \quad 0 \quad 1 \quad 2 \\
\end{align*}
\]

(8) Blinking gap 8:

\[
\begin{align*}
(F1)(a_2) & \quad 3 \quad 3 \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \\
(F2)(b_5) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(F3)(c_3) & \quad - \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \\
(F4)(d_5) & \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(a_5 \text{ or } b_5)(F1) & \quad - \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(a_6 \text{ or } b_6)(F2) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(c_5 \text{ or } d_5)(F3) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
(c_6 \text{ or } d_6)(F4) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad - \\
\end{align*}
\]

\[
\begin{align*}
v^— & \quad 2 \quad 3 \quad x_1 \quad x_2 \quad x_3 \quad 5 \quad 0 \quad 1 \quad 2 \\
\end{align*}
\]
assertion follows from the two claims. Hence we may assume that $v$ mating nodes edges or loops with weights in $\{6, 7, 8, 11\}$ can be used. Moreover, edges of weight 11 or 8 emanating nodes or in the digraph cannot be used, since in the first six iterations starting from those local configurations $v$ is not pulled by the leaf. Deleting all those edges, we obtain the following digraph which should generate the blinking sequence of $v$:

\[(7 \text{ and } 6) \text{ Blinking gaps } 7 \text{ and } 6:\]

\[
\begin{align*}
(F1)(a_1) & \quad 3 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \\
(F2)(b_2) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(F3)(c_2) & \quad - \quad - \quad - \quad - \quad - \quad 5 \quad 0 \\
(F4)(d_2) & \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
(a_1)(a_0) & \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
(c_2)(c_0) & \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
\end{align*}
\]

For instance, $(F1)(a_2)$ is the subsequence of $(F1)$ from the first blink of $v^-$ to the second blink $a_2 = 2$; $(a_2)(F1)$ is the subsequence of $(F1)(F1)$ from $a_2 = 2$ to the first blink of $v^+$ in the second $(F1)$. Note that the last three sequences for gap 6 are contradictory, so only the first two are possible.

**Proposition 6.2.** Let $(T, X_0)$ be as before. Suppose that each connected component of $T_{v^+ - v^-}$ is either a singleton or fractal. Further assume that at least one such component which is fractal. Then $v^-$ has at least two leaves.

**Proof.** Suppose for contrary that $v^-$ has at most one leaf. By Proposition 6.1, blinking gaps of $v^-$ are at most 11. Note that blinking gap 10 is impossible, since it requires at least three external pulls during the first six iterations while $v^-$ has at most two external neighbors.

We first claim that $v^-$ needs to have at least two leaves in order to have blinking gap 9. Since gap 9 requires at least one leaf for $v^-$, we may assume for contrary that $v^-$ has exactly one leaf. Note that $(a_2$ or $b_2)(F2)$ is necessary to generate gap 9 with $v^-$ since otherwise blinking gap 9 would require at least three external neighbors. Among the sequences which generate gaps $\in \{6, 7, 8, 11\}$, $(F2)(b_2)$ for gap 8 is the only sequence which can follow. Similarly, the sequence $(b_2)(F1)$ for gap 8 can only follow this. Hence we only need to rule out consecutive gaps 8-8. Note that in Figure 6, $a \rightarrow c \rightarrow d$ and $e \rightarrow b \rightarrow d$ are the only possible walks that generate blinking sequence 8-8 for $v^-$. But note that during the second transition in each walk, the center is not pulled by the leaf. This makes the second blinking gap 8 for $v^-$ during $(b_2)(F1)$ impossible. This shows the second claim. Thus we may assume that $g_i(v^-) \in \{6, 7, 8, 11\}$ for all $i \geq 1$.

Our second claim is that the assertion holds assuming $v^-$ has blinking gaps $\leq 8$ only. By ruling out sequences above which cannot be concatenated by any other to the right or left, we find that the local dynamics of $v$ should be given by repeating the following sequences:

\[
(a_0)(F2) - (F2)(b_2) - (b_2)(F1) - (F1)(a_1) - (a_1)(a_0); \quad 8 - 8 - 8 - 7 - 7
\]

which induce stings of blinking gaps of $v^-$ as indicated on the right. By the asymmetry of such strings and minimality, this yields that $T_v$ is the unique fractal subtree rooted at $v^-$. However, this means that whenever $v^-$ has blinking gap 8 induced by sequence $(F2)(b_2)$, in which $v$ does not pull $v^-$, $v^-$ needs two external pulls, a contradiction. This shows the second claim.

Now we show the assertion. If $v^-$ has no leaf, then blinking gap 11 is impossible so the assertion follows from the two claims. Hence we may assume that $v^-$ has one leaf. By Proposition 6.3, the blinking sequence of $v^-$ is generated by the digraph in Figure 6. In fact, only the edges or loops with weights in $\{6, 7, 8, 11\}$ can be used. Moreover, edges of weight 11 or 8 emanating nodes $b$ or $c$ in the digraph cannot be used, since in the first six iterations starting from those local configurations $v^-$ is not pulled by the leaf. Deleting all those edges, we obtain the following digraph which should generate the blinking sequence of $v^-$:

\[
\begin{align*}
(F1)(a_1) & \quad 3 \quad 3 \quad - \quad - \quad - \quad 5 \quad 0 \\
(F2)(b_2) & \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(F3)(c_2) & \quad - \quad - \quad - \quad - \quad - \quad 5 \quad 0 \\
(F4)(d_2) & \quad - \quad - \quad - \quad 5 \quad 0 \quad 1 \quad 2 \\
(a_1)(a_0) & \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
(c_2)(c_0) & \quad 0 \quad 1 \quad 2 \quad 3 \quad - \quad - \quad - \\
\end{align*}
\]
By the second claim, \( v^- \) must have blinking gap 11, and the only closed walk in the above digraph containing an edge of weight 11 is the one alternating between node \( a \) and \( e \). Hence the blinking sequence of \( v^- \) should alternate 11 and 7, but no sequences for gap 7 can be followed by any sequence for 11. This shows the assertion.

**Proposition 6.3.** Let \((T, X_0)\) be as before. Suppose that each connected component of \( T_{v^-} - v^- \) is either a singleton or fractal. Further assume that at least one such component which is fractal.

Then the blinking sequence of \( v^- \) alternates 10 and 9 or 11 and 8.

**Proof.** By Proposition 6.1, we know that the blinking gaps of \( v^- \) are at most 11. We first show that the blinking sequence of \( v^- \) alternates 8 and 11 or 9 and 10. By Proposition 6.2, \( v^- \) has at least two leaves. By Proposition 4.3, \( v^- \) never have blinking gap 6. If \( v^- \) has three leaves, then by Proposition 4.3 the local dynamics on the 3-star centered at \( v^- \) should be given by Figure 8 (c), so \( v^- \) has blinking gap alternating 10 and 9. Hence we may assume that \( v^- \) has exactly two leaves. Next, we show that string 9-9 also cannot appear in the blinking sequence of \( v^- \). The two leaves of \( v^- \) forces that the following blinking gap of \( v^- \) after 9-9 should be either 10 or 11. From the list of subsequences generating blinking gap 9, we see that \( v^- \) must blink at \( a_3 \) or \( c_4 \) at the end of second blinking gap 9. But no subsequence for gap 11 begins with \( v^- \): this requires \( v^- \) to have four external pulls during a blinking gap 10, a contradiction.

Next, we rule out blinking gap 7 for \( v^- \). Suppose \( v^- \) does have blinking gap 7. Then the local dynamics on the 2-star centered at \( v^- \) is given by digraph (b) in Figure 8. Moreover, since gap 9-9 does not appear, the only possible local configurations for this 2-star are Figure 8(b3) and (b4). This forces that gap 7 is always followed and preceded by gap 9. Observe that for string 9-7-9 in the blinking sequence of \( v^- \), by considering possible concatenations of the subsequences in Figure 4.4 one see that the second 9 after 7 should be given by \((b_4)(F1)\). Thus the second 9 cannot be followed by 7, since otherwise the second 7 is given by \((F1)(a_2)\), but no sequence for 9 begins with \( a_2 \). Thus the blinking sequence of \( v^- \) must contain the string 9-7-9-10-9-7-9. However, the second 7 in this string must end with \((a_1)\), but no sequence for gap 9 begins with \((a_1)\). Thus \( v^- \) does not have blinking gap 7 if it has at least two leaves.

Now we may assume that \( v^- \) has exactly two leaves only the gaps 8,9,10, and 11 appears. If the 2-star centered at \( v^- \) has local dynamics confined in digraphs (b) or (c) in Figure 8 then we are done. Hence we may assume that the local dynamics are given by digraph (d) in Figure 8. We want to show that the local configuration on the 2-star alternates local configurations...
Since we have shown that 9-9 does not appear in the blinking sequence of \( v^- \), it suffices to rule out the string 8-9-11 and 11-11. First, observe that the former is uniquely generated by \((F4)(d_2)-(d_5)(F3)-(c_3)\). Hence \( v^- \) has at most one fractal branch rooted at it, and needs at least four external pulls during the last blinking gap of 11, a contradiction. To rule out the string 11-11, observe that there are five sequences that generates consecutive blinking gap 11 of \( v^- \):

\[
(F1)(a_5)-(a_5)(F2) \\
(a_2)(F1)-(F1)(a_5) \\
(a_5 \text{ or } b_5)(F2)-(F2)(b_5) \\
(c_3 \text{ or } d_3)(F3)-(F3)(c_6) \\
(c_5 \text{ or } d_5)(F4)-(F4)(d_4)
\]

Since the blinking sequence of \( v^- \) is generated by the digraph (d) in Figure 4.3, the next blinking gap after 11-11 should be either 9 or 11. Note that no subsequences generating those blinking gaps could be concatenated after the last three sequences above. This yields that \( v^- \) could have at most two fractal subtrees rooted at itself whose local dynamics during the second blinking gap 11 should be given by subsequences \((a_3)(F2)\) or \((F1)(a_5)\). But then the second blinking gap 11 of \( v^- \) requires at least four external pulls, which is impossible with only two leaves for \( v^- \). This shows the assertion.

Now we are ready to prove Lemma 4.5.

**Proof of Lemma 4.5.** By Propositions 6.1, 6.2, and 6.3, we may assume that \( v^- \) has at least two leaves and its blinking sequence alternates 10 and 9 or 11 and 8. To show that \( T_{v^-} \) is fractal, we need to show that \( v^- \in V(T) \) and it provides external pulls on \( v^- \) at right place. First let us analyze the blinking sequence that alternates 8 and 11. In this case, by Proposition 4.3 we may assume that \( v^- \) has exactly two leaves. By Proposition 4.3, the 2-star centered at \( v^- \) should alternate between the two local configurations in Figure 8, \( d_2 \) and \( d_3 \). Hence when \( v^- \) blinks at the beginning of a gap 11, its two leaves must have colors 3 and 4. Consider the following 11 iterations during a gap 11:

\[
\begin{array}{cccccccccc}
\text{leaves} & 2 & 3 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 5 & 0 & 1 & 2 \\
34 & 34 & 45 & 50 & 01 & 12 & 23 & 45 & 50 & 01 & 12 & 23 \\
\end{array}
\]

From the list of possible sequence for gap 11, \( v^- \) is not pulled by any of its internal neighbors (i.e., centers of fractal subtrees rooted at \( v^- \)) during the transition \( x_1 \to x_3 \). In order to make gap 11, \( v^- \) needs to be pulled by an external neighbor during \( x_1 \to x_3 \). Thus \( v^- \in V(T) \) and \( 2 \in \{y_2, y_3, y_4\} \). Since \( y_4 = 2 \) yields \( y_3 = 5 \) which is a contradiction, we have \( 2 \in \{y_2, y_4\} \). This shows \( T_{v^-} \) is fractal of type 11/8, as desired.

Now we assume that the blinking sequence of \( v^- \) alternates 10 and 9. The argument is similar as before. By Proposition 4.3, the k-star (\( k \in \{2,3\} \)) centered at \( v^- \) should alternate local configurations \( b_1 \) and \( b_2 \) or \( c_1 \) and \( c_2 \) in Figure 8. In the first case, \( v^- \) has two leaves which have colors 0 and 4 when \( v^- \) blinks at the beginning of gap 10 as in Figure 4.3 \( b_2 \); in second case it has three leaves of color 0, 3, and 4 at such instant as in Figure 8 \( c_1 \). Following sequence shows ten iterations during a blinking gap 10 together with all possible local dynamics on the leaves of \( v^- \):

\[
\begin{array}{cccccccccc}
\text{leaves} & 2 & 3 & x_1 & x_2 & x_3 & x_4 & x_5 & 5 & 0 & 1 & 2 \\
04 & 14 & 25 & 30 & 41 & 02 & 13 & 25 & 30 & 41 & 52 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{leaves} & 034 & 134 & 245 & 350 & 401 & 012 & 123 & 235 & 340 & 451 & 502 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{leaves} & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
26 &
\end{array}
\]
By the list of sequences giving blinking gap 10 for $v^-$, there is no internal pulls on $v^-$ during the first six iterations in the above sequence. Hence $v^-$ still needs one extra external pulls, and this yields $v^- \in V(T)$ with $2 \in \{y_2, y_4, y_5\}$. Since $y_4 \neq 2$ for similar reason this shows that $T_{v^-}$ is fractal of type $10/9$. This shows the assertion. ■

**Proof of Proposition 6.1.** By Proposition 4.5, the maximum possible blinking gap of $v^-$ is 22 generated by $(F_1)(F_2)$ without secondary blink within $(F_1)$, but this requires at least 5 external pulls during six iterations, so it cannot occur. Blinking gap 20 arises from $(F_3)(F_4)$ but impossible for similar reason, and there is no subsequence which gives blinking gap 20 (e.g., see Figure 10). We rule out large blinking gaps from 19 to 12 below.

(19) There are only four subsequences giving gap 19, namely, $(F_i)(F_i)$ for $1 \leq i \leq 4$ without $v^-$ blinking more than once in the first sequence $(F_i)(F_i)$. Hence if $v^-$ has blinking gap 19, then $v^-$ can have at most four fractal subtrees rooted at $v^-$. We overlap all four sequences to see the least number of required external pulls:

\[
(F_1)(F_1) \quad 3 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \\
(F_2)(F_2) \quad \star \quad \star \quad \star \quad \star \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \\
(F_3)(F_3) \quad \star \quad \star \quad \star \quad \star \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \\
(F_4)(F_4) \quad \star \quad \star \quad \star \quad \star \quad \star \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \\
\]

$v^- \in V(T)$ with $2 \in \{y_2, y_4, y_5\}$.

(18) Impossible

(17) Gap 17 arises uniquely from $(F_4)(F_3)$, so in this case $v^-$ can have at most one fractal subtree. In the following sequence during blinking gap 17, $v^-$ needs to get at least 5 external pulls during the first six iterations, which is impossible.

\[
(F_4)(F_3) \quad \star \quad \star \quad \star \quad \star \quad \star \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \\
\]

(16) Gap 16 arises uniquely from $(F_2)(F_1)$, so $v^-$ can have at most one fractal subtree. Consider the following sequence during blinking gap 16:

\[
(F_2)(F_1) \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \quad 3 \quad \star \quad \star \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \\
\]

If $x_5 = 5$, then $v^-$ needs five external pulls in a row after $x_5$, which is impossible. Thus $3 \leq x_5 \leq 4$. Hence $v^-$ needs at least four external pulls for the first six iterations in the above sequence. So it has exactly three leaves. By Proposition 4.3, the 3-star centered at $v^-$ must have local configuration Figure 8 \[c_1\] in order to match with blinking gap 16. Inserting the three leaves according to such local configuration, the first six iterations in the above sequence looks as follows:

\[
(F_2)(F_1) \quad \star \quad \star \quad \star \quad 5 \quad 0 \quad 1 \quad 2 \\
\]

leaves 012 123 234 345 450 501 012

27
But since \( x_3 \leq 4 \), this requires \( v^- \) to be pulled by its only remaining external neighbor, namely its parent, twice for the last three iterations, a contradiction.

(15) Impossible.

(14) Within the sequence (F1), we could have blinking gap 14 if \( a_7 = 2 \). Also possible is \( a_2 = 2 \) and (F1) is followed by (F2). Last possibility is that \( c_1 = 2 \) in (F3) and (F4) follows. Denote these three cases by (F1)(a7), (a2)(F2), and (c1)(F4). This gives following sequence for blinking gap 14:

\[
\begin{align*}
\text{(F1)(a7)} & : 3 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{(a2)(F2)} & : 1 & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{(c1)(F4)} & : 0 & 1 & 2 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{v^-} & : 2 & 3 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\end{align*}
\]

Hence \( v^- \) requires at least four external pulls during the five iterations from \( x_2 \) to \( x_7 \). Hence \( v^- \) has exactly three leaves. But neither digraphs (a) or (c) in Figure 8 can generate gap 14, a contradiction.

(13) There are four possibilities for gap 13 as below, which clearly requires \( v^- \) to have at least two leaves. But blinking gap 13 is not generated from any digraphs in Figure 8, contrary to Proposition 4.3.

\[
\begin{align*}
\text{(F1)(a6)} & : 3 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{(F3)(c8)} & : 1 & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{(c3)(F4)} & : 1 & 2 & 3 & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{(d3)(F4)} & : 1 & 2 & 3 & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{v^-} & : 2 & 3 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\end{align*}
\]

(12) \( v^- \) could have at most six fractal subtrees generating blinking gap 12. First observe that \( v^- \) cannot have three leaves. To see this, note that by Proposition 4.3, its blinking gaps should be of the form \( 12 + 6k_i \) for \( k_i \geq 0 \); since we have seen that \( v^- \) does not have a blinking gap 18, its blinking gap should be 12 constantly. But a fractal subtree rooted at \( v^- \) makes this impossible (e.g., no subsequence is possible between the six possibilities below). Second, suppose \( v^- \) has two leaves. Then by Proposition 4.3, the 2-star centered at \( v^- \) must have local configuration Figure 8(b1) or (b3). Their dynamics are inserted in the following matrix:

\[
\begin{align*}
\text{(a1)(F1)} & : 0 & 1 & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{(a2)(F2)} & : 3 & - & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{(b1)(F2)} & : 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{(c2)(F3)} & : 0 & 1 & 2 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\text{(c1)(F4)} & : 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{(d1)(F4)} & : 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \\
\text{v^-} & : 2 & 3 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & 5 & 0 & 1 & 2 & 3 & - & - & - \\
\end{align*}
\]

Note that we cannot have both of the last two rows at the same time. Considering each case separately, we see that \( v^- \) still needs at least two external pulls, a contradiction. The above matrix also shows that \( v^- \) needs at least two leaves, so blinking gap 12 is impossible.

This shows the assertion. ■
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References

[1] Arora, A., Dolev, S., Gouda, M., 1992. Maintaining digital clocks in step. In: Distributed Algorithms. Springer, pp. 71–79.
[2] Boulisier, C., Petit, F., Villain, V., 2006. Toward a time-optimal odd phase clock unison in trees. In: Stabilization, Safety, and Security of Distributed Systems. Springer, pp. 137–151.
[3] Bramson, M., Griffeath, D., 1989. Flux and fixation in cyclic particle systems. The Annals of Probability, 26–45.
[4] Buck, J. B., 1938. Synchronous rhythmic flashing of fireflies. The Quarterly Review of Biology 13 (3), 301–314.
[5] Dolev, S., 2000. Self-stabilization. MIT press.
[6] Dorfler, F., Bullo, F., 2012. Synchronization and transient stability in power networks and nonuniform kuramoto oscillators. SIAM Journal on Control and Optimization 50 (3), 1616–1642.
[7] Enright, J. T., 1980. Temporal precision in circadian systems: a reliable neuronal clock from unreliable components? Science 209 (4464), 1542–1545.
[8] Gravner, J., Lyu, H., Sivakoff, D., 2016. Limiting behavior of 3-color excitable media on arbitrary graphs. arXiv:1610.07320.
[9] Greenberg, J. M., Hastings, S., 1978. Spatial patterns for discrete models of diffusion in excitable media. SIAM Journal on Applied Mathematics 34 (3), 515–523.
[10] Herman, T., Ghosh, S., 1995. Stabilizing phase-clocks. Information Processing Letters 54 (5), 259–265.
[11] Hong, Y.-W., Scaglione, A., 2005. A scalable synchronization protocol for large scale sensor networks and its applications. Selected Areas in Communications, IEEE Journal on 23 (5), 1085–1099.
[12] Lampert, L., 1978. Time, clocks, and the ordering of events in a distributed system. Communications of the ACM 21 (7), 558–565.
[13] Lyu, H., 2015. Synchronization of finite-state pulse-coupled oscillators. Physica D: Nonlinear Phenomena 303, 26–36.
[14] Lyu, H., Sivakoff, D., 2017. Synchronization of finite-state pulse-coupled oscillators on Z. arXiv:1701.00319.
[15] Mesbahi, M., Egerstedt, M., 2010. Graph theoretic methods in multiagent networks. Princeton University Press.
[16] Nair, S., Leonard, N. E., 2007. Stable synchronization of rigid body networks. Networks and Heterogeneous Media 2 (4), 597.
[17] Pagliari, R., Scaglione, A., 2011. Scalable network synchronization with pulse-coupled oscillators. Mobile Computing, IEEE Transactions on 10 (3), 392–405.
[18] Strogatz, S. H., 2000. From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators. Physica D: Nonlinear Phenomena 143 (1), 1–20.
[19] Strogatz, S. H., 2001. Exploring complex networks. Nature 410 (6825), 268–276.
[20] Wang, Y., Núñez, E., Doyle, F. J., 2013. Increasing sync rate of pulse-coupled oscillators via phase response function design: theory and application to wireless networks. Control Systems Technology, IEEE Transactions on 21 (4), 1455–1462.
[21] Wang, Y., Núñez, E., Doyle III, F. J., 2012. Energy-efficient pulse-coupled synchronization strategy design for wireless sensor networks through reduced idle listening. Signal Processing, IEEE Transactions on 60 (10), 5293–5306.