Duality Group for Calabi–Yau 2–Moduli Space†

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ABSTRACT

We present an efficient method for computing the duality group \( \Gamma \) of the moduli space \( \mathcal{M} \) for strings compactified on a Calabi–Yau manifold described by a two-moduli deformation of the quintic polynomial immersed in \( \mathbb{C}P(4) \), \( \mathcal{W} = \frac{1}{5}(y_1^5 + \cdots + y_5^5) - ay_1^2y_2^3 - by_1^2y_3^2 \). We show that \( \Gamma \) is given by a 3-dimensional representation of a central extension of \( B_5 \), the braid group on five strands.

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1. Introduction

Generic string models are believed to exhibit target space duality\(^1\), which is a discrete symmetry acting on the moduli space of the underlying superconformal field theory (SCFT), or of the relevant compactified 6-dimensional manifold. Duality symmetry, whose origin is strictly related to the fact that the string is a 1-dimensional object, leaves invariant the spectrum and supposedly the interactions, and quite generally describes quantum symmetries of the low energy effective action. Such symmetry has been explicitly found in simple models like toroidal compactifications and their orbifolds, where it generalizes the \(R \to 1/R\) symmetry of the bosonic string compactified on a circle. The most celebrated example is the effective Lagrangian obtained by compactification of the heterotic string on a 6–torus in the large radius limit\(^2\). The Kähler class untwisted modulus corresponding to the volume size \(t = 2(R^2 + i\sqrt{g})\) parametrizes the homogeneous space \(SU(1,1)/U(1)\), so that this sector of the theory is invariant under \(PSL(2,\mathbb{R})\). However, when other (twisted) sectors are introduced and/or quantum corrections computed, the residual symmetry is given by the duality group \(SL(2,\mathbb{Z})\) with generators \(t \to t + \frac{1}{7}\) and \(t \to -\frac{1}{7}\). The determination of the duality symmetry group \(\Gamma\) for Calabi–Yau compactifications, or in broader sense for \((2,2)\) \(c = 9\) SCFT, is in general a very difficult problem, so far solved only for few examples in which the moduli space is one-dimensional. In particular, in a seminal paper\(^3\), Candelas et al. treated in detail the class of Calabi–Yau 3–folds immersed in \(\mathbb{CP}(4)\) given by the one parameter deformation of the quintic polynomial \(W_0\)

\[
W = W_0 - \psi y_1 y_2 y_3 y_4 y_5, \quad W_0 = \frac{1}{5} \left( y_1^5 + y_2^5 + y_3^5 + y_4^5 + y_5^5 \right) \quad (1.1)
\]

where \(y_i\) are homogeneous coordinates on \(\mathbb{CP}(4)\). Using the monodromy properties of the solutions of the Picard–Fuchs equations, these authors were able to reconstruct the full duality group of the 1–dimensional space parametrized by \(\psi\). It was found that \(\Gamma\) has two generators, \(\{A, T\}\) of order 5 and \(\infty\) respectively, and their representation in terms of \(4 \times 4\)–matrices acting on the periods was given. After
that, several other examples of 1–dimensional deformations of other Calabi–Yau manifolds defined by polynomials in weighted projective space have been developed along the same lines\(^4\).

The general computation of the duality group for more than one modulus has not been successfully attempted so far, because of the mathematical complexity that is encountered in going from the 1–dimensional case to the higher dimensional ones. However, a 2–moduli case is presently under investigation\(^5\). Moreover, it has been observed\(^6\) that for any number of moduli, the translational symmetry on the special variables \(t^a \to t^a + n^a, n^a \in \mathbb{Z}\), is always a subgroup of the duality group.

In this paper we present an efficient method for determining the duality group for a 2–moduli deformation of the quintic polynomial \(\mathcal{W}_0\),

\[
\mathcal{W} = \mathcal{W}_0 - a y_4^3 y_5^2 - b y_4^3 y_5^2,
\]

This example of Calabi–Yau manifold is a subspace of the general 101–dimensional deformation of \(\mathcal{W}_0\) which gives rise to zero Yukawa couplings for the associated effective Lagrangian. Because of that, if taken by itself, it constitutes more of a toy model as far as the low energy Lagrangian is concerned. However, it provides a 2-moduli example for which the duality group can be determined completely, displaying the power of some general techniques of algebraic geometry which were developed and applied to the study of the monodromy groups of Feynman integrals many years ago\(^7,8\).

Our result for the duality group is surprisingly simple: \(\Gamma\) is given by an \(U(1,2)\) valued (projective) representation of \(B_5\), the braid group on five strands. The representation acts on the periods associated to the uniquely defined holomorphic three–form \(\Omega\) which always exists on a Calabi–Yau manifold. In terms of the defining polynomial \(\mathcal{W}\), the fundamental period is defined by the integral representation\(^16,17\)

\[
\omega_0(a, b) = \oint_{\Gamma} \frac{\omega}{\mathcal{W}(y; a, b)},
\]

(1.3)
where $\omega$ is the volume element

$$\omega = \sum (-1)^i y_i \, dy_1 \wedge \cdots \wedge dy_i \wedge \cdots \wedge dy_5 \quad (1.4)$$

(the hat means that the corresponding differential must be omitted), and $\Gamma$ is an element of the basis for the homology cycles of $H(4)(\mathbb{CP}(4) - \mathbb{W}; \mathbb{Z})$. There are as many independent integrals $\omega^I_0$ as there are elements of the basis, $\Gamma^I \subset H(4)(\mathbb{CP}(4) - \mathbb{W}; \mathbb{Z})$.

Quite generally, if $L^{N-1}$ is the singularity locus of an algebraic curve $\mathbb{W}$ parametrized by $N$ moduli, the monodromy group $\Gamma$ acting on the periods of $\mathbb{W}$ is given by a representation of the fundamental group $\pi_1$ of the embedding space $\mathbb{CP}(N)$.

The computation of the homotopy group $\pi_1$ is based on the use of the following two theorems$^9,10$.

**Theorem 1** (Picard-Severi). *Let $L^{N-1}$ be the $N - 1$ complex dimensional singular locus of a given algebraic curve. If the (complex) projective line $\mathbb{CP}(1) \subset \mathbb{CP}(N)$ is generic with respect to $L^{N-1}$ (i.e., it avoids all singular points of $L^{N-1}$), then we have the isomorphism

$$\pi_1 \left( \mathbb{CP}(1) - (\mathbb{CP}(1) \cap L^{N-1}); B \right) / G \approx \pi_1 \left( \mathbb{CP}(N) - L^{N-1}; B \right) \quad (1.5)$$

where $B$ is the base point and $G$ is an invariant subgroup of $\pi_1(\mathbb{CP}(1) - (\mathbb{CP}(1) \cap L^{N-1}); B)$.

In other words, $\pi_1(\mathbb{CP}(N) - L^{N-1}; B)$ is obtained from $\pi_1(\mathbb{CP}(1) - (\mathbb{CP}(1) \cap L^{N-1}); B)$ by adding the identities $\gamma = \mathbb{1} \ \forall \gamma \in G$.

A method for deriving such identities has been provided by Van Kampen$^{11}$, and we shall use it in our particular case to obtain the isomorphism of eq. (1.5). As the singular locus of the algebraic curve $\mathbb{W}$ is given by an equation of the form $L(a,b) = 0$, we are interested in the case $N = 2$. 
The second theorem, due to Zariski\(^{10}\), allows to understand that, as far as the identification of the fundamental group \(\pi_1\) is concerned, the case of more than two moduli can be essentially reduced to the \(N = 2\) case, so that the general computation of \(\pi_1\) in will not be more difficult than the one under study. However, for several variables it is in general much harder to find the singular locus and the behaviour of the algebraic curve in its neighbourhood, and therefore the determination of the monodromy group can be more involved.

**Theorem 2** (Zariski). If the complex projective plane \(\mathbb{P}(2)\) is generic with respect to \(L^{N-1}\) and if \(B \in (\mathbb{P}(2) - \mathbb{P}(2) \cap L^{N-1})\), then the map

\[
\pi_1 \left( \mathbb{P}(2) - \mathbb{P}(2) \cap L^{N-1}; B \right) \to \pi_1 \left( \mathbb{P}(N) - L^{(N-1)}; B \right) \quad (1.6)
\]

is an isomorphism.

We see that, in virtue of Zariski theorem, the study of the homotopy group of \(\mathbb{P}(N) - L^{N-1}\) is reduced to the study of the homotopy of the complement of the 1–dimensional curve \(L^1 = \mathbb{P}(2) \cap L^{N-1}\) on a generic two-dimensional section. Since the singular locus of the curve (1.2) is already one-dimensional (\(N = 2\)), theorem 1. is sufficient for our present purposes.

Before proceeding to the actual determination of the duality group, we recall what is the local geometry associated to the moduli space \(\mathcal{M}\) of \(\mathcal{W}\). The two-moduli family of Calabi–Yau deformations given by eq. (1.2) was first introduced in ref. 14 as a particular tensor product example of five copies of minimal models with \(k = 3\) and \(c = 9/5\). It was observed that in such model there are restrictions given by charge conservation which enforce the condition \(W_{\alpha\beta\gamma} = 0\) for the Yukawa couplings. Because of that, the constraint of Special Geometry\(^{12}\)

\[
R_{\alpha\beta\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\gamma\beta} - e^{2K}W_{\alpha\gamma\delta}W_{\beta\alpha\delta}g^{\rho\sigma}g^{\rho\sigma} \quad (1.7)
\]

reduces to

\[
R_{\alpha\beta\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\gamma\beta} \quad , \quad \alpha, \beta, \gamma, \delta = 1, 2 \quad (1.8)
\]

Thus, the local geometry of \(\mathcal{M}\) is given by a 2–dimensional Kähler manifold with
constant curvature, which according to the classification of ref. 15, corresponds to the coset space $U(1,2)/(U(1)\otimes U(2))$. The global structure of such moduli space is determined by modding out the discrete isometries given by the duality group $\mathbb{Z}$.

In principle, for the 2–moduli case, one would expect a 6–dimensional representation of the modular group. In fact, the dimension of the $H^{(3)}$ cohomology group is given by $2h_{21} + 2$, where $h_{21} = \dim H^{(2,1)}$, the number of complex structure moduli of the Calabi–Yau manifold. It turns out, however, that our representation is only 3–dimensional, as this specific model is singular due to the vanishing of the Yukawa couplings $W_{\alpha\beta\gamma}$. In such case, the Picard-Fuchs equations for the periods of $\mathcal{W}$ are of second order rather than fourth-order\(^{13}\). The 6–dimensional representation, which in the symplectic basis is valued in $Sp(6,\mathbb{Z})$, splits into two 3–dimensional representations of $U(1,2)$ according to the embedding $6 = 3 + \overline{3}$ of $U(1,2)$ in $Sp(6)$. Later, we shall verify explicitly that the 3–dimensional representation of $B_5$ is actually valued in $U(1,2)$.

2. The Fundamental Group of $\mathcal{W}(y; a, b)$.

In this section we compute the fundamental group $\pi_1(CP(2) - L; B)$ of the algebraic curve $\mathcal{W} = 0$. The first step consists of finding the singular locus $L$ of eq. (1.2), which is given by solving simultaneously the equations

$$\frac{\partial \mathcal{W}}{\partial y_i} = 0 \quad i = 1, \ldots, 5. \quad (2.1)$$

A straightforward computation yields the 1–dimensional complex curve

$$L(a, b) = 108(a^5 + b^5) - 80a^3b^3 - 165a^2b^2 + 30ab - 1 = 0 \quad (2.2)$$

which represents the locus of the complex points of the original curve $\mathcal{W} = 0$ where two or more of the roots coincide.
For the derivation of the Van Kampen relations among the homotopy generators around the various branches of \( L(a, b) \), it is important to know where \( L(a, b) \) itself has multiple points. These are found by solving the equations 
\[
L(a, b) = \frac{\partial L}{\partial a} = \frac{\partial L}{\partial b} = 0,
\]
which give the location of the multiple roots 
\[
(a, b) = (\rho^k, \rho^{-k}) \quad k = 0, \ldots, 4
\]
where \( \rho = e^{2\pi i/5} \). The first set of roots in (2.3) corresponds to nodes with two distinct complex conjugate tangents (which are isolated points for the real section of (2.2) represented by real values of \( a \) and \( b \)). The second set instead represents (second order) cusps, since the Hessian \( \frac{\partial^2 L}{\partial a \partial b} \) is degenerate at these points.

We may obtain a more elegant and geometrically intuitive representation of the curve \( L(a, b) = 0 \) by choosing new coordinates \( (p, q) \) such that the real section corresponding to real values of \( p \) and \( q \) exhibits the previous singular points in the real \( (p, q) \) plane. It is sufficient to set 
\[
\begin{align*}
a &= p + i q \\
b &= p - i q
\end{align*}
\]
and we find 
\[
L(p, q) = -1 + 30(p^2 + q^2) - 165(p^4 + q^4) - 80(p^6 + q^6) + 216p^5 \\
-330p^2q^2 - 2160p^3q^2 - 240(p^4q^2 + p^2q^4) + 1080pq^4.
\]
In the real plane of \( (p, q) \) the multiple points (2.3) take now the real values 
\[
(p, q) = (\cos \frac{2\pi k}{5}, \sin \frac{2\pi k}{5})
\]
\[
(p, q) = -\frac{1}{4}(\cos \frac{2\pi k}{5}, \sin \frac{2\pi k}{5})
\]
respectively. Actually, the curve \( L(p, q) = 0 \) can be put in a parametric form by
setting

\[
\begin{cases}
  p = \frac{1}{5}(3 \cos 2t + 2 \cos 3t) \\
  q = \frac{1}{5}(3 \sin 2t - 2 \sin 3t)
\end{cases}
\quad 0 \leq t \leq 2\pi
\tag{2.7}
\]

and can be recognised as a pentacuspidal hypocycloid (the curve described by a point of a circle of radius \( R = 1/5 \) rolling inside a circle of \( R = 1 \)), whose graph is represented in fig. 1.

According to Picard-Severi theorem, we now take a generic line \( C \) through the base point \( B = (0, 0) \) which intersects the real branch of the hypocycloid in four (finite) points \( P_i \) (fig. 2). To each of such representative points, we attach a generator of \( \pi_1(\mathcal{C} - \mathcal{C} \cap L; B) \) by constructing a loop which leaves \( B \) along a straight line, makes an infinitesimal loop around \( P_i \) counterclockwise in the complex plane \( \mathcal{C} \) containing the real line, and comes back to \( B \) again in the opposite direction. If the straight line encounters a real branch of \( L \), it will be taken to undercross it by describing a small semicircle in the complex plane. If by varying the angular coefficient of the straight line no critical point of \( L \) is encountered, the corresponding points give rise to equivalent loops. Generally inequivalent loops are obtained if two straight lines intersect \( L \) along points belonging to two different real branches of \( L \) emanating from the critical points. We thus obtain 15 loops, 5 of which go around the sides of the pentagonal figure described by \( L \) and 10 going around the branches emanating from the cusps (5 of them are shown in fig. 3.).

We now quote the Van Kampen relations\(^\text{11}\) between loops which, added to the free group generated by the above mentioned 15 generators, make it isomorphic to \( \pi_1(\mathcal{C} \setminus \mathcal{C} - L; B) \).

For the nodes corresponding to transversal intersections of the real branches of \( L \), we have (see fig. 4)

\[
\alpha \beta = \beta \alpha \quad ; \quad \alpha = \alpha' \quad ; \quad \beta = \beta'
\tag{2.8}
\]

\( i.e., \) we can slide the representative loops across the node without any change, and
the loops around two branches commute. In this way, the 15 generators reduce to five independent ones.

For each cusp we have the relation (see fig. 5)

$$\alpha \beta \alpha = \beta \alpha \beta$$

(2.9)

Let us enumerate in increasing order the five branches of the hypocycloid described successively by the point of the small circle rolling inside the big circle (fig. 2). Then, denoting by $\alpha_i$, $i = 1, \ldots, 5$ the loops winding around the five branches of fig. 1, we have the following set of relations among the generators

$$\begin{align*}
\alpha_1 \alpha_3 &= \alpha_3 \alpha_1 \\
\alpha_1 \alpha_4 &= \alpha_4 \alpha_1 \\
\alpha_2 \alpha_4 &= \alpha_4 \alpha_2 \\
\alpha_2 \alpha_5 &= \alpha_5 \alpha_2 \\
\alpha_3 \alpha_5 &= \alpha_5 \alpha_3
\end{align*}$$

(2.10)

We note that the subset of relations (2.10) not involving $\alpha_5$ coincide with the defining relations of the braid group $B_5$, with four generators $\alpha_i$, $i = 1, \ldots, 4$,

$$\begin{align*}
\alpha_i \alpha_j &= \alpha_j \alpha_i & |i - j| \geq 2 \\
\alpha_i \alpha_{i+1} \alpha_i &= \alpha_{i+1} \alpha_i \alpha_{i+1} & i = 1, \ldots, 3
\end{align*}$$

(2.11)

However, it may be easily verified that the element of $B_5$ exchanging the first and the fifth strand can be written in terms of the four generators $\alpha_i$ by the word

$$\alpha_5 = (\alpha_4 \alpha_3 \alpha_2) \alpha_1 (\alpha_4 \alpha_3 \alpha_2)^{-1}$$

(2.12)

With a little effort, using the Van Kampen relations for $\alpha_1, \ldots, \alpha_4$ one checks that the extra relations of (2.10) involving $\alpha_5$ are indeed verified. Therefore, we come to the conclusion that, if the relation (2.12) also holds among the monodromy generators $\alpha_i$ ($i = 1, \ldots, 5$) of the hypocicloyd, then $\pi_1(C(2) - L; B)$ is isomorphic to $B_5$. 

9
The reason why we do not find eq. (2.12) among the Van Kampen relations is that we have not considered the critical point of $L$ at $\infty$ and the associated generator. Rather than studying such critical point, we shall give evidence in the sequel that eq. (2.12) must be satisfied by the monodromy generators, so that indeed $B_5$ coincides with the fundamental group associated to $\mathcal{W}$.

3. Behaviour of the Periods Around the Singular Curve.

Till now we have only considered the abstract presentation of the fundamental group in terms of its generators $\alpha_i$. To obtain an explicit realization on the periods of $\mathcal{W}$, it is necessary to consider their leading behaviour in the neighbourhood of the singularity locus $L(a, b)$. Let us evaluate the integral defined in (1.3) on a suitable contour. Setting $y_3 = 1$, we may rewrite it in the following way

$$\omega_0 = 5 \oint dy_2 dy_4 dy_5 \frac{dy_1}{y_1^5 + f(a, b, y_2, y_4, y_5)}, \quad (3.1)$$

where

$$f = y_2^5 + y_4^5 + y_5^5 + 1 - 5a y_4^3 y_5^2 - 5b y_4^2 y_5^3. \quad (3.2)$$

Performing the last integration on the cycle $|y_1| = \text{const}$ (posing $y_1^5 = t$), gives immediately $\frac{2\pi i}{5} (-f)^{4/5}$, so that we may write

$$\omega_0 = 2\pi i \oint \frac{dy_2 dy_4 dy_5}{(y_2^5 + y_4^5 + y_5^5 + 1 - 5a y_4^3 y_5^2 - 5b y_4^2 y_5^3)^{4/5}} \quad (3.3)$$

$$= 2\pi i \oint dy_4 dy_5 \frac{dy_2}{[y_2^5 + g(a, b, y_4, y_5)]^{4/5}}$$

where

$$g = y_4^5 + y_5^5 + 1 - 5a y_4^3 y_5^2 - 5b y_4^2 y_5^3. \quad (3.4)$$

Again, with $y_2 = g^{1/5} u$, the last integral on the cycle $|y_2| = \text{const}$ yields

$$g^{-3/5} \oint \frac{du}{(u^5 + 1)^{4/5}} \quad (3.5)$$
which is a number independent of $a$ and $b$, so that we arrive to

$$\omega_0 = \text{const} \times \oint \frac{dy_4 dy_5}{g^{3/5}} \quad (3.6)$$

After the further change of variables

$$y_4 = \sigma \tau^{1/2} \quad ; \quad y_5 = \sigma \tau^{-1/2} \quad (3.7)$$

we find

$$\omega_0 = \text{const} \times \oint \frac{d\tau}{\tau} \oint \frac{\sigma d\sigma}{[1 + \sigma^5 h(\tau)]^{3/5}} \quad (3.8)$$

where $h(\tau) = \tau^{5/2} + \tau^{-5/2} - 5a\tau^{1/2} - 5b\tau^{-1/2}$. Setting at last $\xi = \sigma h^{1/5}$, the $\sigma$ cyclic integral gives $h^{2/5}$ times a purely numerical integral. Therefore we obtain that the periods $\omega_0$ can be written as a simple 1–dimensional integral

$$\omega_{ij} = \text{const} \times \oint_{\gamma_{ij}} \frac{d\tau}{(\tau^5 - 5a \tau^3 - 5b \tau^2 + 1)^{2/5}} \quad (3.9)$$

along a set of suitably chosen contours $\gamma_{ij}$. The quintic polynomial $P(\tau) = \tau^5 - 5a \tau^3 - 5b \tau^2 + 1$ has five roots $\tau_i$, and the possible singularities of the integral (3.9) arise as pinching singularities due to the coincidence of two roots encircled by a figure-eight contour (see fig. 6.).

One can check that the singular locus of the singularities of $P(\tau)$ is given again by the curve $L(a, b) = 0$ of eq. (2.2) by computing the resultant between $P(\tau)$ and $\frac{dP}{d\tau}$.

At this point, we may conclude that the monodromy group is exactly $B_5$. In fact we note that the braid group $B_n$ can be identified with the fundamental group of the space of all unordered sets of $n$ distinct complex numbers $\tau_i \; i = 1 \ldots, n$. More precisely, $B_n = \pi_1(\mathbb{C}^n/S_n \setminus S/S_n)$, where $S_n$ is the permutation group on five elements and $S$ is the union of the hyperplanes $\tau_i = \tau_j \forall(i, j)^{18}$. Identifying
the numbers \( \tau_i \) with the roots of a polynomial \( P_n(\tau) \) we see that in our case the monodromy group of the periods \( \omega_0 \) must be a subgroup of \( B_5 \), since \( P(\tau) \) is a quintic polynomial. On the other hand, \( B_5 \) contains the element given by (2.12) exchanging the strands 1 and 5. Since

\[
\alpha_5 = \alpha_4 \alpha_3 \alpha_2 \alpha_1 \alpha_2^{-1} \alpha_3^{-1} \alpha_4^{-1}
\]  

satisfies the relations (2.10) involving \( \alpha_5 \), the group whose presentation is given by (2.10) must actually coincide with \( B_5 \).

There is yet another independent argument leading to the same conclusions. Given a general polynomial of 5–th degree, we may always fix 3 of its 5 coefficients by a Mőbius transformation to arbitrary values. As \( P(\tau) \) contains only two parameters, it is in fact a gauge fixed form of a generic quintic polynomial, and therefore the associated subgroup of \( B_5 \) is really \( B_5 \) itself.

One can show that the set of all possible figure eight contours \( \gamma_{ij} \) encircling a couple of roots can be expressed linearly in terms of three of them, thus confirming that the number of linearly independent periods of \( \mathcal{W} \) is indeed equal to three. To see this, let us denote by \( \gamma_{i,i+1} \) the figure–eight contour encircling two consecutive roots \( \tau_i, \tau_{i+1} \) (loops encircling non consecutive roots are easily written as products of the \( \gamma_{i,i+1} \)'s, e.g. \( \gamma_{i,i+2} = \gamma_{i,i+1} \circ \gamma_{i+1,i+2} \), etc.), and by \( \omega_{i,i+1} \) the corresponding period. Only three of them are independent, since they satisfy the following two relations

\[
\sum_{k=1}^{5} \omega_{k,k+1} = 0
\]

\[
\sum_{k=1}^{5} e^{i(k-1)4\pi/5} \omega_{k,k+1} = 0
\]

The first relation easily follows from the fact that \( \gamma_{1,2} \circ \gamma_{2,3} \circ \gamma_{3,4} \circ \gamma_{4,5} \circ \gamma_{5,1} \) is homotopic to a single loop encircling all the five roots of \( P(\tau) \), and the integral is
regular at \( \infty \). The second relation can be obtained by observing that

\[
\omega_{i,i+1} = I_{i+1} - e^{-4\pi i/5} I_i
\]  

(3.12)

where \( I_i \) is the integral around a loop winding counterclockwise around the simple root \( \tau_i \). Therefore, the representation of \( B_5 \) on the \( \omega_{i,i+1} \) is 3–dimensional, thus confirming the observation made in the introduction that the vanishing of the Yukawa couplings reduces the 6–dimensional \( Sp(6, \mathbb{Z}) \)–valued representation of the periods into the \( 3 + \mathbb{Z} \)- representation which will be later shown to belong to \( U(1,2) \).

The behaviour of \( \omega_{ij}(a,b) \) around the \( L(a,b) = 0 \) singularity can be now obtained by expanding \( P(\tau) \) in the neighbourhood of a point \( \tau_0 \) where \( P(\tau) \) vanishes together with its first derivative:

\[
P(\tau) = (\tau - \tau_0)^2 + H(\tau,a,b)
\]

\[
\lim_{\tau \to \tau_0} H(\tau,a,b) = L(a,b)
\]  

(3.13)

If we now set \( \tau - \tau_0 = H^{1/2} \eta \), then we find

\[
\omega_{ij} = \text{const} \, L^{1/10} \int \frac{d\eta}{(\eta^2 + 1)^{3/5}} \equiv \text{const} \, L^{1/10}(a,b)
\]  

(3.14)

thus finding that, upon performing a loop around a branch of the hypocycloid, the integral acquires a phase \( e^{i\pi/5} \equiv z \).

An alternative way of reaching the same conclusion is to consider the effect of analytic continuation of \( \omega_{i,i+1} \) around the loop \( \alpha_i \in B_5 \) in the \((a,b)\)–plane,

\[
\omega_{i,i+1} \xrightarrow{\alpha_i} e^{i\pi/5} \omega_{i,i+1}
\]  

(3.15)

Indeed, when the path \( \alpha_i \) winds around the \( i \)–th branch of the hypocycloid, the roots \( \tau_i(a,b), \tau_{i+1}(a,b) \) of the polynomial \( P(\tau) \) are exchanged. In the \( \tau \) plane, \( \gamma_{i,i+1} \) gets deformed as in fig.7. The final result is a new circuit followed in opposite direction where the base point \( B \) is on a different Riemann sheet. This gives the same integral as before except for a phase \(- (e^{2\pi i})^{-2/5} = z \).
The correctness of this result can also be ascertained by studying the leading behaviour of the period $\omega_0$ from the Picard-Fuchs equations obeyed by it. Using the methods of 16–17, one derives the following set of three partial differential equations

\[ \frac{\partial^2 \omega_0}{\partial a^2} = \frac{1}{1-4ab} \left[ 6b^2 \frac{\partial^2 \omega_0}{\partial a \partial b} + 3a \frac{\partial^2 \omega_0}{\partial b^2} + 12b \frac{1-2ab}{1-4ab} \frac{\partial \omega_0}{\partial a} + \frac{12a}{1-4ab} \frac{\partial \omega_0}{\partial b} \right] \]

\[ \frac{\partial^2 \omega_0}{\partial b^2} = \frac{1}{1-4ab} \left[ 6a^2 \frac{\partial^2 \omega_0}{\partial a \partial b} + 3b \frac{\partial^2 \omega_0}{\partial a^2} + 12a \frac{1-2ab}{1-4ab} \frac{\partial \omega_0}{\partial b} + \frac{12b}{1-4ab} \frac{\partial \omega_0}{\partial a} \right] \]

\[ \frac{\partial^2 \omega_0}{\partial a \partial b} = \frac{1}{1-13ab} \left[ 6a^2 \frac{\partial^2 \omega_0}{\partial a^2} + 6b^2 \frac{\partial^2 \omega_0}{\partial b^2} + 11a \frac{\partial \omega_0}{\partial a} + 11b \frac{\partial \omega_0}{\partial b} + \omega_0 \right] \]

(3.16)

The study of the singularities of eqs. (3.16) is better achieved by writing the associated linear system. Setting

\[ \omega_1 = \frac{\partial \omega_0}{\partial a} = \oint_{\gamma} \frac{y^3}{W^2} \omega \]

\[ \omega_2 = \frac{\partial \omega_0}{\partial b} = \oint_{\gamma} \frac{y^2}{W^2} \omega \]

(3.17)

and by elimination of the mixed derivatives in (3.16) we find

\[ \frac{\partial}{\partial a} \Pi = A(a,b)\Pi \quad , \quad \frac{\partial}{\partial b} \Pi = B(a,b)\Pi \quad , \quad \Pi = \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{pmatrix} \]

(3.18)

where the $3 \times 3$ matrices $A(a,b)$ and $B(a,b)$ are rational functions of $a$ and $b$ whose denominator contains the singular locus $L(a,b)$. Actually, most of the matrix elements of $A$ and $B$ also contain an extra factor of $(1-4ab)$ whose appearance would imply the extra singular locus $4ab = 1$. However, such singularity is a gauge artifact, as the two systems in (3.18) are covariant under gauge transformations

\[ A \rightarrow N^{-1}AN - N^{-1}\partial N \]

(3.19)

(and similarly for $B$), where $N$ belongs to the Borel subgroup of lower triangular matrices of $GL(3,\mathbb{C})$. The exam of the Picard-Fuchs system in the neighbourhood
of the variable \( v = ab = 1/4 \) gives in fact perfectly regular solutions in the new basis

\[
\begin{align*}
\tilde{\omega}_0 &\equiv \omega_0 \\
\tilde{\omega}_1 &= \omega_1 + 8a^5\omega_2 \\
\tilde{\omega}_2 &= 2(64a^5 - 5)\omega_1 - (1 + 32a^5)\omega_2 
\end{align*}
\] (3.20)

The same conclusion is found by replacing the two linear systems in (3.18) by two 3rd order ordinary differential equations in \( a \) and \( b \) for \( \omega_0 \), with coefficients depending on the other variable, where the singularity \( ab = 1/4 \) is absent. Furthermore, the Fuchsian analysis of the linear system (3.18) or of the two 3rd order differential equations in the neighbourhood of \( L(a, b)=0 \) gives again the behaviour \( L^{1/10} \) around the singularity, thus confirming our previous analysis.

4. The monodromy generators

In this section we determine the representation \( \mathcal{R}(\alpha_i) \subset GL(3,\mathbb{C}) \) of the fundamental group acting on \( V \), the vector space spanned by three independent periods. The computation is made by extending the representation of the group \( B_5 \) to a representation on the group ring over the complex field \( \mathbb{C} \). Noting that \( \alpha_i \) acts on the period \( \omega_{i,i+1} \) as an analytic continuation around the \( i \)-th branch of \( L(a, b) = 0 \), the corresponding discontinuities are given by

\[
\begin{align*}
(\mathcal{R}(\alpha_i) - 1)\omega_{i,i+1} &= (z - 1)\omega_{i,i+1} \\
(\mathcal{R}(\alpha_i) - 1)\omega_{j,j+1} &= 0 \quad i \neq j, j + 1 
\end{align*}
\] (4.1)

Hence we introduce the (Picard–Lefschetz) discontinuity operators\(^{18}\)

\[
\mathcal{R}(\alpha_i) = (z - 1)u_i + 1
\] (4.2)

where \( u_i \) are 1-dimensional projection operators obeying \( u_i^2 = u_i \). The first set of
relations (2.10) imply

\[ u_i u_j = u_j u_i \]  \hspace{1cm} (4.3)

where \( i, j \) are non contiguous indices and we are considering 1, 2, 3, 4, 5 cyclicly ordered so that 1 and 5 are contiguous. By right multiplication with \( u_j \) and left multiplication with \( u_i \) we get

\[ u_i u_j = u_j u_i u_j \equiv \lambda u_j \]
\[ u_i u_j = u_i u_j u_i \equiv \lambda u_i \] \hspace{1cm} (4.4)

where we have used the fact that since the \( u_i \) are 1-dimensional projection operators, for any operator \( \mathcal{O} \),

\[ u_i \mathcal{O} u_i = \lambda \mathcal{O} u_i \] \hspace{1cm} (4.5)

Eqs. (4.4) then imply

\[ u_i u_j = u_j u_i = 0 \] \hspace{1cm} (4.6)

which is a relation much stronger than (4.3). Indeed, eq. (4.6) has an intuitive meaning since e.g. \( u_1, u_3 \) correspond to projection of the integral (3.9) around the disconnected figure eight circuits winding the roots \( \tau_1, \tau_3 \) and \( \tau_3, \tau_4 \) respectively.

From the second set of (2.10) we get

\[ (z - 1)^2 u_i u_{i+1} u_i + z u_i = (z - 1)^2 u_{i+1} u_i u_{i+1} + z u_{i+1} \] \hspace{1cm} (4.7)

Again, since the \( u_i \) are 1-dimensional projection operators, we have

\[ u_i u_{i+1} u_i = \rho u_i \]
\[ u_{i+1} u_i u_{i+1} = \sigma u_{i+1} \] \hspace{1cm} (4.8)

Multiplying the two equations in (4.8) by \( u_{i+1} \) on the right and \( u_i \) on the left
respectively, we find $\rho = \sigma$, so that (4.7) gives

$$u_i((z - 1)^2 \rho + z) = u_{i+1}((z - 1)^2 \rho + z)$$  \hspace{1cm} (4.9)

or

$$\rho = -\frac{z}{(z - 1)^2}$$  \hspace{1cm} (4.10)

Using the relation (3.10) in the form $\alpha_5 \alpha_1 \alpha_3 \alpha_2 = \alpha_4 \alpha_3 \alpha_2 \alpha_1$, we find from (4.2)

$$(z - 1)^3 u_5 u_4 u_3 u_2 + (z - 1)^2 u_5 u_4 u_3 + (z - 1) u_5 u_4 + u_5 =$$

$$= (z - 1)^3 u_4 u_3 u_2 u_1 + (z - 1)^2 u_3 u_2 u_1 + (z - 1) u_2 u_1 + u_1$$  \hspace{1cm} (4.11)

By right multiplication with $u_2$ and left multiplication with $u_5$, using (4.8) – (4.10) we get

$$(z - 1) z^2 u_5 u_4 u_3 u_2 = u_5 u_1 u_2$$  \hspace{1cm} (4.12)

where the final form of the coefficient on the l.h.s. of (4.12) is obtained by using the relations obeyed by the tenth roots of unity $z = e^{i \pi/5}$ (e.g. $1 - z + z^2 - z^3 + z^4 \equiv 0$).

Obviously, equations analogous to (4.12) are also obeyed by similar products of $u_i$’s with indices cyclically permuted. Such relations allow us to replace a product of four contiguous $u_i$ operators in decreasing order with (a coefficient times) the product of three contiguous $u_i$’s in increasing order, the first and the last factors being the same in both expressions.

We are now ready to construct the explicit representation for the $\alpha_i$’s. We select three arbitrary linearly independent basis vectors $\omega_{51}, \omega_{12}, \omega_{45}$ defined as the eigenvectors of $u_5, u_1, u_4$ corresponding to eigenvalue one

$$u_5 \omega_{51} = \omega_{51} \equiv \Psi_{51}$$

$$u_1 \omega_{51} = \frac{1}{z - 1} \omega_{12} \equiv \Psi_{12}$$

$$u_4 \omega_{51} = \frac{1 - z}{z} \omega_{45} \equiv \Psi_{45}$$  \hspace{1cm} (4.13)

where the factors in front of $\omega_{ij}$ have been chosen in such a way that the action of the cyclic permutation $Z$ of $\mathbb{Z}_5 \subset B_5$ gives $Z \omega_{i,i+1} = \omega_{i+1,i+2}$ with no extra phase.
The application of \( u_i, i = 1, \ldots, 5 \) to any basis vector gives a linear combination of them, namely

\[
\begin{align*}
    u_i u_k \Psi_{51} &= p_i \Psi_{51} + q_i \Psi_{12} + r_i \Psi_{45} & \text{for } k = 5, 1, 4 \quad (4.14)
\end{align*}
\]

It is now easy to compute the coefficients \( p_i, q_i, r_i \) by repeated use of the formulae (4.8)-(4.11). For instance, if we take \( i = 3 \), then

\[
\begin{align*}
    u_3 \Psi_{51} &= u_3 u_5 \Psi_{51} = 0 \\
    u_3 \Psi_{45} &= u_3 u_4 \Psi_{51} = p \Psi_{51} + q u_1 \Psi_{51} + r u_4 \Psi_{51} \\
\end{align*}
\]

On the other hand,

\[
    u_3 u_4 \Psi_{51} = u_3 u_4 u_5 \Psi_{51} \quad (4.15)
\]

and using (4.12)

\[
    u_3 u_4 u_5 \Psi_{51} = (z - 1) z^2 u_3 u_2 u_1 u_5 \Psi_{51} = p \Psi_{51} + q u_1 \Psi_{51} + r u_4 \Psi_{51} \\
\]

Multiplying the last two sides by \( u_2 \) on the left we obtain

\[
\begin{align*}
    - (z - 1) z^2 \frac{z}{(z - 1)^2} u_2 u_1 \Psi_{51} &= qu_2 u_1 \Psi_{51} \\
    \rightarrow q &= \frac{-z^3}{z - 1} \\
\end{align*}
\]

Applying now \( u_3 \) on the left of (4.15) , we find

\[
    u_3 u_4 \Psi_{51} = r u_3 u_4 \Psi_{51} \rightarrow r = 1 \quad (4.16)
\]

Finally, acting with \( u_4 \) in (4.15) we have

\[
\begin{align*}
    u_4 u_3 u_4 \Psi_{51} &= (p + r) u_4 \Psi_{51} \\
    \rightarrow -\frac{z}{(z - 1)^2} u_4 \Psi_{51} &= (p + r) u_4 \Psi_{51} \\
    \rightarrow p &= \frac{1 - z^3}{(z - 1)^3} \\
\end{align*}
\]

In the same way one can compute the coefficients for the action of any other projection operator \( u_i \). The final result for the monodromy operator \( \alpha_i \) on the
basis \{\omega_45, \omega_51, \omega_12\} is

\[
R(\alpha_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & z \end{pmatrix}, \quad R(\alpha_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ z^4 & -1 + z - z^2 & z \end{pmatrix}, \\
R(\alpha_3) = \begin{pmatrix} z & z^2(1 - z) & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R(\alpha_4) = \begin{pmatrix} -z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
R(\alpha_5) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & z & 0 \\ 0 & -z & 1 \end{pmatrix}.
\]

(4.21)

From eq.s (4.21) we may verify that indeed \(\alpha_5\) satisfies the relation (2.12). Furthermore, we can use (4.21) to compute the monodromy generator around \(\infty\), which we have disregarded until now. It can be shown that the generator \(\alpha_\infty\) can be written by the following word

\[
\alpha_\infty = \alpha_4\alpha_2\alpha_3\alpha_5\alpha_1\alpha_5
\]

(4.22)

and we obtain

\[
R(\alpha_\infty) = \begin{pmatrix} z^2 & 1 + z^2 & z(1 + z^2) \\ -z^2 & -1 - z^2 & -z^3 \\ -1 & -1 & -1 - z \end{pmatrix}
\]

(4.23)

We note that the eigenvalues of \(R(\alpha_\infty)\) are \{-1, -1, -z\} thus showing the presence of a singularity at \(\infty\) with critical exponent \(-\frac{2}{5}\). This can also be confirmed by the behaviour of the integral (3.9) for large values of \(a\) and \(b\). We find in this case

\[
\omega_0^{a,b \to \infty} \sim \int \frac{d\tau}{(5a\tau^3 + 5b\tau^2)^{2/5}}
\]

(4.24)

and by the rescaling \(a \to \lambda \xi, b \to \lambda \eta\) we find

\[
\omega_0 \sim \int \frac{d\tau}{(\xi \tau^3 + \eta \tau^2)^{-2/5}} \lambda^{-2/5} = \text{const} \lambda^{-2/5}
\]

(4.25)

thus confirming the critical behaviour computed from (4.23).
5. The Duality Group

It is known that the full duality group of the moduli space is given not only by the monodromy group of the periods, \( \Gamma_M \), but also by the symmetry group of the defining polynomial \( \mathcal{W} \), \( \Gamma_W \). We now want to show that the symmetries of the defining polynomial \( \mathcal{W}(y; a, b) = 0 \) give at most a central extension for the monodromy group \( B_5 \) acting on the 3–dimensional basis of the periods.

It is easily seen that the transformations leaving invariant \( \mathcal{W} \), are given by

\[
\begin{aligned}
    a &\rightarrow \rho a \\
    b &\rightarrow \rho^{-1} b
\end{aligned}
\]

with \( \rho^5 = 1 \), as they can be undone by the linear coordinate transformation

\[
\begin{aligned}
    y_4 &\rightarrow \rho y_4 \\
    y_5 &\rightarrow \rho^{-1} y_5
\end{aligned}
\]

Since there is apparently no other action with this property, we conclude that the duality group of the superpotential is given by \( \mathbb{Z}_5 \).

In order to find the representation \( U \) of the transformations (5.1) on the periods, we observe that on any integral, say \( \omega_{51} \), the transformation (5.1) can be compensated in the integrand by the map

\[
\tau \rightarrow \rho^3 \tau
\]

On the other hand, choosing \( \rho = e^{4\pi i/5} \), the transformations (5.1) on the \( (p, q) \) real plane correspond to a rotation of an angle \( 4\pi / 5 \), mapping the 5–th branch of the hypocycloid into the 1–st, so that \( \gamma_{51} \) is mapped into \( \gamma_{12} \). Taking into account that \( d\tau \rightarrow \rho^3 d\tau \), we find

\[
U : \omega_{51} \rightarrow e^{2\pi i/5} \omega_{12}
\]

and analogous relations for cyclically permuted indices. We now observe that the
transformation $\omega_{5,1} \rightarrow \omega_{1,2}$ is realized by the monodromy operator

$$Z = R(\alpha_3 \alpha_2 \alpha_1 \alpha_5) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^4 & -1 + z - z^2 & z - 1 \end{pmatrix} \quad (5.5)$$

which corresponds to a generator of the cyclic subgroup $\mathbb{Z}_5 \subset B_5$. It follows that

$$UZ = e^{2\pi i/5}I \quad (5.6)$$

on any period $\omega_{i,i+1}$ and therefore also on the selected basis \{\omega_{45}, \omega_{51}, \omega_{12}\}. Thus we conclude that, unless there is some element of $B_5$ represented by $UZ$, the $U$–transformation gives a central extension of the braid group $B_5$. The above central extension gives the full duality group of the moduli space of the Calabi–Yau manifold.

Notice that our result differs from the previously studied one–dimensional examples, where the full duality group $\Gamma$ was given by the semidirect product $\Gamma_M \rtimes \Gamma_W$ of the monodromy group of the periods and the symmetry group of $W$, as suggested in 17. We further remark that, since the moduli space is 2–dimensional, we may take as coordinates the ratios $t_1 = \frac{\omega_{12}}{\omega_{51}}, t_2 = \frac{\omega_{45}}{\omega_{51}}$ which correspond to a linear combination of the “special” variables of Special Geometry. Hence, on $t_1, t_2$, the action of the full duality group is given by a faithful projective 3–dimensional representation of $B_5$.

Recalling that the full symmetry of the moduli space is given by modding out by $\Gamma$ the local moduli space, we obtain that the geometry of $\mathcal{M}$ is given by

$$\mathcal{M} = \frac{U(1,2)}{U(1) \otimes U(2)} / \hat{B}_5 \quad (5.7)$$

where $\hat{B}_5$ is the previously introduced central extension of $B_5$. 

21
6. Conclusions

Some comments are in order. We have found a 3-dimensional representation of the monodromy group for the three fundamental periods $\omega_{4,5}, \omega_{5,1}, \omega_{1,2}$ given by the three integrals associated to independent loops of the integral (3.9), or, equivalently, to the top solution of the system of differential equations (3.18). We know that $B_5$ must act as a group of discrete isometries on the local moduli space $U(1) \otimes U(2)$ and therefore our matrices should belong to the $U(1,2)$ group. In fact, it can be shown that the matrices (4.21) satisfy

$$\mathcal{R}(a_i^\dagger)g \mathcal{R}(a_i) = g$$

where $g$ is the metric given by

$$g = \begin{pmatrix}
1 & \frac{1-z}{z} & 0 \\
\frac{1-z}{z} & 2 + z^2 - z^3 & z - 1 \\
0 & z - 1 & 1
\end{pmatrix}$$

Since $g$ has one positive and two negative eigenvalues, indeed $\alpha_i \in U(1,2)$. As we have already remarked, there must exist a canonical basis for the $H_{(3)}$ homology of the Calabi–Yau, where the direct sum of the 3 and $\overline{3}$ representations of $U(1,2)$ given by (4.21) and their complex conjugate take values in $SP(6,\mathbb{Z})$, six being the dimension of $H_{(3)}$. We leave the determination of this change of basis to forthcoming work.

Let us summarize our results. Starting with the family of curves given in eq. (1.2) we have been able to compute exactly the duality group of the periods associated to $\mathcal{W}(y; a, b)$ by means of some very efficient and powerful techniques of algebraic geometry, without resorting to the explicit computation of the periods e.g. via solutions of the Picard-Fuchs equations. Our method is in principle applicable also to more complicated situations where more moduli are present and/or Yukawa couplings are non–vanishing. In fact, the computation of the fundamental group
\( \pi_1(\mathbb{P}(N) - L^{N-1}; B) \) is always possible in virtue of the fundamental theorems of Picard–Severi and Zariski, together with the Van Kampen relations. The actual construction of the monodromy group relies however also on the knowledge of the behaviour of the periods around the singular locus of the defining polynomial. In our case, this computation was derived from the study of the 1–dimensional integral, which simplifies the actual task. It is clear that in general one cannot expect that the periods can always be reduced to such one–dimensional integrals, and the exam of the leading singularity can be more involved. Still, it is important to realize that, as mentioned in chapter 3, the analysis of the singularities can always be done in a systematic way from the linear system of Picard–Fuchs equations using standard techniques of fuchsian analysis.

We hope to extend these techniques for the computation of the duality group to more complicated examples in future publications.

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