How many ways can you make change: Some Easy Proofs

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Abstract

Given a dollar, how many ways are there to make change using pennies, nickels, dimes, and quarters? What if you are given a different amount of money? What if you use different coin denominations? This is a well known problem and formulas are known. We present simpler proofs in several cases. We use recurrences to derive formulas if the coin denominations are \( \{1, x, kx, rx\} \), and we use a simple proof using generating functions to derive a formula for any coin set.

1 Introduction

How many ways are there to make change of a dollar using pennies, nickels, dimes, and quarters? This is a well known question in recreational math, serious math, and computer science. We have observed three types answers in the literature and on the web.

1. There are 242 ways to make change. The author then points to a computer program or to the actual list of ways to do it.

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2. The number of ways to make change for \( n \) cents is the coefficient of \( z^n \) in the power series for
\[
\frac{1}{(1 - z)(1 - z^5)(1 - x^{10})(1 - z^{25})}
\]
which can be worked out.

3. This is known as the problem of finding the Coefficients of the Sylvester Denumerant and is related to the Frobenius problem (given \( n \) coins what is the largest number that you cannot make change of). These papers tend to use advanced mathematics. These papers give different approaches to obtain formulas for any coin set. We will discuss these papers later; however, for now we just list some of them [1, 2, 3, 4, 5, 6, 9, 11, 12].

**Def 1.1** If \( S \) is a set of coin denominations then the change function for \( S \) is the function that, on input \( n \), outputs the number of ways to make change for \( n \) using the coins in \( S \).

In the first part of this paper we use recurrences to obtain simple derivations for the change function for two types of sets: (1) \( S = \{1, s, ks\} \) where \( s, k \geq 2 \), and (2) \( S = \{1, s, ks, rs\} \) where \( s \geq 2 \), and \( 2 \leq k < r \). As a corollary we obtain the case of pennies, nickels, dimes, and quarters. In passing we solve the change-for-a-dollar problem by hand.

In the second part of this paper we use generating functions (in a simple way) to obtain for any finite set \( S \), the change function.

The formulas we derive are known; however, our proofs are simpler than those in the literature.

2 **General Definitions and Theorems**

**Convention 2.1** Let \( S \) be a non empty set of coins. The number of ways to make 0 cents change is 1. For all \( n \leq -1 \) the number of ways to make \( n \) cents change is 0.

**Def 2.2** Let \( S = \{1 < s < t < u\} \).
1. $a_n$ is the number of ways to make change of $n$ cents using pennies. Clearly $(\forall n)[a_n = 1]$.

2. $b_n$ is the number of ways to make change of $n$ cents using the first two coins (pennies and $s$-cent coins). Clearly $(\forall n)[b_n = a_n + b_{n-s}]$. We use that $(\forall n \leq -1)[a_n = 0]$.

3. $c_n$ is the number of ways to make change of $n$ cents using the first three coins (pennies, $s$-cent coins, and $t$-cent coins). Clearly $(\forall n)[c_n = b_n + c_{n-t}]$.

4. $d_n$ is the number of ways to make change of $n$ cents using all four coins (pennies, $s$-cent coins, $t$-cent coins, and $u$-cent coins). Clearly $(\forall n)[d_n = c_n + d_{n-u}]$.

We do one example: Let $S = \{1, 2, 4, 5\}$. What is $d_9$?

1. If one 5-cent coin is used then for the remaining four cents you must use either one 4-cent coin; two 2-cents coins; one 2-cent coin and two pennies; or four pennies.

2. If no 5-cent coins and one 4-cent coin is used then for the remaining five cents you must use either two 2-cents coins and one penny; one 2-cent coin and three pennies; or five pennies.

3. If no 5-cent coins and two 4-cent coins are used then for the remaining one cent you must use one penny.

4. If no 5-cent coins and no 4-cent coins are used then you must use either zero, one, two, three, or four 2-cent coins and the appropriate number of pennies.

Hence $d_9 = 4 + 3 + 1 + 5 = 12$.

Using the recurrence for $b_n$ and $(\forall n)[a_n = 1]$, one can show the following.

**Theorem 2.3** $(\forall n)[b_n = \left\lfloor \frac{n}{s} \right\rfloor + 1]$.

We can now solve the change-for-a-dollar problem by hand. Let $S = \{1, 5, 10, 25\}$. We need to compute $d_{100}$. We use the exact formula for $b_n$ and the recurrences for $c_n$ and $d_n$. 

3
\[d_{100} = c_{100} + c_{75} + c_{50} + c_{25} + c_0\]
\[c_0 = 1\]
\[c_{25} = b_{25} + b_{15} + b_5 = 6 + 4 + 2 = 12\]
\[c_{50} = b_{50} + b_{40} + b_{30} + b_{20} + b_{10} + b_0 = 11 + 9 + 7 + 5 + 3 + 1 = 36\]
\[c_{75} = b_{75} + b_{65} + b_{55} + b_{45} + b_{35} + c_{25} = 16 + 14 + 12 + 10 + 8 + 12 = 72\]
\[c_{100} = b_{100} + b_{90} + b_{80} + b_{70} + b_{60} + c_{50} = 21 + 19 + 17 + 15 + 13 + 36 = 121\]

Hence
\[d_{100} = 1 + 12 + 36 + 72 + 121 = 242.\]

3 **The Coin Set \{1, s, ks\}**

Throughout this section we will be using the coin set \(S = \{1, s, ks\}\) where \(s, k \geq 2\) are fixed natural numbers. The quantities \(a_n, b_n, c_n\) are as in Definition 2.2 with coin set \(S\).

To determine the number of ways to make change, you can always round down to the nearest multiple of \(s\). Formally \(c_sL + L_0 = c_{sL}\). We use this without mention.

Let \(n = sL + L_0\) where \(0 \leq L_0 \leq s - 1\) and \(L \geq 1\). Using the recurrence for \(c_n\) and the formula for \(b_n\) (from Theorem 2.3) we have:

\[c_n = c_{sL} = b_{sL} + c_{s(L-k)}\]
\[= b_{sL} + b_{s(L-k)} + c_{s(L-2k)}\]
\[= b_{s(L-0)} + b_{s(L-k)} + \cdots + b_{s(L-ki)} + c_{s(L-ki-k)}\]
\[= (L + 1) + (L - k + 1) + \cdots + (L - ki + 1) + c_{s(L-ki-k)}\]

Let \(L \equiv j \pmod{k}\). Let \(i = \frac{(L-j-k)}{k}\). Then the last term in the sum is \(c_{sj}\). Since \(j \leq k - 1\), \(sj < sk\). Hence \(c_{sj} = b_{sj} = j + 1\). The resulting sum is an arithmetic series with first term \(j + 1\), last term \(j + 1 + (\frac{L-j}{k})k\), and number of terms \(\frac{L-j}{k} + 1 = \frac{L-j+k}{k}\). Hence after easy algebra we have the following

**Theorem 3.1** Let \(n = sL + L_0\) where \(0 \leq L_0 \leq s - 1\) and \(L \geq 1\). (So that \(L = \left\lfloor \frac{n}{s} \right\rfloor\).)
1. Let \( j \) be such that \( L \equiv j \mod k \). Then

\[
c_n = \frac{L^2 + (k + 2)L + 2k}{2k} + \frac{(k - 2)j - j^2}{2k}.
\]

2. \[ \frac{n^2}{2ks^2} + \frac{n}{2s} - k \leq c_n \leq \frac{n^2}{2ks^2} + \frac{(k+2)n}{2ks} + \frac{(k-2)^2}{8} + 1 \]

**Proof:** Part 1 follows from our work. We prove Part 2

For the lower bound we use that \( L \geq \frac{n-s}{s} \) and note that the last term has min value, as \( 0 \leq j \leq k - 1, \) of \(-k\). For the upper bound we use that \( L \leq \frac{n}{s} \) and note that the last term is max value, as \( 0 \leq j \leq k - 1, \) of \( \frac{(k-2)^2}{8} + 1. \)

**Note 3.2** Theorem 3.1 for the special case of pennies, nickels, and dimes (\( s = 5, \ k = 2 \)) was proven by Deborah Levine’s article [10].

**Note 3.3** One can derive \( c_n = \frac{n^2}{2ks^2} + \Theta\left(\frac{n}{2s}\right) \) from Schur’s theorem [7, 14, 15].

4. **The Coin Set** \( \{1, s, ks, rs\} \)

Throughout this section we will be using the coin set \( S = \{1, s, ks, rs\} \) where \( s, k, r \) are fixed natural numbers with \( s, k, r \geq 2 \) and \( r > k \). The quantities \( a_n, b_n, c_n, d_n \) are as in Definition 2.2 with coin set \( S \).

To determine the number of ways to make change, you can always round down to the nearest multiple of \( s \). Formally \( d_{sL+L_0} = d_{sL} \). We use this without mention.

Let \( n = s(rL + M) + L_0 \) where \( 0 \leq M \leq r - 1, \ 0 \leq L_0 \leq s - 1 \). Using the recurrence for \( d_n \) we have:
\[ d_n = d_s(rL+M) = c_s(rL+M) + d_s(rL+M-rx1) \]
\[ = c_s(rL+M) + c_s(rL+M-rx1) + d_s(rL+M-rx2) \]
\[ = c_s(rL+M) + c_s(rL+M-rx1) + c_s(rL+M-rx2) + \cdots + c_s(M+r) + d_sM \]
\[ = c_s(rL+M) + c_s(rL+M-rx1) + c_s(rL+M-rx2) + \cdots + c_s(M+r) + c_sM \]
\[ = \sum_{i=0}^{L} c_s(ri+M) \]

Using the formula for \(c_n\) from Theorem 3.1 we obtain

\[ d_n = \sum_{i=0}^{L} \frac{(M+ri)^2 + (k+2)(M+ri) + 2k}{2k} + \sum_{i=0}^{L} \frac{(k-2)(ri + M \mod k) - (ri + M \mod k)^2}{2k}. \]

We will evaluate the second sum later. For now we name it:

**Notation 4.1** \(\Delta(L, M) = \sum_{i=0}^{L} \frac{(k-2)(ri + M \mod k) - (ri + M \mod k)^2}{2k} \)

Thus \(d_n\) is

\[ \sum_{i=0}^{L} \frac{(M+ri)^2 + (k+2)(M+ri) + 2k}{2k} + \Delta(L, M). \]

\[ = \frac{1}{2k} \left( (L+1)(M^2 + kM + 2M + 2k) + r(2M + k + 2) \sum_{k=0}^{L} i + r^2 \sum_{i=0}^{L} i^2 \right) + \Delta(L, M) \]
\[ = \frac{1}{12k} \left( (L+1)(2r^2L^2 + (r^2 + 6Mr + 3kr + 6r)L + 6M^2 + (6k + 12)M + 12k) \right) + \Delta(L, M) \]
Lemma 4.2 Let $L, M \geq 1$ and $a \geq 0$.

1. $\sum_{i=0}^{L}(ri + M \mod k)^a = \sum_{j=0}^{k-1}(rj + M \mod k)^a \left\lfloor \frac{L-j+k}{k} \right\rfloor$.

2. $\Delta(L, M) = \frac{1}{2k}(\sum_{j=0}^{k-1}\left(\left\lfloor \frac{L-j}{k} \right\rfloor + 1\right)((k-2)(rj + M \mod k) - (rj + M \mod k)^2)$

3. If $k = 2$ and $r \equiv 0 \pmod{2}$ then $\Delta(L, M) = -\frac{(1+(-1)^{M+1})(L+1)}{8}$.

4. If $k = 2$ and $r \equiv 1 \pmod{2}$ then $\Delta(L, M) = -\frac{2L+(1+(-1)^{L})(1+(-1)^{M+1})+(1+(-1)^{L+1})}{16}$.

**Proof:**

1) We break this sum into parts depending on what $i$ is congruent to mod $k$.

$$
\sum_{i=0}^{L}(ri + M \mod k)^a = \sum_{j=0}^{k-1} \sum_{i=0, i \equiv j \mod k} \sum_{j=0}^{L}(ri + M \mod k)^a \\
= \sum_{j=0}^{k-1} \sum_{i=0, i \equiv j \mod k} \sum_{j=0}^{L}(rj + M \mod k)^a \\
= \sum_{j=0}^{k-1} (rj + M \mod k)^a \sum_{i=0, i \equiv j \mod k} 1 \\
= \sum_{j=0}^{k-1} (rj + M \mod k)^a \left\lfloor \frac{L-j+k}{k} \right\rfloor \\
$$

2) This follows from part 1 using $a = 1$ and $a = 2$.

3 and 4) If $k = 2$ then notice that the expression for $\Delta(L, M)$ is simplified considerably since $k-2 = 0$ and $(rj + M \mod 2)^2 = (rj + M \mod 2)$. Also note that the summation only has two terms ($j = 0$ and $j = 1$). Hence we obtain

$$
\Delta(L, M) = -\frac{1}{4} \left( \left\lfloor \frac{L+2}{2} \right\rfloor (M \mod 2) + \left\lfloor \frac{L+1}{2} \right\rfloor (M \mod 2) \right).
$$

**Case 0:** $r \equiv 0 \pmod{2}$.

If $M \equiv 0 \pmod{2}$ then $\Delta(L, M) = 0$.

If $M \equiv 1 \pmod{2}$ then

$$
\Delta(L, M) = -\frac{1}{4} \left( \left\lfloor \frac{L+2}{2} \right\rfloor + \left\lfloor \frac{L+1}{2} \right\rfloor \right) = -\frac{L+1}{4}.
$$
One can check that $\Delta(L, M) = (1+(-1)^{M+1})(L+1)$.  

**Case 1: $r \equiv 1 \pmod{2}$** Then

$$\Delta(L, M) = -\frac{1}{4} \left( \left\lfloor \frac{L+2}{2} \right\rfloor (M \mod 2) + \left\lfloor \frac{L+1}{2} \right\rfloor (1 + M \mod 2) \right).$$

The following table summarizes what $\Delta(L, M)$ is, given what $L, M$ are mod 2.

| $L \mod 2$ | $M \mod 2$ | $\Delta(L, M)$ |
|------------|------------|----------------|
| 0          | 0          | $-\frac{L}{8}$ |
| 0          | 1          | $-\frac{L+2}{8}$ |
| 1          | 0          | $-\frac{L+1}{8}$ |
| 1          | 1          | $-\frac{L+1}{8}$ |

One can check that $\Delta(L, M) = \frac{2L+(1+(-1)^r)(1+(-1)^{M+1})+(1+(-1)^{L+1})}{16}$.  

Putting this all together we have the following.

**Theorem 4.3** Let $n = s(rL + M) + L_0$ where $0 \leq M \leq r - 1$, $0 \leq L_0 \leq s - 1$, and $L \geq 1$. (So $L = \left\lfloor \frac{n}{rs} \right\rfloor$, $M = \left\lfloor \frac{n \mod rs}{s} \right\rfloor$, and $n \equiv L_0 \pmod{s}$.) Then the following are true.

1. $d_n$ is

$$\frac{1}{12k} \left( (L+1)(2r^2L^2 + (r^2 + 6Mr + 3kr + 6r)L + 6M^2 + (6k + 12)M + 12k) \right)$$

$$+ \frac{1}{2k} \left( \sum_{j=0}^{k-1} \left( \left\lfloor \frac{L - j}{k} \right\rfloor + 1 \right) \right. (k - 2)(rj + M \mod k) - (rj + M \mod k)^2 \left. \right)$$

2. $d_n = \frac{n^4}{6krs^3} + \Theta(n^2)$.  

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3. If \( k = 2 \) and \( r \equiv 0 \pmod{2} \) then \( d_n \) is

\[
\frac{1}{24} \left( (L+1)(2r^2L^2 + (r^2 + 6Mr + 12r)L + 6M^2 + 24M + 24) \right) - \frac{(1 + (-1)^{M+1})(L+1)}{8}
\]

4. If \( k = 2 \) and \( r \equiv 1 \pmod{2} \) then \( d_n \) is

\[
\frac{1}{24} \left( (L+1)(2r^2L^2 + (r^2 + 6Mr + 12r)L + 6M^2 + 24M + 24) \right) \\
+ \frac{2L + (1 + (-1)^L)(1 + (-1)^{M+1}) + (1 + (-1)^{L+1})}{16}.
\]

**Note 4.4** One can derive the asymptotic result (part 2) from Schur’s theorem \([7, 14, 15]\).

As a corollary of Theorem 4.3 we obtain a formula for making change of \( n \) cents using pennies, nickels, dimes, and quarters.

**Corollary 4.5** If \( s = 5, k = 2, \) and \( r = 5 \) then \( d_n \) is

\[
\frac{1}{24} \left( (L+1)(50L^2 + (8530M)L + 6M^2 + 24M + 24) \right) \\
+ \frac{2L + (1 + (-1)^L)(1 + (-1)^{M+1}) + (1 + (-1)^{L+1})}{16}.
\]

5 Any Finite Coin Set

Throughout this section \( S = \{t_1 < t_2 < \cdots < t_v\} \) is our coin set. We will derive the change function for \( S \) using generating functions. The use of generating functions is well known in this area. The basic idea of this proof is from Graham, Knuth, Patashnik \([8]\). The proof we give seems to be new. We discuss the literature after we obtain the result.
It is easy to see that the number of ways to make change of \( n \) cents using the coins in \( S \) is the coefficient of \( z^n \) in

\[
C(z) = \prod_{i=1}^{v} \frac{1}{(1 - z^{t_i})}
\]

Let \( t \) be the least common multiple of \( \{t_1, \ldots, t_v\} \). For \( 1 \leq i \leq v \) let \( f_i \) be the polynomial such that \( (1 - z^t) = (1 - z^{t_i})f_i(z) \). Note for later that

\[
f_i(z) = (1 + z^{t_i} + z^{2t_i} + \cdots + z^t).
\]

It is easy to see that

\[
A(z) = f_1(z) \cdots f_v(z).
\]

Let \( M \) be the degree of \( A(z) \) which is \(((t - t_1) + (t - t_2) + \cdots + (t - t_v)) \leq tv \). Let the coefficient of \( z^j \) in \( A(z) \) be \( a_j \). Using the power series expansion \( \frac{1}{(1-x)^v} = \sum_{i=0}^{\infty} \binom{i+v-1}{v-1} x^i \) and plugging in \( x = z^t \) we obtain

\[
C(z) = \left( \sum_{j=0}^{M} a_j z^j \right) \left( \sum_{i=0}^{\infty} \binom{i+v-1}{v-1} z^{ti} \right) = \sum_{j=0}^{M} \sum_{i=0}^{\infty} a_j \binom{i+v-1}{v-1} z^{ti+j}.
\]

The coefficient of \( z^n \) is

\[
\sum_{0 \leq j \leq M : j \equiv n \mod t} a_j \binom{n-j}{t} + v - 1 \]

How easy is it to find the \( a_j \)'s? Note that since

\[
f_i(z) = (1 + z^{t_i} + z^{2t_i} + \cdots + z^t)
\]

\( a_j \) is the number of ways to make change of \( j \) using coins in \( S \) with the restriction that coin \( t_i \) is
used at most \( \frac{t}{t_i} \) times. A simple dynamic program can find all of the \( a_j \)'s; however, this will take \( t^v \) steps.

How many operations does it take to, given \( n \), find the answer. Note that the binomial coefficients are consecutive in that the top part goes through \( \leq \frac{M}{t} \) consecutive numbers while the bottom part stays the same. Hence this can be done in \( \leq v - 1 + \frac{M}{t} \leq 2v \) multiplications. each one is multiplied by the appropriate \( a_j \) which is another \( \leq \frac{M}{t} \leq v \) multiplications. Hence we have \( \leq 3v \) multiplications. The summation then adds \( \frac{M}{t} \leq v \) additions (one would need to code this up carefully and only visit those \( j \equiv n \mod t \)). This leads to \( 4v \) operations. The number of operations depends only on the number of coins and not their values; however, if the coins had large values that would make each operation take longer.

Putting this all together we have the following.

**Theorem 5.1** Let \( S = \{t_1, \ldots, t_v\} \) and \( t \) be the least common multiple of \( t_1, \ldots, t_v \). Let \( M = ((t - t_1) + (t - t_2) + \cdots + (t - t_v)) \). Then the change function for \( S \) is

\[
\sum_{0 \leq j \leq M : j \equiv n \mod t} a_j \left( \frac{n-j}{t} + v - 1 \right)
\]

where the coefficients \( a_j \) can be found in \( O(t^v) \) steps and the formula itself can be evaluated in \( O(v) \) steps.

**Def 5.2** A function \( f(n) \) is *quasi polynomial* if there exists a polynomial \( g \), a function \( h \), and a number \( B \) such that \( f(n) = g(n) + h(n) \) and \( h(n) \) depends only on \( n \mod B \). The polynomial \( g \) is called the *polynomial part*.

We obtain the following (known) corollary of Theorem 5.1.

**Corollary 5.3** For any set \( S \) the change function of \( S \) is quasi polynomial with \( B \) being the least common multiple of the elements of \( S \).
Corollary 5.3 was known in the days of Sylvester and Cayley (see the references in [6]). E.T. Bell [6] provided a different proof which is fairly simple but uses roots of unity. We believe our proof of Corollary 5.3 is simpler. Komatsu [9] gives a method to determine, given coin set $S$ and the number $n$ how many ways are there to make change. He claims his method is computationally practical. I presume it is faster than our method; however, he is not explicit about how long it takes. The calculations involve roots of unity and partial derivatives. We believe our approach is simpler. Losonek [11] claims to have an exact formula for the case of $v = 3$; however, since the paper is not online, whatever he has will be lost to future generations. Beck, Gessel, and Komatsu [5] derive a general formula for the polynomial part of the change function. Their polynomial depends on Bernoulli numbers. They do not obtain a general formula; hence our result and theirs are incomparable. Baldoni, Berline, De Loera, Dutra, and Vergne [4] have a polynomial time algorithm for the following: for fixed $k$, compute the first highest $k + 1$ coefficients of the change function (they define this carefully). Their algorithm uses rather sophisticated mathematics. Our approach is simpler but our algorithm to obtain the change function is slower. Tripathi [13] gives a simple proof of a general formula. His formula depends on parameters $m_j$ that are the least $N$ such that $N \equiv \text{mod} t_1$ and one can make change for $N$. Our proof and his are different. While we believe ours is simpler, this is debatable.

Can one obtain a formula for the change problem quickly? Since the unbounded knapsack problem (you are allowed to use any item any number of times) is NP-complete, it is unlikely that a formula can be obtained and evaluated quickly.

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