H-Theorem and Generalized Entropies
Within the Framework of Non Linear Kinetics

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In the present effort we consider the most general non linear particle kinetics within the framework of the Fokker-Planck picture. We show that the kinetics imposes the form of the generalized entropy and subsequently we demonstrate the H-theorem. The particle statistical distribution is obtained, both as stationary solution of the non linear evolution equation and as the state which maximizes the generalized entropy. The present approach allows to treat the statistical distributions already known in the literature in a unifying scheme. The present approach allows to treat the statistical distributions already known in the literature in a unifying scheme. As a working example we consider the kinetics, constructed by using the $\kappa$-exponential $\exp_{\kappa}(x) = (\sqrt{1 + \kappa^2 x^2 + \kappa x})^{1/\kappa}$ recently proposed which reduces to the standard exponential as the deformation parameter $\kappa$ approaches to zero and presents the relevant power law asymptotic behaviour $\exp_{\kappa}(x) \sim 2\kappa x^{1/|\kappa|}$. The $\kappa$-kinetics obeys the H-theorem and in the case of Brownian particles, admits as stationary state the distribution $f = Z^{-1} \exp_{\kappa}[-(\beta m v^2 / 2 - \mu)]$ which can be obtained also by maximizing the entropy $S_{\kappa} = \int d^3v \left( c(\kappa) f^{1+\kappa} + c(-\kappa) f^{1-\kappa} \right)$ with $c(\kappa) = -Z^n / [2\kappa(1+\kappa)]$ after properly constrained.

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I. INTRODUCTION

In the last few decades there has been an intensive discussion on non conventional classical or quantum statistics. For instance, there are several experimental evidences of distributions with tails which exhibit a power law decay. Up to now several entropies with the ensuing statistics have been considered. A question which arises spontaneously is if it is possible to obtain the stationary statistical distribution of the various physical systems within the framework of a time dependent scheme. The problem of the non linear kinetics from a more general point of view in the Fokker-Planck picture has been considered only in 1994. In ref.s [1] an evolution equation has been proposed, describing a generic non linear kinetics. Subsequently some properties of this kinetics have been studied in ref.s [2-4].

After 1995, in the Fokker-Planck picture, the anomalous diffusion has been linked with the time dependent Tsallis statistical distribution [1]. Finally, the kinetics described by non linear Fokker-Planck equations has been reconsidered in ref.s [2-4].

Recently in the paper [14] it has been proposed the following new, one-parameter deformation of the exponential and logarithm functions:

$$\exp_{\kappa}(x) = \left( \sqrt{1 + \kappa^2 x^2 + \kappa x} \right)^{1/\kappa}, \quad (1)$$

$$\ln_{\kappa}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}. \quad (2)$$

These $\kappa$-deformed functions reduce to the standard exponential and logarithm as the real deformation parameter $\kappa$ approaches to zero and present a power law asymptotic behaviour: $\exp_{\kappa}(x) \sim 2\kappa x^{1/|\kappa|}$ and $\ln_{\kappa}(x) \sim -2\kappa |x|^{-1}|\kappa|$. We remark that the $\kappa$-exponential is the only function besides the standard exponential, satisfying the condition $\exp_{\kappa}(-x) \exp_{\kappa}(x) = 1$ (both functions decrease for $x \to -\infty$ and increase for $x \to +\infty$ with the same rapidness) and obeying the scale law $[\exp_{\kappa}(x)]^\lambda = \exp_{\kappa'}(\lambda x)$. It is easy to verify that the standard exponential can be obtained by posing $\kappa' = \kappa$ while the $\kappa$-exponential given by Eq. (4) is found after setting $\kappa' = \kappa / \lambda$.

In the same paper [14] a $\kappa$-kinetics has been constructed which obeys the H-theorem and admits as stationary state the distribution $f = Z^{-1} \exp_{\kappa'}[-(\beta m v^2 / 2 - \mu)]$. This distribution can be obtained also by maximizing, after properly constrained, the entropy

$$S_{\kappa} = \int d^3v \left[ c(\kappa) f^{1+\kappa} + c(-\kappa) f^{1-\kappa} \right], \quad (3)$$

being $c(\kappa) = -Z^n / [2\kappa(1+\kappa)]$, which reduces to the standard entropy as the deformation parameter approaches to zero.

In the present paper, starting from the transition probability which defines the most general non linear kinetics in the Fokker-Planck picture, we obtain the associated generalized entropy. Subsequently, after obtaining the statistical distribution both as stationary solution of the non linear evolution equation and as the state which
maximizes the generalized entropy, we demonstrate the H-theorem. Finally, we reconsider some distributions already known in the literature in the frame of the approach here proposed and in particular we consider more extensively the $\kappa$-kinetics.

II. TRANSITION PROBABILITIES FOR THE NON LINEAR KINETICS

Let us consider a particle system interacting with its environment which we consider as a bath. We denote with $f = f(t, \mathbf{v})$ and $f' = f(t, \mathbf{v}')$ the particle densities in the sites $\mathbf{v}$ and $\mathbf{v}'$ respectively, and postulate for the transition probability $\pi(t, \mathbf{v} \rightarrow \mathbf{v}')$ from the site $\mathbf{v}$ to the site $\mathbf{v}'$ the following form

$$\pi(t, \mathbf{v} \rightarrow \mathbf{v}') = W(t, \mathbf{v}, \mathbf{v}') \gamma(f, f').$$  

(4)

The above transition probability is given by a product of two factors. The first, $W(t, \mathbf{v}, \mathbf{v}')$ in (4), is the transition rate which depends on the nature of the interaction between the particle and the bath and is a function of the starting $\mathbf{v}$ and arrival $\mathbf{v}'$ sites.

The second factor $\gamma(f, f')$ in (4) is an arbitrary function of the particle populations of the starting and of arrival sites. This function must satisfy the condition $\gamma(0, f') = 0$ because, if the starting site is empty, the transition probability is equal to zero. The dependence of the function $\gamma(f, f')$ on the particle population $f'$ of the arrival site plays a very important role in the particle kinetics because can stimulate or inhibit the particle transition $\mathbf{v} \rightarrow \mathbf{v}'$ in such a way that interactions originated from collective effects can be taken into account.

The condition $\gamma(f, 0) \neq 0$ requires that in the case the arrival site is empty the transition probability must depends only on the population of the starting site. We note that for the standard linear kinetics the relation $\gamma(f, f') = f$ holds.

In this paper, we will study kinetics coming from the transition probabilities (4) when the function $\gamma$ satisfies the condition

$$\frac{\gamma(f, f')}{\gamma(f', f)} = \frac{\kappa(f)}{\kappa(f')} ,$$  

(5)

where $\kappa(f)$ is a positive real function. This condition implies that $\gamma(f, f')/\kappa(f)$ is a symmetric function. Then we can pose $\gamma(f, f') = \kappa(f)b(f)b(f')c(f, f')$ where $b(f)$ and $c(f, f') = c(f', f)$ are two real arbitrary functions. It will be convenient later on to introduce the real arbitrary function $a(f)$ by means of

$$\kappa(f) = \frac{a(f)}{b(f)} ,$$  

(6)

and write $\gamma(f, f')$ under the guise

$$\gamma(f, f') = a(f)b(f')c(f, f') .$$  

(7)

We claim at this point that $\gamma(f, f')$ given by (7) with $a(f)$ and $b(f')$ linked through (6), is the most general function obeying the condition (5). We wish to note that $\gamma(f, f')$ is given as a product of three factors. The first factor $a(f)$ is an arbitrary function of the particle population of the starting site and satisfies the condition $a(0) = 0$ because if the starting site is empty the transition probability is equal to zero. The second factor $b(f')$ is an arbitrary function of the arrival site particle population. For this function we have the condition $b(0) = 1$ which requires that the transition probability does not depend on the arrival site if, in it, particles are absent. The expression of the function $b(f')$ plays a very important role in the particle kinetics, because it stimulates or inhibits the transition $\mathbf{v} \rightarrow \mathbf{v}'$, allowing in such a way to consider the interactions originated from collective effects. Finally, the third factor $c(f, f')$ takes into account that the populations of the two sites, namely $f$ and $f'$, can eventually influence the transition, collectively and symmetrically.

The function $\gamma(f, f')$ given by (7) defines a special interaction which involves, separately and/or together, the two particle bunches entertained in the starting and arrival sites. We recall that the interaction, depending on the relative distance of the two involved sites, is taken into account in the transition probability by the function $W$ defined in (4). The special interaction defined through $\gamma(f, f')$ given by (7) can be viewed as derived from a principle, the *Kinetical Interaction Principle* (KIP). As we will see in the following sections, the KIP governs the system evolving toward the equilibrium and imposes the stationary state of the system. Particular expressions of this principle are for instance the Pauli exclusion principle [7], the generalized exclusion-inclusion principle [8], the Haldane generalized exclusion principle [9], the principle underlying the nonextensive statistics [10].

III. FOKKER-PLANCK GENERALIZED KINETICS

We consider a classical stochastic marcoffian process in a $n-$dimensional velocity space (of course the same process can be considered in the physical space). It is described by the distribution function $f = f(t, \mathbf{v})$ which obeys the following evolution equation:

$$\frac{\partial f(t, \mathbf{v})}{\partial t} = \int_{\mathbb{R}^{n}} \left[ \pi(t, \mathbf{v}' \rightarrow \mathbf{v}) - \pi(t, \mathbf{v} \rightarrow \mathbf{v}') \right] d\mathbf{v}'$$  

(8)

where the transition probability according to the KIP is given by $\pi(t, \mathbf{v} \rightarrow \mathbf{v}') = W(t, \mathbf{v}, \mathbf{v}') \gamma(f, f')$. Let us write the transition rate as $W(t, \mathbf{v}, \mathbf{v}') = W(t, \mathbf{v}, \mathbf{v}' - \mathbf{v})$, where the second argument in $W$ represents the change of the vector state during the transition. For physical systems evolving very slowly, $w(t, \mathbf{v}, \mathbf{v}' - \mathbf{v})$ decreases very
expeditiously as \( v' - v \) increases. Then we can consider the Kramers-Moyal expansion of Eq. (8):

\[
\frac{\partial f(t, v)}{\partial t} = \sum_{m=1}^{\infty} \left[ \frac{\partial^m}{\partial u_{a_1} \partial u_{a_2} \cdots \partial u_{a_m}} \{ \zeta_{a_1 \cdots a_m}(t, u) \gamma[f(t, u), f(t, v)] \} + \right. \\
\left. ( -1)^{m-1} \zeta_{a_1 \cdots a_m}(t, v) \frac{\partial^m \gamma[f(t, v), f(t, u)]}{\partial u_{a_1} \partial u_{a_2} \cdots \partial u_{a_m}} \right] \mathbf{u} = \mathbf{v} 
\]

where the \( m \)-th order momentum \( \zeta_{a_1 \cdots a_m}(t, \mathbf{v}) \) of the transition rate is defined as:

\[
\zeta_{a_1 \cdots a_m}(t, \mathbf{v}) = \frac{1}{m!} \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} y_{a_1} y_{a_2} \cdots y_{a_m} w(t, \mathbf{v}, \mathbf{y}) d^n \mathbf{y}. 
\]

We remark that from Eq. (9) we can obtain as a particular case Eq. (7) of ref. [1]. In the first neighbor approximation only the first order (drift coefficient) \( \zeta_i(t, \mathbf{v}) \) and the second order (diffusion coefficient) \( \zeta_{ij}(t, \mathbf{v}) \) momenta of the transition rate are considered so that Eq. (4) reduces to the following non linear second order partial differential equation

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ \left( \zeta_i + \frac{\partial \zeta_{ij}}{\partial v_j} \right) \gamma(f) \right. \\
\left. + \zeta_{ij}(f) \lambda(f) \frac{\partial f}{\partial v_j} \right] , 
\]

with \( \gamma(f) = \gamma(f, f) \) and

\[
\lambda(f) = \left[ \frac{\partial}{\partial f} \ln \frac{\gamma(f, f')}{\gamma(f', f)} \right]_{f'=f} .
\]

By taking into account the condition (9), the function \( \lambda(f) \) simplifies as

\[
\lambda(f) = \frac{\partial \ln \kappa(f)}{\partial f} ,
\]

and Eq. (11) becomes

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[ \left( \zeta_i + \frac{\partial \zeta_{ij}}{\partial v_j} \right) \gamma(f) \right. \\
\left. + \gamma(f) \frac{\partial \ln \kappa(f)}{\partial f} \zeta_{ij} \frac{\partial f}{\partial v_j} \right] .
\]

We assume the independence of motion among the \( n \) directions of the homogeneous and isotropic \( n \)-dimensional velocity space and pose \( \zeta_i = J_i \), \( \zeta_{ij} = D \delta_{ij} \), being \( \mathbf{J} = \mathbf{J}(\mathbf{v}) \) and \( D = D(\mathbf{v}) \) the drift and diffusivity coefficients respectively. Moreover we introduce the potential \( U = U(\mathbf{v}) \) by means of

\[
\beta \frac{\partial U}{\partial \mathbf{v}} = \frac{1}{D} \left( \mathbf{J} + \frac{\partial D}{\partial \mathbf{v}} \right) ,
\]

with \( \beta \) a constant. Then we can write Eq. (13) in the form

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \left( D \gamma(f) \frac{\partial}{\partial \mathbf{v}} \left[ \beta(U - \mu') + \ln \kappa(f) \right] \right) ,
\]

where \( \mu' \) is an arbitrary constant. Eq. (15) can be written also as

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \left\{ D a(f) b(f) c(f) \right. \\
\left. \times \frac{\beta(U - \mu') + \ln a(f)}{b(f)} \right\} ,
\]

with \( c(f) = c(f, f) = \lim_{v' \rightarrow v} c(f(v), f(v')) \). From this definition we have that \( c(f) \) can depend also on the derivative of \( f \). For instance we can choose \( c(f) = C[f, (\partial f/\partial \mathbf{v})^2] \) and so on. We note that Eq. (16) in the case \( c(f) = 1 \) reduces to Eq. (10) of ref. [1].

In the following part of this section we will refer to the form (15) for the evolution equation of the distribution function \( f(t, \mathbf{v}) \). First of all we observe that this equation is a non linear continuity equation, namely

\[
\frac{\partial f}{\partial t} + \frac{\partial j}{\partial \mathbf{v}} = 0 ,
\]

where the current \( j = j(\mathbf{v}, f) \) assumes the form

\[
j = D \gamma(f) \mathbf{U}(\mathbf{v}, f) ,
\]

and the field \( \mathbf{U}(\mathbf{v}, f) \) can be derived from the potential \( \Phi(\mathbf{v}, f) = -\beta(U - \mu') - \ln \kappa(f) \):

\[
\mathbf{U}(\mathbf{v}, f) = \frac{\partial \Phi(\mathbf{v}, f)}{\partial \mathbf{v}} .
\]

It is easy to verify that this potential can be written as the functional derivative \( \Phi(\mathbf{v}, f) = \delta \mathcal{K}/\delta f \), of the functional \( \mathcal{K} \) defined as

\[
\mathcal{K} = -\int_{\mathbb{R}^n} d^n \mathbf{v} \left[ \int \ln(\kappa(f)) df + \beta(U - \mu') f \right] .
\]

The current \( j \) becomes

\[
j = D \gamma(f) \frac{\partial}{\partial \mathbf{v}} \frac{\delta \mathcal{K}}{\delta f} ,
\]

and the evolution equation (15) of the distribution function \( f(t, \mathbf{v}) \) assumes the following compact form:

\[
\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{v}} \left[ D \gamma(f) \frac{\partial}{\partial \mathbf{v}} \frac{\delta \mathcal{K}}{\delta f} \right] = 0 .
\]

We observe that from the structure of the evolution equation (22) the stationary distribution \( f_\infty = f(\infty, \mathbf{v}) \) of the system can be obtained from a variational principle:
\[ \frac{\delta K}{\delta f} = 0 \Rightarrow \frac{\partial f}{\partial t} = 0 ; \ f = f_s \ , \]

and assumes the form:

\[ \ln \kappa(f_s) = -\beta(U - \mu') \ , \quad (23) \]

where the constant \( \mu' \) will be calculated taking into account the normalization condition \( \int_R d^n v f = 1 \). We consider the time evolution of the functional \( K = K(t) \) which can write in the form:

\[ K = \int_R d^n v K(f, f_s) \ , \quad (24) \]

where its density is given by

\[ K(f, f_s) = -\int df \ln \frac{\kappa(f)}{\kappa(f_s)} \ . \quad (25) \]

### IV. H-THEOREM WITHIN THE FRAMEWORK OF THE NON LINEAR KINETICS

The time derivative of \( K \) is given by

\[ \frac{dK}{dt} = \int_R d^n v \frac{\delta K}{\delta f} \frac{\partial f}{\partial t} \ , \quad (26) \]

where \( \partial f/\partial t \) can be calculated from (22):

\[ \frac{dK}{dt} = -\int_R d^n v \frac{\delta K}{\delta f} \frac{\partial}{\partial v} \left[ D\gamma(f) \frac{\partial}{\partial v} \right] \ . \]

Performing an integration by parts and assuming the appropriate boundary conditions for the current, we obtain

\[ \frac{dK}{dt} = \int_R d^n v D\gamma(f) \left( \frac{\partial}{\partial v} \frac{\delta K}{\delta f} \right)^2 \ . \quad (27) \]

We may conclude that \( K \) is an increasing function

\[ \frac{dK}{dt} \geq 0 \ . \quad (28) \]

In order to study the behaviour of \( K \) when \( t \to \infty \) we introduce the function

\[ \sigma(f) = -\int df \ln \kappa(f) \ , \quad (29) \]

so that \( \kappa(f) \) can be written as

\[ \kappa(f) = \exp \left[ -\frac{d\sigma(f)}{df} \right] \ . \quad (30) \]

Now we are able to calculate, in the limit \( t \to \infty \), the following difference

\[ K(t) - K(\infty) = \int_R d^n v \left[ \sigma(f) - \sigma(f_s) + (f - f_s) \ln \kappa(f_s) \right] \]

\[ = \int_R d^n v \left[ \sigma(f) - \sigma(f_s) - (f - f_s) \frac{d\sigma(f_s)}{df_s} \right] \]

\[ \approx \int_R d^n v \left[ \frac{1}{2} \frac{d^2 \sigma(f)}{df^2} (f - f_s)^2 \right] \ , \quad (31) \]

and assume that \( d^2 \sigma(f)/df^2 \leq 0 \). This requirement is satisfied if the function \( \kappa(f) \) obeys to the condition

\[ \frac{dk(f)}{df} \geq 0 \ . \quad (32) \]

Consequently we have:

\[ K(t) \leq K(\infty) \ . \quad (33) \]

We discuss now briefly the meaning of \( K \). Firstly, we observe that Eq.s (25) and (23) constitute the H-theorem for the non linear system governed by the evolution equation (24) and implies that \( -K \) is a Lyapunov functional. Secondly, the functional \( K \) given by (24) and (23) can be written as the sum of two terms \( K = S + S_c \). The first term is the free system entropy

\[ S = \int_R d^n v \sigma(f) \ , \quad (34) \]

while the second term, \( S_c = -\beta(E - \mu') \), represents an entropy originated from the constraints imposed by the normalization requirement, and by the mean value of the relevant energy of the system, defined as \( E = \int_R d^n v U f \). With these positions \( K \) can be viewed as the entropy of the constrained system, or alternatively, as the entropy of the universe: system + environment. We observe on the other hand that, by deriving with respect to \( t \) the equation \( K = S + S_c \) and taking into account the expression of \( S_c \), one obtains immediately the Clausius inequality:

\[ \frac{dS}{dt} \geq \beta \frac{dE}{dt} \ . \quad (35) \]

Finally, the definition (24)-(25) and the relationships (28) and (33) suggest that the quantity

\[ D(t) = K(\infty) - K(t) \ , \quad (36) \]

can be viewed as the statistical distance between the two distributions \( f_s \) and \( f \). The maximization of the constrained entropy \( K \) implies the minimization of the distance \( D(t) \). This distance \( D(t) \), in the case of the linear kinetics where \( \kappa(f) = f_s \), reduces to the well known Kullback-Leibler distance [13, 17]:

\[ D(t) = \int_R d^n v f \ln \frac{f}{f_s} \ . \quad (37) \]
In concluding this section we remark that, starting from the function $K = K(f, f_c)$ given by [25], we can consider the class of functionals defined through:

$$\Lambda = \chi_1 \left( \int_R d^m v \, \chi_2(K) \right) , \tag{38}$$

being $\chi_1$ and $\chi_2$ two arbitrary algebraic functions. After indicating with $\chi_1'$ and $\chi_2'$ their first derivatives, one obtains the following expression for the functional derivative of $\Lambda$

$$\frac{\delta \Lambda}{\delta f} = \chi_1' \left( \int_R d^m v \, \chi_2(K) \right) \chi_2'(K) \frac{\delta K}{\delta f} . \tag{39}$$

It is apparent that if we choose $\chi_1$ and $\chi_2$ with the appropriate properties, $-\Lambda$ can be a Lyapunov functional. The quantity $\Lambda$ can be chosen as the entropic functional and then can replace $K$. We recall that, for instance, the Renyi [18] and the Sharma-Mittal [19] entropies, can be obtained naturally in the present frame after choosing properly $\chi''(1)$ or fermion-like ($\chi''(1) = 1$) or boson-like ($\chi''(1) = 0$) statistics. For $\chi''(1)$ holds and we obtain the Bose-Einstein statistics. Finally for $\chi''(1)$ taken into account the Pauli exclusion principle and the stationary distribution define the Fermi-Dirac statistics. For $\chi''(1)$ an inclusion principle holds and we obtain the Bose-Einstein statistics. Finally for $\chi''(1)$ we have a intermediate quantum statistics interpolating between the Bose and Fermi ones.

The boson-like ($\eta = 1$) or fermion-like ($\eta = -1$) quon statistics [3] can be obtained easily by posing $a(f) = [f]_q$ and $b(f') = [1 + \eta f']_q$, where $[x]_q = (q^x - q^{-x})/2 \ln q$ and $q \in \mathbb{R}$. The stationary distribution is given through $[f]_q/[1 + \eta f]_q = \exp(-\epsilon)$. If we choose for simplicity $c(f) = c_q = (2 \ln q)/(q - q^{-1})$, the evolution equation becomes [3]:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left( c_q \, D \beta m v \, f_q [1 + \eta f]_q + D \frac{\partial f}{\partial v} \right) . \tag{41}$$

VI. NONEXTENSIVE STATISTICS

In this section we consider two examples of classical non linear kinetics. After considering briefly the non linear kinetics admitting as stationary distribution the Maxwell-Boltzmann one, we introduce the non extensive Tsallis kinetics within the framework of the theory developed in the sections II, III and IV.

We write Eq. (23) in the case of brownian particles as follows

$$\kappa(f_s) = \frac{1}{Z} \exp \left[ -\beta \left( \frac{1}{2} m v^2 - \mu \right) \right] , \tag{42}$$

where $Z$ is calculated from the condition $\int_R d^m v f_s = 1$. We remark that $f_s$ given by Eq. (22), can be obtained from the variational principle

$$\frac{\delta}{\delta f} \int_R d^m v \left[ - \int f \kappa(f) df - \beta \frac{1}{2} m v^2 f + (\beta \mu - \ln Z) f \right] = 0 , \tag{43}$$

and also as stationary solution of the evolution equation:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left\{ D \gamma(f) \frac{\partial}{\partial v} \left[ \ln \kappa(f) - \ln \kappa(f_s) \right] \right\} . \tag{44}$$

Maxwell-Boltzmann statistics: We start by considering the MB statistics given by $f_s = Z^{-1} \exp(-\beta m v^2/2)$. It is readily seen that the related kinetics is defined posing $a(f) = f$, $b(f) = 1$, while the symmetric function $c(f)$ remains arbitrary. Then we have $\kappa(f) = c(f)$ and $\gamma(f) = c(f)$. The evolution equation becomes

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left\{ D \gamma(f) \frac{\partial}{\partial v} \left[ \ln \left( f/f_s \right) \right] \right\} . \tag{45}$$

We observe that there are infinite ways, one for any choice of $c(f)$, to obtain the MB distribution. This statistical distribution can be obtained also after maximizing the standard additive (extensive) Boltzmann-Gibbs-Shannon entropy

$$S = - \int_R d^m v \, f \ln f , \tag{46}$$

after properly constrained (we set $k_B = 1$).

Tsallis statistics: We consider the non extensive thermodynamics introduced by Tsallis [5] which can be obtained naturally in the present frame after choosing properly $\kappa(f)$ while $\gamma(f)$ remains an arbitrary function.
A) Let us consider the kinetics defined by fixing \( \kappa(f) \) through
\[
\ln[Z\kappa(f)] = \ln_q(Zf) ,
\]
where \( \ln_q(x) = (x^{1-q} - 1)/(1-q) \). We note that \( ds(f)/df \geq 0 \) for all \( q \in R \). The kinetics of the system is described by means of the evolution equation:
\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left\{ D \gamma(f) \frac{\partial}{\partial v} \left[ \ln_q(Zf) - \ln_q(Zf_n) \right] \right\} ,
\]
which, in the case \( \gamma(f) = f \), reproduces the equation proposed in ref. [9]. The stationary solution of Eq. (48) is given by:
\[
f = \frac{1}{Z} \exp_q \left[ -\beta \left( \frac{1}{2}mv^2 - \mu \right) \right] ,
\]
with \( \exp_q(x) = [1 + (1 - q)x]^{1/(1-q)} \) and \( Z = \int_R d^n v \exp_q [-\beta (mv^2/2 - \mu)] \). This distribution can be obtained also from the variational principle given by Eq. (43):
\[
\frac{\delta}{\delta f} \int_R d^n v \left[ -\frac{1}{2-q} \frac{(Zf)^{1-q} - 1}{1-q} + \frac{1}{2-q} \right. \\
\left. -\beta \frac{1}{2}mv^2 + \beta \mu \right] f = 0 .
\]
From Eq. (50) it results apparent that the entropy is given by:
\[
S_q[f] = -\frac{1}{2-q} < \ln_q(Zf) > + \frac{1}{2-q} .
\]
We recall that in the frame of the present non linear kinetics the mean value of \( A \) is defined through
\[
<A> = \int_R d^n v Af ; \int_R d^n v f = 1.
\]
B) Taking into account that \( \exp_q(-x) \neq [\exp_q(x)]^{-1} \), we consider the statistical distribution
\[
f = \frac{1}{Z} \left\{ \exp_q \left[ \beta \left( \frac{1}{2}mv^2 - \mu \right) \right] \right\}^{-1} ,
\]
which can be viewed as the steady state of the kinetics defined by fixing \( \kappa(f) \) through \( \ln[Z\kappa(f)] = -\ln_q(1/Zf) \). Also in this case we have \( d\kappa(f)/df \geq 0 \) for all \( q \in R \). The relation \( [\exp_q(x)]^{-1} = \exp_{2-q}(-x) \) imposes that (53) can be obtained by maximizing the entropy
\[
S_{2-q}[f] = -\frac{1}{q} < \ln_{2-q}(Zf) > + \frac{1}{q} ,
\]
after properly constrained.
C) The relation \( \exp_q(x)q = \exp_{2-1/q}(qx) \) suggests to write the statistical distribution under the form:
\[
f = p^q ; \quad p = \frac{1}{Z} \exp_q \left[ -\beta \left( \frac{1}{2}mv^2 - \mu \right) \right] .
\]
This distribution can be viewed as the steady state of the kinetics defined by imposing \( \ln[Z\kappa(f)] = \ln_q(Zf^{1/q}) \). We remark that \( d\kappa(f)/df \geq 0 \) for \( q > 0 \). In the following we consider \( q > 0 \) in order to keep the H-theorem. Of course the normalization condition \( \int_R d^n v f = 1 \) implies that:
\[
\int_R d^n v p^q = 1 ; \quad \int_R d^n v p \neq 1 ,
\]
and
\[
Z^q = \int_R d^n v \left\{ \exp_q \left[ -\beta \left( \frac{mv^2}{2} - \mu \right) \right] \right\}^q .
\]
The mean value of \( A \) can be calculated as:
\[
<A> = \int_R d^n v Af = \frac{\int_R d^n v Af^q}{\int_R d^n v p^q} = \int_R d^n v Ap^q .
\]
The variational principle expressed by Eq. (43) in the case of the distribution function \( p = f^{1/q} \) can be written as
\[
\frac{\delta}{\delta p} \int_R d^n v \left[ \frac{1-q(Zp)^{-q}}{1-q} \right. \\
\left. -\beta \frac{1}{2}mv^2 + \beta \mu \right] p^q = 0 ,
\]
and the entropy consequently is defined through:
\[
S_q[p] = -q < \ln_q(Zp) > + 1 .
\]
We conclude by remarking that here we have obtained Tsallis statistics, within the framework of a general non linear kinetics, in a new version, different from the ones already known in literature. We note that the present version in terms of escort probabilities (49)-(50) is consistent with the proposal of ref. [20] where the validity of Eq.s (49) has been postulated to solve some open problems in Tsallis statistics.

VII. THE \( \kappa \)-STATISTICS

Let us consider a deformation of the logarithm, namely \( \ln_{(\kappa)}(f) \), obeying the condition \( \ln_{(\kappa)}(f^{-1}) = -\ln_{(\kappa)}(f) \). The most general solution of this last equation is given by \( \ln_{(\kappa)}(f) = [\lambda_{(\kappa)}(f) - \lambda_0(f^{-1})]/2 \) being \( \lambda_{(\kappa)}(f) \) a real arbitrary function. Let us consider only one parameter \( (\kappa) \) deformations and impose that \( \ln_{(\kappa)}(f) \) obeys to the condition \( \ln_{(\kappa)}(f^m) = m \ln_{(\kappa)}(f) \), being \( \kappa' = \kappa'(m,\kappa) \). It is easy to verify that two solutions exist. First trivial solution is: \( \kappa' = \kappa, \ln_{(\kappa)}(f) = \ln(f) \). Second solution is
given by: $\kappa' = m\kappa$, $\ln_{\kappa'}(f) = (f^{\kappa} - f^{-\kappa})/2\kappa$. We note that this one parameter family of solutions contains, as limiting case for $\kappa \to 0$, the trivial undeformed logarithm. The inverse function of the $\kappa$-logarithm is the $\kappa$-exponential, namely $f = \exp_{\kappa}(x)$. Very recently in ref. [4], starting from this deformed exponential function a one-parameter deformed mathematics has been constructed which shows various very interesting symmetries.

In the same reference the $\kappa-$kinetics has been proposed and studied. This deformed kinetics can be introduced naturally by posing that $\gamma(f)$ remains a arbitrary function, while $\kappa(f)$ is given by

$$\ln[Z\kappa(f)] = \ln_{\kappa}(Zf)$$

(61)

We remark that $d\kappa(f)/df \geq 0$ for $\forall \kappa \in R$ so that the H-theorem still holds. The evolution equation becomes

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left\{ D \gamma(f) \frac{\partial}{\partial v} \left[ \ln_{\kappa}(Zf) - \ln_{\kappa}(Zf_{s}) \right] \right\}$$

(62)

and its stationary distribution is given by

$$f_{s} = \frac{1}{Z} \exp_{\kappa} \left[ -\beta \left( \frac{1}{2} mv^{2} - \mu \right) \right]$$

(63)

which, after calculation of $Z$, in the case $\mu = 0$, assumes the form

$$f_{s} = \left[ \frac{\beta m |\kappa|}{\pi^{2}} \right]^{\frac{1}{2}} \left[ 1 + \frac{n}{2} |\kappa| \right]$$

$$\times \frac{\Gamma \left( \frac{1}{2 |\kappa|} + \frac{4}{2} \right)}{\Gamma \left( \frac{1}{2 |\kappa|} - \frac{4}{2} \right)} \exp_{\kappa} \left( -\frac{\beta}{2} mv^{2} \right)$$

(64)

with $|\kappa| < 2/n$ and $n = 1, 2, 3$ being the dimension of the velocity space. The asymptotic behaviour of the distribution $f_{s}$ of (63) is

$$f_{s} \sim_{v \to +\infty} \pi^{-\frac{1}{2}} \left( \frac{\beta m |\kappa|}{\pi^{2}} \right)^{\frac{1}{2} - \frac{n}{2 |\kappa|}} \left[ 1 + \frac{n}{2} |\kappa| \right]$$

$$\times \frac{\Gamma \left( \frac{1}{2 |\kappa|} + \frac{4}{2} \right)}{\Gamma \left( \frac{1}{2 |\kappa|} - \frac{4}{2} \right)} v^{-\frac{m}{2}}$$

(65)

The r-order momentum

$$<v^{r}>_{\kappa} = \frac{\int_{R} d^{n}v v^{r} f}{\int_{R} d^{n}v f}$$

(66)

of the distribution function (64) is finite when $|\kappa| < 2/(n + r)$ and is given by

$$<v^{r}>_{\kappa} = \left( \frac{m \beta |\kappa|}{\pi} \right)^{-\frac{1}{2}} \left[ 1 + \frac{n}{2} \frac{|\kappa|}{2 |\kappa|} \right] \frac{\Gamma \left( \frac{n+1}{2 |\kappa|} \right)}{\Gamma \left( \frac{n}{2 |\kappa|} \right)}$$

$$\times \frac{\Gamma \left( \frac{1}{2 |\kappa|} + \frac{4}{2} \right)}{\Gamma \left( \frac{1}{2 |\kappa|} - \frac{4}{2} \right)} \frac{\Gamma \left( \frac{1}{2 |\kappa|} - \frac{4}{2} \right)}{\Gamma \left( \frac{1}{2 |\kappa|} + \frac{4}{2} \right)}$$

(67)

It is easy to verify that the distribution (64) can be obtained after maximization under the appropriate constraints of the entropy

$$S_{\kappa} = - \frac{1}{2\kappa} \int_{R} d^{n}v \left( \frac{Z_{\kappa}}{1 + \kappa} f^{\kappa} - \frac{Z_{\kappa}}{1 - \kappa} f^{-\kappa} \right)$$

(68)

which reduced to the standard Shannon entropy $S_{0} = - \int_{R} d^{n}v f \ln Zf$ as the the deformation parameter $\kappa \to 0$. We note that the entropy $S_{\kappa}$ and the entropy $S = - \int_{R} d^{n}v f \ln Zf$ are connected through $S_{\kappa} = S - \ln Z$, being $\int d^{n}v f = 1$. The variational equation (63) becomes:

$$\frac{\delta}{\delta f} \left[ S_{\kappa} + \int_{R} d^{n}v \left( - \frac{\beta}{2} mv^{2} + \beta \mu \right) f \right] = 0$$

(69)

Let us define, for real positive functions, the following $\kappa$-product:

$$f \otimes_{\kappa} g = \exp_{\kappa} \left( \ln_{\kappa}(f) + \ln_{\kappa}(g) \right)$$

(70)

which reduces to the ordinary product as $\kappa \to 0$, namely $f \otimes_{0} g = f \cdot g$. The above $\kappa$-product has the same properties of the ordinary one: i) associative law: $(f \otimes_{\kappa} g) \otimes_{\kappa} h = f \otimes_{\kappa} (g \otimes_{\kappa} h)$; ii) neutral element: $f \otimes_{\kappa} 1 = 1 \otimes_{\kappa} f = f$; iii) inverse element: $f \otimes_{\kappa} (1/f) = (1/f) \otimes_{\kappa} f^{\kappa} = 1$; iv) commutative law : $f \otimes_{\kappa} g = g \otimes_{\kappa} f$. Of course, the $\kappa$-division can be defined as $f \otimes_{\kappa} g = f \otimes_{\kappa} (1/g)$. Finally, $f \otimes_{\kappa} 0 = 0 \otimes_{\kappa} f = 0$. We remark the following interesting property of the $\kappa$-exponential:

$$\exp_{\kappa}(x) \otimes_{\kappa} \exp_{\kappa}(y) = \exp_{\kappa}(x + y)$$

(71)

Eq. (71) in the case $y = -x$ reduces to the relationship $\exp_{\kappa}(x) \exp_{\kappa}(-x) = 1$, which is the starting point of ref. [14] for the construction of the $\kappa$-exponential.

We conclude the discussion by noting that the evolution equation (64) can be written in the form

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} \left\{ D \gamma(f) \frac{\partial}{\partial v} \left[ \ln_{\kappa}(Zf) \otimes_{\kappa} (Zf_{s}) \right] \right\}$$

(72)

and appears structurally similar to Eq. (15).

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