Efficient Cross-Device Federated Learning Algorithms for Minimax Problems

Jiahao Xie,1 Chao Zhang,1 Zebang Shen,2 Weijie Liu,1 Hui Qian1

1College of Computer Science and Technology, Zhejiang University
2University of Pennsylvania

xiejh@zju.edu.cn, zczju@zju.edu.cn, zebang@seas.upenn.edu, westonhunter@zju.edu.cn, qianhui@zju.edu.cn

Abstract
In many machine learning applications where massive and privacy-sensitive data are generated on numerous mobile or IoT devices, collecting data in a centralized location may be prohibitive. Thus, it is increasingly attractive to estimate parameters over mobile or IoT devices while keeping data localized. Such learning setting is known as cross-device federated learning. In this paper, we propose the first theoretically guaranteed algorithms for general minimax problems in the cross-device federated learning setting. Our algorithms require only a fraction of devices in each round of training, which overcomes the difficulty introduced by the low availability of devices. The communication overhead is further reduced by performing multiple local update steps on clients before communication with the server, and global gradient estimates are leveraged to correct the bias in local update directions introduced by data heterogeneity. By developing analyses based on novel potential functions, we establish theoretical convergence guarantees for our algorithms. Experimental results on AUC maximization, robust adversarial network training, and GAN training tasks demonstrate the efficiency of our algorithms.

1 Introduction
During the last few years, minimax problems have found a surge of important applications in machine learning. Typical examples include AUC (area under the ROC curve) maximization (Ying, Wen, and Lyu 2016; Liu et al. 2020a), robust adversarial learning (Namkoong and Duchi 2016; Madry et al. 2018), and Generative Adversarial Network (GAN) training (Goodfellow et al. 2014). These applications have sparked a lot of interests in developing efficient algorithms for minimax problems (Rafique et al. 2018; Jin, Netrapalli, and Jordan 2020; Thekumparampil et al. 2019; Nouiehed et al. 2019; Liu et al. 2020b; Lin, Jin, and Jordan 2020; Luo et al. 2020; Qiu et al. 2020; Wang et al. 2020; Xu et al. 2020; Huang et al. 2020; Yang, Kiyavash, and He 2020).

While mainstream minimax algorithms are designed for the single-machine setting, there is a growing trend of minimax optimization in the Federated Learning (FL) paradigm (Mohri, Sivek, and Suresh 2019; Deng, Kamani, and Mahdavi 2020; Reisizadeh et al. 2020; Rasouli, Sun, and Rajagopal 2020; Beznosikov, Samokhin, and Gasnikov 2020; Hou et al. 2021; Guo et al. 2020; Yuan et al. 2021; Deng and Mahdavi 2021). In such regime, multiple clients collaborate to train a machine learning model under the coordination of a central server without transferring any client’s private data (McMahan et al. 2017). Existing federated algorithms for general minimax problems require all clients to participate in each training round. Thus, they are restricted to the cross-silo setting of FL and do not apply to another important setting—the cross-device setting. The former involves a relatively small number of organizations or data centers with reliable network connections (Kairouz et al. 2019). By contrast, in the latter setting, the clients are numerous (up to 1010) unreliable mobile/IoT devices with relatively slow network connections, and only a fraction of devices are eligible for training at any one time (Kairouz et al. 2019, Table 1).

As the volume of data generated by mobile and IoT devices grows rapidly and privacy concerns increase, gathering data in an aggregation center may be prohibitive (McMahan et al. 2017). Therefore, the cross-device FL becomes an increasingly popular distributed computing paradigm (Li et al. 2018). In order to develop an efficient algorithm for minimax problems in the cross-device setting, the following three difficulties need to be overcome.

(i) **Low availability of clients.** Only a portion of devices are available at any one time. For example, a device is temporarily unavailable when it is offline or out of power. This precludes global synchronization (i.e., synchronization over all clients), which is required by mainstream distributed minimax algorithms.

(ii) **The communication bottleneck.** The network connection between the server and a client may be slow due to the limit of bandwidth. Thus, frequently transferring large amount of information over the network may significantly deteriorate the training efficiency.

(iii) **Data heterogeneity.** In general, clients’ data vary significantly and thus a client’s local data distribution may be substantially different from the global distribution. As global synchronization is prohibitive, the data heterogeneity makes it more challenging to train a global model.

**Contribution.** (i) In this paper, we develop two theoretically guaranteed cross-device minimax algorithms, CD-MAGE and CD-MAGE+, to address the above difficulties. In each round of our algorithms, only a small subset of clients are required for training, which provides robustness against the limited availability of clients. To reduce the communication overhead, our algorithms perform multiple local stochas-
tic gradient descent-ascent steps on clients before communication to the server in each round. Two global gradient estimators, the minibatch and recursive momentum based estimators, are adopted for CD-MAGE and CD-MAGE+ separately, to mitigate the bias of local updates introduced by data heterogeneity. (ii) We develop analyses of the proposed algorithms based on novel potential functions, which show that our algorithms converge to a stationary point if the loss function is non-convex w.r.t. to the primal variable and satisfies the Polyak-Łojasiewicz (PL) condition on the dual variable. Specifically, CD-MAGE (resp., CD-MAGE+) achieves an $O(1/T + 1/\sqrt{T})$ (resp., $O(1/T + 1/T^{2/3}S^{1/3})$) convergence rate, where $T$ and $S$ denote the number of communication rounds and the number of clients participating per round, respectively. Note that these rates are the first convergence rates for cross-device federated algorithms on general minimax problems.

Experimental results on AUC maximization, robust adversarial neural network training, and GAN training tasks demonstrate the efficiency of the proposed algorithms.

**Notation.** We use bold lowercase symbols (e.g., $\mathbf{x}$) to denote vectors. For a vector $\mathbf{x}$, we denote its $j$-th coordinate as $[\mathbf{x}]_j$. For a function $f(\cdot, \cdot) : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$, $\nabla_\mathbf{x} f$ and $\nabla_\mathbf{y} f$ denote its partial derivatives with respect to the first and second variables, respectively. We also denote $(\nabla_\mathbf{x} f, \nabla_\mathbf{y} f)$ as $\nabla f$. The Euclidean norm of a vector $\mathbf{x}$ is denoted by $\|\mathbf{x}\|$.

## 2 Related Work

**Minimax Optimization.** Minimax optimization has a long history dating back to (Brown 1951). The vast majority of existing works on minimax problems focus on the convex-concave regime (Korpelevich 1976; Nemirovski 2004; Nedić and Ozdaglar 2009; Palaniappan and Bach 2016; Mokhtari, Ozdaglar, and Pattathil 2019; Chavdarova et al. 2019; Lin, Jin, and Jordan 2020). Recently, there has emerged a surge of studies of more general nonconvex-concave and nonconvex-nonconcave minimax problems (Sanjabi, Razaviyayn, and Lee 2018; Nouiehed et al. 2019; Lin, Jin, and Jordan 2020; Luo et al. 2020; Xu et al. 2020; Liu et al. 2020b; Huang et al. 2020; Qiu et al. 2020; Yang, Kiyavash, and He 2020).

There are generally two types of algorithms for minimax problems in nonconvex-concave and nonconvex-nonconcave regimes: (i) GDmax-type algorithms (Sanjabi, Razaviyayn, and Lee 2018; Nouiehed et al. 2019; Luo et al. 2020; Xu et al. 2020), and (ii) GDA-type algorithms (Lin, Jin, and Jordan 2020; Liu et al. 2020b; Huang et al. 2020; Qiu et al. 2020; Yang, Kiyavash, and He 2020). The former alternate between performing a single update on the primal variable $\mathbf{x}$ and approximately solving a maximization subproblem w.r.t. to the dual variable $\mathbf{y}$. By contrast, the latter algorithms only perform both a gradient descent step on $\mathbf{x}$ and a gradient ascent step on $\mathbf{y}$ per iteration. Hence, the GDA-type methods are usually more computationally practical for real-world problems with complex structures (Qiu et al. 2020).

**Federated learning.** The principal cross-device federated learning algorithm for minimization problems is FedAvg (McMahan et al. 2017). While FedAvg has succeeded in a few applications, it suffers from the client drift issue (i.e., on heterogeneous data, the local variables move towards the individual client optima instead of a global optimum).

Recently, several minimax algorithms for cross-silo federated learning have been proposed (Reisizadeh et al. 2020; Beznosikov, Samokhin, and Gasnikov 2020; Hou et al. 2021; Guo et al. 2020; Yuan et al. 2021; Deng and Mahdavi 2021). However, these algorithms require all clients to participate in each round of training, and thus do not apply to the cross-device setting. Though these algorithms can be forcibly extended to the cross-device setting by involving only a subset of clients in each round, there are no theoretical guarantees for their cross-device versions.

Actually there are two algorithms for tackling a specific minimax problem in the cross-device setting (Mohri, Sivek, and Suresh 2019; Deng, Kamani, and Mahdavi 2020). Indeed, they are both specialized for a distributionally robust problem of the form $\min_\mathbf{x} \max_{\mathbf{y} \in \mathbb{R}^q} \|f_1(\mathbf{x}, \mathbf{y})\|_1 + \sum_{i=1}^N |\mathbf{y}_i|$, where $N$ denotes the total number of clients and $f_i$ is the local loss function of client $i$. As these two algorithms heavily depend on the special problem structure, they do not apply to many other popular minimax problems, e.g., AUC maximization, robust adversarial learning, and generative adversarial network training.

## 3 Problem Setting and Algorithms

We consider the following general minimax optimization problem in the cross-device federated learning setting:

$$\min_{\mathbf{x} \in \mathbb{R}^p} \max_{\mathbf{y} \in \mathbb{R}^q} \{ f(\mathbf{x}, \mathbf{y}) := \mathbb{E}_{\mathbf{t} \sim \mathcal{D}}[f_t(\mathbf{x}, \mathbf{y})] \}, (1)$$

where $f_t(\mathbf{x}, \mathbf{y}) := \frac{1}{n_t} \sum_{j=1}^{n_t} f_i(\mathbf{x}, \mathbf{y}; \xi_{i,j})$ is the local loss function of client $i$, $\{\xi_{i,1}, \ldots, \xi_{i,n_t}\}$ denotes the local data points on client $i$, and $\mathcal{D}$ represents the client distribution. A typical example of $\mathcal{D}$ is the uniform distribution over $N$ clients, which corresponds to the loss function $f = \frac{1}{N} \sum_{i=1}^N f_i$. Generally, $\mathcal{D}$ can be any (nonuniform) distribution over $N$ clients, even with infinite $N$. We note that we do not need access to all clients during the whole optimization procedure as the number of clients may be extremely large in the cross-device setting.

To solve problem (1), we propose two algorithms: (i) CD-MAGE, and (ii) CD-MAGE+. These two algorithms are detailed in Algorithms 1-3. In each round, CD-MAGE and CD-MAGE+ proceed in the following two phases.

(i) **Gradient collection phase.** The server sends $(\mathbf{x}_t, \mathbf{y}_t)$ (and $(\mathbf{x}_{t-1}, \mathbf{y}_{t-1})$ in CD-MAGE+) to a subset of clients, and then proceeds once it receives local gradients from $S$ clients. We denote these $S$ clients as $S'_t$. The server then computes the global gradient estimate $(\mathbf{u}_t, \mathbf{v}_t)$ by aggregating the received local gradients. CD-MAGE and CD-MAGE+ utilize different global gradient estimators. The former incorporates the minibatch gradient estimator which simply averages the local gradients of the clients in $S'_t$. This estimator is the most widely used and the simplest estimator in the optimization literature, but it
Algorithm 1: CD-MAGE/CD-MAGE+ on the server.

**Input:** initial point \((x_{-1}, y_{-1})\), the number of rounds \(T\), and the batch size of clients \(S\).
1. \(x_0 \leftarrow x_{-1}, y_0 \leftarrow y_{-1}\);
2. for \(t = 0, 1, \ldots, T - 1\) do
3. **Gradient collection phase:**
   - Send \(\{x_{i,t}, y_{i,t}\}\) or \(\{x_{i,t}, x_{i-1,t}, y_{t-1}\}\) to a subset of clients;
   - Once local gradient information from \(S\) clients is received, proceed;
   - Compute \(u_t\) and \(v_t\) by Eq. (2) or Eq. (3);
4. **Parameter update phase:**
   - Send \((u_t, v_t)\) and \((x_{i,t}, y_{i,t})\) to a subset of clients;
5. **end for**

Algorithm 2: CD-MAGE/CD-MAGE+ on client \(i\) in the gradient collection phase of the \(t\)-th round.

**Input:** \((x_{t}, y_{t})\) and \((x_{t-1}, y_{t-1})\) for CD-MAGE+.
1. Compute \(\nabla f_i(x_t, y_t)\) or \(\nabla f_i(x_{t-1}, y_{t-1})\) \(\triangleright\) CD-MAGE+ and send it to the server;

usually has a large variance (Johnson and Zhang 2013). It is constructed as
\[
\begin{align*}
    u_t &= \frac{1}{|S_t|} \sum_{i \in S_t} \nabla x f_i(x_t, y_t), \\
    v_t &= \frac{1}{|S_t|} \sum_{i \in S_t} \nabla y f_i(x_t, y_t),
\end{align*}
\]

Inspired by (Cutkosky and Orabona 2019), the CD-MAGE+ algorithm further utilizes a lightweight estimator that aggregates historical gradients to reduce variance. It is constructed in the following recursive manner
\[
\begin{align*}
    u_t &= (1 - \alpha_t) u_{t-1} + \frac{1}{|S_t|} \sum_{i \in S_t} \nabla x f_i(x_t, y_t), \\
    v_t &= (1 - \alpha_t) v_{t-1} + \frac{1}{|S_t|} \sum_{i \in S_t} \nabla y f_i(x_t, y_t),
\end{align*}
\]
where \(\alpha_t\) is a real number in \([0, 1]\), and \(u_0\) and \(v_0\) are initialized as \(\frac{1}{|S_0|} \sum_{i \in S_0} \nabla x f_i(x_0, y_0)\) and \(\frac{1}{|S_0|} \sum_{i \in S_0} \nabla y f_i(x_0, y_0)\), respectively.

(ii) **Parameter update phase.** The server sends \((u_t, v_t)\) and \((x_{i,t}, y_{i,t})\) to a subset of clients that is potentially different from the subset in the previous phase. Each client in this subset performs \(K\) local steps and sends its last iterate to the server. Upon receiving the local iterates from \(S\) clients, the server updates the global parameter by averaging the received local iterates. We denote these \(S\) clients as \(S_t\).

Note that, to ensure at least \(S\) clients successfully respond to the server in the two phases of each round \(t\), the server can

Algorithm 3: CD-MAGE/CD-MAGE+ on client \(i\) in the parameter update phase of the \(t\)-th round.

**Input:** the number of local steps \(K\), step sizes \(\eta_t\) and \(\gamma_t\), \((x_{i,t}, y_{i,t})\), and \((u_t, v_t)\).
1. Initialize local model \(x_{i,0}^{(0)} \leftarrow x_{i,t}, y_{i,0}^{(0)} \leftarrow y_{i,t}\);
2. for \(k = 0, 1, \ldots, K - 1\) do
3. Sample a minibatch \(B_{i,t}^{(k)}\) from local data;
4. Compute \(d_{x,i,t}^{(k)}\) and \(d_{y,i,t}^{(k)}\) by (4);
5. \(x_{i,t}^{(k+1)} \leftarrow x_{i,t}^{(k)} - \eta_t d_{x,i,t}^{(k)}\); \(y_{i,t}^{(k+1)} \leftarrow y_{i,t}^{(k)} + \gamma_t d_{y,i,t}^{(k)}\);
7. end for
8. Send \((x_{i,t}^{(K)}, y_{i,t}^{(K)})\) to the server;

simply communicate to a large enough subset of clients and proceed once it receives information from \(S\) clients.

In each local iteration (line 3 to 6 in Algorithm 3) of our algorithms, we simultaneously take a descent step on the local primal variable and an ascent step on the dual one. Due to the data heterogeneity, simply using local gradients as local update directions may bias the local iterates towards individual local optima. To tackle this issue, we leverage global gradient estimates to correct local update directions, similar to the techniques in (Johnson and Zhang 2013; Karimireddy et al. 2020a). Specifically, the local primal and dual update directions are computed as
\[
\begin{align*}
    d_{x,i,t}^{(k)} &= \nabla_x F_i(x_{i,t}^{(k)}, y_{i,t}^{(K)}; B_{i,t}^{(k)}) - \nabla_x F_i(x_{i,t}^{(k)}, y_{i,t}; B_{i,t}^{(k)}) + u_t, \\
    d_{y,i,t}^{(k)} &= \nabla_y F_i(x_{i,t}^{(k)}, y_{i,t}^{(K)}; B_{i,t}^{(k)}) - \nabla_y F_i(x_{i,t}^{(k)}, y_{i,t}; B_{i,t}^{(k)}) + v_t.
\end{align*}
\]

where \(F_i(x, y; B)\) is the average of local losses over the minibatch \(B_{i,t}^{(k)}\) of data samples, i.e.,
\[
F_i(x, y; B) := |B|^{-1} \sum_{z \in B} F_i(x, y; z).
\]

The update strategy (4) drives the local update directions to be close to global gradients when \((x_{i,t}^{(k)}, y_{i,t})\) is close to \((x_{i,t}, y_{i,t})\) and \(F_i\) is smooth. This mitigates the bias of local updates and thus provides robustness to the data heterogeneity. Besides, since only a subset of clients are involved in each round and multiple local steps are performed before communication, our algorithms are resilient to the limited availability of clients and have relatively low communication overhead.

If we simply use the local stochastic gradient \(\nabla_x F_i(x_{i,t}^{(k)}, y_{i,t}^{(K)}; B_{i,t}^{(k)})\) (resp., \(\nabla_x F_i(x_{i,t}^{(k)}, y_{i,t}^{(K)}; B_{i,t}^{(k)})\)) as the local primal (resp., dual) update direction, we obtain a simpler algorithm. We refer to such algorithm as Cross-device Minimim Averaging (CD-MA), which is detailed in Algorithm 4 in Appendix A. CD-MA can be viewed as a cross-device variant of the Local SGDA algorithm (Deng and Mahdavi 2021), where the latter only applies to the cross-silo setting as it requires the participation of all clients in each round. As we shall see in Section 4, the convergence rates of CD-MAGE and CD-MAGE+ are superior to CD-MA, which demonstrates the advantage of utilizing global gradient estimates in local updates.
4 Convergence Analysis

In this section, we establish convergence guarantees for the proposed algorithms. We note that the key of our analyses is to bound the following three terms: the gradient estimation error, the bias of local updates, and the gap between $f(x_i, y_i)$ and $\max_y f(x_i, y)$. However, these terms twist together as the algorithm proceeds and thus are difficult to control, which makes our convergence analyses particularly challenging. To address this, we construct delicate potential functions that properly handle these terms. Note that all missing proofs are deferred to the Appendix due to the limit of space. We also make the following common assumptions throughout.

Assumption 1 (Bounded gradient dissimilarity). There exist two positive constants $\sigma_1 > 0$ and $\sigma_2 > 0$ such that $\forall x \in \mathbb{R}^p$, $\forall y \in \mathbb{R}^q$,
\[
\mathbb{E}_{i \sim D}[\|\nabla_x f_i(x, y) - \nabla_x f(x, y)\|^2] \leq \sigma_1^2,
\]
\[
\mathbb{E}_{i \sim D}[\|\nabla_y f_i(x, y) - \nabla_y f(x, y)\|^2] \leq \sigma_2^2.
\]

Assumption 2 (Lipschitz continuous gradient). There exist four positive constants $L_1, L_2, L_{12}, L_{21} > 0$ such that for any $\zeta \in \{\zeta_i, \ldots, \zeta_{i,n}\}$, $x_1, x_2 \in \mathbb{R}^p$ and $y_1, y_2 \in \mathbb{R}^q$, the following inequalities hold
\[
\|\nabla_x F_i(x_1, y_1; \zeta) - \nabla_x F_i(x_2, y_1; \zeta)\| \leq L_1 \|x_1 - x_2\|,
\]
\[
\|\nabla_y F_i(x_1, y_1; \zeta) - \nabla_y F_i(x_2, y_1; \zeta)\| \leq L_{12} \|y_1 - y_2\|,
\]
\[
\|\nabla_y F_i(x_1, y_1; \zeta) - \nabla_y F_i(x_2, y_2; \zeta)\| \leq L_{21} \|y_1 - y_2\|.
\]

Assumption 3 (Polyak-Lojasiewicz (PL) condition). There exists a constant $\mu > 0$ such that $\forall x \in \mathbb{R}^p, y \in \mathbb{R}^q$,
\[
\|\nabla_y f(x, y)\|^2 \geq 2\mu \max_{y \in \mathbb{R}^q} f(x, y') - f(x, y).
\]

The PL condition is a well-studied and common condition in the optimization literature and many practical problems satisfy this assumption. Actually, it holds for any strongly concave functions (e.g., the minimax formulation of the AUC maximization problem (Liu et al. 2020a)), and many non-convex-concave functions arising in over-parameterized deep networks, deep networks with linear activations, one-hidden-layer networks with Leaky ReLU activation (Du et al. 2019; Allen-Zhu, Li, and Song 2019; Charles and Papailiopoulos 2018), to name a few.

Besides, we also assume that each client in $S'_i$ or $S_i$ is randomly drawn from the underlying distribution $D$, which is a common assumption in the federated learning literature (Li et al. 2018, 2020; Karimireddy et al. 2020a). For ease of notation, we denote $x := (x, y)$ for any $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. The notation $y^*(x)$ denotes the projection of $y$ onto the solution set of $\max_y f(x, y)$. We also denote $\Phi(x) := \max_{y \in \mathbb{R}^q} f(x, y)$. As stated in Section 3, for simplicity we assume $|S'_i| = |S_i| = |S|$ in our algorithms.

Under Assumptions 2 and 3, both $f$ and $\Phi$ have Lipschitz continuous gradients, as shown in the two lemmas below.

Lemma 1. Under Assumption 2, $\forall \zeta \in \{\zeta_i, \ldots, \zeta_{i,n}\}$, $F_i(x; y; \zeta)$ is $L_f$-smooth with respect to $(x, y)$, i.e., $\forall x_1, x_2 \in \mathbb{R}^p$ and $y_1, y_2 \in \mathbb{R}^q$,
\[
\|\nabla_x F_i(x, y_1; \zeta) - \nabla_x F_i(x, y_2; \zeta)\| \leq L_f \|x_1 - x_2\|,
\]
\[
\|\nabla_y F_i(x_1, y; \zeta) - \nabla_y F_i(x_2, y; \zeta)\| \leq L_f \|y_1 - y_2\|,
\]
where $L_f := \max\{\sqrt{2(L_1^2 + L_{12}^2)}, \sqrt{2(L_2^2 + L_{21}^2)}\}$. Furthermore, both $f_i$ and $f$ are also $L_f$-smooth.

Lemma 2. (Nouiehed et al. 2019, Lemma 22) Under Assumptions 2 and 3, the function $\Phi(x)$ is differentiable, and its gradient is given by
\[
\nabla \Phi(x) = \nabla_x f(x, y^*),
\]
where $y^* \in \arg\max_{y \in \mathbb{R}^q} f(x, y)$. Moreover, $\Phi$ has $L_f$-Lipschitz continuous gradients with $L_f := L_1 + \frac{L_{12}L_{12}}{2\mu}$.

Having established that $\Phi(\cdot)$ is a smooth function, a natural metric for measuring the performance of an algorithm on problem (1) is the squared gradient norm of $\Phi$, i.e., $\|\nabla \Phi(x)\|^2$. This metric is commonly used in analyzing algorithms for non-convex-PL or non-convex-strongly-concave min-max problems (Reisizadeh et al. 2020; Deng and Mahdavi 2021; Lin, Jin, and Jordan 2020; Luo et al. 2020). Note that $\Phi(x)$ only measures the performance of $x$. Once an approximate primal solution $x_T$ is obtained, one can efficiently approximate $\max_y f(x_T, y)$. Such approach is common in non-convex minimax optimization.

4.1 Analysis of CD-MAGE

To establish the convergence of CD-MAGE, we first provide several key lemmas. The lemma below reveals the relationship between two consecutive global iterates in CD-MAGE.

Lemma 3. The global iterates of CD-MAGE satisfy
\[
\begin{align*}
x_{t+1} &= x_t - \eta_t K(u_t + e_{x,t}), \\
y_{t+1} &= y_t + \gamma_t K(v_t + e_{y,t}),
\end{align*}
\]
where $e_{x,t}$ and $e_{y,t}$ are defined as $e_{x,t} := \frac{1}{\sqrt{K}} \sum_{i \in S_t} \sum_{k=0}^{K-1} (\nabla_x F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_t))$ and $e_{y,t} := \frac{1}{\sqrt{K}} \sum_{i \in S_t} \sum_{k=0}^{K-1} (\nabla_y F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_y F_i(z_t; B_t))$, respectively.

The next lemma shows that the expected loss value $\mathbb{E}[\Phi(x_t)]$ decreases in each round as long as the primal variable update error $\mathbb{E}[\|e_{x,t}\|^2]$ and the dual variable update error $\mathbb{E}[\Phi(x_t) - f(z_t)]$ are controlled in a desired level.

Lemma 4. Suppose that $\eta_t \leq \frac{1}{4\sqrt{\mu}}$ in CD-MAGE. Then,
\[
\mathbb{E}[\Phi(x_{t+1}) - \Phi(x_t)] \\
\leq \frac{5\eta_t K}{4} \mathbb{E}[\|e_{x,t}\|^2] + \frac{\eta_t K L_1^2}{2\mu} \mathbb{E}[\Phi(x_t) - f(z_t)] + \frac{L_f \eta_t^2 K^2}{S} \sigma_1^2,
\]
where $e_{x,t}$ is defined in Lemma 3.

However, these two errors are coupled as the algorithm proceeds, which makes it difficult to directly control these error terms. To circumvent this difficulty, we define a new potential function
\[
L_t = \mathbb{E}[\Phi(x_t) - \Phi(x^*) + \frac{1}{2\theta}(\Phi(x_t) - f(x_t, y_t))],
\]
where $x^* \in \arg\min_{x \in \mathbb{R}^q} \Phi(x)$. Note that $L_t \geq 0$ since $\Phi(x_t) \geq \Phi(x^*)$ and $\Phi(x_t) \geq f(x_t, y_t)$. If $L_t = 0$, then $x_t$ is a minimizer of $\Phi$ and $y_t \in \arg\max_{y \in \mathbb{R}^q} f(x_t, y)$. The following lemma provides an upper bound of $L_{t+1}$. 


Lemma 5. Suppose that the step sizes \( \eta_t, \gamma_t \) of CD-MAGE satisfy \( \eta_t \leq \min \left\{ \frac{1}{36 L_f K}, \frac{L_f^2}{4 T} \sqrt{- \frac{1}{2 T^2}}, \frac{1}{2 L_f K} \right\} \) and \( \gamma_t \leq \frac{1}{8 T L_f K} \).

Then, we have
\[
L_{t+1} \leq L_t - \frac{21 \eta_t K}{40} \mathbb{E}\left[ \|\nabla \Phi(x_t)\|^2 \right]
+ \frac{\eta_t^2 K^2}{S} \left( 1.05 L_f + 0.81 L_f \right) \sigma_t^2
+ 3 \gamma_t^2 K^2 L_f \frac{\sigma_t^2}{4S}. \]

By recursively applying Lemma 5 and selecting appropriate step sizes, we derive the convergence rate of CD-MAGE as stated in the following theorem.

Theorem 1. Define the step sizes \( \eta_t \) and \( \gamma_t \) of CD-MAGE as
\[
\eta_t = \min \left\{ \frac{1}{36 L_f K}, \frac{L_f^2}{4 T} \sqrt{- \frac{1}{2 T^2}}, \frac{1}{2 L_f K} \right\}
\quad \text{and}
\gamma_t = \min \left\{ \frac{1}{8 T L_f K}, \sqrt{- \frac{4L_0 S}{3T L_f K + \sigma_t^2}} \right\}
\]
where \( L_0 := \Phi(x_0) - \Phi(x^*) + \frac{1}{20} (\Phi(x_0) - f(z_0)) \).

Then
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[ \|\nabla \Phi(x_t)\|^2 \right] \leq 4 \left( \frac{40 L_0 (L_f + L_f^2) \sigma_t^2}{21 ST} \right) + \frac{20}{7} L_f \sqrt{L_0 L_f \frac{\sigma_t^2}{3T}}
+ \frac{8 L_f + \left( \frac{1827 L_f^3}{2 T} + 72 \right) L_f}{21 T} \frac{40 L_0}{L_f}. \]

Remark 1. The rate in Theorem 1 can be simplified as \( O(1/T + 1/\sqrt{ST}) \), if we treat \( L_0, L_1, L_f, \mu, L_0, \sigma_1, \) and \( \sigma_2 \) as constants. In comparison, the non-local counterpart of CD-MAGE, i.e., Parallel SGDA, which directly parallelizes the single-machine SGD algorithm (Lin, Jin, and Jordan 2020) over the clients, has an \( O(1/T + 1/\sqrt{ST}) \) convergence rate (with step sizes depending on \( T \)) under similar assumptions. Both of these two rates are inferior to that of CD-MAGE when \( S < T \), and all of them are equal to \( O(1/T) \) when \( S > T \). Thus, the communication cost of CD-MAGE with any \( K \geq 1 \) is comparable to or better than that of its non-local counterpart Parallel SGDA. In contrast, most existing rates of FL algorithms at best match those of their non-local counterparts only at certain values of \( K \), given the same number of gradient evaluations.

4.2 Analysis of CD-MAGE+

The analysis of CD-MAGE+ is more sophisticated than that of CD-MAGE. Due to the limited space, we defer the detailed analysis to the Appendix and summarize the main result in the following theorem.

Theorem 2. Define the parameters \( \eta_t, \gamma_t, \) and \( \alpha_t \) of CD-MAGE+ as \( \eta_t = \min \left\{ \frac{1}{2 L_f}, \frac{1}{2 L_f}, \frac{2 \mu^2 \gamma_t}{30 L_f K} \right\} \), \( \gamma_t = \min \left\{ \frac{1}{100 L_f K}, \frac{1}{L_f (K+1)}, \frac{1}{2 L_f K} \right\} \), and \( \alpha_t = \frac{8192 L_f^2 K^2 \gamma_t^2}{T} \).

Then
\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[ \|\nabla \Phi(x_t)\|^2 \right] \leq 10 \sqrt{L_0 (L_f + L_f^2) \frac{\sigma_t^2}{ST}} + 9 L_f^2 \frac{\sigma_t^2}{\mu^2} \left( \frac{L_0 L_f + L_0}{ST} \right)
+ \frac{9 L_f^2}{\mu^2} \left( \frac{\sigma_t^2}{10 (\sigma_t^2 + \sigma_2^2)^{1/3}} + 3 (\sigma_t^2 + \sigma_2^2)^{1/3} \right) \left( \frac{L_0 L_f}{T} \right)^{2/3}
+ \left( \frac{37 + 13 L_f^2}{\mu^2} \right) (\sigma_t^2 + \sigma_2^2) \left( \frac{L_0 L_f}{T} \right)^{2/3}.
\]

Remark 4. As shown in Theorem 3, CD-MA attains an \( O(1/T^{2/3} + 1/\sqrt{ST}) \) convergence rate, which is inferior to that of CD-MAGE and CD-MAGE+. Our theory shows that
CD-MAGE and CD-MAGE+ are less affected by the data heterogeneity compared to CD-MA. Specifically, in the convergence rate of CD-MAGE (resp., CD-MAGE+), the terms depending on the gradient dissimilarity (σ1 or σ2) are of the order $O(1/\sqrt{ST})$ (resp., $O(1/(\tau + 1/\sqrt{ST})^{2/3})$), whereas CD-MA has a higher order of $O(1/\sqrt{ST} + 1/\tau^{2/3})$. Note that the gradient dissimilarity is commonly used to quantify data heterogeneity in the FL literature (Kairouz et al. 2019; Rasouli, Sun, and Rajagopal 2020; Reisizadeh et al. 2020; Hou et al. 2021; Yuan et al. 2021; Deng and Mahdavi 2021). Actually most existing FL algorithms are affected by data heterogeneity. The Scaffold method (Karimireddy et al. 2020b) is an exception, but its convergence rate depends on $N/S$, where $N$ is the total number of clients.

5 Experiments

To demonstrate the efficiency of the proposed algorithms, we conduct experiments on three tasks: (i) AUC maximization, (ii) robust adversarial neural network training, and (iii) generative adversarial network training. We compare the proposed algorithms with Parallel SGD, which is the parallel version of the single-machine SGD algorithm (Lin, Jin, and Jordan 2020), and the cross-device versions of Extra Step Local SGD (Beznosikov, Samokhin, and Gasnikov 2020), Local SGD++ (Deng and Mahdavi 2021), Catalyst-Scaffold-S (Hou et al. 2021), CODA+, and CODASCA (Yuan et al. 2021). We note that Local SGD++, Extra Step Local SGD, Catalyst-Scaffold-S, CODA+, and CODASCA are state-of-the-art federated minimax algorithms which are required to the participation of all clients in each round and thus only apply to the cross-silo setting. Here, we use their cross-device variants by only involving a subset of clients in each round, which are actually not theoretically guaranteed. We exclude the CODA algorithm (Guo et al. 2020) from our baselines as we have already included CODA+, which is an improved version of CODA (Yuan et al. 2021).

In our experimental setting, there are 500 clients in total and the client distribution $\mathcal{D}$ is the uniform distribution (check the discussion below the problem formulation (1)). Since communication is often the major bottleneck in cross-device federated learning and clients generally have much slower upload than download bandwidth (Kairouz et al. 2019), we compare the algorithms given the same amount of data transferred from clients to the server. Specifically, for CD-MA, Parallel SGD, Local SGD++, Extra Step Local SGD and CODA++, which are only required to send parameter updates from clients to the server in each round, we fix the number of clients participating at each time to $S = 10$. For the rest algorithms, we fix $S = 5$ because they require to send both parameter updates and gradient estimates to the server in each round. In this way, all algorithms have equal amount of client communication to the server in each round.

Throughout the experiments, we used the same random seed for all algorithms. We did a grid-search on hyper-parameters for the algorithms and selected the best hyper-parameters. The ranges of the tuning values are listed in Appendix B.1. All experiments were implemented in Pytorch and run on a workstation with 2 Intel E5-2680 v4 CPUs (28 cores and 56 threads) and 378GB memory, where each client is assigned to a single process.

5.1 AUC Maximization

In the first experiment, we consider the $\ell_2$-relaxed AUC maximization problem in the minimax formulation (Ying, Wen, and Lyu 2016; Liu et al. 2020a). AUC is a widely used metric for binary classification on imbalanced data. For completeness, we refer the reader to Appendix (?) for the $\ell_2$-relaxed AUC maximization problem and its minimax formulation.

We use two datasets: CIFAR-10 (Krizhevsky, Hinton et al. 2009) and MNIST (LeCun et al. 1998). To make the data imbalanced, we split the original CIFAR-10 dataset (resp., MNIST) into two classes by treating the original “airplane” (resp., 0) class as the positive class and the remaining as negative. In addition, we preprocess the data by rescaling each pixel value to the range $[-1, 1]$. To make the data heterogeneous, the training data is first sorted according to the original class label and then equally partitioned into 500 clients so that all data points on one client are from the same class. We use a convolutional neural network from (TensorFlow team 2021) as the classification model for CIFAR-10 following (McMahan et al. 2017) and LeNet5 (LeCun et al. 1998) for MNIST. The parameters of these networks are randomly initialized by default in Pytorch.
We observe that CD-MAGE+ and CD-MAGE are superior where \( \ell_0 \) norm subject to the \( \ell_2 \) regularization term. The same as in Section 5.1 and each pixel value of the image is rescaled to \([-1, 1]\). Following the setting in (Deng and Mahdavi 2021), we use a 2-layer MLP with the ReLU activation, where each layer consists of 200 neurons. The performance measure is the robust training loss defined as 

\[
\max_y E_{t \sim T} \left[ \frac{1}{m} \sum_{j=1}^m \ell(h_{x_t} (a_{i,j} + y), b_{i,j}) - \frac{1}{2} \|y\|^2 \right].
\]

To compute the robust training loss which involves solving a maximization problem, we run the gradient ascent algorithm for a few epochs following (Deng and Mahdavi 2021). Similar to the previous experiment, each algorithm runs with the minibatch size \( B = 10 \) for exactly one epoch (12 local steps) on clients. The results are shown in Figure 2. We can see that both CD-MAGE+ and CD-MAGE outperform other algorithms. In addition, CD-MAGE+ performs much better than CD-MAGE, which shows the effectiveness of variance reduction in global gradient estimates.

### 5.3 GAN Training

In the last experiment, we test the performance of our algorithms on training GANs (Goodfellow, Shlens, and Szegedy 2014). Denote the generator network parameterized by \( x \) as \( G_x \) and the discriminator network parameterized by \( y \) as \( D_y \). The local loss function is defined as

\[
f_i(x, y) = \frac{1}{n_i} \sum_{j=1}^{n_i} \log D_y(a_{i,j}+y, b_{i,j}),
\]

where \( \ell \) denotes the cross entropy function, and \( y \) is the adversarial noise subject to the \( \ell_2 \)-norm ball \( \{y : \|y\|_2 \leq 1\} \). We relax this constrained minimax problem to an unconstrained one by replacing the constraint set with an \( \ell_2 \) regularization term \( -\frac{\lambda}{2} \|y\|^2 \), where \( \lambda \) is set to 0.001. We use MNIST and Fashion MNIST (Xiao, Rasul, and Vollgraf 2017) datasets in this experiment. The data partitioning scheme is the same as in Section 5.1 and each pixel value of the images is rescaled to \([-1, 1]\). Following the setting in (Deng and Mahdavi 2021), we use a 2-layer MLP with the ReLU activation, where each layer consists of 200 neurons. The performance measure is the robust training loss defined as

\[
\max_y E_{t \sim T} \left[ \frac{1}{m} \sum_{j=1}^m \ell(h_{x_t} (a_{i,j} + y), b_{i,j}) - \frac{1}{2} \|y\|^2 \right].
\]
Figure 3: Results on the GAN training task (left: MNIST, right: Fashion MNIST). The horizontal axis stands for the number of rounds and the vertical axis represents the Inception score.

Appendix B.4 depict samples generated by the compared algorithms after training, which also show that CD-MAGE and CD-MAGE+ generate high-quality images.

6 Conclusion

We present the first theoretically guaranteed algorithms for general minimax problems in the cross-device federated learning setting. Our algorithms have the advantages of being communication-efficient and being robust to the limited availability and the data heterogeneity of clients. Theoretical analyses and experimental results show the efficiency of our algorithms. Directions for future work include the development and analyses of variants with adaptive step sizes, and the study of lower bounds for minimax problems in the cross-device federated learning setting.

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Appendix of Efficient Cross-Device Federated Learning Algorithms for Minimax Problems

A The CD-MA Algorithm

The CD-MA algorithm described in Section 3 is detailed in Algorithm 4.

Algorithm 4: Cross-device Minimax Averaging (CD-MA)

Input: initial point \((x_0, y_0)\), the number of rounds \(T\), step sizes \(\{\eta_t\}_{t=0}^{T-1}\), \(\{\gamma_t\}_{t=0}^{T-1}\), and the batch size of clients \(S\).

Server executes:
1. \(\text{for } t = 0, 1, \ldots, T - 1 \text{ do} \)
2. Send \((x_t, y_t)\) to a subset of clients;
3. Once local iterates \((x_{t,i}^{(K)}, y_{t,i}^{(K)})\) from \(S\) clients are received, proceed;
4. \(x_{t+1} \leftarrow \frac{1}{|S_t|} \sum_{i \in S_t} x_{t,i}^{(K)}, y_{t+1} \leftarrow \frac{1}{|S_t|} \sum_{i \in S_t} y_{t,i}^{(K)}, \) where \(S_t\) is the \(S\) clients in the above step;
5. \(\text{end for} \)

Client \(i \in S_t\) executes:
6. Initialize local model \(x_{t,i}^{(0)} \leftarrow x_t, y_{t,i}^{(0)} \leftarrow y_t; \)
7. \(\text{for } k = 0, 1, \ldots, K - 1 \text{ do} \)
8. Sample a minibatch \(B_{t,i}^{(k)}\) from local data;
9. \(x_{t,i}^{(k+1)} \leftarrow x_{t,i}^{(k)} - \eta_t \nabla_x F_t(x_{t,i}^{(k)}, y_{t,i}^{(k)}, B_{t,i}^{(k)}); \)
10. \(y_{t,i}^{(k+1)} \leftarrow y_{t,i}^{(k)} + \gamma_t \nabla_y F_t(x_{t,i}^{(k)}, y_{t,i}^{(k)}, B_{t,i}^{(k)}); \)
11. \(\text{end for} \)
12. Send \((x_{t,i}^{(K)}, y_{t,i}^{(K)})\) to the server;

B Additional Details of Experiments

B.1 Specification of Hyper-parameters

The hyper-parameters of CD-MAGE, CD-MA, Parallel SGDA, and Local SGDA+ are the constant step sizes \(\eta\) and \(\gamma\). The Extra Step Local SGD algorithm has only one hyper-parameter, the local step size \(\eta_t\). CODA+ involves three hyper-parameters: the local step size \(\eta_t\), the weight coefficient \(\theta\) of the proximal-point subproblem, and the interval \(K_0\) (in terms of the number of local updates) between two proximal point updates. CODASCA and Catalyst-Scaffold-S have all the three hyper-parameters of CODA+ as well as an additional global step size \(\eta_g\). Following Hou et al. (2021), we set \(\eta_g = \eta e\) for Catalyst-Scaffold-S. The hyper-parameters of CD-MAGE+ are \(\eta_t, \gamma_t\) and \(\alpha_t\). In our implementation, we set \(\eta_t = \frac{e_0}{(t+1)^p}, \gamma_t = \frac{e_1}{(t+1)^p}\), and \(\alpha_t = \min\{1, \frac{1}{(t+1)^p}\}\), where \(e_0, e_1, e_3\), and \(e\) are tunable parameters. We note that Catalyst-Scaffold-S communicates with probability \(p\) at each local step, while other algorithms communicate after every \(K\) local steps. We set \(p = 1/K\) so that the average number of local steps of Catalyst-Scaffold-S in each round is equal to \(K\), where \(K\) has been defined in Section 5.1-5.3.

Hyper-parameters for AUC Maximization

For the AUC maximization task, the hyper-parameters are chosen as follows.

- For CD-MAGE, CD-MA, Parallel SGDA, and Local SGDA+, \(\eta\) is chosen from \(\{1, 3.162e-1, 1e-1\}\) and \(\gamma\) is chosen from \(\{1, 3.162e-1, 1e-1\}\) (1, 3.162e-2, 1e-2, 3.162e-3, 1e-3, 3.162e-4, 1e-4\).
- The step size \(\eta_t\) of Extra Step Local SGD is tuned from \(\{1, 3.162e-1, 1e-1, 3.162e-2, 1e-2, 3.162e-3, 1e-3, 3.162e-4, 1e-4\}\) for CD-MAGE+.
- For CODA+, \(c_\eta \in \{1, 3.162e-1, 1e-1\}\) and \(c_\gamma \in \{1, 3.162e-1, 1e-1, 3.162e-2, 1e-2, 3.162e-3, 1e-3\}\), \(c_\alpha \in \{5, 10\}\), and \(p \in \{1/5, 1/3\}\). We note that while our theory suggests that \(p = 1/3\), in practice CD-MAGE+ with \(p = 1/5\) performs better in some cases.
- The hyper-parameters \(\eta_t, \theta,\) and \(K_0\) of CODA+, and Catalyst-Scaffold-S are chosen from \(\{1, 3.162e-1, 1e-1, 3.162e-2, 1e-2\}\), \(\{0, 1e-1, 1e1\}\), and \(\{2e3, 4e3\}\), respectively.
- For CODASCA, \(\eta_t, \theta,\) and \(K_0\) are chosen in the same way as those of CODA+, and \(\eta_g\) is tuned from \(\{1.1, 1.0, 0.99\}\) following Yuan et al. (2021).

Table 1 shows the best hyper-parameters for each algorithm on the AUC maximization experiment.

\(^{1}\)Note that 3.162e-1 = 10^{-0.5}.\)
Table 1: Best hyper-parameters on the AUC maximization experiment.

| Algorithm       | Best hyper-parameters on CIFAR-10          | Best hyper-parameters on MNIST               |
|-----------------|-------------------------------------------|---------------------------------------------|
| CD-MAGE         | $\eta=1e^{-1}, \gamma=3.162e-1$           | $\eta=3.162e-1, \gamma=3.162e-2$           |
| CD-MA           | $\eta=3.162e-1, \gamma=3.162e-2$           | $\eta=1, \gamma=3.162e-1$                  |
| Parallel SGDA  | $\eta=1, \gamma=3.162e-2$                  | $\eta=1, \gamma=1$                         |
| Local SGDA+     | $\eta=1e-1, \gamma=1e-2$                  | $\eta=1, \gamma=3.162e-1$                  |
| Extra Step Local SGD | $\eta_\ell=1e-1$                       | $\eta_\ell=3.162e-1$                       |
| CD-MAGE+        | $c_\eta=3.162e-1, c_\gamma=10, \rho=1/5$ | $c_\eta=3.162e-1, c_\gamma=1e-2, c_\alpha=5, \rho=1/5$ |
| CODA+           | $\eta_\ell=1e-1, \theta=0, K_0=2e3$      | $\eta_\ell=3.162e-1, \theta=0, K_0=2e3$  |
| Catalyst-Scaffold-S | $\eta_\ell=1e-1, \theta=0, K_0=2e3$   | $\eta_\ell=1e-1, \theta=0, K_0=2e3$  |
| CODASCA         | $\eta_\ell=3.162e-1, \eta_g=1, \theta=0, K_0=2e3$ | $\eta_\ell=1, \eta_g=1.1, \theta=0, K_0=2e3$ |

Hyper-parameters for Robust Adversarial Network Training  For the robust adversarial network training task, the hyper-parameters are chosen as follows.

- For CD-MAGE, CD-MA, Parallel SGDA, and Local SGDA+, $\eta$ is chosen from $\{3.162e-2, 1e-2, 3.162e-3, 1e-2\}$ and $\gamma$ is chosen from $\{1, 3.162e-1, 1e-1, 3.162e-2, 1e-3\}$.
- The step size $\eta_\ell$ of Extra Step Local SGD is tuned from $\{1, 3.162e-1, 1e-1, 3.162e-2, 1e-2, 3.162e-3, 1e-3, 3.162e-4, 1e-4\}$.
- For CD-MAGE+, $c_\eta \in \{3.162e-2, 1e-2, 3.162e-3, 1e-2\}$ and $c_\gamma \in \{1, 3.162e-1, 1e-1, 3.162e-2, 1e-2\}$, $c_\alpha \in \{5, 10\}$, and $\rho \in \{1/5, 1/3\}$.
- The hyper-parameters $\eta_\ell$, $\theta$, and $K_0$ of CODA+, and Catalyst-Scaffold-S are chosen from $\{1e-1, 3.162e-2, 1e-2, 3.162e-3, 1e-3\}$, $\{0, 1e-1, 1e1\}$, and $\{2e3, 4e3\}$, respectively.
- For CODASCA, $\eta_\ell$, $\theta$, and $K_0$ are chosen in the same way as those of CODA+, and $\eta_g$ is tuned from $\{1.1, 1.0, 0.99\}$ following Yuan et al. (2021).

Table 2 shows the best hyper-parameters for each algorithm on the robust adversarial network training experiment.

Table 2: Best hyper-parameters on the robust adversarial network training experiment.

| Algorithm       | Best hyper-parameters on MNIST          | Best hyper-parameters on Fashion MNIST     |
|-----------------|-----------------------------------------|--------------------------------------------|
| CD-MAGE         | $\eta=1e-3, \gamma=3.162e-1$            | $\eta=1e-3, \gamma=1e-1$                  |
| CD-MA           | $\eta=1e-3, \gamma=3.162e-2$            | $\eta=1e-3, \gamma=3.162e-2$              |
| Parallel SGDA  | $\eta=1e-2, \gamma=1$                  | $\eta=1e-2, \gamma=3.162e-1$              |
| Local SGDA+     | $\eta=1e-3, \gamma=3.162e-2$            | $\eta=1e-3, \gamma=3.162e-2$              |
| Extra Step Local SGD | $\eta_\ell=3.162e-4$               | $\eta_\ell=1e-3$                         |
| CD-MAGE+        | $c_\eta=1e-2, c_\gamma=1, c_\alpha=5, \rho=1/3$ | $c_\eta=1e-2, c_\gamma=1, c_\alpha=5, \rho=1/3$ |
| CODA+           | $\eta_\ell=1e-2, \theta=1e-1, K_0=2e3$ | $\eta_\ell=1e-3, \theta=0, K_0=2e3$     |
| Catalyst-Scaffold-S | $\eta_\ell=1e-3, \theta=1e1, K_0=2e3$ | $\eta_\ell=1e-3, \theta=1e-1, K_0=4e3$ |
| CODASCA         | $\eta_\ell=3.162e-3, \eta_g=1, \theta=1e1, K_0=2e3$ | $\eta_\ell=3.162e-3, \eta_g=1.1, \theta=1e-1, K_0=4e3$ |

Hyper-parameters for GAN Training  For the GAN training task, the hyper-parameters are chosen as follows.

- For CD-MAGE, CD-MA, Parallel SGDA, and Local SGDA+, both $\eta$ and $\gamma$ are chosen from $\{1e-2, 1e-3, 1e-4\}$.
- The step size $\eta_\ell$ of Extra Step Local SGD is tuned from $\{1e-2, 1e-3, 1e-4, 1e-5\}$.
- For CD-MAGE+, both $c_\eta$ and $c_\gamma$ are selected from $\{1e-2, 1e-3, 1e-4\}$, $c_\alpha \in \{5, 10\}$, and $\rho \in \{1/5, 1/3\}$.
- The hyper-parameters $\eta_\ell$, $\gamma$, and $K_0$ of CODA+, and Catalyst-Scaffold-S are chosen from $\{1e-2, 1e-3, 1e-4\}$, $\{0, 1e-1, 1e1\}$, and $\{2e3, 4e3\}$, respectively.
- For CODASCA, $\eta_\ell$, $\gamma$, and $K_0$ are chosen in the same way as those of CODA+, and $\eta_g$ is tuned from $\{1.1, 1.0, 0.99\}$ following Yuan et al. (2021).

Table 3 summarizes the best hyper-parameters for each algorithm on MNIST and Fashion MNIST.
where $m$ denotes the number of classes.

The test results of the AUC maximization task in Section 5.1 are shown in Figure 4. These results are similar to the training results in Figure 2.

### B.2 The $\ell_2$-relaxed AUC Maximization Problem and its Equivalent Minimax Formulation

Suppose that we are given a dataset $\{(\mathbf{w}_i, \ell_i)\}_{i=1}^m$, where $\mathbf{w}_i$ denotes a feature vector and $\ell_i \in \{-1, +1\}$ denotes the corresponding label. For a scoring function $h_\theta$ of a classification model parameterized by $\theta \in \mathbb{R}^p$, the AUC is defined as

$$\max_{\theta} \frac{1}{m^+m^-} \sum_{\ell_i=+1, \ell_j=-1} \mathbb{I}_{\{h_\theta(\mathbf{w}_i) \geq h_\theta(\mathbf{w}_j)\}},$$

where $m^+ (m^-)$ denotes the number of positive (negative) samples and $\mathbb{I}$ denotes the indicator function. Since directly maximizing the AUC is NP hard in general, the indicator function in (6) is usually replaced by a surrogate such as the squared loss in practice:

$$\max_{\theta} \frac{1}{m^+m^-} \sum_{\ell_i=+1, \ell_j=-1} (1 - h_\theta(\mathbf{w}_i) + h_\theta(\mathbf{w}_j))^2.$$  

(7)

It has been shown that (7) has the following equivalent minimax formulation (Ying, Wen, and Lyu 2016; Liu et al. 2020a)

$$\min_{(\theta, a, b, \lambda) \in \mathbb{R}^{p+2}} \max_{\lambda \in \mathbb{R}} f(\theta, a, b, \lambda) := \frac{1}{m} \sum_{i=1}^m \left\{ (1 - \tau)(h_\theta(\mathbf{w}_i) - a)^2 \mathbb{I}_{\{\ell_i=1\}} - \tau(1 - \tau)\lambda^2 + \tau(h_\theta(\mathbf{w}_i) - b)^2 \mathbb{I}_{\{\ell_i=-1\}} \right\}$$

$$+ 2(1 + \lambda)\tau h_\theta(\mathbf{w}_i) \mathbb{I}_{\{\ell_i=-1\}} - 2(1 + \lambda)(1 - \tau)h_\theta(\mathbf{w}_i) \mathbb{I}_{\{\ell_i=1\}},$$

where $\tau := m^+/(m^+ + m^-)$ is the ratio of positive data. Note that $f(\theta, a, b, \cdot)$ is strongly convex for any $(\theta, a, b) \in \mathbb{R}^{p+2}$ and thus satisfies Assumption 3.

### B.3 Additional Results of the AUC maximization and robust adversarial Network training tasks

The test results of the AUC maximization task in Section 5.1 are shown in Figure 4. These results are similar to the training results in Figure 1.

![Figure 4: Test results on the AUC maximization task (left: CIFAR-10, right: MNIST).](image-url)

Figure 5 presents the robust test loss of the robust adversarial network training experiment in Section 5.1. We observe that the test results are basically the same as the training results in Figure 2.
**B.4 Additional Results of the GAN training task in Section 5.3**

Table 4 shows the final Inception score of each algorithm after 400 (resp., 800) rounds of training on the MNIST (resp., Fashion MNIST) dataset. We note that in the evaluation of the Inception score, we used the Resnet18 network pretrained on MNIST/Fashion MNIST following (Chavdarova et al. 2019), instead of using the original Inception network pretrained on ImageNet. The reason is that both MNIST and Fashion MNIST consist of grayscale images, whereas the ImageNet dataset consists of colored images. We also visualize sample images generated by the algorithms in Figure 6 and Figure 7. We can see that CD-MAGE and CD-MAGE+ generate relatively high-quality images.

| Algorithm            | IS on MNIST | IS on Fashion MNIST |
|----------------------|-------------|----------------------|
| CD-MAGE              | 8.58 ± 0.030 | 8.76 ± 0.012         |
| CD-MAGE+             | 8.47 ± 0.026 | 7.93 ± 0.022         |
| CD-MA                | 7.79 ± 0.061 | 7.34 ± 0.014         |
| Parallel SGDA       | 6.08 ± 0.051 | 6.34 ± 0.038         |
| CODASCA              | 8.10 ± 0.059 | 6.31 ± 0.033         |
| CODA+                | 8.28 ± 0.076 | 7.34 ± 0.041         |
| Catalyst-Scaffold-S | 8.15 ± 0.026 | 7.67 ± 0.040         |
| Local SGDA+          | 8.08 ± 0.009 | 7.39 ± 0.022         |
| Extra Step Local SGD | 7.39 ± 0.025 | 7.46 ± 0.034         |

Table 4: The final Inception Score (IS) of the compared algorithms on MNIST and Fashion MNIST.
Figure 6: Generative samples of the compared algorithms after 400 rounds of training on MNIST. The samples are generated using the same set of random noise vectors.
Figure 7: Generative samples of the compared algorithms after 800 rounds of training on Fashion MNIST. The samples are generated using the same set of random noise vectors.
C Deferred proofs

In our proofs, we denote the pair of primal and dual variables \((\mathbf{x}, \mathbf{y})\) as \(\mathbf{z}\) for ease of notation. For example, \(\mathbf{z}_{t,i}^{(k)}\) represents client \(i\)'s local iterate \((x_{t,i}^{(k)}, y_{t,i}^{(k)})\) at the \(k\)-th local iteration in round \(t\).

C.1 Proof of Lemma 1

Proof. Denote \(\mathbf{z}_1 := (\mathbf{x}_1, \mathbf{y}_1)\) and \(\mathbf{z}_2 := (\mathbf{x}_2, \mathbf{y}_2)\). By Assumption 2, we have

\[
\begin{align*}
&\|\nabla F_i(\mathbf{z}_1; \mathbf{z}_2) - \nabla F_i(\mathbf{z}_2; \mathbf{z}_2)\|^2 = \|\nabla_x F_i(x_1; y_1; \mathbf{z}_2) - \nabla_x F_i(x_2; y_2; \mathbf{z}_2)\|^2 + \|\nabla_y F_i(x_1; y_1; \mathbf{z}_2) - \nabla_y F_i(x_2; y_2; \mathbf{z}_2)\|^2 \\
&\leq 2\|\nabla_x F_i(x_1; y_1; \mathbf{z}_2) - \nabla_x F_i(x_1; y_1; \mathbf{z}_1)\|^2 + 2\|\nabla_y F_i(x_1; y_1; \mathbf{z}_2) - \nabla_y F_i(x_1; y_1; \mathbf{z}_1)\|^2 \\
&\quad + 2\|\nabla_x F_i(x_2; y_2; \mathbf{z}_2) - \nabla_x F_i(x_2; y_2; \mathbf{z}_1)\|^2 + 2\|\nabla_y F_i(x_2; y_2; \mathbf{z}_2) - \nabla_y F_i(x_2; y_2; \mathbf{z}_1)\|^2 \\
&\leq 2(L_1^2 + L_2^2)\|\mathbf{x}_1 - \mathbf{x}_2\|^2 + 2(L_2^2 + L_1^2)\|\mathbf{y}_1 - \mathbf{y}_2\|^2 \\
&\leq \max\{2(L_1^2 + L_2^2), 2(L_2^2 + L_1^2)\}\|\mathbf{z}_1 - \mathbf{z}_2\|^2 = L_f^2\|\mathbf{z}_1 - \mathbf{z}_2\|^2.
\end{align*}
\]

Thus, \(F_i(\mathbf{x}, \mathbf{y}, \zeta)\) is \(L_f\)-smooth w.r.t. \((\mathbf{x}, \mathbf{y})\). Recall that \(f_i = \frac{1}{n_i}\sum_{j=1}^{n_i} f_i(\mathbf{x}, y; \zeta_{i,j})\) and \(f(\mathbf{x}, \mathbf{y}) = E_{i\sim D}[f_i(\mathbf{x}, \mathbf{y})]\). By Jensen’s inequality and the convexity of \(\|\cdot\|^2\), both \(f_i\) and \(f\) are \(L_f\)-smooth. □

C.2 Proofs of the results in Section 4.1

Proof of Lemma 3

Proof. The global primal variable \(x_{t+1}\) can be rewritten as

\[
x_{t+1} = x_t - \eta_t K\left(\mathbf{u}_t + \frac{1}{S\bar{K}} \sum_{i\in S_t} \sum_{k=0}^{K-1} \left(\nabla_x F_i(x_{t,i}^{(k)}, B_{t,i}^{(k)}) - \nabla_x f_i(z_i; B_{t,i}^{(k)})\right)\right)
\]

Similarly, the global dual variable \(y_{t+1}\) can be rewritten as \(y_{t+1} = y_t + \gamma_t K(v_t + e_{x,t})\).

Proof of Lemma 4

Proof. By the smoothness of \(L_{\Phi}\), we have

\[
E[\Phi(x_{t+1}) - \Phi(x_t)] \leq E\left[\left(\nabla \Phi(x_t), x_{t+1} - x_t\right) + \frac{L_{\Phi} \eta_t^2 K^2}{2}\|x_{t+1} - x_t\|^2\right]
\]

\[
= -\eta_t K E[\nabla \Phi(x_t), \mathbf{u}_t + e_{x,t}] + \frac{L_{\Phi} \eta_t^2 K^2}{2} E[\|\mathbf{u}_t + e_{x,t}\|^2]
\]

\[
\leq \frac{\eta_t K}{2} E[\|\nabla_x f(z_t) + e_{x,t}\| + \|\nabla_y f(x_t)\|] - \frac{\eta_t K}{2} E[\|\nabla_x f(z_t) + e_{x,t}\|^2] + \frac{\eta_t K}{2} E[\|\nabla_y f(x_t)\|^2]
\]

\[
\leq \frac{\eta_t K}{2} E[\|\nabla_x f(z_t) - \nabla \Phi(x_t)\|^2 + \|\nabla_y f(x_t)\|^2] - \frac{\eta_t K}{2} E[\|\nabla_y f(x_t)\|^2]
\]

\[
\leq \frac{\eta_t K}{2} E[\|\nabla_x f(z_t)\|^2 + \|\nabla_y f(x_t)\|^2] + \frac{L_{\Phi} \eta_t^2 K^2}{2} E[\|\nabla_x f(z_t)\|^2 + \|\nabla_y f(x_t)\|^2]
\]

\[
\leq \frac{L_{\Phi} \eta_t^2 K^2}{2} E[\|\nabla_x f(z_t)\|^2 + \|\nabla_y f(x_t)\|^2] + \frac{\eta_t K}{4} E[\|e_{x,t}\|^2],
\]

where (a) follows from Lemma 3, (b) holds because of the unbiasedness of \(\mathbf{u}_t\), (c) follows from Assumption 1, and (d) follows from Assumption 2, the condition \(\eta_t \leq 1/(4L_{\Phi} K)\) and the fact that \(|a + b|^2 \geq \frac{1}{2} |a|^2 - |b|^2\). By (Karimi, Nutini, and Schmidt 2016, Theorem 2), the PL condition (Assumption 3) implies that \(v(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{p \times q}\),

\[
\Phi(\mathbf{x}) - f(\mathbf{x}, \mathbf{y}) \geq 2\mu\|\mathbf{y} - \Phi^*(\mathbf{x})\|^2. \tag{9}
\]

Combining the above two inequalities leads to the desired result. □
Proof of Lemma 5. To prove Lemma 5, we provide upper bounds of the error terms $\mathbb{E}[\|e_{x,t}\|^2]$ and $\mathbb{E}[\Phi(x_{t+1}) - f(z_t)]$ in the next two lemmas, respectively.

Lemma 6. Suppose that $\eta_t, \gamma_t \leq \frac{1}{2KL_f}$ in CD-MAGE. Then, both $\mathbb{E}[\|e_{x,t}\|^2]$ and $\mathbb{E}[\|e_{y,t}\|^2]$ are bounded from above by

$$18L_f^2 \left( \eta_t^2 K^2 \mathbb{E}[\|\nabla_x f(z_t)\|^2] + \gamma_t^2 K^2 \mathbb{E}[\|\nabla_y f(z_t)\|^2] \right) + \frac{\eta_t^2 K^2}{S}\sigma_1^2 + \frac{\gamma_t^2 K^2}{S}\sigma_2^2.$$

Proof. First, we observe that

$$\mathbb{E}[\|e_{x,t}\|^2] = \mathbb{E} \left[ \frac{1}{SK} \sum_{i \in S} \sum_{k=0}^{K-1} \left\| \nabla_x F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)}) \right\|^2 \right]$$

$$\leq \frac{1}{SK} \sum_{i,k} \mathbb{E} \left[ \left\| \nabla_x F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)}) \right\|^2 \right] \leq \frac{L_f^2}{SK} \sum_{i,k} \mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2],$$

where the first inequality follows from Cauchy-Schwarz and the last inequality follows from Lemma 1. Similarly, we have $\mathbb{E}[\|e_{y,t}\|^2] \leq \frac{L_f^2}{SK} \sum_{i,k} \mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2].$

If $K = 1$, then $\mathbb{E}[\|z_{t,i}^{(1)} - z_t\|^2] = \eta_t^2 \mathbb{E}[\|u_i\|^2] + \gamma_t^2 \mathbb{E}[\|v_i\|^2]$. Suppose that $K \geq 2$, then for any $k \in \{0, \ldots, K-1\}$,

$$\mathbb{E}[\|z_{t,i}^{(k+1)} - z_t\|^2] = \mathbb{E}[\|z_{t,i}^{(k)} - x_t - \eta_t (\nabla_x F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)}) + u_i)\|^2]$$

$$+ \mathbb{E}[\|z_{t,i}^{(k)} - y_t + \gamma_t (\nabla_y F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_y F_i(z_t; B_{t,i}^{(k)}) + v_i)\|^2]$$

$$\leq (1 + \frac{1}{K-1}) \mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2] + K \eta_t^2 \mathbb{E}[\|\nabla_x F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)}) + u_i\|^2]$$

$$+ K \gamma_t^2 \mathbb{E}[\|\nabla_y F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_y F_i(z_t; B_{t,i}^{(k)}) + v_i\|^2]$$

$$\leq (1 + \frac{1}{K-1}) \mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2] + \frac{2K}{4KL_f^2} \mathbb{E}[\|\nabla_x F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)})\|^2]$$

$$+ \frac{2}{K^2L_f^2} \mathbb{E}[\|\nabla_y F_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_y F_i(z_t; B_{t,i}^{(k)})\|^2] + 2K \eta_t^2 \mathbb{E}[\|u_i\|^2] + 2K \gamma_t^2 \mathbb{E}[\|v_i\|^2]$$

$$\leq (1 + \frac{2}{K-1}) \mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2] + 2K \eta_t^2 \mathbb{E}[\|u_i\|^2] + 2K \gamma_t^2 \mathbb{E}[\|v_i\|^2],$$

where the first inequality follows from Young's inequality, the second inequality follows from the condition $\eta_t, \gamma_t \leq \frac{1}{(2L_f K)}$, and the last inequality follows from Lemma 1. Recursively applying the above inequality yields

$$\mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2] \leq \left( 2K \eta_t^2 \mathbb{E}[\|u_i\|^2] + 2K \gamma_t^2 \mathbb{E}[\|v_i\|^2] \right) \sum_{k'=0}^{k-1} \left( 1 + \frac{2}{K-1} \right)^{k'-1}$$

$$\leq 18\eta_t^2 K^2 \mathbb{E}[\|u_i\|^2] + 18\gamma_t^2 K^2 \mathbb{E}[\|v_i\|^2]$$

$$\leq 18\eta_t^2 K^2 \mathbb{E}[\|\nabla_x f(z_t)\|^2] + 18\gamma_t^2 K^2 \mathbb{E}[\|\nabla_y f(z_t)\|^2] + \frac{18\eta_t^2 K^2}{S} \sigma_1^2 + \frac{18\gamma_t^2 K^2}{S} \sigma_2^2,$$

where the last inequality follows from Assumption 1. $\square$

Lemma 7. Let $\eta_t \leq \min\left\{ 1/(2L_f K), 1/(4L_f K) \right\}$ and $\gamma_t \leq \min\left\{ 1/(4L_f K), 1/(2L_f K) \right\}$ in CD-MAGE. Then

$$\mathbb{E}[\Phi(x_{t+1}) - f(z_{t+1})] \leq (1 - \frac{\mu}{2} \frac{\gamma_t K}{S}) \mathbb{E}[\Phi(x_t) - f(z_t)] + \frac{\gamma_t K}{2} \mathbb{E}[\|e_{x,t}\|^2] + \frac{11\eta_t K}{S} + 2\eta_t^2 K^2 L_f \mathbb{E}[\|\nabla_x f(z_t)\|^2]$$

$$+ \frac{\gamma_t^2 K^2 L_f}{S} \sigma_2^2 + \left( \frac{7\eta_t K}{4} + 2\eta_t^2 K^2 L_f \right) \mathbb{E}[\|e_{y,t}\|^2] + \frac{\eta_t^2 K^2 (L_f + L_f)}{S} \sigma_1^2$$

$$- \frac{\gamma_t K}{4} \mathbb{E}[\|\nabla_y f(z_t)\|^2] - \frac{\eta_t K}{2} \mathbb{E}[\|\nabla \Phi(x_t)\|^2].$$

Proof. First, we rewrite $\Phi(x_{t+1}) - f(z_{t+1})$ as

$$\Phi(x_{t+1}) - f(z_{t+1}) = \Phi(x_{t+1}) - \Phi(x_t) + \Phi(x_t) - f(z_t) + f(z_t) - f(z_{t+1}).$$
By the smoothness of $f$, we have

$$
\begin{align*}
\mathbb{E}[f(z_t) - f(z_{t+1})] & \leq \mathbb{E} \left[ \eta_t K \langle \nabla_x f(z_t), u_t + e_{xt} \rangle - \gamma_t K \langle \nabla_y f(z_t), v_t + e_{yt} \rangle \right] \\
& \quad + \frac{L_f}{2} \left( \eta_t^2 K^2 \|u_t + e_{xt}\|^2 + \gamma_t^2 K^2 \|v_t + e_{yt}\|^2 \right) \\
& \quad + \mathbb{E} \left[ \frac{3}{2} \eta_t^2 K^2 \|\nabla_x f(z_t)\|^2 + \frac{\eta_t K}{2} \|e_{xt}\|^2 + \gamma_t K \|\nabla_y f(z_t)\|^2 - \gamma_t \frac{K}{2} \|\nabla_y f(z_t) + e_{yt}\|^2 \right] \\
& \quad + \mathbb{E} \left[ \frac{3}{2} \eta_t^2 K^2 \|\nabla_y f(z_t)\|^2 + \frac{\eta_t K}{2} \|e_{yt}\|^2 + \gamma_t K \|\nabla_x f(z_t)\|^2 - \gamma_t \frac{K}{2} \|\nabla_x f(z_t) + e_{xt}\|^2 \right]
\end{align*}
$$

where (a) follows from Lemma 3 and Lemma 1; (b) holds because $\mathbb{E}[\|\nabla_x f(z_t), u_t\|] = \mathbb{E}[\|\nabla_x f(z_t)|^2], \mathbb{E}[\|\nabla_y f(z_t), v_t\|] = \mathbb{E}[\|\nabla_y f(z_t)|^2];$ (c) follows from Assumption 1; (d) follows from the condition $\gamma_t \leq 1/(4L_f K)$.

Combining the above inequalities and Lemmas 4 and 5, we have

$$
\begin{align*}
\mathbb{E} \left[ \Phi(x_{t+1}) - f(z_{t+1}) \right] & \leq (1 + \frac{n_t K L_f}{2\mu}) \mathbb{E} \left[ \Phi(x_t) - f(z_t) \right] + \left( \frac{11}{8} \frac{K L_f}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \mathbb{E}[\|\nabla_x f(z_t)|^2] \\
& \quad + \left( \frac{7}{4} \frac{K L_f^2}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \mathbb{E}[\|e_{xt}\|^2] + \gamma_t \left( \frac{K}{2} \mathbb{E}[\|e_{yt}\|^2] \right) \\
& \quad - \frac{\mu \gamma_t K}{2} \mathbb{E}[\Phi(x_t) - f(z_t)] - \left( \frac{7}{4} \frac{K L_f^2}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \mathbb{E}[\|\nabla_x f(z_t)|^2] \\
& \quad + \left( \frac{7}{4} \frac{K L_f^2}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \mathbb{E}[\|e_{yt}\|^2] + \left( \frac{11}{8} \frac{K L_f}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \mathbb{E}[\|\nabla_y f(z_t)|^2]
\end{align*}
$$

where the first inequality follows from Assumption 3.

Now, we are ready to prove Lemma 5.

**Proof of Lemma 5.** Recall that we define the potential function $L_t$ as

$$
L_t = \mathbb{E} \left[ \Phi(x_t) - \Phi(x^*) + \frac{1}{20} (\Phi(x_t) - f(x_t, y_t)) \right].
$$

Combining Lemma 4 and Lemma 7, we have

$$
\begin{align*}
\mathbb{E} \left[ \Phi(x_{t+1}) - \Phi(x^*) + \frac{1}{20} (\Phi(x_{t+1}) - f(x_{t+1}, y_{t+1})) \right] \\
& \leq \mathbb{E} \left[ \Phi(x_t) - \Phi(x^*) + \left( \frac{1}{20} - \frac{\mu \gamma_t K}{40} + \frac{211 \eta_t K L_f^2}{40 \mu} \right) \mathbb{E}[\Phi(x_t) - f(x_t, y_t)] \\
& \quad - \frac{21 \eta_t K}{40} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] - \left( \frac{7}{4} \frac{K L_f^2}{\eta_t} + \frac{1}{20} \left( \frac{11}{8} \frac{K L_f}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \right) \mathbb{E}[\|\nabla_x f(z_t)|^2] \\
& \quad + \left( \frac{5}{4} \frac{K L_f^2}{\eta_t} + \frac{1}{20} \left( \frac{7}{4} \frac{K L_f^2}{\eta_t} + 2 \eta_t^2 K^2 L_f \right) \right) \mathbb{E}[\|e_{xt}\|^2] + \frac{\gamma_t K}{40} \mathbb{E}[\|e_{yt}\|^2] \\
& \quad + \frac{\eta_t^2 K^2}{80} \left( \frac{211 L_f}{20} + \frac{L_f}{20} \right) \mathbb{E}[\|\nabla_y f(z_t)|^2]
\end{align*}
$$
where the second inequality follows from Lemma 6 and the last inequality follows from the conditions $\eta_t \leq \min \left\{ \frac{1}{36L_f \gamma}, \sqrt{\frac{4\epsilon \gamma}{3L_f KT^2 \sigma_f^2}} \right\}$ and $\gamma_t \leq \frac{1}{8\gamma}$. 

Proof of Theorem 1

Proof. Denote $\gamma_t = \gamma := \min \left\{ \frac{1}{36L_f \gamma}, \sqrt{\frac{4\epsilon \gamma}{3L_f KT^2 \sigma_f^2}} \right\}$ and $\eta_t = \eta := \min \left\{ \frac{1}{36L_f \gamma}, \sqrt{\frac{4\epsilon \gamma}{3L_f KT^2 \sigma_f^2}} \right\}$. Recursively applying the inequality in Lemma 5 yields

$$\mathcal{L}_T \leq \mathcal{L}_0 - \frac{21\gamma_t K}{40} \sum_{i=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_i)\|^2] + \frac{21T}{20S} \eta_t^2 K^2 (L_f + L_f) T \sigma_f^2 + \frac{3T}{4S} \gamma_t^2 K^2 L_f \sigma_f^2.$$ 

Plugging the values of $\eta$ and $\gamma$ to the above inequality and rearranging terms, we arrive at

$$\frac{1}{T} \sum_{i=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_i)\|^2] \leq \frac{40}{21\gamma_t KT} \mathcal{L}_0 - \frac{21\gamma_t K}{40} \sum_{i=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_i)\|^2] + \frac{21T}{20S} \eta_t^2 K^2 (L_f + L_f) T \sigma_f^2 + \frac{3T}{4S} \gamma_t^2 K^2 L_f \sigma_f^2.$$ 

which is the desired result. 

C.3 Proofs of the results in Section 4.2

The analysis of CD-MAGE+ is more sophisticated than that of CD-MAGE. For ease of notation, we omit the subscript $t, i$ of the iterates in CD-MAGE+ and denote $x^{(t)}$ (resp., $y^{(t)}$) as $x^{(k)}$ (resp., $y^{(k)}$). To prove Theorem 2, we first define some auxiliary quantities and establish several lemmas. We denote the error of gradient estimations in CD-MAGE+ as

$$\mathbb{E} \left[ \| \mathbf{u}_t - \nabla_x f(x_t, y_t) \|^2 \right],$$
$$\mathbb{E} \left[ \| \mathbf{v}_t - \nabla_y f(x_t, y_t) \|^2 \right].$$
where the expectation is taken over all randomness. These two quantities can be bounded as follows.

**Lemma 8.** The gradient estimation errors \( \varepsilon_{x,t+1} \) and \( \varepsilon_{y,t+1} \) can be bounded by

\[
\begin{align*}
\varepsilon_{x,t+1} &\leq (1 - \alpha_{t+1})^2 \varepsilon_{x,t} + \frac{2\alpha_t^2 \sigma_x^2}{S} + \frac{8(1-\alpha_{t+1})^2 L_y^2}{S} E[\|z_{t+1} - z_t\|^2], \\
\varepsilon_{y,t+1} &\leq (1 - \alpha_{t+1})^2 \varepsilon_{y,t} + \frac{2\alpha_t^2 \sigma_y^2}{S} + \frac{8(1-\alpha_{t+1})^2 L_y^2}{S} E[\|z_{t+1} - z_t\|^2].
\end{align*}
\]

**Proof.** By the definition of \( \varepsilon_{x,t+1} \), we have

\[
\varepsilon_{x,t+1} = E[\|u_{t+1} - \nabla_x f(z_{t+1})\|^2]
\]

\[
= E\left[\left(1 - \alpha_{t+1}\right)(u_t - \nabla_x f(z_t)) + \alpha_{t+1}\left(\frac{1}{S} \sum_{i \in S_{t+1}} (\nabla_x f_i(z_{t+1}) - \nabla_x f(z_t)) + (\nabla_x f_i(z_{t+1}) - \nabla_x f(z_t))\right)^2\right]
\]

\[
= (1 - \alpha_{t+1})^2 \varepsilon_{x,t} + E\left[\left(\frac{1}{S} \sum_{i \in S_{t+1}} (\nabla_x f_i(z_{t+1}) - \nabla_x f(z_t))\right)^2\right]
\]

\[
\leq (1 - \alpha_{t+1})^2 \varepsilon_{x,t} + 2\alpha_t^2 E\left[\left(\frac{1}{S} \sum_{i \in S_{t+1}} (\nabla_x f_i(z_{t+1}) - \nabla_x f(z_t))\right)^2\right]
\]

\[
\leq (1 - \alpha_{t+1})^2 \varepsilon_{x,t} + \frac{8(1-\alpha_{t+1})^2 L_y^2}{S} E[\|z_{t+1} - z_t\|^2] + \frac{2\alpha_t^2 \sigma_x^2}{S},
\]

where (a) holds because \( E_{S_{t+1}}[\frac{1}{S} \sum_{i \in S_{t+1}} f_i(z)] = f(z) \) and \( S_{t+1} \) is independent of \( \varepsilon_{x,t} \); (b) follows from Assumption 1 and Lemma 1. This completes the first part of Lemma 8. The second part of the lemma follows from the same argument. \( \Box \)

The following lemma relates the approximation error of \( d_x^{(k)} \) (resp., \( d_y^{(k)} \)) defined in (4) to \( \varepsilon_{x,t} \) (resp., \( \varepsilon_{y,t} \)).

**Lemma 9.** Consider CD-MAGE+. Under Assumptions 2-3, we have

\[
\begin{align*}
E[\|d_x^{(k)} - \nabla \Phi(x^{(k)})\|^2] &\leq \frac{3L_y^2}{2}\|\Phi(x^{(k)}) - f(z^{(k)})\| + 12L_y^2 E[\|z^{(k)} - z_t\|^2] + 3\varepsilon_{x,t}, \\
E[\|d_x^{(k)} - \nabla_x f(z^{(k)})\|^2] &\leq 8L_y^2 E[\|z^{(k)} - z_t\|^2] + 2\varepsilon_{x,t}, \\
E[\|d_y^{(k)} - \nabla_y f(z^{(k)})\|^2] &\leq 8L_y^2 E[\|z^{(k)} - z_t\|^2] + 2\varepsilon_{y,t}.
\end{align*}
\]

**Proof.** By Lemma 1, both \( F_i(\cdot, \cdot; B^{(k)}) \) and \( f(\cdot, \cdot) \) are \( L_f \)-smooth. Thus, we can bound

\[
E[\|\nabla_x F_i(z^{(k)}; B^{(k)}) - \nabla_x F_i(z; B^{(k)}) + u_t - \nabla \Phi(x^{(k)})\|^2]
\]

\[
= E[\|\nabla_x F_i(z^{(k)}; B^{(k)}) - \nabla_x F_i(z_t; B^{(k)}) - \nabla_x f(z_t) + \nabla_x f(z_t) - \nabla_x f(z_t) + u_t - \nabla \Phi(x^{(k)})\|^2]
\]

\[
\leq 3E[\|\nabla_x F_i(z^{(k)}; B^{(k)}) - \nabla_x F_i(z_t; B^{(k)})\|^2] + \|\nabla_x f(z_t) - \nabla_x f(z_t)\|^2 + 3\varepsilon_{x,t} + 3E[\|\nabla_x f(z^{(k)}) - \nabla \Phi(x^{(k)})\|^2]
\]

\[
\leq 6E[\|\nabla_x F_i(z^{(k)}; B^{(k)}) - \nabla_x F_i(z_t; B^{(k)})\|^2] + \|\nabla_x f(z_t) - \nabla_x f(z_t)\|^2 + 3\varepsilon_{x,t} + 3E[\|\nabla_x f(z^{(k)}) - \nabla \Phi(x^{(k)})\|^2]
\]

\[
\leq 12L_y^2 E[\|z^{(k)} - z_t\|^2] + 3\varepsilon_{x,t} + 3E[\|\nabla_x f(z^{(k)}) - \nabla \Phi(x^{(k)})\|^2]
\]

\[
\leq 12L_y^2 E[\|z^{(k)} - z_t\|^2] + 3\varepsilon_{x,t} + 3L_y^2 E[\|y^{(k)} - y^*(x^{(k)})\|^2]
\]

\[
\leq 12L_y^2 E[\|z^{(k)} - z_t\|^2] + 3\varepsilon_{x,t} + \frac{3L_y^2}{2}\|\Phi(x^{(k)}) - f(z^{(k)})\|,
\]

where (a) holds because \( E_{S_{t+1}}[\frac{1}{S} \sum_{i \in S_{t+1}} f_i(z)] = f(z) \) and \( S_{t+1} \) is independent of \( \varepsilon_{x,t} \); (b) follows from Assumption 1 and Lemma 1.
where (a) follows from Lemma 1, (b) follows from Assumption 2, and (c) follows from (9). This proves the first inequality of Lemma 9. Similarly, we can bound
\[
E[\|d^k_x - \nabla_x f(z^k)\|^2] = E[\|\nabla_x F_t(z^k; B^k) - \nabla_x F_t(z_t; B^k) - \nabla_x f(z^k) + \nabla_x f(z_t) + v_t - \nabla_x f(z_t)\|^2] 
\leq 8L_2^2E[\|z^k - z_t\|^2] + 2\varepsilon_{x,t},
\]
\[
E[\|d^k_y - \nabla_y f(z^k)\|^2] = E[\|\nabla_y F_t(z^k; B^k) - \nabla_y F_t(z_t; B^k) - \nabla_y f(z^k) + \nabla_y f(z_t) + v_t - \nabla_y f(z_t)\|^2] 
\leq 8L_2^2E[\|z^k - z_t\|^2] + 2\varepsilon_{y,t}.
\]

The next lemma shows that the expected loss value \(E[\Phi(x^k)]\) decreases as \(k\) increases as long as the errors \(E[\Phi(x^k) - f(z^k)], E[\|z^k - z_t\|^2]\), and \(\varepsilon_{x,t}\) are controlled in a desirable level.

**Lemma 10 (One-iteration descent).** Suppose that \(\eta_t \leq \frac{1}{2\pi L}\). Under Assumptions 1-3, \(\forall k \in \{0, \ldots, K-1\}\), we have
\[
E[\Phi(x^{k+1}) - \Phi(x^k)] \leq -\eta_t E[\|\nabla \Phi(x^k)\|^2] + \frac{3L_2^2}{4\mu} \eta_t E[\Phi(x^k) - f(z^k)] 
+ 7L_2^2 \eta_t E[\|z^k - z_t\|^2] + \frac{7\eta_t}{4} \varepsilon_{x,t} - \frac{\eta_t}{16} E[\|\nabla_x f(z^k)\|^2] - \frac{\eta_t}{8} E[\|d^k_y\|^2].
\]

Proof. By Lemma 2, we have
\[
E[\Phi(x^{k+1}) - \Phi(x^k)] \leq E[\langle\nabla \Phi(x^k), x^{k+1} - x^k\rangle] + \frac{L_2}{2} \|x^{k+1} - x^k\|^2 
= -\eta_t E[\langle\nabla \Phi(x^k), d^k_x\rangle] + \frac{L_2\eta_t^2}{2} E[\|d^k_x\|^2] 
= \frac{\eta_t}{2} E[\|d^k_x\|^2] - \frac{\eta_t}{2} E[\|\nabla \Phi(x^k)\|^2] - \frac{\eta_t}{4} E[\|\nabla \Phi(x^k)\|^2] - \frac{L_2\eta_t^2}{2} E[\|d^k_x\|^2] 
\leq \frac{\eta_t}{2} E[\|d^k_x\|^2 - \|\nabla \Phi(x^k)\|^2] - \frac{\eta_t}{2} E[\|\nabla \Phi(x^k)\|^2] - \frac{\eta_t}{4} E[\|\nabla \Phi(x^k)\|^2] 
- \frac{\eta_t}{16} E[\|\nabla_x f(x^k, y^k)\|^2] + \frac{\eta_t}{8} E[\|d^k_x - \nabla_x f(x^k, y^k)\|^2],
\]
where the first equality follows from the update rule \(x^{k+1} = x^k - \eta_t d^k_x\) and the second equality follows from the fact that \(-2\langle a, b \rangle = \|a - b\|^2 - \|a\|^2 - \|b\|^2\). Combining (11) and Lemma 9 completes the proof. □

We provide upper bounds of the error terms \(E[\|z^k - z_t\|^2]\) and \(E[\Phi(x^k) - f(z^k)]\) in the next two lemmas.

**Lemma 11.** Under Assumptions 1-3, for any \(k \in \{0, \ldots, K-1\}\), we have
\[
\begin{align*}
E[\|x^{k+1} - x^k\|^2] &\leq (1 + \frac{1}{K})E[\|x^k - x_t\|^2] + (1 + K)\eta_t^2 E[\|d^k_x\|^2], \\
E[\|x^{k+1} - y^k\|^2] &\leq (1 + \frac{1}{K})E[\|y^k - y_t\|^2] + (1 + K)\gamma_t^2 E[\|d^k_y\|^2].
\end{align*}
\]

Proof. First, we bound \(E[\|x^{k+1} - x^k\|^2]\) as follows
\[
E[\|x^{k+1} - x^k\|^2] = E[\|x^k - \eta_t d^k_x - x_t\|^2] \leq (1 + \frac{1}{K})E[\|x^k - x_t\|^2] + (1 + K)\eta_t^2 E[\|d^k_x\|^2],
\]
where the inequality follows from Young’s inequality. Similarly, \(E[\|y^{k+1} - y^k\|^2]\) can be bounded by
\[
E[\|y^{k+1} - y^k\|^2] \leq (1 + \frac{1}{K})E[\|y^k - y_t\|^2] + (1 + K)\gamma_t^2 E[\|d^k_y\|^2].
\]

□

**Lemma 12.** Suppose that \(\eta_t \leq \min\{\frac{1}{2\pi L}, \frac{1}{2\pi \gamma}\}\) and \(\gamma_t \leq \frac{1}{2\gamma}\). Under Assumptions 1-3, for any \(k \in \{0, \ldots, K-1\}\), we have
\[
E[\Phi(x^{k+1}) - f(z^{k+1})] \leq (1 - \mu \eta_t)^2 + \frac{3L_2^2}{4\mu} \eta_t E[\|\nabla \Phi(x^k)\|^2] - \frac{\eta_t}{2} E[\|\nabla \Phi(x^k)\|^2] 
+ (7\eta_t + 4\gamma_t)L_2^2 E[\|z^k - z_t\|^2] + \frac{7\eta_t}{16} E[\|\nabla_x f(z^k)\|^2] + \frac{5\eta_t}{8} E[\|d^k_x\|^2] + \frac{7\eta_t}{4} \varepsilon_{x,t} 
+ \gamma_t \varepsilon_{y,t} - \frac{\gamma_t}{4} E[\|d^k_y\|^2] - \frac{\gamma_t}{4} E[\|\nabla_y f(z^k)\|^2].
\]
Proof. First, we notice that

\[ \mathbb{E}[\Phi(x^{(k+1)}) - f(z^{(k+1)})] = \mathbb{E}[\Phi(x^{(k+1)}) - \Phi(x^{(k)}) + \Phi(x^{(k)}) - f(z^{(k)}) + f(z^{(k)}) - f(z^{(k+1)})]. \]

In what follows, we bound \( \mathbb{E}[f(z^{(k)}) - f(z^{(k+1)})] \).

\[
\mathbb{E}[f(z^{(k)}) - f(z^{(k+1)})] \\
\leq \mathbb{E} \left[ \langle \nabla_x f(z^{(k)}), x^{(k)} - x^{(k+1)} \rangle + \langle \nabla_y f(z^{(k)}), y^{(k)} - y^{(k+1)} \rangle + \frac{L_f}{2} \left( \|x^{(k)} - x^{(k+1)}\|^2 + \|y^{(k)} - y^{(k+1)}\|^2 \right) \right] \\
= \mathbb{E} \left[ \eta_t \langle \nabla_x f(z^{(k)}), d_x^{(k)} \rangle - \gamma_t \langle \nabla_y f(z^{(k)}), d_y^{(k)} \rangle + \frac{L_f \eta_t^2}{2} \|d_x^{(k)}\|^2 + \frac{L_f \gamma_t^2}{2} \|d_y^{(k)}\|^2 \right] \\
\leq \mathbb{E} \left[ \frac{\eta_t}{2} \|\nabla_x f(z^{(k)})\|^2 + \frac{\eta_t (1 + \eta_t L_f)}{2} \|d_x^{(k)}\|^2 - \frac{\gamma_t}{2} \|\nabla_y f(z^{(k)})\|^2 - \left( \frac{\gamma_t}{2} - \frac{L_f \gamma_t^2}{2} \right) \|d_y^{(k)}\|^2 + \frac{\gamma_t}{2} \|d_y^{(k)}\|^2 - \frac{\gamma_t L_f^2}{2} \mathbb{E}[\|z^{(k)} - z_t\|^2] + \gamma_t \xi_{x,y,t} \right],
\]

where the last inequality follows from the condition \( \gamma_t \leq 1/(2L_f) \) and Lemma 9. Combining the above two inequalities and Lemma 10, we obtain

\[
\mathbb{E}[\Phi(x^{(k+1)}) - f(z^{(k+1)})] \\
\leq \left( 1 + \frac{3L_f^2 \eta_t}{4\mu} \right) \mathbb{E}[\Phi(x^{(k)}) - f(z^{(k)})] - \frac{\eta_t}{2} \mathbb{E}[\|\nabla \Phi(x^{(k)})\|^2] \\
+ 7L_f^2 \eta_t \mathbb{E}[\|z^{(k)} - z_t\|^2] - \frac{7\eta_t}{8} \mathbb{E}[\|\nabla_x f(z^{(k)})\|^2] + \frac{7\eta_t}{8} \mathbb{E}[\|d_x^{(k)}\|^2] \\
+ \mathbb{E} \left[ \frac{\eta_t}{2} \|\nabla_x f(z^{(k)})\|^2 + \frac{\eta_t (1 + \eta_t L_f)}{2} \|d_x^{(k)}\|^2 - \frac{\gamma_t}{2} \|\nabla_y f(z^{(k)})\|^2 + \frac{\gamma_t}{2} \|d_y^{(k)}\|^2 + 4\gamma_t L_f^2 \mathbb{E}[\|z^{(k)} - z_t\|^2] + \gamma_t \xi_{x,y,t} \right] \\
\leq (1 - \frac{\mu \gamma_t}{2} + \frac{3L_f^2 \eta_t}{4\mu}) \mathbb{E}[\Phi(x^{(k)}) - f(z^{(k)})] - \frac{\eta_t}{2} \mathbb{E}[\|\nabla \Phi(x^{(k)})\|^2] + (\eta_t + 4\gamma_t) L_f^2 \mathbb{E}[\|z^{(k)} - z_t\|^2] \\
+ \frac{7\eta_t}{8} \mathbb{E}[\|\nabla_x f(z^{(k)})\|^2] + \frac{5\eta_t}{8} \mathbb{E}[\|d_x^{(k)}\|^2] + \frac{7\eta_t}{8} \mathbb{E}[\|d_y^{(k)}\|^2] \\
+ \gamma_t \xi_{x,y,t} + \gamma_t \xi_{x,y,t} - \frac{\gamma_t}{4} \mathbb{E}[\|\nabla_y f(z^{(k)})\|^2] - \frac{\gamma_t}{4} \mathbb{E}[\|\nabla_x f(z^{(k)})\|^2],
\]

where the last inequality follows from the condition \( \eta_t \leq 1/(2L_f) \) and Assumption 3. This completes the proof. \( \square \)

Proof of Theorem 2

Proof. First, we define the following potential function

\[
\mathcal{L}^{(k)} := \mathbb{E}[\Phi(x^{(k)}) - \Phi(x^*)] + \frac{1}{10} \mathbb{E}[\Phi(x^{(k)}) - f(z^{(k)})] \\
+ \frac{64(1.1L_f + 0.1L_f)\gamma_t}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k} \mathbb{E}[\|x^{(k)} - x_t\|^2] + 64L_f K \frac{K}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k} \mathbb{E}[\|y^{(k)} - y_t\|^2].
\]

Combining Lemma 10, 11, 12, and 8, we can bound

\[
\mathcal{L}^{(k+1)} + \frac{4\eta_{t+1}}{\alpha_{t+1}} \varepsilon_{x,t+1} + \frac{2\gamma_{t+1}}{5\alpha_{t+1}} \varepsilon_{y,t+1} \\
\leq \mathbb{E}[\Phi(x^{(k)}) - \Phi(x^*)] + c_1 \mathbb{E}[\Phi(x^{(k)}) - f(z^{(k)})] - \frac{11\eta_{t}}{20} \mathbb{E}[\|\nabla_x \Phi(x^{(k)})\|^2] + c_2 \mathbb{E}[\|x^{(k)} - x_t\|^2] \\
+ c_3 \mathbb{E}[\|y^{(k)} - x_t\|^2] + c_4 \varepsilon_{x,t} + c_5 \varepsilon_{y,t} - \frac{3}{160} \mathbb{E}[\|\nabla_x f(z^{(k)})\|^2] - \frac{\gamma_{t}}{40} \mathbb{E}[\|\nabla_y f(z^{(k)})\|^2] + c_6 \mathbb{E}[\|d_x^{(k)}\|^2] \\
+ c_7 \mathbb{E}[\|d_y^{(k)}\|^2] + c_8 \mathbb{E}[\|z_{t+1} - z_t\|^2] + \frac{8\eta_{t+1}\alpha_{t+1}}{S} \sigma_t^2 + \frac{4\gamma_{t+1}\alpha_{t+1}}{5S} \sigma_t^2,
\]

where \( c_1, \ldots, c_8 \) are given by

\[
c_1 := \frac{1}{10} \left( 1 - \frac{\mu \gamma_t}{2} + \frac{3L_f^2 \eta_t}{4\mu} \right),
\]

\[
c_2 := 7L_f^2 \eta_t + 0.1(7\eta_t + 4\gamma_t) L_f^2 + \frac{64(1.1L_f + 0.1L_f)^2 K \gamma_t}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k+1} \left( 1 + \frac{1}{K} \right),
\]

(12)
\[ c_3 := 7L_f^2 \eta_t + 0.1(7\eta_t + 4\gamma_t)L_f^2 + \frac{6.4L_f^2 K \gamma_t}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k+1} \left( 1 + \frac{1}{K} \right), \]
\[ c_4 := \frac{77\eta_t}{40} + \frac{4\eta_t}{\alpha_t+1}(1 - \alpha_{t+1})^2, \]
\[ c_5 := 0.1\gamma_t + \frac{2\eta_t}{5\alpha_t+1}(1 - \alpha_{t+1})^2, \]
\[ c_6 := -\frac{\eta_t}{8} + \frac{5\eta_t}{80} + \frac{(1.1L_\Phi + 0.1L_f)2K\gamma_t}{10\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k+1} \left( 1 + K \right)\eta_t^2, \]
\[ c_7 := \frac{\gamma_t}{40} + \frac{6.4L_f^2 K \gamma_t}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k+1} \left( 1 + K \right)\gamma_t^2, \]
\[ c_8 := \frac{32\eta_t+1(1 - \alpha_{t+1})^2 L_f^2}{S\alpha_t+1} + \frac{16\gamma_t+1(1 - \alpha_{t+1})^2 L_f^2}{5S\alpha_t+1}. \]

Notice that \( 1 \leq (1 + \frac{2}{K})^{K-k} \leq 8, \eta_t/2 \leq \eta_{t+1} \leq \eta_t, \) and \( \gamma_t/2 \leq \gamma_{t+1} \leq \gamma_t. \) Applying our choice of \((\eta_t, \gamma_t, \alpha_t, \alpha_t),\) we have
\[ c_1 \leq 0.1, \quad c_2 \leq \frac{64(1.1L_\Phi + 0.1L_f)K\gamma_t}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k}, \quad c_3 \leq \frac{2L_f K \gamma_t}{\alpha_t} \left( 1 + \frac{2}{K} \right)^{K-k}, \]
\[ c_4 \leq \frac{4\eta_t}{\alpha_t}, \quad c_5 \leq \frac{2\gamma_t}{5\alpha_t}, \quad c_7 \leq 0, \quad c_8 \leq \frac{64(1.1L_\Phi + 0.1L_f)2K\gamma_t}{S\alpha_t} + \frac{2L_f^2 K \gamma_t}{5S\alpha_t}. \]

Plugging the bounds of \(c_1, \ldots, c_8\) into (12), we get
\[ \tilde{L}^{(k+1)} + \frac{4\eta_t}{\alpha_t+1} \varepsilon_{x,t+1} + \frac{2\gamma_t}{5\alpha_t+1} \varepsilon_{y,t+1} \leq \tilde{L}^{(k)} - \frac{\eta_t}{2} \mathbb{E}[[\nabla_x \Phi(x^{(k)})]^2] + \frac{4\eta_t}{\alpha_t} \varepsilon_{x,t} + \frac{2\gamma_t}{5\alpha_t} \varepsilon_{y,t} + \frac{8\eta_t+1\alpha_t+1}{S} \sigma_1^2 + \frac{64(1.1L_\Phi + 0.1L_f)2K\gamma_t}{S\alpha_t} \mathbb{E}[[\varepsilon_{y,t}]^2] + \frac{2L_f^2 K \gamma_t}{5S\alpha_t} \mathbb{E}[[\varepsilon_{x,t}]^2]. \]

Recall that \(f(x, y)\) is \(L_f\)-smooth w.r.t. \((x, y)\) and \(\Phi(x)\) is \(L_\Phi\)-smooth w.r.t. \(x.\) Besides, by our choice of \((\alpha_t, \alpha_t, \gamma_t),\) we have
\[ \frac{64(1.1L_\Phi + 0.1L_f)2K\gamma_t}{\alpha_t} \geq \frac{1.1L_\Phi + 0.1L_f}{2}, \quad \text{and} \quad \frac{2L_f^2 K \gamma_t}{5\alpha_t} \leq \frac{L_f}{20}. \] Therefore, the function \(\Phi(x) - \Phi(x^*) + 0.1(\Phi(x) - f(x, y)) + \frac{64(1.1L_\Phi + 0.1L_f)2K\gamma_t}{\alpha_t} \mathbb{E}[[\varepsilon_{x,t}]^2] + \frac{2L_f^2 K \gamma_t}{5\alpha_t} \mathbb{E}[[\varepsilon_{y,t}]^2] \) is convex w.r.t. \((x, y),\) which implies that
\[ \frac{1}{S} \sum_{i \in S_t} \left( \Phi(x^{(k)}) - \Phi(x^*) + 0.1(\Phi(x^{(k)}) - f(z^{(k)})) + \frac{64(1.1L_\Phi + 0.1L_f)2K\gamma_t}{\alpha_t} \mathbb{E}[[\varepsilon_{x,t}]^2] + \frac{2L_f^2 K \gamma_t}{5\alpha_t} \mathbb{E}[[\varepsilon_{y,t}]^2] \right) \geq \Phi(x_{t+1}) - \Phi(x^*) + 0.1((x_{t+1}) - f(z_{t+1})) + \frac{64(1.1L_\Phi + 0.1L_f)2K\gamma_t}{\alpha_t} \mathbb{E}[[\varepsilon_{x,t}]^2] + \frac{2L_f^2 K \gamma_t}{5\alpha_t} \mathbb{E}[[\varepsilon_{y,t}]^2]. \]

Combining (13) and (14) yields
\[ \frac{\eta_t}{2SK} \sum_{i \in S_t} \sum_{k=0}^{K-1} \mathbb{E}[[\nabla_x \Phi(x_{t+k}^{(k)})]^2] \leq \left( \frac{1}{R} \mathbb{E}[\Phi(x_t) - \Phi(x^*)] + \frac{1}{10K} \mathbb{E}[\Phi(x_t) - f(z_t)] + \frac{4\eta_t}{\alpha_t} \varepsilon_{x,t} + \frac{2\gamma_t}{5\alpha_t} \varepsilon_{y,t} + \frac{8\eta_t+1\alpha_t+1}{S} \sigma_1^2 + \frac{4\gamma_t+1\alpha_t+1}{5S} \sigma_2^2 \right) - \left( \frac{1}{R} \mathbb{E}[\Phi(x_{t+1}) - \Phi(x^*)] + \frac{1}{10K} \mathbb{E}[\Phi(x_{t+1}) - f(z_{t+1})] + \frac{4\eta_t+1}{\alpha_t+1} \varepsilon_{x,t+1} + \frac{2\gamma_t+1}{5\alpha_t+1} \varepsilon_{y,t+1} \right). \]
Telescoping the above inequality yields
\[
\frac{1}{SKT} \sum_{t=0}^{T-1} \sum_{i \in S_t} \sum_{k=0}^{K-1} \mathbb{E}[\|\nabla \Phi(x_{t,i}^{(k)})\|^2]
\leq \frac{2\tilde{C}_0}{KT\eta_{t-1}} + 8\eta_0\sigma_2^2 + \frac{4\gamma_0\sigma_2^2}{5S\alpha_0T\eta_{t-1}} + \frac{16\sigma_2^2}{ST\eta_{t-1}} \sum_{i=0}^{T-1} \eta_{t+1} + \frac{8\sigma_2^2}{5ST\eta_{t-1}} \sum_{i=0}^{T-1} \gamma_{t+1}\alpha_{t+1}
\leq O\left(\left(\frac{L_\Phi}{TK} + \frac{L_2^2}{\mu^2} \frac{L_fK}{T} + \frac{L_fK}{S1/3T2/3}\right) \times \left(\tilde{C}_0 + (\sigma_2^2 + \sigma_2^2) \log(T + 1)\right)\right)
= O\left(\left(\frac{L_\Phi}{TK} + \frac{L_2^2}{\mu^2} \frac{1}{T} + \frac{1}{S1/3T2/3}\right) \times \left(\tilde{C}_0 + (\sigma_2^2 + \sigma_2^2) \log(T + 1)\right)\right),
\]
which is the desired result. \(\square\)

C.4 Proofs of the results in Section 4.3
The analysis is similar to that of CD-MAGE. Before we prove Theorem 3, we first present and prove some useful lemmas.

**Lemma 13.** The update rule of CD-MA can be written as
\[
\begin{align*}
\{x_{t+1} &= x_t - \eta_t K \left(\frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t) + e_{x,t}^{(x,t)}\right), \\
y_{t+1} &= y_t + \eta_t K \left(\frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t) + e_{y,t}^{(y,t)}\right),
\end{align*}
\]

where \(e_{x,t}^{(x,t)}\) and \(e_{y,t}^{(y,t)}\) are defined as
\[
\begin{align*}
e_{x,t}^{(x,t)} &= \frac{1}{SK} \sum_{i \in S_t} \sum_{k=0}^{K-1} \left(\nabla_x F_i(x_{t,i}^{(k)}; B_{t,i}^{(k)}) - \frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t)\right), \\
e_{y,t}^{(y,t)} &= \frac{1}{SK} \sum_{i \in S_t} \sum_{k=0}^{K-1} \left(\nabla_y F_i(x_{t,i}^{(k)}; B_{t,i}^{(k)}) - \frac{1}{S} \sum_{i \in S_t} \nabla_y f_i(z_t)\right).
\end{align*}
\]

**Proof.** The primal global variable \(x_{t+1}\) can be written as
\[
x_{t+1} = x_t - \eta_t K \left(\frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t) + \frac{1}{SK} \sum_{i \in S_t} \nabla_x F_i(x_{t,i}^{(k)}; B_{t,i}^{(k)})\right)
= x_t - \eta_t K \left(\frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t) - \frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t) + \frac{1}{SK} \sum_{i \in S_t} \sum_{k=0}^{K-1} \nabla_x F_i(x_{t,i}^{(k)}; B_{t,i}^{(k)})\right)
= x_t - \eta_t K \left(\frac{1}{S} \sum_{i \in S_t} \nabla_x f_i(z_t) + e_{x,t}^{(x,t)}\right).
\]

Similarly, \(y_{t+1} = y_t + \eta_t K \left(\frac{1}{S} \sum_{i \in S_t} \nabla_y f_i(z_t) + e_{y,t}^{(y,t)}\right).\) \(\square\)

**Lemma 14.** Suppose that \(\eta_t, \gamma_t \leq \frac{1}{2KL_f}\) in CD-MA. If the minibatches \(B_{t,i}^{(0)}, \ldots, B_{t,i}^{(K-1)}\) in CD-MA are drawn in a random reshuffling manner and \(K\) is an integer multiple of the epoch length, then both \(\mathbb{E}[\|e_{x,t}^{(x,t)}\|^2]\) and \(\mathbb{E}[\|e_{y,t}^{(y,t)}\|^2]\) are bounded from above by \(\frac{4L_\Phi^2}{3\eta_t^2} K^2 (\mathbb{E}[\|\nabla_x f(z_t)\|^2] + \sigma_2^2 + G_2^2) + \frac{4L_\Phi^2}{3} \gamma_t^2 K^2 (\mathbb{E}[\|\nabla_y f(z_t)\|^2] + \sigma_2^2 + G_2^2).\)

**Proof.** By the definition of \(e_{x,t}^{(x,t)}\), we have
\[
\mathbb{E}[\|e_{x,t}^{(x,t)}\|^2] = \mathbb{E} \left[ \left\| \frac{1}{SK} \sum_{i,k} \left(\nabla_x F_i(x_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)})\right) + \frac{1}{S} \sum_{i \in S_t} \left(\frac{1}{K} \sum_{k=0}^{K-1} \nabla_x F_i(z_t; B_{t,i}^{(k)}) - \nabla_x f_i(z_t)\right) \right\|^2 \right]
\leq \frac{1}{SK} \sum_{i,k} \mathbb{E} \left[ \left\| \nabla_x F_i(x_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x F_i(z_t; B_{t,i}^{(k)}) \right\|^2 \right] \leq \frac{L_2^2}{SK} \sum_{i,k} \mathbb{E}[\|z_{t,i}^{(k)} - z_t\|^2],
\]
where the second equality follows from the condition that the minibatches $B_{t,i}^{(1)}, \ldots, B_{t,i}^{(K-1)}$ in CD-MA are drawn in a random reshuffling manner and $K$ is an integral multiple of the epoch length. Similarly, we have $\mathbb{E}[\|y_{t,i}\|^2] \leq \frac{L^2}{2K} \sum_{i,k} \mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2]$.

If $k \geq 1$, then
\[
\mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2] = \frac{1}{T} \sum_{i=0}^{T} \mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2] + \gamma_k^2 \mathbb{E}[\|\sum_{i=0}^{T} \nabla_y f_i(z_{t,i}^{(k)}; B_{t,i}^{(k)})\|^2].
\] (15)

The first term on the RHS of (15) can be bounded by
\[
\mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_x f_i(z_{t,i}^{(k)}; B_{t,i}^{(k)})\right\|^2\right] 
\leq 2 \mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \left(\nabla_x f_i(z_{t,i}^{(k)}; B_{t,i}^{(k)}) - \nabla_x f_i(z_{t,i})\right)\right\|^2\right] + 2 \mathbb{E}\left[\left\|\nabla_x f_i(z_{t,i})\right\|^2\right].
\]

Recall that $\nabla_x f_i(z_{t,i}; B_{t,i}^{(k)}) = \frac{1}{B} \sum_{\zeta \in B_{t,i}^{(k)}} \nabla_x f_i(z_{t,i}; \zeta)$ and the minibatches $B_{t,i}^{(0)}, \ldots, B_{t,i}^{(K-1)}$ are drawn in a random reshuffling manner. By Assumption 4 and (Mishchenko, Khaled Ragab Bayoumi, and Richtárik 2020, Lemma 1), we can bound
\[
\mathbb{E}\left[\left\|\nabla_x f_i(z_{t,i})\right\|^2\right] 
\leq \frac{n_i - 1 - \text{mod}(kB - 1, n_i)}{(\text{mod}(kB - 1, n_i) + 1)(n_i - 1)} G_i^2 \leq G_i^2.
\]

Plugging the above inequality into (16) yields
\[
\mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_x f_i(z_{t,i}^{(k)}; B_{t,i}^{(k)})\right\|^2\right] 
\leq 2kL_f^2 \mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_x f_i(z_{t,i}^{(k)}) - \nabla_x f_i(z_{t,i})\right\|^2\right] + 2k^2G_i^2
\]
\[
\leq 2kL_f^2 \mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_x f_i(z_{t,i}^{(k)}) - \nabla_x f_i(z_{t,i})\right\|^2\right] + 2k^2G_i^2
\]
\[
\leq 2kL_f^2 \mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_x f_i(z_{t,i}^{(k)}) - \nabla_x f_i(z_{t,i})\right\|^2\right] + 2k^2G_i^2.
\]

where the last inequality follows from Assumption 1. Similarly, we can show that
\[
\mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_y f_i(z_{t,i}^{(k)}; B_{t,i}^{(k)})\right\|^2\right] 
\leq 2kL_f^2 \mathbb{E}\left[\left\|\sum_{k=0}^{T-1} \nabla_y f_i(z_{t,i}^{(k)}) - \nabla_y f_i(z_{t,i})\right\|^2\right] + 2k^2G_i^2.
\]

Combining the above two inequalities, we have
\[
\mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2] \leq 2kL_f^2 (\sigma_k^2 + \gamma_k^2) \mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2] + 2n_i^2k^2 \mathbb{E}[\|\nabla_x f_i(z_{t,i})\|^2] + 2\gamma_k^2k^2 \mathbb{E}[\|\nabla_y f_i(z_{t,i})\|^2]
\]
\[ + 2n_i^2k^2(\sigma_k^2 + G_i^2) + 2\gamma_k^2k^2(\sigma_k^2 + G_i^2).
\]

Summing the above inequality from $k = 1$ to $k = K - 1$ yields
\[
\sum_{k=0}^{K-1} \mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2] \leq L_f^2 K(K - 1)(\eta_k^2 + \gamma_k^2) \sum_{k=0}^{K-1} \mathbb{E}[\|z_{t,i}^{(k)} - z_{t,i}\|^2] + \frac{n_i^2}{3} (K - 1)K(2K - 1) \mathbb{E}[\|\nabla_x f_i(z_{t,i})\|^2] + \sigma_k^2 + G_i^2
\]
\[ + \gamma_k^2(2K - 1)(\mathbb{E}[\|\nabla_y f_i(z_{t,i})\|^2] + \sigma_k^2 + G_i^2).
\]
Plugging the condition \( \eta_t, \gamma_t \leq 1/(2L_f K) \) into the above inequality and rearranging terms, we obtain
\[
\frac{1}{K} \sum_{k=0}^{K-1} \mathbb{E}[||x_{i,t}^{(k)} - z_t||^2] \leq \frac{4}{3} \eta_t^2 K^2 (\mathbb{E}[||\nabla_x f(z_t)||^2] + \sigma_1^2 + G_1^2) + \frac{4}{3} \gamma_t^2 K^2 (\mathbb{E}[||\nabla_y f(z_t)||^2] + \sigma_2^2 + G_2^2).
\]
Summing the above inequality for \( i \in S_t \) leads to the desired result. \( \Box \)

**Proof of Theorem 3**

Proof. The proof is similar to that of Theorem 1. Denote \( \gamma_t = \gamma := \min \left\{ \sqrt{\frac{20L_0 S}{L_f T \sigma_1^2}}, \left( \frac{3L_0}{L_f (\sigma_2^2 + G^2 T) K^2} \right)^{1/3}, \frac{1}{8\mu L_f K} \right\} \) and \( \eta_t = \eta := \min \left\{ \sqrt{\frac{20L_0 S}{(L_f + L_f) K^2 \sigma_1^2}}, \frac{3L_0}{L_f (\sigma_2^2 + G^2 T) K^2} \right\} \). We note that Lemma 4 and Lemma 7 also apply to CD-MA with \( e_{x,t} \) and \( e_{y,t} \) replaced by \( e_{x,t}^\prime \) and \( e_{y,t}^\prime \), respectively. Following the same argument as (10), we have
\[
\mathbb{E}[\Phi(x_{t+1}) - \Phi(x^*)] \leq \left( \frac{1}{20} \mathbb{E}[\Phi(x_t) - f(x_t, y_t)] - \frac{21\eta K}{40} \mathbb{E}[||\nabla f(z_t)||^2] \right) \left( \frac{1}{20} \mathbb{E}[\Phi(x_t) - f(x_t, y_t)] - \frac{21\eta K}{40} \mathbb{E}[||\nabla f(z_t)||^2] \right)
\]
where the second inequality follows from Lemma 6 and the last inequality follows from the conditions \( \eta_t \leq \min \left\{ \frac{1}{36L_f K}, \frac{4L_0}{10L_f K}, \frac{\mu^2 L_f}{21L_f^2} \right\} \) and \( \gamma_t \leq \frac{1}{8\mu L_f K} \). Recursively applying the above inequality yields
\[
\mathbb{E}[\Phi(x_T) - \Phi(x^*)] \leq L_0 - \frac{21\eta K}{40} \sum_{t=0}^{T-1} \mathbb{E}[||\nabla f(z_t)||^2] + \frac{21\eta^2 K^2 T}{20S} (L_\Phi + L_f) \sigma_1^2 + (\eta^2 K^3 L_f^3 + \frac{\gamma_\eta^2 K^3 L_f^3}{30})(\sigma_1^2 + G_1^2)T
\]

\[
+ \frac{21\eta^2 K^2 T}{20S} \sigma_2^2 + (\eta^2 K^3 L_f^3 + \frac{\gamma_\eta^2 K^3 L_f^3}{30})(\sigma_2^2 + G_2^2)T.
\]
Plugging the values of \( \eta \) and \( \gamma \) to the above inequality and rearranging terms, we arrive at

\[
\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}[\|\nabla \Phi(x_t)\|^2] \\
\leq \frac{40}{21\eta K T} \mathcal{L}_0 + \frac{2\eta K (L_f + L_f)}{S} \sigma_1^2 + \frac{2\gamma^2 K L_f}{21 \eta S} \sigma_2^2 + \left( \frac{40 \eta^2 K^2 L_f^2}{21} + \frac{4 \gamma^2 K^2 L_f^2}{63} \right) (\sigma_1^2 + G_1^2) \\
+ \left( \frac{4 \gamma^2 K^2 L_f^2}{21} + \frac{4 \gamma^3 K^2 L_f^2}{63 \eta} \right) (\sigma_2^2 + G_2^2) \\
\leq (36 L_f + 4 \Phi) \frac{40 \mathcal{L}_0}{7T} + \frac{2 L_{12}^2 L_f \sigma_2^2}{\mu^2 S} \gamma K + \frac{4 L_{12}^2 L_f^2}{3 \mu^2} (\sigma_2^2 + G_2^2)^2 \gamma^2 K^2 + 8 \sqrt{\frac{5 \mathcal{L}_0 (L_f + \Phi) \sigma_1^2}{7 T}} \\
+ \frac{3}{10} \left( \frac{\mathcal{L}_0 L_f (\sigma_2^2 + G_2^2)}{(\sigma_2^2 + G_2^2)^{1/2} T} \right)^{2/3} + 8 \left( \frac{\mathcal{L}_0 L_f (\sigma_1^2 + G_1^2)^{1/2}}{T} \right)^{2/3} + 37 \left( \frac{\mathcal{L}_0 L_f (\sigma_2^2 + G_2^2)^{1/2}}{T} \right)^{2/3} \\
\leq \frac{160 (9 L_f + \Phi) \mathcal{L}_0}{7 T} + 7 \sigma_1 \sqrt{\frac{\mathcal{L}_0 (L_f + \Phi) \sigma_1^2}{ST}} + \frac{9 L_{12}^2 \sigma_2^2}{\mu^2} \sqrt{\frac{\mathcal{L}_0 L_f}{ST}} \\
+ \left( 8 (\sigma_1^2 + G_1^2)^{1/3} + \frac{3 (\sigma_1^2 + G_1^2)^{2/3}}{10 (\sigma_2^2 + G_2^2)^{1/3}} + (37 + \frac{13 L_{12}^2}{\mu^2}) (\sigma_2^2 + G_2^2)^{1/3} \right) \left( \frac{\mathcal{L}_0 L_f}{T} \right)^{2/3} ,
\]

which is the desired result. \( \square \)