ON CONNECTION BETWEEN THE NUMBERS OF PERMUTATIONS AND FULL CYCLES WITH SOME RESTRICTIONS ON POSITIONS AND UP-DOWN STRUCTURE

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Abstract. We discuss both simple and more subtle connections between the numbers of permutations and full cycles with some restrictions, in particular, between the numbers of permutations and full cycles with prescribed up-down structure.

1. Introduction

The following theorem is very well known.

Theorem 1. (cf. e.g., [6]). The number of full cycles of \( n \) elements \( 1, 2, \ldots, n \) equals to \((n - 1)!\) i.e. the number of all permutations of \( n - 1 \) elements \( 1, 2, \ldots, n - 1 \).

Let \( A \) be a quadratic (0,1)-matrix of order \( n \). For a permutation \( \sigma \) of elements \( 1, 2, \ldots, n \) denote \( H_\sigma \) the incidence (0,1)-matrix of \( \sigma \). Let us consider the set \( B(A) \) of permutations \( \sigma \) for which \( H_\sigma \leq A \). \( B(A) \) is the class of permutations with restriction on positions which is defined by zeros of the matrix \( A \). It is well known that \(|B(A)| = \text{per} A\). Furthermore, let us consider a matrix function which is defined in a similar way as the permanent by the formula

\[
\text{perf } A = \sum_{\sigma}\sum_{i=1}^{n}a_{i,\sigma(i)}
\]

where the external sum is over all full cycles of elements \( 1, 2, \ldots, n \). In particular,

\[
\text{perf}(a_{11}) = a_{11},
\]

\[
\text{perf} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{12}a_{21},
\]
From our results\cite{7} where we considered a system of "partial permanents" including \( \text{perf}\, \mathbf{A} \) with notation \( \text{pper} \mathbf{A}_1 \), follows an expansion of \( \text{perf}\, \mathbf{A} \) by the first row of \( n \times n \) quadratic matrix \( \mathbf{A} \).

**Theorem 2.** \( \text{perf}\, \mathbf{A} = \sum_{j=2}^{n} a_{1j} \text{perf}\, \mathbf{A}_{1,j}^* \), where \( \mathbf{A}_{1,j} \) is obtained from \( \mathbf{A} \) by deletion of the first row and the \( j \) –th column and \( \mathbf{A}_{1,j}^* \) is obtained from \( \mathbf{A}_{1,j} \) by the permutation of its first \( j-1 \) columns by the rule:

\[
i \rightarrow i + 1 \quad (\mod (j-1)), \quad i = 1, 2, \ldots, j - 1,
\]

with the positive minimal residues modulo \( j - 1 \).

For example,

\[
\text{perf}\begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix} = a_{12} \text{perf}\begin{pmatrix}
a_{21} & a_{23} & a_{24} \\
a_{31} & a_{33} & a_{34} \\
a_{41} & a_{43} & a_{44}
\end{pmatrix} + \\
+ a_{13} \text{perf}\begin{pmatrix}
a_{22} & a_{21} & a_{24} \\
a_{32} & a_{31} & a_{34} \\
a_{42} & a_{41} & a_{44}
\end{pmatrix} + a_{14} \text{perf}\begin{pmatrix}
a_{22} & a_{23} & a_{21} \\
a_{32} & a_{33} & a_{31} \\
a_{42} & a_{43} & a_{41}
\end{pmatrix}.
\]

The algorithm of Theorem 2 is easily realized by computer. The observations obtained using this algorithm allowed us to formulate the following conjecture (1993) which is a generalization of Theorem 1.

**Conjecture 1**\cite{8}. \( P_n^{(i)} \) be quadratic \((0,1)\) matrix of order \( n \) with only \( 1 \)’s on places \((1,1+i), (2,2+i), \ldots, (n-i,n), \) \( 0 \leq i \leq n-1 \), \( J_n \) be \( n \times n \) matrix composed of \( 1 \)’s. Then

\[
(2) \quad \text{perf}(J_n - \sum_{j=1}^{k} P_n^{(j)}) = \text{perf}(J_{n-1} - \sum_{i=0}^{k-1} P_n^{(i)}).
\]

It is clear that in the case of \( k = 0 \) we obtain Theorem 1 in the form
On connection between the numbers of permutations

(3) \[ \operatorname{perf}(J_n) = \operatorname{perf}(J_{n-1}) = (n-1)!. \]

In 1994, our conjecture was proved independently by Ira M. Gessel using ideals of his paper [1] and Richard P. Stanley which gave a direct proof (private correspondences, unpublished).

In this paper we discuss quite another intriguing connections between the numbers of permutations and full cycles with prescribed up-down structure.

2. On up-down basis polynomials

Basis polynomial with up-down index \( k \), denoted by \( \{ n \atop k \} \) is the number of permutations \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \) of elements 1, 2, \ldots, \( n \) with the condition \( \pi_1 < \pi_{i+1} (\pi_i > \pi_{i+1}) \), \( i = 1, 2, \ldots, n-1 \), if in the \((n-1)\)-digit binary expansion of \( k \) the \( i \)-th digit equals to zero (unit) [10].

Let \( k \in [2^{t-1}, 2^t) \) and the \((n-1)\)-digit binary expansion of \( k \) has a form:

(4) \[ k = 0 \ldots 0 1 \underbrace{0 \ldots 0}_{n-t-1} 1 \underbrace{0 \ldots 0}_{s_2-s_1-1} 1 \underbrace{0 \ldots 0}_{s_3-s_2-1} \ldots 1 \underbrace{0 \ldots 0}_{s_m-s_{m-1}-1} 1 \underbrace{0 \ldots 0}_{t-s_m} \]

where

\[ 1 = s_1 < s_2 < \ldots < s_m \]

are places of 1’s after \( n-t-1 \) 0’s before the first 1.

In [9] using the fundamental Niven’s result [5] the following formula was proved.

**Theorem 3.**

(5) \[
\left\{ n \atop k \right\} = (-1)^m + \sum_{p=1}^{m} (-1)^{m-p} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq m} \left( \begin{array}{c} n \\ t + 1 - s_{i_p} \end{array} \right) \prod_{r=2}^{p} \left( s_{i_r} - s_{i_{r-1}} - 1 \right).
\]

Let us write (4) in the form

(6) \[ k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}, \ t_1 > t_2 > \ldots > t_m \geq 1. \]

Comparing (4) and (6) we find

(7) \[ t_i = t - s_i + 1, \ i = 1, 2, \ldots, m. \]
It is easy to check directly the following identity

\[
\binom{n}{t_{ip}} \prod_{r=2}^{p} \left( \binom{n-t_{ir}}{t_{ir} - t_{ir-1}} \right) = \left( \binom{n}{t_1} \right) \left( \binom{t_1}{t_2} \right) \left( \binom{t_2}{t_3} \right) \ldots \left( \binom{t_{ip-1}}{t_{ip}} \right).
\]

Now by (5), (7) and (8) we obtain \( \binom{n}{k} \) as a linear combinations of binomial coefficients.

**Theorem 4.** (10). For \( k \) (6) we have

\[
\binom{n}{k} = (-1)^m + \sum_{p=1}^{m} (-1)^{m-p} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq m} \binom{n}{t_{i_1}} \prod_{r=2}^{p} \binom{t_{i_r-1}}{t_{i_r}}.
\]

Below, as in [10] we consider \( \binom{n}{k} \) from the formal (wider than only combinatorial) point of view: according to (9) it is a polynomial in \( n \) of degree \( t_1 = \lfloor \log_2(2k) \rfloor \). In particular,

\[
\binom{0}{k} = (-1)^m = \tau_k,
\]

where \( \tau_k, \ k = 0, 1, 2, \ldots \), is the Thue-Morse sequence [4], [2].

The following theorem is equivalent to Theorem 10 [10]. Here we give a more detailed proof of this theorem.

**Theorem 5.** For \( k \) (7) we have

\[
\binom{n}{k} = \left| \begin{array}{cccccccc}
(1) & (2) & (3) & \cdots & (m-1) & (m) & 1 \\
1 & (1) & (2) & \cdots & (m-1) & (m) & 1 \\
0 & 1 & (2) & \cdots & (m-1) & (m) & 1 \\
0 & 0 & 1 & \cdots & (m-1) & (m) & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 & (m-1) & 1 \\
0 & 0 & 0 & \cdots & 0 & (m) & 1 \\
\end{array} \right|.
\]
**Proof.** The number of diagonals of matrix \( \Pi \) having no 0’s equals to permanent of the following \((m+1) \times (m+1)\) matrix

\[
C_{m+1} = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 \\
0 & 1 & 1 & \ldots & 1 & 1 & 1 \\
0 & 0 & 1 & \ldots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 1 & 1 \\
0 & 0 & 0 & \ldots & 0 & 1 & 1
\end{pmatrix}.
\]

Decomposing \( \text{per} C_{m+1} \) by the last row we find

\[
\text{per} C_{m+1} = 2 \text{per} C_m = 2^2 \text{per} C_{m-1} = \ldots = 2^m.
\]

Denote \( A \) \((m+1) \times (m+1)\) matrix \( \Pi \) and consider \((m \times m)\) upper-triangle submatrix \( T \) with the main diagonal composed of 1’s. Let us choose \( p \) 1’s of the main diagonal of \( T \) in its rows \((1 \leq i_1 < i_2 < \ldots < i_p \leq m)\). To such choice corresponds a diagonal of \( A \) composed of the other \((m-p)\) 1’s of the main diagonal of \( T \) + the unit in the last column of \( A \) which is the continuation of the \( i_p \)-th row of \( T \) + elements \( \binom{n}{t_{i_1}}, \binom{t_{i_2}}{t_{i_2}}, \ldots, \binom{t_{i_p-1}}{t_{i_p}} \) such that in all we have \( m - p + 1 + p = m + 1 \) elements of \( A \) which are in different rows and columns. As a result, we obtain

\[
\sum_{p=0}^{m} \binom{m}{p} = 2^m \text{ i.e. all diagonals of } A \text{ having no 0’s (note that, to } p = 0 \text{ corresponds the choice of the empty subset of 1’s of the main diagonal of } T, \text{ i.e. all these 1’s and the unit in the first row of } A \text{ form in this case the only diagonal of 1’s)}.\]

Therefore,

\[
\text{per } A = 1 + \sum_{p=1}^{m} \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq m} \binom{n}{t_{i_1}} \prod_{r=2}^{p} \binom{t_{i_{r-1}}}{t_{i_r}}
\]

and according to \( \Pi \) it is left to notice that in the chosen diagonals the number of transpositions equals to \( m - p \), \( p = 0, 1, \ldots, m \), such that

\[
\binom{n}{k} = \det A. \blacksquare
\]

Many different properties of \( \binom{n}{k} \) were proved in \( \Pi \). Let us prove an additional interesting property.
Theorem 6. (cf. our comment to sequence A060351 [11]).

If \( k \equiv 0 \pmod{4} \) then

\[
\begin{array}{l}
\sum_{i=0}^{m} \binom{n}{t_i} = 0
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k} - \binom{n}{k+1} + \binom{n}{k+2} - \binom{n}{k+3} = 0
\end{array}
\]

\[
\begin{array}{l}
k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}, \ t_1 > t_2 > \ldots > t_m \geq 3
\end{array}
\]

and by (11)

\[
\begin{array}{l}
\binom{n}{k+3} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k+2} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k+1} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]

Proof. By the condition,

\[
\begin{array}{l}
k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}, \ t_1 > t_2 > \ldots > t_m \geq 3
\end{array}
\]  

and by (11)

\[
\begin{array}{l}
\binom{n}{k+3} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k+2} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k+1} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]  

\[
\begin{array}{l}
\binom{n}{k} = \binom{n}{t_1} \binom{n}{t_2} \ldots \binom{n}{t_m} \binom{n}{2} \binom{n}{1}
\end{array}
\]
Furthermore,

\[
\left\{ \begin{array}{c}
\binom{n}{k + 2} = - \binom{n}{k + 2} \\
\end{array} \right.
\]

\[
\begin{array}{c}
\left( \begin{array}{ccccccc}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{n}{1} \\
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 1 & t_{m-1} & t_{m-1} \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{array} \right) + \\
\left( \begin{array}{ccccccc}
\binom{n}{t_1} \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} \binom{n}{t_m} & \binom{n}{1} \\
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 1 & t_{m-1} & t_{m-1} \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{array} \right)
\]

(13) + 2

Furthermore,

\[
\left\{ \begin{array}{c}
\binom{n}{k + 2} \\
\end{array} \right.
\]

\[
\begin{array}{c}
\left( \begin{array}{ccccccc}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{n}{2} \\
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 1 & t_{m-1} & t_{m-1} \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{array} \right) = \\
\left( \begin{array}{ccccccc}
\binom{n}{t_1} \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} \binom{n}{t_m} & \binom{n}{2} \\
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & \cdots & 1 & t_{m-1} & t_{m-1} \\
0 & 0 & \cdots & 0 & 1 & 1 \\
\end{array} \right)
\]


\[
\begin{align*}
\left( \binom{n}{k} \right) &= - \left\{ \binom{n}{k} \right\} + \left( \begin{array}{cccccc}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{n}{2} \\
t_1 & t_2 & \cdots & t_{m-1} & t_m & 1 \\
1 & t_1 & \cdots & t_{m-1} & t_m & \binom{1}{2} \\
0 & 1 & \cdots & t_{m-1} & t_m & \binom{1}{2} \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & \binom{t_{m-1}}{t_m} & \binom{t_{m-1}}{t_m} \\
0 & 0 & \cdots & 0 & 1 & \binom{t_m}{1}
\end{array} \right). \\
\end{align*}
\]

Thus, by (13) and (14) we have

\[
\left\{ \binom{n}{k+3} \right\} = \left\{ \binom{n}{k} \right\} - \left( \begin{array}{cccccc}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{1}{1} \\
t_1 & t_2 & \cdots & t_{m-1} & t_m & 1 \\
1 & t_1 & \cdots & t_{m-1} & t_m & \binom{1}{1} \\
0 & 1 & \cdots & t_{m-1} & t_m & \binom{1}{1} \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & \binom{t_{m-1}}{t_m} & \binom{t_{m-1}}{1} \\
0 & 0 & \cdots & 0 & 1 & \binom{t_m}{1}
\end{array} \right) + \left( \begin{array}{cccccc}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{n}{2} \\
t_1 & t_2 & \cdots & t_{m-1} & t_m & \binom{2}{1} \\
1 & t_1 & \cdots & t_{m-1} & t_m & \binom{2}{1} \\
0 & 1 & \cdots & t_{m-1} & t_m & \binom{2}{1} \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & \binom{t_{m-1}}{t_m} & \binom{t_{m-1}}{t_m} \\
0 & 0 & \cdots & 0 & 1 & \binom{t_m}{2}
\end{array} \right). 
\]
at last,

\[
\begin{align*}
\left\{ \binom{n}{k+1} \right\} &= \begin{vmatrix}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{n}{1} \\
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & \cdots & t_2 & t_m & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 & t_m \\
0 & 0 & \cdots & 0 & 0 & 1 \\
\end{vmatrix} = \\
\begin{vmatrix}
\binom{n}{t_1} & \binom{n}{t_2} & \cdots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & \binom{n}{1} \\
1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & \cdots & t_2 & t_m & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 1 & t_m \\
0 & 0 & \cdots & 0 & 0 & 1 \\
\end{vmatrix}
\end{align*}
\]

(16) = \left\{ \binom{n}{k} \right\}

Subtracting (16) from (14) and after that subtracting the result from (15) we find

\[
\left\{ \binom{n}{k+3} \right\} - \left( \left\{ \binom{n}{k+2} \right\} - \left\{ \binom{n}{k+1} \right\} \right) = \left\{ \binom{n}{k} \right\}
\]

and (12) follows. ■

Note that in particular we have

\[
\sum_{k=0}^{2^{r-1}} (-1)^k \binom{n}{k} = 0, \quad r \geq 2.
\]

Taking into account that (10)

\[
\sum_{k=0}^{2^{r-1}} \binom{n}{k} = n(n-1)\cdots(n-r+1), \quad 1 \leq r \leq n-1,
\]
we find as well that

\begin{equation}
\sum_{i=0}^{2^r-1} \binom{n}{2i} = \sum_{i=0}^{2^r-1} \binom{n}{2i+1} = \frac{1}{2} n(n-1) \ldots (n-r+1), \quad 2 \leq r \leq n-1,
\end{equation}

and, in particular,

\begin{equation}
\sum_{i=0}^{2^{n-1}-1} \binom{n}{2i} = \sum_{i=0}^{2^{n-1}-1} \binom{n}{2i+1} = \frac{n!}{2}, \quad n \geq 3.
\end{equation}

3. Main conjecture

Denote by \( \langle n \rangle \) the number of all full cycles which have up-down index \( k \). Many observations show that \( \langle n \rangle \approx \frac{1}{n} \binom{n}{k} \) with highly good approximation. Moreover, we think that the following conjecture is true.

**Conjecture 2.** Let \( t_1 = t_1(k) \) be defined by (6). If all the divisors of \( n \geq 3 \), which are different from 1, are larger than \( t_1 \) then exactly

\begin{equation}
\frac{1}{n} \left( \binom{n}{k} - 0 \right) = \frac{1}{n} \left( \binom{n}{p} - 1 - (-1) \right) = \frac{(n-1)(n-2) \ldots (n-p+1)}{p!},
\end{equation}

where according to (10), \( \binom{0}{k} = \tau_k \).

Note that, if the conditions of Conjecture 2 are not satisfied then, generally speaking, the fraction in (19) is not an integer. E.g., if \( k = 2^{p-1} \) then \( t_1 = p \) and if despite of the conditions of Conjecture 2, \( p|n \) then using (11) we have

\[ \frac{1}{n} \left( \binom{n}{2^{p-1}} - \binom{0}{2^{p-1}} \right) = \frac{1}{n} \left( \binom{n}{p} - 1 - (-1) \right) = \frac{(n-1)(n-2) \ldots (n-p+1)}{p!}. \]
ON CONNECTION BETWEEN THE NUMBERS OF PERMUTATIONS

The latter fraction is not integer for \( p | n \). On the other hand, in the conditions of Conjecture 2 the fraction in (19) is an integer. Indeed, from (11) we find

\[
\binom{n}{k} - \binom{0}{k} = \begin{vmatrix}
\begin{array}{cccccc}
(n) & (n) & (n) & \ldots & (n) & (n) \\
t_1 & t_2 & t_3 & \ldots & t_{m-1} & t_m \\
1 & 1 & 1 & \ldots & 1 & 1 \\
0 & 0 & 0 & \ldots & 1 & t_{m-1} \\
0 & 0 & 0 & \ldots & 0 & 1 \\
& \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\end{vmatrix}_0 = (-1)^m \tau_k = \binom{0}{k}.
\]

Matrix (20) differs from matrix (11) only in the last element of the first row. To this element corresponds the only diagonal in matrix (11) composed of 1's. The corresponding term in determinant (11) is

\[-1 \cdot t_m^{p-1} = (-1)^m \tau_k = \binom{0}{k}.\]

In the conditions of Conjecture 2 all elements of the first row of matrix (20) are divided by \( n \). Therefore, the fraction in (19) is an integer. This property of \( \binom{n}{k} \) as a matter of fact was known since 1996 [9].

As a corollary of these arguments we obtain, e.g., the following statement.

**Theorem 7.** If Conjecture 2 is true then for a fixed prime \( p \) the sequence \( \left( \binom{n}{2^{p-1}} \right)_{n \geq 0} \) is not a polynomial sequence.

**Proof.** Let the sequence \( \left( \binom{n}{2^{p-1}} \right)_{n \geq 0} \) be \( P(n) \)-polynomial. By (20) the fraction \( \frac{1}{n} \left( \binom{n}{2^{p-1}} - \binom{0}{2^{p-1}} \right) \) is a polynomial \( Q(n) \) of degree \( p - 1 \). If \( n \) assumes prime values larger than \( p \) then the values

\[
\binom{n}{2^{p-1}} = \frac{1}{n} \left( \binom{n}{2^{p-1}} - \binom{0}{2^{p-1}} \right)
\]
by Conjecture 2 deduce the identity $P(n) \equiv Q(n)$, while as we saw for values of $n$ which are multiples of $p$, $Q(n)$ is not an integer. Therefore, $P(n) \equiv Q(n)$ does not equal to $\left\langle \frac{n}{2^{p-1}} \right\rangle$.

Furthermore, in connection with Theorem 6 note that if Conjecture 2 is true then in its conditions for $k \equiv 0 \pmod{4}$ we also have

\begin{equation}
\left\langle \frac{n}{k} \right\rangle - \left\langle \frac{n}{k+1} \right\rangle + \left\langle \frac{n}{k+2} \right\rangle - \left\langle \frac{n}{k+3} \right\rangle = 0, \quad n \geq 3.
\end{equation}

In particular, (21) is true for $n$ being an odd prime.

4. An analog of sequence A360651 for full cycles

Put for $k \geq 1$

\begin{equation}
g(k) = \lfloor \log_2 k \rfloor + 1, \quad h(k) = k - 2^{g(k)-1}.
\end{equation}

Sequence A360051 [11] is the sequence

\begin{align*}
\{1\} & ; \{2\} ; \{2\} ; \{3\} ; \{3\} ; \{3\} ; \{3\} , \\
\{4\} & ; \{4\} ; \{4\} ; \{4\} ; \{4\} ; \{4\} ; \{4\} ; \{4\} ,
\end{align*}

i.e. the sequence [10]

\begin{equation}
\left( \begin{array}{c}
g(k) \\
h(k)
\end{array} \right)_{k+1}.
\end{equation}

The most simple algorithm for evaluation of $\left\{ \frac{n}{k} \right\}$ is the following recursion which is directly obtained from Theorem 16 [10].

**Theorem 8.** We have

\begin{equation}
\left\{ \frac{n}{k} \right\} = \left\{ \frac{g(k)}{h(k)} \right\} \left( \frac{n}{g(k)} \right) - \left\{ \frac{n}{h(k)} \right\}, \quad k \geq 1
\end{equation}

with the initial condition $\left\{ \frac{n}{0} \right\} = 1$. 
An analog of sequence A360051 for full cycles is the sequence
\[
\langle 1 \rangle_0, \langle 2 \rangle_0, \langle 2 \rangle_1, \langle 3 \rangle_0, \langle 3 \rangle_1, \langle 3 \rangle_2, \langle 3 \rangle_3, \ldots \\
\langle 4 \rangle_0, \langle 4 \rangle_1, \langle 4 \rangle_2, \langle 4 \rangle_3, \langle 4 \rangle_4, \langle 4 \rangle_5, \langle 4 \rangle_6, \langle 4 \rangle_7, \ldots,
\]
(26)
i.e. the sequence
\[
\left( \frac{g(k)}{h(k)} \right)_{k=1}^{\infty},
\]
(27)
We take \( \langle 1 \rangle_0 = 1 \). Further, for \( n = 2 \) the only cycle is \((2, 1)\) with \( k = 0 \), i.e. \( \langle 2 \rangle_0 = 1, \langle 2 \rangle_1 = 0 \). Note that, for \( n \geq 3 \) \( \langle n \rangle_0 = 0 \), since the only permutation corresponding to this case, has more than one cycle.
Furthermore, similar to \( \left\{ \binom{n}{k} \right\}_1^{10} \) one can prove that for \( n \geq 3 \)
\[
\binom{n}{k} = \binom{2^{n-1} - 1 - k}{n}, \quad k \in [0, 2^{n-2} - 1].
\]
(28)
In particular, each block of sequence (26) begins and ends with 0:
\[
\binom{2^{n-1} - 1}{n} = \binom{n}{0} = 0, \quad n \geq 3.
\]
(29)
Note that the conditions of Conjecture 2 are satisfied for a whole block
\[
\left( \binom{n}{k} \right)_{k=0}^{2^{n-1} - 1}
\]
if and only if \( n \) is an odd prime. For example, for \( n = 3 \) we have only two full cycles \((2,3,1)\) and \((3,1,2)\) with \( k = 2 \) and \( k = 1 \) correspondingly. Thus, this block in (26) has the form: 0, 1, 1, 0. This conforms to (20). Indeed,
\[
\langle 3 \rangle_1 = \frac{1}{3} \begin{vmatrix} 3 \\ 1 \end{vmatrix} 1 = 1, \quad \langle 3 \rangle_2 = \frac{1}{3} \begin{vmatrix} 3 \\ 2 \end{vmatrix} 1 = 1,
\]
\[
\begin{pmatrix}
3 \\
3
\end{pmatrix} = \frac{1}{3} \begin{pmatrix}
\binom{3}{2} & \binom{3}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 0.
\]

For \( n = 5 \) we have the only cycle \((5, 4, 2, 1, 3)\) with \( k = 1 \), two cycles \((4, 3, 1, 5, 2)\) and \((5, 4, 1, 3, 2)\) with \( k = 2 \), only cycle \((5, 3, 1, 2, 4)\) with \( k = 3 \), two cycles \((4, 1, 5, 3, 2)\) and \((4, 3, 5, 2, 1)\) with \( k = 4 \), three cycles \((3, 1, 5, 2, 4), (5, 1, 4, 2, 3)\) and \((5, 3, 4, 2, 1)\) with \( k = 5 \), two cycles \((3, 1, 4, 5, 2)\) and \((4, 1, 2, 5, 3)\) with \( k = 6 \), only cycle \((5, 1, 2, 3, 4)\) with \( k = 7 \) and the numbers of cycles with \( k = 8, 9, \ldots, 14 \) conform to \( (28) \). They are:

\[
(3, 5, 4, 2, 1); (2, 5, 4, 1, 2), (3, 5, 2, 1, 4); (2, 4, 1, 5, 3), (3, 4, 2, 5, 1),
(4, 5, 2, 3, 1); (2, 5, 1, 3, 4), (4, 5, 1, 2, 3); (2, 4, 5, 3, 1); (3, 4, 5, 1, 2),
(2, 3, 5, 1, 4); (2, 3, 4, 5, 1).
\]

Thus, this block in \( (26) \) has the form: \( 0, 1, 2, 1, 2, 3, 2, 1, 1, 2, 3, 2, 1, 2, 1, 0 \). This conforms to \( (19), (20) \) and \( (21) \). Indeed,

\[
\begin{pmatrix}
5 \\
1
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
\binom{5}{2} & \binom{5}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 1, \quad \begin{pmatrix}
5 \\
2
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
\binom{5}{2} & \binom{5}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 2,
\]

\[
\begin{pmatrix}
5 \\
3
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
\binom{5}{2} & \binom{5}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 1, \quad \begin{pmatrix}
5 \\
4
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
\binom{5}{3} & \binom{5}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 2,
\]

\[
\begin{pmatrix}
5 \\
5
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
\binom{5}{3} & \binom{5}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 1, \quad \begin{pmatrix}
5 \\
6
\end{pmatrix} = \frac{1}{5} \begin{pmatrix}
\binom{5}{3} & \binom{5}{1} & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix} = 2.
\]
At the same time, for \( n = 4 \) which is not a prime, (20) is satisfied, generally speaking, only approximately. Indeed, here we have only permutations with indices
\[
k = 1, 2, \ldots, 6 : (4, 3, 1, 2), (3, 1, 4, 2), (4, 1, 2, 3)(3, 4, 2, 1), (2, 4, 1, 3), (2, 3, 4, 1)
\]
correspondingly. I.e. this block in (26) hast the form 0, 1, 1, 1, 1, 1, 0, while according to (20) we have 0, 1, \( \frac{3}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2} \).

This, the first numbers of sequence (26) are:

\[
(30) \quad 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 2, 1, 2, 3, 2, 1, 1, 2, 3, 2, 1, 2, 1, 0, 0, \ldots
\]

5. SOME OTHER OPEN PROBLEMS

1. It is very interesting to estimate the remainder term of approximation (19) in the general case.

2. Let in the block "n" \((n \geq 3)\) in (26) for every \( k \in [0, 2^{n-1} - 1] \) which is divided by 4, (21) be satisfied. We conjecture that in this case \( n \) is a prime.

3. It is known that the sequence \((a_n)\) of the numbers of the alternating permutations of elements 1, 2, \ldots, \( n \) for which \( \pi_1 < \pi_2 > \pi_3 < \ldots \) for \( n \geq 1 \) is \((A000111[11])\),

\[
(31) \quad 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, 353792, 2702765, \ldots
\]

The corresponding sequence \((f_n)\) of the numbers of alternating full cycles is (3):

\[
(32) \quad 1, 0, 1, 1, 3, 10, 39, 173, 882, 5052, 32163, 225230, \ldots
\]

It is naturally to conjecture (3, 1996) that

\[
(33) \quad f_n \approx \frac{a_n}{n}.
\]
Indeed, the sequence \( \left( \frac{a_n}{n} \right) \) gives a highly good approximation of \((32)\):

\[
(34) \\
1, 0.5, 0.7, 1.3, 3.3, 10.2, 38.9, 173.1, 881.8, 5052.1, 32162.9, 225230.4, \ldots
\]

It is interesting to prove \((33)\) with an estimate of the remainder term.

4. A difficult combinatorial problem - to enumerate the alternating permutations and antialternating permutations for which \( \pi_1 > \pi_2 < \pi_3 > \ldots \) without fixed points or, the same, without cycles of length 1. The first numbers of these sequences \((b_n)\) and \((b_n^*)\) for \( n \geq 1 \) are

\[
(35) \quad 0, 0, 1, 2, 6, 22, 102, 506, 2952, 18502, 131112, 991226, \ldots
\]

and

\[
(36) \quad 0, 1, 1, 2, 6, 24, 102, 528, 2952, 19008, 131112, 1009728, \ldots
\]

It is not difficult to prove that

\[
(37) \quad b_{2n-1}^* = b_{2n-1}, \quad n \geq 1,
\]

\[
(38) \quad b_{2n}^* = b_{2n} + b_{2n-2}, \quad n \geq 2.
\]

**Conjecture 3.**

\[
(39) \quad \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{a_n}{b_n^*} = e.
\]

Moreover, we conjecture that since \( n = 3 \)

\[
(40) \quad \frac{a_{2n}}{b_{2n}^*} < e < \frac{a_{2n}}{b_{2n}},
\]

such that \( \frac{a_{2n}}{b_{2n}} \) increases and \( \frac{a_{2n}}{b_{2n}^*} \) decreases. So, for \( n = 3 \)

\[
\frac{61}{24} = 2.541 \ldots < e < \frac{61}{22} = 2.772 \ldots
\]

for \( n=4 \)

\[
\frac{1385}{528} = 2.623 \ldots < e < \frac{1385}{506} = 2.737 \ldots
\]
for \( n = 5 \)
\[
\frac{50521}{19008} = 2.657 \ldots < e < \frac{50521}{18502} = 2.730 \ldots
\]
for \( n = 6 \)
\[
\frac{2702765}{1009728} = 2.676 \ldots < e < \frac{2702765}{991226} = 2.726 \ldots \text{ etc.}
\]
If to consider the concatenation sequence similar to (26) for permutations having no fixed points with up-down index \( k \geq 0 \) then we obtain a sequence asymmetric in its blocks (except \( n = 3 \)) with the following first numbers:

\[
0, 1, 0, 0, 1, 1, 0, 1, 1, 2, 1, 1, 0, 0, 2,
\]
(41)
\[
4, 2, 5, 5, 4, 1, 2, 4, 6, 3, 2, 3, 1, 0, \ldots
\]
5. An algorithm for calculating the cyclic indicators for the alternating and antialternating permutations with restricted positions given by any \((0,1)\) matrix \( A \) was obtained in [3] with its realization in Turbo-Pascal 6.0

For example, if \( A = J_6 \) (i.e. without restrictions on positions) the "alternating" indicator has the form

\[
10t_6 + 12t_1t_5 + 10t_1t_2t_3 + 8t_1^2t_4 + 7t_2t_4 + 4t_1^2t_2^2 + 4t_3^2 + 4t_1^3t_3 + t_2^3 + t_1^4t_2
\]
while the "antialternating" indicator has the form

\[
10t_6 + 12t_1t_5 + 12t_1t_2t_3 + 7t_1^2t_4 + 8t_2t_4 + 4t_1^2t_2^2 + 4t_3^2 + 2t_1^3t_3 + 2t_2^3,
\]
such that the difference between these indicators is

\[
(42) \quad R_6 = (t_1^2 - t_2)(t_4 + 2t_1t_3 + t_2(t_1^2 + t_2))
\]
Analogously we have

\[
(43) \quad R_2 = t_1^2 - t_2, \quad R_4 = 0,
\]
\[
(44) \quad R_8 = (t_1^2 - t_2)(10t_6 + 12t_1t_8 + 8t_1t_2t_3 + 4t_3^2 + 2t_1^2t_2^2 + (t_1^2 + t_2)(7t_4 + 2t_1t_3 + t_2^2))
\]
\[ R_{10} = (t_1^2 - t_2)(173t_8 + 198t_1t_7 + 120t_1t_3t_4 + 96t_3t_5 + \]
\[ + 96t_1t_2t_5 + 43t_4^2 + 39t_1^2t_2t_4 + 26t_1t_2^2t_3 + 9t_1t_2^3 + \]
\[ +(t_1^2 + t_2)(110t_6 + 40t_3^2 + 36t_1t_5 + 34t_2t_4 + 30t_1t_2t_3 + t_2^3) + \]
\[ + (t_1^4 + t_2^2t_4 + t_2^3)(3t_2^2 + 6t_4)), \text{ etc.} \]

(45)

In addition, it is easy to see that for odd \( n \), \( R_n = 0 \). In the case \( t_1 = 0, t_2 = \ldots = t_{2n} = 1 \) we have \( R_{2n} = b_{2n-2} \) (cf. (35) for \( n \geq 2 \)). It follows from (38).

**Conjecture 4.** Polynomial \( R_{2n} \) is divided by \( t_1^2 - t_2 \) and, moreover, all coefficients of the polynomial \( \frac{R_{2n}}{t_1^2 - t_2} \) are positive.

Note that the sequence of the maximal coefficients of polynomials \( \frac{R_{2n}}{t_1^2 - t_2} \), \( n = 1, 2, \ldots \), is

(46) \[ 1, 0, 2, 12, 198, \ldots \]

Whether is true that the maximal coefficient of the polynomial \( \frac{R_{2n}}{t_1^2 - t_2} \) is always the coefficient of \( t_1t_{2n-3} \)?

Note that if Conjecture 4 is true then the numbers of the alternating and the antialternating full cycles are equal for \( n \geq 3 \). Indeed, if \( t_1 = t_2 = 0 \) then by Conjecture 4 always \( R_n = 0 \).

Finally, if Conjecture 4 is true then for \( t_1^2 = t_2 = t_2^2 \) we have

(47) \[ R_{2n}(t, t_1^2, t_3, t_4, \ldots) = 0. \]

The latter means that the numbers of all alternating and all antialternating permutations of elements \( 1, 2, \ldots, 2n \) having the same given summary length of cycles of length 1 and 2 and the same given numbers of cycles of length \( i \), \( i = 3, 4, \ldots, 2n \), are equal.

E.g., in the case of \( n = 10 \) the sum of the coefficients of \( t_1^6t_4, t_1^4t_2t_4, t_1^2t_2^2t_4 \) and \( t_2^3t_4 \) in the ”alternating” indicator is

\[ 6 + 241 + 770 + 248 = 1265 \]
and in the ”antialternating” indicator it is

\[ 0 + 168 + 809 + 288 = 1265. \]

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