Non-zero Contact and Sasakian Reduction

Oana Dragulete
Liviu Ornea

Vienna, Preprint ESI 1501 (2004) August 10, 2004

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via anonymous ftp from FTP.ESI.AC.AT
or via WWW, URL: http://www.esi.ac.at
NON-ZERO CONTACT AND SASAKIAN REDUCTION

OANA DRÅGULETE AND LIVIU ORNEA

ABSTRACT. We complete the reduction of Sasakian manifolds with the non-zero case by showing that Willett’s contact reduction is compatible with the Sasakian structure. We then prove the compatibility of the non-zero Sasakian (in particular, contact) reduction with the reduction of the Kähler (in particular, symplectic) cone. We provide examples obtained by toric actions on Sasakian spheres and make some comments concerning the curvature of the quotients.

1. Introduction

1.1. Sasakian manifolds. We start by briefly recalling the notion of a Sasakian manifold, sending to [6] and [7] for more details and examples.

Definition 1.1. A Sasakian manifold is a $(2n+1)$-dimensional Riemannian manifold $(M,g)$ endowed with a unitary Killing vector field $\xi$ such that the curvature tensor of $g$ satisfies the equation:

\[ R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi \]

where $\eta$ is the metric dual 1-form of $\xi$: $\eta(X) = g(\xi,X)$.

It can be seen that $\eta$ is a contact form (with Reeb field $\xi$). Using the Killing property of $\xi$ and equation (1.1), one defines an almost complex structure on the contact distribution $\text{Ker} \, \eta$, by (the restriction of) $\varphi = \nabla \xi$, where $\nabla$ is the Levi-Civita connection of $g$.

The following formulae are then easily deduced:

\[ \varphi^2 = 0, \quad g(\varphi Y,\varphi Z) = g(Y,Z) - \eta(Y)\eta(Z). \]

The simplest compact example is the round sphere $S^{2n-1} \subset \mathbb{C}^n$, with the metric induced by the flat one of $\mathbb{C}^n$. The characteristic Killing vector field is $\xi_\varphi = -i \varphi$, $i$ being the imaginary unit. More general Sasakian structures on the sphere can be obtained by deforming this standard structure as follows. Let $\eta_A = \frac{1}{\sum a_j |z_j|^2} \eta_0$, for $0 < a_1 \leq a_2 \cdots \leq a_n$. Its Reeb field is $R_A =$
\[ \sum a_j(x_j \partial y_j - y_j \partial x_j). \] Clearly, \( \eta_0 \) and \( \eta_A \) underly the same contact structure. Define the metric \( g_A \) by the conditions:

- \( g_A(X,Y) = \frac{1}{2} d\eta_A(IX, Y) \) on the contact distribution (here \( I \) is the standard complex structure of \( \mathbb{C}^n \));
- \( R_A \) is normal to the contact distribution and has unit length.

It can be seen that \( S^{2n-1}_A := (S^{2n-1}, g_A) \) is a Sasakian manifold (cf. [11]). It has recently been shown in [14] that each compact Sasakian manifold admits a CR-immersion in a \( S^{2n-1}_A \).

Sasakian manifolds, especially the Sasakian-Einstein ones, seem to be more and more important in physical theories (connected with the Malda-
cena conjecture). Many new examples appeared lately, especially in the
work of Ch.P. Boyer, K. Galicki and their collaborators.

This growing importance of Sasakian structures was the first motivation
for extending in [10] the contact (zero) reduction to this metric setting, by
showing that the contact reduction is compatible with the Sasakian data.

A good procedure for contact reduction away from zero was not available
when the paper [10] was written. We here complete the missing picture by
showing that Willett’s recently defined non-zero reduction introduced in [16]
is compatible with the Sasakian data.

Acknowledgements. The authors thank P. Gauduchon, E. Lerman and
T.S. Ratiu for their interest in this work and for very useful suggestions and
criticism.

1.2. Contact reduction.

1.2.1. Contact reduction at 0 following [2], [9]. Let \( (M^{2n-1}, \eta) \) be an exact
contact manifold: this means that \( \eta \) is a contact form \( (\eta \wedge (d\eta)^n) \neq 0) \), hence
its kernel is a contact structure on \( M \).

Let \( R \) be the Reeb vector field, characterized by the conditions \( \eta(R) = 1 \) and
\( d\eta(R, \cdot) = 0 \). The flow of the (nowhere vanishing) Reeb vector field
preserves the contact form \( \eta \).

Let \( \Phi : G \times M \rightarrow M \) be an action by strong contactomorphisms of a (finite
dimensional) Lie group on \( M \): for any \( f \in G \), \( f^* \eta = \eta^1 \). Such a \( G \)-action by
strong contactomorphisms on \( (M, \eta) \) always admits an equivariant momentum
map \( J : M \rightarrow \mathfrak{g}^* \) given by evaluating the contact form on fundamental
fields: \( \langle J, \xi \rangle = \eta(\xi_M)^2 \). Note the main difference towards the symplectic
case: an action by contactomorphisms is automatically Hamiltonian.

It can be seen that \( 0 \in \mathfrak{g}^* \) is a regular value for \( J \) if and only if the fundamental
fields induced by the action do not vanish on the zero level set of \( J \). In this case, the pull back of the contact form to \( J^{-1}(0) \) is basic. Let

---

1If the action is proper or \( G \) is compact, this is not more restrictive than asking \( G \) to
preserve only the contact structure: in the first case, one uses a Palais type argument, in
the second case an invariant contact form can be found by averaging.

2Here and in the sequel, for a \( X \in \mathfrak{g} \), \( X_M \) denotes the fundamental field it induces on
\( M \).
Consider $\pi_0: J^{-1}(0) \to J^{-1}(0)/G$ and $\iota_0: J^{-1}(0) \hookrightarrow M$ be the canonical projection (we shall always suppose that the considered actions are free and proper, although these hypothesis can be relaxed to deal with the category of orbifolds) and inclusion respectively. Albert’s reduction theorem assures the existence of a unique contact form $\eta_0$ on $J^{-1}(0)/G$ such that $\pi_0^*\eta = \iota_0^*\eta$. It can be seen that the contact structure of the quotient depends only on the contact structure on $M$.

The Sasakian version of this result states (cf. [10]) that if $M$ is Sasakian and $G$ acts by isometric strong contactomorphisms, then the metric also projects to the contact quotient and the whole structure is Sasakian.

1.2.2. Contact reduction away from zero following [16]. For $\mu \neq 0$, the restriction of the Reeb field is no more basic on $J^{-1}(\mu)$ with respect to the action of $G_\mu$, hence the above scheme does not apply. This situation was corrected by Albert, but in an unsatisfactory way, see [16] for examples. Willett’s method, that we now describe, is more appropriate and was already used in [8] to extend the cotangent reduction theorems in the contact context. In the above setting, for a $\mu \in \mathfrak{g}^*$, Willett calls the kernel group of $\mu$, the connected Lie subgroup $K_\mu$ of $G_\mu$ with Lie algebra $\mathfrak{k}_\mu = \ker(\mu|_{\mathfrak{g}_\mu})$. One can see that $\mathfrak{k}_\mu$ is an ideal in $\mathfrak{g}_\mu$, hence $K_\mu$ is a connected normal subgroup of $G_\mu$. The contact quotient of $M$ by $G$ at $\mu$ is defined by Willett as $M_\mu := J^{-1}(\mathbb{R}_+\mu)/K_\mu$.

If $K_\mu$ acts freely and properly on $J^{-1}(\mathbb{R}_+\mu)$, then $J$ is transversal to $\mathbb{R}_+\mu$ and the pull back of $\eta$ to $J^{-1}(\mathbb{R}_+\mu)$ is basic relative to the $K_\mu$-action on $J^{-1}(\mathbb{R}_+\mu)$, thus inducing a 1-form $\eta_\mu$ on the quotient $M_\mu$. If, in addition, $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$ then the form $\eta_\mu$ is also a contact form. It is characterized, as usual, by the identity $\pi_\mu^*\eta_\mu = i_{\mu}\eta$, where $\pi_\mu: J^{-1}(\mathbb{R}_+\mu) \to M_\mu$ is the canonical projection and $i_\mu: J^{-1}(\mathbb{R}_+\mu) \hookrightarrow N$ is the inclusion.

**Remark 1.1.** For $\mu = 0$, Albert’s and Willett’s quotients coincide.

In the next section we prove the compatibility of this procedure with the metric context.

2. Main results

2.1. The reduction theorem.

**Theorem 2.1.** Let $(M, g, \xi, \eta)$ be a $(2n-1)$ dimensional Sasakian manifold, let $G$ be a Lie group of dimension $d$ acting on $M$ by strong contactomorphisms. Let $J: M \to \mathfrak{g}^*$ be the momentum map associated to the action of $G$ and let $\mu$ be an element of the dual $\mathfrak{g}^*$. We assume that:

1. $\ker \mu + \mathfrak{g}_\mu = \mathfrak{g}$.
2. The action of $K_\mu$ on $J^{-1}(\mathbb{R}_+\mu)$ is proper and by isometries.
3. $J$ is transverse to $\mathbb{R}_+\mu$. 
4 OANA DRĂGULETE AND LIVIU ORNEA

Then the contact quotient

\[ M_\mu = J^{-1}(\mathbb{R}_+\mu)/K_\mu \]

is a Sasakian manifold with respect to the projected metric and Reeb field.

Proof. We already know that the reduced space \( M_\mu \) is a contact manifold (see [16]). What is left to be proved is that the metric \( g \) and the Reeb field \( \xi \) project on \( M_\mu \), the latter in a Killing field such that the curvature tensor of the projected metric satisfies formula \( (1.1) \).

From the transversality condition satisfied by the momentum map one knows that \( J^{-1}(\mathbb{R}_+\mu) \) is an isometric Riemannian submanifold of \( M \) (which induced metric we also denote by \( g \)). As the flow of the Reeb field leaves invariant the level sets of the momentum \( J \), one derives that the restriction of \( \xi \) is still a unit Killing field on \( J^{-1}(\mathbb{R}_+\mu) \).

In order to establish the metric properties of the canonical projection \( \pi_\mu : J^{-1}(\mathbb{R}_+\mu) \rightarrow M_\mu \), we have to understand the extrinsic geometry of the submanifold \( J^{-1}(\mathbb{R}_+\mu) \subset M \). The first step is to find a basis in the normal bundle of \( J^{-1}(\mathbb{R}_+\mu) \). To this end we look at the direct sum \( \mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{m} \) where \( \mu |_\mathfrak{m} = 0 \) (such a decomposition exists, because \( \ker \mu + \mathfrak{g}_\mu = \mathfrak{g} \)). Let \( \mathfrak{m}_M = \{ X_M | X \in \mathfrak{m} \} \) and recall that (see [16, Theorem 1]):

\[ (2.1) \ (T_xJ^{-1}(\mathbb{R}_+\mu) \cap \ker \eta_\mu) \oplus \mathbb{R}\xi_x \oplus \mathfrak{m}_M(x) = (T_x\Phi^{-1}(0) \cap \ker \eta_\mu) \oplus \mathbb{R}\xi_x, \]

for any \( x \in J^{-1}(\mathbb{R}_+\mu) \), where \( \Phi \) is the momentum map associated to the action of \( K_\mu \) on \( M \).

Let now \( \{ X_1, \ldots, X_k \} \) and \( \{ Y_1, \ldots, Y_m \} \) be two bases in \( \mathfrak{t}_\mu \) and, respectively, \( \mathfrak{m} \). Without loss of generality, one may suppose that the fundamental fields \( \{ Y_M \}_{i=1}^m \) form an orthogonal basis of \( \mathfrak{m}_M \), \( g \)-orthogonal on \( TJ^{-1}(\mathbb{R}_+\mu) \cap \ker \eta \) and that \( \{ X_M \}_{i=1}^k \) are mutually orthogonal.

With these hypotheses, one derives that \( \{ \varphi X_iM, \varphi Y_jM \} \) are linearly independent in each \( x \in M \) and

\[ g(\varphi Y_jM, W) = g(\varphi X_iM, W) = d\eta(W, X_iM) = -(dJ(W), X_i) = \langle r\mu, X_i \rangle = 0 \]

for any vector field \( W \) tangent to \( J^{-1}(\mathbb{R}_+\mu) \). Therefore, for any \( i, j \), the fields \( \{ \varphi X_iM, \varphi Y_jM \} \) belong to the normal bundle of \( J^{-1}(\mathbb{R}_+\mu) \). A simple counting of the dimensions in the relation \( (2.1) \), together with the fact that \( \{ \varphi X_iM \} \) is a basis in the normal bundle of \( T\Phi^{-1}(0) \) (see the proof of [10, Theorem 3.1]), imply that \( \{ \varphi X_iM, \varphi Y_jM \} \) is indeed a basis of the normal bundle of \( J^{-1}(\mathbb{R}_+\mu) \)\(^3\).

Let \( \nabla, \nabla^M \) be the Levi-Civita covariant derivatives of \( J^{-1}(\mathbb{R}_+\mu) \) and \( M \) respectively and let \( A_i, A_j \) be the Weingarten operators associated to the unitary normal sections \( \varphi X_iM/\|X_iM\|, 1 \leq i \leq k, \varphi Y_jM/\|Y_jM\|, 1 \leq j \leq m \). By applying the Weingarten formula and the relation \( (1.1) \), one obtains, for

\(^3\{\varphi X_iM, Y_jM\} \) is also a basis for \( T^+J^{-1}(\mathbb{R}_+\mu) \) Our choice is only technically motivated.
any $X, Y, Z$ tangent to $J^{-1}(\mathbb{R}_+, \mu)$:

\[
g(A, Y, Z) = \|X_{iM}\|^{-1}\{g(X_{iM}, Y)\eta(Z) - g(\varphi\nabla_{Y}^{M}X_{iM}, Z)\},
\]

\[
g(A, Y, Z) = \|Y_{jM}\|^{-1}\{g(Y_{jM}, Y)\eta(Z) - g(\varphi\nabla_{Y}^{M}Y_{jM}, Z)\}.
\]

As $K_{\mu}$ acts by strong contact isometries, the metric $g$ projects on a metric $g^{M_{\mu}}$ on $M_{\mu}$ with respect to which the canonical projection $\pi_{\mu}$ becomes a Riemannian submersion. We now show that the vertical distribution $V$ is locally generated by the vector fields $\{X_{iM}\}$. We have indeed:

\[
T_{x}\pi_{\mu}(X_{iM}(x)) = T_{x}\pi_{\mu}(\dot{c}(0)) = (\pi_{\mu} \circ c)(0)
\]

where $c(t) = \Phi(\exp tX_{iM}, x)$.

But $(\pi_{\mu} \circ c)(t) = \pi_{\mu}(x)$ for any $t$ and then

\[
T_{x}\pi_{\mu}(X_{iM}(x)) = 0 \quad \text{for any } x \in J^{-1}(\mathbb{R}_+, \mu).
\]

This proves that $\{X_{iM}\}_{1 \leq i \leq k} \subset V_{x}$ and, as $\dim V_{x} = k$, it implies that $\{X_{iM}\}$ generate $V$.

The formulae $L_{X_{iM}}\xi = 0$ for $i = 1, \ldots, k$ prove that $\xi$ is a projectable vector field and its projection $\zeta$ is a unit Killing field on the reduced space $M_{\mu}$.

Let $X, Y, Z$ be vector fields orthogonal to $\zeta$. Using O’Neill’s formulae (see [5, (9.28f)]) we derive:

\[
g^{M_{\mu}}(R^{M_{\mu}}(X, \zeta)Y, Z) = g(R(X^{h}, \xi)Y^{h}, Z^{h}) + 2g(A(X^{h}, \xi), A(Y^{h}, Z^{h})
\]

\[
- g(A(\xi, Y^{h}), A(X^{h}, Z^{h})) + g(A(X^{h}, Y^{h}), A(\xi, Z^{h}))
\]

where $X^{h}$ denotes the horizontal lift of the vector field $X$, $A$ is O’Neill’s (1,2) tensor field given by the relation: $A(Z^{h}, X^{h}) = \textit{vertical part of } \nabla_{Z^{h}}^{M}X^{h}$ and $R$ the curvature tensor of the connection $\nabla$ on $J^{-1}(\mathbb{R}_+, \mu)$. On the other hand:

\[
g(\nabla_{Z^{h}}\xi, X_{iM}) = g(\varphi Z^{h}, X_{iM}) = d\eta(X_{iM}, Z^{h}) = \langle dJ(Z^{h}), X_{iM} \rangle
\]

\[
= r\langle \mu, X_{iM} \rangle = 0,
\]

and hence:

\[
R^{M_{\mu}}(X, \zeta)Y = R(X^{h}, \xi)Y^{h}.
\]

This completes the proof.

\[
\square
\]

**Remark 2.1.** Under the hypothesis of the above theorem, the dimension of the reduced space is $2n - d - m - k + 1$.

### 2.2. Compatibility with the Kähler reduction

We now analyze the compatibility of the non-zero Sasakian reduction with Kähler reduction using the cone construction. In particular, we obtain a relation between non-zero contact reduction and symplectic reduction.

Let $C(M) = M \times \mathbb{R}_+$ be the cone over $M$ endowed with the Kähler metric $r^2 g + dr^2$. The action of $G$ on $M$ lifts to an action on $C(M)$ by holomorphic
isometries which commute with the translations along the generators (see [10] e.g.). Similarly, the action of $K_{\mu}$ lifts to the cone, the lifted action being the restriction of the above.

Let $J_s : C(M) \to g^*$, resp. $\Phi_s : C(M) \to \xi_{\mu}^*$ be the symplectic momentum map associated to the $G$-action, resp. $K_{\mu}$-action on the cone. The differentials of these two momentum maps are related by the transpose $i^t$ of the natural inclusion $i : \xi_{\mu} \hookrightarrow g$, namely $T\Phi_s = i^t T J_s$.

We now embed $M$ in the cone as $M \times \{1\}$ and observe that the contact momentum maps are the restrictions of the symplectic ones: $J = J_s|_{M \times \{1\}}$, resp. $\Phi = \Phi_s|_{M \times \{1\}}$. Clearly $J$ and $\Phi$ are the contact momentum maps associated to the $G$, resp. $K_{\mu}$-action on $M \times \{1\}$. Moreover, we have

\[(2.2) \ \Phi = i^t \circ J.\]

On the other hand, we recall (see [10]) that the reduced space at 0 of the Kähler cone is the Kähler cone of the Sasakian reduced space at 0:

$$\Phi_s^{-1}(0)/K_{\mu} = C(\Phi^{-1}(0)/K_{\mu}).$$

We are now prepared to prove:

**Theorem 2.2.** Let $(M, g, \xi, \eta)$ be a Sasakian manifold, let $G$ be a Lie group acting on $M$ by strong contactomorphisms and $\mu$ an element of $g^*$. Suppose that:

- $0$ is a regular value for $J_s$.
- $\text{Ker} \mu + g_{\mu} = g$.
- $K_{\mu}$ acts properly and by isometries on $\Phi^{-1}(0)$.
- $J$ is transverse to $\mathbb{R}_+ \mu$ and to $\mathbb{R}_- \mu$.

Then the cone over the Sasakian quotient of $M$ at 0 with respect to the $K_{\mu}$ action is the disjoint union of the Kähler cones over the Sasakian quotients $M_{\mu}$ and $M_{-\mu}$ and the cone over the co-isotropic submanifold $J^{-1}(0)/K_{\mu}$:

$$C(M)_0 = C(\Phi^{-1}(0)/K_{\mu}) = C(M_{\mu}) \cup C(J^{-1}(0)/K_{\mu}) \cup C(M_{-\mu}).$$

**Proof.** Since $i^t$ is surjective, from (2.2) and from 0 being a regular value for $J_s$ (and hence also for $J$) it follows that 0 is a regular value for $\Phi_s$ and hence for $\Phi$.

As $\Phi^{-1}(0) = J^{-1}(\mathbb{R}_\mu)$ (cf. [16, proof of Theorem 2]) and $K_{\mu}$ acts on $J^{-1}(0)$ and on $J^{-1}(\mathbb{R}_+ \mu)$, we have the partition

$$\Phi^{-1}(0)/K_{\mu} = J^{-1}(\mathbb{R}_+ \mu)/K_{\mu} \cup J^{-1}(0)/K_{\mu} \cup J^{-1}(\mathbb{R}_- \mu)/K_{\mu}.$$

We note that $(M, g, -\xi)$ is also a Sasakian manifold on which $G$ acts by Sasakian automorphisms and the associated momentum map is $-J$. Then, if the quotient $M_{-\mu} := J^{-1}(\mathbb{R}_- \mu)/K_{\mu}$ exists (or, equivalently, the quotient of $(M, g, \xi)$ at $-\mu$), it will be a Sasakian manifold according to our previous theorem. But note that two of the hypothesis of the theorem are not automatically satisfied in both cases: if $K_{\mu}$ acts properly on $J^{-1}(\mathbb{R}_+ \mu)$ it does not necessarily act properly on $J^{-1}(\mathbb{R}_- \mu)$ and similarly for the transversality condition.
Now \( J^{-1}(0)/K_{\mu} \) is a manifold on which the one-form \( \eta \) is projected. However, it is no more a contact form, nor, in general, a symplectic one. Indeed, using Albert [2, Propositions 1,2], \( J^{-1}(0)/K_{\mu} \) is contact or symplectic if and only if
\[
T_x(K_{\mu} \cdot x) = \text{Ker} (T_{\eta_x}|_{T_xJ^{-1}(0)\cap \text{Ker} \eta_x}).
\]
But, in general, one has \( \text{Ker} (T_{\eta_x}|_{T_xJ^{-1}(0)\cap \text{Ker} \eta_x}) = T_x(G \cdot x) \). However, this implies that \( J^{-1}(0)/K_{\mu} \) is a co-isotropic submanifold with respect to the contact form of \( \Phi^{-1}(0)/K_{\mu} \).

**Remark 2.2.** Forgetting the metric and *mutatis mutandis*, the result of Theorem 2.2 remains valid for contact manifolds.

3. Examples: actions of tori on spheres.

In Willett’s reduction scheme, the smallest dimension of \( G \) which produces non-trivial examples is 2. We here present some complete computations for various actions of \( G = T^2 \) on \( M = S^7 \) with the standard Sasakian structure given by the contact form \( \eta = \sum (x_j dy_j - y_j dx_j) \). When possible, we briefly discuss also the reduction at zero with the same group and the cone construction (the notations for the momentum maps will be the ones used in the previous section). Generalizations to \( S^{2n+1} \) are also indicated.

Note that our examples show the dependence of the dimension of the quotient on the choice of \( \mu \).

**Example 3.1.** Let first \( T^2 \) act on \( S^7 \) by
\[
((e^{i\theta_0}, e^{i\theta_1}), (z_0, ..., z_3)) \mapsto (e^{i\theta_0} z_0, e^{i\theta_1} z_1, e^{i\theta_1} z_2, e^{i\theta_1} z_3).
\]
Since \( G \) is commutative, \( \mathfrak{g}_\mu = \mathfrak{g} = \mathbb{R}^2 \).

For any \((r_1, r_2) \in \mathfrak{g}\) the associated infinitesimal generator is given by
\[
(r_1, r_2) \cdot (z) = r_1(-y_0 \partial_{x_0} + x_0 \partial_{y_0}) + r_1(-y_1 \partial_{x_1} + x_1 \partial_{y_1})
+ r_2(-y_2 \partial_{x_2} + x_2 \partial_{y_2}) + r_2(-y_3 \partial_{x_3} + x_3 \partial_{y_3})
\]
and the momentum map \( J : S^7 \rightarrow (\mathbb{R}^2)^* \) reads \( J(z) = (|z_0|^2 + |z_1|^2, |z_2|^2 + |z_3|^2, \cdot) \).

Let \( \mu : \mathbb{R}^2 \rightarrow \mathbb{R}, \mu = \langle v, \cdot \rangle, v \in \mathbb{R}^2 \setminus \{0\} \) fixed. Then:
\[
J^{-1}(\mathbb{R}^+ \mu) = \begin{cases} 
S^3(\sqrt{\frac{v_1}{v_1 + v_2}}) \times S^3(\sqrt{\frac{v_2}{v_1 + v_2}}), & \text{if } v_1, v_2 > 0 \\
S^3(\sqrt{\frac{v_1}{v_1 + v_2}}), & \text{if } v_1 > 0, v_2 = 0 \\
S^3(\sqrt{\frac{v_2}{v_1 + v_2}}), & \text{if } v_1 = 0, v_2 > 0 
\end{cases}
\]

For \( v = (1, 0) \) \( J^{-1}(\mathbb{R}^+ \mu) = S^3 \), \( \text{Ker} \mu = \mathfrak{f}_\mu = \{0\} \times \mathbb{R} \), \( K_{\mu} = \{e\} \times S^1 \).

The action of \( K_{\mu} \) on \( J^{-1}(\mathbb{R}^+ \mu) \) is trivial and hence \( M_{\mu} = S^3 \). In this case 0 is not a regular value of \( \Phi \)-the momentum map associated to the \( K_{\mu} \) action but, nevertheless, \( \Phi^{-1}(0) \) is a submanifold of \( S^7 \) and hence the reduced space at zero, \( \Phi^{-1}(0)/K_{\mu} \) is a Sasaki manifold. As \( \Phi^{-1}(0) = S^3 \) and \( C(S^n) = \mathbb{R}^{n+1} \setminus \{0\} \), we obtain that \( (C(S^7))_0 = \mathbb{R}^4 \setminus \{0\} \). Note that
for this choice of $\mu$ reducing and taking the cone are commuting operations exactly as in the zero case.

For $v = (1,1)$ we obtain: $J^{-1}(\mathbb{R}_+\mu) = S^3(\sqrt{\frac{1}{2}}) \times S^3(\sqrt{\frac{1}{2}})$, $\ell_\mu = \{(x,x)|x \in \mathbb{R}\}$, $K_\mu = \{(e^{-it}, e^{it})|e^{it} \in S^1\}$. The action of $K_\mu$ on $J^{-1}(\mathbb{R}_+\mu)$ is given by

$$((e^{-it}, e^{it}), z) \mapsto (e^{-it}z_0, e^{-it}z_1, e^{it}z_2, e^{it}z_3),$$

thus $M_\mu = S^2 \times S^3$.

We can generalize this example for $M = S^{2n+1}$ by considering the action

$$((e^{it_0}, e^{it_1}), z) = (e^{it_0}z_0, e^{it_1}z_1, e^{it_1}z_2, \ldots, e^{it_n}z_n).$$

Now the momentum map is $J(z) = (|z_0|^2 + |z_1|^2, \sum |z_k|^2, \cdot)$. For $\mu$ as above, we have:

$$J^{-1}(\mathbb{R}_+\mu) = \begin{cases} 
S^3(\sqrt{\frac{v_1}{v_1+v_2}}) \times S^{2n-3}(\sqrt{\frac{v_2}{v_1+v_2}}) & \text{if } v_1, v_2 > 0 \\
S^3(\sqrt{\frac{v_1}{v_1+v_2}}) & \text{if } v_1 > 0, v_2 = 0 \\
S^{2n-3}(\sqrt{\frac{v_1}{v_1+v_2}}) & \text{if } v_1 = 0, v_2 > 0
\end{cases}$$

For the same particular choices of $\mu$ as above, we obtain as reduced spaces respectively $S^3$, $S^{2n-3}$ or $S^3 \times \mathbb{C}P^{n-2}$.

**Example 3.2.** Let now the action be given by

$$((e^{it_0}, e^{it_1}), z) \mapsto (e^{-it_0}z_0, e^{it_0}z_1, e^{it_1}z_2, e^{it_1}z_3).$$

The infinitesimal generator of the action will be

$$(r_1, r_2)_{S^7}(z) = r_1(y_0\partial_{x_0} - x_0\partial_{y_0}) + r_1(-y_1\partial_{x_1} + x_1\partial_{y_1}) + r_2(-y_2\partial_{x_2} + x_2\partial_{y_2}) + r_2(-y_3\partial_{x_3} + x_3\partial_{y_3}).$$

The momentum map is $J(z) = (|z_1|^2 - |z_0|^2, |z_2|^2 + |z_3|^2, \cdot)$ and

$$J^{-1}(\mathbb{R}_+\mu) = \left\{ z \in S^7 \mid \exists s > 0 \text{ such that } \begin{cases} |z_1|^2 - |z_0|^2 - sv_1 = 0, \\
|z_2|^2 + |z_3|^2 - sv_2 = 0. \end{cases} \right\}$$

For $v = (1,0)$ we obtain

$$J^{-1}(\mathbb{R}_+\mu) = \{ z \in S^7 | z_2 = z_3 = 0, |z_1| > |z_0| \} = S^3 \setminus \{ |z_1| \leq |z_0| \}.$$

The action of $K_\mu = \{ e \} \times S^1$ on $J^{-1}(\mathbb{R}_+\mu)$ is trivial, thus $M_\mu = S^3 \setminus \{ |z_1| \leq |z_0| \}$, an open submanifold of $S^3$. For $v = (1,1)$, solving for $s$ the equations in (3.1) gives $s \in (0, \frac{1}{2}]$. Hence:

$$J^{-1}(\mathbb{R}_+\mu) \simeq \left( S^1(\frac{1}{\sqrt{2}}) \times S^5(\frac{1}{\sqrt{2}}) \right) \setminus \left\{ z \in S^7 | |z_0|^2 = \frac{1}{2} \right\} \simeq S^1(\frac{1}{\sqrt{2}}) \times (S^5(\frac{1}{\sqrt{2}}) \setminus S^1(\frac{1}{\sqrt{2}}))$$

an open submanifold of the product of spheres.
The action of $K_\mu$ on $J^{-1}(\mathbb{R}_+\mu)$ is given by
$$(e^{-it}, e^{it}), z) \mapsto (e^{it}z_0, e^{-it}z_1, e^{it}z_2, e^{it}z_3).$$
Let $A$ denote the set $\{z \in S^7(\sqrt{2}) \mid 0 < |z_2|^2 + |z_3|^2 \leq 1\}$. Obviously, the above action of $K_\mu$ can be understood on the whole $\mathbb{C}^4$ and, as such, restricts to an action on $A$. Then $M_\mu$ is diffeomorphic with $(S^1 \times S^5) \cap A/K_\mu$. To identify the quotient, let $g : (S^1 \times S^5) \cap A \rightarrow (S^1 \times S^5) \cap A$ be given by
$$(z_0, z_1, z_2, z_3) \mapsto (z_0, z_1^{-1}, z_2, z_3).$$
$g$ induces a map from $((S^1 \times S^5) \cap A)S^1$ (with respect to the diagonal action of $S^1$) to $((S^1 \times S^5) \cap A)/K_\mu$. The map
$$(z_0, ..., z_3) \mapsto (\tilde{z}_1 z_0, z_1, \tilde{z}_1 z_2, \tilde{z}_1 z_3)$$
is a diffeomorphism of $(S^1 \times S^5) \cap A$ equivariant with respect to the diagonal action of $S^1$ and the action of $S^1$ on the first factor. Hence $M_\mu$ is diffeomorphic to $S^5(\frac{1}{\sqrt{2}}) \setminus \text{pr} \{z \in S^7 \mid |z_0|^2 = \frac{1}{2}\} \cong S^5(\frac{1}{\sqrt{2}}) \setminus S^1(\frac{1}{\sqrt{2}})$, where
$$\text{pr} : \mathbb{C}^4 \rightarrow \mathbb{C}^3, \text{pr}(z_0, ..., z_3) = (z_0, z_2, z_3).$$
If we change the action on $z_0$ with $e^{-ikt}z_0$, the reduced space will be the above one quotiented by $\mathbb{Z}^k$ (see also [10, Example 4.2]).

**Example 3.3.** Let us take this time:
$$(e^{it}, e^{-it}), z) \mapsto (e^{it}z_0, e^{-it}z_1, e^{it}z_2, e^{-it}z_3),$$
whose infinitesimal generator is
$$(r_1, r_2)s^7(z) = r_1(-y_0 \partial x_0 + x_0 \partial y_0) + r_2 \sum_{j=1}^{3} (-y_j \partial x_j + x_j \partial y_j).$$
The momentum map is:
$$J(z) = \langle (|z_0|^2, |z_1|^2 + |z_2|^2 + |z_3|^2) \rangle.$$
For $J^{-1}(\mathbb{R}_+\mu)$ we obtain the following possibilities:

$$(3.2) \quad J^{-1}(\mathbb{R}_+\mu) = \begin{cases} 
S^1(\sqrt{\frac{v_1}{v_1 + v_2}}) \times S^5(\sqrt{\frac{v_2}{v_1 + v_2}}), & \text{if } v_1, v_2 > 0 \\
S^5(\sqrt{\frac{v_2}{v_1 + v_2}}), & \text{if } v_1 = 0, v_2 > 0 \\
S^1(\sqrt{\frac{v_1}{v_1 + v_2}}), & \text{if } v_2 = 0, v_1 > 0 
\end{cases}$$
In particular, for $v = (1, 0)$, $M_\mu = S^1$, for $v = (0, 1)$, $M_\mu = S^5$ and for $v = (1, 1)$ one obtains the same quotient as in the preceding example.

**Example 3.4.** Considering the weighted action of $T^2$ on $S^7$ given this time by
$$(e^{it}, e^{-it}), z) \mapsto (e^{it}\lambda_0 z_0, e^{it}\lambda_1 z_1, z_2, z_3),$$
one obtains the momentum map
$$J(z) = \langle (\lambda_0|z_0|^2, \lambda_1|z_1|^2) \rangle.$$
where $\text{pr}$. and the reduced space be fixed in $S^7 \setminus S^5$ if $\lambda_0 \neq 0$ and $S^7 \setminus S^5$ if $\lambda_0 = 0$.

The cone construction is verified in this case. Indeed, $J^{-1}(0) = S^3$ and

$$\mathcal{C}(S^7) \setminus 0 \simeq \mathcal{C}(S^5) = \mathcal{C}(S^3) \cup \mathcal{C}(S^5 \setminus S^3).$$

If $v = (1, 1)$ and $\lambda_0, \lambda_1 > 0$,

$$(3.3) \quad J^{-1}(\mathbb{R}_+, \mu) = \left\{ z \in S^7 \mid |z_1| = \sqrt{\frac{\lambda_0}{\lambda_1}} |z_0|, z_0 \neq 0 \right\} = S^7 \cap (\mathbb{C}^* \times A)$$

where $A$ is the ellipsoid of equation

$$|z_1|^2 (1 + \frac{\lambda_1}{\lambda_0}) + |z_2|^2 + |z_3|^2 = 1.$$  

The action of $K_\mu$ on $J^{-1}(\mathbb{R}_+, \mu)$ is given by

$$(e^{-it}, e^{it}) \mapsto (e^{-it\lambda_0} z_0, e^{it\lambda_1} z_1, z_2, z_3)$$

and the reduced space

$$M_\mu = \bigcup_{(z_2, z_3) \in \text{pr}(J^{-1}(\mathbb{R}_+, \mu))} S^1(\beta^{-\lambda_0} \alpha^{\lambda_1}) \times \{ (z_2, z_3) \}$$

where $\text{pr} : \mathbb{C}^4 \to \mathbb{C}^2$, $\text{pr}(z_0, \ldots, z_3) = (z_2, z_3)$, $\beta = \sqrt{\frac{\lambda_0(1 - |z_2|^2 - |z_3|^2)}{\lambda_0 + \lambda_1}}$ and $\alpha = \sqrt{\frac{\lambda_1(1 - |z_2|^2 - |z_3|^2)}{\lambda_0 + \lambda_1}}$.

If $[z] = [z']$ in the reduced space then $z_2 = z'_2$ and $z_3 = z'_3$. So let $(z_2, z_3)$ be fixed in $\text{pr}(J^{-1}(\mathbb{R}_+, \mu))$. $z \in J^{-1}(\mathbb{R}_+, \mu)$ and $\text{pr}(z) = (z_2, z_3)$ imply $|z_0| = \alpha$ and $|z_1| = \beta$. The action of $K_\mu$ on $J^{-1}(\mathbb{R}_+, \mu)$ is in fact the diagonal action of $S^1$ on the first two coordinates. Let $f : (S^1(\alpha) \times S^1(\beta) \times \{ (z_2, z_3) \}) / S^1 \to S^1(\alpha^{\lambda_1} \beta^{-\lambda_0})$ be the map given by

$$[z] \mapsto z_0^{\lambda_1} z_1^{-\lambda_0}.$$  

One can easily check that $f$ is a diffeomorphism.

In the previous examples, the Reeb flow on the reduced space is the restriction of the canonical one of the standard sphere. In this latter case, we obtain a non-standard Reeb flow.

We now write the flow of the Reeb field of the reduced contact form on $M_\mu$ (for $v = (1, 1)$). Let $r(t) = (\cos t, \sin t)$, $Z = (z_0^2, z_3^3)^t$. Then the flow is written as

$$\varphi_t = (A e^{i(a + bt)}, R(t) Z),$$

where

$$A = \|z_0^0\|^{\lambda_1} \|z_0^1\|^{-\lambda_0},$$

$$a = \lambda_1 v_0 + \lambda_0 v_1, \text{ with } v_0 = \arg(z_0^0), v_1 = \arg(z_0^1),$$

$$b = \lambda_1 + \lambda_0,$$

$$R(t) = \text{diag}(r(t), r(t)).$$
4. The sectional curvature of the quotient

4.1. Contact CR submanifolds. In order to evaluate the sectional curvature of the Sasakian reduced space, both at 0 and away from 0, it will be convenient to place ourselves in a slightly more general situation. We first recall the following definition (see e.g. [4]):

**Definition 4.1.** Let $(M, g_M, \xi)$ be a Sasakian manifold. An isometric submanifold $N$ is called contact CR or semi-invariant if it admits two mutually orthogonal distributions $D$ and $D^\perp$, such that:

1. $TN$ decomposes orthogonally as: $TN = D \oplus D^\perp \oplus \langle \xi \rangle$ and
2. $\varphi D = D$, $\varphi D^\perp \subseteq T^\perp N$.

We see that, in general, the normal bundle of the submanifold also splits into two orthogonal distributions: $\varphi D^\perp$ and its orthogonal complement that we denote by $\nu$ and which is invariant at the action of $\varphi$. We then have:

$$TM|_N = D \oplus D^\perp \oplus \langle \xi \rangle \oplus \varphi D^\perp \oplus \nu.$$ 

For a vector field $V$ normal to $N$ we shall denote $\tilde{V}$, respectively $\tilde{V}$ its component in $\varphi D^\perp$, respectively in $\nu$.

Such submanifolds have been extensively studied in the last thirty years. Obviously, very natural examples are the level sets of Sasakian momentum maps. To better mimic our situation, we moreover make the following:

**Assumption.** There exists a Riemannian submersion $\pi : N \rightarrow P$ over a Sasakian manifold $(P, g_P, \zeta)$ such that:

1. $D \oplus \langle \xi \rangle$ represents the horizontal distribution of the submersion; (and hence $D^\perp$ represents the vertical distribution of the submersion);
2. The two Reeb fields are $\pi$-related: $\xi$ is basic and projects over $\zeta$.

This situation was already considered by Papaghiuc in [15], on the model of Kobayashi’s paper [12] where the similar setting was discussed in Kählerian context.

Let $\phi := \nabla^P \zeta$ and observe that in our assumption we have $(\phi X)^h = \varphi X^h$.

We want to relate the sectional curvature of planes generated by orthonormal pairs $\{X, \phi X\}$, respectively $\{X^h, \varphi X^h\}$. This is usually known as $\varphi$-sectional curvature, the analogue in Sasakian geometry of holomorphic sectional curvature; it completely determines the curvature tensor, cf. [6], so it is worth having information about it.

We first apply (as in the proof of Theorem 2.1) O’Neill’s formula to relate the curvatures of $N$ and $P$. For $X$ tangent to $P$ and orthogonal to $\zeta$, (this is not restrictive, as the planes passing through the Reeb field have sectional curvature 1 on a Sasakian manifold), using the anti-symmetry of the tensor $A$, we obtain:

\begin{equation}
R^N(X^h, \varphi X^h, X^h, \varphi X^h) - R^P(X, \phi X, X, \phi X) = -3\|A(X^h, \varphi X^h)\|^2_N,
\end{equation}

where the sub-index refers to the norm with respect to $g^N$. 

In the general case, from (4.3) we easily derive:

\begin{equation}
R^h(X^h,\varphi X^h, X^h, \varphi X^h) - R^N(X^h, \varphi X^h, X^h, \varphi X^h)
= \|h(X^h, \varphi X^h)\|^2_M - g_M(h(X^h, X^h), h(\varphi X^h, \varphi X^h)).
\end{equation}

We now need to relate the tensors $A$ and $h$. To this end, we write $hE$, respectively $vE$ for the horizontal, respectively vertical part of a tangent (to $N$) vector field $E$ and we first decompose

$$\nabla^M_X(\varphi Y^h) = h\nabla^M_X(\varphi Y^h) + A(X^h, \varphi Y^h) + h(X^h, \varphi Y^h).$$

Then we use the formulae ($\nabla^M_E\varphi F = \eta(F)E - g_M(E, F)\xi$ (see [6]) and ($\nabla^M_E\varphi F = \nabla^M_E(\varphi F) - \varphi\nabla^M_E F$) to express $\nabla^M_X(\varphi Y^h)$. Finally, equaling the tangent and normal parts in the equation we obtain this way, we arrive at the following relations:

\begin{equation}
A(X^h, \varphi Y^h) = v\varphi h(X^h, Y^h),
\end{equation}
\begin{equation}
h(X^h, \varphi Y^h) = \varphi A(X^h, Y^h) + \varphi h(\widetilde{X^h}, \widetilde{Y^h}).
\end{equation}

Note that if $\varphi D^\perp = T^\perp N$ (i.e. $\nu = \{0\}$), and this is the case when $N$ is the zero level set of a Sasakian momentum map, the above relations simplify to:

\begin{equation}
A(X^h, \varphi Y^h) = \varphi h(X^h, Y^h),
\end{equation}
\begin{equation}
h(X^h, \varphi Y^h) = \varphi A(X^h, Y^h).
\end{equation}

In the general case, from (4.3) we easily derive:

$$h(\varphi X^h, \varphi Y^h) = \overline{h(X^h, Y^h)} - h(\widetilde{X^h}, \widetilde{Y^h}),$$

and hence

\begin{equation}
g_M(h(\varphi X^h, \varphi Y^h), h(X^h, Y^h)) = \|h(X^h, Y^h)\|^2_M - \|h(\widetilde{X^h}, \widetilde{Y^h})\|^2_M.
\end{equation}

From equation (1.2) it follows that on the orthogonal complement of $\xi$, the tensor $\varphi$ acts like an isometry. Therefore, using again (4.3), we derive:

\begin{equation}
\|h(X^h, \varphi Y^h)\|^2_M = \|A(X^h, Y^h)\|^2_M + \|h(\widetilde{X^h}, \widetilde{Y^h})\|^2_M,
\end{equation}

\begin{equation}
\|A(X^h, \varphi Y^h)\|^2_M = \|h(X^h, Y^h)\|^2_M.
\end{equation}

Let us denote $K^F_\varphi(X)$, respectively $K^M_\varphi(X^h)$ the sectional curvature of the plane $\{X, \phi X\}$, respectively $\{X^h, \varphi X^h\}$. Adding equations (4.1), (4.2) and using (4.5), (4.6), we finally obtain (taking again into account the anti-symmetry of $A$):

\begin{equation}
K^F_\varphi(X) = K^M_\varphi(X^h) + 4\|h(X^h, X^h)\|^2_M - 2\|h(\widetilde{X^h}, \widetilde{X^h})\|^2_M.
\end{equation}
4.2. The curvature of the quotient. In general, from equation (4.7) one hopes to deduce the positivity of the $\varphi$-sectional curvature of the quotient. This depends on the extrinsic geometry of the level set, which is a data additional to the reduction scheme: the second fundamental form of the level set cannot be entirely expressed in terms of the action. But in some particular cases, one is able to derive a conclusion.

Obviously the simplest situation occurs when $J^{-1}(\mathbb{R}_+\mu)$ is totally geodesic in $M$: then the $\varphi$-sectional curvatures of $M$ and $M_\mu$ are equal. But even our examples show that this is not always the case. In fact, one is only interested in the vanishing of $h(X^h, Y^h)$, which, by the first equation in (4.3), is implied by the vanishing of O'Neill’s integrability tensor $A$. This is a rather strong condition, implying that $J^{-1}(\mathbb{R}_+\mu)$ is a locally (not necessarily Riemannian) product and cannot be predicted by the action. Other conditions on the second fundamental form which are common in Riemannian and Cauchy-Riemann submanifold theory, see e.g. [4], (mixed totally geodesic, (contact)-totally umbilical, extrinsic sphere etc.) and permit some speculations in (4.7) or even the computation of the Ricci curvature of the quotient, seem to be artificial in this context, as not directly expressible in terms of the action.

We apply the above computation for $N$ being $J^{-1}(0)$ and for $P$ being the respective reduced space. Then equation (4.7) implies:

**Proposition 4.1.** The reduced space at 0 of a Sasakian manifold with positive $\varphi$-sectional curvature (in particular of an odd sphere with the standard Sasakian structure) has strictly positive $\varphi$-sectional curvature.

**References**

[1] R. Abraham, J. Marsden, *Foundations of Mechanics*, second edition, New York, Benjamin/Cummings, 1978.
[2] C. Albert, *Le théorème de réduction de Marsden-Weinstein en géométrie cosymplectique et de contact*, J. Geom. Physics, 6 (1989), 627–649.
[3] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Verlag, 1984.
[4] A. Bejancu, *The geometry of CR submanifolds*, Mathematics and its Applications (East European Series), 23. D. Reidel Publishing Co., Dordrecht, 1986.
[5] A. Besse, *Einstein manifolds*, Springer Verlag (1987).
[6] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. 203, Birkhäuser, Boston, Basel, 2002.
[7] Ch.P. Boyer, K. Galicki, *On Sasakian-Einstein geometry*, Internat. J. Math. 11 (2000), 873–909.
[8] O. Drăgulete, L. Ornea, T.S. Ratiu, *Cosphere bundle reduction in contact geometry*, J. Symplectic Geom. (to appear).
[9] H. Geiges, *Constructions of contact manifolds*, Math. Proc. Cambridge Philos. Soc., 121 (1997), 455–464.
[10] G. Grantcharov, L. Ornea, *Reduction of Sasakian manifolds*, J. Math. Physics, 42 (2001), 3808–3816.
[11] Y. Kamishima, L. Ornea, *Geometric flow on compact locally conformal Kähler manifolds*, Tohoku Math. J. (to appear).
[12] S. Kobayashi, *Submersions of CR submanifolds*, Tohoku Math. J. 39 (1987), 95–100.
[13] J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Springer Texts in Appl. Math. **17**, Second edition, 1999.

[14] L. Ornea, M. Verbitsky, *Immersion theorem for compact Vaisman manifolds*, Math. Ann. (to appear), available at math.AG/0306077.

[15] N. Papaghiuc, *Submersions of semi-invariant submanifolds of a Sasakian manifold*, An. Științ. Univ. "Al. I. Cuza Iași", Sect. I Mat. **35** (1989), 281–288.

[16] C. Willett, *Contact reduction*, Trans. Amer. Math. Soc. **354** (2002), 4245–4260.

Section de mathématiques, EPFL, CH-1015 Lausanne, Switzerland

and

Department of Mathematics,
University "Politehnica" of Bucharest, Romania

E-mail address: oana.dragulete@epfl.ch

University of Bucharest, Faculty of Mathematics

14 Academiei str., 70109 Bucharest, Romania

E-mail address: Liviu.Ornea@imar.ro