On a Non-linear Interacting New Holographic Dark Energy Models: observational constraints

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Abstract

In this paper, we study interacting DGP braneworld Holographic Dark Energy model in a spatially flat FRW universe. Mainly, in this study we concentrate our attention on 4 different forms of non-linear and non-gravitational interaction. The study shows that the equation of state and the deceleration parameter depict an accelerated universe for all variety of interactions. On the other hand, the StateFinder analysis of all models shows that all models behave similar to both quintessence and phantom dark energy, but for their present value only the first model with $3bH(\rho_D + \rho_D^2 / (\rho_D + \rho_m))$ interaction term obeys the behavior of phantom dark energy. Moreover, the result of $Om$-diagnostic is an emphasis on the result of the equation of state showing that the current model has a Phantom-like behavior. By the use of the squared sound speed $v_s^2$ we find that the present model is stable compared to the other holographic models of dark energy such as Ghost Dark Energy (GDE), Standard Holographic Dark Energy (HDE), Sharma-Mittal Holographic Dark Energy (SMHDE) and Agegraphic Dark Energy (ADE) which are instable against the background perturbation. In order to obtain the best fit values of the parameters in this work we used the latest observational data (cJLA, Boss DR12, Planck 2015, OHD, SGL) implementing Metropolis-Hastings algorithm with $1\sigma$ and $2\sigma$ confidential levels. For choosing the best model we employ Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). We conclude that the proposed model in the presence of the interaction $3bH(\rho_D + \rho_D^2 / (\rho_D + \rho_m))$ is compatible with the recent observational data. We obtain the extra density of DGP and braneworld $\Omega_{rc}$ less than 0.003 which has suitable compatibility with the amount of dark components of the universe. Using the modified version of the Einstein-Boltzmann CAMB code, we plot CMB power spectrum $C_l^{TT}$ we found that in situation of existing curvature the main difference between four models and the $\Lambda$CDM model is in the late ISW and in low-$l$ area. This plot demonstrates deviations between the models and $\Lambda$CDM. In spite of having the biggest deviation among the models, the first model is the best one and independent of $\Lambda$CDM behavior.

Keywords: Interacting dark energy models, accelerated expanding universe, observational constrains

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I INTRODUCTION

Dark energy - raised in 1998 [1] - has become one of the main issues in modern cosmology and many models have been proposed to investigate this new concepts of cosmology. In spite of many proposed models, dark energy still remains one of the open issues in cosmology [2],[3],[4],[5],[6],[7],[8],[9],[10] (to mention a few). Among these models, the holographic dark energy (HDE) has attracted more attention in recent years [11],[12],[13],[14]. This model is stemmed from the holographic principle related to which all of the information in a specific area of space can be drawn out from its boundary region and are constrained by an IR cutoff [15],[16]. The energy density of HDE can be shown as\[ \rho_D = 3c^2M_P^2/L^2 \] [17],[18],[19]. Using HDE compared to other models is more appropriate to investigate the problems of dark energy [16],[20],[21],[17],[22],[23]. Studying of HDE, also help to avoid the formation of Black Holes which can be investigated by paying attention to values of the equation of state (less than -1) [24],[25],[26],[27],[28],[29],[30],[31]. This issue was discussed also by Nojiri and Odintsov (and their collaborators) who checked the possibility and probability of the universe having an equation of state with phantom behavior [28],[31],[32],[33],[34]. On the other hand, the use of \( Q \) as a non-gravitational interaction term between dark matter and dark energy is a way to avoid the coincidence problem [35],[36],[37],[38],[39],[40]. The observations support also a possible energy flow direction change which in literature is known as sign changing non-gravitational interactions. Moreover, non-linear interaction terms has been proposed in some works [29],[41],[42],[43],[44]. The non-linear interaction as a phenomenological approach can be used to study dark energy related problems. Therefore, physicists have open hands for selection and comparison of linear and nonlinear models. On the other hand, the profound relation between the gravitational terms which describe in the bulk and the first law of thermodynamic can lead to different ideas of holography [45]. In recent years, brane theories embedded in a higher dimensional space-time has attracted more attention [46],[47],[48],[49]. In these theories the cosmic evolution is explained by a Friedmann equation interacting with the bulk’s effects onto the brane. The most popular model in the framework of braneworld has proposed as DGP which stands for DvaliGabadadze-Porrati [50]. In DGP model the four dimensional universe is turned to five dimensional Minkowskian bulk. The self-accelerating characteristic of DGP model is able to convey the late time cosmic speed up without relation to dark energy [50],[51]. On the other hand, this characteristic of DGP model cannot satisfy the phantom line crossing and for this issue adding an energy feature on the brane is required [45]. Regarding this, an added dark energy component to the brane models lead to emergence of a novel way of explanation for late time acceleration and also better compatibility with observational points [45]. To check the usability of various models in the context of different cosmological frameworks, one can study the types of evolution and also behavior of the models under the accurate conditions. Despite that the evolution of cosmic expansion defined by Hubble parameter (\( H \)) and the rate of acceleration and deceleration of this expansion are defined by \( q \) and \( \omega_D \), we are not able intelligibly to identify variety of dark energy models by the use of these two parameters since for all cases \( H > 0 \) or \( q < 0 \). Hence, in order to have accurate calculations about this issue and due to the development in observational data during the recent two decades a new geometrical diagnostic pair-known as the StateFinder pair- for tracking the dark energy models has been proposed [52],[53].

\[
\begin{align*}
  r &= \frac{\dddot{a}}{aH^3} = 1 + \frac{\dddot{H}}{H^3} + 3\frac{\ddot{H}}{H^2} \\
  s &= \frac{r - 1}{3(q - \frac{1}{2})}.
\end{align*}
\] (1)

This tool open a new way to specify the features of dark energy and check the distance from the main HDE models. By the use of this advantageous tool, cosmologists trace the path of current models. To find out the behavior of the dark energy models, also one can use \( Om \)-diagnostic tool. The \( Om \)-diagnostic tool due to its dependency on expansion rate specifies more easier from observations than StateFinder pair [54]. The plot of this tool has two parts: Phantom-like part for positive trajectories and quintessence for negative trajectories. The \( Om \)-diagnostic term can be written as

\[ Om(x) = \frac{h(x)^2 - 1}{x^3 - 1}, \] (2)
where $h(x) = H(x)/H_0$ and $x = \ln(z + 1)^{-1}$. The mentioned discussion of the diagnostic tools has been made for understanding the behavior of a new dark energy model, but it cannot give us any advantageous information about the situation of stability of the model. From this, by employing the squared sound speed $v_s^2$ [3], checking the stability of the models against perturbations of the background will be achievable.

In this paper, motivated from aforementioned cases we would like to study a new model of HDE (NHDE) based on DGP braneworld with consideration of a non-gravitational and non-linear interaction between dark energy and dark matter. We investigate the behavior of present model in the context of the deceleration parameter and the equation of state. We also use the StateFinder pair and OM-diagnostic tool for investigation of the new HDE presented in this work and taking into account four types of nonlinear interactions. Moreover, we test the stability of the present models using the squared sound speed. In particular, the analysis of the models with the help of $\chi^2_{min}$ and Markov chain Monte Carlo (MCMC) method using BAO, cJLA, CMB, SGL and OHD observational data is performed. We can see that the present interactions are compatible with observations and make stable models in order to investigation of dark energy behavior. We can also see that the phantom behavior is accessible in these models. The study shows that the extra component of DGP framework should have a limited value to be consisted with the observational data. For studying the CMB power spectrum in the linear regime, we modified the standard Einstein-Boltzmann CAMB code [55, 56].

The structure of this paper is as follows. In the next section, we introduce the New Holographic Dark Energy model (NHDE). In section III, we present four phenomenological interactions to reach the proper terms for Hubble and dark energy for checking the evolution of the Universe. In section IV, employing the Om-diagnostic tool and the StateFinder pair, we investigate characteristics of the models. In section V, we extend the study to check the stability of the models. Finally, in the section VI, using the latest observational data free parameters in four different models will be constrained and also the appropriate cosmological model will be selected using Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). In addition to this, an appropriate modification of CAMB code is used to compare the behavior of the models with $\Lambda$CDM model. The last section is devoted to concluding and remarks.

II Background Evolution

It is well-known that a homogeneous and isotropic Friedmann-Roberston-Walker universe can be described by

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right),$$

where $k = 0, 1, -1$ denote a flat, closed and open universe respectively. According to the equation above and calculation of [10] the Friedmann equations in DGP braneworld can be written as

$$H^2 + \frac{k}{a^2} = \left(\sqrt{\frac{\rho}{3M_P^2}} + \frac{1}{4r_c^2} + \frac{\epsilon}{2r_c}\right)^2,$$

where $M_P^2 = 8\pi G$ and $\rho = \rho_D + \rho_m$. In the consideration of limitation $r_c \rightarrow \infty$, the ordinary Friedmann equation is recovered. In the modern cosmology, by the use of modern observations, we know that the Universe is spatially flat. Hence, a flat Friedmann-Roberston-Walker equation for $r_c \gg 1$ can be written as

$$H^2 - \frac{\epsilon}{r_c}H = \frac{\rho}{3M_P^2},$$

where $r_c = \frac{M_P^2}{2M^2} = \frac{G \rho}{2\pi}$ stands for the crossover length scale determining the transition from 4D to 5D behavior, $\epsilon = \pm 1$ corresponds to the two branches (self-accelerated and normal) of solution [45].
The $\epsilon = +1$ is related to the self-accelerating solution in which the universe may enter an accelerating phase in the late time without additional dark energy component. The $\epsilon = -1$ corresponds to the universe which is accelerated provided that the dark energy component is set on the brane. Using these concepts of the DGP braneworld model, many authors have analyzed the physical behavior of the universe in order to constrain the cosmic parameters and check the changes in different models of dark energy [57], [58], [59], [60], [61], [62], [63]. Assuming the following terms

$$\Omega_m = \frac{\rho_m}{3M_p^2 H^2}, \quad \Omega_D = \frac{\rho_D}{3M_p^2 H^2}, \quad \Omega_{rc} = \frac{1}{3r_c^2 H^2},$$

the Eq. 5 changes as follows

$$\Omega_m + \Omega_D = 1 - 2\epsilon \sqrt{\Omega_{rc}}.$$

The effect of limit consideration of $r_c$ in comparison with Hubble scale can be found in the dimensionless parameter $\Omega_{rc}$ Eq. 6 similar to $\Omega_D$ and $\Omega_m$ and even $\Omega_k = -K/H_0^2$. As it mentioned for $r_c \rightarrow \infty$ the Eq. 5 tend to standard cosmology. This component tied with the amount of dark energy and dark matter in the universe in this framework as the following term

$$\Omega_D + \Omega_m + 2\sqrt{\Omega_{rc}} = 1,$$

in which $\Omega_{rc} = (4r_c^2 H^2)^{-1}$. According to this equation, considering $\Omega_{rc}$ approximately bigger than 0.1 could result in the lack of proportion between amount of the dark energy and dark matter. For instance, let us to consider $\Omega_{rc} = 0.1$. According to this assumption we can easily observe $\Omega_D + \Omega_m = 0.36$ which cannot satisfy the issues of accelerating universe, formation of galaxies, the speed of the galaxies’ rotation and even large scale structure. In this study, as will be demonstrated in future, we obtained $\Omega_{rc}$ less than 0.003 which results in $\Omega_D + \Omega_m > 0.89$. The modified form of Eqs. 4 and 5 would be

$$\Omega_k + \left(\sqrt{\Omega_{rc} + \Omega_D} \right) = 1,$$

in which $\Omega = \frac{8\pi G \rho}{3H^2}$ and one can study the behavior of the model under the existence of the curvature. The energy density of the new holographic dark energy (NHDE) is given by the following relation [45]

$$\rho_D = \frac{3 c^2}{L^2} \left(1 - L \epsilon \frac{r_c}{3} \right), \quad \Omega_D = \epsilon^2 \left(1 - 2 \epsilon \sqrt{\Omega_{rc}} \right),$$

in which $L = H^{-1}$ is the Hubble horizon as the system’s IR cutoff. In what follows, by this choice for the system’s IR cutoff and constraining the present model by use of the latest observational data, we study the evolution of equation of state (EoS) and the deceleration parameter. We also survey the physical aspects of the current model by the use of two diagnostic tools known as Om-Diagnostic and StateFinder pair and we examine the stability of the model.

### III Interacting NHDE

In this section, following recent work [39] we would like to introduce the forms of non-gravitational interactions considered in this paper. But before, we would like to mention, that in modern cosmology, the non-gravitational interaction between any kind of dark energy and dark matter is understood in the following way

$$\dot{\rho}_m + 3H \rho_m = Q,$$

$$\dot{\rho}_D + 3H (\rho_D + P_D) = -Q,$$
which $Q$ is the phenomenological interaction term indicating energy flow between the components. Now let us consider the form of non-gravitational interaction \[Q\] as

\[Q = 3Hb q^n \left( \rho + \frac{\rho_D^2}{\rho_D + \rho_m} \right), \tag{13}\]

where $n$ is a positive constant, $q$ is the deceleration parameter with $-1 - \frac{\dot{H}}{H^2}$ definition term, $H$ is the Hubble parameter and $\rho$ would be either the dark energy density and dark matter density. This kind of interactions are very common types of non-linear interactions (as we see in [64], [65]). Choosing $n = 0$ in the Eq. 13 the interactions will have the fixed sign during whole evolution of the Universe. In particular, we will consider the following 4 forms of the interaction

\[Q_1 = 3bH \left( \rho_D + \frac{\rho_D^2}{\rho_D + \rho_m} \right), \tag{14}\]

\[Q_2 = 3bH \left( \rho_D + \rho_m + \frac{\rho_m^2}{\rho_D + \rho_m} \right), \tag{15}\]

\[Q_3 = 3bH \left( \rho_D + \rho_m + \frac{\rho_D^2}{\rho_D + \rho_m} \right), \tag{16}\]

\[Q_4 = 3bH \left( \rho_m + \frac{\rho_m^2}{\rho_D + \rho_m} \right), \tag{17}\]

identifying them in the rest of the paper as Model 1, Model 2, Model 3 and Model 4, respectively, where $b$ is the coupling constant, $H$ is the Hubble parameter, $\rho_D$ is the density of dark energy of the present work (NHDE) and $\rho_m$ is the density of dark matter. To simplify future calculations and due to the consideration only different types of interaction it is reasonable to obtain some mathematical pattern, which can be used for any form of interaction term. In this regards, taking time derivative of Eq. 5 and using Eqs. 5, 11 and 12 yields

\[P_D = \frac{-2}{3} \frac{\dot{H}}{H^2} \rho_m - \frac{c M_P^2}{H} \frac{\dot{H}}{H} - \rho_m. \tag{18}\]

Combining the Eqs. 9, 10, 12 and 18 we have

\[\dot{\Omega}_D + \left( 2 \Omega_D - 2 - 3 \left( \frac{\Omega_D - c^2}{c^2} \right) \right) \frac{\dot{H}}{H} + 3 \left( -1 + 2 \Omega_D - 3 \left( \frac{\Omega_D - c^2}{c^2} \right) + \frac{\Omega_i}{3} \right) H = 0, \tag{19}\]

in which $\Omega_i = Q(3M_P^2 H^3)^{-1}$. Now, for scrutinizing the evolution of the universe using Eqs. 10 and 19 we have the following two differential equations

\[\frac{dH(z)}{dz} = \frac{H(z)}{1 + z} \left( \frac{6 + 3 \Omega_D - 9 \left( \frac{\Omega_D - c^2}{c^2} \right) - \Omega_i}{1 + c^2 + \Omega_D \left( 1 - \frac{3}{c^2} \right)} \right), \tag{20}\]

\[\frac{d\Omega_D(z)}{dz} = \frac{\left( c^2 - \Omega_D \right)}{1 + z} \left( \frac{6 + 3 \Omega_D - 9 \left( \frac{\Omega_D - c^2}{c^2} \right) - \Omega_i}{1 + c^2 + \Omega_D \left( 1 - \frac{3}{c^2} \right)} \right). \tag{21}\]

For clarification of the calculations in the next parts one can write the Eq. 20

\[\frac{\dot{H}}{H^2} = \left( \frac{-6 - 3 \Omega_D + 9 \left( \frac{\Omega_D - c^2}{c^2} \right) + \Omega_i}{1 + c^2 + \Omega_D \left( 1 - \frac{3}{c^2} \right)} \right). \tag{22}\]

These equations explain the behavior of Hubble parameter and the energy desity of NHDE and will be used in the calculations by solving numerically.
IV State of the Universe

In this section, we would like to present and discuss the behavior of the models using the deceleration parameter and the equation of state. In order to simplify discussion we have organized two subsections namely, the deceleration parameter and the equation of state.

IV.I The deceleration parameter

The deceleration parameter is defined by

\[ q = -\frac{\ddot{a}}{a^2} = -1 - \frac{\dot{H}}{H^2}, \]  \hspace{1cm} (23)

where \( a \) is the scale factor of the universe, \( H \) is the Hubble parameter and dots indicate the time derivative. The expansion of the universe will be accelerated if \( \ddot{a} > 0 \) and in this case the deceleration parameter turns to be negative. Using the Eq. \( 22 \) we find

\[ q = -1 - \left( \frac{-6 - 3\Omega_D + 9(\Omega_{DC^2}) + \Omega_i}{1 + c^2 + \Omega_D (1 - \frac{3}{c^2})} \right). \]  \hspace{1cm} (24)

In Fig. 1 we plotted the deceleration parameter for all nonlinear interaction models. Using the deceleration parameter we can find the time of shifting from decelerating to accelerating universe[66]. All models due to the imposing an interaction between dark energy and dark matter show an accelerating universe \( q < 0 \) shifting from matter dominated to dark energy dominated era. Observations suggest that the transition point from decelerating to accelerating time in the redshift range of \( z \approx 0.6 \) and in range of \( 0.45 < z < 1 \) \[67,68,69,63,70,71\]. According to this we can see that the Model 1 can satisfy the condition of accelerating universe in the suitable era at redshift \( z = 0.55 \). While, three other models enter in the dark energy dominated era after \( z = 0.3 \). In conclusion, these models are reliable to check the deceleration parameter in various dark energy models.

IV.II The equation of state

Taking time derivative of Eq. \( 10 \) and using Eqs. \( 12 \) and \( 20 \) we can study the evolution of EoS

\[ \omega_D = -1 - \frac{1}{3\Omega_D} \left( \Omega_i + (3c^2 - \Omega_D) \frac{\dot{H}}{H^2} \right), \]  \hspace{1cm} (25)

Using the Eq. \( 20 \) we easily reach the following term

\[ \omega_D = -1 - \frac{1}{3\Omega_D} \left( \Omega_i + (3c^2 - \Omega_D) \left( \frac{-6 - 3\Omega_D + 9\left(\frac{\Omega_D}{c^2}\right) + \Omega_i}{1 + c^2 + \Omega_D (1 - \frac{3}{c^2})} \right) \right). \]  \hspace{1cm} (26)

Regarding to the Fig. 2 we can discuss the equation of state for all models. As this figure shows, according to the relation between the deceleration parameter and the equation of state, we can see that only model 1 cross the line of \( \omega_D = -0.33 \) in redshift range \( 0.45 < z < 1 \) which is the transition phase \[67,68,69,63,70,71\]. Of course, all models except for model 4 have the ability of crossing the phantom divided line \( \omega_D = -1 \) at the late time \( z < 0 \). For further explanations, if we consider the term \( \rho_m + \rho_D \) in the denominator of all models as the introducing term and affectless for distinguishing the behavior of models from each other, we can see that removing the dark energy component from the interactions results in the inability of phantom line crossing. This phenomenon can be seen in model 4 and similar to the linear interaction \( 3\rho_m^2 \rho_m \). In this figure, it can be observed that the present model (NHDE) has the behavior similar to \( \Lambda \)CDM model for non-interacting model. The interacting model also has a tendency to \( \Lambda \)CDM model.
Figure 1: The evolution of the deceleration parameter in terms of redshift for four types of interaction. Model 1, Model 2, Model 3 and Model 4 also correspond to \( Q_1 = 3bH \left( \rho_D + \rho_D^2 / (\rho_D + \rho_m) \right) \), \( Q_2 = 3bH \left( \rho_D + \rho_m + \rho_m^2 / (\rho_D + \rho_m) \right) \), \( Q_3 = 3bH \left( \rho_D + \rho_m + \rho_D^2 / (\rho_D + \rho_m) \right) \), \( Q_4 = 3bH \left( \rho_m + \rho_m^2 / (\rho_D + \rho_m) \right) \), respectively. Dashed line indicates the non-interacting and solid line indicates the interacting model according to the best fitted value of the decoupling constant inserted in Table 1.

V Diagnostic recognition

In this section we are going to present and discuss the behavior of the models using StateFinder and \( Om \) analysis. In order to simplify future discussion we have organized two subsections namely, the StateFinder pair and the \( Om \)-diagnostic tool.

V.I The StateFinder pair

Using the \( q \), we have plotted the StateFinder pair \((s \text{ in terms of } r)\) in Fig. 3. For \( \left( \frac{\ddot{H}}{H^3} - 2 \frac{\dot{H}}{H^2} \right) \) in Eq. 1 by taking the time derivative of both sides of Eq. 20, we have

\[
\frac{\ddot{H}}{H^3} = \left( -3\Omega_D \left( \frac{3}{c^2} - 1 \right) + 3b\Omega_D \right) \left( \Omega_D \left( \frac{3}{c^2} - 1 \right) - c^2 - 1 \right) + 3\Omega_D \left( \frac{3}{c^2} - 1 \right) \left( 1 + \Omega_D \left( \frac{3}{c^2} - 1 \right) - 3 + 3b\Omega_D \right) \times \left( \Omega_D \left( \frac{3}{c^2} - 1 \right) - c^2 - 1 \right)^{-2} + 2 \left( \frac{\dot{H}}{H^2} \right)^2.
\]  
(27)
Figure 2: The evolution of the equation of state in terms of redshift for all models. Model 1, Model 2, Model 3 and Model 4 also correspond to $Q_1 = 3bH \left( \rho_D + \rho_D^2 / (\rho_D + \rho_m) \right)$, $Q_2 = 3bH (\rho_D + \rho_m + \rho_D^2 / (\rho_D + \rho_m))$, $Q_3 = 3bH (\rho_D + \rho_m + \rho_D^2 / (\rho_D + \rho_m))$, $Q_4 = 3bH (\rho_m + \rho_D^2 / (\rho_D + \rho_m))$, respectively. Dashed line indicates the non-interacting and solid line indicates the interacting model according to the best fitted value of the decoupling constant inserted in the Table 1.

in which $\Omega_D' = \frac{\dot{\Omega}_D}{H}$ and taking time derivative of Eq. 10 yields

$$\dot{\Omega}_D = \frac{2cc^2 \sqrt{\Omega_{cc}} \dot{H}}{3 \dot{H}}.$$  

(28)

It can be seen that the in all models, as the universe expands, by increasing the value of parameter $r$, the parameter $s$ moves from positive to negative values. The fixed point $(r, s) = (1, 0)$ represents the $\Lambda$CDM scenario. Checking the track of each case shows us that Model 1, 2 and 3 has the Chaplygin gas behavior ($s < 0, r > 1$) and model 4 has the quintessence behavior ($s > 0, r < 1$). All models’ trajectory meet the fixed point $(1, 0)$ indicating the evolution from quintessence to phantom-like behavior as the universe expands. The behavior of Model 1 also by far in comparison to the other models is close to $\Lambda$CDM. Moreover, for simple power law evolution of the scale factor $a(t) \approx t^{0.66\alpha}$, it can be easily found $r = (1 - 3\alpha) (1 - 1.5\alpha)$ and $s = \alpha$ [72]. Accordingly, $s < 0$ corresponds to a phantom-like dark energy appearing in Models 1, 2 and 3. This is an affirmation on the equation of state results.
Figure 3: The $s$ in terms of $r$ for four types of interaction. Model 1, Model 2, Model 3 and Model 4 also correspond to $Q_1 = 3bH \left( \rho_D + \rho_D^2 / (\rho_D + \rho_m) \right)$, $Q_2 = 3bH \left( \rho_D + \rho_m + \rho_m^2 / (\rho_D + \rho_m) \right)$, $Q_3 = 3bH \left( \rho_D + \rho_m + \rho_D^2 / (\rho_D + \rho_m) \right)$, $Q_4 = 3bH \left( \rho_m + \rho_m^2 / (\rho_D + \rho_m) \right)$, respectively. Dashed line indicates the non-interacting and solid line indicates the interacting model according to the best fitted value of the decoupling constant inserted in the Table 1. Star symbol denotes the $\Lambda$CDM model and bold dot represents the present value in each case.

Figure 4 shows the $\Omega_m$-diagnostic trajectories for all models. The advantage of the $\Omega_m$-diagnostic is its less dependency on the matter density relative to the equation of state of dark energy. In this figure we can analyze the results according to two sight of view. Firstly, if the $\Omega_m$ of the models are bigger than $\Omega_0$ the model has the quintessence behavior. If the $\Omega_m$ of the models are less than $\Omega_{m0}$ the model has the phantom-like behavior. Here we consider the $\Omega_{m0} = 0.287$. Secondly, If the value of $\Omega_{m0}$ is not exactly known, since the positive, null and negative correspond to phantom ($\omega < -1$), $\Lambda$CDM ($\omega = -1$) and quintessence($\omega > -1$), respectively. According to the both cases, one can see the 4 models convey a phantom-like behavior.

V.II The $\Omega_m$-diagnostic tool

Fig. 4 shows the $\Omega_m$-diagnostic trajectories for all models. The advantage of the $\Omega_m$-diagnostic is its less dependency on the matter density relative to the equation of state of dark energy. In this figure we can analyze the results according to two sight of view. Firstly, if the $\Omega_m$ of the models are bigger than $\Omega_0$ the model has the quintessence behavior. If the $\Omega_m$ of the models are less than $\Omega_{m0}$ the model has the phantom-like behavior. Here we consider the $\Omega_{m0} = 0.287$. Secondly, If the value of $\Omega_{m0}$ is not exactly known, since the positive, null and negative correspond to phantom ($\omega < -1$), $\Lambda$CDM ($\omega = -1$) and quintessence($\omega > -1$), respectively. According to the both cases, one can see the 4 models convey a phantom-like behavior.
VI Stability

In order to test viability of a new dark energy model we refer to investigate the stability of the model against perturbation. The behavior of square sound speed \( v_s^2 \) \([5]\) as an approach to check the stability of a new dark energy model can be studied. It is claimed that the sign of \( v_s^2 \) is important to specify the stability of background evolution. The signs of squared sound speed \( v_s^2 > 0 \) and \( v_s^2 < 0 \) denote the a stable and instable universe against perturbation respectively. The perturbed energy density of the background in a linear perturbation structure is

\[
\rho(x,t) = \rho(t) + \delta \rho(x,t), \tag{29}
\]

in which \( \rho(t) \) is unperturbed energy density of the background. The equation of energy conservation is \([5]\)

\[
\delta \dot{\rho} = v_s^2 \nabla^2 \delta \rho(x,t). \tag{30}
\]
For positive sign of squared sound speed the Eq. 30 will be a regular wave equation which its solution can be obtained as 
\[ \delta \rho = \delta \rho_0 e^{-i\omega_0 t + ikx} \]
indicating a propagation state for density perturbation. It is easy to see that the squared sound speed can be written as

\[ \frac{v^2_s}{\rho} = \frac{\dot{\rho}}{\rho} = \frac{\dot{\omega}D\rho}{\rho} + \omega_D, \] (31)

taking time derivative of Eqs. 10 and 25 and combining with Eqs. 27 and 31 one can plot the evolution of \( v^2_s \) in terms of redshift as it is shown in Fig. 5. During the cosmic evolution, the four models in comparison with GDE [73, 74], SMHDE [75], ADE [76] and also HDE in the standard cosmology which are instable against perturbations [77] show stability against background perturbations in early time, present and late time.

Figure 5: The evolution of \( v^2_s \) versus redshift. The positive value of trajectory for each model shows the stability against perturbation of the background. The Model 1, Model 2, Model 3 and Model 4 correspond \( Q_1 = 3bH(\rho_D + \rho_D^2/(\rho_D + \rho_m)), \)
\( Q_2 = 3bH(\rho_D + \rho_m + \rho_m^2/(\rho_D + \rho_m)), \)
\( Q_3 = 3bH(\rho_D + \rho_m + \rho_D^2/(\rho_D + \rho_m)), \)
\( Q_4 = 3bH(\rho_m + \rho_m^2/(\rho_D + \rho_m)), \) respectively. Dashed line indicates the non-interacting and solid line indicates the interacting model according to the best fitted value of the decoupling constant inserted in the Table 1.
Table 1: Best fit values of free parameters and the corresponding 1σ and 2σ intervals. The best fit values are inferred from minimizing $\chi^2$ in the local MCMC chains. AIC and BIC stand for Akaike Information Criterion and Bayesian Information Criterion respectively. $\Delta$AIC = AIC$_{i}$ - AIC$_{min}$, $\Delta$BIC = BIC$_{i}$ - BIC$_{min}$. From this not only the $\text{Eqs.21 and } \Omega_{cJLA + BOSS DR12 + CMB + OHD + SGL}$ parameters, but also $\text{Eqs.24 and } \Omega_{b}$, $\Omega_{c}$, $H_{0}$, $\Omega_{\Lambda}$ should be used for obtaining the exact behavior of the current model with various interactions.

| Model 1 | Model 2 | Model 3 | Model 4 |
|---------|---------|---------|---------|
| 3.4978  | 5.7101  | 3.5183  | 1.7453  |
| 197.9325| 213.7137| 198.1869| 180.4568|
| 0.1232  | 0.1127  | 0.1502  | 0.1264  |
| 0.2688  | 0.3346  | 0.3371  | 0.2435  |
| 0.00067 | 0.00065 | 0.00021 | 0.00014 |

**Note:** The table contains a mix of numbers and text, but the main focus is on the parameters and their uncertainties. The table is designed to compare different models and their associated information criteria (AIC and BIC) to determine the best fit. The models include variations in parameter values and their corresponding intervals. The best fit values are inferred from minimizing $\chi^2$. The models are compared using $\Delta$AIC and $\Delta$BIC to evaluate the performance of each model.
VII Data Analysis Methods

To analyze the models and to obtain the best fit values for the model parameters, in this paper we combine the latest observational data including SN Ia, BAO, CMB and OHD to fit the free parameters. For this purpose, we implement the Metropolis-Hastings algorithm to perform MCMC simulation in order to fit the cosmological parameters for 1σ and 2σ confidence area. This method also provides reliable error estimates on the measured variables.

For the Supernova type Ia (SNIa) we consider the compressed Joint Light Analysis (cJLA) data set of 31 check points with the redshift range $0.01 < z < 1.3$ [78]. The $\chi^2$ function for SNIa as a good approximation of the full JLA likelihood is

$$\chi^2_{SNIa} = r^t C_b^{-1} r, \quad (32)$$

in which

$$r = \mu_b - M - 5 \log_{10} d_L, \quad (33)$$

in which $C_b$ is the covariance matrix of $\mu_b$ [78]. The $\mu_b$ denotes the observational distance modulus, $M$ is a free normalization parameter which should be fitted. The dimensionless luminosity distance may be expressed as

$$d_L = \frac{c(1 + z)}{H_0} \int_0^{z'} \frac{dz'}{H(z)}. \quad (34)$$

We use the BOSS DR12 [79] including six data points of BAO as the latest observational data for BAO. The $\chi^2_{BAO}$ can be explained as

$$\chi^2_{BAO} = X^t C_{BAO}^{-1} X, \quad (35)$$

where $X$ for six data points is

$$X = \begin{pmatrix}
D_M(0.38)r_s(z_d) & \frac{H(0.38)}{r_s(z_d)} & -1512.39 \\
D_M(0.51)r_s(z_d) & \frac{H(0.51)}{r_s(z_d)} & -81.208 \\
D_M(0.61)r_s(z_d) & \frac{H(0.61)}{r_s(z_d)} & -90.9 \\
D_M(0.38)r_s(z_d) & \frac{H(0.38)}{r_s(z_d)} & 1975.22 \\
D_M(0.51)r_s(z_d) & \frac{H(0.51)}{r_s(z_d)} & 98.964 \\
D_M(0.61)r_s(z_d) & \frac{H(0.61)}{r_s(z_d)} & 2306.68
\end{pmatrix}, \quad (36)$$
and \( r_{s,fid} = 147.78 \) Mpc is the sound horizon of fiducial model, \( D_M(z) = (1 + z) D_A(z) \) is the comoving angular diameter distance. The sound horizon at the decoupling time \( r_s(z_d) \) is defined as

\[
r_s(z_d) = \int_{z_d}^{\infty} \frac{c_s(z)}{H(z)} dz,
\]

in which \( c_s = 1/\sqrt{3(1 + R_b/(1 + z))} \) is the sound speed with \( R_b = 31500\Omega_b h^2 (2.726/2.7)^{-4} \). The covariance matrix \( Cov_{BAO} \) is:

\[
C_{BAO}^{-1} =
\begin{pmatrix}
624.707 & 23.729 & 325.332 & 8.34963 & 157.386 & 3.57778 \\
23.729 & 5.60873 & 11.6429 & 2.33996 & 6.39263 & 0.968056 \\
325.332 & 11.6429 & 905.777 & 29.3392 & 515.271 & 14.1013 \\
8.34963 & 2.33996 & 29.3392 & 5.42327 & 16.1422 & 2.85334 \\
157.386 & 6.39263 & 515.271 & 16.1422 & 1375.12 & 40.4327 \\
3.57778 & 0.968056 & 14.1013 & 2.85334 & 40.4327 & 6.25936
\end{pmatrix}.
\]

(37)

Descovering the expansion history of the universe, we check Cosmic Microwave Background (CMB). For this, we use the data of Planck 2015 [80]. The \( \chi^2_{CMB} \) function may be explained as

\[
\chi^2_{CMB} = q_i - q_i^{data} Cov_{CMB}^{-1}(q_i, q_j),
\]

(39)

where \( q_1 = R(z_*) \), \( q_2 = l_A(z_*) \) and \( q_3 = \omega_b \) and \( Cov_{CMB} \) is the covariance matrix [80]. The data of Planck 2015 are

\[
q_1^{data} = 1.7382, \\
q_2^{data} = 301.63, \\
q_3^{data} = 0.02262.
\]

(40, 41, 42)

The acoustic scale \( l_A \) is

\[
l_A = \frac{3.14 d_L(z_*)}{(1 + z) r_s(z_*)},
\]

(43)

in which \( r_s(z_*) \) is the comoving sound horizon at the drag epoch \( (z_*) \). The function of redshift at the drag epoch is [81]

\[
z_* = 1048 \left[ 1 + 0.00124 (\Omega_b h^2)^{-0.738} \right] \left[ 1 + g_1 (\Omega_m h^2)^{g_2} \right],
\]

(44)

where

\[
g_1 = \frac{0.0783 (\Omega_b h^2)^{-0.238}}{1 + 39.5 (\Omega_b h^2)^{-0.763}}, \\
g_2 = \frac{0.560}{1 + 21.1 (\Omega_b h^2)^{1.81}}.
\]

(45)

The CMB shift parameter is [82]

\[
R = \sqrt{\Omega_m} \frac{H_0}{c} r_s(z_*).
\]

(46)

The reader should notice that the usage of CMB data does not provide the full Planck information but it is an optimum way of studying wide range of dark energy models.

In order to study the expansion time line of the universe, the determination of Hubbe parameter using observational data is the other important part of fitting parameters. The \( \chi^2_{OHD} \) is

\[
\chi^2_{OHD} = \sum_{i=1}^{n} \frac{[H_{obs}(z_i) - H_{th}(z_i)]^2}{\sigma_i^2},
\]

(47)
We use 43 data points in the redshift range $0 < z < 2.5$. The data for BAO and CMB could be found in the online source of latest version of MontePython. For data of cJLA please refer to Joint Light-curve Analysis online sources. Using minimized $\chi^2_{\text{min}}$, we can constrain and obtain the best-fit values of the free parameters.

$$\chi^2_{\text{min}} = \chi^2_{\text{SNIa}} + \chi^2_{\text{CMB}} + \chi^2_{\text{BAO}} + \chi^2_{\text{SGL}} + \chi^2_{\text{OHD}}.$$  

(48)

The best-fit values of $\Omega_D$, $H_0$, $\Omega_{m0}$, $c$, $b^2$ and $M$ by consideration of the $1\sigma$ and $2\sigma$ confidence level are shown in the Table 1. As the powerful probe in the study of cosmology Strong Gravitational Lensing is used for fitting free parameters combining with other observational data too. Despite the fact that $\chi^2$ is known as the effective way of understanding the best values of free parameters, it cannot be only used to determine the best model between variety of models. Hence, for this issue Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) have been proposed. For further information see [84], [85], [86], [87]. The AIC was based on the Kullback-Leibler in formation entropy can be written as

$$AIC = -2 \ln \mathcal{L}_{\text{max}} + 2k,$$

(49)

where $-2 \ln \mathcal{L}_{\text{max}} = \chi^2_{\text{min}}$ is the highest likelihood, $k$ is the number of free parameters (4 for $\Lambda$CDM and 6 for Models 1 to 4) and $N$ is the number of data points used in the analysis. The BIC is similar to AIC with different second term

$$BIC = -2 \ln \mathcal{L}_{\text{max}} + k \ln N.$$

(50)

In spite of the fact that the differences between models in the deceleration parameter, the equation of state and stability studied in this work are marginal, one can see the best model among them is the Model 1 in terms of good compatibility with observational data. Constraints on free parameters also are shown in Figs 9, 10, 11, 12 and the Table 1. We can see from the results (also, supplementary depiction Fig 7) that Model 1 is by far favored by the AIC and BIC as the chi square

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*http://baudren.github.io/montepython.html
†http://supernovae.in2p3.fr/sdss_snls_jla/ReadMe.html
Moreover, using the modified version of the Boltzmann code CAMB [55, 56] we compare the nonlinear interactions of the present work with ΛCDM model. Using the flat universe and flectless Ω_{rc}, all models coincide on the ΛCDM line which shows the compatibility with the equation of state plots. Once we add curvature term (including Ω_{rc}), we can see the difference between the models. The main difference is in the late ISW before l < 50. It is clear that the Model 1 and Model 4 behaviors result in a suppression of CMB temperature power which means a smaller $C_l^{TT}$ due to the late ISW effect. The $C_l$ is the power spectra and $T$ denotes the temperature. The behavior of the Models 2 and 3 could be neglected due to their similarities to ΛCDM model which do not add any significant result to the studying of interaction’s field. While the peaks did not witness any change, the differences after early ISW and the deviation between models are stemmed from the value of Ω_{rc} or curvature structure of the DGP model which has an explicit effect on the curvature conditions. The results are similar to the results of [90]. It is interesting to note that the Model 1 has the biggest deviation in comparison with ΛCDM as it is obvious in the lower panel, but it is the best model according to observational data. It should be mentioned that the deviation between models is stemmed from the value of Ω_{rc} which has an explicit effect on the curvature condition.

Figure 8: The CMB temperature spectra $C_l^{TT}$. Comparing the four nonlinear interactions in the context of NHDE with ΛCDM model. Here we use the decoupling constant known as the interaction component according to bets fit values of Table 1. The plot of Model 2 and Model 3 due to their small quantity of Ω_{rc} are close to ΛCDM.

V. CONCLUSIONS

In the present work, we studied a New Holographic Dark Energy model (NHDE) with Hubble horizon as IR cutoff in the framework of the flat FRW with taking into account the interaction between dark matter and holographic dark energy. We chose the non-linear type of interaction in order to
test an alternative way of study the holographic dark energy models. In particular we considered and mentioned the forms of the interactions. We use the latest observational data sets, namely SNIa compressed Joint Light-Analysis (cJLA) compilation, Baryon Acoustic Oscillations (BAO) from BOSS DR12 and the Cosmic Microwave Background (CMB) of Planck 2015, OHD and Strong Gravitational lensing (SGL). We found that for all phenomenological interaction models the corresponding universe is expanding and also accelerating. We found that the StateFinder trajectory for all models embrace $\Lambda$CDM model $(r, s) = (1, 0)$ and also behave similar to the both quintessence and Chaplygin gas dark energy models. Using the $\Omega m$-diagnostic tool by taking $x = \ln(z + 1)^{-1}$, the evolution in terms of redshift shows positive values which implies the Phantom-like behavior similar to the results of the equation of state which is the important issue for avoiding the creation of black hole’s mass. We also found that removing dark matter component in interaction terms leads to inability of phantom line crossing. For further investigation, we studied the stability of the considered models using the evolution of the squared sound speed $v_s^2$. In spite of the growth of background perturbations, the Models show suitable stability. The mentioned results have been obtained using the fitted free parameters of the present model. We used MCMC (MH) with consideration of the equation of state and the deceleration parameter with $2 \times 10^6$ steps in order to reach the proper results. And for finding the best model with the help of AIC and BIC criteria penalizing the introduction of additional parameters the Model 1, $Q_1 = 3bH(\rho_D + \rho_D^2/(\rho_D + \rho_m))$ has been chosen as the preferable one. This should note that the differences between models without AIC and BIC comparison are marginal but the best one is the Model 1 as it is mentioned. In addition, the deceleration parameter for the Model 1 showed an accelerating universe undergoes from matter to dark energy dominated era roughly at $z = 0.6$ which has a good compatibility with recent observational data $0.4 < z < 1$. In addition in this study we found that the $\Omega_{rc}$ has the value less than 0.003 leading to $\Omega_D + \Omega_m > 0.89$. On the other hand, using the Einstein-Boltzmann CAMB code, we studied the behavior of the models in comparison with $\Lambda$CDM. We found that the consideration of the flat FRW for the models has the behavior similar to $\Lambda$CDM. However, we added curvature with extra density $\Omega_{rc}$ and it can be seen that the main difference between the models and the $\Lambda$CDM is in $l < 50$ at low-$l$ area. Despite the high deviation for the Model 1 compared to $\Lambda$CDM according to Fig.8, it can be seen that this model is chosen as the favorite one by the observational data. Hence, it should be stated that the Model 1 as the different model in comparison with $\Lambda$CDM can be study as a reliable and authentic model for the future studies.

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Figure 9: The contour maps of $H_0, \Omega_D, c, b$ and $\Omega_{orc}$ with 1$\sigma$ (68.3%) and 2$\sigma$ (95.4%) confidence level for the Model 1 with the interaction $Q_1 = 3bH \left( \rho_D + \rho_D^2 / (\rho_D + \rho_m) \right)$ term.
Figure 10: The contour maps of $H_0, \Omega_D, c, b$ and $\Omega_{arc}$ with $1\sigma$ (68.3\%) and $2\sigma$ (95.4\%) confidence level for the Model 2 with the interaction $Q_2 = 3bH \left(\rho_D + \rho_m + \rho_m^2 / (\rho_D + \rho_m)\right)$ term.

Figure 11: The contour maps of $H_0, \Omega_D, c, b$ and $\Omega_{arc}$ with $1\sigma$ (68.3\%) and $2\sigma$ (95.4\%) confidence level for the Model 3 with the interaction $Q_3 = 3bH \left(\rho_D + \rho_m + \rho_m^2 / (\rho_D + \rho_m)\right)$ term.
Figure 12: The contour maps of $H_0$, $\Omega_D$, $c$, $b$ and $\Omega_{arc}$ with $1\sigma$ (68.3\%) and $2\sigma$ (95.4\%) confidence level for the Model 4 with the interaction $Q_4 = 3bH(\rho_m + \rho_m^2 / (\rho_D + \rho_m))$ term.

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