On the combinatorics of several integrable hierarchies

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Abstract

We demonstrate that statistics of certain classes of set partitions are described by generating functions related to the Burgers, Ibragimov–Shabat and Korteweg–de Vries integrable hierarchies.

Keywords: set partition, B type partition, non-overlapping partition, generating function, Bell polynomial, Dowling numbers, Bessel numbers

1. Introduction

In recent decades numerous interrelations have been discovered between combinatorics and the theory of integrable systems. Mainly, these links involve solutions; either special ones, such as the Painlevé transcendents [6] and solitons [10], or generic ones, such as the tau function of the Kadomtsev–Petviashvili hierarchy [1].

However, equations themselves exhibit a certain combinatorial nature due to the recurrence relations which govern the higher symmetries and conservation laws of integrable hierarchies. This aspect has so far been paid less attention, although quite a simple description of the language of set partitions for the Burgers hierarchy has been known for a long time [13, 14]. We reproduce this combinatorial interpretation for the sake of completeness and as a base for further generalizations. New results obtained in the paper are related to the Ibragimov–Shabat and KdV hierarchies, see table 1. In these cases the combinatorics becomes more complicated, since the ordinary set partitions are replaced by special ones which are characterized by additional restrictions. Moreover, this combinatorics comes in disguise: for instance, in the Burgers case we consider the generating function intermediately for the higher flows, but in the KdV case we have to consider a formal series for the logarithmic derivative of the ψ function which solves the Riccati equation (inversion of the Miura map, see e.g. [9]). The flows and conservation laws of the hierarchy are related to this generating function by simple algebraic relations. In the Ibragimov–Shabat case a natural choice of the generating function is dictated by the linearization procedure.
Although we are not interested in ‘explicit’ formulae for the coefficients of generating functions here, it should be mentioned that such formulae for the potential KdV flows actually do exist. One of them, obtained already in \[9\], represents the coefficient of a given monomial as a certain multiple integral. Other expressions found in \[2, 17, 18\] are more combinatorial, but remain very complicated. Only in the case of the pot-Burgers hierarchy can the formula for the coefficients be considered as a truly explicit one.

2. Potential burgers hierarchy

The pot-Burgers hierarchy is obtained from the linear heat equation hierarchy

\[ \psi_n = \psi_t \]

by means of the change of dependent variable \( \psi = \psi^t \). This yields

\[ \nu_n = e^{-\nu} D^n (e^\nu) = (D + \nu)^n(1) = Y_n (\nu_1, ..., \nu_n), \quad n = 0, 1, 2, ... \]

Here and further on we denote the derivatives as follows: \( \nu_n = D^n(\nu), D = \partial/\partial x, \nu_n = \partial \nu/\partial \nu_n. \)

Several first equations (2) are shown in the table 2.

It is easy to see that \( Y_n \) are polynomials with integer coefficients, homogeneous with respect to the weight \( w(\nu_j) = j \). These polynomials play a fundamental role in combinatorics and are known under the name of (complete exponential) Bell polynomials \[5\]. An equivalent definition through the exponential generating functions reads

\[ \sum_{n=0}^{\infty} \frac{Y_n}{n!} x^n = e^{-x} \sum_{n=0}^{\infty} \frac{D^n (e^x)}{n!} = e^{x (x + 1 - y(x))} = \exp \left( \sum_{n=1}^{\infty} \frac{Y_n}{n!} x^n \right) \]

and this immediately implies the explicit formula

\[ Y_n = \sum_{k_1 + 2k_2 + ... + rk_r = n} \frac{n!}{(1!)^{k_1} ... (r!)^{k_r} k_1! ... k_r!} \nu_{1}^{k_1} ... \nu_{r}^{k_r}. \]

Its combinatorial interpretation is obvious:

- monomials correspond to partitions of the number \( n \);  
- coefficients of monomials count partitions of the set \( [n] = \{1, ..., n\} \) into the subsets (or blocks) of prescribed size.
For example, let us list all set partitions for \( n = 2, 3, 4 \):

\[
\begin{array}{c|c|c|c|c|c}
\text{n} & \text{v}_1 & \text{v}_1^2 & \text{v}_2 & \text{v}_3 & \text{v}_4 \\
2 & 1 & 1 & 2 & 4 & 1 + 3 \\
3 & 2 & 1 & 12 & 1234 & 1 + 1 + 2 \\
4 & 1 & 2 & 3 & 12 & 1 + 1 + + +
\end{array}
\]

Recall, that each set partition is considered as an unordered set (with blocks as the elements), that is, ordering of the blocks does not matter. However, it is often useful to enumerate the blocks somehow. For the sake of definiteness, we will adopt the enumeration corresponding to the ordering of the minimal elements in the blocks.

We see that the combinatorics behind the hierarchy (2) is quite simple. The following statement is well known, see e.g. [13, 14].

**Theorem 1.** In the potential Burgers hierarchy, the coefficient of the monomial \( v_1^{k_1} \cdots v_r^{k_r} \) is equal to the number of partitions of the set of \( n = k_1 + 2k_2 + \ldots + rk_r \) elements into \( k_1 \) blocks with 1 element, \( k_2 \) blocks with 2 elements, ..., \( k_r \) blocks with \( r \) elements.

**Proof.** One proof follows intermediately from the explicit formula (3) for the coefficients. However, we will not always have such a formula at hand. The following reasoning provides a more conceptual proof.

Let \( \Pi_{n,k} \) denote the set of all partitions of the set \([n]\) into \( k \) blocks and \( \Pi_n \) denote the set of all partitions of \([n]\). Let us consider operations

\[
d_j : \Pi_{n,k} \to \Pi_{n+1,k}, \quad j = 1, \ldots, k, \quad M : \Pi_{n,k} \to \Pi_{n+1,k+1}
\]
defined, respectively, as appending the element \( n + 1 \) to the \( j \)th block or adding it to the partition as a new singleton. This can be visualized by the following diagram, where rows represent the blocks ordered by their minimal elements and the column on the right shows the available slots for the new element:

(Notice, that in such pictorial representations it is common to allow several blocks to occupy one row, provided that the corresponding intervals do not intersect. Under this convention, the above example could be represented by a diagram with just three rows; however, for our purpose it is more convenient to reserve one row per block.) Starting from the partition \( \{ \emptyset \} \) of the set \( [0] = \emptyset \) and applying operations \( d_j, M \), one can generate any partition of \( [n] \), in a unique way. Indeed, the required sequence of operations is uniquely recovered by deleting elements in the inverse order from \( n \) to 1.

In the theorem, a set partition \( \pi \) with \( k_1 \) 1-blocks, \( ..., k_r \) \( r \)-blocks corresponds to the monomial \( p(\pi) = v_1^{k_1} ... v_r^{k_r} \). The differentiation \( D(p(\pi)) \) by the Leibnitz rule amounts to replacing \( v_i \) with \( v_i + 1 \) for each factor in turn, taking the multiplicity into account. In the partition language, this means that we add the new element to each block in turn. As a result, we obtain the sum of monomials \( p(d_j(\pi)) \) for all admissible values of \( j \). Multiplication of the monomial \( p(\pi) \) by \( v_1 \) gives the monomial \( p(M(\pi)) \). Thus, the polynomials

\[
P_n = \sum_{\pi \in \Pi_n} p(\pi)
\]

are related by the recurrence relation \( P_{n+1} = (D + v_1)(P_n) \) and since \( P_1 = v_1 \), hence \( P_n = Y_n(v_1, ..., v_n) \).

Less detailed statistics are obtained if we forget about sizes of blocks and consider just their number in a given partition. Obviously, this corresponds to summing up the coefficients of terms of the same degree, which gives us the Bell polynomials of one variable

\[
B_n(u) = Y_n(u, ..., u) = (u d_u + u)^n(1) = \sum_{k=0}^{n} \binom{n}{k} u^k.
\]

The coefficient \( \binom{n}{k} \) of \( u^k \), that is, the number of partitions of \( [n] \) into \( k \) blocks, is called the Stirling number of the second kind [16, A048993]:

1 1
0 1 1
0 1 3 1
0 1 7 6 1
0 1 15 25 10 1
0 1 31 90 65 15 1
0 1 63 301 350 140 211 877
By definition, \( \{ \binom{n}{1} \} = 0 \) at \( n > 0 \) and \( \{ \binom{0}{0} \} = \# \{ \emptyset \} = 1 \). The total numbers of set partitions with \( n \) elements, the Bell or the exponential numbers [16, A000110], are given by the sums of the rows:

\[
B_n = B_n(1) = Y_n(1, \ldots, 1) = \sum_{k=0}^{n} \binom{n}{k}, \quad \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = e^{e^z-1}.
\]

**Remark 1.** It is easy to see that the scaling of dependent and independent variables amounts to introducing the factor \( C^k \) for the terms of the \( k \)th degree (this defines the partial Bell polynomials [5]), that is, this transformation is useful in the study of partial or simplified statistics. However, more general substitutions destroy the picture. In the theory of integrability, it is natural to consider equations related by point or contact transformations as equivalent ones, because these transformations do not affect such properties as existence of the higher symmetries, conservation laws and Lax pairs. In contrast, we have seen in this section how a meaningful combinatorics appears just from nothing, by a simple change of variables between equations (1) and (2). The next section demonstrates the changing of combinatorics under a differential substitution. This means that combinatorial interpretation is a more subtle, noninvariant property which is related to a particular form of integrable hierarchy.

### 3. Burgers hierarchy

The right hand sides of equations (2) do not contain \( v \) and this makes the substitution \( u = v_1 \) possible. This brings us to the Burgers hierarchy

\[
u_{1n} = D \left( Y_n(u, \ldots, u_{n-1}) \right), \quad n = 1, 2, \ldots
\]

which is homogeneous with respect to the weight \( w(u_j) = j + 1 \). Several first equations are written down in table 3. What is the combinatorial interpretation in this case? This can be easily understood by the following example, for \( n = 3 \):

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**Table 3.** Burgers hierarchy (weight \( w(u_j) = j + 1 \)).

| Term | Equation |
|------|----------|
| \( u_1 \) | \( u_1 \) |
| \( u_2 \) | \( u_2 + 2uu_1 \) |
| \( u_3 \) | \( u_3 + (3uu_2 + 3u_1^2) + 3u_1u_1 \) |
| \( u_4 \) | \( u_4 + (4uu_3 + 10u_1u_2) + (6u^2u_2 + 12uu_1^2) + 4u^3u_1 \) |
| \( u_5 \) | \( u_5 + (5uu_4 + 15u_1u_3 + 10u_2^2) + (10u^2u_3 + 50uu_1u_2 + 15u^3u_1) + (10u^3u_2 + 30u^2u_1^2) + 5u^4u_1 \) |

---
Certainly, renaming \( v_j \rightarrow u_{j-1} \) does not change the combinatorics. The differentiation amounts to appending the new element to all blocks in turn, however, now we do not add it as a new block. Therefore, the partitions under consideration are constructed as in theorem 1, but we do not apply the operation \( M \) at the last step. As a result, all partitions \( \Pi_n \) are mapped onto some subset of partitions \( \Pi_{n+1} \), namely, those partitions where the element \( n+1 \) does not appear as a singleton. We arrive at the following combinatorial interpretation of equations (4).

**Theorem 2.** In the Burgers hierarchy, the coefficient of the monomial \( u^{k_0} u_1^{k_1} \ldots u_r^{k_r} \) is equal to the number of partitions of the set with one distinguished element into \( k_0 \) blocks with 1 element, \( \ldots \), \( k_r \) blocks with \((r+1)\) element and such that the distinguished element does not constitute 1-block.

As before, one can consider more rough statistics. For instance, setting \( u = 1 \) gives us the total number of partitions under consideration of the set \([n+1]\):

\[
D(Y_n(u, \ldots, u_{n-1}))|_{u_j=1} = B'_n(1) = \sum_{k=1}^{n} \binom{n}{k} \cdot n \geq 1.
\]

The sequence of these numbers (2-Bell numbers) starts

1, 3, 10, 37, 151, 674, 3263, 17007, 94828, 562595, ...

According to [16, A005493], it can also be characterized in many other ways, in particular, as the number of partitions of \([n]\) with distinguished block or as the total number of blocks in all set partitions of \([n]\). These interpretations are obvious as well, since the distinguished blocks can be identified with the blocks enlarged by the operations \( d_j \), and these operations are applied exactly as many times as there are blocks in all partitions.
Table 4 displays the sequence of point changes and substitutions between equation \(\psi_3 = \psi_5\) and the Ibragimov–Shabat equation \([11]\)
\[
\psi_3 = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1. 
\]

Although this transformation looks quite harmless, it partially destroys the symmetry algebra: in the variables \(\psi_i\), it consists of equations (1) of arbitrary order, while only odd order equations survive in the variables \(u\). Indeed, the change \(\psi^2 = s\) brings to equation \(s_n = \ldots\) where the right hand side is a full derivative only if \(n\) is odd:
\[
s_n = 2\psi\psi_\mu = D \left( 2\psi\psi_{\mu-1} - 2\psi_\mu\psi_{\mu-2} + 2\psi_\mu\psi_{\mu-3} + \ldots \pm \psi^{2}_{(n-1)/2} \right). \tag{5}
\]

In the analogous equation for even \(n\) the term \(\psi^{2}_{n/2}\) remains outside the parentheses, that is, \(s_n \not\in \text{Im } D\), and therefore the further substitution \(s = q_1\) leads out of the class of evolutionary equations. The structure of odd flows is described by the following statement.

**Statement 3.** Let us denote \(D_i = \partial_i + z^2\partial_i + z^4\partial_i + \ldots\), \(A = A(z) = a_0 + a_1z + a_2z^2 + \ldots\), \(\bar{A} = A(-z)\), then the Ibragimov–Shabat hierarchy is equivalent to equations
\[
D_i(u) = \frac{1}{2u} D(\bar{A}\bar{A}) = \frac{1}{2z} (A - \bar{A}) - uA\bar{A}, \tag{6}
\]
\[
z (D + u^2)(\lambda) = A - u. \tag{7}
\]

**Proof.** Let us consider the generating function
\[
\Psi = \psi + \psi_1 z + \psi_2 z^2 + \ldots
\]
and set \(\Psi = \sqrt{2} e^u A\). Equation (7) follows from the relations
\[
z D(\Psi) = \Psi - \psi, \quad \psi = \sqrt{q_1} = \sqrt{2e^{2w}} w_1 = \sqrt{2} e^w u.
\]

**4. Ibragimov–Shabat hierarchy**

**4.1. Recurrence relations**

Table 5. Polynomials \(a_n\) (weight \(w(u) = 2j + 1\)).

| \(a_0\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) |
|---|---|---|---|---|---|
| \(u\) | \(u_1 + u^3\) | \(u_2 + 4u^2u_1 + u^5\) | \(u_3 + (5u^2u_2 + 8uu_1^2) + 9u^4u_1 + u^7\) | \(u_4 + (6u^2u_3 + 26uu_1u_2 + 8uu_1^3) + (14u^4u_2 + 44u^3u_1^2) + 16u^6u_1 + u^9\) | \(u_5 + (7u^2u_4 + 38uu_1u_3 + 26uu_1^2u_2 + 50u^2u_1^3u_2) + (20u^4u_3 + 170uu_1u_2u_2 + 140u^4u_1^2) + (30u^6u_2 + 140u^5u_1^3) + 25u^8u_1 + u^{11}\) |

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Next, let $\bar{\Psi} = \Psi(-z)$, then (cf (5))

$$D(\bar{\Psi}\Psi) = z^{-1}(\Psi - \psi)\bar{\Psi} - z^{-1}\Psi(\bar{\Psi} - \psi) = z^{-1}\Psi(\bar{\Psi} - \hat{\Psi}) = 2\psi D_1(\psi) = D_1(s).$$

Applying $D^{-1}$ yields $\Psi \bar{\Psi} = 2e^{2w}D_1(w)$, wherefrom

$$2u D_1(u) = D_1(v) = DD_1(w) = \frac{1}{2}D(e^{-2w}\Psi \bar{\Psi}) = D(A\bar{A}).$$

The second equality in (6) follows after the elimination of derivatives by use of (7).

Equation (7) is equivalent to recurrence relations

$$a_0 = u, \quad a_n = a_n(u, \ldots, u_n) = (D + u^2)(a_{n-1}), \quad n = 1, 2, \ldots$$

which are our object of study. Let us try to find a combinatorial interpretation for this recursion.

Several first polynomials $a_n$ are presented in the table 5. Given these data as a prescribed statistics, our goal is to figure out a definition of the corresponding combinatorial objects, that is, to solve a kind of inverse problem of the enumerative combinatorics. In contrast to the Burgers hierarchy case, here we do not know an explicit formula for the coefficients, but this is not too important; the main problem is to guess what the objects are that we are counting.

An invaluable aid in such an ill-posed problem may be obtained by comparison with the known data collected in the encyclopedia of integer sequences [16]. Let us pass to the less detailed statistics by gluing together terms of the same degree. Polynomials of one variable $a_n(u, \ldots, u) = (ua_n + u^2)^n(u)$ contain only odd powers of $u$ and their coefficients constitute the triangle

|      |     |     |     |     |     |     |     |     |     |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|      | 1   |     |     |     |     |     |     |     |     |
| 1    | 1   | 2   |     |     |     |     |     |     |     |
| 1    | 4   | 1   | 6   |     |     |     |     |     |     |
| 1    | 13  | 9   | 1   | 24  |     |     |     |     |     |
| 1    | 40  | 58  | 16  | 1   | 116 |     |     |     |     |
| 1    | 121 | 330 | 170 | 25  | 1   | 648 |     |     |     |
| 1    | 1364| 1771| 1520| 395 | 36  | 1   | 4088|     |     |
| 1    | 1093| 9219| 12411| 5075| 791 | 49  | 1   | 28640|     |

which turns out to be known: according to [16, A039755] this is the triangle of analogs of Stirling numbers of the second kind for the so-called $B$ type set partitions. The sums of numbers in rows, that is, the total sums of the coefficients $a_n(1, \ldots, 1)$ form the sequence [16, A007405] of the Dowling numbers, or $B$-analog of the Bell numbers. This gives us a broad hint at a possible connection between polynomials (8) and $B$ type partitions. This guess is proved in the next section.

4.2. Generating operations for type $B$ set partitions

Special classes of set partitions appear when one takes into account some additional structure of the set. Set partitions of $B$ type (or signed set partitions, $\mathcal{E}_2$-partitions) [7], see also e.g. [3, 19] make use of the reflection $j \rightarrow -j$.

**Definition 1.** A partition $\pi$ of the set $\{-n, \ldots, n\}$ is called the $B_n$ type partition if:
(1) $\pi = -\pi$, that is for each block $\beta \in \pi$ also $-\beta \in \pi$;
(2) $\pi$ contains only one block $\pi_0 \in \pi$ such that $\pi_0 = -\pi_0$.

We will denote by $\Pi_{n}^{B}$ the set of all such partitions and by $\Pi_{n,k}^{B}$ those partitions which contain $k$ block pairs.

In a brief notation for $B$ type partitions, the negative elements of the 0-block are omitted, and only that block of each pair is displayed for which the element with minimal absolute value is positive; the minus signs are denoted by over bars. For example, the partition $-5, -4| -3, 0, 3| -2, 1| -1, 2|4, 5$ is represented in this notation as $0|3|12|45$. The pictorial representation is clear from the diagram

\[ \text{Diagram} \]

where, for instance, the second row on the right represents the block 12 with the negative element marked with the white disk.

Now let us define the map $p$ from $\Pi_{n}^{B}$ into the set of monomials on the variables $u_j$. Let $|\beta|$ denote the number of positive elements in the block $\beta$:

$|\beta| = \# \{ i \in \beta : i > 0 \}$.

It is clear that the number of negative elements in the block is $|\bar{\beta}|$. Let a set partition $\pi \in \Pi_{n,k}^{B}$ consist of 0-block $\pi_0$ and block pairs $\pi_1, \pi_2, ..., \pi_k$, such that the element of $\pi_j$ with minimal absolute value is positive. For such a partition, let

$p(\pi) = u_{|\pi_0|} \cdot u_{|\pi_1|} \cdot u_{|\pi_2|} \cdot \cdots \cdot u_{|\pi_k|}$.

As an example, let us write down $\Pi_{3}^{B}$ partitions, collecting together all partitions corresponding to the same monomial:

\[
\begin{align*}
&u_5 5u_2^2u_2 8uu_1^3 9u^4u_1 u_7 \\
&0123 0|123 0|12^23 0|12|3 1|2|3 4 \\
&0|1\bar{2}3 0|1\bar{2}3 0|1\bar{2}|3 \bar{2} \\
&012|3 0|1\bar{2}3 0|1\bar{3}|2 \\
&013|2 0|1\bar{2}3 0|1\bar{3}|2 \\
&023|1 02|1\bar{3} 0|2\bar{3}|1 \\
&02|1\bar{3} 0|2\bar{3}|1 \\
&03|1\bar{2} 01|2|3 \\
&03|1\bar{2} 02|1|3 \\
&03|1\bar{2} \\
\end{align*}
\]

The resulting polynomial is exactly $a_7$. The following theorem demonstrates that this is not just a coincidence and the polynomials $a_n$ are, indeed, the $\mathbb{Z}_2$-analogs of the complete exponential Bell polynomials $Y_n$. 

Theorem 4. The polynomials (8) are equal to

\[ a_n = \sum_{\pi \in \Pi_n} p(\pi). \]

Proof. Let us denote the sum in the right hand side \( p_n \). Obviously, \( p_0 = u = a_0 \), so we only have to prove that \( p_n \) satisfy the same recurrence relations as \( a_n \), that is, \( p_n = (D + u^2)(p_{n-1}) \).

Notice that the deletion of elements \( \pm n \) from any \( B_n \) type set partition gives us a \( B_{n-1} \) type set partition. Therefore, \( \Pi_n^B \) is constructed from \( \Pi_{n-1}^B \) by adding \( \pm n \) in all possible ways. It is easy to see that this is done by the following operations:

- \( d_0 : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k}^B \), insertion of both elements \( \pm n \) into 0-block;
- \( d_j : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k}^B \), \( j = 1, \ldots, k \), insertion of \( \pm n \) into blocks \( \pm \pi_j \);
- \( \bar{d}_j : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k}^B \), \( j = 1, \ldots, k \), insertion of \( \pm n \) into blocks \( \mp \pi_j \);
- \( M : \Pi_{n-1,k}^B \rightarrow \Pi_{n,k+1}^B \), adding of the new block pair \( \{-n\}, \{n\} \).

Starting from the trivial partition of the set \( \{0\} \) and applying these operations, one can obtain, in a unique way, any \( B \) type set partition. Let us keep track of the monomial \( p(\pi) \), \( \pi \in \Pi_{n-1,k}^B \) under these operations:

- \( d_0 : \) the factor \( u_{|\pi|} \) is replaced with \( u_{|\pi|+1} \);
- \( d_j : \) the factor \( u_{|\pi_j|-1} \) is replaced with \( u_{|\pi_j|} \);
- \( \bar{d}_j : \) the factor \( u_{|\pi_j|} \) is replaced with \( u_{|\pi_j|+1} \);
- \( M : \) two new factors \( u \) are added.

Therefore, application of all possible operations maps the monomial \( p(\pi) \) to the sum of monomials \( (D + u^2)(p(\pi)) \). \( \square \)

5. Korteweg–de Vries hierarchy

5.1. Recurrence relations

Let us recall (for a proof, see e.g. [9]) a computation method of the KdV conservation laws and flows, based on the solution of the Riccati equation
\[
D(f) + f^2 = \lambda - u, \quad \lambda = z^2/4
\]  
by the formal power series
\[
f(z) = -\frac{z}{2} + f_1(u) + \frac{f_2(u, u_1)}{z^2} + \ldots + \frac{f_n(u, \ldots, u_{n-1})}{z^n} + \ldots.
\]
Equation (9) is equivalent to the recurrence relations
\[
f_1 = u, \quad f_{n+1} = D(f_n) + \sum_{s=1}^{n-1} f_s f_{n-s}, \quad n = 1, 2, \ldots
\]  
which will be the main object of our study. Several polynomials \(f_n\) are written down in table 6.

The flows are computed from the polynomials with odd subscripts: let
\[
g(z) = -\frac{1}{2z} - \frac{g_1}{z^3} - \frac{g_3}{z^5} - \ldots - \frac{g_{2m-1}}{z^{2m+1}} = \ldots \equiv \frac{1}{2(f(z) - f(-z))}
\]
which is equivalent to the recurrence relations
\[
g_1 = u, \quad g_{2m+1} = f_{2m+1} + 2 \sum_{s=1}^{m} g_{2s-1} f_{2m-2s+1}, \quad m = 1, 2, \ldots
\]
then the KdV hierarchy reads
\[
U_{2m+1} = D(g_{2m+1}) = u_{2m+1} + \ldots, \quad m = 0, 1, 2, \ldots
\]
Moreover, polynomials (10) with odd subscripts serve as common conserved densities for all these flows.

One interpretation of the polynomials \(f_n\) can be seen immediately from the recurrence relations (10). Let us consider expressions \(\varphi\) built from the variable \(u\) and operations \(M(a, b), d_j(a), 1 \leq j \leq \deg a\) where \(\deg a\) is equal to the number of instances of \(u\) in \(a\). Such expressions can be called ‘unexpanded monomials’. For any expression \(\varphi\) its value \(\text{expand}(\varphi)\) is computed according to the following rules:

— independently of the order of operations, all \(d_j\) are applied before \(M\);
— the action of \(d_j(a)\) amounts to replacing the \(j\)th instance of \(u_i\) in \(a\) with \(u_{i+1}\) (\(u\) is identified with \(u_0\), as usual);
— \(M(a, b)\) is replaced by the product \(ab\).

Let \(Q_n\) denote the set of all expressions with the total number of symbols \(u, d, M\) equal to \(n\). For instance:

| Unexpanded monomials | Expanded monomials |
|----------------------|--------------------|
| \(n = 1\) | \(u\) | \(u\) |
| \(n = 2\) | \(d_1(u)\) | \(u_1\) |
| \(n = 3\) | \(d_1(d_1(u)), M(u, u)\) | \(u_2, u^2\) |
| \(n = 4\) | \(d_1(d_1(d_1(u))), \ldots, d_3(M(u, u)), d_2(M(u, u))\) | \(u_3, uu_1\) |
| \(M(d_1(u), u), M(u, d_1(u))\) | \(uu_1, uu_1\) |

**Theorem 5.** The number of different expressions built from symbols \(M, d_j, u\) with the same monomial as their value is equal to the coefficient of this monomial in polynomials \(f_n\). In other
words,

\[ f_n = \sum_{\varphi \in \Phi_n} \text{expand}(\varphi), \quad (11) \]

**Proof.** Any expression from \( \Phi_{n+1}, n > 0 \) is either of the form \( d_j(a) \) where \( a \in \Phi_n \), \( 1 \leq j \leq \deg a \) or of the form \( M(a, b) \) where \( a \in \Phi_1, b \in \Phi_{n-1} \). Taking into account the obvious properties

\[
\sum_{j=1}^{\deg a} \text{expand}(d_j(a)) = D(\text{expand}(a)),
\]

\[
\text{expand}(M(a, b)) = \text{expand}(a)\text{expand}(b),
\]

this implies that polynomials (11) satisfy the recurrence relation (10).

□

This interpretation is fairly intuitive, but it is desirable to compare it with something more standard. As before, let us pass to polynomials of one variable by collecting together terms of the same degrees. This brings us to a number triangle which apparently is not in the OEIS:

\[
\begin{array}{cccccccc}
1 & 1 \\
1 & 1 & 2 \\
1 & 4 & 5 & 14 \\
1 & 11 & 2 & 14 & 43 \\
1 & 26 & 16 & 57 & 80 & 143 \\
1 & 120 & 324 & 64 & 509 & 14 & 1922 \\
1 & 502 & 3948 & 2944 & 256 & 7651 & 1013 & 12776 & 15403 & 2730 & 42 & 31965 & (12)
\end{array}
\]

Nevertheless, the sequence of coefficients sum totals turns out to be known: \( f_{n+1}[1] \) is equal to the number of non-overlapping partitions of the set \([n]\), or the Bessel number \( B_n^* \) [16, A006789]. Notice that identifying all \( u_j \) results in the Riccati equation \( u\partial u + f^2 = \lambda - u \) which is, indeed, equivalent to the Bessel equation. Moreover, one can see in the triangle the Euler numbers [16, A000295], the Catalan numbers [16, A000108] and powers of 4.

5.2. Generating operations for non-overlapping partitions

This class of set partitions was introduced in [8], see also [4, 12]. Its definition engages the order relation on the partitioned set \([n] = \{1, \ldots, n\}\).

**Definition 2.** Blocks \( \alpha \) and \( \beta \) of a set partition \( \pi \) overlap if

\[
\min \alpha < \min \beta < \max \alpha < \max \beta.
\]

The set partition is called non-overlapping (NOP) if any two blocks in it do not overlap. All NOPs of the set \([n]\) will be denoted \( \Pi_n^* \).

The interval \([\min \alpha, \max \alpha]\) is called the support of the block \( \alpha \). The above definition of NOP is equivalent to the property that the supports of any two blocks either do not intersect or
lie one in another. The left diagram below shows a pair of overlapping blocks and the right diagram shows non-overlapping ones:

\[
\begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}
\end{array}
\]

Remark 2. A neighbour class of non-crossing partitions is characterized by a more restrictive condition which forbids the pattern \(\alpha_1 < \beta_1 < \alpha_2 < \beta_2\) for any elements of any two blocks. It is under active study in combinatorics as well; moreover, it makes sense to combine such types of restrictions with symmetries like the reflection for the \(B\) type partitions, see e.g. [15]. It is an open question whether some integrable hierarchies may be associated with such kinds of objects.

Some simple properties of NOPs are:

— At \(n = 0, 1, 2, 3\) we have \(\Pi_n^* = \Pi_n\) and there is only one overlapping partition \(13|24\) in \(\Pi_4\).
— Singletons do not overlap with any block.
— NOPs containing only doublets can be easily identified with the balanced sets of parentheses:

\[
\begin{array}{c}
\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}
\end{array}
\rightarrow \quad ( ( ) ( ) ) ( )
\]

The last property explains where the Catalan numbers appear from in the triangle (12). The deletion of differentiation in equation (10) brings to the recursion for the ‘dispersionless terms’:

\[
f_1 = u, \quad f_{n+1} = \sum_{s=1}^{n-1} f_s f_{n-s} \rightarrow u, 0, u^2, 0, 2u^3, 0, 5u^4, 0, \ldots
\]

In order to establish a correspondence with the general polynomials \(f_n\), let us identify the variable \(u\) with the set partition \(\{\emptyset\}\) and define the action of the operations \(M\) and \(d_j\) on the NOPs, in such a way that expressions \(\Phi_{n+1}\) are in a one-to-one correspondence with \(\Pi_n^*\).

Degree. Let \(\deg \pi = k\) if \(\pi\) contains \(k - 1\) multplets (blocks with more than one element).

Operation \(M\). Let \(\rho \in \Pi_n^*, \sigma \in \Pi_n^*.\) Denote by \((\sigma)_{r+1}\) the partition of the set \([r + 2, r + s + 1]\) obtained from \(\sigma\) by adding \(r + 1\) to each element, and define

\[
M(\rho, \sigma) = \rho \cup \{ [r + 1, r + s + 2] \} \cup (\sigma)_{r+1} \in \Pi_{r+r+2}^*.
\]

This can be illustrated by the diagram
In particular, if \( \rho = \{\emptyset\} \) then \((\sigma_1)\) is bounded by the doublet \( \{1, s + 2\}\), and if \( \sigma = \{\emptyset\} \) then the doublet \( \{r + 1, r + 2\}\) is appended to \( \rho \). Notice that \( \deg \langle \rho, \sigma \rangle = \deg \rho \deg \sigma \).

**Operation \( d_j \).** This consists of adding one element \( n_1 + 1 \) to \( \pi \in \Pi_n^* \). If \( j = 1 \) then the element is added just as a singleton. For \( 1 < j \leq k = \deg \pi \), the operation requires a detailed description. Let us denote by \( \mu_1, \ldots, \mu_k \) all multiplets in \( \pi \), ordered by increase of their minimal elements. Assume that all blocks with support containing \( \mu_j \) are enumerated by a sequence \( j_1 < \ldots < j_k = j \). Let us divide each of these blocks into the left and right parts with respect to \( m = \max \mu_j \):

\[
\mu_\mu = \left\{ i \in \mu_j : i < m \right\}, \quad \mu_\mu = \left\{ i \in \mu_j : i \geq m \right\}
\]

and form the new blocks

\[
\beta_\mu = \mu_\mu \cup \{m, n + 1\}, \quad \mu_\mu = \mu_\mu \cup \mu_\mu, \quad r = 2, \ldots, s
\]

as shown on the following diagram. The rest blocks of the partition do not change under this operation.

**Theorem 6.** The operations \( M, d_j \) generate any non-overlapping partition in a unique way.

**Proof.** The last operation on a given partition is uniquely defined by consideration of the block \( \beta \) containing the maximal element of the partition. If it is a singleton, then the last operation was \( d_1 \); if it is a doublet, then it was \( M \); if it is a multiplet, then the operation was \( d_j \) where \( j \) is the maximal number such that the support of multiplet \( \mu_j \) contains the last but one element of \( \beta \). In each case, applying inverse operation achieves NOPs with fewer elements.

Taking the theorem 5 into account, the established bijection allows us to associate a certain monomial with each NOP, although not in a particularly effective way, because first we have to build an expression \( \varphi \in \Phi_n \) corresponding to \( \pi \in \Pi_n^* \) and then compute \( \text{expand}(\varphi) \):

\[
\Phi_n \leftrightarrow \Pi_{n-1}^*, \quad \text{expand} \downarrow \checkmark \quad f_n
\]
Nevertheless, it is easy to trace the degree of monomial under this correspondence; it is one more than the number of multiplets in the partition. This gives us the following interpretation of the number triangle (12).

**Corollary 7.** The number of NOPs of \( n \) elements containing \( k \) multiplets is equal to the number in the \( n \)th row and \( k \)th column of the number triangle (12), starting their enumeration from 0. This number is equal to the coefficient of \( u^{k+1} \) in the polynomial \( F_{n+1}(u) = f_{n+1}(u, \ldots, u) \) defined by the recurrence relations

\[
F_0 = u, \quad F_{n+1} = u \partial_u (F_n) + \sum_{s=1}^{n-1} F_s F_{n-s}, \quad n = 1, 2, \ldots
\]

6. Conclusion

We have established a relation between several classes of set partitions and generating functions for integrable hierarchies. Hopefully, this observation may become useful for both theories. Of course, we have too few examples at the moment to come to far-reaching conclusions. A conjecture is that each integrable hierarchy has an underlying generating function which may be interpreted as statistics for some kind of combinatorial objects (possibly unknown). As further steps, it would be interesting to reveal the combinatorics associated with the mKdV equation, KdV-like equations of fifth order, the nonlinear Schrödinger equation and so on.

However, the objects studied in the combinatorics are so plentiful and diverse that it seems doubtful that any one of them can be associated with an integrable hierarchy. In all likelihood, this is a very special property, so it would be interesting to understand what is integrability intermediately in combinatorial terms (rather than on the level of generating functions). For instance, we can try to obtin a proof of commutativity of the flows of a hierarchy based on their combinatorial interpretation.

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