Characterization of circulant graphs having perfect state transfer

Milan Bašić
Faculty of Sciences and Mathematics, University of Niš,
Višegradska 33, 18000 Niš, Serbia
E-mail: basic.milan@yahoo.com

Abstract

In this paper we answer the question of when circulant quantum spin networks with nearest-neighbor couplings can give perfect state transfer. The network is described by a circulant graph \(G\), which is characterized by its circulant adjacency matrix \(A\). Formally, we say that there exists a perfect state transfer (PST) between vertices \(a, b \in V(G)\) if \(|F(\tau)_{ab}| = 1\), for some positive real number \(\tau\), where \(F(t) = \exp(iAt)\). Saxena, Severini and Shparlinski (International Journal of Quantum Information 5 (2007), 417–430) proved that \(|F(\tau)_{aa}| = 1\) for some \(a \in V(G)\) and \(\tau \in \mathbb{R}^+\) if and only if all eigenvalues of \(G\) are integer (that is, the graph is integral). The integral circulant graph \(ICG_n(D)\) has the vertex set \(\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}\) and vertices \(a\) and \(b\) are adjacent if \(\gcd(a - b, n) \in D\), where \(D \subseteq \{d \in \mathbb{Z} | d \mid n, 1 \leq d < n\}\). These graphs are highly symmetric and have important applications in chemical graph theory. We show that \(ICG_n(D)\) has PST if and only if \(n \in 4\mathbb{N}\) and \(D = \tilde{D}_3 \cup D_2 \cup 2D_2 \cup 4D_2 \cup \{n/2^a\}\), where \(\tilde{D}_3 = \{d \in D | n/d \in 8\mathbb{N}\}\), \(D_2 = \{d \in D | n/d \in 8\mathbb{N} + 4\}\) \(\setminus \{n/4\}\) and \(a \in \{1, 2\}\). We have thus answered the question of complete characterization of perfect state transfer in integral circulant graphs raised in Quantum Information and Computation, Vol. 10, No. 3&4 (2010) 0325-0342 by Angeles-Canul et al. Furthermore, we also calculate perfect quantum communication distance (distance between vertices where PST occurs) and describe the spectra of integral circulant graphs having PST. We conclude by giving a closed form expression calculating the number of integral circulant graphs of a given order having PST.

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1 Introduction

The transfer of a quantum state from one location to another is a crucial ingredient for many quantum information processing protocols. There are various physical systems that can serve as quantum channels, one of them being a quantum spin network. These networks consist of \(n\) qubits where some pairs of qubits are coupled via XY-interaction. The perfect transfer of quantum states from one qubit to another in such networks was first considered in [7]. There are two special qubits \(A\) and \(B\) representing the input and output qubit, respectively. The transfer is implemented by setting the qubit \(A\) in a prescribed quantum state and by retrieving the state from the output qubit \(B\) after some time. The transfer is called perfect state transfer (transfer with unit fidelity) if the initial state of the qubit \(A\) and the final state of the qubit \(B\) are equal up to a local phase rotation.

Every quantum spin network with fixed nearest-neighbor couplings is uniquely described by an undirected graph \(G\) on a vertex set \(V(G) = \{1, 2, \ldots, n\}\). The edges of the graph \(G\) specify which qubits are coupled. In other words, there is an edge between vertices \(i\) and \(j\) if \(i\)-th and \(j\)-th qubit are coupled.

In [7] a simple XY coupling is considered such that the Hamiltonian of the system has the form

\[
H_G = \frac{1}{2} \sum_{(i,j) \in E(G)} \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y.
\]
and $\sigma_i^x, \sigma_i^y$ and $\sigma_i^z$ are Pauli matrices acting on $i$-th qubit. The standard basis chosen for an individual qubit is $\{|0\rangle, |1\rangle\}$ and it is assumed that all spins initially point down ($|0\rangle$) along the prescribed $z$ axis. In other words, the initial state of the network is $|0\rangle = |0_1 \ldots 0_n\rangle$. This is an eigenstate of Hamiltonian $H_G$ corresponding to zero energy. The Hilbert space $H_G$ associated with a network is spanned by the vectors $|e_1 e_2 \ldots e_n\rangle$ where $e_i \in \{0, 1\}$ and, therefore, its dimension is $2^n$.

The process of transmitting a quantum state from $A$ to $B$ begins with the creation of the initial state $\alpha|0_1 A_0 \ldots 0_B\rangle + \beta|1_1 A_0 \ldots 0_B\rangle$ of the network. Since $|0\rangle$ is a zero-energy eigenstate of $H_G$, the coefficient $\alpha$ will not change in time. Since the operator of total $z$ component of the spin $\sigma_i^z = \sum_{i=1}^{n} \sigma_i^z$ commutes with $H_G$, state $|1_1 A_0 \ldots 0_B\rangle$ must evolve into a superposition of the states $|i\rangle = |0 \ldots 01_1, 0, \ldots, 0\rangle$ for $i = 1, \ldots, n$. Denote by $S_G$ the subspace of $H_G$ spanned by the vectors $|i\rangle$, $i = 1, \ldots, n$. Hence, the initial state of network evolves in time $t$ into the state

$$\alpha|0\rangle + \sum_{i=1}^{n} \beta_i(t)|i\rangle \in S_G.$$ 

The previous equation shows that system dynamics is completely determined by the evolution in $n$-dimensional space $S_G$. The restriction of the Hamiltonian $H_G$ to the subspace $S_G$ is an $n \times n$ matrix identical to the adjacency matrix $A_G$ of the graph $G$.

Thus, the time evolution operator can be written in the form $F(t) = \exp(iA_G t)$. The matrix exponential $\exp(M)$ is defined as usual

$$\exp(M) = \sum_{n=0}^{+\infty} \frac{1}{n!} M^n.$$ 

Perfect state transfer (PST) between different vertices (qubits) $a$ and $b$ $(1 \leq a, b \leq n)$ is obtained in time $\tau$, if $\langle a|F(t)\rangle|b\rangle = |F(\tau)_{ab}\rangle = 1$. The graph (network) is periodic at $a$ if $|F(\tau)_{aa}| = 1$ for some $\tau$. A graph is periodic if it is periodic at each vertex $a$.

The existence of PST for some network topologies has already been considered in the literature. For example, Christandl et al. [8] proved that PST occurs in paths of length one and two between their end-vertices and also in Cartesian powers of these graphs between vertices at maximal distance. In the recent paper [11], Godsil constructed a class of distance-regular graphs of diameter three, with PST. Some properties of quantum dynamics on circulant graphs were studied in [1]. Saxena, Severini and Shparlinski [18] considered circulant graphs as potential candidates for modeling quantum spin networks having PST. They show that a circulant graph is periodic if and only if all eigenvalues of the graph are integers (i.e. graph is integral). Since periodicity is a necessary condition for PST existence [18], circulant graphs having PST must be integral circulant graphs. A simple and general characterization of the existence of PST in an integral circulant graph, in terms of its eigenvalues, was given by Bašić, Petković and Stevanović in [4]. Furthermore, it was shown that for odd number of vertices, there is no PST, and that among the class of unitary Cayley graphs a subclass of integral circulant graphs, only $K_2$ (the complete graph with two nodes) and $C_4$ (the cycle of length four) have PST. In the recent paper [17], it was proven that there exists an integral circulant graph with $n$ vertices having PST if and only if $4 \mid n$. Several classes of integral circulant graphs having PST were found as well and several others in [15].

In all known classes of graphs having PST perfect quantum communication distances (i.e. the distances between vertices where PST occurs) are considerably small compared to the order of the graph. One idea for the distance enlargement, is to consider networks with fixed but different couplings between qubits. These networks correspond to graphs with weighted adjacency matrices. For example, in [7, 8] the authors showed that PST can be achieved over arbitrarily long distances in a weighted linear paths. Many recent papers have proposed such an approach [14, 15, 16].

Studying PST in integral circulant graphs can also be interpreted as a contribution to the spectral theory of integral graphs. These graphs are highly symmetric and have some remarkable properties connecting graph theory and number theory. The term ‘integral circulant graph’ first appears in the work of So [20], where a nice characterization of these graphs in terms of their symbol set is given. The upper bounds on the number of vertices and the diameter of integral circulant graphs were given in [18]. Furthermore, Stevanović, Petković and Bašić [19] improved the upper bound. Various other properties of unitary Cayley graphs were recently investigated. For example, Berrizbeitia and Giudici [6] and Fuchs [9] established the lower and upper bound on the size of the longest induced cycle. Klotz and Sander [13] determined the diameter, clique
number, chromatic number and eigenvalues of unitary Cayley graphs. Bašić and Ilić [2] calculated the clique number of integral circulant graphs with exactly one and two divisors and also provided an inequality for the general case.

In this paper we proceed with the study of circulant networks supporting PST initiated in [3, 4, 15, 17, 18]. First we give some properties of the spectra of ICG$_n$($D_1$) where $D_1 = \{d \in D \mid 4 \nmid n/d\}$. In that subsection we present some preliminary results which are used in the sequel of the section. We show that ICG$_n$(D) has PST if and only if $n/2 \in D$ or equal to two if $n/4 \in D$. Moreover, we describe the spectra of integral circulant graphs having PST. The paper is concluded with a formula for the number of ICG$_n$(D) having PST as a function of the number of vertices $n$. This results answer questions posed in [15, 17, 18].

2 Integral circulant graphs

The circulant graph $G(n; S)$ is a graph on vertices $Z_n = \{0, 1, \ldots, n-1\}$ such that each vertex $i$ is adjacent to vertices $i +_ns$ for all $s \in S$. The set $S \subseteq Z_n$ is called the symbol of the graph $G(n; S)$ and $+_n$ denotes addition modulo $n$. Note that the degree of $G(n; S)$ is $\#S$. A graph is integral if all its eigenvalues are integers. Wasin So has characterized integral circulant graphs [20] in the following theorem:

Theorem 1 [20] A circulant graph $G(n; S)$ is integral if and only if

$$S = \bigcup_{d \in D} G_n(d),$$

for some set of divisors $D \subseteq D_n$. Here $G_n(d) = \{k : \gcd(k, n) = d, \ 1 \leq k \leq n-1\}$, and $D_n$ is the set of all divisors of $n$, different from $n$.

Therefore an integral circulant graph (ICG) $G(n; S)$ is defined by its order $n$ and the set of divisors $D$. Such graphs are also known as gcd-graphs (see for example [13]). An integral circulant graph with $n$ vertices, defined by the set of divisors $D \subseteq D_n$ will be denoted by ICG$_n$(D). From Theorem 1 we have that the degree of an integral circulant graph is $\deg$ICG$_n$(D) = $\sum_{d \in D} \varphi(n/d)$. Here $\varphi(n)$ denotes the Euler-phi function [12].

The eigenvalues and eigenvectors of ICG$_n$(D) are given in [18] as

$$\lambda_j = \sum_{s \in S} \omega_n^{js}, \quad v_j = [1, \omega_n^s, \omega_n^{2s}, \ldots, \omega_n^{(n-1)s}], \quad (1)$$

where $\omega_n = \exp(i2\pi/n)$ is the $n$-th root of unity. Denote by $c(n, j)$ the following expression

$$c(j, n) = \mu(t_{n, j}) \frac{\varphi(n)}{\varphi(t_{n, j})}, \quad t_{n, j} = n / \gcd(n, j), \quad (2)$$

where $\mu$ is the Möbius function defined as

$$\mu(n) = \begin{cases} 
1, & \text{if } n = 1 \\
0, & \text{if } n \text{ is not square–free} \\
(-1)^k, & \text{if } n \text{ is product of } k \text{ distinct prime numbers.} 
\end{cases} \quad (3)$$

The expression $c(j, n)$ is known as the Ramanujan function ([12, p. 55]). Eigenvalues $\lambda_j$ can be expressed in terms of the Ramanujan function as follows ([13, Theorem 16])

$$\lambda_j = \sum_{d \in D} c(j, n/d). \quad (4)$$

Let us observe that the Ramanujan function has the following basic properties which we will make use of in the paper.
Proposition 2 For any positive integers \( n, j \) and \( d \) such that \( d \mid n \), holds

\[
\begin{align*}
c(0, n) & = \varphi(n), \\
c(1, n) & = \mu(n), \\
c(2, n) & = \begin{cases} 
\mu(n), & n \in 2\mathbb{N} + 1 \\
\mu(n/2), & n \in 4\mathbb{N} + 2 \\
2\mu(n/2), & n \in 4\mathbb{N}
\end{cases} \\
c(n/2, n/d) & = \begin{cases} 
\varphi(n/d), & d \in 2\mathbb{N} \\
-\varphi(n/d), & d \in 2\mathbb{N} + 1
\end{cases}
\end{align*}
\]

(5) (6) (7) (8)

Proof. Directly using relation (2). \( \square \)

The integral circulant graph \( ICG_n(D) \) is connected if and only if \( \gcd(n, d_1, \ldots, d_k) = 1 \) where \( D = \{d_1, \ldots, d_k\} \). In the rest of the paper we will only consider connected integral circulant graphs.

3 Perfect state transfer

Let \( G \) be an undirected graph and denote by \( A_G \) its adjacency matrix. Let \( F(t) = \exp(iA_Gt) \). There is a perfect state transfer (PST) in graph \( G \) \([7, 11, 18]\) if there are distinct vertices \( a \) and \( b \) and a positive real number \( t \) such that \( |F(t)_{ab}| = 1 \).

Let \( \lambda_0, \lambda_2, \ldots, \lambda_{n-1} \) be the eigenvalues (not necessarily distinct) of \( A_G \) and \( u_0, u_1, \ldots, u_{n-1} \) be the corresponding normalized eigenvectors. We use spectral decomposition of the real symmetric matrix \( A_G \) (see for example \([10]\) (Theorem 5.5.1) for more details). The matrix function \( F(t) \) can be represented as

\[
F(t) = \sum_{k=0}^{n-1} \exp(i\lambda_k t)u_k u_k^*.
\]

(9)

Now let \( G = ICG_n(D) \) be an integral circulant graph. By simple calculation and using (1), we see that \( \|v_k\| = \sqrt{n} \) and hence \( u_k = v_k/\sqrt{n} \). Expression (9) now becomes

\[
F(t) = \frac{1}{n} \sum_{k=0}^{n-1} \exp(i\lambda_k t)v_k v_k^*.
\]

In particular, from the last expression and (1) it directly follows

\[
F(t)_{ab} = \frac{1}{n} \sum_{k=0}^{n-1} \exp(i\lambda_k t)\omega_n^{k(a-b)}.
\]

This expression is given in \([18]\) (Proposition 1). Finally, our goal is to check whether there exist distinct integers \( a, b \in \mathbb{Z}_n \) and a positive real number \( t \) such that \( |F(t)_{ab}| = 1 \), i.e.

\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} \exp(i\lambda_k t)\omega_n^{k(a-b)} \right| = 1.
\]

(10)

Since the left-hand side of (10) depends on \( a \) and \( b \) only as a function of \( a - b \) we can, without any loss of generality, assume that \( b = 0 \). Therefore, throughout the paper we consider the existence of PST only between vertices \( a \) and \( 0 \).

We restate some results proved in \([4]\). These results establish necessary and sufficient conditions for (10).

Theorem 3 \([4]\) There exists PST in \( ICG_n(D) \) between vertices \( a \) and \( 0 \) if and only if there are integers \( p \) and \( q \) such that \( \gcd(p, q) = 1 \) and

\[
\frac{p}{q}(\lambda_{j+1} - \lambda_j) + \frac{a}{n} \in \mathbb{Z},
\]

(11)

for all \( j = 0, \ldots, n-2 \).
In this section we deal with integral circulant graphs ICG\(_n\) just say that there exists PST in ICG\(_n\) between vertices \(a\) and 0, then \(a = n/2\).

According to Theorem 4, PST may exist in ICG\(_n\) only between vertices \(n/2\) and 0 (i.e., between \(b\) and \(n/2 + b\) as mentioned in [18]). Hence we will avoid referring to the input and output vertex and will just say that there exists PST in ICG\(_n\).

The next corollary is derived from Theorem 4 and is further used as the criterion for a nonexistence of PST.

**Corollary 5** [4] If \(\lambda_j = \lambda_{j+1}\) for some \(j = 0, \ldots, n - 2\) then there is no PST in ICG\(_n\).

For a given prime number \(p\) and an integer \(n \in \mathbb{N}\) denote by \(S_p(n)\) the maximal number \(\alpha\) such that \(p^\alpha \mid n\) if \(n \in \mathbb{N}\), and \(S_p(0) = +\infty\). The following result was proven in [4] and is further used as a criterion for the existence of PST.

**Lemma 6** [4] There exists PST in ICG\(_n\), if and only if there exists a number \(m \in \mathbb{N}_0\) such that the following holds for all \(j = 0, 1, \ldots, n - 2\)

\[
S_2(\lambda_{j+1} - \lambda_j) = m. \tag{12}
\]

The following corollary follows directly from Lemma 6.

**Corollary 7** Let ICG\(_n\) have PST. One of the following two statements must hold

1. \(\lambda_j \equiv \lambda_{j+1} \pmod{2}\) for every \(0 \leq j \leq n - 1\) (i.e., all eigenvalues \(\lambda_j\) have the same parity).
2. \(\lambda_j \equiv \lambda_{j+1} + 1 \pmod{2}\) for every \(0 \leq j \leq n - 1\) (i.e., \(\lambda_j\) are alternatively odd and even).

We end this section with the following result concerning nonexistence of PST in ICG\(_n\), where \(n \in 4\mathbb{N} + 2\).

**Theorem 8** [17] There is no PST in ICG\(_n\) for an arbitrary set of divisors \(D\) for \(n \in 4\mathbb{N} + 2\).

### 4 Integral circulant graphs having PST

Let ICG\(_n\) be an arbitrary integral circulant graph. We define sets \(D_i \subseteq D\) for \(0 \leq i \leq l\), where \(l = S_2(n)\), in the following way

\[
D_i = \{d \in D \mid S_2(n/d) = i\}.
\]

For simplicity of notation we also define sets \(\bar{D}_1, \bar{D}_3 \subseteq D\) to be \(\bar{D}_1 = D_0 \cup D_1\) and \(\bar{D}_3 = \cup_{i \geq 3} D_i\).

Let us introduce the notation \(kD\) for the set \(\{kd \mid d \in D\}\) for a positive integer \(k\).

#### 4.1 Spectrum of the integral circulant graph ICG\(_n(\bar{D}_1)\)

In this section we deal with integral circulant graphs ICG\(_n(D)\) such that for each divisor \(d \in D\) it holds that \(4 \not\mid \frac{n}{d}\). In the rest of the section we will consider only such classes of graphs unless otherwise stated.

In Lemmas 9 and 10 we present some properties of the Ramanujan function.

Throughout the section, we let \(n = 2^\alpha_0 p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}\), where \(p_1 < p_2 < \ldots < p_k\) are distinct primes, and \(\alpha_i \geq 1\) for \(1 \leq i \leq k\) and \(\alpha_0 \geq 0\).

**Lemma 9** For \(n \geq 2\) it holds that \(c(j, n) \in 2\mathbb{N} + 1\) if and only if \(4 \nmid n\) and \(j = p_1^{\alpha_1 - 1} \cdots p_k^{\alpha_k - 1} J\) for some integer \(J\) such that \(\gcd(J, n) \in \{1, 2\}\).

\(^1\mathbb{N}_0 = \mathbb{N} \cup \{0\}\)
Proof.

(⇒:) Suppose that \(c(j, n)\) is an odd integer. Since \(c(j, n) = \mu(t_{n,j})\varphi(n)/\varphi(t_{n,j})\), it holds that \(\mu(t_{n,j}) = ±1\), i.e. \(t_{n,j}\) is square-free and \(\varphi(n)/\varphi(t_{n,j})\) is an odd integer.

Suppose that for some odd \(p_i\) it holds that \(p_i \nmid t_{n,j}\). Let \(n' = n/p_i\). Since \(t_{n,j} \mid n'\) and so \(\varphi(t_{n,j}) \mid \varphi(n')\) we obtain that

\[
c(j, n) = \pm \frac{\varphi(n)}{\varphi(t_{n,j})} = \pm \frac{\varphi(p_i^a)\varphi(n')}{\varphi(t_{n,j})} = \pm p_i^{a - 1}(p_i - 1) \frac{\varphi(n')}{\varphi(t_{n,j})}.
\]

The last equation implies that \(c(j, n)\) is even since \(p_i - 1\) is even. This is a contradiction and we can conclude that \(p_i \mid t_{n,j}\) for every \(1 \leq i \leq k\).

Now we have that \(\varphi(t_{n,j}) = (p_1 - 1) \cdots (p_k - 1)\) and thus

\[
c(j, n) = 2^{a_0 - 1}p_1^{a_0 - 1}p_2^{a_0 - 1} \cdots p_k^{a_0 - 1}.
\]

Since \(c(j, n)\) is odd it holds that \(0 \leq a_0 \leq 1\) or equivalently \(4 \nmid n\).

If \(n \in 2\mathbb{N} + 1\) it must hold that \(t_{n,j} = p_1 \cdots p_k\) since \(t_{n,j}\) is square-free. If \(n \in 4\mathbb{N} + 2\), we have two possibilities for \(t_{n,j} = p_1 \cdots p_k\) or \(t_{n,j} = 2p_1 \cdots p_k\) depending on the parity of \(j\).

Furthermore, using \(n = \gcd(n, j)t_{n,j}\) we obtain that \(\gcd(n, j) = p_1^{a_0 - 1} \cdots p_k^{a_0 - 1} (t_{n,j} and n have the same parity) or \(\gcd(n, j) = 2p_1^{a_0 - 1} \cdots p_k^{a_0 - 1}\) (otherwise). This implication of the lemma is now straightforward.

(⇐:) Since \(\gcd(n, j) = p_1^{a_0 - 1} \cdots p_k^{a_0 - 1}\) \(\gcd(J, n)\) and \(\gcd(J, n) = \{1, 2\}\), it holds that \(t_{n,j} = p_1 \cdots p_k\) or \(t_{n,j} = 2p_1 \cdots p_k\). In either case it holds that \(\varphi(t_{n,j}) = (p_1 - 1) \cdots (p_k - 1)\). Now since

\[
c(j, n) = \mu(t_{n,j})\frac{\varphi(n)}{\varphi(t_{n,j})} = \pm p_1^{a_0 - 1} \cdots p_k^{a_0 - 1},
\]

we conclude that \(c(j, n) \in 2\mathbb{N} + 1\).

\[\square\]

Lemma 10 Let \(d\) be an arbitrary divisor of \(n\) such that \(n/d \in 2\mathbb{N} + 1\) and \(0 \leq j \leq n - 1\) be an arbitrary integer, then \(c(j, n/d) = -c(j, 2n/d)\) for \(j \in 2\mathbb{N} + 1\) and \(c(j, n/d) = c(j, 2n/d)\) for \(j \in 2\mathbb{N}\).

Proof. As \(n/d \in 2\mathbb{N} + 1\) we conclude that \(\varphi(2n/d) = \varphi(n/d)\).

Suppose that \(j \in 2\mathbb{N} + 1\). Then \(\gcd(2n/d, j) = \gcd(n/d, j)\) and

\[
t_{2n/d,j} = \frac{2n}{d\gcd(2n/d, j)} = \frac{n}{d\gcd(n/d, j)} = 2t_{n/d,j}.
\]

Furthermore, it holds that \(\varphi(t_{2n/d,j}) = \varphi(t_{n/d,j})\) since \(t_{n/d,j}\) is odd. Also we have that \(t_{2n/d,j}\) is square-free if and only if \(t_{n/d,j}\) is square-free, and \(\mu(t_{2n/d,j}) = -\mu(t_{n/d,j})\). Now we can directly conclude that

\[
c(j, 2n/d) = \mu(t_{2n/d,j})\frac{\varphi(2n/d)}{\varphi(t_{2n/d,j})} = -\mu(t_{n/d,j})\frac{\varphi(n/d)}{\varphi(t_{n/d,j})} = -c(j, n/d).
\]

Now suppose that \(j \in 2\mathbb{N}\). We have \(\gcd(2n/d, j) = 2\gcd(n/d, j)\) and also \(t_{2n/d,j} = t_{n/d,j}\). This yields directly that \(c(j, 2n/d) = c(j, n/d)\).

\[\square\]

Theorem 11 For an arbitrary integral circulant graph \(ICG_n(D)\) there exists an odd number \(0 \leq j \leq n - 1\) with \(\lambda_j\) also odd if and only if there is a divisor \(d \in D\) satisfying \(d/2 \notin D\) and \(2d \notin D\).

Proof.

(⇒:) Suppose that for every \(d \in D\) it holds that either \(2d \in D\) or \(d/2 \in D\). It follows that \(D = D_1 \cup 2D_1\). Let \(j \in 2\mathbb{N} + 1\). According to Lemma 10 we have \(c(j, n/d) = -c(j, 2n/d)\) for any \(d \in D\) such that \(n/d \in 2\mathbb{N} + 1\) and therefore

\[
\lambda_j = \sum_{d \in D_1 \cup 2D_1} c(j, n/d) = \sum_{d \in D_1} c(j, n/d) + c(j, 2n/d) = 0 \in 2\mathbb{N}.
\]
We conclude that all eigenvalues with odd indices are even.

\((\Leftarrow)\) Let \(D' = \{d \in D \mid \gcd(n/d, 2) = 1\}\) and \(D'' = D' \cup 2D'\).

Let \(D' = D \setminus D''\). By the assumption, \(D'\) is a non-empty set. According to Lemma 11, it holds that 
\[
\gcd(n/d) + c(j, 2n/d) = 2n + 1
\]
for every \(d \in D''\) such that \(n/d \notin 2\mathbb{N} + 1\). Denote by \(d'_{\text{max}} = \max D'\). Since \(d'_{\text{max}}\)
is a divisor of \(n\), it can be represented in the form \(d'_{\text{max}} = 2^\beta s_{\text{max}}\) where \(0 \leq \beta_i \leq \alpha_i\) for \(i = 1, \ldots, k\).

Without loss of generality, we can suppose that there exists \(1 \leq s \leq k\) such that \(\beta_i < \alpha_i\) for \(i = 1, \ldots, s\) and \(\beta_i = \alpha_i\) for \(i = s + 1, \ldots, k\). Then we can write \(n/d'_{\text{max}} = 2^\alpha \beta_0 p_1^{\alpha_1 - \beta_1} \cdots p_s^{\alpha_s - \beta_s}\). Denote by
\[
j_0 = p_1^{\alpha_1 - \beta_1} \cdots p_s^{\alpha_s - \beta_s - 1} p_{s+1}^{\alpha_{s+1}} \cdots p_k^{\alpha_k}.
\]

It holds trivially that \(0 \leq j_0 \leq n - 1\). Lemma 8 yields directly that \(c(j_0, n/d'_{\text{max}})\) is odd since \(n/d'_{\text{max}} > 2\).

Suppose that \(D' \setminus \{d'_{\text{max}}\} \neq \emptyset\) and let \(d \in D' \setminus \{d'_{\text{max}}\}\) be an arbitrary divisor with its prime factorization \(d = 2^\gamma p_1^{\gamma_1} \cdots p_k^{\gamma_k}\).

We will show that there exists \(1 \leq i \leq k\) such that \(0 \leq \gamma_i < \beta_i \leq \alpha_i\). Suppose this is not the case, which means that \(0 \leq \beta_i < \gamma_i \leq \alpha_i\) for \(1 \leq i \leq k\). If \(\beta_0 \leq \gamma_0\) then \(d'_{\text{max}}\) is a divisor of \(d\), which is a contradiction. Similarly, if \(\beta_0 = \gamma_0 + 1\) then it holds that \(d'_{\text{max}}|2d\), which implies \(d = d'_{\text{max}}/2\), providing a contradiction with the definition of the set \(D'\).

Let \(i\) be an arbitrary index such that \(\gamma_i < \beta_i\). Suppose that \(\alpha_i - \beta_i \geq 1\). Then \(i \leq s\) and \(S_{p_i}(n/d) = \alpha_i - \gamma_i \geq 2\). Since \(S_{p_i}(j_0) = \alpha_i - \beta_i - 1 > \alpha_i - \gamma_i - 1 = S_{p_i}(n/d) - 1\), from Lemma 9 we can conclude that \(c(j_0, n/d)\) is even.

Now suppose that \(\alpha_i = \beta_i\). Then \(i > s\) and \(S_{p_i}(n/d) = \alpha_i - \gamma_i \geq 1\). Again since \(S_{p_i}(j_0) = \alpha_i - \gamma_i - 1 = S_{p_i}(n/d) - 1\), Lemma 9 yields that \(c(j_0, n/d)\) is even.

This implies that there is an odd index \(j_0\) such that \(c(j_0, n/d'_{\text{max}})\) is odd and \(c(j_0, n/d)\) is even for every \(d \in D' \setminus \{d'_{\text{max}}\}\). Now we have
\[
\lambda_{j_0} = c(j_0, n/d'_{\text{max}}) + \sum_{d \in D' \setminus \{d'_{\text{max}}\}} c(j_0, n/d) + \sum_{d \in D''} c(j_0, n/d) \in 2\mathbb{N} + 1,
\]
since both sums in the last expression are even.

If \(d'_{\text{max}}\) is the only divisor contained in \(D\), the above sum is reduced to
\[
\lambda_{j_0} = c(j_0, n/d'_{\text{max}}) + \sum_{d \in D''} c(j_0, n/d) \in 2\mathbb{N} + 1,
\]
and it is still even. \(\square\)

Let us mention two direct consequences of the previous theorem. The first one is actually the contrapositive of the assertion of the theorem.

**Lemma 12** All eigenvalues of \(\text{ICG}_n(D)\) on odd positions are even if and only if \(D = D_1 \cup 2D_1\).

**Lemma 13** Let \(n/2 \notin D\). All eigenvalues of \(\text{ICG}_n(D)\) on odd positions are odd if and only if \(D = D_1^* \cup 2D_1^* \cup \{n/2\}\) where \(D_1^* = D_1 \setminus \{n/2\}\).

**Proof.** Suppose that all eigenvalues of \(\text{ICG}_n(D)\) on odd positions are odd. Denote by \(\lambda_j\) the eigenvalues of the graph. Now, consider the integral circulant graph \(\text{ICG}_n(D')\) where \(D' = D \setminus \{n/2\}\). Denote by \(\lambda'_j\) the eigenvalues of integral circulant graph \(\text{ICG}_n(D')\). Since
\[
t_{2,j} = \frac{2}{\gcd(2,j)} = \begin{cases} 2, & j \equiv 0 \pmod{2} \\ 1, & j \equiv 1 \pmod{2} \end{cases}, \quad c(j, 2) = \begin{cases} -1, & j \equiv 0 \pmod{2} \\ 1, & j \equiv 1 \pmod{2} \end{cases}
\]
(13)
it holds that
\[
\lambda_j = \begin{cases} \lambda'_j + 1, & j \equiv 0 \pmod{2} \\ \lambda'_j - 1, & j \equiv 1 \pmod{2} \end{cases}.
\]
(14)
This gives that all eigenvalues of $ICG(D')$ on odd positions are even. According to Lemma 12 we have that $D' = D_1^* \cup 2D_1$ where $D_1^* = D_1 \setminus \{n/2\}$ which completes the first part of the proof.

The converse of the assertion can be proven analogously. Let $D = D_1^* \cup 2D_1^* \cup \{n/2\}$ and $D' = D \setminus \{n/2\}$. According to Lemma 12 all eigenvalues of $ICG_n(D')$ on odd positions are even. Furthermore, using the relation (14) we have that all eigenvalues of $ICG_n(D)$ on odd positions are odd. \(\square\)

### 4.2 Perfect quantum distance, spectrum and characterization of $ICG_n(D)$ having PST

The main result of this section is characterization of $ICG_n(D)$ having PST. In addition, we also calculate perfect quantum distance of integral circulants having PST. Thus, according to Theorem 8 from now on we assume that $n$ is divisible by four.

**Theorem 14** Let $ICG_n(D)$ have PST. If $n/2 \in D$ then all eigenvalues are odd, otherwise they are even. Moreover, in the first case the eigenvalues on odd positions are equal to $-1$ and in the second one the eigenvalues on odd positions are equal to $0$.

**Proof.** According to Proposition 2 we have $\lambda_1 = \sum_{d \in D} \mu(n/d)$. Since $4 \mid n/d$ for $d \in D_2 \cup \tilde{D}_3$ we conclude that $\mu(n/d) = 0$ and therefore $\lambda_1 = \sum_{d \in D_1} \mu(n/d)$. Using Proposition 2 once again we see that $\lambda_2 = \sum_{d \in D_0} \mu(n/d) + \sum_{d \in D_1} \mu(n/2d) + \sum_{d \in D_2 \cup \tilde{D}_3} 2\mu(n/2d)$. For $d \in \tilde{D}_3$ we have $4 \mid n/2d$, which yields

$$\lambda_2 - \lambda_1 = \sum_{d \in D_1} (\mu(n/2d) - \mu(n/d)) + 2 \sum_{d \in D_2} \mu(n/2d) \in 2\mathbb{N}.$$ 

By Lemma 6 all the differences $\lambda_{j+1} - \lambda_j \in 2\mathbb{N}$ for $0 \leq i \leq n - 2$, since $ICG_n(D)$ has PST. If $n/2 \notin D$ then $\lambda_0 \in 2\mathbb{N}$ and thus all the eigenvalues are even, otherwise $\lambda_0 \in 2\mathbb{N} + 1$ and all the eigenvalues are odd.

Let $j \in 2\mathbb{N} + 1$. For $d \in D_2 \cup \tilde{D}_3$ we have that $4 \mid t_{n/d,j}$ and thus $c(j, n/d) = 0$. This reduces the formula for the $j$-th eigenvalue to

$$\lambda_j = \sum_{d \in \tilde{D}_1} c(j, n/d).$$

The condition $n/2 \notin D$ yields that $\lambda_0 \in 2\mathbb{N}$ and according to the first part of the proof all the eigenvalues are even.

Let $\mu_j, 0 \leq j \leq n - 1$, be the eigenvalues of the integral circulant graph $ICG_n(D_1)$. Thus $\lambda_j = \mu_j$ for any odd $0 \leq j \leq n - 1$, so $\mu_j \in 2\mathbb{N}$ for $j \in 2\mathbb{N} + 1$. From Lemma 12 we conclude that all the eigenvalues $\mu_j$ on odd positions are even if and only if $D_0 = 2D_1$ and $\tilde{D}_1 = D_1 \cup 2D_1$. By the first part of the proof of Theorem 14 we deduce that $\mu_j = 0$ for $j \in 2\mathbb{N} + 1$ and consequently $\lambda_j = 0$ for $j \in 2\mathbb{N} + 1$.

Analogously, if $n/2 \in D$ we obtain that $\lambda_j \in 2\mathbb{N} + 1$ for $1 \leq j \leq n - 1$ and so $\mu_j \in 2\mathbb{N} + 1$ for odd $0 \leq j \leq n - 1$. Now, Lemma 13 yields $D = D_1^* \cup 2D_1^* \cup \{n/2\}$ where $D_1^* = D_1 \setminus \{n/2\}$.

Now, consider the integral circulant graph $ICG_n(D')$ where $D' = D \setminus \{n/2\} = D_1^* \cup 2D_1^*$. Denote by $\lambda_j'$ the eigenvalues of the integral circulant graph $ICG_n(D')$. From (14) we obtain that $\lambda_j = \lambda_j' - 1$ for $j \in 2\mathbb{N} + 1$. But, according to the first part of the proof we have that $\lambda_j' = 0$ for $j \in 2\mathbb{N} + 1$ and so $\lambda_j = -1$ for $j \in 2\mathbb{N} + 1$.

\(\square\)

From the proof of the last theorem we can derive the following important corollary

**Corollary 15** If $ICG_n(D)$ has PST then $D_0 = 2(D_1 \setminus \{n/2\})$.

Using Theorem 14 we can establish a more precise criterion for the characterization of integral circulant graphs having PST, than the one given by Corollary 7.

**Lemma 16** $ICG_n(D)$ has PST if and only if there exists an integer $k \geq 1$ such that one of the following conditions are satisfied

i) $S_2(\lambda_{2j}) = k$ and $\lambda_{2j+1} = 0$, if $n/2 \notin D$
Lemma 17 Let $n$ be an even number, $d$ be a divisor of $n$ and $n_1 = n/2$. For an even number $0 \leq j \leq n - 1$ the following equalities hold

1. $c(j,n/d) = c(j/2, \frac{n_1}{d/2})$, if $d \in D_0$
2. $c(j,n/d) = c(j/2,n_1/d)$, if $d \in D_1$
3. $c(j,n/d) = 2c(j/2,n_1/d)$, if $d \in D_2 \cup \tilde{D}_3$.

Proof.

1. Suppose that $d \in D_0$. Then $\gcd(n/d,j) = \gcd(\frac{n}{d/2}, j/2)$ and

$$t_{n/d,j} = \frac{n}{d \gcd(n/d,j)} = \frac{2n_1}{2 \gcd(\frac{n}{d/2}, j/2)} = \frac{n_1}{\frac{n}{d/2}j/2}.$$ 

Furthermore, it holds that $\varphi(n/d) = \varphi(\frac{n_1}{d/2})$ and so $c(j,n/d) = c(j/2, \frac{n_1}{d/2})$.

2. Suppose now $d \in D_1$. Then $\gcd(n/d,j) = 2\gcd(n_1/d,j/2)$ and

$$t_{n/d,j} = \frac{n}{d \gcd(n/d,j)} = \frac{2n_1}{2 \gcd(n_1/d,j/2)} = \frac{n_1}{n_1d/j/2}.$$ 

We also conclude that $\varphi(n/d) = \varphi(2\frac{n_1}{d}) = \varphi(n_1/d)$ and so $c(n/d,j) = c(n_1/d,j/2)$.

3. Suppose now $d \in D_2 \cup \tilde{D}_3$. By the preceding case we obtain that $\gcd(n/d,j) = 2\gcd(n_1/d,j/2)$ and $t_{n/d,j} = t_{n_1/d,j/2}$. Let $k = S_2(n/d)$ and $n'$ be an integer such $n/d = 2^k n'/d$. Notice that $k \geq 2$, since $d \in D_2 \cup \tilde{D}_3$. Also, $n'/d$ is odd and thus $\gcd(2^k, n'/d) = 1$ which yields $\varphi(n/d) = \varphi(2^k \frac{n'}{d}) = 2^{k-1} \varphi(\frac{n'}{d})$.

Furthermore we have

$$\varphi(n/d) = 2^{k-1} \varphi(\frac{n'}{d}) = 2 \varphi(2^{k-1}) \varphi(\frac{n'}{d}) = 2 \varphi(2^{k-1} \frac{n'}{d}) = 2 \varphi(n_1/d)$$

and so $c(j,n/d) = 2c(j/2,n_1/d)$, if $d \in D_2 \cup \tilde{D}_3$.

\[\square\]

Lemma 18 If $\text{ICG}_n(D)$ has PST then $D_1 = 2(D_2 \setminus \{n/4\})$.

Proof.

Let $\lambda_j$ be an eigenvalue of $\text{ICG}_n(D)$ where $0 \leq j \leq n - 1$.

According to Corollary [15] it holds that $D_0 = 2(D_1 \setminus \{n/2\})$ and so

$$\lambda_j = \sum_{d \in 2(D_1 \setminus \{n/2\})} c(j,n/d) + \sum_{d \in D_1} c(j,n/d) + \sum_{d \in D_2 \cup \tilde{D}_3} c(j,n/d).$$

From the relation [13] we have $c(j,2) = 1$ for $j \in 2\mathbb{N}$.

For $d \in 2(D_1 \setminus \{n/2\})$, let $d = 2d'$ where $d' \in D_1 \setminus \{n/2\}$. Then $c(j,n/d) = c(j,n_1/d')$ and according to Lemma [17] (part 1.), we have $c(j,n_1/d') = c(j/2, \frac{n_1/2}{d'/2}) = c(j/2,n_1/d')$.

If $d \in D_1$ using Lemma [17] (part 2.) it holds that $c(j,n/d) = c(j/2,n_1/d)$.

Finally, if $d \in D_2 \cup \tilde{D}_3$ using Lemma [17] (part 3.) it holds that $c(j,n/d) = 2c(j/2,n_1/d)$.

Taking the discussion above into account we obtain
\[ \lambda_j = \begin{cases} c(j, 2) + 2 \sum_{d \in D_1\setminus\{n/2\} \cup D_2 \cup \tilde{D}_3} c(j/2, n_1/d) = 2 \lambda'_{j/2} + 1, & n/2 \in D \\ 2 \sum_{d \in D_1\setminus\{n/2\} \cup D_2 \cup \tilde{D}_3} c(j/2, n_1/d) = 2 \lambda'_{j/2}, & n/2 \notin D \end{cases} \] where \( n_1 = n/2 \) and \( \lambda'_j \) are the eigenvalues of the integral circulant graph \( \text{ICG}_{n_1}(D') \) where \( D' = (D_1 \setminus \{n/2\}) \cup D_2 \cup \tilde{D}_3 \).

We conclude that \( \gcd(n/d, j) = 2 \) for \( j \in 4N + 2 \) and \( d \in \tilde{D}_3 \), hence that \( 4 \mid t_{n/d, j} \), and finally that \( c(j, n/d) = 0 \). This yields that

\[ \lambda_j = \sum_{d \in \tilde{D}_3} c(j, n/d) + \sum_{d \in D_2} c(j, n/d) = \begin{cases} 2 \lambda'_{j/2} + 1, & n/2 \in D \\ 2 \lambda'_{j/2}, & n/2 \notin D \end{cases} \]

where \( \lambda'_j = \sum_{d \in D_1\setminus\{n/2\}} c(j, n_1/d) + \sum_{d \in D_2} c(j, n_1/d) \). This means that the eigenvalues \( \lambda'_j \) for odd \( 0 \leq j \leq n_1 - 1 \) coincide with the eigenvalues on odd positions of \( \text{ICG}_{n_1}(D_1 \cup D_2 \setminus \{n/2\}) \). Let us denote by \( \mu_j \) the eigenvalues of \( \text{ICG}_{n_1}(D_1 \cup D_2 \setminus \{n/2\}) \), for \( 0 \leq j \leq n_1 - 1 \).

From Lemma \[16] it follows that \( S_2(\lambda_j) = k \) or \( S_2(\lambda_j + 1) = k \), for \( j \in 4N + 2 \) and some integer \( k \geq 1 \), depending on whether \( n/2 \notin D \) or \( n/2 \in D \). Furthermore, we have that either \( S_2(\lambda'_{j/2}) = k - 1 \) or \( S_2(\lambda'_{j/2} + 1) = k - 1 \) for odd \( 0 \leq j/2 \leq n_1 - 1 \). Since \( \mu_j = \lambda'_j \) for odd \( 0 \leq j \leq n_1 - 1 \) (eigenvalues \( \mu_j \) on odd positions have the same parity), Lemma \[12] and Lemma \[13\] yield that \( D_1 = 2(D_2 \setminus \{n_1/2\}) \), which completes the proof.

\[ \square \]

**Theorem 19** If \( \text{ICG}_n(D) \) has PST then either \( n/2 \notin D \) or \( n/4 \notin D \).

**Proof.**

Case 1. \( n/2 \notin D \). Suppose also that \( n/4 \notin D \). Since \( n/2 \notin D \), according to Corollary \[15\] it holds that \( D_0 = 2D_1 \) and using Proposition \[2\] we have \( \lambda_2 = \sum_{d \in 2D_1} \mu(n/d) + \sum_{d \in D_1} \mu(n/2d) + 2 \sum_{d \in D_2 \cup \tilde{D}_3} \mu(n/2d) \).

For \( d \in \tilde{D}_3 \) we conclude that \( 4 \mid n/2d \) and so \( \sum_{d \in \tilde{D}_3} \mu(n/2d) = 0 \). Now, the formula for the eigenvalue \( \lambda_2 \) becomes \( \lambda_2 = \sum_{d \in D_1} \mu(n/2d) + \sum_{d \in D_2} \mu(n/2d) + 2 \sum_{d \in \tilde{D}_3} \mu(n/2d) \). Since \( n/4 \notin D \), Lemma \[18\] now leads to

\[ \lambda_2 = 2\left( \sum_{d \in D_2} \mu(n/2d) + \sum_{d \in \tilde{D}_3} \mu(n/2d) \right) = 2\left( \sum_{d \in D_2} \mu(n/4d) + 2 \mu(n/2d) \right) = 0. \]

**Case 2.** \( n/2 \in D \). Suppose also that \( n/4 \in D \). This gives \( D_0 = 2(D_1 \setminus \{n/2\}) \) and \( D_1 = 2(D_2 \setminus \{n/4\}) \), which follows from Corollary \[15\] and Lemma \[18\]. In this case \( \lambda_2 \) can be written as follows

\[ \lambda_2 = \sum_{d \in D'_1} \mu(n/d) + \sum_{d \in D_1 \setminus \{n/2\}} \mu(n/2d) + 2 \sum_{d \in D_2 \setminus \{n/4\}} \mu(n/2d) + \mu(1) + 2 \mu(2) = 0 + 1 - 2 = -1. \]

Notice that \( \sum_{d \in D_0} \mu(n/d) + \sum_{d \in D_1 \setminus \{n/2\}} \mu(n/2d) + 2 \sum_{d \in D_2 \setminus \{n/4\}} \mu(n/2d) = 0 \) according to Case 1 of the proof. In both cases we conclude \( \lambda_1 = \lambda_2 \) (according to Lemma \[16\]) and this is a contradiction according to Corollary \[5\].

\[ \square \]

**Perfect quantum communication distance (PQCD)** of an arbitrary pair of vertices \( a \) and \( b \) is the distance \( d(a, b) \) if a perfect state transfer exists between them. If we consider a circulant network with identical couplings PST occurs only between vertices \( b \) and \( b + n/2 \) for \( 0 \leq b \leq n/2 - 1 \) (Theorem \[3\]). For the integral circulant graph \( \text{ICG}_n(D) \) PQCD of \( b \) and \( b + n/2 \) is equal to one, if \( n/2 \in D \). Otherwise, we have that \( n/4 \in D \) (Theorem \[19\]) and thus the path \( b, b + n/4, b + n/2 \) shows that PQCD is equal to two. In both cases PQCD is independent of the order of the graph.

Now, we are ready to describe the spectrum of integral circulant graphs. The criterion for existence of PST in integral circulant graphs that we will use in the next two theorems is given by the following lemma.
Lemma 20  ICG$_n(D)$ has PST if and only if one of the following conditions holds

i)  $\lambda_{2j} \in 4\mathbb{N} + 2$ and $\lambda_{2j+1} = 0$, if $n/2 \notin D$

ii)  $\lambda_{2j} \in 4\mathbb{N} + 1$ and $\lambda_{2j+1} = -1$, if $n/2 \in D$

for $0 \leq j \leq n/2$.

Proof. In the proof we will use the same notation as in the proof of Lemma 18. Suppose that ICG$_n(D)$ has PST.

From Lemma 16 it follows that $\lambda_j = 0$ or $\lambda_j = -1$ depending on whether $n/2 \notin D$ or $n/2 \in D$, for any odd $0 \leq j \leq n - 1$.

Now suppose that $n/4 \notin D$. According to Theorem 19 we have that $n/2 \notin D$. This implies that $\lambda_0 \in 2\mathbb{N}$ and hence all the eigenvalues of ICG$_n(D)$ are even for $j \in 2\mathbb{N}$, which follows from Lemma 16. Now, we proceed using the proof of Lemma 18. We have that

$$\lambda_j = 2\lambda'_j/2$$

where $\lambda'_j$ are the eigenvalues of the integral circulant graph ICG$_{n/2}(D')$ where $D' = D_1 \cup D_2 \cup D_3 \setminus \{n/2\}$ and $n_1 = n/2$.

Since $n/4 = n_1/2 \in D$ it follows that $\lambda'_0 \in 2\mathbb{N} + 1$ which further implies that $\lambda_0 \in 4\mathbb{N} + 2$. Finally, from Lemma 16 we conclude that $\lambda_j \in 4\mathbb{N} + 2$ for $j \in 2\mathbb{N}$.

If $n/2 \in D$ then $\lambda_0 \in 2\mathbb{N} + 1$ and hence $\lambda_j \in 2\mathbb{N} + 1$ for $j \in 2\mathbb{N}$. This further implies that $\lambda_j = 2\lambda'_j/2 + 1$. According to Theorem 19 it holds that $n/4 = n_1/2 \notin D$ and so $\lambda'_0 \in 2\mathbb{N}$. This implies that $\lambda_0 \in 4\mathbb{N} + 1$. Finally, from Lemma 16 we conclude that $\lambda_j \in 4\mathbb{N} + 1$ for $j \in 2\mathbb{N}$.

If any of i) or ii) holds, it can easily be seen that $\lambda_{j+1} - \lambda_j \in 4\mathbb{N} + 2$ which implies that ICG$_n(D)$ has PST, according to Lemma 6.

$\square$

Theorem 21  Let $D$ be a set of divisors of $n$ such that $n/2, n/4 \notin D$. Then ICG$_n(D \cup \{n/4\})$ has PST if and only if ICG$_n(D \cup \{n/2\})$ has PST.

Proof.

Let $\lambda_j$, $\mu_j$ and $\nu_j$ be the eigenvalues of graphs ICG$_n(D \cup \{n/4\})$, ICG$_n(D \cup \{n/2\})$ and ICG$_n(D)$, respectively. We have the following relations between these eigenvalues: $\lambda_j = \nu_j + c(j, 4)$ and $\mu_j = \nu_j + c(j, 2)$. This yields that $\mu_j = \lambda_j - c(j, 4) + c(j, 2)$ for $0 \leq j \leq n - 1$.

By direct computation we show that

$$t_{4,j} = \frac{4}{\gcd(4,j)} = \begin{cases} 4, & j \equiv 0 \pmod{2}, \quad S_2(j) = 1, \quad c(j, 4) = \begin{cases} 0, & j \in 2\mathbb{N} + 1 \\ -2, & j \in 4\mathbb{N} + 2 \\ 2, & j \in 4\mathbb{N} \end{cases} \end{cases}$$

From this it follows that

$$\mu_j = \begin{cases} \lambda_j - 1, & j \in 2\mathbb{N} + 1 \\ \lambda_j + 3, & j \in 4\mathbb{N} + 2 \\ \lambda_j - 1, & j \in 4\mathbb{N} \end{cases}$$

The following two facts can now be easily deduced: for $j \in 2\mathbb{N} + 1$, $\lambda_j = 0$ if and only if $\mu_j = -1$ and for $j \in 2\mathbb{N}$, $\lambda_j \in 4\mathbb{N} + 2$ if and only if $\mu_j \in 4\mathbb{N} + 1$. To complete the proof it only remains to apply Lemma 20.

$\square$

Finally we can state the next of our main results.

Theorem 22  ICG$_n(D)$ has PST if and only if $n \in 4\mathbb{N}$, $D_1^* = 2D_2^*$, $D_0 = 4D_2^*$ and either $n/4 \in D$ or $n/2 \in D$, where $D_2^* = D_2 \setminus \{n/4\}$ and $D_1^* = D_1 \setminus \{n/2\}$.  

11
Proof.

(⇒:) This is an easy consequence of Theorem 8, Lemma 18, Corollary 15, and Theorem 19.

(⇐:) According to Theorem 21, this implication is sufficient to prove for \( n/4 \in D \). Furthermore, by Theorem 19 we have \( n/2 \not\in D \).

Let \( 0 \leq j \leq n - 1 \) be an odd number. For \( d \in D_2 \cup \overline{D_3} \), we conclude that \( c(j, n/d) = 0 \), which follows from the fact that \( 4 \mid t_{n/d,j} \). This implies that

\[
\lambda_j = \sum_{d \in 2D_1} c(j, n/d) + \sum_{d \in D_1} c(j, n/d) = \sum_{d \in D_1} c(j, n/2d) + c(j, n/d) = 0.
\]

The last equality follows from Lemma 10.

Let \( 0 \leq j \leq n - 1 \) be an even number. We have

\[
\lambda_j = \sum_{d \in 2D_1} c(j, n/d) + \sum_{d \in D_1} c(j, n/d) + \sum_{d \in D_2 \setminus \{n/4\}} c(j, n/d) + c(j, 4) + \sum_{d \in \overline{D_3}} c(j, n/d).
\]

From Lemma 10 and relation (17) it follows that

\[
\lambda_j = 2 \sum_{d \in D_1} c(j, n/d) + \sum_{d \in D_2 \setminus \{n/4\}} c(j, n/d) + \sum_{d \in \overline{D_3}} c(j, n/d) \pm 2.
\]

Now using Lemma 17 we obtain

\[
\lambda_j = 2 \sum_{d \in D_1 = 2(D_2 \setminus \{n/4\})} c(j/2, n_1/d) + 2 \sum_{d \in D_2 \setminus \{n/4\}} c(j/2, n_1/d) + 2 \sum_{d \in \overline{D_3}} c(j/2, n_1/d) \pm 2
\]

\[
= 2 \sum_{d \in D_2 \setminus \{n/4\}} (c(j/2, n_1/d) + c(j/2, n_1/2d)) + 2 \sum_{d \in \overline{D_3}} c(j/2, n_1/d) \pm 2,
\]

where \( n_1 = n/2 \).

Let \( j \in 4\mathbb{N} + 2 \). By Lemma 10 we obtain that \( c(j/2, n_1/d) + c(j/2, n_1/2d) = 0 \), since \( j/2 \in 2\mathbb{N} + 1 \). For \( d \in \overline{D_3} \) we conclude that \( 4 \mid t_{n_1, j/2} \) and \( c(j/2, n_1/d) = 0 \). Finally, we conclude that \( \lambda_j = c(j, 4) = -2 \).

If \( j \in 4\mathbb{N} \), according to Lemma 10 we show that

\[
\lambda_j = 4 \sum_{d \in D_2 \setminus \{n/4\}} c(j/2, n_1/d) + 2 \sum_{d \in \overline{D_3}} c(j/2, n_1/d) + 2.
\]

Furthermore, using Lemma 17 we have \( c(j/2, n_1/d) = 2c(j/4, n_1/2d) \). In either case we conclude that \( \lambda_j \in 4\mathbb{N} + 2 \) for \( j \in 2\mathbb{N} \). Now, direct application of Lemma 20 completes the proof. \( \square \)

According to the previous result, we notice that graph \( \text{ICG}_n(D_n \setminus \{n/2\}) \) has PST for \( n \in 4\mathbb{N} \). This class of circulant graphs is known as cocktail-party graphs (see 10).

5 Conclusion

In this paper we continue to address the question of when circulant graphs can have perfect state transfer, and improve the necessary condition of \( n \) being divisible by 4 given in 17. Theorem 22 completely characterizes the graphs \( \text{ICG}_n(D) \) having PST. This result includes the classes of graphs having PST found in 8, 15, 17.

From the above characterization we can calculate the number of integral circulant graphs of a given order having PST. If \( n \in 8\mathbb{N} \), by the rule of product, the number is equal to the product of the cardinalities of the power sets of \( \{d : d \mid n, n/d \in 8\mathbb{N}\} \) and \( \{d : d \mid n, n/d \in 8\mathbb{N} + 4\} \setminus \{n/4\} \) times two. If \( n \in 8\mathbb{N} + 4 \) the number is equal to the cardinality of the power set of \( \{d : d \mid n, n/d \in 8\mathbb{N} + 4\} \setminus \{n/4\} \) times two. In either of cases we have two possibilities since either \( n/2 \in D \) or \( n/4 \in D \). Thus, for a given number \( n \) the number of integral circulant graphs \( \text{ICG}_n(D) \) having PST is given by the following formula

\[
\text{Number of PST graphs} = 2^{\left| \{d : d \mid n, n/d \in 8\mathbb{N}\} \right| \times 2} \times 2^{\left| \{d : d \mid n, n/d \in 8\mathbb{N} + 4\} \setminus \{n/4\} \right| \times 2}
\]
\[ | \text{ICG}_n(D) | = \begin{cases} \
 \frac{2^{\tau(n)}}{\tau(2^4)} , & n \in 8\mathbb{N} + 4, \\
 \frac{2^{\tau(n)} \tau(n)}{\tau(2^8)} , & n \in 8\mathbb{N} \
 \end{cases} \]

where \( \tau(n) \) denotes the number of the divisors of \( n \).

We can see from the formula that for some values of \( n \in 8\mathbb{N} \) (for example \( n = 96, 120, 144, 160, 168, 192, \ldots \)) there is a great number of graphs having PST, while for some other values \( n \in 8\mathbb{N} + 4 \), there are only 2 such graphs. However, the number of \( \text{ICG}_n(D) \) having PST is asymptotically equal to the number of \( \text{ICG}_n(D) \) of a given order \( n \). The last conclusion follows from Corollary 7.2 given in [20], where it was shown that there are at most \( 2^{\tau(n)-1} \) integral circulant graphs on \( n \) vertices.

It is worth mentioning that the maximum value of the perfect quantum communication distance (i.e. the maximal distance between vertices where a perfect state transfer occurs) is equal to 2 for every \( n \in 4\mathbb{N} \). Thus, it is still an open problem whether one can construct a network with identical couplings in which any quantum state can be perfectly transferred over a larger distance than \( 2\log_3 n \), obtained in [7, 8] for two-link hypercubes with \( n \) vertices.

An improvement of the perfect quantum communication distance is made in [7] by considering fixed but different nearest-neighbor couplings. A similar approach used on circulant graphs (having a weighted adjacency matrix) might also enlarge the perfect quantum communication distance. Many recent papers propose such an approach [11,14,15,16]. First results concerning characterization and finding new classes of weighted circulant graphs are given in [5]. Characterization of integral circulant graphs (moreover circulant graphs) having PST is the first step in describing the more general class of weighted integral circulant graphs having PST. The approach to this problem should, in our opinion, use the interplay of graph and number theory.

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