Models with three Higgs doublets in the triplet representations of $A_4$ or $S_4$

R. González Felipe, H. Serôdio, and João P. Silva

1 Instituto Superior de Engenharia de Lisboa - ISEL, 1959-007 Lisboa, Portugal
2 Centro de Física Teórica de Partículas (CFTP), Instituto Superior Técnico, Universidade Técnica de Lisboa, 1049-001 Lisboa, Portugal
3 Departament de Física Teòrica and IFIC, Universitat de València-CSIC, E-46100, Burjassot, Spain

(Dated: February 15, 2013)

We consider the quark sector of theories containing three scalar $SU(2)_L$ doublets in the triplet representation of $A_4$ (or of $S_4$) and three generations of quarks in arbitrary $A_4$ (or $S_4$) representations. We show that, for all possible choices of quark field representations and for all possible alignments of the Higgs vacuum expectation values that can constitute global minima of the scalar potential, it is not possible to obtain simultaneously non-vanishing quark masses and a non-vanishing CP-violating phase in the CKM quark mixing matrix. As a result, in this minimal form, models with three scalar fields in the triplet representation of $A_4$ or of $S_4$ cannot be extended to the quark sector in a way consistent with experiment.

PACS numbers: 12.60.Fr, 14.80.Ec, 11.30.Qc, 11.30.Ly

I. INTRODUCTION

There is a long history of articles considering discrete symmetries in the study of the lepton sector (see for instance the recent reviews [1–4] and references therein), including many models predicting tri-bimaximal lepton mixing [5], now disfavored by the measurement of a large mixing angle $\theta_{13}$ [6,7]. In the quark sector, models based on the $A_4$ symmetry as a possible family symmetry were first introduced in Refs. [10,11]. After the impact of the symmetry on the Yukawa matrices is known, some structure for the vacuum expectation values (vev) has to be assumed before moving on to the mass matrices and respective phenomenological predictions. Occasionally, this has been performed without a full study of the scalar sector and without ensuring properly whether the assumed vacuum structure indeed corresponds to the global minimum. This may occur in part because finding local minima is easy (one just has to show that the gradient of the potential vanishes), while ensuring that there is no other, lower-lying, minimum is often rather difficult. Recently, Degee, Ivanov, and Keus [12] have introduced a geometrical procedure to minimize highly symmetric scalar potentials, and solved the problem for a three Higgs doublet model (3HDM) potential with an $A_4$ or an $S_4$ symmetry. Although it is not explicitly stated, Ref. [12] refers to a set of three Higgs fields in a triplet representation of the group $A_4$. This is a crucial point since, if one were to place each of the three Higgs fields in a singlet representation, then one would end up with the most general 3HDM potential. It is found that the possible vev alignments for the $A_4$ symmetric potential [14] which may correspond to a global minimum are

\[
\begin{align*}
&v(1,0,0), \\
&v(1,1,1), \\
&v(\pm 1,\eta,\eta^* ) \text{ with } \eta = e^{i\pi/3}, \\
&v(1,e^{i\alpha},0) \text{ with any } \alpha.
\end{align*}
\]  

Similarly, the possible vev alignments corresponding to global minima in the $S_4$ symmetric potential are

\[
\begin{align*}
&v(1,0,0), \\
&v(1,1,1), \\
&v(\pm 1,\eta,\eta^* ) \text{ with } \eta = e^{i\pi/3}, \\
&v(1,i,0).
\end{align*}
\]  

In each case, a vev corresponding to some permutation of the fields is also a possible global minimum. Any other solution of the stationarity conditions may be a saddle point, a local maximum, or even a local minimum, but never the global minimum.

Besides a correct identification of global minima, one must also consider whether the specific discrete symmetry under study can be extended to the whole Lagrangian of the theory, in a way consistent with known data. In particular, in the quark sector there should be no massless quarks, no diagonal blocks in the CKM matrix, and/or no vanishing CP-violating phase. As shown by Ferreira and Silva [15], these constraints place stringent limits on the type of mass matrices obtainable from Abelian symmetries in the 2HDM.

In this article, we consider models with three Higgs doublets $\Phi$ in a triplet representation of $A_4$ (Sec. I), or in a triplet representation of $S_4$ (Sec. II). This ensures that the only possible global vev structures are those in Eqs. (1) and (2), respectively. The models contain only three generations of left-handed quark doublets $Q_L$, right-handed up-type quark singlets $u_R$, and right-handed down-type quark singlets $d_R$. Our conclusions are briefly summarized in Sec. IV.

---

* E-mail: ricardo.felipe@ist.utl.pt
† E-mail: hugo.serodio@ific.uv.es
‡ E-mail: jpsilva@cftp.ist.utl.pt
3 To be precise, the three scalar fields must be in a faithful representation of the group $A_4$. 

---

Program: hep-ph
II. THE A₄ CASE

A₄ is the group of the even permutations of four objects and it has 12 elements divided into four irreducible representations, namely, three singlets 1, 1', 1'' and one triplet 3. The multiplication rules are

\[
\begin{align*}
1 \otimes \text{any} & = \text{any}, \\
1' \otimes 1' & = 1'', \\
1' \otimes 1'' & = 1, \\
1' \otimes 3 & = 3, \\
1'' \otimes 1'' & = 1', \\
1'' \otimes 3 & = 3, \\
3 \otimes 3 & = 1 \oplus 1' \oplus 1'' \oplus 3_a \oplus 3_b.
\end{align*}
\]

We recall that, for the corresponding entry of the Yukawa coupling matrix to be non-vanishing, the Yukawa Lagrangian must be in the invariant singlet representation 1. Since the three Higgs doublets are in the representation 3, we see from Eqs. (3), that the product of left-handed and right-handed fermions must also be in a triplet representation. This means that at least one of the fermion fields in each charge sector must be in a triplet representation. The possibilities for the representations of the left-handed quark fields and for the up and down right-handed quarks are listed in Table I.

| Q_L | u_R | d_R |
|-----|-----|-----|
| 3   | 3   | 3   |
| 3   | 3   | three singlets |
| 3   | three singlets | 3 |
| 3   | three singlets | 3 |

TABLE I. Possible representations of the left-handed quark doublets (Q_L), the right-handed up quark singlets (u_R), and the right-handed down quark singlets (d_R), when the three Higgs doublets are in a triplet representation 3.

Since permutations of the three fields in each sector do not lead to new structures for the Yukawa matrices, the notation “three singlets” stands for the following independent possibilities for the fields in each of the three generations:

\[
\begin{align*}
(1, 1, 1), & \quad (1, 1', 1''), \\
(1, 1, 1'), & \quad (1', 1', 1'), \\
(1, 1', 1'), & \quad (1', 1', 1''), \\
(1, 1, 1''), & \quad (1', 1'', 1''), \\
(1, 1'', 1''), & \quad (1'', 1', 1'').
\end{align*}
\]

In order to use the vevs given in Eq. (1), one must be sure to use a representation of the group that is consistent with the basis in which those vevs were obtained in Ref. [12]. Indeed, if one starts from Higgs fields with the vevs of Eq. (1), and one changes the scalar fields by a unitary transformation U, i.e.

\[
\begin{pmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{pmatrix}
\rightarrow
U
\begin{pmatrix}
\Phi_1 \\
\Phi_2 \\
\Phi_3
\end{pmatrix},
\]

then the vevs also transform as

\[
\begin{pmatrix}
\langle \Phi_1 \rangle \\
\langle \Phi_2 \rangle \\
\langle \Phi_3 \rangle
\end{pmatrix}
\rightarrow
U
\begin{pmatrix}
\langle \Phi_1 \rangle \\
\langle \Phi_2 \rangle \\
\langle \Phi_3 \rangle
\end{pmatrix},
\]

and, in general, will no longer have the form in Eq. (1).

A suitable basis for the triplet representation of A₄ is given by

\[
S = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad T = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
\]

In the notation of Sec. 6.4 of Ref. [16], \(a_1 = S, b = T, \) and \(a_2 = T^{-1}ST\) is redundant. These matrices satisfy \(S^2 = T^3 = (ST)^3 = 1,\) showing that they indeed generate the group A₄. Equations (7) also coincide with the basis used in Ref. [17].

One way to confirm that we are indeed using a basis consistent with Ref. [12] is to check that imposing S and T on the 3HDM potential, we recover

\[
V = -\frac{M_0}{\sqrt{3}} (|\langle \Phi_1 \rangle|^2 + |\langle \Phi_2 \rangle|^2 + |\langle \Phi_3 \rangle|^2)
\]

\[
+ \frac{\Lambda_0}{3} (|\langle \Phi_1 \rangle|^2 + |\langle \Phi_2 \rangle|^2 + |\langle \Phi_3 \rangle|^2)^2
\]

\[
+ \frac{\Lambda_3}{3} (|\langle \Phi_1 \rangle|^4 + |\langle \Phi_2 \rangle|^4 + |\langle \Phi_3 \rangle|^4 - |\langle \Phi_1 \rangle|^2 |\langle \Phi_2 \rangle|^2 - |\langle \Phi_2 \rangle|^2 |\langle \Phi_3 \rangle|^2 - |\langle \Phi_3 \rangle|^2 |\langle \Phi_1 \rangle|^2)
\]

\[
+ \Lambda_1 \left( (Re\Phi_1^* \Phi_2) + (Re\Phi_2^* \Phi_3) + (Re\Phi_3^* \Phi_1) \right)^2
\]

\[
+ \Lambda_2 \left( (Im\Phi_1^* \Phi_2) + (Im\Phi_2^* \Phi_3) + (Im\Phi_3^* \Phi_1) \right)^2
\]

\[
+ \Lambda_4 \left[ (Re\Phi_1^* \Phi_2)(Im\Phi_1 \Phi_2) + (Re\Phi_2^* \Phi_3)(Im\Phi_2 \Phi_3)
\right. \]

\[
+ \left. (Re\Phi_3^* \Phi_1)(Im\Phi_3 \Phi_1) \right],
\]

as in Eq. (9) of Ref. [12].

In A₄, with the basis of Eq. (7), the product of two triplets, \(\alpha = (a_1, a_2, a_3)\) and \(\beta = (b_1, b_2, b_3),\) gives

\[
\begin{align*}
(a \otimes b)_1 & = a_1 b_1 + a_2 b_2 + a_3 b_3, \\
(a \otimes b)_1' & = a_1 b_1 + a_2 b_2 + a_3 b_3, \\
(a \otimes b)_1'' & = a_1 b_1 + a_2 b_2 + a_3 b_3, \\
(a \otimes b)_3 & = (a_2 b_3 + a_3 b_2, a_3 b_1 + a_1 b_3, a_1 b_2 + a_2 b_1), \\
(a \otimes b)_3_a & = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1),
\end{align*}
\]

2 Equation (9) of Ref. [12] coincides with the sum of Eqs. (38) and (39) of Ref. [16], with the substitutions \(\Lambda_0 = 3\lambda + \lambda', \) \(\Lambda_1 = \lambda'' + 2\text{Re}(\lambda), \) \(\Lambda_2 = \lambda'' - 2\text{Re}(\lambda), \) \(\Lambda_3 = -\lambda', \) \(\Lambda_4 = -4\text{Im}(\lambda).\)
where $\omega = e^{2i\pi/3}$, and $s, a$ stand for the symmetric and anti-symmetric triplet components, respectively.

We will also need the product of three triplets, $a, b,$ and $c = (c_1, c_2, c_3)$:

$$
(a \otimes b \otimes c)_s = a_1(b_2c_3 + b_3c_2) + a_2(b_3c_1 + b_1c_3) + a_3(b_1c_2 + b_2c_1),
$$

$$
(a \otimes b \otimes c)_a = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).
$$

We are now ready to construct the Yukawa matrices for the various cases. We have built a program to test all possibilities automatically. As a first example, let us consider the case $\Phi \sim 3, (Q_{L1}, Q_{L2}, Q_{L3}) \sim (1, 1, 1'), d_R \sim 3$, and $u_R \sim 3$. We start with the down sector. Since $Q_{L1}$ is in the $1'$ representation, it must couple to the $(\Phi \otimes d_R)_1'$ combination obtained from Eq. (9). The same is true for $Q_{L2}, Q_{L3}$, with an independent coefficient. This leads to the Yukawa terms

$$
\alpha_1 \overline{Q}_{L1} [\Phi_1 d_{R1} + \Phi_2 d_{R2} + \Phi_3 d_{R3}]
+ \alpha_2 \overline{Q}_{L2} [\Phi_1 d_{R1} + \Phi_2 d_{R2} + \Phi_3 d_{R3}].
$$

Once the fields $\Phi_i$ are substituted by their vevs $v_i$, these terms give the first and second row of the down-type quark mass matrix, $M_d$, respectively. Since $Q_{L3}$ is in the $1'$ representation, we can only obtain a singlet with the $1''$ combination $(\Phi \otimes d_R)_{1''}$ in Eq. (9). This leads to a term

$$
\alpha_3 \overline{Q}_{L3} [\Phi_1 d_{R1} + \omega \Phi_2 d_{R2} + \omega^2 \Phi_3 d_{R3}],
$$

which will fill the third row of $M_d$. Thus, the down-type quark mass matrix reads

$$
M_d = \begin{pmatrix} 
\alpha_1 v_1 & \alpha_1 v_2 & \alpha_1 v_3 \\
\alpha_2 v_1 & \alpha_2 v_2 & \alpha_2 v_3 \\
\alpha_3 v_1 & \omega \alpha_3 v_2 & \omega^2 \alpha_3 v_3
\end{pmatrix},
$$

with arbitrary complex constants $\alpha_i$.

Recalling that the up-quark Yukawa terms involve the combinations $\overline{Q}_L \Phi u_R$, a similar analysis of the up-type quark sector yields

$$
M_u = \begin{pmatrix} 
\beta_1 v_1^* & \beta_1 v_2^* & \beta_1 v_3^* \\
\beta_2 v_1^* & \beta_2 v_2^* & \beta_2 v_3^* \\
\beta_3 v_1^* & \omega \beta_3 v_2^* & \omega^2 \beta_3 v_3^*
\end{pmatrix},
$$

where $\beta_i$ are arbitrary complex constants.

In order to find the most relevant features of the quark sector, we define the Hermitian matrices

$$
H_d = M_d M_d^\dagger, \quad H_u = M_u M_u^\dagger,
$$

whose eigenvalues coincide with the squared masses in each quark sector. Moreover, the CKM CP-violating phase is proportional to the determinant

$$
J = \text{Det}(H_d H_u - H_u H_d).
$$

We must now substitute $(v_1, v_2, v_3)$ by each of the possible vev alignments in Eq. (1), including all possible permutations, and study the properties of $H_d, H_u,$ and $J$. As an example, consider the possibility that $(v_1, v_2, v_3) = v(1, e^{i\alpha}, 0)$, for any phase $\alpha$. Then

$$
M_d = v \begin{pmatrix} 
\alpha_1 & \alpha_1 e^{i\alpha} & 0 \\
\alpha_2 & \alpha_2 e^{i\alpha} & 0 \\
\alpha_3 & \omega \alpha_3 e^{i\alpha} & 0
\end{pmatrix},
$$

$$
M_u = v \begin{pmatrix} 
\beta_1 & \beta_1 e^{-i\alpha} & 0 \\
\beta_2 & \beta_2 e^{-i\alpha} & 0 \\
\beta_3 & \omega \beta_3 e^{-i\alpha} & 0
\end{pmatrix}.
$$

As a result, we predict one massless quark with charge $-1/3$ and one massless quark with charge $2/3$, contrary to experimental evidence. It is interesting to note that, in this case, $H_d$ and $H_u$ do not depend on $\alpha$ but, nevertheless, $J \neq 0$. This means that the model predicts one massless quark in each charge sector but displays explicit CP violation in the CKM matrix.

As a second example, let us consider the case $\Phi \sim 3, (Q_{L1}, Q_{L2}, Q_{L3}) \sim (1, 1', 1''), d_R \sim 3$, and $u_R \sim 3$. We find

$$
M_d = \begin{pmatrix} 
\alpha_1 v_1 & \alpha_1 v_2 & \alpha_1 v_3 \\
\alpha_2 v_1 & \alpha_2 v_2 & \alpha_2 v_3 \\
\alpha_3 v_1 & \omega \alpha_3 v_2 & \omega^2 \alpha_3 v_3
\end{pmatrix},
$$

$$
M_u = \begin{pmatrix} 
\beta_1 v_1^* & \beta_1 v_2^* & \beta_1 v_3^* \\
\beta_2 v_1^* & \beta_2 v_2^* & \beta_2 v_3^* \\
\beta_3 v_1^* & \omega \beta_3 v_2^* & \omega^2 \beta_3 v_3^*
\end{pmatrix}.
$$

For the vev alignments $v(1, 1, 1)$ and $v(\pm 1, \eta, \eta^*)$ of Eq. (1), this leads to

$$
H_d = 3v^2 \begin{pmatrix} |\alpha_1|^2 & 0 & 0 \\
0 & |\alpha_2|^2 & 0 \\
0 & 0 & |\alpha_3|^2
\end{pmatrix},
$$

$$
H_u = 3v^2 \begin{pmatrix} |\beta_1|^2 & 0 & 0 \\
0 & |\beta_2|^2 & 0 \\
0 & 0 & |\beta_3|^2
\end{pmatrix},
$$

meaning that, in these cases, all quark masses are non-vanishing and non-degenerate. However, we find a diagonal CKM matrix and no CP-violation, in blatant contradiction with experiment.

The particular case where $Q_{L1}, u_R$, and $d_R$ (in addition to $\Phi$) are all in a triplet representation of $A_4$ has been considered in Refs. [10, 11] for the first three vevs given in Eq. (1). Ref. [10] solves the problem by adding a fourth scalar as a singlet of $A_4$; Ref. [11] considers symmetry breaking in stages.

---

3 One could envisage a more complicated setup where the light quark masses appear radiatively.
Having gone through all cases in Table I and all possible vev alignments in Eq. (11) (including permutations), we find that in all situations one obtains either massless quarks or a vanishing CKM phase.

In Table II we present, for each choice of representations and for each vev alignment given in Eq. (11), the different quark mass spectra and the number of CKM mixing angles not predicted by the discrete symmetry, i.e. the number of parameter-dependent mixing angles (PDMA).

| vev  | $Q_L$ | $u_R$ | $d_R$ | Number of PDMA | Mass spectrum |
|------|-------|-------|-------|----------------|--------------|
| 3 3 3 | 0    | $(0, m_{u,d}, m'_{u,d})$ | | | |
| 3 3 s | 0    | $(0, m_u, m_u')$ | | | |
| 3 s 3 | 0    | $(0, m_u)$ | | | |
| 3 s s | 0    | $(0, m_d, m_d')$ | | | |
| s 3 3 | 2    | $(0, 0, m_d)$ | | | |
| 3 3 3 | 0    | $(m_{u,d}, m'_u, m'_d)$ | | | |
| 3 3 s | 0    | $(m_u, m_u', m_u'')$ | | | |
| 3 s 3 | 0    | $(x, x, m_u)$ | | | |
| 3 s s | 0    | $(x, x, m_d)$ | | | |
| s 3 3 | 1    | $(0, m_{u,d}, m'_{u,d})$ | 2 | $(0, 0, m_d)$ | |
| 3 3 3 | 1    | $(0, m_{u,d}, m_{u,d})$ | | | |
| 3 3 s | 1    | $(0, m_u, m_u')$ | | | |
| 3 s 3 | 1    | $(0, x, m_u)$ | | | |
| 3 s s | 1    | $(0, x, m_d)$ | | | |
| s 3 3 | 3    | $(0, m_{u,d}, m'_{u,d})$ | 2 | $(0, 0, m_d)$ | |

TABLE II. Quark mass spectra and number of arbitrary CKM parameter-dependent mixing angles (PDMA) in the $A_4$ case. The symbol $x$ stands for 0 or $m_i \neq 0$; $s$ stands for $1, 1'$ or $1''$. Requiring non-vanishing quarks by itself, restricts the representations of $\{Q_L, u_R, d_R\}$ to the five possibilities $\{s; 3, 3\}$, $\{3; s, s\}$, $\{3; 3, 3\}$, $\{3; 3, 3\}$, and $\{3; 3, 3\}$, where $s$ stands for $(1, 1', 1'')$, with the vevs restricted to $v(1, 1, 1)$ or $v(\pm 1, \eta, \eta^*)$. In all these special cases, the CKM matrix equals the unit matrix. Thus, it is not possible to extend the $A_4$ symmetry to the quark sector, with only three generations of quarks and the three scalar fields in a triplet of $A_4$.

It is conceivable that this problem can be evaded by adding quark generations. More commonly, one considers other representations for the three scalar fields and/or one adds extra scalars to the theory in other representations of $A_4$. But, in such cases one must prove that the local minimum does indeed correspond to a global minimum. One can see from the treatment of $A_4$ that this endeavor is far from trivial [12].

III. THE $S_4$ CASE

$S_4$ is the group of all permutations of four objects. It has 24 elements divided into five irreducible representations: two singlets $1_1$, $1_2$, one doublet $2_1$ and two triplets $3_1, 3_2$. The multiplication rules are:

\[1_2 \otimes 1_2 = 1_1,\]
\[1_2 \otimes 2 = 2,\]
\[1_2 \otimes 3_1 = 3_2,\]
\[1_2 \otimes 3_2 = 3_1,\]
\[2 \otimes 2 = 1_1 \oplus 1_2 \oplus 2,\]
\[2 \otimes 3_1 = 3_1 \oplus 3_2,\]
\[2 \otimes 3_2 = 3_1 \oplus 3_2,\]
\[3_1 \otimes 3_1 = 1_1 \oplus 2 \oplus 3_1 \oplus 3_2,\]
\[3_1 \otimes 3_2 = 1_2 \oplus 2 \oplus 3_1 \oplus 3_2,\]
\[3_2 \otimes 3_2 = 1_1 \oplus 2 \oplus 3_1 \oplus 3_2.\]

Since $A_4$ is a subgroup of $S_4$, this case will have at least the same unphysical restrictions. Yet, for model building, it is useful to go through the analysis in detail, uncovering the specific constraints that should be corrected when enlarging the model.

Let us start by assuming that the three Higgs doublets are in the representation $3_1$. By looking at Eqs. (23), we see that the product of left-handed and right-handed fermions must also be in a $3_1$ representation (or else, the Yukawa Lagrangian would not be in the invariant of the up and down right-handed quarks are listed in Table III when $Q_L$ is in a triplet representation.

When two of the $Q_L$ are in the doublet $2$ representation, the possibilities are $(Q_L, u_R, d_R) \sim (2, 3_1, 3_1), (2, 3_1, 3_2), (2, 3_2, 3_1), (2, 3_2, 3_2)$. Similarly, when one of the $Q_L$ is in a singlet representation, there are only two possibilities: either $(Q_L, u_R, d_R) \sim (1_1, 3_1, 3_1)$, or $(Q_L, u_R, d_R) \sim (1_2, 3_2, 3_2)$. But, in this case, the third $Q_L$ field must be in a singlet representation that yields a Yukawa Lagrangian in the singlet representation. Otherwise, the mass matrix would have a row of zeros, and there would be a massless quark. As a result, when two of the $Q_L$ are in the doublet $2$ representation, the only viable possibilities for $u_R$ and $d_R$ are the ones listed in Table IV.

Finally, requiring that there are no massless quarks, when all the $Q_L$ are in a singlet representation, the possibilities for $u_R$ and $d_R$ are listed in Table V.

A suitable basis for the $3_1$ representation of $S_4$, consistent with the notation of Ref. [12], can be found in
ample, the product of two

$F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. \hspace{1cm} (24)

Notice that $G_3$ coincides with $T$ in Eq. (7). Imposing $F_3$ and $G_3$ on the 3HDM potential we recover Eq. (8), with $\Lambda_4 = 0$. The $3_2$ representation of $S_4$ can be identified with the matrices $-F_3$ and $G_3$. These matrices satisfy $F_3^2 = G_3^2 = (F_3G_3)^4 = 1$, showing that they indeed generate the group $S_4$. As for the explicit form of the tensor products, we will use the Appendix of Ref. [19]. For example, the product of two $3_1$ triplets, $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, gives

\begin{align*}
(a \otimes b)_{11} &= a_1 b_1 + a_2 b_2 + a_3 b_3, \\
(a \otimes b)_{2} &= (a_1 b_1 + a_2 b_2 + \omega^2 a_3 b_3), \\
(a \otimes b)_{3} &= (a_2 b_3 + a_3 b_1, a_1 b_3, a_1 b_2 + a_2 b_1), \\
(a \otimes b)_{3_2} &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).
\end{align*}

For illustration, let us consider the case $\Phi \sim 3_1$, $(Q_{L1}, Q_{L2}) \sim 2$, $Q_{L3} \sim 1_1$, $d_R \sim 3_1$, and $u_R \sim 3_1$. We start with the down sector. The fact that $(Q_{L1}, Q_{L2})$ is in the doublet representation 2, means that we must pick up the doublet combination $(\Phi \otimes d_R)_2$ obtained from Eq. (25), leading to

\begin{align*}
\alpha_1 \overline{Q}_{L1} [\Phi_1 d_{R1} + \omega \Phi_2 d_{R2} + \omega^2 \Phi_3 d_{R3}] \\
+ \alpha_1 \overline{Q}_{L2} [\Phi_1 d_{R1} + \omega^2 \Phi_2 d_{R2} + \omega \Phi_3 d_{R3}] .
\end{align*}

(26)

On the other hand, $\overline{Q}_{L3} \sim 1_1$ couples to $(\Phi \otimes d_R)_{11}$ in Eq. (25), yielding

\begin{align*}
\alpha_2 \overline{Q}_{L3} (\Phi_1 d_{R1} + d_{R2} + \Phi_3 d_{R3}).
\end{align*}

(27)

Hence,

\begin{align*}
M_d = \begin{pmatrix} \alpha_1 v_1 \omega \alpha_1 v_2 \omega^2 \alpha_1 v_3 \\ \alpha_1 \omega \alpha_1 v_2 \omega v_3 \\ \alpha_2 v_1 \alpha_2 v_2 \alpha_2 v_3 \end{pmatrix} .
\end{align*}

(28)

Similarly,

\begin{align*}
M_u = \begin{pmatrix} \beta_1 v_1^* \omega \beta_1 v_2 \omega^2 \beta_1 v_3^* \\ \beta_1 \omega \beta_1 v_2 \omega \beta_1 v_3^* \\ \beta_2 v_1^* \beta_2 v_2 \beta_2 v_3^* \end{pmatrix} .
\end{align*}

(29)

The predictions for the physical observables should now be found for all the possible global minima presented in Eq. (2). Let us test the case with the vev alignment $v_1, v_2, v_3$. We find

\begin{align*}
H_u = 3v^2 \begin{pmatrix} |\beta_1|^2 & 0 & 0 \\ 0 & |\beta_1|^2 & 0 \\ 0 & 0 & |\beta_2|^2 \end{pmatrix} ,
\end{align*}

(30)

Although this case does not exhibit massless quarks, it has a pair of degenerate quarks in each sector, the CKM is the unit matrix and, of course, there is no CP violation.

The analysis for $\Phi \sim 3_2$ leads to a new set of cases obtained trivially from Tables II, III, and IV by noting that $3_2 = 3_1 \otimes 1_2$. As we did for $A_4$, we have also built a program to test all $S_4$ possibilities automatically. In all cases, there is no CP violation in the CKM matrix ($J = 0$) and, in the absence of massless quarks, there will always be one pair of degenerate quarks in each sector.
The restrictions on the physical parameters obtained for each choice of representations and for each vev alignment in Eq. (2), can be found in Tables VI-VIII. This may help model builders in identifying what features need to be corrected when adding extra fields to the theory.

| vev | $Q_L$ | $u_R$ | $d_R$ | Number of PDMA | Mass spectrum |
|-----|-------|-------|-------|----------------|---------------|
| $1_1$ | $1_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $1_1$ | $1_1$ | $2_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $1_1$ | $3_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $1_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $1_1$ | $2_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $1_1$ | $3_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $3_1$ | $1_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $3_1$ | $2_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $3_1$ | $3_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $3_1$ | $3_1$ | $1_1$ | 1 | $(0,0,m_{u,d})$ |
| $1_1$ | $3_1$ | $3_1$ | $1_1$ | 2 | $(0,0,m_{u,d})$ |

Table VI. Quark mass spectra and number of arbitrary CKM parameter-dependent mixing angles (PDMA) in the $S_4$ case, for the vev $v(1,0,0)$. In all cases, $\Phi \sim 3_1$.

| vev | $Q_L$ | $u_R$ | $d_R$ | Number of PDMA | Mass spectrum |
|-----|-------|-------|-------|----------------|---------------|
| $1_1$ | $1_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $1_1$ | $1_1$ | $2_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $1_1$ | $3_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $1_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $1_1$ | $2_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $1_1$ | $3_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $3_1$ | $1_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $3_1$ | $2_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $3_1$ | $3_1$ | $1_1$ | $1_1$ | 0 | $(0,0,m_{u,d})$ |
| $2_1$ | $3_1$ | $3_1$ | $1_1$ | 1 | $(0,0,m_{u,d})$ |
| $1_1$ | $3_1$ | $3_1$ | $1_1$ | 2 | $(0,0,m_{u,d})$ |

Table VII. As in Table VI for the vev $v(1,1,1)$ and $v(1,1,1)$. 

IV. CONCLUSIONS

We have studied the possibility of generating the quark masses and CKM mixing in the context of three Higgs doublet models extended by a discrete $A_4$ or $S_4$ symmetry. Assuming that the Higgs fields are in the triplet (faithful) representation of the discrete group, we have shown that none of the possible vev alignments that corresponds to a global minimum of the scalar potential leads to phenomenologically viable mass matrices for the three generations of quarks of the Standard Model and, simultaneously, to a non-vanishing CKM phase. Clearly, these conclusions can be evaded by extending the field content with extra scalars and/or fermions.

Our analysis can be applied straightforwardly to the leptonic sector of the theory, if neutrinos are Dirac particles. In that case, one massless neutrino or lack of leptonic CP violation would not contradict current experiments.

ACKNOWLEDGMENTS

The work of R.G.F. and J.P.S. was partially supported by Portuguese national funds through FCT - Fundação para a Ciência e a Tecnologia, under the projects PEst-OE/FIS/UI0777/2011 and CERN/FP/116328/2010, and by the EU RTN project Marie Curie PITN-GA-2009-237920. The work of H.S. is funded by the European FEDER, Spanish MINECO, under the grant FPA2011-23596.
[1] G. Altarelli and F. Feruglio, Rev. Mod. Phys. 82, 2701 (2010).
[2] H. Ishimori, T. Kobayashi, H. Okada, Y. Shimizu, H. Ohki and M. Tanimoto, Prog. Theor. Phys. Suppl. 183, 1 (2010).
[3] W. Grimus and P. O. Ludl, J. Phys. A 45, 233001 (2012).
[4] S. F. King and C. Luhn. arXiv:1301.1340 [hep-ph].
[5] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002).
[6] K. Abe et al. [T2K Collaboration], Phys. Rev. Lett. 107, 041801 (2011).
[7] Y. Abe et al. [DOUBLE-CHOOZ Collaboration], Phys. Rev. Lett. 108, 131801 (2012).
[8] F. P. An et al. [DAYA-BAY Collaboration], Phys. Rev. Lett. 108, 171803 (2012).
[9] J. K. Ahn et al. [RENO Collaboration], Phys. Rev. Lett. 108, 191802 (2012).
[10] D. Wyler, Phys. Rev. D 19, 3369 (1979).
[11] G. C. Branco, H. P. Nilles and V. Rittenberg, Phys. Rev. D 21, 3417 (1980).
[12] A. Degee, I. P. Ivanov and V. Keus, arXiv:1211.4980 [hep-ph].
[13] I. P. Ivanov, private communication.
[14] Implications of minima (including local) in $A_4$ models have been studied in R. de Adelhart Toorop, F. Bazzocchi, L. Merlo and A. Paris, JHEP 1103, 035 (2011) [Erratum-ibid. 1301, 098 (2013)]; ibid, JHEP 1103, 040 (2011).
[15] P. M. Ferreira and J. P. Silva, Phys. Rev. D 83, 065026 (2011).
[16] I. P. Ivanov and E. Vdovin, arXiv:1210.6553 [hep-ph].
[17] G. C. Branco, R. González Felipe, M. N. Rebelo, and H. Serêdio, Phys. Rev. D 79, 093008 (2009).
[18] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985); I. Dunietz, O. W. Greenberg, and D.-D. Wu, Phys. Rev. Lett. 55, 2935 (1985); F. J. Botella and L. -L. Chau, Phys. Lett. B 181, 123 (1986); J. Bernabéu, G. C. Branco, and M. Gronau, Phys. Lett. B 169, 243 (1986).
[19] I. d. M. Varzielas and L. Lavoura, arXiv:1212.3247 [hep-ph].