On the slope of the moduli space of genus 15 and 16 curves

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Abstract

We revisit the work of Chang and Ran on bounding the slopes of $\mathcal{M}_{15}$ and $\mathcal{M}_{16}$, correct one of the formulas used at the conclusion of the argument, and recompute the lower bounds on the slopes, yielding $s(\mathcal{M}_{15}) > 6.5$ but not for $\mathcal{M}_{16}$. Our contribution only involves plugging in formulas.

1 Introduction

The slope of the moduli space of curves is an important invariant, giving consequence for the birational geometry of $\mathcal{M}_g$ [CFM13]. In particular, Chang and Ran used 1-parameter families of space curves constructed using monads to show the slopes of $\mathcal{M}_{15}$ and $\mathcal{M}_{16}$ exceed 6.5 [CR86, CR91]. The main result of [BDPP13], together with the slope bounds of Chang and Ran, would imply $\mathcal{M}_{15}$ and $\mathcal{M}_{16}$ are uniruled (see also [Far09a, Theorem 2.7]).

Our goal is to correct the computation at the conclusion of the argument in [CR86, Section 3] of the slope of the family of space curves $Y \subset \mathbb{P}^1 \times \mathbb{P}^3$ given as the degeneracy locus of a vector bundle. We find $s(\mathcal{M}_{15}) > 6.53$ instead of 6.66 as originally claimed. Therefore, the qualitative result that $\mathcal{M}_{15}$ is uniruled remains unchanged. In fact, it has since been shown that $\mathcal{M}_{15}$ is rationally connected [BV05]. However, the recomputed lower bound for $s(\mathcal{M}_{16})$ using [CR91] is only about 6 instead of 6.567 as originally claimed, so the question of the uniruledness of $\mathcal{M}_{16}$ is still open.

1.1 Acknowledgements

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2 Computation

We begin with a correction of the formula in [CR86, page 219]. It is a special case of the Chern numbers of degeneracy loci computed in [HT84].

Theorem 2.1 (corrected form of [CR86, page 219]). Let $M$ be a smooth variety of dimension 4 and $f : A \to B$ be a homomorphism between vector bundles of rank $a$ and $a + 1$, respectively. Suppose the locus $Z \subset M$, where $f$ has rank $< a$, is a locally complete intersection surface. Then, the virtual Chern numbers of $Z$ are given by

$$c_1(Z)^2 = (c_1(M) - c_1)^2 c_2 - 2(c_1(M) - c_1)c_3 + c_4$$

$$c_2(Z) = (c_2(M) - c_1(M)c_1 + c_2(A) - c_2(B) + c_1(B)^2 - c_1(A)c_1(B))c_2 +$$

$$+ (-c_1(M) + 2c_1)c_3 + c_4.$$

where $c_i := c_i(B-A)$.

1 The main difference between the formula in Theorem 2.1 and the original is that each instance of $c_1c_2$ and $c_1^2c_2$ is replaced by $c_3$ and $c_4$ respectively. Note, however, the sign of $c_1(M)$ in $(-c_1(M) + 2c_1)c_1$ is also flipped in the corrected version.

2 There are two relevant sign errors in [HT84]. First, [HT84, 1.4] is valid if you replace $x_1, \ldots, x_m$ with the dual Chern roots, as the proof in Section 2 immediately defines the $x_i$ to be the dual Chern roots (this typo is also mentioned in [Far09b, page 833]). Also, the sign in front of $c_1(M)$ in $(-c_1(M) + 2c_1)c_1$ is flipped in [HT84, page 474], which I suspect is why the sign is also flipped in [CR86, page 219].
Theorem 2.2. The slope of $\mathcal{M}_{15}$ is at least $\frac{65}{14} \approx 4.65$, so in particular $\mathcal{M}_{15}$ has Kodaira dimension $-\infty$.

Proof. Applying Theorem 2.1 to the case $M = \mathbb{P}^1 \times \mathbb{P}^3$, $c(A) = c(\mathcal{O}^4)$ and $B = E(2)$, where $E$ is given as

$$0 \rightarrow E \rightarrow \mathcal{O}(1,0)^8 \oplus \mathcal{O}(0,-1) \rightarrow \mathcal{O}(1,1)^4 \rightarrow 0$$

as in [CR86] Example 1.6, we find

$$c_1(Z)^2 = 216 \quad c_2(Z) = 336$$

$$\kappa = 328 \quad \delta = 392 \quad \lambda = 60,$$

giving the claimed lower bound to the slope of $\mathcal{M}_{15}$. □

However, this is not sufficient for the application to $\mathcal{M}_{16}$ given in [CR91]. Instead, one gets

Theorem 2.3. The slope of $\mathcal{M}_{16}$ is at least $\frac{1472}{245} \approx 6.008$

Proof. We will refer the reader to [CR91] for the details of the proof. We will just check one computation here. This is just Type $\beta$ family in [CR91] page 271, but there are typos in the formulas. Specifically, the second and fourth line of [CR91] (1.3)] should read

$$\beta(F, A_1, A_2) = m_1 m_2 F \cdot \delta_j \quad \text{for } j \neq 0, 1, i$$

$$\beta(F, A_1, A_2) = m_1 m_2 F \cdot \delta_0 + \sum_{\ell=1}^2 \left( m_2 - \ell(m_4(2h-2) - 2g(A_\ell) - A_\ell \cdot A_\ell) - A_1 \cdot A_2 \right).$$

In spite of this, our recomputed correction term $\beta(F, A_1, A_2) = \frac{\Delta_{-\ell}}{\epsilon_{m_1+1}} \delta - m_1 m_2 F \cdot \delta$ specialized to our case agrees with the correction term $-2(14 \cdot 220 + 16) + 16$ found in the formula for $F_{0,16} \cdot \delta$ on [CR91] page 273.

From the proof of Theorem 2.2, Chang and Ran construct a surface $\mathcal{Y} \subset \mathbb{P}^1 \times \mathbb{P}^3$, viewed as a family of curves over $\mathbb{P}^1$. Each member of $\mathcal{Y} \rightarrow \mathbb{P}^1$ is a degree 14 space curve of genus 15, and the image of $\mathcal{Y} \rightarrow \mathbb{P}^3$ is a degree 16 surface [CR91] page 273.

By pulling back generic hyperplanes in $\mathbb{P}^3$, we get two smooth multisections $A_1$ and $A_2$ of $\mathcal{Y} \rightarrow \mathbb{P}^1$ of degree 14 meeting transversely with $A_1^2 = A_2^2 = A_1 \cdot A_2 = 16$. We can also assume $A_1$ and $A_2$ do not meet at points where either multisection is tangent to the fiber. By base changing under $B := A_1 \times_{\mathbb{P}^1} A_2 \rightarrow \mathbb{P}^1$, we get a family $\pi^* \mathcal{Y} \rightarrow B$ with two sections $\sigma_1, \sigma_2$ mapping isomorphically onto $A_1, A_2 \subset \mathcal{Y}$. Blowing up $\pi^* \mathcal{Y}$ at the (reduced) points of intersection of $\sigma_1$ with $\sigma_2$, we get nonintersecting sections $\tilde{\sigma}_1, \tilde{\sigma}_2$ of a family $\mathcal{Y} \rightarrow B$.

Now, we want to determine the slope of the map $\phi_B : B \rightarrow \mathcal{M}_{16}$ given by $\mathcal{Y}$ in terms of the map $\phi_{\mathbb{P}^1} : \mathbb{P}^1 \rightarrow \mathcal{M}_{15}$ given by $\mathcal{Y}$. We see

$$\phi_B^* \lambda = (14)^2 \phi_{\mathbb{P}^1}^* \lambda \quad \phi_B^* \delta_1 = 16 \quad \phi_B^* \delta_i = (14)^2 \phi_{\mathbb{P}^1}^* \delta_i = 0 \quad \text{for } i > 1,$$

so the only intersection left is $\phi_B^* \delta_0$. This differs from $(14)^2 \phi_{\mathbb{P}^1}^* \delta_0$ by the sum of the chern numbers of the normal bundles of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ [HM98] page 147. To do this, we see that

$$(\tilde{\sigma}_1)^2 = -\tilde{\sigma}_1^* \omega_{\mathcal{Y}} / B = -\sigma_1^* \omega_{\mathcal{Y}} / B - A_1 \cdot A_2 = 14(-A_1 \cdot \omega_{\mathcal{Y}} / \mathbb{P}^1) - 16,$$

where $A_1 \cdot \omega_{\mathcal{Y}} / \mathbb{P}^1$ can be computed using adjunction on $\mathcal{Y}$ to be

$$A_1 \cdot \omega_{\mathcal{Y}} = (14)(c_1(\omega_{\mathbb{P}^1})) = (2g(A_1) - 2) - A_1^2 - (14)(-2).$$

Therefore, $\phi_B^* \lambda = 60 \cdot 14^2$ and $(\tilde{\sigma}_1)^2 = 14(-15 \cdot 14 - 2) + (2 \cdot 0 - 2) \cdot 14 + 16 - 16 = -3096$, and

$$\frac{\phi_B^* \delta}{\phi_B^* \lambda} = \frac{\phi_B^* \delta_0 + \phi_B^* \delta_1}{\phi_B^* \lambda} = \frac{(14^2 \phi_{\mathbb{P}^1}^* \delta_0 + (\tilde{\sigma}_1)^2 + (\tilde{\sigma}_2)^2) + 16}{60 \cdot 14^2} = \frac{1472}{245}. \square
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