WHEN EVERY PRINCIPAL IDEAL IS FLAT

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ABSTRACT. This paper deals with well-known notion of $PF$-rings, that is, rings in which principal ideals are flat. We give a new characterization of $PF$-rings. Also, we provide a necessary and sufficient condition for $R \triangleright I$ (resp., $R/I$ when $R$ is a Dedekind domain or $I$ is a primary ideal) to be $PF$-ring. The article includes a brief discussion of the scope and precision of our results.

1. Introduction

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary. We start by recalling some definitions.

A ring $R$ is called a $PF$-ring if principal ideals of $R$ are flat. Recall that $R$ is a $PF$-ring if and only if $R_Q$ is a domain for every prime (resp., maximal) ideal $Q$ of $R$. For example, any domain and any semihereditary ring is a $PF$-ring (since a localization of a semihereditary ring by a prime (resp., maximal) ideal is a valuation domain). Note that a $PF$-ring is reduced by [12, Theorem 4.2.2, p. 114]. See for instance [12, 13].

An $R$-module $M$ is called $P$-flat if, for any $(s,x) \in R \times M$ such that $sx = 0$, then $x \in (0 : s)M$. If $M$ is flat, then $M$ is naturally $P$-flat. When $R$ is a domain, $M$ is $P$-flat if and only if it is torsion-free. When $R$ is an arithmetical ring, then any $P$-flat module is flat (by [5, p. 236]). Also, every $P$-flat cyclic module is flat (by [5, Proposition 1(2)]). See for instance [5, 12].

The amalgamated duplication of a ring $R$ along an ideal $I$ is a ring that is defined as the following subring with unit element $(1, 1)$ of $R \times R$:

$$R \bowtie I = \{(r, r + i)/r \in R, i \in I\}.$$ This construction has been studied, in the general case, and from the different point of view of pullbacks, by D’Anna and Fontana [8]. Also, in [7], they have considered the case of the amalgamated duplication of a ring, in not necessarily Noetherian setting, along a multiplicative canonical ideal in the sense of [14]. In [6] D’Anna has studied some properties of $R \bowtie I$, in
order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and has applied this construction to curve singularities. On the other hand, Maimani and Yassemi, in [16], have studied the diameter and girth of the zero-divisor of the ring $R \bowtie I$. Some references are [7, 8, 9, 10, 16].

Let $A$ and $B$ be rings and let $\varphi : A \to B$ be a ring homomorphism making $B$ an $A$-module. We say that $A$ is a module retract of $B$ if there exists a ring homomorphism $\psi : B \to A$ such that $\psi \circ \varphi = \text{id}_A$. $\psi$ is called retraction of $\varphi$. See for instance [12].

Our first main result in this paper is Theorem 2 which gives us a new characterization of $PF$-rings. Also, we provide a necessary and sufficient condition for $R \bowtie I$ (resp., $R/I$ when $R$ is a Dedekind domain or $I$ is a primary ideal) to be $PF$-ring. Our results generate new and original examples which enrich the current literature with new families of $PF$-rings with zero-divisors.

2. Main Results

Recall that an $R$-module $M$ is called $P$-flat if, for any $(s, x) \in R \times M$ such that $sx = 0$, then $x \in (0 : s)M$. Now, we give a new characterization for a class of $PF$-rings, which is the first main result of this paper.

Theorem 2.1. Let $R$ be a commutative ring. Then the following conditions are equivalent:

1. Every ideal of $R$ is $P$-flat.
2. Every principal ideal of $R$ is $P$-flat.
3. $R$ is a $PF$-ring, that is every principal ideal of $R$ is flat.
4. For any elements $(s, x) \in R^2$ such that $sx = 0$, there exists $\alpha \in (0 : s)$ such that $x = \alpha x$.

Proof. (1) $\implies$ (2) Clear.
(2) $\implies$ (3) Let $Ra$ be a principal ideal of $R$ generated by $a$. Our aim is to show that $Ra$ is flat.
Let $J$ be an ideal of $R$. We must show that $u : Ra \otimes J \to Ra \otimes R$, where $u(a \otimes x) = ax$, is injective. Let $a \in R$ and $x \in J$ such that $ax = 0$. Hence, there exists $\beta \in (0 : x)$ and $\lambda \in R$ such that $a = \beta \lambda a$ (since $Ra$ is $P$-flat).
Therefore, $a \otimes x = \beta \lambda a \otimes x = \lambda a \otimes \beta x = 0$, as desired.
(3) $\implies$ (4) Let $(s, x)$ be an element of $R^2$ such that $sx = 0$. Our aim is to show that there exists $\beta \in (0 : s)$ such that $x = \beta x$. The principal ideal generated by $x$ is $P$-flat (since it is flat), so there exists $\alpha \in (0 : s)$ and $r \in R$ such that $x = \alpha rx = \beta x$ with $\beta = \alpha r \in (0 : s)$.
(4) $\implies$ (1) Let $I$ be an ideal of $R$. Let $(s, x) \in R \times I$ such that $sx = 0$. 
Hence, there exists $\alpha \in (0 : s)$ such that $x = \alpha x$ and so $x \in (0 : s)I$. Therefore, $I$ is P-flat, as desired. \hfill \square

**Corollary 2.2.** Let $R$ be a ring. The following conditions are equivalent:

1. Every ideal of $R$ is P-flat.
2. Every ideal of $R_Q$ is P-flat for every prime ideal $Q$ of $R$.
3. Every ideal of $R_m$ is P-flat for every maximal ideal $m$ of $R$.
4. $R_Q$ is a domain for every prime ideal $Q$ of $R$.
5. $R_m$ is a domain for every maximal ideal $m$ of $R$.

**Proof.** By Theorem 2.1 and [12, Theorem 4.2.2]. \hfill \square

Recall that a ring $R$ is called an arithmetical ring if the lattice formed by its ideals is distributive. If $\text{wgldim}(R) \leq 1$, then $R$ is an arithmetical ring. See for instance [2, 3].

Now, we add a condition with arithmetical in order to have equivalence between arithmetical and $\text{wgldim}(R) \leq 1$.

**Proposition 2.3.** Let $R$ be a ring. Then the following conditions are equivalent:

1. $\text{wgldim}(R) \leq 1$.
2. $R$ is arithmetical and a PF-ring.
3. $R$ is arithmetical and every principal ideal of $R$ is flat.
4. $R$ is arithmetical and every principal ideal of $R$ is P-flat.
5. $R$ is arithmetical and every ideal of $R$ is P-flat.

**Proof.** $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 5)$. By Theorem 2.1. $5) \Rightarrow 1)$. Assume that the ring $R$ is arithmetical and every ideal of $R$ is P-flat. Our aim is to show that $\text{wgldim}(R) \leq 1$. Let $I$ be a finitely generated ideal of $R$. Hence, $I$ is P-flat and so $I$ is flat (since $R$ is arithmetical by [5, p. 236]) and this completes the proof. \hfill \square

Now we show that the localization of a PF-ring is always a PF-ring.

**Proposition 2.4.** Let $R$ be a PF-ring and let $S$ be a multiplicative subset of $R$. Then $S^{-1}(R)$ is a PF-ring.

**Proof.** Assume that $R$ is a PF-ring and let $J$ be a principal ideal of $S^{-1}(R)$. We claim that $J$ is flat. Indeed, since $J$ is a principal ideal of $S^{-1}R$, then there exists an element $\frac{a}{b}$ of $J$ such that $J = S^{-1}(R)\frac{a}{b}$. Set $I = Ra$. Hence, $I$ is flat since $R$ is a PF-ring and so $J (= S^{-1}(I))$ is a flat ideal of $S^{-1}R$. It follows that $S^{-1}(R)$ is a PF-ring. \hfill \square
Now, we study the transfer of $PF$-ring property to the direct product.

**Proposition 2.5.** Let $(R_i)_{i \in I}$ be a family of commutative rings. Then $R = \prod_{i \in I} R_i$ is a $PF$-ring if and only if $R_i$ is a $PF$-ring for all $i \in I$.

**Proof.** Assume that $R_i$ is a $PF$-ring for each $i \in I$ and set $R = \prod_{i \in I} R_i$. Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be two elements of $R$ such that $xy = 0$. Then, for every $i \in I$, there exists $\alpha_i \in (0 : x_i)$ such that $y_i = \alpha_i y_i$ (since $R_i$ is a $PF$-ring). Hence, $(y_i)_{i \in I} = (\alpha_i)_{i \in I}(y_i)_{i \in I}$ and $(\alpha_i)_{i \in I}(x_i)_{i \in I} = (\alpha_i x_i)_{i \in I} = 0$. Therefore, $R$ is a $PF$-ring.

Conversely, assume that $R = \prod_{i \in I} R_i$ is a $PF$-ring and we claim that $R_i$ is a $PF$-ring for every $i \in I$. Indeed, let $i \in I$ and let $x_i$, $y_i$ be two elements of $R_i$ such that $x_i y_i = 0$. Consider $x = (a_j)_{j \in I}$, with $\begin{cases} a_i = x_i, \\ a_j = 0 \text{ for } j \neq i. \end{cases}$ and $y = (b_j)_{j \in I}$, with $\begin{cases} b_i = y_i, \\ b_j = 0 \text{ for } i \neq j. \end{cases}$ Since $R$ is a $PF$-ring, then there exists $\alpha \in (0 : x)$ such that $y = \alpha y$ (that is, for all $j \in I$, $b_j = \alpha_j b_j$ and $\alpha_j a_j = 0$). Hence, $y_i = \alpha_i y_i$ with $\alpha_i \in (0 : x_i)$. Therefore, $R_i$ is a $PF$-ring for all $i \in I$ and this completes the proof. $\square$

Next we study the transfer of $PF$-ring property to homomorphic image. First, the following example shows that the homomorphic image of a $PF$-ring is not always a $PF$-ring.

**Example 2.6.** Let $A$ be a domain and let $R = A[X]$. Then:
1. $R$ is a $PF$-ring since it is a domain.
2. $R/(X^n)$ (for $n \geq 2$) is not a $PF$-ring since $X^n = 0$ and $X \neq 0$.

Recall that if $R$ is a Dedekind domain and $I$ is a nonzero ideal of $R$, then $I = P_1^{\alpha_1}...P_n^{\alpha_n}$ for some distinct prime ideals $P_1,...,P_n$ uniquely determined by $I$ and some positive integers $\alpha_1,...,\alpha_n$ uniquely determined by $I$ (by [[11], Theorem 3.14]]).

Now, when $R$ is a Dedekind domain or $I$ is a primary ideal, we give a characterization of $R$ and $I$ such that $R/I$ is a $PF$-ring.

**Theorem 2.7.** Let $R$ be a ring and let $I$ be an ideal of $R$. Then:
1. Assume that $R$ is a Dedekind domain and $I = P_1^{\alpha_1}...P_n^{\alpha_n}$ a nonzero ideal of $R$, where $P_1,...,P_n$ are the prime ideals defined by $I$. Then $R/I$ is a $PF$-ring if and only if $\alpha_i = 1$ for all $i \in \{1,...,n\}$. 
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(2) $I$ is a primary ideal of $R$ and $R/I$ is a PF-ring if and only if $I$ is a prime ideal of $R$.

**Proof.** 1) Let $R$ be a Dedekind domain and let $I = P_1^\alpha_1 \cdots P_n^\alpha_n$ for $P_1, \ldots, P_n$ be a nonzero prime ideals of $R$, then $R/I \cong \prod_{i=1}^n (R/P_i^\alpha_i)$.
Assume that $\alpha_i = 1$ for all $1 \leq i \leq n$. Hence, $R/P_i$ is a PF-ring since $R/P_i$ is an integral domain, and so $R/I \cong \prod_{i=1}^n (R/P_i^\alpha_i)$ is a PF-ring by Proposition 2.5.

Conversely, assume that $R/I = \prod_{i=1}^n (R/P_i^\alpha_i)$ is a PF-ring. Let $i \in \{1, \ldots, n\}$. Then $R/P_i^\alpha_i$ is a PF-ring by Proposition 2.5. Hence, $R/P_i^\alpha_i$ is reduced and so the intersection of all prime ideals $Q$ of $R/P_i^\alpha_i$ is zero (i.e $\cap_{Q \in \text{spec}(R/P_i^\alpha_i)} Q = \{0\}$) by [1, Proposition 1.8]. On the other hand for every prime ideals $Q$ of $R/P_i^\alpha_i$, there exist a prime ideal $Q'$ of $R$ such that $P_i^\alpha_i \subset Q'$ and $Q = Q'/P_i^\alpha_i$, then, $P_i/P_i^\alpha_i \subset Q$. It follows that $\{0\} = \cap_{Q \in \text{spec}(R/P_i^\alpha_i)} Q = R/P_i^\alpha_i$ and so $P_i = P_i^\alpha_i$, since $R$ is Dedekind domain then, $\alpha_i = 1$.

2) It’s obvious that if $I$ is a prime ideal, then $R/I$ is a PF-ring and $I$ is a primary ideal. Conversely, assume that $I$ is a primary ideal and $R/I$ is a PF-ring. Our aim is to show that $I$ is a prime ideal of $R$. Let $x,y \in R$ such that $xy \in I$. We claim that $x \in I$ or $y \in I$. Without loss of generality, we may assume that $x \notin I$. Since $xy \in I$, then there exists an integer $n > 0$ such that $y^n \in I$ (since $I$ is a primary ideal). Hence, $y^n = 0$ and so $\overline{y} = 0$ since $R/I$ is a PF-ring; that is $y \in I$. Therefore, $x \in I$ or $y \in I$ and so $I$ is a prime ideal of $R$, as desired.

Now, we are able to give examples of PF-rings and non-PF-rings.

**Example 2.8.** (1) $\mathbb{Z}/4\mathbb{Z}$ is not a PF-ring by Theorem 2.8(1).
(2) $\mathbb{Z}/30\mathbb{Z}$ is a PF-ring by Theorem 2.8(1).

Now, we study the transfer of a PF-property to amalgamated duplication of a ring $R$ along an ideal $I$.

**Theorem 2.9.** Let $R$ be a ring, and let $I$ be an ideal of $R$. Then, $R \bowtie I$ is a PF-ring if and only if $R$ is a PF and $I$ is pure.

We need the following lemma before proving this Theorem.
Lemma 2.10. Let $R$ and $S$ be a rings and let $\varphi : R \rightarrow S$ be a ring homomorphism making $R$ a module retract of $S$. If $S$ is a PF-ring, then so is $R$.

Proof. Let $\varphi : R \rightarrow S$ be a ring homomorphism and let $\psi : S \rightarrow R$ be a ring homomorphism such that $\psi \circ \varphi = id_R$. Let $(x,y) \in R^2$ such that $xy = 0$. Then $\varphi(x)\varphi(y) = \varphi(xy) = 0$. Hence, there exists an element $\alpha \in S$ such that $\alpha \varphi(x) = 0$ and $\varphi(y) = \alpha \varphi(y)$ (since $S$ is a PF-ring) and so $y = \psi(\varphi(y)) = \psi(\alpha)\psi(y)$ and $\psi(\alpha)x = \psi(\alpha \varphi(x)) = \psi(0) = 0$, as desired.

Proof. of Theorem 2.9.
Assume that $R \bowtie I$ is a PF-ring and we must to show that $R$ is a PF-ring and $I$ is a pure ideal of $R$. We can easily show that $R$ is a module retract of $R \bowtie I$ where the retraction map $\varphi$ is defined by $\varphi(r, r + i) = r$ and so $R$ is a PF-ring by Lemma 2.10.

We claim that $I_m \in \{0, R_m\}$ for every maximal ideal $m$ of $R$. Let $m$ be an arbitrary maximal ideal of $R$, we have: $I \subseteq m$ or $I \subseteq m^c$. If $I \subseteq m$ then, $I_m = R_m$. If $I \subseteq m^c$. Deny. $I_m \notin \{0, R_m\}$ and so $(R \bowtie I)_M = R_m \bowtie I_m$, where $M$ a maximal ideal of $R \bowtie I$ such that $M \cap R = m$. Since $R_m$ is a domain, then $R_m \bowtie I_m$ is reduced and $O_1(= \{0\} \times I_m)$ and $O_2(= I_m \times \{0\})$ are the only minimal prime ideals of $(R \bowtie I)_M$ by [8] Proposition 2.1]; hence it is not a PF-ring by [12] Theorem 4.2.2 (since $(R \bowtie I)_M$ is local), a desired contradiction. Therefore, $I_m \in \{0, R_m\}$ for every maximal ideal $m$ of $R$.

Conversely, assume that $R$ is a PF-ring and $I$ is a pure ideal of $R$, i.e. $I_m \in \{0, R_m\}$ for every maximal ideal $m$ of $R$. Our aim is to prove that $R \bowtie I$ is a PF-ring. Using Corollary 2.2, we need to prove that $(R \bowtie I)_M$ is a PF-ring whenever $M$ is a maximal ideal of $R \bowtie I$. Let $M$ be an arbitrary maximal ideal of $R \bowtie I$ and set $m = M \cap R$. Then, necessarily $M \in \{M_1, M_2\}$, where $M_1 = \{(r, r + i)/r \in m, i \in I\}$ and $M_2 = \{(r + i, r)/r \in m, r \in I\}$, by [7] Theorem 3.5. On the other hand, $I_m \in \{0, R_m\}$. Then, testing all cases of [8] Proposition 7, we have two cases:

(a) $(R \bowtie I)_M \cong R_m$ if $I_m = 0$ or $I \subseteq m$.
(b) $(R \bowtie I)_M \cong R_m \times R_m$ if $I_m = R_m$ and $I \subseteq m$.

Since $R_m$ is a PF-ring (by Corollary 2.2), then so is $R_m \times R_m$ by Proposition 2.5 and hence $(R \bowtie I)_M$ is a PF-ring.

Corollary 2.11. Let $R$ be a domain and let $I$ be a proper ideal of $R$. Then $R \bowtie I$ is never a PF-ring.

Corollary 2.12. Let $(R, m)$ be a local ring and let $I$ be a proper ideal of $R$. Then $R \bowtie I$ is never a PF-ring.
Now we are able to construct a class of \(PF\)-rings.

**Example 2.13.** Let \(R\) be a \(PF\)-ring and let \(I = Re\), where \(e\) is an idempotent element of \(R\). Then \(R \bowtie I\) is a \(PF\)-ring by Theorem 2.9.

The following example shows that a subring of \(PF\)-ring is not always a \(PF\)-ring. For any ring \(R\), we denote by \(T(R)\) the total ring of quotients of \(R\).

**Example 2.14.** Let \(R\) be an integral domain, \(I\) a proper ideal of \(R\) and let \(S = R \bowtie I\). Then:

1. \(S(= R \bowtie I)\) is not a \(PF\)-ring by Corollary 2.11.
2. \(R \bowtie I \subseteq R \times R\) and \(R \times R\) is a \(PF\)-ring by Proposition 2.5 (since \(R\) is a \(PF\)-ring).
3. \(T(S) = T(R \times R) = K \times K\), where \(K = T(R)\).

We end this paper by showing that the transfer of \(PF\)-ring property to Pullback is not always a \(PF\)-ring.

**Example 2.15.** Let \(R\) be a domain and \(I\) a proper ideal of \(R\). Then:

1. The ring \(R \bowtie I\) can be obtained as a pullback of \(R\) and \(R \times R\) over \(R \times (R/I)\).
2. The ring \(R \bowtie I\) is not a \(PF\)-ring by Corollary 2.11.
3. The rings \(R\) and \(R \times R\) are \(PF\)-rings.

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