ON THE MEHLIG-WILKINSON REPRESENTATION OF METAPLECTIC OPERATORS

Maurice A. de Gosson

March 29, 2022

Abstract

We study the Weyl representation of metaplectic operators suggested by earlier work of Mehlig and Wilkinson. We give precise calculations for the associated Maslov indices; these intervene in a crucial way in the Gutzwiller formula of semiclassical mechanics.

1 Introduction

In an interesting paper [6] the physicists Mehlig and Wilkinson introduce, in connection with their study of the Gutzwiller semiclassical trace formula, a class of unitary operators \( \hat{S} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \). These operators are defined as follows: let \( S \in Sp(n) \) have no eigenvalue equal to one; to \( S \) one associates the Weyl operator

\[
\hat{R}(S) = \left( \frac{1}{2\pi} \right)^n \frac{i^{\nu}}{\sqrt{|\det(S-I)|}} \int e^{i\hat{T}(z_0)\hat{S}z_0} d^{2n}z_0
\]  

(1)

where \( \hat{T}(z_0) \) is the Weyl–Heisenberg operator and

\[
M_S = \frac{1}{2}J(S + I)(S - I)^{-1}
\]

(2)

\( I \) being the identity and \( J \) the standard symplectic matrix (see below). The index \( \nu \) is an integer related to the sign of \( \det(S - I) \), and which is not studied in the general case in [6]. Mehlig and Wilkinson moreover show that

\[
\hat{R}(SS') = \pm \hat{R}(S)\hat{R}(S')
\]

(3)

for all \( S, S' \) for which both sides are defined. They claim that these operators belong to the metaplectic group. This property is however not quite obvious; what is acceptably “obvious” is that \( \hat{R}(S) \) is a multiple by a scalar factor of modulus one of any of the two metaplectic operators \( \pm\hat{S} \) associated to \( Mp(n) \):
This is achieved using the metaplectic covariance of the Heisenberg–Weyl operators (see below). The purpose of this paper is to precise Mehlig and Wilkinson’s statement by comparing explicitly the integer \( \nu \) in (1) with the Maslov indices on the metaplectic group we have studied in a previous work [3]. This is indeed important—and not just an academic exercise—since the ultimate goal in [6] is to apply formula (1) to give a new proof of Gutzwiller’s trace formula for chaotic systems. It is well-known that the calculation of the associated “Maslov indices” is notoriously difficult: it suffices to have a look on the impressive bibliography devoted to that embarrassingly subtle topic. We will, in addition, give a semiclassical interpretation of \( \hat{R}(S) \), expressed in terms of the phase space wavefunctions we introduced in [2, 4].

**Remark 1** An alternative approach to the results of this paper would be to use Howe’s beautiful “oscillator group” method [5] (see [1] for a review); this would however in our case lead to unnecessary technical complications.

**Notations**

We denote by \( \sigma \) the canonical symplectic form on \( \mathbb{R}_z^{2n} = \mathbb{R}_x^n \times \mathbb{R}_p^n \)

\[
\sigma(z, z') = \langle p, x' \rangle - \langle p', x \rangle \quad \text{if} \quad z = (x, p), \ z' = (x', p')
\]

that is

\[
\sigma(z, z') = \langle Jz, z' \rangle, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.
\]

The real symplectic group \( Sp(n) \) consists of all linear automorphisms \( S : \mathbb{R}_z^{2n} \rightarrow \mathbb{R}_z^{2n} \) such that \( \sigma(Sz, Sz') = \sigma(z, z') \) for all \( z, z' \). It is a connected Lie group. We denote by \( \ell_X \) and \( \ell_P \) the Lagrangian planes \( \mathbb{R}_x^n \times 0 \) and \( 0 \times \mathbb{R}_p^n \), respectively. \( \mathcal{S}(\mathbb{R}^n) \) is the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^n \), and its dual \( \mathcal{S}'(\mathbb{R}^n) \) the space of tempered distributions.

**2 Prerequisites**

**2.1 Standard theory of \( Mp(n) \): Review**

The material of this first subsection is quite classical; see for instance [1, 3] and the references therein.

Every \( S \in Mp(n) \) is the product of two “quadratic Fourier transforms”, which are operators \( S_{W,m} \) defined on \( \mathcal{S}(X) \) by

\[
S_{W,m}f(x) = \left( \frac{1}{2\pi i} \right)^n i^{m} \sqrt{|\det L|} \int e^{iW(x,x')} f(x')d^n x'
\]

where \( W \) is a quadratic form in the variables \( x, x' \) of the type

\[
W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle
\]
with \( P = P^T, \) \( Q = Q^T, \) \( \det L \neq 0. \) The integer \( m \) appearing in (4) corresponds to a choice of \( \arg \det L: \)

\[ m\pi \equiv \arg \det L \mod 2\pi \]

and to every \( W \) there thus corresponds two different choices of \( m \) modulo 4: if \( m \) is one choice, then \( m + 2 \) is the other (this of course reflects the fact that \( Mp(n) \) is a two-fold covering of \( Sp(n) \)). The projection \( \pi : Mp(n) \to Sp(n) \) is entirely specified by the datum of each \( \pi(S_{W,m}) \), and we have \( \pi(S_{W,m}) = S_{W,n} \)
where

\[ (x,p) = S_{W}(x',p') \iff p = \partial_x W(x,x') \text{ and } p' = -\partial_{x'} W(x,x'). \]

In particular,

\[ S_{W} = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - LT & PL^{-1} \end{pmatrix} \quad (6) \]

is the free symplectic automorphism generated by the quadratic form \( W \); observe that \( S_{W} \ell_p \cap \ell_p = 0 \) for every \( W \). The inverse \( S_{W,m}^{-1} = \tilde{S}_{W,m}^{*} \) of \( S_{W,m} \) is the operator \( S_{W,-m} \) where \( W^{*}(x,x') = -W(x,x') \) and \( m' = n - m, \) mod 4. Note that if conversely \( S \) is a free symplectic matrix

\[ S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n), \text{ } \det B \neq 0 \quad (7) \]

then \( S = S_{W} \) with \( P = B^{-1}A, \) \( L = B^{-1}, \) \( Q = DB^{-1}. \)

### 2.2 Heisenberg–Weyl operators

For \( z_0 = (x_0,p_0) \) we denote by \( T(z_0) \) the translation \( z \mapsto z + z_0; \) it acts on functions by push-forward: \( T(z_0)f(z) = f(z - z_0). \) We denote by \( \hat{T}(z_0) \) the corresponding Heisenberg–Weyl operator: for \( f \in \mathcal{S}(\mathbb{R}^n) \) we have

\[ \hat{T}(z_0) = e^{i(p_0 \cdot x) - \frac{1}{2}(p_0 \cdot x_0)} f(x - x_0). \]

The operators \( \hat{T}(z_0) \) satisfy the metaplectic covariance formula:

\[ \hat{S} \hat{T}(z) = \hat{T}(Sz) \hat{S} \quad (S = \pi(\hat{S})) \quad (8) \]

for every \( \hat{S} \in Mp(n) \) and \( z. \) In fact, the metaplectic operators are the only unitary operators, up to a \( \pi \) factor in \( S^1 \) satisfying (3): *For every \( S \in Sp(n) \) there exists a unitary transformation \( \hat{U} \) in \( L^2(\mathbb{R}^n) \) satisfying (3) and \( \hat{U} \) is uniquely determined apart from a constant factor of modulus one.*

The Heisenberg–Weyl operators moreover satisfy the relations

\[ \hat{T}(z_0)\hat{T}(z_1) = e^{-i\pi(z_0,z_1)}\hat{T}(z_1)\hat{T}(z_0) \quad (9) \]

\[ \hat{T}(z_0 + z_1) = e^{-\frac{i}{2}\pi(z_0,z_1)}\hat{T}(z_0)\hat{T}(z_1) \quad (10) \]

as is easily seen from the definition of these operators.
2.3 Weyl operators

Let \( a^w \) be the Weyl operator with symbol \( a \):

\[
a^w f(x) = \left( \frac{1}{2\pi} \right)^n \int e^{(p,x-y)} a(\frac{1}{2}(x+y),p) f(y) d^n y d^n p;
\]

where \( f \in \mathcal{S}(\mathbb{R}^n) \) equivalently

\[
a^w = \int a_\sigma(z_0) \hat{T}(z_0) d^n z_0
\]

where \( a_\sigma \) is the symplectic Fourier transform \( F_\sigma a \) defined by

\[
F_\sigma a(z) = \left( \frac{1}{2\pi} \right)^n \int e^{i\sigma(z,z')} a(z') d^{2n} z'.
\]

The kernel of \( a^w \) is related to \( a \) by the formula

\[
a(x,p) = \int e^{-i(p,y)} K(x + \frac{1}{2} y, x - \frac{1}{2} y) d^n y.
\]

The Mehlig–Wilkinson operator (11) is the Weyl operator with twisted Weyl symbol

\[
a_\sigma(z) = \left( \frac{1}{2\pi} \right)^n \frac{i^\nu}{\sqrt{|\det(S-I)|}} e^{-\frac{i}{2} \langle (M^2 z_0, z_0) \rangle}.
\] (11)

2.4 Generalized Fresnel Formula

We will use the following formula, generalizing the usual Fresnel integral to complex Gaussians. Let \( M \) be a real symmetric \( n \times n \) matrix. If \( M \) is invertible then the Fourier transform of the exponential \( \exp(i \langle Mx, x \rangle / 2) \) is given by the formula

\[
\left( \frac{1}{2\pi} \right)^{n/2} \int e^{-i(p,x)} e^{\frac{i}{2} \langle Mx, x \rangle} d^n x = |\det M|^{-1/2} e^{\frac{i}{4} \langle (M^{-1}x, x) \rangle} \sgn M e^{-\frac{i}{2} \langle (M^{-1}x, x) \rangle}
\] (12)

where \( \sgn M \), the “signature” of \( M \), is the number of \( > 0 \) eigenvalues of \( M \) minus the number of \( < 0 \) eigenvalues.

For a proof see for instance [1], App. A.

3 Discussion of the Mehlig–Wilkinson Formula

The Mehlig–Wilkinson operators \( \hat{R}(S) \) are Weyl operators with twisted Weyl symbol

\[
a_\sigma(z) = \left( \frac{1}{2\pi} \right)^n \frac{i^\nu}{\sqrt{|\det(S-I)|}} e^{-\frac{i}{2} \langle (M z_0, z_0) \rangle}.
\]

We begin by giving two straightforward alternative formulations of these operators.
3.1 Equivalent formulations

We begin by remarking that the matrix \( M_S = \frac{1}{2} J(S + I)(S - I)^{-1} \) is symmetric; this immediately follows from the conditions

\[
S \in Sp(n) \iff S^T J S = J \iff SJS^T = J.
\]

Notice that (2) can be “solved” in \( S \), yielding \( S = (2M - J)^{-1}(2M + J) \).

**Proposition 2** The operator

\[
\hat{R}(S) = \left( \frac{1}{2\pi} \right)^n \frac{i^n}{\sqrt{\text{det}(S - I)}} \int e^{i\langle MSz_0, z_0 \rangle} \hat{T}(z_0) d^{2n}z_0
\]

(13)

can be written in the following alternative two forms:

\[
\hat{R}(S) = \left( \frac{1}{2\pi} \right)^n \frac{i^n}{\sqrt{\text{det}(S - I)}} \int e^{-i\sigma(z_0, z_0)} \hat{T}((S - I)z_0) d^{2n}z_0
\]

(14)

\[
\hat{R}(S) = \left( \frac{1}{2\pi} \right)^n i^n \sqrt{\text{det}(S - I)} \int \hat{T}(S_{z_0}) \hat{T}(-z_0) d^{2n}z_0
\]

(15)

for \( \text{det}(S - I) \neq 0 \).

**Proof.** We have

\[
\frac{1}{2} J(S + I)(S - I)^{-1} = \frac{1}{2} J + J(S - I)^{-1}
\]

hence, in view of the antisymmetry of \( J \),

\[
\langle MSz_0, z_0 \rangle = \langle J(S - I)^{-1}z_0, z_0 \rangle = \sigma((S - I)^{-1}z_0, z_0)
\]

Performing the change of variables \( z_0 \mapsto (S - I)^{-1}z_0 \) we can rewrite the integral in the right hand side of (13) as

\[
\int e^{i\langle MSz_0, z_0 \rangle} \hat{T}(z) d^{2n}z_0 = \int e^{i\sigma(z_0, (S - I)z_0)} \hat{T}((S - I)z_0) d^{2n}z_0
\]

\[
= \int e^{-i\sigma(z_0, z_0)} \hat{T}((S - I)z_0) d^{2n}z_0
\]

hence (14). Taking into account the relation (10) we have

\[
\hat{T}((S - I)z_0) = e^{-i\sigma(S_{z_0}, z_0)} \hat{T}(S_{z_0}) \hat{T}(-z_0)
\]

and formula (15) follows. ■

**Corollary 3** We have \( \hat{R}(S) = c_S \hat{S}_{W,m} \) where \( c \) is a complex constant with \( |c| = 1 \).

**Proof.** We begin by noting that \( \hat{R}(S) \) satisfies the metaplectic covariance relation

\[
\hat{R}(S) \hat{T}(z_0) = \hat{T}(S_{z_0}) \hat{R}(S)
\]

as immediately follows from the alternative form (15) of \( \hat{R}(S) \). On the other hand, a straightforward calculation using formula (14) shows that \( \hat{R}(S) \) is unitary, hence the claim. ■
3.2 The case $\hat{S} = \hat{S}_{W,m}$

We are going to show that the Mehlig–Wilkinson operators coincide with the metaplectic operators $\hat{S}_{W,m}$ when $S = S_W$ and we will thereafter determine the correct choice for $\nu$; we will see that it is related by a simple formula to the usual Maslov index as defined in [3].

Let us first prove the following technical result:

**Lemma 4** Let $S_W$ be a free symplectic matrix (7). We have

$$\det(S_W - I) = \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1})$$

that is, when $S$ is written in the form (6):

$$\det(S_W - I) = \det(L^{-1}) \det(P + Q - L - L^T).$$

**Proof.** We begin by noting that since $B$ is invertible we can write $S - I$ as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix}$$

hence

$$\det(S_W - I) = \det B \det(C - (D - I)B^{-1}(A - I)).$$

Since $S$ is symplectic we have $C - DB^{-1}A = -(B^T)^{-1}$ (use for instance the fact that $S^TJS = JSJ^T = J$) and hence

$$C - (D - I)B^{-1}(A - I) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1};$$

the Lemma follows. $\blacksquare$

**Proposition 5** Let $S$ be a free symplectic matrix (7) and $\hat{R}(S)$ the corresponding Mehlig–Wilkinson operator. We have $\hat{R}(S) = \hat{S}_{W,m}$ provided that $\nu$ is chosen so that

$$\nu \equiv m - \text{Inert}(P + Q - L - L^T) \mod 4$$

(\text{Inert}(P + Q - L - L^T) the number of } < 0 \text{ eigenvalues of the symmetric matrix } P + Q - L - L^T).$$

**Proof.** Recall that we have shown that $\hat{R}(S) = c_S \hat{S}_{W,m}$ where $c_S$ is a complex constant with $|c_S| = 1$. Let us determine that constant. Let $\delta \in \mathcal{S}'(\mathbb{R}^n)$ be the Dirac distribution centered at $x = 0$; setting

$$C = \left(\frac{1}{2\pi}\right)^n \frac{i^\nu}{\sqrt{\det(S_W - I)}}$$

we have, by definition of $\hat{R}(S)$,

$$\hat{R}(S)\delta(x) = C \int e^{\frac{i}{\hbar}(M_S z_0, z_0)} e^{i((p_0, x) - \frac{1}{\hbar}(p_0, x_0))} \delta(x - x_0) d^{2n}z_0$$

$$= C \int e^{\frac{i}{\hbar}(M_S (x, p_0), (x, p_0))} e^{\frac{i}{\hbar}(p, x)} \delta(x - x_0) d^{2n}z_0$$
hence, setting $x = 0$,
\[
\hat{R}(S)\delta(0) = C \int e^{\frac{i}{2} (M_S(0,p_0), (0,p_0))} \delta(-x_0) d^2 x_0
\]
that is, since $\int \delta(-x_0) d^n x_0 = 1$,
\[
\hat{R}(S)\delta(0) = \left( \frac{1}{2\pi} \right)^n e^{i \nu \sqrt{\frac{1}{\det(S - I)}}} \int e^{\frac{i}{2} (M_S(0,p_0), (0,p_0))} d^n p_0. \tag{19}
\]
Let us calculate the scalar product
\[
\langle M_S(0,p_0), (0,p_0) \rangle = \sigma((S-I)^{-1}0, 0, (0,p_0))
\]
The relation $(x, p) = (S-I)^{-1}(0, p_0)$ is equivalent to $S(x, p) = (x, p + p_0)$ that is to $p + p_0 = \partial_x W(x, x)$ and $p = -\partial_x W(x, x)$.
Using the explicit form (5) of $W$ together with Lemma 4 these relations yield
\[
x = (P + Q - L - L^T)^{-1}p_0 ; \quad p = (L - Q)(P + Q - L - L^T)^{-1}p_0
\]
and hence
\[
\langle M_S(0,p_0), (0,p_0) \rangle = -\langle (P + Q - L - L^T)^{-1}p_0, p_0 \rangle. \tag{20}
\]
Applying Fresnel’s formula (12) we get
\[
\left( \frac{1}{2\pi} \right)^n \int e^{\frac{i}{2} (M_S(0,p_0), (0,p_0))} d^n p_0 = e^{-\frac{i\nu}{4} \det(P + Q - L - L^T)} |\det(P + Q - L - L^T)|^{1/2};
\]
since
\[
\frac{1}{\sqrt{\det(S - I)}} = |\det L|^{1/2} |\det(P + Q - L - L^T)|^{1/2}
\]
in view of (17) in Lemma 4 we thus have
\[
\hat{R}(S)\delta(0) = \left( \frac{1}{2\pi} \right)^n i^{\nu \frac{1}{4} \det(P + Q - L - L^T)} |\det L|^{1/2}.
\]
Now, by definition of $\hat{S}_{W,m}$ we have
\[
\hat{S}_{W,m}\delta(0) = \left( \frac{1}{2\pi} \right)^n i^{m-n/2} |\det L|^{1/2}
\]
hence
\[
i^{\nu \frac{1}{4} \det(P + Q - L - L^T)} = i^{m-n/2}.
\]
It follows that we have
\[
\nu - \frac{1}{2} \text{sgn}(P + Q - L - L^T) \equiv m - \frac{n}{2} \mod 4
\]
which is the same thing as (18) since $P + Q - L - L^T$ has rank $n$. ■
3.3 The general case

Recall that we established in Lemma 4 the equality
\[
\det(S_W - I) = \det(L^{-1} \det(P + Q - L - L^T)).
\] (21)
valid for all free matrices \(S_W \in Sp(n)\). Also recall that every \(\hat{S} \in Mp(n)\) can be written (in infinitely many ways) as a product \(\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'}\). We are going to show that \(\hat{S}_{W,m}\) and \(\hat{S}_{W',m'}\) in addition always can be chosen such that \(\det(\hat{S}_{W,m} - I) \neq 0\) and \(\det(\hat{S}_{W',m'} - I) \neq 0\). For that purpose we need the following straightforward factorization result (see [3]):

Lemma 6 Let \(W\) be given by (5); then \(\hat{S}_{W,m} = \hat{V}_{-P} \hat{M}_{L,m} \hat{J} \hat{V}_{-Q}\) \(\) (22)
where
\[
\hat{V}_{-P} f(x) = e^{\hat{P}(P x,x)} f(x) \; ; \; \hat{M}_{L,m} f(x) = i^m \sqrt{\det L} f(Lx)
\]
and \(\hat{J}\) is the modified Fourier transform given by
\[
\hat{J} f(x) = \left( \frac{1}{2\pi i} \right)^{n/2} \int e^{-i(x,x')} f(x') d^n x'.
\]

Let us now state and prove the main result of this section:

Proposition 7 Every \(\hat{S} \in Mp(n)\) is the product of two Mehlig–Wilkinson operators; these operators thus generate Mp(n).

Proof. Let us write \(\hat{S} = \hat{S}_{W,m} \hat{S}_{W',m'}\) and apply (22) to each of the factors; this yields (with obvious notations)
\[
\hat{S} = \hat{V}_{-P} \hat{M}_{L,m} \hat{J} \hat{V}_{-(P' + Q)} \hat{M}_{L',m'} \hat{J} \hat{V}_{-Q'}.
\] (23)
We claim that \(\hat{S}_{W,m}\) and \(\hat{S}_{W',m'}\) can be chosen in such a way that \(\det(\hat{S}_{W,m} - I) \neq 0\) and \(\det(\hat{S}_{W',m'} - I) \neq 0\) that is,
\[
\det(P + Q - L - L^T) \neq 0 \quad \text{and} \quad \det(P' + Q' - L' - L'^T) \neq 0.
\]
This will prove the assertion in view of (21). We first remark that the right hand-side of (23) obviously does not change if we replace \(P'\) by \(P' + \lambda I\) and \(Q\) by \(Q - \lambda I\) where \(\lambda \in \mathbb{R}\). Choose now \(\lambda\) such that it is not an eigenvalue of \(P + Q - L - L^T\) and \(-\lambda\) is not an eigenvalue of \(P' + Q' - L' - L'^T\); then
\[
\det(P + Q - \lambda I - L - L^T) \neq 0 \quad \text{and} \quad \det(P' + \lambda I + Q' - L - L^T) \neq 0.
\]
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