The Gap between the Chromatic number and the Distinguishing Chromatic Number

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Abstract

The *Distinguishing Chromatic Number* of a graph $G$, denoted $\chi_D(G)$, was first defined in [5] as the minimum number of colors needed to properly color $G$ such that no non-trivial automorphism $\phi$ of the graph $G$ fixes each color class of $G$. In this paper, we consider certain ‘natural’ families of bipartite graphs that have reasonably large automorphism groups and we show that in all those cases, the distinguishing chromatic number is precisely 3. We also consider random Cayley graphs $\Gamma(A,S)$ defined over certain abelian groups $A$ and show that with high probability, $\chi_D(\Gamma) \leq \chi(\Gamma) + 1$.

**Keywords:** Distinguishing Chromatic Number of Levi graph, Projective planes, Random Cayley graphs.

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1 Introduction

Let $G$ be a graph and let $\text{Aut}(G)$ denote its full automorphism group. Albertson and Collins introduced the notion of the *Distinguishing number of a graph* in [2].

**Definition 1.** A labeling of vertices of a graph $G$, $h : V(G) \rightarrow \{1, \ldots, r\}$ is said to be distinguishing (or $r$-distinguishing) provided no nontrivial automorphism of the graph preserves all of the vertex labels. The distinguishing number of a graph $G$, denoted by $D(G)$, is the minimum $r$ such that $G$ has an $r$-distinguishing labeling.

Later, Collins and Trenk introduced the notion of the Distinguishing Chromatic Number in [5].
Definition 2. A labeling of vertices of a graph $G,h : V(G) \to \{1, \ldots, r\}$ is said to be proper distinguishing (or proper $r$-distinguishing) provided the labeling is proper and distinguishing. The distinguishing chromatic number of a graph $G, \chi_D(G)$, is the minimum $r$ such that $G$ has a proper $r$-distinguishing labeling.

In other words, the Distinguishing Chromatic Number is the least integer $r$ such that the vertex set can be partitioned into sets $V_1, V_2, \ldots, V_r$ such that each $V_i$ is independent in $G$, and for every $1 \neq \pi \in \text{Aut}(G)$ there exists some color class $V_i$ such that $\pi(V_i) \neq V_i$.

Clearly, the notion of the distinguishing chromatic number begins to get more interesting only if the graph admits a large group of automorphisms, in which case, it can vary substantially from the usual chromatic number. It is easy to see that for any $2 \leq k \leq l$ there exist graphs $G$ with $\chi(G) = k, \chi_D(G) = l$. Furthermore, it is an easy observation that adding disjoint copies of complete multipartite graphs tend to increase the group of automorphisms, so it is more interesting to consider ‘natural’ families of graphs that admit large automorphism groups, and consider the question of how much larger the distinguishing chromatic number can be.

As it turns out, there are some natural families of graphs admitting a large group of automorphisms, for which this question has been addressed. For instance the distinguishing chromatic number of the Kneser graphs $K(n,r)$ are determined in [4]. It turns out that $\chi_D(K(n,r)) \leq \chi(K(n,r)) + 1$ with equality for all $r \geq 2$, i.e., for $r \geq 3$ we have $\chi_D(K(n,r)) = \chi(K(n,r))$ whenever $n \geq 2r + 1$.

In this paper, we consider incidence graphs for certain regular combinatorial structures, which admit a large automorphism group so that the naturally defined incidence bipartite graphs associated with these regular structures also admit large automorphism groups. Here, we restrict our attention to Desarguesian projective planes and consider the Levi graphs of these projective planes, which are the bipartite incidence graphs corresponding to the set of points and lines of the projective plane. It is well known [9] that the theorem of Desargues is valid in a projective plane if and only if the plane can be constructed from a three dimensional vector space over a skew field, which in the finite case reduces to the three dimensional vector spaces over finite fields.

In order to describe the graphs we are interested in, we set up some notation. Let $F_q$ denote the finite field of order $q$, and let us denote the vector space $F_q^3$ over $F_q$ by $V$. Let $P$ be the set of 1-dimensional subspaces of $V$ and $L$, the set of 2-dimensional subspaces of $V$. We shall refer to the members of these sets by points and lines, respectively. The Levi graph of order $q$, denoted by $LG_q$ is a bipartite graph defined as follows: $V(LG_q) = P \cup L$, where this describes the partition of the vertex set; a point $p$ is adjacent to a line $l$ if and only if $p \in l$.

The Fundamental theorem of Projective Geometry [9] states that the full group of automorphisms of the projective plane $PG(2,F_q)$ is induced by the group of all non-singular semi-linear transformations $\text{PTL}(V)$ of $V$ onto $V$, where $V$ is the corresponding vector space of $PG(2,F_q)$. If $q = p^n$ for a prime number $p$, $\text{PTL}(V) \cong PGL(V) \times Gal(F_q)$, where $Gal(F_q)$ is the Galois group of $K := F_q$ over $k := F_p$. In particular, if $q$ is a prime, we have $\text{PTL}(V) \cong PGL(V)$. The upshot is that $LG_q$ admits a large group of automorphisms, namely, $\text{PTL}(V)$.\footnote{It follows that this group is contained in the full automorphism group. The full group is only a little larger.}
A related natural family of bipartite graphs corresponds to the case \( q = 1 \), i.e., the set case. More precisely, suppose \( n > k \) are positive integers, and consider the bipartite graphs \( G = G(L,R,E) \) where \( L := \binom{[n]}{k} \) corresponds to the set of \( k \) subsets of \([n]\), \( R := \binom{[n]}{k-1} \) corresponds to the \( k \) subsets of \([n]\), and for \( A \subseteq L, B \subseteq R \) we have \( A \) and \( B \) adjacent if and only if \( A \subset B \). We shall refer to these graphs as **Levi Graphs of order 1**. It is easy to see that \( S_n \) acts as a group of automorphisms of \( LG_1 \).

Another natural family of graphs with potentially large automorphism groups arise as Cayley graphs of groups. Let \( A \) be a finite group with cardinality \( n \). Let \( S \) be a subset of \( A \) such that \( 1 \notin S \) and suppose that \( S \) is inverse closed, i.e., let \( S = S^{-1} \) where \( S^{-1} := \{ g^{-1} : g \in S \} \). The Cayley graph of \( A \) with respect to \( S \), denoted by \( \Gamma(A,S) \) or simply \( \Gamma \), is a graph with \( V(\Gamma(A,S)) = A \) and \( E(\Gamma(A,S)) = \{(g,gh) : g \in A, h \in S\} \).

For a given group \( A \), a random Cayley graph is chosen as follows. Let \( 0 < p < 1 \). Each element \( g \in A \) of order \( 2 \) is chosen with probability \( p \) and for any other \( x \in A \), the pair \((x,x^{-1})\) is chosen with probability \( p \) and all these random choices are made independently.

The rest of the paper is organized as follows. In section 2 we consider the problem of determining \( \chi_D(LG_q) \) for all prime powers \( q \). In section 3, we consider the graphs \( LG_1 \). Our results show that

\[ \chi_D(LG_q) = 3 \text{ for all } q \geq 1, q \neq 2, 3, 4 \]

and

\[ \chi_D(LG_2) = 4, \chi_D(LG_3) \leq 5, \chi_D(LG_4) \leq 4. \]

In section 4 we consider random Cayley graphs for certain abelian groups and again, our main result is

\[ \chi_D(\Gamma) \leq \chi(\Gamma) + 1 \text{ with high probability (whp).} \]

By the phrase ‘with high probability’ we mean that the probability of said event occurs with probability at least \( 1 - (n^{-\Omega(\log n)}) \), where \( n = |V(G)| \). The final section contains some remarks and some open questions for further enquiry.

## 2 Levi graphs \( LG_q \) and their Distinguishing Chromatic Number

We first consider the case \( q > 1 \).

Firstly, we remark that the upper bound \( \chi_D(G) \leq 2\Delta - 2 \) whenever \( G \) is bipartite and \( G \neq K_{\Delta-1,\Delta}, K_{\Delta,\Delta} \), which appears in [11], gives \( \chi_D(LG_q) \leq 2q \). In particular, \( \chi_D(LG_2) \leq 4 \). It is not too hard to show that in fact for any \( q, \chi_D(LG_q) \leq 6 \). We shall however obtain more precise results in this section. To set up some notation, let \( \{e_1,e_2,e_3\} \) be the standard basis of the vector space \( V \) with \( e_1 = (1,0,0), e_2 = (0,1,0) \) and \( e_3 = (0,0,1) \). For \( g, h, k \in F_q \), a vector \( v \in V \) is denoted by \((g,h,k)\) if \( v = ge_1 + he_2 + ke_3 \). A point \( p \in \mathcal{P} \) is denoted by \((g,h,k)\) if \( p = 2ge_1 + he_2 + ke_3 \). Thus, there are \( q^2 \) points in the form \((1,h,k)\) such that \( h, k \in F_q \), \( q \) points in the form of \((0,1,k)\) such that \( k \in F_q \) and finally the point \((0,0,1)\) to account for a total of \( q^2 + q + 1 \) points in \( PG(2,F_q) \).
2.1 The Levi Graph $LG_2$

We shall firstly show that $\chi_D(LG_2) = 4$. We start with the following definition.

**Definition 3.** A coloring of the Levi graph is said to be **Monochromatic** if all the vertices in one set of the vertex partition have the same color.

**Lemma 4.** $LG_2$ does not have a proper distinguishing monochromatic 3-coloring.

**Proof.** Assume that $LG_2$ has a proper distinguishing monochromatic 3-coloring. Without loss of generality let the line set $\mathcal{L}$ be colored with a single color, say red. Call the remaining two colors blue and green, say, which are the colors assigned to the vertices in $\mathcal{P}$. We shall refer to the set of points that are assigned a particular color, say green, as the color class $Green$. By rank of a color class $C$ (denoted $r(C)$), we mean the rank of the vector subspace generated by $C$. Observe that a nontrivial linear map $T$ that fixes the color class $Green$, must necessarily also fix the color class $Blue$, so any such linear map would correspond to an automorphism that preserves each color class.

For any 2-coloring of $\mathcal{P}$ (which has 7 points), one of the two color class has fewer than four points. Without loss of generality, assume that this is the color class $Green$. Firstly, if $r(Green) \leq 2$ then consider a basis $B$ of $V$ which contains a maximal linearly independent set of points in color class $Green$. If $r(Green) = 2$, then the linear map $T$ obtained by swapping the elements of the color class $Green$ in $B$, and fixing every other basis element is a non-trivial linear transformation of $V$ which necessarily fixes the color class $Green$. If $r(Green) = 1$, then consider the map $T$ which fixes the green point of $B$ and swaps the other two (necessarily blue) is a nontrivial linear transform that fixes the color class $Green$. Finally, if $r(Green) = 3$, then let $T$ be the map that swaps two of them and fixes the third. Again, this map is a nontrivial linear map that fixes every color class. □

We now set up some notation. Denote the Points in $LG_2$ by \{e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3, e_1 + e_2 + e_3\} and denote the lines in the following way:

1. $l_1 : \langle e_1, e_2 \rangle$ the line (two dimensional subspace) spanned by $e_1$ and $e_2$.
2. $l_2 : \langle e_1, e_3 \rangle$.
3. $l_3 : \langle e_2, e_3 \rangle$.
4. $l_4 : \langle e_1, e_2 + e_3 \rangle$.
5. $l_5 : \langle e_2, e_1 + e_3 \rangle$.
6. $l_6 : \langle e_3, e_1 + e_2 \rangle$.
7. $l_7 : \langle e_1 + e_3, e_2 + e_3 \rangle$.

**Theorem 5.** $\chi_D(LG_2) = 4$.

**Proof.** By the remark at the beginning of the section, we have $\chi_D(LG_2) \leq 4$, so it suffices to show $\chi_D(LG_2) > 3$. 

4
We first claim that if $LG_2$ has a proper distinguishing 3-coloring, then three linearly independent points (points corresponding to three linearly independent vectors) get the same color. Suppose the claim is false. Then each monochrome set $C$ of points satisfies $r(C) \leq 2$. Since any set of four points contains three linearly independent points and $|V(LG_2)| = 7$, a 3-coloring yields a monochrome set of points of size exactly three. Denote this set by $E$ and observe that $E$ in fact corresponds to a line $l_E \in \mathcal{L}$. Since any two lines intersect, no line is colored the same as the points of $E$. If $p, p' \in P \setminus E$ are colored differently, then the line $l_{p,p'}$ cannot be colored by any of the three colors contradicting the assumption. Consequently, every point in $P \setminus E$ must be colored the same if the coloring were to be proper. But then this gives a color class with four points which contains three linearly independent points contradicting that the claim was false.

Without loss of generality, suppose $e_1, e_2, e_3$ are all colored red. Since $l_7$ contains the points $e_1 + e_2, e_2 + e_3$ and $e_1 + e_3$, these three points cannot all have different colors. Hence at least two of these three points are in the same color class. Without loss of generality, assume that $e_1 + e_2$ and $e_2 + e_3$ have the same color.

Now observe that the map $\sigma$ defined by $\sigma(e_1) = e_3, \sigma(e_3) = e_1, \sigma(e_2) = e_2$ induces an automorphism of $LG_2$ that fixes every color class within $\mathcal{P}$. Furthermore $\sigma$ swaps $l_1$ with $l_3$ and $l_4$ with $l_6$ and fixes all the other lines. If the sets of lines $\{l_4, l_6\}$ and $\{l_1, l_3\}$ are both monochrome in $\mathcal{L}$, then note that $\sigma$ fixes every color class contradicting that the coloring in question is distinguishing. Thus we consider the alternative, i.e., the possibilities that the lines $l_1$ and $l_3$ (resp. $l_4$ and $l_6$) are in different color classes, and in each of those cases produce a non-trivial automorphism fixing every color class.

Case I: $l_4$ and $l_6$ have different colors, say blue and green respectively. In this case, the point set witnesses at most two colors and none of the points of $\mathcal{P} \setminus \{e_1 + e_3\}$ can be colored blue or green. Moreover, by lemma 4 all the seven points cannot be colored red (note that $e_1, e_2, e_3$ are colored red). Consequently, $e_1 + e_3$ is colored, say blue, and all the other points are colored red. The $l_7, l_5$ and $l_2$ are all colored green since all these three lines contain the point $e_1 + e_3$.

As mentioned above, we shall in every case that may arise, describe a non-trivial automorphism $\sigma$ that fixes each color class. As before, we shall only describe its action on the set $\{e_1, e_2, e_3\}$.

Sub case 1: $l_1$ is colored blue. Then $\sigma(e_1) = e_1, \sigma(e_2) = e_2 + e_3, \sigma(e_3) = e_3$ fixes $e_1 + e_3$, swaps $l_1$ with $l_4$ and fixes $l_3$. Consequently, it fixes every color class.

Sub case 2: $l_1$ is colored green and $l_3$ is colored blue. In this case, $\sigma(e_1) = e_2, \sigma(e_2) = e_1, \sigma(e_3) = e_1 + e_2 + e_3$ does the job.

Sub case 3: $l_1$ and $l_3$ are both colored green. In this case, the only line which is colored blue is $l_4$. Then $\sigma(e_1) = e_2 + e_3, \sigma(e_2) = e_2, \sigma(e_3) = e_1 + e_2$, does the job.

From the above it follows that $l_4$ and $l_6$ cannot be in different color classes. So, we now consider the other possibility, namely that $l_1$ and $l_3$ are in different color classes.

Case II: $l_6$ and $l_4$ have the same color but $l_1$ and $l_3$ are in different color classes, say blue and green respectively. Here we first note that $e_1 + e_2$ and $e_2 + e_3$ are necessarily red because they
belong to $l_1$ and $l_3$ respectively. Again, we are led to three subcases:

**Sub case 1:** $e_1 + e_3$ and $e_1 + e_2 + e_3$ are both colored blue. Here, it is a straightforward check to see that every $l \neq l_1$ is colored green. Then, one can check that $\sigma(e_1) = e_1 + e_2, \sigma(e_2) = e_2, \sigma(e_3) = e_3$ fixes every color class.

**Sub case 2:** The point $e_1 + e_3$ is colored red and $e_1 + e_2 + e_3$ is colored blue. Again, one can check in a straightforward manner, that for all $3 \leq i \leq 6, l_i$ is colored green. If $l_2$ is blue then $\sigma(e_2) = e_3, \sigma(e_3) = e_2, \sigma(e_1) = e_1$ does the job. If $l_2$ is colored green, $\sigma(e_1) = e_2, \sigma(e_2) = e_1, \sigma(e_3) = e_3$ does the job.

**Sub case 3:** $e_1 + e_2 + e_3$ is colored red and $e_1 + e_3$ is colored blue. Here we first observe that $l_2, l_3, l_5, l_7$ are all necessarily green. Also, by the underlying assumption (characterizing Case II), $l_4, l_6$ bear the same color. In this case, $\sigma(e_1) = e_1 + e_2, \sigma(e_3) = e_2 + e_3, \sigma(e_2) = e_2$, does the job.

This exhausts all the possibilities, and hence we are through.

**2.2 The Levi graph $LG_3$**

As remarked earlier, it is not too hard to show that $\chi(LG_q) \leq 6$, so the same holds for $q = 3$ as well. The next proposition shows an improvement on this result.

**Theorem 6.** $\chi_D(LG_3) \leq 5$.

**Proof.** As indicated earlier we denote the points $p \in \mathcal{P}$ as mentioned in the beginning of this section. A line corresponding to the subspace $\{(x, y, z) \in \mathcal{P} : ax + by + cz = 0\}$ is denoted $(a, b, c)$. We color the graph using the colors 1, 2, 3, 4, 5 as in figure 1 (the color is indicated in a rectangular box corresponding to the vertex) It is straightforward to check that the coloring is proper. For an easy check we provide below, a table containing adjacencies of each $p \in \mathcal{P}$.

| Points → | 100 | 110 | 010 | 120 | 112 | 121 | 012 | 122 | 011 | 111 | 110 | 101 | 102 | 001 |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Lines    | 001 | 001 | 001 | 001 | 011 | 011 | 011 | 012 | 012 | 012 | 010 | 010 | 010 | 010 |
| ↓        | 011 | 120 | 100 | 110 | 120 | 121 | 122 | 122 | 121 | 120 | 122 | 121 | 120 | 120 |
|          | 012 | 121 | 101 | 111 | 101 | 102 | 100 | 101 | 100 | 102 | 102 | 101 | 100 | 100 |
|          | 010 | 122 | 102 | 112 | 112 | 110 | 111 | 110 | 112 | 111 | 112 | 111 | 110 | 110 |

Here the first row lists all the points in the projective plane of order 3. The column corresponding to the vertex $p \in \mathcal{P}$ lists the set of lines $l \in \mathcal{L}$ such that $p \in l$, so that the columns are the adjacency lists for the vertices in $\mathcal{P}$.

To see that this coloring is distinguishing, firstly, observe that the line 001 is the only vertex with color 1. Therefore, any automorphism $\phi$ that fixes every color class necessarily fixes this line.
Consequently, the points on 001 are mapped by \( \phi \) onto themselves. Since each point on 001 bears a different color, it follows that \( \phi \) fixes each \( p \in 001 \). In particular, for \( 1 \leq i \leq 4 \), \( \phi \) maps each set \( \{l_{i1}, l_{i2}, l_{i3}\} \) onto itself. Here, \( \{l_{ij}, 1 \leq j \leq 3\} \) denotes the set of lines adjacent to the \( i^{th} \) point of 001. But again note that by the coloring indicated, the vertices \( l_{ij} \) and \( l_{ij}' \) have different colors for each \( i \), so \( \phi(l_{ij}) = l_{ij} \) for each pair \((i, j)\) with \( 1 \leq i \leq 4, 1 \leq j \leq 3 \). Now it is a straightforward check to see that \( \phi = I \).

### 2.3 Levi graphs of order at least 5

In this section we consider the graphs \( LG_q \), for all prime number \( q \geq 5 \). We start with a definition.

**Definition 7.** The Orbit of a point \( p \) with respect to an automorphism \( A \) is the set \( \text{Orb}_A(p) := \{p, Ap, A^2p, \ldots, A^{k-1}p\} \) where \( A^k p = p \).

**Theorem 8.** \( \chi_D(LG_q) = 3 \) for all prime number \( q \geq 5 \).

**Proof.** For each \( p \in \mathcal{P} \), pick uniformly and independently, an element in \( \{1, 2, \ldots, t\} \) and color \( p \) using that color. Finally, color the line set \( \mathcal{L} \) using a different color \( t+1 \). This gives a \( t+1 \)-coloring of \( LG_q \). Since our coloring scheme colors all the lines with the same color any automorphism that preserves each color class must necessarily correspond to an automorphism of \( PG(2, F_q) \), it suffices to restrict our attention to the elements of \( PGL(3, F_q) \). Let \( A \) be a non-trivial automorphism of \( LG_q \) and let \( B_A \) denote the event that \( A \) fixes every color class. Observe that if \( A \) fixes a color class containing a point \( p \), then the remaining elements of \( \text{Orb}_A(p) \) are also in the same color class. The probability that \( \text{Orb}_A(p) \) is in the color class of \( p \), equals \( t^{1 - |\text{Orb}_A(p)|} \).
Let \( \theta_A \) denote the total number of distinct orbits induced by the automorphism \( A \). Then

\[
\mathbb{P}(B_A) = \prod_{\theta_A} \frac{1}{t^{|\text{Orb}_A(p)|}-1} = \frac{1}{t^{q^2+q+1-\theta_A}}
\]

Hence,

\[
\mathbb{P}\left( \bigcup_{A \in PGL(3, \mathbb{F}_q)} B_A \right) \leq \sum_{A \in PGL(3, \mathbb{F}_q)} \mathbb{P}(B_A) = \sum_{A \in PGL(3, \mathbb{F}_q)} \frac{1}{t^{q^2+q+1-\theta_A}}.
\]

**Case 1**: \( q \geq 7 \).

Observe that, each \( A \in PGL(3, \mathbb{F}_q) \) fixes at most \( q + 2 \) points of \( LG_q \). Hence

\[
\theta_A \leq q + 2 + \frac{(q^2 + q + 1) - (q + 2)}{2} = \frac{q^2 + 2q + 3}{2}.
\]

Consequently,

\[
\sum_{A \in PGL(3, \mathbb{F}_q)} \mathbb{P}(B_A) \leq \frac{q^8 - q^6 - q^5 + q^3}{t^{q^2+1}}.
\]

Setting \( q = 7, t = 2 \), gives \( \mathbb{P}(B) \approx 0.3 \). Since the right hand side in the above inequality is decreasing in \( q \), it follows that \( \mathbb{P}(B) < 1 \) for \( q \geq 7 \), hence a proper distinguishing 3–coloring exists. In particular, \( \chi_D(LG_q) = 3 \), for \( q \geq 7 \), since clearly, \( \chi_D(LG_q) > 2 \).

**Case 2**: \( q = 5 \).

In this case, for \( t = 2 \) we actually calculate \( \sum_{A \in PGL(3, \mathbb{F}_5)} \mathbb{P}(B_A) \) using the open source Mathematica software SAGE to obtain \( \mathbb{P}(B) \approx 0.2 \). Therefore, again in this case we have \( \chi_D(LG_5) = 3 \).

### 2.4 Levi graphs of prime power order

In this section, we consider the graphs \( LG_q \) where \( q = p^n \) with \( n \geq 2 \), with \( p \) prime.

**Theorem 9.** \( \chi_D(LG_q) = 3 \) when for all prime powers \( q \geq 8 \).

**Proof.** When \( q = p^n, n \geq 2 \), the cardinality of the automorphism group of \( PG(2, \mathbb{F}_q) \) equals

\[
n|PGL(3, \mathbb{F}_p)| \leq \log_2(q)|PGL(3, \mathbb{F}_q)|.
\]
We argue in the same manner as in the prime case, upper bounding
\[ \sum_{A \in PGL(3,F_q)} \mathbb{P}(B_A) \leq \frac{\log_2 q(q^3 - q^6 - q^9 + q^{12})}{t^{2t+1}}. \]

For \( q = 8 \) and \( t = 2 \) the right hand side is approximately 0.01. Since \( \mathbb{P}(B) \) is a decreasing function of \( q \), it follows that \( \chi_D(LG_8) = 3 \).

For \( q = 4 \) we are not quite able to give the same result though similar methods (using SAGE to make the actual computation) gives us \( \chi_D(LG_4) \leq 4 \). More precisely, the calculations give us \( \mathbb{P}(B) \approx 0.2 \) for \( q = 4 \), and \( t = 3 \).

We conjecture that \( \chi_D(LG_4) = 3 \) though again, our methods fall short of proving this.

3 The Distinguishing Chromatic Number of Levi graphs of order one

Suppose \( n, k \in \mathbb{N} \) and \( 2k < n \). The graph \( LG_1(k, n) := G(L, R, E) \) is a bipartite graph with vertex sets \( L = (\binom{n}{k}) \), \( R = (\binom{n}{k}) \), and \( u \in L, v \in R \) are adjacent if and only if \( u \subset v \). Note that for each \( u \in L, v \in R \) we have \( d(u) = n - k + 1 \) and \( d(v) = k \).

For \( \sigma \in S_n \) let \( F_\sigma := \{ v \in R : \sigma(v) = v \} \).

**Lemma 10.** For \( n > 4 \), \( |F_\sigma| \leq \binom{n-2}{k-2} + \binom{n-2}{k} \) and equality is attained if and only if \( \sigma \) swaps \( i \) and \( j \) for some \( i \neq j \) and fixes every other \( l \in [n] \).

**Proof.** Firstly, it is easy to see that the if \( \sigma = (12) \) then \( |F_\sigma| = \binom{n-2}{k-2} + \binom{n-2}{k} \), so it suffices to show that for any \( \pi \) that is not of the above form, \( |F_\pi| < |F_\sigma| \), where \( \sigma = (12) \).

Suppose not, i.e., suppose \( \pi \in S_n \) is not an involution and \( |F_\pi| \) is maximum. Write \( \pi = C_1C_2 \ldots C_t \) as a product of disjoint cycles with \( |C_1| \geq |C_2| \geq \cdots \geq |C_t| \). Then either \( |C_1| > 2 \), or \( |C_1| = |C_2| = 2 \). If \( |C_1| > 2 \), then suppose without loss of generality, let \( C_1 = (123 \cdots ) \) if \( h \in F_\pi \) then either \( \{1,2\} \subset h \) or \( \{1,2\} \cap h = \emptyset \). In either case we observe that \( h \in F_\sigma \) as well. Therefore \( F_\pi \subset F_\sigma \). Furthermore, note that \( \sigma \) fixes the set \( g = \{1,2,4,\ldots ,k+1\} \), while \( \pi \) does not. Hence \( |F_\sigma| > |F_\pi| \), contradicting that \( |F_\pi| \) is maximum. If \( |C_1| = |C_2| = 2 \), again without loss of generality let \( C_1 = (12), C_2 = (34) \). Again, \( h \in F_\pi \) implies that either \( \{1,2\} \subset h \) or \( \{1,2\} \cap h = \emptyset \), so once again, \( h \in F_\sigma \Rightarrow h \in F_\pi \). Furthermore, \( \{1,2,3,5,\ldots ,k+1\} \in F_\sigma \cap F_\pi \), which contradicts the maximality of \( |F_\pi| \). \( \square \)

For \( k \geq 2 \) define \( n_0(k) := 2k + 1 \) for \( k \geq 3 \) and \( n_0(2) := 6 \).

**Theorem 11.** \( \chi_D(LG_1(k, n)) = 3 \) for \( k \geq 2 \) for \( n \geq n_0(k) \).
Proof. We deal with the cases \( k = 2, k = 3 \) first, and then consider the general case of \( k > 3 \).

For \( k = 2 \), let \( A = \{(1, 2), (2, 3), (2, 4), (3, 4), (4, 5), (5, 6), \ldots, (n - 1, n)\} \), and consider the coloring with the color classes being \( L, A, R \setminus A \). Consider the graph \( G \) with \( V(G) = [n] \) and \( E(G) = A \). Observe that the only automorphism \( G \) admits is the identity. Since a nontrivial automorphism that preserves all the color classes of this coloring must in fact be a nontrivial automorphism of \( G \), it follows that the coloring described is indeed distinguishing. If \( k = 3 \), note that the color classes described by the sets \( R, A, L \setminus A \) is proper and distinguishing for the very same reason.

Suppose now that \( k \geq 4 \). For each vertex in \( R \), assign an element of \( \{1, 2, \ldots, t\} \) each chosen at random, uniformly and independently. These define the new color classes for vertices in \( R \). Assign color \( t + 1 \) to all vertices in \( L = \binom{[n]}{k-1} \).

As in the proof of theorem \( \mathbf{8} \) we have

\[
\mathbb{P}(B) \leq \sum_{A \in \text{Aut}(\Gamma)} \frac{1}{f(n) - \theta_A}
\]

(1)

where \( B \) is the event : A non-trivial automorphism fixes every color class. Here, as before, \( \theta_A \) is the number of distinct orbits induced by the automorphism \( A \).

It is clear that \( S_n \subset \text{Aut}(LG_1) \). We claim that in fact \( \text{Aut}(LG_1) = S_n \). To see this we define an auxiliary graph \( H \) with vertex set \( R \) in which two vertices \( v_1, v_2 \) in \( R \) are adjacent in \( H \) if and only if \( |v_1 \cap v_2| = k - 1 \). Note that this is precisely the Johnson graph \( J(n, k, k - 1) \). It is well known that \( \text{Aut}(J(n, k, k - 1)) = S_n \), the symmetric group on \( n \) elements.

Since \( \phi \in \text{Aut}(LG_1) \) necessarily preserves distances in \( LG_1 \), it follows that if \( u, v \in R \) such that \( d(u, v) = 2 \) then the same holds for the pair \( (\phi(u), \phi(v)) \) as well. But \( d(u, v) = 2 \) in \( LG_1 \) if and only if \( u, v \) are adjacent in \( H \). In particular, \( \phi \) induces an automorphism of \( H \), so that we have a group homomorphism \( \phi : \text{Aut}(LG_1) \to S_n \). To determine it’s kernel, suppose \( \phi \) induces the identity map on \( R \). Since for every \( u \in L \) there is a pair \( x, y \in R \) such that \( u \) is the unique common neighbor of both \( x, y \) in \( LG_1 \), it follows that \( \phi \) induces the identity on \( L \) as well. But since \( S_n \subset \text{Aut}(LG_1) \) the claim follows.

Arguing as in the proof of theorem \( \mathbf{8} \) and using lemma \( \mathbf{10} \) we have

\[
\theta_A \leq |F_A| + \frac{\binom{n}{k} - |F_A|}{2} \leq \frac{\binom{n}{k} - \binom{n-2}{k-2} + \binom{n-2}{k-2}}{2}.
\]

Setting \( t = 2 \) gives

\[
\mathbb{P}(B) \leq \frac{n!}{2^K},
\]

where \( K = \frac{\binom{n}{k} - \binom{n-2}{k-2} - \binom{n-2}{k-2}}{2} \). For \( n > 2k \) it is not hard to show that \( \frac{n!}{2^K} < 1 \) for \( n \geq n_0(k) \). This completes the proof. \( \square \)

Remark: One can show that \( \chi_D(LG_1(2, 5)) \leq 4 \) by constructing four color classes as follows.

Let \( C_1 = \{1\}, C_2 = \{2, 3, 4, 5\}, C_3 = \{12, 23, 24, 35\} \) and \( C_4 = \{13, 25, 14, 15, 34, 45\} \). It suffices to show that this is a distinguishing coloring. Observe that, any nontrivial automorphism that fixes each \( C_i \) fixes the element 1. But then the graph whose edges constitute the set \( C_3 \) is asymmetric, and hence it follows that this coloring is distinguishing.
4 Random Cayley graphs of abelian groups

In this section we consider Cayley graphs of groups and examine the difference between the distinguishing chromatic number and the chromatic number. Recall that the Cayley graph of a (finite) group $A$ with respect to an inverse-closed subset $S$, denoted by $\Gamma(A, S)$ or simply $\Gamma$, is the graph with $V(\Gamma(A, S)) = A$ and $E(\Gamma(A, S)) = \{(g, gh) : g \in A, h \in S\}$. It is straightforward to see that $\Gamma(A, S)$ admits the group $A$ as a group of automorphisms that act regularly on $\Gamma$.

The full automorphism group can of course be much larger, and determining it clearly depends on the set $S$. In what follows we shall restrict our attention to the case when $A$ is abelian, (with the group operation expressed additively) and consider random Cayley subgraphs on $A$. In the case of $A$ abelian, the following map $i(g) = -g$ is also an automorphism of $A$ which is distinct from any of the automorphisms induced by the member of $A$ unless $A \cong \mathbb{F}_2^r$ for some $r$. We say that a Cayley graph $\Gamma$ on an abelian group $A$ has automorphism group as small as possible if $\text{Aut}(\Gamma) = A \rtimes \langle i \rangle$, where $i$ is the map described above.

In a recent paper by Dobson, Spiga and Veret [7], the authors have shown that if $A$ is an abelian group of order $n$, the proportion of inverse closed subsets $S$ for which for the corresponding Cayley graph $\Gamma(A, S)$ has automorphism group as small as possible is $1 - o(1)$ as $n$ goes to infinity. In other words, if an inverse closed set is picked uniformly at random then asymptotically almost surely, the corresponding random Cayley graph has automorphism group as small as possible.

In this section, we deal with random Cayley graphs $\Gamma_p(A, S)$ which are defined as follows. Suppose $|A| = n$, and let $0 < p(n) < 1$. Each element $g \in A$ of order 2 is chosen with probability $p$ and for any other $x \in A$, the pair $(x, -x)$ is chosen with probability $p$ and all these random choices are made independently. The number of elements of the group $A$ whose order is at most two, is denoted by $m$. In what follows, we shall restrict our focus to the following random Cayley graphs:

1. The random Cayley graph $\Gamma_p(A, S)$, with $(|A|, 6) = 1$.
2. The random Cayley graph $\Gamma_p(A, S)$, where $A \cong \mathbb{Z}_2^r \times N$, and $N$ is an odd order group which is not cyclic.

We shall call these as abelian groups of Type I and Type II respectively. The specific restrictions on $A$ may be relaxed, but the results get a little messier, so we restrict our attention to these families of random Cayley graphs.

Firstly, we show that the results of [7] may be extended to the model of random graphs we are interested in, using very similar ideas, for a wider range of $p(n)$. We make no attempt to obtain the best possible constants that would make the following results work. We shall implicitly assume that $n$ is sufficiently large whenever the need arises.

We write $f(n) \ll g(n)$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$. By log we shall mean $\log_2$ in the rest of this section.
Lemma 12. If \( \frac{3}{2} \leq c_1, c_2 \leq n \) satisfy \( c_1 c_2 \geq \frac{n}{27} \), and \( p \in \left[ \frac{25(\log n)^2}{n}, 1 - \frac{25(\log n)^2}{n} \right] \), then

\[
n^{\log n} (p^{c_1} + (1 - p)^{c_1})^{c_2} \leq n^{-\Omega(\log n)}.
\]

Proof. Consider \( f(p) = (p^{c_1} + (1 - p)^{c_1})^{c_2} \) on the interval \([0, 1]\). It follows (by standard calculus) that \( f(p) \) attains minimum at \( p = \frac{1}{2} \). In particular, for \( p \in \left[ \frac{25(\log n)^2}{n}, 1 - \frac{25(\log n)^2}{n} \right] \), \( f(p) \) attains its maximum value at the endpoints. Therefore, it suffices to prove the statement when \( p = \frac{25(\log n)^2}{n} \) since \( f \) is symmetric about \( 1/2 \).

Now,

\[
(p^{c_1} + (1 - p)^{c_1})^{c_2} \leq e^{-pc_1 c_2} e^{c_2 y^{c_1}}
\]

where \( y = \frac{p}{1-p} \).

For \( p \in \left[ \frac{25(\log n)^2}{n}, 1/2 \right] \), we first observe that for any \( 1 < c_1, c_2 \leq n \), the expression \( c_2 y^{c_1} \) is bounded. Indeed,

\[
c_2 y^{c_1} \leq c_2 \left( \frac{25(\log n)^2}{n - 25(\log n)^2} \right)^{c_1} \leq 1. \tag{2}
\]

The last inequality follows from the fact that \( 3/2 \leq c_1, c_2 \leq n \). Therefore

\[
n^{\log n} (p^{c_1} + (1 - p)^{c_1})^{c_2} \leq e^{(\log n)^2 - pc_1 c_2}.
\]

Since \( c_1 c_2 \geq n/24 \) the right hand side in the last inequality is at most \( \exp(-\frac{\log^2 n}{24}) \). This completes the proof. \( \square \)

In what follows, unless otherwise mentioned, \( p \in \left[ \frac{25(\log n)^2}{n}, 1 - \frac{25(\log n)^2}{n} \right] \).

Lemma 13. Suppose \( A \) is an abelian group which is not a 2-group, and suppose \( S \) is chosen randomly by picking each pair \( (x, -x) \) independently with probability \( p \) where \( \frac{25(\log n)^2}{n} \leq p \leq \frac{1}{2} \). Then,

\[ \mathbb{P}(\text{There exist } 1 < H \leq K < A \text{ such that } S \setminus K \text{ is a union of } H - \text{cosets}) \leq O(\exp(-\log^2 n)). \]

Proof. Observe that since \( H \subset K \subset A \), \( A \setminus K \) is also a union of \( H - \)cosets, and let the set of these cosets be denoted \( H \). Write \( A' := A \setminus K \) and \( S' := S \setminus K \). We shall denote the order of an element \( a \) by \( o(a) \) and \( a + a \) is denoted to \( 2a \).

Define

\[
O_2 := \{ a \in A : o(a) \leq 2 \}, \quad J := K \cap O_2, \quad I := \{ a \in A' : 2a \in H \}, \quad I' := A \setminus (K \cup I), \quad L := \{ a \in H : o(a) = 2 \}.
\]

\[ \text{If } x = -x \text{ then the pair is just the singleton } \{x\}. \]
Let \(|H| = h, |I| = i, |K| = k, |J| = j\) and \(|L| = l\). We have \(|O_2| = m\).

The probability that \(S'\) is a union of \(H\)-cosets is precisely

\[
P \left( \bigcap_{g + H \in H} \left\{ (g + H \subseteq S) \text{ or } (g + H \cap S = \emptyset) \right\} \right).
\]

Let \(g \in I'\). If \(h_1 \in H\), and if possible \(g + h_1 \in K \cup I\) then, it follows that \(g + h_1 \in I\) so that \(g \in I\) contradicting that \(g \in I'\). Therefore, if \(g \in I'\) then, \(g + H \subseteq I'\). Moreover \(-g \notin g + H\) which implies that \(g + H \neq -g + H\). Also, observe that \(I' \cap O_2 = \emptyset\). Since each pair \((g, -g)\) is independently picked with probability \(p\) into \(S\) we have that

\[
g + H \subseteq S' \iff -g + H \subseteq S'.
\]

Since there are \(\frac{n-k-i}{2h}\) pairs of cosets in \(I'\) of the type \((g + H, -g + H)\), the probability that for every \(g + H \in I'\) either \(g + H \subseteq S\) or \(g + H \cap S = \emptyset\) is exactly \((p \cdot 1 - (1 - p)h)^{\frac{n-k-i}{2h}}\).

Suppose that \(g \in I\). In this case note that \(g + H = -g + H\). Suppose \(o(g) = 2\). Then for \(h \in H\), we have \(2(g + h) = 0\) if and only if \(o(h) = 2\). In particular, the number of order 2 elements in \(g + H\) is precisely the number of order two elements in \(H\). Since there are \(l\) elements in \(g + H\) of order two and \(h - l\) elements of order greater than two, and since the number of \(H\) cosets \(g + H\) with \(g \in I\), \(o(g) = 2\) that contain order two elements is precisely \(\frac{m-i}{l}\), the probability that every coset \(g + H\) with \(g \in O_2 \cap I\) satisfies that \(g + H \cap S = \emptyset\) or \(g + H \subseteq S\) is precisely \((p \cdot 2 + (1 - p)\cdot 2)^{\frac{m-i}{l}}\).

Finally, now suppose that \(g \in I\) and \(o(g) > 2\). In this case it follows that \(g + H\) has no element of order two. There are exactly \(i - \frac{m-i}{l} \cdot h\) elements \(g \in I\) of this type and furthermore, the set of these elements must also necessarily be the union of \(\frac{1}{h}(i - \frac{m-i}{l} h)\) \(H\)-cosets. If \(g + H \subseteq S'\), one need to include the \(\frac{h}{2}\) pairs \((x, -x)\) of the coset into \(S\), so the probability that every \(g + H\) with \(o(g) > 2\) is either disjoint with \(S\) or is contained in \(S\) is precisely \((p \cdot \frac{h}{2} + (1 - p)\cdot \frac{h}{2})^{(\frac{h}{2} - \frac{m-i}{l})}\).

Again, as in the previous lemma, set \(y := \frac{p}{1 - p}\). Then, from the above discussions, for a fixed \(H \subseteq K\), we have,

\[
P(S' = \text{ union of } H\text{-cosets}) = (p^h (1-p)^h)^{\frac{n-k-i}{2h}} \cdot (p^h + (1-p)^h)^{\frac{m-i}{l}} \cdot (p^\frac{h}{2} + (1-p)^\frac{h}{2})^{(\frac{h}{2} - \frac{m-i}{l})}
\]

\[
\leq (1 - p)\frac{h}{2} \exp(\frac{n-k-i}{2h}y^h) \exp(\frac{m-j}{l} \cdot y^\frac{h}{2}) \exp((\frac{i}{h} - \frac{m-j}{l})y^\frac{h}{2})
\]

The last inequality is obtained by using the facts that \(k \leq \frac{n}{2}\) and \(j \leq m\). Furthermore, note that we may without loss of generality assume that \(p \in [\frac{25(\log n)^2}{n}, \frac{1}{2}]\). We shall now show that each of \(\exp(\frac{n-k-i}{2h}y^h), \exp(\frac{m-j}{l} y^\frac{h}{2})\), \(\exp((\frac{i}{h} - \frac{m-j}{l})y^\frac{h}{2})\) is bounded.

If \(h > 2\), then, using inequality [2] of lemma [12] and taking \(c_1 = h, c_2 = \frac{n-k-i}{2h}\), it follows that \(\exp(\frac{n-k-i}{2h}y^h)\) is bounded. Again using the same inequality, and taking \(c_1 = \frac{h+i}{2} > 1\) and \(c_2 = \frac{m}{l} \leq n\) it follows that \(\exp(\frac{m-j}{l} y^\frac{h}{2})\) is bounded. As for \(\exp((\frac{i}{h} - \frac{m-j}{l})y^\frac{h}{2})\), we set \(c_1 = \frac{h}{2} > 1\) and \(c_2 = (\frac{i}{h} - \frac{m-j}{l}) < n\). To pick a pair of non-trivial subgroups \(H\) and \(K\), it suffices to only pick
sets of generators for these groups which can be done in at most \( (n \log n)^2 = 2^{2 \log^2 n} \) ways. Hence
\[
\mathbb{P}(\text{There exist } 1 < H \leq K < A : |H| > 2, S \setminus K = \text{ union of } H - \text{cosets}) \leq O\left(2^{2(\log n)^2} (1 - p)^{\frac{n}{2}}\right).
\]
By lemma 12 we have \( 2^{2(\log n)^2} (1 - p)^{\frac{n}{2}} \leq \exp(-\frac{1}{4}(\log n)^2) \) for \( p \in \left[\frac{125(\log n)^2}{n}, \frac{1}{2}\right] \).

If \( h = 2 \), then, firstly note that if \( g \) satisfies \( 2g \in H \) then \( o(g)|4 \), so \( g \) lies in the Sylow 2-subgroup of \( A \). Since \( A \) is not a 2-group by assumption, it follows that \( i \leq n/3 \). Hence using that \( j \leq m, k \leq \frac{n}{2} \) we have
\[
\mathbb{P}(S \setminus K \text{ is a union of } H - \text{cosets}) = (p^2 + (1 - p)^2)^{\frac{n-k-j}{4}} (p^{\frac{2}{2} + (1 - p)^{\frac{2}{2}}} )^{m-j}\]
\[
\leq (1 - p)^{\frac{n}{2}} \exp\left(\frac{n-k-j}{4} y^2\right) \exp\left(\frac{m-j}{l} y^{\frac{2}{2} + (1 - p)^{\frac{2}{2}}} \right) \tag{3}
\]
As before, the boundedness of \( \exp\left(\frac{n-k-j}{4} y^2\right) \) follows by setting \( c_1 = 2 \) and \( c_2 = \frac{n-k-j}{4} < n \) and the boundedness of \( \exp\left(\frac{m-j}{l} y^{\frac{2}{2} + (1 - p)^{\frac{2}{2}}} \right) \) follows by setting \( c_1 = \frac{2}{2} + (1 - p)^{\frac{2}{2}} > 1, c_2 = \frac{m-j}{l} < n \). Again,
\[
\mathbb{P}(\text{There exist } 1 < H \leq K < A : |H| = 2, S \setminus K = \text{ union of } H - \text{cosets}) \leq O\left(2^{2(\log n)^2} (1 - p)^{\frac{n}{2}}\right)
\]
and by lemma 12 this is at most \( \exp(-\frac{1}{4}(\log n)^2) \). \( \square \)

The next lemma again is an extension of a result of [7]. For \( S \subset A \) and \( \phi \in \text{Aut}(A) \), we say that \( \phi \) normalizes \( S \) if \( \phi(S) = S \).

**Lemma 14.** Suppose \( A \) is abelian, and let \( S \) be a random inverse closed subset of \( A \) with each pair \((x, -x)\) picked with probability \( p \). Let \( i : A \to A \) be the automorphism of \( A \) defined by \( i : x \to -x \). Then the probability that there exists \( \phi \in \text{Aut}(A) \setminus \{1, i\} \) such that \( S \) is normalized by \( \phi \) is at most \( O(\exp(-\frac{2}{4}(\log n)^2)) \).

**Proof.** Fix \( \phi \in \text{Aut}(A) \) and suppose that \( \phi \) normalizes \( S \). Since \( |i| = 2 \), we have \( m = |C_A(i)| \) where \( C_A(i) \) is the centralizer of \( i \) in \( A \). Let \( |C_A(\phi)| = c \) and \( |C_A(i, \phi)| = k \).

Suppose that \( |\phi| \) is divisible by an odd prime \( q \).

In this case, without loss of generality we assume \( |\phi| = q \), otherwise we may replace \( \phi \) with a suitable power. Observe that, if \( a \in S \) then \( \{a, \phi(a), \ldots, \phi^{q-1}(a)\} \subseteq S \). Therefore,
\[
\mathbb{P}(\phi(S) \subset S) = (p^q + (1 - p)^q)^{\frac{m-k}{q}} (p^q + (1 - p)^q)^{\frac{n-(c-qk)}{2q}} \leq (p^q + (1 - p)^q)^{\frac{n}{2q}}.
\]
The last inequality follows by using \( k \leq m, c \leq \frac{n}{2} \) and \( (p^q + (1 - p)^q) \leq 1 \). Since \( |\text{Aut}(A)| \leq n^{\log_2 n} \), it follows that the probability that there exists \( \phi \in \text{Aut}(A) \setminus \{1, i\} \) such that \( \phi(S) = S \) is at most \( n^{\log_2 n} (p^q + (1 - p)^q)^{\frac{n}{2q}} \). We use lemma 12 by setting \( c_1 = q \) and \( c_2 = \frac{n}{2q} \) to see that this probability is \( O(\exp(-\frac{2}{4}(\log n)^2)) \).

Now suppose \( |\phi| \) is a power of two. Two cases arise:

**Case 1:** \( i \in \langle \phi \rangle \)

By replacing \( \phi \) by a suitable power, we may assume that \( \phi^2 = i \). Then, similar to the case 1,
\[
\mathbb{P}(\phi(S) \subset S) = (p^2 + (1 - p)^2)^{\frac{m-c}{2}} (p^2 + (1 - p)^2)^{\frac{n-m}{4}} \leq (p^2 + (1 - p)^2)^{\frac{n}{8}}.
\]

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The last inequality is obtained by using $m \leq \frac{n}{2}$ and $c \leq m$. Again, we use lemma 12 with $c_1 = 2$ and $c_2 = \frac{n}{8}$ to see that the above probability is at most $2^{(\log n)^2}(p^2 + (1 - p)^2)^{\frac{m}{2}} \leq O(\exp(-\frac{21}{4}(\log n)^2))$.

**Case 2: $i \notin \langle \phi \rangle$**

In this case
\[ P(\phi(S) \subset S) = (p^2 + (1 - p)^2)^{\frac{m}{2}} (p^2 + (1 - p)^2)^{\frac{m+n}{2}} = (p^2 + (1 - p)^2)^{\frac{m+n}{2}}. \]

Again, setting $c_1 = 2, c_2 = \frac{m+n}{4}$ and applying lemma 12 we see that the above probability is at most $O(\exp(-\frac{23}{2}(\log n)^2))$.

The following lemma is proved for the abelian groups of Types I and II.

**Lemma 15.** Let $A$ be an abelian group of Type I or Type II. Let $C$ be a cyclic group, and $Z$ an elementary abelian 2 group. For a subset $S \subset A$, we call a pair of subgroups $(C, Z)$ of $A$, good for $S$, if

1. $A = C \times Z$.
2. $|C| = t \geq 4$.
3. There exist $S' \in \{\emptyset, \{0\}, A \setminus \{0\}\}$, and $S'' \subset Z$ such that $S = S' \times S''$.

For a random inverse-closed subset $S \subset A$, the probability that there exists a pair $(C, Z)$ good for $S$ is at most $O \left( \exp(-\frac{25(\log n)^2(n-1)}{2n}) \right)$.

**Proof.** The lemma is trivial in the case $A \cong \mathbb{Z}_2^k \times N$, where $N$ is an odd order group which is not cyclic. Let $A$ be abelian with $(|A|, 6) = 1$. For a fixed $S \subset A$ which is inverse-closed, if $(C, Z)$ is good for $S$, then $A \cong C$, $Z$ is trivial, and furthermore, $S' \in \{\emptyset, A, \{0\}, A \setminus \{0\}\}, S'' \in \{\emptyset, \{0\}\}$. Since $S$ is inverse-closed and $0 \notin S$, there are two possibilities: $S = \emptyset$ or $S = A \setminus \{0\}$. In either case, it is easy to check that the probability that there exist $(C, Z, S', S'')$, satisfying the hypotheses is at most $\exp(-\frac{25(\log n)^2(n-1)}{2n})$.

For the abelian groups mentioned in the beginning of this section, we state the extended version of Theorem 1.5 from [7]. The proof is along the same lines as the proof that appears in [7], so we skip the details.

**Theorem 16.** Let $\Gamma_p := \Gamma_p(A, S)$ be the random Cayley graph with $\frac{25(\log n)^2}{n} \leq p \leq 1 - \frac{25(\log n)^2}{n}$. Then,
\[ P(\text{Aut}(\Gamma_p) \nmid A \rtimes \langle i \rangle) \leq O(\exp(-\log^2 n)), \]
where $i : A \to A$ is the automorphism $i(x) = -x$.

We first consider abelian groups $A$ with $(|A|, 6) = 1$. Set $|A| = n$. We adopt the convention that an event $E$ occurs in the random Cayley graph $\Gamma_p(A, S)$ with high probability (whp for short) if $P(E) \geq 1 - n^{-\Omega(\log n)}$. 

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**Theorem 17.** Let $\Gamma_p := \Gamma_p(A, S)$ be the random Cayley graph with
\[
\frac{25(\log n)^2}{n} \leq p \leq 1 - \left(\frac{10 \log n}{n}\right)^{2/3},
\]
where $A$ is an abelian group with order co-prime to six. Then, $\chi_D(\Gamma) \leq \chi(\Gamma) + 1$ with high probability.

**Proof.** Our main probabilistic tool here is Janson’s inequality. To set the notation up, we first give the setup and state Janson’s inequality.

Let $R \subset \Omega$ be a random subset where each $r \in \Omega$ is chosen into $R$ independently with probability $p_r$. Let $X_i \subset \Omega$ for $i = 1, 2, \ldots, t$ and let $B_i$ denote the event: $X_i \subset R$. Let $N = \#\{i : X_i \subset R\}$, $\mu := E(N)$, $\Delta := \sum_{i \sim j} P(B_i \land B_j)$, where $i \sim j$ if $X_i \cap X_j \neq \emptyset$. Then,
\[
P(N = 0) \leq \exp\left(-\frac{\mu^2}{2\Delta}\right)
\]
if $\mu \leq \Delta$.

The random process of picking $S$ is equivalent to rejecting each pair $(x, -x)$ in $A$ (for $x \neq 0$) independently with probability $q = 1 - p$.

Let $T := \{(x, y, z) \subset A : x + y + z = 0, x \neq 0, y \neq 0, z \neq 0\}$ and for each $T \in \mathcal{T}$, let $D(T) := \{\pm(x - y), \pm(y - z), \pm(x - z)\}$.

First, observe that $|T| = \frac{(n-5)(n-1)}{6}$. Indeed, there are $n - 1$ choices for $x$ with $x \neq 0$, and since $y \notin \{0, x, -x, 2x\}, 2y \neq -x$, there are $n - 5$ choices for $y$ and $z$ is consequently determined uniquely, so that gives $(n - 1)(n - 5)$ ordered triples $(x, y, z)$ satisfying the conditions of the sets in $\mathcal{T}$.

Consider the events $B_T$: $D(T) \subset S$, and let $N = \#\{T \in \mathcal{T} : D(T) \subset S\}$. Then
\[
E(N) = |T|q^3 = \frac{(n-5)(n-1)}{6}q^3.
\]

Observe that $T \sim U$ if and only if $|D(T) \cap D(U)| \neq 0$ since otherwise the choices for the sets $T, U \in \mathcal{T}$ are decided over disjoint sets of inverse-closed pairs. Set
\[
\Delta = \sum_{|D(T) \cap D(U)| \neq 0} P(B_T \land B_U) \tag{4}
\]
We shall find a suitable upper bound for $\Delta$ and in order to do that, we shall count the number of $U \in \mathcal{T}$ with $U \sim T$ for a fixed $T \in \mathcal{T}$.

Suppose that $|D(T) \cap D(U)| = 2$. Let $T = \{x, y, z\}$ and $U = \{u, v, w\}$. If one of $(x - y), -(x - y) \in U$, say $x - y = u - v$, then it follows that $\{u, v, w\} = \{u, u - (x - y), -2u + (x - y)\}$ for
some $0 \neq u \in A$. In particular, for a given $T \in \mathcal{T}$ there are $3(n - 1)$ choices for $U$ such that $|D(T) \cap D(U)| = 2$. One can check (by a straightforward calculation; we skip the details) that there is at most one set $U$ with $|D(T) \cap D(U)| = 4$, and that for any $T \in \mathcal{T}$, $-T \neq T$, and $U = -T$ is the unique member of $\mathcal{T}$ satisfying $|D(T) \cap D(U)| = 6$.

Therefore, we have

$$\Delta < 3n|T|q^5 + |T|q^4 + |T|q^3,$$

so by Janson’s inequality, it follows that

$$\mathbb{P}(N = 0) < \exp\left(\frac{-|T|q^3}{2(3nq^2 + q + 1)}\right) = e^{-\Omega(\log^2 n)}$$

for $q \geq \left(\frac{17 \log n}{n}\right)^{2/3}$.

Suppose $\sigma \in A \times \langle i \rangle$ is non-trivial and $\sigma(T) = T$ for some $T \in \mathcal{T}$. If $\sigma = (g,1)$ for some $g \in A$, and if $\sigma(x) = y, \sigma(y) = z, \sigma(z) = x$, say, then by the action of $(g,1)$ on $A$, it follows that $3g = 0$ contradicting that $\sigma$ is non-trivial. If $\sigma(x) = y, \sigma(y) = x$ and $\sigma(z) = z$, say, then it similarly follows that $2g = 0$, contradicting that $\sigma$ is non-trivial. If $\sigma = (g,i)$ for some $g \in A$, and if $\sigma(x) = y, \sigma(y) = z$ and $\sigma(z) = x$, then since $(g,i)(x) = g - x$, it follows that $x = y = z$ contradicting that $\{x,y,z\} \in T$. Again, if $\sigma(x) = y, \sigma(y) = x$ and $\sigma(z) = z$, then it follows that $2z - x = y$ and since $x + y + z = 0$, we have $2z = 0$, again, a contradiction to the assumption that $\{x,y,z\} \in T$. The upshot is that no non-trivial $\sigma \in A \times \langle i \rangle$ fixes any $T \in \mathcal{T}$.

By theorem \([10]\) the full automorphism group of this random Cayley graph is isomorphic to $A \times \langle i \rangle$ whp. From the preceding discussions, it follows that the random Cayley graph $\Gamma_p(A,S)$, contains a 3-element independent set $\{x,y,z\}$ which is not fixed by any non-trivial automorphism $\sigma \in Aut(\Gamma)$ whp. Color this set with a new color and the rest of the graph using as few colors as possible. This coloring is both proper and distinguishing.

\[ \square \]

The next theorem deals with the other case of abelian groups as indicated in the beginning of this section. But firstly we shall need a general lemma.

**Lemma 18.** Let $A \cong \mathbb{Z}_5 \times N$, where $N$ is a non-cyclic group of odd order and let $\Gamma = \Gamma(A,S)$ be a Cayley graph on $A$. Suppose that $Aut(\Gamma) \cong A \rtimes \langle i \rangle$. If $m$ is the number of elements in $A$ of order at most 2, and $\chi(\Gamma) \leq \frac{n^2}{m + 2\log(2n)}$, then $\chi_D(\Gamma) \leq \chi(\Gamma) + 1$.

**Proof.** Let us denote $\chi(\Gamma) = \chi$ and let $C$ be a maximum sized color class in a proper coloring of $\Gamma$ using $\chi$ colors, so that $|C| \geq n/\chi$. For each $x \in C$, let us randomly assign a number from $\{1,2,\ldots,t\}$ uniformly and independently, for some $t$ to be determined later.

Observe that a non-trivial automorphism which fixes any vertex of $\Gamma$ is necessarily of the form $(g,i)$ for some $g \in A$. Moreover, $(g,i)$ fixes a vertex $h \in \Gamma$ if and only if $g = 2h$ in $A$. It follows that any non-trivial automorphism $\sigma$ fixes at most $m$ vertices in $\Gamma$. As in the proof of theorem \([8]\), we shall partition $C$ into at most $t$ other color classes, so we wish to calculate the probability that some non-trivial automorphism fixes each of the new classes. By similar arguments, it follows that
this probability is at most $2nt^{-\alpha}$ where $\alpha := \frac{n/\chi - m}{2}$. Now observe that
$$t := \lceil (2n)^{\frac{2\chi}{n-m\chi}} \rceil \implies 2n < t^\alpha.$$ Hence there exists a proper $\chi + t - 1$ coloring of $\Gamma$ that is also distinguishing. In particular, if $\chi < \frac{n}{m+2\log(2n)}$ we may take $t = 2$, and this proves the lemma.

Finally we have the corresponding theorem for random Cayley graph $\Gamma_p(A, S)$ for $A \simeq \mathbb{Z}_r \times N$ with $N$ being a non-cyclic group of odd order.

**Theorem 19.** Suppose $A$ is a Type II abelian group of order $n$ and suppose that $m \ll \frac{n}{\log^2 n}$. Let $\Gamma_p := \Gamma_p(A, S)$ be the random Cayley graph, with $\frac{25(\log n)^2}{n} \leq p \leq \frac{7}{13(n+2\log 2n)}$. Then whp
$$\chi_D(\Gamma_p) \leq \chi(\Gamma_p) + 1.$$

**Proof.** Let
$$X' := \sum_{x: 2x = 0 \atop x \neq 0} 1_{x \in S} \quad X'' := \sum_{(x,-x) \atop x \neq -x} 1_{x,-x \in S}$$
so $|S| = X' + 2X''$. Then $X', X''$ are binomial random variables with parameters $(m - 1, p)$ and $(\frac{n-m}{2}, p)$ respectively. Then
$$\mathbb{E}(|S|) = (n-1)p < np.$$ By the concentration of binomial random variables (see theorem 2.1 in [10]) we have
$$\mathbb{P}(|S| \geq \mathbb{E}(|S|) + 3t) \leq \mathbb{P}(X' \geq \mathbb{E}(X') + t) + \mathbb{P}(X'' \geq \mathbb{E}(X'') + t) \leq \exp \left( -\frac{t^2}{2((m-1)p + \frac{t}{3})} \right) + \exp \left( -\frac{t^2}{2(\frac{n-m}{2}p + \frac{t}{3})} \right)$$

Set $t = \frac{2n}{13(m+2\log 2n)}$. Since $m \ll \frac{n}{\log^2 n}$ it follows that for
$$\frac{25\log^2 n}{n} \leq p < \frac{7}{13(m+2\log 2n)} < 1 - \frac{25\log^2 n}{n}$$
the right hand side of (5) is at most $e^{-\Omega(\log^2 n)}$, so that whp $|S| \leq \frac{13np}{7} < \frac{n}{m+2\log(2n)}$. Hence by theorem 16 and lemma 18, and the fact that $\chi(G) \leq \Delta(G) + 1$ for any graph $G$, it follows that $\chi_D(\Gamma_p) \leq \chi(\Gamma_p) + 1$ whp.

5 Concluding Remarks

1. Most of the results of this paper basically espouse the following theme: If the automorphism group of $G$ is ‘not substantially larger’ than $|G|$, then $\chi_D(G) \leq \chi(G) + 1$. The theme of bounding the distinguishing chromatic number has been considered in [6] but our perspective is a little different from theirs. It is a singular observation that all known instances of graphs
with \( \chi_D(G) \) being significantly larger than \( \chi(G) \) admit automorphism groups that are 'super large' in the size of \( G \). It would be interesting to construct graphs where this is not the case, and where the distinguishing chromatic number is much larger than the chromatic number. We believe that such instances may only be sporadic.

2. It is possible to consider other Levi graphs arising out of other projective geometries (affine planes, incidence bipartite graphs of 1-dimensional subspaces versus \( k \) dimensional subspaces in an \( n \) dimensional vector space for some \( k \) etc). Many of our results and methods work in those contexts as well and it should be possible to prove similar results there as well, as long as the full automorphism group is not substantially larger. For instance, in the case of the incidence graphs of \( k \) sets versus \( l \)-sets of \( [n] \), it is widely believed (see chapter 1, [8]) that in most cases, the full automorphism group of the generalized Johnson graphs is indeed \( S_n \) though it is not known with certainty.

3. As stated earlier, we believe that \( \chi_D(LG_3) = 4, \chi_D(LG_4) = 3 \) though we haven't been able to show the same. One can, by tedious arguments considering several cases, show that a monochromatic 3-coloring of \( LG_3 \) is not a proper distinguishing coloring.

4. All our results regarding random Cayley graphs in fact hold with probability \( 1 - n^{-\Omega(\log n)} \). However, if we wish to only prove that certain results hold asymptotically almost surely, i.e., with probability \( 1 - o(1) \), then improvements on some of the results is not difficult. For instance, Alon proved in [1] that if we pick \( k \leq n/2 \) subsets uniformly at random and then complete them to inverse-closed sets, then a.a.s \( \chi(\Gamma(A,S)) \leq O\left(\frac{k}{\log k}\right) \). So for \( A \simeq \mathbb{Z}_q^2 \times N \) with \( N \) a non-cyclic group of odd order with \( n^{3/4} \log n \ll m \ll \frac{n}{\log n} \), one can prove by minor modifications, that a.a.s \( \chi_D(\Gamma_p) \leq \chi(\Gamma_p) + 1 \) if \( \frac{c\log^2 n}{n} \leq p \leq \frac{C\log n}{m+2\log 2n} \) for suitable constants \( c,C \). We skip the details.

5. It is possible to extend some of the methods in the study of \( \chi_D(\Gamma_p(A,S)) \) to other abelian groups as well. For non abelian groups \( A \), it is a yet-unsettled conjecture of Babai, Godsil, Imrich, and Lovász (see [2] for details and a proof of the conjecture for nilpotent non-abelian groups), that for any group which is not generalized dihedral, almost surely \( Aut(\Gamma_{1/2}(A,S)) \simeq A \) as \( |A| \to \infty \). Thus, for all such graphs it is clear that \( \chi_D(G) \leq \chi(G) + 1 \) since one can pick an arbitrary non-identity vertex and color it using a distinct color, and color the rest of the graph using at most \( \chi(G) \) colors. Since \( A \) acts regularly, it follows that this coloring is distinguishing as well. We in fact believe that something stronger is true, viz., that for almost all Cayley graphs, \( \chi_D(G) = \chi(G) \). At the moment, we are only able to show the same in certain non-abelian \( q \)-groups, for \( q \) a large enough prime. Indeed, by the result of [1], for \( p = 1/2 \), a random Cayley graph \( \Gamma = \Gamma_{1/2}(A,S) \) almost surely has full automorphism group isomorphic to \( A \), when \( A \) is a nilpotent group. Furthermore, from a result of [1], we have \( \chi(\Gamma) = \Omega\left(\frac{n}{\log n}\right) \). Suppose \( |A| = q^r \) for a fixed \( r \), and \( q \) a sufficiently large prime. If \( \phi = \phi_q \) for \( q \in A \) is an automorphism that fixes every color class of this coloring, then note that each color class has at least \( q \) elements, so that \( \chi(\Gamma) \leq q^{r-1} \). But this contradicts the result of [1] since \( q \gg \Omega_{c}(\log^2 q) \). The same arguments work over a slightly larger range for \( p = \Omega(1) \) along the same lines as discussed above.

6. One of the motifs of this paper suggests that that if the automorphism group of a graph is not too large, then the distinguishing chromatic number is very close to it's chromatic number.

\footnote{It requires a very small tweak but the proof runs through without any major changes}

\footnote{Again, the proof in [1] can be followed as it is in our random Cayley graph model to get the same result.}
The converse question is one that is tempting, i.e., if the size of the full automorphism group of a graph $G$ is, say, at least exponential in $|V(G)|$, then one might wonder if $\chi_D(G) > \chi(G)$. As was proven in [4], this is not true as witnessed by the Kneser graphs $K(n, r)$ with $r \geq 3$. However, one might also expect that in such cases, distinguishing proper colorings are perhaps rare, or at the very least, that there do exist minimal proper, non-distinguishing colorings of $G$. It turns out that even this is not true, as we shall show to be the case with the complement of the Kneser graphs $\overline{K(n, r)}$, for $n \geq 2r$ and $r \geq 3$. Recall that the vertices of $K(n, r)$ correspond to $r$-element subsets of $[n]$ and two vertices are adjacent if and only if their intersection is non-empty. Since $\text{Aut}(K(n, r)) \cong S_n$ for $n \geq 2r$, it follows that the full automorphism group of $K(n, r)$ is also $S_n$. Consider a proper coloring $c$ of $K(n, r)$ into color classes $C_1, C_2, \ldots, C_t$. Note that for any two vertices $v_1, v_2$ in the same color class, $v_1 \cap v_2 = \emptyset$.

If possible, let $\sigma \in S_n$ be a non-trivial automorphism which fixes $C_i$ for each $i$. Without loss of generality let $\sigma(1) = 2$. Observe that for the vertex $v_1 = (1, 2, \ldots, r)$, its color class has no other vertex containing 1 or 2, so $\sigma$ maps $\{1, 2, \ldots, r\}$ to $\{1, 2, \ldots, r\}$. Again, with the vertex $v_2 = \{1, 3, \ldots, r+1\}$, which is in color class $C_2 \neq C_1$, $\sigma$ maps $v_2$ into $\{2, \sigma(3), \ldots, \sigma(r+1)\} \neq v_2$, so $\sigma(v_2) \cap v_2 = \emptyset$ by assumption. However, since $\sigma(i) \in \{1, 2, \ldots, r\}$ for each $3 \leq i \leq r$ this yields a contradiction.

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