ON THE VOLUME OF LOCALLY CONFORMALLY FLAT 4 DIMENSIONAL CLOSED HYPERSURFACE

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ABSTRACT. Let $M$ be a 5 dimensional Riemannian manifold with $\text{Sec}_M \in [0, 1]$, $\Sigma$ be a locally conformally flat closed hypersurface in $M$ with mean curvature function $H$. We prove that, there exists $\varepsilon_0 > 0$, such that

$$\int_\Sigma (1 + H^2)^2 \geq \frac{4 \pi^2}{3} \chi(\Sigma),$$

provided $|H| \leq \varepsilon_0$, where $\chi(\Sigma)$ is the Euler number of $\Sigma$. In particular, if $\Sigma$ is a locally conformally flat minimal hypersphere in $M$, then $\text{Vol}(\Sigma) \geq 8 \pi^2 / 3$, which partially answer a question proposed by Mazet and Rosenberg [6]. Moreover, we show that if $M$ is (some special but large class) rotationally symmetric, the inequality (1) holds for all $H$.

1. Introduction

Let $M$ be a 2-sphere with a smooth Riemannian metric such that the curvature is between 0 and 1. It is known (see [3] or [28]) that the length of an embedded closed geodesic in $M$ is at least $2\pi$, which is the length of the standard circle in Euclidean plane. When $M$ is a Riemannian 3-manifold with sectional curvature between 0 and 1, one can easily apply Gauss equation and Gauss-Bonnet theorem to obtain that an embedded minimal sphere $\Sigma$ in $M$ has area at least $4\pi$, that is

$$4\pi = \int_\Sigma \text{Sec}_\Sigma = \int_\Sigma R_{1212} = \int_\Sigma \left( R_{1212} - \frac{1}{2} |A|^2 \right) \leq \int_\Sigma \overline{R}_{1212} \leq \text{Vol}(\Sigma),$$

where $R$ and $\overline{R}$ denote the curvature tensors of $\Sigma$ and $M$, $A$ denotes the second fundamental form of $\Sigma$ in $M$.

In [6], Mazet and Rosenberg study the equality case and get a rigidity theorem for $M$. The authors also put forward two very interesting questions, one of them is, if $M$ is an $(n+1)$-Riemannian manifold with $\text{Sec}_M \in [0, 1]$, does an embedded minimal hypersphere (i.e., minimal hypersurface diffeomorphic to the standard Euclidean $n$-sphere $S^n$) has volume at least the volume of $S^n$? In 1974, Hoffman and Spruck [4] studied the isoperimetric inequality and showed, if $M$ is a simply connected Riemannian $(n+1)$-manifold with $\text{Sec}_M \in [1/4, 1]$, then any closed minimal hypersurface has at least the volume of $S^n$. Therefore, if the answer of the Mazet and Rosenberg’s question is true, it can be seen as a generalization (with topological restricted) of Hoffman and Spruck’s result. We would like to point out that, if $\text{Sec}_M \in [0, 1]$, the topological restriction on $\Sigma$ is necessary. Actually, given $\varepsilon > 0$, let $\Sigma$ be a flat $n$–torus with $\text{Vol}(\Sigma) \leq \varepsilon$ (which can be done by passing a dilation), then $\Sigma$ is a totally geodesic closed hypersurface embedded in $\Sigma \times \mathbb{R}$ whose sectional curvature is 0.

Note that in the case of $n = 2$, every surface admits an isothermal coordinates and therefore is locally conformally flat. It is seems natural to add the condition “locally conformally flat” on the hypersurface in high dimensional case. In this paper, we focus our attention on the case of $n = 4$, pose the assumption...
that $\Sigma$ is locally conformally flat, and partially answer the question proposed by Mazet and Rosenberg. Actually, we get a more general result as follows.

**Theorem 1.1.** Let $M$ be a 5-dimensional Riemannian manifold with $\text{Sec}_M \in [0, 1]$, and $\Sigma$ be an embedded locally conformally flat closed hypersurface in $M$ with mean curvature function $H$. Then we have

$$
\int_{\Sigma} \left( (1 + H^2)^2 + |H| f(|H|) \right) \geq \frac{4\pi^2}{3} \chi(\Sigma),
$$

where $f$ is a nonnegative function defined in Section 2, and $\chi(\Sigma)$ is the Euler number of $\Sigma$.

Moreover, there exists $\varepsilon_0 > 0$, such that if $|H| \leq \varepsilon_0$, we obtain,

$$
\int_{\Sigma} (1 + H^2)^2 \geq \frac{4\pi^2}{3} \chi(\Sigma).
$$

The equality holds if and only if the mean curvature $H$ is constant, $\Sigma$ is totally umbilic and isometric to $S^4\left(\frac{1}{1+H^2}\right)$.

As an immediate corollary of theorem [1.1] the following result partially answer the question proposed by Mazet and Rosenberg.

**Theorem 1.2.** Let $M$ be a 5-dimensional Riemannian manifold with $\text{Sec}_M \in [0, 1]$, and $\Sigma$ be an embedded locally conformally flat minimal hypersphere in $M$. Then

$$
\text{Vol}(\Sigma) \geq \frac{8\pi^2}{3} = \text{Vol}(S^4).
$$

The equality holds if and only if $\Sigma$ is totally geodesic and isometric to $S^4$.

This paper is organized as follows. In Section 2, we list some notations and known formulas, and give the proof of Theorem [1.1]. In Section 3, we deal with a special case when $M$ is rotationally symmetric and get the lower bound volume for all $H$, see Theorem [1.2].

2. Preliminary and proof of Theorem [1.1]

Let $(M, \bar{g})$ be an $n + 1$ dimensional Riemannian manifold, and $(\Sigma, g)$ be a hypersurface isometric immersed in $M$. If there is no ambiguity, $\langle \cdot, \cdot \rangle$ will denote both $\bar{g}$ and $g$. Let $\bar{\nabla}$ and $\nabla$ be the Levi-Civita connection induced by metric $\bar{g}$ and $g$ respectively. Let $R$ be the curvature tensor on $\Sigma$ defined by, for all $X, Y, Z, W \in \mathfrak{X}(T\Sigma)$,

$$
R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle,
$$

where $R(X, Y) = -\nabla_X \nabla_Y + \nabla_Y \nabla_X + \nabla_{[X, Y]}$. Also let $\bar{R}$ be the curvature tensor on $M$ which is defined similarly.

Let $e_1, \cdots, e_n$ be a local orthonormal frame on $\Sigma$. For all $1 \leq i, j, k, l \leq n$, write

$$
R_{ijkl} = \langle e_i, e_j, e_k, e_l \rangle, \quad \bar{R}_{ijkl} = \langle e_i, e_j, e_k, e_l \rangle.
$$

The sectional curvature will be

$$
\text{Sec}_M(e_i \wedge e_j) = R_{ijij}, \quad \text{Sec}_M(e_i \wedge e_j) = \bar{R}_{ijij}.
$$

Let $A$ be the second fundamental form of $\Sigma$ in $M$, $h_{ij} = \langle A(e_i), e_j \rangle$ be the coefficients of $A$. Then the Gauss equation can be written as

$$
R_{ijkl} = \bar{R}_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}.
$$

We also denote by $S^n$ be the standard unit $n$ sphere in $n + 1$ Euclidean space and by $S^n(r)$ be the round $n$-sphere with radius $r$. Now we will prove Theorem [1.1]
Proof of Theorem 1.1. The Gauss-Bonnet-Chern formula for a closed 4-manifold \( \Sigma \) is (see [2] or [3])

\[
4\pi^2 \chi(\Sigma) = \int_{\Sigma} \left( \frac{S^2}{12} - \frac{|Ric|^2}{4} + \frac{|W|^2}{8} \right),
\]

where \( \chi(\Sigma) \) is the Euler characteristic of \( \Sigma \), \( S \) is the scalar curvature, \( Ric \) is the Ricci tensor and \( W \) is the Weyl tensor. It is well known that, when dimension greater than 3, locally conformally flatness equivalent to Weyl tensor vanishing. Therefore, to prove the first part of the theorem, it is sufficient to prove, pointwisely,

Claim. \( Q := \frac{S^2}{12} - \frac{|Ric|^2}{4} \leq 3(1 + H^2)^2 + 3|H| f(|H|) \).

Next we will consider our problem at one point \( p \in \Sigma \) (in the calculations, we omit the letter "p" for simplicity). Throughout this proof, \( i, j, k, l \) will range from 1 to 4 if there is no special explanation.

Firstly, by the Gauss equation, we obtain,

\[
S^2 = \left( \sum_{i,j} R_{ijij} \right)^2 = \left( \sum_{i,j} R_{ijij} + 16H^2 - |A|^2 \right)^2 = \left( \sigma + 12H^2 - |\hat{A}|^2 \right)^2 = \sigma^2 + 144H^4 + |\hat{A}|^4 + 24\sigma H^2 - 2|\hat{A}|^2 - 24H^2 |\hat{A}|^2,
\]

where

\[
\sigma := \sum_{i,j} R_{ijij}, \quad \hat{A} := A - HI,
\]

i.e., \( \hat{A} \) is the traceless part of \( A \).

For simplicity, let \( e_1, e_2, e_3, e_4 \) be the principal directions at the point \( p \), and \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) be the corresponding principal curvatures, we have

\[
|Ric|^2 = \sum_{i,j} \left( \sum_k R_{ikjk} \right)^2 = \sum_{i,j} \left( \sum_k R_{ikjk} + 4\delta_{ij}\lambda_i H - \delta_{ij}\lambda_i \lambda_j \right)^2
\]

For simplifying \( |Ric|^2 \), we need to introduce some notations as follows:

\[
a_{ij} := \sum_k R_{ikjk}, \quad \tilde{a}_{ij} := \sum_k R_{ikjk} - \frac{\sigma}{4} \delta_{ij}.
\]

Note that \( \sigma \) is the trace of \( (a_{ij}) \) and \( (\tilde{a}_{ij}) \) is the traceless part of \( (a_{ij}) \). Using these notations, we get

\[
|Ric|^2 = \sum_{i,j} (a_{ij} + 4\delta_{ij}\lambda_i H - \delta_{ij}\lambda_i \lambda_j)^2
\]

\[
= \sum_{i,j} a_{ij}^2 + 16H^2 |A|^2 + \sum_i \lambda^4_i
\]

\[
+ 8H \sum_i \lambda_i a_{ii} - 2 \sum_i \left( \lambda_i^2 a_{ii} \right) - 8H \sum_i \lambda_i^3
\]

\[
= \frac{\sigma^2}{4} + |\hat{a}|^2 + 16H^2 |\hat{A}|^2 + 64H^4 + \sum_i \lambda^4_i
\]

\[
- 2 \sum_i \left( \lambda_i^2 - 4H\lambda_i \right) a_{ii} - 8H \sum_i \lambda_i^3,
\]
where $|\hat{A}|^2 = \sum_{i,j} \hat{A}_{ij}^2$. Next we set $\mu_i = \lambda_i - H$ which are the eigenvalue of $\hat{A}$. Then by a direct computation, we have

$$
\sum_i \lambda_i^4 = \sum_i \mu_i^4 + 12H^4 - 6H^2 |\hat{A}|^2 + 4H \sum_i \lambda_i^3
$$

Next we set $\mu_i = \lambda_i - H$ which are the eigenvalue of $\hat{A}$. Then by a direct computation, we have

$$
\sum_i \lambda_i^4 = \sum_i \mu_i^4 - 12H^4 - 6H^2 |\hat{A}|^2 + 4H \sum_i \lambda_i^3,
$$

and

$$
\sum_i \lambda_i^3 = \sum_i \mu_i^3 + 4H^3 + 3H|\hat{A}|^2.
$$

Combined (5), (6), (7) and (8) we obtain

$$
Q = \frac{1}{12} \left( \frac{\sigma^2}{4} + 6\sigma H^2 + 36H^4 + |\hat{A}|^4 - 3 \sum_i \mu_i^4 + 6 \sum_i \mu_i^2 \left( a_{ii} - \frac{\sigma}{3} \right) 
- 12H \sum_i \mu_i a_{ii} - 18H^2 |\hat{A}|^2 + 12H \sum_i \mu_i^3 - 3|\hat{A}|^2 \right).
$$

We will divide our proof of the Claim into two main cases. We will see that the locally conformally flatness will play a key role in the estimate of $Q$.

(i) At the point $p$, $|\hat{A}|^2(p) \leq 12 + 24H^2(p)$.

By the Gauss equation,

$$
S = \sigma + 12H^2 - |\hat{A}|^2,
$$

where $\sigma$ is defined in (5). Since $0 \leq \sigma \leq 12$, on one hand

$$
S = \sigma + 12H^2 - |\hat{A}|^2 \leq 12 + 12H^2.
$$

On the other hand,

$$
S = \sigma + 12H^2 - |\hat{A}|^2 \geq 12H^2 - (12 + 24H^2) = -12(H^2 + 1).
$$

The above two inequalities yield $S^2 \leq 144(1 + H^2)^2$. Therefore,

$$
Q = \frac{S^2}{12} - \frac{|\text{Ric}|^2}{4} = \frac{S^2}{48} - \frac{|E|^2}{4} \leq \frac{S^2}{48} \leq 3(H^2 + 1)^2,
$$

where $E$ is the traceless part of the Ricci tensor, namely the Einstein tensor.

(ii) At the point $p$, $|\hat{A}|^2(p) \geq 12 + 24H^2(p)$.

The proof of this case is more difficult than case (i). To prove the claim, we need to estimate $Q$ by using the equality (9).

First note that for a fixed $i$, the term $a_{ii} - \frac{\sigma}{3}$ is bounded above by 1. We take $i = 1$ for example:

$$
a_{11} - \frac{\sigma}{3} = \sum_k \mathcal{R}_{1kk} - \frac{\sigma}{3} = \sum_k \mathcal{R}_{1kk} - \frac{1}{3} \sum_{ij} \mathcal{R}_{ijij}
= \frac{1}{3} \left( \mathcal{R}_{1212} + \mathcal{R}_{1313} + \mathcal{R}_{1414} \right) - \frac{2}{3} \left( \mathcal{R}_{2323} + \mathcal{R}_{2424} + \mathcal{R}_{3434} \right) \leq 1
$$

where we have used the curvature condition that $0 \leq \mathcal{R}_{ijij} \leq 1$ for all $i \neq j$. 

By a direct computation, we have

\[
\sum_i \mu_i^4 = \sum_{i=1}^3 \mu_i^4 + \left( \sum_{i=1}^3 \mu_i \right)^4 \\
= \frac{1}{2} \left( \sum_{i=1}^3 \mu_i^2 + \left( \sum_{i=1}^3 \mu_i \right)^2 \right)^2 + 4 \mu_1 \mu_2 \mu_3 \left( \sum_{i=1}^3 \mu_i \right) \\
= \frac{1}{2} |\hat{A}|^4 - 4 \prod_i \mu_i := \frac{1}{2} |\hat{A}|^4 - 4 \mathcal{K},
\]

where \( \mathcal{K} = \prod_i \mu_i \) is the Gauss-Kronecker curvature of \( \hat{A} \).

Observe that \( \sum_i \mu_i = 0 \), we get

\[
-12H \sum_i \mu_i a_{ii} - 3 |\hat{a}|^2 \\
= -12H \sum_i \mu_i \hat{a}_{ii} - 3 |\hat{a}|^2 \\
= -3 \sum_i \left( \hat{a}_{ii}^2 - 4H \mu_i \hat{a}_{ii} + 4H^2 \mu_i^2 \right) + 12H^2 |\hat{A}|^2 - 3 \sum_{i \neq j} \hat{a}_{ij}^2 \\
\leq 12H^2 |\hat{A}|^2.
\]

Combined (9), (10), (11), (12) and the fact that \( 0 \leq \sigma \leq 12 \), we have

\[
Q \leq 3 \left( H^2 \right)^2 \\
+ \frac{1}{12} \left( -\frac{1}{2} |\hat{A}|^4 + 12H \sum_i \mu_i^3 + 6(1 - H^2) |\hat{A}|^2 + 12 \mathcal{K} \right).
\]

Next we will take the Weyl tensor into consideration. The Weyl tensor defined in a coordinate chart is given by (see e.g. [1], p117)

\[
W_{ijkl} = R_{ijkl} - \frac{1}{2} \left( R_{ik} g_{jl} - R_{il} g_{jk} + R_{jl} g_{ik} - R_{jk} g_{il} \right) \\
+ \frac{S}{6} \left( g_{jk} g_{il} - g_{jl} g_{ik} \right),
\]

where \( R_{ij} = \sum_k R_{ikjk} \) is the Ricci tensor. Therefore, when \( i \neq j \), we have,

\[
W_{ijij} = R_{ijij} - \frac{1}{2} \left( R_{ii} + R_{jj} \right) + \frac{S}{6}.
\]

Now we fix \( i = 1 \) and \( j = 2 \) for example, and get

\[
\frac{S}{6} - W_{1212} \\
= \frac{1}{2} (R_{11} + R_{22}) - R_{1212} \\
= \frac{1}{2} (R_{1313} + R_{1414} + R_{2323} + R_{2424}) \\
= \frac{1}{2} (R_{1212} + R_{1313} + R_{1414} + R_{2323} + R_{2424} + R_{3434}) - \frac{1}{2} (R_{1212} + R_{3434}) \\
= \frac{S}{4} - \frac{1}{2} \left( R_{1212} + R_{3434} \right).
\]
As a consequence, for \( \{i, j, k, l\} = \{1, 2, 3, 4\} \), we obtain,

\[
\frac{S}{6} = R_{ijij} + R_{klkl} - 2W_{ijij}
\]

\[
= R_{ijij} + R_{klkl} - 2W_{ijij} + \lambda_i \lambda_j + \lambda_k \lambda_l
\]

\[
= R_{ijij} + R_{klkl} - 2W_{ijij} + 2H^2
\]

\[
+ (\lambda_i - H)(\lambda_j - H) + (\lambda_k - H)(\lambda_l - H).
\]

\[
= R_{ijij} + R_{klkl} - 2W_{ijij} + 2H^2 + \mu_i \mu_j + \mu_k \mu_l.
\]

Note that the above formula has no summation on \( i, j, k, l \).

In what follows, without lose of generality, we assume, at the point \( p \),

\[
\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4.
\]

Next we will split our proof of case (ii) into three parts according to the values of \( K \) (defined in (11)) and \( \mu_i \).

(a) \( K(p) \geq 0, \mu_1 \geq \mu_2 \geq 0 \geq \mu_3 \geq \mu_4 \).

Let \( i = 1 \) and \( j = 2 \) in (15), and since locally conformally flatness implies \( W \equiv 0 \), we get

\[
\frac{S}{6} = R_{ijij} + R_{klkl} + 2H^2 + \mu_1 \mu_2 + \mu_3 \mu_4 \geq 2H^2.
\]

Consequently, \( S \geq 12H^2 \geq -12(1 + H^2) \). The remain proof of this part is similar with case (i).

(b) \( K(p) < 0 \) and \( \mu_1 \geq \mu_2 \geq \mu_3 > 0 > \mu_4 \).

In this part, a direct computation gives

\[
\sum_i \mu^3_i = \sum_{i=1}^{3} \mu_i - \left( \sum_{i=1}^{3} \mu_i \right)^3 \leq 0.
\]

Without loss of generality, we assume \( H(p) \geq 0 \) (otherwise the term "12H \sum \lambda_i^3" in (16) will be nonnegative, this case can be dealt with a similarly method as next part (c)). Therefore, combined (13), (16) and the assumption \( K \leq 0 \), we obtain

\[
Q \leq 3(1 + H^2)^2 - \frac{1}{24} |\hat{A}|^2 \left( |\hat{A}|^2 - 12(1 - H^2) \right).
\]

Note that in case (ii)

\[
|\hat{A}|^2 \geq 12 + 24H^2 \geq 12(1 - H^2).
\]

Thus the second term in the right hand side of (17) is nonpositive, and consequently we have \( Q \leq 3(1 + H^2)^2 \).

(c) \( K(p) < 0 \) and \( \mu_1 > 0 \geq \mu_2 \geq \mu_3 \geq \mu_4 \).

In this part, inequality (13) is not enough for our estimate, and we will go back into the equality (9) and estimate term by term.

Firstly, \( \sum_i \mu^3_i \geq 0 \), and we will use the following inequality (see [9, Lemma 1])

\[
\sum_i \mu^3_i \leq \frac{1}{\sqrt{3}} |\hat{A}|^3.
\]
Secondly, for the term $6 \sum_i \mu_i^2 (a_{ii} - \frac{\sigma}{3})$, under the assumption of this part, we will use a more accurate (than (10)) estimate as follows.

\begin{align*}
3 \sum_i \mu_i^2 \left( a_{ii} - \frac{\sigma}{3} \right) \\
= \mu_1^2 \left( R_{1212} + R_{1313} + R_{1414} - 2 \left( R_{2323} + R_{2424} + R_{3434} \right) \right) \\
+ \mu_2^2 \left( R_{2121} + R_{2323} + R_{2424} - 2 \left( R_{1313} + R_{1414} + R_{3434} \right) \right) \\
+ \mu_3^2 \left( R_{3131} + R_{3232} + R_{3434} - 2 \left( R_{1212} + R_{1414} + R_{2424} \right) \right) \\
+ \mu_4^2 \left( R_{4141} + R_{4242} + R_{4343} - 2 \left( R_{1212} + R_{1313} + R_{2323} \right) \right) \\
= (\mu_1^2 + \mu_2^2 - 2(\mu_3^2 + \mu_4^2))R_{1212} + (\mu_1^2 + \mu_3^2 - 2(\mu_2^2 + \mu_4^2))R_{1313} \\
+ (\mu_2^2 + \mu_3^2 - 2(\mu_1^2 + \mu_4^2))R_{2323} + (\mu_2^2 + \mu_3^2 - 2(\mu_1^2 + \mu_4^2))R_{2424} \\
+ (\mu_3^2 + \mu_4^2 - 2(\mu_1^2 + \mu_2^2))R_{3434} \\
\leq (\mu_1^2 + \mu_2^2 - 2(\mu_3^2 + \mu_4^2))R_{1212} + (\mu_1^2 + \mu_3^2 - 2(\mu_2^2 + \mu_4^2))R_{1313} \\
+ (\mu_2^2 + \mu_3^2 - 2(\mu_1^2 + \mu_4^2))R_{2323} + (\mu_2^2 + \mu_3^2 - 2(\mu_1^2 + \mu_4^2))R_{2424} \\
+ (\mu_3^2 + \mu_4^2 - 2(\mu_1^2 + \mu_2^2))R_{3434} \\
\leq -2\mu_1\mu_2 \leq \frac{3}{2} |\bar{A}|^2,
\end{align*}

where we have used the facts that

\[ \mu_1 > 0 > \mu_2 \geq \mu_3 \geq \mu_4, \quad \sum_i \mu_i = 0, \]

and the inequality

\[ |\bar{A}|^2 = \sum_i \mu_i^2 \geq \mu_1^2 + \frac{(\mu_2 + \mu_3 + \mu_4)^2}{3} = \frac{4}{3} \bar{A}^2. \]

Combined (9), (11), (12), (13), (19) and the fact that $0 \leq \sigma \leq 12$, we obtain

\begin{align*}
Q \leq 3 \left( 1 + H^2 \right)^2 \\
\quad + \frac{1}{12} \left( -\frac{1}{2} |\bar{A}|^4 + 4\sqrt{3}|H||\bar{A}|^3 + 3(1 - 2H^2)|\bar{A}|^2 \right) \\
:= 3 \left( 1 + H^2 \right)^2 + F(|\bar{A}|),
\end{align*}

where

\begin{align*}
F(|\bar{A}|) = \frac{1}{12} \left( -\frac{1}{2} |\bar{A}|^4 + 4\sqrt{3}|H||\bar{A}|^3 + 3(1 - 2H^2)|\bar{A}|^2 \right) \\
= \frac{1}{12} \left( -\frac{1}{2} |\bar{A}|^4 - 6|\bar{A}|^2 + |H|(4\sqrt{3}|\bar{A}|^3 - 6|H||\bar{A}|^2) \right).
\end{align*}

It is easy to see $F(x)$ attains its maximum at $x_0 = 3\sqrt{3}|H| + \sqrt{3 + 21H^2}$ and decreasing when $x \geq x_0$. Keep in mind that this part is one of the three parts of case (ii), which assumes that

\[ |\bar{A}|^2 \geq 12 + 24H^2. \]
Therefore, if \( x_0 \leq \sqrt{12 + 24H^2} \), we have
\[
F(|\tilde{A}|) \leq F(\sqrt{12 + 24H^2})
\]
\[
= -12H^2(2H^2 + 1) + |H| \left( \frac{\sqrt{3}}{3}x^3 - \frac{1}{2}|H|x^2 \right) \bigg|_{x=\sqrt{12 + 24H^2}}
\]
\[
\leq |H|f_1(|H|),
\]
where
\[
f_1(|H|) = \left( \frac{\sqrt{3}}{3}x^3 - \frac{1}{2}|H|x^2 \right) \bigg|_{x=\sqrt{12 + 24H^2}} \geq 0.
\]
If \( x_0 \geq \sqrt{12 + 24H^2} \), we obtain
\[
F(|\tilde{A}|) \leq F(x_0)
\]
\[
= \frac{1}{12} \left( -\frac{1}{2}(x^4 - 6x^2) \big|_{x=x_0} + |H|(4\sqrt{3}x^3 - 6|H|x^2) \big|_{x=x_0} \right)
\]
\[
\leq \frac{1}{12} \left( -\frac{1}{2}(x^4 - 6x^2) \big|_{x=\sqrt{12 + 24H^2}} \right) + |H|f_2(|H|),
\]
\[
\leq -12H^2(2H^2 + 1) + |H|f_2(|H|)
\]
\[
\leq |H|f_2(|H|),
\]
where
\[
f_2(|H|) = \left( \frac{\sqrt{3}}{3}x^3 - \frac{1}{2}|H|x^2 \right) \bigg|_{x=x_0} \geq 0.
\]
Combined (20), (21), (22) and (23), we get
\[
Q \leq 3(1 + H^2)^2 + 3|H|f(|H|),
\]
where \( f(|H|) \) is a function of \(|H|\) defined by
\[
3f(|H|) := \begin{cases} 
  f_1(|H|), & x_0 \leq \sqrt{12 + 24H^2}, \\
  f_2(|H|), & x_0 \geq \sqrt{12 + 24H^2}.
\end{cases}
\]
To sum up the above two cases, we have proved the Claim, and inequality (3) follows immediately.

Next we will show if \(|H|\) is small, inequality (3) holds. Check all the cases in the proof of the Claim, we find inequality (3) holds except for the case (ii) (c). Thus, it is enough to show, if \(|H|\) is small, inequality (3) holds in the case (ii) (c). By (20), it is sufficient to show \( F(|\tilde{A}|) \leq 0 \) when \(|H|\) is small. Observe that \( F(|\tilde{A}|) \) can be decomposed as
\[
F(|\tilde{A}|) = -\frac{|\tilde{A}|^2}{24} \left( |\tilde{A}| - \eta_1 \right) \left( |\tilde{A}| - \eta_2 \right),
\]
where \( \eta_1 = 4\sqrt{3}|H| - \sqrt{6 + 36H^2}, \quad \eta_2 = 4\sqrt{3}|H| + \sqrt{6 + 36H^2} \). Remember that in case (ii), \( |\tilde{A}| \geq \sqrt{12 + 24H^2} \). It is easy to see, if \(|H|\) is small, say \(|H| \leq \varepsilon_0 \) for some constant \( \varepsilon_0 \),
\[
|\tilde{A}| \geq \sqrt{12 + 24H^2} \geq \eta_2 > \eta_1,
\]
which implies \( F(|\tilde{A}|) \leq 0 \).

Check the above arguments step by step, we find the equality holds in (3) if and only if
\[
\sigma = \sum_{i,j} R_{ijj} = 12, \quad \tilde{A} \equiv 0,
\]
which implies \( \Sigma \) is totally umbilic and
\[
\mathcal{R}_{iijj} = 1, \quad \text{for all } i \neq j.
\]
Therefore, by the Gauss equation, we get, for all \( i \neq j \),
\[
R_{iijj} = \mathcal{R}_{iijj} + \lambda_i \lambda_j = 1 + (\mu_i + H)(\mu_j + H) = 1 + H^2,
\]
which means the sectional curvature of \( \Sigma \) at one point \( p \) is the same for all tangent plane \( \pi \in T_p \Sigma \). By Schur’s lemma, \( \text{sec}_{\Sigma} \) is constant. Hence, by (27), \( H \) is constant and \( \text{sec}_{\Sigma} \equiv 1 + H^2 \). Therefore, \( \Sigma \) is isometric to \( S^4(\frac{1}{1+H^2}) \).

\[\square\]

**Remark.** The condition \( |H| \leq \varepsilon_0 \) is just a technical condition. The constant \( \varepsilon_0 \) can be taken to be \( \sqrt{\frac{368\sqrt{3} - 598}{46}} \). But this is not the best number. Actually, after a long calculation similar as (19), we can get a better estimate than (12) and finally improve \( \varepsilon_0 \). We believe the condition \( |H| \leq \varepsilon_0 \) is not necessary for inequality (3). Actually, in next section, we study a special case when \( M \) is rotationally symmetric and show that inequality (3) holds for all \( H \).

### 3. A special case

In this section, we will deal with a special case, the ambient manifold is rotationally symmetric, i.e., \( M = \mathbb{R} \times_{\phi} S^n \) with the metric
\[
g = dt^2 + q^2(t)ds_n^2,
\]
where \( q(t) \) is a smooth positive function, and \( ds_n^2 \) is the standard metric of \( S^n \). Denote by \( \partial_t \) the unit vector in the \( \mathbb{R} \) direction, and assume \( X,Y \) are two vectors tangent to \( S^n \), then the curvature tensor is given by (see [22 section 4.2.3])
\[
\mathcal{R}(X \wedge \partial_t) = -\frac{\phi}{q} X \wedge \partial_t, \quad \mathcal{R}(X \wedge Y) = \frac{1 - \frac{q^2}{q^2}}{q^2} X \wedge Y.
\]
For simplicity we write
\[
\kappa_1 := -\frac{\phi}{q}, \quad \kappa_2 := \frac{1 - \frac{q^2}{q^2}}{q^2}.
\]
Let \( \Sigma \) be a hypersurfaces in \( M \), \( T \) be the tangential (with \( \Sigma \)) part of \( \partial_t, e_1, \ldots, e_n \) be the local orthonormal frame on \( \Sigma \). Write \( T_i = g(T,e_i) \). Decomposed each \( e_i \) into two parts
\[
e_i = e_i' + g(e_i, \partial_t) = e_i' + T_i,
\]
where \( e_i' \) tangent to \( S^n \). A direct computation, by using (29), (31) and the multilinearity of the curvature tensor, gives
\[
\mathcal{R}_{ijkl} = \kappa_2 (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (\kappa_1 - \kappa_2) \left( T_i T_k \delta_{jl} + T_j T_l \delta_{ik} - T_i T_l \delta_{jk} - T_j T_k \delta_{il} \right).
\]
Therefore, for \( i \neq j \), we have
\[
R_{iijj} = \mathcal{R}_{iijj} + \lambda_i \lambda_j = \kappa_2 + (\kappa_1 - \kappa_2) \left( T_i^2 + T_j^2 \right) + \lambda_i \lambda_j.
\]

We need the following lemma, which was first proved by Cartan (we appreciate Professor Marcos Dajczer pointing this fact out to us). For completeness, we give a direct proof here.

**Lemma 3.1.** Let \( M \) be an \( n+1 \) (\( n \geq 4 \)) dimensional rotationally symmetric Riemannian manifold with metric (28), and \( \Sigma \) be a hypersurface in \( M \). Then \( \Sigma \) is locally conformally flat if and only if at each point \( p \in \Sigma \), there are at most two distinct principal curvatures, one of them has multiplicity \( n - 1 \).
Proof. We will adopt the notations in section 2. In this proof, \(i, j, k, l\) will range from 1 to \(n\). By using the Weyl tensor formula ([1, p117]), we have

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{ij} + R_{ij}) + \frac{5}{(n-1)(n-2)} \\
= \kappa_2 + (\kappa_1 - \kappa_2) \left( T_{i}^2 + T_{j}^2 \right) + \lambda_i \lambda_j \\
- \frac{1}{n-2} \sum_{k \neq i} \left( \kappa_2 + (\kappa_1 - \kappa_2) \left( T_{i}^2 + T_{j}^2 \right) + \lambda_i \lambda_k \right) \\
- \frac{1}{n-2} \sum_{k \neq j} \left( \kappa_2 + (\kappa_1 - \kappa_2) \left( T_{i}^2 + T_{j}^2 \right) + \lambda_j \lambda_k \right) \\
+ \sum_{k \neq l} \left( \kappa_2 + (\kappa_1 - \kappa_2) \left( T_{i}^2 + T_{j}^2 \right) + n^2 H^2 - |A|^2 \right) \\
= \lambda_i \lambda_j - \frac{1}{n-2} \left( \sum_{k \neq i} \lambda_i \lambda_k + \sum_{k \neq j} \lambda_j \lambda_k \right) + \frac{n^2 H^2 - |A|^2}{(n-1)(n-2)}.
\]

Using the relations
\[
\lambda_k = \mu_k + H, \quad |A|^2 = |\hat{A}|^2 + n H^2,
\]
we substitute \(\mu_k\) for \(\lambda_k\) in the above equality and obtain

\[
W_{ijkl} = \frac{(\mu_i + \mu_j)^2 + (n - 4) \mu_i \mu_j}{n-2} - \frac{|\hat{A}|^2}{(n-1)(n-2)}.
\]

Therefore, \(\Sigma\) is locally conformally flat \(\iff W \equiv 0\)

(at each point) \(\iff (\mu_i + \mu_j)^2 + (n - 4) \mu_i \mu_j = \frac{|\hat{A}|^2}{(n-1)}\), \(\forall i \neq j\)

\[
(\sum_{i} \mu_i = 0, |\hat{A}|^2 = \sum_{i} \mu_i^2) \iff \mu_i = \mu_j \text{ or } \mu_i = -(n-1) \mu_j, \forall i \neq j.
\]

Thus \(\{\mu_1, \ldots, \mu_n\} = \{\mu, -(n-1) \mu\}\), and \(\mu\) has multiplicity \(n-1\). Consequently,

\[
\{\lambda_1, \ldots, \lambda_n\} = \{\mu + H, -(n-1) \mu + H\},
\]

and \(\mu + H\) has multiplicity \(n-1\).

With the aid of the above lemma, we can prove the following theorem.

**Theorem 3.2.** Let \(M\) be a rotationally symmetric Riemannian 5-manifold with \(0 \leq \kappa_1 \leq \kappa_2 \leq 1\) (\(\kappa_1\) and \(\kappa_2\) are defined in ([30]), and \(\Sigma\) be a locally conformally flat closed hypersurface embedded in \(M\) with mean curvature \(H\), then

\[
\int_{\Sigma} (1 + H^2)^2 \geq \frac{4\pi^2}{3} \chi(\Sigma).
\]

The equality holds if and only if \(H\) is constant, \(\Sigma\) is totally umbilic and isometric to \(S^4 \left( \frac{1}{1 + H^2} \right)\).

Proof. We will adopt the same notations as in the proof of Theorem [1,1]. By Lemma [3,1] we have \(|\mu_1| = 3|\mu_i|, i = 2, 3, 4\). Therefore,

\[
\sum_i \mu_i^4 = \frac{7}{12} |\hat{A}|^4.
\]
Direct computations by using (32) yield
\[ \sigma = 12\kappa_2 + 6(\kappa_1 - \kappa_2)|T|^2, \]
\[ a_{ii} = 3\kappa_2 + (\kappa_1 - \kappa_2)(2T_i^2 + |T|^2), \]
\[ |\hat{a}|^2 = 3(\kappa_1 - \kappa_2)^2|T|^4. \]

Insert the above equalities into (9), we obtain,
\[ Q = 3(\kappa_2 + H^2)^2 + \frac{1}{12} \left( -\frac{3}{4} |\hat{A}|^4 + 12H \sum_i \mu_i^3 - 6(\kappa_2 + 3H^2) |\hat{A}|^2 \right) \]
\[ + \left( \kappa_1 - \kappa_2 \right) \frac{2}{2} \left( 6 + |\hat{A}|^2 + 4H^2 \right) \left| T \right|^2 + 2 \sum_i T_i^2 (\mu_i - H)^2 \]
\[ \leq 3(\kappa_2 + H^2)^2 + \frac{1}{12} \left( -\frac{3}{4} |\hat{A}|^4 + 12H \sum_i \mu_i^3 - 6(\kappa_2 + 3H^2) |\hat{A}|^2 \right) \]
\[ \leq 3(\kappa_2 + H^2)^2 + \frac{1}{12} \left( -\frac{3}{4} |\hat{A}|^4 + 4\sqrt{3}|H| |\hat{A}|^3 - 6(\kappa_2 + 3H^2) |\hat{A}|^2 \right) \]
\[ \leq 3(\kappa_2 + H^2)^2 \leq 3(1 + H^2)^2. \]

The remain proof is similar as the proof of Theorem 1.1. □

**Remark.** The assumption "0 ≤ κ_1 ≤ κ_2 ≤ 1" is reasonable for many manifolds. For example,
- if we take ϕ(t) = sin(t), then M = S^5 and κ_1 = κ_2 ≡ 1;
- if we take ϕ(t) ≡ 1, then M = S^4 × R and 0 ≡ κ_1 < κ_2 ≡ 1.

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