The algebraic structure of certain theta constants

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Abstract  In this paper, we investigated the algebraic structure of certain theta constants with rational characteristics.

Key words theta functions; theta constants; rational characteristic; resultant

MSC(2010) 14K25

1 Introduction

In this paper, following Farkas and Kra [2], we introduce the theta function with characteristic \( \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] \in \mathbb{R}^2 \), which is defined by

\[
\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\zeta, \tau) = \theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (0) := \sum_{n \in \mathbb{Z}} \exp \left( 2\pi i \left( \frac{1}{2} \left( n + \frac{\epsilon}{2} \right)^2 \tau + \left( n + \frac{\epsilon}{2} \right) (\zeta + \frac{\epsilon'}{2}) \right) \right),
\]

which uniformly and absolutely converges on compact subsets of \( \mathbb{C} \times \mathbb{H}^2 \), where \( \mathbb{H}^2 \) is the upper half-plane. The theta constants are defined by

\[
\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] := \theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (0, \tau), \quad \theta' \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] := \frac{\partial}{\partial \zeta} \theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\zeta, \tau) \bigg|_{\zeta=0}.
\]

Farkas and Kra [2] treated the theta constants with rational characteristics, that is, the case where \( \epsilon \) and \( \epsilon' \) are both rational numbers, and derived a number of interesting theta constant identities.

One of the most famous theta constant identities is Jacobi’s quartic, which is given by

\[
\theta^4 \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \theta^4 \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + \theta^4 \left[ \begin{array}{c} 0 \\ 1 \end{array} \right].
\]

This identity suggests that at most two of \( \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] \), \( \theta \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \), \( \theta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \) are algebraically independent.
Based on the uniformization theory, Farkas and Kra [1] proved that for every $\tau \in \mathbb{H}^2$,
\[
\theta^3 \left[ \frac{1}{3} \right] + \theta^3 \left[ \frac{1}{5} \right] = \theta^3 \left[ \frac{1}{1} \right],
\]
and
\[
\exp \left( \frac{\pi i}{3} \right) \theta^3 \left[ \frac{1}{3} \right] + \exp \left( \frac{2\pi i}{3} \right) \theta^3 \left[ \frac{1}{5} \right] = \theta^3 \left[ \frac{1}{1} \right],
\]
which implies that at most two of $\theta \left[ \frac{1}{3} \right]$, $\theta \left[ \frac{1}{5} \right]$, and $\theta \left[ \frac{1}{1} \right]$ are algebraically independent. In the Appendix, we give elementary proof of equations (1.1) and (1.2).

In this paper, we investigated the algebraic structure of $\theta \left[ \frac{j}{15} \right]$, $\theta \left[ \frac{1}{j} \right]$, where $j = 1, 3, 5, 7, 9$. Our main theorem is as follows:

**Theorem 1.1.**

1. $\theta \left[ \frac{1}{5} \right] / \theta \left[ \frac{1}{3} \right]$ is given by a rational expression of four of $\theta \left[ \frac{1}{5} \right]$, where $k = 1, 3, 5, 7, 9$.

2. $\theta \left[ \frac{1}{5} \right] / \theta \left[ \frac{1}{3} \right]$ is given by a rational expression of four of $\theta \left[ \frac{3}{5} \right]$, where $k = 1, 3, 5, 7, 9$.

3. One of $\theta \left[ \frac{1}{5} \right]$, $(k = 1, 3, 5, 7, 9)$, is given by two rational expressions of the other four theta constants.

4. One of $\theta \left[ \frac{3}{5} \right]$, $(k = 1, 3, 5, 7, 9)$, is given by two rational expressions of the other four theta constants.

5. At most three of $\theta \left[ \frac{1}{5} \right]$, $(k = 1, 3, 5, 7, 9)$, are algebraically independent.

6. At most three of $\theta \left[ \frac{3}{5} \right]$, $(k = 1, 3, 5, 7, 9)$, are algebraically independent.

7. At most six of $\theta \left[ \frac{j}{15} \right]$, $\theta \left[ \frac{1}{j} \right]$, $(j = 1, 3, k = 1, 3, 5, 7, 9)$, are algebraically independent.

This paper is organized as follows. In Section 2, we review the properties of the theta functions. In Section 3, we prove Theorem 1.1 (1) and (2). For this purpose, we derive the theta constant identities by the classical method based on the consideration of linear algebra.
In Section 4, we prove Theorem 1.1 (3) and (4). For this purpose, considering the determinant structure, we obtain the theta constant identities. This method is based on the method of Matsuda [4].

In Section 5, we prove Theorem 1.1 (5) and (6). For this purpose, we use the theta constant identities of Section 4.

In Section 7, we prove Theorem 1.1 (7). For this purpose, we use the theta constant identities of Section 3.

In the Appendix, we prove equations (1.1) and (1.2). This method is the classical one based on the consideration of linear algebra.

Remark

By the identity (1.1), Farkas [3] showed that for $n \in \mathbb{N}_0$,

$$\sigma(3n + 2) = 3 \sum_{k=0}^{n} \delta(3k + 1)\delta(3(n - k) + 1),$$

where for $n \in \mathbb{N}$,

$$\sigma(n) = \sum_{d|n} d,$$

and

$$\delta(n) = d_{1,3}(n) - d_{2,3}(n).$$

We believe that the theta constant identities of this paper can be applied to number theory.

Acknowledgments

We are grateful to Professor H. Watanabe for his useful comments.

2 The properties of the theta functions

We first note that for $m, n \in \mathbb{Z}$,

$$\theta \left[ \frac{\epsilon}{\epsilon'} \right] (\zeta + n + m\tau, \tau) = \exp(2\pi i) \left[ \frac{n\epsilon - m\epsilon'}{2} - mz - \frac{m^2\tau}{2} \right] \theta \left[ \frac{\epsilon}{\epsilon'} \right] (\zeta, \tau), \quad (2.1)$$

and

$$\theta \left[ \frac{\epsilon + 2m}{\epsilon' + 2n} \right] (\zeta, \tau) = \exp(\pi i n) \theta \left[ \frac{\epsilon}{\epsilon'} \right] (\zeta, \tau). \quad (2.2)$$

Furthermore,

$$\theta \left[ \frac{-\epsilon}{-\epsilon'} \right] (\zeta, \tau) = \theta \left[ \frac{\epsilon}{\epsilon'} \right] (-\zeta, \tau) \quad \text{and} \quad \theta' \left[ \frac{-\epsilon}{-\epsilon'} \right] (\zeta, \tau) = -\theta' \left[ \frac{\epsilon}{\epsilon'} \right] (-\zeta, \tau).$$
For $m, n \in \mathbb{R}$, we see that

$$
\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\zeta + \frac{n + m\tau}{2}, \tau) = \exp(2\pi i) \left[ -\frac{m\zeta}{2} - \frac{m^2\tau}{8} - \frac{m(\epsilon' + n)}{4} \right] \theta \left[ \begin{array}{c} \epsilon + m \\ \epsilon' + n \end{array} \right] (\zeta, \tau). \tag{2.3}
$$

We note that $\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\zeta, \tau)$ has only one zero in the fundamental parallelogram, which is given by

$$
\zeta = \frac{1 - \epsilon}{2}\tau + \frac{1 - \epsilon'}{2}.
$$

All the theta functions have infinite product expansions, which are given by

$$
\theta \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] (\zeta, \tau) = \exp \left( \frac{\pi i\epsilon\epsilon'}{2} \right) x^{\frac{\epsilon^2}{\tau}} z^{\frac{\epsilon'}{\tau}} \prod_{n=1}^{\infty} (1 - x^{2n})(1 + e^{\pi i\epsilon' x^{2n-1+\epsilon}})(1 + e^{-\pi i\epsilon' x^{2n-1-\epsilon}}/z), \tag{2.4}
$$

where $x = \exp(\pi i \tau)$, $z = \exp(2\pi i \zeta)$.

Following Farkas and Kra [2], we last define $\mathcal{F}_N \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right]$ to be the set of the entire functions $f$ that satisfy the two functional equations,

$$
f(\zeta + 1) = \exp(\pi i \epsilon) f(\zeta),
$$

and

$$
f(\zeta + \tau) = \exp(-\pi i)[\epsilon' + 2N\zeta + N\tau] f(\zeta), \quad \zeta \in \mathbb{C}, \quad \tau \in \mathbb{H}^2,
$$

where $N$ is a positive integer and $\left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] \in \mathbb{R}^2$. This set of functions is called the space of $N$-th order $\theta$-functions with characteristic $\left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right]$. Note that

$$
\dim \mathcal{F}_N \left[ \begin{array}{c} \epsilon \\ \epsilon' \end{array} \right] = N.
$$

For its proof, see Farkas and Kra [2] pp.131].
3 Proof of Theorem 1.1 (1) and (2)

3.1 Proof of Theorem 1.1 (1)

Proposition 3.1. For every \((\zeta, \tau) \in \mathbb{C} \times \mathbb{H}^2\), we have

\[
\theta \left[ \frac{1}{3} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{3} \right] (\zeta, \tau) + \zeta^2 \theta \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) (\zeta, \tau)
\]

\[
+ \zeta_5^2 \theta \left[ \frac{1}{3} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{3} \right] (\zeta, \tau) - \zeta^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) = 0, \quad (3.1)
\]

\[
\theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) - \zeta^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau)
\]

\[
+ \zeta_5^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) + \zeta^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) = 0, \quad (3.2)
\]

\[
\theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) + \zeta^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau)
\]

\[
- \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) + \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) = 0, \quad (3.3)
\]

\[
\theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) - \zeta^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau)
\]

\[
+ \zeta_5^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) - \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) = 0. \quad (3.5)
\]

Proof. We treat equation (3.1). The others can be proved in the same way. For this purpose, we first note that \(\dim \mathcal{F}_3 \left[ \frac{3}{5} \right] = 3\) and

\[
\theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau), \ \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau)
\]

\[
\theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau), \ \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) \in \mathcal{F}_3 \left[ \frac{3}{5} \right].
\]
Therefore, there exist complex numbers, $x_1, x_2, x_3, x_4$, not all zero such that

$$x_1 \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) + x_2 \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) + x_3 \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) + x_4 \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) = 0.$$

By substituting

$$\zeta = \frac{2\tau \pm 2}{5}, \frac{2\tau \pm 1}{5},$$

we have the following system of equations:

$$x_2 \zeta_5^2 \theta \left[ \frac{1}{3/5} \right] + x_3 \theta \left[ \frac{1}{1/5} \right] = 0,$$

$$x_3 \theta \left[ \frac{1}{1/5} \right] - x_4 \zeta_5^2 \theta \left[ \frac{1}{3/5} \right] = 0,$$

$$x_1 \zeta_5^2 \theta \left[ \frac{1}{1/5} \right] + x_4 \theta \left[ \frac{1}{3/5} \right] = 0,$$

$$x_1 \zeta_5^2 \theta \left[ \frac{1}{1/5} \right] - x_2 \theta \left[ \frac{1}{3/5} \right] = 0.$$

Solving this system of equations, we obtain

$$(x_1, x_2, x_3, x_4) = \alpha \left( \theta \left[ \frac{1}{3/5} \right], \zeta_5^2 \theta \left[ \frac{1}{1/5} \right], -\zeta_5^4 \theta \left[ \frac{1}{3/5} \right], -\zeta_5^2 \theta \left[ \frac{1}{1/5} \right] \right),$$

for some nonzero complex number $\alpha \in \mathbb{C}^*$, which proves the proposition. \(\square\)
Theorem 3.2. For every $\tau \in \mathbb{H}^2$, we have
\[
\begin{align*}
\theta &\left[ \frac{1}{5} \right] (0, \tau) = \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) - \zeta_5^4 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau), \\
\theta &\left[ \frac{1}{3} \right] (0, \tau) = \zeta_5^2 \theta^2 &\left[ \frac{1}{3} \right] (0, \tau) \theta &\left[ \frac{1}{3} \right] (0, \tau) - \zeta_5^2 \theta^2 &\left[ \frac{1}{3} \right] (0, \tau) \theta &\left[ \frac{1}{3} \right] (0, \tau), \\
\theta &\left[ \frac{1}{5} \right] (0, \tau) = \zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) + \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau), \\
\theta &\left[ \frac{1}{3} \right] (0, \tau) = \theta^2 &\left[ \frac{1}{3} \right] (0, \tau) \theta &\left[ \frac{1}{3} \right] (0, \tau) + \zeta_5^2 \theta^2 &\left[ \frac{1}{3} \right] (0, \tau) \theta &\left[ \frac{1}{3} \right] (0, \tau), \\
\theta &\left[ \frac{1}{5} \right] (0, \tau) = -\theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) + \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau), \\
\theta &\left[ \frac{1}{3} \right] (0, \tau) = \zeta_5^2 \theta^2 &\left[ \frac{1}{3} \right] (0, \tau) \theta &\left[ \frac{1}{3} \right] (0, \tau) + \theta^2 &\left[ \frac{1}{3} \right] (0, \tau) \theta &\left[ \frac{1}{3} \right] (0, \tau), \\
\theta &\left[ \frac{1}{5} \right] (0, \tau) = \zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) + \zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau).
\end{align*}
\]  

(3.6) (3.7) (3.8) (3.9) (3.10)

Proof. We deal with equation (3.6). The others can be proved in the same way. For this purpose, setting $\zeta = 0$ in equation (3.11), we have
\[
\begin{align*}
\theta &\left[ \frac{1}{5} \right] (0, \tau) \left\{ \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) - \zeta_5^4 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) \right\} \\
= &\theta &\left[ \frac{1}{5} \right] (0, \tau) \left\{ \zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) - \zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) \right\}.
\end{align*}
\]
Suppose that for every $\tau \in \mathbb{H}^2$,
\[
\zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) - \zeta_5^2 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) \equiv 0,
\]
which implies that
\[
\theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) - \zeta_5^4 \theta^2 &\left[ \frac{1}{5} \right] (0, \tau) \theta &\left[ \frac{1}{5} \right] (0, \tau) \equiv 0.
\]
Thus, we have
\[ \zeta_5^2 - \left( \frac{\theta \left[ \frac{1}{2} \frac{3}{5} \right]}{\theta \left[ \frac{1}{2} \frac{3}{5} \right]} (0, \tau) \right)^5 \equiv 0, \]
which implies that \( \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, \tau) \) is a constant. Taking the limit \( \tau \to i\infty \), we obtain
\[ \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, 100\tau) = \exp \left( \frac{8\pi i}{50} \right) \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, 100\tau), \]
where we used the Jacobi triple product identity (2.4). Since
\[ \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, 100\tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi i \cdot \frac{1}{10} n) \frac{9}{10} x^{10n+1} , \]
comparing the coefficients of \( x^{121} \), we obtain \( \zeta_5^3 = 1 \), which is impossible.
Thus, it follows that
\[ \zeta_5^2 \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, \tau) \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, \tau) - \zeta_5^2 \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, \tau) \theta \left[ \frac{1}{2} \frac{3}{5} \right] (0, \tau) \neq 0, \]
which proves the theorem. \( \square \)

3.2 Proof of Theorem 1.1 (2)

In the same way, we can prove the following proposition and theorem:

Proposition 3.3. For every \((\zeta, \tau) \in \mathbb{C} \times \mathbb{H}^2\), we have
\[ \theta \left[ \frac{1}{3} \frac{2}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) + \zeta_5 \theta \left[ \frac{1}{3} \frac{1}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) = 0, \]
\[ -\zeta_5^2 \theta \left[ \frac{1}{3} \frac{2}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) + \zeta_5 \theta \left[ \frac{1}{3} \frac{1}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) = 0, \]
\[ \theta \left[ \frac{1}{5} \frac{1}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) - \zeta_5 \theta \left[ \frac{1}{5} \frac{1}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) = 0, \]
\[ + \zeta_5 \theta \left[ \frac{1}{5} \frac{1}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) - \theta \left[ \frac{1}{5} \frac{1}{5} \right] \theta^2 \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \frac{3}{5} \right] (\zeta, \tau) = 0, \]
Theorem 3.4. For every $\tau \in \mathbb{H}^2$, we have

\[
\begin{align*}
\theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] & \theta^2 \left[ \begin{array}{c}
3/10 \\
3/5 
\end{array} \right] (\zeta, \tau) \theta \left[ \begin{array}{c}
3/4 \\
3/5 
\end{array} \right] \theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] (\zeta, \tau) - \zeta_5 \theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] \theta^2 \left[ \begin{array}{c}
3/10 \\
3/5 
\end{array} \right] (\zeta, \tau) \theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] (\zeta, \tau) = 0, \quad (3.13)
\end{align*}
\]

\[
\begin{align*}
\theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] & \theta^2 \left[ \begin{array}{c}
3/10 \\
3/5 
\end{array} \right] (\zeta, \tau) \theta \left[ \begin{array}{c}
3/4 \\
3/5 
\end{array} \right] (\zeta, \tau) + \zeta_5 \theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] \theta^2 \left[ \begin{array}{c}
3/10 \\
3/5 
\end{array} \right] (\zeta, \tau) \theta \left[ \begin{array}{c}
3/4 \\
3/5 
\end{array} \right] (\zeta, \tau) = 0, \quad (3.14)
\end{align*}
\]

\[
\begin{align*}
\theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] & \theta^2 \left[ \begin{array}{c}
3/10 \\
3/5 
\end{array} \right] (\zeta, \tau) \theta \left[ \begin{array}{c}
3/4 \\
3/5 
\end{array} \right] (\zeta, \tau) - \zeta_5 \theta \left[ \begin{array}{c}
1/3 \\
1/5 
\end{array} \right] \theta^2 \left[ \begin{array}{c}
3/10 \\
3/5 
\end{array} \right] (\zeta, \tau) \theta \left[ \begin{array}{c}
3/4 \\
3/5 
\end{array} \right] (\zeta, \tau) = 0. \quad (3.15)
\end{align*}
\]
4 Proof of Theorem 1.1 (3) and (4)

4.1 Review of the resultant

Set

\[ f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n, \]
\[ g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m. \]

We then define the resultant \( R(f, g) \) by the following \((m + n) \times (m + n)\) determinant:

\[
R(f, g) = \begin{vmatrix}
  a_0 & a_1 & \cdots & a_n & 0 \\
  a_0 & a_1 & \cdots & a_n & 0 \\
  0 & \ddots & \ddots & \ddots & \ddots \\
  b_0 & b_1 & \cdots & b_m & 0 \\
  b_0 & b_1 & \cdots & b_m & 0 \\
  0 & \ddots & \ddots & \ddots & \ddots \\
  b_0 & b_1 & \cdots & b_m & 0 \\
\end{vmatrix}.
\]

\( R(f, g) \) shows us the condition for \( f(x) \) and \( g(x) \) to have a common zero.

**Proposition 4.1.** \( f(x) \) and \( g(x) \) both have a common zero if and only if \( R(f, g) = 0 \).

Especially, when

\[ f(x) = a_0 x^2 + a_1 x + a_2, \text{ and } g(x) = b_0 x^2 + b_1 x + b_2, \]

the resultant is given by

\[
R(f, g) = \begin{vmatrix}
  a_0 & a_1 & a_2 & 0 \\
  a_0 & a_1 & a_2 & 0 \\
  0 & a_0 & a_1 & a_2 \\
  b_0 & b_1 & b_2 & 0 \\
  b_0 & b_1 & b_2 & 0 \\
\end{vmatrix} = (a_0 b_2 - a_2 b_0)^2 - (a_0 b_1 - a_1 b_0)(a_1 b_2 - a_2 b_1).
\]

4.2 Proof of Theorem 1.1 (3)

**Proposition 4.2.** For every \((\zeta, \tau) \in \mathbb{C} \times \mathbb{H}^2\), we have

\[
\theta^2 \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{5} \right] (\zeta, \tau) \theta \left[ \frac{1}{5} \right] (\zeta, \tau) - \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{1}{3} \right] (\zeta, \tau) + \theta \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] (\zeta, \tau) = 0,
\]

(4.1)
Proof. Equation (4.1) was proved by Matsuda [4], who proved it in the same way as Proposition 3.1. The others can be also proved in the same way.
Theorem 4.3. For every $\tau \in \mathbb{H}^2$, we have

$$
\theta \left[ \frac{1}{\tau} \right] = \left\{ \begin{array}{l}
\theta \left[ \frac{1}{\tau} \right] \theta^3 \left[ \frac{1}{\tau} \right] - \theta^3 \left[ \frac{1}{\tau} \right] \theta \left[ \frac{1}{\tau} \right] \theta^2 \left[ \frac{1}{\tau} \right] \theta \left[ \frac{1}{\tau} \right] + \zeta \theta \left[ \frac{1}{\tau} \right] \theta^2 \left[ \frac{1}{\tau} \right] \right\}, \\
(4.6)
\end{array} \right.
$$
\[
\theta \left[ \frac{1}{5} \right] = \left\{ \theta \left[ \frac{1}{3} \right] \theta^2 \left[ \frac{1}{5} \right] \left[ z, \tau \right] - \theta^2 \left[ \frac{1}{5} \right] \left[ z, \tau \right] - \theta \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{5} \right] \left[ z, \tau \right] + \zeta_5^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{3} \right] \right\} \theta^2 \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{5} \right] \left[ z, \tau \right].
\] (4.15)

**Proof.** By equations (4.1) and (4.3), we prove equation (4.6). The others can be proved by two equations of Proposition 4.2 in the same way.

Equations (4.1) and (4.3) show that for arbitrary complex numbers \( z, w \in \mathbb{C} \),

\[
\theta^2 \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{5} \right] \left[ z, \tau \right] - \theta^2 \left[ \frac{1}{5} \right] \left[ z, \tau \right] + \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{3} \right] \left[ z, \tau \right] = 0,
\]

and

\[
\theta^2 \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{5} \right] \left[ w, \tau \right] - \zeta_5^2 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{3} \right] \left[ w, \tau \right] = 0,
\]

which implies that \( \theta \left[ \frac{1}{3} \right] / \theta \left[ \frac{1}{5} \right] \) satisfies the following two algebraic equations:

\[
f(x) = x^2 \theta \left[ \frac{1}{3} \right] (z) \theta \left[ \frac{1}{5} \right] (z) - x \theta^2 \left[ \frac{1}{5} \right] (z) - \theta \left[ \frac{1}{3} \right] ((z) \theta \left[ \frac{1}{5} \right] (z) = 0,
\]

\[
g(x) = x^2 \zeta_5^2 \theta \left[ \frac{1}{5} \right] (w) \theta \left[ \frac{1}{3} \right] (w) - x \zeta_5^2 \theta^2 \left[ \frac{1}{3} \right] \left[ w, \tau \right] \left[ \frac{1}{5} \right] (w) + \theta \left[ \frac{1}{5} \right] \left[ w, \tau \right] \left[ \frac{1}{3} \right] (w) = 0.
\]

Since \( f(x) \) and \( g(x) \) both have the same solution, it follows that

\[
R(f, g) = \begin{vmatrix}
\theta \left[ \frac{1}{3} \right] (z) & -\theta^2 \left[ \frac{1}{5} \right] (z) & -\theta \left[ \frac{1}{3} \right] (z) & 0 \\
0 & \theta \left[ \frac{1}{3} \right] (z) & -\theta \left[ \frac{1}{5} \right] (z) & -\theta \left[ \frac{1}{3} \right] (z) \\
\zeta_5^2 \theta \left[ \frac{1}{5} \right] (w) & -\zeta_5^2 \theta^2 \left[ \frac{1}{3} \right] (w) & \theta \left[ \frac{1}{5} \right] (w) & 0 \\
0 & \zeta_5^2 \theta \left[ \frac{1}{5} \right] (w) & -\zeta_5^2 \theta^2 \left[ \frac{1}{3} \right] (w) & \theta \left[ \frac{1}{3} \right] (w)
\end{vmatrix} = 0,
\]

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which shows that
\[
\left\{ \zeta_5^3 \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (w) \theta \left[ \frac{1}{5} \right] (w) + \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (w) \theta \left[ \frac{1}{5} \right] (w) \right\}^2
\]

\[
= \left\{ \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (w) - \theta^2 \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (w) \theta \left[ \frac{1}{5} \right] (w) \right\}
\]

\[
\times \left\{ \zeta_5^3 \theta^2 \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (w) \theta \left[ \frac{1}{5} \right] (w) + \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (z) \theta \left[ \frac{1}{5} \right] (w) \right\}.
\]

Setting \( z = w = 0 \), we have
\[
\theta \left[ \frac{1}{5} \right] \left\{ \zeta_5^3 \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] + \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] \right\}^2
\]

\[
= \left\{ \theta \left[ \frac{1}{5} \right] \theta^3 \left[ \frac{1}{5} \right] - \theta^3 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] \right\} \left\{ \zeta_5^3 \theta \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] + \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] \right\}.
\]

Let us show that
\[
\zeta_5^3 \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] + \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] \neq 0.
\]

For this purpose, suppose that for every \( \tau \in \mathbb{H} \),
\[
\zeta_5^3 \theta^2 \left[ \frac{1}{5} \right] (0, \tau) \theta \left[ \frac{1}{5} \right] (0, \tau) + \theta \left[ \frac{1}{5} \right] (0, \tau) \theta^2 \left[ \frac{1}{5} \right] (0, \tau) \equiv 0,
\]

which implies that
\[
\theta \left[ \frac{1}{5} \right] (0, \tau) \theta^3 \left[ \frac{1}{5} \right] (0, \tau) - \theta^3 \left[ \frac{1}{5} \right] (0, \tau) \theta \left[ \frac{1}{5} \right] (0, \tau) \equiv 0,
\]

or
\[
\zeta_5^3 \theta \left[ \frac{1}{5} \right] (0, \tau) \theta^2 \left[ \frac{1}{5} \right] (0, \tau) + \theta^2 \left[ \frac{1}{5} \right] (0, \tau) \theta \left[ \frac{1}{5} \right] (0, \tau) \equiv 0.
\]

Thus, it follows that
\[
\zeta_5 + \left( \frac{\theta \left[ \frac{1}{5} \right] (0, \tau)}{\theta \left[ \frac{1}{5} \right] (0, \tau)} \right)^5 \equiv 0,
\]
which implies that \( \theta \left[ \frac{1}{5} \right] (0, \tau) / \theta \left[ \frac{1}{5} \right] (0, \tau) \) is a constant. Taking the limit \( \tau \to i\infty \), we have

\[
\theta \left[ \frac{1}{5} \right] (0, 100\tau) = \exp \left( \frac{6\pi i}{50} \right) \theta \left[ \frac{1}{5} \right] (0, 100\tau),
\]

in which we used Jacobi's triple product identity \((2.4)\). Since

\[
\theta \left[ \frac{1}{5} \right] (0, 100\tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi i \left( \frac{n+1}{10} \right) \frac{9}{10} x) x^{(10n+1)^2}
\]

and

\[
\theta \left[ \frac{1}{5} \right] (0, 100\tau) = \sum_{n \in \mathbb{Z}} \exp(2\pi i \left( n + \frac{1}{10} \right) \frac{3}{10} x) x^{(10n+1)^2}, \quad x = \exp(\pi i \tau),
\]

comparing the coefficients of \( x^{121} \), we have \( \zeta_5^3 = 1 \), which is impossible.

Therefore, we have

\[
\theta \left[ \frac{1}{5} \right] = \left\{ \theta \left[ \frac{1}{5} \right] \theta^3 \left[ \frac{1}{5} \right] - \theta^3 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] \right\} \left\{ \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] + \zeta_5^3 \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] \right\}^2
\]

\[
\left\{ \zeta_5^3 \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] + \theta \left[ \frac{1}{5} \right] \theta^2 \left[ \frac{1}{5} \right] \right\}
\]

\[
\Box
\]

4.3 Proof of Theorem 1.1 (4)

In the same way, we can prove the following proposition and theorem:

**Proposition 4.4.** For every \((\zeta, \tau) \in \mathbb{C} \times \mathbb{H}^2\), we have

\[
\theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \right] (\zeta, \tau) = \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau) - \theta^2 \left[ \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau)
\]

\[
+ \theta \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau) = 0,
\]

\[
(4.16)
\]

\[
\zeta_5 \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \right] (\zeta, \tau) - \zeta_5 \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau) \theta \left[ \frac{3}{5} \right] (\zeta, \tau)
\]

\[
+ \theta \left[ \frac{1}{5} \right] \theta \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] (\zeta, \tau) = 0,
\]

\[
(4.17)
\]
Theorem 4.5. For every $\tau \in \mathbb{H}^2$, we have

\begin{align*}
\theta^2 \left[ \frac{1}{3} \right] \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) + \zeta_5 \theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) = 0, \\
\theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) + \zeta_5 \theta^2 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) = 0, \\
\theta^2 \left[ \frac{1}{5} \right] \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) - \theta^2 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \left( \zeta, \tau \right) = 0.
\end{align*}

(4.21)

(4.22)

(4.23)

(4.24)

(4.25)
Theorem 5.1.

Proof of Theorem 1.1 (5) and (6)

5.1 Proof of Theorem 1.1 (5)

Theorem 5.1. For every $\tau \in \mathbb{H}^2$, we have

\[
\theta \left[ \frac{4}{11} \right] = \left\{ -\zeta_5 \theta^2 \left[ \frac{2}{11} \right] \theta \left[ \frac{7}{11} \right] + \zeta_5 \theta \left[ \frac{4}{11} \right] \theta^2 \left[ \frac{4}{11} \right] \theta \left[ \frac{4}{11} \right] \right\}\left\{ -\zeta_5^2 \theta^3 \left[ \frac{4}{11} \right] \theta \left[ \frac{4}{11} \right] + \theta \left[ \frac{4}{11} \right] \theta^3 \left[ \frac{4}{11} \right] \theta \left[ \frac{4}{11} \right] \right\},
\]

(4.26)

\[
\theta \left[ \frac{3}{11} \right] = \left\{ \zeta_5 \theta \left[ \frac{3}{11} \right] \theta^2 \left[ \frac{3}{11} \right] + \theta \left[ \frac{3}{11} \right] \theta^2 \left[ \frac{3}{11} \right] \theta \left[ \frac{3}{11} \right] \right\}\left\{ -\theta^2 \left[ \frac{3}{11} \right] \theta \left[ \frac{1}{11} \right] + \theta \left[ \frac{3}{11} \right] \theta^2 \left[ \frac{3}{11} \right] \theta \left[ \frac{3}{11} \right] \right\},
\]

(4.27)

\[
\theta \left[ \frac{2}{11} \right] = \left\{ \theta^3 \left[ \frac{2}{11} \right] \theta \left[ \frac{3}{11} \right] + \zeta_5 \theta \left[ \frac{2}{11} \right] \theta \left[ \frac{3}{11} \right] \right\}\left\{ -\theta^3 \left[ \frac{2}{11} \right] \theta \left[ \frac{3}{11} \right] + \theta \left[ \frac{3}{11} \right] \theta^3 \left[ \frac{3}{11} \right] \theta \left[ \frac{3}{11} \right] \right\},
\]

(4.28)

\[
\theta \left[ \frac{1}{11} \right] = \left\{ \theta \left[ \frac{1}{11} \right] \theta^2 \left[ \frac{1}{11} \right] + \zeta_5^2 \theta \left[ \frac{1}{11} \right] \theta \left[ \frac{1}{11} \right] \right\}\left\{ -\theta \left[ \frac{1}{11} \right] \theta^3 \left[ \frac{1}{11} \right] + \zeta_5 \theta \left[ \frac{1}{11} \right] \theta^3 \left[ \frac{1}{11} \right] \theta \left[ \frac{1}{11} \right] \right\},
\]

(4.29)

\[
\theta \left[ \frac{0}{11} \right] = \left\{ \theta \left[ \frac{0}{11} \right] \theta^2 \left[ \frac{0}{11} \right] + \zeta_5^3 \theta \left[ \frac{0}{11} \right] \theta \left[ \frac{0}{11} \right] \right\}\left\{ -\theta \left[ \frac{0}{11} \right] \theta^3 \left[ \frac{0}{11} \right] + \zeta_5^2 \theta \left[ \frac{0}{11} \right] \theta^3 \left[ \frac{0}{11} \right] \theta \left[ \frac{0}{11} \right] \right\},
\]

(4.30)
Proof. The first equation follows from equations (4.6) and (4.7). The second equation is obtained by equations (4.8) and (4.9). The third equation is derived from equations (4.10) and (4.11). The fourth equation follows from equations (4.12) and (4.13). The fifth equation follows from equations (4.14) and (4.15).

Theorem 1.1 (5) follows from Theorems 4.3 and 5.1.

5.2 Proof of Theorem 1.1 (6)

In the same way, we can prove the following theorem and Theorem 1.1 (6):

Theorem 5.2. For every $\tau \in \mathbb{H}^2$, we have

$$
\left\{ \begin{array}{l}
\theta \left[ \frac{3}{5} \right] \theta^2 \left[ \frac{3}{5} \right] \theta \left[ \frac{1}{5} \right] + \zeta_5^2 \theta^2 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \theta^3 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] - \zeta_5^3 \theta^3 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \\
\end{array} \right.
$$

$$
= \left\{ \begin{array}{l}
\zeta_5^2 \theta^2 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \theta^3 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] - \zeta_5^3 \theta^3 \left[ \frac{3}{5} \right] \theta \left[ \frac{3}{5} \right] \\
\end{array} \right.
$$

where $H^2$ is the hyperbolic plane.
6 Proof of Theorem 1.1 (7)

Theorem 6.1. For every $\tau \in \mathbb{H}^2$, we have

$$\begin{align*}
\left\{ \theta^2 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] + \zeta_5^4 \theta^2 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} & \left\{ \theta^3 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] + \zeta_5^4 \theta^3 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} \\
= \left\{ \theta \left[ \frac{3}{1} \right] \theta^2 \left[ \frac{3}{1} \right] - \theta \left[ \frac{3}{1} \right] \theta^2 \left[ \frac{3}{1} \right] \right\} & \left\{ \zeta_5 \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] + \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] \right\},
\end{align*}$$

$$\begin{align*}
\left\{ \zeta_5^2 \theta \left[ \frac{3}{1} \right] \theta^2 \left[ \frac{3}{1} \right] - \theta^2 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} & \left\{ \zeta_5^3 \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] - \zeta_5^3 \theta^3 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} \\
= \left\{ \theta^2 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} & \left\{ \zeta_5^3 \theta \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] - \zeta_5 \theta \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\},
\end{align*}$$

$$\begin{align*}
\left\{ \zeta_5 \theta \left[ \frac{3}{1} \right] \theta^2 \left[ \frac{3}{1} \right] + \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] \right\} & \left\{ \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] + \zeta_5^4 \theta^3 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} \\
= \left\{ \theta^2 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\} & \left\{ \zeta_5 \theta \left[ \frac{3}{1} \right] \theta^3 \left[ \frac{3}{1} \right] \theta \left[ \frac{3}{1} \right] \right\}.
\end{align*}$$

\[ \tag{6.1} \]
\[
\left\{ \zeta \theta^2 \left[ \begin{array}{c} 1 \\ 3 \\ 5 \\ 1 \end{array} \right] \theta \left[ \begin{array}{c} 1 \\ 3 \\ 5 \\ 1 \end{array} \right] + \theta^2 \left[ \begin{array}{c} 1 \\ 3 \\ 5 \\ 1 \end{array} \right] \theta \left[ \begin{array}{c} 1 \\ 3 \\ 5 \\ 1 \end{array} \right] \right\} - \zeta \left[ \begin{array}{c} 1 \\ 3 \\ 5 \\ 1 \end{array} \right] \theta \left[ \begin{array}{c} 1 \\ 3 \\ 5 \\ 1 \end{array} \right] = 0,
\]

Equation (6.1) follows from equations (3.6) and (3.16). Equation (6.2) is obtained by equations (3.7) and (3.17). Equation (6.3) is derived from equations (3.8) and (3.18). Equation (6.4) follows from equations (3.9) and (3.19). Equation (6.5) is obtained from equations (3.10) and (3.20).

**Proof.** Equation (6.1) follows from equations (3.6) and (3.16). Equation (6.2) is obtained by equations (3.7) and (3.17). Equation (6.3) is derived from equations (3.8) and (3.18). Equation (6.4) follows from equations (3.9) and (3.19). Equation (6.5) is obtained from equations (3.10) and (3.20).

Theorem 1.1 (7) follows from Theorems 1.1 (1)–(6) and 6.1.

### A Elementary proof of the cubic identities (1.1) and (1.2) of Farkas and Kra

**Proposition A.1.** For every \((\zeta, \tau) \in \mathbb{C} \times \mathbb{H}^2\), we have

\[
\theta \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \theta \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} (\zeta, \tau) - \zeta_0^2 \theta \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \theta \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} (\zeta, \tau) - \zeta_0^2 \theta \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \theta \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} (\zeta, \tau) = 0.
\]
and

\[
\theta \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{3} \right] \theta^2 \left[ \frac{1}{3} \right] (\zeta, \tau) - \zeta_6^5 \theta^2 \left[ \frac{1}{4 \cdot 3} \right] \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{1}{4 \cdot 3} \right] (\zeta, \tau)
\]

\[
- \theta^2 \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{1}{4 \cdot 3} \right] (\zeta, \tau) = 0,
\]

(A.2)

where \( \zeta_6 = \exp(2\pi i/6) \).

\textbf{Proof.} We treat equation (A.1). Equation (A.2) can be proved in the same way.

We first note that \( \dim F_2 \left[ \frac{2}{3} \right] = 2 \) and

\[
\theta^2 \left[ \frac{1}{3} \right] (\zeta, \tau), \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{1}{3} \right] (\zeta, \tau), \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{1}{4 \cdot 3} \right] (\zeta, \tau) \in F_2 \left[ \frac{2}{3} \right].
\]

Therefore, there exists complex numbers, \( x_1, x_2, x_3, \) not all zero, such that

\[
x_1 \theta^2 \left[ \frac{1}{3} \right] (\zeta, \tau) + x_2 \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{1}{3} \right] (\zeta, \tau)
\]

\[
+ x_3 \theta \left[ \frac{1}{3} \right] (\zeta, \tau) \theta \left[ \frac{5}{4 \cdot 3} \right] (\zeta, \tau) = 0.
\]

Substituting \( \zeta = \frac{\tau + 1}{3}, \frac{\tau}{3} \) and \( \frac{1}{3} \), we have

\[
x_1 \zeta_6^2 \left[ \frac{1}{3} \right] + x_2 \theta^2 \left[ \frac{1}{3} \right] + x_3 \zeta_6 \theta^2 \left[ \frac{1}{3} \right] = 0,
\]

\[
x_1 \zeta_6^2 \left[ \frac{1}{3} \right] + x_3 \theta \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{4 \cdot 3} \right] = 0,
\]

\[
x_1 \theta^2 \left[ \frac{1}{3} \right] + x_2 \zeta_6 \theta \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{4 \cdot 3} \right] = 0.
\]

Solving this system of equations, we obtain

\[
(x_1, x_2, x_3) = \alpha \left( \theta \left[ \frac{1}{3} \right] \theta \left[ \frac{1}{3} \right], -\zeta_6^5 \theta^2 \left[ \frac{1}{3} \right], -\zeta_6 \theta^2 \left[ \frac{1}{3} \right] \right), \alpha \in \mathbb{C}^*,
\]

which proves the proposition. \( \square \)
Setting $\zeta = 0$ in equations (A.1) and (A.2), we obtain

$$\theta^3 \left[ \frac{1}{3} \right] + \exp \left( \frac{2\pi i}{3} \right) \theta^3 \left[ \frac{1}{3} \right] = \theta^3 \left[ \frac{1}{1} \right],$$

and

$$\theta^3 \left[ \frac{1}{3} \right] + \theta^3 \left[ \frac{1}{3} \right] = \exp \left( \frac{2\pi i}{6} \right) \theta^3 \left[ \frac{1}{3} \right],$$

which implies that

$$\theta^3 \left[ \frac{1}{3} \right] + \theta^3 \left[ \frac{1}{3} \right] = \theta^3 \left[ \frac{1}{1} \right],$$

and

$$\exp \left( \frac{\pi i}{3} \right) \theta^3 \left[ \frac{1}{3} \right] + \exp \left( \frac{2\pi i}{3} \right) \theta^3 \left[ \frac{1}{3} \right] = \theta^3 \left[ \frac{1}{3} \right].$$

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