FREE SEMIDEFINITE REPRESENTATION OF MATRIX ROOT FUNCTIONS

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Abstract. Consider the matrix root function $X^p$ defined over the cone of positive definite matrices $S_{++}^n$. It is known that $X^p$ is convex over $S_{++}^n$ if $p \in [-1, 0] \cup [1, 2]$ and $X^p$ is concave over $S_{++}^n$ if $p \in [0, 1]$. We show that the hypograph of $X^p$ admits a free semidefinite representation if $p \in [0, 1]$ is rational, and the epigraph of $X^p$ admits a free semidefinite representation if $p \in [-1, 0] \cup [1, 2]$ is rational.

1. Introduction

Let $S^n$ be the space of real symmetric $n \times n$ matrices, and $S^n_+$ (resp. $S^n_{++}$) be the cone of positive semidefinite (resp. definite) matrices in $S^n$. The matrix root function $X^p$ on $S^n$ is defined as $X^p = Q^T \Lambda^p Q$, with $X = Q^T \Lambda Q$ an orthogonal spectral decomposition. It is well known that (cf. [B97, pp. 147]) that

1. $X^p$ is convex over $S_{++}^n$ if $p \in [-1, 0] \cup [1, 2]$, and
2. $X^p$ is concave over $S_{++}^n$ if $p \in [0, 1]$.

Here, the concavity and convexity are defined as usual for functions of matrices. The goal of this paper is to give a free semidefinite representation (i.e., in terms of linear matrix inequalities whose construction is independent of the matrix dimension $n$) for the epigraph or hypograph of the matrix root function $X^p$ for a range of rational exponents $p$.

1.1. Convex and concave matrix-valued functions. Let $D$ be a convex subset of the space of the cartesian product $(S^n)^g$, with $g > 0$ an integer. A matrix-valued function $f : D \to S^n$ is called convex if

$$f(tX + (1-t)Y) \preceq tf(X) + (1-t)f(Y), \quad \forall t \in [0, 1]$$

for all $X, Y \in D$. If $-f$ is convex, we say that $f$ is concave. The epigraph (resp. hypograph) of $f$ is then defined as

$$\{(X, Y) : f(X) \preceq Y, X \in D\} \quad (resp. \quad \{(X, Y) : f(X) \succeq Y, X \in D\}).$$

The following is a straightforward but useful fact. Due to lackness of a suitable reference in case of matrix-valued functions, we include a short proof here.

Lemma 1.1. Suppose $D$ is a convex set. Then $f$ is convex over $D$ if and only if its epigraph is convex. Similarly, $f$ is concave over $D$ if and only if its hypograph is convex.
Proof. We will prove only the first half of the proposition as the second half clearly follows from the first.

\(\Rightarrow\) If \((X, W)\) and \((Y, Z)\) are in the epigraph of \(f\) and if \(t \in [0, 1]\), then by the convexity of \(f\),
\[
f(tX + (1-t)Y) \preceq tf(X) + (1-t)f(Y) \preceq tW + (1-t)Z
\]
so that \((tX + (1-t)Y, tW + (1-t)Z)\) is in the epigraph of \(f\).

\(\Leftarrow\) If \(X, Y \in \mathcal{D}\) and \(t \in \mathbb{R}\), then \((X, f(X))\) and \((Y, f(Y))\) are in the epigraph of \(f\). Since the epigraph is convex, \((tX + (1-t)Y, tf(X) + (1-t)f(Y))\) is in the epigraph as well. But this says that \(f(tX + (1-t)Y) \preceq tf(X) + (1-t)f(Y)\). \(\square\)

In the case that \(f(X) \succeq 0\) for all \(X \in \mathcal{D}\), we are often only interested in the pairs \((X, Y)\) from the hypograph of \(f\) with \(Y \succeq 0\). Thus, in this case, we slightly abuse terminology and refer to
\[
\{ (X, Y) : f(X) \succeq Y \succeq 0, X \in \mathcal{D} \}
\]
as the hypograph of \(f\). Note that Lemma 1.1 remains true with this definition of hypograph.

1.2. Linear pencils and free semidefinite representation. Given positive integers \(n\) and \(g\), let \((S^n)^g\) denote the set of \(g\)-tuples of matrices in \(S^n\). Let \(\otimes\) denote the Kroneker, i.e. tensor, product of two matrices. If \(A = (A_0, \ldots, A_g) \in (S^l)^{g+1}\), we define the linear pencil \(L_A\), which acts on \((S^n)^g\) \(n = 1, 2, \ldots\) as
\[
L_A(X) := A_0 \otimes I_n + \sum_{j=1}^g A_j \otimes X_j.
\]
For instance, if
\[
A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 8 & 9 \end{pmatrix}, \quad X = (X_1, X_2)
\]
with \(X_1\) and \(X_2\) being \(n \times n\) matrices, then
\[
L_A(X) = \begin{bmatrix} I_n + 4X_1 + 7X_2 & 2I_n + 5X_1 + 8X_2 \\ 2I_n + 5X_1 + 8X_2 & 3I_n + 6X_1 + 9X_2 \end{bmatrix}
\]
is a \(2n \times 2n\) matrix.

A spectrahedron in \((S^n)^g\) is a set of the form
\[
\mathcal{D}_{L_A} := \{ X \in (S^n)^g : L_A(X) \succeq 0 \}
\]
where \(L_A\) is a linear pencil. An inequality of the form \(L_A(X) \succeq 0\) is called a linear matrix inequality (LMI).

We now begin the discussion of projected spectrahedra. If \(A \in (S^l)^{g+g'+1}, X \in (S^n)^g\) and \(W \in (S^n)^{g'}\) then define
\[
L_A(X, W) := A_0 \otimes I_n + \sum_{j=1}^g A_j \otimes X_j + \sum_{j=g+1}^{g+g'} A_j \otimes W_{j-g}.
\]
We then define the projection onto the \(X\)-space as
\[
P_X \mathcal{D}_{L_A} := \{ X \in (S^n)^g : \exists W \in (S^n)^{g'}, L_A(X, W) \succeq 0 \}.
\]
Let $\mathcal{F}$ be a set in the Cartesian product $\prod_{n=1}^{\infty}(S^n)^g$. Every element $Z$ of $\mathcal{F}$ is an $\infty$-tuple in the form

$$Z = (Z(1), Z(2), \ldots, Z(n), \ldots), \quad Z(n) \in (S^n)^g, \; n = 1, 2, \ldots.$$  

The $n$-th section of $\mathcal{F}$ is defined as

$$\mathcal{F}|_n := \{Z(n) : (Z(1), Z(2), \ldots, Z(n), \ldots) \in \mathcal{F}, \; Z(i) \in (S^i)^g, \; i = 1, 2, \ldots\}.$$  

A set $\mathcal{F}$ in $\prod_{n=1}^{\infty}(S^n)^g$ is said to have a **free semidefinite representation** (free SDr) if there exists a linear pencil $L_A$, in tuples $X$ and $W$, such that for all $n = 1, 2, \ldots$

$$\mathcal{F}|_n = \{X \in (S^n)^g : \exists W \in (S^n)^g', \; L_A(X, W) \succeq 0\}.$$  

In the above, the set $\mathcal{G} \subseteq \prod_{n=1}^{\infty}(S^n)^g \times \prod_{n=1}^{\infty}(S^n)^g'$ defined such that, for all $n = 1, 2, \ldots$

$$\mathcal{G}|_n = \{(X, W) \in (S^n)^g \times (S^n)^g' : L_A(X, W) \succeq 0\}$$

is called a **free LMI lift** of $\mathcal{F}$. We emphasize that the key virtue of free SDr is that one representor $L_A$ works for all dimensions $n$ of matrix tuples $X, Z$.

**1.3. Contributions.** We consider the matrix root function $f(X) := X^p$. It is defined over the cone of positive semidefinite matrices for all $p \geq 0$, and defined over the cone of positive definite matrices for all $p$. By definition, the epigraph and hypograph of $f$ are naturally sets in $\prod_{n=1}^{\infty}(S^n)^2$. For convenience, they are respectively denoted as $\text{epi}(f)$ and $\text{hyp}(f)$. Then, for all $n = 1, 2, \ldots$

$$\text{epi}(f)|_n = \{(X, Y) \in (S^+_n)^2 : f(X) \preceq Y\},$$

$$\text{hyp}(f)|_n = \{(X, Y) \in (S^+_n)^2 : f(X) \succeq Y\}.$$  

Our main result is the following theorem.

**Theorem 1.2.** Let $f(X) = X^p$ be the matrix root function defined over the cone of positive semidefinite matrices. If $p \in [0, 1]$ is rational, then the hypograph of $f$ has a free semidefinite representation; if $p \in [1, 2]$ is rational, then the epigraph of $f$ has a free semidefinite representation. Furthermore, if $p \in [-1, 0]$ (restricting the domain to positive definite matrices), then the epigraph of $f$ has a free semidefinite representation.

To see how this relates to the polynomial case, we note that, by Theorem [HM04], cf. [HKM11], any polynomial in matrices with convex epigraph for each dimension has degree 2 or less. Also, sets of symmetric matrices of the form

$$\mathcal{C} := \{X : f(X) \succeq 0\}$$

which are convex and bounded all have the form $\mathcal{C} := \{X : \Lambda(X) \succeq 0\}$ for some monic linear pencil $\Lambda$. As a consequence, if such a set $\mathcal{C}$ is semidefinite representable, then it is “LMI representable”, thus lifting offers no advantages. All of this is true even in several matrix variables, for details, see [HM12]. For treatments of rational functions of matrices see [KV09]. While we have focused on representing sets with LMI lifts (lifts of sets of matrices are used in engineering to “convexify” problems) that is building convex supersets of a given set. There is no systematic theory of this and it involves great cleverness (cf. [OGB02], [GO10]).

We should mention the classical SDr literature concerning variables $x$ which are not matrices but which are scalar variables. Firstly, there exists a similar result
for scalar root functions $x^p$ by Ben-Tal and Nemirovski [BTN]. The role of SDr in optimization and perspective appear in Nemirovski [N06]. For recent advances in SDr, we refer to Lasserre [Las09a, Las09b, Las10], Helton and Nie [HN09, HN10], Netzer [Net10], Nie [Nie11, Nie12], Gouveia, Parrilo and Thomas [GPT10].

1.4. Ingredients of the proof and guide. The existence of a free SDr for rational powers of matrices is done by a sequence of constructions which use variables, denoted by $W, Z$ and $U$. This takes the remainder of the paper. We first build a free SDr for $X^{1/2}$, and then we recursively build constructions for $X^{1/m}$ for $m \in \mathbb{N}$. This is done in §2. In §3, we build on these in order to construct a free SDr for $X^{s/t}$ for rational $−1 < s/t < 2$. The proof the Theorem 1.2 concludes in §3.3. For an overview see the book [BPT].

Before continuing, we collect facts which we will use throughout the proof:

**Lemma 1.3.** (Löwner-Heinz inequality) [B97, pp. 123] If $\alpha \in [0, 1]$ and $A, B \in S^n$ such that $A \succeq B \succeq 0$, then $A^\alpha \succeq B^\alpha \succeq 0$.

Recall the Moore-Penrose pseudoinverse $C^\dagger$ of a symmetric matrix $C$ is the symmetric matrix satisfying

$$CC^\dagger = C^\dagger C = P$$

where $P$ is the orthogonal projection onto the range of $C$, denoted $\text{Range}(C)$, see [D06].

**Lemma 1.4.** (Schur complements for positive semidefinite matrices) [Lemma 12.19 in [D06]]

If $A, B, C \in S^n$, then the block matrix

$$
\begin{bmatrix}
A & B \\
B & C
\end{bmatrix}
$$

is positive semidefinite if and only if $A \succeq BC^\dagger B$, $\text{Range}(B) \subseteq \text{Range}(C)$ and $C \succeq 0$.

Now we list some additional useful facts. If $A \succeq B$, then $M^T AM \succeq M^T BM$. If $C \succeq D \succeq 0$ and $\text{Range}(C) = \text{Range}(D)$, then $D^\dagger \succeq C^\dagger \succeq 0$. Indeed, this is true if $\text{Range}(C) = \mathbb{R}^p$. Otherwise, we can view $C, D$ as operators mapping into the space $\text{Range}(C)$. As a reminder, $X^p$ is only defined for symmetric $X$ such that $X \succeq 0$. Additionally, all matrices throughout the paper are assumed to be symmetric.

2. Proof and construction for $X^p$ with $p = 1/m$

Throughout this and the next section $p$ will always denote a rational number. This section is devoted to the following proposition.

**Proposition 2.1.** For any positive integer $m$ the hypograph of $X^{1/m}$ has a free SDr representation.

The proof consumes this section.

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1To prove this factor $C = K^T K, D = R^T R$. Then $K^T K \succeq R^T R$, so $I \succeq K^{-1T} R^T R K^{-1}$ and consequently $I \succeq R K^{-1} K^{-1T} R^T$. This implies $D^{-1} = R^{-1} R^{-1T} \succeq K^{-1} K^{-1T} = C^{-1}$. 


2.1. $p = 1/2$. Consider the hypograph

$$\mathcal{H}_{1/2} := \{(X, Y) : X^{1/2} \succeq Y \succeq 0\}.$$ 

Define the free SDr set

$$\mathcal{L}_{1/2} := \{(X, Y) : \exists W \begin{bmatrix} X & W \\ W & I \end{bmatrix} \succeq 0, \, W \succeq Y \succeq 0\}.$$ 

**Lemma 2.2.** $\mathcal{H}_{1/2} = \mathcal{L}_{1/2}$

*Proof.* Using Schur complements (Lemma 1.4), we see that

$$\mathcal{L}_{1/2} := \{(X, Y) : \exists W \begin{bmatrix} X & W \\ W & I \end{bmatrix} \succeq 0, \, W \succeq Y \succeq 0\}.$$ 

Clearly, it holds that $\mathcal{H}_{1/2} \subseteq \mathcal{L}_{1/2}$ by letting $W = X^{1/2}$. Now we prove the reverse containment. Note that by the Löwner-Heinz inequality

$$X \succeq W^2 \Rightarrow X^{1/2} \succeq W \Rightarrow X^{1/2} \succeq Y.$$

□

2.2. $p = 1/m$ for $m > 2$ a positive integer. Our treatment splits in two parts: when $m$ is even and when $m$ is odd. Each case will use a recursive construction for semidefinite representability. The $\frac{1}{m}$ case relies on $\frac{1}{m}$ being free SDr and the $\frac{1}{2d+1}$ case relies on $\frac{1}{2d+1}$ being free SDr. In other words, if viewed as an algorithm starting with $m$ as the denominator, we move to the case where the denominator is $m/2$ if $m$ is even whereas we move to the case where the denominator is $(m+1)/2$ if $m$ is odd. Clearly this will end in the case $m = 2$ in finitely many steps.

Consider the hypograph, for each integer $m \geq 0$,

$$\mathcal{H}_{1/m} := \{(X, Y) : X^{1/m} \succeq Y \succeq 0\}.$$ 

First we consider even $m$, that is, $p = 1/2d$.

**Lemma 2.3.** The following holds.

\begin{align*}
(2.6) \quad \mathcal{H}_{1/2d} &= \{(X, Y) : \exists W \begin{bmatrix} X & W \\ W & I \end{bmatrix} \succeq 0, \, (W, Y) \in \mathcal{H}_{1/d}\} \\
(2.7) \quad &= \{(X, Y) : \exists W \begin{bmatrix} X & W \\ W & I \end{bmatrix} \succeq 0, \, (W, Y) \in \mathcal{H}_{1/d}\}
\end{align*}

Consequently, if $\mathcal{H}_{1/d}$ is a free SDr set, then so is $\mathcal{H}_{1/2d}$.

*Proof.* Define

\begin{align*}
(2.8) \quad \tilde{\mathcal{H}}_{1/2d} := \{(X, Y) : \exists W \begin{bmatrix} X & W \\ W & I \end{bmatrix} \succeq 0, \, (W, Y) \in \mathcal{H}_{1/d}\} \\
(2.9) \quad &= \{(X, Y) : \exists W \begin{bmatrix} X & W \\ W & I \end{bmatrix} \succeq 0, \, W^{1/d} \succeq Y \succeq 0\}
\end{align*}

Clearly, it holds that $\mathcal{H}_{1/2d} \subseteq \tilde{\mathcal{H}}_{1/2d}$ by letting $W = X^{1/2}$. Conversely, if $(X, Y) \in \tilde{\mathcal{H}}_{1/2d}$ from the Löwner-Heinz inequality, we get

$$X^{1/2d} \succeq W^{1/d} \succeq Y.$$ 

Thus $\mathcal{H}_{1/2d} = \tilde{\mathcal{H}}_{1/2d}$ □
2.3. \( p = \frac{1}{2d+1} \) for \( d \) a positive integer. An odd \( m \) satisfies a recursive equation.

**Lemma 2.4.**

\[ \mathcal{H}_{\frac{1}{2d+1}} = \left\{ (X, Y) : \exists (W, Z) \begin{bmatrix} X & W \\ W & Z \end{bmatrix} \succeq 0, \ (W, Z) \in \mathcal{H}_{\frac{1}{d+1}}, \ Z \succeq Y \succeq 0 \right\} \]

From this we see: if \( \mathcal{H}_{\frac{1}{2d+1}} \) is a free SDr set, then so is \( \mathcal{H}_{\frac{1}{2d+1}} \).

**Proof.** Define

\[ \tilde{\mathcal{H}}_{\frac{1}{2d+1}} = \left\{ (X, Y) : \exists (W, Z) \begin{bmatrix} X & W \\ W & Z \end{bmatrix} \succeq 0, \ (W, Z) \in \mathcal{H}_{\frac{1}{d+1}}, \ Z \succeq Y \succeq 0 \right\} \]

Note that

\[ \tilde{\mathcal{H}}_{\frac{1}{2d+1}} := \left\{ (X, Y) : \exists (W, Z) \ X \succeq WZ^\dagger W, \ Range(W) \subseteq Range(Z), \ W^{\frac{1}{d+1}} \succeq Z \succeq Y \succeq 0 \right\} \]

by Lemma 1.4. The fact that \( Range(W) \subseteq Range(Z) \) and \( W^{\frac{1}{d+1}} \succeq Z \succeq Y \succeq 0 \)

imply \( Range(W) = Range(Z) \). Clearly, it holds that \( \mathcal{H}_{\frac{1}{2d+1}} \subseteq \mathcal{H}_{\frac{1}{2d+1}} \) by letting \( Z = X^{\frac{d+1}{2d+1}} \) and \( W = X^{\frac{2d}{2d+1}} \). Now we prove that \( \tilde{\mathcal{H}}_{\frac{1}{2d+1}} \subseteq \mathcal{H}_{\frac{1}{2d+1}} \). Suppose \( (X, Y) \in \tilde{\mathcal{H}}_{\frac{1}{2d+1}} \). Note that

\[ W^{\frac{1}{d+1}} \succeq Z \succeq 0 \implies Z^{\dagger} \succeq (W^{\frac{1}{d+1}})^\dagger \implies W^{\frac{1}{d+1}} Z^{\dagger} W^{\frac{1}{d+1}} \succeq W^{\frac{1}{d+1}}. \]

(The first implication uses the fact \( Range(W) = Range(Z) \).) Then it holds that

\[ X \succeq WZ^\dagger W = W^{\frac{1}{d+1}} (W^{\frac{1}{d+1}} Z^{\dagger} W^{\frac{1}{d+1}}) W^{\frac{1}{d+1}} \succeq W^{\frac{2d}{2d+1}}. \]

By the Löwner-Heinz inequality, one gets

\[ X^{\frac{2}{2d+1}} \succeq W^{\frac{1}{2d+1}} \succeq Y. \]

\[ \square \]

2.4. **Proof of Proposition 2.1.** Given \( 1/m \) the recursions in the lemmas above reduce having a free SDr representation to successively smaller \( m \). For example, if \( p = 1/14 \), then the recursion is \( 1/14, 1/7, 1/4, 1/2 \). This terminates in \( m = 2 \).

However, we saw that the hypograph of \( X^{\frac{1}{2}} \) has a free SDr representation. \( \square \)

3. **Proof and construction for \( X^p \) with \(-1 < p < 2 \) rational.**

The next stage of the proof of Theorem 1.2 is slightly more involved than the previous \( X^{1/m} \) stage. Though it has similarities, the recursion steps are not as obvious. For this reason, we explicitly formulate a recursion defining free SDr sets \( \mathcal{H}_p \) followed by showing these sets are actually equal to the hypographs

\[ \mathcal{H}_p := \{(X, Y) : X^p \succeq Y \succeq 0\} \]

for \( 0 < p < 1 \); see \( \underline{3.1} \). After that, it is relatively easy to broaden the range of \( p \) to \( -1 < p < 2 \). In particular, we show (in \( \underline{3.2} \)) that the epigraph

\[ \mathcal{E}_p := \{(X, Y) : X^p \preceq Y, X \succeq 0\} \]

is free SDr for \( 1 < p < 2 \) and free SDr for \(-1 < p < 0 \).

3.1. **\( \mathcal{H}_p \) for \( 0 < p < 1 \) is free SDr.**
3.1.1. Preliminaries on rational numbers $0 < p < 1$. Define

$$p' := 2 - \frac{1}{p}. $$

Then

$$0 < p' < 1 \quad \text{iff} \quad 1/2 < p < 1.$$

In particular, $p' = 1/2$ iff $p = 2/3$ and $0 < p' < 1/2$ iff $1/2 < p < 2/3$.

**Lemma 3.1.** For $1/2 < p < 1$ we have

1. $p' < p$
2. denominator $p' <$ denominator $p$,
3. numerator $p' <$ numerator $p$,

$$p' = 2 - t/s = (2s - t)/s = (s - (t - s))/s.$$

**Proof.** (1): Trivial calculation.

(2) and (3): Denote $p = s/t$ with $t < 2s < 2t$ and $s, t$ relatively prime. We have

$$p' = 2 - t/s = (2s - t)/s = (s - (t - s))/s$$

with (2) saying $s < t$ and (3) holding because $(t - s) > 0$. $\square$

Suppose $0 < p < 1/2$. There exists an integer $d$ satisfying

$$1/2 \leq dp < 1$$

let $d(p)$ denote the smallest such $d$. Clearly denominator $p =$ denominator $d(p)p$.

3.1.2. Construction of sequence of rational $p_i$ for $0 < p < 1$. Given a rational number $0 < p < 1$, there is a list $S(p) := \{p_0, p_1, p_2, \ldots, p_m = \frac{1}{2}\}$ of rational numbers with $0 < p_i < 1$ such that

(a) $p_0 = p$,
(b) $\mathcal{H}_{p_{i-1}}$ can be written as the intersection of a free SDr set and $\mathcal{H}_{p_i}$

Now we turn to how $S(p)$ is constructed: If $p_i = 1/2$, the list terminates. Otherwise define $p_{i+1}$ as follows

1. if $0 < p_i < 1/2$, then: $p_{i+1} := p_id(p_i)$.
2. if $1/2 < p_i < 1$, then: $p_{i+1} := 2 - \frac{1}{p_i}.$

**Example**  Consider $p_0 = 7/11$.

A. Use (2) to get $p_1 = 2 - 11/7 = 3/7$.
B. Use (1): we have $d(p_1) = 2$, so $p_2 = 6/7$.
C. Use (2) to get $p_3 = 2 - 7/6 = 5/6$ and again to get $p_4 = 4/5$ and again to get $p_5 = 3/4$ and again $p_6 = 2/3$ and again $p_7 = 2 - 3/2 = 1/2$. Stop.

**Lemma 3.2.** This algorithm, when run automatically, stops in a finite number of steps. The final $p_m$ which occurs is $1/2$.

**Proof.** Everytime (2) is invoked the denominator strictly decreases. Also (1) never increases the denominator. For the facts above see Lemma 3.1. Immediately after (1) is invoked (2) is always invoked. Hence the denominators decrease until one obtains $p_m$ whose denominator is 2. Since $0 < p_m < 1$, we get $p_m = 1/2$. Here the recursion stops. $\square$
3.1.3. The recursion on $H_p$ for $0 < p < 1$. We now show that if $p_{i−1}, p_i$ are on the list $S(p_0)$, then $H_{p_{i−1}}$ is free SDr provided $H_{p_i}$ is. This fact immediately follows from the lemmas below which give alternate characterizations of the sets $H_p$.

**Lemma 3.3.** Suppose $1/2 < p < 1$. Then $H_p = \tilde{H}_p$ where $\tilde{H}_p$ is defined to be

$$\tilde{H}_p = \left\{ (X, Y) : \exists (W, Z) \in H_{2−\frac{1}{p}}, \begin{bmatrix} X & W \\ W & Z \end{bmatrix} \succeq 0, \; W \succeq Y \succeq 0 \right\}.$$ 

Moreover, $\tilde{H}_p$ is

$$\{(X, Y) : \exists (W, Z) \; W^{2−\frac{1}{p}} \succeq Z \succeq 0, \; \text{Range}(W) = \text{Range}(Z), \; X \succeq W Z^d W, \; W \succeq Y \succeq 0\}.$$

**Proof.** Note the two formulas in the lemma for $\tilde{H}_p$ indeed give the same set, by Lemma 2.1 and the fact $\text{Range}(W) = \text{Range}(Z)$. (The same argument as Lemma 2.4)

It holds that $\tilde{H}_p \subseteq H_p$ by letting $W = X^p$ and $Z = X^{2p−1}$. Now we prove that $H_p \subseteq \tilde{H}_p$. Start with $(X, Y) \in H_p$, then there are $W, Z$ with $\text{Range}(Z) = \text{Range}(W)$, satisfying

$$W^{2−\frac{1}{p}} \succeq Z \succ 0 \implies Z^d \succeq (W^{2−\frac{1}{p}})^d \implies W^{2−\frac{1}{p}} Z^d W^{2−\frac{1}{p}} \succeq W^\frac{1}{p}.$$ 

Thus

$$X \succeq W Z^d W = (W^{1−\frac{1}{p}})^d (W^{2−\frac{1}{p}} Z^d W^{2−\frac{1}{p}})^d \succeq W^\frac{1}{p}. $$

By the Löwner-Heinz inequality, one gets

$$X^p \succeq W \succeq Y \succeq 0.$$ 

Hence $\tilde{H}_p = H_p$. \hfill \Box

**Lemma 3.4.** Suppose $0 < p < 1/2$. Let

$$\tilde{H}_p := \{(X, Y) : \exists W \; X^{1/d(p)} \succeq W, \; W^{d(p)p} \succeq Y \succeq 0\}$$

$$= \{(X, Y) : \exists W \; (X, W) \in H_{1/d(p)}, \; (W, Y) \in H_{d(p)p}\}.$$ 

Then $H_p = \tilde{H}_p$.

**Proof.** Observe that $H_p \subseteq \tilde{H}_p$, by letting $W = X^{1/d(p)}$. Now we prove the reverse containment. From the Löwner-Heinz inequality, we get

$$X^p = \left(X^{1/d(p)}\right)^{d(p)p} \succeq \left(W\right)^{d(p)p} \succeq Y.$$ 

\hfill \Box

3.1.4. $H_p$ is free SDr for $0 < p < 1$. Consider the list $S(p)$ of rational numbers constructed in 3.1.2. Proposition 2.1 and Lemmas 3.3 and 3.4 tell us that $H_{p_{i−1}}$ is free SDr if $H_{p_i}$ is. By 2.1 we have $H_{1/2}$ is free SDr and thus $H_{p_j}$ is free SDr for all $0 \leq j \leq m$. In particular $H_{p_0}$ is free SDr where $p_0 = p$. This completes the proof that $H_p$ is a free SDr set for all $0 < p < 1$.

3.2. Broadening the range of $p$ to $−1 < p < 2$. 

3.2.1. $1 < p < 2$. Consider the epigraph
\[ \mathcal{E}_p := \{(X, Y) : X^p \preceq Y, X \succeq 0\}. \]

Define the free SDr set
\[ \tilde{\mathcal{E}}_p := \{(X, Y) : \exists Z \begin{bmatrix} Y & X \\ X & Z \end{bmatrix} \succeq 0, (X, Z) \in H_{2-p}, X \succeq 0\}. \]

By §3.1.4, we have that $H_{2-p}$ is free SDr (since $0 < 2 - p < 1$).

**Lemma 3.5.** $\mathcal{E}_p = \tilde{\mathcal{E}}_p$

**Proof.** First note that
\[ \tilde{\mathcal{E}}_p = \{(X, Y) : \exists Z \begin{bmatrix} Y & X \\ X & Z \end{bmatrix} \succeq 0, (X^2)^{2-p} \succeq Z \succeq 0, X \succeq 0\}. \]

By Lemma 1.4, it holds that $\mathcal{E}_p \subseteq \tilde{\mathcal{E}}_p$ by letting $Z = X^{2-p}$. Now we prove that $\tilde{\mathcal{E}}_p \subseteq \mathcal{E}_p$. From the L"owner-Heinz inequality, we get
\[ X^{2-p} \succeq Z \succeq 0 \Rightarrow Z \preceq (X^{2-p})^\dagger \Rightarrow XZ \preceq X. \]

Therefore $Y \succeq X^p$.

3.2.2. $-1 < p < 0$. Consider the epigraph
\[ \mathcal{E}_p := \{(X, Y) : Y \succeq X^p \succ 0\}. \]

Define the free SDr set
\[ \tilde{\mathcal{E}}_p := \{(X, Y) : \exists Z (X, Z) \in H_{-p}, \begin{bmatrix} Z & I \\ I & Y \end{bmatrix} \succeq 0, X \succ 0\}. \]

**Lemma 3.6.** $\mathcal{E}_p = \tilde{\mathcal{E}}_p$.

**Proof.** Note that in this case
\[ \mathcal{E}_p = \{(X, Y) : X^p \preceq Y, X \succ 0\} = \{X^{-p} \preceq Y^{-1}, X \succ 0, Y \succ 0\}. \]

Now by §3.1.4, we have that $H_{-p}$ is free SDr $(0 < -p < 1)$ and that
\[ \tilde{\mathcal{E}}_p := \{(X, Y) : \exists Z X^{-p} \preceq Z, \begin{bmatrix} Z & I \\ I & Y \end{bmatrix} \succeq 0, X \succeq 0\}. \]

Clearly, it holds that $\mathcal{E}_p = \{(X, Y) \in \tilde{\mathcal{E}}_p : X, Y \succ 0\}$ (letting $Z = Y^{-1}$ on one hand and using Schur complements on the other).

3.3. **Proof of Theorem 1.2.** Now we put the results together for rational numbers $p$ in $-1 < p < 2$. From §3.1.4, we have that the hypograph of $X^p$ ($0 < p < 1$) is free SDr with the domain $S_n^+$. From §3.2.1, the epigraph of $X^p$ ($1 < p < 2$) is free SDr again with the domain $S_n^+$. Shrinking the domain to $S_{n+1}^+$, §3.2.2 shows the epigraph of $X^p$ ($-1 < p < 0$) is free SDr. This proves Theorem 1.2.
4. Matrix concavity in many variables

We proved above that the function which takes the root of a single matrix variable is concave. We can attempt to generalize this to the case for symmetric multivariable matrix functions. A natural case to consider is the root function

\[ q(X) = \left( X_0^{p_0/2} X_1^{p_1/2} X_2^{p_2} \cdots X_g^{p_g/2} \right)^{1/k} \]

with \( k \geq p_0 + p_1 + \cdots + p_g \) and \( p_j \in \mathbb{Q} \) (i.e. we are taking a root of a simple symmetric multivariable polynomial) where \( q \) is defined on \( g \)-tuples of positive semidefinite symmetric matrices (i.e. for \( X = (X_1, \ldots, X_g) \in (\mathbb{S}^n)^g \)). Unlike the single variable case, even the simplest function of this kind is not concave. For instance, the set

\[ \{(X_0, X_1) : (X_1X_0X_1)^{1/3} \succeq I\} \]

which is the same as the set

\[ \{(X_0, X_1) : X_1X_0X_1 \succeq I\} \]

is not convex. If it were, then fixing

\[ X_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}^{-1}, \quad A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \]

and letting \( X_1 = X \) would imply that the set

\[ Q = \{X : X^2 \succeq A, X \succeq 0\} \]

is convex. However, letting

\[ X_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 3 & 1 \\ 1 & 133/64 \end{pmatrix} \]

we have that \( X_1, X_2 \in Q \) but that \( Z = (X_1 + X_2)/2 \notin Q \). This is because the matrix

\[ E := Z^2 - A = \begin{pmatrix} 5/2 & 517/256 \\ 517/256 & 26521/16384 \end{pmatrix} \]

is not positive semidefinite (its determinant is \(-2079/65536 < 0\)). Thus, our natural generalization of the single variable root function does not preserve concavity when more variables are added.

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