A conditional Entropy Power Inequality for dependent variables

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Abstract

We provide a condition under which a version of Shannon’s Entropy Power Inequality will hold for dependent variables. We provide information inequalities extending those found in the independent case.

Shannon’s Entropy Power Inequality states that

**Theorem 1** For independent random variables $X,Y$ with densities, the entropy of the sum satisfies:

$$2^{2H(X+Y)} \geq 2^{2H(X)} + 2^{2H(Y)},$$

with equality if and only if $X,Y$ are normal.

Apart from its intrinsic interest, it provides a sub-additive inequality for sums of random variables and is thus an important part of the entropy-theoretic proof of the Central Limit Theorem [1]. Whilst Shannon’s proof [5] seems incomplete, in that he only checks that the necessary conditions for a local maximum are satisfied, a rigorous proof is provided by Stam [6] (see Blachman [2]). This proof is based on a related inequality concerning Fisher information:

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Lemma 2  For $X,Y$ with differentiable densities:

\[
\frac{1}{J(X+Y)} \geq \frac{1}{J(X)} + \frac{1}{J(Y)}.
\]

The relationship between Theorem 1 and Lemma 2 comes via de Bruijn’s identity, which expresses entropy as an integral of Fisher informations.

Now, Takano [7] provided conditions on the random variables $X,Y$, such that Theorem 1 would still hold for weakly dependent variables. In contrast, we change the equation, replacing entropies by conditional entropies, providing alternative conditions for this related result. Our approach is again to develop a Fisher information inequality, and to use an integral form of that to deduce the full result.

We consider random variables $X,Y$ with joint density $p(x,y)$ and marginal densities $p_X(x), p_Y(y)$. We need to refer to score functions and Fisher informations. Write $\rho_X(x) = p_X'(x)/p_X(x)$ and $\rho_Y(y) = p_Y'(y)/p_Y(y)$. We write $p^{(1)}(x,y)$ for $\partial p(x,y)/\partial x$, and similarly for $p^{(2)}(x,y)$, and $\rho^{(1)}(x,y) = p^{(1)}(x,y)/p(x,y)$, and similarly for $\rho^{(2)}(x,y)$. Now, we can define $J(X) = \mathbb{E}\rho_X(X)^2$ and $J(Y) = \mathbb{E}\rho_Y(Y)^2$ for the Fisher informations of $X$ and $Y$, and $J_{XX} = \mathbb{E}\rho^{(1)}(X,Y)^2$, $J_{YY} = \mathbb{E}\rho^{(2)}(X,Y)^2$, $J_{XY} = \mathbb{E}\rho^{(1)}(X,Y)\rho^{(2)}(X,Y)$, similarly. We will need to consider terms of the form: $M_{a,b}(x,y) = a(\rho^{(1)}(x,y) - \rho_X(x)) + b(\rho^{(2)}(x,y) - \rho_Y(y))$.

Lemma 3 (Takano) As in the independent case, we can express the score function $\rho_W$ of the sum $W = X+Y$ as a conditional expectation of $\rho_X(X,Y)$.

\[
\rho_W(w) = \mathbb{E}(\rho^{(2)}(X,Y)|X+Y = w) = \mathbb{E}(\rho^{(2)}(X,Y)|X+Y = w).
\]

Proof  Since $W = X+Y$ has density $p_W(w) = \int p(x, w-x)dx = \int p(w-y,y)dy$, we know that:

\[
\rho_Z(z) = \frac{p_W'(z)}{p_W(z)} = \int \frac{p^{(1)}(w-y,y)}{p_W(w)}dy = \int \frac{p^{(1)}(w-y,y) p(w-y,y)}{p(w-y,y) p_W(w)}dy,
\]

hence the result follows. 

Using this, we establish the following proposition, the equivalent of Lemma 2 for dependent variables, and which reduces to Lemma 2 in the independent case:
Proposition 4  For random variables $X,Y$ with differentiable densities:
\[
\frac{1}{J(X+Y) - J_{XY}} \geq \frac{1}{J_{XX} - J_{XY}} + \frac{1}{J_{YY} - J_{XY}}.
\]
Equality holds when $X,Y$ are multivariate normal.

Proof  Using the conditional representation, Lemma 3, for any $a,b$:
\[
0 \leq \mathbb{E} \left( ap^{(1)}(X,Y) + bp^{(2)}(X,Y) - (a+b)\hat{\rho}(X+Y) \right)^2
= a^2 J_{XX} + 2ab J_{XY} + b^2 J_{YY} - (a+b)^2 J(X+Y).
\]
Now, motivated by the choice of $a,b$ that give equality in the Gaussian case, we take $a = J_{YY} - J_{XY}$, $b = J_{XX} - J_{XY}$, and rearranging, we obtain that:
\[
J(X+Y) \leq \frac{J_{XX}J_{YY} - J_{XY}^2}{J_{XX} + J_{YY} - 2J_{XY}},
\]
and subtracting $J_{XY}$ from both sides we obtain the result. \(\square\)

Lemma 5  If $(X_t, Y_t) = X + Z_{Ct}$, where $Z_{Ct} \sim N(0, Ct)$, and $W_t = X_t + Y_t$ then writing $a = J_{XX} - J_{XY}$, $b = J_{YY} - J_{XY}$:
\[
\frac{\partial}{\partial t} (2H(X_t, Y_t) - 2H(W_t)) \geq \frac{a^2 C_{11} - 2ab C_{12} + b^2 C_{22}}{a + b} \geq 0.
\]
Proof  Johnson and Suhov \[4\] prove the multivariate de Bruijn identity:
\[
\frac{\partial H}{\partial t}(X_t) = \frac{1}{2} \sum_{ij} C_{ij} J_{ij}(X + Z_{Ct}),
\]
where $J$ is the Fisher matrix $\mathbb{E} \rho^T \rho$, with $\rho$, the score vector equal to $\nabla f/f$.
By Proposition 4 we deduce that:
\[
\frac{\partial}{\partial t} (2H(X_t, Y_t) - 2H(W_t))
\begin{align*}
&= C_{11} J_{XX} + 2C_{12} J_{XY} + C_{22} J_{YY} - (C_{11} + 2C_{12} + C_{22}) J(W_t) \\
&\geq C_{11} J_{XX} + 2C_{12} J_{XY} + C_{22} J_{YY} \\
&\quad - (C_{11} + 2C_{12} + C_{22}) \left( \frac{J_{XX}J_{YY} - J_{XY}^2}{J_{XX} + J_{YY} - 2J_{XY}} \right)
\end{align*}
\]
Now, for functions $f(t), g(t)$, we can define $(X_t, Y_t) = (X, Y) + (Z_1, Z_2)$, where $Z_1, Z_2$ are independent, with $Z_1 \sim N(0, f(t)), Z_2 \sim N(0, g(t))$. We write $v_{X_t}$, $p_{X_t}$, and $\rho_{X_t}$ for the variance, density and score function of $X_t$. This perturbation ensures that densities are smooth and allows us to use the 2-dimensional version of the de Bruijn identity:

**Condition 1** For all $t$, $\mathbb{E} \rho_{X_t}(X_t) \rho_{Y_t}(Y_t) \geq 0$.

Compare Condition 1 with Takano’s condition [7], involving the same term:

**Condition 2** For all $t$, $\mathbb{E} \rho_{X_t}(X_t) \rho_{Y_t}(Y_t) \geq \mathbb{E} M_2^{\lambda, \lambda - 1}$, where $\lambda = \sqrt{\frac{J(X_t)}{J(Y_t)}}$.

Takano shows that Condition 2 implies that the original Entropy Power Inequality, Theorem 1, holds. With our weaker condition, we provide a weaker, though still interesting, result.

**Theorem 6 (Conditional Entropy Power Inequality)** If Condition 1 holds then:

$$2^{2H(X+Y)} \geq 2^{2H(X|Y)} + 2^{2H(Y|X)}.$$ 

**Proof** Taking $f, g$ defined by $f' = 2^{2H(X_t|Y_t)}, g' = 2^{2H(Y_t|X_t)}$ and defining $s(t) = (2^{2H(X_t|Y_t)} + 2^{2H(Y_t|X_t)})/2^{2H(W_t)}$,

s'(t) \geq \frac{1}{2^{2H(W_t)}} \left( \left( 2^{2H(X_t|Y_t)} + 2^{2H(Y_t|X_t)} \right) \frac{A^2 f' + B^2 g'}{A + B} - f'g'(J(X_t) + J(Y_t)) \right) 

\geq \frac{1}{2^{2H(W_t)}} \left( \frac{(Af' - Bg')^2}{A + B} + f'g'(A + B - J(X_t) - J(Y_t)) \right) 

= \frac{1}{2^{2H(W_t)}} \left( \frac{(Af' - Bg')^2}{A + B} + f'g'\mathbb{E} M_{i-1}^2 + 2f'g'\mathbb{E} \rho_{X_t}\rho_{Y_t} \right) \geq 0,

since $0 \leq \mathbb{E} M_{i-1}^2 = J_{XX} - 2J_{XY} + J_{YY} - J(X_t) - J(Y_t) - 2\mathbb{E} \rho_{X_t}\rho_{Y_t}$. Hence $s(t)$ is an increasing function of $t$. Now as $t \to \infty$, $s(t) \to 1$, since $(X, Y)$ tends to an independent pair of normals. Hence $s(0) \leq 1$ and the result follows. 

\qed
Cover and Zhang [3] provide a bound on the entropy $H(X + Y)$, under the condition that $X$ and $Y$ have the same marginal density $f$. They show that $H(X + Y) \leq H(2X)$ if and only if $f$ is log-concave (that is, the score function is decreasing). Notice that our Condition 1 holds if $X, Y$ are FKG variables with log-concave densities.

We write $\psi(X, Y) = \sup_{x,y} |p_{X,Y}(x, y)/p_X(x)p_Y(y) - 1|$, the so-called $\psi$-mixing coefficient. Note that since $E_{\rho_X} E_{\rho_Y} = \psi(X_t, Y_t)$, we know that if $X, Y$ have the same marginals (and wlog variance 1), then $\psi(X_t, Y_t) \leq (J^2(X) - 1)/(1 + f(t)J(X))$. Thus, we require that:

$$\psi(X_t, Y_t) \leq \frac{\psi(X, Y) \sqrt{1 + f(t)J(X)}}{1 + f(t)} \geq \psi(X_t, Y_t)(J^2(X) - 1).$$

From $f = 0$, we deduce that we need $J(X) \leq \sqrt{\psi(X, Y) + 1}$, and if $\lim_{t \to \infty} f(t)^{1/2} \psi(X_t, Y_t) = 0$, we are through.

Although we know that $\psi(X_t, Y_t) \leq \psi(X, Y)$, we need some theory of convexity of mixing coefficients to provide the most natural conditions.

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