Single-brane world with stabilized extra dimension

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Abstract

We present a model describing a single brane with tension embedded into a five-dimensional space-time with compact extra dimension, which can be easily stabilized. We examine the linearized gravity in the model and obtain an expression for the four-dimensional Planck mass on the brane in terms of the model parameters. It is also shown that the scalar sector of the effective four-dimensional theory contains a tachyonic mode, and we discuss the problem of stability of the model.

1 Introduction

Brane world models and their phenomenology have been widely discussed in the last years. One of the most interesting brane world models is the Randall-Sundrum model with two branes, – the RS1 model [1]. This model solves the hierarchy problem due to the warp factor in the metric and predicts an interesting new physics in the TeV range of energies.

Most of the brane world models with one compact extra dimension and thin branes with tension demand the existence of at least two branes. At the same time the matter located on the brane, which is not ”our” brane, can strongly affect the world located on ”our” brane. For the case of the RS1 model it was shown in [2]. So it would be quite interesting to find out, whether it is possible to construct a model with only one tensionful brane in a compact extra dimension, admitting a solution to the hierarchy problem in the way analogous to that proposed in [1].

A characteristic feature of models with single brane is the presence of at least one tachyonic mode in the perturbative linearized theory [3]. At the same time the linearized theory, as well as the five-dimensional effective action describing a brane world model, is valid for the energy range of the order of the fundamental energy scale of the theory, defined by the five-dimensional gravity (we suppose that this scale is of the order of $10^{16}$ TeV). Thus, if the masses of the tachyonic modes are far beyond the energy range of its applicability, their influence on the theory cannot be accessed in the linear approximation, and one needs to consider the nonlinear effects.

Some solutions with single brane in a compact extra dimension, interesting from the cosmological point of view, were obtained in [4]. But the energy-momentum tensors used for obtaining these solution are ”phenomenological”, i.e. they are added to the action ”by hand”.

Here we present a model describing the scalar field minimally coupled to gravity in a five-dimensional space-time, admitting the existence of a single brane quite naturally and being of interest from the point of view of the hierarchy problem. Moreover, the size of the extra dimension in this model can be easily stabilized. Thus, the model appears to be devoid of the main flaw of the original Randall-Sundrum model – the existence of the massless scalar mode, called the radion, which arises due to the fluctuations of the branes with respect to each other and whose interactions contradict the existing experimental data. We argue that the
stabilization of the size of extra dimension is made in the same way as in [3]. This method is free from the main disadvantage of the approach proposed in [6], where the backreaction of the scalar field on the background metric is not taken into account. There is only one tachyonic mode in the model with the mass of the order of the four-dimensional Planck mass \(\sim 10^{19}\text{GeV}\), thus lying far beyond the applicability range of the theory.

The paper is organized as follows. In Section 2 we present the background solution and the method of its stabilization. In Section 3 we obtain gauge conditions and equations of motion for linearized gravity in the model. In Section 4 we consider the tensor modes and obtain an expression for the four-dimensional Planck mass on the brane in terms of the model parameters. In Section 5 we consider the scalar sector of the theory, discuss the stability of the model and obtain the estimates for the mass of the lowest scalar mode and its coupling constant to matter on the brane. And finally, we discuss the obtained results.

2 The model

Let us denote the coordinates in five-dimensional space-time \(E = M_4 \times S^1\) by \(\{x^N\} \equiv \{x^\mu, y\}\), \(N = 0, 1, 2, 3, 4, \mu = 0, 1, 2, 3\), the coordinate \(x^4 \equiv y, -L \leq y \leq L\) parametrizing the fifth dimension with identified points \(-L\) and \(L\). The brane is located at the point \(y = L\).

The action of stabilized brane world model can be written as

\[
S = \int d^4x \int_{-L}^L dy \sqrt{-g} \left[ 2M^3 R - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \right] - \int_{y=L} d^4x \sqrt{-\tilde{g}} \lambda(\phi),
\]

(1)

Here \(V(\phi)\) is a bulk scalar field potential and \(\lambda(\phi)\) is the brane scalar field potential, \(\tilde{g} = \text{det} \tilde{g}_{\mu\nu}\), and \(\tilde{g}_{\mu\nu}\) denotes the metric induced on the brane. The signature of the metric \(g_{MN}\) is chosen to be \((-+, +, +, +, +)\).

The standard ansatz for the metric and the scalar field, which preserves the Poincaré invariance in any four-dimensional subspace \(y = \text{const}\), looks like

\[
ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \equiv \gamma_{MN}(y) dx^M dx^N,
\]

\[
\phi(x, y) = \phi(y),
\]

(2)

(3)

\(\eta_{\mu\nu}\) denoting the flat Minkowski metric. If one substitutes this ansatz into the equations corresponding to action (1), one gets a rather complicated system of nonlinear differential equations for functions \(A(y), \phi(y)\):

\[
\frac{dV}{d\phi} + \frac{d\lambda}{d\phi} \delta(y - L) = -4A' \phi' + \phi'',
\]

\[
12M^3 (A')^2 + \frac{1}{2} \left( V - \frac{1}{2} (\phi')^2 \right) = 0,
\]

\[
\frac{1}{2} \left( \frac{1}{2} (\phi')^2 + V + \lambda \delta(y - L) \right) = -2M^3 \left( -3A'' + 6(A')^2 \right). \tag{4}
\]

Here and below \(\cdot' \equiv d/\text{dy}\). An interesting conclusion following from these equations is that the relation

\[
A''(y) = \frac{1}{12M^3} \phi'^2 \tag{5}
\]

holds in the bulk for any potential \(V(\phi)\), and thus \(A'' \geq 0\) in the bulk. This inequality was also obtained in [7] from the weaker energy condition.
To find an analytic solution to this system we will use the results of [5, 8]. Let us consider a special class of potentials, which can be represented as

\[ V(\phi) = \frac{1}{8} \left( \frac{dW}{d\phi} \right)^2 - \frac{1}{24M^3} W^2(\phi). \]

Let us also suppose that

\[ W(\phi) = \frac{8\gamma}{3} \phi^\frac{3}{2}. \]  

(6)

In this case the scalar field potential takes a simple polynomial form

\[ V(\phi) = \gamma^2 \left( 2\phi - \left( \frac{2}{3M} \right)^3 \phi^3 \right), \]

(7)

and the corresponding continuous background solution can be easily found (with the help of the procedure described in [5, 8]):

\[ \phi = (\gamma y)^2, \]

(8)

\[ A = \frac{1}{36M^3} \left( (\gamma y)^4 - (\gamma L)^4 \right). \]

The additive constant in the solution for \( A(y) \) is chosen in such a way that the coordinates \( \{x^\mu\} \) are Galilean on the brane (see [9, 10] for details). We will refer all the energy parameters, which appear in the theory, to this Galilean coordinate system on the brane.

In order the equations of motion be valid on the brane too, one needs to finetune the brane potential \( \lambda(\phi) \). We choose

\[ \lambda(\phi) = -W(\phi) = -\frac{8\gamma}{3} \phi^\frac{3}{2}. \]

(9)

In this case the brane appears to be of the BPS type. The size of the extra dimension is not defined by the solution yet.

To stabilize the size of the extra dimension, let us add the following term to the scalar field potential on the brane:

\[ \Delta \lambda(\phi) = \beta^2 (\phi - \phi_0)^2. \]

(10)

Such an addition will not affect the equations of motion provided

\[ \phi \big|_{y=L} = \phi_0, \]

(11)

which means that

\[ L = \frac{\sqrt{\phi_0}}{\gamma}. \]

(12)

Thus, we see that the size of the extra dimension is stabilized.

It is necessary to note that the background solution presented above was obtained without imposing \( Z_2 \) orbifold symmetry, which is inherent to the most brane world models, although the solution itself possesses reflection symmetry with respect to the point \( y = 0 \).

We also suppose that the parameters of the potentials \( \gamma, \phi_0, \beta \), when made dimensionless by the fundamental five-dimensional energy scale of the theory \( M \), should be positive quantities of the order \( O(1) \), i.e. there should be no hierarchical difference in the parameters. We note that action (1) and the corresponding four-dimensional effective theory can be used only at the energy scales \( E \lesssim M \) measured in Galilean coordinates on the brane.
3 Linearized gravity

Now let us turn to the examination of linearized gravity in the model. We represent the metric and the scalar field as

\[
g_{MN}(x, y) = \gamma_{MN}(y) + \frac{1}{\sqrt{2M^3}} h_{MN}(x, y),
\]

\[
\phi(x, y) = \phi(y) + \frac{1}{\sqrt{2M^3}} f(x, y).
\]

To simplify the analysis, let us impose $Z_2$ orbifold symmetry conditions (although this symmetry is not necessary for obtaining the background solution). Correspondingly, the metric $g_{MN}$ and the scalar field $\phi$ satisfy the orbifold symmetry conditions

\[
g_{\mu\nu}(x, -y) = g_{\mu\nu}(x, y), \quad g_{\mu4}(x, -y) = -g_{\mu4}(x, y),
\]

\[
g_{44}(x, -y) = g_{44}(x, y), \quad \phi(x, -y) = \phi(x, y).
\]

We realize that imposing $Z_2$ orbifold symmetry is a rather artificial procedure. But a consistent and thorough analysis of linearized gravity without this symmetry, i.e. taking into all the degrees of freedom coming from the metric, is a very complicated problem (for example, we cannot impose the gauge conditions which will be used later). At the same time, a theory with the orbifold symmetry makes sense and was studied, for example, in [3]. Moreover, we have a developed formalism for studying linearized gravity in brane world models stabilized by the bulk scalar field and with extra dimension forming the orbifold $S^1/Z_2$ – see [11]. The only difference from this case is that all the fields should have a ”good” behavior at the point $y = 0$, i.e. the fields should be smooth at $y = 0$, which corresponds to the absence of the brane as a physical object at this point. For these reasons, in this paper we restrict ourselves to the case with $Z_2$ orbifold symmetry conditions.

Substituting representation (13) and (14) into action (1) and keeping the terms of the second order in $h_{MN}$ and $f$, we get the second variation Lagrangian of this action [11]. This Lagrangian is invariant under the gauge transformations

\[
h_{MN}^{(t)}(x, y) = h_{MN}(x, y) - (\nabla_M \xi_N + \nabla_N \xi_M),
\]

\[
f^{(t)}(x, y) = f(x, y) - \phi' \xi_4,
\]

where $\nabla_M$ is the covariant derivative with respect to the background metric, provided $\xi_M(x, y)$ satisfy the orbifold symmetry conditions

\[
\xi_\mu(x, -y) = \xi_\mu(x, y), \quad \xi_4(x, -y) = -\xi_4(x, y).
\]

These gauge transformations are a generalization of the gauge transformations in the unstabilized RS1 model [9,10]. We will use them to isolate the physical degrees of freedom of the fields $h_{MN}$ and $f$. We also note that since $\xi_4|_{y=L} = 0$, the brane appears to be straight (the disadvantages of bent-brane formalism were discussed in [12]).

It was shown in [11] that with the help of these gauge transformations one can impose the gauge

\[
(e^{-2A} h_{44})' - \frac{1}{3M^3} e^{-2A} \phi' f = 0,
\]

\[
h_{44} = 0,
\]

\[
(16)
\]
after which there remain the gauge transformations satisfying
\[(e^{2A} \xi_\mu)\prime = 0. \tag{17}\]

A substitution
\[h_{\mu\nu} = b_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} h_{44}\tag{18}\]
allows us to decouple the equations of motion, following from the second variation Lagrangian, in gauge \[16\]. Gauge transformations satisfying \[17\] allow one to impose the traceless-transverse gauge condition on the field \[b_{\mu\nu} \tag{10, 11}\]
\[\tilde{b} = \gamma_{\mu\nu} b_{\mu\nu} = 0, \quad \partial^\nu b_{\mu\nu} = 0, \tag{19}\]
the residual gauge transformations now being
\[\xi_\mu = e^{-2A} \epsilon_\mu(x), \quad \partial^\nu \epsilon_\nu(x) = 0, \quad \Box \epsilon_\nu = 0, \tag{20}\]
where \[\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu\]. Transformations \[20\] act only on the massless mode of the field \[b_{\mu\nu}\] and provide the correct number of degrees of freedom of the massless graviton \[10\].

Finally, we get the equations of motion in the interval \((0, L)\) with corresponding boundary conditions at the points \(y = 0, y = L\) for the field \(b_{\mu\nu}\)
\[\frac{1}{2} \left( e^{2A(y)} \Box b_{\mu\nu} + \frac{\partial^2 b_{\mu\nu}}{\partial y^2} \right) - b_{\mu\nu} \left( 2(A')^2 - A'' \right) = 0, \tag{21}\]
\[b'_{\mu\nu} \mid_{y=+0} = 0, \tag{22}\]
\[b'_{\mu\nu} + 2A'b_{\mu\nu} \mid_{y=L-0} = 0, \tag{23}\]
and for the field \(g = e^{-2A(y)} h_{44}(x, y)\)
\[g'' + 2g' \left( A' - \frac{\phi''}{\phi'} \right) - \frac{(\phi')^2}{6M^2} g + \partial_\mu \partial^\mu g = 0, \tag{24}\]
\[g' \mid_{y=+0} = 0, \tag{25}\]
\[\beta^2 g' - \partial_\mu \partial^\mu g \mid_{y=L-0} = 0, \tag{26}\]
see \[11\] for details.

### 4 Tensor modes and the hierarchy problem

Let us study first the modes of the tensor field \(b_{\mu\nu}(x, y)\), which satisfies Eq. \[21\]. Substituting into this equation
\[b_{\mu\nu}(x, y) = c_{\mu\nu} e^{ipx} \psi_n(y), \quad c_{\mu\nu} = \text{const}, \quad p^2 = -m_n^2, \tag{27}\]
we get:

\[
\frac{d^2 \psi_n}{dy^2} - 2(2(A')^2 - A'') \psi_n = -m_n^2 e^{2A} \psi_n,
\]

\[
\psi_n'|_{y=0} = \psi_n'|_{y=L-0} + 2A' \psi_n|_{y=L-0} = 0.
\]  

(25)

The boundary conditions suggest a substitution \( \psi_n = \exp(-2A) \omega_n \) (note that \( A'|_{y=0} = 0 \)), which turns this equation into

\[
\frac{d}{dy} \left( e^{-4A} \omega'_n \right) = -m_n^2 e^{-2A} \omega_n,
\]

\[
\omega'_n|_{y=0} = \omega'_n|_{y=L-0} = 0.
\]  

(26)

We see that the eigenfunctions \( \omega_n \) are solutions of a Sturm-Liouville problem with von Neumann boundary conditions. In accordance with the general theory [13], the problem at hand has no negative eigenvalues for arbitrary \( A \), only one zero eigenvalue, corresponding to \( \omega_0 = \text{const} \).

The eigenfunctions \( \{ \psi_n(y) \} \) of eigenvalue problem (25) build a complete orthonormal set, the eigenfunction of the zero mode being

\[
\psi_0(y) = Ne^{-2A(y)}.
\]  

(27)

Expanding \( b_{\mu \nu} \) in this system

\[
b_{\mu \nu} = \sum_{n=0}^{\infty} b_{\mu \nu}^n(x) \psi_n(y),
\]  

(28)

we get four-dimensional tensor fields \( b_{\mu \nu}^n(x) \) with definite masses.

A standard technique gives us an expression for the four-dimensional Planck mass on the brane

\[
M_{Pl}^2 = M^3 \int_{-L}^{L} e^{-2A} dy \simeq M^3 2e^{\frac{(\gamma L)^4}{18M^3}} \int_{0}^{\infty} e^{-\frac{(\gamma y)^4}{18M^3}} dy =
\]

\[
= 2M^3 e^{\frac{(\gamma L)^4}{18M^3}} \left( \frac{1}{4} \right) \Gamma \left( \frac{1}{4} \right) \approx 3.7 \cdot M^3 e^{\frac{(\gamma L)^4}{18M^3}} \frac{M^2}{\gamma}
\]  

(29)

and

\[
M_{Pl} \approx 2M \sqrt{\frac{M^2}{\gamma}} e^{\frac{(\gamma L)^4}{36M^3}}.
\]  

(30)

Let us suppose that all fundamental parameters of the theory lie in the TeV range. To have the hierarchy problem solved, one should take

\[
\frac{\gamma^4 L^4}{36M^2} = \frac{\phi_0^2}{36M^3} \approx 36,
\]  

(31)

which means that

\[
\phi_0 \approx 36M^{\frac{3}{2}}
\]  

(32)

and

\[
L \approx \frac{6M^{\frac{3}{2}}}{\gamma}.
\]  

(33)

Although Eq. (26) cannot be solved analytically for \( n \neq 0 \), it is reasonable to suppose that the lowest masses of the four-dimensional tensor excitations \( b_{\mu \nu}^n(x) \) are of the order of \( L^{-1} \).
5 Scalar sector and stability

In order to find the mass spectrum of the scalar particles described by Eq. (23) let us substitute
\[ g(x, y) = e^{ipx}g_n(y), \quad p^2 = -\mu_n^2, \]
into this equation:
\[ g''_n + 2A'g'_n - 2\phi''\phi'g_n - \frac{(\phi')^2}{6M^3}g_n = -\mu_n^2 e^{2A}g_n, \]  \hspace{1cm} (34)
\[ g'_n|_{y=+0} = 0, \]
\[ \beta^2 g'_n - \mu_n^2 e^{2A}g_n|_{y=-0} = 0. \]  \hspace{1cm} (35)
\[ \beta^2 g'_n - \mu_n^2 e^{2A}g_n|_{y=-0} = 0. \]  \hspace{1cm} (36)

It is necessary to note that since the field \( f \) should be smooth at the point \( y = 0 \), from (16) it follows that the value \((g'_n/\phi')\) should be continuous at \( y = 0 \) too.

First, let us solve Eq. (34) for the case \( \mu = 0 \), i.e. for the zero mode. In the case of background solution \( \phi \) the wave function \( g_0 \), satisfying boundary condition at \( y = 0 \), has the form
\[ g_0 \sim e^{-\frac{\gamma}{18M^4}t^4} + \frac{\gamma^3}{(18M^3)^{\frac{1}{2}}} |y|^3 \int_0^{\frac{\gamma}{18M^3}t^4} q^{-\frac{1}{2}} e^{-q} dq. \]  \hspace{1cm} (37)

It is not difficult to check that \( g'_0|_{y=L} \neq 0 \). Thus, the scalar zero mode is absent in the model.

Now let us examine, whether there are scalar tachyons in the model. To this end we denote \( \tilde{\mu}^2 = -\mu^2 > 0 \) (here and below we omit the subscript \( n \)) and introduce a new dimensionless variable
\[ t = \frac{\gamma}{M^4} y. \]

In this case Eq. (34) and boundary conditions take the form
\[ \ddot{g} + 2\dot{g} \left( \frac{t^3}{9} - \frac{1}{t} \right) - \frac{2}{3}t^2 g - \tilde{\mu}^2 \exp \left( \frac{t^4}{18} \right) g = 0, \]  \hspace{1cm} (38)
\[ \frac{M^4}{\gamma} \dot{g} + \tilde{\mu}^2 \exp \left( \frac{t^4}{18} \right) g \bigg|_{t=\frac{\gamma}{M^4}L} = 0, \]  \hspace{1cm} (39)
where \( \tilde{\mu} = \frac{\mu M_4}{\gamma} \exp \left( -\frac{(\gamma L)^3}{36M^4} \right) \), \( \frac{\gamma L}{M^4} \approx 6 \) (see (33)) and \( \dot{g} \equiv dg/dt \).

Unfortunately we cannot solve Eq. (38) analytically. Numerical analysis (see Appendix A) shows that for \( \tilde{\mu} \leq 0.9507 \), \( t = 3.2 \): \( g(t) > 0 \) and \( \dot{g}(t) > 0 \) (see examples on Figs. 1, 2). At the same time for \( \tilde{\mu} \geq 0.9509 \), \( t = 3.2 \): \( g(t) < 0 \) and \( \dot{g}(t) < 0 \) (see examples on Figs. 3, 4). Fig. 4 is shown for \( t \leq 2 \), but one can check that for \( \tilde{\mu} = 1.5 \) and \( t = 3.2 \) \( g(t) < 0 \) and \( \dot{g}(t) < 0 \). The graphs on Figs. 1, 2, 3 are shown for \( t \leq 3.3 \), it is made to show the behavior of \( g(t) \) in the interval \( t \in [0, 3.2] \), especially for the cases \( \tilde{\mu} = 0.9507 \) and \( \tilde{\mu} = 0.9509 \). For \( t > 3.2 \): \( \dot{g}/g > 0 \), it can be easily seen from the structure of Eq. (38). Indeed, let us divide (38) by \( g \) and pass to the equation for \( q(t) = \dot{g}/g \), which takes the form of a Riccati equation
\[ \dot{q} + q^2 + 2 \left( \frac{t^3}{9} - \frac{1}{t} \right) q = 2 \frac{2}{3}t^2 + \tilde{\mu}^2 \exp \left( \frac{t^4}{18} \right). \]  \hspace{1cm} (41)
If initially for some value of \( t = t_q > 0 \): \( q > 0 \), then \( q \) will remain positive for any \( t > t_q \). Indeed, the function \( q \) should pass through zero to change the sign. But if \( 0 < q \ll 1 \), then from Eq. (11) it follows that \( \dot{q} > 0 \), \( q \) appears to be a growing function and thus remains positive. Thus, for \( t > 3.2 \): \( g(t) > 0, \) \( g(t) > 0 \) or \( \dot{g}(t) < 0, \) \( \dot{g}(t) < 0 \) depending on the sign of \( g(t) \) at \( t = 3.2 \). In both cases (39) is not satisfied, since \( M > 0, \gamma > 0 \) and \( \beta^2 > 0 \).

Eq. (39) can be satisfied if \( \dot{g}(t) = g(t) = 0 \) at the point \( t = \frac{\gamma L}{M^\frac{4}{3}} \approx 6 \). But such boundary conditions imply that \( g(t) \equiv 0 \) in the interval \( t \in \left[ e^{-\frac{\gamma L}{M^\frac{4}{3}}} \right] \) for any \( \epsilon > 0 \) – this conclusion follows from the theorem about the existence and uniqueness of solution for the Cauchy problem – see, for example, [14]. Finally, due to the continuity of the function \( g(t) \), we get \( g(t) \equiv 0 \) in the interval \( t \in \left[ 0, \frac{\gamma L}{M^\frac{4}{3}} \right] \). Of course, the same conclusion can be made for any \( 0 < t_1 < \frac{\gamma L}{M^\frac{4}{3}} \) such that \( \dot{g}(t_1) = g(t_1) = 0 \).

As for the region \( 0.9507 < \bar{\mu} < 0.9509 \), we have made a large number of numerical simulations with different values of \( \bar{\mu} \). The behavior of the corresponding solutions is such that for \( t \geq 3.6 \): \( g(t) > 0, \) \( g(t) > 0 \) or \( \dot{g}(t) < 0, \) \( \dot{g}(t) < 0 \) respectively, analogous to the behavior of solutions presented on Figs. 2, 3. A simple qualitative explanation of this fact can be given. For \( 0.9507 < \bar{\mu} < 0.9509 \) and \( t > 3.6 \) the coefficient \( \bar{\mu}^2 \exp \left( \frac{t^3}{18} \right) \) in (38) grows rapidly, which leads to the growth of the absolute value of function \( g(t) \) with coordinate \( t \) for \( t > 3.6 \).

Nevertheless, for some value of \( \bar{\mu} \) such that \( 0.9507 < \bar{\mu} < 0.9509 \) there exists a solution, which satisfies condition (39). Indeed, let us define a function \( F(\bar{\mu}) = \frac{M^\frac{4}{3}\beta^2}{\gamma} \bar{\mu}^2 \exp \left( \frac{t^3}{18} \right) g(t) \big|_{t = \frac{\gamma L}{M^\frac{4}{3}}} \). For \( \bar{\mu} < 0.9507 \) it is positive, whereas for \( 0.9509 < \bar{\mu} \) it is negative (see Figs. 1, 2, 3, 4). Thus, it is reasonable to suppose that there exists an appropriate value \( \bar{\mu}^* \) (\( 0.9507 < \bar{\mu}^* < 0.9509 \)) such that \( F(\bar{\mu}^*) \big|_{t = \frac{\gamma L}{M^\frac{4}{3}}} = 0 \), which corresponds to a tachyonic mode. It appears to be very difficult to find the exact value of the tachyonic mass numerically. At the same time the physical mass of the tachyon is such that

\[
\mu^* \approx -\bar{\mu}^2 M^\frac{4}{3} \gamma \frac{3}{3.7} M^\frac{4}{3} \approx -\frac{0.9}{3.7} M^2 \epsilon \approx -\left( 0.5 \cdot 10^9 GeV \right)^2
\]

(42)

for the given values of the model parameters (we suppose that \( \gamma \approx M^\frac{4}{3} \)). Such energy scale lies outside the range of validity of our effective theory, described by action (1) (because \( |\mu^*| \sim E \gg M \), see Section 2). From the classical point of view it can be understood as follows: the tachyonic mode should behave as \( e^{\mu^* x^0} \). The time derivative of the tachyon field \( \sim \mu^* e^{\mu^* x^0} \), i.e. it is enhanced by the large value of \( \mu^* \sim M^2 \epsilon \) in comparison with the tachyon field itself. Thus, even if the value of the tachyon field is small, its time derivative would lead to breakdown of perturbative approach and corresponding nonlinear effects, coming from the five-dimensional curvature (through substitution (18)). Another remarkable thing is that the wave function of the tachyon is such that if \( g|_{y=0} = 1 \), then in the leading order \( g|_{y=L} \sim \exp \left( -\mu^* \exp \left( \frac{\gamma L^4}{30 M^\frac{4}{3}} \right) \right) \approx \exp (-\mu^* \exp (36)) \). It means that when the nonlinear effects and (or) effects of the underlaying fundamental theory begin to affect the behavior of the theory in the bulk, the theory on the brane remain intact, because the coupling constant of the tachyon to matter on the brane, which is proportional to the value of the wave function on the brane, is negligibly small - much smaller than the coupling constant of the massless tensor graviton. Thus, the runaway of the scalar field can be stopped in the bulk because of the nonlinear effects coming from action (1) or from the underlaying fundamental theory. Of course, we cannot argue that it is indeed so,
Figure 1: Numerical solution for $g(t)$, $\bar{\mu} = 0.5$.

Figure 2: Numerical solution for $g(t)$, $\bar{\mu} = 0.9507$. 
Figure 3: Numerical solution for $g(t)$, $\bar{\mu} = 0.9509$

Figure 4: Numerical solution for $g(t)$, $\bar{\mu} = 1.5$
but such situation can be realized.

Of course, our examination is not explicit since it is based on the numerical calculations. But we think that the analysis made testifies in favor of absence of scalar tachyons in the model below the energy scale of our effective theory \[1\]. As for the ghosts, the form of the effective action for the scalar modes ensures the proper signs of the appropriate kinetic terms \[11\].

The form of Eq. (38) allows us to estimate the mass of the lowest scalar excitation and its coupling to matter on the brane. Indeed, let us suppose that the lowest mass \(\mu_1\) (see (34)) is such that \(\mu_1/M \approx O(1)\). In this case we can neglect the last term in Eq. (34) in comparison with the last but one term of this equation, and the solution of the resulting equation takes the form

\[
g_1(y) \approx A_1 \left( e^{-\frac{4}{18 M^3}} + \frac{\gamma^3}{(18 M^3)^{2.5}} |y|^3 \int_0^{\gamma^3 / 18 M^3} q^{-2.5} e^{-q} dq \right) \sim g_0(y),
\]

(43)

where \(A_1\) is a normalization constant. Let us suppose that the size of the extra dimension is such that \(\gamma L/M^{3/4} = 6\), see (33). The values \(g_1(L)\) and \(g_1'(L)\) can be easily calculated, which gives us

\[
g_1(L) \approx A_1 \cdot 89.6,
\]

(44)

\[
g_1'(L) \approx A_1 \cdot 44.8 \frac{\gamma}{M^{3/4}}.
\]

(45)

Substituting (44) and (45) into (36) we easily get

\[
\mu_1^2 \approx \frac{\beta^2 \gamma}{2 M^{3/4}}.
\]

(46)

For example, if \(\beta^2 \approx M, \gamma \approx M^{7/4}\) and \(M \approx 10\, TeV\), the lowest mass \(\mu_1 \approx 7\, TeV\). Of course, it can be even smaller depending on the values of the parameters \(\beta, \gamma\) and \(M\).

It is also necessary to note that the analysis carried out with the help of the numerical solution of Eq. (34) for such small \(\mu_1\) reproduces the results obtained using (43) with a very good accuracy (of the order of 1 -- 2\%).

Now let us calculate the coupling constant of the first scalar mode to matter on the brane. To this end we need to calculate the normalization constant \(A_1\). The normalization condition for the scalar modes takes the form \[11\]

\[
\int_0^L dy e^{2A} \left( g_1^2 + \frac{6 M^3}{(\phi')^2} g_1'^2 \right) = \frac{2}{3}. 
\]

(47)

It is more convenient to pass to the variable \(t = \frac{\gamma}{M^{3/4}} y\):

\[
e^{-72} \int_0^6 dt e^{-t^4} \left( g_1^2 + \frac{6}{4t^2} g_1'^2 \right) = \frac{2 \gamma}{3 M^{3/4}}.
\]

(48)

The integral in (48) can be evaluated numerically, which gives us

\[
A_1^2 \approx 0.004 \frac{\gamma}{M^{3/4}}.
\]

(49)

Now we can calculate the coupling constant of the lightest scalar mode to matter on the brane (see \[11\]):

\[
\epsilon_1 = - \frac{g_1(L)}{2 \sqrt{8 M^3}} \approx - \frac{A_1 \cdot 89.6}{2 \sqrt{8 M^3}} \approx - \sqrt{\frac{\gamma}{M^{3/4}}}.
\]

(50)
One can see that $\epsilon_1 \approx \frac{1}{10^{14}}$ for the given values of the fundamental parameters $\gamma$ and $M$.

Unfortunately it is impossible to calculate even the lowest mass of the tensor excitations using the method described above. One should carry out a very precise numerical analysis to get an information about the spectrum of the tensor modes.

6 Conclusion

In this paper a model describing the scalar field minimally coupled to gravity in the spacetime with one compact extra dimension is proposed. It admits the existence of a single tensionful brane, contrary to the most brane world models with one compact extra dimension demanding the existence of at least two branes (of course, except the simplest case of the ADD model [15, 16] with tensionless branes). We also showed that the model could be interesting in view of the hierarchy problem.

The linearized gravity in the model was studied under the assumption of the $Z_2$ orbifold symmetry. We obtained the expression for the four-dimensional Planck mass on the brane in terms of the fundamental five-dimensional parameters of the theory. We also made a stability analysis of the model, – analytical for the tensor modes and numerical for the scalar modes, which resulted in the conclusion that the scalar sector of the model contain one tachyon, which corresponds to the result obtained in [3], its "mass" being of the order of the four-dimensional Planck mass. Thus, the model as it is, at least in the linear approximation, is unstable and its "lifetime" is of the order of the four-dimensional Planck time. Nevertheless, the energy scale of the tachyon is such that multidimensional nonlinear fundamental underlaying theory can come to play and "lift up" the scalar sector from falling down. Of course it is not necessarily so, but in principle it seems to be possible.

The background solution can also be used to describe the world with two branes. Indeed, the second brane can be placed at the point $y = L_0$, $0 < L_0 < L$. The results of [11] suggest that such a system is totally devoid of tachyons.

It is very interesting to carry out a numerical calculation of the coupling constants and the masses of the tensor modes and a complete description of the scalar sector of the model, as well as the model without $Z_2$ orbifold symmetry (in the latter case there should appear antisymmetric modes). One can also use the model discussed in this paper (for example, the stable configuration with two branes) as a basis for constructing models with universal extra dimensions. These tasks deserve additional thorough investigation.

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Appendix: note on numerical analysis

We solve Eq. (38) numerically with the following initial conditions on the "time" variable $t$:

\begin{align}
  g(t)|_{t=0} &= 1, \\
  \dot{g}(t)|_{t=0} &= 0. 
\end{align}

But since the coefficient
\[ \frac{t^3}{9} - \frac{1}{t} \]
in (38) is not defined at the point $t = 0$, it is inconvenient to use the point $t = 0$ as the initial point for numerical calculations. To bypass this problem, we find an approximate analytical solution of Eq. (38) in the vicinity of the point $t = 0$:

\[ g(t) \approx 1 - \frac{\bar{\mu}^2}{2} t^2. \]

Now we choose the point $t_0 = 10^{-11}$ as the initial point instead of $t = 0$. The corresponding initial conditions take the form

\begin{align}
  g(t)|_{t=t_0} &= 1 - \frac{\bar{\mu}^2}{2} 10^{-22}, \\
  \dot{g}(t)|_{t=t_0} &= -\bar{\mu}^2 10^{-11}. 
\end{align}

The numerical analysis for a large number of different values of $\bar{\mu}$ was made using the program package Mathematica, version 5.2. Selected solutions are presented on Figs. 1, 2, 3, 4.

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