ON A RESONANT AND SUPERLINEAR ELLIPTIC SYSTEM

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Abstract. We prove existence of solutions for a class of nonhomogeneous elliptic system with asymmetric nonlinearities that are resonant at $-\infty$ and superlinear at $+\infty$. The proof is based on topological degree arguments. A priori bounds for the solutions are obtained by adapting the method of Brezis-Turner.

1. Introduction. Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$. We are concerned with the solvability of the following system of semilinear elliptic equations

$$\begin{aligned}
-\Delta u &= au + bv + u^p + f(x) \quad x \in \Omega \\
-\Delta v &= bu + cv + v^q + g(x) \quad x \in \Omega \\
u = v &= 0 \quad x \in \partial \Omega.
\end{aligned}$$

(1)

Throughout the paper we assume the following conditions

$$\max\{a, c\} > 0, \quad b > 0 \quad \text{and} \quad 1 < p, q < \frac{N + 1}{N - 1}.$$  \hspace{1cm} (2)

Moreover, we also assume that the functions $f$ and $g$ are such that

$$f, g \in L^r \quad \text{with} \quad r > N.$$  \hspace{1cm} (3)

The above condition implies, by regularity theory, that all weak solutions of (1) are strong solutions, i.e., it belongs to $W^{2,r}(\Omega)$. Moreover, we recall that $W^{2,r}(\Omega) \subset C^1(\Omega)$, and the standard notation $u_+ = \max\{u, 0\}$. In the sequel, we will present our resonance assumption. We begin writing the system (1) in an equivalent matricial form

$$\begin{aligned}
-\Delta U &= AU + G(U) + F(x) \quad x \in \Omega \\
U &= 0 \quad x \in \partial \Omega,
\end{aligned}$$

(4)

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where
\[ U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad G(U) = \begin{pmatrix} u^p \\ v^q \end{pmatrix} \quad \text{and} \quad F(x) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \]

The matrix A has real eigenvalues
\[ \xi = \frac{a + c}{2} + \sqrt{\frac{(a - c)^2}{4} + b^2} \quad \text{and} \quad \eta = \frac{a + c}{2} - \sqrt{\frac{(a - c)^2}{4} + b^2}. \]

Denote by \( \lambda_1 \) the first eigenvalue of \( (-\Delta, H_0^1(\Omega)) \) and \( \phi_1 \) is the positive eigenfunction associated to \( \lambda_1 \) with \( |\phi_1|_{L^2} = 1 \). We will assume that \( \lambda_1 = \xi \). This assumption is called resonance hypothesis. Note that, in this case \( \lambda_1 > a \) and \( \Phi = (b\phi_1, (\lambda_1 - a)\phi_1) \) is the positive eigenfunction of the following eigenvalue problem
\[
\begin{cases}
-\Delta U = \lambda AU & x \in \Omega \\
U = 0 & x \in \partial \Omega,
\end{cases}
\]
associated to the eigenvalue \( \lambda_1(A) = 1 \). The above eigenvalue problem is studied in the Appendix (see [3] for more details).

Our main result is

**Theorem 1.** Assume that (2) and (3) hold. In addition, suppose that \( \lambda_1 = \xi \) and
\[
b \int_{\Omega} f\phi_1 + (\lambda_1 - a) \int_{\Omega} g\phi_1 < 0. \tag{5}
\]

Then system (1) possesses at least one strong solution.

Our proof is based upon ideas found in [6]. In this paper, the authors get a solution of the following superlinear elliptic equation
\[
\begin{cases}
-\Delta u = \lambda_1 u + u^p + f(x) & x \in \Omega \\
u = 0 & x \in \partial \Omega,
\end{cases}
\]
under the assumptions \( 1 < p < \frac{N+1}{N-1} \), \( f \in L^r \) with \( r > N \) and \( \int_{\Omega} f\phi_1 < 0 \). The proof of the main result in [6] uses the technique introduced in [1]. The method consists in getting a priori bounds, using Hardy-Sobolev type inequalities, with topological degree arguments. This approach was also used in [12] to study an Ambrosetti-Prodi type problem. Similar problems, under Dirichlet and Neumann boundary condition, can be found in [4, 11, 13] (see also [5] for a different method to those in [6]). We also mention that in [6], the authors consider the Hamiltonian systems
\[-\Delta u = \lambda_1 u + v^q + f(x), \quad -\Delta v = \lambda_1 u + u^p + g(x), \]
under appropriate conditions.

Let \( H \) be the Hilbert space \( H_0^1(\Omega) \times H_0^1(\Omega) \) equipped with the norm
\[
\| U \|^2 = \| u \|^2 + \| v \|^2 = \int_{\Omega} |\nabla u|^2 + \int_{\Omega} |\nabla v|^2
\]
for \( (u, v) \in H \). The space \( C_0^1(\Omega) \) is defined as
\[
C_0^1(\Omega) = \{ u \in C^1(\Omega); u = 0 \text{ on } \partial \Omega \}
\]
and we equip the Banach space \( C_0^1(\Omega) \times C_0^1(\Omega) \) with the norm
\[
\| U \|_{C_0^1(\Omega)} = \| u \|_{C_0^1(\Omega)} + \| v \|_{C_0^1(\Omega)}.
\]
2. **A priori estimates.** Let us first remark the following lemma based on the Hardy-Sobolev inequality. The proof can be found in [6].

**Lemma 1.** Let $1 < p < \frac{N+1}{N-1}$. Then there is a constant $C = C(p, \Omega)$ such that, for all $u, v \in H^1_0(\Omega)$ with $|u| \leq v \ a.e.$, it holds

$$
\int_{\Omega} |u|^p v \leq C \left( \int_{\Omega} |u|^p \right)^\alpha \left( \int_{\Omega} |\nabla v|^2 \right)^{\beta/2}
$$

where \( \alpha = 1 - \frac{N}{2+2N-(N-2)p} \in (0, 1) \) and \( \beta = 1 + \frac{Np}{2+2N-(N-2)p} \in (1, 2) \).

**Theorem 2.** Under the assumptions of Theorem 1, there is an increasing continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on $p$, $q$ and $\Omega$, such that $\rho(0) = 0$ and

$$
\|U\|_{C^1_0(\Omega)} \leq \rho(\|F\|_r),
$$

for all $U \in H$ solution of (4).

**Proof.** Let $U \in H$ be a weak solution of (4). Using $\Phi = (b \phi_1, (\lambda_1 - a) \phi_1)$ as a test function in (4), we find

$$
\int_{\Omega} (G(U), \Phi) = - \int_{\Omega} (F(x), \Phi) \leq C \|F(x)\|_r \quad . \tag{6}
$$

If $U$ is decomposed in the form $U = t \Phi + U_1$ with $U_1 = (u_1, v_1)$ orthogonal to $\Phi$ in $H$. It follows that

$$
\int_{\Omega} \langle U, \Phi \rangle = t \int_{\Omega} \langle \Phi, \Phi \rangle + \int_{\Omega} \langle U_1, \Phi \rangle,
$$

thus

$$
t = C \int_{\Omega} \langle U, \Phi \rangle \leq C \left( \int_{\Omega} b u_1 \phi_1 + (\lambda_1 - a) \int_{\Omega} v_1 \phi_1 \right).
$$

By Hölder inequality and using (6), it results

$$
t \leq C \left( \int_{\Omega} u_1^p \phi_1 \right)^{1/p} + C \left( \int_{\Omega} v_1^q \phi_1 \right)^{1/q} \leq C \left( \|F\|_{r}^{1/p} + \|F\|_{r}^{1/q} \right) . \tag{7}
$$

Now, the proof will be divided in two parts, according to the sign of $t$.

**Case 1.** $t \geq 0$. By (7) we obtain a bound for $t$. We need to find an estimate for $U_1$.

Multiplying the equation (4) by $U_1$ and using the decomposition of $U$, we obtain

$$
\|U_1\|^2 = \int_{\Omega} \langle AU_1, U_1 \rangle + \int_{\Omega} \langle G(U), U_1 \rangle + \int_{\Omega} \langle F(x), U_1 \rangle .
$$

The variational characterization of the eigenvalues implies that

$$
\left( 1 - \frac{1}{\lambda_2(A)} \right) \|U_1\|^2 \leq \int_{\Omega} \langle G(U), U_1 \rangle + \int_{\Omega} \langle F(x), U_1 \rangle .
$$

By Hölder inequality and using the Sobolev embedding theorem, we have

$$
\|U_1\|^2 \leq C \int_{\Omega} \langle G(U), U_1 \rangle + C \|F\|_r \|U_1\| . \tag{8}
$$

Now, we will estimate the integral in the right-hand side of (8):

$$
\int_{\Omega} \langle G(U), U_1 \rangle = \int_{\Omega} u_1^p u_1 + v_1^q v_1 \leq \int_{\Omega} u_1^p (u_1)_+ + \int_{\Omega} v_1^q (v_1)_+ .
$$
Using the fact that \((u_1)_+ \leq u_+\) and \((v_1)_+ \leq v_+\), we get
\[
\int_\Omega \langle G(U), U_1 \rangle \leq \int_\Omega u_+^{p+1} + \int_\Omega v_+^{q+1}.
\]
By applying Lemma 1, we obtain
\[
\int_\Omega \langle G(U), U_1 \rangle \leq \left( \int_\Omega u_+^a \phi_1 \right)^\alpha \left( \int_\Omega |\nabla u_+|^2 \right)^{\delta/2} + \left( \int_\Omega v_+^a \phi_1 \right)^{\alpha'} \left( \int_\Omega |\nabla v_+|^2 \right)^{\delta'/2},
\]
with \(\alpha, \alpha' \in (0, 1)\) and \(\delta, \delta' \in (1, 2)\). Using the inequality (6),
\[
\int_\Omega \langle G(U), U_1 \rangle \leq C \left( \|F\|^\alpha_r \|u\|^\delta + \|F\|^{\alpha' \delta}/p \right) \leq C \left( \|F\|^\alpha_r \|U\|^\delta + \|F\|^{\alpha' \delta}/p \right) \leq C \|F\|^\alpha_r \|U\|^\delta + C \|F\|^{\alpha' \delta}/p \|U\|^\delta.
\]
By inequality (7), we have
\[
\int_\Omega \langle G(U), U_1 \rangle \leq C \|F\|^{\alpha_r} \left( \|F\|^{1/p} + \|F\|^{1/q} \right) \leq C \|F\|^{\alpha'_r} \left( \|F\|^{1/p} + \|F\|^{1/q} \right) \leq C \|F\|^{\alpha'_r} \|U\|^\delta + C \|F\|^{\alpha'_r} \|U_1\|\|U\|^\delta.
\]
Replacing the estimate (9) in (8), we get
\[
\|U_1\|^2 \leq C \left( \|F\|^{\alpha+r + \delta}/p + \|F\|^{\alpha+r + \delta}/q + \|F\|^{\alpha'+r + \delta'/p} + \|F\|^{\alpha'+r + \delta'/q} \right) + C \|F\|^{\alpha'_r} \|U\|^\delta + C \|F\|^{\alpha'_r} \|U_1\|^\delta + C \|F\|_r \|U_1\|.
\]
By Young’s inequality, we deduce that
\[
\|U_1\| \leq C \left( \|F\|^{\alpha+r + \delta}/p + \|F\|^{\alpha+r + \delta}/q + \|F\|^{\alpha'+r + \delta'/p} + \|F\|^{\alpha'+r + \delta'/q} \right)^{1/2} + C \|F\|^{\alpha_r/(2-\delta)} + C \|F\|^{\alpha'_r/(2-\delta') + C \|F\|_r}.
\]
Therefore
\[
\|U\| \leq C \left( \|F\|^{1/p} + \|F\|^{1/q} \right) + C \left( \|F\|^{\alpha+r + \delta}/p + \|F\|^{\alpha+r + \delta}/q + \|F\|^{\alpha'+r + \delta'/p} + \|F\|^{\alpha'+r + \delta'/q} \right)^{1/2} + C \|F\|^{\alpha'_r/2-\delta} + C \|F\|^{\alpha'_r/2-\delta'} + C \|F\|_r.
\]
Replacing the values \(\alpha, \alpha', \delta\) and \(\delta'\), given by Lemma 1, we conclude
\[
\|U\| \leq C \max \left\{ \|F\|^{1/p}, \|F\|^{1/q}, \|F\|^{\alpha'-r}_{\alpha'-r}, \|F\|^{\alpha'_r/2-\delta} \right\}.
\]
A bootstrap argument applied to (1) and the previous bound of \(U\) in \(H\)-norm will give us the claim of the Theorem.

**Case 2.** \(t < 0\). By Hopf’s Maximum Principle, there is \(\epsilon > 0\) such that
\[
w \in B_{C_0(\Omega)}(\phi_1, \epsilon) \Rightarrow w > 0 \text{ in } \Omega \text{ and } \frac{\partial w}{\partial \eta} < 0 \text{ on } \partial \Omega,
\]
where \(\eta\) denotes the exterior normal derivative at the boundary of \(\Omega\). Let us remember that our solution \(U = (u, v)\) of problem (1), as well as \(U_1 = (u_1, v_1)\), belongs to
Hence, let \( \epsilon_0 \) be the supremum of the \( \epsilon \)'s above. We claim that
\[
-\frac{u_1}{bt} \notin B_{C_0}(\Omega, \epsilon_0) \quad \text{and} \quad -\frac{v_1}{(\lambda_1 - a)t} \notin B_{C_0}(\Omega, \epsilon_0).
\]
Otherwise, we would have
\[
\frac{u}{bt} = \phi_1 - \left( -\frac{u_1}{bt} \right) \in B_{C_0}(\Omega, \epsilon_0)
\]
and
\[
\frac{v}{(\lambda_1 - a)t} = \phi_1 - \left( -\frac{v_1}{(\lambda_1 - a)t} \right) \in B_{C_0}(\Omega, \epsilon_0).
\]
This is a contradiction with the fact that \( u_+, v_+ \neq 0, b > 0, \lambda_1 > a \) and \( t < 0 \).

Hence,
\[
|t| \leq \frac{1}{b\epsilon_0} \|u_1\|_{C_0} \quad \text{and} \quad |t| \leq \frac{1}{(\lambda_1 - a)\epsilon_0} \|v_1\|_{C_0},
\]
and we conclude that,
\[
|t| \leq C \|u_1\|_{C_0} + C \|v_1\|_{C_0} = C \|U_1\|_{C_0}.
\]
Thus it is enough to find an a priori bound for \( \|U_1\|_{C_0} \).

We note that inequality (8) remains true for \( t < 0 \). Thus, in this case \( u_+ < |u_1| \) and \( v_+ < |v_1| \). Hence, by Lemma 1, we have
\[
\left| \int_{\Omega} G(U, U_1) \right| \leq \left( \int_{\Omega} u_+^r \phi_1 \right)^\alpha \left( \int_{\Omega} |\nabla u_1| \right)^{s/2} + \left( \int_{\Omega} v_+^r \phi_1 \right)^\alpha \left( \int_{\Omega} |\nabla v_1| \right)^{s'/2}.
\]

And using (6) we get
\[
\left| \int_{\Omega} G(U, U_1) \right| \leq C \|F\|_r^\alpha \|u_1\|^\delta + C \|F\|_r^\alpha' \|v_1\|^{s'}\]
\[
\leq C \|F\|_r^\alpha \|U_1\|^{\delta} + C \|F\|_r^\alpha' \|U_1\|^{s'}.
\]
Replacing (12) in (8), we find
\[
\|U_1\|^2 \leq C \|F\|_r^\alpha \|U_1\|^\delta + C \|F\|_r^\alpha' \|U_1\|^{s'} + C \|F\|_r \|U_1\|.
\]
Since \( \alpha, \alpha' \in (0, 1) \) and \( \delta, s' \in (1, 2) \) by Young’s inequality we deduce that
\[
\|U_1\| \leq C \|F\|_r^\alpha \|U_1\|^\delta + C \|F\|_r^\alpha' \|U_1\|^{s'} + C \|F\|_r.
\]
We now use that \( U_1 \) solves the problem
\[
\begin{cases}
-\Delta U_1 = AU_1 + G(U) + F(x) & x \in \Omega \\
U_1 = 0 & x \in \partial \Omega,
\end{cases}
\]
a bootstrap argument to this equation gives us that
\[
\|U_1\|_{2, \rho} \leq C (\|F\|_r^\gamma + \|U_1\|^\eta),
\]
with constants \( \gamma, \eta \geq 1 \). Replacing (13) in (14), result
\[
\|U_1\|_{2, \rho} \leq \rho(\|F\|_r).
\]
Recalling that $W^{2,r}(\Omega) \hookrightarrow C^1(\overline{\Omega})$ because $r > N$, we obtain
\[ \|U_1\|_{C^1(\overline{\Omega})} \leq \rho(\|F\|_r). \] (15)

Using the decomposition $U = U_1 + t\Phi$ and inequalities (11) and (15), we conclude that
\[ \|U\|_{C^1(\overline{\Omega})} \leq \rho(\|F\|_r), \]
which completes the proof.

3. Some lemmas. In this section we prove two lemmas. The first lemma will be used to prove that all solutions of the system are non-degenerate, since $f$ and $g$ are small enough. The second will be used in the computation of the index of such solutions. In order to do that we need to consider the linearization of problem (1) at some solution $(u_0, v_0)$, which is
\[
\begin{aligned}
-\Delta w &= aw + bz + p(u_0^+)^{p-1}w \quad x \in \Omega \\
-\Delta z &= bw + cz + q(v_0^+)^{q-1}z \quad x \in \Omega \\
w &= z = 0 \quad x \in \partial\Omega.
\end{aligned}
\] (16)

**Lemma 2.** There exists $\epsilon > 0$ such that for any $m, k \in L^\infty$, such that $m, k \geq 0$ a.e., $m > 0$ or $k > 0$ in a set of positive measure, and $\|m\|_{L^\infty}, \|k\|_{L^\infty} < \epsilon$, then the system
\[
\begin{aligned}
-\Delta w &= aw + bz + m(x)w \quad x \in \Omega \\
-\Delta z &= bw + cz + k(x)z \quad x \in \Omega \\
w &= z = 0 \quad x \in \partial\Omega,
\end{aligned}
\] (17)
possesses only the trivial solution $w = z = 0$.

**Proof.** We may write (17) in vector form as
\[
\begin{aligned}
-\Delta W &= A(x)W \quad x \in \Omega \\
W &= 0 \quad x \in \partial\Omega,
\end{aligned}
\]
with
\[
A(x) = \begin{pmatrix}
a + m(x) & b \\
b & c + k(x)
\end{pmatrix}
\]
and
\[
W = \begin{pmatrix}
w \\
z
\end{pmatrix}.
\]
Set $(x, y) \in \Omega \times \mathbb{R}^2$ with $y = (y_1, y_2)$. The conditions given on the functions $m$ and $k$ assure that the expression
\[
\left\langle \left( A(x) - A \right) y, y \right\rangle = m(x)y_1^2 + k(x)y_2^2
\]
is nonnegative in $\Omega \times \mathbb{R}^2$ and nontrivial if $y_1 \neq 0$ and $y_2 \neq 0$. We conclude that $A \prec A(x)$, (Definition 1). Moreover, the eigenfunction $\Phi_1^A$ associated to eigenvalue $\lambda_1(A)$ fills the Unique Continuation Property, see Definition 2. Hence, using the Proposition 1, we obtain $\lambda_1(\tilde{A}) < \lambda_1(A) = 1$.

On the other hand, each component of the matrix $\tilde{A}(x)$ converges in $L^\infty$ to that of $A$. Using the continuity of the eigenvalues relative to the weights, we conclude that $\lambda_j(\tilde{A}) \to \lambda_j(A)$ for $j = 1, 2, \ldots$. In particular for $j = 2$ we have that $\lambda_2(\tilde{A}) \to \lambda_2(A) > \lambda_1(A) = 1$. Hence, $\lambda_1(A) < 1 < \lambda_2(\tilde{A})$ and $w = z = 0$ is the only solution of the problem (17).
Lemma 3. There exists $\epsilon > 0$ such that for any $m, k \in L^\infty$ satisfying the conditions in the previous lemma, the following eigenvalue problem

$$
\begin{aligned}
-\Delta w &= \mu (aw + bz + m(x)w) & x \in \Omega \\
-\Delta z &= \mu (bw + cz + k(x)z) & x \in \Omega \\
w &= z = 0 & x \in \partial \Omega.
\end{aligned}
$$

(18)

has only one eigenvalue $\mu$ in the interval $(0, 1)$.

Proof. We may write (18) in vector form as

$$
\begin{aligned}
-\Delta W &= \lambda \tilde{A} W & x \in \Omega \\
W &= 0 & x \in \partial \Omega,
\end{aligned}
$$

with $\tilde{A} = \left( \begin{array}{cc} a + m(x) & b \\ b & c + k(x) \end{array} \right)$ and $W = \left( \begin{array}{c} w \\ z \end{array} \right)$.

Similarly to the previous result $A \prec \tilde{A}$. Using Proposition (1) we obtain $\lambda_1 (\tilde{A}) < \lambda_1 (A) = 1$. Now we take $s$ small enough such that the matrix $\tilde{A}$ converge term by term in $L^\infty$ for the terms of $A$. Using the continuity of the eigenvalues relative to the weights, we conclude that $\lambda_2 (\tilde{A}) \rightarrow \lambda_2 (A) > \lambda_1 (A) = 1$. Therefore $\lambda_1 (\tilde{A}) < 1 \prec \lambda_2 (\tilde{A})$.

4. Proof of the Theorem 1. Let $T_F : C^1_0 (\Omega)^2 \rightarrow C^1_0 (\Omega)^2$ map such that

$$
T_F (U) = (-\Delta)^{-1} (AU + G (U) + F (x))
$$

$$
= ((-\Delta)^{-1} (au + bv + w^p) + f (x)) , (-\Delta)^{-1} (bu + cv + v^q + g (x))).
$$

Notice that, $T_F$ is continuous, compact and $T_F (u, v) = (u, v)$ if, and only if, $U = (u, v)$ solves (1). Choose $F_1 = (f_1, g_1)$ such that

$$
f_1 = -(\gamma b \phi_1)^p \quad \text{and} \quad g_1 = - (\gamma (\lambda_1 - a) \phi_1)^q
$$

with $\gamma > 0$. Then, it is easy to see that, $U_1 = (\gamma b \phi_1, \gamma (\lambda_1 - a) \phi_1)$ is a solution of problem (1) with $F = F_1$. Moreover, from the a-priori estimates of Theorem 2, taking $\gamma$ small enough, we can apply Lemma 2 for

$$
m(x) = pu^{p-1}_+ \quad \text{and} \quad k(x) = q v^{q-1}_+,$
$$

where $(u, v)$ is any arbitrary solution of (1) with $F = F_1$. It can be conclude, as a consequence of Lemma 2, that $(u, v)$ is non-degenerate. In addition, for some $R_{(u,v)}$ sufficiently small, we have

$$
\deg \left( I - T_{F_1}, B_{C^1_0 (\Omega)^2}, (u, v), R_{(u,v)} \right, 0 \right) = (-1)^{\beta},
$$

where $\beta$ is a number of characteristic values, between 0 and 1, of the eigenvalue problem

$$
\begin{aligned}
-\Delta w &= \mu (aw + bz + pu_{+}^{p-1}w) & x \in \Omega \\
-\Delta z &= \mu (bw + cz + qv_{+}^{q-1}z) & x \in \Omega \\
w &= z = 0 & x \in \partial \Omega.
\end{aligned}
$$

(19)

By Lemma 3, we can assume that that $\beta = 1$. We can also conclude that $(u, v)$ is an isolated solution. Moreover, the number of the solutions is finite. Then for $R$ large enough

$$
\deg (I - T_{F_1}, B_{C^1_0 (\Omega)^2}, (0, R), 0) = \sum (-1) \neq 0.
$$
Now, consider the following homotopy \( H : [0, 1] \times C^1_0(\Omega)^2 \to C^1_0(\Omega)^2 \) with 
\[
H(\tau, U) = ((I - (-\Delta)^{-1})(AU + G(U) + (1 - \tau)F(x) + \tau F_1(x))).
\]
Then \( H(\tau, U) = 0 \) if, and only if, \( U \) is a solution of the system
\[
\begin{cases}
-\Delta U = AU + G(U) + (1 - \tau)F(x) + \tau F_1(x) & x \in \Omega \\
U = 0 & x \in \partial\Omega.
\end{cases}
\]
By the a priori bounds, Theorem 2, all solutions of the previous system are uniformly bounded in \( C^1_0(\Omega) \times C^1_0(\Omega) \). Hence for \( R > 0 \) large enough, we have \( H(\tau, U) \neq 0 \) for all \((\tau, U) \in [0, 1] \times \partial B_{C^1_0(\Omega)}^2(0, R) \). It follows that
\[
\deg(H(0, U), B_{C^1_0(\Omega)^2}(0, R), 0) = \deg(H(1, U), B_{C^1_0(\Omega)^2}(0, R), 0).
\]
As \( H(0, U) = I - T_F \) and \( H(1, U) = I - T_{F_1} \), we conclude that
\[
\deg(I - T_F, B_{C^1_0(\Omega)^2}(0, R), 0) = \deg(I - T_{F_1}, B_{C^1_0(\Omega)^2}(0, R), 0) \neq 0.
\]
This finish the proof.

**Appendix.** Let \( M_2(\Omega) \) be the set of all symmetric matrices of the form
\[
A(x) = \begin{pmatrix}
a(x) & b(x) \\
b(x) & c(x)
\end{pmatrix},
\]
where the functions \( a, b, c \in C(\bar{\Omega}, \mathbb{R}) \) satisfy
1. \( b(x) \geq 0 \) for all \( x \in \bar{\Omega} \) and \( b \neq 0 \);
2. \( \max_{x \in \Omega} \max\{a(x), c(x)\} > 0 \).

We can consider the weighted eigenvalue problem
\[
\begin{cases}
-\Delta U = \lambda A(x)U & x \in \Omega \\
U = 0 & x \in \partial\Omega.
\end{cases}
\]
and obtain a sequence of eigenvalues
\[
0 < \lambda_1(A) < \lambda_2(A) \leq ... \leq \lambda_k(A) \leq ...
\]
such that
\[
\frac{1}{\lambda_k(A)} = \sup_{F_k} \inf_{y \in F_k} \int_{\Omega} \langle A(x)y, y \rangle; y \in F_k \ \text{e} \ \|y\| = 1, \tag{20}
\]
where \( F_k \) varies over all \( k \)-dimensional subspaces of \( H \). Moreover, \( \lambda_k(A) \to \infty \) when \( k \to \infty \). The eigenvalue \( \lambda_1(A) \) is positive, simple, isolated and the associated eigenfunction can be choose with positive components. Additional, if we set \( V_k = \text{span}\{\Phi^A_1, ..., \Phi^A_k\} \), with \( \Phi^A_k \) the associated eigenfunction, we can decompose \( H \) as \( H = V_k \oplus V_k^\perp \).

Furthermore, the followings variational inequalities hold
\[
\|y\|^2 \leq \lambda_k(A) \int_{\Omega} \langle Ay, y \rangle \quad \forall y \in V_k, \tag{21}
\]
and
\[
\|y\|^2 \geq \lambda_{k+1}(A) \int_{\Omega} \langle Ay, y \rangle \quad \forall y \in V_k^\perp. \tag{22}
\]
These results can be found in detail in [2, 7, 10].
Definition 1. Let $A(x), B(x) \in M_2(\Omega)$ with $\Omega \in \mathbb{R}^N$. We say that $A \leq B$ if
\[
\langle A(x)y, y \rangle \leq \langle B(x)y, y \rangle,
\]
for all $(x, y) \in \Omega \times \mathbb{R}^2$. Moreover, it defines the relation $A \prec B$ if $A \leq B$ and $B - A$ is positive definite in some $\tilde{\Omega} \subseteq \Omega$ with $|\tilde{\Omega}| > 0$, this is,
\[
\langle (B(x) - A(x)y), y \rangle > 0
\]
for a.e. $x \in \tilde{\Omega}$ and $y = (y_1, y_2)$ with $y_1 \neq 0$ and $y_2 \neq 0$.

Definition 2. We say that a family of vector functions fills the Unique Continuation Property, (U.C.P.), if no component function vanishes on a set of positive measure.

Proposition 1. Let $A$ e $B \in M_2(\Omega)$ be two weights with $A \prec B$. If the eigenfunctions associated to $\lambda_j(A)$, with $j \in \mathbb{Z}_0$, fills the U.C.P., then $\lambda_j(A) > \lambda_j(B)$.

Proof. We argue as in [8]. Let $j > 0$, since the extreme in (20) is achieved, exists $F_j \in H$ such that
\[
\frac{1}{\lambda_j(A)} = \inf \left\{ \int_{\Omega} \langle Az, z \rangle ; z \in F_j \text{ and } \|z\| = 1 \right\}. \tag{23}
\]
Pick $z \in F_j$ with $\|z\| = 1$, in this moment we have two cases:

Case 1. $y$ achieves the infimum in (23). Then $z$ is an eigenfunction associated to $\mu_j(A)$ and so, by the unique continuation property we obtain
\[
\frac{1}{\lambda_j(A)} = \int_{\Omega} \langle Ay, y \rangle < \int_{\Omega} \langle By, y \rangle.
\]

Case 2. $y$ does not achieve the infimum in (23). Then
\[
\frac{1}{\lambda_j(A)} = \inf \left\{ \int_{\Omega} \langle Ay, y \rangle ; y \in F_j \text{ and } \|y\| = 1 \right\} < \int_{\Omega} \langle Ay, y \rangle \leq \int_{\Omega} \langle By, y \rangle.
\]
Thus, in any case,
\[
\frac{1}{\lambda_j(A)} < \int_{\Omega} \langle By, y \rangle.
\]
By a compactness argument, we conclude that
\[
\frac{1}{\lambda_j(A)} < \sup_{F_j} \inf \left\{ \int_{\Omega} \langle By, y \rangle ; y \in F_j \text{ and } \|y\| = 1 \right\} = \frac{1}{\lambda_j(B)}.
\]

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