Gaussian density estimates for solutions of fully coupled forward-backward SDEs

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Abstract
We obtain upper and lower Gaussian density estimates for the laws of each component of the solution to a one-dimensional fully coupled forward-backward SDE. Our approach relies on the link between FBSDEs and quasilinear parabolic PDEs, and is fully based on the use of classical results on PDEs rather than on manipulation of FBSDEs, compared to other papers on this topic. This essentially simplifies the analysis.

KEYWORDS
density estimates, forward-backward SDEs, Malliavin calculus, Nourdin–Viens’ formula

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1 | INTRODUCTION

Forward-backward stochastic differential equations (FBSDEs) have numerous applications in stochastic control theory and mathematical finance (see, for instance, [3, 7, 13, 14]). Several recent papers [1, 2, 11] studied existence of densities and density estimates for the laws of solutions of one-dimensional backward SDEs (BSDEs) ([1, 2, 11]). To the best of authors’ knowledge, the aforementioned problem has never been addressed in connection to the laws of solutions to fully coupled FBSDEs.

In this paper, we are concerned with the fully coupled one-dimensional FBSDE

\[
\begin{align*}
X_t &= x + \int_0^t f(s, X_s, Y_s, Z_s) \, ds + \int_0^t \sigma(s, X_s, Y_s) \, dB_s, \\
Y_t &= h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dB_s,
\end{align*}
\]

where \(B_t\) is a one-dimensional standard Brownian motion, and \(f, \sigma, g, h\) are functions defined on appropriate spaces and taking values in \(\mathbb{R}\). It is known (see, e.g., [10, 13, 14], and also references therein) that there exists a unique \(\mathcal{F}_t\)-adapted solution \((X_t, Y_t, Z_t)\) of the system (1.1) under some appropriate smoothness and boundedness conditions on its coefficients, where \(\mathcal{F}_t\) is the augmented filtration generated by the Brownian motion \(B_t\).

Our goal is to provide conditions that guarantee the existence of the densities for the laws of \(X_t, Y_t,\) and \(Z_t\), and that allow Gaussian estimates of these densities. Additionally, we obtain estimates of the tail probabilities of the laws of the solution components. Our approach works for a large class of the FBSDE coefficients. In particular, the BSDE generator \(g\) is not assumed to depend just on some of the spatial variables (unlike [2, 11]), or to be linear in \(Z_s\) (unlike [1]). To add even more generality, we obtain our density estimates in the situation when the generator \(g\) has the quadratic growth in the last variable,
and hence, it is not necessarily Lipschitz in $Z_s$. Our method relies on the analysis of the quasilinear parabolic PDE associated to FBSDE (1.1):

$$\begin{cases}
\frac{1}{2} \sigma^2(t, x, u) \partial^2_{xx} u + f(t, x, u, \sigma(t, x, u) \partial_x u) \partial_x u + g(t, x, u, \sigma(t, x, u) \partial_x u) + \partial_t u = 0, \\
u(T, x) = h(x),
\end{cases} \tag{1.2}$$

where $u, \partial_t u, \partial_x u$, and $\partial^2_{xx} u$ are everywhere evaluated at $(t, x)$. It is well known (see, e.g., [10]) that if $u$ is the $C^{1,2}_b$-solution to the final value problem (1.2), then it is related to the solution of the FBSDE (1.1) by the formulas

$$Y_t = u(t, X_t), \quad Z_t = \partial_x u(t, X_t) \sigma(t, X_t, u(t, X_t)). \tag{1.3}$$

where $X_t$ is the unique $F_t$-adapted solution to the SDE

$$X_t = x + \int_0^t \tilde{f}(t, X_t) + \int_0^t \tilde{\sigma}(t, X_t) \, dB_t \quad \text{with}$$

$$\tilde{f}(t, x) = f(t, x, u(t, x), \partial_x u(t, x) \sigma(t, x, u(t, x))), \quad \tilde{\sigma}(t, x) = \sigma(t, x, u(t, x)). \tag{1.4}$$

Since the Malliavin differentiability and the existence of bounds for $D_x X_t$ are well known facts (see, e.g., [4, 12]), then, provided that the coefficients of the PDE (1.2) are sufficiently smooth, the Malliavin differentiability of $Y_t$ and $Z_t$ follows immediately, and, moreover, the existence of bounds for $D_y Y_t$ and $D_z Z_t$ is reduced to the existence of positive lower bounds for $\partial_x u$ and $\partial_x (\partial_x u \sigma)$. This can be done by the classical comparison theorem for PDEs (see, e.g., [6]).

Let us remark that our assumptions allow the BSDE generators $g(t, x, u, p)$ to have the quadratic growth in $p$. It happens because the four step scheme, developed in [10], also works for a quadratic BSDE, provided that it is one-dimensional. This results are valid for PDEs of several space variables.

Overall, compared to the previous works, our analysis is simpler, many of the assumptions are dropped or easier formulated (cf. [11]), while the FBSDE itself is, overall, more general (in particular, fully coupled) and the density estimates hold on the entire real line.

## 2 | PRELIMINARIES

For simplicity, all PDEs considered in this section are one-dimensional and with respect to one space variable, although all the results are valid for PDEs of several space variables.

### 2.1 | Useful function spaces

We start by defining some function spaces used in this paper.

The Hölder space $C^{2+\beta}_b(\mathbb{R})$, $\beta \in (0, 1)$, is understood as the (Banach) space with the norm

$$\|\phi\|_{C^{2+\beta}_b(\mathbb{R})} = \|\phi\|_{C^2(\mathbb{R})} + [\phi'']_{\beta'},$$

where $[\phi]_{\beta} = \sup_{x, y \in \mathbb{R}, 0 < |x-y| < 1} \frac{|\phi(x) - \phi(y)|}{|x-y|^\beta}$,

and $C^2(\mathbb{R})$ denotes the space of twice continuously differentiable functions on $\mathbb{R}$ with bounded derivatives up to the second order.

For a function $\phi(x, \xi)$ of more than one variable, the Hölder constant with respect to $x$ is defined by

$$[\phi]_{\beta}^x = \sup_{x, x' \in \mathbb{R}, 0 < |x-x'| < 1} \frac{|\phi(x, \xi) - \phi(x', \xi)|}{|x-x'|^\beta},$$

i.e., it is understood as a function of $\xi$. 
The Hölder spaces $C^{1+\frac{\beta}{2},2+\beta}_{b}(\{0,T\} \times \mathbb{R})$, $C_{b}^{\frac{\beta}{2},\beta}(\{0,T\} \times \mathbb{R})$, $C_{b}^{\frac{\beta}{2},1+\beta}(\{0,T\} \times \mathbb{R})$, and $C_{b}^{0,\beta}(\{0,T\} \times \mathbb{R})$ (\(\beta \in (0,1)\)) are defined, respectively, to be Banach spaces of functions $\phi(t,x)$ possessing the finite norms

$$\|\phi\|_{C^{1+\frac{\beta}{2},2+\beta}_{b}(\{0,T\} \times \mathbb{R})} = \|\phi\|_{C^{1+\frac{\beta}{2},2+\beta}_{b}(\{0,T\} \times \mathbb{R})} + \sup_{t \in [0,T]} \left( \partial_{t} \phi \right)^{\frac{\beta}{2}} + \sup_{t \in [0,T]} \left( \partial_{x}^{2} \phi \right)^{\frac{\beta}{2}} + \sup_{x \in \mathbb{R}} \left( \partial_{x} \phi \right)^{1+\beta} + \sup_{x \in \mathbb{R}} \left( \partial_{x} \phi \right)^{1+\beta}$$

where $C^{1,\frac{\beta}{2}}_{b}(\{0,T\} \times \mathbb{R})$ is the space of bounded continuous functions whose derivatives up to the first order in $t \in [0,T]$ and the second order in $x \in \mathbb{R}$ are bounded and continuous on $[0,T] \times \mathbb{R}$, and $C_{b}([0,T] \times \mathbb{R})$ is the space of bounded continuous functions.

\[\textbf{2.2 Some results on quasilinear parabolic PDEs}\]

Here we formulate some results on linear and quasilinear parabolic PDEs which will be useful in the next section.

Consider the Cauchy problem for a one-dimensional PDE of one space variable

$$\begin{cases}
    a(t,x,u)\partial_{x}^{2}u + f(t,x,u,\partial_{x}u)\partial_{x}u + g(t,x,u,\partial_{u}u) - \partial_{t}u = 0, \\
    u(0,x) = h(x),
\end{cases}$$

(2.1)

where $u$, $\partial_{t}u$, $\partial_{x}u$, and $\partial_{x}^{2}u$ are everywhere evaluated at $(t,x)$.

In what follows, $(t,x,u,p)$ denotes the element of $[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $f$ and $g$ are functions of $(t,x,u,p)$, and $a$ is a function of $(t,x,u)$. Further, $\partial_{t}$, $\partial_{x}$, $\partial_{u}$, and $\partial_{p}$ denote the partial derivatives w.r.t. $t$, $x$, $u$, and $p$, respectively.

The theorem below, proved in [8] (Theorem 8.1, Section V, p. 495), provides the existence and uniqueness of solution to the problem (2.1).

\textbf{Theorem 2.1.} Assume conditions (i)–(vii) below:

(i) for all $(t,x,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, $\nu(|u|) \leq a(t,x,u) \leq \mu(|u|)$, where $\nu$ and $\mu$ are non-increasing and, respectively, non-decreasing positive functions;

(ii) for all $(t,x,u,p) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $g(t,x,u,p)u \leq c_{1} + c_{2}|u|^{2}$, where $c_{1}$ and $c_{2}$ are positive constants;

(iii) the function $h$ is of class $C^{2+\beta}_{b}(\mathbb{R})$, $\beta \in (0,1)$.

(iv) $\partial_{x}a$ and $\partial_{u}a$ exist and $|\partial_{x}a| + |\partial_{u}a| \leq a$, where $a > 0$ is a constant;

(v) there exists a positive non-decreasing function $\tilde{\mu}$ such that $|f| \leq \tilde{\mu}(|u|)(1 + |p|)$ and $|g| \leq \tilde{\mu}(|u|)(1 + |p|^{2})$ everywhere on $[0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$;

(vi) the functions $a$, $\partial_{t}a$, $\partial_{x}a$, $\partial_{u}a$, $f$, and $g$ are Hölder continuous in $t$, $x$, $u$, and $p$ with exponents $\frac{\beta}{2}$, $\beta$, $\beta$, and $\beta$, respectively, and globally bounded Hölder constants;

(vii) the derivatives $\partial_{x}f$, $\partial_{x}g$, $\partial_{p}f$, $\partial_{p}g$ exist and $\sup_{|u| + |p| \leq N} (|\partial_{x}f| + |\partial_{x}g| + |\partial_{p}f| + |\partial_{p}g|) \leq \gamma(N)$, where $\gamma(N)$ is a positive constant depending on $N$.

Then, there exists a unique $C^{1+\frac{\beta}{2},2+\beta}_{b}(\{0,T\} \times \mathbb{R})$-solution to the problem (2.1).

Now consider a Cauchy problem for a linear PDE:

$$\begin{cases}
    a(t,x)\partial_{x}^{2}u + b(t,x)\partial_{x}u + c(t,x)u + g(t,x) - \partial_{t}u = 0, \\
    u(0,x) = \phi(x),
\end{cases}$$

(2.2)
We have the following result, proved in [6] (Theorem 12, p. 25, and Theorem 10, p. 44), on the solvability of the problem (2.2) and the representation of its solution via the fundamental solution \( \Gamma(t, x, s, z) \) to the PDE (2.2).

**Theorem 2.2.** Let the PDE (2.2) be uniformly parabolic, and let the coefficient \( a \) of (2.2) be of class \( C_b^{\beta, \beta}([0, T] \times \mathbb{R}) \), \( \beta \in (0, 1) \). Further let the coefficients \( b, c \), and the function \( g \) be of class \( C_b^{0, \beta}([0, T] \times \mathbb{R}) \), and the initial condition \( \varphi \) be of class \( C_b(\mathbb{R}) \). Then, there exists a unique \( C_b^{1, 2}([0, T] \times \mathbb{R}) \)-solution to the problem (2.2). Moreover, this solution takes the form

\[
\Psi(x) = \int_{\mathbb{R}} \Gamma(t, x, 0, z) \varphi(z) \, dz - \int_0^t \int_{\mathbb{R}} \Gamma(t, x, s, z) g(s, z) \, ds \, dz.
\]

Introduce the linear differential operator

\[
L = a(t, x) \partial_{xx}^2 u + b(t, x) \partial_x u + c(t, x) u - \partial_t u.
\]

Theorem 2.3 below, provides conditions when the solution \( u \) to (2.2) belongs to class \( C_b^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{R}) \). The theorem was proved in [8] (Theorem 5.1, p. 320).

**Theorem 2.3.** Let the PDE (2.2) be uniformly parabolic, and let the coefficients of the operator \( L \) and the function \( g \) belong to class \( C_b^{\frac{\beta}{2}, -\beta}([0, T] \times \mathbb{R}) \). Further let the initial condition \( \varphi \) belong to class \( C_b^{2+\beta}(\mathbb{R}) \). Then, the problem (2.2) has a unique \( C_b^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{R}) \)-solution \( u(t, x) \).

The following below comparison theorem, proved in [6] (Theorem 9, p. 43), will be an important tool in the next section.

**Theorem 2.4.** Let the coefficients of \( L \) be bounded and continuous on \([0, T] \times \mathbb{R}\). Assume that \( L u \leq 0 \) on \([0, T] \times \mathbb{R}\) and that \( u \) is bounded. If \( \varphi(x) \geq 0 \) on \( \mathbb{R} \), then \( u(t, x) \geq 0 \) on \([0, T] \times \mathbb{R}\).

### 2.3 A link between FBSDEs and quasilinear parabolic PDEs

It is well known that there is a link between the FBSDE (1.1) and a quasilinear parabolic PDE of form (2.1) (see, e.g., [10]). Specifically, the final value problem for the PDE associated to the FBSDE (1.1) takes the form (1.2). By introducing the time-changed function \( \theta(t, x) = \Psi(T - t, x) \), we transform (1.2) to the Cauchy problem

\[
\frac{1}{2} \sigma^2(T - t, x, \theta) \partial_{xx}^2 \theta + f(T - t, x, \theta, \sigma(T - t, x, \theta) \partial_x \theta) \partial_x \theta + g(T - t, x, \theta, \sigma(T - t, x, \theta) \partial_x \theta) - \partial_t \theta = 0,
\]

\[
\theta(0, x) = h(x).
\]

Remark that under assumptions (i)–(vii) of Theorem 2.1, the existence and uniqueness of a \( C_b^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{R}) \)-solution to (2.3) is established, and is equivalent to the existence and uniqueness of a \( C_b^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{R}) \)-solution \( u \) to the final value problem (1.2). The theorem below provides an explicit solution to the FBSDE (1.1) via the solution \( u \).

**Theorem 2.5.** Let the functions \( f, g, \) and \( h \) satisfy assumptions (ii), (iii), (v)–(vii) of Theorem 2.1. Further let the function \( \sigma \) satisfy assumptions (i), (iv), and (vi) of the same theorem in the place of the function \( a \). Then, there exists a unique \( \mathcal{F}_t \)-adapted solution \((X_t, Y_t, Z_t)\) to the FBSDE (1.1). Moreover, this solution takes the form (1.3) with \( u \) being the unique \( C_b^{1+\frac{\beta}{2}, 2+\beta}([0, T] \times \mathbb{R}) \)-solution, \( \beta \in (0, 1) \), to the problem (1.2).

**Remark 2.6.** The solution to the FBSDE (1.1) is understood as in [10].

The proof of Theorem 2.5 is exactly the same as the proof of Theorem 4.1 in [10], where the latter result is known as the four step scheme. It relies exceptionally on the existence of the unique \( C_b^{1, 2} \)-solution to the Cauchy problem (2.3). This implies that the assumptions of Theorem 2.1 guarantee the existence of a unique solution to the FBSDE (1.1). These assumptions turn out to be more general than in [10], but they are restricted to the case of just one PDE. Remark, that the Cauchy problem for systems of PDEs was not actually solved in [8], so the authors of [10] had to fill this gap imposing own assumptions. However, for the case of just one PDE, the Cauchy problem is solved in [8], and the result is represented by Theorem 8.1 in Section V (p. 495), so we make use of its more general assumptions.
2.4 | The Malliavin derivative

Here we describe the elements from the Malliavin calculus that we need in the paper. We refer the reader to [12] for a more complete exposition.

Consider $\mathcal{H}$ to be a real separable Hilbert space and to be $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbb{E} (B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$.

We denote by $D$ the Malliavin derivative operator that acts on smooth functions of the form $F = g (B(\varphi_1), \ldots, B(\varphi_n))$ ($g$ is a smooth function with compact support and $\varphi_i \in \mathcal{H}, i = 1, \ldots, n$):

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial X_i} (B(\varphi_1), \ldots, B(\varphi_n)) \varphi_i.$$  

It can be checked that the operator $D$ is closable from $S$ (the space of smooth functionals as above) into $L^2(\Omega; \mathcal{H})$ and it can be extended to the space $\mathcal{D}^{1,2}$ which is the closure of $S$ with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|DF\|_{\mathcal{H}}^p.$$

2.5 | Gaussian density estimates

Theorem 2.7 below is an important tool that we will use to obtain the existence of densities and density estimates. It was proved in [5, Theorem 2.4].

**Theorem 2.7.** Let $F \in D^{1,2}$ be a random variable such that

$$0 < l \leq \int_0^\infty D_s F \mathbb{E}[D_s F | \mathcal{F}_s] \, ds \leq L \quad a.s.,$$  

where $l$ and $L$ are constants. Then, $F$ possesses a density $p_F$ with respect to the Lebesgue measure. Moreover, for almost all $x \in \mathbb{R}$, the density $p_F$ satisfies

$$\frac{\mathbb{E}|F - \mathbb{E}[F]|}{2L} \exp \left( -\frac{(x - \mathbb{E}[F])^2}{2l} \right) \leq p_F(x) \leq \frac{\mathbb{E}|F - \mathbb{E}[F]|}{2L} \exp \left( -\frac{(x - \mathbb{E}[F])^2}{2L} \right).$$

Furthermore, for all $x > 0$, the tail probabilities satisfy

$$\mathbb{P}(F \geq x) \leq \exp \left( -\frac{(x - \mathbb{E}[F])^2}{2L} \right) \quad \text{and} \quad \mathbb{P}(F \leq -x) \leq \exp \left( -\frac{(x + \mathbb{E}[F])^2}{2L} \right).$$

**Remark 2.8.** Theorem 2.7 was, in fact, obtained in [5] for centered random variables $F$. However, since $p_F(x) = p_{F-\mathbb{E}[F]}(x - \mathbb{E}[F])$, where $p_{F-\mathbb{E}[F]}$ is the density function for $F - \mathbb{E}[F]$, and condition (2.4) does not change if we replace $F$ with $F - \mathbb{E}[F]$, the statement of Theorem 2.7 follows immediately.

2.6 | Malliavin derivatives of solutions to SDEs

Consider the SDE (1.4), where the coefficients are given by (1.5) and $u(t, x)$ is the unique $C_b^{1+\frac{\beta}{2}, \frac{\beta+2}{2}}([0, T] \times \mathbb{R})$-solution to the problem (1.2). It is known that (see, e.g., [12]) if the coefficients of an SDE are differentiable with bounded derivatives, its solution is Malliavin differentiable. It is also known that if, additionally, $\tilde{\sigma}$ is bounded away from zero, then, by means of Lamperti’s transform $\eta(t, x) = \int_0^x \frac{1}{\tilde{\sigma}(t, x)} \, d\xi$ ([9]), the Malliavin derivative of $X_t$ can be explicitly computed. The algorithm is well known (see, e.g., [4]), so we skip the computation, and write the final result:

$$D_r X_t = \tilde{\sigma}(t, X_t) \psi(s, X_t) \, ds,$$  

where

$$\psi(s, x) = \frac{2f(x)\partial_x \tilde{\sigma}(s, x)}{\tilde{\sigma}^2(s, x)} - \frac{\partial_x \tilde{f}(s, x)\partial_x \tilde{\sigma}(s, x) + \partial_x \tilde{\sigma}(s, x) + \partial_x \tilde{\sigma}(s, x)}{\tilde{\sigma}(s, x)} - \frac{1}{2} \partial_x^2 \tilde{\sigma}(s, x)\tilde{\sigma}(s, x).$$
3 | RESULTS

In this section, we prove that the laws of $X_t$, $Y_t$, and $Z_t$ possess densities with respect to the Lebesgue measure, and obtain Gaussian estimates for the densities and tail probabilities of these laws.

In what follows, we will make use of assumptions (A1)–(A9) below. Assumptions (A1)–(A3) are required to obtain density estimates for the law of $X_t$.

(A1) For all $(t,x,u) \in [0,T] \times \mathbb{R} \times \mathbb{R}$, $\nu(|u|) \leq \sigma(t,x,u) \leq \mu(|u|)$, where $\nu$ and $\mu$ are non-increasing and, respectively, non-decreasing positive functions;

(A2) the functions $f$, $g$, and $h$ satisfy conditions (ii), (iii), and (v)–(vii) of Theorem 2.1;

(A3) the derivatives $\partial_t \sigma$, $\partial_u \sigma$, exist and are Hölder continuous in $t$, $x$, $u$ with exponents $\frac{\beta}{2}$, $\beta$, $\beta$, respectively, and globally bounded Hölder constants; further, $\partial_t \sigma$ exists, and $|\sigma| + |\partial_t \sigma| + |\partial_u \sigma| + |\partial_u \sigma| \leq \alpha$ for some constant $\alpha > 0$.

Assumptions (A4) and (A5) below should be added to (A1)–(A3) to obtain density estimates for the law of $Y_t$. Remark that under (A1)–(A3), the solution $u$ to the problem (1.2) possesses a bound for $|\partial_u u|$. This bound will be denoted by $M_1$. Also, we recall that the bound for $|u|$ is denoted by $M$.

(A4) In the region $[0,T] \times \mathbb{R}$, where $\mathcal{R} = \mathbb{R} \times \{ |u| \leq M \} \times \{ |p| \leq M_1 \}$, $\partial_t f$, $\partial_x f$, $\partial_u f$, $\partial_x g$, $\partial_u g$, $\partial_p g$, $\partial_x g$, $\partial_u g$, $\partial_p g$, $\partial_x g$, $\partial_u g$, exist and are bounded Hölder continuous in $t$, $x$, $u$, $p$ with exponents $\frac{\beta}{2}$, $\beta$, $\beta$, respectively, and bounded Hölder constants;

(A5) either (a) or (b) holds:

(a) $h' \geq 0$ and $\inf_{(x,u,p)\in \mathcal{R}} \partial_x g(t,x,u,p) > 0$ for all $t \in (0,T]$;

(b) $h' \leq 0$ and $\sup_{(x,u,p)\in \mathcal{R}} \partial_x g(t,x,u,p) < 0$ for all $t \in (0,T]$.

Finally, to estimate the density of the law of $Z_t$, assumption (A5) should be replaced with assumption (A5') below, and, additionally, (A6)–(A9) should be in force.

(A5') For all $(t,x,u,p) \in (0,T] \times \mathcal{R}$, $\partial_x g \geq 0$, $h'' \geq 0$.

Further, (A6)–(A9) read:

(A6) $\partial_t \sigma \geq 0$, $\partial_u \sigma \geq 0$ on $[0,T] \times \mathbb{R} \times \{ |u| \leq M \}$;

(A7) $\partial^2_{xx} f$, $\partial^2_{uu} f$, $\partial^2_{pp} f$, $\partial^2_{xx} g$, $\partial^2_{uu} g$, $\partial^2_{pp} g$, $\partial^2_{xx g}$, $\partial^2_{uu g}$, $\partial^2_{pp g}$, $\partial^2_{xx g}$, $\partial^2_{uu g}$, $\partial^2_{pp g}$, $\partial^2_{uu g}$ exist on $[0,T] \times \mathcal{R}$, are bounded and Hölder continuous in $t$, $x$, $u$, $p$ with exponents $\frac{\beta}{2}$, $\beta$, $\beta$, respectively, and bounded Hölder constants;

(A8) for all $t \in (0,T]$, $\inf_{(x,u,p)\in \mathcal{R}} \partial^2_{xx g} > 0$ and $h'' \geq 0$;

(A9) the following inequalities hold on $[0,T] \times \mathcal{R}$:

\[
\begin{aligned}
\partial^2_{xx} f + 2 \partial^2_{xx g} + \partial_p g \partial^2_{xx} \sigma + \partial^2_{pp} g \partial_u \sigma \geq 0, \\
\partial^2_{uu} g + 2 \partial^2_{uu} g + \partial_p g \partial^2_{uu} \sigma + \partial^2_{pp} g \partial_u \sigma \geq 0, \\
\partial^2_{uu} f + 2 \partial^2_{uu} f + \partial_p g \partial^2_{uu} \sigma + \partial^2_{pp} g \partial_u \sigma \geq 0, \\
\partial^2_{uu} f + 2 \partial^2_{uu} f + \partial_p g \partial^2_{uu} \sigma + \partial^2_{pp} g \partial_u \sigma \geq 0, \\
2 \partial^2_{uu} f \partial_u \sigma + 2 \partial^2_{pp} f \partial_u \sigma + \partial^2_{pp} f \partial_u \sigma + 2 \partial^2_{uu} g \partial_u \sigma + \partial^2_{pp} g \partial_u \sigma + \partial^2_{pp} g \partial_u \sigma \geq 0, \\
\partial^2_{pp} f \partial_u \sigma + 2 \partial^2_{pp} f \partial_u \sigma + \partial^2_{pp} f \partial_u \sigma + \partial^2_{pp} f \partial_u \sigma \geq 0.
\end{aligned}
\]

3.1 | Density estimates for the law of $X_t$

Theorem 3.1. Let (A1)–(A3) hold. Then, the law of $X_t$ has a density $p_{X_t}$ with respect to the Lebesgue measure. Moreover, for almost all $x \in \mathbb{R}$, $p_{X_t}$ satisfies the estimate

\[
\frac{\mathbb{E}[X_t - \mathbb{E}[X_t]]}{2\Xi(t)} \exp \left( -\frac{(x - \mathbb{E}[X_t])^2}{2\xi(t)} \right) \leq p_{X_t}(x) \leq \frac{\mathbb{E}[X_t - \mathbb{E}[X_t]]}{2\xi(t)} \exp \left( -\frac{(x - \mathbb{E}[X_t])^2}{2\Xi(t)} \right),
\]

(3.1)
where \( \xi(t) \) and \( \Xi(t) \) are positive functions that can be computed explicitly. Further, for all \( x > 0 \), the tail probabilities of \( X_t \) satisfy
\[
\mathbb{P}(X_t > x) \leq \exp \left( -\frac{(x - \mathbb{E}[X_t])^2}{2\Xi(t)} \right) \quad \text{and} \quad \mathbb{P}(X_t < -x) \leq \exp \left( -\frac{(x + \mathbb{E}[X_t])^2}{2\Xi(t)} \right).
\]

**Proof.** Note that, under (A1)–(A3), the solution \( u \) to the problem (1.2) and its derivatives \( \partial_x u, \partial_x^2 u, \partial_x^3 u, \) and \( \partial_x^4 u \) are bounded. Hence, \( f, \sigma, \partial_x f, \partial_x \sigma, \partial_x^2 \sigma, \) and \( \partial_x^3 \sigma \) are bounded as well (the functions \( f \) and \( \sigma \) are defined by (1.5)). Further, by (A3), on \([0, T] \times \mathbb{R}, \sigma(t, x) \geq v(M)\), where by \( M \) is the bound for \( |u| \). Therefore, the function \( \psi \) in (2.5) is bounded. Let \( M_\psi \) be its bound. The formula (2.5) allows us to estimate \( D_t X_t \) as follows
\[
v(M)e^{-M_\psi t} \leq D_t X_t \leq \mu(M)e^{M_\psi t} \quad \text{a.s.}
\]
This implies that
\[
t v(M)^2 e^{-2M_\psi t} \leq \int_0^t D_t X_t \mathbb{E}[D_t X_t | F_t] \, dr \leq t \mu(M)^2 e^{2M_\psi t}.
\]
Remark that \( D_t X_t = 0 \) if \( r > t \). By Theorem 2.7, the law of \( X_t \) has a density with respect to the Lebesgue measure and estimate (3.1) holds with \( \xi(t) = tv(M)^2 e^{-2M_\psi t} \) and \( \Xi(t) = t \mu(M)^2 e^{2M_\psi t} \). Moreover, the tail probabilities of \( X_t \) satisfy (3.2). □

### 3.2 Density estimates for the law of \( Y_t \)

To estimate the density for \( Y_t \), we will use the formula \( Y_t = u(t, X_t) \), where \( u \) is the unique \( C^{1+\frac{\beta}{2}, 2+\beta} \)-solution to the problem (1.2). This formula immediately implies that \( Y_t \) is Malliavin differentiable and \( D_t Y_t = \partial_x u(t, X_t) D_t X_t \).

Below, we prove that under (A1)–(A5), there exists a positive function \( m(t), t \in [0, T] \), such that
\[
\text{either} \quad \inf_{x \in \mathbb{R}} \partial_x u(t, x) \geq m(t) \quad \text{for all} \quad t \in [0, T] \quad \text{or} \quad \sup_{x \in \mathbb{R}} \partial_x u(t, x) \leq -m(t) \quad \text{for all} \quad t \in [0, T].
\]

To this end, we obtain a PDE for the function \( v = \partial_x u \). We start by considering the linear PDE (2.2) and prove that we can differentiate it with respect to \( x \). The following result can be viewed as a corollary of Theorem 2.2.

**Proposition 3.2.** Assume that the PDE (2.2) is uniformly parabolic. Let the coefficients of \( L \) and the function \( g \) be of class \( C^{1+\frac{\beta}{2}, 2+\beta}_b([0, T] \times \mathbb{R}), \beta \in (0, 1). \) Further, let the initial condition \( \varphi \) be of class \( C^1_b([0, T] \times \mathbb{R}) \). Then, the solution \( u(t, x) \) of (2.2), whose existence was established by Theorem 2.2, belongs to class \( C^{1,3}_b([0, T] \times \mathbb{R}) \), and its derivative \( v(t, x) = \partial_x u(t, x) \) is the unique solution to
\[
\begin{align*}
Lv + \partial_x a \partial_{xx}^2 u + \partial_x b \partial_x^2 u + \partial_x c \partial_x u + \partial_x g &= 0, \\
v(0, x) &= \varphi'(x).
\end{align*}
\]

In particular, we can differentiate the PDE (2.2) w.r.t. \( x \).

**Proof.** Introduce the function
\[
u(t, x) = \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x}.
\]
Since \( u \) is a solution to (2.2), the linear PDE for \( u_\Delta \) takes the form
\[
(Lu_\Delta)(t, x) = -\left( \tilde{\Delta}_2 a(t, x) \partial_{xx}^2 u(t, x + \Delta x) + \tilde{\Delta}_1 b(t, x) \partial_x u(t, x + \Delta x) + \tilde{\Delta}_0 c(t, x) u(t, x + \Delta x) + \tilde{\Delta}_0 g(t, x) \right).
\]
where \( \tilde{\phi}(x) = \int_0^1 \partial_x \phi(x + \lambda \Delta x) \, d\lambda \). Remark that, by Theorem 2.3, \( u \) is of class \( C_0^{1+\frac{\beta}{2}+\gamma} \), and, therefore, by assumptions, the right-hand side of (3.6) is of class \( C_0^{\beta} \). By Theorem 2.2, the Cauchy problem consisting of the PDE (3.6) and the initial condition \( u_\Delta(0, x) = \frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x} \) has a unique solution which takes the form

\[
uu(t, x, z)\right)\( \partial_x a(s, z)\partial_{xx}^2 u(s, z + \Delta x) + \partial_x b(s, z)\partial_x u(s, z + \Delta x) + \partial_x c(s, z)u(s, z + \Delta x) + \tilde{\phi}(s, z) \) \right) ds \, dz.
\]

On the other hand, consider problem (3.5) w.r.t. \( v \). By Theorem 2.2, (3.5) has a unique solution \( v(t, x) \) which takes the form

\[
v(t, x) = \int_\mathbb{R} \Gamma(t, x, 0, z)\phi'(z) \, dz - \int_0^t \int_\mathbb{R} \Gamma(t, x, s, z)\left( \partial_x a(s, z)\partial_{xx}^2 u(s, z) + \partial_x b(s, z)\partial_x u(s, z) + \partial_x c(s, z)u(s, z) + \partial_x g(s, z) \right) \, ds \, dz.
\]

Recalling that the fundamental solution \( \Gamma(t, x, s, z) \) possesses bounds by Gaussian densities [6], we conclude that as \( \Delta x \to 0 \), \( u_\Delta(t, x) \to v(t, x) \). This means that \( v = \partial_x u \). In particular, it means that the derivatives \( \partial_{xxx}^3 u \) and \( \partial_{xx}^2 u \) exist, and we can differentiate the PDE (2.2) w.r.t. \( x \).

**Lemma 3.3.** Let (A1)–(A5) hold, and let \( u \) be the solution to the problem (1.2) (whose existence, together with the existence of the bound \( M_1 \) for its gradient \( \partial_x u \), was established under (A1)–(A3)). Then, there exists a positive function \( m(t) \), such that one of the alternatives in (3.4) is fulfilled.

**Proof.** The problem (1.2) can be rewritten as a linear problem as follows

\[
\begin{align*}
\frac{1}{2} \partial_x^2(t, x)\partial_{xx}^2 u + \tilde{f}(t, x)\partial_x u + \tilde{g}(t, x) + \partial_t u &= 0, \\
u(T, x) &= h(x),
\end{align*}
\]

(3.7)

where \( \tilde{g}(t, x) = g(t, x, u(t, x), \partial_t u(t, x)\sigma(t, x, u(t, x))) \), and \( \tilde{\sigma} \) and \( \tilde{f} \) are defined by (1.5). By Proposition 3.2, we can differentiate the PDE (3.7) w.r.t. \( x \). By doing so, we obtain the following PDE for \( v(t, x) = \partial_x \theta(t, x) = \partial_x u(T - t, x) \)

\[
a(t, x, \theta) \partial_{xx}^2 v + b(t, x, \theta, \partial_x \theta) \partial_x v + c(t, x, \theta, \partial_x \theta) v - \partial_t v = -\partial_x g(t, x, \theta, \partial_x \theta),
\]

(3.8)

where \( \theta(t, x) = u(T - t, x) \), and the functions \( v, \theta, \partial_x v, \) and \( \partial_x \theta \) are everywhere evaluated at \( (t, x) \). Furthermore, \( a, b, \) and \( c \) are defined as follows

\[
\begin{align*}
a(t, \ldots) &= \frac{1}{2} \sigma^2(T - t, \ldots); \\
b(t, \ldots) &= \left( \partial_x a + \partial_p f \sigma \partial_x u + \partial_p g \sigma + f \right)(T - t, \ldots); \\
c(t, \ldots) &= \left( \partial_x f + \partial_p g \tilde{\sigma} + \partial_p g \partial_x \tilde{\sigma} + \partial_x f \partial_x u + \partial_p f \partial_x \tilde{\sigma} \partial_x u \right)(T - t, \ldots),
\end{align*}
\]

(3.9)

where the dots are used to simplify notation and are to be substituted with \( x, \theta(t, x), \partial_x \theta(t, x) \). Let \( \mathcal{L} \) be the partial differential operator defined by the left-hand side of (3.8), i.e.,

\[
\mathcal{L} v = a \partial_{xx}^2 v + b \partial_x v + c v - \partial_t v.
\]

If (A5)-(a) is in force, define the function \( \tilde{v}(t, x) = v(t, x) - m(t) \), where \( m(t) = \int_0^t \tilde{m}(s) \, ds \) and \( \tilde{m}(s) \) is a positive sufficiently small function whose choice is explained below. Then,

\[
\mathcal{L} \tilde{v} = -\partial_x g - c m(t) + \tilde{m}(t) \leq -\inf_{(x, u, p) \in \mathcal{R}} \partial_x g(t, x, u, p) - c m(t) + \tilde{m}(t).
\]
To estimate the density for \( s(0) \) and \( m(t) \) are sufficiently small, then \( \mathcal{L} s(0) \leq 0 \). Further, since \( m(0) = 0 \), then \( v(0, x) \geq 0 \). By Theorem 2.4, \( v(t, x) \geq 0 \), and, therefore \( v(t, x) \geq m(t) \) on \([0, T] \times \mathbb{R}\). If (A5)-(b) is in force, then the function \( v(t, x) = m(t) - s(0) \). By a similar argument, we conclude that \( v(t, x) \leq -m(t) \) on \([0, T] \times \mathbb{R}\). The lemma is proved.

As a corollary of Theorem 2.7 and Lemma 3.3, we obtain Gaussian estimates for the density of the law of \( Y_t \).

**Theorem 3.4.** Let (A1)–(A5) hold. Then, the distribution of \( Y_t \) has a density \( p_{Y_t} \) with respect to the Lebesgue measure. Moreover, for almost all \( x \in \mathbb{R} \), this satisfies the estimate

\[
\mathbb{E}[Y_t - \mathbb{E}[Y_t]]^2 \exp \left( -\frac{(x - \mathbb{E}[Y_t])^2}{2\lambda(t)} \right) \leq p_{Y_t}(x) \leq \mathbb{E}[Y_t - \mathbb{E}[Y_t]]^2 \exp \left( -\frac{(x + \mathbb{E}[Y_t])^2}{2\lambda(t)} \right),
\]

where \( \lambda(t) \) and \( \Lambda(t) \) are positive functions that can be computed explicitly. Further, for all \( x > 0 \), the tail probabilities of \( Y_t \) satisfy

\[
\mathbb{P}(Y_t > x) \leq \exp \left( -\frac{(x - \mathbb{E}[Y_t])^2}{2\Lambda(t)} \right) \quad \text{and} \quad \mathbb{P}(Y_t < -x) \leq \exp \left( -\frac{(x + \mathbb{E}[Y_t])^2}{2\Lambda(t)} \right).
\]

**Proof.** Since \( D_x Y_t = \partial_x u(t, X_t) D_x X_t \), by (3.3) and Lemma 3.3,

\[
\left\{
\begin{array}{ll}
\text{either} & \mu(M) e^{-M_{t'}} \leq D_x Y_t \leq M_1 \mu(M) e^{M_{t'}} \quad \text{a.s.} \\
\text{or} & \mu(M) e^{-M_{t'}} \leq -D_x Y_t \leq M_1 \mu(M) e^{M_{t'}} \quad \text{a.s.},
\end{array}
\right.
\]

where \( M_1 \) is the bound for \( \partial_x u \). Taking into account that \( D_x X_t = 0 \) if \( r > t \), we obtain

\[
\lambda(t) = t(\mu(M) e^{-M_{t'}})^2 \leq \int_0^t D_x Y_t \mathbb{E}[D_x Y_t | F_r] \, dr < t(\mu(M) e^{M_{t'}})^2 = \Lambda(t).
\]

By Theorem 2.7, \( Y_t \) has a density with respect to the Lebesgue measure, and estimate (3.10) holds. Also, we have estimates for the tail probabilities of \( Y_t \), given by (3.11).

### 3.3 Density estimates for the law of \( Z_t \)

To estimate the density for \( Z_t \), we recall that \( Z_t = \partial_x u(t, X_t) \sigma(t, X_t, u(t, X_t)) \). This immediately implies that \( Z_t \) is Malliavin differentiable, and

\[
D_x Z_t = \left( \partial_x u(t, X_t) \partial_x \sigma(t, X_t) + \partial^2_{xx} u(t, X_t) \sigma(t, X_t) \right) D_x X_t,
\]

where \( \sigma(t, x) = \sigma(t, x, u(t, x)) \). Lemma 3.5 below provides a lower bound for the derivative \( \partial^2_{xx} u \).

**Lemma 3.5.** Let (A1)–(A4), (A5'), and (A7)–(A9) hold, and let \( u \) be the solution to the problem (1.2). Then there exists a positive function \( \rho(t) \) such that \( \partial^2_{xx} u \geq \rho(t) \) for all \( (t, x) \in [0, T] \times \mathbb{R} \).

**Proof.** Remark that the linear PDE (3.8) takes the form (2.2) with \( a, b, \) and \( c \) given by (3.9). Since \( \theta(t, x) = u(T - t, x) \) is of class \( C^{1+\frac{\beta}{2}}([0, T] \times \mathbb{R}) \), then, by (A7), the coefficients \( a(t, x, \theta(t, x), \partial_x \theta(t, x)), b(t, x, \theta(t, x), \partial^2_x \theta(t, x)), \) and \( c(t, x, \theta(t, x), \partial_x \theta(t, x)) \) of the PDE (3.8) and its right-hand side \( -\partial_x g(t, x, \theta(t, x), \partial_x \theta(t, x)) \) are of class \( C^{\beta,1+\beta}_b \) as functions of \( (t, x) \). By Proposition 3.2, the solution \( v = \partial_x u \to (3.8) \) is of class \( C^{1+\beta}_b \) (and, therefore, \( u \) is of class \( C^{1,4}_b \), and we can differentiate the PDE (3.8) w.r.t. \( x \). Defining \( w = \partial^2_{xx} u \) and replacing \( \partial^2_{xx} u, \partial^3_{xxx} u, \) and \( \partial^4_{xxxx} u \) by \( w, \partial_x w, \) and \( \partial^2_{xx} w, \) respectively, everywhere where it is possible, we obtain the following PDE w.r.t. \( w \)

\[
\partial^2_{xx} w + b \partial_x w + Pw - \partial_t w = -\partial^2_{xx} g - \Psi_1 \partial_x u - \Psi_2 (\partial_x u)^2 - \Psi_3 (\partial_x u)^3 - \Psi_4 (\partial_x u)^4 - \Psi_5 (\partial_x u)^5,
\]

where \( \Psi_i \) are positive functions that can be computed explicitly. Further, for all \( x > 0 \), the tail probabilities of \( Z_t \) satisfy

\[
\mathbb{P}(Z_t > x) \leq \exp \left( -\frac{(x - \mathbb{E}[Z_t])^2}{2\Lambda(t)} \right) \quad \text{and} \quad \mathbb{P}(Z_t < -x) \leq \exp \left( -\frac{(x + \mathbb{E}[Z_t])^2}{2\Lambda(t)} \right).
\]
where \( P \) is a polynomial of \( \sigma, f, g \), all their first and second order derivatives w.r.t. \( x, u, p \), and, additionally, of \( \partial_u u \). Further, the functions \( \Psi_i, i = 1, 2, 3, 4, 5 \), are defined by the right-hand sides of the first, second, third, fourth, and the fifth inequalities, respectively, in assumption (A9). Let \( \mathcal{L}_1 \) denote the partial differential operator defined by the left-hand side of (3.14). We proceed with the same argument as in Lemma 3.3, that is, define the function \( \tilde{w}(t, x) = w(t, x) - \rho(t) \), where \( \rho(t) = \int_0^t \tilde{\rho}(s) \, ds \) and \( \tilde{\rho}(s) \) is a sufficiently small positive function. Then,

\[
\mathcal{L}_1 \tilde{w} = -\partial_{xx}^2 g - \sum_{n=1}^5 \Psi_n \left( \partial_u u \right)^n - P \rho(t) + \tilde{\rho}(t).
\]

Remark that under (A5'), \( \partial_u u \geq 0 \) on \([0, T] \times \mathbb{R} \). Indeed, this follows from the proof of Lemma 3.3, to which we have to apply Theorem 2.4 with \( m(t) = 0 \). Hence, by (A9), \( \sum_{n=1}^5 \Psi_n \left( \partial_u u \right)^n \geq 0 \). Further, by (A2)–(A4), \( P \) is bounded. Therefore, (A7) implies that if \( \tilde{\rho}(t) \) and \( \rho(t) \) are sufficiently small, then \( \mathcal{L}_1 \tilde{w} \leq 0 \). Since \( \tilde{\rho}(t) \geq 0 \), by Theorem 2.4, we obtain that \( w(t, x) \geq \rho(t) \) for all \( (t, x) \in [0, T] \times \mathbb{R} \).

**Theorem 3.6.** Let (A1)–(A4), (A5'), and (A6)–(A9) hold. Then, the distribution of \( Z_t \) has a density \( p_{Z_t} \) with respect to the Lebesgue measure. Moreover, for almost all \( x \in \mathbb{R} \), this density satisfies

\[
\frac{\mathbb{E}[Z_t - \mathbb{E}[Z_t]]}{2\Sigma(t)} \exp \left( -\frac{(x - \mathbb{E}[Z_t])^2}{2\Sigma(t)} \right) \leq p_{Z_t}(x) \leq \frac{\mathbb{E}[Z_t - \mathbb{E}[Z_t]]}{2\Sigma(t)} \exp \left( -\frac{(x - \mathbb{E}[Z_t])^2}{2\Sigma(t)} \right),
\]

(3.15)

where \( \zeta(t) \) and \( \Sigma(t) \) are positive functions which can be computed explicitly. Further, for all \( x > 0 \), the tail probabilities of \( Z_t \) satisfy

\[
\mathbb{P}(Z_t > x) \leq \exp \left( -\frac{(x - \mathbb{E}[Z_t])^2}{2\Sigma(t)} \right) \quad \text{and} \quad \mathbb{P}(Z_t < -x) \leq \exp \left( -\frac{(x + \mathbb{E}[Z_t])^2}{2\Sigma(t)} \right),
\]

(3.16)

**Proof.** Assumptions (A5') and (A6)–(A9) provide the lower bound for the function

\[
\partial_x u \partial_x \sigma + \left( \partial_u u \right)^2 \partial_x \sigma + \partial_{xx}^2 u \sigma \quad (3.17)
\]

on the right-hand side of (3.13). Indeed, \( \partial_x u \partial_x \sigma + \left( \partial_u u \right)^2 \partial_x \sigma \geq 0 \) by (A5') and (A6). Finally, from (A1) and (A7)–(A9), by virtue of Lemma 3.5, it follows that \( \partial_x^2 u \sigma \geq \rho(t) \nu(M) \), where \( \rho(t) \) is the positive function defined in Lemma 3.5 and \( \nu(\cdot) \) is the function from (A1). Now taking into account that \( D_x X_t \) possesses upper and lower positive bounds, provided by (3.3), we obtain that the Malliavin derivative \( D_x Z_t \) satisfies

\[
\nu(M)^2 \rho(t)e^{-M_{\nu^t}} \leq D_x Z_t \leq \mu(M) \gamma e^{M_{\nu^t}} \quad \text{a.s.},
\]

where \( \gamma \) is an upper bound for (3.18). This bound, indeed, exists by (A3) and since the solution \( u \) has bounded derivatives. Now by the same argument as in Theorem 3.4, we obtain that the law of \( Z_t \) possesses a density \( p_{Z_t} \) w.r.t. the Lebesgue measure, and (3.16) holds with \( \Sigma = \mu(M)^2 \gamma^2 e^{2M_{\nu^t}} \) and \( \zeta(t) = \nu(M)^4 \rho(t)^2 e^{-2M_{\nu^t}} \). Moreover, we obtain estimates (3.17) for the tail probabilities of \( Z_t \).

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**REFERENCES**

[1] O. Aboura and S. Bourguin, *Density estimates for solutions to one dimensional backward SDEs*, Potential Anal. **38** (2013), 573–587.
[2] F. Antonelli and A. Kohatsu-Higa, *Densities of one-dimensional backward SDEs*, Potential Anal. **22** (2005), no. 3, 263–287.
[3] J. Cvitanic and J. Ma, *Hedging options for a large investor and forward-backward SDEs*, Ann. Appl. Probab. **6** (1996), 370–398.
[4] J. Detemple, R. Garcia, and M. Rindisbacher, *Representation formulas for Malliavin derivatives of diffusion processes*, Finance Stoch. 9 (2005), 349–367.

[5] N. T. Dung, N. Privault, and G. L. Torrisi, *Gaussian estimates for the solutions of some one-dimensional stochastic equations*, Potential Anal. 43 (2015), 289–311.

[6] A. Friedman, *Partial differential equations of parabolic type*, Robert E. Krieger Publishing Company, 1983.

[7] N. El Karoui, S. Peng, and M. C. Quenez, *Backward stochastic differential equations in finance*, Math. Finance 7 (1997), 1–71.

[8] O. Ladyzenskaja, V. Solonnikov, and N. N. Uralceva, *Linear and quasi-linear equations of parabolic type*, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, R.I., 1968.

[9] J. Lamperti, *A simple construction of certain diffusion processes*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 32 (1964), 1–76.

[10] J. Ma, P. Protter, and J. Yong, *Solving forward backward stochastic differential equations explicitly: a four step scheme*, Probab. Theory Related Fields 98 (1994), 339–359.

[11] T. Mastrolia, D. Possamaï, and A. Réveillac, *Density analysis for FBSDEs*, Ann. Probab. 44 (2016), no. 4, 2817–2857.

[12] D. Nualart, *The Malliavin calculus and related topics*, Springer Science & Business Media, 2006.

[13] E. Pardoux and S. Peng, *Backward stochastic differential equations and quasilinear parabolic partial differential equations*, Stochastic Partial Differential Equations and Their Applications, Lecture Notes in Control and Inform. Sci., vol. 176, Springer, 1992, pp. 200–217.

[14] J. Yong and X. Y. Zhou, *Stochastic controls. Hamiltonian systems and HJB equations*, Springer, New York, 1999.

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