Computations of Expansions for the Maximum Likelihood Estimator and Its Distribution Function

Shanti Venetiaan

Abstract. In this paper, insight is given in the techniques used to compute asymptotic expansions. In a broad fashion the technique is described. Most of the results apply to the paper "An expansion for the maximum likelihood estimator and its distribution function", which will be submitted.

Key words. asymptotic expansion, maximum likelihood estimator of location

AMS subject classifications. 62E20

1. Expansions. In the paper "An expansion for the maximum likelihood estimator and its distribution function", which will be submitted, many expansions are being calculated. Because the technique itself is not so difficult and the outcomes take a lot of space to present, many steps of the computation of the expansions will be omitted in that paper. However if one wants to check the computations, it may be useful to have some insight in how the expansions were obtained. None of the applied techniques are claimed by the author. They are just written down. We will give the full outcome of all the steps needed to construct the needed expansions. We will not give any proofs, just the method of obtaining the expansions. This means that certain rest terms will be omitted. Also note that the results are only valid for the maximum likelihood estimator of location.

2. An expansion for the maximum likelihood estimator. We define the maximum likelihood estimator \( \hat{\theta}_n \) by

\[
L_n(\hat{\theta}_n) = \inf_{\theta \in \mathbb{R}} L_n(\theta),
\]

where \( L_n(\theta) = n^{-1} \sum_{i=1}^{n} \rho(X_i - \theta) \), where \( \rho(\cdot) = -\log(\cdot) \). It may be proved that (cf. Chibisov (1973)) \( \hat{\theta}_n \) is the solution of the equation

\[
L'_n(\theta) = 0,
\]

with probability \( 1 + o(n^{-3/2}) \). We expand \( L'_n(\theta) \) with a Taylor expansion and get

\[
L'_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \rho'(X_i - \theta) = \frac{1}{n} \sum_{i=1}^{n} \rho'(X_i) - \theta \frac{1}{n} \sum_{i=1}^{n} \rho''(X_i) + \frac{\theta^2}{2n} \sum_{i=1}^{n} \rho^{(3)}(X_i) \]
\[ - \frac{\theta^3}{6n} \sum_{i=1}^{n} \rho^{(4)}(X_i) + \frac{\theta^4}{24n} \sum_{i=1}^{n} \rho^{(5)}(X_i - \theta'), \]

where \( |\theta'| \leq |\theta| \).

Faculty of Technology, Anton de Kom Universiteit van Suriname, Paramaribo, Suriname
We introduce the notation

\begin{equation}
(2.4) \quad \xi_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\rho^{(j)}(X_i) - a_j), \quad a_j = E_0 \rho^{(j)}(X) \quad \text{for} \quad j = 1, \ldots, 5.
\end{equation}

Note that the \( \xi_j \)'s are normalized sums and that \( a_1 = 0 \). Equation (2.5) becomes

\begin{equation}
L''_n(\theta) = \frac{\xi_{1n}}{\sqrt{n}} - \theta (\frac{\xi_{2n}}{\sqrt{n}} + a_2) + \frac{\theta^2}{2} (\frac{\xi_{3n}}{\sqrt{n}} + a_3)
\end{equation}

\begin{equation}
- \frac{\theta^3}{6} (\frac{\xi_{4n}}{\sqrt{n}} + a_4) + \frac{\theta^4}{24} (\frac{\xi_{5n}}{\sqrt{n}} + a_5) + \cdots,
\end{equation}

To find the expansion for \( \hat{\theta}_n \), we put

\begin{equation}
(2.6) \quad \hat{\theta}_n = B_1/\sqrt{n} + B_2/n + B_3/n^{3/2} + B_4/n^2.
\end{equation}

Substituting this into (2.6) leads to

\begin{equation}
(2.7) \quad (-a_2 B_1 + \xi_1)/n^{1/2} + (-\xi_2 B_1 - a_2 B_2 + \frac{1}{2} a_3 B_1^2)/n
\end{equation}

\begin{equation}
+ (-\frac{1}{6} a_4 B_1^3 + \frac{1}{2} a_3 B_1 B_2 - a_2 B_3)/n^{3/2}
\end{equation}

\begin{equation}
+ \frac{1}{2} a_4 B_1^2 B_2 + \frac{1}{2} a_3 B_2^2 + \frac{1}{24} a_5 B_1^4 - \frac{1}{2} a_4 \xi B_1 B_2 - \frac{1}{6} \xi B_1 B_2 - \frac{1}{6} \xi B_1 B_2 - a_2 B_4)/n^2
\end{equation}

Now we take the first term of (2.7) and put it equal to 0. We get

\begin{equation}
(2.8) \quad -a_2 B_1 + \xi_1 = 0 \Rightarrow B_1 = \xi_1/a_2.
\end{equation}

Substituting the obtained \( B_1 \) in (2.7) gives

\begin{equation}
(2.9) \quad (-\xi_2 \xi_1/a_2 + \frac{1}{2} a_3 \xi_2^2/a_2 - a_2 B_2)/n
\end{equation}

\begin{equation}
+ (-a_2 B_3 + a_3 \xi_1 B_2/a_2 - \xi_2 B_2 + \frac{1}{2} a_4 \xi_1^2/a_2 - \frac{1}{6} a_4 \xi_1^3/a_2^3)/n^{3/2}
\end{equation}

\begin{equation}
+ (a_3 \xi_1 B_3/a_2 + \frac{1}{24} a_5 \xi_1^4/a_2 - \frac{1}{2} a_4 \xi_1^2 B_2/a_2 - \frac{1}{6} a_5 \xi_2/a_2 + \frac{1}{2} a_3 B_2^2 + \xi_4 \xi_1 B_2/a_2 - \xi_2 B_3 - a_2 B_4)/n^2
\end{equation}

Note that the \( 1/\sqrt{n} \) term has vanished. We take the first term of (2.9) and put it equal to 0. This results in

\begin{equation}
(2.10) \quad B_2 = -\xi_1 \xi_2/a_2^2 + \frac{1}{2} \xi_2^2 a_3/a_2^2.
\end{equation}

We substitute this \( B_2 \) in (2.9) and get

\begin{equation}
(2.11) \quad (-a_2 B_3 + \frac{1}{6} a_4 \xi_1^3/a_2 - \frac{1}{2} a_3 \xi_1^2/a_2 - \frac{3}{2} \xi_2 \xi_3 a_3/a_2^2 + \xi_4 \xi_1/a_2 + \frac{1}{2} \xi_1^3 a_3^2/a_2^4)/n^{3/2}
\end{equation}

\begin{equation}
+ (-\frac{1}{6} \xi_2 \xi_1^2/a_2 + a_3 \xi_1 B_2/a_2 + \frac{1}{8} \xi_1 a_3^2/a_2 + \frac{1}{2} a_3 \xi_1 a_3/a_2^2)
\end{equation}

\begin{equation}
- \frac{1}{2} \xi_1 \xi_2 a_3/a_2 - \xi_2 B_3 - \xi_2 \xi_2 a_3/a_2^2
\end{equation}

\begin{equation}
+ \frac{1}{24} a_5 \xi_1^4/a_2 + \frac{1}{2} a_3 \xi_1^2/a_2 + \frac{1}{2} a_4 \xi_1^2 a_2 - a_2 B_2 - \frac{1}{4} a_4 \xi_1^3 a_3^2/a_2^5)/n^2
\end{equation}
Again we put the first term of the result to 0 and get

\[(2.12) \quad B_3 = \xi_1 \xi_2^2 / a_2^3 - 6 \xi_1^3 / a_2^2 + 1 / 2 \xi_1^2 \xi_3 / a_2^2 - 3 / 2 \xi_1 \xi_2 a_3 / a_2^4 + 1 / 2 \xi_1^2 a_3 / a_2^2.\]

We substitute \(B_3\) in (2.11) to obtain

\[(2.13) \quad -a_2 B_4 - 5 / 2 \xi_2^3 a_2^3 / a_2^2 + 1 / 2 \xi_1 a_3^3 / a_2^2 + 3 \xi_1 \xi_2 a_3 / a_2^4 - 3 / 2 \xi_1 \xi_2^3 / a_2^3 \]
\[+ 3 / 2 \xi_1^3 \xi_3 a_2 / a_2^4 + 1 / 12 a_4 \xi_4^3 / a_2^6 + 3 \xi_1 \xi_2 a_3 / a_2^4 - 5 / 12 a_4 \xi_4 a_3 / a_2^6.\]

This is put equal to 0 and at last we obtain

\[(2.14) \quad B_4 = \xi_1 \xi_2 a_3 / a_2^4 - 5 / 2 \xi_2^3 a_2 / a_2^2 + 5 / 8 \xi_1 a_3^3 / a_2^2 - \xi_2 a_2 / a_2^4 - 6 / 6 \xi_2 a_4 / a_2^6 \]
\[+ 3 / 2 \xi_1 \xi_3 a_2 / a_2^4 + 2 / 3 \xi_2 a_2 a_3 / a_2^2 + 1 / 24 a_4 \xi_4 / a_2^6 + 3 \xi_1 \xi_2 a_3 / a_2^4 - 5 / 12 a_4 \xi_4 a_3 / a_2^6.\]

Eventually

\[(2.15) \quad \sqrt{n}(\hat{\theta}_n) = \frac{\xi_1}{a_2} + \frac{1}{\sqrt{n}} \left( \frac{-\xi_1 \xi_2}{a_2^2} + \frac{a_2^3}{2 a_2^2} + \frac{1}{n} \left( \frac{\xi_1 \xi_3}{a_2^2} - \frac{3 a_4 \xi_2 a_4}{2 a_2^2} - \frac{\xi_2 a_2}{2 a_2^2} + \frac{\xi_2^2 \xi_2}{a_2^2} + \frac{a_4 \xi_4}{6 a_2^2} \right) \right) + \cdots.\]

3. Expansion for the distribution function of the maximum likelihood estimator. The estimator in (2.15) fits the model of Hall (1992), Section 2.3. This means that the cumulants of \(\sqrt{n} \hat{\theta}_n\) will determine the expansion for the distribution function. First we note that the cumulants will be of the form (cf. Hall(1992), Section 2.3)

\[(3.1) \quad \kappa_1 = 0 + k_{12} / \sqrt{n} + k_{13} / n^{3/2} + \cdots \]
\[\kappa_2 = k_{22} + k_{23} / n + \cdots \]
\[\kappa_3 = k_{33} / \sqrt{n} + k_{34} / n^{3/2} + \cdots \]
\[\kappa_4 = k_{44} / n + \cdots \]
\[\kappa_5 = k_{55} / n^{3/2} + \cdots.\]

Furthermore (cf. Hall(1992), Section 2.2)

\[(3.2) \quad \kappa_1 = E S_n \]
\[\kappa_2 = E S_n^2 - (E S_n)^2 \]
\[\kappa_3 = E (S_n - E S_n)^3 = E S_n^3 - 3E S_n^2 E S_n + 2(E S_n)^3 \]
\[\kappa_4 = E (S_n - E S_n)^4 - 3 \kappa_2^2 \]
\[\kappa_5 = E (S_n - E S_n)^5 - 10 \kappa_2 \kappa_3.\]
3.1. Computation of the kappa’s. We will now compute the kappa’s. For the computation of the kappa’s we need to calculate the expectation of the normalized sums, the so called \( \xi \)'s. Terms that become too small will be omitted. First we will introduce the notation

\[
\psi_i(\cdot) = f_i(\cdot)
\]

\[
\eta_2 = E(\psi_i^2(X_i)), \quad \eta_3 = E(\psi_i^3(X_i)), \quad \eta_4 = E(\psi_i^4(X_i)), \quad \eta_5 = E(\psi_i^5(X_i)), \quad \eta_6 = E(\psi_2\psi_3(X_i))
\]

the above results in

\[
a_1 = 0, \quad a_2 = E(\psi_i^2(X_i)), \quad \text{without loss of generality we put} \quad a_2 = 1,
\]

\[
a_3 = -\frac{1}{2} \eta_3, \quad a_4 = \frac{2}{3} \eta_4 - \eta_2, \quad a_5 = 5\eta_6 - \frac{3}{2} \eta_5.
\]

and that

\[
E(\psi_1\psi_2) = \frac{1}{2} \eta_3, \quad E(\psi_1\psi_3) = -\eta_2 + \frac{2}{3} \eta_4, \quad E(\psi_1\psi_4) = -5\eta_6 + \frac{3}{2} \eta_5
\]

\[
E(\psi_i^2\psi_2) = \frac{2}{3} \eta_4, \quad E(\psi_1\psi_i^2) = 2\eta_6, \quad E(\psi_i^3\psi_2) = \frac{3}{4} \eta_5, \quad E(\psi_i^3\psi_3) = -4\eta_6 + \frac{3}{2} \eta_5
\]

Furthermore

\[
w_j = -(\rho^{(j)}(X_i) - a_j), \quad \text{for} \quad j = 1, \ldots, 5.
\]

Consequently,

\[
w_1 = \psi_1,
\]

\[
w_2 = \psi_2 - \psi_1^2 + 1,
\]

\[
w_3 = \psi_3 - 3\psi_1\psi_2 + 2\psi_1^3 + a_3,
\]

\[
w_4 = \psi_4 - 4\psi_1\psi_3 + 12\psi_1^2\psi_2 - 3\psi_2^2 - 6\psi_1^4 + a_4
\]

\[
w_5 = \psi_5 - 5\psi_1\psi_4 + 20\psi_1^2\psi_3 - 10\psi_2\psi_3 + 30\psi_1\psi_2^2 - 60\psi_1^3\psi_2 + 24\psi_1^5 + a_5
\]

Note that \( E(w_j) = 0 \) for \( j = 1, \ldots, 5 \) and that \( E(w_1^2) = 1 \).

\[
E(w_1w_2) = -\frac{1}{2} \eta_3, \quad E(w_1w_3) = \frac{2}{3} \eta_4 - \eta_2, \quad E(w_1w_4) = 5\eta_6 - \frac{3}{2} \eta_5
\]

\[
E(w_2w_2) = -\frac{1}{3} \eta_4 + 1, \quad E(w_2^2w_3) = -4\eta_6 + \frac{5}{4} \eta_5 - \frac{1}{2} \eta_3
\]

\[
E(w_1^3w_2) = -\frac{1}{4} \eta_5 + \eta_3, \quad E(w_1^2w_2^2) = 2\eta_6 - \frac{1}{2} \eta_5 - \eta_3
\]
CALCULATION OF EXPANSIONS FOR THE MLE AND ITS DISTRIBUTION FUNCTION

\[ E(w_2^2) = \eta_2 - \frac{1}{3}\eta_4 - 1, \quad E(w_2w_3) = -\eta_6 + \frac{1}{4}\eta_5 + \frac{1}{2}\eta_3 \]

We will give an example here to illustrate how the expectations of the terms of \( \sqrt{n}\hat{\theta}_n \) are obtained.

\[
E(\xi_1^2) = E(\frac{1}{\sqrt{n}} \sum_i w_{1i})^2
= E\left\{ \frac{1}{n} \left( \sum_i w_{1i}^2 + 2 \sum_{i<j} w_{1i}w_{1j} \right) \right\}
= \frac{1}{n} (nEw_1^2 + \frac{2n(n-1)}{2}Ew_1Ew_1) = Ew_1^2 + 0 = Ew_1^2 = 1.
\]

\[
(3.7)
E(\xi_1^8) = E(\frac{1}{\sqrt{n}} \sum_i w_{1i})^8
= \frac{1}{n^4} E \left[ \sum_i w_{1i}^8 + 8 \sum_{i \neq j} w_{1i}^7w_{1j} + \binom{8}{2} \sum_{i \neq j} w_{1i}^6w_{1j}^2 \\
\quad + \binom{8}{3} \sum_{i \neq j} w_{1i}^5w_{1j}^3 + \binom{8}{4} \sum_{i \neq j} w_{1i}^4w_{1j}^4 \\
\quad + \binom{8}{4,2,2} \sum_{i \neq j} \sum_{k \neq i} w_{1i}^4w_{1j}^2w_{1k}^2 + \binom{8}{2,3,3} \sum_{i \neq j} \sum_{k \neq i} w_{1i}^2w_{1j}^3w_{1k}^3 \\
\quad + \binom{8}{2,2,2,2} \sum_{i \neq j} \sum_{k \neq i} \sum_{l \neq i} w_{1i}^2w_{1j}^2w_{1k}^2w_{1l}^2 \right]
= \frac{1}{n^4} \left[ \binom{8}{2,2,2,2} \frac{n(n-1)(n-2)(n-3)}{4!} (Ew_1^2)^4 \\
+ \binom{8}{2,2,4} \frac{n(n-1)(n-2)}{2!} (Ew_1^2)^2 \\
+ \binom{8}{2,3,3} \frac{n(n-1)(n-2)}{2!} (Ew_1^3)^2Ew_1^2 + \ldots \right]
= \frac{1}{n^4} \left[ 105(n^3 - 6n^3 + 11n^2 - 6n)(Ew_1^2)^4 + 210(n^3 - 3n^2 + 2n)Ew_1^4(Ew_1^2)^2 \\
+ 280(n^3 - 3n^2 + 2n)(Ew_1^3)^2Ew_1^2 + \ldots \right]
= 105(Ew_1^2)^4 + \frac{1}{n}[-630(Ew_1^2)^4 + 210Ew_1^4(Ew_1^2)^2 + 280(Ew_1^3)^2Ew_1^2] + \ldots
\]

The following equations give formula's for the other expectation that have to be computed.
(3.9) \[ E(\xi_1^{2k}) = \left(\frac{2k}{2,\ldots,2}\right) \frac{1}{k!} (E(w_1^2))^{k} + \frac{1}{n} \left[ -\left(\frac{2k}{2,\ldots,2}\right) \frac{k}{k!} (Ew_1^2)^{k} \right. \\
+ \left(\frac{2k}{4,2,\ldots,2}\right) \frac{1}{(k-2)!} Ew_1^4 (Ew_1^2)^{k-2} \\
+ \left(\frac{2k}{3,3,2,\ldots,2}\right) \frac{1}{(k-3)!} (Ew_1^3)^2 (Ew_1^2)^{k-3} \right] + \ldots \]

(3.10) \[ E(\xi_1^{2k+1}) = -\frac{1}{\sqrt{n}} \left[ \left(\frac{2k+1}{3,2,\ldots,2}\right) \frac{1}{(k-1)!} (Ew_1^2)^{k-1} Ew_1^3 \right. \\
- \frac{1}{n\sqrt{n}} \left[ \left(\frac{k}{2}\right) \left(\frac{2k+1}{3,2,\ldots,2}\right) \frac{1}{(k-1)!} (Ew_1^2)^{k-1} Ew_1^3 \right. \\
+ \left(\frac{2k+1}{5,2,\ldots,2}\right) \frac{1}{(k-2)!} (Ew_1^2)^{k-2} Ew_1^5 \\
+ \left(\frac{2k+1}{4,3,2,\ldots,2}\right) \frac{1}{(k-3)!} (Ew_1^2)^{k-3} Ew_1^4 Ew_1^3 \right] + \ldots \]

For \( j = 2, \ldots, 4 \) we have

(3.11) \[ E(\xi_1^{2k} \xi_j) = -\frac{1}{\sqrt{n}} \left[ \left(\frac{2k}{2,\ldots,2}\right) \frac{1}{(k-1)!} Ew_1^2 w_j (Ew_1^2)^{k-1} ight. \\
+ \left(\frac{2k}{1,3,2,\ldots,2}\right) \frac{1}{(k-2)!} Ew_1^3 Ew_1 w_j (Ew_1^2)^{k-2} \\
- \frac{1}{n\sqrt{n}} \left[ \left(\frac{k}{2}\right) \left(\frac{2k}{2,\ldots,2}\right) \frac{1}{(k-1)!} Ew_1^2 w_j (Ew_1^2)^{k-1} \right. \\
- \left(\frac{k}{2}\right) \left(\frac{2k}{1,3,2,\ldots,2}\right) \frac{1}{(k-2)!} Ew_1^3 Ew_1 w_j (Ew_1^2)^{k-2} \\
+ \left(\frac{2k}{5,1,2,\ldots,2}\right) \frac{1}{(k-3)!} Ew_1^4 Ew_1^2 w_j (Ew_1^2)^{k-3} \\
+ \left(\frac{2k}{4,2,\ldots,2}\right) \frac{1}{(k-3)!} Ew_1^4 Ew_1^2 w_j (Ew_1^2)^{k-3} \\
+ \left(\frac{2k}{3,3,2,\ldots,2}\right) \frac{1}{(k-3)!} Ew_1^4 Ew_1^2 w_j (Ew_1^2)^{k-3} \\
+ \left(\frac{2k}{4,2,\ldots,2}\right) \frac{1}{(k-2)!} Ew_1^4 w_j (Ew_1^2)^{k-2} \right] + \ldots \]

(3.12) \[ E(\xi_1^{2k+1} \xi_2) = (2k+1)(2k-1) \cdots 1 (E(w_1^2))^k Ew_1 w_2 \\
+ \frac{1}{n} \left[ -\left(\frac{k+1}{2}\right) \left(\frac{2k+1}{1,2,\ldots,2}\right) Ew_1 w_2 (Ew_1^2)^k \right. \\
+ \frac{1}{(k-2)!} \left(\frac{2k+1}{4,1,\ldots,2}\right) Ew_1^4 Ew_1 w_2 (Ew_1^2)^{k-2} \right] + \ldots \]
\[
E(\xi_1^{2k}\xi_2^2) = \left( \begin{array}{c}
2k \\
2, \ldots, 2
\end{array} \right) \frac{1}{k!} (E\xi_1^2)^k E\xi_2^2 \\
+ \left( \begin{array}{c}
2k \\
1, 1, 2, \ldots, 2
\end{array} \right) \frac{1}{(k-1)!} (E\xi_1^2)^{k-1} (E\xi_1 w_1)^2 \\
+ \frac{1}{n} \left( \begin{array}{c}
2k \\
4, 2, \ldots, 2
\end{array} \right) \frac{1}{(k-2)!} E\xi_1^4 (E\xi_1^2)^{k-2} E\xi_2^2 \\
+ \left( \begin{array}{c}
2k \\
3, 3, 2, \ldots, 2
\end{array} \right) \frac{1}{2 \cdot (k-3)!} (E\xi_1^3)^2 (E\xi_1^2)^{k-3} E\xi_2^2 \\
+ \left( \begin{array}{c}
2k \\
4, 1, 1, 2, \ldots, 2
\end{array} \right) \frac{1}{(k-3)!} E\xi_1^4 (E\xi_1 w_2)^2 (E\xi_1^2)^{k-3} \\
+ \left( \begin{array}{c}
2k \\
1, 3, 2, \ldots, 2
\end{array} \right) \frac{1}{(k-2)!} E\xi_1^3 E\xi_1 w_2^2 (E\xi_1^2)^{k-2} \\
+ \left( \begin{array}{c}
2k \\
1, 3, 2, \ldots, 2
\end{array} \right) \left( \begin{array}{c}
2 \\
2, \ldots, 2
\end{array} \right) \frac{(k)}{(k-2)!} E\xi_1^3 w_2 E\xi_1^2 w_1 (E\xi_1^2)^{k-2} \\
+ \left( \begin{array}{c}
2k \\
1, 3, 2, \ldots, 2
\end{array} \right) \left( \begin{array}{c}
2 \\
2, \ldots, 2
\end{array} \right) \frac{(k)}{(k-2)!} E\xi_1^3 w_2 E\xi_1^2 w_1 (E\xi_1^2)^{k-2} \\
- \left( \begin{array}{c}
k+1 \\
2, \ldots, 2
\end{array} \right) \frac{1}{k!} (E\xi_1^2)^k E\xi_2^2 \\
+ \left( \begin{array}{c}
2k \\
2, \ldots, 2
\end{array} \right) \frac{(k)}{2(k-2)!} (E\xi_1^2 w_2)^2 (E\xi_2^2)^{k-2} \\
+ \left( \begin{array}{c}
2k \\
1, 1, 2, \ldots, 2
\end{array} \right) \frac{(k+1)}{(k-1)!} (E\xi_1^2)^{k-1} (E\xi_1 w_2)^2 \\
+ \left( \begin{array}{c}
2k \\
2, \ldots, 2
\end{array} \right) \frac{1}{(k-1)!} (E\xi_1^2 w_2^2 (E\xi_2^2)^{k-1} \right] + \ldots
\]

(3.13)

\[
E(\xi_1^{2k}\xi_2^2) = \left( \begin{array}{c}
2k \\
2, \ldots, 2
\end{array} \right) \frac{1}{k!} (E\xi_1^2)^k E\xi_2^2 w_3 \\
+ \frac{1}{(k-1)!} \left( \begin{array}{c}
2k \\
1, 1, 2 \ldots 2
\end{array} \right) E\xi_1 w_2 E\xi_1 w_3 + \ldots
\]

(3.14)

\[
E(\xi_1^{2k}\xi_2^3) = -\frac{1}{\sqrt{n}} \left[ \left( \begin{array}{c}
2k + 1 \\
3, 2, \ldots, 2
\end{array} \right) \frac{1}{(k-1)!} E\xi_1^3 E\xi_2 w_3 (E\xi_1^2)^{k-1} \right]
\]

(3.15)
\[ E(\xi_1^{2k+1}, \xi_2^{2k+1}) = -\frac{1}{\sqrt{n}} \left( \frac{2k+1}{3, 2, \ldots, 2} \right) \frac{1}{(k-2)!} Ew_1^3 Ew_2 Ew_3 (Ew_1^2)^{k-2} \]
\[ + \left( \frac{2k+1}{1, 1, 3, 2, \ldots, 2} \right) \frac{1}{(k-1)!} Ew_1^3 Ew_2 Ew_3 (Ew_1^2)^{k-1} \]
\[ + \left( \frac{2k+1}{1, 2, \ldots, 2} \right) \frac{1}{(k-1)!} Ew_1 w_2 (Ew_1^2)^{k-1} \]
\[ + \left( \frac{2k+1}{1, 2, \ldots, 2} \right) \frac{1}{k!} Ew_1 w_2 (Ew_1^2)^{k} \]
\[ + \cdots. \]

(3.16)

\[ E(\xi_1^{3k+2}) = \left( \frac{2k+1}{3, 2, \ldots, 2} \right) \frac{1}{(k-2)!} Ew_1^3 Ew_2 (Ew_1^2)^{k-2} \]
\[ + \left( \frac{2k+1}{1, 1, 3, 2, \ldots, 2} \right) \frac{1}{(k-2)!} Ew_1^3 (Ew_1 w_2)^2 (Ew_1^2)^{k-2} \]
\[ + \left( \frac{2k+1}{1, 2, \ldots, 2} \right) \frac{1}{(k-2)!} Ew_1^3 (Ew_1^2)^{k-1} \]
\[ + \cdots. \]

(3.17)

\[ E(\xi_1^{2k+1}) = \left( \frac{2k+1}{1, 2, \ldots, 2} \right) \frac{1}{3!} Ew_1 w_2 Ew_1^2 (Ew_1^2)^{k} \]
\[ + \left( \frac{2k+1}{1, 1, 1, 2, \ldots, 2} \right) \frac{1}{3!} (Ew_1 w_2)^3 (Ew_1^2)^{k-1} + \cdots. \]

(3.18)

\[ E(\xi_1^{2k}) = -\frac{1}{\sqrt{n}} \left( \frac{2k}{1, 3, 2, \ldots, 2} \right) \frac{1}{(k-2)!} Ew_1^2 Ew_2 Ew_3 (Ew_1^2)^{k-2} \]
\[ + \frac{1}{k!} \left( \frac{2k}{2, \ldots, 2} \right) Ew_1^3 (Ew_1^2)^{k} \]
\[ + \left( \frac{2k}{2, \ldots, 2} \right) \frac{3}{(k-1)!} Ew_1^2 w_2 Ew_1^3 (Ew_1^2)^{k-1} \]
\[ + \frac{1}{(k-3)!} \left( \frac{2k}{1, 1, 1, 3, 2, \ldots, 2} \right) Ew_1^2 (Ew_1 w_2)^3 (Ew_1^2)^{k-3} \]
\[ + \left( \frac{2k}{1, 1, 2, \ldots, 2} \right) \frac{3}{(k-1)!} (Ew_1^2)^{k-1} Ew_1 w_2 \]
\[ + \left( \frac{2k}{1, 1, 2, \ldots, 2} \right) \frac{3}{(k-2)!} (Ew_1^2)^{k-2} Ew_1 w_2 (Ew_1 w_2)^2 \]
\[ + \cdots. \]

(3.19)

\[ E(\xi_1^{2k}) = \left( \frac{2k}{2, \ldots, 2} \right) \frac{3}{k!} (Ew_1^2)^{2} (Ew_1^2)^{k} \]
which make up the expansion of the distribution function.

By using (3.1) we get

\[
\frac{2k}{(1, 1, 2, \ldots, 2)} \frac{6}{(k - 1)!} (Ew_1^2)^{k-1} Ew_2^2 (Ew_1 w_2)^2
\]

\[
\frac{2k}{(1, 1, 1, 1, \ldots, 2)} \frac{1}{(k - 2)!} (Ew_1^2)^{k-2} (Ew_1 w_2)^4 + \cdots
\]

Further computations lead to

\[
(3.20) \quad ES_n = \frac{\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left( \frac{1}{9} \eta_4 \eta_3 + \frac{1}{16} \eta_5 - \frac{1}{4} \eta_3 - \frac{5}{24} \eta_3 \eta_2 - \frac{11}{64} \eta_3^2 - \frac{3}{8} \eta_6 \right) + \cdots
\]

\[
ES_n^2 = 1 + \frac{1}{n} \left( -\frac{1}{16} \eta_3^2 - \frac{1}{3} \eta_4 + \eta_2 + 1 \right) + \cdots
\]

\[
ES_n^3 = \frac{5\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left( -\frac{5}{12} \eta_4 \eta_3 + \frac{35}{8} \eta_3 \eta_2 - \frac{15}{4} \eta_3^2 - \frac{15}{8} \eta_4 - \frac{11}{16} \eta_6 \right) + \cdots
\]

\[
ES_n^4 = 3 + \frac{1}{n} \left( 10\eta_2 - 9 + \frac{1}{8} \eta_5^2 - \frac{11}{3} \eta_4 \right) + \cdots
\]

\[
ES_n^5 = \frac{35\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left( -\frac{175}{12} \eta_4 \eta_3 - \frac{525}{8} \eta_6 - \frac{875}{64} \eta_3^3 + \frac{259}{16} \eta_5 + \frac{525}{8} \eta_3 \eta_2 - \frac{105}{2} \eta_3 \right) + \cdots
\]

Finally, by using (3.2) we get the cumulants,

\[
(3.21) \quad \kappa_1 = \frac{\eta_3}{4\sqrt{n}} + \frac{1}{n^{3/2}} \left( \frac{1}{9} \eta_4 \eta_3 + \frac{1}{16} \eta_5 - \frac{1}{4} \eta_3 - \frac{5}{24} \eta_3 \eta_2 - \frac{11}{64} \eta_3^2 - \frac{3}{8} \eta_6 \right) + \cdots
\]

\[
\kappa_2 = 1 + \frac{1}{n} \left( -\frac{1}{16} \eta_3^2 - \frac{1}{3} \eta_4 + \eta_2 \right) + \cdots
\]

\[
\kappa_3 = \frac{\eta_3}{2\sqrt{n}} + \frac{1}{n^{3/2}} \left( -\frac{9}{4} \eta_4 + 3 \eta_3 \eta_2 - \frac{1}{2} \eta_3 \eta_4 + \frac{9}{8} \eta_6 - \frac{13}{16} \eta_3^2 - \frac{9}{2} \eta_6 \right) + \cdots
\]

\[
\kappa_4 = \frac{1}{n} \left( -3 - \frac{5}{3} \eta_4 + 4 \eta_2 \right) + \cdots
\]

\[
\kappa_5 = \frac{1}{n^{3/2}} \left( -15 \eta_6 - 10 \eta_3 - 5 \eta_3 \eta_4 + 15 \eta_3 \eta_2 + 4 \eta_5 - \frac{15}{8} \eta_3 \right) + \cdots
\]

Note that indeed they are of the form in (3.1).

3.2. Finding the polynomials of the expansion. We will now find the polynomials which make up the expansion of the distribution function.

We view the characteristic function as

\[
\exp \{ \kappa_1(it) + \frac{1}{2} \kappa_2(it)^2 + \frac{1}{6} \kappa_3(it)^3 + \frac{1}{24} \kappa_4(it)^4 + \frac{1}{120} \kappa_5(it)^5 + \ldots \}
\]

By using (3.1) we get

\[
\exp \{ (it) \left( \frac{k_{12}}{\sqrt{n}} + \frac{k_{13}}{n^{3/2}} \right) + \frac{1}{2} (it)^2 (1 + \frac{k_{22}}{n}) + \frac{1}{6} (it)^3 (\frac{k_{31}}{\sqrt{n}} + \frac{k_{32}}{n^{3/2}}) \}
\]
Consequently, $\varepsilon^3 = \frac{1}{2} k_{12}(it) + k_{31}(it) + \frac{1}{6} k_{32}(it) + \frac{1}{120} k_{51}(it)$. 

Constructing a Taylor expansion for the above gives

\begin{align}
(3.23) & \quad \exp\left\{-\frac{1}{2} t^2\right\} \exp\left\{-\frac{1}{\sqrt{n}} (k_{12}(it) + \frac{1}{6} k_{31}(it))^3\right\} + \frac{1}{n} \left\{\frac{1}{2} k_{22}(it)^2 \right\} \\
& + \frac{1}{24} k_{41}(it) (k_{12}(it) + \frac{1}{6} k_{31}(it))^3 + \frac{1}{120} k_{51}(it)^5 \\
& = \exp\left\{-\frac{1}{2} t^2\right\} \left[1 + \frac{1}{\sqrt{n}} (k_{12}(it) + \frac{1}{6} k_{31}(it))^3\right] + \frac{1}{n} \left\{\frac{1}{2} k_{22}(it)^2 \right\} \\
& + \frac{1}{24} k_{41}(it) (k_{12}(it) + \frac{1}{6} k_{31}(it))^3 + \frac{1}{120} k_{51}(it)^5 \\
& + \frac{1}{2} \left(k_{12}(it) + \frac{1}{6} k_{31}(it)^3\right)^2 \\
& + \frac{2}{n^{3/2}} (k_{12}(it) + \frac{1}{6} k_{31}(it)^3) \left(k_{22}(it)^2 + \frac{1}{24} k_{41}(it)^4 + \ldots \right) \\
& + \frac{1}{6} \left(k_{12}(it) + \frac{1}{6} k_{31}(it)^3\right)^3 \\
& = \exp\left\{-\frac{1}{2} t^2\right\} \left(1 + \frac{r_1(it)}{\sqrt{n}} + \frac{r_2(it)}{n} + \frac{r_3(it)}{n^{3/2}}\right).
\end{align}

Consequently,

\begin{align}
(3.24) & \quad r_1(it) = (it) k_{12} + \frac{1}{6} (it)^3 k_{31} \\
(3.25) & \quad r_2(it) = \left(\frac{1}{2} (it)^2 k_{12}^2 + \frac{17}{2} (it)^2 k_{31} + \frac{1}{6} (it)^4 k_{12} k_{31} + \frac{1}{24} (it)^4 k_{41} + \frac{1}{2} (it)^2 k_{22}\right) \\
(3.26) & \quad r_3(it) = \left(\frac{1}{72} (it)^7 k_{12} k_{31}^2 + \frac{1}{24} (it)^5 k_{12} k_{31} + \frac{1}{12} (it)^5 k_{22} k_{31} + \frac{1}{144} (it)^7 k_{31} k_{41}\right) \\
& \quad + \frac{1}{120} k_{51}(it)^5 + \frac{1}{12} (it)^3 k_{12}^2 k_{31} + \frac{1}{2} (it)^3 k_{12} k_{22} \\
& \quad + \frac{1}{6} (it)^3 k_{32} + (it) k_{13} + \frac{1}{6} (it)^3 k_{12} + \frac{1}{1296} (it)^3 k_{31}^3
\end{align}

The polynomials of the expansions are now (cf. Hall(1992), Section 2.2)

\begin{align}
(3.27) & \quad p_1 = -\frac{1}{12} \eta_3 x^2 + 2 \\
(3.28) & \quad p_2 = -\frac{1}{288} \eta_3^2 x^5 + \left(\frac{1}{12} \eta_3^2 + \frac{1}{8} \eta_4 - \frac{5}{72} \eta_2 \eta_3 - \frac{1}{6} \eta_3^2 \right) x^3 + \left(\frac{1}{8} + \frac{1}{24} \eta_3^2 - \frac{1}{24} \eta_4 \right) x \\
(3.29) & \quad p_3 = -\frac{1}{10368} \eta_3^3 x^8 + \left(\frac{1}{192} \eta_3 + \frac{1}{10368} \eta_3^3 - \frac{1}{72} \eta_3 \eta_2 + \frac{5}{864} \eta_2 \eta_3 \right) x^6
\end{align}
CALCULATION OF EXPANSIONS FOR THE MLE AND ITS DISTRIBUTION FUNCTION

\[
+ \left( \frac{19}{1728} \eta_3^3 - \frac{1}{30} \eta_5 + \frac{1}{8} \eta_6 - \frac{1}{72} \eta_4 \eta_3 \right) x^4 + \left( -\frac{5}{96} \eta_4 \eta_3 + \frac{35}{864} \eta_5^2 + \frac{1}{32} \eta_3 + \frac{1}{80} \eta_5 \right) x^2
\]

\[
+ \frac{35}{432} \eta_3^3 - \frac{5}{48} \eta_4 \eta_3 + \frac{1}{40} \eta_5 + \frac{1}{16} \eta_3
\]

The expansion for the distribution function of the maximum likelihood estimator is now

\[
G_n(x) = \Phi(x) + \frac{1}{\sqrt{n}} p_1(x) \phi(x) + \frac{1}{n} p_2(x) \phi(x) + \frac{1}{n^{3/2}} p_3(x) \phi(x) + o(n^{-3/2}),
\]

with \( G_n(x) = P_0(\sqrt{n} \hat{\theta}_n \leq x) \).

3.3. Finding the Cornish-Fisher expansion for \( G_n^{-1}(u) \). We will now find a Cornish-Fisher expansion for \( G_n^{-1}(\cdot) \).

Assume that \( G_n^{-1} \) is of the form

\[
G_n^{-1}(u) = z_u + \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{3/2}},
\]

where \( z_u \) denotes \( \Phi^{-1}(u) \). We construct Taylor expansions for the terms at the right hand side of (3.30) and plug \( G_n^{-1}(u) \) into these expansions.

The first term of (3.30)

\[
\Phi(G_n^{-1}(u)) = \Phi(z_u + \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{3/2}})
\]

\[
= \Phi(z_u) + \left( \frac{A}{\sqrt{n}} + \frac{B}{n} + \frac{C}{n^{3/2}} \right) \phi(z_u) + \frac{1}{2} (\frac{A}{\sqrt{n}} + \frac{B}{n})^2 \cdot -z_u \phi(z_u)
\]

\[
+ \frac{1}{6} (\frac{A}{\sqrt{n}})^3 (z_u^2 - 1) \phi(z_u)
\]

\[
= u + \frac{A}{\sqrt{n}} + \frac{B}{n} (-\frac{1}{2} z_u A^2 + B)
\]

\[
+ \frac{1}{n^{3/2}} (-\frac{1}{6} A^3 + \frac{1}{6} A^3 z_u^2 - AB z_u + C) \phi(z_u)
\]

The second term of (3.30)

\[
\frac{1}{\sqrt{n}} p_1(z_u + \frac{A}{\sqrt{n}} + \frac{B}{n} \phi(z_u + \frac{A}{\sqrt{n}} + \frac{B}{n})
\]

\[
= \frac{1}{\sqrt{n}} \left( -\frac{1}{12} \eta_3 ((z_u + \frac{A}{\sqrt{n}} + \frac{B}{n})^2 + 2) \right) [1 - z_u (\frac{A}{\sqrt{n}} + \frac{B}{n}) + \frac{1}{2} z_u^2 - 1 (\frac{A}{\sqrt{n}})^2 \phi(z_u)]
\]

\[
= \left[ \frac{1}{\sqrt{n}} \left( -\frac{1}{12} \eta_3 z_u^2 + \frac{1}{6} \eta_3 \right) + \frac{z_u A \eta_3}{12n} + \left( -\frac{1}{72} z_u^2 A^2 \eta_3 + \frac{1}{4} z_u A^2 \eta_3 \beta_3 + \frac{1}{12} z_u^3 B \eta_3 \right) \right] \phi(z_u)
\]

The third term of (3.30)
(3.34) \[ \frac{1}{n} \phi(z_u + \frac{A}{\sqrt{n}}) \phi(z_u + \frac{A}{\sqrt{n}}) \]
\[ = \frac{1}{n} \left(-\frac{1}{288} \eta_3^3 (z_u + \frac{A}{\sqrt{n}})^5 + \frac{1}{72} \eta_3^2 + \frac{5}{72} \eta_4 - \frac{1}{6} \eta_2 \right) (z_u + \frac{A}{\sqrt{n}})^3 \]
\[ + \frac{1}{8} + \frac{1}{24} \eta_3 (z_u + \frac{A}{\sqrt{n}}) (1 - z_u (\frac{A}{\sqrt{n}})) \phi(z_u) \]
\[ = \frac{1}{n} \left(-\frac{1}{288} \eta_3^3 z_u^5 + \frac{1}{72} \eta_3^2 + \frac{5}{72} \eta_4 - \frac{1}{6} \eta_2 \right) z_u^3 + \left( \frac{1}{8} + \frac{1}{24} \eta_3^2 - \frac{1}{8} \right) z_u^4 \]
\[ + \frac{1}{4} + \frac{1}{4} \eta_4 - \frac{1}{2} \eta_2 z_u^2 + \left( \frac{1}{8} + \frac{1}{24} \eta_3^2 - \frac{1}{24} \eta_4 \right) A \phi(z_u) \]

The last term of (3.30)

(3.35) \[ \frac{1}{n^{3/2}} p_3(z_u) \phi(z_u) \]
\[ = \left( -\frac{1}{10368} \eta_3^3 z_u^8 + \frac{1}{96} \eta_3^7 + \frac{19}{10368} \eta_3^3 \eta_2 - \frac{1}{72} \eta_3 \eta_2 + \frac{5}{864} \eta_4 \eta_3 \right) z_u^6 \]
\[ + \left( \frac{19}{1728} \eta_3^3 - \frac{1}{30} \eta_3^5 + \frac{1}{8} \eta_6 - \frac{1}{72} \eta_4 \eta_3 \right) z_u^4 + \left( -\frac{5}{96} \eta_4 \eta_3 + \frac{35}{864} \eta_3^3 + \frac{1}{32} \eta_3 + \frac{1}{80} \eta_5 \right) z_u^2 \]
\[ + \frac{35}{432} \eta_3^3 - \frac{5}{48} \eta_4 \eta_3 + \frac{1}{40} \eta_5 + \frac{1}{16} \eta_3 \phi(z_u) \]

We take all the terms of the order $1/\sqrt{n}$ together to find $A$.

(3.36) \[ A = \frac{\eta_3}{12} (z_u^2 + 2). \]

Next, we take all the terms of order $1/n$, plug in the found $A$, to get $B$

(3.37) \[ B = \left( -\frac{1}{8} - \frac{1}{72} \eta_3^2 - \frac{5}{72} \eta_4 + \frac{1}{6} \eta_2 \right) z_u^3 + \left( -\frac{1}{36} \eta_3^2 - \frac{1}{8} + \frac{1}{24} \eta_4 \right) z_u \]

By plugging $A$ and $B$ in (3.32), (3.33), (3.31), (3.33) and taking all the $n^{-3/2}$ terms together we find

(3.38) \[ C = \left( -\frac{1}{48} \eta_3 - \frac{1}{144} \eta_3 \eta_3 + \frac{1}{24} \eta_3 \eta_2 + \frac{1}{30} \eta_5 - \frac{1}{8} \eta_6 - \frac{19}{1728} \eta_3 \right) z_u^4 \]
\[ + \left( \frac{5}{48} \eta_3 + \frac{1}{12} \eta_3 \eta_2 - \frac{1}{80} \eta_5 - \frac{67}{1296} \eta_3^3 + \frac{1}{48} \eta_4 \eta_3 \right) z_u^2 \]
\[ - \frac{1}{12} \eta_3 - \frac{1}{40} \eta_5 + \frac{1}{9} \eta_3 \eta_3 - \frac{113}{1296} \eta_3^3. \]
REFERENCES

[1] M. Akahira, Third order efficiency implies fourth order efficiency: a resolution of the conjecture of J.K. Ghosh, Ann. Inst. Statist. Math., vol 48, No. 2 (1996), 365 - 380.

[2] D. Chibisov, An asymptotic expansion for a class of estimators containing maximum likelihood estimators, Theory Probab. Appl. vol 18, (1973), 295 - 303.

[3] P. Hall, The bootstrap and Edgeworth expansion, Springer-Verlag 1992.

[4] S. Venetiaan, Bootstrap bounds, Ph.D. thesis, University of Amsterdam 1994.

[5] S. Venetiaan, An expansion for the maximum likelihood estimator of location and its distribution function, to be submitted for publication.