CRITICAL SCHRÖDINGER–HARDY SYSTEMS IN THE HEISENBERG GROUP

PATRIZIA PUCCI
Department of Mathematics and Informatics
University of Perugia
Via Vanvitelli, 1
06123 Perugia, Italy

Dedicated to Professor Vicentiu D. Radulescu on the occasion of his 60th birthday,
with high feelings of admiration for his notable contributions
in Mathematics and great affection

ABSTRACT. The paper is focused on existence of nontrivial solutions of a Schrödinger–Hardy system in the Heisenberg group, involving critical nonlinearities. Existence is obtained by an application of the mountain pass theorem and the Ekeland variational principle, but there are several difficulties arising in the framework of Heisenberg groups, also due to the presence of the Hardy terms as well as critical nonlinearities.

1. Introduction. In this paper we prove existence of nontrivial solutions for the elliptic Schrödinger–Hardy system in the Heisenberg group \( \mathbb{H}^n \)

\[
\begin{align*}
-\Delta_{\mathbb{H}^n}^p u + V(q)|u|^{p-2}u - \gamma |u|^p \frac{|u|^{p-2}u}{r(q)^p} &= \lambda H_u(q, u, v) + \frac{\alpha}{p^*}|v|^\beta |u|^\alpha u, \\
-\Delta_{\mathbb{H}^n}^p v + V(q)|v|^{p-2}v - \gamma |v|^p \frac{|v|^{p-2}v}{r(q)^p} &= \lambda H_v(q, u, v) + \frac{\beta}{p^*}|u|^\alpha |v|^\beta v,
\end{align*}
\]

(1.1)

where \( \gamma \) and \( \lambda \) are real parameters, \( Q = 2n + 2 \) is the homogeneous dimension of the Heisenberg group \( \mathbb{H}^n \), \( 1 < p < Q \), the exponents \( \alpha > 1 \) and \( \beta > 1 \) are such that \( \alpha + \beta = p^* \), \( p^* = pQ/(Q - p) \), and \( \Delta_{\mathbb{H}^n}^p \) is the \( p \)-Laplacian operator on \( \mathbb{H}^n \), which is defined by

\[
\Delta_{\mathbb{H}^n}^p \varphi = \text{div}_H(|D_{\mathbb{H}^n} \varphi|^{p-2}D_{\mathbb{H}^n} \varphi)
\]

along any \( \varphi \in C_0^\infty(\mathbb{H}^n) \), that is \( \Delta_{\mathbb{H}^n}^p \) is the familiar horizontal \( p \)-Laplacian operator.

The potential function \( V \) verifies

\( \text{(V)} \quad V \in C(\mathbb{H}^n) \) and \( \inf_{x \in \mathbb{H}^n} V(x) = V_0 > 0 \).

Moreover, \( r \) denotes the Heisenberg norm \( r(q) = r(z, t) = (|z|^4 + t^2)^{1/4} \), \( z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n \), \( t \in \mathbb{R} \), \( |z| \) the Euclidean norm in \( \mathbb{R}^{2n} \),

\[
D_{\mathbb{H}^n} u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u)
\]

2010 Mathematics Subject Classification. Primary: 35R03, 35H20, 35J70; Secondary: 35B33, 35A15.

Key words and phrases. Heisenberg group, entire solutions, Schrödinger–Hardy systems, subelliptic critical systems.
the horizontal gradient as in (2.2), \( \{X_j, Y_j\}_{j=1}^n \) the basis of horizontal left invariant vector fields on \( \mathbb{H}^n \), that is

\[
X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},
\]

for \( j = 1, \ldots, n \). The weight function \( \psi \) appearing in (1.1) is defined as \( \psi = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n} \). We emphasize that \( \psi \) is identically 1 in the Euclidean canonical case and we refer to Section 2 for further details.

The nonlinearities \( H_u \) and \( H_v \) denote the partial derivatives of \( H \) with respect to the second variable and the third variable, respectively, and \( H \) satisfies

\( (\mathcal{H}) \quad H \in C^1(\mathbb{H}^n \times \mathbb{R}^2, \mathbb{R}^+), H_z(q,0,0) = 0 \) for all \( q \in \mathbb{H}^n \) and there exist \( \mu \) and \( s \) such that \( p < \mu \leq s < p^* \) and for every \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) for which the inequality

\[
|H_w(q, w)| \leq \mu |w|^{\mu - 1} + q C_\varepsilon |w|^{s - 1}, \quad w = (u, v), \quad |w| = \sqrt{u^2 + v^2},
\]

holds for any \( (q, w) \in \mathbb{H}^n \times \mathbb{R}^2 \), where \( H_w = (H_u, H_v) \), and also

\[
0 \leq \mu H(q, w) \leq H_w(q, w) \cdot w \quad \text{for all } (q, w) \in \mathbb{H}^n \times \mathbb{R}^2
\]

is valid. Finally, for all measurable set \( E \) of \( \mathbb{H}^n \), with positive measure, \( H(q,u,v) > 0 \) for all \( q \in E \) and \( (u,v) \in \mathbb{R}^+ \times \mathbb{R}^+ \).

Throughout the paper, statements involving measure theory are always understood to be with respect to Haar measure on \( \mathbb{H}^n \), which coincides with the \((2n+1)\)-dimensional Lebesgue measure. We refer to Section 2 for further details.

A similar problem was recently studied in [13] and [27], for the fractional \( p \)-Laplacian operator, in the context of the Euclidean space. In [27], the Hardy terms were not considered.

In order to handle system (1.1), it is crucial to introduce the Hardy–Sobolev inequality. Since \( 1 < p < Q \), by [14, 15, 31, 32], we know that for all \( \varphi \in C_0^\infty(\mathbb{H}^n) \)

\[
\|\varphi\|_{p^*} \leq C_Q,p \|D_{\mathbb{H}^n} \varphi\|_p, \quad p^* = \frac{pQ}{Q-p}, \tag{1.2}
\]

where \( C_{Q,p} \) is a positive constant depending only on \( Q \) and \( p \). Theorem 1 of [28] gives that

\[
\int_{\mathbb{H}^n} \psi^p \frac{|\varphi|^p}{r^p} dq \leq \left( \frac{p}{Q-p} \right)^p \int_{\mathbb{H}^n} |D_{\mathbb{H}^n} \varphi|^p_{\mathbb{H}^n} dq \tag{1.3}
\]

for all \( \varphi \in C_0^\infty(\mathbb{H}^n \setminus \{O\}) \), with \( O = (0,0) \) the natural origin in \( \mathbb{H}^n \). The above Hardy inequality was obtained in [18] when \( p = 2 \) and, in another version, in [10] for all \( p > 1 \). The best Hardy–Sobolev constant \( H_p = H(p,q) \) is given by

\[
H_p = \inf_{u \in S^{1,p}(\mathbb{H}^n) \setminus \{0\}} \frac{\|D_{\mathbb{H}^n} u\|_p^{2}}{\|u\|_{H_p}^{2}}, \quad \|u\|_{H_p} = \int_{\mathbb{H}^n} \psi^p \frac{|u|^p}{r^p} dq, \tag{1.4}
\]

where \( S^{1,p}(\mathbb{H}^n) \) is the Folland–Stein space, defined as the completion of \( C_0^\infty(\mathbb{H}^n) \) with respect to the norm

\[
\|D_{\mathbb{H}^n} u\|_p = \left( \int_{\mathbb{H}^n} |D_{\mathbb{H}^n} u|^p_{\mathbb{H}^n} dq \right)^{1/p}.
\]

Clearly, \( H_p > 0 \) thanks to (1.3).
Let us introduce some notation. Define

\[ E_{p,V} = \left\{ u \in S^1_1(\mathbb{H}^n) : \int_{\mathbb{H}^n} V(q)|u(q)|^p dq < \infty \right\}, \]

equipped with the norm

\[ \|u\|_{E_{p,V}} = \left( \|D_{\mathbb{H}^n} u\|_p^p + \|u\|_{p,V}^p \right)^{1/p}, \]

where \( \|u\|_{p,V} = \left( \int_{\mathbb{H}^n} V(q)|u|^p dq \right)^{1/p} \). The natural solution space for (1.1) is

\[ W = E_{p,V} \times E_{p,V}, \]

with associated norm

\[ \|(u, v)\| = \left( \|u\|_{E_{p,V}}^p + \|v\|_{E_{p,V}}^p \right)^{1/p}. \]

Much interest has grown on systems involving Hardy terms and critical exponents. We refer to the recent papers [12, 13, 17] for a large bibliography on the topic in the Euclidean setting, and to [5] in the Heisenberg frame.

**Theorem 1.1.** Suppose that \( V \) satisfies \((V)\) and that \( H \) fulfills \((H)\). Then for any \( \gamma \in (-\infty, H_p) \) there exists \( \lambda^* = \lambda^*(\gamma) \) such that system (1.1) admits at least one nontrivial solution \( (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \) in \( W \) for all \( \lambda \geq \lambda^* \). Moreover,

\[ \lim_{\lambda \to \infty} \|(u_{\gamma, \lambda}, v_{\gamma, \lambda})\| = 0 \]  \hspace{1cm} (1.5)

holds.

The proof of Theorem 1.1 somehow follows [13], but there are some technical difficulties due to the more general setting considered in this paper, as well as to the presence of the Hardy terms and the critical nonlinearities. The main compactness result, Theorem 2.2, is proved following the argument of Lemma 2.2 in [7] given for the Euclidean case. We refer also to Theorem 2.1 in [27] for further details and comments. In any case, the new key result, of independent interest, is given by the crucial Lemma 3.4.

Taking inspiration from [13], we treat also the sublinear case, that is when the exponent \( s \in (1, p) \), and when \( H \) is of the special separated form \( H(q, u, v) = h(q)f(u, v) \). Hence, we deal with the new system in \( \mathbb{H}^n \)

\[
\begin{cases}
\Delta_{\mathbb{H}^n}^p u + V(q)|u|^{p-2} u - \gamma \psi^p \frac{|u|^{p-2} u}{r(q)^p} = h(q)f_u(u, v) + \sigma \frac{\alpha}{p^*} |v|^\beta |u|^{\alpha-2} u, \\
\Delta_{\mathbb{H}^n}^p v + V(q)|v|^{p-2} v - \gamma \psi^p \frac{|v|^{p-2} v}{r(q)^p} = h(q)f_v(u, v) + \sigma \frac{\beta}{p^*} |u|^\alpha |v|^{\beta-2} v,
\end{cases} \]  \hspace{1cm} (1.6)

where \( V \) satisfies \((V)\), \( \sigma > 0 \) and \( f \) verifies

\((f_1)\) \( f \in C^1(\mathbb{R}^2, \mathbb{R}^+) \) and there exist \( C > 0 \) and \( s \in (1, p) \) such that

\[ |f_w(w)| \leq C|w|^{s-1} \quad \text{for all } w = (u, v) \in \mathbb{R}^2, \]

where \( f_w = (f_u, f_v) \) and \( f_u, f_v \) denote the partial derivatives of \( f \) with respect to the first and second variable;

\((f_2)\) there exist \( a_0 > 0, \delta > 0 \) and \( s_1 \in (1, p) \) such that

\[ f(w) \geq a_0 |w|^{s_1} \quad \text{for all } w \in \mathbb{R}^2, \text{ with } |w| \leq \delta. \]

Concerning the function \( h \) in (1.6), we assume from now on that \( h \) verifies
Clearly, conditions (h) simply require that $h$ is not trivial.

In order to cover the more interesting case when $\sigma > 0$ in (1.6), we need a further assumption on $h$. Fix $\gamma < H_p$ and set

$$\eta(t) = \frac{1}{2p} \left( 1 - \frac{\gamma^+}{H_p} \right) t^p - \frac{C_{Q,p}^p}{p^*} t^p,$$

for all $t \geq 0$, where $C$ is introduced in (f1) and $C_{Q,p}^p > 0$ in (1.2). Since $p < p^*$ the positive number $\rho_0 = \left( (H_p - \gamma^+)/2H_p C_{Q,p}^p \right)^{1/(p^*-p)}$ is such that

$$\eta(\rho_0) = \max_{t \geq 0} \eta(t) = \left( \frac{1}{2p} - \frac{1}{2p^*} \right) \left( 1 - \frac{\gamma^+}{H_p} \right)^{p^*/(p^*-p)} (2C_{Q,p}^p)^{p/(p^*-p)} > 0.$$

We are now able to state the existence result for (1.6).

**Theorem 1.2.** Assume that $V$ satisfies (V), that $f$ fulfills (f1)-(f2) and that $h$ satisfies (h). Let $\gamma$ be in $(-\infty,H_p)$. Then (1.6) admits at least one nontrivial solution $(u_{\gamma,\sigma},v_{\gamma,\sigma})$ in $W$ for all $\sigma \leq 0$.

If $\sigma > 0$ and $h$, depending on $\gamma^+$, satisfies

$$\eta(\rho_0) > \left( \frac{1}{2p} - \frac{1}{2p^*} \right)^{s/(s-p)} \left( C_{Q,p}^p \| h \|_{p^*-s} \right)^{p/(p^*-s)}, \quad (1.7)$$

where $s$ is the exponent in (f1), then there exists a threshold $\sigma^* = \sigma^*(\gamma,h) > 0$ such that system (1.6) admits at least one nontrivial solution $(u_{\gamma,\sigma},v_{\gamma,\sigma})$ in $W$ for all $\sigma \in (0,\sigma^*)$.

Theorem 1.2 was recently established in the Euclidean setting in Theorem 1.3 of [13] and in Theorem 1.2 of [27] when $\gamma = 0$, that is without the Hardy terms. Again, the Heisenberg setting makes Theorem 1.2 more difficult to handle than in [13, 27]. As far as we know, Theorems 1.1 and 1.2 are new even when $p = 2$.

Indeed, Theorem 1.2 generalizes the existence results obtained in [8] in several directions, as well as in the papers cited in [13].

The paper is organized as follows. In Section 2 we recall the main notations and definitions related to the Heisenberg group and for the natural solution space $W$ of (1.1) and (1.6) we prove the key compactness theorems, particularly helpful for the next sections. In Section 3, using the mountain pass of Ambrosetti and Rabinowitz, we obtain the existence of nontrivial solutions for (1.1), that is we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2 via the Ekeland variational principle.

2. Notations and preliminaries. We briefly recall the relevant definitions and notations related to the Heisenberg group functional setting. For a complete treatment, we refer to [9, 18, 23, 24].

Let $H^n$ be the Heisenberg group of topological dimension $2n + 1$, that is the Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$, endowed with the non-Abelian group law

$$q \circ q' = \left( z + z', t + t' + 2 \sum_{i=1}^{n} (y_i x_i' - x_i y_i') \right)$$
for all $q, q' \in \mathbb{H}^n$, with
\[
q = (z, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t), \quad q' = (z', t') = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, t').
\]
In $\mathbb{H}^n$ the natural origin is denoted by $O = (0, 0)$. Define
\[
|q| = r(q) = |(z|^4 + t^2)^{1/4} \quad \text{for all } q = (z, t) \in \mathbb{H}^n,
\]
where $| \cdot |$ stands for the Euclidean norm in $\mathbb{R}^{2n}$. The Korányi norm is homogeneous of degree 1, with respect to the dilations $\delta_R : (z, t) \mapsto (Rz, R^2t)$, $R > 0$. Indeed, for all $q = (z, t) \in \mathbb{H}^n$
\[
r(\delta_R(q)) = r(Rz, R^2t) = (|Rz|^4 + R^4t^2)^{1/4} = R r(q).
\]
Hence, the Korányi distance, is
\[
d_K(q, q') = r(q^{-1} \circ q') \quad \text{for all } (q, q') \in \mathbb{H}^n \times \mathbb{H}^n,
\]
and the Korányi open ball of radius $R$ centered at $q_0$ is
\[
B_R(q_0) = \{ q \in \mathbb{H}^n : d_K(q, q_0) < R \}.
\]
For simplicity $B_R$ denotes the ball of radius $R$ centered at $q_0 = O$.

The Jacobian determinant of $\delta_R$ is $R^{2n+2}$. The natural number $Q = 2n + 2$, which is the so-called homogeneous dimension of $\mathbb{H}^n$, plays a role analogous to the topological dimension in the Euclidean context, see [24] and the references therein.

The Haar measure on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. It is invariant under left translations and $Q$–homogeneous with respect to dilations. Hence, as noted in [23], the topological dimension $2n + 1$ of $\mathbb{H}^n$ is strictly less than its Hausdorff dimension $Q = 2n + 2$. We denote by $|E|$ the $(2n + 1)$–dimensional Lebesgue measure of any measurable set $E \subset \mathbb{H}^n$. Then,
\[
|\delta_R(E)| = R^Q|E|, \quad d(\delta_Rq) = R^Q dq.
\]

In particular, if $E = B_R$, then $|B_R| = |B_1| R^Q$.

The vector fields for $j = 1, \ldots, n$
\[
X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad (2.1)
\]
constitute a basis for the real Lie algebra of left–invariant vector fields on $\mathbb{H}^n$. This basis satisfies the Heisenberg canonical commutation relations for position and momentum $[X_j, Y_k] = -4\delta_{jk} \frac{\partial}{\partial t}$, all other commutators being zero. A vector field in the span of $\{X_j, Y_j\}_{j=1}^n$ will be called horizontal.

Let $u \in C^1(\mathbb{H}^n)$ be fixed. The horizontal gradient $D_{\mathbb{H}^n} u$ is
\[
D_{\mathbb{H}^n} u = \sum_{j=1}^n [(X_j u) X_j + (Y_j u) Y_j], \quad (2.2)
\]
that is it is an element of the span of $\{X_j, Y_j\}_{j=1}^n$. Furthermore, if $f \in C^1(\mathbb{R})$, then $D_{\mathbb{H}^n} f(u) = f'(u) D_{\mathbb{H}^n} u$.

The natural inner product in the span of $\{X_j, Y_j\}_{j=1}^n$
\[
(W, Z)_{\mathbb{H}^n} = \sum_{j=1}^n (w^j z^j + \bar{w}^j \bar{z}^j)
\]
for $W = \{w^j X_j + \bar{w}^j Y_j\}_{j=1}^n$ and $Z = \{z^j X_j + \bar{z}^j Y_j\}_{j=1}^n$ produces the Hilbertian norm
\[
|D_{\mathbb{H}^n} u|_{\mathbb{H}^n} = \sqrt{(D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} u)_{\mathbb{H}^n}}.
\]
for the horizontal vector field $D_{\mathbb{H}^n}u$. Moreover, if also $v \in C^1(\mathbb{H}^n)$ then the Cauchy–Schwarz inequality

$$\left|\langle D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}v \rangle_{\mathbb{H}^n} \right| \leq |D_{\mathbb{H}^n}u|_{\mathbb{H}^n} |D_{\mathbb{H}^n}v|_{\mathbb{H}^n}$$

continues to be valid.

For any horizontal vector field $W = \{w^jX_j + \bar{w}^jY_j\}_{j=1}^n$ of class $C^1(\mathbb{H}^n; \mathbb{R}^{2n})$ the horizontal divergence is defined by

$$\operatorname{div}_H W = \sum_{j=1}^n [X_j(w^j) + Y_j(\bar{w}^j)].$$

If furthermore $g \in C^1(\mathbb{R})$, then the Leibnitz formula holds, namely

$$\operatorname{div}_H (gW) = g\operatorname{div}_H(W) + (D_{\mathbb{H}^n} g, W)_{\mathbb{H}^n}.$$  

Similarly, if $u \in C^2(\mathbb{H}^n)$, then the Kohn–Spencer Laplacian, or equivalently the horizontal Laplacian in $\mathbb{H}^n$, of $u$ is defined as follows

$$\Delta_{\mathbb{H}^n} u = \sum_{j=1}^n (X_j^2 + Y_j^2) u$$

$$= \sum_{j=1}^n \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4y_j \frac{\partial^2}{\partial x_j \partial t} - 4x_j \frac{\partial^2}{\partial y_j \partial t} \right) u + 4|z|^2 \frac{\partial^2 u}{\partial t^2},$$

and $\Delta_{\mathbb{H}^n}$ is hypoelliptic according to the celebrated Theorem 1.1 due to Hörmander in [21]. In particular, $\Delta_{\mathbb{H}^n} u = \operatorname{div}_H D_{\mathbb{H}^n} u$ for each $u \in C^2(\mathbb{H}^n)$.

The main geometrical function $\psi$ in (1.1) is defined by

$$\psi(q) = |D_{\mathbb{H}^n} r|_{\mathbb{H}^n} = \frac{|z|}{r(q)}$$

for all $q = (z, t) \in \mathbb{H}^n$, with $q \neq O$, (2.3)

and $0 \leq \psi \leq 1$, $\psi(0, t) \equiv 0$, $\psi(z, 0) \equiv 1$. Furthermore, $\psi^2$ is the density function, which is homogeneous of degree 0, with respect to the dilatation $\delta_R$. Direct calculations show that

$$\Delta_{\mathbb{H}^n} r = \frac{2n + 1}{r} \psi^2$$

in $\mathbb{H}^n \setminus \{O\}$.

For details we refer to Section 2.1 of [25].

A well known generalization of the Kohn–Spencer Laplacian is the horizontal $p$–Laplacian on the Heisenberg group, defined by

$$\Delta_{\mathbb{H}^n}^p \varphi = \operatorname{div}_H(|D_{\mathbb{H}^n} \varphi|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} \varphi), \quad p \in (1, \infty),$$

for all $\varphi \in C_0^\infty(\mathbb{H}^n)$.

We recall that $1 < p < Q$. First, $E_{p, V} = (E_{p, V}, \| \cdot \|_{E_{p, V}})$ is a separable, reflexive Banach space. A proof of this fact is given in the Euclidean setting in Lemma 10 of [29] and can be extended, with obvious changes, in the Heisenberg context. Hence, by Theorem 1.12 of [1], the main solution space $W = E_{p, V} \times E_{p, V}$, with \( \| (u, v) \| = (\| u \|_{E_{p, V}} + \| v \|_{E_{p, V}})^{1/p} \), is a real separable reflexive Banach space.

From now on $HW^{1, p}(\mathbb{H}^n)$ denotes the horizontal Sobolev space of the functions $u \in L^p(\mathbb{H}^n)$ such that $D_{\mathbb{H}^n} u$ exists in the sense of distributions and $|D_{\mathbb{H}^n} u|_{\mathbb{H}^n}$ is in $L^p(\mathbb{H}^n)$, endowed with the natural norm

$$\| u \|_{HW^{1, p}(\mathbb{H}^n)} = \left( \int_{\mathbb{H}^n} |D_{\mathbb{H}^n} u|_{\mathbb{H}^n}^p \, dq + \int_{\mathbb{H}^n} |u|^p \, dq \right)^{1/p}.$$
The embedding
\[ HW^{1,p}(\mathbb{H}^n) \hookrightarrow L^\nu(\mathbb{H}^n) \]  
(2.4)
is continuous for any \( \nu \in [p,p^*) \) by (1.2) and the interpolation inequality. Furthermore, by [19, 22, 33] we know that, if \( \Omega \) is a bounded Poincaré–Sobolev domain in \( \mathbb{H}^n \), the embedding
\[ HW^{1,p}(\Omega) \hookrightarrow L^\nu(\Omega) \]  
(2.5)
is compact provided that \( 1 \leq \nu < p^* \). This result holds for Carnot–Carathéodory balls, since by [16, 22, 33] such sets are Poincaré–Sobolev domains. In the proofs of the main compactness results below, we apply (2.5) to the Korányi balls \( B_R \). Indeed, it is well-known that the Carnot–Carathéodory distance and the Korányi distance are equivalent on \( \mathbb{H}^n \), see [3, 26, 30].

The next result was established in Lemma 2.1 of [5] in the Heisenberg context, and is an adaptation of Lemma 4.1 of [6] and Lemma 1 of [29], where the Euclidean space \( \mathbb{R}^n \) is replaced by \( \mathbb{H}^n \).

**Lemma 2.1.** Let \( V \) satisfy (\( \mathcal{V} \)) and let \( \nu \in [p,p^*) \). Then the embeddings
\[ E_{p,V} \hookrightarrow HW^{1,p}(\mathbb{H}^n) \hookrightarrow L^\nu(\mathbb{H}^n) \]
are continuous. In particular,
\[ W \hookrightarrow HW^{1,p}(\mathbb{H}^n) \times HW^{1,p}(\mathbb{H}^n) \hookrightarrow L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n) \]
are continuous and there exists a constant \( C_\nu \) such that
\[ \| (u,v) \|_\nu = \| (u,v) \|_{L^\nu(\mathbb{H}^n)} \leq C_\nu \| (u,v) \| \quad \text{for all } (u,v) \in W, \]  
(2.6)
where \( C_\nu \) depends on \( \nu \), \( n \) and \( p \).

If \( \nu \in [1,p^*) \), then the embedding \( E_{p,V} \hookrightarrow L^\nu(B_R) \) is compact for any \( R > 0 \).

The proof of the next result relies on Lemma 2.2 in [7], see also Theorem 2.1 in [27], for the Euclidean case. The extension to the Heisenberg setting can be derived proceeding as in [7, 27], with obvious changes. For a different proof in \( \mathbb{H}^n \) we refer also to Theorem 3.1 and Lemma 3.6 of [5], which extends to the Heisenberg case Theorem 2.1 in [29] and Lemma 4.4 in [6].

**Lemma 2.2.** Suppose that \( V \) satisfies (\( \mathcal{V} \)). Let \( \{(u_k,v_k)\}_k \) and \((u,v)\) be in \( W \), such that \( (u_k,v_k) \rightharpoonup (u,v) \) weakly in \( W \), and \( (u_k,v_k) \to (u,v) \) a.e. in \( \mathbb{H}^n \). Then \( (u_k,v_k) \to (u,v) \) strongly in \( L^\nu(\mathbb{H}^n) \times L^\nu(\mathbb{H}^n) \) as \( k \to \infty \) for any \( \nu \in (p,p^*) \).

From Lemma 3.5 of [5] we know that if \( \{(u_k,v_k)\}_k \) and \((u,v)\) be in \( W \), such that \( (u_k,v_k) \rightharpoonup (u,v) \) weakly in \( W \), then, up to a subsequence, \( (u_k,v_k) \to (u,v) \) a.e. in \( \mathbb{H}^n \) as \( k \to \infty \). Lemma 3.5 of [5] is an extension to the Heisenberg setting of Proposition A.10 of [2].

Since \( \alpha > 1 \) and \( \beta > 1 \) in (1.1) and (1.6) are such that \( \alpha + \beta = p^* \), then the Hölder inequality and (1.2) yield
\[ \int_{\mathbb{H}^n} |u|^\alpha |v|^\beta \, dq \leq \left( \int_{\mathbb{H}^n} |u|^{p^*} \, dq \right)^{\alpha/p^*} \left( \int_{\mathbb{H}^n} |v|^{p^*} \, dq \right)^{\beta/p^*} \]  
(2.7)
\[ \leq C_{Q,p}^\nu \| D_{\mathbb{H}^n} u \|_p \| D_{\mathbb{H}^n} v \|_p \leq C_{Q,p}^\nu \| (D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} v) \|_p \]
for all \((u,v)\) in \( W \). The next results is essential for the main proofs of Sections 3 and 4.
Lemma 2.3. Suppose that $V$ satisfies (V). Let $\alpha > 1$ and $\beta > 1$, with $\alpha + \beta = p^*$, and let $\{(u_k, v_k)\}_k$ and $(u, v)$ be in $W$, such that $(u_k, v_k) \rightharpoonup (u, v)$ weakly in $W$, and $(u_k, v_k) \rightarrow (u, v)$ a.e. in $\mathbb{H}^n$. Then,

$$
\lim_{k \rightarrow \infty} \int_{\mathbb{H}^n} |u_k - u|^\alpha |v_k - v|^\beta dq = \lim_{k \rightarrow \infty} \int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta dq - \int_{\mathbb{H}^n} |u|^\alpha |v|^\beta dq,
$$

$$
\lim_{k \rightarrow \infty} \int_{\mathbb{H}^n} |v_k|^\beta |u|^{\alpha - 2} u(u_k - u) dq = 0,
$$

(2.8)

$$
\lim_{k \rightarrow \infty} \int_{\mathbb{H}^n} |u_k|^{\alpha}\beta - 2 v(v_k - v) dq = 0.
$$

Proof. The first part of (2.8) can be obtained just adapting the proof of Lemma 2.1 to the Heisenberg group.

For the second part of (2.8), let us fix any measurable subset $U \subset \mathbb{H}^n$. Clearly, the sequences $(u_k)_k$ and $(v_k)_k$ are bounded in $HW^{1, p}(\mathbb{H}^n)$, so that the Hölder inequality gives

$$
\int_U ||u|^{\alpha - 2} u(u_k - u)|^{p^*/\alpha} dq \leq \left( \int_U |u|^{p^*} dq \right)^{(\alpha - 1)/\alpha p^*} \|u_k - u\|^{(\alpha - 1)/\alpha}.
$$

and, similarly,

$$
\int_U ||v|^{\beta - 2} v(v_k - v)|^{p^*/\beta} dq \leq \left( \int_U |v|^{p^*} dq \right)^{(\beta - 1)/\beta p^*} \|v_k - v\|^{(\beta - 1)/\beta}.
$$

Moreover, $(u, v) \in L^{p^*}(\mathbb{H}^n) \times L^{p^*}(\mathbb{H}^n)$ by Lemma 2.1. Thus, the sequences

$$
\left( ||u|^{\alpha - 2} u(u_k - u)|^{p^*/\alpha} \right)_k \quad \text{and} \quad \left( ||v|^{\beta - 2} v(v_k - v)|^{p^*/\beta} \right)_k
$$

are equi–integrable in $\mathbb{H}^n$. Clearly,

$$
|u|^{\alpha - 2} u(u_k - u) \rightarrow 0 \quad \text{and} \quad |v|^{\beta - 2} v(v_k - v) \rightarrow 0 \quad \text{a.e. in } \mathbb{H}^n,
$$

since $(u_k, v_k) \rightarrow (u, v)$ a.e. in $\mathbb{H}^n$ by assumption. Hence, the Vitali convergence theorem yields

$$
|u|^{\alpha - 2} u(u_k - u) \rightarrow 0 \quad \text{strongly in } L^{p^*/\alpha}(\mathbb{H}^n),
$$

$$
|v|^{\beta - 2} v(v_k - v) \rightarrow 0 \quad \text{strongly in } L^{p^*/\beta}(\mathbb{H}^n).
$$

(2.9)

Obviously, $(|u_k|)_k$ is bounded in $L^{p^*}(\mathbb{H}^n)$ and $(|v_k|)_k$ is bounded in $L^{p^*}(\mathbb{H}^n)$ by Lemma 2.1. Therefore, the Hölder inequality, the fact that $\alpha + \beta = p^*$ and (2.9) give as $k \rightarrow \infty$

$$
\left| \int_{\mathbb{H}^n} |v_k|^\beta |u|^{\alpha - 2} u(u_k - u) dq \right| \leq \|v_k\|^{\beta}_{p^*} \|u\|^{\alpha - 2} u(u_k - u))\|_{p^*/\alpha} = o(1),
$$

and similarly

$$
\left| \int_{\mathbb{H}^n} |u_k|^\alpha |v|^{\beta - 2} v(v_k - v) dq \right| \leq \|u_k\|^{\alpha}_{p^*} \|v\|^{\beta - 2} v(v_k - v)\|_{p^*/\beta} = o(1).
$$

This completes the proof of (2.8).  \(\square\)
3. Proof of Theorem 1.1. In this section, we assume, without further mentioning, that the assumptions required in Theorem 1.1 are satisfied.

System (1.1) has a variational structure and to prove Theorem 1.1 we use the celebrated mountain pass theorem of Ambrosetti and Rabinowitz at a special critical level. The Euler–Lagrange functional, \( I = I_{\gamma,\lambda} \), associated to (1.1) is
\[
I(u,v) = \frac{1}{p} \|u,v\|^p - \frac{\gamma}{p} (\|u\|_{H^p}^p + \|v\|_{H^p}^p) - \lambda \int_{\mathbb{R}^n} H(q,u,v) dq - \frac{1}{p} \int_{\mathbb{R}^n} |u|^\alpha|v|^\beta dq.
\]
Clearly, the functional \( I \) is well-defined on \( W \). Under condition (H), it is easy to see that \( I \) is of class \( C^1(W) \), and for \((u,v) \in W\)
\[
\langle I'(u,v), (\Phi,\Psi) \rangle = \langle (u,v), (\Phi,\Psi) \rangle - \gamma \langle (u,v), H_u \rangle - \langle v,\Psi \rangle.
\]
for all \((\Phi,\Psi) \in W \). From here on \langle \cdot,\cdot \rangle simply denotes the dual pairing between \( W \) and its dual space \( W' \), that is \( \langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle_{W',W} \). Hence, the critical points of \( I \) in \( W \) are exactly the (weak) solutions of (1.1).

Indeed, we say that \((u,v) \in W\) is an entire (weak) solution of problem (1.1) if
\[
\langle (u,v), (\Phi,\Psi) \rangle - \gamma \langle (u,v), H_u \rangle - \langle v,\Psi \rangle = \lambda \int_{\mathbb{R}^n} [H_u(q,u,v)\Phi + H_v(q,u,v)\Psi] dq + \frac{\alpha}{p^*} \int_{\mathbb{R}^n} |u|^\alpha-2 |v|^\beta \Phi dq + \frac{\beta}{p^*} \int_{\mathbb{R}^n} |u|^\alpha|v|^\beta-2 v\Psi dq
\]
for any \((\Phi,\Psi) \in W \), where
\[
\langle (u,v), (\Phi,\Psi) \rangle_p = \int_{\mathbb{R}^n} \left\{ \langle D_{H^p} u \rangle_{|H^p|}^2 D_{H^p} u, D_{H^p} \Phi \rangle_{H^p} + \langle D_{H^p} v \rangle_{|H^p|}^2 D_{H^p} v, D_{H^p} \Psi \rangle_{H^p} \right\} dq,
\]
\[
\langle (u,v), (\Phi,\Psi) \rangle = \langle (u,v), (\Phi,\Psi) \rangle_{H^p} + \int_{\mathbb{R}^n} V(q) \left( |u|^p-2 u\Phi + |v|^p-2 v\Psi \right) dq,
\]
\[
\langle u,\Phi \rangle_{H^p} = \int_{\mathbb{R}^n} |u(q)|^p-2 u(q)\Phi(q) dq, \quad \langle v,\Psi \rangle_{H^p} = \int_{\mathbb{R}^n} |v(q)|^p-2 v(q)\Psi(q) dq.
\]
The simplified notation is reasonable, since \( \langle (u,v), (\cdot,\cdot) \rangle_{H^p} \) and \( \langle v,\cdot \rangle_{H^p} \) are linear bounded functionals on \( W \) for all \((u,v) \in W\).

In this section, we first prove that the functional \( I \) has the geometric features required to apply the mountain pass theorem of Ambrosetti and Rabinowitz.

Lemma 3.1. Fix \( \gamma \in (-\infty, H_p) \) and any \( \lambda > 0 \). Then there exists a couple \((e_1,e_2) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \), independent of \( \gamma^+ \) and \( \lambda \), such that \( I(e_1,e_2) < 0, \|e_1,e_2\| \geq 2 \) and \( \int_{\mathbb{R}^n} |e_1|^\alpha|e_2|^\beta dq > 0 \). Furthermore, there exist numbers \( j = j(\gamma,\lambda) > 0 \) and \( \rho = \rho(\gamma,\lambda) \in (0,1) \) such that \( I(u,v) \geq j \) for any \((u,v) \in W \), with \( \| (u,v) \| = \rho \).

Proof. Fix \( \gamma \in (-\infty, H_p) \) and \( \lambda > 0 \). Now \((u,v) \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \) such that \( \| (u,v) \| = 1 \) and \( \int_{\mathbb{R}^n} |u|^\alpha|v|^\beta dx > 0 \). Thus, by (H), (2.7) and the fact that \( \lambda > 0 \), we have for \( t \to \infty \)
\[
I(tu,tv) \leq \frac{tp}{p} + \gamma \frac{tp}{p} \left( \|u\|_{H^p}^p + \|v\|_{H^p}^p \right) - \frac{tp^*}{p^*} \int_{\mathbb{R}^n} |u|^\alpha|v|^\beta dx \to -\infty, \quad (3.3)
\]
since $1 < p < p^*$. Hence, taking $(e_1, e_2) = \tau_0(u,v)$ with $\tau_0 > 0$ sufficiently large, we obtain at once that $\|(e_1, e_2)\| \geq 2$, $\int_{\mathbb{H}^n} |e_1|^p |e_2|^{p^*} dq > 0$ and $I(e_1, e_2) < 0$, as stated.

For the second part, we first note that $(H)$ gives for any $\varepsilon > 0$ the existence of $C_\varepsilon > 0$ such that
\[
|H(q, w)| \leq \varepsilon \|w\|^\mu + C_\varepsilon \|w\|^s \quad \text{for all } (q, w) \in \mathbb{H}^n \times \mathbb{R}^2
\] (3.4)
holds. Hence, (1.4), Lemma 2.1 and (2.7) imply that for all $(u, v) \in \mathbb{H}^n$, we have
\[
I(u, v) \geq \frac{1}{p} \|(u, v)\|^p - \frac{\gamma^+}{pH_p} \left( \|D_{\mathbb{H}^n} u\|^p + \|D_{\mathbb{H}^n} v\|^p \right)
- \lambda \int_{\mathbb{H}^n} \varepsilon (u^2 + v^2)^{n/2} dq - \lambda \int_{\mathbb{H}^n} C_\varepsilon (u^2 + v^2)^{s/2} dq - \frac{1}{p^*} \|(u, v)\|_{p^*}^{p^*},
\]
\[
\geq \frac{1}{p} \left( 1 - \frac{\gamma^+}{H_p} \right) \|(u, v)\|^p - \lambda \varepsilon C_\mu^p \|(u, v)\|^\mu - \lambda C_\varepsilon C_s^s \|(u, v)\|^s - \frac{C_\mu^{p^*}}{p^*} \|(u, v)\|^{p^*}.
\]
Clearly, there exists $\rho \in (0, 1)$ such that
\[
\max_{t \in [0, 1]} y(t) = y(\rho) > 0, \quad \text{where } y(t) = \frac{1}{p} \left( 1 - \frac{\gamma^+}{H_p} \right) t^p - \lambda \varepsilon C_\mu^p t^p - \lambda C_\varepsilon C_s^s t^s - \frac{C_\mu^{p^*}}{p^*} t^{p^*},
\]
\[
since p < \mu \leq s < p^* and \gamma < H_p. Consequently, I(u, v) \geq y(\rho) = j for all (u, v) \in W, with \|(u, v)\| = \rho, as desired. This concludes the proof. \qed

We recall in passing that, if $X$ is a real Banach space, a $C^1(X)$ functional $J$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ if any Palais–Smale sequence $(u_k)_k$ at level $c$, that is such that
\[
J(u_k) \to c \quad \text{and} \quad J'(u_k) \to 0 \quad \text{in } X' \text{ as } k \to \infty,
\] (3.5)
admits a convergent subsequence in $X$.

Now we discuss the compactness property for the functional $I$, given by the Palais–Smale condition at a suitable level. To this aim, fix $\gamma \in (0, H_p)$, $\lambda > 0$ and set
\[
c_{\gamma, \lambda} = \inf_{\xi \in \Gamma} \max_{t \in [0, 1]} I(\xi(t)),
\] (3.6)
where
\[
\Gamma = \{ \xi \in C([0, 1]; W) : \xi(0) = (0,0), \ I(\xi(1)) < 0 \}.
\]
Obviously, $c_{\gamma, \lambda} > 0$ thanks to Lemma 3.1, since in particular $\|(e_1, e_2)\| > \rho$. Before proving that $I$ satisfies the Palais–Smale condition at level $c_{\gamma, \lambda}$, we introduce an asymptotic condition for the level $c_{\gamma, \lambda}$. This result was proved in Lemma 2.3 of [6] in the scalar Euclidean case and will be crucial to overcome the lack of compactness due to the presence of Hardy terms and critical nonlinearities. For other comments on this key idea we refer to [6] and the references therein.

**Lemma 3.2.** For any $\gamma \in (0, H_p)$ it results
\[
\lim_{\lambda \to \infty} c_{\gamma, \lambda} = 0.
\]
**Proof.** Fix $\gamma \in (0, H_p)$ and $\lambda > 0$. Let $(e_1, e_2)$ be the couple determined in Lemma 3.1, which is independent of $\gamma^+$ and $\lambda$. Since $I$ satisfies the mountain pass geometry at the points $(0,0)$ and $(e_1, e_2)$ of $W$, there exists $t_{\gamma, \lambda} > 0$, with
\( I(t, e_1, e_2) = \max_{t \geq 0} I(te_1, te_2) \). Therefore, \( \langle I'(t, e_1, e_2), (e_1, e_2) \rangle = 0 \).

Thus,

\[
\begin{align*}
& t^{p-1}_{\gamma, \lambda} (\|e_1, e_2\|^p - \gamma\|e_1\|^p_{H_p} - \gamma\|e_2\|^p_{H_p}) = \lambda \int_{\mathbb{H}^n} H_u(q, t, e_1, e_2) e_1 dq \\
& \quad + \lambda \int_{\mathbb{H}^n} H_v(q, t, e_1, e_2) e_2 dq + t^{p-1}_{\gamma, \lambda} \int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta dq \quad (3.7)
\end{align*}
\]

by \((\mathcal{H})\), being \( \lambda > 0 \).

We claim that \( \{t, \lambda\}_\lambda \) is bounded in \( \mathbb{R} \). Indeed, putting

\[ \Lambda = \{ \lambda \geq 0 : t, \lambda \|e_1, e_2\| \geq 1 \} \]

from (1.4) we derive that

\[
\begin{align*}
& t^{p}_{\gamma, \lambda} (\|e_1, e_2\|^p - \gamma\|e_1\|^p_{H_p} - \gamma\|e_2\|^p_{H_p}) \\
& \quad \leq t^{p}_{\gamma, \lambda} (\|e_1, e_2\|^p) + \frac{\gamma}{t^{\alpha}_{\gamma, \lambda}} (\|D_{\mathbb{H}^n} e_1\|^p_{H_p} + \|D_{\mathbb{H}^n} e_2\|^p_{H_p}) \quad (3.8)
\end{align*}
\]

for any \( \lambda \in \Lambda \). Therefore, (3.7) and (3.8) imply that

\[ \left(1 + \frac{\gamma}{t^{\alpha}_{\gamma, \lambda}}\right) (\|e_1, e_2\|^p \geq t^{p-\alpha}_{\gamma, \lambda} \int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta dq \quad \text{for any} \quad \lambda \in \Lambda, \]

which yields that \( \{t, \lambda\}_{\lambda \in \Lambda} \) is bounded, since \( 1 < p < p^* \) and \( \int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta dq > 0 \) by Lemma 3.1. It follows at once that \( \{t, \lambda\}_{\lambda > 0} \) is bounded. This proves the claim.

Fix now a sequence \( \{\lambda_k\}_k \subset (0, ) \) such that \( \lambda_k \to \infty \) as \( k \to \infty \). Obviously \( \{t, \lambda_k\}_k \) is bounded. Thus, there exist a \( \tau_0 \geq 0 \) and a subsequence of \( \{\lambda_k\}_k \), still denoted by \( \{\lambda_k\}_k \), such that \( t, \lambda_k \to \tau_0 \). By (3.7) there exists \( C_{\tau_0} > 0 \) such that, for any \( k \in \mathbb{N} \)

\[
\lambda_k \left( \int_{\mathbb{H}^n} H_u(q, t, e_1, e_2) e_1 dq + \int_{\mathbb{H}^n} H_v(q, t, e_1, e_2) e_2 dq \right)
\]

\[ + t^{p-1}_{\sigma, \lambda_k} \int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta dq \leq C_{\tau_0}. \quad (3.9) \]

We assert that \( \tau_0 = 0 \). Otherwise, the dominated convergence theorem yields

\[
\int_{\mathbb{H}^n} H_u(q, t, e_1, e_2) e_1 dq \to \int_{\mathbb{H}^n} H_u(q, \tau_0 e_1, \tau_0 e_2) e_1 dq,
\]

\[
\int_{\mathbb{H}^n} H_v(q, t, e_1, e_2) e_2 dq \to \int_{\mathbb{H}^n} H_v(q, \tau_0 e_1, \tau_0 e_2) e_2 dq
\]

as \( k \to \infty \). In particular, as \( k \to \infty \)

\[
\int_{\mathbb{H}^n} \{ H_u(q, t, e_1, e_2) e_1 + \int_{\mathbb{H}^n} H_u(q, t, e_1, e_2) e_1 dq \} dq \\
\to \int_{\mathbb{H}^n} \{ H_u(q, \tau_0 e_1, \tau_0 e_2) e_1 + H_u(q, \tau_0 e_1, \tau_0 e_2) e_2 \} dq > 0
\]
by \((H)\) and the fact that \(\int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta \, dq > 0\) as stated in Lemma 3.1. Recalling that \(\lambda_k \to \infty\), we get
\[
\lim_{k \to \infty} \left[ \lambda_k \left( \int_{\mathbb{H}^n} H_u(q, t_{\gamma, \lambda_k} e_1, t_{\gamma, \lambda_k} e_2) e_1 \, dq + \int_{\mathbb{H}^n} H_v(q, t_{\gamma, \lambda_k} e_1, t_{\gamma, \lambda_k} e_2) e_2 \, dq \right) + t_{\gamma, \lambda_k}^{p^*-1} \int_{\mathbb{H}^n} |e_1|^\alpha |e_2|^\beta \, dq \right] = \infty,
\]
which contradicts (3.9). Thus \(\tau_0 = 0\) and \(t_{\gamma, \lambda} \to 0\) as \(\lambda \to \infty\), since the sequence \((\lambda_k)_k\) is arbitrary.

Now the path \(\xi(t) = t(e_1, e_2), t \in [0, 1]\), belongs to \(\Gamma\), so that Lemma 3.1 gives
\[
0 < c_{\gamma, \lambda} \leq \max_{t \in [0, 1]} I(\xi(t)) \leq I(t_{\gamma, \lambda} e_1, t_{\gamma, \lambda} e_2) \leq \frac{\gamma}{p} \left( \|e_1\|_{H^p}^p + \|e_2\|_{H^p}^p \right).
\]
Hence the assertion is proved, since \(t_{\gamma, \lambda} \to 0\) as \(\lambda \to \infty\) and \((e_1, e_2)\) does not depend on \(\lambda\).

Before verifying that \(I\) satisfies the Palais-Smale condition at level \(c_{\gamma, \lambda}\) let us prove an essential lemma, which is inspired by Lemma 3.8 of [2], see also Lemma 4.2 of [5].

**Lemma 3.3.** Let \(\gamma\) and \(\lambda\) be two fixed parameters and let \(\{(u_k, v_k)\}_k\) be a bounded sequence in \(W\). Consider the sequences \((g_k)_k\) and \((h_k)_k\), defined for all \(k\) and all \(q \in \mathbb{H}^n\) by
\[
g_k(q) = -V(q)|u_k|^{p-2}u_k + \gamma \psi(q)^p \cdot \frac{|u_k|^{p-2}u_k}{r^p} + \lambda H_u(q, u_k, v_k) + \frac{\alpha}{p^*}|v_k|^\beta |u_k|^{\alpha-2}u_k,
\]
\[
h_k(q) = -V(q)|v_k|^{p-2}v_k + \gamma \psi(q)^p \cdot \frac{|v_k|^{p-2}v_k}{r^p} + \lambda H_v(q, u_k, v_k) + \frac{\beta}{p^*}|u_k|^\alpha |v_k|^{\beta-2}v_k.
\]
(3.10)

For all compact set \(K\) of \(\mathbb{H}^n\) there exists \(C_K > 0\) such that
\[
\sup_k \int_K (|g_k(q)| + |h_k(q)|) \, dq \leq C_K.
\]

**Proof.** Fix \(\gamma\), \(\lambda\) and \(\{(u_k, v_k)\}_k\) as in the statement. Let \(K\) be a compact set of \(\mathbb{H}^n\). Concerning the first and the fifth term, by Hölder’s inequality
\[
\int_K (V(q)|u_k|^{p-1} + V(q)|v_k|^{p-1}) \, dq \leq \|V\|_{x,K} \left( \sup_k \|u_k\|_{p^*}^{p-1} + \sup_k \|v_k\|_{p^*}^{p-1} \right) = c_1,
\]
where \(c_1 = c_1(K), x = p^*/(p^* - p + 1)\), since \(V \in C(\mathbb{H}^n)\) and Lemma 2.1 can be applied, being \(1 < p < p^*\). The second and the sixth term can be similarly evaluated, as
\[
\int_K \left( \frac{\psi}{r} \right)^p |u_k|^{p-1} + \left( \frac{\psi}{r} \right)^p |v_k|^{p-1} \, dq \leq \|\psi/r\|_p \left( \sup_k \|u_k\|_{H^p}^{p-1} + \sup_k \|v_k\|_{H^p}^{p-1} \right) = c_2,
\]
and \(c_2 = c_2(K)\) by (1.3). Indeed, \(\psi^p r^{-p} \in L^1_{\text{loc}}(\mathbb{H}^n)\), since \(\psi = |\psi| \leq 1\), the Jacobian determinant is \(r^Q\) and \(1 < p < Q\). Elementary inequalities and \((\mathcal{H})\), with \(\varepsilon = 1\), give

\[
\int_K |H_u(q, u_k, v_k) + H_v(q, u_k, v_k)| dq \leq \sqrt{2} \int_K \left| (u_k, v_k) \right|^\mu - 1 + C_1 \left| (u_k, v_k) \right|^\mu - 1 dq
\]

\[
\leq \sqrt{2} \left| K \right|^{1/\gamma} \sup_k \left| (u_k, v_k) \right|^\mu - 1 + C_1 \left| K \right|^{1/\tilde{g}} \sup_k \left| (u_k, v_k) \right|^\mu - 1\]
\]

where \(c_3 = c_3(K)\), \(y = p^*/(p^* - \mu + 1) > 1\) and \(\tilde{y} = p^*/(p^* - s + 1) > 1\), since \(p < \mu \leq s < p^*\) by \((\mathcal{H})\).

Finally, the fourth and the eighth term can be treated in the same way, since \(\alpha > 1\), \(\beta > 1\) and \(\alpha + \beta = p^*\), that is

\[
\int_K (|u_k|^\alpha |v_k|^\beta - 1 + |u_k|^{\alpha - 1} |v_k|^{\beta - 1}) dq \leq |K|^{1/p^*} \left( \sup_k \left| u_k \right|_p^\alpha \left| v_k \right|_p^{\beta - 1} + \sup_k \left| u_k \right|_p^{\alpha - 1} \left| v_k \right|_p^{\beta} \right) = c_4,
\]

where \(c_4 = c_4(K)\). This completes the proof.

The next crucial lemma relies on the proofs of Theorem 2.1 of [4], of Lemma 2 of [11] and of Step 1 of Theorem 4.4 of [2] in the Euclidean context. We also refer to proof of Lemma 4.3 of [5] for the Heisenberg setting.

**Lemma 3.4.** Suppose that the structural assumptions of either Theorem 1.1 or Theorem 1.2 hold. Let \(\{(u_k, v_k)\}_k\) and \((u, v)\) be in \(W\), such that \((u_k, v_k) \rightarrow (u, v)\) weakly in \(W\), \((u_k, v_k) \rightarrow (u, v)\) a.e. in \(\mathbb{H}^n\) and \(f'(u_k, v_k) \rightarrow 0\) strongly in \(W^\prime\). Assume furthermore that there exist two vector field functions \(\Theta\) and \(\Lambda\) in \(\mathbb{H}^n\) of class \(L^p(\mathbb{H}^n; \mathbb{R}^{2n})\), and such that

\[
|D_{\mathbb{H}^n}u_k|^\mu - 2 D_{\mathbb{H}^n}u_k \rightarrow \Theta\quad \text{and}\quad |D_{\mathbb{H}^n}v_k|^\mu - 2 D_{\mathbb{H}^n}v_k \rightarrow \Lambda
\]

weakly in \(L^p(\mathbb{H}^n; \mathbb{R}^{2n})\). Then, up to a subsequence, if necessary,

\[
D_{\mathbb{H}^n}u_k \rightarrow D_{\mathbb{H}^n}u\quad \text{and}\quad D_{\mathbb{H}^n}v_k \rightarrow D_{\mathbb{H}^n}v\quad \text{a.e. in } \mathbb{H}^n.
\]

**Proof.** Fix \(R > 0\). Let \(\varphi_R \in C^\infty_0(\mathbb{H}^n)\) be such that \(0 \leq \varphi_R \leq 1\) in \(\mathbb{H}^n\) and \(\varphi_R \equiv 1\) in \(B_R\). Given \(\varepsilon > 0\) define for each \(\tau \in \mathbb{R}\)

\[
\eta_\varepsilon(\tau) = \begin{cases} 
\tau, & \text{if } |\tau| < \varepsilon, \\
\varepsilon \frac{\tau}{|\tau|}, & \text{if } |\tau| \geq \varepsilon.
\end{cases}
\]

Put \(\phi_k = \varphi_R \eta_\varepsilon \circ (u_k - u)\) and \(\psi_k = \varphi_R \eta_\varepsilon \circ (v_k - v)\) so that \(\phi_k\) and similarly \(\psi_k\) are in \(HW^{1,p}(\mathbb{H}^n)\) by Lemma 2.1. Taking \(\tilde{\Phi} = \phi_k\) and \(\tilde{\Psi} = \psi_k\) in (3.1), we get

\[
\int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n}u_k|^\mu - 2 D_{\mathbb{H}^n}u_k - |D_{\mathbb{H}^n}u|^\mu - 2 D_{\mathbb{H}^n}u, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u)) \right)_{\mathbb{H}^n} dq
\]

\[
+ \int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n}v_k|^\mu - 2 D_{\mathbb{H}^n}v_k - |D_{\mathbb{H}^n}v|^\mu - 2 D_{\mathbb{H}^n}v, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (v_k - v)) \right)_{\mathbb{H}^n} dq
\]

\[
= - \int_{\mathbb{H}^n} \eta_\varepsilon \circ (u_k - u) \left( |D_{\mathbb{H}^n}u_k|^\mu - 2 D_{\mathbb{H}^n}u_k, D_{\mathbb{H}^n} \varphi_R \right)_{\mathbb{H}^n} dq
\]

\[
- \int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n}u_k|^\mu - 2 D_{\mathbb{H}^n}u_k, D_{\mathbb{H}^n}(\eta_\varepsilon \circ (u_k - u)) \right)_{\mathbb{H}^n} dq
\]
\[
- \int_{\mathbb{H}^n} \eta_e \circ (v_k - v) \left( |D_{\mathbb{H}^n} v_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v_k, D_{\mathbb{H}^n} \varphi_R \right)_{\mathbb{H}^n} dq \\
- \int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} v|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (\eta_e \circ (v_k - v)) \right)_{\mathbb{H}^n} \, dq \\
+ \langle I'(u_k, v_k), (\phi_k, \psi_k) \rangle + \int_{\mathbb{H}^n} (g_k \phi_k + h_k \psi_k) \, dq,
\]
where \((g_k)_k\) and \((h_k)_k\) are the sequences associated to \(\{(u_k, v_k)\}_k\) and defined in (3.10). Now, as \(k \to \infty\):

\[
\int_{\mathbb{H}^n} \eta_e \circ (u_k - u) \left( |D_{\mathbb{H}^n} u_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u_k, D_{\mathbb{H}^n} \varphi_R \right)_{\mathbb{H}^n} \, dq \to 0,
\]
\[
\int_{\mathbb{H}^n} \eta_e \circ (v_k - v) \left( |D_{\mathbb{H}^n} v_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v_k, D_{\mathbb{H}^n} \varphi_R \right)_{\mathbb{H}^n} \, dq \to 0,
\]
since \(\eta_e \circ (u_k - u) D_{\mathbb{H}^n} \varphi_R \big|_{\mathbb{H}^n} \to 0\) and \(\eta_e \circ (v_k - v) D_{\mathbb{H}^n} \varphi_R \big|_{\mathbb{H}^n} \to 0\) in \(L^p(\text{supp}\varphi_R)\) by (2.5), being \(\text{supp}\varphi_R\) contained in a suitable ball of \(\mathbb{H}^n\).

Furthermore, \(D_{\mathbb{H}^n} (\eta_e \circ (u_k - u)) \to 0\) and \(D_{\mathbb{H}^n} (\eta_e \circ (v_k - v)) \to 0\) in \(L^p(\mathbb{H}^n; \mathbb{R}^{2n})\), since \(u_k \to u\) and \(v_k \to v\) in \(HW^{1,p}(\mathbb{H}^n)\) by Lemma 2.1. Consequently, as \(k \to \infty\):

\[
\int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} u|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} (\eta_e \circ (u_k - u)) \right)_{\mathbb{H}^n} \, dq \to 0,
\]
\[
\int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} v|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (\eta_e \circ (v_k - v)) \right)_{\mathbb{H}^n} \, dq \to 0,
\]
since \(|D_{\mathbb{H}^n} u|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u\) and \(|D_{\mathbb{H}^n} v|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v\) are in \(L^p(\mathbb{H}^n; \mathbb{R}^{2n})\). Moreover,

\[
\langle I'(u_k, v_k), (\phi_k, \psi_k) \rangle \to 0
\]
as \(k \to \infty\), since \(I'(u_k, v_k) \to 0\) in \(W'\) by assumption and \((\phi_k, \psi_k) \to 0\) in \(W\) as \(k \to \infty\) by construction.

In conclusion, the first five terms in the right hand side of (3.12) go to zero as \(k \to \infty\). Now, recalling that \(0 \leq \varphi_R \leq 1\) in \(\mathbb{H}^n\), we have

\[
\int_{\mathbb{H}^n} (g_k \phi_k + h_k \psi_k) \, dq \leq \int_{\text{supp} \varphi_R} \left( |g_k| \cdot |\eta_e \circ (u_k - u)| + |h_k| \cdot |\eta_e \circ (v_k - v)| \right) \, dq
\]
\[
\leq \varepsilon \int_{\text{supp} \varphi_R} (|g_k| + |h_k|) \, dq \leq \varepsilon C_R,
\]
since \((g_k)_k\) and \((h_k)_k\) are bounded in \(L^1_{\text{loc}}(\mathbb{H}^n)\) by Lemma 3.3. By the definitions of \(\varphi_R\) and \(\eta_e\),

\[
\varphi_R \left( |D_{\mathbb{H}^n} u_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} (\eta_e \circ (u_k - u)) \right)_{\mathbb{H}^n} \geq 0,
\]
\[
\varphi_R \left( |D_{\mathbb{H}^n} v_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v_k - |D_{\mathbb{H}^n} v|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (\eta_e \circ (v_k - v)) \right)_{\mathbb{H}^n} \geq 0
\]
a.e. in \(\mathbb{H}^n\), and in turn

\[
\int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} u_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} (\eta_e \circ (u_k - u)) \right)_{\mathbb{H}^n} \, dq
\]
\[
+ \int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} v_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v_k - |D_{\mathbb{H}^n} v|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (\eta_e \circ (v_k - v)) \right)_{\mathbb{H}^n} \, dq
\]
\[
\leq \int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} u_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} (\eta_e \circ (u_k - u)) \right)_{\mathbb{H}^n} \, dq
\]
\[
+ \int_{\mathbb{H}^n} \varphi_R \left( |D_{\mathbb{H}^n} v_k|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v_k - |D_{\mathbb{H}^n} v|_{L^2_{\mathbb{H}^n}}^{p-2} D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (\eta_e \circ (v_k - v)) \right)_{\mathbb{H}^n} \, dq.
\]
Moreover, by Hölder's inequality, since \( \eta \circ (u_k - u) \) are bounded in \( L^1(\mathbb{H}^n) \), we have

\[
\limsup_{k \to \infty} \left( \int_{B_R} \left( |D_{\mathbb{H}^n} u_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} (\eta \circ (u_k - u)) \right)_{L^2_{\mathbb{H}^n}} \, dq \right) + \int_{B_R} \left( |D_{\mathbb{H}^n} v_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v_k - |D_{\mathbb{H}^n} v|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (\eta \circ (v_k - v)) \right)_{L^2_{\mathbb{H}^n}} \, dq \leq \varepsilon C_R,
\]

since \( \varphi_R \equiv 1 \) in \( B_R \). Define the function \( e_k = e_k(q) \) by

\[
e_k = e_{u,k} + e_{v,k}, \quad e_{u,k} = (|D_{\mathbb{H}^n} u_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u, D_{\mathbb{H}^n} (u_k - u)),
\]

and

\[
e_{v,k} = (|D_{\mathbb{H}^n} v_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v_k - |D_{\mathbb{H}^n} v|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v, D_{\mathbb{H}^n} (v_k - v)).
\]

Clearly, \( e_k \) is nonnegative a.e. in \( \mathbb{H}^n \) for all \( k \). Moreover, \( (e_k)_k \) is bounded in \( L^1(\mathbb{H}^n) \). Indeed,

\[
0 \leq \int_{B_R} e_k(q) \, dq \leq \left\| |D_{\mathbb{H}^n} u_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u_k - |D_{\mathbb{H}^n} u|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u \right\|_{L^p_{\mathbb{H}^n}} \left\| D_{\mathbb{H}^n} u_k - D_{\mathbb{H}^n} u \right\|_{L^p_{\mathbb{H}^n}} + \left\| |D_{\mathbb{H}^n} v_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v_k - |D_{\mathbb{H}^n} v|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v \right\|_{L^p_{\mathbb{H}^n}} \left\| D_{\mathbb{H}^n} v_k - D_{\mathbb{H}^n} v \right\|_{L^p_{\mathbb{H}^n}} \leq C_0
\]

where \( C_0 \) is an appropriate constant, independent of \( k \), since \( (D_{\mathbb{H}^n} u_k)_k \) and \( (D_{\mathbb{H}^n} v_k)_k \) are bounded in \( L^p(\mathbb{H}^n; \mathbb{R}^{2n}) \), and similarly

\[
\left\| |D_{\mathbb{H}^n} u_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} u_k \right\|_{L^p_{\mathbb{H}^n}} \left\| D_{\mathbb{H}^n} u_k - D_{\mathbb{H}^n} u \right\|_{L^p_{\mathbb{H}^n}} \quad \text{and} \quad \left\| |D_{\mathbb{H}^n} v_k|_{L^{p-2}_{\mathbb{H}^n}}^2 D_{\mathbb{H}^n} v_k \right\|_{L^p_{\mathbb{H}^n}} \left\| D_{\mathbb{H}^n} v_k - D_{\mathbb{H}^n} v \right\|_{L^p_{\mathbb{H}^n}}
\]

are bounded weakly in \( L^p(\mathbb{H}^n; \mathbb{R}^{2n}) \), since \( (D_{\mathbb{H}^n} u_k)_k \) and \( (D_{\mathbb{H}^n} v_k)_k \) converge weakly in \( L^p(\mathbb{H}^n; \mathbb{R}^{2n}) \) by assumption.

Fix \( \theta \in (0, 1) \). Split the ball \( B_R \) into four sets

\[
S_{u,k}(R) = \{ q \in B_R : |u_k(q) - u(q)| \leq \varepsilon \}, \quad G_{u,k}(R) = B_R \setminus S_{u,k}(R),
\]

\[
S_{v,k}(R) = \{ q \in B_R : |v_k(q) - v(q)| \leq \varepsilon \}, \quad G_{v,k}(R) = B_R \setminus S_{v,k}(R).
\]

By Hölder's inequality, by (3.13), since \( D_{\mathbb{H}^n} (\eta \circ (u_k - u)) = D_{\mathbb{H}^n} (u_k - u) \) in \( S_{u,k}^c(R) \), \( D_{\mathbb{H}^n} (\eta \circ (v_k - v)) = D_{\mathbb{H}^n} (v_k - v) \) in \( S_{v,k}^c(R) \), and finally by (3.14), we obtain

\[
\int_{B_R} e_k \, dq \leq \int_{B_R} e_{u,k} \, dq + \int_{B_R} e_{v,k} \, dq = \int_{S_{u,k}^c(R)} e_{u,k} \, dq + \int_{G_{u,k}(R)} e_{u,k} \, dq + \int_{S_{v,k}^c(R)} e_{v,k} \, dq + \int_{G_{v,k}(R)} e_{v,k} \, dq \leq \left( \int_{S_{u,k}^c(R)} e_{u,k} \, dq \right)^\theta |S_{u,k}^c(R)|^{1-\theta} + \left( \int_{G_{u,k}(R)} e_{k} \, dq \right)^\theta |G_{u,k}(R)|^{1-\theta}
\]

\[
+ \left( \int_{S_{v,k}^c(R)} e_{v,k} \, dq \right)^\theta |S_{v,k}^c(R)|^{1-\theta} + \left( \int_{G_{v,k}(R)} e_{k} \, dq \right)^\theta |G_{v,k}(R)|^{1-\theta} \leq (\varepsilon C_R)^\theta \left( |S_{u,k}^c(R)|^{1-\theta} + |S_{v,k}^c(R)|^{1-\theta} \right) + C_0 \left( |G_{u,k}(R)|^{1-\theta} + |G_{v,k}(R)|^{1-\theta} \right).
\]

Moreover, \( |G_{u,k}(R)| \) and \( |G_{v,k}(R)| \) tend to zero as \( k \to \infty \) by the Severini–Egorov theorem in \( B_R \), since \( u_k \to u \) and \( v_k \to v \) a.e. in \( \mathbb{H}^n \) by assumption. Hence

\[
0 \leq \limsup_{k \to \infty} \int_{B_R} e_k \, dq \leq (\varepsilon C_R)^\theta |B_R|^{1-\theta}.
\]
Letting ε tend to 0+ we find that $e_k^0 \to 0$ in $L^1(B_R)$ and so, since $R > 0$ is arbitrary, we deduce that

$$e_k \to 0 \text{ a.e. in } \mathbb{H}^n$$

up to a subsequence, if necessary. From Lemma 3 of [11] it follows the validity of (3.11) and this completes the proof. \(\square\)

Now we are ready to prove the compactness property of $I$ at the special level $c_{\gamma, \lambda}$ introduced in (3.6).

**Lemma 3.5.** For any $\gamma \in (-\infty, H_\mu)$ there exists $\lambda^* = \lambda^*(\gamma) > 0$ such that for any $\lambda \geq \lambda^*$ the functional $I$ satisfies the Palais–Smale condition at level $c_{\gamma, \lambda}$.

**Proof.** Fix $\gamma \in (-\infty, H_\mu)$ and let $\{(u_k, v_k)\}_k \subset W$ be a Palais–Smale sequence of $I$ at level $c_{\gamma, \lambda}$ for any $\lambda > 0$. By (H), we get

$$I(u_k, v_k) - \frac{1}{\mu} \langle I'(u_k, v_k), (u_k, v_k) \rangle = \frac{1}{p} \|(u_k, v_k)\|^p - \frac{1}{\mu} \|(u_k, v_k)\|^p$$

$$- \gamma \left( \frac{1}{p} - \frac{1}{\mu} \right) \left( \|u_k\|_{H_p}^p + \|v_k\|_{H_p}^p \right)$$

$$- \lambda \int_{\mathbb{H}^n} \left( H(q, u_k, v_k) - \frac{1}{\mu} H_u(q, u_k, v_k) u_k - \frac{1}{\mu} H_v(q, u_k, v_k) v_k \right) dq$$

$$+ \left( \frac{1}{p} - \frac{1}{\mu} \right) \int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^{\beta} dq$$

$$\geq \left( \frac{1}{p} - \frac{1}{\mu} \right) \|(u_k, v_k)\|^p - \frac{\gamma^*}{H_p} \left( \frac{1}{p} - \frac{1}{\mu} \right) \|(u_k, v_k)\|^p,$$

since $p < \mu < p^*$. Therefore, thanks to (1.4) and (3.5), there exists $\sigma_{\gamma, \lambda} > 0$ such that as $k \to \infty$

$$c_{\gamma, \lambda} + \sigma_{\gamma, \lambda} \|(u_k, v_k)\| + o(1) \geq \nu_{\gamma} \|(u_k, v_k)\|^p,$$

$$\nu_{\gamma} = \left( \frac{1}{p} - \frac{1}{\mu} \right) \left( 1 - \frac{\gamma^*}{H_p} \right) > 0,$$

(3.15)

since $\gamma < H_\mu$. Therefore $\{(u_k, v_k)\}_k$ is bounded in the reflexive Banach space $W$.

Now $\alpha > 1$, $\beta > 1$ and $\alpha + \beta = p^*$, so that the Hölder inequality and (1.2) give

$$\int_{\mathbb{H}^n} |u_k|^{\alpha-1} |v_k|^\beta |v_k|^{p^*-1} dq \leq \|u_k\|_{p^{\alpha-1}} \|v_k\|_{p^*} \|v_k\|_{p^{\alpha-1}}$$

$$\leq C_{Q, p}^\alpha \|D_\mathbb{H} u_k\|_{p^{\alpha-1}} \|D_\mathbb{H} v_k\|_{p^{\alpha-1}} \|D_\mathbb{H} v_k\|_{p^*}$$

$$\leq C_{Q, p}^\alpha \|D_\mathbb{H} v_k\|_{p^*} \leq C,$$

for a suitable constant $C > 0$. Similarly,

$$\int_{\mathbb{H}^n} |u_k|^{\alpha} |v_k|^{\beta-1} |v_k|^{p^*-1} dq \leq C.$$

Thus, since $\{(u_k, v_k)\}_k$ is bounded in the reflexive Banach space $W$, there exist $(u_{\gamma, \lambda}, v_{\gamma, \lambda}) \in W$, two vector field functions $\Theta$ and $\Lambda$ in $\mathbb{H}^n$ of class $L^p(\mathbb{H}^n; \mathbb{R}^{2n})$, and nonnegative numbers $\kappa_{\gamma, \lambda}$, $\ell_{\gamma, \lambda}$, $\ell_{\gamma, \lambda}$ and $\delta_{\gamma, \lambda}$, and two functions $g_{\mu} \in L^p(\mathbb{H}^n)$ and $g_{\delta} \in L^p(\mathbb{H}^n)$ such that such that, up to a subsequence, still denoted by $\{(u_k, v_k)\}_k,$
we have
\[(u_k, v_k) \to (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \text{ in } W, \quad \|(u_k, v_k)\| \to \kappa_{\gamma, \lambda},\]
\[u_k \to u_{\gamma, \lambda} \text{ in } L^p(\mathbb{H}^n, (\psi/r)^p), \quad \|u_k - u_{\gamma, \lambda}\|_{H^p} \to \ell_{\gamma, \lambda},\]
\[v_k \to v_{\gamma, \lambda} \text{ in } L^p(\mathbb{H}^n, (\psi/r)^p), \quad \|v_k - v_{\gamma, \lambda}\|_{H^p} \to \ell_{\gamma, \lambda},\]
\[(u_k, v_k) \to (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \text{ in } L^p(\mathbb{H}^n) \times L^p(\mathbb{H}^n),\]
\[(u_k, v_k) \to (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \text{ a.e. in } \mathbb{H}^n,\]
\[
\|(u_k, v_k)\| \leq g_\mu \text{ a.e. in } \mathbb{H}^n, \quad \|(u_k, v_k)\| \leq g_\nu \text{ a.e. in } \mathbb{H}^n \text{ and all } k,
\]
\[
\int_{\mathbb{H}^n} |u_k - u_{\gamma, \lambda}|^\alpha |v_k - v_{\gamma, \lambda}|^\beta \, dq \to \delta_{\gamma, \lambda},
\]
\[
|u_k|^{\alpha - 2} u_k |v_k|^\beta \to |u_{\gamma, \lambda}|^{\alpha - 2} u_{\gamma, \lambda} |v_{\gamma, \lambda}|^\beta \text{ in } L^{p/\alpha} (\mathbb{H}^n),
\]
\[
|u_k|^{\beta - 2} v_k \to |u_{\gamma, \lambda}|^{\beta - 2} v_{\gamma, \lambda} \text{ in } L^{p/\beta} (\mathbb{H}^n),
\]
\[
|D_{\mathbb{H}^n} u_k|^{p - 2} D_{\mathbb{H}^n} u_k \to \Theta \text{ in } L^p (\mathbb{H}^n; \mathbb{R}^{2n})
\]
\[
|D_{\mathbb{H}^n} v_k|^{p - 2} D_{\mathbb{H}^n} v_k \to \Lambda \text{ in } L^p (\mathbb{H}^n; \mathbb{R}^{2n})
\]

with \(\nu \in (p, p^*)\), by (1.4), (2.7), Lemmas 2.2 and 2.3, and finally Proposition A.8 of [2], since \((u_k, v_k) \to (u, v)\) a.e. in \(\mathbb{H}^n\) and \((u_k, v_k) \to (u, v)\) weakly in \(L^p(\mathbb{H}^n)\), as shown above.

By (3.5), we have shown that
\[
c_{\gamma, \lambda} + o(1) \geq \nu_\gamma \|(u_k, v_k)\|^p + \left(\frac{1}{p} - \frac{1}{p^*}\right) \int_{\mathbb{H}^n} |u_k|^\alpha |v_k|^\beta \, dq,
\]
where \(\nu_\gamma\) is given in (3.15).

First we assert that
\[
\lim_{\lambda \to \infty} \kappa_{\gamma, \lambda} = 0.
\]

Otherwise, \(\limsup_{\lambda \to \infty} \kappa_{\gamma, \lambda} = \kappa_\gamma > 0\). Hence there is a sequence \(j \mapsto \lambda_j \uparrow \infty\) such that \(\kappa_{\gamma, \lambda_j} \to \kappa_\gamma\) as \(j \to \infty\). Then, letting \(j \to \infty\) we get from (3.17) and Lemma 3.2 that
\[
0 \geq \nu_\gamma \kappa_\gamma^p > 0.
\]

This contradiction proves the assertion (3.18). Moreover,
\[
\|(u_{\gamma, \lambda}, v_{\gamma, \lambda})\| \leq \kappa_{\gamma, \lambda},
\]
since \((u_k, v_k) \to (u_{\gamma, \lambda}, v_{\gamma, \lambda})\) in \(W\), so that \((\mathcal{V})\), (1.4), (2.7) and (3.18) imply that
\[
\lim_{\lambda \to \infty} \|u_{\gamma, \lambda}\|_{H^p} = \lim_{\lambda \to \infty} \|v_{\gamma, \lambda}\|_{H^p} = \lim_{\lambda \to \infty} \int_{\mathbb{H}^n} |u_{\gamma, \lambda}|^\alpha |v_{\gamma, \lambda}|^\beta \, dq
\]
\[
= \lim_{\lambda \to \infty} \|(u_{\gamma, \lambda}, v_{\gamma, \lambda})\| = 0.
\]

Let us prove that \(\{(u_k, v_k)\}_k\), up to a possibly further beyond subsequence, converges strongly to \((u_{\gamma, \lambda}, v_{\gamma, \lambda})\) in \(W\). First, by (3.16) Lemma 3.4 can be applied so that (3.11) holds. In particular, \(|D_{\mathbb{H}^n} u_k|^{p - 2} D_{\mathbb{H}^n} u_k| \to |D_{\mathbb{H}^n} u_{\gamma, \lambda}|^{p - 2} D_{\mathbb{H}^n} u_{\gamma, \lambda}\) and \(|D_{\mathbb{H}^n} v_k|^{p - 2} D_{\mathbb{H}^n} v_k| \to |D_{\mathbb{H}^n} v_{\gamma, \lambda}|^{p - 2} D_{\mathbb{H}^n} v_{\gamma, \lambda}\) a.e. in \(\mathbb{H}^n\). Hence, Proposition A.7 of [2] implies that \(\Theta = |D_{\mathbb{H}^n} u_{\gamma, \lambda}|^{p - 2} D_{\mathbb{H}^n} u_{\gamma, \lambda}\) and \(\Lambda = |D_{\mathbb{H}^n} v_{\gamma, \lambda}|^{p - 2} D_{\mathbb{H}^n} v_{\gamma, \lambda}\) a.e.
in $\mathbb{H}^n$ in (3.16). Consequently for all $(\Phi, \Psi) \in W$

$$\int_{\mathbb{H}^n} |(D_{\mathbb{H}^n} u_k)^{p-2} D_{\mathbb{H}^n} u_k, D_{\mathbb{H}^n} \Phi|_{\mathbb{H}^n} \, dq \to \int_{\mathbb{H}^n} |(D_{\mathbb{H}^n} u_{\gamma, \lambda})^{p-2} D_{\mathbb{H}^n} u_{\gamma, \lambda}, D_{\mathbb{H}^n} \Phi|_{\mathbb{H}^n} \, dq,$$

$$\int_{\mathbb{H}^n} |(D_{\mathbb{H}^n} v_k)^{p-2} D_{\mathbb{H}^n} v_k, D_{\mathbb{H}^n} \Psi|_{\mathbb{H}^n} \, dq \to \int_{\mathbb{H}^n} |(D_{\mathbb{H}^n} v_{\gamma, \lambda})^{p-2} D_{\mathbb{H}^n} v_{\gamma, \lambda}, D_{\mathbb{H}^n} \Psi|_{\mathbb{H}^n} \, dq$$

(3.20)

as $k \to \infty$, since $|D_{\mathbb{H}^n} u_k|^{p-2} D_{\mathbb{H}^n} u_k \to |D_{\mathbb{H}^n} u_{\gamma, \lambda}|^{p-2} D_{\mathbb{H}^n} u_{\gamma, \lambda}$ in $L^p(\mathbb{H}^n; \mathbb{R}^{2n})$ and also $|D_{\mathbb{H}^n} v_k|^{p-2} D_{\mathbb{H}^n} v_k \to |D_{\mathbb{H}^n} v_{\gamma, \lambda}|^{p-2} D_{\mathbb{H}^n} v_{\gamma, \lambda}$ in $L^p(\mathbb{H}^n; \mathbb{R}^{2n})$.

Furthermore, $|u_k|^{p-2} u_k \to |u_{\gamma, \lambda}|^{p-2} u_{\gamma, \lambda}$ in $L^p(\mathbb{H}^n, V)$ as $k \to \infty$ by (3.16) and Proposition A.8 of [2]. Therefore,

$$\int_{\mathbb{H}^n} V(q)|u_k|^{p-2} u_k \Phi dq \to \int_{\mathbb{H}^n} V(q)|u_{\gamma, \lambda}|^{p-2} u_{\gamma, \lambda} \Phi dq$$

(3.21)

for any $\Phi \in E_{p, V}(\mathbb{H}^n)$, since $\Phi \in L^p(\mathbb{H}^n, V)$. In the same way, (3.16) and Proposition A.8 of [2] imply that $|u_k|^{p-2} u_k \to |u_{\gamma, \lambda}|^{p-2} u_{\gamma, \lambda}$ in $L^p(\mathbb{H}^n, (\psi/r)^p)$ as $k \to \infty$. Indeed, Proposition 3.3. of [5] can be applied by (3.16) and by the fact that the weight function $(\psi/r)^p$ is of class $L^1_{\text{loc}}(\mathbb{H}^n)$, as explained in the proof of Lemma 3.3.

Therefore,

$$\langle u_k, \Phi \rangle_{H^p} \to \langle u_{\gamma, \lambda}, \Phi \rangle_{H^p}$$

(3.22)

for any $\Phi \in E_{p, V}$, since $\Phi \in L^p(\mathbb{H}^n, (\psi/r)^p)$ by (1.3).

A similar argument shows that

$$\int_{\mathbb{H}^n} V(q)|v_k|^{p-2} v_k \Psi dq \to \int_{\mathbb{H}^n} V(q)|v_{\gamma, \lambda}|^{p-2} v_{\gamma, \lambda} \Psi dq, \quad \langle v_k, \Psi \rangle_{H^p} \to \langle v_{\gamma, \lambda}, \Psi \rangle_{H^p}$$

(3.23)

for all $\Psi \in E_{p, V}$.

By $(\mathcal{H})$, with $\varepsilon = 1$, and (3.16), the H"older inequality gives

$$\left| \int_{\mathbb{H}^n} H_u(q, u_k, v_k)(u_k - u_{\gamma, \lambda}) + H_v(q, u_k, v_k)(v_k - u_{\gamma, \lambda}) dq \right|$$

$$\leq \int_{\mathbb{H}^n} |(u_k, v_k)|^{\mu-1}(u_k, v_k) - (u_{\gamma, \lambda}, v_{\gamma, \lambda})| + C_1 |(u_k, v_k)|^{s-1}|(u_k, v_k) - (u_{\gamma, \lambda}, v_{\gamma, \lambda})| dq$$

$$\leq C \left( \|(u_k, v_k) - (u_{\gamma, \lambda}, v_{\gamma, \lambda})\|_\mu + \|(u_k, v_k) - (u_{\gamma, \lambda}, v_{\gamma, \lambda})\|_s \right)$$

(3.24)

as $k \to \infty$, by Lemma 2.3 since $p < \mu \leq s < p'$, for a suitable constant $C > 0$. While $(\mathcal{H})$ and the use of the dominated convergence theorem yield for any $(\Phi, \Psi) \in W$

$$\int_{\mathbb{H}^n} H_u(q, u_k, v_k) \Phi dq \to \int_{\mathbb{H}^n} H_u(q, u_{\gamma, \lambda}, v_{\gamma, \lambda}) \Phi dq,$$

$$\int_{\mathbb{H}^n} H_v(q, u_k, v_k) \Psi dq \to \int_{\mathbb{H}^n} H_v(q, u_{\gamma, \lambda}, v_{\gamma, \lambda}) \Psi dq$$

(3.25)

as $k \to \infty$. Consequently, (3.5), (3.16), (3.20)–(3.23) and (3.25) give at once that $(u_{\gamma, \lambda}, v_{\gamma, \lambda})$ satisfies the identity

$$\langle (u_{\gamma, \lambda}, v_{\gamma, \lambda}), (\varphi, \Psi) \rangle - \gamma \left( \langle u_{\gamma, \lambda}, \Phi \rangle_{H^p} + \langle v_{\gamma, \lambda}, \Psi \rangle_{H^p} \right)$$

$$= \int_{\mathbb{H}^n} [H_u(q, u_{\gamma, \lambda}, v_{\gamma, \lambda}) \Phi + H_v(q, u_{\gamma, \lambda}, v_{\gamma, \lambda}) \Psi] dq$$

$$+ \frac{\alpha}{p^*} \int_{\mathbb{H}^n} |u_{\gamma, \lambda}|^{p_0-2} u_{\gamma, \lambda} |\varphi|^{p_0} dq + \frac{\beta}{p^*} \int_{\mathbb{H}^n} |v_{\gamma, \lambda}|^{p_0-2} v_{\gamma, \lambda} |\Psi|^{p_0} dq,$$

for any $(\Phi, \Psi) \in W$. In other words, $(u_{\gamma, \lambda}, v_{\gamma, \lambda})$ is a critical point of $I$ in $W$. 
Since \( \{(u_k, v_k)\}_k \) is bounded in \( W \) and \( (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \) is a critical point of \( I \) in \( W \), we deduce from (3.5), (3.16), (3.20)–(3.24) and (2.8) that

\[
o(1) = \left( I'(u_k, v_k) - I'(u_{\gamma, \lambda}, v_{\gamma, \lambda}), (u_k, v_k) - (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \right) \\
= \| (u_k, v_k) \|^p - \| (u_k, v_k), (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \| \\
+ \| (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \|^p - \| (u_{\gamma, \lambda}, v_{\gamma, \lambda}), (u_k, v_k) \| \\
- \gamma \int_{\mathbb{R}^n} \psi^p \left( |u_k|^{p-2} u_k - |u_{\gamma, \lambda}|^{p-2} u_{\gamma, \lambda} \right) (u_k - u_{\gamma, \lambda}) dq \\
- \gamma \int_{\mathbb{R}^n} \psi^p \left( |v_k|^{p-2} v_k - |v_{\gamma, \lambda}|^{p-2} v_{\gamma, \lambda} \right) (v_k - v_{\gamma, \lambda}) dq \\
- \lambda \int_{\mathbb{R}^n} \left[ H_a(q, u_k, v_k) - H_a(q, u_{\gamma, \lambda}, v_{\gamma, \lambda}) \right] (u_k - u_{\gamma, \lambda}) dq \\
- \lambda \int_{\mathbb{R}^n} \left[ H_c(q, u_k, v_k) - H_c(q, u_{\gamma, \lambda}, v_{\gamma, \lambda}) \right] (v_k - v_{\gamma, \lambda}) dq \\
- \frac{\alpha}{p^*} \int_{\mathbb{R}^n} (|u_n|^\beta |u_k|^{\alpha} - |v_{\gamma, \lambda}|^{\alpha} |u_{\gamma, \lambda}|^{\alpha} - |u_{\gamma, \lambda}|^{\alpha} |u_k|^{\alpha} - |v_{\gamma, \lambda}|^{\alpha} |u_{\gamma, \lambda}|^{\alpha} - |u_{\gamma, \lambda}|^{\alpha} |u_k|^{\alpha} - |v_{\gamma, \lambda}|^{\alpha} |u_{\gamma, \lambda}|^{\alpha}) (u_k - u_{\gamma, \lambda}) dq \\
- \frac{\beta}{p^*} \int_{\mathbb{R}^n} (|v_n|^\beta |u_k|^{\beta} - |v_{\gamma, \lambda}|^{\beta} |u_{\gamma, \lambda}|^{\beta} - |u_{\gamma, \lambda}|^{\beta} |u_k|^{\beta} - |v_{\gamma, \lambda}|^{\beta} |u_{\gamma, \lambda}|^{\beta} - |u_{\gamma, \lambda}|^{\beta} |u_k|^{\beta} - |v_{\gamma, \lambda}|^{\beta} |u_{\gamma, \lambda}|^{\beta}) (v_k - v_{\gamma, \lambda}) dq \\
= \left[ \kappa_{\gamma, \lambda} - \| (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \|^p \right] - \gamma \left[ \| u_k \|^p_{H^p} + \| v_k \|^p_{H^p} - \| u_{\gamma, \lambda} \|^p_{H^p} - \| u_{\gamma, \lambda} \|^p_{H^p} \right] \\
- \int_{\mathbb{R}^n} |u_k|^\alpha |v_k|^{\beta} dq + \int_{\mathbb{R}^n} |u_{\gamma, \lambda}|^\alpha |v_{\gamma, \lambda}|^{\beta} dq + o(1),
\]

since \( \| (u_k, v_k) \| \to \kappa_{\gamma, \lambda} \) as \( k \to \infty \), and \( \alpha + \beta = p^* \). Therefore as \( k \to \infty \)

\[
o(1) = \left[ \kappa_{\gamma, \lambda} - \| (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \|^p \right] - \gamma \left[ \| u_k \|^p_{H^p} + \| v_k \|^p_{H^p} - \| u_{\gamma, \lambda} \|^p_{H^p} - \| u_{\gamma, \lambda} \|^p_{H^p} \right] \\
- \int_{\mathbb{R}^n} |u_k|^\alpha |v_k|^{\beta} dq + \int_{\mathbb{R}^n} |u_{\gamma, \lambda}|^\alpha |v_{\gamma, \lambda}|^{\beta} dq + o(1), \tag{3.26}
\]

Furthermore, (3.16) and the celebrated Brézis and Lieb lemma give

\[
\| u_k \|^p_{H^p} = \| u_k - u_{\gamma, \lambda} \|^p_{H^p} + \| u_{\gamma, \lambda} \|^p_{H^p} + o(1), \\
\| v_k \|^p_{H^p} = \| v_k - v_{\gamma, \lambda} \|^p_{H^p} + \| v_{\gamma, \lambda} \|^p_{H^p} + o(1) \tag{3.27}
\]
as \( k \to \infty \), while again (3.16) and Lemma 3.4

\[
\| (u_k, v_k) \|^p = \| (u_k, v_k) - (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \|^p + \| (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \|^p + o(1) \tag{3.28}
\]
as \( k \to \infty \). Hence, from (3.16), (3.26)–(3.28) and Lemma 2.3 we obtain

\[
\lim_{k \to \infty} \left[ \| u_k - (u_{\gamma, \lambda}, v_{\gamma, \lambda}) \|^p \right] = \gamma \lim_{n \to \infty} \left( \| u_k - u_{\gamma, \lambda} \|^p_{H^p} + \| v_k - v_{\gamma, \lambda} \|^p_{H^p} \right) \\
+ \lim_{n \to \infty} \int_{\mathbb{R}^n} |u_k - u_{\gamma, \lambda}|^\alpha |v_k - v_{\gamma, \lambda}|^\beta dq \tag{3.29}
\]
as \( k \to \infty \). By (3.17) and Lemma 2.3 again, we get as \( k \to \infty \)

\[
c_{\gamma, \lambda} + o(1) \geq \left( \frac{1}{\mu} - \frac{1}{p^*} \right) \left[ \delta_{\gamma, \lambda} + \int_{\mathbb{R}^n} |u_{\gamma, \lambda}|^\alpha |v_{\gamma, \lambda}|^\beta dq \right].
\]

Then, Lemma 3.2 and (3.19) imply that

\[
\lim_{\lambda \to \infty} \delta_{\gamma, \lambda} = 0. \tag{3.30}
\]
Since \( \gamma < H_p \) there exists \( c \in [0,1) \) such that \( \gamma^+ = c H_p \). Of course, (3.29) can be rewritten as
\[
(1-c) \lim_{k \to \infty} \|u_k - u_{\gamma,\lambda}, v_k - v_{\gamma,\lambda}\|^p + c \lim_{k \to \infty} \|u_k - u_{\gamma,\lambda}, v_k - v_{\gamma,\lambda}\|^p
= \gamma^+(p^+_{\gamma,\lambda} + \delta_{\gamma,\lambda}).
\]
Thus, by (1.4) and (V), using (2.7) with \( (u,v) = (u_k - u_{\gamma,\lambda}, v_k - v_{\gamma,\lambda}) \), we get
\[
\delta_{\gamma,\lambda} + \gamma^+(p^+_{\gamma,\lambda} + \delta_{\gamma,\lambda}) \geq (1-c) C Q_p \delta_{\gamma,\lambda} \leq c H_p (p^+_{\gamma,\lambda} + \delta_{\gamma,\lambda})
\]
for all \( \lambda > 0 \), being \( c \in [0,1) \). Therefore, since \( \gamma^+ = c H_p \),
\[
\delta_{\gamma,\lambda} \geq (1-c) C Q_p \delta_{\gamma,\lambda}.
\]
Consequently, (3.19) and (3.31) imply at once that there exists \( \lambda^* = \lambda^*(\gamma) > 0 \) such that \( \delta_{\gamma,\lambda} = 0 \) for all \( \lambda \geq \lambda^* \). In other words,
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \|u_k - u_{\gamma,\lambda}\|^a |v_k - v_{\gamma,\lambda}|^b dq = 0
\]
for all \( \lambda \geq \lambda^* \).
Now, assume by contradiction that there exists \( \lambda \geq \lambda^* \) such that \( v_{\gamma,\lambda} + \ell_{\gamma,\lambda} > 0 \). Then, by (1.4), since \( \gamma < H_p \), we have
\[
\lim_{n \to \infty} \|u_k - u_{\gamma,\lambda}, v_k - v_{\gamma,\lambda}\|^p \leq \gamma \lim_{k \to \infty} \|u_k - u_{\gamma,\lambda}\|^p_{H_p} + \|v_k - v_{\gamma,\lambda}\|^p_{H_p}
< H_p \lim_{k \to \infty} \|u_k - u_{\gamma,\lambda}\|^p_{H_p} + \|v_k - v_{\gamma,\lambda}\|^p_{H_p}
\leq \lim_{n \to \infty} \|u_k - u_{\gamma,\lambda}, v_k - v_{\gamma,\lambda}\|^p,
\]
which gives an obvious contradiction.
Thus, \( v_{\gamma,\lambda} + \ell_{\gamma,\lambda} = 0 \) for all \( \lambda \geq \lambda^* \). By (3.29) this yields
\[
\lim_{n \to \infty} \|u_k - u_{\gamma,\lambda}\| = 0.
\]
In conclusion, \( (u_k,v_k) \to (u_{\gamma,\lambda},v_{\gamma,\lambda}) \) as \( k \to \infty \) in \( W \), as required.

\textbf{Proof of Theorem 1.1.} Lemmas 3.1 and 3.5 guarantee that for any \( \gamma \in (-\infty, H_p) \) there exists \( \lambda_* = \lambda_*(\gamma) > 0 \) such that for any \( \lambda \geq \lambda_* \) the functional \( I \) satisfies all assumptions of the mountain pass theorem at the level \( c_{\gamma,\lambda} \). Hence, there exists a critical point \( (u_{\gamma,\lambda},v_{\gamma,\lambda}) \in W \) of \( I \) at level \( c_{\gamma,\lambda} \). Clearly, \( (u_{\gamma,\lambda},v_{\gamma,\lambda}) \neq (0,0) \), since \( I(u_{\gamma,\lambda},v_{\gamma,\lambda}) = c_{\gamma,\lambda} > 0 = I(0,0) \). Moreover, the asymptotic behavior (1.5) is a direct consequence of (3.19).

\textbf{4. Proof of Theorem 1.2.} In this section, we assume, without further mentioning, that the assumptions required in Theorem 1.2 are satisfied.

System (1.6) has a variational structure and the underlying functional is
\[
I(u,v) = \frac{1}{p} \|(u,v)\|^p - \frac{\gamma}{p} \|u\|^p_{H_p} + \|v\|^p_{H_p} - \int_{\mathbb{R}^n} h(q)(u,v) dq - \frac{\sigma}{p} \int_{\mathbb{R}^n} |u|^\alpha |v|^\beta dq,
\]
with \( I : W \to \mathbb{R} \). Clearly, (V), (f_1) and (h) imply that \( I \) is of class \( C^1(W) \). We first show that \( I \) has a useful geometrical profile, and recall that \( \gamma \in (-\infty, H_p) \) and that, when \( \sigma > 0 \), we require also (1.7) on \( h \), that is \( h \) may depend on \( \gamma^+ \).
Lemma 4.1. For any γ ∈ (−∞, Hp) and σ ≤ 1 there exist positive numbers ρ0 and j such that \( \mathcal{I}(u, v) \geq j \) for any \((u, v) \in W, \) with \( \| (u, v) \| = \rho_0, \) for any function \( h, \) depending only on \( \gamma^+ \) and satisfying (h1)–(h2). Moreover,
\[
m_{\gamma, \sigma} = \inf_{(u, v) \in B_{\rho_0}} \mathcal{I}(u, v) < 0,
\]
where \( B_{\rho_0} = \{(u, v) \in W : \|(u, v)\| < \rho_0\}, \) and there exist a sequence \( \{(u_k, v_k)\}_k \) in \( B_{\rho_0} \) and a function \( (u, \gamma, v, \sigma) \) in \( B_{\rho_0} \) such that for all \( k \)
\[
\|(u_k, v_k)\| < \rho_0, \quad m_{\gamma, \sigma} \leq \mathcal{I}(u_k, v_k) \leq m_{\gamma, \sigma} + \frac{1}{k},
\]
(4.1)
\[ (u_k, v_k) \rightharpoonup (u, \gamma, v, \sigma) \] in \( W, \) \( (u_k, v_k) \rightarrow (u, \gamma, v, \sigma) \) a.e. in \( \mathbb{H}^n, \)
\[ \mathcal{I}(u_k, v_k) \rightarrow 0 \) in \( W'. \)

Proof. Fix \( \gamma \in (-\infty, Hp) \) and \( \sigma \leq 1. \) By (f1), (h), (1.4), Lemma 2.1 and (2.7) we obtain for all \((u, v) \in W\)
\[
\mathcal{I}(u, v) \geq \frac{1}{p} \|(u, v)\|^p - \frac{\gamma^+}{pH_p} (\|D_{H^n}u\|^p_p + \|D_{H^n}v\|^p_p)
\]
\[
- C \int_{H^n} h(q)\|(u, v)\|^s dq - \frac{\sigma^+}{p^*}\|u\|^{\sigma_0}_p \|v\|^{\beta}_p \geq \frac{1}{p} \left(1 - \frac{\gamma^+}{H_p^p}\right) \|(u, v)\|^p - CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}} \|(u, v)\|^s - \frac{\sigma^+}{p^*} CC_{Q,p}^s \|(u, v)\|^{p^*}. 
\]

Therefore, if \( \sigma \leq 0, \) for \( \rho_0 > 0 \) sufficiently large we have
\[
\mathcal{I}(u, v) \geq \rho_0 \left[ \frac{1}{p} \left(1 - \frac{\gamma^+}{H_p^p}\right) \rho_0^{p-s} - CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}} \right] = j > 0
\]
for all \((u, v) \in W, \) with \( \|(u, v)\| = \rho_0, \) since \( 1 < s < p. \) If \( \sigma \in (0, 1], \) then the Young inequality yields for any \( \varepsilon > 0 \)
\[
CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}} \|(u, v)\|^s \leq \varepsilon \|(u, v)\|^p + \varepsilon^{-\frac{s}{p-s}} CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}},
\]
being \( 1 < s < p. \) Thus, for \( \varepsilon = (H_p - \gamma^+)/2pH_p > 0 \) it follows that
\[
\mathcal{I}(u, v) \geq \varepsilon \|(u, v)\|^p - \varepsilon^{s/(s-p)} CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}} \left(CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}}\right)^{p/(p-s)} - \frac{CC_{Q,p}^s}{p^*} \|(u, v)\|^{p^*}
\]
for all \((u, v) \in W, \) since \( 0 < \sigma \leq 1. \) Let us consider the function
\[
\eta(t) = \varepsilon t^p - \frac{CC_{Q,p}^s}{p^*} t^{p^*}, \quad t \geq 0.
\]
Now the number \( \rho_0 = \left(\frac{(H_p - \gamma^+)/2pH_p CC_{Q,p}^s}{1/(p^*-p)}\right)^{1/(p^*-p)} > 0 \) is such that
\[
\eta(\rho_0) = \max_{t \geq 0} \eta(t) = \left(\frac{1}{2p} - \frac{1}{2p^*}\right) \left(1 - \frac{\gamma^+}{H_p^p}\right) \left(2CC_{Q,p}^s\right)^{p/(p-p^*)} > 0
\]
because \( 1 < p < p^*. \) Therefore, since \( h \) satisfies (1.7), we obtain for any \((u, v) \in W, \) with \( \|(u, v)\| = \rho_0, \)
\[
\mathcal{I}(u, v) \geq \eta(\rho_0) - \left[\frac{1}{2p} \left(1 - \frac{\gamma^+}{H_p^p}\right)^{s/(s-p)} CC_{Q,p}^s \|h\|_{\frac{p^*}{p-s}}\right]^{p/(p-s)} > 0,
\]
which concludes the proof of the first part.
Hence, we consider Lemma 4.2. For any $0 < m < 1$ there exists a sequence $\{v_k\}$ such that $h(q) \leq m$ and $\|\Psi\|_\infty < \rho_0$, and $\int_{\Omega} |(\Phi, \Psi)|^s dq > 0$. Let $\delta > 0$ be the number given in (f2). For all $t \in (0, \delta)$ then (f2) and (h) yield

$$I(t\Phi, t\Psi) \leq \frac{1}{p} \int_{\Omega} |t(\Phi, \Psi)|^p + \frac{\gamma}{p} \int_{\Omega} |(\Phi, \Psi)|^{p^*} dq - \frac{\gamma}{p} \int_{\Omega} \frac{t(\Phi, \Psi)}{p^*} \int_{\Omega} |(\Phi, \Psi)|^{p^*} dq$$

$$+ \frac{\gamma}{p} \int_{\Omega} |(\Phi, \Psi)|^{p^*} dq + \sigma \int_{\Omega} |(\Phi, \Psi)|^{p^*} dq.$$

Hence, $I(t\Phi, t\Psi) < 0$ for $t \in (0, \delta)$ sufficiently small, since $1 < s_1 < p < p^*$ by (f2). This shows that $m_{\gamma, \sigma} < 0$ and completes the proof of the first part.

Applying the Ekeland variational principle in $\overline{B_{\rho_0}}$, the first part of the lemma, there exists a sequence $\{(u_k, v_k)\}_k$ in $B_{\rho_0}$ such that

$$m_{\gamma, \sigma} \leq I(u_k, v_k) \leq m_{\gamma, \sigma} + \frac{1}{k},$$

$$I(u, v) \geq I(u_k, v_k) - \frac{1}{k} \|u - v\|$$

for all $(u, v) \in \overline{B_{\rho_0}}$. A standard procedure gives that $I'(u_k, v_k) \to 0$ in $W'$ as $k \to \infty$ and clearly, up to a subsequence, the bounded sequence $\{(u_k, v_k)\}_k \subset B_{\rho_0}$ weakly converges to some $(u_{\gamma, \sigma}, v_{\gamma, \sigma}) \in \overline{B_{\rho_0}}$ and $(u_k, v_k) \to (u_{\gamma, \sigma}, v_{\gamma, \sigma})$ a.e. in $H^n$. This completes the proof of (4.1) and of the lemma.

Clearly, (4.1) of Lemma 4.1 implies that the bounded sequence $\{(u_k, v_k)\}_k$ is a Palais–Smale sequence of $I$ in $W$ at level $m_{\gamma, \sigma}$.

**Lemma 4.2.** For any $\gamma \in (-\infty, H_p)$ there exists $\sigma^* = \sigma^*(\gamma) \in (0, 1]$ such that, up to a subsequence, $\{(u_k, v_k)\}_k$ strongly converges to some $(u_{\gamma, \sigma}, v_{\gamma, \sigma})$ in $W$ for all $\sigma < \sigma^*$.

**Proof.** Fix $\gamma \in (-\infty, H_p)$ and $\sigma \leq 1$. By (4.1) of Lemma 4.1, passing up to a further subsequence, if necessary, $\{(u_k, v_k)\}_k$ and $(u_{\gamma, \sigma}, v_{\gamma, \sigma}) \in \overline{B_{\rho_0}}$ continue to satisfy (3.16). We claim that

$$\int_{\mathbb{R}^n} h(q)(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})^s dq \to 0. \quad (4.2)$$

Fix $\varepsilon > 0$. Since $h \in L^{p^*/s}(\mathbb{R}^n)$ and $\{(u_k, v_k)\}_k$ is bounded in $W$, there exists $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_R} h(q)(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})^s dq \leq \left( \int_{\mathbb{R}^n \setminus B_R} |h(q)|^{p^*/(p^*-s)} dq \right)^{(p^*-s)/p^*} \times \|u_k - (u_{\gamma, \sigma}, v_{\gamma, \sigma})\|_{p^*}^s \leq \frac{\varepsilon}{2},$$
where $B_R$ is the ball in $\mathbb{H}^n$ with radius $R > 0$ centered at point $O$ of $\mathbb{H}^n$. Furthermore, for any measurable subset $U \subset B_R$, by the Hölder inequality

$$\int_U h(q)|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})|^s dq \leq C \left( \int_U |h(q)|^{p^*/(p^*-s)} dq \right)^{(p^*-s)/p^*}.$$  

Hence, $\{h(q)(u_k - u_{\gamma, \sigma}, v_k - u_{\gamma, \sigma})\}_k$ is equi-integrable and uniformly bounded in $L^1(B_R)$, thanks to (h). Thus by (3.16) and the Vitali convergence theorem, for all $\varepsilon > 0$ there exists $k_0 > 0$ such that

$$\int_{B_R} h(q)|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})|^s dq \leq \varepsilon$$

for all $k \geq k_0$. Therefore,

$$\int_{\mathbb{H}^n} h(q)|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})|^s dq \leq \int_{\mathbb{H}^n \setminus B_R} h(q)|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})|^s dq + \int_{B_R} h(q)|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})|^s dq \leq \varepsilon$$

for all $k \geq k_0$. This proves the claim and the validity of (4.2).

Now $(f_1)$ and the Hölder inequality give

$$\left| \int_{\mathbb{H}^n} \{h(q)\alpha(u_k, v_k)(u_k - u_{\gamma, \sigma}) + h(q)\beta(u_k, v_k)(v_k - v_{\gamma, \sigma})\} dq \right|$$

$$\leq C \int_{\mathbb{H}^n} h(q)|(u_k, v_k)|^{s-1}|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})| dq$$

$$\leq C \left( \int_{\mathbb{H}^n} h(q)|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})|^s dq \right)^{1/s},$$

for a suitable constant $C > 0$. Thus, by (4.2) it follows that

$$\lim_{k \to \infty} \int_{\mathbb{H}^n} \{h(q)\alpha(u_k, v_k)(u_k - u_{\gamma, \sigma}) + h(q)\beta(u_k, v_k)(v_k - v_{\gamma, \sigma})\} dq = 0. \quad (4.3)$$

Similarly, by using again (h) and $(f_1)$ we have as $k \to \infty$

$$\int_{\mathbb{H}^n} h(q)\alpha(u_k, v_k)\Phi dq \to \int_{\mathbb{H}^n} h(q)\alpha(u_{\gamma, \sigma}, v_{\gamma, \sigma})\Phi dq,$$

$$\int_{\mathbb{H}^n} h(q)\beta(u_k, v_k)\Psi dq \to \int_{\mathbb{H}^n} h(q)\beta(u_{\gamma, \sigma}, v_{\gamma, \sigma})\Psi dq \quad (4.4)$$

for any $(\Phi, \Psi) \in W$.

As already justified in the proof of Lemma 3.5, thanks to (3.16) Lemma 3.4 can be applied so that (3.11) holds. Thus, $|D_{\mathbb{H}^n} u_k|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_k \to |D_{\mathbb{H}^n} u_{\gamma, \sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma, \sigma}$ a.e. in $\mathbb{H}^n$ and Proposition A.7 of [2] gives that $\Theta = |D_{\mathbb{H}^n} u_{\gamma, \sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} u_{\gamma, \sigma}$ and $\Lambda = |D_{\mathbb{H}^n} v_{\gamma, \sigma}|_{\mathbb{H}^n}^{p-2} D_{\mathbb{H}^n} v_{\gamma, \sigma}$ a.e. in $\mathbb{H}^n$ in (3.16). Consequently, as in Lemma 3.5, we get (3.20)–(3.23) also in this context. Hence, (3.16) and (4.4) yield that $(u_{\gamma, \sigma}, v_{\gamma, \sigma})$ is a critical point of $I$ in $W$. Thus, as $k \to \infty$ in (4.1) and by (3.16) and (4.3), we get (3.26) from which, by the Brézis and Lieb lemma, we have (3.27). By (3.16) and Lemma 3.4 we again have (3.28) and so, combining all, we obtain the main formula

$$\lim_{k \to \infty} \|(u_k, v_k) - (u_{\gamma, \sigma}, v_{\gamma, \sigma})\|^p = \gamma \lim_{k \to \infty} \left( \|u_k - u_{\gamma, \sigma}\|_{H^p} + \|v_k - v_{\gamma, \sigma}\|_{H^p} \right)$$
where $\gamma\in\mathbb{R}^n$ and $\sigma\geq 0$. Assume by contradiction that $v_{\gamma,\sigma}^p + \ell_{p,\gamma,\sigma} > 0$. Then, from (4.5) we get

$$\lim_{k\to\infty} \| (u_k - u_{\gamma,\sigma}, v_k - v_{\gamma,\sigma}) \|^p \leq \gamma \lim_{k\to\infty} (\| u_k - u_{\gamma,\sigma} \|_{H^p} + \| v_k - v_{\gamma,\sigma} \|_{H^p})$$

$$< H_p \lim_{k\to\infty} (\| u_k - u_{\gamma,\sigma} \|_{H^p} + \| v_k - v_{\gamma,\sigma} \|_{H^p})$$

$$\leq \lim_{k\to\infty} \| (u_k - u_{\gamma,\sigma}, v_k - v_{\gamma,\sigma}) \|^p,$$

since $\gamma < H_p$. This is impossible. Therefore, $v_{\gamma,\sigma}^p + \ell_{p,\gamma,\sigma} = 0$ for all $\sigma \leq 0$ and so (4.5) implies that

$$\lim_{k\to\infty} \| (u_k, v_k) - (u_{\gamma,\sigma}, v_{\gamma,\sigma}) \| = 0,$$

as required.

Let us now consider the case $\sigma \in (0, 1]$. Since $\gamma < H_p$ there exists $c \in (0, 1)$ such that $\gamma^+ = c H_p$. Clearly, (4.5) can be rewritten as

$$(1 - c) \lim_{k\to\infty} \| (u_k - u_{\gamma,\sigma}, v_k - v_{\gamma,\sigma}) \|^p + c \lim_{k\to\infty} \| (u_k - u_{\gamma,\sigma}, v_k - v_{\gamma,\sigma}) \|^p$$

$$= \gamma (v_{\gamma,\sigma}^p + \ell_{p,\gamma,\sigma}) + \sigma \delta_{\gamma,\sigma}.$$

Thus, by (1.4) and (V), using (2.7) with $(u, v) = (u_k - u_{\gamma,\sigma}, v_k - v_{\gamma,\sigma})$, we get

$$\sigma \delta_{\gamma,\sigma} + \gamma^+ (v_{\gamma,\sigma}^p + \ell_{p,\gamma,\sigma}) \geq (1 - c) C_{Q,p}^\gamma \sigma^p \gamma^p + c H_p (v_{\gamma,\sigma}^p + \ell_{p,\gamma,\sigma})$$

for all $\sigma \in (0, 1]$, being $c \in (0, 1)$. Therefore,

$$\sigma \delta_{\gamma,\sigma} \geq (1 - c) C_{Q,p}^\gamma \sigma^p \gamma^p,$$

since $\gamma^+ = c H_p$.

Let us define

$$\sigma^* = \begin{cases} \inf \{ \sigma \in (0, 1] : \delta_{\gamma,\sigma} > 0 \} & \text{if there exists } \sigma \in (0, 1] \text{ such that } \delta_{\gamma,\sigma} > 0, \\ 1 & \text{if } \delta_{\gamma,\sigma} = 0 \text{ for all } \sigma \in (0, 1]. \end{cases}$$

We claim that $\sigma^* > 0$ if there exists $\sigma \in (0, 1]$ such that $\delta_{\gamma,\sigma} > 0$. Otherwise, there exists a sequence $\{\sigma_k\}_k$ with $\delta_{\gamma,\sigma_k} > 0$, such that $\sigma_k \to 0$ as $k \to \infty$. Thus, (4.7) implies that

$$\sigma_k \delta_{\gamma,\sigma_k}^\gamma \sigma_k^p \gamma^p \geq (1 - c) C_{Q,p}^\gamma \sigma_k^p \gamma^p > 0.$$
which gives a contradiction. Thus, $v_{\gamma,\sigma} + \ell_{\gamma,\sigma} = 0$ for any $\sigma \in (0, \sigma^*)$. Now (3.29) implies again the validity of (4.6).

In conclusion, $(u_k, v_k) \to (u_{\gamma,\sigma}, v_{\gamma,\sigma})$ as $k \to \infty$ in $W$, as required. \hfill \Box

Proof of Theorem 1.2. Fix $\gamma \in (-\infty, H_p)$. For any $\sigma \in (0,1]$ Lemma 4.1 and the Ekeland variational principle give the existence of a a Palais–Smale sequence $\{ (u_k, v_k) \}_k$ in $W$ at level $m_{\gamma,\sigma}$. Moreover, by Lemma 4.2 there exists $\sigma^* > 0$ such that, up to a subsequence, $\{ (u_k, v_k) \}_k$ strongly converges to $(u_{\gamma,\sigma}, v_{\gamma,\sigma})$ in $W$, with $m_{\gamma,\sigma} = I(u_{\gamma,\sigma}, v_{\gamma,\sigma}) < 0$ and $I'(u_{\gamma,\sigma}, v_{\gamma,\sigma}) = 0$ for any $\sigma \in (0, \sigma^*)$. Consequently, $(u_{\gamma,\sigma}, v_{\gamma,\sigma})$ is a nontrivial solution of system (1.6). \hfill \Box

Acknowledgments. The author was partly supported by the Italian MIUR project Variational methods, with applications to problems in mathematical physics and geometry (2015KB9WPT,009) and is a member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilit`a e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The manuscript was realized within the auspices of the INdAM–GNAMPA Project 2018 titled Problemi non lineari alle derivate parziali (Prot,U-UFBMBAZ-2018-000384), and of the Fondo Ricerca di Base di Ateneo – Esercizio 2015 of the University of Perugia, titled PDEs e Analisi Non-lineare.

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, second eds., Academic Press, New York–London, 2003.
[2] G. Autuori and P. Pucci, Existence of entire solutions for a class of quasilinear elliptic equations, NoDEA Nonlinear Differential Equations Appl., 20 (2013), 977–1009.
[3] Z. M. Balogh and A. Kristály, Lions–type compactness and Rubik actions on the Heisenberg group, Calc. Var. Partial Differential Equations, 48 (2015), 89–109.
[4] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal., 19 (1992), 581–597.
[5] S. Bordonaro and P. Pucci, Schrödinger–Hardy systems involving two Laplacian operators in the Heisenberg group, Bull. Sci. Math., 146 (2018), 50-88.
[6] M. Caponi and P. Pucci, Schrödinger–Hardy systems involving two Laplacian operators in the Heisenberg group, Bull. Sci. Math., 146 (2018), 50-88.
[7] C. Chen, Infinitely many solutions to a class of quasilinear Schrödinger system in $\mathbb{R}^N$, Appl. Math. Lett., 52 (2016), 176–182.
[8] W. Chen and M. Squassina, Critical nonlocal systems with concave–convex powers, Adv. Nonlinear Stud., 16 (2016), 821–842.
[9] J. Y. Chu, Z. W. Wei and Q. Y. Wu, $L^p$ and BMO bounds for weighted Hardy operators on the Heisenberg group, J. Inequal. Appl., (2016), Paper No. 292, 12 pp.
[10] L. D’Ambrosio, Hardy-type inequalities related to degenerate elliptic differential operators, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 4 (2005), 451–486.
[11] F. Demengel and E. Hebey, On some nonlinear equations on compact Riemannian manifolds, Adv. Differential Equations, 3 (1998), 533–574.
[12] A. Fiscella, P. Pucci and S. Saldi, Existence of entire solutions for Schrödinger–Hardy systems involving two fractional operators, Nonlinear Anal., 158 (2017), 109–131.
[13] A. Fiscella, P. Pucci and B. Zhang, $p$-fractional Hardy–Schrödinger–Krichhoff Systems with Critical Nonlinearities, submitted for publication, pages 22.
[14] G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups, Ark. Math., 13 (1975), 161–207.
[15] G. B. Folland and E. M. Stein, Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group, Comm. Pure Appl. Math., 27 (1974), 429–522.
400 PATRIZIA PUCCI

[16] B. Franchi, C. Gutierrez and R. L. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators, Comm. PDE, 19 (1994), 523–604.
[17] Y. Fu, H. Li and P. Pucci, Existence of nonnegative solutions for a class of systems involving fractional ($p,q$)–Laplacian operators, Chin. Ann. Math. Ser. B, 39 (2018), 357–372.
[18] N. Garofalo and E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, Ann. Inst. Fourier, 40 (1990), 313–356.
[19] N. Garofalo and D.–M. Nhieu, Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces, Comm. Pure Appl. Math., 49 (1996), 1081–1144.
[20] P. Han, The effect of the domain topology on the number of positive solutions of an elliptic system involving critical Sobolev exponents, Houston J. Math., 32 (2006), 1241–1257.
[21] L. Hörmander, Hypoelliptic second order differential equations, Acta Math., 119 (1967), 147–171.
[22] S.P. Ivanov, D.N. Vassilev, Extremals for the Sobolev Inequality and the Quaternionic Contact Yanoake Problem, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, xviii+219 pp., 2011.
[23] G. P. Leonardi and S. Masnou, On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$, Ann. Mat. Pura Appl., (4) 184 (2005), 533–553.
[24] A. Loiudice, Improved Sobolev inequalities on the Heisenberg group, Nonlinear Anal., 62 (2005), 953–962.
[25] M. Magliaro, L. Mari, P. Mastroniu and M. Rigoli, Keller–Osserman type conditions for differential inequalities with gradient terms on the Heisenberg group, J. Diff. Equations, 250 (2011), 2643–2670.
[26] G. Mingione, A. Zatorska–Goldstein and X. Zhong, Gradient regularity for elliptic equations in the Heisenberg group, Adv. Math., 222 (2009), 62–129.
[27] X. Mingqi, V. Radulescu and B. Zhang, Combined effects for fractional Schrödinger–Kirchhoff systems with critical nonlinearities, ESAIM Control Optim. Calc. Var., (2017), pages 28.
[28] P. Niu, H. Zhang and Y. Wang, Hardy–type and Rellich type inequalities on the Heisenberg group, Proc. Amer. Math. Soc., 129 (2001), 3623–3630.
[29] P. Pucci, M. Q. Xiang and B. L. Zhang, Multiple solutions for nonhomogeneous Schrödinger–Kirchhoff type equations involving the fractional $p$–Laplacian in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, 54 (2015), 2785–2806.
[30] D. Ricciotti, $p$–Laplace Equation in the Heisenberg Group. Regularity of Solutions, Springer Briefs in Mathematics, BCAM Basque Center for Applied Mathematics, Bilbao, xiv+87 pp., 2015.
[31] N. Varopoulos, Analysis on nilpotent Lie groups, J. Funct. Anal., 66 (1986), 406–431.
[32] N. Varopoulos, Sobolev inequalities on Lie groups and symmetric spaces, J. Funct. Anal., 86 (1989), 19–40.
[33] D. Vassilev, Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups, Pacific J. Math., 227 (2006), 361–397.

Received May 2017; revised December 2017.

E-mail address: patrizia.pucci@unipg.it