Green’s function on lattices

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ABSTRACT

A method to calculate exact Green’s functions on lattices in various dimensions is presented. Expressions in terms of generalized hypergeometric functions in one or more variables are obtained for various examples by relating the resolvent to a contour integral, evaluated using residues. Different ways of arranging the series leads to different combinations of hypergeometric functions providing identities involving generalized hypergeometric functions. The method is shown to be useful for computing Green’s functions with next-nearest neighbor hopping as well.

1 Introduction

A variety of physical situations call for studying Green’s function on lattices. Examples include crystal Physics [1], electrical circuits [2, 3], statistical Physics [4–6], lattice gauge theory [7, 8] etc. to mention a few. Various aspects of lattice Green’s functions have been studied on a variety of lattices in diverse dimensions [9–15].

In this article we present a method to compute the exact form of Green’s functions on lattices with a few examples. Given a lattice, the Green’s function, defined as the kernel of the Laplacian on the same is first expressed as Fourier integrals over closed intervals, as many as the dimension of the lattice in number. Each of these closed domains is expressed in turn as the unit circle in an appropriately defined complex plane, promoting the Fourier integral to a contour integral. The latter can be evaluated by the method of residues. The Green’s function thus obtained is, more often than not, singular. We resort to evaluating the resolvent of the Laplacian which is, by the same token, a contour integral containing an auxiliary complex parameter, the spectral parameter. Computation of residue yields a series in the spectral parameter, whose domain is restricted by the radius of convergence of the series. It is usually possible to analytically continue the result to ulterior domains, appropriate to the specificity of physical situations. Indeed, in most cases we are able to express the resolvent in terms of generalized hypergeometric functions for which rigorous results of analytic continuation are available [16–18].

In the following sections we study the Laplacian on lattices in various dimensions. In the rest of this section let us recall the relevant definitions in order to fix notation. Let Λ denote an infinitely
extended lattice in the $D$-dimensional Euclidean space $\mathbb{R}^D$, with a point marked as the origin. Let $\mathcal{N}$ denote the set of points on the lattice adjacent to the origin, that is the set of nearest neighbors of the origin, according to the canonical Euclidean metric on $\mathbb{R}^D$. Let $N$ denote the the cardinality of $\mathcal{N}$, that is the number of nearest neighbors to any point of $\Lambda$, called the co-ordination number of the lattice. For any element $a$ of $\mathcal{N}$, let $\nabla_a$ denote the difference operator on the space of complex-valued functions in $\mathbb{R}^D$, that is

$$\nabla_a f(\xi) = f(\xi + a) - f(\xi),$$ (1)

where $f : \mathbb{R}^D \to \mathbb{C}$ and $\xi \equiv (\xi_1, \cdots, \xi_D) \in \mathbb{R}^D$. The second order difference operator on $\Lambda$ is then

$$\nabla^2 f(\xi) = \sum_{a,b \in \mathcal{N}} \nabla_a \nabla_b f(\xi) = \sum_{a,b \in \mathcal{N}} f(\xi + a + b) - 2N \sum_{a \in \mathcal{N}} f(\xi + a) + N^2 f(\xi).$$ (2)

Shifting $\xi$ to $\xi - a$ and dividing by $N$ leads to the Laplacian $\triangle$ defined as

$$\triangle f(x) = \frac{1}{2} \sum_{a \in \mathcal{N}} \left( f(\xi + a) + f(\xi - a) - 2 f(\xi) \right).$$ (3)

The normalizing factor of half is introduced to match with the standard expression on lattices for which both $a$ and $-a$ belong to $\mathcal{N}$. The Green’s function $G_a$ is then obtained as the kernel of the Laplacian $\triangle$, so that

$$\triangle G_a(\xi) = -\delta(\xi).$$ (4)

In order to find the Green’s function let us write any complex-valued function $f(x)$ as a Fourier integral,

$$f(\xi) = \frac{1}{(2\pi)^D} \int_{0}^{2\pi} d^D \theta e^{i\xi \cdot \theta} g(\theta),$$ (5)

where $\theta \in \mathbb{R}^D$ and $\cdot$ denotes the dot product in $\mathbb{R}^D$. Substituting in (3) we obtain

$$\triangle f(\xi) = \frac{1}{(2\pi)^D} \int_{0}^{2\pi} d^D \theta e^{i\xi \cdot \theta} g(\theta) \left( \frac{1}{2} \sum_{a \in \mathcal{N}} \left( e^{ia \cdot \theta} + e^{-ia \cdot \theta} \right) - N \right),$$ (6)

which reduces to $-\delta(\xi)$ if and only if

$$g(\theta) = \frac{1}{N} - \frac{1}{2} \sum_{a \in \mathcal{N}} \left( e^{ia \cdot \theta} + e^{-ia \cdot \theta} \right).$$ (7)

The two-point correlation function for a pair of lattice points $\alpha, \beta$ is defined as

$$\mathcal{G}(\alpha, \beta) = G(\alpha, \beta) - G(0, 0).$$ (8)

Subtraction of $G(0, 0)$ is to impose the translational symmetry of the lattice $\Lambda$, required since we started with the origin marked in $\Lambda$. Introducing the vector $r = \alpha - \beta$ with integer components when
expanded in the basis of the lattice vectors, \( r = (r_1, r_2, \cdots, r_D) \) we write

\[
G_{r_1, r_2, \cdots, r_D} = G(\alpha, \beta) = \frac{1}{(2\pi)^D} \int_0^{2\pi} d^D \theta \frac{e^{i\gamma \theta}}{N - \frac{1}{2} \sum_{a \in \mathcal{N}} (e^{ia \theta} + e^{-ia \theta})},
\]

(9)

The integral as defined is usually singular. We recourse to computing the resolvent

\[
H_r(t) = \frac{1}{(2\pi)^D} \int_0^{2\pi} d^D \theta \frac{e^{i\gamma \theta}}{t - \sum_{a \in \mathcal{N}} \cos a \cdot \theta}.
\]

(10)

where \( t \) is a complex parameter, the spectral parameter. By abuse of notation we shall continue denoting the resolvent by \( H(t) \) even though \( t \) is normalized differently in different examples for convenience. The general strategy adopted here to evaluate the resolvent is to first introduce a complex variable for every component of \( \theta \). The integral over the angular variables are then interpreted as a contour integral over the unit circle in each of these complex planes. The contour integral is then evaluated using the method of residues [19]. Let us now employ the strategy in a variety of examples in various dimensions.

2 One dimension

The set of nearest neighbors to the origin in a one-dimensional lattice is \( \mathcal{N} = \{-1, 1\} \). The coordination number is \( N = 2 \). The Green’s function is obtained from (9) to be

\[
G^{(1)}_r = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{e^{i\gamma \theta}}{2 - 2 \cos \theta}.
\]

(11)

The resolvent (10) is written as a contour integral in terms of a complex variable \( x = e^{i\theta} \) as

\[
H_r(t) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \frac{x^r}{2t - (x + 1/x)}.
\]

(12)

The Green’s function is \( G^{(1)}_r = H_r(1) \). In order to evaluate the contour integral we rewrite it by first pulling out \( 2t \) from the denominator and then expanding the denominator in a geometric series to obtain

\[
H_r(t) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} x^r \sum_{n=0}^{\infty} (2t)^{-1-n} (x + 1/x)^n
\]

\[
= \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \sum_{n=0}^{\infty} (2t)^{-1-n} \sum_{a=0}^{n} \binom{n}{a} x^{2a-n+r},
\]

(13)
where we have used the binomial theorem to expand \((x + 1/x)^n\) in the second step. Writing the factorials in the binomial coefficients as gamma functions the integral takes the form

\[
H_{r,s}(t) = \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} \sum_{n=0}^{\infty} \sum_{a=0}^{\infty} (2t)^{-1-n} \frac{\Gamma(1+n)}{\Gamma(1+n-a)\Gamma(1+a)} x^{2a-n+r},
\]

(14)

where the domain of the \(a\) is extended taking into account the poles of gamma function for non-positive integers appearing in the denominator. The integral is then evaluated as the residue of the integrand. Since the measure is \(dx/x\), the residue is given by the constant, that is, the totality of \(x\)-independent terms of the integrand, with

\[
2a - n + r = 0.
\]

(15)

This equation can be solved for \(a\) as \(a = (n - r)/2\), leading to

\[
H_r(t) = \sum_{n=0}^{\infty} \left( \frac{1}{2t} \right)^{1+n} \frac{\Gamma(1+n)}{\Gamma(1+n-r/2)\Gamma(1+n+r/2)}.
\]

(16)

Splitting the sum over all positive integers into even and odd parts as

\[
\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} f(2n) + \sum_{n=0}^{\infty} f(2n+1)
\]

(17)

for any function \(f\) and using the duplication formula

\[
\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma(x+1/2),
\]

(18)

the sum is written as

\[
H_r(t) = \frac{1}{2\sqrt{\pi} t} \sum_{n=0}^{\infty} \left( \frac{1}{t^2} \right)^n \frac{\Gamma(1/2+n)\Gamma(1+n)}{\Gamma(3/2+n)\Gamma(1+n-r/2)\Gamma(1+n+r/2)} + \frac{1}{2\sqrt{\pi} t^2} \sum_{n=0}^{\infty} \left( \frac{1}{t^2} \right)^n \frac{\Gamma(3/2+n)\Gamma(1+n)}{\Gamma(3/2+n)\Gamma(3/2+r/2)\Gamma(3/2-r/2)}.
\]

(19)

This can be expressed in terms of hypergeometric functions as

\[
H_r(t) = \frac{\Gamma(1/2)^2\Gamma(1/2)}{2\sqrt{\pi} t} \, _3F_2 \left( \begin{array}{c} 1, 1, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{array} \right) \frac{1}{t^2} + \frac{\Gamma(3/2)^2\Gamma(3/2)}{2\sqrt{\pi} t^2} \, _3F_2 \left( \begin{array}{c} 1, 1, \frac{3}{2} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{array} \right) \frac{1}{t^2}.
\]

(20)

Using the relations \(\Gamma(1-z)\Gamma(z) = \pi/\sin \pi z\), \(\Gamma(z+1) = z\Gamma(z)\), for \(Re z > 0\), the values \(\Gamma(1/2) = \sqrt{\pi}\), \(\Gamma(1) = 1\) leading to the expressions

\[
\Gamma \left( \frac{1 + r}{2} \right) \Gamma \left( \frac{1 - r}{2} \right) = \frac{\pi r}{2 \sin \frac{\pi r}{2}},
\]

(21)

\[
\Gamma \left( \frac{3 + r}{2} \right) \Gamma \left( \frac{3 - r}{2} \right) = \frac{\pi (1 - r^2)}{4 \cos \frac{\pi r}{2}},
\]

(22)
we finally arrive at
\[
H_r(t) = \frac{1}{\pi r t} \sin \left( \frac{\pi r}{2} \right) _3 F_2 \left( \frac{1}{2}, 1, r, \frac{1}{2} \right) + \frac{1}{\pi (1 - r^2) t^2} \cos \left( \frac{\pi r}{2} \right) _3 F_2 \left( \frac{3}{2}, 1, 2, \frac{3}{2}, \frac{3}{2} \right) .
\] (23)

The Green's function is given by the value of \( H_r \) at \( t = 1 \).

The expression is manifestly symmetric under the reflection \( r \to -r \), which is a symmetry of the one-dimensional lattice. It is not best-suited for analyzing the asymptotics, however. The appearence of gamma functions with both signs of \( r \) in their arguments makes it difficult to take limits of \( t \). In order to obtain a formula better adapted for such studies, let us solve (15) as
\[
n = r + 2a, \quad r \geq 0,
\] (24)
which leads to the series
\[
H_r(t) = \sum_{a=0}^{\infty} \left( \frac{1}{2t} \right)^{1+r+2a} \frac{\Gamma(2a + 1 + r)}{\Gamma(1 + a) \Gamma(1 + r + a)}.
\] (25)
Using the duplication formula (18) in the numerator we obtain
\[
H_r(t) = \left( \frac{1}{2t} \right)^{1+r} _2 F_1 \left( 1 + \frac{r}{2}, \frac{1+r}{2}, \frac{1}{t^2} \right),
\] (26)
or, equivalently [18, 15.4.18],
\[
H_r(t) = \frac{1}{\sqrt{t^2 - 1}} \frac{1}{(t + \sqrt{t^2 - 1})^r}.
\] (27)
This reproduces the well-know formula for \( r = 0 \). Equating the expressions (23) and (26) furnishes an identity of hypergeometric functions. Asymptotic behavior of \( H \) and the Green’s function can be studied using existing results [18, 20, 21].

3 Two dimensions

There are various kinds of lattices in two dimensions. We consider Green’s function on three such in this section. Using existing results on analytic continuation of hypergeometric functions we demonstrate the cancellation of singularity in (8) for the square lattice.

3.1 Square lattice

For the square lattice, the set of nearest neighbors is \( \mathcal{N} = \{(\pm 1, 0), (0, \pm 1)\} \) with cardinality \( N = 4 \).

The Green’s function is
\[
G_{p,q}^{\square} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{i p \theta_1 + i q \theta_2} d\theta_1 d\theta_2 \frac{4 - 2 \cos \theta_1 - 2 \cos \theta_2}{4 - 2 \cos \theta_1 - 2 \cos \theta_2}.
\] (28)
Isotropy of the square lattice requires the Green’s function to satisfy

\[ G_{q,p} = G_{-p,q} = G_{p,-q} = G_{-p,-q} \]  (29)

We deal with the resolvent written as a contour integral

\[ H_{r,s}(t) = \frac{1}{(2\pi i)^2} \oint_{|x|=1} \oint_{|y|=1} \frac{dx}{x} \frac{dy}{y} \frac{x^r y^s}{4t - (x + 1/x)(y + 1/y)} \]  (30)

in two complex variables \( x \) and \( y \), where the contour is the union of unit circles in each complex plane. Parametrizing the circles as phases with \( x = e^{i(\theta_1 + \theta_2)/2} \) and \( y = e^{i(\theta_1 - \theta_2)/2} \) relates the resolvent and the Green’s function through

\[ G_{p,q} = \frac{1}{8\pi^2} H_{p+q,p-q}(1). \]  (31)

The integral \( H_{r,s}(t) \) is evaluated by computing the residue of the integrand at the poles inside the unit circles. By first pulling out \( 4t \) from the denominator and then expanding the denominator in a binomial series we have

\[ H_{r,s}(t) = \sum_{n=0}^{\infty} (4t)^{-1-n} \oint_{|x|=1} \oint_{|y|=1} \frac{dx}{x} \frac{dy}{y} (x + 1/x)^n (y + 1/y)^n x^r y^s. \]  (32)

Expanding the factors \( (x + 1/x)^n \) and \( (y + 1/y)^n \) in binomial series then leads to

\[ H_{r,s}(t) = \oint_{|x|=1} \oint_{|y|=1} \frac{dx}{x} \frac{dy}{y} \sum_{n=0}^{\infty} (4t)^{-1-n} \sum_{a,b=0}^{n} \binom{n}{a} \binom{n}{b} x^{r+2a-n} y^{s+2b-n}. \]  (33)

Due to the presence of inverses of \( x \) and \( y \) in the measure the residue of the integrand is given by the terms of the sum which are independent of \( x \) and \( y \), as before. These constant terms lead to the resolvent

\[ H_{r,s}(t) = \sum_{n,a,b=0}^{\infty} \left( \frac{1}{4t} \right)^{1+n} \frac{\Gamma (1+n)^2}{\Gamma (1+a)\Gamma (1+b)\Gamma (1+n-a)\Gamma (1+n-b)}, \]  (34)

with the indices satisfying the constraints

\[ r + 2a - n = 0, \]
\[ s + 2b - n = 0. \]  (35)

The domain of the indices \( a \) and \( b \) are extended as before since gamma functions having poles for non-positive integers appear in the denominator for the extra terms. Solving these for \( a \) and \( b \) with \( a = (n - r)/2 \) and \( b = (n - s)/2 \) yields

\[ H_{r,s}(t) = \sum_{n=0}^{\infty} \left( \frac{1}{4t} \right)^{1+n} \frac{\Gamma (1+n)^2}{\Gamma \left(1 + \frac{n+r}{2}\right)\Gamma \left(1 + \frac{n-r}{2}\right)\Gamma \left(1 + \frac{n+s}{2}\right)\Gamma \left(1 + \frac{n-s}{2}\right)}. \]  (36)
Splitting the sum over all positive integers into even and odd parts as in (17) and using the duplication formula (18) as in the previous section, the above expression is written in terms of the zero-balanced generalized hypergeometric function \( _5F_4 \) as

\[
H_{r,s}(t) = \frac{1}{4\pi t} \frac{\Gamma(1)\Gamma(\frac{1}{2})^2}{\Gamma(1+r)\Gamma(1+\frac{r}{2})\Gamma(1+s)\Gamma(1+\frac{s}{2})} _5F_4 \left(\begin{array}{c} 1, 1, 1, \frac{1}{2}, \frac{1}{2} \\ \frac{2+r}{2}, \frac{2-r}{2}, \frac{2+s}{2}, \frac{2-s}{2} \end{array} \right) \frac{1}{t^2}
\]

\[
+ \frac{1}{4\pi t^2} \frac{\Gamma(1)\Gamma(\frac{3-r}{2})^2}{\Gamma(1-r)\Gamma(1+\frac{3-r}{2})\Gamma(1+s)\Gamma(1+\frac{s}{2})} _5F_4 \left(\begin{array}{c} 1, 1, 1, \frac{3}{2}, \frac{3}{2} \\ \frac{3+p}{2}, \frac{3-p}{2}, \frac{3+q}{2}, \frac{3-q}{2} \end{array} \right) \frac{1}{t^2}
\]

in the domain \(|t| > 1\). Substituting \(r = p + q\) and \(s = p - q\) we obtain

\[
H_{p+q,p-q}(t) = \frac{1}{2\pi^2 t} \frac{\cos(\pi p) - \cos(\pi q)}{p^2 - q^2} _5F_4 \left(\begin{array}{c} 1, 1, 1, \frac{1}{2}, \frac{1}{2} \\ \frac{2+p+q}{2}, \frac{2-p-q}{2}, \frac{2+p-q}{2}, \frac{2-p+q}{2} \end{array} \right) \frac{1}{t^2}
\]

\[
+ \frac{1}{2\pi^2 t^2} \frac{\cos(\pi p) + \cos(\pi q)}{((p+q)^2 - 1)((p-q)^2 - 1)} _5F_4 \left(\begin{array}{c} 1, 1, 1, \frac{3}{2}, \frac{3}{2} \\ \frac{3+p+q}{2}, \frac{3-p-q}{2}, \frac{3+p-q}{2}, \frac{3-p+q}{2} \end{array} \right) \frac{1}{t^2}
\]

This gives the Green’s function \(G_{p,q}\) by (31), possessing the symmetries (29), which is singular as the hypergeometric function is singular at \(t = 1\). The singularity is cancelled by subtracting \(G_{0,0}\). To see this we first analytically continue \(H_{r,s}(t)\) to a neighborhood of \(t = 1\) to isolate the logarithmic singularity. We then verify that the coefficient of the logarithmic term is independent of \(p, q\). The expression (37) continues to [16]

\[
H_{r,s}(t) = \sum_{n=0}^{\infty} (1 - \frac{1}{t^2})^n \left[ \sum_{k=0}^{n} \frac{(-n)_k}{k!k!} \left( \frac{A_{k}^{(4)}}{4\pi t} + \frac{A_{k}^{(4)}}{4\pi t^2} \right) \left( \psi(1 + n - k) - \psi(1 + n) - \log(1 - \frac{1}{t^2}) \right) \right] + (-1)^n n! \sum_{k=n+1}^{\infty} \frac{(k - n - 1)!}{k!k!} \left( \frac{A_{k}^{(4)}}{4\pi t} + \frac{A_{k}^{(4)}}{4\pi t^2} \right),
\]

in a neighborhood of \(t = 1\), where we used the Pochhammer symbol

\[
(\ell)_n = \ell(\ell + 1) \cdots (\ell + n - 1) = \Gamma(\ell + n)/\Gamma(\ell),
\]

which for positive integral \(n\) satisfies \((1)_n = n!, (\ell)_0 = 1\) for all \(\ell\), \((0)_0 = 1\), in particular, and \((0)_n = 0\) for non-vanishing \(n\). The dependence on \(r\) and \(s\) are solely in the coefficients \(A\) and \(A'\), defined as [16]

\[
A_{k}^{(4)} = \frac{(1 + \frac{r}{2})k(1 - \frac{r}{2})k}{k!} \sum_{\ell=0}^{\infty} \frac{(\frac{1}{2})_{\ell}(1)_{\ell}}{\ell!(1 + \frac{r}{2})_{\ell}(1 - \frac{r}{2})_{\ell}} F_2 \left(\begin{array}{c} 1+s, -1+s, -\ell \\ \frac{1}{2}, 1 \end{array} \right),
\]

\[
A_{k}^{(4)} = \frac{(1 + \frac{r}{2})k(-1 + \frac{r}{2})k}{k!} \sum_{\ell=0}^{\infty} \frac{(\frac{1}{2})_{\ell}(0)_{\ell}(-k)_{\ell}}{\ell!(1 + \frac{r}{2})_{\ell}(-\frac{1+r}{2})_{\ell}} F_2 \left(\begin{array}{c} \frac{s}{2}, -\frac{s}{2}, -\ell \\ \frac{1}{2}, 0 \end{array} \right)
\]

(41)
From (39) we note that the logarithmic singularity of $H_{r,s}$ arises from the $n = 0$ term in the sum. The coefficient of $\log(1 - 1/t^2)$, namely,

$$-\frac{A_0^{(4)}}{4\pi t} - \frac{A_0^r}{4\pi t^2},$$

(42)
evaluates to $-1/2\pi$ as $t$ tends to unity, independent of $r$ or $s$. Hence the Green’s function (8) defined as the limit

$$\mathcal{G}_{p,q} = \lim_{t \to 1} \frac{1}{8\pi^2} \left( H_{p+q,p-q}(t) - H_{0,0}(t) \right)$$

(43)
is non-singular.

As in the one-dimensional case, the expression (37), while possessing the symmetries of the lattice, is not amenable to asymptotic studies. We can derive another expression for the resolvent by solving (35) for $n$ as $n = 2a + r$, leading to

$$b = a + \frac{r - s}{2}, \quad n - b = a + \frac{r + s}{2}.$$  

(44)

Substituting these in (36) and using the duplication formula (18) we obtain $H_{r,s}$ in terms of another zero-balanced generalized hypergeometric function

$$H_{r,s}(t) = \left( \frac{1}{4t} \right)^{1+r} \sum_{n=0}^{\infty} \frac{\Gamma(1 + r + 2n)^2}{\Gamma(1 + r + n) \Gamma(1 + \frac{r + s}{2} + n) \Gamma(1 + \frac{r - s}{2} + n)} \left( \frac{1}{16t^2} \right)^n \frac{1}{n!}$$

(45)

$$= \frac{\Gamma(1 + r)}{\Gamma(1 + \frac{r + s}{2}) \Gamma(1 + \frac{r - s}{2})} \left( \frac{1}{2t} \right)^{1+r} _{4}F_{3} \left( \begin{array}{c} 1 + \frac{r}{2}, 1 + \frac{r}{2}, 1 + \frac{r+s}{2}, 1 + \frac{r-s}{2} \\ 1, 1 + \frac{r+s}{2}, 1 + \frac{r-s}{2} \end{array} \right| \frac{1}{2t^2} \right).$$  

(46)

Isotropy (29) of the lattice is not manifest in the expression for the resolvent $H_{p+q,p-q}(t)$. For $r = s = 0$ this yields

$$H_{0,0}(t) = \frac{1}{\pi t^2} K(1/t),$$

(47)

where the identification of the hypergeometric function and the complete elliptic integral of the first kind, namely, $\frac{\Gamma(1/2)}{\int_0^\infty K(x)} = \frac{\Gamma(1/2)}{\pi} K(1/2)$, is used. For large distances, $G_{p,p}$ is obtained, according to (31), as $H_{2p,0}(1)$ for $p \gg 1$. Equating the two expressions (37) and (45) for the same resolvent furnishes another identity for generalized hypergeometric functions. Asymptotic logarithmic behavior of the Green’s function $\mathcal{G}_{p,q}$ has been studied in detail [4, 22].

### 3.2 Triangular and honeycomb lattices

Another class of two-dimensional lattices that we consider next are the triangular and honeycomb lattices, which are closely related. Indeed, they can be obtained from the same resolvent by changing the definition of the angular variables. In order to compute the Green’s function on triangular and honeycomb lattices [15, 24, 25] let us start from the integral

$$H_{r,s}(t) = \frac{1}{(2\pi \xi)^2} \oint_{|x| = 1, |y| = 1} \oint_{|x| = 1, |y| = 1} \frac{dx \ dy}{x \ y \ t - (x + y + 1/xy)(1/x + 1/y + xy)} x^r y^s.$$

(48)
over unit circles in the $x$- and $y$-planes. For the honeycomb lattice with $N = \{(1, 0), (0, 1), (-1, -1)\}$ and $N = 3$. Choosing angular variables $\theta_1$ and $\theta_2$ such that

$$e^{i\theta_1} = x/y, \quad e^{i\theta_2} = xy^2,$$

we have $x^2y = e^{i(\theta_1+\theta_2)}$. Consequently, the Green’s function on a honeycomb lattice

$$G_{p,q}^\bigcirc = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{ip\theta_1+iq\theta_2}d\theta_1d\theta_2}{3 - (\cos \theta_1 + \cos \theta_2 + \cos(\theta_1 + \theta_2))}$$

is written in terms of (48) as

$$G_{p,q}^\bigcirc = -6H_{p+q,2q-p}(9).$$

Similarly, for the triangular lattice with $N = \{(-2, 0), (-1, \pm 1)\}$ and $N = 6$. With the choice of angular variables as

$$x^2y = e^{2i\theta_1}, \quad xy^2 = e^{i(\theta_1+\theta_2)},$$

we have $x/y = e^{i(\theta_1-\theta_2)}$. The Green’s function on the triangular lattice

$$G_{p,q}^\bigtriangleup = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{ip\theta_1+iq\theta_2}d\theta_1d\theta_2}{6 - 2(\cos 2\theta_1 + \cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2))}$$

is related to (48) by

$$G_{p,q}^\bigtriangleup = -\frac{3}{2}H_{p,\frac{p+3a}{2}}(9).$$

It thus suffices to consider the integral (48) which is again evaluated by first expanding the denominator as

$$H_{r,s}(t) = (2\pi i)^2 \sum_{n=0}^{\infty} t^{-1-n} \oint_{|x|=1} \oint_{|y|=1} \frac{dx}{x} \frac{dy}{y} x^r y^s (x + y + \frac{1}{xy})^n (\frac{1}{x} + \frac{1}{y} + xy)^n$$

$$= (2\pi i)^2 \sum_{n=0}^{\infty} \sum_{i+j \leq n, a+b \leq n} t^{-1-n} \oint_{|x|=1} \oint_{|y|=1} \frac{dx}{x} \frac{dy}{y} x^{r+2a+b-2i-j} y^{s+a+2b-i-2j}$$

and evaluating residue by collecting the constant part of the integrand by choosing

$$i = 3a + (2r - s)/3, \quad j = b + (2s - r)/3,$$

to derive

$$H_{r,s}(t) = \sum_{n=0}^{\infty} \sum_{i+j \leq n} t^{-1-n} \frac{n!n!}{i!j!(n-i-j)!(i+s-2a)!(j+r-2b)!(n-i-j+r+s)!}.$$ (57)

Inserting the appropriate values of $r$ and $s$ we then have,

$$H_{p,q,2q-p}(t) = \sum_{n=0}^{\infty} \sum_{i+j \leq n} t^{-1-n} \frac{n!n!}{i!j!(n-i-j)!(i-p)!(j+p-q)!(n-i-j+q)!}.$$ (58)
for the honeycomb lattice and
\[ H_{p,\frac{p+q}{2}}(t) = \sum_{n=0}^{\infty} \sum_{i,j=0}^{n} t^{-1-n} \frac{n! n!}{i! j! (n-i-j)! (j-q)! (i+q-p/2)! (n-i-j+p+q/2)!} \] (59)

for the triangular lattice. The latter can further be written as \( H_{p',q',2q'-p'} \), with \( p' = q \) and \( q' = (p+q)/2 \), by swapping \( i \) and \( j \) in the sum. Hence we only need to consider \( H_{p+q,2q-p}(t) \), which can be written in terms of gamma functions as
\[ H_{p+q,2q-p}(t) = \sum_{i,j,k=0}^{\infty} \frac{\Gamma(1+i+j+k)^2}{i! j! k! \Gamma(1-p+i)\Gamma(1+p-q+j)\Gamma(1+q+k)} t^{-1-i-j-k}, \] (60)

where the domain of the summing indices are extended to infinity taking into account the poles of gamma functions with non-positive arguments as before. The series can be rewritten as
\[ H_{p+q,2q-p}(t) = \frac{1}{\Gamma(1-p)\Gamma(1+q)\Gamma(1+p-q)} \frac{1}{t^{1-p-q}} \frac{1}{t^{1+1-q}} \frac{1}{t^{1+p}} \frac{1}{F_C^{(3)}} \left( \frac{1,1}{1-t,1-t,1-t} \right), \] (61)

for \(|t| > 3\), where \( F_C^{(3)} \) denotes the third Lauricella function in three variables. While the expression is not in the domain of interest for the evaluation of the Green’s function, its values near \( t = 1 \) may be obtained by analytic continuation.

The result for Kagome and dice lattices are obtained by the same \( H \) at other values of \( t \) [26].

### 4 Higher dimensions

The method generalizes to three and higher dimensions as well. In the case of a body-centered cubic lattice in \( D \) dimensions, for example, with \( \mathcal{N} = \{\pm 1, \pm 1, \cdots, \pm 1\} \) and \( N = 2^D \), the Green’s function (9) is
\[ G_{r_1,r_2,\cdots,r_D} = \left( \frac{1}{2\pi} \right)^D \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_D \frac{e^{i p \cdot \theta}}{2^D - 2^D \prod_{i=0}^{D-1} \cos \theta_i}. \] (62)

Defining angular variables as \( x_i = e^{\theta_i} \), for \( i = 1, 2, \cdots D \) we write this as
\[ G_{r_1,r_2,\cdots,r_D} = H_{r_1,r_2,\cdots,r_D}(1), \] (63)

with the resolvent defined as a contour integral as before, namely,
\[ H_{r_1,\cdots,r_D}(t) = \left( \frac{1}{2\pi i} \right)^D \oint_{|x_1|=1} \cdots \oint_{|x_D|=1} \frac{dx_1}{x_1} \cdots \frac{dx_D}{x_D} \frac{x_1^{r_1} \cdots x_D^{r_D}}{2^D t - \prod_{i=0}^{D-1} (x_i + 1/x_i)}. \] (64)

9
Evaluating the contour integral in a similar manner as in the previous cases through expansions using
the geometric series and binomial series seriatim, finally calculating the residue yields

\[ H_{r_1, \ldots, r_D}(t) = \frac{1}{\pi^D t} \prod_{i=1}^{D} \frac{1}{r_i} \cos \left( \frac{\pi r_i}{2} \right) 2D+1F_{2D} \left( \begin{array}{c} D+1 \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2} \end{array} \mid \frac{1}{t^2} \right) \]

\[ + \frac{1}{\pi^D t^2} \prod_{i=1}^{D} \frac{1}{1-r_i^2} \cos \left( \frac{\pi r_i}{2} \right) 2D+1F_{2D} \left( \begin{array}{c} D+1 \\ \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \ldots, \frac{3}{2} \end{array} \mid \frac{1}{4t^2} \right) \]

in terms of the hypergeometric function.

## 5 Inclusion of next-nearest neighbor hopping

The scope of the method under consideration is even more general. It depends on writing cosines of
angular variables in terms of phases so as to interpret the integration over each angular variable as a
contour integration on a complex plane. This aspect generalizes to cases with longer range hopping
as well. Let us demonstrate this by evaluating the Green’s function on a one-dimensional lattice with
next-nearest neighbor hopping, namely,

\[ G_{\text{NNN}}(\tau_1, \tau_2) = \frac{1}{2\pi} \int_0^{2\pi} e^{ir\theta} d\theta \]

where \( \tau_1 \) and \( \tau_2 \) are the hopping strengths of the nearest and the next-nearest neighbors, respectively.
Similar to the previous cases the Green’s function can be written in terms of the contour integral

\[ H_{\text{NNN}}(\tau_1, \tau_2) = \frac{1}{\tau_1} \oint_{|x|=1} \frac{dx}{x} \frac{dx'}{x'} \frac{x^r}{x^2 - (x + 1) - \frac{\tau_2}{\tau_1}(x^2 + 1)} \]

as \( G_{\text{NNN}}(\tau_1, \tau_2) = \frac{1}{\tau_1} H_{\text{NNN}}(\tau_1, \tau_2) \). Following our method of evaluating the contour integral as the
coefficient of the constant term in a series expansion of the integrand we obtain

\[ H_{\text{NNN}}(\tau_1, \tau_2) = \sum_{n,n',b,a=0}^{\infty} \left( \frac{\tau_2}{\tau_1} \right)^{n'} \left( \frac{\tau_1}{2} \right)^{1+n} \frac{\Gamma(1+n)}{\Gamma(1+b)\Gamma(1+a)\Gamma(1+n'-b)\Gamma(1+n-n')}, \]

with the indices satisfying the constraint

\[ r + 4b + 2a - n' - n = 0. \]

Writing \( n - n' - b = c \) and \( n' - b = n'' \) the constraint is solved for \( n'' \) as

\[ n'' = b + \frac{r + a - c}{2}. \]
Substituting this in the series (68) and using the duplication formula (18) in the numerator we obtain the expression for the resolvent as

\[ H_{r}^{\text{NNN}}(\tau_{1}, \tau_{2}) = \frac{\tau_{2}^{r/2} \tau_{1}^{2}}{2\sqrt{\pi}} \sum_{a, b, c = 0}^{\infty} \frac{(\tau_{1}^{2} \tau_{2})^{a}}{a!b!c!} \left( \frac{\tau_{2}}{\tau_{1}} \right)^{2b} \left( 1 + \frac{3a+c+r}{4} + b \right) \Gamma \left( \frac{1}{2} + \frac{3a+c+r}{4} + b \right) \Gamma \left( 1 + \frac{a+c}{2} + b \right). \]  

(71)

This series can be expressed in terms of hypergeometric functions. In particular, effecting the sum over \( b \) it can be expressed as an infinite series in the Gauss’ hypergeometric function \( _2F_1 \). However, the particular manner of arranging the terms affects the domain of convergence of the series, which in turn depends on the physical situation under consideration. The Green’s function for various limits of the coupling parameters \( \tau_{1} \) and \( \tau_{2} \) can be studied using the series by expressing it as Barnes’ integral and deforming contours.

6 Discussion

In this article we have studied Green’s function on lattices as the kernel of the discrete Laplacian operator. We present a means to evaluate the resolvent exemplified by square lattices in any dimension as well as the triangular and the honeycomb lattices in two dimensions. The Green’s function, written as a Fourier transform is first expressed in terms of a contour integral in as many variables as the dimension of the lattice. The contour integral is evaluated as the residue of the integrand. The residue, defined as usual as the coefficient of the terms with inverses of the integration variables in the Laurent expansion of the integrand is obtained as a series in every case. We identify these series as combination of generalized hypergeometric functions for the square lattices. For the two-dimensional triangular and honeycomb lattices the Greens’ function is expressed in terms of the Lauricella function of the third kind, which is a generalization of hypergeometric functions in three variables. As a consequence of being able to express the Green’s function in terms of generalized hypergeometric functions, for the two-dimensional square lattice we exhibit the cancellation of the singularity of the self-energy term, which is subtracted from the Green’s function to impose translational symmetry, using existing results on analytic continuation of zero-balanced generalized hypergeometric functions.

We show that depending on the way of arranging terms of the series the same Green’s function may be expressed as different combinations hypergeometric functions. This provide identities between hypergeometric functions. Physically, moreover, it corresponds to respecting or breaking the symmetries of the lattice by the Green’s function. The form obtained by breaking the isotropy of the lattice is better-suited for studying the asymptotics, namely, the behavior of the Green’s function at large distances [22, 27]. Since the resolvents are written in terms of hypergeometric functions in all the cases considered here, it is possible to write them as solutions to certain Fuchsian equations in the spectral parameter using the differential equations satisfied by the corresponding hypergeometric functions. It will be interesting to relate these to geometry by interpreting the contour integrals as periods of appropriate algebraic varieties [15, 19] and the Fuchsian equation as the equation for existence of flat connection in the moduli space.

We expect the method presented here to be of use in various other cases as well.
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References

[1] E. W. Montroll, “Theory of the Vibration of Simple Cubic Lattices with Nearest Neighbor Interactions”, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 3: Contributions to Astronomy and Physics, University of California Press, 1956, pp. 209–246.

[2] D. Atkinson and F. J. van Steenwijk, “Infinite resistive lattices”, Am. J. Phys. 67, 486 (1999).

[3] J. Cserti, “Application of the lattice Greens function for calculating the resistance of an infinite network of resistors”, Am. J. Phys. 68, 896 (2000).

[4] C. Itzykson, J.-M. Drouffe “Statistical Field Theory”, Vol. 1, CUP, 1989.

[5] E. V. Ivashkevich and V. B. Priezzhev, “Introduction to the sandpile model”, Physica A 254 (1998) 97.

[6] V. S. Poghosyan, S. Y. Grigorev, V. B. Priezzhev, P. Ruelle, “Logarithmic two-point correlators in the Abelian sandpile model”, J.Stat.Mech.1007:P07025,2010.

[7] M. Lüscher, P. Weisz, “Coordinate space methods for the evaluation of Feynman diagrams in lattice field theory”, Nucl.Phys. B445 (1995) 429; arxiv:hep-lat/9502017.

[8] S. Necco, R. Sommer, “The $N_f = 0$ heavy quark potential from short to intermediate distances”, Nucl.Phys. B622 (2002) 328; arXiv:hep-lat/0108008.

[9] M. Möller et. al., “Efficient computation of lattice Green functions for models with longer range hopping”, J. Phys. A: Math. Theor. 45 (2012) 115206.

[10] G. S. Joyce and R. Delves, “On the Exact Evaluation of the Face-Centred Cubic Lattice Green Function”, J. Stat. Phys. 145 (2011) 613.

[11] M. Berciu, “On computing the square lattice Greens function without any integrations”, J. Phys. A: Math. Theor. 42 (2009) 395207.

[12] B. A. Mamedov and I. M. Askerov, “Accurate Evaluation of the Cubic Lattice Green Functions Using Binominal Expansion Theorems”, Int J Theor Phys 47 (2008) 2945.

[13] Z. Maassarani, “Series expansions for lattice Green functions”, J.Phys. A33 (2000) 5675.

[14] S. Hollos and R. Hollos, “Some Square Lattice Green Function Formulas”, arXiv:cond-mat/0508779.
[15] A. J. Guttmann, “Lattice Green’s functions in all dimensions”, J. Phys. A: Math. Theor. 43 (2010) 305205.

[16] W. Bühring and H. M. Srivastava, “Analytic continuation of the generalized hypergeometric series near unit argument with emphasis on the zero-balanced series”, Themistocles M. Rassias (Ed.), Approximation Theory and Applications, Hadronic Press, Palm Harbor, FL 34682-1577, U.S.A., ISBN 1-57485-041-5, 1998, pp. 17-35. arXiv:math/0102032.

[17] S. L. Skorokhodov, “Symbolic Transformations in the Problem of Analytic Continuation of the Hypergeometric Function $pF_{p-1}(z)$ to the Neighborhood of the Point $z = 1$ in the Logarithmic Case”, Program. Comput. Softw. 30 (2004) 150.

[18] “NIST Handbook of Mathematical Functions”. Also: NIST Digital Library of Mathematical Functions. url: http://dlmf.nist.gov,

[19] K. Ray and S. Sen, “An Algebraic Geometry Method for Calculating DOS for 2D tight binding models”, Adv. Theor. Math. Phys. 17 (2013) 1343. arXiv:1205.1122 [cond-mat.other]

[20] N. M. Temme, “Large Parameter Cases of the Gauss Hypergeometric Function”, arXiv:math/020506

[21] R. B. Paris, “Asymptotics of the Gauss hypergeometric function with large parameters, I”, J. Class. Anal. 2 (2013) 183. “Asymptotics of the Gauss hypergeometric function with large parameters, II”, ibid 3 (2013) 1

[22] S. Mahieu and P. Ruelle, “Scaling fields in the two-dimensional Abelian sandpile model”, Phys. Rev. E 64 (2001) 066130

[23] E. T. Whittaker and G. N. Watson, “A course of modern analysis”, 4th ed. §§13.6.

[24] N. Azimi-Tafreshi, H. Dashti-Naserabadi, S. Moghimi-Araghi, P. Ruelle “Abelian Sandpile Model on the Honeycomb Lattice”, J.Stat Mech.1002:P02004,2010.

[25] V. V. Papoyan, A. M. Povolotsky “Renormalization Group Study of Sandpile on the Triangular Lattice”, Physica A, 246(1997)241.

[26] R. Shrock and F. Y. Wu, “Spanning trees on graphs and lattices in d dimensions”, J. Phys. A: Math. Gen. 33 (2000) 3881.

[27] P.A. Martin, “Discrete scattering theory: Greens function for a square lattice ”, Wave Motion 43 (2006) 619.