A free-group valued invariant of free knots

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Abstract

The aim of the present paper is to construct series of invariants of free knots (flat virtual knots, virtual knots) valued in free groups (and also free products of cyclic groups)

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1 Introduction. Preliminaries

Knots are encoded by Gauss diagrams modulo Reidemeister moves. For classical knots one imposes the planarity restriction on the set of Gauss diagrams. Once we forget planarity, we get virtual knots. Gauss diagrams have all chords endowed with two bits of information: local writhe number and the direction from one point (“over”) to the other point (“under”). If we abandon these two bits of information, we get arbitrary Gauss diagrams modulo the three Reidemeister moves as depicted in Fig. 1, top.

The other way to encode Gauss diagrams is to use 1-component framed 4-graphs. For more details, see [5]. In the bottom part of Fig. 1 we show the Reidemeister moves in the language of framed 4-graphs.

One can also define long free knots by breaking a framed 4-graph at a point and pulling the two ends to the infinity; certainly, for long knots it is not allowed to perform moves over the infinity.

We shall often switch from free knots to long free knots; the main advantage of free knots is the presence of a reference points; one can naturally see that if we associate a word to a free knot, we can associate a cyclic word to a (compact free knot); yet another advantage of free knots is the possibilities to fix some “coordinate system” which allows to “justify” the names of letters; actually, for compact free knots the word associated to letters will be sometimes well defined

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only up to some renaming letters, which is some outer automorphism of the group in question; for long free knots there is no such problem.

Here by components of a framed 4-graph we mean unicursal curves. The same can be used to define free links: we take similar (Gauss) chord diagrams with many core circles and arbitrary framed 4-graphs modulo Reidemeister moves.

Free knots (initially introduced by Turaev [9]) under the name of homotopy classes of Gauss words were first conjectured to be trivial; it was disproved in [5], see also [2].

The parity theory turns out to be a very powerful tool in the study of free knots and links, and in [5] (and later in [3], [7]) we demonstrated the following features of parity.

1. One can construct picture-valued invariants of free knots (by pictures we mean linear combinations of free knot diagrams)

2. There is an invariant called parity bracket, $[\cdot]$, which for some free knot diagrams $K$ gives us

   $$[K] = K.$$

3. The above formula gives rise to the following principle: if a diagram is complicated enough then it realises itself. The simplest variant of this principle is: If all crossings of the diagram $K$ are odd and there is no way to apply the second decreasing move to $K$ then any diagram $K'$ equivalent to $K$ admits a smoothing which is identical to $K$. 

Figure 1: Gauss diagrams and Reidemeister moves
4. We often see that locally minimal free knot diagrams (those which do not admit decreasing Reidemeister moves) are globally minimal in a strong sense (e.g., are contained in any equivalent diagrams).

5. Such results can be partially extended to the realm of cobordisms: if free knots can be capped by some “folded” disc then they can be capped by a disc in a simple way; for more details see \cite{I}.

6. Knots with parity behave like links: one can explicitly construct 2-fold coverings over free/flat/virtual/pseudo/quasi knots by links. \cite{3}.

Actually, the parity appears whenever one has some additional “$\mathbb{Z}_2$-homology” condition: roughly speaking, one has “even” and “odd” homology classes associated to crossings in a certain way.

The property “if a diagram is complicated enough then it realises itself” makes free knots similar to free groups (see \cite{7}). Indeed, if we have a free group $F^n$ in generators, say, $a_1, \cdots, a_n$ then for any reduced word $W$ any word $W'$ equivalent to $W$ contains $W$ as a “subword”, i.e., $W$ can be obtained from $W'$ by iterative cancellation of some pairs of $a_j a_j^{-1}$ or $a_j^{-1} a_j$.

Hence, there is an obvious analogy between free knots and free words; here we have:

1. Reductions $\iff$ second decreasing Reidemeister moves;
2. Reduced words $\iff$ minimal representatives of diagrams.

Certainly, the above correspondence is not $1 \leftrightarrow 1$. We want to exploit this correspondence in both directions: we try to study groups by using knots and knots by using groups.

Here we concentrate on the

1. Free knots $\rightarrow$ elements of a free group;
2. (Compact) free knots $\rightarrow$ conjugacy classes of a free group.

The general plan to be realised for topological problems is as follows:

1. First, we look at some “homological” conditions;
2. These conditions allow one to get a certain “labels” $l_1, l_2, \cdots, l_n$
3. Construct the free group $\mathbb{Z}_{l_1} \ast \cdots \ast \mathbb{Z}_{l_n}$ or the free product $\mathbb{Z}_2 \ast \cdots \ast \mathbb{Z}_n$ of generators $a_1, \cdots, a_n$ corresponding to $l_1 \cdots l_n$.
4. Try to prove that when we apply equivalence (say, Reidemeister moves) to the initial topological objects (say, knots) then different generators in the image do not commute.
In fact, the last step is the most complicated one. Actually, it is not too
difficult to grasp some homological information for crossings so that the two
crossings related by the second Reidemeister move are cancelled as follows:

\[ a_i a_i^{-1} = 1. \]

Having some non-trivial homological information, one can avoid writing down
generators corresponding to the first Reidemeister moves: if there were such
generators, say, \( a_j \), they would give rise to the painful relation

\[ a_j = 1. \]

The hardest problem here is the existence of the third Reidemeister move. It
gives rise to the same transformation of a word in three places.

In each of these three places the word is transformed by the commutativity
relation \( a_j a_i = a_i a_j \) which does not allow one to construct a genuine free group invariant.

In the present paper, we shall show the very first instance how this difficulty
can be overcome and how one can get

1. From homology to homotopy;
2. From free abelian groups to free groups.

In the present paper, we shall construct various types of letters
which are all due to the existence of just one source of homological information, the Gaussian parity, which exists for free knots.

This paper is one of the series of papers having the goal to study classical objects (knots, links, their cobordisms) by using virtual knot methods. This is in some sense “the last piece of the whole puzzle”: we get invariants of free knots. The first pieces of the puzzle will start from classical knots and transform them to “virtual” objects having non-trivial parity etc.

Here we note that knots in the cylinder already possess enough structure to use virtual knot theory methods; on the other hand, cylinders naturally appear in the study of classical knots and links (For more details, see [7]).

\[ ^1 \text{Actually, the relation } a_j^2 \text{ is less painful, but sometimes we may count say, only left ends and not right chord ends} \]
1.1 The structure of the paper

The paper is organised as follows. In Section 2, we construct (some old) the invariants of free knots valued in free groups and prove its invariance.

Also, we informally discuss various ways of generalisation for this invariant.

In Section 3, we calculate the main example which shows that free knots can give rise to “highly non-commutative” words in the free group.

We conclude the paper by a discussion of further directions like cobordisms and Legendrian knots.

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2 The construction of the invariants.

The invariance proof

In the present section, we describe the whole procedure how to construct the invariant of free knots valued in free groups. To make the process clear and accessible for other types of parities (say, homological), we describe it step-by-step.

1. The first step will consist in construction of some integer-valued invariant which can be reduced to some count: we count the number of chords which are “non-trivial” by some reasons.

2. The second step will give rise to the invariant valued in the infinite dihedral group which we write down as: $G = \langle a, b, b'|a^2 = b^2 = (b')^2 = 1, ab = b'a \rangle$.

3. The third step will give rise to an invariant valued in a true group of exponential growth. The key ingredient is the possibility to get rid of the relations $b^2 = b'^{-2}$ by renaming letters and rewriting words.\(^2\)

2.1 The first step

Let $K$ be a Gauss diagram; we say that two chords $c, c'$ are linked if the endpoints of $c$ alternate with the endpoints of $c'$. Sometimes we shall write $\langle c, d \rangle = 1 \in \mathbb{Z}_2$ if $c, d$ are linked and $\langle c, d \rangle = 0 \in \mathbb{Z}_2$, otherwise.

\(^2\)Actually, this procedure can be accurately defined by using wreath products of groups with $\mathbb{Z}_2$ but we don’t want to give much detail in this short note.
A chord $c$ is **even** if the number of chords it is linked with, is **even**, and **odd** if it is odd.

Now, the reader can immediately see that:

1. the chord does not change its parity when a Reidemeister-3 move is applied;
2. if two chords are cancelled by a Reidemeister-2 move then they have the same parity;
3. A chord taking part in the first Reidemeister move is even.
4. The number of odd chords of a Gauss diagram is even.

We say that an even chord $c$ has **first type** if it is linked with oddly many odd chords, and **of the second type** if it is linked with evenly many odd chords.

Let $K$ be an oriented long knot diagram. We enumerate the endpoints as they appear according to the orientation; we say that an odd chord is **of the first type** if both endpoints of it are in odd positions and it is linked with evenly many even chords or if both endpoints of it are even and it is linked with oddly many even chords.

Otherwise we say that the odd chord is **of the second type**.

Actually, the notion of **type** is an instance of the notion of **hierarchy** where odd chords are of hierarchy 0, even chords of the first type are of hierarchy 2 and other even chords of type 2 have hierarchy $n$, where $n \in \{2, 3, \cdots \} \cup \{\infty\}$. [12].

The results of the present paper can be definitely extended if we use more information about the hierarchy, but here the type will be sufficient.

Now, we define $l$ to be twice the number of odd chords of the first type minus twice the number of odd chords of the second type.

**Theorem 1.** If $K_1$, $K_2$ represent the same long free knot then $l(K_1) = l(K_2)$. If two long free knots are obtained from some free knot $K$ by breaking at two different points then $l(K_1) = \pm l(K_2)$.

The proof follows from the step-by-step check of the Reidemeister moves. For more details, see [6].

Note that the here $l$ is nothing but just bare count (if we deal with compact knots, not long ones, we get just relative signs).

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We take twice the number of chords in order to agree with [6].
2.2 The second step

Now, we undertake an attempt to encode the above information about the bare count of odd chords by means of a (non-abelian) group. Let $K$ be a long Gauss diagram. We create the word $\phi(K)$ by using the following rule: we follow the orientation of $K$ and whenever we meet an even chord, we write $a$, when we have an odd chord, we write $b$ for ends of chords of the first type or $b'$ for ends of chords of the second type see Fig. 4 below.

Theorem 2. If $K_1, K_2$ are equivalent as long knots then $\phi(K_1) = \phi(K_2)$ in $G$.

Proof. One just has to check that the first Reidemeister move gives: $aa = 1$, the second Reidemeister move gives $aa = 1$ or $bb = 1$ or $b'b' = 1$; finally, the third Reidemeister move gives rise to three substitutions each of which is either identical or $ab = b'a$ or $ab' = ba$.

Hence, we see that the information from a group obtained here is not stronger than just the invariant $l$.

Indeed, when we look at the Cayley graph of the group $\mathbb{Z}$, one can see that the group is commensurable with the infinite dihedral group. However, the number $l$ now turns into a geometrical interpretation as the distance in the Cayley graph.

2.3 The third step

Let us write more carefully at the map constructed on the second step. Can it be enhanced? In fact, we are very much restricted in our attempt to write down all maps in terms of groups and group homomorphisms.

The problem is that when we write down the relation $b'a = ab$, we want this relation to be a group relation, i.e., we want this relation to give rise to a group homomorphism, in particular, this relation yields $bab'a = 1$. 

Figure 4: A long free knot $K$ and the word $bababb'ab'ab'ab$
The same about $bb' = 1$; this relation follows from the fact that we want two chord ends for the Reidemeister-2 move to be cancelled.

Actually, we may treat $b$ in different positions differently; to describe this completely from the algebraic point of view, one should use lots of wreath products and more elaborate machinery; we want to omit it for the first short paper.

However, one can consider the following group

$$G' = \langle a, b | a^2 = 1, ab = (b')^{-1}a \rangle$$

Note that unlike $G$, the group $G'$ has exponential growth, and $b, b'$ do not commute.

Now, the invariant

$$\psi(K)$$

is obtained from $\phi(K)$ by using the rule:

- replace all letters $b'$ in even positions with $(b')^{-1}$ and replace $b$ in even positions with $b^{-1}$.

**Remark 1.** Actually, we can go even a bit further and get rid of the relation $a^2 = 1$ if we pass from any **even chords** to special **even chords**, but for our purposes (exponential growth) it suffices just to work with $G'$.

**Theorem 3.** If two long knots $K_1$ and $K_2$ are equivalent then $\psi(K_1) = \psi(K_2)$ as elements of the group $G'$.

**Proof.** The proof follows from the direct check of the Reidemeister moves (actually, this is a minor modification of Theorem ??).

Looking at Fig. 4, we readily see that for the knot $K$ given there we have

$$\psi(K) = bababab(b')^{-1}a(b')^{-1}a(b')^{-1}a(b')^{-1}$$

## 3 An example

Let us consider the free knot $K$ given in Fig. 4.

The invariant $l = \pm 8$ says that it is not trivial and it is **not slice** because if has four chords of the first type (if we look at the compact knot, we say that we have four chords of the same type), so they can not be cancelled by means of Reidemeister moves and can not be spanned by a disc. On the Cayley graph these eight chord ends sum up and give 8; this number 8 is invariant.

The group $G$ actually gives the same information: though it is not abelian, it has an abelian group of finite index, hence the only information one can get in this group is just $(bb')^k$ (in our case $k = 4$).

From the construction of the group $G'$ we immediately get

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4There are various cobordism obstructions which originate from the count of some “odd things” with writhe numbers serving signs [4], however, here odd chords themselves have other signs coming from other reasonings; the reader can readily expand these invariant for the case of virtual knots.
Theorem 4. Let $K_1, K_2$ be two long free knots. Then for the connected sum $\psi(K_1 \# K_2) = \psi(K_1) \cdot \lambda \psi(K_2)$, where $\lambda$ is some automorphism of the group $G'$.

Here one can naturally see that the invariant $\psi$ can easily detect

1. Free knot invertibility;
2. Any sorts of free knot mutations.

Of course, having this done for free knots, we have lots of consequences for virtual knots. Certainly, the next goal will be to do the same for classical knots and for knots in 3-manifolds.

4 Further directions

Here we sketch some directions of further investigation we are undertaking now. In each of the problems we highlight the main problem which seems to be tractable by using the approach given in the present paper

1. First, it is rather obvious that free knots almost never commute. Indeed, the group $G'$ is non-abelian; to show that for long knots $K_1, K_2$ we have $K_1 \# K_2 \neq K_2 \# K_1$ we have to look carefully at the group $G'$ constructed above.

2. Certainly, the groups constructed here can be enhanced by large if we use not only parity but also parity hierarchy; one can construct invariants of free knots valued in free products of groups whose generators are “responsible for” the existence of certain “sub-knot structures”.

3. The invariant $l$ (counted with $\pm$ sign) is certainly a sliceness obstruction for free knots but it can’t say anything about the slice genus of free knots. The problem is that having a non-zero value of $l$ (say, $l = 8$) and applying cut-and-paste techniques, we can “roughly” split 8 into the sum 4 + 4 and then paste 4 with $-4$ and get 0.

Having groups which are close to free groups one can use commutator length techniques and estimate from below the number of operations one needs to cut the word into, in order to compose a trivial element.

4. Certainly, the approach given here gives a lot for virtual objects (upgrades from free knots to virtual knots can be done straightforwardly).

We expect to have invariants for classical knots. This is to be done in a sequel of the present paper.

The key idea is to concentrate on knots inside the cylinder; this will give rise to parities, indices and other non-trivial structures. As a result, we expect maps (not necessarily homomorphic) from classical knots to free groups.
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