Ladder operators and coherent states for the trigonometric Pöschl-Teller potential

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Abstract

In this work we make use of deformed operators to construct the coherent states of some nonlinear systems by generalization of two definitions: i) As eigenstates of a deformed annihilation operator and ii) by application of a deformed displacement operator to the vacuum state. We also construct the coherent states for the same systems using the ladder operators obtained by traditional methods with the knowledge of the eigenfunctions and eigenvalues of the corresponding Schrödinger equation. We show that both methods yield coherent states with identical algebraic structure.

1 Introduction

In this work we make use of two different methods to construct ladder operators for different potentials and exemplify the methodology with a well known example. On the one hand, we make use of the knowledge of the eigenfunctions of the corresponding Schrödinger equation and apply a standard procedure for their construction [1,2]. On the other hand, we make
use of the idea of deformed oscillators introduced by Man’ko [3] and, by
making the appropriate choice of the deformation function, a Hamiltonian of
the harmonic oscillator form written in terms of deformed operators yields
the eigenvalues of the corresponding Schrödinger equation. Once we have
constructed the ladder operators with each one of these methodologies, we
construct their coherent states as eigenstates of the annihilation operator and
by the application of the displacement operator to the vacuum state.

2 Ladder operators

In this section we use a standard procedure for the construction of ladder
operators for a system whose analytical solution is known. As an example
we consider a trigonometric potential given by:

\[ V(x) = U_0 \tan^2(ax) \]

where \( U_0 \) is the potential’s strength and \( a \) its range. The number of bound
states for this potential is infinite. It’s eigenfunctions and eigenvalues are [4]:

\[ \psi_n^\lambda(x) = \sqrt{\frac{a(\lambda + n)\Gamma(2\lambda + n)}{\Gamma(n + 1)}} (\cos(ax))^{1/2} P_{n+\lambda-1/2}^{1/2-\lambda}(\sin(ax)), \]

\[ E_n = \frac{\hbar^2 a^2}{2\mu} (n^2 + 2n\lambda + \lambda) = \hbar \omega(n + \frac{1}{2} + \frac{n^2}{2\lambda}), \]

where \( \mu \) is the mass of the particle, \( \omega = \hbar \lambda a^2/\mu \) and the parameter \( \lambda \)
is related to the potential strength and range by \( \lambda(\lambda + 1) = 2\mu U_0/h^2 a^2 \).

In the harmonic limit \( \lambda \to \infty \) and \( a \to 0 \) with \( \lambda a^2 = \mu \omega/\hbar \).

Making the change of variable \( u = \sin(ax) \) we can write the eigenfunctions
as

\[ \psi_n^\lambda(u) = \sqrt{\frac{a(\lambda + n)\Gamma(2\lambda + n)}{\Gamma(n + 1)}} (1 - u^2)^{1/4} P_{n+\lambda-1/2}^{1/2-\lambda}(u). \]

The action of the differential operator \( d/du \) acting upon the eigenfunctions
\( \psi_n^\lambda(u) \) can be obtained from its action upon the Legendre polynomials:

\[ (1 - x^2)\frac{dP_\nu^\mu(x)}{dx} = (1 + \nu)xP_\nu^\mu(x) - (\nu - \mu + 1)P_{\nu+1}^\mu(x) \]
Calling
\[ N_n^\lambda = \sqrt{\frac{a(\lambda + n)\Gamma(2\lambda + n)}{\Gamma(n + 1)}} = \sqrt{\frac{a(\lambda + n)\Gamma(2\lambda + n)}{n!}} \]
we get:
\[ \frac{d\psi_n^\lambda(u)}{du} = \left( \frac{n + \lambda}{1 - u^2} \right) \psi_n^\lambda(u) - \frac{2\lambda + n}{1 - u^2} \frac{N_n^\lambda}{N_{n+1}^\lambda} \psi_{n+1}^\lambda(u) \quad (6) \]
rearranging terms, substituting the normalization constant and defining \( \epsilon = \lambda + n \) we obtain the following relation between the eigenfunctions \( \psi_n^\lambda(u) \) and \( \psi_{n+1}^\lambda(u) \):
\[ \frac{1}{1 - u^2} \left( -\frac{d}{du} + \frac{\epsilon}{1 - u^2} \right) \sqrt{\frac{\epsilon + 1}{\epsilon}} \psi_n^\lambda(u) = \sqrt{(n + 1)(2\lambda + n)} \psi_{n+1}^\lambda(u) \quad (7) \]
so that, the operator \( M_+ \) defined by
\[ M_+ = (1 - u^2) \left( -\frac{d}{du} + \frac{\epsilon}{1 - u^2} \right) \sqrt{\frac{\epsilon + 1}{\epsilon}} \quad (8) \]
acting upon the function \( \psi_n^\lambda(u) \) yields
\[ M_+ \psi_n^\lambda(u) = m_+ \psi_{n+1}^\lambda(u), \quad m_+ = \sqrt{(n + 1)(2\lambda + n)}. \quad (9) \]
In order to construct the annihilation operator \( M_- \) we first make use of the relationship [19]
\[ (1 - x^2) \frac{dP_\nu^\mu(x)}{dx} = -\nu x P_\nu^\mu(x) + (\nu + \mu) P_{\nu-1}^\mu(x). \]
Acting upon the eigenfunctions \( \psi_n^\lambda(u) \) we obtain
\[ \frac{d\psi_n^\lambda(u)}{du} = -\frac{(n + \lambda)}{1 - u^2} \psi_n^\lambda(u) + \frac{n}{1 - u^2} \frac{N_n^\lambda}{N_{n-1}^\lambda} \psi_{n-1}^\lambda(u) \quad (10) \]
rearranging terms and substituting the explicit form of the normalization constants we get:
\[ (1 - u^2) \left( \frac{d}{du} + \frac{n + \lambda}{1 - u^2} \right) \sqrt{\frac{\lambda + n - 1}{\lambda + n}} \psi_n^\lambda(u) = \sqrt{n(2\lambda + n - 1)} \psi_{n-1}^\lambda(u) \quad (11) \]
and we can define the annihilation operator

\[ M_- = (1 - u^2) \left( \frac{d}{du} + \frac{u}{1-u^2} \epsilon \right) \sqrt{\frac{\epsilon - 1}{\epsilon}} \tag{12} \]

whose effect acting upon the wavefunction \( \psi_{n}^\lambda(u) \) is

\[ M_- \psi_{n}^\lambda(u) = m_- \psi_{n-1}^\lambda(u), \quad m_- = \sqrt{n(2\lambda + n - 1)} \tag{13} \]

In order to recover the harmonic limit we define the operators \( \hat{b}^\dagger, \hat{b} \) as:

\[ \hat{b}^\dagger = \frac{1}{\sqrt{2\lambda}} M_+, \quad \hat{b} = \frac{1}{\sqrt{2\lambda}} M_- \tag{14} \]

whose action upon the eigenfunctions \( \psi_{n}^\lambda \) is:

\[ \hat{b} \psi_{n}^\lambda = \sqrt{\frac{n(2\lambda + n - 1)}{2\lambda}} \psi_{n-1}^\lambda = b_- \psi_{n-1}^\lambda \tag{15} \]

and

\[ \hat{b}^\dagger \psi_{n}^\lambda = \sqrt{\frac{(n + 1)(2\lambda + n)}{2\lambda}} \psi_{n+1}^\lambda = b_+ \psi_{n+1}^\lambda. \tag{16} \]

The commutation relations between the operators \( \hat{b}^\dagger, \hat{b} \) are obtained from their action upon the eigenfunctions \( \psi_{n}^\lambda \):

\[ [\hat{b}^\dagger, \hat{b}] \psi_{n}^\lambda = \left( \frac{n(2\lambda + n - 1)}{2\lambda} - \frac{(n + 1)(2\lambda + n)}{2\lambda} \right) \psi_{n}^\lambda = -2\frac{(\lambda + n)}{2\lambda} \psi_{n}^\lambda \tag{17} \]

calling the operator

\[ \hat{b}_0 = \frac{\lambda + \hat{n}}{\lambda} = 1 + \frac{\hat{n}}{\lambda} \]

we obtain the commutation relations:

\[ [\hat{b}, \hat{b}^\dagger] = \hat{b}_0, \quad [\hat{b}, \hat{b}_0] = \frac{1}{\lambda} \hat{b}, \quad [\hat{b}^\dagger, \hat{b}_0] = -\frac{1}{\lambda} \hat{b}^\dagger. \tag{18} \]

In the harmonic limit the operators \( \hat{b}^\dagger, \hat{b} \) go into

\[ \hat{b}^\dagger \rightarrow -\sqrt{\frac{\hbar}{2\mu\omega}} \frac{d}{dx} + \sqrt{\frac{\mu\omega}{2\hbar}} x \tag{19} \]

and

\[ \hat{b} \rightarrow \sqrt{\frac{\hbar}{2\mu\omega}} \frac{d}{dx} + \sqrt{\frac{\mu\omega}{2\hbar}} x \tag{20} \]
as they should.
3 Deformed Oscillator

In this section we introduce deformed boson creation and annihilation operators $\hat{A}^\dagger$, $\hat{A}$ which differ from the usual harmonic oscillator operators $\hat{a}$, $\hat{a}^\dagger$ by a deformation function of the number operator, that is,

$$\hat{A} = \hat{a} f(\hat{n}) = f(\hat{n} + 1)\hat{a}, \quad \hat{A}^\dagger = f(\hat{n})\hat{a}^\dagger = \hat{a}^\dagger f(\hat{n} + 1). \quad (21)$$

In this work we will consider real and positive functions of the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$.

A Hamiltonian of the Harmonic oscillator form written in terms of the deformed operators becomes

$$H_D = \frac{\hbar \Omega}{2} \left( \hat{A}^\dagger \hat{A} + \hat{A} \hat{A}^\dagger \right) = \frac{\hbar \Omega}{2} \left( \hat{n} f^2(\hat{n}) + (\hat{n} + 1) f^2(\hat{n} + 1) \right). \quad (22)$$

The explicit form of the energy spectra can be fixed when one specifies the deformation function. As a first example let us consider the trigonometric Pöschl-Teller energy spectra given by Eq. 3. If we choose the deformation function as:

$$f^2(\hat{n}) = \frac{\hbar a^2}{2 \mu \Omega} (\hat{n} + 2\lambda - 1) \quad (23)$$

the deformed Hamiltonian given by Eq. 22 becomes

$$H_D = \frac{\hbar^2 a^2}{2 \mu} \left( \hat{n}^2 + 2\lambda \hat{n} + \lambda \right) \quad (24)$$

whose eigenvalues are identical with those of the trigonometric Pöschl-Teller potential given in Eq. 3.

The action of these operators over the states of number $|n\rangle$ give as result

$$\hat{A}|n\rangle = \sqrt{n(2\lambda + n - 1)}|n - 1\rangle, \quad \hat{A}^\dagger |n\rangle = \sqrt{(n + 1)(2\lambda + n)}|n + 1\rangle \quad (25)$$

notice the similarity with Eqs. 15 and 16. Because of this, if we take a hamiltonian of the harmonic oscillator form in terms of the operators $\hat{b}$, $\hat{b}^\dagger$ the eigenvalues will be the same as those shown in Eq. 3 i.e., they are isospectral.

Once we have fixed the deformation function, the operators $\hat{A}$, $\hat{A}^\dagger$ are determined and the algebraic properties of the system can be explored. The commutation relations between the deformed operators are

$$[\hat{A}, \hat{A}^\dagger] = \frac{\hbar a^2}{\mu \Omega} (\hat{n} + \lambda), \quad [\hat{A}, \hat{n}] = \hat{A}, \quad [\hat{A}^\dagger, \hat{n}] = -\hat{A}^\dagger, \quad (26)$$
notice that the set of operators \( \{ \hat{A}, \hat{A}^\dagger, \hat{n}, 1 \} \) is closed under the operation of commutation.

4 Coherent states

The first proposal for the construction of what is now known as a coherent state was done by Schrödinger in 1926 [5] in connection with the classical states of the quantum harmonic oscillator. Much later, in 1963 Glauber [6] constructed the eigenstates of the annihilation operator of the harmonic oscillator in order to study the electromagnetic correlation functions and showed that such states are enormously useful for quantum optics. At about the same time, Klauder [7] developed a set of continuous states in which the basic ideas of coherent states for arbitrary Lie groups were contained. In the early 70’s the complete construction of coherent states of Lie groups was achieved by Perelomov [8] and Gilmore [9]. The basic theme of this development was to connect the coherent states with the dynamical group for each physical problem. Since then, the idea of generalizing the concept of coherent states to arbitrary systems has been considered, one of the difficulties being that the different generalizations lead to different results [10, 11, 12]. For the harmonic oscillator there are three alternative definitions for the construction of its coherent states: i) as those states that saturate the dispersion relations \( \Delta x \Delta p = \hbar/2 \), ii) as eigenstates of the annihilation operator \( \hat{a} |\alpha\rangle = \alpha |\alpha\rangle \) and iii) by displacement of the vacuum state \( D(\alpha) |0\rangle = |\alpha\rangle \). In this work we consider the generalization of two of the abovementioned definitions.

4.1 Coherent states as eigenstates of the annihilation operator

Coherent states for the trigonometric Pöschl-Teller potential can be obtained as eigenstates of the annihilation operator \( \hat{b} \)

\[
\hat{b} |\alpha, \lambda\rangle = \alpha |\alpha, \lambda\rangle \tag{27}
\]

The states \( |n, \lambda\rangle \) form a complete set, then we can write the coherent state \( |\alpha, \lambda\rangle \) as:

\[
|\alpha, \lambda\rangle = \sum_{n=0}^{\infty} C_n |n, \lambda\rangle \tag{28}
\]
We have then, by application of the annihilation operator $\hat{b}$ to the coherent state
\[ \hat{b}|\alpha, \lambda\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n(1 + \frac{n-1}{2\lambda})}|n-1, \lambda\rangle = \alpha \sum_{n=1}^{\infty} C_{n-1}|n-1, \lambda\rangle \]
and we thus find the relationship between the coefficients $C_n$ and $C_{n-1}$
\[ C_n \sqrt{n \left(1 + \frac{n-1}{2\lambda}\right)} = \alpha C_{n-1}. \]
Further applications of the annihilation operator yield relations between $C_n$ and $C_{n-2}, C_{n-3}, \ldots$ until after $n$ applications we finally get a relationship between $C_n$ and $C_0$:
\[ C_n \sqrt{n(n-1) \cdots (1)(2\lambda + n - 1)(2\lambda + n - 2) \cdots (2\lambda)} = \alpha^n C_0. \]
Then, substituting into Eq. 28 the coherent state $|\alpha, \lambda\rangle$ can be written as:
\[ |\alpha, \lambda\rangle = C_0 \sum_{n=0}^{\infty} \alpha^n \frac{(2\lambda)^n \Gamma(2\lambda)}{n!\Gamma(2\lambda + n)}|n, \lambda\rangle \]
with $C_0$ a normalization constant.

4.2 Displacement Operator Coherent States

In this section we construct the coherent states by application of the generalized displacement operator upon the vacuum state. In order to do that it is necessary to make use of the commutation relations between the ladder operators and the number operator. With the ladder operators we constructed in section 1 we have the commutation relations:
\[ [\hat{b}, \hat{b}^\dagger] = \hat{b}_0 = 1 + \frac{\hat{n}}{\lambda}, \quad [\hat{b}, \hat{b}_0] = \frac{1}{\lambda} \hat{b}, \quad [\hat{b}^\dagger, \hat{b}_0] = -\frac{1}{\lambda} \hat{b}^\dagger, \]
since the set of operators $\{\hat{b}, \hat{b}^\dagger, \hat{b}_0\}$ is closed under commutation we can write:
\[ D_D(\alpha) = \exp[\alpha \hat{b}^\dagger - \alpha^* \hat{b}] = e^{\alpha \hat{b}^\dagger} e^{\alpha \hat{b}_0} e^{\alpha^* \hat{b}} \]
with complex functions $\alpha_+, \alpha_0, \alpha_-$ to be determined. Using the results of Ref. [14] we obtain:

$$D_D(\alpha) = \exp \left[ \frac{\alpha \tanh(|\alpha|/\sqrt{2\lambda})}{|\alpha|/\sqrt{2\lambda}} \hat{b} \right] \exp \left[ -2\lambda \ln(\cosh(|\alpha|/\sqrt{2\lambda})) \hat{b}_0 \right] \times \exp \left[ -\alpha^* \tanh(|\alpha|/\sqrt{2\lambda}) \hat{b} \right]$$

with $\alpha = |\alpha|e^{i\phi}$ and $\zeta = e^{i\phi} \tanh(|\alpha|/\sqrt{2\lambda})$, the deformed displacement operator can be written as:

$$D_D(\zeta) = \exp \left[ \zeta \sqrt{2\lambda} \hat{b} \right] (1 - |\zeta|^2)^{\lambda_0} \exp \left[ -\zeta^* \sqrt{2\lambda} \hat{b} \right]$$

(34)

Applying it to the vacuum state we obtain the coherent states

$$|\zeta\rangle = D_D(\zeta)|0, \lambda\rangle = (1 - |\zeta|^2)^{\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n! \Gamma(2\lambda)} \zeta^n |n, \lambda\rangle$$

(35)

4.3 Coherent states as eigenstates of the deformed annihilation operator

Let us now construct their coherent states as eigenstates of the deformed annihilation operator [13, 14]

$$\hat{A} |\alpha, f\rangle = \alpha |\alpha, f\rangle$$

(36)

As in section 4.1 we express them as an expansion in the basis of number states $\{|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots\}$

$$|\alpha, f\rangle = N_f \sum_{n=0}^{\infty} c_n^f |n\rangle,$$

(37)

substitution into Eq. 36 yields the following relation between the coefficients $c_n^f$ and $c_{n-1}^f$:

$$c_{n-1}^f |n\rangle \sqrt{n} = \alpha c_n^f$$

(38)

applying the annihilation operator $n$ times we obtain:

$$c_n^f |n\rangle \sqrt{n}! = \alpha^n c_0^f$$

(39)
where \( f(n)! = f(n)f(n-1) \cdots f(0) \).

Then the coherent state is given by:

\[
|\alpha, f \rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!f(n)!}} |n\rangle.
\]  

(40)

Substitution of the explicit form of the deformation function yields

\[
|\alpha, f \rangle = N_f \sum_{n=0}^{\infty} \sqrt{\left(2\lambda_0\right)^n \Gamma(2\lambda) \frac{\alpha^n}{n!\Gamma(2\lambda+n)}} |n\rangle
\]  

(41)

with \( N_f \) a normalization constant. Notice that the coherent states we have constructed with the knowledge of the deformation function have an identical structure as those obtained via the ladder operators in section 4.1.

4.4 Coherent states obtained via the deformed displacement operator

From the commutation relations given by Eqs. 26 we notice that the set of operators \( \{\hat{A}, \hat{A}^\dagger, \hat{n}, 1\} \) constitute a finite Lie algebra. For convenience we define

\[ g(\hat{n}; a, \lambda) = \hbar a^2(\lambda + \hat{n})/\mu\Omega = \frac{\lambda + \hat{n}}{\lambda_0} \]

with \( \lambda_0 = \mu\Omega/\hbar a^2 \) so that the commutation relations given in Eqs. 26 become

\[
[\hat{A}, g(\hat{n}; a, \lambda)] = \frac{\hat{A}}{\lambda_0}, \quad [\hat{A}^\dagger, g(\hat{n}; a, \lambda)] = -\frac{\hat{A}^\dagger}{\lambda_0}, \quad [\hat{A}, \hat{A}^\dagger] = g(\hat{n}; a, \lambda).
\]  

(42)

clearly, the set of operators \( \{\hat{A}, \hat{A}^\dagger, g(\hat{n}; a, \lambda)\} \) is closed under commutation and the deformed displacement operator \( D_D(\alpha) \) obtained by the replacement of the usual harmonic oscillator operators \( \hat{a}, \hat{a}^\dagger \) by their deformed counterparts can be expressed in terms of a product of exponentials. That is,

\[
D_D(\alpha) = \exp \left[ \alpha \frac{\tanh(\frac{\lambda}{\sqrt{2\lambda_0}})}{\sqrt{2\lambda_0}} \hat{A}^\dagger \right] \exp \left[ -2\lambda_0 \ln(\cosh(\frac{\lambda}{\sqrt{2\lambda_0}})) g(\hat{n}; a, \lambda) \right] \times \exp \left[ -\alpha^* \frac{\tanh(\frac{\lambda}{\sqrt{2\lambda_0}})}{\sqrt{2\lambda_0}} \hat{A} \right].
\]  

(43)
if we now define \( \zeta_0 = e^{i\phi_0} \tanh(|\alpha|/\sqrt{2\lambda_0}) \) and write \( \alpha = |\alpha|e^{i\phi_0} \) we obtain after a little algebra:

\[
D_D(\zeta_0) = \exp \left[ \zeta_0 \sqrt{2\lambda_0} \hat{A}^\dagger \right] \left( 1 - |\zeta_0|^2 \right)^{\lambda \log(\hat{a},a,\lambda)} \exp \left[ -\zeta_0^* \sqrt{2\lambda_0} \hat{A} \right] . \tag{44}
\]

The coherent state obtained by application of this deformed displacement operator upon the vacuum state yields

\[
|\zeta_0\rangle = D_D(\zeta_0)|0\rangle = (1 - |\zeta_0|^2)^\lambda \sum_{n=0}^{\infty} \frac{2^\lambda (2\lambda_0)^{n/2}}{\sqrt{n!}} f(n)|n\rangle \tag{45}
\]

replacing the explicit form of the deformation function we obtain finally

\[
|\zeta_0\rangle = (1 - |\zeta_0|^2)^\lambda \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}} \zeta_0^n |n\rangle . \tag{46}
\]

These coherent states have exactly the same algebraic structure as those obtained with the ladder operators \( \hat{b}, \hat{b}^\dagger \) constructed in section 4.2.

### 4.5 Other examples

We now consider a harmonic oscillator with an inverse square potential in two dimensions. The potential has the form

\[
V(r) = \frac{1}{2} \mu \omega^2 r^2 + \frac{\hbar^2}{2\mu} \frac{\alpha}{r^2}
\]

where \( \omega \) is the frequency of the oscillator and \( \alpha \) is the potential strength. As shown in [16], the eigenfunctions are

\[
R_n(\rho) = N_n \rho^s e^{-\frac{\rho^2}{4}} L_n^{2s}(\rho), \quad N_n = \sqrt{\frac{2n!}{\Gamma(n+2s+1)}}
\]

and the eigenvalues are

\[
E_n = 2(n + s + \frac{1}{2}),
\]

where \( s = \sqrt{\alpha + m^2}/2 \) with \( m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots \), \( L_n^{2s}(\rho) \) is the associated Laguerre Polynomial and the units used were \( \hbar = \mu = \omega = 1 \).
The ladder operators obtained after the application of the usual procedure are:

\[ \hat{L}_- = -\rho \frac{d}{d\rho} + s + \hat{n} - \frac{\rho}{2}, \quad \hat{L}_+ = \rho \frac{d}{d\rho} + s + \hat{n} + 1 - \frac{\rho}{2} \]  

(47)

whose action upon the eigenfunctions is:

\[ \hat{L}_- |n, \rho\rangle = \sqrt{n(n + 2s)} |n - 1, \rho\rangle, \quad \hat{L}_+ |n, \rho\rangle = \sqrt{(n + 1)(n + 2s + 1)} |n + 1, \rho\rangle. \]  

(48)

and their commutation relations are

\[ [\hat{L}_-, \hat{L}_+] = 2\hat{L}_0, \quad [\hat{L}_0, \hat{L}_-] = -\hat{L}_+, \quad [\hat{L}_0, \hat{L}_+] = \hat{L}_-. \]  

(49)

where the operator \( \hat{L}_0 = \hat{n} + s + \frac{1}{2} \).

Following the methodology of section 4.1 we obtain the coherent states:

\[ |\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!(2s+n)(2s+n-1)\cdots(2s+1)}} |n, \rho\rangle \]  

(50)

\[ = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!(2s+n+1)\Gamma(2s+1)}} \frac{\Gamma(2s+1)}{\Gamma(2s+n+1)} |n, \rho\rangle. \]

When using deformed operators for the trigonometric potential we found that a deformation function of the form given by Eq. 23 could reproduce the energy spectra. In this case there is no quadratic term in the spectra so we will take a antisymmetric combination for the factorization of the Hamiltonian. That is:

\[ H_D = \hat{A}\hat{A}^\dagger - \hat{A}^\dagger\hat{A} = (\hat{n} + 1)f^2(\hat{n} + 1) - \hat{n}f^2(\hat{n}). \]  

(51)

We propose a deformation function of the form

\[ f^2(\hat{n}) = a\hat{n} + b \]  

(52)

with \( a \) and \( b \) real constants to be determined. Substitution into Eq. 51 and comparaison with the eigenvalues

\[ (\hat{n} + 1)f^2(\hat{n} + 1) - \hat{n}f^2(\hat{n}) = 2(\hat{n}a + \frac{a + b}{2}) = 2(\hat{n} + s + \frac{1}{2}) \]

yield \( a = 1, b = 2s \) so that the deformation function for this potential is:

\[ f^2(\hat{n}) = \hat{n} + 2s. \]  

(53)
Once we have selected the deformation function, the deformed operators are specified. Their commutation relations are:

\[
[\hat{A}, \hat{A}^\dagger] = 2(\hat{n} + s + \frac{1}{2}) = 2\hat{A}_0, \quad [\hat{A}_0, \hat{A}] = -\hat{A}, \quad [\hat{A}_0, \hat{A}^\dagger] = \hat{A}^\dagger.
\] (54)

The coherent states obtained as eigenstates of the annihilation operator can be obtained by direct application of Eq. 40. The result is:

\[
|\alpha, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!(n+2s)(n+2s-1)\cdots(2s+1)}} |n\rangle
\] (55)

which have exactly the same structure as those obtained as eigenstates of the annihilation operator \(\hat{L}_-\). Since the algebraic properties of the operators \(\{\hat{L}_+, \hat{L}_-, \hat{L}_0\}\) and the deformed operators \(\{\hat{A}, \hat{A}^\dagger, \hat{A}_0\}\) are the same, the coherent states obtained by application of the generalized displacement operator with either set will have the same structure.

5 Discussion

In this work we constructed the ladder operators corresponding to a trigonometric potential and to a harmonic oscillator with an inverse square potential and with them we found their coherent states by generalization of two definitions. As eigenstates of the annihilation operator and as those states obtained by the displacement of the vacuum state. We also made use of deformed operators with the deformation function chosen in order to reproduce the energy spectra of each case. For the trigonometric potential the deformation function was such that a Hamiltonian of the harmonic oscillator form written in terms of these operators gave the same energy spectra. For the other case we used a antisymmetric combination of the deformed operators in order to reproduce the energy spectra. With these operators we also constructed the coherent states using the two generalizations mentioned above. We found that the coherent states obtained by the two methods have identical algebraic structure. We have also considered other potentials whose eigenfunctions and eigenvalues are known and whose ladder operators have
been obtained as for instance the Morse potential, the hyperbolic Pöschl-Teller potential and the ring-shaped non spherical oscillator \[17, 2, 18\]. From the knowledge of the energy spectra we have obtained the deformation function and the corresponding deformed operators. In all the above mentioned cases the coherent states obtained with the deformed operators and those obtained with the ladder operators have identical structure.

We have to mention that the deformed operators method, which we use for obtain coherent states of the systems analyze, is much faster, powerful and elegant than the ladder operators method, constructed from the eigenfunctions of the corresponding Schrödinger equation. Due to the first one doesn’t need to know the explicit form of these, just enough to know the energy spectra of the system and it comes down to find the deformation function that reproduce the spectrum.

References

[1] Shi-Hai Dong, R. Lemus, International Journal of Quantum Chemistry 86, 265-272 (2002).

[2] Shi-Hai Dong, Fundamental Theories in Physics 150 Factorization Methods in Quantum Mechanics, (Springer, 2007)

[3] V. I. Man’ko, G. Marmo, E. C. G. Sudarshan and F. Zaccaria, Phys. Scr. 55, 528 (1997); V. I. Man’ko, G. Marmo, F. Zaccaria, E. C. G. Sudarshan, \textit{f-oscillators}, Proceedings of the IV Wigner Symposium, Ed. N. M. Atakishiyev, T. H. Seligman and K. B. Wolf (World Scientific, Singapore, 1996).

[4] Nieto M. M., Phys. Rev. A 17, 1273 (1978).

[5] Schrödinger E, Naturwissenschaften 14, 664 (1926).

[6] Glauber R. J., Phys. Rev. Lett. 10, 84 (1963)

[7] Klauder J. R., J. Math. Phys. 4,1055 (Part I); 1058 (Part II) (1963)

[8] Perelomov A. M., Commun. Math. Phys. 26, 222 (1972).

[9] Gilmore R., Ann. Phys. 1974, 391 (1972)
[10] Wei-Min Zhang, Da Hsuan Feng, Robert Gilmore, Rev. Mod. Phys. 62, 868 (1990); R. Gilmore, Lie Groups, Lie Algebras and Some of Their Applications (Wiley, New York).

[11] Gazeau J. P., Klauder, J., J. Phys. A: Math. Gen. 32, 123 (1999).

[12] J. Récamier, M. Gorayeb, W. L. Mochán and J. L. Paz, Int. J. Theor. Phys. 47, 673-683 (2008).

[13] Man’ko V. I., Marmo G., Zaccaria F and Sudarshan E.C.G., 1996 Proc. 4th. Wigner Symp. ed N M Atakishiyev, T H Seligman, and K B Wolf (Singapore: World Scientific) p 421.

[14] O. de los Santos-Sánchez, J. Récamier, J. Phys. A: Math. Theor. 44 (2011) 145307

[15] Récamier J., de León P. G., Jáuregui R., Frank A., Castaños O., International Journal of Quantum Chemistry 89, (2002) 494-502.

[16] Shi-Hai Dong, Guo-Hua Sun, M. Lozada-Cassou, International Journal of Modern Physics A 20, (24) (2005) 5663-5670.

[17] Morse R. Lemus, R. Bernal, Chem. Phys. (?)

[18] Shi-Hai Dong, Guo-Hua Sun, M. Lozada-Cassou, Phys. Lett. A 328 (2004) 299-305

[19] Gradshteyn, I. S.; Ryzhik, I. M. Tables of Integrals, Series, and Products, 5th ed.; Academic Press: New York, 1994.