Global attractors for Kirchhoff wave equation with nonlinear damping and memory

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Abstract
In this paper, we prove that the existence of global attractors for a Kirchhoff wave equation with nonlinear damping and memory.

Keywords: Memory; Nonlinear damping; Global attractors

1 Introduction
Let Ω be an open bounded subset of \( \mathbb{R}^N \) with sufficiently smooth boundary ∂Ω, we consider the following Kirchhoff wave equation with nonlinear damping and linear memory:

\[
\begin{align*}
    u_{tt} - M(\|\nabla u\|^2) \Delta u + a(x)g(u_t) - k(0)\Delta u - \int_0^\infty k'(s)\Delta u(t - s) \, ds \\
    + f(u) &= h(x), \quad \text{in } \Omega \times \mathbb{R}^+, \\
    u|_{x \in \partial \Omega}(x, t) &= 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \\
    u(x, t) &= u_0(x, t), \quad x \in \Omega, \quad t \leq 0,
\end{align*}
\]

where \( M(s) = 1 + s^m \), \( m \geq 1 \), \( k(0), k(\infty) > 0 \) and \( k'(s) \leq 0 \) for every \( s \in \mathbb{R}^+ \), and the assumptions on nonlinear functions \( f(u), g(u_t), a(x) \) and external force term \( h(x) \) will be specified later.

This kind of wave models goes back to Kirchhoff. In 1883, Kirchhoff [1] firstly introduced the following equation to describe small vibrations of an elastic stretch string:

\[
u_{tt} - M(\|\nabla u\|^2) \Delta u = h,
\]

where \( M(s) = a + bs \). There has been much research on global attractors; Lazo studied the existence for the IBVP of the Kirchhoff equation with memory term [2]

\[
u_{tt} - M(\|\nabla u\|^2) \Delta u + \int_0^t g(t - \tau) \Delta u(x, \tau) \, d\tau = 0.
\]

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Chueshov [3] studied the well-posedness and the global attractors of the Kirchhoff equation with strong nonlinear damping

$$u_{tt} - \sigma \left( \| \nabla u \|^2 \right) \Delta u_t - \phi \left( \| \nabla u \|^2 \right) \Delta^2 u + g(u) = h(x), \quad \frac{1}{2} \leq \theta < 1.$$  

Next, Chueshov [4] also studied the Kirchhoff equation with strong nonlinear damping in nature space $\mathcal{H} = H_0^1(\Omega) \cap L^{p+1}(\Omega) \times L^2(\Omega)$ as $\theta = 1$. For related work on the Kirchhoff wave equations with strong damping, see [5, 6] and the references therein.

When $M(s) = 0$, Eq. (1) become the well-known wave equation. Ma and Zhong [7] showed the existence of global attractors for the hyperbolic equation with memory

$$u_{tt} + \alpha u_t - K(0) \Delta u - \int_0^\infty K'(s) \Delta u(t-s) \, ds + g(u) = f.$$  

Recently, Park and Kang [8] studied the existence of global attractors for the semilinear hyperbolic with nonlinear damping and memory

$$u_{tt} + a(x) g(u_t) + \lambda u - K(0) \Delta u - \int_0^\infty K'(s) \Delta u(t-s) \, ds + f(u) = h(x).$$  

In [9], Kang and Rivera showed the existence of global attractors for the beam equation localized nonlinear damping and memory

$$u_{tt} + a(x) g(u_t) + \Delta^2 u - K(0) \left( 1 + \| \nabla u \|^2 \right) \Delta u - \int_0^\infty K'(s) \Delta u(t-s) \, ds + f(u) = h.$$  

Motivated by [5, 7–9], we will prove the existence of global attractors for Eq. (1). Following the framework proposed in [7], we shall add a new variable $\eta$ to the system, which corresponds to the relative displacement history. Let us define

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s).$$  

By differentiation, we have

$$\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t).$$  

Let $\mu(s) = -k'(s), k(\infty) = 1$, (1) transforms into the following system:

$$u_{tt} - \left( 1 + \| \nabla u \|^m \right) \Delta u_t + a(x) g(u_t) - \int_0^\infty \mu(s) \Delta \eta(x,s) \, ds + f(u) = h,$$

$$\eta_t = -\eta_s + u_t,$$  

with boundary condition

$$u = 0, \quad \text{on} \ \Gamma \times \mathbb{R}^+, \quad \eta = 0, \quad \text{on} \ \partial \Omega \times \mathbb{R}^+ \times \mathbb{R},$$  

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \eta^t(x,0) = 0, \quad \eta^0(x,s) = \eta_0(x,s).$$  

This paper is organized as follows. In Sect. 2, we introduce some preliminaries. In Sect. 3, we show the existence of a bounded absorbing set in \( \mathcal{H} \). In Sect. 4, we give the existence of global attractors of problems (6)–(9).

### 2 Preliminaries

We first state some assumptions, which will be used in this paper.

**Assumption (1)** The memory kernel \( \mu \) is required to satisfy the following hypotheses:

\( h1 \) \( \mu(s) \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}), \forall s \in \mathbb{R}^+; \)

\( h2 \) \( \int_0^{\infty} \mu(s) \, ds = k(0); \)

\( h3 \) \( \mu(s) \geq 0, \mu'(s) \leq 0; \)

\( h4 \) \( \mu'(s) + k_1 \mu(s) \leq 0, \forall s \in \mathbb{R}^+, \) for some \( k_1 > 0. \)

**Assumption (2)** The function \( a(x) \) satisfies

\[ a(x) \in L^\infty(\Omega), \quad a(x) \geq a_0 > 0, \tag{10} \]

where \( a_0 \) is a constant.

**Assumption (3)** The function \( f \in C^1(\mathbb{R}) \) satisfies

\[ |f'(s)| \leq C_1 (1 + |s|^p), \tag{11} \]

\[ \lim_{|s| \to \infty} \inf \frac{f(s)}{s} > -\lambda_1, \tag{12} \]

where \( 0 < p < \infty \), if \( n \leq 2 \), and \( 0 < p \leq \frac{2}{n-2} \) if \( n \leq 2 \). \( \lambda_1 \) is the constant in the Poincáre type inequality \( \| \nabla u \|^2 \geq \lambda_1 \| u \|^2. \)

**Assumption (4)** The damping function \( g \in C^1(\mathbb{R}) \) satisfies

\[ g(0) = 0, \quad g \text{ is strictly increasing, and } \liminf_{|s| \to \infty} g'(s) > 0, \tag{13} \]

\[ |g(s)| \leq C_2 (1 + |s|^q), \tag{14} \]

with \( 1 \leq q < \infty \) if \( n \leq 2 \), and \( 1 \leq q \leq \frac{n+2}{n-2} \) if \( n > 2 \).

In order to consider the relative displacement \( \eta \) as a new variable, we introduce the weighted \( L^2 \)-space

\[ \mathcal{M} = L^2_{\mu}(\mathbb{R}^+; H^1_0(\Omega)) = \left\{ \xi : \mathbb{R}^+ \to H^1_0(\Omega) \left| \int_0^{\infty} \mu(s) \| \nabla \xi(s) \|^2 \, ds < \infty \right. \right\}, \]

which is a Hilbert space endowed with inner product and norm

\[ (\xi, \zeta)_\mathcal{M} = \int_0^{\infty} \mu(s) \left( \int_\Omega \nabla \xi(s) \nabla \zeta(s) \, dx \right) \, ds \quad \text{and} \quad \| \xi \|^2_\mathcal{M} = \int_0^{\infty} \mu(s) \| \nabla \xi \|^2 \, ds, \]

respectively.
Our analysis is given on the phase space
\[ \mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times \mathcal{M}, \]
which is equipped with the norm
\[ \| (u, v, \eta) \|_{\mathcal{H}}^2 = \| \nabla u \|_2^2 + \| v \|_2^2 + \| \eta \|_M^2. \]

In order to obtain the global attractors of the problems (6)–(9), we need the following theorem of existence, uniqueness of solution and continuous dependence on the initial data.

**Theorem 2.1** ([9]) Let assumptions (1)–(4) hold, if \( z_0 = (u_0, v_0, \eta_0) \in \mathcal{H} \), then there exists a unique solution \( z = (u, u_t, \eta) \) of (6)–(9) such that
\[ z \in C([0, T], \mathcal{H}) \text{ for all } T > 0. \]

Next, we recall the simple compactness criterion stated in [9, 10].

**Definition 2.1** ([9, 10]) Let \( X \) be a Banach space and \( B \) be a bounded subset of \( X \), we call a function \( \Phi(\cdot, \cdot) \) which defined on \( X \times X \), is a contractive on \( B \times B \) if for any sequence \( \{ x_n \}_{n=1}^{\infty} \subset B \), there is a subsequence \( \{ x_{n_k} \}_{k=1}^{\infty} \subset \{ x_n \}_{n=1}^{\infty} \) such that
\[ \lim_{k \to \infty} \lim_{l \to \infty} \Phi(T) x_{n_k}, x_{n_l} = 0. \]

Denote all such contractive functions on \( B \times B \) by \( C(B) \).

**Theorem 2.2** ([9, 10]) Let \( \{ s(t) \}_{t \geq 0} \) be a semigroup on a Banach space \( (X, \| \cdot \|) \) and has a bounded absorbing set \( B_0 \). Moreover, assume that for any \( \epsilon \geq 0 \) there exist \( T = T(B_0, \epsilon) \) and \( \Phi(\cdot, \cdot) \in C(B) \) such that
\[ \| S(T) x - S(T) y \| \leq \epsilon + \Phi_T(x, y) \text{ for all } x, y \in B_0, \]
where \( \Phi_T \) depends on \( T \). Then \( \{ s(t) \}_{t \geq 0} \) is asymptotically compact in \( X \), i.e., for any bounded sequence \( \{ y_n \}_{n=1}^{\infty} \subset X \) and \( \{ t_n \} \) with \( t_n \to \infty \), \( \{ S(t_n) y_n \}_{n=1}^{\infty} \) is compact in \( X \).

**Lemma 2.1** ([11]) Let \( g(\cdot) \) satisfy condition (13). Then for any \( \delta > 0 \) there exists \( c(\delta) > 0 \), such that
\[ |u - v|^2 \leq \delta + C(\delta) (g(u) - g(v))(u - v), \text{ for all } u, v \in \mathbb{R}. \]

### 3 Absorbing set in \( \mathcal{H} \)

In this section, we prove the existence of the bounded absorbing set in \( \mathcal{H} \). We use \( C_i \) to denote several positive constants.

**Lemma 3.1** Under assumptions (1)–(4), the semigroup \( \{ S(t) \}_{t \geq 0} \) corresponding to problems (6)–(9) has a bounded absorbing set in \( \mathcal{H} \).
Proof we take the scalar product in $L^2$ of system (6) with $u_t$ and (7) with $\eta$, respectively, we have
\[
\frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{m+2} \|\nabla u\|^{m+2} + \frac{1}{2} \|\eta\|^2_{\mathcal{M}} + \int_{\Omega} (F(u) - hu) \, dx \right) 
+ (\eta, \eta)_{\mathcal{M}} + (a(x)g(u_t)u_t) = 0,
\]
(17)
where $F(u) = \int_0^u f(s) \, ds$. As in [7]
\[
(\eta, \eta)_{\mathcal{M}} = \frac{1}{2} \int_0^\infty \mu(s) \frac{d}{ds} \|\nabla \eta(s)\|^2 \, ds = -\frac{1}{2} \int_0^\infty \mu'(s) \|\nabla \eta(s)\|^2 \, ds \geq \frac{k_1}{2} \|\eta\|^2_{\mathcal{M}}.
\]
(18)
We set
\[
E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{m+2} \|\nabla u\|^{m+2} + \frac{1}{2} \|\eta\|^2_{\mathcal{M}} + \int_{\Omega} (F(u) - hu) \, dx.
\]
Then from (17) and (18) we obtain
\[
\frac{d}{dt} E(t) + \frac{k_1}{2} \|\eta\|^2_{\mathcal{M}} + \int_{\Omega} (a(x)g(u_t)u_t) \, dx \leq 0.
\]
(19)
From (10), (13) we obtain
\[
E(t) \leq E(0), \quad t \geq 0.
\]
(20)
By the hypothesis (12) we know that there are $\lambda > \lambda_1 > 0$ and $C_0$ such that
\[
(f(u), u) > \frac{\lambda}{2} \|u\|^2 - C_0 \text{mes}(\Omega), \quad \int_{\Omega} F(u) \, dx > -\frac{\lambda}{4} \|u\|^2 - C_0 \text{mes}(\Omega).
\]
(21)
Using the Young inequality, we have
\[
-\int_{\Omega} hudx \geq -\varepsilon \|u\|^2 - \frac{1}{4\varepsilon} \|h\|^2,
\]
we choose proper $\lambda$ and $\varepsilon$ small enough so that $\frac{1}{2} - \frac{1}{4\lambda_1} - \varepsilon > \frac{1}{6}$, and we have
\[
E(0) \geq E(t) \geq \frac{1}{2} \|u_t\|^2 + \frac{1}{m+2} \|\nabla u\|^{m+2} + \frac{1}{2} \|\eta\|^2_{\mathcal{M}} - C_1 (\text{mes}(\Omega) + \|h\|^2)
\geq -C_1 (\text{mes}(\Omega) + \|h\|^2),
\]
(22)
combining (19) with (22), we have
\[
\int_0^t \int_{\Omega} (a(x)g(u_t)u_t) \, dx \leq E(0) - E(t) \leq E(0) + C_1 (\text{mes}(\Omega) + \|h\|^2), \quad \forall t \geq 0.
\]
(23)
Taking the scalar product in $L^2$ of (6) with $v = u_t + \varepsilon u$, we obtain

$$
\frac{d}{dt} \left( \frac{1}{2} \| v \|^2 + \frac{1}{m+2} \| \nabla u \|^2 + \frac{1}{m+2} \| \nabla v \|^2 + \frac{1}{2} \| \eta \|_{L^2}^2 - \frac{\varepsilon^2}{2} \| u \|^2 + \varepsilon \| \nabla u \|^2 \right) + \int_{\Omega} (F(u) - hu) \, dx + \varepsilon \| \nabla u \|_{L^2}^2 + \frac{k_1}{2} \| \eta \|_{L^2}^2 + \varepsilon (f(u), u) + (a(x) g(u_t) - \varepsilon u_t, u_t) + \varepsilon (a(x) g(u_t), u) \leq \varepsilon (\eta, u)_M. \tag{24}
$$

Let

$$
F(t) = \frac{1}{2} \| v \|^2 + \frac{1}{m+2} \| \nabla u \|^2 + \frac{1}{m+2} \| \nabla v \|^2 + \frac{1}{2} \| \eta \|_{L^2}^2 - \frac{\varepsilon^2}{2} \| u \|^2 + \int_{\Omega} (F(u) - hu) \, dx,
$$

$$
G(t) = \varepsilon \| \nabla u \|_{L^2}^2 + \varepsilon \| \nabla v \|_{L^2}^2 + \frac{k_1}{2} \| \eta \|_{L^2}^2 + \varepsilon (f(u), u) - \varepsilon (h, u) - \varepsilon (\eta, u)_M + (a(x) g(u_t) - \varepsilon u_t, u_t) + \varepsilon (a(x) g(u_t), u),
$$

so

$$
\frac{d}{dt} F(t) + G(t) \leq 0. \tag{25}
$$

Similarly, using (21), the Poincaré inequality and the Young inequality, choosing proper $\lambda$ and $\varepsilon$ small enough so that \( \frac{1}{2} - \frac{\varepsilon^2}{2\lambda_1} - \frac{\lambda_1}{4} - \varepsilon > \frac{1}{8} \), we have

$$
F(t) \geq \frac{1}{8} \| v \|^2 + \frac{1}{m+2} \| \nabla u \|^2 + \frac{1}{m+2} \| \nabla v \|^2 + \frac{1}{2} \| \eta \|_{L^2}^2 - \frac{\varepsilon^2}{2} \| u \|^2 - C (\text{mes}(\Omega) + \| h \|^2). \tag{26}
$$

It is obvious that (10) and (13) imply that there are $\varepsilon > 0$ and $C > 0$ such that

\[
\begin{align*}
(a(x) g(u_t), u_t) &\geq 2\varepsilon \| u_t \|^2 - C \varepsilon \text{mes}(\Omega), \\
(a(x) g(u_t) - \varepsilon u_t, u_t) &\geq \varepsilon \| u_t \|^2 - C(\varepsilon) \text{mes}(\Omega).
\end{align*}
\tag{27}
\]

Due to the Young inequality we have

$$
\varepsilon (\eta, u)_M \geq - \frac{k_1}{4} \| \eta \|_{L^2}^2 - \frac{k(0)\varepsilon^2}{k_1} \| \nabla u \|^2. \tag{28}
$$

Using (13) and (14) yields

$$
|g(s)|^{\frac{4}{m+1}} = |g(s)|^{\frac{2}{m+1}} |g(s)| \leq C(1 + |s|) |g(s)|,
$$

so

\[
\begin{align*}
|g(s)|^{\frac{4}{m+1}} &\leq C, & |s| \leq 1; \\
|g(s)|^{\frac{4}{m+1}} &\leq 2C|g(s)|, & |s| \geq 1,
\end{align*}
\tag{29}
\]

where $C$ is a constant which is independent of $s$. 
Then from (29), using the Hölder inequality, the Young inequality and the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{q+1}(\Omega)$, we obtain

\[
\left| \int_{\Omega} a(x)g(u_t)u \, dx \right| \\
\leq \int_{\Omega[|u_t| \leq 1]} |a(x)g(u_t)| u \, dx + \int_{\Omega[|u_t| \geq 1]} |a(x)g(u_t)| u \, dx \\
\leq \int_{\Omega[|u_t| \leq 1]} C|a(x)u| \, dx \\
+ \left( \int_{\Omega[|u_t| \geq 1]} a(x)|g(u_t)|^{\frac{q+1}{q}} \, dx \right)^{\frac{q}{q+1}} \left( \int_{\Omega[|u_t| \geq 1]} a(x)|u|^{q+1} \, dx \right)^{\frac{1}{q+1}} \\
\leq \int_{\Omega[|u_t| \leq 1]} C|a(x)u| \, dx \\
+ 2C \left( \int_{\Omega[|u_t| \geq 1]} a(x)|g(u_t)u_t| \, dx \right)^{\frac{q}{q+1}} \left( \int_{\Omega[|u_t| \geq 1]} a(x)|u|^{q+1} \, dx \right)^{\frac{1}{q+1}} \\
\leq \frac{C}{4\gamma} \int_{\Omega} \frac{|a(x)|^2}{a_0} \, dx \\
+ C\gamma a_0^2 \|u\|^2 + C\gamma \left( \int_{\Omega[|u_t| \geq 1]} a(x)|g(u_t)u_t| \, dx \right) \|u\|_{q+1}^{\frac{q+1}{q}} + \eta \|u\|_{q+1}^{q+1} \\
\leq \frac{C}{4\gamma} \text{mes}(\Omega) + C\gamma a_0^2 \|u\|^2 + C\gamma \|\nabla u\|^2 \int_{\Omega} a(x)|g(u_t)u_t| \, dx + \gamma C\|\nabla u\|^2,
\] (30)

where $a_0 = \sup_{x \in \Omega} a(x)$, and $\gamma$ is a constant. From (21), (27), (28), (30) we have

\[
G(t) \geq \varepsilon \|u_t\|^2 + \varepsilon \|\nabla u\|^{m+2} + \frac{k_1}{4} \|\eta\|_{\mathcal{M}}^2 \\
+ \varepsilon \left( \frac{1}{2} - \frac{k(0)e^2}{k_1} + C \right) \|\nabla u\|^2 - \left( \varepsilon C\gamma a_0^2 + \frac{\varepsilon k^2}{4} \right) \|u\|^2 \\
- \varepsilon C \|\nabla u\|^{\frac{q+1}{q}} \int_{\Omega} a(x)|g(u_t)u_t| \, dx - C_\varepsilon (\text{mes}(\Omega) + \|h\|^2),
\]

we choose $\varepsilon$ and $C$ small enough so that $\frac{1}{2} - \frac{k(0)e^2}{k_1} + C > \frac{1}{4}$, we get

\[
G(t) \geq \frac{\varepsilon}{4} \left( \|u_t\|^2 + \|\nabla u\|^2 \right) + \frac{k_1}{4} \|\eta\|_{\mathcal{M}}^2 \\
- C_{E(0)} \int_{\Omega} a(x)|g(u_t)u_t| \, dx - C_\varepsilon (\text{mes}(\Omega) + \|h\|^2),
\] (31)

where $C_{E(0)}$ is a constant which depends on $\varepsilon$, $\gamma$, $C$ and $E(0)$, $C_\varepsilon$ is a constant depending on $\varepsilon$, $C_\varepsilon$ and $C$.

We have

\[
\|u_t\|^2 + \|\nabla u\|^2 + \|\eta\|_{\mathcal{M}}^2 = \|u_t + \delta u - \delta u\|^2 + \|\nabla u\|^2 + \|\eta\|_{\mathcal{M}}^2
\]
\[
\leq 2\|v\|^2 + \left(\frac{2\delta^2}{\lambda_1} + 1\right)\|\nabla u\|^2 + \|\eta\|^2
\]
\[
\leq C_0(\|v\|^2 + \|\nabla u\|^2 + \|\eta\|^2_M),
\]
where \(C_0 = \max\{2, 1 + \frac{2\delta^2}{\lambda_1}\} \).

Integrating (25), combining with (23), (26), (31), yields

\[
\|u_t(t_0)\|^2 + \|\nabla u(t_0)\|^2 + \|\eta^{t_0}(t)\|^2_M \leq C_0 (\|v\|^2 + \|\nabla u\|^2 + \|\eta\|^2_M).
\]

Therefore, for any \(\rho > 4C_0^2C_0\mu_0(\mu_1 + \|h\|^2)\) there exists \(t_0\) such that

\[
\|u_t(t_0)\|^2 + \|\nabla u(t_0)\|^2 + \|\eta^{t_0}(t)\|^2_M \leq \rho.
\]

Set

\[
B_0 = \{(u_0, v_0, \eta_0) \in \mathcal{H} \mid \|\nabla u_0\|^2 + \|v_0\|^2 + \|\eta_0\|^2_M \leq \rho\},
\]

then we see \(B_0\) is a bounded absorbing set. Define

\[
B_1 = \bigcup_{t \geq 0} S(t)B_0,
\]

so \(B_1\) is also a bounded absorbing set.

\section{4 Existence of the global attractor in \(\mathcal{H}\)}

### 4.1 A priori estimate

Firstly, we use the prior estimates to obtain the asymptotic compactness following the standard energy method. In this section, \(C_i\) are positive constants.

Let \((u, u_t, \eta)\) and \((v, v_t, \xi)\) be two solutions to systems (6)–(9), and \((u, u_t, \eta)\) and \((v, v_t, \xi)\) \(\in B_1\), \(\omega(t) = u(t) - v(t), \zeta = \eta - \xi\). Then \(\omega(t), \zeta\) satisfy

\[
\begin{aligned}
\omega_{tt} - \|\nabla u\|^m \Delta u + \|\nabla v\|^m \Delta v - \Delta \omega - \int_0^\infty \mu(s) \Delta \xi(s) ds \\
+ a(x)\xi(u_t) - a(x)\xi(u_{2t}) + f(u_1) - f(u_2) = 0,
\end{aligned}
\]

\[
\zeta_t = -\zeta_t + \omega_t,
\]

and

\[
\omega(t) = u(t) - v(t), \quad \zeta = \eta - \xi.
\]

Then \(\omega(t), \zeta\) satisfy

\[
\begin{aligned}
\omega_{tt} - \|\nabla u\|^m \Delta u + \|\nabla v\|^m \Delta v - \Delta \omega - \int_0^\infty \mu(s) \Delta \xi(s) ds \\
+ a(x)\xi(u_t) - a(x)\xi(u_{2t}) + f(u_1) - f(u_2) = 0,
\end{aligned}
\]

\[
\zeta_t = -\zeta_t + \omega_t,
\]

and

\[
\omega(t) = u(t) - v(t), \quad \zeta = \eta - \xi.
\]
firstly, taking the scalar product in $L^2$ of (35) with $\omega$ and integrating over $[0, T]$, we get

\[
\int_0^T \| \nabla \omega(s) \|^2 \, ds = \int_\Omega \omega_t(0) \omega(0) \, dx - \int_\Omega \omega_t(T) \omega(T) \, dx + \int_0^T \| \omega_t(s) \|^2 \, ds \\
- \int_0^T \| \nabla u(s) \|^m \| \nabla \omega(s) \|^2 \, ds - \int_0^T (\xi, \omega)_{\mathcal{M}} \, ds \\
- \int_0^T \int_\Omega (\| \nabla u(s) \|^m - \| \nabla v(s) \|^m) \nabla v(s) \nabla \omega(s) \, dx \, ds \\
- \int_0^T \int_\Omega a(x)(g(u_t(s)) - g(v_t(s))) \omega(s) \, dx \, ds \\
- \int_0^T \int_\Omega (f(\omega(s)) - f(v(s))) \omega(s) \, dx \, ds. \tag{37}
\]

Using the Young inequality and (h3), we obtain

\[
(\xi, \omega)_{\mathcal{M}} \geq -\frac{1}{2} \| \nabla \omega \|^2 - \frac{k(0)}{2} \| \xi \|^2_{\mathcal{M}}. \tag{38}
\]

Secondly, taking the scalar product in $L^2$ of (35), (36) with $\omega_t$ and integrating over $[0, T]$, we get

\[
\frac{d}{dt} \left( \frac{1}{2} \| \omega_t \|^2 + \frac{1}{2} \| \nabla \omega \|^2 + \frac{1}{2} \| \xi \|^2_{\mathcal{M}} \right) + \int_\Omega (\| \nabla u \|^m - \| \nabla v \|^m) \nabla v \nabla \omega_t \, dx \\
+ (\xi, \omega_t)_{\mathcal{M}} + \int_\Omega (f(u) - f(v)) \omega_t \, dx + \int_\Omega a(x)(g(u_t) - g(v_t)) \omega_t \, dx = 0. \tag{39}
\]

Let

\[
E_\omega(t) = \frac{1}{2} \| \omega_t \|^2 + \frac{1}{2} \| \nabla \omega \|^2 + \frac{1}{2} \| \xi \|^2_{\mathcal{M}}.
\]

Integrating (39) over $(s, T]$ and combining with (38), where $s \in [0, T]$, we have

\[
E_\omega(t) + \frac{k_1}{2} \int_s^T \| \xi \|^2_{\mathcal{M}} + \frac{1}{2} \int_s^T \int_\Omega a(x)(g(u_t(\tau)) - g(v_t(\tau))) \omega_t(\tau) \, dx \, d\tau \\
+ \frac{1}{2} \int_\Omega \| \nabla u(T) \|^m \| \nabla \omega(T) \|^2 \, dx \\
\leq E_\omega(s) + \frac{1}{2} \int_\Omega \| \nabla u(s) \|^m \| \nabla \omega(s) \|^2 \, dx \\
+ \frac{m}{2} \int_s^T \int_\Omega \| \nabla \omega(\tau) \|^2 \| \nabla u(\tau) \|^{m-1} \nabla u_t(\tau) \, dx \, d\tau \\
- \int_s^T \int_\Omega (\| \nabla u(\tau) \|^m - \| \nabla v(\tau) \|^m) \nabla v(\tau) \nabla \omega_t(\tau) \, dx \, d\tau \\
- \int_s^T \int_\Omega (f(\omega(\tau)) - f(v(\tau))) \omega_t(\tau) \, dx \, d\tau. \tag{40}
\]
Integrating (40) over [0, T] with respect to s, we get

\[
TE_\omega(t) \leq \int_0^T E_\omega(s) \, ds + \frac{1}{2} \int_0^T \int_\Omega \| \nabla u(s) \| \| \nabla \omega(s) \|^2 \, dx \\
+ \frac{m}{2} \int_0^T \int_\Omega \int_0^T \| \nabla \omega(t) \|^2 \| \nabla u(t) \|^m \| \nabla u_\omega(t) \| \, dx \, dt \\
- \int_0^T \int_\Omega \int_0^T (\| \nabla u(t) \|^m - \| \nabla \nu(t) \|^m) \nabla \nu(t) \nabla \omega(t) \, dx \, dt \\
- \int_0^T \int_\Omega \int_0^T (f(u(t)) - f(\nu(t))) \omega_\nu(t) \, dx \, dt.
\]

Due to (10), (40), and Lemma 2.1, we obtain, for any \( \delta > 0 \),

\[
\int_0^T \| \xi^\tau \|^2 \, d\tau + \int_0^T \| \omega_\nu(t) \|^2 \, d\tau \\
\leq C_2 E_\omega(0) + C_2 \int_0^T \int_\Omega (f(u(t)) - f(\nu(t))) \omega_\nu(t) \, dx \, dt \\
+ \delta T \text{ mes} (\Omega) - C_2 \int_0^T \int_\Omega \| \nabla u(T) \|^m \| \nabla \omega(T) \|^2 \, dx \\
- C_2 \int_0^T \int_\Omega (\| \nabla u(t) \|^m - \| \nabla \nu(t) \|^m) \nabla \nu(t) \nabla \omega_\nu(t) \, dx \, dt \\
- C_2 \int_0^T \int_\Omega (f(u(t)) - f(\nu(t))) \omega_\nu(t) \, dx \, dt \\
+ \frac{m C_2}{2} \int_0^T \int_\Omega \| \nabla \omega(t) \|^2 \| \nabla u(t) \|^m \| \nabla u_\omega(t) \| \, dx \, dt,
\]

where \( C_2 \) is a constant which depends on \( \delta, \alpha_0 \) and \( k_1 \).

Thus, from (37), (38) and (42) we have

\[
\int_0^T E_\omega(t) \, dt \leq C_3 \delta T \text{ mes} (\Omega) + C_2 C_3 E_\omega(0) - \frac{C_2 C_3}{2} \int_\Omega \| \nabla u(T) \|^m \| \nabla \omega(T) \|^2 \, dx \\
+ \frac{C_2 C_3}{2} \int_\Omega \| \nabla u(0) \|^m \| \nabla \omega(0) \|^2 \, dx \\
- C_2 C_3 \int_0^T \int_\Omega (\| \nabla u(t) \|^m - \| \nabla \nu(t) \|^m) \nabla \nu(t) \nabla \omega_\nu(t) \, dx \, dt \\
- C_2 C_3 \int_0^T \int_\Omega (f(u(t)) - f(\nu(t))) \omega_\nu(t) \, dx \, dt \\
+ \frac{m C_2 C_3}{2} \int_0^T \int_\Omega \| \nabla \omega(t) \|^2 \| \nabla u(t) \|^m \| \nabla u_\omega(t) \| \, dx \, dt \\
+ \int_\Omega \omega_\nu(0) \omega(0) \, dx - \int_\Omega \omega_\nu(T) \omega(T) \, dx \\
- \int_0^T \| \nabla u(s) \|^m \| \nabla \omega(s) \|^2 \, ds \\
- \int_0^T (\| \nabla u(s) \|^m - \| \nabla \nu(s) \|^m) \nabla \nu(s) \nabla \omega(s) \, ds.
\]
\[
- \int_0^T \int_{\Omega} a(x)(g(u_t(s)) - g(v_t(s))) \omega(s) \, dx \, ds \\
- \int_0^T \int_{\Omega} (f(u(s)) - f(v(s))) \omega(s) \, dx \, ds,
\]
(43)

where \( C_3 = \max \{ \frac{3}{2}, \frac{k(0)+1}{2} \} \). From (23) and the existence of the absorbing set, we get

\[
\int_0^T \int_{\Omega} a(x)(g(u_t)) u_t \, dx \, ds \leq C_\rho, \tag{44}
\]
\[
\int_0^T \int_{\Omega} a(x)(g(v_t)) v_t \, dx \, ds \leq C_\rho, \tag{45}
\]

where \( C_\rho \) is a constant which depends on \( \text{mes}(\Omega) \), \( \|h\|^2 \) and the size of \( B_0 \). By a similar method to that of (30) and (43), (44), we have

\[
\left| \int_0^T \int_{\Omega} a(x)g(u_t(s)) \omega(s) \, dx \, ds \right| \\
\leq C_3^{\frac{q}{n+1}} \int_0^T \int_{\Omega[|u_t| \leq 1]} |a(x)| \omega \, dx \, ds \\
+ (2C)^{\frac{q}{n+1}} \left( \int_0^T \int_{\Omega[|u_t| \geq 1]} a(x)g(u_t) u_t \, dx \, ds \right)^{\frac{q}{n+1}} \\
\times \left( \int_0^T \int_{\Omega[|\omega| \geq 1]} a(x)|\omega|^{q+1} \, dx \, ds \right)^{\frac{1}{q+1}} \\
\leq C_3^{\frac{q}{n+1}} \int_0^T \int_{\Omega} a(x)|\omega| \, dx \, ds + C_\rho T^{\frac{1}{n+1}}, \tag{46}
\]

similarly

\[
\left| \int_0^T \int_{\Omega} a(x)g(v_t(s)) \omega(s) \, dx \, ds \right| \leq C_3^{\frac{q}{n+1}} \int_0^T \int_{\Omega} a(x)|\omega| \, dx \, ds + C_\rho T^{\frac{1}{n+1}}, \tag{47}
\]

combining (41), (43), (46), (47), we have

\[
TE_{\omega}(T) \leq C_B + \Phi_T(z_0^1, z_0^2), \tag{48}
\]

where

\[
C_B = C_3 \delta T \text{mes}(\Omega) + C_2 C_3 E_{\omega}(0) + \int_{\Omega} \omega_t(0)\omega(0) \, dx - \int_{\Omega} \omega_t(T)\omega(T) \, dx + 2C_\rho T^{\frac{1}{n+1}} \\
+ \frac{C_2 C_3}{2} \int_{\Omega} \|\nabla u(0)\|^m \|\nabla \omega(0)\|^2 \, dx - \frac{C_2 C_3}{2} \int_{\Omega} \|\nabla u(T)\|^m \|\nabla \omega(T)\|^2 \, dx, \tag{49}
\]
Under assumptions Theorem 4.1 can choose that \( \delta \) is small enough so that the semigroup is asymptotically compact in \( \mathcal{H} \).

Then we have

\[
E_\omega(T) \leq \frac{C_B}{T} + \frac{1}{T} \Phi_T(z_0^1, z_0^2). \tag{51}
\]

**4.2 Asymptotic compactness**

In this subsection, following the argument in [9, 10], we will prove the asymptotic compactness of the semigroup \( \{S(t)\}_{t \geq 0} \) in \( \mathcal{H} \), which is given in the following theorem.

**Theorem 4.1** Under assumptions (1)–(4), the semigroup \( \{S(t)\}_{t \geq 0} \) to systems (6)–(9) is asymptotically compact in \( \mathcal{H} \).

**Proof** since the semigroup \( \{S(t)\}_{t \geq 0} \) has a bounded absorbing set, for every fixed \( \varepsilon > 0 \), we can choose that \( \varepsilon \leq \frac{\varepsilon}{2C_B \max_{\Omega} |\tau|} \), and then let \( T \) become so large that

\[
\frac{C_B}{T} \leq \varepsilon. \tag{52}
\]

Hence, thanks to Theorem 2.2, we only need to verify that the function \( \Phi_T(z_0^1, z_0^2) \) defined in (50) belongs to \( C(B_1) \) for each fixed \( T \). and we claim that

\[
\| S(t)z_0^1 - S(t)z_0^2 \|_{\mathcal{H}} \leq \varepsilon + \Phi_T(z_0^1, z_0^2), \quad \forall z_0^1, z_0^2 \in B. \tag{53}
\]

Here \( (u(t), u_i(t), \eta) = S(t)z_0^1 \) and \( (v(t), v_i(t), \xi) = S(t)z_0^2 \) are the solutions of (6)–(9) with respect to initial \( z_0^1, z_0^2 \in B_1 \). Then, since \( C(B_1) \) is a bounded positively invariant set in \( \mathcal{H} \), it follows that \( (u_n, u_i, \eta) \) is uniformly bounded in \( \mathcal{H} \). We have

\[
u_n \rightarrow u \quad \text{weakly star in } L^\infty(0, T; H_0^1(\Omega)), \tag{54}
\]
\[ u_{n_l} \rightharpoonup u_t \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)). \] (55)

Then, by the compact embedding \( H^1_0(\Omega) \hookrightarrow L^k(\Omega), \) we have

\[ u_n \rightharpoonup u \quad \text{strongly in } L^2(0, T; L^2(\Omega)), \] (56)

\[ u_n \rightharpoonup u \quad \text{strongly in } L^k(0, T; L^k(\Omega)), \] (57)

where \( k \leq \frac{2n}{n-2}, \) therefore from (56) we have

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \int_\Omega \left( f(u_l(\tau)) - f(u_k(\tau)) \right) (u_l(\tau) - u_k(\tau)) \, dx \, d\tau = 0, \] (58)

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \int_\Omega \left( f(u_l(\tau)) - f(u_k(\tau)) \right) (u_l(\tau) - u_k(\tau)) \, dx \, d\tau = 0, \] (59)

then from (57) and (10), we obtain

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \int_\Omega a(x) |u_l(s) - u_k(s)| \, dx \, ds = 0. \] (60)

Finally, we follow a similar argument to the ones given in [9, 10]. We have

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \int_\Omega \| \nabla u_l(\tau) - \nabla u_k(\tau) \|^2 \nabla u_l(\tau) \nabla u_k(\tau) \, dx \, d\tau = 0, \] (61)

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \int_\Omega \left( \| \nabla u_l(\tau) \|^2 - \| \nabla u_k(\tau) \|^2 \right) \nabla u_l(\tau) (\nabla u_l - \nabla u_k) \, dx \, d\tau = 0, \] (62)

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \| \nabla u(t) \|^2 \| \nabla u_k(t) \|^2 \, dt = 0, \] (63)

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \| u_l(t) - u_k(t) \|^2 \, dt = 0, \] (64)

\[ \lim_{l \to \infty} \lim_{k \to \infty} \int_0^T \| u_l(t) - u_k(t) \|^2 \, dt = 0. \] (65)

Finally, combining (58)–(65) we get \( \Phi(\cdot, \cdot) \in C(B_1). \)

4.3 Existence of global attractor

**Theorem 4.2** Under assumptions (1)–(4), then problems (6)–(9) have a global attractor in \( \mathcal{H}, \) which is invariant and compact.

**Proof** Lemma 3.1 and Theorem 4.1 imply the existence of the global attractor.

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Authors’ contributions
This paper is mainly completed by SZ, JZ and HW dealt with the nonlinear damping term as proving the existence of a bounded absorbing set. All authors read and approved the final manuscript.

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