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**Spectral Theory For Strongly Continuous Cosine**

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**Abstract:** Let \((C(t))_{t \in \mathbb{R}}\) be a strongly continuous cosine family and \(A\) be its infinitesimal generator. In this work, we prove that, if \(C(t) - \cosh \lambda t\) is semi-Fredholm (resp. semi-Browder, Drazin invertible, left essentially Drazin and right essentially Drazin invertible) operator and \(\lambda t \not\in i\mathbb{Z}\), then \(A - \lambda^2\) is also. We show by counter-example that the converse is false in general.

**Keywords:** Cosine, semi-Fredholm, Drazin invertible, semi-Browder, left essentially Drazin invertible, right essentially Drazin invertible.

**MSC:** 47D09; 47A11

1 Introduction

Let \(X\) be a complex Banach space, \(\mathcal{B}(X)\) denote the algebra of all bounded linear operators on \(X\) and \(\mathcal{C}(X)\) the set of all linear closed operators from \(X\) to \(X\). We write \(\mathcal{D}(T), \mathcal{R}(T), \mathcal{N}(T), \rho(T), \sigma(T), \sigma_p(T), \sigma_{ap}(T)\) and \(\sigma_e(T)\) respectively for the domain, the range, the kernel, the resolvent, the spectrum, the essential spectrum, the approximate point spectrum and residual spectrum of an operator \(T \in \mathcal{C}(X)\). The function resolvent of \(T \in \mathcal{C}(X)\) is defined for all \(\lambda \in \rho(T)\) by \(R(\lambda, T) = (\lambda - T)^{-1}\). The ascent \(a(T)\), the descent \(d(T)\), the essential ascent \(a_e(T)\) and the essential descent \(d_e(T)\) of an operator \(T \in \mathcal{C}(X)\) are defined respectively by

\[
a(T) = \inf\{k \in \mathbb{N} : \mathcal{N}(T^k) = \mathcal{N}(T^{k+1})\},
\]

\[
d(T) = \inf\{k \in \mathbb{N} : \mathcal{R}(T^k) = \mathcal{R}(T^{k+1})\},
\]

\[
a_e(T) = \min\{k \in \mathbb{N} : \dim \mathcal{N}(T^{k+1})/\mathcal{N}(T^k) < \infty\},
\]

\[
d_e(T) = \min\{k \in \mathbb{N} : \dim \mathcal{R}(T^k)/\mathcal{R}(T^{k+1}) < \infty\},
\]

with the convention \(\inf(\emptyset) = \infty\), see ([5, 13]). For \(T \in \mathcal{C}(X)\), if there is an operator \(S \in \mathcal{B}(X)\) with \(\mathcal{R}(S) \subseteq \mathcal{D}(T)\) such that \(STS = S, TSx = STx\) for all \(x \in \mathcal{D}(T)\), and \(T^k(I - TS) = 0\) for some \(k \in \mathbb{N}\), then \(S\) is called a Drazin inverse of \(T\). Note that an operator \(T \in \mathcal{C}(X)\) has a Drazin inverse if and only if there exists \(k \in \mathbb{N}\) such that \(a(T) = d(T) = k\) and \(X = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k)\), (see [12]). An operator \(T \in \mathcal{C}(X)\) is a left essentially Drazin invertible operator if \(a_e(T) < \infty\) and \(\mathcal{R}(T^{a_e(T)})\) is closed. If \(d_e(T) < \infty\) and \(\mathcal{R}(T^{d_e(T)})\) is closed, then \(T\) is called right essentially Drazin invertible. The Drazin inverse, left essentially Drazin and right essentially Drazin invertible spectra are defined by

\[
\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Drazin invertible }\},
\]

\[
\sigma_e'(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left essentially Drazin invertible }\},
\]

\[
\sigma_e''(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not right essentially Drazin invertible }\}.
\]

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An operator $T \in \mathcal{C}(X)$ is called upper semi-Fredholm (resp. lower semi-Fredholm) if the range $\mathcal{R}(T)$ is closed and $\dim \mathcal{N}(T) < \infty$ (resp. $\text{codim} \mathcal{R}(T) < \infty$). If $T$ is either upper or lower semi-Fredholm, then $T$ is called a semi-Fredholm operator. If $T$ is both upper and lower semi-Fredholm, then $T$ is called a Fredholm operator, see ([11]). The upper semi-Fredholm spectrum $\sigma_{uf}(T)$, the lower semi-Fredholm spectrum $\sigma_{lf}(T)$, the spectrum semi-Fredholm $\sigma_{sf}(T)$ and the Fredholm spectrum $\sigma_f(T)$ of $T$ are defined by

$$\sigma_{uf}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not upper semi-Fredholm} \},$$
$$\sigma_{lf}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not lower semi-Fredholm} \},$$
$$\sigma_{sf}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not semi-Fredholm} \},$$
$$\sigma_f(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm} \}.$$

We say that an operator $T \in \mathcal{C}(X)$ is upper semi-Browder if it is upper semi-Fredholm and has finite ascent. Similarly, $T$ is lower semi-Browder if it is lower semi-Fredholm and has finite descent. An operator $T$ is Browder if it is both lower and upper semi-Browder, see ([13]). The upper semi-Browder spectrum $\sigma_{ub}(T)$, the lower semi-Browder spectrum $\sigma_{lb}(T)$, the spectrum semi-Browder $\sigma_{sb}(T)$ and the Browder spectrum $\sigma_b(T)$ of $T$ are defined by

$$\sigma_{ub}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not upper semi-Browder} \},$$
$$\sigma_{lb}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not lower semi-Browder} \},$$
$$\sigma_{sb}(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not semi-Browder} \},$$
$$\sigma_b(T) = \{ \lambda \in \mathbb{C} : \lambda - T \text{ is not Browder} \}.$$

Consider in $X$ the well-posed Cauchy problem

$$\begin{cases}
  u''(t) = Au(t), & t \in \mathbb{R} \\
  u(0) = u_0 \\
  u'(0) = u_1
\end{cases} \quad \text{(*)}$$

Where $A : X \rightarrow X$ is a densely defined closed operator with nonempty resolvent set $\rho(A)$. The problem (*) is (see [4, 10]) well-posed if and only if $A$ generates a strongly continuous cosine operator function $(C(t))_{t \in \mathbb{R}}$, i.e., a family of operators satisfying the following conditions:

1. $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.
2. $C(0) = I$ (the identity operator).
3. $C(t)x$ is strongly continuous with respect to $t$ for any fixed $x \in X$.

There exist some $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

If $(C(t))_{t \in \mathbb{R}}$ is a strongly continuous cosine operator function, then the infinitesimal generating operator $A$ is defined by

$$\mathcal{D}(A) = \left\{ x \in X : \lim_{s \rightarrow 0} \frac{2(C(s)x - x)}{s^2} \text{ exists} \right\}$$

and

$$Ax = \lim_{s \rightarrow 0} \frac{2(C(s)x - x)}{s^2}.$$

A solution of problem (*) is given with the help of a strongly continuous cosine operator function by the formula $u(t) = C(t)u_0 + S(t)u_1$ for $t \in \mathbb{R}$, where $S(t)$ is the sine operator function associated with the $(C(t))_{t \in \mathbb{R}}$ and is defined as $S(t)x := \int_0^t C(s)x ds$, $t \in \mathbb{R}$, $x \in X$. In this work we will use the theory of integration in the sense of Bochner.

If $(C(t))_{t \in \mathbb{R}}$ is a uniformly continuous operator cosine function then there is an $A \in \mathcal{B}(X)$ with $C(t) = \cosh t\sqrt{A}$, $t \in \mathbb{R}$. We have $A = \lim_{s \rightarrow 0} \frac{2(C(s) - I)}{s^2}$ in the uniform operator topology, see [6, Theorem.2.18].
For $t \in \mathbb{R}$, the function $f : z \in \mathbb{C} \mapsto \cosh t \sqrt{z}$ defines an entire function. Thus, according to the spectral mapping theorem, we have $\cosh t \sqrt{\sigma(A)} = \sigma(C(t))$, for all $t \in \mathbb{R}$, with $\sigma$ the spectrum corresponding to regularity in the sense of V. Müller \cite[Definition 6.1]{Muller}.

In the context of a strongly continuous cosine the following spectral inclusion $\cosh t \sqrt{\sigma(A)} \subseteq \sigma(C(t))$, $t \in \mathbb{R}$ was obtained by B. Nagy; he also gave an example where the reverse inclusion fails \cite{Nagy} and he showed that $\sigma(C(t)) = \cosh t \sqrt{\sigma(A)}$, $t \in \mathbb{R}$, with $* \in \{p, r\}$.

However there are several large classes of generators $A$ for which the spectrum of $C(t)$ can be expressed in terms of $\sigma(A)$, namely $\sigma(C(t)) = \cosh t \sqrt{\sigma(A)}$, $t \in \mathbb{R}$, if $A$ is the generator of a uniformly bounded cosine function on a Hilbert space \cite{Nagy} or of a cosine function of normal operators \cite{Drazin}.

In this paper, we continue to study the spectral theory of strongly continuous cosine operator function. We investigate the relationships between the different spectra of a strongly continuous cosine operator function and their generators, precisely we prove that

$$\cosh t \sqrt{\sigma(A)} \cup \{ -1, 1 \} \subseteq \sigma_*(C(t)) \cup \{ -1, 1 \},$$

where $\sigma_*$ denotes the upper and lower semi-Fredholm, semi-Fredholm, Fredholm, Drazin, upper and lower semi-Browder, semi-Browder, Browder, essential ascent and descent spectra. We show by counter-example that these inclusions are strict in general.

## 2 Main results

The following lemmas are among the most widely used results of this paper.

**Lemma 2.1.** \cite[Lemma. 4]{Nagy} Let $A$ be the generator of the cosine operator function $C$. For $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, let

$$S_A(t)x := \int_0^t \sinh \lambda(t - s)C(s)xds, \quad x \in X.$$  

Then $S_A(t) \in \mathcal{B}(X)$ is an operator that commutes with $A$, and

$$(A - \lambda^2)S_A(t)x = \lambda(C(t) - \cosh \lambda t)x,$$

for all $x \in X$.

**Lemma 2.2.** Let $A$ be the generator of a cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin \mathbb{Z}$, there exist two operators $L_A(t), G_A(t) \in \mathcal{B}(X)$ such that

$$(A - \lambda^2)L_A(t) + G_A(t)S_A(t) = I.$$  

Moreover, the operators $L_A(t), S_A(t), G_A(t)$ and $A - \lambda^2$ are mutually commuting.

**Proof.** For all $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \notin \mathbb{Z}$, let $K_A(t)x := \int_0^t \sinh \lambda(t - s)S_A(s)xds$. It is clear that $K_A(t)$ is a bounded linear operator of $X$. We consider the function $f : s \in [0, t] \mapsto \sinh \lambda(t - s)S_A(s)x$, then $f$ is Bochner integrable and $f(s) \in \mathcal{D}(A)$ for all $s \in [0, t]$. Moreover, $(A - \lambda^2)f(s) = \lambda \sinh \lambda(t - s)(C(t) - \cosh \lambda t)I$ is Bochner.
We put integrable. From [2, Proposition 1.1.7], $K_\lambda(t)x \in \mathcal{D}(A)$ and
\[
(A - \lambda^2)K_\lambda(t)x = \int_0^t \lambda \sinh(\lambda(t-s))C(s) - \cosh \lambda s)x ds = \lambda \int_0^t \lambda \sinh(\lambda(t-s))C(s) ds - \lambda \int_0^t \sinh \lambda(t-s) \cosh \lambda s x ds = A S_\lambda(t)x - t \lambda \sinh \lambda x.
\]
We put $F_\lambda(t) := (-t \lambda \sinh \lambda t)^{-1} K_\lambda(t)$ and $G_\lambda(t) := (t \lambda \sinh \lambda t)^{-1} I$. Then we have $(A - \lambda^2)L_n(t) + G_\lambda(t)S_\lambda(t) = I$. Furthermore, it is clear that the operators $L_n(t), S_n(t), G_\lambda(t)$ and $A - \lambda^2$ are mutually commuting. 

**Lemma 2.3.** Let $A$ be the generator of a cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $n \in \mathbb{N}^*$, $t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \not\in i \pi \mathbb{Z}$, there exist two operators $F_{\lambda,n}(t), H_{\lambda,n}(t) \in \mathcal{B}(X)$ such that,
\[
(A - \lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S_\lambda^n(t) = I.
\]
Moreover, the operators $F_{\lambda,n}(t), H_{\lambda,n}(t), S_\lambda^n(t)$ and $(A - \lambda^2)^n$ are mutually commuting.

**Proof.** By Lemma 2.2, there exist two operators $L_n(t), G_\lambda(t) \in \mathcal{B}(X)$ such that
\[
(A - \lambda^2)L_n(t) + G_\lambda(t)S_\lambda(t) = I.
\]
For all $n \geq 1$ and $x \in X$, we have $L_n^\ast(t)x \in \mathcal{D}(A^n)$. In fact, the proof is by induction. For $n = 1$, from lemma 2.2 $L_n(t)x \in \mathcal{D}(A)$, suppose that $L_n^{n-1}(t)x \in \mathcal{D}(A^{n-1})$, so $L_n(t)x \in \mathcal{D}(A^{n-1})$ and
\[
(A - \lambda^2)^{n-1}L_n^\ast(t)x = [(A - \lambda^2)L_n(t)]^{n-1}L_n(t)x = L_n(t)[(A - \lambda^2)L_n(t)]^{n-1}x \in \mathcal{D}(A),
\]
hence, $L_n(t)x \in \mathcal{D}(A^n)$. Furthermore,
\[
(A - \lambda^2)^n L_n^\ast(t) = [(A - \lambda^2)L_n(t)]^n = [I - G_\lambda(t)S_\lambda(t)]^n = I - T_{\lambda,n}(t)S_\lambda(t),
\]
with $T_{\lambda,n}(t) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} G_\lambda^k(t)S_\lambda^{k-1}(t)$. So $(A - \lambda^2)^n L_n^\ast(t) + T_{\lambda,n}S_\lambda(t) = I$. Similarly, we have
\[
T_{\lambda,n}^\ast(t)S_\lambda^n(t) = I - (A - \lambda^2)^n L_n^\ast(t) = I - (A - \lambda^2)^n \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (A - \lambda^2)^{n(k-1)}L_n^{nk}(t).
\]
We define $F_{\lambda,n}(t) = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (A - \lambda^2)^{n(k-1)}L_n^{nk}(t)$ and $H_{\lambda,n}(t) = T_{\lambda,n}(t)$. Then $(A - \lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S_\lambda^n(t) = I$. Moreover the operators $(A - \lambda^2)^n, F_{\lambda,n}(t), H_{\lambda,n}(t)$ and $S_\lambda^n(t)$ are pairwise commuting. 

**Lemma 2.4.** Let $A$ be the generator of a cosine operator function $(C(t))_{t \in \mathbb{R}}$. Then for all $q \in \mathbb{N}, t \neq 0$ and $\lambda \in \mathbb{C}$ with $\lambda t \not\in i \pi \mathbb{Z}$, if $\Re (C(t) - \cosh \lambda t)^q$ is closed, then $\Re (A - \lambda^2)^q$ is also closed.

**Proof.** Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in $\Re (A - \lambda^2)^q$ converging to $y \in X$. Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ of $\mathcal{D}(A^q)$ satisfying $(A - \lambda^2)^q x_n = y_n$. By Lemma 2.2, we obtain
\[
(A - \lambda^2)^q F_{\lambda,q}(t)y_n + H_{\lambda,q}(t)S_\lambda^q(t)y_n = y_n.
\]
Hence, we conclude that
\[
\lambda^q (C(t) - \cosh \lambda t)^q H_{\lambda, q}^q(t) x_n = S_n^q(t)(A - \lambda^2)^q H_{\lambda, q}^q(t) x_n
\]
\[
= H_{\lambda, q}^q(t) S_n^q(t)(A - \lambda^2)^q x_n
\]
\[
= H_{\lambda, q}^q(t) S_n^q(t)y_n
\]
\[
= y_n - (A - \lambda^2)^q F_{\lambda, q}(t)y_n.
\]
Thus, \(y_n - (A - \lambda^2)^q F_{\lambda, q}(t)y_n \in \mathcal{R}(C(t) - \cosh \lambda t)^q\). Since \(\mathcal{R}(C(t) - \cosh \lambda t)^q\) is closed and \((A - \lambda^2)^q F_{\lambda, q}(t)\) is bounded linear operator, it follows that the sequence \(y_n - (A - \lambda^2)^q F_{\lambda, q}(t)y_n\) converges to \(y - (A - \lambda^2)^q F_{\lambda, q}(t)y\) as \(n\) tends to \(\infty\) and
\[
y - (A - \lambda^2)^q F_{\lambda, q}(t)y \in \mathcal{R}(C(t) - \cosh \lambda t)^q \subseteq \mathcal{R}(A - \lambda^2)^q.
\]
We obtain \(y \in \mathcal{R}(A - \lambda^2)^q\), which completes the proof.

**Theorem 2.1.** Let \(A\) be the generator of a cosine operator function \((C(t))_{t \in \mathbb{R}}\). Then for all \(t \neq 0\),
\[
cosh t\sqrt{\sigma(A)} \cup \{-1, 1\} \subseteq \sigma(C(t)) \cup \{-1, 1\},
\]
with \(* \in \{uf, lf, sf, f\}.

**Proof.** Suppose that \(C(t) - \cosh tA\) is upper semi-Fredholm, then \(N(C(t) - \cosh \lambda t)\) is finite dimensional and \(\mathcal{R}(C(t) - \cosh \lambda t)\) is closed. By Lemma 2.4, we obtain \(R(A - \lambda^2)\) is closed. Since \(N(A - \lambda^2) \subseteq N(C(t) - \cosh \lambda t)\), then \(N(A - \lambda^2)\) is finite dimensional. Therefore \(A - \lambda^2\) is upper semi-Fredholm.

2. Let \(C(t) - \cosh tA\) is lower semi-Fredholm, then \(\mathcal{R}(C(t) - \cosh \lambda t)\) is a subspace of \(X\) of finite codimension. Since \(\mathcal{R}(C(t) - \cosh \lambda t) \subseteq \mathcal{R}(A - \lambda^2)\), then \(\mathcal{R}(A - \lambda^2)\) is a subspace of \(X\) of finite codimension. Therefore \(A - \lambda^2\) is lower semi-Fredholm.

3. It is easy by the previous assertions of this theorem.

4. Obvious.

**Theorem 2.2.** Let \((C(t))_{t \in \mathbb{R}}\) be a strongly continuous cosine function of operators with infinitesimal generator \(A\). Then for all \(t \neq 0\),
\[
cosh t\sqrt{\sigma_D(A)} \cup \{-1, 1\} \subseteq \sigma_D(C(t)) \cup \{-1, 1\}
\]
We need the following lemma, which will also be useful later.

**Lemma 2.5.** Let \(A\) be the generator of a cosine operator function \((C(t))_{t \in \mathbb{R}}\). Then for all \(t \neq 0\) and \(\lambda \in \mathbb{C}\) with \(\lambda t \notin in\mathbb{Z}\),

1. if \(d(C(t) - \cosh \lambda t) = n\), then \(d(A - \lambda^2) \leq n\);
2. if \(a(C(t) - \cosh \lambda t) = n\), then \(a(A - \lambda^2) \leq n\).

**Proof.** If \(d(C(t) - \cosh \lambda t) = n\), then \(\mathcal{R}(C(t) - \cosh \lambda t)^n = \mathcal{R}(C(t) - \cosh \lambda t)^{n+1}\). Let \(y \in \mathcal{R}(A - \lambda^2)^n\), then there exists \(x \in \mathcal{D}(A^n)\) such that \((A - \lambda^2)^n x = y\). Then there exists \(z \in X\) such that \((C(t) - \cosh \lambda t)^n z = (C(t) - \cosh \lambda t)^{n+1} z\). By Lemma 2.2 we have
\[
(A - \lambda^2)^n (A - \lambda^2)^n F_{\lambda, n}(t) + H_{\lambda, n}(t)S_n(t) = I.
\]
Then,
\[
y = (A - \lambda^2)^n ((A - \lambda^2)^n F_{\lambda, n}(t) + H_{\lambda, n}(t)S_n(t)) x,
\]
\[
= (A - \lambda^2)^2 F_{\lambda, n}(t) x + \lambda^n H_{\lambda, n}(t)(C(t) - \cosh \lambda t)^n x,
\]
\[
= (A - \lambda^2)^n F_{\lambda, n}(t) x + \lambda^n H_{\lambda, n}(t)(C(t) - \cosh \lambda t)^{n+1} z,
\]
\[
= (A - \lambda^2)^n F_{\lambda, n}(t) x + \lambda^n H_{\lambda, n}(t)(A - \lambda^2)^n S_{\lambda, n}(t) z,
\]
\[
= (A - \lambda^2)^{n+1} F_{\lambda, n}(t) x + \lambda^n H_{\lambda, n}(t)S_{\lambda, n+1}(t) z.
\]
So \(y \in \mathcal{R}(A - \lambda^2)^{n+1}\), hence \(\mathcal{R}(A - \lambda^2)^n = \mathcal{R}(A - \lambda^2)^{n+1}\). Finally \(d(A - \lambda^2) \leq n\).
2. If \( a(C(t) - \cosh \lambda t) = n \) then \( N(C(t) - \cosh \lambda t)^n = N(C(t) - \cosh \lambda t)^{n+1} \). Let \( x \in N(\lambda - A)^{n+1} \). From Lemma 2.1, \( x \in N(C(t) - \cosh \lambda t)^n \). Then,

\[
(A - \lambda^2)^n x = (A - \lambda^2)^n [(A - \lambda^2)^n F_{\lambda,n}(t) + H_{\lambda,n}(t)S^n(t)]x,
\]

\[
= (A - \lambda^2)^{n+2} F_{\lambda,n}(t)x + (A - \lambda^2)^n H_{\lambda,n}(t)S^n(t)x,
\]

\[
= (A - \lambda^2)^{n+1} F_{\lambda,n}(t)x + \lambda^n H_{\lambda,n}(t)(C(t) - \cosh \lambda t)^n x,
\]

\[
= (A - \lambda^2)^{n+1} F_{\lambda,n}(t)\lambda(n+1)x,
\]

\[
= 0.
\]

Therefore, \( x \in N(A - \lambda^2)^n \) and hence \( a(A - \lambda^2) \leq n \).

\[\square\]

Theorem 2.1 and Lemma 2.5 imply the following corollary:

**Corollary 2.1.** Let \( (C(t))_{t \in \mathbb{R}} \) be a strongly continuous cosine function of operators with infinitesimal generator \( A \). Then for all \( t \neq 0 \),

\[
cosh t \sqrt{\sigma^*(A)} \cup \{-1, 1\} \subseteq \sigma^*(C(t)) \cup \{-1, 1\},
\]

with \(* \in \{ub, lb, sb, b\} \).

**Proof of Theorem 2.2.** If \( C(t) - \cosh \lambda t \) is of invertible Drazin, then the descent and the ascent of \( C(t) - \cosh \lambda t \) are finite and equal to \( n \). From [7, Theorem 20.4], we have \( R(C(t) - \cosh \lambda t)^n \) is closed. By Lemma 2.4, \( \Re(C(t) - \cosh \lambda t)^n \) is closed. According to Lemma 2.5, the descent and the ascent \( \lambda - \lambda^2 \) are finite and from [12, Theorem 21.1], we have \( a(A - \lambda^2) \leq d(A - \lambda^2) \leq n \). It is easy to see that \( N(A - \lambda^2)^n \cap N(A - \lambda^2)^n = \{0\} \). Indeed if \( u \in N(A - \lambda^2)^n \cap N(A - \lambda^2)^n \), then there exists \( v \in D(A - \lambda^2)^n \) such that \( u = (A - \lambda^2)^n v \). And as \( u \in N(A - \lambda^2)^n \), we see that \( v \in N(A - \lambda^2)^n \). Let us show that \( x = R(A - \lambda^2) + N(A - \lambda^2)^n \). As \( d(A - \lambda^2) \leq n \), then in particular we have \( R(A - \lambda^2)^n = R(A - \lambda^2)^{2n} \). So if \( u \in D(A - \lambda^2)^n \), then \( (A - \lambda^2)^n u \in R(A - \lambda^2)^n = R(A - \lambda^2)^{2n} \). So there exists \( v \in D(A - \lambda^2)^n \) such that \( A^n u = (A - \lambda^2)^{2n} v \). Hence \( u - (A - \lambda^2)^n v \in N(A - \lambda^2)^n \). Consequently \( u \in R(A - \lambda^2)^n + N(A - \lambda^2)^n \) and therefore \( D(A - \lambda^2)^n \subseteq R(A - \lambda^2)^n + N(A - \lambda^2)^n \). According [2, Theorem 3.14.17], \( A \) generates a strongly continuous semigroup and from [3, 18 Proposition], \( D(A - \lambda^2)^n \) is dense in \( X \), since \( R(A - \lambda^2)^n + N(A - \lambda^2)^n \) is closed, then \( X = \Re(A - \lambda^2)^n + N(A - \lambda^2)^n \). From [1, Theorem 1.35], we have \( A - \lambda^2 \) is of invertible Drazin, which finishes the proof.

**Lemma 2.6.** Let \( (C(t))_{t \in \mathbb{R}} \) be a strongly continuous cosine function of operators with infinitesimal generator \( A \). Then for all \( t \neq 0 \) and \( \lambda \in \mathbb{C} \) with \( \lambda t \notin in \mathbb{Z} \),

1. if \( d_e(C(t) - \cosh \lambda t) = n \), then \( d_e(A - \lambda^2) \leq n \).
2. If \( a_e(C(t) - \cosh \lambda t) = n \), then \( a_e(A - \lambda^2) \leq n \).

**Proof.** Suppose that \( C(t) - \cosh \lambda t \) has finite essential descent, then there exists \( n \in \mathbb{N} \) such that \( R(C(t) - \cosh \lambda t)^n / R(C(t) - \cosh \lambda t)^{n+1} \) is finite dimensional. Let

\[
\phi : \Re(A - \lambda^2)^n \rightarrow \Re(C(t) - \cosh \lambda t)^n / \Re(C(t) - \cosh \lambda t)^{n+1}
\]

the mapping defined by

\[
\phi((A - \lambda)^n x) = (C(t) - \cosh \lambda t)^n x + R(C(t) - \cosh \lambda t)^{n+1}.
\]

Thus, by isomorphism Theorem, we obtain \( \Re(A - \lambda^2)^n / N(\phi) \) is isomorphic to \( \Re(C(t) - \cosh \lambda t)^n / R(C(t) - \cosh \lambda t)^{n+1} \). Therefore \( \Re(A - \lambda^2)^n / N(\phi) \) is finite dimensional. Since \( N(\phi) \subseteq R(C(t) - \cosh \lambda t)^n \subseteq \Re(A - \lambda^2)^{n+1} \), then \( \Re(A - \lambda^2)^n / R(C(t) - \cosh \lambda t)^n \subseteq \Re(A - \lambda^2)^{n+1} / N(\phi) \). Finally, \( A - \lambda^2 \) has finite essential descent.
2. Suppose that \( \text{C}(t) - \cosh \lambda t \) has finite essential ascent. Then \( \mathcal{N}(\text{C}(t) - \cosh \lambda t)^{n+1} / \mathcal{N}(\text{C}(t) - \cosh \lambda t)^n \) is finite dimensional. Let 
\[
\psi : \mathcal{N}(\lambda - A)^{n+1} \rightarrow \mathcal{N}(\text{C}(t) - \cosh \lambda t)^{n+1} / \mathcal{N}(\text{C}(t) - \cosh \lambda t)^n
\]
the mapping defined by 
\[
\psi(x) = x + \mathcal{N}(\text{C}(t) - \cosh \lambda t)^n.
\]
By isomorphism Theorem, \( \mathcal{N}(\lambda - A)^{n+1}/\mathcal{N}(\psi) \) is isomorphic to \( \mathcal{R}(\psi) \), since \( \mathcal{R}(\psi) \subseteq \mathcal{N}(\text{C}(t) - \cosh \lambda t)^{n+1} / \mathcal{N}(\text{C}(t) - \cosh \lambda t)^n \), Then \( \mathcal{N}(\lambda - A)^{n+1}/\mathcal{N}(\psi) \) is finite dimensional. From Lemma 2.3, we have, 
\[
\mathcal{N}(\psi) \subseteq \mathcal{N}(\lambda - A)^n \cap \mathcal{N}(\text{C}(t) - \cosh \lambda t)^n \subseteq \mathcal{N}(\lambda - A)^n,
\]
hence, 
\[
\mathcal{N}(\lambda - A)^n/\mathcal{N}(\lambda - A)^n \subseteq \mathcal{N}(\lambda - A)^{n+1}/\mathcal{N}(\psi).
\]
Finally, \( \mathcal{N}(\lambda - A)^{n+1}/\mathcal{N}(\lambda - A)^n \) is finite dimensional.

**Theorem 2.3.** Let \( \text{C}(t)_{t \in \mathbb{R}} \) be a strongly continuous cosine function of operators with infinitesimal generator \( A \). Then for all \( t \neq 0 \),
\[
\cosh t \sqrt{\sigma(A)} \cup \{-1, 1\} \subseteq \sigma(\text{C}(t)) \cup \{-1, 1\},
\]
with \( \sigma_\ast \in \{\sigma_{ld}, \sigma_{rd}\} \).

**Proof.** The conclusion follows from Lemma 2.4 and Lemma 2.6

**Remark 2.1.** Let \( X \) be the complex \( l_2 \) space, and for \( (z_n)_{n \in \mathbb{N}} \in l_2, s \in \mathbb{R} \) put \( \text{C}(s)(z_n)_n = (\cos(ns)z_n)_n \). Then \( A(z_n)_n = (-n^2 z_n)_n \) with \( \text{D}(A) = \{(z_n)_n \in l_2 : \sum_{n=1}^{\infty} n^2 |z_n|^2 < \infty \} \) and \( \sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}^+\} \). Then \( \cosh \sqrt{\sigma(A)} \) is countable, with \( \sigma = \{\sigma_{uf}, \sigma_{lf}, \sigma_{fd}, \sigma_{ul}, \sigma_{ub}, \sigma_{ld}, \sigma_{rb}, \sigma_{rd}, \sigma_{rb}^e, \sigma_{rd}^e\} \). From \[8\], \( \sigma(\text{C}(1)) \) contains the set \([-1, 1]\) \( \{\cos n : n \in \mathbb{N}^+\} \) which is uncountable set. Then \( \sigma(\text{C}(1)) \) is uncountable. By \[9\], Corollary 2.10, we have \( \sigma_\ast(\text{C}(1)) \) is uncountable, with \( \sigma_\ast \in \{\sigma_{uf}, \sigma_{lf}, \sigma_{fd}, \sigma_{ul}, \sigma_{ub}, \sigma_{ld}, \sigma_{rb}, \sigma_{rd}, \sigma_{rb}^e, \sigma_{rd}^e\} \). This shows that all of the above inclusions are strict.

**Question 2.1.** Under what conditions were equal in the previous inclusions?

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