Quantifying the Capacity Gains in Coarsely Quantized SISO Systems with Nonlinear Analog Operators

Farhad Shirani†, Hamidreza Aghasi‡
†Florida International University, ‡University of California, Irvine,
Email: fshirani@fiu.edu, haghasi@uci.edu

Abstract—The power consumption of high-speed, high-resolution analog to digital converters (ADCs) is a limiting factor in implementing large-bandwidth mm-wave communication systems. A mitigating solution, which has drawn considerable recent interest, is to use a few low-resolution ADCs at the receiver. While reducing the number and resolution of the ADCs decreases power consumption, it also leads to a reduction in channel capacity due to the information loss induced by coarse quantization. This implies a rate-energy tradeoff governed by the number and resolution of ADCs. Recently, it was shown that given a fixed number of low-resolution ADCs, the application of practically implementable nonlinear analog operators, prior to sampling and quantization, may significantly reduce the aforementioned rate-loss. Building upon these observations, this work focuses on single-input single-output (SISO) communication scenarios, and i) characterizes capacity expressions under various assumptions on the set of implementable nonlinear analog functions, ii) provides computational methods to calculate the channel capacity numerically, and iii) quantifies the gains due to the use of nonlinear operators in SISO receiver terminals. Furthermore, circuit-level simulations, using a 65 nm Bulk CMOS technology, are provided to show the implementability of the desired nonlinear operators in the analog domain. The power requirements of the proposed circuits are quantified for various analog operators.

I. INTRODUCTION

In order to satisfy the ever-growing demand for higher data-rates, the fifth generation (5G) of wireless networks operate in a spectrum which includes frequencies above 6 GHz especially the millimeter wave (mm-wave) bands. This allows for larger channel bandwidths compared to earlier generation radio frequency (RF) systems which operate in lower frequency bands. The energy consumption of components such as analog to digital converters (ADCs) increases significantly with bandwidth [1]. For instance, the power consumption of current commercial high-speed (≥ 20 GSample/s), high-resolution (e.g. 8-12 bits) ADCs is around 500 mW per ADC [2]. In the standard ADC design, the power consumption is proportional to the number of quantization bins and hence grows exponentially in the number of output bits [1]. As a result, one method which has been proposed to address high power consumption in mm-wave systems is to use a few low-resolution ADCs at the receiver [3]–[10]. The application of low-resolution ADCs poses fundamental questions in the design of receiver architectures, coding strategies, and capacity analysis of the resulting communication systems.

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This work focuses on the receiver architectures and set of achievable rates in single-input single-output (SISO) systems equipped with low resolution ADCs. The setup has been considered extensively in prior works. An important initial result was due to the elegant approach proposed by Witsenhausen [11], which implies that, under peak power constraints, the capacity of a SISO system with one K-bit ADC is achieved by a discrete input distribution with at most $K+1$ mass points. Later, this extended to SISO scenarios with average input power constraints [12]. For multiple-input multiple-output (MIMO) systems with low-resolution ADCs under peak power constraints, it was shown that the optimal input distribution has a finite discrete alphabet [13]. Similarly, in multiterminal communications, for multiple-access channels (MAC) with a single antenna at each terminal and a single one-bit ADC at the receiver, input cardinality bounds were derived under peak power constraints [14]. It should be noted that although in these scenarios the input distribution has a discrete and finite alphabet with known cardinality, the optimization in the channel capacity expression is complex since each choice of input mass points and quantization thresholds yields a different set of channel transition probabilities. As a result, deriving analytical expressions for the channel capacity is challenging and often computational methods are proposed to evaluate the capacity, e.g. the cutting-plane algorithm [12], [15].

Recently, we considered MIMO communication systems with one-bit ADCs, and showed that the use of nonlinear analog operators, whose output is a polynomial function of their input, prior to sampling and quantization at the ADCs may significantly reduce the rate-loss due to coarse quantization [16]. The receiver setup is shown in Fig. 1. Furthermore, we
introduced an analog circuit design which produces a quadratic function of its input signal. The underlying idea in the circuit design is to leverage the nonlinearities of analog components to produce harmonics of the input signal, which are then extracted via frequency filtering techniques. It should be noted that the circuit complexity and power consumption increases with the degree of the desired polynomial. As a result, there are practical constraints on the degree of polynomials which are implementable under a given power budget. In this work, we build upon the observations in [16] and investigate transceiver design and the resulting channel capacity in SISO systems.

The main contributions of this work are summarized below:

- To characterize the high SNR SISO channel capacity as a function of the number of ADCs, $n_q$, number of output levels of each ADC, $\ell$, and maximum polynomial degree which is implementable using analog circuits, $\delta$.
- To provide computational methods for finding the channel capacity and quantifying the gains due to nonlinear analog processing, and to provide explanations of how these gains change as SNR, $n_q$, $\ell$, and $\delta$ are changed.
- To provide circuit designs and associated performance simulations for implementing polynomials of degree up to four; and to evaluate their power consumption.

It should be noted that for MIMO scenarios with one-bit ADCs closed-form capacity expressions are derived in [16] in terms of single-letter information measures. However, these expressions involve optimization steps which may not be computable in general, or have high computational complexity. In contrast, in this work, we consider general low resolution ADCs — as opposed to one-bit ADCs — and provide computational methods to quantify the resulting channel capacity.

**Notation:** The set $\{1, 2, \ldots, n\}$ is represented by $[n]$. The vector $(x_1, x_2, \ldots, x_n)$ is written as $x(1:n)$ and $x^i$, interchangeably. Similarly, we interchange $x(i)$ and $x_i$. The vector $(x_k, x_{k+1}, \ldots, x_n)$ is denoted by $x(k:n)$. We write $\| \cdot \|_2$ to denote the $L_2$-norm. An $n \times m$ matrix is written as $h(1:n, 1:m) = [h_{ij}]_{i,j\in[m]}$, its $j$th column is $h(:,j)$, $j \in [m]$, and its $i$th row is $h(i,:)$. We write $f$ and $h$ instead of $f(1:n)$ and $h(1:n, 1:m)$, respectively, when the dimension is clear from context. Sets are denoted by calligraphic letters such as $\mathcal{X}$, families of sets by sans-serif letters such as $\mathcal{X}$, and collections of families of sets by $\mathcal{X}$. $\Phi$ represent the empty set. For the set $\mathcal{A} \subset \mathbb{R}^n$, the set $\partial \mathcal{A}_k$ denotes its boundary. $\mathbb{B}$ denotes the Borel $\sigma$-field. For the event $E$, the variable $\mathbb{I}(E)$ denotes the indicator of the event. The notation $N \sim N(0, 1)$ indicates that $N$ is a zero-mean, unit-variance Gaussian distribution.

### II. System Model

We consider a SISO channel, whose input and output $(X, Y) \in \mathbb{R}^2$ are related through $Y = hX + N$, where $N \sim N(0, 1)$, and $h \in \mathbb{R}$ is the (fixed) channel gain coefficient. We assume that the transmitter and receiver have perfect knowledge of $h$, and the channel input has average power constraint $P$, i.e., $E\|X\|^2 \leq P$. Let the message $M$ be chosen uniformly from $[\Theta]$, where $\Theta \subset \mathbb{N}$. The communication blocklength is $n \in \mathbb{N}$ and the communication rate is $\frac{1}{n} \log \Theta$. The transmitter produces $e(M) = X^n$, where $e : [\Theta] \rightarrow \mathbb{R}^n$ is the encoding function. At the $i$th channel-use, the input $X(i), i \in [n]$ is transmitted and the receiver receives $Y(i) = hX(i) + N(i)$. The receiver produces the message reconstruction $\hat{M} = d(Y^n)$, where $d : \mathbb{R}^n \rightarrow [\Theta]$ is the decoding function. The choice of the decoding function $d(\cdot)$ is restricted by the limitations on the number of low-resolution threshold ADCs, $n_q \in \mathbb{N}$, the number of output levels of the ADCs, $\ell \in \mathbb{N}$, and the set of implementable nonlinear analog functions:

$$\mathcal{F}_a = \{f(x) = \sum_{i=0}^{\delta} a_i x^i, x \in \mathbb{R} | a_i \in \mathbb{R}, i = 0, 1, \ldots, \delta\},$$

which consists of all polynomials of degree at most $\delta \in \mathbb{N}$. The restriction to low-degree polynomial functions is due to limitations in analog circuit design, and the implementability of such functions is justified by the circuit designs and simulations provided in Section V.

The receiver architecture, shown in Figure 1, consists of:

i) A set of elementwise analog processing functions $f_{a,j} \in \mathcal{F}_a, j \in [n_q]$ operating on channel output $Y$ and producing the vector $W(1:n_q)$, where $W(j) = f_{a,j}(Y), j \in [n_q]$.

ii) A set of $n_q$ ADCs, each with $\ell$ output levels and threshold vectors $t(j, 1: \ell - 1) \in \mathbb{R}^{\ell-1}$, $j \in [n_q]$ operating on the vector $W(1:n_q)$ and producing $\hat{W}(1:n_q)$, where

$$\hat{W}(j) = k \quad \text{if} \quad W(j) \in \{t(j, k), t(j, k+1)\}, k \in [0, \ell - 1],$$

where $j \in [n_q]$ and we have defined $t(0, 0) = -\infty$ and $t(\ell, \ell) = \infty$. We call $t(1:n_q, 1: \ell - 1)$ the threshold matrix.

iii) A digital processing module represented by $f_d : \{0, 1, \ldots, \ell - 1\}^{n_q} \rightarrow [\Theta]$, operating on the block of ADC outputs after $n$-channel use of $\hat{W}(1:n, 1:n_q)$. After the $n$th channel-use, the digital processing module produces the message reconstruction $\hat{M} = f_d(\hat{W}(1:n, 1:n_q))$. The communication system is characterized by $(P, h, n_q, \ell, \delta)$, and the transmission system by $(n, \Theta, e, f_{a,q}^{e}, t(1:n_q, 1: \ell - 1), f_d)$, where $f_{a,q}^{e} = (f_{a,1}, f_{a,2}, \ldots, f_{a,n_q})$ and $f_{a,j}, j \in [n_q]$ are polynomials with degree at most $\delta$, and $e(\cdot)$ is such that the channel input satisfies the average power constraint. Achievability and probability of error are defined in the standard sense. The capacity maximized over all implementable analog functions is denoted by $C_G(P, h, n_q, \delta, \ell)$.

### III. Communication Strategies and Achievable Rates

In this section, we investigate the SISO channel capacity for a given communication system parametrized by $(P, h, n_q, \ell, \delta)$.

#### A. Preliminaries

Let us consider the scenario where $n_q$ one-bit ADCs are used at the receiver, where $n_q > 1$, and the set of implementable analog functions $\mathcal{F}_a$ is restricted to quadratic functions, i.e., $\delta = \ell = 2$. In [16, Th. 4], the high SNR capacity was derived for MIMO systems with one-bit ADCs. The result implies that the high SNR SISO capacity is equal to $1 + \log n_q$ bit/channel-use, and is strictly greater than $\log (1 + n_q)$ bit/channel-use, which is the hybrid beamforming capacity where linear analog processing is used. The proof relies on a geometric
argument. To elaborate, it was argued that the number of messages transmitted per channel-use is equal to the number of partition regions of the output space imposed by the ADC quantization process. For one-bit threshold ADCs with linear analog processing, the number of partition regions is equal to \( n_q + 1 \), hence the high SNR capacity is \( \log(1 + n_q) \). Whereas when quadratic functions are used for analog processing, the maximum number of partition regions is equal to \( 2n_q \), yielding the capacity of \( 1 + \log n_q \). The latter statement is proved by counting the number of partition regions imposed on the one-dimensional manifold \( \{(Y, Y^2) | Y \in \mathbb{R} \} \) in partitions of \( \mathbb{R}^2 \) by \( n_q \) lines. This geometric argument does not extend naturally if ADCs with more than two output levels and higher degree polynomial functions are used, i.e., \( \delta > 2 \) and \( \ell > 2 \). In the sequel, we provide an alternative proof of [16, Th. 4] for scenarios with \( \delta = 2 \) and \( \ell = 2 \). We build upon this to derive capacity expressions for \( \delta, \ell \in \mathbb{N} \). To this end, we first introduce some useful terminology and preliminary results.

The quantization process at the receiver is modeled by two sets of functions. The analog processing functions \( f_{a,j}(\cdot), j \in [n_q] \) and ADC threshold matrix \( t(1 : n_q, 1 : \ell - 1) \).

**Definition 1 (Quantizer).** A quantizer \( Q : \mathbb{R} \rightarrow [\ell]^n_q \) characterized by the tuple \( (\ell, \delta, n_q, f_{a,j}(\cdot), t(1 : n_q, 1 : \ell - 1)) \) is defined as \( Q(y) = (Q_1(y), \ldots, Q_{n_q}(y)) \), where \( Q_j(y) = k \) if \( f_{a,j}(y) \in \{t(j,k), t(j,k+1)\}, j \in [n_q] \), the functions \( f_{a,j}(\cdot) \), \( j \in [n_q] \) are polynomials of degree at most \( \delta \), and \( t(1 : n_q, 1 : \ell - 1) \in \mathbb{R}^{n_q \times (\ell - 1)} \). The associated partition of \( Q(\cdot) \) is:

\[
P = \{P_1, j \in [\ell]^n_q \} - \Phi, \quad \text{where} \quad P_1 = \{y \in \mathbb{R} | Q(y) = i\}, i \in [\ell]^n_q.
\]

For a quantizer \( Q(\cdot) \), we call \( y \in \mathbb{R} \) a point of transition if the value of \( Q(\cdot) \) changes at input \( y \), i.e., if it is a point of discontinuity of \( Q(\cdot) \). Let \( r \) be a point of transition of \( Q(\cdot) \). Then, there must exist output vectors \( c \neq c' \) and \( \epsilon > 0 \) such that \( Q(y) = c, y \in (r - \epsilon, r) \) and \( Q(y) = c', y \in (r, r + \epsilon) \). So, there exists \( j \in [n_q] \) and \( k \in [\ell - 1] \) such that \( f_{a,j}(y) < t(j,k), y \in (r - \epsilon, r) \) and \( f_{a,j}(y) \geq t(j,k), y \in (r, r + \epsilon) \), or vice versa; or \( r \) is a root of the polynomial \( f_{a,j}(\cdot) \). Let \( r_1, r_2, \ldots, r_{(\ell-1)n_q} \) be the sequence of roots of polynomials \( f_{a,j}(\cdot) \) in \( [n_q] \), \( k \in [\ell - 1] \) (including repeated roots), written in non-decreasing order, and let \( C = (c_0, c_1, \ldots, c_{(\ell-1)n_q}) \) be the corresponding quantizer outputs, i.e., \( c_{i-1} = \lim_{y \rightarrow r^-} Q(y), i \in [(\ell - 1)n_q] \) and \( c_{(\ell-1)n_q} = \lim_{y \rightarrow \infty} Q(y) \). We call \( C \) the code associated with the quantizer and it plays an important role in the analysis provided in the sequel. Note that the associated code is an ordered set of vectors. The size of the code \( |C| \) is defined as the number of unique vectors in \( C \). Each \( c_j = (c_{1,j}, c_{2,j}, \ldots, c_{(\ell-1)n_q}, j \in [0, 1, \ldots, (\ell - 1)n_q] \) is called a codeword. For a fixed \( j \in [n_q] \), the transition count of position \( j \) is the number of codeword indices where the value of the \( j \)th element changes, and it is denoted by \( \kappa_j \), i.e.,

\[
\kappa_j = \sum_{k=1}^{(\ell-1)n_q} \mathbb{1}(c_{k,j} = c_{k+1,j} \neq c_{k,j}).
\]

It is straightforward to see that \( |P| = |C| \) since both cardinalities are equal to the number of unique outputs the quantizer produces. The following example clarifies the definitions given above.

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**Example 1 (Associated Code).** Let \( n_q = \delta = 2 \) and \( \ell = 3 \) and consider a quantizer characterized by polynomials \( f_{a,1}(y) = y^2 + 2y \) and \( f_{a,2}(y) = y^2 + 3y \), \( y \in \mathbb{R} \), and thresholds

\[
(r_1, 1) = 3, \quad (r_2, 1) = 0, \quad (r_1, 2) = 10, \quad (r_2, 2) = 18.
\]

We have:

\[
\begin{align*}
f_{a,1,1}(y) &= y^2 + 2y - 3, \quad f_{a,1,2}(y) = y^2 + 2y \\
f_{a,2,1}(y) &= y^2 + 3y - 10, \quad f_{a,2,2}(y) = y^2 + 3y - 18.
\end{align*}
\]

The ordered root sequence is \( (r_1, r_2, \ldots, r_8) = (-6, -5, -3, 2, 0, 1, 2, 3) \). The associated partition is:

\[
P = \{[-\infty, -6], (-6, -5), (-5, -3), (-3, -2), (-2, 0), (0, 1), (1, 2), (2, 3), (3, \infty)\}.
\]

The size of the code is \( |C| = 5 \). The high SNR capacity of a SISO channel using this quantizer at the receiver is \( \log|P| = \log|C| = \log 5 \).

**B. SISO Systems with One-bit ADCs and Quadratic Functions**

To illustrate the usefulness of the notion of associated code of a quantizer, introduced in the prequel, let us prove the high SNR SISO capacity result for \( \ell = 2 \) given in [16, Th. 4] using the framework introduced in Section III-A.

**Proposition 1.** Let \( h \in \mathbb{R} \) and \( n_q > 1 \). Then

\[
\lim_{P \rightarrow \infty} C_0(P, h, n_q, 2) = 1 + \log n_q.
\]

**Proof.** For a given quantizer, the high SNR achievable rate is equal to \( \log|P| = \log|C| \). So, finding the capacity is equivalent to finding the maximum \( |C| \) over all choices of \( Q(\cdot) \). First, let us prove the converse result. Note that \( |C| \leq 2n_q \) since \( c_0 = c_{2n_q} \).

The reason is that for the quadratic function \( f_{a,j}(\cdot), j \in [n_q] \), we have

\[
\lim_{y \rightarrow \infty} f_{a,j}(y) = \lim_{y \rightarrow -\infty} f_{a,j}(y) = \{-(\ell - 1)n_q, -n_q + 1, \ldots, -1, 1, 2, \ldots, n_q\}.
\]

As a result, \( \log|C| \leq 1 + \log n_q \). Next, we prove achievability. Let \( t_j = 0, j \in [n_q] \) and \( f_{a,j}(y) = -(y + n_q + 1 - j)(y - j), j \in [n_q] \). Then, \( (r_1, r_2, \ldots, r_{2n_q}) = (-n_q, -n_q + 1, \ldots, -1, 1, 2, \ldots, n_q) \) and

\[
e(i, j) = \begin{cases} 1 & \text{if } i \leq n_q, \\ 0 & \text{otherwise.} \end{cases}
\]

For instance for \( n_q = 3 \), we have \( C = \{000, 010, 110, 111, 110, 100, 000\} \). It is straightforward to see that the only
Towards this, the following proposition states several useful results provided by considering symmetric quantizers.

**Theorem 1.** Consider a SISO system parametrized by \((P, h, n_q, \delta, \ell)\), where \(P > 0, h \in \mathbb{R}, n_q > 1\), and \(\delta = \ell = 2\). Then, the capacity is given by:

\[
C_{\Omega}(P, h, n_q, \delta, \ell) = \sup_{\vec{X} \in \mathcal{X}^{2n_q}} \sup_{P \in \mathcal{P}(P)} \sup \frac{1}{P(X|Y)} I(X; \vec{Y}),
\]

where \(\vec{Y} = \frac{Q((hX + N))}{\sqrt{n_q}}\), \(N \sim N(0, 1)\), \(\mathcal{P}(P)\) is the set of probability distributions defined on \(\{x_1, x_2, \ldots, x_{2n_q+1}\}\) such that \(\mathbb{E}(X^2) \leq P\), and \(Q(y) = k\) if \(y \in [t_k, t_{k+1}), k \in [1, \ldots, 2n_q]\), and \(Q(y) = 0\) if \(y > t_{2n_q}\) or \(y < t_1\).

**Proof.** Please refer to [17].

**Remark 1.** The capacity expression in Equation 1 can be evaluated numerically, e.g., via the cutting plane algorithm [12], [15], or the extension of Blahut-Arimoto algorithm in [18]. This is investigated in Section IV, where inner-bounds are provided by considering symmetric quantizers.

**C. Low-resolution ADCs and Low-degree Polynomials**

We wish to extend our analysis to SISO systems with \(\delta, \ell > 2\). Towards this, the following proposition states several useful properties for the code associated with a quantizer \(Q(\cdot)\). These are straightforward extensions of the properties shown in the proof of Theorem 1 and their proof is omitted for brevity.

**Proposition 2 (Properties of the Associated Code).** Consider a quantizer \(Q(\cdot)\) with threshold matrix \(t(1 : n_q, 1 : \ell - 1)\) and associated polynomials \(f_{a,j}(\cdot), j \in [n_q]\), such that \(f_{a,j}(\cdot) \neq f_{a,j}(\cdot) - t(j, k), j \in [n_q], k \in [\ell - 1]\) do not have repeated roots. The associated code \(C\) satisfies the following:

1. The number of codewords in \(C\) is equal to \(\gamma \geq (\ell - 1)\delta n_q + 1\), i.e. \(C = \{c_0, c_1, \ldots, c_{\gamma-1}\}\).
2. All elements of the first codeword \(c_0\) are either equal to \(\ell - 1\) or equal to 0, i.e. \(c_{i,0} \in \{0, 1, \ldots, \gamma - 1\}\) or \(c_{i,0} = \ell - 1, i \in [0, 1, \ldots, \gamma - 1]\).
3. Consecutive codewords differ in only one position, and their \(L_1\) distance is equal to one, i.e. \(\sum_{j=1}^{n_q} |c_{i,j} - c_{i+1,j}| = 1, i \in [0, 1, \ldots, \gamma - 1]\).
4. The transition count at every position is \(\kappa_j = \frac{\gamma}{n_q} = (\ell - 1)\delta, j \in [n_q]\).
5. Let \(i_1, i_2, \ldots, i_k\) be the non-decreasingly ordered indices of codewords where the \(j\)th element has value-transitions. Then, the sequence \((c_{i_1,j}, c_{i_2,j}, \ldots, c_{i_k,j})\) is periodic, in each period it takes all values between 0 and \(\ell - 1\), and \(c_{i,j} - c_{i+1,j} = 1, k \in [\ell - 1]\). Furthermore, \(c_{i_1,j} \in [0, \ell - 1]\).
6. If \(\delta\) is even, then \(|C| \leq \min(\ell n_q, (\ell - 1)\delta n_q)\) and if \(\delta\) is odd, then \(|C| \leq \min(\ell n_q, (\ell - 1)\delta n_q + 1)\).

Next, we study the capacity region for SISO systems when \(\ell = 2\) and \(\delta\) is even. First we prove two useful propositions. The first one proves that given an ordered set \(C\) satisfying the properties in Proposition 2, one can always construct a quantizer whose associated code is equal to \(C\). The second proposition provides conditions under which there exists a code satisfying the properties in Proposition 2. The proof ideas follow techniques used in study of balanced and locally balanced gray codes [19], [20]. Combining the two results allows us to characterize the necessary and sufficient conditions for existence of quantizers with desirable properties.

**Proposition 3 (Quantizer Construction).** Let \(\ell = 2, n_q \in \mathbb{N}\) and \(\delta\) be an even number. Given an ordered set \(C \subset [0, 1]^{n_q}\) satisfying properties 1)-5) in Proposition 2, and a sequence of non-decreasing real numbers \(r_1, r_2, \ldots, r_{|C|}\), where \(\gamma = \delta n_q\). There exists a quantizer \(Q(\cdot)\) with zero threshold vector and associated polynomials \(f_{a,j}(\cdot), j \in [n_q]\) such that its associated code is \(C\), and \(r_1, r_2, \ldots, r_{|C|}\) is the non-decreasing sequence of roots of its associated polynomials \(f_{a,j}(\cdot), j \in [n_q]\).

**Proof.** Without loss of generality, let us assume that \(c_0\) is the all-zero sequence. Let \(\gamma\) be the number of codewords in \(C\). Note that in general \(\gamma\) may be larger than \(|C|\) since there might be repeated codeword sequences. Let \(t_1, t_2, \ldots, t_{|C|}\) be the transition sequence of \(C\). That is, \(i_k, k \in [1, \ldots, \gamma - 1]\) is the bit position which is different between \(c_{k-1}\) and \(c_k\). Consider a quantizer \(Q(\cdot)\) with zero threshold and associated polynomials \(f_{a,j}(\cdot) \neq f_{a,j}(\cdot) - t(j, k), j \in [n_q]\). Then, \(r_1, r_2, \ldots, r_{|C|}\) are the non-decreasing sequence of roots of \(f_{a,j}(\cdot), j \in [n_q]\), and the associated code of the quantizer \(Q(\cdot)\) is \(C\) as desired.

**Proposition 4. (Code Construction)** Let \(\ell = 2, n_q \in \mathbb{N}\), and \(k_1, k_2, \ldots, k_n\) be even numbers such that \(|k_j - k_j| \leq 2, j, j' \in [n_q]\). Then, there exists a code \(C\) with transition count at position \(\gamma\) equal to \(k_j, j \in [n_q]\) satisfying properties 1), 2), 3), and 5) in Proposition 2 such that \(|C| = \min(2^{k_1}, \sum_{j=1}^{n_q} k_j)\). Particularly, if \(k_j = \delta, j \in [n_q]\), then there exists \(C\) with \(|C| = \min(2^{n_q}, \delta n_q)\) satisfying properties 1)-5) in Proposition 2.

**Proof.** Please see [17].

Using Propositions 3 and 4, we characterize the SISO capacity for \(\ell = 2\) and even-valued \(\delta\).

**Theorem 2.** Consider a SISO system parametrized by \((P, h, n_q, \delta, \ell)\), where \(P > 0, h \in \mathbb{R}, n_q \in \mathbb{N}, \delta \in [2, 4, 6, \ldots]\), and \(\ell = 2\). Then, the capacity is given by:

\[
C_{\Omega}(P, h, n_q, \delta, \ell) = \sup_{\vec{X} \in \mathcal{X}^{2n_q}} \sup_{P \in \mathcal{P}(P)} \sup \frac{1}{P(X|Y)} I(X; \vec{Y}),
\]

where \(\gamma \geq \min(2^{k_1}, (\ell - 1)\delta n_q + 1)\), \(\vec{Y} = Q((hX + N))\), \(N \sim N(0, 1)\), \(\mathcal{P}(P)\) is the set of distributions on \(\{x_1, x_2, \ldots, x_P\}\) such that \(\mathbb{E}(X^2) \leq P\), and \(Q(y) = k\) if \(y \in [t_k, t_{k+1}, k \in [1, \ldots, \Gamma - 1]\) and \(Q(y) = 0\) if \(y > t_{\Gamma - 1}\) or \(y < t_1\).

The proof follows by similar arguments as in the proof of Theorem 1. The converse follows from Proposition 2 Item 4). Achievability follows from Proposition 4.

Furthermore, using property 6) in Proposition 2 along with the proof of Theorem 1, we derive the following upper and lower bounds to the case when \(\delta\) is an odd number.
Theorem 3. Consider a SISO system parametrized by \((P, h, n_q, \delta, \ell)\), where \(P > 0, h \in \mathbb{R}, n_q \in \mathbb{N}\), \(\delta \in \{1, 3, 5, \cdots\}\), and \(\ell = 2\). Then, the capacity satisfies:

\[
\sup_{x \in \Xi^l} \sup_{P \in \mathcal{P}_h(P)} \sup_{\gamma \in \mathcal{G}(P)} I(X; \gamma) \leq C_0(P, h, n_q, \delta, \ell) \leq \sup_{x \in \Xi^l} \sup_{P \in \mathcal{P}_h(P)} \sup_{\gamma \in \mathcal{G}(P)} I(X; \gamma),
\]

where \(\Gamma = \min(2^{n_q}, 2^{n_q} \delta n_q)\) and \(\Gamma' = \min(2^{n_q}, 2^{n_q} \delta n_q + 1)\).

Lastly, for scenarios with \(\ell > 2\) the following theorem characterizes the channel capacity. The proof follows from Propositions 2 and 4 similar to the arguments given in the proof of Theorem 1.

Theorem 4. Consider a SISO system parametrized by \((P, h, n_q, \delta, \ell)\), where \(P > 0, h \in \mathbb{R}, n_q \in \mathbb{N}\), and \(\ell, \delta \in \mathbb{N}\. Let \(\Gamma\) be the maximum size of codes satisfying condition 1)-5) in Proposition 2. Then,

\[
C_0(P, h, n_q, \delta, \ell) = \sup_{x \in \Xi^l} \sup_{P \in \mathcal{P}_h(P)} \sup_{\gamma \in \mathcal{G}(P)} I(X; \gamma).
\]

Optimizing (4) requires calculating \(\Gamma\). The number of codes satisfying conditions 1)-5) in Proposition 2 is bounded from above by \(\left((\ell - 1)\delta, (\ell - 1)\delta, \cdots, (\ell - 1)\delta\right)\). Hence, for SISO systems with a few low resolution ADCs and low degree polynomials (small \(\ell, n_q\), and \(\delta\), one can find \(\Gamma\) by searching over all such codes.

IV. NUMERICAL ANALYSIS OF SISO CHANNEL CAPACITY

In this section, we provide a numerical analysis of the capacity bounds derived in Section III and evaluate the gains due to the use of nonlinear analog components in the receiver terminal. In particular, we compute inner-bounds to the capacity expression in Section III using the extension of the Blahut-Arimoto algorithm to discrete memoryless channels with input cost constraints given in [18] to find the best input distribution, and then we conduct a brute-force search over all possible symmetric threshold vectors, where a vector \(t\) is symmetric if \(t = -t\) [12]. To find the mass points of \(X\), we discretize the real-line using a grid with step-size 0.1, and optimize the distribution over the resulting discrete space. Fig. 3 shows the resulting achievable rates for SNRs in the range of 0 to 30 dB for various values of \((n_q, \ell, \delta)\). It can be observed that the performance improvements due to the use of higher degree polynomials are more significant at high SNRs. Furthermore, it can be observed that the set of achievable rates only depends on \(\min(n_q, (\ell - 1)\delta n_q + 1)\). As a result, for instance the achievable rate when \(n_q = 1, \ell = 2, \delta = 2\) is the same as that of \(n_q = 1, \ell = 2, \delta = 1\) as shown in the figure. So, in this case, using higher degree polynomials does not lead to rate improvements. On the other hand, the achievable rate for \(n_q = 3, \ell = 2, \delta = 1\) is lower than that of \(n_q = 3, \ell = 2, \delta = 2\) as shown in the figure. So, using higher degree polynomials does lead to rate improvements in this scenario.

V. CIRCUIT DESIGN FOR POLYNOMIALS OF DEGREE UP TO FOUR

In [16], we considered a single-carrier system, where the baseband input signal is a sinc\((\cdot)\) function and showed the feasibility of implementing achievable rates for various values of \((n_q, \ell, \delta)\).

![Fig. 3. The set of achievable rates for various values of \((n_q, \ell, \delta)\).](image_url)
gain drops at higher frequencies. These results are based on CMOS 65nm technology. The power consumption can be further improved by transitioning into smaller transistor nodes.

Theorem 2 shows that the channel capacity depends on the number of ADCs through \( n_q + 1 \), so that the use of a quadratic analog operator instead of a linear operator (\( \delta : 1 \rightarrow 2 \)) has an equivalent effect on capacity as that of doubling the number of ADCs \( n_q \). This fact, along with the values given in Figure 4(b) justify the use of nonlinear analog operators. It should be noted that power consumption is dependent on circuit configuration, transistor size, and passive quality factors. These simulations serve as a proof-of-concept to justify the effectiveness of the proposed receiver architecture designs.

VI. CONCLUSION

Application of nonlinear analog operations, prior to sampling and quantization, in SISO receivers was considered. Capacity expressions under various assumptions on the set of implementable analog functions were derived. Furthermore, circuit-level simulations, using a 65 nm Bulk CMOS technology, were provided to show the implementability of the desired nonlinear analog operators with practical power budgets.

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