BIASING IN GAUSSIAN RANDOM FIELDS AND GALAXY CORRELATIONS

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ABSTRACT

In this Letter, we show that in a Gaussian random field, the correlation length—the typical size of correlated structures—does not change with biasing. We interpret the amplification of the correlation functions of subsets identified by different thresholds as being caused by the increasing sparseness of peaks over threshold. This clarifies a long-standing misconception in the literature. We also argue that this effect does not explain the observed increase of the amplitude of the correlation function ξ(r) when galaxies of brighter luminosity or galaxy clusters of increasing richness are considered.

Subject headings: galaxies: general — galaxies: statistics — large-scale structure of universe

We first explain, in mathematical terms, the notion of biasing for a Gaussian random field. Here we follow the ideas of Kaiser (1984; developed further in Bardeen et al. 1986). We then calculate biasing for some examples and clarify the physical meaning of bias in the context of Kaiser (1984). Finally, we comment on the significance of our findings for the correlations of galaxies and clusters.

We consider a homogeneous, isotropic, and correlated continuous Gaussian random field δ(x), with mean zero and variance σ2 = ⟨(δ(x))2⟩ in a volume V. The application of the following discussion to a discrete set of points is straightforward considering the effect of a smoothing length. The marginal one-point probability density function of δ is

\[ P(δ) = \frac{1}{\sqrt{2πσ}} e^{-δ^2/2σ^2}. \]

Using P, we calculate the fraction of the volume V with δ(x) ≥ νσ, \( P_v = \int_ν P(δ)δ(δ) \).

The correlation function between two values of δ(x) in two points separated by a distance r is given by ξ(r) = ⟨δ(x)δ(x + r)⟩. By definition, ξ(0) = σ2. In this context, homogeneity means that the variance σ2 and the correlation function ξ(r) do not depend on x. Isotropy means that ξ(r) does not depend on the direction n.4 An important application we have in mind are cosmological density fluctuations, δ(x) = [ρ(x) − ρ0]ρ0, where ρ0 = ⟨ρ⟩ is the mean density; but the following arguments are completely general.5 Here and in what follows we assume that the average density ρ0 is a well-defined positive quantity. This is not so if the distribution is fractal (Pietronero 1987).

Our goal is to determine the correlation function of local maxima from the correlation function of the underlying density field. Like Kaiser (1984), we simplify the problem by computing the correlations of regions above a certain threshold νσ instead of the correlations of maxima. However, these quantities are closely related for values of ν significantly larger than 1. We define the threshold density \( \theta_v(ν) \) by

\[ \theta_v(ν) = \theta(δ(δ) ≥ νσ) = \begin{cases} 1 & \text{if } δ(δ) ≥ νσ, \\ 0 & \text{otherwise}. \end{cases} \]

Note the qualitative difference between δ, which is a weighted density field, and \( \theta_v \), which just defines a set, all points having equal weight. We note the following simple facts concerning the threshold density \( \theta_v \), due only to its definition, independently on the correlation properties of δ(x):

\[ \theta_v(ν) = P_v(ν) ≤ 1, \quad \{δ(δ) ≥ νσ\} = \theta_v(ν), \]

\[ \frac{\{δ(δ) ≥ νσ\}}{P_v(ν)^2} = 1 \equiv ξ_v(r) ≤ ξ_v(0) = \frac{1}{P_v(ν)} - 1, \]

\[ ξ_v(0) > ξ_v(0) \quad \text{for } ν' > ν, \]

\[ ξ_v(0) > ξ_v(0) \quad \text{for } ν' > ν. \]

The difference between \( \theta_v \) for different values of ν is called biasing. The enhancement of \( ξ_v(0) \) for higher thresholds has clearly nothing to do with how “strongly clustered” the peaks are but is entirely due to the fact that the larger ν, the lower the fraction of points above the threshold [i.e., \( P_v(ν') < P_v(ν) \) for \( ν' > ν \)]. If we consider the trivial case of white Gaussian noise \( ξ(0) = 0 \), the peaks are just spikes. When a threshold νσ is considered, the number of spikes decreases and hence \( ξ_v(0) \) is amplified because they are much more sparse and not because they are “more strongly clustered”: we show in the following that also in the case of a correlated field \( ξ(0) ≠ 0 \) the importance of sparseness is crucial in order to explain the amplification of \( ξ_v(0) \).

In the context of cosmological density fluctuations, if the average density of matter is a well-defined positive constant, the amplitude of ξ_v(0) of matter distribution is very important, since its integral over a given radius is proportional to the over density on this scale,

\[ \sigma(R) = 3R^3 \int_0^R ξ_v(r)r^2dr. \]

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4 In other words, we assume δ(x) to be a so-called “stationary normal stochastic process” (Feller 1965).
5 Clearly, cosmological density fluctuations can never be perfectly Gaussian, since ρ(x) ≥ 0 and thus δ(x) ≥ −1, but for small fluctuations, a Gaussian can be a good approximation. Furthermore, our results remain at least qualitatively correct also in the non-Gaussian case.
The scale $R_\gamma$, where $\sigma(R_\gamma) \sim 1$, separates large, nonlinear fluctuations from small ones (Gaite, Domínguez, & Pérez-Mercader 1999). It is very important to stress the following point: from the knowledge of the functions $\xi(r)$ for two different subsets of the density field obtained from two different values $\nu$ and $\nu'$ of the threshold, it is not possible to predict the amplitude of the fluctuations of the original density field at any scale if we do not know the underlying values of $\nu$, $\nu'$, and $\sigma$. On the other hand, as we are going to show, the only feature of the original field that can be inferred by the behavior of $\xi(r)$ is the large-scale behavior of the correlation function $\xi(r)$, in particular the correlation length (if this length is finite in the statistical physics terminology). The correlation length $r_c$ can be defined as (Gaite et al. 1999)

$$r_c^2 = \frac{1}{2} \left| \frac{\nabla^2 P(k)}{P(k)} \right|_{k=0},$$

where $P(k)$ is the Fourier transform of $\xi(r)$. Note that if $r_c$ is independent of any multiplying constant in $\xi(r)$, then it is not related to its amplitude. This correlation length is that used in statistical physics and field theory (Ma 1984) and gives the length scale beyond which $\xi(r)$ decays rapidly to zero (e.g., exponentially). Roughly, this implies that the fluctuations of the field are organized in structures up to a scale $r_c$ (Gaite et al. 1999). However, in cosmology the correlation length has been defined historically (Peebles 1980) through the amplitude of $\xi(r)$ by looking at the distance $r_c$ at which it is equal to 1. Provided that a constant positive density $\rho_0$ of the field exists, $r_c$ gives the scale beyond which the fluctuations become small with respect to $\rho_0$ (then it is analogous to the previously defined $R_\gamma$), and hence it provides also the minimal size of a sample of the field giving a good estimate of the intrinsic $\rho_0$. The confusion between $r_c$ and $r_\gamma$ (see also Gaite et al. 1999) is at the basis of the misinterpretation of the concept of bias, as we are going to show.

The joint two-point probability density $P_2(\delta, \delta'; r)$ depends on the distance $r$ between $x$ and $x'$, where $\delta = \delta(x)$ and $\delta' = \delta(x')$. For Gaussian fields, $P_2(r)$ is entirely determined by the two-point correlation function $\xi(r)$ (Rise 1954; Feller 1965):

$$P_2(\delta, \delta'; r) = \frac{1}{2\pi \sqrt{\sigma^2 - \xi^2(r)}} \times \exp \left\{ -\frac{\sigma^2 (\delta^2 + \delta'^2) - 2\xi(r) \delta \delta'}{2[\sigma^2 - \xi^2(r)]} \right\}.$$

By definition

$$\xi(r) \equiv \langle \delta(x + r n) \delta(x) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\delta d\delta' \delta \delta' P_2(\delta, \delta'; r).$$

The probability that both $\delta$ and $\delta'$ are larger than $\nu \sigma$ is

$$P(\nu, r) = \int_{\nu\sigma}^{\infty} \int_{\nu\sigma}^{\infty} P_2(\delta, \delta', r) d\delta d\delta' \equiv \langle \theta(\nu) \theta(\nu) \rangle.$$

The conditional probability that $\delta(y) \geq \nu \sigma$, given $\delta(x) \geq \nu \sigma$, where $|x - y| = r$, is then just $P(\nu, r)/P(\nu)$. The two-point correlation function for the stochastic variable $\theta(\nu)\theta(\nu)$ introduced above can be expressed in terms of $P_1$ and $P_2$ by

$$\xi(\nu, r) = \frac{P(\nu, r)}{P_2(\nu, r)} - 1.$$

Defining $\xi(\nu) = \xi(\nu)/\sigma^2$, we obtain

$$P(\nu)^2 [\xi(\nu) + 1] = \frac{1}{2\pi \sqrt{1 - \xi^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dx' x^2 = 2\xi(\nu) xx'.\]$$

It is worth noting that the amplitude of $\xi(r)$ does not give information about how large the fluctuations are with respect to $\rho_0$, but it rather describes the “fluctuations of the fluctuations,” that is the fluctuations of the new variable $\theta(\nu)$ around its average $P(\nu)$. Similar arguments to those introduced for the original field can now be developed to characterize the typical scales of the new set defined by $\theta(\nu)$. In particular, one can define a correlation length $r_c(\nu)$ using the analog of equation (4) by replacing $\xi(r)$ with $\xi(\nu)$.

$$P(\nu)^2 [\xi(\nu) + 1] = \frac{1}{2\pi \sqrt{1 - \xi^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dx' x^2 = 2\xi(\nu) xx'.\]$$

One would thus expect an exponential enhancement on scales such that (Politzer & Wise 1984), $\xi(r) \ll 1$ (Politzer & Wise 1984).

$$\xi(r) = \exp [\nu^2 \xi(r)] - 1.$$

This is the relation derived by Kaiser (1984). He only states the condition $\xi(r) \ll 1$ and separately $\nu \gg 1$, which is significantly weaker than the required $r^2 \xi(r) \ll 1$, especially around the correlation length where $\xi$ is not yet very small.

It is important to note that in the cosmologically relevant regime, $\xi \gg 1$, the Kaiser relation (eq. [11]) does not apply and $\xi(\nu)$ is actually exponentially enhanced. If this mechanism would be the cause for the observed cluster correlation function, one would thus expect an exponential enhancement on scales...
at which $\xi_{\infty} \approx 1$, i.e., $R \approx 20 \ h^{-1} \ \text{Mpc}$. This is in contradiction with observations (Bahcall & Soneira 1983)\footnote{One might argue that nonlinearities which are important when the fluctuations are large can “rescue” the Kaiser relation (eq. [11]) also into the regime $\xi > 1$. There are two objections against this: First of all, as we pointed out above, $\xi > 1$ does not imply large fluctuations of the original density field. Actually, most cosmologists would agree that on $R \sim 20 \ h^{-1} \ \text{Mpc}$, where the cluster correlation function $\xi_c \sim 1$, fluctuations are linear. Second, it seems very unphysical that Newtonian clustering should act as to change the exponential relation (eq. [10]) into a linear one (eq. [11]).}

If, within a range of scales, $\xi(r)$ can be approximated by a power law $\xi = (r/r_0)^{-\gamma}$, and if the threshold $\nu$ is such that equation (11) holds, which implies $\nu \ll 1$, we have $\xi = [r/(r_0(\nu))^\gamma].$ The scales $r_0(\nu)$ for different biases are related by $r_0(\nu) = r_0(\nu') (\nu'/\nu)^{2/3}$. For that reason Kaiser, who first derived equation (11), interpreted it as an increase in the “correlation length” $r_0(\nu)$, in which our language is the homogeneity scale of the set $\theta(x)$.

In order to clarify the meaning of the two length scales $r_0(\nu)$ and $r_0(\nu)$, we first study an example of a Gaussian density field with finite correlation length $r_c$ and which is well approximated by a power law on a certain range of scales. The case in which $r_c \to \infty$ is straightforward. We set

$$\xi(r) = \frac{\sigma^2 \exp(-r/r_c)}{1 + (r/k)^2}$$

with $k \ll r_c$, $k_c \ll r$, and $r_c$ is the correlation length of the system does not change for the sets above the threshold.

For relatively small values of the threshold, $\nu \ll \nu_c \approx (k, r)^{2/3}$, one finds in this case $r_0(\nu) \ll r_c$ and $r_0(\nu) \sim k_c \nu$. On the other hand, if $\nu \gg \nu_c$, we have $r_0(\nu) \sim r_c \log(\nu)$, and in this case the statistics are dominated by shot noise (see below).

\[ \xi(r) = \frac{\sigma^2 \exp(-r/r_c)}{1 + (r/k_c)^2}. \]
and roughly independent of $\nu$. This behavior is very different from the one found in galaxy catalogs!

Let us now clarify how the amplification of $\xi(r)$ is related to the increase of the peak sparseness with the threshold $\nu$. For a Gaussian random field, the mean peak size $D_\nu(n)$ and the mean peak distance $L_\nu$ are, respectively (Vanmarcke 1983; Coles 1986), $D_\nu(n) \approx D_\nu(k_p, r_p)/\nu$ and $L_\nu(n) \approx D_\nu(k_p, r_p) \times \exp(\nu^2/6)\nu^{-2/3}$ so that

$$L_\nu/D_\nu = \nu^{\nu/3} \exp(\nu^2/6) \text{ for } \nu \gg 1. \quad (13)$$

$D_\nu(k_p, r_p)$ is given by

$$D_\nu^2 = \frac{\int_0^\infty dk P_1(k)}{\int_0^\infty dk k^2 P_1(k)}, \quad (14)$$

where $P_1(k)$ is the Fourier transform of $\xi(r)$ along a line in space [in $d = 1$, it coincides with $P(k)$]. Equation (13) shows the strong enhancement of the sparseness of peaks (object) with increasing $\nu$. It is this increase of sparseness that is at the origin of the amplification by biasing. In light of equations (10), (11), and (13), we see that increasing $\nu$ corresponds to a very particular sampling of fluctuations: the typical size of the surviving peaks $D_\nu$ is slowly varying with $\nu$, while the average distance between peaks $L_\nu$ is more than exponentially amplified, and finally the scale $r_\nu(\nu)$, over which the fluctuations are structured, is practically unchanged.

We have argued that bias does not influence the correlation length [$r_\nu(\nu) \approx r_p$], it amplifies the correlation function by the fact that the mean density $P(\nu)$ is reduced more strongly than the conditional density $P(\nu, r)/P(\nu)$. According to equation (10), this amplification is strongly nonlinear at $\xi(r)$ (exponential) at scales at which $\nu^2\xi(r) \geq 1$ and thus $\xi(r) > 1$.

Consequently, as we want to stress once more, the biasing mechanism introduced by Kaiser and discussed in this work cannot lead to a relation of the form $\xi_p(r) = \alpha_p \xi_n(r)$ over a range of scales $r_1 < r < r_2$ such that $1 < \xi_p(r_1)$ and $\xi_p(r_2) < 1$. But exactly this behavior is found in galaxy and cluster catalogs. For example, in Bahcall & Soneira (1983) or Benoist et al. (1996), a constant biasing factor $\alpha_p$ over a range from about 1 to 20 $h^{-1}$ Mpc is observed for correlation amplitudes varying from about 20 to 0.1. We therefore conclude that the explanation by Kaiser (1984) cannot be at the origin of the difference of the correlation functions observed in the distribution of galaxies with different intrinsic magnitude or in the distribution of clusters with different richness.

This result appears at first disappointing, since it invalidates an explanation without proposing a new one. On the other hand, the search for an explanation of an observed phenomenon is not only motivated if we are fully aware of the fact that we do not already have one.

Last but not least, we want to point out that fractal density fluctuations together with the fact that more luminous objects are seen out to larger distances do actually induce an increase in the amplitude of the correlation function $\xi(r)$ similar to the one observed in real galaxy catalogs (Pietronero 1987; Sylos Labini, Montuori, & Pietronero 1998). In this explanation, the linear amplification found for the correlation function has nothing to do with a correlation length but is a pure finite size effect, and the distribution of galaxies does not have any intrinsic characteristic scale.

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