Quantum-mechanical corrections to the Schwarzschild black-hole metric

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Abstract – Motivated by quantum-mechanical corrections to the Newtonian potential, which can be translated into a \( \hbar \)-correction to the \( g_{00} \) component of the Schwarzschild metric, we construct a quantum-mechanically corrected metric assuming \( -g_{00} = g^{rr} \). We show how the Bekenstein black-hole entropy \( S \) receives its logarithmic contribution provided the quantum-mechanical corrections to the metric are negative. In this case the standard horizon at the Schwarzschild radius \( r_S \) increases by small terms proportional to \( \hbar \) and a remnant of the order of Planck mass emerges.

Introduction. – The full theory of quantum gravity is one of the last unsolved challenges in fundamental science and is still eluding us. Nevertheless, some effects of quantum theory do enter the context of the theory of quantum gravity and can be handled in a rigorous way without the knowledge of the full fledged theory. Such is the case of the Hawking radiation [1] and/or the Unruh effect [2]. Apart from these paradigms there are some other interesting quantum effects related to gravity like the absence of stable orbits of fermions around a black hole [3], the quantum correction to the Bekenstein entropy \( S \) of black holes and of other black objects using different approaches to quantum gravity [4–30] and the quantum correction to the Newtonian potential or metrics [31–49] (for some applications of the new corrections see [50–52]). Indeed, the results regarding the corrected Newtonian potential \( \Phi \) spread over a period of the last forty-five years starting with the early seventies, whereas the corrections to \( S \) are a relatively new undertaking. Whichever model one uses, it turns out that \( S \) receives corrections proportional to the logarithm of the black-hole area and, in some models, also proportional to the square root of this quantity. This is also the finding of our approach starting from a different context. We will make a connection between the \( \hbar \)-corrected metric and the quantum-mechanical corrections to the entropy. We will construct our quantum-mechanically corrected metric by demanding i) that it reproduces the \( \hbar \)-corrected Newtonian limit, ii) that it reproduces the standard result for the entropy of a black hole (BH) including, in addition, the \( \hbar \)-corrections which are similar to results established elsewhere and iii) that it passes some consistency checks regarding the geodesic motion of a test particle moving in this metric. Point i), which has to do with weak gravity, can be easily accommodated by invoking the classical connection between the \( g_{00} \) metric component and the Newtonian potential. The second point requires the determination of the horizons and probes into the strong-regime of gravity. In principle, we cannot infer the strong gravity effects from results zeroing around the weak regime as is the case of quantum corrections to the Newtonian potential. However, we let ourselves be guided by the fact that in the most radially symmetric metrics the time component is the inverse of the radial one. We will take over this fact to the quantum-mechanically corrected metric and show that this step is sufficient to derive the quantum correction to the Bekenstein entropy. Strictly speaking, this step is justified after having obtained the right result. Finally, we check how the equation of motion of a test particle gets affected by the quantum corrections. If overall the new metric is in accordance with observational facts including the classical tests of general relativity (GR), we can then consider such a result as consistent.

We note that we arrive at the standard results for the black-hole entropy (logarithmic corrections) obtained in different ways elsewhere. This gives us some confidence about the \( \hbar \)-corrections to the metric and the way we
handle the calculation. Again, our results show that it is not necessary to invoke the full machinery of a particular quantum gravity theory to derive a valid quantum-mechanical result in gravity. Indeed, the quantum corrections to $\Phi$ have been obtained by treating gravity as an effective field theory, which is a conventional approach.

The paper is organized as follows. In the next section we motivate the metric and present its full form. We give the first insight into the horizons connected with the metric. The third section is devoted to the thermodynamics of the black hole governed by the quantum-mechanically corrected metric. Here we calculate the corrections to the Bekenstein black-hole entropy. In the subsequent section we show by means of the heat capacity that a black-hole remnant emerges. This is followed by a section in which we compare our results with results obtained in the literature. The sixth section discusses the geodesic equation of motion resulting from the new metric. This serves as a consistency check to show that no unwanted features will appear in the motion of a test particle. In the last section we draw our conclusions.

$h$-correction to the metric. – In the field-theoretical language of an effective field theory of gravity the Fourier transform of an elastic scattering amplitude gives the potential in $r$. The one-loop correction being always proportional to $h$ represents then (after the Fourier transform) the quantum-mechanical correction to the potential under discussion. The terms considered in such a calculation are non-analytic terms which come from the propagation of two or more massless particles (gravitons) in the Feynman diagrams. These terms dominate over the analytic contributions in the low-energy limit of the effective theory [37]. They are of the type $1/\sqrt{-q^2}$ and $\ln(q^2)$, where $q^2$ is the momentum transfer. After the Fourier transform the first term emerges as proportional to $1/r^2$, whereas the second one gives $1/r^3$ terms. The full result is often written in the form

$$\Phi(r) = -\frac{GM_1M_2}{r} \left[ 1 + \lambda \frac{GM_1 + M_2}{r^2c^2} - \gamma \frac{Gh}{r^3c^4} + \ldots \right],$$

(1)

where the $\lambda$ and $\gamma$ are parameters which take different values depending on the framework under consideration. Partly, we can attribute the reason for the discrepancies in $\lambda$ to the precise coordinate definition used in the calculation [45]. The question about the ambiguity of this potential, due to the lack of clarity in the coordinates, has also been put forward in some related articles [37,45,53]. It is argued that a redefinition $r \rightarrow r' = r(1 + aGM/r)$ would change the parameter $\lambda$ without affecting the observables. The general consensus is that we can write the corrected potential as [45,46]

$$\Phi(r) = -\frac{GM_1M_2}{r} \left[ 1 - \gamma \frac{Gh}{r^3c^3} + \ldots \right]$$

(2)

taking $r$ to be the distance between two objects with masses $M_1$ and $M_2$, namely the static Schwarzschild

Table 1: Different values of $\gamma$ found in the literature.

| Year | Reference | $\gamma$ |
|------|-----------|----------|
| 1994 | [32]      | $-\frac{17}{25}$ |
| 1995 | [33]      | $-\frac{121}{197}$ |
| 1995 | [34]      | $-\frac{17}{25}$ |
| 1998 | [35]      | $-\frac{197}{305}$ |
| 2002 | [36]      | $-\frac{121}{197}$ |
| 2003 | [37]      | $-\frac{41}{107}$ |
| 2003 | [38]      | $-\frac{197}{305}$ |
| 2007 | [39]      | $-\frac{41}{107}$ |
| 2007 | [40]      | $-\frac{197}{305}$ |
| 2012 | [41]      | $-\frac{41}{107}$ |
| 2015 | [42]      | $-\frac{41}{107}$ |

$r$ [36,54] and not the gauges considered in other references. Hence, we note that the terms proportional to $\lambda$ in (1) are not additional relativistic corrections, but artifacts of the different choice of coordinates. The aforementioned reparametrization freedom still cannot account for all the discrepancies of the different $\gamma$’s found in the literature. A number of errors have been identified [32,37,55], but it is not clear if this accounts for all the different values available. It is therefore fair to list some of the results (see table 1). In table 1 we have collected the different values for $\gamma$ which also vary in sign. It is also worth noting that the value in [33] differs from the result in [43] not because of an error which occurred in the calculation, but due to a deliberate choice of the amplitude: ref. [33] considers only the one-particle reducible amplitude in contrast to, e.g., [43] where the full amplitude is taken into account.

Given the history of the subject, our approach here will be to take the latest value (see, however, [47]), as the correct one, i.e., $\gamma = -41/10$. By requiring absence of negative norm states it has been argued in [56,57] that modified gravity cannot become weaker than the pure Einstein gravity. This would hint towards a negative $\gamma$. We can make the argument more conclusive by using the lower bound on the dimensionless weak coupling $\alpha_g$ which, according to [56,58], reads $\alpha_g(p) \geq p^2/M_p$, where $M_p = 1/\sqrt{16\pi G}$ (reduced Planck mass) and $p$ is a typical momentum in the process which we interpret here as the momentum transfer $q$. Since $q \sim 1/r$ we can tentatively estimate that $\alpha_g(r) \geq 1/(r^2M_p)$. In pure gravity the dimensionless quantity $\alpha_g(r)$ is simply $G/r^2$ and the Newtonian potential reads $\Phi = -\alpha_g(r)M_1M_2r$. In the quantum-corrected gravity we have $\alpha_g(r) = (G/r^2)[1 - \gamma(l_p/r)^2]$ ($l_p$ denotes the Planck length defined by $l_p^2 = Gh/c^5$) and the inequality from above tells us that $-\gamma(l_p/r)^2 \geq 0$ which implies a negative $\gamma$.

At the level of the amplitude, unitarity is violated at the energy $E_{\text{CM}}^2 = 20(GN)^{-1}$, where $N$ is a sum of scalars,
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fermions and vector particles [59]. This would indicate that the validity of the approach breaks down as we are approaching the Planck scale. It is argued in [60] that the violation of unitarity at the tree level is “self-healing” through iteration of vacuum polarization, but the effect on the correction to the Newtonian potential remains open. Therefore, we will mention it explicitly if our conclusions touch the Planck region. We kept the presentation on the horizon that this assumption bears some non-perturbative elements. Indeed, it is not anymore of the form: classical corrections to the Bekenstein entropy of the black holes. We assume that the corrected metric is such that the correction to the Newtonian potential remains open. We kept here the limits of integration open as they depend on whether or not we have a black-hole remnant which we will discuss below. After expanding the expression inside the integral to first order in $\bar{h}$, we will arrive to

$$D = \frac{\beta^2}{4} - \frac{\beta}{27},$$

gives us the necessary information on the number of real roots [61]. We recall that if $D > 0$ the polynomial in $\xi$ has only one real solution, if $D < 0$ it has three real solutions or if $D = 0$ it has two real solutions. In the following we will consider the physical case, namely when $\gamma$ is negative.

**Thermodynamics.** After taking $\beta = -|\beta|$, which is equivalent to having $\gamma < 0$, the reduced third order polynomial in $\xi$ can be written as

$$\xi^3 - \frac{1}{3} \xi - \frac{2}{27} - |\beta| = 0,$$  \hspace{1cm} (6)

and the discriminant $D = |\beta|^2 - |\beta|^3$ is always positive definite. Hence, only one real solution exists to the polynomial in $\xi$, i.e., $\xi_1$. Expanded in $\beta$, $\xi_1$ reads

$$\xi_1 = \frac{2}{3} + |\beta| + \mathcal{O}(\beta^{3/2}).$$  \hspace{1cm} (7)

The horizon receives a quantum-mechanical correction of the form

$$r_{nh} = r_s + |\beta| r_s = r_s + \frac{l_p^2}{r_s} |\gamma|.$$  \hspace{1cm} (8)

Within our previous assumption, $-g_{00} = g^{rr}$, the temperature of the black hole is given by $T = \frac{\hbar}{2\pi c} \gamma$, where we take $k_B = 1$ and $\kappa$ is the surface gravity defined as

$$\kappa = \frac{1}{2} \lim_{r \to r_{nh}} \frac{\partial g_{00}}{\partial r} \bigg|_{g_{00} = 0}.$$  \hspace{1cm} (9)

After some algebraic manipulations we find the surface gravity to be

$$\kappa = \frac{2GM}{(2c^2 e^{\frac{\hbar}{2GM} |\gamma|})^2 + \frac{3G^2 M \hbar}{c^5} \frac{|\gamma|}{(2c^2 e^{\frac{\hbar}{2GM} |\gamma|})^2}}.$$  \hspace{1cm} (10)

The black-hole temperature suitably expanded in $\hbar$ takes the simple expression

$$T = \frac{\hbar}{2\pi c} \left( \frac{c_4}{4GM} + \frac{c_5 h}{16G^2 M^2 |\gamma|} + \mathcal{O}(\hbar^2) \right).$$  \hspace{1cm} (11)

The black-hole entropy is computed using the first law of black-hole thermodynamics [62] $dS = c^2 dM / T$. In integral form it is written as [63]

$$\int dS = \frac{2\pi c^3}{\hbar} \int \frac{dM}{\left( \frac{4GM}{c^4} + \frac{c^5 h}{16G^2 M^2 |\gamma|} + \mathcal{O}(\hbar^2) \right)}.$$  \hspace{1cm} (12)

We left here the limits of integration open as they depend on whether or not we have a black-hole remnant which we will discuss below. After expanding the expression inside the integral to first order in $\hbar$,

$$S = \frac{2\pi c^3}{\hbar} \int dM \left[ \frac{4GM}{c^4} - \frac{\hbar}{c^3 M |\gamma|} + \mathcal{O}(\hbar^2) \right].$$  \hspace{1cm} (13)

Going back to our usual variables we get the final result

$$S = S_{BH} - \pi |\gamma| \ln \left[ \frac{A}{4l_p^2} \right] + \pi |\gamma| \ln \left[ 4\pi \right] + \mathcal{O}(\hbar^2),$$  \hspace{1cm} (14)
where we introduced the Schwarzschild black-hole area, i.e., $A = 4\pi r_s^2 = \frac{16\pi G^2 M^2}{c^4}$ and the classical expression obtained by Bekenstein [64] is $S_{BH} = A/4\pi^2$.

Furthermore, eq. (12) has an exact solution given by $S = S_{BH} - \pi|\gamma|\ln\frac{4m}{\bar{\hbar}^2\pi^2} + |\gamma|\ln A + \pi|\gamma| |\gamma| + O(\hbar)^2$.

We will comment on this result in the fifth section, but we note already here that there exists an overwhelming agreement in the literature on logarithmic corrections to the black-hole entropy [4–12,22–30]. Whereas most of the results use a model for quantum gravity, we have obtained corrections of the same form by analyzing $\hbar$-corrections to the Newtonian potential via an effective theory of gravity.

**Heat capacity and the black-hole remnant.** Let us compute the heat capacity of the black-hole using the standard expressions $C = c^2\frac{dA}{dT}$. From eq. (11) we can expand $T^{-1}$ up to order one in $\hbar$ and deduce two solutions for $M$ as a function of $T^{-1}$:

$$M_{\pm} = \frac{1}{T} \pm \sqrt{\left(\frac{1}{T}\right)^2 + 64\frac{\pi^2 G}{\hbar c^4}|\gamma|}. \quad (15)$$

We note that taking $\gamma = 0$ forces us to consider the positive sign in order to recover the usual case for the Schwarzschild black hole. Therefore, the heat capacity turns out to be

$$C = -\frac{2\pi}{m_p^2} \left[\frac{-4M^2 + m_p^2|\gamma|}{4M^2 + m_p^2|\gamma|}\right]. \quad (16)$$

At this point, let us define the remnant mass, $M_r$, by $C(M_r) = 0$ [65–69]. Thus, when $M_r$ is reached, the black-hole evaporation stops. From eq. (16) we obtain $M_r = \sqrt{|\gamma|}m_p$, which is of the order of the Planck mass $m_p$. We can relate this remnant mass to a maximum temperature, by taking equation (11) and replacing the mass by the value of the remnant mass, $M_r$. This yields

$$T(M_r) \equiv T_{max} = \frac{c^2}{4\pi m_p^2} \left[\frac{1}{|\gamma|^{1/2}} + \frac{1}{|\gamma|^{3/2}}\right]. \quad (17)$$

which is of the order of $T_0 = c^2m_p$, a number suggested by Sakharov [70] for the maximum temperature of thermal radiation. We have basel the black-hole remnant on the zero value of the heat capacity. In a different context a black-hole remnant emerges when the temperature becomes complex [13,71] where the maximum temperature is of the same order as here.

As we have already mentioned the validity of the approach might break down at the Planck scales due to the lack of unitarity of the amplitude at these scales. Higher-order corrections might become important here. Therefore, the above results are to be taken with caution. We have included them because the one-loop result often shows the right tendency of the final phenomenological effect. Secondly, as mentioned before the metric assumption is essentially non-pertubative and should, in principle, take some of the higher-order effects into account. Indeed, both, the remnant black-hole mass of the order of the Planck mass and the maximum temperature have analogies in the literature. We point out that the existence of a remnant in the $\gamma < 0$ case is in complete agreement with similar conclusions obtained within the quadratic GUP formalism [65,66]. Hawking radiation formulated within the formalism of a generalized uncertainty relation also indicates a black-hole remnant as shown in [71]. Including the cosmological constant $\Lambda$ the generalized uncertainty relation not only gives the maximum temperature and minimum mass, but, in addition, also a minimum temperature of the order of $\sqrt{\Lambda}$ and a maximum mass proportional to $m_p^2/\sqrt{\Lambda}$ [72].

**Logarithmic corrections to the black-hole entropy in different models.** Different approaches to quantum gravity have predicted corrections to the Bekenstein-Hawking entropy for black holes of the form [4–12]

$$S = \frac{A}{4\pi^2} + c_0 \ln \left(\frac{A}{4\pi^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4\pi^2}\right)^{-n}. \quad (18)$$

where the $c_n$ coefficients are parameters which depend on the specific model considered. Interestingly, loop quantum gravity calculations are used to fix $c_0 = -1/2$ [7]. Moreover, the deformed commutation relations giving place to a generalized uncertainty principle (GUP) have been also used to compute the effects of the GUP on the black-hole entropy from different perspectives (see, for example, [5,13–16]). In this case the entropy reads

$$S = \frac{A}{4\pi^2} + \sqrt{\pi \alpha_0} \frac{A}{4\sqrt{\pi^2}} - \pi \alpha_0^2 \ln \left(\frac{A}{4\pi^2}\right) + O(\frac{\hbar^2}{c^4}). \quad (19)$$

where $\alpha_0$ measures the deviation for the standard Heisenberg case, i.e.,

$$x_i = x_0; \quad p_i = p_0 \left(1 - \alpha p_0 + 2\alpha^2 p_0^2\right), \quad (20)$$

where $[x_0, p_0] = i\hbar\delta_{ij}$ and $p_0^2 = \sum_{j=1}^{3} p_{0j}p_{0j}$ and $\alpha = \alpha_0/m_p c$, with $\alpha_0$ being a dimensionless constant [73,74]. Even more, polymerization (a non-standard representation of quantum mechanics that was inspired by loop quantum gravity (LQG)) also predicts logarithmic corrections to the black-hole entropy [17] (it has been shown that polymerization and quadratic GUP are equivalent provided $\alpha_0$ and the polymerization parameter are proportional [75]). In fact, the leading-order corrections to the entropy of any thermodynamic system due to small statistical fluctuations around equilibrium, when applied to black holes, are shown to be of the form $\ln|A|$ [8]. Of course, the four-dimensional Schwarzschild BH has a negative heat capacity and to apply the previous idea, the authors of [8] considered an AdS-Schwarzschild BH and take the limit of large $\Lambda$ to account for a stable system.
Once this is realized, the leading corrections are shown to be logarithmic. In this sense, the authors of [8] claim that log-corrections are universal.

Therefore, the corrections given by eq. (14), obtained from one-loop calculations, are consistent with different approaches which incorporate, in some sense, some quantum-gravitational considerations. Specifically, the correct sign for the logarithmic term is obtained for the $\gamma < 0$ case. As we commented before, this is the case of LQG and of quadratic GUP. Therefore, our approach is consistent with both of them provided $|\gamma| = (2\pi)^{-1} = \alpha_0^2/64$. Interestingly, we note that, in a recent work [76], the authors propose a technique to compute the deformation parameter of the GUP by using the leading quantum corrections to the Newtonian potential, somehow in the same spirit of our work.

We would like to note that our analysis uses macroscopic black-hole properties related by the first law. In this sense, no use of properties of the underlying microscopical theory, whatever it will be, are used. Although we are aware of the thermodynamical instability of the Schwarzschild black hole without putting it in a box, we expect that the case here considered turn to be thermodynamically stable by adding a cosmological constant. Therefore, following [8], the corrections to the canonical entropy due to small statistical fluctuations around equilibrium will be shown to be of the form $\ln[A]$. Even more, although the coefficient of the logarithmic corrections (and even their sign) depend on the particular scheme one used to define the entropy [77], the logarithmic dependence seems to be universal.

In this sense, we think that our comparison with other models will not be complete unless Euclidean path integral methods are taken into account [78]. As is well known, in the stationary phase and one-loop approximation, the leading correction to the partition function is given by the metric which is an Euclidean classical solution and the one-loop contribution comes from the action due to matter fields. In this sense, the physical interpretation of the logarithmic corrections in this approach is clear: they come from the spectrum of massless fields and their coupling to the gravitational background.

In particular, the first Euclidean gravity calculation on the log-correction for a Schwarzschild BH has been recently reported by Sen [79]. Let us recall that the exact numerical prefactor accompanying the log-correction depends on the employed ensemble and on the number of fields (other than gravity) included in the calculation. However, the log-correction is recovered in all cases. Moreover, as pointed out by Sen, the macroscopic results for logarithmic corrections seem quite robust since loops (higher than one) do not give any log-correction.

We are aware that the statistical interpretation provided for the log-correction (which arises in this context even without matter simply by graviton fluctuations) by the path integral method is not obvious in our approach. However, although the particular numerical prefactor of these approaches does not agree in general (the same lack of agreement is also present in the path integral and loop quantum gravity comparison), we think that in our work we show explicitly the universality of log-corrections to BH entropy from an effective field theory approach. This idea can be traced back to a recent work by Bjerrum-Bohr, Donoghue and Vanhove [80]. Specifically, as recently expressed by the authors of this work, we stress that our results are universal and thus will hold in any quantum theory of gravity with the same low-energy degrees of freedom as we are considering. Therefore, both the quantum correction to the Newtonian potential, the corresponding correction to the Schwarzschild metric [55] and the logarithmic correction to the entropy are among these universal results.

In retrospect, the agreement with other findings on the corrections to the black-hole entropy gives us some confidence about the quantum-mechanical corrections to the Newtonian potential and the conclusions drawn from it.

Finally, we note that there exist other approaches to the quantum aspects of the Schwarzschild metric and we refer the reader to the original literature (see [81] and references therein).

The geodesic equation of motion. – One of the key observables in GR is the particle trajectory given by the metric. We check whether the quantum-mechanical corrections proposed above will change the standard predictions in a drastic way. This would be the case if, for instance, new circular (stable or unstable) orbits would appear leading to new phenomenological results for the particle trajectory. Even if these corrections to the geodesic equation of motion are proportional to $\ell_p$, there is no a priori guarantee that all observables will receive small corrections and that no new features will emerge. Small quantum effects on the three-body Lagrangian points were recently found using the same corrections to the Newtonian potential [82]. Mixing of scales can lead to new results as it happens, e.g., in the Schwarzschild-de Sitter metric where scales of the cosmological constant combined with the Schwarzschild radius reveal new aspects of the effective potential [83].

The quantum corrections to the effective potential. Using the metric given above, one can cast the equation of motion in the form $\frac{\dot{r}}{r} + V_{\text{eff}}(r) = 0$. Here the effective potential can be split as two terms indicating the classical and the quantum part $V_{\text{eff}}(r) = V_{\text{eff}}^{(h)}(r) + V_{\text{eff}}^{(b)}(r)$, i.e.,

$$V_{\text{eff}}^{(h)}(r) = \begin{cases} \frac{GM}{r}, & \text{if } m \neq 0, \\ \frac{\gamma}{2} - \frac{GM^2}{c^2r^3}, & \text{if } m = 0, \end{cases}$$

and

$$V_{\text{eff}}^{(b)}(r) = \begin{cases} \frac{G^2Mh}{c^3r^5} + \frac{G^2M^2h}{c^2r^5} \gamma, & \text{if } m \neq 0, \\ \frac{G^2M^2h}{c^3r^5} \gamma, & \text{if } m = 0, \end{cases}$$

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with $m$ the mass of the test particle and $I$ the angular momentum per mass. We will study the extrema and zeros of the corrected effective potential. If no new zeros and local extrema emerge as a result of the quantum corrections and the new zeros and extrema receive corrections of the order of $\hbar$, we can consider the theory as consistent and in accordance with observational facts. In the antipodal case of additional extrema this would imply new stable and unstable circular orbits.

The massless case. Based on the $m = 0$ case in eqs. (21) and (22) we look for the zeros $(r_0)$ of this function. With $r_0 \neq 0$ we obtain $r_0^2 - r_s r_0 + \beta^2 = 0$. Defining $x = \frac{r_0}{r_s} \beta = \frac{r_0}{r_s} \gamma$ and setting $y = x - \frac{1}{2}$ we arrive at a third-order polynomial, i.e., $y^3 - \frac{3}{2} y - \frac{1}{2} + \beta = 0$. The discriminant of this equation is given by $D = \frac{1}{2} \beta^2 - \frac{1}{2} \beta$. For $\gamma < 0$ it turns out that $D > 0$, which implies one real solution of the cubic polynomial. The only real zero in this case is

$$r_0 = r_s - \frac{l^2}{r_s} \gamma + O(l_p)^3.$$  \hfill (23)

To find the extreme we put the derivative of the effective potential to zero, which results in a third-order equation in $r_{\text{max}}$. The latter can be transformed using $\xi = \frac{r_{\text{max}}}{r_s} - \frac{1}{2}$ to $\xi^3 - \frac{3}{2} \xi - \frac{1}{2} + \frac{3}{2} \beta = 0$. The discriminant in this case is $D_{\text{max}} = 25 \beta^2 - \frac{3}{2} \beta$. The relevant case is $D_{\text{max}} > 0$ if $(\gamma < 0 \text{ or } r_s^2 < 5 l_p^2)$, meaning only one real solution. The radius of the unstable photon circular orbit receives a correction proportional to $\hbar$,

$$r_{\text{max}} = \frac{3}{2} r_s - \frac{10 l_s^2}{9 r_s} \gamma + O(l_p)^3.$$  \hfill (24)

The first term in $r_{\text{max}}$ is the standard result in GR.

The massive case. We use the effective potential from the $m \neq 0$ case in eqs. (21) and (22). The search for the zeros and extrema gives a fourth-order polynomial. To be pragmatic we skip the details and quote the final result obtained with the help of MATHEMATICA. The physically relevant zeros are

$$r_0^{1,2} = \frac{1}{2} \left[ \frac{l^2}{c^2 r_s} \pm \sqrt{\Delta_0} - \left( 1 \pm \frac{l^2}{\sqrt{\Delta_0}} \right) \frac{1}{r_s} \gamma \right] + O(l_p),$$  \hfill (25)

where $\Delta_0 = \frac{4 l_s^4}{r_s^4} - 4 \frac{l^2}{r_s}$. As $\hbar \rightarrow 0$ we recover the usual roots. The extrema are located at

$$r_{\text{max}}^{1,2} = \frac{l^2}{c^2 r_s} \pm \sqrt{\Delta_{\text{max}}} - \frac{1}{9} \left. \left( 5 \pm \frac{12 r_s + 10 l_s^2}{\sqrt{\Delta_{\text{max}}}} \right) \frac{l^2}{r_s} \gamma \right|_{r_{\text{max}}},$$  \hfill (26)

where $\Delta_{\text{max}} = \frac{4 l_s^4}{r_s^4} - 3 \frac{l^2}{r_s}$. For real $r_0$ values we impose $l^2 > 4 r_s^2$ and, for real $r_{\text{max}}$ values, we have $\frac{l^2}{r_s} > 3 r_s^2$. In the classical case of the Schwarzschild metric, only respecting $l < l_{\text{crit}} = \sqrt{3} r_s c$ yields no local extrema.

Conclusions. – We have explored the consequences of quantum-mechanical corrections to the Newtonian potential. This correction in tandem with $-g_{00} = g^{rr}$ fixes the metric. We probe into the physics around the horizon of this metric. We find a corrected Schwarzschild horizon where the correction is proportional to $\hbar$. This was used to infer the corrections to the black-hole entropy. We derived logarithmic corrections in agreement with many other approaches. A black-hole remnant of the order of Planck mass emerges in this case. Finally, we examine the consequences of the $h$-correction in the geodesic equation of motion and find that that classical tests of GR will be affected only marginally.

In conclusion, the simple quantum-mechanical correction to the Newtonian potential taken together with a reasonable assumption on the $g^{rr}$ component has remarkable consequences. Whether Hawking radiation or Bekenstein entropy, the quantum mechanics in the gravity of a black hole is focused at the horizon. We added to this list a quantum mechanical correction of the horizon and connected it with the correction to the entropy.

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REFERENCES

[1] HAWKING S. W., Commun. Math. Phys., 43 (1975) 199; 46 (1976) 206.
[2] Unruh W. G., Phys. Rev. D, 14 (1976) 870.
[3] Lasenby A., Doran C., Pritchard J., Caceres A. and Dolan S., Phys. Rev. D, 72 (2005) 105014; Batic D. and Nowakowski M., Class. Quantum Grav., 25 (2008) 225022; Batic D., Morgan K., Nowakowski M. and Bravo Medina S., Dirac equation in the Kerr-de Sitter metric, arXiv:1509.00452; Batic D. and Nowakowski M., Universe, 2 (2016) 31.
[4] Kaul R. K. and Majumdar P., Phys. Rev. Lett., 84 (2000) 5255.
[5] Medved A. J. and Vagenas E. C., Phys. Rev. D, 70 (2004) 124021.
[6] Amelino-Camelia G., Arzano M. and Procacci A., Phys. Rev. D, 70 (2004) 107501.
[7] Meissner K. A., Class. Quantum Grav., 21 (2004) 5245.
[8] Das S., Majumdar P. and Bhaduri R. K., Class. Quantum Grav., 19 (2002) 2355.
[9] Chatterjee A. and Majumdar P., Phys. Rev. Lett., 92 (2004) 141301.
[10] Åkbar M. M. and Das S., Class. Quantum Grav., 21 (2004) 1383.
[11] Myung Y. S., Phys. Lett. B, 579 (2004) 205.
Quantum-mechanical corrections to the Schwarzschild black-hole metric

[12] Chatterjee A. and Majumdar P., Phys. Rev. D, 71 (2005) 024003.
[13] Adler R. J., Chen P. and Santiago D. I., Gen. Relativ. Gravit., 33 (2001) 2101.
[14] Amelino-Camelia G., Arzano M., Ling Y. and Mandanici G., Class. Quantum Grav., 23 (2006) 2585.
[15] Majumder B., Phys. Lett. B, 703 (2011) 402.
[16] Majumder B., Gen. Relativ. Gravit., 45 (2013) 2403.
[17] Gorji M. A., Nozari K. and Vakili B., Phys. Lett. B, 735 (2014) 62.
[18] Pourhassan B. and Faizal M., EPL, 111 (2015) 40006.
[19] Pourhassan B. and Faizal M., Phys. Lett. B, 751 (2015) 487.
[20] Pourhassan B. and Faizal M. and Dearnth U., Eur. Phys. J. C, 76 (2016) 145.
[21] Pourhassan B. and Faizal M., Phys. Lett. B, 755 (2016) 444.
[22] Carlip S., Class. Quantum Grav., 17 (2000) 4175.
[23] Gho A. and Mitra P., Phys. Rev. D, 71 (2005) 027502.
[24] Nojiri S., Odintsov S. D. and Ogushi S., Int. Mod. Phys. A, 18 (2003) 3395.
[25] Kaul R. K., SIGMA, 8 (2012) 005.
[26] Hod S., Class. Quantum Grav., L97 (2004).
[27] Hai-Xia Z., Huai-Fan L., Shuang-Qi H. and Ren Z., Commun. Theor. Phys., 48 (2007) 465.
[28] Tawfiq A. N. and El Daibab A., Int. J. Mod. Phys. A, 30 (2015) 1550030.
[29] Mukherji S. and Pal S. S., JHEP, 05 (2002) 026.
[30] Wang F. J., Gui Y. X. and Ma C. R., Phys. Lett. B, 660 (2008) 144.
[31] Donoghue J. F., Phys. Rev. Lett., 72 (1994) 2996.
[32] Donoghue J. F., Phys. Rev. D, 50 (1994) 3874.
[33] Hamber H. W. and Liu S., Phys. Lett. B, 357 (1995) 51.
[34] Muzinich I. J. and Vokos S., Phys. Rev. D, 52 (1995) 3472.
[35] Akhundov A., Bellucci S. and Shiekh A., Phys. Lett. B, 395 (1998) 19.
[36] Kirilovich I. B. and Kirilan G. G., J. Exp. Theor. Phys., 95 (2002) 981.
[37] Bjerrum-Bohr N. E. J., Donoghue J. F. and Holstein B. R., Phys. Rev. D, 67 (2003) 084033.
[38] Bjerrum-Bohr N. E. J., Donoghue J. F. and Holstein B. R., Phys. Rev. D, 68 (2003) 084005.
[39] Ross A. and Holstein B. R., J. Phys. A: Math. Theor., 40 (2007) 6973.
[40] Kirilan G. G., Phys. Rev. D, 75 (2007) 108501.
[41] Haranas I. I. and Mioc V., Rom. Astron. J., 20 (2010) 153.
[42] Donoghue J. F., AIP Conf. Proc., 1483 (2012) 73.
[43] Bjerrum-Bohr N. E. J., Donoghue J. F., Holstein B. R., Plante L. and Vanhove P., Phys. Rev. Lett., 114 (2015) 061301.
[44] De Lorenzo T., Pacifico C., Rovelli C. and Speziale S., Gen. Relativ. Gravit., 47 (2015).
[45] Burgess C. P., Living Rev. Rel., 7 (2004) 5.
[46] Akhundov A. and Shiekh A., Electron. J. Theor. Phys., 5, issue No. 17 (2008) 1.
[47] Wang C. L. and Woodward R. P., Phys. Rev. D, 92 (2015) 084008.
[48] Radkowski A. F., Ann. Phys., 56 (1970) 319.
[49] Capper D. M., Duff M. J. and Halpern L., Phys. Rev. D, 10 (1974) 461.
[50] Burns D. and Pilaftsis A., Phys. Rev. D, 91 (2015) 064047.
[51] Haranas I. I., Ragos O., Gkioktzis I. and Kotsireas I., Astrophys. Space. Sci., 358 (2015) 12.
[52] Faller S., Phys. Rev. D, 77 (2008) 124030.
[53] Hida K. and Okamura H., Prog. Theor. Phys., 47 (1972) 1743.
[54] Duff M. J., Phys. Rev. D, 9 (1974) 1837.
[55] Donoghue J. F. and Holstein B. R., J. Phys. G: Nucl. Part. Phys., 42 (2015) 103102.
[56] Dvali G., Folkerts S. and Germani C., Phys. Rev. D, 84 (2011) 024039.
[57] Brustein R., Dvali G. and Veneziano G., JHEP, 10 (2009) 085.
[58] Dvali G. and Gómez C., arXiv:1005.3497.
[59] Han T. and Willenbrock S., Phys. Lett. B, 616 (2005) 215.
[60] Aydemir U., Anber M. M. and Donoghue J. F., Phys. Rev. D, 86 (2012) 014025.
[61] Bronshtein I. N., Semendyayev K. A., Musiol G. and Muehlig H., Handbook of Mathematics (Springer) 2005.
[62] Bardeen J. M., Carter B. and Hawking S. W., Commun. Math. Phys., 31 (1973) 161.
[63] Suskind L. and Lindeay J., An Introduction to Black Holes, Information and the String Theory Revolution (World Scientific) 2005, p. 51.
[64] Bekenstein J., Phys. Rev. D, 7 (1973) 2333.
[65] Bajerjee R. and Gho S., Phys. Lett. B, 688 (2010) 224.
[66] Dutta A. and Gangopadhyay S., Gen. Relativ. Gravit., 46 (2014) 1747.
[67] Gangopadhyay S., Dutta A. and Faizal M., EPL, 112 (2015) 20006.
[68] Xiang L., Phys. Lett. B, 647 (2007) 207.
[69] Xiang L. and Wen X. Q., JHEP, 10 (2009) 046.
[70] Sakharov A. D., JETP Lett., 3 (1966) 288.
[71] Adler R. J., Chen P. and Santiago D. I., Mod. Phys. Lett. A, 14 (1999) 1371.
[72] Arraut L., Batie D. and Nowakowski M., Class. Quantum Grav., 26 (2006) 125006.
[73] Ali A. F., Das S. and Vagenas E. C., Phys. Lett. B, 678 (2009) 497.
[74] Ali A. F., Das S. and Vagenas E. C., Phys. Rev. D, 84 (2011) 044013.
[75] Bargueño P. and Vagenas E. C., Phys. Lett. B, 742 (2015) 15.
[76] Scardigli F., Lambiase G. and Vagens E. C., Phys. Lett. B, 767 (2017) 242.
[77] Page D. N., New. J. Phys., 7 (2005) 203.
[78] Gibbons G. W. and Hawking S. W., Phys. Rev. D, 15 (1977) 2752.
[79] Sen A., JHEP, 04 (2013) 156.
[80] Bjerrum-Bohr N. E. J., Donoghue J. F. and Vanhove P., JHEP, 02 (2014) 111.
[81] Lake M. J. and Carr B., JHEP, 11 (2015) 105.
[82] Battista E., Dell’Agnello S., Esposito G. and Simo J., Phys. Rev. D, 91 (2015) 084041.
[83] Balaguera-Antolínez A., Boehmer C. G. and Nowakowski M., Class. Quantum Grav., 23 (2006) 485.

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