Where are the Mirror Manifolds?

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ABSTRACT

The recent classification of Landau–Ginzburg potentials and their abelian symmetries focuses attention on a number of models with large positive Euler number for which no mirror partner is known. All of these models are related to Calabi–Yau manifolds in weighted \( \mathbb{P}_4 \), with a characteristic structure of the defining polynomials. A closer look at these potentials suggests a series of non-linear transformations, which relate the models to configurations for which a construction of the mirror is known, though only at certain points in moduli space. A special case of these transformations generalizes the \( \mathbb{Z}_2 \) orbifold representation of the \( D \) invariant, implying a hidden symmetry in tensor products of minimal models.

CERN-TH.6802/93
February 1993

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1 Introduction

Mirror symmetry [1,2,3], which relates string compactifications with exchanged numbers of 27 and 27 representations of $E_6$, provides a powerful guiding principle in striving for completeness in the classification of string vacua. In $N = 2$ superconformal field theory this symmetry is realized by the natural operation of redefining the right-moving $U(1)$ charge [4]. For various explicit constructions, as for example Calabi–Yau manifolds [5], on the other hand, the existence of such a mirror partner is a highly non-trivial matter.

A large class of string vacua with $(2,2)$ superconformal invariance can be constructed from Landau–Ginzburg models [6,7], which are particularly useful in case of $N = 2$ supersymmetry because of a non-renormalization theorem for the superpotential. Recently the non-degenerate potentials that give rise to $N = 2$ superconformal models with central charge $c = 9$, as needed for the internal sector, and all their abelian symmetries have been classified by construction [8, 9, 10, 11]. This requires consideration of up to 9 chiral superfields; the models with up to 5 fields, which already represent 70% of all configurations, are related to Calabi–Yau manifolds in weighted $\mathbb{P}_4$ with the same defining polynomial [12,13].

For a subclass of the non-degenerate potentials a construction of the mirror model was given by Berglund and Hübsch [14]. Their method was fully confirmed, for the relevant class of polynomials, by the calculation of all abelian Landau–Ginzburg orbifolds [11]. The full set of models, however, features a striking lack of mirror symmetry: 810 of the 3837 orbifold spectra have no mirror. In fact, this problem was apparent already in the untwisted case [9]. The new spectra from orbifolding all have Euler numbers $\chi \leq 480$, whereas there is a remarkable set of potentials with large positive Euler numbers, for which no mirror model is known yet. These “singlet spectra” range up to $\chi = 840$, already close to the maximal value of $\chi = 960$.

It is this set of polynomials that we want to analyse in the present paper. All of them have 4 or 5 non-trivial fields, and thus define Calabi–Yau manifolds. As there is only a handful of models with, say, $\chi > 500$, we expect that it is easier in this class to find some structure that may be relevant for the lack of mirror symmetry. Indeed, 7 of the 9 singlet spectra in this range come from a polynomial of class VI in Arnold’s classification [15]; the remaining two require only a slight modification, with couplings among four fields. Moreover, each spectrum can be obtained from a number of different configurations and orbifolds, which, however, look very similar. It is thus easy to find non-linear transformations with constant determinant [12], which indicate that they are, in fact, in (almost) all cases equivalent.

In section 2 we consider the model with $\chi = 840$ in some detail. Here we already find the essential structures and the non-linear transformations relating different configurations. In this case, they can even be used to represent the model in a Fermat configuration, so that for a deformation of an orbifold we would know how to construct the mirror.

In section 3 we then list all the singlet models along decreasing (modulo of the) Euler number until we hit, at $|\chi| = 450$, the first singlet spectrum with negative $\chi$. Most of these models exactly follow the pattern found in section 2. Only at $\chi = 540$ do we find a new type of polynomial. Furthermore, this spectrum apparently comes from two inequivalent configurations. A systematic search for a non-linear relation, instead, reveals a non-linear symmetry, present in each of the two models. Nevertheless, a connection between the two
configurations, involving deformations, orbifolding, and a mirror map, can be found.

In section 4 we generalize these non-linear transformations to arbitrary non-degenerate polynomials and give a simple algorithm for checking the conditions that have to be fulfilled by the exponents. Then we briefly discuss the implied hidden symmetries in the special case of tensor products of minimal models. In this case the transformation has already been found in [16]. Sections 3 and 4 are almost independent and could be exchanged. Section 5 contains our conclusions.

\section{Cutting loops and trees}

In this section we analyse the model with Euler number 840. But first we need to introduce some notation.

A configuration $\mathcal{C}_{(n_1,\ldots,n_I)}[d]$ denotes the set of non-degenerate polynomials that are quasi-homogeneous of degree $d$ in the superfields $X_i$ with respect to weights $n_i$. In weighted $\mathbb{P}_I$, $I$ must be 5 and the condition for vanishing first Chern class is $\sum n_i = d$. For Landau–Ginzburg models we can have an arbitrary number $I$ of fields, but the central charge $c = 3 \sum (1 - 2n_i/d)$ is required to be 9 for the internal sector of a heterotic string. This coincides with the Calabi–Yau condition for $I = 5$; for $I = 4$ we have to add a trivial (massive) field with $n_i = d/2$ to make contact with Calabi–Yau manifolds. For convenience, however, we will often omit such fields in the following; their appropriate transformation under symmetry groups to make determinants positive should be understood implicitly.

In both frameworks we need to require that the quasi-homogeneous polynomial $W(X_i)$ is non-degenerate (or transversal), i.e. that the origin is the only place where all gradients vanish. This implies, in particular, that for each field $X_i$ there must be a monomial of the form $X_i^{a_i} X_j$ [15] (the coefficients can be normalized to 1). We call the sum of these $I$ terms the skeleton of $W$, and say that $X_i$ points at $X_j$ if $i \neq j$. If there is more than one pointer with the same target, then non-degeneracy requires the presence of additional monomials, which we call links (the details will not be important in the following and can be found in ref. [8]). Note that the skeleton already determines the configuration. A given configuration, on the other hand, in general admits a number of inequivalent skeletons.

The Berglund–Hübsch construction of the mirror manifold [14] now applies exactly if no links are required, i.e. if the polynomial is equal to its skeleton. We call such a polynomial invertible; the mirror can then be constructed as a particular orbifold with the exponents $a_i + \delta_{ij}$ transposed along each chain of pointers in the skeleton.

For constructing a heterotic string with space-time supersymmetry we need to project the Landau–Ginzburg model to integer charges, i.e. mod out the symmetry $\mathbb{Z}_d(n_1,\ldots,n_I)$, whose generator acts by multiplication with a phase $\exp(2\pi i n_i/d)$ on the field $X_i$ [17]. This projection will always be assumed and it is only in case of additional twists that we will use the term orbifold. The gauge non-singlet particle spectrum at low energies is then determined by the numbers $n_{27} = b_{12}$ and $n_{17} = b_{11}$ of chiral primary fields with charges $(Q_L, Q_R) = (1,1)$ and $(1,2)$, respectively [18], where $b_{ij}$ are the Hodge numbers of the corresponding Calabi–Yau manifold, if it exists.
The spectra with the largest values of the Euler number $\chi = 2(n_{27} - n_{27})$ found in refs. $[9,10]$ are $(n_{27}, n_{27}, \chi) = (11,491;960), (12,462;900), (13,433;840), (14,416;804), \ldots$, of which already the third one does not have a known mirror partner. This singlet spectrum comes from the three configurations $\mathfrak{C}(24,31,244,567,866)[1732], \mathfrak{C}(36,31,366,866,1299)[2598]$, and $\mathfrak{C}(18,31,183,634,433)[1299]$, each of which has a unique skeleton:

$$X_1^{62}X_3 + X_2^{48}X_3 + X_3^7X_1 + X_4^3X_2 + X_5^2 + \varepsilon X_1^{67-31k}X_2^{4+24k}, \quad (1)$$

$$X_1^{62}X_3 + X_2^{37}X_3 + X_3^7X_1 + X_4^3 + X_5^2 + \varepsilon X_1^{67-31k}X_2^{6+36k}, \quad (2)$$

$$X_1^{62}X_3 + X_2^{36}X_3 + X_3^3X_1 + X_4^3X_2 + X_5^3 + \varepsilon X_1^{67-31k}X_2^{3+18k}. \quad (3)$$

Here we have added the simplest link monomials that make the polynomials non-degenerate (the number of monomials of degree $d$ is 28 in all three cases). A graphical representation of the potentials, with a dot for each field, an arrow for each pointer, and a dashed line for the link, is given in fig. 1A for the polynomials (1) and (3), and in fig. 1B for the second case.

The structure and the exponents of the polynomials (1)–(3) almost coincide, suggesting that the respective conformal field theories might be identical. Greene, Vafa and Warner [12] have argued that Landau–Ginzburg models should flow to equivalent renormalization group fixed points if the potentials can be related by a change of variables with constant determinant, provided that multiple coverings are taken into account by appropriate orbifolding. Indeed, such a map is easily found:

$$X_2 \rightarrow X_2^{-\frac{n}{2}}, \quad X_4 \rightarrow X_4 X_2^{-\frac{n-1}{n}}, \quad (4)$$

transforms (1) and (3) into (4), where $n = 3$ and $n = 2$, respectively. The effect on the skeletons is to “cut” the pointer from $X_4$ to $X_2$ and to change the exponent of the target field of that pointer. The transformation has constant determinant for arbitrary $n$ and is $n - 1$ to $n$. These multiplicities, however, are automatically taken care of by the canonical $\mathbb{Z}_d$ twist, as $d = 4p, 6p$ and $3p$, respectively, where $p = 433$ is the 86th prime number.

Having seen that all three representations of the model with $\chi = 840$ are related by non-linear transformations, with the correct multiplicities accidentally provided by the ratios of the degrees, it is natural to look for orbifold representations of this model. Of course, the fact that abelian orbifolds did not provide any new spectra with $\chi > 480$ (1) does not exclude this possibility. Indeed, there is even an orbifold realization in the Fermat configuration $\mathfrak{C}(1,1,12,28)[84]$, which has the spectrum $(11,491;960)$ with the largest value of the Euler number. We cannot find $\chi = 840$ for an invertible skeleton in that configuration, though, because then we would know the mirror. So we have to start from the polynomial

$$Y_1^{72}Y_3 + Y_2^{72}Y_3 + Y_3^7 + Y_4^3 + \varepsilon Y_1^{78-36k}Y_2^{6+36k} \quad (5)$$
and mod out the product of the groups $\mathbb{Z}_2(0,1,0,0)$, $\mathbb{Z}_3(1,1,0,1)$, $\mathbb{Z}_7(3,3,1,0)$, $\mathbb{Z}_8(1,7,0,0)$, and $\mathbb{Z}_9(1,8,0,0)$. Here the generator $g_{84}$ of the canonical $\mathbb{Z}_d$ is given by the product $g_{84} = g_2g_3g_7(g_8)^2$ of the respective generators.¹

Again, there is a striking similarity between the polynomials (4) and (2), which suggests to us to look for a non-linear relation. A straightforward calculation shows that

$$X_1 = Y_1^{504}, \quad X_2 = Y_2Y_1^{433}, \quad X_3 = Y_3Y_1^{433}, \quad X_4 = Y_4$$

(6)
does the job and has constant determinant. This transformation is 433 to 504, so that the $\mathbb{Z}_d$ orbifold of (4) is indeed mapped onto the above $\mathbb{Z}_{504} \times \mathbb{Z}_6$ orbifold of (4). Here the effect of the transformation is to cut the pointer from $X_3$ to $X_1$, i.e. to cut the loop, as is seen in fig. 1C, the graphical representation of (4).

The construction of the mirror manifold for a Fermat polynomial is well established [1], so for a deformation of an orbifold of our model we would know how to proceed (the untwisted model can be considered as an orbifold with respect to the quantum symmetry [19] of the orbifold). The trouble is, however, that we do not know how to mod a quantum symmetry of a Calabi–Yau manifold. In the Landau–Ginzburg framework, on the other hand, where these symmetries usually are accessible by discrete torsion [20], it is not clear how to deform the model by moduli that are not polynomial deformations but come from twisted sectors.

Note that these non-linear relations have non-trivial implications for the underlying conformal field theories. If the different orbifold models indeed flow to the same conformal field theory, then that theory should have all of the respective quantum symmetries. Unfortunately, however, in each of our Landau–Ginzburg formulations, only part of that full symmetry would be manifest.

### 3 More missing mirror models

In this section we list the singlet models with $\chi \geq 450$ along decreasing Euler number. All of the corresponding 30 configurations with 13 different spectra only admit a unique skeleton. Furthermore, all of these skeletons contain a loop and for 9 of the spectra the structure of the polynomial is the one shown in fig. 1. These 9 models are listed in table 1: For each spectrum the respective configurations are printed with a superscript (A) or (B), indicating which of the graphs in fig. 1 applies. Then we list the exponents of the fields in the skeleton. In all cases with more than one configuration the different polynomials are related by the transformation (4).

As for the model with $\chi = 840$ we are interested in constructing an equivalent orbifold representation in an invertible configuration (i.e. a configuration admitting an invertible skeleton). In most cases this can be done by using a transformation like (6) to cut the pointer from $X_3$ to $X_1$. Note that it is not possible to cut one of the pointers at $X_3$, because then we

¹The configuration $C_{1,1,12,28}[84]$ also accommodates the transpose of $X_8^{84}X_2 + X_8^{83} + X_7^3 + X_4^3$. This polynomial belongs to the configuration with the maximal degree $d = 3486$ and the minimal Euler number $\chi = -960$. 


Table 1: Singlet spectra with the skeleton of fig. 1.

| $\chi$ | $n_{27}$ | $n_{27}$ | Configuration | Exponents | Orbifold | Twist | $b_1$ |
|--------|----------|----------|---------------|-----------|----------|--------|-------|
| 840    | 13       | 433      | $C_{(24,31,244,567)}$ | 1732 | $C_{(2,3,24,55)}$ | 168 | 433/504 | 72   |
|        |          |          | $C_{(36,31,366,866)}$ | 2598 | $C_{(1,1,12,28)}$ | 84 | 433/504 | 72   |
|        |          |          | $C_{(18,31,183,634,433)}$ | 1299 | $C_{(1,2,12,41,28)}$ | 84 | 433/504 | 72   |
| 648    | 17       | 341      | $C_{(24,19,444,191)}$ | 1356 | $C_{(6,7,168,71)}$ | 504 | 113/168 | 56   |
|        |          |          | $C_{(28,19,518,226)}$ | 1582 | $C_{(1,1,28,12)}$ | 84 | 113/168 | 56   |
|        |          |          | $C_{(14,19,259,386,113)}$ | 791 | $C_{(1,2,28,41,12)}$ | 84 | 113/168 | 56   |
| 612    | 20       | 326      | $C_{(12,31,184,423)}$ | 1300 | $C_{(1,3,18,41)}$ | 126 | 325/378 | 108  |
|        |          |          | $C_{(18,31,276,650)}$ | 1950 | $C_{(1,2,18,42)}$ | 126 | 325/378 | 108  |
|        |          |          | $C_{(9,31,138,472,325)}$ | 975 | $C_{(1,4,18,61,42)}$ | 126 | 325/378 | 108  |
| 576    | 7        | 295      | $C_{(28,37,144,381)}$ | 1180 | $C_{(2,3,12,31)}$ | 96 | 295/336 | 42   |
|        |          |          | $C_{(43,192,61,590)}$ | 1770 | $C_{(1,1,16,16)}$ | 48 | 295/336 | 42   |
|        |          |          | $C_{(21,37,108,424,295)}$ | 885 | $C_{(1,2,6,23,16)}$ | 48 | 295/336 | 42   |
| 528    | 27       | 291      | $C_{(8,31,164,375)}$ | 1156 | $C_{(2,9,48,109)}$ | 336 | 289/336 | 144  |
|        |          |          | $C_{(12,31,246,578)}$ | 1734 | $C_{(1,3,24,50)}$ | 168 | 289/336 | 144  |
|        |          |          | $C_{(6,31,123,418,289)}$ | 867 | $C_{(1,6,24,81,50)}$ | 168 | 289/336 | 144  |
| 516    | 36       | 294      | $C_{(7,247,41,590)}$ | 1770 | $C_{(1,36,6,80)}$ | 258 | 295/301 | 252  |
| 510    | 38       | 293      | $C_{(4,494,41,343,147)}$ | 1029 | $C_{(2,252,21,175,75)}$ | 525 | 49/50 | 252  |
| 468    | 36       | 270      | $C_{(6,31,154,351)}$ | 1084 | $C_{(1,6,30,68)}$ | 210 | 271/315 | 180  |
|        |          |          | $C_{(9,31,231,542)}$ | 1626 | $C_{(1,4,30,70)}$ | 210 | 271/315 | 180  |
| 456    | 37       | 265      | $C_{(18,7,24,351)}$ | 1060 | $C_{(1,4,30,70)}$ | 210 | 271/315 | 180  |
would arrive at an invertible skeleton and would know how to construct the mirror. The last 3 columns of table 1 show the results of this computation: First we give the configuration of the orbifold, with the superscript indicating if it is invertible or of the Fermat type. The ratio $m/n$ in the column labelled “Twist” means that the map is $m$ to $n$, so that $nd/m$ is the order of the twist group in the orbifold representation (which must be a multiple of the degree of that configuration); $b_I$ is the new exponent of $X_1$ and the graph of the skeleton is shown in fig. 1C, where an additional pointer from $X$ to $X_4$ should be supplemented if we start from fig. 1A (for more details on these transformations see section 4 below).

Unfortunately, the lower right corner of table 1 is empty: The spectrum with $\chi = 456$ is, in fact, the only case where we do not find any possibility to cut the loop, let alone a transformation to an invertible configuration. A search of the results of ref. [11] for orbifold representations of the spectrum confirms that indeed such a transformation does not exist.

The first deviation from the above pattern occurs at Euler number 540. Again all skeletons are unique, but this time the loop contains 3 fields. The graphical representation is shown in fig. 2. The same graphs also account for the remaining 3 spectra with $\chi \geq 450$, which are listed in table 2. Here most of the entries are the same as in table 1. The only modification is that the orbifold representation can now be obtained by cutting the pointer at $X_2$ or the pointer at $X_3$, as is shown in figs. 2C and 2D, respectively. This is indicated by a superscript of the orbifold configuration; $b_I$ is the new exponent of $X_2$ or $X_3$ in the two cases. The model with $\chi = 512$ is the only one for which both pointers can be cut. Thus we find 3 different orbifold representations, where $C \cap D$ indicates that both pointers have been cut (see section 4 for more details). As before, an additional pointer from $X_5$ to $X_1$ has to be supplemented in the graphs 2C and 2D if the orbifold configuration descends from 2A. Each of the models in

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2The singlet spectrum (278, 53; −450) with the smallest Euler number, on the other hand, is quite different: It comes from the configuration $\mathfrak{f}_{(1,4,27,63,94)}[189]$, which admits 5 different skeletons and does not require a loop.
Table 2 can be realized in at least one invertible configuration, but none of them do we find in a Fermat configuration.

For the spectrum with $\chi = 540$ we have two entries in table 2, because in this case there is no obvious non-linear transformation relating the following two polynomials:

$$X_1^{44} X_4 + X_2^{38} X_4 + X_3^5 X_2 + X_4^3 X_3 + \varepsilon X_1^{30} X_2^{26},$$

(7)

$$X_1^{52} X_4 + X_2^{38} X_4 + X_3^3 X_2 + X_4^5 X_3 + \varepsilon X_1^{30} X_2^{22}.$$  

(8)

The two configurations with 5 fields are of course related to (7), (8) by the transformation (4), with $n = 2$ and a relabelling of the fields. To search for a relation between (7) and (8) I tried a general ansatz, allowing the monomials of the skeleton of one of these polynomials to transform into an arbitrary monomial of degree $d$ in the target configuration. As there are 21 such monomials, this required the computation of $\frac{21!}{17!} = 143640$ determinants (which Mathematica did in 14 hours). Unfortunately, none of these determinants is constant and non-vanishing.

As a simple check of the program I also used it to search for non-linear symmetries of the above skeletons. It should have found at least the identity, of course, but it also found an additional symmetry:

$$X_i \to X_i \rho^c_i, \quad \rho = \left(\frac{X_2^{19}}{X_1^{22}}\right)^{\frac{1}{571}}, \quad \{c_i\} = \{26, -30, 6, -2\},$$

(9)

where $p = 571$ is the 105$^{th}$ prime number. This transformation leaves (7) invariant, whereas for (8) we have to use $\rho^{571} = X_2^{19}/X_1^{26}$ and $\{c_i\} = \{22, -30, 10, -2\}$. In both cases we have a non-linear $\mathbb{Z}_2$ symmetry of the skeleton. Of course, as for all our non-linear transformations, we also have to check that there exists a system of links that is consistent with the transformation. In (7), (8) the simplest invariant link that makes the polynomial non-degenerate is already included. Surprisingly, most of the monomials of degree $d$ that exist in these configurations are invariant (the counting is done without trivial fields): For the configuration of

Fig. 2: Graphical representation of the models in table 2.
there are 15 invariant monomials of degree $d$, whereas the following three pairs of monomials are transposed by the non-linear $\mathbb{Z}_2$ symmetry: $(X_1^{14} X_4, X_2^{38} X_4)$, $(X_1^{12} X_2^7, X_1^8 X_2^{45})$, and $(X_1^{40} X_2^2 X_3, X_1^2 X_2^{40} X_3)$. In the case of (8) we have 18 invariant monomials and the two transforming pairs $(X_1^{52} X_4, X_2^{38} X_4)$ and $(X_1^{60} X_2^3, X_1^4 X_2^{41})$.

For the time being we thus have to consider the Landau–Ginzburg models based on (2) and (8) as distinct. It cannot be excluded that there exists some identification of the underlying conformal field theories, for example as an orbifold with respect to the non-linear $\mathbb{Z}_2$ symmetry, but I do not know how to check for this possibility. Still, we can find some relation: As above, we can construct orbifold representations of the models in the invertible configurations $\mathcal{C}_{(1,1,12,16)}[60]$ and $\mathcal{C}_{(1,1,8,20)}[60]$ by cutting the loops. We can then deform the potentials such that they become the transpose of each other. Thus further orbifolding and a mirror map complete the connection.

Let us finally note that almost all polynomials in table 1 have exponents 3 and 7, whereas those in table 2 generically have exponents 3 and 5. This is not surprising, as the smallest values for the central charge larger than the critical value $c = 3$ enter the derivation of limits on the exponents in non-degenerate potentials [3]. The first of these come from the polynomials $X^3 + Y^7$, $X^3 + XY^5 \sim X^5 + XY^3$, $X^3 + Y^8$, and $X^3 Y + XY^4$, with $c/3 - 1 = 1/21, 1/15, 1/12$, and $1/11$, respectively. It is then also not surprising that the models with an exponent 5 need an additional pointer. The presence of these particular exponents is thus related to the requirement of a large Euler number rather than to the absence of mirror symmetry. What appears to be significant, then, is that even in these exceptional cases there are plenty of non-linear relations, and of their associated (hidden) symmetries.

4 Another look at non-linear transformations

It is straightforward to generalize the non-linear transformations encountered above to arbitrary skeletons. With the only exception of the non-linear symmetry (9), all of them had the effect of eliminating one pointer and changing one exponent. So let us first consider the case of a pure loop and make the ansatz

$$X_1^{a_1} X_2 + X_2^{a_2} X_3 + \ldots + X_n^{a_n} X_1 = Y_1^{b_1} Y_2 + Y_2^{a_2} Y_3 + \ldots + Y_n^{a_n},$$

with $Y_i = X_i X_1^{c_i}$. Then $c_i = (-1)^{n-i}/(a_i \ldots a_n)$ for $i > 1$ and $b_1 = (a_1 - c_2)/(1 + c_1)$, where $c_1$ is to be computed from the condition $\sum_{i=1}^{n-1} c_i = 0$ for constant determinant. Because of our ansatz for the transformation, this condition is equivalent to not changing the central charge. The transformation makes sense if the exponent $b_1$, as computed from these formulae, is integer.

Now we try to cut the pointer at $X_I$ in an unbranched tree of length $n$,

$$X_1^{a_1} X_2 + \ldots + X_n^{a_n} = Y_1^{a_1} Y_2 + \ldots Y_I^{a_{I-1}} + Y_I^{b_I} Y_{I+1} + \ldots + Y_n^{a_n},$$

\[3\text{ Any accumulation point can only be approximated from below [5]. So there must be an interval above } c = 3 \text{ with a finite spectrum, providing, in a sense, a continuation of the “exceptional” cases } E_6, E_7 \text{ and } E_8 [5].\]
with the same ansatz for the transformation. Here we find \( c_i = (-1)^{I-i-1}/(a_i \ldots a_{I-1}) \) for \( i < I \) and \( c_i = 0 \) for \( i > I \). Again, \( c_I \) has to be computed from the determinant condition \( \sum c_i = 0 \) and the new exponent \( b_I = a_I/(1 + c_I) \) should be integer.

For a given field \( X_I \) in a general skeleton we can have a number of pointers at \( X_I \), which we take to be \( X_1, \ldots, X_N \), and \( X_I \) can point at \( X_{I+1} \) or can be of the Fermat type. In the latter case we formally set \( X_{I+1} = 1 \) and \( c_{I+1} = 0 \). Then the \( X_I \)-dependent part of the potential is

\[
\sum_{j=1}^{N} X_j^{a_j} X_I + X_I^{a_I} X_{I+1} = \sum_{j=1}^{J} Y_j^{a_j} + \sum_{j=J}^{N} Y_j^{a_j} Y_I + Y_I^{b_I} Y_{I+1},
\]

where we try, without loss of generality, to cut the pointers from \( Y_1, \ldots, Y_J \) and to keep the remaining ones. From our ansatz we find

\[
b_I = \frac{a_I - c_{I+1}}{1 + c_I}, \quad c_j = \begin{cases} 1/a_j & j \leq J \\ -c_I/a_j & j > J. \end{cases}
\]

For any field \( X_i \rightarrow X_k \) with a line of pointers connecting it to \( X_I \) the corresponding \( c_i \) is given recursively by \( c_i = -c_k/a_i \). If there is no such line of pointers, then \( c_i = 0 \). Thus, in particular, \( c_{I+1} = 0 \) if \( X_I \) is not in a loop. Eventually, \( c_I \) has to be computed from the condition that the sum of all \( c_i \) has to vanish and the new exponent \( b_I \) should be integer.

With these formulae it is straightforward to check all possibilities to cut a pointer in an arbitrary graph by a transformation of the form

\[
X_i \rightarrow X_i \circ X_i,
\]

where \( X_i \) is the target of that pointer. In fact, I computed the orbifold part of tables 1 and 2 by writing a simple program, which does just that (the entry “Twist” in the tables is given by the number \( 1 + c_I \)). In the above sections we found only one transformation that is not an iteration of (14), namely the symmetry (14), but even that case is of the form \( X_i \rightarrow \rho^a X_i \). Of course, in case of branchings, one always has to make sure that the additional monomials required for non-degeneracy can be chosen in a consistent way [for (14) it is sufficient to check the \( X_I \)-dependent links].

As non-linear transformations with constant determinant play such a prominent role in Landau–Ginzburg models, it would be important to have more checks of the conjectured equivalence of the corresponding orbifolds. Such a check should be possible in the simplest case, namely with two fields and a single pointer. Here the condition that the pointer can be cut implies that \( W = X^{(n-1)a} + XY^n \). The transformed potential \( \bar{W} = \bar{X}^{na} + \bar{Y}^n \) is of the Fermat type and thus a tensor product of minimal models (this case has already been studied in ref. [18]). As the transformation \( X \rightarrow X^n/(n-1), Y \rightarrow Y/X^{1/(n-1)} \) is \( n \) to \( n - 1 \), the \( \mathbb{Z}_n \) orbifold of the exactly solvable Fermat model should have a hidden \( \mathbb{Z}_{n-1} \) symmetry if the conformal field theories \( W/\mathbb{Z}_{n-1} \) and \( W/\mathbb{Z}_n \) are indeed identical.

It is easy to check that the charge degeneracies of the chiral ring agree in the two cases, which both have central charge \( c/6 = 1 - (a + 1)/(na) \) [17]: In the pointer case the charges \( q_i = n_i/d \) of the chiral fields are \( q_X = 1/((n-1)a) \) and \( q_Y = (na - a - 1)/(n(n-1)a) \). The projected untwisted sector of the \((c,c)\) ring of the orbifold is generated by

\[
\mathcal{R}^u = \langle \{ Y^{n-1} \} \cup \{ (XY)^j X^{(n-1)k} \mid j \leq n-2, \ k \leq a-1 \} \rangle,
\]

(15)
and in addition we have \( n - 2 \) twisted \((c,c)\) states, all of which have charge \( c/6 \). The quantum \( \mathbb{Z}_{n-1} \) symmetry, which goes with the orbifolding [13], only acts on these twisted states. In the Fermat case, on the other hand, the charges are \( q_X = 1/na \) and \( q_Y = 1/n \). The untwisted sector is generated by

\[
\mathcal{R}^u = \langle \{ (\bar{X}\bar{Y})^j \bar{X}^{nk} \mid j \leq n - 2, k \leq a - 1 \} \rangle,
\]

and there are \( n - 1 \) twisted chiral primary fields with non-trivial action of the quantum \( \mathbb{Z}_n \). The charges of the monomials in (15) and (16) coincide as functions of \( j \) and \( k \), but in the pointer case there is the additional invariant contribution \( Y^{n-1} \) to the chiral ring. This field has indeed the right charge to replace one of the twisted states in the Fermat case and should correspond to a linear combination of them. We thus have some indications of how the hidden quantum \( \mathbb{Z}_{n-1} \) symmetry should act in the Fermat case, but we must leave the full investigation for future work.

5 Discussion

We have analysed a number of models with large positive Euler number for which no mirror is known. All of them have surprisingly similar structures, which helped to find a large class of non-linear transformations with constant determinants. Although our models can be obtained from a number of different configurations and orbifolds, the transformations could be used, in most cases, to identify all of the apparently different representations. For the only exception, the first spectrum in table 2, which has two apparently different families of representations, we instead found a non-linear \( \mathbb{Z}_2 \) symmetry. Another rule with one exception (the last spectrum in table 1) is that each of the models can be obtained as an orbifold in an invertible configuration. For a deformation, which however has different symmetries, we would thus know how to construct the mirror.

Non-linear transformations could of course be used to reduce considerably the number of polynomials that have to be investigated, for example, in the classification of orbifolds. With the present state of technology this would, however, be of limited value, as we would lose many symmetries that are manifest only in the redundant configurations. Further studies are therefore in order: It would be important to find out whether the related orbifolds indeed correspond to identical conformal field theories. And if this is the case, we would need a technology for exploiting the implied hidden and non-linear symmetries.

It may be hoped that further investigation of our set of models will also be of help in the search for their mirror partners. But so far, unfortunately, the question remains: Where are the mirror manifolds?

Note added: After submitting the present work for publication I was kindly informed by M.R. Plesser about ref. [21], where related ideas are pursued. That paper, in particular, clarifies the meaning of non-linear field transformations with constant Jacobian in the context of toric geometry. Furthermore, it is argued that the corresponding field theories flow to fixed points with the same complex structure, but different values of the Kähler moduli. For a number of examples this is confirmed by identifying the implied classical symmetries of the mirror manifolds.
Acknowledgements. It is a pleasure to thank Per Berglund, Philip Candelas and Jürgen Fuchs for helpful discussions, Jan Louis for triggering the present investigation, and in particular Harald Skarke for the enjoyable collaboration on the results of which this work is based.

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