On Poles of Twisted Tensor L-functions\(^*)\)

By Yuval Z. Flicker and Dmitrii Zinoviev

Department of Mathematics, The Ohio State University, U. S. A.

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Abstract: It is shown that the only possible pole of the twisted tensor L-functions in \(\text{Re}(s) \geq 1\) is located at \(s = 1\) for all quadratic extensions of global fields.

0. Introduction. Let \(E\) be a quadratic separable field extension of a global field \(F\). Denote by \(A_E, A_F\) the corresponding rings of adeles. Put \(G_n\) for \(\text{GL}_n\) and \(Z_n\) for its center. Then \(Z_n(A_E)\) is the group \(A_E^n\) of ideles of \(A_E\). Fix a cuspidal representation \(\pi\) of the adele group \(G_\mathbb{R}\). Without lost of generality, we may assume that the central character of \(\pi\) is trivial on the split component of \(A_E^n\). This is the multiplicative group \(\mathbb{R}^\times\) of the field of real numbers embedded in \(A_E^n\) via \(x \mapsto (x, \ldots, x, 1, \ldots)\) (\(x\) in the archimedean, 1 in the finite components). Let \(S\) be a finite set of places of \(F\) (depending on \(\pi\)), including the places where \(E/F\) ramify, and the archimedean places, such that for each place \(v'\) of \(E\) above a place \(v\) outside \(S\) the component \(\pi_v\) of \(\pi\) is unramified. Following [1], let \(r\) be the twisted tensor representation of \(\bar{G} = [G_\mathbb{R}(n, \mathbb{C}) \times \text{GL}(n, \mathbb{C})] \times \text{Gal}(E/F)\) on \(\mathbb{C}^n \otimes \mathbb{C}^n\). It acts by \(r((a, b)) (x \otimes y) = ax \otimes by\) and \(r(\sigma)(x \otimes y) = y \otimes x\) (\(\sigma \in \text{Gal}(E/F), \sigma \neq 1\)). Let \(q_v\) be the cardinality of the residue field \(R_v/\pi_v R_v\) of the ring \(R_v\) of integers in \(F_v\). We define the twisted tensor L-function to be the Euler product

\[
L(s, r(\pi), S) = \prod_{v \in S} \det [1 - q_v^{-s}r(t_v)]^{-1}.
\]

The representation \(\pi\) is called distinguished if its central character is trivial on \(A_E^n\) and there is an automorphic form \(\phi \in \pi\) in \(L^2(G_n(E) / Z_n(A_E))\), such that \(\int \phi(g) dg \neq 0\). The integral is taken over the closed subspace \(G_n(F)Z_n(A_f) / G_n(A_F)\) of \(G_n(E)Z_n(A_E) / G_n(A_E)\).

The following theorem is proven in [1, p. 309] for a quadratic extension \(E/F\) of global fields, such that each archimedean place of \(F\) splits in \(E\). We prove it for any quadratic extension of global fields, i.e. also for number fields with completions \(E_v/F_v = \mathbb{C}/\mathbb{R}\).

Theorem. The product \(L(s, r(\pi), S)\) converges absolutely, uniformly in compact subsets, in some right half-plane. It has analytic continuation as a meromorphic function to the right half plane \(\text{Re}(s) > 1 - \varepsilon\), for some small \(\varepsilon > 0\). The only possible pole of \(L(s, r(\pi), S)\) in \(\text{Re}(s) > 1 - \varepsilon\) is simple, located at \(s = 1\). The function \(L(s, r(\pi), S)\) has a pole at \(s = 1\) if and only if \(\pi\) is distinguished.

Proof. The proof of this theorem is the same as that of the Theorem of [1, §4], pp. 309-310. On lines 14 and 18 of page 310 of [1], we use the proposition below. It holds in the non-split archimedean case too. Hence the restriction put in [1] on the extension \(E/F\) can be removed.

For the functional equation satisfied by \(L(s, r(\pi), S)\), see [1]. For the local L-factors at all non-archimedean places of \(F\), see [2]. The non-vanishing of this L-function on the edge \(\text{Re}(s) = 1\) of the critical strip has been shown by Shahidi [6]. Twisted tensor L-functions are used in the study (see Kon-no [5]) of the residual spectrum of unitary groups.

1. Local computations. From now on, we consider the local case only. Let \(E/F\) be a quadratic extension of local fields. Thus in the archimedean case \(E/F = \mathbb{C}/\mathbb{R}\). Denote by \(x \mapsto \tilde{x}\) the non-trivial automorphism of \(E\) over \(F\). Let \(\xi \neq 0\) be an element of \(E\), such that \(\tilde{\xi} = -\xi\). Put \(G_n\) for \(\text{GL}_n\). The groups of \(F\) and \(E\)-points are denoted by \(G_n(F)\) and \(G_n(E)\). Denote by \(N_n\) the unipotent radical of the upper triangular subgroup of \(G_n\), and by \(A_n\) the diagonal subgroup. Let \(\phi_0\) be a non trivial additive character of \(F\). For example, if \(F = \mathbb{R}\) then \(\phi_0(x) = e^{2\pi i x}\). Let \(\phi\)

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be the (non-trivial) character \( \phi(z) = \phi_\nu((z - \bar{z})/i) \) of \( E \). It is trivial on \( F \). For \( u \in N_n(E) \), set 
\[
\theta(u) = \phi(\sum_{i=1}^{n-1} u_{i,i+1}).
\]

Fix an irreducible admissible representation \( \pi \) of \( G_n(E) \) on a complex vector space \( V \). The representation \( \pi \) is called \textit{generic} if there exists a non-zero linear form \( \lambda \) on \( V \), such that \( \lambda(\pi(u)v) = \theta(u)\lambda(v) \) for all \( v \) in \( V \) and \( u \) in \( N_n(E) \). The dimension of the space of such \( \lambda \) is bounded by one. Let \( W(\pi; \theta) \) be the space of functions \( W \) on \( G_n(E) \) of the form \( W(g) = \lambda(\pi(g)v) \), where \( v \) in \( V \). We have \( W(ug) = \theta(u)W(g) \) \( (g \in G_n(E), u \in N_n(E)) \). Denote by \( W(\pi; \theta) \) those functions in \( W(\pi; \theta) \) whose corresponding vectors \( v \) are in the space of \( K \)-finite vectors, where \( K = K_n(E) \) is the standard maximal compact subgroup of \( G_n(E) \).

For \( \Phi \in S(F^n) \), define the integral 
\[
\mathcal{U}(s, \Phi, W) = \int_{N_n(F)\backslash G_n(F)} W(g) \Phi(\varepsilon g) | \det g |^s dg,
\]
where \( \varepsilon = (0, 0, \ldots, 0, 1) \) is a row vector of size \( n \).

**Proposition.** (i) There exists some small constant \( \varepsilon > 0 \), such that the integral \( \mathcal{U}(s, \Phi, W) \) converges absolutely, uniformly in compact subsets, for \( \text{Re}(s) > 1 - \varepsilon \):

(ii) There exists \( W \) in \( W(\pi; \theta) \) and \( \Phi \) in \( S(F^n) \), such that \( \mathcal{U}(s, \Phi, W) \neq 0 \).

**Proof.** When \( E/F \) is an extension of non-archimedean local fields, (i) and (ii) are treated in the Proposition of [1], §4, p. 308. We prove (i) in general, including the case \((E, F) = (C, R)\), following Jacquet and Shalika [3], pp. 204–206.

Using the Iwasawa decomposition \( G_n(F) = N_n(F)A_n(F)K_n(F) \), and the associated measure decomposition, we need to show the convergence of the integral 
\[
\int_{N_n(F)\backslash G_n(F)} | W(ak) | | \det a |^{s-1} \delta_{n-p}^{-1}(a) | \det a |^k da dk.
\]
Here \( a = \text{diag}(a_1, a_2, \ldots, a_{n-1}, 1) \). Recall that 
\[
\delta_{n-p}(a) = \delta_{n-1,p}(a) | \det a | = | \det a | \prod_{1 \leq i < j \leq n-1} | a_j |,
\]
and (see e.g. [1], p. 307) that 
\[
\delta_{n,p}(a) = \theta_{n,p}(a).
\]

By Proposition 3 of Jacquet and Shalika [3, §4] there is a finite set \( X \) of finite functions in \( n - 1 \) variables such that \( | W(ak) | \) is bounded by a finite sum of expressions of the form 
\[
C_x \delta_{n-1,p}^{-1/2}(a) \Phi \left( \frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, a_{n-1} \right).
\]
Here \( C_x \) is the absolute value of some element of \( X \) and \( \Phi \geq 0 \) is in \( S(F^{n-1}) \). Thus, it suffices to show that the integral obtained by replacing \( W \) by this estimate is convergent. Using that 
\[
\delta_{n-1,p}^{-1/2}(a) \delta_{n,p}(a) = \delta_{n-1,p}(a) \delta_{n,p}^{-1}(a) = | \det a |^{-1},
\]
we arrive at the finite sum of integrals 
\[
\int C_x \Phi \left( \frac{a_1}{a_2}, \frac{a_2}{a_3}, \ldots, a_{n-1} \right) | \det a |^{s-1} da.
\]
The change of variables \( a_1 = t_1 \ldots t_{n-1}, a_2 = t_2 \ldots t_{n-2}, \ldots, a_{n-1} = t_{n-1} \), has the Jacobian \( t_2 t_3^2 \ldots t_{n-2}^2 \). We obtain a sum of expressions of the form 
\[
\int C_x \Phi(t_1, t_2, \ldots, t_{n-1}) t_1^{n-2} t_2^{n-3} \ldots t_{n-2} dt.
\]
Again, by Proposition 3 of Jacquet and Shalika [3, §4] the set \( X \) is such that any \( \chi \) in it is the product of (1) a polynomial in the logarithms of the absolute values of the variables, and (2) a character of the form 
\[
\chi_1(t_1) \chi_2(t_2) \ldots \chi_{n-1}(t_{n-1}),
\]
with \( \text{Re}(\chi_i) > 0 \), for each \( i \). It follows that the above integral converges uniformly in compact subsets of \( \text{Re}(s) > 1 - \varepsilon \), for some small \( \varepsilon > 0 \). This completes the proof of (i).

For (ii) we will follow the proof of Proposition 7.3 of Jacquet and Shalika [3]. Assume that \( \mathcal{U}(s, \Phi, W) = 0 \) for all choices of \( W \) in \( W(\pi; \theta) \) and \( \Phi \) in \( S(F^n) \). We will show that \( W(e) = 0 \) for all \( W \), a contradiction which will imply (ii) of the lemma. Since \( \Phi \) is arbitrary, it follows that for all \( W \) we have 
\[
\int_{N_{n-1}(F)\backslash G_{n-1}(F)} W \left[ \begin{array}{c} g \\ 0 \\ 0 \end{array} \right] | \det g |^s dg = 0.
\]
Define 
\[
I_k(W) = \int_{N_k(F)\backslash G_k(F)} W \left[ \begin{array}{c} g \\ 0 \\ 0 \end{array} \right] | \det g |^s dg.
\]
We claim that \( I_k(W) \) is zero for all \( W \) and all \( k \) with \( 0 \leq k \leq n - 1 \). The lemma would then follow, since \( W(e) = I_0(W) \). We will show this claim by descending induction on \( k \). We have just seen that \( I_{n-1}(W) = 0 \). So fix \( k \leq n - 1 \) with \( I_k(W) = 0 \) for all \( W \). We proceed to show that \( I_{k-1}(W) = 0 \) for all \( W \).

We apply the fact that \( I_k(W) = 0 \) to the function \( W_\Phi \) defined by 
\[
W_\Phi(g) = \int_{N_k(F)} W \left[ \begin{array}{c} g \\ 0 \\ 0 \end{array} \right] \Phi(u) du.
\]
Here \( u \) is a column of size \( k \), \( \Phi \in S(F^k) \) and \( W \in W_\Phi(\pi; \theta) \). Proposition 2.4 of Jacquet and
Shalika [4; II], p. 784, and the remark following it (top of p. 786), assure us that this function is in the space $W_0(\pi; \theta)$.

Note that
\[
W_\phi \left[ \begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix} \right] = W \left[ \begin{pmatrix} g & 0 \\ 0 & 1_{n-k} \end{pmatrix} \right] \Phi(\varepsilon_k g),
\]
where $\Phi(\varepsilon_k g) = \int_{\mathbb{F}_k^*} \Phi(u) \phi_0(y \cdot u) \, du$ denotes the Fourier transform of $\Phi \in S(F^+)$. Indeed
\[
(\varepsilon_k g) = \int_{\mathbb{F}_k^*} \Phi(u) \phi_0 (\varepsilon_k g \cdot u) \, du = \int_{\mathbb{F}_k^*} \Phi(u) \phi_0 \left( \sum_{j=1}^k g_{kj} u_j \right) \, du.
\]
Further, since
\[
\begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} = \begin{pmatrix} 1_k & \varepsilon_k g & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix}
\]
we have
\[
W \left[ \begin{pmatrix} 1_k & \varepsilon_k g & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \right] = \theta \left( \begin{pmatrix} 1_k & \varepsilon_k g & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \right) W \left[ \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \right] = \phi_0 \left( \sum_{j=1}^k g_{kj} u_j \right) W \left[ \begin{pmatrix} g & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1_{n-k-1} \end{pmatrix} \right]
\]
Now substituting $W_\phi$ for $W$ in $I_k(W) = 0$, we obtain
\[
\int_{N_k(\mathbb{F}) \backslash G_k(\mathbb{F})} \phi_0 (\varepsilon_k g) \det g \, dg = 0
\]
for all $\Phi \in S(F^+)$ and all $W \in W_0(\pi; \theta)$. In this integral $\phi_0$ can be replaced by any element of $S(F^+)$. Hence $I_{k-1}(W) = 0$ for all $W$ and we are done.

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