On Weighted $L^p$ – Approximation by Weighted Bernstein-Durrmeyer Operators

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Abstract. In the present paper, we establish direct and converse theorems for weighted Bernstein-Durrmeyer operators under weighted $L^p$ – norm with Jacobi weight $w(x) = x^\alpha (1-x)^\beta$. All the results involved have no restriction $\alpha, \beta < 1 - \frac{1}{p}$, which indicates that the weighted Bernstein-Durrmeyer operators have some better approximation properties than the usual Bernstein-Durrmeyer operators.

Key Words: Weighted $L^p$–approximation, weighted Bernstein-Durrmeyer operators, direct and converse theorems.

AMS Subject Classifications: 41A10, 41A25

1 Introduction

Let

$$w(x) = x^\alpha (1-x)^\beta, \quad \alpha, \beta > -1, \quad 0 \leq x \leq 1,$$

be the classical Jacobi weights. Let

$$L_w^p := \begin{cases} \{f : w f \in L^p (0,1)\}, & 1 \leq p < \infty, \\ \{f : f \in C (0,1), \lim_{x(1-x) \to 0} (w f) (x) = 0\}, & p = \infty. \end{cases}$$

Set

$$\|f\|_{p,w,I} = \begin{cases} \left( \int_I |(w f)(x)|^p dx \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{x \in I} |(w f)(x)|, & p = \infty. \end{cases}$$

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When \( I = [0,1] \), we briefly write \( \| f \|_{p,w} \) instead of \( \| f \|_{p,w,[0,1]} \). Obviously, \( \| f \|_{p,w} \) is the norm of \( L^p_w \) spaces.

For any \( f \in L^p([0,1]), 1 \leq p \leq \infty \), the corresponding Bernstein-Durrmeyer operators \( M_n(f,x) \) are defined as follows:

\[
M_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt, \quad x \in [0,1],
\]

where

\[
p_{n,k}(x) = \binom{k}{n} x^k (1-x)^{n-k}, \quad x \in [0,1], \quad k = 0, 1, \ldots, n.
\]

The approximation properties of \( M_n(f,x) \) in \( L^p_w \) were also studied by Zhang (see [9]). Some approximation results were given under the restrictions \( -\frac{1}{p} < \alpha, \beta < 1 - \frac{1}{p} \) on the weight parameters. Generally speaking, the restrictions can not be eliminated for the approximation by \( M_n(f,x) \). For the weighted approximation by Kantorovich-Bernstein operators defined by

\[
K_n(f,x) := \sum_{k=0}^{n} \frac{(n+1)}{\pi^n} f(u) du p_{nk}(x),
\]

the situation is similar (see [5]). Recently, Della Vecchia, Mastroianni and Szabados (see [2]) introduced a weighted generalization of the \( K_n(f,x) \) as follows:

\[
K_n^w(f,x) := \sum_{k=0}^{n} \frac{1}{\pi^n} \frac{f(u)}{u} du p_{nk}(x), \quad x \in [0,1]. \tag{1.1}
\]

When \( \alpha = \beta = 0 \), \( K_n^w(f,x) \) reduces to the classical Kantorovich-Bernstein operator \( K_n(f,x) \). Della Vecchia, Mastroianni and Szabados obtained the direct and converse theorems and a Voronovskaya-type relation in [2], and solved the saturation problem of the operator in [3]. Their results showed that \( K_n^w(f,x) \) allows a wider class of functions than the operator \( K_n(f,x) \). In fact, they dropped the restrictions \( \alpha, \beta < 1 - \frac{1}{p} \) on the weight parameters. Later, Yu (see [8]) introduced another kind of modified Bernstein-Kantorvich operators, and established direct and converse results on weighted approximation which also have no restrictions \( \alpha, \beta < 1 - \frac{1}{p} \).

Then, a natural question is: can we modified the Bernstein-Durrmeyer operators properly such that the restrictions \( \alpha, \beta < 1 - \frac{1}{p} \) on weighted approximation can be dropped? In the present paper, we will show that the weighted Bernstein-Durrmeyer operator
introduced by Berens and Xu (see [1]) is the one we need. The weighted Bernstein-Durrmeyer operator is defined as follows:

\[ M^*_n(f,x) = \sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_0^1 p_{n,k}(t) w(t) f(t) dt, \quad x \in [0,1], \quad f \in L^1_w, \]

where

\[ C_{n,k} = \left( \int_0^1 p_{n,k}(t) w(t) dt \right)^{-1}, \quad k = 0,1,\ldots,n. \]

Define

\[ W^p_{2,w} := \{ f \in L^p_w : f' \in AC(0,1), \| q^2 f'' \|_{p,w} < \infty \}, \]

where \( q(x) = \sqrt{x(1-x)} \), and \( AC(I) \) is the set of all absolutely continuous functions on \( I \).

For \( f \in L^p_w \), define the weighted modulus of smoothness by

\[ \omega^2_{q}(f,t)_{p,w} := \sup_{0 < h \leq t} \left\{ \| \Delta^2_h f \|_{p,w,[Ch^2,1-Ch^2]} + \| \Delta^2_h f \|_{p,w,[0,Ch^2]} + \| \Delta^2_h f \|_{p,w,[1-Ch^2,1]} \right\}, \]

with

\[ \Delta^2_h f(x) = f(x + h \frac{q(x)}{2}) - 2f(x) + f(x - h \frac{q(x)}{2}), \]

\[ \Delta^2_h f(x) = f(x + 2h) - 2f(x + h) + f(x), \]

\[ \Delta^2_h f(x) = f(x) - 2f(x - h) + f(x - 2h). \]

Define

\[ E_0(f)_{p,w} := \inf_{C \in \mathbb{R}} \| f - C \|_{p,w} \]

to be the best approximation of \( f \) in weighted \( L^p_w \) spaces by constants.

The main results of the present paper are the following:

Theorem 1.1. If \( f \in L^p_w, 1 \leq p \leq \infty \), then

\[ \| f - M^*_n(f) \|_{p,w} \leq C \left( \omega^2_{q}(f,\frac{1}{\sqrt{n}})_{p,w} + \frac{E_0(f)_{p,w}}{n} \right). \]  

(1.2)

Theorem 1.2. If \( f \in L^p_w, 1 \leq p \leq \infty \), then

\[ \| f - M^*_n(f) \|_{p,w} = O \left( n^{-\gamma/2} \right) \iff \omega^2_{q}(f,h)_{p,w} = O(h^\gamma), \quad 0 < \gamma < 2. \]  

(1.3)
2 Auxiliary lemmas

We need the following inequalities:

\[ \int_0^1 p_{n,k}(x)dx = \frac{1}{n+1}, \quad \text{(see [7])}, \quad (2.1a) \]

\[ \sum_{k=0}^n \left( \frac{k^*}{n} \right)^{-u} \left( 1 - \frac{k^*}{n} \right)^{-v} p_{n,k}(x) \leq C x^{-u} (1-x)^{-v}, \quad u,v \geq 0, \quad \text{(see [8])}, \quad (2.1b) \]

\[ \sum_{k=0}^n p_{n,k}(x) \left| \frac{k}{n} - x \right|^\gamma \leq C n^{-\frac{1}{2}} \varphi^\gamma(x), \quad \gamma \geq 0, \quad \text{(see [7])}, \quad (2.1c) \]

\[ \int_0^1 w^p(x) \varphi^{-2}(x) p_{n,k}(x) |k-nx|^2 dx \leq C w^p \left( \frac{k^*}{n} \right), \quad \text{(see [2])}, \quad (2.1d) \]

where

\[ k^* := \begin{cases} 
1, & k = 0, \\
 k, & 1 \leq k \leq n-1, \\
n-1, & k = n.
\end{cases} \quad (2.2) \]

It should be noted that (2.1d) is contained in the first inequality of [2, pp. 9].

**Lemma 2.1.** For \( 1 \leq p < \infty, 0 \leq k \leq n \) and \( n \geq 3 \), we have

\[ \int_0^1 w^p(x) p_{n,k}(x)dx \sim n^{-1} w^p \left( \frac{k^*}{n} \right), \quad (2.3) \]

where \( k^* \) is defined by (2.2).

**Proof.** By the fact that (see [3])

\[ \frac{\Gamma(n+a)}{n^a \Gamma(n)} = 1 + O \left( \frac{1}{n} \right), \quad a > -1, \]

we deduce that

\[ \int_0^1 w^p(x) p_{n,k}(x)dx = \left( \frac{n}{k} \right) \int_0^1 x^{k+a p} (1-x)^{n-k+\beta p} dx \]

\[ = \left( \frac{n}{k} \right) \frac{\Gamma(k+a p+1) \Gamma(n-k+\beta p+1)}{\Gamma(n+a p+\beta p+2)} \]

\[ = \frac{(n+1)^{a p+\beta p+1} \Gamma(n+1) \Gamma(k+a p+1)}{(k+1)^{a p} \Gamma(k+1)} \]

\[ \left( \frac{n}{k} \right) \Gamma(k+a p+1) \Gamma(n-k+\beta p+1) \]

\[ = \frac{(n+1)^{a p+\beta p+1} \Gamma(n+1) \Gamma(k+a p+1)}{(k+1)^{a p} \Gamma(k+1)} \]
\[
\Gamma(n-k+\beta p+1) \\
(n-k+1)\beta p(n-k+1)^{\beta p} \\
(k+1)^{\beta p} \Gamma(n-k+1) \\
(n+1)^{\beta p+\beta p+1} \\
= \left(1+\Theta\left(\frac{1}{n}\right)\right) \frac{(k+1)^{\beta p} (n-k+1)^{\beta p}}{(n+1)^{\beta p+\beta p+1}} \\
\sim n^{-1} w^p \left(\frac{k^*}{n}\right).
\]

Thus, we complete the proof. \(\square\)

Especially, by taking \(p = 1\) in (2.3), we get

\[
C_{n,k}^{-1} = \int_0^1 w(x) p_{n,k}(x) dx \sim n^{-1} w \left(\frac{k^*}{n}\right), \quad k = 0,1,\ldots,n. \tag{2.4}
\]

**Lemma 2.2.** For any \(f \in L_{\beta w}^p\), \(1 \leq p \leq \infty\), we have

\[
\|M_n^*(f)\|_{\beta w} \leq C\|f\|_{\beta w}. \tag{2.5}
\]

**Proof.** When \(p = \infty\), by (2.1a), (2.4) and (2.1b), we get

\[
|w(x) M_n^*(f,x)| \leq w(x) \sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_0^1 p_{n,k}(t) |w(t) f(t)| dt \\
\leq \|f\|_{\infty, w} w(x) \sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_0^1 p_{n,k}(t) dt \\
\leq \|f\|_{\infty, w} w(x) \sum_{k=0}^n p_{n,k}(x) w^{-1} \left(\frac{k^*}{n}\right) \\
= C \|f\|_{\infty, w}. \tag{2.5}
\]

When \(1 < p < \infty\), by using Hölder’s inequality, (2.4) and (2.3), we have

\[
\|M_n^*(f)\|_{\beta w}^p \leq \int_0^1 w^p(x) \left(\sum_{k=0}^n p_{n,k}(x) C_{n,k}^p \int_0^1 p_{n,k}(t) w(t) f(t) dt\right)^p \left(\sum_{k=0}^n p_{n,k}(x)\right)^{p-1} dx \\
\leq \frac{1}{(n+1)^p-1} \int_0^1 w^p(x) \left(\sum_{k=0}^n p_{n,k}(x) C_{n,k}^p \int_0^1 p_{n,k}(t) |w(t) f(t)|^p dt dx \right) \int_0^1 p_{n,k}(t) |w(t) f(t)|^p dt \\
\leq Cn \sum_{k=0}^n \left(\int_0^1 w^p(x) p_{n,k}(x) dx\right) w^{-p} \left(\frac{k^*}{n}\right) \int_0^1 p_{n,k}(t) |w(t) f(t)|^p dt \\
\leq C \sum_{k=0}^n \int_0^1 p_{n,k}(t) |w(t) f(t)|^p dt \\
= C \|f\|_{\beta w}^p. \tag{2.6}
\]

By a similar and more simpler deduction, we see that (2.6) also holds for \(p = 1\).

Combining (2.5) and (2.6), Lemma 2.2 is proved. \(\square\)
Lemma 2.3. If \( f \in W_{w}^{2,p} \), then
\[
\|q^2 M_n''(f)\|_{p,w} \leq C \|q^2 f''\|_{p,w} \quad 1 \leq p \leq \infty.
\]

Proof. Direct calculations yield that (see [4, pp. 331-332]),
\[
M_n''(f,x) = \frac{n!}{(n-2)!} \sum_{j=0}^{2} p_{n-2,k-j}(x)((-1)^{2-j}C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)f(t)dt
\]
\[
= \frac{n!}{(n-2)!} \sum_{j=0}^{2} p_{n-2,k}(x) \sum_{j=0}^{2} \left( \begin{array}{c} 2 \\ j \end{array} \right) (-1)^{2-j}C_{n,k+j}p_{n,k+j}(t)w(t)f(t)dt
\]
\[
= \frac{n!}{(n-2)!} \sum_{j=0}^{2} p_{n-2,k}(x) \sum_{j=0}^{2} \left( \begin{array}{c} 2 \\ j \end{array} \right) (-1)^{2-j}C_{n,k+j}p_{n,k+j}(t)w(t)f(t)dt
\]
\[
= \frac{n!}{(n-2)!} \sum_{j=0}^{2} p_{n-2,k}(x) \sum_{j=0}^{2} \left( \begin{array}{c} 2 \\ j \end{array} \right) (-1)^{2-j}C_{n,k+j}p_{n,k+j}(t)w(t)f(t)dt. \tag{2.7}
\]
Therefore,
\[
\|q^2 M_n''(f)\|_{p,w} \leq C \left\| \sum_{k=0}^{n-2} p_{n,k+1}(x)C_{n+2,k+2} \int_{0}^{1} p_{n,k+1}(t)w(t)q^2(t)f''(t)dt \right\|_{p,w}
\]
\[
\leq C \left\| \sum_{k=0}^{n-2} p_{n,k+1}(x)C_{n,k+1} \int_{0}^{1} p_{n,k+1}(t)w(t)|q^2(t)f''(t)|dt \right\|_{p,w}
\]
\[
\leq C \left\| \sum_{k=0}^{n} p_{n,k}(x)C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)|q^2(t)f''(t)|dt \right\|_{p,w}.
\]

For \( p = \infty \), by (2.1a), (2.1b) and (2.1c), we have
\[
|w(x)q^2(x)M_n''(f,x)| \leq w(x) \sum_{k=0}^{n} p_{n,k}(x)C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)|q^2(t)f''(t)|dt
\]
\[
\leq \|wq^2f''\|_{\infty,w} \sum_{k=0}^{n} p_{n,k}(x)w^{-1} \left( \frac{k^4}{n^4} \right)
\]
\[
\leq C \|q^2 f''\|_{\infty,w}. \tag{2.8}
\]

For \( 1 < p < \infty \) (for \( p = 1 \), it can be treated similarly and more simpler), by using Hölder’s inequality, (2.1a), (2.3) and (2.4), we have
\[
\|q^2 M_n''(f)\|_{p,w}^p
\]
\[
\leq C \int_{0}^{1} \sum_{k=0}^{n} p_{n,k}(x)C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)|q^2(t)f''(t)|dt \|w^p(x)dx
\]
\[
\leq C \int_{0}^{1} \sum_{k=0}^{n} p_{n,k}(x)C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)|q^2(t)f''(t)|p dt \left( \int_{0}^{1} p_{n,k}(tdt) \right)^{p-1} \|w^p(x)dx.
\]
Proof. We prove the result by estimating the integral on two intervals

We finish Lemma 2.3 by combining (2.8) and (2.9).

Lemma 2.4. If \( f \in L_w^p \), then

\[
\|q^2M_n^{\prime\prime\prime}(f)\|_{p,w} \leq Cn\|f\|_{p,w}, \quad 1 \leq p \leq \infty.
\]

Proof. We prove the result by estimating the integral on two intervals \( E_n = \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] \) and \( E_n = \{0,1\} \) \( E_n \) respectively.

Simple calculation leads to

\[
q^2(x)M_n^{\prime\prime\prime}(f,x) = \frac{n^2}{q^2(x)} \sum_{k=0}^{n} p_n,k(x) \left( \frac{k}{n} - x \right)^2 C_{n,k} \int_0^1 p_n,k(t)w(t)f(t)dt - nM_n^1(f,x)
\]

\[
- \frac{d}{dx} (q^2(x)) \frac{n}{q^2(x)} \sum_{k=0}^{n} p_n,k(x) \left( \frac{k}{n} - x \right) C_{n,k} \int_0^1 p_n,k(t)w(t)f(t)dt
\]

\[
= : I_1(n,x) + I_2(n,x) + I_3(n,x).
\] (2.10)

For \( I_1(n,x) \), when \( p = \infty \), by applying (2.1a)-(2.1c), (2.4) and Cauchy’s inequality, we have

\[
|w(x)I_1(n,x)| \leq n^2 \frac{w(x)}{q^2(x)} \sum_{k=0}^{n} p_n,k(x) \left( \frac{k}{n} - x \right)^2 C_{n,k} \left| \int_0^1 p_n,k(t)w(t)f(t)dt \right|
\]

\[
\leq n^2 \|f\|_{\infty,w} \frac{w(x)}{q^2(x)} \sum_{k=0}^{n} p_n,k(x) \left( \frac{k}{n} - x \right)^2 C_{n,k} \int_0^1 p_n,k(t)dt
\]

\[
\leq C n^2 \|f\|_{\infty,w} \frac{w(x)}{q^2(x)} \sum_{k=0}^{n} p_n,k(x) \left( \frac{k}{n} - x \right)^2 w^{-1} \left( \frac{k^+}{n} \right)
\]

\[
\leq C n^2 \|f\|_{\infty,w} \frac{w(x)}{q^2(x)} \left( \sum_{k=0}^{n} p_n,k(x) \left( \frac{k}{n} - x \right)^4 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{n} p_n,k(x)w^{-2} \left( \frac{k^+}{n} \right) \right)^{\frac{1}{2}}
\]

\[
\leq C n^2 \|f\|_{\infty,w} \frac{w(x)}{q^2(x)} \left( \sum_{k=0}^{n} p_n,k(x)w^{-2} \left( \frac{k^+}{n} \right) \right)^{\frac{1}{2}}
\]

\[
= Cn\|f\|_{\infty,w}.
\] (2.11)
When $1 \leq p < \infty$, by using Hölder’s inequality twice for $p > 1$ ($p = 1$ is more direct), (2.1c), (2.1d), (2.4) and (2.1a),

\[
\begin{align*}
&\int_0^1 |w(x) I_1(n,x)|^p dx \\
&\leq \int_0^1 n^{2p} \frac{w^p(x)}{q^p(x)} \left( \sum_{k=0}^{n} p_{n,k}(x) \left( \frac{k}{n} - x \right)^2 \right)^{p-1} \sum_{k=0}^{n} p_{n,k}(x) \left( \frac{k}{n} - x \right)^2 C_{n,k}^p \\
&\times \left( \int_0^1 p_{n,k}(t)|w(t)f(t)|dt \right)^p dx \\
&\leq Cn^{p+1} \int_0^1 w^p(x) q^{-2}(x) \sum_{k=0}^{n} p_{n,k}(x) \left( \frac{k}{n} - x \right)^2 C_{n,k}^p \left( \int_0^1 p_{n,k}(t)w(t)f(t)dt \right)^p dx \\
&\leq Cn^{p+1} \int_0^1 w^p(x) q^{-2}(x) \sum_{k=0}^{n} p_{n,k}(x) \left( \frac{k}{n} - x \right)^2 C_{n,k}^p \left( \int_0^1 p_{n,k}(t)w(t)f(t)dt \right)^p \\
&\leq Cn^{p+1} \int_0^1 p_{n,k}(t)|w(t)f(t)|^p dt \left( \int_0^1 p_{n,k}(t)dt \right)^{p-1}
\end{align*}
\]

(2.12)

For $I_2(n,x)$, by Lemma 2.2, we have

\[
|I_2|_{p,w} \leq Cn||f||_{p,w}, \quad 1 \leq p \leq \infty.
\]

(2.13)

For $I_3(n,x)$, when $p = \infty$, by (2.1a)-(2.1d) and (2.4), we have

\[
|w(x) I_3(n,x)| \leq n \frac{w(x)}{q^2(x)} \sum_{k=0}^{n} p_{n,k}(x) \left| \frac{k}{n} - x \right| C_{n,k} \int_0^1 p_{n,k}(t)|w(t)f(t)|dt \\
\leq ||f||_{\infty,w} \frac{w(x)}{q^2(x)} \sum_{k=0}^{n} p_{n,k}(x) \left| \frac{k}{n} - x \right| C_{n,k} \\
\leq ||f||_{\infty,w} \frac{w(x)}{q^2(x)} \left( \sum_{k=0}^{n} p_{n,k}(x) \left( \frac{k}{n} - x \right)^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{n} p_{n,k}(x)C_{n,k}^2 \right)^{\frac{1}{2}} \\
\leq C||f||_{\infty,w} \frac{n^{\frac{1}{2}}}{q^2(x)} \leq Cn||f||_{\infty,w}
\]

(2.14)

where in the last inequality, we used the fact $1/n^{\frac{1}{2}} q(x) \leq C$, $x \in E_n$.

When $1 \leq p < \infty$, by using Hölder’s inequality, (2.1a), (2.1c), (2.1d), (2.4), and the fact
\[ 1/n^2 \varphi(x) \leq C, x \in E_n \] again, we deduce that

\[
\int_{E_n} |w(x)I_3(n,x)|^p dx 
\leq \int_{E_n} w^p(x) \frac{n^p}{\varphi^p(x)} \sum_{k=0}^{n} p_{n,k}(x) \left( k/n - x \right) C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)f(t)dt \left| dx \right|^p
\] (2.15)

\[
\leq \int_{E_n} w^p(x) \frac{n^p}{\varphi^p(x)} \left( \sum_{k=0}^{n} p_{n,k}(x) \left| k/n - x \right| \right)^{p-1} \times \sum_{k=0}^{n} p_{n,k}(x) \left| k/n - x \right| C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)f(t)dt \left| dx \right|^p
\]

\[
\leq C \int_{E_n} w^p(x) \frac{n^{-2/2}p}{\varphi^{p+1}(x)} \sum_{k=0}^{n} p_{n,k}(x) \left| k/n - x \right| C_{n,k} \int_{0}^{1} p_{n,k}(t)w(t)f(t)\left| t \right|^p dt dx
\]

\[
\leq Cn \sum_{k=0}^{n} C_{n,k}^{p} \left( \int_{E_n} w^p(x) \varphi^{-2}(x)p_{n,k}(x) \left( k/n - x \right)^2 dx \right)^{1/2}
\times \left( \int_{E_n} w^p(x) \varphi^{-2}(x)p_{n,k}(x)dx \right)^{1/2} \int_{0}^{1} p_{n,k}(t)w(t)f(t)\left| t \right|^p dt
\]

\[
\leq Cn^{3/2} \sum_{k=0}^{n} C_{n,k}^{p} \left( n^{-2}w^p \left( k/n \right) \right)^{1/2} \left( \int_{E_n} w^p(x) p_{n,k}(x)dx \right)^{1/2} \int_{0}^{1} p_{n,k}(t)w(t)f(t)\left| t \right|^p dt
\]

\[
= Cn^{p}\left| f \right|_{p,w}^{p}. \tag{2.16}
\]

By combining (2.10)-(2.15), we already have

\[
\left\| \varphi^2 M_n^{n''}(f) \right\|_{p,w,E_n} \leq Cn\left\| f \right\|_{p,w}, \quad 1 \leq p \leq \infty. \tag{2.17}
\]

Now, we estimate the integral on \( E_0^c \). By (2.7), we have

\[
M_n^{n''}(f,x) = \frac{n!}{(n-2)!} \sum_{k=0}^{n-2} p_{n-2,k}(x) \int_{0}^{1} \sum_{j=0}^{2} (2j)! \left(-1\right)^{2-j} C_{n,k+j} p_{n,k+j}(t)w(t)f(t)dt.
\]

When \( 1 \leq p < \infty \), noting that \( n \varphi^2(x) \leq C \) for \( x \in E_n^c \), by Hölder’s inequality, (2.3) and (2.4),
we have

\[ \| q^2 M_n^{x''} f \|_{p,W,E_n}^p \]
\[ \leq \int_{E_n} \left| w(x) q^2(x) n^2 \sum_{k=0}^{n-2} p_{n-2,k}(x) \int_0^1 \sum_{j=0}^1 \frac{1}{(2j)!} (-1)^{2j} C_{n,k+j}(t) w(t) f(t) dt \right|^p dx \]
\[ \leq C n^p \sum_{j=0}^2 \int_{E_n} w(x) q^2(x) \sum_{k=0}^{n-2} p_{n-2,k}(x) C_{n,k+j}^{p'} \left| \int_0^1 p_{n,k+j}(t) w(t) f(t) dt \right|^p dx \]
\[ \leq C n^p \sum_{j=0}^2 \int_{E_n} w(x) q^2(x) \sum_{k=0}^{n-2} p_{n-2,k}(x) n^p w^{-p} \left( \frac{k^+}{n} \right) \]
\[ \times \left[ \int_0^1 p_{n,k+j}(t) |w(t)| f(t) |t|^p dt \right]^{p-1} \left( \int_0^1 p_{n,k+j}(t) dt \right) dx \]
\[ \leq C n^{p+1} \sum_{j=0}^2 \sum_{k=0}^{n-2} w^{-p} \left( \frac{k^+}{n} \right) \int_{E_n} w(x) p_{n-2,k}(x) dx \int_0^1 p_{n,k+j}(t) |w(t)| f(t) |t|^p dt \]
\[ \leq C n^{p+1} \sum_{j=0}^2 \sum_{k=0}^{n-2} n^{-1} w^{-p} \left( \frac{k^+}{n} \right) w^{-p} \left( \frac{k^+}{n} \right) \int_0^1 p_{n,k+j}(t) |w(t)| f(t) |t|^p dt \]
\[ \leq C n^p \int_0^1 \left( \sum_{j=0}^2 \sum_{k=0}^{n-2} p_{n,k+j}(t) |w(t)| f(t) |t|^p dt \right) \]
\[ \leq C n^p \| f \|_{p,W}^p. \quad (2.18) \]

When \( p = \infty \), for \( x \in E_n \), by (2.1a) and (2.1b),

\[ \left| w(x) q^2(x) M_n^{x''}(f, x) \right| \]
\[ \leq C n^2 \| f \|_{\infty,W} \sum_{j=0}^2 \sum_{k=0}^{n-2} w(x) q^2(x) p_{n-2,k}(x) C_{n,k+j} \int_0^1 p_{n,k+j}(t) dt \]
\[ \leq C \| f \|_{\infty,W} \sum_{j=0}^2 \sum_{k=0}^{n-2} w(x) p_{n-2,k}(x) C_{n,k+j} \]
\[ \leq C \| f \|_{\infty,W} w(x) \sum_{k=0}^{n-2} p_{n-2,k}(x) n w^{-1} \left( \frac{k^+}{n-2} \right) \]
\[ \leq C n \| f \|_{\infty,W}. \quad (2.19) \]

By (2.18) and (2.19), we see that

\[ \| q^2 M_n^{x''}(f) \|_{p,W,E_n} \leq C n \| f \|_{p,W}, \quad 1 \leq p \leq \infty. \quad (2.20) \]
By (2.17) and (2.20), we complete the proof of Lemma 2.4.

Lemma 2.5. For any nonnegative integer $m$, set

$$T_n^m((x-t)^m,x) = \sum_{k=0}^{n} p_{nk}(x) C_{n,k} \int_0^1 p_{nk}(t) w(t) (x-t)^m dt.$$ 

Then

$$T_{n,2m} = \sum_{i=0}^{m} p_{i,m,n,a,b}(x) \left( \frac{q^2(x)}{n} \right)^{m-i} n^{-2i}$$

and

$$T_{n,2m-1} = \sum_{i=0}^{m-1} p_{i,m,n,a,b}(x) \left( \frac{q^2(x)}{n} \right)^{m-i-1} n^{-2i+1},$$

where $p_{i,m,n,a,b}(x)$ are polynomials in $x$ of fixed degree with coefficients that are bounded uniformly for all $n$.

Proof. Analogue to [4], we have the recursion relation:

$$(n + m + 2) T_{n,m+1}(x) = x(1-x) (2m T_{n,m-1}(x) - T'_{nm}(x)) - (1-2x)(m+1) T_{nm}(x).$$

Direct calculations yield that

$$T_{n0}(x) = 1,$$

and

$$T_{n1}(x) = \sum_{k=0}^{n} p_{nk}(x) C_{n,k} \int_0^1 w(t) p_{nk}(t) t dt - x$$

$$= \sum_{k=0}^{n} \frac{B((k+a+1)+1,n-k+b+1)}{B(k+a+b+1,n-k+b+1)} p_{nk}(x) - x$$

$$= \sum_{k=0}^{n} \frac{k+a+1}{n+a+b+2} p_{nk}(x) - x$$

$$= \frac{nx+a+1}{n+a+b+2} - x$$

$$= \frac{\alpha+1-(\alpha+b+2)x}{n+a+b+2},$$

where $B(p,q) := \int_0^1 x^{p-1} (1-x)^{q-1} dx$, $p,q > 0$.

By (2.23)-(2.25) and a simple induction process, we obtain (2.21) and (2.22).

By (2.21), we have
Lemma 2.6. For any given m, it holds that

\[ T_{n,2m}(x) \leq Cn^{-m} \left( \phi^2(x) + \frac{1}{n} \right)^m. \]  

(2.26)

Lemma 2.7. For \( 1 \leq p < \infty \), \( f \in W^{2,p}_w \), there is a positive constant \( C \) such that

\[ \| M_n(R_2(f,t,x),x) \|_{p,w,E_n} \leq \frac{C}{n} \| \phi^2(x) f'' \|_{p,w}, \]  

(2.27)

where

\[ R_2(f,t,x) := \int_{\mathcal{E}} (t-v) f''(v)dv. \]

Proof. Firstly, we consider the case \( p = 1 \). Set \( g(v) = w(v) \phi^2(v) f''(v) \). By the inequality (see [5]):

\[ \frac{|t-u|}{\phi^2(u)} \leq \frac{|t-x|}{\phi^2(x)} \]  

for any \( u \) between \( x \) and \( t \),

we have

\[
\begin{align*}
&\int_{E_n} w(x) |M_n(R_2(f,t,x),x)|dx \\
&\leq \int_{E_n} w(x) \sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{0}^{1} p_{n,k}(t) w(t) \left[ \int_{x}^{t} |v| f''(v)dv \right] dt dx \\
&\leq \int_{E_n} w(x) \sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{0}^{1} p_{n,k}(t) w(t) \left[ \int_{x}^{t} g(v)dv \right] |t-x| \left( \frac{1}{w(x)} + \frac{1}{w(t)} \right) dt dx \\
&\leq \int_{E_n} \phi^{-2}(x) \sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{0}^{1} p_{n,k}(t) w(t) \left[ \int_{x}^{t} g(v)dv \right] |t-x| dt dx \\
&\quad + \int_{E_n} w(x) \phi^{-2}(x) \sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{0}^{1} p_{n,k}(t) \left[ \int_{x}^{t} g(v)dv \right] |t-x| dt dx \\
&= : I_1 + I_2. \tag{2.28}
\end{align*}
\]

Set

\[ D(l,n,x) := \{ t: \ln^{1-l} \phi(x) \leq |t-x| \leq (l+1)n^{-1} \phi(x) \}, \]

\[ F(l,x) := \{ v: v \in (0,1), |v-x| \leq (l+1)n^{-1} \phi(x) \}, \]

\[ G(l,v) := \{ x: x \in E_n, v \in F(l,x) \}. \]
For $l \geq 1$, by (2.1b), (2.4), and (2.26) with $w \equiv 1$, we deduce that

$$\sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{D(l,n,x)} p_{n,k}(t) |t-x| dt$$

$$\leq \frac{Cn^2}{l^4 q^4(x)} \sum_{k=0}^{n} p_{n,k}(x) n w^{-1} \left( \frac{k^*}{n} \right) \int_{D(l,n,x)} p_{n,k}(t) |t-x|^5 dt$$

$$\leq \frac{Cn^3}{l^4 q^4(x)} \left( \sum_{k=0}^{n} p_{n,k}(x) n w^{-2} \left( \frac{k^*}{n} \right) \right)^{\frac{1}{2}} \left( \sum_{k=0}^{n} p_{n,k}(x) \left( \int_{0}^{l} p_{n,k}(t) |t-x|^5 dt \right)^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{Cn^2}{l^4 q^4(x) w(x)} \left( \sum_{k=0}^{n} p_{n,k}(x)(n+1) \int_{0}^{l} p_{n,k}(t) |t-x|^{10} dt \right)^{\frac{1}{2}}$$

$$\leq \frac{Cn^2}{l^4 q^4(x) w(x)} \left( n - \frac{1}{2} q(x) \right) \frac{C}{(l+1)^4 w(x)}.$$

For $l = 0$, by (2.1b) and (2.4),

$$\sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{D(l,n,x)} p_{n,k}(t) |t-x| dt \leq Cn^{-\frac{1}{2}} q(x) \sum_{k=0}^{n} p_{n,k}(x) w^{-1} \left( \frac{k^*}{n} \right) \leq C \frac{n^{-\frac{1}{2}} q(x)}{w(x)}.$$

Therefore,

$$I_2 \leq \int_{E_n} w(x) q^{-2}(x) \sum_{l=0}^{n} \sum_{k=0}^{n} p_{n,k}(x) C_{n,k} \int_{D(l,n,x)} p_{n,k}(t) \left| \int_{x}^{l} g(v) dv \right| |t-x| dt dx$$

$$\leq C n^{-1} \sum_{l=0}^{n} \frac{1}{(l+1)^4} \int_{E_n} q(x) \int_{f(l,x)} g(v) dv dx$$

$$\leq C n^{-1} \sum_{l=0}^{n} \frac{1}{(l+1)^4} n^{\frac{1}{2}} \int_{0}^{1} g(v) \left\{ \int_{G(l,v)} q^{-1}(x) dx \right\} dv.$$

Noting that $G(l,v) \subset E_n \subset [0,1]$ for $l \geq n^{\frac{1}{2}}$ and $\int_{0}^{1} q^{-1}(x) dx \leq C$, we have

$$n^{\frac{1}{2}} \sum_{l \geq n^{\frac{1}{2}}} \frac{1}{(l+1)^4} \int_{0}^{1} g(v) \left\{ \int_{G(l,v)} q^{-1}(x) dx \right\} dv \leq C \|q^2 f''\|_{1,w}.$$

Since (see [5])

$$\int_{\{x: |v-x| \leq h q(x)\}} q^{-1}(x) dx \leq Ch,$$

then (by taking $h = (l+1)n^{\frac{1}{2}}$)

$$\int_{G(l,v)} q^{-1}(x) dx \leq C(l+1)n^{\frac{1}{2}}.$$
Therefore,
\[
\frac{1}{n^4} \sum_{0 \leq l \leq n^4} \frac{1}{(l+1)^4} \int_0^1 g(v) \left\{ \int_{G(l,v)} \varphi^{-1}(x) dx \right\} dv \leq C \| \varphi^2 f'' \|_{1,w}.
\]

Thus, we can conclude that
\[
I_2 \leq \frac{C}{n} \| \varphi^2 f'' \|_{1,w}. \quad (2.29)
\]

Now, we begin to prove the following
\[
I_1 \leq \frac{C}{n} \| \varphi^2 f'' \|_{1,w}. \quad (2.30)
\]

For \( l \geq 1 \), by (2.4) and (2.26), we deduce that
\[
\sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_{D(l,n,x)} p_{n,k}(t) w(t) |t-x| dt \\
\leq C \frac{n^2}{l^4 \varphi^4(x)} \sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_0^1 p_{n,k}(t) w(t) |t-x|^{10} dt \\
\leq C \frac{n^2}{l^4 \varphi^4(x)} \left[ \sum_{k=0}^n p_{n,k}(x) C_{n,k} \left( \int_0^1 p_{n,k}(t) w(t) |t-x|^{5} dt \right)^2 \right]^\frac{1}{2} \\
\leq C \frac{n^2}{l^4 \varphi^4(x)} \left[ \sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_0^1 p_{n,k}(t) w(t) |t-x|^{10} dt \right]^\frac{1}{2} \\
\leq C \frac{n^2}{l^4 \varphi^4(x)} \left( n^{-5} \varphi^{10}(x) \right)^\frac{1}{2} \leq \frac{n^{-\frac{1}{2}} \varphi(x)}{(l+1)^4}. \tag{2.30}
\]

For \( l = 0 \), by (2.1b), (2.3) and (2.4), we have
\[
\sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_{D(l,n,x)} p_{n,k}(t) w(t) |t-x| dt \\
\leq C n^{-\frac{1}{2}} \varphi(x) \sum_{k=0}^n p_{n,k} w^{-1} \left( \frac{k^2}{n} \right) \int_0^1 p_{n,k}(t) w(t) dt \\
\leq C n^{-\frac{1}{2}} \varphi(x).
\]

Then, we can derive (2.30) in a similar way to the proof of (2.29).

By combining (2.28)-(2.30), we obtain Lemma 2.7 for \( p = 1 \).

Finally, we prove Lemma 2.7 for \( 1 < p \leq \infty \).

Set
\[
G(g,x) = \sup_l \left| \frac{1}{l-x} \int_x^l g(v) dv \right|.
\]
The following maximal function inequality are well known
\[ \| G(g) \|_p \leq C \| g \|_p. \]
Since \( 1/w(v) \leq C(1/w(t) + 1/w(x)) \) for any \( v \) between \( x \) and \( t \), by the maximal function inequality, we have
\[
\| M^*_n(R_3(f,t,x),x) \|_{p,w,E_n} 
\leq C \left\| \varphi^{-2}(x) M^*_n \left( |t-x| \left( \frac{1}{w(t)} + \frac{1}{w(x)} \right) \int_x^t w(v) \varphi^2(v) f''(v) dv, x \right) \right\|_{p,w,E_n}
\leq C \left\| \varphi^{-2}(x) M^*_n \left( (t-x)^2 \left( \frac{1}{w(t)} + \frac{1}{w(x)} \right) G(g,x), x \right) \right\|_{p,w,E_n}
\leq C \| g \|_p \left\| \varphi^{-2}(x) M^*_n \left( (t-x)^2 \left( \frac{1}{w(t)} + \frac{1}{w(x)} \right), x \right) \right\|_{\infty,w,E_n}.
\]
Therefore, we only need to prove that
\[ \| K \|_{\infty,E_n} = \left\| \frac{w(x)}{\varphi^2(x)} M^*_n \left( (t-x)^2 \left( \frac{1}{w(t)} + \frac{1}{w(x)} \right), x \right) \right\|_{\infty,E_n} \leq \frac{C}{n}, \tag{2.31} \]
where \( \| K \|_{\infty,E_n} \) is the usual supremum norm of \( K \) on \( E_n \), and
\[ K = \frac{1}{\varphi^2(x)} M^*_n((t-x)^2,x) + \frac{w(x)}{\varphi^2(x)} M^*_n \left( \frac{(t-x)^2}{w(t)}, x \right). \tag{2.32} \]
For the first part of \( K \), by (2.26),
\[ \frac{1}{\varphi^2(x)} M^*_n((t-x)^2,x) \leq \frac{C}{n}, \quad x \in E_n. \tag{2.33} \]
For the second part of \( K \), by (2.4), (2.1b) and (2.26) (with \( w = 1 \)),
\[
= \frac{w(x)}{\varphi^2(x)} \sum_{k=0}^n p_{n,k}(x) C_{n,k} \int_0^1 p_{n,k}(t)(t-x)^2 dt
\leq \frac{w(x)}{\varphi^2(x)} \left[ \sum_{k=0}^n p_{n,k}(x) w^{-2} \left( \frac{k^2}{n} \right) \sum_{k=0}^n p_{n,k}(x) n^2 \left( \int_0^1 p_{n,k}(t)(t-x)^2 dt \right)^2 \right]^{\frac{1}{2}}
\leq \frac{1}{\varphi^2(x)} \left[ \sum_{k=0}^n p_{n,k}(x) (n+1) \int_0^1 p_{n,k}(t)(t-x)^4 dt \right]^{\frac{1}{2}}
\leq \frac{1}{\varphi^2(x)} \left( n^{-\frac{1}{2}} \varphi(x) \right)^2 \leq \frac{C}{n}. \tag{2.34}
\]
By (2.33) and (2.34), we get (2.31), and thus Lemma 2.7 is valid for \( 1 < p \leq \infty. \)
3 Proofs of theorems

Proof of Theorem 1.1. It is sufficient to prove that
\[
\| M_n^* (f) - f \|_{p,w} \leq C_n \left( \| \phi^2 f'' \|_{p,w} + \| f' \|_{p,w} \right)
\]  
(3.1)

for \( \phi^2 f'' \in L^p_w \). By the Taylor’s formula
\[
f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-v)f''(v)dv,
\]
we have
\[
w(x)(M_n^* (f,x) - f(x)) = w(x)f'(x)M_n^* ((t-x),x) + w(x)M_n^* (R_2 (f,t,x),x).
\]
Then, by (2.25) and (2.27), we get (3.1) immediately. □

Proof of Theorem 1.2. The “\( \Leftarrow \)” part follows from Theorem 1.1. The “\( \Rightarrow \)” part can be done by using the argument of proof of Theorem 9.3.2 in [5], we omit the details here. □

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