On expectile-assisted inverse regression estimation for sufficient dimension reduction

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Abstract

Moment-based sufficient dimension reduction methods such as sliced inverse regression may not work well in the presence of heteroscedasticity. We propose to first estimate the expectiles through kernel expectile regression, and then carry out dimension reduction based on random projections of the regression expectiles. Several popular inverse regression methods in the literature are extended under this general framework. The proposed expectile-assisted methods outperform existing moment-based dimension reduction methods in both numerical studies and an analysis of the Big Mac data.

Keywords: asymmetric least squares, directional regression, kernel expectile regression, projective resampling, sliced average variance estimation, sliced inverse regression

1. Introduction

Since its inception about three decades ago, sufficient dimension reduction (Li, 1991; Cook, 1998a) has become a very important tool for modern multivariate analysis. For predictor $X \in \mathbb{R}^p$ and response $Y \in \mathbb{R}$, the goal of sufficient dimension reduction is to find $B \in \mathbb{R}^{p \times d}$ with $d \leq p$ such that

$$Y \perp\!
\!
\perp X|B^\top X,$$

where $\perp\!
\!
\perp$ means statistical independence. The column space of $B$ is known as a dimension reduction space. Under mild conditions, Yin, Li and Cook

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(2008) showed that the intersection of all dimension reduction spaces is still a dimension reduction space, and it is referred to as the central space for the regression $Y$ on $X$. We denote the central space by $S_{Y|X}$. The dimension of the central space is known as the structural dimension.

There are many sufficient dimension reduction methods in the literature. Moment-based estimators include sliced inverse regression (SIR) (Li, 1991), sliced average variance estimation (SAVE) (Cook and Weisberg, 1991), principal Hessian directions (Li, 1992; Cook, 1998b), and sliced average third-moment estimation (Yin and Cook, 2003). Semiparametric estimators include minimum average variance estimation (MAVE) (Xia et al., 2002), and semiparametric dimension reduction (Ma and Zhu, 2012; Luo, Li and Yin, 2014). Sparse dimension reduction estimators include sparse SIR (Li, 2007; Tan, Shi and Yu, 2019), sparse MAVE (Wang and Yin, 2008), coordinate-independent sparse estimation (Chen, Zou and Cook, 2010), and sparse semiparametric estimation (Yu et al., 2013). Other sufficient dimension reduction methods include ensemble sufficient dimension reduction (Yin and Li, 2011), nonlinear sufficient dimension reduction (Li, Artemiou and Li, 2011; Lee, Li and Chiaromonte, 2013), groupwise sufficient dimension reduction (Li, Li and Zhu, 2010; Guo et al., 2015), and post dimension reduction inference (Kim et al., 2019). For general reviews, one can refer to Cook (2007), Ma and Zhu (2013), and Dong (2019). An excellent reference is the recent book by Li (2018).

Due to their ease of implementation, SIR and SAVE are two of the most popular sufficient dimension reduction methods. One well-known limitation of SIR and SAVE is that they are not very efficient in the presence of heteroscedasticity. Quantile-based methods are proposed by Wang, Shin and Wu (2018) and Kim, Wu and Shin (2019) to address this limitation, and their proposals work better than SIR or SAVE with heteroscedastic error. However, another well-known limitation of SIR and SAVE is that they may be sensitive to specific link functions between the response and the predictor. In particular, SIR does not work well when the link function is symmetric, and SAVE is not efficient with monotone link functions. Since the quantile-based methods are extensions of SIR and SAVE, they inherit the limitation of their moment-based counterparts and may still have uneven performances with various link functions.

We propose expectile-assisted inverse regression in this paper. Our contribution is two-fold. First, we provide a general framework to extend moment-based dimension reduction methods to their expectile-based counterparts,
such as expectile-assisted SIR, expectile-assisted SAVE, and expectile-assisted directional regression. Similar to the quantile-based methods, our expectile-based proposals utilize the information across different levels of the conditional distribution of $Y$ given $X$, and perform better than the corresponding moment-based methods in the presence of heteroscedasticity. Since directional regression (Li and Wang, 2007) is known to perform well for a wide range of link functions, the expectile-assisted directional regression enjoys the additional benefit that it is no longer sensitive to the specific forms of the unknown link functions. Furthermore, to combine the information across different quantile levels, existing quantile-based methods such as the quantile-slicing mean estimation (Kim, Wu and Shin, 2019) rely on intricate weights, and it is not clear how the choice of different weights may affect the final estimation. We propose to combine the information across different quantile levels through random projection, which has roots in the projected resampling approach for multiple response sufficient dimension reduction (Li, Wen and Zhu, 2008). Our proposed expectile-assisted estimators outperform existing methods in both simulation studies and a real data analysis.

The rest of the paper is organized as follows. In Sections 2 and 3, we provide the population level and the sample level development of expectile-assisted SIR, respectively. Further extensions to expectile-assisted SAVE and expectile-assisted directional regression are described in Section 4. Some practical issues such as tuning parameter selection are discussed in Section 5. Extensive simulation studies are provided in Section 6 and we conclude the paper with a real data analysis in Section 7. All proofs are relegated to the Appendix.

2. Population level development of expectile-assisted SIR

Expectiles were first introduced by Newey and Powell (1987) in the seminal asymmetric least squares paper. It has gained popularity in finance and risk management for estimating the expected shortfall and value at risk. See, for example, Kim and Lee (2016), Daouia, Girard and Stupfler (2018), and Chen (2018). For $0 < \tau < 1$, denote $f_\tau(X)$ as the $\tau$-th expectile of the conditional distribution of $Y$ given $X$. Then

$$f_\tau(x) = \arg \min_a E\{\phi_\tau(Y-a)|X=x\}, \quad (2)$$
where $\phi_\tau(\cdot)$ is known as the asymmetric loss function and is defined as

$$
\phi_\tau(c) = \begin{cases} 
(1 - \tau)c^2, & \text{if } c \leq 0, \\
\tau c^2, & \text{if } c > 0.
\end{cases}
$$

**Proposition 1.** For $0 < \tau_1 < \cdots < \tau_k < 1$, let $\xi_X = (f_{\tau_1}(X), \ldots, f_{\tau_k}(X))^\top$. Then $S_{\xi_X|X} \subseteq S_{Y|X}$.

Proposition 1 suggests that we can recover the central space $S_{Y|X}$ through estimation of the central space for the regression of $\xi_X$ on $X$. We implicitly assume that $f_{\tau_\ell}(X)$ from (2) is well-defined for $\ell = 1, \ldots, k$.

For $\xi_X \in \mathbb{R}^k$, the original SIR can not be applied directly due to the multivariate response. Let $T \in \mathbb{R}^k$ be a random vector. We follow Li, Wen, and Zhu (2008) and apply SIR for the regression between $\xi_X^\top T$ and $X$ instead. Let $E(X) = \mu$ and $\text{Var}(X) = \Sigma$. Then $Z = \Sigma^{-1/2}(X - \mu)$ denotes the standardized predictor. Let $J_1(T), \ldots, J_H(T)$ be the partition of the support of $\xi_X^\top T$. For $h = 1, \ldots, H$, denote $I_h(T)$ as the indicator function of $\xi_X^\top T \in J_h(T)$. Define

$$
M(T) = \sum_{h=1}^H p_h(T) \mu_h(T)^\top 
$$

where $p_h(T) = E\{I_h(T)\}$ and $\mu_h(T) = E\{Z|\xi_X^\top T \in J_h(T)\}$.

Before we state the next result, we need the following linear conditional mean (LCM) assumption, which is a common assumption in the sufficient dimension reduction literature.

**Assumption 1.** $E(X|B^\top X)$ is a linear function of $B^\top X$, where $B$ is a basis of $S_{Y|X}$.

**Proposition 2.** Let $T$ be a random vector uniformly distributed on the unit sphere $\mathbb{S}^k$. Then under Assumption 1, $\text{span}(\Sigma^{-1/2}E\{M(T)\}) \subseteq S_{Y|X}$.

Here $\text{span}(\cdot)$ denotes the column space, and the expectation $E\{M(T)\}$ is over the distribution of $T$. We remark that the LCM assumption is not needed in Proposition 1, and it is only needed in Proposition 2 because the classical SIR requires the LCM assumption.
3. Sample level algorithm of expectile-assisted SIR

3.1. Kernel expectile regression

Given an i.i.d. sample \( \{(X_i, Y_i)\}, i = 1, \ldots, n \), we explain how to estimate the \( \tau \)-th expectile \( f_\tau(X) \) of the conditional distribution of \( Y \) given \( X \) in this section. This step is the same for expectile-assisted SIR and the other expectile-assisted inverse regression methods to be discussed in Section 4. The original estimator in Newey and Powell (1987) focused on expectiles in linear regression. To estimate the conditional expectiles in nonlinear models, Yao and Tong (1996) proposed a local linear polynomial estimator. More recently, Yang, Zhang and Zou (2018) developed a reproducing kernel Hilbert space (RKHS) estimator for flexible expectile regression. We adapt the RKHS estimator with the following Gaussian radial basis kernel

\[
K(X_i, X_j) = \exp(-r \|X_i - X_j\|^2),
\]

where \( r \) is a tuning parameter and \( \| \cdot \| \) denotes the Euclidean norm. Let \( \mathbb{H}_K \) be the RKHS generated from the kernel function (4). As an element of \( \mathbb{H}_K \), \( f_\tau(X) \) evaluated at \( X = X_i \) can be estimated by

\[
\hat{f}_\tau(X_i) = \hat{\alpha}_{0,\tau} + \sum_{j=1}^{n} \hat{\alpha}_{j,\tau} K(X_i, X_j).
\]

Let \( \hat{\alpha}_\tau = (\hat{\alpha}_{0,\tau}, \hat{\alpha}_{1,\tau}, \ldots, \hat{\alpha}_{n,\tau}) \) and \( \alpha_\tau = (\alpha_{0,\tau}, \alpha_{1,\tau}, \ldots, \alpha_{n,\tau}) \). Then \( \hat{\alpha}_\tau \) in (5) is the minimizer of the regularized empirical risk function on \( \mathbb{H}_K \)

\[
\hat{\alpha}_\tau = \arg\min_{\alpha_\tau} \sum_{i=1}^{n} \phi_\tau\left(Y_i - \alpha_{0,\tau} - \sum_{j=1}^{n} \alpha_{j,\tau} K(X_i, X_j)\right) + \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,\tau} \alpha_{j,\tau} K(X_i, X_j),
\]

where \( \phi_\tau(\cdot) \) is defined in (2) and \( \lambda \) is a tuning parameter. The optimization (6) and the evaluation (5) can be done very efficiently in the KERE package in R. The choices for the tuning parameters \( r \) in (4) and \( \lambda \) in (6) are discussed in Section 5.
3.2. Projective resampling for multiple response SIR

Given an i.i.d. sample \( \{(X_i, Y_i)\}, i = 1, \ldots, n \), the sample level expectile-assisted SIR algorithm is as follows.

1. For a given integer \( k \), specify \( 0 < \tau_1 < \cdots < \tau_k < 1 \). For \( i = 1, \ldots, n \), calculate \( \hat{\xi}_X = (\hat{f}_{\tau_1}(X_i), \cdots, \hat{f}_{\tau_k}(X_i))^\top \), where the \( \ell \)-th component of \( \hat{\xi}_X \) is given by (5) with \( \tau = \tau_\ell \).

2. Let \( \hat{\mu} = n^{-1} \sum_{i=1}^n X_i \) and \( \hat{\Sigma} = n^{-1} \sum_{i=1}^n (X_i - \hat{\mu})(X_i - \hat{\mu})^\top \). Calculate standardized predictors \( \hat{Z}_i = \hat{\Sigma}^{-1/2}(X_i - \hat{\mu}) \) for \( i = 1, \ldots, n \).

3. For a given integer \( N \), generate an i.i.d. sample \( t^{(1)}, \ldots, t^{(N)} \) from the uniform distribution on the unit sphere \( S^k \). For \( j = 1, \ldots, N \), let \( J_{1}(t^{(j)}), \ldots, J_{H}(t^{(j)}) \) be the partition of the support of \( \hat{\xi}_X^\top t^{(j)} \). For \( h = 1, \ldots, H \), denote \( I_{hi}(t^{(j)}) \) as the indicator function of \( \hat{\xi}_X^\top t^{(j)} \in J_{h}(t^{(j)}) \).

4. We now calculate the sample version of \( E\{M(T)\} \).

   4.1 For \( j = 1, \ldots, N \), let \( \hat{p}_h(t^{(j)}) = n^{-1} \sum_{i=1}^n I_{hi}(t^{(j)}) \) and \( \hat{\mu}_h(t^{(j)}) = (\hat{p}_h(t^{(j)}))^{-1} \sum_{i=1}^n \hat{Z}_i I_{hi}(t^{(j)}) \). Then the sample estimator of (3) with \( T = t^{(j)} \) becomes

\[
\hat{M}(t^{(j)}) = \sum_{h=1}^H \hat{p}_h(t^{(j)}) \hat{\mu}_h(t^{(j)}) \hat{\mu}_h^\top(t^{(j)}).
\]

4.2 Calculate \( \hat{M}(T) = N^{-1} \sum_{j=1}^N \hat{M}(t^{(j)}) \).

5. For a given structural dimension \( d \), let \( (\hat{v}_1, \ldots, \hat{v}_d) \) be the eigenvectors corresponding to the \( d \) leading eigenvalues of \( \hat{M}(T) \). The final estimator of \( \mathcal{S}_{Y|X} \) is then \( \text{span}(\hat{B}) \), where \( \hat{B} = (\Sigma^{-1/2}\hat{v}_1, \ldots, \Sigma^{-1/2}\hat{v}_d) \).

Note that the first three steps above are the same for all expectile-assisted estimators. In the numerical studies, we fix \( N = 1000, k = 9 \), and set \( \tau_\ell = 10^{-1}\ell \) for \( \ell = 1, \ldots, 9 \). Our experience suggests that the proposed method is not very sensitive to the choice of \( N \) and \( k \).
4. Extensions of SAVE and directional regression

Expectile-assisted dimension reduction is a very general framework, and can be readily generalized to other moment-based methods such as SAVE and directional regression. We focus on the population level development of expectile-assisted SAVE and expectile-assisted directional regression in this section.

Recall that $I_h(T)$ denotes the indicator function of $\xi^\top X T \in J_h(T)$, $p_h(T) = E\{I_h(T)\}$, and $\mu_h(T) = E\{Z | \xi^\top X T \in J_h(T)\}$. Define

$$G(T) = \sum_{h=1}^H p_h(T)\{V_h(T) - \mu_h(T)\mu_h^\top(T)\}^2,$$

where $V_h(T) = E\{ZZ^\top - I_p|\xi^\top X T \in J_h(T)\}$. In addition to the LCM assumption, we need the constant conditional variance assumption as follows

Assumption 2. Var$(X|B^\top X)$ is a nonrandom matrix, where $B$ is a basis of $S_{Y|X}$.

Proposition 3. Let $T$ be a random vector uniformly distributed on the unit sphere $S^k$. Then under Assumptions 1 and 2, $\text{span}(\Sigma^{-1/2}E\{G(T)\}) \subset S_{Y|X}$.

In a similar fashion, define

$$F(T) = 2 \sum_{h=1}^H p_h(T)V_h(T)V_h(T) + 2 \left\{ \sum_{h=1}^H p_h(T)\mu_h(T)\mu_h^\top(T) \right\}^2 + 2 \left\{ \sum_{h=1}^H p_h(T)\mu_h^\top(T)\mu_h(T) \right\}\left\{ \sum_{h=1}^H p_h(T)\mu_h(T)\mu_h^\top(T) \right\},$$

and we have

Proposition 4. Let $T$ be a random vector uniformly distributed on the unit sphere $S^k$. Then under Assumptions 1 and 2, $\text{span}(\Sigma^{-1/2}E\{F(T)\}) \subset S_{Y|X}$.

Based on Proposition 3 and Proposition 4, we may update step 4 and step 5 of the expectile-assisted SIR algorithm to get the sample estimators of $\Sigma^{-1/2}E\{G(T)\}$ and $\Sigma^{-1/2}E\{F(T)\}$. We refer to them as the expectile-assisted SAVE estimator and the expectile-assisted directional regression estimator, respectively.
5. Additional issues

5.1. Selecting tuning parameters

We first discuss the choice of \( r \) in (4). For the Gaussian radial basis kernel, Li, Artemiou and Li (2011) suggested using

\[
\gamma = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \| X_i - X_j \|,
\]

where \((X_1, \ldots, X_n)\) is an i.i.d. sample. This seems to work well in our numerical studies.

Now we turn our attention to selecting \( \lambda \) in (6). For a given \( \lambda \), denote the final estimator from the algorithm in Section 3.2 as \( \hat{B}_\lambda \). Partial distance correlation (Székely and Rizzo, 2014) is known be a good measure of conditional independence. Denote \( \rho(Y, X | B^\top X) \) as the partial distance correlation between \( Y \) and \( X \) conditional on \( B^\top X \). From (1), we know that the true central space basis \( B \) satisfies \( Y \perp X | B^\top X \), and thus \( \rho(Y, X | B^\top X) = 0 \). Given an i.i.d. sample \{\((X_i, Y_i), i = 1, \ldots, n\}\} and \( \hat{B}_\lambda \), the sample partial distance correlation between \( Y \) and \( X \) conditional on \( \hat{B}_\lambda^\top X \) is denoted as \( \hat{\rho}(Y, X | \hat{B}_\lambda^\top X) \). From a set of candidate values for \( \lambda \), we choose \( \lambda \) such that the absolute value of the sample partial distance correlation \( |\hat{\rho}(Y, X | \hat{B}_\lambda^\top X)| \) is minimized.

5.2. Pooled marginal estimators

In Cook and Setodji (2003) and Yin and Bura (2006), pooled marginal estimators are proposed for sufficient dimension reduction with multiple responses. Without loss of generality, we propose the pooled marginal expectile-assisted SIR in this section. The extensions to SAVE and directional regression are similar and thus omitted.

Recall that for \( \ell = 1, \ldots, k \), \( f_{\tau_\ell}(X) \) denotes the \( \tau_\ell \)-th conditional expectile of \( Y \) given \( X \). Let \( J_{1, \ell}, \ldots, J_{H, \ell} \) be a partition for the support of \( f_{\tau_\ell}(X) \). Let \( p_{h, \ell} = E\{ f_{\tau_\ell}(X) \in J_{h, \ell} \} \), \( \mu_{h, \ell} = E\{ Z | f_{\tau_\ell}(X) \in J_{h, \ell} \} \), and \( M_\ell = \sum_{h=1}^{H} p_{h, \ell} \mu_{h, \ell} \mu_{h, \ell}^\top \). Define \( \tilde{M} = (M_1, \ldots, M_k) \), and we have

**Proposition 5.** Under Assumption 1, \( \text{span}(\tilde{\Sigma}^{-1/2} \tilde{M}) \subseteq S_{Y|X} \).

The marginal approach essentially considers \( k \) univariate response sufficient dimension reduction problems separately and then assemble the individual
estimators for each response to get the final estimator. At the sample level, the estimator of the central space consists of the left singular vectors of the sample version of $\Sigma^{-1/2}\tilde{M}$. We refer to it as the pooled marginal expectile-assisted SIR estimator.

6. Simulation studies

We examine the empirical performances of our proposals through synthetic examples in this section. The predictor $X$ is generated from $N(0, I_p)$ with $p = 6$ or $p = 20$. The first six components of $\beta_1 \in \mathbb{R}^p$ is $(1,1,1,0,0,0)$, and the first six components of $\beta_2 \in \mathbb{R}^p$ is $(1,0,0,0,1,3)$. The remaining components of $\beta_1$ and $\beta_2$ are all zero when $p = 20$. The response $Y$ is generated as follows:

$I : Y = 0.4(\beta_1^T X)^2 + 3\sin(\beta_2^T X/4) + \sigma \epsilon$,

$II : Y = 3\sin(\beta_1^T X/4) + 3\sin(\beta_2^T X/4) + \sigma \epsilon$,

$III : Y = 0.4(\beta_1^T X)^2 + |\beta_2^T X|^{1/2} + \sigma \epsilon$,

$IV : Y = 3\sin(\beta_2^T X/4) + [1 + (\beta_1^T X)^2] \sigma \epsilon$,

where $\sigma = 0.2$, $\epsilon \sim N(0,1)$, and $\epsilon$ is independent of $X$. These are the same models used in Li and Wang (2007). Following Li and Wang (2007), two sample size settings are considered. For $n = 100$, we set $p = 6$ and number of slices $H = 5$. For $n = 500$, we set $p = 20$ and $H = 10$.

First, we compare SIR, SAVE, directional regression (DR), expectile-assisted SIR (EA-SIR), expectile-assisted SAVE (EA-SAVE), and expectile-assisted directional regression (EA-DR). Quantile-slicing mean estimation (QUME) (Kim, Wu and Shin, 2019) is also included for the comparison. In all four models, the basis for the central space is $B = (\beta_1, \beta_2)$. For estimator $\hat{B}$, we measure its performance by $\Delta = \|P_B - \hat{P}_B\|_F$. Here $P_B = B(B^T B)^{-1}B^T$, $P_B = \hat{B}(\hat{B}^T \hat{B})^{-1}\hat{B}^T$, and $\| \cdot \|_F$ denotes the matrix Frobenius norm.

For the $(n,p) = (100,6)$ setting, we summarize the simulation results based on 100 repetitions in Table 1. We notice that the expectile-assisted estimators are consistently better than their moment-based counterparts in all four models. For model IV with heteroscedastic error, the improvement is the most significant. As an extension of SIR, QUME does not work as well as EA-SIR. SIR does not work well with symmetric link functions, and EA-SIR
Table 1: Results based on \((n, p) = (100, 6)\). The average of \(\Delta\) and its standard error (in parentheses) are reported based on 100 repetitions.

| Model | SIR    | SAVE   | DR     | EA-SIR | EA-SAVE | EA-DR  | QUME  |
|-------|--------|--------|--------|--------|---------|--------|-------|
| I     | 1.648  | 0.626  | 0.384  | 1.336  | 0.571   | 0.356  | 1.559 |
|       | (0.429)| (0.588)| (0.411)| (0.630)| (0.478) | (0.329)| (0.498)|
| II    | 1.591  | 1.565  | 1.492  | 1.519  | 1.558   | 1.491  | 1.542 |
|       | (0.464)| (0.469)| (0.513)| (0.466)| (0.477) | (0.497)| (0.453)|
| III   | 2.620  | 0.652  | 0.638  | 2.456  | 0.559   | 0.540  | 2.669 |
|       | (0.635)| (0.499)| (0.513)| (0.466)| (0.482) | (0.502)| (0.692)|
| IV    | 1.700  | 1.598  | 1.557  | 1.490  | 1.414   | 1.324  | 1.520 |
|       | (0.340)| (0.455)| (0.462)| (0.562)| (0.535) | (0.496)| (0.494)|

Table 2: Results based on \((n, p) = (500, 20)\). The average of \(\Delta\) and its standard error (in parentheses) are reported based on 100 repetitions.

| Model | SIR    | SAVE   | DR     | EA-SIR | EA-SAVE | EA-DR  | QUME  |
|-------|--------|--------|--------|--------|---------|--------|-------|
| I     | 1.845  | 1.114  | 0.245  | 1.698  | 0.453   | 0.253  | 1.715 |
|       | (0.257)| (0.615)| (0.073)| (0.368)| (0.145) | (0.077)| (0.325)|
| II    | 1.564  | 1.796  | 1.710  | 1.586  | 1.826   | 1.645  | 1.792 |
|       | (0.358)| (0.226)| (0.288)| (0.338)| (0.165) | (0.337)| (0.243)|
| III   | 3.594  | 0.451  | 0.443  | 3.376  | 0.378   | 0.365  | 3.499 |
|       | (0.281)| (0.163)| (0.149)| (0.407)| (0.128) | (0.134)| (0.346)|
| IV    | 1.908  | 1.747  | 1.584  | 1.814  | 1.499   | 1.511  | 1.823 |
|       | (0.158)| (0.401)| (0.411)| (0.278)| (0.452) | (0.441)| (0.231)|

inherits this limitation. SIR and EA-SIR do not work well for models I, III, and IV, where at least one of the two link functions is symmetric. On the other hand, SAVE and EA-SAVE are not very efficient with monotone link functions in model II. DR is very competitive across all four models as it is not sensitive to the shape of the link functions. EA-DR further improves over DR and enjoys the best overall performance.

The simulation results for the \((n, p) = (500, 20)\) setting are summarized in Table 2. The expectile-assisted methods improve over their moment-based counterpart in three out of the four models. For model IV with heteroscedastic error, the expectile-assisted methods are consistently better than their moment-based counterparts. DR is very competitive across all four models, and EA-DR again has the best overall performance.
Table 3: Results based on $p = 6$. The average of $\Delta$ and its standard error (in parentheses) are reported based on 100 repetitions.

| Model | $n$ | EA-SIR  | EA-SAVE | EA-DR  | mEA-SIR | mEA-SAVE | mEA-DR  |
|-------|-----|---------|---------|--------|---------|----------|---------|
| I     | 50  | 1.509 (0.592) | 1.812 (0.681) | 0.939 (0.560) | 1.729 (0.524) | 1.817 (0.667) | 1.502 (0.543) |
|       | 150 | 1.214 (0.610) | 0.255 (0.262) | 0.175 (0.134) | 1.499 (0.572) | 0.278 (0.289) | 0.182 (0.142) |
| IV    | 50  | 1.503 (0.520) | 1.780 (0.662) | 1.489 (0.478) | 1.684 (0.437) | 1.832 (0.691) | 1.565 (0.459) |
|       | 150 | 1.424 (0.530) | 0.933 (0.599) | 0.898 (0.584) | 1.585 (0.485) | 1.443 (0.479) | 1.452 (0.502) |

Next we compare our proposed expectile-assisted methods based on random projections with the pooled marginal estimators described in Section 5.2. We denote the pooled marginal expectile-assisted SIR as mEA-SIR. Similarly, mEA-SAVE and mEA-DR denote the corresponding pooled marginal estimators for SAVE and DR. For this comparison, we fix $p = 6$, $H = 5$, and consider $n = 50$ or $n = 150$. The results based on 100 repetitions are summarized in Table 3. For both model I and model IV, the pooled marginal estimators are outperformed by the corresponding projective resampling estimators in all settings. This confirms the finding in Li, Wen and Zhu (2008) that projective resampling is more efficient than the pooled marginal estimators with multivariate response. By comparing the same method across different sample sizes, we see that EA-SAVE and EA-DR improve a lot when sample size increases from $n = 50$ to $n = 150$. Li and Wang (2007) commented that SAVE and DR are not very efficient with monotone link functions when the sample size is small. As both model I and model IV include a monotone link function, we observe the same inefficiency for EA-SAVE and EA-DR with $n = 50$.

7. Analysis of the Big Mac data

The Big Mac data contains 10 economic variables from 45 cities around the world in 1991. The data can be downloaded at [http://www.stat.umn.edu/RegGraph/data/BigMac.lsp](http://www.stat.umn.edu/RegGraph/data/BigMac.lsp). The response $Y$ is the minutes of labor needed to buy a Big Mac. The detailed description of the predictors can be found at the above website.
Following the discussions in Li (2008) (page 92), the scatter plot matrix reveals that the joint distribution of the predictors is not elliptical, and the LCM assumption may be violated. Thus we apply the optimal Box-Cox transformation (Box and Cox, 1964) before we compare different dimension reduction methods. Denote the predictors after the Box-Cox transformation as $X = (X_1, \ldots, X_9)^\top$.

We compare the performances of five methods: SIR, DR, EA-SIR, EA-DR, and QUME. As suggested in Li (2008) (page 139), we use structural dimension $d = 1$ for this data. We randomly split the $n = 45$ total observations into 24 observations for the training set and 21 observations for the testing set. First we get $\hat{\beta} \in \mathbb{R}^9$ for each method based on the training set. Then we calculate the sample partial distance correlation between $Y$ and $X$ conditional on $\hat{\beta}^\top X$ based on the testing set. We repeat this procedure 100 times, and the boxplots of the absolute sample partial distance correlation $|\hat{\rho}(Y, X | \hat{\beta}^\top X)|$ for each method are provided in Figure 1.

From scatter plot of the response $Y$ versus $\hat{\beta}^\top X$ (not reported), we can see a clear monotone trend. As we have seen in the simulation study, DR and
EA-DR are not expected to work well with monotone link functions when the sample size is small. As extensions of SIR, both EA-SIR and QUME improve over the original SIR and have smaller values of $|\hat{\rho}|$ than all the other methods. EA-SIR with 4 slices has the best performance with the smallest median $|\hat{\rho}|$ as well as the smallest variation across different splits.

**Appendix**

**Proof of Proposition 1.** Let $B$ be a basis of $\mathcal{S}_{Y|X}$. Then we have

$$E\{ \phi_\tau(Y - a)|X = x \} = E\{ \phi_\tau(Y - a)|B^T X = B^T x \}$$

because $Y \perp X|B^T X$. By the definition in (2), $f_\tau(X)$ evaluated at $x$ becomes

$$f_\tau(x) = \arg \min_a E\{ \phi_\tau(Y - a)|B^T X = B^T x \}.$$ 

This implies that $f_\tau(X)$ is a function of $B^T X$ and $f_\tau(X) \perp X|B^T X$ for any fixed $\tau$. It follows that $\xi_X \perp X|B^T X$. By the definition of the central space, we have $\mathcal{S}_{\xi_X|X} \subseteq \text{span}(B) = \mathcal{S}_{Y|X}$.

**Proof of Proposition 2.** The proof follows directly from Theorem 3.1 of Li, Wen and Zhu (2008), and is thus omitted.

**Proof of Proposition 3.** The proof follows directly from Theorem 3.1 of Li, Wen and Zhu (2008), and is thus omitted.

**Proof of Proposition 4.** The proof follows directly from Theorem 3.1 of Li, Wen and Zhu (2008) and Theorem 2 of Li and Wang (2007), and is thus omitted.

**Proof of Proposition 5.** From Theorem 3.1 of Li (1991), we have

$$\text{span}(\Sigma^{-1/2} M_\ell) \subseteq \mathcal{S}_{f_\tau(X)|X} \text{ for } \ell = 1, \ldots, k.$$ 

From the proof of Proposition 1, we know $\mathcal{S}_{f_\tau(X)|X} \subseteq \mathcal{S}_{Y|X}$. Together we have $\text{span}(\Sigma^{-1/2} \tilde{M}) = \text{span}(\Sigma^{-1/2} M_1, \ldots, \Sigma^{-1/2} M_k) \subseteq \mathcal{S}_{Y|X}$.

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