PRODUCT OF ALMOST-HERMITIAN MANIFOLDS

XU-QIAN FAN\textsuperscript{1}, LUEN-FAI TAM\textsuperscript{2}, AND CHENGJIE YU\textsuperscript{3}

Abstract. This is a continuous work about the nonexistence of some complete metrics on the product of two manifolds studied by Tam-Yu \cite{11}. Motivated by the result of Tossati \cite{12}. We generalize the corresponding results of Tam-Yu \cite{11} to the almost-Hermitian case.

1. Introduction

In \cite{14}, Yang proved the nonexistence of complete Kähler metrics with holomorphic bisectional curvature bounded between two negative constants on the polydisc. Later, Seshadri \cite{8} and Seshadri-Zheng \cite{9} extended Yang’s result onto the product of two complex manifolds of positive dimensions. Indeed, they showed that there is no complete Hermitian metrics with holomorphic bisectional curvature bounded between two negative constants and bounded torsion on the product of two complex manifolds of positive dimensions. In \cite{11}, Tam-Yu relaxed the curvature bounds of the result of Seshadri-Zheng \cite{9} to a reasonable curvature decay or growth rate in the Kähler category. In \cite{12}, Tosatti generalized the result of Seshadri-Zheng \cite{9} onto the product two almost complex manifolds. Indeed, Tosatti obtained the following result:

**Theorem 1.1** (Tosatti). Let $M = X \times Y$ be a product of almost complex manifolds of positive dimensions. Then, there is no complete almost Hermitian metric on $M$ satisfying the following conditions:

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The holomorphic bisectional is bounded between two negative constants;
(2) The \((2,0)\) part of the curvature tensor is bounded;
(3) The torsion is bounded.

In this paper, motivated by the result of Tosatti, we generalize the results of Tam-Yu \([11]\) onto the product of two almost complex manifolds of positive dimensions. The main results we obtain are the follows.

**Theorem 1.2.** Let \(X^{2m}, Y^{2n}\) be two almost complex manifolds of real dimension \(2m, 2n\) respectively, \(m, n \geq 1\). Then there is no complete almost Hermitian metric on \(M = X \times Y\) satisfying the following:

1. second Ricci curvature \(\geq -A(1 + r)^2\);
2. holomorphic bisectional curvature \(\leq -B < 0\);
3. torsion bounded by \(A(1 + r)\);
4. \((2,0)\) part of the curvature tensor bounded by \(A(1 + r)^2\),

where \(r(x) = d(x, o)\) is the distance between \(x\) and a fixed point \(o \in M\), and \(A, B\) are two positive constants.

**Theorem 1.3.** Let \(M = X^{2m} \times Y^{2n}\) be the product of two almost complex manifolds with positive dimension. Then there is no complete almost Hermitian metric on \(M\) satisfying the following:

1. second Ricci curvature \(\geq -A(1 + r^2)^{\gamma}\);
2. holomorphic bisectional curvature \(\leq -B(1 + r^2)^{-\delta}\);
3. nonpositive sectional curvature for the Levi-Civita connection;
4. torsion is bounded by \(A(1 + r^2)^{\gamma/2}\);
5. \((2,0)\) part of the curvature tensor is bounded by \(A(1 + r^2)^{\gamma}\),

where \(\gamma \geq 0, \delta > 0\) such that \(\gamma + 2\delta < 1\), \(A, B\) are some positive constants, and \(r(x) = d(x, o)\) is the distance of \(x\) and a fixed point \(o \in M\).

Clearly, if \((M, J, g)\) is Kähler, then the torsion and \((2,0)\) part of the curvature tensor are zero, and second Ricci curvature is just the Ricci curvature, so these theorems cover Theorem 1.2 and Theorem 1.3 in \([11]\) respectively.

The techniques using for proving the main results are mainly the same as in Tam-Yu \([11]\). However, for the almost Hermitian case, we don’t have at hand a simple formula similar to the Kähler or Hermitian case for computing the curvature tensor, so the computation in \([11]\) can not be extended directly to the almost Hermitian case. In this paper, we use a general Ricci identity (Lemma \([2.3]\)) to handle this difficulty. Generally speaking, this is a Bochner technique on almost Hermitian manifolds. Another difference with the complex case is that
we don’t have holomorphic coordinates in the almost Hermitian case so that computations can be performed on the coordinate since the complex structure may not be integrable. In this paper, we introduce local coordinates that play similar roles of holomorphic coordinates on almost complex manifolds so that similar computations as in Kähler geometry can also be performed on almost Hermitian manifolds.

The contents of this paper are arranged as follows. In section 2, we recall some preliminary definitions and results about almost Hermitian manifolds. In section 3, we give a proof of Theorem 1.2. In section 4, we give a proof of Theorem 1.3.

2. Preliminaries on almost Hermitian manifolds

For convenience, let us recall some notations and basic results about almost-Hermitian manifolds, please see e.g. [4, 13, 12].

We say that \((M^{2n}, J, g)\) is an almost-Hermitian manifold of real dimension \(2n\) if \(J\) is an almost complex structure on \(M\) and \(g\) is a Riemannian metric which is \(J\) invariant. For a point \(p \in M\), let \(T^C_pM = T_pM \otimes \mathbb{C}\), and decompose it as \(T^C_pM = T'_pM \oplus T''_pM\) where \(T'_pM\) and \(T''_pM\) are the eigenspaces of \(J\) corresponding to the eigenvalues \(\sqrt{-1}\) and \(-\sqrt{-1}\) respectively.

An affine connection \(\nabla\) on \(TM\) which is extended linearly to \(T^C M\) is called an almost-Hermitian connection if \(\nabla J = \nabla g = 0\). Let \(\tau\) be the torsion of the connection \(\nabla\) which is defined by

\[
\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]
\]

for \(X, Y \in T^C M\). One has the following result (see, e.g. [4, 7]).

**Lemma 2.1.** There exists a unique almost-Hermitian connection \(\nabla\) on \((M, J, g)\) such that the torsion \(\tau\) has vanishing \((1, 1)\) part.

This connection is called the canonical connection. It is first introduced by Ehresmann and Libermann in [3], and if \(J\) is integrable it is the connection defined in [2] by Chern. In this work, we always denote the canonical connection by \(\nabla\) and the Levi-Civita connection by \(D\). For the difference between the canonical connection and the Levi-Civita connection, we have the following conclusion, see [4].

**Lemma 2.2.** On an almost Hermitian manifold \((M, J, g)\):

\[
\langle D_Y X - \nabla_Y X, Z \rangle = \frac{1}{2}[\langle \tau(X, Y), Z \rangle + \langle \tau(Y, Z), X \rangle - \langle \tau(Z, X), Y \rangle]
\]

**Proof.** Note that

\[
(2.1) \quad \langle D_X Y, Z \rangle + \langle D_Y Z, Y \rangle = X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle \nabla_X Z, Y \rangle
\]
Hence
\begin{equation}
\langle D_X Y, Z \rangle + \langle D_Z X, Y \rangle = \langle \nabla_X Y, Z \rangle + \langle \nabla_Z X, Y \rangle - \langle \tau(Z, X), Y \rangle
\end{equation}

Similarly, we have
\begin{equation}
\langle D_Z X, Y \rangle + \langle D_Y Z, X \rangle = \langle \nabla_Z X, Y \rangle + \langle \nabla_Y Z, X \rangle - \langle \tau(Y, Z), X \rangle
\end{equation}
and
\begin{equation}
\langle D_Y Z, X \rangle + \langle D_X Y, Z \rangle = \langle \nabla_Y Z, X \rangle + \langle \nabla_X Y, Z \rangle - \langle \tau(X, Y), Z \rangle
\end{equation}

Adding (2.2) and (2.4), and subtracting (2.3), we get
\begin{equation}
\langle D_X Y - \nabla_X Y, Z \rangle = \frac{1}{2} \left( \langle \tau(Y, X), Z \rangle + \langle \tau(X, Z), Y \rangle - \langle \tau(Z, Y), X \rangle \right).
\end{equation}

This completes the proof. \(\square\)

In local frame \(e_a\), \(1 \leq a \leq 2n\), we have

**Corollary 2.1.** \((\gamma^e_{ab} - \Gamma^e_{ab})g_{ec} = \frac{1}{2}(\tau^e_{ab}g_{ec} + \tau^e_{be}g_{ea} - \tau^e_{ca}g_{eb})\) where \(\gamma^e_{ab}\)'s are the Christoffel symbol of the Riemannian connection and \(\nabla_{ea}e_b = \Gamma^c_{ab}e_c\).

**Corollary 2.2.** Let \(f\) be a smooth function on \(M\), then
\begin{equation}
\nabla^2 f(X, Y) - D^2 f(X, Y) = \frac{1}{2} \left( (\tau(X, Y), \nabla f) + (\tau(Y, \nabla f), X) - (\tau(\nabla f, X), Y) \right)
\end{equation}

In local frame \(e_a\),
\begin{equation}
f_{ab} - f_{ba} = \frac{1}{2}(\tau^e_{ab}f_c + \tau^e_{be}g^{cd}f_dg_{ea} - \tau^e_{ca}g^{cd}f_dg_{eb})
\end{equation}

where “;” means taking covariant derivatives with respect to the Levi-Civita connection.

If \(\{e_1, \cdots, e_n\}\) is a local frames of \(T'M\), then
\begin{equation}
f_{ij} - f_{ji} = \frac{1}{2}(\tau^k_{i\lambda}g^{\mu\lambda}f_{\mu k} + \tau^k_{j\lambda}g^{\lambda\mu}f_{\mu k}).
\end{equation}

In particular, \(f_{ab} - f_{ba} = \tau^c_{ab}f_c\) and \(f_{ij} = f_{ji}\).

The last assertion follows from the fact that the (1,1) part of \(\tau\) is zero.

Taking trace of the above gives us, see [12].

**Corollary 2.3.** \(\Delta f - \Delta^L f = \tau^a_{ab}g^{bc}f_c\), where \(\Delta f = g^{ab}\nabla^2 f(e_a, e_b)\) and \(\Delta^L f = g^{ab}D^2 f(e_a, e_b)\) is the Laplacian with respect to the Levi-Civita connection.
In general, let $M$ be a manifold with connection $\nabla$, and $E$ be a vector bundle over $M$ with connection $D$. Let $s$ be a section of $E$. Then $Ds$ is a section of $TM \otimes E$. To compute more derivatives, we need the connection $\nabla$ on $M$. Let $\tau$ be the torsion of $\nabla$. Then, we have following Ricci identity.

**Lemma 2.3.** $D^2 s(X,Y) - D^2 s(Y,X) = - R(X,Y) s + D\tau(X,Y)s$

**Proof.**

$$D^2 s(X,Y) - D^2 s(Y,X) = (D_Y Ds)(X) - (D_X Ds)(Y)$$

(2.9)

$$= D_Y D_X s - D_X D_Y s + Ds(\nabla_X Y)$$

$$= - R(X,Y)s + D\tau(X,Y)s$$

□

Now let us recall some definitions about the curvature. At a point $p$, choose a local unitary frame $\{e_1, \cdots, e_n\}$ for $T'_p(M)$, and denote $\{\theta^1, \cdots, \theta^n\}$ as a dual coframe. Denote $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, $R(e_C,e_D)e_A = R^E_{A,C,D} e_E$ and $R_{ABCD} = R(e_A,e_B,e_C,e_D) = \langle R(e_C,e_D)e_A,e_B \rangle = R^F_{A,C,D} g_{E,F}$. Here $A, B, C, D$ can be taken $1, 1, \cdots, n, n$. Define the second Ricci curvature as $R'_{k\bar{l}} = R^i_{k i \bar{l}}$, the holomorphic bisectional curvature in the directions $X$ and $Y$ as

$$B(X,Y) = \frac{R(X,\overline{X},Y,\overline{Y})}{\|X\|^2 \|Y\|^2},$$

and the $(2,0)$ part of the curvature as $R_{j k i}^i \theta^k \wedge \theta^i$.

Similar to [12], we say that the holomorphic bisectional curvature is bounded from above by $K$ if

$$B(X,Y) \leq K$$

for all $X, Y \in T'M$. The second Ricci curvature is bounded from below by $-A_1$ if

$$R'_{k\bar{l}} X^k \overline{X}^\ell \geq -A_1 \|X\|^2$$

for all $X \in T'M$. The torsion is bounded by $A_2 > 0$ if

$$|\tau(X,Y)| \leq A_2 \|X\| \|Y\|$$

for all $X, Y \in T'M$. The $(2,0)$ part of the curvature is bounded by $A_3 > 0$ if

$$|R(X,\overline{X},Y,\overline{Y})| \leq A_3 \|X\|^2 \|Y\|^2$$
for all $X, Y \in T'M$.

3. PROOF OF THEOREM 1.2

We will prove Theorem 1.2 by contradiction as in [11]. Suppose $M = X^{2n} \times Y^{2m}$ is a product of two almost complex manifolds of positive dimensions satisfying the conditions in Theorem 1.2. Fix a point $q \in Y$, we will show that the volume growth of $X \times \{q\}$ has some upper estimate. On the other hand we show that this upper estimate is not possible because of the following maximum principle which is similar to Theorem 1.1 in [10].

Lemma 3.1. Let $(M, g)$ be a complete non-compact Riemannian manifold, $r(x)$ be the distance function from a fixed point $o \in M$. Let $u$ be a smooth function on $M$ satisfying the inequality

\[ \Delta L u \geq C_1 u^2 - C_2 (1 + r) |\nabla u| \]

on $\{u > \delta\} \neq \emptyset$ for some $C_1, C_2, \delta > 0$, where $\Delta L$ is the Laplace operator with respect to the Levi-Civita connection, then

\[ \liminf_{t \to +\infty} \frac{\log V_o(t)}{t^2} = +\infty \]

where $V_o(t)$ is the volume of the geodesic ball of radius $t$ centered at the point $o \in M$.

Proof. We will adapt the proof of Theorem 2.1 in [10]. For simplicity, in the proof of this lemma, we write $\Delta$ instead of $\Delta L$. Firstly, we may assume that $\sup_M u = +\infty$ satisfying the differential inequality (3.1) with a different $C_1$. Otherwise, suppose that $\sup_M u = u^*$. By differential inequality (3.1) satisfied by $u$, $u^*$ can not be attained. Let

$v = \frac{1}{u^*-u}$, we have

\[ \Delta v = \frac{\Delta u}{(u^*-u)^2} + \frac{2|\nabla u|^2}{(u^*-u)^3} \geq C_1 \delta^2 v^2 - C_2 (1 + \rho) |\nabla v| \]

on $\{v > 1/(u^* - \delta)\} = \{u > \delta\}$. Now $\sup_M u = \infty$, for any number $Q > \delta$, we can assume that $\{u > Q\}$ is not empty. Replace $u$ by $u/Q$, we know that

$\Delta u \geq C_1 Q u^2 - C_2 (1 + \rho) |\nabla u|$ on $\{u > 1\}$. So we conclude that, for any constant $\beta > C_1 \delta$, there is a smooth function $u$ on $M$ such that

(3.3) $\Delta u \geq \beta u^2 - C_2 (1 + r) |\nabla u|$ on the nonempty set $M^* = \{u > 1\}$. We will choose $\beta$ to be large enough later. Note that $C_2$ is independent of $\beta$. 

The same as [10], let \(0 \leq \sigma \leq 1\) be a smooth function on \(\mathbb{R}\) such that \(\sigma = 0\) on \(t \leq 1\), \(\sigma = 1\) on \(t \geq 2\) and \(\sigma > 0\) for \(t > 1\), \(\sigma' \geq 0\). Let
\[
\lambda(t) = \int_{-\infty}^{t} \sigma(s) ds.
\]
Then
\[
\left\{
\begin{array}{ll}
\lambda(t) \equiv 0 & \text{if } t \leq 1 \\
\lambda(t) > 0, \lambda'(t) > 0, \lambda''(t) \geq 0 & \text{if } t > 1 \\
\lambda'(t) \equiv 1 & \text{if } t \geq 2.
\end{array}
\right.
\]
For \(\rho > 0\), let \(\omega\) be a Lipschitz continuous on \(M\) such that
\[
\left\{
\begin{array}{ll}
0 \leq \omega \leq 1, \quad |\nabla \omega| \leq 1/\rho \\
\text{Supp}(\omega) \subset B_\rho(2\rho) \\
\omega \equiv 1 \text{ on } B_\rho(\rho).
\end{array}
\right.
\]
For any positive constants \(p, q, \epsilon\), by (3.3), we have
\[(3.4)
\text{div}(\omega^{2q} \nabla \lambda(u^p)) = \langle \nabla \omega^{2q}, \nabla \lambda(u^p) \rangle + \omega^{2q} \Delta \lambda(u^p)
\]
\[= 2qp\lambda' \omega^{2q-1} u^{p-1} \langle \nabla \omega, \nabla u \rangle
\]
\[+ \omega^{2q} [\lambda''(pu^{p-1})^2 |\nabla u|^2 + p\lambda'(p-1)u^{p-2} |\nabla u|^2 + \lambda pu^{p-1} \Delta u]
\]
\[\geq p\lambda' \left[ - \epsilon \omega^{2q} u^{p-2} |\nabla u|^2 - \frac{q^2}{\epsilon} \omega^{2q(p-1)} u^p |\nabla \omega|^2 + \omega^{2q} (p-1)u^{p-2} |\nabla u|^2 
\]
\[+ \beta \omega^{2q} u^{p+1} - \omega^{2q} u^p - C_2^2 (1 + \rho)^2 \omega^{2q} u^{p-2} |\nabla u|^2 \right]
\]
\[= p\lambda'(\beta - 1) \omega^{2q} u^{p+1} + (p - 1 - C_2^2 (1 + \rho)^2 - \epsilon) \omega^{2q} u^{p-2} |\nabla u|^2 - \frac{q^2}{\epsilon} \omega^{2q(p-1)} u^p |\nabla \omega|^2 \]
in \(B_\rho(2\rho)\), provided \(u > 1\). Since \(\lambda' = 0\) if \(t \leq 1\) and \(\omega\) has support in \(B_\rho(2\rho)\), the above inequality is true in \(M\).
Choosing \(p = p(\rho)\) such that
\[(3.5) \quad p - 1 = 2C_2^2 (1 + \rho)^2, \quad \epsilon = \frac{p - 1}{2}, q = p + 1\]
Since \(\omega\) has compact support, assume further that \(\beta > 1\)
\[
\int_{B_\rho(\rho)} \lambda' \leq \int_{B_\rho(\rho)} \lambda' (\beta - 1) u^{p+1}
\]
\[\leq \int_{B_\rho(2\rho)} \lambda' (\beta - 1) \omega^{2q} u^{p+1}\]
On the other hand, let \( q = p + 1 \), by (3.4)

\[
(\beta - 1) \int_{B_o(2\rho)} \lambda' \omega^{2q} u^{p+1} = (\beta - 1) \int_{B_o(2\rho)} \lambda' \omega^{2(q+1)} u^{p+1}
\]

\[
\leq \frac{2(p + 1)^2}{p - 1} \int_{B_o(2\rho)} \lambda' \omega^{2p} u^p |\nabla \omega|^2
\]

\[
\leq \frac{2(p + 1)^2}{(p - 1)\rho^2} \left( \int_{B_o(2\rho)} \lambda' \omega^{2(p+1)} u^{p+1} \right)^{\frac{p}{p+1}} \left( \int_{B_o(2\rho)} \lambda' \right)^{\frac{1}{p+1}}.
\]

Hence

\[
\int_{B_o(\rho)} \lambda' \leq \left( \frac{2(p + 1)^2}{(\beta - 1)(p - 1)\rho^2} \right)^p \cdot \frac{2(p + 1)^2}{(p - 1)\rho^2} \int_{B_o(2\rho)} \lambda'.
\]

By the definition of \( p = p(\rho) \), choose \( \beta \) such that \( \beta - 1 > 16(C_2^2 + 1)^2 \).

There is \( \rho_0 \) such that if \( \rho \geq \rho_0 \),

\[
\frac{2(p + 1)^2}{(\beta - 1)(p - 1)\rho^2} < \frac{1}{2(C_2^2 + 1)}.
\]

Hence for \( \rho \geq \rho_0 \),

\[
\int_{B_o(\rho)} \lambda' (u^{p(\rho)}) \leq \left( \frac{1}{2} \right)^p \int_{B_o(2\rho)} \lambda' (u^{p(\rho)}) \leq \int_{B_o(2\rho)} \lambda' (u^{p(2\rho)}).
\]

for some \( k \) which is independent of \( \rho \). Here we have used the fact that \( \lambda' \) is nondecreasing, \( p(2\rho) > p(\rho) \) and \( \lambda' = 0 \) if \( u \leq 1 \). Let

\[
F(\rho) = \int_{B_o(\rho)} \lambda' (u^{p(\rho)}).
\]

We have

\[
F(\rho) \leq \left( \frac{1}{2} \right)^p F(2\rho).
\]

By iterating, we have

\[
F(\rho_0) \leq \left( \frac{1}{2} \right)^{C_3 \rho^2} F(\rho) \leq \left( \frac{1}{2} \right)^{C_3 \rho^2} V_o(\rho)
\]

for some \( C_3 > 0 \), for \( \rho > 0 \), because \( \lambda' \leq 1 \). Since \( \{ u > 1 \} \) is nonempty, if \( \rho_0 \) is chosen large enough, \( F(\rho_0) > 0 \). From this it is easy to see that the lemma is true.

To estimate the volume growth, we will use the following result due to Tosatti [12, Theorem 4.2]:
Lemma 3.2 (Tossati). Suppose \((M^m, J, g)\) is a complete almost-Hermitian manifolds with real dimension \(m\), \(B_o(R)\) is a geodesic ball centered at \(o \in M\) of radius \(R\). If the second Ricci curvature of \(B_o(R)\) is bounded from below by \(-K_1\), the torsion bounded by \(A_2\) and \((2, 0)\) part of the curvature bounded by \(A_3\) on \(B_o(R)\) for some positive constants \(A_1, A_2, K\), then

\[
\Delta r \leq \frac{m}{r} + c
\]

where \(r\) is the distance function from \(o\), \(\Delta\) is the Laplace operator with respect to the canonical connection, \(c = c_1\alpha\), \(\alpha = (A_2 + \sqrt{K_1} + \sqrt{A_3})\), and \(c_1\) is a positive constant depending on \(m\).

From this, one has

Corollary 3.1. Under the same notations and assumptions as in Lemma 3.2, we have for any fixed \(0 < t_0 < R\),

\[
V_o(t) \leq V_o(t_0) \left(\frac{t}{t_0}\right)^{m+1} e^{C\alpha t} \quad \text{for} \quad R \geq t \geq t_0,
\]

where \(C\) is a constant depending only on \(m\). In particular,

\[
V_o(R) \leq V_o(t_0) \left(\frac{R}{t_0}\right)^{m+1} e^{C\alpha R}.
\]

Proof. By Corollary 2.3 or Lemma 3.2 in [12], and by Lemma 3.2 we have

\[
\Delta r \leq \frac{m}{r} + c
\]

where \(c\) is the constant in Lemma 3.2. So

\[
\Delta^L r \leq \frac{m}{r} + C\alpha
\]

where \(C\) is a positive constant depending only on \(m\). Multiplying \(r\) to the both sides of inequality above, we have

\[
r \Delta^L r \leq m + rC\alpha.
\]

So

\[
\int_{B_o(t)} r \Delta^L r \leq \int_{B_o(t)} \left(m + rC\alpha\right).
\]

Hence

\[
t A_o(t) \leq (m + 1 + C\alpha) V_o(t).
\]

That is

\[
(ln V_o(t))' \leq t^{-1} (m + 1 + C\alpha).
\]
Integrating both sides from $t_0$ to $r$, we have
\[
V_o(t) \leq V_o(t_0) \left( \frac{t}{t_0} \right)^{m+1} e^{C \alpha t}.
\]

This completes the proof of the lemma. \[\square\]

Similar to Lemma 2.1 in [11], we have

**Lemma 3.3.** Suppose $(M^m, J, g)$ and $(N^n, \tilde{J}, h)$ are two complete almost-Hermitian manifolds. Let $f$ be a non-constant almost-complex map from $M$ to $N$. Let $o \in M$ and let $R > 0$. If the second Ricci curvature of $B_o(2R)$ is bounded from below by $-K_1$, the torsion bounded by $A_2$ and $(2, 0)$ part of the curvature bounded by $A_3$ on $B_o(2R)$, and the bisectional curvature in $f(B_o(2R))$ is bounded above by $-K_2$, where $A_2, A_3, K_1, K_2$ are positive constants, then on $B_o(R)$,
\[
f^*h \leq \frac{2K_1 + CR^{-2}(1 + cR)}{2K_2} g
\]
where $C$ is a constant depending only on $m$ and $c$ is the same as in (3.6).

**Proof.** Let $u = \text{tr}_g(f^*h)$. Since the second Ricci curvature of $B_o(2R)$ is bounded from below by $-K_1$ and the bisectional curvature in $f(B_o(2R))$ is bounded above by $-K_2$, by the result of [12, page 1081], one has
\[
(3.7) \quad \Delta u \geq 2K_2 u^2 - 2K_1 u
\]
on $B_o(2R)$, where $\Delta$ denotes the Laplacian with respect to the canonical connection on $M$.

Similar to the proof of Lemma 2.1 in [11]. Let $\eta \geq 0$ be a smooth function on $\mathbb{R}$ such that (1) $\eta(t) = 1$ for $t \leq 1$, (2) $\eta(t) = 0$ for $t \geq 2$, (3) $-C_3 \leq \eta'/\eta^{1/2} \leq 0$ for all $t \in \mathbb{R}$, and (4) $|\eta''(t)| \leq C_3$ for all $t \in \mathbb{R}$ for some absolute constant $C_3 > 0$. Let $\phi = \eta(r/R)$, where $r$ is the distance function from $o$.

Suppose $\phi u$ attains maximum at $\bar{x} \in B_o(2R)$, then $\phi(\bar{x}) > 0$. Using an argument of Calabi as in [1], we may assume that $\phi u$ is smooth at $\bar{x}$. Then we have (1) $\nabla(\phi u)(\bar{x}) = 0$ which implies that at $\bar{x}$, $\nabla u =$

\[ -u\phi^{-1}\nabla\phi, \] 
(2) \( \Delta(\phi u)(\bar{x}) \leq 0 \). Using Corollary 2.3 at \( \bar{x} \), we have 
\[ 0 \geq \Delta(\phi u) \] 
(3.8) 
\[ = \phi\Delta u + u\Delta\phi + 2\langle \nabla\phi, \nabla u \rangle \] 
\[ = \phi\Delta u + u(\eta''R^{-2} + \eta'R^{-1}\Delta r) + 2\langle \nabla\phi, \nabla u \rangle \] 
\[ \geq \phi\Delta u - 2C_1uR^{-2} + u(\eta''R^{-2} + \eta'R^{-1}\Delta r) \] 
\[ \geq \phi(2K_2u^2 - 2K_1u) - 2C_1uR^{-2} - C_3uR^{-2} + u\eta'R^{-1}\Delta r \] 
(by (3.7)). 

So 
\[ 2\phi K_2u^2 \leq 2K_1\phi u + 2C_1uR^{-2} + C_3uR^{-2} - u\eta'R^{-1}\Delta r. \] 

By Lemma 3.2, we have 
\[ \Delta r \leq \frac{m}{r} + c \] 
where \( c \) is the same as in (3.6). So we can get 
\[ 2K_2\phi u^2 \leq 2K_1\phi u + 2C_1uR^{-2} + C_3uR^{-2} + C_3R^{-1}u \left( \frac{m}{R} + c \right) \] 
(3.9) 
\[ \leq u[2K_1 + R^{-2}(2C_1^2 + C_3 + C_3m + C_3cR)]. \] 

Hence 
\[ (3.10) \quad \sup_{B_o(R)} u \leq \sup_{B_o(2R)} (\phi u) \leq \frac{2K_1 + CR^{-2}(1 + cR)}{2K_2} \] 
where \( C \) is a constant depending on \( m \) and \( C_1 \), and \( c \) is the same as in (3.6). Therefore the lemma holds. \( \square \)

We have the following volume growth estimate of geodesic ball on the submanifold \( X \).

**Lemma 3.4.** Let \( M, X, Y \) as in Theorem 1.2. Suppose there is a complete almost Hermitian metric on \( X \times Y \) satisfying the assumptions in the theorem, that is:

1. second Ricci curvature \( \geq -A(1 + r)^2 \)  
2. holomorphic bisectional curvature \( \leq -B < 0 \)  
3. torsion bounded by \( A(1 + r) \)  
4. \((2,0)\) part of the curvature tensor bounded by \( A(1 + r)^2 \)

for some positive constants \( A, B \), where \( r(x) = d(x, o) \) is the distance of \( x \) and a fixed point \( o = (p, q) \in X \times Y \). Let \( V^X_p(t) \) be the volume of
the geodesic ball of radius $t$ with center at $p$ with respect to the induced metric $g^q$ on $X_q = X \times \{q\}$. Then

$$V_p^{X_q}(t) \leq C_4 \exp(C_4t^2)$$

for some positive constant $C_4$ independent of $t$.

Proof. The proof is similar to the proof of Lemma 2.2 in [11]. By [6], the bisectional curvature of the canonical connection of an almost complex submanifold is not bigger than the one of the ambient space. So for any point $y_0 \in Y$, then the bisectional curvature of $X_{y_0}$ is also bounded from above by $-B$. The same holds for $Y_{x_0}$ where $Y_{x_0} = \{x_0\} \times Y$. By Lemma 3.3 there is a constant $C_5$ independent of $x_0, y_0$ such that

$$(\pi''_{x_0})^*(g_{x_0}(x, y)) \leq C_5 (1 + r(x, y))^2 g_{(x, y)}$$

and

$$(\pi'_{y_0})^*(g_{y_0}(x, y)) \leq C_5 (1 + r(x, y))^2 g_{(x, y)}$$

for $(x, y) \in M$, where $\pi'_{y_0}$ is the projection from $M$ onto $X_{y_0}$ defined by $\pi'_{y_0}(x, y) = (x, y_0)$, $\pi''_{x_0}$ is the projection from $M$ onto $Y_{x_0}$ defined by $\pi''_{x_0}(x, y) = (x_0, y)$, $g_{x_0}, g_{y_0}$ are the induced metrics on $X_{x_0}, X_{y_0}$ respectively. By Corollary 3.1 we have $V_0(2R) \leq \exp(C(1 + R)^2)$ for some constant $C$. In the rest of the proof, we can follow the corresponding argument of the proof of Lemma 2.2 in [11] to get the conclusion of this lemma. \[\square\]

Next we want to find a function on $X_q$ satisfying the differential inequality in Lemma 3.1. For convenience, we will introduce a special coordinate near a point on almost complex manifold.

**Definition 3.1.** Let $(M^{2m}, J)$ be an almost complex manifold. Let $p \in M$, $U$ be an open neighborhood of $p$. Let $\phi : U \to \Omega \subset \mathbb{C}^m$ be a diffeomorphism. Then, $(U, \phi)$ is called a complex coordinate.

Let $\phi = (z^1, z^2, \ldots, z^m)$ be a complex coordinate, and suppose that $z^i = x^i + \sqrt{-1}y^i$. Then $(x^1, y^1, \ldots, x^m, y^m)$ is a local coordinate. As usual, we define

**(3.11)**

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right)$$

and

**(3.12)**

$$\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left( \frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i} \right)$$

as vectors in $TM \otimes \mathbb{C}$. 
Definition 3.2. Let $(M, J)$ be an almost complex manifold. Let $(z^1, z^2, \ldots, z^m)$ be a complex coordinate at $p$. It is called almost holomorphic at $p$ if

\begin{equation}
J(\frac{\partial}{\partial z^i})(p) = \sqrt{-1}\frac{\partial}{\partial z^i}(p)
\end{equation}

for all $i = 1, 2, \ldots, m$.

Lemma 3.5. Let $(M, J)$ be an almost complex manifold. Then for any $p \in M$, there is a local complex coordinate $(z^1, z^2, \ldots, z^n)$ that is almost holomorphic at $p$ such that

\begin{equation}
\partial_i J^k_j(p) = \partial_i J^k_j(p) = \partial_i J^k_j(p) = 0
\end{equation}

where

\begin{equation}
J(\frac{\partial}{\partial z^i}) = J^j_i \frac{\partial}{\partial z^j} + J^j_i \frac{\partial}{\partial z^j}.
\end{equation}

Proof. Let $(z^1, z^2, \ldots, z^n)$ be a local complex coordinate that is almost holomorphic at $p$. Suppose that

\begin{equation}
J(\frac{\partial}{\partial z^i}) = J^j_i \frac{\partial}{\partial z^j} + J^j_i \frac{\partial}{\partial z^j}
\end{equation}

Then

\begin{equation}
J^i_j(p) = \sqrt{-1}\delta_{ij} \text{ and } J^i_j(p) = 0
\end{equation}

By that $J^2 = -id$, we know that

\begin{equation}
J^i_j J^k_j + J^j_i J^k_i = -\delta_{ik}.
\end{equation}

Taking partial differentiations of (3.18), we know that

\begin{equation}
\partial_j J^k_i(p) = \partial_j J^k_i(p) = \partial_j J^k_i(p) = 0
\end{equation}

for all $i, j, k = 1, 2, \ldots, m$.

Let $(w^1, w^2, \ldots, w^n)$ be a coordinate change of $(z^1, z^2, \ldots, z^n)$ such that

\begin{equation}
\frac{\partial w^i}{\partial z^j}(0) = \delta_{ij} \text{ and } \frac{\partial w^i}{\partial z^j}(0) = 0.
\end{equation}

Suppose that

\begin{equation}
J(\frac{\partial}{\partial w^i}) = J^j_i \frac{\partial}{\partial w^j} + J^j_i \frac{\partial}{\partial w^j}.
\end{equation}

By a straight forward computation, we have

\begin{equation}
J^i_j = \frac{\partial z^j}{\partial w^i} J^k_j \frac{\partial w^i}{\partial z^k} + \frac{\partial z^j}{\partial w^i} J^k_j \frac{\partial w^i}{\partial z^k} + \frac{\partial z^j}{\partial w^i} J^k_j \frac{\partial w^i}{\partial z^k} + \frac{\partial z^j}{\partial w^i} J^k_j \frac{\partial w^i}{\partial z^k}.
\end{equation}
Then, by using (3.19) and (3.20), we have

\[
\frac{\partial}{\partial w^\alpha} \tilde{J}_i^j(p) = \sqrt{-1} \frac{\partial}{\partial w^\alpha}(\frac{\partial z^i}{\partial z^j}) - \sqrt{-1} \frac{\partial}{\partial w^\alpha}(\frac{\partial z^i}{\partial w^j}) + \frac{\partial}{\partial z^\alpha}(\tilde{J}_i^j)
\]

(3.23)

\[
= -2\sqrt{-1} \frac{\partial z^i}{\partial w^\alpha}(0) + \frac{\partial}{\partial z^\alpha}(\tilde{J}_i^j)(p)
\]

So, if we choose \((w^1, w^2, \ldots, w^n)\) such that

(3.24)

\[
\frac{\partial z^i}{\partial w^\alpha}(0) = \frac{1}{2\sqrt{-1}} \frac{\partial}{\partial z^\alpha}(\tilde{J}_i^j)(p)
\]

and (3.21) are both true. Then

(3.25)

\[
\frac{\partial}{\partial w^\alpha} \tilde{J}_i^j(p) = 0
\]

and \((w^1, w^2, \ldots, w^n)\) is a complex coordinate that is almost holomorphic at \(p\). This completes the proof.

**Definition 3.3.** We call the local coordinate in the last lemma an holomorphic coordinate at \(p\).

**Corollary 3.2.** Let \(M = X \times Y\) be a product of two almost complex manifolds. Let \((z^1, z^2, \ldots, z^k)\) be a local holomorphic coordinate for \(X\) at \(x\) and \((w^1, w^2, \ldots, w^l)\) be a local holomorphic coordinate for \(Y\) at \(y\). Then \((z^1, z^2, \ldots, z^k, w^1, \ldots, w^l)\) is a local holomorphic coordinate at \(p = (x, y)\).

**Proof.** Let \(J_X\) and \(J_Y\) be the almost complex structures on \(X\) and \(Y\) respectively. Since the almost complex structure on \(M = X \times Y\) is a product of \(J_X\) and \(J_Y\), we have

(3.26)

\[
J(\frac{\partial}{\partial z^i}) = J_X(\frac{\partial}{\partial z^i}) = (J_X)_i^j(z) \frac{\partial}{\partial z^j} + (J_X)_i^j(z) \frac{\partial}{\partial z^j}
\]

and

(3.27)

\[
J(\frac{\partial}{\partial w^\alpha}) = J_Y(\frac{\partial}{\partial w^\alpha}) = (J_Y)_\alpha^\beta(w) \frac{\partial}{\partial w^\beta} + (J_Y)_\alpha^\beta(w) \frac{\partial}{\partial w^\beta}.
\]

Then the conclusion comes directly by a simple computation. \(\square\)

For almost-Hermitian manifold with canonical connection, we have

**Lemma 3.6.** Let \((M^{2n}, J, g)\) be an almost Hermitian complex manifold and \(\nabla\) be the canonical connection. Then, for each point \(p \in M\) and any holomorphic coordinate \((z^1, z^2, \ldots, z^n)\) at \(p\), we have

(3.28)

\[
\nabla \frac{\partial}{\partial z^i}(p) = 0.
\]
Proof. Since $\nabla J = 0$, 

\begin{equation}
J \left( \nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} (p) \right)
\end{equation}

\begin{equation}
= \nabla \frac{\partial}{\partial z^j} \left( J \frac{\partial}{\partial z^i} \right) (p)
\end{equation}

\begin{equation}
= \nabla \frac{\partial}{\partial z^i} \left( J^k_j \frac{\partial}{\partial z^k} + J^k_j \frac{\partial}{\partial z^k} \right) (p)
\end{equation}

\begin{equation}
= J^k_j \nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^k} (p) + J^k_j \nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^k} (p), \text{ (by the definition of holomorphic coordinates)}
\end{equation}

\begin{equation}
= \sqrt{-1} \nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} (p)
\end{equation}

because $J^k_i = \sqrt{-1} \delta_{ij}$, $J^j_i = 0$ at $p$. Hence $\nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} (p)$ is a $(1,0)$ vector.

Similarly, we can show that $\nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} (p)$ is a $(0,1)$ vector. On the other hand,

\begin{equation}
\nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} (p) - \nabla \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^j} (p) = \tau \left( \frac{\partial}{\partial z^i} (p), \frac{\partial}{\partial z^j} (p) \right) = 0.
\end{equation}

Hence the conclusion follows.

\[\square\]

From the lemma, one can get the following.

**Corollary 3.3.** Let $(z^1, z^2, \ldots, z^n)$ be a holomorphic coordinate at $p$ on an almost Hermitian manifold $(M^{2n}, J, g)$, then

\begin{equation}
u_{ij}(p) = \partial_i \partial_j u(p)
\end{equation}

where $u_{ij} = \nabla^2 u(\partial_i, \partial_j)$ is the complex Hessian with respect to the canonical connection.

We also need the following facts on submanifolds. Let $(M, J, g)$ be an almost Hermitian manifold and $\nabla$ the canonical connection, and $\bar{\tau}$ be the torsion of $\nabla$. Let $N$ be a submanifold of $M$. Define the connection on $N$

\begin{equation}
\nabla_X Y = (\nabla_X Y)^T
\end{equation}

We will also need the following result about the torsion of submanifold.

**Lemma 3.7.** (a) $\nabla$ is the canonical connection of the induced almost Hermitian manifold $(N, J, g)$ with torsion

\begin{equation}
\tau(X, Y) = \bar{\tau}(X, Y)^T
\end{equation}

for any $X, Y \in TN$. 

(b) $h(X, \overline{Y}) = h(\overline{Y}, X) = 0$ for $X, Y \in T'(N)$, where $h(U, V) = - (\nabla_U V)\top$, $U, V \in T^C(N)$.

(c) Let $f$ be a smooth function on $M$, then

$$\nabla^2 f(X, \overline{Y}) = \nabla^2 f(X, \overline{Y})$$

for $X, Y \in T'(N)$.

Proof. (a) By the definitions of the torsion and the connection $\nabla$, for $U, V \in T^C(N)$, we have

$$\tau(U, V) = \nabla_U V - \nabla_V U - [U, V]$$

$$= (\nabla_U V)\top - (\nabla_V U)\top - [U, V]$$

$$= \overline{\tau(U, V)}\top.$$

Clearly $\nabla$ is also a canonical connection on $N$.

(b) Noting that $JW \in T^C N$ for $W \in T^C N$, we can get

(3.34) $h(U, JV) = J(h(U, V))$

for $U, V \in T^C N$. Since $\tau(X, \overline{Y}) = 0$, we have

$$h(\overline{Y}, X) - h(X, \overline{Y}) = (\overline{\nabla_X Y})\top - (\overline{\nabla_Y X})\top$$

$$= (\nabla_X Y - \nabla_Y X)\top$$

$$= ([X, \overline{Y}] + \tau(X, \overline{Y}))\top = 0.$$

So

(3.35) $h(X, \overline{Y}) = h(\overline{Y}, X).$

Let $\{e_1, \ldots, e_n\}$ be a unitary frame on $T'N$ where $n$ is the complex dimension of $N$, by (3.34), we can get

(3.36) $\langle h(e_i, e_j), e_k \rangle = \langle Jh(e_i, e_j), Je_k \rangle$

and by (3.35),

(3.37) $\langle h(e_i, e_j), e_k \rangle = \langle h(e_j, e_i), e_k \rangle$

so $\langle h(e_i, e_j), e_k \rangle = 0 = \langle h(e_i, e_j), e_k \rangle$, here $i, j, k \in \{1, \ldots, n\}$. Hence

(3.38) $h(X, \overline{Y}) = h(\overline{Y}, X) = 0$
for any $X, Y \in T'N$.

\begin{equation}
\nabla^2 f(X, \bar{Y}) = \bar{Y} X(f) - \nabla \bar{Y} X(f) = \bar{Y} X(f) - \nabla \bar{Y} X(f) + h(\bar{Y}, X)(f)
\end{equation}

Therefore the lemma is true. \hfill \Box

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We proceed by contradiction. Let $g$ be a complete almost Hermitian metric on $X^{2m} \times Y^{2n}$ satisfying the assumptions, $\nabla$ be the canonical connection.

Denote the fixed point $o$ as $(p, q) \in X \times Y$. Consider the inclusion map: $i : X_q \hookrightarrow X \times Y$ defined by $i(x) = (x, q)$, and pull back the tangent bundle $T(X \times Y)$ by $i$ on $X_q$. Let $u \in T_q' Y$, we can get a section $V$ of $i^* T(X \times Y)$ on $X_q$ such that $V(x) = u$ for all $x \in X_q$. For simplicity, let $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}\}$ be a unitary frame on $T^q M$ such that $\{e_1, \ldots, e_m\}$ is a frame on $T' X_q$. In the rest of this proof, we will take $\alpha \in \{1, \ldots, m\}$ and $i, j \in \{1, \ldots, m+n\}$. Since $u$ is a $(1, 0)$ vector, we can write $V = V^i e_i$. By the Ricci identity (Lemma 2.3), see also e.g. [12, page 1075] and Lemma 3.1 therein, we have

\begin{equation}
\frac{1}{2} \Delta_{X_q} \|V\|^2 = \langle V^i \bar{V}^i \rangle_{\alpha \bar{\alpha}} + \sum_{i, \alpha} V^i_{\alpha \bar{\alpha}} \bar{V}^i_{\alpha \bar{\alpha}} + V^i_\alpha \bar{V}^i_\alpha + V^i_{\bar{\alpha}} \bar{V}^i_{\bar{\alpha}} \geq mB \|V\|^2
\end{equation}

where we have used Corollary 3.2 which implies that $V^i_{\bar{\alpha}} = 0$ and hence $V^i_{\alpha \bar{\alpha}} = 0$.

Moreover, by Corollary 2.3 (See also Lemma 3.2 in [12]), and the assumption of the torsion, we have

\begin{equation}
\Delta_{X_q} \|V\|^2 \geq 2mB \|V\|^2 - C(m, A) (1 + \rho) \|\nabla \|V\|^2\|,
\end{equation}

where $\rho$ is the distance function from $p$ on $X_q$. Similar to the proof (2.8) in [11] using the Schwartz lemma (Lemma 3.3), one can get that $\|V\|^2$ is a positive bounded function. By Lemma 3.1 and Lemma 3.4, we have a contradiction because $|V| > 0$. \hfill \Box
4. Proof of Theorem 1.3

We need a lemma similar to Lemma 3.1 in [11]. Since $M$ may not be Kähler, we need Corollary 3.3 and Lemma 3.7 in our computations.

**Lemma 4.1.** Let $M = X^{2m} \times Y^{2n}$ be the product of two almost complex manifolds with positive dimensions. Assume that $M$ is simply connected. Suppose there is a complete Hermitian metric $g$ on $M$ satisfying the assumptions in Theorem 1.3, that is:

1. the second Ricci curvature $\geq -A(1 + r^2)^\gamma$;
2. the holomorphic bisectional curvature $\leq -B(1 + r^2)^{-\delta}$;
3. sectional curvature for the Levi-Civita connection is nonpositive;
4. torsion is bounded by $A(1 + r^2)^{\gamma/2}$;
5. $(2,0)$ part of the curvature tensor is bounded by $A(1 + r^2)^\gamma$.

where $\gamma \geq 0$, $\delta > 0$ such that $\gamma + 2\delta < 1$, $A, B$ are some positive constants, and $r(x, y) = d(o, (x, y))$ is the distance of $(x, y) \in X \times Y = M$ from a fixed point $o \in M$. Then there is a positive constant $C$ depending only on $m, n, \gamma, \delta, A$ and $B$, such that

$$g|_{(x_0, y)}(u, \bar{u}) \leq C(1 + r^2(x, y))^{\gamma}(1 + r^2(x_0, y))^\delta g|_{(x, y)}(u, \bar{u})$$

for any $x_0, x \in X$, $y \in Y$ and $u \in T'_y(Y)$.

**Proof.** Let $\pi : X \times Y \to \{x_0\} \times Y = Y_{x_0}$ be the natural projection. We only need to prove that

$$u(x, y) \leq C(1 + r^2(x, y))^{\gamma}(1 + r^2(x_0, y))^\delta$$

where $u(x, y)$ is the energy density of $\pi$.

By equation (5.9) in [12], the assumptions (1) and (2), and the fact that the bisectional curvature of the canonical connection of an almost complex submanifold is not bigger than the one of the ambient space [6], we have at $(x, y) \in M$,

$$\Delta u \geq -2A(1 + r^2(x, y))^{\gamma}u + 2B(1 + r^2(x_0, y))^{-\delta}u^2.$$  

(4.3)

Let $(x, y) \in M$. $T'_{(x, y)}(M) = T'_x(M) \oplus T'_y(M)$, by Lemma 3.6 we can choose a holomorphic coordinate $(z^1, z^2, \ldots, z^m)$ of $X$ at $x$ and a holomorphic coordinate $(z^{m+1}, \ldots, z^{m+n})$ of $Y$ at $y$. Then, by Corollary 3.2, $(z^1, \ldots, z^{m+n})$ is a holomorphic coordinate at $(x, y)$ in $M$. Note that

$$u(x, y) = g_{\alpha\bar{\beta}}(x_0, y)g^{\bar{\beta} \alpha}(x, y)$$

where $\alpha, \beta \in \{m + 1, \ldots, m + n\}$. Let $f(y) = r(x_0, y)$, $y \in Y_{x_0}$. By abusing notations, we also denote the function $f \circ \pi$ on $M$ be $f$. Since
$M$ is simply connected with nonpositive Riemannian curvature, $f^2$ is a smooth function. Then
\[ |\nabla f|^2(x, y) \leq e(\pi)|\nabla_{Y_{x_0}} f|^2(y) \leq u(x, y). \]

(4.5)

Near $x$, choose a frame $\{e_1, \cdots, e_m\}$ on $T'(X)$ with dual co-frame $\omega_1, \ldots, \omega_m$, and near $y$ choose a frame $\{e_{m+1}, \ldots, e_{m+n}\}$ on $T'(Y)$ with the dual co-frame $\{\omega^{m+1}, \ldots, \omega^{m+n}\}$ satisfying that at $y$, $\langle \omega^\alpha, \omega^\beta \rangle|_{(x, y)} = \delta_{\alpha\beta}$. Then $e_1, \ldots, e_{m+n}$ is a frame near $(x, y)$ with coframe $\omega^1, \ldots, \omega^{m+n}$.

Moreover, at $(x, y)$, $e_1, \ldots, e_m$ are linear combinations of $\partial/\partial x^1, \ldots, \partial/\partial x^m$ and $e_{m+1}, \ldots, e_{m+n}$ are linear combinations of $\partial/\partial r^1, \ldots, \partial/\partial r^{m+n}$. Without loss of generality we may assume that $e_a = \partial/\partial x^a$ for all $a$ at the point $(x, y)$.

By Corollary 3.3, the fact that $f^2$ is independence of $x$, and Corollary 2.2 we have
\[
\begin{align*}
\Delta f^2|_{(x, y)} &= 2g^{ba}(x, y) \partial_a \partial_b f^2|_{(x, y)} \\
&= 2g^{\bar{b}a}(x, y) \partial_a \partial_{\bar{b}} f^2|_{(x, y)} \\
&= 2g^{\bar{b}a}(x, y) (r^2)_{,a\bar{b}}(x_0, y) \\
&\quad + 2g^{\bar{b}a}(x, y) \left[ \frac{1}{2} g^{b\bar{c}} g^{kt} \partial_k \partial_t (r^2) + \frac{1}{2} g^{b\bar{c}} g^{kt} \partial_k \partial_t (r^2) \right]|_{(x_0, y)} \\
&= 2g^{\bar{b}a}(x, y) (r^2)_{,a\bar{b}}(x_0, y) \\
&\quad + g^{\bar{b}a}(x, y) \left[ g^{b\bar{c}} g^{kt} \partial_k \partial_t (r^2) + g^{b\bar{c}} g^{kt} \partial_k \partial_t (r^2) \right]|_{(x_0, y)}.
\end{align*}
\]

(4.6)

where $(r^2)_{,a\bar{b}}$ means the Hessian of $r^2(x, y)$ with respect to the Riemannian connection, $a, b, h, k, t \in \{1, \cdots, m+n\}$. First of all, we want to show that
\[
2g^{\bar{b}a}(x, y) (r^2)_{,a\bar{b}}(x_0, y) \leq u(x, y)(\Delta^L_{M_y} r^2)(x_0, y).
\]

(4.7)

Note that, by our choices of frames,
\[
2g^{\bar{b}a}(x, y) (r^2)_{,a\bar{b}}(x_0, y) = 2(r^2)_{,a\bar{a}}(x_0, y).
\]

(4.8)

By the assumption (3), the sectional curvature for the Levi-Civita connection is nonpositive, we know $(r^2)_{,a\bar{b}}(x_0, y)$ is positive definite, please see [3], then for any fixed $\alpha \in \{1, \cdots, m\}$,
\[
2(D^2 r^2)|_{(x_0, y)}(e_\alpha, e_\bar{\alpha}) \leq 2\text{trace}((D^2 r^2)|_{(x_0, y)}) g(e_\alpha, e_\bar{\alpha})|_{(x_0, y)} = g_{a\bar{a}}(x_0, y)(\Delta^L_{M_y} r^2)(x_0, y).
\]

(4.9)
Combining (4.8) and (4.9), we can get (4.7). By Lemma 3.2 in [12] and Lemma 3.2 in this paper, under the assumptions of the curvature and the torsion, we can get

\[
(\Delta_{M}r^{2})(x_{0}, y) \leq C(m, n, A)(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}.
\]

Here \( A \) is the same one as in the assumptions. Submitting this to (4.7), we can get

\[
2\delta g^{\alpha}(x, y)(r^{2}),\alpha\beta(x_{0}, y) \leq C(m, n, A)u(x, y)(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}.
\]

Now we want to estimate the second term in the last equality of (4.6). Denoting \( Q_{\alpha\beta} = [\tau_{\alpha}^{b}g^{kt}g_{\beta\delta}(r^{2}) + \tau_{\beta}^{b}g^{kt}g_{\alpha\delta}(r^{2})]_{(x_{0}, y)} \), it is a 2-tensor on \( T_{y}^{(1,0)}Y \). Choose a unitary basis \( \{s_{m+1}, \cdots, s_{m+n}\} \) on \( T_{y}^{(1,0)}Y \), and extent it to \( \{s_{1}, \cdots, s_{m}, s_{m+1}, \cdots, s_{m+n}\} \) as a unitary basis on \( T_{(x_{0}, y)}X \times Y \). Taking a vector \( W = W^{\alpha}s_{\alpha} \in T_{y}^{(1,0)}Y \), we have

\[
|W^{\alpha}Q_{\alpha\beta}W^{\bar{\beta}}| = |\langle\tau(W, s_{t}), s_{b}\rangle\langle s_{b}, W\rangle\tau(r^{2})\rangle|_{(x_{0}, y)} \leq |\langle\tau(W, s_{t}), W\rangle\tau(r^{2})\rangle|_{(x_{0}, y)} \leq 2nA(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}\|W\|_{g^{2}}^{}.
\]

Here we have used the assumption (4) about the restriction on the torsion with respect to the canonical connection. So we get

\[
Q_{\alpha\beta} \leq 2nA(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}g_{\alpha\beta}(x_{0}, y).
\]

Hence

\[
g^{\alpha}(x, y)Q_{\alpha\beta} \leq 2nA(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}g^{\alpha}(x, y)g_{\alpha\beta}(x_{0}, y) = 2nAu(x, y)(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}.
\]

Combining this with (4.6) (4.11), we have

\[
\Delta f^{2}|_{(x, y)} \leq C_{6}u(x, y)(1 + r^{2}(x_{0}, y))^{\frac{\alpha+1}{2}}
\]

for some constant \( C_{6} \) depends on \( A, m, n \).

Let

\[
\Delta w = (C_{0} + v)^{-\delta}\Delta u - 2\delta(C_{0} + v)^{-1-\delta}\langle \nabla u, \nabla v \rangle - u\delta(C_{0} + v)^{-1-\delta}\Delta v + u\delta(\delta + 1)(C_{0} + v)^{-2-\delta}\|\nabla v\|^{2}
\]

\[
\geq (C_{0} + v)^{-\delta}\Delta u - u\delta(C_{0} + v)^{-1-\delta}\Delta v - 2\delta(C_{0} + v)^{-1}(\nabla w, \nabla v)
\]

\[
\geq (C_{0} + v)^{-\delta}\Delta u - u\delta(C_{0} + v)^{-1-\delta}\Delta v - 2\delta(C_{0} + v)^{-1}\|\nabla w\| \cdot \|\nabla v\|.
\]
Since $C_0 \geq 1$, submitting the estimations of (4.3) (4.5) and (4.12) to this inequality, we can get
\[
\Delta w(x, y) \\
\geq -2A(1 + r^2(x, y))^\gamma w(x, y) + 2B(1 + v)^{-\delta}(C_0 + v)^\delta w^2(x, y) \\
- 4\delta|\nabla w|(x, y)w^{\frac{1}{2}}(x, y) - C_0\delta(C_0 + v)^{-1+\delta}(1 + v)^{\frac{2\delta}{1+\gamma}}w^2(x, y) \\
\geq -2A(1 + r^2(x, y))^\gamma w(x, y) + \left(2B - C_0\delta(C_0 + v)^{-\frac{1+\gamma}{1-2\delta}}\right)w^2(x, y) \\
- 4\delta|\nabla w|(x, y)w^{\frac{1}{2}}(x, y).
\]

Since $\gamma + 2\delta < 1$, we can choose $C_0$ large enough depending on $C_6, \delta, \gamma, B$ such that
\[
(4.14) \quad \Delta w \geq Bw^2 - 2A(1 + r^2)^\gamma w - 4\delta\|\nabla w\|w^{\frac{1}{2}}.
\]
So, a similar cut-off argument in the proof of Lemma 3.3 will imply that
\[
(4.15) \quad w \leq C_7(1 + r^2)^\gamma
\]
where $C_7$ is positive constant depending on $A, B, m, n, \delta$. \qed

**Proof of Theorem 1.3.** First of all, we may assume $M$ is simply connected because the distance function in the universal cover of $M$ is no less than the distance function of $M$. Suppose there is a complete metric $g$ on $M$ satisfying the assumptions of the theorem. Let us choose the section $V$ as in the proof of Theorem 1.2, and set $f(x) = |V|^2_{g(x, q)}$.

By assumption (2) and the result of [6], the holomorphic bisectional curvature of $X_q$ at $x$ is also less than or equal to $-B(1 + r^2(x, q))^{-\delta}$.

By (3.40), we have
\[
(4.16) \quad \Delta_{X_q} f(x) \geq 2mB(1 + r^2(x, q))^{-\delta} f(x).
\]

By Lemma 4.1, we can get
\[
(4.17) \quad 0 < f(x) \leq C(1 + r^2(x, q))^\delta.
\]

Now we want to show that
\[
(4.18) \quad \Delta_{X_q} r^2(x, q) \leq C_8(1 + r^2(x, q))^{\frac{1+\gamma}{2}}
\]
for some positive constant $C_8$ independent of $x$. For any fixed point $x \in X_q$, choose an holomorphic coordinate $(z^1, \cdots, z^m)$ of $X_q$ at $x$ such that the induced metric $g^q$ on $X_q$ satisfies $g^q_{\alpha\beta}(x) = \delta_{\alpha\beta}$, here $\alpha, \beta \in \{1, \cdots, m\}$. Setting $\varphi(x, y) = r^2(x, q)$, from Lemma 3.7 and

\[
\text{Product of almost-Hermitian manifolds 21}
\]
Corollary 2.3, we can get
\[
\Delta_{X_q} \varphi = 2 \sum_{\alpha=1}^{m} (\varphi)_{\alpha\bar{\alpha}} \tag{4.19}
\]

\[
= 2 \sum_{\alpha=1}^{m} (\varphi)_{\alpha\bar{\alpha}} + [\tau^\alpha_{\alpha\beta} \partial_\beta (\varphi) + \tau^\bar{\alpha}_{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} (\varphi)].
\]

Clearly
\[
|\tau^\alpha_{\alpha\beta} \partial_\beta (\varphi) + \tau^\bar{\alpha}_{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} (\varphi)| \leq C (1 + \varphi)^{\frac{1+\gamma}{2}} \tag{4.20}
\]

for some constant \( C \) independent of \( x \). Noting that the sectional curvature for the Levi-Civita connection is nonpositive, by \([5]\), we can get
\[
2 \sum_{\alpha=1}^{m} (\varphi)_{\alpha\bar{\alpha}} \leq \Delta^L_M \varphi.
\]

By (4.10), at \((x, q)\) we have \( \Delta^L_M \varphi \leq C (1 + \varphi)^{\frac{1+\gamma}{2}} \) for some constant \( C \) independent of \( x \). Combining this with (4.19) and (4.20), we can get (4.18).

Let \( h(x) = \log f(x) - 2\delta \log (C_9 + r^2(x, q)) \) where \( C_9 > 1 \) is some constant. Follow the proof of (3.10) in \([11]\), from (4.17) (4.18) and the assumption \( \gamma + 2\delta < 1 \), we can get if \( C_9 \) is big enough, then at a maximum point \((\bar{x}, q) \in X_q\)
\[
0 \geq \Delta_{X_q} h(\bar{x}) > 0. \tag{4.21}
\]

Hence we have a contradiction. Therefore Theorem 1.3 holds. \( \square \)

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**Department of Mathematics, Jinan University, Guangzhou, 510632, China**

*E-mail address: txqfan@jnu.edu.cn*

**The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.**

*E-mail address: lftam@math.cuhk.edu.hk*

**Department of Mathematics, Shantou University, Shantou, Guangdong, 515063, China**

*E-mail address: cjyu@stu.edu.cn*