On generalized CIR equations

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Abstract

The paper is concerned with stochastic equations for the short rate process

\[ dR(t) = F(R(t))dt + G(R(t-))dZ(t), \]

in the affine model of the bond prices. The equation is driven by a Lévy martingale \( Z \). It is shown that the discounted bond prices are local martingales if either \( Z \) is a stable process of index \( \alpha \in (1,2] \), \( F(x) = ax + b, b \geq 0 \), \( G(x) = cx^{1/\alpha}, c > 0 \) or \( Z \) must be a Lévy martingale with positive jumps and trajectories of bounded variation, \( F(x) = ax + b, b \geq 0 \) and \( G \) is a constant. The result generalizes the well known Cox-Ingersoll-Ross result from [5] and extends the Vasićek result, see [19], to non-negative short rates.

Key words: CIR model, bond market, HJM condition, stable martingales.

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1 Introduction

The Cox-Ingersoll-Ross (CIR) equation is a stochastic equation for a non-negative process \( R(t), t \geq 0 \), of the form

\[ dR(t) = (aR(t) + b)dt + c\sqrt{R(t)}dW(t), \quad R(0) = x \geq 0, \]

(1.1)

where \( a, b, c \) are constants, \( b \geq 0, c \geq 0 \), and \( W \) is a Wiener process. It was introduced in the paper [5] to model short rate process \( R(t) \) for which the bond prices \( P(t,T), 0 \leq t \leq T < +\infty \) have an affine structure, that is

\[ P(t,T) = e^{-A(T-t)-B(T-t)R(t)}, \quad 0 \leq t \leq T, \]

(1.2)
where $A$ and $B$ are smooth, non-negative, deterministic functions and the discounted bond prices are local martingales. That is the following martingale property is satisfied:

For each $T > 0$, the discounted bond price process

$$(MP) \quad \hat{P}(t,T) = P(t,T)e^{-\int_0^T R(s)ds}, \quad t \in [0,T],$$

is a local martingale.

This property implies that the market with bonds priced by (1.2) does not allow arbitrage. In fact a stronger result is true, see e.g. Filipović [8]. If one requires that the short rate process $R$ is only a time homogeneous path continuous Markov process and the bond prices (1.2), with some functions $A$ and $B$, satisfy the martingale property, then it can be identified with the solution of the CIR equation.

In the present paper we want to find all equations of the form

$$dR(t) = F(R(t))dt + G(R(t-))dZ(t), \quad R(0) = x \geq 0, \quad (1.3)$$

where $Z$ is a real Lévy martingale, such that the solution $R$ is non-negative and determines affine bond prices with the martingale property (MP). Although there exists a precise description of the infinitesimal generator of such process due to Filipović, see [8], the analog of the CIR equation, in a more general setting of càdlàg short rate process $R$, is appearing in the present paper for the first time. Our main theorems give a precise characterization of all those equations (1.3) for which the martingale property holds. In our approach we do not use the Filipović characterization but examine the problem via the Heat-Jarrow-Morton (HJM) conditions for the absence of arbitrage.

The choice of a Lévy process in the equation (1.3) is motivated by the requirement that solutions to (1.3) should be Markov processes. It is, however, natural, in the light of a meta-theorem, that Markov processes can be represented as solutions to stochastic equation with the noise being Lévy process but, in general, infinite dimensional, see the book [18].

In the present paper we allow only a one dimensional noise process. For the multidimensional Lévy noise $Z = (Z_1, Z_2, \ldots, Z_d)$ the problem of determining all equations

$$dR(t) = F(R(t))dt + \sum_{j=1}^d G_j(R(s-))dZ_j(s), \quad R(0) = x \geq 0, \quad (1.4)$$

for which the martingale property holds, is open. We have only partial answers which can be found in [1]. It seems that even equations with multidimensional noise are not enough to represent all Markovian short rate processes with the martingale property.

As a part of the main proofs, in the section devoted to auxiliary results, we derive rather general results on positive invariance of stochastic equations and on equations with the martingale property, see Proposition 3.1, Proposition 3.2, Proposition 3.4. They might be helpful for an attempt to cover the multidimensional case.

### 2 Results

Our aim is to determine functions $F, G$ and a Lévy process $Z$ in (1.3) as well as the functions $A, B$ in (1.2) such that solutions $R$ to the equation (1.3) starting from arbitrary $x \geq 0$ are non-negative and the processes $\hat{P}(t,T), t \in [0,T]$ are local martingales for arbitrary $T > 0$. If
this is the case then we say that the equation (1.3) determines an affine arbitrage free market or equivalently that equation (1.3) has the martingale property (MP).

It is natural to assume that

\[ 0 < P(t, T) \leq 1, \quad P(T, T) = 1, \quad 0 \leq t \leq T, \]

and that \( P(t, T) \) is a decreasing function of \( T \). Since \( R(t), t \geq 0 \), should be a non-negative process, \( A \) and \( B \) should be increasing, non-negative functions starting from zero, i.e.

\[ A(0) = 0, \quad B(0) = 0. \quad (2.1) \]

In the sequel we will argue that necessarily

\[ A'(0) = 0, \quad B'(0) = 1, \quad (2.2) \]

see the reasoning preceding formula (3.16) in the sequel. Without loss of generality we will look for functions \( G \) such that

\[ \exists \bar{x} > 0 \quad G(\bar{x}) > 0. \quad (2.3) \]

We can reduce the general consideration to this case by multiplying \( G \) and \( Z \) by \(-1\), if \( G \) is always non-positive. We also assume that the Lévy process is integrable with mean zero, i.e.

\[ \mathbb{E} |Z(t)| < +\infty, \quad \mathbb{E}Z(t) = 0, \quad t \geq 0. \quad (2.4) \]

Clearly, (2.4) means that \( Z \) is a martingale. Note that if \( \mathbb{E}Z(t) \neq 0 \), then \( \mathbb{E}Z(t) = at, t \geq 0 \) for some \( a \neq 0 \), and replacing \( Z \) by \( \bar{Z}(t) = Z(t) - at, t \geq 0 \), one arrives at the, equivalent to (1.3), equation

\[ dR(t) = (F(R(t)) + aG(R(t)))dt + G(R(t-))d\bar{Z}(t), \quad R(0) = x \geq 0. \]

By \( J \) we denote the Laplace exponent of the process \( Z \), that is

\[ \mathbb{E}[e^{-\lambda Z(t)}] = e^{J(\lambda)}, \quad t \geq 0, \lambda \in \Lambda, \quad (2.5) \]

where \( \Lambda \) is the set for which the left side of (2.5) is well defined. It is well known that

\[ J(\lambda) = -a\lambda + \frac{q}{2} \lambda^2 + \int_{-\infty}^{+\infty} (e^{-\lambda y} - 1 + \lambda y 1_{(-1,1)}(y))\nu(dy), \quad (2.6) \]

where \( a \in \mathbb{R}, \ q \geq 0 \) are constants and \( \nu \) is a measure on \( \mathbb{R} \setminus \{0\} \) such that

\[ \int_{-\infty}^{+\infty} (y^2 \wedge 1)\nu(dy) < +\infty, \]

and that

\[ \Lambda = \{ \lambda \in \mathbb{R} : \int_{|y|\geq 1} e^{-\lambda y}\nu(dy) < +\infty \}. \]

The measure \( \nu \) is called the Lévy measure of the process \( Z \) and this measure together with the constants \( a, q \) in (2.6) determine the process \( Z \) in a unique way. The fact that \( Z \) is a martingale is equivalent to:

\[ \int_{|y|>1} |y| \nu(dy) < +\infty, \quad a + \int_{|y|>1} y\nu(dy) = 0. \quad (2.7) \]
If the Lévy measure of \( Z \) is given by
\[
\nu(dy) = \frac{1}{y^{1+\alpha}} 1_{(0, +\infty)}(y) dy,
\]
then \( J(\lambda) = c_\alpha \lambda^\alpha, \ z \geq 0, \) where \( \alpha \in (1, 2) \) and \( c_\alpha := \frac{1}{\alpha(a-1)} \Gamma(2 - \alpha) > 0. \) Here \( \Gamma \) stands for the Gamma function. Then \( Z \) is a stable martingale with index \( \alpha \), providing that \( a \) is such that \((2.7)\) holds. This martingale will be denoted by \( Z^\alpha \) for \( \alpha \in (1, 2) \) and \( Z^\alpha \) with \( \alpha = 2 \) stands for the Wiener process.

**Theorem 2.1 [Necessity]**

Assume that the equation \((1.3)\) with functions \( F, G \) which are continuous on \([0, +\infty)\) has the martingale property (MP) with some functions \( A, B \) satisfying \((2.1)\) and \((2.2)\).

1) If \( G \) is differentiable on \((0, +\infty)\) and \( G(\bar{x}) > 0, \) \( G'(\bar{x}) \neq 0 \) for some \( \bar{x} > 0 \), then
   a) \( Z = Z^\alpha \) is a stable Lévy process with index \( \alpha \in (1, 2] \) with positive jumps only,
   b) \( F(x) = ax + b \) with \( a \in \mathbb{R}, \ b \geq 0, \ x \geq 0, \)
   c) \( G(x) = c^2 x^\frac{1}{\alpha}, \ c > 0, \ x \geq 0. \)
2) If \( G \) is a positive constant \( \sigma \), then
   d) \( Z \) has no Wiener part, i.e. \( q \) in \((2.6)\) disappears,
   e) the martingale \( Z \) has positive jumps only and \( \int_0^{+\infty} y\nu(dy) < +\infty, \)
   f) \( F(x) = ax + b, \ \ x \geq 0, \) with \( a \in \mathbb{R}, \ b \geq \sigma \int_0^{+\infty} y\nu(dy). \)

Note that if there exists a number \( \bar{x} > 0 \) such that \( G(\bar{x}) > 0, \) but for all such numbers \( G'(\bar{x}) = 0, \) then the function \( G \) should be a positive constant on \([0, +\infty)\). This case is covered by Part II) of Theorem 2.1.

**Theorem 2.2 [Sufficiency]**

1) The equation
\[
dR(t) = (aR(t) + b)dt + c_\alpha R(t)\frac{1}{\tau}\hat{\nu}(Z)(t), \quad R(0) = x \geq 0, \quad a \in \mathbb{R}, \ b \geq 0, \ c > 0, \quad (2.8)
\]
has a unique non-negative strong solution and satisfies the martingale property (MP). The functions \( A, B \) in \((1.2)\) are such that \( B \) solves the equation
\[
B'(v) = -cc_\alpha B^\alpha(v) + aB(v) + 1, \quad v \geq 0, \quad B(0) = 0, \quad (2.9)
\]
and \( A \) is given by \( A'(v) = bB(v), \ v \geq 0, \ A(0) = 0. \)
2) If \( G \) is a positive constant \( \sigma \) and \((d), (e), (f)\) in Theorem 2.1 hold, then the equation
\[
dR(t) = (aR(t) + b) + \sigma dZ(t), \quad R(0) = x \geq 0, \quad t > 0,
\]
has the martingale property (MP) and its solutions are non-negative processes. Moreover, \( A, B \) are given by
\[
B'(v) = B(v)a + 1, \quad B(0) = 0, \quad (2.10)
\]
\[
A'(v) = B(v)(b - \sigma \int_0^{+\infty} y\nu(dy)) + \int_0^{+\infty} (1 - e^{-\sigma B(v)y})\nu(dy), \quad A(0) = 0. \quad (2.11)
\]
The equation (2.9), can be solved explicitly for several values of α. If α = 2 the equation (2.8) becomes the CIR equation and (2.9) boils down to the Riccati equation.

**Remark 2.3** It is an open problem to characterize equations

\[ dR(t) = F(R(t))dt + < G(R(s−)), dZ(s) >, \quad R(0) = x ≥ 0, \]  

(2.12)

where \( Z \) is a Lévy martingale of a given, greater than 1, dimension, which solutions determine affine term structure with the martingale property.

It is easy to check that if \( α, β \in (1, 2], \quad Z^α, Z^β \) are independent stable martingales with indexes \( α, β \) respectively and if \( b ≥ 0 \) then the equation

\[ dR(t) = (aR(t) + b)dt + R(t−)^{\frac{1}{α}}dZ^α(t) + R(t−)^{\frac{1}{β}}dZ^β(t), \quad R(0) = x ≥ 0, \]  

(2.13)

has a unique non-negative solution which determines an affine term structure with the martingale property.

### 3 Auxiliary results

Here we derive some auxiliary results used in the proof of Theorem 2.1.

Short rates should be non-negative processes. It is not surprising that in the proof of necessary conditions in Theorem 2.1 we will need some results on equations which solutions are positive invariant. The positive invariance has consequences for the structure of the Lévy process \( Z \), see Proposition 3.1 and on the diffusion coefficient \( G \), see Proposition 3.2. We also derive analytic conditions on the coefficients of the equation (1.3) implied by the martingale property, see Proposition 3.4.

#### 3.1 Positive invariance and the noise process

We show that if all solutions of (1.3) starting from non-negative points are non-negative then the noise process \( Z \) can’t have arbitrarily large negative jumps. This in turn affects properties of the Laplace exponent of \( Z \).

Below \( \text{supp}(\nu) \) stands for the closed support of \( \nu \).

**Proposition 3.1** Assume that \( G(\bar{x}) > 0 \) for some \( \bar{x} > 0 \) and solutions to (1.3) with all possible initial values \( x ≥ 0 \) are non-negative. Then

a) the support of \( \nu \) satisfies

\[ \text{supp}(\nu) \subseteq (b, +\infty), \quad \text{for some} \quad b \in \mathbb{R}, \]  

(3.1)

b) the domain \( \Lambda \) of the Laplace exponent of \( Z \) satisfies

\[ \Lambda \supseteq [0, +\infty), \]

c) \( J'(\lambda) \) is finite for \( \lambda > 0 \) and

\[ J'(0+) = 0. \]
Proof: a) For arbitrary negative $a$ let us consider the decomposition
\[ \nu(dy) = \nu_1(dy) + \nu_2(dy), \quad \nu_2(dy) = 1_{(-\infty,0)}(y)\nu(dy). \]
The process $Z$ can be represented as a sum of two independent Lévy processes $Z_1$, $Z_2$ such that $Z_2$ is a compound Poisson process with Lévy measure $\nu_2$. The process $Z_2$ is a pure jump process with all jumps smaller than $a$. Let $X_1$ be a solution to the equation
\[ dX_1(t) = F(X_1(t))dt + G(X_1(t-))dZ_1(t), \quad X_1(0) = \bar{x}, \]
and let $c, r$ be positive numbers such that
\[ G(x) \geq c, \quad \text{for} \ x \in I := \{ z \geq 0 : \bar{x} - r < z < \bar{x} + r \}. \]
Let $\tau_1$ and $\tau_2$ be, respectively, the first exit time of $X_1$ from the interval $I$ and the moment of the first jump of the process $Z_2$. There exists $T > 0$ such that
\[ P(\tau_1 > T) > 0, \quad P(\tau_2 < T) > 0. \]
Since the events $\{\tau_1 > T\}$, $\{\tau_2 < T\}$ are independent,
\[ P(\{\tau_1 > T\} \cap \{\tau_2 < T\}) = P(\tau_1 > T) \cdot P(\tau_2 < T) > 0. \]
Note that the solution of
\[ dX(t) = F(X(t))dt + G(X(t-))d(Z_1(t) + Z_2(t)), \quad X(0) = \bar{x}, \]
satisfies
\[ X(t) = X_1(t), \quad \text{for} \ t < \tau_2. \]
However, on the set $\{\tau_1 > T\} \cap \{\tau_2 < T\}$ we have
\[ \triangle X(\tau_2) = G(X(\tau_2-))\triangle Z(\tau_2) \]
\[ = G(X(\tau_2-))\triangle Z_2(\tau_2), \]
and
\[ X(\tau_2) = X_1(\tau_2-) + G(X_1(\tau_2-))\triangle Z_2(\tau_2) \]
\[ \leq \bar{x} + r + ca. \]
Thus, since $a$ can be chosen arbitrarily, we arrive to the condition $X(\tau_2) < 0$, which is a contradiction.

Proofs of b) and c). Since the jumps of $Z$ are bounded from below by $b$, the integral
\[ \int_{|y|>1} e^{-\lambda y} \nu(dy) = \int_{|y|>1, y>b} e^{-\lambda y} \nu(dy), \]
is finite for $\lambda \geq 0$. Consequently, the Laplace exponent $J(\lambda)$ is well defined for $\lambda \geq 0$. Moreover, by (2.7) also
\[ J'(\lambda) = -a + q\lambda + \int_b^{+\infty} y(1_{|y|>1} - e^{-\lambda y})\nu(dy), \]
is finite for $\lambda > 0$ and
\[ \lim_{\lambda \downarrow 0} \frac{J(\lambda)}{\lambda} = J'(0+) = 0. \]
\[ \square \]
3.2 Positive invariance and the diffusion coefficient

We concentrate now on the equation

\[ dR(t) = F(R(t))dt + G(R(t-))dZ^\alpha(t), \quad R(0) = x \geq 0, \]  

(3.2)

where \( Z^\alpha \) is a stable martingale with index \( \alpha \in (1, 2) \) and positive jumps only. Recall, its Lévy measure has the form

\[ \nu(dy) = 1_{(0, +\infty)} \frac{1}{y^{1+\alpha}} dy. \]

We will prove the following result:

**Proposition 3.2** If \( G \) is a Lipschitz function and the equation (3.2) is positive invariant, then \( G(0) = 0 \).

In the proof we use the classical maximal inequality

\[ \mathbb{P}( \sup_{s \in [0,t]} |X(s)| \geq r) \leq \frac{3}{r} \mathbb{E} |X(t)|, \quad t > 0, \]  

(3.3)

where \( X \) is a càdlàg submartingale, see Proposition 7.12 in [16].

We will also need the following lemma:

**Lemma 3.3** For \( 2 \geq p > \alpha > 1 \)

\[ \mathbb{E} | \int_0^t g(s)dZ^\alpha_0(s) |^p \leq \frac{c_p}{p - \alpha} \mathbb{E} \int_0^t |g(s)|^p ds, \quad t \geq 0, \]  

(3.4)

with some \( c_p > 0 \).

Here \( Z^\alpha_0 \) is a modified \( \alpha \)-stable martingale \( Z^\alpha \) with Lévy measure

\[ \nu(dy) = 1_{(0,1)} \frac{1}{y^{1+\alpha}} dy. \]

Its jumps are thus bounded by 1 and it is identical with the process \( Z^\alpha \) on the interval \([0, \tau_1)\), where \( \tau_1 \) is the first jump of \( Z^\alpha \) exceeding 1.

**Proof of Lemma 3.3**: Since the quadratic variation of the integral \( \int g(s)dZ^\alpha_0(s) \) equals

\[ \left[ \int g(s)dZ^\alpha_0(s) \right](t) = \int_0^t \int_0^1 g^2(s)y^2 \pi_0(ds, dy), \]

where \( \pi_0 \) stands for the jump measure of \( Z^\alpha_0 \), by the Burkholder-Davis-Gundy inequality we obtain, for some \( c_p > 0 \),

\[ \mathbb{E} | \int_0^t g(s)dZ^\alpha_0(s) |^p \leq c_p \mathbb{E} \left[ \left( \int g(s)dZ^\alpha_0(s) \right)^2 \right](t) \]

\[ = c_p \mathbb{E} \left( \int_0^t \int_0^1 g^2(s)y^2 \pi_0(ds, dy) \right)^{\frac{p}{2}}, \]
and further, since \(p/2 \leq 1\),

\[
\mathbb{E} \left| \int_0^t g(s)dZ_0^\alpha(s) \right|^p \leq c_p \mathbb{E} \int_0^t \int_0^1 |g(s)|^p \frac{1}{y^{1+\alpha}} dy ds \int_0^1 \frac{y^p}{y^{1+\alpha}} dz \\
\leq c_p \mathbb{E} \int_0^t |g(s)|^p \int_0^1 \frac{y^p}{y^{1+\alpha}} dy ds \int_0^1 \frac{y^p}{y^{1+\alpha-p}} dy \\
\leq \frac{c_p}{p - \alpha} \mathbb{E} \int_0^t |g(s)|^p ds.
\]

\(\Box\)

**Proof of Proposition 3.2** We adopt the proof of Milian \[17\] for the Wiener noise, which goes back to Gihman, Skorohod \[13\]. Let us consider (3.2) with \(x = 0\). Then we can write \(R\) in the form

\[
R(t) = \int_0^t F(R(s))ds + \int_0^t G(R(s))dZ^\alpha(s) + G(0)Z^\alpha(t), \quad t > 0.
\]

Dividing by \(t^{\frac{1}{\alpha}}\) yields

\[
\frac{1}{t^{\frac{1}{\alpha}}} R(t) = \frac{1}{t^{\frac{1}{\alpha}}} \int_0^t F(R(s))ds + \frac{1}{t^{\frac{1}{\alpha}}} \int_0^t G(R(s))dZ^\alpha(s) + \frac{1}{t^{\frac{1}{\alpha}}} G(0)Z^\alpha(t), \quad t > 0.
\] (3.5)

Since

\[
\liminf_{t \to 0} \frac{1}{t^{\frac{1}{\alpha}}} Z^\alpha(t) = -\infty, \quad \limsup_{t \to 0} \frac{1}{t^{\frac{1}{\alpha}}} Z^\alpha(t) = +\infty,
\]

see \[2\], Theorem 5 in Section VIII, the last term in (3.5) becomes negative for some sequence \(t_n \downarrow 0\) providing that \(G(0) \neq 0\). Since

\[
\frac{1}{t^{\frac{1}{\alpha}}} \int_0^t F(R(s))ds \xrightarrow{t \to 0} 0,
\]

the assertion is true if we show that

\[
\frac{1}{t^{\frac{1}{\alpha}}} \int_0^t G(R(s))dZ^\alpha(s) \xrightarrow{t \to 0} 0.
\]

Let us denote \(g(s) := G(R(s)) - G(0)\). In the neighborhood of zero we can replace \(Z^\alpha\) by \(Z_0^\alpha\). Then, by (3.3), for the submartingale \(|\int_0^t g(u)dZ_0^\alpha(u)|^p\), with \(2 > p > \alpha > 1\), we have

\[
\mathbb{P} \left( \sup_{0 \leq s \leq t} \frac{1}{t^{\frac{1}{\alpha}}} \left| \int_0^s g(u)dZ_0^\alpha(u) \right| > \varepsilon \right) = \mathbb{P} \left( \sup_{0 \leq s \leq t} \left| \int_0^s g(u)dZ_0^\alpha(u) \right| > (\varepsilon t^{\frac{1}{\alpha}})^p \right) \\
\leq \frac{3}{(\varepsilon t^{\frac{1}{\alpha}})^p} \mathbb{E} \left| \int_0^t g(u)dZ_0^\alpha(u) \right|^p.
\] (3.6)

It follows from (3.3) that

\[
\frac{3}{(\varepsilon t^{\frac{1}{\alpha}})^p} \mathbb{E} \left| \int_0^t g(u)dZ_0^\alpha(u) \right|^p \leq \frac{3c_p}{(\varepsilon t^{\frac{1}{\alpha}})^p(p - \alpha)} \int_0^t \mathbb{E} |g(u)|^p du.
\] (3.7)
Since $G$ is Lipschitz, so
\[
\mathbb{E} | g(\alpha) |^p = \mathbb{E} [G(R(\alpha)) - G(R(0))] |^p \leq K \cdot \mathbb{E} | R(\alpha) |^p,
\] (3.8)
with some constant $K > 0$. By (3.6), (3.7) and (3.8) we obtain thus
\[
P \left( \sup_{0 \leq s \leq t} \frac{1}{s^{\alpha}} \left| \int_0^s g(u) dZ_0^\alpha(u) \right| > \varepsilon \right) \leq \frac{3Kc_p}{(\varepsilon t^{\frac{1}{p}})^p (p-\alpha)} \int_0^t \mathbb{E} | R(\alpha) |^p du.
\] (3.9)
Therefore, for a sequence $\{a_k\}$ we obtain
\[
H(k) := P \left( \sup_{2^{-k} \leq s \leq 2^{-k+1}} \frac{1}{s^{\alpha}} \left| \int_0^s g(u) dZ_0^\alpha(u) \right| > a_k \right)
\leq P \left( \sup_{2^{-k} \leq s \leq 2^{-k+1}} \frac{1}{(2^{-k+1})^{\frac{1}{p}}} \left( \frac{2^{-k+1}}{s} \right)^{\frac{1}{p}} \left| \int_0^s g(u) dZ_0^\alpha(u) \right| > a_k \right)
\leq P \left( \sup_{0 \leq s \leq 2^{-k+1}} \frac{2^{\frac{1}{p}}}{(2^{-k+1})^{\frac{1}{p}}} \left| \int_0^s g(u) dZ_0^\alpha(u) \right| > a_k \right)
\leq P \left( \sup_{0 \leq s \leq 2^{-k+1}} \frac{1}{(2^{-k+1})^{\frac{1}{p}}} \left| \int_0^s g(u) dZ_0^\alpha(u) \right| > \frac{a_k}{2^{\frac{1}{p}}} \right), \quad k = 0, 1, \ldots
\] and, consequently, by (3.9),
\[
H(k) \leq \frac{3Kc_p}{\left( \frac{a_k}{2^{\frac{1}{p}}} \right)^p (p-\alpha)} \int_0^{2^{-k+1}} \mathbb{E} | R(\alpha) |^p du, \quad k = 0, 1, \ldots
\] (3.10)
Now we estimate the integral $\int_0^t \mathbb{E} | R(\alpha) |^p du$ for $t > 0$. We can assume that $F$ and $G$ are bounded because we investigate the behaviour of $R$ before it leaves a neighborhood of zero. Then
\[
|R(t)|^p \leq 2^{p-1} \left( \int_0^t |F(R(s))| ds \right)^p + \int_0^t |G(R(s))| dZ_0^\alpha(s) |^p
\] and, consequently,
\[
\mathbb{E} | R(t) |^p \leq 2^{p-1} (ct^p + \mathbb{E} \int_0^t |G(R(s))|^p ds) \leq \tilde{c}t,
\]
with some constants $c, \tilde{c}$. Hence
\[
\int_0^t \mathbb{E} | R(\alpha) |^p du = \int_0^t \mathbb{E} | R(\alpha) |^p du \leq \tilde{c} \int_0^t ds = \frac{\tilde{c}}{2} t^2, \quad t > 0.
\] (3.11)
By (3.10) and (3.11) we obtain finally
\[
H(k) \leq \frac{3Kc_p}{\left( \frac{a_k}{2^{\frac{1}{p}}} \right)^p (p-\alpha)} \frac{\tilde{c}}{2} (2^{-k+1})^2
\]
\[
= \frac{3Kc_p \tilde{c} 2^{\frac{p}{p-1}}}{p-\alpha} \cdot \frac{1}{a_k^{p-1}} (2^{-k+1})^{2-\frac{p}{a_k}}, \quad k = 0, 1, \ldots
\]
Taking \( a_k = \frac{1}{k} \) and \( \delta := 2 - p/\alpha > 0 \) we obtain that
\[
\sum_{k=0}^{+\infty} H_k < +\infty,
\]
and, by the Borel-Cantelli lemma,
\[
\frac{1}{t^\alpha} \int_0^t G(R(s)) - G(0) dZ(s) \xrightarrow{t \to 0} 0,
\]
as required.

\[\square\]

3.3 Analytic HJM condition

In the Heath-Jarrow-Morton (HJM) model of the bond market, see [14], the bond prices are written in the form:
\[
P(t, T) = e^{-\int_t^T f(t,s) ds}, \quad 0 \leq t \leq T,
\]
where \( f(t, T), 0 \leq t \leq T \) is the so-called forward rate process given by
\[
df(t, T) = \alpha(t, T) dt + \sigma(t, T) dZ(t), \quad f(0, T) = f_0(T), \quad 0 \leq t \leq T.
\]
The processes \( \alpha \) and \( \sigma \) are called respectively drift and volatility of \( f \). Moreover, the short rate process equals
\[
R(t) = f(t,t), \quad t \geq 0.
\]
In the model introduced by Heath, Jarrow and Morton, \( Z \) was a Wiener process. Extensions to models with discontinuous \( Z \) were discussed by many authors, see, for instance, [3], [4], [15]. In particular, it was shown that the discounted bond prices, with \( T > 0 \),
\[
\hat{P}(t, T) = e^{-\int_t^T R(s) ds} e^{-\int_t^T f(t,s) ds}, \quad t \in [0, T],
\]
are local martingales if and only if
\[
A(t, T) = J(\Sigma(t, T)), \quad 0 \leq t \leq T,
\]
for each \( T > 0 \), almost all \( t \in [0, T] \) almost \( \omega \)-surely. Above \( A(t, T) := \int_t^T \alpha(t, s) ds \) and \( \Sigma(t, T) := \int_t^T \sigma(t, s) ds \).

We transform now the affine model (1.2) to the HJM framework and examine the martingale property via (3.14). Comparing the exponents in (1.2) and (3.12) yields
\[
A(T - t) + B(T - t) R(t) = \int_t^T f(t, s) ds, \quad 0 \leq t \leq T,
\]
which determines the forward rate in affine model
\[
f(t, T) = A'(T - t) + B'(T - t) R(t), \quad 0 \leq t \leq T.
\]
It follows then by setting \( T = t \) and (3.13) that
\[
A'(0) = 0, \quad B'(0) = 1,
\]
which, together with (2.1), gives preliminary requirements for the functions $A$ and $B$.

Let us define the interval $I := ((a, b),$ with $0 \leq a < b,$ such that

$$\bar{x} \in I, \quad G(x) > 0 \quad \text{for} \quad x \in I,$$

and such that $\bar{I}$ is the maximal interval satisfying (3.17).

**Proposition 3.4** Assume that, for arbitrary $x \geq 0$, the non-negative process $R$ is a strong solution of (1.3) with continuous coefficients $F, G$. Let $A, B$ be twice continuously differentiable functions satisfying (2.1) and (3.16). Then the affine model (1.2) has the martingale property if and only if

$$J\left(G(x)B(v)\right) = -A'(v) - [B'(v) - 1]x + B(v)F(x), \quad x \in \bar{I}, \quad v \geq 0. \quad (3.18)$$

**Proof:** We convert the model to the HJM framework. Applying Itô’s formula to (3.15) and taking into account (2.1) we obtain by (3.14) that the affine model has the martingale property if and only if

$$J\left(G(x)B(v)\right) = -A'(v) - [B'(v) - 1]x + B(v)F(x), \quad x \in \bar{I}, \quad v \geq 0. \quad (3.18)$$

Consequently, for $t < T$,

$$A(t, T) = \int_t^T \alpha(t, s)ds = \int_t^T \left(F(R(t))B'(s - t) - A''(s - t) - B''(s - t)R(t)\right)ds$$

$$= F(R(t))[B(T - t) - B(0)] - [A'(T - t) - A'(0)] - [B'(T - t) - B'(0)]R(t),$$

$$\Sigma(t, T) = \int_t^T \sigma(t, s)ds = \int_t^T B'(s - t)G(R(t - \cdot))ds = G(R(t - \cdot))[B(T - t) - B(0)].$$

Taking into account (2.1) and (3.16) we obtain by (3.14) that the affine model has the martingale property if and only if

$$J\left(G(R(t - \cdot))B(T - t)\right) = -A'(T - t) - [B'(T - t) - 1]R(t) + B(T - t)F(R(t)) \quad (3.21)$$

for each $T > 0$, $\mathbb{P}$-almost surely, for almost all $t \in [0, T]$.

Now we prove that (3.21) is equivalent to (3.18). Since, for almost all $t \geq 0$, $R(t - \cdot) = R(t)$ one can replace $R(t - \cdot)$ by $R(t)$ in (3.21). So, it is clear that (3.18) is sufficient for (3.21). Now we show necessity. Assume to the contrary that for some $\bar{x} > 0$ and $\bar{v} > 0$

$$J(G(\bar{x})B(\bar{v})) > -A'(\bar{v}) - [B'(\bar{v}) - 1]\bar{x} + B(\bar{v})F(\bar{x}).$$

Then there exists $\delta > 0$ such that

$$J(G(x)B(v)) > -A'(v) - [B'(v) - 1]x + B(v)F(x),$$

for $x \in (\bar{x} - 2\delta, \bar{x} + \delta)$ and $v \in (\bar{v} - 2\delta, \bar{v} + \delta)$. Let us consider the solution $R$ of (1.3) starting from $\bar{x}$ and let us define

$$\tau := \inf\{t \geq 0 : R(t) - \bar{x} > \delta\}.$$

Then for $t \in (0, \tau)$ and $T$ such that $T - t \in (\bar{v} - 2\delta, \bar{v} + \delta)$

$$J(G(R(t - \cdot))B(T - t)) > -A'(T - t) - [B'(T - t) - 1]R(t) + B(T - t)F(R(t)),$$

which is a contradiction. \qed
4 Proofs of the main results

4.1 Proof of Part (I) of Theorem 2.1

The proof is rather involved and therefore is divided into several steps. The general idea is as follows. First one proves the affine formula for $F$ on some subinterval of $[0, +\infty)$. Then one establishes that the process $Z$ is an $\alpha$-stable martingale with $\alpha \in (1, 2]$. For this one proves first that on an open subinterval of $(0, +\infty)$, $G(x) = c(x + B)^\gamma$, with some constants $c, B, \gamma$, and then one deduces that $J(\lambda) = c_\alpha \lambda^\alpha$ first locally and then globally on the whole $[0, +\infty)$. In the final step one shows that $G(0) = 0$ and thus that $B = 0$.

Let $\bar{I} := (\bar{a}, \bar{b})$, $0 \leq \bar{a} < \bar{b}$, be the maximal interval for which
\[ \bar{x} \in \bar{I}, \quad G(x) > 0 \quad \text{for} \quad x \in \bar{I}. \]  
(4.1)

If the affine model with short rate $R$ and functions $A, B$ has the martingale property then it follows from Theorem 3.4 that
\[ J(G(x)B(v)) = -A'(v) - [B'(v) - 1]x + B(v)F(x), \quad v \geq 0, \quad x \in \bar{I}. \]  
(4.2)

**Step 1:** We prove the linear form of $F$ on the interval $\bar{I}$.

Differentiation of (4.2) yields
\[ J'(G(x)B(v))G(x)B'(v) = -A''(v) - B''(v)x + B'(v)F(x), \quad v \geq 0, \quad x \in \bar{I}. \]

Putting $v = 0$ yields
\[ J'(0+)G(x) = -A''(0) - B''(0)x + F(x), \quad x \in \bar{I}. \]

Since $J'(0+) = 0$ we obtain the formula for $F$
\[ F(x) = A''(0) + B''(0)x := ax + b, \quad x \in \bar{I}. \]  
(4.3)

To show that $b \geq 0$, assume, by contradiction, that $b < 0$ and consider solution $R$ of the equation starting from 0. Since $R$ is non-negative, we have:
\[ R(t) = b \int_0^t e^{a(t-s)} ds + \int_0^t e^{a(t-s)} G(R(s-))dZ(s) \geq 0, \quad t \geq 0. \]

Hence
\[ -|b| \int_0^t e^{-as} ds + \int_0^t e^{-as} G(R(s-))dZ(s) \geq 0, \quad t \geq 0, \]
or equivalently
\[ \int_0^t e^{-as} G(R(s-))dZ(s) \geq |b| \int_0^t e^{-as} ds \quad t \geq 0. \]

Since the above stochastic integral is a local non-negative martingale starting from 0 it must be identically 0. Thus the process
\[ R(t) = b \int_0^t e^{a(t-s)} ds, \quad t \in [0, T] \]
is strictly negative and we have contradiction.
**Step 2:** We prove that $G$ satisfies

$$
\frac{G'(x)}{G(x)} = \frac{\bar{A}}{B + x}, \quad x \in \bar{I}, \quad x \neq \bar{B},
$$

(4.4)

with some constants $\bar{A}, \bar{B} \in \mathbb{R}, \bar{A} \neq 0$.

Since $F(x) = ax + b, x \in \bar{I}$, (4.2) yields

$$
J(G(x)B(v)) = -A'(v) + B(v)b + x[B(v)a + 1 - B'(v)], \quad v \geq 0, \quad x \in \bar{I}.
$$

(4.5)

Differentiation of (4.5) over $x$ and $v$ leads to

$$
J'(G(x)B(v))G'(x)B(v) = 0, \quad v \geq 0, \quad x \in \bar{I},
$$

(4.6)

and

$$
J'(G(x)B(v))G(x)B'(v) = -A''(v) + B'(v)b + x[B'(v)a - B''(v)], \quad v \geq 0, \quad x \in \bar{I}.
$$

(4.7)

Since $B$ is continuously differentiable and $B'(0) = 1$, we can find an $\varepsilon > 0$ such that

$$
B(v) > 0, \quad B'(v) > 0, \quad v \in (0, \varepsilon).
$$

Let us assume that the right side of (4.6) equals zero for $v \in (0, \varepsilon)$. Then $B$ solves

$$
B'(v) = aB(v) + 1, \quad B(0) = 0, \quad v \in (0, \varepsilon),
$$

so is, on the interval $(0, \varepsilon)$, of the form $B(v) = \frac{1}{a}(e^{av} - 1)$ if $a \neq 0$ or $B(v) = v$ if $a = 0$. Since the left side of (4.6) equals zero and $B(v) > 0$ for $v \in (0, \varepsilon)$ and $G'(\bar{x}) \neq 0$, we obtain

$$
J'(G(\bar{x})B(v)) = 0, \quad v \in (0, \varepsilon).
$$

Hence $J'$ disappears on some interval and consequently must disappear on $[0, +\infty)$ as an analytic function. Since $J(0) = 0$, this implies that $J(\lambda) = 0$ for $\lambda \in [0, +\infty)$, which is impossible. It follows thus that the right side of (4.6) is different from zero for some $\bar{v} \in (0, \varepsilon)$. This implies that

$$
B(\bar{v}) \neq 0, \quad G'(x) \neq 0, \quad J'(G(x)B(\bar{v})) \neq 0, \quad x \in \bar{I}.
$$

Hence, by (4.6),

$$
J'(G(x)B(\bar{v})) = \frac{B(\bar{v})a + 1 - B'(\bar{v})}{G'(x)B(\bar{v})}, \quad x \in \bar{I}.
$$

(4.8)

Putting this into (4.7) with $v = \bar{v}$ yields

$$
\frac{B(\bar{v})a + 1 - B'(\bar{v})}{G'(x)B(\bar{v})} \cdot G(x)B'(\bar{v}) = -A''(\bar{v}) + B'(\bar{v})b + x[B'(\bar{v})a - B''(\bar{v})], \quad x \in \bar{I},
$$

and, consequently,

$$
\frac{G(x)}{G'(x)} = \frac{-A''(\bar{v}) + B'(\bar{v})b + x[B'(\bar{v})a - B''(\bar{v})]}{B(\bar{v})a + 1 - B'(\bar{v})}, \quad x \in \bar{I}.
$$

(4.8)

Hence the quotient $G(x)/G'(x)$ is a linear function of $x$. If $B'(\bar{v})a - B''(\bar{v}) = 0$ then

$$
\frac{G(x)}{G'(x)} = c, \quad x \in \bar{I},
$$

and, consequently,
with $c = G(\tilde{x})/G'(\tilde{x}) \neq 0$. Thus in this case $G(x) = k e^{\tilde{x}}$, $x \in \tilde{I}$ with $k > 0$. Using this form of $G$ in (4.3) gives

$$J(ke^{\tilde{x}}B(v)) = -A'(v) + B(v)b + x[B(v)a + 1 - B'(v)], \quad v \geq 0, \ x \in \tilde{I},$$

and allows us to determine $J$ by

$$J(\lambda) = -A'(v) + B(v)b + \ln \lambda \frac{c}{B(v)k}[B(v)a + 1 - B'(v)], \quad \lambda > 0, \ v \geq 0.$$

The right side above must be independent of $v$ and $J(0+) = \pm \infty$ or $J(0) \equiv 0$. Both situations are not possible, so we conclude that $B'(\tilde{v})a - B''(\tilde{v}) \neq 0$ and thus (4.4) holds.

**Step 3:** We prove that $Z$ is a stable martingale with index $\alpha \in (0, 2]$ by examining (4.4) with non-negative and negative $\bar{B}$.

(a) If $\bar{B} \geq 0$ then (4.4) can be written in the form

$$\frac{d}{dx} \ln(G(x)) = \bar{A} \frac{d}{dx} \ln(\bar{B} + x), \quad x \in \tilde{I}, \ x \neq \bar{B},$$

which yields

$$\ln(G(x)) - \ln(\bar{B} + x)^{\bar{A}} = k, \quad x \in \tilde{I}, \ x \neq \bar{B},$$

for some $k \in \mathbb{R}$. Consequently,

$$G(x) = K(\bar{B} + x)^{\bar{A}}, \quad x \in \tilde{I}, \ x \neq \bar{B}, \quad (4.9)$$

with $K := e^k$. Now we put (4.9) into (4.5). This yields

$$J\left(K(\bar{B} + x)^{\bar{A}}B(v)\right) = -A'(v) + B(v)b + x[B(v)a + 1 - B'(v)], \quad v \geq 0, \ x \in \tilde{I}, \ x \neq \bar{B}.$$

We fix $v = \tilde{v}$ such that $B(\tilde{v}) \neq 0$ and introduce $z := K(\bar{B} + x)^{\bar{A}}B(\tilde{v})$. Then $x = (KB(\tilde{v}))^{-\frac{1}{\bar{A}}} z^{\frac{1}{\bar{A}}} - \bar{B}$ and consequently

$$J(z) = k_1z^{\frac{1}{\bar{A}}} + k_2, \quad z \in \bar{J} := (K(\bar{B} + a)^{\bar{A}}B(\tilde{v}), K(\bar{B} + b)^{\bar{A}}B(\tilde{v}))$$

with some constants $k_1, k_2$. Since $J(0) = 0$ and $J'(0+) < +\infty$ we obtain that

$$J(z) = k_1z^{\frac{1}{\bar{A}}}, \quad z \in \bar{J},$$

and $\alpha := \frac{1}{\bar{A}} > 1$. In fact, $\alpha$ can not be greater than 2, which we show below. By (2.6), for $Z$ without negative jumps,

$$J(z) = -az + \frac{1}{2}qz^2 + \int_0^{+\infty} \left(e^{-zy} - 1 + zy1_{(-1,1)}(y)\right) \nu(dy)$$

$$= -az + \frac{1}{2}qz^2 + \int_0^1 \left(e^{-zy} - 1 + zy\right) \nu(dy) + \int_1^{+\infty} \left(e^{-zy} - 1\right) \nu(dy)$$

$$= -az + \frac{1}{2}qz^2 + z^2 \int_0^1 \frac{e^{-zy} - 1 + zy}{(zy)^2} \nu(dy) + \int_1^{+\infty} (e^{-zy} - 1) \nu(dy), \quad z \geq 0.$$

Since the function

$$x \to \frac{e^{-x} - 1 + x}{x^2}, \quad x \geq 0,$$
We prove that $\overline{S}$ solving (4.12) contradicts the differentiability of $0$. Since, by definition, \( \lim_{x \to \pm \infty} y \nu(dy) \), we see actually that $J(z) \leq ax^2 + bz + c$ for some positive constants $a, b, c$. Since $J$ is analytic on $[0, +\infty)$ we obtain finally

\[ J(z) = c_\alpha z^\alpha, \quad z \in [0, +\infty), \]

where $c_\alpha = k_1$. Therefore $Z$ is a stable process with index $\alpha \in (1, 2]$.

(b) If $\bar{B} < 0$ and $-\bar{B} \leq \bar{x}$ then we can examine (4.10) on the set $\{x : x \in \bar{I}, x > -\bar{B}\}$ in the same way as in (a). If $\bar{B} < 0$ and $-\bar{B} > \bar{x}$ then we examine (4.10) on the set $\{x : x \in (\bar{x}, -\bar{B} \land \bar{b})\}$ and write it in the form

\[ \frac{G'(x)}{G(x)} = \frac{-\bar{A}}{\bar{B} - x}, \quad x \in (\bar{x}, -\bar{B} \land \bar{b}). \]

It follows that

\[ \frac{d}{dx} \ln(G(x)) = \bar{A} \frac{d}{dx} \ln(-\bar{B} - x), \quad x \in (\bar{x}, -\bar{B} \land \bar{b}) \]

and, consequently

\[ G(x) = K(-\bar{B} - x)^{\bar{A}}, \quad x \in (\bar{x}, -\bar{B} \land \bar{b}), \quad (4.10) \]

with some $K > 0$. Putting this formula into (4.10) yields

\[ J\left( K(-\bar{B} - x)^{\bar{A}}B(v) \right) = -A'(v) + B(v)\bar{b} + x|B(v)\bar{a} + 1 - B'(v)|, \quad v \geq 0, \quad x \in (\bar{x}, -\bar{B} \land \bar{b}). \]

Now, like in (a) one shows that $J(z) = c_\alpha z^\alpha, z \in [0, +\infty)$ with $\alpha := 1/\bar{A} \in (1, 2]$.

**Step 4:** We prove that $\bar{I} = (0, +\infty)$ and that

\[ G(x) = c_\alpha x^{\frac{1}{\alpha}}, \quad c > 0, \quad x \in [0, +\infty). \quad (4.11) \]

Since we know from Step 3 that $J(z) = c_\alpha z^\alpha, z \in [0, +\infty), \alpha \in (1, 2]$, it follows from (4.10) that

\[ \alpha c_\alpha G^{\alpha - 1}(x)G'(x)B^\alpha(v) = B(v)\bar{a} + 1 - B'(v), \quad x \in \bar{I}, \quad v \geq 0. \]

We can find $\bar{v} > 0$ such that $B(\bar{v}) \neq 0$. Then

\[ \alpha c_\alpha G^{\alpha - 1}(x)G'(x) = M, \quad x \in \bar{I}, \quad (4.12) \]

with $M := (B(\bar{v})\bar{a} + 1 - B'(\bar{v}))/B^\alpha(\bar{v})$. Now we show that $\bar{I} = (0, +\infty)$. Assume that $\bar{a} > 0$. Since, by definition, $\lim_{x \to \bar{I}} G(x) = 0$, we see from (4.12) that $\lim_{x \to \bar{I}} G'(x) = \pm \infty$, which contradicts the differentiability of $G$ on $(0, +\infty)$. Similarly one can exclude the case $\bar{b} < +\infty$. Solving (4.12) we obtain

\[ G(x) = \left( G(\bar{x}) - \frac{M}{c_\alpha \bar{x} + \frac{M}{c_\alpha}} \right)^{\frac{1}{\alpha}} := (m_1 + m_2x)^{\frac{1}{\alpha}}, \quad x \in (0, +\infty), \]

with $m_1 \geq 0, m_2 > 0$. If $m_1 > 0$ then $G$ is Lipschitz at zero and by Proposition 3.2 $G(0) = 0$ which is a contradiction. Hence (4.11) follows with $c := m_2$. \qed
4.2 Proof of Part (II) of Theorem 2.1

By elementary arguments, positivity of the solutions to the stochastic equation (1.3) with \(G(x) \equiv \sigma\) implies that \(Z\) has no Wiener part and can have only positive jumps. Repeating the arguments from the proof of Part (I) one can show that that \(F(x) = ax + b, x \geq 0, \text{ and } b \geq 0\). We will establish now that

\[
\int_{0}^{+\infty} y\nu(dy) < +\infty, \quad b \geq \sigma \int_{0}^{+\infty} y\nu(dy).
\]  

(4.13)

Let \(\tilde{\pi}\) be the compensated jump measure corresponding to the martingale \(Z\). Then for \(\epsilon > 0\),

\[
Z(t) = \int_{0}^{t} \int_{0}^{+\infty} y\tilde{\pi}(ds, dy) = \int_{0}^{t} \int_{0}^{\epsilon} y\tilde{\pi}(ds, dy) + \int_{0}^{t} \int_{\epsilon}^{+\infty} y\tilde{\pi}(ds, dy)
\]  

(4.14)

\[= Z_{\epsilon}(t) + P_{\epsilon}(t) - t \int_{\epsilon}^{+\infty} y\nu(dy). \]  

(4.15)

Here \(Z_{\epsilon}\) is a Lévy martingale with positive jumps bounded by \(\epsilon\), \(P_{\epsilon}\) is a compound Poisson process with the Lévy measure \(\nu\) restricted to the interval \([\epsilon, +\infty)\). For the solution \(R\) of the stochastic equation, starting from 0, we have for all \(t \in [0, T]\):

\[
e^{-at} R(t) = b \int_{0}^{t} e^{-as}ds + \sigma \int_{0}^{t} e^{-as}dZ(s)
\]  

[\(= (b - \sigma \int_{\epsilon}^{+\infty} y\nu(dy)) \int_{0}^{t} e^{-as}ds + \sigma \int_{\epsilon}^{t} e^{-as}dZ_{\epsilon}(s) + \int_{\epsilon}^{t} e^{-as}dP_{\epsilon}(s) \geq 0\).  

(4.16)

If \(\int_{0}^{+\infty} y\nu(dy) = +\infty\), then by taking \(\epsilon\) close to 0, the number \((b - \sigma \int_{\epsilon}^{+\infty} y\nu(dy))\) can be made arbitrary small negative. The stochastic integrals with respect to \(Z_{\epsilon}, P_{\epsilon}\) are independent processes. The former one can be made, with positive probability, uniformly smaller on \([0, T]\), than given in advance number and the latter one , is 0 on \([0, T]\) also with positive probability. Thus \(e^{-at} R(t)\) is negative for some \(t \in [0, T]\), with positive probability, which is a contradiction. Now we show that \(b \geq \sigma \int_{0}^{+\infty} y\nu(dy)\). In the opposite case we have that the difference

\[b - \sigma \int_{\epsilon}^{+\infty} y\nu(dy),\]

is negative for sufficiently small \(\epsilon > 0\) and decreases as \(\epsilon \downarrow 0\). It follows from the Markov inequality that for any \(\gamma > 0\) and \(t > 0\)

\[
\mathbb{P}(\sigma \int_{0}^{t} e^{-as}dZ_{\epsilon}(s) > \gamma) \leq \frac{\sigma^{2} \mathbb{E}(\int_{0}^{t} e^{-as}dZ_{\epsilon}(s))^{2}}{\gamma^{2}} = \frac{\sigma^{2} \int_{0}^{t} e^{-2as}ds\nu(dy)}{\gamma^{2}} \xrightarrow{\epsilon \to 0} 0,
\]

and consequently

\[
\mathbb{P}(\sigma \int_{0}^{t} e^{-as}dZ_{\epsilon}(s) \leq \gamma) \xrightarrow{\epsilon \to 0} 1.
\]

Since the integral over \(P_{\epsilon}\) disappears with positive probability, we have by (4.16) that \(R(t) < 0\) which is a contradiction. \(\square\)

4.3 Proof of Part (I) of Theorem 2.2

It was shown in [12] that equation (2.8) actually has a unique non-negative strong solution. Now we use Theorem 3.4 with \(J(\lambda) = c_{\alpha}x^{\lambda}, F(x) = ax + b\) and \(G(x) = c^\frac{1}{\alpha}x^\frac{1}{\alpha}\). Then (3.18) boils down to

\[
c_{\alpha} \left(c^\frac{1}{\alpha}x^\frac{1}{\alpha}B(v)\right)^{\alpha} = -A'(v) - [B'(v) - 1]x + B(v)[ax + b], \quad x \geq 0, \quad v \geq 0.
\]
Consequently,
\[c_{\alpha}cxB^\alpha(v) = (aB(v) - B'(v) + 1)x + bB(v) - A'(v), \quad x \geq 0, \quad v \geq 0.\]  \hspace{1cm} (4.17)

Putting \(x = 0\) yields
\[bB(v) - A'(v) = 0, \quad v \geq 0,
\]
which is the required formula for \(A\). It follows from (4.17) that
\[c_{\alpha}cxB^\alpha(v) = aB(v) - B'(v) + 1, \quad v \geq 0,
\]
which yields the equation for \(B\). \(\square\)

### 4.4 Proof of Part (II) of Theorem 2.2

Note that functions \(A, B\) should satisfy, for all \(x \geq 0, v \geq 0\), the equation
\[J(\sigma B(v)) = -A'(v) - (B'(v) - 1)x + B(v)(ax + b)
\]
\[= x(B(v)a - B'(v) + 1) + B(v)b - A'(v).
\]
Consequently
\[B'(v) = B(v)a + 1, \quad B(0) = 0
\]
\[A'(v) = B(v)b - J(\sigma B(v)), \quad A(0) = 0.
\]
It remains to show that \(A\) is an increasing function that is, that is \(B(v)b - J(\sigma B(v)) \geq 0\).

However,
\[J(\lambda) = \int_0^{+\infty} (e^{-\lambda y} - 1)\nu(dy) + \lambda \int_0^{+\infty} y\nu(dy)
\]
\[=:J_0(\lambda) + \lambda \int_0^{+\infty} y\nu(dy).
\]  \hspace{1cm} (4.18)
\[=J_0(\lambda) + \lambda \int_0^{+\infty} y\nu(dy).
\]  \hspace{1cm} (4.19)

It is clear that
\[B(v)(b - \sigma \int_0^{+\infty} y\nu(dy)) - J_0(\sigma B(v)) \geq 0,
\]
and the result follows. \(\square\)

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