On Exact and Efficient Inference for Many Normal Means

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Abstract

Inference about the unknown means \( \theta = (\theta_1, ..., \theta_n)' \in \mathbb{R}^n \) in the sampling model \( X \sim N_n(\theta, I) \) from the observed \( X \), known as the many-normal-means problem, has proven to be fundamental and yet challenging inasmuch as a satisfactory understanding remains elusive.

To tackle exact and efficient inference about \( \theta \), in this paper we propose innovative formulations of Inferential Models for two kinds of this problem: the classic kind given as is and the empirical-Bayes kind where \( \theta \)'s are further assumed to be an unobservable sample from an unknown non-parametric distribution \( \mathcal{G} \). The formulation for the empirical-Bayes kind via numerical deconvolution allows for prior-free probabilistic inference with over-parameterization for the non-parametric model \( \mathcal{G} \), whereas the formation for the first kind utilizes a latent random permutation and as a result provides a sound reasoning with uncertainty toward a deeper understanding. For uncertainty quantification with the more familiar frequentist inference framework, the method of maximum plausibility estimation is used for point estimation. Exact but conservative interval estimation is obtained based on plausibility, with a Monte Carlo based adaptive-adjustment approach to constructing shorter confidence intervals with targeted coverage. These methods are illustrated via simulation studies and a real-data example. The numerical results show that for interval estimation, adaptive intervals are satisfactory in both coverage and efficiency and that for point estimation, the proposed methods outperform the traditional James-Stein and Efron’s g-modeling in terms of mean square error. The paper concludes with a few remarks, including future developments and extensions of the proposed methods.

Key Words: Empirical Bayes, Inferential Models, Latent permutation, Numerical deconvolution, Stein’s paradox.

1 Introduction

The many-normal-means problem concerns simultaneous inference about a collection of unknown normal means. More precisely, one has the observed data \( X_1, ..., X_n \) that are assumed to have the sampling model

\[
X_i = \theta_i + Z_i \quad (Z_i \text{ i.i.d. } \sim N(0, 1); \ i = 1, ..., n)
\]  

(1.1)

This problem has proven to be fundamental in statistics. The most noticeable contribution, perhaps, dates back to [Stein (1956)], on the remarkable discovery that the usual maximum likelihood estimates \( \hat{\theta}_i = X_i \) are inadmissible under quadratic loss for \( n \geq 3 \). Understanding of and developing methods for this problem have been resulting in successful research in multi-parameter or high-dimensional problems, including, for example, the James-Stein shrinkage method [James and Stein (1961)], the empirical Bayes method of Efron and Morris (1972), the wavelet shrinkage method of Donoho and Johnstone (1995) for non-parametric regression, and LASSO of Tibshirani (1996).
for linear regression. The literature on the many normal means problem itself and methods built up on shrinkage is massive and is, thereby, impossible to provide a complete review in this note. We refer to the problem setting given above as is as the first or classic kind, reflecting Stein’s discovery. It is perhaps somewhat surprising that the understanding of the Stein phenomenon has not been very satisfactory, at least not to a level that would help develop efficient methods based on a sound reasoning with uncertainty, as Stigler (1990) wrote:

*The best heuristic explanation that has been offered is a Bayesian argument ... Another explanation that has been offered is that [the James-Stein estimator] can be viewed as a relative of a “pre-test” estimator ... But neither of these explanations is fully satisfactory (although both help render the result more plausible); the first because it requires special a priori assumptions where Stein did not, the second because it corresponds to the result only in the loosest qualitative way. The difficulty of understanding the Stein paradox is compounded by the fact that its proof usually depends on explicit computation of the risk function or the theory of complete sufficient statistics, by a process that convinces us of its truth without really illuminating the reasons that it works.*

Attributing back to Francis Galton’s knowledge in the 1880’s and viewing Stein estimation as an inverse regression problem $E(\theta | X)$, Stigler (1990) provided a nice interpretation of the James-Stein (shrinking toward zero) and Efron-Morris (shrinking toward the sample mean $\bar{X}$) estimators of $\theta$. The fundamental problem of this classic kind has however remained unsolved, due to the difficulty as recognized by Stigler (1990):

*With no distributional assumptions about the $\theta$’s, we are of course prevented from looking for an optimal estimate of “$E(\theta | X)$”.*

Our innovative formulation in Section 4 via introducing a latent permutation in addition to unobserved $Z$’s in (1.1) leads to an inferentially valid $E(\theta | X)$ and thus offers a new understanding, shedding light on this challenging problem from an Inferential Models perspective.

A popular variant of the classic kind is obtained by modeling the unknown $\theta_i$’s as an unobserved sample from a non-parametric distribution $G$ with the cumulative distribution function (cdf) $G(.)$. In this case, integrating out the unobservable $\theta_i$’s leads to inference about the mixture distribution

$$F_G(x) = \int \phi(x - \theta) dG(\theta)$$

from the observed $X$’s. Note that in this case, $X$’s is necessarily a sample from the convoluted distribution $F_G(x)$. Inference about $\theta$’s is then carried out with their conditional distribution, given $X_i$’s and estimates of $G$. The central problem in this case is thus to estimate $G(.)$. There is a considerable literature on this deconvolution problem, as in Laird (1978), Carroll and Hall (1988), Stefanski and Carroll (1990), Fan (1991), Zhang (1997), Hall and Meister (2007), Butucea and Comte (2009), and Efron (2014, 2016). For convenience, we refer to inference via this non-parametric partial Bayes model as the second or (non-parametric) empirical-Bayes kind, in contrast to the classic kind.

Here, we consider exact and efficient inference for the two kinds of the many-normal-means problems by taking the Inferential Models (IMs) approach, which is to be briefly reviewed in Section 2. The empirical-Bayes kind and classic kind are investigated in Sections 3 and 4, respectively. The development of IMs for the empirical-Bayes kind appears relatively straightforward because of the nice independence structure imposed. Nevertheless, following the discretization idea in Efron (2014, 2016) but not going further for $g$-modeling, our approach is to make inference via over-parameterization or numerical deconvolution without effectively having to impose more restrictions.
on $G$. It should be noted that unlike traditional Frequentist and Bayesian methods, IMs can provides honest inference even if the unknown parameters for $G$ are unidentifiable.

For the classic kind, we introduce a sorted standard normal sample with a random permutation as the so-called auxiliary variable. This novel approach, with a full IMs argument, leads to a prior-free probabilistic inference, for example, by making use of the quantity $E(\theta|X)$ for point estimation. As a result, it sheds a new light on a satisfactory understanding Stein estimation.

Predictive random sets for IMs inference are constructed to produce probabilistic inference in terms of belief and plausibility. It is well known that valid probabilistic inference says more than but can be used for point and interval estimations. As a method for point estimation, maximum plausibility estimation is discussed in detail. This adaptive method is so straightforward in the context of frequency calibration that it is simply presented as a technical remark; see Remark 3.4. This is illustrated in Section 5.2 for a real-data example.

To evaluate the performance of the proposed methods, a simulation study is conducted in Section 5 with the computational methods provided in Appendices. The numerical results show that the proposed methods outperform the traditional James-Stein and Efron’s g-modeling (Narasimhan and Efron, 2020), and are comparable to the auto-modeling method proposed recently by Jiang and Liu (2022). Section 6 concludes the paper with a few remarks, including a brief discussion of other kinds of the many-normal-means problem, future potential developments, and extensions.

2 Inferential models: a brief review

IMs was proposed in Martin and Liu (2013) as an inferential framework for prior-free probabilistic inference. It makes use of the so-called unobserved but predictable auxiliary random variables, or simply, auxiliary variables $U$ that has a known distribution $P_U$ on $U$. The setup for IMs is a system of equations, called association, representing a statistical model with unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^p$ for the observed data $X \in \mathcal{X}$ via an auxiliary variables $U$. For the sake of clarity, we write this association as

$$X = a(U, \theta) \quad (X \in \mathcal{X}, \theta \in \Theta, U \sim P_U) \quad (2.1)$$

IMs relies on so-called valid or frequency-calibrated predictive random sets (PRS) $S$ for predicting the unobserved $U$ to produce valid uncertainty quantification on unknown quantities of interest. Formally, a PRS $S$ is said to be valid for $U$ if it is independent of $U$ and

$$\text{Prob}(\text{Prob}(S \not\ni U) \geq 1 - \alpha) \leq \alpha \quad (2.2)$$

for all $\alpha \in (0, 1)$. Typically, $S$ is chosen in such a way that $\text{Prob}(S \not\ni U) \sim \text{Uniform}(0,1)$, the standard uniform distribution. For example, the $S(U) = \{u : |u| \leq |U|\}, U \sim N(0,1)$, is valid for predicting an unobserved realization from $N(0,1)$.

Given a PRS $S$ for predicting $U$ in (2.1), the inverse set-valued mapping

$$\Theta_X(S) = \{\theta : X = a(u, \theta) \text{ for some } u \in S\} \quad (2.3)$$

defines a random set on $\Theta$. For simplicity, here we assume $\Theta_X(S) \neq \emptyset$ for all values of $S$ (c.f. Leaf and Liu (2012) for the case where $\text{Prob}(S = \emptyset) > 0$). IMs produces uncertainty assessments on assertions of interest, $A \subset \Theta$, by computing the following lower and upper probabilities

$$\text{bel}_X(A) = \text{Prob}(\Theta_X(S) \subseteq A) \quad (2.4)$$
and

\[ \text{pl}_X(A) = 1 - \text{bel}_X(A^c) = \text{Prob}(\Theta_X(S) \not\subseteq A^c) \]  \hspace{1cm} (2.5)

The probabilities (2.4) and (2.5) are also known as the belief function at \( A \) and plausibility of \( A \), respectively. A recent discussion of these IMs lower and upper probabilities and other existing ones is given in Liu and Martin (2021); see also references therein.

As described above, IMs is rooted in R. A. Fisher’s fiducial argument in the sense to make prior-free probabilistic inference about parameters by using auxiliary variable to represent the sample model. Avoiding the difficulties of R. A. Fisher’s fiducial argument and the related theories (Liu and Martin, 2015; Martin et al., 2010; Zhang and Liu, 2011), however, IMs emphasizes frequency-calibrated/valid and efficient probabilistic inference. Together with Conditional IMs and marginal IMs (Martin and Liu, 2015a and Martin and Liu, 2015b), subject to further developments, this theory, if so desired, also provides as a new mathematical tool for developing exact and efficient frequentist methods. For example, for the familiar Neyman-Pearson school of thought, it offers its plausibility regions for unknown parameters as exact Neyman-Pearson confidence regions (Martin and Liu, 2013), and for the Fisherian school of thought, its plausibility for assertions of interest provides a plausibility interpretation of or a probabilistic alternative to Fisher’s P-value (Martin and Liu, 2014).

Nevertheless, research on foundations of statistical inference has been challenging. Like other existing approaches other than the standard Bayes and frequentist schools of thought, including generalized fiducial (Cui and Hannig, 2022; Hannig, 2009), the method of confidence distribution (Schweder and Hjort, 2016; Xie et al., 2011), and the Dempster-Shafer theory of belief functions (Dempster, 2008; Martin et al., 2010; Shafer, 1976, 1992), IMs is subject to scrutiny and acceptance. This is expected to take time.

3 IMs for MNMs of the empirical Bayes kind

3.1 The sampling model and IMs association

Without real loss of generality, our jumping-off point is to take a numerical approach to approximate \( G(\cdot) \) in (1.2) via discretization; see also Efron (2014, 2016). Denote by \( \vartheta_1 < \ldots < \vartheta_K \) the known space of \( \theta \), and denote by \( \gamma_1 \) the unknown probability mass at \( \vartheta_k \), where

\[ \gamma \equiv (\gamma_1, \ldots, \gamma_K) \in \left\{ (\gamma_1, \ldots, \gamma_K) : \gamma_k \geq 0 \text{ for all } k = 1, \ldots, K; \sum_{k=1}^{K} \gamma_k = 1 \right\} \]  \hspace{1cm} (3.1)

**Remark 3.1.** Take \( \vartheta_k - \vartheta_{k-1} = \delta \) for all \( k = 1, \ldots, K \). This discrete approximation can also be view as numerical approach to mixed distributions, including both discrete and continuous distributions.

In this scenario, \( G(\cdot) \) is represented by the \( K \)-dimensional \( \gamma \) vector in (3.1), that is, in terms of cdf

\[ G(\theta) = \sum_{\vartheta_k \leq \theta} \gamma_k \]

and \( X_1, \ldots, X_i \) is a sample from the convolution of \( G(\cdot) \) and \( N(0, 1) \):

\[ F_{\vartheta, \gamma}(x) = \sum_{k=1}^{K} \gamma_k \Phi(x - \vartheta_k) \]  \hspace{1cm} (3.2)
where $\Phi(.)$ denotes the cdf of the standard norm distribution $N(0,1)$. Furthermore, given $\gamma$ and $X$, $\theta_i$’s are independent of each other with

$$
\Pr(\theta_i = \theta_j | \gamma, X) = \frac{\gamma_k \phi(X_i - \theta_j)}{\sum_{k=1}^K \gamma_k \phi(X_i - \theta_k)} \quad (j = 1, ..., K)
$$

(3.3)

where $\phi(.)$ stands for the probability density function of the standard norm distribution $N(0,1)$.

From (3.2) and (3.3), we see that inference about $\theta_j$ is reduced to inference about $\gamma_k$’s. It is known that $F_{\theta,\gamma}(X_i)$’s form a sample of size $n$ from $F_{\theta,\gamma}(x)$. For an IMs approach, we take a sorted uniforms of size of $n$, denoted by $U_{(1)} < ..., < U_{(n)}$, as auxiliary variables and specify the association as

$$
U_{(i)} = F_{\theta,\gamma}(X_{(i)}) = \sum_{k=1}^K \gamma_k \Phi(X_{(i)} - \theta_k)
$$

(3.4)

### 3.2 A PRS for sorted uniforms

Let $\mathbb{S}_n$ denote the sample space of $U_{(i)}$’s, the simplex space of size $n$:

$$
\mathbb{S}_n = \{(u_1, ..., u_n) : 0 < u_1 < ... < u_n < 1\}
$$

Intuitively, we aim to constructing a PRS for $U_{(i)}$’s that should work well for the well-known non-parametric one-sample test, where it is known that the cdf values at the observations serve as the sufficient statistics and form a sample of uniforms. For simplicity, we use a type of predictive random set similar to that of Liu and Xie (2014) and Zhang (2010) for $(\Phi(Z_{(1)}), ..., \Phi(Z_{(n)}))$,

$$
\mathcal{U} = \{u : u \in \mathbb{S}_n, B(u) \leq B(U)\}
$$

(3.5)

where $U = (U_{(1)}, ..., U_{(n)})$, the vector of sorted sample of size $n$ from Uniform(0,1) and

$$
B(u) = -\sum_{i=1}^n [a_i \ln(u_i) + b_i \ln(1 - u_i)]
$$

(3.6)

with $a_i = \frac{1}{n-i+c_n}$, $b_i = \frac{1}{1+c_n}$, and $c_n \approx \frac{2}{3}$. Zhang (2010) considered this PRS with $c_n = 0.7$ as an IMs approach to the non-parametric one-sample problem and showed that the resulting method is either comparable to or better than traditional methods such as the Kolmogorov-Smirnov test.

Recall the familiar results on sorted uniforms regarding the marginal distribution of $U_{(i)}$. We have $U_{(i)} \sim$ Beta($i, n-i+1$) which has its mean and variance given by $\frac{i}{n+1}$ and $\frac{i(n-i+1)}{(n+1)^2(n+2)}$. The PRS defined by (3.5) and (3.6) is constructed as a weighted logarithms of the marginal densities with a minor modification so that the medians are approximately most likely (c.f. Kerman 2011) for the choice of $c_n = \frac{2}{3}$. The weights are set to be about inversely proportional to their variances and, thereby, pay more attentions to the tail areas of the underlying sampling distribution. This leads to a slightly more general class of PRSs is to use alternative weight schemes:

$$
a_i = w_i(i - 1 + c_n), \ b_i = w_i(n - i + c_n)
$$

(3.7)

with

$$
w_i = \frac{[i(n-i+1)]^{-\nu/2}}{\sum_{j=1}^n [j(n-j+1)]^{-\nu/2}}
$$

(3.8)

for $i = 1, ..., n$, where $\nu \in [0, 2]$. The PRS (3.7) with $\nu = 2$ is used in the numeric examples in Section 5.
3.3 Induced random sets in the $\gamma$ space

A realization of random set $\{3.5\}$ induced the random sets in the $\gamma$ space through the convolution $\{3.9\}$:

$$\Gamma(U) = \left\{ (\gamma_1, \ldots, \gamma_K) : B \left( \sum_{k=1}^{K} \gamma_k \Phi(X(1) - \vartheta_k), \ldots, \sum_{k=1}^{K} \gamma_k \Phi(X(n) - \vartheta_k) \right) \leq B(U) \right\}$$

This random set for $\gamma$ defines the set of numerical approximations to the unknown distribution $G(.)$. The usual IMs uncertainty quantification about $G(.)$ can be carried out as prescribed in Section 2. In the context of multiple testing, for example, inference may be made about fractional number of outliers. For MNMs problem, inference about $\theta_i$’s is often of interest and obtained by making use of $\{3.9\}$ and $\{3.3\}$. This is discussed in detail below in next two subsections.

3.4 Maximum plausibility estimation

A straightforward application of the above IMs is to produce point estimation. Given an estimate of $G$, the usual Bayes posterior mean

$$\hat{\theta}_i(\gamma) = \frac{\sum_{k=1}^{K} \gamma_k \phi(X_i - \vartheta_k) \vartheta_k}{\sum_{k=1}^{K} \gamma_k \phi(X_i - \vartheta_k)}$$

provides the most efficient estimator [Brown, 1971], in terms of mean square error (MSE):

$$\text{MSE}(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^{n} (\hat{\theta}_i - \theta_i)^2$$

Here we consider MPE of $\gamma$. The smallest PRS with the nested focal elements, obtained with the method of (elastic) for validity, is given by

$$U_{\text{min}} = \{ u : B(u) \leq B_{\text{min}} \}$$

where

$$B_{\text{min}} = \min_{\gamma} B \left( \sum_{k=1}^{K} \gamma_k \Phi(X(1) - \vartheta_k), \ldots, \sum_{k=1}^{K} \gamma_k \Phi(X(n) - \vartheta_k) \right)$$

This induces the maximum plausibility set for $\gamma$

$$\Gamma_{\text{min}} = \left\{ (\gamma_1, \ldots, \gamma_K) : B \left( \sum_{k=1}^{K} \gamma_k \Phi(X(1) - \vartheta_k), \ldots, \sum_{k=1}^{K} \gamma_k \Phi(X(n) - \vartheta_k) \right) \leq B_{\text{min}} \right\}$$

It follows that the maximum plausibility is necessarily one and is achieved for all $\gamma \in \Gamma_{\text{min}}$. Since $\Gamma_{\text{min}}$ given in $\{3.14\}$ is a set, we have a set of MPEs of $\theta_i$, one for each $\gamma \in \Gamma_{\text{min}}$.

For simple practical applications, we can summarize the above set of MPEs by two extremes, the lower and upper posterior expectations

$$\hat{\theta}_i^{(\text{lower})} = \min_{\gamma \in \Gamma_{\text{min}}} \hat{\theta}_i(\gamma) \quad \text{and} \quad \hat{\theta}_i^{(\text{upper})} = \max_{\gamma \in \Gamma_{\text{min}}} \hat{\theta}_i(\gamma)$$

for $i = 1, \ldots, n$. When a single value estimate is desired, the mean of the two extremes may be used, that is,

$$\hat{\theta}_i = \frac{\hat{\theta}_i^{(\text{lower})} + \hat{\theta}_i^{(\text{upper})}}{2}$$

This one is used to evaluate PRS performance in the simulation study in Section 5.
3.5 Plausibility intervals

Interval estimation can be made to have desirable frequency-calibration property. Suppose that one is interested in the simultaneous interval estimation, one for each $\theta_i$. Denote by $C_i^{(1-\alpha)}(X)$ the interval estimate of $\theta_i$ with coefficient $1 - \alpha$. A frequency-calibration property commonly used in practice is that

$$E\left(\frac{1}{n} \sum_{i=1}^{n} I\{\theta_i \in C_i^{(1-\alpha)}(X)\}\right) \geq 1 - \alpha. \quad (3.17)$$

In words, it requires on average at least $1 - \alpha$ percent of the true values of $\theta_i$’s will be covered, in a frequentist sense within and cross experiments. This, more precisely, the case with only the equality constraint in (3.18) has been a popular standard in practice (Brown et al., 2011; Wahba, 1983).

The validity of IMs that guarantees frequency-calibration has been established in Martin and Liu (2013). Formally, we have the following result.

**Theorem 1 (IMs Validity).** Let $B_{1-\pi}$ be the $(1-\pi)$-th quantile of $B(U)$ and denote by $\Gamma_{1-\pi}$ the $\alpha$-plausibility region

$$\Gamma_{1-\pi} = \left\{(\gamma_1, \ldots, \gamma_K) : B\left(\sum_{k=1}^{K} \gamma_k \phi(X_i - \vartheta_k), \ldots, \sum_{k=1}^{K} \gamma_k \phi(X_{(n)} - \vartheta_k)\right) \leq B_{1-\pi}\right\}. \quad (3.18)$$

Then, $E((\gamma_1, \ldots, \gamma_n) \in \Gamma_{1-\gamma}) \geq 1 - \pi$.

Suppose that for each $\gamma \in \Gamma_{1-\pi}$, we construct the two-sided “confidence interval” with the left-end point

$$\theta^{(\text{lower})}_{\alpha/2}(\gamma) = \max_{i} \left\{ \theta_i : \hat{\theta}_i(\gamma) = \frac{\sum_{k=1}^{t} \gamma_k \phi(X_i - \vartheta_k)}{\sum_{k=1}^{K} \gamma_k \phi(X_i - \vartheta_k)} \leq \frac{\alpha}{2} \right\} \quad (3.19)$$

and the right-end point

$$\theta^{(\text{upper})}_{\alpha/2}(\gamma) = \min_{i} \left\{ \theta_i : \hat{\theta}_i(\gamma) = \frac{\sum_{k=1}^{r} \gamma_k \phi(X_i - \vartheta_k)}{\sum_{k=1}^{K} \gamma_k \phi(X_i - \vartheta_k)} \leq \frac{\alpha}{2} \right\}. \quad (3.20)$$

This leads to the following plausibility interval

$$C_i(1-\pi, 1-\alpha) \equiv \left[ \min_{\gamma \in \Gamma_{1-\pi}} \theta^{(\text{lower})}_{\alpha/2}(\gamma), \ max_{\gamma \in \Gamma_{1-\pi}} \theta^{(\text{upper})}_{\alpha/2}(\gamma) \right]. \quad (3.21)$$

The following theorem establishes a lower bound for the “confidence” coefficient of (3.21).

**Theorem 2 (Coverage Property).** The coverage probability of the confidence interval (3.21) is at least $(1-\gamma)(1-\pi)$, i.e.,

$$\Pr\left(C_i(1-\pi, 1-\alpha) \ni \theta_i \right) \geq (1-\pi)(1-\alpha)$$

for all $\pi \in (0, 1)$ and $\alpha \in (0, 1)$.

**Proof.** The IMs validity says that $\gamma$ is covered by $\Gamma_{1-\pi}$ $1-\pi$ percent. In such a case, the interval defined by the two end points (3.19) and (3.20) covers the true value, generated from $G$, with probability $1 - \alpha$. The claimed result follows from the multiplication rule of probability. \qed
Theorem 2 provides a way of constructing exact confidence intervals. Because the set for $G$ or $\gamma$ can be large, rendering conservative intervals, we provide some relevant comments on this issue, aiming to a method of adaptive construction of shorter intervals with approximate coverage via Monte Carlo simulations.

Remark 3.2 (Within-experiment Validation). Within-experiment frequency evaluation of confidence interval is possible when $n$ is large. The fractional number of $\theta_i$’s falling outside of their $1 - \alpha$ confidence intervals is expected to be $n\alpha$ and in such cases, the corresponding observed $X_i$’s have probability 0.5 to fall outside of the intervals as well. On the basis of this argument, we expect to see that the number of $X_i$ falling outside of corresponding individual $1 - \alpha$ confidence intervals for $\theta_i$ is about $n\alpha/2$.

Remark 3.3 (Selection of $\pi$ as a Way of Modeling). When confidence intervals are excessively large, modeling can be useful. Efron’s g-modeling, which assumes $\gamma \propto e^{Q\zeta}$ for some fixed $(K \times p)$ matrix $Q$ with unknown $\zeta \in \mathbb{R}^p$, serves a good example. Alternatively, one can also choose $\pi$ and attribute the ignored relevant error to the modeling error.

Remark 3.4 (Adaptive Adjustment). Efficient confidence intervals can be constructed via Monte Carlo as follows. Without loss of generality, suppose that 95% CIs are of interest. So, take $1 - \alpha = \sqrt{0.95}$, evaluate coverage via Monte Carlo simulation with $\gamma \in \ldots 1 - \pi = \sqrt{0.95}$ for a sequence of $m$, say, $m = 10$, quantiles $B(U)$ with probabilities $(s/m)\sqrt{0.95}$, $s = 1, \ldots, m$. The MPE estimate of $G$, for example, can be used to generate data for this Monte Carlo approach. The approximate intervals for the real data are the $(s/m)\sqrt{0.95}$ intervals, where $s$ is the index of the simulation intervals with their actual coverage in the simulated data matching the targeted coverage. This approach is used in the real-data example in Section 5.2.

4 IMs for MNMs of the classic kind

4.1 Constructing an association via augmenting latent auxiliary variables

The Stein phenomenon implies that when performance assessment of inferential methods involves simultaneous inference about $\theta_i$’s, such as mean square errors for point estimation, efficient methods can be developed by allowing the individual estimators of $\theta_i$’s to share risk with each other. To see this intuitively from an IMs perspective, we write $Z_1, \ldots, Z_n$ in (1.1) in terms of their sorted values $Z_{(1)}, \ldots, Z_{(n)}$ with an auxiliary random permutation of $(1, \ldots, n)$, denoted by $(\tau_1, \ldots, \tau_n)$:

$$X_i = \theta_i + Z_{(\tau_i)}$$  \hspace{1cm} (4.1)

Since everyone of the $n!$ permutations is equally likely, the sampling models (1.1) and (4.1) are equivalent to each other. Clearly, efficient simultaneous inference can be obtained, conditional on $(\tau_1, \ldots, \tau_n)$. Here we develop efficient methods by estimating the unknown permutation $\tau$ as a way of controlling the level of risk sharing. Before proceeding we record a result that is useful in what follows, where optimization over $(Z_{(1)}, \ldots, Z_{(n)})$ and $\tau = (\tau_1, \ldots, \tau_n)$ can be carried out over $\theta \in \mathbb{R}^n$.

**Proposition 1.** The observed data $X_1, \ldots, X_n$, the auxiliary variables $(Z_{(1)}, \ldots, Z_{(n)})$ and $\tau = (\tau_1, \ldots, \tau_n)$ determine and are determined by $\theta_i$’s.

What is interesting here for IMs is that the number of auxiliary variables is larger than that of the observed data. For this reason, we call the auxiliary permutation $\tau$ a latent auxiliary variable. For technical clarity, we denote by RP the abbreviation of “random permutation” and summarize the above discussion into the following lemma concerning the equivalence of three sampling associations, including (1.1) and (4.1).
Lemma 1 (Equivalence of Three Sampling Associations). The sampling associations (4.1), (4.1), and
\[ X_{\tau_i} = \theta_{\tau_i} + Z_i \quad (\tau \sim \text{RP}(1, \ldots, n); \ Z_i \overset{iid}{\sim} N(0, 1)) \] (4.2)
are all equivalent in the sense that the induced sampling distributions of X’s, given \( \theta \)’s, are the same, i.e., \( X_i \overset{iid}{\sim} N(\theta_i, 1) \) for all \( i = 1, \ldots, n \).

Lemma 1 can be easily established with standard technical arguments in terms of sampling distributions. This result allows for flexibility in formulating IMs for prior-free probabilistic inference. For example, Equation (4.2) implies that the sampling scheme in the classic kind can be viewed as follows. There is a collection of \( \theta \)'s, denoted by \( \{\theta(k) : k = 1, \ldots, n\} \). One takes \( \theta \)'s from \( \{\theta(k) : k = 1, \ldots, n\} \) without replacement, draws a sample of size of \( n \) from \( N(0, 1) \), and creates \( X \)'s via either (4.2) or \( X_{\tau_i} = \theta_{\tau_i} + Z(i) \) for \( i = 1, \ldots, n \).

Enforcing risk sharing clearly prevents \( \tau \) from being integrated out. Note that different permutations produce observed data samples as a collection of data points looking like each other or, more precisely, having the same distribution. Given the observed data, however, the permutation that was chosen to generate the data would be the best to be conditioned on for inference via predicting \( Z(\tau_1), \ldots, Z(\tau_n) \). This suggests that the collection of data points \( X_i \)'s be used to establish an association, which we call an augmented association. In other words, we can make use of the sampling model of the order and sufficient statistics \( X(1) \leq \ldots \leq X(n) \) to create such an augmented association.

Note that the marginal distribution for a random draw from \( \{X(1), \ldots, X(n)\} \) is given by
\[ F_\theta(x) = \sum_{k=1}^{n} \frac{1}{n} \Phi(x - \theta_k) \] (4.3)
In terms of the true realizations of the auxiliary variables we can also write (4.3) as
\[ F_\theta(x) = \sum_{k=1}^{n} \frac{1}{n} \Phi(x - [X_k - Z(\tau_k)]) \] (4.4)
With Equation (4.4) we have the following association
\[ F_\theta(X(i)) = \sum_{k=1}^{n} \frac{1}{n} \Phi(X(i) - [X_k - Z(\tau_k)]) \] (4.5)
for \( i = 1, \ldots, n \). Before proceeding we make the following two remarks, which motivates the PRS construction in the next section.

Remark 4.1 (Dependence of Latent Auxiliary Variables). Comparison of (4.1) and (4.5) reveals that (4.5) unsurprisingly depends on \( \Phi(Z_i) \)'s. This can be seen clearly, for example, by the extreme case where all the \( \theta \)'s are the same and, thereby, in this extreme case (4.1) and (4.5) are identical.

Remark 4.2 (Sampling W/O Replacement). Although \( F_\theta(\cdot) \) defined in (4.3) is the distribution function for all \( X_i \)'s, \( \{X(1), \ldots, X(n)\} \) is not a sample from \( F_\theta(\cdot) \). This is because that \( \{X(1), \ldots, X(n)\} \) is equivalent to the ordered sample of observations \( X_i \)'s taken from the \( n \) subpopulations \( N(\theta_i, 1) \) without replacement (of subpopulations), whereas a sample from \( F_\theta(\cdot) \), i.e., the \( n \) subpopulations \( N(\theta_i, 1) \), is necessarily drawn with replacement.
4.2 Point and interval estimations

The PRS defined in Section 3 for the sorted uniform samples can be used for sorted sample of \( \Phi(Z_i) \)'s. The remaining challenges concern a specification of a PRS for the latent permutation \( \tau \), which is to be utilized for inference through \( \Theta(B) \). Intuitively, for largest possible risk sharing, it is important to control the accuracy of predicting \( \tau \) using PRS. For the extreme case in this way of reasoning, we would ignore predicting \( Z(i) \)'s, at least not predicting directly, i.e., formally predicting \( Z(i) \)'s with their entire probability sample space. In other words, predicting \( \tau \) alone effectively estimates the collection of \( \theta \)'s. What is needed is to create a random bound \( B \) on \( B(F_\theta(X(1)), ..., F_\theta(X(n))) \), working in the same way as the basic IMs to induce a random set for \( \theta \)'s. Theoretically, valid inference can be made as long as we use a random bound \( B \) that is stochastically not smaller than \( B(F_\theta(X(1)), ..., F_\theta(X(n))) \). This is supported rigorously by the following theorem.

**Theorem 3** (Validity of conservative PRSs). Suppose that \( B \) is a random variable that is stochastically larger than \( B(F_\theta(X(1)), ..., F_\theta(X(n))) \) with \( X \) and \( \theta \) pairs given in \( 4.1 \). Then inference on \( \theta \) in the Inferential Models framework through the induced random set on the parameter space

\[
\Theta(B) = \{ \theta : \theta \in \mathbb{R}^n; B(F_\theta(X(1)), ..., F_\theta(X(n))) \leq B \}
\]

is valid.

*Proof.* This result can be proved by using arguments similar to those in the proof of Theorem 2 of Martin and Liu (2013); see also Martin (2015). \( \square \)

In the simulation study of Section 5, we use \( B = B(U(1), ..., U(n)) \), where \( U(1), ..., U(n) \) is the sorted \( n \) uniforms. A heuristic argument for this is that the sorted sample of \( X \)'s without replacement (of \( \theta \)'s spreads more evenly than that with replacement from the same marginal distribution \( 4.3 \). Together with \( 4.3 \), this implies that the sorted \( X \)s have about the same component-wise locations but smaller variability. As a result, \( B = B(U(1), ..., U(n)) \) serves as a random bound in Theorem 3 for producing valid inference on \( \theta \), at least for values of \( B(F_\theta(X(1)), ..., F_\theta(X(n))) \) that are not large; see Figure 2 for a pictorial illustration that supports this argument. Although it guarantees validity, the use of \( B = B(U(1), ..., U(n)) \) can lead to conservative inference. Since the distribution of \( B(F_\theta(X(1)), ..., F_\theta(X(n))) \) depends on \( \theta \)'s, efficient inference can be made by using the so-called assertion-specific PRS (Martin and Liu 2015a). More precisely, we summarize this approach into the following theorem, followed by remark on its potential applications and computational challenges.

**Theorem 4** (Validity of Assertion-specific PRSs for MPE and Constructing CIs). Suppose that the assertion \( \{ \theta_k(k) : k = 1, ..., n \} \in \mathbb{R}^n \) is of interest. Define the induced random set on \( \theta \)

\[
\Theta(X') = \{ \theta : \theta \in \mathbb{R}^n; B(F_\theta(X(1)), ..., F_\theta(X(n))) \leq B(F_{\theta'}(X'(1)), ..., F_{\theta'}(X'(n))) \}
\]

where \( X'_i \overset{iid}{\sim} N(\theta'_i, 1) \) for \( i = 1, ..., n \). Then inference based on \( \Theta(X') \) in the IMs framework is valid.

*Proof.* This result can also be proved by using arguments similar to those in the proof of Theorem 2 of Martin and Liu (2013). \( \square \)

**Remark 4.3.** According to Theorems 3 and 4, a way of constructing efficient confidence regions is to find the minimum plausibility for any given \( \theta \in \Theta(B) \). This minimum plausibility is the plausibility of \( \Theta(B) \). Evaluation of this plausibility appears intensive. It would be interesting to develop efficient computational methods for this approach.
The theory established so far provides inference about the $\theta$’s as a collection. Recall that given the $\theta$ collection, $X$’s were generated by drawing $\theta$’s without replacement. Standard algebraic operations of applying the Bayes theorem lead to the following results.

**Theorem 5** (Conditional Distribution of $\theta$). Suppose $n \geq 2$ and assume that the value of $\theta$, $\{\vartheta_k : \vartheta_k = \theta(k), k = 1, ..., n\}$, are known. Then the partial conditional distribution of $\theta_i$ given $X_i$ has the probability mass function (pmf)

$$Pr(\theta_i = \vartheta_j | X_i) = \frac{\phi(X_i - \vartheta_j)}{\sum_{k=1}^{n} \phi(X_i - \vartheta_k)}$$  \hspace{1cm} (4.8)

and the full conditional distribution of $\theta$ given $X$ has the pmf

$$Pr(\theta_1 = \vartheta_{\tau_1}, ..., \theta_n = \vartheta_{\tau_n} | X_1, ..., X_n) = \frac{1}{n!} \prod_{j=1}^{n} \phi(X_j - \vartheta_{\tau_j}) \sum_{\tau'} \frac{1}{n!} \prod_{j=1}^{n} \phi(X_j - \vartheta_{\tau'j})$$  \hspace{1cm} (4.9)

Theorem 5 makes it possible to estimate $\theta$’s via $E(\theta | X)$, fulfilling the required operation to overcome the fundamental difficulty of making inference for the classic kind (c.f., Section 1 and Stigler (1990)). While it is appealing to use the full conditional distribution (4.9) for MPE, for computational simplicity here we use the following simple estimator, obtained from (4.8),

$$\hat{\theta}_i = \frac{\sum_{k=1}^{n} \vartheta_k \phi(X_i - \vartheta_k)}{\sum_{k=1}^{n} \phi(X_i - \vartheta_k)}$$  \hspace{1cm} (4.10)

for $i = 1, ..., n$, with $\{\vartheta(k) : k = 1, ..., n\}$ obtained from the approach in either Theorem 3 or Theorem 4. The computational strategies for the empirical-Bayes kind provided in Appendices can be applied here for the classic kind. More discussion is given in Section 6 on partial conditioning versus full conditioning in Theorem 5 for estimation $\theta$.

5 Numerical illustration

5.1 A simulation study

In this simulation study, we consider the three scenarios used in the simulation study of Jiang and Liu (2022) for their many-normal-means example:

- **Single-Mode.** $\theta_i \overset{iid}{\sim} N(0, 0.01)$, the case in favor of James-Stein estimator;
- **Two-Mode.** $\theta_i \overset{iid}{\sim} \frac{1}{2} N(-2, 0.01) + \frac{1}{2} N(2, 0.01)$; and
- **Single-Mode with Outliers.** $\theta_i \overset{iid}{\sim} 0.9\delta_0 + 0.1N(-3, 1)$, where $\delta_0$ denotes the degenerated distribution at 0.

For all simulation studies, three different sample sizes, $n = 10, 20, \text{ and } 50$ are considered. The numerical deconvolution uses $K = 1000$ points equally spaced in the range from $X_{(1)} + \Phi^{-1}(0.0001)$ to from $X_{(n)} + \Phi^{-1}(0.9999)$, where $\Phi^{-1}(.)$ stands for the inverse mapping of the standard normal cdf $\Phi(.)$. For each case, $M = 200$ data sets are generated to compute the Monte Carlo approximations to, for example, mean square error (MSE).

The MSE performance results of different methods for point estimation are summarized into Table 1. The results for Efron’s $g$-modeling, the auto-modeling method of Jiang and Liu (2022), the method of maximum likelihood estimation (MLE), and the James-Stein shrinkage estimator
Table 1: MSE results in three simulation studies with different methods.

| Problem Kind | Method       | N(0, 0.01)         | \(\frac{1}{2}N(-2, 0.01) + \frac{1}{2}N(2, 0.01)\) | 0.96_0 + 0.1N(-3, 1) |
|--------------|--------------|--------------------|--------------------------------------------------|----------------------|
|              | n = 10       | n = 20             | n = 50                                           | n = 10               | n = 20             | n = 50               |
| **The EB Kind** | g-modeling   | 0.419              | 0.748                                            | 0.554                | 0.552              | 0.364                |
|              | Auto-modeling| 0.199              | 0.600                                            | 0.420                | 0.418              | 0.312                |
|              | IMs          | 0.219              | 0.678                                            | 0.462                | 0.370              | 0.361                |
| **Classic Kind** | MLE          | 1.022              | 0.990                                            | 0.971                | 1.021              | 0.983                |
|              | James-Stein  | 0.300              | 0.876                                            | 0.521                | 0.516              | 0.482                |
|              | IMs          | 0.312              | 0.719                                            | 0.484                | 0.376              | 0.317                |

Table 2: Actual coverage and average interval length (in parentheses) obtained by the IMs method for the empirical-Bayes kind in three simulation studies.

| Targeted Coverage | N(0, 0.01)         | \(\frac{1}{2}N(-2, 0.01) + \frac{1}{2}N(2, 0.01)\) | 0.96_0 + 0.1N(-3, 1) |
|-------------------|--------------------|--------------------------------------------------|----------------------|
|                   | n = 10             | n = 20                                           | n = 10               | n = 20             | n = 50               |
| MPE               | 83.77 (1.57)       | 91.58 (2.95)                                    | 96.94 (2.96)         | 86.09 (2.00)       | 93.84 (1.93)         |
| 50%               | 93.43 (2.14)       | 97.91 (4.34)                                    | 99.08 (3.42)         | 94.72 (2.67)       | 95.73 (2.36)         |
| 75%               | 97.56 (2.47)       | 98.56 (4.68)                                    | 99.23 (3.97)         | 96.06 (3.03)       | 97.01 (2.56)         |
| 90%               | 99.18 (2.84)       | 98.42 (4.89)                                    | 99.44 (4.50)         | 97.97 (3.43)       | 98.41 (2.79)         |
| 95%               | 99.50 (3.19)       | 98.71 (5.09)                                    | 99.29 (4.64)         | 98.28 (3.76)       | 98.54 (2.97)         |
Figure 1: A typical example of IMs for the empirical-Bayes kind in the two-modal case with $n = 20$, where X — $X$’s, curves — lower and upper $E(\theta_i|X)$, circles — true values of $\theta$’s, and vertical segments — confidence intervals computed at the MPE (set). The coefficients of the confidence intervals is about 97%, according to the simulation results given in Table 2.

Figure 2: Illustration of conservative PRSs in four different collections of $n = 20$ $\theta$’s: (1) one simulated values from the single mode model, (2) one simulated values from the two-mode model, and (3) one simulated values from the single mode model with outliers.

are taken from Jiang and Liu (2022). From these simulation-based numerical results, we see that IMs for the classic kind outperforms the James-Stein estimator. This is not surprising because of the difficulty recognized, for example, by Stigler (1990) and the sound reasoning with uncertainty introduced in this paper for the classic kind. Figure 1 displays the IMs results for a typical data set generated from the two-mode model. It is interesting to notice that IMs “shrinks” toward local modes, whereas the James-Stein estimates are of course linear function of $X$’s.

The results also show that our IMs for the empirical-Bayes kind performs better than Efron’s $g$-modeling method. Possible explanations may be that Efron’s $g$-modeling method is subject to a larger model error due to the $g$-modeling. The use of the method of (penalized) MLE for estimation
of the hyper-parameters introduced in the g-modeling could also be a contributing factor.

In addition, it is encouraging to observe that the IMs are comparable with the auto-modeling method of Jiang and Liu (2022). Good performance is expected for the IMs, although they may be further improved a bit when better PRSs are used. An intuitive explanation for the similar good performance for both the IMs and the auto-modeling method of Jiang and Liu (2022) is that both methods take it into consideration that generated data from the estimated model should look like the observed. The IMs formulated this paper does it through a PRS on \( F_\theta(X_i) \)'s, whereas auto-modeling explicitly makes this clear from a modeling perspective.

The results for interval estimation produced by the IMs for the empirical-Bayes kind are summarized into Table 2 for \( n = 10 \) and \( n = 20 \). The “Targeted Coverage” is the nominal coefficients, except for the row of \( MPE \). The values in the row of \( MPE \) are computed with respect to all \( \gamma \)'s in the maximum plausibility region. The confidence coefficient for \( \theta \)'s conditional on \( \gamma \)'s is set to be \( 1 - \alpha = 0.95^{1/2} = 0.9747 \). The plausibility coefficient \( 1 - \pi \) is taken to fulfill the nominal level, by making use of Theorem 2. We see that these intervals have more coverage than their nominal interval coefficients. This is somewhat expected because of the implicit use of “confidence” region for the high 1,000-dimensional \( \gamma \). More efficient intervals can be obtained via adaptive construction of shorter intervals; see Remark 3.4. This can be in fact seen clearly from able 2. This is also used in Section 5.2 for the real-data example.

5.2 A real-data example

| School (i) | Estimated Treatment-Effects (y_i) | Associated Standard-Error (s_i) |
|------------|----------------------------------|--------------------------------|
| 1          | 28.39                            | 14.9                           |
| 2          | 7.94                             | 10.2                           |
| 3          | -2.75                            | 16.3                           |
| 4          | 6.82                             | 11.0                           |
| 5          | -0.64                            | 9.4                            |
| 6          | 0.63                             | 11.4                           |
| 7          | 18.01                            | 10.4                           |
| 8          | 12.16                            | 17.6                           |

Rubin (1981) considered assessing effects of SAT coaching programs based on \( n = 8 \) parallel randomized experiments in 8 schools. The estimated individual effects and the associated standard deviations are tabulated in Table 3. It is arguable to use the following sampling model

\[
y_i^{\text{ind}} \sim N(\mu_i, s_i^2) \quad (i = 1, \ldots, n),
\]

because each \( y_i \) was obtained from a data set that is large enough for the acceptance of the normality assumption and the assumption that \( s_i^2 \)'s are known.

Let \( X_i = y_i/s_i \) and let \( \theta_i = \mu_i/s_i \) for \( i = 1, \ldots, 8 \). Inference about \( \mu_i \) from \( y_i \)'s via (5.1) can be obtained from inference about \( \theta_i \)'s from \( X_i \)'s based on the sampling model (1.1). In this case, the Efron-Morris shrinkage estimator over-shrinks \( \theta_i \)'s due to the fact that the sample variance of \( X_i \)'s is small, resulting a strong negative correlation between \( X_i \)'s and the shrinkage estimates of their
corresponding $\theta$s. As suggested in the literature, this can be of course easily modified in the same way of creating the James-Stein positive-part estimator (James and Stein, 1961).

![Figure 3](image_url)

**Figure 3:** The results for the real data example by the IMs for the empirical-Bayes kind, where $X$ — $X$s, curves — lower and upper $E(\theta_i|X)$, circles — Efron-Morris shrinkage estimates, and vertical segments — confidence intervals computed at the MPE (set). The coefficients of the confidence intervals is about 97%, suggested by an application of the method of adaptive adjustment; see Remark 3.4.

IMs results based on the empirical-Bayes kind are shown in Figure 3. Similar results can be extended to $\mu$'s because of the simple scaling transformation $\mu_i = s_i \theta_i$. The coefficients of the confidence intervals is about 97%, suggested by an application of the method of adaptive adjustment; see Remark 3.4. It is interesting to see that the MPEs of $\theta$'s are about the same. To verify this directly, consider the mean-removed statistic

\[ S^2 = X'(I - \frac{1}{n}J)X = 4.2899 \]

where $I$ is the $n \times n$ identity matrix and $J$ is the $n \times n$ matrix of ones. Note that the sampling distribution of $S^2$ is the non-central chi-square with 7 degrees of freedom and non-centrality parameter $\delta^2 = \theta'(I - \frac{1}{n}J)\theta$. The cdf value of $S^2$ under the central chi-square distribution is 0.2542. This doesn’t provide much strong evidence against the assertion that $\delta^2 = 0$ or, equivalently, all the $\theta$s are the same. So, it is not surprising to see that the MPEs shown in Figure 3 are about the same; see Leaf et al. (2015) for more discussion from an IMs perspective.

### 6 Discussion

Taking an IMs approach, we have investigated on making exact and efficient inference for the two kinds of many-normal-means. For the empirical Bayes kind, with IMs one can aim to producing efficient inference by balancing validity and flexibility. Since the IMs framework offers prior-free probabilistic inference, which says more than confidence, IMs plausibility can be used for point and interval estimations. In addition to the desired exactness, efficiency of the resulting methods
is satisfactory, as shown in the simulation study. Furthermore, in the case further \( g \)-modeling is preferred, the IMs results could be useful for model building inasmuch as valid non-parametric methods are useful for model checking; see Tukey (1962) for more discussion along this line.

It should be noted that IMs works well for the over-parameterized \( G \). This can be viewed as a good property of IMs, unlike frequentist and Bayesian schools of thought. Frequentist statisticians often do inference by first constructing point estimators and then develop confidence methods based on point estimators. For over-parameterized models, it is often that some kinds of regularization via penalty methods are used. Bayesians have to fully specify a probability model for everything, making it identifiable in a probabilistic sense with the usual theory of probability. With IMs, our way of reasoning is done with respect to the auxiliary variables. Non-identifiability is no longer an issue, because we work with set-valued functions/mappings.

There are also other kinds of the many-normal-means problem. A popular kind is obviously the so-called normal-normal empirical Bayes, where the unobserved \( \theta \)'s are assumed to have followed a normal distribution with unknown hyper-parameters. This model has been widely used in applied statistics; see, for example, Rubin (1980), Morris (1983), Gelman et al. (1995), Qiu et al. (2018), and references therein.

Our understanding of the classic kind from the IMs perspective motivated the new idea of introducing latent permutation auxiliary variable. While the proposed methods are satisfactory in the context of our numerical results for illustration and comparison, there are remaining theoretical challenges as far as optimal solutions are of interest. This is mainly due to the technical difficulties in dealing with data collected via sampling without replacement, whereas investigations of the empirical Bayes kind enjoys the relative simplicity offered by data collection through sampling with replacement. In particular, it would be interesting to develop efficient computational algorithms for evaluation of the full conditional expectation in Theorem 5. Nevertheless, it is expected that there is a large difference between partial and full conditional expectations for small \( n \) and not so for large \( n \).

The importance of the many-normal-means problem suggests immediately that our new methods can be developed for the many-normal-means type of problems, such as linear and non-parametric regression. These new methods can also be potentially extended to situations beyond normal models, including Poisson and Binomial models. Incidentally, the idea of considering latent associations would inspire new research on developments of the IMs theory itself as well as methods of other schools of thought. For example, an alternative to popular penalty methods for maximum likelihood estimation is to make use of the idea of using latent associations as a data-adaptive way of regularization, which has been more or less a standard method in contemporary high-dimensional statistics. We plan to work on these topics and report results elsewhere.

Appendices

A Maximum plausibility estimation

The MPE \( \hat{\theta}_i \) in (3.16) can be computed in the following three steps:

*Step 1.* Find \( B_{\text{min}} \) by solving the constrained optimization problem (3.13) with the constraints in (3.1) on \( \gamma \);

*Step 2.* Shift \( \gamma \) to the left-most and right-most positions within \( \Gamma_{1-\pi} \) as the starting values for the next step;
Step 3. Find $\hat{\theta}_i^{(\text{lower})}$ and $\hat{\theta}_i^{(\text{upper})}$ by solving the two corresponding constrained optimization problems with the objective function (3.10) and the constraints in both (3.1) and (3.14).

These constrained optimization problems are known as nonlinear programming. Standard numerical methods for such problems are widely available, including the R package `nloptr` by (Johnson, 2022, and coauthors).

The algorithms of Svanberg (2002), Conn et al. (1991), and Birgin and Martinez (2008), made available as options NLOPT_LN_MMA and NLOPT_LN_AUGLAG of the `nloptr` function in Johnson (2022), are used for the simulation study in Section 5. These algorithms performed well with starting values for Step 1 set as follows:

**Step 1.** Set $\gamma_k = 1/K$ for all $k = 1, \ldots, K$.

**Step 2.** For each $i = 1, \ldots, n$, increase $\gamma_k$ by $1/n$, where $k = \min_{j \in \{1, \ldots, K\}} |\theta_j - X_i|$, followed by the renormalization $\gamma_k = \gamma_k / \sum_{j=1}^K \gamma_j$. To be conservative, we also used different starting values set as above but for resampled $X_i$’s with replacement. No significant improvements could be found in the simulation study.

**B Plausibility intervals**

The confidence intervals can be computed as plausibility intervals in the following three steps:

**Step 1.** Find $B_{1-\pi}$ in (3.18);

**Step 2.** Apply the methods in Appendix A to find the extreme $\gamma$ values for computing extreme means.

**Step 3.** Take the extreme $\gamma$ values computed in Step 2 as starting values, and find the end points (3.19) and (3.20) of the confidence interval.

These constrained optimization problems are known as nonlinear programming. Standard numerical methods for such problems are widely available, including the R package `nloptr` by (Johnson, 2022, and coauthors). The algorithms of Svanberg (2002) and Runarsson and Yao (2005), made available as options NLOPT_LN_MMA and NLOPT_LN_ISRES of the `nloptr` function in Johnson (2022), are used for the simulation study in Section 5.

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