Quintom Cosmology: theoretical implications and observations

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Abstract

We review the paradigm of quintom cosmology. This scenario is motivated by the observational indications that the equation of state of dark energy across the cosmological constant boundary is mildly favored, although the data are still far from being conclusive. As a theoretical setup we introduce a no-go theorem existing in quintom cosmology, and based on it we discuss the conditions for the equation of state of dark energy realizing the quintom scenario. The simplest quintom model can be achieved by introducing two scalar fields with one being quintessence and the other phantom. Based on the double-field quintom model we perform a detailed analysis of dark energy perturbations and we discuss their effects on current observations. This type of scenarios usually suffer from a manifest problem due to the existence of a ghost degree of freedom, and thus we review various alternative realizations of the quintom paradigm. The developments in particle physics and string theory provide potential clues indicating that a quintom scenario may be obtained from scalar systems with higher derivative terms, as well as from non-scalar systems. Additionally, we construct a quintom realization in the framework of braneworld cosmology, where the cosmic acceleration and the phantom divide crossing result from the combined effects of the field evolution on the brane and the competition between four and five dimensional gravity. Finally, we study the outsets and fates of a universe in quintom cosmology. In a scenario with null energy condition violation one may obtain a bouncing solution at early times and therefore avoid the Big Bang singularity. Furthermore, if this occurs periodically, we obtain a realization of an oscillating universe. Lastly, we comment on several open issues in quintom cosmology and their connection to future investigations.
Key words: Dark energy, Quintom scenario, Null energy condition, Bouncing cosmology.
PACS: 95.36.+x, 98.80.-k

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References
1 Introduction

Accompanied by the recent developments on distance detection techniques, such as balloons, telescopes and satellites, our knowledge about cosmology has been greatly enriched. These new discoveries in astrophysical experiments have brought many challenges for the current theory of cosmology. The most distinguished event is that two independent observational signals on distant Type Ia supernovae (SNIa) in 1998 have revealed the speeding up of our universe [1,2]. This acceleration implies that if the theory of Einstein’s gravity is reliable on cosmological scales, then our universe is dominated by a mysterious form of matter. This unknown component possesses some remarkable features, for instance it is not clustered on large length scales and its pressure must be negative in order to be able to drive the current acceleration of the universe. This matter content is called “dark energy” (DE). Observations show that the energy density of DE occupies about 70% of today’s universe. However, at early cosmological epochs DE could not have dominated since it would have destroyed the formation of the observed large scale structure. These features have significantly challenged our thoughts about Nature. People begin to ask questions like: What is the constitution of DE? Why it dominates the evolution of our universe today? What is the relation among DE, dark matter and particle physics, which is successfully constructed?

The simplest solution to the above questions is a cosmological constant $\Lambda$ [3,4,5,6]. As required by observations, the energy density of this constant has to be $\rho_\Lambda \sim (10^{-3} eV)^4$, which seems un-physically small comparing to other physical constants in Einstein’s gravity. At the classical level this value does not suffer from any problems and we can measure it with progressively higher accuracy by accumulated observational data. However, questioning about the origin of a cosmological constant, given by the energy stored in the vacuum, does not lead to a reasonable answer. Since in particle physics the vacuum-energy is associated with phase transitions and symmetry breaking, the vacuum of quantum electrodynamics for instance implies a $\rho_\Lambda$ about 120 orders of magnitude larger than what has been observed. This is the worst fine-tuning problem of physics.

Since the fundamental theory of nature that could explain the microscopic physics of DE is unknown at present, phenomenologists take delight in constructing various models based on its macroscopic behavior. There have been a number of review articles on theoretical developments and phenomenological studies of dark energy and acceleration and here we would like to refer to Refs. [7,8,9,10,11,12,13,14] as the background for the current paper. Note that, the most powerful quantity of DE is its equation of state (EoS) effectively defined as $w_{DE} \equiv p_{DE}/\rho_{DE}$, where $p_{DE}$ and $\rho_{DE}$ are the pressure and energy density respectively. If we restrict ourselves in four dimensional Einstein’s gravity, nearly all DE models can be classified by the behaviors of equations of state as following:

- **Cosmological constant**: its EoS is exactly equal to $w_\Lambda = -1$.
- **Quintessence**: its EoS remains above the cosmological constant boundary, that is $w_Q \geq -1$ [15,16].
- **Phantom**: its EoS lies below the cosmological constant boundary, that is $w_P \leq -1$ [17,18].
- **Quintom**: its EoS is able to evolve across the cosmological constant boundary [19].
With the accumulated observational data, such as SNIa, Wilkinson Microwave Anisotropy Probe observations (WMAP), Sloan Digital Sky Survey (SDSS) and forthcoming Planck etc., it becomes possible in the recent and coming years to probe the dynamics of DE by using parameterizations of its EoS, constraining the corresponding models. Although the recent data-fits show a remarkable agreement with the cosmological constant and the general belief is that the data are far from being conclusive, it is worth noting that some data analyses suggest the cosmological constant boundary (or phantom divide) is crossed\cite{19,20}, which corresponds to a class of dynamical models with EoS across $-1$, dubbed \textit{quintom}. This potential experimental signature introduced an additional big challenge to theoretical cosmology. As far as we know all consistent theories in physics satisfy the so-called Null Energy Condition (NEC), which requires the EoS of normal matter not to be smaller than the cosmological constant boundary, otherwise the theory might be unstable and unbounded. Therefore, the construction of the quintom paradigm is a very hard task theoretically. As first pointed out in Ref. \cite{19}, and later proven in Ref. \cite{21} (see also Refs. \cite{22,23,24,25,26}), for a single fluid or a single scalar field with a generic lagrangian of form $\mathcal{L}(\phi, \partial_{\mu}\phi \partial^{\mu}\phi)$ there exists in general a no-go theorem forbidding the EoS crossing over the cosmological constant. Hence, at this level, a quintom scenario of DE is designed to enlighten the nature of NEC violation. Due to this unique feature, quintom cosmology differs from any other paradigm in the determination of the cosmological evolution and the fate of the universe.

This review is primarily intended to present the current status of research on quintom cosmology, including theoretical constructions of quintom models, its perturbation theory and predictions on observations. Moreover, we include the discussions about quintom cosmology and NEC, in order to make the nature of DE more transparent. Finally, we examine the application of quintom in the early universe, which leads to a nonsingular bouncing solution.

This work is organized as follows. In Section 2 we begin with the basics of Friedmann-Robertson-Walker cosmology and we introduce a concordant model of $\Lambda$CDM, referring briefly to scenarios beyond $\Lambda$CDM. In Section 3 we present the theoretical setup of quintom cosmology and we discuss the conditions for the DE EoS crossing $-1$. In Section 4 we introduce the simplest quintom scenario, which involves two scalar fields, and we extract its basic properties. Section 5 is devoted to the discussion of the perturbation theory in quintom cosmology and to the examination of its potential signatures on cosmological observations. Due to the existence in quintom cosmology of a degree-of-freedom violating NEC, the aforementioned simplest model often suffers from a quantum instability inherited from phantom behavior. Therefore, in Section 6 we extend to a class of quintom models involving higher derivative terms, since these constructions might be inspired by fundamental theories such as string theory. In Section 7 we present the constructions of quintom behavior in non-scalar models, such are cosmological systems driven by a spinor or vector field, while in Section 8 we turn to the discussion of quintom-realizations in modified (or extended) Einstein’s gravity. In Section 9 we examine the violation of NEC in quintom cosmology. Additionally, we apply it to the early universe, obtaining a nonsingular bouncing solution in four dimensional Einstein’s gravity, and we further give an example of an exactly cyclic solution in quintom cosmology which is completely free of spacetime singularities. Finally, in Section 10 we conclude, summarize and outline future prospects of quintom cosmology. Lastly, we finish our work by addressing some unsettled issues of quintom cosmology. Throughout the review we use the normalization of natural units $c = \hbar = 1$ and we define $\kappa^2 = 8\pi G = M_p^{-2}$. 
2 Basic cosmology

Modern cosmology is based on the assumptions of large-scale homogeneity and isotropy of the universe, associated with the assumption of Einstein’s general relativity validity on cosmological scales. In this section we briefly report on current observational status of our universe and in particular of DE, and we review the elements of FRW cosmology.

2.1 Observational evidence for DE

With the accumulation of observational data from Cosmic Microwave Background measurements (CMB), Large Scale Structure surveys (LSS) and Supernovae observations, and the improvement of the data quality, the cosmological observations play a crucial role in our understanding of the universe. In this subsection, we briefly review the observational evidence for DE.

2.1.1 Supernovae Ia

In 1998 two groups \[1,2\] independently discovered the accelerating expansion of our current universe, based on the analysis of SNIa observations of the redshift-distance relations. The luminosity distance \(d_L\), which is very important in astronomy, is related to the apparent magnitude \(m\) of the source with an absolute magnitude \(M\) through

\[
\mu \equiv m - M = 5 \log_{10} \left( \frac{d_L}{\text{Mpc}} \right) + 25,
\]

where \(\mu\) is the distance module. In flat FRW universe the luminosity distance is given by:

\[
d_L = \frac{1 + z}{H_0} \int_0^z \frac{dz'}{E(z')},
\]

with

\[
E(z) \equiv \frac{H(z)}{H_0} = \left\{ \Omega_{m0}(1 + z)^3 + \Omega_{r0}(1 + z)^4 + \Omega_{\Lambda0} \exp \left[ 3 \int_0^z \frac{1 + w(z')}{1 + z'} dz' \right] \right\}^{1/2},
\]

where \(H_0\) is the Hubble constant, \(w\) is the EoS of DE component, \(z\) is the redshift, and \(\Omega_i0 = 8\pi G \rho_i0/(3H_0^2)\) is the density parameter of each component (with \(\Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda0} = 1\), i.e its energy density divided by the critical density at the present epoch.

In 1998 both the the high-z supernova team (HSST) \[1\] and the supernova cosmology project (SCP) \[2\], had found that distant SNIa are dimmer than they would be in dark-matter dominated, decelerating universe. When assuming the flat FRW cosmology (\(\Omega_{m0} + \Omega_{\Lambda0} = 1\)), Perlmutter \textit{et al.} found that at present the energy density of dark-matter component is:

\[
\Omega_{m0}^{\text{flat}} = 0.28^{+0.09}_{-0.08} \ (1\sigma \text{ statistical}) \quad 0.05^{+0.05}_{-0.04} \ (\text{identified systematics}),
\]
based on the analysis of 42 SNIa at redshifts between 0.18 and 0.83 [2]. In Fig. 1 we present the Hubble diagram of SNIa measured by SCP and HSST groups, together with the theoretical curves of three cosmological models. One can observe that the data are inconsistent with a $\Lambda = 0$ flat universe and strongly support a nonzero and positive cosmological constant at greater than 99% confidence. See Ref. [28,29,30,31,32,33] for recent progress on the SNIa data analysis.

2.1.2 CMB and LSS

Besides the SNIa observations, the CMB and LSS measurements also provide evidences for a dark-energy dominated universe. The recent WMAP data [34] are in good agreement with a Gaussian, adiabatic, and scale-invariant primordial spectrum, which are consistent with single-field slow-roll inflation predictions. Moreover, the positions and amplitudes of the acoustic peaks indicate that the universe is spatially flat, with $-0.0179 < \Omega_{K0} < 0.0081$ (95%CL).

Under the assumption of DE EoS being $w \equiv -1$, the so-called $\Lambda$CDM model, the 5-year WMAP
data, when they combine with the distance measurements from the SNIa and the Baryon Acoustic Oscillations (BAO) in the distribution of galaxies, imply that the DE density parameter is \( \Omega_{\Lambda 0} = 0.726 \pm 0.015 \) at present. In Fig. 2 we illustrate the two dimensional contour plots of the vacuum energy density \( \Omega_\Lambda \) and the spatial curvature parameter \( \Omega_K \). One can see that the flat universe without the cosmological constant is obviously ruled out. Furthermore, the LSS data of the SDSS [35,36,37] agree with the WMAP data in favoring the flat universe dominated by the DE component.

### 2.1.3 The age of the universe

An additional evidence arises from the comparison of the age of the universe with other age-independent estimations of the oldest stellar objects. In flat FRW cosmology the age of the universe is given by:

\[
t_u = \int_0^{t_u} \frac{dt'}{H(1+z)} = \int_0^\infty \frac{dz}{H_0(1+z)E(z)}.
\]

(2.5)

In the dark-matter dominated flat universe \( (\Omega_K = 0, \Omega_{m0} = 1) \), relation (2.5) leads to a universe-age:

\[
t_u = \int_0^\infty \frac{dz}{H_0(1+z)\sqrt{\Omega_{m0}(1+z)^3}} = \frac{2}{3H_0}.
\]

(2.6)

We have neglected the radiation contribution, setting \( \Omega_{\gamma0} = 0 \), since the radiation dominated period is much shorter than the total age of the universe. Combining with the present constraints on the Hubble constant from the Hubble Space Telescope Key project (HST) [38], \( h = 0.72 \pm 0.08 \), the age of universe is 7.4Gyr < \( t_u < 10.1 \)Gyr.

On the other hand, many groups have independently measured the oldest stellar objects and have extracted the corresponding constraints the age of the universe: 11Gyr < \( t_u < 15 \)Gyr.
Fig. 3. (Color online) The age of the universe $t_u$ (in units of $H_0^{-1}$) versus $\Omega_m$. From Ref. [9].

Moreover, the present 5-year WMAP data provide the age-limit of $t_u = 13.72 \pm 0.12$ Gyr, when assuming the $\Lambda$CDM model. Thus, one can conclude that the age of a flat universe dominated by dark matter is inconsistent with these age limits.

The above contradiction can be easily resolved by the flat universe with a cosmological constant. Including the DE component, the age of the universe reads:

$$t_u = \int_0^\infty \frac{dz}{H_0(1+z)\sqrt{\Omega_m(1+z)^3 + \Omega_{\Lambda 0}}}.$$ (2.7)

Therefore, the age of universe will increase when $\Omega_{\Lambda 0}$ becomes large. In Fig 3 we depict the universe age $t_u$ (in units of $H_0^{-1}$) versus $\Omega_m$. In this case, $t_u = 1/H_0 \approx 13.58$ Gyr when $\Omega_m = 0.26$ and $\Omega_{\Lambda 0} = 0.74$. As we observe, the age limits from the oldest stars are safely satisfied.

2.2 A concordant model: $\Lambda$CDM

From observations such as the large scale distribution of galaxies and the CMB near-uniform temperature, we have deduced that our universe is nearly homogeneous and isotropic. Within this assumption we can express the background metric in the FRW form,

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2d\Omega_r^2 \right].$$ (2.8)
where $t$ is the cosmic time, $r$ is the spatial radius coordinate, $\Omega_2$ is the 2-dimensional unit sphere volume, and the quantity $k$ characterizes the curvature of 3-dimensional space of which $k = -1, 0, 1$ corresponds to open, flat and closed universe respectively. Finally, as usual, $a(t)$ is the scale factor, describing the expansion of the universe.

A concordant model only involves radiation $\rho_r$, baryon matter $\rho_b$, cold dark matter $\rho_{dm}$ and a cosmological constant $\Lambda$. Thus, this class consists the so-called $\Lambda$CDM models. In the frame of standard Einstein’s gravity, the background evolution is determined by the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\left(\rho_r + \rho_b + \rho_{dm} + \rho_k + \rho_\Lambda\right),$$

(2.9)

where $H \equiv \frac{\dot{a}}{a}$ is the Hubble parameter, and we have defined the effective energy densities of spatial curvature and cosmological constant $\rho_k \equiv -\frac{3k}{8\pi Ga^2}$ and $\rho_\Lambda = \frac{\Lambda}{8\pi G}$, respectively. Furthermore, the continuity equations for the various matter components write as

$$\dot{\rho}_i + 3H(1 + w_i)\rho_i = 0,$$

(2.10)

where the equation-of-state parameters $w_i \equiv \frac{p_i}{\rho_i}$ are defined as the ratio of pressure to energy density. In particular, they read $w_r = \frac{1}{3}$ for radiation, $w_b = 0$ for baryon matter, $w_{dm} = 0$ for cold dark matter, $w_k = -\frac{1}{3}$ for spatial curvature, and $w_\Lambda = -1$ for cosmological constant. One can generalize the EoS of the $i$-th component as a function of the redshift $w_i(z)$, with the redshift given as $1 + z = \frac{1}{a}$. Therefore, the evolutions of the various energy densities are given by

$$\rho_i = \rho_{i0} \exp\left\{3\int_0^z [1 + w_i(\tilde{z})]d\ln(1 + \tilde{z})\right\}.$$

(2.11)

From observations we deduce that $\Lambda$ is of the order of the present Hubble parameter $H_0$, of which the energy density is given by

$$\rho_\Lambda \sim 10^{-47}\text{GeV}^4.$$  

(2.12)

This provides a critical energy scale

$$M_\Lambda \sim \rho_\Lambda^{\frac{1}{4}} \sim 10^{-3}\text{eV}.$$  

(2.13)

As far as we know, this energy scale is far below any cut-off or symmetry-breaking scales in quantum field theory. Therefore, if the cosmological constant originates from a vacuum energy density in quantum field theory, we need to find another constant to cancel this vacuum energy density but leave the rest slightly deviated from vanishing. This is a well-known fine-tuning problem in $\Lambda$CDM cosmology (for example see Ref. [43] for a comprehensive discussion).

Attempts on alleviating the fine-tuning problem of $\Lambda$CDM have been intensively addressed in the frame of string theory, namely, Bousso-Polchinski (BP) mechanism [44], Kachru-Kallosh-Linde-Trivedi (KKLT) scenario [45], and anthropic selection in multiverse [46].
2.3 Beyond $\Lambda$CDM

DE scenario constructed by a cosmological constant corresponds to a perfect fluid with EoS $w_\Lambda = -1$. Phenomenologically, one can construct a model of DE with a dynamical component, such as the quintessence, phantom, K-essence, or quintom. With accumulated astronomical data of higher precision, it becomes possible in recent years to probe the current and even the early behavior of DE, by using parameterizations for its EoS, and additionally to constrain its dynamical behavior. In particular, the new released 5-year WMAP data have given the most precise probe on the CMB radiations so far. The recent data-fits of the combination of 5-year WMAP with other cosmological observational data, remarkably show the consistency of the cosmological constant scenario. However, it is worth noting that dynamical DE models are still allowed, and especially the subclass of dynamical models with EoS across $-1$.

Quintessence is regarded as a DE scenario with EoS larger than $-1$. One can use a canonical scalar field to construct such models, which action is written as [15,17,48,49],

$$S_Q = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right].$$

By varying the action with respect to the metric, one gets the energy density and pressure of quintessence,

$$\rho_Q = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_Q = \frac{1}{2} \dot{\phi}^2 - V(\phi),$$

and correspondingly the EoS is given by

$$w_Q = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V}. \quad (2.16)$$

However, recent observation data indicate that the contour of the EoS for DE includes a regime with $w < -1$. The simplest scenario extending into this regime uses a scalar field with a negative kinetic term, which is also referred to be a ghost [17]. Its action takes the form

$$S_P = \int d^4x \sqrt{-g} \left[ - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right],$$

and thus the DE EoS is

$$w_P = \frac{\dot{\phi}^2 + 2V}{\dot{\phi}^2 - 2V}. \quad (2.18)$$

Although this model can realize the EoS to be below the cosmological constant boundary $w = -1$, it suffers from the problem of quantum instability with its energy state unbounded from below [50,51]. Moreover, if there is no maximal value of its potential, this scenario is even unstable at the classical level, which is referred as a Big Rip singularity [18].

Another class of dynamical DE model is K-essence [52,53,54], with its Lagrangian $P$ being a general function of the kinetic term [55,56]. One can define the kinetic variable $X \equiv \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$,
obtaining
\[ S_K = \int d^4x \sqrt{-g} P(\phi, X) , \quad (2.19) \]
and the DE energy density and pressure are given by,
\[ \rho_K = 2X \frac{\partial P}{\partial X} - P , \quad p = P(\phi, X) . \quad (2.20) \]
Consequently, its EoS is expressed as
\[ w_K = -1 + \frac{2XP_x}{2XP_x - P} , \quad (2.21) \]
where the subscript ",X" denotes the derivative with respect to X. By requiring a positive energy density, this model can realize either \( w > -1 \) or \( w < -1 \), but cannot provide a consistent realization of the \(-1\)-crossing, as will be discussed in the next section.

Regarding the above three classes of DE scenarios together with the cosmological constant model, there is a common question needed to be answered, namely why the universe enters a period of cosmic acceleration around today, in which the DE energy density is still comparable to that of the rest components. In particular, if its domination was stronger than observed then the cosmic acceleration would begin earlier and thus large-scale structures such as galaxies would never have the time to form. Only a few dynamical DE models present the so-called tracker behavior\[49,57,9\], which avoids this coincidence problem. In these models, DE presents an energy density which initially closely tracks the radiation energy density until matter-radiation equality, it then tracks the dark matter density until recently, and finally it behaves as the observed DE.

### 2.4 Observational Evidence for Quintom DE Scenario

In this subsection we briefly review the recent observational evidence that mildly favor the quintom DE scenario.

In 2004, with the accumulation of supernovae Ia data, the time variation of DE EoS was allowed to be constrained. In Ref.\[20\] the authors produced uncorrelated and nearly model independent band power estimates (basing on the principal component analysis\[58\]) of the EoS of DE and its density as a function of redshift, by fitting to the SNIa data. They found marginal (2\(\sigma\)) evidence for \( w(z) < -1 \) at \( z < 0.2 \), which is consistent with other results in the literature\[59,60,61,62,63,64\].

This result implied that the EoS of DE could vary with time. Two type of parameterizations for \( w_{DE} \) are usually considered. One form (Model A) is:
\[ w_{DE} = w_0 + w'z , \quad (2.22) \]
where \( w_0 \) is the EoS at present and \( w' \) characterizes the running of \( w_{DE} \). However, this parametrization is only valid in the low redshift, suffering from a severe divergence problem at high redshift, such as the last scattering surface \( z \sim 1100 \). Therefore, an alternative form (Model B) was
proposed by Ref. [65,66]:

\[ w_{\text{DE}} = w_0 + w_1 (1 - a) = w_0 + w_1 \frac{z}{1 + z}, \]  

(2.23)

where \( a \) is the scale factor and \( w_1 = -\frac{dw}{da} \). This parametrization exhibits a very good behavior at high redshifts.

In Ref. [19] the authors used the “gold” sample of 157 SNIa, the low limit of cosmic ages and the HST prior, as well as the uniform weak prior on \( \Omega_M h^2 \), to constrain the free parameters of the aforementioned two DE parameterizations. As shown in Fig.4 they found that the data seem to favor an evolving DE with EoS being below \(-1\) at present, evolved from \( w > -1 \) in the past. The best fit value of the current EoS is \( w_0 < -1 \), with its running being larger than 0.

Apart from the SNIa data, CMB and LSS data can be also used to study the variation of DE EoS. In Ref. [67], the authors used the first year WMAP, SDSS and 2dFGRS data to constrain the DE models, and they found that the data evidently favor a strongly time-dependent \( w_{\text{DE}} \) at present, which is consistent with other results in the literature [68,69,70,71,72,73,74,75,76,77]. As observed in Fig.5 using the latest 5-year WMAP data, combined with SNIa and BAO data, the constraints on the DE parameters of Model B are: \( w_0 = -1.06 \pm 0.14 \) and \( w_1 = 0.36 \pm 0.62 \) [34,78,79]. Thus, one deduces that current observational data mildly favor \( w_{\text{DE}} \) crossing the phantom divide during the evolution of universe. However, the \( \Lambda \)CDM model still fits the data.
in great agreement.

3 Quintom cosmology: Theoretical basics

The scenario that the EoS of DE crosses the cosmological constant boundary is referred as a “Quintom” scenario. Its appearance has brought another question, namely why does the universe enter a period of cosmic super-acceleration just today. The discussion of this second coincidence problem has been carried out extensively in a number of works [19,80,81,82]. However, the explicit construction of Quintom scenario is more difficult than other dynamical DE models, due to a no-go theorem.

3.1 A No-Go theorem

In this subsection, we proceed to the detailed presentation and proof of the “No-Go” theorem which forbids the EoS parameter of a single perfect fluid or a single scalar field to cross the \(-1\) boundary. This theorem states that in a DE theory described by a single perfect fluid, or a single scalar field \(\phi\) minimally coupled to Einstein Gravity with a lagrangian \(\mathcal{L} = P(\phi, X)\), in FRW geometry, the DE EoS \(w\) cannot cross over the cosmological constant boundary. It has been proved or discussed in various approaches in the literature [21,22,23,24,25,26], and in the following we would desire to present a proof from the viewpoint of stability of DE perturbations.
Let us first consider the fluid-case. Generally, a perfect fluid, without viscosity and conduct heat, can be described by parameters such as pressure $p$, energy density $\rho$ and entropy $S$, satisfying a general EoS $p = p(\rho, S)$. According to the fluid properties, such a perfect fluid can be classified into two classes, namely barotropic and non-barotropic ones.

Working in the conformal Newtonian gauge\textsuperscript{1}, one can describe the DE perturbations in Fourier space as

\begin{equation}
\delta' = -(1 + w)(\theta - 3\Phi') - 3\mathcal{H}(c_s^2 - w)\delta, \tag{3.1}
\end{equation}

\begin{equation}
\theta' = -\mathcal{H}(1 - 3w)\theta - \frac{w'}{1 + w}\theta + k^2 \left( \frac{c_s^2\delta}{1 + w} + \Psi \right), \tag{3.2}
\end{equation}

where the prime denotes the derivative with respect to conformal time defined as $\eta \equiv \int dt/a$, $\mathcal{H}$ is the conformal Hubble parameter, $c_s^2 \equiv \delta p/\delta \rho$ is the sound speed and

\begin{equation}
\delta \equiv \delta \rho/\rho, \quad \theta \equiv \frac{ik^j\delta T^0_j}{\rho + p}, \tag{3.3}
\end{equation}

are the relative density and velocity perturbations respectively. Finally, as usual the variables $\Phi$ and $\Psi$ represent metric perturbations of scalar type.

If the fluid is barotropic, the iso-pressure surface is identical with the iso-density surface, thus the pressure depends only on its density, namely $p = p(\rho)$. The adiabatic sound speed is determined by

\begin{equation}
c_a^2 \equiv c_{s|\text{adiabatic}}^2 = \frac{p'}{\rho'} = w - \frac{w'}{3\mathcal{H}(1 + w)}. \tag{3.4}
\end{equation}

From this relation we can see that the sound speed of a single perfect fluid is apparently divergent when $w$ crosses $-1$, which leads to instability in DE perturbation.

If the fluid is non-barotropic, the pressure depends generally on both its density and entropy, $p = p(\rho, S)$. The simple form of the sound speed given in Eq.\textsuperscript{(3.3)} is not well-defined. From the fundamental definition of the sound speed, that is taking gravitational gauge invariance into consideration, we can obtain a more general relationship between the pressure and the energy density as follows:

\begin{equation}
\delta \dot{\rho} = \hat{c}_s^2 \delta \dot{\rho}, \tag{3.5}
\end{equation}

with the hat denoting gauge invariance. A gauge-invariant form of density fluctuation can be written as

\textsuperscript{1} We refer to Ref. \textsuperscript{83} for a comprehensive study on cosmological perturbation theory.
\[ \delta \rho = \delta \rho + 3 \mathcal{H} (\rho + p) \frac{\theta}{k^2}, \]  
(3.6)
and correspondingly a gauge-invariant perturbation of pressure writes

\[ \delta \hat{p} = \delta \rho + 3 \mathcal{H} c_a^2 (\rho + p) \frac{\theta}{k^2}. \]  
(3.7)

Finally, the gauge-invariant intrinsic entropy perturbation \( \Gamma \) can be described as

\[ \Gamma = \frac{1}{w \rho}(\delta \rho - c_a^2 \delta \rho) = \frac{1}{w \rho} \left( \delta \hat{p} - c_a^2 \delta \rho \right). \]  
(3.8)

Combining Eqs. (3.5)-(3.7), we obtain the following expression,

\[ \delta \rho = \hat{c}_a^2 \delta \rho + \frac{3 \mathcal{H}\rho \theta(1 + w)(\hat{c}_a^2 - c_a^2)}{k^2} \]
\[ = \hat{c}_a^2 \delta \rho + \hat{c}_a^2 \frac{3 \mathcal{H}\rho \theta(1 + w)}{k^2} - \frac{3 \mathcal{H}\rho(1 + w)}{k^2} \rho \theta w' + \frac{\rho \theta w'}{k^2}. \]  
(3.9)

From Eq. (3.3), one can see that \( \theta \) will be divergent when \( w \) crosses \(-1\), unless it satisfies the condition

\[ \theta w' = k^2 \frac{\delta p}{\rho}. \]  
(3.10)

Substituting the definition of adiabatic sound speed \( c_a^2 \) (relation (3.4)) and the condition (3.10) into (3.7), we obtain \( \delta \hat{p} = 0 \). Since \( \Gamma = \frac{1}{w \rho}(\delta \rho - c_a^2 \delta \hat{p}) \), it is obvious that due to the divergence of \( c_a^2 \) at the crossing point, we have to require \( \delta \rho = 0 \) to maintain a finite \( \Gamma \). Thus, we deduce the last possibility, that is \( \delta \hat{p} = 0 \) and \( \delta \hat{\rho} = 0 \). From relations (3.6), (3.7) this case requires that \( \delta p = -c_a^2 \frac{3 \mathcal{H}\rho \theta(1 + w)}{k^2} \) and \( \delta \rho = -\frac{3 \mathcal{H}\rho(1 + w)}{k^2} \), and thus \( \delta \rho = c_a^2 \delta \rho \). Therefore, it returns to the case of adiabatic perturbation, which is divergent as mentioned above.

In conclusion, from classical stability analysis we demonstrated that there is no possibility for a single perfect fluid to realize \( w \) crossing \(-1\). For other proofs, see Refs. [23,26].

Let us now discuss the case of a general single scalar field. The analysis is an extension of the discussion in Ref. [25]. The action of the field is given by

\[ S = \int d^4 x \sqrt{-g} P(\phi, X), \]  
(3.11)

where \( g \) is the determinant of the metric \( g_{\mu \nu} \). In order to study the DE EoS we first write down its energy-momentum tensor. By definition \( \delta g_{\mu \nu} S = -\int d^4 x \sqrt{-g} T^{\mu \nu} \delta g_{\mu \nu} \) and comparing with the fluid definition in the FRW universe one obtains:
\[ p = -T_i^i = \mathcal{L}, \tag{3.12} \]
\[ \rho = T_0^0 = 2X_{p,X} - p, \tag{3.13} \]

where \( \cdot_{\text{X}} \) stands for \( \frac{\partial}{\partial \text{X}} \). Using the formulae above, the EoS is given by

\[ w = \frac{p}{\rho} = \frac{p}{2X_{p,X} - p} = -1 + \frac{2X_{p,X}}{2X_{p,X} - p}. \tag{3.14} \]

This means that, at the crossing point \( t^* \), \( X_{p,X}|_{t^*} = 0 \). Since \( w \) needs to cross \(-1\), it is required that \( X_{p,X} \) changes sign before and after the crossing point. That is, in the neighborhood of \( t^* \), \( (t^* - \epsilon, t^* + \epsilon) \), we have

\[ X_{p,X}|_{t^* - \epsilon} \cdot X_{p,X}|_{t^* + \epsilon} < 0. \tag{3.15} \]

Since \( X = \frac{1}{2}\dot{\phi}^2 \) is non-negative, relation (3.15) can be simplified as \( p_{,X}|_{t^* - \epsilon} \cdot p_{,X}|_{t^* + \epsilon} < 0 \). Thus, due to the continuity of perturbation during the crossing epoch, we acquire \( p_{,X}|_{t^*} = 0 \).

Let us now consider the perturbations of the field. We calculate the perturbation equation with respect to conformal time \( \eta \) as

\[ u'' - c_s^2 \nabla^2 u - \left[ \frac{z''}{z} + 3c_s^2(\mathcal{H}' - \mathcal{H}^2) \right] u = 0, \tag{3.16} \]

where we have defined

\[ u \equiv a z \frac{\delta \phi}{\phi}, \quad z \equiv \sqrt{\phi'^2|\rho_{,X}|}. \tag{3.17} \]

A Fourier expansion of the perturbation function \( u \) leads to the dispersion relation:

\[ \omega^2 = c_s^2 k^2 - \frac{z''}{z} - 3c_s^2(\mathcal{H}' - \mathcal{H}^2), \tag{3.18} \]

with \( c_s^2 \) defined as \( \frac{p_{,X}}{\rho_{,X}} \). Stability requires \( c_s^2 > 0 \). Since at the crossing point \( p_{,X} = 0 \) and \( p_{,X} \) changes sign, one can always find a small region where \( c_s^2 < 0 \), unless \( \rho_{,X} \) also becomes zero with a similar behavior as \( p_{,X} \). Therefore, the parameter \( z \) vanishes at the crossing and thus \( \mathcal{H}' - \mathcal{H}^2 \) in (3.18) is finite. Hence, when assuming that the universe is fulfilled by that scalar field it turns out to be zero, since at the crossing point we have \( \rho_{,X} = 0 \), \( z = 0 \). Consequently, if \( z'' \neq 0 \) at the crossing point the term \( \frac{z''}{z} \) will be divergent. Note that even if \( z'' = 0 \) this conclusion is still valid, since in this case \( \frac{z''}{z} = \frac{z'''}{z^2} \). But \( z \) is a non-negative parameter with \( z = 0 \) being its minimum and thus \( z' \) must vanish, therefore \( \frac{z''}{z} \) is either divergent or equal to \( \frac{z^{(n)}}{z^{(n-2)}} \), where \( z'' \) is also equal to zero as discussed above. Along this way, if we assume that the first \( (n - 1) \)-th derivatives of \( z \) with respect to \( \eta \) vanish at the crossing point and that \( z^{(n)} \neq 0 \), we can always use the L’Hospital theorem until we find that \( \frac{z''}{z} = \frac{z^{(n)}}{z^{(n-2)}} \), which will still be divergent. Therefore, the dispersion
relation will be divergent at the crossing point as well, and hence the perturbation will also be unstable.

In summary, we have analyzed the most general case of a single scalar field described by a lagrangian of the form $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi \partial^\mu \phi)$ and we have studied different possibilities of $w$ crossing the cosmological constant boundary. As we have shown, these cases can either lead to a negative effective sound speed $c_s^2$, or lead to a divergent dispersion relation which makes the system unstable.

### 3.2 Conditions of the $-1$-crossing

Let us close this section by examining the realization conditions of quintom scenario. Dynamically, the necessary condition can be expressed as follows: when the EoS is close to the cosmological constant boundary $w = -1$ we must have $w'|_{w=-1} \neq 0$. Under this condition we additionally require both the background and the perturbations to be stable and to cross the boundary smoothly. Therefore, we can achieve a quintom scenario by breaking certain constraints appearing in the no-go theorem.

Since we have seen that a single fluid or a single scalar field cannot give rise to quintom, we can introduce an additional degree of freedom to realize it. Namely, we can construct a model in terms of two scalars with one being quintessence and the other a ghost field. For each component separately the EoS does not need to cross the cosmological constant boundary and so their classical perturbations are stable. However, the combination of these two components can lead to a quintom scenario. An alternative way to introduce an extra degree of freedom is to involve higher derivative operators in the action. The realization of quintom scenario has also been discussed within models of modified gravity, in which we can define an effective EoS to mimic the dynamical behavior of DE observed today. Furthermore, a few attempts have been addressed in the frame of string theory, however, models of this type suffers from the problem of relating stringy scale with DE. We will describes these quintom models in the following sections.

Here we should stress again that in a realistic quintom construction one ought to consider the perturbation aspects carefully, since conventionally, dangerous instabilities do appear. The concordance cosmology is based on precise observations, many of which are tightly connected to the growth of perturbations, and thus we must ensure their stability. If we start merely with parameterizations of the scale factor or EoS to realize a quintom scenario, it will become too arbitrary without the considerations of perturbations. On the other hand, if we begin with a scenario described by a concrete action which leads to an EoS across $w = -1$, we can make a judgement on the model by considering both its background dynamics and the stability of its perturbations.
4 The simplest quintom model with double fields

As we proved in the previous section, a single fluid or scalar field cannot realize a viable quintom model in conventional cases. Consequently, one must introduce extra degrees of freedom or introduce the non-minimal couplings or modify the Einstein gravity. In recent years there has been a large amount of research in constructing models with $w$ crossing $-1$.

4.1 The model

To begin with, let us construct the simple quintom cosmological paradigm. It requires the simultaneous consideration of two fields, namely one canonical $\phi$ and one phantom $\sigma$, and the DE is attributed to their combination [19]. The action of a universe constituted of a such two fields is [81,82]:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V_{\phi}(\phi) + \frac{1}{2} g^{\mu\nu} \partial_{\mu} \sigma \partial_{\nu} \sigma - V_{\sigma}(\sigma) + \mathcal{L}_M \right], \quad (4.1)$$

where we have set $\kappa^2 \equiv 8\pi G$ as the gravitational coupling. The term $\mathcal{L}_M$ accounts for the (dark) matter content of the universe, which energy density $\rho_M$ and pressure $p_M$ are connected by the EoS $\rho_M = w_M p_M$. Finally, although we could straightforwardly include baryonic matter and radiation in the model, for simplicity reasons we neglect them.

In a flat geometry, the Friedmann equations read [81,82]:

$$H^2 = \frac{\kappa^2}{3} \left( \rho_M + \rho_\phi + \rho_\sigma \right), \quad (4.2)$$

$$\dot{H} = -\frac{\kappa^2}{2} \left( \rho_M + p_M + \rho_\phi + p_\phi + \rho_\sigma + p_\sigma \right). \quad (4.3)$$

The evolution equations for the canonical and the phantom fields are:

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \quad (4.4)$$

$$\dot{\rho}_\sigma + 3H(\rho_\sigma + p_\sigma) = 0, \quad (4.5)$$

where $H = \dot{a}/a$ is the Hubble parameter.

In these expressions, $p_\phi$ and $\rho_\phi$ are respectively the pressure and density of the canonical field, while $p_\sigma$ and $\rho_\sigma$ are the corresponding quantities for the phantom field. They are given by:
\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V_\phi(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V_\phi(\phi), \quad \rho_\sigma = -\frac{1}{2} \dot{\sigma}^2 + V_\sigma(\sigma), \quad p_\sigma = -\frac{1}{2} \dot{\sigma}^2 - V_\sigma(\sigma), \quad (4.6) \]

where \( V_\phi(\phi), V_\sigma(\sigma) \) are the potentials for the canonical and phantom field respectively. Therefore, we can equivalently write the evolution equations for the two constituents of the quintom model in field terms:

\[ \ddot{\phi} + 3H \dot{\phi} + \frac{\partial V_\phi(\phi)}{\partial \phi} = 0 \quad (4.8) \]
\[ \ddot{\sigma} + 3H \dot{\sigma} - \frac{\partial V_\sigma(\sigma)}{\partial \sigma} = 0. \quad (4.9) \]

Finally, the equations close by considering the evolution of the matter density:

\[ \dot{\rho}_M + 3H(\rho_M + p_M) = 0. \quad (4.10) \]

As we have mentioned, in a double-field quintom model, the DE is attributed to the combination of the canonical and phantom fields:

\[ \rho_{DE} \equiv \rho_\phi + \rho_\sigma, \quad p_{DE} \equiv p_\phi + p_\sigma, \quad (4.11) \]

and its EoS is given by

\[ w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} = \frac{p_\phi + p_\sigma}{\rho_\phi + \rho_\sigma}. \quad (4.12) \]

Alternatively, we could introduce the “total” energy density \( \rho_{tot} \equiv \rho_M + \rho_\phi + \rho_\sigma \), obtaining:

\[ \dot{\rho}_{tot} + 3H(1 + w_{tot})\rho_{tot} = 0, \quad (4.13) \]

with

\[ w_{tot} = \frac{p_\phi + p_\sigma + p_M}{\rho_\phi + \rho_\sigma + \rho_M} = w_\phi \Omega_\phi + w_\sigma \Omega_\sigma + w_M \Omega_M, \quad (4.14) \]

with the individual EoS parameters defined as

\[ w_\phi = \frac{p_\phi}{\rho_\phi}, \quad w_\sigma = \frac{p_\sigma}{\rho_\sigma}, \quad w_M = \frac{p_M}{\rho_M}, \quad (4.15) \]

and
\[
\Omega_\phi \equiv \frac{\rho_\phi}{\rho_{tot}}, \quad \Omega_\sigma \equiv \frac{\rho_\sigma}{\rho_{tot}}, \quad \Omega_M \equiv \frac{\rho_M}{\rho_{tot}}, \quad (4.16)
\]

are the corresponding individual densities. These constitute:

\[
\Omega_\phi + \Omega_\sigma \equiv \Omega_{DE}, \quad (4.17)
\]

and

\[
\Omega_\phi + \Omega_\sigma + \Omega_M = 1. \quad (4.18)
\]

### 4.2 Phase space analysis

In order to investigate the properties of the constructed simple quintom model, we proceed to a phase-space analysis. To perform such a phase-space and stability analysis of the phantom model at hand, we have to transform the aforementioned dynamical system into its autonomous form [9, 11, 12, 13]. This will be achieved by introducing the auxiliary variables:

\[
\begin{align*}
x_\phi &\equiv \frac{\kappa \dot{\phi}}{\sqrt{6H}}, \quad x_\sigma \equiv \frac{\kappa \dot{\sigma}}{\sqrt{6H}}, \\
y_\phi &\equiv \frac{\kappa \sqrt{V_\phi(\phi)}}{\sqrt{3H}}, \quad y_\sigma \equiv \frac{\kappa \sqrt{V_\sigma(\sigma)}}{\sqrt{3H}}, \\
z &\equiv \frac{\kappa \sqrt{\rho_M}}{\sqrt{3H}},
\end{align*}
\quad (4.19)
\]

together with \( N = \ln a \). Thus, it is easy to see that for every quantity \( F \) we acquire \( \dot{F} = H \frac{dF}{dN} \).

Using these variables we straightforwardly obtain:

\[
\begin{align*}
\Omega_\phi &\equiv \frac{\kappa^2 \rho_\phi}{3H^2} = x_\phi^2 + y_\phi^2, \\
\Omega_\sigma &\equiv \frac{\kappa^2 \rho_\sigma}{3H^2} = -x_\sigma^2 + y_\sigma^2, \\
\Omega_{DE} &\equiv \frac{\kappa^2 (\rho_\phi + \rho_\sigma)}{3H^2} = x_\phi^2 + y_\phi^2 - x_\sigma^2 + y_\sigma^2,
\end{align*}
\quad (4.20, 4.21, 4.22)
\]

and

\[
\begin{align*}
w_\phi &\equiv \frac{x_\phi^2 - y_\phi^2}{x_\phi^2 + y_\phi^2}, \quad w_\sigma = \frac{-x_\sigma^2 - y_\sigma^2}{-x_\sigma^2 + y_\sigma^2}, \\
w_{DE} &\equiv \frac{x_\phi^2 - y_\phi^2 - x_\sigma^2 - y_\sigma^2}{x_\phi^2 + y_\phi^2 - x_\sigma^2 + y_\sigma^2}.
\end{align*}
\quad (4.23, 4.24)
\]

For \( w_{tot} \) we acquire:
where we have introduced the barotropic form for the matter EoS, defining \( w_M \equiv \gamma - 1 \). Finally, the Friedmann constraint (4.2) becomes:

\[
x_\phi^2 + y_\phi^2 - x_\sigma^2 + y_\sigma^2 + z^2 = 1.
\] (4.26)

A final assumption must be made in order to handle the potential derivatives that are present in (4.8) and (4.9). The usual assumption in the literature is to assume an exponential potential of the form

\[
V_\phi = V_{\phi_0} e^{-\kappa \lambda_\phi \phi},
\]

\[
V_\sigma = V_{\sigma_0} e^{-\kappa \lambda_\sigma \sigma},
\] (4.27)

since exponential potentials are known to be significant in various cosmological models [141,142,9]. Note that equivalently, but more generally, we could consider potentials satisfying

\[
\lambda_\phi = -\frac{1}{V_{\phi_0}(\phi)} \frac{\partial V_\phi(\phi)}{\partial \phi} \approx \text{const}
\]

and similarly \( \lambda_\sigma = -\frac{1}{V_{\sigma_0}(\sigma)} \frac{\partial V_\sigma(\sigma)}{\partial \sigma} \approx \text{const} \) (for example this relation is valid for arbitrary but nearly flat potentials [144]).

Using the auxiliary variables (4.19), the equations of motion (4.2), (4.3), (4.8), (4.9) and (4.10) can be transformed to an autonomous system containing the variables \( x_\phi, x_\sigma, y_\phi, y_\sigma, z \) and their derivatives with respect to \( N = \ln a \). Thus, we obtain:

\[
X' = f(X),
\] (4.28)

where \( X \) is the column vector constituted by the auxiliary variables, \( f(X) \) the corresponding column vector of the autonomous equations, and prime denotes derivative with respect to \( N = \ln a \). Then, we can extract its critical points \( X_c \) satisfying \( X' = 0 \). In order to determine the stability properties of these critical points, we expand (4.28) around \( X_c \), setting \( X = X_c + U \) with \( U \) the perturbations of the variables considered as a column vector. Thus, for each critical point we expand the equations for the perturbations up to the first order as:

\[
U' = \Xi \cdot U,
\] (4.29)

where the matrix \( \Xi \) contains the coefficients of the perturbation equations. Thus, for each critical point, the eigenvalues of \( \Xi \) determine its type and stability.

In particular, the autonomous form of the cosmological system is [81,96,104,111,112,116,124,127,131,135]:
initially dominated by the quintessence field, it will eventually evolve into the phantom-dominated true even if there exists the interaction between the two fields \[82\]. Thus, if this coupled system is

\[ \frac{\dot{\phi}}{\phi} = 3 \left( 1 - \frac{\gamma}{2} \right) - \lambda \phi \frac{\sqrt{6}}{2} \phi_\sigma, \]

\[ \frac{\dot{\sigma}}{\sigma} = 3 \left( 1 + x_\sigma - \frac{\gamma}{2} \right) + \lambda \phi \frac{\sqrt{6}}{2} \phi_\sigma, \]

\[ \frac{\dot{\alpha}}{\alpha} = -3 x_\sigma \left( 1 + x_\sigma - \frac{\gamma}{2} \right) - \lambda \phi \frac{\sqrt{6}}{2} \phi_\sigma, \]

\[ \frac{\dot{\beta}}{\beta} = 3 y_\phi \left( -x_\sigma + x_\phi + \gamma \frac{\sqrt{6}}{2} \phi_\sigma - \lambda \phi \frac{\sqrt{6}}{2} \phi, \right), \]

\[ \frac{\dot{\gamma}}{\gamma} = 3 y_\phi \left( -x_\sigma + x_\phi + \gamma \frac{\sqrt{6}}{2} \phi_\sigma - \lambda \phi \frac{\sqrt{6}}{2} \phi, \right), \]

\[ \frac{\dot{\delta}}{\delta} = 3 z \left( -x_\sigma + x_\phi + \gamma \frac{\sqrt{6}}{2} \phi_\sigma - \lambda \phi \frac{\sqrt{6}}{2} \phi, \right). \]

The real and physically meaningful (i.e. corresponding to \( y_i > 0 \) and \( 0 \leq \Omega_i \leq 1 \)) of them are presented in Table \( 1 \). In Table \( 2 \) we present the eigenvalues of the corresponding matrix \( \mathbf{B} \) for these critical points, which determine their stability properties (a stable point requires negative real parts of all eigenvalues).

From Tables \( 1 \) and \( 2 \), one can see that the phantom-dominated solution is always a late-time stable attractor (stable point \( B \)). In particular, this late-time solution corresponds to \( \Omega_{DE} = \Omega_\sigma = 1 \) and \( w_{DE} = w_\sigma = -1 - \lambda_\sigma^2 / 3 \) (see relations \( (4.22) \) and \( (4.24) \)), i.e. to a complete DE domination. This is true even if there exists the interaction between the two fields \[82\]. Thus, if this coupled system is initially dominated by the quintessence field, it will eventually evolve into the phantom-dominated

| Label | \( x_{\sigma c} \) | \( y_{\sigma c} \) | \( x_{\phi c} \) | \( y_{\phi c} \) | \( z_c \) |
|-------|------------------|------------------|------------------|------------------|------------------|
| \( A \) | \( x_\phi^2 - x_\sigma^2 = 1 \) | 0 | 0 | 0 | 0 |
| \( B \) | \( -\frac{\lambda_\phi}{\sqrt{6}} \sqrt{1 + \frac{\lambda_\phi^2}{6}} \) | 0 | 0 | 0 | 0 |
| \( C \) | 0 | 0 | \( \frac{\lambda_\phi}{\sqrt{6}} \sqrt{1 - \frac{\lambda_\phi^2}{6}} \) | 0 | 0 |
| \( D \) | 0 | 0 | 0 | 0 | 1 |
| \( E \) | 0 | 0 | \( \frac{3\gamma}{\sqrt{6}} \sqrt{\frac{3(2 - \gamma)}{2\lambda_\phi^2}} \) | \( \sqrt{1 - \frac{3\gamma}{\lambda_\phi^2}} \) | 0 |

Table 1
The list of the critical points of the simplest quintom model.

| Label | \( m_1 \) | \( m_2 \) | \( m_3 \) | \( m_4 \) | Stability |
|-------|------------------|------------------|------------------|------------------|-----------|
| \( A \) | \(-6(1 - \frac{\gamma}{2})x_\sigma^2 \) | \( 3(1 - \frac{\sqrt{6}}{6})\lambda_\sigma x_\sigma \) | \( 6(1 - \frac{\gamma}{2})x_\phi^2 \) | \( 3(1 - \frac{\sqrt{6}}{6})\lambda_\phi x_\phi \) | unstable |
| \( B \) | \(-\frac{\lambda_\phi^2}{2} \) | \(-\frac{1}{2}(6 + \lambda_\phi^2) \) | \(-\frac{1}{2}(6 + \lambda_\phi^2) \) | \(-3\gamma - \lambda_\phi^2 \) | stable |
| \( C \) | \( \frac{\lambda_\phi^2}{2} \) | \(-3(1 - \frac{\lambda_\phi^2}{6}) \) | \(-3(1 - \frac{\lambda_\phi^2}{6}) \) | \(-3\gamma + \lambda_\phi^2 \) | unstable |
| \( D \) | \( \frac{3\gamma}{2} \) | \( \frac{3\gamma}{2} \) | \(-3(1 - \frac{\gamma}{2}) \) | \(-3(1 - \frac{\gamma}{2}) \) | unstable |
| \( E \) | \( \frac{3\gamma}{2} \) | \((-3 - \frac{3\gamma}{2}) \) | \( \frac{3(\gamma - 2)}{2} \left[ 1 + \frac{6 - 3\gamma - 24\lambda_\phi^2}{2\lambda_\phi^2 - 3\lambda_\phi^2} \right] \) | \( \frac{3(\gamma - 2)}{2} \left[ 1 - \frac{6 - 3\gamma - 24\lambda_\phi^2}{2\lambda_\phi^2 - 3\lambda_\phi^2} \right] \) | unstable |

Table 2
The eigenvalues and stability of the critical points of the simplest quintom model.
phase and the crossing through the phantom divide is inevitable. This behavior provides a natural realization of the quintom scenario, and was the motive of the present cosmological paradigm. Finally, we mention that during its motion to the late-time attractor, the dynamical system could present an oscillating behavior for $w_{DE}$, with observationally testable effects [82].

4.3 State-finder diagnosis

In this paragraph we are going to present a way to discriminate between the various quintom scenarios, in a model independent manner, following [145] and based in the seminal works [146,147]. In those works, the authors proposed a cosmological diagnostic pair $\{r, s\}$ called statefinder, which are defined as

$$r \equiv \frac{\ddot{a}}{aH^3}, ~ s \equiv \frac{r - 1}{3(q - 1/2)},$$

(4.35)

to differentiate between different forms of DE. In these expressions, $q$ is the deceleration parameter $q \equiv -\dot{a}/a^2 = -\ddot{a}/aH^2$, $r$ forms the next step in the hierarchy of geometrical cosmological parameters beyond $H$ and $q$, and $s$ is a linear combination of $r$ and $q$. Apparently, the statefinder parameters depend only on $a$ and its derivatives, and thus it is a geometrical diagnostic. Since different quintom cosmological models exhibit qualitatively different evolution trajectories in the $s - r$ plane, this statefinder diagnostic can differentiate between them.

However, let us use an alternative form of statefinder parameters defined as [145]:

$$r = 1 + \frac{9}{2}\frac{(\dot{\rho}_\text{tot} + \dot{p}_\text{tot})}{\rho_\text{tot}} \frac{\dot{p}_\text{tot}}{\dot{\rho}_\text{tot}}, ~ s = \frac{(\dot{\rho}_\text{tot} + \dot{p}_\text{tot})}{\rho_\text{tot}} \frac{\dot{\rho}_\text{tot}}{\dot{p}_\text{tot}}.$$

(4.36)

Here $\rho_\text{tot} \equiv \rho_M + \rho_\phi + \rho_\sigma + \rho_\text{rad}$ is the total energy density (note that in order to be more general we have also added a radiation part) and $p_\text{tot}$ is the corresponding total pressure in the universe. Since the total energy, the DE (quintom) energy and radiation are conserved separately, we have $\dot{\rho}_\text{tot} = -3H(\rho_\text{tot} + p_\text{tot})$, $\dot{\rho}_\text{DE} = -3H(1 + w_{DE})\rho_\text{DE}$ and $\dot{\rho}_\text{rad} = -4H\rho_\text{rad}$, respectively. Thus, we can obtain

$$r = 1 - \frac{3}{2}[\dot{w}_{DE}/H - 3w_{DE}(1 + w_{DE})] \Omega_{DE} + 2\Omega_\text{rad},$$

(4.37)

$$s = -\frac{3[\dot{w}_{DE}/H - 3w_{DE}(1 + w_{DE})] \Omega_{DE} + 4\Omega_\text{rad}}{9w_{DE}\Omega_{DE} + 3\Omega_\text{rad}},$$

(4.38)

and

$$q = \frac{1}{2}(1 + 3w_{DE}\Omega_{DE} + \Omega_\text{rad}),$$

(4.39)

where as usual $\Omega_{DE} = \rho_{DE}/\rho_\text{tot}$ and $\Omega_\text{rad} = \rho_\text{rad}/\rho_\text{tot}$.
The $r - s$ diagram of no direct coupling exponential potential $V(\sigma, \phi) = V_\sigma e^{-\alpha \sigma} + V_\phi e^{-\beta \phi}$, with $V_\sigma = 0.3\rho_0$ and $V_\phi = 0.6\rho_0$, $\beta = 3$ and $\alpha = 1, 1.5, 2$ (solid line, dashed line and dot-dashed line respectively), and $\rho_0$ the present energy density of our universe. The curves $r(s)$ evolve in the time interval $\frac{t}{t_0} \in [0.5, 4]$ where $t_0$ is the present time. Dots locate the current values of the statefinder parameters. From Ref. [145].

We discuss the statefinder for the quintom scenario, imposing various potentials $V(\phi, \sigma) \equiv V_\phi(\phi) + V_\sigma(\sigma)$. Firstly we assume that there is no direct coupling between the phantom scalar field and the normal scalar field, that is we assume the simple exponential potentials of the previous subsection: $V(\sigma, \phi) = V_\sigma e^{-\alpha \sigma} + V_\phi e^{-\beta \phi}$, where $\alpha$ and $\beta$ are constants. As we have shown, in this case the universe is moving towards the phantom dominated, late time attractor [81,82,137,148]. In Fig. 6 it is depicted the time evolution of statefinder pair $\{r, s\}$ in the time interval $\frac{t}{t_0} \in [0.5, 4]$ where $t_0$ is the present time [145]. As we can see, in the past and future the $r - s$ curve is almost linear, which means that the deceleration parameter changes from one constant to another nearly with the increasing of time, and the parameters will pass the fixed point of LCDM in the future. These trajectories of $r(s)$ are different from other DE models discussed in [147,149,150,151,152,153].

Similarly, in Fig. 7 we can see the $r(s)$ curves for the potentials $V(\sigma, \phi) = V_\sigma e^{-\alpha \sigma} + V_\phi e^{-\beta \phi} + V_0 e^{-\kappa (\sigma + \phi)}$ and $V(\sigma, \phi) = V_\sigma e^{-\alpha \sigma^2} + V_\phi e^{-\beta \phi^2}$, which lead to late-time attractors. The former potential leads to a Big Rip attractor while the latter one to a de Sitter attractor [81,82]. Apparently the upper figure is very similar to Fig. 6. This shows that for uncoupling and coupling exponential potentials the evolutions of our universe are very similar in the time interval we consider here. The lower figure is different from Fig. 6 but has a common characteristic with the phantom with power-law potential [148], quintessence with inverse power-law potential and Chaplygin gas model [149] that it reaches the point of LCDM with the increasing of time. This is due to the fact that they all lead to the same fate of the universe—de Sitter expansion, but the trajectories to LCDM in the future. These trajectories of $r(s)$ are different from other DE models discussed in [147,149,150,151,152,153].

Let us investigate the case of linear coupling potential $V(\sigma, \phi) = \kappa (\sigma + \phi) + \lambda \sigma \phi$, where $\kappa$ and $\lambda$ are two constants. The scalar field with a linear potential was first studied in [154] and it has been argued that such a potential is favored by anthropic principle considerations [155,156,157] and can solve the coincidence problem [158]. In addition, if the universe is dominated by quintessence...
In the upper figure curves \( r(s) \) evolve in the time interval \( \frac{t}{t_0} \in [0.5, 4] \) where \( t_0 \) is the present time for the potential \( V(\sigma, \phi) = V_{\sigma_0}e^{-\alpha \sigma} + V_{\phi_0}e^{-\beta \phi} + V_0e^{-\kappa(\sigma+\phi)} \). The model parameters are chosen as \( V_{\sigma_0} = 0.3 \rho_0, V_{\phi_0} = 0.6 \rho_0, V_0 = 0.3 \rho_0, \alpha = 1, \beta = 1 \) and \( \kappa = 1, 2 \) (solid line and dot-dashed line respectively). In the lower figure the potential is \( V(\sigma, \phi) = V_{\sigma_0}e^{-\alpha \sigma^2} + V_{\phi_0}e^{-\beta \phi^2} \). The model parameter are chosen as \( V_{\sigma_0} = 0.3 \rho_0, V_{\phi_0} = 0.6 \rho_0, \alpha = 1 \) and \( \beta = 1, 3 \) (solid line and dot-dashed line respectively). Dots locate the current values of the statefinder parameters. From Ref. [145].

(phantom) with this potential, it ends with a Big Crunch (Big Rip) [18, 159]. The time evolutions of statefinder pair \( \{r, s\} \) in the time interval \( \frac{t}{t_0} \in [0.5, 4] \) are shown in Fig.8. We see that in the case of negative coupling the diagram is very similar to Fig.6 and the upper graph in Fig.7, which shows that in these cases the evolutions of our universe are similar in the time interval we consider here.

In conclusion, the statefinder diagnostic can differentiate the quintom model with other DE models, but it is not very helpful in order to differentiate quintom DE models with some different kinds of potentials which lead to a similar evolution of our universe in the time interval we consider.

4.4 Cosmic duality

As we have seen in paragraph 4.1 the simplest quintom model consists of two scalar fields, one canonical, quintessence-like and one phantom one [81, 82]. In [104] it was shown that there exists two basic categories of such quintom models: one is quintom-A type, where the canonical field dominates at early times while the phantom one dominates at late times and thus the phantom divide crossing is realized from above to below, and the other is quintom-B type for which the EoS is arranged to change from below \(-1\) to above \(-1\). As we have analyzed in paragraph 4.1
The $r - s$ diagrams for the linear potential case $V(\sigma, \phi) = \kappa(\sigma + \phi) + \lambda \sigma \phi$. The curves $r(s)$ evolve in the time interval $\frac{t}{t_0} \in [0.5, 4]$. In the upper graph $\kappa = 0.8$ and $\lambda = -0.2$, $-0.8$ (solid line and dashed line respectively). In the lower graph $\kappa = 0.8$ and $\lambda = 0.2, 0.8$ (solid line and dashed line respectively). Dots locate the current values of the statefinder parameters. From Ref. [145].

the exponential, as well as other simple potential forms, belong to quintom-A type, which is realized easily. However, if one desires to construct a specific model of quintom-B type then he has to use more sophisticated or fine-tuned potentials, or to add more degrees of freedom [104]. Alternatively he can include higher derivative terms (see for example [87]), or add suitably constructed interactive terms which lead to a transition from phantom-to-quintessence domination (see [82]). Generally speaking, both quintom-A and quintom-B types could be consistent with current observational data.

One question arises naturally: is there a cosmic duality between these two quintom types? Dualities in field and string theory have been widely studied, predicting many interesting phenomena [160]. The authors of [161,162] have considered a possible transformation with the Hubble parameter and they have studied the relevant issues with the cosmic duality [163,164]. Specifically, in [165] a link between standard cosmology with quintessence matter and contracting cosmology with phantom, has been shown. Later on this duality was generalized into more complicated DE models, where it has been shown to exist too [166,167,168,169,170,171]. In [172,173] the authors have studied this cosmic duality and its connection to the fates of the universe, while in [174] the author has also discussed the possibility of realizing the aforementioned duality in braneworld cosmological paradigm. A common feature of these studies is that the EoS parameter does not cross $-1$. Therefore, it would be interesting to study the implications of this cosmic duality in quintom models of DE, and in particular between quintom-A and quintom-B types.

We consider the simple quintom model constructed in 4.1. Following [105,165] we can construct...
a form-invariant transformation by defining a group of new quantities $\bar{H}$, $\bar{\rho}$, $\bar{p}$ and $\bar{w}$ which keep Einstein equations invariant:

$$\bar{\rho} = \bar{\rho} (\rho_{DE}) ,$$  \hspace{1cm} (4.40) $$\bar{H} = - \left( \frac{\bar{\rho}}{\rho_{DE}} \right)^{\frac{1}{2}} H .$$  \hspace{1cm} (4.41)

Under this transformation, we obtain the corresponding changes for the pressure $p_{DE}$ and the EoS $w_{DE}$,

$$\bar{p} = -\bar{\rho} - \left( \frac{\rho_{DE}}{\bar{\rho}} \right)^{\frac{1}{2}} (\rho_{DE} + p_{DE}) \frac{d\bar{\rho}}{d\rho_{DE}} ,$$  \hspace{1cm} (4.42) $$\bar{w} = -1 - \left( \frac{\rho_{DE}}{\bar{\rho}} \right)^{\frac{1}{2}} \frac{d\bar{\rho}}{d\rho_{DE}} (1 + w_{DE}) .$$  \hspace{1cm} (4.43)

From relations (4.42) and (4.43) one can see that for a positive $\frac{d\bar{\rho}}{d\rho_{DE}}$, one would be able to establish a connection between the quintom-A and quintom-B types. Assuming without loss of generality, and as an example for a detailed discussion, that $\bar{\rho} = \rho_{DE}$ in (4.42) and (4.43), we can obtain the dual transformation:

$$\bar{H} = -H ,$$  \hspace{1cm} (4.44) $$\bar{p} = -2\rho_{DE} - p_{DE} ,$$  \hspace{1cm} (4.45) $$\bar{w} = -2 - w_{DE} .$$  \hspace{1cm} (4.46)

Consequently, using the canonical and phantom energy density definitions (4.6) and (4.7), we can extract the dual form of the (quintom) DE Lagrangian:

$$\bar{\mathcal{L}} = \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \delta \mathcal{L}_1(\phi) - \delta \mathcal{L}_2(\sigma) ,$$  \hspace{1cm} (4.47)

where $\delta \mathcal{L}_1$ and $\delta \mathcal{L}_2$ are

$$\delta \mathcal{L}_1 = V_\phi(\phi) + \dot{\phi}^2 ,$$  \hspace{1cm} (4.48) $$\delta \mathcal{L}_2 = V_\sigma(\sigma) - \dot{\sigma}^2 .$$  \hspace{1cm} (4.49)

Therefore, we can easily see that if the original Lagrangian is a quintom-A type then the dual one is a quintom-B type, and vice versa. Thus, under this duality one expects a general connection amongst different fates of the universe, and it might be possible that the early universe is linked to its subsequent epochs.

For a specific discussion let us impose a special form for the potentials [105]:
\[ V_\phi (\phi) \propto -3\sqrt{2}\phi + 2e^{-\sqrt{2}\phi} , \quad (4.50) \]
\[ V_\sigma (\sigma) \propto \frac{3}{2} \sigma^2 + 4\sigma . \quad (4.51) \]

Solving explicitly the cosmological equation (4.2), (4.3), (4.8), (4.9), neglecting the matter density, we study the two periods of the universe evolution. For early times where \(|t| \ll 1\) (in Planck mass units), we choose the initial conditions by fixing \(\phi \to -\infty\) and \(\sigma \to 0\). With these initial conditions we can see that the dominant component in DE density is the exponential term of \(V_\phi (\phi)\) in (4.50), and thus the contribution from the phantom potential in (4.51) and the linear part of quintessence potential in (4.50) can be neglected. Therefore, the universe behaves like being dominated by the quintessence component \(\phi\) and it evolves following the approximate analytical solution [105]:

\[ \phi \sim \sqrt{2} \ln |t| , \quad \sigma \sim \frac{1}{2} t^2 , \quad H \sim \frac{1}{t} . \quad (4.52) \]

Thus, we see that the scale factor in this period would variate with respect to time as: \(a \propto \pm t\), in which the signal is determined by the positive definite form of the scale factor. Therefore, the scale factor here corresponds to the Big Bang or Big Crunch of quintessence-dominated universe.

The dual form of the solution above is a description of a universe dominated by a phantom component with a Lagrangian given by (4.47) and

\[ \delta L_1 + \delta L_2 = \dot{\phi}^2 - \dot{\sigma}^2 + V_\phi (\phi) + V_\sigma (\sigma) \propto \left( -3\sqrt{2}\phi + 4e^{-\sqrt{2}\phi} + 2\sigma + \frac{3}{2} \sigma^2 \right) . \quad (4.53) \]

In addition, the dual Hubble parameter is of the form \(\dot{H} \sim -\frac{1}{t}\) and for the scale factor we acquire \(a \propto \pm \frac{1}{t}\). Accordingly, the scale factor of the dual form is tending towards infinity in the beginning or the end of universe. From what we have investigated so far, we can see that, for the positive branch there is a duality between an expanding universe with initial singularity at \(t = 0^+\) and a contracting one that begins with an infinite scale factor at \(t = 0^-\). However, for the negative branch there is a duality between a contracting universe ending in a big crunch at \(t = 0^-\) and an expanding one that ends in a final Big Rip at \(t = 0^-\). The latter is dominated by a phantom component. Besides, in general, under phantom domination the contracting solution is not stable, because the phantom universe will evolve into Big Rip or Big Sudden or will expand forever approaching to a de Sitter solution, and thus it will not be able to stay in the contracting phase forever [175,176]. This problem, however, can be avoided in quintom cosmology since in the dual universe with quintom-B DE the increase of kinetic energy of phantom during the contraction can be set off by that of quintessence at late time.

For times where \(|t| \gg 1\), the phantom component in the first quintom model will dominate and the universe will expand. For the specific potentials (4.50) and (4.51) the mass term in \(V_\sigma (\sigma)\) will gradually play an important role in the evolution of DE. Proceeding as above we obtain:
\[ \phi \sim \sqrt{2} \ln |t|, \quad \sigma \sim \sqrt{2} t, \quad H \sim t, \quad (4.54) \]

where we note that the scale factor is \( a \propto \exp \left( \frac{t^2}{2} \right) \). Consequently, the scale factor would increase towards infinity rapidly for the positive branch, while it would start from infinity for the negative branch. In this case, the transformed Lagrangian is \((4.47)\) with

\[ \delta L_1 + \delta L_2 \propto \left( -3\sqrt{2}\phi + 4e^{-\sqrt{2}\phi} + 4\sigma + \frac{3}{2} \sigma^2 - 2 \right). \quad (4.55) \]

The universe is evolving with a Hubble parameter \( \dot{H} \sim -t \) and a scale factor \( a \propto \exp \left( -\frac{t^2}{2} \right) \), which is close to singularity related to the origin and the fate of universe. Finally, the component which dominates the evolution of the universe is \( \sigma \), that is the scalar field resembling quintessence in \((4.47)\). Consequently we conclude that, for the positive branch, there is a dual relation between an expanding universe with a fate of expanding for ever with \( t \to +\infty \) and a contracting universe with a destiny of shrinking for ever with \( t \to +\infty \). Meanwhile, for the negative one, there is a duality connecting a contracting universe starting from near infinity with \( t \to -\infty \) and an expanding universe originating from infinity with \( t \to -\infty \).

Having presented the analytical arguments for the duality between quintom A and quintom B types, we proceed to numerical investigation. In Fig.9 we depict the evolution of the EoS parameter of the quintom model and its dual. One can see from this figure that under the framework of the duality studied above, the EoS of the quintom model and its dual are symmetric around \( w = -1 \). Accordingly, in this case quintom A is dual to quintom B rigorously, which supports our analytical arguments above.

Fig. 9. (Color Online) Evolution of EoS parameter \( w \) of the quintom model and its dual as a function of the scale factor \( \ln a \) for \( V = -3\sqrt{2}M^3\phi + 2M^4 e^{-\sqrt{2}\phi} + \frac{3}{2}M^2\sigma^2 + 4M^3\sigma \) and \( M \) is the Planck mass. From Ref. [105].
In Fig. [10] we assume the potentials $V_\phi(\phi)$ and $V_\sigma(\sigma)$ to be exponentials and one can see that the EoS parameter for quintom A approaches to a fixed value which corresponds to the attractor solution of this type of model [82]. Through the duality we can see that there exists a corresponding attractor of the quintom B model dual to the former one. Finally, in Fig. [11] we provide another example for the duality.

Fig. 10. (Color Online) Evolution of EoS parameter $w$ of the quintom model and its dual as a function of the scale factor $\ln a$ for $V = V_0(e^{-\phi M} + e^{-2\sigma M})$ and $M$ is the planck mass. From Ref. [105].

In summary, we have shown that the quintom model has its dual partner, specifically the quintom A model is dual to the quintom B one, while the cosmological equations are form-invariant. These two models describe two different behaviors of the universe evolution with one in the expanding phase and the other in the contracting one, depending on the imposed initial conditions. The cosmic duality, which connects the two totally different scenarios of universe evolution, preserves the energy density of the universe unchanged but it transforms the Hubble parameter.

5 Perturbation theory and current observational constraints

In this section we investigate the perturbations of quintom DE scenario and the effects of these perturbations on current observations. As proved in Section 3, it is forbidden for a single fluid or scalar field to realize a quintom scenario in conventional cases, and thus one is led to add extra degrees of freedom. Therefore, it is important to check the consistency of this extension at the classical level and in particular to analyze the behavior of perturbations when the EoS crosses the cosmological constant boundary [25].
Fig. 11. (Color Online) Evolution of EoS parameter $w$ of the quintom model and its dual as a function of the scale factor $\ln a$ for $V = m_1^2 \phi^2 + m_2^2 \sigma^2$ where $m_1$ corresponds to the quintessence component mass and $m_2$ the phantom component mass. From Ref. [105].

5.1 Analysis of perturbations in quintom cosmology

As we have argued above, the quintom scenario needs extra degrees of freedom to the conventional models of a single scalar field, and the simplest realization of the quintom is a model with two scalar fields or two “effective” scalar fields in the case of higher-derivative operators. In the following discussions on quintom perturbations we will restrict ourselves to the double-field model of quintom with the lagrangian

$$\mathcal{L} = \mathcal{L}_Q + \mathcal{L}_P, \quad (5.1)$$

where

$$\mathcal{L}_Q = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 - V_1(\phi_1) \quad (5.2)$$

describes the quintessence component, and

$$\mathcal{L}_P = -\frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - V_2(\phi_2) \quad (5.3)$$

for the phantom component. The background equations of motion for the two scalar fields $\phi_i (i = 1, 2)$ are

$$\ddot{\phi}_i + 2H \dot{\phi}_i \pm a^2 \frac{\partial V_i}{\partial \phi_i} = 0, \quad \quad (5.4)$$

where the positive sign is for the quintessence and the minus sign for the phantom. In general there will be couplings between the two scalar fields, but for simplicity we neglect them.

For a complete perturbation study, both the fluctuations of the fields, as well as those of the metric, need to be considered. In the conformal Newtonian gauge the perturbed metric is given
by
\[ ds^2 = a^2(\tau)[(1 + 2\Psi)d\tau^2 - (1 - 2\Phi)dx^i dx_i] . \] (5.5)

Using the notations of [83], the perturbation equations satisfied by each component of the two-field quintom model read:

\[
\dot{\delta}_i = -(1 + w_i)(\theta_i - 3\dot{\Phi}) - 3\mathcal{H} \left( \frac{\delta P_i}{\delta \rho_i} - w_i \right) \delta_i , \]

\[
\dot{\theta}_i = -\mathcal{H}(1 - 3w_i) \theta_i - \frac{\dot{w}_i}{1 + w_i} \theta_i + k^2 \left( \frac{\delta P_i / \delta \rho_i}{1 + w_i} \delta_i - \sigma_i + \Psi \right) , \]

where

\[
\theta_i = (k^2/\dot{\phi}_i)\delta\phi_i , \quad \sigma_i = 0 , \quad w_i = \frac{P_i}{\rho_i} ,
\]

and

\[
\delta P_i = \delta \rho_i - 2V_i' \delta \phi_i = \delta \rho_i + \frac{\rho_i \theta_i}{k^2} \left[ 3\mathcal{H}(1 - w_i^2) + \dot{w}_i \right] . \]

Combining Eqs. (5.6), (5.7) and (5.9), we have

\[
\dot{\theta}_i = 2\mathcal{H} \theta_i + \frac{k^2}{1 + w_i} \delta_i + k^2 \Psi , \]

\[
\dot{\delta}_i = -(1 + w_i)(\theta_i - 3\dot{\Phi}) - 3\mathcal{H}(1 - w_i)\delta_i - 3\mathcal{H} \left[ \frac{\dot{\theta}_i + 3\mathcal{H}(1 - w_i^2)}{k^2} \right] \theta_i . \]

Since the simple two-field quintom model is essentially a combination of a quintessence and a phantom field, one obtains the perturbation equations of quintom by combining the above equations. The corresponding variables for the quintom system are

\[
w_{\text{quintom}} = \frac{\sum_i P_i}{\sum_i \rho_i} ,
\]

\[
\delta_{\text{quintom}} = \frac{\sum_i \rho_i \delta_i}{\sum_i \rho_i} ,
\]

\[
\theta_{\text{quintom}} = \frac{\sum_i (\rho_i + P_i) \theta_i}{\sum_i (\rho_i + P_i)} .
\]

Note that for the quintessence component, \(-1 \leq w_1 \leq 1\), while for the phantom component, \(w_2 \leq -1\).

The two-field quintom model is characterized by the potentials \(V_i\). Let us consider \(V_i(\phi_i) = \frac{1}{2}m_i^2\phi_i^2\). In general the perturbations of \(\phi_i\) today stem from two origins, the adiabatic and the isocurvature modes. If we use the gauge invariant variable \(\zeta_i = -\Phi - \mathcal{H}\frac{\delta \rho_i}{\rho_i}\) instead of \(\delta_i\), and the
relation $\Phi = \Psi$ in the universe without anisotropic stress, the equations (5.11) and (5.10) can be rewritten as,

$$
\dot{\zeta}_i = -\frac{\theta_i}{3} - C_i \left( \zeta_i + \Phi + \frac{H}{k^2} \theta_i \right),
$$

(5.15)

$$
\dot{\theta}_i = 2H \theta_i + k^2 (3\zeta_i + 4\Phi),
$$

(5.16)

where

$$
C_i = \frac{\dot{w}_i}{1 + w_i} + 3H(1 - w_i) = \partial_0 [\ln (a^6 |p_i + p_i|)].
$$

(5.17)

$\zeta_\alpha$ is the curvature perturbation on the uniform-density hypersurfaces for the $\alpha$-component in the universe $[177]$. Usually, the isocurvature perturbations of $\phi_i$ are characterized by the differences between the curvature perturbation of the uniform-$\phi_i$-density hypersurfaces and that of the uniform-radiation-density hypersurfaces,

$$
S_{ir} \equiv 3(\zeta_i - \zeta_r),
$$

(5.18)

where the subscript $r$ represents radiation. We assume there are no matter isocurvature perturbations and thus $\zeta_M = \zeta_r$. Eliminating $\zeta_i$ from equations (5.15) and (5.16), we obtain a second order equation for $\theta_i$, namely

$$
\ddot{\theta}_i + (C_i - 2H) \dot{\theta}_i + (C_i H - 2\dot{H} + k^2) \theta_i = k^2 (4\dot{\Phi} + C_i \Phi).
$$

(5.19)

This is an inhomogeneous differential equation and its general solution is the sum of the general solution of its homogeneous part and a special integration. In the following, we will show that the special integration corresponds to the adiabatic perturbation. Before the era of DE domination, the universe is dominated by some background fluids, for instance, the radiation or the matter. The perturbation equations of the background fluid are,

$$
\dot{\zeta}_f = -\theta_f/3,
$$

$$
\dot{\theta}_f = -H(1 - 3w_f) \theta_f + k^2 [3w_f \zeta_f + (1 + 3w_f)\Phi].
$$

(5.20)

From the Poisson equation

$$
-\frac{k^2}{H^2} \Phi = \frac{9}{2} \sum_\alpha \Omega_\alpha (1 + w_\alpha) \left( \zeta_\alpha + \Phi + \frac{H}{k^2} \theta_\alpha \right)
$$

$$
\simeq \frac{9}{2} (1 + w_f) \left( \zeta_f + \Phi + \frac{H}{k^2} \theta_f \right),
$$

(5.21)

on large scales we approximately acquire:
\[ \Phi \simeq -\zeta_f - \frac{\mathcal{H}}{k^2} \theta_f . \]  

(5.22)

Combining these equations with \( \mathcal{H} = 2/[(1 + 3w_f)\tau] \), we obtain (note numerically \( \theta_f \sim \mathcal{O}(k^2)\zeta_f \))

\[ \begin{align*}
\zeta_f &= -\frac{5 + 3w_f}{3(1 + w_f)} \Phi = \text{const}. , \\
\theta_f &= \frac{k^2(1 + 3w_f)}{3(1 + w_f)} \Phi \tau .
\end{align*} \]  

(5.23)

Thus, we can see from Eq. (5.19) that there is a special solution, which is given approximately on large scales by \( \theta_{i}^{ad} = \theta_f \), and from Eq. (5.16) we have \( \zeta_{i}^{ad} = \zeta_f \). This indicates that the special integration of (5.19) corresponds to the adiabatic perturbation. Hence, concerning the isocurvature perturbations of \( \phi_i \), we can consider only the solution of the homogeneous part of (5.19):

\[ \ddot{\theta}_i + (C_i - 2\mathcal{H})\dot{\theta}_i + (C_i \mathcal{H} - 2\dot{\mathcal{H}} + k^2)\theta_i = 0 . \]  

(5.24)

These solutions are represented by \( \theta_{i}^{iso} \) and \( \zeta_{i}^{iso} \) and the relation between them is

\[ \begin{align*}
\zeta_{i}^{iso} &= \frac{\dot{\theta}_{i}^{iso} - 2\mathcal{H}\theta_{i}^{iso}}{3k^2} .
\end{align*} \]  

(5.25)

Since the general solution of \( \zeta_i \) is \( \zeta_i = \zeta_{i}^{ad} + \zeta_{i}^{iso} = \zeta_r + \zeta_{i}^{iso} \), the isocurvature perturbations are simply \( S_{ir} = 3\zeta_{i}^{iso} \).

In order to solve Eq. (5.24) we need to know the forms of \( C_i \) and \( \mathcal{H} \) as functions of time \( \tau \). For this purpose, we solve the background equations (5.4). In radiation dominated period, \( a = A\tau \), \( \mathcal{H} = 1/\tau \) and we have

\[ \begin{align*}
\phi_1 &= \tau^{-1/2} \left[ A_1 J_{1/4}(\frac{A}{2}m_1\tau^2) + A_2 J_{-1/4}(\frac{A}{2}m_1\tau^2) \right] , \\
\phi_2 &= \tau^{-1/2} \left[ \tilde{A}_1 I_{1/4}(\frac{A}{2}m_2\tau^2) + \tilde{A}_2 I_{-1/4}(\frac{A}{2}m_2\tau^2) \right] ,
\end{align*} \]  

(5.26-27)

respectively, where \( A, A_i \) and \( \tilde{A}_i \) are constants, \( J_\nu(x) \) is the \( \nu \)th order Bessel function and \( I_\nu(x) \) is the \( \nu \)th order modified Bessel function. Usually the masses are small in comparison with the expansion rate in the early universe \( m_i \ll \mathcal{H}/a \), and we can approximate the (modified) Bessel functions as \( J_\nu(x) \sim x^\nu(\tilde{c}_1 + \tilde{c}_2x^2) \) and \( I_\nu(x) \sim x^\nu(\tilde{c}_1 + \tilde{c}_2x^2) \). We note that \( J_{-1/4} \) and \( I_{-1/4} \) are divergent when \( x \to 0 \). Given these arguments one can see that this requires large initial values of \( \phi_i \) and \( \dot{\phi}_i \) if \( A_2 \) and \( \tilde{A}_2 \) are not vanished.

If we choose small initial values, which is the natural choice if the DE fields are assumed to survive after inflation, only \( A_1 \) and \( \tilde{A}_1 \) modes exist, so \( \dot{\phi}_i \) will be proportional to \( \tau^3 \) in the leading order.
Thus, the parameters $C_i$ in equation (5.17) will be $C_i = 10/\tau$ (we have used $|\rho_i + p_i| = \dot{\phi}_i^2/a^2$). Therefore, we obtain the solution of Eq. (5.24) as

$$
\theta_i^{iso} = \tau^{-4} \left[ D_{i1} \cos(k\tau) + D_{i2} \sin(k\tau) \right].
$$

That is, $\theta_i^{iso}$ presents an oscillatory behavior, with an amplitude damping with the expansion of the universe. The isocurvature perturbations $\zeta_i^{iso}$ decrease rapidly.

On the other hand, if we choose large initial values for $\phi_i$ and $\dot{\phi}_i$, $A_2$ and $\tilde{A}_2$ modes are present, $\dot{\phi}_i$ will be proportional to $\tau^{-2}$ in the leading order and $C_i = 0$. Thus, the solution of (5.24) is

$$
\theta_i^{iso} = \tau \left[ D_{i1} \cos(k\tau) + D_{i2} \sin(k\tau) \right].
$$

Therefore, $\theta_i^{iso}$ will oscillate with an increasing amplitude, so $\zeta_i^{iso}$ remains constant on large scales.

Similarly, during matter dominated era, $a = B\tau^2$, $\mathcal{H} = 2/\tau$, and the solutions for the fields $\phi_i$ read:

$$
\phi_1 = \tau^{-3} \left[ B_1 \sin\left(\frac{B}{3} m_1 \tau^3\right) + B_2 \cos\left(\frac{B}{3} m_1 \tau^3\right) \right],
$$

$$
\phi_2 = \tau^{-3} \left[ \tilde{B}_1 \sinh\left(\frac{B}{3} m_2 \tau^3\right) + \tilde{B}_2 \cosh\left(\frac{B}{3} m_2 \tau^3\right) \right].
$$

We acquire the same conclusions as those reached by the aforementioned analysis for the radiation dominated era. If we choose small initial values at the beginning of the matter domination we find that the isocurvature perturbations in $\phi_i$ decrease with time. On the contrary, for large initial values the isocurvature perturbations remain constant at large scales. This conclusion is expectable. In the case of large initial velocity, the energy density in the scalar field is dominated by the kinetic term and it behaves like the fluid with $w = 1$. The isocurvature perturbation in such a fluid remains constant on large scales. In the opposite case, however, the energy density in the scalar field will be dominated by the potential energy due to the slow rolling. It behaves like a cosmological constant, and there are only tiny isocurvature perturbations in it.

In summary, we have seen that the isocurvature perturbations in quintessence-like or phantom-like field with quadratic potential, decrease or remain constant on large scales depending on the initial velocities. In this sense the isocurvature perturbations are stable on large scales. The amplitude of these perturbations will be proportional to the value of Hubble rate evaluated during the period of inflation $H_{inf}$, if their quantum origins are from inflation. For a large $H_{inf}$ the isocurvature DE perturbations may be non-negligible and will contribute to the observed CMB anisotropy [178,179]. In the cases discussed here, however, these isocurvature perturbations are negligible. Firstly, large initial velocities are not possible if these fields survive after inflation as mentioned above. Secondly, even if the initial velocities are large at the beginning of the radiation domination, they will be reduced to a small value due to the small masses and the damping...
effect of Hubble expansion. In general the contributions of DE isocurvature perturbations are not very large \cite{180}, and here for simplicity we have assumed that $H_{inf}$ is small enough and thus the isocurvature contributions are negligible. Therefore, in the next subsection we focus on the effects of the adiabatic perturbations of the quintom model with two scalars fields.

### 5.2 Signatures of perturbations in quintom DE

Based on perturbation equations (5.13) and (5.14), we modify the code of CAMB \cite{181} and we study preliminarily in this subsection the quintom observational signatures. For simplicity we impose a flat geometry as a background, although this is not necessary. Moreover, we assume the fiducial background parameters to be $\Omega_b = 0.042$, $\Omega_{DM} = 0.231$, $\Omega_{DE} = 0.727$, where $b$ stands for baryons, $DM$ for dark matter and $DE$ for dark energy, while today’s Hubble constant is fixed at $H_0 = 69.255$ km/s Mpc$^{-2}$. We will calculate the effects of perturbed quintom on CMB and LSS.

In the two-field quintom model there are two parameters, namely the quintessence and phantom masses. When the quintessence mass is larger than the Hubble parameter, the field starts to oscillate and consequently one obtains an oscillating quintom. In the numerical analysis we fix the phantom mass to be $m_P \sim 4 \times 10^{-61} M_p$, and we vary the quintessence mass with the typical values being $m_Q = 2 \times 10^{-61} M_p$ and $8 \times 10^{-61} M_p$ respectively. In Fig. [12] we depict the equation-of-state parameters as a function of the scale factor, for the aforementioned two parameter-sets, and additionally their corresponding effects on observations. We clearly observe the quintom oscillating behavior as the mass of quintessence component increases. After reaching the $w = -1$ pivot for several times, $w$ crosses $-1$ consequently with the phantom-component domination in dark energy. As a result, the quintom fields modify the metric perturbations $\delta g_{00} = 2a^2 \Psi$, $\delta g_{ii} = 2a^2 \Phi \delta_{ij}$, and consequently they contribute to the late-time Integrated Sachs-Wolfe (ISW) effect. The ISW effect is an integrant of $\dot{\Phi} + \dot{\Psi}$ over conformal time and wavenumber $k$. The above two specific quintom models yield quite different evolving $\Phi + \Psi$ as shown in the right graph of Fig. [12] where the scale is $k \sim 10^{-3}$ Mpc$^{-1}$. As we can see, the late time ISW effects differ significantly when DE perturbations are taken into account (solid lines) or not (dashed lines).

ISW effects play an important role on large angular scales of CMB and on the matter power spectrum of LSS. For a constant EoS of phantom the authors of \cite{182} have shown that the low multipoles of CMB will be significantly enhanced when DE perturbations are neglected. On the other hand, for a matter-like scalar field where the EoS is around zero, perturbations will also play an important role on the large scales of CMB, as shown in \cite{183}. The results on CMB and LSS reflect the two combined effects of phantom and oscillating quintessence. Note that in the early studies of quintessence effects on CMB, one usually considers a constant $w_{eff}$ instead

$$w_{eff} \equiv \frac{\int da \Omega(a)w(a)}{\int da \Omega(a)} , \quad (5.32)$$

however this is not enough for the study of effects on SNIa, nor for CMB, when the EoS of DE has a very large variation with redshift, such as the model of oscillating quintom considered above.
Fig. 12. (Color online) Effects of the two-field oscillating quintom on the observables. The mass of the phantom field is fixed at \(4 \times 10^{-61} M_p\) and the mass of the quintessence field are \(2 \times 10^{-61} M_p\) (thicker line) and \(8 \times 10^{-61} M_p\) (thinner line) respectively. The upper right graph illustrates the evolution of the metric perturbations \(\Phi + \Psi\) of the two models, with (solid lines) and without (dashed lines) DE perturbations. The scale is \(k \sim 10^{-3} \text{ Mpc}^{-1}\). The lower left graph shows the CMB effects and the lower right graph delineates the effects on the matter power spectrum, with (solid lines) and without (dashed lines) DE perturbations. From Ref. [25].

To analyze the oscillating quintom-model under the current observations, we perform a preliminary fitting to the first year WMAP TT and the TE temperature–polarization cross-power spectrum, as well as the recently released 157 “Gold” SNIa data [28]. Following Refs. [184,185] in all the fittings below we fix \(r = 0.17\), \(\Omega_M h^2 = 0.135\) and \(\Omega_b h^2 = 0.022\), setting the spectral index as \(n_S = 0.95\), and using the amplitude of the primordial spectrum as a continuous parameter. In the fittings of oscillating quintom we have fixed the phantom-mass to be \(m_p \sim 1.2 \times 10^{-61} M_p\). Fig.13 delineates 3\(\sigma\) WMAP and SNIa constraints on the two-field quintom model, and in addition it shows the corresponding best fit values. In the left graph of Fig.13 we present the separate WMAP and SNIa constraints. The green (shaded) area marks WMAP constraints on models where DE perturbations have been included, while the blue area (contour with solid
Fig. 13. (Color online) $3\sigma$ WMAP and SNIa constraints on two-field quintom model, shown together with the best fit values. $m_Q$ and $m_P$ denote the quintessence and phantom mass respectively. We have fixed $m_P \sim 1.2 \times 10^{-61} M_p$ and we have varied the value of $m_Q$. Left graph: separate WMAP and SNIa constraints. The green (shaded) area marks the WMAP constraints on models where DE perturbations have been included, while the blue area (contour with solid lines) corresponds to the case where DE perturbations have not been taken into account. Right graph: combined WMAP and SNIa constraints on the two-field quintom model with perturbations (green/shaded region) and without perturbations (red region/contour with solid lines). From Ref. [25].

lines) is the corresponding area without DE perturbations. The perturbations of DE have no effects on the geometric constraint of SNIa. The right graph shows the combined WMAP and SNIa constraints on the two-field quintom model with perturbations (green/shaded region) and without perturbations (red region/contour with solid lines). We conclude that the confidence regions indeed present a large difference, if the DE perturbations have been taken into account or not.

So far we have investigated the imprints of oscillating quintom on CMB and LSS. Now we consider another example, in which $w$ crosses $-1$ smoothly without oscillation. It is interesting to study the effects of this type of quintom model, with its effective EoS defined in (5.32) exactly equal to $-1$, on CMB and matter power-spectrum. This investigation will help to distinguish the quintom
model from the cosmological constant. We have realized such a quintom model in the lower right graph of Fig. 14, which can be easily given in the two-field model with lighter quintessence mass. In this example we have set $m_Q \sim 5.2 \times 10^{-62} M_p$ and $m_P \sim 1.2 \times 10^{-61} M_p$. Additionally, we assume that there is no initial kinetic energy. The initial value of the quintessence component is set to $\phi_1 = 0.045 M_p$, while for the phantom part we impose $\phi_2 = 1.32 \times 10^{-3} M_p$. We find that the EOS of quintom crosses $-1$ at $z \sim 0.15$, which is consistent with the latest SNIa results.

The quintom scenario, which is mildly favored by current SNIa data, needs to be confronted with other observations in the framework of concordance cosmology. Since SNIa offer the only direct detection of DE, this model is the most promising to be distinguished from the cosmological constant and other dynamical DE models which do not get across $-1$, by future SNIa projects on the low redshift (for illustrations see e.g. [20]). This is also the case for quintom scenario in the

Fig. 14. (Color online) Comparison of the effects of the two-field quintom model with $w_{\text{eff}} = -1$ and of the simple cosmological constant, in CMB (WMAP), the metric perturbations $\Phi + \Psi$ (the scale is $k \sim 10^{-3} \ Mpc^{-1}$) and the linear growth factor. The binned error bars in the upper right graph are WMAP TT and TE data [186, 187]. From Ref. [25].
full parameter space: it can be most directly tested in low redshift Type Ia supernova surveys. In the upper left graph of Fig. [14] we delineate the different ISW effects among the cosmological constant (red/light solid) and the quintom model which exhibits $w_{\text{eff}} = -1$, with (blue/dark solid) and without (blue dashed) perturbations. Similarly to the previous oscillating case, the difference is big when switching off quintom perturbations and much smaller when including the perturbations. In the upper right graph we find that the quintom model cannot be distinguished from a cosmological constant in light of WMAP. The two models give almost exactly the same results in CMB TT and TE power-spectra when including the perturbations. We deduce that the difference in CMB is hardly distinguishable even by cosmic variance.

So far we have seen that CMB observations cannot distinguish between a quintom model with $w_{\text{eff}} = -1$ and a cosmological constant. Thus, in order to acquire distinctive signatures, we have to rely on other observations. To achieve that we need to consider the physical observables which can be affected by the evolving $w$ sensitively. In comparison with the cosmological constant, such a quintom model exhibits a different evolution of the universe’s expansion history, and in particular it gives rise to a different epoch of matter-radiation equality. The Hubble expansion rate $H$ is

$$H \equiv \frac{\dot{a}}{a^2} = H_0[\Omega_M a^{-3} + \Omega_r a^{-4} + X]^{1/2},$$

(5.33)

where $X$, the energy density ratio of DE between the early times and today, is quite different between the quintom-CDM and ΛCDM. In the ΛCDM scenario $X$ is simply a constant, while in general for DE models with varying energy density or EoS,

$$X = \Omega_{DE} a^{-3} e^{-3 \int w(a) d\ln a}.$$  

(5.34)

The two models will give different Hubble expansion rates. This is also the case between the quintom model with $w_{\text{eff}} = -1$ in the left graph of Fig. [14] and a cosmological constant.

Finally, we mention that different $H$ leads directly to different behaviors of the growth factor. In particular, according to the linear perturbation theory all Fourier modes of the matter density perturbations grow at the same rate, that is the matter density perturbations are independent of $k$:

$$\ddot{\delta}_k + \mathcal{H} \dot{\delta}_k - 4\pi G a^2 \rho_M \delta_k = 0.$$  

(5.35)

The growth factor $D_1(a)$ characterizes the growth of the matter density perturbations: $D_1(a) = \delta_k(a)/\delta_k(a = 1)$ and it is normalized to unity today. In the matter-dominated epoch we have $D_1(a) = a$. Analytically $D_1(a)$ is often approximated by the Meszaros equation \[188\]²:

$$D_1(a) = \frac{5\Omega_M H(a)}{2H_0} \int_0^a \frac{da'}{(a'H(a')/H_0)^3}.$$  

(5.36)

Therefore, we can easily observe the difference between the quintom and cosmological constant scenarios, due to the different Hubble expansion rates. More strictly one needs to solve Eq. (5.35) numerically. In the lower left graph of Fig. [14] we show the difference of $D_1(a)$ between the

² One should notice that the Meszaros equation is an exact solution to the differential equation for the linear-theory growth factor only when dark energy is a cosmological constant or it is absent. However, it is still a good approximation in the case of dynamical dark energy models on small length scales of the universe, as in the analysis of Section 3.1 of Ref. [189].
quintom with $w_{\text{eff}} = -1$ and the cosmological constant. The difference in the linear growth function is considerably large in the late-time evolution, and thus possibly distinguishable in future LSS surveys and in weak gravitational lensing (WGL) observations. WGL has emerged with a direct mapping of cosmic structures and it has been recently shown that the method of cosmic magnification tomography can be extremely efficient [190,191,192], which leaves a promising future for breaking the degeneracy between quintom and a cosmological constant.

6 Quintom models with higher derivative terms

As we demonstrated previously, we usually need to introduce new degrees of freedom into a normal lagrangian in order to obtain a viable quintom scenario. One approach is to construct a double-field quintom model introduced in Section IV. In this section we provide an alternative possibility of introducing extra degrees of freedom for the realization of the transition between quintessence phase and phantom phase [193]. This model is originally proposed by Lee and Wick to address on hierarchy problem in particle physics in the 70’s [194,195], and recently applied in [87] involving higher derivative operators in the Lagrangian.

6.1 Lee-Wick model

In this subsection we present a single field model with the inclusion of higher derivatives, following [87]. We consider a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}[\phi, X, (\Box \phi)(\Box \phi), (\nabla_\mu \phi)(\nabla^\mu \phi)],$$

where $X = 1/2 \nabla_\mu \phi \nabla^\mu \phi$ and $\Box \equiv \nabla_\mu \nabla^\mu$ is the d’Alembertian operator. We mention that this model is just an effective theory and one assumes that the operators associated with the higher derivatives can be derived from some fundamental theories, for instance due to the quantum corrections or the non-local physics in string theory [196,197,198]. Additionally, with the inclusion of higher derivative terms to the Einstein gravity, the theory is shown to be renormalizable [199]. Finally, higher derivative operators have been shown to be capable of stabilizing the linear fluctuations in the scenario of “ghost condensation” [200,201,202,203,204,205].

We consider the simple model with

$$\mathcal{L} = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{c}{2M^2}(\Box \phi)^2 - V(\phi),$$

where $M$ is a constant with mass dimension and $c$ is a dimensionless constant. The energy-momentum tensor reads [87]:
\[ T^{\mu \nu} = g^{\mu \nu} \left[ \frac{1}{2} \nabla_\rho \phi \nabla^\rho \phi + \frac{c}{2 M^2} (\Box \phi)^2 + \frac{c}{M^2} \nabla_\rho \phi \nabla^\rho (\Box \phi) + V \right] - \nabla^\mu \phi \nabla^\nu \phi - \frac{c}{M^2} \left[ \nabla^\mu \phi \nabla^\nu (\Box \phi) + \nabla^\mu \phi \nabla^\nu (\Box \phi) \right], \]  
(6.3)

and the equation of motion is given by

\[- \Box \phi - \frac{c}{M^2} \Box^2 \phi + \frac{dV}{d\phi} = 0. \]  
(6.4)

However, the energy-momentum tensor (6.3) can be rewritten as

\[ T^{\mu \nu} = \nabla^\mu \chi \nabla^\nu \chi - \nabla^\mu \psi \nabla^\nu \psi + \frac{1}{2} \left[ \nabla_\rho \psi \nabla^\rho \psi - \nabla_\rho \chi \nabla^\rho \chi + 2V(\psi - \chi) + \frac{M^2}{c^2} \chi^2 \right] g^{\mu \nu}, \]  
(6.5)

where \( \chi \) and \( \psi \) are defined by

\[ \chi = \frac{c}{M^2} \Box \phi, \quad \psi = \phi + \chi. \]  
(6.6)

It is not difficult to see that the energy-momentum tensor (6.5) can be derived from the following Lagrangian

\[ \mathcal{L} = -\frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi + \frac{1}{2} \nabla_\mu \chi \nabla^\mu \chi - V(\psi - \chi) - \frac{M^2}{2c} \chi^2, \]  
(6.7)

with \( \psi \) and \( \chi \) being two independent fields. The variations of the Lagrangian (6.7), with respect to the fields \( \psi \) and \( \chi \) respectively, give rise to the equations of motions

\[- \Box \psi - V' = 0, \]  
(6.8)

\[- \Box \chi + \frac{M^2}{c} \chi - V' = 0, \]  
(6.9)

where the prime denotes the derivative with respect to \( \psi - \chi \).

When one imposes \( \psi = \phi + \chi \) in (6.7) and (6.8), he recovers the equation of motion (6.4) of the single field model and the transformation equation (6.6). Thus, one can see that the single field model (6.2) is equivalent to the one with two fields (6.7), where one is the canonical scalar field \( \chi \) and the other is the phantom field \( \psi \). Therefore, assuming that \( \chi \) dominates the DE sector in the early time, the EoS parameter will be larger than \(-1\), but when the phantom mode becomes dominant the EoS parameter will become less than \(-1\), crossing the cosmological constant boundary at an intermediate redshift.

In general the potential term \( V(\psi - \chi) \) (equivalently \( V(\phi) \) in our single field model) should include interactions between the two fields \( \psi \) and \( \chi \). However, for some specific choices of the
potential $V$ these two modes could decouple. As an example we consider $V = (1/2) m^2 \phi^2$. Then the Lagrangian (6.7) can be “diagonalized” as:

$$L = \frac{1}{2} \nabla_\mu \phi_1 \nabla^\mu \phi_1 - \frac{1}{2} \nabla_\mu \phi_2 \nabla^\mu \phi_2 - \frac{1}{2} m_1^2 \phi_1^2 - \frac{1}{2} m_2^2 \phi_2^2,$$

(6.10)

through the transformation

$$\begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

(6.11)

where

$$a_1 = \frac{1}{2} \left( 1 + \frac{4cm^2}{M^2} \right)^{-1/4} \left( \sqrt{1 + \frac{4cm^2}{M^2}} - 1 \right),$$

$$a_2 = \frac{1}{2} \left( 1 + \frac{4cm^2}{M^2} \right)^{-1/4} \left( \sqrt{1 + \frac{4cm^2}{M^2}} + 1 \right),$$

(6.12)

and

$$m_1^2 = \frac{M^2}{2c} \left( \sqrt{1 + \frac{4cm^2}{M^2}} + 1 \right),$$

$$m_2^2 = \frac{M^2}{2c} \left( \sqrt{1 + \frac{4cm^2}{M^2}} - 1 \right).$$

(6.13)

Hence, one can see that the model (6.2) with $V = (1/2) m^2 \phi^2$ is equivalent to the uncoupled system (6.10). The two modes $\phi_1$ and $\phi_2$ evolve independently in the universe. The positivity of the parameter $c$ guarantees the positivity of $m_1^2$ ($m_2^2$ is always positive as long as $1 + \frac{4cm^2}{M^2} > 0$).

In the limit $m \ll M$, the masses of the two modes are approximately $m_1 \simeq M/\sqrt{c}$ and $m_2 \simeq m$. In fact, $\phi_1$ and $\phi_2$ are the eigenfunctions of the d’Alembertian operator $\Box$, with the eigenvalues $-m_1^2$ and $m_2^2$ respectively. The solution to the equation of motion (6.4) $\phi$, is decomposed by these eigenfunctions as $\phi = (1 + \frac{4cm^2}{M^2})^{-1/4}(\phi_1 + \phi_2)$. Finally, we mention that the perturbations in the phantom component $\phi_2$ do not present any unphysical instabilities at the classical level. Thus, the model is free from the difficulties of the singularity and the gravitational instabilities of the general k-essence-like scenarios [87].

In Fig. 15 we present the EoS parameter of the single field model as a function of $\ln a$, where $w$ clearly crosses $-1$ during evolution. On the same figure it is also depicted the corresponding evolution of EoS parameter of the equivalent two-field model. Indeed, one can see that the two models coincide. Therefore, the high derivative single scalar field model of DE presents the $-1$-crossing, and it is equivalent to the usual two-field quintom paradigm. However, note that when we consider the interactions between these two fields in the potential, they may possibly show
Fig. 15. (Color online) Evolution of the EoS parameters for the high derivative single-field model and the equivalent two-field model with one quintessence and one phantom field. From Ref. [87].

some different behaviors.

One can easily generalize the aforementioned model considering the Lagrangian [97]:

\[
\mathcal{L} = \frac{1}{2} A(\phi) \nabla_\mu \phi \nabla_\mu \phi + \frac{C(\phi)}{2 M_p^2} (\Box \phi)^2 - V(\phi),
\]

(6.14)

where \( \chi = \frac{C(\phi)}{M_p^2} \Box \phi \) and \( \phi = \phi(\psi, \chi) \). Using \( A(\phi) = -1, V(\phi) = 0 \) this Lagrangian is simplified as

\[
\mathcal{L} = -\frac{1}{2} \nabla_\mu \psi \nabla^\mu \psi + \frac{1}{2} \nabla_\mu \chi \nabla^\mu \chi - \frac{M_p^2 \chi^2}{2 C[\phi(\psi, \chi)]},
\]

(6.15)

where \( \phi = -(\psi + \chi) \). The equations of motion for the two fields are:

\[
\Box \psi + \frac{M_p^2 C'}{2 C^2} \chi^2 = 0,
\]

\[
\Box \chi - \frac{M_p^2 C'}{2 C^2} \chi^2 + \frac{M_p^2}{C} \chi = 0,
\]

(6.16)

where the prime is the derivative with respect to \( \psi + \chi \). In FRW universe, \( \Box = \partial^2 / \partial t^2 + 3 H \partial / \partial t \) with \( H \) being the Hubble expansion rate. Furthermore, one can easily extract the density and the pressure of DE as
\[
\rho = -\frac{\dot{\psi}^2}{2} + \frac{\dot{\chi}^2}{2} + \frac{M_p^2 \chi^2}{2C}, \tag{6.17}
\]
\[
p = -\frac{\dot{\psi}^2}{2} + \frac{\dot{\chi}^2}{2} - \frac{M_p^2 \chi^2}{2C}, \tag{6.18}
\]
and thus the EoS parameter writes
\[
w = \frac{p}{\rho} = -1 + \frac{2(\dot{\chi}^2 - \dot{\psi}^2)}{\dot{\chi}^2 - \dot{\psi}^2 + \frac{M_p^2 \chi^2}{C}}, \tag{6.19}
\]
where the dot represents the derivative with respect to the cosmic time. Again, \(\chi\) behaves as a canonical field while \(\psi\) as a phantom one. However, in this case they couple to each other through an effective potential, that is the last term in the right hand side of equation (6.15):
\[
V_{\text{eff}} = \frac{M_p^2 \chi^2}{2C[\phi(\psi, \chi)]}. \tag{6.20}
\]
As an example one can consider \[97\]:
\[
C[\phi(\psi, \chi)] = C_0 \left\{ \frac{\pi}{2} + \arctan \left[ \frac{\chi(\psi + \chi)}{M_p} \right] \right\}. \tag{6.21}
\]
Thus, \(C\) is almost constant when \(|\psi + \chi| \gg 0\), and \(\psi\) and \(\chi\) are nearly decoupled at these regimes, which are dubbed “weak coupling” regimes. By contrast, the two fields couple tightly in the “strong coupling” regime where \(|\psi + \chi| \sim 0\). In the weak coupling regime, as shown in (6.16), the phantom-like field \(\psi\) behaves as a massless scalar field and its energy density \(- (1/2) \dot{\psi}^2\) dilutes as \(a^{-6}\), with \(a\) is the scale factor of the universe. The quintessence-like field \(\chi\) has a mass term with \(m_\chi = M_p/\sqrt{C(\psi + \chi)}\). Its behavior is determined by the ratio of \(m_\chi/H\). If \(m_\chi \ll H\), \(\chi\) is slow-rolling and it behaves like a cosmological constant. On the other hand, in cases \(m_\chi \gg H\) the kinetic term and potential oscillate coherently and their assembly evolves as \(a^{-3}\), just like that of collisionless dust.

The EoS parameter of such a DE model is depicted in Fig. 16. The initial conditions are chosen in order for the DE to start in the strong coupling regime with \(\chi > 0\) at high red-shift. As we observe, DE is frozen quickly at initial times, and then it becomes significant around redshift \(z \sim 1\), and the phantom kinetic term \(\dot{\psi}^2\) becomes larger than the quintessence one \(\dot{\chi}^2\). The DE crosses the phantom divide from above to below. In the future, the system evolves into the weak coupling regime, \(\dot{\psi}^2\) dilutes quickly and \(\chi\) gets a large mass. The whole system behaves as cold matter and its EoS oscillate around the point of \(w = 0\). So, the universe will exit from the phantom phase and end in the state of matter-domination. This result resembles the late-time behavior of “B-inflation” model \[206\]. Therefore, the present cosmological scenario differs from the simple quintom DE model, since it leads to a double crossing of the phantom divide and it can give a nice exit from phantom phase, in which it stays just for a finite time, avoiding the realization of a Big Rip \[18, 129, 207, 208, 209\]. In addition, at late times \(w\) oscillates with
Fig. 16. (Color online) The EoS parameter $w$ as a function of $\ln a$ for $C = 6.0 \times 10^{121}$, $\lambda = 25$, and the initial values are $\psi_i = -0.052 M_p$, $\dot{\psi}_i = 1.40 \times 10^{-63} M_p^2$, $\chi_i = 0.050 M_p$, $\dot{\chi}_i = -1.44 \times 10^{-63} M_p^2$, with $\Omega_{DE0} \approx 0.73$. From Ref. [97].

high frequency around zero, so the universe will end in an accelerated expansion at some time. Finally, note that the quintom model based on higher derivative theory possesses also ghost degrees of freedom, and thus it could face the problem of quantum instability [51,50]. However, as pointed out in [210], the problem of quantum instability arises because $\phi$ and $\Box \phi$ are quantized in canonical way independently. In fact, in the present model both of them are determined by $\phi$, and a more appropriate quantization method seems to be possible to avoid the instability [97].

6.2 Quintom DE inspired by string theory

In recent years, there have been active studies on relating quintom scenario with fundamental physics, and in particular with string theory. In such a context a rolling tachyon field in the world volume theory of the open string, stretched between a D-brane and an anti-D-brane or a non-BPS D-brane, plays the role of the scalar field in the quintom model [101,211,212,213,214,215,216,217]. The effective action used in the study of tachyon cosmology consists of the standard Einstein-Hilbert action plus an effective action for the tachyon field on unstable D-brane or (D-brane)-(anti-D-brane) system. Such a single-field effective action, to the lowest order in $\nabla_{\mu} \phi \nabla^{\mu} \phi$ around the top of the tachyon potential, can be obtained by the stringy computations for either a D3 brane in bosonic theory [218,219] or a non-BPS D3 brane in supersymmetric theory [220] (see also [221] for an application in inflation). What distinguishes the tachyon action from the standard Klein-Gordon form for a scalar field, is that the former is non-standard but it is of the “Dirac-
Born-Infeld” form \cite{222,223,224}. The tachyon potential is derived from string theory itself and thus it has to satisfy some definite properties and requirements, such as to describe the tachyon condensation.

6.2.1 A model inspired by string theory

In this paragraph we construct a specific quintom model inspired by string theory. One considers an action of the form \cite{110}:

$$S = \int d^4x \sqrt{-g} \left[ -V(\phi) \sqrt{1 - \alpha' \nabla_\mu \phi \nabla^\mu \phi + \beta' \phi \Box \phi} \right], \quad (6.22)$$

which generalizes the usual “Born-Infeld-type” action, for the effective description of tachyon dynamics, by adding a term $\phi^2 \phi$ to the usual $\nabla_\mu \phi \nabla^\mu \phi$ in the square root. The two parameters $\alpha'$ and $\beta'$ in (6.22) can be also made arbitrary when the background flux is turned on \cite{225}. Note that we have defined $\alpha' = \alpha/M^4$ and $\beta' = \beta/M^4$ with $\alpha$ and $\beta$ being the dimensionless parameters respectively, and $M$ an energy scale used to make the “kinetic energy terms” dimensionless.

$V(\phi)$ is the potential of scalar field $\phi$ (e.g., a tachyon) with dimension of $[\text{mass}]^4$, with a usual behavior in general, that is bounded and reaching its minimum asymptotically. Note that, $\sqrt{1 - \alpha' \nabla_\mu \phi \nabla^\mu \phi + \beta' \phi \Box \phi} = \sqrt{g^\mu_\rho \nabla_\mu \phi \nabla^\rho \phi}$, and therefore in (6.22) the terms $\nabla_\mu \phi \nabla^\mu \phi$ and $\phi \Box \phi$ both involve two fields and two derivatives.

The equation of motion for the scalar field $\phi$ reads:

$$\frac{\beta}{2} \Box \left( \frac{V(\phi)}{f} \right) + \alpha \nabla_\mu \left( \frac{V \nabla^\mu \phi}{f} \right) + M^4 V \phi f + \frac{\beta V}{2f} \phi = 0 , \quad (6.23)$$

where $f = \sqrt{1 - \alpha' \nabla_\mu \phi \nabla^\mu \phi + \beta' \phi \Box \phi}$ and $V_\phi = dV/d\phi$. Following the convention of \cite{87}, the energy-momentum tensor $T^{\mu\nu}$ is given by the standard definition $\delta g_{\mu\nu} S \equiv -\int d^4x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}$, thus:

$$T^{\mu\nu} = g^{\mu\nu} \left[ V(\phi) f + \nabla_\rho (\psi \nabla^\rho \phi) \right] + \frac{\alpha V(\phi)}{M^4} f \nabla^\mu \phi \nabla^\nu \phi - \nabla^\mu \psi \nabla^\nu \phi - \nabla^\nu \psi \nabla^\mu \phi, \quad (6.24)$$

where $\psi \equiv \frac{\partial L}{\partial \dot{\phi}} = -\frac{V \beta \phi}{2M^4 f}$.

For a flat FRW universe and a homogenous scalar field $\phi$, the equation of motion (6.23) can be solved equivalently by the following two equations

$$\ddot{\phi} + 3H \dot{\phi} = \frac{\beta V^2 \phi}{4M^4 \psi^2} - \frac{M^4}{\beta \phi} \phi^2 , \quad (6.25)$$

$$\ddot{\psi} + 3H \dot{\psi} = (2\alpha + \beta) \left( \frac{M^4 \psi^2}{\beta^2 \phi^2} - \frac{V^2}{4M^4 \psi} \right) - (2\alpha - \beta) \frac{\alpha \psi}{\beta^2 \phi^2} \phi^2 - \frac{2\alpha}{\beta \phi} \dot{\phi} \dot{\psi} - \frac{\beta V \phi}{2M^4 \psi} \phi , \quad (6.26)$$

where we have made use of $\psi$ as defined above. Therefore, (6.25) is just the defining equation for $\psi$ in terms of $\phi$ and its derivatives. Finally, from (6.24) we obtain the energy density and
The DE EoS parameter as a function of $\ln a$ for $V(\phi) = V_0 e^{-\lambda \phi^2}$ with $V_0 = 0.8$, $\lambda = 1$, $\alpha = 1$, and $\beta = -0.8$, and for initial conditions $\phi_i = 0.9$, $\dot{\phi}_i = 0.6$, $\langle \Box \phi \rangle_i = \frac{4}{f}(\Box \phi)_i = 0$. From Ref. [110].

Numerically one sees that $\dot{\phi} = 0$ when $w$ crosses $-1$.

In Fig. 18 we assume a different potential, namely $V(\phi) = \frac{V_0}{e^{0.6 e^{-0.8 \phi}}}$. One observes again that the EoS of the model crosses $-1$ during the evolution. Finally, in Fig. 19 the potential has been also chosen as $V(\phi) = \frac{V_0}{e^{0.6 e^{-0.8 \phi}}}$, but taking a positive $\beta$ we observe that the EoS parameter starts with $w < -1$, it crosses $-1$ from below to above and then transits again to $w < -1$.

One can generalize the aforementioned model in the case of a tachyon, non-minimally coupled to gravity [226]. Thus, one adds an extra term $T \Box T$ to the usual terms in the square root and the non-minimal coupling $R f(T)$ [119,227]:

$$S = \int d^4 x \sqrt{-g} \left[ \frac{M_p^2}{2} R f(T) - AV(T) \sqrt{1 - \alpha' g^{\mu \nu} \partial_\mu T \partial_\nu T + \beta T \Box T} \right],$$
Fig. 18. (Color online) The DE EoS parameter as a function of $\ln a$, for $V(\phi) = \frac{V_0}{e^{\lambda \phi} + e^{-\lambda \phi}}$ with $V_0 = 0.5$, $\lambda = 1$, $\alpha = 1$, and $\beta = -0.8$, and for initial conditions $\phi_i = 0.9$, $\dot{\phi}_i = 0.6$, $(\Box \phi)_i = \frac{d}{dt}(\Box \phi)_i = 0$. From Ref. [110].

Fig. 19. (Color online) The DE EoS parameter as a function of $\ln a$, for $V(\phi) = \frac{V_0}{e^{\lambda \phi} + e^{-\lambda \phi}}$ with $V_0 = 0.5$, $\lambda = 1$, $\alpha = 1$, and $\beta = 0.8$, and for initial conditions $\phi_i = 0.9$, $\dot{\phi}_i = 0.6$, $(\Box \phi)_i = \frac{d}{dt}(\Box \phi)_i = 0$. From Ref. [110].

where $M_p$ is the reduced Planck mass. Action (6.29) can be brought to a simpler form for the
derivation of the equations of motion by performing a conformal transformation \( g_{\mu \nu} \rightarrow f(T) g_{\mu \nu} \):

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M_p^2}{2} \left( R - \frac{3f''}{2f^2} \partial_\mu T \partial^\mu T \right) - A \tilde{V}(T) \sqrt{1 - (\alpha' f(T) - 2\beta' f'(T) T) \partial_\mu T \partial^\mu T + \beta' f(T) T \Box T} \right],
\]

(6.30)

where \( \tilde{V}(T) = \frac{V(T)}{f(T)} \) is the effective potential of the tachyon. For a flat FRW universe and a homogenous scalar field \( T \), the equations of motion can be solved equivalently by the following two equations,

\[
\ddot{\psi} + 3H \dot{\psi} = \left( \frac{2\beta' f' T - \alpha' f}{f T} \right) \dot{\psi} T - \frac{A^2 \beta' f \tilde{V} T}{2\psi} \tilde{V}',
- \frac{3M_p^2}{2} \left( \frac{f' f'' - f^3}{f^3} \right) \dot{T}^2 - \left[ \frac{(1 - \beta')(\alpha' - 2\beta') f'}{\beta' f} - \frac{\alpha'}{T} \right] \psi \dot{T}^2 T,
\]

(6.31)

\[
\dot{T} + 3HT = \left[ \frac{2 \left( \frac{f'' + \beta' f'}{T^2} \right) T \dot{T}^2 - 2 \left( \alpha' - 2\beta' \frac{f'}{f} \right) \dot{H} \dot{T} \right] 1 + \frac{2\alpha'}{\beta'} - 3 \frac{f'}{f} T - \frac{3M_p^2}{2} \left( \frac{f'}{f} \right)^2 \frac{T}{\psi},
\]

(6.32)

where

\[
\dot{\psi} = \frac{\partial L}{\partial \dot{\phi}} = -A \beta' \tilde{V} f T \frac{\dot{T}}{2h},
\]

(6.33)

\[ h = \sqrt{1 - (\alpha' f - 2\beta' f'T) \partial_\mu T \partial^\mu T + \beta' f T \Box T}, \]

\[ \tilde{V}' = \frac{d\tilde{V}}{dT}. \]

Finally, the energy density and pressure read [119,227]:

\[
\rho = A \tilde{V} h + \frac{d}{a^3 dt} \left( a^3 \psi \dot{T} \right) + \left( \alpha' f - 2\beta' f'T \right) \frac{A \tilde{V}}{h} \dot{T}^2 - 2 \dot{\psi} \dot{T} + \frac{3M_p^2}{4} \left( \frac{f'}{f} \right) \dot{T}^2,
\]

(6.34)

\[
p = -A \tilde{V} h - \frac{d}{a^3 dt} \left( a^3 \psi \dot{T} \right) + \frac{3M_p^2}{4} \left( \frac{f'}{f} \right) \dot{T}^2.
\]

(6.35)

The numerical exploration of the constructed model has been performed in [119]. In Fig. 20 we present the EoS parameter of DE for the (motivated by string theory) potential \( V(T) = e^{-\lambda T^2} \) and \( f(T) = 1 + \sum_{i=1}^n c_i T^{2i} \). Thus, the model experiences the \(-1\)-crossing, before the tachyon potential reaches asymptotically to its minimum.

### 6.2.2 Analysis on perturbations

Let us examine the stability of the aforementioned model by considering quadratic perturbations. Consider a small perturbation \( \pi \) around the background \( \phi \),
Fig. 20. (Color online) The plot of EoS parameter for the potential $V(T) = e^{-\lambda T^2}$, for $\alpha = -2$ and $\beta = 2.2$. The initial values are $\phi = 1$ and $\dot{\phi} = 3$. From Ref. [119].

$$\phi \rightarrow \phi(t) + \pi(t, x) ,$$

(6.36)

where the background field $\phi$ in the FRW cosmology is spatially homogenous. After this shift of the field, and a tedious calculation, we obtain the following terms for the quadratic perturbations of the action [110]:

$$(2) S \sim \int d^4x \sqrt{-g} \left[ \left( \frac{\alpha'}{2} V f^{-1} + \frac{\alpha'^2}{2} V f^{-3} \dot{\phi}^2 + \frac{\beta'}{2} V f^{-1} + \frac{\beta'}{2} \phi V f^{-1} - \frac{\beta'^2}{4} \phi \square \phi V f^{-3} \right) \dot{\pi}^2 
- \left( \frac{\alpha'}{2} V f^{-1} + \beta' \phi V f^{-1} + \beta' \phi V f^{-1} - \frac{\beta'^2}{4} \phi \square \phi V f^{-3} \right) (\nabla \pi)^2 + ... \right] .$$

(6.37)

Interestingly, we notice that due to the positivity of the term $\frac{\alpha'^2}{2} V f^{-3} \dot{\phi}^2$ in this model if the coefficient of $(\nabla \pi)^2$ is positive, the term in front of $\dot{\pi}^2$ is guaranteed to be positively valued. The sound speed characterizing the stability property of the perturbations is given by

$$c_s^2 = \frac{\frac{\alpha'}{2} V f^{-1} + \frac{\alpha'^2}{2} V f^{-3} \dot{\phi}^2 + \frac{\beta'}{2} \phi V f^{-1} - \frac{\beta'^2}{4} \phi \square \phi V f^{-3}}{\frac{\alpha'}{2} V f^{-1} + \frac{\alpha'^2}{2} V f^{-3} \dot{\phi}^2 + \frac{\beta'}{2} \phi V f^{-1} + \frac{\beta'}{2} \phi V f^{-1} - \frac{\beta'^2}{4} \phi \square \phi V f^{-3}} .$$

(6.38)

If the $c_s^2$ lies in the range between 0 and 1 and the coefficient of $\dot{\pi}^2$ remains positive, then the model will be stable. In Fig. 21 and Fig. 22 we present $c_s^2$ and the coefficients of $\dot{\pi}^2$ respectively, for the models of Figs. 17, 18, and 19. We observe that for these models $c_s^2$ lies in the range [0, 1] and the coefficients of $\dot{\pi}^2$ are positive, thus the models are indeed stable.
Fig. 21. (Color online) Evolution of the sound speeds (from left to right) for the three models considered in the text. From Ref. [110].

Fig. 22. (Color online) Evolution of the coefficient of $\dot{\pi}^2$ (from left to right) for the three models considered in the text. From Ref. [110].
7 Realizations of quintom scenario in non-scalar systems

7.1 Spinor quintom

The quintom models described so far have been constructed by scalar fields, which are able to accommodate a rich variety of phenomenological behaviors, but the ghost field may lead to quantum instabilities. To solve this problem one may consider the linearized perturbations in the quantum-corrected effective field equation at two-loop order \cite{228,229,230} and obtain a super-acceleration phase induced by these quantum effects \cite{231,232}. However, there is the alternative possibility where the acceleration of the universe is driven by a classical homogeneous spinor field. Some earlier studies on applications of spinor fields in cosmology are given in Refs. \cite{233,234,235}. In recent years there are many works on studying spinor fields as gravitational sources in cosmology. For example see Refs. \cite{236,237,238} for inflation and cyclic universe driven by spinor fields, Refs. \cite{239} for spinor matter in Bianchi Type I spacetime, Refs. \cite{240,241} for a DE model with spinor matter etc.

7.1.1 Algebra of a spinor field in FRW cosmology

Recently, a delicate quintom model was proposed by making use of spinor matter \cite{122}. This scenario can realize the quintom behavior, with an EoS crossing $-1$, without introducing a ghost field. Instead, the derivative of its potential with respect to the scalar bilinear $\bar{\psi}\psi$, which is defined as the mass term, becomes negative when the spinor field lies in the phantom-like phase. If this model can realize its EoS across $-1$ more than one time, the total EoS of the universe can satisfy $w \geq -1$ during the whole evolution, as it is required by NEC \cite{242,243}, and we expect to treat this process as a phase transition merely existing for a short while.

To begin with, we simply review the dynamics of a spinor field which is minimally coupled to Einstein gravity (see Refs. \cite{244,245,246} for a detailed introduction). Following the general covariance principle, a connection between the metric $g_{\mu\nu}$ and the vierbein is given by

$$
g_{\mu\nu}e^a_\mu e^b_\nu = \eta_{ab},
$$

(7.1)

where $e^a_\mu$ denotes the vierbein, $g_{\mu\nu}$ is the space-time metric, and $\eta_{ab}$ is the Minkowski metric with $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. Note that the Latin indices represents the local inertial frame, while the Greek indices represents the space-time frame.

We choose the Dirac-Pauli representation as

$$
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},
$$

(7.2)

where $\sigma_i$ is Pauli matrices. One can see that the $4 \times 4 \gamma^a$ satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta_{ab}$. 

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The $\gamma^a$ and $e_\mu^a$ provide the definition of a new set of Gamma matrices

$$\Gamma^\mu = e_\mu^a \gamma^a,$$  \hspace{1cm} (7.3)

which satisfy the algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2g_{\mu\nu}$. The generators of the Spinor representation of the Lorentz group can be written as $\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$. Therefore, the covariant derivatives are given by

$$D_\mu \psi = (\partial_\mu + \Omega^\mu_\mu) \psi$$
$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi} \Omega^\mu_\mu ,$$  \hspace{1cm} (7.4)

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi} \Omega^\mu_\mu ,$$  \hspace{1cm} (7.5)

where the Dirac adjoint $\bar{\psi}$ is defined as $\psi^+ \gamma^0$. The $4 \times 4$ matrix $\Omega^\mu_\mu = \frac{1}{2}\omega^{\mu}_{\mu ab} \Sigma^{ab}$ is the spin connection, where $\omega^{\mu}_{\mu ab} = e_\nu^a \nabla_\mu e_\nu^b$ are the Ricci spin coefficients.

Using the aforementioned algebra we can write the following Dirac action in a curved space-time background \cite{237,240,247}:

$$S_\psi = \int d^4x \; e \left[ \frac{i}{2} \left( \bar{\psi} \Gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \Gamma^\mu \psi \right) - V \right].$$  \hspace{1cm} (7.6)

Here, $e$ is the determinant of the vierbein $e_\mu^a$ and $V$ stands for the potential of the spinor field $\psi$ and its adjoint $\bar{\psi}$. Due to the covariance requirement, the potential $V$ depends only on the scalar bilinear $\bar{\psi} \psi$ and on the “pseudo-scalar” term $\bar{\psi} \gamma^5 \psi$. For simplicity we neglect the latter term and assume that $V = V(\bar{\psi} \psi)$.

Varying the action with respect to the vierbein $e_\mu^a$, we obtain the energy-momentum-tensor

$$T^\mu_\nu = \frac{e_{\mu a} \delta S_\psi}{e \delta e_\nu^a}$$

$$= \frac{i}{4} \left[ \bar{\psi} \Gamma^\nu D_\mu \psi + \bar{\psi} \Gamma^\mu D_\nu \psi - D_\mu \bar{\psi} \Gamma^\nu \psi - D_\nu \bar{\psi} \Gamma^\mu \psi \right] - g^\mu_\nu \mathcal{L}_\psi .$$  \hspace{1cm} (7.7)

On the other hand, variation of the action with respect to the fields $\bar{\psi}$, $\psi$ respectively yields the equation of motion of the spinor

$$i \Gamma^\mu D_\mu \psi - \frac{\partial V}{\partial \bar{\psi}} = 0 , \quad i D_\mu \bar{\psi} \Gamma^\mu + \frac{\partial V}{\partial \psi} = 0 .$$  \hspace{1cm} (7.8)

As usual, we deal with the homogeneous and isotropic FRW metric. Correspondingly, the vierbein are given by

$$e_0^\mu = \delta_0^\mu , \quad e_i^\mu = \frac{1}{a} \delta_i^\mu .$$  \hspace{1cm} (7.9)
Assuming that the spinor field is space-independent, the equations of motion read:

\[ \dot{\bar{\psi}} + \frac{3}{2} H \bar{\psi} + i \gamma^0 V' \bar{\psi} = 0 \quad (7.10) \]
\[ \dot{\psi} + \frac{3}{2} H \psi - i \gamma^0 V' \psi = 0 \quad , \quad (7.11) \]

where a dot denotes a time derivative \( \frac{d}{dt} \), a prime denotes a derivative with respect to \( \bar{\psi} \bar{\psi} \), and \( H \) is the Hubble parameter. Taking a further derivative, we can obtain:

\[ \bar{\psi} \bar{\psi} = \frac{N}{a^3} \quad , \quad (7.12) \]

where \( N \) is a positive time-independent constant, defined as the value of \( \bar{\psi} \bar{\psi} \) at present.

From the expression of the energy-momentum tensor in (7.7), we obtain the energy density and pressure of the spinor field:

\[ \rho_\psi = T^{00} = V \quad (7.13) \]
\[ p_\psi = -T^i_i = V' \bar{\psi} \psi - V \quad , \quad (7.14) \]

where Eqs. (7.10) and (7.11) have been applied. The EoS of the spinor field, defined as the ratio of its pressure to energy density, is given by

\[ w_\psi \equiv \frac{p_\psi}{\rho_\psi} = -1 + \frac{V' \bar{\psi} \psi}{V} \quad . \quad (7.15) \]

If we assume the potential to be a power law as \( V = V_0 \left( \frac{\bar{\psi} \psi}{N} \right)^\alpha \), with \( \alpha \) as a nonzero constant, we obtain a constant EoS:

\[ w_\psi = -1 + \alpha \quad . \quad (7.16) \]

In this case, the spinor matter behaves like a linear-barotropic-like perfect fluid. For example, if \( \alpha = \frac{4}{3} \), we can acquire \( \rho_\psi \sim a^{-4} \) and \( w_\psi = \frac{1}{3} \), which is the same as radiation. If \( \alpha = 1 \), then \( \rho_\psi \sim a^{-3} \) and \( w_\psi = 0 \), thus this component behaves like normal matter.

It is interesting to see that the spinor matter is able to realize the acceleration of the universe if \( \alpha < \frac{2}{3} \). Thus, it provides a potential motivation to construct a dynamical DE model with the spinor matter. In the next subsection we focus our investigation on constructing the subclass of quintom DE models using the spinor field, which is called Spinor Quintom.
7.1.2 Evolutions of Spinor Quintom

In the model of the previous subsection $V$ is positive in order for the energy density to be positive (see (7.13)). Therefore, from (7.15) we deduce that $w_\psi > -1$ when $V' > 0$ and $w_\psi < -1$ when $V' < 0$. The former corresponds to a quintessence-like phase and the latter to a phantom-like phase. Therefore, the potential derivative $V'$ is required to change its sign if one desires to obtain a quintom picture. In terms of the variation of $V'$ there are three categories of evolutions in spinor quintom:

(i) $V' > 0 \rightarrow V' < 0$,

which gives a Quintom-A scenario by describing the universe evolving from quintessence-like phase with $w_\psi > -1$ to phantom-like phase with $w_\psi < -1$;

(ii) $V' < 0 \rightarrow V' > 0$,

which gives a Quintom-B scenario for which the EoS is arranged to change from below $-1$ to above $-1$;

(iii) $V'$ changes its sign more than one time, thus one obtains a new Quintom scenario with its EoS crossing $-1$ more than one time, dubbed Quintom-C scenario.

In the following, we assume different potentials in order to realize the three types of evolutions mentioned above.

To begin with, we shall investigate Case (i) and provide a Quintom-A model. We use the potential $V = V_0[(\bar{\psi}\psi - b)^2 + c]$, where $V_0$, $b$, $c$ are undefined parameters. Then we obtain $V' = 2V_0(\bar{\psi}\psi - b)$ and the EoS reads:

$$w_\psi = \frac{(\bar{\psi}\psi)^2 - b^2 - c}{(\bar{\psi}\psi)^2 - 2b\bar{\psi}\psi + b^2 + c}.$$  \hspace{1cm} (7.17)

According to (7.12) one finds that $\bar{\psi}\psi$ is decreasing with increasing scale factor $a$ during the expansion of the universe. From the formula of $V'$ we deduce that at the beginning of the evolution the scale factor $a$ is very small, so $\bar{\psi}\psi$ becomes very large and ensures that initially $V' > 0$. Then $\bar{\psi}\psi$ decreases with expanding $a$. At the moment where $\bar{\psi}\psi = b$, one can see that $V' = 0$, which results in the EoS $w_\psi = -1$. After that $V'$ becomes less than 0 and thus the universe enters a phantom-like phase. Finally, the universe approaches a de-Sitter spacetime. This behavior is also obtained by numerical elaboration and it is shown in Fig. 23. From this figure one observes that the EoS $w_\psi$ starts the evolution from 1, then mildly increases to a maximum and then begins to decrease. When $\bar{\psi}\psi = b$, it reaches the point $w_\psi = -1$ and crosses $-1$ from above to below smoothly. After that, the EoS progressively decreases to a minimal value, then

\footnote{Note that we choose the potentials phenomenologically without any constraints from quantum field theory or other consensus. From the phenomenological viewpoint it is allowed to treat the background classically, while dealing with the perturbations in quantum level, like what it is usual in inflation theory.}
Evolution of the EoS in Case (i) as a function of time. For numerical elaboration we consider the potential of the spinor field as \( V = V_0[(\bar{\psi}\psi - b)^2 + c] \), where we choose \( V_0 = 1.0909 \times 10^{-117} \), \( b = 0.05 \) and \( c = 10^{-3} \) for the model parameters. For the initial condition we take \( N = 0.051 \). From Ref. [122].

Fig. 23. (Color online)

In Case (ii) we choose the potential as \( V = V_0[−(\bar{\psi}\psi - b)\bar{\psi}\psi + c] \). Then one can obtain \( V' = V_0(−2\bar{\psi}\psi + b) \) and the EoS

\[
\frac{d\bar{\psi}}{dt} = \frac{-(\bar{\psi}\psi)^2 - c}{(\bar{\psi}\psi)^2 + b\bar{\psi}\psi + c}.
\]

Initially \( V' \) is negative due to the large values of \( \bar{\psi}\psi \). Then it increases to 0 when \( \bar{\psi}\psi = \frac{b}{2} \), whereafter it changes its sign and becomes larger than 0, in correspondence with the Case (ii).

Fig. 24.

In the third case we explore a quintom scenario which presents the \(-1\)-crossing for two times, using the potential \( V = V_0[(\bar{\psi}\psi - b)^2\bar{\psi}\psi + c] \). Thus, we acquire \( V' = V_0(3\bar{\psi}\psi - b) \) and the EoS becomes

\[
\frac{d\bar{\psi}}{dt} = \frac{2(\bar{\psi}\psi)^3 - 2b(\bar{\psi}\psi)^2 - c}{(\bar{\psi}\psi)^3 - 2b(\bar{\psi}\psi)^2 + b^2(\bar{\psi}\psi) + c}.
\]

Evidently, the equation \( V' = 0 \) has two solutions which are \( \bar{\psi}\psi = b \) and \( \bar{\psi}\psi = \frac{b}{3} \), thus \( V' \) changes its sign two times. From the EoS expression (7.19), we find that initially \( w_{\psi} > -1 \). When the value of \( \bar{\psi}\psi \) becomes equal to \( b \), it crosses \( -1 \) for the first time. After the first crossing, it
Evolution of the EoS in Case (ii) as a function of time. For numerical elaboration we consider the potential of the spinor field as 
\[ V = V_0[-(\dot{\psi}\psi - b)\bar{\psi}\psi + c], \]
where we choose 
\[ V_0 = 1.0909 \times 10^{-117}, \]
\[ b = 0.05 \]
and 
\[ c = 10^{-3} \]
for the model parameters. For the initial condition we take 
\[ N = 0.051. \]
From Ref. [122].

technology state, it continuously descends until it passes through its minimum, then it ascends to \( \bar{\psi}\psi = b^3 \) and then it experiences the second crossing, moving eventually to the quintessence-like phase. This behavior is depicted in Fig. 25. One can see that in this case the Big Rip can be avoided.

Moreover, considering the dark-matter component with energy density \( \rho_M \propto 1/a^3 \) and \( w_M = 0 \), we can see that the EoS of the universe \( w_u = w_\psi \frac{\rho_\psi}{\rho_\psi + \rho_M} \) satisfies the relation \( w_u \geq -1 \) in this case. As it is argued in Ref. [243], NEC might be satisfied in such a model, since \( w_\psi < -1 \) is only obtained for a short period during the evolution of the universe.

As mentioned in previous sections, one important issue of a DE model is the analysis of its perturbations. Performing such an investigation we might be able to learn to what degree the system is stable both at quantum and classical level. Usually systems with \( w < -1 \) present instabilities, for example a phantom universe suffers from a Big Rip singularity. Although this classical singularity can be avoided in most quintom models, which usually enter a de-Sitter expansion in the final epoch, all the scalar quintom models suffer from quantum instabilities since there are negative kinetic modes from ghost fields. Here we would like to perform the perturbation theory of spinor quintom. Since we do not introduce any ghost fields in our model, the phantom divide crossing is achieved by the spinor field itself and thus it does not result to any particular instability.

In order to simplify the derivation, we would like to redefine the spinor as \( \psi_N \equiv a^{3/2}\psi \). Then perturbing the spinor field we extract the perturbation equation as
Fig. 25. (Color online) Evolution of the EoS in Case (iii) as a function of time. The magenta solid line stands for the EoS of spinor quintom and the green dot line stands for that of the whole universe. For numerical elaboration we consider the potential of the spinor field as
\[ V = V_0[(\bar{\psi}\psi - b)^2\bar{\psi}\psi + c], \]
where we choose \( V_0 = 1.0909 \times 10^{-117} \), \( b = 0.05 \) and \( c = 10^{-3} \) for the model parameters. For the initial condition we take \( N = 0.051 \). From Ref. [122].

\[
\frac{d^2}{d\tau^2} \delta \psi_N - \nabla^2 \delta \psi_N + a^2 \left[ V'' + i\gamma^0(HV' - 3HV''\bar{\psi}\psi) \right] \delta \psi_N =
-2a^2V''\delta(\bar{\psi}\psi)\psi_N - i\gamma^\mu \partial_\mu[aV''\delta(\bar{\psi}\psi)]\psi_N ,
\]
(7.20)

where \( \tau \) is the conformal time defined by \( d\tau \equiv dt/a \). Since the right hand side of the equation decays proportionally to \( a^{-3} \) or even faster, we can neglect those terms during the late-time evolution of the universe for simplicity.

From the perturbation equation above, we can read that the sound speed is equal to 1, and this eliminates the instability of the system in short wavelength. Furthermore, when the EoS \( w \) crosses \(-1\) we have \( V' = 0 \) and thus the eigenfunction of the solution to Eq. (7.20) in momentum space is a Hankel function with an index \( \frac{1}{2} \). Therefore, the perturbations of the spinor field oscillate inside the hubble radius. This is an interesting result, because in this way we might be able to establish the quantum theory of the spinor perturbations, just as what it is done in inflation theory.

Note that the above derivation does not mean that spinor quintom is able to avoid all instabilities. We still have not examined the effects of the right hand side of Eq. (7.20), which could destroy the system stability under particular occasions. Finally, another possible instability could arise from quantum effects, if this model is non-renormalizable.
7.1.3 A unified model of Quintom and Chaplygin gas

As we have analyzed so far, a spinor field with a power-law-like potential behaves like a perfect fluid with a constant EoS. However, it is still obscure to establish a concrete scenario to explain how a universe dominated by matter evolves to the current stage, that is dominated by DE. The last years, an alternative interesting perfect fluid, with an exotic EoS $p = -c/\rho$, has been applied into cosmology [248] in the aim of unifying a matter-dominated phase where $\rho \propto 1/a^3$, and a de-Sitter phase where $p = -\rho$, which describes the transition from a universe filled with dust-like matter to an exponentially expanding universe. This so-called Chaplygin gas [248] and its generalization [249] have been intensively studied in the literature.

Some possibilities of this scenario, motivated by field theory, are investigated in [250]. Interestingly, a Chaplygin gas model can be viewed as an effective fluid associated with D-branes [251,252], and it can be also obtained from the Born-Infeld action [253,254]. The combination of quintom and Chaplygin gas has been realized by the interacting Chaplygin gas scenario [255], as well as in the framework of Randall-Sundrum braneworld [256]. Finally, the Chaplygin gas model has been thoroughly investigated for its impact on the cosmic expansion history. A considerable range of models was found to be consistent with SNIa data [257], the CMBR [258], the gamma-ray bursts [259], the X-ray gas mass fraction of clusters [260], the large-scale structure [261], etc.

Here, we propose a new model constructed by spinor quintom, which combines the property of a Chaplygin gas. The generic expression of the potential is given by

$$V = \sqrt[1+\beta]{f(\bar{\psi}\psi)} + c,$$

(7.21)

where $f(\bar{\psi}\psi)$ is an arbitrary function of $\bar{\psi}\psi$. Altering the form of $f(\bar{\psi}\psi)$, one can realize both the Chaplygin gas and quintom scenario in a spinor field.

Firstly, let us show how this model recovers a picture of generalized Chaplygin gas. Assuming $f(\bar{\psi}\psi) = V_0(\bar{\psi}\psi)^{1+\beta}$ the potential becomes

$$V = \sqrt[1+\beta]{V_0(\bar{\psi}\psi)^{1+\beta}} + c,$$

(7.22)

and therefore, the energy density and pressure of (7.13), (7.14) read

$$\rho_\psi = \sqrt[1+\beta]{V_0(\bar{\psi}\psi)^{1+\beta}} + c,$$

(7.23)

$$p_\psi = -c[V_0(\bar{\psi}\psi)^{1+\beta} + c]^{-\frac{\beta}{1+\beta}}.$$

(7.24)

Thus, this spinor model behaves like a generalized Chaplygin fluid, satisfying the exotic relation $p_\psi = -\frac{c}{\rho_\psi}$. Being more explicit we assume $\beta = 1$, obtaining the expressions of energy density and pressure as
\[ \rho_\psi = \sqrt{\frac{N^2}{a^6} + c}, \quad p_\psi = -\frac{c}{\rho_\psi}. \] (7.25)

In this case a perfect fluid of Chaplygin gas is given by a spinor field. Based on the above analysis, we conclude that this simple and elegant model is able to mimic different behaviors of a perfect fluid and therefore it accommodates a large variety of evolutions phenomenologically.

In succession, we will use this model to realize a combination of a Chaplygin gas and a Quintom-A model, which is mildly favored by observations. Choosing \( f(\bar{\psi}\psi) \) to be \( f(\bar{\psi}\psi) = V_0(\bar{\psi}\psi - b)^2 \) we acquire the potential

\[ V = \sqrt{V_0(\bar{\psi}\psi - b)^2 + c}, \] (7.26)

where \( V_0, b, c \) are undetermined parameters. Its derivative reads

\[ V' = \frac{V_0(\bar{\psi}\psi - b)}{\sqrt{V_0(\bar{\psi}\psi - b)^2 + c}}, \] (7.27)

and thus the EoS becomes

\[ w_\psi = -1 + \frac{V_0\bar{\psi}\psi(\bar{\psi}\psi - b)}{V_0(\bar{\psi}\psi - b)^2 + c}. \] (7.28)

with the \(-1\)-crossing taking place when \( \bar{\psi}\psi = b \). During the expansion \( \bar{\psi}\psi \) is decreasing following (7.12), with \( \bar{\psi}\psi \to \infty \) at the beginning of the evolution where \( a \to 0 \). Therefore, \( w_\psi \) increases from 0 to a maximum value and then it starts to decrease. When \( \bar{\psi}\psi = b \) the EoS arrives at the cosmological constant boundary \( w_\psi = -1 \) and then it crosses it. Due to the existence of \( c \)-term, the EoS eventually approaches the cosmological constant boundary. In this case the universe finally becomes a de-Sitter space-time.

Considering a universe filled with dark matter and the aforementioned spinor quintom matter, we perform a numerical analysis and we present the evolution of the EoS in Fig. 26. Moreover, in Fig. 27 we display the evolution of the density parameters of dark matter \( \Omega_M \equiv \rho_M/(\rho_M + \rho_\psi) \) and spinor quintom \( \Omega_\psi \equiv \rho_\psi/(\rho_M + \rho_\psi) \). It is evident that the ratio of these two components from the beginning of evolution is of order one, in relief of the fine-tuning and coincidence problems. During the evolution DE overtakes dark matter, driving the universe into an accelerating expansion at present, and eventually it dominates the universe completely, leading to an asymptotic de-Sitter expansion.

Summarizing the results of this section we see that while a scalar-type quintom model leads to quantum instabilities (due to the ghost field with a negative kinetic term), the class of quintom models in virtue of a spinor field can avoid such a problem. As shown above, such a scenario could behave as a generalized Chaplygin gas, and give rise to \(-1\)-crossing during the evolution, caused by the sign-flip of the \( V' \)-term. Compared with other models experiencing the \(-1\)-crossing, this scenario is also economical, in the sense that it involves a single spinor field.
Fig. 26. (Color online) *Evolution of the EoS of the unified model with potential (7.26) as a function of time. For the numerical analysis we assume \( V_0 = 3.0909 \times 10^{-239} \), \( b = 0.05 \) and \( c = 9 \times 10^{-241} \), while for the initial conditions we take \( N = 0.051 \). From Ref. [122].*

Fig. 27. (Color online) *Evolution of the density parameters of DE (violet solid line) and dark matter (red dash line) as a function of time. From Ref. [122].*
7.2 Other non-scalar models

We finish this section mentioning that non-scalar systems may give rise to a quintom scenario under certain cases. For example, developed from a vector-like DE model [262,263], Ref. [264] has shown that an unconventional cosmological vector field in absence of gauge symmetry can realize its EoS across $-1$. This model and its extensions have also been shown to be possible candidates for DE [265,266,267,268,269,270,271], and even solve the coincidence problem [272,273,274]. However, see [275] for a different viewpoint.

Finally, recently it has been noticed that the quintom scenario can be achieved in a model of non-relativistic gravity [276,277,278], but this is realized at a high energy scale, far away from that of DE. The possible successes of this model has led to the exploration of the potential relations among quintom scenario and phenomena of Lorentz symmetry breaking [279,280,281,282,283].

8 Quintom scenario in the braneworld

An alternative way of explaining the observed acceleration of the late universe is to modify gravity at large scales. A well-studied model of modified gravity is the braneworld model. Although the exciting idea that we live in a fundamentally higher-dimensional spacetime which is greatly curved by vacuum energy was older [284,285,286,287,288,289,290,291,292,293,294,295], the new class of “warped” geometries offered a simple way of localizing the low energy gravitons on the 4D submanifold (brane) [296,297,298]. Motivated by string/M theory, AdS/CFT correspondence, and the hierarchy problem of particle physics, braneworld models were studied actively in recent years [299,300,301,302,303,304,305], and have been shown to admit a much wider range of possibilities for dark energy [306]. A well studied model is the Dvali-Gabadadze-Porrati (DGP) one [307,308] where the braneworld is embedded in the flat bulk with infinite extra dimension. In this model, gravity appears 4D at short distances but it is altered at distance larger than some freely adjustable crossover scale $r_c$, through the slow evaporation of the graviton off our 4D braneworld universe into the fifth dimension. The inclusion of a graviton kinetic term on the brane recovers the usual gravitational force law scaling, $1/r^2$, at short distances, but at large distances it asymptotes to the 5D scaling, $1/r^3$. The matter particles cannot freely propagate in the extra dimensions, and are constrained to live on the (tensionless) brane. In such a model, late-time self-acceleration solutions appears naturally [309,310,311], since they are driven by the manifestation of the excruciatingly slow leakage of gravity into the extra dimension.

8.1 Quintom DE in DGP scenario

In this subsection we investigate a simple model of a single scalar field in the framework of DGP braneworld, following [108]. Although the behavior of the effective DE on a DGP brane with a cosmological constant and dust has been studied in [102,312], from the unified theoretic point of view we can assume that the gravitational action is not necessarily the Einstein-Hilbert action. In fact, string theory suggests that the dimensionally reduced effective action includes not only
higher-order curvature terms but also dilatonic gravitational scalar fields. Thus, at the level of
the low-energy 5D theory (we restrict to 5D braneworld models although even higher-dimensional
are also possible \cite{313,314,315,316,317}), it is naturally expected that there appears a dilaton-like
scalar field in addition to the Einstein-Hilbert action \cite{318,319,320}. Hence, it is interesting to
investigate how such a scalar field in the 5D theory affects the braneworld \cite{321,322}. Since we
consider a single-field model, the accelerated expansion of the universe will be a result of the
combined effect of the field evolution and the competition between 4D gravity and 5D gravity.

Let us start from the action of the DGP model

\[ S = S_{\text{bulk}} + S_{\text{brane}}, \quad (8.1) \]

where

\[ S_{\text{bulk}} = \int_M d^5X \sqrt{-g} \frac{1}{2 \kappa_5^2} (5) R, \quad (8.2) \]

and

\[ S_{\text{brane}} = \int_M d^4x \sqrt{-g} \left[ \frac{1}{\kappa_5^2} K^\pm + L_{\text{brane}}(g_{\alpha\beta}, \psi) \right]. \quad (8.3) \]

Here \( \kappa_5^2 \) is the 5-dimensional gravitational constant, \( (5) R \) is the 5-dimensional curvature scalar,
\( x^\mu (\mu = 0, 1, 2, 3) \) are the induced 4D coordinates on the brane and \( K^\pm \) is the trace of extrinsic
curvature on either side of the brane. Finally, \( L_{\text{brane}}(g_{\alpha\beta}, \psi) \) is the effective 4D Lagrangian, which
is given by a generic functional of the brane metric \( g_{\alpha\beta} \) and brane matter fields \( \psi \).

We consider a brane Lagrangian consisting of the following terms

\[ L_{\text{brane}} = \frac{\mu^2}{2} R + L_m + L_\phi, \quad (8.4) \]

where \( \mu \) is the 4D reduced Planck mass, \( R \) denotes the curvature scalar on the brane, \( L_m \) stands
for the Lagrangian of other matters on the brane, and \( L_\phi \) represents the lagrangian of a scalar
confined to the brane. Assuming a mirror symmetry in the bulk we obtain the Friedmann equation
on the brane \cite{309,310,311,89}:

\[ H^2 + \frac{k}{a^2} = \frac{1}{3\mu^2} \left[ \rho + \rho_0 + \theta \rho_0 \left( 1 + \frac{2\rho}{\rho_0} \right)^{1/2} \right], \quad (8.5) \]

where \( k \) is the spatial curvature of the three dimensional maximally symmetric space in the FRW
metric on the brane, and \( \theta = \pm 1 \) denotes the two branches of DGP model. \( \rho \) denotes the total
energy density on the brane, including dark matter and the scalar field \( (\rho = \rho_\phi + \rho_{dm}) \), and the
term \( \rho_0 \) relates the strength of the 5D gravity with respect to 4D gravity, that is \( \rho_0 = \frac{6\mu^2}{r_c^2} \),
where the cross radius is defined as \( r_c \equiv \kappa_5^2\mu^2 \). Comparing the modified Friedmann equation on the
brane with the standard one:

\[ H^2 + \frac{k}{a^2} = \frac{1}{3\mu^2} \left( \rho_{dm} + \rho_{de} \right), \quad (8.6) \]

one obtains the density of the effective 4D dark energy as

\[ \rho_{de} = \rho_\phi + \rho_0 + \theta \rho_0 \left[ \rho + \rho_0 + \theta \rho_0 \left( 1 + \frac{2\rho}{\rho_0} \right)^{1/2} \right]. \quad (8.7) \]
As usual DE satisfies the continuity equation:

$$\frac{d\rho_{de}}{dt} + 3H(\rho_{de} + p_{eff}) = 0, \quad (8.8)$$

where $p_{eff}$ denotes the effective pressure of DE. Then we can express the EoS of DE as

$$w_{de} = \frac{p_{eff}}{\rho_{de}} = -1 + \frac{1}{3} \left( \frac{d\ln \rho_{de}}{d\ln(1 + z)} \right). \quad (8.9)$$

where, from (8.8)

$$\frac{d\ln \rho_{de}}{d\ln(1 + z)} = \frac{3}{\rho_{de}} \left\{ \rho_{de} + p_{de} + \theta \left[ 1 + 2 \left( \frac{\rho_{de} + \rho_{dm}}{\rho_0} \right) \right]^{-1/2} \left( \rho_{de} + \rho_{dm} + p_{de} \right) \right\}. \quad (8.10)$$

We mention that the DE pressure $p_{de}$ is different from $p_{eff}$.

Clearly, if $\rho_{de}$ decreases and then increases with respect to redshift, or increases and then decreases, it is implied that DE experiences a phantom-divide crossing. Equations like (8.5) and (8.8) coincide with the subclass of inhomogeneous EoS of FRW universe [90] or FRW universe with general EoS [323], which can give rise to the phantom-divide crossing.

### 8.1.1 Canonical scalar field

For a canonical scalar with an exponential potential of the form: $V = V_0 e^{-\lambda \phi}$ the effective DE evolves as [108]:

$$\frac{d\rho_{de}}{d\ln(1 + z)} = 3 \left( \dot{\phi}^2 + \theta \left[ 1 + \frac{\dot{\phi}^2 + 2V + 2\rho_{dm}}{\rho_0} \right]^{-1/2} \left( \dot{\phi}^2 + \rho_{dm} \right) \right). \quad (8.11)$$

If $\theta = 1$ both terms on the RHS are positive and thus it never goes to zero at finite time, while in the case $\theta = -1$ the two terms of the RHS have opposite sign, therefore it is possible to obtain the $-1$-crossing. Hence, we restrict our study to $\theta = -1$.

In order to transform the aforementioned dynamical system into its autonomous form [91,141,142], we introduce the auxiliary variables:

$$x \equiv \frac{\dot{\phi}}{\sqrt{6\mu H}}, \quad y \equiv \frac{\sqrt{V}}{\sqrt{3\mu H}}, \quad l \equiv \frac{\sqrt{\rho_M}}{\sqrt{3\mu H}}, \quad b \equiv \frac{\sqrt{\rho_0}}{\sqrt{3\mu H}}. \quad (8.12)$$

Thus, the dynamics of the system can be described by:
\[ x' = -\frac{3}{2} \alpha x(2x^2 + l^2) + 3x - \frac{\sqrt{6}}{2} \lambda y^2, \] (8.13)
\[ y' = -\frac{3}{2} \alpha y(2x^2 + l^2) + \frac{\sqrt{6}}{2} \lambda xy, \] (8.14)
\[ l' = -\frac{3}{2} \alpha l(2x^2 + l^2) + \frac{3}{2} l, \] (8.15)
\[ b' = -\frac{3}{2} \alpha b(2x^2 + l^2), \] (8.16)

where \( \alpha \equiv 1 - (1 + 2x^2 + y^2 + l^2)^{-1/2} \), and here a prime stands for derivation with respect to \( s \equiv -\ln(1 + z) \), and we have set \( k = 0 \). One can see that this system degenerates to a quintessence with dust matter in standard general relativity. Note that the four equations (8.13), (8.14), (8.15), (8.16) are not independent, since the constraint

\[ x^2 + y^2 + l^2 + b^2 - b^2 \left(1 + 2 \frac{x^2 + y^2 + l^2}{b^2}\right)^{1/2} = 1, \] (8.17)

arising from the Friedmann equation, reduces the number of independent equations to three. There are two critical points of this system, satisfying \( x' = y' = l' = b' = 0 \), appearing at

\[ x = y = l = 0, \quad b = \text{constant}; \] (8.18)
\[ x = y = l = b = 0. \] (8.19)

However, neither of them satisfies the Friedmann constraint (8.17). Hence one proves that there is no (kinetic energy)-(potential energy) scaling solution or (kinetic energy)-(potential energy)-(dust matter) scaling solution on a DGP brane with quintessence and dust matter.

The most significant parameter from the viewpoint of observations is the deceleration parameter \( q \), which carries the total effects of cosmic fluids and it is defined as \( q = -\frac{\ddot{a}}{a^2} = -1 + \frac{3}{2} \alpha (2x^2 + l^2) \). In addition, for convenience we introduce the dimensionless density as

\[ \beta = \frac{\rho_{de}}{\rho_c} = \frac{\Omega_{r_c}}{b^2} \left\{ x^2 + y^2 + b^2 - b^2 \left[ 1 + 2 \left( \frac{x^2 + y^2 + l^2}{b^2} \right) \right]^{1/2} \right\}, \] (8.20)

where \( \rho_c \) denotes the present critical density of the universe, and the rate of change with respect to redshift of DE:

\[ \gamma = \frac{1}{\rho_c \Omega_{r_c}} \frac{d\rho_{de}}{ds} = 3 \left\{ 1 + 2 \left( \frac{x^2 + y^2 + l^2}{b^2} \right) \right\}^{-1/2} \left( 2x^2 + l^2 \right) - 2x^2 \}. \] (8.21)

In Fig. 28 we present a concrete numerical example of the \(-1\)-crossing, depicting \( \beta, \gamma \) and \( q \) as a function of \( s \equiv -\ln(1 + z) \). Thus, contrary to the conventional 4D cosmology, where a single canonical field cannot experience the \(-1\)-crossing, in the present model this is possible due to the induced term \( \rho_0 \) of the “energy density” of \( r_c \). Only a small component of \( \rho_0 \), i.e. \( \Omega_{r_c} = 0.01, \)
Fig. 28. (a) \( \beta \) (solid curve) and \( \gamma \) (dotted curve) as functions of \( s \equiv -\ln(1+z) \) for the canonical field case of the DGP model. The EoS parameter of DE crosses \(-1\) at about \( s = -1.6 \), i.e. at \( z = 3.9 \). (b) The corresponding deceleration parameter \( q \) vs \( s \), which crosses 0 at about \( s = -0.52 \), i.e. at \( z = 0.68 \). For this figure, \( \Omega_{ki} = 0.01 \), \( \Omega_M = 0.3 \), and \( \Omega_{\rho_c} \equiv \rho_0/\rho_c = 0.01 \), \( \lambda = 0.05 \). From Ref. [108].

is capable of making the EoS of DE cross \(-1\). At the same time the deceleration parameter is consistent with observations.

8.1.2 Phantom scalar field

For a phantom scalar with an exponential potential of the form: \( V = V_0e^{-\lambda \phi} \) the effective DE evolves as [108]:

\[
\frac{d\rho_{de}}{d \ln(1+z)} = 3 \left[ -\dot{\phi}^2 + \theta \left( 1 + \frac{-\dot{\phi}^2 + 2V + 2\rho_{dm}}{\rho_0} \right)^{-1/2} \left( -\dot{\phi}^2 + \rho_{dm} \right) \right].
\]  

(8.22)

If \( \theta = -1 \), both terms of RHS are negative and it never goes to zero at finite time. Contrarily, if \( \theta = 1 \) the two terms of RHS have opposite sign and the phantom-divide crossing is possible. In the following we consider the branch of \( \theta = 1 \). The dynamics is described by the following autonomous system:

\[
x' = -\frac{3}{2} \alpha' x (-2x^2 + l^2) + 3x - \frac{\sqrt{6}}{2} \lambda y^2,
\]

(8.23)

\[
y' = -\frac{3}{2} \alpha' y (-2x^2 + l^2) + \frac{\sqrt{6}}{2} \lambda xy,
\]

(8.24)

\[
l' = -\frac{3}{2} \alpha' l (-2x^2 + l^2) + \frac{3}{2},
\]

(8.25)

\[
b' = -\frac{3}{2} \alpha' b (-2x^2 + l^2),
\]

(8.26)

where \( \alpha' \equiv 1 + (1 + 2\frac{-x^2+y^2+z^2}{\rho_0})^{-1/2} \), and we have also adopted the spatial flatness condition. Now the Friedmann constraint becomes
Fig. 29. (a) $\beta$ (solid curve) and $\gamma$ (dotted curve) as functions of $s \equiv - \ln(1 + z)$ for the phantom field case of the DGP model. The EoS parameter of DE crosses $-1$ at about $s = -1.25$, i.e. at $z = 1.49$. (b) The corresponding deceleration parameter $q$ vs $s$, which crosses 0 at about $s = -0.50$, i.e. at $z = 0.65$. For this figure, $\Omega_{ki} = 0.01$, $\Omega_M = 0.3$, and $\Omega_{rc} \equiv \rho_0/\rho_c = 0.01$, $\lambda = 0.01$. From Ref. [108].

\[-x^2 + y^2 + l^2 + b^2 + b^2 \left( 1 + 2 \frac{-x^2 + y^2 + l^2}{b^2} \right)^{1/2} = 1, \]

with which there are three independent equations left in the system. Through a similar analysis as in the case of a canonical scalar field, it can be shown that there is no (kinetic energy)-(potential energy) scaling solution or (kinetic energy)-(potential energy)-(dust matter) scaling solution on a DGP brane with phantom and dust dark matter [108]. The deceleration parameter $q$ becomes

\[q = -1 + \frac{3}{2} \alpha'(-2x^2 + l^2),\]

and the dimensionless density and rate of change with respect to redshift of DE become,

\[\beta = \frac{\Omega_{rc}}{b^2} \left\{ -x^2 + y^2 + b^2 + b^2 \left[ 1 + 2 \left( \frac{-x^2 + y^2 + l^2}{b^2} \right)^{1/2} \right] \right\}, \]

\[\gamma = 3 \left\{ 1 + 2 \left( \frac{-x^2 + y^2 + l^2}{b^2} \right)^{-1/2} (-2x^2 + l^2) + 2x^2 \right\}. \]

In Fig. [29] we depict $\beta$, $\gamma$ and $q$ as a function of $s \equiv - \ln(1 + z)$. As we observe, the (effective) EoS parameter of DE crosses $-1$ as expected. This is also in contrast with conventional 4D cosmology, where a single phantom field lies always below $-1$. The 5D gravity plays a critical role in the realization of the $-1$-crossing, and at the same time the deceleration parameter is consistent with observations.

Finally, we stress that the most important difference of the dynamics between an ordinary and a phantom field is that while for the canonical field the $-1$-crossing takes place from above to below, in the phantom case it takes place from below to above (see $\gamma$ vs $s$ in Figs. [28] and [29]).

8.2 Quintom DE in the braneworld

Let us now investigate the effects of the bulk quintom field in the DGP braneworld scenario, and in particular whether it is possible or not to have a late-time accelerated phase on the brane for
i) a tensionless brane with a quintom fluid in the bulk and ii) a quintom DE fluid on the brane and an empty bulk. We consider a 5D metric of the form

$$ds^2 = -n^2(t,y)dt^2 + a^2(t,y)\gamma_{ij}dx^i dx^j + b^2(t,y)dy^2,$$

where \(y\) is the coordinate of the fifth dimension and \(\gamma_{ij}\) is a maximally symmetric 3-dimensional metric. We will use \(k\) to parameterize the spatial curvature and assume that the brane is a hypersurface located at \(y = 0\). We shall be interested in the model described by the action

$$S = \int d^5 x \sqrt{-g} \left( \frac{1}{2\kappa_5^2} R^{(5)} - \Lambda + L_B^{\text{mat}} \right) + \int d^4 x \sqrt{-q}(-\delta_b + L_B^{\text{mat}}),$$

where \(R^{(5)}\) is the curvature scalar of the 5D metric \(g_{AB}\), \(\Lambda\) is the bulk cosmological constant, \(\delta_b\) is the brane tension, \(\kappa_5^2 = 8\pi G_5\), and \(q_{AB} = g_{AB} - n_A n_B\) (\(n_A\) is the unit vector normal to the brane and \(A, B = 0, 1, 2, 3, 5\)) is the induced metric on the 3-brane. The 5D Einstein equation can be written as

$$R^{(5)}_{AB} - \frac{1}{2} g_{AB} R^{(5)} = \kappa_5^2 [-\Lambda g_{AB} + T_{AB} + S_{\mu\nu} \delta_i^A \delta_j^B \delta(y_b)].$$

Here \(\delta(y_b) = \frac{\delta(y)}{b}\), \(T_{AB}\) is the energy momentum tensor of the bulk matter, and the last term corresponds to the matter content in the brane

$$S_{\mu\nu} = -\delta_b g_{\mu\nu} + \tau_{\mu\nu}.$$ (8.33)

The non-zero components of the 5D Einstein equation read

$$3 \left\{ - \frac{\dot{a}}{n^2 a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) + \frac{1}{b^2} \left[ \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right] - \frac{k}{a^2} \right\} = \kappa_5^2 \left[ -\Lambda + T_{0}^0 + S_{0}^0 \delta(y_b) \right],$$

$$\frac{1}{b^2} \delta_j^i \left\{ \frac{a'}{a} \left[ \frac{n'}{n} + 2 \left( \frac{a'}{a} \right) \right] + \frac{a''}{a} - \frac{n'n}{n^2} \right\} + \frac{1}{b} = \kappa_5^2 \left[ -\Lambda + T_j^i + S_j^i \delta(y_b) \right],$$

$$3 \left\{ \frac{n'}{n} \frac{\dot{a}}{a} + \frac{\dot{b}}{b} - \frac{\dot{a'}}{a} \right\} = \kappa_5^2 T_{05},$$

$$3 \left\{ \frac{a'}{ab^2} \left( \frac{\dot{a}}{a} + \frac{n'}{n} \right) - \frac{1}{b} \left[ \frac{\dot{a}}{a} \left( \dot{n} - \frac{n}{n} \right) + \frac{\dot{a}}{a} \right] - \frac{k}{a^2} \right\} = \kappa_5^2 \left[ -\Lambda + T_5^5 \right],$$

where primes indicate derivatives with respect to \(y\) and dots derivatives with respect to \(t\). On the brane we assume a perfect fluid

$$\tau_{\mu}^\nu = \text{diag}(-\rho_b, p_b, p_b, p_b).$$
In the bulk space we consider a quintom field, constituted from the normal scalar field \( \phi(t, y) \) and the negative-kinetic scalar field \( \sigma(t, y) \), with a Lagrangian of the form:

\[
L_{\text{mat}}^B = \frac{1}{2} g^{AB}(\phi, A\phi, B - \sigma, A\sigma, B) + V(\phi, \sigma).
\] (8.39)

According to this action, the energy momentum tensor of the bulk quintom field is given by

\[
T_{AB} = \phi, A\phi, B - \sigma, A\sigma, B - g_{AB} \left[ \frac{1}{2} g^{CD}(\phi, C\phi, D - \sigma, C\sigma, D) + V(\phi, \sigma) \right].
\] (8.40)

Finally, the equations of motion of the scalar fields \( \phi \) and \( \sigma \) in the bulk space write:

\[
-\ddot{\phi} - 3\dot{a} \dot{\phi} + \left( n' + \frac{3a'}{a} \right) \phi' + \phi'' - \frac{\partial V(\phi, \sigma)}{\partial \phi} = \frac{\delta L_{\text{mat}}^B}{\delta \phi} \delta(y_b),
\] (8.41)

\[
-\ddot{\sigma} - 3\dot{a} \dot{\sigma} + \left( n' + \frac{3a'}{a} \right) \sigma' + \sigma'' + \frac{\partial V(\phi, \sigma)}{\partial \sigma} = -\frac{\delta L_{\text{mat}}^B}{\delta \sigma} \delta(y_b).
\] (8.42)

We are interested in studying the Einstein equations in the presence of a bulk quintom field at the location of the brane. Without losing generality we choose \( b(t, y) = 1 \) and \( n(t, 0) = 1 \), which can be achieved by scaling the time coordinate. As is well known, the presence of the brane leads to a singular term proportional to \( \delta \)-function in \( y \) on the right-hand sides of the Einstein equations (8.34) and (8.35) and the equation of motions (8.41),(8.42), which have to be matched by singularity in the second derivatives in \( y \) on the left-hand side. Since all fields under consideration are symmetric under the orbifold symmetry \( Z_2 \), these jumps in the first derivatives in \( y \) fix the first derivatives at \( y = 0 \). In our case, the junction conditions read

\[
a' \big|_{y=0} = -\frac{\kappa_5^2}{6} (\rho_b + \delta_b),
\]

\[
n' \big|_{y=0} = \frac{\kappa_5^2}{6} (3\rho_b + 2\rho_b - \delta_b),
\] (8.43)

and

\[
\phi' \big|_{y=0} = \frac{1}{2} \frac{\delta L_{\text{mat}}^B}{\delta \phi}, \quad \sigma' \big|_{y=0} = -\frac{1}{2} \frac{\delta L_{\text{mat}}^B}{\delta \sigma}.
\] (8.44)

Using the 00 and 55 components of the Einstein equations in the bulk, one obtains

\[
F' = \frac{2\kappa_5^2}{3} \left( \Lambda - T_0^0 \right) a^3 a' - \frac{2\kappa_5^2}{3} T_5^0 a^3 \dot{a},
\] (8.45)

\[
\dot{F} = \frac{2\kappa_5^2}{3} \left( \Lambda - T_5^5 \right) a^3 \dot{a} - \frac{2\kappa_5^2}{3} n^2 T_5^0 a^3 a',
\] (8.46)

where \( F \) is a function of \( t \) and \( y \) defined by

68
\[ F(t, y) = \frac{(\dot{a}a)^2}{n^2} - (a'a)^2 + k\alpha^2. \]  

(8.47)

In order to find the solution for \( F \), we assume that the quintom field in the bulk is independent of \( y \). Under this assumption the non-vanishing components of the quintom energy momentum tensor take the form

\[
T_{0}^{0} = -\rho_B = -\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\sigma}^2 - V(\phi, \sigma) \\
T_{i}^{i} = p_B = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 - V(\phi, \sigma) \\
T_{05} = \dot{\phi} \dot{\sigma}' - \dot{\phi}' \dot{\sigma} = 0 \\
T_{5}^{5} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 - V(\phi, \sigma),
\]

(8.48)

valid also on the brane. As we observe, in the case of \( y \)-independent bulk quintom field, \( T_{05} \) vanishes, thus there is no matter-flow along the fifth dimension. Furthermore, it is obvious from (8.44) that the quintom field cannot appear in the matter content on the brane [321,322].

Solving equation (8.45) leads to the first integral of the 00 component of Einstein equation as

\[
\frac{k^2_5}{6} \left( \Lambda + \rho_B \right) + \frac{C}{a^4} - \frac{a^2}{a^2 n^2} + \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = 0,
\]

(8.49)

where \( C \) is a constant of integration usually referred as dark radiation [337]. Substituting the junction conditions (8.43) into (8.49), we arrive at the generalized Friedmann equation on the brane as 

\[
H^2 + \frac{k}{a^2} = \frac{k^2_5}{6} \left( \Lambda + \frac{k^2_5}{6} \delta b \right) + \frac{k^4_5}{18} \delta b \rho_b + \frac{k^2_5}{6} \rho_B + \frac{k^4_5}{36} \rho_b^2 + \frac{C}{a^4}.
\]

(8.50)

As one can see from (8.50), in the absence of the bulk matter field, cosmological constant and tension, the equation gives rise to a Friedmann equation of the form \( H \propto \rho_b \) instead of \( H \propto \sqrt{\rho_b} \), which is inconsistent with cosmological observations. This problem can be solved by either considering the cosmological constant and tension on the brane or considering a matter field in the bulk [339,340].

Recalling the junction conditions (8.43), the 05 component of Einstein equations and the field equation (8.41), (8.42) acquire the following forms on the brane:

\[
\dot{\rho}_b + 3H(\rho_b + p_b) = 0, \quad \dot{\phi} + 3 \left( \frac{\dot{a}}{a} \right) \phi + \frac{\partial V(\phi, \sigma)}{\partial \phi} = 0, \quad \dot{\sigma} + 3 \left( \frac{\dot{a}}{a} \right) \sigma - \frac{\partial V(\phi, \sigma)}{\partial \sigma} = 0.
\]

(8.51)  
(8.52)  
(8.53)

It should be noted that if the scalar fields \( \phi \) and \( \sigma \) satisfy the field equations (8.52) and (8.53) respectively, the bulk energy momentum tensor is conserved and we obtain
\[ \dot{\rho}_B + 3H(\rho_B + p_B) = 0. \quad (8.54) \]

We are interested in studying the acceleration condition for a universe with quintom field in the bulk. The condition for acceleration can be obtained from (8.50) by using the conservation equation of the brane and bulk matter ((8.51) and (8.54) respectively):

\[ \ddot{a} = \frac{\kappa_5^2}{6} \left( \Lambda + \frac{\kappa_5^2}{6} \delta_b^2 \right) - \frac{k_5^4}{36} \delta_b \left( \rho_b + 3p_b \right) - \frac{\kappa_5^2}{12} \left( \rho_B + 3p_B \right) - \frac{k_5^4}{36} \left( 2\rho_B^2 + 3\rho_B p_b \right) - \frac{C}{a^4}. \quad (8.55) \]

In order to study the role of bulk quintom field in the late-time acceleration phase on the brane, we neglect the effect of tension, brane matter and dark radiation, and therefore the aforementioned equation becomes:

\[ \ddot{a} = \frac{\kappa_5^2}{6} \Lambda - \frac{\kappa_5^2}{12} \left( \rho_B + 3p_B \right). \quad (8.56) \]

Thus, the acceleration condition for a universe with quintom DE in the bulk reads:

\[ p_B < \frac{1}{3} (\Lambda - \rho_B) \quad \text{or} \quad \dot{\phi}^2 - \dot{\sigma}^2 < \Lambda/2 + V(\phi, \sigma). \quad (8.57) \]

Now we consider the 55-component of Einstein equations at the position of the brane, which leads to the Raychaudhuri equation

\[ \ddot{a} + H^2 + \frac{k}{a^2} = \frac{\kappa_5^2}{3} \left( \Lambda + \frac{\kappa_5^2}{6} \delta_b^2 \right) - \frac{\kappa_5^2}{36} \delta_b \left( 3p_b - \rho_b \right) + \rho_b \left( 3p_b + \rho_b \right) - \frac{\kappa_5^2}{3} T_5^5. \quad (8.58) \]

Using (8.50) one can rewrite the above equation as

\[ \ddot{a} = \frac{\kappa_5^2}{6} \left( \Lambda + \frac{\kappa_5^2}{6} \delta_b^2 \right) - \frac{k_5^4}{36} \delta_b \left( \rho_b + 3p_b \right) - \frac{\kappa_5^2}{6} \rho_B - \frac{k_5^4}{36} \left( 2\rho_B^2 + 3\rho_B p_b \right) - \frac{C}{a^4} - \frac{k_5^4}{3} T_5^5. \quad (8.59) \]

Comparing equation (8.55) with (8.59) we provide a constraint on the bulk energy momentum tensor:

\[ (3p_B - \rho_B) = 4T_5^5, \quad (8.60) \]

which for the quintom field with the energy-momentum tensor (8.38) leads to

\[ \dot{\phi}^2 - \dot{\sigma}^2 = 0. \quad (8.61) \]

Therefore, we deduce that the time-dependent bulk quintom field influences the brane as a time-dependent cosmological constant, which can be written in terms of the quintom potential: \( p_B = -\rho_B = -V(\phi) \). Thus, in order to acquire an accelerating universe the potential \( V \) must be a positive function of time, and in order to be compatible with observations it should be decreasing with time. For
the particular solution of (8.61) in which $\phi$ and $\sigma$ are constant on the brane, we arrive at the natural cosmological constant on the brane, which in this case is induced by time-dependent bulk fields.

Let us now consider a $y$-independent bulk quintom field with Lagrangian (8.39) \[130,341\]. The generalized Friedmann equation in this case reads:

\[
\left(1 + \frac{\delta_b \kappa^2}{6 \mu^2}\right)\left(H^2 + \frac{k}{a^2}\right) - \frac{\kappa^2}{6} \left(\Lambda + \frac{\kappa^2}{6} \delta_b\right) - \frac{k^4}{18} \delta_b \rho_b - \frac{\kappa^2}{6} \rho_B + \frac{C}{a^4} = \frac{k^5}{36 \mu^4} \left[ -\mu^2 \rho_b + 3 \left(\frac{H^2 + k}{a^2}\right)^2 \right],
\]

(8.62)

where $\mu^2 = 8 \pi G_4$ since $r_c^2 = \frac{\kappa^2}{2 \mu^2}$. The brane Friedmann equation (8.50) can be derived from (8.62) by letting $\mu$ go to infinity. We are interested in studying the effect of the quintom field on the brane. Ignoring the matter on the brane and the cosmological constant $\Lambda$, (8.62) can straightforwardly be rewritten as:

\[
H^2 + \frac{k}{a^2} = \frac{1}{2 r_c^2} \left(1 + \epsilon \sqrt{1 - \frac{2 \kappa^2}{3} \rho_B \tau_c^2}\right),
\]

(8.63)

where $\rho_B$ is the energy density of the bulk quintom field on the brane, given by the first relation of (8.48). The two different possible $\epsilon$-values ($\epsilon = \pm 1$), correspond to two different embeddings of the brane into the bulk spacetime \[342,343\]. Since the bulk quintom field satisfies the usual energy momentum conservation law on the brane (8.51), we have $\rho_B \propto a^{-3(w+1)}$. Integrating (8.63) for $k = 0$ and $w \geq -1$ (that is $\dot{\phi} > \dot{\sigma}$), shows that the scale factor $a$ diverges at late times. Thus, the energy density of the bulk matter goes to zero for late times and thus it reaches a regime where it is small in comparison with $1/\tau_c^2$. In the case $w < -1$ (that is $\dot{\phi} < \dot{\sigma}$), integrating (8.63) indicates a vanishing scale factor $a$ at late times, therefore the matter density goes to zero and we can use $\kappa^2 \rho_B \ll 1/\tau_c^2$. In summary, in DGP model with a quintom DE fluid in the bulk space, one can expand (8.63) under the condition that $\kappa^2 \rho_B \ll 1/\tau_c^2$ for the whole $w$-range.

At zeroth order and for spatially flat metric, two different results, depending on the value of $\epsilon$, can be derived. Considering the case $\epsilon = -1$ yields

\[
H^2 = 0,
\]

(8.64)

which describes an asymptotically static universe. Considering $\epsilon = 1$ leads to

\[
H^2 = \frac{1}{\tau_c^2}, \quad \text{or} \quad a(t) \propto \exp\left(\frac{t}{\tau_c}\right).
\]

(8.65)

This branch provides the self acceleration solution in the late universe and it is the most important aspect of the model at hand. Therefore, the late-time behavior of the universe does not change even if we ignore the matter field on the brane and consider a scenario with the bulk quintom fields only.

Let us now consider the “opposite” case, and neglect the bulk matter, considering only quintom fields confined on the brane, in DGP framework, with a Lagrangian as

\[
\mathcal{L}^\text{mat}_b = \frac{\kappa^2}{2} g^\mu\nu (\phi_\mu \phi_\nu - \sigma_\mu \sigma_\nu) + \tilde{V}(\phi, \sigma).
\]

(8.66)
The energy-momentum tensor of the quintom fields on the brane is given by

\[ \tau_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \sigma_{,\mu}\sigma_{,\nu} - q_{\mu\nu} \left[ \frac{1}{2} q^{\alpha\beta} (\phi_{,\alpha}\phi_{,\beta} - \sigma_{,\alpha}\sigma_{,\beta}) + \tilde{V}(\phi, \sigma) \right]. \] (8.67)

In absence of bulk field and brane tension the first integral of the 00 component of (8.67) leads to

\[ \sqrt{H^2 + \frac{k}{a^2}} = \frac{1}{2r_c} \left( \epsilon + \sqrt{1 + \frac{4\mu^2}{3} \rho_b r_c^2} \right), \] (8.68)

in which \( \rho_b \) is quintom energy density obtained by (8.67). Since the energy-momentum tensor on the brane is conserved, we could use the aforementioned procedure to examine the late-time cosmology on the brane. Integrating (8.68) for a flat geometry indicates that for \( w \geq -1 \) the scale factor \( a \) diverges at late times \[309\], while for \( w < -1 \) the scale factor \( a \) vanishes at late times \[130\]. Thus, the energy density of quintom DE goes to zero for late times and reaches a regime where it is small in comparison with \( 1/r_c^2 \). Expanding equation (8.68) under the condition \( \mu^2 \rho_b \ll 1/r_c^2 \) provides an asymptotically static universe, \( H = 0 \), in the case \( \epsilon = -1 \) and a self accelerated phase, \( H = \frac{1}{r_c} \), in the case \( \epsilon = 1 \).

In summary, the presence of quintom DE on the brane or in the bulk does not change the late time behavior of the universe. In both cases and for the whole range of \( w \) (\( w < -1 \) or \( w > -1 \)), one can derive the self accelerating universe at late times.

### 8.3 Other modified-gravity models

We close this section mentioning that a quintom scenario can also be obtained in many other modified gravity models, namely, gravitational models with Gauss-Bonnet corrections \[89,345-346,347,348,349,350\], \( f(R) \) models with singular \( w = -1 \) crossing \[351,352,353\], scalar-tensor models with nonminimal gravitational couplings \[128,354-355,356\], gravitational ghost condensate models \[200,357,358,359,360\] (see Ref. \[361\] for a review). The analysis of the perturbations of these models can be studied by directly expanding metric fluctuations \[362,363\] or by the Parameterized Post-Friedmann approach \[364,365,366\] under certain cases, when one wants to check their rationalities. In the framework of these modified gravity models, the current cosmic acceleration may be achieved naturally.

### 9 Energy conditions and quintom cosmology in the early universe

In this section we investigate the violations of energy conditions in quintom cosmology, and we study its implications in the early universe.
9.1 Null Energy Condition

It is well known that energy conditions play an important role in classical theory of general relativity and thermodynamics \[367\]. In classical general relativity it is usually convenient and efficient to restrict a physical system to satisfy one or more of energy conditions, for example in the proof of Hawking-Penrose singularity theorem \[368,369\], the positive mass theorem \[370,371\] etc. Furthermore, in thermodynamics the energy conditions are the bases for obtaining entropy bounds \[372,373\]. Among those energy conditions, the null energy condition (NEC) is the weakest one, and it states that for any null vector \(n^\mu\) the stress-energy tensor \(T_{\mu\nu}\) should satisfy the relation

\[
T_{\mu\nu}n^\mu n^\nu \geq 0 .
\] 

(9.1)

Usually, the violation of NEC may lead to the breakdown of causality in general relativity and the violation of the second law of thermodynamics \[374\]. These pathologies suggest that the total stress tensor in a physical spacetime manifold needs to obey the NEC. In the framework of standard 4D FRW cosmology the NEC implies \(\rho + p \geq 0\), which in turn gives rise to the constraint \(w_u \geq -1\) on the EoS of the universe \(w_u\), defined as the ratio of pressure to energy density.

In the epochs of universe evolution in which radiation is dominant the EoS of the universe \(w_u\) is approximately equal to 1/3, while in the matter-dominated period \(w_u\) is nearly zero, thus NEC is always satisfied. However, when the DE component is not negligible, with the NEC being satisfied, we acquire

\[
w_u = w_M \Omega_M + w_{DE} \Omega_{DE} \geq -1 ,
\] 

(9.2)

where the subscripts ‘\(M\)’ and ‘\(DE\)’ stand for matter and DE, respectively. With \(w_M = 0\), inequality (9.2) becomes

\[
w_{DE} \Omega_{DE} \geq -1 .
\] 

(9.3)

From this inequality we can deduce that it is impossible for a universe to satisfy NEC if one of its component does not. In this case, a quintom scenario could be obtained if NEC is violated for a short period of time during the evolution of the universe.

9.2 Quintom Bounce

A bouncing universe with an initial contraction to a non-vanishing minimal radius and a subsequent expanding phase, provides a possible solution to the singularity problem of the standard Big Bang cosmology. For a successful bounce, it can be shown that within the framework of standard 4D FRW cosmology with Einstein gravity, the NEC is violated for a period of time around the bouncing point. Moreover, for the universe entering into the hot Big Bang era after the bouncing, the EoS must transit from \(w < -1\) to \(w > -1\).

We start with an examination on the necessary conditions required for a successful bounce \[375,376,377,378,379,380,381\].
(we refer to [395] for a review on bounce cosmology). During the contracting phase the scale factor $a(t)$ decreases ($\dot{a} < 0$), while in the expanding phase it increases ($\dot{a} > 0$). At the bouncing point, $\dot{a} = 0$, and around this point $\ddot{a} > 0$ for a period of time. Equivalently, in the bouncing cosmology the hubble parameter $H$ changes from $H < 0$ to $H > 0$, with $H = 0$ at the bouncing point. A successful bounce requires

$$\dot{H} = -\frac{1}{2M_p^2}(\rho + P) = -\frac{1}{2M_p^2}\rho\left(1 + w\right) > 0,$$  

(9.4)

around this point and thus $w < -1$ in its neighborhood. After the bounce the universe needs to enter into the hot Big Bang era, otherwise it will reach the Big Rip singularity similarly to the phantom DE scenario [18]. This requirement leads the EoS to transit from $w < -1$ to $w > -1$, which is exactly a quintom scenario [391].

9.2.1 A phenomenological analysis

Let us examine the possibility of obtaining the bouncing solution in a phenomenological quintom models, in which the EoS is described by:

$$w(t) = -r - \frac{s}{t^2}.$$  

(9.5)

In (9.5) $r$ and $s$ are parameters and we require that $r < 1$ and $s > 0$. One can see from (9.5) that $w$ runs from negative infinity at $t = 0$ to the cosmological constant boundary at $t = \sqrt{\frac{s}{1 - r}}$ and then it crosses this boundary. Assuming that the universe is dominated by the matter with the EoS given by (9.5), we solve the Friedmann equation, obtaining the corresponding evolution of hubble parameter $H(t)$ and scale factor $a(t)$ as:

$$H(t) = \frac{2}{3}\left[\frac{t}{(1 - r)t^2 + s}\right]^{1/(1 - r)}.$$  

(9.6)

$$a(t) = \left[t^2 + \frac{s}{1 - r}\right]^{\frac{1}{3(1 - r)}}.$$  

(9.7)

Here we choose $t = 0$ as the bouncing point and we normalize $a = 1$ at this point. Thus, our solution provides a universe evolution with contracting (for $t < 0$), bouncing (at $t = 0$) and expanding (for $t > 0$) phases. In Fig. 30 we depict the evolution of the EoS, the Hubble parameter and the scale factor, as it arises from numerical elaboration. As we observe, a non-singular bounce is realized at $t = 0$, at a minimal non-vanishing scale factor $a$, with the Hubble parameter $H$ running across zero. At the bouncing point $w$ approaches negative infinity.

9.2.2 Double-field quintom model

Having presented the bouncing solution with the phenomenological quintom matter, we now study the bounce in a two-field quintom model with the action given by
Fig. 30. (Color online) Evolution of the EoS $w$, the Hubble parameter $H$ and the scale factor $a$ as a function of the cosmic time $t$. For the numerical elaboration we take $r = 0.6$ and $s = 1$. From Ref. [391].

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V(\phi, \sigma) \right],$$

where $\phi$ is the canonical and $\sigma$ the phantom field. In the framework of FRW cosmology as usual we have $\rho = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 + V$ and $p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 - V$ and the cosmological equations read

$$H^2 = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 + V \right) \tag{9.8}$$

$$\ddot{\phi} + 3H \dot{\phi} + \frac{dV}{d\phi} = 0 \tag{9.9}$$

$$\ddot{\sigma} + 3H \dot{\sigma} - \frac{dV}{d\sigma} = 0. \tag{9.10}$$

Thus, we see that $\dot{\sigma}^2 = \dot{\phi}^2 + 2V$ when $H$ crosses zero, and moreover from (9.4) we deduce that $\dot{\sigma}^2 = \dot{\phi}^2$ when $w$ crosses $-1$. These constraints can be easily satisfied in the parameter space of the model at hand.

An interesting scenario realized by this model is a universe originated from a contraction, that has undergone a smooth bounce, then an inflationary period, and finally it returns to the standard thermal history. Models of this class have been studied in Refs. [385,391,396,397] and the dynamics of their perturbations have been analyzed in Refs. [398,399,400,401,402]. In the following we will provide explicitly one example to illustrate this scenario.

We consider a double-field model in which the potential is only a function of the field $\phi$ and of Coleman-Weinberg form [403]:
\[ V = \frac{1}{4} \lambda \phi^4 \left( \ln \left| \frac{\phi}{v} - \frac{1}{4} \right| + \frac{1}{16} \lambda v^4, \right), \]  
(9.11)

which takes its maximum value \( \lambda v^4/16 \) at \( \phi = 0 \) and vanishes at the minima where \( \phi = \pm v \). Therefore, the scalar field \( \sigma \) affects the evolution only around the bounce but it decreases quickly away from it.

In order to discuss the perturbations explicitly, we first examine the evolution of the background. In this model a contracting universe can be driven to reach a minimal size during which the universe evolves like a matter-dominated one, and then a quasi-exponential expansion is following. The process to link the contraction and expansion is a smooth bounce, and the evolution of the hubble parameter can be treated as a linear function of the cosmic time approximately.

For the background initial conditions we assume that \( \phi \) lies at one vacuum (for instance \(-v\)) when the universe is contracting, and moreover that \( \dot{\sigma} \) is small enough in order to be neglected. In this case the field \( \phi \) oscillates around \(-v\) leading the universe EoS to oscillate around \( w = 0 \), and so this state is in average similar to a matter-dominated one. Thus, the useful expressions of the background evolution write

\[ a \sim (-\eta)^2, \quad \mathcal{H} = \frac{2}{\eta}, \quad |\dot{\phi}| \sim \eta^{-3}, \]  
(9.12)

where \( \mathcal{H} \equiv a'/a \) is the comoving hubble parameter and the prime denotes the derivative with respect to the comoving time \( \eta \). Moreover, since in the contracting phase the universe is dominated by the regular field and then we have an approximation \( \phi'' \simeq 2(\mathcal{H}^2 - \mathcal{H}') \), we can obtain another useful relation

\[ \frac{\phi''}{\phi'} = \frac{2\mathcal{H}\mathcal{H}' - \mathcal{H}''}{2(\mathcal{H}^2 - \mathcal{H}')}, \]  
(9.13)

which will be used to calculate the metric perturbations.

When the universe is contracting, the amplitude of \( \phi \)-oscillations increases, while the contribution of the \( \sigma \)-field grows rapidly. When the \( \phi \)-field reaches the plateau, the bounce starts at the moment \( t_B^- \). During the bounce we use the parametrization \( H(t) = \alpha(t - t_B) \) around the bounce point \( t_B \), and the coefficient \( \alpha \) is a positive constant determined by numerical elaboration. This parametrization leads to a scale factor of the form \( a = \frac{1}{\sqrt{1 - \frac{\alpha a}{4\pi G}}} \). In the bouncing phase the kinetic term of \( \sigma \) reaches the maximal value, and from the equation of motion we deduce that \( \ddot{\phi}/\dot{\phi} = -3\mathcal{H} \dot{\sigma}^2/(\dot{\sigma}^2 - \frac{\alpha}{4\pi G}) \simeq -3H \) when \( \alpha \) is not very large. Finally, we obtain the approximate relations

\[ \mathcal{H} \simeq \frac{y}{2}(\eta - \eta_B), \quad \phi'' \simeq -2\mathcal{H}\phi', \quad |\dot{\phi}| \sim e^{-\frac{y}{4}(\eta - \eta_B)^2}, \]  
(9.14)

where we have defined \( y \equiv 8\alpha a_B^2/\pi \).

After the bounce, as the field \( \phi \) moves forward slowly along the plateau, the universe enters into an expanding phase at the moment \( t_{B^+} \), and its EoS is approximately \(-1\). Thus, the universe expands with its scale factor growing almost exponentially. In this phase we have the well-known relations for the background evolution
Finally, when the $\phi$-field reaches the vacuum state $+v$, it will oscillate again and the EoS of the universe will oscillate around zero as it happens before the bounce.

In order to present this scenario explicitly, we perform a numerical elaboration and the results are depicted in Fig. 31.

Finally, we mention that if the potential of the $\phi$-field is not flat enough, the inflationary stage could be very short. An extreme case of this scenario is that, after a matter-like contraction, the universe undergoes a smooth bounce and enters a normal expanding phase directly. This is the so-called “Matter Bounce” scenario, which has been intensively studied in the literature.

9.3 A cyclic scenario and oscillating universe

The idea of cyclic universe was initially introduced in 1930’s by Richard Tolman [409]. Since then there have been various proposals in the literature. The authors of Refs. [410,411,412,413] introduced a cyclic model in high dimensional string theory with an infinite and flat universe. Within a modified Friedmann equation the cyclic evolution of the universe can also be realized [414,415,416,417,418]. In Ref. [419], it is shown that in the framework of loop quantum cosmology (LQC) a cyclic universe can be obtained
with the quintom matter. In this section, however, we will study the solution of oscillating universe in the absence of modifications of the standard 4D Einstein Gravity, within a flat universe.

To begin with, let us examine in detail the conditions required for an oscillating solution. The basic picture for the evolution of the cyclic universe can be shown below:

\[ \text{bounce} \xrightarrow{\text{expanding}} \text{turn-around} \xrightarrow{\text{contracting}} \text{bounce} \ldots \]

In the 4D FRW framework the Einstein equations can be written as:

\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{\rho}{3M_p^2} \quad \text{and} \quad \ddot{a} = -\frac{\rho + 3p}{6M_p^2},
\]

where we have defined \( M_p^2 \equiv \frac{1}{8\pi G} \). \( H \) stands for the Hubble parameter, while \( \rho \) and \( p \) represent the energy density and pressure of the universe respectively. By definition, for a pivot (bounce or turn-around) process to occur, one must require that at the pivot point \( \dot{a} = 0 \) and \( \ddot{a} > 0 \) around the bouncing point, while \( \ddot{a} < 0 \) around the turn-around point. According to (9.16), one obtains

\[
\rho = 0, \quad p < 0 \text{ (or } p > 0) \quad \text{for the bounce (or turn-around)},
\]

or equivalently \( w \equiv \frac{p}{\rho} \rightarrow -\infty \) (or \(+ \infty\)) at the bounce (or turn-around) point, with the parameter \( w \) being the EoS of the content of the universe. Thus, when the universe undergoes from bounce to turn-around \( w \) evolves from \(-\infty\) to \(+\infty\), while in the reverse case \( w \) goes from \(+\infty\) to \(-\infty\). This behavior requires \( w \) to cross over the cosmological constant boundary \( (w = -1) \) in these processes, which interestingly implies the necessity of the quintom matter in order to achieve the realization of the oscillating universe in 4D Einstein Gravity.

An additional interesting feature of an oscillating universe is that it undergoes accelerations periodically, avoiding the Big Rip and Big Crunch, and furthermore we are able to unify inflation and current acceleration. We mention that the scale factor keeps increasing from one period to another and thus we are naturally led to a highly flat universe. This scenario was first proposed in Ref. \[80\], in which a parameterized quintom model was used and the coincidence problem was argued to be reconciled.

### 9.3.1 A solution of oscillating universe in the double-field quintom model

We consider the simplest quintom model consisting of two scalars with one being quintessence-like and another the phantom-like, with the usual action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - V(\phi, \sigma) \right].
\]

As mentioned in previous sections the energy density and pressure are \( \rho = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 + V(\phi, \sigma) \) and \( p = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\sigma}^2 - V(\phi, \sigma) \), and the equations of motion for these two fields write: \( \ddot{\phi} + 3H \dot{\phi} + V_{,\phi} = 0 \) and \( \ddot{\sigma} + 3H \dot{\sigma} - V_{,\sigma} = 0 \).

Phenomenologically, a general potential form for a renormalizable model includes operators with dimension 4 or less, including various powers of the scalar fields. We impose a \( Z_2 \) symmetry, that is the potential remains invariant under the simultaneous transformations \( \phi \rightarrow -\phi \) and \( \sigma \rightarrow -\sigma \). Thus, the
potential of the model writes:

$$V(\phi, \sigma) = V_0 + \frac{1}{2}m_1^2\phi^2 + \frac{1}{2}m_2^2\sigma^2 + \gamma_1\phi^4 + \gamma_2\sigma^4 + g_1\phi\sigma + g_2\phi^3\sigma^2 + g_3\phi^2\sigma^2 + g_4\phi^3\sigma.$$  \hfill (9.19)

From condition (9.17) we can see that at both the bounce and the turn-around point $\dot{\sigma}^2 = \dot{\phi}^2 + 2V$. Similarly, we deduce that $p = -2V$. When the universe undergoes a bounce the pressure is required to be negative, which implies the potential to be positive. However, when a turn-around takes place, the pressure of the universe is required to be positive, and consequently the potential must be negative. Therefore, the potential must contain a negative term in order to give rise to an oscillating scenario.

In the model at hand the two scalar fields dominate the universe alternately, since the evolution from the bounce to the turn-around requires the transition from the phantom-dominated phase into the quintessence-dominated one, or vice versa. However, as pointed out in [81,82] this process does not happen if the two fields are decoupled, and thus interaction between the two fields is crucial. For a detailed quantitative study we consider

$$V(\phi, \sigma) = (\Lambda_0 + \lambda\phi\sigma)^2 + \frac{1}{2}m^2\phi^2 - \frac{1}{2}m^2\sigma^2.$$  

This potential acquires a negative value when $\phi$ is near the origin and $\sigma$ large. However, due to the interaction the potential is still bounded from below and it is positive definite when the fields are both away from zero. This potential allows for the analytic solution

$$\phi = \sqrt{A_0}\cos mt, \quad \sigma = \sqrt{A_0}\sin mt$$  \hfill (9.20)

with $\lambda = \sqrt{\frac{3m}{2M_p}}$, and with the parameter $A_0$ describing the oscillation amplitude. In Fig. 32 we depict the evolution of the energy density and scale factor.

### 9.3.2 Classifications of the solutions

Let us study the detailed cosmological evolutions of the cyclic quintom model. First of all, from (9.16) one acquires the Hubble parameter

$$H = \sqrt{\frac{3}{3M_p}}(\Lambda_0 + \Lambda_1\sin 2mt),$$  \hfill (9.21)

where $\Lambda_1 \equiv \sqrt{\frac{3m}{2M_p}}A_0$. For different parameters, the evolution of the universe can be classified into five cases. Taking $M_p = 1$ we have the following:

Case (I): $\Lambda_0 = 0$. In this case the Hubble parameter is given by $H = \frac{ma_0}{3M_p}\sin 2mt$. Thus, the scale factor is

$$\ln a \propto \cos 2mt,$$  \hfill (9.22)

and therefore there is no spacetime singularity. In Fig. 33 we depict the evolution of the scale factor, the Hubble parameter, the energy density and the EoS.

Case (II): $0 < \Lambda_0 < \Lambda_1$. In this case the scale factor is

$$\ln a \propto C_1 t + C_2 \cos 2mt,$$  \hfill (9.23)
where $C_1 = \Lambda_0 / \sqrt{3} M_p$, $C_2 = -A_0 / 8 M_p^2$. This solution also describes a cyclic universe, but both the minimal and maximal values of the scale factor increase cycle by cycle. Therefore, the average size of the universe is growing up gradually without Big Crunch or Big Rip singularities, although its scale factor experiences contractions and expansions alternately. Finally, the backward time evolution cannot lead to a shrinking scale factor either. These features are presented in Fig. 34. The graph for energy density and Hubble parameter is similar to Fig. 33, but the oscillating amplitude of the energy density in the expanding phase is larger than that of the contracting one.

Case (III): $\Lambda_0 \geq \Lambda_1$. The solution for the scale factor is the same as Case (II), however the evolution of the universe is different. The universe lies in the expanding period forever and there is no contracting phase, that is the accelerating expansion is periodical, without any bounce or turn-around. Since the EoS $w$ is oscillating around $-1$, the Big Rip can be avoided and the energy density and Hubble parameter are always positive. Furthermore, for reasonable parameters one can unify the inflationary period and the late-time acceleration. This behavior is shown in Fig. 35.

Case (IV): $-|\Lambda_1| < \Lambda_0 < 0$. This case corresponds to a cyclic universe with decreasing minimal and maximal scale factor for each epoch. Contrary to Case (II), the total tendency is that the scale factor decreases cycle by cycle, since the contracting phase is longer than the expanding phase, but it never reaches zero. These features are depicted in Fig. 36.

Case (V): $\Lambda_0 < -|\Lambda_1|$. This case describes an enterally contracting universe, which corresponds to the reverse of Case (III), and thus it is unphysical.

Having presented the possible solutions, we make some final comments. One may notice that the EoS of the model oscillates around the cosmological constant boundary, and every time it approaches $w = -1$ it remains close to that for some time. Correspondingly, the Hubble parameter evolves near its maximal

Fig. 32. (Color online) Evolution of the energy density and scale factor in a cyclic quintom universe. For each cycle the quintessence-like and phantom-like components dominate alternately. From Ref. [120].
Fig. 33. (Color online) Evolution for Case (I). This figure presents an exactly cyclic universe. The scale factor oscillates between the minimal and maximal value. In the numerical elaboration we take $m = 3 \times 10^{-3}$ and $A_0 = 300$ or equivalently $\Lambda_1 \simeq 0.39$. The Hubble parameter is depicted by the purple line, while the energy density by the red line. From Ref. [120].

Fig. 34. (Color online) Evolution for Case (II). $m$ and $A_0$ are the same as those used in Fig. 33 and $\Lambda_0 = 0.10$. From Ref. [120].

value and makes the scale factor expand exponentially. This period corresponds to an inflationary stage after the bounce. It is a significant process since it dilutes the relics created in the last cycle. Furthermore, entropy can be diluted too, providing a solution to the problem of enteral entropy increase in cyclic cosmology. Additionally, some of the primordial perturbations are able to exit the horizon and re-enter the horizon when the inflationary stage ceases. When these perturbations re-enter they lead to new
Fig. 35. (Color online) 

Evolution for Case (III). The parameters $m$ and $A_0$ are taken the same as in Fig. 33, and $\Lambda_0 = 0.402$. From Ref. [120].

Lastly, let us constrain the model parameters $(m, \Lambda_0, A_0)$. First of all, we require the oscillation period of the Hubble parameter to be no less than 2 times the age of our universe which is of the order of the present Hubble time. Thus, from (9.21) we obtain the period $T = \frac{2}{m} \sim \mathcal{O}(H_0^{-1})$, where $H_0^{-1} \sim 10^{60}M_p^{-1}$, and therefore $m \sim \mathcal{O}(10^{-60})M_p$. Secondly we require the maximal value of the Hubble parameter to be able
to reach the inflationary energy scale. The maximal value of $H$ is given by $(\Lambda_0 + \Lambda_1)$, when $(\sin 2mt)$ reaches its maximum. For case (I), (III) and (IV), $|\Lambda_0| \leq \Lambda_1$, so we have $(\Lambda_0 + \Lambda_1) \sim O\left(\frac{m_A}{M_p}\right)$. If we consider inflation with energy density, for example around $O\left(10^{-20}\right)M_p^4$, we find that $A_0$ must be of $O\left(10^{50}\right)M_p^2$. Such a large value of $A_0$ indicates that the scalar fields $\phi$ and $\sigma$ take values above the Planck scale, which makes our effective lagrangian description invalid. One possibility of solving this problem is to consider a model with a large number of quintom fields. In this case the Hubble parameter is amplified by a pre-factor $\sqrt{N}$ with $N$ the number of quintom fields. If $N$ is larger than $10^{50}$, $A_0$ can be relaxed to $O(1)M_p^2$ or less.

### 10 Concluding remarks

Since discovered in 1998, the nature of DE has become one of the most intriguing puzzles of modern physics and it has been widely investigated. The simplest candidate of DE is a cosmological constant, but it suffers from the well-known fine-tuning and coincidence problems. Alternatively, dynamical DE models have been proposed, such as quintessence, phantom, and k-essence. Since at present we know very little about the theoretical aspects of DE, the cosmological observations play a crucial role in enlightening our picture. In particular, since astronomical observations have shown a mild preference for an EoS of DE smaller than $-1$ at present, a phantom model appeared in the literature.

In these lines, with accumulated observational data a scenario of quintom, with DE EoS crossing $-1$ during evolution, has been constructed in the literature. If such a class of dynamical DE scenario were verified by future observations, it would be a challenge to the model-building of DE. This is because of a No-Go theorem, which forbids dynamical models with a single scalar field to lead to an EoS crossing over the cosmological constant boundary in the frame of Einstein’s gravity. The current work aims at presenting a review of successful examples of quintom models and the corresponding observational consequences.

We studied in detail the simplest quintom model, which involves two scalar fields, with one being quintessence and the other being phantom. We have shown that these models possess stable critical points in the phase space, and in addition certain dualities appear, with form-invariant cosmological equations.

As a consistent consideration we analyzed the behavior of quintom perturbations. In the example of the simplest quintom model there are two degrees of freedom in its perturbations, which are finite and stable. It should be noticed that since the quintom scenario usually involves extra degrees of freedom, additional relative pressure perturbations are inevitably produced in order to seed entropy perturbations.

We also provided alternative approaches to realize the quintom scenario in a scalar field system within standard Einstein’s gravity, in which higher derivative terms were introduced. Specifically, we constructed a Lee-Wick and a string inspired quintom model. For the latter, the perturbation propagations are quite different than usual, with a time-dependent sound speed due to the non-canonical kinetic term.

Usually, a quintom model constructed by a double-scalar system and a single scale with higher derivatives, suffers from the problem of ambiguous quantum behavior inherited from the DE phantom model. This feature offered a motivation to study the possibilities of realizing a quintom scenario without a ghost mode. An attempt on solving this problem is to construct a string-inspired quintom model, but it is still unclear how to be quantized since it involves highly non-linear terms.
Furthermore, we considered the model-building in the framework of a non-scalar system. A model of spinor quintom is able to evade the drawbacks of considering a phantom field, since its kinetic term is well defined, and the sign-change of its potential derivative with respect to the scalar bilinear $\bar{\psi}\psi$ can realize the $-1$-crossing. However, it is non-renormalizable if we consider higher order effects at the quantum level.

Another interesting issue is the investigation of a quintom scenario in braneworld cosmology, which is motivated by higher dimensional constructions, closer to fundamental theories. Braneworld models were studied actively in recent years, offering a new path to describe DE. In this review we presented the possibility of realizing a quintom scenario in DGP braneworld model. In such a construction, solutions of late-time super-acceleration can be obtained when matter components are driven by a manifestation of the excruciatingly slow leakage of gravity into the extra dimension. In particular, we showed that for a canonical field living on the brane the $-1$-crossing takes place from above to below, while in the phantom case it takes place from below to above.

Concerning the theoretical implications of quintom cosmology, an important issue is the violation of energy conditions. Such conditions usually play a crucial role in classical theory of general relativity. An important predication of general relativity is that a singularity cannot be avoided under assumptions of certain energy conditions, which was originally proved by Penrose and Hawking and later developed into cosmological framework by Borde, Vilenkin and Guth [420]. Therefore, in a quintom scenario with NEC violation, one may obtain a bouncing solution at early times of the universe. This may provide a possible solution to the singularity problem of standard Big Bang cosmology. Specifically, we studied bouncing cosmology realized by a model of double-field quintom. To extend, we also provided a scenario of oscillating universe, in which it experiences expansions and contractions periodically.

As an end, we would like to comment on some unsettled issues in quintom cosmology. Although the quintom paradigm has been studied intensively in recent years, its nature is still unclear. In the following, we list three main questions, which are crucial to the developments of quintom cosmology.

- (a) how to combine quintom cosmology with the well-known particle physics or fundamental theories?
- (b) how to construct a ghost-free quintom scenario with robust quantum behavior?
- (c) why the DE EoS crosses the cosmological constant boundary around today (the so-called second coincidence problem)?

Recently, physicists have made many attempts on partially addressing the above questions in quintom cosmology. For example, a quintom scenario can be realized by a rolling tachyon in the cubic string field theory [101,421,422] due to a non-local effect [423]. Interestingly, motivated by studies on thermodynamics of a cosmological structure [424,425,426,427,428,429,430], a cosmic holographic bound (originally suggested by [431,432] and comprehensively reviewed in [433]) on dark energy dynamics allows a period of quintom scenario with NEC violation [434,435,436,437,438,439].

However, the three aforementioned questions remain open, requiring to be faced by the coming theoretical studies on quintom cosmology. In the meantime, cosmological observations of higher accuracy are needed in order to determine whether the DE EoS has crossed the cosmological constant boundary. If this is indeed true, cosmology will enter in a very interesting and challenging era.
Acknowledgement

We are grateful to Hao Wei, Xin Zhang, Xinmin Zhang, Yang Zhang, Wen Zhao for giving us very useful comments on the manuscript. We also thank Hong Li, Mingzhe Li, Jian-Xin Lu, Yun-Song Piao, Taoqiao Qiu, Jing Wang, Hua-Hui Xiong, Hong-Sheng Zhang, Xiao-Fei Zhang, Gongbo Zhao, and Zong-Hong Zhu for permissions to include figures from their works.

Part of the numerical elaboration was performed on the MagicCube of Shanghai Supercomputer Center (SSC). E.N.S wishes to thank Institut de Physique Théorique, CEA, for the hospitality during the preparation of the present work. The work of M.R.S. is financially supported by Research Institute for Astronomy and Astrophysics of Maragha, Iran. The researches of Y.F.C. and J.Q.X. are partly supported by National Science Foundation of China under Grant Nos. 10803001, 10533010, 10821063 and 10675136, and the 973 program No. 2007CB815401, and by the Chinese Academy of Science under Grant No. KJCX3-SYW-N2.

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