On deeply critical oriented cliques

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Abstract
In this work we consider arc criticality in colourings of oriented graphs. We study deeply critical oriented graphs, those graphs for which the removal of any arc results in a decrease of the oriented chromatic number by 2. We prove the existence of deeply critical oriented cliques of every odd order \( n \geq 9 \), closing an open question posed by Borodin et al. Additionally, we prove the nonexistence of deeply critical oriented cliques among the family of circulant oriented cliques of even order.

Keywords
critical graphs, deeply critical graphs, oriented chromatic number, oriented graph

1 | INTRODUCTION

In 1994, Courcelle [3] defined oriented colouring as part of his seminal work on the monadic second-order logic of graphs in which he established the illustrious Courcelle’s Theorem [4]. In the years following, oriented colouring and the oriented chromatic number gained popularity and developed into an independent field of research. We refer the reader to Sopena’s updated survey [7] for a broad overview of the state of the art.

An oriented graph \( \overrightarrow{G} \) is a directed graph without any directed cycle of length one or two. That is, it is a directed graph that is irreflexive and antisymmetric. We denote the set of vertices and arcs of an oriented graph \( \overrightarrow{G} \) by \( V(\overrightarrow{G}) \) and \( A(\overrightarrow{G}) \), respectively.

By generalising to oriented graphs the interpretation of graph colouring as homomorphism to a complete graph, one arrives at the following definition of oriented graph colouring.
An oriented $k$-colouring of an oriented graph $\vec{G}$ is a function $\phi : V(\vec{G}) \to \{1, 2, \ldots, k\}$ so that

(i) $\phi(x) \neq \phi(y)$ for all $\vec{xy} \in A(\vec{G})$; and
(ii) for all distinct $\vec{xy}, \vec{uv} \in A(\vec{G})$, if $\phi(x) = \phi(v)$, then $\phi(y) \neq \phi(u)$.

The vertices of the target of the homomorphism correspond to the colours. Condition (i) ensures that the target of the homomorphism is irreflexive. Condition (ii) ensures that the target of the homomorphism is antisymmetric.

Condition (ii) applied for $yu \neq u$ implies vertices connected by a directed path of length 2 (i.e., a 2-dipath) must receive different colours.

The oriented chromatic number $\chi_0(\vec{G})$ of an oriented graph $\vec{G}$ is the minimum $k$ for which $\vec{G}$ admits an oriented $k$-colouring.

A major theme in oriented colourings research is the study of analogous versions of graph colouring concepts for oriented graphs. In 2004 Klostermeyer and MacGillivray [5] generalised the notion of clique to oriented colouring, studying those oriented graphs for which $\chi_0(\vec{G}) = |V(\vec{G})|$. This work was continued by Nandi, Sen and Sopena [6]. Such oriented graphs are called absolute oriented cliques and admit the following classification.

**Theorem 1.1** (Klostermeyer and MacGillivray [5]). An oriented graph is an absolute oriented clique if and only if every pair of nonadjacent vertices is connected by a 2-dipath.

For convenience, we say that “a vertex $u$ sees a vertex $v$” if $u$ and $v$ are either adjacent or connected by a 2-dipath. Note that, an absolute oriented clique is different from an orientation of a clique or a complete graph.

In 2001, Borodin et al. [2] extended the notion of arc criticality for graph colouring to oriented colourings. Notably they gave examples of oriented graphs for which the removal of any arc decreases the oriented chromatic number by 2, the maximum possible [2]. Formally, a deeply critical oriented graph is an oriented graph $\vec{G}$ for which

$$\chi_0(\vec{G} - \vec{xy}) = \chi_0(\vec{G}) - 2$$

for each arc $\vec{xy} \in A(\vec{G})$.

Borodin et al. [2] gave an infinite family of deeply critical oriented graphs that were also absolute oriented cliques. For convenience, we refer to such an oriented graph as a deeply critical oriented clique. By way of example, we invite the reader to verify that the directed cycle on five vertices is a deeply critical oriented clique.

**Theorem 1.2** (Borodin et al. [2]). There exists a deeply critical oriented clique of order $n$ for every $n = 2 \cdot 3^m - 1$, where $m \geq 1$.

In their work Borodin et al. [2] speculated the existence of deeply critical oriented cliques of odd order $n$ for all $n \geq 33$. We close this long-standing open problem by proving the following result.

**Theorem 1.3.** Let $n \geq 1$ be an odd integer. There exists a deeply critical oriented clique of order $n$, if and only if $n \geq 5$, and $n \neq 7$. 
We prove this theorem in Section 2.

A remarkable aspect of the study of deeply critical oriented cliques is a lack of examples of such oriented graphs of even order, despite intensive computer search. We conjecture such deeply critical oriented cliques not to exist.

**Conjecture 1.4.** There exists a deeply critical oriented clique of order $n$, if and only if $n$ is odd, $n \geq 5$, and $n \neq 7$.

Let $n$ be an integer and let $S \subseteq \mathbb{Z}_n$ so that for all $k \in \mathbb{Z}_n$ if $k \in S$, then $-k \notin S$. The oriented circulant graph $\mathcal{C}(n, S)$ is the oriented graph with vertex set $V_i, i \in \mathbb{Z}_n$ so that $V_iV_j \in A(\mathcal{C}(n, S))$ when $j - i$ is congruent modulo $n$ to an element of $S$.

We provide further evidence towards Conjecture 1.4 by proving no deeply critical oriented clique appears among the family of oriented circulant graphs of even order.

**Theorem 1.5.** There does not exist any circulant deeply critical oriented clique of even order.

The significance of the above result is highlighted by the fact that all the examples of deeply critical oriented cliques used to prove Theorem 1.2 are, in particular, circulant oriented graphs.

Our work proceeds as follows. We prove Theorems 1.3 and 1.5 in Sections 2 and 3, respectively. In the former, we provide a method to construct a deeply critical oriented clique for any odd integer $n \geq 5$, exclusive of $n = 7$ and in the latter we give a full classification of deeply critical oriented circulant cliques. We provide concluding remarks and suggestions for future work in Section 4. We refer the reader to [1] for definitions of standard graph theoretic terminology and notation not defined herein.

## 2 | PROOF OF THEOREM 1.3

We begin by defining the following terms and notations. Let $\mathcal{G}$ be an oriented graph. For any vertex $u \in V(\mathcal{G})$, the set of out-neighbours (resp., in-neighbours) of $u$, denoted by $N^+(u)$ (resp., $N^-(u)$), is the set of all vertices $v$ in $V(\mathcal{G})$, such that $\mathcal{G}uv$ (resp., $\mathcal{G}vu$) is an arc in $\mathcal{G}$. An extending partition of $\mathcal{G}$ is a partition of its set of vertices $V(\mathcal{G}) = X_1 \uplus X_2 \uplus X_3$ so that

(i) there is no arc from a vertex of $X_{i+1}$ to a vertex of $X_i$, for all $i \in \{1, 2, 3\}$;
(ii) for each $u$ in $X_i$, there exists a unique out-neighbour, denoted by $out(u)$, in $X_{i+1}$, and there exists a unique in-neighbour, denoted by $in(u)$, in $X_{i-1}$, for all $i \in \{1, 2, 3\}$,

where addition in indices is taken modulo 3. Observe that $out(in(u)) = u$ for any vertex $u \in V(\mathcal{G})$.

We say an oriented graph is extendable if it admits an extending partition. For such graphs we define the following supergraphs. Let $\mathcal{G}$ be an oriented graph with extending partition...
V(\overrightarrow{G}) = X_1 \uplus X_2 \uplus X_3. The 6-extension of \overrightarrow{G} is the graph \overrightarrow{G}_6 constructed from \overrightarrow{G} as follows (see Figure 1):

- Include six new vertices \(x_1^+, x_1^-, x_2^+, x_2^-, x_3^+, x_3^-\) to the graph \overrightarrow{G}, and add the arcs \(\overrightarrow{x_1^-x_2^+}, \overrightarrow{x_1^+x_2^-}, \overrightarrow{x_1^-x_3^+}, \overrightarrow{x_1^+x_3^-}, \overrightarrow{x_2^+x_3^-}\) and \(\overrightarrow{x_2^-x_1^+}\).
- Add all the arcs of the form \(\overrightarrow{x_i^-x}\) and \(\overrightarrow{xx_i^+}\) for all \(x \in X_i, i \in \{1, 2, 3\}\).

Following the definition of 6-extension, we define two further extensions of \overrightarrow{G}, which arise as induced subgraphs of \overrightarrow{G}_6. The 4-extension \overrightarrow{G}_4 of \overrightarrow{G} is the graph obtained from \overrightarrow{G}_6 by deleting the vertices \(x_1^+\) and \(x_2^-\) from \overrightarrow{G}_6. The 2-extension \overrightarrow{G}_2 of \overrightarrow{G} is the graph obtained from \overrightarrow{G}_6 deleting the vertices \(x_3^+, x_3^-, x_2^-\) and \(x_3^+\) from \overrightarrow{G}_6.

**Lemma 2.1.** Let \overrightarrow{G} be an extendable deeply critical oriented clique. The 2-, 4- and 6-extensions of \overrightarrow{G} are deeply critical oriented cliques.

**Proof.** Let \overrightarrow{G} be a deeply critical oriented clique having an extending partition \(V(\overrightarrow{G}) = X_1 \uplus X_2 \uplus X_3\). \overrightarrow{G} is an oriented clique, and the addition of the new arcs does not disturb any adjacencies or 2-dipaths between the vertices of \(X_1, X_2\) and \(X_3\). Thus, by Theorem 1.1, to verify whether the 6-extension \overrightarrow{G}_6 of \overrightarrow{G} is an oriented clique, we need to only check whether the newly added vertices see every other vertex. This is made clear in Table 1. The entries of the table can be determined in most cases by direct observation of Figure 1, while the remaining entries are determined by applying the properties of

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**FIGURE 1** Construction of \(\overrightarrow{G}_6\). Thickened arcs indicate the existence of all possible arcs between the vertex and the set \(X_i\). Arcs between \(X_i\)’s are not shown.
extending partitions. For example, $x_1^-$ sees each vertex $x_2$ of $X_2$ through $\text{in}(x_2)$ in $X_1$ due to property (2) of the definition of extending partition. Since, Table 1 has entries corresponding to every possible pair of the newly added vertices, $G_6$ is an oriented clique.

To verify that the induced subgraphs $G_2$ or $G_4$ are also oriented cliques, it is enough to observe that if we restrict the rows and columns in Table 1 to the ones corresponding to the vertices of $G_2$ or $G_4$, then the entries in the table are only elements of $G_2$ or $G_4$, respectively.

Since, $G$ is deeply critical, and the addition of the new vertices does not create any new adjacencies or 2-dipaths between the vertices of $X_1$, $X_2$ and $X_3$, to verify whether $G_6$ is deeply critical we need to only check whether removing the newly added arcs decreases the chromatic number of $G_6$ by 2. Table 2 lists how this can be done for each case, and the $n - 4$ vertices that are unaffected for a particular arc get distinct colours. As previously, direct observation of Figure 1 or the application of properties of extending partitions justifies all the entries in the table.

To verify that the induced subgraphs $G_2$ or $G_4$ are also deeply critical, it is enough to observe that if we restrict the rows and columns in Table 1 to the ones corresponding to the vertices of $G_2$ or $G_4$, then the entries in the table complete the argument for the arcs in $G_2$ or $G_4$, respectively.

Lemma 2.1 implies that given an extendable deeply critical oriented clique on $n$ vertices, one may construct deeply critical oriented cliques on $n + 2$, $n + 4$ and $n + 6$ vertices. We note, however that computer search yields many examples of deeply critical oriented cliques that do not arise as an extension of a smaller deeply critical oriented clique. Figure 2 gives such an example. Curiously, though generated by computer search, the oriented graph in Figure 2 does arise as a 6-extension of a directed three cycle (shown using thickened arcs in the figure).

Next, we analyse the 6-extension, $G_6$ of $G$. 

| $x_1^+$ | $x_2^-$ | $x_2^+$ | $x_3^+$ | $x_3^-$ | $x_1$ | $x_2$ | $x_3$ |
|---------|---------|---------|---------|---------|-------|-------|-------|
| $x_1^+$ | $x_1^- x_1^+$ | $x_2^- x_2^+$ | $x_3^- x_3^+$ | $x_1^- x_2^- x_2^+$ | $x_1^- x_2^- x_3^- x_3^+$ | $x_1^- x_1^+ x_1^-$ | $x_1^- x_2^- x_2^+$ |

Note: An entry in a cell of the form $ab$ means that there is an arc $\overrightarrow{ab} \in A(G)$ and an entry of the form $abc$, means that $\overrightarrow{ab}, \overrightarrow{bc} \in A(G)$, forming a 2-dipath from $a$ to $c$. 

### TABLE 1

Each row heading vertex sees each column heading vertex: The row (column) heading $x_i$ refers to an arbitrary vertex in $X_i$. 

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Lemma 2.2. The 6-extension of an extendable deeply critical oriented clique is extendable.

Proof. Let $\vec{G}$ be a deeply critical oriented clique having an extending partition $V(\vec{G}) = X_1 \uplus X_2 \uplus X_3$. We claim that $\vec{G}_6$, which is also a deeply critical oriented clique due to Lemma 2.1, admits an extending partition $V(\vec{G}_6) = X'_1 \uplus X'_2 \uplus X'_3$, where we have $X'_i = X_i \uplus \{x_i^{-}, x_i^{+}\}$ for all $i \in \{1, 2, 3\}$.

Claim 1: There is no arc from a vertex of $X'_{i+1}$ to a vertex of $X'_i$, for all $i \in \{1, 2, 3\}$.
Proof of the claim. We know that $X'_i = X_i \cup \{x_i^-, x_i^+\}$. By property (2) of the definition of extending partitions, there are no arcs from $X_{i+1}$ to $X_i$ and by also taking into account the construction of $G_6$, the claim is obvious.

**Claim 2:** For each $u \in X'_i$, there exists a vertex $v \in X'_{i+1}$ such that $N^-(v) \cap X'_i = \{u\}$, for all $i \in \{1, 2, 3\}$.

**Proof of the claim.** We observe that $N^-(x^-_{i+1}) \cap X'_i = \{x^+_i\}$ and $N^-(x^+_i) \cap X'_i = \{x^-_i\}$ by construction and together with property (2) of the definition of extending partition applied to $X_i$ and $X_{i+1}$, we have our claim.

**Claim 3:** For each $v \in X'_{i+1}$, there exists a vertex $u \in X'_i$ such that $N^+(u) \cap X'_{i+1} = \{v\}$, for all $i \in \{1, 2, 3\}$.

**Proof of the claim.** We observe that $N^+(x^+_i) \cap X'_{i+1} = \{x^-_{i+1}\}$ and $N^+(x^-_i) \cap X'_{i+1} = \{x^+_i\}$ by construction and together with property (2) of the definition of extending partition applied to $X_i$ and $X_{i+1}$, we have our claim.

Therefore, $G_6$ is extendable.

With these two lemmas in place, we provide a proof of Theorem 1.3.

**Proof of Theorem 1.3.** The directed cycle on five vertices is a deeply critical oriented clique. By computer search there is no deeply critical oriented clique on seven vertices (See https://github.com/rbsandeep/dcoc for the detailed code used).

The oriented graph given in Figure 2 is a deeply critical oriented clique with nine vertices. This oriented graph admits the following extending partition: $X_1 = \{6, 2, 7\}$, $X_2 = \{1, 5, 0\}$, $X_3 = \{4, 8, 3\}$. The result now follows inductively from the following observation: By Lemmas 2.1 and 2.2, if $G$ is an extendable deeply critical oriented clique with $n$ vertices, then there exists a deeply critical oriented clique on $n + 2$ and $n + 4$ vertices, and an extendable deeply critical oriented clique on $n + 6$ vertices.

3 | PROOF OF THEOREM 1.5

We provide a proof of Theorem 1.5 by first giving a full classification of deeply critical oriented circulant cliques.

**Lemma 3.1.** The circulant graph $\overrightarrow{C}(n, S)$ is a deeply critical oriented clique if and only if for every $k \in \mathbb{Z}_n$

(a) there exists $x, y \in S \cup \{0\}$ so that $k \equiv x + y \pmod{n}$ or $k \equiv -(x + y) \pmod{n}$; and

(b) if $k$ is even and $\frac{k}{2} \in S$ then the only way to express $k$ as $k \equiv x + y \pmod{n}$ or $k \equiv -(x + y) \pmod{n}$ is by taking $x = y = \frac{k}{2}$.
Proof. First let us prove the “if” part. Therefore, assume that $\overrightarrow{C}(n, S)$ satisfies the given conditions. The indices mentioned hereafter are assumed to be taken (mod $n$).

Note that, for any $x \in S$, there is an arc from $v_{i-x}$ to $v_i$ and from $v_i$ to $v_{i+x}$. Thus, for any values of $x, y \in S$, there exists a 2-dipath from $v_{i-(x+y)}$ to $v_i$ and from $v_i$ to $v_{i+(x+y)}$. Thus condition (a) of the statement implies that, for any two vertices of $\overrightarrow{C}(n, S)$ there is either an arc or a 2-dipath connecting them. Hence, $\overrightarrow{C}(n, S)$ is an oriented absolute clique due to Theorem 1.1.

Next we will show that $\overrightarrow{C}(n, S)$ is deeply critical. To do so, we need to show that the removal of any arc $\overrightarrow{v_iv_{i+x}}$ will decrease the oriented chromatic number by exactly 2. We know that removal of an arc can decrease the oriented chromatic number by at most 2, so it will be enough to show that there exists an oriented $(n - 2)$-colouring of $\overrightarrow{C}(n, S) - \overrightarrow{v_iv_{i+x}}$ for each $x \in S$. Moreover, as $\overrightarrow{C}(n, S)$ is vertex transitive, it is enough to consider $i = 0$, in particular for our proof.

Thus, let $\overrightarrow{C}'$ be the oriented graph obtained by deleting the arc $\overrightarrow{v_0v_x}$ from $\overrightarrow{C}(n, S)$, for some $x \in S$. Therefore, due to condition (b) of the statement, we know that there is no arc or 2-dipath connecting the pairs of vertices $(v_{i-x}, v_x)$ and $(v_0, v_{2x})$. Note that, the oriented colouring of $\overrightarrow{C}'$ given by assigning colour 1 to $v_{x}, v_{2x}$, colour 2 to $v_0, v_{2x}$ and $(n - 4)$ distinct colours to the other vertices is an oriented $(n - 2)$-colouring of $\overrightarrow{C}'$. Thus, we have proved the “if” part of the statement.

Now we will prove the “only if” part of the proof. Assume that $\overrightarrow{C}(n, S)$ is a deeply critical oriented clique.

First note that condition (a) of the statement is trivial for $k = 0$. Suppose that condition (a) is not satisfied for some integer $k \neq 0$, then we can say that $v_0$ and $v_k$ are neither adjacent nor connected by a 2-dipath. This contradicts the fact that $\overrightarrow{C}(n, S)$ is an oriented absolute clique.

Next suppose that condition (b) of the statement is false for some even $k \in S$. Observe that, surely $k \equiv (\frac{1}{2} + \frac{k}{2})$ (mod $n$) is one way to express $k$. However, as condition (b) is false, there is another way of expressing $k$ according to condition (a), that is, $k = x + y$ (mod $n$) or $k = -(x + y)$ (mod $n$), for some $x, y \in S$.

Let $\overrightarrow{C}'$ be the oriented graph obtained by deleting the arc $\overrightarrow{v_0v_{k/2}}$ from $\overrightarrow{C}(n, S)$. Note that, all the adjacent vertices of $v_0$ in $\overrightarrow{C}(n, S)$, except $v_{k/2}$, remain adjacent in $\overrightarrow{C}'$. Suppose, $v_\ell$ is a vertex which is not adjacent to $v_0$ in $\overrightarrow{C}(n, S)$. As we know that $\overrightarrow{C}(n, S)$ is an oriented absolute clique, due to Theorem 1.1, there must exist a 2-dipath between $v_0$ and $v_\ell$, say, $v_0v_{i+j}$, where $i + j = \ell$, which also exists in $\overrightarrow{C}'$ unless $i = k/2$. If $i = k/2$ and $j \neq k/2$, then $v_0v_{i+j}$ is a 2-dipath connecting $v_0$ and $v_\ell$ in $\overrightarrow{C}'$. If $i = j = k/2$, then $\ell = k$ and $v_0v_{k/2}$ is a 2-dipath from $v_0$ to $v_\ell$ in $\overrightarrow{C}'$. The argument for the case if the 2-dipath is from $v_\ell$ to $v_0$ is similar.

Therefore, even after removing the arc $\overrightarrow{v_0v_{k/2}}$, all vertices except maybe $v_{k/2}$ will remain adjacent or connected by a 2-dipath with $v_0$. As any pair of vertices not including $v_0$ or $v_{k/2}$ have not used the arc $\overrightarrow{v_0v_{k/2}}$ for being adjacent or connected by a 2-dipath in $\overrightarrow{C}(n, S)$, such pairs remain adjacent or connected by a 2-dipath in $\overrightarrow{C}'$. Thus, $\overrightarrow{C}' - v_{k/2}$ is
an oriented absolute clique and the oriented chromatic number of $\vec{C}'$ must be at least $(n - 1)$. This contradicts the fact that $\vec{C}(n, S)$ is a deeply critical oriented graph. Therefore, the converse is also proved.

Now, we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** Let $\vec{C}(n, S)$ be an oriented circulant graph so that $\vec{C}(n, S)$ is an absolute clique and $n$ is even. As $n$ is even we have $\frac{n}{2} \equiv -\frac{n}{2} \pmod{n}$. On the other hand, by the definition of oriented circulant graphs, if $k \in S$, then $-k \notin S$. Therefore $\frac{n}{2} \notin S$. Subsequently, vertices with indices 0 and $\frac{n}{2}$ are not adjacent in $\vec{C}(n, S)$.

Since $\vec{C}(n, S)$ is an absolute clique, by Theorem 1.1 there is a 2-dipath connecting vertices with indices 0 and $\frac{n}{2}$. Thus, there exists $x, y \in S$ satisfying $(x + y) \equiv \frac{n}{2} \pmod{n}$ or $-(x + y) \equiv \frac{n}{2} \pmod{n}$. Therefore, $2x \equiv x + x \equiv -(y + y) \pmod{n}$ or $-2x \equiv -x - x \equiv y + y \pmod{n}$, violating part (b) of Lemma 3.1. Therefore, $\vec{C}(n, S)$ is not a deeply critical oriented clique.

4 | CONCLUSIONS AND OUTLOOK

Work in [2] and herein provides examples of infinite families of deeply critical oriented cliques. These constructions and extensive computer search have yielded no examples of deeply critical oriented cliques of even order. These observations together with the result of Theorem 1.5 lend support to the statement of Conjecture 1.4.

Our computer search has yielded surprising insight into the density of deeply critical oriented cliques among the family of absolute oriented cliques. We identified 9917 examples of previously unknown sporadic deeply critical oriented cliques on up to 17 vertices. In addition to these examples, our search of oriented circulant graphs found 28 examples of previously unknown deeply critical circulant oriented cliques on up to 49 vertices. A classification of odd orders for which there exists a deeply critical circulant oriented clique remains open.

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