Dynamical Instability of Spherical Anisotropic Sources in $f(R, T, R_{\mu\nu}T^{\mu\nu})$ Gravity

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Abstract

In this paper, we study the effects of modification of gravity on the problem of dynamical instability of the spherical relativistic anisotropic interiors. We have considered non-zero influence of expansion scalar throughout during the evolutionary phases of spherical geometry that led to the use of fluid stiffness parameter. The modified hydrostatic equation for the stellar anisotropic matter distributions is constructed and then solved by using radial perturbation scheme. Such a differential equation can be further used to obtain instability constraints at both weak field and post-Newtonian approximations after considering a particular Harrison-Wheeler equation of state. This approach allows us to deal with the effects of usual and effective matter variables on the stability exotic stellar of self-gravitating structures.

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1 Introduction

General relativity (GR) is regarded as the foundations of relativistic astrophysics and cosmology. The observational outcomes coming from some cosmic models like Λ-cold dark matter turn out to consistent with various cosmological issues besides some discrepancies such as fine-tuning and cosmic coincidence \[1\]. The accelerated cosmic expansion is strongly evidenced from the surveys of cosmic microwave background radiations, redshift, Supernovae Type Ia and large scale structures \[2\]. These observations claimed the role of some mysterious force (dubbed as dark energy (DE)) behind this expansion. Several mathematical models have been introduced to modify GR with the aim to explore DE and dark matter (DM). Qadir et al. \[3\] proposed that modification of Einstein gravity could be considered as a workable toy model for various cosmological issues, like quantum gravity and DM problem.

The models of modified gravity came into their existence by modifying the geometric portion of the Einstein-Hilbert (EH) action (for further reviews on DE and modified gravity, see, for instance, \[4–8\])). Nojiri and Odintsov \[9\], in this direction, presented some \(f(R)\) gravity that are theoretically well-consistent for the case of our accelerating universe. Modified gravity theories includes \(f(R)\) \[10\], Einstein-Λ \[11\], \(f(R, T)\) \[12\], \(f(G)\) \[13\] gravity (here \(R, T\) and \(G\) represent Ricci scalar, trace of stress energy tensor and Gauss-Bonnet Invariant, respectively) and \(f(R, T, Q)\) \[14, 15\] theories (where \(Q = R_{\lambda\sigma}T^{\lambda\sigma}\)) etc. which includes non-minimal coupling relating matter and geometry.

The isotropic and anisotropic nature of fluid configurations have utmost relevance in the evolution and formation of compact stars. Initially, Chandrasekhar \[16\] investigated the dynamical instability conditions of an oscillating perfect spherically symmetric celestial object. Later on, this problem has been probed under various complicated backgrounds of relativistic matter and geometry. Herrera et al. \[17\] examined this problem by considering non-adiabatic nature of relativistic spherical structures with weak field approximations. Chan et al. \[18\] and Chan \[19\] analyzed the role of locally anisotropcity as well as heat radiation in the formulation of dynamical instability constraints of the shearing viscous spherical matter content at both Newtonian and post-Newtonian (pN) eras. The dynamical instability of expansion-free locally anisotropic spherical stars have been analyzed by Herrera at el. \[20\] through perturbation scheme.

Odintsov and Sáez-Gómes \[14\] studied various cosmological aspects in \(f(R, T, Q)\) gravity through construction of viable cosmological models. Haghani et al. \[21\] evaluated \(f(R, T, Q)\) equations of motion for the case of massive test particles by using the method of Lagrange multiplier. After assuming special case of conservation of the stress-energy tensor in this theory, they evaluated a the corresponding class of field equations. Ayuso et al. \[22\] obtained some consistent results from the nonminimally coupled \(f(R, T, Q)\) gravity and claimed that such theoretical models could be helpful to remove ceratin types of pathologies related
with the equations of higher order class of theories. Elizalde and Vacaru \cite{23} suggested that some mathematical formulations of $f(R, T, Q)$ gravity along with their non-diagonal non-holonomic equivalents could produce captivating connections between viable quantum gravity theories. Recently, Yousaf et al. \cite{24} analyzed the impact of particular $f(R, T, Q)$ models on the evolutionary phases of collapsing relativistic systems.

Dynamical stability is the characteristic of an object or system to retain its stable position, whenever it is disturbed due to fluctuations. The instability/stability of celestial bodies has been discussed not only in GR but also in modified gravitational theories. In order to analyze the dynamics of massive objects, one can use N and pN approximations in the corresponding hydrodynamical equation. Gravitational collapse (GC) of massive or dense stars and stability/instability investigation of massive objects have attained great interest of researchers in relativistic astronomy \cite{25-41}. Adhav \cite{42} and Sahoo et al. \cite{43} evaluated some exact analytical solution by assuming a particular class $f(R, T)$ models. The discussion of inhomogeneous energy density in the regime of Einstein theory has also been discussed widely by Herrera et al. \cite{44}.

Capozziello et al. \cite{45} studied GC of a dust cloud through dispersion relations and perturbation scheme and calculated certain unstable limits for the onset of collapsing phenomenon. Cembranos et al. \cite{46} analyzed GC of an inhomogeneous stellar objects in order to check large scale structure formation with the help different early time $f(R)$ models. Certain modified gravity models are likely to host supermassive structures with comparatively smaller radii as that in \cite{47}. Yousaf et al. \cite{48} inferred that dark source terms induced from some models of $f(R, T)$ gravity could be treated as effective tools to study collapse of non-interacting particles. Baffou et al. \cite{49} performed stability analysis with the help of de-Sitter and power law solution in $f(R, T, Q)$ gravity and found that extra curvature $f(R, T, Q)$ terms could help to understand early evolutionary cosmic stages.

The aim of this paper is to check the role of $f(R, T, Q)$ gravity on the stability of self-gravitating celestial body. The format of paper is as follows. We present the fundamental formalism to form the corresponding field equations in section 2. Section 3 is devoted to formulate static as well as non-static perturbed field as well as conservation laws. After using particular choice of equation of state, we have explored hydrodynamical equation and instability constraints with both N and pN approximations. In the last section, we conclude our main findings.
2 $f(R, T, R_{\mu\nu}T^{\mu\nu})$ Theory of Gravity and Field Equations

The theories with generic functions of the form $f(R, T, R_{\mu\nu}T^{\mu\nu})$ in the action have attracted the attention of several theoretical astrophysicists. Odintsov and Sáez-Gómez \[14\] claimed that such theories could provide some useful insights provided by Hořava-like gravity under some conditions. Thus, such theories could be considered as a theoretical bridge between Hořava-Lifshitz gravity and modified theories of gravity. This section is devoted to discuss some basic formulations of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ theory. We will also formulate the corresponding field equations for spherical anisotropic self-gravitating systems.

2.1 Basic Formalisms of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ Equations

The formulation of $f(R, T, R_{\gamma\delta}T^{\gamma\delta})$ gravity is based on the contribution of strong association between geometry and matter where the Ricci scalar in usual EH action, is replaced with generic function of $R, T$ and $R_{\gamma\delta}T^{\gamma\delta}$. The modified action can be written as

$$I_{f(R,T,Q)} = \frac{1}{2} \int d^4x \sqrt{-g} [f(R, T, Q) + L_m], \quad (1)$$

where, $L_m$ is the relative matter Lagrangian density. The expression for energy-momentum tensor is given by the following relation

$$T^{(m)}_{\lambda\sigma} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{\lambda\sigma}}. \quad (2)$$

After applying variations in the above equation with respect to $g_{\lambda\sigma}$, we get

$$- G_{\lambda\sigma} (f_Q L_m - f_R) - g_{\lambda\sigma} \left\{ \frac{f}{2} - \Box f_R - \frac{R}{2} f_R - \frac{1}{2} \nabla^\pi \nabla^\rho (f_Q T^{\pi\rho}) - L_m f_T \right\}$$

$$+ 2 f_Q R^\pi_{(\lambda} T^\sigma) \frac{1}{2} \Box (f_Q T_{\lambda\sigma}) - \nabla^\pi \nabla^\rho [T_{\pi\rho}] f_Q \} - 2 (f_T g^{\pi\rho} + f_Q R^{\pi\rho}) \frac{\partial^2 L_m}{\partial g^{\lambda\sigma} \partial g^{\rho\sigma}}$$

$$- T^{(m)}_{\lambda\sigma} (f_T + R/2 f_Q + T(f_T + 1) - \nabla^\lambda \nabla^\sigma f_R = 0, \quad (3)$$

where $\nabla^\pi$, $G_{\lambda\sigma}$ describes covariant derivative and Einstein tensor, respectively, while $\Box = g^{\lambda\sigma} \nabla^\lambda \nabla^\sigma$ corresponds to d’Alembert’s operator. Further, the quantities $f_R$, $f_T$ and $f_Q$ represent the partial differentiation of function $f$ with respect to $R$, $T$ and $Q$, respectively. From Eq.(3), the expression of trace is obtained as

$$3 \Box f_R + \frac{1}{2} \Box (f_Q T) - T(f_T + 1) + \nabla^\pi \nabla^\rho (f_Q T^{\pi\rho}) + R(f_R - T/2 f_Q)$$

4
\[ (Rf_Q + 4f_T) L_m - 2f + 2R_{\pi\rho} T^{\pi\rho} f_Q - 2 \frac{\partial^2 L_m}{\partial g^{\lambda\sigma} \partial g^{\pi\rho}} (f_T g^{\pi\rho} + f_Q R^{\pi\rho}) . \]

The case \( Q = 0 \) boils down \( f(R, T, Q) \) theory to \( f(R, T) \) theory of gravity. However, the vacuum case of \( f(R, T, Q) \) theory leads to \( f(R) \) gravity theory. It is important to note that \( L_m \) is in the form of second or superior orders, whereas in case of relativistic frame which are connected by a specific matter collection, the second variation of \( L_m \) can be neglected. In frame work of [50], the matter Lagrangian has no specific distinction for ideal fluid, also second variation was taken to be negligible. In GR prospective, Eq.(3) can be demonstrated as

\[ R_{\lambda\sigma} - \frac{R}{2} g_{\lambda\sigma} = G_{\lambda\sigma} = T^{\text{eff}}_{\lambda\sigma}, \]

where

\[
T^{\text{eff}}_{\lambda\sigma} = \frac{1}{(f_R - f_Q L_m)} \left[ (f_T + \frac{1}{2} Rf_Q + 1) T^{(m)}_{\lambda\sigma} + \left\{ \frac{R}{2} \left( \frac{f}{R} - f_R \right) - L_m f_T - \frac{1}{2} \right. \right.
\]

\[
\times \nabla_\pi \nabla_\rho (f_Q T^{\pi\rho}) \} g_{\lambda\sigma} - \frac{1}{2} \Box (f_Q T_{\lambda\sigma}) - (g_{\lambda\sigma} \Box - \nabla_\lambda \nabla_\sigma) f_R - 2f_Q R_{\pi\sigma} T^{\pi}_{\sigma}
\]

\[
+ \nabla_\pi \nabla_\lambda \left[ T^{\pi}_{\sigma} f_Q \right] + 2 (f_Q R^{\pi\rho} + f_T g^{\pi\rho}) \frac{\partial^2 L_m}{\partial g^{\lambda\sigma} \partial g^{\pi\rho}} \].

2.2 Spherical Anisotropic Source

We assume our relativistic stellar objects to be in spherical shape with the following line element

\[ ds^2_\perp = -A^2(t, r) dt^2 + B^2(t, r) dr^2 + C^2(t, r) (d\theta^2 + \sin^2 \theta d\phi^2), \]

In spherical geometry, we consider the distribution of anisotropic fluid which collapses adiabatically and has mathematical form as

\[ T_{\lambda\sigma} = (P_\perp + \mu) V_{\lambda} V_{\sigma} + P g_{\lambda\sigma} - \chi_{\lambda\sigma}(P_\perp - P_r) \]

where, \( P_\perp \), \( P_r \) and \( \mu \) describe the tangential, radial pressure and energy density of the fluid, respectively. For comoving frame of reference, the four-vectors are defined as \( V^\lambda = A^{-1} \delta^\lambda_0 \) and \( \chi^\lambda = B^{-1} \delta^\lambda_1 \), obeying

\[ V^\lambda V_\lambda = 1, \quad \chi^\lambda \chi_\lambda = -1, \quad \chi^\lambda V_\lambda = 0. \]

The expansion scalar, \( (\Theta = V^\lambda V_\lambda) \), for our system leads to

\[ \Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{2C'}{C} \right). \]
The modified field equations for our line element associated with matter distribution (6) become

\[
\mu_{\text{eff}} = \frac{1}{(f_R + f_Q \mu)} \left[ -\mu f_T + f''_R \frac{B'}{B^2} + \left( \frac{2C'}{C} - \frac{B'}{B} \right) f'_R \frac{B'}{B^2} - \left( \frac{\dot{B}}{B} + \frac{2\dot{C}}{C} \right) f_R \frac{B'}{B^2} + \mu \chi_1 \\
- \frac{R}{2} \left( f_R - f_R \right) + \chi_2 \mu + \chi_3 \mu' + \frac{f_Q}{2A^2} \mu'' + \chi_4 P_r + \left( \frac{f'_R}{B^2} - \frac{5}{2} f_Q B' \right) \right] \\
\times P'_R + \frac{f_Q}{2B^2} P''_R - \frac{f_Q}{2A^2 B} \dot{B} \dot{P}_r + \chi_5 P_{\perp} - \frac{3 f_Q}{2A^2 C} \dot{P}_\perp \dot{C} - \frac{3 f_Q}{2B^2 C} P_{\perp}'' \right], \tag{8}
\]

\[
P_{\text{eff}}^R = \frac{1}{(f_R + f_Q \mu)} \left[ \mu f_T + f''_R \frac{A'}{A^2} + \left( \frac{2\dot{C}}{C} - \frac{\dot{A}}{A} \right) f'_R \frac{A'}{A^2} - \left( \frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) f_R \frac{A'}{A^2} + \chi_6 P_r \\
+ \frac{R}{2} \left( f_R - f_R \right) + \chi_7 P'_R + \chi_8 \dot{P}_r + \chi_9 P_{\perp} + \frac{f_Q}{2A^2 C} \dot{P}_\perp - \frac{f_Q}{2A^2} \mu' + \frac{f_Q}{2B^2} A' \mu' \\
+ \chi_{10} \mu + \chi_{11} \dot{\mu} + \frac{f_Q}{2B^2} C' P_{\perp}' \right], \tag{9}
\]

\[
P_{\text{eff}} \perp = \frac{1}{(f_R + f_Q \mu)} \left[ \mu f_T + f''_R \frac{B'}{B^2} + \left( \frac{B'}{B} - \frac{A'}{A} - \frac{C'}{C} \right) f'_R \frac{B'}{B^2} - \frac{f_Q}{2A^2} \dot{P}_\perp + \frac{f_Q}{2B^2} P''_\perp \\
+ \chi_{14} P'_R + \frac{R}{2} \left( f_R - f_R \right) + \left( \frac{B'}{B} - \frac{A'}{A} - \frac{C'}{C} \right) f'_R \frac{B'}{B^2} - \frac{f_Q}{2A^2} \dot{P}_\perp + \frac{f_Q}{2B^2} P''_\perp \\
+ \chi_{16} \mu + \frac{f_Q}{2A^2} \dot{B} \dot{P}_r + \frac{5 f_Q}{2B^2} B' P'_R - \frac{P'_R f_Q}{2B^2} P''_R + \left( \frac{5 f_Q \dot{A}}{2A^2} - \frac{f_Q}{A^2} \right) \dot{\mu} \\
- \frac{f_Q}{2A^2} \mu' + \frac{f_Q}{2B^2} A' \mu' + \chi_{15} P_r \right], \tag{10}
\]

\[
T_{01}^{\text{eff}} = \frac{1}{(f_R + f_Q \mu)} \left[ \dot{f}_R - \frac{A'}{A} \dot{f}_R - \frac{\dot{B}}{B} f'_R \right] \equiv H, \tag{11}
\]

where the quantities \( \chi_i \)'s consists of metric variables and their derivatives, and their expressions are given in Appendix. The derivatives with respect to time and radial coordinate are shown with the help of \( \cdot \) and \( ' \) notations, respectively. The value of the Ricci scalar \( R \) is given as

\[
R = R(t, r) = \left( \frac{2B'}{B} - \frac{C'}{C} - \frac{2A'}{A} \right) \frac{2C'}{CB'^2} - \frac{2}{B^2} \left( \frac{2C''}{C} - \frac{B'A'}{BA} + \frac{2A'}{A} \right) - 2 \frac{\dot{C}}{CA^2}
\]

6
\[ \times \left( \frac{2\dot{A}}{A} - \frac{\dot{C}}{C} - \frac{2\dot{B}}{B} \right) - \frac{2}{C^2} + \frac{2}{A^2} \left( \frac{\ddot{B}}{B} - \frac{A\dot{B}}{AB} + 2\frac{\ddot{C}}{C} \right). \quad (12) \]

3 The Perturbative Scheme and Collapse Equation

Here, we wish to calculate \( f(R, T, Q) \) field as well as dynamical equations through perturbation technique. This technique would be helpful to compute the instability zones for analytical models of spherical geometry in \( f(R, T, Q) \) theory of gravity.

3.1 Mass Function and Divergence of Effective Energy Momentum Tensor

Here, our aim is to explore the expression of hydrodynamical equation. For this purpose, we shall define modified versions of dynamical equations and a viable collapsing model in the framework. The mass function for spherical geometry is found by using Misner-Sharp formalism as \[ m(t, r) = \frac{C}{2} \left( 1 - \frac{C''}{B^2} + \frac{\dot{C}^2}{A^2} \right). \quad (13) \]

Its radial derivative is obtained as

\[ m' = \frac{C''C^2}{2A^2} \left( T_{00}^{\text{eff}} + \kappa \mu A^2 \right) - \frac{C^2\dot{C}}{A^2} T_{01}^{\text{eff}}. \]

The integration of the above equation gives rise to

\[ m = \frac{1}{2} \int_0^r \left( T_{00}^{\text{eff}} \frac{C'}{A^2} + \kappa \mu C' - T_{01}^{\text{eff}} \frac{\dot{C}}{A^2} \right) C^2 dr, \]

where we have considered the case under which \( m(t, 0) = 0 \).

The divergence of effective energy momentum tensor in this modified theory gives rise to

\[
\nabla^\lambda T_{\lambda\sigma} = \frac{2}{Rf_Q + 2f_T + 1} \left[ \nabla_\sigma (L_m f_T) + \nabla_\sigma (f_Q R^\pi_\lambda T_{\pi\sigma}) - \frac{1}{2} (f_T g_{\pi\rho} + f_Q R_{\pi\rho}) \right. \\
\left. \times \nabla_\sigma T^\pi_{\rho} - G_{\lambda\sigma} \nabla^\lambda (f_Q L_m) \right],
\]
This expression would give us a set of two equations as our matter variables depend upon \( t \) and \( r \) variables. By using \( G^{\lambda\sigma} = 0 \) and Eqs. (8)-(11), this correspondence with \( \lambda = 0, 1 \) assigns

\[
\left[ \mu_{\text{eff}} + P_{r\text{eff}} + \mu_{\text{eff}} \right] \frac{\dot{B}}{B} + 2\left( \mu_{\text{eff}} + P_{\perp\text{eff}} \right) \frac{\dot{C}}{C} \frac{1}{A} + AH' + AH \left( \frac{3A'}{A} + \frac{3B'}{B} + \frac{2C'}{C} \right) = 0, \quad (15)
\]

\[
\left[ P_{r\text{eff}} + P_{r\text{eff}} + \mu_{\text{eff}} \right] \frac{A'}{A} + 2 \left( P_{r\text{eff}} - P_{\perp\text{eff}} \right) \frac{C'}{C} \frac{1}{B} + B\dot{H} + BH \left( \frac{\dot{A}}{A} + \frac{3\dot{B}}{B} + \frac{2\dot{C}}{C} \right) = 0. \quad (16)
\]

The above dynamical equations could help to understand the hydrodynamics of locally anisotropic spherical relativistic massive bodies. Here, the superscript \( \text{eff} \) shows the presence of \( f(R, T, Q) \) dark sources in the corresponding matter quantities. Now, we perform our analysis of dynamical instability by using the following choice of \( f(R, T, Q) \) formulations \cite{22} as

\[
f(R, T, Q) = \beta R(1 + \alpha Q), \quad (17)
\]

in which the quantities \( \alpha \) and \( \beta \) are constant numbers. The particular values of these constants provide modified correction for some particular cases. For instance, non-zero values of \( \alpha \) and \( \beta \) give rise to redefinition of gravitational field, thereby presenting this to be physically viable model. First term in the model leads to GR results.

### 3.2 Perturbations

The perturbation approach assists one to convert non-linear and non-solvable relations to linear and solvable. This scheme is based on the non-zero and very small perturbation parameter denoted by \( \epsilon \) with the assumption that \( 0 < \epsilon \ll 1 \). We shall perturb our equations up to first order in \( \epsilon \). It is assumed that initially, the celestial system was in the phase of hydrostatic equilibrium but with the passage of time it undergoes a periodic motion with frequency rate \( \xi \). Therefore, all the material and metric functions depend upon the time parameter \( \omega(t) \) at that instant. The perturbed configuration is expressed as \cite{52},

\[
Y(t, r) = y_o(r) + \epsilon\omega(t)y(r), \quad Z(t, r) = z_o(r) + \epsilon z(t, r), \quad (18)
\]

where \( Y \) and \( Z \) indicate the metric and material functions, respectively. Applying this technique on Eq. (12), the solution of the second order partial differential equation can be written as

\[
\omega = \omega(t) = -\exp(\xi t), \quad (19)
\]
where the frequency $\xi$ of the anisotropic spherical body using Schwarzschild radius, is calculated as

$$\xi^2 = \frac{1}{r(2cB_o^2 + bn)} \left[ 2c' + 2rbA_o' - 2c\epsilon^a' + 2c'rA_o' - 2\epsilon rA_o' - 3b 
+ 4a - 2b \epsilon a' A_o'^2 - 2c\epsilon a' A_o'^2 - 2c'rA_o' - 6aB_o' - 2cr^2A_o'B_o' + 2cr^2 
- 2ar^2A_o'B_o' - A_o'^2B_o^2 - A_o'^2B_o^2 + r^2a' A_o'^2 + 2r^2bA_o'^2 - 3a + 2crA_o' 
+ \frac{R_o}{2} \left( 3a + \frac{3b}{rA_o'^2B_o^2} + \frac{2c}{rA_o'^2B_o^2} \right) \right] B_o^2A_o'. \tag{20}$$

By using Eqs. (18), the perturbed $f(R, T, Q)$ gravity model, is

$$f = R_o(1 + \alpha Q_o) + \omega(t)[d + \alpha(R_o g + dQ_o)], \tag{21}$$

where, $d = d(r)$, $g = g(r)$. The static background of $f(R, T, Q)$ field equations obtained from perturbation scheme are

$$\mu_o^{\text{eff}} = \frac{1}{1 + \alpha(Q_o + \mu_o R_o)} \left[ \frac{\alpha}{B_o^2} \left( Q'' - \frac{B_o Q'_o}{B_o} + \frac{2Q'_o}{r} \right) + \mu_o \chi_{1o} + \mu_o' \chi_{3o} 
+ P_{\perp o}(\chi_{5o}) + \frac{\alpha R_o}{2B_o^2} \left( \mu_o'' + P_{ro}'' - \frac{3P_{ro}'}{r} + 2P_{ro}'' R_o' R_o \right) - \frac{5\alpha}{2} R_o B_o' P_{ro}' \right], \tag{22}$$

$$P_{ro}^{\text{eff}} = \frac{1}{1 + \alpha(Q_o + \mu_o R_o)} \left[ -\frac{\alpha Q'_o}{B_o^2} \left( A_o' + \frac{2}{r} \right) + \mu_o \chi_{1o} + P_{ro} \chi_{3o} + \mu_o' \chi_{7o} 
+ P_{\perp o}(\chi_{9o}) + \frac{\alpha}{2B_o^2} \left( \mu_o' R_o' A_o' + 2R_o \frac{P_{ro}'}{r} \right) \right], \tag{23}$$

$$P_{\perp o}^{\text{eff}} = \frac{1}{1 + \alpha(Q_o + \mu_o R_o)} \left[ -\frac{\alpha Q_o''}{B_o^2} \left( Q'' + Q'_o A_o' - B_o^2 - \frac{1}{r} \right) + \mu_o \chi_{16o} 
+ P_{ro} \chi_{15o} + P_{\perp o}(\chi_{12o}) + \frac{\alpha R_o}{2B_o^2} \left( P_{ro}'' - P_{ro}'' R_o' - 5P_{ro} R_o' B_o'' \right) B_o \right.
+ \left. \mu_o' \frac{A_o'}{A_o} - 2P_{ro} R_o' R_o \right], \tag{24}$$

while the non-static field equations under this strategy have the following form

$$\ddot{\mu}^{\text{eff}} = \frac{1}{1 + \alpha(Q_o + \mu_o R_o)} \left[ \alpha \omega B_o^2 \left( g'' - 2b Q_o'' - Q'_o B_o' + 2b Q_o' B_o' - 4b Q_o' B_o' \right) \right. \tag{25}$$
\( + 2Q' \left( \frac{c}{r} \right)' - g' \frac{B_o'}{B_o} + 2g' \frac{1}{r} + \frac{d \mu'_{o}}{2} + P' \rho d' - \frac{3d}{2r} P'_{\perp o} + \frac{d P'_{\rho o}}{2} \) + \( \omega (\mu_o x_1 + \mu'_o x_3 + P_{\rho o} x_4 + P_{\perp o} x_5) + \mu_\chi_{10} + \dot{\mu}_\chi_{10} + \mu_\chi_{20} + \dot{\mu}_\chi_{20} + P_{\rho \chi_{40}} + \bar{P}_{\perp \chi_{50}} \) \\
+ \( \frac{\alpha R_o}{B_o^2} \left\{ \frac{\bar{\mu}'}{A_o^2} + \frac{\bar{\mu}''}{B_o^2} + \frac{\bar{P}'' \rho o}{2} \right\} - \frac{\alpha \omega b R_o}{B_o} \left\{ \frac{\mu''_{o}}{B_o} + \frac{2}{\rho} \right\} + \frac{P'_{\rho o} R_o}{R_o} - \frac{3}{R_o} \frac{P'_{\rho o}}{B_o} \) \\
+ \( \frac{3P'_{\perp o}}{2b} \left( \frac{c}{r} \right)' \) - \( \frac{\alpha \omega}{2} \left( \frac{5b^2 B_o' P'_{\rho o}}{2} + 5b R_o' P_{\rho o} \right) - \frac{\alpha R_o}{2} \left\{ \frac{5}{B_o} \bar{P}'' + \frac{3}{B_o^2} \right\} \\
+ 2 \frac{R_o' \bar{P}''}{B_o^2 R_o} - \alpha \mu'_{o} (R_o \bar{\mu} + \omega (g + d \mu_o) ) \right] ,

\( \bar{P} \equiv \frac{1}{1 + \alpha (Q_o + \mu_o R_o) + \alpha (Q_o - \mu_o R_o) \left\{ \frac{\alpha \omega}{B_o^2} - \frac{\alpha \omega}{B_o} \left\{ \frac{g''}{B_o} - \frac{3b Q_o' B_o'}{B_o} - \frac{b Q_o' B_o'}{B_o} - \frac{2}{B_o} \right\} + \omega (P_{\perp o} x_{12} + P_{\perp o} t_{14} + \mu_o x_{16} + P_{\perp o} \chi_{120} + \bar{P}_{\perp o} \chi_{120} + \bar{P}_{\perp o} \chi_{140} + \bar{P}_{\perp o} \chi_{150} + \bar{\mu} \chi_{160} - \frac{3b}{2} \frac{Q_o B_o'}{B_o} - \frac{b Q_o B_o'}{B_o} + Q_o' \left( \frac{c}{r} \right)' \right\} \left\{ \frac{\mu''_{o}}{A_o} - \frac{\mu''_{o}}{B_o^2 A_o} - \frac{P_{\perp o}}{2 B_o^2} \right\} - \alpha \mu_{\rho o} \right\} \times \left( R_o \bar{\mu} + \omega (g + d \mu_o) \right) ,

\( \bar{P} \equiv \frac{1}{1 + \alpha (Q_o + \mu_o R_o) \left\{ \frac{\alpha \omega}{B_o^2} - \frac{\alpha \omega}{B_o} \left\{ \frac{g''}{B_o} - \frac{3b Q_o' B_o'}{B_o} - \frac{b Q_o' B_o'}{B_o} - \frac{2}{B_o} \right\} + \omega (P_{\perp o} x_{12} + P_{\perp o} t_{14} + \mu_o x_{16} + P_{\perp o} \chi_{120} + \bar{P}_{\perp o} \chi_{120} + \bar{P}_{\perp o} \chi_{140} + \bar{P}_{\perp o} \chi_{150} + \bar{\mu} \chi_{160} - \frac{3b}{2} \frac{Q_o B_o'}{B_o} - \frac{b Q_o B_o'}{B_o} + Q_o' \left( \frac{c}{r} \right)' \right\} \left\{ \frac{\mu''_{o}}{A_o} - \frac{\mu''_{o}}{B_o^2 A_o} - \frac{P_{\perp o}}{2 B_o^2} \right\} - \alpha \mu_{\rho o} \right\} \times \left( R_o \bar{\mu} + \omega (g + d \mu_o) \right) ,

\( \frac{\omega}{B_o} \left\{ \frac{d}{2} (P_{\perp o} - P_{\rho o} + \mu_o A_o + 5 B_o' P_{\rho o}) - d' P_{\rho o} \right\} - \alpha \mu_{\rho o} (R_o \bar{\mu} + \omega (g + d \mu_o)) \right] .
In hydrostatic equilibrium position, the second conservation law has the static form
\[
\frac{1}{B_o} \left[ P_{r0}^{\text{eff}} + (\mu_o^{\text{eff}} + P_{r0}^{\text{eff}}) A_o' \frac{A_o}{A_o} - \frac{2}{r} \left( P_{\perp o}^{\text{eff}} - P_{r0}^{\text{eff}} \right) \right] = 0.
\]

The non-static perturbed configurations of Eqs. (15) and (16) are
\[
\frac{1}{A_o} \left[ \dot{\mu}^{\text{eff}} + \frac{b \omega}{B_o} (\mu_o^{\text{eff}} + P_{r0}^{\text{eff}}) + 2 \frac{c \omega}{r} (P_{\perp o}^{\text{eff}} + \mu_o^{\text{eff}}) \right] + \omega A_o h' + h \omega \left( \frac{2 A_o}{r} + 3 A_o' \right) = 0, \quad (26)
\]
\[
\frac{1}{B_o} \left[ \ddot{P}_{r} + A_o' (\dot{P}_{r}^{\text{eff}} + \mu^{\text{eff}}) + \left( \frac{a}{A_o} \right)' (P_{r0}^{\text{eff}} + \mu_o^{\text{eff}}) \omega - 2 \omega \left( \frac{c}{r} \right)' (P_{\perp o}^{\text{eff}} - P_{r0}^{\text{eff}}) - \frac{2}{r} (\ddot{P}_{\perp} - \ddot{P}_{r}^{\text{eff}}) \right] + h \dot{\omega} = 0. \quad (27)
\]

From Eq. (11), we can find the relation between \(b\) and \(g\) as follows
\[
\frac{\alpha \omega}{1 + \alpha (Q_o + \mu_o R_o)} \left[ g' - \frac{g A_o'}{A_o} - \frac{b Q_o'}{B_o} \right] = 0.
\]

In spherical geometry, the matter content described by Misner and Sharp, in static and non-static positions give
\[
m_o = m_o(r) = \left( 1 - \frac{1}{B_o^2} \right) \frac{r}{2}, \quad \bar{m} = \bar{m}(t, r) = \frac{r \omega}{B_o^2} \left\{ \left( \frac{b}{B_o} - c' \right) - \frac{c}{2r} (1 - B_o^2) \right\}. \quad (28)
\]

### 3.3 Stability Analysis

Dynamical stability is the characteristics of an object or system to retain its stable position, whenever it is subjected to perturbations. The dynamical stability has utmost relevance in structure formation and evolution of self-gravitating bodies. The instability/stability of celestial bodies has been discussed not only in the framework of general relativity but also in different modified gravity theories. In order to analyze the dynamics of massive objects, one can calculate its stability condition with \(N\) and \(pN\) approximations. It is interesting to analyze what happens when the phase of equilibrium of stellar structures is disturbed? Will this perturbation be relaxed (stable state) or will it grow (unstable state). In this respect, one needs to take into account the dynamical instability problem or thermal instability issue.

It is found that under hydrostatic equilibrium phase, the stability criterion can easily be achieved by making linearized field equations as well as conservation equations against
radial perturbation. We remark that the realistic object moves, during evolution, via several evolutionary phases determined by instability/stability degrees of freedom. This suggests that relativistic systems can be stable at one instant but not at the other. Thus, one needs to understand the dynamical behavior of self-gravitating systems by calculating instability regions at both N as well as pN regimes. Such epochs have vital role in the discussion of gravitational collapse of compact objects.

Now, we discuss the stability of local anisotropic spherical dense objects using the equations developed in the previous section. One can understand the notion of instability of relativistic interiors via adiabatic index ($\Gamma_1$). Equation of state suggested by Harrison et al. [52] provides a relationship between energy density and pressure of the source which measures a change in pressure corresponding to a given change in energy density. This is

$$\dot{P}_\text{eff} = \Gamma_1 \frac{P_{\text{ro}}}{\mu_0 + P_{\text{ro}}} \dot{\mu}.$$  \hfill (29)

We have adopted Harrison et al. equation of state [52] as it measures the stiffness of the fluid. As we are interested in examining the role played by matter variables on the stability of spherical system in the background of a particular modified gravity. Therefore, we have chosen such equation of state with the assumption that adiabatic index is constant throughout the matter distribution or, at least, within the observed region of spacetime. Also, we are exploring the stability conditions for non-static anisotropic spherical geometry. For this purpose, we take the equation of state in the scenario of second law of thermodynamics, where both $\dot{P}$ and $\dot{\mu}$ are functions of $t$ and $r$ obtained after first order perturbation. It is not possible to use this scheme to investigate the instability range with an equation of state which is not perturbed or does not contain non-static terms. Consequently, the other equation of state would lead to different kind of investigation for stability conditions which have also been discussed in literature [53]. Equation (29) can be re-casted as

$$\dot{\mu}_\text{eff} = - \left[ \frac{b}{B_\text{ro}} (P_{\text{eff} \text{ro}} + \mu_\text{eff} \text{ro}) + \frac{2c}{r} (\mu_\text{eff} + P_{\text{eff} \perp}) \right] \dot{\omega} - \omega J_1,$$  \hfill (30)

where

$$J_1 = h' - 3\alpha \frac{A'_o}{A_o} - \frac{2h}{r}.$$ 

Using value of $B_\text{ro}^2 = \frac{r}{r - 2m_{\text{ro}}}$, in hydrostatic part of 11 field equation, we have

$$\frac{A'_o}{A_o} = \frac{1}{(r - 2m_{\text{ro}}) (2\alpha \mu_\text{ro} R'_\text{ro} - \alpha Q'_\text{ro})} \left[ P_{\text{eff} \text{ro}} (r + \alpha r (Q_{\text{ro}} + \mu_\text{ro} R_{\text{ro}})) + 2\alpha Q'_\text{ro} \right.$$
\[-4\alpha Q_o^{m_o \over r} - r(p_{\rho\varphi}\chi_{\rho\varphi} + p_{\rho\varphi}\chi_{7\varphi} + p_{\perp\varphi}\chi_{9\varphi} + \mu_o\chi_{10\varphi}) - \alpha R_0 p_{\perp\varphi}^{m_o \over r} + 2\alpha R_0 p_{\perp\varphi}^{m_o \over r} \].

The static profile of the Ricci scalar is

\[ R_o = \left(1 - \frac{1}{B_o^2}\right) \frac{2}{r^2} - 2 \frac{A_o'}{A_o B_o^2} \left(\frac{2}{r} - B_o'\right)^2 - 2 \frac{B_o''}{B_o^2} \left(\frac{A_o''}{A_o^2} - \frac{2B_o'}{rB_o}\right). \]

Taking integration of Eq.(30) with \( t \), we have

\[ \bar{\mu}_{\text{eff}} = -J\omega, \quad (31) \]

where

\[ J = \left[ \frac{b}{B_o} \left(P_{\rho\varphi}^{\text{eff}} + \mu_o^{\text{eff}}\right) + \frac{2c}{r} \left(\mu_o^{\text{eff}} + P_{\perp\varphi}^{\text{eff}}\right) \right] \frac{J_1}{\xi}. \]

Putting the expression of \( \bar{\mu}_{\text{eff}} = -J\omega \) in Eq.(29), one can find

\[ P_{\rho\varphi}^{\text{eff}} = -\Gamma_1 \frac{P_{\rho\varphi}^{\text{eff}} J\omega}{(\mu_o^{\text{eff}} + P_{\rho\varphi}^{\text{eff}})}, \quad P_{\perp\varphi}^{\text{eff}} = -\Gamma_1 \frac{P_{\perp\varphi}^{\text{eff}} J\omega}{(\mu_o^{\text{eff}} + P_{\perp\varphi}^{\text{eff}})}. \quad (32) \]

By making use of Eqs.(31), (32) and (27), the corresponding hydrodynamical equation turns out to be

\[ \frac{2\omega}{B_o} \left(\frac{P_{\rho\varphi}^{\text{eff}}}{P_{\rho\varphi}^{\text{eff}}} - \frac{P_{\perp\varphi}^{\text{eff}}}{P_{\perp\varphi}^{\text{eff}}}\right) \left(\frac{c}{r}\right)' - \Gamma_1 \frac{J\omega}{B_o} \frac{P_{\rho\varphi}^{\text{eff}}}{(\mu_o^{\text{eff}} + P_{\rho\varphi}^{\text{eff}})} + \Gamma_1 \omega \frac{J}{B_o} \frac{\mu_o^{\text{eff}} P_{\rho\varphi}^{\text{eff}}}{(\mu_o^{\text{eff}} + P_{\rho\varphi}^{\text{eff}})^2} + \Gamma_1 \frac{J\omega}{B_o} \]

\[ \times \left(\frac{P_{\rho\varphi}^{\text{eff}}}{(\mu_o^{\text{eff}} + P_{\rho\varphi}^{\text{eff}})^2}\right)' - \Gamma_1 \frac{J}{B_o} \frac{P_{\rho\varphi}^{\text{eff}}}{(\mu_o^{\text{eff}} + P_{\rho\varphi}^{\text{eff}})} + \frac{2\omega J}{B_o} \frac{P_{\rho\varphi}^{\text{eff}}}{(\mu_o^{\text{eff}} + P_{\rho\varphi}^{\text{eff}})} - \frac{2\omega J}{B_o} \frac{P_{\perp\varphi}^{\text{eff}}}{(\mu_o^{\text{eff}} + P_{\perp\varphi}^{\text{eff}})} \quad (33) \]

\[ + \left(\xi\omega\right)^2 \left(\alpha g' - \alpha \left(g' \frac{A_o'}{A_o} + b Q_o' \frac{B_o'}{B_o}\right)\right) = 0. \]

The above equation is also known as modified version of collapse equation, in which the matter variables are related with stiffness parameter This equation yield the effects of counter gravity and pressure gradients in a single expression. The rest of the entries are the originator of the gravity forces. The effects, generated by \( f(R, T, Q) \) gravity terms and principal stresses mediated by perfect fluid are of having pivotal role in order to understand gravitational forces. We will analyze the collapse rate of the celestial model in spherical geometry, in that case the dynamical quantity, i.e., \( \Gamma_1 \) is positive only which would make the stable hydrostatic environment among gravitational forces and principal stresses.

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4 Newtonian Approximations

Now, we find the stability conditions for the spherical locally anisotropic interiors with N limit. For this purpose, we take flat background metric which leads to weak field approximations as follows

$$\mu_0 \gg P_0, \quad A_0 = 1, \quad B_0 = 1.$$ 

We have assumed that the contribution of energy density of matter distribution is much much greater than its pressure components. In order to achieve the stability regions of anisotropic spherical compact stellar system, we need to consider that each term in the the collapse equation to be positive then the expression of corresponding hydro-dynamical equation takes the form

$$\Gamma_1 F = a' \mu_o^{\text{eff}} - \xi_N^2(\alpha g' - \alpha b Q_o),$$  \hspace{1cm} (34)

where $Z_N = \frac{J_1}{\xi_N}$ with

$$F = \left[ \frac{2}{r} \left( b + \frac{2c}{r} \right) \left( P_{r o}^{\text{eff}} - P_{\perp o}^{\text{eff}} \right) - \left( b + \frac{2c}{r} \right) P_{r o}^{\text{eff}} - P_{r o}^{\text{eff}} Z_N \right],$$

$$\xi_N = \left[ \frac{4}{r^3 (br + 2c)} \left( c' + a'r - rb' + c''r + \frac{a''r^2}{2} - b \right) \right]^\frac{1}{2},$$

and $J_1$ consists the dark sources terms mediated from $\beta R(1 + \alpha Q)$ model. Equations (34) provides the following values of adiabatic index

$$\Gamma_1 = \frac{a' \mu_o^{\text{eff}} - \alpha \omega \xi_N (g' - b Q_o')}{F},$$  \hspace{1cm} (35)

This provides the hydrostatic condition which implies that the system enters in the stable window for

$$\Gamma_1 > \frac{a' \mu_o^{\text{eff}} - \alpha \omega \xi_N (g' - b Q_o')}{F}. $$ \hspace{1cm} (36)

In order to keep $\Gamma_1 > 0$, we need to consider $|a' \mu_o^{\text{eff}} - \alpha \omega \xi_N (g' - b Q_o')|$ and $|F|$. Thus the system remains in the stable phase as long as it obeys inequality (36). This represents that the instability range (36), that has been calculated through adiabatic index, depends upon the pressure components as well as anti-gravitational force coupled with adiabatic index and gravitational force. These variable quantities eventually depends upon the radial profiles of the energy density, anisotropicity and $f(R, T, Q)$ curvature terms. It is pertinent to note that the presence of anisotropicity in the fluid pressure has greatly influenced the instability
regimes of the spherical relativistic structure at N epoch as described by inequality (36). By
keeping the absolute values of denominator, we noticed that effective pressure anisotropicity
tend to decreases the stability regions or tends to remove hindrances for the system to move
in the collapsing phase. This result is well-consistent with [18]. One can easily notice that
$f(R, T, Q)$ terms appearing in the expression (36) tends to decrease the stability range as
these some of these terms are appearing in the numerator with negative sign. It has been seen
that this expression contains effective forms of matter variables that shows that terms coming
from the coupling of matter and geometry have greatly modify the instability constraints
due to their non-attractive nature. We now briefly describe our results as follows:

1. If the gravitational forces $|a'\mu_o^{\text{eff}}-\alpha\omega\xi_N(g'-bQ'_o)|$ are balanced by the anti-gravitational
and effective pressure forces $|F|$, (thereby boiling down inequality (35) to $\Gamma_1 = 1$) then
the system will rest in the window of hydrostatic equilibrium.

2. If the modified gravity forces produced by $|a'\mu_o^{\text{eff}}-\alpha\omega\xi_N(g'-bQ'_o)|$ are greater than
that of $|F|$, then the system will enter in the stable phase instead of collapsing, i.e,
counter gravitational forces as well as effective principal pressures give the stability
constraint $\Gamma_1 > 1$.

3. The celestial system will be in unstable state whenever it achieve contribution from
$|\mu_o^{\text{eff}}-\alpha\omega\xi_N(g'-bQ'_o)|$ to be lesser than from $|F|$. This assigns the range of $\Gamma_1$ belonging
to the open interval $(0, 1)$.

The GR limit, $f(R, T, Q) = R$, converts all the effective fluid variables appearing in inequality
(36) to usual matter variables, i.e., $P^{\text{eff}}_r \rightarrow P_r$, $P^{\text{eff}}_\perp \rightarrow P_\perp$, and $\mu^{\text{eff}}_o \rightarrow \mu_o$. Furthermore,
under this limit, the quantity $Z_N$ vanishes, thereby recovery the whole dynamics in the
framework of GR. Thus, the stability constraint becomes

$$\Gamma_1 > \frac{|a'\mu_o|}{\frac{2}{3}(P_r - P_\perp)(b + \frac{3}{2}) - (b + \frac{5}{2}) P'_r}. \quad (37)$$

This expression exactly match with that obtained in [57], under certain conditions. However,
for isotropic spherical system, the stability constraint with N approximations boils down to

$$\Gamma_1 > \frac{|a'\mu_o|}{|(b + \frac{5}{2}) P'_r|}. \quad (38)$$

5 Post-Newtonian Approximation

Different aspects of various gravitational framework can upraise particular problems in ap-
lications of practical interest. These contain the nonlinear nature of equations of motion
and the non-existence of a background geometry that can be utilized to discuss physically interesting quantities, for example, energy and momentum. In this direction, some approximations techniques are applied to construct physically concerning predictions. The example of such approximation scheme is a linearized gravity where the non linear parts of spacetime metric are ignored, that eventually give rise to some useful approximate outcomes. As a result of this scheme, linearized field equations describing weak gravitational field can easily be governed. Thus, N and pN limits are considered as the approximations for the weak field of relativistic gravitational theory, in which the corresponding equations of motion and metric variables are approximated in the inverse power of the light speed.

In the realm of gravitational theories, both N and pN approximations narrate the order of small perturbations/deviations of any local system from its isotropic flat and homogeneous environment. One can evaluate these approximations by expanding metric functions through Taylor series as

\[ g_{\gamma\delta} \approx \eta_{\gamma\delta} + h_{\gamma\delta}, \quad |h_{\gamma\delta}| \ll 1 \]

with

\[ h_{00} \approx h_{00}^{(II)} + h_{00}^{(IV)} + ... , \quad h_{0i} \approx h_{0i}^{(III)} + h_{0i}^{(V)} + ... , \quad h_{ij} \approx h_{ij}^{(II)} + h_{ij}^{(IV)} + ... , \quad i, j = 1, 2, 3 \]

where the superscripts (II), (III), (IV) describe approximation orders up to \((\frac{1}{c^2})\), \((\frac{1}{c^3})\) and \((\frac{1}{c^4})\), while \(\eta_{\mu\nu}\) stands for the Minkowski spacetime that represents isotropic and homogeneous flat environment of \(g_{\gamma\delta}\) and \(h_{\mu\nu}\) indicates perturbation of metric tensor \(g_{\mu\nu}\) from \(\eta_{\mu\nu}\) (background values). The approximations \(g_{\gamma\delta} \sim \eta_{\gamma\delta} + h_{\gamma\delta}^{(II)}, g_{ij} \sim \eta_{ij} + h_{ij}^{(II)}\) yields N limit, while the pN limits need the information of \(g_{\gamma\delta} \sim \eta_{\gamma\delta} + h_{\gamma\delta}^{(II)} + h_{\gamma\delta}^{(IV)}, g_{0i} \sim h_{0i}^{(III)}, g_{0j} \sim h_{0j}^{(II)}, g_{ij} \sim \eta_{ij} + h_{ij}^{(II)}\). This describes the peculiar connection between the calculations of N and pN limits for any toy relativistic model.

To achieve pN instability limits, we take \(A_0(r) = 1 - \phi, B_0(r) = 1 + \phi\), with linear \(O(\phi)\) and \(\phi(r) = \frac{m_o}{r}\). In this aspect, the value of stiffness parameter through the collapse equation can be given as

\[ \Gamma_1 = \frac{E_{pN}}{\psi J'_{pN} - kJ_{pN}}, \quad (39) \]

where

\[
E_{pN} = P_{ro}^{\text{eff}} \left( \gamma - 2 \left(1 - \frac{2m_o}{r} \right) \right) + (\mu_o^{\text{eff}}) \left( \gamma - 2 \left(1 - \frac{m_o}{r} \right) \right) + \left[(\frac{m_o}{r})'(-\frac{2m_o}{r})\right] Z_{pN} \\
- \frac{2c}{r} \left(1 - \frac{m_o}{r} \right)' - 2 \left(1 - \frac{2m_o}{r} \right)' - \alpha \omega \xi_{pN}^2 \left( g' + g \left(1 + \frac{m_o}{r} \right) - bQ' \left(1 - \frac{m_o}{r} \right) \right)
\]
The locally anisotropic spherical compact structure will move in the window of stable regime, if modified gravitational forces mediated by $|\psi J'_{pN} - k J_{pN}|$ are lesser than produced by $|E_{pN}|$. Then, the stability of the spherical fluids at pN epoch can be checked through

$$\Gamma_1 > \frac{E_{pN}}{\psi J'_{pN} - k J_{pN}}.$$  

(40)

If the relativistic interior is able to accomplish the state satisfying $|E| = |\psi J'_{pN} - k J_{pN}|$, then the system reverts itself in its initial hydrostatic equilibrium state. This situation can be dealt with the help of Eq.(39). The system will enter into dynamical instability window, if the influence of $|E|$ is less than $|\psi J'_{pN} - k J_{pN}|$. This gives rise to

$$\Gamma_1 < \frac{E_{pN}}{\psi J'_{pN} - k J_{pN}}.$$  

For the case of isotropic spherically symmetric system, the instability regime with pN approximations turns out to be

$$\Gamma_1 < \frac{E_{pN}}{\psi J'_{pN} - k J_{pN}},$$

where

$$J_{pN} = \left\{ b \left( 1 + \frac{2 m_o}{r} \right) + \frac{2 c}{r} \right\} (P^o_{\text{eff}} + \mu^o_{\text{eff}}) + \frac{J_1}{\xi}, \quad \tilde{\psi} = \frac{r - 2 m_o}{r(\mu_o + P_o)} P_o$$

$$E_{pN} = \mu^o_{\text{eff}} \left[ \gamma - \frac{2 c}{r} \left( \frac{m_o}{r} \right)' \left( 1 - \frac{m_o}{r} \right) \right] + P^o_{\text{eff}} \left[ \left( \frac{m_o}{r} \right)' \left( - \frac{2 m_o}{r} \right) Z_{pN} - 4 \left( \frac{c}{r} \right)' \left( 1 - \frac{2 m_o}{r} \right) r \right]$$

$$- \frac{2 c}{r} \left( \frac{m_o}{r} \right)' \left( 1 - \frac{m_o}{r} \right) + \gamma - \alpha \omega \xi_{pN}^2 \left\{ g' + g \left( \frac{m_o}{r} \right)' \left( 1 + \frac{m_o}{r} \right) - b Q_o \left( 1 - \frac{m_o}{r} \right) \right\}. \]$$

Under GR limit, i.e., when $f(R, T, Q) = R$, the instability limit provided by the collapse equation through $\Gamma_1$ at pN limits boils down to

$$\Gamma_1 < \frac{E_{GR}}{\psi J'_{pN} - k J_{pN}}$$  

(41)

where

$$E_{GR} = P_{ro} \left( \gamma - 2 \left( \frac{c}{r} \right)' \left( 1 - \frac{2 m_o}{r} \right) \right) + \mu_o \left\{ \gamma - \frac{2 c}{r} \left( \frac{m_o}{r} \right)' \left( 1 - \frac{m_o}{r} \right) \right\}' \]$$

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\[ + P_{\perp o} \left[ - \frac{2c}{r} \left( \frac{m_o}{r} \right) \left( 1 - \frac{m_o}{r} \right) - 2\left( \frac{c}{r} \right) \left( 1 - \frac{2m_o}{r} \right) \right], \]

\[ J_{pN} = \frac{b}{1 + \frac{m_o}{r}} \left[ P_{ro} + \mu_o + \frac{2c}{r} \left( \mu_o + P_{ro} \right) \right], \]

\[ \gamma = -b \left( \frac{m_o}{r} \right)' \left( 1 - \frac{2m_o}{r} \right) - a' \left( 1 + \frac{m_o}{r} \right) - a \left( \frac{m_o}{r} \right)', \]

\[ \psi = \frac{r - 2m_o}{r(\mu_o + P_{ro})} P_{ro}, \quad Z_{pN} = 0. \]

This reveals the significance of static profiles of relative matter variables and stiffness parameter. This constraint exactly match with that already obtained in [57] under certain conditions.

### 6 Conclusions

The stability problem of dense objects in the field of modified gravitational has come into sight as a main concern. In this setting, the instability eras for the locally anisotropic self-gravitating spherical configurations are examined with a particular formulation \( f(R, T, Q) \) gravity. We have calculated the corresponding equations of motion for the locally anisotropic matter filled in spherical irrotational symmetry. The conservation laws are explored after using the contracted formulations of Bianchi identities with the background of effective energy momentum tensor. The radial perturbation scheme is applied on main equations and then static as well as non static profiles of field and dynamical expressions are presented.

We first assume our relativistic sphere rests in the window of hydrostatic phase at the initial times. But, as time passes, the evolving system starts to enter in the window of perturbation background. The resulting equations, after implication of perturbation strategy, are then used to construct \( f(R, T, Q) \) collapse equation. Then, we have used well-known Harrison-Wheeler state equation that has related peculiarly the profiles of energy density and pressure components via stiffness of fluid content. After considering a viable configurations of \( f(R, T, Q) \) model, we have examined its impact in the definitions of modified hydrodynamical equation. The corresponding constraints at both \( N \) and \( pN \) are evaluated. We observed that extra degrees of freedom induced from \( f(R, T, Q) \) gravity try to produce obstacles in the evolutionary phases of anisotropic compact star, thereby pushes the system to enter in unstable window.

Chandrasekhar [16] calculated a specific value of the adiabatic index, i.e., 4/3, for the stability regime of the locally isotropic spherical relativistic objects. After this, many astrophysicists investigated the these regions by taking various choices of matter as well as
geometric configurations. We have pointed out a key role of stiffness parameter in the main-
tenance of stable backgrounds. We explore that the adiabatic index have the influence of
extra curvature ingredients due to matter curvature coupling in static background. It is
found the self-gravitating celestial object remains in stable state until it satisfies (36) and
(41) for N and PN regimes, respectively. Once, the system fail to comply with the prescribed
ranges, it will enter into the unstable regime. We conclude that the extra curvature terms
due to $f(R, T, Q)$ theory makes the system more stable with the evolution of time, thereby
slowing down the collapse rate. It is noted that with the zero existence of non-minimal
coupling of matter and geometry, these outcomes supports the results obtained in $f(R, T)$
findings [54].

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The quantities $\chi_i$'s of field equations are

\[
\chi_1 = 1 + f_T - \frac{3R}{2} f_Q - \frac{\dot{A} f_Q}{2A^3} + \frac{4A^2}{A^4} f_Q - \frac{7\dot{A}}{2A^3} f_Q + \frac{2A' f'_Q}{AB^2} + \frac{A'' f_Q}{AB^2} + \frac{A'''}{AB^2} - \frac{\dot{B} f_Q}{2A^2B} - \frac{\dot{A} \dot{B}}{A^3B} f_Q - \frac{B' f'_Q}{2B^3} + \frac{A' B'}{AB^3} f_Q - \frac{\dot{C} f_Q}{A^2C} - \frac{2\dot{A} C}{A^3C} f_Q + \frac{C' f'_Q}{C B^2} + \frac{2A' C'}{AB^2 C} f_Q,
\]

\[
\chi_2 = -\frac{11}{2A^3} f_Q \dot{A} - \frac{f_Q \dot{B}}{2A^2 B} - \frac{f_Q \dot{C}}{A^2 C}, \quad \chi_3 = \frac{f_Q}{C B^2} C' + \frac{f'_Q}{2B^2} - \frac{2A' f_Q}{AB^2} - \frac{B' f_Q}{2B^3},
\]

\[
\chi_4 = \frac{f''_Q}{2B^2} + \frac{4B^2 f_Q}{B^4} - \frac{B''}{B^3} f_Q - \frac{5B' f'_Q}{2B^3} - \frac{\dot{B} f_Q}{A^2 B} + \frac{\dot{B}^2 f_Q}{A^2 B^2},
\]

\[
\chi_5 = \frac{3C^2 f_Q}{A^2 C^2} - \frac{3C f'_Q}{2B^2 C} - \frac{3C' f'_Q}{2B^2 C} + \frac{3C'^2 f_Q}{B^2 C^4},
\]

\[
\chi_6 = 1 + f_T - \frac{3}{2} R f_Q - \frac{3B'^2 f_Q}{B^3} + \frac{2B''}{B^3} f_Q + \frac{B'}{B^3} f'_Q - \frac{5\dot{B} f_Q}{2A^2 B} - \frac{3B^2 f_Q}{2A^2 B^2} - \frac{R f_Q}{2A^2 B} - \frac{\dot{A} f_Q}{2A^3} + \frac{\dot{A} \dot{B} f_Q}{A^3 B} f_Q + \frac{A' f'_Q}{2AB^2},
\]

\[
\chi_7 = \frac{4B'}{B^3} f_Q - \frac{A'}{2 AB^2} f_Q, \quad \chi_8 = \frac{B}{2 A^2 B} f_Q + \frac{\dot{A} f_Q}{2A^3} - \frac{5\dot{B} f_Q}{2A^2 B},
\]

\[
\chi_9 = \frac{C f'_R}{A^2 C} - \frac{2C}{A^2 C^2} f_Q, \quad \chi_{11} = \frac{5\dot{A} f_Q}{2A^3} - \frac{f_Q}{A^2},
\]
\[ \chi_{10} = \frac{5\dot{A}f_Q}{2A^3} - \frac{4\dot{A}^2}{A^4}f_Q + \frac{\ddot{A}}{A^3}f_Q - \frac{\ddot{f}_Q}{2A^2} + \frac{A'f'_Q}{2AB^2} - \frac{A''f_Q}{A^2B^2}, \]

\[ \chi_{12} = 1 + f_T - \frac{3}{2}Rf_Q - \frac{5\dddot{C}^2}{A^2C^2}f_Q + \frac{2C'}{CB^2}f_Q - \frac{\dddot{C}f_Q}{A^2C} - \frac{\dddot{f}_Q}{2A^2} + \frac{\dddot{A}f_Q}{2A^3} + \frac{\dot{A}C}{CA^3}f_Q + \frac{A'f'_Q}{2AB^2} + \frac{A'C'}{ACB^2}f_Q + \frac{2C''f'_Q}{B^2C} + \frac{C''f_Q}{B^2C^2} + \frac{C'''}{B^2C^3}f_Q - \frac{\dot{B}f_Q}{2A^2B} \]

\[ + \frac{\dddot{C}B}{CA^3}f_Q - \frac{\dddot{B}'f'_Q}{2B^3} - \frac{\dddot{C}'B'}{CB^3}f_Q, \]

\[ \chi_{13} = \frac{\dot{C}A}{2A^3C^2}f_Q - \frac{2\dddot{C}}{A^2C}f_Q - \frac{\dddot{f}_Q}{2A^2} - \frac{\dddot{B}f_Q}{2A^2B}; \]

\[ \chi_{14} = \frac{2C'}{B^2C}f_Q + \frac{A'f_Q}{2AB^2} + \frac{2C'f_Q}{B^2C} + \frac{\dddot{f}_Q}{2B^3} - \frac{B'f_Q}{2B^3}; \]

\[ \chi_{15} = \frac{5B'f'_Q}{2B^3} - \frac{4B''}{B^4}f_Q + \frac{B''}{B^3}f_Q - \frac{\dddot{f}_Q}{2B^2} + \frac{\dddot{B}f_Q}{2A^2B} - \frac{\dddot{C}f_Q}{A^2B^2}; \]

\[ \chi_{16} = \frac{5\dot{A}}{2A^3}f_Q - \frac{4\dot{A}^2}{A^4}f_Q + \frac{\dddot{A}}{A^3}f_Q - \frac{\dddot{f}_Q}{2A^2} - \frac{A''f_Q}{A^2B^2} + \frac{A'f'_Q}{2AB^2}. \]