String Modular Motives of Mirrors of Rigid Calabi-Yau Varieties

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Abstract. The modular properties of some higher dimensional varieties of special Fano type are analyzed by computing the L-function of their Ω−motives. It is shown that the emerging modular forms are string theoretic in origin, derived from the characters of the underlying rational conformal field theory. The definition of the class of Fano varieties of special type is motivated by the goal to find candidates for a geometric realization of the mirrors of rigid Calabi-Yau varieties. We consider explicitly the cubic sevenfold and the quartic fivefold, and show that their motivic L-functions agree with the L-functions of their rigid mirror Calabi-Yau varieties. We also show that the cubic fourfold is string theoretic, with a modular form that is determined by that of an exactly solvable K3 surface.

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1. Introduction

In this paper we continue the program of applying methods from arithmetic geometry to the problem of understanding the question how spacetime

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emerges in string theory, more precisely, the relation between the physics on the worldsheet and the nontrivial geometric component of spacetime. The focus here will be on a class of varieties that have been discussed in the context of mirror symmetry for rigid Calabi-Yau spaces [26, 6, 27]. These manifolds are higher dimensional spaces, characterized by the existence of a nontrivial cohomology group $H^{n-(Q-1)(Q-1)}(X)$, where $n = \dim \mathbb{C} X$ and $Q \geq 1$ is an integer. In particular $H^{n-i,i}(X) = 0$ for $0 \leq i < Q - 1$ if $Q > 1$, and for $Q = 1$ this class of varieties reduces to manifolds Calabi-Yau type. For $Q > 1$ spaces of this type were called Fano manifolds of special type in [26, 27], and generalized Calabi-Yau manifolds in [6]. The latter name has in the meantime generally been adopted for Hitchin’s generalization of Calabi-Yau varieties. We will call these spaces special Fano manifolds of charge $Q$. In the context of exactly solvable models manifolds of this type are distinguished by the fact that the number of tensor factors of $N = 2$ supersymmetric minimal models exceeds the number of variables normally associated with the standard relation between the central charge and the dimension of the variety.

Our goal is to show that, in complete analogy to the case of Calabi-Yau varieties considered in [30, 31] and references therein, the arithmetic geometry of Fano varieties of special type encodes features of the underlying conformal field theories. In particular the L-function of the $\Omega$—motive, considered in [30, 32] in the context of Calabi-Yau varieties, generalizes to the class of Fano varieties of special type. We will show that for those members in this class that are expected to be mirrors of rigid Calabi-Yau manifolds the inverse Mellin transform of the L-function of the $\Omega$—motive with a Tate twist of charge $(Q - 1)$ is a modular form of weight four which decomposes into factors derived from the underlying conformal field theory.

In the context of finding mirrors of rigid Calabi-Yau manifolds two varieties are of particular interest. The first is the cubic seven fold

$$X_7^3 = \left\{ (z_0 : \cdots : z_8) \in \mathbb{P}_8 \mid \sum_{i=0}^{8} z_i^3 = 0 \right\}$$

whose Hodge decomposition of the intermediate cohomology is given by

$$h^{7,0}(X_7^3) = h^{6,1}(X_7^3) = 0$$
$$h^{5,2}(X_7^3) = 1$$
$$h^{4,3}(X_7^3) = 84$$

(1.2)

The second variety is the quartic fivefold

$$X_5^4 = \left\{ (z_0 : \cdots : z_6) \in \mathbb{P}_{(1,1,1,1,1,1,2)} \mid \sum_{i=0}^{5} z_i^4 + z_6^2 = 0 \right\},$$

(1.3)
with a Hodge decomposition of the intermediate cohomology that is given by

\[ h^{5,0}(X_5^4) = 0, \]
\[ h^{4,1}(X_5^4) = 1, \]
\[ h^{3,2}(X_5^4) = 90. \]

(1.4)

The motivation for considering these hypersurfaces is that their cohomology suggests that they are related to conformal field theories at central charge \( c = 9 \). The geometric cohomology projection of \([26, 27]\) and the period computation of \([6]\) furthermore suggest that they are of relevance in the context of rigid mirrors of Calabi-Yau threefolds.

We first show that the Tate twisted \( \Omega - \)motives of these varieties are modular in terms of Hecke eigenforms of weight four which are induced by string theoretic modular forms of elliptic type. The latter factor into Hecke indefinite modular forms of weight one associated to the affine Lie algebra \( A^{(1)}_1 \). Let \( X^d_n \) be a hypersurface of degree \( d \) and dimension \( n \), and denote by \( L^{Q-1}_{\Omega}(X^d_n, s) \) the L-function of the \( \Omega - \)motive \( M_{\Omega}(X^d_n) \) of the variety \( X^d_n \) with the necessary Tate twist of charge \((Q - 1)\). Denote by \( \Theta_{k,m}^\ell(\tau) = \eta^3(\tau)c_{k,m}^\ell(\tau) \) the theta functions defined in terms of the Dedekind eta function \( \eta(\tau) \) and the Kac-Peterson string functions \( c_{k,m}^\ell(\tau) \). The Hecke indefinite modular forms \( \Theta_{k,m}^\ell(\tau) \) are associated to the affine Lie algebra \( A^{(1)}_1 \) of the underlying rational conformal field theory. Finally, we denote by \( \chi_2 \) denote the Legendre character, and set \( q = e^{2\pi i \tau} \).

**Theorem 1.** Let \((n, d) \in \{(7, 3), (5, 4)\} \). The inverse Mellin transforms \( f^{Q-1}_{\Omega}(X^d_n, q) \) of the L-functions \( L^{Q-1}_{\Omega}(X^d_n, s) \) of the \( \Omega - \)motive with a Tate twist of charge \((Q - 1)\) are modular forms in \( S_4(\Gamma_0(N)) \) with \( N \in \{9, 64\} \), described by the algebraic Hecke characters \( \psi_{d,3} \)

\[ f^{Q-1}_{\Omega}(X^d_n, q) = f(\psi_{d,3}^\ell, q). \]

(1.5)

The modular forms \( f(\psi_{d,3}, q) \in S_2(\Gamma_0(d^2)) \) are of elliptic type, and factor into string theoretic Hecke indefinite modular forms as

\[ f(\psi_{27}, q) = \Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9) \]
\[ f(\psi_{64}, q) = \Theta_{1,1}^2(q^4)^2 \otimes \chi_2. \]

(1.6)

This result provides an interpretation of the L-function of the twisted \( \Omega - \)motive of these varieties in terms of modular forms on the string worldsheet theory of exactly solvable Gepner models at central charge \( c = 9 \), with nine and six minimal \( N = 2 \) supersymmetric factors, respectively. The twist character \( \chi_2 \) is the quadratic character of the field of quantum dimensions of the underlying affine Lie algebra \( A^{(1)}_1 \) \([28, 23, 29]\).

The characters \( \psi_{27} \) and \( \psi_{64} \) are the complex multiplication Hecke characters associated to the elliptic Brieskorn-Pham curves \( E^3 \subset \mathbb{P}_2 \) and \( E^4 \subset \mathbb{P}_{(1,1,2)} \) of conductor 27 and 64, respectively. These tori are exactly solvable.
and their string theoretic modular forms have been determined in [33] and [23], respectively, to factor as
\[
\begin{align*}
  f(E^3, q) &= \Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9) \\
  f(E^4, q) &= \Theta_{1,1}^2(q^4)^2 \otimes \chi_2.
\end{align*}
\] (1.7)

Their appearance in the context of the two Fano varieties \(X^3_7\) and \(X^4_5\) suggests to consider resolutions \(X^d_3\) of quotients of triple products of elliptic curves \((E^d)^3\) for \(d = 3, 4\), and to compare their L-functions with those of the special Fano varieties determined in Theorem 1. Define \(Z_d := \mathbb{Z}/d\mathbb{Z}\). It turns out that there are actions of \(Z_d \times Z_d\) on \((E^d)^3\) such that the resolved manifolds \(X^d_3\) of the quotient threefolds \((E^d)^3/\mathbb{Z}_d \times \mathbb{Z}_d\) have the appropriate mirror cohomology of the special Fano varieties \(X^d_n\) for \(d = 3, 4\)
\[
\begin{align*}
  h^{1,1}(X^3_3) &= 84 \\
  h^{1,1}(X^4_3) &= 90.
\end{align*}
\] (1.8)

Combining Theorem 1 with results by Cynk and Hulek [7] leads to the following corollary.

**Theorem 2.** The varieties \(X^d_3\), \(d = 3, 4\), are rigid Calabi-Yau manifolds whose L-functions of the intermediate cohomology are modular and agree with the L-functions of the special Fano varieties \(X^d_n\) with \((d, n) \in \{(3, 7), (4, 5)\}\), respectively.

We interpret the string theoretic modularity of the special Fano varieties \(X^3_7\) and \(X^4_5\), together with the agreement of their motivic L-functions with those of their corresponding rigid Calabi-Yau spaces, as further support for the interpretation of certain higher dimensional Fano varieties of special type as mirrors of rigid Calabi-Yau manifolds. It would be of interest to extend our arithmetic analysis of the mirror pairs \((X^d_n, X^d_3)\) to families by using the methods developed in [3, 4], and further considered in [19, 20, 21, 5].

The modular analysis described here generalizes to other dimensions and central charges. To illustrate this we consider a third Fano variety of special type, defined by the cubic fourfold
\[
(1.9) \quad X^3_4 = \left\{ (z_0 : \cdots : z_5) \in \mathbb{P}_5 \mid \sum_{i=0}^5 z_i^3 = 0 \right\},
\]
whose Hodge decomposition of the intermediate cohomology is given by
\[
\begin{align*}
  h^{4,0}(X^3_4) &= 0 \\
  h^{3,1}(X^3_4) &= 1 \\
  h^{2,2}(X^3_4) &= 21.
\end{align*}
\] (1.10)

For this variety we obtain the following string theoretic interpretation. Let \(\vartheta(q)\) denote a modular form of weight one to be defined later in the paper.

**Theorem 3.** The inverse Mellin transform \(f^\Omega_{1}(X^3_4, q)\) of the L-series of the Tate twisted \(\Omega\)-motive of the cubic fourfold \(X^3_4\) is a modular form of weight
three and level 27, which factors into Hecke indefinite modular forms as
\[(1.11) \quad f_\Omega^1(X_3^4, q) = \vartheta(q^3)\Theta_1^{1,1}(q^3)\Theta_1^{1,1}(q^3) \in S_3(\Gamma_0(27)).\]
This form is identical to the modular form of the $\Omega$-motive of the K3 surface
\[(1.12) \quad S^6 = \{(z_0 : \cdots : z_3) \in \mathbb{P}_{(1,1,1,3)} \mid z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0\},
\]
i.e. $f_\Omega^1(X_3^3, q) = f_\Omega(S_6, q)$. It has complex multiplication and is determined by
the Hecke character $\psi_{27}$ of weight 2 associated to the Eisenstein field $\mathbb{Q}(\mu_3)$
\[(1.13) \quad f_\Omega^1(X_3^3, q) = f(\psi_{27}^2, q).\]
The modularity of the cubic fourfold has independently been considered by
Goto [9] and also by Hulek and Kloosterman [12]. The string theoretic in-
terpretation based on the affine algebra $A^{(1)}_{1}$ follows from the results of [30].

2. Fano varieties of special type

In this section we briefly review the characterization of Fano varieties of
special type introduced in [26, 6, 27], adopting the notation of [26, 27]. Let
$Q, D_{\text{crit}} \in \mathbb{N}$ and set $s = D_{\text{crit}} + 2Q - 1$. We consider the class of hypersurfaces
in weighted projective space $\mathbb{P}_{(k_0, \ldots, k_s)}$ with weights $k_i$ chosen such that
\[(2.1) \quad d = \frac{1}{Q} \sum_{i=0}^{s} k_i\]
is an integer, defining the degree of the polynomial that determines the variety
\[(2.2) \quad X^d_{s-1} = \{(z_0 : \cdots : z_s) \in \mathbb{P}_{(k_0, \ldots, k_s)} \mid p(z_0, \ldots, z_s) = 0, \deg p = d\}.
\]
The integer $Q$ has the physical interpretation of a charge of the underlying
physical theory, and $D_{\text{crit}}$ can be viewed as the critical complex dimension
of the corresponding Calabi-Yau manifold in those cases where it can be
constructed. It is determined by the central charge $c$ of the conformal field
theory as $c = 3D_{\text{crit}}$.

Alternatively, varieties $X$ of this type can be characterized by their first
Chern class
\[(2.3) \quad c_1(X) = (Q - 1)c_1(\mathcal{N}),\]
where $c_1(\mathcal{N})$ is the first Chern class of the normal bundle $\mathcal{N}$ of $X \subset \mathbb{P}_{(k_0, \ldots, k_s)}$. For $Q = 1$ this construction therefore recovers the class of Calabi-Yau hypersurfaces for arbitrary dimensions, while for $Q > 1$ these hypersurfaces have positive first Chern class and hence do not a priori describe consistent string vacua, but instead define a sub-class of Fano varieties. It was shown in
[26, 27] that these varieties nevertheless are closely related to string vacua
in the critical dimension $D_{\text{crit}} = s + 1 - 2Q$ in the sense that it is possible
to derive the massless spectrum of critical vacua from the cohomology of the
special Fano $(s-1)$-folds via a geometric projection. For particular classes of
these special Fano varieties it is also possible to construct the critical Calabi-
Yau manifold explicitly via this projection, in which case the charge \( Q \) has
the meaning of the codimension of the critical manifold. More intrinsically,
we can therefore write

\[
D_{\text{crit}}(X) = \dim \mathbb{C}X + 2(1 - Q).
\]

The essential ingredient of our arithmetic analysis below is the existence
of a nontrivial cohomology group \( H^{n-(Q-1)}(X) \). We therefore use this
feature as the characterizing property.

**Definition.** A Fano variety is called special and of charge \( Q \), if it has a
nonvanishing cohomology group \( H^{n-(Q-1)}(X) \).

Interest in this class of manifolds arose originally because they provide
a geometric framework for \( \mathcal{N} = 2 \) exactly solvable Gepner models which do
not have Kähler deformations, hence they provide candidates for the mirrors
of rigid Calabi-Yau varieties \([26, 6, 27]\). A toric description of these special
Fano spaces was subsequently given by Batyrev and Borisov \([1]\), and Batyrev
and Dais \([2]\). A discussion of Lagrangian subvarieties in the context of the
Strominger-Yau-Zaslow mirror conjecture \([36]\) was given by Leung \([22]\).

### 3. String theoretic modular forms

**3.1** The simplest class of \( \mathcal{N} = 2 \) supersymmetric exactly solvable theories
are built in terms of the affine Lie algebras. A construction of such non-
twisted affine Kac-Moody algebras, is provided by the extension \([16]\)

\[
\hat{G} = LG \oplus \mathbb{C}k \oplus \mathbb{C}d
\]
of the loop algebra

\[
LG = G \otimes \mathbb{C}[t, t^{-1}]
\]

by the central extension \( k \) and \( d = t \frac{d}{dt} \). In terms of the generators \( J^a \otimes t^m \)
the algebra takes the form

\[
[J^a \otimes t^m, J^b \otimes t^n] = if^{ab}_{\ c}J^c \otimes t^{m+n} + km\delta^{ab}\delta_{m+n,0}.
\]

The representations of this algebra can be parametrized by affine roots \( \hat{\lambda} = (\lambda, k, n) \) of the Cartan-Weyl subalgebra \( \{H_0, E_0, L_0\} \), with \( i = 1, ..., r \), where
\( r \) denotes the rank of the underlying Lie algebra \( G \). For fixed affine level \( k \) of
the theory the characters \( \chi_{\hat{\lambda}} \) are essentially parametrized by the weight \( \lambda \) of
the representation. For the reduced characters of an affine Lie algebra \( \hat{G} \) at
level \( k \) the characters transform as

\[
\chi_{\hat{\lambda}}(-1/\tau) = \sum_{\hat{\mu} \in P_+^k} S_{\hat{\lambda},\hat{\mu}} \chi_{\hat{\mu}}(\tau),
\]

where the modular \( S \)-matrix takes the form

\[
S_{\hat{\lambda},\hat{\mu}} = \frac{i^{\frac{1}{2}(|\Delta_+|}{\sqrt{|P/Q\|((k + g)')} \sum_{w \in W} \epsilon(w)e^{-2\pi i \frac{<\omega(\hat{\lambda} + \hat{\mu}), \mu + \rho>}{k + g}}.}
\]
Here $P = \sum_i \mathbb{Z} \omega_i$ denotes the lattice with fundamental weights $\omega_i$ defined by $\langle \omega_i, \alpha^\vee_j \rangle = \delta_{ij}$ via co-roots $\alpha^\vee_j = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. $Q^\vee = \sum_i \mathbb{Z} \alpha^\vee_i$ is the co-root lattice and $P/Q^\vee$ denotes the lattice points of $P$ lying in an elementary cell of $Q^\vee$, while $|P/Q^\vee|$ describes the number of points in this set. $P^+_k$ is the set of all dominant weights at level $k$, and $\epsilon(w) = (-1)^{\ell(w)}$ is the signature of the Weyl group element $w$, where $\ell(w)$ is the minimum number of simple Weyl reflections that $w$ decomposes into. $\Delta_+$ is the number of positive roots in the Lie algebra $G$.

3.2 Supersymmetric string models can be constructed in terms of conformal field theories with $N = 2$ supersymmetry. For the present discussion the important structure is determined by the affine Lie algebra $A_1^{(1)}$, which provides the essential building block of the supersymmetric theory. It turns out that of particular importance are the Hecke indefinite modular forms which can be defined as

$$\Theta_{k, \ell, m}(\tau) = \sum_{\substack{-|x| < |y| \leq |x| \atop (x, y) \text{ or } (\frac{1}{2} - x, \frac{1}{2} + y) \in \mathbb{Z}^2 + \left(\frac{k+1}{2}, \frac{m}{2}\right)}} \text{sign}(x) e^{2\pi i ((k+2)x^2 - ky^2)}$$

Related to these forms are the string functions $c_{k, \ell, m}(\tau) = \Theta_{k, \ell, m}(\tau)/\eta^3(\tau)$ of Kac-Peterson [17, 18], which are of immediate physical relevance because they appear in the $N = 2$ superconformal characters $\chi_{k, q, s}(\tau)$

$$\chi_{k, q, s}(\tau, z, u) = \sum_{j \geq 0} c_{k, q + 4j - s}(\tau) \theta_{2q + (4j - s)(k+2), 2k(k+2)}(\tau, z, u),$$

where the theta functions $\theta_m$ are defined as

$$\theta_{n, m}(\tau, z, u) = e^{-2\pi i m u} \sum_{\ell \in \mathbb{Z} + \frac{n}{2m}} e^{2\pi i m \ell^2 \tau + 2\pi i \ell z}.$$

These characters in turn define the partition function of the conformal field theory.

The theta functions $\Theta_{k, \ell, m}(\tau)$ are associated to quadratic number fields determined by the level of the affine theory. These are modular forms of weight one and cannot, therefore, be identified with the geometric modular form. It turns out, however, that appropriate products do lead to interesting forms [33, 23, 29, 30, 31].

4. $L$–functions of $\Omega$–motives

For a general smooth algebraic variety $X$ reduced mod $p$ the congruence zeta function of $X/\mathbb{F}_p$ is defined by

$$Z(X/\mathbb{F}_p, t) = \exp \left( \sum_{r \in \mathbb{N}} \#(X/\mathbb{F}_p^r) \frac{t^r}{r} \right).$$
Here the sum is over all finite extensions $\mathbb{F}_{p^r}$ of $\mathbb{F}_p$ of degree $r$. Per definition $Z(X/\mathbb{F}_p, t) \in 1 + \mathbb{Q}[[t]]$, but the expansion can be shown to be integer valued by writing it as an Euler product.

The proof by Grothendieck [10] of part of the Weil conjectures [40] asserts that the zeta function is a rational function determined by the cohomology of the variety

$$Z(X/\mathbb{F}_p, t) = \frac{\prod_{j=1}^{n} P_{p}^{2j-1}(t)}{\prod_{j=0}^{n} P_{p}^{2j}(t)},$$

where $\dim_{\mathbb{C}} X = n$, and $P_{p}^{i}(t)$ is a polynomial $P_{p}^{i}(t) = \sum_{j=0}^{b_i} \beta_{ji}(p)t^j$ associated to the $i$th cohomology group with degrees $b_i$ given by the $i$th Betti number $b_i = \dim H^i(X)$. This result motivates the introduction of $L$-functions associated to the polynomials $P_{p}^{i}(t)$, thereby reducing the complexity of the congruent zeta function.

In the context of the string theoretic modularity problem the $L$-function associated to the full cohomology group of a variety is a physically useful object only in the case of elliptic curves. In higher dimensions these functions are in general not modular. This motivates the consideration of a factorization of $L$-functions, and to ask whether modular forms emerge from the emerging pieces, and if so, whether these modular forms admit a Kac-Moody theoretic interpretation.

It was shown in refs. [30, 32], in the context of establishing string theoretic modularity of certain Calabi-Yau varieties, that a useful way to factorize the cohomological $L$-functions of Calabi-Yau varieties is to consider subspaces of the cohomology defined by Galois orbits of the holomorphic forms $\Omega$, $n = \dim_{\mathbb{C}} X$, where the Galois groups are defined in an inherent way by the arithmetic of the variety. The resulting $L$-functions have rational coefficients, and therefore can in principle admit a string theoretic interpretation along the lines discussed in [33, 23, 29, 30, 31, 32]. In this paper we generalize this strategy to the case of Fano varieties of special type by considering the orbit motive of a Galois representation determined by the generator of the cohomology group $H^{n-(Q-1),Q-1}(X)$ with $Q > 1$. We continue to denote the generator of this group by $\Omega$, and denote the motive by $M_{\Omega}(X)$. The goal is to compute the $L$-series associated to this motive, and to find a string theoretic interpretation of this function, following the strategy used in [30, 32]. More precisely, we need to consider motives with Tate twists, introduced by Tate in the context of the Tate conjectures, which assert that certain subgroups of the $i$th twists $H^{2i}(X)(i)$ of the cohomology groups $H^{2i}(X)$ are generated by classes of algebraic cycles [37, 38] (see also [34]). In the present context we consider for any special Fano variety of charge $Q$ the twisted omega motive $M_{\Omega}(X)(Q-1)$ and its $L$-function, which we will denote by $L_{M_{\Omega}}^{Q-1}(X, s) = L(M_{\Omega}(X)(Q-1), s)$. We will call $M_{\Omega}(X)(Q-1)$ the $\Omega$-motive with a Tate twist of charge $(Q-1)$, or simply the omega motive of charge $(Q-1)$.
In analogy to the general considerations in \cite{30,32} for the class of Calabi-Yau varieties in arbitrary dimensions, we can ask the following for the class of Fano varieties of special type.

**Question 1.** Is the L-function \( \text{L}_{\Omega}^{-1}(X, s) \) of the \( \Omega \)-motive \( M_{\Omega}(X)(Q - 1) \) of charge \( (Q - 1) \) of a special Fano variety of charge \( Q \) always modular?

**Question 2.** Can modular inverse Mellin transforms of \( \text{L}_{\Omega}^{-1}(X, s) \) be expressed in terms of Hecke indefinite modular forms associated to Kac-Moody algebras?

It will become clear below that the answer to this question is affirmative, at least for certain examples. In such cases the L-function can be viewed as a map that takes motives and turns them into conformal field theoretic objects. More challenging, in part because of the lack of definition of some of the terms, therefore would be the following picture.

**Conjecture:**
The L-function is a functor from the additive category of motives of special Fano type to the multiplicative category of \( N = 2 \) supersymmetric conformal field theories.

The basic outline described below is a generalization of the method based on Jacobi sums of weighted projective spaces described in \cite{30} in the context of K3 surface modularity, and in \cite{32} for higher dimensional Calabi-Yau varieties.

### 5. Modularity of Hecke L-series

The modularity of the L-series determined in this paper follows from the fact that they can be interpreted in terms of Hecke L-series associated to Größencharacters, defined by Jacobi sums. A Größencharacter can be associated to any number field. Hecke’s modularity discussion \cite{11} has been extended by Shimura \cite{35} and Ribet \cite{24}.

Let \( K \) be a number field and \( \sigma : K \rightarrow \mathbb{C} \) denote an embedding.

**Definition.** A Größencharacter is a homomorphism \( \psi : I_m(K) \rightarrow \mathbb{C}^\times \) of the fractional ideals of \( K \) prime to the congruence integral ideal \( m \) such that \( \psi((z)) = \sigma(z)^{w-1}, \forall K^\times \ni z \equiv 1(\text{mod}^\times m) \).

In the present case the cyclotomic Jacobi sums computed above arise from imaginary quadratic fields, in which case the structure of these characters simplifies. Consider an imaginary quadratic field \( K \) with discriminant \(-D\) and denote by \( \varphi \) the Dirichlet character associated to \( K \), viewed as a mod \( D \) character. Define a second Dirichlet character \( \lambda \) defined mod \( Nm \) by setting

\[
\lambda(a) = \psi((a))/\sigma(a)^{w-1}, \quad a \in \mathbb{Z}.
\]

The product of \( \varphi \) and \( \lambda \) then determines the nebentypus character of the modular form determined by Hecke character \( \psi \).
Denote by $N_p$ the norm of prime ideal $p$, and by
\begin{equation}
L(\psi, s) = \prod_{p \in \text{Spec} \mathcal{O}_K} (1 - \psi(p)N_p^{-s})^{-1}
\end{equation}
the Hecke L-series associated to the character $\psi$. The modularity of the corresponding $q$-series $f(\psi, q) = \sum_n a_n q^n$ associated to the L-series via the Mellin transform is characterized by the following result of Hecke.

**Theorem 4.** The power series $f(\psi, q)$ is a cusp form of weight $w$ and character $\epsilon = \varphi \lambda$ on $\Gamma_0(D\mathfrak{N}m)$. If $p \mid D\mathfrak{N}m$ then $f \mid T_p = a_pf$, where $T_p$ are the Hecke operators.

### 6. Special Fano varieties with critical dimension three

The focus in this paper will be on Fano varieties of Brieskorn-Pham type, hence the computation of their L-functions can proceed via their Jacobi sums \[40, 41\].

**Theorem 5.** For a smooth weighted projective hypersurface with degree vector $\mathbf{n} = (n_0, \ldots, n_s)$
\begin{equation}
X^\mathbf{n} = \{b_0z_0^{n_0} + b_1z_1^{n_1} + \cdots + b_s z_s^{n_s} = 0\} \subset \mathbb{P}_{(k_0, k_1, \ldots, k_s)},
\end{equation}
defined over the finite field $\mathbb{F}_q$, for $q = p^r, r \in \mathbb{N}$, set $d_i = (n_i, q - 1)$ and denote by $\mathcal{A}_s^{\mathbf{n}}$ the set of rational vectors $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_s)$ given by
\begin{equation}
\mathcal{A}_s^{\mathbf{n}} = \left\{ \alpha \in \mathbb{Q}^{s+1} \mid 0 < \alpha_i < 1, d_i \alpha_i = 0 \text{ mod } 1, \sum_{i=0}^{s} \alpha_i = 0 \text{ (mod 1)} \right\}.
\end{equation}
For each $(s + 1)-$tuple $\alpha$ define the Jacobi sum
\begin{equation}
J_q(\alpha) = \frac{1}{q-1} \sum_{u_i \in \mathbb{F}_q \atop u_0 + u_1 + \cdots + u_s = 0} \chi_{\alpha_0}(u_0)\chi_{\alpha_1}(u_1)\cdots\chi_{\alpha_s}(u_s),
\end{equation}
where $\chi_{\alpha_i}(u_i) = e^{2\pi i \alpha_i m_i}$ with integers $m_i$ determined via $u_i = g^{m_i}$, where $g \in \mathbb{F}_q$ is a generator. Then the cardinality of $X^\mathbf{n}/\mathbb{F}_q$ is given by
\begin{equation}
N_q(X^\mathbf{n}) = 1 + q + \cdots + q^{s-1} + \sum_{\alpha \in \mathcal{A}_s^{\mathbf{n}}} J_q(\alpha).
\end{equation}

With these Jacobi sums one defines for primes $p$ such that $d \mid (p^f - 1)$ for some $f \in \mathbb{N}$ the polynomials
\begin{equation}
P_p^{s-1}(t) = \prod_{\alpha \in \mathcal{A}_s^{\mathbf{n}}} \left(1 - (-1)^{s-1}J_p^f(\alpha) \prod_i \chi_{\alpha_i}(b_i)t\right)^{1/f}
\end{equation}
and the associated L-function is given as a product over all good primes $p$
\begin{equation}
L^{(j)}(X, s) = \prod_p (P_p(p^{-s}))^{-1}.
\end{equation}
The completion of the L-function at the bad primes will not be of relevance in the following.

The two varieties considered here in the context of $D_{\text{crit}}$, i.e. Gepner models with central charge $c = 9$, are odd-dimensional, hence the rational form of the congruent zeta function for a projective hypersurface of odd dimension $s$ takes the form

$$Z(X_{s-1}/\mathbb{F}_p, t) = \frac{\mathcal{P}_p^{s-1}(t)}{(1-t)(1-pt)\cdots(1-p^{s-1}t)}.$$  

(6.7)

We also consider a Fano variety corresponding to a K3 surface, i.e. a Gepner model with central charge $c = 6$. In this case the smooth Fano variety is of complex dimension four and the zeta function is of the form

$$Z(X_3^3/\mathbb{F}_p, t) = \frac{1}{(1-t)(1-pt)\mathcal{P}_p^3(t)(1-p^3t)(1-p^4t)}.$$  

(6.8)

In all these examples there is therefore only one interesting cohomological L-function. It turns out that these L-functions are not themselves of interest and that it is more useful to consider the L-function of the $\Omega$-motive. In all three examples the order of the Galois group is two, hence the motive is of dimension two. Denoting the Jacobi sum associated to the $\Omega$-form by $j_p(\alpha_\Omega)$ the polynomial of the $\Omega$-motive takes the form

$$\mathcal{P}_p^\Omega(t) = (1 - j_{p^n}(\alpha_\Omega)t^f)^{1/f} - (1 - j_{p^n}(\bar{\alpha}_\Omega)t^f)^{1/f},$$  

(6.9)

where $\bar{\alpha}_\Omega := 1 - \alpha_\Omega$ is the dual vector and $1$ is the unit vector. For primes $p$ such that $d|(p-1)$ this simplifies and the local L-function factors of the twisted omega motive of charge $(Q-1)$ take the form $L_{\Omega,p}^{Q-1}(X,s) = (\mathcal{P}_p^\Omega,\Omega^{-1}(p^{-s}))^{-1}$ with

$$\mathcal{P}_p^{\Omega, Q-1}(t) = 1 + \beta(p)p^{-(Q-1)t} + p^{n-2(Q-1)t^2},$$  

where $\beta(p) = j_p(\alpha_\Omega) + j_p(\bar{\alpha}_\Omega)$.

6.1. The cubic Fermat sevenfold $X_7^3$.

6.1.1. The motivic L-function. For the cubic Fermat sevenfold the Hodge decomposition takes the form $h^{i,j}(X_7^3) = 1$, $i = 0, 1, 2, 3$ with all other $h^{i,j}(X_7^3) = 0$ for $i, j < 4, i \neq j$. The intermediate cohomology decomposes as in [122], i.e. we have $Q = 3$ and $D_{\text{crit}} = 3$. The zeta function therefore has the form

$$Z(X_7^3/\mathbb{F}_p, t) = \mathcal{P}_p^7(t)/\prod_{i=0}^7(1-p^it),$$  

leading to the L-function

$$L(X_7^3, s) := \prod_{p} 1/\mathcal{P}_p^7(p^{-s}).$$  

(6.10)

The cyclotomic field defined by this manifold is $\mathbb{Q}(\mu_3)$, hence we have $\text{Gal}(\mathbb{Q}(\mu_3)/\mathbb{Q}) = (\mathbb{Z}/3\mathbb{Z})^\times$, and the L-series of the $\Omega$-motive of $X_7^3$ can therefore be computed from the orbit of the Jacobi sums $j_p(\alpha_\Omega)$ of length two, with $\alpha_\Omega = \frac{1}{3}(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. The L-series of the twisted $\Omega$-motive of charge two of $X_7^3$ is defined as $L_\Omega^2(X_7^3, s) = \prod_p 1/\mathcal{P}_p^{\Omega, 2}(p^{-s})$ with $\mathcal{P}_p^{\Omega, 2}(t) =$
1 + a_p t + p^3 t^2 for n|(p-1), \beta(p) = (j_p(\alpha_\Omega) + j_p(\alpha_\Omega)) and a_p = \beta(p)/p^2 because the Tate twist is of charge two. The computation of these sums for the first few nontrivial primes is sufficient to determine the L-function of the Tate twisted Ω-motive

\begin{equation}
L^2_{\Omega}(X^3_7, s) = 1 + \frac{20}{7^s} - \frac{70}{13^s} + \frac{56}{19^s} + \frac{308}{31^s} + \cdots
\end{equation}

and the associated q-series

\begin{equation}
f^2_\Omega(X^3_7, q) = q + 20q^7 - 70q^{13} + 56q^{19} + 308q^{31} + \cdots
\end{equation}

completely.

It will become clear below that this q-series describes a modular cusp form of weight four and modular level nine with respect to the Hecke congruence group.

It was suggested in [26, 6, 27] that the cubic Fermat sevenfold can be interpreted as a geometric model associated to the tensor model $1^{e^9}$ of nine copies of $N = 2$ superconformal models [8]. If this is indeed the case, then we would expect, in analogy to the results obtained in [33, 23, 29, 30, 31, 32], that we can express the q-series of the Ω-motive in terms of the Hecke indefinite modular forms $\Theta^k_{\ell,m}(\tau)$ associated to the underlying conformal field theory model. At the affine level $k = 1$ there is a single such form, hence this is a make or break situation which is very constrained, much like the first example established in this framework [33]. It turns out that the geometric modular form of the Ω-motive of the cubic sevenfold does admit a string theoretic interpretation. There are several ways this can be described. The most direct is that the modular form $f^2_\Omega(X^3_7, q)$ can be written as

\begin{equation}
f^2_\Omega(X^3_7, q) = \Theta^1_4(q^3)^4 \in S_4(\Gamma_0(9)).
\end{equation}

This modular form has the expected weight of a Calabi-Yau threefold, consistent with the central charge of the model. The absence of a twist character in this result is consistent with the fact that the field of quantum dimensions of the underlying field theory is trivial [28].

6.1.2. CM interpretation of the L-series and modularity. The motivic modular form $f^2_\Omega(X^3_7, q)$ of the cubic sevenfold is a form with complex multiplication by the Eisenstein field $\mathbb{Q}(\sqrt{-3})$, and therefore can be described by a Hecke character associated to this field. Define a character $\psi_{27}$ by the following congruence condition

\begin{equation}
\psi_{27}(p) = \alpha_p, \quad \alpha_p \equiv 1(\text{mod } 3),
\end{equation}

where the generator of the prime ideal $p = (\alpha_p)$ is determined uniquely by the congruence constraint. As described in the previous section, associated to any Hecke character $\psi$ is a Hecke series $L(\psi, s)$, and we find

\begin{equation}
L^2_{\Omega}(X^3_7, s) = L(\psi^3_{27}, s).
\end{equation}
It follows from Hecke’s theorem that the $q$–series of this $L$-function is modular, hence the data provided above is sufficient to prove that the Mellin transform of the $L$-function $L_{\Omega}^{2}(X_{7}^{3},s)$ is a modular form, and therefore the Tate twisted $\Omega$–motive of the cubic sevenfold is modular.

This result shows that the basic arithmetic building block of the cubic sevenfold is the elliptic cubic Fermat curve

$$E^{3} = \{(z_{0} : z_{1} : z_{2}) \in \mathbb{P}_{2} \mid z_{0}^{3} + z_{1}^{3} + z_{2}^{3} = 0\}.$$  

The Hasse-Weil $L$-series of $E^{3}$ agrees with the $L$-series of the algebraic Hecke character $\psi_{27}$

$$L_{HW}(E^{3}, s) = L(\psi_{27}, s).$$

The Mellin transform of $L_{HW}(E^{3}, s)$ has weight 2 and level 27, and it factors into string theoretically induced modular forms as

$$f(E^{3}, q) = \Theta_{1,1}^{1}(q^{3})\Theta_{1,1}^{1}(q^{9}).$$

This proves the part of Theorem 1 concerned with the cubic Fermat sevenfold.

The relation between twisted $\Omega$–motivic $L$-series of $X_{7}^{3}$ and the $L$-series of the elliptic curve $E^{3}$ that follows from these considerations can be formulated in a more direct way. Consider the expansions $L_{\Omega}^{2}(X_{7}^{3},s) = \sum_{n} a_{n}n^{-s}$ for the sevenfold, and $L_{HW}(E^{3},s) = \sum_{n} b_{n}n^{-s}$ for the elliptic curve. The Hecke character interpretation means that $a_{p} = \alpha_{p}^{3} + \bar{\alpha}_{p}^{3}$ and $b_{p} = \alpha_{p} + \bar{\alpha}_{p}$. Therefore

$$a_{p} = b_{p}^{3} - 3pb_{p}.$$  

6.1.3. $X_{7}^{3}$ as the mirror of a rigid Calabi-Yau variety. The fact that the CM character of the elliptic curve $E^{3} \subset \mathbb{P}_{2}$ is the building block of the $L$-function of the cubic sevenfold suggests that its $\Omega$–motivic $L$-function should be identical to that of the resolution of a quotient of the triple product $(E^{3})^{3}$ which produces the correct mirror cohomology of the critical part of the spectrum of the sevenfold. Such a quotient variety can be obtained by considering the $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$ action defined on $(E^{3})^{3}$ as

$$Z_{3} \times Z_{3} \ni g_{1} \times g_{2} : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

with obvious notation for the generators $g_{1}, g_{2}$. The resolution $X_{3}^{3}$ of the quotient variety $(E^{3})^{3}/\mathbb{Z}_{3} \times \mathbb{Z}_{3}$ is rigid and has $h^{1,1}(X_{3}^{3}) = 84$, as needed for the mirror cohomology of the relevant part of the cohomology of the sevenfold. Higher dimensional varieties defined by the resolution of quotients of the type $E^{n}/G$, for elliptic curves and certain finite groups $G$ were considered by Cynk and Hulek [7].

**Theorem 6.** Let $n \in \mathbb{N}$ be an odd integer, $E$ be the elliptic curve with an automorphism of order 3, and denote by $\tilde{X}_{n}$ the quotient of $E^{n}$ by the action
of the group
\[
\left\{ (\xi^{a_1} \times \cdots \times \xi^{a_n}) \in \text{End}(E^n) \mid \sum_{i=1}^{n} a_i = 0 \text{ (mod 3)} \right\}.
\]

Then $\tilde{X}_n$ has a smooth model $X_n$, which is a Calabi-Yau manifold such that $\dim H^n(X_n) = 2$ and $L(X_n, s) = L(g_{n+1}, s)$, where $g_{n+1}$ is the weight $n + 1$ cusp form with complex multiplication by $\mathbb{Q}(\sqrt{-3})$, associated to the $n$th power of the Größencharacter of $E$.

It follows by combining this result with those of Theorem 1 that the $L$-function of the $\Omega$–motive of the cubic sevenfold, with a Tate twist of charge two, is identical to the $L$-function of the intermediate cohomology of the variety $X_3^3$. This completes the proof of the part of Theorem 2 concerned with the cubic Fermat sevenfold.

6.1.4. Relatives and correspondences. The agreement between the motivic $L$-function of $X_3^3$ and the $L$-function of the intermediate cohomology of $X_3^3$ identifies these two varieties as "relatives" in the sense of ref. [13]. A variety $X_1$ with a 2-dimensional Galois representation $\rho_1$ is defined to be a relative of a variety $X_2$ if $\rho_1$ occurs in the cohomology of $X_2$. This notion identifies $X_3^3$ as a relative of $X_3^7$. Other relatives of Calabi-Yau type for $X_3^3$ have been considered by van Geemen in [42], Saito-Yui [25], Verrill [39], and Yui [43].

A stronger relation is predicted by the Tate conjecture, which in the present context can be formulated as a statement that an isomorphism between two 2-dimensional Galois representations which occur in the étale cohomology of two varieties defined over $\mathbb{Q}$ should be induced by a correspondence between the two varieties defined over $\mathbb{Q}$, i.e. there should be an algebraic cycle on the product of the two varieties [43]. The Tate conjecture therefore implies that there should exist a correspondence between $X_3^3$ and $X_3^7$. An explicit construction of such a correspondence should provide better insight into the precise relation between the pair of manifolds $(X_3^3, X_3^7)$.

The fact that physically the $L$-function is determined by the modular form $\Theta^1_{1,1}(\tau)$ of the underlying conformal field theory on the worldsheet leads to the conclusion that from a fourdimensional point of view these two varieties contain motivic sectors that are indistinguishable.

6.2. The quartic Brieskorn-Pham fivefold $X_5^4$.

6.2.1. The $L$-series of the $\Omega$–motive. The quartic fivefold $X_5^4$ defined in (1.3) has a Hodge decomposition of the form $h^{i,i}(X_5^4) = 1, i = 0, 1, 2$, with all other $H^{i,j}(X_5^4) = 0$ for $i, j < 5$ and $i \neq j$. The decomposition of the intermediate cohomology group is given by (1.4), i.e. $Q = 2$ and $D_{crit} = 3$.

The congruence zeta function therefore takes the form $Z(X_5^4/\mathbb{F}_p, t) = P_5^5(t)/\prod_{i=0}^{5}(1 - p^i t)$, leading to the $L$-function $L(X_5^4, s) = \prod_p (P_5^5(p^{-s}))^{-1}$. We are interested in the factor describing the twisted $\Omega$–motive of charge one, which is determined by the Jacobi sums $j_p(\alpha_\Omega)$ with $\alpha_\Omega = \frac{1}{4}(1, 1, 1, 1, 1, 1, 2)$. 
The computation of the first few Jacobi sums for low primes leads to the $q$--expansion of the L-function of the twisted $\Omega$--motive of the quartic fivefold given by

$$f_{\Omega}^1(X_4^5, q) \doteq q - 22q^5 + 18q^{13} - 94q^{17} + 130q^{29} + \cdots$$

This is a modular cusp form of weight four and level 64, $f_{\Omega}^1(X_4^5, q) \in S_4(\Gamma_0(64))$.

6.2.2. **CM interpretation of the L-series and modularity.** The motivic modular form $f_{\Omega}^1(X_4^5, q)$ has CM by the Gauss field $\mathbb{Q}(\sqrt{-1})$, and therefore can be described by a Hecke character associated to this field as well. Consider the twisted character $\psi_{64} = \psi_{32}\chi_2$ where $\psi_{32}$ is defined by

$$\psi_{32}(p) = \alpha_p, \quad \alpha_p \equiv 1 \pmod{(2 + 2i)},$$

where the generator of the prime ideal $p = (\alpha_p)$ is determined uniquely by the constraint.

The Hecke series associated to $\psi_{64}$ then is again the building block of the geometric L-series

$$L^{1}_{\Omega}(X_5^4, s) = L(\psi_{64}^3, s)$$

hence modularity of this L-function follows. Similar to the cubic sevenfold this result shows that the quartic fivefold has an elliptic building block, in this case the Brieskorn-Pham type curve

$$E^4 = \left\{ (z_0 : z_1 : z_2) \in \mathbb{P}^{(1,1,2)} \left| z_0^4 + z_1^4 + z_2^2 = 0 \right. \right\}.$$

It can be checked that the Hasse-Weil L-series of $E^4$ is determined by the character $\psi_{64}$

$$L_{HW}(E^4, s) = L(\psi_{64}, s)$$

and it was shown in [23] that the Mellin transform of $L_{HW}(E^4, s)$ factors into string theoretic Hecke indefinite modular forms as

$$f(E^4, q) = \Theta^{2}_{1,1}(q^4)^2 \otimes \chi_2.$$

This completes the proof of Theorem 1.

6.2.3. **A rigid Calabi-Yau mirror.** The appearance of the quartic elliptic curve $E^4$ again suggests to consider a resolved quotient of the triple product $(E^4)^3$, defined by some discrete symmetry group action such that the resolved manifold has a Hodge decomposition that is mirror to the critical part of the spectrum of the quartic fivefold. It turns out that the group that produces such a resolution is given by $\mathbb{Z}_4 \times \mathbb{Z}_4$, with the action

$$\mathbb{Z}_4 \times \mathbb{Z}_4 \ni g_1 \times g_2 : \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 \end{bmatrix}.$$

Denote by $X_4^5$ the resolution of the quotient variety $(E^4)^3/\mathbb{Z}_4 \times \mathbb{Z}_4$. This is a rigid Calabi-Yau variety which has $h^{1,1}(X_4^5) = 90$, as needed for the mirror cohomology of the relevant part of the cohomology of the fivefold. Again we can apply result of Cynk and Hulek [7] to the mirror just constructed.
Theorem 7. Let \( n \in \mathbb{N} \) be an odd integer, \( E \) be the elliptic curve with an automorphism of order 4, and denote by \( \bar{X}_n \) the quotient of \( E^n \) by the action of the group
\[
\left\{ (\xi^{a_1} \times \cdots \times \xi^{a_n}) \in \text{End}(E^n) \mid \sum_{i=1}^n a_i = 0 \text{ (mod 4)} \right\}.
\]
Then \( \bar{X}_n \) has a smooth model \( X_n \), which is a Calabi-Yau manifold such that \( \dim H^2(X_n) = 2 \) and \( L(X_n, s) = L(g_{n+1}, s) \), where \( g_{n+1} \) is the weight \( n+1 \) cusp form with complex multiplication by \( \mathbb{Q}(\sqrt{-1}) \), associated to the \( n^{th} \) power of the Größencharakter of \( E \).

It follows from Theorems 1 and 7 that the L-functions of the quartic fivefold and the variety \( X_3^3 \) are identical. This completes the proof of Theorem 2.

6.2.4. Relatives and correspondences. As in the case of the mirror pair \((X_3^3, X_3^3)\), the pair of varieties \((X_4^4, X_4^4)\) form a pair of relatives in that the Galois representation of \( X_4^4 \) is contained in the cohomology of the quartic fivefold. Again, the Tate conjecture implies that there is a correspondence between \( X_4^4 \) and \( X_4^4 \). The fact that the L-function has a physical interpretation in terms of the worldsheet modular form \( \Theta_{1,1}^{2m}(\tau) \) leads again to the conclusion that from a low energy perspective the corresponding motives of these two varieties are indistinguishable.

7. A special Fano variety of critical dimension two

7.1. Arithmetic of the cubic Fermat fourfold \( X_4^3 \). The relation between special Fano varieties and critical string models manifolds exists in all dimensions. An interesting lower-dimensional case is that of K3 surfaces. Among the Gepner models there exists an example that is the direct analog of the cubic 7-fold, defined by the cubic fourfold \((1.9)\). The Hodge diamond structure of this variety is \( h^{i,j}(X_3^3) = 1 \), \( i = 0, 1 \), with all other \( H^{i,j}(X_3^3) = 0 \) for \( i, j < 4 \) and \( i \neq j \), while the intermediate cohomology group decomposes as described above in eq. \((1.10)\), i.e. \( Q = 2 \) and \( D_{\text{crit}} = 2 \).

The middle dimensional cohomology therefore is that of the K3 surface, modulo the sector inherited from the ambient space \( \mathbb{P}_5 \). The zeta function of this cubic fourfold therefore takes the form
\[
Z(X_4^3, t) = \frac{1}{(1 - t)(1 - pt)(1 - p^3t)(1 - p^4t)},
\]
and it is natural to ask whether the \( \Omega \)-motive of this variety leads to the L-function of an \( \Omega \)-motive of K3-type.

The Galois group associated to this variety is again \((\mathbb{Z}/3\mathbb{Z})^x\) and the computation of the Tate twisted \( \Omega \)-motive obtained from the Jacobi sums \( j_p(\alpha_\Omega) \) with \( \alpha_\Omega = \frac{1}{3}(1, 1, 1, 1) \) leads to \( q \)-expansion
\[
f^1_\Omega(X_4^3, q) \equiv q - 13q^7 - q^{13} + 11q^{19} - 46q^{31} + \cdots
\]
This series describes a weight 3 modular form of level 27 which can be described in closed form as
\[
\frac{1}{2} \vartheta(q^3) \Theta_{1,1}(q^3) \vartheta_1 \vartheta_{11} \vartheta_{21} \vartheta_{31} \in S_3(\Gamma_0(27)),
\]
where \( \vartheta(q) \) is the Eisenstein series
\[
\vartheta(q) = \sum_{z \in \mathcal{O}_K} q^{Nz}
\]
associated to the field \( K = \mathbb{Q}(\sqrt{-3}) \).

It was shown in [30] that (7.2) is precisely the form which describes the Mellin transform of the L-series of the \( \Omega \) motive of the degree six K3 surface
\[
S^6 = \left\{ (z_0 : \cdots : z_3) \in \mathbb{P}^{(1,1,1,3)} \left| z_0^6 + z_1^6 + z_2^6 + z_3^2 = 0 \right. \right\}.
\]

Hence we have
\[
\frac{1}{2} \vartheta(q^3) \Theta_{1,1}(q^3) \vartheta_1 \vartheta_{11} \vartheta_{21} \vartheta_{31} = \vartheta(q^3) \Theta_{1,1}(q^3) \vartheta_1 \vartheta_{11} \vartheta_{21} \vartheta_{31}.
\]

It is therefore natural to ask whether these two varieties can be related in some explicit way. This can be analyzed via the twist map.

### 7.2. Twist map construction.

In the notation of [15, 14] consider the map
\[
\Phi : \mathbb{P}_{(w_0, \ldots, w_m)} \times \mathbb{P}_{(v_0, \ldots, v_n)} \rightarrow \mathbb{P}_{(w_0 w_1, \ldots, w_0 w_m, w_0 v_1, \ldots, w_0 v_n)}
\]
defined as
\[
((x_0, \ldots, x_m), (y_0, \ldots, y_n)) \mapsto (y_0^{w_0/w_1} x_1, \ldots, y_0^{w_0/w_m} x_m, x_0^{v_0/v_1} y_1, \ldots, x_0^{v_0/v_n} y_n)
\]

This map restricts on the subvarieties
\[
\begin{align*}
X_1 &= \{ x_0^\ell + p(x_i) = 0 \} \subset \mathbb{P}_{(w_0, \ldots, w_m)} \\
X_2 &= \{ y_0^\ell + q(y_j) = 0 \} \subset \mathbb{P}_{(v_0, \ldots, v_n)}
\end{align*}
\]

to the hypersurface
\[
X = \{ p(z_i) - q(t_j) = 0 \} \subset \mathbb{P}_{(v_0 w_1, \ldots, v_0 w_m, w_0 v_1, \ldots, w_0 v_n)}
\]
as a finite map. The degrees of the hypersurfaces \( X_i \) are given by
\[
\begin{align*}
\deg X_1 &= w_0 \ell, \\
\deg X_2 &= v_0 \ell,
\end{align*}
\]
leading to the degree \( \deg X = v_0 w_0 \ell \).

Applying this construction to the hypersurfaces
\[
S^6_\pm = \left\{ (z_0 : \cdots : z_3) \in \mathbb{P}^{(1,1,1,3)} \left| z_0^6 + z_1^6 + z_2^6 \pm z_3^2 = 0 \right. \right\}
\]
leads to the cubic fourfold embedded in \( \mathbb{P}_5 \), explaining geometrically the L-function result.
7.3. CM interpretation of the L-series and modularity. The modular form $f_1^1(X^3, q)$ of the twisted omega motive of the cubic fourfold has complex multiplication by the Eisenstein field $\mathbb{Q}(\sqrt{-3})$ with the same underlying basic character $\psi_{27}$ defined in (6.14) as the cubic sevenfold, except that in the present case we have to consider the square of the character

\[(7.13) \quad L_1^1(X^3, s) = L(\psi_{27}^2, s).\]

Hence the $\Omega$-motive of the cubic fourfold is modular. The basic arithmetic building block is again the elliptic curve $E^3$, described in string theoretic terms in §6.

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