Triggering Wave-Like Convergence of Tail-biting Spatially Coupled LDPC Codes: Single and Dual-Channel Setup

Sebastian Cammerer*, Vahid Aref†, Laurent Schmalen†, and Stephan ten Brink*

* Institute of Telecommunications, Universität Stuttgart, Stuttgart, Germany
† Bell Laboratories, Alcatel-Lucent AG, Stuttgart, Germany

Abstract—Spatially coupled low-density parity-check (SC-LDPC) codes can achieve the channel capacity under low-complexity belief propagation (BP) decoding. For practical finite coupling lengths however, there is a non-negligible rate-loss because of termination effects. In this paper, we focus on tail-biting SC-LDPC codes which do not require termination and study how we can approach the performance of terminated SC-LDPC codes by random shortening the code. We find the minimum required rate-loss in order to achieve the same performance than terminated codes. In the second part of this paper, we study the use of tail-biting SC-LDPC codes for transmission over parallel channels, e.g., bit-interleaved-coded-modulation (BICM), multi-level coding, or multi-carrier transmission. We investigate how the distribution of the coded bits between two parallel binary erasure channels can change the performance of the code. We show that if the code bits are carefully interleaved between channels, a tail-biting SC-LDPC code can be operated with BP decoding almost anywhere within the achievable region of MAP decoding.

I. INTRODUCTION

Their excellent performance under low-complexity belief propagation (BP) decoding renders spatially coupled LDPC codes (SC-LDPC) attractive for error correction subsystems in upcoming communication systems such as, e.g., long-haul optical fiber transceivers [1]. The effect of spatial coupling was first observed in LDPC convolutional codes. It was numerically shown that these codes can approach the channel capacity under BP decoding [2]. Considering a general spatially coupled structure, it was later proven in [3], [4], [5], [6] that SC-LDPC codes can achieve the capacity of binary-input, memoryless, symmetric-output (BMS) channels under BP decoding. In particular, it was proven that the BP threshold of an SC-LDPC ensemble asymptotically converges to the threshold of the underlying LDPC ensemble under maximum-a-posteriori (MAP) decoding. This phenomenon is termed threshold saturation. Moreover, the MAP threshold of LDPC ensembles such as regular LDPC codes converges to the capacity of the underlying channel when the average variable node degree grows large.

The idea of spatial coupling is to take a replication of \( L \gg 1 \) copies of an LDPC code and place them next to each other along a spatial dimension. Then, the individual LDPC codes are connected by carefully swapping edges in a way that the resulted code keeps the same local graphical structure as the original one. If the structure is terminated in an effective way, the threshold saturation effect occurs. The termination imposes additional constraints on the coded bits such that the BP algorithm encounters an “easier” problem at some specific part of code structure. Thus, the BP algorithm can successfully decode the code bits of this specific part. The decoding then continues in a successive manner along the spatial dimension: the knowledge of recovered code bits helps the BP algorithm to decode their spatially adjacent code bits, and so on.

The most common way of termination is to shorten the code by setting all the code bits of a few LDPC copies at the boundaries of the spatial chain to zero [2], [3], [4]. This approach reduces the total rate of the code. Although the rate-loss vanishes as \( L \to \infty \), it is not negligible for practical finite values of \( L \). By careful modification of the code structure, a more efficient termination for regular SC-LDPC codes is proposed in [7]. This approach decreases the rate-loss but entails a degradation of the codes’ finite-length performance.

In this paper, we consider tail-biting SC-LDPC codes [8] which do not require termination. We first study the minimum rate-loss required for threshold saturation over a binary erasure channel (BEC) and use shortening to trigger the decoding wave of threshold saturation. In particular, we use the technique of density evolution to find the best way of shortening code bits over all underlying LDPC copies, i.e., we minimize the total number of shortened code bits. Our approach has also been suggested for rate-loss mitigation in [9], without giving however an in-depth analysis of the minimum required shortening. In this work, we compare the rate-loss for different shortening strategies.

We additionally study how to mitigate the rate-loss when the transmission takes place over parallel channels. Examples of parallel channels include bit-interleaved coded-modulation (BICM) for high spectral efficiency modulation formats and multi-carrier transmissions, such as transmission over two polarizations, frequently occurring in optical communications. The performance of SC-LDPC codes with BICM has already been studied in [10], [11], [12], [13], [14]. Threshold saturation has been shown in [11], [12]. Optimized bit interleavers for BICM using both terminated and tail-biting SC-LDPC codes have been studied in [10], [11], [12], [13], [14]. It has been shown that the optimized interleavers decrease the number of decoding iterations and also improve the decoding performance. In this paper, we study how to exploit the fact that the different channels have varying entropy and realize codes without rate-loss but that still show threshold saturation. For simplicity, we consider transmission over two parallel BECs. We show that if the code bits are carefully interleaved between...
channels, a tail-biting SC-LDPC code under BP decoding can be operated almost anywhere within the achievable region of MAP decoding.

II. TAIL-BITING SPATIALLY COUPLED LDPC CODES

Tail-biting SC-LDPC codes are a subset of block codes with sparse parity check matrices. We first lay out a set of spatial positions indexed by integers \( z \in [0, L) \) on a circle, where \( L \) denotes the replication factor of the code. We fix a “smoothing parameter” which is an integer \( w \geq 1 \). At each position \( z \), we consider \( n \) code bits, thus in total, \( N = Ln \) code bits, and \( m \) parity checks to satisfy, thus \( M = Lm \) parity constraints in total. We usually represent such a sparse code by a bipartite graph called Tanner graph. We assign a node, called variable node, to each code bit and assign a node called check node to each parity constraint. We connect variable node \( i \) to check node \( a \) by an edge if and only if the code bit \( i \) participates in the corresponding constraint. Here, we consider code ensembles with regular degree distributions: each variable node is connected to \( d_v \) check nodes, and each check node is connected to \( d_c \) variable nodes.

To construct a random instance of the tail-biting SC-LDPC code ensemble, we connect the variable nodes and the check nodes in the following manner: Each variable node at position \( z \) is connected randomly to \( d_v \) check nodes lying within the range \( \{ \lfloor z + w - 1 \rfloor \} \), where \( (x)_L \) returns the remainder of the integer division of \( x \) by \( L \) (“\( x \) modulo \( L \)”). Equivalently, each check node at position \( z \) is connected randomly to \( d_c \) variable nodes in the range \( \{ \lfloor z - w + 1, z \rfloor \} \). Note that positions \( z \notin [0, L) \) are circularly shifted to \( z \in [0, L) \), which is taken care of by the \( (i)_L \) operator. For additional details, we refer the reader to [3], [15]. If all the code bits in position \( z \in [0, w - 2] \) are set to zero, the code ensemble becomes a (terminated) SC-LDPC ensemble [3].

We use the technique of density evolution (DE) to study the asymptotic performance of the code ensemble when \( n \to \infty \). Let us assume that the transmission takes place over a BEC or parallel BECs. Let \( \varepsilon_z \) denote the average erasure probability of code bits at spatial position \( z \). Let \( x_z^{(t)} \) denote the average erasure probability of the outgoing messages from code bits in position \( z \) and at iteration \( t \). We initialize \( x_z^{(0)} = 1 \) for all \( z \in [0, L) \). The densities are updated as follows:

\[
x_z^{(t+1)} = \varepsilon_z \left( 1 - \varepsilon_z \sum_{i=0}^{w-1} \left( 1 - \varepsilon_z \sum_{j=0}^{w-1} x_{(z+i-j)}^{(t)} \right)^{d_c-1} \right)^{d_v-1}
\]

(1)

The average erasure probability of code bits after \( T \) iterations is given by

\[
P_e(T) = \frac{1}{L} \sum_{z=0}^{L-1} \varepsilon_z \left( 1 - \varepsilon_z \sum_{i=0}^{w-1} \left( 1 - \varepsilon_z \sum_{j=0}^{w-1} x_{(z+i-j)}^{(T)} \right)^{d_c-1} \right)^{d_v-1}
\]

(2)

For transmission over a BEC, \( \varepsilon_z = \varepsilon \), the erasure probability of the channel. If we additionally set \( \varepsilon_z = 0 \) for \( z \in [0, w-2] \), we have the DE equation of the terminated SC-LDPC ensemble over a BEC.

III. RATE-LOSS MITIGATION OVER THE BEC

Consider the tail-biting SC-LDPC code ensemble. It is known that the BP threshold and the MAP threshold of this ensemble are equal to the BP threshold and the MAP threshold of the underlying LDPC code ensemble [3], [16]. With a properly selected shortening, the BP threshold of the spatially coupled ensemble saturates to the MAP threshold. We shorten the code by setting some code bits to zero (or to a known feasible value). Thus, we inject some prior knowledge that can be used to trigger BP decoding.

Let \( \alpha_z \) be the fraction of code bits set to zero for \( z \in [0, L) \) (see Figure 1). For instance, we have the terminated SC-LDPC code ensemble, if we let \( \alpha_z = 1 \) for \( 0 \leq z \leq w - 2 \). For the analysis, we randomly select a fraction of \( \alpha_z \) code bits and set them to zero. Thus, the design rate of the code is decreased from \( R = 1 - d_v/d_c \) to

\[
R(\alpha) = 1 - \frac{d_v}{d_c} \cdot \frac{L - \sum_{z=0}^{L-1} \left( \frac{1}{w} \sum_{j=0}^{w-1} \alpha_{z-j} \right)^{d_v}}{L - \sum_{z=0}^{L-1} \alpha_z}
\]

(3)

where \( \alpha = (\alpha_0, \ldots, \alpha_{L-1}) \). For large \( d_c \), we have

\[
R(\alpha) \approx 1 - \frac{d_v}{d_c} \cdot \frac{L}{L - \sum_{z=0}^{L-1} \alpha_z}
\]

Consider the transmission over a BEC with erasure probability \( \varepsilon \). At position \( z \), the average fraction of code bits erased...
by the channel is \( \varepsilon_z = (1 - \alpha_z) \varepsilon \). Let \( \varepsilon_{BP} \) and \( \varepsilon_{MAP} \) denote the BP threshold and the MAP threshold of the uncoupled LDPC\((d_v, d_c)\) ensemble. For any \( \varepsilon \in (\varepsilon_{BP}, \varepsilon_{MAP}) \), our goal is to find \( \alpha \) which maximizes \( R(\alpha) \) under successful BP decoding, i.e.,

\[
\alpha_{opt}(\varepsilon) = \arg \max_{\alpha} \{ R(\alpha) \mid \lim_{T \to \infty} P_e(T) = 0 \}. \tag{4}
\]

For our numerical computation, we relax the above maximization as follows:

(i) For a given \( \alpha \), define \( T_\alpha(\varepsilon) \) as the smallest iteration \( t \) such that \( \frac{1}{L} \sum_{z=0}^{L-1} |x_z^{(t-1)} - x_z^{(t)}|^2 < \delta \).

(ii) We find \( \alpha^*(\varepsilon) = \arg \max_{\alpha} \{ R(\alpha) \mid P_e(T_\alpha(\alpha)) < \delta \}. \)

We set \( \delta = 10^{-7} \) in our optimization. To find \( \alpha^*(\varepsilon) \), we apply two sub-optimal algorithms: (i) For small \( L \), an exhaustive search is carried out over the discretized space of \( \alpha \). Each \( \alpha_z \) is quantized with resolution \( \Delta = 10^{-3} \). (ii) The sub-optimal differential evolution algorithm \([17]\) is also used for large \( L \) which is generally much faster than an exhaustive search. We observe a good consistency between the results of both algorithms. Figure 3 shows the evolution of densities for both algorithms. Figure 2 illustrates the optimization results for the tail-biting SC-LDPC\((d_v = 3, d_c = 6, L = 50, w = 3)\) ensemble when it is shortened by the optimized \( \alpha^* \) and used over a BEC with \( \varepsilon = 0.48 \).

We also consider a simpler scheme denoted “uniform shortening” and defined as

\[
\alpha_{uni}(B) = \left\{ \begin{array}{ll}
\alpha_z = \alpha, & 0 < z < B \\
\alpha_z = 0, & \text{otherwise}
\end{array} \right.
\]

where \( 0 \leq B < L \). Similarly, we can find

\[
\alpha^*_{uni}(\varepsilon) = \arg \max_{B, \alpha_{uni}(B)} \{ R(\alpha) \mid P_e(T_\alpha(\alpha)) < \delta \}. \tag{5}
\]

The above optimization is much simpler. For each \( B \), it is a one-dimensional optimization over a bounded interval. The BP performance is monotonic in terms of \( \alpha \), and thus, we can simply use algorithms such as the bisection method to find the best \( \alpha_{uni}(B) \). Then we change \( B \) to find \( \alpha^*_{uni}(\varepsilon) \). Figure 3 illustrates the optimization results for the tail-biting SC-LDPC\((3, 6, 50, 3)\) ensemble. The maximum design rate is computed for both \( \alpha^*(\varepsilon) \) and \( \alpha^*_{uni}(\varepsilon) \), and for \( \varepsilon \in (\varepsilon_{BP}, \varepsilon_{MAP}) \). We observe that both optimized shortenings reduce the rate-loss by more than 50%. Moreover, there is only a very small difference between the rate of \( \alpha^*(\varepsilon) \) and \( \alpha^*_{uni}(\varepsilon) \). We observe the same behavior for tail-biting ensembles of LDPC\((4, 8)\) and LDPC\((5, 10)\) codes. Note that the error probability of BP decoding is monotonically increasing in terms of \( \varepsilon \). Therefore, \( \alpha^*(\varepsilon_{MAP}) \) is also a feasible shortening for \( \varepsilon \leq \varepsilon_{MAP} \). This \( \alpha^*(\varepsilon_{MAP}) \) can be universally used for all \( \varepsilon \in (\varepsilon_{BP}, \varepsilon_{MAP}) \) with more than 50% rate-loss gain, even though it is not, of course, the best \( \alpha \) for \( \varepsilon < \varepsilon_{MAP} \).

Figure 4 shows the average bit erasure probability \( P_e \) under BP decoding for the tail-biting SC-LDPC\((3, 6, 50, 3)\) code shortened by \( \alpha^*(\varepsilon_{MAP}) \). For comparison, we also plot the erasure probability curve of the terminated SC-LDPC\((3, 6, 50, 3)\) code, and the tail-biting code (without shortening). For all codes, \( n = 2000 \). We observe that the terminated SC-LDPC code has much smaller \( P_e \) than the tail-biting SC-LDPC code as the BP threshold of SC-LDPC code is increased to the MAP threshold. With shortening, not only the performance of the tail-biting SC-LDPC code improves to the performance of the terminated SC-LDPC code but also the shortened tail-biting SC-LDPC code has a larger rate than the SC-LDPC code. We also plot the performance of these codes over a binary additive white Gaussian noise (BAWGN) channel in Figure 3. The left sub-plot shows the bit error rate (BER) in terms of signal-to-noise (SNR) ratios \( E_b/N_0 \). We observe the similar behavior as for BEC in Figure 5. In order to see the gain in coding rate, the same BER values are plotted in terms of \( E_b/N_0 = E_s/(R N_0) \) in the right sub-plot. It shows that the larger rate leads to an additional net coding gain of \( \sim 0.1 \text{ dB} \).

IV. TAIL-BITING SC-LDPC CODES OVER TWO PARALLEL BECs

In this section, we study the performance of tail-biting SC-LDPC\((d_v, d_c, L, w)\) ensembles when the transmission takes place over two independent parallel channels. A practical example of such a scheme are the different bit-channels in BICM with 4-ASK modulation format (or, 16-QAM which can be perceived as two orthogonal 4-ASK signals). As a simplified model, we assume that the channels are independent BECs. We also assume that the transmitter knows the channel parameters \( \varepsilon_1 \) and \( \varepsilon_2 \). Without loss of generality, we assume that \( \varepsilon_1 \leq \varepsilon_2 \). The code bits are transmitted over either the
for some \( B < L \). However, the uniform shortening is much becomes similar to the best shortening optimization presented then it is equivalent to transmitting over a BEC with erasure bits in each spatial position are passed through each channel, \( z \). We show that we can exceed this bound if we carefully interleave the code bits of different spatial positions between channels (see also [13], [14]). In particular, we can seed an effective termination for threshold saturation by using more of the better channel in some specific positions.

Let \( \beta \) denote the fraction of code bits in spatial position \( z \) transmitted over the BEC with \( \varepsilon_1 \). Clearly, \( \sum_{z=0}^{L-1} \beta_z = L/2 \). The average erasure probability at position \( z \) is \( \varepsilon_z = \beta_z \varepsilon_1 + (1 - \beta_z) \varepsilon_2 \). Now, the problem of finding optimal \( \beta \) becomes similar to the best shortening optimization presented in Section II. We already observed in Figure 3 that there is a very small difference between uniform shortening and non-uniform shortening. However, the uniform shortening is much simpler to study and we follow this method here. In fact, we consider the following set of inequalities in Fig. 6

\[
\varepsilon = \begin{cases} 
\mu_1, & 0 \leq z < B, \\
\mu_2, & z \geq B
\end{cases}
\]

for some \( B < L/2 \) provided that

(i) \( \varepsilon_1 \leq \mu_1 \), and \( \mu_2 \leq \varepsilon_2 \),

(ii) \( \mu_2 = \frac{2B}{L} (\varepsilon_1 + \varepsilon_2) - \frac{L-B}{L} \mu_2 \).

From these conditions, it follows that \( \mu_2 \leq \frac{\varepsilon_1 + \varepsilon_2}{(1-B/L)} \). If we know all the feasible triples \((\mu_1, \mu_2, B)\), we can determine the achievable region \((\varepsilon_1, \varepsilon_2)\). The idea is as follows: if \( \mu_1 < \varepsilon_{\text{ap}} \) and \( B \) is large enough, the BP algorithm can decode some code bits lying in the range \( z < B \). In particular, the “computation graph” of some code bits has enough non-erased variable nodes for the BP algorithm to recover them. Those recovered code bits then play the role of effective termination for the coupled ensemble if \( \mu_2 \) is small enough (and \( \mu_2 < \varepsilon_{\text{map}} \)). Note that successful BP decoding is not attained by merely interleaving if either \( \varepsilon_2 \geq \varepsilon_1 \geq \varepsilon_{\text{ap}} \) or \( \varepsilon_1 + \varepsilon_2 \geq 2 \varepsilon_{\text{map}} \).

For a given \((\mu_2, B)\), let \( f(\mu_2, B) \) denote the largest \( \mu_1 \) for which BP decoding is successful (see [5]). Thus, we have the following set of inequalities

\[
\varepsilon_1 \leq \varepsilon_{\text{ap}}, \\
\varepsilon_1 + \varepsilon_2 \leq \frac{2B}{L} f(\mu_2, B) + 2(1 - \frac{B}{L}) \mu_2 = g(\mu_2, B), \\
(1 - \frac{2B}{L}) \varepsilon_1 + \varepsilon_2 \geq 2(1 - \frac{B}{L}) \mu_2.
\]

The DE equation (1) is monotonically increasing in terms of the channel erasures \( \varepsilon_2 \). Therefore, if \((\varepsilon_1, \varepsilon_2)\) is achievable under BP decoding, then for all \( \varepsilon_1' \leq \varepsilon_1 \) and \( \varepsilon_2' \leq \varepsilon_2 \), \((\varepsilon_1', \varepsilon_2')\) is also achievable under BP decoding (with different triple \((\mu_1, \mu_2, B)\)). Figure 8 illustrates \( g(\mu_2, B) \) for the tail-biting SC-LDPC(3, 6, 50, 3) ensemble numerically evaluated by density evolution. We observe that \( g(\mu_2, B) \) is an increasing function of \( \mu_2 \). Using (6), we can numerically calculate the maximal achievable region of \((\varepsilon_1, \varepsilon_2)\) which can be decoded successfully in the limit of the code-length. We plot this achievable region in Figure 8 for the tail-biting SC-LDPC(3, 6, 50, 3) ensemble and in the inset the tail-biting SC-LDPC(4, 8, 5, 3) ensemble with \( L = 25, 50 \) and 100. The achievable region under MAP decoding is \( \varepsilon_1 + \varepsilon_2 \leq 2 \varepsilon_{\text{map}} \). We observe that by suitable channel interleaving, the achievable region under BP decoding almost covers the entire MAP achievable region except for a small triangular region in which \( \varepsilon_1 \geq \varepsilon_{\text{ap}} \) and \( \varepsilon_2 \geq \varepsilon_{\text{ap}} \). As \( L \) grows, the coverage saturates to the MAP achievable region (except for \( \varepsilon_1 \geq \varepsilon_{\text{ap}} \) and \( \varepsilon_2 \geq \varepsilon_{\text{ap}} \)). Let us summarize the results by partitioning the \((\varepsilon_1, \varepsilon_2)\) region:

(i) \( \varepsilon_1 + \varepsilon_2 \geq 2 \varepsilon_{\text{map}} \): The code bits cannot be fully recovered neither under MAP decoding, nor under BP decoding.

(ii) \( \varepsilon_1 + \varepsilon_2 \leq 2 \varepsilon_{\text{map}} \): The code bits can be recovered under BP decoding if the code bits in each position are equally divided between channels.

(iii) \( \varepsilon_1 + \varepsilon_2 \leq 2 \varepsilon_{\text{map}} \), and \( \varepsilon_1 < \varepsilon_{\text{ap}} < \varepsilon_2 \): The code bits can be recovered under BP decoding by suitable channel interleaving. The mapping must satisfy (6). Many such mappings are feasible. This gives an additional degree of freedom to choose the best one in terms of decoding iterations. Large \( L \) is required when \( \varepsilon_1 + \varepsilon_2 \) gets close to \( 2 \varepsilon_{\text{map}} \). We have the same results, if \( \varepsilon_2 < \varepsilon_{\text{ap}} < \varepsilon_1 \).

(iv) \( \varepsilon_1 + \varepsilon_2 \leq 2 \varepsilon_{\text{map}} \), and \( \varepsilon_1 > \varepsilon_{\text{ap}} \) and \( \varepsilon_2 > \varepsilon_{\text{ap}} \): Interleaving over both channels is not enough for successful BP decoding. Shortening in the range \( z \in [0, B) \) is then
\( \varepsilon \) in the limit of information bits which must be provided for successful BP decoding scheme numerically estimates the minimum amount of 50% by a suitable shortening pattern. Our random shortening also required for effective termination leading to an inevitable rate loss.

![Fig. 7. \( g(\mu_2, B) \) of the tail-biting SC-LDPC(3, 6, 50, 3) ensemble.](image)

![Fig. 8. The achievable region under BP decoding for the tail-biting SC-LDPC(3, 6, L, 3) ensemble via suitable channel interleaving. The region is plotted for \( L = 25, 50 \) and 100. The inset figure shows the achievable region of tail-biting SC-LDPC(4, 8, L, 4) ensemble under BP decoding.](image)

\section{V. Conclusion}

Rate-loss mitigation of spatially coupled codes is one of the major challenges towards a practical implementation of this class of codes. We have shown that a significant rate-loss reduction can be obtained from shortening the tail-biting SC-LDPC codes. For the case of SC-LDPC(\( d_1 = 3, d_c = 6, L, w = 3 \)) codes, we can reduce the rate-loss by more than 50% by a suitable shortening pattern. Our random shortening scheme numerically estimates the minimum amount of information bits which must be provided for successful BP decoding. Therefore, it is a lower bound which can be achieved in the limit of \( n \). Our simulations show that—for finite \( n \)—we need to shorten a larger fraction of code-bits in order to obtain the error performance of the terminated SC-LDPC codes. Finding the minimal required fraction of shortened bits for finite \( n \) is an interesting open problem.

We also observe that the shortened tail-biting SC-LDPC codes can outperform the terminated SC-LDPC codes (in terms of \( E_b/N_0 \)) for transmission over AWGN channels.

Rate-loss mitigation can be fulfilled effectively when the transmission takes place over more than one binary-input channel, as is the case in BICM. The extra available channel dimensions can be exploited by properly interleaving code bits in different spatial positions among channels. For the case of two parallel BECs, we show that if the code bits are carefully interleaved between channels, a tail-biting SC-LDPC code under BP decoding can be operated almost anywhere within the achievable region of MAP decoding.

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