FORMAL SYMPLECTIC GROUPOID OF A DEFORMATION QUANTIZATION

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Abstract. We give a self-contained algebraic description of a formal symplectic groupoid over a Poisson manifold $M$. To each natural star product on $M$ we then associate a canonical formal symplectic groupoid over $M$. Finally, we construct a unique formal symplectic groupoid ‘with separation of variables’ over an arbitrary Kähler-Poisson manifold.

1. Introduction

Symplectic groupoids are semiclassical geometric objects whose heuristic quantum counterparts are associative algebras treated as quantum objects. In [4], [14], and [15] evidence was given that the star algebra of a deformation quantization gives rise to a formal analogue of a symplectic groupoid. In this paper we give a global definition of a formal symplectic groupoid and show that to each natural deformation quantization (in the sense of Gutt and Rawnsley, [10]) there corresponds a canonical formal symplectic groupoid.

Symplectic groupoids were introduced independently by Karasëv [16], Weinstein [23], and Zakrzewski [25]. Recall that a local symplectic groupoid is an object that has the properties of a symplectic groupoid in which the multiplication is local, being only defined in a neighborhood of the unit space. It was proved in [16] and [23] that for any Poisson manifold $M$ there exists a local symplectic groupoid over $M$ that ‘integrates’ it. In [4] A. S. Cattaneo, B. Dherin, and G. Felder considered the formal integration problem for $\mathbb{R}^n$ endowed with an arbitrary Poisson structure, whose solution is given by a formal symplectic groupoid. They start with the zero Poisson structure on $\mathbb{R}^n$, the corresponding trivial symplectic groupoid $T^*\mathbb{R}^n$, and a generating function of the Lagrangian product space of this groupoid. A formal symplectic groupoid is then defined in terms of a formal deformation...

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of that trivial generating function. One of the main results of [4] is an explicit formula for a generating function that delivers the formal symplectic groupoid related to the Kontsevich star product. The approach to formal symplectic groupoids developed in [4] demonstrates the relationship between geometric and algebraic deformations described in [22].

One can take an alternative approach to the definition of a formal symplectic groupoid over a Poisson manifold \( M \) (which leads to the same object) by replacing the symplectic manifold \( \Sigma \) on which a (local) symplectic groupoid over \( M \) is defined, with the formal neighborhood \((\Sigma, \Lambda)\) of its unit space \( \Lambda \) (see the definition of a formal neighborhood in Section 2). We use a simple model of the algebra of formal functions on \((\Sigma, \Lambda)\) which is reminiscent of the Hopf algebroid constructed by Vainerman in [21]. This model provides effective means to check the axioms of a formal symplectic groupoid and to do the calculations.

In Section 2 we state formal analogues of the axioms of a symplectic groupoid and give a definition of a formal symplectic groupoid over a given Poisson manifold \( M \). Such a formal groupoid is defined on the formal neighborhood of a Lagrangian submanifold of a symplectic manifold. In Section 3 we give a self-contained algebraic description of a formal symplectic groupoid and show that a strict formal symplectic realization of an arbitrary Poisson manifold \( M \) gives rise to a unique formal symplectic groupoid over \( M \) whose source mapping is given by that formal symplectic realization. In Section 4 we describe the space of all formal symplectic groupoids over \( M \) which are defined on a given formal neighborhood of a Lagrangian submanifold of a symplectic manifold. In Section 5 we relate to each natural deformation quantization on \( M \) a canonical formal symplectic groupoid. In Section 6 we prove that any deformation quantization with separation of variables on a Kähler-Poisson manifold \( M \) is natural and show that its canonical formal symplectic groupoid has a property which we call ‘separation of variables’. Finally, in Section 7 we prove that for an arbitrary Kähler-Poisson manifold \( M \) there exists a unique formal symplectic groupoid with separation of variables over \( M \).

2. Definition of a formal symplectic groupoid

A symplectic groupoid over a Poisson manifold \((M, \{\cdot , \cdot \}_M)\) is a symplectic manifold \( \Sigma \) endowed with the associated Poisson source mapping \( s : \Sigma \to M \), the anti-Poisson target mapping \( t : \Sigma \to M \), which both are surjective submersions, the antisymplectic involutive inverse mapping \( i : \Sigma \to \Sigma \), and the unit mapping \( \epsilon : M \to \Sigma \),
which is an embedding. The image $\Lambda = \epsilon(M)$ of the unit mapping is the Lagrangian unit space of the symplectic groupoid. Denote by $\Sigma^n$ the Cartesian product of $n$ copies of the manifold $\Sigma$ and by $\Sigma_n$ the submanifold of $\Sigma^n$ formed by the $n$-tuples $(\alpha_1, \ldots, \alpha_n) \in \Sigma^n$ such that $t(\alpha_k) = s(\alpha_{k+1})$, $1 \leq k \leq n-1$. The coisotropic submanifold $\Sigma_2$ of $\Sigma \times \Sigma$ is the domain of the groupoid multiplication $m : \Sigma_2 \to \Sigma$. For $\alpha, \beta \in \Sigma_2$ we write $m(\alpha, \beta) = \alpha\beta$. The groupoid multiplication is associative. For $(\alpha, \beta, \gamma) \in \Sigma_3$ the associativity condition $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ holds. The graph $\Gamma = \{(\alpha, \beta, \gamma) | (\beta, \gamma) \in \Sigma_2, \alpha = \beta\gamma\}$ of the groupoid multiplication (the product space) is a Lagrangian submanifold of $\Sigma \times \Sigma \times \Sigma$, where $\Sigma$ is a copy of the manifold $\Sigma$ endowed with the opposite symplectic structure.

The groupoid operations satisfy the following axioms. For any composable $\alpha, \beta \in \Sigma$ and $x \in M$

\begin{align*}
(A1) \quad & s(\alpha\beta) = s(\alpha), \quad (A2) \quad t(\alpha\beta) = t(\beta), \quad (A3) \quad s \circ \epsilon = \text{id}_M, \\
(A4) \quad & t \circ \epsilon = \text{id}_M, \quad (A5) \quad \epsilon(s(\alpha))\alpha = \alpha, \quad (A6) \quad \alpha \epsilon(t(\alpha)) = \alpha, \\
(A7) \quad & s(i(\alpha)) = t(\alpha), \quad (A8) \quad \alpha i(\alpha) = \epsilon(s(\alpha)), \quad (A9) \quad i(\alpha)\alpha = \epsilon(t(\alpha)).
\end{align*}

Recall the definition of the formal neighborhood $(X,Y)$ of a submanifold $Y$ of a manifold $X$. Let $Y$ be a closed $k$-dimensional submanifold of a real $n$-dimensional manifold $X$ and $I_Y \subset C^\infty(X)$ be the ideal of smooth functions on $X$ vanishing on $Y$. Then the quotient algebra $C^\infty(X,Y) := C^\infty(X)/I_X^\infty$, where $I_X^\infty = \cap_{l=1}^\infty I_Y^l$, can be thought of as the algebra of smooth functions on the formal neighborhood $(X,Y)$ of the submanifold $Y$ in $X$. If $U \subset X$ is a local coordinate chart on $X$ with coordinates $\{x^i\}$ such that $U \cap Y$ is given by the equations $x^{k+1} = 0, \ldots, x^n = 0$, then $C^\infty(U, U \cap Y)$ is isomorphic to $C^\infty(U \cap Y)[[x^{k+1}, \ldots, x^n]]$, where the isomorphism is established via the formal Taylor expansion of the functions on $U$ in the variables $x^{k+1}, \ldots, x^n$. Thus the formal neighborhood $(X,Y)$ of $Y \subset X$ is the ringed space $Y$ with the sheaf of rings whose global sections form the algebra $C^\infty(X,Y)$. Let $Y_i$ be a submanifold of a manifold $X_i$ for $i = 1,2$. If $f : X_1 \to X_2$ is a mapping such that $f(Y_1) \subset Y_2$, then $f^*(I_{Y_2}) \subset I_{Y_1}$. Therefore the mapping $f$ induces the dual morphism of algebras $f^* : C^\infty(X_2, Y_2) \to C^\infty(X_1, Y_1)$.

Denote by $\Lambda^n$ the Cartesian product of $n$ copies of the manifold $\Lambda$ and by $\Lambda_n$ the diagonal of $\Lambda^n$. Notice that $\Sigma_n \cap \Lambda^n = \Lambda_n$. The algebra $C^\infty(\Sigma)$ is a Poisson algebra with respect to the natural Poisson bracket $\{\cdot, \cdot\}_\Sigma$ on $\Sigma$. The space $C^\infty(\Sigma, \Lambda)$ inherits a structure of Poisson algebra from $C^\infty(\Sigma)$. We will use the same notation $\{\cdot, \cdot\}_\Sigma$ for the induced Poisson bracket on $C^\infty(\Sigma, \Lambda)$. Similarly, denote by $\{\cdot, \cdot\}_{\Sigma^n}$ the Poisson
bracket on $C^\infty(\Sigma^n)$ corresponding to the product Poisson structure, and the induced bracket on $C^\infty(\Sigma^n,\Lambda^n)$. Let $\iota : \Sigma_2 \to \Sigma \times \Sigma$ be the inclusion mapping. We will say that functions $F, G \in C^\infty(\Sigma)$ such that $m^* F = \iota^* G$ agree on $\Sigma_2$. The functions $F \in C^\infty(\Sigma)$ and $G \in C^\infty(\Sigma \times \Sigma)$ agree on $\Sigma_2$ if and only if the function $F \otimes 1 \otimes 1 - 1 \otimes G \in C^\infty(\Sigma \times \Sigma \times \Sigma)$ vanishes on the product space $\Gamma$. Since $\Gamma$ is a Lagrangian submanifold of $\Sigma \times \Sigma \times \Sigma$, the Poisson bracket of two functions vanishing on $\Gamma$ also vanishes on $\Gamma$. For functions $F_i \in C^\infty(\Sigma)$ and $G_i \in C^\infty(\Sigma \times \Sigma), i = 1, 2$, the Poisson bracket of $F_1 \otimes 1 \otimes 1 - 1 \otimes G_1$ and $F_2 \otimes 1 \otimes 1 - 1 \otimes G_2$ equals
\[
\{ F_1, F_2 \}_\Sigma \otimes 1 \otimes 1 - 1 \otimes \{ G_1, G_2 \}_\Sigma^2,
\]
whence we obtain the following lemma.

**Lemma 1.** If functions $F_i \in C^\infty(\Sigma)$ and $G_i \in C^\infty(\Sigma \times \Sigma), i = 1, 2$, agree on $\Sigma_2$, then the Poisson brackets $\{ F_1, F_2 \}_\Sigma$ and $\{ G_1, G_2 \}_\Sigma^2$ also agree on $\Sigma_2$.

The multiplication $m : \Sigma_2 \to \Sigma$ identifies $\Lambda_2$ with $\Lambda$ and thus induces the comultiplication mapping
\[
m^* : C^\infty(\Sigma, \Lambda) \to C^\infty(\Sigma_2, \Lambda_2).
\]
Denote by $\iota_n : \Sigma_n \to \Sigma^n$ the inclusion mapping. In particular, $\iota = \iota_2$. Since the mapping $\iota_n$ maps $\Lambda_n$ to $\Lambda^n$, it induces the algebra morphism
\[
\iota_n^* : C^\infty(\Sigma^n, \Lambda^n) \to C^\infty(\Sigma_n, \Lambda_n).
\]
We say that elements $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $C^\infty(\Sigma_2, \Lambda_2)$ if $m^* F = \iota^* G$ in $C^\infty(\Sigma_2, \Lambda_2)$. It follows from Lemma 1 that if $F_i \in C^\infty(\Sigma, \Lambda)$ agrees with $G_i \in C^\infty(\Sigma^2, \Lambda^2)$ on $C^\infty(\Sigma_2, \Lambda_2)$ for $i = 1, 2$, then $\{ F_1, F_2 \}_\Sigma$ agrees with $\{ G_1, G_2 \}_\Sigma^2$ on $C^\infty(\Sigma_2, \Lambda_2)$ as well. We will call this property of comultiplication *Property P* and use it in the definition of a formal symplectic groupoid. The mappings $s, t : \Sigma \to M$ induce the algebra morphisms
\[
S, T : C^\infty(M) \to C^\infty(\Sigma, \Lambda).
\]
The source mapping $S$ is a Poisson morphism and the target mapping $T$ is an anti-Poisson morphism. For any $f, g \in C^\infty(M)$ the elements $Sf, Tg \in C^\infty(\Sigma, \Lambda)$ Poisson commute. The unit mapping $\epsilon : M \to \Sigma$ identifies $M$ with $\Lambda$ and thus induces the algebra morphism $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$. Axioms (A3) and (A4) imply that
\[
(1) \quad E S = id_{C^\infty(M)} \text{ and } ET = id_{C^\infty(M)}.
\]
The inverse mapping $i : \Sigma \to \Sigma$ leaves fixed the elements of $\Lambda$ and therefore induces the antisymplectic involutive algebra morphism $I :
It follows from Axiom (A7) that

\[ IS = T. \]

To find the formal analogue of multiplication in a symplectic groupoid, we need a different description of the algebra \( C^\infty(\Sigma_n, \Lambda_n) \). For \( f \in C^\infty(M) \) introduce functions \( S^k_n f, T^k_n f \in C^\infty(\Sigma^n, \Lambda^n) \) by the following formulas:

\[
S^k_n f = 1 \otimes \ldots \otimes Sf \otimes \ldots \otimes 1, \quad T^k_n f = 1 \otimes \ldots \otimes Tf \otimes \ldots \otimes 1.
\]

Denote by \( \mathcal{I}_n \) the ideal in \( C^\infty(\Sigma^n, \Lambda_n) \) generated by the functions \( S^k_{n+1} f - T^k_n f, f \in C^\infty(M), \ 1 \leq k \leq n - 1 \). Taking into account that \( \Sigma_n \cap \Lambda^n = \Lambda_n \), we see that the inclusion of \( \Sigma_n \) into \( \Sigma^n \) induces the following exact sequence of algebras:

\[
0 \rightarrow \mathcal{I}_n \rightarrow C^\infty(\Sigma^n, \Lambda^n) \rightarrow C^\infty(\Sigma_n, \Lambda_n) \rightarrow 0,
\]

whence \( C^\infty(\Sigma_n, \Lambda_n) \) is canonically isomorphic to the quotient algebra \( C^\infty(\Sigma^n, \Lambda^n) / \mathcal{I}_n \). Denote

\[ \mathcal{E}_n := C^\infty(\Sigma^n, \Lambda^n) / \mathcal{I}_n. \]

Notice that \( \mathcal{E}_1 = C^\infty(\Sigma, \Lambda) \).

**Remark.** A formal neighborhood \((X, Y)\) is the simplest example of a formal manifold which, in general, should be defined as a ringed space on \( Y \). If a formal symplectic groupoid is defined as a formal neighborhood \((\Sigma, \Lambda)\), it is too restrictive to require the existence of the manifolds \( \Sigma_n \) for \( n \geq 2 \). This is why from now on we will automatically replace the algebra \( C^\infty(\Sigma_n, \Lambda_n) \) with \( \mathcal{E}_n \) for \( n \geq 2 \). In particular, we consider the comultiplication mapping \( m^* \) as a mapping from \( C^\infty(\Sigma, \Lambda) \) to \( \mathcal{E}_2 \) and the algebra morphism \( i^*_n \) as the quotient mapping from \( C^\infty(\Sigma^n, \Lambda^n) \) to \( \mathcal{E}_n \).

Axioms (A1) and (A2) imply the following identities in the algebra \( \mathcal{E}_2 \): for \( f \in C^\infty(M) \)

\[
m^* (Sf) = i^* (Sf \otimes 1)
\]

and

\[
m^* (Tf) = i^* (1 \otimes Tf),
\]

respectively.

In order to state the formal analogues of axioms (A5), (A6), (A8), and (A9) we need one more mapping. Denote by \( \delta : \Sigma \rightarrow \Sigma \times \Sigma \)

\[ C^\infty(\Sigma, \Lambda) \rightarrow C^\infty(\Sigma, \Lambda). \]
the diagonal inclusion of $\Sigma$. Since $\delta(\Lambda)$ is the diagonal of $\Lambda \times \Lambda$, the mapping $\delta$ induces the dual morphism

$$\delta^*: C^\infty(\Sigma^2, \Lambda^2) \to C^\infty(\Sigma, \Lambda).$$

In what follows $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $\mathcal{E}_2$, i.e., $m^*F = \iota^*G$ in $\mathcal{E}_2$. Axiom (A5) implies that $F(\alpha) = G(\epsilon(s(\alpha)), \alpha)$, whence

$$F = (\delta^* \circ (SE \otimes 1))G.$$  \hspace{1cm} (7)

Similarly, it follows from Axiom (A6) that

$$F = (\delta^* \circ (1 \otimes TE))G.$$  \hspace{1cm} (8)

Axioms (A8) and (A9) imply that

$$F = (\delta^* \circ (1 \otimes I))G$$ \hspace{1cm} (SE)F = (\delta^* \circ (1 \otimes I))G\text{ and } (TE)F = (\delta^* \circ (I \otimes 1))G,$$ respectively.

Now we need to state the formal analogue of the associativity of the groupoid multiplication. The mapping $m^* \otimes 1$ maps $F \in C^\infty(\Sigma^2, \Lambda^2)$ to a coset in $C^\infty(\Sigma^3, \Lambda^3)$ of the ideal generated by the functions $Tf \otimes 1 \otimes 1 - 1 \otimes Sf \otimes 1$. This ideal belongs to the ideal $\mathcal{I}_3$. Therefore the image of $F$ with respect to the mapping $m^* \otimes 1$ is a well defined element of $\mathcal{E}_3$. It can be checked using formula (6) that the homomorphism $m^* \otimes 1$ maps the ideal $\mathcal{I}_2$ to $\mathcal{I}_3$. This implies that the mapping $m^* \otimes 1$ induces a well defined mapping from $\mathcal{E}_2$ to $\mathcal{E}_3$, which we will denote $(m^*_1)^*$. Similarly, we construct the mapping $(m^*_2)^*: \mathcal{E}_2 \to \mathcal{E}_3$ induced by $1 \otimes m^*$. The associativity of the groupoid multiplication implies that

$$(m^*_1)^* \circ m^* = (m^*_2)^* \circ m^*.$$  \hspace{1cm} (10)

To define a formal symplectic groupoid over a Poisson manifold $M$, we begin with a collection of the following data: a symplectic manifold $\Sigma$, a Lagrangian manifold $\Lambda \subseteq \Sigma$, an embedding $\epsilon: M \to \Sigma$ such that $\epsilon(M) = \Lambda$, its dual $E: C^\infty(\Sigma, \Lambda) \to C^\infty(M)$, a Poisson morphism $S: C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ and an anti-Poisson morphism $T: C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ such that $Sf$ and $Tg$ Poisson commute for any $f, g \in C^\infty(M)$, and an involutive antisymplectic automorphism $I: C^\infty(\Sigma, \Lambda) \to C^\infty(\Sigma, \Lambda)$. For $f \in C^\infty(M)$ introduce the functions $S_k^hf, T_k^hf \in C^\infty(\Sigma^n, \Lambda^n)$ by formulas (3). For each $n$ define the ideal $I_n$ in $C^\infty(\Sigma^n, \Lambda^n)$ generated by the functions $S_k^hf - T_k^hf$, where $f \in C^\infty(M)$ and $1 \leq k \leq n - 1$, and the quotient algebra $\mathcal{E}_n = C^\infty(\Sigma^n, \Lambda^n)/I_n$ as above. Denote by $\iota_n^*: C^\infty(\Sigma^n, \Lambda^n) \to \mathcal{E}_n$ the quotient mapping. There should exist a comultiplication mapping $m^*: C^\infty(\Sigma, \Lambda) \to \mathcal{E}_2$ which has Property P and satisfies the formal
analyses of axioms (A1) - (A9) given by formulas (5), (6), (1), (7), (8), (2), and (9), respectively. It should generate the mappings 

$$(m_1^2)^*, (m_2^2)^*: \mathcal{E}_2 \to \mathcal{E}_3$$

as above so that the coassociativity condition (10) is satisfied. In what follows we will refer to the formal analogues of axioms (A1) - (A9) as to axioms (FA1) - (FA9).

3. Formal symplectic realization of a Poisson manifold

If $\Sigma$ is a symplectic manifold, $M$ a Poisson manifold, $s : \Sigma \to M$ a surjective submersion which is a Poisson mapping, and $\epsilon : M \to \Sigma$ an embedding such that $s \circ \epsilon = \text{id}_M$ and $\Lambda = \epsilon(M) \subset \Sigma$ is a Lagrangian manifold, then $\Sigma$ is called a strict symplectic realization of the Poisson manifold $M$ (see [6]). It is known that, given a strict symplectic realization $\Sigma$ of the Poisson manifold $M$, there exists a canonical local symplectic groupoid over the manifold $M$ defined on a neighborhood of $\Lambda$ in $\Sigma$, such that $s$ is its source mapping ([6], Thm. 1.2 on page 44).

In this section we will prove a formal version of this theorem. Let $\Sigma$ be a symplectic manifold, $M$ a Poisson manifold, and $\epsilon : M \to \Sigma$ an embedding such that $s \circ \epsilon = \text{id}_M$ and $\Lambda = \epsilon(M) \subset \Sigma$ is a Lagrangian manifold. Denote by $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$ the dual mapping of $\epsilon$. Then, if there is given a formal Poisson morphism $S : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ such that $ES = \text{id}_{C^\infty(M)}$, we say that the formal neighborhood $(\Sigma, \Lambda)$ is a formal strict symplectic realization of the Poisson manifold $M$.

**Theorem 1.** Given a formal strict symplectic realization of a Poisson manifold $M$ on the formal neighborhood $(\Sigma, \Lambda)$ of a Lagrangian submanifold $\Lambda$ of a symplectic manifold $\Sigma$ via a formal Poisson morphism $S : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$, there exists a unique formal symplectic groupoid on $(\Sigma, \Lambda)$ over the manifold $M$ such that $S$ is its source mapping.

Assume there is a formal strict symplectic realization $(\Sigma, \Lambda)$ of the Poisson manifold $M$. Given an element $F \in C^\infty(\Sigma, \Lambda)$, denote by $H_F = \{F, \cdot\}_\Sigma$ the formal Hamiltonian vector field corresponding to the formal Hamiltonian $F$. Denote by $\lambda$ the representation of the Lie algebra $\mathfrak{g} := (C^\infty(M), \{\cdot, \cdot\}_M)$ on the space $C^\infty(\Sigma, \Lambda)$ given by the formula

$$\lambda(f) = H_{Sf},$$
where $f \in \mathfrak{g}$. Extend the representation $\lambda$ to the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ and define a mapping

$$\langle F \rangle : U(\mathfrak{g}) \to C^\infty(M)$$

by the formula

$$\langle F \rangle (u) = E(\lambda(u)F),$$

where $u \in U(\mathfrak{g})$. Denote the multiplication in the algebra $U(\mathfrak{g})$ by $\bullet$, so that $f \bullet g - g \bullet f = \{f, g\}_M$ for $f, g \in \mathfrak{g}$. We will often work in the following local framework on $\Sigma$. Let $U$ be a Darboux chart on $\Sigma$ with the local coordinates $\{x^k, \xi_l\}$ such that $\Lambda \cap U$ is given by the equations $\xi = 0$ and for $F, G \in C^\infty(U)$

$$\{F, G\}_\Sigma = \partial^k F \partial_k G - \partial^k G \partial_k F,$$

(11)

where $\partial_k = \partial/\partial x^k$ and $\partial^i = \partial/\partial \xi_i$. We will say that these Darboux coordinates are standard. In this framework a formal function $F \in C^\infty(\Sigma, \Lambda)$ will be represented on the formal neighborhood $(U, \Lambda \cap U)$ as an element of $C^\infty(\Lambda \cap U) \llbracket \xi \rrbracket$ and the coordinates $\xi_i$ will be treated as formal variables. We identify $M$ and $\Lambda$ via the mapping $\epsilon$, so that $\{x^k\}$ are used also as coordinates on $\epsilon^{-1}(\Lambda \cap U)$. In particular, for $F = F(x, \xi)$ we have $E(F)(x) = F(x, 0)$. A local expression for the Poisson bracket on $M$ is

$$\{f, g\}_M = \eta^{ij} \partial_i f \partial_j g,$$

(12)

where $f, g \in C^\infty(M)$.

**Lemma 2.** Given a function $f \in C^\infty(M)$, the element $Sf \in C^\infty(\Sigma, \Lambda)$ can be written in standard local coordinates $(x, \xi)$ on a Darboux chart $U \subset \Sigma$ as

$$Sf(x, \xi) = f(x) + \alpha^{ij}(x) \partial_i f \xi_j \pmod{\xi^2}$$

(13)

for some function $\alpha^{ij}(x)$ such that $\alpha^{ij} - \alpha^{ji} = \eta^{ij}$.

**Proof.** Denote $s^i = Sx^i$. Since $E(s^i) = x^i$, expanding $s^i(x, \xi)$ with respect to the formal variables $\xi$ we get that $s^i = x^i + \alpha^{ij}(x)\xi_j \pmod{\xi^2}$ for some function $\alpha^{ij}(x)$. It follows from the fact that $S$ is an algebra morphism, that

$$Sf(x, \xi) = f(s(x, \xi)) = f(x) + \alpha^{ij}(x) \partial_i f \xi_j \pmod{\xi^2}.$$

Notice that the ‘substitution’ $f(s(x, \xi))$ is understood as a composition of formal series. Since $S$ is a Poisson morphism, we have that $\{Sf, Sg\}_\Sigma = S(\{f, g\}_M)$ for any $f, g \in C^\infty(M)$. On the one hand, according to formulas (11) and (13),

$$\partial^k(Sf) \partial_k(Sg) - \partial^k(Sf) \partial_k(Sg) = \alpha^{jk} \partial_j f \partial_k g - \alpha^{jk} \partial_j g \partial_k f \pmod{\xi}.$$
On the other hand, \( S(\{ f, g \}_M) = \eta^{ij} \partial_i f \partial_j g \pmod{\xi} \), which concludes the proof. \( \Box \)

**Lemma 3.** For any \( F \in C^\infty(\Sigma, \Lambda) \) and \( u \in U(\mathfrak{g}) \) the mapping \( C^\infty(M) \ni f \mapsto \langle F \rangle(f \cdot u) \) is a derivation on \( C^\infty(M) \).

**Proof.** Let us show that the mapping \( C^\infty(M) \ni f \mapsto \chi(f \cdot u) \) is a derivation. Using Lemma 2 and formula (13), we obtain that in local Darboux coordinates

\[
E(H_Sf) = E(\{Sf, F\}_\Sigma) = E(\partial^k(Sf)\partial_k F - \partial^k F\partial_k(Sf)) = \alpha^{ik}\partial_i f E(\partial_k F) - \partial_k f E(\partial^k F).
\]

(14)

To prove the statement of the Lemma, the element \( F \) should be replaced with \( \lambda(u) F \). \( \Box \)

Denote by \( C \) the space of linear mappings \( C : U(\mathfrak{g}) \to C^\infty(M) \) such that for any \( u \in U(\mathfrak{g}) \) the mapping \( C^\infty(M) \ni f \mapsto \chi(f \cdot u) \) is a derivation on \( C^\infty(M) \). Lemma 3 implies that the mapping

\[
\chi : F \mapsto \langle F \rangle
\]

maps \( C^\infty(\Sigma, \Lambda) \) to \( C \). We will prove that the mapping \( \chi : C^\infty(\Sigma, \Lambda) \to C \) is actually a bijection. Each element \( C \in C \) is completely determined by the family of polydifferential operators \( \{C_n\}, n \geq 0, \) on \( M \), where \( C_n \) is the \( n \)-differential operator such that

\[
C_n(f_1, \ldots, f_n) = C(f_1 \cdot \ldots \cdot f_n).
\]

(15)

The operators \( \{C_n\} \) enjoy the following two properties.

**Property A.** Each operator \( C_n, n \geq 0, \) is a derivation in the first argument.

**Property B.** For any \( k, n \) such that \( 1 \leq k \leq n - 1 \)

\[
C_n(f_1, \ldots, f_k, f_{k+1}, \ldots, f_n) - C_n(f_1, \ldots, f_{k+1}, f_k, \ldots, f_n) = \sum_{i=1}^{k} C_{n-1}(f_1, \ldots, \{f_i, f_{k+1}\}_M, \ldots, f_n).
\]

(16)

We will call a family \( \{C_n(f_1, \ldots, f_n)\}, n \geq 0, \) of polydifferential operators on \( M \) coherent if it has Properties A and B. The correspondence \( C \mapsto \{C_n\} \) given by formula (15) is a bijection between the space \( C \) and the set of all coherent families.

It is easy to show that each operator \( C_n \) from a coherent family annihilates constants (i.e., \( C_n(f_1, \ldots, f_n) = 0 \) if \( f_k = 1 \) for at least one index \( k \)) and is of order not greater than \( k \) in the \( k \)th argument for \( 1 \leq k \leq n \).

It is important to notice that if \( \{C_n\}, n \geq 0, \) is a coherent family on \( M \) and \( \phi \in C^\infty(M) \), then the operators \( \{\phi \cdot C_n\}, n \geq 0, \) also form a
coherent family. This observation means that one can apply partition of unity arguments to the coherent families.

The standard increasing filtration on the universal enveloping algebra $\mathcal{U}(g)$ induces the dual decreasing filtration $\{C^{(n)}\}$ on $\mathcal{C}$, i.e., $C^{(n)}$ consists of all operators $C$ such that the corresponding coherent family $\{C_k\}$ satisfies the condition $C_k = 0$ for $0 \leq k \leq n - 1$. The following lemma is an immediate consequence of Properties A and B of the coherent families.

**Lemma 4.** If $C \in C^{(n)}$, then $C_n(f_1, \ldots, f_n)$ is a symmetric multiderivation on $M$ (i.e., of order one in each argument and null on constants).

We will also consider finite coherent families $\{C_k\}, 0 \leq k \leq n$. It turns out that any $n$-element coherent family can be extended to an $(n+1)$-element coherent family.

**Theorem 2.** Any $n$-element coherent family $\{C_k\}, 0 \leq k \leq n - 1$, can be extended to an $(n+1)$-element coherent family $\{C_k\}, 0 \leq k \leq n$. The operator $C_n$ is unique up to an arbitrary symmetric multiderivation.

We will prove Theorem 2 in the Appendix.

Given an element $F \in C^\infty(\Sigma, \Lambda)$, set $C = \langle F \rangle = \chi(F)$. We will denote the corresponding operator $C_n$ by $\langle F \rangle_n$, so that

$$\langle F \rangle_n(f_1, \ldots, f_n) = E(HS_{f_1} \cdots HS_{f_n} F),$$

where $f_i \in C^\infty(M)$. Denote by $\mathcal{J} = I_\Lambda/I_\Lambda^\infty$ the kernel of the mapping $E : C^\infty(\Sigma, \Lambda) \to C^\infty(M)$, i.e., the ideal of formal functions on $(\Sigma, \Lambda)$ vanishing on $\Lambda$. The powers of this ideal, $\{\mathcal{J}^n\}$, form a decreasing filtration on the algebra $C^\infty(\Sigma, \Lambda)$. Consider an element $F \in \mathcal{J}^n$. The operator $\langle F \rangle_n$ vanishes if $k < n$, therefore $\langle F \rangle \in C^{(n)}$. Thus the mapping $\chi : C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ is a morphism of filtered spaces. Notice that the filtrations on $C^\infty(\Sigma, \Lambda)$ and $\mathcal{C}$ are complete and separated.

**Lemma 5.** Let $F$ be an arbitrary element in $\mathcal{J}^n$. Then

$$\langle F \rangle_n(f_1, \ldots, f_n) = E(HS_{f_1} \cdots HS_{f_n} F),$$

$f_i \in C^\infty(M)$, is a symmetric multiderivation which does not depend on the choice of the source mapping $S$. The mapping $\chi : F \mapsto \langle F \rangle_n$ induces an isomorphism of $\mathcal{J}^n/\mathcal{J}^{n+1}$ onto the space of symmetric $n$-derivations on $M$.

**Proof.** In standard local Darboux coordinates $(x^k, \xi_i)$ on $\Sigma$ a function $F \in \mathcal{J}^n$ can be written as $F(x, \xi) = F^{i_1\cdots i_n}(x)\xi_{i_1} \cdots \xi_{i_n} \pmod{\xi^{n+1}}$, where $F^{i_1\cdots i_n}(x)$ is symmetric in $i_1, \ldots, i_n$. Taking into account formula (11) and that $Sf = f \pmod{\xi}$, we get that

$$\langle F \rangle_n(f_1, \ldots, f_n) = E(HS_{f_1} \cdots HS_{f_n} F) = (-1)^n n! F^{i_1\cdots i_n} \partial_{i_1} f_1 \cdots \partial_{i_n} f_n.$$
This calculation shows that $F^{i_1\ldots i_n}(x)$ is a symmetric tensor which does not depend on the choice of the source mapping $S$ and that the mapping $\chi : F \mapsto \langle F \rangle_n$ induces an isomorphism of $\mathcal{J}^n/\mathcal{J}^{n+1}$ onto the space of symmetric $n$-derivations on $M$. □

Remark. One can describe the tensor $F^{i_1\ldots i_n}(x)$ (and the corresponding multiderivation) independently, regardless the existence of the source mapping $S$. The description is based upon the identification of the conormal bundle of $\Lambda \subset \Sigma$ with its tangent bundle $T\Lambda$.

Proposition 1. The mapping $\chi : C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ is a bijection.

Proof. The mapping $\chi$ is a morphism of complete Hausdorff filtered spaces. According to Theorem 2, the quotient space $\mathcal{C}(n)/\mathcal{C}(n+1)$ is isomorphic to the space of symmetric $n$-derivations on $M$, the isomorphism being induced by the mapping $\mathcal{C}(n) \ni C \mapsto C_n$. Lemma 5 thus shows that the mapping $\chi$ induces an isomorphism of $\mathcal{J}^n/\mathcal{J}^{n+1}$ onto $\mathcal{C}(n)/\mathcal{C}(n+1)$, whence the Proposition follows. □

Using Proposition 1 we will transfer the structure of Poisson algebra from $C^\infty(\Sigma, \Lambda)$ to $\mathcal{C}$ via the mapping $\chi$. It turns out that the resulting Poisson algebra structure on $\mathcal{C}$ does not depend on the mapping $S$ and can be described canonically and intrinsically in terms of the Poisson structure on $M$.

Denote by $\delta_\mathcal{U} : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ the standard cocommutative coproduct on $\mathcal{U}(\mathfrak{g})$, so that $\delta_\mathcal{U}(f) = f \otimes 1 + 1 \otimes f$ for $f \in \mathfrak{g}$.

For $F, G \in C^\infty(\Sigma, \Lambda)$, $u \in \mathcal{U}(\mathfrak{g})$, we have

$$
\langle FG \rangle(u) = E(\lambda(u)(FG)) = \sum_i E(\lambda(u_i')F\lambda(u_i'')G) = \sum_i E(\lambda(u_i')F)E(\lambda(u_i'')G) = \sum_i \langle F \rangle(u_i')\langle G \rangle(u_i'').
$$

(17)

Here, as well as in the rest of the paper, we use the notation

$$
\delta_\mathcal{U}(u) = \sum_i u_i' \otimes u_i''.
$$

For $A, B \in \mathcal{C}$ denote by $AB$ their convolution product, so that

$$
(AB)(u) = \sum_i A(u_i')B(u_i'').
$$

(18)

This product is commutative since $\delta_\mathcal{U}$ is cocommutative. Formula (17) shows that the mapping $\chi$ is an algebra isomorphism from $C^\infty(\Sigma, \Lambda)$ to $\mathcal{C}$ endowed with the convolution product.
For $F, G \in C^\infty(\Sigma, \Lambda)$ we obtain by setting $f = E(G)$ in formula (14) that
\begin{equation}
E(H_{(SE)(G)}F) = \alpha^{ik}E(\partial_iG)E(\partial_kF) - E(\partial_kF)E(\partial_iG).
\end{equation}
Swapping $F$ and $G$ in (19) and subtracting the resulting equation from (19) we get, taking into account formulas (11), (12), and Lemma 2, that
\begin{equation}
E(\{F, G\}_\Sigma) = E(H_{(SE)(F)}G) - E(H_{(SE)(G)}F) - \{E(F), E(G)\}_M.
\end{equation}
For $u, v \in U(\mathfrak{g})$
\begin{equation}
E(H_{(SE)(\lambda(u))}\lambda(v)G) = E(H_{S((F)(u))}\lambda(v)G) = E(\lambda(\langle F\rangle(u))\lambda(v)G) =
\end{equation}
\begin{equation}
E(\lambda(\langle F\rangle(u) \bullet v)G) = \langle G\rangle(\langle F\rangle(u) \bullet v).
\end{equation}
Using the Jacobi identity, formulas (20) and (21), we obtain that
\begin{equation}
\langle\{F, G\}_\Sigma\rangle(u) = E(\lambda(u)\{F, G\}_\Sigma) = 
\sum_i E(\{\lambda(u_i')F, \lambda(u''_i)G\}_\Sigma) =
\sum_i (\langle G\rangle(\langle F\rangle(u_i') \bullet u''_i) - \langle F\rangle(\langle G\rangle(u''_i) \bullet u_i') - \{\langle F\rangle(u_i'), \langle G\rangle(u''_i)\}_M).
\end{equation}
Notice that in formula (22) the functions $\langle F\rangle(u_i'), \langle G\rangle(u''_i) \in C^\infty(M)$ are used as elements of the Lie algebra $\mathfrak{g}$. Formula (22) shows that the mapping $\chi$ transfers the Poisson bracket from $C^\infty(\Sigma, \Lambda)$ to the following Poisson bracket on $\mathcal{C}$:
\begin{equation}
\{A, B\}_\mathcal{C}(u) = \sum_i \left( B(A(u_i') \bullet u''_i) - A(B(u''_i) \bullet u_i') - \{A(u_i'), B(u''_i)\}_M \right),
\end{equation}
where $A, B \in \mathcal{C}$. We see that the bracket (23) is defined intrinsically in terms of the Poisson structure on $M$. One can prove that the bracket (23) defines a Poisson algebra structure on the algebra $\mathcal{C}$ regardless the existence of the mapping $S$.

Now we can construct an anti-Poisson morphism $T : C^\infty(M) \to C^\infty(\Sigma, \Lambda)$ such that $ET = id_{C^\infty(M)}$ and the formal functions $Sf$ and $Tg$ Poisson commute for any $f, g \in C^\infty(M)$.

Denote by $\epsilon_U : U(\mathfrak{g}) \to \mathbb{C}$ the counit mapping of the algebra $U(\mathfrak{g})$, so that $\epsilon_U(1) = 1$ and $\epsilon_U(f) = 0$ for $f \in \mathfrak{g}$. Here $1$ is the unity in the algebra $U(\mathfrak{g})$. Let $k$ denote the trivial representation of the algebra $U(\mathfrak{g})$ on $C^\infty(M)$, i.e., such that
\begin{equation}
k(u)f = \epsilon_U(u) \cdot f
\end{equation}
for \( u \in \mathcal{U}(\mathfrak{g}), f \in C^\infty(M) \). For \( f \in C^\infty(M) \) consider a mapping \( X_f \in \mathcal{C} \) such that

\[
X_f(u) = k(u)f,
\]

where \( u \in \mathcal{U}(\mathfrak{g}) \). For \( f, g \in C^\infty(M) \) and \( u \in \mathcal{U}(\mathfrak{g}) \) we get from formula (23):

\[
\{X_f, X_g\}_{C}(u) = - \left( \sum_i \epsilon_\mathcal{U}(u_i') \cdot \epsilon_\mathcal{U}(u_i'') \right) \{f, g\}_M = -\epsilon_\mathcal{U}(u)f - \epsilon_\mathcal{U}(u)\{f, g\}_M = -X_{\{f, g\}}(u).
\]

Thus the mapping \( f \mapsto X_f \) is an anti-Poisson morphism from \( C^\infty(M) \) to \( \mathcal{C} \).

Let \( h \) denote the representation of the Lie algebra \( \mathfrak{g} \) on \( C^\infty(M) \) by the Hamiltonian vector fields, \( h(f) = \{f, \cdot\}_M, f \in \mathfrak{g} \). Extend it to \( \mathcal{U}(\mathfrak{g}) \). It follows from the fact that \( ES = \text{id}_{C^\infty(M)} \), that

\[
\langle Sf \rangle(u) = h(u)f.
\]

For \( f, g \in C^\infty(M) \) we get from formula (23) that \( \langle Sf \rangle \) Poisson commutes with \( X_g \):

\[
\{\langle Sf \rangle, X_g\}_{C}(u) = \sum_i (-\epsilon_\mathcal{U}(u_i''') h(g \cdot u_i') f - \epsilon_\mathcal{U}(u_i'') \{h(u_i')f, g\}_M) = \sum_i (-\epsilon_\mathcal{U}(u_i''') h(g) h(u_i') f + \epsilon_\mathcal{U}(u_i'') \{g, h(u_i')f\}_M) = 0.
\]

Taking into account that the mapping \( \chi \) is a Poisson algebra isomorphism of \( C^\infty(\Sigma, \Lambda) \) onto \( \mathcal{C} \), we define the mapping \( T : C^\infty(M) \to C^\infty(\Sigma, \Lambda) \) as follows. For \( f \in C^\infty(M) \) \( Tf \) is chosen to be the unique element of \( C^\infty(\Sigma, \Lambda) \) such that

\[
\langle Tf \rangle = X_f.
\]

We see that the mapping \( T : C^\infty(M) \to C^\infty(\Sigma, \Lambda) \) is an anti-Poisson morphism and for any \( f, g \in C^\infty(M) \) the formal functions \( Sf \) and \( Tg \) Poisson commute. Thus the mapping \( T \) enjoys the properties of the target mapping. On the other hand, if \( T \) is the target mapping of a formal symplectic groupoid on \( (\Sigma, \Lambda) \) whose source mapping is \( S \), it is straightforward that

\[
\langle Tf \rangle(u) = k(u)f.
\]

It means that the target mapping \( T \) is uniquely determined by the source mapping \( S \).

In order to construct the inverse mapping \( I \) and the comultiplication \( m^* \) from the mappings \( S \) and \( T \), we will consider mappings from tensor powers of \( \mathcal{U}(\mathfrak{g}) \) to \( C^\infty(M) \) which generalize the mappings from \( \mathcal{C} \). The
space $\text{Hom}(\mathcal{U}(\mathfrak{g})^\otimes n, C^\infty(M))$ is endowed with the convolution product defined on its elements $A, B$ as follows:

$$(AB)(u_1 \otimes \ldots \otimes u_n) = \sum_{i_1, \ldots, i_n} A(u'_{i_1} \otimes \ldots \otimes u'_{n_{i_n}})B(u''_{i_1} \otimes \ldots \otimes u''_{n_{i_n}}),$$

where

$$\delta_u(u_k) = \sum_i u'_{ki} \otimes u''_{ki}.$$

Denote by $\{\cdot, \cdot\}_\Sigma$ the Poisson bracket on $C^\infty(\Sigma^n)$ (and on $C^\infty(\Sigma^n, \Lambda^n)$) corresponding to the product Poisson structure. For $F \in C^\infty(\Sigma^n, \Lambda^n)$ let $H_F = \{F, \cdot\}_\Sigma$ denote the corresponding formal Hamiltonian vector field on $(\Sigma^n, \Lambda^n)$. Introduce representations $\lambda_n^k$, $0 \leq k \leq n$, of the Lie algebra $\mathfrak{g}$ on $C^\infty(\Sigma^n, \Lambda^n)$ by the following formulas:

$$\lambda_n^0(f) = H_{S_n f}, \quad \lambda_n^1(f) = -H_{T_n f}, \quad \text{and} \quad \lambda_n^k(f) = H_{(s_n^{k+1} - T_n^k)f},$$

for $1 \leq k \leq n - 1$, where the functions $S_n^k, T_n^k \in C^\infty(\Sigma^n, \Lambda^n)$ are given by formulas (3). These representations pairwise commute. Notice that in these notations the representation $\lambda$ is denoted $\lambda_0^1$. Denote the representation $\lambda_1^1$ by $\rho$ so that $\rho(f) = -H_{T f}$ for $f \in \mathfrak{g}$. Extend the representations $\lambda_n^k$ to the algebra $\mathcal{U}(\mathfrak{g})$. For $u \in \mathcal{U}(\mathfrak{g})$

$$\lambda_n^0(u) = \lambda(u) \otimes 1 \otimes \ldots \otimes 1, \quad \lambda_n^1(u) = 1 \otimes \ldots \otimes 1 \otimes \rho(u),$$

and

$$\lambda_n^k(u) = \sum_{i} 1 \otimes \ldots \otimes \rho(u'_i) \otimes \lambda(u''_i) \otimes \ldots \otimes 1,$$

where $1 \leq k \leq n - 1$. Let $\epsilon_n : M \to \Sigma^n$ denote the composition of the identification mapping from $M$ onto $\Lambda_n$ with the inclusion of $\Lambda_n$ into $\Sigma^n$. Since $\Lambda_n \subset \Lambda^n$, the mapping $\epsilon_n$ induces the algebra morphism $E_n : C^\infty(\Sigma^n, \Lambda^n) \to C^\infty(M)$. In particular, $\epsilon = \epsilon_1$ and $E = E_1$. After some preparations we will show that the morphism $E_n$ intertwines the representations $\mathfrak{h}$ and $\sum_{k=0}^n \lambda_n^k$.

We cover the submanifold $\Lambda^n \subset \Sigma^n$ by Cartesian products $U_1 \times \ldots \times U_n$ of standard Darboux charts $U_i \subset \Sigma$ and use the coordinates $\{x^{[k]}_i, \xi^{[k]}_j\}$ on the $k$-th factor. In particular, in local coordinates $S_n^k f = (S f)(x^{[k]}_i, \xi^{[k]}_j)$ and $T_n^k f = (T f)(x^{[k]}_i, \xi^{[k]}_j)$. For a function $F = F(x^{[1]}_i, \xi^{[1]}_j, \ldots, x^{[n]}_i, \xi^{[n]}_j)$ on $U_1 \times \ldots \times U_n$ we have $E_n(F) =$
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$F(x, 0, \ldots, x, 0)$. We will use below the following obvious formulas,

$$E_n(f(x_k)F) = f(x)E_n(F) \text{ and } \partial_i E_n(F) = \sum_{k=1}^{n} E_n(\partial_{i[k]} F),$$

where $\partial_{i[k]} = \partial / \partial x_{i[k]}$. It can be proved as in Lemma 2 that in local Darboux coordinates $(x, \xi)$

$$T f(x, \xi) = f(x) + \alpha^{ij}(x) \partial_i f \xi_j \pmod{\xi^2},$$

where the function $\alpha^{ij}(x)$ is the same as in formula (13).

**Lemma 6.** For $f \in C^\infty(M)$ and $F \in C^\infty(\Sigma^n, \Lambda^n)$

$$h(f) E_n(F) = \sum_{k=0}^{n} E_n(\lambda_n^k(f) F).$$

**Proof.** Using formulas (11),(12), (13),(29), and Lemma 2 we get:

$$E(H_{(Sf - T f)} F) = E(\{S f - T f, F\}_\Sigma) = E(\partial^k(\eta^{ij} \partial_i f \xi_j) \partial_k F) = h(h) E(F).$$

Now the Lemma follows from formulas (28) and the fact that

$$\sum_{k=0}^{n} \lambda_n^k(f) = \sum_{k=1}^{n} H_{(Sf - T f)}.$$

$\square$

Denote by $C_n$ the subspace of $\text{Hom}(U(\mathfrak{g})^{\otimes(n+1)}, C^\infty(M))$ of the mappings $C$ such that

- for any $C \in C_n$, $u_i \in U(\mathfrak{g}), 0 \leq i \leq n$, and $k$ satisfying $0 \leq k \leq n$, the mapping

  $$C^\infty(M) \ni f \mapsto C(u_0 \otimes \ldots \otimes f \bullet u_k \otimes \ldots \otimes u_n)$$

  is a derivation on $C^\infty(M)$; and

- for any $f \in C^\infty(M)$

$$h(f) C(u_0 \otimes \ldots \otimes u_n) = \sum_{k=0}^{n} C(u_0 \otimes \ldots \otimes f \bullet u_k \otimes \ldots \otimes u_n).$$

The space $C_n$ is closed under the convolution product and thus is an algebra. For an element $F \in C^\infty(\Sigma^n, \Lambda^n)$ define a mapping

$$\langle\langle F \rangle\rangle \in \text{Hom}(U(\mathfrak{g})^{\otimes(n+1)}, C^\infty(M))$$

such that

$$\langle\langle F \rangle\rangle(u_0 \otimes \ldots \otimes u_n) = E_n(\lambda_n^0(u_0) \ldots \lambda_n^n(u_n) F).$$
A straightforward generalization of the proof of Lemma 3 shows that for each $k$ satisfying $0 \leq k \leq n$ the mapping 

$$C^\infty(M) \ni f \mapsto \langle\langle F \rangle\rangle(u_0 \otimes \cdots \otimes f \cdot u_k \otimes \cdots \otimes u_n)$$

is a derivation on $C^\infty(M)$. It follows from Lemma 6 that the mapping $C = \langle\langle F \rangle\rangle$ satisfies formula (30). Thus the mapping 

$$C^\infty(\Sigma^n, \Lambda^n) \ni F \mapsto \langle\langle F \rangle\rangle$$

maps $C^\infty(\Sigma^n, \Lambda^n)$ to $C_n$. Denote this mapping by $\chi_n$. A simple calculation shows that $\chi_n : C^\infty(\Sigma^n, \Lambda^n) \rightarrow C_n$ is an algebra homomorphism.

Denote by $\tilde{\mathcal{C}}_n$ the subspace of $\text{Hom}(\mathcal{U}(\mathfrak{g}) \otimes \Sigma^n, C^\infty(M))$ consisting of the elements $C \in \tilde{\mathcal{C}}_n$ such that for any $u_i \in \mathcal{U}(\mathfrak{g}), 1 \leq i \leq n$, and $k$ satisfying $1 \leq k \leq n$, the mapping

$$C^\infty(M) \ni f \mapsto C(u_1 \otimes \cdots \otimes f \cdot u_k \otimes \cdots \otimes u_n)$$

is a derivation on $C^\infty(M)$. Notice that in these notations $\mathcal{C} = \tilde{\mathcal{C}}_1$. The space $\tilde{\mathcal{C}}_n$ is also an algebra with respect to the convolution product.

Consider a reduction mapping $C \mapsto \tilde{C}$ from $\mathcal{C}_n$ to $\tilde{\mathcal{C}}_n$ defined as follows:

$$\tilde{C}(u_1 \otimes \cdots \otimes u_n) = C(u_1 \otimes \cdots \otimes u_n \otimes 1),$$

where $1$ is the unity in the algebra $\mathcal{U}(\mathfrak{g})$ (which should not be confused with the unit constant $1 \in \mathfrak{g}$). Formula (30) implies that the reduction mapping $C \mapsto \tilde{C}$ is a bijection of $\mathcal{C}_n$ onto $\tilde{\mathcal{C}}_n$. It is easy to check that the reduction mapping $C \mapsto \tilde{C}$ is an algebra isomorphism of $\mathcal{C}_n$ onto $\tilde{\mathcal{C}}_n$. A straightforward calculation shows that the reduction mapping pulls back the Poisson bracket (23) on $\mathcal{C} = \tilde{\mathcal{C}}_1$ to the Poisson bracket $\{\cdot, \cdot\}_{C_1}$ on $\mathcal{C}_1$ defined as follows. For $A, B \in \mathcal{C}_1$ and $u, v \in \mathcal{U}(\mathfrak{g})$

$$\{A, B\}_{C_1}(u \otimes v) = -\sum_{i,j} \left( A(B(u^{''}_i \otimes v'_j) \cdot u'_i) \otimes v'_j \right) + B(u^{''}_i \otimes (A(u'_i \otimes v'_j) \cdot v''_j)), \tag{32}$$

where

$$\delta_U(u) = \sum_i u'_i \otimes u^{''}_i \text{ and } \delta_U(v) = \sum_j v'_j \otimes v''_j.$$ 

Recall that in (32) the functions $A(u'_i \otimes v'_j), B(u^{''}_i \otimes v''_j) \in C^\infty(M)$ are treated as elements of the Lie algebra $\mathfrak{g}$. The right-hand side of formula (32) is skew-symmetric due to formula (30) and cocommutativity of the coproduct $\delta_U$.

For $F \in C^\infty(\Sigma, \Lambda)$ the mapping $\langle\langle F \rangle\rangle \in \mathcal{C}_1$ such that

$$\langle\langle F \rangle\rangle(u \otimes v) = E(\lambda(u) \rho(v) F)$$
for \( u, v \in U(\mathfrak{g}) \) is completely determined by its reduction \( \langle F \rangle(u) = E(\lambda(u)F) \). Thus the mapping \( \chi : C^\infty(\Sigma, \Lambda) \to C \) is a Poisson algebra isomorphism. This isomorphism will be used to introduce the inverse mapping \( I \) on \( C^\infty(\Sigma, \Lambda) \) in the most transparent way.

A simple calculation shows that for \( f \in C^\infty(M) \) and \( u, v \in U(\mathfrak{g}) \)

\[
\langle \langle Sf \rangle \rangle(u \otimes v) = h(u)k(v)f \quad \text{and} \quad \langle \langle Tf \rangle \rangle(u \otimes v) = h(v)k(u)f.
\]

Given a mapping \( C : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to C^\infty(M) \), denote by \( C^\dagger \) the mapping from \( U(\mathfrak{g}) \otimes U(\mathfrak{g}) \) to \( C^\infty(M) \) such that

\[
C^\dagger(u \otimes v) = C(v \otimes u)
\]

for \( u, v \in U(\mathfrak{g}) \). It is easy to check that the mapping \( C \mapsto C^\dagger \) leaves invariant the space \( C^1 \). Formulas (33) indicate that

\[
\langle \langle Sf \rangle \rangle^\dagger = \langle \langle Tf \rangle \rangle.
\]

Using formulas (26) for \( n = 2 \) and (32) one can readily show that the mapping \( C \mapsto C^\dagger \) induces an involutive anti-Poisson automorphism of the Poisson algebra \( C_1 \). Define a unique mapping \( I \) on \( C^\infty(\Sigma, \Lambda) \) such that

\[
\langle \langle I(F) \rangle \rangle(u \otimes v) = \langle \langle F \rangle \rangle(v \otimes u).
\]

It follows that the mapping \( I \) is an involutive anti-Poisson automorphism of \( C^\infty(\Sigma, \Lambda) \) such that

\[
IS = T \quad \text{and} \quad IT = S.
\]

Now assume that \( I \) is the inverse mapping of a formal symplectic groupoid on \((\Sigma, \Lambda)\) over \( M \) with the source mapping \( S \) (and target mapping \( T \)). Then for \( f \in C^\infty(M) \) and \( F \in C^\infty(\Sigma, \Lambda) \)

\[
I(\lambda(f)F) = \{ Sf, F \}_\Sigma = \{ ISf, I(F) \}_\Sigma = \{ Tf, I(F) \}_\Sigma = \rho(f)I(F).
\]

Therefore \( I \circ \lambda(u) = \rho(u) \circ I \) for \( u \in U(\mathfrak{g}) \). Since \( I \) is involutive, \( I \circ \rho(u) = \lambda(u) \circ I \). One can derive from the groupoid axioms that \( i \circ \epsilon = \epsilon \). Similarly, for a formal symplectic groupoid, the formula

\[
(EI)(F) = E(F),
\]

where \( F \in C^\infty(\Sigma, \Lambda) \), holds. Now,

\[
\langle \langle I(F) \rangle \rangle(u \otimes v) = E(\lambda(u)\rho(v)I(F)) = E(I(\rho(u)\lambda(v)F)) = E(\rho(u)\lambda(v)F) = E(\lambda(v)\rho(u)F) = \langle \langle F \rangle \rangle(v \otimes u),
\]

which means that the inverse mapping \( I \) is uniquely determined by the source mapping \( S \).
Our next task is to construct the comultiplication of the formal symplectic groupoid from the source and target mappings. Denote by $I_n$, as above, the ideal in $C^\infty(\Sigma^n, \Lambda^n)$ generated by the functions
\begin{equation}
S_n^{k+1}f - T_n^k f,
\end{equation}
where $f \in C^\infty(M)$, $1 \leq k \leq n - 1$, and set $E_n = C^\infty(\Sigma^n, \Lambda^n)/I_n$ as in formula (4).

Lemma 7. The representations $\lambda_n^k$ leave invariant the ideal $I_n$. The ideal $I_n$ is in the kernel of the algebra morphism $\chi_n : C^\infty(\Sigma^n, \Lambda^n) \to C_n$.

Proof. For $f, g \in C^\infty(M)$
\[ \lambda_n^k(f)(S_n^{l+1} - T_n^l)g = \begin{cases} (S_n^{k+1} - T_n^k)\{f, g\}_M & \text{if } k = l \\ 0 & \text{otherwise,} \end{cases} \]
whence we see that the representations $\lambda_n^k$ leave invariant the ideal $I_n$. Since $E_n(S_n^k f) = f$ and $E_n(T_n^k f) = f$, we get that $E_n(S_n^{k+1} f - T_n^k f) = 0$. Therefore the ideal $I_n$ is in the kernel of the algebra morphism $E_n : C^\infty(\Sigma^n, \Lambda^n) \to C^\infty(M)$. Now the Lemma follows from formula (31).

Lemma 7 implies that the homomorphism $\chi_n$ factors through $E_n$. Denote by $\psi_n$ the induced homomorphism from $E_n$ to $C_n$. Notice that $E_1 = C^\infty(\Sigma, \Lambda)$ and $\psi_1 = \chi_1$. It can be obtained by a straightforward generalization of the proof of Proposition 1 that the induced homomorphism $\psi_n : E_n \to C_n$ is, actually, an isomorphism. Introduce a mapping $\theta : C_1 \to C_2$ as follows. For $C \in C_1$ set
\begin{equation}
\theta[C](u \otimes v \otimes w) = k(v)C(u \otimes w).
\end{equation}
We define the comultiplication $m^* : E_1 \to E_2$ as a pullback of the mapping $\theta$ with respect to the isomorphisms $\psi_1, \psi_2$:
\[ m^* := \psi_2^{-1} \circ \theta \circ \psi_1. \]
Assume that $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $E_2$, i.e., $m^*F = \iota^*G$ in $E_2$. This is equivalent to the condition that $\psi_2(m^*F) = \psi_2(\iota^*G)$ in $C_2$, where $\iota^* : C^\infty(\Sigma^2, \Lambda^2) \to E_2$ is the quotient mapping. On the one hand, $\psi_2(\iota^*G) = \chi_2(G)$. On the other hand, $\psi_2(m^*F) = \theta[\psi_1(F)] = \theta[\chi_1(F)]$. Thus $F$ and $G$ agree on $E_2$ iff
\begin{equation}
\langle \langle G \rangle \rangle(u \otimes v \otimes w) = k(v)\langle \langle F \rangle \rangle(u \otimes w)
\end{equation}
for any $u, v, w \in \mathcal{U}(g)$.

Now we will check formula (5), i.e., Axiom (FA1). For $f \in C^\infty(M)$ we need to show that $m^*(Sf) = \iota^*(Sf \otimes 1)$ or, equivalently, that for
An easy calculation with the use of formulas (27) and (33) shows that both sides of (39) equal $h(u)k(v)k(w)f$, whence the statement follows.

Axiom (FA3), i.e., the identity $ES = id_{C^\infty(M)}$, is a part of the definition of a formal strict symplectic realization of the Poisson manifold $M$, and the target mapping $T$ was constructed to satisfy the identity $ET = id_{C^\infty(M)}$, which is Axiom (FA4).

Our next goal is to check formula (7), i.e., Axiom (FA5). We start with a pair of functions $F \in C^\infty(\Sigma,\Lambda)$ and $G \in C^\infty(\Sigma^2,\Lambda^2)$ which agree on $E_2$, i.e., satisfy condition (38).

We need to check that formula (7) holds. Applying the isomorphism $\chi$ to the both sides of formula (7), we obtain an equivalent condition:

$$\langle\langle F \rangle\rangle(u \otimes v) = E(\lambda(u)\rho(v)(\delta^* \circ (SE \otimes 1))G).$$

It is straightforward that

$$\lambda(u) \circ S = S \circ h(u), \quad \rho(u) \circ S = S \circ k(u),$$

and Lemma (6), we see that

$$(\lambda(u)\rho(v)) \circ (SE) = \epsilon_U(v) \cdot \sum_i (SE) \circ (\lambda(u'_i)\rho(u''_i)).$$

Finally, taking into account that $E \circ \delta^* = E_2$, $E_2 \circ (SE \otimes 1) = E_2$, and formula (27), we obtain that

$$\sum_{ij} k(v'_j)E_2((\lambda(u'_i)\rho(u''_i)) \otimes (\lambda(u''_i)\rho(v'_j))G) = \sum_i \langle\langle G \rangle\rangle(u'_i \otimes u''_i \otimes v),$$

where we have used the following notation:

$$(\delta_U \otimes 1) \circ \delta_U)(u) = (1 \otimes \delta_U \circ \delta_U)(u) = \sum_i u'_i \otimes u''_i \otimes u'''_i.$$

Thus condition (40) is equivalent to the following one:

$$\langle\langle F \rangle\rangle(u \otimes v) = \sum_i \langle\langle G \rangle\rangle(u'_i \otimes u''_i \otimes v).$$

Formula (41) is an immediate consequence of (38).
The remaining axioms of a formal symplectic groupoid can be checked along the same lines.

In order to check Property P of the comultiplication we need the following lemma.

**Lemma 8.** If elements \( F \in C^\infty(\Sigma, \Lambda) \) and \( G \in C^\infty(\Sigma^2, \Lambda^2) \) agree on \( \mathcal{E}_2 \), then for any \( u, v, w \in \mathcal{U}(\mathfrak{g}) \) the elements \( \tilde{F} = \epsilon_{\tilde{u}}(v) \cdot (\lambda(u)\rho(v)F) \) and \( \tilde{G} = \lambda^0_2(u)\lambda^1_2(v)\lambda^2_2(w)G \) agree on \( \mathcal{E}_2 \) as well.

**Proof.** We have to show that
\[
\langle \langle \tilde{G} \rangle \rangle (\tilde{u} \otimes \tilde{v} \otimes \tilde{w}) = k(v)\langle \langle \tilde{F} \rangle \rangle (\tilde{u} \otimes \tilde{w})
\]
for any \( \tilde{u}, \tilde{v}, \tilde{w} \in \mathcal{U}(\mathfrak{g}) \). It follows immediately from the fact that the representations \( \lambda^n_k \), \( 0 \leq k \leq n \), pairwise commute. \( \square \)

Assume that elements \( F_i \in C^\infty(\Sigma, \Lambda) \) and \( G_i \in C^\infty(\Sigma^2, \Lambda^2) \) agree on \( \mathcal{E}_2 \) for \( i = 1, 2 \). To check Property P we need to prove that
\[
\langle \langle \{G_1, G_2\}_\Sigma^2 \rangle \rangle (u \otimes v \otimes w) = k(v)\langle \langle \{F_1, F_2\}_\Sigma \rangle \rangle (u \otimes w).
\]

A straightforward calculation with the use of formulas (13), (29), and (28) applied to condition (38) with \( F = F_i, G = G_i \), where \( i = 1, 2 \), shows that
\[
E(\{F_1, F_2\}_\Sigma) = E_2(\{G_1, G_2\}_\Sigma^2).
\]
Then it remains to use the Jacobi identity and Lemma 8.

In order to check the coassociativity of the comultiplication \( m^* \) we consider the mappings
\[
(m^1_2)^*, (m^2_2)^* : \mathcal{E}_2 \rightarrow \mathcal{E}_3
\]
induced by \( m^* \otimes 1 \) and \( 1 \otimes m^* \) as in Section 2. These mappings are well defined due to Axioms (FA1) and (FA2) given by formulas (5) and (6) respectively. Pushing forward the mappings \( (m^1_2)^* \) and \( (m^2_2)^* \) via the isomorphisms \( \psi_2, \psi_3 \) we obtain the mappings \( \theta_2^1, \theta_2^3 : \mathcal{C}_2 \rightarrow \mathcal{C}_3 \) such that
\[
\theta_2^1 = \psi_3 \circ (m^1_2)^* \circ \psi_2^{-1}, \quad \theta_2^3 = \psi_3 \circ (m^2_2)^* \circ \psi_2^{-1}.
\]
These mappings act on an element \( C \in \mathcal{C}_2 \) as follows:
\[
\theta_2^1[C](u \otimes v \otimes w \otimes z) = k(v)C(u \otimes w \otimes z),
\]
\[
\theta_2^3[C](u \otimes v \otimes w \otimes z) = k(w)C(u \otimes v \otimes z).
\]
Now, both \( \theta_2^1 \circ \theta \) and \( \theta_2^3 \circ \theta \) map \( B \in \mathcal{C}_1 \) to an element \( D \in \mathcal{C}_3 \) such that
\[
D(u \otimes v \otimes w \otimes z) = k(v)k(w)B(u \otimes z),
\]
which implies the coassociativity of the coproduct \( m^* \).
Assume that there is given a formal symplectic groupoid on $(\Sigma, \Lambda)$ over the Poisson manifold $M$ with the source mapping $S$ and comultiplication $m^*$. To conclude the proof of Theorem 1 we need to prove the following statements.

**Lemma 9.** If elements $F \in C^\infty(\Sigma, \Lambda)$ and $G \in C^\infty(\Sigma^2, \Lambda^2)$ agree on $E_2$, then $E(F) = E_2(G)$.

**Proof.** Axiom (FA5) given by (7) and formula (13) imply that

$$E(F) = E(\delta^* \circ (SE \otimes 1))G = E_2((SE \otimes 1))G = E_2(G).$$

□

**Proposition 2.** The mapping $\psi_2 \circ m^* \circ \psi_1^{-1}$ coincides with the mapping $\theta$, given by formula (37)

**Proof.** Axiom (FA1) of a formal symplectic groupoid given by formula (5) means that the formal functions $Sf$ and $Sf \otimes 1$ agree for all $f \in C^\infty(M)$. Similarly, Axiom (FA2) given by formula (6) means that $Tf$ agrees with $1 \otimes Tf$. Finally, zero constant $0$ agrees with $1 \otimes Sf - Tf \otimes 1$, since the function $1 \otimes Sf - Tf \otimes 1$ is in the ideal $I_2$ which is the kernel of the mapping $\iota^*$. Property P implies that if $F \in C^\infty(\Sigma, \Lambda)$ agrees with $G \in C^\infty(\Sigma^2, \Lambda^2)$, then

$$m^*(\lambda(f)F) = \iota^*(\lambda^0_2(f)G), \ m^*(\rho(f)F) = \iota^*(\lambda^2_2(f)G), \ i^*(\lambda^1_2(f)G) = 0.$$

Thus for $u, v, w \in U(\mathfrak{g})$

$$\epsilon(u) m^*(\lambda(u)\rho(w)F) = \iota^*(\lambda^0_2(u)\lambda^1_2(v)\lambda^2_2(w)G).$$

Taking into account Lemma 9 we obtain from (42) that

$$k(v)(\langle F \rangle)(u \otimes w) = \langle G \rangle(u \otimes v \otimes w),$$

whence the Proposition follows. □

Proposition 2 shows that the comultiplication $m^*$ is uniquely defined by the source mapping $S$. This concludes the proof of Theorem 1.

**Remark.** Let $M$ be a symplectic manifold. Denote by $\tilde{M}$ a copy of the manifold $M$ endowed with the opposite symplectic structure and by $M_{\text{diag}}$ the diagonal of $M \times \tilde{M}$. It follows from the results obtained in [14] that, given a formal symplectic groupoid $G$ on $(\Sigma, \Lambda)$ over a symplectic manifold $M$ with the source mapping $S$ and target mapping $T$, then the mapping

$$S \otimes T : C^\infty(M \times \tilde{M}, M_{\text{diag}}) \to C^\infty(\Sigma, \Lambda)$$

is a formal symplectic isomorphism. It can be easily checked that the mapping $S \otimes T$ induces an isomorphism of the formal pair symplectic groupoid on $(M \times \tilde{M}, M_{\text{diag}})$ over $M$ with the groupoid $G$. 
4. ISOMORPHISMS OF FORMAL SYMPLECTIC GROUPOIDS

Let $\Sigma$ be a symplectic manifold and $\Lambda$ its Lagrangian submanifold which is a copy of a given Poisson manifold $M$. In this section we will consider the formal symplectic groupoids on the formal neighborhood $(\Sigma, \Lambda)$ over $M$. It is known that there exists a local symplectic groupoid over $M$ defined on a symplectic manifold $\Sigma'$. Its unit space $\Lambda'$ is a copy of $\Lambda$. One can find a symplectomorphism of a neighborhood $V$ of $\Lambda$ in $\Sigma$ onto a neighborhood $V'$ of $\Lambda'$ in $\Sigma'$ which identifies $\Lambda$ with $\Lambda'$. One can then transfer the local symplectic groupoid on $V'$ to $V$ and induce a formal symplectic groupoid on $(\Sigma, \Lambda)$ over $M$. We are going to describe the space of all formal symplectic groupoids on $(\Sigma, \Lambda)$ over $M$ as a principal homogeneous space of a certain pronilpotent infinite dimensional Lie group.

Let $G$ and $G'$ be two formal symplectic groupoids on $(\Sigma, \Lambda)$ over $M$ with the source mappings $S, S'$: $C^\infty(M) \to C^\infty(\Sigma, \Lambda)$, target mappings $T, T'$, and inverse mappings $I, I'$ respectively. Denote by $\chi, \chi': C^\infty(\Sigma, \Lambda) \to \mathcal{C}$ and by $\chi_1, \chi'_1: C^\infty(\Sigma, \Lambda) \to \mathcal{C}_1$ the corresponding Poisson isomorphisms, as introduced in Section 3. For $F \in C^\infty(\Sigma, \Lambda)$ we use the notations $\langle F \rangle = \chi(F)$, $\langle F' \rangle = \chi'(F)$. There exists a unique Poisson automorphism $Q$ of $C^\infty(\Sigma, \Lambda)$ such that

$$\chi' = \chi \circ Q.$$  

It follows from formulas (24) and (25) that for $f \in C^\infty(M)$

$$\langle Sf \rangle = \langle S'f \rangle' \quad \text{and} \quad \langle Tf \rangle = \langle T'f \rangle',$$

whence

$$S = Q \circ S' \quad \text{and} \quad T = Q \circ T'.$$

The isomorphisms $\chi_1, \chi'_1: C^\infty(\Sigma, \Lambda) \to \mathcal{C}_1$ push forward the corresponding inverse mappings $I$ and $I'$ of the formal symplectic groupoids $G, G'$ to the same mapping $C \mapsto C^\dagger$ on $\mathcal{C}_1$. Therefore

$$QI' = IQ.$$

We want to describe the structure of the automorphism $Q$. The isomorphisms $\chi, \chi'$ respect the filtrations on $C^\infty(\Sigma, \Lambda)$ and $\mathcal{C}$. Therefore, the automorphism $Q$ respects the filtration on $C^\infty(\Sigma, \Lambda)$, i.e., $Q(J^n) \subset J^{n'}$, $n \geq 0$, where $J = I_\Lambda/I_\infty$ is the kernel of the unit mapping $E: C^\infty(\Sigma, \Lambda) \to C^\infty(M)$ and $J^0 := C^\infty(\Sigma, \Lambda)$. One can prove a stronger statement.
Proposition 3. The operator $Q^{-1} : C^\infty(\Sigma, \Lambda) \to C^\infty(\Sigma, \Lambda)$ increases the filtration degree by one, i.e., $(Q - 1)\mathcal{J}^n \subset \mathcal{J}^{n+1}$, $n \geq 0$.

Proof. For an arbitrary element $G \in \mathcal{J}^n$ set $F = Q(G) \in \mathcal{J}^n$. We have that $\langle F \rangle = \langle G' \rangle$ and $\langle F \rangle_k = \langle G' \rangle_k = 0$ for all $k < n$. According to Lemma 5, $\langle G \rangle_n = \langle G' \rangle_n$, whence $\langle F \rangle_k = \langle G \rangle_k$ for all $k \leq n$. Therefore $(Q - 1)G = F - G \in \mathcal{J}^{n+1}$, which concludes the proof. □

For $G \in C^\infty(\Sigma, \Lambda)$ set $F = Q(G)$. Using that $\chi(F) = \chi'(G)$, it is easy to check that in standard local Darboux coordinates $(x, \xi)$ on $\Sigma$

$$E(\partial^a F) = \Phi_{\gamma}^{\alpha\beta}(x) \partial_\beta E(\partial^\gamma G),$$

where $\alpha, \beta, \gamma$ are multi-indices (recall that $\partial_i = \partial/\partial x^i$ and $\partial^i = \partial/\partial \xi_j$). We see that locally $Q = \Psi_{\beta}^{\alpha}(x, \xi) \partial_\alpha \partial^\beta$, i.e., $Q$ is a formal differential operator on the formal neighborhood $(\Sigma, \Lambda)$. Proposition 3 implies that the operator

$$H := \log Q = \log \left(1 + (Q - 1)\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (Q - 1)^n.$$

on $C^\infty(\Sigma, \Lambda)$ is correctly defined via a $\mathcal{J}$-adically convergent series and increases the filtration degree by one. Since $Q$ is a Poisson automorphism of $C^\infty(\Sigma, \Lambda)$, the operator $H$ is a derivation of $C^\infty(\Sigma, \Lambda)$ which respects the Poisson bracket. The operator $H$ is a formal vector field on $(\Sigma, \Lambda)$ locally given by the formula

$$(44) \quad H = a^i(x, \xi) \partial_i + b_j(x, \xi) \partial^j,$$

where $a^i = 0 \pmod{\xi}$ and $b_j = 0 \pmod{\xi^2}$, since $H$ increases the filtration degree by one. We want to show that $H$ is a formal Hamiltonian vector field on $(\Sigma, \Lambda)$.

Lemma 10. A formal vector field $H$ on $(\Sigma, \Lambda)$ respects the Poisson bracket $\{\cdot, \cdot\}_\Sigma$ and increases by one the filtration degree in $C^\infty(\Sigma, \Lambda)$ if and only if there exists a formal Hamiltonian $F \in \mathcal{J}^2$ such that $H = H_F$. If $H = H_F$ for some formal Hamiltonian $F \in \mathcal{J}^2$, then $F$ is defined uniquely.

Proof. Assume that $H$ is given in standard Darboux coordinates by formula (44). The condition that $H$ respects the Poisson bracket $\{\cdot, \cdot\}_\Sigma$ can be expressed in local coordinates as follows:

$$\partial^i a^j = \partial^j a^i, \quad \partial_i b_j = \partial_j b_i, \quad \partial_i a^j = -\partial^j b_i,$$

which is equivalent to the fact that the formal one-form $A = a^i d\xi_i - b_j dx^j$ is closed. Introduce a grading $| \cdot |$ on the differential forms in
the variables $x, \xi$ such that $|x| = 0, |dx| = 0, |\xi| = 1, |d\xi| = 1$. The differential $d = \partial_i dx^i + \partial^i d\xi_i$ respects the grading. Denote by $A_q$ the homogeneous component of degree $q$ of the form $A$. Then

\begin{equation}
A_q = a^i_{q-1} dx^i - b_{jq} dx^j,
\end{equation}

where $a^i_q$ and $b_{jq}$ denote the homogeneous components of $a^i$ and $b_j$ of degree $q$, respectively. Since $a^i = 0 \pmod{\xi}$ and $b_j = 0 \pmod{\xi^2}$, we see from formula (45) that the series $A = \sum A_q$ starts with the term $A_2$. The form $A$ is closed iff each homogeneous component $A_q$ is closed. Using the standard homotopy argument involving the Euler operator $\xi_j \partial^j$ related to the grading, we get that if $A_q$ is closed, there exists a unique function $F_q(x, \xi)$ homogeneous of degree $q$ in $\xi$ such that $A_q = dF_q$. Now, $F = F_2 + F_3 + \ldots$ is the unique element of $J^2$ such that $A = dF$, or, equivalently, such that $H = H_F$. \hfill \Box

It follows from Lemma 10 that there exists a unique formal function $F \in J^2$ such that $Q = \exp H_F$. Now assume that $G$ is a formal symplectic groupoid on $(\Sigma, \Lambda)$ over $M$ with the source mapping $S$.

**Lemma 11.** Let $W$ be a Poisson automorphism of $C^\infty(\Sigma, \Lambda)$ such that $E \circ W = E$ and $W \circ S = S$. Then $W$ is the identity automorphism, $W = 1$.

**Proof.** Since $W$ is a Poisson automorphism of $C^\infty(\Sigma, \Lambda)$ and $W \circ S = S$, we get for $f \in C^\infty(M)$ and $F \in C^\infty(\Sigma, \Lambda)$ that $W(H_{Sf}F) = W(\{Sf, F\}_\Sigma) = \{WSf, WF\}_\Sigma = \{Sf, WF\}_\Sigma = H_{Sf}W(F)$. Therefore $W \circ \lambda(u) = \lambda(u) \circ W$ for any $u \in U(\mathfrak{g})$. Taking into account that $E \circ W = E$, we obtain that

\[ \langle F \rangle(u) = E(\lambda(u)F) = E(W \lambda(u)F) = E(\lambda(u)WF) = \langle W(F) \rangle(u). \]

Proposition 1 implies that $W = 1$, which concludes the proof. \hfill \Box

Take an arbitrary element $F \in J^2$. The operator $H_F$ on $C^\infty(\Sigma, \Lambda)$ increases the filtration degree by one, therefore there is a Poisson automorphism $Q = \exp H_F$ of $C^\infty(\Sigma, \Lambda)$ such that $E \circ Q = E$. The mapping $S'$ uniquely determined by the equation $S = Q \circ S'$ is a Poisson morphism from $C^\infty(M)$ to $C^\infty(\Sigma, \Lambda)$ with the property that $ES' = \text{id}_{C^\infty(M)}$. Therefore it determines a unique formal symplectic groupoid $G'$ on $(\Sigma, \Lambda)$ over $M$ whose source mapping is $S'$. Take $F' \in J^2$ and set $Q' = \exp H_{F'}$. Lemma 11 implies that if $S = Q' \circ S'$, then $Q = Q'$ and $F = F'$.

The automorphism $Q$ such that $S = Q \circ S'$ plays the role of the equivalence morphism of the groupoids $G$ and $G'$. 
Denote by $g_{\Sigma}$ the pronilpotent Lie algebra $(J^2, \{\cdot, \cdot\}_\Sigma)$ and by $G_{\Sigma} = \exp g_{\Sigma}$ the corresponding pronilpotent Lie group. The results of this Section can be combined in the following theorem.

**Theorem 3.** The space of formal symplectic groupoids over a Poisson manifold $M$ defined on the formal symplectic neighborhood $(\Sigma, \Lambda)$ of a Lagrangian submanifold $\Lambda$ of a symplectic manifold $\Sigma$ is a principal homogeneous space of the group $G_{\Sigma}$ of formal symplectic automorphisms of $C^\infty(\Sigma, \Lambda)$.

Let $G$ be a formal symplectic groupoid over a Poisson manifold $M$ defined on the formal neighborhood $(T^*M, Z)$ of the zero section $Z$ of the cotangent bundle $T^*M$. Denote by $\tau$ the antisymplectic involutive automorphism of $T^*M$ given by the formula $\tau : (x, \xi) \mapsto (x, -\xi)$, where $\{x^i\}$ are local coordinates on $M$ lifted to $T^*M$ and $\{\xi^j\}$ the dual fibre coordinates on $T^*M$. It induces the dual antisymplectic involutive morphism $\tau^* : C^\infty(T^*M, Z) \rightarrow C^\infty(T^*M, Z)$. Let $S, T, I$ be the source, target, and inverse mappings of the groupoid $G$, respectively. Since $T : C^\infty(M) \rightarrow C^\infty(T^*M, Z)$ is an anti-Poisson morphism such that $E_T = \text{id}_{C^\infty(M)}$, the mapping

$$S = \tau^* \circ T$$

is a Poisson morphism from $C^\infty(M)$ to $C^\infty(T^*M, Z)$ such that $E \tilde{S} = \text{id}_{C^\infty(M)}$. Therefore there exists a unique formal symplectic groupoid $\tilde{G}$ on $(T^*M, Z)$ over $M$ whose source mapping is $\tilde{S}$. We call $\tilde{G}$ the dual formal symplectic groupoid of $G$. Theorem 3 implies that there exists a unique symplectic automorphism $Q \in G_{\Sigma}$ such that

$$S = Q \circ \tilde{S}.$$  

The automorphism $Q$ is uniquely represented as $Q = \exp H_F$ for some element $F \in J^2$. Since $T = IS$, we get from formulas (46) and (47) that

$$S = Q \circ \tau^* \circ I \circ S.$$

Set $W := Q \circ \tau^* \circ I$. One can check that $E \circ Q = E$, $E \circ I = E$, and $E \circ \tau^* = E$, whence $E \circ W = E$. Since $W$ is a Poisson automorphism of $C^\infty(T^*M, Z)$, it follows from Lemma 11 that $Q \circ \tau^* \circ I = W = 1$. Taking into account that the inverse mapping $I$ is involutive, we obtain that

$$I = Q \circ \tau^* = \exp H_F \circ \tau^*.$$

The Hamiltonian $F$ is canonically related to the formal groupoid $G$. Since $\tau^*$ is involutive, we get that $Q \circ \tau^* = \tau^* \circ Q^{-1}$, whence $H_F \circ \tau^* =
-τ* ◦ H_F, which means that

\( \tau^* F = F, \)

i.e., that \( F(x, \xi) = F(x, -\xi). \)

5. Canonical formal symplectic groupoid of a natural deformation quantization

Let \((M, \{\cdot, \cdot\}_M)\) be a Poisson manifold. Denote by \(C^\infty(M)[[\nu]]\) the space of formal series in \(\nu\) with coefficients from \(C^\infty(M)\). As introduced in [1], a formal differentiable deformation quantization on \(M\) is an associative algebra structure on \(C^\infty(M)[[\nu]]\) with the \(\nu\)-linear and \(\nu\)-adically continuous product \(*\) (named star-product) given on \(f, g \in C^\infty(M)\) by the formula

\[
\sum_{r=0}^{\infty} \nu^r C_r(f, g),
\]

where \(C_r, r \geq 0\), are bidifferential operators on \(M\), \(C_0(f, g) = fg\) and \(C_1(f, g) - C_1(g, f) = \{f, g\}\). We adopt the convention that the unity of a star-product is the unit constant. Two differentiable star-products \(*, *'\) on a Poisson manifold \((M, \{\cdot, \cdot\}_M)\) are called equivalent if there exists an isomorphism of algebras \(B : (C^\infty(M)[[\nu]], *) \rightarrow (C^\infty(M)[[\nu]], *)\) of the form \(B = 1 + \nu B_1 + \nu^2 B_2 + \ldots\), where \(B_r, r \geq 1\), are differential operators on \(M\). The existence and classification problem for deformation quantization was first solved in the non-degenerate (symplectic) case (see [5], [20], [8] for existence proofs and [9], [18], [7], [2], [24] for classification) and then Kontsevich [17] showed that every Poisson manifold admits a deformation quantization and that the equivalence classes of deformation quantizations can be parameterized by the formal deformations of the Poisson structure.

All the explicit constructions of star-products enjoy the following property: for all \(r \geq 0\) the bidifferential operator \(C_r\) in (50) is of order not greater than \(r\) in each argument (most important examples are Fedosov’s star-products on symplectic manifolds and Kontsevich’s star-product on \(\mathbb{R}^n\) endowed with an arbitrary Poisson bracket). The star-products with this property were called natural by Gutt and Rawnsley in [10], where general properties of such star-products were studied.

Let \(\mathcal{D} = \mathcal{D}(M)\) be the algebra of differential operators with smooth complex-valued coefficients and \(\mathcal{D}[[\nu]]\) be the algebra of formal differential operators on \(M\). The algebra \(\mathcal{D}\) has a natural increasing filtration \(\{\mathcal{D}_r\}\), where \(\mathcal{D}_r\) is the space of differential operators of order not greater than \(r\). We call a formal differential operator \(A = A_0 + \nu A_1 + \cdots \in \mathcal{D}[[\nu]]\) a formal differential operator with smooth complex-valued coefficients and \(\mathcal{D}[[\nu]]\) be the algebra of formal differential operators on \(M\). The algebra \(\mathcal{D}\) has a natural increasing filtration \(\{\mathcal{D}_r\}\), where \(\mathcal{D}_r\) is the space of differential operators of order not greater than \(r\). We call a formal differential operator \(A = A_0 + \nu A_1 + \cdots \in \mathcal{D}[[\nu]]\) a formal differential operator.
$\mathcal{D}[[\nu]]$ natural if $A_r \in \mathcal{D}_r$ for any $r \geq 0$. The natural formal differential operators form an algebra which we denote by $\mathcal{N}$.

Let $T^*M$ be the cotangent bundle of the manifold $M$ and $Z$ be its zero section. Denote by $\epsilon : M \to T^*M$ the composition of the identifying mapping from $M$ onto $Z$ with the inclusion mapping of $Z$ into $T^*M$. It induces the dual mapping $E : C^\infty(T^*M, Z) \to C^\infty(M)$.

If $\{x^k\}$ are local coordinates on $M$ and $\{\xi_k\}$ are the dual fibre coordinates on $T^*M$, then the principal symbol of an operator $A \in \mathcal{D}_r$, whose leading term is $a^{i_1...i_r}(x)\partial_{i_1}...\partial_{i_r}$, is given by the formula $\text{Symb}_r(A) = a^{i_1...i_r}(x)\xi_{i_1}...\xi_{i_r}$. It is globally defined on $T^*M$ and fibrewise is a homogeneous polynomial of degree $r$. We define a $\sigma$-symbol of a natural formal differential operator $A = A_0 + i\nu A_1 + (i\nu)^2 A_2 + ...$ as the formal series $\sigma(A) = \text{Symb}_0(A_0) + \text{Symb}_1(A_1) + ...$. Such a formal series can be treated as a formal function from $C^\infty(T^*M, Z)$. The mapping $\sigma : A \mapsto \sigma(A)$ is an algebra morphism from $\mathcal{N}$ to $C^\infty(T^*M, Z)$. Moreover, for $A, B \in \mathcal{N}$ the operator $\frac{1}{\nu}[A, B]$ is also natural and

$$\sigma\left(\frac{1}{\nu}[A, B]\right) = \{\sigma(A), \sigma(B)\}_T^*M,$$

where $\{\cdot, \cdot\}_{T^*M}$ denotes the standard Poisson bracket on $T^*M$ and the induced bracket on $C^\infty(T^*M, Z)$ given locally by the formula

$$\{\Phi, \Psi\}_{T^*M} = \partial^i\Phi \partial_i\Psi - \partial^i\Psi \partial_i\Phi.$$

For $f, g \in C^\infty(M)[[\nu]]$ denote by $L_f$ and $R_g$ the operators of $*$-multiplication by $f$ from the left and of $*$-multiplication by $g$ from the right respectively, so that $L_f g = f * g = R_g$. The associativity of $*$ is equivalent to the fact that $[L_f, R_g] = 0$. A star-product $*$ on $M$ is natural iff for any $f, g \in C^\infty(M)[[\nu]]$ the operators $L_f, R_g$ are natural. It was proved in [14] that the mappings

$$S, T : C^\infty(M) \to C^\infty(T^*M, Z)$$

defined by the formulas

$$S f = \sigma(L_f), \ T f = \sigma(R_f),$$

where $f \in C^\infty(M)$, are a Poisson and an anti-Poisson morphisms, respectively, which satisfy the formulas $E S = \text{id}_{C^\infty(M)}$ and $E T = \text{id}_{C^\infty(M)}$. Moreover, for $f, g \in C^\infty(M)$ the formal functions $S f, T g$ Poisson commute. For each natural deformation quantization on $M$ we constructed in [15] an involutive antisymplectic automorphism $I$ of the Poisson algebra $C^\infty(T^*M, Z)$ such that $I S = T$ and $I T = S$. It follows from Theorem 1 that there exists a canonical formal symplectic groupoid on $(T^*M, Z)$ over $M$ with the source mapping $S$, target
mapping $T$, and inverse mapping $I$. We call it the formal symplectic groupoid of the natural deformation quantization.

If $\ast$ and $\ast'$ are two equivalent natural star products on $M$, it was proved in [10] that any equivalence operator $B$ of these star products satisfying the identity

$$Bf \ast Bg = B(f \ast' g)$$

can be represented as $B = \exp \frac{1}{\nu}X$, where $X$ is a natural operator such that $X = 0 \pmod{\nu^2}$. Let $G$ and $G'$ be the formal symplectic groupoids of the star products $\ast$ and $\ast'$ with the source mappings $S$ and $S'$, respectively. It is easy to check that if $Q$ is the equivalence morphism of these groupoids such that $S = Q \circ S'$, then

$$Q = \exp H_{\sigma(X)}.$$

6. Deformation quantizations with separation of variables

Let $M$ be a complex manifold endowed with a Poisson tensor $\eta$ of type (1,1) with respect to the complex structure. We call such manifolds Kähler-Poisson. If $\eta$ is nondegenerate, $M$ is a Kähler manifold.

If $U \subset M$ is a coordinate chart with local holomorphic coordinates $\{z^k, \bar{z}^l\}$, we will write $\eta = g^{lk} \bar{\partial}_l \partial_k$ on $U$, where $\partial_k = \partial/\partial z^k$ and $\bar{\partial}_l = \partial/\partial \bar{z}^l$. The condition that $\eta$ is a Poisson tensor is expressed in terms of $g^{lk}$ as follows:

$$g^{lk} \bar{\partial}_l \partial_k g^{jm} = g^{lk} \bar{\partial}_l \partial_k g^{jm} \quad \text{and} \quad g^{lk} \bar{\partial}_l \partial_k g^{jm} = g^{lk} \bar{\partial}_l \partial_k g^{jm}.$$

The corresponding Poisson bracket on $M$ is given locally as

$$\{\phi, \psi\}_M = g^{lk} (\bar{\partial}_l \phi \partial_k \psi - \bar{\partial}_l \psi \partial_k \phi).$$

We say that a star-product (50) on a Kähler-Poisson manifold $M$ defines a deformation quantization with separation of variables on $M$ if the bidifferential operators $C_r$ differentiate their first argument in antiholomorphic directions and its second argument in holomorphic ones.

With the assumption that the unit constant 1 is the unity of the star-algebra $(C^\infty(M)[[\nu]], \ast)$, the condition that $\ast$ is a star-product with separation of variables can be restated as follows. For any local holomorphic function $a$ and antiholomorphic function $b$ the operators $L_a$ and $R_b$ are the operators of point-wise multiplication by the functions $a$ and $b$ respectively, $L_a = a, R_b = b$. In such a case it is easy to check that $C_1(\phi, \psi) = g^{lk} \bar{\partial}_l \phi \partial_k \psi$, so that

$$\phi \ast \psi = \phi \psi + \nu g^{lk} \bar{\partial}_l \phi \partial_k \psi + \ldots$$
Deformation quantizations with separation of variables on a Kähler manifold $M$ (also known as deformation quantizations of the Wick type, see \[3\]) are completely described and parameterized by the formal deformations of the Kähler form on $M$ in \[11\]. If $\left( g^{lk} \right)$ is an arbitrary matrix with constant entries, the formula

$$\left( \phi \ast \psi \right)(z, \bar{z}) = \left( \exp \nu g^{lk} \frac{\partial}{\partial \bar{v}^l} \frac{\partial}{\partial v^k} \right) \phi(z, \bar{v}) \psi(v, \bar{z}) \mid_{v=z, \bar{v} = \bar{z}}$$

defines a star-product with separation of variables on the Kähler-Poisson manifold $(\mathbb{C}^d, g^{lk} \bar{\partial}_l \wedge \partial_k)$. One can give more elaborate examples of deformation quantizations with separation of variables on Kähler-Poisson manifolds. We conjecture that star-products with separation of variables exist on an arbitrary Kähler-Poisson manifold and they can be parameterized by the formal deformations of the Kähler-Poisson tensor $\eta$ (not the equivalence classes, but the star-products themselves). The nature of this parameterization must be very different from that of the parameterization by the formal deformations of the Kähler form in the Kähler case (see also \[13\]).

For a given star-product with separation of variables $\ast$ on $M$ there exists a unique formal differential operator $B$ on $M$ such that

$$B(ab) = b \ast a \quad (55)$$

for any local holomorphic function $a$ and antiholomorphic function $b$. The operator $B$ is called the formal Berezin transform (see \[12\]). One can check that the operator $\Delta$ defined locally by the formula $g^{lk} \partial_l \bar{\partial}_k$ is coordinate invariant and thus globally defined on $M$ and that

$$B = 1 + \nu \Delta + \ldots \quad (56)$$

In particular, $B$ is invertible. Introduce a dual star product $\hat{\ast}$ on $M$ by the formula

$$\phi \hat{\ast} \psi = B^{-1}(B \psi \ast B \phi) \quad (57)$$

We will show that $\hat{\ast}$ is a deformation quantization with separation of variables on the complex manifold $M$ endowed with the opposite Poisson tensor $-\eta$. This statement was proved in the Kähler case in \[12\], but the proof does not work in the Kähler-Poisson case.

It follows from (55) that

$$Ba = a \text{ and } Bb = b \quad (58)$$

In particular, $B1 = 1$. 

Lemma 12. For any local holomorphic function $a$ and antiholomorphic function $b$

$$BaB^{-1} = R_a \text{ and } BbB^{-1} = L_b.$$  

Proof. We need to show that $BaB^{-1}f = f \ast a$ for any formal function $f$. Since $B$ is invertible, the function $f$ can be represented as $f = Bg$ for some formal function $g$. Now we need to check that $B(ag) = Bg \ast a$ for an arbitrary formal function $g$. It suffices to check it only for $g$ of the form $g = \tilde{a}b$, where $\tilde{a}$ is a local holomorphic function and $b$ a local antiholomorphic function. We have

$$B(a\tilde{a}b) = b \ast (\tilde{a}a) = (b \ast \tilde{a}) \ast a = B(\tilde{a}b) \ast a.$$  

The formula $BbB^{-1} = L_b$ can be proved similarly. \hfill $\Box$

Denote by $\tilde{L}_\phi$ the operator of star-multiplication by a function $\phi$ from the left and by $\tilde{R}_\psi$ the operator of star-multiplication by a function $\psi$ from the right with respect to the star-product $\tilde{\ast}$. It follows from (57) that

$$(59) \quad \tilde{L}_\phi = B^{-1}R_{B\phi}B \quad \text{and} \quad \tilde{R}_\psi = B^{-1}L_{B\psi}B.$$  

Proposition 4. The dual star-product $\tilde{\ast}$ given by formula (57) is a deformation quantization with separation of variables on the manifold $M$ endowed with the same complex structure but with the opposite Poisson tensor $-\eta$.

Proof. Lemma 12 and formulas (58) and (59) imply that for any local holomorphic function $a$

$$\tilde{L}_a = B^{-1}R_{Ba}B = B^{-1}R_aB = B^{-1}(BaB^{-1})B = a.$$  

Similarly, $\tilde{R}_b = b$ for any local antiholomorphic function $b$. Thus $\tilde{\ast}$ is a star-product with separation of variables. Using formulas (54), (56), and (57) we get that

$$\phi \tilde{\ast} \psi = \phi \psi - \nu g^{lk} \partial_l \phi \partial_k \psi + \ldots,$$

which implies that $\tilde{\ast}$ is a star-product on the Kähler-Poisson manifold $(M,-\eta)$. \hfill $\Box$

Lemma 12 and formula (58) imply that for any local holomorphic functions $a, \tilde{a}$ and antiholomorphic functions $b, \tilde{b}$

$$(60) \quad [BaB^{-1}, \tilde{a}] = [R_a, L_{\tilde{a}}] = 0 \quad \text{and} \quad [BbB^{-1}, \tilde{b}] = [L_{\tilde{b}}, R_b] = 0.$$  

It follows from formula (56) that $B = \exp \left( \frac{1}{\nu} X \right)$ for some formal differential operator

$$(61) \quad X = \nu^2 X_2 + \nu^3 X_3 + \ldots,$$
where $X_2 = \Delta$. We want to show that the operator $X$ is natural. To this end we need the following technical lemma. If $U$ is a holomorphic chart on $M$ with local coordinates $\{z^k, \bar{z}^l\}$ we denote by $\{\zeta^k, \bar{\zeta}^l\}$ the dual fibre coordinates on $T^*U$ and set $\partial^k = \partial/\partial \zeta^k$ and $\bar{\partial}^l = \partial/\partial \bar{\zeta}^l$.

**Lemma 13.** Given an integer $n \geq 2$, let $X$ be a nonzero differential operator on a holomorphic chart $U$ with coordinates $\{z^k, \bar{z}^l\}$, such that the operators $[[X, z^i], z^k]$ and $[[X, \bar{z}^j], \bar{z}^l]$ are of order not greater than $n - 2$ for any $i,j,k,l$. Then the operator $X$ is of order not greater than $n$.

**Proof.** Assume that $X$ is a differential operator of order $N > n$. Its principal symbol $p(\zeta, \bar{\zeta})$ is a nonzero homogeneous polynomial of degree $N$ with respect to the fibre coordinates $\{\zeta^k, \bar{\zeta}^l\}$. The condition that the operator $[[X, z^i], z^k]$ is of order not greater than $n - 2$ means that the function $\partial^i \partial^k p$ is a polynomial of order not greater than $n - 2$ in the variables $\zeta, \bar{\zeta}$. On the other hand, $\partial^i \partial^k p$ is of order $N - 2 > n - 2$ which means that $\partial^i \partial^k p = 0$ for any $i,k$. Similarly, $\bar{\partial}^j \bar{\partial}^l p = 0$ for any $j,l$. Since $N \geq 3$, at least one of the partial derivatives $\partial^i \partial^k p$ or $\bar{\partial}^j \bar{\partial}^l p$ should be nonzero. Thus the assumption that $N > n$ leads to a contradiction. \qed

Formula (58) implies that for any $n$ the operator $X_n$ in (61) annihilates holomorphic and antiholomorphic functions. In particular, $X_n 1 = 0$. We get from formula (60) that

\[
\left[ \exp \left( \frac{1}{\nu} \text{ad} X \right) a, \bar{a} \right] = [B a B^{-1}, \bar{a}] = 0 \quad \text{and} \quad \left[ \exp \left( \frac{1}{\nu} \text{ad} X \right) b, \bar{b} \right] = [B b B^{-1}, \bar{b}] = 0.
\]

Expanding the left-hand sides of formulas (62) in the formal series in the parameter $\nu$ and equating the coefficient at $\nu^{n-1}$ to zero, we get

\[
\sum_{k \geq 1} \frac{1}{k!} \sum_{i_1 + \ldots + i_k - k = n-1} [[X_{i_1}, \ldots, [X_{i_k}, a] \ldots], \bar{a}] = 0
\]

for $n \geq 2$. Since all the indices $i_j$ in (63) satisfy the condition $i_j \geq 2$, we have that $n - 1 = i_1 + \ldots + i_k - k \geq k$. Thus we obtain from (63) that

\[
[[X_n, a], \bar{a}] = -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1 + \ldots + i_k - k = n-1} [[X_{i_1}, \ldots, [X_{i_k}, a] \ldots], \bar{a}].
\]
Similarly,

\[
[X_n, b], \tilde{b} = -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1 + \ldots + i_k = n-k-1} [[X_{i_1}, \ldots, [X_{i_k}, b], \ldots], \tilde{b}].
\]

The right-hand sides of equations (64) and (65) depend only on \(X_k\) with \(k < n\). We know that \(X_2 = \Delta\) is of order (not greater than) two. Assume that we have proved that \(X_k\) is of order not greater than \(k\) for all \(k < n\). We see from (64) and (65) that \([X_n, a], \tilde{a}\) and \([X_n, b], \tilde{b}\) are of order not greater than \(n - 2\). It follows from Lemma 13 that \(X_n\) is of order not greater than \(n\). The induction shows that \(X\) is indeed a natural operator. We have proved the following proposition.

**Proposition 5.** The formal Berezin transform \(B\) of a deformation quantization with separation of variables on a Kähler-Poisson manifold is of the form \(B = \exp \frac{1}{\nu} X\), where \(X\) is a natural differential operator such that \(X = 0 \pmod{\nu^2}\).

It follows from Proposition 5 that the conjugation of the formal differential operators with respect to the formal Berezin transform, \(A \mapsto BAB^{-1}\), leaves invariant the algebra \(\mathcal{N}\) of natural differential operators. In particular, the operators \(R_a = B a B^{-1}\) and \(L_b = B b B^{-1}\) are natural. Now, if \(f = ab = a * b\) we see that \(L_f = L_{a*b} = L_a L_b = a L_b\) and \(R_f = R_{a*b} = R_b R_a = b R_a\) are natural differential operators. Using the same arguments as in Proposition 1 of [15] we can prove the following theorem.

**Theorem 4.** Any deformation quantization with separation of variables on a Kähler-Poisson manifold is natural.

Theorem 4 was proved in [3] and [19] in the Kähler case.

It follows from Theorem 4 that to any deformation quantization with separation of variables on a Kähler-Poisson manifold \(M\) there corresponds a canonical formal symplectic groupoid on \((T^* M, Z)\) over \(M\). Since for any deformation quantization with separation of variables \(L_a = a\) and \(R_b = b\), we see that \(S a = \sigma(L_a) = \sigma(a) = a\) and, similarly, \(T b = b\) (abusing notations we denote by \(a\) and \(b\) both local functions on \(M\) and their lifts to \(T^* M\) with respect to the standard bundle projection).

Given a Kähler-Poisson manifold \(M\), we call a formal symplectic groupoid on \((T^* M, Z)\) over \(M\) such that \(S a = a\) and \(T b = b\) for any local holomorphic function \(a\) and antiholomorphic function \(b\), a formal symplectic groupoid with separation of variables.
7. Formal symplectic groupoid with separation of variables

In this section we will show that for any Kähler-Poisson manifold $M$ there is a unique formal symplectic groupoid with separation of variables over $M$. Let $U \subset M$ be an arbitrary coordinate chart with local holomorphic coordinates $\{z^k, \bar{z}^l\}$. Introduce differential operators $D^k, \bar{D}^l$ on $U$ by the formulas

$$D^k \psi = g^{k\ell} \partial_{\ell} \psi = -\{z^k, \psi\}_M$$

and $\bar{D}^l \psi = \bar{g}^{k\ell} \partial_{\ell} \psi = \{\bar{z}^l, \psi\}_M$,

where the Poisson bracket $\{\cdot, \cdot\}_M$ is given by formula (53). Conditions (52) are equivalent to the statement that

$$[D^k, D^m] = 0 \text{ and } [\bar{D}^l, \bar{D}^n] = 0$$

for any $k, l, m, n$. Using the operators $D^k, \bar{D}^l$ we can write

$$\{\phi, \psi\}_M = D^k \phi \partial_k \psi - D^k \psi \partial_k \phi = \bar{D}^l \phi \partial_l \psi - \bar{D}^l \psi \partial_l \phi.$$

Denote by $\{\zeta_k, \bar{\zeta}_l\}$ the fibre coordinates on $T^*U$ dual to $\{z^k, \bar{z}^l\}$. The standard Poisson bracket on $T^*M$ can be written locally as

$$\{\Phi, \Psi\}_{T^*M} = \partial^k \Phi \partial_k \Psi - \partial^k \Psi \partial_k \Phi + \bar{\partial}^l \Phi \partial_l \Psi - \bar{\partial}^l \Psi \partial_l \Phi,$$

where $\partial^k = \partial/\partial \zeta_k$, $\bar{\partial}^l = \partial/\partial \bar{\zeta}_l$. The Poisson bracket on $T^*M$ induces a Poisson bracket on $C^\infty(T^*M, Z)$ which will be denoted also by $\{\cdot, \cdot\}_{T^*M}$. Introduce mappings $S, T : C^\infty(U) \to C^\infty(T^*U, Z \cap U)$ by the formulas

$$(S\phi)(z, \bar{z}, \zeta) = e^{\zeta_k D^k} \phi, \quad (T\psi)(z, \bar{z}, \bar{\zeta}) = e^{\bar{\zeta}_l \bar{D}^l} \psi,$$

where $\phi, \psi \in C^\infty(M)$, the variables $\zeta, \bar{\zeta}$ are used as formal parameters, and the exponentials are defined via formal Taylor series.

**Proposition 6.** The mappings

$$S, T : (C^\infty(U), \{\cdot, \cdot\}_M) \to (C^\infty(T^*U, Z \cap U), \{\cdot, \cdot\}_{T^*M})$$

are a Poisson and an anti-Poisson morphisms, respectively. For any $\phi, \psi \in C^\infty(U)$ the elements $S\phi, T\psi \in C^\infty(T^*U, Z \cap U)$ Poisson commute.

**Proof.** Since $D^k, \bar{D}^l$ are derivations of the algebra $C^\infty(T^*U, Z \cap U)$, the operators $e^{\zeta_k D^k}, e^{\bar{\zeta}_l \bar{D}^l}$ are automorphisms of this algebra which implies that $S, T$ are algebra homomorphisms. We see from (66) and (69) that

$$\partial^k (S\phi) = D^k (S\phi) \quad \text{and} \quad \bar{\partial}^l (T\psi) = \bar{D}^l (T\psi).$$

Fix arbitrary functions $\phi, \psi \in C^\infty(U)$ and introduce an element $u(\zeta) \in C^\infty(T^*U, Z \cap U)$ by the formula

$$u(\zeta) = \{S\phi, S\psi\}_{T^*M}.$$
In order to show that $S$ is a Poisson morphism we need to prove that $u(\zeta) = \{S, \phi\}_M = e^{\zeta D^k \{\phi, \psi\}_M}$. This amounts to checking that $u(0) = \{\phi, \psi\}_M$ and that $\partial^m u = D^m u$. Using (68), (69), and (70) we get

$$u(\zeta) = \{S, \phi\}_M = \partial^k S \partial_k (S \phi) - \partial^k S \partial_k (S \phi)$$

$$= D^k (S \phi) \partial_k (S \phi) - D^k (S \phi) \partial_k (S \phi).$$

(71)

It follows from (67) and (71) that

$$u(0) = D^k \phi \partial_k \psi - D^k \psi \partial_k \phi = \{\phi, \psi\}_M.$$

Now, taking into account (52) and (66), we obtain from (71) that

$$\partial^m u - D^m u = (D^m D^k \phi \partial_k (S \phi) - D^m D^k (S \phi) \partial_k (S \phi) + D^k (S \phi) \partial_k (D^m S \phi) - D^k (S \phi) \partial_k (D^m S \phi)) - \partial^m (D^k (S \phi) \partial_k (S \phi) - D^k (S \phi) \partial_k (S \phi)) + D^k (S \phi) \partial_k (S \phi) - D^k (S \phi) \partial_k (S \phi) = D^k (S \phi) \partial_k (D^m S \phi) - D^k (S \phi) \partial_k (D^m S \phi) = g^\bar{\iota} k g^\bar{\iota} m \partial h (S \phi) \partial (S \phi) - g^\bar{\iota} k g^\bar{\iota} m \partial h (S \psi) \partial (S \phi) = g^\bar{\iota} k g^\bar{\iota} m \partial h (S \phi) \partial (S \phi) - g^\bar{\iota} k g^\bar{\iota} m \partial h (S \phi) \partial (S \psi) = 0,$$

which concludes the check that $S$ is a Poisson morphism. The proof that $T$ is an anti-Poisson morphism is similar. It remains to show that $\{S, T, \psi\}_M = 0$. It follows from (68), (69), and (70) that

$$\{S, T, \psi\}_M = \partial^k S \partial_k T \partial h (S \phi) - \partial^k T \partial h (S \phi) \partial_k T \partial h (S \phi) = D^k (S \phi) \partial k T \partial h (S \phi) - g^\bar{\iota} k \partial h (S \phi) \partial k T \partial h (S \phi) = 0.$$

\[ \square \]

According to Theorem 1 there exists a canonical formal symplectic groupoid $G_U$ on the formal neighborhood $(T^* U, Z \cap U)$ such that the mappings $S, T$ are the source and target maps for $G_U$ respectively. The mapping $\tau : (z, \bar{z}, \zeta, \bar{\zeta}) \mapsto (z, \bar{z}, -\zeta, -\bar{\zeta})$ is a global anti-Poisson involutive automorphism of $T^* M$. It induces an anti-Poisson involutive automorphism of the Poisson algebra $C^\infty (T^* M, Z)$. Set $\tilde{S} = \tau^* T$ and $\tilde{T} = \tau^* S$. Thus for $\phi, \psi \in C^\infty (U)$

$$\tilde{S}(z, \bar{z}, \zeta, \bar{\zeta}) = e^{-\zeta t D^k \phi}, \quad (\tilde{T} \psi)(z, \bar{z}, \zeta, \bar{\zeta}) = e^{-\zeta t D^k \psi}.$$

It follows from Proposition 6 that the mappings

$$(\tilde{S}, \tilde{T}) : (C^\infty (U), \{\cdot, \cdot\}_M) \to (C^\infty (T^* U, Z \cap U), \{\cdot, \cdot\}_{T^* M})$$
are a Poisson and an anti-Poisson morphisms, respectively. Moreover, for any \( \phi, \psi \in \mathcal{C}^\infty(U) \) the elements \( \tilde{S}\phi, \tilde{T}\psi \in \mathcal{C}^\infty(T^*U, Z \cap U) \) Poisson commute. Now, there is a canonical formal symplectic groupoid \( \tilde{\mathcal{G}}_U \) on \( (T^*U, Z \cap U) \) (the dual of \( \mathcal{G}_U \)) such that the mappings \( \tilde{S}, \tilde{T} \) are the source and target maps of \( \tilde{\mathcal{G}}_U \), respectively. According to formula (43) there is a unique formal symplectic automorphism \( Q \) of \( \mathcal{C}^\infty(T^*U, Z \cap U) \) such that

\[
S = Q \tilde{S} \quad \text{and} \quad T = Q \tilde{T}.
\]

Let \( a, \tilde{a} \) be arbitrary holomorphic functions and \( b, \tilde{b} \) arbitrary antiholomorphic functions on \( U \). It follows from formulas (69) and (73) that

\[
S a = a, \quad T b = b, \quad \tilde{S} b = b, \quad \text{and} \quad \tilde{T} a = a,
\]

whence we see that \( \mathcal{G}_U \) is a formal symplectic groupoid with separation of variables over \( M \) and that the dual formal groupoid \( \tilde{\mathcal{G}}_U \) is a formal symplectic groupoid with separation of variables with respect to the opposite complex structure on \( M \). Proposition 6, formulas (74) and (75) imply that

\[
\{Qa, \tilde{a}\}_{T^*M} = \{Q \tilde{T} a, \tilde{a}\}_{T^*M} = \{Ta, S \tilde{a}\}_{T^*M} = 0 \quad \text{and} \quad \{Qb, \tilde{b}\}_{T^*M} = \{Q \tilde{S} b, \tilde{b}\}_{T^*M} = \{S b, T \tilde{b}\}_{T^*M} = 0.
\]

We would like to draw the reader’s attention to the analogy between formulas (62) and (76). There exists a unique element \( F \in \mathcal{J}^2 \) such that \( Q = \exp H_F \). Represent it as

\[
F = F_2 + F_3 + \ldots,
\]

where \( F_q \) is the homogeneous component of \( F \) of degree \( q \) with respect to the variables \( \zeta_k, \bar{\zeta}_l \). Extracting the homogeneous components of degree \( n - 2 \) of the left-hand sides of (76) and equating them to zero we obtain the following formulas where we drop the subscript \( T^*M \) in all the Poisson brackets:

\[
\begin{align*}
\{\{F_n, a\}, \tilde{a}\} &= -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1 + \ldots + i_k = n-1} \{\{F_{i_1}, \ldots, \{F_{i_k}, a\} \ldots\}, \tilde{a}\},
\{\{F_n, b\}, \tilde{b}\} &= -\sum_{k=2}^{n-1} \frac{1}{k!} \sum_{i_1 + \ldots + i_k = n-1} \{\{F_{i_1}, \ldots, \{F_{i_k}, b\} \ldots\}, \tilde{b}\}.
\end{align*}
\]

The right-hand sides of (78) depend only on \( F_q \) for \( q < n \) and are assumed to be equal to zero for \( n = 2 \).
Lemma 14. Let $\Phi_q = \Phi(z, \bar{z}, \zeta, \bar{\zeta})$ be a homogeneous function of degree $q$ in the variables $\zeta, \bar{\zeta}$ on $T^*U$ such that $\{\{\Phi_q, z^i\}_{T^*M}, z^k\}_{T^*M} = 0$ and $\{\{\Phi_q, \bar{z}^j\}_{T^*M}, \bar{z}^l\}_{T^*M} = 0$ for any $i, j, k, l$. Then $\Phi_2 = \phi^{\bar{z}k}(z, \bar{z})\zeta_k \bar{\zeta}_l$ for some function $\phi^{\bar{z}k}$ on $U$ and $\Phi_q = 0$ for $q \geq 3$.

Proof. Using formula (68) we get that $\{\{\Phi_q, z^i\}_{T^*M}, z^k\}_{T^*M} = \partial^i \partial^k \Phi_q = 0$ and $\{\{\Phi_q, \bar{z}^j\}_{T^*M}, \bar{z}^l\}_{T^*M} = \bar{\partial}^j \bar{\partial}^l \Phi_q = 0$, whence the Lemma follows. □

Lemma 14 applied to formulas (78) implies that function (77) is uniquely determined by the term $F_2$ which is of the form $F_2 = \phi^{\bar{z}k}(z, \bar{z})\zeta_k \bar{\zeta}_l$. We can find $F_2$ explicitly using formulas (69), (73), and (74). For an arbitrary $f = f(z, \bar{z}) \in C^\infty(U)$ calculate the both sides of the formula $Sf = Q(Sf)$ modulo $J^2$:

$$\begin{align*}
(1 + \zeta_k D_k)f &= (1 + H_{F_2})(1 - \zeta_l D_l)f \quad (\text{mod } J^2).
\end{align*}$$

It follows from formulas (68) and (79) that $\partial^k F_2 = g^{\bar{z}l}\zeta_l$, whence we obtain that $\phi^{\bar{z}k} = g^{\bar{z}k}$ and therefore

$$F_2 = g^{\bar{z}k}\zeta_k \bar{\zeta}_l.$$

The remaining terms of series (77) can be found recursively from (78) in local coordinates. Formula (49) implies that $F_k = 0$ for the odd values of $k$. We conclude that the function $F$ and the automorphism $Q = \exp H_F$ are uniquely determined by the Kähler-Poisson tensor $g^{\bar{z}k}$. Since condition (76) on $Q$ is coordinate independent, both $F$ and $Q$ are globally defined on $(T^*M, Z)$. It follows from formulas (74) and (75) that for $f(z, \bar{z}) = a(z)b(\bar{z})$

$$Sf = S(ab) = Sa \cdot Sb = a \cdot Qb$$

is completely determined by $Q$ which means that the source mapping $S$ is uniquely defined and global on $M$. The following theorem is a consequence of Theorem 1.

Theorem 5. For any Kähler-Poisson manifold $M$ there exists a unique formal symplectic groupoid with separation of variables on $(T^*M, Z)$ over $M$. Its source and target mappings are given locally by formulas (69).

Now let $\ast$ be a star product with separation of variables on a Kähler-Poisson manifold $M$. Theorem 4 states that it is natural. The formal symplectic groupoid of the star product $\ast$ is the unique formal symplectic groupoid with separation of variables on $(T^*M, Z)$ over $M$. According to Proposition 5 the formal Berezin transform $B$ of the star product $\ast$ is of the form $B = \exp \frac{1}{\nu}X$, where $X$ is a natural formal
differential operator on $M$. Using formula (51) we can derive from (62) and (76) that
\[ \sigma(X) = F, \]
where $F = F_2 + F_4 + \ldots$ is determined by the condition that $F_2 = \text{Symb}_2(\Delta) = g^{ik} \zeta_k \zeta_l$ and equations (78).

8. Appendix

In this section we give a proof of Theorem 2. To this end we need some preparations.

Let $K = (i_1, \ldots, i_n)$ be a multi-index. Denote by $K' = (i_2, i_1, \ldots, i_n)$ the multi-index obtained from $K$ by permuting $i_1$ and $i_2$, and by $\tilde{K} = (j_1, \ldots, j_n)$ the multi-index such that $j_1 = i_1$ and $j_2 \leq \ldots \leq j_n$ is the ordering permutation of $i_2, \ldots, i_n$. If $u_K = u_{i_1 \ldots i_n}$ is a tensor symmetric in $i_2, \ldots, i_n$ then the tensor
\[ v^K = u^K - u^{K'} \]
is skew symmetric in $i_1, i_2$, symmetric in $i_3, \ldots, i_n$ and its cyclic sum over $i_1, i_2, i_3$ is zero.

Lemma 15. Suppose that $v^K = v^{i_1 \ldots i_n}$ is a tensor skew symmetric in $i_1, i_2$, symmetric in $i_3, \ldots, i_n$ and its cyclic sum over $i_1, i_2, i_3$ is zero. There exists a unique tensor $u^K$ symmetric in $i_2, \ldots, i_n$ that satisfies (80) and such that $u^K = 0$ if $i_1 \leq \ldots \leq i_n$.

Proof. To define $u^K$, consider $\tilde{K} = (j_1, \ldots, j_n)$. Set $u^K = 0$ if $j_1 \leq j_2$ and $u^K = v^K$ if $j_1 \geq j_2$ (these conditions agree if $j_1 = j_2$). Thus $u^K = u^{\tilde{K}}$ which implies that $u^K$ is symmetric in $i_2, \ldots, i_n$. In order to show that $u^K$ is well defined we need to check condition (80). For the multi-index $K$ in (80) we can assume without loss of generality that $i_2 \leq i_1$ and that $i_3 = \min\{i_3, \ldots, i_n\}$. If $i_2 \leq i_3$ then $u^K = v^K$ and $u^{K'} = 0$, so (80) holds. If $i_3 < i_2$ then $u^K = v^{i_3 i_2 \ldots}$, $u^{K'} = v^{i_2 i_3 \ldots}$ where the order of the remaining indices does not matter. Now (80) holds since the cyclic sum of the tensor $v^K$ over $i_1, i_2, i_3$ is zero. \qed

For a coherent family $\{C_n\}$ and any $f_i, \phi \in C^\infty(M)$ one can prove the following formula using Property B.
\[ C_n(\phi, f_2, \ldots, f_n) = C_n(f_2, \ldots, f_n, \phi) + \sum_{i=2}^{n} C_{n-1}(f_2, \ldots, \{\phi, f_i\}, f_n). \]

(81)

Let $(U, \{x^i\})$ be an arbitrary coordinate chart on $M$. We will construct an operator $C_n$ locally on $U$ using induction on $n$. Assume that
one can extend by one element any $k$-element coherent family for all $k < n$. Consider an $n$-element coherent family $\{C_k\}, 0 \leq k \leq n - 1$. Then for each index $i$ and $k < n - 1$ the operators

$$D^i_k(f_1, \ldots, f_k) = C_{k+1}(f_1, \ldots, f_k, x^i)$$

form a coherent family. By induction this family can be extended by an operator $D^i_{n-1}$ so that

$$D^i_{n-1}(f_2, \ldots, f_k, f_{k+1}, \ldots f_n) - D^i_{n-1}(f_2, \ldots, f_k, f_{k+1}, f_k, \ldots f_n) = C_{n-1}(f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n, x^i),$$

Introduce the following auxiliary operator

$$D_{n}(f_1, \ldots, f_n) = \left(D^i_{n-1}(f_2, \ldots, f_n) + \sum_{j=2}^{n} C_{n-1}(f_2, \ldots, \{x^i, f_j\}, \ldots f_n) \frac{\partial f_1}{\partial x^i}\right).$$

The operator $D_n$ annihilates constants and is of order one in the first argument. We will show that for any $k \geq 2$

$$D_n(f_1, \ldots, f_k, f_{k+1}, \ldots f_n) - D_n(f_1, \ldots, f_{k+1}, f_k, \ldots f_n) =$$

$$C_{n-1}(f_1, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n).$$

Using that a derivation $A(f)$ on $U$ can be written as $A(x^i) \frac{\partial f}{\partial x^i}$, Property A, and formula (82) we can show that equation (84) is a consequence of the following one:

$$C_{n-1}(f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n, x^i) +$$

$$\sum_{j=2}^{k-1} C_{n-2}(f_2, \ldots, \{x^i, f_j\}, \ldots, f_k, f_{k+1}, \ldots f_n) +$$

$$\left(C_{n-2}(f_2, \ldots, \{x^i, f_k\}, f_{k+1}, \ldots f_n) + C_{n-2}(f_2, \ldots, \{f_k, \{x^i, f_{k+1}\}\}, \ldots, f_n) + \right.$$\n
$$\sum_{j=k+2}^{n} C_{n-2}(f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, \{x^i, f_j\}, \ldots f_n) =$$

$$C_{n-1}(x^i, f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n).$$

Using the Jacobi identity, replace the sum in the parentheses in (85) with $C_{n-2}(f_2, \ldots, \{x^i, \{f_k, f_{k+1}\}\}, \ldots f_n)$. The resulting identity follows from formula (81).

We will construct the operator $C_n$ on the coordinate chart $U$ in the form $C_n = D_n + E_n$, where $E_n(f_1, \ldots, f_n)$ is a multiderivation.
symmetric in $f_2, \ldots, f_n$. The operator $E_n$ must be chosen so that $C_n$ would satisfy Property B for $k = 1$ (all other conditions on $C_n$ are already satisfied). This condition can be written in the form

\begin{equation}
V_n(f_1, f_2, \ldots, f_n) = E_n(f_2, f_1, \ldots, f_n) - E_n(f_1, f_2, \ldots, f_n),
\end{equation}

where the operator $V_n$ is given by the formula

\begin{equation}
V_n(f_1, f_2, \ldots, f_n) = D_n(f_1, f_2, \ldots, f_n) - D_n(f_2, f_1, \ldots, f_n) - C_{n-1}(\{f_1, f_2\}, \ldots, f_n).
\end{equation}

According to Lemma 15, an operator $E_n$ with the required properties exists if $V_n(f_1, f_2, \ldots, f_n)$ is a multiderivation skew symmetric in $f_1, f_2$, symmetric in $f_3, \ldots, f_n$, and such that the cyclic sum of $V_n$ over $f_1, f_2, f_3$ is zero. We will show that the operator $V_n$ enjoys all these properties. Check that the operator $V_n$ is a derivation in the second argument. Substituting formula (83) in (87) and taking into account Property A we see that it remains to check that the operator

\begin{equation}
C_{n-1}(\{x^i, f_2\}, f_3, \ldots, f_n) \frac{\partial f_1}{\partial x^i} - C_{n-1}(\{f_1, f_2\}, f_3, \ldots, f_n)
\end{equation}

is a derivation in $f_2$. Formula (88) can be rewritten as follows,

\begin{equation}
C_{n-1}(x^i, f_3, \ldots, f_n) \left( \frac{\partial f_1}{\partial x^i} \frac{\partial}{\partial x^j} \{x^i, f_2\} - \frac{\partial}{\partial x^j} \{f_1, f_2\} \right).
\end{equation}

In local coordinates $\{f, g\} = \eta^{kl} \frac{\partial f}{\partial x^k} \frac{\partial g}{\partial x^l}$, where $\eta^{kl}$ is a Poisson tensor. The second factor in (89) equals

\begin{equation}
\frac{\partial f_1}{\partial x^k} \frac{\partial}{\partial x^j} \left( \eta^{kl} \frac{\partial f_2}{\partial x^l} \right) - \frac{\partial}{\partial x^j} \left( \eta^{kl} \frac{\partial f_1}{\partial x^k} \frac{\partial f_2}{\partial x^l} \right) = -\eta^{kl} \frac{\partial^2 f_1}{\partial x^j \partial x^k} \frac{\partial f_2}{\partial x^l}.
\end{equation}

Thus $V_n(f_1, f_2, \ldots, f_n)$ is a derivation in $f_2$. Since it is obviously skew symmetric in $f_1, f_2$, it is also a derivation in $f_1$.

We will prove that $V_n(f_1, f_2, \ldots, f_n)$ is symmetric in $f_3, \ldots, f_n$ using formula (84). For $k \geq 3$

\begin{equation}
V_n(f_1, \ldots, f_k, f_{k+1}, \ldots, f_n) - V_n(f_1, \ldots, f_{k+1}, f_k, \ldots, f_n) = \nonumber
C_{n-1}(f_1, f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) - C_{n-1}(f_1, f_2, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) - C_{n-2}(\{f_1, f_2\}, \ldots, \{f_k, f_{k+1}\}, \ldots, f_n) = 0.
\end{equation}
It remains to show that $V_n(f_1, f_2, \ldots, f_n)$ is a derivation in $f_3$ and that its cyclic sum over $f_1, f_2, f_3$ is zero. We have, using formula (84), that

$$V_n(f_1, f_2, f_3, \ldots, f_n) = D_n(f_1, f_2, f_3, \ldots, f_n) - D_n(f_2, f_1, f_3, \ldots, f_n) - C_{n-1}(\{f_1, f_2\}, f_3, \ldots, f_n) =$$

$$D_n(f_1, f_3, f_2, \ldots, f_n) + C_{n-1}(f_1, \{f_2, f_3\}, \ldots, f_n) - D_n(f_2, f_1, f_3, \ldots, f_n) - C_{n-1}(\{f_1, f_2\}, f_3, \ldots, f_n) =$$

$$D_n(f_1, f_3, f_2, \ldots, f_n) - D_n(f_2, f_1, f_3, \ldots, f_n) + C_{n-2}(\{f_1, \{f_2, f_3\}\}, \ldots, f_n).$$

We see that the cyclic sum of $V_n$ over $f_1, f_2, f_3$ is zero due to the Jacobi identity. Therefore,

$$V_n(f_1, f_2, f_3, \ldots, f_n) = -V_n(f_2, f_3, f_1, \ldots, f_n) - V_n(f_3, f_1, f_2, \ldots, f_n).$$

We have already proved that $V_n$ is a derivation in the first two arguments, whence the right hand side and therefore the left hand side of (90) are derivations in $f_3$. Since $V_n(f_1, \ldots, f_n)$ was shown to be symmetric in $f_3, \ldots, f_n$, this implies that $V_n$ is a multiderivation. This concludes the proof of all the properties of the operator $V_n$ and provides a local construction of the operator $C_n$. Finally, we use partition of unity to construct $C_n$ globally on $M$.

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