Chapter 1

Spatial structure of Cooper pairs in nuclei

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We discuss the spatial structure of the Cooper pair in dilute neutron matter and neutron-rich nuclei by means of the BCS theory and the Skyrme-Hartree-Fock-Bogoliubov model, respectively. The neutron pairing in dilute neutron matter is close to the region of the BCS-BEC crossover in a wide density range, giving rise to spatially compact Cooper pair whose size is smaller than the average interparticle distance. This behavior extends to moderate low density (∼10^{-1} of the saturation density) where the Cooper pair size becomes smallerst (∼5 fm). The Cooper pair in finite nuclei also exhibits the spatial correlation favoring the coupling of neutrons at small relative distances r<∼3 fm with large probability. Neutron-rich nuclei having small neutron separation energy may provide us opportunity to probe the spatial correlation since the neutron pairing and the spatial correlation persists also in an area of low-density neutron distribution extending from the surface to far outside the nucleus.

1. Introduction

The formation and the condensation of the Cooper pairs are the essence of superconductivity and superfluidity in many-Fermion systems. The binding energy of the Cooper pair is closely related to the pairing gap Δ. The spatial size of the Cooper pair is identified to the coherence length ξ of the superconductors, which plays important roles in many aspects, for instance, in distinguishing the type I and type II superconductors. What is the size of the Cooper pair in the superfluidity of nuclear systems? A simple estimate of the coherence length ξ, based on the uncertainty principle in uniform matter, leads to ξ ~ ℏv_F / 2Δ with v_F being the Fermi velocity. If one considers saturated nuclear matter as a simplification of finite nuclei, and
adopts the typical value of the pairing gap $\Delta \approx 12/\sqrt{A} \sim 1$ MeV appropriate for heavy nuclei, the estimate gives $\xi \sim 20$ fm which is much larger than the radius of nuclei $R \approx 1.2A^{1/3} \sim 3-7$ fm or interparticle distance $\sim 2.5$ fm in saturated matter. However, if one considers extreme situations, such as dilute neutron matter and exotic nuclei with large neutron excess, there appear new features of the nuclear pairing that can be related to the spatial structure of the Cooper pair. It is the aim of this article to illustrate it using a few examples.

2. Dilute neutron matter

The superfluidity in neutron matter is density dependent. The pairing gap can be obtained by solving the BCS equations for the bare nuclear force in the $^1S$ channel at each neutron density $\rho = k_F^3/3\pi^2$ or the Fermi momentum $k_F$. The gap is small $\Delta \ll 1$ MeV at $k_F = 1.36$ fm$^{-1}$ ($\rho/\rho_0 = 1$, the neutron density at saturation $\rho_0 = 0.08$ fm$^{-3}$). With decreasing the density it first increases, reaching the maximum $\Delta \approx 3$ MeV around $k_F \approx 0.8$ fm$^{-1}$ ($\rho/\rho_0 \approx 0.2$), then decreases and approaches to zero at the low-density limit. Other many-body medium effects which are beyond the BCS approximation reduce the gap, but the predictions vary depending on the theoretical methods. Recent ab initio Monte Carlo calculations, on the other hand, predict rather modest reduction by less than 50%, and the qualitative features of the density dependence is kept. Having these reservations in mind, let us consider the structure of the neutron Cooper pair in the BCS approximation.

The Cooper pair wave function can be defined, apart from the normalization, as an expectation value of the pair operator with respect to the BCS state:

$$\Psi_{\text{pair}}(r_1, r_2) = \langle \psi(r_1 \uparrow)\psi(r_2 \downarrow) \rangle = \sum_k u_k v_k e^{i k \cdot r}.$$ (1)

It is a function of the relative coordinate $r = r_2 - r_1$ of the two neutrons, and in the momentum space it is a product of the $u$ and $v$ factors. Examples of the Cooper pair wave functions are shown in Fig. 1 for two different densities. The wave function exhibits an oscillatory behavior characterized by the Fermi wave length $2\pi/k_F$ and an overall decay profile whose asymptotic form is exponential $\sim \exp(-r\Delta/\hbar v_F)$ (for large relative distance $r = |r_2 - r_1|$) whose length scale is nothing but the coherence length, or the size of the Cooper pair. More precisely, the coherence
length can be calculated as the rms radius of the Cooper pair $\xi = \sqrt{\langle r^2 \rangle}$ with 
$$\langle r^2 \rangle = \int d^3r r^2 |\Psi_{\text{pair}}(r)|^2 / \int d^3r |\Psi_{\text{pair}}(r)|^2.$$ 

An interesting feature of the neutron Cooper pair in superfluid neutron matter is that its size also varies significantly with changing the neutron density (See Fig.1(a)). From a very large value $\xi = 46$ fm at $\rho/\rho_0 = 1$, the coherence length $\xi$ decreases sharply with decreasing the density. The coherence length takes the smallest values $\xi = 5 - 8$ fm for a rather wide range of the density $\rho/\rho_0 = 0.2 - 10^{-2}$, and it increases gradually with decreasing the density.

![Fig. 1. (a) The coherence length $\xi$ and the average interparticle distance $d$ in superfluid neutron matter, plotted as a function of the neutron density $\rho/\rho_0$ (with $\rho_0 = 0.08$ fm$^{-3}$). (b)(c) The Cooper pair wave function $r^2 |\Psi_{\text{pair}}(r)|^2$ at densities $\rho/\rho_0 = 1$ and 1/8. The bare force $G_{3RS}^{14}$ is used in the present BCS calculation.](image)

The Cooper pair wave function at densities where the coherence length is the smallest is very different from that of the electron Cooper pair in the traditional metal superconductors. An example is shown in Fig.1 (c), which is for $\rho/\rho_0 = 1/8$ where the coherence length $\xi = 4.9$ fm is close to the minimum value. It is seen that the oscillatory behavior is strongly suppressed. The probability distribution is concentrated ($\sim 80\%$) at small relative distances within the first node $r < \pi/k_F \approx 4.5$ fm, and the probability at the second and third bumps is very small. This is because the size of the Cooper pair ($\xi = 4.9$ fm) is almost equal to the position of the first node $\pi/k_F$ which is nothing but the average interparticle distance $d = \rho^{-1/3} \approx \pi/k_F$. The size of the Cooper pair is "small" in this sense. This is quite contrasting to the metal superconductors where the Cooper pair size $\xi$ is thousands times larger than the average interparticle distance $d$. The situation of the "small" Cooper pair $\xi \ll d$ is seen in a wide interval of densities $\rho/\rho_0 = 10^{-4} - 10^{-1}$ (Fig.1(a)). The shape of the Cooper pair wave
function at these densities is similar to that of Fig.1(c), and the probability is even more concentrated in the first bump although the absolute size is larger at very low densities $\rho/\rho_0 \approx 10^{-4} - 10^{-2}$. It is noted that the Cooper pair at moderate low densities $\rho/\rho_0 \approx 10^{-1} - 0.5$ exhibits also the strong spatial correlation at small relative distances. The wave function at $\rho/\rho_0 = 0.5$ is shown in Fig.1(b). In this case the calculated coherence length $\xi = 11$ fm is a few times larger than the average interparticle distance $d = 2.8$ fm. Nevertheless the concentration of the probability within the relative distance $r < \sim 3$ fm (in the first bump) is significant, and the probability in $r < 3$ fm reaches as large as $\sim 50\%$.

The situation of the small Cooper pair $\xi/d \lesssim 1$ is related to the so-called BCS-BEC crossover phenomenon\textsuperscript{16,17} which has been discussed intensively in ultra-cold Fermi atom gas in a trap.\textsuperscript{20,21} It is a phenomenon which can occur generally in any kind of many-Fermion superfluid systems by changing the strength of the interparticle attractive force or the density. In a situation of the weak interaction, which the original BCS theory has dealt with, the bound pair (the Cooper pair) can be formed only in the medium. However, if the interaction is as strong as to form a bound pair (a composite boson) even in the free space, the condensed phase is more close to a condensate of the composite bosons, i.e. the Bose-Einstein condensate (BEC). The BCS-BEC crossover is characterized by the ratio $\xi/d$ of the coherence length and the average interparticle distance and the ratio $\Delta/e_F$ of the pairing gap and the Fermi energy. The weak-coupling BCS and the BEC limits correspond to $\xi/d \gg 1$, $\Delta/e_F \ll 1$ and $\xi/d \ll 1$, $\Delta/e_F \gg 1$, respectively while the region of the crossover may be related to $0.2 \lesssim \xi/d \lesssim 1.2$ and $0.2 \lesssim \Delta/e_F \lesssim 1.3$.\textsuperscript{17,19} At the midway of the crossover, called the unitarity limit, the interaction strength is on the threshold to form the isolated two-particle bound state, and the values are $\xi/d = 0.36$, $\Delta/e_F = 0.69$. In the BCS calculation discussed above\textsuperscript{16} small $\xi/d$ ratio $0.7 - 1.2$ and large $\Delta/e_F$ ratio $0.2 - 0.4$ is realized at $\rho/\rho_0 \sim 10^{-4} - 10^{-1}$. (Note that also in an ab initio calculation\textsuperscript{13} the large gap ratio $\Delta/e_F \sim 0.2 - 0.3$ is obtained in approximately the same but slightly small density region.) We can regard dilute neutron matter in the wide low-density interval $\rho/\rho_0 = 10^{-4} - 10^{-1}$ (or in slightly narrower interval) as being in the crossover region. We note here that the nuclear force in the $^1S$ channel has a large scattering length $a = -18$ fm, indicating that the interaction strength is very close to the threshold to form a two-neutron bound state. The small Cooper pair $\xi/d \lesssim 1$ at low densities originates from the nature of the nuclear force.
3. Cooper pair in neutron-rich nuclei

Let us consider the spatial structure of the Cooper pair in finite nuclei.

The spatial structure of the correlated two neutrons has been discussed intensively for two neutrons in the light two-neutron halo nuclei $^9\text{Li}$ and $^6\text{He}$ in (inert or active) core plus two neutron models.\(^{22-30}\) A common prediction is that the valence halo neutrons exhibit a spatial correlation favoring the 'di-neutron' configuration with two neutrons coupled at small relative distances. The spatial correlation is also discussed in stable heavy nuclei with closed-shell core plus two neutrons, e.g. $^{206,210}\text{Pb}$, by means of shell model approaches.\(^{31-35}\) One can generalize these findings by using the Hartree-Fock-Bogoliubov (HFB) method, which can be applied to a wide class of open shell nuclei including isotopes very close to the drip-line and also to non-uniform matter.

Let us start defining the wave function of the Cooper pair in finite nuclei. It may be given by

$$\Psi_{\text{pair}}(r_1, r_2) = \langle \Phi_{A-2} | \psi(r_1 \uparrow) \psi(r_2 \downarrow) | \Phi_A \rangle \quad (2)$$

using the pair correlated ground states $\Phi_A$ and $\Phi_{A-2}$. This represents the probability amplitude of removing two neutrons (positioned at $r_1$ and $r_2$) from the ground state $\Phi_A$, and leaving the remaining system in the ground state $\Phi_{A-2}$. Provided that the ground state is described within the HFB framework, where the ground states with different nucleon numbers are represented by a single HFB state $\Phi_{\text{HFB}}$, the definition Eq. (2) can be replaced with the expectation value as in Eq. (1). Then, since the HFB state is a generalized Slater determinant consisting of the Bogoliubov quasiparticle states, this quantity is evaluated\(^{36}\) as a sum over all quasiparticle states $i$

$$\Psi_{\text{pair}}(r_1, r_2) = \langle \Phi_{\text{HFB}} | \psi(r_1 \uparrow) \psi(r_2 \downarrow) | \Phi_{\text{HFB}} \rangle = \sum_i \varphi_i^{(1)}(r_1 \uparrow) \varphi_i^{(2)*}(r_2 \downarrow) \quad (3)$$

using the first and the second components of the quasiparticle wave function $\phi_i(r\sigma) = (\varphi_i^{(1)}(r\sigma), \varphi_i^{(2)}(r\sigma))$. In the following we show the results of our HFB calculation, which adopts the Skryme functional and the density-dependent contact interaction as a phenomenological pairing force.\(^{37,38}\) The parameter set of the pairing interaction is such that it reproduces the scattering length $a = -18$ fm in the low-density limit, and reproduces the average pairing gap in known nuclei.\(^{37,38}\)

An example calculated for $^{142}\text{Sn}$ is shown in Fig.2. Here one neutron is fixed at the position slightly outside the nucleus $r_1 = 7$ fm and the prob-
Fig. 2. (Left) Cooper pair wave function $|\Psi_{pair}(r_1, r_2)|^2 / \rho_n(r_1)$ in neutron-rich nucleus $^{142}\text{Sn}$, plotted as a function of $r_2$ on the $xz$ plane while the coordinate $r_1$ is fixed to $(0, 0, 7)$ fm located slightly outside the surface. (Right) The same but plotted along the $z$-axis. Different curves are results obtained by putting cut-off’s with respect to the orbital angular momentum $l$ of the single-particle orbits.

ability distribution $|\Psi_{pair}(r_1, r_2)|^2$ is plotted as a function of $r_2$. It shows that the second neutron has a large probability ($\sim 50\%$) to be correlated at small relative distances $|r_1 - r_2| \lesssim 3$ fm to the partner neutron. The spatial correlation seen here is generic in a sense that it is seen systematically in Ca, Ni, and Sn isotopes including both stable and neutron-rich nuclei. The strong spatial correlation is also seen in other HFB calculations which adopt the finite-range Gogny force as the effective pairing force. In order to describe the correlation with the length scale $D \sim 3$ fm, the single-particle basis needs to cover a momentum range up to $p_{\text{max}} \sim h/D$, which corresponds to a maximal energy $e_{\text{max}} \sim p_{\text{max}}^2/2m \sim 80$ MeV, or a maximal angular momentum $l_{\text{max}} \sim R p_{\text{max}} \sim 10\hbar$ (for the nuclear radius $R \sim 5$ fm). This is demonstrated in Fig.2(right), where the summation over the quasiparticle states $i$ in Eq.(3) is truncated by introducing a cut-off with respect to the orbital angular momentum $l$. Single-particle orbits with large angular momentum up to $l_{\text{max}} \sim 10$ have sizable contributions. Note that in $^{142}\text{Sn}$ with $N = 92$ the Fermi energy is around the $3p_{3/2}$ orbit, and the maximal orbital angular momentum of the orbits occupied in the independent particle limit is $l = 5$. The single-particle states with $l = 5 - 10$ lie high above the Fermi energy. If one uses the harmonic oscillator basis, it should include $\sim 10$ oscillator quanta. In fact, all the HFB calculations where the strong spatial correlation in the Cooper pair wave functions is demonstrated adopt such a large single-particle space.

Equivalently, a small single-particle space is insufficient. If we restrict ourselves to a single-$j$ shell ($nlj$), i.e., the sum in Eq.(3) is restricted to the
magnetic substates of the orbit \((nlj)\), we obtain the angular correlation \(P_1(\theta_{12})\) for small relative angles \(\theta_{12} \lesssim 1/l\), but the correlation with respect to the radial direction is not produced. Inclusion of all the orbits in one oscillator shell still has deficiency. The Cooper pair wave function in this case exhibits an artificial symmetry \(\Psi_{\text{pair}}(r_1, r_2) = \pm \Psi_{\text{pair}}(-r_1, r_2)\) because of the common single-particle parity, and the probability appears not only around \(r_2 \sim r_1\), but also around the mirror reflected position \(r_2 \sim -r_1\).

The spatial correlation of neutron Cooper pairs plays an important role if we consider neutron-rich nuclei with small neutron separation energy. Nuclei of this kind often accompany low-density distribution of neutrons, called skin or halo, extending from the nuclear surface toward the outside. Figure 3(a) is an example of the pair potential \(\Delta(r)\) for the very neutron rich nucleus \(^{142}\text{Sn}\) obtained in the same Skyrme-HFB calculation as in Fig.2. The pair potential \(\Delta(r)\) exhibits significant enhancement around \(r \sim 5 - 8\) fm, which is slightly outside the nuclear surface (the corresponding neutron density there is about 1/2-1/10 of the central density). The pair potential decreases rather slowly with moving outside the surface region, and it is about to diminish only at very large distances \(r \gtrsim 12\) fm. It is much more extended than the neutron density. Furthermore the spatial correlation persists in this low density region as shown in Fig.3(b). We note that the spatial correlation is present also in stable open shell nuclei and it is enhanced around the nuclear surface. However the nucleons (and hence the Cooper pairs) do not penetrate far outside the surface in stable isotopes (cf bottom panel of Fig.3(b)). The pair correlations in the dilute surrounding
is a unique feature of weakly bound nuclei.

4. Probing the spatially correlated Cooper pair

4.1. Soft modes

If spatially correlated di-neutrons exist in nuclei, especially in the low-density skin/halo region, there may emerge new modes of excitation reflecting the motion of di-neutron(s). This simple idea has been a focus of theoretical and experimental studies of the soft dipole excitation in two-neutron halo nuclei. Although the reality is not that simple, the core+n+n models of $^{11}$Li explain the observed large E1 strength of the soft dipole excitation in terms of the pairing and the spatial correlation of the valence halo neutrons. It is interesting to explore possibility of similar excitation modes in heavier mass neutron-rich nuclei, where more than two weakly bound neutrons contribute to the pair correlation.

A useful scheme to describe excitation modes built on the pair correlated ground state is the quasiparticle random phase approximation (QRPA). Let us take the formulation based on the same Skyrme-HFB model that is used for the description of the ground state. Having a QRPA excited state $|n,LM\rangle$, one can calculate the two-particle amplitude $\langle n,LM|\psi^\dagger(r_1 \uparrow)\psi^\dagger(r_2 \downarrow)|0_{gs}\rangle$, which tells us how two particles move in the excited state $|n,LM\rangle$ in reference to the ground state (of the $N−2$ system). For simplicity let us look at the zero-range part at $r_1 = r_2$ of the amplitude:

$$P_{pair_n}^{pair}(r) = \langle n,LM|\psi^\dagger(r \uparrow)\psi^\dagger(r \downarrow)|0_{gs}\rangle,$$

which is called the pair transition density.

Figure 4 is an example of soft dipole excitation which suggests motion of the spatially correlated di-neutrons. The soft dipole excitation is seen here as a bump of the E1 strength which lies just above the neutron separation energies ($S_{1n}, S_{2n} = 1.9, 2.4$ MeV). In neutron-rich Ni isotopes beyond the $N = 50$ shell closure both of the one- and the two-neutron separation energies are calculated to be very low $S_n, S_{2n} \approx 1−3$ MeV. In such weakly bound nuclei, the low-lying dipole modes appear just above the separation energy since it is possible to excite a bound neutron to unbound orbits in the continuum, letting the neutron escape from the nucleus. If the pair correlation is taken into account, however, the mode is dominated by the pair motion rather than by a simple particle-hole (or independent two-quasiparticle excitation). Consequently the pair transition density $P_{pair_n}^{pair}(r)$
Fig. 4. (a) The $B(E1)$ strength function in neutron-rich nucleus $^{84}\text{Ni}$, obtained with the Skyrme-HFB + continuum QRPA method\textsuperscript{44}. The large strengths around $E = 10 - 20$ MeV are the giant dipole resonance (GDR) while some amount of strength is distributed just above the one- and two-neutron separation threshold energies (the small arrows). (b) The neutron pair transition density $P_{\text{pair}}(r)$ of the soft dipole mode evaluated at $E = 3$ MeV (marked with the big arrow in (a)). Figures taken from Ref.\textsuperscript{44}

has larger amplitude, especially for $r > R_{\text{surf}}$, as seen in Fig.4(b). It is not explicit in this figure whether the neutron pair in the excited state is spatially correlated, but we can infer it from the observation that a large number of orbital angular momenta $l$ reaching more than $10\hbar$ have significant and coherent contributions to the pair transition density. As we discussed above (cf. Fig.2), large $l$ implies a spatial correlation at small distances between the two neutrons.

A similar mode of excitation having the character of di-neutron motion is predicted also in the octupole response in the same isotopes $^{80}\text{Ni}$ beyond $N = 50$\textsuperscript{44}. It is a smooth distribution of neutron strength lying just above the threshold energy (like the soft dipole mode), and it coexists from the octupole surface vibrational mode of the isoscalar character seen in many of stable nuclei.

In contrast to the light two-neutron halo nuclei, the presence of the spatial correlation does not influence strongly the $E1$ strength of soft dipole excitation in heavy neutron-rich nuclei such as $^{84}\text{Ni}$. We need other probes which are directly connected to the pair transition density. Since the soft dipole excitation in $^{11}\text{Li}$ and in $^{80}\text{Ni}$ is located above the two-neutron separation energy, one can expect that momentum distribution/correlation of two neutrons emitted from the soft mode may carry information on the spatial correlation of the neutron pair. Quantitative theoretical description of the two-neutron correlation is achieved only for the core+n+n models\textsuperscript{46,47} for $^{11}\text{Li}$ and $^{6}\text{He}$, and experimental information
is very scarce so far. It is possible to describe the two-neutron correlation in the continuum also in the framework of the QRPA since the information on the directions of two neutrons are contained in the pair transition density $\langle n, LM | \psi^\dagger(r_1 \uparrow)\psi^\dagger(r_2 \downarrow) | 0_{gs} \rangle$ especially in its asymptotic form at $|r_1|, |r_2| \to \infty$. It is an interesting future subject to study in heavier neutron-rich nuclei such as $^{80}$Ni using the HFB+QRPA formalism.

4.2. Two-neutron transfer

The two-neutron transfer reactions such as (p,t) and (t,p) are known as a good probe to the pair correlation in the ground state. More precisely it can be regarded as a probe of the Cooper pair wave function, especially its behavior at small relative distances between the paired neutrons. Consider the (p,t) reaction populating the ground state of the neighboring $N-2$ nucleus in the single-step DWBA and the zero-range approximation. Then the transition matrix elements involves the form factor $F(R) = \int dr \langle 0_{gs,N-2} | \psi(R + r/2 \uparrow)\psi(R - r/2 \downarrow) | 0_{gs,N} \rangle \phi(r)$

which is the convolution of the Cooper pair wave function $\Psi_{pair}(R + r/2, R - r/2)$ with the two-particle wave function $\phi(r)$ in the triton. Noting the small radius of the triton $\sim 2$ fm, we see immediately the form factor picks up the correlation at small relative distances in the Cooper pair wave functions. It is then not a very bad approximation to utilize the Cooper pair wave function at zero relative distance $r = 0$, i.e. $\Psi_{pair}(R, R) = \langle \psi(R \uparrow)\psi(R \downarrow) \rangle \equiv P_{pair}(R)$ as a substitute of the form factor assuming $F(R) \propto P_{pair}(R)$. $P_{pair}(R)$ is nothing but the pair density $\rho(R)$ implemented automatically in the Skyrme-HFB model using the pairing force of the contact type.

An example of the calculated pair transition density $P_{pair}(R)$ is shown in Fig.5(a) for Sn isotopes covering from stable isotopes to very neutron-rich $^{150}$Sn. It is seen that the radial dependence of $P_{pair}(R)$ suddenly changes at the $N = 82$ shell closure (at $^{132}$Sn). In neutron-rich isotopes beyond $N = 82$, the amplitude extends far outside the nuclear surface $r > R_{surf} + 3$ fm ($\gtrsim 9$ fm). This happens because neutron single-particle orbits above the $N = 82$ shell gap are bound only weakly, and the weakly bound neutrons have density distributions extended far outside the nuclear surface. (The one-neutron separation energy is the order of $\sim 2 - 3$ MeV for $A > 132$, while it is more than 8 MeV in isotopes with $A \lesssim 132$.) Consequently both the pair potential $\Delta(r)$ and the Cooper pair wave function keep non-
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Fig. 5. (a) Neutron pair transition density $P_{\text{pair}}(R)$ for the ground state transition, evaluated for even-even Sn isotopes for $A = 120 - 130$ (thin solid curve), $A = 134 - 140$ (dotted), and $A = 142 - 150$ (thick solid). (b) The two-neutron transfer strength $B(P0)$ for the ground state transition (filled diamond), the strength for two-neutron addition transfer for the excited $0^+$ state (small triangle), and the squared pairing gap $\Delta^2$ (open circle) for the even-even Sn isotopes. The horizontal axis is the mass number $A$.

negligible magnitude even far outside (See also Fig.3). As seen in the figure the amplitude $P_{\text{pair}}(R)$ extends up to $r \sim 12$ fm for the isotopes $A > 140$.

The above observation leads to an expectation that the (p,t) and (t,p) cross sections may be enhanced considerably as the neutron separation energy becomes small\cite{negligible}. An estimate of the isotopic trend, much simpler than the DWBA calculation, is shown in Fig.5(b). Here is plotted the 'strength' which is defined by $B(P0) = \left| \int dR P_{\text{pair}}(R) \right|^2$. It is illuminating to compare it with the isotopic trends of the pairing gap $\Delta$ squared ($\Delta$ being an average value of the pair potential $\Delta(r)$). If the pair potential and the pair transition density are confined in the nuclear volume, a proportionality relation $B(P0) \propto \Delta^2$ is expected\cite{example} in analogy with the $B(E2)$ of the deformed rotor since the pair gap is a deformation parameter. We see in Fig.5(b) that the proportionality $B(P0) \propto \Delta^2$ valid for $100 < A < 132$ is violated for $A > 132$ and especially $A > 140$, where the strength $B(P0)$ significantly increases. The two-neutron transfer reaction on the neutron-rich isotopes (e.g. the Sn isotopes with $A > 132$) thus provides us a tool to probe the Cooper pair wave function in the low-density region far outside the nuclear surface. It is predicted\cite{prediction} also that the isotopes $^{134-140}$Sn with $A = 134 - 140$ exhibit a precursor phenomenon, i.e., an anomalously large two-neutron transfer strength of $(t,p)$ type for the transitions to the excited $0^+$ states (Fig.5(b)). Recently two-neutron transfer experiment on the halo nucleus $^{11}$Li has become available, and the crucial role of the pair correlation is demonstrated\cite{experiment}. We wait for a future experiment using the radioactive beams of neutron-rich Sn isotopes with $A > 132$. 
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