Parameterized Viscosity Solutions of Convex Hamiltonian Systems with Time Periodic Damping

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Abstract
In this article we develop an analogue of Aubry-Mather theory for time periodic dissipative equation
\[
\begin{align*}
\dot{x} &= \partial_p H(x, p, t), \\
\dot{p} &= -\partial_x H(x, p, t) - f(t) p
\end{align*}
\]
with \((x, p, t) \in T^*M \times \mathbb{T}\) (compact manifold \(M\) without boundary). We discuss the asymptotic behaviors of viscosity solutions for the associated Hamilton-Jacobi equation
\[
\partial_t u + f(t) u + H(x, \partial_x u, t) = 0, \quad (x, t) \in M \times \mathbb{T}
\]
w.r.t. certain parameters, and analyze the meanings in controlling the global dynamics. We also discuss the prospect of applying our conclusions to many physical models.

Keywords Viscosity solution · Aubry Mather theory · Convex hamiltonian · Global attractor · Rotation number

Mathematics Subject Classification (2010) 37J50 · 37J55 · 35B40 · 49L25
1 Introduction

For a smooth compact Riemannian manifold $M$ without boundary, the Hamiltonian $H$ is usually characterized as a $C^{r \geq 2}$ smooth function on the cotangent bundle $T^*M$, with the associated Hamilton’s equation defined by

$$(1)\begin{cases} \dot{x} = \partial_p H(x, p) \\ \dot{p} = -\partial_x H(x, p) \end{cases}$$

for each initial point $(x, p) \in T^*M$. From the physical aspect, the Hamilton’s equation describes the movement of particles with conservative energy, since the Hamiltonian $H(x, p)$ verifies to be a FirstIntegral of (1). In particular, if the potential periodically depends on the time $t$ (for systems with periodic propulsion or procession), we can introduce an augmented Hamiltonian

$$(2)\tilde{H}(x, p, t, l) = I + H(x, p, t), \quad (x, p, t, l) \in T^*M \times T^*\mathbb{T}$$

such that the associated Hamilton’s equation

$$(3)\begin{cases} \dot{x} = \partial_p H(x, p, t) \\ \dot{p} = -\partial_x H(x, p, t) \\ \dot{t} = 1 \\ \dot{l} = -\partial_t H(x, p, t) \end{cases}$$

still preserves $\tilde{H}$.

However, the realistic motion of the masses inevitably sustains a dissipation of energy, due to the friction from the environment, e.g. the wind, the fluid, interface, etc. That urges us to make rational modifications of previous equations. In the current paper, the damping is assumed to be time-periodically proportional to the momentum. Precisely, we modify (3) into

$$(4)\begin{cases} \dot{x} = \partial_p H(x, p, t) \\ \dot{p} = -\partial_x H(x, p, t) - f(t)p \\ \dot{t} = 1 \\ \dot{l} = -\partial_t H(x, p, t) - f'(t)u - f(t)I \\ \dot{u} = \langle H_p, p \rangle - H + \alpha - f(t)u \end{cases}$$

with $\alpha \in \mathbb{R}$ being a constant of initial energy and $f \in C^r(\mathbb{T} := \mathbb{R}/[0, 1], \mathbb{R})$. Notice that the former three equations of (4) is decoupled with the latter two, so we can denote the flow of the former three equations in (4) by $\varphi_H^t$ and by $\tilde{\varphi}_H^t$ the flow of the whole (4). The following individual cases of $f(t)$ will be considered:

- (H0) $[f] := \int_0^1 f(t)dt > 0$
- (H1) $[f] = 0$
- (H2) $[f] < 0$

Besides, we propose the following standing assumptions for the Hamiltonian:

- (H1) [Smoothness] $H : TM \times \mathbb{T} \to \mathbb{R}$ is $C^r$ smooth;
- (H2) [Convexity] For any $(x, t) \in M \times \mathbb{T}$, $H(x, \cdot, t)$ is strictly convex on $T^*_xM$;
(H3) [Superlinearity] For any \((x, t) \in M \times \mathbb{T}\), \(\lim_{|p|_x \rightarrow +\infty} H(x, p, t)/|p|_x = +\infty\) where \(|\cdot|_x\) is the norm deduced from the Riemannian metric.

(H4) [Completeness] For any \((x, p, \theta) \in T^*M \times \mathbb{T}\), the flow \(\varphi^t_H(x, p, \theta)\) exists for all \(t \in \mathbb{R}\).

Remark 1.1  

i) As we can see, the three different cases of (H0) respectively leads to a dissipation, acceleration and periodic conservation of energy along \(\tilde{\phi}_H^t\) in the forward time, if we take

\[
\hat{H}(x, p, t, I, u) = \tilde{H}(x, p, t, I) + f(t)u - \alpha.
\]  

See (40) of Section 4 for the proof.

ii) (H1-H3) are usually called Tonelli conditions. As for (H4), the completeness of \(\varphi^t_H\) is actually equivalent to the completeness of \(\tilde{\phi}_H^t\). A sufficient condition to (H4) is the following:

\[
|H_x| \leq \kappa(1 + |p|_x) \text{ for all } (x, p, t) \in T^*M \times \mathbb{T}
\]

for some constant \(\kappa\).

iii) Observe that the time-1 map \(\varphi^1_H : \{(x, p, t(\text{mod } 1) = 0)\} \rightarrow \{(x, p, t(\text{mod } 1) = 0)\}\) is conformally symplectic, i.e.

\[
(\varphi^1_H)^*dp \wedge dx = e^{[f]}dp \wedge dx.
\]

Such maps have wide applications in astronomy [9], optimal transport [32], biological physics [8] and economics [1], etc (see Sect. 2 for more details).

iv) For \(f \equiv 0\), (4) degenerates to (3), and that is just the case concerned by Mather’s variational theory [23]. Besides, Maro and Sorrentino considered another special setting: \(f \equiv \lambda > 0\) in [21]. They established an analogue of Aubry-Mather theory under such a setting. Simultaneously, [4, 29, 30] explored the variational properties for contact Hamiltonian systems which was originally proposed by Lions in [18].

In earlier works [2, 11], the authors considered a time-periodic Lagrangian system, which can be considered as a special case of equation (4) with \(f \equiv 0\). As the extension of their works, in the current paper we focus on the dynamical mechanisms of damping (or accelerating) phenomena and obtain the attractive (or repulsive) invariant sets for the case \([f] \neq 0\). Such a consideration is quite different.

1.1 Variational Principle and Hamilton-Jacobi Equation

As the dual of the Hamiltonian, the Lagrangian can be defined as the following

\[
L(x, v, t) := \max_{p \in T^*_xM} \langle p, v \rangle - H(x, p, t), \quad (x, v, t) \in TM \times \mathbb{T}.
\]  

of which the maximum is achieved at \(v = H_p(x, p, t) \in T_xM\), once (H1-H3) are assumed. Therefore, the Legendre transformation

\[
\mathcal{L} : T^*M \times \mathbb{T} \rightarrow TM \times \mathbb{T}, \quad \text{via } (x, p, t) \rightarrow (x, H_p(x, p, t), t)
\]  

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is a diffeomorphism. Notice that the Lagrangian $L : TM \times T\to \mathbb{R}$ is also $C^r-$smooth and convex, superlinear in $p \in T_x M$, so by a slight abuse of notions we say it satisfies (H1-H3) as well. As a conjugation of $\varphi_H^t$, the Euler-Lagrangian flow $\varphi^t_L$ is defined by

$$\begin{cases}
\dot{x} = u, \\
\frac{d}{dt} L_u (x, u, t) = L_x (x, u, t) - f(t) L_u (x, u, t)
\end{cases} \tag{E-L}$$

It has equivalently effective in exploring the dynamics of (4), and the completeness of $\varphi^t_L$ is equivalent to the completeness of $\varphi^t_H$. To present (E-L) a variational characterization, we introduce a minimal variation on the absolutely continuous curves with fixed endpoints

$$h^s_t (x, y) = \inf_{\gamma \in C^a([s,t], M)} \int_s^t e^{F(\tau)} (L(\gamma, \dot{\gamma}, \tau) + \alpha) d\tau,$$

where $F(t) = \int_0^t f(\tau) d\tau$ and $\alpha \in \mathbb{R}$. It is a classical result in the calculus of variations that the infimum is always available for all $s < t \in \mathbb{R}$, which is actually $C^r-$ smooth and satisfies (E-L), once (H4) is assumed (due to the Weierstrass Theorem in [23] or Theorem 3.7.1 of [15]).

**Theorem 1.2 (main 1).** For $f(t)$ satisfying (H0$^-$), $H(x, p, t)$ satisfying (H1-H4) and any $\alpha \in \mathbb{R}$, the following

$$u^-_\alpha (x, t) := \inf_{\gamma \in C^a((-\infty, t], \mathbb{R})} \int_{-\infty}^t e^{F(s) - F(t)} (L(\gamma(s), \dot{\gamma}(s), s) + \alpha) ds$$

is well defined for $(x, t) \in M \times \mathbb{R}$ and satisfies

1. (Periodicity) $u^-_\alpha (x, t + 1) = u^-_\alpha (x, t)$ for any $x \in M$ and $t \in \mathbb{R}$. By taking $\bar{t} \in [0, 1)$ with $t \equiv \bar{t} \mod 1$ for any $t \in \mathbb{R}$, we can interpret $u^-_\alpha$ as a function on $M \times \mathbb{T}$.
2. (Lipshitzness) $u^-_\alpha : M \times \mathbb{T} \to \mathbb{R}$ is Lipschitz, with the Lipschitz constant depending on $L$ and $f$;
3. (Domination$^1$) For any absolutely continuous curve $\gamma : [s, t] \to M$ connecting $(x, \bar{s}) \in M \times \mathbb{T}$ and $(y, \bar{t}) \in M \times \mathbb{T}$, we have

$$e^{F(t)} u^-_\alpha (y, \bar{t}) - e^{F(s)} u^-_\alpha (x, \bar{s}) \leq \int_s^t e^{F(\tau)} (L(\gamma, \dot{\gamma}, \tau) + \alpha) d\tau. \tag{8}$$

4. (Calibration) For any $(x, \theta) \in M \times \mathbb{T}$, there exists a backward calibrated curve $\gamma^-_x, \theta : (-\infty, \theta) \to M$, $C^r-$smooth and $\gamma^-_x, \theta (\theta) = x$, such that for all $s \leq t \leq \theta$, we have

$$e^{F(t)} u^-_\alpha (\gamma^-_x, \theta (t), \bar{t}) - e^{F(s)} u^-_\alpha (\gamma^-_x, \theta (s), \bar{s}) = \int_s^t e^{F(\tau)} (L(\gamma^-_x, \theta, \dot{\gamma}^-_x, \theta, \tau) + \alpha) d\tau. \tag{9}$$

$^1$ Any function $\omega \in C(M \times \mathbb{T}, \mathbb{R})$ satisfying (8) is called a (viscosity) subsolution of (HJ$_+$) and denoted by $\omega <_f L + \alpha$. 

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(5) (Viscosity) \( u^-_\alpha : M \times \mathbb{T} \to \mathbb{R} \) is a viscosity solution of the following Stationary Hamilton-Jacobi equation (with time periodic damping):

\[
(HJ_+) \quad \partial_t u + f(t)u + H(x, \partial_x u, t) = \alpha, \quad (x, t) \in M \times \mathbb{T}, \ \alpha \in \mathbb{R}.
\]

**Theorem 1.3** (main 1'). For \( f(t) \) satisfying \((H0^0)\) and \( H(x, p, t) \) satisfying \((H1-H4)\), there exists a unique \( c(H) \in \mathbb{R} \) (Mañé Critical Value) such that

\[
u^-_{\alpha, \xi}(x, \tilde{t}) := \lim_{t-\xi \equiv \tilde{t} \equiv t \equiv (\mathbb{Z}, M)} \inf_{\gamma \in C^\infty([\xi, \tilde{t}], M)} \int_{\xi}^{\tilde{t}} e^{f(\tau) - F(\tau)} \left( L(\gamma, \dot{\gamma}, \tau) + c(H) \right) d\tau
\]

is well defined on \( M \times \mathbb{T} \) (for any fixed \( z, \xi \) \in \( M \times \mathbb{T} \)) and satisfies

(1) (Lipschitzness) \( \nu^-_{\alpha, \xi} : M \times \mathbb{T} \to \mathbb{R} \) is Lipschitz.

(2) (Domination) For any absolutely continuous curve \( \gamma : [s, t] \to M \) connecting \( (x, \tilde{s}) \in M \times \mathbb{T} \) and \( (y, \tilde{t}) \in M \times \mathbb{T} \), we have

\[
e^{F(t)} \nu^-_{\alpha, \xi}(y, \tilde{t}) - e^{F(s)} \nu^-_{\alpha, \xi}(x, \tilde{s}) \leq \int_s^t e^{F(\tau)} \left( L(\gamma, \dot{\gamma}, \tau) + c(H) \right) d\tau.
\]

Namely, \( \nu^-_{\alpha, \xi} \prec f \ L + c(H) \).

(3) (Calibration) For any \((x, \theta) \in M \times \mathbb{T}\), there exists a \( C^r \) curve \( \gamma_{x, \theta}^- : (-\infty, \theta] \to M \) with \( \gamma_{x, \theta}^-(\theta) = x \), such that for all \( s \leq t \leq \theta \), we have

\[
e^{F(t)} \nu^-_{\alpha, \xi}(\gamma_{x, \theta}^-(t), \tilde{t}) - e^{F(s)} \nu^-_{\alpha, \xi}(\gamma_{x, \theta}^-(s), \tilde{s}) = \int_s^t e^{F(\tau)} \left( L(\gamma_{x, \theta}^-, \dot{\gamma}_{x, \theta}^-, \tau) + c(H) \right) d\tau.
\]

(4) (Viscosity) \( \nu^-_{\alpha, \xi} \) is a viscosity solution of

\[
(HJ_0) \quad \partial_t u + f(t)u + H(x, \partial_x u, t) = c(H), \quad (x, t) \in M \times \mathbb{T}.
\]

**Remark 1.4**

i) For \( f \) satisfying \((H0^+)\), we can similarly define

\[
u^+_\alpha(x, t) := \sup_{\gamma \in C^\infty([t, +\infty), M)} \int_t^{+\infty} -e^{F(s) - F(t)} \left( L(\gamma(s), \dot{\gamma}(s), s) + \alpha \right) ds,
\]

and verify similar properties (1)-(4) as in Theorem (1.2) for it. Moreover, \( -u^+_{\alpha}(x, -t) \) is a viscosity solution of the following reverse equation of \((HJ_+)\):

\[
\partial_t u - f(-t)u + H(x, -\partial_x u, -t) = \alpha.
\]

ii) Following the terminologies in [15, 21], it’s appropriate to call the function given in Theorem 1.2 (resp. Theorem 1.3) a (backward) weak KAM solution. Such a solution can be used to pick up different types of invariant sets with variational meanings of \((4)\):

**Theorem 1.5** (main 2). For \( f(t) \) satisfying \((H0^-)\), \( H(x, p, t) \) satisfying \((H1-H4)\) and any \( \alpha \in \mathbb{R} \), we can get the following sets:
• (Aubry Set) \( \gamma : \mathbb{R} \to M \) is called globally calibrated, if for any \( s < t \in \mathbb{R} \), (9) holds on \([s, t]\). There exists a \( \varphi_{\nu}^t \)–invariant set defined by

\[
\tilde{A} := \{ (\gamma(t), \dot{\gamma}(t), t) \in TM \times T| \gamma \text{ is globally calibrated} \}
\]

with the following properties:

- \( \tilde{A} \) is a Lipschitz graph over the projected Aubry set \( \mathcal{A} := \pi \tilde{A} \subset M \times T \), where \( \pi : T^*M \times T \to M \times T \) is the standard projection.
- \( \tilde{A} \) is upper semicontinuous w.r.t. \( L : TM \times T \to \mathbb{R} \).
- \( u_\alpha^- \) is differentiable on \( \mathcal{A} \).

• (Mather Set) Suppose \( \mathcal{M}_L \) is the set of all \( \varphi_{\mu}^t \)–invariant probability measure, then \( \tilde{\mu} \in \mathcal{M}_L \) is called a Mather measure if it minimizes

\[
\min_{\tilde{\nu} \in \mathcal{M}_L} \int_{TM \times T} L + \alpha - f(t)u_\alpha^- d\tilde{\nu}.
\]

Let’s denote by \( \mathcal{M}_m \) the set of all Mather measures. Accordingly, the Mather set is defined by

\[
\tilde{M} := \bigcup \{ \text{supp} \tilde{\mu} | \tilde{\mu} \in \mathcal{M}_m \}
\]

which satisfies

(1) \( \tilde{M} \neq \emptyset \) and \( \tilde{M} \subset \tilde{A} \).

(2) \( \tilde{M} \) is a Lipschitz graph over the projected Mather set \( \mathcal{M} := \pi \tilde{M} \subset M \times T \).

• (Maximal Global Attractor) Define

\[
\mathcal{H}_0 := \{ (x, p, s, \alpha - f(s)u - H(x, p, s), u) \in T^*M \times T^*T \times \mathbb{R} | u > u_\alpha^-(x, s) \}
\]

and

\[
\mathcal{H} := \{ (x, p, s, \alpha - f(s)u - H(x, p, s), u) \in T^*M \times T^*T \times \mathbb{R} | u = u_\alpha^-(x, s) \},
\]

then \( \Omega := \bigcap_{t \geq 0} \mathcal{H}^t (\mathcal{H} \cup \mathcal{H}^0) \) is the maximal \( \varphi_{\nu}^t \)–invariant set, which satisfies:

(1) If the \( p \)–component of \( \Omega \) is bounded, then the \( u \)– and \( I \)–component of \( \Omega \) are also bounded.

(2) If \( \Omega \) is compact, it has to be a global attractor in the sense that for any point \( (x, p, s, I, u) \in T^*M \times T^*T \times \mathbb{R} \) and any open neighborhood \( U \supseteq \Omega \), there exists a \( T_\Omega(U) \) such that for all \( t \geq T_\Omega(U) \), \( \varphi_{\nu}^t(x, p, s, I, u) \in U \). Besides, the followings hold:

- \( \Omega \) is a maximal attractor, i.e. it isn’t strictly contained in any other global attractor;
- \( \tilde{A} \) is the maximal invariant set contained in \( \mathcal{H}_0 \), where

\[
\tilde{A} := \{ (\mathcal{L}(x, \partial_x u_\alpha^-(x, s), \bar{s}), \partial_x u_\alpha^-(x, s), u_\alpha^-(x, s), u_\alpha^-(x, s)) \in TM \times T^*T \times \mathbb{R} | (x, \bar{s}) \in \mathcal{A} \}.
\]

Remark 1.6 For \( f(t) \) satisfying \((H0^+)\), we can also define the associated Aubry sets and Mather sets and the maximal global attractor by using the function \( u_\alpha^+ \) in Remark 1.4. The procedure is similar with the proof of Theorem 1.5 (see Sect. 4 for more details).

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**Theorem 1.7** (main 2'). For $f(t)$ satisfying $(H\theta^0)$ and $H(x, p, t)$ satisfying $(H1$-$H4)$, the Mañé Critical Value $c(H)$ has an alternative expression

$$-c(H) = \inf_{\tilde{\mu}\in\mathfrak{M}_L} \int_{TM\times T} e^{F(t)} L(x, \nu, t) d\tilde{\mu} / \int_0^1 e^{F(t)} dt. \quad (13)$$

Moreover, the minimizer achieving the right side of (13) has to be a Mather measure. Similarly we can define the Mather set $\tilde{\mathcal{M}}$ as the union of the support sets of all the Mather measures, which is Lipschitz-graphic over the projected Mather set $\mathcal{M} := \pi\tilde{\mathcal{M}}$.

### 1.2 Parametrized Viscosity Solutions and Asymptotic Dynamics

In this section we deal with two kinds of parametrized viscosity solutions with practical meanings (see Sect. 2 for more physical viewpoints). The first case corresponds to a Hamiltonian

$$\hat{H}_\delta(x, p, t, I, u) := I + H(x, p, t) + f_\delta(t)u, \quad (14)$$

with $(x, p, \tilde{I}, I, u) \in T^*M \times T^*T \times \mathbb{R}$ and $f_\delta \in C^\ell(T, \mathbb{R})$ continuous of $\delta \in \mathbb{R}$. For suitable $\alpha \in \mathbb{R}$, we can seek the weak KAM solution of

$$\partial_t u_\delta(x, t) + H(x, \partial_x u_\delta, t) + f_\delta(t)u_\delta = \alpha \quad (15)$$

as we did in previous theorems. Consequently, it’s natural to explore the convergence of viscosity solutions w.r.t. the parameter $\delta$:

**Theorem 1.8** (main 3). Suppose $f_\delta$ converges to $f_0$ w.r.t. the uniform norm as $\delta \to 0_+$ such that $[f_0] = 0$ and the right derivative of $f_\delta$ w.r.t. $\delta$ exists at 0, i.e.

$$f_1(t) := \lim_{\delta \to 0_+} \frac{f_\delta(t) - f_0(t)}{\delta} > 0. \quad (16)$$

If $H(x, p, t)$ satisfies $(H1$-$H4)$, then there exists a unique $c(H) \in \mathbb{R}$ given by (13) and a $\delta_0 > 0$, such that the weak KAM solution $u_\delta^-(x, t)$ of (15) associated with $f_\delta$ and $\alpha_\delta \equiv c(H)$ for all $\delta \in (0, \delta_0)$ converges to a uniquely identified viscosity solution of

$$\partial_t u(x, t) + H(x, \partial_x u, t) + f_0(t)u = c(H), \quad (17)$$

which equals

$$\sup\left\{ u < f_0 , L + c(H) \left| \int_{TM\times T} e^{F_0(t)} f_1(t) \cdot u(x, t) d\tilde{\mu} \leq 0, \forall \tilde{\mu} \in \mathfrak{M}_m(\delta = 0) \right. \right\}$$

with $F_0(t) = \int_0^t f_0(\tau) d\tau$ and $\mathfrak{M}_m(0)$ being the set of Mather measures for the system with $\delta = 0$.

**Remark 1.9** The convergence of the viscosity solutions for several kinds of $1^{st}$ order PDEs was recently explored in [10, 13, 31, 33]. Such a viscous approximation problem was initiated by Lions, Papanicolaou and Varadhan in [19]. Usually in such a problem the Comparison principle is necessarily used to guarantee the uniqueness of viscosity solution for (15). However, in our case $f_\delta$ could be negative, which invalidates this principle and brings new difficulties to prove the equi-boundedness and equi-Lipschitzness of $\{u_\delta^-(\cdot, \cdot)\}_{\delta > 0}$. Nonetheless, by analyzing the properties of the Lax-Oleinik semigroups associated with $\hat{H}_\delta$ systems, these difficulties can be overcome (see Sect. 5 for more details).
The second parametrized problem we concern takes \(M = \mathbb{T}\) and a mechanical \(H(x, p, t)\). Let \(H^1(\mathbb{T}, \mathbb{R})\) be the first order cohomology group of manifold \(\mathbb{T}\). We can involve a cohomology parameter \(c \in H^1(\mathbb{T}, \mathbb{R})\) to

\[
\hat{H}(x, p, t, I, u) = I + \frac{1}{2}(p + c)^2 + V(x, t) + f(t)u
\]

(18)
of which \(H(x, p, t)\) surely satisfies (H1-H4), then (4) becomes

\[
\begin{align*}
\dot{x} &= p + c \\
\dot{p} &= -V_x - f(t)p \\
\dot{i} &= 1 \\
\dot{I} &= -V_t - f'(t)u - f(t)I \\
\dot{u} &= \frac{1}{2}(p^2 - c^2) - V(x, t) - f(t)u.
\end{align*}
\]

(19)

In physical models, the former three equations of (19) is usually condensed into a single equation

\[
\ddot{x} + V_x(x, t) + f(t)(\dot{x} - c) = 0, \quad (x, t) \in M \times \mathbb{T}.
\]

(20)

**Theorem 1.10** (main 4). For \(f(t)\) satisfying (H0), the following conclusions hold for equation (20):

- For any \(c \in H^1(\mathbb{T}, \mathbb{R})\), there exists a unified rotation number of \(\tilde{A}(c)\), which is defined by

\[
\rho(c) := \lim_{T \to +\infty} \frac{1}{T} \int_0^T \, d\gamma, \quad \forall \text{ globally calibrated curve } \gamma.
\]

- \(\rho(c)\) is continuous of \(c \in H^1(\mathbb{T}, \mathbb{R})\). Moreover, we have

\[
|\rho(c) - c| \leq \varsigma([f]) \cdot \|V(x, t)\|_{C^1}
\]

(21)

for some constant \(\varsigma\) depending only on \([f]\). Consequently, for any \(p/q \in \mathbb{Q}\) irreducible, there always exists a \(c_{p/q}\) such that \(\rho(c_{p/q}) = p/q\).

- There exists an compact maximal global attractor \(\Omega \subset T^*\mathbb{T} \times T^*\mathbb{T} \times \mathbb{R}\) of the flow \(\hat{\varphi}_t\).

**Organization of the article** The paper is organized as follows: In Sect. 2, we exhibit a list of physical models with time periodic damping. For these models, we state some notable dynamic phenomena and show how these phenomena can be linked to our main conclusions. In Sect. 3, we prove Theorems 1.2 and 1.3. In Sect. 4, we get an analogue of Aubry-Mather theory for systems satisfying condition (H0), and prove Theorem 1.5. Besides, we also prove Theorem 1.7 for systems satisfying condition (H0'). In Sect. 5, we discuss the parametrized viscosity solutions of (15), and prove the convergence of them. In Sect. 6, for 1-D mechanical systems with time periodic damping, we prove Theorem 1.10, which is related to the dynamic phenomena of the models in Sect. 2. For the consistency of the proof, parts of preliminary conclusions are postponed to the Appendix.
2 Zoo of Practical Models

In this section we display a bunch of physical models with time-periodic damping, and introduce some practical problems (related with our main conclusions) around them.

2.1 Conformally Symplectic Systems

For \( f(t) \equiv \lambda > 0 \) being constant, we get a so called conformally symplectic system (or discount system). The associated ODE becomes

\[
\begin{aligned}
\dot{x} &= \partial_p H(x, p, t), \\
\dot{p} &= -\partial_x H(x, p, t) - \lambda p.
\end{aligned}
\] (22)

This kind of systems has been considered in [3, 13, 21], although earlier results on Aubry-Mather sets have been discussed by Le Calvez [17] and Casdagli [7] for \( M = \mathbb{T} \). Besides, we need to specify that the Duffing equation with viscous damping also conforms to this case, which concerns all kinds of oscillations widely found in electromagnetics [22] and elastomechanics [24].

A significant property this kind of systems possess is that

\[
(\varphi_H^1)^* dp \wedge dx = e^{\lambda} dp \wedge dx.
\]

When \( H(x, p, t) \) is mechanical, the equation usually describes the low velocity oscillation of a solid in a fluid medium (see Fig. 1), which can be formally expressed as

\[
\ddot{x} + \lambda \dot{x} + \partial_x V(x, t) = 0, \quad x \in \mathbb{T}, \quad \lambda > 0.
\] (23)

Chaos and bifurcations topics of this setting has ever been rather popular in 1970s [16].

Fig. 1 A dissipative pendulum with \( \lambda = 1/5 \) and \( V(t, x) = 1 - \cos x \)
2.2 Tidal Torque Model

The tidal torque model was firstly introduced by [26], describing the motion of a rigid satellite $S$ under the gravitational influence of a point-mass planet $P$. Due to the internal non-rigidity of the body, a tidal torque will cause a time-periodic dissipative to the motion of $S$, which can be formalized by

$$
\ddot{x} + \varepsilon V_x(x, e, t) + \kappa \eta(e, t)(\dot{x} - c(e)) = 0, \quad (x, t) \in \mathbb{T}^2, \tag{24}
$$

with the parameter $e$ is the eccentricity of the elliptic motion of $S$ around $P$. Due to the astronomical observation, $\varepsilon$ is the equatorial ellipticity of the satellite and

$$
\kappa \propto \frac{1}{a^3} \cdot \frac{m_P}{m_S},
$$

with $a$ being the semi-major and $m_P$ (resp. $m_S$) being the mass respectively. Although this model might seem very special, there are several examples in the solar system for which such a model yields a good description of the motion, at least in a first approximation, and anyhow represents a first step toward the understanding of the problem. For instance, in the pairs Moon-Earth, Enceladus-Saturn, Dione-Saturn, Rhea-Saturn even Mercury-Sun this model is available. Besides, we need to specify that usually $\kappa \ll \varepsilon$ in all these occasions.

A few interesting phenomena have been explained by numerical approaches, e.g. the $1 : 1$ resonance for Moon-Earth system which make the people can only see one side of the moon from the earth. However, the Mercury-Sun model shows a different $3 : 2$ resonance because of the large eccentricity, see Fig. 2.

Due to Theorems 1.5 and 1.8, such a resonance seems to be explained by the following aspect: any trajectory within the global attractor $\Omega$ of (24) has a longtime stability of velocity, namely, the average velocity is close to certain rotation number, or even

![Fig. 2 A tidal torque model for Moon-Earth and Mercury-Sun](image-url)
asymptotic to it. In Sect. 6 we will show that variational minimal trajectories indeed match this description.

Remark 2.1 As a further simplification, a spin-orbit model with $\eta(e)$ being a constant is also widely concerned, which is actually a conformally symplectic system. In [3] they further discussed the existence of KAM torus for this model and proved the local attraction of the KAM torus.

2.3 Pumping of the Swing

The pumping of a swing is usually modeled as a rigid object forced to rotate back and forth at the lower ends of supporting ropes. After a series of approximations and reasonable simplifications, the pumping of the swing can be characterized as a harmonic oscillator with driving and parametric terms [8]. Therefore, this model has a typical meaning in understanding the dynamics of motors.

As shown in Fig. 3, the length of the rope supporting the swinger is $l$, and $s$ is the distance between the center of mass of the swinger to the lower ends of the rope. The angle of the supporting rope to the vertical position is denoted by $\phi$, and the angle between the symmetric axis of the swinger and the rope is $\theta$, which varies as $\theta = \theta_0 \cos \omega t$. So we get the equation of the motion by

\[
(l^2 - 2ls \cos \theta + s^2 + R^2)\ddot{\phi} = -gl \sin \phi + gs \sin(\phi + \theta) - ls \sin \theta \dot{\theta}^2 \\
+ (ls \cos \theta - s^2 - R^2) \ddot{\theta} - 2ls \sin \theta \dot{\theta} \dot{\phi}, \quad \phi \in \mathbb{T}
\]
where $g$ is the gravity index and $mR^2$ is the moment of inertia of the center ($m$ is the mass of swinger). We can see that by reasonable adjustment of $l$, $s$, $\omega$ parameters, this system can be dissipative, accelerative or critical.

Notice that numerical research of this equation for $|\phi| \ll 1$ has been done by numerical experts in a bunch of papers, see [27] for a survey of that. Those results successfully simulate the swinging at small to modest amplitudes. As the amplitude grows, these results become less and less accurate, and that’s why we resort to a theoretical analysis in this paper.

3 Weak KAM Solution of (HJ$_+$)

Due to the superlinearity of $L(x, v, t)$, for each $k \geq 0$, there exists $C(k) \geq 0$, such that

$$L(x, v, t) \geq k|v| - C(k), \quad k > 0, \quad x \in M.$$  

Moreover, the compactness of $M$ implies that for each $k > 0$, there exists $C_k > 0$ such that

$$\max_{(x, t) \in M \times \mathbb{T}} L(x, v, t) \leq C_k.$$  

3.1 Weak KAM Solution of (HJ$_+$) in the Condition (H0$^-$)

Note that $[f] > 0$. The following conclusion can be easily checked.

**Lemma 3.1** Assume $t > s$, then

1. $F(s) - F(t) \leq 2k_0 - (t - s - 1)[f]$;
2. $\int_s^t e^{F(\tau) - F(t)} d\tau \leq \frac{e^{2k_0 + [f]}}{[f]} (1 - e^{-(t-s)[f]})$;
3. $\int_{-\infty}^t e^{F(\tau) - F(t)} d\tau \leq \frac{e^{2k_0 + [f]}}{[f]},$

where $k_0 = \max_{t \in [0, 2]} |\int_0^t f(\tau) d\tau|$.

Now we define a function $u^-_\alpha : M \times \mathbb{R} \to \mathbb{R}$ by

$$u^-_\alpha (x, t) := \inf \int_{-\infty}^t e^{F(s) - F(t)} (L(\gamma(s), \dot{\gamma}(s), s) + \alpha) ds$$  \hspace{1cm} (26)

where the infimum is taken for all $\gamma \in C^{ac}((-\infty, t], M)^2$ with $\gamma(t) = x$. We can easily prove this function is bounded, since

$$-|C(k = 0) - \alpha| \cdot \frac{e^{2k_0 + [f]}}{[f]} \leq u^-_\alpha (x, t) \leq |C_{k=0} + \alpha| \cdot \frac{e^{2k_0 + [f]}}{[f]},$$

where $C(0)$ and $C_0$ have been defined in the beginning of Sect. 3. Consequently, (1) and (3) of Theorem 1.2 can be easily achieved.

**Lemma 3.2** For each $(x, t) \in M \times \mathbb{R}$ and $s < t$, it holds

$$e^{F(t)} u^-_\alpha (x, t)$$

$$= \inf_{\gamma \in C^{ac}((s,t], M)} \left\{ e^{F(s)} u^-_\alpha (\gamma(s), s) + \int_s^t e^{F(\tau)} (L(\gamma(\tau), \dot{\gamma}(\tau), \tau) + \alpha) d\tau \right\}.$$

2 absolutely continuous curves.
Moreover, the infimum in (27) can be achieved by a $C^r$ smooth minimizer.

**Proof** Equality (27) can be proved by a same method as employed in the proof of Proposition 6.5 in [13]. Therefore, we can find a sequence of absolutely continuous curve $\{\gamma_n\}$ with $\gamma_n(t) = x$ such that

$$e^{F(t)}u_\alpha(x, t) = \lim_{n \to \infty} \left\{ e^{F(s)}u_\alpha(\gamma_n(s), s) + \int_s^t e^{F(\tau)}(L(\gamma_n, \dot{\gamma}_n, \tau) + \alpha)d\tau \right\}.$$

Hence, there exists a constant $c$ independent of $n$, such that

$$\int_s^t e^{F(\tau)}(L(\gamma_n, \dot{\gamma}_n, \tau) + \alpha)d\tau \leq c. \quad (28)$$

Due to Dunford-Petti Theorem (Theorem 6.4 in [13]), there exists a subsequence $\{\gamma_{n_k}\}$ converging to a curve $\gamma_*$ in the space $C_c([s, t], M)$ endowed with metric $d_0(\gamma_1, \gamma_2) = \sup\{d(\gamma_1(\sigma), \gamma_2(\sigma)) : \sigma \in [s, t]\}$, where $d(\cdot, \cdot)$ is the distance function induced by the Riemannian metric on $M$, such that

$$\int_s^t e^{F(\tau)}(L(\gamma_{n_k}, \dot{\gamma}_{n_k}, \tau) + \alpha)d\tau \leq \lim_{k \to \infty} \int_s^t e^{F(\tau)}(L(\gamma_{n_k}, \dot{\gamma}_{n_k}, \tau) + \alpha)d\tau. \quad (29)$$

Hence, the infimum in (27) can be achieved at $\gamma_* : [s, t] \to M$, which definitely solves the Euler-Lagrange equation (E-L). Due to the Weierstrass Theorem in [23], $\gamma_*$ is $C^r$ smooth.□

For any $n \in \mathbb{Z}_+$, we apply Lemma 3.2 on the interval $[t - n, t] \subset \mathbb{R}$ and achieve the infimum curve $\gamma_n$. By Dunford-Petti Theorem and a diagonal argument, the uniform limit of $\gamma_n$ (up extraction of a subsequence) exists as a $C^r$ calibrated curve on $(-\infty, t]$.

**Lemma 3.3** [(4 of Theorem 1.2)] For each $\alpha \in \mathbb{R}$ and $(x, t) \in M \times \mathbb{R}$, there exists a $C^r$ curve $\gamma_{x,t}^- : (-\infty, t] \to M$ with $\gamma_{x,t}^-(t) = x$ such that for each $t_1 < t_2 \leq t$,

$$e^{F(t_2)}u_\alpha(\gamma_{x,t}^-(t_2), t_2) - e^{F(t_1)}u_\alpha(\gamma_{x,t}^-(t_1), t_1) = \int_{t_1}^{t_2} e^{F(\tau)}(L(\gamma_{x,t}^-(\tau), \dot{\gamma}_{x,t}^-(\tau), \tau) + \alpha)d\tau. \quad (30)$$

**Remark 3.4** Due to the boundedness of $u_\alpha$, by making $t_1 \to -\infty$ in (30) we instantly get

$$u_\alpha(x, t) = \int_{-\infty}^{t} e^{F(\tau) - F(t)}(L(\gamma_{x,t}^-(\tau), \dot{\gamma}_{x,t}^-(\tau), \tau) + \alpha)d\tau,$$

i.e. the infimum in (26) is achieved at $\gamma_{x,t}^- : (-\infty, t] \to M$.

**Lemma 3.5** Suppose $\gamma_{x,\theta}^- : (-\infty, \theta] \to M$ is a backward calibrated curve ending with $x$ of $u_\alpha(x, \theta)$, then

$$|\dot{\gamma}_{x,\theta}^-(\tau)| \leq \kappa_0, \quad \forall (x, \theta) \in M \times T, \tau < \theta.$$

for a constant $\kappa_0$ depending on $L$ and $\alpha$. That implies $\gamma_{x,\theta}^-$ is Lipschitz on $(-\infty, \theta]$.

**Proof** Let $s_1, s_2 \leq \theta$ and $s_2 - s_1 = 1$. Due to Lemma 3.3,

$$e^{F(s_2)}u_\alpha(\gamma_{x,\theta}^-(s_2), s_2) - e^{F(s_1)}u_\alpha(\gamma_{x,\theta}^-(s_1), s_1) = \int_{s_1}^{s_2} e^{F(\tau)}(L(\gamma_{x,\theta}^-(\tau), \dot{\gamma}_{x,\theta}^-(\tau), \tau) + \alpha)d\tau \geq \int_{s_1}^{s_2} e^{F(\tau)}(|\dot{\gamma}_{x,\theta}^-(\tau)| - C(1 + \alpha)d\tau.$$

\(\square\) Springer
On the other hand, let \( \beta : [s_1, s_2] \to M \) be a geodesic satisfying \( \beta(s_1) = \gamma_{x,\theta}(s_1), \beta(s_2) = \gamma_{x,\theta}(s_2) \), and \( |\dot{\beta}(\tau)| \leq \text{diam}(M) =: k_1 \). Then
\[
e^{-F(s_2)}u^-_{\alpha}(\gamma_{x,\theta}(s_2), s_2) - e^{-F(s_1)}u^-_{\alpha}(\gamma_{x,\theta}(s_1), s_1) \\
\leq \int_{s_1}^{s_2} e^{F(\tau)}(L(\beta(\tau), \dot{\beta}(\tau), \tau) + \alpha) \, d\tau \\
\leq \int_{s_1}^{s_2} e^{F(\tau)}(C_{k_1} + \alpha) \, d\tau.
\]

Hence,
\[
\int_{s_1}^{s_2} e^{F(\tau)}|\dot{\gamma}_{x,\theta}(\tau)| \, d\tau \leq \int_{s_1}^{s_2} e^{F(\tau)}(C_{k_1} + (1)) \, d\tau.
\]

Due to the continuity of \( \dot{\gamma}_{x,\theta}(\tau) \), there exists \( s_0 \in (s_1, s_2) \) such that
\[
|\dot{\gamma}_{x,\theta}(s_0)| \leq C_{k_1} + (1).
\]

Note that \( \gamma_{x,\theta} \) solves (E-L), so \( |\dot{\gamma}_{x,\theta}(\tau)| \) is uniformly bounded for \( (x, \theta) \in M \times \mathbb{T} \) and \( \tau \in (-\infty, \theta] \).

\[\square\]

**Lemma 3.6** [(2) of Theorem 1.2] For each \( \alpha \in \mathbb{R} \), \( u^-_{\alpha} \) is Lipschitz on \( M \times \mathbb{T} \).

**Proof** First of all, we prove \( u^-_{\alpha}(-\cdot, \theta) : M \to \mathbb{R} \) is uniformly Lipschitz w.r.t. \( \theta \in \mathbb{T} \). Let \( x, y \in M, \Delta t = d(x, y) \), and \( \gamma_{x,\theta} : (-\infty, \theta] \to M \) be a minimizer of \( u^-_{\alpha}(x, \theta) \). Define \( \tilde{\gamma} : (-\infty, \theta] \to M \) by
\[
\tilde{\gamma}(s) = \begin{cases} 
\gamma_{x,\theta}(s), & s \in (-\infty, \theta - \Delta t), \\
\beta(s), & s \in [\theta - \Delta t, \theta],
\end{cases}
\]
where \( \beta : [\theta - \Delta t, \theta] \to M \) is a geodesic satisfying \( \beta(\theta - \Delta t) = \gamma_{x,\theta}(\theta - \Delta t), \beta(\theta) = y \), and
\[
|\dot{\beta}(s)| \equiv \frac{d(y_{x,\theta}(\theta - \Delta t), y)}{\Delta t} \leq \frac{d(y_{x,\theta}(\theta - \Delta t), x)}{\Delta t} + 1 \leq \kappa_0 + 1.
\]

Then,
\[
u^-_{\alpha}(y, \theta) - u^-_{\alpha}(x, \theta) \leq \int_{\theta - \Delta t}^{\theta} e^{F(\tau) - F(\theta)}(L(\beta, \dot{\beta}, \tau) - L(\gamma_{x,\theta}, \dot{\gamma}_{x,\theta}, \tau)) \, d\tau \\
\leq (C_{\kappa_0 + 1} + C(0)) \int_{\theta - \Delta t}^{\theta} e^{F(\tau) - F(\theta)} \, d\tau \\
\leq (C_{\kappa_0 + 1} + C(0))e^{2\kappa_0 +|f|} \cdot d(x, y).
\]

By a similar approach, we derive the opposite inequality holds. Hence,
\[
|u^-_{\alpha}(y, \theta) - u^-_{\alpha}(x, \theta)| \leq \rho_\ast \cdot d(x, y),
\]
where \( \rho_\ast = (C_{\kappa_0 + 1} + C(0))e^{2\kappa_0 +|f|} \).

Next, we prove \( u^-_{\alpha}(x, \cdot) \) is uniformly Lipschitz continuous for \( x \in M \). Let \( \tilde{t}, \tilde{t}' \in \mathbb{T}, d(\tilde{t}, \tilde{t}') = t' - t, \) and \( t \in [0, 1] \). Then, \( t' \in [0, 2] \). A curve \( \eta : (-\infty, t'] \to M \) is defined by
\[
\eta(s) = \begin{cases} 
\gamma_{x,t}(s), & s \in (-\infty, t], \\
x, & s \in (t, t'].
\end{cases}
\]
Then,
\[
e^{F(t')}u^\alpha_-(x, t') - e^{F(t)}u^\alpha_-(x, t)
\leq \int_{-\infty}^{t'} e^{F(\tau)}(L(\tilde{\eta}, \tilde{\eta}', \tau) + \alpha) d\tau - \int_{-\infty}^{t} e^{F(\tau)}(L(\gamma_{x, t}', \gamma_{x, t}', \tau) + \alpha) d\tau
\leq \int_{t}^{t'} e^{F(\tau)}(C_0 + \alpha) d\tau \leq (C_0 + |\alpha|) \max_{\tau \in [0, 2]} e^{F(\tau)} \cdot |t' - t|.
\]

On the other hand, we write \(\Delta t = d(\tilde{\eta'}, \tilde{\eta})\) and define \(\eta_1 \in C^{ac}((-\infty, t], M)\) by
\[
\eta_1(s) = \begin{cases}
\gamma_{x, t'}(s), & s \in (-\infty, t - \Delta t], \\
\gamma_{x, t'}(2(s - t) + t'), & s \in (t - \Delta t, t].
\end{cases}
\]

It is easy to check \(\eta_1(t) = x\), and \(|\eta_1(\tau)| \leq 2\kappa_0\), where \(\kappa_0\) is a Lipschitz constant of \(\gamma_{x, t'}\).

\[
e^{F(t)}u^\alpha_-(x, t) \leq \int_{-\infty}^{t} e^{F(\tau)}(L(\eta_1(\tau), \dot{\eta}_1(\tau), \tau) + \alpha) d\tau
\leq \int_{t - \Delta t}^{t} e^{F(\tau)}(L(\eta_1(\tau), \dot{\eta}_1(\tau), \tau) + \alpha) d\tau
\leq \int_{t - \Delta t}^{t} e^{F(\tau)}(L(\gamma_{x, t'}(\tau), \gamma_{x, t'}(\tau), \tau) + \alpha) d\tau.
\]

Note that \(\gamma_{x, t'}\) is a minimizer of \(u^\alpha_-(x, t')\). We derive that
\[
e^{F(t)}u^\alpha_-(x, t) - e^{F(t')}u^\alpha_-(x, t')
\leq \int_{t - \Delta t}^{t} e^{F(\tau)}(L(\eta_1(\tau), \dot{\eta}_1(\tau), \tau) + \alpha) d\tau
\leq \int_{t - \Delta t}^{t} e^{F(\tau)}(L(\gamma_{x, t'}(\tau), \gamma_{x, t'}(\tau), \tau) + \alpha) d\tau
\leq (C_{2\kappa_0} + 2C(0) + |\alpha|) \max_{\tau \in [0, 2]} e^{F(\tau)} \cdot d(\tilde{\eta'}, \tilde{\eta}).
\]

We have proved the map \(t \mapsto e^{F(t')}u^\alpha_-(x, t)\) is uniformly Lipschitz for \(x \in M\), with Lipschitz constant depends only on \(L, f\) and \(\alpha\). Note that \(F(t)\) is \(C^{r+1}\) and \(F'(t) = f(t)\) is 1-periodic. We derive \(u^\alpha_-(x, \cdot)\) is uniformly Lipschitz for \(x \in M\) with Lipschitz constant \(\rho_0\) depending on \(L, f\) and \(\alpha\). It follows that
\[
|u^\alpha_-(x', \theta') - u^\alpha_-(x, \theta)| \leq |u^\alpha_-(x', \theta') - u^\alpha_-(x, \theta')| + |u^\alpha_-(x, \theta') - u^\alpha_-(x, \theta)|
\leq \rho_0 d(x', x) + \rho_0^* d(\theta', \theta)
\]
so we finish the proof. \(\Box\)

**Lemma 3.7** [(5) of Theorem 1.2] The function \(u^\alpha_-(x, t)\) defined by (26) is a viscosity solution of \((HJ_+).\)

**Proof** By a standard argument as in Proposition 7.2.7 in [15] (or Proposition 6.3 in [10]), \(u^\alpha_-(x, t)\) is viscosity solution can be derived from the weak KAM properties of \(u^\alpha_-(x, t)\). \(\Box\)

As a complement, the following result, which is similar to Proposition 6 of [21], will be useful in the following sections:
Proposition 3.8 The weak KAM solution \( u^-_\alpha \) of \((HJ_+)^\prime\) is differentiable at \((\gamma^-_{x,t}(s), \bar{s})\) for any \( \mathbb{R} \ni s < t \), where \( \gamma_{x,t}^- : (-\infty, t] \to M \) is a backward calibrated curve ending with \( x \). In other words, we have

\[
\partial_t u^-_\alpha(\gamma^-_{x,t}(s), s) + H(\gamma^-_{x,t}(s), \partial_s u^-_\alpha(\gamma^-_{x,t}(s), s), s) + f(s)u^-_\alpha(\gamma^-_{x,t}(s), s) = \alpha
\]

and

\[
(\gamma^-_{x,t}(s), \dot{\gamma}^-_{x,t}(s), \bar{s}) = \mathcal{L}\left(\gamma^-_{x,t}(s), \partial_s u^-_\alpha(\gamma^-_{x,t}(s), s), \bar{s}\right)
\]

for all \( \mathbb{R} \ni s < t \).

Proof By Theorem B.4, we derive \( u^-_\alpha(x, t) \) is semiconcave. Let \( s \in (-\infty, t) \) and \( \bar{p} = (p_x, p_t) \in D^+ u^-_\alpha(\gamma^-_{x,t}(s), s) \). For \( \Delta s > 0 \),

\[
e^{F(s+\Delta s)} u^-_\alpha(\gamma^-_{x,t}(s + \Delta s), s + \Delta s) - e^{F(s)} u^-_\alpha(\gamma^-_{x,t}(s), s) \]

\[
\frac{\Delta s}{\Delta s} = 1 \int_s^{s+\Delta s} e^{F(\tau)} (L(\gamma^-_{x,t}(\tau), \dot{\gamma}^-_{x,t}(\tau), \tau)) d\tau.
\]

Then

\[
\lim_{\Delta s \to 0^+} \frac{u^-_\alpha(\gamma^-_{x,t}(s + \Delta s), s + \Delta s) - u^-_\alpha(\gamma^-_{x,t}(s), s)}{\Delta s} = L(\gamma^-_{x,t}(s), \dot{\gamma}^-_{x,t}(s), s) + \alpha - f(s) u^-_\alpha(\gamma^-_{x,t}(s), s).
\]

By Proposition B.3.,

\[
\lim_{\Delta s \to 0^+} \frac{u^-_\alpha(\gamma^-_{x,t}(s + \Delta s), s + \Delta s) - u^-_\alpha(\gamma^-_{x,t}(s), s)}{\Delta s} \leq p_x \cdot \dot{\gamma}^-_{x,t}(s) + p_t,
\]

which implies

\[
p_t + H(\gamma^-_{x,t}(s), p_x, s) + f(s) u^-_\alpha(\gamma^-_{x,t}(s), s) \geq \alpha.
\]

On the other hand, \( u^-_\alpha \) is a viscosity solution of \((HJ_+)^\prime\). Hence, for each \( (p_x, p_t) \in D^+ u^-_\alpha(\gamma^-_{x,t}(s), s) \),

\[
p_t + H(\gamma^-_{x,t}(s), p_x, s) + f(s) u^-_\alpha(\gamma^-_{x,t}(s), s) = \alpha.
\]

Note that \( H(x, p, t) \) is strictly convex with respect to \( p \). By (34), We derive \( D^+ u^-_\alpha(\gamma^-_{x,t}(s), s) \) is a singleton. By Proposition B.3., \( u^-_\alpha(x, t) \) is differentiable at \((\gamma^-_{x,t}(s), s)\).

3.2 Weak KAM Solution of \((HJ_0)^\prime\) in the Condition \((H0^0)\)

Now \([f] = 0\), so \( F(t) := \int_0^t f(\tau) d\tau \) is 1-periodic. Let \( \ell = \int_0^1 e^{F(\tau)} d\tau \), then we define a new Lagrangian \( \mathbf{L} : TM \times \mathbb{T} \to \mathbb{R} \) by

\[
\mathbf{L}(x, v, t) := e^{F(t)} \mathbf{L}(x, v, t).
\]

For such a \( \mathbf{L} \), the Peierls Barrier \( h^-_\alpha \) is defined as

\[
h^-_\alpha(x, \bar{s}, y, \bar{t}) = \liminf_{t \equiv \ell, s \equiv 3 \ (\text{mod} \ 1)} \inf_{\gamma \in C^a([s, \bar{t}], M)} \int_s^\bar{t} \mathbf{L}(y, \dot{y}, \tau) + \alpha \cdot \ell d\tau.
\]
is well-defined, once \( \alpha \) is uniquely established by

\[
c(H) = \inf\{ \alpha \in \mathbb{R} | \int_s^t L(\gamma, \dot{\gamma}, \tau) + \alpha \cdot \ell d\tau \geq 0, \forall \gamma \in C \}
\] (35)

with \( C = \{ \gamma \in C^{ac}([s, t], M) | \gamma(s) = \gamma(t) \text{ and } t - s \in \mathbb{Z}_+ \} \), due to Proposition 2 of [11]. Moreover, the following properties were proved in [11]:

**Proposition 3.9**

(i): If \( \alpha < c(H) \), \( h^\infty_\alpha \equiv -\infty \).

(ii): If \( \alpha > c(H) \), \( h^\infty_\alpha \equiv +\infty \).

(iii): \( h^\infty_{c(H)} \) is finite.

(iv): \( h^\infty_{c(H)} \) is Lipschitz.

(v): For each \( \gamma \in C^{ac}([s, t], M) \) with \( \gamma(s) = x, \gamma(t) = y \),

\[
h^\infty_{c(H)}(z, \xi, y, \bar{t}) - h^\infty_{c(H)}(z, \xi, x, \bar{s}) \leq \int_s^t L(\gamma, \dot{\gamma}, \tau) + c(H) \cdot \ell d\tau.
\]

Consequently, for any \((z, \xi) \in M \times \mathbb{T}\) fixed, we construct a function \( u^-_{z, \xi} : M \times \mathbb{T} \rightarrow \mathbb{R} \) by

\[
u^-_{z, \xi}(x, \bar{t}) = e^{-F(\bar{t})} \left( h^\infty_{c(H)}(z, \xi, x, \bar{t}) + c(H) \cdot \int_{\xi}^{\bar{t}} e^{F(\tau)} - \ell d\tau \right).
\] (36)

**Proof of Theorem 1.3:**

(1) Due to (iv) of Proposition 3.9, \( u^-_{z, \xi} \) is also Lipschitz.

(2) The domination property of \( u^-_{z, \xi} \) can be achieved immediately by (v) of Proposition 3.9.

(3) By Tonelli Theorem and the definition of \( u^-_{z, \xi} \), there exists a sequence \( \xi_k \) tending to \(-\infty\) and a sequence \( \gamma_k \in C^{ac}([\xi_k, \theta], M) \) with \( \gamma_k(\xi_k) = z, \gamma_k(\theta) = x \), such that \( \gamma_k \) minimizes the action function

\[
\mathcal{F}(\beta) = \inf_{\beta \in C^{ac}([\xi_k, \theta], M) | \beta(\xi_k) = z, \beta(\theta) = x} \int_{\xi_k}^{\theta} e^{F(\tau)} (L(\beta, \dot{\beta}, \tau) + c(H)) d\tau
\]

and

\[
e^{F(\theta)} u^-_{z, \xi}(x, \theta) = \lim_{k \to +\infty} \int_{\xi_k}^{\theta} e^{F(\tau)} (L(\gamma_k, \dot{\gamma}_k, \tau) + c(H)) d\tau.
\]

Since each \( \gamma_k \) solves (E-L), which implies \( \gamma_k \) is \( C^r \). By a standard way, there exists \( \kappa_0 \) independent of the choice of \( k \), such that \(|\gamma_k| \leq \kappa_0\), when \( \theta - \xi_k \geq 1 \). By Ascoli Theorem, there exists a subsequence of \( \{\gamma_k\} \) (denoted still by \( \gamma_k \)) and an absolutely continuous curve \( \gamma^-_{x, \theta} : (-\infty, \theta] \rightarrow M \) such that \( \gamma_k \) converges uniformly to \( \gamma^-_{x, \theta} \) on each compact subset
of \((-\infty, \theta]\) and \(y_{x,\theta}^{-}(\theta) = x\). Then, for each \(s < \theta\),

\[
e^{E(\theta)}u_{z,\xi}^{-}(x, \theta) = \lim_{k \to +\infty} \left( \int_{s_k}^{s} e^{F(\tau)}(L(y_k, \dot{y}_k, \tau) + c(H))d\tau \right.
\nonumber
\]

\[
+ \int_{s}^{\theta} e^{F(\tau)}(L(y_k, \dot{y}_k, \tau) + c(H))d\tau
\nonumber
\]

\[
\geq \lim \inf_{k \to +\infty} \int_{s_k}^{s} e^{F(\tau)}(L(y_k, \dot{y}_k, \tau) + c(H))d\tau
\nonumber
\]

\[
\geq \lim \inf_{k \to +\infty} \int_{s_k}^{s} e^{F(\tau)}(L(y_k, \dot{y}_k, \tau) + c(H))d\tau
\nonumber
\]

\[
\geq e^{F(s)}u_{z,\xi}^{-}(y_{x,\theta}^{-}(s), s) + \int_{s}^{\theta} e^{F(\tau)}(L(y_x^{-}, \dot{y}_x^{-}, \tau) + c(H))d\tau
\nonumber
\]

which implies \(y_{x,\theta}^{-}\) is a calibrated curve by \(u_{z,\xi}^{-}\).

(4) By a similar approach of the proof of Lemma 3.7, we derive \(u_{z,\xi}^{-}\) is also a viscosity solution of \((HJ_0)\). \qed}

## 4 Various Properties of Variational Invariant Sets

### 4.1 Aubry Set in the Condition \((H0^-)\)

Due to Theorem 1.2 and Proposition 3.8, for any \((x, \bar{s}) \in M \times \mathbb{T}\) we can find a backward calibrated curve

\[
\bar{\gamma}_{x,s}^{-} := \left( y_{x,s}^{-}(t) \right) : t \in (-\infty, s] \to M \times \mathbb{T}
\]

(37)

ending with it, such that the associated backward orbit \(\varphi_{L}^{-s}(y_{x,s}^{-}(s), y_{x,s}^{-}(s), s)\) has an \(\alpha\)-limit set \(\bar{A}_{x,s} \subset TM \times \mathbb{T}\), which is invariant and graphic over \(A_{x,s} := \pi \bar{A}_{x,s}\). Therefore, any critical curve \(\bar{\gamma}_{x,s}^{-}\) in \(A_{x,s}\) has to be a globally calibrated curve, namely

\[
\bar{A}_{x,s} \subset \bar{A}, \quad \text{(resp.} A_{x,s} \subset A). \nonumber
\]

So \(\bar{A} \neq \emptyset\).

Recall that any critical curve in \(A\) is globally calibrated, then due to Proposition 3.8, that implies for any \((x, \bar{s}) \in A\), the critical curve \(\bar{\gamma}_{x,s}^{-}\) passing it is unique. In other words, \(\pi^{-1} : A \to \bar{A}\) is a graph, and

\[
\gamma_{x,s}^{-}(t) = \partial_{p}H(d\gamma^{-}(\gamma_{x,s}^{-}(t), t), t), \quad \forall \ t \in \mathbb{R}.
\]

That indicates that \(d\gamma^{-} : A \to TM\) coincides with \(\partial_{v}L \circ (\pi |_{\bar{A}})^{-1}\). On the other side, \(\|\gamma_{x,s}^{-}(t)\| \leq A < +\infty\) for all \(t \in \mathbb{R}\) due to Lemma 3.5, so \(\partial_{v}L \circ (\pi |_{\bar{A}})^{-1}\) has to be Lipschitz. So \(\bar{A}\) is Lipschitz over \(A\). This is an analogue of Theorem 4.11.5 of [15] and a.4) of [21], which is known as Mather’s graph theorem in more earlier works [23] for conservative Hamiltonian systems.

**Lemma 4.1** \(\bar{A}\) has an equivalent expression

\[
\bar{A} := \{(y(t), \dot{y}(t), \bar{t}) \in TM \times \mathbb{T} | \forall \ a < b \in \mathbb{R}, \ \gamma \text{achieves} h_{a,b}^{\alpha}(\gamma(a), \gamma(b))\}. \quad (38)
\]

(Springer)
Proof Let \( \gamma : \mathbb{R} \to M \) be a globally calibrated curve by \( u^-_a \). Due to (3) and (4) of Theorem 1.2, for \( a < b \in \mathbb{R} \),

\[
\int_a^b e^{F(\tau)}(L(\gamma, \dot{\gamma}, \tau) + \alpha) d\tau = e^{F(b)}u^-_a(\gamma(b), b) - e^{F(a)}u^-_a(\gamma(a), a) \\
\leq h^a_{a,b}(\gamma(a), \gamma(b)).
\]

Due to the definition of \( h^a_{a,b}(\gamma(a), \gamma(b)) \), we derive \( \gamma \) achieves \( h^a_{a,b}(\gamma(a), \gamma(b)) \) for all \( a < b \in \mathbb{R} \).

To prove the lemma, it suffices to show any curve \( \gamma : \mathbb{R} \to M \) achieving \( h^a_{a,b}(\gamma(a), \gamma(b)) \) for all \( a < b \in \mathbb{R} \) is a calibrated curve by \( u^-_a \). We claim

\[
\lim_{s \to -\infty} h^{s,t}_{a}(z, x) = e^{F(t)}u^-_a(x, t), \quad \forall x, z \in M, t \in \mathbb{R}.
\]

Due to (3) of Theorem 1.2, for \( s < t \),

\[
e^{F(t)}u^-_a(x, t) - h^{s,t}_{a}(z, x) \leq e^{F(s)}u^-_a(z, s) \to 0, s \to -\infty.
\]

On the other hand, we assume \( \gamma_{s,t} \) is a globally calibrated curve by \( u^-_a \) with \( \gamma_{s,t}(t) = x \) and \( s + 1 < t \). Let \( \beta : [s, s+1] \to M \) be a geodesic with \( \beta(s) = x, \beta(s+1) = \gamma_{s,t}(s+1) \) satisfying \( |\dot{\beta}| \leq k_1 := \text{diam}(M) \). Then,

\[
h^{s,t}_{a}(z, x) \leq \int_s^{s+1} e^{F(\tau)}(L(\beta, \dot{\beta}, \tau) + \alpha) d\tau + \int_{s+1}^t e^{F(\tau)}(L(\gamma_{s,t}, \dot{\gamma}_{s,t}, \tau) + \alpha) d\tau \\
\leq (C_{k_1} + \alpha)e^{f}[f] + e^{F(t)}u^-_a(x, t) - e^{F(s+1)}u^-_a(\gamma_{s,t}(s+1), s+1).
\]

Hence,

\[
h^{s,t}_{a}(z, x) - e^{F(t)}u^-_a(x, t) \leq (C_{k_1} + \alpha)e^{f}[f] + e^{F(s+1)}u^-_a(\gamma_{s,t}(s+1), s+1).
\]

From \( [f] > 0 \), it follows that the right side of the inequality above tending to 0, as \( s \to -\infty \). Hence, (39) holds. Actually, the limit in (39) is uniform for \( x, z \in M \) and \( t \in \mathbb{R} \).

If \( \gamma \) achieves \( h^a_{a,b}(\gamma(a), \gamma(b)) \) for \( a < b \in \mathbb{R} \), then

\[
h^a_{a,b}(\gamma(s), \gamma(b)) - h^a_{a,b}(\gamma(s), \gamma(a)) = \int_a^b e^{F(\tau)}(L(\gamma, \dot{\gamma}, \tau) + \alpha) d\tau, \forall s < a.
\]

Taking \( s \to -\infty \), we derive \( \gamma \) is also a calibrated curve by \( u^-_a \).

With the help of (38), the following Lemma can be proved:

Lemma 4.2 (Upper Semi-continuity). The set valued function

\[
L \in C^{f;2}(TM \times \mathbb{T}, \mathbb{R}) \longrightarrow \tilde{A} \subset TM \times \mathbb{T}
\]

is upper semi-continuous. Here \( \| \cdot \|_{C^f} \) is the \( C^f \) -norm and \( d_M \) is the Hausdorff distance.

Proof It suffices to prove that for any \( L_n \to L \) w.r.t. \( \| \cdot \|_{C^f} \) -norm, the accumulating curve of any sequence of curves \( \gamma_n \) in \( \mathcal{A}(L_n) \) should lie in \( \mathcal{A}(L) \). Due to Lemma 3.5, for any \( n \in \mathbb{Z}_+ \)
such that $\|L_n - L\|_{C^r} \leq 1$, $\tilde{A}(L_n)$ is uniformly compact in the phase space. Therefore, for any sequence $\{\tilde{\gamma}_n\}$ each of which is globally minimal, the accumulating curve $\tilde{\gamma}_s$ satisfies

$$\int_t^s e^F(\tau) (L(\gamma_s, \dot{\gamma}_s, \tau) + \alpha) \, d\tau \leq \lim_{n \to +\infty} \int_t^s e^F(\tau) (L_n(\gamma_n, \dot{\gamma}_n, \tau) + \alpha) \, d\tau$$

$$\leq \lim_{n \to +\infty} \int_t^s e^F(\tau) (L_n(\eta_n, \dot{\eta}_n, \tau) + \alpha) \, d\tau$$

for any absolutely continuous $\eta_n : [t, s] \to M$ ending with $\gamma_n(t)$ and $\gamma_n(s)$. Since for any absolutely continuous $\eta : [t, s] \to M$ ending with $\gamma_s(t)$ and $\gamma_s(s)$, we can find such a sequence $\eta_n : [t, s] \to M$ converging to $\eta$ uniformly, then we get

$$\int_t^s e^F(\tau) (L(\gamma_s, \dot{\gamma}_s, \tau) + \alpha) \, d\tau \leq \inf_{\eta \in \mathcal{C}^c([t, s], M)} \int_t^s e^F(\tau) (L(\eta, \dot{\eta}, \tau) + \alpha) \, d\tau$$

for any $t < s \in \mathbb{R}$, which implies $\gamma_s$ satisfies the Euler-Lagrange equation. Due to Theorem 1.2, the weak KAM solution $u^-_s$ associated with $L$ is unique, so $\gamma_s$ is globally minimal, then globally calibrated by $u^-_s$, i.e. $\tilde{\gamma}_s \in \mathcal{A}(L)$.

**4.2 Mather Set in the Condition (H0−)**

For any globally calibrated curve $\tilde{\gamma}$, we can always find a sequence $T_n > 0$, such that a $\varphi'_{L}^{-1}$-invariant measure $\tilde{\mu}$ can be found by

$$\int_{T M \times T} f(x, v, t) d\tilde{\mu} = \lim_{n \to +\infty} \frac{1}{T_n} \int_0^{T_n} f(\gamma, \dot{\gamma}, t) \, dt, \quad \forall f \in \mathcal{C}_c(TM \times T, \mathbb{R}).$$

So the set of $\varphi'_{L}^{-1}$-invariant measures $\mathcal{M}_L$ is not empty.

**Proposition 4.3** For all $\tilde{\nu} \in \mathcal{M}_L$ and $\alpha \in \mathbb{R}$, we have

$$\int_{T M \times T} L + \alpha - f(t) u^-_a d\tilde{\nu} \geq 0.$$

Besides,

$$\int_{T M \times T} L + \alpha - f(t) u^-_a d\tilde{\nu} = 0 \iff \text{supp}(\tilde{\nu}) \subset \tilde{A}$$
The Aubry set, we can similarly get that
\[ \pi \]

\[ \hat{\gamma} \]

\[ 4.3 \]

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\[ \begin{align*}
\int_{TM \times T} f(t)u_{\alpha}^{-}(x, t)d\tilde{v} \\
&= \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(t)u_{\alpha}^{-}(\gamma(t), t)dt \\
&\leq \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(t) \int_{-\infty}^{t} e^{F(s)-F(t)}[L(\gamma(s), \dot{\gamma}(s), s) + \alpha]ds dt \\
&= \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(t)e^{-F(t)} \int_{-\infty}^{t} e^{F(s)}[L(\gamma(s), \dot{\gamma}(s), s) + \alpha]ds dt \\
&= \lim_{T \to +\infty} -\frac{1}{T} \int_{0}^{T} \left( \int_{-\infty}^{t} e^{F(s)}[L(\gamma(s), \dot{\gamma}(s), s) + \alpha]ds \right) e^{-F(t)} dt \\
&= \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} L(\gamma(t), \dot{\gamma}(t), t) + \alpha dt \\
&= \int_{TM \times T} L(x, u, \alpha) d\tilde{v},
\end{align*} \]

which is an equality only when \( \gamma \) is a backward calibrated curve of \((-\infty, t]\) for all \( t \in \mathbb{R} \),

which implies \( \gamma \) is globally calibrated. \( \square \)

Due to this Proposition we can easily show that \( \emptyset \neq \widetilde{\mathcal{M}} \subset \tilde{\mathcal{A}} \). Moreover, as we did for the Aubry set, we can similarly get that \( \pi^{-1} : \mathcal{M} \to \widetilde{\mathcal{M}} \) is a Lipschitz function.

4.3 Maximal Global Attractor in the Condition (H0\(^{-}\))

Note that \( \hat{H}(x, p, t, I, u) = H(x, p, t) + I + f(t)u - \alpha \) and equation(4). We derive

\[ \frac{d}{dt} \hat{H} = \frac{d}{dt} \left( H(x, p, t) + I + f(t)u - \alpha \right) \]

\[ = \partial_x H \cdot \dot{x} + \partial_p H \cdot \dot{p} + \partial_t H + \hat{I} + f'(t)u + f(t)\hat{u} \]

\[ = \partial_x H \cdot \partial_p H + \partial_p H \cdot (-\partial_x H - f(t)p) + \partial_t H + (-\partial_t H - f(t)I) + f'(t)u + f(t)\hat{u} \]

\[ = -f(t)(H + I + f(t)u - \alpha) \]

\[ = -f(t)\hat{H}. \]

\[ ^3 \text{Here } \pi_x, \pi_t, \pi_u \text{ is the standard projection to the space } M, T, \mathbb{R} \text{ respectively.} \]
From Remark 1.1 and \( |f| > 0 \), it follows that for any initial point \((x, p, s, I, u)\), the \( \omega \)-limit of trajectory \( \tilde{\varphi}_H^t(x, p, \tilde{s}, I, u) \) lies in
\[
\hat{\Sigma}_H := \{ \tilde{H}(x, p, \tilde{s}, I, u) = 0 \} \subset T^* M \times T^* \times \mathbb{R}. \tag{41}
\]

**Lemma 4.4** For any point \( Z := (x, p, \tilde{s}, \alpha - f(s)u - H(x, p, s), u) \in \hat{\Sigma}_H \) with \( u \leq u^-_\alpha(x, s) \), if
\[
\liminf_{t \to -\infty} e^{F(t)} |\pi_u \tilde{\varphi}_H^t(Z)| = 0,
\]
then \( \pi_s \tilde{\varphi}_H^t(Z) \) is a backward calibrated curve for \( t \leq 0 \).

**Proof** From the equation \( \dot{u} = \langle H_p, p \rangle - H - \alpha - f(t)u \), we derive
\[
e^{F(s)} \pi_u Z = \int_{-\infty}^0 \frac{d}{dt} e^{F(t+s)} \pi_u \tilde{\varphi}_H^t(Z) dt
= \int_{-\infty}^s e^{F(t)} \left( L(L(\tilde{\varphi}_H^{t-s}(x, p, \tilde{s}))) + \alpha \right) dt \leq u^-_\alpha(x, s),
\]
then due to the expression of \( u^-_\alpha \) in (26), \( \pi_s \tilde{\varphi}_H^t(Z) \) is a backward calibrated curve for \( t \leq 0 \).\( \square \)

This Lemma inspires us to decompose \( \hat{\Sigma}_H \) further:
\[
\begin{align*}
\hat{\Sigma}_H^- & := \{ (x, p, \tilde{s}, \alpha - f(s)u - H(x, p, s), u) | u > u^-_\alpha(x, s) \}, \\
\hat{\Sigma}_H^0 & := \{ (x, p, \tilde{s}, \alpha - f(s)u - H(x, p, s), u) | u = u^-_\alpha(x, s) \}, \\
\hat{\Sigma}_H^+ & := \{ (x, p, \tilde{s}, \alpha - f(s)u - H(x, p, s), u) | u < u^-_\alpha(x, s) \}.
\end{align*}
\]

**Lemma 4.5** For any \( Z := (x, p, \tilde{s}, \alpha - f(s)u - H(x, p, s), u) \in \hat{\Sigma}_H \), we have
\[
\begin{align*}
\partial^+_t \left( u^-_\alpha(\pi_x, \tilde{\varphi}_H^t(Z)) - \pi_u \tilde{\varphi}_H^t(Z) \right)
\leq -f(t + s) \left( u^-_\alpha(\pi_x, \tilde{\varphi}_H^t(Z)) - \pi_u \tilde{\varphi}_H^t(Z) \right).
\end{align*}
\tag{42}
\]
Consequently, \( \lim_{t \to +\infty} \tilde{\varphi}_H^t(Z) \in \hat{\Sigma}_H^- \cup \hat{\Sigma}_H^0 \).

**Proof** As \( \tilde{\varphi}_H^t(Z) = (x(t), p(t), \bar{t} + s, -f(s + t)u(t) - H(x(t), p(t), s + t), u(t)) \), then
\[
\begin{align*}
\partial^+_t [u^-_\alpha(x(t), s + t) - u(t)]
\leq & \max \{ \partial^+_s u^-_\alpha(x(t), s + t), \dot{x}(t) \} + \partial^+_s u^-_\alpha(x(t), s + t) - \dot{u}(t) \\
\leq & H(x(t), \partial^+_s u^-_\alpha(x(t), s + t), s + t) + L(x(t), \dot{x}(t), s + t) \\
+ & \partial^+_s u^-_\alpha(x(t), s + t) - \langle H_p(x(t), p(t), t + s), p(t) \rangle \\
+ & f(t + s)u(t) + H(x(t), p(t), s + t) - \alpha \\
= & \max \{ H(x(t), \partial^+_s u^-_\alpha(x(t), s + t), s + t) + \partial^+_s u^-_\alpha(x(t), s + t) \\
+ & f(t + s)u(t) - \alpha \} \\
\leq & f(t + s)u(t) - u^-_\alpha(x(t), t + s) \}
\end{align*}
\]
where the ‘max’ is about all the element \( (\partial^+_s u^-_\alpha(x(t), s + t), \partial^+_s u^-_\alpha(x(t), s + t)) \) in \( D^* u^-_\alpha(x(t), s + t) \) (see Theorem B.5 for the definition). So \( \lim_{t \to +\infty} \tilde{\varphi}_H^t(Z) \in \hat{\Sigma}_H^- \cup \hat{\Sigma}_H^0 \). \( \square \)
Proposition 4.6 \( \Omega := \bigcap_{t \geq 0} \hat{\varphi}^t_H (\hat{\Sigma}^-_H \cup \hat{\Sigma}^0_H) \) is the maximal invariant set contained in \( \hat{\Sigma}^-_H \cup \hat{\Sigma}^0_H \).

Proof Due to (42), \( \hat{\Sigma}^-_H \cup \hat{\Sigma}^0_H \) is forward invariant. Besides, any invariant set in \( \hat{\Sigma}_H \) has to lie in \( \hat{\Sigma}^-_H \cup \hat{\Sigma}^0_H \). So \( \Omega \) is the maximal invariant set in \( \hat{\Sigma}^-_H \cup \hat{\Sigma}^0_H \). \( \square \)

Lemma 4.7 If the \( p \)-component of \( \Omega \) is bounded, then the \( u, I \)-components of \( \Omega \) are also bounded.

Proof It suffices to prove that for any \( (x_0, p_0, \tilde{t}_0, I_0, u_0) \in T^*M \times T^*T \times \mathbb{R} \), there exists a time \( T(x_0, p_0, \tilde{t}_0, I_0, u_0) > 0 \) such that for any \( t \geq T \),

\[
\| \pi_{u,I} \hat{\varphi}^t_H (x_0, p_0, \tilde{t}_0, I_0, u_0) \| \leq C \tag{*}
\]

for a uniform constant \( C = C(\pi_p \Omega) \). Since \( \pi_p \Omega \) is bounded, due to the definition of \( \Omega \), for any \( (x_0, p_0, \tilde{t}_0, I_0, u_0) \in T^*M \times T^*T \times \mathbb{R} \), there always exists a time \( T'(x_0, p_0, \tilde{t}_0, I_0, u_0) > 0 \) such that for any \( t \geq T' \),

\[
\| \pi_p \hat{\varphi}^t_H (x_0, p_0, \tilde{t}_0, I_0, u_0) \| \leq C' = \frac{3}{2} \text{diam}(\pi_p \Omega)
\]

On the other side, the \( u \)-equation of (4) implies that for any \( t > 0 \),

\[
\| \pi_u \hat{\varphi}^{t+T'}_H (x_0, p_0, \tilde{t}_0, I_0, u_0) \| \\
\leq e^{\int_0^t e^{F(s+0+T') - F(t+0+T')} |\pi_u \hat{\varphi}^{s+T'}_H (x_0, p_0, \tilde{t}_0, I_0, u_0)| ds} + \int_0^t e^{F(s+0+T') - F(t+0+T')} \left| H_p - H \right|_{\hat{\varphi}^{t+T'}_H (x_0, p_0, \tilde{t}_0, I_0, u_0)} ds
\]

where the first term of the right hand side will tend to zero as \( t \to +\infty \), and the second term has a uniform bound depending only on \( |f| \), \( C' \). Therefore, there exists a time \( T''(x_0, p_0, \tilde{t}_0, I_0, u_0) \) such that for any \( t \geq T' + T'' \), there exists a constant \( C'' = C''(C', |f|) \) such that

\[
\| \pi_u \hat{\varphi}^t_H (x_0, p_0, \tilde{t}_0, I_0, u_0) \| \leq C''
\]

Benefiting from the boundedness of \( u \)-component, we can repeat aforementioned scheme to the \( I \)-equation of (4), then prove (*). \( \square \)

Once \( \Omega \) is compact, it has to be the maximal global attractor of \( \hat{\varphi}^t_H \) in the whole phase space \( T^*M \times T^*T \times \mathbb{R} \). Then due to Proposition 3.8, any backward calibrated curve \( \gamma_{X,s} : (-\infty, s] \to M \) decides a unique trajectory

\[
\hat{\varphi}^t_H \left( L^{-1}(x, \lim_{\varsigma \to s-} \hat{\gamma}_{X,s}(\varsigma), s), \alpha - f(s)u^-_a(x,s), -H(L^{-1}(x, \lim_{\varsigma \to s-} \hat{\gamma}_{X,s}(\varsigma), s), u^-_a(x,s)) \right)
\]

for \( t \in \mathbb{R} \), which lies in \( \hat{\Sigma}_H \). Furthermore,

\[
\hat{A} := \{ (L^{-1}(x, \partial_x u^-_a(x,t), t), \partial_t u^-_a(x,t), u^-_a(x,t)) \} \subset \Omega
\]

because \( \Omega \) is the maximal invariant set in \( \hat{\Sigma}_H \).

Lemma 4.8 \( \hat{A} \) is the maximal invariant set contained in \( \hat{\Sigma}^0_H \).
**Proof** If $\mathcal{I}$ is an invariant set contained in $\tilde{\Sigma}_H$, then $\pi_\mu(\tilde{\varphi}_H^t(\mathcal{I}))$ is always bounded. Due to Lemma 4.4, any trajectory in $\mathcal{I}$ has to be backward calibrated. As $\mathcal{I}$ is invariant, any trajectory in it has to be contained in $\hat{\mathcal{A}}$.

**Proof of Theorem 1.7:** Let $\tilde{\mu} \in \mathfrak{M}_L$ be ergodic, then we can find $(x_0, v_0, t_0) \in TM \times \mathbb{T}$ such that

$$\int_{TM \times \mathbb{T}} e^{F(t)}(L(x, v, t) + c(H))d\tilde{\mu} = \lim_{T \to +\infty} \frac{1}{T} \int_{-T}^{0} e^{F(\tau)}(L(\varphi_L^\tau(x_0, v_0, t_0)) + c(H))d\tau.$$ 

Therefore, for any weak KAM solution $u^-_c : M \times \mathbb{T} \to \mathbb{R}$ of (HJ0), we have

$$e^{F(0)}u^-_c(x_0, t_0) - e^{F(-\tau)}u^-_c(\pi_x, \varphi_L^{-\tau}(x_0, v_0, t_0)) \leq \int_{-\tau}^{0} e^{F(\tau)}(L(\varphi_L^\tau(x_0, v_0, t_0)) + c(H))d\tau,$$

which implies

$$\lim_{T \to +\infty} \frac{1}{T} \int_{-\tau}^{0} e^{F(\tau)}(L(\varphi_L^\tau(x_0, v_0, t_0)) + c(H))d\tau = \lim_{T \to +\infty} \frac{1}{T} (e^{F(0)}u^-_c(x_0 , t_0) - e^{F(-\tau)}u^-_c(\pi_x, \varphi_L^{-\tau}(x_0, v_0, t_0))) = 0.$$

Hence,

$$\int_{TM \times \mathbb{T}} e^{F(t)}(L(x, v, t) + c(H))d\tilde{\mu} \geq 0.$$ 

That further implies

$$\inf_{\mu \in \mathfrak{M}_L} \frac{\int_{TM \times \mathbb{T}} e^{F(t)}L(x, v, t)d\mu}{\int_{0}^{1} e^{F(\tau)}d\tau} \geq -c(H).$$

On the other side, for any $(x, 0) \in M \times \mathbb{T}$ fixed, the backward calibrated curve $\gamma^-_{x,0} : (0, -\infty) \to M$ satisfies

$$e^{F(0)}u_c(\gamma_{x,0}^-(0), 0) - e^{F(-n)}u_c(\gamma_{x,0}^-(n), -n) = \int_{-n}^{0} e^{F(\tau)}(L(\gamma_{x,0}^-(\tau), \dot{\gamma}_{x,0}^-(\tau), \tau)) + c)d\tau$$

for any $n \in \mathbb{Z}_+$. By the Resize Representation Theorem, the time average w.r.t. $\gamma_{x,0}^-[n, -n] : [-n, -n] \to M$ decides a sequence of Borel probability measures $\mu_n$. Due to Lemma 3.5, we can always find a subsequence $\{\tilde{\mu}_{n_k}\}$ converging to a unique Borel probability measure $\tilde{\mu}^*$, i.e.

$$\int_{TM \times \mathbb{T}} g(x, v, t)d\tilde{\mu}^* = \lim_{k \to \infty} \frac{1}{n_k} \int_{TM \times \mathbb{T}} g(x, v, t)d\tilde{\mu}_{n_k} = \lim_{k \to \infty} \frac{1}{n_k} \int_{-n_k}^{0} g(\gamma_{x,0}^-(\tau), \dot{\gamma}_{x,0}^-(\tau), \tilde{\tau})d\tau.$$
for any \( g \in C_c(TM \times \mathbb{T}, \mathbb{R}) \). Besides, we can easily prove that \( \tilde{\mu}^* \in \mathcal{M}_L \) and
\[
\int_{TM \times \mathbb{T}} e^{F(t)}(L(x, v, t) + c(H)) d\tilde{\mu}^* \\
= \lim_{k \to \infty} \frac{1}{n_k} \int_{-n_k}^0 e^{F(\tau)}(L(\gamma_{x,0}^-(-\tau), \dot{\gamma}_{x,0}^-(-\tau), \tau) + c(H)) d\tau \\
= \lim_{k \to \infty} \frac{1}{n_k} \left( u^-_c(\gamma_{x,0}^-(-0), 0) - u^-_c(\gamma_{x,0}^-(-n_k), -n_k) \right) = 0.
\]

Then,
\[
-c(H) = \inf_{\tilde{\mu} \in \mathcal{M}_L} \int_{TM \times \mathbb{T}} e^{F(t)} L(x, v, t) d\tilde{\mu}.
\]

Gathering all the infimum of the right side of previous equality, we get a set of Mather measures \( \mathcal{M}_m \). Due to the Crossing Lemma in [23], the Mather set
\[
\tilde{\mathcal{M}} := \bigcup_{\tilde{\mu} \in \mathcal{M}_m} \text{supp}(\tilde{\mu})
\]
is a Lipschitz graph over \( \mathcal{M} := \pi \tilde{\mathcal{M}} \).

\section{5 Convergence of Parameterized Viscosity Solutions}

In this section we deal with the convergence of weak KAM solution \( u_\delta^- \) for system (14) as \( \delta \to 0_+ \). Recall that \( [f_0] = 0 \) and
\[
f_1(t) := \lim_{\delta \to 0_+} \frac{f_\delta(t) - f_0(t)}{\delta} > 0,
\]
there must exist a \( \delta_0 > 0 \) such that
\[
f_\delta(t) > f_0(t), \quad \forall \ t \in \mathbb{T}
\]
for all \( \delta \in [0, \delta_0] \). Due to Theorem 1.3 there exists a unique \( c(H) \), such that the weak KAM solutions \( u_\delta^- \) of (17) with \( \alpha = c(H) \) exist. For each \( (x, t) \in M \times \mathbb{R} \) and \( s < t \), the Lax–Oleinik operator
\[
T_{\delta}^-\gamma(x, t) = \inf_{\gamma \in C^{ac}_{[s,t]}(x, M)} \int_s^t e^{F_\delta(\tau) - F_\delta(s)}(L(\gamma(\tau), \dot{\gamma}(\tau), \tau) + c(H)) d\tau
\]
is well defined, of which the following Lemma holds:

**Lemma 5.1** For each \( \delta \geq 0 \) and \( T_{\delta}^-\gamma(x, t) \) converges uniformly to \( u_-\gamma(x, t) \) on each compact subset of \( M \times \mathbb{R} \) as \( s \to -\infty \).

**Proof** Let \( \gamma_{\delta,x,t}^- : (-\infty, t] \to M \) be a calibrated curve of \( u_\delta^- (x, t) \). Then,
\[
e^{F_\delta(t)}u_\delta^-(x, t) = e^{F_\delta(s)}u_\delta^-(\gamma_{\delta,x,t}^-(s), s) + \int_s^t e^{F_\delta(\tau)}(L(\gamma_{\delta,x,t}^-(\tau), \dot{\gamma}_{\delta,x,t}^-(\tau), \tau) + c(H)) d\tau
\]
and
\[
e^{F_\delta(t)}T_{\delta}^-\gamma(x, t) \leq \int_s^t e^{F_\delta(\tau)}(L(\gamma_{\delta,x,t}^-(\tau), \dot{\gamma}_{\delta,x,t}^-(\tau), \tau) + c(H)) d\tau.
\]
Then,
\[ T_s^{\delta,-}(x, t) - u_\delta^-(x, t) \leq -e^{F_3(s) - F_3(t)} u_\delta^-(\gamma_{0,x,t}(s), s). \quad (43) \]

On the other hand, let \( \gamma_0 : [s, t] \to M \) be a minimizer of \( T_s^{\delta,-}(x, t) \). Then,
\[ e^{F_3(t)} T_s^{\delta,-}(x, t) = \int_s^t e^{F_3(\tau)} (L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H))d\tau \]
and
\[ e^{F_3(t)} u_\delta^-(x, t) - e^{F_3(s)} u_\delta^-(\gamma_0(s), s) \leq \int_s^t e^{F_3(\tau)} (L(\gamma_0, \dot{\gamma}_0, \tau) + c(H))d\tau. \]

Hence,
\[ u_\delta^-(x, t) - T_s^{\delta,-}(x, t) \leq e^{F_3(s) - F_3(t)} u_\delta^-(\gamma_0(s), s). \quad (44) \]

From (43) and (44), it follows
\[ |u_\delta^-(x, t) - T_s^{\delta,-}(x, t)| \leq e^{F_3(s) - F_3(t)} \max u_\delta^- \]
which means \( T_s^{\delta,-}(x, t) \) converges uniformly to \( u_\delta^-(x, t) \) on each compact subset of \( M \times \mathbb{R} \).

**Lemma 5.2** \( u_\delta^- : M \times \mathbb{T} \to \mathbb{R} \) are equi-bounded and equi-Lipschitz w.r.t. \( \delta \in (0, \delta_0] \).

**Proof** To show \( u_\delta^- \) are equi-bounded from below, it suffices to show
\[ \{ T_s^{\delta,-}(x, t)|(x, t) \in M \times [0, 1], s \leq 0, \delta \in (0, \delta_0]\} \]
is bounded from below. Let \( \gamma_0 : [s, t] \to M \) be a minimizer of \( T_s^{\delta,-}(x, t) \), \( u_\delta(\tau) := T_s^{\delta,-}(\gamma_0(\tau), \tau) \), and \( \bar{u}_\delta(\tau) := e^{F_3(\tau)} u_\delta(\tau), \tau \in [s, t] \). Then,
\[ \frac{d\bar{u}_\delta(\tau)}{d\tau} = e^{F_3(\tau)} (L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H)). \]

Hence,
\[ \frac{d\bar{u}_\delta(\tau)}{d\tau} = L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H) - f_\delta(\tau) u_\delta(\tau). \]

We could assume \( T_s^{\delta,-}(x, t) < 0 \) for some \( \delta \in (0, \delta_0] \), \( (x, t) \in M \times [0, 1], s \leq 0 \), otherwise 0 is a uniform lower bound of \( \{ T_s^{\delta,-}(x, t)|(x, t) \in M \times [0, 1], s \leq 0, \delta \in (0, \delta_0]\} \). Note that \( u_\delta(\cdot) \) are continuous and \( u_\delta(s) = 0 \). There exists \( s_0 \in [s, t] \) such that \( u_\delta(s_0) = 0 \) and \( u_\delta(\tau) < 0, \tau \in (s_0, t] \). From \( f_\delta > f_0 \), it follows that
\[ \frac{d\bar{u}_\delta(\tau)}{d\tau} \geq L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H) - f_0(\tau) u_\delta(\tau), \tau \in [s_0, t]. \]

Hence,
\[ \frac{d}{d\tau} (e^{F_0(\tau)} u_\delta(\tau)) \geq e^{F_0(\tau)} (L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H)), \]
where \( F_0(\tau) = f_0^{\tau} f_0(\sigma)d\sigma \). Integrating on \([s_0, t] \), it holds that
\[ e^{F_0(t)} u_\delta(t) \geq \int_{s_0}^t e^{F_0(\tau)} (L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H))d\tau. \quad (45) \]

Let \( \beta : [t, t+2-\frac{1}{k_1}] \to M \) be a geodesic with \( \beta(t) = \gamma_0(t), \beta(t+2-\frac{1}{k_1} = \gamma_0(s_0) \), and
\[ |\dot{\beta}(t)| = \frac{d(\gamma_0(s_0), \gamma_0(t))}{2 - \frac{1}{k_1}} \leq \text{diam}(M) =: k_1. \]
Due to the definition of $c(H)$ in (35), we derive
\[ \int_{s_0}^t e^{F_0(\tau)} (L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H)) d\tau \]
\[ + \int_t^{t+2-t-s_0} e^{F_0(\tau)} (L(\beta(\tau), \dot{\beta}(\tau), \tau) + c(H)) d\tau \geq 0. \]

Note that
\[ \int_{t}^{t+2-t-s_0} e^{F_0(\tau)} (L(\beta(\tau), \dot{\beta}(\tau), \tau) + c(H)) d\tau \]
\[ \leq \int_{t}^{t+2-t-s_0} e^{F_0(\tau)} (C_k + c(H)) d\tau \leq 2(C_k + c(H)) e^{\max_f e\tau} F_0(t). \]

Hence,
\[ \int_{s_0}^t e^{F_0(\tau)} (L(\gamma_0(\tau), \dot{\gamma}_0(\tau), \tau) + c(H)) d\tau \geq -2(C_k + c(H)) e^{\max F_0}. \]

Combining (45), we derive
\[ u_\delta(t) \geq -2[C_k + c(H)] e^{\max F_0 - \min F_0}. \]

Next, we prove $u_\delta^-(x, t)$ are equi-bounded from above. It suffices to show $(T_\delta^{-}(x, t))(x, t) \in M \times [0, 1], s \leq 0, \delta \in (0, \delta_0])$ is bounded from above. We could assume $T_\delta^{-}(x, t) > 0$ for some $\delta \in (0, \delta_0], (x, t) \in M \times [0, 1], s \leq 0, \delta \in (0, \delta_0])$.

Let $u_0^-(x, t)$ be a weak KAM solution of
\[ \partial_t u + H(x, \partial_x u, t) + f_0(t)u = c(H), \]

and $y_{x,t}^- : (\infty, t] \to M$ be a calibrated curve of $u_0^-(x, t)$. Let
\[ v_\delta(\tau) := T_\delta^-(y_{x,t}^-(\tau), \tau), \tau \in [s, t]. \]

Then
\[ e^{F_3(\tau+\Delta \tau)} v_\delta(\tau + \Delta \tau) - e^{F_3(\tau)} v_\delta(\tau) = \frac{1}{\Delta \tau} \int_{\tau}^{\tau+\Delta \tau} e^{F_3(\sigma)} (L(y_{x,t}^-(\sigma), \dot{y}_{x,t}^-(\sigma), \sigma) + c(H)) d\sigma. \]

Note that
\[ \lim_{\Delta \tau \to 0} \frac{e^{F_3(\tau+\Delta \tau)} v_\delta(\tau + \Delta \tau) - e^{F_3(\tau)} v_\delta(\tau)}{\Delta \tau} = \lim_{\Delta \tau \to 0} e^{F_3(\tau+\Delta \tau)} v_\delta(\tau + \Delta \tau) - e^{F_3(\tau+\Delta \tau)} v_\delta(\tau) + e^{F_3(\tau+\Delta \tau)} v_\delta(\tau) - e^{F_3(\tau)} v_\delta(\tau) \]
\[ = e^{F_3(\tau)} \lim_{\Delta \tau \to 0} \left( \frac{v_\delta(\tau + \Delta \tau) - v_\delta(\tau)}{\Delta \tau} \right) + e^{F_3(\tau)} f_\delta(v_\delta(\tau)). \]

Hence,
\[ \lim_{\Delta \tau \to 0} \left( \frac{v_\delta(\tau + \Delta \tau) - v_\delta(\tau)}{\Delta \tau} \right) \leq L(y_{x,t}^-(\tau), \dot{y}_{x,t}^-(\tau), \tau) + c(H) - f_\delta(v_\delta(\tau)). \]
Since \( v_\delta(s) = 0 \) and \( v_\delta(\tau) \) is continuous, there exists \( s_1 \in [s, t) \) such that \( v_\delta(s_1) = 0 \) and \( v_\delta(\tau) > 0, \tau \in (s_1, t] \).

For \( \tau \in (s_1, t] \),
\[
\lim_{\Delta \tau \to 0} \frac{v_\delta(\tau + \Delta \tau) - v_\delta(\tau)}{\Delta \tau} \leq L(\gamma_{x,t}^- (\tau), \dot{\gamma}_{x,t}^- (\tau), \tau) + c(H) - f_\delta(\tau) v_\delta(\tau) \\
\leq L(\gamma_{x,t}^- (\tau), \dot{\gamma}_{x,t}^- (\tau), \tau) + c(H) - f_0(\tau) v_\delta(\tau).
\]

Then,
\[
\lim_{\Delta \tau \to 0} \left( \frac{e^{F_0(\tau + \Delta \tau)} v_\delta(\tau + \Delta \tau) - e^{F_0(\tau)} v_\delta(\tau)}{\Delta \tau} \right) \leq e^{F_0(\tau)} (L(\gamma_{x,t}^- (\tau), \dot{\gamma}_{x,t}^- (\tau), \tau) + c(H)).
\]

From \( v_\delta(s_1) = 0 \), it follows that
\[
e^{F_0(t)} v_\delta(t) \leq \int_{s_1}^{t} e^{F_0(\tau)} (L(\gamma_{x,t}^- (\tau), \dot{\gamma}_{x,t}^- (\tau), \tau) + c(H)) \, d\tau \\
= e^{F_0(t)} u_0^- (x, t) - e^{F_0(s_1)} u_0^- (\gamma_{x,t}^- (s_1), s_1).
\]

Then,
\[
v_\delta(t) \leq 2 \max \{u_0^- \} \cdot e^{\max F_0 - \min F_0}.
\]

Note that \( u_\delta^- (x, t) \) are equi-bounded. By a similar approach of the proof of Lemma 3.6, we derive that \( u_\delta^- \) are equi-Lipschitz.

\[\square\]

**Lemma 5.3** For any \( \delta \in (0, \delta_0] \) and any \( (x, \tilde{s}) \in M \times T \), the backward calibrated curve \( \gamma_{\delta,x,s}^- : (-\infty, s] \to M \) associated with \( u_\delta^- \) has a uniformly bounded velocity, i.e. there exists a constant \( K > 0 \), such that
\[
|\dot{\gamma}_{\delta,x,s}^- (t)| \leq K, \quad \forall \delta \in (0, 1) \text{ and } t \in (-\infty, s).
\]

**Proof** By a similar way in the proof of Lemma 3.5, there exists \( s_0 \) in each interval with length 1, such that
\[
|\dot{\gamma}_{\delta,x,s}^- (s_0)| \leq C_k_1 + C(1),
\]
where \( k_1 = \text{diam}(M) \). Note that \( f_\delta \) depends continuously on \( \delta \) and is 1-periodic. We derive the Lagrangian flow \( (\gamma_{\delta,x,s}^- (\tau), \dot{\gamma}_{\delta,x,s}^- (\tau), \tau) \) is 1-periodic and depends continuously on the parameter \( \delta \). Hence, there exists \( K > 0 \) depends only on \( L, k_1, \) and \( \delta_0 \), such that \( |\dot{\gamma}_{\delta,x,s}^-| < K \).

\[\square\]

**Proposition 5.4** For any ergodic measure \( \mu \in M_m(0) \) and any \( 0 < \delta \leq \delta_0 \), we have
\[
\int_{TM \times T} e^{F_0(t)} \frac{f_\delta(t) - f_0(t)}{\delta} u_\delta^- (x, t) \, d\tilde{\mu}(x, v, t) \leq 0.
\]

**Proof** Since \( \{u_\delta^- \}_{\delta \in (0, \delta_0]} \) is uniformly bounded and \([f_0] = 0\), then
\[
\lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_{TM \times T} u_\delta^- (\gamma(t), t) \, d\mu_\delta(t) = \int_{TM \times T} u_\delta^- (x, t) f_0(t) e^{F_0(t)} \, d\tilde{\mu}(x, v, t)
\]
for any regular curve \( \gamma(t) = (\gamma(t), \tilde{t}) : t \in \mathbb{R} \to M \times T \) contained in \( M(\delta) \). Due to Proposition 3.8,
\[
\frac{1}{T} \int_0^T u_\delta^- (\gamma(t), t) \, d\mu_\delta(t) = \frac{1}{T} u_\delta^- (\gamma(t), t) e^{F_0(t)} T_0 - \frac{1}{T} \int_0^T e^{F_0(t)} [\partial_t u_\delta^- (\gamma(t), t)] \, dt
\]
\[
+ \langle \dot{\gamma}(t), \partial_x u_\delta^- (\gamma(t), t) \rangle \]
and
\[ \frac{1}{T} \int_0^T e^{F_0(t)} \left[ \partial_t u_{\delta}(\gamma(t), t) + \langle \gamma(t), \partial_x u_{\delta}(\gamma(t), t) \rangle \right] dt \]
\[ \leq \frac{1}{T} \int_0^T e^{F_0(t)} \left[ L(\gamma, \dot{\gamma}, t) + H(\gamma(t), \partial_x u_{\delta}(\gamma(t), t), t) + \partial_t u_{\delta}(\gamma(t), t) \right] dt \]
\[ \leq \frac{1}{T} \int_0^T e^{F_0(t)} \left[ L(\gamma, \dot{\gamma}, t) + c(H) - f_\delta(t) u_{\delta}(\gamma(t), t) \right] dt, \]
by taking \( T \rightarrow +\infty \) and dividing both sides by \( \delta \) we get the conclusion.

**Definition 5.5** Let’s denote by \( F_+ \) the set of all viscosity subsolutions \( \omega : M \times T \rightarrow \mathbb{R} \) of (15) with \( \delta = 0 \) such that
\[ \int_{TM \times T} f_1(t) \omega(x, t) e^{F_0(t)} d\tilde{\mu} \leq 0, \quad \forall \tilde{\mu} \in \mathcal{M}_m(0). \] (47)

**Lemma 5.6** The set \( F_- \) is uniformly bounded from above, i.e.
\[ \sup \{ u(x) \mid \forall x \in M, \ u \in F_- \} < +\infty. \]

**Proof** By an analogy of Lemma 10 of [11], all the functions in the set
\[ \left\{ e^{F_0(t)} \omega : M \times T \rightarrow \mathbb{R} \mid \omega < f_0 \ L + c(H) \right\} \]
are uniformly Lipschitz with a Lipschitz constant \( \kappa > 0 \). For any \( \tilde{\mu} \in \mathcal{M}_m(0) \) and \( u \in F_- \)
\[ \min_{(x, t) \in M \times T} u(x, t) e^{F_0(t)} = \frac{\int_{TM \times T} f_1(t) \min_{(x, t) \in M \times T} u(x, t) e^{F_0(t)} d\tilde{\mu}}{\int_{TM \times T} f_1(t) d\tilde{\mu}} \]
\[ = \frac{\int_{TM \times T} f_1(t) \min_{(x, t) \in M \times T} u(x, t) e^{F_0(t)} d\tilde{\mu}}{\int_0^1 f_1(t) dt} \]
\[ \leq \left( \frac{\int_M f_1(t) u(x, t) e^{F_0(t)} d\tilde{\mu}}{\int_0^1 f_1(t) dt} \right) \leq 0. \]
Then,
\[ \max_{(x, t) \in M \times T} u(x, t) e^{F_0(t)} \leq \max_{(x, t) \in M \times T} u(x, t) e^{F_0(t)} - \min_{(x, t) \in M \times T} u(x, t) e^{F_0(t)} \]
\[ \leq \kappa \ \text{diam}(M \times T) < +\infty. \]
As a result,
\[ \max_{(x, t) \in M \times T} u(x, t) \leq \frac{\max_{(x, t) \in M \times T} u(x, t) e^{F_0(t)}}{\min_{t \in T} e^{F_0(t)}} < +\infty \]
so we finish the proof. \( \square \)

As \( F_- \) is now upper bounded, we can define a supreme subsolution by
\[ u_0^* := \sup_{u \in F_-} u. \] (48)
Later we will see that this is indeed a viscosity solution of (15) for \( \delta = 0, \alpha = c(H) \) and is the unique accumulating function of \( u_{\delta}^- \) as \( \delta \rightarrow 0_+ \). 

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Proposition 5.7 For any $\delta > 0$, any viscosity subsolution $\omega : M \times \mathbb{T} \to \mathbb{R}$ of (15) with $\delta = 0$, $\alpha = c(H)$ and any point $(x, s) \in M \times \mathbb{T}$, there exists a $\varphi'_L$—backward invariant finite measure $\mu^\delta_{x,s} : TM \times \mathbb{T} \to \mathbb{R}$ such that

\begin{equation}
 u^\delta_\omega(x, s) \geq \omega(x, s) - \int_{TM \times \mathbb{T}} \omega(y, t)e^{F_0(t)}f_1(t)d\mu^\delta_{x,s}(y, v, t) \tag{49}
\end{equation}

where

$$
\int_{TM \times \mathbb{T}} g(y, t)d\mu^\delta_{x,s}(y, v, t) \tag{50.1}
$$

$$
:= \int_{-\infty}^{s} g(\gamma_{\delta,x,s}^-(t), t) \cdot \frac{d}{dt} \left( e^{F_1(t)} - e^{F_0(t)} \right) dt, \forall g \in C(M \times \mathbb{T}, \mathbb{R}).
$$

Proof For any $(x, s) \in M \times \mathbb{T}$ and any $\delta \in (0, \delta_0]$, there exists a backward calibrated curve $\gamma_{\delta,x,s}^-$ ending with $x$, such that the viscosity solution $u^\delta_\omega$ is differentiable along $(\gamma_{\delta,x,s}^-(t), t)$ for all $t \in (-\infty, s)$ due to Proposition 3.8. Precisely, for all $t \in (-\infty, s)$

$$
d\left( e^{F_1}(t)u^\delta_\omega(\gamma_{\delta,x,s}^-(t), t) \right) = e^{F_1}(t) \left( L(\gamma_{\delta,x,s}^-(t), R_{\delta,x,s}(t), t) + c(H) \right).
$$

Integrating on $[s, -T]$,

$$
e^{F_3(s)}u^\delta_\omega(x, s) - e^{F_3(-T)}u^\delta_\omega(\gamma_{\delta,x,s}^-(T), -T) = \int_{-T}^{s} e^{F_3(t)} \left[ L(\gamma_{\delta,x,s}^-(t), R_{\delta,x,s}(t), t) + c(H) \right] dt
$$

for any $T > 0$, where $F_3(t) := \int_0^t f_3(\tau) d\tau$. On the other side,

$$
\partial_t \omega(x, t) + H(x, \partial_x \omega(x, t)) + f_0(t) \omega(x, t) \leq c(H), \quad a.e. (x, t) \in M \times \mathbb{T}
$$

since $\omega$ is also a subsolution of (15) (with $\delta = 0$), then

$$
e^{F_3(s)}u^\delta_\omega(x, s) - e^{F_3(-T)}u^\delta_\omega(\gamma_{\delta,x,s}^-(T), -T)
\geq \int_{-T}^{s} e^{F_3(t)} \left[ L(\gamma_{\delta,x,s}^-(t), R_{\delta,x,s}(t), t) + H(\gamma_{\delta,x,s}^-(t), \partial_x \omega(\gamma_{\delta,x,s}^-(t), t)) \right] dt
\geq \int_{-T}^{s} e^{F_3(t)} \left[ \frac{d}{dt} \omega(\gamma_{\delta,x,s}^-(t), t) + f_0(t) \omega(\gamma_{\delta,x,s}^-(t), t) \right] dt
\geq e^{F_3(s)} \omega(x, s) - e^{F_3(-T)} \omega(\gamma_{\delta,x,s}^-(T), -T) - \int_{-T}^{s} \omega(\gamma_{\delta,x,s}^-(t), t)e^{F_3(t)}(f_3(t) - f_0(t)) dt.
$$

By taking $T \to +\infty$ we finally get

$$
e^{F_3(s)}u^\delta_\omega(x, s) - e^{F_3(s)} \omega(x, s) \geq -\int_{-\infty}^{s} \omega(\gamma_{\delta,x,s}^-(t), t)e^{F_3(t)}(f_3(t) - f_0(t)) dt.
$$
By a suitable transformation,
\[
\begin{align*}
    u_\delta^c(x, s) & \geq \omega(x, s) - \int_{-\infty}^s \omega(\gamma_{\delta,x,s}^-, t) e^{F_0(t)} e^{F_\delta(t) - F_0(t)} \left( f_\delta(t) - f_0(t) \right) dt \\
    &= \omega(x, s) - \int_{-\infty}^s \omega(\gamma_{\delta,x,s}^-, t) e^{F_0(t)} dF_\delta(t) - F_0(t) \\
    &= \omega(x, s) - \int_{-\infty}^s \omega(\gamma_{\delta,x,s}^-, t) e^{F_0(t)} f_1(t) \frac{dF_\delta(t) - F_0(t)}{f_1(t)}.
\end{align*}
\]

Then for any \( g \in C(M \times \mathbb{T}, \mathbb{R}) \), the measure \( \widetilde{\mu}_{x,s}^\delta \) defined by
\[
\int_{TM \times \mathbb{T}} g(y, \tau) \, d\widetilde{\mu}_{x,s}^\delta(y, v_y, \tau) := \int_{-\infty}^s g(\gamma_{\delta,x,s}^-, t) \frac{dF_\delta(t) - F_0(t)}{f_1(t)}
\]
is just the desired one. \( \square \)

**Lemma 5.8** Any weak limit of the normalized measure
\[
\hat{\mu}_{x,s}^\delta := \frac{\tilde{\mu}_{x,s}^\delta}{\int_{TM \times \mathbb{T}} d\tilde{\mu}_{x,s}^\delta}
\]
as \( \delta \to 0_+ \) is contained in \( \mathcal{M}_m(0) \), i.e. a Mather measure.

**Proof** As is proved in Proposition 5.7, \( \tilde{\mu}_{x,s}^\delta \) are uniformly bounded w.r.t. \( \delta \in (0, \delta_0] \). Therefore, it suffices to prove that any weak limit \( \mu_{x,s} \) of \( \tilde{\mu}_{x,s}^\delta \) as \( \delta \to 0_+ \) satisfies the following two conclusions:

First, we show \( \mu_{x,s} \) is a closed measure. It is equivalent to show that for any \( \phi(\cdot) \in C^1(M \times \mathbb{T}, \mathbb{R}) \),
\[
\lim_{\delta \to 0_+} \int_{-\infty}^s \frac{d}{dt} \phi(\gamma_{\delta,x,s}^-, t) \frac{dF_\delta(t) - F_0(t)}{f_1(t)} = 0.
\]
Indeed, we have
\[
\begin{align*}
    \lim_{\delta \to 0_+} \int_{-\infty}^s \frac{d}{dt} \phi(\gamma_{\delta,x,s}^-, t) \frac{dF_\delta(t) - F_0(t)}{f_1(t)} &= \lim_{\delta \to 0_+} \int_{-\infty}^s e^{F_\delta(t) - F_0(t)} f_\delta(t) - f_0(t) \frac{f_1(t)}{f_1(t)} \phi(\gamma_{\delta,x,s}^-, t) dt \\
    &= \lim_{\delta \to 0_+} \int_{-\infty}^s f_\delta(t) - f_0(t) \frac{f_1(t)}{f_1(t)} e^{F_\delta(t) - F_0(t)} \phi(\gamma_{\delta,x,s}^-, t) dt \\
    &\quad - \lim_{\delta \to 0_+} \int_{-\infty}^s \phi(\gamma_{\delta,x,s}^-, t) \cdot d \left( \frac{f_\delta(t) - f_0(t)}{f_1(t)} e^{F_\delta(t) - F_0(t)} \right) = 0
\end{align*}
\]
because \( f_\delta \to f_0 \) uniformly as \( \delta \to 0_+ \).

Next, we can show that
\[
\lim_{\delta \to 0_+} \int_{-\infty}^s e^{F_\delta(t)} \left[ L(\gamma_{\delta,x,s}^-, \dot{\gamma}_{\delta,x,s}^-) + c(H) \right] \frac{dF_\delta(t) - F_0(t)}{f_1(t)} = 0.
\]
Note that
\[
\frac{d}{dt} \left( e^{F_\delta(t)} u_\delta^c(\gamma_{\delta,x,s}^-, t) \right) = e^{F_\delta(t)} \left( L(\gamma_{\delta,x,s}^-, \dot{\gamma}_{\delta,x,s}^-) + c(H) \right).
\]
We derive
\[
\lim_{\delta \to 0+} \int_{-\infty}^{s} e^{F_\delta(t)} \left[ L(\gamma_{\delta,x,s}(t), \dot{\gamma}_{\delta,x,s}(t), t) + c(H) \right] \frac{\delta e^{F_\delta(t)} - F_0(t)}{f_1(t)} = 0,
\]
since \( u_\delta^- \) is differentiable along \( (\gamma_{\delta,x,s}(t), \tilde{t}) \) for all \( t \in (-\infty, s) \) and \( \tilde{\mu}_{x,s} \) is closed. So we finish the proof.

**Proof of Theorem 1.8:** Due to the stability of viscosity solution (see Theorem 1.4 in [12]), any accumulating function \( u_0^- \) of \( u_\delta^- \) as \( \delta \to 0_+ \) is a viscosity solution of (15) with \( \delta = 0 \). Therefore, Proposition 5.4 indicates \( u_0^- \in \mathcal{F}_- \), so \( u_0^- \leq u_0^n \). On the other side, Proposition 5.7 implies \( u_0^- \geq \omega \) for any \( \omega \in \mathcal{F}_- \) as \( \delta \to 0_+ \), since any weak limit of \( \tilde{\mu}_{x,s}^\delta \) as \( \delta \to 0_+ \) proves to be a Mather measure in Lemma 5.8. So we have \( u_0^- \geq u_0^n \). \( \square \)

6 Asymptotic Behaviors of Trajectories of 1-D Mechanical Systems

**Lemma 6.1** For system (20), \( \rho(c) \) is continuous of \( c \in H^1(\mathbb{T}, \mathbb{R}) \).

**Proof** Firstly, all the orbits in \( \tilde{\mathcal{A}}(c) \) should have the unified rotation number. This is because \( \pi^{-1} : \mathcal{A}(c) \to \tilde{\mathcal{A}}(c) \) is a Lipschitz graph and \( \dim(M) = 1 \). Secondly, \( \lim_{c' \to c} \tilde{\mathcal{A}}(c') \subset \tilde{\mathcal{A}}(c) \) due to Lemma 4.2. That further indicates \( \lim_{c' \to c} \rho(c') = \rho(c) \).

**Lemma 6.2** For system (20), the rotation number \( \rho(c) \) can be dominated by
\[
-\|V\|_{C^1} \cdot \varsigma - c \leq \rho(c) \leq \|V\|_{C^1} \cdot \varsigma - c \quad (51)
\]
where \( \varsigma = \varsigma([f]) > 0 \) tends to infinity as \( [f] \to 0_+ \).

**Proof** Recall that
\[
\dot{p} = -V_x(x, t) - f(t)p,
\]
then starting from any point \( (x_0, p_0, \tilde{t}_0) \in T^*M \times \mathbb{T} \), we get
\[
p(t) = e^{-F(t)} p_0 - e^{-F(t)} \int_0^t e^{F(s)} V_x(x(s), s) ds, \quad t > 0.
\]
As \( t \to +\infty \), we have
\[
\lim_{t \to +\infty} |p(t)| \leq \|V\|_{C^1} \cdot \limsup_{t \to +\infty} e^{-F(t)} \int_0^t e^{F(s)} ds \leq \varsigma(\|f\|) \cdot \|V\|_{C^1} \quad (52)
\]
for a constant \( \varsigma(\|f\|) > 0 \) depending only on \( f \). As a consequence,
\[
-\|V\|_{C^1} \cdot \varsigma \leq \pi_p \tilde{\mathcal{A}}(c) \leq \|V\|_{C^1} \cdot \varsigma \quad (53)
\]
dominate the \( p \)-component of \( \tilde{\mathcal{A}}(c) \). \( \square \)

**Proof of Theorem 1.10:** The first two items have been proved in previous Lemmas 6.1 and 6.2. As for the third item, Lemma 6.2 has shown the boundedness of \( p \)-component of \( \Omega \), then due to Theorem 1.5, we get the compactness of \( \Omega \). \( \square \)
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**Appendix A: Mather Measure of Convex Lagrangians with $[f] = 0$**

For a Tonelli Hamiltonian $H(x, p, t)$, the conjugated Lagrangian $L(x, v, t)$ can be established by (6), which is also Tonelli. On the other side, for $[f] = 0$, the following Lagrangian $	ilde{L}(x, v, t)$ can be established by (6), which is also Tonelli. On the other side, for $[f] = 0$, the following Lagrangian

$$
\tilde{L}(x, v, t) := e^{F(t)} L(x, v, t), \quad (x, v, t) \in TM \times \mathbb{T}
$$

with

$$
F(t) := \int_0^t f(s) ds
$$

is still time-periodic as the case considered in [23]. Besides, the Euler-Lagrange equation associated with $\tilde{L}$ is the same with (E-L). So Mañé’s approach to get a Mather measure in [20] is still available for us. As his approach doesn’t rely on the E-L flow, that supplies us with great convenience.

Let $X$ be a metric separable space. A probability measure on $X$ is a nonnegative, countably additive set function $\mu$ defined on the $\sigma$-algebra $B(X)$ of Borel subsets of $X$ such that $\mu(X) = 1$. In this paper, $X = TM \times \mathbb{T}$. We say that a sequence of probability measures $\{\tilde{\mu}_n\}_{n \in \mathbb{N}}$ (weakly) converges to a probability measure $\tilde{\mu}$ on $TM \times \mathbb{T}$ if

$$
\lim_{n \to +\infty} \int_{TM \times \mathbb{T}} h(x, v, t) d\tilde{\mu}_n(x, v, t) = \int_{TM \times \mathbb{T}} h(x, v, t) d\tilde{\mu}(x, v, t)
$$

for any $h \in C_c(TM \times \mathbb{T}, \mathbb{R})$.

**Definition A.1.** A probability measure $\tilde{\mu}$ on $TM \times \mathbb{T}$ is called closed if it satisfies:

1. $\int_{TM \times \mathbb{T}} |v| d\tilde{\mu}(x, v, t) < +\infty$;
2. $\int_{TM \times \mathbb{T}} (\delta_x \phi(x, t), v) + \delta_t \phi(x, t) d\tilde{\mu}(x, v, t) = 0$ for every $\phi \in C^1(M \times \mathbb{T}, \mathbb{R})$.

Let’s denote by $P_c(TM \times \mathbb{T})$ the set of all closed measures on $TM \times \mathbb{T}$, then the following conclusion is proved in [20]:

**Theorem A.2.**

$$
\min_{\tilde{\mu} \in P_c(TM \times \mathbb{T})} \int_{TM} \tilde{L}(x, v, t) d\tilde{\mu}(x, v, t) = -c(H).
$$

Moreover, the minimizer $\tilde{\mu}_{\min}$ must be a Mather measure, i.e. $\tilde{\mu}_{\min}$ is invariant w.r.t. the Euler-Lagrange flow (E-L).

**Proof** This conclusion is a direct adaption of Proposition 1.3 of [20] to our system $\tilde{L}(x, v, t)$, with the $c(H)$ already given in (13). 

**Appendix B: Semiconcave Functions**

Here we attach a series of conclusions about the semiconcave functions which can be found in [5], for the use of Proposition 3.8.
Definition B.1. Assume $S$ is a subset of $\mathbb{R}^n$. A function $u: S \to \mathbb{R}$ is called semiconcave, if there exists a nondecreasing upper semicontinuous function $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\rho \to 0^+} \omega(\rho) = 0$ and
\[
\lambda u(x) - (1 - \lambda)u(y) - u(\lambda x + (1 - \lambda)y) \leq \lambda(1 - \lambda)|x - y|\omega(|x - y|), \forall \lambda \in [0, 1]. \tag{54}
\]
We call $\omega$ a modulus of semiconcavity for $u$ in $S$.

Definition B.2. [Definition 3.1.1 in [5]] For any $x \in S$, the set
\[
D^+ u(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\}
\]
is called the Fréchet superdifferential of $u$ at $x$. We shall give some properties of $D^+ u(x)$, which can be found in Chapter 3 of [5].

Proposition B.3. Assume $A \subset \mathbb{R}^n$ is open. Let $u: A \to \mathbb{R}$ be a semiconcave function with modulus $\omega$ and $x \in A$. Then,
\begin{itemize}
  \item $D^+ u(x) \neq \emptyset$.
  \item $D^+ u(x)$ is a closed, convex set of $T^*_x A \cong \mathbb{R}^n$
  \item If $D^+ u(x)$ is a singleton, then $u$ is differentiable at $x$.
  \item If $A$ is also convex, $p \in D^+ u(x)$ if and only if
    \[
    u(y) - u(x) - \langle p, y - x \rangle \leq |y - x|\omega(|y - x|)
    \]
    for each $y \in A$.
\end{itemize}

Theorem B.4. [Theorem 3.2 in [6]] Let $u \in Lip_{loc}(\Omega \times (0, T))$ be a viscosity solution of
\[
\partial_t u + G(x, \partial_x u, t, u) = 0 \tag{55}
\]
where $G \in Lip_{loc}(\Omega \times \mathbb{R}^n \times (0, T) \times \mathbb{R})$ is strictly convex in the second group of variables. Then $u$ is locally semiconcave in $\Omega \times (0, T)$.

Theorem B.5. [Proposition 3.3.4, Theorem 3.3.6 in [5]] For any $(x, t) \in \Omega \times (0, T)$, we define the reachable derivative set of any viscosity solution $u$ of (55) by
\[
D^* u(x, t) := \left\{ (p_x, p_t) = \lim_{n \to +\infty} (\partial_x u(x_n, t_n), \partial_t u(x_n, t_n)) \in T^*_x \Omega \times T^*_t (0, T) \mid \exists (x_n, t_n)_{n \in \mathbb{Z}^+} \in \Omega \times (0, T) \text{converging to } (x, t), \text{at which } u \text{ is differentiable} \right\}.
\]
Consequently, $D^+ u(x, t) = co(D^* u(x, t))$ i.e. any superdifferential of $u$ at $(x, t)$ is a convex combination of elements in $D^* u(x, t)$.

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