An integrable generalization of the super Kaup-Newell soliton hierarchy and its bi-Hamiltonian structure

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Abstract

An integrable generalization of the super Kaup-Newell (KN) isospectral problem is introduced and its corresponding generalized super KN soliton hierarchy are established based on a Lie super-algebra B(0,1) and super-trace identity in this paper. And the resulting super soliton hierarchy can be put into a super bi-Hamiltonian form. In addition, a generalized super KN soliton hierarchy with self-consistent sources is also presented.

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1 Introduction

As we all know, With the development of soliton theory, super integrable systems associated with Lie super algebra have been paid growing attention, many classical integrable equations have been extended to be the super completely integrable equations [1-12]. Among those, Hu \textsuperscript{12} and Ma \textsuperscript{13} has made a great contribution. Hu \textsuperscript{12} proposed the super-trace identity, which is an effective tool to constructing super Hamiltonian structures of super integrable equations. In 2008, Ma given the proof of the super-trace identity and the super Hamiltonian structure of many super integrable equations is established by the super-trace identity \textsuperscript{13, 14}.

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The soliton equation with self-consistent sources play an important role in discussing the integrability for soliton hierarchy. They are relevant to some problems related to hydrodynamics, solid state physics, plasma physics, and they are also usually used to describe interactions between different solitary waves [16-18]. Very recently, self-consistent sources for super integrable equation hierarchy are constructed [19-30].

In Ref.[30], Yan considered a hierarchy of generalized KN equations, where the spatial spectral problem is given by

$$\phi_x = U \phi, U = \begin{pmatrix} \lambda + \mu qr & q \\ r & -\lambda - \mu qr \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (1.1)$$

where $q$ and $r$ are both scalar potentials, $\lambda$ is the spectral parameter, and $\mu$ is an arbitrary constant. The case of $\mu = 0$ Eq.(1.1) reduces to the well-know Kaup-Newell spectral problem [31], and $\mu = -\frac{1}{2}$ the Eq.(1.1) becomes the one considered by Qiao[32] by using the spectral gradient method and nonlinearization approach. Another three versions of generalized KN equations were also discussed in refs.[34-39]. The same idea to generalize the AKNS hierarchy [40-42] and the Wadati-Konno-Ichikaw(WKI) hierarchy [42], whose bi-Hamiltonian structures were constructed. Inspired by those generalizations, we would, in this paper, like to construct a generalized super KN hierarchy.

Organization of this paper. In Section 2, we shall construct a generalized super KN hierarchy based on a Lie super-algebra. In Section 3, the super bi-Hamiltonian form will be presented for the obtained generalized super KN hierarchy by making use of the super trace identity and a generalized super KN hierarchy with self-consistent sources generated from the variational derivative of spectra. And some conclusions and discussions are given in the last Section.

2 A hierarchy of generalized super KN equations

In this section, we shall construct a generalized super KN hierarchy starting from a Lie super-algebra. Consider the following spatial spectral problem

$$\phi_x = M \phi, M = \begin{pmatrix} \lambda + \omega & q & \alpha \\ \lambda r & -\lambda - \omega & \lambda \beta \\ \lambda \beta & -\alpha & 0 \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, u = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}, \quad (2.1)$$

where $\omega = \mu(qr + 2\alpha \beta)$ with $\mu$ is an arbitrary even constant, $\lambda$ is the spectral parameter, $q$ and $r$ are even potentials, and $\alpha$ and $\beta$ are odd potentials. Obviously, the spatial spectral problem (2.1) with $\mu = 0$ reduces to the standard super KN case [8, 21].
And associated with the Lie super-algebra \(G_1 = \{\sum_{i=1}^{5} \lambda_i, \lambda_i \in A, i = 1, 2, 3, 4, 5\}\).

\[
e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

which satisfy the following relationship

\[
[e_1, e_2] = -2e_2, [e_1, e_3] = 2e_3, [e_2, e_3] = -e_1,
\]

\[
[e_5, e_1] = [e_2, e_4] = e_5, [e_3, e_4] = [e_2, e_5] = 0, [e_3, e_5] = [e_1, e_4] = e_4,
\]

\[
[e_4, e_4]_+ = -2e_3, [e_5, e_5]_+ = 2e_2, [e_4, e_5]_+ = [e_5, e_4]_+ = e_1. \quad (2.2)
\]

where \(e_1, e_2, e_3\) are even and \(e_4, e_5\) are odd, \([., .]\) and \([., .]_+\) denote the commutator and the anticommutator, meanwhile \(q, r\) are even elements and \(\alpha, \beta\) are odd elements in (2.2).

From the Tu format, We setting

\[
N = \begin{pmatrix} A & B & \rho \\ \lambda C & -A & \lambda \delta \\ -\rho & 0 & 0 \end{pmatrix} = \sum_{m=0}^{n} \begin{pmatrix} a_m & b_m & \rho_m \\ \lambda c_m & -a_m & \lambda \delta_m \\ \lambda \delta_m & -\rho_m & 0 \end{pmatrix} \lambda^{-m}, \quad (2.3)
\]

the corresponding \(A, B, C\) are even elements and \(\rho, \delta\) are odd elements, if we want to get the super integrable system, we solve the stationary zero curvature equation at first

\[
N_x = [M, N]. \quad (2.4)
\]

Substituting \(M\) in (2.1) and \(N\) in (2.3) into Eq.(2.4) and comparing the coefficients of \(\lambda^{-m}(m \geq 0)\), we obtain

\[
\begin{align*}
& a_{m,x} = q c_{m+1} - r b_{m+1} + \alpha \delta_{m+1} + \beta \rho_{m+1},
& b_{m+1} = \frac{1}{2} b_{m,x} + q a_m + \alpha \rho_m - \omega b_m,
& c_{m+1} = -\frac{1}{2} c_{m,x} + r a_m + \beta \delta_{m+1} - \omega c_m,
& \rho_{m+1} = \rho_{m,x} + \alpha a_m + \beta b_{m+1} - q \delta_{m+1} - \omega \rho_m,
& \delta_{m+1} = -\delta_{m,x} + \beta a_m - \alpha c_m + r \rho_m - \omega \delta_m.
\end{align*} \quad (2.5)
\]

which leads to a recursive relationship

\[
\begin{align*}
& (c_{m+1}, b_{m+1}, \delta_{m+1}, \rho_{m+1})^T = L(c_m, b_m, \delta_m, \rho_m)^T,
& a_m = \partial^{-1}\left(-\frac{1}{2} q c_{m,x} - \frac{1}{2} r b_{m,x} - \alpha \delta_{m,x} + \beta \rho_{m,x} - q \omega c_m + r \omega b_m - \alpha \omega \delta_m - \beta \omega \rho_m\right).
\end{align*} \quad (2.6)
\]
Where the recursion operator $L$ has the following form

$$L = (L_{ij})_{4 \times 4}, \quad i, j = 1, 2, 3, 4,$$

with

$$L_{11} = -\beta \alpha - (\omega + \frac{1}{2} \partial) - r \partial^{-1} q(\omega + \frac{1}{2} \partial), \quad L_{12} = r \partial^{-1} q(\omega - \frac{1}{2} \partial),$$

$$L_{13} = -r \partial^{-1} \alpha(\partial + \omega) - \beta(\partial + \omega), \quad L_{14} = r \partial^{-1} \beta(\partial - \omega) + r \beta, \quad L_{21} = -q \partial^{-1} q(\omega + \frac{1}{2} \partial),$$

$$L_{22} = q \partial^{-1} r(\omega - \frac{1}{2} \partial) + \frac{1}{2} \partial - \omega, \quad L_{23} = -q \partial^{-1} \alpha(\partial + \omega), \quad L_{24} = q \partial^{-1} \beta(\partial - \omega) + \alpha,$$

$$L_{31} = -\beta \partial^{-1} q(\omega + \frac{1}{2} \partial) - \alpha, \quad L_{32} = \beta \partial^{-1} r(\omega - \frac{1}{2} \partial), \quad L_{33} = -\beta \partial^{-1} \alpha(\partial + \omega) - (\partial + \omega),$$

$$L_{34} = \beta \partial^{-1} \beta(\partial - \omega) + r, \quad L_{41} = q \alpha - \alpha \partial^{-1} q(\omega + \frac{1}{2} \partial), \quad L_{42} = \alpha \partial^{-1} r(\omega - \frac{1}{2} \partial) - \beta(\omega - \frac{1}{2} \partial),$$

$$L_{43} = -\alpha \partial^{-1} \alpha(\partial + \omega) + q(\partial + \omega), \quad L_{44} = \alpha \partial^{-1} \beta(\partial - \omega) + (\partial - \omega) - qr. \quad (2.7)$$

For a given initial value $a_0 = k_0 \neq 0, b_0 = c_0 = \rho_0 = \delta_0 = 0$, the $a_j, b_j, c_j, \rho_j, \delta_j (j \geq 1)$ can be calculated by the recursion relation (2.6). Here we list the several values

$$a_1 = -\frac{1}{2} k_0 (qr + 2 \alpha \beta), \quad b_1 = k_0 q, \quad c_1 = k_0 r, \quad \rho_1 = k_0 \alpha, \quad \delta_1 = k_0 \beta,$$

$$a_2 = k_0 \left[ \frac{3}{8} q^2 r^2 + \frac{3}{2} q r \alpha \beta + (qr + 2 \alpha \beta) \omega + \frac{1}{4} (qr_x - q_x r) + (\alpha \beta_x - \alpha_x \beta) + \frac{3}{2} q \beta \beta_x \right],$$

$$b_2 = k_0 \left[ \frac{1}{2} q_x - \frac{1}{2} q (qr + 2 \alpha \beta) - q \omega \right], \quad c_2 = -k_0 \left[ \frac{1}{2} r_x + \frac{1}{2} r (qr + 2 \alpha \beta) + r \omega + \beta \beta_x \right],$$

$$\rho_2 = k_0 (\alpha_x - \frac{1}{2} q r + \frac{1}{2} \beta q_x + q \beta_x - \omega \alpha), \quad \delta_2 = -k_0 (\beta_x + \frac{1}{2} \beta q r + \omega \beta), \ldots \ldots .$$

Then, consider the auxiliary spectral problem associated with the spectral problem (2.1)

$$\phi_{tn} = N^{(n)} \phi$$

where

$$N^{(n)} = N^{(n)}_+ + \Delta_n = \sum_{m=0}^{n} \begin{pmatrix} a_m & b_m & \rho_m \\ \lambda c_m & -a_m & \lambda \delta_m \\ \lambda \delta_m & -\rho_m & 0 \end{pmatrix} \lambda^{n-m} + \begin{pmatrix} a & b & e \\ c & -a & f \\ f & -c & 0 \end{pmatrix}, \quad (2.8)$$

with $\Delta_n$ being the modification term, substituting Eq.(2.1) and Eq.(2.8) into the following zero curvature equation

$$M_{tn} - N^{(n)}_+ + [M, N^{(n)}] = 0, \quad (2.9)$$
where \( n \geq 0 \). Making use of Eq. (2.9), we have

\[
\begin{cases}
\omega_{t_n} = a_x, b = c = e = f = 0, \\
q_{t_n} = b_{n+1} + 2q_n + 2\alpha \rho_n - 2\omega b_n + 2q \rho = 2b_{n+1} + 2q \rho, \\
r_{t_n} = c_{n+1} - 2r_n + 2\beta \delta_{n+1} + 2\omega c_n - 2r \rho = -2c_{n+1} + 2r \rho, \\
\alpha_{t_n} = \rho_{n+1} + \alpha a_n + \beta_{n+1} - q \delta_{n+1} - \omega \rho_n + \alpha = \rho_{n+1} + \alpha a, \\
\beta_{t_n} = \delta_{n+1} - \beta a_n + \alpha c_n - r \rho_n + \omega \delta_n - \beta a = -\delta_{n+1} - \beta a.
\end{cases}
\tag{2.10}
\]

which guarantees the following identity:

\[
(qr + 2\alpha \beta)_{t_n} = -2(qc_{n+1} - rb_{n+1} + \alpha \delta_{n+1} + \beta \rho_{n+1}) = -2a_{n,x}
\tag{2.11}
\]

Choosing \( a = -2\mu a_n \), we can obtain the following hierarchy:

\[
u_{t_n} = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2b_{n+1} - 4\mu q \rho_n \\ -2c_{n+1} + 4\mu r \rho_n \\ \rho_{n+1} - 2\mu \alpha a_n \\ -\delta_{n+1} + 2\mu \beta a_n \end{pmatrix}.
\tag{2.12}
\]

where \( n \geq 0 \). The case of Eq. (2.12) with \( \mu = 0 \) is exactly the super KN hierarchy [8 21]. Therefore, Eq. (2.12) is called the generalized super KN hierarchy.

Taking \( k_0 = 2, n = 2 \) in Eq. (2.12) and by using symbolic computation software (Maple), we obtain the first non-trivial flow is given by as follows:

\[
\begin{cases}
q_{t_2} = q_{xx} - 3qq_x r + \frac{3}{2} q^2 r^2 + 4\alpha \alpha x + 8q \alpha \beta x - 4 \alpha q \beta x + 6q^2 \beta \beta x + 6q^2 \alpha \beta \\
+ \mu(3q^3 r^2 - 4q^2 r_x - 4qv r - q^2 \alpha \beta x - 8q \alpha \beta x + 8q \alpha \beta + 12q^2 \beta \beta x \\
+ 12q^2 \alpha \beta - 4\mu q^2 r (qr + 4\alpha \beta), \\
r_{t_2} = -r_{xx} - 3qq r_x - \frac{3}{2} q^2 r^2 - 6\beta \beta x - 2q^2 \alpha \beta - 10qr \beta \beta x - 4r \alpha \beta \\
+ \mu(-3q^2 r^2 - 4q r_x r - 4qr r - 8r_x \alpha \beta + 8r \alpha \beta x - 8q \alpha \beta x + 4qr \beta \beta x \\
- 4q^2 \alpha \beta + 4\mu q^2 r (qr + 4\alpha \beta), \\
\alpha_{t_2} = 2\alpha_{xx} + 2q \alpha \beta_x + \frac{3}{2} q_{xx} \beta - \frac{3}{2} q r_x \alpha - \frac{3}{2} qr \alpha - \frac{3}{2} q^2 \alpha \beta x - 2qr \alpha x - 3q \alpha \beta x - 3q x \beta + \mu(\frac{3}{2} q^2 r^2 \alpha - 4q^2 \beta \beta x - 2qr \alpha x \\
- 3q r_x \alpha - q r x \alpha - 3q^2 r_x \beta - 6qq r \beta - 14q \beta \alpha \beta) - 2\mu q^2 r^2 \beta, \\
\beta_{t_2} = -2\beta_{xx} - r_{xx} \alpha - 2r \alpha \beta - 2qr \beta x - \frac{3}{2} qr \alpha \beta - \frac{3}{2} qr \beta x - \frac{3}{2} q^2 r^2 \beta + 4q^2 \beta \beta x \\
+ \mu(-3q r_x \beta - qr \beta x - 2qr \beta x - \frac{3}{2} q^2 r^2 \beta + 2\mu q^2 r^2 \beta, \\
\end{cases}
\tag{2.13}
\]

whose Lax pair are \( M \) in Eq. (2.11) and \( N^{(2)} \) has the following form

\[
N^{(2)} = \begin{pmatrix}
N_{11}^{(2)} & N_{12}^{(2)} & N_{13}^{(2)} \\
N_{12}^{(2)} & -N_{11}^{(2)} & N_{13}^{(2)} \\
N_{21}^{(2)} & -N_{11}^{(2)} & N_{23}^{(2)} \\
N_{23}^{(2)} & -N_{13}^{(2)} & 0
\end{pmatrix}
\tag{2.14}
\]
with
\[
\begin{align*}
N^{(2)}_{11} &= 2\lambda^2 - \lambda(qr + 2\alpha\beta) + (2 - 4\mu)[\frac{3}{8}q^2r^2 + \frac{3}{2}qr\alpha\beta + (qr + 2\alpha\beta)\omega \\
+ \frac{1}{2}(qr_x - q_xr) + (\alpha\beta_x - \alpha_x\beta) + \frac{3}{2}q\beta_x], \\
N^{(2)}_{12} &= 2q\lambda + q_x - q(qr + 2\alpha\beta) - 2q\omega, \\
N^{(2)}_{13} &= 2\lambda\alpha + 2\alpha_x - qr\alpha + q\beta_x + 2q\beta_x - 2\omega\alpha, \\
N^{(2)}_{21} &= 2\lambda^2r - \lambda(r_x + r(qr + 2\alpha\beta) + 2r\omega + 2\beta_x), \\
N^{(2)}_{23} &= 2\lambda^2\beta - \lambda(2\beta_x + qr\beta + 2\omega\beta).
\end{align*}
\]
when \( \mu = \beta = \alpha = 0 \) and \( t_2 = t \), Eq. (2.13) just reduces to the well-know KN equation hierarchy \[43\]
\[
\begin{align*}
q_t &= q_{xx} - 3qq_x r + \frac{3}{2}q^3 r^2, \\
r_t &= -r_{xx} - 3qrr_x - \frac{3}{2}q^2 r^3.
\end{align*}
\]

3 Super bi-Hamiltonian structures

In what follows we shall find super bi-Hamiltonian structures of the generalized super KN hierarchy \[2.12\]. To this end, we shall use the super trace identity, which proposed by Hu in \[12\] and rigorously proved by Ma et al. in ref. \[13\]:
\[
\frac{\delta}{\delta u} \int Str(N \frac{\partial M}{\partial \lambda}) dx = (\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma) Str(\frac{\partial M}{\partial u} N),
\]
where \( Str \) denotes the super trace. It is not difficult to find that
\[
Str(N \frac{\partial M}{\partial \lambda}) = 2A + rB, Str(\frac{\partial M}{\partial q} N) = 2\mu rA + \lambda C, Str(\frac{\partial M}{\partial r} N) = 2\mu qA + \lambda B,
\]
\[
Str(\frac{\partial M}{\partial \alpha} N) = 4\mu \beta A + 2\lambda \delta, Str(\frac{\partial M}{\partial \beta} N) = 4\mu \alpha A - 2\lambda \rho,
\]
Substituting Eq. (3.2) into Eq. (3.1), and comparing the coefficient of \( \lambda^{-n-1} \) of both sides of Eq. (3.1) yields
\[
\left( \begin{array}{c}
\frac{\delta}{\delta q} \\
\frac{\delta}{\delta r} \\
\frac{\delta}{\delta \alpha} \\
\frac{\delta}{\delta \beta}
\end{array} \right) \int (2a_{n+1} + rb_{n+1}) dx = (\gamma - n) \left( \begin{array}{c}
c_{n+1} + 2\mu r a_n \\
b_{n+1} + 2\mu q a_n \\
2\delta_{n+1} + 4\mu \beta a_n \\
-2\rho_{n+1} + 4\mu \alpha a_n
\end{array} \right).
\]
To fix the vaule of \( \gamma \), we let \( n = 0 \) in Eq. (3.3) and find that (1) when \( \mu = -\frac{1}{2} \), \( \gamma \) is arbitrary constant; (2) when \( \mu \neq -\frac{1}{2} \), \( \gamma = 0 \). Thus, we taking \( \mu \neq -\frac{1}{2} \) yields
\[
\left( \begin{array}{c}
c_{n+1} + 2\mu r a_n \\
b_{n+1} + 2\mu q a_n \\
2\delta_{n+1} + 4\mu \beta a_n \\
-2\rho_{n+1} + 4\mu \alpha a_n
\end{array} \right) = \frac{\delta H_n}{\delta u}, \quad \tilde{H}_n = \int \frac{2a_{n+1} + rb_{n+1}}{n} dx.
\]
Moreover, it is easy to know that

\[
\begin{pmatrix}
c_{n+1} \\
b_{n+1} \\
\delta_{n+1} \\
\rho_{n+1}
\end{pmatrix} = R \begin{pmatrix}
c_{n+1} + 2\mu r a_n \\
b_{n+1} + 2\mu q a_n \\
2\delta_{n+1} + 4\mu \beta a_n \\
-2\rho_{n+1} + 4\mu \alpha a_n
\end{pmatrix}
\]  \tag{3.5}

where \( R \) is given by

\[
R = \begin{pmatrix}
1 - 2\mu r \partial^{-1} q & 2\mu r \partial^{-1} r & -\mu r \partial^{-1} \alpha & \mu r \partial^{-1} \beta \\
-2\mu q \partial^{-1} q & 1 + 2\mu q \partial^{-1} r & -\mu q \partial^{-1} \alpha & \mu q \partial^{-1} \beta \\
-2\mu \alpha \partial^{-1} q & 2\mu \beta \partial^{-1} r & \frac{1}{2} - \mu \beta \partial^{-1} \alpha & \mu \beta \partial^{-1} \beta \\
-2\mu \alpha \partial^{-1} q & 2\mu \alpha \partial^{-1} r & -\mu \alpha \partial^{-1} \alpha & -\frac{1}{2} + \mu \alpha \partial^{-1} \beta
\end{pmatrix}
\]

Thus, the hierarchy of generalized super KN \( (2.12) \) possesses the following super-Hamiltonian structure

\[
u_{t_n} = Q R = Q \begin{pmatrix}
c_{n+1} + 2\mu r a_n \\
b_{n+1} + 2\mu q a_n \\
2\delta_{n+1} + 4\mu \beta a_n \\
-2\rho_{n+1} + 4\mu \alpha a_n
\end{pmatrix} = J \frac{\delta \tilde{H}_n}{\delta u}, n \geq 1. \tag{3.6}
\]

where

\[
Q = \begin{pmatrix}
-4\mu q \partial^{-1} q & 2 + 4\mu q \partial^{-1} r & -4\mu q \partial^{-1} \alpha & -4\mu q \partial^{-1} \beta \\
-2 + 4\mu r \partial^{-1} q & -4\mu r \partial^{-1} r & 4\mu r \partial^{-1} \alpha & 4\mu r \partial^{-1} \beta \\
-2\mu \alpha \partial^{-1} q & 2\mu \alpha \partial^{-1} r & -2\mu \alpha \partial^{-1} \alpha & 1 - 2\mu \alpha \partial^{-1} \beta \\
2\mu \beta \partial^{-1} q & -2\mu \beta \partial^{-1} r & -1 + 2\mu \beta \partial^{-1} \alpha & 2\mu \beta \partial^{-1} \beta
\end{pmatrix}
\]

and

\[
J = QR = \begin{pmatrix}
-8\mu q \partial^{-1} q & 2 + 8\mu q \partial^{-1} r & -4\mu q \partial^{-1} \alpha & 4\mu q \partial^{-1} \beta \\
-2 + 8\mu r \partial^{-1} q & -8\mu r \partial^{-1} r & 4\mu r \partial^{-1} \alpha & -4\mu r \partial^{-1} \beta \\
-4\mu \alpha \partial^{-1} q & 4\mu \alpha \partial^{-1} r & -2\mu \alpha \partial^{-1} \alpha & -\frac{1}{2} + 2\mu \alpha \partial^{-1} \beta \\
4\mu \beta \partial^{-1} q & -4\mu \beta \partial^{-1} r & -\frac{1}{2} + 2\mu \beta \partial^{-1} \alpha & -2\mu \beta \partial^{-1} \beta
\end{pmatrix} \tag{3.7}
\]

here \( J \) is a super Hamiltonian operator.

Specially, by making use of the recursive relationship \( (2.6) \), the generalized super KN hierarchy \( (2.12) \) possesses the following super-bi-Hamiltonian structure

\[
u_{t_n} = Q L \begin{pmatrix}
c_n \\
b_n \\
\delta_n \\
\rho_n
\end{pmatrix} = Q LR \begin{pmatrix}
c_{n+1} + 2\mu r a_n \\
b_{n+1} + 2\mu q a_n \\
2\delta_{n+1} + 4\mu \beta a_n \\
-2\rho_{n+1} + 4\mu \alpha a_n
\end{pmatrix} = P \frac{\delta \tilde{H}_{n-1}}{\delta u}, n \geq 2. \tag{3.8}
\]
where the second compatible super-Hamiltonian operator $P = QLR = (P_{ij})_{4 \times 4}, i, j = 1, 2, 3, 4$, is given by

\[
P_{11} = 4(\omega - \frac{1}{2}\partial)\mu q^{-1}q + 2q\Delta_1, \quad P_{12} = -4(\omega - \frac{1}{2}\partial)\mu q^{-1}r - 2(\omega - \frac{1}{2}\partial) - 2q\Delta_2, \\
P_{13} = 2(\omega - \frac{1}{2}\partial)\mu q^{-1}\alpha + q\Delta_3, \quad P_{14} = -2(\omega - \frac{1}{2}\partial)\mu q^{-1}\beta - \alpha - q\Delta_4, \\
P_{21} = -4(\omega + \frac{1}{2}\partial)\mu r^{-1}q - 4\mu_3 \beta q^{-1}q + 2(\omega + \frac{1}{2}\partial) - 2\alpha\beta - 2r\Delta_1, \\
P_{22} = 4(\omega + \frac{1}{2}\partial)\mu r^{-1}r + 4\mu_3 \beta q^{-1}r + 2r\Delta_2, \\
P_{23} = -2(\omega + \frac{1}{2}\partial)\mu r^{-1}\alpha - 2\mu_3 \beta q^{-1}\alpha + \beta(\omega + \partial) - r\Delta_3, \\
P_{24} = 2(\omega + \frac{1}{2}\partial)\mu r^{-1}\beta + 2\mu_3 \beta q^{-1}\beta + r\beta + r\Delta_4, \\
P_{31} = q\alpha + 2\mu_\omega \alpha^{-1}q - \mu_3 \beta q^{-1}q - 2\mu q^{-1}q - 2\mu\alpha\alpha^{-1} + \alpha\Delta_1, \\
P_{32} = -\beta(\omega - \frac{1}{2}\partial) - 2\mu_\omega \alpha^{-1}r + \mu_3 \beta q^{-1}r - 2\mu q^{-1}r + 2\mu\alpha \beta^{-1}r - \alpha\Delta_2, \\
P_{33} = \frac{1}{2}q(\omega + \partial) + \mu_\omega \alpha^{-1}\alpha - \frac{1}{2}\mu_3 \beta q^{-1}\alpha - \mu q^{-1}q^{-1}\alpha - \mu_\alpha\alpha^{-1}\alpha + \frac{1}{2}\alpha\Delta_3, \\
P_{34} = \frac{1}{2}(\omega - \partial) + \frac{1}{2}\mu r - \mu_\omega \alpha^{-1}\beta + \frac{1}{2}\mu_3 \beta q^{-1}\beta + \mu q^{-1}q^{-1}\beta + \mu_\alpha\alpha^{-1}\beta - \frac{1}{2}\alpha\Delta_4, \\
P_{41} = -2\mu\beta\alpha^{-1}q - 2\mu_\omega \beta^{-1}q + \alpha - \beta\Delta_1, \quad P_{42} = 2\mu\beta\alpha^{-1}r - 2\mu_\omega \beta^{-1}r - \beta\Delta_2, \\
P_{43} = -\mu\beta\alpha^{-1}\alpha - \mu\omega \beta^{-1}\alpha - \frac{1}{2}\beta\Delta_3, \quad P_{44} = \mu_\beta \beta^{-1}\beta + \mu_\omega \beta^{-1}\beta + \frac{1}{2}r - \frac{1}{2}\beta\Delta_4.
\]

with

\[
\Delta_1 = (2\mu - 1)\partial^{-1}q(\omega + \frac{1}{2}\partial) - \mu(2\mu - 1)\Delta\partial^{-1}q, \\
\Delta_2 = (2\mu - 1)\partial^{-1}r(\omega - \frac{1}{2}\partial) - \mu(2\mu - 1)\Delta\partial^{-1}r, \\
\Delta_3 = (2\mu - 1)\partial^{-1}\alpha(\omega + \partial) - \mu(2\mu - 1)\Delta\partial^{-1}\alpha, \\
\Delta_4 = (2\mu - 1)\partial^{-1}\beta(\omega - \partial) - \mu(2\mu - 1)\Delta\partial^{-1}\beta,
\]

and

\[
\Delta = \partial^{-1}q\partial r + \partial^{-1}r\partial q + 2\partial^{-1}\alpha\partial\beta - 2\partial^{-1}\beta\partial\alpha.
\]

Next, we are construct the generalized super KN hierarchy with self-consistent sources. At the super-isospectral problem

\[
\phi_x = M\phi, \quad \phi_t = N\phi. \quad (3.9)
\]

Let $\lambda = \lambda_j$, the spectral vector corresponding $\phi$ remember to $\phi_j$, we obtain the the linear
system as following

\[
\begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix}_x = M_j \begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix},
\begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix}_t = N_j \begin{pmatrix}
\phi_{1j} \\
\phi_{2j} \\
\phi_{3j}
\end{pmatrix},
\]

(3.10)

where \(M_j = M|_{\lambda=\lambda_j}, N_j = N|_{\lambda=\lambda_j}, j = 1, 2...N\). By

\[
\frac{\delta \hat{H}_n}{\delta u} = \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = \sum_{j=1}^{N} \begin{pmatrix}
Str(\Psi_j \frac{\delta \lambda_j}{\delta q}) \\
Str(\Psi_j \frac{\delta \lambda_j}{\delta r}) \\
Str(\Psi_j \frac{\delta \lambda_j}{\delta \alpha}) \\
Str(\Psi_j \frac{\delta \lambda_j}{\delta \beta})
\end{pmatrix} = \begin{pmatrix}
< \Phi_2, \Phi_2 > +2\mu r < \Phi_1, \Phi_2 > \\
-< \Lambda \Phi_1, \Phi_1 > +2\mu q < \Phi_1, \Phi_2 > \\
-2 < \Phi_2, \Phi_3 > +4\mu \beta < \Phi_1, \Phi_2 > \\
2 < \Lambda \Phi_1, \Phi_3 > +4\mu \alpha < \Phi_1, \Phi_2 >
\end{pmatrix},
\]

(3.11)

where \(\Phi_j = (\phi_{j1}, \cdots, \phi_{jN})^T, j = 1, 2, 3\). So the generalized super KN hierarchy with self-consistent sources is proposed

\[
u_t = \begin{pmatrix}
q \\
r \\
\alpha \\
\beta
\end{pmatrix} = J \begin{pmatrix}
c_{n+1} +2\mu r a_n \\
b_{n+1} +2\mu q a_n \\
2\delta_{n+1} +4\mu \beta a_n \\
-2\rho_{n+1} +4\mu \alpha a_n
\end{pmatrix} + J \begin{pmatrix}
< \Phi_2, \Phi_2 > +2\mu r < \Phi_1, \Phi_2 > \\
-< \Lambda \Phi_1, \Phi_1 > +2\mu q < \Phi_1, \Phi_2 > \\
-2 < \Phi_2, \Phi_3 > +4\mu \beta < \Phi_1, \Phi_2 > \\
2 < \Lambda \Phi_1, \Phi_3 > +4\mu \alpha < \Phi_1, \Phi_2 >
\end{pmatrix},
\]

(3.12)

where \(J\) is a super Hamiltonian operator given by in (3.7).

4 Conclusion and discussions

Starting from Lie super algebras, we may get super equation hierarchy. With the help of variational identity, the Hamiltonian structure can also be presented. Based on Lie super algebra, the self-consistent sources of a generalized super Kaup-Newell hierarchy can be obtained. It enriched the content of self-consistent sources of super soliton hierarchy. The methods in this study can be applied to other super soliton hierarchy to get more super hierarchies with self-consistent will be discussed in our future work.

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