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Does any convex quadrilateral have circumscribed ellipses?

New insight into innate connection of conic sections and beauty of geometry

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Abstract: The past decades have witnessed several well-known beautiful conclusions on four con-cyclic points. With highly promising research value, we profoundly studied circumscribed ellipses of convex quadrilaterals in this paper. Using tools of parallel projective transformation and analytic geometry, we derived several theorems including the proof of the existence of circumscribed ellipses of convex quadrilaterals, the properties of its minimal coverage area, and locus center, respectively. This simple approach lays a solid foundation for its application to three-dimensional situations, which is namely the circumscribed quadric surface of a solid figure and its wide-range utility in construction engineering. Meanwhile, we have a new insight into innate connection of conic sections as well as a taste of beauty and harmony of geometry.

Keywords: Convex quadrilateral, Circumscribed ellipse, Parallel projective transformation, Analytical geometry, Conic section

MSC: 51M04, 51M05, 51M09, 51N15, 51N20

1 Introduction

Since the development of Euclidean geometry, circumscribed and inscribed circle related problems have been at the center of widespread interests. Several well-known beautiful conclusions such as the circle-power theorem and Ptolemy’s theorem were of particular interest [1, 2]. Con-cyclic points, from four to six, were proposed to extend the original solutions [3, 4]. Enlightened by these elegant studies, we took a step further to unravel the relationship between ellipse and polygon [5–9]. Herein, the properties of circumscribed ellipses of convex quadrilaterals were studied using tools of parallel projective transformation [10, 11] and analytic geometry [12, 13], and the procedures were always from the particular to the general, from square, parallelogram, trapezium to a general convex quadrilateral. Moreover, for the sake of integrity, we also focused on the cases of other two kinds of conics and concave quadrilaterals.
It is universally acknowledged that there is no ellipse crossing four points if these four points form a concave quadrilateral or any three points of them is on a line. However, does any convex quadrilateral have circumscribed ellipses [14]?

**Theorem 1.** There exist circumscribed ellipses of any convex quadrilateral.

Given a convex quadrilateral $ABDC$ on Plane $\alpha$, we first define the following sets (Figure 1a):

- $M = \{ D \in \alpha | \overrightarrow{AD} = a\overrightarrow{AC}, \ a > 0, \ b > 0, \ a + b > 1 \}$
- $M_1 = \{ \sigma \subset \alpha | \sigma \text{ is a circumscribed ellipse of quadrilateral } ABDC \}$
- $\tilde{M} = \bigcup_{\sigma \in M_1} \sigma$

Then for any point $P$ on $\alpha$, we have

1. $A, B, C, P$ are vertices of a convex quadrilateral $\iff P \in M$.
2. If $P \in M \cap \tilde{M}$, then there exists a circumscribed ellipse of quadrilateral $ABPC$.

Hence **Theorem 1.** can be restated as for any $\Delta ABC$, $M \subset \tilde{M}$.

For simplicity, we first prove the following result.

**Lemma 1.** For any scalene triangle $\Delta ABC$, there exists a parallel projective transformation which maps $\Delta ABC$ onto an isosceles triangle.

**Proof of Lemma 1.** Let $\Delta ABC$ be a scalene triangle and $\Delta AB'C'$ be its projection on Plane $Q$. $BB'$, $CC'$ are both perpendicular to Plane $Q$ (Figure 1b). Let $\frac{\cos \angle BAB'}{\cos \angle CAC'} = \frac{AC'}{AB'}$, then we have $AB' = AC'$.
Proof of Theorem 1. Note that in Lemma 1, Area $M$ of $\triangle ABC$ is projected to be that of $\triangle A'B'C'$, and the circumscribed ellipses of $\triangle ABC$ are projected to be those of $\triangle A'B'C'$. Thus, without loss of generality, we assume $AB = AC$ in $\triangle ABC$.

Set up a rectangular coordinate system in which $A$ is the origin and $B$, $C$ are symmetric about $x$-axis (Figure 1c). Let $B(x_1, y_1)$, $C(x_1, -y_1)$. For any $D(x_2, y_2) \in M$ fixed, without loss of generality, assume $x_2 > x_1 > 0$, $y_1 > 0, y_2 > 0$ and $\frac{y_1}{x_1} < \frac{y_2}{x_2}$.

For a general quadratic equation in two variables $x$ and $y$

$$a_1 x^2 + 2a_{12} xy + a_{22} y^2 + 2b_1 x + 2b_2 y + c = 0 \quad (2.1)$$

where $a_{11}, a_{12}, a_{22}, b_1, b_2, c$ are constants,

define its discriminants \(L_1 = a_{11} + a_{22}, I_2 = a_{11} a_{22} - a_{12}^2, I_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{12} & a_{22} & b_2 \\ b_1 & b_2 & c \end{vmatrix}.\)

Let $\xi = \frac{x_1}{y_1}$, the problem is discussed on three cases with respect to coordinates of $D$.

Case 1 $\frac{y_2}{x_2} > \xi$,

Define $\sigma_1 : A_1 x^2 + C_1 y^2 + x = 0$,

$$\begin{vmatrix} x_1 & y_1^2 \\ x_2 & y_2^2 \end{vmatrix}, \quad C_1 = -\begin{vmatrix} x_2 & x_1 \\ y_2^2 & y_1^2 \end{vmatrix}.$$ 

Claim: $\sigma_1$ is a circumscribed ellipse of quadrilateral $ABDC$.

Since $K \in M$, we have $x_1^2 y_1^2 / x_2^2 y_2^2 < 0$.

First, we check $A, B, D, C \in \sigma_1$.

For $A$, it’s trivial.

For $B$ and $D$, it suffices to show that

$$\begin{vmatrix} x_1 & x_2^2 & y_1^2 \\ x_2 & x_1 & y_2^2 \end{vmatrix} = x_1^2 x_2 y_1^2 + y_1^2 x_1 x_2 = \begin{vmatrix} x_1^2 & 0 & y_1^2 \\ x_2^2 & x_1 & y_1^2 \end{vmatrix}. \quad (2.2)$$

In Equation (2.2), RHS = $\begin{vmatrix} x_1^2 & 0 & y_1^2 \\ x_2^2 & x_1 & y_1^2 \end{vmatrix}$

LHS.

Hence $B, C \in \sigma_1$.

For $D$, it suffices to show that

$$\begin{vmatrix} x_2 & x_1^2 & y_1^2 \\ x_1 & x_2^2 & y_2^2 \end{vmatrix} = x_2^2 x_1 y_1^2 + y_1^2 x_2 x_1 = \begin{vmatrix} x_2^2 & 0 & y_1^2 \\ x_1^2 & x_2 & y_1^2 \end{vmatrix}. \quad (2.3)$$

In Equation (2.3), RHS = $\begin{vmatrix} x_2^2 & 0 & y_1^2 \\ x_1^2 & x_2 & y_1^2 \end{vmatrix}$

LHS.

Hence, $D \in \sigma_1$.

Next, we show that $\sigma_1$ is an ellipse by checking its discriminants.

Since

$$I_1 = A_1 + C_1 < 0, I_2 = A_1 C_1 > 0, I_3 = \begin{vmatrix} A_1 & 0 & \frac{1}{2} \\ 0 & C_1 & 0 \end{vmatrix} > 0, \quad (2.4)$$

Does any convex quadrilateral have circumscribed ellipses?
\( \sigma_1 \) is an ellipse.

In conclusion, if \( D(x_2, y_2) \in \mathbb{M} \) and \( \frac{x_2}{y_2} > \xi \), then \( D(x_2, y_2) \in \hat{\mathbb{M}} \). (Figure 1d)

Case 2 \( \frac{x_2}{y_2} < \xi \)

Without loss of generality, assume \( y_2 \geq 0 \).

According to Equation (2.1), we have

\[
\begin{align*}
    a_{11} &= \frac{2(x_1y_2^2-x_2y_1^2)b_1+2y_1^2y_2^2-x_2^2b_2}{x_1^2y_1^2-x_2^2y_2^2}, \\
    a_{12} &= -\frac{2x_1y_2}{x_1^2y_1^2-x_2^2y_2^2} \\
    a_{22} &= \frac{2x_1(x_2-x_1)(y_1b_1+y_2b_2)}{x_1^2y_1^2-x_2^2y_2^2}
\end{align*}
\]

(2.5)

Assume \( b_1 > 0 \), from \( \frac{y_2}{y_1} < \frac{x_2}{x_1} < \frac{y_2}{y_1} \) we have \( \frac{x_2y_2^2-x_1y_1^2}{x_1^2y_1^2-x_2^2y_2^2} > -\frac{x_1y_2}{y_1^2} \). Let \( \frac{x_1y_2^2-x_2y_1^2}{x_1^2y_1^2-x_2^2y_2^2} b_1 > b_2 > -\frac{y_2}{y_1} b_1 \), we have, \( a_{11} < 0, a_{22} < 0 \). Then

\[
I_1 = a_{11} + a_{22} < 0, I_2 = a_{11}a_{22} - a_{12}^2 > 0, I_3 = -\frac{2b_1b_2^2}{x_1} - a_{22}b_1^2 - a_{11}b_2^2 > 0
\]

(2.6)

Therefore, for \( D(x_2, y_2) \in \mathbb{M} \), if \( \frac{x_2}{y_2} < \xi \), then \( D \in \hat{\mathbb{M}} \).

Case 3 \( \frac{x_2}{y_2} = \xi \)

Similarly, we have

\[
\begin{align*}
    a_{11} &= \frac{3y_1^2y_2^2-x_1^2b_1}{x_1^2y_1^2-x_2^2y_2^2}, \\
    a_{12} &= -\frac{6x_1y_2}{x_1^2y_1^2-x_2^2y_2^2} \\
    a_{22} &= \frac{2x_1(x_2-x_1)(y_1b_1+y_2b_2)}{x_1^2y_1^2-x_2^2y_2^2}
\end{align*}
\]

(2.7)

Assume \( b_1 > 0 \), \( -\frac{x_2}{y_2} b_1 < b_2 < 0 \). Then

\[
I_1 = a_{11} + a_{22} < 0, I_2 = a_{11}a_{22} - a_{12}^2, I_3 = -\frac{2b_1b_2^2}{x_1} - a_{22}b_1^2 - a_{11}b_2^2 > 0
\]

(2.8)

Therefore, for \( D(x_2, y_2) \in \mathbb{M} \), if \( \frac{x_2}{y_2} = \xi \), then \( D \in \hat{\mathbb{M}} \).

From the above discussion, we have that for any \( D(x_2, y_2) \in \mathbb{M} \), \( D \in \hat{\mathbb{M}} \). I.e., \( \mathbb{M} \subset \hat{\mathbb{M}} \).

3 The Coverage Area of Circumscribed Ellipses

The necessary and sufficient condition for four con-elliptic points is that they can form a convex quadrilateral. Here we discuss the situation of five points on the same ellipse. If four of them are fixed, we can describe this problem as the coverage area of the circumscribed ellipse of a convex quadrilateral.

3.1 Parallelogram

We simplified the problem to study the coverage area of the circumscribed ellipse of a parallelogram. It is common knowledge that any parallelogram can be an oblique section of a column with a square bottom (since the ratio between line segments is certain in a parallel projection, the two diagonals bisecting each other can make it), we can take a step further and change it into a simpler and more specific case, which is known as a square, by using the powerful tool of parallel projective transformation.

Set up a rectangular coordinate system in which \( A \) is the origin and the axes are parallel to the sides of the square (Figure 2a). Set \( B(x_0, x_0) \), and the equation of circumscribed ellipses is expressed by

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

(3.1)
Does any convex quadrilateral have circumscribed ellipses?

Figure 2: The coverage area of circumscribed ellipses. (a) Set up a rectangular coordinate system for a square. (b) The coverage area of the circumscribed ellipses of a square is the shaded part. (c) The coverage area of the circumscribed ellipses of a parallelogram is the shaded part. (d) Set up a rectangular coordinate system for an isosceles trapezium. (e) The coverage area of the circumscribed ellipses of an isosceles trapezium is the light grey part. (f) The coverage area of the circumscribed ellipses of a trapezium is the light grey part. (g) Set up a rectangular coordinate system for a general convex quadrilateral. (h) The coverage area of the circumscribed ellipses of a general convex quadrilateral is the light grey part.

Set the ellipse crosses the fifth point $E(x_1, y_1)$, and we have

$$
\begin{align*}
a^2 &= \frac{x_1^2 y_1^2 - x_2^2 y_2^2}{x_1^2 - x_2^2} > 0 \\
b^2 &= \frac{x_1^2 y_1^2 - y_2^2}{x_1^2 - x_2^2} > 0
\end{align*}
$$

According to Equation (3.2), we have $(x_1^2 y_1^2 - x_2^2 y_2^2)(y_1^2 - y_2^2)$. Therefore, the area of point $E$, which also means the coverage area of the circumscribed ellipses of the square, is shown as the shaded part in Figure 2b. Through parallel projective transformation, ellipses are still ellipses and squares change into any parallelograms, the coverage area of the circumscribed ellipses of a parallelogram is shown as the shaded part in Figure 2c.

### 3.2 Trapezium

Similarly, we studied properties of trapezium first on a simplified shape: isosceles trapezium. Let the coordinates of the four vertices of an isosceles trapezium be $P_1(x_1, y_1), P_2(x_2, -y_1), P_3(-x_2, -y_1), P_4(-x_1, y_1)$ (Figure 2d). We might as well assume $x_2 > x_1 > 0, y_1 > 0$, and the equations of the four sides are $P_1P_2 : 2y_1x + (x_2 - x_1)y - (x_1 + x_2)y_1 = 0, P_2P_3 : y + y_1 = 0, P_1P_4 : y - y_1 = 0, \text{ and } P_3P_4 : 2y_1x + (x_2 - x_1)y + (x_1 + x_2)y_1 = 0.$
From the theorems of analytic geometry, if four points \(P_1, P_2, P_3\) and \(P_4\) are given, the family of conic sections crossing these four points can be expressed by
\[
[2y_1x + (x_2 - x_1)y - (x_1 + x_2)y_1][2y_1x - (x_1 - x_2)y + (x_1 + x_2)y_1] + m(y - y_1)(y + y_1) = 0 \tag{3.3}
\]

Then plug the fifth point into \(P_5(x_5, y_5)\) Equation (3.3), we have
\[
m = \frac{4y_1^2x_5^2 - (x_1 - x_2)^2y_5^2 + 2(x_1^2 - x_2^2)y_1y_5 - (x_1 + x_2)^2y_1^2}{y_1^2 - y_5^2} \tag{3.4}
\]

Plug Equation (3.4) into Equation (3.3) and rearrange it, we have the general quadratic equation. Since the locus of the Equation (3.6) is an ellipse, we have an inequation for it by using the discriminant \(I_2 = AC = 4y_1^2 + \frac{4y_1x_5^2 + 20x_5^2y_1y_5 - 20(x_1^2 + x_2^2)y_1^2}{y_1^2 - y_5^2} > 0\). Then we have that if \(-y_1 < y_5 < y_1, y_5 > \frac{2y_1}{x_1 - x_2} \frac{x_5^2}{x_1^2 - x_2^2} y_1\), if \(y_5 < -y_1\) or \(y_5 > y_1\), we have
\[
y_5 < -y_1\text{ or } y_5 > y_1,
\]

Hence, for any \(P_5\) in the light grey part shown in Figure 2e, we have \(I_3 = (a_{11}(a_{22}c - b_2^2) < 0\). Now we can prove that the coverage area of the circumscribed ellipses of the isosceles trapezium is the light grey part in Figure 2e. Parallelism does not change through parallel projective transformation, the coverage area of circumscribed ellipses of trapezium is shown as the light grey part in Figure 2f.

### 3.3 General Convex Quadrilateral

From the discussion above, the coverage area of the circumscribed ellipses of the two kinds of special convex quadrilaterals, we find if a pair of opposite sides is parallel, the boundary of the coverage area takes place on the parallel lines where the two sides are located. On the other hand, if the two opposite sides are not parallel, the boundary is the parabola crossing its four vertices. In this case, we can regard the parallel lines as a degenerated form of a parabola. Therefore, we guess that for a general convex quadrilateral whose opposite sides are both not parallel, the boundaries of its coverage area are two parabolas crossing its four vertices.

Let the coordinates of the four vertices of a general convex quadrilateral be \(P_1(x_1, y_1), P_2(x_2, 0), P_3(0, 0), P_4(x_4, y_4), y_1 \neq y_4, \frac{y_4}{x_4} \neq \frac{y_1}{x_1}\) (Figure 2g). Based on the same procedure, we first listed the equations of the four sides, \(P_1 : y_1x + (x_2 - x_1)y - x_2y_1 = 0, P_2 : y_4x - x_4y = 0, P_3 : x = 0, P_4 : (y_1 - y_4)x - (x_1 - x_4)y - x_4y_1 + y_4x_1 = 0\). Then, the family of conic sections crossing these four points can be expressed by
\[
[y_1x + (x_2 - x_1)y - x_2y_1][y_4x - x_4y] + m[y_1 - y_4]x - (x_1 - x_4)y - x_4y_1 + y_4x_1 = 0 \tag{3.5}
\]

Then plug the fifth point \(P_5(x_5, y_5)\) into Equation (3.5), and we have
\[
m = \frac{-[y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5]}{y_5[y_1 - y_4]x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1} \tag{3.6}
\]

Since the locus of the Equation (3.6) is an ellipse, we have an inequation for it by using the discriminant \(I_2 = y_1y_4[x_4(x_1 - x_2) - m(x_1 - x_4)] - \frac{1}{2}(-x_4y_1 - y_4x_1 + y_4x_2 + m(y_1 - y_4)) > 0\). Namely, we have
\[
(y_1 - y_4)^2m^2 + 2(-x_4y_1 - y_4x_1 + y_4x_2)(y_1 - y_4) + 4y_1y_4(x_1 - x_4)[y_1 - y_4] + (-x_4y_1 - y_4x_1 + y_4x_2)^2
\]
\[-4y_1y_4x_4(x_1 - x_2) < 0 \tag{3.7}
\]
Table 1: Using discriminants \( I_1, I_2 \) and \( I_3 \) to determine the locus of a general quadratic equation containing two variables \( x \) and \( y \).

| Condition of Discriminant \( I_2 \) | Condition of Discriminants \( I_1 \) and \( I_3 \) | Geometry Interpretation |
|------------------------------------|-----------------------------------------------|--------------------------|
| \( I_2 > 0 \)                       | \( I_1I_3 < 0 \)                              | Ellipse                  |
|                                    | \( I_2 \neq 0 \)                              | Parabola                 |
| \( I_2 = 0 \)                       | \( I_3 = 0 \)                                | Two Parallel Lines       |
|                                    | \( \bar{I}_2 \neq 0 \)                       | Hyperbola                |
| \( I_2 < 0 \)                       | \( \bar{I}_3 = 0 \)                          | Two Intersecting Lines   |

Since \( y_1 \neq y_6 \), Inequation (3.7) is a quadratic one of \( m \). Because of the existence of circumscribed ellipse of any convex quadrilateral, Inequation (3.7) has solution. As the coefficient of the quadratic term \((y_1 - y_4)^2 > 0\), we can assume the solution of Inequation (3.7) is \( k_1 < m < k_2 \).

Since \( \frac{y_5}{y_6} \neq \frac{y_1}{x_1} \), the constant term \((-x_4y_1 - y_4x_2)^2 - 4y_1y_4x_4(x_1 - x_2) \neq 0 \). Hence, \( k_1, k_2 \neq 0 \).

Plug Equation (3.6) into \( I_2 < 0 \), we have \( k_1 < -\frac{|y_1x_5(x_2 - x_1)y_5 - x_2y_1||y_4x_5 - x_4y_5|}{\sqrt{(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1}} < k_2 \).

If \( y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] > 0 \), which means \( P_5 \) is above \( P_1P_4 \) or below \( P_2P_3 \), we have

\[
y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] < 0 \quad (3.8)
\]

\[
y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_2y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] > 0 \quad (3.9)
\]

Since Equation (3.5) is the quadratic curve crossing four points, if \( m = k_1, k_2, I_2 = 0 \). Hence, the left side of Inequation (3.8) and Inequation (3.9) represent the two parabolas crossing \( P_1, P_2, P_3 \) and \( P_4 \). In other words, \( P_5 \) is in the parabola whose equation is

\[
y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_1y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] = 0 \quad (3.10)
\]

and \( P_5 \) is also out of the parabola whose equation is

\[
y_1x_5 + (x_2 - x_1)y_5 - x_2y_1][y_4x_5 - x_4y_5] + k_2y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] = 0 \quad (3.11)
\]

It is worth mentioning that since \( I_2 = 0 \) is a quadratic equation of \( m \), it has only two roots, which means there are only two parabolas crossing the four vertices of the convex quadrilateral.

Using the same argument, when \( y_5[(y_1 - y_4)x_5 - (x_1 - x_4)y_5 - x_4y_1 + y_4x_1] < 0 \), which means \( P_5 \) is between \( P_1P_4 \) and \( P_2P_3 \), we can have the same conclusion.

In conclusion, the coverage area of circumscribed ellipses of general convex quadrilateral is shown as the light grey part in Figure 2h. Similar to the case of trapezium, the area enclosed by two parabolas and four sides cannot be covered.

### 3.4 Circumscribed Conics of Convex Quadrilaterals

Through the studies above, we conclude that the boundaries of the coverage area of circumscribed ellipses are another conic, which is parabola. Here we discuss the third conic, hyperbola. We will generalize the conclusion about the coverage area of circumscribed ellipses to the coverage area of circumscribed conics.

For a general quadratic equation in two variables, we can use discriminants to determine its locus (Table 1). We conclude that the light grey part in Figure 2h is the coverage area of the circumscribed ellipses, where \( I_2 > 0 \). Therefore, the area which cannot be covered by the circumscribed ellipse (except the boundary region) can be covered by the circumscribed hyperbolas (including the degenerated condition, namely two intersecting lines), where \( I_2 < 0 \) (as shown the shaded part in Figure 3a). Besides, the boundary of these two coverage areas is the parabola crossing the four vertices, where \( I_2 = 0 \).

So far, we can divide the plane where a convex quadrilateral is located into three parts. One part is con-elliptic with its four vertices (Figure 2h). The second part is con-parabolic with the vertices (namely two
Figure 3: Circumscribed conics of convex quadrilaterals. (a) The coverage area of the circumscribed hyperbolas of a general convex quadrilateral is the shaded part. (b,c) “2+2” hyperbola, which means two vertices are on one branch while the other two are on the other branch. (d) “4+0” hyperbola, which means all of four vertices are on one branch. (e) There exists no “3+1” hyperbola which is circumscribed about a convex quadrilateral. (f) The “2+2” hyperbolas cover the light grey area, while the “4+0” hyperbolas cover the dark grey area.

parabolas crossing the four vertices of the convex quadrilateral). The third part is con-hyperbolic with the vertices (Figure 3a).

Normally, the circumscribed hyperbolas should be classified into three categories (The demo video was provided in Supplementary 1):

“2+2” hyperbola: Two vertices are on one branch of the hyperbola, while the other two are on the other branch (Figure 3b).

“3+1” hyperbola: Three vertices are on one branch of the hyperbola, while the forth is on the other branch (Figure 3c)

“4+0” hyperbola: All of four vertices are on one branch of the hyperbola (Figure 3d).

However, not all of these three kinds can be circumscribed hyperbolas of a convex quadrilateral. So the following proposition was proposed:

Proposition 1. There exists no “3+1” hyperbola which is circumscribed about a convex quadrilateral.

Proof Suppose the vertex of \( P_1 \) is on the left branch of the hyperbola, and the other three vertices \( P_2, P_3 \) and \( P_4 \) are on the right branch. According to the definition of convex quadrilaterals, we can infer that \( P_4 \) must be located in the area enclosed by rays formed by any two of the sides of \( \Delta P_1P_2P_3 \) and its third side (the light grey part in Figure 3e). Each line has and only has two intersections with the hyperbola, and the slope of \( P_2P_3 \) is greater than the asymptote while that of \( P_1P_3 \) is smaller than it, we can infer that the right branch must be out of the light grey area which is to the right of \( P_1P_3 \) (as shown in Figure 3e). We can prove the rest in the same way. Therefore, the right branch must be outside of the light grey area, and therefore the proposition holds true.

In conclusion, the circumscribed hyperbolas only can be classified into two categories: “2+2” hyperbolas (Figure 3b, 3c) & “4+0” hyperbolas (Figure 3d).

Furthermore, we have discussed the coverage area of these two kinds of hyperbolas by using limit thought, and the results are shown in Figure 3f. The “2+2” hyperbolas cover the light grey area, while the “4+0” hyperbolas cover the dark grey area.
Does any convex quadrilateral have circumscribed ellipses?

Figure 4: The locus of the center of circumscribed conics. (a) The locus of the center of circumscribed conics of a convex quadrilateral is its nine-point curve, which is a hyperbola. (b) The axes of symmetry of a general convex quadrilateral’s two circumscribed parabolas are parallel to the asymptotic lines of its nine-point curve, respectively. (c) The center O of the circumscribed conics of a concave quadrilateral is on its nine-point curve, which is an ellipse.

4 The Locus of the Center of Circumscribed Conics

4.1 Convex Quadrilateral

For a parallelogram, the center of its circumscribed conics is necessarily the center of the parallelogram. For an isosceles trapezium, since conics and isosceles trapeziums are both mirror-symmetrical graphs, the center of its circumscribed conics must locate at the line, which crosses the midpoints of its bases. Through parallel projective projection, the ratio of segments and parallelism do not change, the center of circumscribed conics of a trapezium must locate at the line, which crosses the midpoints of its bases. Next, we try to figure out the locus of the center of circumscribed conics of a general convex quadrilateral.

Theorem 2. The locus of the center of circumscribed conics of a convex quadrilateral is its nine-point curve, which is a hyperbola. Moreover, the locus of the center of circumscribed ellipses is a branch of it, and the locus of the center of circumscribed hyperbolas is the other branch (Figure 4a) (The demo video was provided in Supplementary 2).

For a quadrilateral, the intersection of the diagonals, two intersections of the opposite sides, four midpoints of the sides and two midpoints of diagonals are necessarily on a conic, which is called the nine-point curve of it. The nine-point curve is always a centered conic. Whether it is a hyperbola or an ellipse depends on whether the quadrilateral is convex or concave. Moreover, the center of the nine-point curve is the barycenter of the quadrilateral.

The following situations are two special cases of nine-point curve: When the quadrilateral has a circumscribed circle, its nine-point curve is a rectangular hyperbola; When the four vertices of the quadrilateral form an orthocentric system, its nine-point curve is a circle, which is known as the nine-point circle [15, 16].

To prove the theorem, we can use Equation (3.5) to express the circumscribed conics of the quadrilateral, and figure out the equation of the center of the conics. The equation is quadratic, and the nine special points, which the nine-point curve crosses, satisfy the equation. Since the nine point can and only can determine one quadratic curve, the locus of the center of circumscribed conics is the nine-point curve.

Corollary 1. The axes of symmetry of a general convex quadrilateral’s two circumscribed parabolas are parallel to the asymptotic lines of its nine-point curve, respectively (Figure 4b).

Proof A parabola is a non-centered conic, and it can also be regarded as the critical situation of an ellipse or a hyperbola. Hence, if we also regard it as a centered conic, we can have the following definition:

Definition The center of a parabola is the point at infinity, which is determined by its axis of symmetry.

From Theorem 2, we know that the locus of the center of a general convex quadrilateral is its nine-point curve. Therefore, the centers of the two circumscribed parabolas are necessarily on the nine-point curve. On the other hand, since the nine-point curve of a convex quadrilateral is a hyperbola, it is infinitely near
its asymptotic lines at infinity. Therefore, the asymptotic lines cross the centers of the two circumscribed parabolas.

In conclusion, the axes of symmetry of the two circumscribed parabolas of a general convex quadrilateral are parallel to the asymptotic lines of its nine-point curve respectively.

\[ \square \]

4.2 Concave Quadrilateral

As mentioned before, a concave quadrilateral also has a nine-point curve, which is an ellipse. Taking a step further, we generalized Theorem 2 to concave quadrilaterals.

**Theorem 3.** The locus of the center of circumscribed conic of a concave quadrilateral is its nine-point curve, which is an ellipse (Figure 4c) (The demo video was provided in Supplementary 3).

It is worth mentioning that the circumscribed conics of a concave quadrilateral can only be “3+1” hyperbolas, while the circumscribed conics of a convex quadrilateral only cannot be “3+1” hyperbolas, which is proved before.

5 The Minimal Area of Circumscribed Ellipses

5.1 Parallelogram

It is well known that among the inscribed parallelograms of an ellipse, the area of the parallelogram is maximal when its diagonals are the conjugate diameters of the ellipse. In that way, among the circumscribed ellipses of a parallelogram, it is doubted that whether the area of the ellipse is minimal when its conjugate diameters are the diagonals of the parallelogram.

**Theorem 4.** When the conjugate diameters of the circumscribed ellipse are the diagonals of the parallelogram, the area of the circumscribed ellipse is minimal.

**Proof** As mentioned before, any parallelogram can be an oblique section of a column with a square bottom. Let \( \theta \) be the dihedral angle between the oblique section and the bottom and \( \theta \) is unique. Suppose \( E \) is a circumscribed ellipse of the parallelogram, so its projection on the oblique section \( E' \) is a circumscribed ellipse of the square, which is the bottom. According to the fundamental theorem in projective geometry, we know that \( S_{E'} = S_E \cdot \cos \theta \). Therefore, if and only if \( S_{E'} \) is minimal, \( S_E \) is minimal. Thus we drew the following lemma.

**Lemma 2.** Among the circumscribed ellipses of a square, the area of the circumscribed circle is the minimum.

**Proof** Since squares and ellipses are both central symmetry, the center of its circumscribed conics is necessarily the center of the parallelogram. Suppose a square \( ABCD \), and its center is \( O \). Set up a rectangular coordinate system in which the center \( O \) is the origin. We first consider the situation where the axes of the circumscribed ellipses are not parallel to the square’s sides (Figure 5a). Let \( A(x_1, y_1), B(x_2, y_2) \), and the circumscribed ellipses are expressed by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Since \( A, B \) are on the ellipse, we have

\[
\begin{align*}
\begin{cases}
a^2 &= \frac{x_1^2y_2^2 - x_2^2y_1^2}{y_2^2 - y_1^2} \\
b^2 &= \frac{x_1^2y_2^2 - x_2^2y_1^2}{x_2^2 - x_1^2}
\end{cases}
\end{align*}
\]  

(5.1)

Plug \( x_1^2 + y_1^2 = x_2^2 + y_2^2 \) into Equations (5.1), and we have \( a^2 = b^2 \).
Does any convex quadrilateral have circumscribed ellipses?

Figure 5: The minimal area of circumscribed ellipses. (a) The axes of the circumscribed ellipses are not parallel to the square's sides. (b) The axes of the circumscribed ellipses are parallel to the square's sides. (c) If the projections of an ellipse's two diameters on its bottom are perpendicular, these two diameters must be conjugate. (d) The drawing method of the minimal circumscribed ellipse of a parallelogram. (e) The minimal circumscribed ellipse of a trapezium. (f) Set up a rectangular coordinate system for an isosceles trapezium. (g) The minimal circumscribed ellipse of an isosceles trapezium. (h) Set up a rectangular coordinate system for a cyclic quadrilateral.

Hence, there exists one and only one circumscribed ellipse of the square, which is the circumscribed circle. Next, we consider the situation where the axes of the circumscribed ellipses are parallel to the square’s sides (Figure 5b). Let \( B(x_0, y_0) \) and the circumscribed ellipses are expressed by \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). By using the fundamental inequations, we have \( \frac{1}{a^2} = \frac{1}{b^2} + \frac{1}{a^2} \geq 2\sqrt{\frac{1}{a^2b^2}} = \frac{2}{ab} \). Namely, we have \( ab \geq 2x_0^2 \) so \( S = \pi ab \geq 2\pi x_0^2 \). If and only if \( a = b = x_0 \), the equality holds, which namely means the area is the minimum. At this time, the circumscribed ellipse of the square is the circumscribed circle. In conclusion, among the circumscribed ellipses of a square, the area of the circumscribed circle is the minimum.

It is obvious that the diagonals of the square are the perpendicular diameters of its circumscribed circle. So what about their projections on the section?

**Lemma 3.** If the projections of an ellipse's two diameters on its bottom are perpendicular, these two diameters must be conjugate (Figure 5c).

**Proof** Suppose there is a cylinder. Let \( A'B' \) and \( C'D' \) be the perpendicular diameters of the bottom. Draw \( M'N' || A'B' \). Let \( K' \) be the midpoint of chord \( M'N' \), so its projection \( K \) is the midpoint of \( MN \). Since parallelism does not change through projection, we also have \( MN || AB \). Thus \( AB \) and \( CD \) are conjugate.

From Lemma 3, we can conclude that the projections of the circle’s two diameters on its section are conjugate. Hence, we can prove Theorem 4. Moreover, to figure out the value of the minimal area, we drew the following theorem.
Theorem 5. The minimal area is $\frac{\pi}{2}$ times the area of the parallelogram.

Proof From Lemma 2, when $E$ is the minimal circumscribed ellipse of the parallelogram, which is a section, its projection $E'$ is the circumscribed circle of the square, which is the bottom. Let the area of the parallelogram be $S$, while the area of the square is $S'$. According to the fundamental theorem in projective geometry, we know that $S_E = S_E' \cdot \cos \theta$. Thus, we have $\frac{S}{S'} = \frac{S_E}{S_E'} = \frac{a^2}{2r} = \frac{\pi}{2}$.

The drawing method of the minimal circumscribed ellipse of a parallelogram was provided:

Since we have found out the minimal circumscribed ellipse of parallelograms, we need to find the drawing method. According to Theorem 4, this proposition is equivalent to drawing an ellipse of which the conjugate diameters are the diagonals of the parallelogram. So next we will point out the drawing method of the ellipse when we know a pair of conjugate diameters of it (Figure 5d).

If we set up a corresponding relationship between $AD$, $BC$ and a pair of conjugate diameters of a circle, the corresponding ellipse to the circle is namely what we want.

1. Let $BC$ coincide with $B'C'$ and draw another diameter $A'D' \perp B'C'$. Then we can determine the corresponding relationship between the circle and the ellipse by the axis $BC(B'C')$ and a pair of corresponding points $A - A'$.

2. Through any point $N'$ on the circle, draw $\Delta N'NM \sim \Delta A'AO$. $N$ is on the ellipse and $M$ is on the axis $BC$.

3. Same as above, draw a series of points similar to point $N$ and join them together with a smooth curve. This is namely the ellipse we want.

Theorem 5. There exists one and only one ellipse of which the conjugate diameters are the diagonals of a parallelogram.

Corollary 2. There exists one and only one minimal circumscribed ellipse of a parallelogram.

5.2 Trapezium

Theorem 6. Let $M$, $N$ be the midpoints of the bases of a trapezium. The center $O$ of the minimal circumscribed ellipse of a trapezium is on $MN$, and it satisfies the ratio $\frac{MO}{ON} = \frac{1+2m^2}{\sqrt{m^4-2m^2+1}}$ where $m$ is the ratio between two bases (Figure 5e).

Proof If an isosceles trapezium and the shape of its circumscribed ellipse (namely the eccentricity) are given, the ellipse is determined. In order to facilitate the research, the paper uses the ratio between major and minor axes to describe the shape of a circumscribed ellipse. Suppose the equation of the circumscribed ellipses is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and $k = \frac{b}{a}(k \in R^+)$. Due to the fact that ellipses and isosceles trapeziums are both graphs of axial symmetry, the symmetry axis of the isosceles trapezium must be the major or minor axis of the circumscribed ellipses. We might as well assume $y$-axis is its symmetry axis (Figure 5f). Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(-x_2, y_2), P_4(-x_1, y_1)$ and $x_2 > x_1 > 0, y_1 > 0, y_2 < 0$. In this case, $x_1$ and $x_2$, which are half of the two bases of the isosceles trapezium, are fixed values, $y_1$ and $y_2$ are non-fixed values, while they must satisfy $y_1 - y_2 = y_0$. Here $y_0$ is the height of the isosceles trapezium, which is also a fixed value.

Since the four vertices are on the ellipse, we have $\frac{y_1}{b} - \frac{y_2}{b} = \frac{y_2}{b} = \sqrt{1 - \frac{x_1^2}{a^2} + \sqrt{1 - \frac{x_1^2}{a^2}}}$. Plug $b = ak$ into it, we have $a^2 = \frac{y_1^2 + k^2(x_1-x_2)^2 + 2k^2y_0^2(x_1^2 + x_2^2)}{4k^2y_0^2}$. Thus the area of the circumscribed ellipses can be expressed by $S = \pi ab = \pi \frac{y_1^2 + k^2(x_1-x_2)^2 + 2k^2y_0^2(x_1^2 + x_2^2)}{4k^2y_0^2}$.

Set $f(k) = \frac{3\sqrt{y_0^2}}{a^2}S = (x_2^2 - x_1^2)k^2 + 2y_0^2(x_1^2 + x_2^2)k + \frac{y_0^4}{a^2}$, to minimize $S$, we calculate derivative of $f$:

$$f'(k) = 3(x_2^2 - x_1^2)k^2 + 2y_0^2(x_1^2 + x_2^2) - \frac{y_0^4}{k^2} = 0$$

Solve Equation (5.2) and we have

$$k^2 = \frac{-y_0^2(x_2^2 + x_1^2) + 2y_0^2\sqrt{x_1^4 + x_2^4 - x_1^2x_2^2}}{3(x_2^2 - x_1^2)^2}$$
On the other hand, \( \frac{x_1^2 - x_2^2}{a^2} = \frac{y_1^2 - y_2^2}{b^2} = \frac{y_1 - y_2}{a^2 k_1} = \frac{y_2 (x_2^2 + y_2^2)}{a^2 k_1} \). Since \( y_1 - y_2 = y_0, \) we have
\[
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix} = \begin{bmatrix}
    y_0^2 + k^2 (x_2^2 - x_1^2) \\
    y_0^2 - k^2 (x_2^2 - x_1^2)
\end{bmatrix}
\]
Let \( m = \frac{y_1}{x_2} \) and plug \( m \), Equation (5.3), Equation (5.4) together, we have
\[
\begin{bmatrix}
    y_1 \\
    y_2
\end{bmatrix} = \begin{bmatrix}
    (1 - 2m^2) + \sqrt{m^4 - m^2 + 1} \\
    (2 - m^2) - \sqrt{m^4 - m^2 + 1}
\end{bmatrix}
\]
For an isosceles trapezium, the center of its circumscribed conics must locate at the line which crosses the midpoints of its bases. Therefore, the value of \( \frac{p_1}{p_2} \) indicates the location of the center of the minimal circumscribed ellipse. Now we can prove \textbf{Theorem 6} holds in this special case of isosceles trapeziums (Figure 5g). Because parallelism and ratio between line segments do not change through parallel projective transformation, we can prove \textbf{Theorem 6} holds in all cases of trapeziums.

### 6 Circumscribed Ellipses of Cyclic Quadrilaterals

Since any convex quadrilateral can be transformed to a cyclic quadrilateral through parallel projection, it is helpful to research the connection between cyclic quadrilaterals and their circumscribed ellipses.

\textbf{Theorem 7.} Let \( A, B, C, D \) be four points on the given ellipse of which the major diameters are parallel to the coordinate axis. If \( A, B, C, D \) are cyclic, the opposite sides of the quadrilateral formed by these four points will locate on two lines of which the slope angles are complementary (Figure 5h).

\textit{Proof} For a given ellipse, set up a rectangular coordinate system of which the coordinate axis is parallel to its major axis (Figure 5h), then the equation of the ellipse can be expressed by \( A_1 x^2 + C_1 y^2 + D_1 x + E_1 y + F_1 = 0 \). Suppose a cyclic quadrilateral \( ABCD \), whose vertices are on the ellipse above. Let the four vertices be on the circle \( x^2 + y^2 + D_0 x + E_0 y + F_0 = 0 \). Then, we can express the quadratic curve crossing these four points by
\[
A_1 x^2 + C_1 y^2 + D_1 x + E_1 y + F_1 + A(x^2 + y^2 + D_0 x + E_0 y + F_0) = 0
\]
On the other hand, we can express the equation of a quadratic curve crossing the four points by
\[
(a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) = 0
\]
Obviously, there are no terms containing \( xy \) in Equation (6.1). By comparing Equation (6.1) and (6.2), we have \( a_1 b_2 + a_2 b_1 = 0 \). Since the curve expressed in Equation (6.1) crosses all the four given points \( A, B, C, D \) and there are no collinear points, the two lines expressed by Equation (3.4) each respectively crosses two of the points. In brief, Equation (6.2) can express any quadratic curve made up of the two lines where any pair of opposite sides of the quadrilateral \( A, B, C, D \) is located.

Therefore, we can know that any pair of opposite sides of the complete quadrilateral \( A, B, C, D \) locates on two lines of which the slope angles are complementary. And the slopes of the two lines are opposite numbers if they exist, namely \( \frac{a_1}{b_1} = \frac{a_2}{b_2} \).

From the theorem above, we can infer two corollaries as following:

\textbf{Corollary 3.} For a circumscribed ellipse of a cyclic quadrilateral, the included angles between its axis and the opposite sides of the quadrilateral are equal.

\textbf{Corollary 4.} For the circumscribed ellipses of a cyclic quadrilateral, their major axes are parallel to each other.

### 7 Conclusion

To summarize, the properties of circumscribed ellipses of convex quadrilaterals were unraveled in this research, using tools of parallel projective transformation and analytic geometry. A totally novel geometric
proof of the existence of circumscribed ellipses of convex quadrilaterals was derived considering three different cases. From the particular to the general, from square, parallelogram, trapezium to a general convex quadrilateral, the coverage area of the circumscribed conics calculated, respectively, which is equivalent to dividing the plane where a convex quadrilateral is located into three parts: con-elliptic, con-parabolic and con-hyperbolic with the four vertices of the quadrilateral. We found the locus of the center of circumscribed conics, both of convex quadrilaterals and concave quadrilaterals, respectively. What's more, we found the minimal area of circumscribed ellipse of convex quadrilaterals. We hope to extend the above results to three-dimensional situations, which is namely the circumscribed quadric surface of a solid figure. Since quadrics are widely applied to construction engineering, we believe such a study has bright perspective in the future. Meanwhile, we have an insight into the innate connection of conic sections as well as a taste of the beauty and harmony of geometry.

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Supplementary Materials

Movies S1# Demo video of circumscribed hyperbolas (“2+2” hyperbolas and “4+0” hyperbolas).

Movies S2# Demo video of the locus of the center of circumscribed conics of a convex quadrilateral (its nine-point curve formed a hyperbola)

Movies S3# Demo video of the locus of the center of circumscribed conic of a concave quadrilateral (its nine-point curve formed an ellipse)

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