Microscopic Origin of the Shear Relaxation Time in Causal Dissipative Fluid Dynamics

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In this paper we show how to compute the shear relaxation time from an underlying microscopic theory. We prove that the shear relaxation time in Israel-Stewart-type theories is given by the inverse of the pole of the corresponding retarded Green’s function, which is nearest to the origin in the complex energy plane. Consequently, the relaxation time in such theories is a microscopic, and not a macroscopic, i.e., fluid-dynamical time scale.

Keywords: Dissipative Relativistic Fluids; Transient dynamics; Relaxation time.

1. Introduction

Relativistic fluid dynamics has been applied with success to describe the dynamics of hot and dense matter created in relativistic heavy-ion collisions at RHIC, and recently, at LHC.\(^1\) Despite this success, the derivation of relativistic fluid dynamics from an underlying microscopic theory is still an open problem and is under intense theoretical investigation. This is mainly due to nontrivial consequences imposed by causality and stability in relativistic fluids.\(^3,4\)
In this paper we discuss a general method to compute all linear transport coefficients (such as the shear viscosity and its corresponding relaxation time) associated with the dissipative behavior of fluids. In particular, we prove that this prescription gives a value for the shear relaxation time that, under the 14-moment approximation, coincides with the one derived from kinetic theory. This shows that transient dynamics is determined by the slowest microscopic and not by the fastest fluid-dynamical time scale. Our approach\(^5\) depends only on generic analytical properties of retarded Green’s functions.

2. Definitions and Power-Counting Scheme

Let us consider the following linear relation between a dissipative current \( J(X) \) and its thermodynamical force \( F(X) \) in Fourier space,

\[
\tilde{J}(Q) = \tilde{G}_R(Q) \tilde{F}(Q),
\]

where we define the Fourier transformation in the following way,

\[
\tilde{A}(Q) = \int d^4X \exp(iQ \cdot X) A(X),
\]

\[
A(X) = \int \frac{d^4Q}{(2\pi)^4} \exp(-iQ \cdot X) \tilde{A}(Q).
\]

Equation (1) contains all the information about the linearized underlying microscopic theory in the form of the retarded Green’s function \( \tilde{G}_R(Q) \). When the system exhibits a clear separation between the typical microscopic and macroscopic scales, \( \lambda \) and \( \ell \), respectively, it is generally assumed that the dynamics of the system can be described in terms of a finite number of macroscopic variables. The macroscopic scale \( \ell \) is associated with the variations of the thermodynamic force \( F \sim \ell^{-1} \) and its derivatives \( \partial^n F \sim \ell^{-n-1} \). The microscopic scale \( \lambda \) is contained in the poles and derivatives of \( \tilde{G}_R(Q) \) and would be, for example, the mean-free path in dilute gases. The challenge is to determine the equation of motion satisfied by the current \( J \) in terms of the microscopic information contained in the retarded Green’s function \( \tilde{G}_R(Q) \) when such a separation of scales exists.
3. Gradient Expansion

The separation between microscopic and macroscopic scales is normally used to determine the macroscopic dynamics of the dissipative current $J$ in terms of an expansion in powers of Knudsen number, $Kn ≡ \lambda/\ell \ll 1$. The simplest realization of this idea is known as the gradient expansion. In the gradient expansion the dissipative currents are expressed solely in terms of gradients of fluid-dynamical variables. In practice, the gradient expansion is implemented by expanding the retarded Green’s function, $\tilde{G}_R(Q)$, in a Taylor series around the origin, $Q = 0$,

$$\tilde{G}_R(Q) \sim \tilde{G}_R(0, 0) + \partial_\omega \tilde{G}_R(0, 0) \omega + \ldots,$$

where the terms resulting from the expansion in $q$ are not explicitly denoted. This expansion leads to the following equation of motion for $J$

$$J(X) = CF(X) + C_1 \partial_t F(X) + \ldots,$$

where $C = \tilde{G}_R(0, 0)$, $C_1 = \partial_\omega \tilde{G}_R(Q)\big|_{Q=0} + \ldots$. We remark that the gradient expansion usually does not involve time derivatives of the thermodynamic force. This is not a problem since one can in principle use the conservation laws to replace time derivatives by spatial gradients.

Apart from overall factors which ensure the correct dimensionality, $C \sim \lambda$ and $C_1 \sim \lambda^2$, respectively. Thus, the term $CF$ is of first order in $Kn$ while $C_1 \partial_t F(t, x)$ is of second order in the Knudsen number, $Kn^2$. If $Kn \ll 1$, higher-order gradients can in principle be omitted and one obtains a closed macroscopic theory for $J$. Navier-Stokes theory corresponds to truncating the series at first order while keeping higher-order corrections leads to the Burnett equations.

This method, however, leads to unstable fluid-dynamical theories. In the non-relativistic case, the second-order truncation, i.e., the Burnett equation, is unstable. In the relativistic case, Navier-Stokes theory itself is unstable. Recently, it has been shown that this instability in relativistic theories is intimately related to the violation of causality. Such problems prohibit the application of these theories to describe any realistic system.

4. Computing the Relaxation Time from the Poles of $\tilde{G}_R$

In the derivation of Eq. (5), two main assumptions were employed: the separation between the microscopic and macroscopic scales and the gradient expansion itself. The first assumption is essential for the coarse-graining procedure that defines fluid dynamics and cannot be disregarded. On the
other hand, the second assumption is solely based on the idea that the Taylor expansion is able to describe the retarded Green’s function in the whole macroscopic/mesoscopic frequency domain. This need not be true, since it is known that Green’s functions are not necessarily analytic in the whole complex plane. In fact, these functions can have singularities in the complex $\omega$ plane, which necessarily restricts the domain of validity of the Taylor expansion.

In order to illustrate the physical meaning of such singularities, it is convenient to consider the generalized diffusion equation for $J$, also known as the Maxwell-Cattaneo equation.$^9$ In this case, as in the Israel-Stewart theories of fluid dynamics,$^{10}$ the dissipative current satisfies a relaxation equation. In the linear domain, this can be written as,

$$
\tau_R \frac{\partial}{\partial t} J + J = DF. \tag{6}
$$

In these theories there are two transport coefficients: the relaxation time $\tau_R$ and the diffusion coefficient $D$. In Fourier space, Eq. (6) can be written in the form $\tilde{J}(\omega) = \tilde{G}_R(\omega) \tilde{F}(\omega)$, where

$$
\tilde{G}_R(\omega) = \frac{D}{\tau_R} \frac{i}{\omega + i/\tau_R}. \tag{7}
$$

It is important to notice the existence of a pole of $\tilde{G}_R(\omega)$ on the imaginary axis. The existence of this pole is directly related to the existence of a non-zero relaxation time and, consequently, to the causal structure of the theory.$^4$ It is also clear from Eq. (6) that Fick’s law and, consequently, the diffusion equation are recovered in the limit of vanishing relaxation time, $\tau_R \to 0$. Mathematically, this limit pushes the pole to infinity, rendering $\tilde{G}_R(\omega)$ an analytic function in the whole complex $\omega$ plane. Only in this case a Taylor expansion near the origin has an infinite radius of convergence and is able to provide a good description of $\tilde{G}_R(\omega)$.

However, in transient theories of fluid dynamics, where the relaxation time is necessarily nonzero, can a Taylor expansion still be used? Strictly speaking, the range of the Taylor expansion is limited by the existence of the pole. As was previously shown, the pole is directly related to the relaxation time and, consequently, the Taylor expansion is unable to probe time scales of the order of the relaxation time.

5. The Role of the Analytical Structure of $\tilde{G}_R(\omega)$

Next, we consider a general analytical structure for $\tilde{G}_R(\omega)$. We assume that $\tilde{G}_R(\omega)$ has $N$ poles in the complex $\omega$ plane, $\omega_1(\mathbf{q}), \ldots, \omega_N(\mathbf{q})$, while the
structure in the \( q \) plane is analytic. Note that, in principle, the number of singularities does not need to be finite. Also, in many cases \( \tilde{G}_R(Q) \) contains branch cuts, especially in the UV region, \( \omega/\omega_1 \gg 1 \), which is sensitive to temperature-independent properties of the system. We only consider the region in the complex \( \omega \) plane without branch cuts. Then, \( \tilde{G}_R(Q) \) can be expressed in the following way

\[
\tilde{G}_R(Q) = \sum_{i=1}^{N} \frac{f_i(Q)}{\omega - \omega_i(q)} = \frac{\Xi(Q)}{[\omega - \omega_1(q)] \cdots [\omega - \omega_N(q)]}, \tag{8}
\]

where \( f_i(Q) \) and \( \Xi(Q) \) are analytic functions in the complex \( \omega \) plane. We write the right-hand side of Eq. (8) in a polynomial form,

\[
\left[ 1 + \Phi_1(q)(-i\omega) + \ldots + \Phi_N(q)(-i\omega)^N \right] \tilde{G}_R(Q) = \frac{(-1)^N \Xi(Q)}{\omega_1(q) \cdots \omega_N(q)}, \tag{9}
\]

where the \( \omega_i(q) \) are the zeros of the polynomial on the left-hand side and we introduced the functions \( \Phi_m(q) \)

\[
\Phi_m(q) = (-i)^m \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq N} \frac{1}{\omega_{i_1}(q) \cdots \omega_{i_m}(q)}. \tag{10}
\]

Combining Eqs. (8) and (9), we obtain

\[
\left[ 1 + \Phi_1(q)(-i\omega) + \ldots + \Phi_N(q)(-i\omega)^N \right] \tilde{G}_R(Q) = \frac{(-1)^N \Xi(Q)}{\omega_1(q) \cdots \omega_N(q)}. \tag{11}
\]

After taking the inverse Fourier transform and expanding the functions \( \Phi_m(q) \) in a Taylor series around \( q = 0 \), we obtain a differential equation satisfied by \( G_R(X - Y) \). It should be clear that this will be a linear differential equation of order \( N \) in time. The equation of motion for \( J \) can be obtained by multiplying Eq. (11) by \( \tilde{J} \) and taking the inverse Fourier transform. See Ref.\(^5\) for details. The result is

\[
\chi_N \partial_t^N J + \ldots + \chi_1 \partial_t J + J = D_0 F + \ldots + D_N \partial_t^N F + O \left( \partial_t^{N+1} F, \partial_x \right). \tag{12}
\]

Here, we omitted all terms involving spatial derivatives. We introduced the transport coefficients \( \chi_m \) and \( D_m \),

\[
\chi_m = \Phi_m(0) = (-i)^m \sum_{1 \leq i_1 < i_2 < \ldots < i_m \leq N} \frac{1}{\omega_{i_1}(0) \cdots \omega_{i_m}(0)}, \tag{13}
\]

\[
D_m = i^m \frac{(-1)^N}{m!} \frac{\partial^m \Xi(\omega,0)|_{\omega=0}}{\omega_1(0) \cdots \omega_N(0)}. \tag{14}
\]
If there is a clear separation of scales, i.e., \( \text{Kn} \ll 1 \), it becomes possible to simplify the right-hand side of Eq. (12). In this case, \( D_0 F \gg D_1 \partial_t F \gg D_2 \partial_t^2 F \), and higher derivatives of \( F \) can be neglected. However, in order to obtain a relaxation-type equation of motion for \( J \), additional approximations are required. For this purpose, one must impose the limiting procedure, \( \chi_i \to 0 \), \( i \geq 2 \). In such a limiting procedure, all the poles of the retarded Green’s function except the pole nearest to the origin (in the following referred to as the “first pole”) are pushed to infinity.\(^5\) Assuming that this can be done, we obtain the following equation of motion for \( J \)

\[
\tau_R \partial_t J + J = D_0 F + D_1 \partial_t F + D_2 \partial_t^2 F + O \left( D_3 \partial_t^3 F, \partial_x \right), \tag{15}
\]

where one can show that

\[
\tau_R = \frac{1}{i \omega_1(0)},
\]

\[
D_0 = \tilde{G}_R(0, 0),
\]

\[
D_1 = i \partial_\omega \tilde{G}_R(\omega, 0) \bigg|_{\omega = 0} + D_0 \tau_R. \tag{16}
\]

Note that the time reversal symmetry of the retarded Green’s function requires the first pole to lie on the imaginary axis, such that \( \tau_R \) is real, as it should be. It is also important to remark that Eq. (15) describes the long- (but not asymptotically long-) time evolution of the system. This is the physical origin of the limiting procedure previously implemented and it is the only limit in which a relaxation equation can be obtained and, consequently, a relaxation time can be defined. If one wants to describe even shorter time scales, it becomes necessary to include more terms \( \sim \chi_i \) in the discussion. Then, the equations of motion will become more complicated. Naturally, the gradient expansion is recovered when all poles are pushed to infinity and, in this sense, the gradient expansion can be interpreted as an asymptotic solution of the more general equation of motion.\(^5\)

Note that this limiting procedure is only allowed when the first pole lies on the imaginary axis. On the other hand, if the first pole and, for reasons of symmetry, its counterpart on the other side of the imaginary axis, have nonzero real parts one cannot disregard one of the poles while keeping the other. Then, the long-time evolution of the dissipative current will be characterized by a second-order differential equation and will display oscillatory motion. Therefore, in this case the fluid-dynamical motion cannot be reduced to a simple relaxation equation, even in the small-frequency domain.
6. Application: The Linearized Boltzmann Equation

The discussion presented above is general and can, in principle, be applied to any type of theory. In this section, we apply this formalism to kinetic theory. In particular, we calculate the shear viscosity and relaxation time coefficients for a weakly coupled classical gas of hard spheres via the Boltzmann equation.

We perform all our calculations in the local rest frame of the fluid element, where \( u^\mu = (1, 0, 0, 0) \). Since we restrict our discussion to shear viscosity, we consider the simplified scenario in which the fluid does not accelerate, \( u^\mu \partial_\mu u^\nu \equiv 0 \), and does not expand, \( \partial_\mu u^\mu \equiv 0 \), and where temperature, \( T = \beta_0^{-1} \), and chemical potential, \( \mu = \alpha_0 / \beta_0 \), are constant. With all these simplifications, the linearized Boltzmann equation can be written as

\[
\partial_t \delta f_k + \mathbf{v} \cdot \nabla f_k - \hat{C} \delta f_k = S(X, K),
\]

where \( \delta f_k = f_k - f_{0k} \), with \( f_{0k} = \exp(\alpha_0 - \beta_0 E_k) \), \( \mathbf{v} = k/E_k \), and \( S = \beta_0 f_{0k} E_k^{-1} k^{(\mu} k^{\nu)} \sigma_{\mu\nu} \). Here, we introduced the collision operator, \( \hat{C} \), defined as

\[
\hat{C} \delta f_k = \frac{1}{\nu E_k} \int dK' dP dP' W_{kk' \rightarrow pp'} f_{0k'} f_{0p} f_{0p'} \left( \frac{1}{f_{0p'}} \delta f_{p'} + \frac{1}{f_{0p}} \delta f_p - \frac{1}{f_{0k'}} \delta f_{k'} - \frac{1}{f_{0k}} \delta f_k \right).
\]

Using Eqs. (17) and (18), it is possible to express the Fourier transform of the shear stress tensor, \( \tilde{\pi}^{\mu\nu}(Q) \), in terms of the Fourier transform of the shear tensor, \( \tilde{\sigma}^{\alpha\beta}(Q) \), in the form \( \tilde{\pi}^{\mu\nu} = G_R^{\mu\nu\alpha\beta} \tilde{\sigma}^{\alpha\beta} \), where

\[
G_R^{\mu\nu\alpha\beta} (\omega, q) = \int dK k^{(\mu} k^{\nu)} \frac{1}{i \omega - i \mathbf{v} \cdot \mathbf{q} - \hat{C}} \beta_0 f_{0k} E_k^{-1} k^{(\alpha} k^{\beta)} f_{0k}.
\]

In order to compute this Green’s function, we define a function \( B^{\alpha\beta}(Q, K) \) which satisfies

\[
(i \omega - i \mathbf{v} \cdot \mathbf{q} - \hat{C}) B^{\alpha\beta}(Q, K) = S(Q, K) = \beta_0 E_k^{-1} k^{(\alpha} k^{\beta)} f_{0k}.
\]

As shown in the previous section, in order to extract the shear viscosity and relaxation time coefficients, it is sufficient to consider the case \( \mathbf{q} = 0 \). Then, the dependence of \( B^{\alpha\beta} \) on \( K \) can be expressed via the following expansion,

\[
B^{\alpha\beta}(\omega, K) = f_{0k} k^{(\alpha} k^{\beta)} \sum_{n=0}^{\infty} a_n(\omega) E_k^n.
\]
Substituting Eq. (21) into Eq. (19), we obtain the following expression for \( \tilde{G}^{\mu\nu\alpha\beta}(\omega, 0) \),

\[
\tilde{G}^{\mu\nu\alpha\beta}(\omega, 0) = \sum_{n=0}^{\infty} a_n(\omega) \int dK k^{(\mu}k^{\nu)}k^{(\alpha}k^{\beta)}E_k^n f_{0k}
= 2\Delta^{\mu\nu\alpha\beta} \sum_{n=0}^{\infty} I_{n+4,2} a_n(\omega), \tag{22}
\]

where we introduced the thermodynamic integral

\[
I_{nq} = \frac{1}{(2q+1)!!} \int dK f_{0k} E_k^{n-2q} (m^2 - E_k^2)^q. \tag{23}
\]

Thus, the relation between \( \tilde{\pi}^{\mu\nu} \) and \( \tilde{\sigma}^{\mu\nu} \) can be cast into a more convenient form \( \tilde{\pi}^{\mu\nu}(\omega, 0) = 2 \tilde{G}^R(\omega, 0) \tilde{\sigma}^{\mu\nu}(\omega, 0) \), with \( \tilde{G}^R(\omega, 0) \) being given by

\[
\tilde{G}^R(\omega, 0) = \sum_{n=0}^{\infty} I_{n+4,2} a_n(\omega). \tag{24}
\]

The whole frequency dependence of \( \tilde{G}^R(\omega, 0) \) is contained in the functions \( a_n(\omega) \). The problem is then reduced to determining the analytic properties of \( a_n(\omega) \). The way to solve this problem is to substitute the expansion (21) into Eq. (20), multiply by \( E_k^m k^{(\mu}k^{\nu)} \), and integrate over \( dK \). Then one obtains

\[
\sum_{n=0}^{\infty} (-i\omega D^{mn} + A^{mn}) a_n(\omega) = \beta_0 I_{m+3,2}, \tag{25}
\]

where we defined the matrices

\[
A^{mn} \Delta^{\mu\nu\alpha\beta} = -\frac{1}{2} \int dK E_k^m k^{(\mu}k^{\nu)} \hat{C}(K) f_{0k} E_k^n k^{(\alpha}k^{\beta)}, \tag{26}
\]

\[
D^{mn} = \frac{1}{5!!} \int dK f_{0k} E_k^{m+n} (m^2 - E_k^2)^2. \tag{27}
\]

The solution of this equation can be formally expressed as,

\[
a_m(\omega) = \beta_0 \sum_{n=0}^{\infty} \left[ (-i\omega D + A)^{-1} \right]^{mn} I_{n+3,2}. \tag{28}
\]

Thus, we obtain

\[
\tilde{G}^R(\omega, 0) = \beta_0 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} I_{m+4,2} \left[ (-i\omega D + A)^{-1} \right]^{mn} I_{n+3,2}. \tag{29}
\]
The poles of this function can be obtained from the roots of the determinant \( \det (-i\omega D + A) = 0 \). Thus, the relaxation times and viscosity coefficients are determined by inverting and finding eigenvalues of infinite-dimensional matrices. In practice, however, the expansion (21) has to be truncated and one never actually deals with infinite matrices. This is a common approach and happens quite often when dealing with the collision operator. Let us try the simplest case and consider only one term in the expansion (this corresponds to using the Israel-Stewart 14-moment approximation in the moments method). Then all matrices become numbers and everything simplifies considerably. In this case, the retarded Green’s function has only one pole, \( \omega_0 = -iA^{00}/I_{42} \).

Since in this case the thermodynamic force is given by \( F = 2\sigma^{\mu\nu} \), the shear viscosity coefficient is given by \( D_0 \). Thus, the shear viscosity and relaxation time are given by

\[
\begin{align*}
\eta &= \tilde{G}_R (0, 0) = \beta_0 \frac{I_{42} I_{32}}{A^{00}}, \\
\tau_\pi &= \frac{1}{i\omega_0} = \frac{I_{42}}{A^{00}}.
\end{align*}
\]

This implies that \( \eta/\tau_\pi = \beta_0 I_{32} \). In the massless limit, for a gas of hard spheres, \( A^{00} = (3/5) I_{42} n_0 \sigma \), with \( \sigma \) being the total cross-section.

Exactly the same results were obtained in Ref.\(^{12}\) within a completely different approach. This clearly demonstrates that the relaxation time in Israel-Stewart theories, which determines the time scale related to transient dynamics, is indeed a microscopic scale. In other words, the relaxation time is determined by the inter-particle scattering rate and not by an arbitrary fluid-dynamical time scale.

7. Conclusion

In this work we proved that the existence of singularities in the retarded Green’s functions considerably restricts the applicability of the gradient expansion as a method to extract macroscopic dynamics. As a matter of fact, the causal structure of relativistic fluid dynamics hinges on the presence of poles in the Green’s functions.

We computed the first pole of the retarded Green’s function associated with shear stress in a weakly coupled classical gas of hard spheres. We proved that the inverse of this pole is directly related to the relaxation time that appears in Israel-Stewart theory. Consequently, the relaxation time in Israel-Stewart theories corresponds to a microscopic time scale and such theories are indeed describing the real transient dynamics of the fluid.
In summary, the true relaxation time is always given by the first pole of the retarded Green’s function. In general, the location of this pole cannot be found using a truncated Taylor expansion around the origin or, in other words, via the gradient expansion. This result should be expected since the transient behavior of a given system cannot be unambiguously extracted from its asymptotic solution given by the gradient expansion.

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References

1. See, for instance, P. Huovinen, P. V. Ruuskanen, Ann. Rev. Nucl. Part. Sci. 56, 163-206 (2006) and refs. therein.
2. L. D. Landau and E. M. Lifshitz, Fluid Mechanics, (Pergamon; Addison-Wesley, London, U.K.; Reading, U.S.A., 1959).
3. W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) 151, 466 (1983), Phys. Rev. D 31, 725 (1985), Phys. Rev. D 35, 3723 (1987), Phys. Lett. A 131, 509 (1988), Phys. Lett. A 131, 509 (1988); S. Pu, T. Koide, D. Rischke, Phys. Rev. D 81, 114039 (2010).
4. G. S. Denicol, T. Kodama, T. Koide and Ph. Mota, J. Phys. G 35, 115102 (2008).
5. G. S. Denicol, H. Niemi, J. Noronha and D. H. Rischke, arXiv:1102.4780 [hep-th] (2011).
6. S. Chapman and T. G. Cowling, The mathematical theory of non-uniform gases, 3rd edition (Cambridge University Press, Cambridge, 1970).
7. D. Burnett, Proc. Lond. Math. Soc. 39, 385–430 (1935); Proc. Lond. Math. Soc. 40, 382–435 (1936).
8. A. V. Bobylev, Sov. Phys. Dokl. 27, 29 (1982).
9. C. Cattaneo, Atti. Semin. Mat. Fis. Univ. Modena 3, 83 (1948).
10. W. Israel and J. M. Stewart, Phys. Lett. 58A, 213 (1976); Ann. Phys. (N.Y.) 118, 341 (1979); Proc. Roy. Soc. London A 365, 43 (1979).
11. S. R. de Groot, W. A. van Leeuwen and Ch. G. van Weerl, Relativistic kinetic theory - Principles and applications, (North-Holland, 1980).
12. G. S. Denicol, T. Koide and D. H. Rischke, Phys. Rev. Lett. 105, 162501 (2010).