Black brane entropy and hydrodynamics: 
the boost-invariant case

Ivan Booth
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, Newfoundland and Labrador, A1C 5S7, Canada

Michal P. Heller
Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Cracow, Poland

Michal Spalinski
Soltan Institute for Nuclear Studies, ul. Hoża 69, 00-681 Warsaw, Poland
and Physics Department, University of Białystok, 15-424 Białystok, Poland.

The framework of slowly evolving horizons is generalized to the case of black branes in asymptotically anti-de Sitter spaces in arbitrary dimensions. The results are used to analyze the behavior of both event and apparent horizons in the gravity dual to boost-invariant flow. These considerations are motivated by the fact that at second order in the gradient expansion the hydrodynamic entropy current in the dual Yang-Mills theory appears to contain an ambiguity. This ambiguity, in the case of boost-invariant flow, is linked with a similar freedom on the gravity side. This leads to a phenomenological definition of the entropy of black branes. Some insights on fluid/gravity duality and the definition of entropy in a time-dependent setting are elucidated.

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I. INTRODUCTION

One of the most important and intriguing recent developments in theoretical physics is the AdS/CFT correspondence. An example of holographic gauge/string theory duality [1] asserts the complete physical equivalence between particular string vacua and ordinary quantum field theories. In certain regimes it relates strongly coupled field theories to weakly curved string theory, which in the leading order reduces to supergravity. A very interesting class of applications involves systems at finite temperature. The holographic duality relates thermodynamic notions in quantum field theory to black hole mechanics in the bulk description. Even more fascinating are non-equilibrium phenomena which cannot be described by thermodynamics or static black hole solutions. There is very strong motivation for such studies on both sides of AdS/CFT duality.
While there are several frameworks that provide fundamental explanations for the entropy of a stationary black hole, the entropy of dynamical black holes is an even more difficult problem. This issue, in view of AdS/CFT duality, is directly connected with the physics of non-equilibrium processes in strongly coupled quantum field theory. Progress in this area will have important bearing on problems in black hole as well as particle physics.

The past few years have seen tremendous activity applying gauge/gravity duality to study gauge theories beyond the perturbation series. One obvious application area where such a tool is sorely needed is the investigation of non-perturbative dynamical QCD plasma. The case of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory has been the focus of attention in this context, since its holographic representation in terms of string theory on anti-de Sitter spacetimes is the best understood example of gauge/gravity duality. Furthermore, it appears that the significant differences between this theory and real-world QCD do not play a major role in a particular range of temperatures relevant to the heavy ion experiments currently in progress at RHIC and soon to start at the LHC. Thus the studies of Yang-Mills plasma at finite temperature using tools provided by string theory (such as [2, 3, 4, 5, 6, 7, 8, 9]) are currently of great practical importance and attract a lot of attention from the heavy ion community.

It is now fairly well established that there is a regime where the dynamics of quark-gluon plasma is well described by relativistic hydrodynamics. The hydrodynamic description itself is well understood despite the fact that the underlying quantum field theory is strongly coupled. It is based on the idea of an expansion in the number of gradients, much in the spirit of effective field theory. One spectacular application of AdS/CFT in recent years was to show that in the case of $\mathcal{N} = 4$ SYM a hydrodynamic description can be derived by considering time-dependent black-brane solutions in AdS spacetime (particularly useful review is [10]). Further applications to quark-gluon plasma require a better understanding of the duality at finite temperature and in a dynamical setting.

A byproduct of the hydrodynamic construction is the notion of entropy current. Recent advances in relativistic conformal hydrodynamics show that a phenomenologically introduced hydrodynamic entropy current constructed order by order in the gradient expansion involves an ambiguity. At second order a 4-parameter family of currents was identified in [8]. Subsequently some important considerations on this topic have appeared in the literature [11], which suggest that it may be possible to reduce this freedom to just one parameter. It seems however that there is a real ambiguity in the hydrodynamic entropy current and in the light of AdS/CFT duality it is natural to ask if one can understand its origins on the gravity side. One of the goals of the present study is to address this issue.

At the root of this question lies the identification of the area increase theorems in general relativity with the second law of thermodynamics. Thus if one hopes to understand this ambiguity on the gravity side, the first step is to carefully examine and understand how horizon areas increase. In doing this one may build on the experience gained in the study of standard black holes in asymptotically flat four-dimensional spacetime. Generalizing these results to five-dimensional anti-de Sitter spacetime is not difficult, but the deep questions encountered earlier remain.

The most “conservative” definition of entropy identifies it with the area of a spatial section of the event horizon. This notion has the drawback of being a teleological (and thus global) concept. In the context of AdS/CFT duality

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1 Many of the results hold also for sectors of other large $N_c$ strongly coupled gauge theories which have gravity dual described by the Einstein-Hilbert action with negative cosmological constant. This is another hint pointing towards possible (limited) applicability of AdS/CFT results to the real world.

2 In general there might be a 5-parameter ambiguity, but one of these parameters is connected with parity-odd effects. This article considers only the parity-even case.
this leads to acausality in the field theory\cite{12, 13}. In classical general relativity, the problems raised by the global character of the event horizon have lead to alternative, quasilocal notions of black holes. These include trapping\cite{14}, isolated\cite{15}, and dynamical\cite{16} horizons. These programmes are closely related to each other and motivated by historical ideas about trapped surfaces\cite{17} and apparent horizons\cite{18}.

These quasilocal horizons have a problem of their own, namely non-uniqueness: unlike event horizons, in dynamical black-hole spacetimes, there are many possible time-evolved apparent (or trapping or dynamical) horizons. In the case of apparent horizons, each foliation of the spacetime will give rise to a different time-evolved horizon\cite{18} while trapping/dynamical horizons are also subject to deformations (see, for example, \cite{19, 20}). Though this is generally thought to be a bad thing, in the context of AdS/CFT it is natural to suspect that the ambiguity of the hydrodynamic entropy current may be related to ambiguities of this type.

In this paper it is shown that the non-uniqueness of apparent horizons is not the source of the ambiguity in the entropy current. Instead, the already existing uncertainty as to whether the event or apparent horizon determines the entropy is embraced and a more general approach is advocated which mimics the phenomenological construction of the boundary entropy current. In the spirit of the membrane paradigm\cite{21} we consider “horizons” that are made up from families of (not necessarily trapped) codimension-two surfaces that satisfy properties such as area increase, asymptoting to the correct equilibrium limits, and being “almost” apparent horizons. In general such families are hard to deal with, but the situation becomes much more manageable close to equilibrium. We consider spacetimes in which a small parameter can be used to formulate a perturbative expansion around equilibrium (this does not necessarily imply a hydrodynamic description\cite{12}). Such a formalism – slowly evolving geometry – has been developed in the context of regular four-dimensional black holes\cite{20, 22, 23} and it is adapted to the case of black brane mechanics as well as generalized to the case of not-quite-apparent horizons. This formalism also makes it possible to see the conditions under which the first law of thermodynamics can be formulated in a dynamical setting.

It is conjectured here that the ambiguity in the above definition of black brane entropy corresponds to the known ambiguity of the hydrodynamic entropy current in the appropriate regime on the gravity side. Verifying this claim in the general case is beyond the scope of this paper, which explores this question in a special, highly symmetric case – the gravity dual to Bjorken (boost-invariant) flow. Besides being tractable, this geometry has the important feature that there is a unique apparent horizon consistent with symmetries of boundary flow. This means that the ambiguity in the hydrodynamic entropy current in case of Bjorken flow cannot be interpreted as a consequence of slicing dependence.

Boost-invariant flow is one of the simplest, yet phenomenologically interesting examples of boundary dynamics. It is a one-dimensional expansion mimicking the dynamics of plasma created in the heavy ion collision with an additional assumption of boost-invariance along the collision axis. This is based on Bjorken’s observation\cite{24} that multiplicity spectra when expressed in proper time and rapidity variables are approximately independent of rapidity in the mid-rapidity region. While this approximation may be somewhat rough, it provides a dramatic simplification. After taking into account all the symmetries in this problem, it turns out that all the physical quantities depend on single variable - the proper time. In the pioneering work\cite{2} Janik and Peschanski considered the gravity dual to boost-invariant flow in the regime of large proper time and showed that regularity of the bulk geometry forces perfect fluid hydrodynamics on the boundary. Subsequent developments included calculating subleading corrections to the large proper time geometry, which correspond to dissipative terms in the boundary energy-momentum tensor\cite{3, 4} (see\cite{25} for a useful review). Understanding the gravity dual to Bjorken flow within the framework of fluid-gravity duality
makes it possible to address issues of black brane mechanics in this case\(^3\).

One can also look at this paper from another perspective. In understanding any mathematical formalism, it is very useful to have concrete examples with which to work. There are very few exact (or even perturbative) dynamical black holes solutions on which to test ideas such as those about slowly evolving horizons. Most of the already known examples were examined in \(23\). These included spherically symmetric spacetimes where the expansion was driven by matter flows and perturbations of Schwarzschild where the expansion was driven by shears (gravitational waves) – though unfortunately in this case the order of the perturbation was such that one could see the shears driving the expansion but not the expansion itself. Then, from this point of view, the current work presents a new non-numerical example of a near-equilibrium horizon and it is the first one for which a shear driven expansion can actually be directly observed. This allows equations to be checked in previously inaccessible regimes and also helps to build a better understanding of the formalism.

The structure of the paper is the following. Section two reviews the various definitions of black hole and then develops black hole mechanics in AdS\(_{n+1}\) spacetime, generalizing \(20, 22\) and keeping things as general as possible with a view to further applications of these results. The third section gives a brief introduction to both field theory and gravity aspects of the boost-invariant flow and the fourth section applies methods developed in section two to the boost-invariant situation. The fifth section contains the analysis of the freedom in the definition of entropy on both sides of the AdS/CFT correspondence in the case of Bjorken flow and shows that there is precise match. The final section offers conclusions and signals some directions for further research.

\section{II. BLACK BRANE MECHANICS}

\subsection{A brief review of black holes and horizons}

We begin with a review of three ways of thinking about black holes. Since our main interest in this paper are the dynamically evolving black branes dual to Bjorken flows, we will focus in particular on dynamical black holes. While the various definitions of black holes generally agree for stationary black holes, they diverge away from equilibrium. For example, they identify different surfaces in spacetime as the boundary of the black hole region and these surfaces have different surface areas. This will be significant in later sections when we compare the thermodynamics on the CFT side with the corresponding black hole mechanics on the gravity side. Then, as noted in the introduction, different surface areas would suggest different values of entropy.

\textit{Event horizons: causally defined black holes}

It is well known that causally defined black holes and event horizons are not locally defined. Paraphrasing, a classically-defined black hole \(18\) is a region of spacetime from which nothing can ever escape. At first glance this seems like a perfectly reasonable definition, however a little consideration quickly turns up problems and these arise from the

\(^3\) It is critical to have a smooth description of the horizon region.
FIG. 1: A schematic demonstrating the non-local nature of event horizon evolution for a spherically symmetric spacetime with the angular dimensions suppressed. Horizontal location measures the radius of the associated spherical shell while time is (roughly) vertical. The shaded gray region represents infalling null dust.

concepts of “ever” and “escape”. To identify a black hole region one must essentially sit at infinity and wait forever to make sure that all escaping signals are identified and further that those that initially look like they might escape really do make it to infinity. Equivalently (but more rigorously) the black hole region is the complement of the causal past of $\mathcal{I}^+$ (future null infinity). The boundary of the black hole region is the congruence of null geodesics known as the event horizon.

To better understand this definition and its associated peculiarities consider Figure 1 which represents the evolution of an initially isolated Schwarzschild black hole which is later irradiated by an infalling shell of null dust (see [28] for more discussion of the Vaidya spacetimes used to generate this example). In the diagram the inward null direction is horizontal while the outward null directions are tangent to the light gray dashed lines. Thus in general the future of any point is “up-and-to-the-left” with the light cones pointing more and more towards the singularity as one approaches it.

In this case one simply tracks the evolution of radial null geodesics to find the event horizon – if they fall into the singularity they are inside the black hole but if they are still heading outwards after the shell of matter passes then they are deemed to escape to infinity. Initially, without any knowledge of the future arrival of the shell of matter, one would guess that the heavily dashed line at $r = 2M_o$ ($M_o$ the mass parameter in the initial Schwarzschild spacetime) would be the event horizon: null geodesics inside it (labelled A in the diagram) clearly head inwards towards the singularity while $r = 2M_o$ itself keeps a constant area. However, with the arrival of the matter shell one finds that this false event horizon also terminates at $r = 0$ as do apparently escaping geodesics like B. It then turns out that the true event horizon was a congruence of geodesics that initially appeared to be escaping but was later “almost caught” by the increased gravitational field arriving with the shell. After the passage of the shell these become stationary at $r = 2(M_o + \Delta M)$, where $M_o + \Delta M$ is the new Schwarzschild mass.

This example nicely demonstrates the consequences of a causal structure definition of black holes. While the evolution
of any congruence of null geodesics is certainly causal, the identification of the set corresponding to the event horizon depends on events in the far future. As a result non-omniscient observers cannot precisely locate them. Further even if identified they will evolve in non-intuitive ways: in the example, the arrival of matter didn’t cause the event horizon to expand but instead curtailed an expansion that began earlier (in a sense in anticipation of the mass increase).

Apparent horizons: geometrically defined black holes

Given these peculiarities, it is often more useful to turn to alternative definitions of black holes and their boundaries which leave aside the causal structure of spacetime and instead focus on the strong gravitational fields characterized by the existence of trapped surfaces. For regular four-dimensional astrophysical black holes, trapped surfaces are closed and spacelike two-surfaces which have the property that all families of null geodesics that intersect them orthogonally must converge into the future. To understand this intuitively, consider a transparent spherical shell that is covered with light bulbs and sitting in empty space. Then if the bulbs are quickly turned on and then off again, two spherical light fronts will be generated — an outwards moving one that expands in area and an inwards moving one that contracts. By contrast, if the shell is transported so that it lies inside a Schwarzschild black hole, concentric with the horizon and enclosing the singularity, then both light fronts will fall towards the centre of the black hole and contract in area; again consider Figure 1 where both outward and inward falling congruences contract in area. This is the canonical example of a trapped surface.

More mathematically, if \( \ell^a \) and \( n^a \) are respectively the outward and inward pointing null normals to a two-surface \( S \) then one can write

\[
\theta(\ell) < 0 \text{ and } \theta(n) < 0,
\]

where \( \theta(\ell) \) and \( \theta(n) \) are the expansions of the null normals and are geometrically analogous to (traces of) extrinsic curvatures

\[
\theta(\ell) = \tilde{\eta}^{ab} \nabla_a \ell_b \text{ and } \theta(n) = \tilde{\eta}^{ab} \nabla_a n_b
\]

with

\[
\tilde{\eta}_{ab} = g_{ab} + \ell_a n_b + n_a \ell_b
\]

being the induced (spacelike) metric on the two-dimensional surface. Alternatively given outward and inward congruences of null geodesics which have tangents \( \ell^a \) and \( n^a \) on \( S \) one can show that

\[
\sqrt{\tilde{\eta} \theta(\ell)} = \frac{1}{2} \mathcal{L}_\ell \sqrt{\tilde{\eta}} \text{ and } \sqrt{\tilde{\eta} \theta(n)} = \frac{1}{2} \mathcal{L}_n \sqrt{\tilde{\eta}},
\]

where \( \sqrt{\tilde{\eta}} \) is the area element on \( S \) and \( \mathcal{L} \) indicates is the Lie derivative operator. Then it is clear that the sign of the expansion determines whether the congruence is expanding or contracting.

More generally, given the energy conditions the mere existence of a trapped surface in an asymptotically flat spacetime implies both: 1) a singularity somewhere in its centre (future) and 2) that it is necessarily contained in a causal black hole and so event horizon. Indeed, for the standard Kerr family of (stationary) black hole solutions, the set of all points contained on some trapped surface coincides exactly with the black hole region. Thus it is not
FIG. 2: An “instant” $\Sigma_t$ along with some of its trapped surfaces (small black circles), the associated trapped region (dark gray) and the apparent horizon (thick dashed line).

unreasonable to consider the existence of trapped surfaces as being the key characterizing feature of a black hole region and this is the basis of the alternative definitions of black holes.

The original such definition was the *apparent horizon*. This begins with the foliation of a spacetime into spacelike hypersurfaces – essentially instants in time. Then at a given instant, the *trapped region* is the union of all the trapped surfaces contained in the hypersurface and the boundary of that region is the *apparent horizon*. It can be shown [18], that on the apparent horizon $\theta(\ell) = 0$ and $\theta(n) < 0$. More generally, any such surface satisfying these conditions is referred to as *marginally trapped*.

This observation that apparent horizons are marginally trapped then motivates the various modern notions of quasilocal horizons such as *trapping* [14], *isolated* [15], and *dynamical* horizons [16] (or see review articles such as [29] and [30]). Though there are significant technicalities, the idea is that the identification of marginally trapped surfaces, which under arbitrarily small deformations become fully trapped, is sufficient to signal the presence of a black hole boundary – even without going through the process of foliating the spacetime and finding apparent horizons on each slice. In fact, this idea is so pervasive that in numerical relativity [31], the term “apparent horizon” has been co-opted to refer to the outermost surface $\theta(\ell) = 0$ on a given slice of spacetime. The study of these ideas is an active and developing area of research with a fairly complicated system of nomenclature but for the purposes of this paper, quasilocal black hole (or brane) horizons in a $(n + 1)$-spacetime dimensions will be understood as $n$-dimensional hypersurfaces that are foliated by $(n - 1)$-dimensional marginally trapped spacelike surfaces. The requirement that there be fully-trapped surfaces “just inside” the horizon can be mathematically written as:

$$\mathcal{L}_n\theta(\ell) < 0,$$

where in this case $n^a$ is any extension of the null normal $n^a$ into a neighbourhood of the putative horizon [20]. In what follows we will usually adopt Hayward’s nomenclature and refer to such structures as future outer trapping horizons (FOTH) [14]. Occasionally however we will refer to time-evolved apparent horizons (which are examples of FOTHs) or dynamical horizons [16] which are almost equivalent.

Mathematically the properties of these surfaces (such as existence, uniqueness, and evolution) may be studied using standard techniques from differential geometry. From a physical perspective however, the key advantages of these generalized apparent horizons include: 1) they are defined by the existence of strong gravitational fields, 2) as for fully trapped surfaces their existence is sufficient to imply the existence of singularities and event horizons, 3) they

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4 Unfortunately there is some inconsistency in the literature about the usage of this phrase. That said, our usage here is probably the most common one – for more details on usage see [29].
FIG. 3: A simulation similar to that of Figure 1 though this time two distinct shells fall into the black hole. Both the apparent and event horizons are plotted.

may be identified without reference to the far future (if one thinks of them as time-evolved apparent horizons), and
4) their evolution is similarly local. The local nature of this evolution is demonstrated in Figure 3. In that case, the
expansion of the event horizon continues to occur in anticipation of the arrival of infalling matter, with the actual
arrival of the matter slowing or ending that expansion. By contrast the apparent horizon evolves in the expected way
in response to the infalling matter – it expands in and only in the presence of actual matter crossing the horizon.

On the downside, it is well-known that quasilocal horizons are not uniquely defined. For classically defined apparent
horizons this is easily be seen. Given a foliation of spacetime we can define a time-evolved apparent horizon \( \triangle \) as the
union of the apparent horizons on each surface. Then, it is clear that different foliations will sample different subsets
of all the possible trapped surfaces. Thus, different foliations will define different \( \triangle \). In the extreme case it is known
that certain slicings of Schwarzschild spacetime contain no trapped surfaces at all and so no apparent horizon \([32]\].
We will return to this lack of uniqueness in later sections.

The membrane paradigm: a physical approach to black holes

A third way of looking at black holes focuses not on causality or geometry but rather on how black holes interact with
their environment. By definition event horizons cannot directly affect their surroundings (they are not in causal contact
with any point outside themselves) and neither can apparent horizons (they are contained within event horizons). All
that either can do is impose restrictions on the behaviour of surfaces “near” the horizons that are in causal contact
with the outside. The membrane paradigm \([21]\) studies the physics of \( n \)-dimensional timelike hypersurfaces “just
outside” the event horizon. For astrophysical purposes, it turns out that one can view these as the time evolutions
of \((n - 1)\)-dimensional spacelike viscous fluid surfaces which carry energy, angular momentum and entropy and can
exchange these with their surroundings. The details of this formalism are not important here but it is important
to understand that for some purposes surfaces besides the usual horizons can usefully be thought of as physically
characterizing black holes.
FIG. 4: A schematic of an $n$-tube $\triangle$ with compact foliation surfaces $S_\lambda$ along with the outward and inward pointing null normals to those surfaces. $\mathcal{V}^a$ is the future-pointing tangent to $\triangle$ that is simultaneously normal to the $S_\lambda$.

### B. The geometry of $n$-tubes

With the motivation of the last section in mind, the mathematics of $n$-tubes, a class of geometric structures that includes both FOTHs and event horizons as well as the time-like hypersurfaces of the membrane paradigm, will be examined here. In an $(n + 1)$-dimensional spacetime $n$-tubes are $n$-dimensional surfaces which can be foliated by $(n - 1)$-dimensional spacelike surfaces $S_\lambda$. The term “tube” comes from $(3 + 1)$-dimensions where the $S_\lambda$ for horizons are compact and generally diffeomorphic to $S^2$ (Figure 4), however this nomenclature will be kept even for black branes where the $S_\lambda$ are diffeomorphic to $\mathbb{R}^3$ (or in the case of $AdS_{n+1}$ with $\mathbb{R}^{n-1}$) and so certainly not compact.

Foliated event horizons, time-evolved apparent horizons and membranes are clearly examples of such structures. Event horizons are $n$-tubes of null signature which have the correct causal properties as discussed in the previous section. Time-evolved apparent horizons are FOTHs whose $S_\lambda$ are the apparent horizons found in individual space-time slices. As will be clear in moment, these are either null (if isolated and in equilibrium) or spacelike (if dynamical and expanding). Finally in the membrane paradigm the $\triangle$ is a timelike surface and the $S_\lambda$ can be thought of as the evolving configurations of a fluid.

To begin, consider the basic geometry of $n$-tubes and in particular focus on the spacelike $S_\lambda$. First, the co-dimension of the $S_\lambda$ is two and the normal space has Minkowski signature so, as in the brief review, one can always find null normals $\ell_a$ and $n_a$ which span that normal space. $\ell^a$ is taken to be outward-pointing (and so is tangent to $\triangle$ if it is null) while $n^a$ points inwards towards the singularity; again see Figure 4. For convenience the normals are usually cross-normalized so that $\ell \cdot n = -1$ which leaves a single scaling degree of freedom in their definition

$$\ell^a \rightarrow f \ell^a \text{ and } n^a \rightarrow \frac{1}{f} n^a \quad (6)$$

for any positive function $f$.

The intrinsic geometry of the $S_\lambda$ is defined by the induced metric $\tilde{q}_{ab}$ (3) while the extrinsic geometry is characterized
by the derivatives of the normal vectors in the directions tangent to $S_\lambda$. These include the extrinsic curvatures:

$$k_{ab}^{(\ell)} = \tilde{q}_a \tilde{q}_b \nabla_c \ell^c \text{ and } k_{ab}^{(n)} = \tilde{q}_a \tilde{q}_b \nabla_c n^c,$$

which decompose into traces $\theta^{(\ell)}$ and $\theta^{(n)}$ as well as trace-free parts $\sigma_{ab}^{(\ell)}$ and $\sigma_{ab}^{(n)}$:

$$k_{ab}^{(\ell)} = \frac{1}{(n-1)} \theta^{(\ell)} \tilde{q}_{ab} + \sigma_{ab}^{(\ell)} \text{ and } k_{ab}^{(n)} = \frac{1}{(n-1)} \theta^{(n)} \tilde{q}_{ab} + \sigma_{ab}^{(n)}.$$  

(7)

Physically these are respectively the expansions and shears of congruences of null curves which have tangents $\ell^a$ and $n^a$ as they intersect $S_\lambda$.

The rest of the extrinsic geometry of the $S_\lambda$ is described by the connection on the normal bundle:

$$\tilde{\omega}_a = -\tilde{q}_a b \nabla_b \ell^c.$$  

(9)

The geometric information contained in $\tilde{\omega}_a$ is somewhat obscured by its gauge-dependence on the scaling $f$:

$$\ell \rightarrow f \ell \text{ and } n \rightarrow \frac{1}{f} n \implies \tilde{\omega}_a \rightarrow \tilde{\omega}_a + d_a \ln f,$$

(10)

where $d_a$ is the $(n-1)$-dimensional gradient operator on $S_\lambda$. In the usual way for gauge potentials, the invariant information is contained in its curvature: in this case this is the curvature of the normal bundle

$$\Omega_{ab} = d_a \tilde{\omega}_b - d_b \tilde{\omega}_a.$$  

(11)

These normal-bundle quantities will arise in the following discussions and physically are closely related to angular momentum (see for example the extensive discussion in [20]). That said, in the actual application of this work to our boost-invariant spacetimes, $\tilde{\omega}_a$ will vanish thanks to the symmetry of the black-branes.

The next natural step is to consider how the $S_\lambda$ fit together to form $\triangle$. Let $V^a$ denote the evolution vector field tangent to $\triangle$ that maps leaves of the foliation into each other ($L_{V^a} = 1$) and is normal to each of the $S_\lambda$. Then $f$ can always be chosen so that

$$V^a = \ell^a - C n^a.$$  

(12)

for some function $C$. This ties the scaling of the null vectors to the foliation and the freedom is reduced to that in the foliation labelling: that is $f = f(\lambda)$. In particular, from (10) it is straightforward to see that this fixes $\tilde{\omega}_a$, removing its gauge dependence.

Clearly thanks to the normalization $\ell \cdot n = -1$, $C > 0 \iff \triangle$ is spacelike, while $C = 0 \iff \triangle$ is null, and $C < 0 \iff \triangle$ is timelike. Further the “time”-rate of change of the area element is

$$L_{V} \sqrt{\tilde{q}} = \sqrt{\tilde{q}} (\theta^{(\ell)} - C \theta^{(n)}).$$  

(13)

Thus, for apparent horizons with $\theta^{(\ell)} = 0$ and $\theta^{(n)} < 0$, $C$ also characterizes the evolution of the horizons and is often referred to as the expansion parameter. Specifically in such cases

$$C < 0 \iff \sqrt{\tilde{q}} \text{ is decreasing } \iff V^a \text{ is timelike},$$

$$C = 0 \iff \sqrt{\tilde{q}} \text{ is unchanging } \iff V^a \text{ is null},$$

$$C > 0 \iff \sqrt{\tilde{q}} \text{ is increasing } \iff V^a \text{ is spacelike}.$$  

(14)

5 Quotation marks are used around the word time since if $V^a$ is spacelike then this is a coordinate rather than physical notion of time.
In contrast, for event horizons $C = 0$ but away from equilibrium $\theta_\ell > 0$ (thanks to the second law) and so the horizon can still expand. For the timelike surfaces of the membrane paradigm none of $\theta_\ell, \theta_n$ or $C$ vanish. Equation \[13\] will be very important in the following, also in cases when $\theta_\ell$ is not exactly zero.

Another gauge quantity

$$\kappa_V = -\mathcal{V}^a n_b \nabla_a \ell^b$$

(15)

describes how the null normal evolves up $\Delta$. Under rescalings

$$\ell \rightarrow f(\lambda)\ell \implies \kappa_V \rightarrow f \kappa_V + \frac{df}{d\lambda}.$$  

(16)

In the case of an isolated horizon $\kappa_V = \kappa_\ell$, the familiar surface gravity that appears in the first law of black hole mechanics and it is not hard to show that a zeroth law of black hole mechanics holds: $\kappa_\ell$ is necessarily constant over the horizons \[15\] (or section II C of this paper), though its exact value is fixed by the chosen normalization of the null vectors.

**Constraints on the geometry**

These geometric quantities are not all independent. Instead they are linked by a set of constraint equations that ensure that the $S_\lambda$ fit smoothly together to form an $n$-tube and further that that tube must be embeddable in a larger $(n + 1)$-dimensional spacetime (on which the Einstein equations hold). We now consider some of these constraints.

First, the “time” rate of change of the two-metric on the $S_\lambda$ can be written in terms of the extrinsic curvature

$$\mathcal{L}_V \tilde{q}_{ab} = \tilde{q}_{ab}^d \mathcal{L}_V \tilde{q}_{cd} = 2(k_\ell^{(\ell)} - Ck_n^{(n)}) = (\theta_\ell - C\theta_n)\tilde{q}_{ab} + 2(\sigma_\ell^{(\ell)} - C\sigma_n^{(n)})$$

(17)

and this is consistent with the already discussed \[13\] if one takes the trace of both sides.

More involved are the equations relating the derivatives of extrinsic curvature quantities to the rest of the geometry. Here we just list some of these results that will be used in our upcoming discussion of mechanics. Those interested in more details of the derivations of these results can consult Appendix A. First, for $\theta_\ell$

$$\mathcal{L}_V \theta_\ell = \kappa_V \theta_\ell - \theta_\ell^2 C + 2\tilde{\omega}^a d_a C - C \left[ \|\tilde{\omega}\|^2 - d_a \tilde{\omega}^a - \tilde{R}/2 + G_{ab}\ell^n b - \theta_\ell \theta_n \right]$$

$$- \left[ \frac{\theta_\ell^2}{n-1} + \|\sigma_\ell\|^2 + G_{ab}\ell^n b \right]$$

(18)

and

$$d_\alpha \theta_\ell = \theta_\ell \tilde{\omega}_a 2(d_b - \tilde{\omega}_b)\sigma_\ell^{(\ell)} b - \frac{1}{n-1} d_a G_{bc}c \ell^c - 2\tilde{q}_{ab}C_{bcde}\ell^d n^e.$$  

(19)

For $\theta_n$

$$\mathcal{L}_V \theta_n = -\kappa_V \theta_n + \left[ \|\tilde{\omega}\|^2 + d_a \tilde{\omega}^a - \tilde{R}/2 + G_{ab}\ell^n b - \theta_\ell \theta_n \right]$$

$$+ C \left[ \frac{\theta_n^2}{n-1} + \|\sigma_n\|^2 + G_{ab}\ell^n b \right]$$

(20)
and

\[ d_a \theta_{(n)} = -\theta_{(n)} \tilde{\omega}_a 2 (d_b + \tilde{\omega}_b) \sigma_a^{(n)b} - \frac{1}{n-1} \tilde{\omega}_a n^c + 2 \tilde{\omega}_a C_{bced} n^c \ell^d n^e. \]  

(21)

In these equations, \( \| \tilde{\omega} \| = \tilde{\omega}^a \tilde{\omega}_a, \| \sigma^{( \ell, n)} \| = \sigma_{ab}^{( \ell, n)} \sigma_a^{(n)b}, \tilde{R} \) is the Ricci scalar on the \((n-1)\)-surfaces, \(G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}\) is the Einstein tensor while \(C_{abcd}\) is the Weyl tensor. The only other equation that we will need is the “time” rate of change of \( \tilde{\omega}_a \):

\[
\mathcal{L}_V \tilde{\omega}_a = d_a k_V - k_{ab} \omega_b - k_a^{(n)} [d^b C - \tilde{\omega}^b C] \\
+ \frac{1}{(n-1)} G_{bc} (\ell^c + C n^c) - C_{bced} \mathcal{V}^c \ell^d n^e.
\]

(22)

As an immediate application of equation Eq. (18), it can be used to (infinitesimally) quantify the lack of uniqueness of apparent horizons. Consider a particular marginally trapped surface \( S_o \) and fix a scaling of \( \ell^a \). Then one can try to solve (18) for \( C \) to find a direction in which \( S_o \) may be evolved while maintaining \( \theta_{(\ell)} = 0 \). If a solution exists then it is unique, but different scalings of \( \ell^a \) will give rise to different \( C \)s and so different possible directions of evolution; in general, \( S_o \) will be a member of an infinite number of different marginally trapped \( n \)-tubes. That said, (18) does strongly constrain the geometry of these possible evolutions from \( S_o \). It can be shown that if one chooses a particular \( \Delta_o \), all other possible evolutions must lie partly in the causal past and partly in the causal future of \( \Delta_o \). Equivalently, all other \( \Delta \) must “interweave” with the original one (the original proof of this result may be found in [19], but also see [20] for discussion in a language closer to that used here). In particular if \( \Delta_o \) is highly symmetric, then other \( \Delta \) cannot share that symmetry. Thus, in highly symmetric spacetimes it is usually possible to select a preferred marginally trapped \( n \)-tube that shares those symmetries.

### C. Equilibrium states and the zeroth law

We now begin an examination of how the geometric calculations of the last section give rise to the laws of black hole mechanics. First consider equilibrium states and the zeroth law.

Equilibrium states are characterized by null \( n \)-tubes on which \( \theta_{(\ell)} = 0 \) and \(-G_a^a \ell^b\) is future-pointing and causal (a condition slightly weaker than the dominant energy condition). These are a type of isolated horizons known as non-expanding horizons [15] and as we now shall see include apparent horizons that are in (possibly temporary) equilibrium with their surroundings as well as event horizons in eternal equilibrium (such as the Kerr family of black holes).

First if \( \Delta \) is null, then \( \mathcal{V}^a = \ell^a \) and \( C = 0 \). Then (18) reduces to the Raychaudhuri equation

\[
\mathcal{L}_q \theta_{(\ell)} = k_V \theta_{(\ell)} - \| \sigma^{(\ell)} \|^2 - G_{ab} \ell^a \ell^b - \theta_{(\ell)}^2 / (n-1),
\]

(23)

or with \( \theta_{(\ell)} = 0 \):

\[
\| \sigma^{(\ell)} \|^2 + G_{ab} \ell^a \ell^b = 0.
\]

(24)

On the geometry side it is clear that \( \sigma_{ab}^{(\ell)} = 0 \) and so the intrinsic geometry of \( \Delta \) is time invariant

\[
\mathcal{L}_q \tilde{q}_{ab} = \frac{1}{2} k_{ab} = \frac{1}{2} \theta_{(\ell)} \tilde{q}_{ab} + \sigma_{ab}^{(\ell)} = 0.
\]

(25)
Meanwhile, on the energy side, Raychaudhuri implies that $G_{ab} \ell^a \ell^b = 0$ and so by the energy condition $- G^{a}_{\ b} \ell^b = \mu \ell^a$ for some function $\mu$: there are no matter flows across the horizon.

If one further assumes that the $\dot{\omega}_a$ and $\kappa_\ell$ components of the extrinsic geometry are time-invariant ($\mathcal{L}_\ell \dot{\omega}_a = 0$ and $\mathcal{L}_\ell \kappa_\ell = 0$) we have a weakly isolated horizon. Then, by (19) $\tilde{q}^b_c \mathcal{G}_{bcde} \ell^d n^e = 0$ and so gathering many of the previous results together, (23) implies that $d_\ell \kappa_\ell = 0$ and so the surface gravity is constant over the horizon. This is the zeroth law of black hole mechanics.

There is also a phase space version of the first law of black hole mechanics which examines how physical quantities change across the phase space of isolated horizons [15]. It follows from a careful Hamiltonian analysis of isolated horizons however it is beyond the scope of this paper. Here we will instead focus on dynamic versions of the first and second law, starting with event horizons.

**D. The first and second law for event horizon mechanics**

Next consider dynamic event horizons – null $n$-tubes with $\theta_\ell(t) \neq 0$ which satisfy appropriate teleological boundary conditions. Then the Raychaudhuri equation (23) multiplied by the area element can be rewritten as

$$\left( \kappa_\ell + \left[ \frac{n - 2}{n - 1} \right] \theta_\ell(t) \right) \mathcal{L}_\ell \sqrt{q} = \mathcal{L}_\ell \left( \mathcal{L}_\ell \sqrt{q} \right) + \sqrt{q} \left( \| \sigma(t) \|^2 + G_{ab} \ell^a \ell^b \right).$$

(26)

Forms of the first law of black hole mechanics may be derived from this in two distinct ways. First, the teleological first law. On integrating (26) over the horizon between the slices $v = v_1$ and $v = v_2$, it becomes

$$\int_{v_1}^{v_2} d\lambda \int d^2x \left( \kappa_\ell + \left[ \frac{n - 2}{n - 1} \right] \theta_\ell(t) \right) \mathcal{L}_\ell \sqrt{q} = \frac{d a}{d v} \bigg|_{S_2}^{S_1} + \int_{v_1}^{v_2} d\lambda \int \sqrt{q} \left( \| \sigma(t) \|^2 + G_{ab} \ell^a \ell^b \right),$$

(27)

where $a$ is the area of the horizon. In the case where the event horizon transitions between (near) equilibrium states on which $\dot{a} \approx 0$ and does so slowly (so that $\theta_\ell(t)$ is small relative to $\kappa_\ell$) this gives the integrated first law

$$\int_{v_1}^{v_2} d\lambda \int d^2x \left( \kappa_\ell \mathcal{L}_\ell \sqrt{q} \right) \approx \int_{v_1}^{v_2} d\lambda \int \sqrt{q} \left( \| \sigma(t) \|^2 + G_{ab} \ell^a \ell^b \right).$$

(28)

As we saw in section II A, contrary to initial intuition, influxes of matter or gravitational radiation do not drive event horizons expansions, but instead curtail existing expansions (this can also be seen in the original Raychaudhuri equation (23) where an increasing matter or shear flux clearly results in a decrease in $\mathcal{L}_\ell \theta_\ell(t)$). In this version of the first law, the integration erases this feature and leaves behind what appears to be a more intuitive first law (first derived in [32]).

That said, there is a regime where a more intuitive version of the first law holds. This is the second version of the first law which we will refer to as the *causal first law* since in this case fluxes appear to drive the expansion. If

$$\mathcal{L}_\ell \left( \mathcal{L}_\ell \sqrt{q} \right) \ll \mathcal{L}_\ell \sqrt{q} \ll \sqrt{q} \kappa_\ell \iff \mathcal{L}_\ell \theta_\ell(t) \ll \theta_\ell(t) \ll \kappa_\ell,$$

(29)

then (23) reduces to:

$$\kappa_\ell \mathcal{L}_\ell \sqrt{q} \approx \sqrt{q} \left( \| \sigma(t) \|^2 + G_{ab} \ell^a \ell^b \right),$$

(30)

and a point-by-point version of the first law (approximately) holds where the fluxes (appear to) drive the expansion. Note that (29) assumes that area rates of change are small and the second derivative of area is much smaller than
the first. So in cases where a derivative expansion can be made in the area element (which will be the case in section \[\text{IVD}\]), this local version of the first law holds for event horizons.

It is important to emphasize that both of these versions of the first law apply to event horizons that are evolving slowly. In the first case we assumed the expansion \(\theta(\ell) \ll \kappa(\ell)\) while in the second we assumed the even more restrictive \(29\).

In this sense, both laws apply to horizons that are close to equilibrium. This should not come as a surprise. In regular thermodynamics the \(dE = TdS\) form of the first law (as opposed to a more general statement about conservation of energy) applies only in quasi-equilibrium situations. As such it is natural to expect the same thing for black hole mechanics. This will be a recurring idea in the rest of this paper.

Finally, the Raychaudhuri equation also lies at the root of the second law. For any congruence of null geodesics we can alway find an affine scaling of the null vectors so that \(\kappa = 0\). Then, if \(\theta(\ell) < 0\) and the null energy condition holds, \(23\) implies that

\[L_\ell \theta(\ell) \leq -\left(\frac{1}{2}\right)^2 \theta^2(\ell)\]

(31)

and so a congruence with \(\theta(\ell) < 0\) necessarily includes caustics. This is the origin of the second law of event horizon mechanics. Other results show that event horizons don’t have caustics and therefore by the preceding argument they must have \(\theta(\ell) \geq 0\) everywhere. In turn this implies that \(L_\ell \sqrt{q} = \sqrt{q} \theta(\ell) \geq 0\); event horizons have non-decreasing area.

\section{The first and second law of apparent horizon/FOTH mechanics}

We now turn to the physics of dynamical \(n\)-tubes which are foliated by \(\theta(\ell) = 0\) surfaces. In contrast to the last section, we begin with the second law.

Just as the Raychaudhuri equation \(23\) was key to event horizon dynamics its generalization Eq.\(18\) is the keystone equation for dynamical apparent horizons. Setting \(\theta(\ell) = 0\) we get a partial differential equation giving allowed values of \(C\) and if one also imposes \(\theta(n) < 0\) and

\[L_n \theta(\ell) = \|\tilde{\omega}\|^2 - d_a \tilde{\omega}^a - \tilde{R}/2 + G_{ab} \ell^a n^b < 0\]

(32)

as motivated in section \[IIA\] many results follow. To begin, the signature of \(\triangle\) is strongly constrained:

1. \(\triangle\) is null (and so isolated) if and only if no matter or gravitational wave flux crosses the horizon: \(C = 0 \Leftrightarrow T_{ab} \ell^a \ell^b = 0\) and \(\sigma^{(t)}_{ab} = 0\).

2. \(\triangle\) is timelike (and so decreasing in area) only if the energy conditions are violated: \(C < 0 \Rightarrow T_{ab} \ell^a \ell^b < 0\). This can happen, for example, in the presence of Hawking radiation.

3. Under all other circumstance, interactions with its environment cause \(\triangle\) to be spacelike (and so expanding in area). In this case, the horizon is said to be dynamical \[16\] and this is the second law of dynamical horizon mechanics.

This set of results was first shown in \[14\] but has since been reproved many times. In the special case of \(C\) constant over each \(S_{\lambda}\) this can be quite easily be seen from \(18\) with the understanding that \(32\) holds. More generally one applies a maximum principle to show these results (see, for example, \[20\]).
Next, the first law. As for event horizon mechanics, we expect this to hold close to equilibrium and so the first task is to define "near" equilibrium for FOTHs. Though the intuition is fairly clear, the implementation is not so straightforward. A near equilibrium FOTH should be: 1) slowly expanding and 2) almost null. Neither of these is trivial to characterize. With respect to the first condition, the second law says that dynamical FOTHs are spacelike and so there is no natural flow of time along the horizon against which one can judge the rate of expansion of the horizon. With respect to the second condition, spacelike and null normal vectors are qualitatively different with squared norms that are respectively positive and zero. While one might naively say that an approach to isolation could be tracked by following how the norm of the normal goes to zero, this runs into difficulties. The only natural way to characterize slow expansion. To strengthen the "almost-null" analogy one should develop a set of tools to track the approach (or departure) of a horizon from equilibrium. Though originally intended for compact horizons in regular four-dimensional spacetimes it is easily adapted to (non-compact) black branes in arbitrary number of dimensions.

As noted in [20], with \( \theta(\ell) = 0 \) the rate of change of horizon area is related to the expansion of the ingoing null normal by

\[
\mathcal{L}_V \sqrt{q} = -C \theta_{(n)} \sqrt{q}.
\]

The definition of \( V^a \) [12] implies that if one relabels the foliation surfaces with \( \lambda \to \tilde{\lambda} \) then \( \ell^a \to f \ell^a \), \( n^a \to n^a / f \) and \( C \to f^2 C \) where \( f = d\tilde{\lambda} / d\lambda \). As such, the scaling dependence of the expansion may be isolated by writing

\[
\mathcal{L}_V \sqrt{q} = \sqrt{q} ||V|| \left( -\sqrt{\frac{C}{2}} \theta_{(n)} \right) \equiv \sqrt{q} ||V|| \theta_{(\tilde{\lambda})},
\]

with the scaling dependence restricted to \( ||V|| = \sqrt{2C} \). The scaling-independent part is the expansion \( \theta(\tilde{\lambda}) \) associated with the unit normalized evolution vector \( \tilde{V}^a = V^a / ||V|| \). Note that even though \( \tilde{V}^a \) is not defined in the null limit, \( \theta(\tilde{\lambda}) \) is well-defined and has the desired limiting behaviour: \( \theta(\tilde{\lambda}) \to 0 \) as \( V^a \) becomes null [34].

Thus there is a reasonable way to characterize slow expansion. To strengthen the "almost-null" analogy one should also require that, to first order, the evolution of the geometry of the foliations of the horizon be characterized by the expansion and shear associated with \( \ell^a \) (as they would be for a truly null surface). Similarly the flux should be approximately that of a null surface. Thus one would like to find the conditions under which

\[
\mathcal{L}_V \tilde{q}_{ab} = (\theta(\ell) - C \theta_{(n)}) \tilde{q}_{ab} + 2(\sigma^{ab}_{(n)} - C a_{ab}) \approx 2 \sigma^{ab}_{(\ell)}, \quad \text{and}
\]

\[
T_{ab} V^a \tau^b = T_{ab}(\ell^a - C n^a)(\ell^b + C n^b) \approx T_{ab} \ell^a \ell^b,
\]

(since \( \theta(\ell) \) vanishes). Further, one would expect these quantities to be small since on a truly isolated horizon both shears vanish.

Following [20] this can all be put on a better mathematical footing by making a few assumptions about the various quantities appearing in Eq. (35). Using the notation \( A \lesssim B \) to indicate that \( A \leq k_o B \) for some \( k_o \) of order one, one assumes that all quantities are bounded relative to some length scale \( \mathcal{L} \):

\[
|\dot{R}|, \, \omega^a \omega_a, \, |d_a \omega^a| \, \text{and} \, |T_{ab} \ell^a n^b| \lesssim \frac{1}{\mathcal{L}},
\]

(37)
and similarly derivatives of $C$ are commensurate

$$\|d_a C\| \lesssim \frac{C_{\text{max}}}{\mathcal{L}} \quad \text{and} \quad \|d_a d_b C\| \lesssim \frac{C_{\text{max}}}{\mathcal{L}^2}.$$  \hfill (38)

These are essentially assumptions that all quantities are of a “reasonable” size and the geometry not be too extreme. They are actually quite weak and, for example, in four-dimensions all members of the Kerr family of solutions satisfy the first condition if $\mathcal{L}$ is taken as the areal radius.

Then, the rate of expansion is characterized by an *evolution parameter* $\epsilon$ defined by:

$$\epsilon^2 / \mathcal{L}^2 = \text{Maximum} \left[ C \left( \|\sigma^{(n)}\|^2 + T_{ab} n^a n^b + \theta^{(n)}_2 / 2 \right) \right].$$  \hfill (39)

Note that if one wishes to allow for energy condition violating matter (so that $C < 0$ is possible) then it is necessary to take the absolute value of the right-hand side to ensure that $\epsilon^2 > 0$.

This quantity can be thought of as a generalization of $\theta^2_\mathcal{V}$ and is independent of the scaling of the null vectors. If $\epsilon \ll 1$, Eq. (38) can be used to show that

$$\|\sigma^{(l)}_{ab}\| \lesssim \frac{\sqrt{C_{\text{max}}}}{\mathcal{L}} \quad \text{while} \quad C\|\sigma^{(n)}_{ab}\| \quad \text{and} \quad C|\theta^{(n)}| \lesssim \epsilon \frac{\sqrt{C_{\text{max}}}}{\mathcal{L}},$$  \hfill (40)

and so Eq. (35) is quantified:

$$\mathcal{L}_V \tilde{q}_{ab} = \underbrace{\sigma^{(l)}_{ab}}_{O(\sqrt{C})} + \underbrace{\left(-C \theta^{(n)}_{ab} - C \sigma^{(n)}_{ab} \right)}_{O(\epsilon \sqrt{C})}.$$  \hfill (41)

These ideas are consolidated to get the first part of the definition of a near-equilibrium FOTH:

**Definition:** Let $\Delta H$ be a section of a $n$-dimensional FOTH foliated by (non-compact) spacelike $(n-1)$-surfaces $S_\lambda$ so that $\Delta H = \{ \cup \lambda S_\lambda : \lambda_1 \leq \lambda \leq \lambda_2 \}$. Further let $\mathcal{V}^a$ be an evolution vector field that generates the foliation so that $\mathcal{L}_V \lambda = \alpha(\lambda)$ for some positive function $\alpha(\lambda)$, and scale the null vectors so that $\mathcal{V}^a = \ell^a - C n^a$. Then $\Delta H$ is a *slowly expanding horizon* relative to the length scale $\mathcal{L}$ if

(i) the *evolution parameter* $\epsilon \ll 1$ where

$$\epsilon^2 / \mathcal{L}^2 = \text{Maximum} \left[ C \left( \|\sigma^{(n)}\|^2 + T_{ab} n^a n^b + \theta^{(n)}_2 / 2 \right) \right],$$  \hfill (42)

(ii) $|\tilde{R}|$ and $|\tilde{\omega}|^2 \lesssim 1 / \mathcal{L}^2$ and $T_{ab} \ell^a n^b \lesssim 1 / \mathcal{L}^2$

(iii) $(n-1)$-surface derivatives of horizon fields are at most of the same order in $\epsilon$ as the (maximum of the) original fields. For example, $\|d_a C\| \lesssim C_{\text{max}} / \mathcal{L}$, where $C_{\text{max}}$ is largest absolute value attained by $C$ on $S_\lambda$.

Compared to the original definition (for three-dimensional horizons in four-dimensional spacetimes [20, 22]), the only changes are in the dimension and switching from compact to non-compact horizons. A secondary change following from these is that the area of the (compact) two-dimensional horizon cross-sections was used in the original definition to set the scale $\mathcal{L}$. For non-compact horizons this is no longer feasible and so the length scale must be set differently. For the black branes of this paper, the only scale is set by the cosmological constant and so we define $\Lambda = -(n-1)n/2 / \mathcal{L}^2$ (the relevant scale is the AdS radius); for the coordinates that we are using is equivalent to setting $\mathcal{L} = 1$.  

To move from geometry to mechanics, the formalism requires that there exist a scaling of the null vectors that satisfies the following (again slightly modified) conditions:

**Definition:** A slowly expanding horizon is said to be *slowly evolving* if there exists a scaling of the null vectors such that \( C \lesssim \epsilon^2 \) and:

1. \( \|L_V \tilde{\omega}_a\| \) and \( |L_V \kappa_V| \lesssim \epsilon/L^2 \) and
2. \( |L_V \theta(n)| \lesssim \epsilon/L^2 \).

Scaling the null vectors so that \( C \lesssim \epsilon^2 \) is motivated by a couple of considerations. First, it means that the norm \( \|V\| = 2C \lesssim \epsilon^2 \) and so in an asymptotic approach to isolation the tangent vector becomes null in an orderly fashion (for example it doesn’t diverge in the limit). Second with this scaling, \( L_V \) “time”-derivatives will properly reflect the slowly evolving nature of the horizon; for example \( L_V \sqrt{\tilde{q}} = -C\theta(n)\sqrt{\tilde{q}} \lesssim \epsilon^2/L \) and so the area expansion is also slow.

The other conditions are motivated by the isolated horizon formalism. For a horizon in equilibrium with its surroundings, foliations can always be found so that both of these vanish and then, as we have seen, the zeroth law of isolated horizon mechanics directly follows. Similarly for slowly evolving, near-equilibrium horizons, these conditions are sufficient to enforce an approximate zeroth law: surface gravity is constant across slices to order \( \epsilon \). The derivation is essentially the same as that for the true first law – one simply applies the assumptions to (22) this time setting quantities to be small rather than zero. Note though that in cases of high symmetry (such as those that will be considered later in this paper), it often turns out that those symmetries will force \( d_a \kappa_V = 0 \) exactly, independently of these considerations.

Finally one can combine (18) and (20) to show that

\[
\kappa_V \theta_V = L_V \theta(\ell) + C L_V \theta(n) + d_a(d^a C - 2C \tilde{\omega}^a) + \sigma^{(\ell)} \sigma^{(V)} + G_{ab} V^a \tau^b + \theta(V) \theta(\tau)/(n-1),
\]

where \( \tau^a = \ell^a + C n^a \). Now for a slowly evolving horizon \( \theta(\ell) \) and \( L_V \theta(\ell) \) vanish and on including the other assumptions and approximations this equation reduces to

\[
\kappa_\alpha L_V \sqrt{\tilde{q}} \approx \sqrt{\tilde{q}} \left( \|\sigma^{(\ell)}\|^2 + T_{ab} \ell^a \ell^b \right),
\]

where \( \kappa_\alpha \) is the lowest order of the surface gravity expansion on the slice: \( \kappa_V \approx \kappa_\alpha + \epsilon \kappa_1 \). This is the first law of slowly evolving FOTHs/dynamical horizons.

**F. The first and second law of alternate “horizon” mechanics**

In the previous subsections we have seen that both apparent and event horizons give rise to reasonable notions of black hole mechanics with the same zeroth law, similar first laws, and different second laws (in that the horizons have different surface areas and so different notions of entropy). Now compared to isolated horizons and stationary black holes, event horizons drop the \( \theta(\ell) = 0 \) requirement while dynamical apparent horizons are no longer null. However both return (at least asymptotically) to become isolated horizons in the equilibrium limit. With two candidates in hand, it is then natural to consider other \( n \)-tubes which might act as horizons: correctly interpolating between...
equilibrium states and obeying the laws of mechanics. Guided by the example of slowly evolving horizons we will continue to consider the near-equilibrium limit: essentially we will consider slowly evolving almost-horizons for which $\theta_{(\ell)} \approx 0$. The guiding principles in defining slowly evolving almost-horizons will be:

P1. The “horizon” $n$-tube should be a one-way membrane in the sense that no causal signal from a trapped surface should be able to cross it in the direction of infinity.

P2. Under equilibrium conditions the $n$-tube should match or at least asymptote to an isolated horizon. That is, the usual notion of equilibrium entropy should be recovered.

P3. Near equilibrium the $n$-tube should be “almost” a slowly evolving horizon (and so “almost” isolated).

P4. Near equilibrium the surface gravity $\kappa_V$ should be almost constant.

P5. Near equilibrium there should be a first law of the form (44).

P6. The area should be non-decreasing.

P7. In the apparent and event horizon limits, the conditions should reduce the known laws for those cases.

Guided by the experience from slowly evolving horizons, we now propose a class of $n$-tubes that satisfy these conditions. As before we will need to be careful to make sure that everything remains independent of the scaling of the null normals but now have the extra challenge when setting conditions that we may have $\theta_{(\ell)} \neq 0$ and $C = 0$ simultaneously. Thus, unlike for slowly evolving horizons we can’t use powers of $C$ to remove scaling invariance. Instead we will use $\theta_{(n)}$ which for standard black hole/brane solutions is non-zero and of finite size when $\theta_{(\ell)} = 0$.

The following structures will meet all of the guiding principles.

**Definition:** Let $\triangle H$ be a section of a $n$-tube foliated by spacelike $(n-1)$-surfaces $S_\lambda$ so that $\triangle H = \{ \cup_\nu S_\lambda : \lambda_1 \leq \lambda \leq \lambda_2 \}$. Further let $V^a$ be an evolution vector field that generates the foliation so that $L_{V^a \lambda} = \alpha(\lambda)$ for some positive function $\alpha(\lambda)$, and scale the null vectors so that $V^a = \ell^a - C n^a$. Then $\triangle H$ is a near-equilibrium $n$-tube (NENT) relative to the length scale $\mathcal{L}$ if:

(i) the evolution parameter $\epsilon \ll 1$ where

$$\epsilon^2 / \mathcal{L}^2 = \text{Maximum} \left[ |\theta_{(n)} \theta_{(\ell)}| + |C| \left( \theta_{(n)}^2 + \|\sigma^{(n)}\|^2 + T_{ab} n^a n^b \right) \right],$$

(ii) $|\tilde{R}|$ and $\|\tilde{\omega}\|^2 \lesssim 1 / \mathcal{L}^2$ and $T_{ab} \ell^a n^b \lesssim 1 / \mathcal{L}^2$

(iii) $(n-1)$-surface derivatives of horizon fields are at most of the same order in $\epsilon$ as the (maximum of the) original fields. For example, $\|d_a C\| \lesssim C_{\text{max}} / \mathcal{L}$, where $C_{\text{max}}$ is largest absolute value attained by $C$ on $S_\lambda$.

Further the scaling of the null normals may be chosen so that

(iv) $\kappa_V, |\theta_{(n)}| \sim 1 / \mathcal{L}$, and $\mathcal{L}_n \theta_{(\ell)} < 0, \theta_{(\ell)} - C \theta_{(n)} > 0$

(v) $|\mathcal{L}_V \theta_{(\ell)}| \lesssim \epsilon^3 / \mathcal{L}^3$
(vi) $\|L\tilde{\omega}_a\|$ and $|L\kappa_V| \lesssim \epsilon/L^2$ and

(vii) $|L\theta_{(n)}| \lesssim \epsilon/L^2$.

Let us consider this definition and how it meets the guiding principles. First if $\theta_{(\ell)} = 0$ on $\Delta H$, then it reduces to the definition of a slowly evolving horizon. Similarly if $C = 0$ it will reduce to the special case of an event horizon for which the point-by-point first law (30) holds. Further if $\epsilon = 0$ then $\theta_{(\ell)} = 0$ and $C = 0$ and we are back to an isolated horizon.

We now consider where this definition differs from that of a slowly evolving horizon and the resulting implications. First (i) contains an extra term and this has the joint purpose of ensuring that the outward null-expansion is small and also that $\epsilon$ is defined even for null horizons where $C = 0$. Further, applying it along with conditions (ii)-(v) to the expression (18) for $L\theta_{(\ell)}$ one can show that

$$||\sigma^{(\ell)}||^2 \lesssim \frac{\epsilon^2}{L^2}$$

and

$$T_{ab}\ell^a\ell^b \approx \frac{\epsilon^2}{L^2}$$

in essentially the same way that these quantities were bounded for isolated and slowly evolving horizons. Note, too that by bounding $|\theta_{(\ell)}\theta_{(n)}|$ to be of the order of $\epsilon^2$ we retain the result that to order $\epsilon$ the horizon will evolve as a $\theta_{(\ell)} = 0$ null surface:

$$L\tilde{\sigma}_{ab} \approx \sigma^{(\ell)}_{ab}.$$  \hspace{1cm} (46)

If $|\theta_{(\ell)}\theta_{(n)}|$ was of order $\epsilon$ then this relation would be lost and a host of difficulties would follow, for example in deriving the first law. Similarly, to order $\epsilon^2$ the matter flux across the horizon will also be that for a null surface:

$$T_{ab}\nu^a\nu^b \approx T_{ab}\ell^a\ell^b.$$  \hspace{1cm} (47)

The scaling condition $\theta_{(n)} \sim -1/L$ is essentially that used in the standard coordinate descriptions of exact black hole solutions and also for spacelike or timelike horizons implies $C \sim \epsilon^2$. The second law is put in by hand in the assumption $\theta_{(\ell)} - C\theta_{(n)} > 0$ however note that if $\theta_{(\ell)} > 0$ (outside a FOTH) then timelike $\Delta H$ with $C < 0$ are allowed. At the same time (vi-vii) guarantee an approximate zeroth law by essentially the same arguments that we saw for isolated and slowly evolving horizons. Similarly, one may rerun the slowly evolving arguments by applying our conditions to (43) to rederive the first law (44) for these “horizons”.

The only condition left to be considered is P1. This is clearly okay if $\Delta H$ is spacelike or null. Intuitively however it should also be okay if $C < 0$. As discussed in the previous paragraph, this is only possible if $\theta_{(\ell)} > 0$ – outside any FOTH. So, signals could pass out of the NENT – but only from a boundary layer outside the trapped region. This is similar to the situation in the membrane paradigm where the boundary is taken to be a timelike surface.

### III. GRAVITY DUAL TO BJORKEN FLOW

#### A. Fluid/gravity duality and “alternate horizons”

One of the most important recent insights within the AdS/CFT correspondence is a gravity dual formulation of (conformal) relativistic hydrodynamics \[13\]. It has been known for many years that black holes are thermal in nature.
In the context of gauge/gravity duality, thermal states of certain strongly coupled quantum field theories have been understood in terms of asymptotically AdS black brane geometries. From the perspective of fluid/gravity duality, hydrodynamics is thought of as long wave-length dynamics of non-equilibrium branes. Dissipative effects in hydrodynamics imply entropy production. On the other hand, entropy on the gravity side had been linked for many years with event horizons and area increase theorems with the second law of thermodynamics. This leads to a unique identification of the entropy also in the context of the gravity picture of hydrodynamics, in contrast with the phenomenological definition of the boundary entropy where some freedom remains. However, as reviewed in the last section, there are many notions of horizon on the gravity side. These lead to distinct definitions of entropy which coincide in the equilibrium situation, which makes it interesting to first investigate the conditions under which a gravitational system is close to equilibrium. The answer can be given with the help of the slowly-evolving formalism generalized to the AdS case in the previous section. After identification of the near-equilibrium situations one may proceed to understand various notions of horizons and identify the freedom of definition of entropy in the gravity dual to hydrodynamics. Since the entropy is an infrared concept, its local identification on the gravity side relies on considerations of surfaces close to the horizon (or, in other words, in the region corresponding to the thermal scale on the gauge theory side). The event horizon itself is a teleological concept, and as such is sometimes hard to deal with. The surfaces which capture the notion of entropy are better characterized by \( \theta(\ell) \approx 0 \) (the “alternate horizons” from the previous section). This uses the fact that close to equilibrium (i.e. in the case of slow evolution) apparent and event horizon should be close to each other.

Understanding this in the general case might be shadowed by significant technical details. However, the fluid/gravity duality links solutions of relativistic conformal hydrodynamics with particular solutions of Einstein equations with negative cosmological constant. One such solution is boost-invariant expansion known as boost-invariant flow. It is complicated enough to capture many non-trivial features of the hydrodynamics and at the same time simple enough to enable efficient treatment of the 5-dimensional geometry in the horizon region. As such it serves as the laboratory for exploring different notions of entropy and the corresponding notions in the hydrodynamics of 4-dimensional supersymmetric Yang-Mills plasma.

### B. Bjorken flow

Boost-invariant flow is an one-dimensional expansion of plasma with the boost symmetry along the expansion axis. If \( x^0 \) denotes lab frame time and \( x^1 \) is the coordinate along the expansion axis, then transformation to more convenient coordinates – proper time \( \tau \) and rapidity \( y \) – takes the form

\[
\begin{align*}
x^0 &= \tau \cosh y, \\
x^1 &= \tau \sinh y.
\end{align*}
\]

The assumption of boost-invariance is based on Bjorken’s observation that multiplicity spectra does not depend on rapidity in the mid-rapidity region. The most general traceless energy-momentum tensor, which is boost-invariant
can be expressed solely in terms of the energy density $\epsilon (\tau)$

$$T_{\mu\nu} = \text{diag} \left\{ \epsilon (\tau), -\tau^3 \epsilon' (\tau), -\tau^2 \epsilon (\tau), \epsilon (\tau) + \frac{1}{2} \tau \epsilon' (\tau), \epsilon (\tau) + \frac{1}{2} \tau \epsilon' (\tau) \right\}$$  \hspace{1cm} (49)

which is a function of proper time only (in boost-invariant setup no physical quantity can depend on rapidity) \[2\].

Applicability of the hydrodynamic description means that the system is fully described by the four-velocity $u^\mu$ ($u_\mu u^\mu = -1$) and temperature $T (\tau)$. Boost-invariance implies that $u^\mu = [\partial / \partial \tau]^\mu$ and, as expected on the basis of the form of the energy momentum tensor, the full dynamics is encoded in the dependence of the energy density on $\tau$. Since for a conformal fluid the equation of state is $p = 1/3 \epsilon$, one has

$$\epsilon = e_0 T^4$$  \hspace{1cm} (50)

$$s = \frac{4}{3} e_0 T^3$$  \hspace{1cm} (51)

where $s$ is the usual thermodynamic entropy density.

The general form of second order hydrodynamics equations (see [35]) is

$$\partial_\tau \epsilon = -\frac{4}{3} \frac{\epsilon}{\tau} + \Phi_0,$$  \hspace{1cm} (52)

$$\tau_\Pi \partial_\tau \Phi = \frac{4}{3} \frac{\eta}{\tau} - \Phi - \frac{4}{3} \frac{\tau_\Pi \Phi}{\tau} - \frac{1}{2} \frac{\lambda_1 \Phi^2}{\tau}.$$  \hspace{1cm} (52)

Here $\Phi$ is yy component of the shear tensor $\Pi^{\mu\nu}$. These equations determine the proper-time evolution of the energy density, which in turn determines the dependence of temperature of $\tau$.

Substituting the form of the energy density (50) (together with the equation of state) into these equations leads to the following solution for the temperature as a function of proper-time:

$$T (\tau) = \frac{A}{\tau^{2/3}} \left\{ 1 - \frac{1}{A \tau^{2/3}} \cdot \frac{\eta_0}{\sqrt{2} \lambda \sqrt{3/\pi}} + \frac{1}{A^2 \tau^{4/3}} \left( \frac{\lambda}_1^{(0)} \frac{\Pi_0^{(0)}}{3 \sqrt{3} \pi^2} - \frac{\eta_0 \Pi_0^{(0)} \lambda_1^{(0)}}{3 \sqrt{3} \pi^2} \right) + \ldots \right\}$$  \hspace{1cm} (53)

where $A$ is a scale fixed by the initial conditions and the only arbitrary number in the construction\[7\]. In much of the literature (such as [2, 4, 5, 6]) the choice $A = \sqrt{2} \sqrt{3/\pi}$ is made. The constants $\eta_0$, $\Pi_0^{(0)}$, $\lambda_1^{(0)}$ are various transport coefficients from the first ($\eta_0$) and second order viscous hydrodynamics \[8\]. They are universal numbers related to the microscopic physics of the underlying quantum field theory. Their presence signals the dissipative nature of the flow – the entropy production.

C. Validity of the hydrodynamic description

The modern view of hydrodynamics is similar to that of effective field theory. It is a phenomenological description of phenomena on scales much larger than those of their microscopic dynamics, constructed as a systematic expansion

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\[7\] It is easy to understand the presence of $\Lambda$ if one considers the dilatation $x^\mu \rightarrow \alpha x^\mu$. Then $\tau \rightarrow \alpha \tau$ and $\Lambda \rightarrow \alpha^{-2/3}$ which leads to $\epsilon (\alpha \tau) = \alpha^{-1} \epsilon (\tau)$ which is the correct scaling

\[8\] More precisely these are dimensionless numbers related to the transport coefficients by the relations

$$\eta = \eta_0 e_0 T^3,$$

$$\Pi_0 = \frac{\tau_0^3}{\Pi_0 T^{-1}},$$

$$\lambda_1 = \lambda_1^0 e_0 T^2,$$  \hspace{1cm} (54)

where $e_0$ is defined as previously by $\epsilon = e_0 T^4$
In gradients. In the context of boost-invariant flow this translates into an expansion in powers of $1/\tau^{2/3}$. As in the case of any perturbative expansion, one needs to observe the regime where the expansion can be reasonably expected to apply. A criterion for this is that the subleading terms in the expansion be smaller than the leading order. In the context of Bjorken flow this can be understood as a condition on the minimal time when the expansion can be trusted. For the first subleading term in (53) to be smaller than the leading order one needs

$$\tau > \tau_{\text{min}}$$

where

$$\frac{1}{\Lambda \tau_{\text{min}}^{2/3}} \cdot \frac{\eta_0}{\sqrt{2} 3^{1/4} \pi} \equiv \alpha < 1$$

(55)

One requires that the expansion of any physical quantity such as energy or entropy density should have this property. Note that including higher order terms does not extend the regime of validity of the hydrodynamic expansion, but rather improves the accuracy within the hydrodynamic window. Moreover, in a general boost-invariant dynamical situation $\tau_{\text{min}}$ does not coincide with thermalization time, since the non-hydrodynamic (exponential) modes do not have to be negligible at that time [36].

D. Holographic description of Bjorken flow

Since the energy-momentum of the gauge theory is related by AdS/CFT dictionary to the five-dimensional asymptotically AdS metric, three independent warp factors are to be expected on the gravity side. This leads to a boost-invariant metric ansatz of the form

$$ds^2 = 2\tilde{g}(\tilde{\tau}, r) d\tilde{\tau} dr + \tilde{h}(\tilde{\tau}, r) dr^2 - r^2 A(\tilde{\tau}, r) d\tilde{\tau}^2 + (1 + r\tilde{\tau})^2 e^{b(\tilde{\tau}, r)} dy^2 + r^2 e^{c(\tilde{\tau}, r)} dx_\perp^2$$

(56)

The functions $g(\tilde{\tau}, r)$ and $h(\tilde{\tau}, r)$ are gauge degrees of freedom and can be chosen conveniently. Setting $g(\tilde{\tau}, r)$ to 0 and $h(\tilde{\tau}, r)$ to $1/r^2$ leads to the Fefferman-Graham-like coordinates, which are particularly useful in obtaining the boundary energy-momentum tensor. These coordinates suffer however from an important drawback: they break down at the locus where $A(\tilde{\tau}, r)$ goes to zero, which is where one would naively expect the location of the horizon. In order to keep a full control over the geometry in this region it is useful to adopt so called ingoing Eddington-Finkelstein coordinates which give the following metric ansatz [26]

$$ds^2 = 2d\tilde{\tau} dr - r^2 A(\tilde{\tau}, r) d\tilde{\tau}^2 + (1 + r\tilde{\tau})^2 e^{b(\tilde{\tau}, r)} dy^2 + r^2 e^{c(\tilde{\tau}, r)} dx_\perp^2$$

(57)

This form of the metric still has a residual gauge freedom $r \rightarrow r + R(\tau)$ [27]. In the following this freedom will usually not be fixed, since it provides a useful cross-check on the correctness of the calculations – physical quantities in the boundary theory should not depend on the choice of gauge in the bulk.

The metric (57) describes the gravity dual to boost-invariant plasma for any $\tau$. It turns out however, that in the regime of large proper time, the boundary energy density $\epsilon(\tau)$ satisfies the equations of hydrodynamics. This highly nontrivial observation was first made in [2] and then developed in [3, 4] using the language of gravity dual.

It is particularly easy to understand $\tau^{-2/3}$ damping of the subleading pieces in the expression of the temperature – they correspond to the gradient expansion. The gradient expansion has to have its counterpart in the gravity dual to

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9 The temperature has been chosen here because it enters the definition of the gradient expansion.
boost-invariant flow in the regime of large proper time. The boundary gradient expansion leads to the introduction of a scaling variable and the large proper time expansion in the scaling limit (keeping \( v = r \cdot \tilde{\tau}^{1/3} \) fixed while taking the limit \( \tilde{\tau} \to \infty \)) on the gravity side

\[
A(\tilde{\tau}, r) = A_0 \left( r \cdot \tilde{\tau}^{1/3} \right) + \frac{1}{\tilde{\tau}^{2/3}} A_1 \left( r \cdot \tilde{\tau}^{1/3} \right) + \frac{1}{\tilde{\tau}^{4/3}} A_2 \left( r \cdot \tilde{\tau}^{1/3} \right) + \ldots,
\]

\[
b(\tilde{\tau}, r) = b_0 \left( r \cdot \tilde{\tau}^{1/3} \right) + \frac{1}{\tilde{\tau}^{2/3}} b_1 \left( r \cdot \tilde{\tau}^{1/3} \right) + \frac{1}{\tilde{\tau}^{4/3}} b_2 \left( r \cdot \tilde{\tau}^{1/3} \right) + \ldots,
\]

\[
c(\tilde{\tau}, r) = c_0 \left( r \cdot \tilde{\tau}^{1/3} \right) + \frac{1}{\tilde{\tau}^{2/3}} c_1 \left( r \cdot \tilde{\tau}^{1/3} \right) + \frac{1}{\tilde{\tau}^{4/3}} c_2 \left( r \cdot \tilde{\tau}^{1/3} \right) + \ldots.
\]  

(58)

The meaning of the scaling limit is that the radial distance in AdS is measured in units of inverse temperature \((r/T \sim r \cdot \tau^{-1/3} \text{ since } T \sim \tau^{-1/3} \text{ in the leading order})\). The large proper time expansion corresponds to the gradient expansion in the hydrodynamics. The metric with \( A_0 \left( r \cdot \tilde{\tau}^{1/3} \right) \) only will be called the zeroth order metric and is dual to perfect fluid hydrodynamics on the boundary side. The first order gravity solution \(- A_1 \left( r \cdot \tilde{\tau}^{1/3} \right), b_1 \left( r \cdot \tilde{\tau}^{1/3} \right) \) and \( c_1 \left( r \cdot \tilde{\tau}^{1/3} \right) \) – mimics boundary viscous effects on the gravity side and is the leading order relevant for the gravitational entropy production, whereas the second order metric corresponds to relaxation time in the boundary theory. The formulae providing the form of the warp factors at zeroth, first and second orders are given below, whereas third order formulae can be found in [38].

At order zero one has

\[
A_0 (v) = 1 - \frac{\pi^4 A^4}{v^4},
\]

\[
b_0 (v) = 0,
\]

\[
c_0 (v) = 0.
\]  

(59)

which is equivalent to the Janik-Peschanski solution [3]. For simplicity, this solution assumes a choice of gauge to bring out the similarity to the static AdS-Schwarzschild black brane (see section IV E).

At first order one finds

\[
A_1 (v) = \frac{2 \pi^3 (v + \pi A) A^3}{3 v^5} + \frac{2 \left( v^4 + \pi^4 A^4 \right) \delta_1}{3 v^5},
\]

\[
b_1 (v) = \frac{2 \arctan \left( \frac{v}{\pi A} \right)}{3 \pi A} - \frac{4 \log (v)}{3 \pi A} + \frac{\log \left( \frac{v^2}{\pi A} + 1 \right)}{3 \pi A} + \frac{2 \log \left( \frac{v}{3 \pi A} + 1 \right)}{3 \pi A},
\]

\[
c_1 (v) = -\frac{1}{2} b_1 (v) + \frac{\delta_1}{v}.
\]  

(60)

The second order formulas can be found in appendix [13].

The zeroth order solution looks like the boosted and dilated black brane. This is in line with the argument of [7], which states that the gravity dual to the general solution of hydrodynamics can be obtained by boosting and dilating black brane and having this as leading order solution solving Einstein equation perturbatively in the gradient expansion. Since boost-invariant flow is a particular solution of the equations of hydrodynamics, this is not a coincidence. The interesting, yet not surprising given its interpretation as one-dimensional expansion, feature of boost-invariant hydrodynamics is that the temperature does not settle down to the constant non-zero value, which gives rise to the question as to whether this solution is covered by an event horizon. In the next sections it will be demonstrated that
this indeed the case, which means that the gravity dual to boost-invariant flow is a well defined geometry from the cosmic censorship hypothesis point of view.

The gravity solution displayed above determines, via holographic renormalization, the boundary energy momentum tensor \[ 39 \], which is of the form \[ 40 \]. Explicitly, one finds

\[
\epsilon(\tau) = \frac{3}{8} N_c^2 \pi^2 \frac{A^4}{\tau^{4/3}} \left\{ 1 - \frac{2}{3\pi A} \cdot \frac{1}{\tau^{2/3}} + \frac{1 + 2 \log(2)}{18\pi^2 A^2} \cdot \frac{1}{\tau^{4/3}} + \frac{-3 + 2\pi^2 + 24\log(2) - 24\log^2(2)}{486\pi^3 A^3 \tau^2} + \ldots \right\}
\]

(61)

Furthermore, the proper-time dependence is exactly what is required by hydrodynamics. The temperature (determined by (61) and (50)) reads

\[
T(\tau) = \frac{\Lambda}{\tau^{1/3}} \left\{ 1 - \frac{1}{6\pi A} \cdot \frac{1}{\tau^{2/3}} + \frac{-1 + \log(2)}{36\pi^2 A^2} \cdot \frac{1}{\tau^{4/3}} + \frac{-21 + 2\pi^2 + 51\log(2) - 24\log^2(2)}{1944\pi^3 A^3} \cdot \frac{1}{\tau^2} \right\}
\]

(62)

Comparison with the static situation determines the coefficient \( e_0 \) in (50) to be \[ 63 \]:

\[
e_0 = \frac{3}{8} N_c^2 \pi^2
\]

(63)

Matching the energy density with (50) and (53) and using the linearized hydrodynamics to obtain the relaxation time from sound wave dispersion relation determines the transport coefficients of N=4 Yang-Mills plasma:

\[
\eta = \frac{1}{4\pi s}
\]

(64)

\[
\tau_\Pi = \frac{2 - \log 2}{2\pi T}
\]

\[
\lambda_1 = \frac{s}{8\pi^2 T}
\]

(65)

These expressions are based on terms up to second order. The formulae (61) and (62) include third order terms, but since the tensorial structure of the third order hydrodynamics has not been yet investigated \[ 10 \], the relevant transport coefficients cannot be calculated.

IV. HORIZONS IN THE BOOST-IN Variant SPACETIME

A. Preliminaries

With the tools from the last section in hand one can now turn to the identification and study of various notions of horizons in the spacetime defined by the bulk metric \[ 57 \]. Since the main goal is to study possible notions of entropy in the dual field theory, the hypersurfaces of interest are those which satisfy the symmetries of the boundary

\[ 10 \] It seems that there is no practical need to do so – the inclusion of second order hydrodynamic terms in RHIC simulations changes the results by just couple of percent \[ 40 \]. The third order terms will give even more suppressed contribution. However there are hints that in some cases resummation of hydrodynamic might be needed \[ 11 \].
dynamics under consideration (Bjorken flow). This singles out spacelike three-surfaces of constant \( \tau \) and \( r \). Such surfaces possess outward and inward pointing null normals

\[
\ell^a = \left[ \frac{\partial}{\partial \tau} \right]^a + \frac{1}{2} r^2 A(\tau, r) \left[ \frac{\partial}{\partial r} \right]^a,
\]

\[
n^a = -\left[ \frac{\partial}{\partial r} \right]^a.
\]

As usual, these vectors have been cross-normalized so that \( \ell \cdot n = -1 \). The rescaling freedom of \( \ell \rightarrow f(\tau) \ell \) and \( n \rightarrow n/f(\tau) \) remains but the specific scaling used here is chosen to be consistent with the flow of time at asymptotic infinity. This will be discussed in more detail below.

The hypersurfaces of interest will all lie within the \( \tau \) = constant hypersurfaces. Given that coefficients of the metric are given by the series \([58]\), solutions can be sought in the form

\[
r_S(\tau) = \frac{1}{\tau^{1/3}} \left( r_0 + \frac{1}{\tau^{2/3}} \right) + \frac{1}{2} \left( 3 \tau^4 (3r_1 + \delta_1 + 1) \Lambda^4 + 2\tau^3 r_0 \Lambda^3 + r_0^2 (3r_1 + \delta_1 + 1) \right).
\]

The coefficients \( r_k \) appearing here will be determined by the conditions imposed, and it turns out that the solutions are unique. Not only is the event horizon uniquely defined, but in this case demanding that the FOTH shares the symmetries of the spacetime means that we can also select a unique FOTH. In this section the focus will be on the FOTH and the event horizon, while in the following section more general surfaces of the form \([67]\) will play an essential role.

**B. The boost-invariant FOTH**

To find marginally trapped three-surfaces within the \( \tau \) = constant hypersurfaces, one needs to solve the equation \( \theta(\ell) = 0 \). Evaluating \( \theta(\ell) \) on a hypersurface of the form \([67]\) yields

\[
\theta(\ell) = \frac{1}{\tau^{1/3}} \left( \frac{3}{2} r_0 \left( 1 - \frac{\pi^4 \Lambda^4}{r_0^4} \right) + \frac{1}{\tau^{2/3}} \right) 3\pi^4 (3r_1 + \delta_1 + 1) \Lambda^4 + 2\tau^3 r_0 \Lambda^3 + r_0^2 (3r_1 + \delta_1 + 1) \right) \right).
\]

Solving \( \theta(\ell) = 0 \) (to third order) shows that there is a unique apparent horizon on the \( \tau \) = constant slices at:

\[
r_{AH}(\tau) = \frac{1}{\tau^{1/3}} \left\{ \frac{\pi \Lambda}{2} + \frac{1}{\tau^{2/3}} + \frac{1}{\Lambda^2 \tau^{4/3}} \left( \frac{\delta_3}{3} - \frac{1}{2} \right) + \frac{1}{\Lambda^2 \tau^{4/3}} \left( \frac{\Lambda \delta_3}{3} + \frac{1}{9\pi} - \frac{1}{24} - \frac{\log(2)}{18\pi} \right) + \frac{1}{\Lambda^4 \tau^{4/3}} \left( \frac{C}{18\pi^2} + \frac{\Lambda \delta_3}{3} - \frac{25}{432\pi} + \frac{25}{81\pi^2} - \frac{7776}{162\pi^2} + \frac{25 \log(2)}{162\pi^2} \right) \right\}.
\]

For the three-surface defined by \( r = r_{AH}(\tau) \), the inward expansion is

\[
\theta(n) = \tau^{1/3} \left\{ \frac{3}{\pi} - \frac{1}{2} \Lambda^2 \tau^{2/3} - \frac{1}{\Lambda^3 \tau^{4/3}} + \frac{1}{\Lambda^4 \tau^{4/3}} \left( \frac{7 \log(2)}{54\pi^4} - \frac{\log(2)}{24\pi^3} + \frac{25 \log(2)}{54\pi^4} - \frac{1}{2592\pi^2} + \frac{C}{6\pi^4} + \frac{35}{216\pi^4} \right) \right\},
\]

where \( C \) is Catalan’s constant. As expected, \([70]\) is independent of the constants \( \delta_k \) which reflect the gauge dependence of \([67]\). It is easy to see numerically that

\[
\theta(n) = -0.95 - 0.051 \Lambda^2 + 0.0010 \Lambda^4 - 0.00039 \Lambda^2\tau^2.
\]
This is clearly negative and will stay negative in a neighbourhood of \( r_{AH} \). Further it is clear from the expression for the outward expansion that for \( r \approx r_{AH}, r > r_{AH} \Rightarrow \theta(\ell) > 0 \) and \( r < r_{AH} \Rightarrow \theta(\ell) < 0 \). That is, there are fully trapped surfaces “just-inside” \( r = r_{AH} \) and so this marginally trapped surface bounds a fully trapped region and so can be identified as a black brane apparent horizon or FOTH.

The remaining geometric quantities discussed in section II can now be calculated. First, requiring that the evolution vector \( Y^a \) be tangent to the horizon, implies that in the large \( \tau \) expansion the expansion parameter \( C \) has the form

\[
C = C_{-1} + \frac{1}{\tau^{5/3}} C_0 + \frac{1}{\tau^{4/3}} C_1 + \frac{1}{\tau^2} C_2 + \frac{1}{\tau^{8/3}} C_3 + O\left(\frac{1}{\tau^{11/3}}\right),
\]

for some set of coefficients \( C_{-1}, \ldots C_3 \). In this case it is straightforward to see that \( C_{-1} = 0 \) identically in consequence of the structure of the large \( \tau \) expansion (even without using the explicit form of the solution). The coefficients \( C_0 \) (perfect fluid) and \( C_1 \) (viscosity) also turn out to vanish, so the leading contribution appears at order \( \frac{1}{\tau^2} \). All in all, on the \( r = r_{AH}(\tau) \) FOTH one finds

\[
C = \left(\frac{1}{9}\right) \frac{1}{\tau^2} - \frac{1}{\Lambda \tau^{8/3}} \left(\frac{\log(2)}{9\pi} - \frac{1}{54}\right) + O\left(\frac{1}{\tau^{11/3}}\right).
\]

This is clearly greater than zero and so the horizon is dynamical: that is spacelike and expanding. This can then be cross-checked in two ways. First one can directly calculate the volume element on the three-surfaces:

\[
\ddot{\varepsilon} = \sqrt{q} dy \wedge d\ell_1 \wedge d\ell_2
\]

where, up to third order

\[
\sqrt{q} = \pi^3 \Lambda^3 - \frac{1}{\tau^{2/3}} \frac{1}{2} \pi^2 \Lambda^2 + \frac{\Lambda}{\tau^{4/3}} \left(\frac{\pi \log(2)}{24} + \frac{\pi}{12}\right)

- \frac{1}{\tau^2} \left(\frac{5}{216} - \frac{\pi}{144} - \frac{\pi^2}{2592} + \frac{5 \log(2)}{216} - \frac{1}{24} \pi \log(2) - \frac{35 \log^2(2)}{216}\right)
\]

Then, to lowest order, the rate of expansion is

\[
\frac{1}{\sqrt{q}} \frac{d\sqrt{q}}{dr} = \frac{1}{\tau^{5/3}} \left\{ \frac{1}{3\pi \Lambda} + \frac{1}{\Lambda^2 \tau^{2/3}} \left( -\frac{\log(2)}{3\pi^2} - \frac{1}{18\pi} + \frac{1}{18\pi^2}\right) \right\}
\]

which is clearly positive. Alternatively

\[
\frac{1}{\sqrt{q}} \frac{d\sqrt{q}}{dr} = \frac{1}{\sqrt{q}} \mathcal{L}_V \sqrt{q} = -C\theta(n),
\]

and substituting in the appropriate values obtains the same result.

Other quantities are the squares of shears in the two null directions:

\[
\sigma^{(l)}_{ab} \sigma^{(l)}_{ab} = \frac{2}{3} \tau^{-2} \left( -\frac{2 \log(2)}{3\pi} - \frac{1}{9}\right)

+ \frac{1}{\Lambda^2 \tau^{10/3}} \left(\frac{35 \log^2(2)}{54\pi^2} + \frac{\log(2)}{6\pi} - \frac{14 \log(2)}{27\pi^2} - \frac{1}{27\pi} + \frac{1}{9\pi^2} + \frac{1}{648}\right)
\]

\[
\sigma^{(n)}_{ab} \sigma^{(n)}_{ab} = \frac{3}{\tau^{2/3}} \frac{1}{8\pi^4 \Lambda^2} + \frac{1}{\Lambda^5 \tau^{4/3}} \left(\frac{3 \log(2)}{8\pi^3} - \frac{1}{16\pi^4} + \frac{1}{2\pi^5}\right)

+ \frac{1}{\Lambda^6 \tau^2} \left(\frac{35 \log^2(2)}{96\pi^6} + \frac{3 \log(2)}{32\pi^5} - \frac{5 \log(2)}{6\pi^6} + \frac{1}{1152\pi^4} - \frac{1}{8\pi^3} + \frac{25}{48\pi^6}\right)
\]
and the accelerations/inaffinities in the two null directions:

\[
\kappa(\ell) = -n_a \ell^b \nabla_b \ell^a = \frac{1}{\tau^{1/3}} \left\{ 2\pi \Lambda - \frac{1}{3} \tau^{2/3} \right\} + \frac{1}{\Lambda \tau^{1/3}} \left( \frac{1}{9\pi} - \log(2) \right) \text{log}(2),
\]

(80)

\[
\kappa(n) = -\ell_a n^b \nabla_b \ell^a = 0
\]

(81)

Since \(\kappa(n)\) vanishes we have

\[
\kappa(V) = \kappa(\ell) - C\kappa(n) = \kappa(\ell).
\]

(82)

Finally, the connection on the normal bundle is

\[
\tilde{\omega}_a = -q_b n_c \nabla_a \ell^c = 0,
\]

(83)

and the (three-dimensional) Ricci scalar of the \(\tau = \text{constant}\) and \(v = v_{AH}\) three-surfaces is

\[
\tilde{R} = 0.
\]

(84)

### C. Boost-invariant flow and slow evolution

Having described the geometry of the FOTH in the boost-invariant spacetime, the next step is to try to understand its physics. Given that the metric (56) is a perturbation of a boosted black brane and that a boosted brane is simply a coordinate transformation of a static black brane spacetime, one would intuitively expect that the horizon should be in a near-equilibrium state. Thus the physics of these horizons should be quasi-equilibrium physics. Quantifying this intuition and then understanding its implications will be the subject of this section.

One can easily check that these conditions hold for the FOTH in the boost invariant black brane spacetimes. First,

\[
\epsilon^2 \approx 2\pi^2 \frac{\Lambda^2}{\tau^{1/3}} \left( 1 + O \left( \frac{1}{\tau^{2/3}} \right) \right)
\]

(85)

and the evolution parameter for the horizon is of the same order as the expansion parameter for the metric. Next (ii) and (iii) are trivially seen to hold: there is no dependence of the geometry or scaling of the null normals on \((w, x, y)\) and all of these quantities vanish. Thus the horizon is slowly expanding to order \(1/\tau^{2/3}\). Though it is a consequence of these conditions, in this case we can also explicitly check that the horizon is almost null in the sense of Eq. (35); the evolution of the three-slices is characterized by the expansion and shear associated with \(\ell^a\), just as they would be for a truly null surface.

One can now check these conditions (and their consequences) for the null scaling (66). Though it is clear that the conditions will hold for a variety of scalings, we make this particular choice so that “time” evolution on the horizon is consistent with that on the boundary (both are proportional to \(\tau\)) and, as a retroactive justification, the surface gravity matches the temperature on CFT side. Explicitly checking the conditions, the first part of (i) holds trivially since \(\tilde{\omega}_a = 0\) by equation (83) while

\[
\kappa_V = -\nabla_b n_a \ell^b = \kappa(\ell)
\]

(86)
by equation (81) and so from (80) the second part holds as well:
\[ \mathcal{L}_V \kappa_V \approx -\frac{2}{3} \pi \Lambda \frac{1}{\tau^{1/3}} \implies |\mathcal{L}_V \kappa_V| \lesssim \epsilon^2. \] (87)

Finally for (ii) a similarly trivial calculation from (70) shows that
\[ \mathcal{L}_V \theta_{(n)} \approx -\frac{1}{\pi \Lambda \tau^{2/3}} \implies |\mathcal{L}_V \theta_{(n)}| \lesssim \epsilon. \] (88)

Thus, the horizon is slowly evolving to order \( \epsilon \sim \tau^{-2/3} \). Then, in addition to it being geometrically “almost” null it is also mechanically close to equilibrium. As already noted, Einstein’s equations are sufficient to imply that the zeroth law of black hole mechanics (almost) holds: the surface gravity \( \kappa_V \) is approximately constant on each slice. Here, the symmetry of the black brane means that a stronger result is valid – from (80) one can see that the surface gravity is constant on each slice (though it evolves slowly up the horizon). The dynamical first law also automatically holds, but again in this case one can also check it explicitly. To order \( \tau^{-2} \) one has:
\[ \kappa_V \mathcal{L}_V \sqrt{\tilde{q}} \approx \sqrt{\tilde{q}} \| \sigma^{(\ell)} \|^2 \approx \frac{2}{3} \pi \Lambda \frac{3}{\tau^2}, \] (89)

and so, as expected, the expansion is driven by the null shears (which would normally suggest incoming gravitational radiation). The dependence on \( \tau \) in this expression matches what is expected in leading order \(^{11}\) on the basis of thermodynamics of Yang-Mills plasma. To see this, note that the volume in the boost invariant setting depends linearly on \( \tau \), so that the entropy of the plasma can be written as \( S = s \tau V_0 \) where \( V_0 \) is a reference volume and \( s \) is the entropy density \(^{51}\). Using the results of the previous section one then finds
\[ T \frac{dS}{d\tau} \sim \frac{1}{\tau^2} \] (90)
which is consistent with (89).

D. Event horizon

It is usually said that the event horizon is somewhat inconvenient to work with, since determining it requires knowing the entire future evolution of the spacetime under consideration. This is indeed an onerous requirement in the typical situation of computing the evolution of spacetime geometry starting from some initial data. The setting explored in this paper is in a sense complementary: the spacetime geometry is constructed order by order in a large proper-time expansion (or gradient expansion) starting in the far future at zeroth order. This circumstance makes it possible to determine the location of the event horizon in the late time regime.

The method of finding the event horizon for boost-invariant flow closely resembles the one presented in \(^{8}\) for the gravity duals to fluid dynamics \(^{7}\). The crucial assumption there was that the metric relaxes to (uniformly boosted) AdS Schwarzschild, where the position of the event horizon is well known. The event horizon for the metrics there was defined as a unique null surface which asymptotically coincides with the event horizon of the static AdS Schwarzschild dual to the uniform flow at constant temperature. Despite the fact that this is not the case for boost-invariant flow, it is still possible to find a unique null surface which can be interpreted as an event horizon. The key observation

\(^{11}\) Obviously leading order means first order in the gradient expansion, since entropy is preserved in the perfect fluid case.
is that the event horizon should coincide with the FOTH in the large-proper time regime and within the scaling limit\(^{12}\). Its radial position in AdS should depend on proper time only, which reflects the boost-invariance (no rapidity dependence) together with translational and rotational symmetry in the perpendicular directions (no \(\vec{x}_\perp\) dependence).

If \(r\) is the radial direction in AdS space, \(\tau\) the proper time and \(r_{EH}(\tau)\) expresses the time evolution of the horizon, then the equation defining the sought co-dimension one surface in AdS takes the form

\[
r - r_{EH}(\tau) = 0. \tag{91}
\]

The covector normal to the surface is \(dr - r'_{EH}(\tau)\,d\tau\) and requiring it is null gives the equation for \(r_{EH}(\tau)\)

\[
A(\tau, r_{EH}) \cdot r'_{EH} - 2r'_{EH} = 0, \tag{92}
\]

where for clarity the dependence of \(r_{EH}\) on \(\tau\) is omitted. This equation can be solved perturbatively in the scaling limit.

Using the solution described in \([26]\) (valid up to third order in the late proper-time expansion) one finds

\[
r_{EH} = \frac{1}{\tau^{1/3}} \left\{ \pi \Lambda - \left( \frac{1}{2} + \frac{\delta_1}{3} \right) \cdot \frac{1}{\tau^{2/3}} + \left( \frac{\delta_2}{3} + \frac{1}{6\pi \Lambda} - \frac{1}{24\Lambda} - \frac{\log(2)}{18\pi \Lambda} \right) \cdot \frac{1}{\tau^{1/3}} + \right. \\
+ \left. \left( \frac{\delta_3}{3} - \frac{\log(\Lambda)}{18\pi \Lambda^2} - \frac{2\log(\Lambda)}{27\pi^2 \Lambda^2} - \frac{29}{432\pi \Lambda^2} + \frac{C}{18\pi^2 \Lambda^2} - \frac{5}{324\pi^2 \Lambda^2} - \frac{1}{7776\Lambda^2} \right) - \right. \\
\right. \left. \frac{\log(\pi)}{18\pi \Lambda^2} + \frac{2\log(\pi)}{27\pi^2 \Lambda^2} + \frac{7\log^2(\pi)}{162\pi^2 \Lambda^2} - \frac{17\log(\pi)}{81\pi^2 \Lambda^2} \right) \cdot \frac{1}{\tau^{2}} \} . \tag{93}
\]

The dependence on arbitrary constants of integration reflects the gauge freedom – the position of the horizon is a gauge-dependent quantity \((r \rightarrow r + f(\tau))\) for arbitrary \(f(\tau))\). Comparing (69) with (93) it turns out that the FOTH coincides with the event horizon in the leading and first subleading orders. This is in agreement with the observation that the constant \(C\) is non-zero only in the second and higher orders in \(1/\tau^{2/3}\) expansion. The second orders differ and the FOTH becomes spatial.

It is also interesting to compare the position of the apparent and event horizon with the naive horizon defined as the hypersurface on which \(A(\tau, r)\) vanishes

\[
r_{\text{naive}} = \frac{1}{\tau^{1/3}} \left\{ \pi \Lambda - \frac{1}{\tau^{2/3}} \left( \frac{1}{3} + \frac{\delta_1}{3} \right) + \ldots \right\} . \tag{94}
\]

Such a hypersurface coincides with the event and apparent horizon in the leading order, but differs in the first subleading order – it is situated between the event horizon and the boundary. This means, for example, that truncating numerical simulations at the point where \(A(\tau, r)\) vanishes is simply wrong.

Finally, observe that in the naive limit \(\tau \rightarrow \infty\) the boost-invariant metric relaxes to the empty AdS\(_5\) metric instead of the static AdS-Schwarzschild solution. However, this is not so strange from the dual CFT point of view, where the fluid is expanding to infinity and its energy density becomes smaller and smaller. It means that the boundary system does not permanently thermalize to non-zero temperature. The interesting feature of the boost-invariant flow is an apparent thermalization, which expresses itself as an applicability of the equations of hydrodynamics in the late stages of the evolution.

\(^{12}\) The scaling limit insures that the higher derivative corrections to the geometry are small (MH thanks Toby Wiseman for discussion on this point)
E. Revisiting the scaling limit

The symmetries of boost-invariant flow make it possible to seek the location of the event horizon considering only the variables $r$ and $\tau$. It is possible then to focus only on the $dr,d\tau$ part of the full metric, which at leading order takes the form

$$ds^2 = 2d\tau dr - r^2 \left\{ 1 - \frac{\pi^4 A^4}{(r+1/3)^4} \right\} d\tau^2 + \ldots$$

(95)

The scaling limit discussed at length in section III involved introducing the scaling variable $v = r\tau^{1/3}$ which is kept fixed as $\tau \to \infty$. This motivates the following change of variables

$$\tau = \left( \frac{2u}{3} \right)^{3/2},$$

$$r = \sqrt{\frac{3}{2u}} v,$$

(96)

which leads to

$$ds^2 = 2dudv - v^2 \left\{ 1 - \frac{\pi^4 A^4}{v^4} \right\} du^2 + O \left( \frac{1}{u} \right) \ldots$$

(97)

This shows that this part of the metric looks precisely the same as the corresponding part of the static black brane metric, with $v$ denoting the radial coordinate and $u$ Eddington-Finkelstein ingoing time coordinate. This means that at leading order in the late-time expansion the problem of determining radial geodesics in the asymptotic boost-invariant geometry is the same as in the static case. It is then not surprising that the naive position of the horizon coincides asymptotically with the actual event horizon.

Note however that these considerations do not imply that the asymptotic geometry is static. Clearly, the remaining terms in the metric are time-dependent even after this coordinate transformation, even though the area of the event horizon remains constant in this regime.

V. PHENOMENOLOGICAL NOTIONS OF ENTROPY

A. Introduction

It is believed that equilibrium states of black holes are thermodynamic in nature. Their entropy is associated with the area of spacelike slices of the event horizon in an unambiguous way and the second law of thermodynamics is linked with Raychaudhuri’s equation. The property that the area of the event horizon is non-decreasing continues to hold in a generic dynamical setting. This prompts the question whether there is a sensible notion of entropy valid in such a non-equilibrium situation. However, as discussed in section III, there are hypersurfaces of non-decreasing area other than the event horizon (which coincide with it in the static case). The notion of entropy thus becomes less clear in these cases, as frequently discussed in the literature. The AdS/CFT correspondence makes it possible to

\footnote{As stressed previously, this is all that is needed to determine the location of the event horizon.}
view this problem from the gauge theory perspective. For example, the teleological nature of the event horizon leads to acausal behaviour of gauge theory entropy associated with it [12].

The bulk description is under control precisely when the field theory is strongly coupled, which in itself makes it hard to analyse directly. However it is well understood how to formulate a hydrodynamic description of the system in the appropriate regime. This description depends only on symmetries and the idea of the gradient expansion, as explained in [7, 35] for the conformal case and in [11] in general.

B. Entropy from second order hydrodynamics

The requirement that entropy should be non-decreasing during hydrodynamic evolution can be expressed in a covariant way in terms of an entropy current whose divergence is non-negative [42]. While the energy momentum tensor is a canonically defined operator, the entropy current is a derived notion. In the spirit of hydrodynamics (or effective field theory) it is also constructed in a gradient expansion as the sum of all possible terms at a given order. The dynamical equations of hydrodynamics are the conservation equations for the expectation value of the energy-momentum tensor. Thus, the coefficients appearing in the gradient expansion of the expectation value of the energy-momentum tensor (the transport coefficients) are the physical parameters of this phenomenological theory, since they figure directly in the evolution equations. They describe physical properties of the underlying quantum field theory. In contrast, the coefficients which appear in the phenomenological expression for the entropy current are constrained only by the requirement that its divergence be non-negative. These parameters are logically independent of the transport coefficients. At the present level of understanding they reflect a real ambiguity in the phenomenological notion of entropy current in hydrodynamics (as explained in the following subsection). This ambiguity is however of no consequence when entropy differences between equilibrium states are considered.

In the case of conformal fluids the most general form of the entropy current was recently constructed [8, 11] up to second order in gradients. The crucial symmetry requirement was Weyl covariance (see [43] for useful explicitly Weyl-covariant formulation of hydrodynamics). Using the notation of [11] (which descends from [8]) one has

$$S_{\mu}^{\text{non-eq}} = u_{\mu} + \frac{A_1}{4} S_1 u^{\mu} + A_2 S_2 u^{\mu} + A_3 \left( 4 S_3 - \frac{1}{2} S_1 + 2 S_2 \right) u^{\mu} + B_1 \left( \frac{1}{2} V_{\mu}^{\nu} + \frac{u^{\mu}}{4} S_1 \right) + B_2 \left( V_{\mu}^{\nu} - u^{\mu} S_2 \right) .$$

(98)

Here $s$ denotes the thermodynamic entropy density [51], and $S_{1,2,3}$ are the 3 possible conformal (Weyl-covariant) scalars

$$S_1 = \sigma_{\mu\nu} \sigma^{\mu\nu} , \quad S_2 = \Omega_{\mu\nu} \Omega^{\mu\nu} ,$$

$$S_3 = c_s^2 \nabla_{\mu} \nabla_{\nu} \ln s + \frac{4}{3} \nabla_{\mu} \ln s \nabla_{\nu} \ln s - \frac{1}{2} u_{\alpha} u_{\beta} R^{\alpha\beta} - \frac{1}{4} R + \frac{1}{3} (\nabla \cdot u)^2 ,$$

(99)

14 In the hydrodynamic formulas $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^{\mu} u^{\nu}$ is projector to the fluid’s local rest frame, whereas $\nabla_{\mu}^{\nu} = \Delta^{\nu\mu} \nabla_{\nu}$. Moreover fluid shear tensor (responsible for dissipation in the first order conformal hydrodynamics) reads $\sigma^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} \left( \nabla_{\alpha} u_{\beta} + \nabla_{\beta} u_{\alpha} \right) - \frac{1}{3} \Delta^{\mu\nu} \nabla_{\alpha} u^{\alpha}$ and vorticity (nonzero for rotating fluid) $\Omega^{\mu\nu} = \Delta^{\mu\alpha} \Delta^{\nu\beta} \left( \nabla_{\alpha} u_{\beta} - \nabla_{\beta} u_{\alpha} \right)$. 

and $V_{1,2}$ are 2 possible conformal vectors$^{15}$

$$V_1^\mu = \nabla_\alpha \sigma^{\alpha \mu} + 2 \epsilon_\alpha^\beta \sigma^{\alpha \mu} \nabla_\alpha \ln s - \frac{u^\mu}{2} \sigma_{\alpha \beta} \sigma^{\alpha \beta}, \quad V_2^\mu = \nabla_\alpha \Omega^{\mu \alpha} + u^\mu \Omega_{\alpha \beta} \Omega^{\alpha \beta},$$

(100)

where $\sigma$ is the hydrodynamic shear tensor and $\Omega$ is the vorticity, and $R$ is the boundary Ricci tensor.

The entropy current depends on 5 constants $A_{1,2,3}$ and $B_{1,2}$ and its divergence reads

$$\nabla_\mu \Sigma_{\text{non-eq}}^\mu = \frac{1}{2} \nabla_\mu \nabla_\nu \sigma^{\mu \nu} (-2A_3 + B_1) + \frac{3}{4} \nabla_\mu \sigma^{\mu \nu} \nabla_\nu \ln s (-2A_3 + B_1)
$$

$$+ \sigma_{\mu \nu} \left[ \frac{\eta}{2T} \sigma^{\mu \nu} + R^{\mu \nu} \left( -\frac{\kappa}{2T} + A_3 \right) + u_\alpha u_\beta R^{\alpha < \mu \nu \beta} \right]
$$

$$+ \frac{1}{4} \sigma^{\lambda \nu} \sigma^{\alpha \lambda} \left( \frac{2\lambda_1 - \eta \tau}{T} + A_1 - 2A_3 \right) + \frac{1}{3} \nabla_\perp \nabla_\perp < \mu \nu > \ln s \left( \frac{\eta \tau}{T} - A_1 - 2A_3 \right)
$$

$$+ \Omega_\alpha \Omega^{\alpha \mu} \left( -\lambda_3 + 2\eta \tau - 2A_2 - 2A_3 + B_1 \right)
$$

$$+ \sigma^{\mu \nu} \left( \nabla \cdot u \right) \left( \frac{2\eta \tau}{T} - 2A_1 + 6A_3 - 5B_1 \right) + \frac{1}{9} \nabla_\perp \nabla_\perp \Omega^{\nu >} \ln s \left( \frac{\eta \tau}{T} + A_1 + B_1 \right).$$

(101)

If the shear tensor is non-vanishing$^{16}$ the positivity of the shear viscosity $\eta$ should guarantee (see however$^{11}$ and the next footnote) that divergence of the entropy current is non-negative: higher order terms cannot change this conclusion as long as the gradient expansion is valid. However, as noted in$^{8}$, it is perfectly reasonable for $\sigma^{\mu \nu}$ to locally vanish (requiring this imposes just 5 conditions for derivatives of the four-velocity) in which case the higher order terms will dominate the entropy production. Positivity of (101) thus requires

$$B_1 = 2A_3.$$

(102)

Since the shear tensor $\sigma^{\mu \nu}$ is multiplying the whole square bracket in (101), in the case when it is 0 the whole contribution from first two orders is absent. At this level there is a real 4-parameter ambiguity in the hydrodynamic construction of the entropy current$^{17}$.

C. The case of Bjorken flow

From the perspective of the AdS/CFT correspondence it is natural to ask whether the ambiguities appearing in the construction of the hydrodynamic entropy current match on both sides of the duality. In a general situation this might be involved, but one may try to gain some insight into this question by considering a particular solution. The Bjorken flow provides a simple, highly symmetric, yet non-trivial example.

The current$^{8}$ evaluated on the boost-invariant solution given by the velocity $u^\mu = [\partial \tau]^\mu$ and temperature $T(\tau)$$^{53}$ takes the form

$$J^\mu = \delta u^\mu$$

(103)

$^{15}$ Note the absence of parity breaking terms present in$^{8}$. For discussion of hydrodynamics with parity breaking terms see$^{44}$.

$^{16}$ Note that $\sigma_{\mu \nu} \sigma^{\mu \nu}$ as a trace of the square of matrix cannot be negative and vanishes if and only if $\sigma^{\mu \nu} = 0$.

$^{17}$ Considerations of the case with an arbitrary small $\sigma^{\mu \nu}$ in$^{11}$ suggest that further constraints on the entropy current may be imposed. In particular, the only freedom left after such consideration is in the parameter $A_1$. It appears that these arguments rest on competition between terms of different orders in the gradient expansion. This paper cannot shed any light on this, since in the boost invariant case the shear is not close to vanishing.
with
\[
\tilde{s}(\tau) = s(T(\tau)) \left\{ 1 + 2 \frac{A_1 - A_3 + B_1}{3\pi^2 T(\tau)^2 \tau^2} \right\} \tag{104}
\]

In general the entropy current does not have to be proportional to the flow velocity beyond leading order (perfect fluid), but in the special case of boost-invariant flow non-leading order effects are captured by the single scalar function \(\tilde{s}(\tau)\). This function involves 3 of the 5 constants appearing in the general phenomenological construction. Changing the value of \(A_1\) has been identified with the freedom in choosing the horizon to boundary map \[8\]. The ambiguity parametrized by \(A_3\) was not interpreted in [8]. One would like to interpret this freedom in terms of allowed definitions of “horizon” on gravity side. Note that the example of Bjorken flow, while rather special, is still rich enough to partially capture this ambiguity.

In quantitative terms this ambiguity can be estimated as follows. In order for the hydrodynamic expansion of (104) to be valid the magnitude of \(|A_1 + A_3|\) should bounded so that the leading term dominates for times larger than \(\tau_{\text{min}}\) defined by (55). Expanding (104) up to second order one has
\[
\tilde{s}(\tau) \sim \frac{\Lambda^3}{\tau} \left\{ 1 - \frac{1}{2\pi \Lambda} \frac{1}{\tau^{2/3}} + \frac{(8(A_1 + A_3) + \log(2))}{12\pi^2 \Lambda^2} \frac{1}{\tau^{4/3}} \right\} \tag{105}
\]
Demanding that the second order contribution be smaller than the first order correction by a factor of \(\alpha \beta\) at \(\tau = \tau_{\text{min}}\) leads to the bound
\[
\frac{1}{8} (\beta - \log(2)) < A_1 + A_3 < \frac{1}{8} (\beta - \log(2)) \tag{106}
\]
where \(\beta\) is at most of order \(1/\alpha\). This provides a rough estimate of the allowed indeterminacy in the phenomenological notion of non-equilibrium entropy as defined by (104):
\[
\tilde{s}(\tau) \sim \frac{\Lambda^3}{\tau} \left\{ 1 - \frac{1}{2\pi \Lambda} \frac{1}{\tau^{2/3}} \pm \frac{\beta}{12\pi^2 \Lambda^2} \frac{1}{\tau^{4/3}} \right\} \tag{107}
\]
One would like to understand this quantitively in terms of the freedom of defining entropy on the gravity side.

D. Entropy from gravity

As reviewed in section II, in a dynamical setting it is no longer clear if there is an appropriate geometrical notion which can be used for the definition of entropy. For example both apparent and event horizons appear to give rise to notions of entropy, which satisfy the second law of thermodynamics and coincide in equilibrium. This provides motivatation to look more generally at the dynamics of hypersurfaces whose area is non-decreasing. The starting point should be equation (13). This formula determines the rate of change of the area element of a general 3-hypersurface in the terms of the expansions \(\theta_{(n)}, \theta_{(\ell)}\) and the expansion parameter \(C\). In the boost-invariant case the 3-hypersurfaces consistent with the boundary symmetry have the form (67). Their area is given by
\[
A = \pi^3 \Lambda^3 \left\{ 1 + \frac{3\hat{r}_1 + 1}{\pi \Lambda \hat{r}^{2/3}} + \frac{36\hat{r}_2^2}{12\pi^2 \Lambda^2 \hat{r}^{4/3}} + 24\hat{r}_1 + 36\hat{r}_2 + 2\pi + 5 \log(2) \right\} \tag{108}
\]
where
\[
\hat{r}_1 = r_1 - \frac{1}{3} \delta_1
\]
\[
\hat{r}_2 = r_2 + \frac{1}{3} \delta_2 \tag{109}
\]
The notion of entropy defined by such hypersurfaces is

$$S = \frac{N_c^2}{2\pi} \Lambda$$  \hspace{1cm} (110)

(since in the units used here (AdS radius set to 1) \( G_N^{-1} = 2\pi^{-1} N_c^2 \)).

Equation (103), which expresses the change of area of the hypersurface sections becomes

$$\mathcal{L}_V \sqrt{q} = \sqrt{q}(\theta(t) - C\theta(u)) = -\frac{6\tilde{r}_1 + 2}{3\pi \Lambda} \frac{1}{\tilde{r}^{5/3}} + \frac{18\tilde{r}_1^2 + 12\tilde{r}_1 - 2\pi(18\tilde{r}_2 + 1) + 6 - 5\log(2)}{9\pi^2 \Lambda^2} \frac{1}{\tilde{r}^{5/3}}$$  \hspace{1cm} (111)

In this formula \( r_0 \) has been set to \( \pi \) to match the thermodynamic entropy when all gradient corrections are discarded. For the leading term (at order \( 1/\tilde{r}^{5/3} \)) to be non-negative one gets the bound \( \tilde{r}_1 < -1/3 \), and then requiring that the following term be smaller gives an allowed range for \( r_2 \). At this level of analysis this is all one gets; \( r_1 \) is not fixed\(^{18}\).

Both notions of entropy (defined via the event or apparent horizon) provide unique \( r_1 \), which lies in the allowed range.

The entropy density\(^{19}\) obtained from the third order expression for the event horizon reads

$$s_{EH}(\tilde{r}) = \frac{1}{2} N_c^2 \pi^2 \Lambda^3 \left\{ 1 - \frac{1}{2\pi \Lambda} \frac{1}{\tilde{r}^{2/3}} + \frac{6 + \pi + 6 \log(2)}{24\pi^2 \Lambda^2} \frac{1}{\tilde{r}^{4/3}} - \frac{420 + 90\pi + \pi^2 + 372 \log(2) + 108\pi \log(2) + 420 \log^2(2)}{2592\pi^3 \Lambda^3 \tilde{r}^2} \right\}$$  \hspace{1cm} (112)

In the dynamical case the result for the entropy density reads

$$s_{AH} = \frac{1}{2} N_c^2 \pi^2 \Lambda^3 \frac{1}{\Lambda} \left\{ 1 - \frac{1}{2\pi \Lambda} \frac{1}{\tilde{r}^{2/3}} + \frac{1}{\Lambda^2 \tilde{r}^{4/3}} \left( \frac{\log(2)}{4\pi^2} + \frac{1}{24\pi} + \frac{1}{12\pi^2} \right) \right\} - \frac{1}{\Lambda^3 \tilde{r}^2} \left( \frac{35 \log^2(2)}{216 \pi^3} + \frac{\log(2)}{24 \pi^2} + \frac{5 \log(2)}{216 \pi^3} \right) - \frac{1}{2592\pi} - \frac{1}{144\pi^2} - \frac{5}{216\pi^3}$$  \hspace{1cm} (113)

Numerically one finds

$$s_{AH} = \frac{1}{2} N_c^2 \pi^2 \Lambda^3 \frac{1}{\Lambda} \left\{ 1 - \frac{1}{\tilde{r}^{2/3}} \frac{0.16}{\Lambda} + \frac{1}{\tilde{r}^{4/3}} \frac{0.039}{\Lambda^2} - \frac{1}{\tilde{r}^2} \frac{0.0065}{\Lambda^3} \right\}$$  \hspace{1cm} (114)

For the event horizon

$$s_{EH} = \frac{1}{2} N_c^2 \pi^2 \Lambda^3 \frac{1}{\Lambda} \left\{ 1 - \frac{0.16}{\Lambda^{2/3}} + \frac{0.056}{\Lambda^{4/3}} - \frac{0.018}{\Lambda^2} \right\}$$  \hspace{1cm} (115)

The key observation is that the event and apparent horizons coincide at the leading and first subleading orders, which is a hint that also in the case of a general surface there should be no ambiguity until the second subleading order. If one is to identify the entropy defined here with the field theory observable, as required by the AdS/CFT correspondence, then it should be Weyl covariant in the boundary sense. To do this explicitly one would need to solve the Einstein equations with the boundary metric given by

$$ds^2 = e^{-2\omega(\tau)} \left\{ -d\tau^2 + \tau^2 dy^2 + dx^2 \right\}$$  \hspace{1cm} (116)

---

\(^{18}\) Note in particular that there are surfaces outside the event horizon which are acceptable from this point of view.  
\(^{19}\) Entropy density is understood as entropy per unit volume, which in the proper-time – rapidity coordinates involves a factor of \( \tau \).
where $\omega(\tau)$ is a conformal factor having the form of an expansion in powers of $1/\tau^{2/3}$. The entropy computed this way would be Weyl covariant (i.e. proportional to the appropriate power of the conformal factor) only for $\tilde{r}_1 = -1/2$, i.e. the value assumed by $\tilde{r}_1$ in the case of the event or apparent horizon (which coincide at this order). The quick way to get this answer is to write the entropy in terms of temperature and velocity, whose transformation rules under Weyl rescalings are known. This procedure parallels the field theory analysis reviewed earlier. The first step is factor out the thermodynamic entropy which sets the Weyl transformation property of the entropy density. This leads to

$$s = \frac{1}{2} N_c^2 \pi^2 T(\tau)^3 \left\{ 1 + \frac{6\tilde{r}_1 + 3}{2\pi^2/3\Lambda} + \frac{36\tilde{r}_1^2 + 42\tilde{r}_1 + 36\pi\tilde{r}_2 + 2\pi + 9 + 4\log(2)}{12\pi^2\tau^{4/3}\Lambda^2} \right\} \quad (117)$$

Since there are no Weyl-covariant scalars nor vectors at first order in derivatives, the only way that this formula can be Weyl covariant is if the first subleading term vanishes, which determines $\tilde{r}_1 = -1/2$. When this result is used in eq. (111), one finds

$$L V \sqrt{q} = \sqrt{q}(\theta(\ell)) - C\theta(n) = \frac{1}{3\pi\Lambda} \frac{1}{\tilde{\tau}^{5/3}} + \frac{9 - 4\pi(18\tilde{r}_2 + 1) - 10\log(2)}{18\pi^2\Lambda^2} \frac{1}{\tilde{\tau}^{7/3}} \quad (118)$$

The leading contribution ensures positive entropy production due to the shear viscosity so there are no further constraints on $\tilde{r}_2$. The appearance of possible freedom in choosing $\tilde{r}_2$ can be understood following again the hydrodynamic argument presented in subsection (V B). In the first place, note that the formula (118) is evaluated using the bulk Eddington-Finkelstein proper-time $\tilde{\tau}$, which raises the question about its relation with boundary proper-time coordinate $\tau$. Such a mapping freedom has been addressed using Weyl-covariant language in [8] and amounts to the trivial mapping in the first two orders of the gradient expansion, with an ambiguity showing up at the second order. In the case of boost-invariant flow the most general mapping (up to second order) takes the form

$$\tilde{\tau} \longrightarrow \tau(1 + \frac{\delta A_1}{\Lambda^2 \tau^{4/3}}) \quad (119)$$

where $\delta A_1$ is a constant parameter multiplying Weyl-covariant scalar $S_1$. For such mappings to make sense within the context of the gradient expansion this parameter must be suitably bounded as explained earlier. This shifts the second order contribution in (114) and (115), so it must be of order $\alpha^2$ in the sense of (55), which leads to

$$\delta A_1 = \pm \frac{\gamma}{32\pi^2} \quad (120)$$

with $\gamma$ of order 1. This mapping freedom accounts only for a part of the ambiguity in the phenomenological construction of the hydrodynamic entropy current. The rest of the ambiguity can be understood following the Weyl analysis of the properties of gravitational entropy. At second order in gradients there are two Weyl covariant scalars ($S_1$ and $S_3$) and one vector ($V_1$) which are non-trivial when evaluated on the boost-invariant solution, with the vector being proportional to velocity in that particular case. Since mapping freedom is identified with the $S_1$ contribution, it is clear the $\tilde{r}_2$ must come from the relevant combination of $S_3$ and $u_\mu V^\mu$. Evaluating (117) with $\tilde{r}_1 = -1/2$ gives

$$s = \frac{1}{2} N_c^2 \pi^2 T(\tau)^3 \left\{ 1 + \frac{-36\pi\tilde{r}_1 + 2\pi + 4\log(2)}{12\pi^2\tau^{4/3}\Lambda^2} \right\} \quad (121)$$

It is also reasonable to stipulate that $A_1$ is partly linked with the mapping freedom and partly with a freedom in choosing between different notions of horizons in the bulk.
Comparing this with (104) one finds
\[ \tilde{r}_2 = \frac{2}{9\pi} (A_1 - A_3 + B_1) + \frac{1}{12\pi} - \frac{1}{18} - \frac{\log(2)}{9\pi} \] (122)

Although this is all one can get from the analysis of the gravity dual to the boost-invariant flow\[^{21}\] it is reassuring that at least in this case the gravity picture is capable of capturing the ambiguities of the boundary phenomenological construction.

### E. A phenomenological definition of black brane entropy

The freedom in the definition of the hydrodynamic entropy current on the gravity side follows not from the various possible notions of horizon, but rather from adopting the phenomenological construction in the bulk, which is analogous to the boundary one. The surfaces considered are not horizons in any of the usual senses, but they do have the property that their area increases, and that is all that is required if one adopts the slowly-evolving geometry approach. In order to understand this in greater generality one would of course need to go beyond the Bjorken flow example and consider the equation (13) evaluated on the gravity dual to the general hydrodynamics. This is certainly possible employing the Weyl-covariant formulation in the bulk. It would be very interesting see this in detail.

### VI. CONCLUSIONS AND OUTLOOK

The main goal of this article was to explore the relationship between the notions of entropy on both sides of AdS/CFT duality. This lead to a phenomenological definition of black brane entropy on the gravity side, which was inspired by the corresponding construction in hydrodynamics. In the case of Bjorken flow the freedom inherent in this definition accounts for the entire ambiguity appearing in the hydrodynamic entropy current in this case. This leads to an understanding why the event horizon coincides in the leading and first non-leading orders of the gradient expansion with the unique apparent horizon compatible with the boundary flow. The origin of this circumstance is the Weyl invariance of the boundary theory. It would be very interesting to understand the form of general hydrodynamic entropy current from the bulk perspective and carefully understand the sources of possible freedom in such definition.

It is natural to ask what is the physical relevance of the potential ambiguity in the definition of entropy current. In the case of local entropy production such an ambiguity signals lack of physical meaning. This however should not be disturbing, because the thermodynamic notion of entropy makes sense only in equilibrium. Since one expects that systems described by hydrodynamics equilibrate due to dissipative effects, the total entropy can be calculated in the late stages of evolution and is given by the thermodynamic entropy. On the gravity side this translates to the notion of isolated horizons as those for which entropy can be defined precisely. The framework of slowly evolving geometry is especially useful here, since it quantifies how far from equilibrium the black brane is by examining the validity of the first law of thermodynamics. This tool can be used as a probe of when and where the strongly coupled boundary quantum field theory is close to equilibrium.

\[^{21}\] Note that in hydrodynamics due to the requirement of positive divergence of the entropy current, \( A_1 \) and \( B_1 \) contributions to the entropy current are not independent, but rather linked by the equation (103). However such considerations were done under the assumption that the shear tensor vanishes locally, which is never the case in the Bjorken picture unless all the dissipative contributions are negligible.
Apart from some more or less obvious generalizations it would be interesting to explore these ideas in the context of equilibration of the boundary quantum field theory perturbed out of equilibrium by localized sources (in the spirit of [12, 36, 45]). This application of the slowly evolving horizons framework is especially interesting because in the case of planar horizons considered here there can be widely separated regions, of which some are in local equilibrium while others are not.

On the gravity side these ideas also raise some interesting questions. Throughout this paper we have often referred to various surfaces being “close” to a future outer trapping horizon (FOTH). In practice this has always meant close along surfaces of constant Eddington-Finkelstein coordinate $v$ in terms of the radial coordinate $r$ of the adapted coordinate system. From a purely geometric point of view this is not really satisfactory; notions of distance shouldn’t be formulated in terms of an arbitrary coordinate system. Thus, more thought needs to be given to properly understanding whether there is an invariant way to define this closeness. One such idea has been addressed in [28] where the authors considered the minimum proper time interval between a spacelike FOTH and a null event horizon. However the results are not entirely satisfactory and work remains to be done. Ideally one would like to obtain a general proof showing that in the vicinity of a slowly evolving horizon there would always be candidate event and alternate horizons which are arbitrarily close in a well-defined manner. How to do this is currently an open problem.

This work also sheds some light on the long-standing discussion as to whether it is more “correct” to consider event or apparent horizons in physical situations. Instead of there being just two choices, we now propose that are many more and the uncertainty as to which one should considered actually reflects a physical ambiguity in the proper definition of entropy and other quantities such as energy as one moves away from equilibrium. Again these issues deserve further investigation.

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APPENDIX A: DERIVATION OF THE CONSTRAINT EQUATIONS

In this appendix we consider the derivations of the geometric constraints mentioned at the end of section II B. A more complete discussion of almost all of the results mentioned in this section can be found in [20].

To begin we write $\mathbf{V}^a$ in the more general form

\[ \mathbf{V}^a = A\ell^a - C_n^a, \]  

(A1)

which will help to display symmetries in the equations. In the main text $A$ is set back to unity.

Now, in finding the equations constraining the derivatives of the extrinsic geometry the key piece of information that
must be used is that \( \ell_a \) and \( n_a \) are always normal to the \( S_\lambda \). Thus

\[
\frac{d}{dt} \mathbf{L}_V \mathbf{\ell}_b = 0 \implies V^b \nabla_b \mathbf{\ell}_a = -(d_a - \tilde{\omega}_a)C + \kappa_V \mathbf{\ell}_a \quad \text{(A2)}
\]

\[
\frac{d}{dt} \mathbf{L}_V \mathbf{n}_b = 0 \implies V^b \nabla_b \mathbf{n}_a = (d_a + \tilde{\omega}_a)A - \kappa_V \mathbf{n}_a \quad \text{(A3)}
\]

The importance of these two relations is that they can be used to replace covariant derivatives in the \( V^a \) direction with derivatives in the tangent direction plus a part proportional to \( \kappa_V \): the \( V^a \) component of the connection on the normal bundle.

It is then possible to apply these relations to find expressions for the rate of change of components of the extrinsic geometry. For \( \theta(\ell) \) one proceeds in the following manner. First,

\[
\mathbf{L}_V \theta(\ell) = \nabla^c \theta(\ell) = (V^c \nabla_c \tilde{q}^{ab}) \nabla_a \mathbf{\ell}_b + \tilde{q}^{ab} V^c (\nabla_c \nabla_a \mathbf{\ell}_b). 
\]

Expanding the first term with \( \text{(3)} \) and then applying \( \text{(A2)} \) and \( \text{(A3)} \) gives

\[
(V^c \nabla_c \tilde{q}^{ab}) \nabla_a \mathbf{\ell}_b = (d^b A + \tilde{\omega}^b A)(\ell^a \nabla_a \mathbf{\ell}_b) - (d^b C - \tilde{\omega}^b C)(n^a \nabla_a \mathbf{\ell}_b) + (d^b C - \tilde{\omega}^b C)\tilde{\omega}_b.
\]

while for the second term, the Riemann tensor can be used to commute the indices of the double derivative to obtain

\[
\tilde{q}^{ab} V^c (\nabla_c \nabla_a \mathbf{\ell}_b) = \tilde{q}^{ab} V^c (R_{cabd} \ell^d + \nabla_a \nabla_c \mathbf{\ell}_b). 
\]

Next, applying the definition of the Einstein tensor \( G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \) along with the Gauss relation

\[
\tilde{q}^{ab} V^c (R_{cabd} \ell^d + \nabla_a \nabla_c \mathbf{\ell}_b) = \tilde{R}_{abcd} + (k_{ac}^{\ell(t)} k_{bd}^{(n)} + k_{ac}^{(n)} k_{bd}^{\ell(t)}) - (k_{bc}^{\ell(t)} k_{ad}^{(n)} + k_{bc}^{(n)} k_{ad}^{\ell(t)}),
\]

where \( \tilde{R}_{ab} \) is Riemann tensor associated with the \( (n - 1) \)-dimensional \( \tilde{q}_{ab} \), the Riemann term of \( \text{A6} \) becomes

\[
\tilde{q}^{ab} V^c (\Delta_{cabd} \ell^d - C G_{ab} \ell^a n^b) + \frac{1}{2} R + \theta(\ell) \theta(n) - k_{ab}^{\ell(t)} k_{ab}^{(n)}. 
\]

Meanwhile, again with the help of \( \text{A2} \), the other term is

\[
\tilde{q}^{ab} V^c \nabla_a \nabla_c \mathbf{\ell}_b = -d^2 A + d_a(C\tilde{\omega}^a) + \kappa_V \theta(\ell) - k_{ab}^{\ell(t)} k_{ab}^{(n)}
\]

\[
- (d^b A + \tilde{\omega}^b A)(\ell^a \nabla_a \mathbf{\ell}_b) + (d^b C - \tilde{\omega}^b C)(n^a \nabla_a \mathbf{\ell}_b).
\]

Combining these results there is some cancellation and so one gets

\[
\mathbf{L}_V \theta(\ell) = \kappa_V \theta(\ell) - d^2 C + 2\tilde{\omega}^a d_a C - C \left[ \|\tilde{\omega}\|^2 - d_a \tilde{\omega}_a - \tilde{R}/2 + G_{ab} \ell^a n^b - \theta(\ell) \theta(n) \right]
\]

\[
- A \left[ k_{ab}^{\ell(t)} k_{ab}^{(n)} + G_{ab} \ell^a n^b \right]. 
\]

where \( \|\tilde{\omega}\|^2 = \tilde{\omega}_a \tilde{\omega}^a \). In this form the relation is independent of the spacetime dimension, however, if one decomposes the extrinsic curvature term on the second line with \( \text{A3} \) the dimension appears

\[
\mathbf{L}_V \theta(\ell) = \kappa_V \theta(\ell) - d^2 C + 2\tilde{\omega}^a d_a C - C \left[ \|\tilde{\omega}\|^2 - d_a \tilde{\omega}_a - \tilde{R}/2 + G_{ab} \ell^a n^b - \theta(\ell) \theta(n) \right]
\]

\[
- A \left[ \frac{\theta^2(\ell)}{n-1} + \|\sigma(\ell)\|^2 + G_{ab} \ell^a n^b \right].
\]
where and \( \| \sigma^{(t)} \|^2 = \sigma_{ab}^{(t)} \sigma_{(t)}^{ab} \). Note in particular that the only quantity not directly defined on \( S_v \) is the gauge-dependent \( \kappa_V \).

The rate of change of the inward expansion may be obtained simply by exchanging \( \ell \) and \( n \) and swapping \( A \leftrightarrow -C \) in \( (18) \). Then

\[
L_V \theta_{(n)} = -\kappa_V \theta_{(n)} + d^2 A + 2 \bar{\omega}^a d_a A + A \left[ \| \bar{\omega} \|^2 + d_a \bar{\omega}^a - \bar{R}/2 + G_{ab} n^a n^b - \theta_{(t)} \theta_{(n)} \right] + C \left[ \theta_{(n)}^2 \frac{n}{n-1} + \| \sigma^{(n)} \|^2 + G_{ab} n^a n^b \right]. 
\]

(A12)

A similar calculation using \( (A2) \) and \( (A3) \) finds the derivatives of the expansions pulled back onto the \( S_v \). First again using the Riemann tensor to commute derivatives one can show the Codazzi relations

\[
\bar{q}_a \bar{q}_b \bar{q}_c \bar{q}_d \bar{R}_{efgh} = (d_a - \bar{\omega}_a) k_{bc}^{(t)} - (d_b - \bar{\omega}_b) k_{ac}^{(t)} \\
\bar{q}_a \bar{q}_b \bar{q}_c \bar{q}_d \bar{R}_{efgh} = (d_a + \bar{\omega}_a) k_{bc}^{(n)} - (d_b + \bar{\omega}_b) k_{ac}^{(n)}.
\]

(A13)

Now, the extrinsic curvatures can be decomposed into expansions and shears and while the Riemann tensor can be decomposed into Weyl and Ricci components using the well-known relation \( [18] \):

\[
\bar{R}_{abcd} = C_{abcd} + \frac{2}{n-1} (g_{a[c} \bar{R}_{d]} + g_{b[d} \bar{R}_{c]} - \frac{2}{n(n-1)} (g_{a[c} g_{d]}),
\]

(A14)

where \( C_{abcd} \) is the Weyl tensor, square brackets indicate anti-symmetrization and the reader should keep in mind that we are considering an \((n + 1)\)-dimensional spacetime. Then contracting with \( \bar{q}^{bc} \) we find

\[
d_a \theta_{(t)} = \theta_{(t)} \bar{\omega}_a 2(d_a - \bar{\omega}_a) \sigma_{a}^{(t)} - \frac{1}{n-1} \bar{q}_a^b G_{bc} \ell^c - 2 \bar{q}_a^b C_{bcde} \ell^c \ell^d n^e \quad \text{and} \\
d_a \theta_{(n)} = -\theta_{(n)} \bar{\omega}_a 2(d_a + \bar{\omega}_a) \sigma_{a}^{(n)} - \frac{1}{n-1} \bar{q}_a^b G_{be} n^c + 2 \bar{q}_a^b C_{bcde} n^c \ell^d n^e.
\]

(A15)

(A16)

The final equation of interest for this paper is the rate of change of the angular momentum one-form up \( \Delta \). This comes from a direct expansion of \( \bar{q}_a^b L_V (n_c \nabla_b \ell^c) \) along with the familiar applications of the Riemann tensor to commute derivatives, \( (A2) \) and \( (A3) \) to enforce the fact that the \( S_v \) fit together smoothly into a surface, and then breaking the Riemann tensor into Weyl and Ricci components. We find

\[
L_V \bar{\omega}_a = d_a \kappa_V - k_{a}^{(t)} \left[ d^b A + \bar{\omega}^b A \right] + k_{a}^{(n)} \left[ d^b C + \bar{\omega}^b C \right] + \bar{q}_a^b \left[ \frac{1}{n-1} G_{bc} (\ell^c + C n^c) - C_{bcde} \ell^e n^c \right],
\]

(A17)

As shown in the main text, these equations lie at the root of black hole mechanics and dynamics.
The second order solution \[26, 27\] for the coefficients in (57) reads

\begin{align}
A_2 (v) &= -\frac{2\pi^2 \log(v) \Lambda^2}{3v^4} + \frac{\pi^2 \log \left( \frac{v^2}{\pi^2 \Lambda^2} + 1 \right) \Lambda^2}{3v^4} - \frac{2\pi^2 \log(\pi \Lambda) \Lambda^2}{3v^4} - \\
&\quad - \frac{\pi^2 \log(2) \Lambda^2}{9v^4} + \frac{(v^4 - 3\pi^4 \Lambda^4) \delta_1^2}{9v^6} - \frac{-2v^4 + \pi^2 \Lambda^2 v^2 + 4\pi^3 \Lambda^3 v + 2\pi^4 \Lambda^4}{6v^6} + \\
&\quad + \frac{2(3v^4 - 2\pi^3 \Lambda^3 v - 3\pi^4 \Lambda^4) \delta_1}{9v^6} - \frac{2 \left( v^4 + \pi^4 \Lambda^4 \right) \delta_2}{3v^5} + \\
&\quad + \frac{(v^4 + \pi^4 \Lambda^4) \tan^{-1} \left( \frac{\Lambda}{v} \right)}{3\pi v^2 \Lambda} \tag{B1}
\end{align}

\begin{align}
b'_2 (v) &= -\frac{8 \log(v) v^2}{9\pi^5 \Lambda^5 - 9\pi v^4 \Lambda} + \frac{8 \log(\pi \Lambda) v^2}{9\pi^5 \Lambda^5 - 9\pi^4 v^4 \Lambda} - \frac{(7v^4 - 4\pi^3 \Lambda^2 v - 3\pi^4 \Lambda^4) \tan^{-1} \left( \frac{\Lambda}{v} \right)}{9\pi \Lambda \left( v^6 - \pi^4 v^4 \Lambda^4 \right)} + \\
&\quad + \frac{4\pi^2 \Lambda^2 \log(2)}{9\pi^4 v^4 \Lambda^4 - 9\pi v^6} + \frac{2 \left( v^2 + \pi \Lambda v + \pi^2 \Lambda^2 \right)}{9\pi \left( v + \pi \Lambda \right) \left( v^2 + \pi^2 \Lambda^2 \right) v} - \frac{2 \left( v^2 - \pi \Lambda v + \pi^2 \Lambda^2 \right) \log \left( \frac{v^2}{\pi^2 \Lambda^2} + 1 \right)}{9\pi \left( v - \pi \Lambda \right) (v^2 + \pi^2 \Lambda^2) v} + \\
&\quad + \frac{4 \left( v^2 + \pi \Lambda v + \pi^2 \Lambda^2 \right) \log \left( \frac{v}{\pi \Lambda} + 1 \right)}{9\pi \left( v + \pi \Lambda \right) \left( v^2 + \pi^2 \Lambda^2 \right) v} + \frac{2\delta_2}{3v^2} + \frac{2\delta_2}{9v^3} + \\
&\quad - \frac{11v^6 + 22\pi \Lambda v^5 + 34\pi^2 \Lambda^2 v^4 + 60\pi^3 \Lambda^3 v^3 + 43\pi^4 \Lambda^4 v^2 + 26\pi^5 \Lambda^5 v + 16\pi^6 \Lambda^6}{9(v + \pi \Lambda)^2 (v^2 + \pi^2 \Lambda^2)^2 v^3} + \\
&\quad + \frac{4 \left( 3v^6 + 6\pi \Lambda v^5 + 9\pi^2 \Lambda^2 v^4 + 7\pi^3 \Lambda^3 v^3 + 5\pi^4 \Lambda^4 v^2 + 3\pi^5 \Lambda^5 v + \pi^6 \Lambda^6 \right) \delta_1}{9(v + \pi \Lambda)^2 (v^2 + \pi^2 \Lambda^2)^2 v^3} \tag{B2}
\end{align}

\begin{align}
c_2 (v) &= d_2 (v) - b_2 (v) \tag{B3}
\end{align}

\begin{align}
d_2 (v) &= -\frac{\delta_2^2}{60v^2} + \frac{\delta_1}{3v^2} - \frac{v - \pi \Lambda}{6\pi v^2 \Lambda} \frac{(v - 3\pi \Lambda) \tan^{-1} \left( \frac{v}{\pi \Lambda} \right)}{6\pi v^2 \Lambda} + \frac{2 \log(v)}{3\pi^2 \Lambda^2} + \\
&\quad + \frac{\log \left( \frac{v^2}{\pi^2 \Lambda^2} + 1 \right)}{4\pi^2 \Lambda^2} + \frac{\log \left( \frac{v}{\pi \Lambda} + 1 \right)}{6\pi v^2 \Lambda} - \frac{2 \log(\pi \Lambda)}{3\pi^2 \Lambda^2} - \frac{\delta_2}{v} + \frac{1}{12\pi \Lambda} \tag{B4}
\end{align}

The expression for $b'_2$ given above can be integrated in terms of polylogarithmic functions, but the explicit form of the integral will not be needed in the sequel. The quantities $\delta_1$ (and $\delta_{2,3,\ldots}$ in higher orders) are integration constants related to residual gauge symmetry $r \to r + R (\tilde{r})$ mentioned earlier.

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