On mesofractal interpretation of the galaxy spatial arrangement

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Abstract. The term mesofractal is used here for a system of random points arrangement of which reveals the fractal character on small scales and looks as uniform on large scales. The way of simulation of such systems is based on the random jump process with Lévy-Feldheim stable distributions taken as transition densities, and the Poisson point process for the origin points of the independent trajectory ensemble. Such model successfully represents main statistical properties of the observed part of the universe and may be useful for testing estimation procedures. This article contains also a short discussion of evolution of cell-count data between fractal and uniform domains.

1. Introduction

When Einstein first applied the general theory of relativity to the Universe, he made one significant simplification: suggestion, that it is homogeneous and isotropic. There are no regions with large density and voids and on large scales the Universe looks approximately the same in all directions and for any chosen coordinate origin. This cosmological principle of Einstein formed the basis of all models of the Universe, including the Big Bang and the stationary model of the Universe, which once competed with it. The discomfort caused by the conflict with the observed heterogeneity of visible matter appearing to the observer in the form of clusters of stars (galaxies) randomly scattered in space, clusters of star clusters, etc., is avoided by the implicit assumption that this principle applies to scales over $10^8$ light years. In general, the term average simplifies perception even more, justifying the transition to a continuous (and even differentiable!) model of the medium, the movement of which can already be described within the framework of the mathematical apparatus of the general theory of relativity. However, the concept of average, like the fluctuations, correlations, etc. related to it by the probable origin, makes sense only when establishing the ensemble over which averaging is performed. The construction of such an ensemble is a very difficult task, although in the molecular theory of gases it often comes down to simply choosing a well-known mathematical model. An example of this kind is a homogeneous Poisson ensemble of random points corresponding to centers of inertia of gas molecules. In astrophysics, it was first used by Chandrasekhar to study gravitational fields in the interstellar medium. It was with him that they began to build their own, subsequently a very popular model of the distribution of galaxies Neumann and Scott. However, in contrast to the stars, the distribution of galaxies, and to an even greater extent, their clusters, are characterized by pronounced distant correlations, and the homogeneous Poisson field had to be modified, choosing the parent galaxy from the homogeneous Poisson ensemble at the first stage, and attaching to it descendants forming...
a cluster.

Qualitatively, the processes occurring in galaxies are now more or less clear, and efforts of researchers are aimed at developing quantitative cosmology based on the latest achievements of instrumental astronomy and computational mathematics. Fundamental questions to be answered include the problem of dark matter. Since its amount in the Universe is five times more than that of ordinary (visible, or baryonic) matter, the formation of large-scale structures must be controlled by dark matter. This is why we are interested in analysing existing and developing new models of the baryonic matter distribution, and first of all, of the galaxy distribution.

We emphasize that in this article we will not talk about modeling the appearance and evolution of the structure of the Universe, but only about on observed distribution of galaxies in space with the goal of compression of information: just like the central limit theorem compresses information about the sum of a large number of random terms, reducing it to only the first two moments of one term, and the random walk scheme with adequately selected distributions of the initial nodes and the distances between them allows us to draw useful conclusions about the whole system.

For the reasons given by Peebles [2] when setting such a problem (the end of item 30), we neglect the curvature of space, the expansion of the Universe and use the usual Euclidean geometry, and we characterize the position of the galaxy as a point in Euclidean space. The actual distribution of galaxies is considered as one of the possible realizations of the probabilistic object called the random point distribution.

2. Fractal set of random points
A large number of physical systems can be approximated by a set of points randomly distributed in the space. Talking about the ideal gas, we keep in mind that its molecules are placed at random points uniformly distributed over a volume independently of each other. Thinking of a real gas or of a fluid, we imagine a set of correlated random points. Looking at the night sky, we see a set of stars non-uniformly distributed over the sky sphere. Telescopes allows as to see distant galaxies and clusters of galaxies which may be considered in a first approximation as point objects randomly placed in the space. The visible galaxy distribution reveals some remarkable feature: the number of neighbors placed at distances from our Galaxy increases like with \( \alpha \in (1, 2) \) at least up to \( \tau_{\text{max}} = 200 \) Mpc (megaparsec). In order to co-ordinate this fact with the Cosmological Principle claiming equivalency of all the mass points (galaxies) with respect to their environment, one has to interpret this equivalence in the statistical sense. The case \( \tau_{\text{max}} = \infty \) is related to the concept of a stochastic fractal. A stochastic fractal is essentially non-homogeneous random structure connected with large fluctuations and clustering at all scales. This idea gave rise a cycle of works related to the fractal cosmology.

However, the most cosmologists believe that the homogeneity must be reached well within the size of the present horizon \( \approx 3000 \) Mpc). This case is related to some modification of the fractal concept – the fractal with a turnover to homogeneity, or the mesofractal.

The use of random jump processes for construction of fractal structures imitating galaxy distribution first was proposed by B.Mandelbrot [1]. Collision events (nodes) of a particle performing infinite number of jumps with the free path distribution density

\[ p_0(r) \propto r^{-\alpha - 1}, \quad r \to \infty, \quad \alpha > 0 \]

(Lévy flights) turn out to be distributed in such a manner that their mean number within radius \( R \) around one of them is

\[ \langle N(R) \rangle \sim AR^D, \quad R \to \infty, \]

where \( D = \alpha \), if \( \alpha < 2 \) and \( D = 2 \), if \( \alpha \geq 2 \). This is the first feature of stochastic fractals. The second feature is their self-similarity in the stochastic sense: the relative fluctuations of the random number \( N(R) \) are the same at all scales.

In computational practice, one can use only finite trajectories that changes the asymptotics of \( \langle N(R) \rangle \):
\( (N(R)) \sim \text{const as } R \to \infty. \) The set of points with such asymptotics is not a fractal anymore.

To simulate mesofractals, a set of independent Lévy-flight trajectories has been offered by a few researchers (see [2]). In this case the problem becomes keener: infinite trajectories yield infinite mean density of the system while finite trajectories do not produce fractals[2].

Three questions will be discussed in this work:
1. What form has the probability distribution of \( N(R) \)?
2. What kind of trajectories must be practically used to hold \( (N(R)) \sim A R^D \)?
3. How can one simulate random mesofractals on the base of Lévy flight trajectories?

3. Infinite trajectories

The generating functional (GF) of set of nodes \( X_t \) of a trajectory with exception of its birth point \( r \)

\[
G(r \to u(\cdot)) = \prod_i u(X_i)
\]

obeys the equation [3]:

\[
G(r \to u(\cdot)) = \int p(r - r') u(r') G(r' \to u(\cdot)) dr'.
\]  
(1)

Calculating the variational derivative from both sides of the equation and taking into account that

\[
\frac{\delta G(r \to u(\cdot))}{\delta u(r_i)}|_{u=1} = g(r \to r_i)
\]

is the mean density of points(nodes) around one of them, we obtain

\[
g(r \to r_i) = \int p(r - r') g(r' \to r_i) dr' + p(r - r_i).
\]  
(2)

The mean number of the points inside the sphere \( U_R \) centered at one of them

\[
\langle N(R) \rangle = \int_{U_R} g(0 \to r) dr.
\]

In case of isotropic Lévy flights

\[
p(r) = \frac{1}{4 \pi r^2} p_0(r) \propto r^{-3}, \quad r \to \infty
\]

and

\[
\langle N(R) \rangle \sim (4 \pi A/\alpha) R^D, \quad R \sim \infty.
\]

4. Factorial moments

Multiple variational differentiation of Eq. (1) leads to the following expression for factorial moments

\[
\langle N^{[k]} \rangle = \langle N(N - 1) ... (N - k + 1) \rangle, \quad k = 1, 2, 3, ...
\]

\[
\langle N^{[k]}(R) \rangle = k! \int_{U_R} ... \int_{U_R} g(0 \to r_1) ... g(r_{k-1} - r_k) dr_1 ... dr_k \sim
\]

\[
~k! (4 \pi A/3)^k R^{ak} K_k(\alpha), \quad R \to \infty,
\]

where

\[
K_0(\alpha) = 1, \quad K_1(\alpha) = (3/4 \pi) \int_{U_1} r^{-3 + \alpha} dr,
\]

\[
K_k(\alpha) = (3/4 \pi)^k \int_{U_1} ... \int_{U_1} (r_1 r_{1,2} ... r_{k-1,k})^{-3 + \alpha} dr_1 dr_2 ... dr_k, \quad k > 1,
\]

\[
r_{ij} = |r_i - r_j|.
\]

A few particular values of \( K_k(\alpha) \) were calculated by Peebles [2]. Below, we shall describe our numerical method and calculate moments for different values of \( \alpha \) and \( k \).

Let us consider a set of functions

\[
\nu_n^{(\alpha)}(r) = \int_{U_1} ... \int_{U_1} (|r - r_1| r_{1,2} ... r_{k-1,k})^{-3 + \alpha} dr_1 dr_2 ... dr_n
\]

so that
\[ K_k(\alpha) = (3/4\pi)^n v_k^{(\alpha)}(0). \]

The functions are recursively interrelated:
\[ v_k^{(\alpha)}(r) = \int_{r_1}^r V^{(\alpha)}(r-r')v_{k-1}^{(\alpha)}(r')dr', \quad V^{(\alpha)}(r) = r^{-3+\alpha}, \quad v_0^{(\alpha)}(r) = 1. \quad (3) \]

Taking into account the spherical symmetry of functions \( V^{(\alpha)}(r) \) and \( v_k^{(\alpha)}(r) \) we have \( dr' = 2\pi r'^2 dr' d\mu \), where \( \mu = \cos \theta \). After integrating \((3)\) with respect to \( \mu \) we obtain
\[ w_k^{(\alpha)}(r) \equiv r v_k^{(\alpha)}(r) = \int_0^1 F^{(\alpha)}(r, r')w_{k-1}^{(\alpha)}(r')dr', \quad (4) \]

where
\[ F^{(\alpha)}(r, r') = \frac{2\pi}{\alpha-1} [(r + r')^{\alpha-1} - |r - r'|^{\alpha-1}], \quad \alpha \neq 1, \]
\[ F^{(1)}(r, r') = 2\pi [\ln(r + r') - \ln|r - r'|]. \]

It is easy to see that \( K_k(\alpha) \) expressed in terms of \( w_k^{(\alpha)} \) takes the form
\[ K_k(\alpha) = (3/4\pi)^k \left[ dw_k^{(\alpha)}(r)/dr \right]_{r=0}. \quad (5) \]

This representation is convenient for sequential numerical calculations of \( K_k(\alpha) \) for \( \alpha > 1 \) when using the standard technique of numerical integration. The situation becomes simpler in the case \( \alpha = 1 \). Equation \((3)\) is transformed into
\[ v_k^{(1)}(r) = \frac{2\pi}{r} \int_0^1 \ln \left| \frac{r + r'}{|r - r|} \right| v_{k-1}^{(1)}(r')dr', \]
\[ v_1^{(1)}(r) = \frac{2\pi}{r} \left\{ r + \frac{1}{2} (1 - r^2) \ln \left| \frac{1 + r}{1 - r} \right| \right\}. \]

Expanding \( v_k^{(1)}(r) \) into power series, we arrive at the recurrent relation
\[ v_k^{(1)}(r) = 2\pi v_{k-1}^{(1)}(0) \left( 2 - \sum_{i=1}^{\infty} r^{2i} \left( \frac{1}{2i-1} - \frac{1}{2i+1} \right) \right) - \]
\[ -(4\pi)^2 v_{k-2}^{(1)}(0) \sum_{i=1}^{\infty} \frac{1}{(2i-1)(2i+1)^2} \]

whence
\[ v_k^{(1)}(0) = 4\pi v_{k-1}^{(1)}(0) - (4\pi)^2 v_{k-2}^{(1)}(0) \sum_{i=1}^{\infty} \frac{1}{(2i-1)(2i+1)^2}. \quad (6) \]

\[ v_0^{(1)}(0) = 1, \quad v_1^{(1)}(0) = 4\pi. \]

In case \( \alpha \leq 1 \) the integrand possesses a singularity at the point \( r' = r \) causing an increase of error during numerical integration. To avoid the trouble we transform \((4)\) into
\[ \int_0^r w_k^{(\alpha)}(r')dr' = W_k^{(\alpha)}(r) - \frac{4\pi}{\alpha(\alpha-1)} \int_0^1 r'^\alpha w_{k-1}^{(\alpha)}(r')dr', \]
\[ W_k^{(\alpha)}(r) = \frac{2\pi}{\alpha(\alpha-1)} \int_0^1 [(r + r')^{\alpha} - (r - r')|r - r'|^{\alpha-1}] w_{k-1}^{(\alpha)}(r')dr'. \]

Eq.\((5)\) becomes
\[ K_k(\alpha) = (3/4\pi)^k \left[ d^2 W_k^{(\alpha)}(r)/dr^2 \right]_{r=0}. \]

The numerical results for \( K_k(\alpha) \) are in agreement with those obtained by Peebles [2].
5. Reconstruction of pdf $\Psi_{\alpha}(z)$

The above results are valid for large values of $\Psi$. In this case

$$\langle N^{[k]} \rangle \sim \langle N^k \rangle \sim k! K_k(\alpha) \langle N(R) \rangle^k$$

and one can introduce a normalized random variable $Z = N(R)/\langle N(R) \rangle$ with moments

$$\langle Z^k \rangle = k! K_k(\alpha).$$

As computation have shown, the moments obey the relation

$$\langle Z^k \rangle = [1 + (k - 1)A(\alpha)] \langle Z^{k-1} \rangle.$$ (7)

This is an aspect of the gamma-distribution

$$\Psi_{\alpha}(z) = \frac{1}{\Gamma(\alpha)} \lambda^z z^{\alpha-1} e^{-\lambda z},$$ (8)

the moments of which are linked in the same manner with $A(\alpha) = (Z^2 - 1) = \sigma_2^2 / \langle N \rangle^2$

has been approximated (with an accuracy to 1%) by the formula

$$A(\alpha) = 0.539 - 0.263\alpha + 0.091\alpha^2$$ (9)

obtained by means of the least-square method. Formulas (7)-(9) are confirmed by direct Monte Carlo simulations[4].

6. Finite trajectories

As mentioned in Introduction, the fractal asymptotics (1) takes place for infinite trajectories only. In other words the solution $g_c(\mathbf{r})$ of integral equation

$$g_c(\mathbf{r}) = c \int p(\mathbf{r} - \mathbf{r}') g_c(\mathbf{r}') d\mathbf{r}' + p(\mathbf{r})$$ (10)

has quite different asymptotics for $c = 1$ (infinite trajectories) and for $c < 1$ (finite trajectories). What will happen as $c \to 1$?

To look at this problem more closely we consider the case when the kernel $p(\mathbf{r}')$ is a 3-dimensional isotropic stable density

$$p(\mathbf{r}) = q(\mathbf{r}; \alpha)$$

having the characteristic function

$$\int q(\mathbf{r}; \alpha) e^{i\mathbf{k} \cdot \mathbf{r}} d\mathbf{r} = e^{-\|\mathbf{k}\|^\alpha}.$$ The density is expanded in the series

$$q(\mathbf{r}; \alpha) = \frac{1}{2\pi r^2} \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \Gamma(n\alpha + 2) \sin(n\alpha \pi/2) r^{-n\alpha}. \quad (11)$$

Representing the solution of (10) by means of Neumann’s series

$$g_c(\mathbf{r}) = \sum_{n=1}^{\infty} c^n q^{(n)}(\mathbf{r}; \alpha)$$

and keeping in mind that for stable distribution

$$q^{(n)}(\mathbf{r}; \alpha) = n^{-3/\alpha} q(n^{-1/\alpha} \mathbf{r}; \alpha)$$

we arrive at the expression

$$g_c(\mathbf{r}) = \sum_{n=1}^{\infty} c^n q(n^{-1/\alpha} \mathbf{r}; \alpha).$$ (12)

Using here (11) and changing the order of summation we get

$$g_c(\mathbf{r}) = (2\pi^2)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \Gamma(n\alpha + 2) \sin(n\alpha \pi/2) \Phi(c, -n, 1) r^{-3-n\alpha}, \quad c < 1, \quad \alpha \neq 0, 1.$$ (13)

where

$$\Phi(c, -n, 1) = \sum_{k=1}^{\infty} c^{k-1} k^n.$$ The asymptotic behavior of (13) far away from origin is governed by the first term of the series:

$$g_c^z(\mathbf{r}) \sim (2\pi^2)^{-1} \Gamma(\alpha + 2) \sin(\alpha \pi/2) (1 - c)^{-2} r^{-3-\alpha}, \quad r \to \infty.$$ As $c \to 1$

$$\Phi(c, -n, 1) \sim n! (1 - c)^{-1-n}$$

and
$$g_c(r) \sim (2\pi^2)^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} \Gamma(n\alpha + 2) \sin(n\alpha \pi/2)(1 - c)^{-n\alpha - 3}$$

However this result is not applicable to the case $c = 1$, which should be considered separately. Putting $c = 1$ in Eq. (12) and applying the Euler-Maclaurin summation formula, we obtain The asymptotic expansion of (13) is thus of the following form:

$$g_1(r) = (2\pi^2)^{-1} \left[ \Gamma(2 - \alpha) \sin(\alpha \pi/2) r^{-3+\alpha} + \sum_{n=1(\text{odd})}^{\infty} \Gamma(n\alpha + 2) \sin(n\alpha \pi/2) A_n r^{-n\alpha - 3} \right]$$

where

$$A_n = \frac{1}{n^2} \frac{1}{(n+1)!} \sum_{m=1}^{(n+1)/2} \frac{\beta_{2m}}{(2m)!(n-2m+1)!}$$

In particular $A_1 = -1/12$, $A_3 = 1/720$, $A_5 = -1/30240$, $A_7 = 1/1209600$, $A_9 = -1/47990160$ and so on. Numerical calculation show that in case when $c$ is close to 1, there exists some region of distances where $g_c(r)$ practically coincides with $g_1(r)$, and the closer $c$ to 1 the wider this region. This phenomenon was named intermediate asymptotics [3].

7. Conclusion

The complete picture of galaxy distribution in space is given by a set of mutually independent Lévy flight trajectories with points of birth placed in a large volume according to the Poisson distribution. Each of the trajectory generates a random fractal and we observe their superposition. The resulting structure can be described in terms of correlation function. They can be derived from the corresponding GF

$$\Phi(u(\cdot)) = \exp\{n_0 \int [G(r \rightarrow u(\cdot)) - 1] dr\}$$

where $n_0$ is the mean density of points of birth. As shown in the article [3] the correspondent correlation functions $\xi_k(r_1, \ldots, r_k)$ are linked with factorial moment densities

$$f_k(r_1, \ldots, r_k) = \frac{\delta^{h_i}(u(\cdot))}{\delta u(r_1) \cdots \delta u(r_k)}$$

by means of the relations [2]:

$$f_2(r_1, r_2) = [1 + \xi_2(r_1, r_2)] f_1(r_1) f_1(r_2),$$

$$f_3(r_1, r_2, r_3) \overset{\Delta}{=} [1 + 3\xi_2(r_2, r_3) + \xi_3(r_1, r_2, r_3)] f_1(r_1) f_1(r_2) f_1(r_3)$$

and so on. Here $\overset{\Delta}{=} \text{means that the right side is symmetrized over all permutations of the arguments.}$

One can find from general relations the total density

$$n = n_0 \int g_c(r) dr = n_0/(1 - c)$$

and correlation functions

$$\xi_2(r) \equiv \xi(r) = 2(c/n) g_c(r),$$

$$\xi_k(r_1, \ldots, r_k) \overset{\Delta}{=} 2^{1-k} k! \xi(r_{12}) \xi(r_{23}) \cdots \xi(r_{k-1,k}).$$

As is clear from above, one can reach a desirable set of points choosing $\alpha, c, n_0$ and the scale factor in an appropriate way. For illustration of these ideas, the Monte Carlo-code was elaborated, and results of 2D-modeling are presented in figures 1–5.

Figure 1. 2D-Poisson random point distribution.
These Figures reflect the main peculiarity of galaxy distribution: raggedness, crowding, clusters intermittent with voids, shown in the right panel. This is completely different from uniform distributions.

What remains on the agenda? These are distant correlations. It is necessary to establish what distances between the measured regions can guarantee the statistical independence of the data, and thereby bring certainty into the measure of the statistical error of cosmological measurements generated by large-scale irregular inhomogeneities. The second question of undoubted interest is the establishment of a quantitative relationship between the characteristics of clusters (their sizes and distances between them) with the model parameters ($\alpha$, scaling and other parameters). Finally, the inclusion of masses in this model, which will allow us to approach the dynamic tasks of statistical cosmology).
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References
[1] Mandelbrot B 1982 The Fractal Geometry of Nature (San Francisco: W. H. Freeman)
[2] Peebles P 1980 The Large-Scale Structure of the Universe (Princeton: Princeton University Press)
[3] Uchaikin V and Gusarov G 1997 J. Math. Phys. 30 2453
[4] Uchaikin V, Gismjatov I, Gusarov G and Svetukhin V 1998 Int. J. of Bifurcation and Chaos 8 977
[5] Jones B J T, Martinez V J, Saar E and Trimble V 2005 Rev. Mod. Phys. 76 1211