Calculations with Characteristic Cycles

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Abstract

We discuss and prove a number of results for calculating characteristic cycles, or graded, enriched characteristic cycles. We concentrate particularly on results related to hypersurfaces.

1 Introduction

Our research into complex analytic singularities began with the study of hypersurfaces. Naturally, we were led to consider the Milnor fibration, and its powerful cohomological manifestation: the vanishing cycles. We developed the Lê cycles and numbers of a singular affine hypersurface [13], and found that they could be characterized in terms of the characteristic cycle of the vanishing cycles [12]. More generally, in [12], we showed that a number of important “invariants” related to singular spaces are related to the characteristic cycles of various constructible complexes.

In fact, as we showed in [17], for many purposes, the data that one needs is not the actual constructible complex of sheaves $F^\bullet$, but rather the associated graded, enriched characteristic cycle $gecc^\bullet(F^\bullet)$. The graded, enriched characteristic cycle is a graded structure which, in a given degree, is a formal sum in which each summand consists of the closure of the conormal space to a stratum with a coefficient given by the Morse module, with coefficients in $F^\bullet$, of the stratum in the given degree (see Definition 2.7). Thus, the (ordinary) characteristic cycle, $CC(F^\bullet)$, of $F^\bullet$ can be recovered from $gecc^\bullet(F^\bullet)$ by taking alternating sums of the ranks of the coefficients of $gecc^\bullet(F^\bullet)$ (provided that the base ring is an integral domain).

In this paper, we show that there is a “calculus” of graded, enriched characteristic cycles. Suppose that $F^\bullet$ is a complex of sheaves on a complex analytic space $X$, and that we know $gecc^\bullet(F^\bullet)$. Let $f : X \to \mathbb{C}$ be a given complex analytic map, and let $V(f) := f^{-1}(0)$. Let $i : X - V(f) \hookrightarrow X$ and $j : V(f) \hookrightarrow X$ denote the respective inclusions. We give results and examples which show how one can start with $gecc^\bullet(F^\bullet)$ and calculate the graded, enriched characteristic cycles of the shifted nearby cycles $\psi_f[-1]F^\bullet$ (Section 5), the complexes $i_!^*F^\bullet$ and $i_*i^*F^\bullet$ (Section 6), and the shifted vanishing cycles $\phi_f[-1]F^\bullet$ (Section 7). In Section 9 we also indicate how to calculate $CC(j_*j^*F^\bullet)$ and $CC(j_!j^!F^\bullet)$.

There have been numerous other works on the computations of characteristic cycles: notably, the papers of Ginsburg [11], Briançon, Maisonobe and Merle [2], and Parusiński and Pragacz [19], plus portions of the books of Kashiwara and Schapira [7] and of Schürmann [20]. However, there are several advantages to the techniques and results presented here.

• The intersection theory that we use is that of properly intersecting cycles inside a complex manifold. For such intersections, there are well-defined intersections cycles, not cycle classes (see [3], Chapter 8).

The fact that we have intersection cycles with fixed underlying analytic sets makes calculations and formulas much easier and algorithmic, and, typically, the amount of genericity that we need in statements

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is merely that the intersections are proper, which is a relatively simple thing to check.

It is an interesting aspect of the theory that, using only enough genericity to obtain proper intersections does not yield objects which are as generic as possible, and it is precisely this lack of really generic genericity that makes formulas work so well.

- While we use an easy intersection theory, we use modules in various degrees for the coefficients of our cycles. This graded, enriched intersection theory adds essentially no difficulty to computations, and yet, almost magically, yields results on the levels of modules, instead of merely giving numerical results.

- In addition to the notion of graded, enriched characteristic cycles (gecc’s), our primary new device involved in the calculus of gecc’s is the graded, enriched relative polar curve (see [18] and Section [4] of this paper). This is a substantial generalization the now-classic relative polar curve introduced by Hamm, Lê, and Teissier in 1973 in [6], [21], [9], and [10].

By giving the “correct” definition of the general polar curve, we are not required to make choices as generically as did Hamm, Lê, and Teissier and, thus, once again, the genericity hypotheses that we need in theorems are simply that certain intersections are proper.

- Our calculation of the gecc of the vanishing cycles uses a generalization of the Lê cycle algorithm that we developed in [13], and so really does allow for explicit calculations in many examples.

We reiterate that, throughout this paper, it is important that, when we state that a choice must be made “generically”, we actually give effective means of checking that the choice is generic enough. This makes the results much more useful when applying them to specific examples, and we give sample calculations to illustrate this point.

2 The Characteristic Cycle and the Graded, Enriched Characteristic Cycle

Throughout this paper, we fix a base ring $R$ that is a regular, Noetherian ring with finite Krull dimension (e.g., $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{C}$). This implies that every finitely-generated $R$-module has finite projective dimension (in fact, it implies that the projective dimension of the module is at most $\dim R$).

We let $\mathcal{U}$ be an open neighborhood of the origin of $\mathbb{C}^{n+1}$, and let $X$ be a closed, analytic subset of $\mathcal{U}$. We let $z := (z_0, \ldots, z_n)$ be coordinates on $\mathcal{U}$. Having fixed the coordinates, we identify the cotangent space $T^{*}\mathcal{U}$ with $\mathcal{U} \times \mathbb{C}^{n+1}$ by mapping $(p, w_0 d_p z_0 + \cdots + w_n d_p z_n)$ to $(p, (w_0, \ldots, w_n))$. Let $\pi : T^{*}\mathcal{U} \to \mathcal{U}$ denote the projection.

Let $\mathcal{S}$ be a complex analytic Whitney stratification of $X$, with connected strata. Let $F^\bullet$ be a bounded complex of sheaves of $R$-modules on $X$, which is constructible with respect to $\mathcal{S}$. For each $S \in \mathcal{S}$, we let $d_S := \dim S$, and let $(\mathbb{N}_S, \mathbb{L}_S)$ denote complex Morse data for $S$ in $X$, consisting of a normal slice and complex link of $S$ in $X$; see, for instance, [5] or [16].

A general reference for the remainder of this section is [17].

**Definition 2.1.** For each $S \in \mathcal{S}$ and each integer $k$, the isomorphism-type of the module $m^k_S(F^\bullet) := \mathbb{H}^{k-d_S}(\mathbb{N}_S, \mathbb{L}_S; F^\bullet)$ is independent of the choice of $(\mathbb{N}_S, \mathbb{L}_S)$; we refer to $m^k_S(F^\bullet)$ as the degree $k$ Morse module of $S$ with respect to $F^\bullet$. 

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Remark 2.2. The shift by $ds$ above is present so that perverse sheaves can have non-zero Morse modules in only degree 0.

We also remark that, up to isomorphism, $m^k_S(F^\bullet)$ can be obtained in terms of vanishing cycles. To accomplish this, select any point $p \in S$. Let $\tilde{g} : (U', p) \to (\mathbb{C}, 0)$ be an analytic function on some open neighborhood of $p$ in $U$ such that $dp\tilde{g}$ is a nondegenerate covector (in the sense of [5]), and such that $p$ is a (complex) nondegenerate critical point of $\tilde{g}|_{U'|_{\mathbb{C}}}$. Let $g := \tilde{g}|_{U'|_X}$. Then, $m^k_S(F^\bullet)$ is isomorphic to the stalk cohomology $H^k(\phi_g[-1]F^\bullet)_p$.

As a final remark, for the reader familiar with (middle perversity) perverse cohomology $^uH^k$, it is a trivial exercise that $m^k_S(F^\bullet) \cong m^n_S(\mu H^k(F^\bullet))$.

For any analytic submanifold $M \subseteq U$, we denote the conormal space $\{(p, \omega) \in T^*U \mid \omega(T_pM) = 0\}$ by $T^*_MU$, and will typically be interested in its closure $\overline{T^*_MU}$ in $T^*U$.

Definition 2.3. Suppose that $R$ is an integral domain.

Define $c_S(F^\bullet) := \sum_{k \in \mathbb{Z}}(-1)^k\text{rank}(m^k_S(F^\bullet))$, and define the characteristic cycle of $F^\bullet$ (in $T^*U$) to be the analytic cycle

$$CC(F^\bullet) = \sum_{S \in \mathcal{S}} c_S(F^\bullet) [\overline{T^*_MU}].$$

We write $c_0(F^\bullet)$ in place of $c_{\{0\}}(F^\bullet)$, and let $c_0(F^\bullet) = 0$ if $\{0\} \not\in \mathcal{S}$.

The underlying set $|CC(F^\bullet)| = \bigcup_{c_S(F^\bullet) \neq 0} \overline{T^*_MU}$ is the characteristic variety of $F^\bullet$ (in $T^*U$).

Throughout this paper, whenever we refer to $c_S(F^\bullet)$ or $CC(F^\bullet)$, we assume that the base ring is an integral domain, even if we do not explicitly state this.

Remark 2.4. We should remark that there are various conventions for the signs involved in the characteristic cycle. In fact, our definition above uses a different convention than we used in our earlier works. Our definition above is the most desirable considering the graded, enriched characteristic cycle that we will define below. In hopes of avoiding confusion with our earlier work, we have also changed our notation for the characteristic cycle.

Note that, using the above convention, the characteristic cycle is not changed by extending $F^\bullet$ by zero to all of $U$.

Three basic easy properties of the characteristic cycle concern how they work with shifting, constant sheaves, distinguished triangles, and the Verdier dual $DF^\bullet$.

Proposition 2.5.

1. $CC(F^\bullet[j]) = (-1)^jCC(F^\bullet)$.

2. If $X$ is a pure-dimensional (e.g., connected) complex manifold, then $CC(R^*_X) = (-1)^{\dim X}[T^*_XU]$, i.e., $CC(R^*_X[\dim X]) = [T^*_XU]$.

3. If $A^\bullet \to B^\bullet \to C^\bullet \xrightarrow{[1]} A^\bullet$ is a distinguished triangle in $D^b(X)$, then $CC(B^\bullet) = CC(A^\bullet) + CC(C^\bullet)$. 

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4. \( \text{CC}(F^*) = \text{CC}(DF^*) \).

For calculating the characteristic cycle of the constant sheaf, the following is very useful:

**Corollary 2.6.** Suppose that \( Y \) and \( Z \) are closed analytic subsets of \( X \) such that \( X = Y \cup Z \). Then,

\[
\text{CC}(R_X^*) = \text{CC}(R_Y^*) + \text{CC}(R_Z^*) - \text{CC}(R_Y^* \cap Z).
\]

**Proof.** Let \( j : Y \hookrightarrow X \), \( k : Z \hookrightarrow X \), and \( l : Y \cap Z \hookrightarrow X \) denote the respective inclusions. Then, there is a canonical distinguished triangle

\[
R_X^* \to j_* j^* R_X^* \oplus k_* k^* R_X^* \to l_* l^* R_X^* \xrightarrow{[1]} R_X^*.
\]

As the pull-back of the constant sheaf is the constant sheaf, and as the characteristic cycle is unaffected by extensions by zero, the desired conclusion follows immediately from Item 2 of Proposition 2.5.

The characteristic cycle uses only the Euler characteristic information from the Morse data to strata. While this makes many calculations far easier, it disposes of a great deal of cohomological data. Hence, we define a formal graded “cycle” with module coefficients (actually, isomorphism classes of modules); we shall discuss such “enriched” cycles more generally in Section 3.

**Definition 2.7.** The graded, enriched characteristic cycle of \( F^* \) in the cotangent bundle \( T^*U \) is defined in degree to be \( k \) to be

\[
\text{gecc}^k(F^*) := \sum_{S \in \mathcal{S}} m_S^k(F^*)[T^*_S U] = \sum_{S \in \mathcal{S}} H^{k-d_S}(N_S, L_S; F^*)[T^*_S U].
\]

The underlying set \( |\text{gecc}^* (F^*)| := \bigcup_{m_S^k(F^*) \neq 0} T^*_S U \) is the microsupport of \( F^* \) (in \( T^*U \)), and is denoted by \( \text{SS}(F^*) \) (see \( \mathbb{C} \)).

Note that, if \( \mathcal{O} \) is a point-stratum, then \( m_0^k(F^*) \cong H^k(\phi_L[-1]F^*)_0 \), where \( \mathcal{L} \) is the restriction to \( X \) of a generic linear form \( \hat{\mathcal{L}} \).

Clearly, \( \text{gecc}^k(F^*[j]) = \text{gecc}^{k+j}(F^*) \), and it is easy to show that \( \supp F^* = \pi(\text{SS}(F^*)) \). We also have the following properties similar to those in Proposition 2.5.

**Proposition 2.8.**

1. If \( X \) is a pure-dimensional (e.g., connected) complex manifold, then \( \text{gecc}^k(R_X^*[\dim X]) = 0 \) if \( k \neq 0 \), and \( \text{gecc}^0(R_X^*[\dim X]) = R[T^*_X U] \).

2. If \( A^* \to B^* \to C^* \xrightarrow{[1]} A^* \) is a distinguished triangle in \( D_c^b(X) \), then, for all \( k \), \( |\text{gecc}^k(B^*)| \subseteq |\text{gecc}^k(A^*)| \cup |\text{gecc}^k(C^*)| \).

3. If \( R \) is Dedekind domain, then, for all \( k \) and for all \( S \in \mathcal{S} \),

\[
m_S^{-k}(DF^*) \cong \text{Hom}(m_S^k(F^*), R) \oplus \text{Ext}(m_S^{k+1}(F^*), R).
\]

In particular, if \( R \) is a field, then \( \text{gecc}^{-k}(DF^*) = \text{gecc}^k(F^*) \).
Proof. Items 1 and 2 are trivial. We will show Item 3.

Let $S \in \mathcal{S}$. Let $p \in S$. Let $g$ be as in Remark $2.2$ so that $m^k_S(F^*) \cong H^k(\phi_g[-1]F^*)_p$, $m^{k+1}_S(F^*) \cong \mathcal{H}^{k+1}(\phi_g[-1]F^*)_p$, and $m^k_S(D(F^*)) \cong H^{-k}(\phi_g[-1]D(F^*)_p \cong H^{-k}(D(\phi_g[-1]F^*)_p$. As the support of $\phi_g[-1]F^*$ is contained in $\{p\}$ and as $R$ is a Dedekind domain, there is a natural split exact sequence

$$0 \to Ext(H^{k+1}(\phi_g[-1]F^*)_p, R) \to H^{-k}(D(\phi_g[-1]F^*)_p \to Hom(H^k(\phi_g[-1]F^*)_p, R) \to 0.$$ 

Item 3 follows. □

While most of our examples will have to wait until we have developed more machinery, we can calculate “bare-handedly” what happens for curves and some basic complexes of sheaves.

Example 2.9. Suppose that $X$ is a curve. The calculations of $\text{CC}(F^*)$ and $\text{gecc}(F^*)$ reduce to calculating what happens at the discrete set of points where $X$ is singular or where $F^*$ is not locally constant. Thus, it suffices to analyze the situation where there is a single zero-dimensional stratum.

Hence, we shall assume that $0 \in X$, and that $X - \{0\}$ is a smooth curve. We assume that each irreducible component of $X$ is homeomorphic to an open disk and, hence, corresponds to an irreducible component of the germ of $X$ at $0$. Let $\{X_i\}_{i \in \Lambda}$ denote the collection of irreducible components of $X$ and, for each $X_i$, let $m_i := \text{mult}_0 X_i$. Let $m := \text{mult}_0 X = \sum_i m_i$. Let $e := |\Lambda|$, i.e., let $e$ be the number of irreducible components of $X$.

Stratify $X$ by using $S_0 := \{0\}$ and $S_i := X_i - \{0\}$ as strata (we are assuming that $0$ is not in the indexing set $\Lambda$). Let $j : \{0\} \to X$ and $i : X - \{0\} \to X$ denote the inclusions. Let $A^* := \mathbb{Z}^*_X[1]$, $B^* := i^! A^*$, $C^* := i_* i^* A^*$, and let $P^*$ be the perverse sheaf given by intersection cohomology with constant $\mathbb{Z}$-coefficients (here, we use the shifts that put all of the possibly non-zero cohomology in non-positive degrees). These are all complexes of sheaves on $X$, which are the constant sheaf, shifted by 1, on $X - \{0\}$.

Let $\tilde{f} : (\mathcal{U}, 0) \to (\mathbb{C}, 0)$ be a complex analytic function, where $\tilde{f}$ may vanish identically on some irreducible components of $X$. Let $f := \tilde{f}|_X$. Consider $P^* := \psi_f[-1]A^*$ and $Q^* := \phi_f[-1]A^*$. These are complexes of sheaves on $V(f) := f^{-1}(0)$.

We wish to calculate the graded, enriched characteristic cycle, and the ordinary characteristic cycle, for each of the six complexes given above.

As $A^*, B^*, C^*$, and $P^*$ are the 1-shifted constant sheaf on $X - \{0\}$, it follows that, if $F^*$ is any of these four complexes, then, for all $t \in \Lambda$, $m^{0}_{S_t}(F^*) \cong \mathbb{Z}$, and $m^{k}_{S_t}(F^*) = 0$ for $k \neq 0$. The question is: what is $m^k_{S_t}(F^*)$?

A normal slice to $S_0$ is simply $\tilde{B}_\varepsilon \cap X$, where $\tilde{B}_\varepsilon$ is a small open ball around the origin. The complex link to $S_0$ is $\tilde{B}_\varepsilon \cap X \cap L^{-1}(a)$, where $L$ is a generic linear form and $0 < |a| \ll \varepsilon$.

$A^*$:

We have

$$m^k_{S_0}(A^*) = \mathbb{H}^k(\tilde{B}_\varepsilon \cap X, \tilde{B}_\varepsilon \cap X \cap L^{-1}(a); \mathbb{Z}^*_X[1]),$$

which is the ordinary degree $k + 1$ integral cohomology of the pair consisting of a contractible space modulo $m$ points in the space. Hence, $m^{0}_{S_0}(A^*) \cong \mathbb{Z}^{(m-1)}$, and $m^{k}_{S_0}(A^*) = 0$ if $k \neq 0$. 

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Thus, we find that, if $k \neq 0$, then $\text{gecc}^k(A^*) = 0$, and

$$\text{gecc}^0(A^*) = \mathbb{Z}^{m-1} [T^*_0 \mathcal{U}] + \sum_i \mathbb{Z} \left[ \frac{\mathcal{T}_{X_i-(0)}^*}{T_{X_i-(0)}^*} \right].$$

It follows that

$$\text{CC}(A^*) = (m - 1) [T^*_0 \mathcal{U}] + \sum_i \left[ \frac{\mathcal{T}_{X_i-(0)}^*}{T_{X_i-(0)}^*} \right].$$

$B^*$:

We have

$$m^k_{S_0}(B^*) = \mathbb{H}^{k+1}(\tilde{\mathcal{B}} \cap X, \tilde{\mathcal{B}} \cap X \cap L^{-1}(a); i_1 \mathbb{Z}^*_{X-(0)}).$$

Using the long exact sequence for the hypercohomology of a pair, and that $\mathbb{H}^*(\tilde{\mathcal{B}} \cap X; i_1 \mathbb{Z}^*_{X-(0)}) = 0$, we find that

$$m^k_{S_0}(B^*) \cong H^k(\tilde{\mathcal{B}} \cap X \cap L^{-1}(a); \mathbb{Z}).$$

Therefore, $m^k_{S_0}(B^*) = 0$ if $k \neq 0$, and $m^0_{S_0}(B^*) \cong \mathbb{Z}^m$.

Thus, we find that, if $k \neq 0$, then $\text{gecc}^k(B^*) = 0$, and

$$\text{gecc}^0(B^*) = \mathbb{Z}^m [T^*_0 \mathcal{U}] + \sum_i \mathbb{Z} \left[ \frac{\mathcal{T}_{X_i-(0)}^*}{T_{X_i-(0)}^*} \right].$$

It follows that

$$\text{CC}(B^*) = m [T^*_0 \mathcal{U}] + \sum_i \left[ \frac{\mathcal{T}_{X_i-(0)}^*}{T_{X_i-(0)}^*} \right].$$

$C^*$:

We have

$$m^k_{S_0}(C^*) = \mathbb{H}^{k+1}(\tilde{\mathcal{B}} \cap X, \tilde{\mathcal{B}} \cap X \cap L^{-1}(a); i_1 \mathbb{Z}^*_{X-(0)}) \cong H^k(\tilde{\mathcal{B}} \cap X \cap L^{-1}(a); \mathbb{Z}).$$

This splits as a direct sum of the degree $k+1$ integral cohomology of pairs consisting of spaces $\tilde{\mathcal{B}} \cap X_1 - \{0\}$, which are homotopy-equivalent to circles, modulo $m_1$ points. As in the $B^*$ case, one easily calculates that $m^k_{S_0}(C^*) = 0$ if $k \neq 0$, and $m^0_{S_0}(C^*) \cong \mathbb{Z}^m$.

Thus, we find that $\text{gecc}^*(B^*) = \text{gecc}^*(C^*)$ and, of course, that $\text{CC}(B^*) = \text{CC}(C^*)$.

$I^*$:

The axioms of intersection cohomology imply that $I^*$ is isomorphic to the direct sum of the extensions by zero of the intersection cohomology on each of the $X_i$. As each $X_i$ is homeomorphic to an open disk, the intersection cohomology complex on $X_i$ is isomorphic to $\mathbb{Z}_{X_i}[1]$.

Thus, we have

$$m^k_{S_0}(I^*) = \bigoplus_{i \in \Lambda} H^k(\tilde{\mathcal{B}} \cap X_i, \tilde{\mathcal{B}} \cap X_i \cap L^{-1}(a); \mathbb{Z}).$$
It follows that \( m^k_{i}(\mathbf{*}) = 0 \) if \( k \neq 0 \), and \( m^0_{i}(\mathbf{*}) \cong \bigoplus_i \mathbb{Z}^{(m_i-1)} \cong \mathbb{Z}^{(m-c)} \).

Thus, we find that, if \( k \neq 0 \), then \( \text{gecc}^k(\mathbf{*}) = 0 \), and

\[
\text{gecc}^0(\mathbf{*}) = \mathbb{Z}^{(m-c)} [T^*_0 U] + \sum_i \mathbb{Z} \left[ \frac{T^*_0 X_i (0) U}{T} \right].
\]

It follows that

\[
\text{CC}(\mathbf{*}) = (m-c) [T^*_0 U] + \sum_i \mathbb{Z} \left[ \frac{T^*_0 X_i (0) U}{T} \right].
\]

The fact that the graded, enriched characteristic cycles of \( A^\bullet, B^\bullet, C^\bullet \), and \( I^\bullet \) are all concentrated in degree zero is equivalent to these complexes being pure with shift 0 (see Definition 7.5.4 of [7]). This is equivalent to saying that the complexes are all perverse ([7], 9.5.2).

We still wish to look at the complexes \( P^\bullet \) and \( Q^\bullet \). Let \( \Lambda_{\leq} := \{ l \in \Lambda \mid f(X_l) \equiv 0 \} \). Hence, \( V(f) = \bigcup_{l \in \Lambda_{\leq}} X_l \), with the convention that, if \( \Lambda_{\leq} \) is empty, then this union is taken as meaning the point-set \( \{ \emptyset \} \). Let \( \Lambda_{\geq} := \Lambda - \Lambda_{\leq} \). For each \( l \in \Lambda_{\geq} \), let \( \eta_l \) equal the intersection multiplicity \( (X_l \cdot V(f))_0 \), and let \( \eta := \sum_{l \in \Lambda_{\geq}} \eta_l \). Let \( m_{\leq} := \sum_{l \in \Lambda_{\leq}} m_{l} \).

\( P^\bullet \):

By definition, \( P^\bullet = \psi[-1]A^\bullet \) is a complex of sheaves on \( V(f) \), but – in our current setting – the support of \( \psi[-1]A^\bullet \) will be contained in \( \{ \emptyset \} \), and the stalk cohomology \( H^*(\psi[-1]A^\bullet)_0 \) is isomorphic to \( \mathbb{Z}^\eta \) in degree 0 and is zero in other degrees.

Thus, we find that, if \( k \neq 0 \), then \( \text{gecc}^k(P^\bullet) = 0 \), and

\[
\text{gecc}^0(P^\bullet) = \mathbb{Z}^\eta [T^*_0 U].
\]

It follows that

\[
\text{CC}(P^\bullet) = \eta [T^*_0 U].
\]

\( Q^\bullet \):

By definition, \( Q^\bullet = \phi[-1]A^\bullet \) is a complex of sheaves on \( V(f) \), and the restriction of \( Q^\bullet \) to \( V(f) - \{ \emptyset \} \) is isomorphic to the 1-shifted constant sheaf. In addition, the stalk cohomology \( H^*(\phi[-1]A^\bullet)_0 \) is isomorphic to \( \mathbb{Z}^{(a-1)} \) in degree 0 and is zero in other degrees. We have

\[
m^k_{i}(Q^\bullet) = \mathbb{H}^k(\overset{\circ}{B}_x \cap \overset{\circ}{X} \cap V(f), \overset{\circ}{B}_x \cap \overset{\circ}{X} \cap V(f) \cap L^{-1}(a); \phi[-1]A^\bullet),
\]

for \( 0 < |a| \ll 1 \). This module fits into the hypercohomology long exact sequence of the pair, in which one has the map induced by inclusion

\[
\mathbb{H}^k(\overset{\circ}{B}_x \cap \overset{\circ}{X} \cap V(f), \phi[-1]A^\bullet) \rightarrow \mathbb{H}^k(\overset{\circ}{B}_x \cap \overset{\circ}{X} \cap V(f) \cap L^{-1}(a); \phi[-1]A^\bullet).
\]

The right-hand term above is clearly isomorphic to \( \bigoplus_{l \in \Lambda_{\leq}} \mathbb{H}^k(\overset{\circ}{B}_x \cap X_l \cap L^{-1}(a); \phi[-1]A^\bullet) \), and (†) can be rewritten as

\[
H^{k+1}(\overset{\circ}{B}_x \cap \overset{\circ}{X}, \overset{\circ}{B}_x \cap \overset{\circ}{X} \cap f^{-1}(b); \mathbb{Z}) \rightarrow \bigoplus_{l \in \Lambda_{\leq}} H^{k+1}(\overset{\circ}{B}_x \cap X_l \cap L^{-1}(a); \mathbb{Z}),
\]
where $0 < |b| \ll |a| \ll 1$. Now, one easily finds from the long exact sequence that $m^k_{S_0}(Q^*) = 0$ for $k \neq 0$, and $m^0_{S_0}(Q^*) \cong \mathbb{Z}^{(m_\epsilon + n - 1)}$.

Thus, we find that, if $k \neq 0$, then $\text{gecc}^k(Q^*) = 0$, and

$\text{gecc}^0(Q^*) = \mathbb{Z}^{(m_\epsilon + n - 1)} [T_0^*U] + \sum_{i \in \Lambda} \mathbb{Z} \left[ \overline{T_{X_i} - \{0\}}U \right]$.

It follows that

$\text{CC}(Q^*) = (m_\epsilon + \eta - 1) [T_0^*U] + \sum_{i \in \Lambda} \left[ \overline{T_{X_i} - \{0\}}U \right]$.

The characteristic cycle calculations for $P^*$ and $Q^*$ can be “checked”. There is the canonical distinguished triangle

$(Z^*_{X_i}[1])_{V(f)}[1] \to \psi_f[-1]Z^*_{X_i}[1] \to \phi_f[-1]Z^*_{X_i}[1] \to (Z^*_{X_i}[1])_{V(f)}[1]$, and so we should find that $\text{CC}(P^*) = \text{CC}(Q^*) + \text{CC}(Z^*_{V(f)}).$

This is easily checked, for $\text{CC}(Z^*_{V(f)}) = -\text{CC}(Z^*_{V(f)}[1])$ and, applying our calculation of $\text{CC}(A^*)$, we find that

$\text{CC}(Z^*_{V(f)}[1]) = (m_\epsilon - 1) [T_0^*U] + \sum_{i \in \Lambda} \left[ \overline{T_{X_i} - \{0\}}U \right]$.

**Example 2.10.** In Example 2.9 all of the graded, enriched characteristic cycles were concentrated in degree 0. As we mentioned, this is a reflection of the fact that each of the complexes that we considered were perverse sheaves. We wish a give an easy example/problem, where the sheaves under consideration are not perverse.

Let $U := \mathbb{C}^3$, and use $x$, $y$, and $z$ as coordinates. Let $X := V(z) \cup V(x, y)$. There are three obvious strata: $S_0 := \{0\}$, $S_1 := V(x, y) - \{0\}$, and $S_2 := V(z) - \{0\}$. Let $j : \{0\} \hookrightarrow X$ and $i : X - \{0\} \hookrightarrow X$ denote the inclusions, and consider the complexes of sheaves $A^* := Z^*_{X_i}[2]$, $B^* := i_*i^*A^*$, and $C^* := i_*i^*A^*$.

We leave it to the reader to verify that:

$\text{gecc}^k(A^*) = 0$ if $k \neq -1, 0$, $\text{gecc}^{-1}(A^*) = \mathbb{Z}[T_0^*U] + \mathbb{Z}[T^*_{V(x, y)}U]$, and $\text{gecc}^0(A^*) = \mathbb{Z}[T^*_{V(z)}U]$.

$\text{gecc}^k(B^*) = 0$ if $k \neq -1, 0$, $\text{gecc}^{-1}(B^*) = \mathbb{Z}^2[T_0^*U] + \mathbb{Z}[T^*_{V(x, y)}U]$, and $\text{gecc}^0(B^*) = \mathbb{Z}[T^*_{V(z)}U]$.

$\text{gecc}^k(C^*) = 0$ if $k \neq -1, 0, 1$, $\text{gecc}^{-1}(C^*) = \mathbb{Z}[T_0^*U] + \mathbb{Z}[T^*_{V(x, y)}U]$, $\text{gecc}^0(C^*) = \mathbb{Z}[T^*_{V(z)}U]$, and $\text{gecc}^1(C^*) = \mathbb{Z}[T_0^*U]$.

**3 Basics of Enriched Cycles**

In Definition 2.7, we defined the graded, enriched characteristic cycle. In this section, we wish to describe graded, enriched cycles more generally and carefully. We also describe the associated intersection theory. The intersection theory that we use is a fairly simple extension of the intersection theory of properly intersecting
cycles in an analytic manifold, as described in section 8.2 of [3]. In this case, one obtains intersection cycles, not merely rational equivalence classes of cycles.

**Definition 3.1.** An enriched cycle, E, in X is a formal, locally finite sum \( \sum V E_V[V] \), where the V’s are irreducible analytic subsets of X and the \( E_V \)’s are finitely-generated R-modules. We refer to the V’s as the components of E, and to \( E_V \) as the V-component module of E. Two enriched cycles are considered the same provided that all of the component modules are isomorphic. The underlying set of E is \( |E| := \cup_{V \neq 0} V \).

If \( C = \sum n_V[V] \) is an ordinary positive cycle in X, i.e., all of the \( n_v \) are non-negative integers, then there is a corresponding enriched cycle \([C]^{\mathrm{enr}}\) in which the V-component module is the free R-module of rank \( n_V \). If \( R \) is an integral domain, so that rank of an R-module is well-defined, then an enriched cycle E yields an ordinary cycle \([E]^{\mathrm{ord}} := \sum (\operatorname{rk}(E_V))[V] \).

If \( q \) is a finitely-generated module and E is an enriched cycle, then we let \( qE := \sum (q \otimes E_V)[V] \); thus, if \( R \) is an integral domain and \( E \) is an enriched cycle, \([qE]^{\mathrm{ord}} = (\operatorname{rk}(q))[E]^{\mathrm{ord}} \) and if \( C \) is an ordinary positive cycle and \( n \) is a positive integer, then \([nC]^{\mathrm{enr}} = R^n[C]^{\mathrm{enr}} \).

The (direct) sum of two enriched cycles \( D \) and \( E \) is given by \( (D + E)_V := D_V + E_V \).

There is a partial ordering on isomorphism classes of finitely-generated R-modules given by \( M \leq Q \) if and only if there exists a finitely-generated R-module \( N \) such that \( M \oplus N \cong Q \). This relation is clearly reflexive and transitive; moreover, anti-symmetry follows from the fact that if \( M \) and \( N \) are Noetherian modules such that \( M \oplus N \cong M \), then \( N = 0 \). This partial ordering extends to a partial ordering on enriched cycles given by: \( D \leq E \) if and only if there exists an enriched cycle \( P \) such that \( D + P = E \). If the base ring \( R \) is a PID and \( D + P = E \), then \( D \) is uniquely determined by \( P \) and \( E \), and we write \( D = E - P \).

If two irreducible analytic subsets \( V \) and \( W \) intersect properly in \( U \), then the (ordinary) intersection cycle \([V] \cdot [W] \) is a well-defined positive cycle; we define the enriched intersection product of \([V]^{\mathrm{enr}} \) and \([W]^{\mathrm{enr}} \) by \([V]^{\mathrm{enr}} \cdot [W]^{\mathrm{enr}} = ([V] \cdot [W])^{\mathrm{enr}} \). If \( D \) and \( E \) are enriched cycles, and every component of \( D \) properly intersects every component of \( E \) in \( U \), then we say that \( D \) and \( E \) intersect properly in \( U \) and we extend the intersection product linearly, i.e., if \( D = \sum_V D_V[V] \) and \( E = \sum_W E_W[W] \), then

\[
D \odot E := \sum_{V,W} (D_V \otimes E_W)[(V] \cdot [W])^{\mathrm{enr}}.
\]

A graded, enriched cycle \( E^i \) is simply an enriched cycle \( E^i \) for \( i \) in some bounded set of integers. An single enriched cycle is considered as a graded enriched cycle by being placed totally in degree zero. The analytic set \( V \) is a component of \( E^i \) if and only if \( V \) is a component of \( E^i \) for some \( i \), and the underlying set of \( E^i \) is \( |E^i| = \bigcup_i |E^i| \). If \( R \) is a domain, then \( E^i \) yields an ordinary cycle \([E^i]^{\mathrm{ord}} = \sum_i (-1)^i (\operatorname{rk}(E^i_V))[V] \).

If \( k \) is an integer, we define the \( k \)-shifted graded, enriched cycle \( E^i[k] \) by \( (E^i[k])^j := E^{i+k} \).

If \( q \) is a finitely-generated module and \( E^i \) is a graded enriched cycle, then we define the graded enriched cycle \( qE^i \) by \( (qE^i)^j := \sum_V (q \otimes E^i_V)[V] \). The (direct) sum of two graded enriched cycles \( D^i \) and \( E^j \) is given by \((D^i + E^j)^i := D^i_V \oplus E^j_V \). If \( D^i \) properly intersects \( E^j \) for all \( i \) and \( j \), then we say that \( D^i \) and \( E^j \) intersect properly and we define the intersection product by

\[
(D^i \odot E^j)^k := \sum_{i+j=k} (D^i \odot E^j).
\]
Whenever we use the enriched intersection product symbol, we mean that we are considering the objects on both sides of \( \circ \) as graded, enriched cycles, even if we do not superscript by \( \text{enr} \) or \( \circ \).

Let \( \tau : W \to Y \) be a proper morphism between analytic spaces. If \( C = \sum n_V[V] \) is an ordinary positive cycle in \( W \), then the proper push-forward \( \tau_*(C) = \sum n_V\tau_*([V]) \) is a well-defined ordinary cycle.

**Definition 3.2.** If \( E^* = \sum_V E^*_V[V] \) is an enriched cycle in \( W \), then we define the **proper push-forward** of \( E^* \) by \( \tau \) to be the graded enriched cycle \( \tau_*(E^*) \) defined by
\[
\tau_*(E^*) := \sum_V E^*_V([V])^\text{enr}.
\]

The ordinary projection formula for divisors ([F], 2.3.c) immediately implies the following enriched version.

**Proposition 3.3.** Let \( E^* \) be a graded enriched cycle in \( X \). Let \( W := |E^*| \). Let \( \tau : W \to Y \) be a proper morphism, and let \( g : Y \to \mathbb{C} \) be an analytic function such that \( g \circ \tau \) is not identically zero on any component of \( E^* \). Then, \( g \) is not identically zero on any component of \( \tau_*(E^*) \) and
\[
\tau_*(E^* \circ V(g \circ \tau)) = \tau_*(E^*) \circ V(g).
\]

## 4 The Relative Polar Curve

We will use the notation established in Section 2. \( \mathcal{U} \) is an open neighborhood of the origin in \( \mathbb{C}^{n+1} \), \( X \) is a closed, analytic subset of \( \mathcal{U} \), \( \mathcal{S} \) is a complex analytic Whitney stratification of \( X \), with connected strata, \( R \) is a base ring (with some technical assumptions), \( F^* \) is a bounded complex of sheaves of \( R \)-modules on \( X \), which is constructible with respect to \( \mathcal{S} \), \( z = (z_0, \ldots, z_n) \) is a set of coordinates on \( \mathcal{U} \), we identify the cotangent space \( T^\ast \mathcal{U} \) with \( \mathcal{U} \times \mathbb{C}^{n+1} \) by mapping \( (p, w_0d_pz_0 + \cdots + w_ndpz_n) \) to \( (p, (w_0, \ldots, w_n)) \), for each \( S \in \mathcal{S} \), \( d_S = \dim S \), and \( (NS, LS) \) is the complex Morse data for \( S \) in \( X \), consisting of a normal slice and complex link of \( S \) in \( X \).

Let \( \tilde{f} \) and \( \tilde{g} \) be analytic functions from \( (\mathcal{U}, 0) \) to \( (\mathbb{C}, 0) \), and let \( f \) and \( g \) denote the restrictions of \( \tilde{f} \) and \( \tilde{g} \), respectively, to \( X \). By refining \( \mathcal{S} \), if necessary, we assume that \( V(f) \) is a union of strata.

**Definition 4.1.** Let \( \mathcal{S}(F^*) := \{ S \in \mathcal{S} \mid H^\ast(NS, LS; F^*) \neq 0 \} \); we refer to the elements of \( \mathcal{S}(F^*) \) as the \( F^* \)-visible strata of \( \mathcal{S} \).

Fix a point \( p \in \mathcal{U} \). In [4, 21], Hamm, Teissier, and Lé define and use the relative polar curve (of \( \tilde{f} \) with respect to \( z_0 \)), \( \Gamma^{1}_{\tilde{f}, z_0} \), to prove a number of topological results related to the Milnor fiber of hypersurface singularities. We shall recall some definitions and results here. We should mention that there are a number of different characterizations of the relative polar curve, all of which agree when \( z_0 \) is sufficiently generic; below, we have selected what we consider the easiest way of describing the relative polar curve as a set, a scheme, and a cycle.

As a set, \( \Gamma^{1}_{\tilde{f}, z_0} \) is the closure of the critical locus of \( (\tilde{f}, z_0) \) minus the critical locus of \( \tilde{f} \), i.e., \( \Gamma^{1}_{\tilde{f}, z_0} \) equals \( \Sigma(\tilde{f}, z_0) - \Sigma \tilde{f} \), as a set. If \( z_0 \) is sufficiently generic for \( \tilde{f} \) at \( p \), then, in a neighborhood of \( p \), \( \Gamma^{1}_{\tilde{f}, z_0} \) will be purely one-dimensional (which includes the possibility of being empty).
It is not difficult to give $\Gamma_{f,z_0}^1$ a scheme structure. If $\Gamma_{f,z_0}^1$ is purely one-dimensional at $p$, then, at points $x$ near, but unequal to, $p$, $\Gamma_{f,z_0}^1$ is given the structure of the scheme $V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. One can also remove “algebraically” any embedded components of $\Gamma_{f,z_0}^1$ at $p$ by using gap sheaves; see Chapter 1 of [13].

In practice, all topological applications of the relative polar curve use only its structure as an analytic cycle (germ), that is, as a locally finite sum of irreducible analytic sets (or germs of sets) counted with integral multiplicities (which will all be non-negative).

If $C$ is a one-dimensional irreducible germ of $\Gamma_{f,z_0}^1$ at $p$, and $x \in C$ is close to, but unequal to, $p$, then the component $C$ appears in the cycle $\Gamma_{f,z_0}^1$, with multiplicity given by the Milnor number of $\tilde{f}_{|H}$ at $x$, where $H$ is a generic affine hyperplane passing through $x$.

Suppose that $M$ is a complex submanifold of $U$. Recall:

**Definition 4.2.** The relative conormal space $T_{f|M}^*U$ is given by

$$T_{f|M}^*U := \{(x, \eta) \in T^*U \mid \eta(T_{x}M \cap \ker \partial f) = 0\}.$$

If $M \subseteq X$, then $T_{f|M}^*U$ depends on $f$, but not on the particular extension $\tilde{f}$. In this case, we write $T_{f|M}^*U$ in place of $T_{\tilde{f}|M}^*U$.

**Definition 4.3.** The graded, enriched relative conormal cycle, $(T_{f|M}^*U)^\bullet$, of $f$, with respect to $F^\bullet$, is defined by

$$(T_{f|M}^*U)^k := \sum_{S \in \mathfrak{S}(F^\bullet)} m_S^k(F^\bullet) \left\{T_{f|M}^*U \right\}.$$

We now wish to define the graded, enriched relative polar curve. We will consider the image, im $\tilde{d}\bar{g}$, of $\tilde{d}\bar{g}$ in $T^*U$; this scheme is defined by

$$V\left(w_0 - \frac{\partial \tilde{g}}{\partial z_0}, \ldots, w_n - \frac{\partial \tilde{g}}{\partial z_n}\right) \subseteq U \times \mathbb{C}^{n+1}.$$ 

We will consider im $\tilde{d}\bar{g}$ as a scheme, an analytic set, an ordinary cycle, and as a graded, enriched cycle; we will denote all of these by simply im $\tilde{d}\bar{g}$, and explicitly state what structure we are using or let the context make the structure clear.

Note that the projection $\pi$ induces an isomorphism from the analytic set im $\tilde{d}\bar{g}$ to $U$. We will use the proper push-forward (Definition 3.2) of the map $\pi$ restricted to im $\tilde{d}\bar{g}$; we will continue to denote this restriction by simply $\pi$.

By our conventions in Section 3, the graded, enriched cycle im $\tilde{d}\bar{g}$ is zero outside of degree 0, and is the enriched cycle $R[\text{im } \tilde{d}\bar{g}]$ in degree 0.

**Definition 4.4.** If $S \in \mathfrak{S}$ and $f_{|S}$ is not constant, we define the relative polar set, $|\Gamma_{f,\tilde{g}}(S)|$, to be $\pi\left(T_{f_{|S}}^*U \cap \text{im } \tilde{d}\bar{g}\right)$; if this set is purely 1-dimensional, so that $T_{f_{|S}}^*U$ and im $\tilde{d}\bar{g}$ intersect properly, we define the (ordinary) relative polar curve, $\Gamma_{f,\tilde{g}}(S)$, to be the cycle $\pi_*(\left[T_{f_{|S}}^*U \right] \cdot [\text{im } \tilde{d}\bar{g}]$).
The relative polar set, $|\Gamma_{f,\tilde{g}}(F^*)|$, is defined by

$$|\Gamma_{f,\tilde{g}}(F^*)| := \pi \left( |(T_{f,\tilde{g}}^*U)^*| \cap \im d\tilde{g} \right).$$

Each 1-dimensional component $C$ of $|\Gamma_{f,\tilde{g}}(F^*)|$ is the image of a component of $|(T_{f,\tilde{g}}^*U)^*| \cap \im d\tilde{g}$ along which $|(T_{f,\tilde{g}}^*U)^*| \cap \im d\tilde{g}$ intersect properly. We give such a component $C$ the structure of the graded, enriched cycle whose underlying set is $C$ and whose graded, enriched cycle structure is given by $\pi^* \left( (T_{f,\tilde{g}}^*U)^* \cap \im d\tilde{g} \right)$ over generic points in $C$. We refer to this as the graded, enriched cycle structure of $C$ in $|\Gamma_{f,\tilde{g}}(F^*)|$.

If $|\Gamma_{f,\tilde{g}}(F^*)|$ is purely 1-dimensional, we say that the graded, enriched relative polar curve, $(\Gamma_{f,\tilde{g}}^1(F^*))^*$, is defined, and is given by

$$(\Gamma_{f,\tilde{g}}^1(F^*))^* := \pi^* \left( (T_{f,\tilde{g}}^*U)^* \cap \im d\tilde{g} \right),$$

i.e.,

$$(\Gamma_{f,\tilde{g}}^1(F^*))^k = \sum_{S \in \Phi(F^*)} m_S^k(F^*) \left( \Gamma_{f,\tilde{g}}^1(S) \right)^{\text{enr}}.$$

**Remark 4.5.** If $\tilde{g} = z_0$ is a generic linear form and $S = U$, then $\Gamma_{f,\tilde{g}}(S)$ is the classical polar curve $\Gamma_{f,z_0}^1$ (as a cycle) of Hamm, Lê, and Teissier.

In the notation for the polar curve, we write $\tilde{g}$, not simply $g$; we do not, in fact, know if $(\Gamma_{f,\tilde{g}}^1(F^*))^*$ is independent of the extension to $\tilde{g}$. However, when $(\Gamma_{f,\tilde{g}}^1(F^*))^*$ is defined and has no component on which $f$ is constant, then $(\Gamma_{f,\tilde{g}}^1(F^*))^*$ is independent of the extension $\tilde{g}$. It is also not difficult to show that the set $|\Gamma_{f,\tilde{g}}(F^*)|$ is independent of the extension of $g$, but we shall not need this result here.

Note that $\overline{T_{f,\tilde{g}}^*U} \cap \im d\tilde{g}$ is at least 1-dimensional at each point of intersection, and so $|\Gamma_{f,\tilde{g}}(F^*)|$ has no isolated points. Also, note that, as $|(T_{f,\tilde{g}}^*U)^*| \cap \im d\tilde{g}$ is a closed subset of $\im d\tilde{g}$, and $\pi$ induces an isomorphism from $\im d\tilde{g}$ to $U$, $|\Gamma_{f,\tilde{g}}(F^*)|$ is a closed subset of $U$.

Finally, if the relative polar set is 1-dimensional, we frequently superscript with a 1 to emphasize that fact.

For the purposes of this paper, we need to recall the following proposition, which is Proposition 3.13 of [18].

**Proposition 4.6.**

1. There exists a non-zero linear form $\tilde{L}$ such that $0 \notin |\Gamma_{f,\tilde{L}}(F^*)|$ if and only if for generic linear $\tilde{L}$, $0 \notin |\Gamma_{f,\tilde{L}}(F^*)|$.  
2. For generic linear $\tilde{L}$, $\dim_0 V(f) \cap |\Gamma_{f,\tilde{L}}(F^*)| \leq 0$ and $\dim_0 V(\tilde{L}) \cap |\Gamma_{f,\tilde{L}}(F^*)| \leq 0$.

**Example 4.7.** In this example, we will calculate another graded, enriched characteristic cycle. We shall use this as a basis for the next two examples, in which we calculate a graded, enriched relative conormal cycle and a graded, enriched relative polar cycle.
Let $f : \mathbb{C}^3 \to \mathbb{C}$ be given by $f(x, y, t) = y(y^2 - x^3 - t^2x^2)$, and let \( X := V(f) = V(y) \cup V(y^2 - x^3 - t^2x^2) \). The singular set of \( X \), \( \Sigma X \), is the 1-dimensional set \( V(x, y) \cup V(x + t^2, y) \). Thus, near the origin (actually, in this specific example, globally),

\[
\mathcal{S} := \{ V(y) - V(y^2 - x^3 - t^2x^2), V(y^2 - x^3 - t^2x^2) - V(y), V(x, y) - \{0\}, V(x + t^2, y) - \{0\} \}
\]

is a Whitney stratification of \( X \) with connected strata. Let \( F^* := \mathbb{Z}_X^*[2] \). We want to calculate the graded, enriched characteristic cycle of \( F^* \).

First, consider the 2-dimensional strata. Let \( S_1 := V(y) - V(y^2 - x^3 - t^2x^2) \). Then, \( \mathbb{N}_{S_1} \) is simply a point, and \( \mathbb{L}_{S_1} \) is empty. Hence, \( H^{k-2}(\mathbb{N}_{S_1}, \mathbb{L}_{S_1}; F^*) = H^k(\mathbb{N}_{S_1}, \mathbb{L}_{S_1}; \mathbb{Z}) \) isomorphic to \( \mathbb{Z} \) if \( k = 0 \), and is 0 if \( k \neq 0 \). The same conclusion holds if \( S_1 \) is replaced by \( S_2 := V(y^2 - x^3 - t^2x^2) - V(y) \).

Now, consider the 1-dimensional strata. Let \( S_3 := V(x, y) - \{0\} \), and \( S_4 := V(x + t^2, y) - \{0\} \). The normal slice \( \mathbb{N}_{S_3} \) is, as a germ, up to analytic isomorphism, three complex lines in \( \mathbb{C}^2 \), which intersect at a point, and \( \mathbb{L}_{S_3} \) is three points. Similarly, the normal slice \( \mathbb{N}_{S_4} \) is, as a germ, up to analytic isomorphism, two complex lines in \( \mathbb{C}^2 \), which intersect at a point, and \( \mathbb{L}_{S_4} \) is two points. Hence, \( H^{k-1}(\mathbb{N}_{S_3}, \mathbb{L}_{S_3}; F^*) = H^{k+1}(\mathbb{N}_{S_3}, \mathbb{L}_{S_3}; \mathbb{Z}) \) isomorphic to \( \mathbb{Z}^2 \) if \( k = 0 \), and is 0 if \( k \neq 0 \). Similarly, \( H^{k-1}(\mathbb{N}_{S_4}, \mathbb{L}_{S_4}; F^*) = H^{k+1}(\mathbb{N}_{S_4}, \mathbb{L}_{S_4}; \mathbb{Z}) \) isomorphic to \( \mathbb{Z} \) if \( k = 0 \), and is 0 if \( k \neq 0 \).

Finally, consider the stratum \( \{0\} \). Then, \( \mathbb{N}_{\{0\}} \) is all of \( X \), intersected with a small ball around the origin. The complex link \( \mathbb{L}_{\{0\}} \) is usually referred to as simply the complex link of \( X \) at \( 0 \). Thus, \( \mathbb{L}_{\{0\}} \) has the homotopy-type of a bouquet of 1-spheres (see [11]), and the number of spheres in this bouquet is equal to the intersection number \( (\Gamma_{f,L}^1 \cdot V(L))_0 \), where \( L \) is any linear form such that \( d_0 L \) is not a degenerate covector from strata of \( X \) at \( 0 \) (see [5]), and the relative polar curve here is the classical one of Lê, Hamm, and Teissier. We claim that we may use \( L := t \) for this calculation.

To see this, first note that \( V(y^2 - x^3 - t^2x^2) \) is the classic example of a space such that the regular part satisfies Whitney’s condition (a) along the \( t \)-axis (or, alternatively, this is an easy exercise). Thus, \( d_0 t \) is not a limit of conormals from \( S_2 \). Now, the closures of \( S_1 \), \( S_3 \), and \( S_4 \) are all smooth, and \( d_0 t \) is not conormal to these closures at the origin.

To find the ordinary cycle \( \Gamma_{f,t}^1 \), we take the components of the cycle below which are not contained in \( \Sigma f \):

\[
V \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = V(y(-3x^2 - 2t^2x), 3y^2 - x^3 - t^2x^2) = V(y, x^2(x + t^2)) + V(3x^2 + 2t^2, 3y^2 - x^3 - t^2x^2) = 2V(x, y) + V(x + t^2, y) + 2V(x, y) + V(3x^2 + 2t^2, 3y^2 - x^3 - t^2x^2).
\]

Thus, \( \Gamma_{f,t}^1 = V(3x^2 + 2t^2, 3y^2 - x^3 - t^2x^2) \), and \( (\Gamma_{f,t}^1 \cdot V(t))_0 = [V(3x^2 + 2t^2, 3y^2 - x^3 - t^2x^2, t)]_0 = 2 \), and \( H^{k-0}(\mathbb{N}_{\{0\}}, \mathbb{L}_{\{0\}}; F^*) = H^{k+2}(\mathbb{N}_{\{0\}}, \mathbb{L}_{\{0\}}; \mathbb{Z}) \) isomorphic to \( \mathbb{Z}^2 \) if \( k = 0 \), and is 0 if \( k \neq 0 \).

Therefore, we find that \( \text{gecc}^0(F^*) = 0 \) if \( k \neq 0 \), and

\[
\text{gecc}^0(F^*) = Z \left[ \frac{T_{S_1}^* C_1^3}{} \right] + Z \left[ \frac{T_{S_2}^* C_2^3}{} \right] + Z^2 \left[ \frac{T_{S_3}^* C_3^3}{} \right] + Z \left[ \frac{T_{S_4}^* C_4^3}{} \right] + Z^2 \left[ \frac{T_{0}^* C_5^3}{} \right].
\]

**Example 4.8.** We continue with the setting of Example [17] where \( X = V(y) \cup V(y^2 - x^3 - t^2x^2) \) and \( F^* = \mathbb{Z}_X^*[2] \). We had Whitney strata consisting of \( \{0\} \), \( S_1 = V(y) - V(y^2 - x^3 - t^2x^2) \), \( S_2 = V(y^2 - x^3 - t^2x^2) - V(y) \), \( S_3 = V(x, y) - \{0\} \), and \( S_4 = V(x + t^2, y) - \{0\} \).
We found that \( \text{gecc}^k(F^\ast) = 0 \) if \( k \neq 0 \), and
\[
\text{gecc}^0(F^\ast) = Z \left[ T^\ast_{S_1} \mathbb{C}^3 \right] + Z \left[ T^\ast_{S_2} \mathbb{C}^3 \right] + Z^2 \left[ T^\ast_{S_3} \mathbb{C}^3 \right] + Z \left[ T^\ast_{S_4} \mathbb{C}^3 \right] + Z^2 \left[ T^\ast_{(0)} \mathbb{C}^3 \right].
\]
We will calculate \( (T^*_{x, p, C^3})^\ast \).

As we said above, we identify \( T^* \mathbb{C}^3 \) with \( \mathbb{C}^3 \times \mathbb{C}^3 \), and will use coordinates \( (w_0, w_1, w_2) \) for cotangent coordinates, so that \( (w_0, w_1, w_2) \) represents \( w_0 dx + w_1 dy + w_2 dt \).

Since \( x \) is identically zero on \( \{0\} \) and \( S_3 \), these two strata are not used in the calculation of \( (T^*_{x, p, C^3})^\ast \).

For the 1-dimensional stratum \( S_4 \), \( \left[ T^\ast_{x_{S_4}} \mathbb{C}^3 \right] \) is the 4-dimensional cycle \( V(x + t^2, y) \subseteq \mathbb{C}^3 \times \mathbb{C}^3 \).

The fiber of \( T^*_{x_{S_1}} \mathbb{C}^3 \) over any \( p \in S_1 \) is
\[
(T^*_{S_1} \mathbb{C}^3)_{p+} < dq > = \{ \omega + adp, x | \omega \in (T^*_{S_1} \mathbb{C}^3)_{p}, a \in \mathbb{C} \} = \{ bd_y + adp, x | a, b \in \mathbb{C} \}.
\]

Hence, \( \left[ T^\ast_{x_{S_1}} \mathbb{C}^3 \right] = V(y, w_2) \).

The fiber of \( T^*_{x_{S_2}} \mathbb{C}^3 \) over any \( p \in S_2 \) which is a regular point of \( x \) restricted to \( S_2 \) is
\[
(T^*_{S_2} \mathbb{C}^3)_{p+} < dq > = \{ \omega + adp, x | \omega \in (T^*_{S_2} \mathbb{C}^3)_{p}, a \in \mathbb{C} \} = \{ b((-3x^2 - 2t^2 x) dp, x + 2y dp, y - 2tx^2 dp, t) + adp, x | a, b \in \mathbb{C} \}.
\]

The form \( w_0 dp, x + w_1 dp, y + w_2 dp, t \) in this set if and only if the determinant of the following matrix is 0:
\[
\begin{bmatrix}
w_0 & w_1 & w_2 \\
-3x^2 - 2t^2 x & 2y & -2tx^2 \\
1 & 0 & 0
\end{bmatrix},
\]
i.e., if and only if \( yw_2 + tx^2 w_1 = 0 \). It is tempting to conclude that
\[
\left[ T^\ast_{x_{S_2}} \mathbb{C}^3 \right] = V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1),
\]
but this is not the case; we must eliminate any components of \( V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1) \) which are contained in \( V(y) \). Obviously, \( V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1) \) is purely 4-dimensional, and one easily shows that any component contained in \( V(y) \) must, in fact, equal \( V(x, y) \) (on the level of sets). Thus, we need to remove any components of \( V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1) \) which are contained in \( V(x, y) \).

Our notation for the resulting scheme (a gap sheaf, see [16], I.1) is
\[
V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1) \rightarrow V(x, y).
\]
Note that, as schemes,
\[
V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1) = V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1, y^2 w_2 + yt x^2 w_1) = V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1, (x^3 + t^2 x^2) w_2 + yt x^2 w_1).
\]
Note the \( x^2 \) factor of the last polynomial listed above, and note that, in the analytic set above, if \( x = 0 \), then \( y \) must be 0, i.e., if a point has \( x = 0 \), the point must be in \( V(x, y) \). Hence, using [16], I.1.3.iv, we find that, as cycles,
\[
V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1) \rightarrow V(x, y) = V(y^2 - x^3 - t^2 x^2, yw_2 + tx^2 w_1, (x + t^2) w_2 + ytw_1).
\]
(This last equality need not be true on the level of schemes, since our generators do not form a regular sequence and, hence, there may be embedded subvarieties.)

Therefore, we find that \((T^*_{x,F^*}C^3)^0\) is 0 unless \(k = 0\), and
\[
(T^*_{x,F^*}C^3)^0 = \mathbb{Z}[V(y, w_2)] + \mathbb{Z}[V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1)] + \mathbb{Z}[V(x + t^2, y)].
\]

**Example 4.9.** We continue with our setting from Example 4.7 and Example 4.8 and consider \(X = V(y) \cup V(y^2 - x^3 - t^2x^2)\) and \(F^* = \mathbb{Z}_X^*\). We will calculate \((\Gamma_{x,t}(F^*))^*\).

Using the isomorphism \(T^*C^3 \cong C^3 \times C^3\) from Example 4.8 im \(dt\) is the scheme
\[
V \left( w_0 - \frac{\partial t}{\partial x}, w_1 - \frac{\partial t}{\partial y}, w_2 - \frac{\partial t}{\partial t} \right) = V(w_0, w_1, w_2 - 1).
\]

In Example 4.8 we found that \((T^*_{x,F^*}C^3)^k\) is 0 unless \(k = 0\), and
\[
(T^*_{x,F^*}C^3)^0 = \mathbb{Z}[V(y, w_2)] + \mathbb{Z}[V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1)] + \mathbb{Z}[V(x + t^2, y)].
\]

Let
\[
E = V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1),
\]
on the level of cycles, throughout the remainder of this example.

Thus, \((\Gamma_{x,t}(F^*))^k\) is 0 unless \(k = 0\) and, to calculate \((\Gamma_{x,t}(F^*))^0\), we need first to calculate the three ordinary cycles
\[
\pi_*(V(y, w_2) \cdot V(w_0, w_1, w_2 - 1)),
\pi_*(E \cdot V(w_0, w_1, w_2 - 1)),
\]
and
\[
\pi_*(V(x + t^2, y) \cdot V(w_0, w_1, w_2 - 1)).
\]

Now, \(V(y, w_2) \cap V(w_0, w_1, w_2 - 1) = \emptyset\), and so \(\pi_*(V(y, w_2) \cdot V(w_0, w_1, w_2 - 1)) = 0\). In addition, it is trivial that there is an equality of cycles \(\pi_*(V(x + t^2, y) \cdot V(w_0, w_1, w_2 - 1)) = V(x + t^2, y)\). However, the remaining cycle is more difficult to calculate.

It is trivial that, as sets,
\[
E \cap V(w_0, w_1, w_2 - 1) = V(x + t^2, y, w_0, w_1, w_2 - 1),
\]
but the difficulty in calculating
\[
\pi_*(E \cdot V(w_0, w_1, w_2 - 1))
\]
is due to the fact that \(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1\) is not a regular sequence. To “fix” this, note that, in Example 4.8, we saw that, as cycles, there is an equality
\[
V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) = C + E,
\]
where the underlying set \(|C| = V(x, y)\). Therefore,
\[
C \cdot V(w_0, w_1, w_2 - 1) + E \cdot V(w_0, w_1, w_2 - 1) = V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1) \cdot V(w_0, w_1, w_2 - 1) =
\]

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Thus, as cycles,
\[ E \cdot V(w_0, w_1, w_2 - 1) = V(x + t^2, y, w_0, w_1, w_2 - 1), \]
and so \( \pi_*(E \cdot V(w_0, w_1, w_2 - 1)) = V(x + t^2, y). \)

Finally, we find that
\[ (\Gamma^1_{x,t}(F^*))^0_0 = \pi_0^0 ((T^*_{f,v}U)^* \circ \text{im} dt) = Z[V(x + t^2, y)] + Z[V(x + t^2, y)] = Z^2[V(x + t^2, y)]. \]

We shall discuss the main results on the graded, enriched relative conormal cycle and the graded, enriched relative polar curve in the following sections.

5 The Nearby Cycles

The following theorem was our primary motivation for defining the graded, enriched conormal cycle. While we state the theorem in the elegant form given in [17], this theorem, in terms of ordinary cycles, is essentially contained in [2].

**Theorem 5.1.** ([17], Theorem 3.3) There is an equality of graded enriched cycles given by
\[ \text{gecc}^*(\psi_f[-1]F^*) = (T^*_{f,v}U)^* \circ (V(f) \times \mathbb{C}^{n+1}). \]

If one knows the irreducible components \( \{ V_j \}_j \) of the underlying set \( \text{SS}(\psi_f[-1]F^*) \), then, by selecting a generic point of each \( V_j \), and taking a normal slice, the calculation of \( \text{gecc}^*(\psi_f[-1]F^*) \) is reduced to calculating the Morse modules of point strata. In other words, if we know \( \text{SS}(\psi_f[-1]F^*) \), then, by taking normal slices, the calculation of \( \text{gecc}^*(\psi_f[-1]F^*) \) reduces to calculating \( m_0^k(\psi_f[-1]F^*), \) i.e., \( H^k(\phi_{\mathcal{L}}[-1]\psi_f[-1]F^*_0) \), where \( \mathcal{L} \) is the restriction to \( V(f) \) of a generic linear form \( \tilde{L} \).

The next result follows from Theorem 4.2 of [14], but is stated as in Theorem 3.12 of [15].

**Theorem 5.2.**
\[ m_0^k(\psi_f[-1]F^*) = \left( (\Gamma^1_{f,\tilde{L}}(F^*))^k \circ V(f) \right)_0 = \bigoplus_{S \in \mathcal{G}(F^*)} m_0^k(\mathcal{S}) \otimes R^{\alpha_S}, \]
where \( \alpha_S := (\Gamma^1_{f,\tilde{L}}(S) \cdot V(f))_0 \) and \( \tilde{L} \) is a generic linear form. Specifically, the amount of genericity that we need is that \( (0, d_0 \tilde{L}) \notin \text{SS}(\psi_f[-1]F^*) - T_0^\circ \mathcal{U} \), which is equivalent to \( \dim_0 |(\Gamma^1_{f,\tilde{L}}(F^*))^* \cap V(f)| \leq 0. \)
Example 5.3. We continue with our example from Example 4.9. For $X = V(y) \cup V(y^2 - x^3 - t^2x^2)$, $F^* = \mathbb{Z}_X[2]$, and we found that $(T_{x,p}^* \mathbb{C}^3)^0$ is 0 unless $k = 0$, and

$$(T_{x,p}^* \mathbb{C}^3)^0 = \mathbb{Z}[V(y, w_2)] + \mathbb{Z}[V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w) + \mathbb{Z}[V(x + t^2, y)]].$$

Let $f := x$.

In light of Theorem 5.1, we find that $\text{gecc}^*(\psi_f[-1]|F^*)$ is concentrated in degree 0, and that

$$(\dagger) \quad \text{gecc}^0(\psi_f[-1]|F^*) = \mathbb{Z}[V(x, y, w_2)] + \mathbb{Z}^2[V(x, y, t)] + \mathbb{Z}[V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w1) \cdot V(x)]^{\text{enr}}.$$

The difficulty is in calculating the cycle $E := [V(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w) \cdot V(x)]$.

The underlying set $|E|$ is easily found to be $V(x, y, t) \cup V(x, y, w_2)$, and we may find the geometric multiplicity of each component in $E$ by moving to generic points.

At a generic point of $V(x, y, t)$, $w_2 \neq 0$ and, at such a point, one easily shows that there is an equality of ideals

$$(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w) = (yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w)$$

and, as $yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w$ is a regular sequence, one easily calculates that, at a point where $w_2 \neq 0$, there are equalities of cycles

$$[V(yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w) \cdot V(x)] = [V(yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w, x)] = [V(y, t^2, x)] = 2[V(x, y, t)].$$

This is the component of $E$ with underlying set $V(x, y, t)$.

At a generic point of $V(x, y, w_2)$, neither $t$ nor $w_1$ is zero, and it follows that $x + t^2$ is not zero. At such a point, one easily shows that there is again an equality of ideals

$$(y^2 - x^3 - t^2x^2, yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w) = (yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w)$$

and one easily calculates that, at a point where neither $t$ nor $w_1$ is zero, there are equalities of cycles

$$[V(yw_2 + tx^2w_1, (x + t^2)w_2 + yt_1w) \cdot V(x)] = [V(yw_2, tw_2 + yw_1, x)] = [V(y, tw_2, x)] + [V(w_2, yw_1, x)] = [V(y, w_2, x)] + [V(w_2, y, x)] = 2[V(x, y, w_2)].$$

This is the component of $E$ with underlying set $V(x, y, w_2)$.

Therefore, $(\dagger)$ tells us that

$$\text{gecc}^0(\psi_f[-1]|F^*) = \mathbb{Z}_2[V(x, y, w_2)] + \mathbb{Z}_2[V(x, y, t)].$$

Note that the component of $\text{gecc}^0(\psi_f[-1]|F^*)$ over the origin agrees with Theorem 5.2 and our calculation in Example 4.9. For

$$(0, d_0t) \not\in \text{SS}(\psi_f[-1]|F^*) - D_0,$$ V(x, y, w_2)$$

and

$$(\Gamma_{x,f}^1(F^*))^0 \cdot V(t) = \mathbb{Z}_2[V(x + t^2, y)] \cdot V(t) = \mathbb{Z}_2[V(x, y, t)].$$

Another example, which we leave as an exercise for the reader, is to recalculate $\text{gecc}^*(\psi_f[-1]|F^*)$ from Example 2.9 by using either Theorem 5.1 or Theorem 5.2.
6 Hypersurface Complements and Restrictions

Let \( i : X - V(f) \hookrightarrow X \) and \( j : V(f) \hookrightarrow X \) denote the inclusions. Recall that we are assuming that \( V(f) \) is a union of strata, and recall the partial ordering on isomorphism classes of finitely-generated \( R \)-modules given in Definition 3.1.

We would like to give an elegant formula for \( \text{gecc}^\bullet (\psi_f[-1] \mathbf{F}^\bullet) \), something along the lines of what we gave for \( \text{gecc}^\bullet (\psi_f[-1] \mathbf{F}^\bullet) \) in Theorem 6.1. We do not quite do this. However, we can do the next best thing; we can give a formula for the set \( \text{SS}(\psi_f[-1] \mathbf{F}^\bullet) \) and a formula for \( m^k_0(\psi_f[-1] \mathbf{F}^\bullet) \).

Once we have these formulas, and so, in principle, know \( \text{gecc}^\bullet (\psi_f[-1] \mathbf{F}^\bullet) \), we can use how the graded, enriched characteristic cycle works with Verdier duals to obtain \( \text{gecc}^\bullet (\psi_f[-1] \mathbf{F}^\bullet) \). In addition, we can use the additivity of ordinary characteristic cycles over distinguished triangles, and the duality formula, to obtain the characteristic cycles of \( j_* j^* [-1] \mathbf{F}^\bullet \) and \( j^! j^*[1] \mathbf{F}^\bullet \) when the base ring is a domain.

The following result is immediate from Theorem 4.2 B of [14] and, for ordinary cycles, is proved in [2].

**Theorem 6.1.**

\[
m^k_0(\psi_f[-1] \mathbf{F}^\bullet) \cong \left( (\Gamma^1_{f, L}(\mathbf{F}^\bullet))^k \otimes V(\mathcal{L}) \right)_0 = \bigoplus_{S \in \mathcal{S}(\mathbf{F}^\bullet)} m^k_S(\mathbf{F}^\bullet) \otimes R^{\beta_S},
\]

where \( \beta_S := \left( \Gamma^1_{f, L}(S) \cdot V(\mathcal{L}) \right)_0 \) and \( \mathcal{L} \) is a generic linear form. Specifically, the amount of genericity that we need is that \( (0, d_0 \mathcal{L}) \not\in \text{SS}(\psi_f[-1] \mathbf{F}^\bullet) - T_0^0 \mathcal{U} \) and \( \dim_0 (\Gamma^1_{f, L}(\mathbf{F}^\bullet))^\ast \cap V(\mathcal{L}) \leq 0 \).

**Corollary 6.2.** If \( S \in \mathcal{S} \) and \( S \not\subseteq V(f) \), then \( m^k_S(\psi_f[-1] \mathbf{F}^\bullet) \cong m^k_S(\mathbf{F}^\bullet) \).

If \( S \in \mathcal{S} \) and \( S \subseteq V(f) \), then \( m^k_S(\psi_f[-1] \mathbf{F}^\bullet) \neq 0 \) if and only if \( m^k_S(\psi_f[-1] \mathbf{F}^\bullet) \neq 0 \), and \( m^k_S(\psi_f[-1] \mathbf{F}^\bullet) \leq m^k_S(\psi_f[-1] \mathbf{F}^\bullet) \).

**Proof.** Outside of \( V(f) \), the complex \( \psi_f[-1] \mathbf{F}^\bullet \) agrees with \( \mathbf{F}^\bullet \); this yields the first statement.

A comparison of Theorem 6.1 with Theorem 6.2 yields the second statement. \( \square \)

**Definition 6.3.** Suppose that \( \mathbf{E}^\bullet \) is a graded, enriched cycle in \( T^* \mathcal{U} \) given by \( E^k = \sum_{S \in \mathcal{S}} E^k_S[T^* \mathcal{U}] \), where \( E^k_S \) is a finitely-generated \( R \)-module.

Let \( (E_{\mathbf{E}(f)})^\bullet \) be the graded, enriched cycle such that \( (E_{\mathbf{E}(f)})^k \) is the sum of those \( E^k_S[T^* \mathcal{U}] \) such that \( S \not\subseteq V(f) \). Similarly, let \( (E_{\mathbf{E}(f)})^\bullet \) be the graded, enriched cycle such that \( (E_{\mathbf{E}(f)})^k \) is the sum of those \( E^k_S[T^* \mathcal{U}] \) such that \( S \subseteq V(f) \).

Let \( |E^k|_{\mathbf{E}(f)} := ||E_{\mathbf{E}(f)}|| \) and \( |E^k|_{\mathbf{E}(f)} := ||E_{\mathbf{E}(f)}|| \).

**Corollary 6.4.**

\[
|\text{gecc}^k(\psi_f[-1] \mathbf{F}^\bullet)| = |\text{gecc}^k(\mathbf{F}^\bullet)|_{\mathbf{E}(f)} \cup |\text{gecc}^k(\psi_f[-1] \mathbf{F}^\bullet)|.
\]
Proof. This is immediate from Corollary 6.2. □

In light of Corollary 6.4 and the hypotheses on $\hat{L}$ in Theorem 6.1, the following proposition is of interest.

**Proposition 6.5.** Suppose that $(0, d_0 \hat{L}) \not\in |SS(F^*)|_{V(f)}$. Then, $\dim_0 \rho |(\Gamma^1_{f, \hat{L}}(F^*))^*| \cap V(L) \leq 0$ if and only if $\dim_0 \rho |(\Gamma^1_{f, \hat{L}}(F^*))^*| \cap V(f) \leq 0$.

**Proof.** Lemma 3.10 of [18] tells us that, if $\dim_0 |(\Gamma^1_{f, \hat{L}}(F^*))^*| \cap V(f) \leq 0$, then $\dim_0 \rho |(\Gamma^1_{f, \hat{L}}(F^*))^*| \cap V(L) \leq 0$.

Suppose then that $\dim_0 |(\Gamma^1_{f, \hat{L}}(F^*))^*| \cap V(f) \leq 0$. Let $p(t)$ be an analytic parametrization of an irreducible component $C$ of $|(\Gamma^1_{f, \hat{L}}(F^*))^*|$ such that $p(0) = 0$. Suppose that $C \subseteq V(\hat{L})$; we wish to derive a contradiction.

Let $S' \in \hat{S}$ be an $F^*$-visible stratum such that $C = \pi(T_{f, \hat{L}}U \cap \text{im} \hat{L})$. Let $S$ denote the stratum of $\hat{S}$ which contains $p(t)$ for $t \neq 0$. Note that neither $S$ nor $S'$ is contained in $V(f)$, since $\dim_0 \rho |(\Gamma^1_{f, \hat{L}}(F^*))^*| \cap V(f) \leq 0$. On the other hand, in a neighborhood of the origin, the stratified critical locus of $\tilde{f}$ is contained in $V(f)$.

It follows that, for all $x \in C - \{0\}$, the fiber $(T_{f, \hat{L}}U)_x$ is equal to $(T_{\hat{L}}U)_x + \{d_x \tilde{f}\}$. Thus, for $t \neq 0$, there exists a complex number $a(t)$ such that

\[
\begin{align*}
(\dagger) \quad & \quad d_{p(t)} \hat{L} + a(t)d_{p(t)} \tilde{f} \in (T_{\hat{L}}U)_{p(t)} \subseteq (T_{\hat{L}}U)_{p(t)}. \\
\end{align*}
\]

By evaluating at $p'(t)$, we find that $(L(p(t)))' + a(t)(f(p(t)))' \equiv 0$. As $C \subseteq V(\hat{L})$, we find that $a(t)(f(p(t)))' \equiv 0$. As $C \not\subseteq V(f)$, we conclude that $a(t) \equiv 0$. From $(\dagger)$, it follows that $d_{p(t)} \hat{L} \in (T_{\hat{L}}U)_{p(t)}$ and, hence, that $(0, d_0 \hat{L}) \not\in T_{\hat{L}}U$. This contradicts the fact that $(0, d_0 \hat{L}) \not\in |SS(F^*)|_{V(f)}$. □

**Theorem 6.6.** For all $k$, $\text{gecc}^k(i_\ast i^*F^*) = \text{gecc}^k(i_\ast i^!F^*)$.

**Proof.** One uses Theorem 6.1 and Corollary 6.4, together with the isomorphisms $i_\ast i^*F^* \cong \mathcal{D} i_\ast i^!DF^*$ and $\mathcal{D} \psi_f[-1] \cong \psi_f[-1] \mathcal{D}$. We leave the proof as an exercise. □

**Remark 6.7.** Note that $\text{gecc}^k(i_\ast i^*F^*)$ and $\text{gecc}^k(i_\ast i^!F^*)$ do not depend on any degree of $\text{gecc}^*F^*$, other than the degree $k$ portion. In particular, if $\text{gecc}^*F^*$ is concentrated in degree 0, then so are $\text{gecc}^*F^*$ and $\text{gecc}^*F^*$. Thus, we recover the well-known fact that, if $F^*$ is a perverse sheaf and $i$ is the inclusion of the complement of the zero locus of a single function, then $i_\ast i^*F^*$ and $i_\ast i^!F^*$ are also perverse.

Recall that we have the closed inclusion $j : V(f) \hookrightarrow X$. Unlike the functors $i_\ast i^*$ and $i_\ast i^!$, discussed above, the functors $j_\ast j^*$ and $j_\ast j^!$ (with or without shifts) do not take perverse sheaves to perverse sheaves. In other words, $\text{gecc}^k(j_\ast j^*F^*)$ and $\text{gecc}^k(j_\ast j^!F^*)$ are not determined by a single degree of $\text{gecc}^*F^*$. Of course, given the canonical distinguished triangles relating $j_\ast j^*$ and $i_\ast i^!$, and $j_\ast j^!$ and $i_\ast i^*$, we immediately obtain:

**Proposition 6.8.** $CC(j_\ast j^*F^*) = CC(F^*) - CC(i_\ast i^!F^*) = CC(F^*) - CC(i_\ast i^*F^*) = CC(j_\ast j^!F^*)$. 

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7 The Vanishing Cycles

Before we can give a formula for \( \text{gecc}^* (\phi_f [\bar{1}] \mathbb{F}^*) \), we must first discuss the graded, enriched exceptional divisor in the blow-up of a graded, enriched cycle along an ideal.

Let us recall the notation established thus far. \( \mathcal{U} \) is an open neighborhood of the origin of \( \mathbb{C}^{n+1} \), \( X \) is a closed, analytic subset of \( \mathcal{U} \), \( z := (z_0, \ldots, z_n) \) are coordinates on \( \mathcal{U} \), we identify the cotangent space \( T^* \mathcal{U} \) with \( \mathcal{U} \times \mathbb{C}^{n+1} \) by mapping \( (p, w_0 d_p z_0 + \cdots + w_n d_p z_n) \) to \( (p, (w_0, \ldots, w_n)) \), and we let \( \pi : T^* \mathcal{U} \to \mathcal{U} \) denote the projection.

Consider a graded, enriched cycle \( D^* \) in \( T^* \mathcal{U} \) given by \( D^k := \sum \mathbb{D}_{\mathbb{F}_{\pi}}^k [V] \). Let \( h_0, \ldots, h_m \) be analytic functions on \( T^* \mathcal{U} \), and let \( I \) be the ideal \( \langle h_0, \ldots, h_m \rangle \). Then, for each \( V \), the blow-up \( \text{Bl}_I \mathcal{V} \) of \( \mathcal{V} \) along \( I \) is naturally a subspace of \( T^* \mathcal{U} \times \mathbb{P}^m \cong \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^m \). Let \( \text{Ex}_I (V) \) denote the exceptional divisor as a cycle.

**Definition 7.1.** The graded, enriched blow-up \( \text{Bl}_I^* (D^*) \) of \( D^* \) along \( I \) in \( T^* \mathcal{U} \times \mathbb{P}^m \) is given by \( \text{Bl}_I^* (D^*) := \sum V \mathbb{D}_{\mathbb{F}_{\pi}}^k [\text{Bl}_I \mathcal{V}] \).

The graded, enriched exceptional divisor \( \text{Ex}_I^* (D^*) \) of \( D^* \) along \( I \) in \( T^* \mathcal{U} \times \mathbb{P}^m \) is given by \( \text{Ex}_I^* (D^*) := \sum V \mathbb{D}_{\mathbb{F}_{\pi}}^k [\text{Ex}_I \mathcal{V}] \).

Instead of subscripting the blow-up and exceptional divisor by the ideal \( I \), it is common to subscript by the analytic scheme \( \mathcal{V} (I) \). In particular, below, we shall blow-up along \( \text{im}(d\tilde{f}) \subseteq \mathcal{U} \times \mathbb{C}^{n+1} \); we remind the reader that is defined by the ideal

\[
\left\langle w_0 - \frac{\partial \tilde{f}}{\partial z_0}, \ldots, w_n - \frac{\partial \tilde{f}}{\partial z_n} \right\rangle.
\]

Let \( \tau : \mathcal{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathcal{U} \times \mathbb{P}^n \) denote the projection, and recall that \( \tau_* \) denotes the proper push-forward. The following is Theorem 3.5 of [17]. Without the graded, enriched structure, we proved this result in [15], and it was also proved independently in [19].

**Theorem 7.2.** There is an equality of closed subsets of \( X \) given by

\[
\bigcup_{v \in \mathbb{C}} \text{supp} \phi_{f-v}[-1] \mathbb{F}^* = \pi (\text{SS}(\mathbb{F}^*) \cap \text{im}(d\tilde{f}))
\]

and, for all \( k \), an equality of graded, enriched cycles given by

\[
\sum_{v \in \mathbb{C}} \mathbb{P}(\text{gecc}^k (\phi_{f-v}[-1] \mathbb{F}^*)) = \tau_* (\text{Ex}_{\text{im}(d\tilde{f})} (\text{gecc}^k (\mathbb{F}^*))).
\]

In particular, for all \( k \), there is an equality of sets

\[
\bigcup_{v \in \mathbb{C}} \pi (|\text{gecc}^k (\phi_{f-v}[-1] \mathbb{F}^*)|) = \pi (|\text{gecc}^k (\mathbb{F}^*)| \cap \text{im}(d\tilde{f})).
\]

**Remark 7.3.** We should remark that, in the above unions and sum over \( v \in \mathbb{C} \), the unions and sum are not merely locally finite, but, in fact, locally over open neighborhoods of points in \( X \), there is only one non-zero (or non-empty) summand (respectively, indexed subset in the union).
The following result follows at once from Theorem 4.2 of [14], and is used in the proof of Theorem 7.2.

**Theorem 7.4.**

\[ m^k_0(\phi_f[-1]F^\bullet) \cong m^k_0(F^\bullet) \oplus \bigoplus_{S \in \mathfrak{S}(F^\bullet)} m^k_S(F^\bullet) \otimes R^{\delta_S}, \]

where \( \delta_S := \left( \Gamma^1_{f, \tilde{L}}(S) \cdot V(F) \right)_0 - \left( \Gamma^1_{f, \tilde{L}}(S) \cdot V(\tilde{L}) \right)_0 \), where \( \tilde{L} \) is a generic linear form; specifically, we need for the following three conditions to hold:

1. \( (0, 0, \tilde{L}) \notin \text{SS}(\phi_f[-1]F^\bullet) - T_0^\mathfrak{U} \);
2. \( \dim_0(\Gamma^1_{f, \tilde{L}}(F^\bullet))^\ast \cap V(\tilde{L}) \leq 0 \), and
3. for all \( F^\bullet \)-visible strata \( S \), not contained in \( V(F) \), for all components \( C \) of \( \Gamma^1_{f, \tilde{L}}(S) \),

\[ \left( C \cdot V(\tilde{F}) \right)_0 \geq \left( C \cdot V(\tilde{L}) \right)_0. \]

From this theorem and Theorem 5.2 it follows immediately that:

**Corollary 7.5.** For all \( k \),

\[ |\text{gecc}^k(F^\bullet)|_{\leq V(F)} \subseteq |\text{gecc}^k(\phi_f[-1]F^\bullet)| \subseteq |\text{gecc}^k(F^\bullet)|_{\leq V(f)} \cup |\text{gecc}^k(\psi_f[-1]F^\bullet)|. \]

**Remark 7.6.** In this long remark, we wish to address how effectively one can calculate \( \text{gecc}^\bullet(\phi_f[-1]F^\bullet) \), given \( f \) and \( \text{gecc}^\bullet(F^\bullet) \). This will require us to discuss much of our work in [12], [16], and [17]. As usual, we will identify \( T^\mathfrak{U} \) with \( \mathfrak{U} \times \mathbb{C}^{n+1} \), and use four different projections: \( \tau : \mathfrak{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathfrak{U} \times \mathbb{P}^n \), \( \eta : \mathfrak{U} \times \mathbb{P}^n \to \mathfrak{U} \), \( \nu : \mathfrak{U} \times \mathbb{C}^{n+1} \times \mathbb{P}^n \to \mathfrak{U} \times \mathbb{C}^{n+1} \), and \( \pi : \mathfrak{U} \times \mathbb{C}^{n+1} \to \mathfrak{U} \).

We shall assume that our base ring \( R \) is a PID, and that \( X \) has codimension at least 1 in \( \mathfrak{U} \) (so that our projectivizations below do not totally discard components).

Assume that we have re-chosen \( \mathfrak{U} \) small enough so that \( \mathbb{P}(\text{gecc}^k(\phi_f[-1]F^\bullet)) \) is the only non-zero summand in Theorem 7.2. Then, Theorem 7.2 gives a nice, elegant algebraic characterization of the projectivized \( \text{gecc} \) of \( \phi_f[-1]F^\bullet \), in terms of blow-ups and exceptional divisors. The problem is that blow-ups and exceptional divisors are not so easy to calculate.

Suppose that \( A^\bullet \) is a bounded complex of sheaves of modules over \( R \), which is constructible with respect to \( \mathfrak{S} \). We shall first describe a general method for “calculating” \( \text{gecc}^\bullet(A^\bullet) \), and then describe in the case of \( \text{gecc}^\bullet(\phi_f[-1]F^\bullet) \) why this really leads to an effective calculation.

First, projectivize and obtain \( \mathbb{P}(\text{gecc}^k(A^\bullet)) = \sum_{S \in \mathfrak{S}} m^k_S(A^\bullet)[\mathbb{P}(T^\mathfrak{U}_{S})] \subseteq \mathfrak{U} \times \mathbb{P}^n \). Recall that our coordinates \( z = (z_0, \ldots, z_n) \) determine our cotangent coordinates \( (w_0, \ldots, w_n) \) and, hence, determine projective coordinates \( [w_0 : \cdots : w_n] \) on \( \mathbb{P}^n \). We assume that \( \mathfrak{U} \) is small enough and that our coordinates \( z \) are generic enough so that, for all \( S \) such that \( T^\mathfrak{U}_{S} \) is a component of \( \text{SS}(A^\bullet) \), for all \( j \) such that \( 0 \leq j \leq n \), the intersection of \( \mathbb{P}(T^\mathfrak{U}_{S}) \) and \( \mathfrak{U} \times \mathbb{P}^j \times \{0\} \) in \( \mathfrak{U} \times \mathbb{P}^n \) is proper, and so is purely \( j \)-dimensional. We claim that the proper push-forwards

\[ \eta_* \circ \mathbb{P}(\text{gecc}^k(A^\bullet)) \circ \mathfrak{U} \times \mathbb{P}^j \times \{0\} \]
completely determine $P(gecc^k(A^*))$ (and, hence, $gecc^k(A^*)$). The $k\Gamma_{A^*}^j$ are the characteristic polar cycles; we refer the reader to Section 5 of [17].

The characteristic polar cycles determine $P(gecc^k(A^*))$ by downward induction on the dimension of strata of $X$. Let $d := \text{dim} X$, which we are assuming is at most $n$.

Consider first a stratum $S$ of dimension $d$. Then, $\overline{T^d_SU}$ appears in $gecc^k(A^*)$ if and only if $\overline{S}$ is a component of $|k\Gamma_{A^*}^d|$. In addition, as

$$\eta_*\left( m^k_S(A^*)[\overline{T^d_SU}] \cap U \times \mathbb{P}^d \times \{0\} \right) = m^k_S(A^*)\left[ \eta_*\left( \overline{T^d SU} \cap U \times \mathbb{P}^d \times \{0\} \right) \right],$$

once we know that $\overline{T^d_SU}$ appears in $gecc^k(A^*)$ and we know $k\Gamma_{A^*}^d$, then we can determine $m^k_S(A^*)$. Note that in this process, we do not actually determine the stratum $S$, but rather a closed analytic set which agrees with $S$ on an open dense set – but this is enough.

Now, suppose that we know the pieces of the enriched cycle $P(gecc^k(A^*))$ for all of those strata of dimension at least $j + 1$. Let us write $D_{\geq j+1}$ for the (enriched) sum of these pieces. Then, one can calculate the enriched cycle $\eta_*\left(D_{\geq j+1} \cap U \times \mathbb{P}^j \times \{0\}\right)$; this cycle is an enriched form of the $j$-dimensional absolute polar varieties of the strata of dimension at least $j + 1$. Now, one can consider the difference (we use that $R$ is a PID here)

$$M^j := k\Gamma_{A^*}^j - \eta_*\left(D_{\geq j+1} \cap U \times \mathbb{P}^j \times \{0\}\right).$$

Suppose that $S$ is a stratum of dimension $j$. Then, one easily sees that $\overline{T^j_SU}$ appears in $gecc^k(A^*)$ if and only if $\overline{S}$ is a component of $|M^j|$. As above, once we know that $\overline{T^j_SU}$ appears in $gecc^k(A^*)$ and we know $M^j$, we can determine $m^k_S(A^*)$ by calculating $\left[ \eta_*\left( \overline{T^j SU} \cap U \times \mathbb{P}^j \times \{0\} \right) \right]$. We have seen above that the characteristic polar cycles determine $gecc^*(A^*)$. The question now is: how does one effectively calculate the characteristic polar cycles in the case where $A^* = \phi_f[-1]F^*$?

Let us adopt the notation $k\Lambda_{\phi_f[-1]F^*}^j := k\Gamma_{\phi_f[-1]F^*}^j$. From our discussion above, we see that we could reconstruct $gecc^*(\phi_f[-1]F^*)$ if we knew the $k\Lambda_{\phi_f[-1]F^*}^j$. The result of Corollary 6.8 of [17] gives an algorithm for calculating the $k\Lambda_{\phi_f[-1]F^*}^j$, assuming that the coordinates $z$ are $\phi_f[-1]F^*$-isolating. Let us put off the discussion of what $\phi_f[-1]F^*$-isolating means; for now, simply assume that the coordinates are generic enough to make true what we write below.

We work in each degree separately; so, fix $k$.

Let $\Pi^{n+1} := gecc^k(F^*)$. Then, $\Pi^{n+1}$ properly intersects $V\left(w_n - \frac{\partial f}{\partial z_n}\right)$, and we may consider the enriched cycle defined by the intersection

$$\sum_{V} M_V[V] := \Pi^{n+1} \cap V\left(w_n - \frac{\partial f}{\partial z_n}\right);$$

this enriched cycle may have some components contained in $\text{im}(df)$ and some components not contained in $\text{im}(df)$. Let $\Pi^n := \sum_{V \subseteq \text{im}(df)} M_V[V]$ and let $\Delta^n := \sum_{V \subseteq \text{im}(df)} M_V[V]$.
Now, proceed inductively: if we have $\Pi^j + 1$, then $V\left(w_j - \frac{\partial f}{\partial z_j}\right)$ properly intersects $\Pi^j + 1$, and we define $\Pi^j$ and $\Delta^j$ by the equality

$$\Pi^j + 1 \cap V\left(w_j - \frac{\partial f}{\partial z_j}\right) = \Pi^j + \Delta^j,$$

where no component of $\Pi^j$ is contained in $\text{im}(d\hat{f})$, and every component of $\Delta^j$ is contained in $\text{im}(d\hat{f})$.

Continue with this process until one obtains $\Pi^0$ and $\Delta^0$.

Then, for all $j$, as germs at $p$, $k\Lambda_j^{\ast} (F^\ast) = \pi_\ast(\Delta^j)$ and

$$\left[k\Lambda_j^{\ast} (F^\ast) \cap V(z_0 - p_0, \ldots, z_{j-1} - p_{j-1})\right]_p \cong H^k(\phi_{z_j - p_j} [1] \psi_{z_{j-1} - p_{j-1} [-1]} \ldots \psi_{z_0 - p_0} [1] \phi_f [1] F^\ast)_p,$$

where, when $j = 0$, we mean

$$\left[k\Lambda_0^{\ast} (F^\ast)\right]_p \cong H^k(\phi_{z_0 - p_0} [1] \phi_f [1] F^\ast)_p.$$

Note that, as we are interested in the end only in the $\Delta^j$, throughout the algorithm above, we may, in each step, discard any components of $\Pi^j$ which do not intersect $\text{im}(d\hat{f})$.

The above works very well for calculating the germs of $k\Lambda_j^{\ast} (F^\ast)$ at $p$, and so $\text{gecc}^\ast (\phi_f [1] F^\ast)$ above a neighborhood of $p$, as long as the coordinates $z$ are $\phi_f [1] F^\ast$-isolating at $p$. In [17], we give two characterizations of $\phi_f [1] F^\ast$-isolating that are relevant here.

Let $s := \dim_p \text{supp} \phi_f [1] F^\ast = \dim_p \pi(\text{ss}(\phi_f [1] F^\ast))$. Then, the coordinates $z$ are $\phi_f [1] F^\ast$-isolating at $p$ if and only if, for all $j$ such that $0 \leq j \leq s - 1$, $p$ is an isolated point in the support of

$$\phi_{z_j - p_j} [-1] \psi_{z_{j-1} - p_{j-1} [-1]} \ldots \psi_{z_0 - p_0} [1] \phi_f [1] F^\ast.$$

This is equivalent to:

for all $j$ such that $0 \leq j \leq s - 1$, there exists an open neighborhood $W$ of $p$ in $U$ such that $P(\text{ss}(\phi_f [1] F^\ast))$ properly intersects $W \times \mathbb{P}^j \times \{0\}$ inside $W \times \mathbb{P}^n$ and

$$\dim_p \left(V(z_0 - p_0, \ldots, z_{j-1} - p_{j-1}) \cap \eta\left(P(\text{ss}(\phi_f [1] F^\ast)) \cap W \times \mathbb{P}^j \times \{0\}\right)\right) \leq 0. \quad (1)$$

When $j = 0$, this condition is interpreted as

$$\dim_p \eta\left(P(\text{ss}(\phi_f [1] F^\ast)) \cap W \times \{1 : 0 : 0 : \cdots : 0\}\right) \leq 0.$$

Are either one of these characterizations of $\phi_f [1] F^\ast$-isolating useful? Yes - the latter one is. Corollary 7.5 tells us that

$$\text{ss}(\phi_f [1] F^\ast) \subseteq (\text{ss}(F^\ast))_{\xi V(f)} \cup \text{ss}(\psi_f [1] F^\ast).$$

So, if our coordinates are generic enough so that Formula 1 above holds with $P(\text{ss}(\phi_f [1] F^\ast))$ replaced by $P((\text{ss}(F^\ast))_{\xi V(f)} \cup P(\text{ss}(\psi_f [1] F^\ast))$, then the entire process above works.
Now we can do an example.

**Example 7.7.** We continue with our earlier situation: \( X = V(y) \cup V(y^2 - x^3 - t^2 x^2) \subseteq \mathbb{C}^3 \), \( \mathbf{F}^* = \mathbb{Z}_X[2] \), and \( \hat{f} := x \). This will give us an easy, but nonetheless, illustrative example of the procedure described in Remark 7.6.

From Example 4.7, we know that \( \text{SS}(\mathbf{F}^*) \subseteq V(f) = T^*_V(x,y) \cup T^*_0 \mathcal{U} \). From Example 5.3, we know that \( \text{SS}(\psi_f[-1] \mathbf{F}^*) \) is also equal to \( T^*_V(x,y) \cup T^*_0 \mathcal{U} \). Therefore, Corollary 7.5 tells us that

\[
\text{SS}(\phi_f[-1] \mathbf{F}^*) \subseteq T^*_V(x,y) \cup T^*_0 \mathcal{U},
\]

(in fact, Corollary 7.5 tells us that this an an equality, though we will not use this stronger fact).

Hence,

\[
\text{supp} \phi_f[-1] \mathbf{F}^* = \pi(\text{SS}(\phi_f[-1] \mathbf{F}^*)) \subseteq \pi(T^*_V(x,y) \cup T^*_0 \mathcal{U}) = V(x,y).
\]

Considering how simple this set is, we could calculate \( \text{gecc}^\bullet(\phi_f[-1] \mathbf{F}^*) \) “barehandedly”, by applying Theorem 7.4 at the origin, and then moving to a generic point on \( V(x,y) \), taking a hyperplane slice, and applying Theorem 7.4 again.

However, we want to demonstrate the procedure that we described in Example 7.6. Hence, we will first determine \( \phi_f[-1] \mathbf{F}^* \)-isolating coordinates at \( 0 \), and then go through the graded, enriched cycle calculation from Example 7.6.

From the above, we see that \( s = \dim_0 \text{supp} \phi_f[-1] \mathbf{F}^* \leq 1 \), and thus our coordinates are \( \phi_f[-1] \mathbf{F}^* \)-isolating at \( 0 \) if Formula 1 holds for \( p = 0 \) and \( j = 0 \); this is the degenerate case mentioned immediately after Formula 1.

It follows that, if we let \((z_0,z_1,z_2) = (t,x,y)\), so that the cotangent coordinates \((w_0,w_1,w_2)\) correspond to \( w_0 dt + w_1 dx + w_2 dy \), then \( \mathbb{P}^0 = \{[1 : 0 : : : 0]\} \) in Formula 1 corresponds to the projective class \([dt]\) and

\[
\mathbb{P}(\text{SS}(\phi_f[-1] \mathbf{F}^*)) \cap \mathcal{W} \times \{[1 : 0 : : : 0]\} \subseteq \left( \mathbb{P}(T^*_V(x,y) \cup \mathbb{P}(T^*_0 \mathcal{U})) \cap \mathcal{W} \times \{[1 : 0 : : : 0]\} \right) = \mathbb{P}^0 \cup \{0,[1 : 0 : : : 0]\}.
\]

Therefore,

\[
\dim_0 \eta \left( \mathbb{P}(\text{SS}(\phi_f[-1] \mathbf{F}^*)) \cap \mathcal{W} \times \{[1 : 0 : : : 0]\} \right) \leq 0,
\]

and the coordinates \((t,x,y)\) are \( \phi_f[-1] \mathbf{F}^* \)-isolating at \( 0 \). Note that this ordering on the coordinates is different from what we used earlier, because we need for \( t \) to come first.

We can now proceed with the enriched cycle calculation as described in Example 7.6.

As we saw in Example 4.7, \( \text{gecc}^k(\mathbf{F}^*) = 0 \) if \( k \neq 0 \); thus, we need calculate only in the fixed degree \( k = 0 \). As we also saw in Example 4.7,

\[
\text{gecc}^0(\mathbf{F}^*) = \mathbb{Z} \left[ \frac{T^*_{S_1} \mathbb{C}^3}{T^*_{S_1} \mathbb{C}^3} \right] + \mathbb{Z} \left[ \frac{T^*_{S_2} \mathbb{C}^3}{T^*_{S_2} \mathbb{C}^3} \right] + \mathbb{Z}^2 \left[ \frac{T^*_{S_3} \mathbb{C}^3}{T^*_{S_3} \mathbb{C}^3} \right] + \mathbb{Z} \left[ \frac{T^*_{S_4} \mathbb{C}^3}{T^*_{S_4} \mathbb{C}^3} \right] + \mathbb{Z}^2 \left[ \frac{T^*_{(0)} \mathbb{C}^3}{T^*_{(0)} \mathbb{C}^3} \right],
\]

where \( S_1 = V(y) - V(y^2 - x^3 - t^2 x^2) \), \( S_2 = V(y^2 - x^3 - t^2 x^2) - V(y) \), \( S_3 = V(x,y) - \{0\} \), and \( S_4 = V(x + t^2,y) - \{0\} \). Using a computer algebra system to find equations defining \( \frac{T^*_{S_2} \mathbb{C}^3}{T^*_{S_2} \mathbb{C}^3} \), we have

\[
\text{gecc}^0(\mathbf{F}^*) = \mathbb{Z} \left[ V(y,w_0,w_1) \right] +
\]
and, hence, Finally, we find

\[ Z^2\left[ V(y, x + t^2, 2tw_1 - w_0, w_2) \right] + Z^2\left[ V(y, x + t^2, 2tw_1 - w_0, w_2) \right] + Z^2\left[ V(t, x, y, w_2) \right] + \text{components which do not intersect } \text{im}(df). \]

Therefore, we have

\[ \Pi^2 = Z^2\left[ V(y, t^2 + x, 2tw_1 - w_0, w_2) \right] + Z^2\left[ V(x, y, w_0, w_2) \right] + Z^2\left[ V(t, x, y, w_2) \right]. \]

We continue with the algorithm. We find

\[ \Pi^2 \circ V \left( w_1 - \frac{\partial x}{\partial x} \right) = \Pi^2 \circ V (w_1 - 1) = \]

\[ Z^2\left[ V(y, t^2 + x, 2t - w_0, w_1 - 1, w_2) \right] + Z^2\left[ V(x, y, w_0, w_1 - 1, w_2) \right] + Z^2\left[ V(t, x, y, w_1 - 1, w_2) \right], \]

and, hence,

\[ \Pi^1 = Z^2\left[ V(y, t^2 + x, 2t - w_0, w_1 - 1, w_2) \right] + Z^2\left[ V(t, x, y, w_1 - 1, w_2) \right] \]

and

\[ \Delta^1 = Z^2\left[ V(x, y, w_0, w_1 - 1, w_2) \right]. \]

Finally, we find

\[ \Pi^1 \circ V \left( w_0 - \frac{\partial x}{\partial t} \right) = \Pi^1 \circ V (w_0) = \]

\[ Z^2\left[ V(y, x, t, w_0, w_1 - 1, w_2) \right] + Z^2\left[ V(t, x, y, w_0, w_1 - 1, w_2) \right] = Z^4\left[ V(t, x, y, w_0, w_1 - 1, w_2) \right] = \Delta^0. \]

Taking proper push-forwards, we find that, inside \( C^3 \),

\[ 0^{A^1}_{f,*}(F^*) = Z^2\left[ V(x, y) \right] \]

and

\[ 0^{A^0}_{f,*}(F^*) = Z^4\left[ V(t, x, y) \right]. \]

We conclude easily now that

\[ \text{gecc}^0(\phi_f[-1]F^*) = Z^2\left[ V(x, y, w_2) \right] + Z^4\left[ V(t, x, y) \right] = Z^2\left[ T_{V(x,y)}^*C^3 \right] + Z^4\left[ T_0^*C^3 \right], \]

which means that the ordinary characteristic cycle is

\[ CC(\phi_f[-1]F^*) = 2\left[ T_{V(x,y)}^*C^3 \right] + 4\left[ T_0^*C^3 \right]. \]
8 Concluding Remarks

It is somewhat annoying in Example 7.7, and in the general algorithm given in Remark 7.6, that, essentially, we first have to know \( \text{SS}(\phi_f[-1]\mathbb{F}^\mathfrak{e}) \) in order to begin the calculation of the cycles \( k\Lambda_{p,s}^j \).

Why do we have to know \( \text{SS}(\phi_f[-1]\mathbb{F}^\mathfrak{e}) \) first? Solely because we need to know that our coordinates are \( \phi_f[-1]\mathbb{F}^\mathfrak{e} \)-isolating. Ideally, we could begin with the calculation of the \( k\Lambda_{p,s}^j \) and check “on-the-fly” that certain intersections are proper, which would then tell us that the coordinate choice is generic enough. This is what happens with the Lê cycles for affine hypersurface singularities; see [13].

Unfortunately, while we suspect that such a result is true more generally, we have yet to find a proof.

The characteristic cycle of the intersection cohomology complex is of great importance in representation theory (see [8] and [1]), and yet, aside from the curve case in Example 2.9, we have not discussed the calculation of the characteristic cycle of intersection cohomology complexes (with constant or twisted coefficients). This is because such a calculation is, not surprisingly, hard, and we have no satisfactory results in this area.

What may be surprising is that the calculation of the characteristic cycle of intersection cohomology, with constant coefficients, is closely related to the relative Milnor monodromy of the constant sheaf along a hypersurface containing the singular set. We will describe this relationship briefly.

Suppose that \( X \) is an analytic space and, as we are happy to work locally at \( 0 \), assume that we have an analytic function \( f : X \to \mathbb{C} \) such that the singular set of \( X \) is contained in \( V(f) \), but that \( f \) does not vanish on any irreducible component of \( X \).

As before, let \( i : X \setminus V(f) \to X \) and \( j : V(f) \to X \) denote the inclusions.

Note that, as intersection cohomology \( \mathbb{I}^\mathfrak{e} \), with constant coefficients, on \( X \) is a perverse sheaf, the graded, enriched characteristic cycle is concentrated in degree 0. Also, the case that is of concern in representation theory is when the base ring is a field. Consequently, we would be satisfied with calculating \( \text{CC}(\mathbb{I}^\mathfrak{e}) \).

Now, the easy, well-known, Proposition 6.8 tells us that we can calculate \( \text{CC}(\mathbb{I}^\mathfrak{e}) \) if we can calculate \( \text{CC}((j_*j^*[-1]\mathbb{I}^\mathfrak{e})) = -\text{CC}(j_*j^*\mathbb{I}^\mathfrak{e}) \) and \( \text{CC}(i^!i^!\mathbb{I}^\mathfrak{e}) \). But \( i^! \mathbb{I}^\mathfrak{e} \) is the restriction of \( \mathbb{I}^\mathfrak{e} \) to a generic subset of the smooth part of \( X \), by our assumptions on \( f \). Thus, \( i^! \mathbb{I}^\mathfrak{e} \) coincides with the restriction of the constant sheaf to a generic smooth subset of \( X \). Hence, \( \text{CC}(i^!i^!\mathbb{I}^\mathfrak{e}) \) can be calculated using Theorem 6.1 and its corollary.

We are left with the problem of calculating \( \text{CC}((j_*j^*[-1]\mathbb{I}^\mathfrak{e})) \), which, after slicing, reduces to the problem of calculating the Morse module \( m^0_{\mathbb{I}}(j_*j^*[-1]\mathbb{I}^\mathfrak{e}) \) or, more precisely, reduces to knowing when this Morse module is zero and when it is not.

There is the fundamental distinguished triangle, relating the nearby and vanishing cycles:

\[
j_*j^*[-1]\mathbb{I}^\mathfrak{e} \to \psi_f[-1]\mathbb{I}^\mathfrak{e} \xrightarrow{\text{can}} \phi_f[-1]\mathbb{I}^\mathfrak{e} \to \psi_f[-1]\mathbb{I}^\mathfrak{e} \xrightarrow{\text{can}} \phi_f[-1]\mathbb{I}^\mathfrak{e},
\]

which is actually a short exact sequence in the Abelian category of perverse sheaves, due to the fact that \( j_*j^*[-1]\mathbb{I}^\mathfrak{e} \) is perverse (which uses that \( \mathbb{I}^\mathfrak{e} \) is intersection cohomology).

There is also the dual variation triangle

\[
\phi_f[-1]\mathbb{I}^\mathfrak{e} \xrightarrow{\text{can}} \psi_f[-1]\mathbb{I}^\mathfrak{e} \to j_!j^*[1]\mathbb{I}^\mathfrak{e} \to \psi_f[-1]\mathbb{I}^\mathfrak{e} \xrightarrow{\text{can}} \phi_f[-1]\mathbb{I}^\mathfrak{e},
\]

which is also a short exact sequence in the Abelian category of perverse sheaves, due to the fact that \( j_!j^*[1]\mathbb{I}^\mathfrak{e} \) is perverse.
This is where the monodromy automorphism $T_f : \psi_f[-1]I^\bullet \to \psi_f[-1]I^\bullet$ comes in. It is well-known that $\text{var} \circ \text{can} = \text{id} - T_f$. It follows that, in the Abelian category of perverse sheaves on $V(f)$, $j_*j^*(-1)I^\bullet \cong \ker\{\text{id} - T_f\}$.

Suppose now that $\tilde{L}$ is a generic linear form, and that $L$ is the restriction of $\tilde{L}$ to $V(f)$. Then, it follows that $\phi_L[-1]j_*j^*[−1]I^\bullet$, which is a finite-dimensional vector space concentrated in degree 0, is isomorphic to the kernel of the map induced by $\text{id} - T_f$ on $\phi_L[-1]\psi_f[-1]I^\bullet$, and so is determined by relative Milnor monodromy.

As $I^\bullet$ agrees with the constant sheaf on the complement of $V(f)$, which is all that $\psi_f[-1]I^\bullet$ cares about, we can calculate $m^0(\psi_f[-1]I^\bullet) \cong \phi_L[-1]\psi_f[-1]I^\bullet$ via Theorem 5.2 in the easy case of the constant sheaf, where the relevant strata are open dense subsets of the smooth parts of the components of $X$. Moreover, the relative monodromy that we need to analyze is also that from the “easy” constant sheaf case.

It is, of course, our hope to analyze the above relative monodromy, and produce a method for calculating, in principle and in practice, characteristic cycles of intersection cohomology.
References

[1] Braden, T. On the Reducibility of Characteristic Varieties. Proc. AMS, 130:2037–2043, 2002.

[2] Briançon, J., Maisonobe, P., and Merle, M. Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom. Invent. Math., 117:531–550, 1994.

[3] Fulton, W. Intersection Theory, volume 2 of Ergeb. Math. Springer-Verlag, 1984.

[4] Ginsburg, V. Characteristic Varieties and Vanishing Cycles. Invent. Math., 84:327–403, 1986.

[5] Goresky, M. and MacPherson, R. Stratified Morse Theory, volume 14 of Ergeb. der Math. Springer-Verlag, 1988.

[6] Hamm, H. and Lê D. T. Un théorème de Zariski du type de Lefschetz. Ann. Sci. Éc. Norm. Sup., 6 (series 4):317–366, 1973.

[7] Kashiwara, M. and Schapira, P. Sheaves on Manifolds, volume 292 of Grund. math. Wissen. Springer-Verlag, 1990.

[8] Kazhdan, D. and Lusztig, G. A topological approach to Springer’s representations. Adv. Math., 38:222–228, 1980.

[9] Lê, D. T. Calcul du Nombre de Cycles Évanouissants d’une Hypersurface Complexe. Ann. Inst. Fourier, Grenoble, 23:261–270, 1973.

[10] Lê, D. T. Topological Use of Polar Curves. Proc. Symp. Pure Math., 29:507–512, 1975.

[11] Lê, D. T. Sur les cycles évanouissants des espaces analytiques. C. R. Acad. Sci. Paris, Sér. A-B, 288:A283–A285, 1979.

[12] Massey, D. Numerical Invariants of Perverse Sheaves. Duke Math. J., 73(2):307–370, 1994.

[13] Massey, D. Lê Cycles and Hypersurface Singularities, volume 1615 of Lecture Notes in Math. Springer-Verlag, 1995.

[14] Massey, D. Hypercohomology of Milnor Fibres. Topology, 35:969–1003, 1996.

[15] Massey, D. Critical Points of Functions on Singular Spaces. Top. and Appl., 103:55–93, 2000.

[16] Massey, D. Numerical Control over Complex Analytic Singularities, volume 778 of Memoirs of the AMS. AMS, 2003.

[17] Massey, D. Singularities and Enriched Cycles. Pacific J. Math., 215, no. 1:35–84, 2004.

[18] Massey, D. Enriched Relative Polar Curves and Discriminants. Contemp. Math., 474:107–144, 2008.

[19] Parusiński, A. and Pragacz, P. Characteristic classes of hypersurfaces and characteristic cycles. J. Alg. Geom., 10(1):63–79, 2001.

[20] Schürmann, J. Topology of Singular Spaces and Constructible Sheaves, volume 63 of Monografie Matematyczne. Birkhäuser, 2004.

[21] Teissier, B. Cycles évanescents, sections planes et conditions de Whitney. Astérisque, 7-8:285–362, 1973.