A NOTE ON TROPICAL CURVES AND THE NEWTON DIAGRAMS OF PLANE CURVE SINGULARITIES

TAKUHIRO TAKAHASHI

Abstract. For a convenient and Newton non-degenerate singularity, the Milnor number is computed from the complement of its Newton diagram in the first quadrant, so-called Kouchnirenko’s formula. In this paper, we consider tropical curves dual to subdivisions of this complement for a plane curve singularity and show the existence of a tropical curve satisfying a certain formula, which looks like a well-known formula for a real morsification due to A’Campo and Gusein-Zade.

1. Introduction

Tropical geometry is rapidly developing as a new study area in mathematics. In [7], Mikhalkin counts nodal curves on toric surfaces, which is an epoch-making result in algebraic geometry. Though there are several studies concerning tropical curves corresponding to singular algebraic curves, see for instance [10, 6, 2], the study of relation between theory of tropical curves and the singularity theory is still underdeveloping in tropical geometry.

In singularity theory, the Newton diagram is an important tool to get information of singularities. Let $f$ be a polynomial of two variables over $\mathbb{C}$, and suppose that $f(0,0) = 0$ and $f$ has an isolated singularity at $0 = (0,0) \in \mathbb{C}^2$. Set $f(x,y) = \sum_{(i,j)} a_{ij} x^i y^j$, and $\Delta_f = \text{Conv}((i,j) \in \mathbb{R}^2; a_{ij} \neq 0)$, where Conv(·) is the convex hull. The convex hull $\Delta_f$ is called the Newton polytope of $f$. Let $\Gamma_- (f)$ be the polyhedron defined by

$$\text{Closure} \left( (\mathbb{R}_{\geq 0})^2 \setminus \text{Conv}((i,j) + (\mathbb{R}_{\geq 0})^2; a_{ij} \neq 0) \right),$$

where Closure(·) is the closure with usual topology of $\mathbb{R}^2$. The Newton boundary $\Gamma (f)$ of $f$ is the union of compact faces of $\text{Conv}((i,j) + (\mathbb{R}_{\geq 0})^2; a_{ij} \neq 0)$. The singularity $(f,0)$ is convenient if $\Gamma_- (f)$ is compact. The singularity $(f,0)$ is Newton non-degenerate if, for any face $\sigma$ in $\Gamma (f)$, the function $f^\sigma (x,y) := \sum_{(i,j) \in \sigma \cap \Delta_f} a_{ij} x^i y^j$ has no singularity in $(\mathbb{C}^*)^2$.

Note that, for a convenient and Newton non-degenerate singularity, the Milnor number of $(f,0)$ is computed from $\Gamma_- (f)$, which is the celebrated theorem of Kouchnirenko [5]. In this paper, we regard $\Gamma_- (f)$ as a part of polyhedrons obtained as a dual subdivision of a tropical curve and give a meaning of $\Gamma_- (f)$ from the viewpoint of tropical geometry.

To state our result, we here introduce some terminologies in tropical geometry. Let $F$ be a polynomial of two variables over the field $K$ of convergent Puiseux series over $\mathbb{C}$. The tropical curve $T_F$ is defined by the image of $F = 0$ by the valuation map, which is a 1-simplicial complex in $\mathbb{R}^2$. The valuation of the coefficients of
\(F\) induces a subdivision \(\text{Sd}(F)\) of the Newton polytope \(\Delta_F\) and it is known that \(\text{Sd}(F)\) is dual to \(T_F\), which is so-called the Duality Theorem (see \(\S 2\)).

Now we consider a union of polygons corresponding to a part of polygons of \(\text{Sd}(F)\). Let \(\Delta'\) be a sub-polyhedron of \(\Delta_F\). A subset \(S\) of \(T_F\) is called the tropical sub-curve with respect to \(\Delta'\) if \(\Delta'\) is a union of sub-polyhedrons of \(\text{Sd}(F)\) and \(S\) is dual to the subdivision of \(\Delta'\) induced by \(\text{Sd}(F)\). We denote it by \(T_F|_{\Delta'}\). Note that if \(\Delta' = \Delta_F\) then \(T_F|_{\Delta'} = T_F\). See Definition 5.1 for the precise definition of \(T_F|_{\Delta'}\).

For plane curve singularities, the real morsification due to A’Campo \(1\) and Gusein-Zade \(3\) gives an explicit way to understand mutual positions of vanishing cycles. Our hope is that we can perform the same observation for tropical curves realized in \(\Gamma_{-}(f)\). The main theorem in this paper asserts that we can see the vanishing cycles of \(f = 0\) on the tropical curve in \(\Gamma_{-}(f)\). For a tropical sub-curve \(T_F|_{\Delta'}\), let \(v(T_F|_{\Delta'})\) denote the number of 4-valent vertices of \(T_F|_{\Delta'}\) and \(r(T_F|_{\Delta'})\) denote the number of regions bounded by \(T_F|_{\Delta'} \subset \mathbb{R}^2\).

**Theorem 1.1.** For any Newton non-degenerate and convenient isolated singularity \((f,0)\), there is a polynomial \(F := F_f \in K[z,w]\) such that \(\Delta_F = \text{Conv}(\Gamma_{-}(f))\) and \(T_F|_{\Gamma_{-}(f)}\) satisfies

\[
\mu(f) = v(T_F|_{\Gamma_{-}(f)}) + r(T_F|_{\Gamma_{-}(f)}).
\]

This result is an analogy of the following equality required for a real morsification:

\[
\mu(f) = \delta(f_s) + r(f_s),
\]

where \(f_s\) is a real morsification of \(f\), \(\delta(f_s)\) is the number of double points of \(f_s = 0\) in a previously fixed small neighborhood \(U\) of the origin, and \(r(f_s)\) is the number of bounded regions of \(\{f_s = 0\}|_{\mathbb{R}^2} \cap U\). As a corollary of Theorem 1.1 we have the equality \(\delta(f) = v(T_F|_{\Gamma_{-}(f)})\), where \(\delta(f)\) is the number of double points of \((f,0)\), see Corollary 3.5.

Remark that the polynomial \(F\) in Theorem 1.1 is given by a patchworking polynomial associated with a subdivision of \(\Delta_F\). A similar observation appears in a paper of Shustin \(9\), where a real polynomial whose critical points has a given index distribution is constructed.

We organize the paper as follows. In section 2, we introduce tropical curves and subdivisions of Newton polytopes induced from the valuation map and state the Duality Theorem. In section 3, we give the definition of tropical sub-curves and prove Theorem 1.1. Two examples will be given before the proof.

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## 2. Preliminaries

First, we give some definitions about polytopes. A **polygon** in \(\mathbb{R}^2\) is the intersection of a finite number of half-spaces in \(\mathbb{R}^2\) whose vertices are contained in the lattice \(\mathbb{Z}^2 \subset \mathbb{R}^2\). A polygon is called a **polytope** if it is compact. In this paper, a **polyhedron** means a union of polytopes which is connected and compact. Thus a polyhedron is not convex generally. A subset in a polyhedron is a **sub-polyhedron** if it is a polyhedron as a subset of \(\mathbb{R}^2\). In particular, if a sub-polyhedron is a polytope then the sub-polyhedron is called a **sub-polytope**.
Let $K := \mathbb{C}\{\{t\}\}$ be the field of convergent Puiseux series over $\mathbb{C}$, and denote the usual non-trivial valuation on $K$ by $\text{val} : K^* \to \mathbb{R}$, that is,

$$\text{val} : K^* \to \mathbb{R} ; \sum_{k=k_0}^{\infty} b_k t^k \mapsto -\frac{k_0}{N},$$

where $K^* := K \setminus \{0\}$ and $b_{k_0} \neq 0$.

For a reduced polynomial

$$F(z, w) = \sum_{(i, j) \in \Delta_F \cap \mathbb{Z}^2} c_{ij} z^i w^j \in K[z, w]$$

over $K$, we denote by $\text{Supp}(F)$ and $\Delta_F$ the support of $F$ and the Newton polytope of $F$, respectively, that is, $\text{Supp}(F) := \{(i, j) \in \mathbb{R}^2 ; c_{ij} \neq 0\}$ and $\Delta_F = \text{Conv}(\text{Supp}(F)) \subset \mathbb{R}^2$. Throughout this paper, we assume that the Newton polytope $\Delta_F$ of $F$ is 2-dimensional.

The valuation map $\text{Val} : (K^*)^2 \to \mathbb{R}^2$ is defined by $(z, w) \mapsto (\text{val}(z), \text{val}(w))$, which is a homomorphism.

**Definition 2.1.** The closure

$$T_F := \text{Closure}(\text{Val}(\{ p \in (K^*)^2 ; F(p) = 0 \})) \subset \mathbb{R}^2$$

with usual topology on $\mathbb{R}^2$ is called the (plane) tropical curve defined by $F$.

Remark that a tropical curve $T_F$ has a structure of a 1-dimensional simplicial complex, cf. [7]. We call a 1-simplex an edge and a 0-simplex a vertex as usual.

Let $F$ be a polynomial over $K$. We introduce the Duality Theorem which gives a correspondence between the tropical curve $T_F$ defined by $F$ and the Newton polytope $\Delta_F$ of $F$. Let $\Delta_\nu(F)$ be the 3-dimensional polygon defined by

$$\Delta_\nu(F) := \text{Conv}\{(i, j, -\text{val}(c_{ij})) \in \mathbb{R}^2 \times \mathbb{R} ; (i, j) \in \text{Supp}(F)\} \subset \mathbb{R}^3$$

and $\nu_F : \Delta_F \to \mathbb{R}$ be the function defined by

$$(i, j) \mapsto \min \{ x \in \mathbb{R} ; (i, j, x) \in \Delta_\nu(F) \},$$

which is a continuous piecewise linear convex function. We can get the following three kinds of sub-polytopes of $\Delta_F$ from $\nu_F$:

- linearity domains of $\nu_F$: $\Delta_1, \ldots, \Delta_N$,
- 1-dimensional polytopes: $\Delta_i \cap \Delta_j \neq \emptyset$ and $\neq \{pt\}$,
- 0-dimensional polytopes: $\Delta_i \cap \Delta_j \cap \Delta_k \neq \emptyset$,

where a linearity domain of $\nu_F$ means a maximal sub-polytope $R$ of the domain $\Delta_F$ such that the restriction $\nu_F|_R$ is an affine linear function. These polytopes give a subdivision of $\Delta_F$, which we denote by $\text{Sd}(F)$. In particular, we call a 1-dimensional and a 0-dimensional polytope a vertex and an edge of $\text{Sd}(F)$, respectively.

For edges of a tropical curve, a certain weight $w : (\text{edges of } T) \to \mathbb{N}$ is defined by directional vectors of edges canonically. We omit the definition since we don’t use it in this paper.

We can find the following claim in §2.5.1 of [4]. A proof in general dimension can be found in [1].

**Theorem 2.2 (Duality Theorem).** The subdivision $\text{Sd}(F)$ is dual to the tropical curve $T_F$ in the following sense:

1. the components of $\mathbb{R}^2 \setminus T_F$ are in 1-to-1 correspondence with the vertices of the subdivision $\text{Sd}(F)$,
the edges of $T_F$ are in 1-to-1 correspondence with the edges of the subdivision $Sd(F)$ so that an edge $E \subset T_F$ is dual to an orthogonal edge of the subdivision $Sd(F)$, having the lattice length equal to $w(E)$, which is a weight of $E$.

3) the vertices of $T_F$ are in 1-to-1 correspondence with the polytopes $Sd(F) : \Delta_1, \ldots, \Delta_N$

so that the valency of a vertex of $T_F$ is equal to the number of sides of the dual polygon.

We call the subdivision $Sd(F)$ the dual subdivision of $T_F$. By Theorem 2.2, we can regard a tropical curve as a dual subdivision of the Newton polytope of its defining polynomial.

3. Main Results

In this section, we first introduce the precise definition of tropical sub-curves which we mentioned in the introduction. Let $F \in K[x, y]$ be a polynomial over $K$ and

$$Sd(F) : \Delta_1, \ldots, \Delta_N$$

be the dual subdivision of $T_F$. Because of the structure theorem in tropical geometry, a tropical hypersurface has the structure of a polyhedral complex. In particular, a plane tropical curve is an embedded plane graph in $\mathbb{R}^2$.

Let $[u, v]$ denote the edge of the tropical curve $T_F$ whose endpoints are 0-cells $u$ and $v$. Set $[u, v] = [u, v] \setminus \{v\}$. We allow that one of the endpoints is at $\infty$. In this case, the other endpoint is contained in the 0-cells of the curve. Note that $[u, \infty] = [u, \infty]$.

Let $Sd'(F) : \Delta_{k_1}, \ldots, \Delta_{k_m}$ be a subset of $Sd(F)$ such that $\bigcup Sd'(F) \setminus Sd^{[0]}(F)$ is connected, where $Sd^{[0]}(F)$ is the set of vertices of $Sd(F)$. Let $\Delta'$ be a sub-polyhedron of $\Delta_F$ given as the union of $Sd'(F)$.

Let $V = \{v_1, \ldots, v_N\}$ and $E = \{[u, v] : u, v \in V\}$ be the set of vertices and edges of $T_F$ respectively.

Definition 3.1. A subset of $T_F$ is called the tropical sub-curve with respect to $\Delta'$ if it has the structure of the metric (open) sub-graph $(V', E')$ of the tropical curve $T_F$ which satisfies the following conditions:

1) the set of vertices $V' \subset V$ is given by $\{v_{k_1}, \ldots, v_{k_m}\}$,
2) the set of edges $E'$ is given by the following manners: for each $[u, v] \in E$,
   (i) if $u, v \in V'$ then $[u, v] \in E'$,
   (ii) if $v = \infty$ and $u \in V'$ then $[u, v] \in E'$,
   (iii) if $u \in V'$ and $v \in V \setminus V'$ then $[u, v'] \in E'$, where $v'$ is taken as the middle point of $[u, v]$.

We denote the tropical sub-curve of $T_F$ with respect to $\Delta'$ by $T_F|_{\Delta'}$.

Example 3.2. (1) Let $F$ be a polynomial over $K$ given by

$$F = 1 + tz + tw + t^3 z^2 + t^2 zw + t^3 w^2 + t^6 z^3 + t^4 z^2 w + t^4 z w^2 + t^6 w^3 + t^{10} z^4 + t^{12} z^2 w + t^{12} z^2 w^2 + t^{12} z w^3 + t^{10} w^4 + t^{15} z^5 + t^{15} w^5.$$
The Newton polytope $\Delta_F$ of $F$ is $\text{Conv}\{(0,0), (0,5), (5,0)\}$. See on the left in Figure 1. This polynomial $F$ is $F_f$ in Theorem 1.1 for the singularity of $x^5+x^2y^2+y^5$ at the origin. The polyhedron $\Delta'$ in the figure is $\Gamma_-(f)$ for the singularity $(f,0)$. The tropical sub-curve $T_F|_{\Gamma_-(f)}$ with respect to $\Gamma_-(f)$ is as shown on the right. Since $\mu(f) = 11$, $v(T_F|_{\Gamma_-(f)}) = 6$ and $r(T_F|_{\Gamma_-(f)}) = 5$, the equality $\mu(f) = v(T_F|_{\Gamma_-(f)}) + r(T_F|_{\Gamma_-(f)})$ in Theorem 1.1 is verified.

(2) Let $F$ be a polynomial over $K$ given by

$$F = 1 + tz + tw + t^3z^2 + t^2zw + t^3w^2 + t^6w^3.$$ 

The Newton polytope $\Delta_F$ of $F$ is $\text{Conv}\{(0,0), (2,0), (0,3)\}$. See on the left in Figure 2. This polynomial $F$ is $F_f$ in Theorem 1.1 for the singularity of $x^2 + y^3$ at the origin. The polyhedron $\Delta_F$ in the figure is $\Gamma_-(f)$ for the singularity $(f,0)$. The tropical sub-curve $T_F|_{\Gamma_-(f)}$ with respect to $\Gamma_-(f)$ is as shown on the right. Since $\mu(f) = 2$, $v(T_F|_{\Gamma_-(f)}) = 1$ and $r(T_F|_{\Gamma_-(f)}) = 1$, the equality in Theorem 1.1 holds.

Suppose that $f$ is convenient. For the lattice points $(i,j) \in \Gamma_-(f) \cap \mathbb{Z}^2$, we define a map $\nu_f|_{\Gamma_-(f) \cap \mathbb{Z}^2} : \Gamma_-(f) \cap \mathbb{Z}^2 \to \mathbb{R}$ by

$$\nu_f(i,j) = a_0 + a_1 + \cdots + a_i + b_0 + b_1 + \cdots + b_j$$
where \( \{a_k\}_{k \in \mathbb{N}}, \{b_k\}_{k \in \mathbb{N}} \) are non-negative strictly increasing sequences of integers.

We then extend it to the whole domain \( \Gamma_{-} (f) \) as a continuous piecewise linear function and obtain a map \( \nu_f : \Gamma_{-} (f) \to \mathbb{R} \). Taking sufficiently large values for \( \nu_f \) at the lattice points of the Newton boundary of \( f \), we may assume that the other sub-polytopes are triangles with area \( 1/2 \).

**Definition 3.3.** We call the subdivision of \( \Gamma_{-} (f) \) defined as above the *special subdivision of \( \Gamma_{-} (f) \) and each square in this subdivision the *special square.*

**Lemma 3.4.** Let \( p, q \in \mathbb{N} \) be coprime integers. The number of special squares in the special subdivision of \( \Delta_{(p,q)} = \text{Conv}\{(0,0), (p,0), (0,q)\} \subset \mathbb{R}^2 \) is \((p-1)(q-1)/2\).

**Proof.** Let \( \hat{\Delta} \) be the rectangle given by
\[
\hat{\Delta} = \text{Conv}\{(0,0), (p,0), (0,q), (p,q)\} \subset \mathbb{R}^2.
\]
We consider the special subdivision of \( \hat{\Delta} \). We decompose it into \( p \) vertical rectangles
\[
\hat{\Delta}_i = ([i, i+1] \times \mathbb{R}) \cap \hat{\Delta} \subset \hat{\Delta}, \quad i = 0, \ldots, p-1.
\]
The special subdivision of \( \hat{\Delta} \) induces a special subdivision of each \( \hat{\Delta}_i \). Let \( \ell \) be the segment connecting \((p,0)\) and \((0,q)\). We denote by \( I \) the number of special squares in \( \hat{\Delta} \) which intersect \( \ell \). Similarly, we denote by \( I_i \) the number of special squares in \( \hat{\Delta}_i \) which intersect \( \ell \). Obviously \( I = \sum_{i=0}^{p-1} I_i \).

Let \( \lambda \) be the number of special squares of \( \Delta_{(p,q)} \). Notice that \( \lambda = \frac{1}{2}(pq-I) \) since \( pq = 2\lambda + I \). Thus it is enough to show \( I = p+q-1 \). Without loss of generality, we may assume \( p < q \). Let \( k, l \) be integers such that \( q = pk + l \) and \( 0 < l < p \). The segment \( \ell \) can be denoted as
\[
(x, -\frac{q}{p}x + q) = (x, (pk + l - kx) - \frac{l}{p}x), \quad x \in [0,p].
\]
Set \( \xi(x) = \frac{x}{p} \). Then, \( I_i \) is calculated as
\[
I_i = pk + l - nk - |\xi(i)| - \left\{pk + l - (i+1)k - |\xi(i+1)| - 1 \right\} \\
= k + 1 - |\xi(i)| + |\xi(i+1)|
\]
for \( i = 1, \ldots, p-2 \) and
\[
I_0 = I_{p-1} = k + 1,
\]
where \( |\alpha| \) means the largest integer not greater than \( \alpha \in \mathbb{R} \). Thus, we obtain
\[
I = \sum_{i=0}^{p-1} I_i = I_0 + (p-2)(k+1) + |\xi(p-1)| - |\xi(1)| + I_{p-1} \\
= pk + p + \left\lfloor \frac{l}{p} \right\rfloor (p-1) = p + q - 1.
\]

**Proof of Theorem 1.1.** Choose a polynomial \( F \) such that the Newton polytope \( \Delta_F \) coincides with \( \text{Conv}(\Gamma_{-} (f)) \). To decide coefficients of \( F \), we take the convex function \( \nu : \Delta_{F} \cap \mathbb{Z}^2 \to \mathbb{R} \) as a linear extension of \( \nu_f \) used in the definition of the special subdivision of \( \Gamma_{-} (f) \), and define \( F \) as the *patchworking polynomial* defined by \( \nu \), that is,
\[
F(z, w) = \sum_{(i,j) \in \Delta_{F} \cap \mathbb{Z}^2} t^{-\nu(i,j)} z^i w^j.
\]
In the rest of the proof, we check $T_F|_{\Gamma_- (f)}$ satisfies the equality in the assertion. To calculate the number of special squares, we decompose $\Gamma_- (f)$ into two subpolyhedrons as follows. Let $p, q \in \mathbb{N}$ be coprime integers. We denote the intersection points of the Newton boundary $\Gamma (f)$ of $f$ and the lattice by

$$\Gamma (f) \cap \mathbb{Z}^2 = \{(0, q), (P_1, Q_1), \ldots, (P_{n-1}, Q_{n-1}), (p, 0)\},$$

where $0 < P_1 < \cdots < P_{n-1} < p$. We set $P_0 = 0, P_n = p, Q_0 = q, Q_n = 0$ and define

$$p_i = |P_i - P_{i-1}|, \quad q_i = |Q_i - Q_{i-1}|, \quad i = 1, \ldots, n.$$ 

Notice that $p = p_1 + \cdots + p_n$ and $q = q_1 + \cdots + q_n$. For $i = 1, \ldots, n$, we define the subset $\Delta_i$ of $\Gamma_- (f)$ as

$$\Delta_i = \text{Conv}\{(P_{i-1}, Q_{i-1}), (P_i, Q_i)\} = \Delta_{(p_i, q_i)}$$

and

$$\Xi_1 := \bigcup_{i=1}^{n} \Delta_i \subset \Gamma_- (f), \quad \Xi_2 := \text{Closure}(\Gamma_- (f) \setminus \Xi_1) \subset \Gamma_- (f).$$

Then, $\Gamma_- (f)$ decomposes as $\Gamma_- (f) = \Xi_1 \cup \Xi_2$. For $i = 1, 2$, we denote by $|\Xi_i|$ the number of special squares contained in the special subdivision of $\Xi_i$ induced by that of $\Gamma_- (f)$. Then, using Lemma 3.4 we have

$$|\Xi_1| = \sum_{i=1}^{n} \frac{1}{2} (p_i - 1)(q_i - 1),$$

$$|\Xi_2| = \sum_{i=1}^{n-1} p_i \cdot (q_i + \cdots + q_n)$$

$$= \sum_{i=1}^{n-1} p_i \cdot \{q - (q_1 + \cdots + q_{i-1})\} = \text{Vol}(\Xi_2).$$

Next we will show the following equalities:

(1) $|\Xi_1| + |\Xi_2| = v(T_F|_{\Gamma_- (f)}),$

(2) $|\Xi_1| + |\Xi_2| - (n - 1) = r(T_F|_{\Gamma_- (f)}).$

By Theorem 2.2, the correspondence between subdivisions and tropical curves, introduced in Theorem 2.2 gives a 1-to-1 correspondence of parallelograms and 4-valent vertices. In our case, any 4-valent vertex corresponds to a special square. Thus, the number of special squares, $|\Xi_1| + |\Xi_2|$, coincides with the number of 4-valent vertices of $T_F|_{\Gamma_- (f)}$. Hence equality (1) holds.

We prove the other equality. There is a 1-to-1 correspondence between

$$\{\text{special square } \otimes \text{ contained in special subdivision of } \Gamma_- (f) \mid V(\otimes) \cap \Gamma (f) = \emptyset\}$$

and

$$\text{int}(\Gamma_- (f)) \cap \mathbb{Z}^2,$$

where $V(\otimes)$ is the set of vertices of a special square $\otimes$ in $\Gamma_- (f)$. Moreover, by the Duality Theorem 2.2 of tropical curves, we have a 1-to-1 correspondence between the bounded regions contained in the complement of $T_F|_{\Gamma_- (f)} \subset \mathbb{R}^2$ and the interior lattice points $\text{int}(\Gamma_- (f)) \cap \mathbb{Z}^2$ in $\Gamma_- (f)$. Since

$$\{\text{special square } \otimes \text{ contained in special subdivision of } \Gamma_- (f) \mid V(\otimes) \cap \Gamma (f) \neq \emptyset\} = n - 1,$$
we get
\[ r(T_F|_{\Gamma^{-}(f)}) = \sharp(\text{int}(\Gamma^{-}(f)) \cap \mathbb{Z}^2) = |\Xi_1| + |\Xi_2| - (n - 1). \]
Thus equality (2) holds.

Set \[ L = \sum_{i=1}^{n} p_i q_i. \] From equality (1), we get \[ L = 2|\Xi_1| + (p + q) - n \]
\[ = \frac{1}{2} (L - (p + q) + n). \]

For the Milnor number \( \mu(f) \), we use Kouchnirenko’s formula in [5]:
\[ \mu(f) = 2V_2 - V_1 + 1, \]
where
\[ V_2 = \frac{L}{2} + |\Xi_2|, \ V_1 = p + q. \]
Thus, we get
\[ \mu(f) = 2V_2 - V_1 + 1 \]
\[ = L + 2|\Xi_2| - (p + q) + 1 \]
\[ = \left\{ 2|\Xi_1| + (p + q) - n \right\} + 2|\Xi_2| - (p + q) + 1 \]
\[ = v(T_F|_{\Gamma^{-}(f)}) + r(T_F|_{\Gamma^{-}(f)}). \]

\[ \square \]

Corollary 3.5. Let \( F := F_f \) be a polynomial obtained in Theorem [1]. Then the number \( \delta(f) \) of double points of \( (f, 0) \) coincides with \( v(T_F|_{\Gamma^{-}(f)}) \).

Proof. In [8], we have \( r(f) = \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 1 \), where \( r(f) \) is the number of local irreducible components of \( f \) at 0. We also have
\[ \mu(f) = 2(|\Xi_1| + |\Xi_2|) - \left\{ \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 2 \right\} \]
\[ = 2v(T_F|_{\Gamma^{-}(f)}) - \left\{ \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 2 \right\} \]
from the argument in the proof of Theorem [1]. Thus
\[ 2\delta(f) = \mu(f) + r(f) - 1 \]
\[ = 2v(T_F|_{\Gamma^{-}(f)}) - \left\{ \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 2 \right\} + \sharp(\mathbb{Z}^2 \cap \Gamma(f)) - 1 - 1 \]
\[ = 2v(T_F|_{\Gamma^{-}(f)}). \]
\[ \square \]

Remark 3.6. As in [1, 3], we can obtain the intersection form of vanishing cycles from the immersed curve of a real morsification. To study the intersection form in our tropical curve we need to fix “framings” on edges of the curve in \( \Gamma^{-}(f) \), though we don’t have any good way to see these “framings”.
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Mathematical Institute, Tohoku University, Aoba, Sendai, Miyagi, 980-8578, Japan

E-mail address: sb3m17@math.tohoku.ac.jp