Structural causal models for macro-variables in time-series

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April 11, 2018

Abstract

We consider a bivariate time series \((X_t, Y_t)\) that is given by a simple linear autoregressive model. Assuming that the equations describing each variable as a linear combination of past values are considered structural equations, there is a clear meaning of how intervening on one particular \(X_t\) influences \(Y_t\) at later times \(t' > t\). In the present work, we describe conditions under which one can define a causal model between variables that are coarse-grained in time, thus admitting statements like ‘setting \(X\) to \(x\) changes \(Y\) in a certain way’ without referring to specific time instances. We show that particularly simple statements follow in the frequency domain, thus providing meaning to interventions on frequencies.

1 Structural equations from dynamical laws

Structural equations, also called ‘functional causal models’ \(^{1}\) are a popular and helpful formalization of causal relations. For a causal directed acyclic graph (DAG) with \(n\) random variables \(X_1, \ldots, X_n\) as nodes they read

\[
X_j = f_j(PA_j, N_j),
\]

where \(PA_j\) denotes the vector of all parent variables and \(N_1, \ldots, N_n\) are jointly independent noise variables. Provided the variables \(X_j\) refer to measurements that are well-localized in time and correspond to time instances \(t_1, \ldots, t_n\), one then assumes that \(PA_j\) contain only those variables \(X_i\) for which \(t_i < t_j\). However, thinking of random variables as measurements that refer to well-defined time instances is too restrictive for many purposes. Random variables may, for instance, describe values attained by a quantity when some system is in its equilibrium state \(^{2,3}\). In that case, intervening on one quantity may change the stationary joint state, and thus also change the values of other quantities.

The authors of \(^{3}\) show how the differential equations describing the dynamics of the system entail, under fairly restrictive conditions, structural equations relating observables in equilibrium. It should be noted, however, that these structural equations may contain causal cycles \(^{4,5,6,7,8,9}\), i.e., unlike the framework in \(^{1}\) they do not correspond to a DAG.

The work \(^{10}\) generalized \(^{3}\), assaying whether the SCM framework can be extended to model systems that do not converge to an equilibrium (cf. also \(^{7}\)), and what assumptions need to be made on the ODE and interventions so that this is possible.

Further, also Granger causality \(^{11}\) yields coarse-grained statements on causality (subject to appropriate assumptions such as causal sufficiency of the time series) by stating that \(X\) causes \(Y\) without reference to specific time instances.

The authors of \(^{12}\) propose an approach for the identification of macro-level causes and effects from high-dimensional micro-level measurements in a scenario that does not refer to time-series.

In the present work, we will elaborate on the following question: suppose we are given a dynamical system that has a clear causal direction when viewed on its elementary time scale. Under which conditions does it also admit a causal model on ‘macro-variables’ that are obtained by coarse-graining variables referring to particular time instances?

2 Causal models for equilibrium values — a negative result

The work \(^{3}\) considered deterministic dynamical systems described by ordinary differential equations and showed that, under particular restrictions, the effect of intervening
Let us assume that $E_t$ to the following equations:

\begin{align}
X_{t+1} &= \alpha X_t + E_t^X \\
Y_{t+1} &= \beta X_t + \gamma Y_t + E_t^Y
\end{align}

Figure 1: AR(1) model where $X$ influences $Y$ but not vice versa.

on some of the variables changes the equilibrium state of the other ones in a way that can be expressed by structural equations among time-less variables, which are derived from the underlying differential equations. Inspired by these results, we consider non-deterministic discrete dynamics\(^2\) as given by autoregressive (AR) models, and ask whether we can define a causal structural equation describing the effect of an intervention on one variable on another one, which, at the same time, reproduces the observed stationary joint distribution. To this end, we consider the following simple AR model of order 1 depicted in Figure 1.

We assume a Markov chain evolving in time according to the following equations:

\begin{align}
X_{t+1} &= \alpha X_t + E_t^X \\
Y_{t+1} &= \beta X_t + \gamma Y_t + E_t^Y
\end{align}

(2)

(3)

Let us assume that $E_t^X, E_t^Y$ are $\mathcal{N}(0,1)$ and i.i.d. random variables. We assume that the chain goes ‘back forever’ such that $(X_t, Y_t)$ are distributed according to the stationary distribution of the Markov chain, and are jointly normal\(^3\).

We want to express the stationary distribution and how it changes under (a restricted set of) interventions using a structural causal model. In this example, we consider interventions do$(X = x)$ and do$(Y = y)$, by which we refer to the sets of interventions do$(X_t = x)$ or do$(Y_t = y)$ for all $t$, respectively.

Let us state our goal more explicitly: we want to derive a structural causal model (SCM) with variables $X$ and $Y$ (and perhaps others) such that the stationary distribution of the Markov chain is the same as the observational distribution on $(X, Y)$ implied by the SCM, and that the stationary distribution of the Markov chain after intervening do$(X_t = x)$ for all $t$ is the same as the SCM distribution after do$(X = x)$ (and similar with interventions on $Y$).

This is informally represented by the diagram shown in Figure 2. We seek a ‘transformation’ $T$ of the original Markov chain (itself an SCM) such that interventions on all $X_t$ can be represented as an intervention on a single variable, and such that the SCM summarises the stationary distributions. (Note that in fact as we will see, we cannot express this in general without extra variables as confounders.) The diagram should commute, compare also [14].

We first compute the stationary joint distribution of $(X, Y)$. Since there is no influence of $Y$ on $X$, we can first compute the distribution of $X$ regardless of its causal link to $Y$. Using

\[ X_{t+1} = E_{t+1}^X + \alpha E_{t-1}^X + \alpha^2 E_{t-2}^X + \ldots + \sum_{k=0}^{\infty} \alpha^k E_{t-k}^X, \]

(4)

and the above conventions

\[ \mathbb{E}[E_t^X] = 0 \quad \text{and} \quad \forall \mathbb{E}[E_t^X] = 1, \]

we then obtain

\[ \mathbb{E}[X_t] = 0 \]

and

\[ \mathbb{V}[X_t] = \mathbb{V} \left[ \sum_{k=0}^{\infty} \alpha^k E_{t-k}^X \right] \]

\[ = \sum_{k=0}^{\infty} \alpha^{2k} \mathbb{V}[E_{t-k}^X] \]

\[ = \sum_{k=0}^{\infty} \alpha^{2k} \]

\[ = \frac{1}{1 - \alpha^2}. \]

\(^2\)Note that [14] considers interventions in stochastic differential equations and provides conditions under which they can be seen as limits of interventions in the ‘discretized version’, such as autoregressive models.

\(^3\)Note that we could assume initial conditions $X_0$ and $Y_0$, in which case the joint distribution $(X_t, Y_t)$ would not be independent of $t$, but would converge to the stationary distribution.

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3 Note that we could assume initial conditions $X_0$ and $Y_0$, in which case the joint distribution $(X_t, Y_t)$ would not be independent of $t$, but would converge to the stationary distribution.
Since the DAG in Figure 1 contains arrows from \( X \) to \( Y \) and none in the opposite direction, one would like to explain this bivariate joint distribution by the causal DAG in Figure 2 (bottom) where \( X \) is causing \( Y \). This would imply \( P(Y|do(X)) = P(Y|X) \). The conditional \( P(Y|X) \) is given by a simple regression which yields

\[
Y = aX + E_Y,
\]

where \( E_Y \) is an independent Gaussian noise variable and \( a \) is the regression coefficient defined by

\[
a := C_{XY}C_{XX}^{-1} = \frac{\alpha\beta}{1 - \alpha\gamma}, \tag{6}
\]

We now show that (6) is not consistent with the effect of interventions on \( X \) when the latter are defined by setting all \( X_t \) to some value \( x \). We refer to this intervention as \( do(X = x) \). The corresponding interventional distribution of \( Y \) reads:

\[
Y_{t+1}^{do(X=x)} = \beta x + \gamma Y_t^{do(X=x)} + E_t^Y = \beta x + \beta\gamma x + \beta\gamma^2 x + \ldots + E_t^Y + \gamma E_{t-1}^Y + \gamma^2 E_{t-2}^Y + \ldots
\]

If the distribution is stationary, we have

\[
Y_{t}^{do(X=x)} = \frac{\beta x}{1 - \gamma} + \sum_{k=0}^{\infty} \gamma^k E_{t-k}^Y.
\]

Hence,

\[
Y_{t}^{do(X=x)} \sim \mathcal{N}\left(\frac{\beta x}{1 - \gamma}, \frac{1}{1 - \gamma^2}\right).
\]

Note that this interventional conditional requires a structural equation whose regression coefficient reads

\[
a' := \frac{\beta x}{1 - \gamma}, \tag{7}
\]

which does not coincide with the coefficient \( a \) given by (6).

We now want to provide an intuition about the mismatch between the regression coefficient \( a \) that would be needed to explain the observed stationary joint distribution and the coefficient \( a' \) describing the true effect of interventions. One the one hand, it should be noted that the conditional of \( Y \) given \( X \) in the stationary distribution refers to observing only the current value \( X_t \). More precisely, \( a \) describes the conditional \( P(Y_t|X_t) \), that is, how the distribution of the current value \( Y_t \) changes after the current value \( X_t \) is observed. In contrast, the interventions we have considered above are of the type set all variables \( X_t \) with \( t' \in \mathbb{Z} \) to some fixed value \( x \). In other words, the intervention is not localized in time while the observation refers to one specific time instance.

where we have used the independence of the noise terms for different \( t \).

For the expectation of \( Y_t \) we get

\[
E[Y_t] = \beta E[X_t] + \gamma E[Y_t] + E[E_t^X] = 0.
\]

To compute the variance of \( Y_t \) we need to sum the variance of all independent noise variables. We obtain (the calculation can be found in the appendix):

\[
V[Y_t] = \frac{1}{1 - \gamma^2} + \frac{\beta^2(1 + \alpha\gamma)}{(1 - \alpha^2)(1 - \alpha\gamma)(1 - \gamma^2)}.
\]

For the covariance of \( X_t \) and \( Y_t \) we get (see also appendix):

\[
\text{Cov}[X_t, Y_t] = \frac{\beta\alpha}{(1 - \alpha\gamma)(1 - \alpha^2)}.
\]

We have thus shown that the stationary joint distribution of \( X, Y \) is

\[
(X, Y) \sim \mathcal{N}(0, C), \tag{5}
\]

where the entries of \( C \) read

\[
\begin{align*}
C_{XX} &= \frac{1}{1 - \alpha^2}, \\
C_{XY} &= \frac{\alpha\beta}{(1 - \alpha\gamma)(1 - \alpha^2)}, \\
C_{YY} &= \frac{1}{1 - \gamma^2} + \frac{\beta^2}{(\alpha - \gamma)^2} \left[ \frac{\alpha^2}{1 - \alpha^2} - \frac{2\alpha\gamma}{1 - \alpha\gamma} + \frac{\gamma^2}{1 - \gamma^2} \right].
\end{align*}
\]
This motivates already the idea of the following section: in order to explain the observational stationary joint distribution by an arrow from \( X \) to \( Y \), we need to define variables that are de-localized in time because in that case observations and interventions are de-localized in time.

3 Non-local variables and frequency analysis

To understand the reason for the negative result of the preceding section, we recall that we compared the interventional conditional variable \( Y_{do(X=x)} \) (where the intervention \( do(X=x) \) referred to all variables \( X_t \)) to the observational conditional \( Y|X_t=x_t \) (where the observation referred only to the current value \( x_t \)). To overcome this mismatch of completely non-local interventions on the one hand and entirely local observations on the other hand, we need to use non-local variables for observations and interventions. This motivates the following. For any functions \( f, g \in L^1(\mathbb{Z}) \) we define the random variables:

\[
X_f := \sum_{t \in \mathbb{Z}} f(t)X_t \quad \text{and} \quad Y_g := \sum_{t \in \mathbb{Z}} g(t)Y_t.
\]

One may think of \( f, g \) as smoothing kernels (for instance, discretized Gaussians). Then \( X_f, Y_g \) may be the result of the above conventions of measurements that perform coarse-graining in time. Alternatively, one could also think of \( f, g \) as trigonometric functions \( \sin, \cos \) restricted to a certain time window. Then \( X_f, Y_g \) are Fourier transforms of the observations in the respective window. In the spirit of [14], \( X_f \) and \( Y_g \) can be thought of as macro-variables derived from the micro-variables \( X_t, Y_t \) by a ‘projection’. We will now show how a simple causal model emerges between the macro-variables provided that we consider appropriate pairs of macro-variables \( X_f, Y_g \).

First, we also define averages over noise variables, which we think of as ‘macro-noise-variables’:

\[
E^X_f := \sum_{t \in \mathbb{Z}} f(t)E^X_t \quad \text{and} \quad E^Y_g := \sum_{t \in \mathbb{Z}} g(t)E^Y_t.
\]

Introducing the shift operator \( S \) by \((Sf)(t) := f(t+1)\) we can rewrite (2) and (3) concisely as

\[
X_f = X_{\alpha S}f + E^X_f \quad \text{and} \quad Y_g = X_{\beta S}g + Y_{\beta S}g + E_g,
\]

which can be transformed to

\[
X_{(1-\alpha S)}f = E^X_f \quad \text{and} \quad Y_{(1-\beta S)}g = X_{\beta S}g + E^Y_g.
\]

and, finally,

\[
X_f = E^X_{(1-\alpha S)^{-1}f} \quad \text{and} \quad Y_g = X_{\gamma S(1-\beta S)^{-1}g} + E^Y_{(1-\beta S)^{-1}g}.
\]

Note that the inverses can be computed from the formal von Neumann series

\[
(I - \alpha)^{-1} = \sum_{j=0}^{\infty} (\alpha S)^j,
\]

and \( \sum_{j=1}^{\infty} (\alpha S)^j f \) converges in \( L^1(\mathbb{Z}) \)-norm for \( \alpha < 1 \) due to \( \|S^j f\|_1 = \|f\|_1 \), and likewise for \( \beta < 1 \). Equation (12) describes how the scalar quantity \( X_f \) is generated from a single scalar noise term that, in turn, is derived from a weighted average over local noise terms. Equation (13) describes how the scalar quantity \( Y_g \) is generated from the scalar \( X_{\gamma S(1-\beta S)^{-1}g} \) and a scalar noise term.

3.1 Making coarse-graining compatible with the causal model

The following observation is crucial for the right choice of pairs of macro-variables: whenever we choose

\[
f_g := \gamma S(I - \beta S)^{-1}g,
\]

equations (12) and (13) turn into the simple form

\[
X_f = E^X_{(1-\alpha S)^{-1}f} \quad \text{and} \quad Y_g = X_{f_s} + E^Y_{(1-\beta S)^{-1}g}.
\]

Equations (15) and (16) describe how the joint distribution of \( (X_f, Y_g) \) can be generated: first, generate \( X_f \), from an appropriate average over noise terms. Then, generate \( Y_g \) from \( X_{f_s} \) plus another averaged noise term. For any \( x \in \mathbb{R} \), the conditional distribution of \( Y_g \), given \( X_f = x \), is therefore identical to the distribution of \( x + E^Y_{(1-\beta S)^{-1}g} \).

We now argue that (15) and (16) can even be read as structural equations, that is, they correctly formalize the effect of interventions. To this end, we consider the following class of interventions. For some arbitrary bounded sequence \( x = (x_t)_{t \in \mathbb{Z}} \) we look at the effect of setting \( X \) to \( x \), that is, setting each \( X_t \) to \( x_t \). Note that this generalizes the intervention \( do(X=x) \) considered in section 2 where each \( X_t \) is set to the same value \( x \in \mathbb{R} \). Using the original structural equation (9) yields

\[
Y_{do(X=x)} = \sum_{t \in \mathbb{Z}} x_t f_g(t) + Y_{\beta S}g + E^Y_g.
\]

Applying the same transformations as above yields

\[
Y_{do(X=x)} = \sum_{t \in \mathbb{Z}} x_t f_g(t) + E^Y_{(1-\beta S)^{-1}g}.
\]

\[\text{[Footnote]}\text{Since } E[|X_t|] \text{ and } E[|Y_t|] \text{ exist, the series converge in } L^1 \text{ norm of the underlying probability space, hence they converge in probability by Markov’s inequality.}\]
Note that the first term on the right hand side is the value attained by the variable \( X_{f,k} \). Hence, the only information about the entire intervention that matters for \( Y_g \) is the value of \( X_{f,k} \). We can thus talk about ‘interventions on \( X_{f,k} \)’ without further specifying what the intervention does with each single variable \( X_t \) and write
\[
Y_{g \text{do}(X_{f,k})} = x + E_{I=\beta S^{-1}g}.
\]
We have thus shown that (15) and (16) also reproduce the effect of interventions and can thus be read as structural equations for the variable pair \((X_{f,k},Y_g)\).

To be more explicit about the distribution of the noise terms in (15) and (19), straightforward computation shows the variance to be given by
\[
\mathbb{V}(E_{(I-\alpha S)^{-1}f}) = \sum_{t \in \mathbb{Z}} \sum_{k,k' \geq 0} \alpha^{k+k'} f(t+k)f(t+k).
\]
(17)

Likewise,
\[
\mathbb{V}(E_{(I-\beta S)^{-1}g}) = \sum_{t \in \mathbb{Z}} \sum_{k,k' \geq 0} \beta^{k+k'} g(t+k')g(t+k).
\]
(18)

We have thus shown the following result.

**Theorem 1 (valid pairs of macro-variables)** Whenever \( f,g \in \mathbb{L}^1(\mathbb{Z}) \) are related by (14), the AR process in (2) and (3) entails the scalar structural equations
\[
X_f = \tilde{E}^X,
\]
(19)
\[
Y_g = X_f + \tilde{E}^Y.
\]
(20)

Here, \( \tilde{E}^X \) and \( \tilde{E}^Y \) are zero mean Gaussians whose variances are given by (17) and (18), respectively.

Equation (20) can be read as a functional ‘causal model’ or ‘structural equation’ in the sense that it describes both the observational conditional of \( Y_g \), given \( X_f \) and the interventional conditional of \( Y_g \), \( \text{do}(X_f = x) \).

In the terminology of [14], the mapping from the entire bivariate process \((X_t,Y_t)_{t \in \mathbb{Z}}\) to the macro-variable pair \((X_f,Y_g)\) thus is an exact transformation if \( f \) and \( g \) are related by (14).

### 3.2 Revisiting the negative result

Theorem [11] provides a simple explanation for our negative result from section 2. To see this, we recall that we considered the distribution of \( Y_t \), which corresponds to the variable \( Y_g \) for \( \nu = (\ldots,0,1,0,\ldots) \), where the number 1 occurs at some arbitrary position \( t \). To compute the corresponding \( f \) according to (14) note that
\[
\gamma S(I-\beta S)^{-1} = \gamma S \sum_{j=0}^{\infty} (\beta S)^j = \frac{\gamma}{\beta} \sum_{j=1}^{\infty} (\beta S)^j.
\]
We thus obtain the following ‘smoothing function’ \( f \) that defines the appropriate coarse graining for \( X \) for which we obtain an exact transformation of causal models:
\[
f = \gamma(\ldots,\beta^2,\beta^3,\beta^0,0,\ldots),
\]
(21)
where the first entry from the right is at position \( t - 1 \), in agreement with the intuition that this \( X_{t-1} \) is the latest value of \( X \) that matters for \( Y_t \).

The intervention \( \text{do}(X = x) \), where all variables \( X_t \) are set to the value \( x \), corresponds to setting \( X_f \) to
\[
x \sum_{t \in \mathbb{Z}} f(t) = x \frac{\gamma}{\beta}.
\]

We thus conclude
\[
Y_{t \text{do}(X=x)} = Y_t | X_f = x \frac{\gamma}{\beta}.
\]

In words, to obtain a valid structural equation that formalizes both interventional and observational conditional we need to condition on \( X_f \) given by (21).

### 3.3 Decoupling of interventions in the frequency domain

Despite the simple relation between \( f \) and \( g \) given by (14), it is somehow disturbing that different coarse-grainings are required for \( X \) and \( Y \). We will now show that \( g \) can be chosen such that \( f \) is almost the same as \( g \) (up to some scalar factor), which leads us to Fourier analysis of the signals.

So far, we have thought of \( f \) and \( g \) as real-valued functions, but for Fourier analysis it is instructive to consider complex waves on some window \([\bar{T},T]\),
\[
g_{\nu,T}(t) := \begin{cases} e^{2\pi i \nu t} & \text{for } t = -T, \ldots, T \text{ otherwise.} \end{cases}
\]

For notational convenience, we also introduce the corresponding vectors \( f \) by
\[
f_{\nu,T} := \gamma S(I-\beta S)^{-1} g_{\nu,T} \]
which are not as simple as \( g_{\nu,T} \). However, for sufficiently large \( T \), the functions \( g_{\nu,T} \) are almost eigenvectors of \( \hat{S} \) with eigenvalue \( z_\nu := e^{2\pi i \nu T} \) since we have
\[
\| S^j g_{\nu,T} - z_\nu^j g_{\nu,T} \| \leq \frac{2j}{\sqrt{2T+1}}
\]
(22)
because the functions differ only at the positions \( -T, \ldots, -T+j-1 \) and \( T+1, \ldots, T+j \). We show in the appendix that this implies
\[
\| f_{\nu,T} - \gamma z_{\nu}(1-z_\nu)^{-1} g_{\nu,T} \| \leq \frac{2}{2T+1} |\beta| (1-\beta)^{-2}.
\]
(23)
that is, \( f_{\nu,T} \) coincides with a complex-valued multiple of \( g_{\nu,T} \) up to an error term that decays with \( O(1/\sqrt{T}) \). Using the abbreviations

\[
E_{\nu,T}^X := E_{(1-\alpha S)^{-1} g_{\nu,T}},
\]

and

\[
E_{\nu,T}^Y := E_{(1-\beta S)^{-1} g_{\nu,T}},
\]

the general structural equations \((15)\) and \((16)\) thus imply the approximate structural equations

\[
X_{g_{\nu,T}} = E_{\nu,T}^X,
\]

\[
Y_{g_{\nu,T}} \approx \gamma e^{2\pi i \nu}(1 - \beta e^{2\pi i \nu})^{-1} X_{g_{\nu,T}} + E_{\nu,T}^Y,
\]

where the error of the approximation \((25)\) is a random variable whose \( L^1 \) norm is bounded by

\[
\frac{2}{\sqrt{2T + 1}} |\beta| |1 - \beta|^{-2} \cdot |\mathbb{E}||X_t||,
\]

due to \((23)\). We conclude with the interpretation that the structural equations for different frequencies perfectly decouple. That is, intervening on one frequency of \( X \) has only effects on the same frequency of \( Y \), as a simple result of linearity and time-invariance of the underlying Markov process.

To phrase this decoupling over frequencies in a precise way, show that \( E_{\nu,T}^X \) and \( E_{\nu,T}^Y \) exist in distribution as complex-valued random variables. It is sufficient to show that the variances and covariances of real and imaginary parts of \( E_{\nu,T}^Y \) converge because both variables are Gaussians with zero mean. We have

\[
\mathbb{V}[\mathbb{R}\{E_{\nu,T}^Y\}] = \frac{1}{4} \mathbb{E}[\mathbb{E}_{\nu,T}^Y + \mathbb{E}_{\nu,T}^Y]^2 = \frac{1}{4} \left( \mathbb{E}[\mathbb{E}_{\nu,T}^Y]^2 + \mathbb{E}[\mathbb{E}_{\nu,T}^Y]^2 + 2 \mathbb{E}[\mathbb{E}_{\nu,T}^Y \mathbb{E}_{\nu,T}^Y] \right).
\]

We obtain

\[
\mathbb{E}[\mathbb{E}_{\nu,T}^Y \mathbb{E}_{\nu,T}^Y] = \mathbb{E}[\mathbb{E}_{(1-\beta S)^{-1} g_{\nu,T}}^Y \mathbb{E}_{(1-\beta S)^{-1} g_{\nu,T}}^Y] = \langle (I + \beta S)^{-1} g_{\nu,T}, (I - \beta S)^{-1} g_{\nu,T} \rangle
\]

\[
\rightarrow |1 - \beta z_{\nu}|^{-2}.
\]

For the first equality, recall that the complex-valued inner product is anti-linear in its first argument. The limit follows from straightforward computations using an analog of \((22)\) for the \( L^2 \) norm,

\[
\|S^i g_{\nu,T} + \bar{z}_i g_{\nu,T}\| \leq \sqrt{\frac{2j}{2T + 1}},
\]

and further algebra akin to the proof of \((23)\) in the appendix.

Moreover,

\[
\mathbb{E}[\mathbb{E}_{\nu,T}^Y]^2 = \mathbb{E}[\mathbb{E}_{(1-\beta S)^{-1} g_{\nu,T}}^Y \mathbb{E}_{(1-\beta S)^{-1} g_{\nu,T}}^Y] = \langle (I + \beta S)^{-1} g_{\nu,T}, (I - \beta S)^{-1} g_{\nu,T} \rangle.
\]

Hence, \( \mathbb{E}[\langle (E_{\nu,T}^Y)^2 \rangle] \) and its conjugate \( \mathbb{E}[\langle \bar{E}_{\nu,T}^Y \rangle^2] \) converge to zero for all \( \nu \neq 0 \) because

\[
\lim_{T \to \infty} \langle (I + \beta S)^{-1} g_{\nu,T}, (I - \beta S)^{-1} g_{\nu,T} \rangle
\]

\[
= (1 - \beta z_{\nu})^{-2} \lim_{T \to \infty} \sum_t g^2_{\nu,T}(t) = 0,
\]

where equality of \((32)\) and \((33)\) follows from \((22)\). Hence, only the mixed term containing both \( E_{\nu,T}^X \) and its conjugate survives the limit. We conclude

\[
\lim_{T \to \infty} \mathbb{V}[\mathbb{R}\{E_{\nu,T}^Y\}] = \frac{1}{2} |(1 - \beta z_{\nu})^{-1}|^2.
\]

Similarly, we can show that \( \mathbb{V}[\mathbb{I}\{E_{\nu,T}^Y\}] \) converges to the same value. Moreover,

\[
\lim_{T \to \infty} \mathbb{Cov}[\mathbb{R}\{E_{\nu,T}^Y\}, \mathbb{I}\{E_{\nu,T}^Y\}] = 0,
\]

because straightforward computation shows that the covariance contains no mixed terms. Hence we can define

\[
E_{\nu} := \lim_{T \to \infty} E_{\nu,T}^Y,
\]

with convergence in distribution. Real and imaginary parts are uncorrelated and their variance read:

\[
\mathbb{V}[\mathbb{R}\{E_{\nu}^Y\}] = \mathbb{V}[\mathbb{I}\{E_{\nu}^Y\}] = \frac{1}{2} |(1 - \beta z_{\nu})^{-1}|^2.
\]

We conclude that the distribution of \( E_{\nu}^Y \) is an isotropic Gaussian in the complex plane, whose components have variance \( \frac{1}{2} |(1 - \beta z_{\nu})^{-1}|^2 \).

To compute the limit of \( E_{\nu,T}^X \) we proceed similarly and observe

\[
\mathbb{E}[E_{\nu,T}^X E_{\nu,T}^Y] = \mathbb{E}[E_{(1-\beta S)^{-1} g_{\nu,T}}^Y \mathbb{E}_{(1-\beta S)^{-1} g_{\nu,T}}^Y] \rightarrow |\gamma z_{\nu}(1 - \beta z_{\nu})^{-2}|^2.
\]

We can therefore define the random variable \( E_{\nu}^X := \lim_{T \to \infty} (\text{again with convergence in distribution}) \)

\[
\mathbb{V}[\mathbb{R}\{E_{\nu}^X\}] = \mathbb{V}[\mathbb{I}\{E_{\nu}^X\}] = \frac{1}{2} \left| \frac{\gamma z_{\nu}}{(1 - \beta z_{\nu})^2} \right|^2.
\]

We can phrase these findings by asymptotic structural equations

\[
X_{\nu} = E_{\nu}^X,
\]

\[
Y_{\nu} = \gamma e^{2\pi i \nu}(1 - \beta e^{2\pi i \nu})^{-1} X_{\nu} + E_{\nu}^Y,
\]

where the variances of real and imaginary parts of \( E_{\nu}^X \) and \( E_{\nu}^Y \) are given by \((37)\) and \((35)\), respectively.


4 Conclusion

We have studied bivariate time series \((X_t, Y_t)\) given by linear autoregressive models, and described conditions under which one can define a causal model between variables that are coarse-grained in time, thus admitting statements like ‘setting \(X\) to \(x\) changes \(Y\) in a certain way’ without referring to specific time instances. We show that particularly elegant statements follow in the frequency domain, thus providing meaning to interventions on frequencies.

References

[1] J. Pearl. *Causality*. Cambridge University Press, 2000.

[2] D. Dash. Restructing dynamic causal systems in equilibrium. In *Proc. Uncertainty in Artificial Intelligence*, 2005.

[3] J. Mooij, D. Janzing, and B. Schölkopf. From ordinary differential equations to structural causal models: the deterministic case. In Nicholson A. and P. Smyth, editors, *Proceedings of the Eleventh Conference on Uncertainty in Artificial Intelligence (UAI)*, pages 440–448, Oregon, USA, 2015. AUAI Press Corvallis.

[4] P. Spirtes. Directed cyclic graphical representations of feedback models. In *Proceedings of the Eleventh Conference on Uncertainty in Artificial Intelligence (UAI)*, pages 440–448, Oregon, USA, 1995. Morgan Kaufmann Publishers Inc.

[5] J. T. A. Koster. Markov properties of nonrecursive causal models. *Annals of Statistics*, 24(5):2148–2177, 1996.

[6] J. Pearl and R. Dechter. Identifying independence in causal graphs with feedback. In *Proceedings of the Twelfth Annual Conference on Uncertainty in Artificial Intelligence (UAI-96)*, pages 420–426, 1996.

[7] M. Voortman, D. Dash, and M. Druzdzel. Learning why things change: The difference-based causality learner. In *Proceedings of the Twenty-Sixth Annual Conference on Uncertainty in Artificial Intelligence (UAI)*, pages 641–650, Corvallis, Oregon, 2010. AUAI Press.

[8] J. Mooij, D. Janzing, B. Schölkopf, and T. Heskes. Causal discovery with cyclic additive noise models. In *Advances in Neural Information Processing Systems 24, Twenty-Fifth Annual Conference on Neural Information Processing Systems (NIPS 2011)*, Curran, pages 639–647, NY, USA, 2011. Red Hook.

[9] A. Hyttinen, F. Eberhardt, and P.O. Hoyer. Learning linear cyclic causal models with latent variables. *Journal for Machine Learning Research*, 13:33873439, November 2012.

[10] P. K. Rubenstein, S. Bongers, J. M. Mooij, and B. Schölkopf. From deterministic ODEs to dynamic structural causal models. *arXiv*, 1608.08028, 2016.

[11] C. W. J. Granger. Investigating causal relations by econometric models and cross-spectral methods. *Econometrica*, 37(3):424–38, July 1969.

[12] K. Chalupka, P. Perona, and F. Eberhardt. Multi-level cause-effect systems. In *Proceedings of the 19th International Conference on Artificial Intelligence and Statistics (AISTATS)*, JMLR: W&CP volume 41, 2016.

[13] N. Hansen and Al. Sokol. Causal interpretation of stochastic differential equations. *Electron. J. Probab.*, 19:124, 2014.

[14] P. K. Rubenstein, S. Weichwald, S. Bongers, J. M. Mooij, D. Janzing, M. Grosse-Wentrup, and B. Schölkopf. Causal consistency of structural equation models. In *Proceedings of the Thirty-Third Conference on Uncertainty in Artificial Intelligence (UAI)*, 2017.

5 Appendix

5.1 Covariance of \(X\) and \(Y\) in the stationary distribution

\[
\text{Cov} [X_{t+1}, Y_{t+1}] = \beta \beta \sum_{k=0}^{\infty} \alpha^{k+2} - \alpha^{k+1} \gamma^{k+1} \alpha^{k+1} - \alpha^{k+1} \gamma^{k+1} \alpha \gamma 1 - \alpha^2 (1 - \alpha \gamma) (1 - \alpha \gamma)
\]
5.2 Approximate eigenvalues of functions of $S$

Using (22) we obtain:

$$
\| \gamma S(I - \beta S)^{-1} g_{\nu, T} - \gamma z_{\nu}(1 - z_{\nu})^{-1} g_{\nu, T} \|_1
$$

$$
= \left\| \frac{2}{\beta} \sum_{j=1}^{\infty} (\beta S)^j g_{\nu, T} - \frac{\gamma}{\beta} \sum_{j=1}^{\infty} (\beta z_{\nu})^j g_{\nu, T} \right\|_1
$$

$$
\leq \frac{2}{\sqrt{2T + 1} |\beta|} \left| \frac{d}{d\beta} \! \left( 1 - \beta \right)^{-1} \right| = \frac{2}{\sqrt{2T + 1} |\beta|} \left| \gamma \right| (1 - \beta)^{-2}.
$$

5.3 Variance of $Y$ in the stationary distribution

$$
\mathbb{V}[Y_{t+1}]
$$

$$
= \mathbb{V} \left[ \sum_{k=0}^{\infty} \gamma^k E_{t-k}^Y \right] + \mathbb{V} \left[ \beta \sum_{k=0}^{\infty} \frac{\alpha^{k+1} - \gamma^{k+1}}{\alpha - \gamma} E_{t-1-k}^X \right]
$$

$$
= \sum_{k=0}^{\infty} \gamma^{2k} + \frac{\beta^2}{(\alpha - \gamma)^2} \sum_{k=0}^{\infty} \alpha^{2k+2} - 2\alpha^{k+1}\gamma^{k+1} - \gamma^{2k+2}
$$

$$
= \frac{1}{1 - \gamma^2} + \frac{\beta^2}{(\alpha - \gamma)^2} \left[ \frac{\alpha^2}{1 - \alpha^2} - 2\alpha\gamma \frac{\gamma}{1 - \alpha\gamma} + \gamma^2 \right]
$$

$$
= \frac{1}{1 - \gamma^2} \left[ \frac{\alpha^2 - \alpha \gamma}{(1 - \alpha\gamma)(1 - \gamma^2)} + \frac{\gamma^2}{(1 - \alpha\gamma)(1 - \gamma^2)} \right]
$$

$$
= \frac{1}{1 - \gamma^2} \left[ \frac{\alpha^2 - \alpha \gamma}{(1 - \alpha^2)(1 - \alpha\gamma)(1 - \gamma^2)} + \frac{2\alpha^2\gamma - 2\alpha \gamma^3 + \gamma^2 - \alpha^2 \gamma^3 - \alpha \gamma^2 + \alpha \gamma^3}{(1 - \alpha^2)(1 - \alpha\gamma)(1 - \gamma^2)} \right]
$$

$$
= \frac{1}{1 - \gamma^2} \left[ \frac{\beta^2}{(\alpha - \gamma)^2} \left[ \frac{\alpha^2 + \alpha \gamma}{(1 - \alpha^2)(1 - \alpha\gamma)(1 - \gamma^2)} \right] \right]
$$

$$
= \frac{1}{1 - \gamma^2} \left[ \frac{\beta^2}{(\alpha - \gamma)^2} \left[ \frac{\alpha^2 - 2\alpha \gamma + \gamma^2}{(1 - \alpha^2)(1 - \alpha\gamma)(1 - \gamma^2)} \right] \right]
$$

$$
= \frac{1}{1 - \gamma^2} \left[ \frac{\beta^2}{(\alpha - \gamma)^2} \left[ \frac{(\alpha - \gamma)^2(1 + \alpha\gamma)}{(1 - \alpha^2)(1 - \alpha\gamma)(1 - \gamma^2)} \right] \right]
$$