MENELAUS RELATION, HIROTA–MIWA EQUATION AND FAY’S TRISECANT FORMULA ARE ASSOCIATIVITY EQUATIONS

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Received 17 March 2010
Accepted 4 May 2010

It is shown that the celebrated Menelaus relation, Hirota–Miwa bilinear equation for KP hierarchy and Fay’s trisecant formula similar to the WDVV equation are associativity conditions for structure constants of certain three-dimensional quasi-algebra.

Keywords: Associativity; Menelaus; Hirota–Miwa; Fay’s trisecant; WDVV equation.

1. Introduction

Associative algebras are fundamental ingredients in a number of theories and constructions in theoretical and mathematical physics. One of the most intriguing and unexpected recent manifestation of their role is due to the discovery of Witten [1] and Dijkgraaf–Verlinde–Verlinde [2]. They showed that the properties of correlation functions $\langle \Phi_j \Phi_k \cdots \rangle$ for the two-dimensional topological field theory are encoded by the algebraic relations of the form

$$
\sum_{m=1}^{N} C_{jk}^{ml} C_{ml} = \sum_{m=1}^{N} C_{jl}^{nm} C_{jm}, \quad j, k, l, n = 1, \ldots, N,
$$

where $C_{jk}^{ml} = \eta^{jm} C_{mkl}$, $\eta$ is the matrix inverse to the matrix of two-points correlation functions $\langle \Phi_j \Phi_k \rangle$ and $C_{mkl} = \langle \Phi_m \Phi_k \Phi_l \rangle$. Moreover, for the deformed model, the three-points correlation function $C_{mkl}(x) = \langle \Phi_m \Phi_k \Phi_l \rangle$ is the third order derivative $C_{mkl} = \frac{\partial^3 F}{\partial x_m \partial x_k \partial x_l}$ and the algebraic equations (1) take the form of the system of partial differential equations for $F$ (WDVV equation) [1, 2].

A remarkable fact is that Eq. (1) are nothing else than the associativity conditions for the structure constants of the algebra of primary fields with the multiplication rule $\Phi_j \cdot \Phi_k = \sum_{m=1}^{N} C_{jk}^{pm} \Phi_m$ [1, 2] and, hence, the WDVV equation is the associativity condition for structure constants with a particular dependence on the deformation parameters $x_j$. 

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This observation was beautifully formalized in [3, 4] as the theory of Frobenius manifolds and then extended to the theory of F-manifolds in [5]. It turned out that the WDVV equation plays a fundamental role in the theory of quantum cohomology and other branches of algebraic geometry (see e.g. [4, 6]). Thus, it was demonstrated that the associativity equation (1) and its deformed forms are fundamental objects encoding important information.

In this paper, we will show that associativity equation plays similar role in three other quite important cases. Two of them are the classical Menelaus relation and Fay’s trisecant formula for Riemann theta function. Separated in time by 2000 years and arozen in a quite different branches of geometry both these formulas are nothing but the associativity condition for the structure constants of a certain triple with commutative multiplication. The same is valid also for the bilinear discrete Hirota–Miwa equation for the KP hierarchy.

The paper is organized as follows. In Sec. 2, we briefly recall the basic formulas for the simplest WDVV equation. Relation between Menelaus configuration and theorem with associativity condition is discussed in Sec. 3. The KP case is considered in Sec. 4. The gauge equivalence of the Menelaus and KP configurations is demonstrated in Sec. 5. In Sec. 6, it is shown that Fay’s trisecant formula also is the associativity equation and a conjecture about the possible role of the quasi-algebra in characterization of Jacobian varieties is formulated.

2. WDVV Equation

Here we will discuss briefly the simplest WDVV equation in order to recall its connection with the associativity condition and in order to use this construction further as a sort of guide. We will derive this equation in a manner (see [7, 8]) which is slightly different from the usual one ([1–6]).

Thus, we consider three-dimensional associative algebra $A$ with the unite element $P_0$. We assume that the algebra possess a commutative basis the elements of which we will denote as $P_0, P_1, P_2$. The table of multiplication $P_j \cdot P_k = P_l, j, k = 0, 1, 2$ and

\[
\begin{align*}
P_0^2 &= AP_0 + BP_1 + CP_2, \\
P_1 P_2 &= P_2 P_1 = DP_0 + EP_1 + GP_2, \\
P_2^2 &= LP_0 + MP_1 + NP_2
\end{align*}
\]

defines the structure constants $A, B, C, \ldots, N$ of the algebra $A$ in this basis. The associativity of the algebra, i.e. the conditions \( [P_j P_k] P_l = P_j (P_k P_l) \), $j, k, l = 0, 1, 2$ (conditions (1)) in this case are equivalent to the following three equations

\[
\begin{align*}
A &= EC + G^2 - BG - CN, \\
D &= CM - GE, \\
L &= E^2 + GM - MB - NE.
\end{align*}
\]
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One of the ways to describe deformations of the structure constants $A, B, \ldots, N$ is to associate the following system of linear differential equations (see e.g. [3, 8])

\[
\begin{align*}
\Psi_{x_1 x_1} &= A \Psi + B \Psi_{x_2} + C \Psi_{x_2}, \\
\Psi_{x_1 x_2} &= D \Psi + E \Psi_{x_1} + F \Psi_{x_2}, \\
\Psi_{x_2 x_2} &= L \Psi + M \Psi_{x_1} + N \Psi_{x_2}
\end{align*}
\]

with the multiplication table (2) (Dirac’s recipe [8]) and require its compatibility, i.e.

\[
\frac{\partial^2}{\partial x_1 \partial x_1} \frac{\partial \Psi}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial^2 \Psi}{\partial x_1 \partial x_2} - \frac{\partial^2 \Psi}{\partial x_2 \partial x_1} \right). 
\]

Here and below $\Psi_{x_k} = \frac{\partial \Psi}{\partial x_k}$, etc. The corresponding system of nonlinear differential equations for the structure constants admits various reductions. One of the distinguished reductions is $C = 1, G = 0, N = 0$. Under this constraint the associativity conditions (3) are reduced to

\[
\begin{align*}
A &= E, \\
D &= M
\end{align*}
\]

and

\[
L = A^2 - DB
\]

while the system of differential equations becomes

\[
\begin{align*}
D_{x_1} - A_{x_2} + A^2 - DB - L &= 0, \\
D_{x_2} - A_{x_2} + L + DB - A^2 &= 0, \\
A_{x_1} - B_{x_2} &= 0, \\
L_{x_1} - D_{x_2} &= 0, \\
E - A &= 0, \\
M - D &= 0.
\end{align*}
\]

This system is equivalent to the algebraic associativity conditions (6), (7) and differential exactness conditions

\[
\begin{align*}
D_{x_1} - A_{x_2} &= 0, \\
A_{x_1} - B_{x_2} &= 0, \\
L_{x_1} - D_{x_2} &= 0.
\end{align*}
\]

Equations (9) imply the existence of a function $F$ such that

\[
\begin{align*}
A &= E = F_{x_1 x_2}, \\
B &= F_{x_1 x_1}, \\
D &= M = F_{x_2 x_2}, \\
L &= F_{x_2 x_2}.
\end{align*}
\]

The remaining associativity condition (7) thus becomes

\[
F_{x_2 x_2} = (F_{x_1 x_1})^2 - F_{x_1 x_1} F_{x_1 x_2}.
\]

It is the famous WDVV equation [1, 2]. Its algebro-geometrical significance is discussed in [4, 6].

Thus, the WDVV equation (11) is nothing but the associativity equation (7) in parametrization (10). We would like to note that the derivation of the WDVV equation
given above shows also that the presence of the algebra \(A\) is not indispensable. To get WDVV equation it is sufficient to consider the triple \(P_0, P_1, P_2\) closed with respect to commutative associative multiplication defined by the relations (2) and \(P_0 \cdot P_j = P_j, \ j = 0, 1, 2\).

3. Menelaus Relation as Associativity Condition

In order to approach the Menelaus relation (see e.g. [9, 10]) in a similar manner one should first choose an appropriate algebraic structure. Thus, we consider a triple \(QA = (P_1, P_2, P_3)\) equipped with the commutative and associative multiplication of distinct elements such that

\[
P_1 P_2 = AP_1 + BP_2, \quad P_1 P_3 = CP_1 + DP_3, \quad P_2 P_3 = EP_2 + GP_3,
\]

where \(A, B, \ldots, G\) are, in general, the complex numbers. \(QA\) is not an associative algebra in the usual sense. However it is its close relative. For instance, the table (12) represents itself the closed sub-table of the table of multiplication for a three-dimensional algebra with the basis elements \(P_1, P_2, P_3\). For this reason one may refer to the triple \(QA\) as the quasi-algebra.

Associativity conditions

\[
P_1(P_2P_3) = P_2(P_1P_3) = P_3(P_1P_2)\]

for the structure constants of such \(QA\) have the form

\[
(A - G)C - EA = 0, \quad (A - G)D + BG = 0, \quad (C - E)B + DE = 0.
\]

Lemma 1. For nonzero \(A, B, \ldots, G\) the associativity conditions (14) are equivalent to the equation

\[
AED + BCG = 0
\]

and one of Eqs. (14), for instance, the equation

\[
(A - G)C - EA = 0.
\]  

Proof. Multiplying the first of Eqs. (14) by \(D\), second by \(C\) and subtracting results, one gets (15). The rest is straightforward.

To describe deformations of the structure constants defined by (12) one should, similar to the WDVV case, apply the Dirac’s recipe to a linear systems which will be realization of the table (12). We choose the realization of \(P_1, P_2, P_3\) by operators of shifts \(P_j = T_j\) where

\[
T_j \Phi(n_1, n_2, n_3) = \Phi(n_1 + 1, n_2, n_3), \quad T_j \Phi(n_1, n_2, n_3) = \Phi(n_1, n_2 + 1, n_3), \quad T_j \Phi(n_1, n_2, n_3) = \Phi(n_1, n_2, n_3 + 1)
\]

and \(n_1, n_2, n_3\) are deformation parameters [11]. The corresponding linear system is [11]

\[
\Phi_{12} = A\Phi_1 + B\Phi_2, \quad \Phi_{13} = C\Phi_1 + D\Phi_3, \quad \Phi_{23} = E\Phi_2 + G\Phi_3.
\]

where \(\Phi_j = T_j^* \Phi, \Phi_{jk} = T_j^* T_k^* \Phi\).
It is the celebrated Menelaus relation (see [9, 10]) which is necessary and sufficient condition for collinearity of the points \( \Phi_i \) with \( i = 1, 2, \ldots, 6 \). Then the relations (15), (16) imply that the points \( \Phi_1, \Phi_2, \Phi_12 \) are collinear as well as the sets of points \( \Phi_{13}, \Phi_{23} \) and \( \Phi_1, \Phi_2, \Phi_3 \). The relations (17) and (18) allow us to express Eq. (17) with \( i = 1, 2, \ldots, G \), obeying associativity conditions (15), (16) define a configuration of six points on the plane.

There are at least two distinguished special configurations among them. The first corresponds to the case when

\[
A + B = 1, \quad C + D = 1, \quad E + G = 1. \tag{18}
\]

For such \( A, B, \ldots, G \) the relations (17), in virtue of the conditions (18), mean that three points \( \Phi_1, \Phi_2, \Phi_12 \) are collinear as well as the sets of points \( \Phi_1, \Phi_3, \Phi_{13} \) and \( \Phi_2, \Phi_3, \Phi_{23} \). Then the relations (15), (16) imply that the points \( \Phi_{12}, \Phi_{13}, \Phi_{23} \) are collinear too, i.e.

\[
\Phi_{12} = \frac{A}{C} \Phi_{11} + \frac{B}{E} \Phi_{23} \tag{19}
\]

with \( \frac{A}{C} + \frac{B}{E} = 1 \). Thus, in the case (18) the relations (17) describe the set of four triples \( (\Phi_1, \Phi_2, \Phi_{12}), (\Phi_1, \Phi_3, \Phi_{13}), (\Phi_2, \Phi_3, \Phi_{23}) \) and \( (\Phi_{12}, \Phi_{13}, \Phi_{23}) \) of collinear points. It is nothing but the celebrated Menelaus configuration of the classical geometry (Fig. 1) (see e.g. [9, 10]).

The relations (17) and (18) allow us to express \( A, B, \ldots, G \) in terms of \( \Phi \). One gets

\[
A = \frac{\Phi_{12}^M - \Phi_{13}^M}{\Phi_{12}^M - \Phi_{23}^M}, \quad B = \frac{\Phi_{13}^M - \Phi_{12}^M}{\Phi_{12}^M - \Phi_{23}^M}, \quad C = \frac{\Phi_{12}^M - \Phi_{13}^M}{\Phi_{13}^M - \Phi_{23}^M}, \quad D = \frac{\Phi_{13}^M - \Phi_{12}^M}{\Phi_{13}^M - \Phi_{23}^M}, \quad E = \frac{\Phi_{12}^M - \Phi_{13}^M}{\Phi_{12}^M - \Phi_{23}^M}, \quad G = \frac{\Phi_{12}^M - \Phi_{13}^M}{\Phi_{12}^M - \Phi_{23}^M}. \tag{20}
\]

where we denote by \( \Phi_i^M \) solution of the system (17), (18). In such a parameterization of \( A, B, \ldots, G \) the associativity conditions (15), (16) are equivalent to the single equation

\[
\left( \Phi_{12}^M - \Phi_{13}^M \right) \left( \Phi_{23}^M - \Phi_{13}^M \right) \left( \Phi_{12}^M - \Phi_{23}^M \right) = -1.
\]

It is the celebrated Menelaus relation (see [9, 10]) which is necessary and sufficient condition for collinearity of the points \( \Phi_{12}, \Phi_{13}, \Phi_{23} \) for any three given points \( \Phi_1, \Phi_2, \Phi_3 \) on the plane.
plane. In our formulation the Menelaus relation (21) is nothing else than the associativity conditions (15), (16) written in terms of $\Phi^M$. Thus, the Menelaus theorem is intimately connected with the associative algebra (12) with the choice (18).

Proposition 1. For a configuration of six points on the plane defined by Eqs. (17) with the relation (18) and three arbitrary points $\Phi_1, \Phi_2, \Phi_3$ the following two conditions are equivalent:

(A) Coefficients $A, B, \ldots, G$ in (17) obey the associativity equations (14) or (15), (16) for the QA (12);

(B) Points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ are collinear.

Proof. An implication $(A) \rightarrow (B)$ has been proved above. Now let us assume that points $\Phi_{12}, \Phi_{13}, \Phi_{23}$ defined by (17) together with (18) are collinear for arbitrary points $\Phi_1, \Phi_2, \Phi_3$. This means that

$$\Phi_{12} = \alpha \Phi_{13} + \beta \Phi_{23}$$

(22)

with $\alpha + \beta = 1$. Substitution of expressions for $\Phi_{12}, \Phi_{13}, \Phi_{23}$ given by (17), (18) into (22) gives the conditions

$$A = \alpha C, \quad B = \beta E, \quad \alpha D + \beta G = 0.$$

These conditions imply that $AE_3 + BC_2 = 0$. Then the substitution of $\alpha = \frac{A}{C}, \beta = \frac{B}{E}$ into the condition $A + B = 1$ gives $AE + BC - CE = 0$. It is easy to check that this equation is equivalent to the first associativity condition (14). Due to Lemma 1 these conditions are equivalent to all associativity conditions (14).

Thus the Menelaus configuration and theorem are just the geometric realizations of the associativity conditions for the QA (12).

Discrete deformations of the Menelaus configurations are governed by discrete equations arising as compatibility conditions

$$\Phi_{3(2)} = \Phi_{1(23)} = \Phi_{2(13)}$$

for the system (17). They are of the form

$$\frac{A_3}{A} = \frac{C_2}{C}, \quad \frac{B_3}{B} = \frac{E_1}{E}, \quad \frac{D_2}{D} = \frac{G_1}{G},$$

(23)

$$\frac{(A_3 - G_1)C - E_1A = 0,}{(A_3 - G_1)A + B_3G = 0,}{(C_2 - E_1)B + D_3E = 0,}$$

(24)

(25)

(26)

One can show that Eqs. (24)–(26) are equivalent, modulo equations (23), to the relation (15) and one of Eqs. (24)–(26), for instance, Eq. (24). Thus, in the Menelaus case the
deformation equations are equivalent to the cubic part of the associativity condition
\[ AED + BCG = 0, \]
(27)
i.e. the Menelaus relation plus deformed quadratic part (24) of the associativity condition and “exactness” Eq. (23). Discrete deformations of the Menelaus configuration given by Eqs. (23)–(26) generate an integrable lattice on the plane [12].

4. KP Configurations, Discrete KP Deformations and Hirota–Miwa Equation

Another distinguished case corresponds to the choice
\[ A + B = 0, \quad C + D = 0, \quad E + G = 0 \]
(28)
for which Eqs. (17) take the form
\[ \Phi_{12} = A(\Phi_1 - \Phi_2), \quad \Phi_{13} = C(\Phi_1 - \Phi_3), \quad \Phi_{23} = E(\Phi_2 - \Phi_3). \]
(29)
With such a choice of \(A,B,...,G\) the relation (15) is a trivial identity and, hence, the associativity conditions are reduced to the single equation
\[ AC + EC - AE = 0. \]
(30)

Geometrical configuration on the plane formed by six points \(\Phi_1, \Phi_2, \Phi_3, \Phi_{12}, \Phi_{23}, \Phi_{13}\) with real \(A,C,E\) is a special one. We, first, observe that the points \(\Phi_{12}, \Phi_{13}, \Phi_{23}\) lie on the straight lines passing through the origin \(0\) and parallel to the straight lines passing through the points \((\Phi_1, \Phi_2), (\Phi_1, \Phi_3), (\Phi_2, \Phi_3)\), respectively. Then, due to the associativity condition (30) the points \(\Phi_{12}, \Phi_{23}, \Phi_{13}\) are collinear. Indeed, Eqs. (29) imply that
\[ \frac{1}{C}\Phi_{13} - \frac{1}{A}\Phi_{12} - \frac{1}{E}\Phi_{23} = 0 \]
(31)
while the relation (30) is equivalent to the condition \(\frac{1}{A} - \frac{1}{C} - \frac{1}{E} = 0\). Thus, the points \(\Phi_1, \Phi_2, \Phi_3, \Phi_{12}, \Phi_{23}, \Phi_{13}\) form the configuration on the complex plane shown in Fig. 2.

![Fig. 2. KP configuration.](image)
The associativity condition (30) provides us also with the relation between the directed lengths for this configuration. Indeed, expressing $A, C, E$ from (29) in terms of $\Phi$ and substituting into (30), one gets

$$\frac{\Phi_1 - \Phi_2}{\Phi_{12}} + \frac{\Phi_2 - \Phi_3}{\Phi_{23}} + \frac{\Phi_3 - \Phi_1}{\Phi_{31}} = 0.$$  

(32)

Since for real $A, C, E$, $\Phi_{12} = |\Phi_1 - \Phi_2|$, etc., the formula (32) represents the relation between the directed lengths $|\Phi_1 - \Phi_2|$ of the interval $(\Phi_1, \Phi_2)$, etc.

We note that the straight line passing through the points $\Phi_{12}, \Phi_{23}, \Phi_{13}$ is a trisecant of the family of three straight lines passing through origin. We will refer to the configuration presented in Fig. 2 as KP configuration by the reason which will be clarified now.

Let us consider discrete deformations of such configurations. They are governed by Eqs. (23)–(26) under the constraint (28).

**Lemma 2.** In the case (28), Eqs. (23)–(26) are equivalent to the associativity condition (30) and equations

$$\frac{A_3}{A} = \frac{C_2}{C} = \frac{E_1}{E}.$$  

(33)

**Proof.** In this case, Eqs. (23) are reduced to Eqs. (33) while Eqs. (24)–(26) are equivalent to the single equation $A_3C + E_1C - AE_1 = 0$. Due to (33) this equation is equivalent to the associativity condition (30).

Equations (33) imply the existence of the function $\tau$ such that

$$A = -\frac{\tau_1\tau_2}{\tau_{12}}, \quad C = -\frac{\tau_1\tau_3}{\tau_{13}}, \quad E = -\frac{\tau_2\tau_3}{\tau_{23}}.$$  

(34)

Substitution of these expressions into (30) gives

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} = 0$$  

(35)

which is the celebrated discrete bilinear Hirota–Miwa equation for the KP hierarchy or the addition formula for KP $\tau$-function [13]. This fact justifies the name of the configuration. We would like to emphasize that the Hirota–Miwa equation (35) is nothing but the associativity condition (30) with the structure constants $A, C, E$ parametrized by $\tau$-function. We note that Eqs. (29) with $A, C, E$ of the form (34) coincide with well-known linear problems for the Hirota–Miwa bilinear equation [13].

Similar to the Menelaus case we thus have

**Proposition 2.** For the six-points configuration on the plane defined by Eqs. (28), (29) and three arbitrary points $\Phi_1, \Phi_2, \Phi_3$ the following conditions are equivalent:

(A) Coefficients $A, E, C$ obey the associativity condition (30);
(B) The points $\Phi_{12}, \Phi_{23}, \Phi_{13}$ are collinear;
(C) Function $\tau$ defined by (34) obeys the discrete Hirota–Miwa equation (35).
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Proof. Implications \( (A) \rightarrow (B) \) and \( (A) \rightarrow (C) \) have been proved above. Implication \( (B) \rightarrow (A) \) is obvious. Then, multiplying Eq. (35) by \( \tau \tau_1 \tau_2 \tau_3 \), one gets (30) that proves \( (C) \rightarrow (A) \) and, hence, \( (C) \rightarrow (B) \).

5. Gauge Equivalence of the Menelaus and KP Configurations

The Menelaus and KP configurations look quite different. For instance, the points \( \Phi_1, \Phi_2, \Phi_{12} \) etc are collinear in the Menelaus case and they are not in the KP case. Nevertheless, they are closely connected, namely, they are gauge equivalent to each other.

A notion of gauge equivalency is a very well-known one in the theory of gauge fields as well in the theory of integrable equations. In our case the linear system (17) is invariant under the gauge transformations \( \Phi \rightarrow \tilde{\Phi} = g^{-1} \Phi \) and

\[
\tilde{A} = \frac{g_1}{g_{12}} A, \quad \tilde{B} = \frac{g_2}{g_{12}} B, \quad \tilde{C} = \frac{g_3}{g_{13}} C, \\
\tilde{D} = \frac{g_3}{g_{13}} D, \quad \tilde{E} = \frac{g_2}{g_{23}} E, \quad \tilde{G} = \frac{g_3}{g_{23}} G,
\]

(36)

where \( g(n_1, n_2, n_3) \) is an arbitrary function. It is a simple check that the system (23)–(26) is invariant under these gauge transformations too.

Lemma 3 [11]. \( \text{The relation (15) is invariant under the gauge transformations (36).} \)

Proof. Indeed, under the gauge transformation (36) one has

\[
\tilde{A} \tilde{E} \tilde{D} + \tilde{B} \tilde{C} \tilde{G} = \frac{g_1 g_2 g_3}{g_{12} g_{23} g_{13}} (AED + BCG).
\]

(37)

So the relation (15) is a characteristic one for orbits of gauge equivalent structure constants.

Gauge invariance of the general system (23)–(26) allows us to choose different gauges. First, we observe that

\[
\tilde{A} + \tilde{B} = \frac{g_1 A + g_2 B}{g_{12}}, \quad \tilde{C} + \tilde{D} = \frac{g_3 C + g_1 D}{g_{13}}, \quad \tilde{E} + \tilde{G} = \frac{g_2 E + g_3 G}{g_{23}}
\]

(38)

So, if for generic \( A, B, \ldots, G \) one chooses the gauge function \( g \) to be a solution \( \tilde{\Phi} \) of the linear system (17) then the gauge transformed \( \tilde{A}, \tilde{B}, \ldots, \tilde{G} \) obey the relations

\[
\tilde{A} + \tilde{B} = 1, \quad \tilde{C} + \tilde{D} = 1, \quad \tilde{E} + \tilde{G} = 1.
\]

(39)

Thus, the relation (18) discussed in Sec. 3 selects a special gauge which is nothing but the distinguished Menelaus gauge.

Solutions of the linear system (17) in this gauge are ratios of two solutions of the generic system (17): \( \Phi = (\Phi_1, \Phi_2, \Phi_{12}) \) as projective homogeneous coordinates, then in terms of affine coordinates \( \frac{\Phi}{c} \) it is the Menelaus system (system (17) with the relations (18)). In other words, the Menelaus configuration is the affine form of the generic configuration of six points defined by the system (17). We note that the gauge transformation (36) geometrically means a local (depending on a point) homothetic transformation.

Now let us begin with the particular KP system (17), i.e. when \( A, B, \ldots, G \) obey the relations (28). In this case after the gauge transformation one has the relations (38) with
January 20, 2011 14:19 WSPC/1402-9251 259-JNMP S1402925110001070

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Let us rewrite this formula in the equivalent form. Using the identity (see e.g. [14, 15])

\[
\int_{\alpha_0}^{\alpha_3} \omega + \int_{\alpha_1}^{\alpha_4} \omega = \int_{\alpha_0}^{\alpha_4} \omega = \int_{\alpha_0 + \alpha_1}^{\alpha_3} \omega,
\]

(42)
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one gets

\[
\theta \left( z + \int_0^\omega \omega \right) \theta \left( z + \int_0^{a_3} \omega - \int_0^{a_1} \omega \right) E(a_0, a_3) E(a_2, a_1) \\
+ \theta \left( z + \int_0^{a_2} \omega - \int_0^{a_1} \omega \right) \theta \left( z + \int_0^{a_1} \omega \right) E(a_2, a_0) E(a_3, a_1) \\
- \theta \left( z + \int_0^{a_1} \omega \right) \theta \left( z + \int_0^{a_0} \omega - \int_0^{a_1} \omega \right) E(a_0, a_1) E(a_2, a_3) = 0.
\]

Then we fix all \( \alpha_j \), denote \( U_j = \int_{a_0}^{a_j} \omega \), \((\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1)\) and introduce shifts \( T_j \) defined by

\[
T_j \theta(z) = \theta(z + U_j), \quad j = 1, 2, 3.
\]

Note that \( U_j \) are commonly used objects connected with the Abel’s map (see e.g. [16]). In these notations the Fay’s formula (43) takes the form

\[
aT_3 \theta(z) \cdot T_1^{-1} T_2 \theta(z) + bT_2 \theta(z) \cdot T_3 \theta(z) + cT_3 \theta(z) \cdot T_3^{-1} T_2 T_3 \theta(z) = 0,
\]

which means that the system implies that the points \( \Phi(z + U_1 + U_2), \Phi(z + U_2 + U_3), \Phi(z + U_1 + U_3) \) are collinear. Thus, we have

**Proposition 3.** The following three conditions are equivalent:

(A) Function \( \theta(z) \) obeys the Fay’s trisecant formula (41).

(B) Structure constants \( A, C, E \) defined by (47) obey the associativity condition

\[
AC + EC - AE = 0,
\]

for \( QA \) (12) in the gauge \( A + B = C + D = E + G = 0 \).

\[\text{FA 1}\]
Three points $\Phi(z + U_1 + U_2), \Phi(z + U_1 + U_3), \Phi(z + U_2 + U_3)$ defined by relations (48) are collinear.

**Proof.** Equivalence of $(A)$ and $(B)$ has been proved above. Equivalence of $(B)$ and $(C)$ is obvious.

Thus, the Fay’s trisecant formula for theta-function of any Riemann surface is nothing but the associativity condition for the structure constants of the QA(12) with parametrization (47). The latter is a consequence of deformation equations (33). We emphasize also that the collinearity of the points $\Phi(z + U_1 + U_2), \Phi(z + U_1 + U_3), \Phi(z + U_2 + U_3)$ is equivalent to the associativity condition.

Applying $T_1^{-1}T_2^{-1}T_3^{-1}$ to the relation (49), one gets

$$\frac{1}{A} T_1^2 T_2^2 T_3^2 \Phi + \frac{1}{E} T_1^{-1} T_2 T_3 \Phi - \frac{1}{C} T_1^2 T_2^2 T_3^2 \Phi = 0,$$

where

$$\frac{1}{A} + \frac{1}{E} - \frac{1}{C} = 0. \quad (52)$$

Thus, the points

$$\Phi \left( z + \frac{1}{2}(U_1 + U_2 - U_3) \right), \quad \Phi \left( z + \frac{1}{2}(-U_1 + U_2 + U_3) \right), \quad \Phi \left( z + \frac{1}{2}(U_1 - U_2 + U_3) \right) \quad (53)$$

are collinear too. Standard Fay’s trisecant formula (41) is equivalent (see e.g. [15, 16]) to the collinearity of the points

$$\varphi \left( z + \frac{1}{2}(U_1 + U_2 - U_3) \right), \quad \varphi \left( z + \frac{1}{2}(-U_1 + U_2 + U_3) \right), \quad \varphi \left( z + \frac{1}{2}(U_1 - U_2 + U_3) \right) \quad (54)$$

in the Kummer variety. It suggests to identify the map $\Phi$ defined by (48) with the Kummer map: $\Phi = \varphi = \theta \begin{bmatrix} 0 \\ \beta_i \end{bmatrix} \in \mathbb{Z}/2\mathbb{Z}$.

Due to the existence of the solutions for the KP $r$-function in terms of the Riemann theta function (see e.g. [16]) and well-known relation between the addition formula for KP hierarchy and Fay’s trisecant formula (see e.g. [14, 15]) the similarity between the KP and Fay cases is quite natural.

In the papers [17–21] it was shown that the existence of a family of trisecants or even only one trisecant characterize Jacobian varieties among indecomposable principally polarized abelian varieties (ppavs). The results of Proposition 3 make it quite natural to conjecture that the existence of the associative three-dimensional QA (12), (28) for ppav such that on the corresponding line bundle equations (48) are valid with shifts $T_j$ defined by (44) is characteristic one for Jacobian varieties.

Equations (46), (48) and (49) can be treated as discrete equations of one takes theta function and function $\Phi$ of the form $\theta(n_1, n_2, n_3; z) = \theta(z + \sum_j U_i n_i)$ and $\Phi(n_1, n_2, n_3; z) = \Phi(z + \sum_j U_i n_i)$ (see e.g. [16] for the theta-function solutions of the Hirota–Miwa equation (35)).
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These equations admit different reductions. For instance, if one freezes the dependence on \(n_3\) then under the constraint \(\Phi = \theta(A + U_1 n_1 + U_2 n_2 + z)\exp(n_1 \alpha + n_2 \beta)\) where fixed \(A = U_3 n_3\) and \(\alpha, \beta\) are constants the first equation (48) coincides with Eq. (1.14) (together with (15), (16)) from the paper [21] which is necessary and sufficient condition for ppav to be Jacobian of a smooth curve of genus \(g\). This fact supports the Conjecture formulated above.

Acknowledgment

The author thanks A. Veselov and A. Nakayashiki for useful and stimulating discussions.

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