Estimation of Semileptonic Decays of $B_c$ Meson to S-wave Charmonia with NRQCD

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We study the semileptonic differential decay rates of $B_c$ meson to S-wave charmonia, $\eta_c$ and $J/\psi$, at the next-to-leading order accuracy in the framework of NRQCD. In the heavy quark limit, $m_h \to \infty$, we obtain analytically the asymptotic expression for the ratio of NLO form factor to LO form factor. Numerical results show that the convergence of the ratio is perfect. At the maximum recoil region, we analyze the differential decay rates in detail with various input parameters and polarization for $J/\psi$, which can now be checked in the LHCb experiment.

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I. INTRODUCTION

Hadron collider provides a large amount of data on $B_c$ events. Wherein the most easily identified decay modes to tag the $B_c$ are: fully reconstructed channel $B_c \to J/\Psi \pi$ and semileptonic decay channel $B_c \to J/\Psi \ell \nu$. The CDF Collaboration made the first observation of the $B_c$ meson by the semileptonic decay channel at the Tevatron fourteen years ago [1]. Latter, the D0 Collaboration performed the same analysis in a sample of 210 pb$^{-1}$ of the Run II data [2]. The cross section of $B_c$ production at the Large Hadron Collider(LHC) is larger than that at the Tevatron by roughly an order of magnitude, which reaches 49.8 nb at the center-of-mass energy $\sqrt{s} = 14$ TeV [3, 4]. This makes the experimental study of the differential branching fraction of $B_c$ meson semileptonic decays to charmonium feasible. We can also obtain the information of the Cabibo-Kobayashi-Maskawa(CKM) matrix element in $B_c$ decays, especially $V_{cb}$ which is not well determined.

Recently, the BABAR collaboration measured the partial branching fraction $\Delta B/\Delta q^2$ in bins of the momentum-transfer squared, with 6 $q^2$ bins for $B^0 \to \pi^- \ell^+\nu$ and 3 $q^2$ bins for $B^0 \to \rho^- \ell^+\nu$ [5]. They found that the partial branching fraction of $B^0 \to \pi^- \ell^+\nu$ decreases as $q^2$ increases, while for $B^0 \to \rho^- \ell^+\nu$ process, the partial branching fraction increases first and then decreases as $q^2$ increases. Actually, we know that all of the five form factors in above two decay channels at the maximum recoil region increases with $q^2$, at the next-to-leading order(NLO) accuracy according to the light cone sum rules calculation [6, 7]. The decrease of $B^0 \to \pi^- \ell^+\nu$ is caused by the phrase space, which counteracts the enhancement from form factors. In this work, we try to make out whether this happens or not in $B_c$ semileptonic decays to charmonia.

There exist several approaches in calculating the $B_c$ meson semileptonic decays to charmonium. Some of them are: the light cone QCD sum rules [8–11], the relativistic quark model [12, 13], the instanton non-relativistic approach to the Bethe-Salpeter equation [14], the non-relativistic constituent quark model [10], the covariant light front model [15], and the QCD potential model [16].

Consider that the $B_c$ meson is constituted by two heavy quarks with different flavors, which masses are much larger than the $\Lambda_{QCD}$, analogous to the situation of heavy quarkonium, the system turns out to be non-relativistic. Hence the relative velocity of heavy quarks within the $B_c$ meson is small, i.e. $v \ll 1$, though bigger than the velocities of quarks in charmonium and bottomonium systems, and the non-relativistic QCD(NRQCD) formalism is applicable to the study of $B_c$ meson semileptonic decays to charmonia. In the NRQCD framework, the matrix elements of the concerned processes can be factorized as

$$\langle J/\psi(\eta_c)\ell\nu|\bar{c}c\Gamma_\mu b\Gamma_\nu|B_c\rangle \approx \sum_{n=0}^{\infty} \psi(0)_B \psi(0)_{J/\psi(\eta_c)} T^n \ . (1)$$

Here, $\Gamma_\mu = \gamma_\mu(1-\gamma_5)$, the nonperturbative parameters $\psi(0)_B$, and $\psi(0)_{J/\psi(\eta_c)}$ are the Schrödinger wave functions at the origin for $b\bar{c}$ and $c\bar{c}$ systems, respectively. $T^n$ are hard scattering kernels which can be calculated perturbatively.

The paper is organized as follows: In section II we present the definition for relevant form factors and work out the expressions of form factors in the NRQCD framework. The dependence of the NLO semileptonic differential decay rates on $q^2$ is also obtained. In section III we study the theoretical uncertainty, and analyze the result in detail of the maximum recoil region. The last section is remained for conclusions.
II. FORM FACTORS AND SEMILEPTONIC DIFFERENTIAL DECAY RATES IN THE NRQCD FRAMEWORK

The $B_c \to J/\psi(\eta_c)$ transition form factors, $f_+, f_0, V$, $A_0, A_1$, and $A_2$ are normally defined as follows [18]

\[
\langle \eta_c(p) | \bar{c} \gamma^\mu b | B_c(P) \rangle = f_+(q^2)(P^\mu + p^\mu - \frac{m_{B_c}^2 - m_{\eta_c}^2}{q^2} q^\mu)
+ f_0(q^2) \frac{m_{B_c}^2 - m_{\eta_c}^2}{q^2} q^\mu,
\]

(2)

\[
\langle J/\psi(p, \epsilon^\star) | \bar{c} \gamma^\mu b | B_c(P) \rangle = \frac{2iV(q^2)}{m_{B_c} + m_{J/\psi}} \epsilon^\mu \epsilon^\nu \rho \sigma \epsilon_\rho P_\sigma,
\]

(3)

\[
\langle J/\psi(p, \epsilon^\star) | \bar{c} \gamma^\mu \gamma_5 b | B_c(P) \rangle = 2m_{J/\psi} A_0(q^2) \frac{\epsilon^* \cdot q}{q^2} q^\mu
- A_2(q^2) \frac{\epsilon^* \cdot q}{m_{B_c} + m_{J/\psi}} (P^\mu + p^\mu - \frac{m_{B_c}^2 - m_{J/\psi}^2}{q^2} q^\mu)
\]

where $z \equiv m_c/m_b$.

In Ref. [19, 20], the form factors of $B_c$ transition to $\eta_c$ with alternative parameterizations had been calculated at the NLO accuracy in the non-relativistic limit. There are three typical scales of the process, which possess the hierarchy of $\Lambda_{QCD} \ll m_c \ll m_b$. Taking into account the work of Ref. [21], where the NLO form factors of $B_c$ transition to $J/\Psi$ are calculated, we expand the ratios of the NLO form factors to the leading order(LO) form factors at first order in $z = m_c/m_b$ expansion in the heavy quark limit $m_b \to \infty$. And the asymptotic expressions of which are then obtained analytically, that can be found in the Appendix A.

For light leptons($\ell = e, \mu$), their masses $m_\ell$ can be readily neglected, hence the semileptonic differential decay rate of $B_c \to \eta_c \ell \nu$ depending on $q^2$ reads

\[
\frac{d\Gamma}{dq^2}(B_c \to \eta_c \ell \nu) = \frac{G_F^2 |V_{cb}|^2}{192\pi^3 m_{B_c}^3} \lambda(q^2)^{3/2} |f_+(q^2)|^2.
\]

(10)

Here we define the momentum transfer $q = P - p$. Note that $f_0(0) = f_+(0)$ at the maximum recoil.

It is straightforward to calculate those form factors at the tree level in the NRQCD. They read

\[
V_{LO}(q^2) = \frac{16\sqrt{2}C_A C_F \pi (3z + 1)\alpha_s(0)\psi(0)_{J/\psi}}{(q^2 - (z - 1)^2)^{3/2} m_{B_c}^3 N_c},
\]

(4)

\[
A_{0,1}^{LO}(q^2) = \frac{16\sqrt{2}C_A C_F \pi (z + 1)^{5/2}\alpha_s(0)\psi(0)_{J/\psi}}{(q^2 - (z - 1)^2)^{3/2} m_{B_c}^3 N_c},
\]

(5)

\[
A_1^{LO}(q^2) = \frac{16\sqrt{2}C_A C_F \pi \sqrt{z + 1} (4z^3 + 5z^2 + 6z - q^2(2z + 1) + 1)\alpha_s(0)\psi(0)_{J/\psi}}{(q^2 - (z - 1)^2)^2 z^{3/2}(3z + 1)m_{B_c}^3 N_c},
\]

(6)

\[
A_2^{LO}(q^2) = \frac{16\sqrt{2}C_A C_F \pi \sqrt{z + 1} (3z + 1)\psi(0)_{B_c} \psi(0)_{J/\psi}}{(q^2 - (z - 1)^2)^2 z^{3/2} m_{B_c}^3 N_c},
\]

(7)

\[
f_+^{LO}(q^2) = \frac{8\sqrt{2}C_A C_F \pi \sqrt{z + 1} (-q^2 + 3z^2 + 2z + 3)\alpha_s(0)\psi(0)_{B_c} \psi(0)_{\eta_c}}{(q^2 - (z - 1)^2)^2 z^{3/2} m_{B_c}^3 N_c},
\]

(8)

\[
f_0^{LO}(q^2) = \frac{8\sqrt{2}C_A C_F \pi \sqrt{z + 1} (9z^3 + 9z^2 + 11z - q^2(5z + 3) + 3)\alpha_s(0)\psi(0)_{B_c} \psi(0)_{\eta_c}}{(q^2 - (z - 1)^2)^2 z^{3/2}(3z + 1)m_{B_c}^3 N_c},
\]

(9)
TABLE I: Theoretical parameters for different sets, with renormalization scale \( \mu = 4.8 \) GeV, the lifetime of the \( B_c \) \( \tau(B_c) = 0.453 \) ps, and \( G_F = 1.1663 \times 10^{-5} \text{ GeV}^{-2} \) \cite{24}, where \( m_b, m_c, \) and \( \Lambda \) are given in GeV, while \( |\psi(0)| \) are given in GeV\(^3/2\) \cite{23, 24}.

| \( m_b \) (GeV) | \( m_c \) (GeV) | \( \Lambda \) (GeV) | \( |\psi(0)|_{J/\Psi} \) | \( |\psi(0)|_{\mu} \) | \( |\psi(0)|_{\eta_c} \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| set 1          | 4.8            | 1.5            | 0.283          | 0.3615         | 0.283          |
| set 2          | 4.9            | 1.4            | 0.10           | 0.283          | 0.283          |
| set 3          | 5.0            | 1.3            |                |                |                |

of Fig. 1. The parameters in the three typical choices can be found in Table I wherein the Schrödinger wave functions at the origin for \( J/\Psi \) are determined through their leptonic decay widths at NLO.

For \( B_c \to \eta_c \ell \nu \) channel, at the maximum recoil point \( q^2 = 0 \), we obtain a value \( 4.67_{-0.58}^{+0.38} \times 10^{-12} |V_{cb}|^2 \) GeV\(^{-1} \) for (10), which is larger than the value of \( 0.65 \times 10^{-12} |V_{cb}|^2 \) GeV\(^{-1} \) obtained in nonrelativistic quark model \cite{10}. Besides, the results away from the maximum recoil point tend to disagree with what in Ref. 16. In NRQCD calculation, the form factors of \( B_c \) to \( \eta_c \) are obviously enhancing with \( q^2 \) increase, than other approaches light cone sum rules and nonrelativistic quark model, and the trend is sharpening at NLO, which counteracts the decrescence from the factor of phase space.

For the channel of \( B_c \to J/\Psi \ell \nu (\ell = e, \mu) \), the decay rates in transverse and longitudinal polarization of vector meson \( J/\Psi \) can be formulated as

\[
\frac{d\Gamma_L}{dq^2} = \frac{G_F^2 \lambda(q^2)^{1/2} |V_{cb}|^2 q^2}{192 \pi^3 m_{B_c}^3} |H_0(q^2)|^2, \tag{11}
\]

\[
\frac{d\Gamma_T}{dq^2} = \frac{G_F^2 \lambda(q^2)^{1/2} |V_{cb}|^2 q^2}{192 \pi^3 m_{B_c}^3} \left( |H_+(q^2)|^2 + |H_-(q^2)|^2 \right), \tag{12}
\]

respectively. Here the helicity amplitudes are expressed as follows:

\[
H_\pm(q^2) = \frac{\lambda(q^2)^{1/2}}{m_{B_c} + m_{J/\Psi}} \left[ V(q^2) + \frac{(m_{B_c} + m_{J/\Psi})^2}{\lambda(q^2)^{1/2}} A_1(q^2) \right],
\]

\[
H_0(q^2) = \frac{1}{2m_{J/\Psi} \sqrt{q^2}} \left[ -\frac{\lambda(q^2)}{m_{B_c} + m_{J/\Psi}} A_2(q^2) + (m_{B_c} + m_{J/\Psi})(m_{B_c}^2 - m_{J/\Psi}^2 - q^2) A_1(q^2) \right]. \tag{14}
\]

While summing up the various polarizations, the semileptonic differential decay rate of \( B_c \to J/\Psi \ell \nu \) over \( q^2 \) is obtained

\[
\frac{d\Gamma}{dq^2}(B_c \to J/\Psi \ell \nu) = \frac{G_F^2 \lambda(q^2)^{1/2} |V_{cb}|^2 q^2}{192 \pi^3 m_{B_c}^3} \times \left( |H_+(q^2)|^2 + |H_-(q^2)|^2 + |H_0(q^2)|^2 \right), \tag{15}
\]

where \( \lambda(q^2) = (m_{B_c}^2 + m_{J/\Psi}^2 - q^2)^2 - 4m_{B_c}^2 m_{J/\Psi}^2 \).

Similar as \( B_c \to \eta_c \ell \nu \), the distribution of NLO differential decay rate on momentum transfer \( q^2 \) for \( B_c \to J/\Psi \ell \nu \) channel with three sets of different values of \( z \) is illustrated in the low diagram of Fig. 1. At the maximum recoil point \( q^2 = 0 \), we obtain a value of \( 3.21_{-0.53}^{+0.37} \times 10^{-12} |V_{cb}|^2 \) GeV\(^{-1} \) for (15), which is larger than the value of \( 0.6 \times 10^{-12} |V_{cb}|^2 \) GeV\(^{-1} \) obtained in nonrelativistic quark model \cite{10}. Except for the enhancement from the NLO K-factor and the NLO Schrödinger wave functions at the origin, the result in LO in NRQCD calculation is intrinsically bigger than what obtained in nonrelativistic quark model.

III. THEORETICAL UNCERTAINTY

In this section, the theoretical uncertainties of the NLO semileptonic decay rates in maximum recoil region are investigated in detail. It is found that the main uncertainties of the concerned processes have two sources, the heavy quark masses and the renormalization scale. In the
evaluation, we vary the charm quark mass $m_c = 1.4$ GeV by $\pm 0.1$ GeV, the bottom quark mass $m_b = 4.9$ GeV by $\pm 0.1$ GeV and the renormalization scale $\mu = 4.8$ GeV by $\pm 1.8$ GeV.

For light leptons ($\ell = e, \mu$), we divide $q^2$ into five bins in maximum recoil region ($0 \leq q^2 \leq 5$ GeV$^2$) and calculate the semileptonic decay rates separately. The results are presented in Table III. We find that at small $q^2$ ($0 \leq q^2 \leq 1$ GeV$^2$), the longitudinally polarized $J/\Psi$ events dominate over the transversally polarized ones by a factor 5.82, and the difference will be reduced as $q^2$ increase. While for lepton $\tau$, we divide $q^2$ into two bins ($m^2_c \leq q^2 \leq 4$, $4 \leq q^2 \leq 5$ GeV$^2$) in maximum recoil region. Here the physical mass of lepton $\tau$ is taken to be $m^2_\tau = 1.776^2$ GeV$^2$ \cite{22}, and the results are shown in Table III.

| bins of $q^2$ (GeV$^2$) | $0 \leq q^2 \leq 1$ | $1 \leq q^2 \leq 2$ | $2 \leq q^2 \leq 3$ | $3 \leq q^2 \leq 4$ | $4 \leq q^2 \leq 5$ |
|--------------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| $\Delta \Gamma(B_c \to \eta_c \ell \nu)$ (10$^{-15}$ GeV) | $8.06^{+1.17+1.96}_{-0.77-0.74}$ | $9.73^{+1.78+2.35}_{-0.61-0.23}$ | $12.0^{+2.74+2.95}_{-1.16-0.89}$ | $15.2^{+4.47+2.73}_{-3.77-1.41}$ | $20.0^{+7.57+5.06}_{-4.36-1.87}$ |
| $\Delta \Gamma(B_c \to J/\Psi (s^\pm) \ell \nu)$ (10$^{-15}$ GeV) | $0.70^{+0.215+0.159}_{-0.141-0.000}$ | $2.64^{+0.95+0.60}_{-0.69-0.23}$ | $5.84^{+2.54+1.34}_{-1.51-0.51}$ | $11.28^{+6.04+1.61}_{-3.25-1.00}$ | $20.97^{+14.14+4.00}_{-7.10-1.87}$ |
| $\Delta \Gamma(B_c \to J/\Psi (s^\pm) \ell \nu)$ (10$^{-15}$ GeV) | $6.01^{+1.144+1.40}_{-0.78-0.53}$ | $7.87^{+0.93+0.28}_{-0.56+0.17}$ | $10.64^{+1.38+1.56}_{-0.27-0.10}$ | $19.96^{+6.18+3.56}_{-3.62-1.30}$ | $22.07^{+12.06+3.31}_{-6.45-2.01}$ |
| $\Delta \Gamma(B_c \to J/\Psi (\ell \nu))$ (10$^{-15}$ GeV) | $6.71^{+0.92-0.59}_{-0.92-0.59}$ | $10.52^{+2.29+2.41}_{-1.88-0.91}$ | $16.49^{+3.83+3.83}_{-4.24-1.47}$ | $26.24^{+6.98+2.35}_{-4.37-1.88}$ | $43.04^{+13.61-3.88}_{-13.61-3.88}$ |

| bins of $q^2$ (GeV$^2$) | $m^2_c \leq q^2 \leq 4$ | $4 \leq q^2 \leq 5$ |
|--------------------------|----------------------|----------------------|
| $\Delta \Gamma(B_c \to \eta_c \ell \nu)$ (10$^{-15}$ GeV) | $2.460^{+0.9245+0.655}_{-0.538-0.241}$ | $17.62^{+8.42+4.70}_{-4.56-1.73}$ |
| $\Delta \Gamma(B_c \to J/\Psi \ell \nu)$ (10$^{-15}$ GeV) | $0.821^{+0.375+0.194}_{-0.213-0.073}$ | $6.922^{+4.017+1.68}_{-2.107-0.625}$ |

### IV. CONCLUSIONS

The NLO semileptonic differential decay rates of $B_c$ meson to charmonia are analyzed in detail with various choices of parameters. The uncertainties of partial decay widths in different bins of momentum transfer $q^2$ are discussed. For $B_c \to J/\Psi \ell \nu$ process, the partial decay widths for transverse and longitudinal polarizations are investigated separately. The distribution in the maximum recoil is found testable in the LHCb experiment, and in turn the NRQCD factorization will be also tested. To be noted that in the minimum recoil region, the NRQCD factorization is spoiled by the infrared divergences and hence is not applicable to the analysis of those processes.

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**Appendix A: The NLO $B_c$ to Charmonia transition form factors**

In this appendix, the QCD NLO $B_c$ to charmonium transition form factors are given at the first order in power of $m_c/m_b$. For compactness of the expressions, we define $z = m_c/m_b$, $s = m^2_c/m^2_z - q^2$, and $\gamma = m^2_c/m^2_z - q^2$. Besides, the form factors at maximum recoil point, i.e. $q^2 = 0$, are also presented, which are in agreement with what given in references \cite{13, 21}.
\[ f^{NLO}(q^2) = f^{LO}(q^2) \]

\[ \frac{f^{NLO}(0)}{f^{LO}(0)} = 1 + \frac{\alpha_s}{4\pi} \left\{ \frac{1}{3} (11C_A - 2n_f) \log \left( \frac{\mu^2}{m_\pi m_e^2} \right) - \frac{10n_f}{9} - \frac{4 \log(2)}{3} \right\}, \]

\[ \frac{f^{NLO}(0)}{f^{LO}(0)} = 1 + \frac{\alpha_s}{4\pi} \left\{ \frac{1}{3} (11C_A - 2n_f) \log \left( \frac{\mu^2}{m_\pi m_e^2} \right) - \frac{10n_f}{9} - \frac{4 \log(2)}{3} \right\}, \]

\[ \frac{V^{NLO}(q^2)}{V^{LO}(q^2)} = 1 + \frac{\alpha_s}{4\pi} \left\{ \frac{1}{3} (11C_A - 2n_f) \log \left( \frac{\mu^2}{m_\pi m_e^2} \right) - \frac{10n_f}{9} \right\} - \frac{C_A}{36s - 18} \left\{ 9s(2s - 1) \log^2(s) + 18(2s \log(2)(2s - 1) + 1) \log(s) \right\}. \]
\[ A_{NLO}^{V(q^2)}(0) = A_{LO}^{V(q^2)}(0) \]

\[ \frac{V_{NLO}(0)}{V_{LO}(0)} = 1 + \frac{\alpha_s}{4\pi} \left( \frac{1}{3} (11C_A - 2n_f) \log \left( \frac{\mu^2}{2\gamma m} \right) - \frac{10n_f}{9} \right) + C_F \left( \log^2(z) + 10 \log(2) \log(z) - 5 \log(z) + 9 \log^2(2) \right) + 7 \log(2) + \frac{\pi^2}{3} - 15 \right) \]

\[ + C_A \left( - \frac{1}{2} \log^2(z) - 2 \log(2) \log(z) - \frac{3}{2} \log(z) - 3 \log^2(2) \right) - \frac{3 \log(2)}{2} - \frac{\pi^2}{3} + \frac{67}{9} \right) \}

\[ \frac{A_{NLO}^{N(q^2)}}{A_{LO}^{N(q^2)}} = \frac{A_{NLO}^{V(q^2)}}{A_{LO}^{V(q^2)}} \]

\[ \frac{\alpha_s}{4\pi} \left( \frac{1}{3} (11C_A - 2n_f) \log \left( \frac{\mu^2}{2\gamma m} \right) - \frac{10n_f}{9} \right) + C_F \left( \log^2(z) + 10 \log(2) \log(z) - 5 \log(z) + 9 \log^2(2) \right) + 7 \log(2) + \frac{\pi^2}{3} - 15 \right) \]

\[ + C_A \left( - \frac{1}{2} \log^2(z) - 2 \log(2) \log(z) - \frac{3}{2} \log(z) - 3 \log^2(2) \right) - \frac{3 \log(2)}{2} - \frac{\pi^2}{3} + \frac{67}{9} \right) \}

\[ \frac{A_{NLO}^{NLO}(0)}{A_{LO}^{NLO}(0)} = 1 + \frac{\alpha_s}{4\pi} \left( \frac{1}{3} (11C_A - 2n_f) \log \left( \frac{\mu^2}{2\gamma m} \right) - \frac{10n_f}{9} \right) + C_F \left( \frac{1}{2} \log^2(z) - \frac{119}{8} \right) + 7 \log(2) \log(z) + \frac{21}{4} \log(z) + 7 \log^2(2) + \frac{15 \log(2)}{4} \right) \]
\[ + C_A \left( - \frac{3}{8} \log^2(z) - \log(2) \log(z) - \frac{9}{8} \log(z) - \frac{7\pi^2}{24} + \frac{67}{9} ight. \] 
\[ - \left. \frac{9 \log^2(2)}{4} + \frac{3 \log(2)}{8} \right) \} \]  

(A9)