Numerical study of superradiant instability for charged stringy black hole-mirror system

Ran Li and Junkun Zhao
Department of Physics, Henan Normal University, Xinxiang 453007, China

We numerically study the superradiant instability of charged massless scalar field in the background of charged stringy black hole with mirror-like boundary condition. We compare the numerical result with the previous analytical result and show the dependencies of this instability upon various parameters of black hole charge $Q$, scalar field charge $q$, and mirror radius $r_m$. Especially, we have observed that imaginary part of BQN frequencies grows with the scalar field charge $q$ rapidly.

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Black hole is one of the most fascinating predictions of general relativity. The classical black hole described by the solution of Einstein field equations is a spacetime region from which gravity prevents anything, including light, from escaping. However, energy can be extracted from rotating black holes by impinging bosonic wave with certain frequency condition. This is the well-known classical effect, superradiance [1–4].

Superradiant effect may lead to instability of system combined by rotating black hole and bosonic field perturbation. The necessary condition of superradiant instability includes two aspects. One is that the black hole should be rotating. Under this condition, bosonic wave impinging on such black hole will undergo classical superradiance process. The second is the existence of a potential well outside the horizon that can trap the metastable bound states. In some cases, the mass of bosonic field or AdS boundary of black hole spacetime may provide such potential well. The superradiant instabilities for these cases have been extensively studied in literatures [5–32].

For a charged scalar field in the background of a charged black hole, if the frequency of scalar field satisfies the superradiant condition, the wave will also undergo superradiant process [33]. However, it is proved by Hod in [34, 35] that, Reissner-Nordström (RN) black holes are stable under the perturbations of massive charged scalar fields.

Soon after, Degollado et al. [36, 37] studied the system composed by RN black hole, reflecting mirror, and charged scalar field. This is the analogous black hole bomb firstly suggested by Press and Teukolsky [38]. The mechanism of black hole bomb seems very simple. If one places a reflecting mirror outside of the black hole, the wave will be bounced back and forth between the event horizon and the mirror amplifying itself each time due to superradiant effect. Meanwhile, the energy of this wave can become sufficiently big in this black hole mirror system until the mirror is destroyed. In [36, 37], they found that the instability in the charged case has a shorter time scale than in the rotating case. This motivates us to investigate whether the other charged black holes have similar properties as RN black hole.

In [39], we shown that the charged stringy black hole is stable against the massive charged scalar perturbation. In [40], we have studied the superradiant instability of the scalar field in the background of charged stringy black hole due to a mirror-like boundary condition. The analytical expression of the unstable superradiant modes is derived by using the asymptotic matching method [5]. In this paper, we will provide a numerical study of the superradiant instability for charged stringy black hole-mirror system, and compare the numerical results with the analytical results.

This black hole is a the static spherical symmetric charged black holes in low energy effective theory of heterotic string theory in four dimensions, which is firstly found by Gibbons and Maeda in [41] and independently found by Garfinkle, Horowitz, and Strominger in [42] a few years later. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

and the electric field and the dilaton field

$$A_t = -\frac{Q}{r},$$

$$e^{2\Phi} = 1 - \frac{Q^2}{Mr}.$$  (2)

The parameters $M$ and $Q$ are the mass and electric charge of the black hole respectively. The event horizon of black hole is located at $r = 2M$.

In this paper, for simplicity, we only consider the charged massless scalar field perturbation in the background of charged stringy black hole. The dynamics of the scalar field is then governed by the Klein-Gordon equation

$$(\nabla^2 - iqA_\nu)(\nabla^\nu - iqA^\nu)\Psi = 0,$$  (3)

where $q$ denotes the charge of the scalar field. By taking the ansatz of the scalar field $\Psi = e^{-i\omega t}R(r)Y_{lm}(\theta, \phi)$,
where $\omega$ is the conserved energy of the mode, $l$ is the spherical harmonic index, and $m$ is the azimuthal harmonic index with $-l \leq m \leq l$, one can deduce the radial wave equation in the form of

$$\Delta \frac{d}{dr}\left(\Delta \frac{dR}{dr}\right) + UR = 0,$$  \hspace{1cm} (4)

where we have introduced a new function $\Delta = (r - r_+)(r - r_-)$ with $r_+ = 2M$ and $r_- = Q^2/M$, and the potential function is given by

$$U = \left(r - \frac{Q^2}{M}\right)^2 (\omega r - qQ)^2 - \Delta(l + 1).$$ \hspace{1cm} (5)

The superradiant condition of the charged scalar field is given by $\Phi_H$,

$$\omega < q\Phi_H,$$ \hspace{1cm} (6)

with $\Phi_H = \frac{Q}{M}$ being the electric potential at the horizon. In this paper, we will impose the mirror’s boundary condition that the scalar field vanishes at the mirror’s location $r_m$, i.e.

$$\Psi(r = r_m) = 0$$ \hspace{1cm} (7)

The complex frequencies satisfying the purely ingoing boundary condition at the black hole horizon and the mirror-like boundary condition are called boxed quasi-normal (BQN) frequencies $\omega_{\text{BQN}}$. In the following, we will present an numerical study of BQN frequencies and compare numerical results with the analytical results.

With the assume that the Compton wavelength of scalar particle is much larger than the typical size of black hole, i.e. $1/\omega \gg M$, we can analytically calculate the BQN frequencies of the system by employing matched asymptotic expansion method. The analytical expressions for the BQN frequencies are obtained in [40], which are given by

$$\omega_{\text{BQN}} = \frac{j_{l+1/2,n}}{r_m} + i\delta,$$ \hspace{1cm} (8)

where the imaginary part $\delta$ is given by

$$\delta = -\gamma \left(\frac{j_{l+1/2,n}}{r_m} - q\Phi_H\right) \frac{(-1)^l J_{l-1/2}(j_{l+1/2,n})}{J'_{l+1/2}(j_{l+1/2,n})},$$ \hspace{1cm} (9)

with

$$\gamma = \frac{2}{(2l + 1)} \frac{l!}{(2l - 1)!} \frac{q^2}{r_m^2} \frac{(r_+(r_+ - r_-)^{2l+1}}{r_m(2l)!} \frac{(\Delta)^{2l+1}}{r_m}. \hspace{1cm} (10)

From these expression, it is easy to see that, in the superradiance regime, $\text{Re}[\omega_{\text{BQN}}] - q\Phi_H < 0$, the imaginary part of the complex BQN frequency $\delta > 0$. This indicates that the BQN frequencies in the superradiant regime is unstable for the charged scalar field with the mirror-like boundary condition in the background of the charged stringy black hole.

The numerical methods employed in this problem are based on the shooting method and numerical minimization, which is also called the direct integration (DI) method [36, 44–46]. The DI method is specially suited to find the unstable modes, since these modes have the positive imaginary parts and therefore decay exponentially at spatial infinity. We can use the DI method to solve the BQN frequencies directly from Eq.(4). First, near the horizon $r = 2M$, we impose the ingoing boundary condition

$$R(r) \sim e^{-i(\omega - q\Phi_H)r^*},$$ \hspace{1cm} (11)

where $r^*$ is the tortoise coordinate defined by $\frac{dr^*}{dr} = \frac{r_+ - r - r^*_2}{r_+ - r}$, and expand the radial function $R(r)$ as a generalized power series in $(r - r_+)$. Then, we can integrate the radial equation (4) with the ingoing boundary conditions from $r = r_+(1 + \epsilon)$ and stop the integration at the radius of the mirror. In this procedure, we have taken the small $\epsilon$ as $10^{-6}$. The procedure can be repeated by varying the frequency $\omega$ until the mirror boundary condition $R(r_m) = 0$ is reached with the desired precision.

We can use a numerical root finder to locate the zeros of the boundary condition in the complex $-\omega$ plan. The obtained frequency is just the BQN frequency.

Since our interest is to study the superradiant instability of this black hole, in which the lower order modes is expected to be stronger, we only focus on the $l = 1$ mode in the following. Firstly, we make a comparison of the numerical and analytical results. In Fig. 1 and Fig. 2, we shown the imaginary and the real parts of BQN frequencies for different values of the mirror’s radius $r_m$ and the charge of scalar field $q$. We observe that in these figures the present imaginary part of numerical results are in good agreement with the analytical ones, while the real part is not. Especially, from Fig. 2, one can see that the numerical results match the analytical very well in the lower frequencies. The reason is that the matching technique employed in the analytical calculation is expected to yield a better approximation in the lower frequency region.

In Fig. 3, we fix the mass and the charge of black hole, i.e. $M = 1$, $Q = 1$. We display the imaginary part of BQN frequencies as a function of mirror radius $r_m$ for different values of scalar charge $q$. We observe that when the charge of scalar field decreases, the magnitude of the imaginary part of BQN frequencies decreases correspondingly. That is to say the instability is hard to generate for the small charge of scalar field. We also observe that, for the given values of $M$, $Q$ and $q$, there exists a critical radius of the mirror. When the radius is smaller than this critical values, there is no instability. Clearly, when the charge of scalar field increases, this critical radius decreases correspondingly. This point has
It should noted that this analytical expression is only valid for the case $qQ \ll 1$. The numerical results indicate that when the scalar charge $q$ or the black hole $Q$ increases, the critical radius decreases, which can also be partly observed in Fig.4. However, the analytical expression Eq.(12) for the critical radius can only partially explain the numerical result because Eq.(12) is only valid in the regime $qQ \ll 1$. So, there requires an analytical calculation of BQN frequencies for other parameter regime. It seems to be difficult to perform such a calculation. In [10], the analytical calculation of BQN frequencies for the Reissner-Nordström black hole in the regime $qQ \gg 1$ is obtained. It will be interesting to perform an analytical treatment of the black hole-mirror system studied in this paper in the $qQ \gg 1$ regime.

FIG. 1: Real and imaginary parts of BQN frequencies for $M = 1$, $Q = 1$, $q = 1$ and for different $r_m$. The Bold dots and the square dots represent the numerical and the analytical results respectively.

FIG. 2: Real and imaginary parts of BQN frequencies for $M = 1$, $Q = 1$, $r_m = 90$ and for different $q$. The Bold dots and the square dots represent the numerical and the analytical results respectively.

FIG. 3: Imaginary parts of BQN frequencies for $M = 1$, $Q = 1$ and for different $q = 0.2, 0.4, 0.6, 0.8, 1.0, 1.2$ from bottom up respectively.
In Fig. 4, we have drawn the imaginary parts of BQN frequencies as a function of the mirror radius $r_m$ for various values of the black hole charge $Q$ and the scalar charge $q$. We took the black hole mass $M = 1$. The scalar charge $q$ is equal to $0.2, 0.5, 0.8$ for the first, second, and third figure. For each figure, the black hole charge $Q$ is equal to $0.8, 0.9, 1.0, 1.1, 1.2$ from right to left.

FIG. 4: Imaginary parts of BQN frequencies drawn as a function of the mirror radius $r_m$ for various values of the black hole charge $Q$ and the scalar charge $q$. We took the black hole mass $M = 1$. The scalar charge $q$ is equal to $0.2, 0.5, 0.8$ for the first, second, and third figure. For each figure, the black hole charge $Q$ is equal to $0.8, 0.9, 1.0, 1.1, 1.2$ from right to left.

From these figures, we firstly observe that, for the small scalar charge $q$, the superradiant instability is very hard to generate. While for the large scalar charge $q$, it is clear the instability is easy to generate. This conclusion has been also obtained from Fig. 3. From the second and third figures, we can also see that when the black hole charge $Q$ increases the critical mirror radius which marks the boundary between the stable and unstable black hole-mirror-scalar field configurations decreases. This verifies the approximate expression Eq.(12) for the critical radius.

At last, we make a comparison of the numerical results for charged stringy black holes with the numerical results for RN black holes in [38]. In Table I, we show the numerical results of BQN frequencies for the RN black hole and the charged stringy black hole. The numerical results of BQN frequencies for the RN black hole is adopted from the Table I in Ref. [38]. It is clearly that the RN black hole-mirror system is more unstable than
that the imaginary part of BQN frequencies grows with the mirror radius \( r \). This shows the dependencies of this instability upon the various conditions in the background of charged stringy black hole. By comparing the numerical results with the analytical approximation, we conclude the analytical calculations are only efficient in the low frequency region. We also show the dependencies of this instability upon the various parameters of black hole charge \( Q \), scalar field charge \( q \), and the mirror radius \( r_m \). Especially, we have observed that the imaginary part of BQN frequencies grows with the scalar charge rapidly. We have also compared the numerical results for charged stringy black holes with the numerical results for charged Reissner-Nordstrom black holes in [30], and found that the Reissner-Nordstrom black hole-mirror system is more unstable. At last, we should point out that the analytical treatment of BQN frequencies for the charged stringy black hole-mirror system in the regime of \( qQ \gg 1 \) is required in the future.

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| \( q \) | \( \omega_{RN} \) | \( \omega_{CS} \) |
|---|---|---|
| 1.2 | \( 0.0605 + 7.1405 \times 10^{-4}i \) | \( 0.0606208 + 3.89657 \times 10^{-4}i \) |
| 1.6 | \( 0.0657 + 3.8595 \times 10^{-5}i \) | \( 0.0658496 + 2.21487 \times 10^{-5}i \) |

TABLE I: The numerical results of RN black hole and charged stringy black hole for \( r_m = 100 \) and \( Q = 0.8 \).