Flow of Jeffrey Fluid over a Horizontal Circular Cylinder with Suspended Nanoparticles and Viscous Dissipation Effect: Buongiorno Model

Syazwani Mohd Zokri1,*, Nur Syamilah Arifin2, Abdul Rahman Mohd Kasim3, Mohd Zuki Salleh3

1 Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (UiTM) Cawangan Terengganu, Kampus Kuala Terengganu, 21080 Terengganu, Malaysia
2 Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (UiTM) Cawangan Johor, Kampus Pasir Gudang, 81750 Masai, Johor, Malaysia
3 Centre for Mathematical Sciences, College for Computing and Applied Sciences, Universiti Malaysia Pahang, 26300 Kuantan, Pahang, Malaysia

ARTICLE INFO

ABSTRACT

Mathematical model of Jeffrey fluid describes the property of viscoelastic that clarifies the two components of relaxation and retardation times. Nevertheless, the poor thermal performance of Jeffrey fluid has been a key issue facing the public. This issue can be accomplished by the use of nanofluid that has superior thermal performance than the conventional fluids. A better cooling rate in industry is in fact not appropriate to attain by the thermal conductivity of the conventional fluids. On that account, the present study aims to delve into the impact of viscous dissipation and suspended nanoparticles on mixed convection flow of Jeffrey fluid from a horizontal circular cylinder. A concise enlightenment on the separation of boundary layer flow is included and discussed starting from the lower stagnation point flow up to the separation point only. The non-dimensional and non-similarity transformation variables are implemented to transform the dimensional nonlinear partial differential equations (PDEs) into two nonlinear PDEs, and then tackled numerically through the Keller-box method. Representation of tabular and graphical results are executed for velocity and temperature profiles as well as the reduced skin friction coefficient, Nusselt number and Sherwood number to investigate the physical insight of emerging parameters. It was found that the incremented ratio of relaxation to retardation, Deborah number and Eckert number have delayed the boundary layer separation up to 120°.

Keywords:
Free convection; Jeffrey nanofluid; horizontal circular cylinder; viscous dissipation

1. Introduction

The concept of nanofluids refers to an innovative idea of engineered heat transfer fluids by dispersing the nanometer-sized particles in the conventional fluids [1]. These particles, which are called as nanoparticles, are usually being composed of oxides (Al2O3, CuO, TiO2, SiO2), metals...
(Al, Cu), nitrides (AlN, SiN), carbides (SiC), or non-metals (graphite and carbon nanotubes) with diameter between 1 and 100 nm. Some examples of conventional fluids are organic liquids such as tri-ethylene-glycols, ethylene and refrigerants, water, polymeric solution, bio-fluids, oil and lubricants, and other liquids. To accomplish the industrial cooling rate requirement, the conventional fluids are found to have limited heat transfer competency attributing to their low thermal conductivity compared to metals. In that capacity, the thermal conductivity of conventional fluids can be conceivably enhanced by suspending it with nanoparticles; nonetheless, subjected to the particles’ shape, size, conductivity, amount of dispersed particles and the conventional fluid itself [2]. A number of works concerning the heat transfer in nanofluids may be found in publication by Zokri et al., [3], Mohamed et al., [4], Zulkifli et al., [5], Azam et al., [6], and Waini et al., [7].

Recent studies have shown that the non-linear rheological fluids had made sizeable progression. This improvement can be tracked down through the complex nature of fluids used in various industrial applications, that a single constitutive equation is inadequate to describe such fluids. Differing to Newtonian fluid, the relationship between the stress and strain rate of non-Newtonian fluids is non-linear because of the dependency of fluid viscosity on time or deformation. The complex nature of fluids has stimulated the development of many non-Newtonian fluid models that can be mathematically recognized by its constitutive equations. Such constitutive equations are more complicated than the Navier-Stokes equations as each of the established models is fundamentally characterized by dissimilar characteristics. Most frequent highlighted non-Newtonian models in the literature comprehend the micropolar fluid model [8], viscoelastic fluid model [9], Jeffrey fluid model [10-12], Casson fluid model [13, 14], Williamson fluid model [15] and second grade fluid model [16]. Amongst all, Jeffrey fluid model has been ascertained as quite successful due to its distinct ability in explaining the dual viscoelastic properties of relaxation and retardation times, which is very much relevant with the polymer industries [17]. The important features of this fluid model include high shear viscosity, shear thinning and yield stress. At very high wall shear stress, this model degenerates to the Newtonian fluids provided that the wall shear stress is much greater than the yield stress.

Free convection flow of an incompressible fluid from a horizontal circular cylinder implicates an imperative problem in many industrial applications, for example in handling hot wire and steam pipe. Merkin [18] attempted the initial investigation on free convection boundary layer flow from a horizontal circular cylinder in a viscous fluid. He presented a complete solution of this problem from the lower stagnation point up to the upper stagnation point of circular cylinder using the Blasius and Gortler series expansion methods coupled with an integral method and finite difference scheme. Soon after, he extended the study on a horizontal cylinder of elliptic cross section when the major axis is horizontal and vertical [19]. Both the constant wall temperature and constant heat flux are incorporated. The free convection problem about a heated horizontal cylinder in a porous medium was addressed by Ingham and Pop [20], while Merkin and Pop [21] utilized a similar method as Merkin [18] to investigate the constant heat flux condition. Following the works of Merkin [18] and Merkin and Pop [21], the non-Newtonian micropolar fluid was included and thoroughly investigated by Nazar et al., [22] under the constant wall temperature.

Ever since, countless investigations have been conducted from a horizontal circular cylinder in both Newtonian and non-Newtonian fluid. This takes in the published study by Molla et al., [23] who utilized the free convection flow of a viscous fluid past an isothermal horizontal circular cylinder. They supposed that the fluid viscosity is proportional to an inverse linear function of the temperature. They applied the Keller-box method to solve the transformed boundary layer equations starting from the lower stagnation point of the cylinder and then proceeded round the cylinder up to the rear stagnation point. In the subsequent year, Molla et al., [24] continued the investigation by incorporating the internal heat generation effect. The transformed equations were solved
numerically using two methods, namely the Keller box method and series solution technique. Again, they observed that the boundary layer proceeds round the cylinder until the upper stagnation point without separating. The surface condition of Newtonian heating was studied by Salleh and Nazar [25] on free convection boundary layer flow in a viscous fluid. Here, the surface heat transfer is assumed to be proportional to the local surface temperature. They concluded that for increasing Prandtl number values, the velocity and temperature profiles were both reduced at the lower stagnation region. The combined effects of MHD, joule heating and heat generation were then presented by Azim and Chowdhury [26] on free convection flow of a viscous fluid with convective boundary conditions. With the help of Keller-box method, they noted that the skin friction along the surface of the cylinder decreases with increasing magnetic parameter and conjugate conduction parameter. Prasad et al., [27] explored the flow of Jeffrey fluid past a horizontal circular cylinder with suction/injection effect. The numerical computation conducted by the Keller-box method has shown that the Deborah number has a reducing impact on the velocity and Nusselt number, but rising impact on the temperature and skin friction coefficient. Later, Makanda et al., [28] deliberated the radiation effect on MHD free convection flow from a cylinder with partial slip in a non-Darcy porous medium of a Casson fluid. The cylinder surface was heated under constant surface temperature, and the partial slip factor was imposed on the surface for both velocity and temperature. The resulting system of equations was solved using the bi-variate quasilinearization method. Mohamed et al., [29] solved the model of nanofluid due to a horizontal circular cylinder with viscous dissipation effect using the Keller-box method. Authors disclosed that the increase of Brownian motion parameter, thermophoresis parameter, Lewis number and Eckert number has increased the skin friction coefficient and Sherwood number, while the Nusselt number decreases.

Rao et al., [30] also applied the Keller-box method to scrutinize the flow of Williamson fluid with Newtonian heating. They reported that the boundary layer separation for skin friction coefficient \((x = 1.5)\) is larger than the Nusselt number \((x = 1.2)\). The convectively heated cylinder in MHD Tangent Hyperbolic Fluid was addressed by Gaffar et al., [31]. It was identified that, for all investigated parameters, the boundary layer flow does not experience singularity. Very recently, the flow of Jeffrey nanofluid at lower stagnation point from a horizontal circular cylinder is addressed by Zokri et al., [32] under the influences of suction/injection, mixed convection and convective boundary conditions.

All of the above cited works were restricted to diverse non-Newtonian fluids flow with two of them concentrated on the Jeffrey fluid. However, none of them was identified to deliberate on free convection flow of Jeffrey nanofluid. Motivated by the published works of Mohamed [29] and Dalir [33], the current investigation aims to solve the free convection flow of Jeffrey nanofluid past a horizontal circular cylinder with viscous dissipation effect.

2. Mathematical Formulation

According to Hayat and Ali [34] and Qasim [35], the constitutive equation for the model of Jeffrey fluid is

\[
\mathbf{\tau} = -p \mathbf{I} + \mathbf{S}, \quad \mathbf{S} = \frac{\mu}{1 + \lambda_1} \left[ \mathbf{R}_1 + \lambda_1 \left( \frac{\partial \mathbf{R}_1}{\partial t} + \nabla \mathbf{V} \right) \right],
\]

where \(\mathbf{\tau}, \mathbf{I}, \mathbf{S}, p\) and \(\mu\) are the Cauchy stress tensor, identity tensor, extra stress tensor, pressure and dynamic viscosity. Furthermore, the material parameters of the Jeffrey fluid are symbolized as \(\lambda_1\) while \(\mathbf{R}_1 = (\nabla \mathbf{V}) + (\nabla \mathbf{V})'\) is the Rivlin-Ericksen tensor. This model is developed with the
The purpose of extending the Maxwell model. The retardation time parameter which appears in Maxwell model is specifically corrected with the time derivative of the strain rate, for which it can measure the required time for the material to react to the deformation.

A steady, two-dimensional and laminar flow of the Jeffrey nanofluid model with uniform ambient temperature \( T_\infty \) and concentration \( C_\infty \) is investigated due to a horizontal circular cylinder. The cylinder is heated at the same constant temperature \( T_w \) and concentration \( C_w \), as exhibited in the flow diagram of Figure 1.

![Schematic diagram of free convection flow in Jeffrey fluid passing over a horizontal circular cylinder](image)

**Fig. 1.** Schematic diagram of free convection flow in Jeffrey fluid passing over a horizontal circular cylinder

The respective \( \bar{x} \) – and \( \bar{y} \) – coordinates are implicated throughout the surface of the cylinder from the lowest point, \( \bar{x} = 0 \) and vertical to it, with \( a \) and \( g \) being the radius of the circular cylinder and gravitational acceleration, respectively. The amalgamated influences of the viscous dissipation and mixed convection are also scrutinized. The law of conservation (after applying the boundary layer approximations) is proposed as the following:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}
\]

\[
\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{\nu}{1+\lambda} \left[ \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \lambda \left( \bar{u} \frac{\partial^3 \bar{u}}{\partial \bar{x} \partial \bar{y}^2} + \bar{v} \frac{\partial^3 \bar{u}}{\partial \bar{x} \partial \bar{y}^3} - \frac{\partial \bar{u}}{\partial \bar{x}} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + \frac{\partial \bar{u}}{\partial \bar{y}} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}} \right) \right] + \bar{g} \beta_\tau (T - T_\infty) \sin \frac{\bar{x}}{a} + g \beta_c (C - C_\infty) \sin \frac{\bar{x}}{a}, \tag{2}
\]

\[
\bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial \bar{y}^2} + \frac{\nu}{C_p (1+\lambda)} \left[ \frac{\partial \bar{u}}{\partial \bar{y}} \right]^2 + \lambda \left( \bar{u} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{v} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}} \right) \frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}} \right] + \left( D_B \frac{\partial C}{\partial \bar{y}} + \frac{D_T}{T_\infty} \left( \frac{\partial T}{\partial \bar{y}} \right)^2 \right), \tag{3}
\]

\[
\bar{u} \frac{\partial \bar{C}}{\partial x} + \bar{v} \frac{\partial \bar{C}}{\partial y} = D_B \frac{\partial^2 \bar{C}}{\partial \bar{y}^2} + \frac{D_T}{T_\infty} \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \tag{4}
\]
In the above equations, the ratio of heat capacity of the nanoparticle to the fluid and the velocity outside the boundary layer are denoted as \( \eta = \tau \rho_c / \rho_{\infty} \), and \( \eta = U_{\infty} \sin(\xi/a) \), respectively, whereas the velocity components along the \( x \) - and \( y \) - coordinates are symbolized as \( \bar{u} \) and \( \bar{v} \), respectively. Besides, the respective ratio of relaxation to retardation times, relaxation time, thermal expansion, concentration expansion, thermal diffusivity, kinematic viscosity, fluid density, local concentration, specific heat capacity at a constant pressure, local temperature, Brownian diffusion coefficient and thermophoretic diffusion coefficient are indicated as \( \lambda, \lambda_1, \beta_\gamma, \beta_c, \alpha, \nu, \rho, C, C_p, T, D_B \) and \( D_T \). Eqs. (1) to (4) are subjected to the following boundary conditions

\[
\bar{u}(x,0) = 0, \quad \bar{v}(x,0) = 0, \quad T(\bar{x},0) = T_w, \quad C(\bar{x},0) = C_w \quad \text{at} \quad \bar{y} = 0
\]

\[
\bar{u}(x,\infty) \to 0, \quad \bar{v}(x,\infty) \to 0, \quad T(\bar{x},\infty) \to T_x, \quad C(\bar{x},\infty) \to C_x \quad \text{as} \quad \bar{y} \to \infty
\]  

The above mathematical model can be furthered non-dimensionalized using the subsequent variables

\[
x = \frac{x}{a}, \quad y = \frac{Gr_\gamma^{1/4} \bar{y}}{a}, \quad u = \frac{a}{v} Gr_\gamma^{-1/2} \bar{u}, \quad v = \frac{a}{v} Gr_\gamma^{-1/4} \bar{v}, \quad \theta(\eta) = \frac{T-T_x}{T_w-T_x}, \quad \phi(\eta) = \frac{C-C_x}{C_w-C_x}
\]  

Using Eq. (6), Eqs. (1) to (5) yield

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{\eta}{\gamma} + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{1}{1+\lambda} \left[ \frac{\partial^2 u}{\partial x^2} + \lambda_2 \left( \frac{\partial^3 u}{\partial x^3} + \lambda_3 \frac{\partial^2 u}{\partial x \partial y^2} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial x \partial y} \right) \right] + (\theta + N\phi) \sin x
\]

\[
\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \left[ \frac{\partial^2 \theta}{\partial x^2} + \lambda_2 \left( \frac{\partial^2 \theta}{\partial x \partial y} + \lambda_3 \frac{\partial^2 \theta}{\partial x \partial y^2} + \frac{\partial \theta}{\partial y} \frac{\partial^2 \theta}{\partial y^2} \right) \right]
\]

\[
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = \frac{1}{Le \Pr} \left[ \frac{\partial^2 \phi}{\partial x^2} + \lambda_2 \left( \frac{\partial^2 \phi}{\partial x \partial y} + \lambda_3 \frac{\partial^2 \phi}{\partial x \partial y^2} \right) \right]
\]

\[
u(x,0) = 0, \quad v(x,0) = 0, \quad \theta(x,0) = 1, \quad \phi(x,0) = 1 \quad \text{at} \quad y = 0
\]

\[
u(x,\infty) \to 0, \quad v(x,\infty) \to 0, \quad \theta(x,\infty) \to 0, \quad \phi(x,\infty) \to 0 \quad \text{as} \quad y \to \infty
\]

In consequence of the above equations, we let \( \Pr, \lambda_2, \gamma, Gr_\gamma, Re, N, Nb, Le \) and \( Nt \) be the Prandtl number, Deborah number, Eckert number, Grashof number, Reynolds number, concentration buoyancy parameter, Brownian motion parameter, Lewis number and thermophoresis diffusion parameter, which can be expressed as below:

\[
\Pr = \frac{\nu}{\alpha}, \quad \lambda_2 = \frac{\lambda_1 Gr_\gamma^{1/2} \nu}{a^2}, \quad Ec = \frac{\nu^2 Gr_\gamma}{\alpha^2 C_p (T_w-T_x)}, \quad Gr_\gamma = \frac{g \beta_c (T_w-T_x) a^3}{\nu^2}, \quad Re = \frac{U_{\infty} a}{\nu}
\]

\[
N = \frac{\beta_c (C_w-C_x)}{\beta_T (T_w-T_x)}, \quad Nb = \frac{\tau D_B (C_w-C_x)}{\nu}, \quad Le = \frac{\alpha}{D_B}, \quad Nt = \frac{\tau D_T (T_w-T_x)}{\nu T_w}
\]
Next, we look for these variables to solve Eqs. (7) to (11): \( \psi = xf(x, y) \), \( \theta = \theta(x, y) \) and \( \phi = \phi(x, y) \), in which the stream function, \( \psi \) is represented by \( u = \partial \psi / \partial y \) and \( v = -\partial \psi / \partial x \). Now, the satisfaction of Eq. (7) is automatically achieved and the resulting PDEs together with the related boundary conditions are

\[
\frac{1}{1 + \lambda} f'' - \left( f' \right)^2 + \sin \frac{x}{\lambda} \left[ \gamma (\theta + N \phi) + \cos x \right] + \frac{\lambda_2}{1 + \lambda} \left[ (f'')^2 - f'^{(iv)} \right] = \frac{x}{f'' - \left( f' \right)^2} - \frac{\lambda_2}{1 + \lambda} \left[ (f'')^2 - f'^{(iv)} \right] \tag{12}
\]

\[
\frac{1}{\Pr} \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - \frac{\lambda_2}{1 + \lambda} \left[ (f'')^2 - f'^{(iv)} \right] \tag{13}
\]

\[
f(x, 0) = 0, \quad f'(x, 0) = 0, \quad \theta(x, 0) = 1, \quad \phi(x, 0) = 1 \text{ at } y = 0 \tag{14}
\]

Note that primes infer the differentiation with respect to the variable \( y \). Also, we found that Eqs. (12) to (15) can be reduced to the mixed convection Newtonian fluid as reported by Mohamed et al., [36], provided the absence of the Jeffrey fluid \( (\lambda = \lambda_2 = 0) \) and nanofluid \( (Nt = Nb = Le = N = 0) \) parameters. At the vicinity of the lower stagnation point \( (x \approx 0) \), the preceding equations (Eqs. (12) to (15)) give rise to the succeeding ordinary differential equations:

\[
\frac{1}{1 + \lambda} f'' - \left( f' \right)^2 + \frac{1}{\Pr} \frac{\partial \psi}{\partial x} = \frac{\lambda_2}{1 + \lambda} \left[ (f'')^2 - f'^{(iv)} \right] = 0, \tag{16}
\]

\[
\frac{1}{\Pr} \frac{\partial \psi}{\partial x} + f \frac{\partial \phi}{\partial x} + Nt \left( \phi' \right)^2 = 0 \tag{17}
\]

\[
f(0) = 0, \quad f'(0) = 0, \quad \theta(0) = 1, \quad \phi(0) = 1 \tag{18}
\]

\[
f'(\infty) \to 0, \quad f''(\infty) \to 0, \quad \theta(\infty) \to 0, \quad \phi(\infty) \to 0 \tag{19}
\]

The non-appearance of parameter \( Ec \) in Eq. (17) clearly signifies that the profiles of velocity, temperature and concentration are no longer being influenced by \( Ec \) at the stagnation point of the cylinder. Further, the local Nusselt and Sherwood numbers are exemplified as follows

\[
C_f = \frac{S_w}{\rho u \sqrt{u}}, \quad S_w = \mu \left[ \frac{\partial u}{\partial y} + \lambda \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \right) \right], \quad Nu = \frac{a q_w}{-k(T - T_0)}, \quad q_w = -k \frac{\partial T}{\partial y} \tag{20}
\]

\[
Sh = \frac{aj_w}{D_h(C_w - C_s)} \tag{20}
\]

\[
j_w = -D_h \left( \frac{\partial C}{\partial y} \right) \tag{20}
\]
The reduced Nusselt and Sherwood numbers are now given by

\[ C_{p, Gr_x^{1/4}} = \frac{X}{1 + \lambda} f''(x, 0), \quad Nu_{x, Gr_x^{-1/4}} = -\theta'(x, 0) \quad \text{and} \quad Sh_{x, Gr_x^{-1/4}} = -\phi'(x, 0) \]  

(21)

3. Results and Discussion

The non-linear PDEs (Eqs. (12) to (14)) with the respective boundary conditions (Eq. (15)) are treated through the Keller-box method. The numerical solutions start at the lower stagnation point, \( x = 0^\circ \) with initial profiles being given by Eqs. (16) to (18) accompanied by boundary conditions (19) and then preceded round the circular cylinder up to the separation point \( x = 120^\circ \). The step size of \( \Delta x = \Delta y = 0.01 \) and the boundary layer thickness, \( y_\infty = 4 \) to 6 are implemented to obtain the numerical results. The results of this study are comprehensively explored and discussed for diverse values of dimensionless governing equations \( \lambda, \lambda_2 \) and \( Ec_1 \), as illustrated in Figures 1 to 9.

In order to authenticate the engaged numerical method, the comparative benchmark of the \( C_{p, Gr_x^{1/4}} \) and \( Nu_{x, Gr_x^{-1/4}} \) values against position of \( x \) are presented through Tables 1 and 2. The limiting results of the current study are matched with the tabulated values of Merkin [18], Nazar [22], Molla [24], Azim and Chowdhury [26] and Mohamed [29], who applied the Keller-box method in solving the free convection flow of viscous, micropolar and nanofluid. A proper match among the comparative values of both tables has manifestly validated the present results. Furthermore, it can be concluded from the comparative values that the \( C_{p, Gr_x^{1/4}} \) rises to a maximum value before declining to a finite value, while the \( Nu_{x, Gr_x^{-1/4}} \) decelerates with increasing position of \( x \).

The graph for velocity \( f'(y) \), temperature \( \theta(y) \) and concentration \( \phi(y) \) profiles are portrayed in Figures 2 to 4 for different values of \( \lambda \) and \( \lambda_2 \). Initially, a rise in \( \lambda \) is noticed to boost the velocity profile; however, the velocity profile starts to deteriorate as the momentum boundary layer thickness increases. Physically, \( \lambda \) is dependent on the retardation time. An increase in \( \lambda \) signifies weaker retardation time while a decrease in \( \lambda \) indicates stronger retardation time. Such change in retardation time leads to the increment and decrement in the momentum boundary layer thickness. Instead, a reversal graph trend is observed for increasing \( \lambda_2 \) values. It is perceived that \( \lambda_2 \) displays a trivial effect at the cylinder surface, but the effect comes to be highly substantial as the thickness of boundary layer increases up to the freestream. Moreover, with increasing value of \( \lambda \), the decrease in temperature profile is found to be slightly significant than the decrease in concentration profile. This outcome goes in the same way as for rising \( \lambda_2 \) values, where a slight significant increase in temperature rather than the concentration profile is spotted. These profiles also decline continuously towards the freestream following the escalation of the boundary layer thickness. The incremented temperature and concentration profiles can be directly related with the behaviour of Deborah number which liable to the changes in retardation time. An increase in retardation time increases the \( \lambda_2 \), which eventually reduces the resistance of fluid motion within the boundary layer. This has subsequently resulted in high impact of fluid motion, which does not only thicken the momentum boundary layer, but also the thermal and concentration boundary layers.

Salient features of skin friction coefficient \( C_{p, Gr_x^{1/4}} \), Nusselt number \( Nu_{x, Gr_x^{-1/4}} \) and Sherwood number \( Sh_{x, Gr_x^{-1/4}} \) are portrayed in Figures 5 to 10 for various values of \( \lambda, \lambda_2 \) and \( Ec \). These figures have demonstrated that the boundary layer separation had occurred at \( x = 120^\circ \), regardless of the
varied parameter values. Figures 5 to 7 demonstrate that the $C_{\mu}Gr_x^{1/4}$ is a lessening function of $\lambda$ and a rising function of $\lambda_2$, while both the $Nu_xGr_x^{-1/4}$ and $Sh_xGr_x^{-1/4}$ perform reversely. It is observed that the heat and nanoparticle concentration transfer rates reduce sequentially as the tangential coordinate value, $x$ increases. Figures 8 to 10 exhibit that, $Ec$ enunciates a rising impact over the $C_{\mu}Gr_x^{1/4}$ and the $Sh_xGr_x^{-1/4}$, but a lessening impact over the $Nu_xGr_x^{-1/4}$. Here, the reversal behaviour of heat transfer transpires as the $Nu_xGr_x^{-1/4}$ values become negative by virtue of escalating $Ec$ from 0 to 2. Such behaviour transpires as a result of dissipative heat effect, thus can be explained as a reversal of the heat flow. One would also expect that the $Nu_xGr_x^{-1/4}$ always gives positive value when $Ec < 0$ and tends to result in negative value when $Ec > 0$. Besides, the impact of $Ec$ for each profile is not plotted here because the graph generates a unique solution. Mathematically, this can also be connected with discontinuation of $Ec$ in the energy equation (Eq. (17)), which subsequently leads to a unique solution of $C_{\mu}Gr_x^{1/4}$, $Nu_xGr_x^{-1/4}$ and $Sh_xGr_x^{-1/4}$ at $x = 0$.

Table 1
Comparative values of $C_{\mu}Gr_x^{1/4}$ for different values of $x$ when $\lambda = 0$, $\lambda_2 \to 0$ (very small), $N = Ec = Nb = Nt = Le = 0$ and $Pr = 1$

| $x$   | Merkin [18] | Nazar [22] | Molla [24] | Azim and Chowdhury [26] | Mohamed [29] | Present |
|-------|-------------|-------------|------------|--------------------------|---------------|---------|
| 0     | 0.0000      | 0.0000      | 0.0000     | 0.0000                   | 0.0000        | 0.0000  |
| $\pi/6$ | 0.4151     | 0.4148      | 0.4145     | 0.4139                   | 0.4121        | 0.4120  |
| $\pi/3$ | 0.7558     | 0.7542      | 0.7539     | 0.7528                   | 0.7538        | 0.7507  |
| $\pi/2$ | 0.9696     | 0.9695      | 0.9641     | 0.9526                   | 0.9653        | 0.9654  |
| $2\pi/3$ | 0.9756    | 0.9698      | 0.9696     | 0.9678                   | 0.9743        | 0.9728  |
| $5\pi/6$ | 0.7822    | 0.7740      | 0.7739     | 0.7718                   | 0.7813        | 0.7761  |
| $\pi$  | 0.3391      | 0.3265      | 0.3264     | 0.3239                   | 0.3371        | 0.3302  |

Table 2
Comparative values of $Nu_xGr_x^{-1/4}$ for different values of $x$ when $\lambda = 0$, $\lambda_2 \to 0$ (very small), $N = Ec = Nb = Nt = Le = 0$ and $Pr = 1$

| $x$   | Merkin [18] | Nazar [22] | Molla [24] | Azim and Chowdhury [26] | Mohamed [29] | Present |
|-------|-------------|-------------|------------|--------------------------|---------------|---------|
| 0     | 0.4214      | 0.4214      | 0.4214     | 0.4216                   | 0.4214        | 0.4214  |
| $\pi/6$ | 0.4161     | 0.4161      | 0.4161     | 0.4163                   | 0.4163        | 0.4162  |
| $\pi/3$ | 0.4007     | 0.4005      | 0.4005     | 0.4006                   | 0.4008        | 0.4009  |
| $\pi/2$ | 0.3745     | 0.3741      | 0.3740     | 0.3742                   | 0.3744        | 0.3743  |
| $2\pi/3$ | 0.3364    | 0.3355      | 0.3355     | 0.3356                   | 0.3364        | 0.3363  |
| $5\pi/6$ | 0.2825    | 0.2811      | 0.2812     | 0.2811                   | 0.2824        | 0.2814  |
| $\pi$  | 0.1945      | 0.1916      | 0.1917     | 0.1912                   | 0.1939        | 0.1932  |
Fig. 2. Variation of $f'(y)$ for several values of $\lambda$ and $\lambda_2$ when $N = Nb = Nt = Ec = 0.1$, $Le = 10$ and $Pr = 7$

Fig. 3. Variation of $\theta(y)$ for several values of $\lambda$ and $\lambda_2$ when $N = Nb = Nt = Ec = 0.1$, $Le = 10$ and $Pr = 7$

Fig. 4. Variation of $\phi(y)$ for several values of $\lambda$ and $\lambda_2$ when $N = Nb = Nt = Ec = 0.1$, $Le = 10$ and $Pr = 7$

Fig. 5. Variation of $C_p Gr^{1/4}$ for several values of $\lambda$ and $\lambda_2$ when $N = Nb = Nt = Ec = 0.1$, $Le = 10$ and $Pr = 7$

Fig. 6. Variation of $Nu_x Gr^{1/4}$ for several values of $\lambda$ and $\lambda_2$ when $N = Nb = Nt = Ec = 0.1$, $Le = 10$ and $Pr = 7$

Fig. 7. Variation of $Sh_x Gr^{1/4}$ for several values of $\lambda$ and $\lambda_2$ when $N = Nb = Nt = Ec = 0.1$, $Le = 10$ and $Pr = 7$
4. Conclusions

The free convection boundary layer flow problem of Jeffrey nanofluid on a horizontal circular cylinder with viscous dissipation effect was deliberated. The effects of Jeffrey fluid parameter and viscous dissipation on the velocity, temperature and concentration profiles as well as the reduced skin friction coefficient, Nusselt number and Sherwood number have been discussed and explained. On the whole, the concise outcome of this investigation is provided as follows:

I. The similar distribution shows the opposite behaviour for both Jeffrey fluid parameters.
II. An increase in $Ec$ shows no effects on the velocity, temperature and concentration profiles at the lower stagnation point. Augmenting $Ec$ has enlarged the skin friction coefficient and Sherwood number, but reduced the Nusselt number.
III. The increase of $\lambda$, $\lambda_2$ and $Ec$ has delayed the boundary layer separation up to $x=120^\circ$. 

---

**Fig. 8.** Variation of $C^\pm_{Gr_s}^{1/4}$ for several values of $Ec$ when $\lambda = \lambda_2 = 0.5$, $N = Nb = Nt = 0.1$, $Le = 10$ and $Pr = 7$

**Fig. 9.** Variation of $Nu^\pm_{Gr_s}^{1/4}$ for several values of $Ec$ when $\lambda = \lambda_2 = 0.5$, $N = Nb = Nt = 0.1$, $Le = 10$ and $Pr = 7$

**Fig. 10.** Variation of $Sh^\pm_{Gr_s}^{1/4}$ for several values of $Ec$ when $\lambda = \lambda_2 = 0.5$, $N = Nb = Nt = 0.1$, $Le = 10$ and $Pr = 7
Acknowledgement
This project has been supported by Ministry of Higher Education under The Fundamental Research Grant Scheme for Research Acculturation of Early Career Researchers (FRGS-RACER) (Ref:RACER/1/2019/STG06/UMP//1) through RDU192602 and Universiti Malaysia Pahang through RDU182307.

References
[1] Choi, Stephen U.S. and Eastman, Jeffrey A., Enhancing Thermal Conductivity of Fluids with Nanoparticles. 1995, Argonne National Lab., IL (United States).
[2] Khlebtsov, N.G., Trachuk, L.A., and Mel'nikov, A.G. "The effect of the size, shape, and structure of metal nanoparticles on the dependence of their optical properties on the refractive index of a disperse medium." Optics and Spectroscopy 98, no. 1 (2005): 77-83. https://doi.org/10.1134/1.1858043
[3] Zokri, Syazwani Mohd, Arifin, Nur Syamilah, Kasim, Abdul Rahman Mohd, Mohammad, Nurul Farahain, and Salleh, Mohd Zuki. "On dissipative MHD mixed convection boundary layer flow of Jeffrey fluid over an inclined stretching sheet with nanoparticles: Buongiorno model." Thermal Science 23, no. 6B (2018): 3817-3832. https://doi.org/10.2298/TSCI171120178M
[4] Mohamed, Muhammad Khairul Anuar, Ong, Huei Ruey, Salleh, Mohd Zuki, and Widodo, Basuki. "Mixed convection boundary layer flow of engine oil nanofluid on a vertical flat plate with viscous dissipation." ASEAN Journal of Automotive Technology 1, no. 1 (2019): 29-38. https://journal.dhuautomotive.edu.my/autojournal/article/view/7
[5] Zulkifli, Siti Norfatihah, Sarif, Norhafizah Md, and Salleh, Mohd Zuki. "Numerical solution of boundary layer flow over a moving plate in a nanofluid with viscous dissipation: A revised model." Journal of Advanced Research in Fluid Mechanics and Thermal Sciences 56, no. 2 (2019): 287-295.
[6] Azam, M., Shakoor, A., Rasool, H.F., and Khan, M. "Numerical simulation for solar energy aspects on unsteady convective flow of MHD Cross nanofluid: A revised approach." International Journal of Heat and Mass Transfer 131 (2019): 495-505. https://doi.org/10.1016/j.ijheatmasstransfer.2018.11.022
[7] Waini, Iskandar, Ishak, Anuar, and Pop, Ioan. "Transpiration effects on hybrid nanofluid flow and heat transfer over a stretching/shrinking sheet with uniform shear flow." Alexandria Engineering Journal 59, no. 1 (2020): 91-99. https://doi.org/10.1016/j.ajegj.2019.12.010
[8] Anwar, M.I., Shafie, Sharidan, Hayat, Tasawar, Shehzad, S.A., and Salleh, Mohd Zuki. "Numerical study for MHD stagnation-point flow of a micropolar nanofluid towards a stretching sheet." Journal of the Brazilian Society of Mechanical Sciences and Engineering 39, no. 1 (2017): 89-100. https://doi.org/10.1007/s40430-016-0610-y
[9] Mohamed, Muhammad Khairul Anuar, Salleh, Mohd Zuki, Ishak, Anuar, and Pop, Ioan. "Stagnation point flow and heat transfer over a stretching/shrinking sheet in a viscoelastic fluid with convective boundary condition and partial slip velocity." The European Physical Journal Plus 130, no. 8 (2015): 171. https://doi.org/10.1140/epjp/i2015-15171-8
[10] Arifin, Nur Syamilah, Zokri, Syazwani Mohd, Kasim, Abdul Rahman Mohd, Salleh, Mohd Zuki, and Arifin, Noor Amalina Nisa. "Jeffrey fluid embedded with dust particles over a shrinking sheet: A numerical investigation." Journal of Advance Research Fluid Mechanics Thermal Science 74, no. 2 (2020): 196-209. https://doi.org/10.37934/arfmts.74.2.196209
[11] Zokri, Syazwani Mohd, Arifin, Nur Syamilah, Mohamed, Muhammad Khairul Anuar, Kasim, Abdul Rahman Mohd, Mohammad, Nurul Farahain, and Salleh, Mohd Zuki. "Mathematical model of mixed convection boundary layer flow over a horizontal circular cylinder filled in a Jeffrey fluid with viscous dissipation effect." Sains Malaysiana 47, no. 7 (2018): 1607-1615. http://dx.doi.org/10.17576/jsm-2018-4707-32
[12] Zokri, Syazwani Mohd, Arifin, Nur Syamilah, Kasim, Abdul Rahman Mohd, and Salleh, Mohd Zuki. "Suspended nanoparticles on mixed convection flow of a Jeffrey fluid due to a horizontal circular cylinder with viscous dissipation." Thermal Science, no. 00 (2019): 106-106. https://doi.org/10.2298/TSCI181027106M
[13] Kasim, Abdul Rahman Mohd, Arifin, Nur Syamilah, Zokri, Syazwani Mohd, Salleh, Mohd Zuki, Mohammad, Nurul Farahain, Ching, Dennis Ling Chuan, Shafie, Sharidan, and Arifin, Noor Amalina Nisa. "Convective transport of
fluid–solid interaction: A study between non-Newtonian Casson model with dust particles." *Crystals* 10, no. 9 (2020): 814. https://doi.org/10.3390/cryst10090814

[14] Kasim, Abdul Rahman Mohd, Ariffin, Nur Syamilah, Ariffin, Noor Amalina Nisa, Salleh, Mohd Zuki, and Anwar, Muhammad Imran. "Mathematical model of simultaneous flow between Casson fluid and dust particle over a vertical stretching sheet." *International Journal of Integrated Engineering* 12, no. 3 (2020): 253-260. https://publsher.uthm.edu.my/ojs/index.php/iije/article/view/5449

[15] Kho, Yap Bing, Hussanan, Abid, Mohamed, Muhammad Khairul Anuar, and Salleh, Mohd Zuki. "Heat and mass transfer analysis on flow of Williamson nanofluid with thermal and velocity slips: Buongiorno model." *Propulsion and Power Research* 8, no. 3 (2019): 243-252. https://doi.org/10.1016/j.jppr.2019.01.011

[16] Rawi, Noraihan Afiqah, Ilias, Mohd Rijal, Isa, Zaiton Mat, and Shafie, Sharidan. "Effect of gravity modulation on mixed convection flow of second grade fluid with different shapes of nanoparticles." *Malaysian Journal of Fundamental and Applied Sciences* 13, no. 2 (2017): 132-136. https://doi.org/10.11113/mjfas.v13n2.643

[17] Prasad, V. Ramachandra, Gaffar, S. Abdul, Reddy, E. Keshava, and Beg, O. Anwar. "Numerical study of non-Newtonian Jeffreys fluid from a permeable horizontal isothermal cylinder in non-Darcy porous medium." *Journal of the Brazilian Society of Mechanical Sciences and Engineering* 37, no. 6 (2015): 1765-1783. https://doi.org/10.1007/s00430-014-0301-5

[18] Merkin, J.H. *Free convection boundary layer on an isothermal horizontal cylinder*. In *Proceedings of the ASME-AIChE, Heat Transfer Conference*. 1976. https://doi.org/10.1007/bf01590813

[19] Merkin, J.H. "Free convection boundary layers on cylinders of elliptic cross section." *Journal of Heat Transfer* 99 (1977): 453-457. https://doi.org/10.1115/1.3450717

[20] Ingham, D.B. and Pop, I. "Natural convection about a heated horizontal cylinder in a porous medium." *Journal of Fluid Mechanics* 184 (1987): 157-181. https://doi.org/10.1017/s0022112087002842

[21] Merkin, J.H. and Pop, Ioan. "A note on the free convection boundary layer on a horizontal circular cylinder with constant heat flux." *Wärme-und Stoffübertragung* 22, no. 1-2 (1988): 79-81. https://doi.org/10.1007/bf01001575

[22] Nazar, Roslinda, Amin, Norsarahaida Saidina, and Pop, Ioan. *Free convection boundary layer on an isothermal horizontal circular cylinder in a micropolar fluid*. In *International Heat Transfer Conference Digital Library*. 2002. Begel House Inc. https://doi.org/10.1615/Ihtc12.2030

[23] Molla, Md Mamun, Hossain, Md Anwar, and Gorla, Rama Subba Reddy. "Natural convection flow from an isothermal horizontal circular cylinder with temperature dependent viscosity." *Heat and Mass Transfer* 41, no. 7 (2005): 594-598. https://doi.org/10.1007/s00231-004-0576-7

[24] Molla, Md Mamun, Hossain, Md Anwar, and Paul, Manosh C. "Natural convection flow from an isothermal horizontal circular cylinder in presence of heat generation." *International Journal of Engineering Science* 44, no. 13-14 (2006): 949-958. https://doi.org/10.1016/j.ijengsci.2006.05.002

[25] Salleh, Mohd Zuki and Nazar, Roslinda. "Free convection boundary layer flow over a horizontal circular cylinder with Newtonian heating." *Sains Malaysia* 39, no. 4 (2010): 671-676. https://doi.org/10.1007/s00231-010-0662-y

[26] Azim, N.H.M. and Chowdhury, M.K. "MHD-conjugate free convection from an isothermal horizontal circular cylinder with joule heating and heat generation." *Journal of Computational Methods in Physics* 2013 (2013): 1-11. https://doi.org/10.1155/2013/180516

[27] Prasad, V. Ramachandra, Gaffar, S. Abdul, Reddy, E. Keshava, and Bég, O. Anwar. "Flow and heat transfer of Jeffrey's non-Newtonian fluid from horizontal circular cylinder." *Journal of Thermophysics and Heat Transfer* 28, no. 4 (2014): 764-770. https://doi.org/10.2514/1.t4253

[28] Makanda, Gilbert, Shaw, Sachin, and Sibanda, Precious. "Effects of radiation on MHD free convection of a Casson fluid from a horizontal circular cylinder with partial slip in non-Darcy porous medium with viscous dissipation." *Boundary Value Problems* 2015, no. 1 (2015): 1-14. https://doi.org/10.1186/s13661-015-0333-5
[29] Mohamed, Muhammad Khairul Anuar, Noar, Nor Aida Zuraimi Md, Salleh, Mohd Zuki, and Ishak, Anuar. "Free convection boundary layer flow on a horizontal circular cylinder in a nanofluid with viscous dissipation." *Sains Malaysiana* 45, no. 2 (2016): 289-296.

[30] Rao, A. Subba, Amanulla, C. H., Nagendra, N., Beg, O. Anwar, and Kadir, A. "Hydromagnetic flow and heat transfer in a Williamson non-Newtonian fluid from a horizontal circular cylinder with Newtonian heating." *International Journal of Applied and Computational Mathematics* 3, no. 4 (2017): 3389-3409. https://doi.org/10.1007/s40819-017-0304-x

[31] Gaffar, S. Abdul, Prasad, V. Ramachandra, and Reddy, E. Keshava. "Magnetohydrodynamic free convection flow and heat transfer of non-Newtonian tangent hyperbolic fluid from horizontal circular cylinder with Biot number effects." *International Journal of Applied and Computational Mathematics* 3, no. 2 (2017): 721-743. https://doi.org/10.1007/s40819-015-0130-y

[32] Zokri, Syazwani Mohd, Salleh, Mohd Zuki, Arifin, Nur Syamilah, and Kasim, Abdul Rahman Mohd. "Lower stagnation point flow of convectively heated horizontal circular cylinder in Jeffrey nanofluid with suction/injection." *Journal of Advanced Research in Fluid Mechanics and Thermal Sciences* 76, no. 1 (2020): 135-144. https://doi.org/10.37934/arfmts.76.1.135144

[33] Dalir, Nemat. "Numerical study of entropy generation for forced convection flow and heat transfer of a Jeffrey fluid over a stretching sheet." *Alexandria Engineering Journal* 53, no. 4 (2014): 769-778. https://doi.org/10.1016/j.aej.2014.08.005

[34] Hayat, Tasawar and Ali, Nasir. "Peristaltic motion of a Jeffrey fluid under the effect of a magnetic field in a tube." *Communications in Nonlinear Science and Numerical Simulation* 13, no. 7 (2008): 1343-1352. https://doi.org/10.1016/j.cnsns.2006.12.009

[35] Qasim, M. "Heat and mass transfer in a Jeffrey fluid over a stretching sheet with heat source/sink." *Alexandria Engineering Journal* 52, no. 4 (2013): 571-575. https://doi.org/10.1016/j.aej.2013.08.004

[36] Mohamed, Muhammad Khairul Anuar, Salleh, Mohd Zuki, Noar, Nor Aida Zuraimi Md, and Ishak, Anuar. "The viscous dissipation effects on the mixed convection boundary layer flow on a horizontal circular cylinder." *Journal Teknologi* 78, no. 4-4 (2016): 73-79. https://doi.org/10.11113/jt.v78.8304