Topological elasticity of non-orientable ribbons

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In this article, we unravel an intimate relationship between two seemingly unrelated concepts: elasticity, that defines the local relations between stress and strain of deformable bodies, and topology that classifies their global shape. Focusing on Möbius strips, we establish that the elastic response of surfaces with non-orientable topology is: non-additive, non-reciprocal and contingent on stress-history. Investigating the elastic instabilities of non-orientable ribbons, we then challenge the very concept of bulk-boundary-correspondence of topological phases. We establish a quantitative connection between the modes found at the interface between inequivalent topological insulators and solitonic bending excitations that freely propagate through the bulk non-orientable ribbons. Beyond the specifics of mechanics, we argue that non-orientability offers a versatile platform to tailor the response of systems as diverse as liquid crystals, photonic and electronic matter.

I. INTRODUCTION

Sewing the first piece of fabric, prehistoric men laid out the first principles of metamaterial design [1]: elementary units assembled into geometrical patterns form structures with mechanical properties that can surpass those of their constituents [2]. In the early 2010’s, building on quantitative analogies with the topological phases of quantum matter, researchers laid out robust design rules for metamaterials supporting mechanical deformations immune from geometrical and material imperfections [2–8]. Today, mechanical analogs of virtually all topological phases of electronic matter have been experimentally realized, or theoretically designed, with mechanical components as simple as coupled gyroscopes or lego pegs [2, 5, 6, 9–12]. The basic strategy consists in connecting mechanical systems with gapped vibrational spectra having topologically distinct eigenspaces [8, 13, 14]. At the interface, this mismatch causes a local gap closing revealed by linear edge modes topologically protected from disorder and backscattering. Until now, as topological mechanics was inspired by analogies with condensed matter, it has been essentially restrained to metamaterials assembled from repeated mechanical units, that inherit robustness from the topology of their abstract vibrational eigenspace [8, 15].

In this article, we elucidate the consequences of real-space topology on the mechanics of homogeneous materials. Firstly, we demonstrate that non-orientability makes Möbius strips’ elasticity: non-additive, non-reciprocal and multistable. In particular, we demonstrate how the static deformations of non-orientable surfaces encode their stress history: Möbius strips have a mechanical memory. Secondly, we address the impact of non-orientability on the paradigmatic Euler elastic instability. We show that the associated buckling patterns propagate as solitary waves on Möbius strips. We finally establish the equivalence between these non-linear bulk excitations and the edge modes found at the interface between inequivalent topological states in one-dimensional topological insulators [13].

II. TOPOLOGICAL ELASTICITY OF NON-ORIENTABLE SURFACES

Simply put, a non-orientable surface is a one-sided thin sheet. A paradigmatic example is given by the Möbius strip shown in Fig. 1 that can be easily replicated by applying a half twist to a band of paper before glueing its two ends. Orientability is indeed a global (topological) property that can be altered only by cutting and gluing back a geometrical surface. In contrast, linear elasticity describes local deformations in response to gentle mechanical stresses. Before introducing a technical framework to relate these two seemingly unrelated concepts, let first us gain some intuition about their relationship. We consider the simple example of a Möbius strip made of an elastic material showed in Fig. 1. The shear deformations of the strip is locally quantified by the angle $\theta^S(s)$, where
s ∈ [0, L] indicates the curvilinear coordinates along the strip centerline. θ^0(s) is a rotation angle defined with respect to the vector \( \mathbf{n}(s) \) normal to the surface. A direct consequence of non-orientability is that no stress distribution can yield homogeneous shear deformations over a Möbius strip. As illustrated in Fig. 1 when transported around the entire strip, \( \mathbf{n}(s) \) changes sign, thereby implying that \( θ^0(0) = -θ^0(L) \), and that the shear angle must vanish at least once along the ribbon. The impossibility to assign an unambiguous orientation to the surface constrains the ribbon to remain undeformed at one point whatever the magnitude of the applied stress. We now account for this topological protection against shear by describing the elasticity of non-orientable ribbons as a \( \mathbb{Z}_2 \) gauge theory.

A. Orientability as a \( \mathbb{Z}_2 \) gauge charge.

For sake of clarity, we restrain ourselves to strips of constant width \( w \) akin to that showed in Figs. 1 and 2. They are defined as ruled surfaces \( S(s, z) = C(s) + zb(s) \) where \( C(s) \) is a base circle of perimeter \( L \), and \( b(s) \) is a unit-vector field normal to the tangent-vector field \( t(s) \), see Fig. 1. Given this definition \( s ∈ [0, L] \) and \( z ∈ [-w/2, w/2] \). We stress that the direction of \( b(s) \) is arbitrary: a local transformation \( b(s) → e(\theta(s))b(s) \), where \( ϵ(s) = ±1 \) leaves the strip geometry unchanged. The tangent to the base circle \( t(s) \) being unambiguously defined, the normal vector \( \mathbf{n}(s) = t(s) × b(s) \) is defined up to the same \( ϵ(s) \) sign factor as \( b(s) \).

By definition, non-orientable strips correspond to shapes where the fields \( ϵ(s)b(s) \) and \( ϵ(s)n(s) \) are discontinuous regardless of the sign convention \( ϵ(s) \). This intrinsic ambiguity in defining the orientation of the (bi)normal vector is better illustrated when discretizing the strip, see Fig. 2. Setting \( s = ia \), where \( a = L/N \) and \( i ∈ [1, N - 1] \), we introduce the \( \mathbb{Z}_2 \) gauge field \( η_{i,i+1} = ϵ_i ϵ_{i+1} \) which represents the connection between adjacent sign conventions. The topological charge \( \mathcal{O} = \prod_{i=1}^{N-1} η_{i,i+1} \), defines the surface orientability: orientable surfaces correspond to \( \mathcal{O} = +1 \) and nonorientable ones to \( \mathcal{O} = -1 \). The independence of \( \mathcal{O} \) with respect to the sign convention becomes clear when applying the series of gauge transformations sketched in Figs. 1b and 2b. Starting from an arbitrary position \( i_G + 1 \) and moving along the base circle, wherever a link with \( η_{i,i+1} = -1 \) is found, we change the sign of \( ϵ_{i+1} \). This transformation reverses simultaneously the signs of both \( η_{i,i+1} \) and \( η_{i+1,i+2} \) thereby leaving \( \mathcal{O} \) unchanged. Moving along the strip and repeating this procedure, we find that the gauge field on all links but the last one can be set to \( η = +1 \). On the last link, it takes the value \( η_{i_G,i_G+1} = \mathcal{O} \). Therefore, when \( \mathcal{O} = -1 \) there is an obstruction to define a homogeneous surface orientation: the surface is non-orientable.

B. Elasticity of twisted elastic strips.

We now make use of this geometric framework to describe the elastic response of a soft Möbius strip having a stress-free equilibrium shape defined by the triad \((t^0(s), b^0(s), n^0(s))\). For sake of simplicity, we do not resort to the full Foppl-von Karman theory of elastic plates. Instead, we consider simplified models to single out the impact of non-orientability on shear, twist and bend deformations leaving a more realistic mechanical description for future work. The amplitude of the pure-shear, \( θ^0(s) \), and pure-twist angles, \( θ^T(s) \), are usually defined from the deformation vector \( u(s) = b(s) - b^0(s) = θ^0(s)t^0(s) + θ^T(s)n^0(s) \). As discussed in the previous section, however, both \( b(s) \) and \( n(s) \) are defined up to a sign convention \( ϵ(s) \), while all physical quantities must be independent of this arbitrary choice. We therefore introduce the orientation-independent deformation field:

\[
(εu) = (εθ^0)t^0 + θ^T(εn^0).
\]

\((εu)\) is invariant upon the orientation transformation: \( \{ε(s) → -ε(s), θ^0(s) → -θ^0(s), θ^T(s) → θ^T(s)\} \). Due to the possibly discontinuous nature of the \( ϵ \) field, we first define the harmonic elasticity associated to \( (εu) \) by resorting to a discretization of the ribbon geometry. The simplest harmonic elasticity is then given by \( E = K/(2a) \sum_i [(εu)]_i^2 \), where \( K \) is an isotropic elastic constant, and \( E \) is readily recast into:

\[
E = \frac{K}{2a} \sum_i [u_{i+1} - η_{i,i+1}u_i]^2.
\]

The invariance of \( (εu_i) \) under orientation transformation translates into a \( \mathbb{Z}_2 \) gauge symmetry of the elastic energy: \( \{η_{i,i+1} → -η_{i,i+1}, θ^0_i → -θ^0_i, θ^T_i → θ^T_i\} \). Following the procedure sketched in Fig. 2, Eq. (2) can be simplified by gauging away the \( η_{i,i+1} \) at all sites but one, at \( i = i_G \).
where \(\eta_{iG,iG+1} = \mathcal{O}\). For this gauge choice, \(\mathcal{E}\) takes the compact form: \(\mathcal{E} = \frac{K}{2a} \sum \left[ (\theta^S_{i+1} - \theta^S_i)^2 + (\theta^T_{i+1} - \theta^T_i)^2 \right] + \frac{K}{a} (1 - \mathcal{O}) \theta^S_{iG} \theta^S_{iG+1}\), where we have implicitly assumed \(w/L\) to be vanishingly small and left finite-size geometrical corrections to future work [17]. The last term of this expression accounts for the coupling between the topological charge \(\mathcal{O}\) and the shear angle at the unspecified site \(i_G\). Continuum elasticity then follows from the limit \(a \to 0\) in Eq. (2):

\[
\mathcal{E}(\{\theta^S, \theta^T\}) = \frac{K}{2} \int_0^L (\partial_s \theta^T)^2 + (\partial_s \theta^S)^2 \, ds \\
+ (1 - \mathcal{O}) \lim_{a \to 0} \frac{K}{a} \left[ (\theta^S)^2 + a \theta^S \partial_s \theta^S \right]_{s=s_G}.
\]

For orientable strips, one recovers the familiar harmonic energy of elastic bodies. In contrast, for Möbius strips where \(\mathcal{O} = -1\), the topological term in Eq. (3) constrains the continuous shear deformations to vanish at \(s = s_G\) [18]. Two comments are in order: Firstly, unlike shear deformations, we find that twist deformations are insensitive to orientability and obey unconstrained harmonic elasticity. Secondly, we stress that the location of the zero-shear point \(s_G\) is an independent and crucial gauge degree of freedom that must be dealt with when computing the fluctuations and mechanical response of non-orientable elastic ribbons as illustrated below.

**C. Non-additive elasticity**

From now on the ribbon elasticity is prescribed by Eq. (3), and the constraint \(\theta^S(s_G) = 0\). It then readily follows that non-orientable ribbons cannot support any homogeneous shear deformation as anticipated in the introduction of Section II and further discussed in Appendix A. The simplest mechanical stress we can consider is a pointwise shear localized at an arbitrary position \(s_1\): \(\sigma^S(s) = \sigma \delta(s - s_1)\). The resulting deformations shown in Fig. 3a are computed minimizing \(\mathcal{E} + W\) with respect to both the shear and gauge degrees of freedom, where \(W = - \int_0^L \sigma^S(s) \theta^S(s) \, ds\) is the work performed by the external stress, see Appendix A. We find a positive elastic response that vanishes at a single point located at maximal distance from the stress source:

\[
\theta_1(s; s_1) = \frac{\sigma}{2K} \left| s - s_1 - \text{sgn}(s - s_1) \frac{L}{2} \right|.
\]

This simple expression has a deep consequence: the response of Möbius strips to shear stresses is intrinsically nonlinear, although the local stress-strain relation is linear. We establish this counter intuitive property by considering the case of two identical stress sources: \(\sigma^S(s) = \sigma \left[ \delta(s - s_1) + \delta(s - s_2) \right]\). The linear superposition of two \(\theta_1\) functions would result in strictly positive shear deformations over the whole strip which is topologically prohibited as \(\theta^S\) must vanish at least at one point \(s_G\). We therefore conclude that the response of Möbius strips to shear stresses is not pairwise additive and therefore nonlinear. This property is illustrated in Fig. 3b where we compare the shear angle \(\theta_2(s; s_1, s_2)\) computed from the minimization of \(\mathcal{E} + W\) with respect to \(\theta^S\) and \(s_G\) to that that derived from a mere superposition principle, see also Appendix A.

We explain below the practical consequence of this topological frustration.

**D. Non-reciprocal elasticity**

The static response of elastic bodies is generically reciprocal. In virtue of the so-called Maxwell-Betti theorem, the deformations measured at a point \(B\), as a result of a force applied at a point \(A\), are identical to the deformations measured at point \(A\) as a result of the same force when applied at point \(B\) [19,21]. The mechanics of non-orientable surfaces, however, is not reciprocal. To prove this counterintuitive results, we consider as a reference state a Möbius strip sheared by a localized source \(\sigma_0 = \sigma \delta(s - s_0)\) causing a deformation \(\theta_1(s; s_0)\). Let us now apply an additional stress \(\sigma\) at \(s_A\), and measure

**FIG. 3. Nonlinear response to shear stress.** Top panels: the color indicates the magnitude of the shear stress along the Möbius strips. The red lines show the positions of the applied stresses, and the dark line the position of \(s_G\). Bottom panels: corresponding plots of \(\theta^S(s)\). a. Response to a point-wise stress source \(\sigma(s) = \sigma \delta(s - 1/4)\) with \(\sigma = K\). The shear angle decays linearly from the stress source and vanishes at \(s = 3/4\). b. Nonlinearity: Response to two point-wise stress sources \(\sigma^S = K[\delta(s - 1/6) - \delta(s - (1/6 + 1/2))]\). The deformations computed from the minimization of the elastic energy as explained in Appendix A vanish at \(s_G = 11/12\). The deformations computed from the minimization of the elastic energy (solid line) are markedly different from the linear superposition of two responses to two individual point sources (dashed line).
Möbius strip is sheared by a localized stress distribution.

The response at $s_B$: $\theta_A(s_B) = \theta_2(s_B; s_0, s_A) - \theta_1(s; s_0)$. We now release the stress applied at $s_A$, apply as stress $\sigma$ at $s_B$, and measure the response at $s_A$: $\theta_B(s_A) = \theta_2(s; s_0, s_B) - \theta_1(s; s_0)$. The two corresponding excess shear angles are shown in Figs. 4a and 4b and are obviously different. Following [21], we plot in Fig. 4c the non-reciprocity factor $\Delta \theta = [\theta_A(s_B) - \theta_A(s_B)]\sigma/\sigma$ as a function of the locations of the two applied stresses ($s_A$ and $s_B$). We find that $\Delta \theta$ is finite over a large fraction of the parameter space and extremal when the stress sources are distant from $L/4$ and $L/2$ from $s_0$: the mechanical response of the strip is non-reciprocal. Two comments are in order. By contrast with the polar metamaterials considered in [21], here non-reciprocity does not rely on non-proportional response. The constitutive relation between stress and strain is linear, the strip is not unstable, and no floppy mode is excited. Non-reciprocity solely stems from the non-additive response of non-orientable strips. We also stress that non-reciprocity does not require any fine-tuning of the strip geometry, or of the applied stresses: Möbius strip mechanics is generically non-reciprocal.

**E. Elastic memory**

In addition to be non-linear and, non-reciprocal, non-orientable elasticity is multistable. This remarkable feature is demonstrated in Fig. 5a showing three equilibrium shear deformations of a strip stressed by the same shear distribution: $\sigma(s) = \sigma_1\delta(s-s_1) + \sigma_2\delta(s-s_2) + \sigma_3\delta(s-s_3)$, with $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$. The difference between the three equilibrium states solely lies in the order used to switch on the three stresses, as illustrated Fig. 5a. The very origin of this elastic multistability stems from the trapping of $s_G$ at different locations between the $s_i$ points.

We now release the stress applied at $s_G$, apply as stress sources: $\frac{3}{2}K\Phi_i^2$, and measure the response at $s_G$. Information is read measuring the shear angle, and deleted releasing the applied stresses.

**III. BUCKLING A MÖBIUS STRIP**

**A. Solitary buckling waves**

We now show how heterogeneous deformations emerge from homogeneous stresses. To do so, we address the consequences of non-orientability on the bending deformations of Möbius strips, see Fig. 3. We consider a simplified description where the strip is modelled by a ladder made of flexible hinges of length $\ell$ as sketched in Fig. 3b. For sake of simplicity, we restrain ourselves to bending deformations along the normal vector which naturally couple to the ribbon orientation. The total elastic energy $E_B$ is composed of three terms: (i) the confomration of the $i^{th}$ hinge is defined by the angle $\Phi_i$ and is associated with a harmonic bending energy $\sim \frac{1}{2}K_{\Phi}^2$, (ii) a harmonic coupling between the hinges adds a contribution $\sim \frac{1}{2\sigma}K_{\Phi}(\Phi_{i+1} - \eta_{i+1}\Phi_i)^2$, and (iii) applying an external compression load $\Sigma_i$ contributes to a mechanical work defined as the scalar product between the applied force and the resulting displacement: $(\Sigma_i\Phi_i - 1)$. We are now equipped to tackle the classical Euler buckling problem: the bending instability of a Möbius strip in response to a homogeneous compression. We first construct a continuum description of $E_B$ following the same procedure as in Section III A, gauging away the $\eta_{i+1}$ variables, taking the continuum limit and restraining ourselves to deformations close to the onset of buckling. $E_B(\{\Phi\})$ then
FIG. 5. Elastic Memory. a. Three different shear deformations are compatible with mechanical equilibrium when point-wise shear stresses of equal strength $K$ are applied at $s = 1/6, 1/2, \text{and } 5/6$. The colors indicate the magnitude of the shear deformations $\theta^s$, same colormap as in Fig. 3. b. Corresponding time variations of the stress amplitude. c. Spatial and temporal variations of the effective potential $U_3(s,t)$. The blue circle indicates the location of the instantaneous minimum of $U_3$ where $s_G$ is trapped.

The buckling patterns minimize $\mathcal{E}_B$ with the constraint $\Phi(s_G) = 0$. This minimization is performed using a dynamical-system analogy elaborated in Appendix B. In short, the strip remains flat until $\Sigma$ exceeds $\Sigma_c = \Sigma_0 \left[ 1 + \left( K_B/K_B' \right)^{2} (\alpha/L)^2 \right]$. Above $\Sigma_c$ it undergoes a buckling transition and deforms into the inhomogeneous pattern illustrated in Fig. 6. The bending angle $\Phi$ remains close to $\Phi_0$ everywhere except in a region of size $\xi/\Phi_0$ around $s_G$ where it vanishes. The buckling pattern is the norm of a $\Phi^4$ kink centered on $s_G$ \cite{22}. In the limit of long strips $\Phi_B$ reduces to:

$$\Phi_B(s - s_G) = \pm \Phi_0 \left| \tanh \left( \frac{\sqrt{2} \Phi_0}{\xi} (s - s_G) \right) \right| + \mathcal{O} \left( \frac{\xi}{L} \right).$$

(7)

where the sign of the solution reflects the arbitrary choice of orientation of the ribbon. The exact solution beyond the very long strip approximation does not bring more insight and is left to Appendix B.

Remarkably, both the $\mathbb{Z}_2$ gauge symmetry and translational invariance are spontaneously broken at the onset of buckling. The ground state of $\mathcal{E}_B$ is continuously degenerate leaving the bending direction and the location of the flat section $s_G$ undetermined. As a consequence the buckling patterns are free to translate around the strips. More quantitatively, having macroscopic systems in mind, we now consider the inertial dynamics of the
buckled strip described by the continuum Hamiltonian:

\[ \mathcal{H}_B = \int \frac{I}{2} \left[ (\partial_t \Phi)^2 + \frac{K_B}{2} (\partial_s \Phi)^2 + U^2(\Phi) \right] ds, \]  

(8)

where \( I \) is the local moment of inertia. As in the static case, \( \mathcal{H}_B \) is complemented by the constraint \( \Phi(s_G(t), t) = 0 \). The existence of solitary waves readily follows from the Lorentz invariance of \( \mathcal{H}_B \). The solitary waves are deduced from Eq. (7) by a Lorentz boost: \( \Phi(s, t) = \Phi_B(\gamma(s - vt)) \), where \( \gamma \equiv \left[ 1 - (v/c)^2 \right]^{-1/2} \) [22], and the propagation speed \( v \) satisfies \( v^2 < c^2 = K_B/I \).

The free propagation of these solitary bending waves restores translational and gauge invariance of Eq. (5): moving the topologically protected section \( s_G \) along the strip corresponds to a gauge transformation which operates at zero energy cost. We note that travelling kinks of the very same nature were first found theoretically and illustrated experimentally in soap films forming non-orientable minimal surfaces [23]. In the context of topological mechanics, spectacular zero-energy mechanisms having a similar solitonic structure were also found at the interface between open one-dimensional isostatic lattices having topologically distinct band spectra [9][24]. In the next section, we show that the latter resemblance is the first hint of a deeper connexion between the topological mechanics of non-orientable ribbons and that of one-dimensional isostatic metamaterials.

B. From buckled Möbius strips to SSH topological insulators

We characterized above the orientability of the ribbon by the invariant \( \mathcal{O} \), and showed that \( \mathcal{O} = -1 \) implies the existence of solitary bending waves. Here, we show that these excitations are characterized by their own topological number \( n \) which we relate to that of interface states between topological insulators. The standard topological characterization of both phononic and electronic excitations was established for lattice models and does not apply to continuous elasticity [8][13]. We circumvent this technical obstacle following Ref. [24]. Resorting to an index theorem applied to the linearized elasticity of the strip, we establish that the soliton carries a topological charge.
$n = 1$ that counts the zero energy translational modes.

In practice, we introduce the linear bending fluctuations of the ribbon around a static buckled state [7]: $\Phi(s, t) = \Phi_B(s - s_G) + \Psi(s, t)$, and deduce the dynamics of $\Psi$ by linearizing Eq. (8):

$$\partial_t^2 \Psi(s, t) = -D \Psi(s, t),$$

where $D = \partial_t^2 - 2U''(\Phi_B) - 2U'(\Phi_B)U''(\Phi_B)$. The topological properties of mechanical vibrations are revealed by the “square root” of the dynamical operator $D$ [3].

In the limit of very long yet finite ribbons, the dynamical properties of the ribbon around a static buckled state (7) is a floppy mode that operates, by definition, at zero energy. This translational mode is reminiscent of the boundary states distributed to all aspects of the research.

We have demonstrated how to surpass the native properties of materials without resorting to geometrical tuning. Constructing a minimal elastic theory for Möbius strips, we have established that non-orientability makes their local mechanics non-linear, non-reciprocal and capable of memorizing its stress history. Investigating their simplest bending instability, we have demonstrated how non-orientability guarantees the existence of a topological phase that supports zero-energy solitons. This mechanical phase, without known condensed-mater counterparts, begs for a generalization of the current bulk-boundary correspondence in topological materials [28-33].

Our main predictions are elaborated building on prototypical models, we therefore expect their experimental implications to extend beyond the specifics of mechanical systems. In particular, the relation between nematic elasticity and $\mathbb{Z}_2$ gauge theories was realized in the early 90’s by Lammert et al. in the context of phase ordering, but to the best of our knowledge has remained virtually uncharted [23]. We stress here that our central equation Eq. (8) also describes the Frank energy of non-orientable nematic films, and can be generalized to describe nematic elasticity around a disclination [35]. A remarkable experimental realization of a non-orientable nematic liquid crystal was provided by self-assembled viral membranes where rod-like units self-organize into Möbius configurations at the membrane edge [36]. Beyond elasticity, we also envision our prediction to be relevant to Möbius configurations of light polarization [37, 38] and to transport in twisted nano crystals [39].

IV. CONCLUSION AND PERSPECTIVES

We have demonstrated how to surpass the native properties of materials without resorting to geometrical tuning. Constructing a minimal elastic theory for Möbius strips, we have established that non-orientability makes their local mechanics non-linear, non-reciprocal and capable of memorizing its stress history. Investigating their simplest bending instability, we have demonstrated how non-orientability guarantees the existence of a topological phase that supports zero-energy solitons. This mechanical phase, without known condensed-mater counterparts, begs for a generalization of the current bulk-boundary correspondence in topological materials [28-33].

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Appendix A: Response to shear

1. No homogeneous shear stress

We showed in Section II C that a Möbius strip cannot support any homogeneous shear deformation. The situation is even more constrained as no uniform shear stress can be applied. The shear stress \( \sigma^S \) and \( \sigma^B \) are conjugated variables, and the mechanical work associated to shear is given by \( W = -\int \sigma^S(s) \delta^S(s) \, ds \). As \( W \) must be independent on the arbitrary definition of the ribbon orientation, \( \sigma^S \) and \( \delta^S \) must obey the same transformation rules upon any change in the ribbon orientation: \( \sigma(s) \) is also topologically constrained to vanish at \( s_G \).

2. Response to a point-wise shear stress

We consider the response to the shear-stress distribution given by: \( \sigma^S = \sigma_1 \delta(s - s_1) \). For sake of clarity, units are here chosen so that \( L = 1 \). The equilibrium configuration is obtained minimizing the total energy \( \mathcal{E} + \mathcal{W} \) defined in Section II C with respect to both \( \theta^S \) and the gauge degree of freedom \( s_G \):

\[
\mathcal{F} = \mathcal{E} + \mathcal{W} = \frac{K}{2} \int_0^1 (\partial_s \theta^S)^2 - \int_0^1 \sigma^S(s) \theta^S(s) \, ds. \quad (A1)
\]

We recall that \( s_G \) is the location of the strip section where \( \theta^B(s) \) is topologically constrained to vanish. Within this framework, the two mechanical equilibrium conditions are:

\[
\frac{\delta \mathcal{F}}{\delta \theta^S} = -K \partial_s^2 \theta^S - \sigma^S(s) = 0, \quad (A2)
\]

\[
\frac{\partial \mathcal{F}}{\partial s_G} = 0. \quad (A3)
\]

These equations are supplemented by the boundary conditions:

\[
\theta^S(s, s_1; s_G) = \theta^S(s + 1, s_1; s_G), \quad (A4)
\]

and the topological constraint

\[
\theta^S(s = s_G, s_1; s_G) = 0. \quad (A5)
\]

The algebra is simplified by redefining the origin of the curvilinear coordinate \( (s \to \tilde{s}) \) such that \( s_G = 0 \). The conditions (A4-A5) then reduce to

\[
\theta^S(\tilde{s} = 0, \tilde{s}_1) = \theta^S(\tilde{s} = 1, \tilde{s}_1). \quad (A6)
\]

In this frame, the gauge degree of freedom then becomes the position of the applied stress \( \tilde{s}_1 = s_1 - s_G \) (mod 1). Solving Eqs. (A2) and (A6) we readily find that the shear deformations are given by \( \theta^S(\tilde{s}; \tilde{s}_1) = \sigma_1 G^{(1)}(\tilde{s}; \tilde{s}_1) \) with

\[
G^{(1)}(\tilde{s}; \tilde{s}_1) = -\frac{1}{2K} (|\tilde{s} - \tilde{s}_1| + (\tilde{s}_1 - 1)\tilde{s} + (\tilde{s} - 1)\tilde{s}_1). \quad (A7)
\]

The corresponding total energy

\[
\mathcal{F} = -\frac{\sigma_1^2}{2K} \delta_1(1 + \delta_1) \quad (A8)
\]

is minimized for \( \delta_1 = \frac{1}{2} \), i.e. for \( s_G = s_1 + \frac{1}{2} \) mod 1. In other words, the point \( s_G \) where the shear deformation vanishes is maximally separated from the applied stress.

Going back to the original frame, the static shear deformations at mechanical equilibrium are easily recast into:

\[
\theta_1(s, s_1) = \frac{\sigma_1}{2K} \left| s - s_1 - \frac{1}{2} \text{sgn}(s - s_1) \right|, \quad (A9)
\]

which corresponds to Eq. 4 in the main text.

3. Response to \( N \) localized shear sources

We now consider the superposition of \( N \) fixed point-wise sources: \( \sigma^S(s) = \sum_{i=1}^{N} \sigma_i \delta(s - s_i) \). The equilibrium condition \( (A2,A3) \) with the boundary conditions \( (A4,A5) \). Working in the frame where \( s_G = 0 \), the solution of this equation is

\[
\theta^S_N(\tilde{s}; \{ \tilde{s}_i \}) = \sum_{i=1}^{N} \sigma_i G^{(1)}(\tilde{s}; \tilde{s}_i), \quad (A10)
\]

where \( \tilde{s}_i = s_i - s_G \) (mod 1), i.e. \( \tilde{s}_i = s_i - s_G + \Theta(s_G - s_i) \) where \( \Theta(s) \) is the Heaviside step function. At first sight Eq. (A10) resembles the mere superposition of independent Green functions and suggests a typical linear response behavior. However we have to keep in mind that the position \( s_G \) is yet to be determined to prescribe the equilibrium deformations. As a shift in the position \( s_G \) corresponds to a uniform translation of all the stress sources, we need to compute the equilibrium value of \( s_G \), keeping all distances \( \tilde{s}_i - \tilde{s}_1 \) fixed. Inspired by the classical calculation of the elastic interactions between inclusions in soft membranes and liquid interfaces (see e.g. [40]), we integrate over the shear degrees of freedom and derive the effective potential \( U_N(s_G) \) that controls the position of \( s_G \) along the strip. To compute \( U_N \), it is convenient to solve a seemingly more complex problem where the strip undergoes thermal fluctuations. The thermal statistics are then defined by the partition function

\[
\mathcal{Z}([\sigma_i]) = \int_0^1 ds_G \int D\theta^S e^{-\beta \int \frac{1}{2}(\partial_s \theta^S)^2 - \int \sigma_i \theta^S(s_i) \}, \quad (A11)
\]

where \( \beta^{-1} = k_B T \) and the field \( \theta^S(\tilde{s}) \) satisfies the condition \( (A6) \). Integrating out the \( \theta^S \) degrees of freedom defines the effective potential \( U_N(s_G) \):

\[
\mathcal{Z} = \int_0^1 ds_G e^{-\beta U_N(s_G)}, \quad (A12)
\]

with \( U_N(s_G) = -\frac{1}{2} \sum_{i,j} \sigma_i \sigma_j G^{(1)}(\tilde{s}_i, \tilde{s}_j) \). \quad (A13)
Minimizing
\[ U(s) = -\frac{\sigma^2}{2K} \left( |s_2 - s_1| + (s_1 + s_2)(\tilde{s}_1 + \tilde{s}_2 - 2) \right) \]  
(A14)

Minimizing \( U_2(s_G) \), we find two local minima satisfying \( \partial_{s_G}U_2 = 0 \) at \( s_G = (s_1 + s_2)/2 \mod (1) \) and \( s_G = (s_1 + s_2)/2 \pm 1/2 \mod (1) \). They are sketched in Fig. 7 and reflect the mirror symmetry of the problem. The lowest energy conformation always corresponds to the value of \( s_G \) the further away from the stress sources. In the symmetric case, where \( \Delta = \frac{1}{2} \), the shear response possesses two degenerate equilibrium positions. With the knowledge of the position \( s_G \) the shear-deformation profile is fully determined. It is given by Eq. (A10) and illustrated in Fig. 3 for various positions of \( s_1, s_2 \).

Appendix B: Buckling patterns and solitary waves, a dynamical-system insight.

We compute the shape of buckled Möbius strips making use of a dynamical system analogy. The expression of the elastic energy given by Eq. (A10) is indeed analogous to the Lagrangian of a classical particle of unit mass, and moving in a potential \( V(\Phi) = -\xi^{-2}(\Phi^2 - \Phi_0^2)^2 \), where \( \Phi \) indicates the particle position, \( s \) the time, and \( \partial_s \Phi \) the particle velocity, see Fig. 3. Both the non-orientability constraint, and the finite size of the strip complexifies the dynamics of this seemingly simple dynamical system. We show below that the trajectories are not periodic and singular at \( s_G \).

Without loss of generality we chose \( s_G = 0 \). Non-orientability therefore implies that \( \Phi(0) = \Phi(L) = 0 \) regardless of the value of the particle speed \( \partial_s \Phi(s_G) \). The trajectory \( \Phi(s) \) is found noting that the mechanical energy \( E_m = \frac{1}{2} |\partial_s \Phi(s)|^2 + V[\Phi(s)] \) is a constant of motion. Noting \( \Phi_m = \max[\Phi(s)] \), the periodicity of the trajectory (reflecting the periodicity of the strip shape) imposes \( |\Phi_m| < \Phi_0 \). Otherwise, one would simultaneously have \( \partial_s \Phi = 0 \) and \( \Phi > \Phi_0 \), thereby leading to runaway solutions. Invariance upon time reversal of the particle Lagrangian also imposes \( \Phi_m = \Phi(L/2) \). Therefore, the conservation of mechanical energy implies:

\[ \partial_s \Phi(s) = \pm \sqrt{2[V(\Phi_m) - V(\Phi(s))]} \]  
(B1)

Let us consider solutions where \( \Phi_m > 0 \). The sign of \( \partial_s \Phi(s) \) in Eq. (B1) is then positive when \( 0 < s < L/2 \) and negative when \( L/2 < s < L \), and the inverse function \( s = \Phi^{-1}(\Phi(s)) \) readily found integrating Eq. (B1) on the two separate intervals:

\[ s = \pm \xi \int_0^\Phi \frac{dx}{\sqrt{2(x^2 - \Phi_m^2)(x^2 + \Phi_m^2 - 2\Phi_0^2)}} = \frac{1}{\sqrt{\Phi_m^2 - 2\Phi_0^2}} F[\arcsin(\Phi_m/k), k], \]  
(B2)

where \( k \equiv \Phi_m^2/(2\Phi_0^2 - \Phi_m^2) \) and \( F(x, k) \) is the complete elliptic integral of the first kind. The final form of the trajectory follows from the definition \( \Phi_m = \Phi(L/2) \) which imposes \( \Phi_m^2 = 2\Phi_0^2 - 2(L^{-1})^2 F[\pi/2, k] \). We stress that our gauge choice constraints \( \phi(s) \) to vanish at \( s_G = 0 \) thereby imposing the derivative of \( \Phi \) to be discontinuous at \( s_G \). The solution corresponds to two symmetric half \( \Phi^4 \) kinks defined on a compact interval is plotted in Fig. 4 in the main text. One last comments is in order. In the limit of large ribbons assembled from very stiff hinges, \( \xi/L \ll 1 \), \( \Phi_m = \Phi_0 \), and the integration of Eq. (B1) results in the usual tanh profiles given by Eq. (7). The buckling pattern corresponds to the symmetrization of the usual \( \Phi^4 \) soliton.

Appendix C: Factorization of the dynamical operator.

We show how to factorize the dynamical operator \( \mathcal{D} = \partial_s^2 - 2U''(\Phi_0) - 2U(\Phi_0)U''(\Phi_0) \) defined in Eq. (5). As discussed above in Appendix B in the limit of infinitely long ribbons, \( \Phi_N^\infty = \Phi_0 \) and \( V(\Phi_N^\infty) = E_m = 0 \). The latter relation simplifies Eq. (B1):

\[ \partial_s \Phi^\infty(s) = \text{sgn}(s - s_G) \sqrt{2U[\Phi^\infty(s)]}. \]  
(C1)
The potential $V(\Phi)$ is plotted versus $\Phi$. At time $s = 0$ the particle is located in the potential well (dark circle). It is however not at rest as its velocity is non-zero; it is kicked uphill and reaches its maximal position at $s = \frac{1}{2}$ (light circle). It then falls back to its initial position at $t = 1$. The particle speed at $s = 0$ and $s = 1$ are opposite. $\Phi_0 = 1/2$, $\xi = 1$. This feature translates into a slope discontinuity of the soliton shape at $s_G$.

Together with the definition of $D$, this relation implies the factorization $D = Q^\dagger Q$, with

$$Q^\dagger = \partial_s + \text{sgn}(s - s_G)\sqrt{2U'(\Phi_0^\infty)} , \quad (C2)$$
$$Q = -\partial_s + \text{sgn}(s - s_G)\sqrt{2U'(\Phi_0^\infty)} , \quad (C3)$$

where $\Phi_0^\infty$ is the shape of the unperturbed buckled ribbon. A $\xi/L$ expansion shows that this form is preserved for very long but finite ribbons. This result is obtained expressing the ribbon shape as a linear perturbation of $\Phi^\infty$: $\Phi_0 = \Phi_0^\infty + (\xi/L)\Phi$. Evaluating $E_m$, and keeping in mind that $U(\Phi_0) = 0$, we find: $E_m = U^2(\Phi_0 + \Phi_m) = U^2(\Phi_0) + 2U(\Phi_0)U(\Phi_m) + O[(\xi/L)^2] = O[(\xi/L)^2]$. The relations $E_m = 0$ and Eq. (C1) are hence preserved at first order in $\xi/L$. Therefore, even though the Hamiltonian $H_P$ defined in Eq. (8) does not enjoy the BPS symmetry of the continuum description of the isostatic chain of linkages introduced in [24], the corresponding dynamical matrix can still be factorized as $D = Q^\dagger Q$, substituting $\Phi^\infty$ by $\Phi$ in Eqs. (C2) and Eqs. (C3).

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