Multiple-Valued Picture Fuzzy Linguistic Set Based on Generalized Heronian Mean Operators and Their Applications in Multiple Attribute Decision Making

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ABSTRACT To better express and handle the increasing complicated fuzzy decision making information, we firstly put forward the concept of multiple-valued picture fuzzy linguistic set (MVPFLS), which is motivated by the advantages of hesitant fuzzy set (HFS), linguistic term set (LTS), and picture fuzzy set (PFS). First, the concepts concerning MVPFLS and the multiple-valued picture fuzzy linguistic element (MVPFLE) are proposed. Second, the operating rules of MVPFLEs on account of linguistic scale functions (LSFs) are given, and the compared functions for MVPFLE are also defined. Third, the traditional generalized Heronian mean (HM) operators capturing the correlation among attributes are extended to multiple-valued picture fuzzy linguistic environment, namely, the multiple-valued picture fuzzy linguistic generalized weighted Heronian mean (MVPFLGWHM) aggregating operator and the multiple-valued picture fuzzy linguistic generalized weighted geometric Heronian mean (MVPFLGWGHM) aggregating operator are proposed, their interesting characteristics and some particular instances are also given. Fourth, the method utilizing the proposed aggregating operators to settle multiple attribute decision making (MADM) issues is provided. Eventually, a practical instance is employed to validate the powerfulness of the new approach, and the sensitivity analysis is performed along with comparative analysis.

INDEX TERMS Multiple-valued picture fuzzy linguistic set, Heronian operator, multiple attribute decision making.

I. INTRODUCTION

Picture fuzzy set (PFS) [1] introduced in 2013 is an extended form of intuitionistic fuzzy set (IFS) [2]. PFS can represent four types of attitudes on decision-maker, namely, positive, neutral, negative, and refusal. The sum of four membership degrees is equal to one. These four types of evaluating information have not been well-described by utilizing the conventional fuzzy set (FS) [3] and IFS. Therefore, a series of achievements on PFS have been made and applied to solve different problems [4]–[14], such as multiple attribute decision making (MADM), clustering, forecasting, medical diagnosis, and so on.

PFS is an effective tool for describing different kinds of quantitative assessment information. Nevertheless, due to the complicacy of the actual situation, people sometimes prefer to employ linguistic term set (LTS) to depict the qualitative assessment information in real life, for instance, the linguistic term of excellent, medium, and weak. Consequently, to better manage qualitative evaluation information, the linguistic variable (LV) [15] was introduced. Due to the advantages of LTS, many scholars have done more work to utilize linguistic variable to handle the MADM problem [16]–[27]. Considering the characteristics of both PFS and LTS, some studies regarding picture fuzzy linguistic information have also been conducted. Wei [28] defined picture uncertain linguistic set (PULS) and expanded conventional Bonferroni mean (BM) operator to MADM under picture...
uncertain linguistic environment. Wei [29], Wei et al. [30] also explored some picture 2-tuple linguistic aggregating operator in MADM problems. Liu and Zhang [31] established the notion of picture fuzzy linguistic set (PFLS), and they also developed the picture fuzzy linguistic weighted arithmetic averaging operator based on Archimedean operations (A-PFLWAA) to cope with multiple criteria group decision-making (MCGDM). Li et al. [32] proposed q-rung picture linguistic set (q-RPLS) and expanded Heronian operator to handle q-rung picture linguistic numbers (q-RPLNs) in MCGDM problems. Zeng et al. [33] extended weighted averaging, ordered weighted averaging, and hybrid averaging operators to picture linguistic environment, and discussed their properties.

Although PFS can adequately express real decision information, there are still some cases where it cannot address uncertain information appropriately. For example, people may feel hesitant when requiring for their opinions concerning the positive degree, neutral degree, negative degree, and refusal degree. Thus, picture hesitant fuzzy set (PHFS) [34] is proposed, where we can give several possible values to represent three types of membership degrees, respectively. PHFS as an extended form of PFS is also called multiple-valued picture fuzzy set (MVPFS), which can better depict the complex and uncertain information. However, it cannot represent the multiple probable membership degrees of positive, neutral, negative, and refusal of an element to a given linguistic term. For instance, several investment projects concerning different criteria need be evaluated. Suppose a project regarding the criterion of risk is weak. Decision-makers consider the possible positive membership degree about linguistic term weak is 0.6 or 0.7, the possible negative membership degree about linguistic term weak is 0.1 or 0.2, and the possible neutral membership degree on linguistic term weak is 0.1. These kinds of evaluation information are unsuitable to be represented by the existing PFS and its variations. To settle this shortcoming, we firstly define multiple-valued picture fuzzy linguistic set (MVPFLS), which is a combination of MVPFS and LTS. Therefore, the issue mentioned above can be represented by \( s_4, ((0.6, 0.7) [0.1] (0.1, 0.2)) \), where \( s_4 \) expresses the linguistic term weak.

In actual MADM problems, the aggregating operator makes an important influence in the procedure of information integration, which can consolidate multiple evaluation values to an overall result. Many aggregating operators have been proposed; for instance, bonferroni mean (BM) operator [35], muirhead mean (MM) operator [36], maclaurin symmetric mean (MSM) operator [37], weighted arithmetic averaging (WAA) operator [38], induced ordered weighted averaging (IOWA) operator [39], power averaging (PA) operator [40], weighted geometric average (WGA) operator [41], heronian mean (HM) operator [42], and so on. Meanwhile, many studies on aggregating operators have been done by scholars [43]–[53] as well. Additionally, there exist commonly related correlations among attributes in real decision-making problems. Thus, under these circumstances, the BM operator and HM operator considering the interrelationship among input values have attracted many attentions. The existing research [54] demonstrates that the HM operator is more superior to the BM operator. To our best knowledge, HM operator has been expanded to various fuzzy sets, such as IFS [55], [56], hesitant fuzzy set [57], linguistic intuitionistic fuzzy set[58], neutrosophic uncertain linguistic set [59], neutrosophic linguistic set [60], single-valued neutrosophic set [61], and picture fuzzy set [62], [63]. Due to HM operator can reflect the correlations between attributes, which commonly exist in real MADM problems. Meanwhile, the HM operator can also reflect the relationship between attribute and itself, and avoid redundant calculation of BM operator. Thus, HM operator has been widely applied to fuse information. Nevertheless, up to now, the HM aggregating operator has not been extended to handle MVPFS and MVPFLS.

Inspired by these, the major achievements of the paper are outlined as below.

1. To better express the complex uncertain decision information in real life, we initially propose the concept of MVPFLS combining PHFS with LTS. The corresponding operational rules on account of linguistic scale function are provided. Meanwhile, the score, accuracy, and certainty functions for comparing MVPFLEs are also defined.

2. The HM operator can take the relationship among input values into account. Thus, it is extended to manage multiple-valued picture fuzzy linguistic information. The multiple-valued picture fuzzy linguistic generalized weighted Heronian mean (MVPFLGWHM) operator and the multiple-valued picture fuzzy linguistic generalized weighted geometric Heronian mean (MVPFLGWHG) operator are investigated, and their desirable characteristic and particular instances are also presented.

3. A new MADM method employing the proposed operators is developed under multiple-valued picture fuzzy linguistic environment. To verify the priority and effectiveness of the novel approach, a numerical example of MADM is presented. Meanwhile, to verify the practicality of the novel approach, the sensitivity analysis is also carried out along with comparative analysis.

The arrangement of this paper is set out as follows. Some related definitions and operations are reviewed in Section 2. Section 3 gives the concepts concerning MVPFLS and MVPFLE. Meanwhile, the operational laws and comparative method are provided as well. In Section 4, the MVPFLGWHM and MVPFLGWHG operator are presented, and their characteristics are discussed. Section 5 develops a novel MADM approach utilizing the above operators with multiple-valued picture fuzzy linguistic information. A numerical example is shown, and the comparative analysis and sensitivity analysis are also performed in Section 6. Section 7 summarizes some outcomes.
II. PRELIMINARIES

Some related definitions are recalled, containing linguistic term set (LTS), linguistic scale function (LSF), picture hesitant fuzzy set (PHFS), and Heronian mean (HM) operators. These concepts will be useful in the remaining of this paper.

A. LTS

Suppose that \( S = \{s_{0}, s_{1}, \ldots, s_{2r}\} \) is an ordered discrete LTS, where \( s_{j} \) expresses a possible linguistic term and \( 2r \) is an even number. Let \( t \) be equal to three, and then the corresponding LTS is represented in the following.

\[
S = \{s_{0}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}\}
\]

\[
= \{\text{pretty poor, slightly poor, poor, medium, good, slightly good, pretty good}\}.
\]

To preserve linguistic information in the computational process, a contiguous LTS \( \tilde{S} = \{s_{j} | j \in [0, \ell]\} \) can be derived by extending LTS mentioned above, and \( \ell > 2t \) is an enough large positive integer.

**Definition 1** [64]: Let \( \tilde{s}_{i}, \tilde{s}_{j} \in \tilde{S} \) be any two LTSs, the basic operational rules are shown as below.

1. \( \lambda \tilde{s}_{i} = \tilde{s}_{j} \theta_{i} \); \( \lambda \geq 0 \);
2. \( \tilde{s}_{i} \oplus \tilde{s}_{j} = \tilde{s}_{i+j} \);
3. \( \tilde{s}_{i} \ominus \tilde{s}_{j} = \tilde{s}_{i-j} \);
4. \( \tilde{s}_{i}^{j} = \tilde{s}_{i+j} \); \( \lambda \geq 0 \);
5. \( \tilde{s}_{i} / \tilde{s}_{j} = \tilde{s}_{i+j} \); \( j \neq 0 \).

B. LINGUISTIC SCALE FUNCTION

Generally, the above operations in terms of the continuous LTS \( \tilde{S} \) directly adopt the subscript of the linguistic term, which may lead to the distortion of decision information under different circumstance. In some cases, decision-makers expect the linguistic term to be assigned different semantic results under different environments. Thus, the linguistic scale function (LSF) is introduced [65], which is a strict monotone increasing function regarding subscript of the linguistic term, and it can map the linguistic term to a crisp value between 0 and 1. The value of LSF denotes the semantic of the linguistic term.

**Definition 2** [65]: Suppose \( \tilde{S} \) be an LTS, and \( S = \{s_{j} | j \in [0, 2t]\} \). The LSF \( f \) represents the mapping from \( s_{j} \) to \( \tilde{S} \), and \( \tilde{S} \) is a value belonging to \([0, 1]\). Subsequently, the LSF \( f \) is given as below. \( f : s_{j} \rightarrow \tilde{S}(j = 0, 1, \ldots, 2t) \).

There are three types of LSF, which will be more flexible in handing linguistic information.

1. \( f_{1}(s_{j}) = \tilde{\theta}_{j} = \frac{j}{2t} (j = 0, 1, \ldots, 2t) \).
2. \( f_{2}(s_{j}) = \tilde{\theta}_{j} = \begin{cases} a^{t} - a^{t-j} & (j = 0, 1, \ldots, t) \\ \frac{2a^{t} - 2a^{t-j}}{2a^{t} - 2} & (j = 0, 1, \ldots, t) \\ a^{t} + a^{t-j} - 2 & (j = 0, 1, \ldots, t) \\ \frac{2a^{t} - 2a^{t-j}}{2a^{t} - 2} & (j = t+1, t+2, \ldots, 2t) \end{cases} \)

(3) \( f_{3}(s_{j}) = \tilde{\theta}_{j} = \begin{cases} t^{a} - (t-j)^{a} & (j = 0, 1, \ldots, t) \\ \frac{2t^{a} - 2}{2t^{a} - 2} & (j = t+1, t+2, \ldots, 2t) \end{cases} \) \( \text{A continuous function } f^{*} : \tilde{S} \rightarrow R^{+} \) is provided through expanding the LSF \( f \) mentioned above, and the corresponding inverse function for \( f^{*} \) can be obtained. For example, suppose there is an LTS \( S = \{s_{0}, s_{1}, s_{2}, s_{3}, s_{4}, s_{5}, s_{6}\} \)

\[ LTS = \{\text{pretty poor, slightly poor, poor, medium, good, slightly good, pretty good}\} \]

The inverse function \( f^{*\rightarrow} \) can be derived as follows.

1. \( f_{1}^{*}(s_{j}) = s_{j} = \frac{j}{6} (j = 0, 1, \ldots, 6) \)
2. \( f_{2}^{*}(s_{j}) = \tilde{\theta}_{j} = \begin{cases} a^{3} - a^{3-j} & (j = 0, 1, 2, 3) \\ \frac{2a^{3} - 2}{2a^{3} - 2} & (j = 4, 5, 6) \end{cases} \)
3. \( f_{3}^{*}(s_{j}) = \tilde{\theta}_{j} = \begin{cases} 3^{a} - 3^{j} & (j = 0, 1, 2, 3) \\ \frac{2}{23} & (j = 4, 5, 6) \end{cases} \)

\( \text{If LTS } S \text{ contains seven elements, } a \in [1.37, 1.4], \alpha = \beta = 0.8 \) [66, 67].

C. PICTURE HESITANT FUZZY SET

**Definition 3** [34]: Suppose that \( X \) is a combination of objects, a picture hesitant fuzzy set (PHFS) \( A \) on \( X \) can be defined:

\[ A = \{(x, \tilde{\mu}_{A}(x), \tilde{\eta}_{A}(x), \tilde{v}_{A}(x)) | x \in X\} \]

where \( \tilde{\mu}_{A}(x) = \{\mu | \mu \in \tilde{\mu}_{A}(x), \tilde{\eta}_{A}(x) = \{\eta | \eta \in \tilde{\eta}_{A}(x)\}, \)

and \( \tilde{v}_{A}(x) = \{v | v \in \tilde{v}_{A}(x)\} \) are three collections of multiple numbers in \([0, 1]\), denoting the probable positive membership degree, neutral membership degree, and negative membership degree respectively, meeting the following conditions \( 0 \leq \mu, \eta, v \leq 10 \leq \sup \tilde{\mu}_{A}(x) + \sup \tilde{\eta}_{A}(x) + \sup \tilde{v}_{A}(x) \leq 1 \), where \( \sup \tilde{\mu}_{A}(x) = \max \{\mu | \mu \in \tilde{\mu}_{A}(x)\}, \sup \tilde{\eta}_{A}(x) = \max \{\eta | \eta \in \tilde{\eta}_{A}(x)\}, \) and \( \sup \tilde{v}_{A}(x) = \max \{v | v \in \tilde{v}_{A}(x)\} \).

Supposing there is only one element in \( X, A \) can be represented by \( A = \{\tilde{\mu}_{A}(x), \tilde{\eta}_{A}(x), \tilde{v}_{A}(x)\} \), which is called a picture hesitant fuzzy element (PHFE) or a multiple-valued image fuzzy element (MVPFE).
D. HERONIAN MEAN OPERATOR

Definition 4 [54]: Let \( p, q \geq 0 \), and \( a_i (i = 1, 2, \cdots, n) \) be a collection of non-negative elements, then the generalized Heronian mean (GHM) operator is given as below.

\[
GHM^{p, q}(a_1, a_2, \cdots, a_n) = \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}}
\]

Definition 5 [55]: Let \( p, q \geq 0 \), and \( a_i (i = 1, 2, \cdots, n) \) be a collection of non-negative elements, then the generalized geometric Heronian mean (GGHM) operator is given as below.

\[
GGHM^{p, q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{i=1}^{n} \prod_{j=1}^{n} (pa_i + qa_j)^{\frac{2}{p+q}}
\]

III. MULTIPLE-VALUED PICTURE FUZZY LINGUISTIC SET

In this section, we initially propose the notions of multiple-valued picture fuzzy linguistic set (MVPFLS) and multiple-valued picture fuzzy linguistic element (MVPFLE), and develop new operational laws for MVPFLS based on the LSF. Furthermore, the compared functions are also presented to sort multiple MVPFLEs.

A. MVPFHS AND ITS OPERATIONS

Definition 6: Let \( X \) be a set of objects, an MVPFLS \( A \) for \( X \) is defined by

\[
A = \{ \langle x, \{ \theta(x), (\tilde{\mu}_A(x), \tilde{\eta}_A(x), \tilde{\upsilon}_A(x)) \} \mid x \in X \} \}
\]

where \( \theta(x) \in \tilde{S} \), \( \tilde{S} \) is a continuous LTS, \( \tilde{\mu}_A(x) = \{ \mu \mid \mu \in \tilde{\mu}_A(x) \} \), \( \tilde{\eta}_A(x) = \{ \eta \mid \eta \in \tilde{\eta}_A(x) \} \), and \( \tilde{\upsilon}_A(x) = \{ v \mid v \in \tilde{\upsilon}_A(x) \} \) are three sets of multiple numbers in \([0, 1]\), denoting the possible positive membership degree, neutral membership degree, and negative membership degree of \( x \in s_{\theta(x)} \) in \( X \). The sum of three membership degrees meets the following condition, \( 0 \leq \mu, \eta, \upsilon \leq 1 \), and \( 0 \leq \sup \tilde{\mu}_A(x) + \sup \tilde{\eta}_A(x) + \sup \tilde{\upsilon}_A(x) \leq 1 \). Meanwhile, the refusal membership degree is represented by \( \overline{\pi}_A(x) = \{ \pi \mid \pi \in \overline{\pi}_A(x) \} \) where \( \pi = 1 - \mu - \eta - \upsilon \). If the fixed set \( X \) contains only one number \( x, \{ \theta(x), (\tilde{\mu}_A(x), \tilde{\eta}_A(x), \tilde{\upsilon}_A(x)) \} \) is called an MVPFLS. Obviously, an MVPFLS is a particular case of MVPFLS.

In real situation, the computation process utilizing LSF is more reasonable than using a linguistic subscript. Since LSF can effectively depict various semantic values when dealing with linguistic information, some new operational rules of MVPFLES in terms of LSF are presented as below.

Definition 7: Let \( a_1 = \{ \theta(a_1), (\tilde{\mu}_1, \tilde{\eta}_1, \tilde{\upsilon}_1) \} \) and \( a_2 = \{ \theta(a_2), (\tilde{\mu}_2, \tilde{\eta}_2, \tilde{\upsilon}_2) \} \) be two MVPFLES, \( \lambda \geq 0 \), \( f^* \) be an LSF which can map \( s_j \) to \( \tilde{\theta}_j \), \( f^{-1} \) is a inverse function which can map \( \tilde{\theta}_j \) to \( s_j \). Then, the operations of MVPFLES are expressed as below.

\[
(1) \ a_1 \oplus a_2 = \left( f^{-1}(f^*(\theta(a_1))) + f^*(\theta(a_2)) \right) \bigcup_{\mu_1 \in \tilde{\mu}_1, \mu_2 \in \tilde{\mu}_2} \{ \mu_1 + \mu_2 - \mu_1 \mu_2 \},
\]

\[
(2) \ a_1 \otimes a_2 = \left( \{ \mu_1 \mu_2 \} \bigcup_{\nu_1 \in \tilde{\upsilon}_1, \nu_2 \in \tilde{\upsilon}_2} \{ \nu_1 \nu_2 \} \right);
\]

\[
(3) \ \lambda a_1 = \left( f^{-1}(f^*(\theta(a_1))) \right) \bigcup_{\eta_1 \in \tilde{\eta}_1} \{ 1 - (1 - \lambda)^{\eta_1} \};
\]

\[
(4) \ a_1^\lambda = \left( f^{-1}(f^*(\theta(a_1))) \right) \bigcup_{\nu_1 \in \tilde{\nu}_1} \{ (1 - \nu_1)^{\lambda} \};
\]

\[
(5) \ nega_1 = \left( f^{-1}(f^*(\theta(a_1))) - f^*(\theta(a_1)) \right) \bigcup_{\eta_1 \in \tilde{\eta}_1} \{ 1 - \eta_1 \};
\]

Obviously, the first sections of the above equations in Definition 7 are still linguistic terms, and the second sections are MVPFLES. Therefore, these operational results in Definition 7 are still MVPFLES, and the following characteristics can be derived based on the aforementioned operational rules.

Theorem 1: Assume that \( a_1 = \{ \theta(a_1), (\tilde{\mu}_1, \tilde{\eta}_1, \tilde{\upsilon}_1) \} \) and \( a_2 = \{ \theta(a_2), (\tilde{\mu}_2, \tilde{\eta}_2, \tilde{\upsilon}_2) \} \) are two arbitrary MVPFLES, and
\(\lambda_1, \lambda_2 \geq 0\), then we can derive

1. \(a_1 \ominus a_2 = a_2 \ominus a_1\);
2. \(a_1 \otimes a_2 = a_2 \otimes a_1\);
3. \(\lambda_1 (a_1 \oplus a_2) = \lambda_1 a_1 \oplus \lambda_1 a_2\);
4. \(\lambda_1 a_1 \oplus \lambda_2 a_1 = (\lambda_1 + \lambda_2) a_1\);
5. \(a_1^{\lambda_1} \otimes a_2^{\lambda_2} = a_1^{\lambda_1 + \lambda_2}\);
6. \(a_1^{\lambda_1} \otimes a_2^{\lambda_2} = (a_1 \otimes a_2)^{\lambda_1}\);

Then, we can prove the Equation (3) is right.

Proof: (3) According to Definition 7, we can get is obtained by the equation as shown at the bottom of this page.

And Therefore, the Equation (3) \(\lambda_1 (a_1 \oplus a_2) = \lambda_1 a_1 \oplus \lambda_1 a_2\) is obtained.

Equations (1)-(2) and (4)-(6) in Theorem 1 can be proved in the same way.

B. COMPARATIVE METHOD

To rank MVPFLEs, we propose the comparative functions, namely, the score, accuracy and certainty functions based on LSF, the calculation formulas are presented as follows.

\[
\begin{align*}
\lambda_1 (a_1 \oplus a_2) &= \left( f^{s-1} \left( f^{s-1} (f^s(s_0(a_1)) + f^s(s_0(a_2))) \right) \right), \\
&= \left( f^{s-1} (f^s(s_0(a_1)) + f^s(s_0(a_2))) \right), \\
&= \left( f^{s-1} (f^s(s_0(a_1)) + f^s(s_0(a_2))) \right),
\end{align*}
\]

\[
\begin{align*}
\lambda_1 a_1 \oplus \lambda_1 a_2 &= \left( f^{s-1} \left( f^{s-1} (f^s(s_0(a_1))) \right) \right), \\
&= \left( f^{s-1} \left( f^{s-1} (f^s(s_0(a_1))) \right) \right), \\
&= \left( f^{s-1} \left( f^{s-1} (f^s(s_0(a_1))) \right) \right).
\end{align*}
\]
Definition 8: Let \( a = \{s_{0(a)}, \mu_{a}, \tilde{\eta}_{a}, \tilde{\upsilon}_{a}\} \) be an MVPFLE, the score function \( E(a) \), the accuracy function \( H(a) \), and the certainty function \( C(a) \) for \( a \) are given as

\[
\begin{align*}
(1) & \quad E(a) = \frac{f^*(s_{0(a)})}{3} \left( \frac{1}{\ell_1} \sum_{\mu \in \mu_a} \mu + \frac{1}{\ell_2} \sum_{\eta \in \tilde{\eta}_a} (1 - \eta) \right) \\
& \quad + \frac{1}{\ell_3} \sum_{\gamma \in \tilde{\upsilon}_a} (1 - \gamma)
(2) & \quad H(a) = \frac{f^*(s_{0(a)})}{\ell_1} \left( \frac{1}{\ell_1} \sum_{\mu \in \mu_a} \mu + \frac{1}{\ell_2} \sum_{\eta \in \tilde{\eta}_a} \eta + \frac{1}{\ell_3} \sum_{\gamma \in \tilde{\upsilon}_a} \gamma \right)
(3) & \quad C(a) = \frac{f^*(s_{0(a)})}{\ell_1} \sum_{\mu \in \mu_a} \mu
\end{align*}
\]

where \( \ell_1, \ell_2 \), and \( \ell_3 \) represent the numbers in \( \mu_a, \tilde{\eta}_a \), and \( \tilde{\upsilon}_a \), respectively.

Obviously, for an MVPFLE \( a \), the bigger the positive membership degree \( \mu_a \) corresponding to the linguistic term \( s_{0(a)} \) is, the smaller the neural membership degree \( \tilde{\eta}_a \) and the negative membership degree \( \tilde{\upsilon}_a \) corresponding to \( s_{0(a)} \) are, the higher the MVPFLE \( a \) is. The bigger the sum of three membership degrees \( \mu_a, \tilde{\eta}_a \), and \( \tilde{\upsilon}_a \) concerning \( s_{0(a)} \) is, then the more affirmitive the statement is. The bigger the sum of three functions \( E(a), H(a) \), and \( C(a) \) are, the higher the corresponding MVPFLE \( a \) is.

According to Definition 8, the comparative method for MVPFLES is given as below.

Theorem 2: Suppose that \( a_1 = \{s_{0(a_1)}, (\mu_{a_1}, \tilde{\eta}_{a_1}, \tilde{\upsilon}_{a_1})\} \) and \( a_2 = \{s_{0(a_2)}, (\mu_{a_2}, \tilde{\eta}_{a_2}, \tilde{\upsilon}_{a_2})\} \) are any two MVPFLES, the ranking order between \( a_1 \) and \( a_2 \) can be expressed as below.

(1) if \( E(a_1) > E(a_2) \), then \( a_1 > a_2 \); 
(2) if \( E(a_1) = E(a_2) \), then,
   (a) if \( H(a_1) > H(a_2) \), then \( a_1 > a_2 \); 
   (b) if \( H(a_1) = H(a_2) \), then \( a_1 \sim a_2 \); 
(3) if \( E(a_1) = E(a_2) \) and \( H(a_1) = H(a_2) \), then,
   (a) if \( C(a_1) > C(a_2) \), then \( a_1 > a_2 \); 
   (b) if \( C(a_1) = C(a_2) \), then \( a_1 \sim a_2 \);

IV. MULTIPLE-VALUED PICTURE FUZZY LINGUISTIC GENERALIZED HERONIAN MEAN OPERATOR

In this subsection, we will extend the traditional generalized Heronian mean (GHM) operators to the multiple-valued picture fuzzy linguistic (MVPFL) environment. We propose the multiple-valued picture fuzzy linguistic generalized weighted Heronian mean (MVPFGLWHM) operator and the multiple-valued picture fuzzy linguistic generalized weighted geometric Heronian mean (MVPFGLGWM) operator to deal with MVPFL information, discuss the corresponding desirable property and special cases of two aggregating operators.

A. MVPFGLWHM OPERATOR

Definition 9: Let \( a_i = \{s_{0(a_i)}, (\mu_{a_i}, \tilde{\eta}_{a_i}, \tilde{\upsilon}_{a_i})\} \) be the set of MVPFLES, \( i = 1, 2, \cdots, n, p, q \geq 0 \), and \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) be the weight vector regarding \( a_i \), meeting the conditions \( \omega_i \in [0, 1] \), and \( \sum_{i=1}^{n} \omega_i = 1 \). Then the MVPFGLWHM operator can be calculated as below.

\[
\text{MVPFGLWHM}^{p,q}(a_1, a_2, \cdots, a_n) = \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega_i a_i)^p \otimes (\omega_j a_j)^q \right)^{\frac{1}{n+q}}
\]

In the light of Definition 7, the following theorem can be derived.

Theorem 3: Let \( a_i = \{s_{0(a_i)}, (\mu_{a_i}, \tilde{\eta}_{a_i}, \tilde{\upsilon}_{a_i})\} \) be the set of MVPFLES, \( i = 1, 2, \cdots, np, q \geq 0 \), and \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) be the weight vector of \( a_i \), meeting the conditions \( \omega_i \in [0, 1] \), and \( \sum_{i=1}^{n} \omega_i = 1 \). Then, the aggregating result by utilizing equation (1) is obtained in the following, and its aggregating result is still an MVPFLE (2), as shown at the bottom of the next page.

Proof: In the light of Definition 7, we have obtained the equation as shown at the bottom of the next page.

And, is obtained by the equation as shown at the bottom of the next page.

Then, is obtained by the equation as shown at the bottom of the page.

Furthermore, is obtained by the equation as shown at the bottom of the page.

Thus, is obtained by the equation as shown at the bottom of the page.

Therefore, is obtained by the equation as shown at the bottom of the page.

Thus, the Equation (2) in Theorem 3 is proved.

Moreover, we will prove the proposed MVPFGLWHM operator satisfies the following property.

Theorem 4 (Monotonicity): Let \( a_i = \{s_{0(a_i)}, (\mu_{a_i}, \tilde{\eta}_{a_i}, \tilde{\upsilon}_{a_i})\} \) and \( b_i = \{s_{0(b_i)}, (\mu_{b_i}, \tilde{\eta}_{b_i}, \tilde{\upsilon}_{b_i})\} \) be two collections of MVPFLES, \( i = 1, 2, \cdots, n \), if \( a_i \leq b_i \) for any \( i \), that is, \( s_{0(a_i)} \leq s_{0(b_i)}, \mu_{a_i} \leq \mu_{b_i}, \tilde{\eta}_{a_i} \geq \tilde{\eta}_{b_i}, \tilde{\upsilon}_{a_i} \geq \tilde{\upsilon}_{b_i} \). Then,

\[
\begin{align*}
\text{MVPFGLWHM}^{p,q}(a_1, a_2, \cdots, a_n) & \leq \text{MVPFGLWHM}^{p,q}(b_1, b_2, \cdots, b_n)
\end{align*}
\]

Proof:

(1) For the part of the linguistic term, 
Since \( s_{0(a_i)} \leq s_{0(b_i)} \) for all \( i, p, q \geq 0 \), \( f^*, f^{*-1} \) are continuous and rigid monotony increasing functions, then we can get 

\[
\begin{align*}
& \quad \text{MVPFGLWHM}^{p,q}(a_1, a_2, \cdots, a_n) \\
& \quad \leq \text{MVPFGLWHM}^{p,q}(b_1, b_2, \cdots, b_n)
\end{align*}
\]
\[
\Rightarrow (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \\
\leq (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \\
\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \\
\leq \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \\
\Rightarrow \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \\
\leq \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \\
\Rightarrow \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \right)^{1/p+q}
\]

MVPFLGWHM\(^{p,q}(a_1, a_2, \ldots, a_n) = \left\{ f^{-1} \left( \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \right)^{1/p+q} \right) \right\},

\bigcup_{\mu_i \in \bar{\mu}_{a_i} \atop \mu_j \in \bar{\mu}_{a_j} \atop \eta_i \in \bar{\eta}_{a_i} \atop \eta_j \in \bar{\eta}_{a_j} \atop \gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ \left( 1 - \prod_{i=1}^{n} (1 - (1 - (1 - (1 - \mu_i)^{\eta_i})^p (1 - (1 - \mu_j)^{\eta_j})^q)^{\frac{2}{\mu_i+\mu_j}} \right) \right\},

\bigcup_{\mu_i \in \bar{\mu}_{a_i} \atop \mu_j \in \bar{\mu}_{a_j} \atop \eta_i \in \bar{\eta}_{a_i} \atop \eta_j \in \bar{\eta}_{a_j} \atop \gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ \left( 1 - \prod_{i=1}^{n} (1 - (1 - (1 - \gamma_i)^{\eta_i})^p (1 - (1 - \gamma_j)^{\eta_j})^q)^{\frac{1}{\mu_i+\mu_j}} \right) \right\},

\bigcup_{\mu_i \in \bar{\mu}_{a_i} \atop \mu_j \in \bar{\mu}_{a_j} \atop \eta_i \in \bar{\eta}_{a_i} \atop \eta_j \in \bar{\eta}_{a_j} \atop \gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ \left( 1 - \prod_{i=1}^{n} (1 - (1 - (1 - \gamma_i)^{\eta_i})^p (1 - (1 - \gamma_j)^{\eta_j})^q)^{\frac{1}{\mu_i+\mu_j}} \right) \right\},

(\omega a_i)^p = \left\{ f^{-1} \left( \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \right)^{1/p+q} \right) \right\},

(\omega a_j)^q = \left\{ f^{-1} \left( \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \right)^{1/p+q} \right) \right\},

(\omega a_i)^p \otimes (\omega a_j)^q = \left\{ f^{-1} \left( \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \right)^{1/p+q} \right) \right\},

(\omega a_j)^q \mathbin{\otimes} (\omega a_j)^q = \left\{ f^{-1} \left( \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p (\omega f^* (s_{(a_j)}))^q \right)^{1/p+q} \right) \right\},

\bigcup_{\eta_i \in \bar{\eta}_{a_i} \atop \eta_j \in \bar{\eta}_{a_j} \atop \gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ (1 - (1 - (1 - \eta_i)^{\eta_i})^p (1 - (1 - \eta_j)^{\eta_j})^q)^{\frac{1}{\mu_i+\mu_j}} \right\},

\bigcup_{\eta_i \in \bar{\eta}_{a_i} \atop \eta_j \in \bar{\eta}_{a_j} \atop \gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ (1 - (1 - (1 - \gamma_i)^{\eta_i})^p (1 - (1 - \gamma_j)^{\eta_j})^q)^{\frac{1}{\mu_i+\mu_j}} \right\},

\bigcup_{\gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ (1 - (1 - (1 - \eta_i)^{\eta_i})^p (1 - (1 - \eta_j)^{\eta_j})^q)^{\frac{1}{\mu_i+\mu_j}} \right\},

\bigcup_{\gamma_i \in \bar{\gamma}_{a_i} \atop \gamma_j \in \bar{\gamma}_{a_j}} \left\{ (1 - (1 - (1 - \gamma_i)^{\eta_i})^p (1 - (1 - \gamma_j)^{\eta_j})^q)^{\frac{1}{\mu_i+\mu_j}} \right\}.
Thus, the linguistic part is right.

(2) For positive membership part,
Since $\mu_{a_i} \leq \tilde{\mu}_b$ for all $i$, then
$$\mu_{a_i} \leq \mu_{b_j}, \quad \tilde{\mu}_a \leq \mu_{b_j} \Rightarrow 1 - \mu_{a_i} \geq 1 - \mu_{b_j}, \quad 1 - \mu_{a_i} \geq 1 - \mu_{b_j} \Rightarrow (1 - \mu_{a_i})^{\omega_a} \geq (1 - \mu_{b_j})^{\omega_a}, \quad (1 - \mu_{a_i})^{\omega_b} \geq (1 - \mu_{b_j})^{\omega_b} \Rightarrow 1 - (1 - \mu_{a_i})^{\omega_b} \leq 1 - (1 - \mu_{b_j})^{\omega_b},$$
$$1 - (1 - \mu_{a_i})^{\omega_a} \leq 1 - (1 - \mu_{b_j})^{\omega_a} \Rightarrow (1 - (1 - \mu_{a_i})^{\omega_a})^p \leq (1 - (1 - \mu_{b_j})^{\omega_a})^p, \quad \Rightarrow (1 - (1 - \mu_{a_i})^{\omega_b})^q \leq (1 - (1 - \mu_{b_j})^{\omega_b})^q \Rightarrow (1 - (1 - \mu_{a_i})^{\omega_b})^q (1 - (1 - \mu_{a_i})^{\omega_b})^q \leq (1 - (1 - \mu_{b_j})^{\omega_b})^q (1 - (1 - \mu_{b_j})^{\omega_b})^q$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (\omega a_i)^p \otimes (\omega j a_j)^q = \left( f^{n+1} - \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p \otimes (\omega f^* (s_{(a_j)}))^q \right) \right),$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - \prod_{j=1}^{n} \right) \left( 1 - (1 - (1 - \mu_{a_i})^{\omega a} (1 - (1 - \mu_{j})^{\omega a})^q \right) \right| \right| \right.,$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - \eta_{i}^{\omega j} \right) (1 - \eta_{j}^{\omega j}) \right) \right| \right| \right.,$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - (1 - v_{i}^{\omega j}) (1 - v_{j}^{\omega j}) \right) \right) \right| \right| \right.,$$
$$\left( f^{n+1} - \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p \otimes (\omega f^* (s_{(a_j)}))^q \right) \right),$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - \prod_{j=1}^{n} \right) \left( 1 - (1 - (1 - \mu_{a_i})^{\omega a} (1 - (1 - \mu_{j})^{\omega a})^q \right) \right) \right| \right| \right.,$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - \eta_{i}^{\omega j} \right) (1 - \eta_{j}^{\omega j}) \right) \right| \right| \right.,$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - (1 - v_{i}^{\omega j}) (1 - v_{j}^{\omega j}) \right) \right) \right| \right| \right.,$$
$$\left( f^{n+1} - \left( \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega f^* (s_{(a_i)}))^p \otimes (\omega f^* (s_{(a_j)}))^q \right) \right),$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - \prod_{j=1}^{n} \right) \left( 1 - (1 - (1 - \mu_{a_i})^{\omega a} (1 - (1 - \mu_{j})^{\omega a})^q \right) \right) \right| \right| \right.,$$
$$\left\{ \left. \prod_{i=1}^{n} \left( 1 - \eta_{i}^{\omega j} \right) (1 - \eta_{j}^{\omega j}) \right) \right| \right| \right.,$$
\[ \left( 1 - (1 - (1 - \mu_a))^{\eta_a} \right)^{\rho} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{q} \leq \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{\rho} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{q} \]

\[ \geq \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{\rho} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{q} \frac{2}{p+q} \]

\[ \left( \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{\rho} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{q} \frac{2}{p+q} \right)^{1} = \left( \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{\rho} \left( 1 - (1 - \mu_a)^{\eta_a} \right)^{q} \frac{2}{p+q} \right)^{1} \]

Therefore, positive membership is right.

For the part of neutral membership,

Since \( \tilde{\eta}_a \geq \tilde{\eta}_b \) for all \( i \), then

\[ \eta_a \geq \eta_b, \quad \eta_a \geq \eta_b \Rightarrow \eta_a^0 \geq \eta_b^0, \quad \eta_a^0 \geq \eta_b^0 \]

\[ \Rightarrow 1 - \eta_a^0 \leq 1 - \eta_b^0, \quad 1 - \eta_a^0 \leq 1 - \eta_b^0 \]

\[ \Rightarrow (1 - \eta_a^0)^p \leq (1 - \eta_b^0)^p, \quad (1 - \eta_a^0)^q \leq (1 - \eta_b^0)^q \]

\[ \Rightarrow (1 - \eta_a^0)^p \left( 1 - \eta_a^0 \right)^q \leq (1 - \eta_b^0)^p \left( 1 - \eta_b^0 \right)^q \]

\[ \Rightarrow (1 - \eta_a^0)^p \left( 1 - \eta_a^0 \right)^q \leq (1 - \eta_b^0)^p \left( 1 - \eta_b^0 \right)^q \]

\[ \geq \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \eta_a^0)^p \left( 1 - \eta_a^0 \right)^q \right) \frac{2}{p+q} \]

\[ \geq \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \eta_a^0)^p \left( 1 - \eta_a^0 \right)^q \right) \frac{2}{p+q} \]

Therefore, neutral membership is right.

Consequently, the part of negative membership is proved as the same way of the neutral membership part, that is,

\[ \Rightarrow 1 - \left( \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \gamma_a^0)^p \left( 1 - \gamma_a^0 \right)^q \right) \frac{2}{p+q} \right)^{1} \]

\[ \geq 1 - \left( \prod_{i=1}^{n} \prod_{j=1}^{n} \left( 1 - (1 - \gamma_a^0)^p \left( 1 - \gamma_a^0 \right)^q \right) \frac{2}{p+q} \right)^{1} \]
Therefore,  
\[
MVPFLGWHM^{p,q}(a_1, a_2, \cdots, a_n) \leq MVPFLGWHM^{p,q}(b_1, b_2, \cdots, b_n)
\]

Furthermore, some particular instances on MVPFLGWHM operator assigned various \( p \) and \( q \) are investigated as below.

1. When \( p = 0 \), then 
\[
MVPFLGWHM^{0,q}(a_1, a_2, \cdots, a_n) = \left( f^{-1} \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (\omega_j f^*(s_{\theta(a_j)}))^{q} \right)^{\frac{1}{q}} \right).
\]

2. When \( q = 0 \), then 
\[
MVPFLGWHM^{p,0}(a_1, a_2, \cdots, a_n) = \left( f^{-1} \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (\omega_j f^*(s_{\theta(a_j)}))^{p} \right)^{\frac{1}{p}} \right).
\]

3. When \( p = q = 1 \), then 
\[
MVPFLGWHM^{1,1}(a_1, a_2, \cdots, a_n) = \left( f^{-1} \left( \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (\omega_j f^*(s_{\theta(a_j)})) \right)^\frac{1}{p} \right) \otimes \left( \omega_j f^*(s_{\theta(a_j)}) \right)^\frac{1}{q}.
\]

\[\bigcup_{\mu_i \in \bar{\mu}_a} \left\{ 1 - \prod_{i=1}^{n} \left( 1 - (1 - \mu_i)^{\omega_i} \right)^{\frac{2}{n(n+1)}} \right\} \bigcup_{\eta_i \in \bar{\eta}_a} \left\{ 1 - \prod_{i=1}^{n} \left( 1 - (1 - \eta_i)^{\omega_i} \right)^{\frac{2}{n(n+1)}} \right\} \bigcup_{\gamma_i \in \bar{\gamma}_a} \left\{ 1 - \prod_{i=1}^{n} \left( 1 - (1 - \gamma_i)^{\omega_i} \right)^{\frac{2}{n(n+1)}} \right\}.
\]

B. MVPFLGWHM OPERATOR

Definition 10: Let \( a_i = \{s_{\theta(a_i)}, (\bar{\mu}_a, \bar{\eta}_a, \bar{\nu}_a)\} \) be the set of MVPFLEs, \( i = 1, 2, \cdots, n, p, q \geq 0 \), and \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) be the weight vector of \( a_i \), meeting the conditions \( \omega_i \in [0, 1] \), and \( \sum_{i=1}^{n} \omega_i = 1 \). Then the MVPFLGWHM operator can be calculated as below.

\[
MVPFLGWHM^{p,q}(a_1, a_2, \cdots, a_n) = \frac{1}{p+q} \prod_{i=1}^{n} \left( (p \omega_i)^{a_0} \oplus (q \omega_i)^{a_0} \right)^{\frac{2}{n(n+1)}}
\]

In the light of Definition 7, the following theorem can be derived.

Theorem 5: Let \( a_i = \{s_{\theta(a_i)}, (\bar{\mu}_a, \bar{\eta}_a, \bar{\nu}_a)\} \) be the set of MVPFLEs, \( i = 1, 2, \cdots, np, q \geq 0 \), and \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) be the weight vector of \( a_i \), meeting conditions \( \omega_i \in [0, 1] \), and \( \sum_{i=1}^{n} \omega_i = 1 \). Then, the aggregating value by utilizing Equation (3) is obtained in the following, and its aggregating result is still an MVPFLE (4), as shown at the bottom of the next page.

Proof: In the light of Definition 7, we have 
\[
p_{\omega_i} = f_{\omega_i}^{-1} \left( (p \omega_i)^{a_0} \right),
\]

\[
\bigcup_{\mu_i \in \bar{\mu}_a} \left\{ 1 - (1 - \mu_i)^{p} \right\} \bigcup_{\eta_i \in \bar{\eta}_a} \left\{ \eta_i^{p} \right\} \bigcup_{\nu_i \in \bar{\nu}_a} \left\{ \nu_i^{p} \right\}
\]
\[ q_{aj} = \left\{ f^{s-1} \left( qf^* (s_{\theta(a_j)}) \right) \right\}, \]

\[
= \left( \bigcup_{\mu_j \in \mu_{aj}} \left\{ 1 - (1 - \mu_j)^q \right\}, \bigcup_{\eta_j \in \eta_{aj}} \left\{ \eta_j^q \right\}, \bigcup_{v_j \in v_{aj}} \left\{ v_j^q \right\} \right) \]

And,

\[
(p_{ai})^{o_j} = \left\{ f^{s-1} \left( (pf^* (s_{\theta(a_j)}) \right) \right\}, \]

\[
= \left( \bigcup_{\mu_i \in \mu_{ai}} \left\{ 1 - (1 - \mu_i)^p \right\}, \bigcup_{\eta_i \in \eta_{ai}} \left\{ 1 - (1 - \eta_i)^q \right\}, \bigcup_{\nu_i \in v_{ai}} \left\{ 1 - (1 - v_i)^q \right\} \right) \]

\[
(qa_j)^{o_j} = \left\{ f^{s-1} \left( (qf^* (s_{\theta(a_j)}) \right) \right\}, \]

\[
= \left( \bigcup_{\mu_j \in \mu_{aj}} \left\{ 1 - (1 - \mu_j)^q \right\}, \bigcup_{\eta_j \in \eta_{aj}} \left\{ 1 - (1 - \eta_j)^q \right\}, \bigcup_{v_j \in v_{aj}} \left\{ 1 - (1 - v_j)^q \right\} \right) \]

Then,

\[
(p_{ai})^{o_j} \oplus (qa_j)^{o_j} = \left\{ f^{s-1} \left( (pf^* (s_{\theta(a_j)}) \right) + (qf^* (s_{\theta(a_j)}) \right) \right\}, \]

Furthermore,

\[
((p_{ai})^{o_j} \oplus (qa_j)^{o_j})^{\frac{2}{n+1}} = \left\{ f^{s-1} \left( (pf^* (s_{\theta(a_j)}) \right) + (qf^* (s_{\theta(a_j)}) \right) \right\}^{\frac{2}{n+1}}, \]

\[
= \left( \bigcup_{\mu_i \in \mu_{ai}} \left\{ 1 - (1 - (1 - \mu_i)^p)^{o_j} \right\}, \bigcup_{\eta_i \in \eta_{ai}} \left\{ 1 - (1 - (1 - \eta_i)^q)^{o_j} \right\}, \bigcup_{\nu_i \in v_{ai}} \left\{ 1 - (1 - (1 - v_i)^q)^{o_j} \right\} \right) \]

\[
MVPFLGWGHM^{p,q}(a_1, a_2, \cdots, a_n) \]

\[
= \left( f^{s-1} \left( \frac{1}{p + q} \prod_{i=1}^{n} \prod_{j=1}^{n} \left( (pf^* (s_{\theta(a_j)}) \right) + (qf^* (s_{\theta(a_j)}) \right) \right) \right) \]

\[
\left\{ \bigcup_{\mu_i \in \mu_{ai}} \left\{ 1 - \left( \prod_{i=1}^{n} (1 - (1 - (1 - \mu_i)^p)^{o_j} \right) \left( 1 - (1 - (1 - \mu_j)^q)^{o_j} \right) \right) \right\}^{\frac{1}{n+1}} \right\}, \]

\[
\left\{ \bigcup_{\eta_i \in \eta_{ai}} \left\{ 1 - \left( \prod_{i=1}^{n} (1 - (1 - \eta_i^p)^{o_j} \right) \left( 1 - (1 - \eta_j^q)^{o_j} \right) \right) \right\}^{\frac{1}{n+1}} \right\}, \]

\[
\left\{ \bigcup_{\nu_i \in v_{ai}} \left\{ 1 - \left( \prod_{i=1}^{n} (1 - (1 - v_i^p)^{o_j} \right) \left( 1 - (1 - v_j^q)^{o_j} \right) \right) \right\}^{\frac{1}{n+1}} \right\} \right) \]
Thus, Equation (4) in Theorem 5 is proved.

Moreover, the proposed operator of MVPFLGWGHM possesses the characteristic of monotonicity.

**Theorem 6 (Monotonicity):** Let \( a_i = \{\tilde{s}(a_i), \tilde{\mu}_a, \tilde{\eta}_a, \tilde{\nu}_a\} \) and \( b_i = \{\tilde{s}(b_i), \tilde{\mu}_b, \tilde{\eta}_b, \tilde{\nu}_b\} \) be two collections of MVPFLEs, \( i = 1, 2, \cdots, n \), if \( a_i \leq b_i \) for any \( i \), that is, \( s_{\tilde{a}(a_i)} \leq s_{\tilde{b}(b_i)} \), \( \tilde{\mu}_a \leq \tilde{\mu}_b \), \( \tilde{\eta}_a \geq \tilde{\eta}_b \), \( \tilde{\nu}_a \geq \tilde{\nu}_b \). Then,

\[
\text{MVPFLGWGHM}^{p, q}(a_1, a_2, \cdots, a_n) \leq \text{MVPFLGWGHM}^{p, q}(b_1, b_2, \cdots, b_n)
\]

Similar to Theorem 4, the proof of Theorem 6 can be obtained in the same way. Thus, the procedure is omitted.

Furthermore, some particular instances regarding the operator of MVPFLGWGHM assigned various \( p \) and \( q \) are investigated.

(1) When \( p = 0 \), then

\[
\text{MVPFLGWGM}^{0, q}(a_1, a_2, \cdots, a_n)
\]
When \( q = 0 \), then

\[
\text{MVPFLGWHM}^{p,0}(a_1, a_2, \cdots, a_n) = \left\{ f^{-1} \left( \frac{1}{p} \sum_{i=1}^{m} \left( \left( f^*(s_{0}(a_i)) \right)^{\eta} \right)^{\frac{2}{m+n+1}} \right) \right\}
\]

(3) When \( p = q = 1 \), then

\[
\text{MVPFLGWHM}^{1,1}(a_1, a_2, \cdots, a_n) = \left\{ f^{-1} \left( \frac{1}{2} \prod_{i=1}^{n} \left( \left( f^*(s_{0}(a_i)) \right)^{\eta} + f^*(s_{0}(a_j))^{\eta} \right)^{\frac{2}{m+n+1}} \right) \right\}
\]

**V. MADM APPROACH UTILIZING PROPOSED OPERATORS**

In this subsection, a typical MADM method under MVPFL environment is constructed, and the proposed MVPFGLGWHM operator and MVPFGLGWHGM operator are applied to cope with MVPFL value in the procedure of decision-making.

Suppose that there are \( m \) alternatives expressed as \( A = \{A_1, A_2, \cdots, A_m\} \) and \( n \) attributes represented as \( C = \{C_1, C_2, \cdots, C_n\} \), and \( \omega = (\omega_1, \omega_2, \cdots, \omega_n) \) is the corresponding weight vector concerning the set of attributes \( C \), meeting the following conditions, \( \omega_j \in [0, 1] \), and \( \sum_{j=1}^{n} \omega_j = 1 \).

To better express the qualitative and quantitative information, meanwhile, consider the hesitancy of people, the assessment value \( a_{ij} \) indicating alternative \( A_i \) on attribute \( C_j \) is provided in the form of MVPFLE by decision-maker. Then, the original MVPFL decision matrix can be denoted by \( R = [a_{ij}]_{m \times n} = [s_{0}(a_{ij}), \tilde{\eta}_{a_{ij}}, s_{\tilde{\eta}_{a_{ij}}}, \tilde{\eta}_{a_{ij}}]_{m \times n} \), where \( \tilde{\eta}_{a_{ij}} \) represents the set of positive membership degrees for alternative \( A_i \) concerning attribute \( C_j \) to the linguistic term value \( s_{0}(a_{ij}) \), \( \tilde{\eta}_{a_{ij}} \) represents the set of neutral membership degrees for alternative \( A_i \) concerning attribute \( C_j \) to the linguistic term value \( s_{0}(a_{ij}) \), \( \tilde{\eta}_{a_{ij}} \) represents the set of negative membership degrees for alternative \( A_i \) concerning attribute \( C_j \) to the linguistic term value \( s_{0}(a_{ij}) \).

The procedure of developed algorithm is provided in Figure 1, and the main steps of alternative selection for the above MADM method are derived as follows.

**FIGURE 1. The procedure of developed algorithm.**

1. Construct multiple-valued picture fuzzy linguistic decision matrix
2. Normalize the decision matrix
3. Aggregate the collective value using MVPFGLGWHM operator or MVPFGLGWHGM operator
4. Compute comparative function value of each alternative
5. Rank the alternatives

**Step 1:** The MVPFL decision matrix is normalized.

There are commonly two kinds of attributes in MADM problem, namely, the minimizing attribute and maximizing attribute. To remove the influence of different sorts of attributes, we can transform the minimizing attribute into the maximizing attribute. That is, the original MVPFL evaluation matrix \( R = [a_{ij}]_{m \times n} \) should be standardized to the
normalized evaluation matrix \( B = [b_{ij}]_{m \times n} \) as shown in the following.

\[
b_{ij} = \begin{cases} 
\langle \theta(a_{ij}), (\mu_{a_{ij}}, \tilde{v}_{a_{ij}}) \rangle & \text{for maximizing} \\
\langle 2\theta(a_{ij}), (\tilde{v}_{a_{ij}}, 1 - \tilde{\eta}_{a_{ij}}, \tilde{\mu}_{a_{ij}}) \rangle & \text{for minimizing} 
\end{cases}
\]

**Step 2:** The collective value for each alternative is aggregated.

In order to obtain the collective evaluation value \( b_i(i = 1, 2, \ldots, m) \) of each alternative \( A_i(i = 1, 2, \ldots, m) \), we can utilize the MVPFLGWHM operator and MVPFLGWGHM operator to fuse the MVPFLEs of the standardized matrix \( B = [b_{ij}]_{m \times n} \) as follows.

\[
b_i = MVPFLGWHM^{p,q} (b_{1i}, b_{2i}, \cdots, b_{ni}) \\
\times (i = 1, 2, \cdots, m)
\]

**Step 3:** The comparison function value is computed.

In terms of the equations given in Definition 8, the score value \( E(b_i) \), the accuracy value \( H(b_i) \), and the certainty value \( C(b_i) \) for the collective value \( b_i(i = 1, 2, \cdots, m) \) can be computed, respectively.

**Step 4:** The alternatives are ranked.

In terms of the comparative method given in Theorem 2, all the alternatives can be ranked on the basis of \( E(b_i) \), \( H(b_i) \), and \( C(b_i) \) \( (i = 1, 2, \cdots, m) \), then the optimal alternative(s) will be gained.

**VI. AN EXAMPLE AND RESULT ANALYSIS**

In this subsection, we will employ an example provided by Wang and Li [34] to validate the effectiveness and powerfulness of the developed MADM method.

A large company decides to choose an appropriate vendor of ERP system. After evaluating the real situation of the market, there are five probable ERP vendors taken into account, and they are \( A_i \) \( (i = 1, 2, 3, 4, 5) \). Four factors are affecting the vendor selection, and they are \( C_j \) \( (j = 1, 2, 3, 4) \), where \( C_1 \) denotes function and technology, \( C_2 \) denotes strategic adaptability, \( C_3 \) denotes vendor’s ability, \( C_4 \) denotes vendor’s reputation. The corresponding weight is \( \omega = (0.2, 0.1, 0.3, 0.4) \). Since the actual problem is complex, the decision-makers can use the form of MVPFL to describe their evaluation information regarding the alternative \( A_i \) \( (i = 1, 2, 3, 4, 5) \) under the attribute \( C_j \) \( (j = 1, 2, 3, 4) \) with the LTS

\[
S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\} = \{\text{pretty poor, slightly poor, poor, medium, good, slightly good, poor good}\}
\]

The assessment values taking the form of PHFE are provided by Wang and Li [34], and we add the linguistic term part to convey qualitative information better. Subsequently, the multiple-valued picture fuzzy linguistic evaluation matrix \( R = [a_{ij}]_{5 \times 4} \) is provided as shown at the bottom of this page.

**A. IMPLEMENTATION**

**Step 1:** Since four attributes belong to the maximizing types, the initial evaluation matrix \( R = [a_{ij}]_{5 \times 4} \) does not need to be transformed.

**Step 2:** The collective value \( a_{ij} \) for alternative \( A_i \) \( (i = 1, 2, 3, 4, 5) \) can be calculated by the MVPFLGWHM operator in Equation (2). Assume \( p = q = 1, f = f_1^1 \), we can derive

\[
a_1 = (0.07127, \langle 0.01077, 0.1253, 0.1163, 0.1335 \rangle, \langle 0.7884, 0.8016, 0.7976, 0.8110 \rangle, \langle 0.4774, 0.4929, 0.4879, 0.5035 \rangle)
\]
Step 3: The score function value \( E(a_i) \) concerning the collective value \( a_i \) is computed according to the formula in Definition 8. Then, we can obtain the results

\[
E(a_1) = 0.0329, \quad E(a_2) = 0.0429, \quad E(a_3) = 0.0605, \\
E(a_4) = 0.0483, \quad E(a_5) = 0.0403.
\]

Step 4: The five alternatives are compared based on the comparative method in Theorem 2. Then, we can get the ranking order shown below.

\[ A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1 \]

Therefore, the best ERP system vendor utilizing MVPFLGWHM operator is \( A_3 \), and the worst is \( A_1 \).

In step 2, the collective value \( a_i \) for alternative \( A_i (i = 1, 2, 3, 4, 5) \) can be calculated using the MVPFLGWGHM operator in Equation (4). Let \( p = q = 1, f^* = f_i^* \), we can derive is obtained by the equation as shown at the bottom of this page.

Furthermore, the score function value \( E(a_i) (i = 1, 2, 3, 4, 5) \) is calculated and presented as below.

\[
E(a_1) = 0.6811, \quad E(a_2) = 0.7193, \quad E(a_3) = 0.7692, \\
E(a_4) = 0.7005, \quad E(a_5) = 0.6848.
\]

Therefore, the ranking results utilizing MVPFLGWGHM operator is \( A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1 \), and the best ERP system vendor is \( A_3 \), and the worst is \( A_1 \).

B. Sensitivity Analysis

The parameters \( p, q \) and the linguistic scale function \( f^* \) may impact ranking orders. In this part, we will explore the effect with different \( f^* \) and parameters \( p, q \). The ranking orders based on different values of \( p, q \) and LSF \( f^* \) by utilizing MVPFLGWHM operator and MVPFLGWGHM operator are listed in Tables 1-6.

From Tables 1-6, it is obvious that the score function values may vary concerning different parameters \( p, q \) or LSF \( f^* \) by utilizing the MVPFLGWHM operator and MVPFLGWGHM operator, and the ranking orders of all alternatives may be various. The values of parameters \( p, q \) and the selection of
LSF $f^*$ can be given by decision-makers in terms of their preferences and actual semantic environment, which illustrates the flexibility of the proposed operators. Meanwhile, it is easily found that the optimal ERP system vendor is always $A_3$ no matter what the aggregating operators are, which demonstrates the constancy of the proposed method.

The sorting results based on different LSF $f^*$ by using MVPFLGWHM operator are presented in Tables 1-3, which illustrates $A_3$ is always the optimal ERP vendor, and $A_1$ or $A_2$ is the worst ERP vendor. No matter what the LSF $f^*$ is, alternative $A_1$ is always the worst vendor excepting the situation where $q$ is assigned to 0 and $p$ is assigned to 0.5 or 1. It can be observed from Table 1 the score function values of alternative $A_i$ $(i = 1, 2, 3, 4, 5)$ increase with the increasing of the parameter $p$ when $q$ is assigned to a fixed value, whereas $p = 0, q = 1$ is an abnormal

| $p, q$ | $E(a_i)$ $(i = 1, 2, 3, 4, 5)$ | Ranking result |
|------|-----------------------------|-----------------|
| $p = 0.5, q = 0$ | $E(a_1) = 0.0252, E(a_2) = 0.0207, E(a_3) = 0.0411, E(a_4) = 0.0320$ | $A_3 > A_4 > A_5 > A_1$ |
| $p = 1, q = 0$ | $E(a_1) = 0.0264, E(a_2) = 0.0261, E(a_3) = 0.0464, E(a_4) = 0.0318$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 2, q = 0$ | $E(a_1) = 0.0292, E(a_2) = 0.0379, E(a_3) = 0.0574, E(a_4) = 0.0407$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 5, q = 0$ | $E(a_1) = 0.0400, E(a_2) = 0.0641, E(a_3) = 0.0887, E(a_4) = 0.0656$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 10, q = 0$ | $E(a_1) = 0.0546, E(a_2) = 0.0869, E(a_3) = 0.1179, E(a_4) = 0.0864$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 0, q = 1$ | $E(a_1) = 0.0397, E(a_2) = 0.0596, E(a_3) = 0.0721, E(a_4) = 0.0446$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 0.5, q = 1$ | $E(a_1) = 0.0340, E(a_2) = 0.0433, E(a_3) = 0.0606, E(a_4) = 0.0394$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 2, q = 1$ | $E(a_1) = 0.0339, E(a_2) = 0.0498, E(a_3) = 0.0671, E(a_4) = 0.0464$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 5, q = 1$ | $E(a_1) = 0.0431, E(a_2) = 0.0702, E(a_3) = 0.0931, E(a_4) = 0.0798, E(a_5) = 0.0667$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = 10, q = 1$ | $E(a_1) = 0.0558, E(a_2) = 0.0901, E(a_3) = 0.1185, E(a_4) = 0.0932, E(a_5) = 0.0852$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = q = 0.5$ | $E(a_1) = 0.0300, E(a_2) = 0.0324, E(a_3) = 0.0506, E(a_4) = 0.0334$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = q = 1$ | $E(a_1) = 0.0329, E(a_2) = 0.0428, E(a_3) = 0.0605, E(a_4) = 0.0303$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = q = 2$ | $E(a_1) = 0.0391, E(a_2) = 0.0609, E(a_3) = 0.0792, E(a_4) = 0.0604$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = q = 5$ | $E(a_1) = 0.0543, E(a_2) = 0.0894, E(a_3) = 0.1144, E(a_4) = 0.0811$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
| $p = q = 10$ | $E(a_1) = 0.0618, E(a_2) = 0.1077, E(a_3) = 0.1367, E(a_4) = 0.0975$ | $A_3 > A_4 > A_5 > A_2 > A_1$ |
TABLE 2. Ranking based on $f^*_2$ with MVPFLGWGHM operator.

| $p, q$  | $E\left(a_i\right)\left( i = 1, 2, 3, 4, 5\right)$ | Ranking result |
|---------|--------------------------------|----------------|
| $p = 0.5, q = 0$ | $E(a_1) = 0.0255, E(a_2) = 0.0216, E(a_3) = 0.0386,$ $E(a_4) = 0.0302, E(a_5) = 0.0285$ | $A_3 > A_4 > A_5 > A_1 > A_2$ |
| $p = 1, q = 0$ | $E(a_1) = 0.0268, E(a_2) = 0.0262, E(a_3) = 0.0437,$ $E(a_4) = 0.0350, E(a_5) = 0.0319$ | $A_3 > A_4 > A_5 > A_1 > A_2$ |
| $p = 2, q = 0$ | $E(a_1) = 0.0298, E(a_2) = 0.0361, E(a_3) = 0.0540,$ $E(a_4) = 0.0454, E(a_5) = 0.0398$ | $A_3 > A_1 > A_5 > A_2 > A_1$ |
| $p = 5, q = 0$ | $E(a_1) = 0.0404, E(a_2) = 0.0591, E(a_3) = 0.0827,$ $E(a_4) = 0.0723, E(a_5) = 0.0612$ | $A_3 > A_1 > A_5 > A_2 > A_1$ |
| $p = 10, q = 0$ | $E(a_1) = 0.0546, E(a_2) = 0.0802, E(a_3) = 0.1097,$ $E(a_4) = 0.0963, E(a_5) = 0.0797$ | $A_2 > A_4 > A_2 > A_1 > A_4$ |
| $p = 0, q = 1$ | $E(a_1) = 0.0407, E(a_2) = 0.0560, E(a_3) = 0.0681,$ $E(a_4) = 0.0515, E(a_5) = 0.0424$ | $A_3 > A_4 > A_2 > A_4 > A_1$ |
| $p = 0.5, q = 1$ | $E(a_1) = 0.0348, E(a_2) = 0.0416, E(a_3) = 0.0572,$ $E(a_4) = 0.0446, E(a_5) = 0.0381$ | $A_3 > A_4 > A_2 > A_4 > A_1$ |
| $p = 2, q = 1$ | $E(a_1) = 0.0346, E(a_2) = 0.0466, E(a_3) = 0.0631,$ $E(a_4) = 0.0521, E(a_5) = 0.0443$ | $A_3 > A_4 > A_2 > A_4 > A_1$ |
| $p = 5, q = 1$ | $E(a_1) = 0.0434, E(a_2) = 0.0648, E(a_3) = 0.0867,$ $E(a_4) = 0.0742, E(a_5) = 0.0620$ | $A_3 > A_4 > A_2 > A_5 > A_1$ |
| $p = 10, q = 1$ | $E(a_1) = 0.0559, E(a_2) = 0.0831, E(a_3) = 0.1103,$ $E(a_4) = 0.0957, E(a_5) = 0.0786$ | $A_3 > A_4 > A_2 > A_5 > A_1$ |
| $p = q = 0.5$ | $E(a_1) = 0.0306, E(a_2) = 0.0320, E(a_3) = 0.0477,$ $E(a_4) = 0.0370, E(a_5) = 0.0328$ | $A_3 > A_4 > A_2 > A_4 > A_1$ |
| $p = q = 1$ | $E(a_1) = 0.0336, E(a_2) = 0.0410, E(a_3) = 0.0570,$ $E(a_4) = 0.0454, E(a_5) = 0.0389$ | $A_3 > A_4 > A_2 > A_4 > A_1$ |
| $p = q = 2$ | $E(a_1) = 0.0398, E(a_2) = 0.0567, E(a_3) = 0.0742,$ $E(a_4) = 0.0615, E(a_5) = 0.0513$ | $A_3 > A_4 > A_2 > A_4 > A_1$ |
| $p = q = 5$ | $E(a_1) = 0.0545, E(a_2) = 0.0824, E(a_3) = 0.1065,$ $E(a_4) = 0.0917, E(a_5) = 0.0750$ | $A_3 > A_4 > A_2 > A_5 > A_1$ |
| $p = q = 10$ | $E(a_1) = 0.0618, E(a_2) = 0.0993, E(a_3) = 0.1272,$ $E(a_4) = 0.1112, E(a_5) = 0.0899$ | $A_3 > A_4 > A_2 > A_5 > A_1$ |

situation. Furthermore, when $p$ and $q$ are given to the same fixed value, the score function values for alternative $A_i\left(i = 1, 2, 3, 4, 5\right)$ also increase with the increasing of parameters $p, q$. The findings mentioned above also arise in Tables 2 and 3.

The sorting results based on different LSF $f^*$ by using MVPFLGWGHM operator are presented in Tables 4-6, which illustrates that $A_3$ is always the optimal ERP vendor. As shown in Table 4, when $q$ is assigned to a fixed value, the score values of each alternative $A_i\left(i = 1, 2, 3, 4, 5\right)$ decrease with the increasing of the parameter $p$, whereas $p = 0, q = 1$ is an exceptional case. Furthermore, when $p$ and $q$ are assigned to the same fixed value, the score function values for alternative $A_i\left(i = 1, 2, 3, 4, 5\right)$ also decrease with the increasing of parameters $p, q$. The above outcomes also appear in Tables 5 and 6.
### TABLE 3. Ranking based on $f^*_3$ with MVPFLGWHM operator.

| $p$, $q$ | $E(\xi_i)$ ($i = 1, 2, 3, 4, 5$) | Ranking result |
|-----------|----------------------------------|-----------------|
| $p = 0.5$, $q = 0$ | $E(\xi_1) = 0.0249$, $E(\xi_2) = 0.0199$, $E(\xi_3) = 0.0429$, $E(\xi_4) = 0.0319$, $E(\xi_5) = 0.0275$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 1$, $q = 0$ | $E(\xi_1) = 0.0261$, $E(\xi_2) = 0.0260$, $E(\xi_3) = 0.0484$, $E(\xi_4) = 0.0376$, $E(\xi_5) = 0.0317$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 2$, $q = 0$ | $E(\xi_1) = 0.0288$, $E(\xi_2) = 0.0394$, $E(\xi_3) = 0.0596$, $E(\xi_4) = 0.0499$, $E(\xi_5) = 0.0416$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 5$, $q = 0$ | $E(\xi_1) = 0.0398$, $E(\xi_2) = 0.0680$, $E(\xi_3) = 0.0917$, $E(\xi_4) = 0.0807$, $E(\xi_5) = 0.0693$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 10$, $q = 0$ | $E(\xi_1) = 0.0546$, $E(\xi_2) = 0.0923$, $E(\xi_3) = 0.1219$, $E(\xi_4) = 0.1071$, $E(\xi_5) = 0.0916$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 0$, $q = 1$ | $E(\xi_1) = 0.0388$, $E(\xi_2) = 0.0625$, $E(\xi_3) = 0.0742$, $E(\xi_4) = 0.0572$, $E(\xi_5) = 0.0463$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 0.5$, $q = 1$ | $E(\xi_1) = 0.0334$, $E(\xi_2) = 0.0445$, $E(\xi_3) = 0.0627$, $E(\xi_4) = 0.0490$, $E(\xi_5) = 0.0405$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 2$, $q = 1$ | $E(\xi_1) = 0.0334$, $E(\xi_2) = 0.0524$, $E(\xi_3) = 0.0694$, $E(\xi_4) = 0.0579$, $E(\xi_5) = 0.0481$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 5$, $q = 1$ | $E(\xi_1) = 0.0429$, $E(\xi_2) = 0.0745$, $E(\xi_3) = 0.0962$, $E(\xi_4) = 0.0827$, $E(\xi_5) = 0.0705$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 10$, $q = 1$ | $E(\xi_1) = 0.0558$, $E(\xi_2) = 0.0957$, $E(\xi_3) = 0.1226$, $E(\xi_4) = 0.1063$, $E(\xi_5) = 0.0904$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 0.5$, $q = 0.5$ | $E(\xi_1) = 0.0402$, $E(\xi_2) = 0.0327$, $E(\xi_3) = 0.0524$, $E(\xi_4) = 0.0405$, $E(\xi_5) = 0.0339$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 1$, $q = 0.5$ | $E(\xi_1) = 0.0500$, $E(\xi_2) = 0.0414$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 2$, $q = 0.5$ | $E(\xi_1) = 0.0387$, $E(\xi_2) = 0.0643$, $E(\xi_3) = 0.0818$, $E(\xi_4) = 0.0685$, $E(\xi_5) = 0.0568$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 5$, $q = 0.5$ | $E(\xi_1) = 0.0542$, $E(\xi_2) = 0.0948$, $E(\xi_3) = 0.1182$, $E(\xi_4) = 0.1020$, $E(\xi_5) = 0.0859$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |
| $p = 10$, $q = 0.5$ | $E(\xi_1) = 0.0618$, $E(\xi_2) = 0.1143$, $E(\xi_3) = 0.1414$, $E(\xi_4) = 0.1235$, $E(\xi_5) = 0.1035$ | $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$ |

**C. COMPARISON ANALYSIS**

To reveal the merits of the novel approach, we perform a comparative analysis between our method and the existing methods [25], [31]. In this part, the method developed by Li et al. [25] is applied to settle the same example with multiple-valued picture fuzzy linguistic information in this manuscript. The comparative results are presented in Table 7.

From Table 7, we can find that the optimal vendor is always $A_3$, while the worst vendor is $A_1$ or $A_3$, and the ranking orders are slightly different between the method proposed by Li et al. [25] and the method proposed in this manuscript. If the aggregating operator based on Algebraic operations developed by Li et al. [25] is used, when $p = q = 1$, the ranking is $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$. Obviously, the optimal vendor is $A_3$, and the worst vendor is $A_1$. If the
TABLE 4. Ranking based on $f^*$ with MVPFLGWGHM operator.

| $p$, $q$ | $E(a_i)$ ($i = 1, 2, 3, 4, 5$) | Ranking result |
|-----------|--------------------------------|-----------------|
| $p = 0.5$, $q = 0$ | $E(a_1) = 1.4097$, $E(a_2) = 1.3851$, $E(a_3) = 1.4986$, $E(a_4) = 1.4469$, $E(a_5) = 1.3977$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 0$, $q = 1$ | $E(a_1) = 0.7555$, $E(a_2) = 0.7292$, $E(a_3) = 0.7978$, $E(a_4) = 0.7532$, $E(a_5) = 0.7358$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 2$, $q = 0$ | $E(a_1) = 0.3877$, $E(a_2) = 0.3656$, $E(a_3) = 0.4041$, $E(a_4) = 0.3735$, $E(a_5) = 0.3670$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 5$, $q = 0$ | $E(a_1) = 0.1519$, $E(a_2) = 0.1404$, $E(a_3) = 0.1579$, $E(a_4) = 0.1436$, $E(a_5) = 0.1402$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 10$, $q = 0$ | $E(a_1) = 0.0736$, $E(a_2) = 0.0687$, $E(a_3) = 0.0788$, $E(a_4) = 0.0711$, $E(a_5) = 0.0686$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 0$, $q = 1$ | $E(a_1) = 0.6088$, $E(a_2) = 0.7051$, $E(a_3) = 0.7394$, $E(a_4) = 0.6427$, $E(a_5) = 0.6326$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 0.5$, $q = 1$ | $E(a_1) = 0.8822$, $E(a_2) = 0.9383$, $E(a_3) = 0.9997$, $E(a_4) = 0.9197$, $E(a_5) = 0.8963$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 2$, $q = 1$ | $E(a_1) = 0.4527$, $E(a_2) = 0.4731$, $E(a_3) = 0.5086$, $E(a_4) = 0.4574$, $E(a_5) = 0.4484$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 5$, $q = 1$ | $E(a_1) = 0.2123$, $E(a_2) = 0.2185$, $E(a_3) = 0.2385$, $E(a_4) = 0.2126$, $E(a_5) = 0.2075$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = 10$, $q = 1$ | $E(a_1) = 0.1079$, $E(a_2) = 0.1102$, $E(a_3) = 0.1229$, $E(a_4) = 0.1094$, $E(a_5) = 0.1057$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = q = 0.5$ | $E(a_1) = 1.2825$, $E(a_2) = 1.3423$, $E(a_3) = 1.4215$, $E(a_4) = 1.3455$, $E(a_5) = 1.3037$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = q = 1$ | $E(a_1) = 0.6811$, $E(a_2) = 0.7193$, $E(a_3) = 0.7692$, $E(a_4) = 0.7005$, $E(a_5) = 0.6848$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = q = 2$ | $E(a_1) = 0.3472$, $E(a_2) = 0.3679$, $E(a_3) = 0.3979$, $E(a_4) = 0.3546$, $E(a_5) = 0.3476$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = q = 5$ | $E(a_1) = 0.1388$, $E(a_2) = 0.1461$, $E(a_3) = 0.1621$, $E(a_4) = 0.1449$, $E(a_5) = 0.1407$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |
| $p = q = 10$ | $E(a_1) = 0.0767$, $E(a_2) = 0.0806$, $E(a_3) = 0.0877$, $E(a_4) = 0.0796$, $E(a_5) = 0.0788$ | $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ |

MVPFLGWGHM operator with different LSF $f^*$ proposed in this manuscript is used, when $p = q = 1$, the ranking is always $A_3 \succ A_4 \succ A_2 \succ A_5 \succ A_1$, which is the same as that employing Li’s method. If the MVPFLGWGHM operator with LSF $f^*_1$ or $f^*_2$ proposed in this manuscript is used, and $p = q = 1$, the ranking is $A_3 \succ A_2 \succ A_4 \succ A_5 \succ A_1$. The result is similar to that of Li’s method, the optimal alternative is $A_3$, and the worst vendor is $A_1$. If the MVPFLGWGHM operator with LSF $f^*_2$ is employed, the ranking is $A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$, the optimal alternative is $A_3$, and the worst alternative is $A_5$.

There are two reasons why the ranking orders are slightly different between the method developed by Li et al. [25] and the method proposed in this manuscript. First, Li et al. [25] introduced MVNLNWBM operator based on Hamacher operations to cope with multi-valued neutrosophic linguistic.
elements. The NWBM operator utilized in Ref. [25] can take the correlations of input arguments into consideration, which is similar to the generalized HM operator in this paper. However, the NWBM operator can result in redundant calculations on interrelationships between attributes, and cannot reflect the relationship between attribute and itself, whereas HM operator can avoid these shortcomings. Thus, the HM operator is superior to the BM operator. Second, the linguistic term part of the collective value for each alternative in Ref. [25] is obtained directly based on the subscripts of the linguistic term, which can distort information sometimes, whereas the three types of LSF provided in this paper can represent different semantic situations and avoid information distortion. Therefore, the developed method in the paper is more agile and practicable in the real decision-making environment.

| $p$, $q$ | $E(a_i)$ for $i = 1, 2, 3, 4, 5$ | Ranking result |
|----------|-----------------------------|----------------|
| $p = 0.5$, $q = 0$ | $E(a_1) = 1.4185$, $E(a_2) = 1.3960$, $E(a_3) = 1.4810$, $E(a_4) = 1.4357$, $E(a_5) = 1.4060$ | $A_5 > A_4 > A_3 > A_6 > A_2$ |
| $p = 1$, $q = 0$ | $E(a_1) = 0.7603$, $E(a_2) = 0.7349$, $E(a_3) = 0.7884$, $E(a_4) = 0.7474$, $E(a_5) = 0.7401$ | $A_6 > A_3 > A_4 > A_5 > A_2$ |
| $p = 2$, $q = 0$ | $E(a_1) = 0.3901$, $E(a_2) = 0.3685$, $E(a_3) = 0.3994$, $E(a_4) = 0.3706$, $E(a_5) = 0.3692$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 5$, $q = 0$ | $E(a_1) = 0.1528$, $E(a_2) = 0.1415$, $E(a_3) = 0.1560$, $E(a_4) = 0.1425$, $E(a_5) = 0.1410$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 10$, $q = 0$ | $E(a_1) = 0.0741$, $E(a_2) = 0.0692$, $E(a_3) = 0.0779$, $E(a_4) = 0.0705$, $E(a_5) = 0.0690$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 0$, $q = 1$ | $E(a_1) = 0.6157$, $E(a_2) = 0.6949$, $E(a_3) = 0.7285$, $E(a_4) = 0.6306$, $E(a_5) = 0.6252$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 0.5$, $q = 1$ | $E(a_1) = 0.8895$, $E(a_2) = 0.9345$, $E(a_3) = 0.9862$, $E(a_4) = 0.9072$, $E(a_5) = 0.8930$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 2$, $q = 1$ | $E(a_1) = 0.4564$, $E(a_2) = 0.4718$, $E(a_3) = 0.5018$, $E(a_4) = 0.4515$, $E(a_5) = 0.4473$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 5$, $q = 1$ | $E(a_1) = 0.2141$, $E(a_2) = 0.2181$, $E(a_3) = 0.2354$, $E(a_4) = 0.2099$, $E(a_5) = 0.2072$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = 10$, $q = 1$ | $E(a_1) = 0.1088$, $E(a_2) = 0.1101$, $E(a_3) = 0.1213$, $E(a_4) = 0.1080$, $E(a_5) = 0.1056$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = q = 0.5$ | $E(a_1) = 1.2928$, $E(a_2) = 1.3387$, $E(a_3) = 1.4062$, $E(a_4) = 1.3283$, $E(a_5) = 1.3002$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = q = 1$ | $E(a_1) = 0.6867$, $E(a_2) = 0.7169$, $E(a_3) = 0.7589$, $E(a_4) = 0.6912$, $E(a_5) = 0.6827$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = q = 2$ | $E(a_1) = 0.3501$, $E(a_2) = 0.3664$, $E(a_3) = 0.3925$, $E(a_4) = 0.3498$, $E(a_5) = 0.3464$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = q = 5$ | $E(a_1) = 0.1400$, $E(a_2) = 0.1457$, $E(a_3) = 0.1509$, $E(a_4) = 0.1429$, $E(a_5) = 0.1401$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
| $p = q = 10$ | $E(a_1) = 0.0774$, $E(a_2) = 0.0802$, $E(a_3) = 0.0865$, $E(a_4) = 0.07849$, $E(a_5) = 0.07848$ | $A_4 > A_3 > A_4 > A_5 > A_2$ |
TABLE 6. Ranking based on $f^*_3$ with MVPFLGWGHM operator.

| $p, q$ | $E(a_i) \{i = 1, 2, 3, 4, 5\}$ | Ranking result |
|--------|---------------------------------|----------------|
| $p = 0.5, q = 0$ | $E(a_4) = 1.4011, E(a_1) = 1.3735, E(a_2) = 1.5104, E(a_3) = 1.3899$ | $A_4 > A_2 > A_1 > A_5 > A_3$ |
| $p = 1, q = 0$   | $E(a_2) = 0.7509, E(a_1) = 0.7230, E(a_3) = 0.8041, E(a_4) = 0.7311$ | $A_2 > A_1 > A_3 > A_5 > A_4$ |
| $p = 2, q = 0$   | $E(a_3) = 0.3853, E(a_1) = 0.3625, E(a_3) = 0.4073, E(a_4) = 0.3647$ | $A_3 > A_1 > A_5 > A_4 > A_2$ |
| $p = 5, q = 0$   | $E(a_4) = 0.1509, E(a_1) = 0.1392, E(a_3) = 0.1591, E(a_4) = 0.1393$ | $A_4 > A_1 > A_5 > A_2 > A_3$ |
| $p = 10, q = 0$  | $E(a_2) = 0.0732, E(a_1) = 0.0681, E(a_3) = 0.0794, E(a_4) = 0.0682$ | $A_2 > A_1 > A_5 > A_4 > A_3$ |
| $p = 0, q = 1$   | $E(a_3) = 0.6023, E(a_1) = 0.7119, E(a_3) = 0.7451, E(a_4) = 0.6377$ | $A_3 > A_2 > A_4 > A_5 > A_1$ |
| $p = 0.5, q = 1$ | $E(a_4) = 0.8754, E(a_1) = 0.9396, E(a_3) = 1.0077, E(a_4) = 0.9270, E(a_2) = 0.8978$ | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| $p = 2, q = 1$   | $E(a_3) = 0.4492, E(a_1) = 0.4732, E(a_3) = 0.5126, E(a_4) = 0.4487$ | $A_3 > A_2 > A_4 > A_5 > A_1$ |
| $p = 5, q = 1$   | $E(a_3) = 0.2107, E(a_1) = 0.2184, E(a_3) = 0.2404, E(a_4) = 0.2075$ | $A_3 > A_2 > A_4 > A_5 > A_1$ |
| $p = 10, q = 1$  | $E(a_3) = 0.1071, E(a_1) = 0.1101, E(a_3) = 0.1238, E(a_4) = 0.1057$ | $A_3 > A_2 > A_4 > A_5 > A_1$ |
| $p = q = 0.5$    | $E(a_1) = 1.2730, E(a_4) = 1.3427, E(a_4) = 1.4363, E(a_1) = 1.3558, E(a_4) = 1.3049$ | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| $p = q = 1$      | $E(a_1) = 0.6759, E(a_4) = 0.7199, E(a_4) = 0.7753, E(a_1) = 0.7060, E(a_4) = 0.6856$ | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| $p = q = 2$      | $E(a_1) = 0.3445, E(a_4) = 0.3684, E(a_4) = 0.4010, E(a_1) = 0.3575, E(a_4) = 0.3481$ | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| $p = q = 5$      | $E(a_1) = 0.1377, E(a_4) = 0.1467, E(a_4) = 0.1634, E(a_1) = 0.1462, E(a_4) = 0.1409$ | $A_4 > A_2 > A_5 > A_3 > A_1$ |
| $p = q = 10$     | $E(a_1) = 0.0761, E(a_4) = 0.0808, E(a_4) = 0.0884, E(a_1) = 0.0803, E(a_4) = 0.0790$ | $A_4 > A_2 > A_5 > A_3 > A_1$ |

Next, the method developed in this manuscript is applied to settle the same instance [31] with single-valued picture fuzzy linguistic information provided by Liu. The comparative results are listed in Table 8.

From Table 8, we can find that the optimal vendor is always $A_4$, while the worst vendor is $A_1$ or $A_2$, or $A_3$, and the ranking orders are different between the method proposed by Liu and Zhang [31] and the method proposed in this manuscript. If the aggregating operator based on Algebraic operations developed by Liu and Zhang [31] is used, when $p = q = 1$, the ranking order is $A_4 > A_2 > A_1 > A_3 > A_5$. Obviously, the optimal vendor is $A_4$, and the worst vendor is $A_1$. If the MVPFLGWGHM operator with $f^*_1$ or $f^*_3$ proposed in this manuscript is used, when $p = q = 1$, the ranking order is $A_4 > A_3 > A_2 > A_1$, the optimal alternative and the worst alternative are the same as that employing Liu’s method. If the
The main characteristics of the paper are summarized as below. (1) The MVPFLS along with the superiority of both MVPFS and LTS is more appropriate to depict the qualitative information and quantitative information in coping with MADM problems, which is a generalization of PFLS, HFS, and IFS. (2) The traditional generalized HM operators extending to the MVPFL environment can consider the interrelationship of attribute values, which is more practicable in actual situation. (3) The flexible LSF $f^*$ provided in the proposed MADM method can be selected to transform linguistic term to semantic value, which is more effective in dealing with different semantic environment.

**VII. CONCLUSION**

According to the existing research achievements, to better describe the complex cognitive information and utilize the advantages of both MVPFS and LTS, the concepts of MVPFLS and MVPFLE are initially proposed in this paper. Moreover, the operational rules and comparative method based on LSF for MVPFLE are investigated. Next, the conventional generalized HM operators are expanded to the MVPFL environment, and the MVPFLGWHM operator and MVPFLGWGHM operator are constructed and proved, the desirable characteristic and some particular instances on the proposed operators are also discussed. What’s more, a new method for coping with MVPFL information is given.
Ultimately, we utilized the developed method to handle an illustrative example, and the sensitivity analysis and comparative analysis are also provided. Obviously, the results illustrate the effectiveness, practicability, and generalization of the introduced method.

In future, we will explore more aggregating operators with MVPFL information and utilize operators to handle MAGDM problems.

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