Global $SU(2)_L \otimes$BRST symmetry and its LSS theorem: Ward-Takahashi identities governing Green’s functions, on-shell T-Matrix elements, and the effective potential, in the scalar-sector of certain spontaneously broken non-Abelian gauge theories

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This work is dedicated to the memory of Raymond Stora (1930-2015). $SU(2)_L$ is the simplest spontaneous symmetry breaking (SSB) non-Abelian gauge theory. Its simplest bosonic representation is a complex scalar doublet $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} H + i\pi_3 \\ -\pi_2 + i\pi_1 \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} H e^{2i\theta}/(H) \\ 1 \end{pmatrix}$ and a vector gauge boson $\tilde{W}^\mu$. In Landau gauge, $\partial_\mu \tilde{W}^\mu = 0$, $\hat{\pi}$ are massless derivatively coupled Nambu-Goldstone bosons (NGB). A global shift symmetry $\tilde{\pi} \rightarrow \tilde{\pi} + \langle H \rangle \tilde{\theta} + \tilde{\pi} \times \tilde{\theta} + O(\theta^2)$ enforces $m_\pi^2 = 0$. We observe that on-shell T-matrix elements of physical states $W^\mu, \phi$ (but not ghosts $\bar{\omega}$ and anti-ghosts $\bar{\eta}$) are independent of anomaly-free global $SU(2)_L$ transformations, a remnant of local anomaly-free $SU(2)_L$ gauge symmetry, and that the associated global current is exactly conserved for amplitudes of physical states. We identify two towers of “1-soft-pion” SSB global Ward-Takahashi Identities (WTI), which govern the $\phi$-sector, and represent a new global symmetry which we call $SU(2)_L \otimes$BRST, a symmetry not of the Lagrangian but of the physical states. The first tower gives relations among $1-\phi-I$ (1 scalar particle irreducible but one $\phi$-sector SSB effective Lagrangian. The second tower governs on-shell T-matrix elements, replaces the Adler self-consistency relations with those for gauge theories, further constrains the effective potential, and guarantees IR finiteness in the scalar sector despite zero NGB mass. These on-shell WTI include a Lee-Stora-Symnzanik (LSS) theorem, which enforces the condition $m_\pi^2 = 0$ (far stronger than $m_\pi^2 = 0$) on the $\hat{\pi}$ and causes all relevant-operator contributions to the effective Lagrangian to vanish exactly. The global $SU(2)_L$ transformations and the nilpotent BRST transformations commute in $R_L$ gauge. $[s, \delta_{SU(2)_L}] = 0$. With the on-shell T-matrix constraints, the physics therefore has more symmetry than does its BRST invariant Lagrangian, i.e. global $SU(2)_L \otimes$BRST symmetry. We also show that the statements made above hold for a $SU(2)_L \otimes U(1)_Y$ gauge theory i.e the electroweak sector of Standard Model bosons.

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I. INTRODUCTION

What are the symmetries driving spontaneously broken $SU(2)_L$ gauge theory physics \textsuperscript{1}? Although the symmetries of the $SU(2)_L$ Lagrangian are well known \textsuperscript{2}, local gauge-invariance is lost, broken by gauge-fixing terms, and replaced with global BRST invariance \textsuperscript{3,5}.

In their seminal work, Elisabeth Kraus and Klaus Sibold \textsuperscript{6} showed important new practicalities of the renormalizability and unitarity (to all-loop-orders) of the spontaneous symmetry breaking (SSB) $U(1)$ Abelian Higgs model (AHM). They did this by deriving rigid invariance from BRST invariance. The SSB case is tricky because the globally BRST-invariant Lagrangian is not $U(1)$ symmetric. But they identified a set of “deformed” (i.e. with no remnant of the original $U(1)$ group symmetry) rigid/global AHM transformations which, after inclusion of well-defined $U(1)$-breaking by quantum loops (e.g. in scalar wavefunction renormalization beyond the classical AHM), are compatible with BRST symmetry.

Kraus and Sibold then constructed deformed Ward-Takahashi identities (WTI) for quantum $U(1)$ Green’s functions, showing them (with appropriate normalization conditions) to obey all-loop-orders renormalizability and unitarity. Because their renormalization relies only on deformed WTI, Kraus and Sibold’s results are independent of regularization scheme, for any acceptable scheme (i.e. if one exists). They did not construct WTI for on-shell T-Matrix elements.

Nevertheless, Slavnov-Taylor identities \textsuperscript{7} prove that the on-shell $S$-Matrix elements of AHM “physical states” $A^\mu, \phi$ (i.e. spin $S = 0$ scalars $h, \pi$ and $S = 1$ gauge bosons $A_\mu$, but not fermionic ghosts $\omega$ or anti-ghosts $\bar{\eta}$) are independent, in the AHM, of the usual undeformed anomaly-free $U(1)$ local/gauge transformations, even though these break the AHM Lagrangian’s BRST symmetry. We observed \textsuperscript{6} that they are therefore also independent of anomaly-free undeformed $U(1)$ global/rigid transformations, resulting in “new” global/rigid currents and appropriate un-deformed $U(1)$ Ward-Takahashi Identities. In this paper, we extend that thinking to non-Abelian gauge theories.

We here distinguish carefully between off-shell Green’s function WTI, which constrain the (un-observable) effec-
tive Lagrangian and action, and on-shell T-Matrix WTI, which further severely constrain observable physics. We show here that, in SSB SU(2)_L, a tower of Ward-Takahashi Identities (WTI) relates all relevant-operator contributions to SU(2)_L physical-scalar-sector observables to one another. An on-shell T-Matrix WTI, i.e. the equivalent of an Adler self-consistency relation but for the SU(2)_L gauge theory, then causes all such contributions to vanish. It does so through its insistence that the scalar mass-squared vanish exactly

\[ m^2_\pi = 0 \]  

in spontaneously broken ((H) \neq 0) theories. We term this the Lee-Stora-Symzanzik (LSS) theorem after the three physicists who recognized its central role in the renormalization of global Linear Sigma Models, and the one who was central to our understanding of its role in the renormalization of gauge theories. In addition to constraining the parameters of the theory, LSS permits us to employ pion-pole-dominance to compute the WTI.

The crucial advance over [13], which considered the global SU(3)_C \times SU(2)_L \times U(1)_Y Linear Sigma Model (LSM), is a proof that the WTI remain in place in a SSB gauge theory, with the LSS theorem playing the same protective role as did the Goldstone Theorem in the global theory [13].

Our new rigid SU(2)_L WTIs govern the scalar-sector of the SU(2)_L gauge theory. They are therefore independent of regularization-scheme (assuming one exists). Although not a gauge-independent procedure, it may help the reader to imagine that loop integrals are cut off at a short-distance finite Euclidean UV scale, \( \Lambda \), never taking \( \Lambda^2 \rightarrow \infty \) limit. Although that cut-off can be imagined to be near the Planck scale \( \Lambda \simeq M_{Pl} \), quantum gravitational loops are not included.

The WTIs derived in this work give relations among one-scalar-particle-irreducible (1-\( \phi \)-I) off-shell Green’s functions, and (separately) among connected on-shell T-matrix elements. Each 1-\( \phi \)-I Green’s function is a sum over an infinite number of one-particle-irreducible (1-P-I) graphs. 1-\( \phi \)-I Green’s functions are the appropriate ones to consider for calculation of the scalar-sector effective potential. Other authors (see [6,19]) have derived WTIs for 1-P-I Green’s functions for SSB gauge theories, but our identities, formulated with respect to a different set of graphs, are fundamentally different from those.

The structure of this paper is as follows:

Section II introduces SU(2)_L \otimes BRST symmetry for a general t Hooft \( R_{\xi} \) gauge, and explains why physical results obey that new symmetry.

Section III concerns the correct renormalization of spontaneously broken SU(2)_L in Landau gauge. We treat SU(2)_L in isolation, as a stand-alone flat-space weak-scale quantum field theory, not embedded or integrated into any higher-scale “Beyond-SU(2)_L” physics.

Section IV extends \( G \otimes \) BRST to gauge theories with certain simple Lie algebraic structure group \( G \).

Section V analyses an example a non-simple Lie algebraic structure group: (SU(2)_L \otimes U(1)_Y) \otimes \) BRST for the standard electroweak model of gauge bosons, complex scalar doublet, ghosts and anti-ghosts.

Section VI extends our results to a CP-conserving version of the Standard Model with only the 3rd generation of quarks and leptons.

Section VII discusses the exacting mathematical rigor that would have fully satisfied Raymond Stora.

Section VIII draws conclusions.

Appendix A reduces the derivation of SU(2)_L gauge-theoretic WTIs in Landau gauge to that of the SU(2)$_{L-R}$ Schwinger LSM.

Appendix B gives a complete and pedagogical derivation of the Landau gauge SU(2)_L WTIs governing the \( \phi \)-sector of SU(2)_L. Our renormalized WTIs include all contributions from virtual transverse gauge bosons; \( \phi \)-scalars; anti-ghosts and ghosts; \( \tilde{W}^\mu; h, \tilde{\eta}_h, \omega_h \) respectively.

Appendix C derives the SU(2)_L current \( J^\mu_L \) in terms of \( \tilde{W}^\mu; h, \tilde{\eta}_h, \omega_h \), together with its divergence and commutators with scalar \( \phi \).

Appendix D derives the SU(2)_L sub-current \( J^\mu_{L,2\otimes 1} \) in the SU(2)_L \times U(1)_Y gauge theory, together with its divergence and commutators with scalar \( \phi \).

Appendix E gives a complete and pedagogical derivation of the Landau gauge SU(2)$_{L-R}$ WTIs governing the \( \phi \)-sector of SU(2)_L \times U(1)_Y. Our renormalized WTIs include all contributions from virtual transverse gauge bosons; \( \phi \)-scalars; anti-ghosts and ghosts; \( \tilde{W}^\mu, B^\mu; h, \tilde{\eta}_B, \omega_B \) respectively.

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1 Note from OG, BWL and GDS: Raymond Stora would never name anything after himself. But we judge that, given the stature of B.W. Lee, R. Stora and K. Symanzik (now all deceased) in the history of SSB physics, the community would refer to that result as the “LSS theorem” anyway.

2 As first noted by Kibble [5], in Landau gauge a relation naively similar in appearance to \( m_{\pi}^2 = 0 \), enforces the masslessness of a Nambu Goldstone Boson (NGB) \( \tilde{\pi} \), i.e. is a Goldstone Theorem [9–11, 13, 14] for the SSB SU(2)_L gauge theory. This is regardless of the fact that the NGBs are not a physical degree of freedom. However, as we describe in greater detail below (cf. equation 17), \( \tilde{\pi} \) is the angular degree of freedom in the Kibble representation of the complex scalar doublet, while \( \bar{\pi} \) is the pseudoscalar degree of freedom in the linear representation. In global Linear Sigma Models (LSM), the masslessness of the NGB and the LSM condition \( H \equiv 0 \) are equivalent. Indeed, B. Lee [12], K. Symanzik [13,14], A. Vassiliev [14] and classic texts [16] advocate that the spontaneously broken (“Goldstone”) mode of a global LSM is to be understood as the zero-explicit-breaking limit (i.e. \( m_{\pi}^2 \rightarrow 0 \)) of the explicit-breaking Partially Conserved Axial-vector Current (PCAC) term, \( L_{PCAC} = \langle H \rangle m_{\pi}^2 H \), included in the Gell-Mann and Lévy LSM [17]. The existence and masslessness of the purely derivatively coupled NGB is a result of and requires the vanishing of the explicit-symmetry-breaking pseudo-scalars’ mass-squared.
II. $SU(2)\otimes$BRST SYMMETRY IN 'T HOOFT $R_\xi$ GAUGES

The BRST-invariant [3] Lagrangian of the $SU(2)_L$ gauge theory may be written, in a general 't Hooft $R_\xi$ gauge, in terms of a transverse vector $\bar{\omega}$, a complex scalar doublet $\phi$, ghosts $\bar{\varphi}$, and anti-ghosts $\bar{\eta}$:

$$L_{\text{SU}(2)_L}^{R_\xi} = L_{\text{GaugeInvariant}}^{SU(2)_L} + L_{\text{GaugeFix;R}_\xi}^{SU(2)_L} + L_{\text{Ghost;R}_\xi}^{SU(2)_L}$$

with

$$L_{\text{GaugeInvariant}}^{SU(2)_L} = -\frac{1}{2} Tr (W_\mu W^{\mu}) + |D_\mu \phi|^2 - V$$

where

$$W_\mu = i [\partial_\mu, W] = -i \lambda_\mu^a \sigma^a$$

and the gauge-fixing function $F_W$

$$F_W = \bar{\eta} \cdot (\bar{\varphi} \phi + M_W \bar{\varphi})$$

after SSB, and $H = h + \langle H \rangle, \bar{H} = \bar{h} + \langle H \rangle$.

In G. 't Hooft's $R_\xi$ gauges, gauge fixing and DeWitt-Faddeev-Popov ghost terms [20, 21] are written in terms of a Nakanishi-Lautrup field $\bar{b}$ [22, 23], the SSB vector mass $M_W$ and the gauge-fixing function $F_W$

$$M_W = \frac{1}{2} g_2 \langle H \rangle$$

$$F_W = \partial_\mu \bar{\varphi}^{\mu} + \xi M_W \bar{\varphi}$$

Under global BRST transformations [3, 5, 22, 24] $s$

$$s\bar{\omega} = \partial_\mu \bar{\varphi}^{\mu} + g_2 \bar{\varphi}^{\mu} \times \bar{\varphi}$$

$$sH = -\frac{1}{2} g_2 \bar{\varphi}^{\mu} \cdot \bar{\varphi}$$

$$s\bar{\varphi} = \frac{1}{2} g_2 \left( H \bar{\varphi}^{\mu} + \bar{\varphi}^{\mu} \times \bar{\varphi} \right)$$

$$s\bar{\varphi} = \bar{b};$$

$$s\bar{\eta} = 0$$

so that the Lagrangian [2] is BRST invariant

$$sL_{\text{SU}(2)_L}^{R_\xi} = 0$$

Now define the properties of the various fields under the usual anomaly-free un-deformed rigid/global $SU(2)_L$ transformations by constant $\Omega$

$$\delta_{SU(2)_L} W_\mu = g_2 \bar{\varphi}^{\mu} \times \Omega$$

$$\delta_{SU(2)_L} H = -\frac{1}{2} g_2 \bar{\varphi}^{\mu} \cdot \Omega$$

$$\delta_{SU(2)_L} \bar{\varphi}^{\mu} = \frac{1}{2} g_2 \left( H \Omega + \Omega \times \bar{\varphi} \right)$$

$$\delta_{SU(2)_L} \bar{\eta} = 0$$

$$\delta_{SU(2)_L} \bar{b} = 0$$

The transformation sets [1] and [9] commute

$$[\delta_{SU(2)_L}, s] \bar{\omega} = 0; \quad [\delta_{SU(2)_L}, s] \bar{\varphi} = 0;$$

$$[\delta_{SU(2)_L}, s] H = 0; \quad [\delta_{SU(2)_L}, s] \bar{\eta} = 0;$$

$$[\delta_{SU(2)_L}, s] \bar{b} = 0;$$

and although $R_\xi$-gauge Lagrangian [2] is not invariant under $SU(2)_L$ transformations

$$\delta_{SU(2)_L} L_{\text{SU}(2)_L}^{R_\xi}$$

$$= s \left( \delta_{SU(2)_L} \left[ \bar{\eta} \cdot \left( \bar{\varphi}^{\mu} + \frac{1}{2} \xi \bar{b} \right) \right] \right)$$

$$= s \left( \bar{\eta} \cdot \left[ g_2 \partial_\mu \bar{\varphi}^{\mu} + \frac{\xi M_W^2}{\langle H \rangle} (H \bar{\varphi}^{\mu} + \bar{\varphi}^{\mu} \times \bar{\varphi}) \right] \right)$$

$$\neq 0$$

with [8] and the nilpotent property $s^2 = 0$,

$$[\delta_{SU(2)_L}, s] L_{\text{SU}(2)_L}^{R_\xi} = 0,$$

and the two separate global symmetries can therefore co-exist in $SU(2)_L$ physics.

In an operational sense, we summarize [10, 12] with the short-hand

$$[\delta_{SU(2)_L}, s] = 0.$$
The reader will have noticed that, although the anti-ghosts $\tilde{\eta}$ transform in the usual way, as a triplet $s\tilde{\eta} = \tilde{b}$ under BRST in (7), we have chosen them to be singlets under global $SU(2)_L$ transformations $\delta_{SU(2)_L} \tilde{\eta} = 0$ in (0). This does not affect the proof of renormalizability and unitarity with Slavnov-Taylor identities [25]: the subject of a future paper, but outside the scope of this paper. This freedom to choose $\tilde{\eta}$ singlets under $SU(2)_L$:

- Renders the ghost Lagrangian without definite $\delta_{SU(2)_L}$ properties.
- Is the genesis of the commutation properties (10) and (12).
- Allows us to build a classical current $\tilde{J}_L^\mu$, conserved up to gauge-fixing terms.

We will show in this paper that, due to (7,9,13), $SU(2)_L$ physics simultaneously obeys both the usual $SU(2)_L$ symmetry and a global $SU(2)_L$ symmetry which controls Green’s functions and on-shell T-Matrix elements. We reason as follows:

- All aspects of the SSB $SU(2)_L$ physics obey BRST symmetry.
- We work in Landau gauge

$$L_{SU(2)_L}^{\text{Landau}} = L_{SU(2)_L}^{\text{GaugeInvariant}}$$

$$- \lim_{\xi \to 0} \frac{1}{2\xi} \left( \partial_\mu \tilde{W}^\mu + \xi M_W \tilde{\pi} \right)^2$$

- Physical states and time-ordered amplitudes of the exact renormalized scalar $\phi = \frac{1}{\sqrt{2}} \left[ \frac{H + i\pi_3}{-\pi_2 + i\pi_1} \right]$ and vector $\tilde{W}_\mu$ obey G. ’t Hooft’s gauge condition [26]

$$\langle 0 | T \left[ \left( \partial_\mu \tilde{W}^\mu (z) \right) \times h(x_1)...h(x_N)\pi_{i_1}(y_1)...\pi_{i_M}(y_M) \right] | 0 \rangle_{\text{connected}} = 0,$$

in Landau gauge. Here we have N external renormalized scalars $h = H - \langle H \rangle$ (coordinates x), and M external ($CP = -1$) renormalized pseudo-scalars $\tilde{\eta}$ (coordinates y, isospin t).

- We prove in Appendix B for $SU(2)_L$ that, in Landau gauge, scalar-sector connected amputated Green’s functions and on-shell T-Matrix elements obey the $SU(2)_L$ symmetry. That is true even though (11) shows that the BRST-invariant $SU(2)_L$ Lagrangian is not invariant under that $SU(2)_L$ symmetry.

### III. $SU(2)_L$ in Landau Gauge

#### A. $SU(2)_L$ in Landau gauge

We form the Lagrangian

$$L_{SU(2)_L}^{\text{Landau}} = L_{SU(2)_L}^{\text{GaugeInvariant}} + L_{SU(2)_L}^{\text{GaugeFix;Landau}} + L_{SU(2)_L}^{\text{Ghost;Landau}},$$

by taking the $\xi \to 0$ limit of (2).

This paper distinguishes carefully between the local BRST-invariant $SU(2)_L$ Lagrangian (16), and its 3 physical modes [12–16]: symmetric Wigner mode, classically scale-invariant point, and physical Goldstone mode.

1) Symmetric Wigner mode $\langle H \rangle = 0$, $M_W^2 = 0$, $m_\phi^2 = m^2_{BEH} = \mu_\phi^2 \neq 0$:

This is $SU(2)_L$ QED with massless photons and massive charged scalars. Thankfully, Nature is not in Wigner mode! Further analysis and renormalization of the Wigner mode lies outside the scope of this paper.

2) Classically scale-invariant point $\langle H \rangle = 0$, $M_W^2 = 0$, $m_\phi^2 = m^2_{BEH} = 0$:

Analysis of the scale-invariant point is also outside the scope of this paper.

3) Spontaneously broken Goldstone mode $\langle H \rangle \neq 0$, $M_W^2 \neq 0$, $m_\phi^2 = 0$, $m^2_{BEH} \neq 0$:

The famous Higgs model, with its Nambu-Goldstone boson (NGB) “eaten” by the Brout-Englert-Higgs mechanism, WTI governed by the LSS theorem [4], is the SSB “Goldstone mode” of the BRST-invariant local Higgs Lagrangian.

We work in Landau gauge and the linear representation for many reasons:

- $\tilde{\pi}$ and $\tilde{\pi}$ do not mix.
- The theory is manifestly renormalizable in the Dyson sense since massive vector propagators fall off as $k^{-2}$ as $k \to \infty$.
- Only in the SSB Goldstone mode of the BRST-invariant Lagrangian (16), and only after first renormalizing in the linear $\phi$ representation, does the renormalized Kibble $\phi$ unitary representation

$$\phi = \frac{1}{\sqrt{2}} \left[ \frac{H + i\pi_3}{-\pi_2 + i\pi_1} \right] = \frac{1}{\sqrt{2}} H e^{2i\tilde{\xi}/\langle H \rangle} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right],$$

make sense. Here $H = \langle H \rangle + h; \tilde{H} = \langle H \rangle + \tilde{h}$.

- We will prove to all-loop-orders the $SU(2)_L$ Lee-Stora-Symanzik theorem [42], an $SU(2)_L$ gauge theory analogue of an old theorem for global $LSM$ [12,13], which forces the $\tilde{\pi}$ mass-squared $m_\phi^2 = 0$.
- We use “pion-pole dominance” (i.e. $m^2 = 0$) arguments to derive $SU(2)_L$ SSB WTI's [41,B15,B23].
• We prove with $SU(2)_L$ Green’s function WTI’s that, in SSB Goldstone mode, $\vec{\pi}$ in $[17]$ is a Nambu-Goldstone boson (NGB), and that the resultant SSB gauge theory has a “shift symmetry” $\vec{\pi} \rightarrow \vec{\pi} + (H)\theta + \vec{\mu} \times \theta + O(\theta^2)$ for constant $\theta$.

Analysis is done in terms of the exact renormalized interacting fields, which asymptotically become the in/out states, i.e. free fields for physical S-Matrix elements.

An important issue is the classification and disposal of relevant operators, in this case the $\vec{\pi}$, h inverse propagators (together with tadpoles). Define the exact renormalized pseudo-scalar propagator in terms of massless $\vec{\pi}$, the Källén-Lehmann [12][27] spectral density $\rho_{SU(2)_L}$, and wavefunction renormalization $Z_{SU(2)_L}^\phi$. In Landau gauge:

$$\Delta_{SU(2)_L}^\phi = \frac{1}{q^2 + i\epsilon} + \int dm^2 \rho_{SU(2)_L}(m^2)$$

Define also the BEH scalar propagator in terms of a BEH scalar pole and the (subtracted) spectral density $\rho_{BEH}$, and the same wavefunction renormalization. We assume h decays weakly, and resembles a resonance:

$$\Delta_{BEH}^\phi = \frac{1}{q^2 - m^2_{BEH; pole} + i\epsilon} + \int dm^2 \rho_{BEH}(m^2)$$

The spectral density parts of the propagators are

$$\Delta_{SU(2)_L}^{\pi; Spectral}(q^2) \equiv \int dm^2 \rho_{SU(2)_L}(m^2)$$

$$\Delta_{SU(2)_L}^{BEH; Spectral}(q^2) \equiv \int dm^2 \rho_{BEH}(m^2)$$

With these choices for renormalization conditions, $\langle H \rangle_{bare} = \sqrt{Z_\phi} \langle H \rangle_{renormalized}$, in agreement with $[28][31]$. Dimensional analysis of the wavefunction renormalizations $[18][19]$, shows that the finite Euclidean cut-off contributes only irrelevant terms $\sim \frac{1}{N^2}$.

### B. Rigid/global $SU(2)_L$, WTI and conserved current (not charge) for physical states in Landau gauge

In their seminal work, E. Kraus and K. Sibold [9] identified, in the Abelian Higgs model (A HM), anomaly-free deformed rigid/global transformations. They are called “deformed” because they have no remnant of the original anomaly-free $U(1)_\gamma$ symmetry due to $U(1)_\gamma$-breaking quantum loops in wavefunction renormalization. The SSB case is tricky because gauge-fixing terms explicitly break both local and global $U(1)_\gamma$ symmetry in the BRST-invariant Lagrangian. Still, Kraus and Sibold constructed deformed global/rigid Ward-Takahashi Identities (WTI) for 1-P-I Green’s functions allowing them to demonstrate (with appropriate normalization conditions) proof of all-loop-orders renormalizability and unitarity for the SSB Abelian Higgs model. Because their renormalization relies only on deformed WTI, Kraus and Sibold’s results are independent of regularization scheme, for any acceptable scheme (i.e. if one exists) $[9]$.

Nevertheless, Slavnov-Taylor identities $[7]$ in the Abelian Higgs Model prove that the on-shell S-Matrix elements of “physical particles” (i.e. spin $S = 0$ scalars $h, \pi$, and $S = 1$ transverse gauge bosons $A_\mu$, but not fermionic ghosts $\omega$ or anti-ghosts $\bar{\eta}$), are independent of the usual undeformed anomaly-free $U(1)_\gamma$ local/gauge transformations, even though these break the Lagrangian’s BRST symmetry. Ref. [11] exploited this fact to derive two towers of WTI’s and an LSS theorem, which represent a new global/rigid $U(1)_\gamma \otimes$ BRST symmetry in the Abelian Higgs Model, and severely constrain its effective potential.

Slavnov-Taylor identities prove the same for SSB $SU(2)_L$ on-shell S-Matrix elements, and for those of any SSB Lie algebraic structure group, $G$ $[7]$. We observe here that SSB $SU(2)_L$ T-Matrix elements are therefore also independent of anomaly-free undeformed $SU(2)_L$ global/rigid transformations, resulting in a “new” global/rigid current, two towers of appropriate undeformed $SU(2)_L$ Ward-Takahashi identities, and a new $SU(2)_L \otimes$ BRST symmetry.

We are interested in rigid-symmetric relations among 1-(h, $\vec{\pi}$)-Irreducible (1-φ-I) (this set of Green’s functions include an infinite number of 1-P-I Green’s functions) connected amputated Green’s functions to use for the scalar-sector effective potential. The 1-(h, $\vec{\pi}$)-Reducible (1-φ-R) connected amputated transition-matrix (T-Matrix) elements $T_{N,M}$, with external φ scalars. Because these are 1-$W_{\mu}$-R in $SU(2)_L$ (i.e. reducible by cutting a $\vec{W}_\mu$), it is convenient to use the powerful old tools (e.g. canonical quantization) from Vintage Quantum Field Theory (Vintage-QFT), a name coined by Ergin Sezgin. The distinction between 1-φ-I Green’s functions and 1-P-I Green’s functions is important. The 1-φ-I (but potentially reducible with respect to other fields) Green’s functions is the correct set of Green’s functions to use for the scalar-sector effective potential, and the WTI’s obtained for those are funda-

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3 E. Kraus and K. Sibold also constructed, in terms of deformed WTI, all-loop-orders renormalized QED, QCD, and the electroweak Standard Model $[22][33]$ to be independent of regularization scheme. From this grew the powerful technology of “Algebraic Renormalization”, used by them, W. Hollik and others $[24]$, to renormalize SUSY QED, SUSY QCD, and the MSSM.
mentally different from the Slavnov-Taylor identities for 1-P-I Green’s functions elsewhere in the literature (e.g., [19]).

We focus on the rigid/global $SU(2)_L$ current constructed with (9) in Appendix C. In Landau gauge, these are

$$
\bar{J}_L^\mu = \bar{J}^\mu_{L;S-W} + \bar{J}^\mu_L,
$$

$$
\bar{J}^\mu_{L;S-W} = \bar{J}^\mu_{L+R;S-W} + \bar{J}^\mu_{L-R;S-W}
$$

(21)

In contrast, we show below that, in Landau gauge, the usual, from the equal-time commutators (B1):

$$
\delta_{SU(2)_L} H(t, \vec{y}) = -i g_2 \Omega_{t^1} \int d^3z \left[ J^\mu_{L;S-W}(t, \vec{z}), H(t, \vec{y}) \right]
$$

(22)

$$
\delta_{SU(2)_L} \pi^t(t, \vec{y}) = -i g_2 \Omega_{t^2} \int d^3z \left[ J^\mu_{L;S-W}(t, \vec{z}), \pi^t(t, \vec{y}) \right]
$$

(23)

so $\bar{J}^\mu_L(t, \vec{z})$ serves as a “proper” local current for commutator purposes.

In contrast, we show below that, in Landau gauge, $SU(2)_L$ has no associated proper global charge because $\frac{d}{dt} \bar{Q}(t) \neq 0$. (See Eqn. (30) below.)

The classical equations of motion reveal a crucial fact: due to gauge-fixing terms in the BRST-invariant Lagrangian [16], the classical current (21) is not conserved. In Landau gauge

$$
\partial_\mu \bar{J}^\mu_L = \frac{1}{2} M_W \left[ \pi \times \partial_\mu \tilde{W}^\mu + H \partial_\mu \tilde{W}^\mu \right].
$$

The global $SU(2)_L$ current (21) is, however, conserved by the physical states, and therefore still qualifies as a “real” current. Strict quantum constraints are imposed that force the relativistically-covariant theory of gauge bosons to propagate only its true number of quantum spin $S = 1$ degrees of freedom. These constraints are, in the modern literature, implemented by use of spin $S = 0$ fermionic DeWitt-Faddeev-Popov ghosts ($\vec{\eta}, \vec{\omega}$). The physical states and their time-ordered products, but not the BRST-invariant Lagrangian [16], then obey G. ’t Hooft’s Landau gauge gauge-fixing condition [15].

Eq. (15) restores conservation of the rigid/global $SU(2)_L$ current for $\phi$-sector connected time-ordered products

$$
\langle 0 | T \left[ \left( \partial_\mu \bar{J}^\mu_L(z) \right) \times h(x_1) \ldots h(x_N) \pi^{t_1}(y_1) \ldots \pi^{t_M}(y_M) \right] | 0 \rangle_{\text{connected}} = 0.
$$

(25)

It is in this “physical” connected-time-ordered product sense that the rigid global $SU(2)_L$ “physical current” is conserved: the physical states, but not the BRST-invariant Lagrangian [16], obey the physical-current conservation equation (25). It is this physical conserved current that generates our $SU(2)_L \otimes$BRST WTI.

Appendices A, B and C derive two towers of quantum $SU(2)_L$ WTI’s that exhaust the information content of [25], severely constrain the dynamics (i.e. the connected time-ordered products) of the $\phi$-sector physical states of SSB $SU(2)_L$, and realize the new $SU(2)_L \otimes$BRST symmetry of Section II.

We might have hoped to also build a conserved charge

$$
\bar{Q}_L(t) = \int d^3z \bar{J}^\mu_L(t, \vec{z})
$$

(26)

in the usual way, but (24) reveals that, because of gauge-fixing terms, classically

$$
\frac{d}{dt} \bar{Q}_L(t) \neq \int d^3z \partial^\mu \bar{J}^\mu_L(t, \vec{z})
$$

(27)

and so $\bar{Q}_L(t)$ is not a classical conserved current.

Hence, having obtained a more restricted form of charge conservation, namely the vanishing of physical connected time-ordered products of the time-derivative of $\bar{Q}_L(t)$. Consider

$$
\langle 0 | T \left[ \left( \frac{d}{dt} \bar{Q}_L(t) \right) \times h(x_1) \ldots h(x_N) \pi^{t_1}(y_1) \ldots \pi^{t_M}(y_M) \right] | 0 \rangle_{\text{connected}}
$$

(28)

where we have used Stokes theorem, and $\vec{z}^{2-\text{surface}}$ is a unit vector normal to the 2-surface. The time-ordered product constrains the 2-surface to lie on or inside the light-cone.

At a given point on the surface of a large enough 3-volume $\int d^3z$ (i.e. the volume of all space), which lies on-or-inside the light cone, all fields on the $z^{2-\text{surface}}$ are
asymptotic in-states and out-states; are properly quantized as free fields; with each field species orthogonal to the others.

Consider first the \( \bar{J}_L^{R} \) contribution (from \( \Phi \)) to the surface integral \( \int d^2z \ z^2 \text{surf face,} \to 0 \)

\[
\langle 0|T \left[ \bar{J}_L^{R} \text{Schwinger} (z) \right] \rangle \text{Connected} \tag{29}
\]

\[
= \int d^2z \ z^2 \text{surf face,} \to 0 \left( \frac{1}{2} \bar{\pi} \partial^2 h - h \partial^2 \bar{\pi} - \langle H \rangle \partial^2 \bar{\pi} \right) (z)
\]

\[
\times h(x_1) \ldots h(x_N) \pi^{t_1}(y_1) \ldots \pi^{t_M}(y_M) \rangle 0 \text{Connected}
\]

The \( (\bar{\pi} \partial^2 h - h \partial^2 \bar{\pi}) \) contribution to the surface integral \( \int d^2z \ z^2 \text{surf face,} \to 0 \) vanishes because \( h \) is massive in spontaneously broken \( SU(2)_L \), \( m_{BEH}^2 \neq 0 \). Propagators connecting \( h \) from points on \( z^2 \text{surf face,} \to 0 \) to the localized interaction points \( (x_1, \ldots, x_N; y_1, \ldots, y_M) \) must stay inside the light-cone, but die off exponentially and are incapable of carrying information to the arbitrarily distant 2-surface.

It is crucially important that this argument fails for the remaining term, \( \frac{1}{2} \langle H \rangle \partial^2 \bar{\pi} (z) \), in \( \int d^2z \ z^2 \text{surf face,} \to 0 \). The \( \bar{\pi} \) are therefore capable of carrying long-ranged pseudo-scalar forces out to the very ends of the light-cone \( (z \to \text{Light Cone} \to \infty) \) (though not inside it). Since it is independent of \( \langle H \rangle \), the \( \bar{J}_L^{R} \) (from \( \Phi \)) contribution to the surface integral \( \int d^2z \ z^2 \text{surf face,} \to 0 \) cannot cancel the \( \bar{J}_L^{R} \) contribution. This shows that, at least in Landau gauge, the spontaneously broken \( SU(2)_L \) charge \( \int \) is not conserved, even in the sense of connected time-ordered products,

\[
\langle 0|T \left[ \frac{d}{dt} \tilde{Q}_L(t) \right] \times h(x_1) \ldots h(x_N) \pi^{t_1}(y_1) \ldots \pi^{t_M}(y_M) \rangle \quad \text{Connected} \tag{30}
\]

\[
\neq 0
\]

dashing all further hope.

The classical proof of the Goldstone theorem \( \Phi \) requires a conserved charge \( \frac{d}{dt} \bar{Q} = 0 \), so that proof fails for spontaneously broken gauge theories. This is a very famous result \( \Phi \), and allows spontaneously broken \( SU(2)_L \) to generate a mass-gap \( m_{W}^{2} \) for the vector \( \tilde{W}^{\mu} \) and avoid massless particles in its observable physical spectrum. This is true, even in \( R_{\xi} (\xi = 0) \) Landau gauge, where there is a Goldstone theorem \( \Phi \), so \( \pi \) are derivatively coupled (hence massless) NGB, and where there is an LSS theorem \( \Phi \), so \( \pi \) is massless.

Massless \( \pi \) (not \( \bar{\pi} \)) is the basis of our pion-pole-dominance-based \( SU(2)_L \) WTI, derived in Appendices \( B \) and \( C \) which give: relations among 1-\( \Phi \) connected amputated \( \Phi \) sector Greens functions \( \Gamma_{N,M} \); 1-soft-pion theorems \( \Phi \), \( \Phi \), \( \Phi \); 1-soft-pion theorems \( \Phi \), \( \Phi \), \( \Phi \); infra-red limitness for \( m_{\pi}^2 = 0 \); \( \Phi \); an LSS theorem \( \Phi \); vanishing 1-\( \Phi \) connected amputated on-shell \( \Phi \) sector T-Matrix elements \( T_{N,M} \); which all realize the full \( SU(2)_L \otimes \text{BRST} \) symmetry of Section II.

C. Construction of the scalar-sector effective Lagrangian from those \( SU(2)_L \) WTIs that govern connected amputated 1-\( \Phi \) Greens functions

In Appendix \( B \) we derive \( SU(2)_L \) "pion-poledominance" 1-\( \Phi \)-R connected amputated T-Matrix WTI \( \Phi \) for CP-conserving SSB \( SU(2)_L \). Their solution is a tower of recursive \( SU(2)_L \) WTI \( \Phi \) that govern 1-\( \Phi \) \( \Phi \)-sector connected amputated Greens functions \( \Gamma_{N,M} \). For \( \pi \) with \( CP = -1 \), the result

\[
\langle H \rangle \Gamma^{t_1 \ldots t_M}_{N,M+1} (p_1 \ldots p_N, q_1 \ldots q_M) = 0
\]

\[
= \sum_{m=1}^{M} \delta^{t_m}_{t_{m+1}} \Gamma^{t_1 \ldots t_{m-1}}_{N+1,M-1} (p_1 \ldots p_N q_m, q_1 \ldots q_{m+1}, q_m)
\]

\[
- \sum_{n=1}^{N} \Gamma^{t_{m+1}}_{N-N-1,M+1} (p_1 \ldots p_n, q_1 \ldots q_{M+1}, q_{M+1})
\]

(31)

is valid for \( N, M \geq 0 \). On the left-hand-side of (31) there are \( N \) renormalized \( h \) external legs (coordinates \( x, \) momenta \( p \)), \( M \) renormalized \( (CP = -1) \) \( \bar{\pi} \) external legs (coordinates \( y, \) momenta \( q, \) isospin \( t \)), and 1 renormalized soft external \( \bar{\pi} (k = 0) \) (coordinates \( z, \) momenta \( k \)). On the right-hand side, "hatted" fields with momenta \( (p_n, q_m) \) are omitted.

The rigid \( SU(2)_L \) WTI 1-soft-pion theorems \( \Phi \) relate a 1-\( \Phi \) Green’s function with \( (N + M + 1) \) external fields (including a zero-momentum \( \bar{\pi} \)) to two 1-\( \Phi \) Green’s functions with \( (N + M) \) external fields.\(^4\)

\(^4\) The rigid \( SU(2)_L \) WTI \( \Phi \) for the \( SU(2)_L \) gauge theory are a generalization of the classic work of B.W. Lee \( \Phi \), who constructed two all-loop-orders renormalized towers of WTIs for the global \( SU(2)_L \otimes SU(2)_R \) Gell-Mann Lévy (GML) model \( \Phi \) with Partially Conserved Axial-vector Currents (PCAC). We replace GML’s strongly-interacting Linear Sigma Model (LSM) with a weakly-interacting BEH LSM, with no explicit PCAC breaking. Replace \( \pi \to H, \bar{\pi} \to \bar{\pi} m_{\pi} \to m_{BEH} \) and \( f_{\pi} \to (H) \), and add local gauge group \( SU(2)_L \). This generates a set of global \( SU(2)_L \) WTI governing relations among weak-interaction 1-\( \Phi \) R T-Matrix elements \( T_{N,M} \). A solution-set of those \( SU(2)_L \) WTI then govern relations among \( SU(2)_L \) 1-\( \Phi \) Green’s functions \( \Gamma_{N,M} \).

As observed by Lee for GML with PCAC, one of those on-shell T-Matrix WTI is equivalent to the Goldstone theorem. This equivalence relies on the ability to incorporate a PCAC term into the global theory, and then retrieve the spontaneously broken theory in the appropriate zero-explicit-breaking limit, namely \( m_{\pi}^2 \to 0 \). In the gauge theory, although explicit-breaking terms are allowed by power-counting, they violate the BRST symmetry and spoil unitarity. Yet, the T-matrix WTI persists and forces \( m_{\pi}^2 = 0 \) in Landau gauge, which is now the new LSS theorem. The Goldstone theorem also persists in Landau gauge, and forces \( m_{\pi}^2 = 0 \). The
Green’s functions $\Gamma^{N,M}_{N,M}(p_1...p_N;q_1...q_M)$ are not themselves gauge-independent. Furthermore, although 1-φ-I, they are 1-$\tilde{W}_\mu$-Reducible (1-$\tilde{W}_\mu$-R) by cutting a transverse $\tilde{W}_\mu$ gauge boson line.

We can now form the $\phi$-sector effective momentum space Lagrangian in Landau gauge:

$$L^{Eff;SU(2)_L}_\phi = \Gamma_{0,0}(0); h + \frac{1}{2!}\Gamma_{1,0}(p_i - p_i); h^2 + \frac{1}{2!}\Gamma_{2,0}(00); h^3 + \frac{1}{4!}\Gamma_{4,0}(0000); h^4 + \frac{1}{4!}\Gamma_{2,2}(00;00)h^3\pi_1\pi_2 + \frac{1}{4!}\Gamma_{4,0}(0000); h^4\pi_1\pi_2 \tag{32}$$

All perturbative quantum loop corrections, to all-loop-orders and including all UVQD, log-divergent and finite contributions, are included in this $\phi$-sector effective Lagrangian: renormalized 1-0-I Green’s functions $\Gamma^{N,M}_{N,M}(p_1...p_N;q_1...q_M)$; wavefunction renormalizations; renormalized $\phi$-scalar propagators [13,19]; the Brout-Englert-Higgs (BEH) VEV ($H$) [28]; all gauge boson and ghost propagators. This includes the full all-loop-orders renormalization of the $SU(2)_L$ $\phi$-sector, originating in quantum loops containing transverse virtual gauge bosons, $\phi$-scalars, anti-ghosts and ghosts (i.e. $\tilde{W}_\mu$, $\tilde{h}$, $\tilde{\pi}$, $\tilde{\omega}$ respectively). Because they arise entirely from global $SU(2)_L$ WTI, our results are independent of regularization scheme [3].

We wish to focus in this paper on finite relevant operators, as well as quadratic and logarithmically divergent operators, arising from $SU(2)_L$ loops. Therefore, in [32], we have separated off three classes of operators that are finite and beyond the scope of this paper:

- Finite $O^{1/2;Irelevent}_{SU(2)_L}$ vanish as $m^2_{Weak}/\Lambda^2 \to 0$;
- $O^{\text{Dim}>4;Light}_{SU(2)_L}$ are finite-dimension $\text{Dim} > 4$ operators, where only the light degrees of freedom ($\tilde{W}_\mu; h, \tilde{\pi}, \tilde{\omega}$) contribute to all-loop-orders renormalization;
- $O^{\text{Dim}\leq 4;NonAnalytic}_{SU(2)_L}$ are finite-dimension $\text{Dim} \leq 4$ operators that are non-analytic in momenta or in a renormalization scale $\mu^2$ (e.g. finite renormalization-group logarithms).

All such operators will be ignored.

$$O^{\text{Ignore}}_{SU(2)_L} = O^{1/2;Irelevent}_{SU(2)_L} + O^{\text{Dim}>4;Light}_{SU(2)_L} + O^{\text{Dim}\leq 4;NonAnalytic}_{SU(2)_L} \tag{33}$$

For clarity, we separate the isospin indices

$$\Gamma_{0,0}(q,q) \equiv \delta^{i_1i_2}\Gamma_{0,0}(q,q), \Gamma_{1,0}(q,0) \equiv \delta^{i_1i_2}\Gamma_{1,0}(q,0), \Gamma_{2,0}(00,00) \equiv \delta^{i_1i_2}\Gamma_{2,0}(00,00), \Gamma_{0,4}(0000) \equiv \Gamma_{0,4}(0000,0000) \tag{34}$$

and write the 1-φ-I $\tilde{h}$ and $h$ (isospin-index-suppressed) inverse propagators as:

$$\Gamma_{0,2}(q, -q) = [\Delta_\pi(q^2)]^{-1} \Gamma_{2,0}(p, -p) = [\Delta_{BEH}(p^2)]^{-1} \tag{35}$$

The quadratic and quartic coupling constants are defined in terms of 2-point and 4-point 1-φ-I connected amputated GF

$$\Gamma_{0,4}(0000,0000) = -2\lambda^2_\phi \Gamma_{0,2}(00,00) = -2m^2_\pi \tag{36}$$

The Ward-Takahashi IDs [31] for Greens functions severely constrain the effective Lagrangian [32]. By using the WTI for $N + M \leq 4$, the all-loop-orders renormalized $\phi$-sector momentum-space effective Lagrangian [32] - constrained only by those $SU(2)_L$ WTI governing Greens functions [31] - may be written:

$$L^{Eff;SU(2)_L;Landau}_\phi = L^{Kinetic;Eff;SU(2)_L;Landau}_\phi - V^{Eff;all-modes}_\phi SU(2)_L = \frac{1}{2} \left( \Gamma_{0,2}(p, -p) - \Gamma_{0,2}(00) \right) h^2 + \frac{1}{2} \left( \Gamma_{0,2}(q, -q) - \Gamma_{0,2}(00) \right) \tilde{h}^2 + \lambda_\phi \left[ \frac{h^2 + \tilde{h}^2}{2} + \langle H \rangle h \right] + \lambda_\phi \left[ \frac{h^2 + \tilde{h}^2}{2} + \langle H \rangle h \right]^2 \tag{37}$$

with finite non-trivial wavefunction renormalizations

$$\Gamma_{0,2}(q, -q) - \Gamma_{0,2}(00) \sim q^2 \tag{38}$$

The $\phi$-sector effective Lagrangian [37] has insufficient boundary conditions to distinguish among the three modes [12,15] of the BRST-invariant Lagrangian $L_{SU(2)_L}$ in [16,37]. For example, the effective potential

$$\frac{1}{4!}\Gamma_{2,2}(00;00) = \frac{1}{4!}\Gamma_{2,2}(00;00) \equiv \frac{1}{4!}\Gamma_{2,2}(00;00) \tag{39}$$

5 The inclusive Gell-Mann Lévy [38] effective potential derived from B.W. Lee’s WTI [12], reduces to the three different effective potentials of the global $SU(2)_L \times SU(2)_R$ Schwinger model [39]: Schwinger Wigner mode ($\langle H \rangle = 0, m^2_\phi = m^2_{BEH} = 0$); Schwinger Scale-Invariant point ($\langle H \rangle = 0, m^2_\phi = m^2_{BEH} = 0$); or Schwinger Goldstone mode ($\langle H \rangle = 0, m^2_\phi = m^2_{BEH} = 0$).
$V_{\text{Eff: all-modes}}^{\text{SU(2)$_L$; Landau}}$ becomes in the appropriate limits:

$$V_{\text{Eff: Wigner}}^{\text{SU(2)$_L$; Landau}} = m_{\pi}^2 \left[ \frac{h^4 + \pi^2}{2} + \lambda_\phi \left( \frac{h^2 + \pi^2}{2} \right)^2 \right],$$

$$V_{\text{Eff: ScaleInvariant}}^{\text{SU(2)$_L$; Landau}} = \lambda_\phi^2 \left[ \frac{h^2 + \pi^2}{2} \right]^2,$$

$$V_{\text{Eff: Goldstone}}^{\text{SU(2)$_L$; Landau}} = \lambda_\phi^2 \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle h \right]^2. \quad (38)$$

Eqn. (37) has exhausted the constraints on the allowed terms in the $\phi$-sector effective Lagrangian due to those $\text{SU}(2)_L$ WTI's governing 1-$\phi$-I $\phi$-sector Green’s functions $\Gamma_{N,M}$ \cite{31,32}. In order to provide boundary conditions that distinguish among the effective potentials in (38), we must turn to the $\text{SU}(2)_L$ WTI's that govern $\phi$-sector 1-$\phi$-R on-shell T-Matrix elements $T_{N,M}$.

D. The Lee-Stora-Symonzik (LSS) Theorem: IR finiteness and automatic tadpole renormalization

"Whether you like it or not, you have to include in the Lagrangian all possible terms consistent with locality and power counting, unless otherwise constrained by Ward identities." Kurt Symanzik, in a private letter to Raymond Stora \cite{30}

Evaluating the effective potential in (37) with $\langle H \rangle \neq 0$, and then in the Kibble representation,

$$V_{\text{Eff: pre-LSS-Goldstone}}^{\text{SU(2)$_L$; Landau}} = m_{\pi}^2 \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle h \right] + \lambda_\phi^2 \left[ \frac{h^2 + \pi^2}{2} + \langle H \rangle h \right]^2$$

$$= m_{\pi}^2 \left[ \phi \phi - \frac{1}{2} \langle H \rangle^2 \right] + \lambda_\phi^2 \left[ \phi \phi - \frac{1}{2} \langle H \rangle^2 \right]^2$$

$$= \frac{m_{\pi}^2}{2} \left[ H_2 - \langle H \rangle^2 \right] + \frac{\lambda_\phi^2}{4} \left[ H_2 - \langle H \rangle^2 \right]^2. \quad (39)$$

As expected, the NGBs $\vec{\pi}$ have disappeared from the effective potential, have purely derivative couplings through their kinetic term, and obey the shift symmetry

$$\vec{\pi} \rightarrow \vec{\pi} + \langle H \rangle \vec{\theta} + \vec{\pi} \times \vec{\theta} + O(\vec{\theta}^2) \quad (40)$$

for constant $\vec{\theta}$. In other words, the Goldstone theorem is already properly enforced.

Eqn. (39) appears at first sight to embrace a disaster: the term linear in $\phi \phi - \frac{1}{2} \langle H \rangle^2$ (a remnant of Wigner mode in (38)) persists, destroying the symmetry of the famous "Mexican hat", and the $\text{SU}(2)_L$ gauge theory is not actually in Goldstone mode!

In strict obedience to K. Symanzik's edict, we now further constrain the allowed terms in the $\phi$-sector effective Lagrangian with those $\text{SU}(2)_L$ Ward-Takahashi identities that govern 1-$\phi$-R on-shell T-Matrix elements $T_{N,M}$.

In Appendix \cite{32} we extend Adler's self-consistency condition (originally written for the global $\text{SU}(2)_L \times \text{SU}(2)_R$ Gel-Man-Lévy Linear Sigma Model with PCAC \cite{32}, but now derived for the CP-conserving $\text{SU}(2)_L$ gauge theory in Landau gauge \cite{30,31}),

$$(H)T_{N,M+1}^{\text{SU}(2)}(p_1 \ldots p_N; q_1 \ldots q_M)\times(2\pi)^4 \delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right) p_1^2 = \ldots = p_N^2 = m_{\pi}^2$$

$$= 0 \quad (41)$$

The T-matrix vanishes as one of the pion moments goes to zero (i.e. these are 1-soft-pion theorems), provided all other physical scalar particles are on mass-shell. Eqn. \cite{32} also

"asserts the absence of infrared (IR) divergences in the scalar-sector (of $\text{SU}(2)_L$) Goldstone mode (in Landau gauge). Although individual Feynman diagrams are IR divergent, those IR divergent parts cancel exactly in each order of perturbation theory. Furthermore, the Goldstone mode amplitude must vanish in the soft-pion limit. B.W. Lee \cite{12} also...

It is crucial to note that the external states in $T_{N,M}$ are $N$ $h$'s and $M$ $\pi$'s, not $\vec{p}$'s. We are working in the soft-$\pi$, not the soft-$\vec{p}$ limit.

The $N = 0, M = 1$ case of (41) is the LSS theorem:

$$(H)T_{0,1}^{\text{SU}(2)}(0; 0) = 0 \quad (42)$$

This looks like the Goldstone Theorem, but, since it involves $\vec{p}$ not $\vec{p}$ it is quite distinct. Since the 2-point

$$\gamma_{\text{PCAC}} = -(H)T_{0,1}^{\text{SU}(2)}(0,0) \quad (43)$$

as the "Goldstone theorem in the presence of PCAC."

The PCAC analogy for Landau-gauge $\text{SU}(2)_L$ would have been

$$\delta_{\mu} T_{\mu}^{\text{SU}(2)} = \gamma_{\text{PCAC}} \vec{\pi} + \langle H \rangle \times \text{a gauge-fixing term} \quad (44)$$

but $\text{SU}(2)_L$ is a local/gauge theory. This requires that $\gamma_{\text{PCAC}}^L = 0$ exactly. SSB current conservation can be broken only softly by gauge-fixing terms as in \cite{34}, in order to preserve renormalizability and unitarity \cite{33}. The Landau-gauge $\text{SU}(2)_L$ LSS theorem therefore reads

$$(H)T_{0,1}^{\text{SU}(2)}(0; 0) = \gamma_{\text{PCAC}}^L \equiv 0 \quad (45)$$

as in \cite{12}. Crucially, with $H \neq 0$ in the SSB Goldstone mode of $\text{SU}(2)_L$ (and of $\text{SU}(2)_L \times U(1)_Y$, and of the $\nu_D \tilde{SM}CP$ in section \cite{34}),

$$0 = T_{0,1}^{\text{SU}(2)}(0; 0) = [\Delta_\pi(0)]^{-1} = -m_{\pi}^2 \quad (46)$$

6 B.W. Lee \cite{12} proves two towers of WTI for the global $\text{SU}(2)_L \times \text{SU}(2)_R$ Gel-Man-Lévy model (GML) \cite{17} in the presence of the Partially Conserved Axial-vector Current (PCAC) hypothesis. PCAC conserves the vector current $\sum_{\mu} T_{\mu}^{\text{GML}} = 0$, but explicitly breaks the axial-vector current, $\sum_{\mu} T_{\mu}^{\text{GML}} = \gamma_{\text{GML}} \vec{\pi}$. Lee identifies the all-loop-orders GML WTI

$$\gamma_{\text{PCAC}} = -(H)T_{0,1}^{\text{GML}}(0,0) \quad (43)$$

as the "Goldstone theorem in the presence of PCAC."

7 The PCAC analogy for Landau-gauge $\text{SU}(2)_L$ would have been

$$\delta_{\mu} T_{\mu}^{\text{SU}(2)} = \gamma_{\text{PCAC}} \vec{\pi} + \langle H \rangle \times \text{a gauge-fixing term} \quad (44)$$

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$$(H)T_{0,1}^{\text{SU}(2)}(0; 0) = \gamma_{\text{PCAC}}^L \equiv 0 \quad (45)$$

as in \cite{12}. Crucially, with $H \neq 0$ in the SSB Goldstone mode of $\text{SU}(2)_L$ (and of $\text{SU}(2)_L \times U(1)_Y$, and of the $\nu_D \tilde{SM}CP$ in section \cite{34}),

$$0 = T_{0,1}^{\text{SU}(2)}(0; 0) = [\Delta_\pi(0)]^{-1} = -m_{\pi}^2 \quad (46)$$
\[ T_{0,2} \text{ is already 1-}\phi \text{-I}\] we may write the LSS theorem as a further constraint on the 1-\phi-I Greens function (and thus the \( \bar{\pi} \) mass)

\[
(H)\Gamma_{0,2}(0; 0) = (H)[\Delta_{\pi}(0)]^{-1} = (H)m_{\pi}^2 = 0. \quad (48)
\]

To the rescue of the effective potential \((39)\), the LSS theorem \((42)\) (and particularly in the form \((48)\), not the shift symmetry \((40)\) and Goldstone theorem, forces the \(SU(2)_L\) gauge theory fully into its true Goldstone mode, with

\[
V_{\text{AHM,\phi, Lorenz}}^{\text{Eff, LSS-Goldstone}} = \frac{\lambda^2_{\phi}}{4} \left[ \bar{\pi}^2 - (H)^2 \right]^2 = \frac{\lambda^2_{\phi}}{4} \left[ \phi^4 - \frac{1}{2}(H)^2 \right] \quad (49)
\]

\[
= \frac{\lambda^2_{\phi}}{4} \left[ \frac{h^2 + \pi^2}{2} + (H)h \right]^2.
\]

A central result of this paper is to recognize that, in order to force \((39)\) to become \((49)\), requires the LSS theorem, a new on-shell T-Matrix symmetry that is not a symmetry of the BRST-invariant \(SU(2)_L\) Lagrangian. \(SU(2)_L\) physics, but not its Lagrangian, has the \(SU(2)_L \otimes \text{BRST}\) symmetry of Section II, a conserved current \((21, 25)\), un-deformed WTIs governing connected amputated Green’s functions \((31)\), and un-deformed WTIs governing connected amputated on-shell T-Matrix elements \((41)\).

A crucial effect of the LSS theorem, together with the \(N = 0, M = 1 SU(2)_L\) Ward-Takahashi Green’s-function identity \((31)\), is to automatically eliminate tadpoles in \((32)\)

\[
\Gamma_{1,0}(0; \cdot) = (H)\Gamma_{0,2}(0; 0) = 0, \quad (50)
\]

so that separate tadpole renormalization is unnecessary. This is in contrast to \([6]\) and \([19]\) where the authors explicitly renormalize the tadpole contributions to 0. However, the LSS theorem tells us that this is unnecessary, when the full symmetries of the theory (in this case the ones concerning on-shell T-matrix elements) are imposed the tadpole contributions vanish exactly.

**E. Further constraints on the \(\phi\)-sector effective Lagrangian:**

\[ m_{\bar{\pi}E H}^2 = 2\lambda^2_{\phi}(H)^2 \]

The \(\phi\)-sector Goldstone-mode coordinate-space Landau-gauge effective Lagrangian of spontaneously broken \(SU(2)_L\), constrained by the Lee-Store-Symanzik theorem, can now be written

\[
L_{\text{Eff,Goldstone}}^{SU(2)_L; \phi; \text{Landau}} = \left| D_{\mu} \phi \right|^2 - \frac{\lambda^2_{\phi}}{2} \left[ \frac{h^2 + \pi^2}{2} + (H)h \right]^2 + \mathcal{O}(SU(2)_L)_{\text{Ignore}}.
\]

The LSS theorem \((42)\) has caused all relevant operators in spontaneously broken \(SU(2)_L\) to vanish! This effective Lagrangian \((41)\):

- is derived from the local BRST-invariant \(SU(2)_L\) Lagrangian \(L_{SU(2)_L}\) \((16)\);
- includes all divergent \(O(\Lambda^2), O(\ln \Lambda^2)\) and finite terms that arise to all perturbative loop-orders in the full \(SU(2)_L\) gauge theory, due to virtual transverse gauge bosons, \(\phi\) scalars, anti-ghosts and ghosts \((\tilde{W}; h, \bar{\pi}; \tilde{q}, \bar{\omega}\) respectively);
- obeys the LSS theorem \((42)\) and all other \(SU(2)_L\) Ward-Takahashi Green’s function and T-Matrix identities;
- obeys the Goldstone theorem in the Landau gauge, having massless derivatively coupled \(\bar{\pi}\) NGBs;
- has its vacuum automatically forced to be at \(H = (\langle \bar{\pi} \rangle = 0)\) by \(SU(2)_L \otimes \text{BRST}\) symmetry, and obeys stationarity \((16)\) of that true minimum;
• preserves the theory’s renormalizability and unitarity, which require that wavefunction renormalization, \( \langle H \rangle_{\text{bare}} = [Z_{\text{AHM}}]^{1/2} \langle H \rangle [16, 18, 27] \), and forbid UVQD, relevant, or any other dimension-2

The Green’s function Ward-Takahashi ID [31] for \( N = 1, M = 1 \), constrained by the LSS theorem [18], relates the BEH mass to the coefficient of the \( h\overline{\phi}^2 \) vertex

\[
\Gamma_{2,0}(00) = \langle H \rangle_{\text{bare}} = (H)_{1,0}(00; 00)\ . \quad (52)
\]

Therefore, the BEH mass-squared

\[
m_{\text{BEH}}^2 = 2\lambda^2 (H)^2 \quad (53)
\]

arises entirely from SSB, as does (together with its \( \mu \) parameters) the resonance pole-mass-squared

\[
m_{\text{BEH: Pole}}^2 = 2\lambda^2 (H)^2 \left[ 1 - 2\lambda^2 (H)^2 \int dm^2 \frac{\rho_{\text{BEH}}(m^2)}{m^2 - i\epsilon} \right]^{-1} + O^\text{ignore}_{S\text{U}(2)\text{L}; \phi^*} \quad (54)
\]

\section{IV. \( \mathcal{G} \otimes \text{BRST Symmetry in Simple Groups in Linear Gauges} \)}

Our results are ubiquitous to gauge theories which spontaneously break a Lie algebraic structure group \( \mathcal{G} \) at a low scale, say for definiteness \( m_{\text{weak}} \). In this Section we prove \( \mathcal{G} \otimes \text{BRST} \) symmetry and motivate its 2 associated towers of 1-soft-pion WTI – one for Green’s functions and another for on-shell T-Matrix elements, including the LSS theorem for \( \mathcal{G} \) at \( m_{\text{weak}} \). These result, as above, in severe WTI constraints on the \( m_{\text{weak}} \)-scalars’ effective potential. We illustrate this in Section V with the specific example of the Standard Model, in which there spontaneous symmetry breaking of \( \text{SU}(2)_L \otimes \text{U}(1)_Y \) to \( \text{U}(1)_{\text{QED}} \otimes \text{BRST} \).

In linear gauges, the Lagrangian, incorporating gauge-fixing and DeWitt-Faddeev-Popov ghost terms [20, 21] is written in terms of gauge, matter, ghost, and anti-ghost fields \( (A_\mu, \Psi, \omega, \overline{\eta}) \), and a Nakanishi-Lautrup field [22, 23] \( b \).

\[
L_{\mathcal{G}} = L_{\mathcal{G}}^{\text{GaugeInvariant}}(A_\mu, \Psi) + s \left( \overline{\eta} \left[ \mathcal{F} + \frac{1}{2} \slashed{b} \right] \right) \quad (55)
\]

There is a gauge coupling \( g \) for the simple gauge group \( \mathcal{G} \), and the gauge-fixing function \( \mathcal{F} \) depends on parameters \( (\xi, M) \).

\[
\mathcal{F} = f \partial_\mu A_\mu + M \Psi \quad (56)
\]

The gauge fields \( A_\mu \), the ghost field \( \omega \), and the parameter \( f \) belong to the adjoint representation; the matter fields \( \Psi \) and the parameter \( M \) belong to the fundamental representation; and the anti-ghost field \( \overline{\eta} \) and the Nakanishi-Lautrup auxiliary field \( b \) belong to the singlet representation of \( \mathcal{G} \). The structure group \( \mathcal{G} \)

\[
[t_\alpha, t_\beta] = i C^\gamma_{\alpha\beta} t_\gamma \quad (57)
\]

is simple, with structure constants \( C^\gamma_{\alpha\beta} \).

Global BRST transformations [3, 13, 22, 24] \( s \) of the fields and parameters are given by

\[
\begin{align*}
& s A_\mu = D_\mu \omega; \quad s \Psi = t(\omega) \Psi; \\
& s \omega = - \frac{1}{2} [\omega, \omega]; \quad s \overline{\eta} = b; \\
& s b = 0; \quad s M = \Lambda; \quad s f = \lambda; \\
& s \lambda = 0; \quad s \xi = 0. \quad (58)
\end{align*}
\]

These ensure that the Lagrangian (55) is BRST invariant \( s L_{\mathcal{G}} = 0 \).

Meanwhile, we define the properties of the fields and parameters under the usual anomaly-free un-deformed rigid/global structure group \( \mathcal{G} \). \( \delta_{\mathcal{G}} \) transforms fields by constant \( \Omega \)

\[
\begin{align*}
& \delta_{\mathcal{G}} A_\mu = [A_\mu, \Omega]; \quad \delta_{\mathcal{G}} \Psi = - t(\Omega) \Psi; \\
& \delta_{\mathcal{G}} \omega = [\omega, \Omega]; \quad \delta_{\mathcal{G}} \overline{\eta} = 0; \\
& \delta_{\mathcal{G}} b = 0; \quad \delta_{\mathcal{G}} M = t(\Omega) M; \quad \delta_{\mathcal{G}} \xi = t(\Omega) \xi; \\
& \delta_{\mathcal{G}} \lambda = t(\Omega) \lambda; \quad \delta_{\mathcal{G}} \omega = \lambda \text{adj}(\Omega). \quad (60)
\end{align*}
\]

The transformation sets (58) and (60) commute

\[
\begin{align*}
& \left[ \delta_{\mathcal{G}}, s \right] A_\mu = 0; \quad \left[ \delta_{\mathcal{G}}, s \right] \Psi = 0; \\
& \left[ \delta_{\mathcal{G}}, s \right] \omega = 0; \quad \left[ \delta_{\mathcal{G}}, s \right] \overline{\eta} = 0; \\
& \left[ \delta_{\mathcal{G}}, s \right] b = 0; \quad \left[ \delta_{\mathcal{G}}, s \right] M = 0; \quad \left[ \delta_{\mathcal{G}}, s \right] f = 0; \\
& \left[ \delta_{\mathcal{G}}, s \right] \Lambda = 0; \quad \left[ \delta_{\mathcal{G}}, s \right] \lambda = 0. \quad (61)
\end{align*}
\]

The Lagrangian (55) is not invariant under \( \mathcal{G} \) transformations:

\[
\delta_{\mathcal{G}} L_{\mathcal{G}} = s \left( \overline{\eta} \delta_{\mathcal{G}} \left[ \mathcal{F} + \frac{1}{2} \slashed{b} \right] \right) \neq 0
\]

Still, with (55, 56) and the nilpotent property \( s^2 = 0 \),

\[
\left[ \delta_{\mathcal{G}}, s \right] L_{\mathcal{G}} = 0, \quad (62)
\]

and the two separate global symmetries can therefore co-exist in \( \mathcal{G} \) physics.

The \( \mathcal{G} \) gauge theory thus simultaneously obeys both the usual BRST symmetry and a global \( \mathcal{G} \) symmetry that controls Green’s functions and on-shell T-Matrix elements. Global \( \mathcal{G} \) symmetry results, as above for \( \text{SU}(2)_L \) in 1-soft-\( \pi \) theorems:

• A tower of rigid SSB \( \mathcal{G} \) WTI governing relations among Green’s functions.
• A tower of rigid SSB \( \mathcal{G} \) WTI which force on-shell 1-soft-pion T-Matrix elements to vanish, including a Lee-Stora-Symanzik theorem.
This is despite the fact that the $\mathcal{G}$ Lagrangian is not invariant under structure group $\mathcal{G}$ transformations!

We choose the gauge-fixing function $\mathcal{F}$ to be the most general function that is a Lorenz scalar, and is linear in the fields and its derivatives. The results discussed here only hold for such linear gauge-fixing functions. The translation to $R_\xi$ gauge for the $SU(2)_L$ theory is:

\[
A^\mu = W^\mu; \quad \Psi = (\pi^1, \pi^2, \pi^3); \quad (63)
\]

\[
f = 1; \quad \mathcal{M} = \frac{1}{2} g_2 \langle H \rangle;
\]
and

\[
\delta_\mathcal{G} \frac{1}{2} g_2 \langle H \rangle = 0; \quad \delta_\mathcal{G} 1 = 0; \quad (64)
\]

where $W^\mu = \tilde{\eta} \cdot \tilde{W}^\mu$. With these choices, $[\delta_\mathcal{G}, s] = 0$ still holds for $\mathcal{G} = SU(2)_L$.

V. SPONTANEOUS SYMMETRY BREAKING
OF THE LANDAU-GAUGE STANDARD
$SU(2)_L \otimes U(1)_Y \otimes BRST$ ELECTROWEAK MODEL:
GAUGE BOSONS, COMPLEX SCALAR
DOUBLET, GHOSTS AND ANTI-GHOSTS

The Lagrangian $L_{2\otimes 1}$ for the spontaneously broken $SU(2)_L \otimes U(1)_Y \otimes BRST$ electroweak model in $R_\xi$ gauge is given in (121). It incorporates: $SU(2)$ and $U(1)$ gauge-boson fields $\tilde{W}_\mu$ and $B_\mu$; a Higgs field $H$, pseudoscalar fields $\bar{\pi}$; $SU(2)$ and $U(1)$ ghosts $\bar{\omega}$ and $\omega_B$, and anti-ghosts $\bar{\eta}$ and $\eta_B$; and $SU(2)$ and $U(1)$ Nakanishi-Lautrup fields $\bar{b}$ and $b_B$.

A. Global $SU(2) \times U(1)_Y$ and BRST transformations

The global BRST transformations $s_{2\otimes 1}$, for the fields in $L_{2\otimes 1}$ are:

\[
s_{2\otimes 1} \tilde{W}_\mu = \partial_\mu \bar{\omega} + g_2 \tilde{W}_\mu \times \bar{\omega}
\]

\[
s_{2\otimes 1} H = -\frac{1}{2} g_2 \bar{\pi} \cdot \bar{\omega} - \tilde{\epsilon} \pi_3 \omega_B
\]

\[
s_{2\otimes 1} \bar{\pi} = \frac{1}{2} g_2 \left( H \bar{\omega} + \bar{\pi} \times \bar{\omega} \right) + \tilde{\epsilon} \left( -\pi_2, \pi_1, H \right) \omega_B
\]

\[
s_{2\otimes 1} \bar{\omega} = -\frac{1}{2} g_2 \bar{\omega} \times \bar{\omega}
\]

\[
s_{2\otimes 1} \bar{\eta} = \bar{b}
\]

\[
s_{2\otimes 1} \bar{b} = 0
\]

\[
s_{2\otimes 1} B_\mu = \partial_\mu \omega_B
\]

\[
s_{2\otimes 1} \omega_B = 0
\]

\[
s_{2\otimes 1} \eta_B = b_B
\]

\[
s_{2\otimes 1} b_B = 0.
\]

These transformation are nilpotent, $s_{2\otimes 1}s_{2\otimes 1} = 0$.

Anomaly-free un-deformed rigid/global $\delta_{SU(2)_L \times U(1)_Y}$ transforms these fields by constant $\bar{\Omega}$ and $\Omega_B$, according to:

\[
\delta_{2\otimes 1} \tilde{W}_\mu = \partial_\mu \bar{\omega} + g_2 \tilde{W}_\mu \times \bar{\omega}
\]

\[
\delta_{2\otimes 1} H = -\frac{1}{2} g_2 \bar{\pi} \cdot \bar{\omega} - \tilde{\epsilon} \pi_3 \omega_B
\]

\[
\delta_{2\otimes 1} \bar{\pi} = \frac{1}{2} g_2 \left( H \bar{\omega} + \bar{\pi} \times \bar{\omega} \right) + \tilde{\epsilon} \left( -\pi_2, \pi_1, H \right) \omega_B
\]

\[
\delta_{2\otimes 1} \bar{\omega} = g_2 \bar{\omega} \times \bar{\omega}
\]

While all other fields $- B_\mu, \omega_B, \bar{\eta}, \eta_B, \bar{b}, b_B$, are singlets.

The transformation sets (65) and (66) have a conserved current $\tilde{J}^\mu_{2\otimes 1}$ still be the most

Although the anti-ghosts $\bar{\eta}$ transform in the usual way, as a triplet $s \bar{\eta} = \bar{b}$ under BRST in (65), we have chosen them to be singlets under global $SU(2)_L$ transformations $\delta_{SU(2)_L} \bar{\eta} = 0$. This does not affect the proof of renormalizability and unitarity with Slavnov-Taylor identities [25]. (The subject of a future paper, but outside the scope of this one.)

This freedom to choose $\bar{\eta} \bar{\eta}$ and $\eta_B$ as singlets under $SU(2)_L \times U(1)_Y$ renders the ghost Lagrangian without undefinite $\delta_{SU(2)_L \times U(1)_Y}$ properties. However, it allows one to build a classical $SU(2)_L \times U(1)_Y$ current $\tilde{J}^\mu_{2\otimes 1}$ and a classical $SU(2)_L$ sub-current $\tilde{J}^\mu_{L:2\otimes 1}$, both conserved up to gauge-fixing terms. Together with CP conservation, this will be the basis of our global $SU(2)_L \times R$ Ward-Takahashi identities for the $SU(2)_L \times U(1)_Y$ gauge theory.

B. Classical global $SU(2)_L$ sub-current

In Appendix D we constructed the $SU(2)_L \times U(1)_Y$ isospin sub-current $\tilde{J}^\mu_{L:2\otimes 1}$, together with its divergence, and commutators with $\phi$ in an arbitrary $R_{2\otimes 1}$ gauge.

Because $SU(2)_L$ is an applicable sub-group of $SU(2)_L \times U(1)_Y$, $\left[ \delta_{SU(2)_L}, s_{2\otimes 1} \right] = 0$ when acting on any of the fields $- \tilde{W}^\mu, \bar{\omega}, H, \bar{\eta}, \bar{\pi}, \bar{b}, \omega_B, \eta_B$ or $b_B$; thus

\[
\left[ \delta_{SU(2)_L}, s_{2\otimes 1} \right] L_{2\otimes 1} = 0.
\]

There is therefore a conserved (up to gauge-fixing terms) $SU(2)_L \times U(1)_Y$ isospin sub-current $\tilde{J}^\mu_{L:2\otimes 1}$. In Landau
gauge the sub-current and its divergence are
\[ J_{L,2}^{\mu} = J_{L,Schwinger}^{\mu} + J_{L,2}^{\mu}, \]
\[ J_{L,2}^{\mu} = \frac{1}{2} \tilde{\pi} \cdot \partial \tilde{\pi} + \frac{1}{2} \left( \tilde{\pi} \partial \tilde{\pi} - H \partial \tilde{\pi} \right), \]
\[ J_{L,2}^{\nu} = \tilde{W}_{\nu} \times \tilde{W}_{\nu}, \]
\[ + \frac{1}{2} g_2 B^\mu \left( \pi_1 \pi_3 - \pi_2 H, \quad \pi_2 \pi_3 + \pi_1 H, \quad \frac{1}{2} \left( H^2 + \pi_3^2 - \pi_1^2 - \pi_2^2 \right) \right) \]
\[ - \frac{1}{4} g_2 \tilde{W}^\mu \left[ H^2 + \pi^2 \right] \]
\[ - \lim_{\xi \to 0} \frac{1}{\xi} \left( \tilde{W}^\mu \times \tilde{F}_W(0) \right) \]
\[ - \partial \tilde{\mu} \cdot \tilde{\omega}, \]
\[ \tilde{F}_W(0) = \partial \tilde{\phi} \tilde{\omega}, \quad F_B(0) = \partial \phi B^\phi \]
\[ M_\phi = \frac{1}{2} g_2^2 (H); \quad M_B = \bar{\varepsilon}(H) \quad (70) \]
\[ \bar{\varepsilon} = \frac{1}{2} Y_\phi g_1 = \frac{1}{2} g_1; \quad M_\phi + M_B = M_\phi. \]
These sub-currents obey commutation relations:
\[ \delta(0 - y_0) \left[ J_{L,2}^{\mu}(z) - J_{L,Schwinger}^{\mu}(z), H(y) \right] = 0, \]
\[ \delta(0 - y_0) \left[ J_{L,2}^{\mu}(z) - J_{L,Schwinger}^{\mu}(z), \tilde{\pi}(y) \right] = 0, \]
\[ \delta(z^0 - y^0) \left[ J_{L,Schwinger}^{\mu}(z), H(y) \right] \]
\[ = - \frac{1}{2} i \delta^t(z - y) \tilde{\pi}(z), \]
\[ \delta(z^0 - y^0) \left[ J_{L,Schwinger}^{\mu}(z), \pi^\mu(y) \right] \]
\[ = \frac{1}{2} i \delta^t(z - y) \times \left( \pi^j t z^j \pi^t z + \delta^t t H(z) \right). \]
The expression in general \( R_\xi \) gauges can be found in Appendix D.

C. \( SU(2)_L \times U(1)_Y \) in Landau gauge

The \( SU(2)_L \times U(1)_Y \) Lagrangian in Landau gauge can be obtained by setting \( \xi = 0 \) in (19). We work in the linear representation of the scalar field in the Goldstone mode of the theory for reasons explained in Section III.A, and again define the exact propagators of the \( \tilde{\pi} \) and \( \tilde{h} \) fields using the Källén-Lehmann representation.

Analysis is again done in terms of the exact renormalized interacting fields, which asymptotically become the in/out states, i.e., free fields for physical S-Matrix elements.

\[ \Delta_{\varphi,1}^{(\mu)}(q^2) = -i(2\pi)^2 (0) T \left[ \pi^\mu(1)(y) \pi^\mu(0) \right] |\text{Fourier Transform} \]
\[ \Delta_{\varphi,1}(q^2) = \frac{1}{q^2 + i\varepsilon} + \int \frac{d^2p_{\varphi,1}(m^2)}{q^2 - m^2 + i\varepsilon} \quad (72) \]
\[ \left[ Z_{\varphi,1}^{\phi} \right]^{-1} = 1 + \int \frac{d^2p_{\varphi,1}(m^2)}{q^2 - m^2 + i\varepsilon}. \]
Define also the BEH scalar propagator in terms of a BEH scalar pole and the (subtracted) spectral density \( \rho_{\varphi,1}^{BEH} \), and the same wavefunction renormalization. We assume \( \hbar \) decays weakly, and resembles a resonance:
\[ \Delta_{\varphi,1}^{BEH}(q^2) = -i(2\pi)^2 (0) T [h(x)h(0)] |\text{Fourier Transform} \]
\[ = \frac{1}{q^2 - m_{BEH}^2 + i\varepsilon} + \int \frac{d^2\rho_{\varphi,1}^{BEH}(m^2)}{q^2 - m^2 + i\varepsilon} \]
\[ \left[ Z_{\varphi,1}^{\phi} \right]^{-1} = 1 + \int \frac{d^2\rho_{\varphi,1}^{BEH}(m^2)}{q^2 - m^2 + i\varepsilon}. \]

The spectral density parts of the propagators are
\[ \Delta_{\varphi,1}^{\mu \pi \text{Spectral}}(q^2) \equiv \int \frac{d^2\rho_{\varphi,1}^{BEH}(m^2)}{q^2 - m^2 + i\varepsilon} + \int \frac{d^2\rho_{\varphi,1}^{BEH}(m^2)}{q^2 - m^2 + i\varepsilon} \]

Dimensional analysis of the wavefunction renormalizations (72, 73), shows that the finite Euclidean cut-off contributes only irrelevant terms \( \sim 1/\bar{x} \).

We are interested in rigid-symmetric relations among 1-\( \phi \)-I connected amputated Green’s functions \( \Gamma_{N,M} \), and among 1-\( \phi \)-R connected amputated transition-matrix (TM-Matrix) elements \( T_{N,M} \), with external \( \phi \) scalars and we separate them as
\[ T_{N,M} = \Gamma_{N,M} + 1 - \phi - R. \quad (74) \]

Because these are 1-\( \tilde{W}_\mu, B_\mu \)-R in \( SU(2)_L \times U(1)_L \) (i.e. reducible by cutting a \( \tilde{W}_\mu \) or \( B_\mu \) line), it is even more convenient to use the powerful old tools from Vintage Quantum Field Theory.

We focus on the rigid/global \( SU(2)_L \) current constructed in Appendix D and displayed in (D17) in Landau gauge.

Rigid/global transformations of the fields arise, as usual, from the equal-time commutators:
\[ \delta_{SU(2)_L} H(t, \bar{y}) = -ig_2 \Omega^1 \int d^3 \bar{z} \left[ J_{L,2}^{\mu}(t, \bar{z}), H(t, \bar{y}) \right] \]
\[ = -\frac{1}{2} g_2 \pi^\mu(t, \bar{y}) \Omega^1 \quad (75) \]
\[ \delta_{SU(2)_L} \pi^\mu(t, \bar{y}) = -ig_2 \Omega^2 \int d^3 \bar{z} \left[ J_{L,2}^{\mu}(t, \bar{z}), \pi^\mu(t, \bar{y}) \right] \]
\[ = \frac{1}{2} g_2 \left[ H(t, \bar{y}) \Omega^1 + \epsilon^t t z^t \pi^t(t, \bar{y}) \Omega^1 \right] \]
so \( J_{\mu,2\otimes 1}^\mu(t,\vec{z}) \) serves as a “proper” local current for commutator purposes.

The classical equations of motion reveal a crucial fact: due to gauge-fixing terms in the BRST-invariant Lagrangian \([D5]\), the classical current \([D17]\) is not conserved. In Landau gauge

\[
\partial_\mu \bar{J}_\mu^\mu = \frac{1}{2} M_W \left[ \bar{\pi} \times \partial_\mu \bar{W}^\mu + H \partial_\mu \bar{W}^\mu \right] + \frac{1}{2} M_B \partial_\mu B^\mu (\omega, \pi_2, \pi_1, H) \tag{76}
\]

The global \( SU(2)_L \) current \([D17]\) is, however, conserved by the physical states, and therefore still qualifies as a “real” current. Strict quantum constraints are imposed that force the relativistically-covariant theory of gauge bosons to propagate only its true number of states, but not the BRST-invariant Lagrangian \([D5]\), then obeys G. ’t Hooft’s gauge-fixing \([26]\) conditions

\[
\langle 0 | T \left[ \left( \partial_0 \bar{W}^\mu(z) \right) \right. \right.
\]
\[
\times h(x_1) \ldots h(x_N) \pi^1(y_1) \ldots \pi^M(y_M) | 0 \rangle_{\text{connected}} = 0
\]
\[
\langle 0 | T \left[ \left( \partial_0 B^\mu(z) \right) \right. \right.
\]
\[
\times h(x_1) \ldots h(x_N) \pi^1(y_1) \ldots \pi^M(y_M) | 0 \rangle_{\text{connected}} = 0 . \tag{77}
\]

Here we have \( N \) external renormalized scalars \( h \) (coordinates \( x, \) momenta \( p \)), and \( M \) external (\( CP = -1 \)) renormalized pseudo-scalars \( \pi \) (coordinates \( y, \) momenta \( q, \) isospin \( t \)).

Eqs. (77) restore conservation of the rigid/global \( SU(2)_L \) current for \( \phi \)-sector connected time-ordered products

\[
\langle 0 | T \left[ \left( \partial_0 \bar{J}_{\mu,2\otimes 1}^\mu(z) \right) \right. \right.
\]
\[
\times h(x_1) \ldots h(x_N) \pi^1(y_1) \ldots \pi^M(y_M) | 0 \rangle_{\text{connected}} = 0 \tag{78}
\]

It is in this “physical” connected-time-ordered product sense that the classical global \( SU(2)_L \) “physical current” is conserved: the physical states, but not the BRST-invariant Lagrangian \([D5]\), obey the physical-current conservation equation \([78]\). It is this physical conserved current that generates our \( SU(2)_L \otimes \text{BRST} \) WTI.

The crucial observation for spontaneously broken \( SU(2)_L \times U(1)_Y \) that is, just as in Section \( [\Pi] \)

\[
\int_{\text{LightCone}} dz \: z_{\text{LightCone}} \langle 0 | T \left[ \left( \frac{1}{2} \{ H, \partial_0 \bar{\pi} \} \right) \right. \right.
\]
\[
\times h(x_1) \ldots h(x_N) \pi^1(y_1) \ldots \pi^M(y_M) | 0 \rangle_{\text{connected}} \neq 0 . \tag{79}
\]

\( \bar{\pi} \) is massless in Landau gauge, capable of carrying (along the light-cone) long-ranged pseudo-scalar forces out to the very ends of the light-cone \( (z_{\text{LightCone}} \to \infty) \), but not.

Eq. \([79]\) shows that the spontaneously broken \( SU(2)_L \) charge

\[
\bar{Q}_{L,2\otimes 1}(t) \equiv \int d^3zd\bar{J}_{\mu,2\otimes 1}^\mu(t,\vec{z}) \tag{80}
\]

is not conserved, even for connected time-ordered-products, in Landau gauge

\[
\langle 0 | T \left[ \left( \frac{d}{dt} \bar{Q}_{L,2\otimes 1}(t) \right) \right. \right.
\]
\[
\times h(x_1) \ldots h(x_N) \pi^1(y_1) \ldots \pi^M(y_M) | 0 \rangle_{\text{Connected}} \neq 0 . \tag{81}
\]

The classical proof of the Goldstone theorem \([S, 10, 11]\) requires a conserved charge \( \frac{d}{dt} \bar{Q} = 0 \), so that proof fails \([S, 35, 37]\) for spontaneously broken gauge theories, allowing spontaneously broken electroweak \( SU(2)_L \times U(1)_Y \) to generate mass-squared-gaps \( M_W^2, M_B^2 \) for the \( W^\pm, Z^0 \), and \( M_B^2 = M_B^2 + M_B^2 = \frac{1}{4} (g_Y^2 + g_Z^2)(\omega^2) \) for the neutral current boson \( Z^0 \), and avoid massless particles apart from the photon \( A^\mu \) in its observable physical spectrum. This is true, even in \( E \) \( (\omega = 0) \) Landau gauge, where there is a Goldstone theorem \([S, 37]\), so \( \bar{\pi} \) are derivatively coupled (hence massless) NGB, and where there is an LSS theorem \([1]\), so \( \bar{\pi} \) is massless.

Massless \( \bar{\pi} \) (not \( \bar{\pi} \)) is the basis of our pion-pole-dominance-based \( SU(2)_L \otimes U(1)_Y \) WTIWs, derived in Appendices \( [D, E] \) which give: relations among 1-\( \phi \)-I connected amputated \( \phi \)-sector Greens functions \( G^{2\otimes 1}_{N,M,E} \); 1-soft-pion theorems \([96, E19, E26]\); infra-red finiteness for \( m^2_\pi = 0 \) \([96, E19]\); an LSS theorem \([97]\); vanishing 1-\( \phi \)-\( R \) connected amputated on-shell \( \phi \)-sector T-Matrix elements \( T^{2\otimes 1}_{N,M} \) \([96, E26]\); which all realize the full \( SU(2)_L \otimes U(1)_Y \otimes \text{BRST} \) symmetry of this Section.

D. Construction of the scalar-sector effective \( SU(2)_L \times U(1)_Y \) Lagrangian from those \( SU(2)_L \otimes R \) WTIs that govern connected amputated 1-\( \phi \)-I Greens functions

We now consider the global axial-vector isospin current \( \bar{J}_{\mu,2\otimes 1}^\mu \) forming time-ordered amplitudes of products of \( \bar{J}_{\mu,2\otimes 1}^\mu \) with \( N \) scalars (coordinates \( x, \) momenta \( p \)) and \( M \) pseudo-scalars (coordinates \( y, \) momenta \( q, \) isospin \( t \))

\[
\langle 0 | T \left[ \left( \bar{J}_{\mu,2\otimes 1}^\mu(z) \right) \right. \right.
\]
\[
\times h(x_1) \ldots h(x_N) \pi^1(y_1) \ldots \pi^M(y_M) | 0 \rangle_{\text{Connected}} \neq 0 . \tag{82}
\]

In Appendix \( [E] \) we derive \( SU(2)_L \) “pion-pole-dominance” 1-\( \phi \)-\( R \) connected amputated T-Matrix WTI
for SSB $SU(2)_L \times U(1)_Y$ from the divergence of the connected amplitudes considered above. Their solution is a tower of recursive $SU(2)_{L-R}$ WTI (27) that govern 1-\phi-i $\phi$-sector connected amputated Greens functions $\Gamma_{N,M}^{(N, M)}$. For $\bar{\pi}$ with $CP = -1$, the result is precisely as (31) for SSB $SU(2)$:

$$
\langle H \rangle^{(2)_N M+1}_{N, M+1}(p_1 \ldots p_N ; q_1 \ldots q_M 0) = \sum_{m=1} M \delta^{tm} \Gamma_{N+1, M-1}^{(2)_N M+1}(p_1 \ldots p_N q_m; q_1 \ldots q_m) - \sum_{n=1} M \Gamma_{N-1, M+1}^{(2)_N M+1}(p_1 \ldots \hat{p}_n \ldots p_N; q_1 \ldots q_M p_n), \tag{83}
$$

valid for $N, M \geq 0$. On the left-hand-side of (83) there are $N$ renormalized $h$ external legs (coordinates $x$, momenta $y$, momenta $p_0$), $M$ renormalized ($CP = -1$) $\bar{\pi}$ external legs (coordinates $y$, momenta $q$, isospin $t$), and $1$ renormalized soft external $\bar{\pi}(k_\mu = 0)$ (coordinates $z$, momenta $k$). “Hatted” fields with momenta $\hat{p}_n$ and $\hat{q}_m$ are omitted.

The rigid $SU(2)_{L-R}$ WTI 1-soft-pion theorems (83) relate a 1-\phi-i Green’s function with $(N+1)\times M$ external fields (which include a zero-momentum $\bar{\pi}$), to two 1-\phi-i Green’s functions with $(N+1)\times M$ external fields.

The Green’s functions $\Gamma_{N,M}^{(N, M)}(p_1 \ldots p_N ; q_1 \ldots q_M )$ are not themselves gauge-independent. Furthermore, although 1-\phi-i, they are 1-(\bar{W}^\nu, B^\mu) by cutting transverse $\bar{W}_\mu$ or $B_\mu$ gauge boson lines.

The 1-\phi-i $\bar{\pi}$ and $h$ inverse propagators are:

$$
\Gamma_{0, 2}^{(2)_N M+1} (q, -q) = \bar{\delta}^{t_2 t_1} \Gamma_{0, 2}^{(2)_N M+1} (q, -q).
$$

We can now form the $\phi$-sector effective momentum space Lagrangian in Landau gauge. All perturbative quantum loop corrections, to all-loop-orders and including all UVQDs, log-divergent and finite contributions, are included in the $\phi$-sector effective Lagrangian: 1-\phi-i Green’s functions $\Gamma_{N,M}^{(2)_N M+1}(p_1 \ldots p_N ; q_1 \ldots q_M )$; wavefunction renormalizations; renormalized $\phi$-scalar propagators (72-78); the Brout-Englert-Higgs (BEH) VEV $\langle H \rangle$ (ES1); all gauge boson and ghost propagators.

We want to classify operators arising in $SU(2)_L \times U(1)_Y$ loops, and separate the finite operators (8) from the divergent ones. We focus on finite relevant operators, as well as quadratic and logarithmically divergent operators.

We classified the irrelevant operators (33) in Section III C and the classification is the same for the $SU(2)_L \otimes U(1)_Y$ case. The only difference now is that the light degrees of freedom include $\bar{W}^\nu, B^\mu, h, \pi, \bar{\eta}_B; \bar{\omega}, \omega_B$. We ignore these operators here as well.

Such finite operators appear throughout the $SU(2)_{L-R}$ Ward-Takahashi IDs (83):

- $N + M \geq 5$ is $O_{20}^{1/A^2: \text{Irrelevant}}$ and $O_{20}^{\text{Dim} > 4: \text{Light}}$;
- the left hand side of (83) for $N + M = 4$ is also $O_{20}^{1/A^2: \text{Irrelevant}}$ and $O_{20}^{\text{Dim} > 4: \text{Light}}$;
- $N + M \leq 4$ operators $O_{20}^{\text{Dim} \leq 4: \text{Non-Analytic}}$ appear in (83).

Finally, there are $N + M \leq 4$ operators that are analytic in momenta. We expand these in powers of momenta, count the resulting dimension of each term in the operator Taylor-series, and ignore $O_{20}^{\text{Dim} > 4: \text{Light}}$ and $O_{20}^{1/A^2: \text{Irrelevant}}$ terms in that series.

As we did in (92) for the $SU(2)_L$ model we suppress gauge fields and form the all-loop-orders renormalized scalar-sector effective Lagrangian with operator dimension less than or equal to 4 for $(h, \overline{\pi})$ with $CP = (1, -1)$

$$
\Gamma_{\phi^{Eff}}^{(2)} = \Gamma_{1, 0}^{(2)} (0 ; h) + \frac{1}{2!} \Gamma_{2, 0}^{(2)} (p, -p) h^2
\frac{1}{3!} \Gamma_{3, 0}^{(2)} (000 ; h) \rho^2 + \frac{1}{4!} \Gamma_{4, 0}^{(2)} (0000 ; h) \rho^4
= \frac{1}{2!} \Gamma_{1, 0}^{(2)} (00 ; 00) h^2 \rho + \frac{1}{4!} \Gamma_{4, 0}^{(2)} (0000 ; h) \rho^4 + O_{20}^{1/A^2}. \tag{85}
$$

The connected amputated Green’s function identities (83) severely constrain the effective Lagrangian (85). For finite operators that arise entirely from SM degrees of freedom that are crucially important for computing experimental observables, the most familiar are the successful 1-loop high precision Standard Model predictions for the top-quark from Z-pole physics (49-50) in 1984 and from the $W^\pm$ mass (51) in 1980, as well as the 2-loop BEH mass from Z-pole physics (49-50) and from the $W^\pm$ mass (51-52). Those precisely predicted the experimentally measured masses of the top quark at FNAL (53), and of BEH scalar at CERN (19-20, 53-55). These operators also include the high-precision electroweak S,T and U (2-11, 56). Such finite operators are not the point of this paper.

8 In the $SU(2)_{L} \times U(1)_Y$ Standard Electro-weak Model, there are finite operators that arise entirely from SM degrees of freedom that are crucially important for computing experimental observables.
pedagogical clarity, we first separate out the isospin indices
\[
\begin{align*}
\Gamma^{2 \otimes 1: t_1 t_2}_2 (q, -q) \equiv & \, \delta^{t_1 t_2} \Gamma^{2 \otimes 1}_0 (q, -q), \\
\Gamma^{2 \otimes 1: t_1 t_2}_1 (q; 0) \equiv & \, \delta^{t_1 t_2} \Gamma^{2 \otimes 1}_1 (q; 0), \\
\Gamma^{2 \otimes 1: t_1 t_2}_0 (0; 0) \equiv & \, \delta^{t_1 t_2} \Gamma^{2 \otimes 1}_2 (0; 0), \\
\Gamma^{2 \otimes 1: t_1 t_2 t_3 t_4}_0 (0; 0; 0) \equiv & \, \Gamma^{2 \otimes 1}_0 (0; 0; 0; 0), \\
\Gamma^{2 \otimes 1: t_1 t_2 t_3 t_4}_0 (0; 0; 0) \times & \left[ \delta^{t_1 t_3} \delta^{t_2 t_4} + \delta^{t_1 t_4} \delta^{t_2 t_3} \right].
\end{align*}
\]

The Ward-Takahashi IDs \([83]\) for Greens functions severely constrain the effective Lagrangian \([85]\):

- **WTI** \( N = 0, M = 1 \)
  \[
  \delta^{t_1 t_2} \Gamma^{2 \otimes 1}_1 (q; q) = (\langle H \rangle)^{2 \otimes 1: t_1 t_2}_2 (q, -q), \\
  \Gamma^{2 \otimes 1}_1 (0; 0) = (\langle H \rangle)^{2 \otimes 1}_2 (0; 0),
  \]
  since no momentum can run into the tadpoles.

- **WTI** \( N = 1, M = 1 \)
  \[
  \delta^{t_1 t_2} \Gamma^{2 \otimes 1}_2 (q, -q) = (\langle H \rangle)^{2 \otimes 1: t_1 t_2}_0 (q, -q), \\
  \Gamma^{2 \otimes 1}_2 (0; 0; 0) = (\langle H \rangle)^{2 \otimes 1}_1 (0; 0; 0).\]

- **WTI** \( N = 2, M = 1 \)
  \[
  (\langle H \rangle)^{2 \otimes 1: t_1 t_2}_1 (0; 0; 0) = 2 \Gamma^{2 \otimes 1: t_1 t_2}_1 (0; 0; 0), \\
  (\langle H \rangle)^{2 \otimes 1}_2 (0; 0; 0) = \Gamma^{2 \otimes 1}_3 (0; 0; 0).\]

- **WTI** \( N = 0, M = 3 \)
  \[
  (\langle H \rangle)^{2 \otimes 1: t_1 t_2 t_3 t_4}_0 (0; 0; 0; 0) = 2 \Gamma^{2 \otimes 1: t_1 t_2 t_3 t_4}_1 (0; 0; 0; 0), \\
  \delta^{t_1 t_3} \Gamma^{2 \otimes 1}_2 (0; 0; 0; 0) = 0, \\
  \Gamma^{2 \otimes 1}_3 (0; 0; 0; 0) = 0.\]

- **WTI** \( N = 1, M = 3 \)
  \[
  - \delta^{t_1 t_2} \Gamma^{2 \otimes 1}_4 (0; 0; 0; 0; 0) = 0, \\
  \delta^{t_1 t_4} \Gamma^{2 \otimes 1}_2 (0; 0; 0; 0; 0) = 0.\]

The quadratic and quartic coupling constants are defined in terms of 2-point and 4-point 1-φ-I connected amputated GF
\[
\Gamma^{2 \otimes 1}_0 (0; 0; 0; 0; 0) \equiv -2 \lambda^2_0
\]
\[(93)\]

The all-loop-orders renormalized φ-sector momentum-space effective Lagrangian \([85]\) - constrained only by those \(SU(2)_L \times U(1)_Y\) WTI governing Greens functions \([83]\) - may be written (for Wigner mode, the scale-invariant point and Goldstone model)
\[
\begin{align*}
L_{\text{Eff: all-modes}}^{\text{2\otimes 1: Landau}} &= L_{\text{Kinetic: Eff: all-modes}}^{\text{2\otimes 1: Landau}} - V^{\text{Eff: all-modes}}_{\text{2\otimes 1: Landau}} + \mathcal{O}^{2 \otimes 1}, \\
L_{\text{Kinetic: Eff: all-modes}}^{\text{2\otimes 1: Landau}} &= \frac{1}{2} \left( \Gamma^{2 \otimes 1}_0 (p, -p) - \Gamma^{2 \otimes 1}_0 (0, 0) \right) h^2 \\
&\quad + \frac{1}{2} \left( \Gamma^{2 \otimes 1}_0 (q, -q) - \Gamma^{2 \otimes 1}_0 (0, 0) \right) \tilde{\pi}^2, \\
V_{\text{Eff: all-modes}}^{\text{2\otimes 1: Landau}} &= \frac{1}{2} \left( \frac{h^2 + \tilde{\pi}^2}{2} + \langle H \rangle \right) h \\
&\quad + \lambda^2 \left( \frac{h^2 + \tilde{\pi}^2}{2} + \langle H \rangle \right)^2.
\end{align*}
\]
\[(94)\]

with finite non-trivial wavefunction renormalization
\[
\Gamma^{2 \otimes 1}_0 (q, -q) - \Gamma^{2 \otimes 1}_0 (0, 0) \sim q^2
\]
\[(95)\]

As in the \(SU(2)_L\) model discussed in Section \[HT\], the φ-sector effective Lagrangian \([94]\) has insufficient boundary conditions to distinguish among the three modes \([12, 15]\) of the BRST-invariant Lagrangian \(L^{2 \otimes 1}\) in \([94]\). The expression for the effective potential in various modes of the theory, but now for the \(SU(2)_L \otimes U(1)_Y\) model, can be found in \([88]\). Eqn. \([94]\) has exhausted the constraints (on the allowed terms in the φ-sector effective Lagrangian) due to those \(SU(2)_L \otimes U(1)_Y\) WTIs that govern 1-φ-I φ-sector Green’s functions \(\Gamma^{2 \otimes 1}_{N,M}\) \([83, B24]\). In order to provide boundary conditions that distinguish among the effective potentials in \([88]\) in the \(SU(2)_L \otimes U(1)_Y\) model, we must turn to the \(SU(2)_L \otimes U(1)_Y\) WTI that govern φ-sector 1-φ-R T-Matrix elements \(T^{2 \otimes 1}_{N,M}\).

E. The Lee-Stora-Symanzik (LSS) Theorem: IR finiteness and automatic tadpole renormalization

In strict obedience to K. Symanzik’s edict “...unless otherwise constrained by Ward Identities...”, we now further constrain the allowed terms in the φ-sector effective Lagrangian with those \(SU(2)_L \times U(1)_Y\) Ward-Takahashi identities that govern 1-φ-R T-Matrix elements \(T^{2 \otimes 1}_{N,M}\).
limit" (B.W. Lee in [12]).

Goldstone mode amplitude must vanish in the soft-pion
in each order of perturbation theory. Furthermore, the

gaines in the scalar-sector (of AHM) Goldstone mode (in

Eqn. (96) "asserts the absence of infrared (IR) diver-

gemines, but, since it involves

As for (42) in

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are

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N,M

, not the soft-˜

π

N,M

(102)

Γ^{2\otimes 1}_{0,0}(0) = \langle H \rangle \Gamma^{2\otimes 1}_{0,2}(0) = 0

so that separate tadpole renormalization is un-necessary.

Eq. (101) is the \phi-sector effective potential of sponta-

neously broken SU(2)_L \times U(1)_Y gauge theory, in Landau
gauge, constrained by the LSS theorem. The effective

Lagrangian [94], with effective potential [101]:

- is derived from the local BRST-invariant standard
electroweak electroweak Lagrangian L^{2\otimes 1}_{0,0} (D19);

- includes all divergent \mathcal{O}(\Lambda^2), \mathcal{O}(\ln \Lambda^2)

and finite terms that arise to all perturbative loop-orders

in the full SU(2)_L \times U(1)_Y gauge theory, due to vir-

tual gauge bosons, \phi scalars, anti-ghosts and ghosts

(W^\pm \mu, Z^\mu, A^\mu; h, \tilde{\pi}, \tilde{\eta}_B; \tilde{\omega}_B respectively);

- obeys the LSS theorem [97]98 and all other

SU(2)_L \times U(1)_Y Ward-Takahashi Greens function and T-Matrix identities;

- obeys the Goldstone theorem in Landau gauge,

having massless derivatively coupled NGBs, \tilde{\pi}.

\textbf{A central result of this paper is to recognize that,}

in order to force [99] to become [101], the LSS theo-

rem incorporates a "new" on-shell T-Matrix symmetry,

which is not a full symmetry of the BRST-invariant elec-
troweak Lagrangian. Electro-weak physics, but not its

Lagrangian, has the SU(2)_L \otimes U(1)_Y \otimes BRST

symmetry of Section V and Appendices D and E, a conserved cur-
rrent \mathcal{J}_L, un-deformed WTI\es governing connected amputated Green's functions [83], un-deformed WTI\es gov-

erning connected amputated on-shell T-Matrix elements

[96], and an LSS theorem [97].

A crucial effect of the LSS theorem [97], together with

the N = 0, M = 1 SU(2)_L \times U(1)_Y Ward-Takahashi

Greens function identity [83], is to automatically elimi-
nate tadpoles in [85]

\mathcal{\Gamma}^{2\otimes 1}_{1,0}(0) = \langle H \rangle \mathcal{\Gamma}^{2\otimes 1}_{0,2}(0) = 0

Imagine we suspected that \tilde{\pi} is not all-loop-orders massless in

Landau gauge in the SSB standard electroweak model, and sim-

ply/naively wrote a mass-squared \nu_{\pi,\text{pole}} into the \tilde{\pi} inverse-

propagator

[\Delta^{2\otimes 1}_{1,0}(0)]^{-1} \equiv -\nu_{\pi,\text{pole}}^2

= -\nu_{\pi,\text{pole}}^2 \int \nu_{\pi,\text{pole}}^2 \frac{\mathcal{\Delta}^{2\otimes 1}_{1,0}(m^2)}{m^2}

However, the LSS theorem [99] insists instead that

[\Delta^{2\otimes 1}_{1,0}(0)]^{-1} \equiv -\nu_{\pi,\text{pole}}^2 = \mathcal{T}^{2\otimes 1}_{1,0}(0) = 0

The \pi-pole-mass vanishes exactly.

\nu_{\pi,\text{pole}}^2 = \nu_{\pi,\text{pole}}^2 \left(1 - \nu_{\pi,\text{pole}}^2 \int \nu_{\pi,\text{pole}}^2 \frac{\mathcal{\Delta}^{2\otimes 1}_{1,0}(m^2)}{m^2}\right) = 0

9

As for the pre-LSS effective potential [39] of SU(2)_L, Eqn. [99]

appears to embrace a disaster: the term linear in \phi^\dagger \phi - \frac{1}{2} (H \omega)^2 persists, destroying the sym-

metry of the famous “Mexican hat”, and implying that

SU(2)_L \times U(1)_Y is not actually in Goldstone mode. As

for SU(2)_L, the LSS theorem [97] (and not the Goldstone

theorem) comes to the rescue, forcing the electroweak

gauge theory fully into its true Goldstone \langle H \rangle \neq 0 mode

\begin{align*}
V^{\text{Eff;pre-LSS-Goldstone}}_{2\otimes 1;0,\phi;\text{Landau}} &= \frac{\nu_{\pi,\text{pole}}^2}{4} \left[ \bar{\nu}^2 \omega^2 - (\langle H \rangle^2)^2 \right] \\
&= \frac{\nu_{\pi,\text{pole}}^2}{4} \left[ \phi^\dagger \phi - \frac{1}{2} (H \omega)^2 \right]^2 \quad \text{(101)} \\
&= \frac{\nu_{\pi,\text{pole}}^2}{4} \left( h^2 + \nu_{\pi,\text{pole}}^2 - \langle H \rangle^2 \right)^2
\end{align*}

As expected, the NGB \tilde{\pi} have disappeared from the effective

potential, have purely derivative couplings through their

kinetic term, and obey the shift symmetry

\tilde{\pi} \rightarrow \tilde{\pi} + \langle H \rangle \theta + \tilde{\pi} \times \tilde{\theta} + \mathcal{O}(\tilde{\theta}^2)

\text{for constant } \tilde{\theta}. \text{ In other words, the Goldstone theorem is already

properly enforced.}

As for the pre-LSS effective potential [39] of SU(2)_L, Eqn. [99]

appears to embrace a disaster: the term linear in \phi^\dagger \phi - \frac{1}{2} (H \omega)^2 persists, destroying the symmetry of the famous “Mexican hat”, and implying that

SU(2)_L \times U(1)_Y is not actually in Goldstone mode. As

for SU(2)_L, the LSS theorem [97] (and not the Goldstone

theorem) comes to the rescue, forcing the electroweak

gauge theory fully into its true Goldstone \langle H \rangle \neq 0 mode

\begin{align*}
V^{\text{Eff;LSS-Goldstone}}_{2\otimes 1;0,\phi;\text{Landau}} &= \frac{\nu_{\pi,\text{pole}}^2}{4} \left[ \bar{\nu}^2 (\omega^2 - \langle H \rangle)^2 \right] \\
&= \frac{\nu_{\pi,\text{pole}}^2}{4} \left[ \phi^\dagger \phi \omega^2 - \frac{1}{2} (H \omega)^2 \right]^2 \quad \text{(101)} \\
&= \frac{\nu_{\pi,\text{pole}}^2}{4} \left( h^2 + \nu_{\pi,\text{pole}}^2 + \langle H \rangle^2 \right)^2
\end{align*}
residues the theory’s renormalizability and unitar-
ity, which requires that wavefunction renomalization
\((\langle H \rangle)_{\text{Bare}} = [Z^\phi_{\text{Bare}}]^{1/2}(\langle H \rangle)\) [16, 18, 27] forbids
UVQD, relevant, or any other dimension-2 operator
corrections to \(\langle H \rangle\).

The LSS theorem [97] has once again caused all relevant
operators to vanish!

The Green’s function Ward-Takahashi ID [89] for \(N =
1, M = 1\), constrained by the LSS theorem [98], relates
the BEH mass to the coefficient of the \(h^2\) vertex
\[\Gamma_{2\to1}^{\phi}(00; \lambda\phi) = \langle H \rangle \Gamma_{2\to1}^{\phi}(00; 00).\] (106)

Therefore, the BEH mass-squared
\[m_{\text{BEH}}^2 = 2\lambda_\phi^2 \langle H \rangle^2\] (107)
arises entirely from SSB, as does (together with its elec-
troweak decays) the resonance pole-mass-squared
\[m_{\text{BEH; Pole}}^2 = 2\lambda_\phi^2 \langle H \rangle^2 \left(1 - 2\lambda_\phi^2 \langle H \rangle^2 \int dm^2 \rho_{\text{BEH}}(m^2) \right)^{-1} \]
\[+ O_{2\to1;\phi}^{\text{Ignore}}.\] (108)

F. Comparison of results with the minimization
procedure

It’s important to compare the results of our LSS theo-
rem to those of the mainstream literature. For pedago-
gical simplicity, in this Subsubsection we suppress men-
vacuum energy and \(O_{2\to1;\phi}^{\text{Ignore}}\). After renormalization,
but before application of the LSS theorem, the effec-
tive potential (94), which is derived entirely from Green’s
function WTIs, can be written
\[V_{2\to1;\phi}^{\text{Eff}} = \left(\frac{\mu_\phi^2 + \lambda_\phi^2 \langle H \rangle^2}{2} + \langle H \rangle h \right)\]
\[+ \lambda_\phi^2 \left(\frac{h^2 + \pi^2}{2} + \langle H \rangle h \right)^2 \] (109)
where \(V_{2\to1;\phi}^{\text{Eff}}, \mu_\phi^2, \lambda_\phi^2, \langle H \rangle^2\) in (109) are all renormalized
quantities. The vanishing of relevant operators in the
effective \(SU(2)_L \times U(1)_Y\) electroweak theory is therefore
not itself controversial:

The literature minimizes (109) to find the vacuum:
\[\frac{\partial}{\partial h} V_{2\to1;\phi}^{\text{Eff}}|_{h=\pi=0} = \langle H \rangle \left(\mu_\phi^2 + \lambda_\phi^2 \langle H \rangle^2\right) = 0.\] (110)

This is interpreted as a calculation of \(\langle H \rangle^2\)
\[\langle H \rangle^2 = -\frac{\mu_\phi^2}{\lambda_\phi^2}.\] (111)
where, in renormalized \(\mu_\phi^2\), UVQD and all other relevant
contributions are regarded as having cancelled against a
bare counter-term \(\delta \mu_{\phi;\text{Bare}}^2\).

In contrast, in this paper we have never minimized
a potential. Instead, we have derived a tower of Adler
self-consistency conditions [96, 119] in Landau gauge in
Appendix 2 derived directly from the exact \(SU(2)_L \times
U(1)_Y\) symmetry obeyed by gauge-independent on-shell
T-Matrix elements. One of these, the \(N = 0, M = 1\)
case, is the LSS theorem:
\[\langle H \rangle m_\pi^2 = \langle H \rangle \left(\mu_\phi^2 + \lambda_\phi^2 \langle H \rangle^2\right) = 0\] (112)
which, for the SSB case, gives
\[m_\pi^2 = \mu_\phi^2 + \lambda_\phi^2 \langle H \rangle^2 = 0.\] (113)

The practical effect of (112) is the same as minimization
of the effective potential. We therefore agree with the
mainstream literature that all relevant operators vanish in
the effective low-energy \(SU(2)_L \times U(1)_Y\) theory, but
there is one vast difference. Although the mainstream
minimization approach regards relevant operators as hav-
ing cancelled, that cancellation is not protected by any
symmetry. In contrast, relevant operators here vanish be-
cause of the exact \(SU(2)_L \times U(1)_Y\) symmetry, represented
by the LSS theorem, remnant in gauge-independent on-
shell T-Matrix elements. Although \(SU(2)_L \times U(1)_Y\) is
not a symmetry of the BRST-invariant Lagrangian, it is
exact for the physics of our new \(SU(2)_L \otimes U(1)_Y \otimes \text{BRST}\)
symmetry.

VI. EXTENSION TO 1-GENERATION
CP-CONSERVING STANDARD MODEL WITH
DIRAC NEUTRINO MASS

We finally extend the theory next to include fermion
matter fields – the 3rd generation of Standard Model
quarks and leptons, together with a right-handed \(\tau\) nu-
trino with Dirac mass \(m_{\nu_\tau}\). This conserves \(CP\).

We also gauge \(SU(3)_C\), adding to our model 8 QCD
gluons \(G^{\mu}_{\alpha}\), so that the local/gauge group is \(SU(3)_C \otimes
SU(2)_L \otimes U(1)_Y\). As usual, the strong \(CP\) problem arises
in a term \(L_\theta = -\frac{\theta}{64\pi^2} \epsilon^{\mu\nu\lambda\sigma} G_{\mu\nu}^a G_{\lambda\sigma}^b\) induced
by instantons, with the \(CP\)-violating parameter \(\theta\). We artifi-
cially set \(\theta = 0\) in order to conserve \(CP\), as \(CP\) conserva-
tion is essential for our proofs of WTI (see eqs. (A18,A19)
and A20).

We call this model \(\nu_D SM_{\text{CP}}\). \(\nu_D SM_{\text{CP}}\) then obeys
various Ward-Takahashi identities. As proved in [13],
the form of the tower of \(SU(2)_L \otimes R\) Green’s function
WTI and that of the on-shell \(SU(2)_L \otimes R\) T-Matrix Adler
self-consistency relations, are unaffected by the addition
of quarks \((q^a_{\alpha;L} = (t^a_{\alpha}, b^a_{\alpha})_{L}, t^a_{\alpha}, b^c_{\alpha;R}, \text{with } c\) the color-
quantum number) and leptons \((\nu^\pm_{\alpha;R}, \nu^R_{\alpha;R}, \tau^R_{\alpha;R})\).
Neither are the form of the tower of \(SU(2)_L \otimes R\) Green’s
function WTI [83] and that of the on-shell \(SU(2)_L \otimes R\) T-
Matrix Adler self-consistency relations [96], derived from
soft-pions, affected by the addition of gluons \(G^{\mu}_{\alpha}\). Mean-
while, vector \(SU(3)_C\) and \(U(1)_{QED}\) WTI add nothing to
our discussion of axial-vector soft-\(\pi\) theorems, just as
conserved vector \(SU(2)_L \otimes R\) in [119] adds nothing to the
discussion of axial-vector soft-pions.
The CP-conserving $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ invariant Lagrangian (summed over color) studied in this work is then

\[
L_{\nu D SM_{\text{CP}}} = L_{QCD} + L_{\nu_D} + L_{\text{h/3}} + L_{\text{Yuk}}
\]

\[
L_{QCD} = -\frac{1}{2} \text{Tr} \left( G_{\mu}^A G_{\mu}^{A'} \right) + i \bar{q}_{L,i} \gamma_\mu D_\mu q_{L,i} + i \bar{q}_{R,i} \gamma_\mu D_\mu q_{R,i}
\]

\[
L_{\nu_D} = \nu_D i \gamma_\mu (\sigma_{\mu\nu} f^{\nu}_{\mu}) R + y_{L}^\nu \bar{q}_{L,i} (\sigma_{\mu\nu} \gamma^\nu f^{\nu}_{\mu}) L
\]

\[
L_{\text{Yuk}} = \nu_D i \gamma_\mu (\sigma_{\mu\nu} f^{\nu}_{\mu}) R + y_{R}^\nu \bar{q}_{R,i} (\sigma_{\mu\nu} \gamma^\nu f^{\nu}_{\mu}) L
\]

In Landau gauge, the $SU(2)_L$ sub-current is

\[
J_{\mu L,\nu_D SM_{\text{CP}}} = \begin{cases}
J_{\mu L+R,v_D SM_{\text{CP}}} + \bar{J}_{\mu L-R,v_D SM_{\text{CP}}} + \bar{J}_{\mu L+2,1} & (115)
\end{cases}
\]

with fermion contributions

\[
2\bar{J}_{\mu L+R,v_D SM_{\text{CP}}} = \sum_i \bar{\ell}_i \gamma_\mu \ell_i \bar{\ell}_i + \sum_{i,c} \bar{q}_{c,i} \gamma_\mu q_{c,i}
\]

\[
2\bar{J}_{\mu L-R,v_D SM_{\text{CP}}} = \sum_i \bar{\ell}_i \gamma_\mu \gamma^5 \ell_i \bar{\ell}_i + \sum_{i,c} \bar{q}_{c,i} \gamma_\mu \gamma^5 q_{c,i}
\]

(Here colors $c = r, w, b$, quark flavors $q_{c,i} = t^c, b^c$, and lepton flavors $\ell_{c,i} = \nu_{\tau_i}(\tau_i)$ $J_{\mu L+2,1}$ is given in [26].)

Since CP is conserved, the fermionic parts of the current are separately conserved for matrix elements

\[
\left\langle J_{\mu}^{\nu_D SM_{\text{CP}}} \right\rangle = 0
\]

\[
\left\langle J_{\mu}^{\nu_D SM_{\text{CP}}} \right\rangle = 0
\]

in the sense of $\text{[A18,A19]}$. We focus on the global isospin current $\bar{J}_{\mu L,v_D SM_{\text{CP}}}$ and examine the divergence of amplitudes such as

\[
\partial_\mu \left\langle 0 | T \left( J_{\mu L,v_D SM_{\text{CP}}} (z) \right) \right. \\
\left. h(x_1) \cdots h(x_N) \pi^{t_1} (y_1) \cdots \pi^{t_M} (y_M) \right\rangle_{\text{Connected}}
\]

We now write down the WTI for the $\nu_D SM_{\text{CP}}$ model, one concerning on-shell T-Matrix elements

\[
\left\langle H \right| T_{N,M+1}^{\nu_D SM_{\text{CP}}; t_1 t_2 \cdots t_M} (p_1 \cdots p_N; q_1 \cdots q_M)
\]

\[
\times (2\pi)^4 \delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)
\]

and the other relating 1-$\phi$-1 (but one $\bar{W}_L^\nu, B_L^\nu, G_A^\nu$ reducible) connected, amputated Greens functions

\[
\Gamma_{N,M}^{\nu_D SM_{\text{CP}}}
\]

\[
\left\langle H \right| \Gamma_{N,M}^{\nu_D SM_{\text{CP}}; t_1 t_2 \cdots t_M} (p_1 \cdots p_N; q_1 \cdots q_M)
\]

\[
= \sum_{m=1}^{M} \delta^{4m} \Gamma_{N+1,M-1}^{\nu_D SM_{\text{CP}}; t_1 t_2 \cdots t_M} (p_1 \cdots p_N; q_1 \cdots q_M)
\]

\[
- \sum_{n=1}^{N} \Gamma_{N-M,1}^{\nu_D SM_{\text{CP}}; t_1 t_2 \cdots t_M} (p_1 \cdots p_N; q_1 \cdots q_M)
\]

These WTI are mathematically identical to the ones in [33,60], but now include all fermion and boson loops, to all perturbative loop-orders, from virtual $q^2_L, t_R^2, b_R^2, l_3, v_{\tau_i}, \tau_R, G_A^\nu, \bar{W}_L^\mu, B_L^\mu, h, h, \bar{h},$ and $\bar{h}$, and the ghosts and anti-ghosts associated with the gauge bosons.

The LSS theorem for $\nu_D SM$ is the $N = 0, M = 1$ case of

\[
\left\langle H \right| T_{0,2}^{\nu_D SM_{\text{CP}}; (0)} (0) = 0 ,
\]

forcing $m_3^2 = 0$, and this is the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ analogue of [77].

The WTI [120] and [121] severely constrain the effective Lagrangian. Considering only operators with $N + M \leq 4$ in [120], and using the LSS theorem [121], the scalar-sector effective potential becomes

\[
V_{\text{Eff; Goldstone}}^{\nu_D SM_{\text{CP}; Landau}} = \lambda(2 \frac{h^2 + \pi^2}{2} + \left\langle H \right| h^2)
\]

As the WTI [120] and the LSS theorem [121] are mathematically the same as their analogues in $SU(2)_L \otimes U(1)_Y$ theory, all the results and conclusions obtained there apply to the CP-conserving $\nu_D SM_{\text{CP}}$ model.

VII. O.G, BWL AND GDS: THIS RESEARCH, VIEWED THROUGH THE PRISM OF MATHEMATICAL RIGOR DEMANDED BY RAYMOND STORA

Raymond Stora regarded vintage QFT as incomplete, fuzzy in its definitions, and primitive in technology. For example, he worried about whether the off-shell T-matrix could be mathematically rigorously defined to exist in Lorenz gauge: e.g. without running into some IR subtlety. The Adler self-consistency conditions proved here guarantee the IR finiteness of the scalar-sector on-shell T-matrix.

Raymond contributed hugely to this work, especially Section [15]. His insight formed part of the foundation of this paper. Because of his contribution, his intentions were to be a co-author on this paper but he passed away before this manuscript reached is final form. He was a perfectionist, as his son Dr. Thierry Stora explained: “However, he was always very critical in accepting to publish scientific results, which often took years before he accepted that they would be submitted to a peer reviewed journal.”
Although he agreed on the correctness of the results presented here, Raymond might complain that we fall short of a strict mathematically rigorous proof (according to his exacting mathematical standards). He reminded us that much has been learned about quantum field theory, via modern path integrals, in the recent ≈45 years. In the time up to his passing, he was intent on improving this work by focusing on the following three issues:

- properly defining and proving the Lorenz-gauge results presented here with modern path integrals;
- tracking our central results directly to SSB, via BRST methods, in an arbitrary manifestly IR finite 't Hooft Rξ gauge, i.e. proving to his satisfaction that they are not an artifact of Lorenz gauge;
- any errors, wrong-headedness, misunderstanding, or misrepresentation appearing in this paper are solely our fault.

VIII. CONCLUSION

\(SU(2)_L \times U(1)_Y\) and \(\nu_DSM_{\text{CP}}\) physics (e.g. on-shell T-Matrix elements) have more symmetry than their BRST-invariant Lagrangians, including

- a new tower of rigid SSB \(SU(2)_{L-R}\) 1-soft-\(\pi\) WTI governing relations among Green’s functions;
- a new tower of rigid SSB \(SU(2)_{L-R}\) WTI's that forces 1-soft-\(\pi\) on-shell T-Matrix elements to vanish, and represents the on-shell behavior of their newly identified \(SU(2)_L \otimes \text{BRST}, SU(2)_L \otimes U(1)_Y \otimes \text{BRST}\) or \(SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes \text{BRST}\) symmetry, respectively.

- A new Lee-Stora-Symanzik theorem.

Our results are ubiquitous to gauge theories that spontaneously break a certain Lie algebraic structure group, \(\mathcal{G}\), at a low scale, say for definiteness \(m_{\text{Weak}}\). We have proved global \(\mathcal{G} \otimes \text{BRST}\) symmetry and motivated: its 2 associated towers of 1-soft-pion WTI (one for Green’s functions, and another for on-shell T-Matrix elements, including the LSS theorem); severe WTI constraints on the \(m_{\text{Weak}}\)-scalars’ effective potential.

Such \(SU(2)_L \otimes U(1)_Y \otimes \text{BRST}\) symmetry may be the reason why the Standard Model viewed as an effective low-energy weak-scale theory, and augmented by classical General Relativity and neutrino mixing, is the most experimentally and observationally successful and accurate known theory of Nature. That “Core Theory” \(^{[57]}\) has no known experimental or observational counter-examples.

IX. ACKNOWLEDGMENTS

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Appendix A: Landau-gauge reduction of $SU(2)_L$

We focus on the global isospin current $\vec{J}_L$. We are interested in global-symmetric relations among 1-scalar-particle-irreducible (1-φ-I) connected amputated Green’s functions (GF), and 1-scalar-particle-Reducible (1-φ-R) connected amputated T-Matrix elements, with external φ scalars. Analysis is done in terms of the exact renormalized 1-soft-pion pseudo-scalars, $\phi$, and the equal-time commutation relations (C20)

in Landau gauge. Here we have N external renormalized scalars $h = H - (\hat{H})$ (coordinates x), and M external (CP = -1) renormalized pseudo-scalars $\bar{\pi}$ (coordinates y, isospin t). We have also thrown away a sum of M terms, proportional to $\langle H \rangle$, that corresponds entirely to disconnected graphs.

A short calculation reveals

$$\partial_\mu \langle 0 | T \left[ J_{L}^{\mu \nu i}(z) h(x_1) \cdots h(x_N) \right. \times \pi^{\nu i}(y_1) \cdots \pi^{\tau M}(y_M) \left. \right] | 0 \rangle_{\text{Connected}}$$

$$= \langle 0 | T \left[ \partial_\mu J_{L}^{\mu \nu i}(z) \right. \times h(x_1) \cdots h(x_N) \pi^{\nu i}(y_1) \cdots \pi^{\tau M}(y_M) \left. \right] | 0 \rangle_{\text{Connected}}$$

(A5)

and the equal-time commutation relations (C20)

$$\delta(z_0 - x_0) \left[ \left( \right. J_{L}^{\mu}(z) - J_{L;\text{Schwinger}}^{\mu}(z), h(x) \right] = 0$$

(A4)

$$\delta(z_0 - x_0) \left[ \left( \right. J_{L}^{\mu}(z) - J_{L;\text{Schwinger}}^{\mu}(z), \bar{\pi}(x) \right] = 0$$

(A3)

We form time-ordered amplitudes of products of $\vec{J}$ and $\pi$ connected amputated $T$-Matrix elements, with external functions (GF), and 1-scalar-particle-Reducible (1-φ-R) connected amputated T-Matrix elements.

We focus on the global isospin current $\vec{J}_L$. We are interested in global-symmetric relations among 1-scalar-particle-irreducible (1-φ-I) connected amputated Green’s functions (GF), and 1-scalar-particle-Reducible (1-φ-R) connected amputated T-Matrix elements, with external φ scalars. Analysis is done in terms of the exact renormalized 1-soft-pion pseudo-scalars, $\phi$, and the equal-time commutation relations (C20)

in Landau gauge. Here we have N external renormalized scalars $h = H - (\hat{H})$ (coordinates x), and M external (CP = -1) renormalized pseudo-scalars $\bar{\pi}$ (coordinates y, isospin t). We have also thrown away a sum of M terms, proportional to $\langle H \rangle$, that corresponds entirely to disconnected graphs.

1. Right-hand side of Master Equation

Making use of current conservation (C20),

$$\partial_\mu \vec{J}_L^{\mu} = \frac{1}{2} M_W \left[ H \partial_\beta \vec{W}^{\beta} + \bar{\pi} \times \partial_\beta \vec{W}^{\beta} \right].$$

(A2)

G. ‘t Hooft’s gauge-fixing [20] condition [21]

$$\langle 0 | T \left[ \left( \partial_\mu \vec{W}^{\mu}(z) \right) \right. \times h(x_1) \cdots h(x_N) \pi^{\nu i}(y_1) \cdots \pi^{\tau M}(y_M) \left. \right] | 0 \rangle_{\text{Connected}} = 0,$$

and the equal-time commutation relations (C20)

$$\delta(z_0 - x_0) \left[ \left( \right. J_{L}^{\mu}(z) - J_{L;\text{Schwinger}}^{\mu}(z), h(x) \right] = 0$$

(A4)

$$\delta(z_0 - x_0) \left[ \left( \right. J_{L}^{\mu}(z) - J_{L;\text{Schwinger}}^{\mu}(z), \bar{\pi}(x) \right] = 0$$

(A3)

We begin by studying the surface integral of the global $SU(2)_L$ current [21] in Landau gauge

$$\vec{J}_L^{\mu} = J_{L;\text{Schwinger}}^{\mu} + \vec{J}_L^{\mu}$$

(A6)

$$\vec{J}_L^{\mu} = -\frac{1}{4} g_2 \vec{W}^{\mu} \left[ H^2 + \pi^2 \right] + \vec{W}^{\nu} \times \vec{W}_\nu$$

$$- \lim_{\xi \to 0} \frac{1}{\xi} \left[ \vec{W}^{\mu} \times \partial_\beta \vec{W}^{\beta} \right] - \partial^\mu \vec{\eta} \times \vec{\omega}$$

In order to transform (A1) into a surface term (and crucially, to later form 1-soft-pion $SU(2)_L$ WTI), we Fourier-transform the current with far-infra-red ultrasoft momentum. We use Stokes theorem, where $\vec{z}_\mu$-surface is a unit vector normal to the 3-surface. The
time-ordered product constrains the 3-surface to lie on, or inside, the light-cone.

In this subsection, we will prove that

\[
0 = \lim_{k_{3-L} \to 0} \int d^4 z e^{i k \cdot z} \partial_\mu \langle 0 | T \left( \left[ \mathcal{J}_\mu (z) \right] \\
\times h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M) \right) | 0 \rangle_{\text{Connected}}
\]

\[
= \int d^4 z \partial_\mu \langle 0 | T \left( \left[ \mathcal{J}_\mu (z) \right] \\
\times h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M) \right) | 0 \rangle_{\text{Connected}}
\]

\[
= \int d^3 z \left[ \mathcal{J}_\mu (z) \right] \langle 0 | T \left( \left[ \mathcal{J}_\mu (z) \right] \\
\times h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M) \right) | 0 \rangle_{\text{Connected}}.
\]

3. LHS of Master Equation: Connected amplitudes linking the $\phi$-sector with external currents

Connected momentum-space amplitudes, with $N$ external BEHs, $M$ external $\pi$s, and a current $\mathcal{J}_{\mu}(z)$, are defined in terms of $\phi$-sector connected time-ordered products

\[
i G_{\mu;N,M}^{t_1 \ldots t_M} \langle J_{\mu}; p_1 \cdots p_N; q_1 \cdots q_M \rangle
\times (2\pi)^4 \delta^4 \left( k + \sum_{n=1}^N p_n + \sum_{m=1}^M q_m \right)
\]

\[
\equiv \int d^4 z e^{i k \cdot z} \prod_{n=1}^N \int d^4 x_n e^{i p_n \cdot x_n} \prod_{m=1}^M \int d^4 y_m e^{i q_m \cdot y_m}
\times (0 | T [J_{\mu}(z)] \\
\times h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M) | 0 \rangle_{\text{Connected}}.
\]

These appear throughout the proof of the WTI\textsuperscript{10} so that

\[
-k^\mu G_{\mu;N,M}^{t_1 \ldots t_M} \langle J_{\mu}; p_1 \cdots p_N; q_1 \cdots q_M \rangle
\times (2\pi)^4 \delta^4 \left( k + \sum_{n=1}^N p_n + \sum_{m=1}^M q_m \right)
\]

\[
\equiv \int d^4 z e^{i k \cdot z} \prod_{n=1}^N \int d^4 x_n e^{i p_n \cdot x_n} \prod_{m=1}^M \int d^4 y_m e^{i q_m \cdot y_m}
\times \partial^\mu (0 | T [J_{\mu}(z)] \\
\times h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M) | 0 \rangle_{\text{Connected}}.
\]

a. Gauge-invariant scalar-sector Lagrangian

The contribution of the appropriate term in \textsuperscript{[A6]} to the LHS of the Master Equation vanishes

\[
-k^\mu G_{\mu;N,M}^{t_1 \ldots t_M} \left( \left[ - \frac{1}{4} g_2 \mathcal{W}_\mu \left( H^2 + \pi^2 \right) \right] ; p_1 \cdots p_N; q_1 \cdots q_M \right)
\sim - \frac{1}{4} g_2 k^\mu \mathcal{W}_\mu (k) \langle \ldots | (k^2) \rangle
\]

\[= 0 ,\]

because the gauge condition\textsuperscript{[15]} obeyed by the states reads, in momentum-space

\[
k^\mu \mathcal{W}_\mu (k) = 0 .\]
b. Gauge fixing Lagrangian $L_{\text{Gauge Fix, Landau}}^{\mu}$

The contribution of the appropriate term in $[A6]$ to the LHS of the Master Equation vanishes

$$-k^\mu \epsilon^{\mu\nu\rho\tau} \left[ \lim_{\xi \to 0} \frac{1}{\xi} \tilde{W}_\mu \times \partial_\nu \tilde{W}_\nu \right] : p_1 \cdots p_N : q_{1 \cdots q_M}$$

$$\sim \lim_{\xi \to 0} \frac{1}{\xi} k^\mu \tilde{W}_\mu (k) \times \left( ik_\nu \tilde{W}_\nu (k) \right)$$

(A14)

because of the momentum-space gauge condition $[A13]$. Two factors of the gauge condition in (A14) make it sufficiently convergent, like $\lim_{\xi \to 0} \frac{1}{\xi} \left( \partial_\mu \tilde{W}_\mu + \xi M_W \pi \right)^2$, as $\xi \to 0$. But $\tilde{W}_\mu$ are also massive and incapable of carrying information to the 3-surface at infinity, so $[A14]$ dies exponentially.

c. Total $SU(2)_L$ Lagrangian $L_{\text{Landau}}^{\mu}$

$$\lim_{k_3 \to 0} \int d^4 z e^{ikz} \partial_\mu \langle 0 | T \left[ J_\mu^{(1)} (z) h (x_1) \cdots h (x_N) \right. \right.$$

$$\left. \times \pi^{(1)} (y_1) \cdots \pi^{(m)} (y_M) \right] | 0 \rangle_{\text{Connected}}$$

(A15)

$$\lim_{k_3 \to 0} \int d^4 z e^{ikz} \partial_\mu \langle 0 | T \left[ J_{\mu \Sigma}^{\Sigma} (z) h (x_1) \cdots h (x_N) \right. \right.$$

$$\left. \times \pi^{(1)} (y_1) \cdots \pi^{(m)} (y_M) \right] | 0 \rangle_{\text{Connected}}.$$

4. Vector $\tilde{J}_{\mu L+R; \text{Schwinger}}$ for $M$ even, Axial-vector $\tilde{J}_{\mu L-R; \text{Schwinger}}$ for $M$ odd

Divide the remainder of the global $SU(2)_L$ isospin current $[A6]$, i.e. the part which survives Subsections A1, A2 and A3, namely $\tilde{J}_{\mu \Sigma}^{\Sigma}$, into vector $SU(2)_L+R$ and axial-vector $SU(2)_L-R$ parts:

$$J_{\mu L+R; \text{Schwinger}} = \tilde{J}_{\mu L+R; \text{Schwinger}} + \tilde{J}_{\mu L-R; \text{Schwinger}}$$

$$\tilde{J}_{\mu L+R; \text{Schwinger}} = \frac{1}{2} \pi \times \partial_\mu \pi$$

(A16)

$$\tilde{J}_{\mu L-R; \text{Schwinger}} = \frac{1}{2} \left( \pi \partial_\mu H - H \partial_\mu \pi \right).$$

The classical equations of motion show that none of these sub-currents is conserved

$$\partial_\mu \tilde{J}_{\mu L+R; \text{Schwinger}} \neq 0;$$

$$\partial_\mu \tilde{J}_{\mu L-R; \text{Schwinger}} \neq 0;$$

(A17)

$$\partial_\mu \tilde{J}_{\mu L; \text{Schwinger}} \neq 0.$$

We are interested only in models where CP is conserved, so that amplitudes connecting the $\phi$-sector with the total $SU(2)_L$ isospin current $[A6]$ conserve CP. Therefore, on-shell and off-shell connected amputated T-matrix elements and Green’s functions of an odd number of $(CP = -1)$ $\pi$s and their derivatives, are zero.

$SU(2)_{L+R}$ and $SU(2)_{L-R}$ are not here strictly applicable sub-groups of $SU(2)_L$: but, while the global vector current $\tilde{J}_{\mu L+R; \text{Schwinger}}$ transforms as an even number of $\pi$s, the global axial-vector current $\tilde{J}_{\mu L-R; \text{Schwinger}}$ transforms as an odd number of $\pi$s. Therefore, for $M$ even, the axial-vector current and its divergence obey

$$\left\langle 0 \left| T \left[ \left( \tilde{J}_{\mu L-R; \text{Schwinger}} (z) \right. \right. \right.$$

$$\left. \times h (x_1) \cdots h (x_N) \pi^{(1)} (y_1) \cdots \pi^{(m)} (y_M) \right] \right| 0 \right\rangle_{\text{Connected}} = 0,$$

(A18)

$$\left\langle 0 \left| T \left[ \left( \partial_\mu \tilde{J}_{\mu L-R; \text{Schwinger}} (z) \right. \right. \right.$$

$$\left. \times h (x_1) \cdots h (x_N) \pi^{(1)} (y_1) \cdots \pi^{(m)} (y_M) \right] \right| 0 \right\rangle_{\text{Connected}} = 0.$$

Meanwhile, for odd $M$, the vector current

$$\left\langle 0 \left| T \left[ \left( \tilde{J}_{\mu L+R; \text{Schwinger}} (z) \right. \right. \right.$$

(A19)

$$\left. \times h (x_1) \cdots h (x_N) \pi^{(1)} (y_1) \cdots \pi^{(m)} (y_M) \right] \right| 0 \right\rangle_{\text{Connected}} = 0,$$

$$\left\langle 0 \left| T \left[ \left( \partial_\mu \tilde{J}_{\mu L+R; \text{Schwinger}} (z) \right. \right. \right.$$

$$\left. \times h (x_1) \cdots h (x_N) \pi^{(1)} (y_1) \cdots \pi^{(m)} (y_M) \right] \right| 0 \right\rangle_{\text{Connected}} = 0.$$

This, plus CP conservation, allows us to write two sets of $SU(2)_L$ Ward-Takahashi identities governing the $\phi$-sector of $SU(2)_L$: one for the vector current $\tilde{J}_{\mu L+R; \text{Schwinger}}$ based on even $M$, and one for the axial-vector current $\tilde{J}_{\mu L-R; \text{Schwinger}}$ based on odd $M$. In this paper, we focus on those strong constraints placed on the scalar-sector of the $SU(2)_L$, by the axial-vector WTI.

5. Axial-vector Master Equation in $\tilde{J}_{\mu L-R; \text{Schwinger}}$

We now assemble the axial-vector Master Equation, from which we derive our “1-soft-$\pi$” $SU(2)_{L-R}$ WTI. The
LHS is from \(\text{(A15)}\), the RHS from \(\text{(A16)}\):

\[
\lim_{k_3 \to 0} \int d^4z e^{ikz} \int d^4z e^{ikz} \partial_\mu \langle 0 | T \left[ \mathcal{J}^\mu_{L-R;\text{Schwinger}}(z) \right] \rangle_{\text{Connected}} \\
\times h(x_1) \cdots h(x_N) \pi^{1 \dagger}(y_1) \cdots \pi^{M \dagger}(y_M) \rangle_0 \\
= \lim_{k_3 \to 0} \int d^4z e^{ikz} \sum_{n=1}^{N} \langle 0 | T \left[ h(x_1) \cdots h(x_{n-1}) \right] \rangle_{\text{Connected}} \\
\times \delta(z^0 - x_n^0) \left[ \mathcal{J}^0_{L-R;\text{Schwinger}}(z), h(x_n) \right] \\
\times h(x_{n+1}) \cdots h(x_N) \pi^{1 \dagger}(y_1) \cdots \pi^{M \dagger}(y_M) \rangle_0 \\
+ \lim_{k_3 \to 0} \int d^4z e^{ikz} \sum_{m=1}^{M} \langle 0 | T \left[ h(x_1) \cdots h(x_N) \right] \rangle_{\text{Connected}} \\
\times \pi^{1 \dagger}(y_1) \cdots \pi^{M \dagger}(y_m-1) \\
\times \delta(z^0 - y_m^0) \left[ \mathcal{J}^0_{L-R;\text{Schwinger}}(z), \pi^{M \dagger}(y_m) \right] \\
\times \pi^{M \dagger}(y_{m+1}) \cdots \pi^{M \dagger}(y_M) \rangle_0 \\
\text{written in terms of the physical states of the complex scalar } \phi. \text{ Here we have } N \text{ external renormalized scalars } h = H - \langle H \rangle (\text{coordinates } x, \text{ momenta } p), \text{ and } M \text{ external } (CP = -1) \text{ renormalized pseudo-scalars } \bar{\pi} (\text{coordinates } y, \text{ momenta } q, \text{ isospin } t).

\text{Eq. } \text{(A20)} \text{ is true for any } M, \text{ odd or even. It is derived from } \text{(A1)} \text{ for } M \text{ odd in } \text{(A15), (A7), (A15). It is also satisfied trivially, because of } \text{(A18), for } M \text{ even.}

\text{This paper is based on the de facto conservation of } \mathcal{J}^\mu_{L-R;\text{Schwinger}}, \text{ for } CP\text{-conserving on-shell and off-shell connected amputated Green’s functions and } T\text{-matrix elements in the } \phi\text{-sector, in the 1-soft-} \pi \text{ limit } \text{(A20)} \text{ of } SU(2)_L.

\textbf{Appendix B: }SU(2)_L \text{ Ward-Takahashi identities}

\text{The purpose of this Appendix is to derive two towers of quantum } SU(2)_L-R \text{ Ward-Takahashi identities, which exhaust the information content of } \text{(A20), and severely constrains the dynamics (i.e. the connected time-ordered products) of the physical states of the spontaneously broken bosonic } SU(2)_L.

\text{For pedagogical completeness, we repeat certain results found elsewhere the paper.}

1) \text{The axial-vector Master Equation } \text{(A20), from which we derive our } SU(2)_L-R \text{ WTII, is derived in Subsection A.5.}

2) \text{We employ Vintage QFT and canonical quantization, imposing equal-time commutators on the exact renormalized fields, yielding at space-time points } y, z:

\[
\delta(z_0 - y_0) \left[ \mathcal{J}^0_{L-R;\text{Schwinger}}(z), H(y) \right] \\
= -\frac{1}{2} \pi^i(y) \delta^4(z - y) \quad \text{(B1)}
\]

\[
\delta(z_0 - y_0) \left[ \mathcal{J}^0_{L-R;\text{Schwinger}}(z), \pi^j(y) \right] \\
= \frac{1}{2} i \delta^{ij} \pi^k(y) \delta^4(z - y)
\]

\text{Field normalization follows from the non-trivial commutators}

\[
\delta(z_0 - y_0) \left[ \partial^0 H(z), H(y) \right] = -i \delta^4(z - y) \quad \text{(B2)}
\]

\[
\delta(z_0 - y_0) \left[ \partial^i \pi^j(z), \pi^k(y) \right] = -i \delta^{ij} \delta^{4k}(z - y).
\]

3) \text{As appropriate to our study of the massless } \bar{\pi}, \text{ we use pion-pole dominance to derive 1-soft-pion theorems, and form the surface integral}

\[
\lim_{k_3 \to 0} \int d^4z e^{ikz} \partial_\mu \langle 0 | T \left[ \left( \frac{1}{2} \left( \bar{\pi} \partial^\mu \pi - h \partial^\mu \bar{\pi} \right) \right) \right] \rangle_{\text{Connected}} \\
\times h(x_1) \cdots h(x_N) \pi^{1 \dagger}(y_1) \cdots \pi^{M \dagger}(y_M) \rangle_0 \\
= \int d^4z \bar{\pi}_{3\text{-surface}} \langle 0 | T \left[ \left( \frac{1}{2} \left( \bar{\pi} \partial^\mu \pi - h \partial^\mu \bar{\pi} \right) \right) \right] \rangle_{\text{Connected}} \\
\times h(x_1) \cdots h(x_N) \pi^{1 \dagger}(y_1) \cdots \pi^{M \dagger}(y_M) \rangle_0 = 0,
\]

\text{where we have used Stokes theorem, and } \bar{\pi}_{3\text{-surface}} \text{ is a unit vector normal to the 3-surface. The time-ordered product constrains the 3-surface to lie on, or inside, the light-cone.}

\text{At a given point on the surface of a large enough 4-volume } \int d^4z (i.e. the volume of all space-time): all fields are asymptotic in-states and out-states, properly quantized as free fields, with each field species orthogonal to the others, and they are evaluated at equal times, making time-ordering un-necessary on the 3-surface at infinity. The surface integral } \text{(B3)} \text{ vanishes because } h \text{ is massive in spontaneously broken } SU(2)_L \text{ with } m_{2BEH}^2 \neq 0. \text{ Propagators connecting } h \text{ from points on the 3-surface at infinity to the localized interaction points } (x_1, x_2, \cdots, y_1, y_M) \text{ must stay inside the light-cone, die off exponentially with mass, and are incapable of carrying information that far.}

\text{It is very important for pion-pole-dominance, and the } SU(2)_L-R \text{ WTII here, that this argument fails for the remaining term in } \mathcal{J}^\mu_{L-R;\text{Schwinger}} \text{ in } \text{(A20):}

\[
\int_{3\text{-surface} \to \infty} d^3z \bar{\pi}_{3\text{-surface}} \langle 0 | T \left[ \left( \frac{1}{2} \left( H \right) \partial^\mu \bar{\pi} \right) \right] \rangle (z) \\
\times h(x_1) \cdots h(x_N) \pi^{1 \dagger}(y_1) \cdots \pi^{M \dagger}(y_M) \rangle_0 \neq 0.
\]
\( \vec{\pi} \) are massless in SSB SU(2)\(_L\) in Landau gauge, capable of carrying (along the light-cone) long-ranged pseudo-scalar forces out to the 2-surface \((z^2-surface \rightarrow \infty)\), i.e. to the very ends of the light-cone (but not inside it). That masslessness is the basis of our pion-pole-dominance-based SU(2)\(_L\)-R WTs, which give 1-soft-pion theorems \([B11]\), infra-red finiteness for \(m^2 \rightarrow 0\) \([B15]\), and an LSS theorem.

4) Using \([B1]\) in \([A20]\) on the right-hand-side, and \([B3]\) in \([A20]\) on the left-hand-side, we rewrite the Master Equation

\[
\lim_{k_\lambda \to 0} \int d^4ze^{ikz} \left\{ - \langle H \rangle \partial_{\mu} \phi \langle 0 | T \left( \partial^\mu \phi^I(z) \right) | 0 \rangle_{\text{Connected}} \right.
\]

where the “hatted” fields \(h(x)\) and \(\pi^m(y)\) are to be removed, and we have suppressed the \(k_\mu - z_\mu\) Fourier transform for clarity of presentation. We have also thrown away a sum of \(M\) terms, proportional to \(\langle H \rangle\), which corresponds entirely to disconnected graphs.

5) Connected momentum-space amplitudes, with \(N\) external BEHs and \(M\) internal \(\pi\)s, are defined in terms of \(\phi\)-sector connected time-space products

\[
iG^{t_1,t_2}_{N,M}(p_1...p_N;q_1...q_M)(2\pi)^4\delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)
\]

\[
= \prod_{n=1}^{N} \int d^4x_n e^{ip_n x_n} \prod_{m=1}^{M} \int d^4y_m e^{iq_m y_m}
\]

The Master Equation \([B5]\) can then be re-written

\[
\lim_{k_\lambda \to 0} \left\{ i\langle H \rangle k^2 i\Delta_\pi(k^2) T_{N+1,M+1}(p_1...p_N;k_1...q_M) \right.
\]

\[
- \sum_{n=1}^{N} G_{N-1,1,M+1}(p_1...p_N; (k + p_n)q_1...q_M)
\]

\[
+ \sum_{m=1}^{M} \delta^{\mu \nu} G^\mu_{N-1,1,M-1}(k + q_m) p_1...p_N; q_1...q_m...q_M
\]

\[
= 0
\]

with the “hatted” momenta \(\hat{p}_n, \hat{q}_m\) and isospin \(\hat{1}_m\) removed in \([B7]\), and an overall momentum conservation factor of \((2\pi)^4\delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)\).

6) Special cases of \([B6]\) are the BEH and \(\vec{\pi}\) propagators

\[
iG_{2,0}(p_1,-p_1) = i \int d^4p_2 \frac{1}{(2\pi)^4} G_{2,0}(p_1, p_2;)
\]

\[
- \int d^4x_1 e^{ip_1 x_1} \langle 0 | T \left( h(x_1) \right) | 0 \rangle
\]

\[
= i \Delta_{BEH}(p_1^2)
\]

\[
iG^{t_2}_{0,2}(q_1,-q_1) = i \int d^4q_2 \frac{1}{(2\pi)^4} G^{t_2}_{0,2}(q_1, q_2)
\]

\[
- \int d^4y_1 e^{iq_1 y_1} \langle 0 | T \left( \pi^T(y_1) \pi^T(0) \right) | 0 \rangle
\]

\[
= i \delta^{t_2} \Delta_\pi(q_1^2)
\]

7) In what follows, isospin indices will become increasingly cumbersome. We therefore adopt B.W. Lee’s \([12]\) convention of suppressing isospin indices, allowing momenta to implicitly carry them.

8) \(\phi\)-sector connected amputated 1-\((h, \pi)\)-Reducible (1-\(\phi\)-\(R\)) transition matrix (T-matrix): With an overall momentum conservation factor \((2\pi)^4\delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)\), the \(\phi\)-sector connected amplitudes are related to \(\phi\)-sector connected amputated T-matrix elements

\[
G_{N,M}(p_1...p_N; q_1...q_M)
\]

so that the Master Equation \([B5]\) can be written

\[
\lim_{k_\lambda \to 0} \left\{ i\langle H \rangle k^2 i\Delta_\pi(k^2) T_{N+1,M+1}(p_1...p_N;k_1...q_M) \right.
\]

\[
- \sum_{n=1}^{N} G_{N-1,1,M+1}(p_1...p_N; (k + p_n)q_1...q_M)
\]

\[
+ \sum_{m=1}^{M} \delta^{\mu \nu} G^\mu_{N-1,1,M-1}(k + q_m) p_1...p_N; q_1...q_m...q_M
\]

\[
= 0
\]

with the “hatted” momenta \(\hat{p}_n, \hat{q}_m\) removed in \([B10]\), and an overall momentum conservation factor of \((2\pi)^4\delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)\).

9) Consider the “soft-pion limit”

\[
\lim_{k_\mu \to 0} k^2 \Delta_\pi(k^2) = 1
\]

where the \(\vec{\pi}\) is hypothesized to be all-loop-orders massless

\[
\Delta_\pi(k^2) = \frac{1}{k^2 + i\epsilon} + \int dm^2 \frac{\rho_{AHM}(m^2)}{k^2 - m^2 + i\epsilon}
\]
The set of 1-soft-pion theorems (B13) have the form

\begin{align}
- \langle H \rangle T_{N,M+1}(p_1\cdots p_N;0q_1\cdots q_M) \\
= \sum_{n=1}^{N} T_{N-1,M+1}(p_1\cdots \hat{p}_n\cdots p_N;pnq_1\cdots q_M) \\
\times \left[i \Delta_\pi(p_n^2) \right] \left[i \Delta_{BEH}(p_n^2) \right]^{-1} \\
- \sum_{m=1}^{M} T_{N+1,M-1}((k+q_m)p_1\cdots p_N;q_1\cdots \hat{q}_m\cdots q_M) \\
\times \left[i \Delta_{BEH}(q_m^2) \right] \left[i \Delta_\pi(q_m^2) \right]^{-1} \\
\end{align}

(B13)

in the 1-soft-pion limit. As usual the “hatted” momenta \((\hat{p}_n, \hat{q}_m)\) and associated fields are removed in \(B13\), and an overall momentum conservation factor \((2\pi)^4 \delta^4(\sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m)\) applied.

The set of 1-soft-pion theorems \(B13\) have the form

\begin{align}
\langle H \rangle T_{N-M+1} \sim T_{N-1,M+1} - T_{N+1,M-1} \\
\end{align}

(B14)

relating, by the addition of a zero-momentum pion, an \(N + M\) + 1-point function to \(N + M\)-point functions.

10) The Adler self-consistency relations, for the \(SU(2)_L\) gauge theory rather than global \(SU(2)_L \times SU(2)_R\) \([41, 42]\), are obtained from \(B13\) by putting the remainder of the particles on mass-shell

\begin{align}
\langle H \rangle T_{N,M+1}(p_1\cdots p_N;0q_1\cdots q_M) \\
\times (2\pi)^4 \delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right) p_1^2=p_2^2=\cdots=p_{N}^2=m_{BEH} \\
q_1^2=q_2^2=\cdots=q_{M}^2=0 \\
= 0 . \quad (B15)
\end{align}

This guarantees the infra-red (IR) finiteness of the \(\phi\)-sector on-shell T-matrix in SSB \(SU(2)_L\) gauge theory in \(R_\xi(\xi = 0)\) Landau gauge, with massless \(\pi\) in the 1-soft-pion limit.

11) \(1-(h, \pi)\) Reducibility \((1-\phi-R)\) and \(1-(h, \pi)\) Irreducibility \((1-\phi-I)\): With some exceptions, the \(\phi\)-sector connected amputated transition matrix \(T_{N,M}\) can be cut in two by cutting an internal \(h\) or \(\pi\) line, and are designated 1-\(\phi-R\). In contrast, the \(\phi\)-sector connected amputated Green’s functions \(\Gamma_{N,M}\) are defined to be 1-\(\phi-I\), i.e. they cannot be cut apart by cutting an internal \(h\) or \(\pi\) line.

\(T_{N,M} = \Gamma_{N,M} + (1 - \phi - R) \quad (B16)\)

Both \(T_{N,M}\) and \(\Gamma_{N,M}\) are 1-(\(\bar{W}_\mu\))-Reducible \((1-\bar{W}_\mu-R)\) i.e. they can be cut in two by cutting an internal transverse \(\bar{W}_\mu\) photon line.

12) The special 2-point functions \(T_{0,2}(p,-q)\) and \(T_{2,0}(p,-p)\), and the 3-point vertex \(T_{1,2}(q;0, -q)\), are 1-\(\phi-I\), i.e. they are not 1-\(\phi-R\). They are therefore equal to the corresponding 1-\(\phi-I\) connected amputated Green’s functions. The 2-point functions

\(T_{2,0}(p,-p) = \Gamma_{2,0}(p,-p) = [\Delta_{BEH}(p^2)]^{-1}\)

\(T_{0,2}(q, -q) = \Gamma_{0,2}(q, -q) = [\Delta_\pi(q^2)]^{-1} \quad (B17)\)

are related to the \((1h, 2\pi)\) 3-point \(h\pi^2\) vertex

\(T_{1,2}(p; q, -p - q) = \Gamma_{1,2}(p; q, -p - q) \quad (B18)\)

by a 1-soft-pion theorem \(B13\)

\(\langle H \rangle T_{1,2}(q;0, -q) - T_{2,0}(q, -q) + T_{0,2}(q, q) \quad (B19)\)

\(= \langle H \rangle T_{1,2}(q;0, -q) - [\Delta_{BEH}(q^2)]^{-1} + [\Delta_\pi(q^2)]^{-1} + \langle H \rangle \Gamma_{1,2}(q;0, -q) - \Gamma_{2,0}(q, -q) - \Gamma_{0,2}(q, -q) = \langle H \rangle \Gamma_{1,2}(q;0, -q) - [\Delta_{BEH}(q^2)]^{-1} + [\Delta_\pi(q^2)]^{-1} = 0 . \quad (B20)\)

13) The Lee-Stora-Symanzik (LSS) theorem, in spontaneously broken \(SU(2)_L\) in \(R_\xi(\xi = 0)\) Landau gauge, is a special case of that SSB gauge theory’s Adler self-consistency relations \(B15\)

\(\langle H \rangle T_{0,2}(0;0) = 0 \quad (B20)\)

\(\langle H \rangle \Gamma_{0,2}(0;0) = 0 \quad (B20)\)

\(\langle H \rangle [\Delta_\pi(0)]^{-1} = -\langle H \rangle m_\pi^2 = 0 , \quad (B20)\)

proving that \(\pi\) is massless. That all-loop-orders renormalized masslessness is protected/guaranteed by the global \(SU(2)_L\) symmetry of the physical states of the gauge theory after spontaneous symmetry breaking.

14) Figure \([\text{I}]\) illustrates \(T_{\text{External},N+1,M+1}\), a \(\phi\)-sector T-Matrix with one soft \(\pi(q_\mu = 0)\) attached to an external-

---

\footnote{The set of 1-\(\phi\)-I graphs includes all the 1-P-I graphs and infinitely many more. Krauss and Sibold\([6]\), and Grass\([13]\) derived WTI’s for 1-P-I graphs for the AHM and for general gauge theories with non-semi-simple gauge groups respectively. As we consider a different set of Green’s functions, the WTI’s derived here are fundamentally different from those derived in [6] [19]. We are interested only in processes involving external scalars and in the scalar-sector effective potential; to calculate those, 1-\(\phi\)-I graphs are the correct set to consider.}

---
so that

\[
\langle H \rangle T_{N,M+1}^{\text{External Leg}}(p_1...p_N; 0q_1...q_M) = \sum_{n=1}^{N} \left[ i(H) \Gamma_{1,2}(p_n, 0_p) \right] [i \Delta_\pi(p_n^2)] \\
\times T_{N-1,M+1}(\bar{p}_n...\bar{p}_N; p_nq_1...q_M) \\
+ \sum_{m=1}^{M} \left[ i(H) \Gamma_{1,2}(q_m, 0_q) \right] [i \Delta_{\text{BEH}}(q_m^2)] \\
\times T_{N+1,M-1}(q_m\bar{p}_m...\bar{p}_N; q_1...\bar{q}_m...q_M) \\
= \sum_{n=1}^{N} \left( 1 - \left[ i \Delta_\pi(p_n^2) \right] \right) [i \Delta_{\text{BEH}}(p_n^2)]^{-1} \\
\times T_{N-1,M+1}(p_1...\bar{p}_n; \bar{p}_N; p_nq_1...q_M) \\
- \sum_{m=1}^{M} \left( 1 - \left[ i \Delta_{\text{BEH}}(q_m^2) \right] \right) [i \Delta_\pi(q_m^2)]^{-1} \\
\times T_{N+1,M-1}(q_m\bar{p}_m...\bar{p}_N; q_1...\bar{q}_m...q_M) \tag{B21}
\]

when we employ (B19). Now separate

\[
T_{N,M+1}(p_1...p_N; 0q_1...q_M) = T_{N,M+1}^{\text{External Leg}}(p_1...p_N; 0q_1...q_M) \\
+ T_{N,M+1}^{\text{Internal Leg}}(p_1...p_N; 0q_1...q_M) \tag{B22}
\]

so that

\[
\langle H \rangle T_{N,M+1}^{\text{Internal Leg}}(p_1...p_N; 0q_1...q_M) = \sum_{m=1}^{M} T_{N+1,M-1}(q_m\bar{p}_m...\bar{p}_N; q_1...\bar{q}_m...q_M) \\
- \sum_{n=1}^{N} T_{N-1,M+1}(p_1...\bar{p}_n; \bar{p}_N; p_nq_1...q_M) \tag{B23}
\]

Removing the 1-\(\phi\)-R graphs from both sides of (B23) yields the recursive \(SU(2)_{L-R}\) WTI for 1-\(\phi\)-I connected amputated Green’s functions \(\Gamma_{n,m}\):

\[
\langle H \rangle \Gamma_{N,M+1}(p_1...p_N; 0q_1...q_M) = \sum_{m=1}^{M} \Gamma_{N+1,M-1}(q_m\bar{p}_m...\bar{p}_N; q_1...\bar{q}_m...q_M) \\
- \sum_{n=1}^{N} \Gamma_{N-1,M+1}(p_1...\bar{p}_n; \bar{p}_N; p_nq_1...q_M). \tag{B24}
\]

B.W. Lee \cite{Lee79} gave an inductive proof for the corresponding recursive \(SU(2)_L \times SU(2)_R\) WTI in the global Gell-Mann Levy model with PCAC \cite{Gell-Mann60}. Specifically, he proved that, given the global \(SU(2)_L \times SU(2)_R\) analogy of (B23), the local \(SU(2)_L\) analogy of (B24) follows. He did this by examining the explicit reducibility/irreducibility of the various Feynman graphs involved.

That proof also works for SSB \(SU(2)_{L-R}\), thus establishing our tower of 1-\(\phi\)-I connected amputated Green’s functions’ recursive \(SU(2)_{L-R}\) WTI (B24) or the local \(SU(2)_{L}\) gauge theory.

Rather than including the lengthy proof here, we paraphrase Witten \cite{Witten79} as follows: (B19) shows that (B24) is true for \((n = 1, m = 1)\). Assume it is true for all \((n, m)\) such that \(n + m < N + M\). Consider (B23) for \((n = N, m = M)\). The two classes of graphs contributing to \(T_{N,M+1}^{\text{Internal Leg}}(p_1...p_N; 0q_1...q_M)\) are displayed in Figure 2.

The top graphs in Figure 2 are 1-\(\phi\)-R. For \((n, m; n + m < N + M)\), we may use (B24), for those 1-\(\phi\)-I Green’s functions \(\Gamma_{n,m}\) that contribute to (B23), to show that the contributions of 1-\(\phi\)-R graphs to both sides of (B23) are identical.

The bottom graphs in Figure 2 are 1-\(\phi\)-I, and so already obey (B24).

16) The LSS theorem makes tadpoles vanish.

\[
\langle 0| h(x = 0) | 0 \rangle_{\text{Connected}} = i \left[ i \Delta_{\text{BEH}}(0) \right] \Gamma_{1,0}(0; 0) \tag{B25}
\]

but the \(N = 0, M = 1\) case of (B24) reads

\[
\Gamma_{1,0}(0; 0) = \langle H \rangle \Gamma_{0,2}(0; 0) = 0, \tag{B26}
\]
where we used (B20). Tadpoles therefore all vanish automatically, and separate tadpole renormalization is unnecessary. Since we can choose the origin of coordinates anywhere we like
\[
\langle 0| h(x)|0 \rangle_{\text{Connected}} = 0. \quad (B27)
\]

17) Renormalized \( \langle H \rangle \) obeys
\[
\langle 0| H(x)|0 \rangle_{\text{Connected}} = \langle 0| h(x)|0 \rangle_{\text{Connected}} + \langle H \rangle
= \langle H \rangle
\partial_\mu \langle H \rangle = 0. \quad (B28)
\]

18) Benjamin W. Lee’s 1970 Cargese summer school lectures’ [12] proof of \( \phi \)-sector WTI focussed on the global \( SU(2)_L \times SU(2)_R \) Gell-Mann Levy theory with PCAC, but it also gave a detailed pedagogical account of the appearance of the Goldstone theorem and massless Nambu-Goldstone bosons in global theories. We include a translation guide in Table 1.
TABLE I. Derivation of Ward-Takahashi IDs

| Property                        | This paper | B.W.Lee [12] |
|---------------------------------|------------|--------------|
| Lagrangian invariant           | BRST       | global group |
| structure group                 | SU(2)_L   | SU(2)_L × SU(2)_R |
| rigid/global group             | SU(2)_L   | SU(2)_L × SU(2)_R |
| local/gauge group              | SU(2)_L   | ——           |
| global currents                 | J^L_µ     | ——           |
| PCAC                            | no         | yes          |
| current divergence              | ∼ ∂_β W^β | 0; f_π m_π^2 π |
| gauge fixing                    | ∂_β W^β = 0 | ——           |
| gauge                           | Landau     | ——           |
| ghosts ̃η, ̃ω                  | ——         | ——           |
| ghosts ̃ω^A, ̃ω^A                    | color      | ——           |
| conserved current               | physical states | Lagrangian |
| physical states                 | h, ̃x, ̃W_µ | s, ̃π |
| unphysical states               | ghosts     | ——           |
| interaction                     | weak       | strong       |
| fields                          | H, ̃x, ̃W_µ, ̃η, ̃ω | σ, ̃π |
| BEH scalar                      | h = H - (H) | s = σ - ⟨σ⟩ |
| VEV                             | (H)        | ⟨σ⟩ = v = f_π |
| loop particles                  | physical & ghosts | all-loop-orders |
| renormalization                 | ——         | all-loop-orders |
| Amplitudes:                     | ——         | ——           |
| no pion pole                    | G          | H            |
| T-Matrix                        | G^N,M      | G^N,M        |
| 1-φ                            | T          | T            |
| 1-φ-sector T^N,M                | T^N,M      | 1-φ-R        |
| 1-φ-sector GF                   | Γ^N,M      | Γ^N,M        |
| connected Γ^N,M                 | amputated  | amputated    |
| connected T^N,M                 | amputated  | amputated    |
| Γ^N,M                           | 1-φ-R, 1-̃W-R, 1-Φ-R | 1-φ-I |
| external leg ̃π                 | T_{External;N,M+1} | T_1 |
| internal ̃π                    | T_{Internal;N,M+1} | T_2 |
| explicit breaking               | m_π^2      | ——           |
| pseudo-NGB mass-squared        | (H) Γ^{0,12}_α (00) = 0 | f_π Γ^{0,12}_α (00) = ε δ^{11} = 0 |
| LSS Theorem                     | 1-D line   | 1-D boundary of 2-D quarter-plane |
| SSB                             | Goldstone Mode | Goldstone Mode |
| NGB after SSB                   | ̃π          | Implied ̃π |
| BEH propagator                  | Δ_{BEH}    | Δ_{BEH}      |
| transverse propagator           | Δ_ν         | Δ_ν          |
| pion propagator                 | Δ_ν         | δ^{11} Δ_ν |
Appendix C: \( R_C \)-gauge renormalized \( SU(2)_L \) for
gauge bosons, complex scalar doublet, ghosts and
anti-ghosts

In this Appendix, we construct, in an arbitrary \( R_C \)
gauge, the \( SU(2)_L \) isospin current \( J^\mu_L \), together with its
divergence and equal-time commutators with fields.

Thirteen renormalized bosons and ghosts appear in
\( SU(2)_L \): a scalar \( H \), three pseudo-scalars \( \bar{\pi} \), three
isospin gauge fields \( \bar{W}_\mu \), three isospin ghosts \( \bar{\omega} \), and three anti-
ghosts \( \bar{\eta} \).

\[
\{ \Phi_i \} = \{ H, \bar{\pi}, \bar{W}_\mu, \bar{\omega}, \bar{\eta} \}; \quad i = 1, 13
\]  
(C1)

1. \( SU(2)_L \) classical current and its divergence

We form the classical current

\[
-J^\mu_L = -(g_2 \Omega) \cdot \bar{J}^\mu_L
\]  
(C2)

and its divergence

\[
-\partial_\mu \bar{J}^\mu_L = -(g_2 \Omega) \cdot \partial_\mu \bar{J}^\mu_L
\]  
(C3)

We have used the equations of motion (EOM) for cer-
tain \( \{ \Phi_i \} \); in particular, the Lagrange-multiplier field \( \bar{b} \)
is constrained,

\[
\bar{b} = -\frac{1}{\xi} F^W \, ,
\]  
(C4)

and the \( \bar{\omega} \) ghost EOM is

\[
0 = s \bar{F}_W
\]  
\[
= \partial^2 \bar{\omega} + g_2 \partial_\mu \left( \bar{W}^\mu \times \bar{\omega} \right) + \xi \frac{M_W^2}{\langle H \rangle} \left( H \bar{\omega} + \bar{\pi} \times \bar{\omega} \right).
\]

Eq. (C5) are used in the construction of the classical
current \( J^\mu_L \) and its divergence \( \partial_\mu J^\mu_L \), but not in the quantum
mechanics of the Lagrangian (26).

2. Equal-time field commutators

We normalize the \( \phi \) field so that

\[
\delta(z_0 - y_0) [\partial^0 H(z), H(y)] = -i \delta^4(z - y)
\]  
(C6)

\[
\delta(z_0 - y_0) [\partial^0 \pi^i(z), \pi^j(y)] = -i \delta^4(z - y)
\]  
(C7)

with similar normalizations for the other fields, as ap-
propriate to gauge bosons and fermionic ghosts. Except
for such momentum-field quantum conditions, all other
equal-time commutators vanish

\[
\delta(z_0 - y_0) [\Phi_i(z), \Phi_j(y)] = 0, \quad \forall i, j
\]  
(C8)

\[
\delta(z_0 - y_0) [\partial^0 \Phi_i(z), \Phi_j(y)] = 0, \quad \forall i, j, \text{with } i \neq j ,
\]  
(C9)

with anti-commutators as appropriate for ghosts.

3. \( SU(2)_L \) currents and commutators in \( R_C \) Gauge

a. Global \( SU(2)_L \) Schwinger model \[39\]

It is useful to remember some results from the global
\( SU(2)_L \) Schwinger model \[39\], with Lagrangian

\[
L_{\text{Schwinger}} = |\partial_\mu \phi|^2 - V
\]  
(C10)

\[
V = \mu_\phi^2 (\phi^\dagger \phi) + \lambda_\phi^2 (\phi^4)^2.
\]

Consider the current

\[
\bar{J}^\mu_{\text{Schwinger}} = \frac{1}{2} \bar{\pi} \times \partial^\mu \bar{\pi} + \frac{1}{2} \left( \bar{\pi} \partial^\mu H - H \partial^\mu \bar{\pi} \right).
\]  
(C11)

Some useful commutators are

\[
\delta(z_0 - y_0) [\bar{J}^0_{\text{Schwinger}}(z), H(y)]
\]  
\[
= \frac{1}{2} \delta^4(z - y) \bar{\pi}(z)
\]  
(C12)

\[
\delta(z_0 - y_0) [\bar{J}^i_{\text{Schwinger}}(z), \pi^{\dagger j}(y)]
\]  
\[
= \frac{1}{2} i \delta^4(z - y) \times \left( \epsilon^{i12} \pi^{\dagger j}(z) + \delta^{i1} H(z) \right).
\]  
(C13)

b. Gauge-invariant scalar Lagrangian \( L^S \)

The gauge-invariant scalar Lagrangian is

\[
L^S = |D_\mu \phi|^2 - V
\]  
(C14)

\[
D_\mu \phi = [\partial_\mu + ig_2 W_\mu] \phi
\]

\[
W_\mu = \bar{\sigma} \cdot \bar{W}_\mu \equiv \frac{1}{2} \bar{\sigma} \cdot \bar{W}_\mu
\]
Consider the current
\[ J_{L}^{\mu \phi} = J_{L}^{\mu \text{Schwinger}} + \bar{J}_{L}^{\mu \phi} \] (C12)
\[ \bar{J}_{L}^{\mu \phi} = -\frac{1}{4} g_{2} \bar{W}^{\mu} [H^{2} + \bar{\pi}] . \]

Some useful commutators are
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu \phi}(z), H(y) \right] = \delta(z_{0}^{} - y_{0}^{}) \left[ \bar{J}_{L}^{\mu \phi}(z), H(y) \right] \]
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu \phi}(z), \pi^{\mu}(y) \right] = \delta(z_{0}^{} - y_{0}^{}) \left[ \bar{J}_{L}^{\mu \phi}(z), \pi^{\mu}(y) \right] . \]

### c. Gauge-invariant isospin Lagrangian \( L^{W} \)

The gauge-invariant isospin Lagrangian is
\[ L^{W} = -\frac{1}{2} \text{Tr} (W_{\mu} W^{\mu \nu}) = -\frac{1}{4} \bar{W}^{\mu \nu} \cdot \bar{W}^{\mu \nu} \] (C13)
\[ W_{\mu \nu} = \partial_{\mu} W_{\nu} - \partial_{\nu} W_{\mu} + ig_{2} [W_{\mu}, W_{\nu}] \]
\[ \bar{W}_{\mu} = \bar{\sigma} \cdot \bar{W}_{\mu} \]
\[ \bar{W}_{\mu \nu} = \partial_{\mu} \bar{W}_{\nu} - \partial_{\nu} \bar{W}_{\mu} - g_{2} \bar{W}_{\mu} \times \bar{W}_{\nu} \]

with Pauli matrices \( \bar{\sigma} \). Consider the current
\[ \bar{J}_{L}^{\mu \phi} = \bar{W}^{\mu \nu} \times \bar{W}_{\nu} . \] (C14)

Useful commutators are:
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{W}(z), H(y) \right] = 0 \]
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{W}(z), \pi^{\mu}(y) \right] = 0 \] (C15)

### d. Isospin gauge-fixing Lagrangian \( L^{R_{\xi}} \)

The isospin gauge-fixing Lagrangian is
\[ L_{R_{\xi}}^{\text{GaugeFix}} = -\frac{1}{2 \xi} \left[ \partial_{\beta} \bar{W}^{\beta} + \xi M_{W} \bar{\pi} \right]^{2} \] (C16)
\[ M_{W} = \frac{1}{2} g_{2} \langle H \rangle ; \]
\[ \bar{F}_{W} = \partial_{\beta} \bar{W}^{\beta} + \xi M_{W} \bar{\pi} . \]

Consider the current
\[ \bar{J}_{L}^{\mu \phi; R_{\xi}}^{\text{GaugeFix}} = -\frac{1}{\xi} \left[ \bar{W}^{\mu} \times \bar{F}_{W} \right] \] (C17)

Useful commutators are
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu \phi; R_{\xi}}^{\text{GaugeFix}}(z), H(y) \right] = 0 \]
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu \phi; R_{\xi}}^{\text{GaugeFix}}(z), \pi^{\mu}(y) \right] = 0 \]

The total \( SU(2)_{L} \) current and commutators

The total \( SU(2)_{L} \) current is therefore
\[ \bar{J}_{L}^{\mu} = \bar{J}_{L; \text{Schwinger}}^{\mu} + \bar{J}_{L}^{\mu} \]
\[ \bar{J}_{L; \text{Schwinger}}^{\mu} = \frac{1}{2} \bar{\pi} \times \partial^{\nu} \bar{\pi} + \frac{1}{2} \left[ \bar{\pi} \partial^{\nu} H - H \partial^{\nu} \bar{\pi} \right] \]
\[ \bar{J}_{L}^{\mu} = -\frac{1}{4} g_{2} \bar{W}^{\mu} [H^{2} + \bar{\pi}] + \bar{W}^{\mu \nu} \times \bar{W}_{\nu} - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \bar{W}^{\mu \nu} \bar{F}_{W}(0) \]
\[ - \partial^{\mu} \bar{\eta} \times \bar{\omega} \]

with
\[ \bar{F}_{W} = \partial_{\beta} \bar{W}^{\beta} + \xi M_{W} \bar{\pi} \]
\[ M_{W} = \frac{1}{2} g_{2} \langle H \rangle ; \]

Its divergence is
\[ \partial_{\mu} \bar{J}_{L}^{\mu} = \frac{1}{2} M_{W} \left[ \bar{\pi} \times \bar{F}_{W} + H \bar{F}_{W} \right] \] (C20)

Useful commutators are
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu}(z) - J_{L}^{\mu; \text{Schwinger}}(z), H(y) \right] = 0 \]
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu}(z) - J_{L}^{\mu; \text{Schwinger}}(z), \pi^{\mu}(y) \right] = 0 \]
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu; \text{Schwinger}}(z), H(y) \right] \]
\[ = -\frac{1}{2} i \delta^{4}(z - y) \bar{\pi}(z) \] (C21)
\[ \delta(z_{0}^{} - y_{0}^{}) \left[ J_{L}^{\mu; \text{Schwinger}}(z), \pi^{\beta}(y) \right] \]
\[ = \frac{1}{2} i \delta^{4}(z - y) \times \left( \epsilon_{\mu \nu \tau \beta} \pi^{\tau}(z) + \delta^{\mu \beta} H(z) \right) \]
5. Total $SU(2)_L$ Lagrangian

The total $SU(2)_L$ Lagrangian is thus

$$L = L^W + L^\phi + L^{Gauge\_Fix}_W + L^{\Ghost}_W$$

(C22)

with

$$L^W = -\frac{1}{2} \text{Tr} (W_{\mu \nu} W^{\mu \nu})$$

$$W_{\mu} = i \cdot \vec{W}_\mu = \frac{i}{2} \sigma \cdot \vec{W}_\mu$$

$$W_{\mu \nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig_2 [W_\mu, W_\nu]$$

$$L^\phi = |D_\mu \phi|^2 - V$$

$$V = \mu_1^2 (\phi \phi) + \lambda_1^2 (\phi^\dagger \phi)^2$$

$$D_\mu \phi = [\partial_\mu + ig_2 W_\mu] \phi$$

(C23)

$$L^{Gauge\_Fix}_W + L^{\Ghost}_W$$

$$= \sum_i \left( \frac{\partial}{\partial (\partial_\mu \Phi_1)} L_{2\otimes 1} \right) \left( \delta_{SU(2)} L \Phi_1 \right)$$

and its divergence

$$-\partial_\mu \vec{J}_L^{2\otimes 1} = - (g_2 \vec{\Omega}) \cdot \vec{J}_L^{2\otimes 1}$$

(D4)

Appendix D: $R_\xi$ gauge renormalized $SU(2)_L \times U(1)_Y$

for gauge bosons, a complex scalar doublet, ghosts and anti-ghosts

In this Appendix, we construct the $SU(2)_L \times U(1)_Y$ isospin sub-current $\vec{J}_L^{2\otimes 1}$, together with its divergence, for bosons, ghosts and anti-ghosts in an arbitrary $R_\xi^{2\otimes 1}$ gauge. The Appendix is pedagogically complete, and repeats certain results found in the body of the paper.

The notation "$2\otimes 1$" is short-hand for $SU(2)_L \times U(1)_Y$.

Sixteen renormalized bosons and ghosts appear in $SU(2)_L \times U(1)_Y$: one scalar $H$, three pseudo-scalars $\vec{\pi}$, three isospin gauge fields $\vec{W}_\mu$, three isospin ghosts $\vec{\omega}$, three anti-ghosts $\vec{\eta}$, a hypercharge gauge field $B_\mu$, a hypercharge ghost $\omega_B$ and anti-ghost $\bar{\eta}_B$.

$$\{ \Phi_i \} = \{ H, \vec{\pi}; \vec{W}_\mu, \vec{\omega}, \vec{\eta}_B; B_\mu, \omega_B, \bar{\eta}_B \}, i = 1, 16 \quad (D1)$$

1. Global $SU(2) \times U(1)_Y$ and BRST transformations

Global BRST transformations $[35][22][23] s_{2\otimes 1}$ appear in [65].

Anomaly-free un-deformed rigid/global $\delta_{SU(2)_L \times U(1)_Y}$ transformations of fields by constant $\vec{\Omega}$ and $\Omega_B$ appear in [66].

The global BRST and $SU(2)_L \times U(1)_Y$ transformation sets [65] and [66] commute

$$\left[ \delta_{2\otimes 1}, s_{2\otimes 1} \right] = 0 \quad (D2)$$

2. $SU(2)_L$ sub-current and its divergence

Because $SU(2)_L$ is an applicable sub-group of $SU(2)_L \times U(1)_Y$,

$$\left[ \delta_{SU(2)_L}, s_{2\otimes 1} \right] = 0 \quad (D3)$$

as in [68], and we can form the classical $SU(2)_L$ sub-current

$$-\vec{J}_{L,2\otimes 1}^{\mu} = -(g_2 \vec{\Omega}) \cdot \vec{J}_{L,2\otimes 1}^{\mu} \quad (D4)$$

and its divergence

$$-\partial_\mu \vec{J}_{L,2\otimes 1}^{\mu} = - (g_2 \vec{\Omega}) \cdot \partial_\mu \vec{J}_{L,2\otimes 1}^{\mu}$$

$$= \sum_i \left( \frac{\partial}{\partial (\partial_\mu \Phi_1)} L_{2\otimes 1} \right) \left( \delta_{SU(2)_L} \Phi_1 \right)$$

(D4)
To arrive at (D5), we have used the EOM for \{\Phi_i\}. In particular, the Lagrange-multiplier fields \(\bar{b}, b_B\) are constrained,

\[
\bar{b} = -\frac{1}{\xi} F^\mu_W; \quad b_B = -\frac{1}{\xi} F_B
\]  
(D6)

and the \(\bar{\omega}, \omega_B\) ghost EOM are

\[
s_{2^\otimes 1} F^\mu_W = \partial^2 \bar{\omega} + g_2 \partial_\mu \left( \bar{W}^\mu \times \bar{\omega} \right) + \xi \frac{M^2_W}{(H)} \left( H \bar{\omega} + \bar{\pi} \times \bar{\omega} \right)
\]  
(D8)

\[
s_{2^\otimes 1} F_B = \partial^2 \omega_B + \xi \frac{M^2_B}{(H)} H \omega_B + \xi \frac{M_B M_W}{(H)} \left[ H \bar{\omega} + \bar{\pi} \times \bar{\omega} \right]_3 = 0
\]

Eq. (D8) are used in the construction of the classical current \(J^\mu_L\) and its divergence \(\partial_\mu J^\mu_L\) but not in the quantum mechanics of the Lagrangian (D19).

3. \(SU(2)_L\) sub-current and commutators

The global \(SU(2)_L\) Schwinger model [39], its relevant current, the divergence of that current, and relevant commutators are found in Subsubsection C.3.a.\(\}

The gauge-invariant isospin Lagrangian \(L^W\), its relevant current, the divergence of that current, and relevant commutators are found in Subsubsection C.3.b.

The isospin gauge-fixing Lagrangian \(L^B_{\text{GaugeFix}}\), its relevant current, the divergence of that current, and relevant commutators are found in Subsubsection C.3.c.

The isospin ghost Lagrangian \(L^B_{\text{Ghost}}\), its relevant current, the divergence of that current, and relevant commutators are found in Subsubsection C.3.d.

a. Gauge-invariant hypercharge Lagrangian \(L^B_{2^\otimes 1}\)

The gauge-invariant hypercharge Lagrangian is

\[
L^B_{2^\otimes 1} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu}
\]  
(D9)

Its relevant current is

\[
J^B_{2^\otimes 1} = 0
\]  
(D10)

and so has commutators

\[
\delta(z^0 - y^0) \left[ J^B_{2^\otimes 1}(z), H(y) \right] = 0
\]

\[
\delta(z^0 - y^0) \left[ J^B_{2^\otimes 1}(z), \bar{\pi}(y) \right] = 0.
\]  
(D11)

b. Gauge-invariant scalar Lagrangian \(L^\phi_{2^\otimes 1}\)

The gauge-invariant scalar Lagrangian is

\[
L^\phi_{2^\otimes 1} = \left| D^\otimes_{\mu} \phi \right|^2 - V
\]

\[
V = \mu^2 \phi^1(\phi)^2 + \lambda^2_3 (\phi^1)^2 (D12)
\]

and its commutators

\[
\delta(z_0 - y_0) \left[ J^\phi_{L;2^\otimes 1}(z), H(y) \right] = 0
\]

\[
\delta(z_0 - y_0) \left[ J^\phi_{L;2^\otimes 1}(z), \bar{\pi}(y) \right] = 0.
\]

C. Global \(SU(2)_L \times U(1)_Y\) gauge-fixing, ghosts and anti-ghosts

The global \(SU(2)_L \times U(1)_Y\) Lagrangian for gauge-fixing, ghosts and anti-ghosts is

\[
L^R_{\text{GaugeFix}} + L^R_{\text{Ghost}}
\]  
(D15)

\[
\bar{F} = \partial_\mu \bar{W}^\mu + \xi \bar{M}_W \bar{\pi}
\]

\[
F = \partial_\mu B^\mu + \xi M_B \pi
\]

Consider the current

\[
J^{R;\mu}_{L;\text{GaugeFix}} + J^{R;\mu}_{L;\text{Ghost}} = J^{R;\mu}_{L;\text{GaugeFix}} + J^{R;\mu}_{L;\text{Ghosts}}
\]  
(D16)

and its commutators

\[
\delta(z_0 - y_0) \left[ J^{R;\mu}_{L;\text{GaugeFix}}(z), H(y) \right] = 0
\]

\[
\delta(z_0 - y_0) \left[ J^{R;\mu}_{L;\text{GaugeFix}}(z), \bar{\pi}(y) \right] = 0
\]

\[
\delta(z_0 - y_0) \left[ J^{R;\mu}_{L;\text{Ghosts}}(z), H(y) \right] = 0
\]

\[
\delta(z_0 - y_0) \left[ J^{R;\mu}_{L;\text{Ghosts}}(z), \bar{\pi}(y) \right] = 0
\]
4. Total $SU(2)_L$ current and commutators

The total $SU(2)_L$ current is

$$\tilde{J}^\mu_{L:2\otimes1} = \tilde{J}^\mu_{L:Schwinger} + \tilde{J}^\mu_{L:2\otimes1} \quad \text{(D17)}$$

$$\tilde{J}^\mu_{Schwinger} = \frac{1}{2} \tilde{\pi} \times \partial^\mu \tilde{\pi} + \frac{1}{2} \left( \pi^0 \partial^\mu H - H \partial^\mu \pi^0 \right)$$

$$\tilde{J}^\mu_{L:2\otimes1} = \tilde{W}^\mu \times \tilde{W}^\mu$$

with

$$L^W = -\frac{1}{2} Tr \left( W_{\mu \nu} W^{\mu \nu} \right)$$

$$W_\mu = \tilde{\tau} \cdot \tilde{W}_\mu \pm \frac{1}{2} \tilde{\sigma} \cdot \tilde{W}_\mu$$

$$W_{\mu \nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + ig_2 [W_\mu, W_\nu]$$

$$L^B_{2\otimes1} = -\frac{1}{2} B_{\mu \nu} B^{\mu \nu} \quad \text{(D20)}$$

$$B_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$L^2_{2\otimes1} = \left[ D^2_{\mu} \phi \right]^2 - V$$

$$V = \mu^2 (\phi^\dagger \phi) + \lambda_2 (\phi^\dagger \phi)^2$$

$$D^2_{\mu} \phi = [\partial_\mu + ig_2 W_\mu + i\tilde{\epsilon} B_\mu] \phi$$

$$L^{GaugeFix}_{2\otimes1} + L^{Ghost}_{2\otimes1}$$

$$= s_{2\otimes1} \left[ \tilde{\eta} \cdot \left[ \tilde{F}_W + \frac{1}{2} \tilde{\xi} \tilde{b}_W \right] + \tilde{\eta} B \left[ \tilde{F}_B + \frac{1}{2} \tilde{\xi} \tilde{b}_B \right] \right].$$

Relevant commutators are

$$\delta(x - y) \left[ \tilde{J}^0_{L:2\otimes1}(z) - \tilde{J}^0_{L:Schwinger}(z), H(y) \right] = 0$$

$$\delta(x - y) \left[ \tilde{J}^0_{L:2\otimes1}(z) - \tilde{J}^0_{L:Schwinger}(z), \tilde{\pi}(y) \right] = 0$$

$$\delta(x - y) \left[ \tilde{J}^0_{Schwinger}(z), H(y) \right] = 0$$

$$\delta(x - y) \left[ \tilde{J}^0_{Schwinger}(z), \tilde{\pi}(y) \right]$$

$$\left[ \tilde{J}^{\delta}_{\mu\nu}_{L:2\otimes1}(z) - \tilde{J}^{\delta}_{\mu\nu}_{L:Schwinger}(z), H(y) \right] = 0$$

$$\left[ \tilde{J}^{\delta}_{\mu\nu}_{L:2\otimes1}(z) - \tilde{J}^{\delta}_{\mu\nu}_{L:Schwinger}(z), \tilde{\pi}(y) \right] = 0$$

$$\left[ \tilde{J}^{\delta}_{\mu\nu}_{Schwinger}(z), H(y) \right]$$

$$\left[ \tilde{J}^{\delta}_{\mu\nu}_{Schwinger}(z), \tilde{\pi}(y) \right]$$

with

$$\tilde{F}_W = \partial_\beta \tilde{W}^\beta + \xi M_W \tilde{\pi}$$

$$F_B = \partial_\beta \tilde{B}^\beta + \xi M_B \tilde{\pi}_3$$

$$M_W = \frac{1}{2} g_2(H); \quad M_B = \tilde{\epsilon}(H)$$

$$\tilde{\epsilon} = \frac{1}{2} \eta \cdot g_1 = -\frac{1}{2} g_1; \quad M_W^2 + M_B^2 = M_Z^2.$$

5. Total $SU(2)_L \times U(1)_Y$ Lagrangian in $R_\xi$ gauge

The total $SU(2)_L \times U(1)_Y$ Lagrangian in $R_\xi$ gauge is

$$L_{2\otimes1} = L^W + L^B_{2\otimes1} + L^2_{2\otimes1} + L^{GaugeFix}_{2\otimes1} + L^{Ghost}_{2\otimes1} \quad \text{(D19)}$$

with

$$\partial_\mu \tilde{J}^\mu_{L:2\otimes1} = \frac{1}{2} M_W \left[ H \partial_\beta \tilde{W}^\beta + \tilde{\pi} \times \partial_\beta \tilde{W}^\beta \right] + \frac{1}{2} M_B \partial_\beta \tilde{B}^\beta (-\pi_2, \pi_1, H) \quad \text{(E2)}$$

Appendix E: $SU(2)_L \times U(1)_Y$ Master equation and WTI's in Landau gauge

We focus on the global isospin current $\tilde{J}^\mu_{L:2\otimes1}$. As usual, we form time-ordered products of $\tilde{J}^\mu_{L:2\otimes1}$, with N scalars and M pseudo-scalars

$$\langle 0 | T \left[ (\tilde{J}^\mu_{L:2\otimes1}(z)) h(x_1) \cdots h(x_N) \pi^{t_1}(y_1) \cdots \pi^{t_M}(y_M) \right] | 0 \rangle$$

and examine the divergence of such connected amplitudes:

$$\partial_\mu \langle 0 | T \left[ (\tilde{J}^\mu_{L:2\otimes1}(z)) \right. \right.$$
G. ’t Hooft’s gauge-fixing [20] conditions (A3),

\[
\langle 0 | T \left[ \left( \partial_\mu \bar{W}^\mu(z) \right) \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}} = 0
\]

\[
\langle 0 | T \left[ \left( \partial_\mu B^\mu(z) \right) \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}} = 0
\]

and the equal-time commutation relations (D17)

\[
\delta(z_0 - x_0) \left( \left( j_{\mu, \text{Schwinger}}^L(z) \right) h(x) \right) = 0
\]

\[
\delta(z_0 - x_0) \left( \left( j_{\mu, \text{Schwinger}}^L(z) \right) \bar{\pi}(x) \right) = 0
\]

a short calculation reveals

\[
\partial_\mu \langle 0 | T \left[ j_{\mu, \text{Schwinger}}^L(z) h(x_1) \cdots h(x_N) \right] \times \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}} = 0
\]

\[
\langle 0 | T \left[ \left( \partial_\mu j_{\mu, \text{Schwinger}}^L(z) \right) \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[
+ \sum_{n=1}^{N} \langle 0 | T \left[ h(x_1) \cdots h(x_{n-1}) \times \delta(z_0 - x_0) \left( j_{\mu, \text{Schwinger}}^L(z), h(x_n) \right) \right. \times h(x_{n+1}) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[= \sum_{n=1}^{N} \langle 0 | T \left[ h(x_1) \cdots h(x_{n-1}) \times \delta(z_0 - x_0) \left( j_{\mu, \text{Schwinger}}^L(z), h(x_n) \right) \right. \times h(x_{n+1}) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[+ \sum_{m=1}^{M} \langle 0 | T \left[ h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi^{m-1} (y_{m-1}) \times \delta(z_0 - x_0) \left( j_{\mu, \text{Schwinger}}^L(z), h(x_m) \right) \right. \times h(x_{m+1}) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[\text{in Landau gauge. Here we have } N \text{ external renormalized scalars } h = H - \langle H \rangle \text{ (coordinates } x) \text{, and } M \text{ external (CP = -1) renormalized pseudo-scalars } \bar{\pi} \text{ (coordinates } y, \text{ isospin } t). \text{ We have also thrown away a sum of } M \text{ terms, proportional to } \langle H \rangle \text{, that corresponds entirely to disconnected graphs.}
\]

2. Left-hand side (LHS) of Master Equation: Surface terms

We begin by studying the surface integral of the global SU(2)_{L;2\oplus 1} sub-current (D17) of the SU(2)_L \times U(1)_Y standard electroweak model in Landau gauge

\[
\hat{J}_L^{\mu, \text{Schwinger}} = \hat{J}_L^{\mu, \text{Schwinger}} + \hat{J}_L^{\mu, \text{W}}
\]

\[
\hat{J}_{L;2\oplus 1}^{\mu} = -\frac{1}{4} g_2 \bar{W}^\mu \left[ H^2 + \bar{\pi}^2 \right]
\]

\[
+ \frac{1}{2} g_1 B^\mu \left( \pi_1 \pi_3 - \pi_2 H, \pi_2 \pi_3 + \pi_1 H, \frac{1}{2} (H^2 - \pi_1^2 - \pi_2^2) \right)
\]

\[
+ \bar{W}^{\mu\nu} \times \bar{W}_\nu - \frac{1}{4} \xi \lim_{\xi \to 0} \left[ \bar{W}^{\mu} \times \partial_\nu \bar{W}^\nu \right]
\]

\[- \partial^\mu \bar{\eta} \times \bar{\omega}
\]

In order to transform the LHS of (E18) into a surface term (and crucially, to later form 1-soft-pion SU(2)_{L;2\oplus 1} WTI), we Fourier-transform the current with far-infrared ultra-soft momentum. We use Stokes theorem, where \(z^3\)-surface is a unit vector normal to the 3-surface. The time-ordered product constrains the 3-surface to lie on, or inside, the light-cone.

In this and the next subsections, we will prove that

\[
\lim_{k_3 \to 0} \int d^4 z e^{ikz} \partial_\mu \langle 0 | T \left[ \left( j_{L;2\oplus 1}^{\mu}(z) \right) \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[= \int d^4 z \partial_\mu \langle 0 | T \left[ \left( j_{L;2\oplus 1}^{\mu}(z) \right) \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[= \int d^3 z \left. \right|_{z^3\text{-surface} \to \infty} \langle 0 | T \left[ \left( j_{L;2\oplus 1}^{\mu}(z) \right) \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[= 0.
\]

a. SU(2)\_ gauge fields’ kinetic and interaction

The surface integral of the 3rd term in \(j_{\mu}^{\text{F}}\) in (E6)

\[
\int d^3 z \left. \right|_{z^3\text{-surface} \to \infty} \langle 0 | T \left[ \left( \bar{W}^{\mu\nu} \times \bar{W}_\nu \right)(z) \right. \times h(x_1) \cdots h(x_N) \pi_t (y_1) \cdots \pi_M (y_M) \right] | 0 \rangle_{\text{Connected}}
\]

\[= 0,
\]
because each term in the current contains at least one massive $\tilde{W}_\beta^\pm$ or $W_\beta^\mp$.

$$\tilde{W}^{\mu \nu} \times \tilde{W}_\nu = -\tilde{W}_\nu \times \partial^\mu \tilde{W}^{\nu} + \tilde{W}_\mu \times \partial^\nu \tilde{W}^{\mu}$$  \hspace{1cm} \text{(E9)}$$

\[ W_{\beta}^\pm \text{ are massive in spontaneously broken } SU(2)_L \times U(1)_Y. \] Propagators connecting $W_{\beta}^\pm$, from points on the 3-surface at infinity to the localized interaction points $(x_1...x_N; y_1...y_M)$, must stay inside the light-cone, but die off exponentially with mass, $M_{W}^2 \neq 0$. They are incapable of carrying information that far.

The surface integral of the last term in $\mathcal{J}_\mu$ in (E6), i.e. ghosts and anti-ghosts, is shown to vanish in Subsection A.2.

3. **LHS of Master Equation: Connected amplitudes linking the \( \phi \)-sector with external currents**

Connected momentum-space amplitudes, with $N$ external BEHs, $M$ external $\bar{\phi}$s, and a current $J_{L;2\oplus 1}^\mu$, are defined in terms of $\phi$-sector connected time-ordered products

$$iG^{\mu l_1...l_M}_{\mu l_1...l_M} \left( J_{L;2\oplus 1}^\mu; p_1...p_N; q_1...q_M \right) \times (2\pi)^4 \delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$$

$$\equiv \int d^4z e^{ikz} \prod_{n=1}^{N} \int d^4x_n e^{ip_n \cdot x_n} \prod_{m=1}^{M} \int d^4y_m e^{iq_m \cdot y_m}$$

$$\times \langle 0 | T \left[ \left( J_{L;2\oplus 1}^\mu (z) \right) \times h(x_1)...h(x_N) \pi^{l_1}(y_1)...\pi^{l_M}(y_M) \right] | 0 \rangle_{\text{Connected}}$$

These appear throughout the proof of the WTI, so that

$$-k^\mu G^{\mu l_1...l_M}_{\mu l_1...l_M} \left( J_{L;2\oplus 1}^\mu; p_1...p_N; q_1...q_M \right) \times (2\pi)^4 \delta^4 \left( k + \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right)$$

$$\equiv \int d^4z e^{ikz} \prod_{n=1}^{N} \int d^4x_n e^{ip_n \cdot x_n} \prod_{m=1}^{M} \int d^4y_m e^{iq_m \cdot y_m}$$

$$\times \partial^\mu \langle 0 | T \left[ \left( J_{L;2\oplus 1}^\mu (z) \right) \times h(x_1)...h(x_N) \pi^{l_1}(y_1)...\pi^{l_M}(y_M) \right] | 0 \rangle_{\text{Connected}}$$

a. **Gauge-invariant scalar-sector Lagrangian**

The contribution of the 1st term in $\mathcal{J}_\mu$ in (E6) to the LHS of the Master Equation vanishes

$$-k^\mu G^{\mu l_1...l_M}_{\mu l_1...l_M} \left( \left[ -\frac{1}{4} g_2 W_\mu^\mu \left( H^2 + \pi^2 \right) \right]; p_1...p_N; q_1...q_M \right)$$

$$\sim -\frac{1}{4} g_2 k^\mu W_\mu^\mu (k) \left( \cdots \right) (k^2)$$

\[ \text{(E12)} \]

because the isospin gauge conditions (E3) obeyed by the states read, in momentum-space

$$k^\mu W_\mu^\mu (k) = 0 \quad \text{(E13)}$$

The contribution of the 2nd term in $\mathcal{J}_\mu$ in (E6) to the LHS of the Master Equation vanishes

$$-k^\mu G^{\mu l_1...l_M}_{\mu l_1...l_M} \left( \left[ \frac{1}{2} g_1 B_\mu \left( \pi_1 \pi_3 - \pi_2 \pi_4, \pi_2 \pi_3 + \pi_1 H, \frac{1}{2} \left( H^2 + \pi^2 - \pi_1^2 - \pi_3^2 \right) \right)^T \right] \right.$$$$\left.; p_1...p_N; q_1...q_M \right) \sim \frac{1}{2} g_1 k^\mu B_\mu (k) \left( \cdots \right) (k^2)$$

$$= 0 \quad \text{(E14)}$$

because the hypercharge gauge condition (E3) obeyed by the states reads, in momentum-space

$$k^\mu B_\mu (k) = 0 \quad \text{(E15)}$$

The contribution of the 4th term in $\mathcal{J}_\mu$ in (E6) to the LHS of the Master Equation is shown to vanish in Subsection A.3.

b. **Total $SU(2)_L$ sub-current contribution to surface terms on LHS of Master equation**

The total $SU(2)_L$ sub-current contribution to surface terms on the LHS of the Master equation is

$$\lim_{k \to 0} \int d^4z e^{ikz} \partial^\mu \langle 0 | T \left[ J_{L;2\oplus 1}^\mu (z) h(x_1) \cdots h(x_N) \times \pi^{l_1}(y_1) \cdots \pi^{l_M}(y_M) \right] | 0 \rangle_{\text{Connected}}$$

$$= \lim_{k \to 0} \int d^4z e^{ikz} \partial^\mu \langle 0 | T \left[ J_{\text{Schwinger}}^\mu (z) h(x_1) \cdots h(x_N) \times \pi^{l_1}(y_1) \cdots \pi^{l_M}(y_M) \right] | 0 \rangle_{\text{Connected}}$$

\[ \text{(E16)} \]

4. **$\tilde{J}_{L+R;Schwinger}, \tilde{J}_{L-R;Schwinger}$ for even & odd $M$**

Amplitudes connecting the $\phi$-sector with the total $SU(2)_L$ isospin sub-current are constructed to conserve $CP$. Therefore, on and off-shell connected amputated T-matrix elements and Green’s functions of an odd number of $(CP = -1)$ $\bar{\phi}$s and their derivatives, are zero.
In analogy with the proof in Subsection A.4 this allows us to write 2 sets of SU(2)_{L,R} Ward-Takahashi identities governing the φ-sector of SU(2)_{L,R} \times U(1)_{Y}: one for the vector current \( J^\mu_{L+R;Schwinger} \) based on \( M \) even, and one for the axial-vector current \( J^\mu_{L-R;Schwinger} \) based on \( M \) odd. In this paper, we focus on those strong constraints placed on the scalar-sector of the SU(2)_{L,R} \times U(1)_{Y}, by the axial-vector WTI.

5. SU(2)_{L} \times U(1)_{Y} axial-vector Master Equation in \( J^\mu_{L-R;Schwinger} \)

We now assemble the axial-vector Master Equation, from which we derive our “1-soft-π” SU(2)_{L-R} WTI. The LHS is from (E16), the RHS from (E5,E6):

\[
\lim_{k_\lambda \to 0} \int d^4z e^{ikz} \int d^4z e^{ikz} \partial_\mu \langle 0 | T \left[ \left( J^\mu_{L-R;Schwinger}(z) \times h(x_1) \cdots h(x_{n-1}) \right) \left( J^\mu_{L-R;Schwinger}(z) \right) \right] | 0 \rangle_{Connected} \\
= \lim_{k_\lambda \to 0} \int d^4z e^{ikz} \sum_{n=N}^{n=M} \langle 0 | T \left[ h(x_1) \cdots h(x_{n-1}) \right] | 0 \rangle_{Connected} \\
+ \lim_{k_\lambda \to 0} \int d^4z e^{ikz} \sum_{m=M}^{m=M} \langle 0 | T \left[ h(x_1) \cdots h(x_{n-1}) \right] | 0 \rangle_{Connected}
\]

Eq. (E17) is true for any \( M \), odd or even. It is derived for \( M \) odd in analogy with (A15). It is also satisfied trivially, in analogy with (A18), for \( M \) even.

This paper is based on the de facto conservation, in the 1-soft-π limit, of \( J^\mu_{L-R;Schwinger} \), for CP-conserving connected amputated Green’s functions and on-shell T-matrix elements, in the φ-sector of SU(2)_{L,R} \times U(1)_{Y}.

6. SU(2)_{L} \times U(1)_{Y} Ward-Takahashi identities

The Master equation (E17) is mathematically identical to that in (A20). This proves that, for each SU(2)_{L,R} WTI that is true in SU(2)_{L,R} \times U(1)_{Y}, an analogous SU(2)_{L,R} WTI is true in SU(2)_{L,R} \times U(1)_{Y}. Appendix B proved SU(2)_{L} WTI relations among 1-φ-R φ-sector T-Matrix elements \( T_{N,M} \), as well as SU(2)_{L} WTI relations among 1-φ-I φ-sector Green’s functions \( \Gamma_{N,M} \), in spontaneously broken SU(2)_{L}. Analogous SU(2)_{L,R} WTI relations among 1-φ-R φ-sector T-Matrix elements \( T_{N,M}^{2\phi} \), as well as analogous SU(2)_{L-R} WTI relations among 1-φ-1 φ-sector Green’s functions \( \Gamma_{N,M}^{2\phi} \), are therefore true for spontaneously broken SU(2)_{L-R} \times U(1)_{Y}.

But there is one huge difference! The renormalization of our SU(2)_{L-R} WTI, governing φ-sector \( T_{N,M}^{2\phi} \) and \( \Gamma_{N,M}^{2\phi} \), now includes the all-loop-orders contributions of virtual isospin and hypercharge gauge bosons, φ-scalars, anti-ghosts, and ghosts, i.e. \( \tilde{W}^\mu \), \( B^\mu \), \( \tilde{\pi}, \tilde{\eta}, \tilde{\eta}_B, \tilde{\omega}, \) and \( \omega_\mu \) respectively.

The SU(2)_{L,R} \times U(1)_{Y} Master equation relates connected time-ordered products, in analogy with (B5):

\[
\lim_{k_\lambda \to 0} \int d^4z e^{ikz} \left\{ - (H) \partial_\mu / |0 \rangle \left[ \left( \partial^\mu \pi^{\pi^\dagger}(z) \times h(x_1) \cdots h(x_N) \pi^{\pi^\dagger}(y_1) \cdots \pi^{\pi^\dagger}(y_M) \right) |0\rangle_{Connected} \\
- \sum_{m=1}^{M} \left( \delta^{m}_m \delta^4(z - y_m) \langle 0 | T \left[ h(z) h(x_1) \cdots h(x_N) \pi^{\pi^\dagger}(y_1) \cdots \pi^{\pi^\dagger}(y_M) \right) |0\rangle_{Connected} \\
+ \sum_{n=1}^{N} \left( \delta^4(z - x_n) \langle 0 | T \left[ h(x_1) \cdots h(x_N) \pi^{\pi^\dagger}(y_1) \cdots \pi^{\pi^\dagger}(y_M) \right) |0\rangle_{Connected} \right) \right\}
\]

= 0

(E18)

where the “hatted” fields \( \hat{h}(x) \) and \( \pi^{\pi^\dagger}(y_M) \) are to be removed.

Isospin indices will become increasingly cumbersome; we therefore again adopt B.W. Lee’s [12] convention of suppressing isospin indices, allowing momenta to implicitly carry them.

The Adler self-consistency relations, but now for the SU(2)_{L,R} \times U(1)_{Y} gauge theory (rather than global SU(2)_{L,R} \times SU(2)_{R}), are obtained by putting the external φ particles on mass-shell:

\[
(H) T_{N,M+1}^{2\phi}(p_1 \cdots p_N; 0 q_1 \cdots q_M) \\
\times (2\pi)^4 \delta^4 \left( \sum_{n=1}^{N} p_n + \sum_{m=1}^{M} q_m \right) q_1^2 = q_2^2 = \cdots = q_M^2 = 0
\]

= 0

(E19)

With some exceptions, the φ-sector connected amputated transition matrix \( T_{N,M}^{2\phi} \) can be split in two by cutting an internalφ or π line, and are designated 1-φ-R. In contrast, the φ-sector connected amputated Green’s functions \( \Gamma_{N,M}^{2\phi} \) are defined to be 1-φ-I, i.e. they cannot be split by cutting an internal φ or π line.

\[
T_{N,M}^{2\phi} = \Gamma_{N,M}^{2\phi} + (1 - \phi - R).
\]

(E20)

Both \( T_{N,M}^{2\phi} \) and \( \Gamma_{N,M}^{2\phi} \) are 1-(\( \tilde{W}_\mu, B_\mu \))-Reducible (1-\( \tilde{W}_\mu, B_\mu \)-R), i.e. they can be split by cutting an internal transverse \( \tilde{W}_\mu \) or \( B_\mu \) line.

The special 2-point functions \( T_{0,2}^{2\phi}(q, -q) \) and \( T_{2,0}^{2\phi}(p, -p) \), and the 3-point vertex \( T_{1,2}^{2\phi}(q; 0, -q) \), are
The corresponding 1-1-1-conserving global $SU_\pi$ proves that massless is protected/guaranteed by the breaking. States of the gauge theory after spontaneous symmetry breaking, is the $N = 1$ case of that SSB gauge, is the $N = 1$ case of that SSB gauge theory’s Adler self-consistency relations. The Lee-Stora-Symanzik (LSS) theorem, in spontaneously broken $SU(2)_L \times U(1)_Y$ in $R^{2\otimes 1}_2 (\xi = 0)$ Landau gauge, is the $N = 0, M = 1$ case of that SSB gauge theory’s Adler self-consistency relations. 

$$\langle H \rangle \Gamma^2_{1,2}(p; 0, -q) = \langle H \rangle \Gamma^2_{0,2}(q; 0, -q) = \langle H \rangle \Gamma^2_{1,2}(p; q, -p - q) = \langle H \rangle \Gamma^2_{0,2}(q; q, -q)$$

(E21)

are related to the $(1h, 2\pi)$ 3-point $h\pi^2$ vertex

$$T^2_{1,2}(p; q, -p - q) = \Gamma^2_{1,2}(p; q, -p - q)$$

(E22)

by a 1-soft-pion theorem analogous with $[B13]$. In analogy with $[B24]$, removing the 1-\phi-R graphs from both sides of $[E26]$ yields the recursive identity

$$\langle H \rangle \Gamma^2_{N,M+1}(p_1 \ldots p_N; 0q_1 \ldots q_M) = \sum_{m=1}^{M} T^2_{N+1,M-1}(qm; q_1 \ldots q_m) + \sum_{m=1}^{N} T^2_{M+1,N-1}(p_1 \ldots \hat{p}_m \ldots p_N; 0q_1 \ldots q_m)$$

(E26)

proving that $\pi$ is massless. That all-loop-orders renormalized masslessness is protected/guaranteed by the $CP$-conserving global $SU(2)_L - R$ symmetry of the physical states of the gauge theory after spontaneous symmetry breaking.

In analogy with $[B22]$, separate

$$T^2_{N,M+1}(p_1 \ldots p_N; 0q_1 \ldots q_M) = T^2_{\text{External}; N,M+1}(p_1 \ldots p_N; 0q_1 \ldots q_M) + T^2_{\text{Internal}; N,M+1}(p_1 \ldots p_N; 0q_1 \ldots q_M)$$

(E25)

so that

$$\langle H \rangle T^2_{\text{Internal}; N,M+1}(p_1 \ldots p_N; 0q_1 \ldots q_M) = \sum_{m=1}^{M} T^2_{N+1,M-1}(qm; q_1 \ldots q_m) + \sum_{m=1}^{N} T^2_{M+1,N-1}(p_1 \ldots \hat{p}_m \ldots p_N; 0q_1 \ldots q_m)$$

(E27)

We observe that the LSS theorem makes tadpoles vanish.

$$\langle 0 | h(x) = 0 | 0 \rangle_{\text{Connected}} = 0$$

(E28)

but the $N = 0, M = 1$ case of $[E27]$ reads

$$\Gamma^2_{1,0}(0; \hat{0}) = \langle H \rangle \Gamma^2_{1,0}(0; \hat{0}) = 0$$

(E29)

so that tadpoles all vanish automatically, and separate tadpole renormalization is un-necessary. Since we can choose the origin of coordinates anywhere we like

$$\langle 0 | h(x) = 0 | 0 \rangle_{\text{Connected}} = 0$$

(E30)

The Renormalized $\langle H \rangle$ obeys

$$\langle 0 | H(x) | 0 \rangle_{\text{Connected}} = \langle 0 | h(x) | 0 \rangle_{\text{Connected}} + \langle H \rangle$$

$$\partial_\mu \langle H \rangle = 0$$

(E31)