The Energy Density of a Gas of Photons
Surrounding a Spherical Mass $M$ at a Non-Zero Temperature

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Abstract

The equations determining the energy density $\rho$ of a gas of photons in thermodynamic equilibrium with a spherical mass $M$ at a non-zero temperature $T_s > 0$ is derived from Einstein’s equations. It is found that for large $r$, $\rho \sim 1/r^2$ where the proportionality constant is a fundamental constant and is the same for all spherical masses at all temperatures.

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§1. Introduction

In the standard Schwarzschild solution [1] of Einstein’s equations for a static, spherical geometry the spacetime outside of a sphere of mass $M$ and radius $R$ is taken to be a vacuum; empty and free of particles. This is certainly true if the sphere is at absolute zero temperature. When, however, the sphere has a non-zero temperature, then we would expect, on a physically basis, a gas of thermal photons to be present. If the system is in thermodynamic equilibrium, then the spacetime surrounding the sphere will not be empty, but will instead be filled with blackbody radiation. As these thermal photons also have a certain non-zero energy, we would expect the presence of this blackbody radiation to also contribute towards determining the geometry of the spacetime.

For any physically reasonable temperatures the energy density of the photons is very small in comparison to the mass density of the sphere and the presence of the photons is ignored in the usual Schwarzschild analysis. This approximation is certainly valid near the spherical body but what happens when one is very far away from the sphere? One should remember that the Schwarzschild solution is asymptotically flat. Due to the mass $M$ of the body being confined within its radius $R$, as one goes further and further away from the sphere the affect of its mass $M$ on the curvature of spacetime becomes less and less. The blackbody radiation, on the other hand, is unconfined. It extends from the surface of the sphere to fill the rest of the spacetime. Although we would expect the energy density of the photons to also decrease as one moves further and further away from the sphere, the important question is how fast it will do so. Let us, for the moment, consider a sphere of radius $r > R$ and the total amount of energy within it. When $r$ is near $R$, we would expect that the fraction $\epsilon_M$ of this energy
which is due to the mass $M$ will be much greater than $\epsilon_\gamma$, the fraction of the total energy which is due to the photons. Consequently, we would expect the mass $M$ to be the dominant factor in determining the geometry of the spacetime within $r$ and would expect the Schwarzschild solutions to be valid in this region. As, however, $r$ increases, $\epsilon_M$ decreases since the mass $M$ of the body is fixed, while $\epsilon_\gamma$ increases since the blackbody radiation extends throughout the spacetime. If the energy density of the photons decreases sufficiently rapidly so that $\epsilon_\gamma \ll \epsilon_M$ for all $r$, then the mass $M$ will always be the dominant factor in determining the geometry of the spacetime. Although the Schwarzschild solution would be modified somewhat by the presence of the photons, we would not expect these modifications to be very drastic. In particular, the spacetime should still be asymptotically flat. If, on the other hand, the energy density does not decrease rapidly enough and $\epsilon_\gamma \gg \epsilon_M$ for $r$ larger than some $r_0$, then we would expect that in this region of spacetime it is the photons which will determine the spacetime geometry. In particular, we would expect the solutions to Einstein’s equations in this regime to be very much different from the Schwarzschild solution. The spacetime may not even be asymptotically flat.

In this paper we shall study some of the affects of non-zero temperatures on the static, spherically symmetric solutions of Einstein’s equations. The system we shall be considering consists of a spherical body with a mass $M$ and a radius $R$ which, due to the sphere being at a temperature $T_s > 0$, is surrounded by blackbody radiation. The system as a whole will be assumed to be in thermodynamic equilibrium with the body serving as the heat reservoir for the system. In particular, this means that the spherical body is assumed to be in thermodynamic equilibrium with the photons surrounding it. It is moreover assumed that the body has not undergone
complete gravitational collapse into a blackhole, is non-rotating, and is not electrically charged. Nor shall there be any other massive objects present in this spacetime. The only difference between this system and the one studied by Schwarzschild is the presence of a non-zero temperature for the sphere.

Our aim in this paper is two fold. First, we shall derive a set of coupled differential equations which will determine the total energy density of the photons. As these are non-linear equations, we shall not be able to solve them analytically. We shall, nonetheless, be able to obtain both asymptotic $r \to \infty$ as well as $r \to R$ solutions to them. Note, however, that we shall be determining the photon’s total energy density in the gravitational field, and not its blackbody spectrum. Second, we shall show that the geometry of this spacetime differs drastically from the Schwarzschild geometry at large $r$. In fact with the presence of the photons the spacetime is no longer asymptotically flat.

In order to avoid working with non-equilibrium systems, we have assumed that the mass $M$ is in thermodynamic equilibrium with the blackbody radiation surrounding it. Unfortunately, even equilibrium quantum statistical mechanics on curved spacetimes has yet to be satisfactorily formulated. The closest that we have come to a complete formalism is that given in [2]. For various reasons, however, it will be difficult, if not impossible, to analyze the system in the manner outlined therein and it is fortunate that all that we shall need is the Tolman-Oppenheimer-Volkoff equation for hydrostatic equilibrium. This, combined with the observation that the energy-momentum tensor of a gas of pure photons is traceless, shall be sufficient to determine the energy density almost uniquely. This method of deriving the energy density has the added advantage of not only taking into account the affects of the curvature of spacetime on the photons, but also the reciprocal affect
of the photons on the spacetime curvature. Photons are not treated as test particles in our analysis. In this, and other, ways our method differs from that described in [2]. The only difficulty that we shall encounter is when we try to identify the temperature of the system. Since we do not have an established formalism which will automatically do this for us, we shall, in the end, have to rely on other physical arguments to do so.

In the formalism given in [2], as indeed in most treatments of thermodynamics in general relativity, the temperature of the system is taken to be a constant throughout the spacetime. This is, it would seem to us, an oversimplification, for the following reason. For massless particles the temperature of the system at equilibrium may be interpreted physically as the most probable energy that any one particle in the statistical ensemble may have. (The case of massive particles is much more complicated and will not be considered here.) In a gravitational field this should include not only its kinetic energy, but also its gravitational energy as well. The two cannot be separated covariantly. Indeed, it is known that the frequency of a photon, and thus its energy, when measured at different points in a gravitational field will either be “redshifted” or “blueshifted” with respect to one another. We would on this basis expect that temperature too should vary from point to point on the manifold.

§2. The Hydrostatic Equation

We begin with an $N$ dimensional manifold $\mathbf{M}$ with a metric $g_{\mu\nu}$ which has a signature of $(-,+,+,+)$. Greek indices shall run from 0 to $N - 1$ and the summation convention is used throughout. It is further assumed that $g_{\mu\nu}$ is static, meaning that there exists a timelike Killing vector $\xi_{\mu}$ for the system. We shall also assume that the system contains only one heat reservoir.
Next, let \( T_{\mu\nu} \) be an energy momentum tensor operator defined on \( \mathbf{M} \). We shall, in a semi-classical approximation, treat \( g_{\mu\nu} \) as a background, classical field. We next denote the thermodynamic average of \( T_{\mu\nu} \) by \( \langle T_{\mu\nu} \rangle \). We shall not need a specific definition of this average, but rather that it satisfy a few basic properties that we would expect from any equilibrium thermodynamic average. First, it should be “time independent”, meaning that

\[
\mathcal{L}_{\xi} \langle T_{\mu\nu} \rangle = 0, \tag{1}
\]

where \( \mathcal{L}_{\xi} \) denotes the Lie derivative along the \( \xi_{\mu} \) direction. It is for this reason that we required \( \mathbf{M} \) to have a timelike Killing vector. Physically, it means that the background field \( g_{\mu\nu} \) cannot change with respect to time the total energy contained in the matter fields so that the system as a whole can be in equilibrium. Second, we require that the average be anomaly-free

\[
\nabla_{\lambda} \langle T_{\mu\nu} \rangle = \langle \nabla_{\lambda} T_{\mu\nu} \rangle, \tag{2}
\]

where \( \nabla_{\lambda} \) denotes the covariant derivative. Third, after the average is taken the energy momentum tensor should have the form \( \langle T_{\mu\nu} \rangle = \rho u_{\mu} u_{\nu} + p (g_{\mu\nu} + u_{\mu} u_{\nu}) \) were \( \rho \) and \( p \) are the proper energy density and pressure, respectively, and \( u_{\mu} \) is a unit velocity vector which must lie in the \( \xi_{\mu} \) direction if (1) is to hold. We shall make this dependency explicit by writting

\[
\langle T_{\mu\nu} \rangle = -\rho \frac{\xi_{\mu} \xi_{\nu}}{\xi^2} + p \left( g_{\mu\nu} - \frac{\xi_{\mu} \xi_{\nu}}{\xi^2} \right). \tag{3}
\]

Then (1) requires that

\[
\xi^{\lambda} \nabla_{\lambda} \rho = 0 \quad , \quad \xi^{\lambda} \nabla_{\lambda} p = 0, \tag{4}
\]

meaning that \( \rho \) and \( p \) are functions of vectors lying in the \( N-1 \) dimensional hypersurface perpendicular to \( \xi_{\mu} \) and/or \( \xi^2 = \xi_{\mu} \xi^{\mu} \). Finally, since the
system is conserved, $\nabla^\mu \langle T_{\mu\nu} \rangle = 0$ and

$$\nabla_\mu p + (\rho + p) \frac{\nabla_\mu |\xi|}{|\xi|} = 0,$$  \hspace{1cm} (5)

where $|\xi| \equiv \sqrt{-\xi^2}$ and we have used (3) and Killing’s equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0,$$  \hspace{1cm} (6)

in obtaining (5). This is the generalization of the Tolman-Oppenheimer-Volkoff [3] equation for hydrostatic equilibrium to general, static spacetimes and it reduces to the usual hydrostatic equation in the case of spherically symmetric spacetimes.

We next consider a region of spacetime which contains only photons. In this region

$$4\pi T_{\mu\nu} = F_{\mu\lambda} F_{\nu}^\lambda - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta},$$  \hspace{1cm} (7)

where $F_{\mu\nu}$ is the field strength tensor. This energy-momentum tensor operator is traceless $T^\mu_\mu = 0$ in four spacetime dimensions. Consequently, $\langle T^\mu_\mu \rangle = 0$ and for photons $\rho = (N - 1)p$. The hydrostatic equation (5) is now trivial to solve yielding

$$\rho = \frac{\sigma}{|\xi|^N},$$  \hspace{1cm} (8)

where $\sigma$ is an arbitrary constant. The average energy momentum tensor for photons is thus given by

$$\langle T_{\mu\nu} \rangle = \frac{1}{(N - 1) |\xi|^N} \left( g_{\mu\nu} - N \frac{\xi_\mu \xi_\nu}{\xi^2} \right).$$  \hspace{1cm} (9)

To determine $\langle T_{\mu\nu} \rangle$ completely one must first determine $\xi_\mu$ for the manifold. As the presence of the photons will also affect the curvature of the spacetime, determining $\xi_\mu$ ultimately involves solving Einstein’s equations using (9) as the source term

$$\frac{1}{(N - 1) |\xi|^N} \left( g_{\mu\nu} - N \frac{\xi_\mu \xi_\nu}{\xi^2} \right) = \frac{1}{8\pi} R_{\mu\nu},$$  \hspace{1cm} (10)
where $R_{\mu\nu}$ is the Ricci tensor and we are using geometrized units in which $G = c = k_B = \hbar = 1$. The $R^\mu_\mu$ term is absent since $\langle T_{\mu\nu} \rangle$ is traceless.

The task now is to interpret (8) physically. Let us consider, for the moment, the case of Minkowski spacetime and enclose the system in a very, very large box which is connected to a heat reservoir at a fixed temperature. Killing’s equation is now a simple partial differential equation and we may choose a coordinate system in which its solution for a timelike Killing vector is $\xi^\mu_f = (-\beta^f, 0, 0, 0)$ where $\beta^f$ is a constant and the superscript $f$ reminds us that this is the Minkowski spacetime. From (8) we find that in four dimensions,

$$\rho^f = \frac{\sigma (\beta^f)^4}{4},$$

which, if we interpret $1/\beta^f$ as the temperature of the heat reservoir, is just Boltzmann’s law for photons. Then $\sigma = \pi^2 k_B^4/(15\hbar^3 c^3)$ is identified with the blackbody radiation constant.

Using the Minkowski spacetime case as motivation, we shall tentatively identify the temperature $T$ of the system as

$$T = \frac{1}{|\xi|},$$

in general, static spacetimes and see whether or not this will make physical sense. First, we note that although $T$ does very with position, it is “time independent”, namely

$$\mathcal{L}_\xi T = \frac{T^3}{2} \xi^\mu \nabla_\mu \xi^2 = 0,$$

as one would expect for a system in equilibrium. Second, we note that variations in $T$ are due solely to the gravitational field. In fact, the temperature at various points of the manifold is related to one another by just the redshift factor,

$$\frac{T(x)}{T(x')} = \left( \frac{-\xi^2(x')}{-\xi^2(x)} \right)^{1/2},$$
which is precisely what one would expect from time dilation and the frequency shift of photons in a gravitational field. Finally, we note that timelike Killing vectors $\xi$ in non-rotating systems are determined by the geometry of the spacetime only up to an overall constant, as can be seen explicitly in Killing’s equation (6). Consequently, we have the freedom to attach an overall constant to the Killing vector which we can then identify as the inverse temperature of the heat reservoir. Indeed, from (14) we see that relative temperatures between two points on the manifold are determined solely by geometry and to determine an absolute temperature requires choosing a reference point on the manifold from which we can measure all subsequent temperatures with respect to. The most natural reference point to choose is the heat reservoir of the system and to measure all other temperatures based on its value.

The procedure for determining the energy density of thermal photons in any static geometry is now clear. First we solve Killing’s equations up to an overall constant for a timelike Killing vector $\xi_{\mu}$ in terms of the components $g_{\mu\nu}$ of the metric of the spacetime. To determine the overall constant, we use as a boundary condition for $\xi_{\mu}$ the temperature $T_{hr}$ of the heat reservoir of the system by evaluating (12) at a point $x_{hr}$ on the surface of the heat reservoir. It is required that the heat reservoir have a constant temperature throughout its surface. Finally, Einstein’s equations (10) are solved with the appropriate boundary conditions to determine $g_{\mu\nu}$. These boundary conditions are usually also given at the heat reservoir. The energy density $\rho$ of the photons and the geometry of the spacetime are thus determined.

§3 Spherical Geometry

We shall now attempt to solve (10) for a static, spherical geometry. Specifically, we shall consider a non-rotating spherical body of mass $M$, and
radius $R$ which at its surface has a temperature $T_s$. The spherical body will serve as the heat reservoir for the system. The most general metric which is static and spherically symmetric has the form [3]

$$ds^2 = -f dt^2 + h dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

(15)

where $f$ and $h$ are functions of $r$ only. Killing’s equation is now straightforward to solve giving $\xi_\mu = (-c_k f(r), 0, 0, 0)$, and $\xi^2 = -c_k^2 f(r)$ for a timelike Killing vector. $c_k$ is an arbitrary constant which is determined by evaluating (12) at the surface of the reservoir: $1/c_k = T_s f(R)^{1/2}$.

As for the boundary conditions for $f$ and $h$, we choose them in the following manner. Take a point $r > R$ and consider the amount of energy contained within a sphere of this radius which is due to the photons. Clearly, because of the presence of the photons, the geometry of the spacetime within $r$ will be different from the Schwarzchild geometry. Let us, however, take $r \to R^+$ so that the amount of energy contained in the sphere due to the photons gradually decreases. Their affect on the geometry of spacetime must also do so correspondingly and just outside the body we would expect the geometry of the manifold to be the same as that of the Schwarzchild geometry. Consequently, we choose as our boundary conditions

$$\lim_{r \to R^+} f(r) = 1 - \frac{2M}{R}, \quad \lim_{r \to R^+} \frac{1}{h(r)} = 1 - \frac{2M}{R}.$$

(16)

Einstein’s equations given by (10) are

$$\frac{8\pi \rho_0}{f^2} = \frac{1}{fh} \left[ \frac{f''}{2} - \frac{f'}{4} \left( \frac{f'}{f} + \frac{h'}{h} \right) + \frac{f'}{r} \right],$$

$$\frac{8\pi \rho_0}{3f^2} = \frac{1}{fh} \left[ -\frac{f''}{2} + \frac{f'}{4} \left( \frac{f'}{f} + \frac{h'}{h} \right) + \frac{fh'}{r} \right],$$

$$\frac{8\pi \rho_0}{3f^2} = -\frac{1}{2rh} \left( \frac{f'}{f} + \frac{h'}{h} \right) + \frac{h'}{r h^2} + \frac{1}{r^2} \left( 1 - \frac{1}{h} \right),$$

(17)
where \( \rho_0 = \sigma T_{\text{e}}^4 f^2(R) \) and the primes denote derivatives with respect to \( r \). They may be reduced to two coupled, nonlinear differential equations

\[
\begin{align*}
\frac{32\pi \rho_0}{3f^2} &= \frac{1}{rh} \left( \frac{f'}{f} + \frac{h'}{h} \right), \\
\frac{8\pi \rho_0}{f^2} &= \frac{h'}{rh^2} + \frac{1}{r^2} \left( 1 - \frac{1}{h} \right).
\end{align*}
\]

It is doubtful that these equations can be solved analytically. There is, nonetheless, one general feature that we can determine from these equations alone. Since we require \( h > 0 \) for all \( r > R \), from (18) we find that \( f \) is a monotonically increasing function of \( r \). Consequently, \( \rho = \rho_0/f^2 \) is a monotonically decreasing function of \( r \). The energy density of the photon gas is at its largest at the surface of the sphere and decreases monotonically as one goes further and further away from it. This is exactly what we would have expected physically. Notice also that since \( T = 1/|\xi| = T_{\text{e}}[f(R)/f(r)]^{1/2} \), the temperature of the photon gas also decreases monotonically with \( r \).

We shall now obtain approximate solutions to (18) in the small and large \( r \) limits. We first consider the near field solutions when \( r \) is near \( R \) and write \( f \approx 1 - \Gamma \), and \( h^{-1} \approx 1 - \Lambda \) for \( \Gamma, \Lambda \ll 1 \). For the boundary conditions to be consistent with this approximation, we shall require \( 2M/R \ll 1 \). Then ignoring terms quadratic in \( \Gamma \) and \( \Lambda \), we find that

\[
\begin{align*}
\frac{1}{h(r)} &\approx 1 - \frac{2M}{r} \left\{ 1 + \frac{4\pi \rho_0 r^3}{3M} \left[ 1 - \left( \frac{R}{r} \right)^4 \right] \right\}, \\
f(r) &\approx 1 - \frac{2M}{r} \left\{ 1 - \frac{4\pi \rho_0 r^3}{3M} \left[ 1 - \left( \frac{R}{r} \right)^2 \right] \right\}, \\
\rho(r) &\approx \rho_0 \left\{ 1 - \frac{2M}{r} + \frac{8\pi}{3} \rho_0 r^2 \left[ 1 - \left( \frac{R}{r} \right)^2 \right] \right\}^{-2}.
\end{align*}
\]

Notice, however, that \( \Gamma \) and \( \Lambda \) increases quadratically with \( r \) and at some point \( r_m \) they will no longer be valid. To estimate \( r_m \), we enforce the
condition that $\Gamma(r) \ll 1$ for $r < r_m$. This gives

$$\frac{4\pi \rho_0}{3} r_m^3 \approx M,$$

(20)
as a determining equation for $r_m$. The near field solutions (19) are valid as long as the total energy of the photons confined in a sphere of radius $r$ is much less than the mass of the spherical body itself.

As for the asymptotic, $r \to \infty$ solutions, we obtain them in the following manner. First we define

$$\rho = \frac{\Delta}{4\pi r^2}, \quad \frac{1}{h} = 1 - 2K.$$  

(21)

Then (18) may be written as

$$\frac{d\Delta}{dy} = -\frac{2\Delta}{1 - 2K} \left( \frac{2}{3}\Delta + 4K - 1 \right),$$

$$\frac{dK}{dy} = \Delta - K,$$

(22)

where $y = \log(r/r_0)$ for some $r_0$. These differential equations have a fix point at

$$\Delta_a = K_a = \frac{3}{14}$$

(23)

where the derivatives of $\Delta$ and $K$ vanish. Then perturbing about this fix point,

$$\frac{d}{dy} \begin{pmatrix} \Delta - \Delta_a \\ K - K_a \end{pmatrix} = \begin{pmatrix} -1/2 & -3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \Delta - \Delta_a \\ K - K_a \end{pmatrix}.$$  

(24)

This matrix has eigenvalues

$$\lambda_\pm = -\frac{3}{4} \pm i\frac{\sqrt{47}}{4}$$

(25)

so that the fix point (23) is stable. Physically, this means that no matter what initial conditions are chosen for $\Delta$ and $K$, both functions will eventually flow to the fix point (23) at large enough $r$. We can see this explicitly
by solving (24)

\[ K(r) = \frac{3}{14} \left\{ 1 + A \left( \frac{r_0}{r} \right)^{3/4} \sin \left( \frac{\sqrt{47}}{4} \log \frac{r}{r_0} \right) \right\} \]

\[ \Delta(r) = \frac{3}{14} \left\{ 1 + A \left( \frac{r_0}{r} \right)^{3/4} \left[ \left( \frac{\sqrt{47}}{4} \cos \left( \frac{\sqrt{47}}{4} \log \frac{r}{r_0} \right) \right) \right. \right. \]

\[ \left. \left. + \frac{1}{4} \sin \left( \frac{\sqrt{47}}{4} \log \frac{r}{r_0} \right) \right\} \right\}. \quad (26) \]

\( A, \) and \( r_0 \) are constants which require matching boundary conditions that are given at small \( r \) to determine. As the solutions to (22) for intermediate \( r \) are not known analytically, we are not able to do this explicitly. Nevertheless, numerical calculations, and a formal perturbative solution of (18) treating \( \rho_0 \) as the perturbation indicates that \( r_0 \sim 1/\sqrt{\rho_0} \) as long as \( 1/\sqrt{\rho_0} > r_m \).

We would therefore expect (26) to hold whenever \( r \gg 1/\sqrt{\rho_0} \). Note also that solutions to (22) approach the fix point (23) very slowly; basicly as \( r^{-3/4} \).

In the very large \( r \) limit solutions to (22) asymptotically approaches \( f_a = (56\pi\rho_0/3)^{1/2}r, \) and \( h_a = 7/4 \) where we have used the subscript \( a \) to denote the asymptotic solutions. Thus the metric at large \( r \) is

\[ ds^2 = -\left( \frac{56\pi\rho_0}{3} \right)^{1/2} r \ dt^2 + \frac{7}{4} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (27) \]

and we can now see explicitly that this spacetime is not asymptotically flat.

Next, we find that the asymptotic energy density is

\[ \rho_a = \frac{3c^4}{56\pi G r^2}, \quad (28) \]

where we have replaced the correct factors of \( c \) and \( G \). At very large \( r \), the energy density decreases as \( 1/r^2 \) with a proportionality constant which is an universal number and is independent of either the mass \( M \) or the temperature \( T_s \) of the sphere. This is once again the consequence of (22)
being a non-linear differential equation and having a stable, non-zero fix point. The temperature of the photon gas in the asymptotic limit is then

\[ k_B T(r) = \left( \frac{45 \hbar^3 c^7}{56 \pi^3 G r^2} \right)^{1/4} = m_{pl} c^2 \left( \frac{45}{56 \pi^3} \frac{l_{pl}^2}{r^2} \right)^{1/4}, \quad \text{(29)} \]

where \( m_{pl} = (\hbar c/G)^{1/2} \) is the Planck mass and \( l_{pl} = m_{pl} G/c^2 \) is the Planck length and we have explicitly used \( \sigma = \pi^2 k_B^4 / (15 \hbar^3 c^3) \). Although \( m_{pl} \) is very large, one should remember that (29) is valid only when \( r \) is also quite large. As \( l_{pl} \sim 10^{-33} \) cm, this ensures that \( k_B T(r) \) will always be very much smaller than the Planck energy. In fact, (18) guarantees that for \( r > R \), \( T(r) \leq T_s \), the temperature at the surface of the sphere. We should also mention that the asymptotic solutions are themselves solutions of (22) at any \( r \), as can be seen explicitly. They do not, unfortunately, satisfy the correct boundary conditions.

We can also calculate the total average energy of the photons

\[ E = \int d^3 x \sqrt{\hbar \rho} = \int_R^\infty \sqrt{\hbar} \frac{dm}{dr} dr, \quad \text{(30)} \]

from (18). Since \( m \sim r \) for large \( r \), we do not expect this \( E \) to be finite. It should instead diverge linearly with \( r \). This divergence is much milder, however, than in the case of flat spacetime where the total average energy diverges as the volume of the system.

\[ \text{§4. Discussion} \]

The spacetime outside of a sphere with temperature \( T_s \) can thus be divided into three regions. In the near field region, \( r \ll r_m \) and the solutions to Einstein’s equations are given by (19). The geometry of the spacetime in this region is dominated by the mass \( M \) at \( r = 0 \) and the presence of the photons will not have an significant affect on it. In the intermediate field region, \( r_m < r < 1/\rho_o \) and the total energy contained in the thermal
photons is now comparable to the mass of the sphere. Both the photons and the mass $M$ together will determine the geometry of the spacetime. In the far field region, $r \gg 1/\sqrt{\rho_0}$ and the asymptotic solutions (26) are now valid. It is now the photons which are dominant over the mass $M$.

We have in this paper considered only systems which are in thermodynamic equilibrium. In particular, this means that the sphere must be in thermodynamic equilibrium with the gas of photons surrounding it. As the spacetime that we have been considering consisted of only the mass $M$ and the photons, this is equivalent to saying the sphere must be in thermodynamic equilibrium with the rest of its universe. Since our universe is filled with the cosmic microwave background radiation which is at a temperature of $\sim 3^o$ K, for a physical body to be in thermodynamic equilibrium with the rest of our universe it must also be at a comparable temperature. There are very few actual physical bodies which are at such low temperatures. Moreover, because the mass $M$ is in equilibrium with the photons surrounding it, the amount of energy radiating away from the sphere must exactly be balanced by the amount of energy impacting on the sphere by the photon gas surrounding it. It is for this reason that the energy density is dependent only on the geometry of the spacetime, and is why the definition of the temperature (12) makes sense. It is also the reason why the intensity of the emitted radiation from the sphere does not have the characteristic $1/r^2$ behavior as one would naively expect.
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