METRICS WITHOUT MORSE INDEX BOUNDS

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0. Introduction

Let \( M^2 \) be a closed orientable surface with curvature \( K \) and \( \gamma \subset M \) a closed geodesic. The Morse index of \( \gamma \) is the index of the critical point \( \gamma \) for the length functional, i.e., the number of negative eigenvalues (counted with multiplicity) of the second derivative of length (throughout curves will always be in \( H^1 \)). Since the second derivative of length at \( \gamma \) in the direction of a normal variation \( u n \) is \( -\int_\gamma u L_\gamma u \) where \( L_\gamma u = u'' + K u \), the Morse index is the number of negative eigenvalues of \( L_\gamma \). (By convention, an eigenfunction \( \phi \) with eigenvalue \( \lambda \) of \( L_\gamma \) is a solution of \( L_\gamma \phi + \lambda \phi = 0 \).) Note that if \( \lambda = 0 \), then \( \phi \) (or \( \phi n \)) is a (normal) Jacobi field. \( \gamma \) is stable if the index is zero. The index of a noncompact geodesic is the dimension of a maximal vector space of compactly supported variations for which the second derivative of length is negative definite. We also say that such a geodesic is stable if the index is 0.

Our main result is:

Theorem 0.1. On any \( M^2 \), there exists a metric with a geodesic lamination with infinitely many unstable leaves. Moreover, there is such a metric with simple closed geodesics of arbitrary high Morse index.

The first part of Theorem 0.1 is relatively easy to achieve and in the proof we do that first.

A codimension one lamination on a surface \( M^2 \) is a collection \( \mathcal{L} \) of smooth disjoint curves (called leaves) such that \( \bigcup_{\ell \in \mathcal{L}} \ell \) is closed. Moreover, for each \( x \in M \) there exists an open neighborhood \( U \) of \( x \) and a \( C^0 \) coordinate chart, \( (U, \Phi) \), with \( \Phi(U) \subset \mathbb{R}^2 \) so that in these coordinates the leaves in \( \mathcal{L} \) pass through \( \Phi(U) \) in slices of the form \( (\mathbb{R} \times \{t\}) \cap \Phi(U) \). A foliation is a lamination for which the union of the leaves is all of \( M \) and a geodesic lamination is a lamination whose leaves are geodesics.

Theorem 0.1 will be proven by first constructing a metric on the disk with convex boundary having no Morse index bounds and then completing the metric to a metric on the given \( M^2 \). By taking the product of this metric on the disk with a circle we get, on a solid torus, a metric with convex boundary and without Morse index bounds for embedded minimal tori, and with a minimal lamination with infinitely many unstable leaves. By completing this metric we get:

Theorem 0.2. On any \( M^3 \), there exists a metric with a minimal lamination with infinitely many unstable leaves. Moreover, there is such a metric with embedded minimal tori of arbitrary high Morse index.

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By construction the embedded minimal tori in Theorem 0.2 and the leaves of the lamination can be taken to be totally geodesic.

Our interest in whether the Morse index is bounded comes from the assertion of J. Pitts and J.H. Rubinstein (see [PiRu], [CM2]) that if one can show that the Morse index of all embedded minimal tori is bounded for a sufficiently large class of metrics on \( S^3 \), then the spherical space form problem can be solved affirmatively.

We will equip the space of metrics on a given manifold with the \( C^\infty \)-topology. A subset of the set of metrics on the manifold is said to be residual if it is a countable intersection of open dense subsets. A metric on a surface is bumpy if each closed geodesic is a nondegenerate critical point, i.e., \( L_\gamma u = 0 \) implies \( u \equiv 0 \). Bumpy metrics are generic, [Ab], [An]; that is the set of bumpy metrics contain a residual set.

To check that any given metric is bumpy is virtually impossible; however it seems that the metric in Theorem 0.1 can be chosen to be bumpy; see also Remark 2.25 below. Thus it seems unlikely that a bumpy metric is enough to ensure a bound for the Morse index of simple closed geodesics on \( M^2 \). What is needed is a nondegeneracy condition for noncompact simple geodesics, rather than one for closed geodesics; cf. [CH1], [CH2].

Bounding the Morse index can be thought of as a purely analytical problem about lower bounds for eigenvalues of a Schrödinger operator. A typical way of getting such lower bounds is in terms of integral bounds for the potential. For instance, if \( Lu = u'' + Ku \) is a Schrödinger operator on a circle \( C \) with \( \text{Length}(C) \int_C \max\{K,0\} \leq C \), then the number of negative eigenvalues counted with multiplicity is bounded by \( N = N(C) \). However, as the next theorem illustrates, bounding the Morse index in this setting is analytically rather subtle; see also [CH1], [CH2].

**Theorem 0.3.** On any \( M^2 \), there exists an open (nonempty) set of metrics so that for each metric there is a sequence of simple closed stable geodesics \( \gamma_i \) with \( \text{Length}(\gamma_i) \to \infty \) and

\[
\inf_i \int_{\gamma_i} \max\{K,0\} \equiv \inf_i \int_{\gamma_i} \max\{K,0\} / \text{Length}(\gamma_i) > 0.
\]  (0.4)

Similarly, for 3-manifolds and minimal surfaces (with second fundamental form \( A \)):

**Theorem 0.5.** On any \( M^3 \), there exists an open (nonempty) set of metrics so that for each metric there is a sequence of embedded stable minimal tori \( \Sigma_i \) with \( \text{Area}(\Sigma_i) \to \infty \) and

\[
\inf_i \int_{\Sigma_i} \max\{|A|^2 + \text{Ric}_M(n,n),0\} > 0.
\]  (0.6)

Theorems 0.3, 0.5 are weaker than Theorems 0.1, 0.2 in the sense that they do not produce examples of metrics with no index bound. But the open set of metrics given in Theorems 0.3, 0.5 means that it is impossible to prove that the Morse index is bounded for a generic metric using only the standard analytic argument mentioned just above Theorem 0.3. It follows that a bound for the Morse bound is not just a simple analytical fact but relies on the (global) dynamics of the situation. This is where a generic condition is needed.

Since the metrics of Theorems 0.3 and 0.5 are much simpler than those of Theorems 0.1 and 0.2, then they are described first in Section 4.

Throughout this paper (except in Theorems 0.2, 0.3) \( M^2 \) is a closed orientable surface with a Riemannian metric, \( \mathcal{L} \) is a geodesic lamination, and when \( x \in M \), \( r_0 > 0 \), and \( D \subset M \),
then we let $B_{r_0}(x)$ denote the ball of radius $r_0$ centered at $x$ and $T_{r_0}(D)$ the $r_0$-tubular neighborhood of $D$. Moreover, if $x, y \in M$, then $\gamma_{x,y} : [0, \text{dist}_M(x,y)] \to M$ will denote a minimal geodesic from $x$ to $y$. Whenever we look at a single geodesic it will always be assumed to be parameterized by arclength.

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1. INDEX OF SCHröDINGER OPERATORS

In this section, we will discuss some of the difficulties with approaching the problem of bounding the Morse index from a purely analytical point of view. In particular, we will show Theorems 0.3 and 0.5.

**Proof.** (of Theorem 0.3). Let $S \subset M$ be a connected planar domain with three interior boundary components and $\nu : [0,1] \to S$ a simple curve connecting two different components of $\partial S$.

It is easy to see from the proof of proposition 1.1 of [CM1] that it is enough to show that the set of metrics $G$ on $S$ so that $S$ has strictly convex boundary and $\min_\nu K > 0$ is nonempty (observe that $G$ can clearly be extended to an open set of metrics on all of $M$). This will ensure that the curve $\gamma_m$ in Figure 1 will cross a region of positive curvature $m$ times, if $\nu$ connects the two lower interior boundary components.

To see that $G$ is nonempty, let $\sigma_1$ and $\sigma_2$ be the two boundary components of $S$ that $\nu$ connects. It is clearly enough to show that we can find a metric on a neighborhood (in $S$) of $\sigma_1 \cup \sigma_2 \cup \nu$ such that $\min_\nu K > 0$ and the outward (to $S$) normal geodesic curvatures of $\sigma_1$ and $\sigma_2$ are positive. This can easily be achieved by letting $B_{r_0}$ be a ball of radius $r_0 < \pi/2$ on the unit sphere $S^2$ and $\nu \subset B_{r_0}$ a geodesic through the center of the ball and thinking of $\sigma_1$ and $\sigma_2$ as two disjoint copies of $\partial B_{r_0}$ each of which intersects $\nu$ in only one end. \qed

Observe that if $\gamma \subset M^2$ is a closed stable geodesic, then by the stability inequality applied to the function $\phi \equiv 1$

$$\int_\gamma K \leq 0. \quad (1.1)$$

Likewise if $\Sigma \subset M^3$ is a closed stable minimal surface, then

$$\int_\Sigma [\|A\|^2 + \text{Ric}_M(n,n)] \leq 0. \quad (1.2)$$

On the other hand, in one dimension (and similarly in two dimensions; cf. Theorem 0.3), easy examples show that if the potentials $K_i$ of a sequence of Schrödinger operators on circles $C_i$ satisfy $\text{Length}(C_i) \int_{C_i} \max\{K_i,0\} \to \infty$ and $\int_{C_i} K_i \leq 0$, then typically there is no uniform bound for the indices.

**Proof.** (of Theorem 0.5). Let again $S$ be a connected planar domain with three interior boundary components and $\nu : [0,1] \to S$ a simple curve connecting two different components $\sigma_1$ and $\sigma_2$ of $\partial S$. Set $\Omega = S \times S^1$. By the proof of Theorem 0.1 of [CM1], it is clearly enough to show that there exists a metric on $\Omega$ with strictly convex boundary and such that $\min_{\nu \times S^1} K_\Omega > 0$. In fact it is enough to show that such a metric exists in a neighborhood of $\sigma_1 \times S^1 \cup \sigma_2 \times S^1 \cup \nu \times S^1$ (in $\Omega$).
To see this observe first that clearly there exists such a metric in a neighborhood of $\nu \times S^1$ (in $\Omega$) - we need to extend this to a metric in a neighborhood of all of $\sigma_1 \times S^1 \cup \sigma_2 \times S^1 \cup \nu \times S^1$ while keeping the surfaces $\sigma_1 \times S^1$ and $\sigma_2 \times S^1$ strictly convex. This can easily be achieved by first choosing any extension to $\sigma_1 \times S^1 \cup \sigma_2 \times S^1$ and then extending it to a small normal neighborhood of $\sigma_1 \times S^1 \cup \sigma_2 \times S^1$ making $\sigma_1 \times S^1 \cup \sigma_2 \times S^1$ strictly convex. □

2. Metrics without Morse index bounds

In this section we will prove Theorems 0.1 and 0.2. As mentioned in the introduction it suffices to construct a metric on the disk with convex boundary that has a geodesic lamination with infinitely many unstable leaves, and show that there are simple closed geodesics of arbitrarily high Morse index. The construction relies on the:

Basic Barrier Principle: Let $R$ be a domain in $M^2$ with piecewise geodesic boundary that is locally convex in $M$; that is, $R$ should locally be the intersection of closed geodesic half-spaces in $M$. Given a simple closed curve $\gamma$ on $R$, there is a simple closed geodesic $\tau$ which is freely homotopic to $\gamma$ on $R$, and which has length less than or equal to that of $\gamma$.

The boundary of $R$ should be thought of as a barrier. The principle is a consequence of a “Lusternik-Schnirelmann curve-shortening process”; that is, of a curve shortening process which does not introduce intersections. There are many versions of such a process; see for instance Grayson’s paper [Gr], and the related work of Angenent [Ang]. With this process a closed curve evolves in the direction of its curvature vector $\kappa n$. Grayson shows that, if the curve does not shrink to a point in finite time, then the flow is defined for all time, and that for every $n \geq 0$,

$$\lim_{t \to \infty} \sup \kappa^{(n)} = 0.$$  
(2.1)
Angenent has shown that the number of intersection points of two evolving curves can only decrease. Using this, we get a slightly stronger principle:

**Second Barrier Principle:** Let $\tilde{M}$ be a covering space of $M$, and let $\tilde{R}$ be a locally convex domain in $\tilde{M}$ as before. (We are not assuming that $\tilde{R}$ is the lift of a set $R$.) Let $\tilde{\gamma}$ be a simple closed curve on $\tilde{R}$ whose image on $M$ is still simple. Then there is a simple closed geodesic $\tilde{\tau}$ in the same free homotopy class as $\tilde{\gamma}$ on $\tilde{R}$ whose image $\tau$ on $M$ is still simple, and which has length less than or equal to that of $\tilde{\gamma}$. Moreover, if $\sigma$ is a geodesic on $\tilde{R}$ (closed, or with boundary on $\partial\tilde{R}$), then $\tilde{\tau}$ intersects $\sigma$ in no more points than did $\tilde{\gamma}$. If the metric is bumpy, then we can assume that $\tilde{\tau}$ is a local minimum of length. If there is no geodesic in $\partial\tilde{R}$ homotopic to $\tilde{\gamma}$, then $\tilde{\tau}$ will lie in the interior of $\tilde{R}$.

*Proof.* This follows since there are no choices involved in the curve-shortening process. □

We start with a surface of revolution about the $z$-axis which is the connected sum of three round spheres using two (sufficiently narrow) necks of negative curvature. It looks like a snowman. It will be useful below to refer to the region between the necks as the “middle-sphere” part of $M$. Let $(r, \theta, z)$ be cylindrical coordinates, and let $\alpha$ be an angle defined on the tangent bundle to the surface of revolution as the angle measured down clockwise from the meridian $\theta=\text{constant}$. (So $\alpha = 0$ for a vector on the surface which points up; $\alpha = \pi/2$ for a vector which points in the direction of increasing $\theta$.) Curves with $dz/d\theta > 0$ will be called *positive*. Later we will alter the metric on our surface. The coordinate functions $\theta$, $z$, and $\alpha$ on the surface will be fixed however, induced from the embedding as a surface of revolution in $\mathbb{R}^3$.

The relative radii of the three balls are not important, but the two necks must have equal radii; this way by Clairaut’s theorem there will be two 1-parameter families of geodesics, which we will call *limiting geodesics*, crossing the (middle-sphere) equator with angles $\pm \alpha_0$, which spiral toward the upper neck as $t \to \infty$ and toward the lower neck as $t \to -\infty$. (Geodesics crossing the equator with $|\alpha| \leq \alpha_0$ will enter the lower and upper spheres; those with $|\alpha| > \alpha_0$ will stay in the center sphere with periodic $z$-values.) It will be convenient to have each neck symmetric about a horizontal plane; this way a geodesic which crosses a neck will emerge as an inverted mirror image of the curve that went in.

Next we add a nose to the top sphere and two to the bottom sphere. For each nose we alter the metric in a small disk away from the $z$-axis in such a way that the disk now supports a strictly stable simple closed geodesic which we will use as a barrier. (The shape of the nose is unimportant so long as we have this barrier geodesic. These barrier geodesics will serve much the same purpose as the interior boundary of $S$ did in the proof of Theorem 0.3.) It will also be useful to have two more simple strictly stable geodesic barriers circling the $z$-axis at the top and bottom of the snowman. To get a disk, we cut away the interior of the lower barrier circling the $z$-axis.

We call this Riemannian surface $M^2_0$. (See Figure 2.)

We need one more alteration of the metric: We will add an infinite number of “bumps” along the equator. Given a radius $\rho$ and a point $P$ on the equator of the middle-sphere, we will make a “bump” of radius $\rho$ at $P$ with the following four properties:

1. The metric remains unchanged except within distance $\rho/2$ of $P$.
2. The metric remains locally rotationally symmetric about $P$.
3. In the new metric, the geodesics through $P$ are longer than they were without the bump.
4. The metric remains smooth after adding an infinite number of disjoint bumps.
This can be done for example by multiplying the metric by a conformal factor \(1 + e^{-1/r} f(r/\rho)\) in coordinates centered at \(P\) (\(r\) is the radial component) for some smooth bump function \(f\) with support in \([0, 1/2]\). In order to prove the first statement in Theorem 0.1, it will be convenient to have \(f\) monotonic. We place the bumps around the equator in such a way that each segment of length \(\mu\) (to be determined below) contains an accumulation point of the bump centers. They should be far enough apart that between any two there is a geodesic which crosses the equator at angle \(\alpha_0\).

This surface with bumps is \(M^2\).

The Geodesics. We digress briefly to say roughly what the geodesics of high index on \(M^2\) look like. First we describe some geodesics on the surface \(M^2_0\). Let \(\eta\) be a simple closed curve with the three noses in its interior and the north- and south-pole barrier geodesics in its exterior. We can assume by the barrier principle that \(\eta\) is geodesic. Let \(R\) be the interior of \(\eta\) with the three barrier holes removed. Figure 1 shows a simple closed curve \(\gamma_m\) on \(R\), which we can again assume is geodesic. Here \(m\) is the number of “strands”, so that \(\gamma_m\) crosses the equator \(2m\) times. Now put a rubber band around \(\eta\), and another around \(\gamma_m\). Pick an integer \(n > 0\). Take the snowman’s head, and rotate it (just the head) counterclockwise through an angle of \(2n\pi\). (Don’t let the rubber bands get unhooked.) This results in new geodesics \(\eta_n\) and \(\gamma_{m,n}\) which wind (and unwind) \(n\) times. Now these geodesics are still local minima (index 0). But if \(n\) is sufficiently large (depending upon \(m\)), we can use a minimax argument to produce a geodesic close to \(\gamma_{m,n}\) on the surface \(M^2\) which goes over a bump each time it crosses the equator, and thus has index \(2m\). The bumps on the equator act as a comb which enables us to separate the strands and move them around the equator. Of course the bumps are arbitrarily small, and we need them high enough to hold the strands.
However as \( n \to \infty \) the strands of \( \gamma_{m,n} \) approach the critical “limiting” angle \( \alpha_0 \) and it takes less and less to hold them.

We will now make all of this precise in the following theorem:

**Theorem 2.2.** For every \( k \geq 0 \), the surface \( M^2 \) has a simple closed geodesic with Morse index in \([k - 2, k]\).

The geodesics we will produce will have local homology in dimension \( k \), and thus (see \([K]\) corollary 2.5.6 and proposition 1.12.3 and also \([C1]\) theorem 5.4 on p. 50) index \( k \leq \text{nullity} + 2 \), since on a surface any closed geodesic has nullity \( \leq 2 \).

The proof of Theorem 2.2 will rely on a number of observations and lemmas. We begin by establishing some basic facts about the curve \( \eta_n \). The curve \( \eta_n \) is a curve on a cylinder \( (M_0^2 \text{ with north pole removed}) \) with 3 holes removed. It wraps around the cylinder \( n \) times, hooks around the top nose, unwraps \( n \) times, and hooks around the bottom two noses. Using curve-shortening, we can assume that \( \eta_n \) is geodesic, and that no other curve in the free homotopy class, and intersecting the equator exactly twice, has length less than that of \( \eta_n \).

Let the barrier circling the north-pole lie at \( z = z_N \) and the upper neck at \( z = z_0 \). Let \( z_1 > z_0 \). After possibly making the necks more narrow, we can assume that there is an \( \epsilon > 0 \) so that if \( \sigma_z(\theta) \) is the curve on \( M_0^2 \) at height \( z \) parameterized by \( \theta \), then \( |\sigma_z'(\theta)| \geq \sigma_z(\theta) + \epsilon \) for every \( z \) with \( z_1 \leq z \leq z_N \). Since \( \eta_n \) has minimal length, for fixed \( z_1 \) the length of the part of \( \eta_n \) which lies above \( z = z_1 \) is bounded as \( n \to \infty \). All the winding around takes place between \( z = z_2 \) and \( z = z_1 \); thus the angles at which \( \eta_n \) crosses the equator both go to \( \alpha_0 \) as \( n \to \infty \).

Let \( R_n \) be the domain bounded by \( \eta_n \) and the 3 hole boundaries. Let \( E_n \) be the intersection of \( R_n \) with the equator. We claim:

\[
\mu = \frac{1}{2} \liminf_{n \to \infty} \text{width}(E_n) > 0 . \tag{2.3}
\]

To see this, note that the two sides of \( \eta_n \) are bounded apart at the equator if and only if they are bounded apart as they cross a level \( z = z_1 \) just above the upper neck, and if they are bounded apart as they cross a level \( z = z_2 \) just below the lower neck. Since they travel a bounded distance outside the region \( z_2 \leq z \leq z_1 \), and since \( \eta_n \) is simple, this will be the case.

Since the set of bumpy metrics is dense (by \([AB]\)) we can let \( g_i \) be a sequence of bumpy metrics with limit the surface \( M^2 \) as \( i \to \infty \). (We introduce the metrics \( g_i \) in order to simplify a Morse theoretic argument which will come later. We will use Morse theory in the metrics \( g_i \) to find simple closed geodesics of high index in the metrics \( g_i \), where the length functional is a nondegenerate Morse function, and then take the limit as \( i \to \infty \).) Using the fact that the 5 barrier geodesics are strictly stable, we can (cf. Lemma A.6, Lemma B.1 below) assume that each metric \( g_i \) has 5 simple closed geodesics with limit as \( i \to \infty \) the 5 barrier geodesics on \( M^2 \). For \( i \) sufficiently large, applying curve-shortening in \( g_i \) metric to the curve \( \eta_n \) will produce a simple closed geodesic \( \eta_{k,n} \) with \( \eta_{k,n} \to \eta_n \) as \( i \to \infty \). Here we are again using the fact that \( \eta_n \) has minimal length; it might be necessary to replace \( \eta_n \) by another simple closed geodesic of the same length, and with all the properties established above. Note \( \eta_n \) will still be simple since on an orientable surface a geodesic which is a limit of simple curves will itself be simple.
Next we show that the bumps do bend geodesics; a geodesic crossing the equator to the right of a bump center will curve to the left at both ends.

**Lemma 2.4.** Consider the surface $M^2_0$ with one bump of radius $\rho$ added at a point $P$ on the equator. If $\rho$ is sufficiently small, there is a geodesic $\lambda$ which meets a ball of radius $\rho/2$ about $P$, and crosses the equator to the right of $P$ at an angle $\alpha \approx \alpha_0$. The “top half” of $\lambda$ will leave the ball of radius $\rho$ about $P$ at an angle more vertical than the critical angle, and will cross each positive limiting geodesic an infinite number of times in the northern part of the middle-sphere before crossing the upper neck. The “bottom half” of $\lambda$ will leave the ball of radius $\rho$ at an angle less vertical than the limiting angle, and will cross each positive limiting geodesic at least twice in the southern part of the middle-sphere before crossing the equator again. If $\tau$ is the limiting geodesic which is tangent to the ball of radius $\rho$ about $P$ on its right hand side, then $\lambda$ stays to the left of $\tau$ in the universal cover of the cylinder.

**Proof.** Let $\zeta$ be the geodesic through $P$ at angle $\alpha_0$. This geodesic is longer than the geodesic $\tau$ to its right which does not meet the bump. Since the two geodesics are asymptotic at both ends, over a large enough distance $\zeta$ is not length minimizing; over a large distance it becomes more efficient to go around the bump. Take two far away points on $\zeta$, and find a minimizing geodesic $\lambda (\neq \zeta)$ in the homotopy class determined by the segment $\zeta$. We can assume that $\lambda$ lies to the right of $\zeta$ and to the left of $\tau$. Extending $\lambda$ at both ends will produce a geodesic as described; after leaving the bump, the geodesic $\lambda$ is described by Clairaut’s theorem. (The hypothesis that $\rho$ is not too large is used to ensure that the bottom half of $\lambda$ does not cross the equator again too soon: In the limit as $\rho \to 0$ the bottom half of $\lambda$ becomes a positive limiting geodesic “followed by” a negative limiting geodesic which will cross each positive limiting geodesic an infinite number of times before crossing the equator again.)

**Corollary 2.5.** The surface $M^2$ has a geodesic lamination with infinitely many unstable leaves.

**Proof.** The limiting geodesics in the metric $M^2_0$ which pass through bump centers are still geodesics on $M^2$, since the bumps are rotationally symmetric. If $f$ is monotonic, it is easy to see that they have positive index. (See Figure 3.) The closure of their union is the desired lamination.

This proves the first part of Theorem 0.1.

We now return to the curves $\eta_n$. Let $n$ and $\rho$ be given, and fix a point $P$ on $E_n$. We consider again the surface $M^2_0$ with a single bump of radius $\rho$ added at $P$. We say the bump of radius $\rho$ at $P$ is high enough to hold $\eta_n$ if 1) there is a geodesic segment $\lambda$ on $R_n$ with both endpoints on the left side of the middle-sphere section of $\eta_n$ which crosses $E_n$ exactly once, to the right of $P$; and if 2) (the mirror image of 1)) there is a geodesic segment $\psi$ on $R_n$ with both endpoints on the right side of the middle-sphere section of $\eta_n$ which crosses $E_n$ exactly once, to the left of $P$. (See Figure 4.) The idea will be to use $\lambda$ and $\eta_n$ together as barriers to get a strand of $\gamma_{m,n}$ to hook around the right side of the bump at $P$.

**Lemma 2.6.** Fix $m \geq 1$. If $n$ is sufficiently large, there are $2m$ bumps on $R_n$ in $M^2$ which are high enough to hold $\eta_n$.

**Proof.** By construction there is a $\epsilon > 0$ so that each interval on the equator of width $\mu$ contains at least $2m$ bumps of radius at least $\epsilon$. By the definition of $\mu$, for $n$ sufficiently
Figure 3. A geodesic lamination on $M$ with infinitely many unstable leaves.

Figure 4. A bump at the point $P$ “high enough to hold $\eta_n$” deflecting the geodesics $\lambda$ and $\psi$. 
large $E_n$ will contain an interval of length $\mu$. Now we use Lemma 2.4 and the fact that the two “sides” of $\eta_n$ are geodesics at angles which approach the limiting angle as $n \to \infty$. Of course Lemma 2.4 works as well to get a geodesic curving around the left side of $P$. $\square$

Next we bring in the curves $\gamma_{m,n}$. To reduce the number of subscripts, fix $m$ and $n$ with $n$ “sufficiently large.” Here is the idea of the rest of the proof. For simplicity assume for the moment that the metric is bumpy. Let $P_1, \cdots, P_{2m}$ be points on $E_n \subset M^2$ which are bump centers for bumps “high enough to hold $\eta_n$,” ordered from left to right. We will show that there is a simple closed geodesic, freely homotopic to $\gamma_{m,n}$ in $R_n$, where we can choose the $j$’th strand (ordered left to right) to cross the equator either to the left or the right of $P_j$. With a bumpy metric this geodesic will be a local minimum of length. This gives, with all possible such choices, $2^{2m}$ distinct simple closed geodesics of index 0, each freely homotopic to $\gamma_{m,n}$. We next (Lemma 2.11) fill in a $2m$-cube of simple closed curves freely homotopic to $\gamma_{m,n}$, with the $2^{2m}$ geodesics as vertices. We will use relative homology to show (Lemma 2.19) that each face of the cube will “lie hanging” on a closed geodesic whose Morse index is equal to the dimension of the face. (Note to get index $>1$ it is not sufficient to have a cube whose vertices are distinct local minima; to get nontrivial topology in higher dimensions we use (2.12) below.)

Let $\xi_1, \xi_2, \xi_3$ be simple curves in $R_n$ starting on the 3 hole boundaries and ending on $\eta_n$. They should not intersect each other or $E_n$. Figure 5 shows a domain homeomorphic to $R_n$ (actually the homeomorphism type does not depend on $n$.) Let $\gamma_{m,n}$ be the curve shown in Figure 4. We assume that $\gamma_{m,n}$ has intersections with the $\xi_i$ as in the figure. Let $S$ be the $2m$-fold covering space of $R_n$ in which $\gamma_{m,n}$ lifts to a closed curve $\tilde{\gamma}_{m,n}$ which crosses a different lift of $E_n$ each time it crosses the equator. (The cover can be constructed as follows: Get $2m$ copies of $R_n$ with the $\xi_i$ marked. Pick one, and start tracing out the curve $\gamma_{m,n}$.)

**Figure 5.** Geodesics in the domain $R_n$. 
Each time you cross one of the $\xi_i$, cut along $\xi_i$ and paste in a new copy of $R_n$. Let $\tilde{E}_n$ be the set in $S$ lying above $E_n$. (It will have $2m$ components.) At this point it should be clear, using the barrier principle, that we can assume that $\tilde{\gamma}_{m,n}$ is geodesic and still has all the same intersections: It crosses each component of $\tilde{E}_n$ once, and its image on $M^2$ crosses each neck $2m$ times. In between the necks its $z$-coordinate is monotone. Label the intersection points of $\tilde{\gamma}_{m,n}$ with $\tilde{E}_n$: $Q_1, \ldots, Q_{2m}$ in such a way that their images in $E_n$ go from left to right. Let $T_j$ be the $2m$ lifts of the portion of $R_n$ which lies between the two necks on $M^2$, labeled so that $T_j$ contains $Q_j$.

Given subsets $I$ and $J$ of $\{1, \ldots, 2m\}$, let

$$\ell_j = \begin{cases} j & \text{if } j \in I, \\ j - 1 & \text{otherwise}, \end{cases} \quad (2.7)$$

$$r_j = \begin{cases} j & \text{if } j \in J, \\ j + 1 & \text{otherwise}. \end{cases} \quad (2.8)$$

We are going to construct a locally convex domain $\tilde{R}_{I,J}$ in $S$ that “forces” the $j$'th strand of a curve homotopic to $\gamma_{m,n}$ to cross $E_n$ strictly between the points $P_{\ell_j}$ and $P_{r_j}$. For each of the bump centers $P_j$ there is (on $M^2$) a geodesic segment $\lambda_j$ with endpoints on the left side of $R_n$ which crosses $E_n$ to the right of $P_j$, and a geodesic segment $\psi_j$ with endpoints on the right side of $R_n$ which crosses $E_n$ to the left of $P_j$. Let $\Lambda_j$ be the part of $R_n$ to the left of $\lambda_j$, and $\Psi_j$ the part to the right of $\psi_j$. By construction $\Lambda_{j-1}$ and $\Psi_j$ have disjoint closures and $E_n \subseteq \Lambda_j \cup \Psi_j$. Let $\Lambda_0$ and $\Psi_{3m+1}$ each be the empty set. We define the domain $\tilde{R}_{I,J}$ to be $S$ with the lifts of $\bigcup_{k \leq \ell_j} \Lambda_k$ and $\bigcup_{k \geq r_j} \Psi_k$ removed from each $T_j$.

It will be important that these unions are monotonically increasing (respectively decreasing) with $j$, and that, by definition,

$$\tilde{R}_{I,J} \cap \tilde{R}_{K,L} = \tilde{R}_{I \cup K, J \cup L}. \quad (2.9)$$

Given $I$ and $J$, subsets of $\{1, \ldots, 2m\}$ again, we define the face

$$F_{I,J} = \{ \epsilon \in [0,1]^{2m} \mid \epsilon_i = 0 \text{ if } i \in I \text{ and } \epsilon_i = 1 \text{ if } i \in J \}. \quad (2.10)$$

Note that $F_{I,J}$ is empty unless $I$ and $J$ are disjoint, and that $F_{I,J}$ is a vertex point if they are disjoint with union $\{1, \ldots, 2m\}$.

**Lemma 2.11.** There is a continuous map $\gamma$ from $[0,1]^{2m}$ to $H^1(S^1, S)$ with the property that if $\epsilon = (\epsilon_1, \ldots, \epsilon_{2m}) \in F_{I,J}$, then $\gamma(\epsilon)$ is a simple closed curve in the interior of $\tilde{R}_{I,J}$, which is freely homotopic in $S$ to $\tilde{\gamma}_{m,n}$, via curves whose images in $M^2$ are simple.

Note that by the definition of $\tilde{R}_{I,J}$, if $\epsilon \in F_{I,J}$, the curve $\gamma(\epsilon)$ described in the lemma will cross the component of $\tilde{E}_n$ containing $Q_j$ at a point whose image in $M^2$ lies (strictly) between $P_{\ell_j}$ and $P_{r_j}$.

**Proof.** (of Lemma 2.11). The strands are of course ordered not consecutively, but in the order in which they cross the equator. Note that $\tilde{R}_{I,J}$ is locally geodesically convex by construction. Without spoiling local convexity we can, by adding an extra geodesic segment to the bottom (respectively top) of $\lambda$ (respectively $\psi$) assume that the coordinate $z$ is monotonic along $\lambda$ and $\psi$. For a fixed vertex point $F_{I,J}$ ($I \cup J = \{1, \ldots, 2m\}$) of the cube, the surface $S$, (where $\tilde{\gamma}_{m,n}$ lies), can be contracted along the curves $z = \text{constant}$ onto the surface $\tilde{R}_{I,J}$.
keeping \( \tilde{R}_{I,J} \) fixed. The image of \( \tilde{\gamma}_{m,n} \) under the contraction will be \( \gamma(\epsilon) \). Observe that the curve \( \gamma(\epsilon) \) agrees with \( \tilde{\gamma}_{m,n} \) except where it is pushed aside by the curves \( \lambda \) and \( \psi \). Since the sets \( \bigcup_{j \leq k} \Lambda_{j+\epsilon_k-1} \) and \( \bigcup_{j \geq k} \Psi_{j+\epsilon_k} \) are monotonically increasing (respectively decreasing) with \( k \), the image of \( \tilde{\gamma}_{m,n} \) under the contraction, while not necessarily a simple curve, can be approximated by simple closed curves. We use linear interpolation along the curves \( z = \text{constant} \) to define \( \gamma \) for non-vertex points \( \epsilon \) in the domain \([0, 1]^{2m} \). It is not difficult to see that linear interpolation will preserve the property that the curves “move to the right” with \( k \). Clearly \( \tilde{\gamma}_{m,n} \) is not homotopic to a curve in \( \partial \tilde{R}_{I,J} \). By turning on the curve-shortening process momentarily, we can assume that the image of \( F_{I,J} \) under \( \gamma \) consists of simple closed curves lying in the interior of \( \tilde{R}_{I,J} \).

We next show that the image of the face \( F_{I,J} \) under the map \( \gamma \) is nontrivial in the appropriate homology group of dimension \( 2m - |I| - |J| \). Since the cube \([0, 1]^{2m} \) is contractible, this will have to be relative homology. The idea will be to show, using induction, that \( \partial \gamma F_{I,J} = \gamma \partial F_{I,J} \) is nontrivial and to thus conclude that \( \gamma F_{I,J} \) is nontrivial. The nontriviality of \( \gamma \partial F_{I,J} \) will follow from the next lemma. Let \( \Omega_{I,J} \) be the set of simple closed curves in the interior of \( \tilde{R}_{I,J} \) which are freely homotopic in \( S \) to \( \tilde{\gamma}_{m,n} \) through curves whose images on \( M^2 \) are simple, and which cross each component of \( \tilde{E}_n \) exactly once, transversely. Thus: The \( j \)-th strand of a curve in \( \Omega_{I,J} \) is constrained to pass to the left of the point \( P_j \) if \( j \in I \), and to the right of \( P_j \) if \( j \in J \); it must pass strictly between \( P_{j-1} \) and \( P_{j+1} \) in any case. We can unambiguously choose a parameterization for all curves in \( \Omega_{I,J} \) by insisting that the curves begin on a particular component of \( \tilde{E}_n \). Note that the sets \( \Omega_{I,J} \) have the same intersection pattern as the faces \( F_{I,J} \) and the domains \( R_{I,J} \), namely

\[
\Omega_{I,J} \cap \Omega_{K,L} = \Omega_{I \cup K, J \cup L}.
\]

(2.12)

Also note that a curve in \( \Omega_{I,J} \) will cross the component of \( \tilde{E}_n \) containing \( Q_j \) at a point whose image in \( M^2 \) lies (strictly) between \( P_{t_j} \) and \( P_{t_{j+1}} \). Finally it will be crucial of course that the sets \( \Omega_{I,J} \) are preserved under the curve shortening process. Let \( \Delta_{I,J} \) be the path space where the boundary of \( F_{I,J} \) lies, namely

\[
\Delta_{I,J} = \cup_{(I,J) \subset (K,L)} \Omega_{K,L},
\]

(2.13)

where we write \( (I, J) \subset (K, L) \) if \( I \subseteq K \) and \( J \subseteq L \), but \( (I, J) \neq (K, L) \).

**Lemma 2.14.** The free Abelian group \( \langle F_{I,J} \rangle_{|I|+|J|=p} \) injects into

\[
\bigoplus_{|I|+|J|=p} H_{2m-p}(\Omega_{I,J}, \Delta_{I,J})
\]

(2.15)

and hence injects into

\[
H_{2m-p}(\bigcup_{|I|+|J|=p} \Omega_{I,J}, \bigcup_{|I|+|J|=p} \Delta_{I,J}).
\]

(2.16)

**Proof.** Induction (downward) on \( p \), using the fact that (inductively) the boundary \( \partial F_{I,J} \) of \( F_{I,J} \) represents a nontrivial element in \( H_{2m-p-1}(\Delta_{I,J}) \), and excision, together with \((2.12)\), for the second injection. To start the induction we use the fact that the sets \( \Omega_{I,J} \) with \( |I| + |J| = 2m \) are disjoint by \((2.12)\).
Now let \( g_i \) be the bumpy metrics close to the given metric on \( M^2 \); for simplicity assume also that distinct closed geodesics have different lengths in the \( g_i \) metric. We are still assuming \( m \) and \( n \) fixed, with \( n \) sufficiently large. If \( i \) is sufficiently large, with the bumpy metric we will (using curve-shortening, and the fact that \( \eta_n \) has minimal length) have a locally convex domain \( R_n \) and geodesics \( \lambda_j \) and \( \psi_j \) with the same pattern of intersections as in the limit \( i \to \infty \). All these barriers will approach the barriers on \( M^2 \) as \( i \to \infty \).

**Lemma 2.17.** Let \( R \) be a locally convex domain, and let \( \tau \) be a geodesic in \( R \) with boundary in \( \partial R \). Let \( \gamma \) be a simple closed curve on \( R \) intersecting \( \tau \) once, transversely. Let \( \Omega \) be the space of (unparameterized) simple closed curves on the interior of \( R \) which are freely homotopic to \( \gamma \), and which intersect \( \tau \) once, transversely. Assume the metric on \( R \) is bumpy, and that different closed geodesics have different lengths. Suppose that \( a \) and \( b \) are not critical values of the length function, and that \( (a, b) \) contains at most one critical value \( c \). Let \( \Omega^a \) denote the curves of length \( \leq a \). Then

\[
H_k(\Omega^b, \Omega^a) = \begin{cases} 
Z & \text{if there is a critical point of index } k, \\
0 & \text{otherwise}. 
\end{cases}
\]  
(2.18)

**Proof.** The curve \( \tau \) is used to choose a parameterization for the curves; we can assume they all start on \( \tau \). Standard Morse theory arguments, for example Theorems 4.1 and 4.2 on p. 34–35 in Chang’s book [Ch], which use the gradient of the energy function on the space of \( H^1 \) curves apply in this context as well, given the previously mentioned properties of the curve-shortening flow, i.e., (2.1). In fact the computation of the local critical group needs no alteration since a sufficiently small neighborhood of a simple closed geodesic will consist entirely of simple curves. \( \square \)

Again let \( m \) and \( n \) be fixed with \( n \) sufficiently large. In the \( g_i \) metric for \( i \) sufficiently large there will be locally convex domains \( R_{I,J} \) and path spaces \( \Omega_{I,J} \). With these hypotheses we have:

**Lemma 2.19.** Each \( \Omega_{I,J} \) contains a (“minimax”) geodesic of index \( 2m - p \), where \(|I| + |J| = p\).

**Proof.** Let \( \delta \) be small but positive. We use downward induction on \( p \). The idea is to push down each cell \( \gamma_{F_{I,J}} \) using the curve-shortening flow, leaving its (already pushed down) boundary fixed. It seems convenient however to work with relative homology groups. The cells \( \gamma_{F_{I,J}} \) can be converted into singular chains using the shuffle homeomorphism ([GgHa], p. 268). All singular chains will have \( \mathbb{Z}_2 \) coefficients. We have the boundary relations

\[
\partial \gamma_{F_{I,J}} = \sum_{(I,J) \subset (K,L), |I| + |J| + 1 = |K| + |L|} \gamma_{F_{K,L}}. 
\]  
(2.20)

We will inductively alter (“pushdown”) the cells \( \gamma_{F_{I,J}} \) by replacing each by a homologous (in \( \Omega_{I,J} \)) chain in \( \Omega_{I,J} \). If we add \( \partial \tau \) to \( \gamma_{F_{K,L}} \), for a chain \( \tau \) in \( \Omega_{K,L} \), then we must add \( \tau \) to each \( \gamma_{F_{I,J}} \) with \( (I, J) \subset (K, L) \) and \(|I| + |J| + 1 = |K| + |L|\); this way the boundary relations (2.20) will be maintained.

The lemma is clear from the Barrier Principle when \( p = 2m \) since \( \Omega_{I,J} \) is nonempty. For each \( I, J \) with \( p = 2m \), the minimum value \( a_{I,J} \) with \( H_0^{a_{I,J}}(\Omega_{I,J}) \neq 0 \) is clearly a critical value of the length function corresponding to a critical point of index 0. We can (using curve-shortening) assume that \( \gamma_{F_{I,J}} \) lies in \( \Omega_{a_{I,J} + \delta}^{I,J} \).
Now fix $I, J$ with $|I| + |J| = p$. Assume that for each $(K, L) \supset (I, J)$ we have $\gamma F_{K, L} \subset \Omega_{K, L}^{a_{K, L}p + \delta}$ and assume that $a_{K, L} - \delta > a_{M, N} + \delta$ if $(K, L) \subset (M, N)$. Assume that in each $\Omega_{K, L} \setminus \Delta_{K, L}$ with $(I, J) \subset (K, L)$, we have a critical point of index $2m - |K| - |L|$ whose critical value $a_{K, L}$ is the infimum $a$ for which the class of $\gamma \partial F_{K, L}$ vanishes in $H_*(\Omega_{K, L})$. This means that the image of $\gamma \partial F_{K, L}$ is nontrivial in $H_*(\Omega_{K, L}^{a_{K, L}p + \delta})$, and that $H_*(\Omega_{K, L}^{\alpha_{K, L}p + \delta}, \Omega_{K, L}^{\alpha_{K, L}p - \delta}) \neq 0$. The assumption that the critical point with critical value $a_{K, L}$ lies in $\Omega_{K, L} \setminus \Delta_{K, L}$, together with the assumption that the different critical points have different critical values, and (2.12), implies that $a_{M, N} \neq a_{K, L}$ if $(M, N) \neq (K, L)$. The lemma will follow from Lemma 2.17 once we establish that

\[
\text{the image of } \gamma \partial F_{I, J} \text{ is nontrivial in } H_*(\Omega_{I, J}^{a + \delta}),
\]

where $a = \max_{(I, J) \subset (K, L)} a_{K, L}$, and

\[
\text{the image of } \gamma \partial F_{I, J} \text{ is nontrivial in } H_*(\Delta_{I, J}).
\]

In order to establish (2.21) we argue as follows. Suppose $a = (\text{say})a_{M, N}$. Line up the long exact sequences for the pairs $(\Delta_{I, J}^{a + \delta}, \Delta_{I, J}^{a - \delta})$ and $(\Omega_{I, J}^{a + \delta}, \Omega_{I, J}^{a - \delta})$. The class of $\gamma F_{M, N}$ in $H_*(\Delta_{I, J}^{a + \delta}, \Delta_{I, J}^{a - \delta})$ is the same as that of $\gamma \partial F_{I, J}$, which comes from $H_*(\Delta_{I, J}^{a + \delta})$. The further image in $H_*(\Omega_{I, J}^{a + \delta}, \Omega_{I, J}^{a - \delta})$ is nonzero by Lemma 2.17 and the inductive hypothesis. Thus the image of $\gamma \partial F_{I, J}$ in $H_*(\Omega_{I, J}^{a + \delta})$ is nonzero. We get (2.22) from (2.16) by looking at the image of

\[
H_*(\Delta_{I, J}) = H_*\left(\bigcup_{(I, J) \subset (K, L)} \Omega_{K, L}\right)
\]

in

\[
H_*\left(\bigcup_{|K| + |L| = p + 1} \Omega_{K, L}, \bigcup_{|K| + |L| = p + 1} \Delta_{K, L}\right).
\]

On the other hand since $\partial \gamma F_{I, J} = \gamma \partial F_{I, J}$, the image of $\gamma \partial F_{I, J}$ in $H_*(\Omega_{I, J})$ is 0; hence for some $b > a$, $H_2m - p(\Omega_{I, J}^{a + \delta}, \Omega_{I, J}^{a - \delta}) \neq 0$, so that we can use (2.18) to get a critical point in $\Omega_{I, J}$ of index $2m - p$. That the critical point does not lie in $\Delta_{I, J}$ follows from (2.22).

\[\square\]

**Proof.** (of Theorem 2.2). For any $m \geq 1$, if $n$ is sufficiently large and $p \leq 2m$, we get a sequence $\{\sigma_i\}$ of simple closed curves with $\sigma_i$ a geodesic in the $g_i$ metric, with index $2m - p$. A subsequence will converge to a closed simple geodesic $\sigma$ with index $2m - p - 2$, $2m - p - 1$, or $2m - p$ and which is freely homotopic in $S$ to $\tilde{\gamma}_{m,n}$. In fact since for different $n$ these are different free homotopy classes in the cylinder-minus-3-holes, for each $k = 2m - p$ there are an infinite number of simple closed geodesics of index $k - 2, k - 1, \text{or } k$.

\[\square\]

**Remark 2.25.** The metric $M_0^2$ described above is not bumpy, as the “middle sphere section” contains a neighborhood of a great circle on the standard sphere. However there is a lot of inessential symmetry in the construction. The curves $\eta_n$ and $\gamma_{m,n}$ will exists for all metrics in a neighborhood of $M_0^2$. In order to get simple closed geodesics of arbitrary high Morse index, all we really need is what comes from Lemma 2.8: For each $m$, if $n$ is sufficiently large we need $2m$ points $P_1, \cdots, P_{2m}$ on $E_n$ and, for each $j < 2m$, a geodesic $\lambda$ on $R_n$ with endpoints on the left side of $R_n$, and crossing $E_n$ once between $P_j$ and $P_{j+1}$; and a geodesic $\psi$ on $R_n$ with endpoints on the right side of $R_n$, and crossing $E_n$ once between $P_j$ and $P_{j+1}$. While we do not know how to prove that there is a bumpy metric with this property, we see
no reason why one should not exist. The construction requires that some noncompact simple geodesic is a limit of unstable geodesics; for this bumpyness of the metric seems irrelevant.

**Appendix A. Convexity of a neighborhood of a strictly stable geodesic**

Let $\gamma \subset M$ be a simple closed geodesic and let $(s, \theta)$ be Fermi coordinates in a neighborhood of $\gamma = \gamma(\theta)$. In these coordinates the metric can be written as $ds^2 + f^2(s, \theta) d\theta^2$. For $\alpha > 1$ and $\phi = \phi(\theta) > 0$ set $F(s, \theta) = s^\alpha \phi^{-\alpha}(\theta)$, then

$$\nabla F = \alpha s^{\alpha-1} \phi^{-\alpha} \frac{\partial}{\partial s} - \alpha s^\alpha \phi \phi^{-\alpha-1} f^{-2} \frac{\partial}{\partial \theta},$$

$$\langle \nabla \frac{\partial}{\partial s} \nabla F, \frac{\partial}{\partial s} \rangle = \alpha (\alpha - 1) s^{\alpha-2} \phi^{-\alpha},$$

$$\langle \nabla \frac{\partial}{\partial \theta} \nabla F, \frac{\partial}{\partial \theta} \rangle = \alpha \phi \phi^{-\alpha+1} s^{\alpha-1} \left( s \frac{f_s}{f} - \alpha \right),$$

$$\langle \nabla \frac{\partial}{\partial s} \nabla F, \frac{\partial}{\partial \theta} \rangle = \alpha s^{\alpha-1} \frac{f_s}{\phi^\alpha} - \alpha s^\alpha \frac{\phi_s}{\phi^{\alpha+1}} + \alpha (\alpha + 1) s^\alpha \frac{\phi^2}{\phi^{\alpha+2}} + \alpha s^\alpha \frac{\phi^2}{\phi^{\alpha+2}} f_s \frac{f}{f}. \tag{A.4}$$

Since $K f = -f_s$ we get by Taylor expansion that

$$\langle \nabla \frac{\partial}{\partial s} \nabla F, \frac{\partial}{\partial \theta} \rangle = -\alpha s^\alpha \frac{L_{\gamma \phi}}{\phi^{\alpha+1}} + \alpha (\alpha + 1) s^\alpha \frac{\phi^2}{\phi^{\alpha+2}} + \alpha \frac{\phi^2}{\phi^{\alpha}} s^\alpha \phi(s^\alpha). \tag{A.5}$$

From this it follows easily that if $\gamma$ is strictly stable and $\phi$ is a positive eigenfunction of $L_{\gamma}$ with eigenvalue $\lambda_1(L_{\gamma}) > 0$, then for $\alpha > 1$ sufficiently large $F$ is convex in a neighborhood $T$ of $\gamma$ and strictly convex in $T \setminus \{\gamma\}$.

Recall that we equip the space of $C^\infty$ metrics on a closed surface $M^2$ with the $C^\infty$-topology and we write $g_i \to g$ if $|g - g_i|_{C^\infty} \to 0$.

An easy consequence of the existence of $F$ is:

**Lemma A.6.** Let $M^2$ be a closed surface with a metric $g$ and suppose that $\gamma \subset M$ is a simple closed strictly stable geodesic. Then there exists $\epsilon > 0$ and an open neighborhood $T$ of $\gamma$ such that if $\tilde{g}$ is a metric on $M$ with $|g - \tilde{g}|_{C^\infty} < \epsilon$, then there exists a unique closed simple geodesic (in the metric $\tilde{g}$) $\tilde{\gamma} \subset T$. Moreover, $\epsilon$ and $T$ can be chosen so that $\tilde{\gamma}$ is strictly stable.

**Proof.** First using the existence of $F$, it follows easily that given a metric $\tilde{g}$ sufficiently close to $g$ there must be a simple closed strictly stable geodesic $\tilde{\gamma}$. Now using $F_{\gamma}$ it follows easily that $\tilde{\gamma}$ is unique. \qed

**Appendix B. Convergence of metrics and geodesics**

**Lemma B.1.** Let $M^2$ be as above with a bumpy metric $g$. For each $L > 0$, there exists at most finitely many closed geodesics of length $< L$. Moreover, if $L$ is not equal to the length of any closed geodesic in $g$, then in a neighborhood of $g$ each metric has precisely as many (simple) closed stable geodesics of length $< L$ as $g$. Finally, if $g_i \to g$ and $\{\gamma_i,k\}$, $\{\gamma_k\}$ are the (simple) closed stable geodesics in $g_i$, $g$, respectively, of length $< L$, then $\gamma_i,k \to \gamma_k$ for $i \to \infty$ and each $k$. 


Proof. If there were an infinite sequence of such (simple) closed geodesics, then it would follow that a subsequence would converge to a closed geodesic with a nontrivial Jacobi field contradicting that the metric is bumpy. In fact it is easy to see (by locally going to a finite cover) that the assumption that the geodesics are simple is not needed for this or anything else in this lemma.

It follows easily from the existence of the convex function $F$ from Appendix A that for each metric in a neighborhood of $g$ there are at least as many (simple) closed stable geodesics of lengths at most $L$ as in $g$. That there are not more (and the last claim) follows from a simple convergence argument together with the assumption that the metric is bumpy. □

**References**

[Ab] R. Abraham, Bumpy metrics, Global Analysis, *Proc. Sympos. Pure Math.* Vol. XIV (1968) 1–3.
[An] D.V. Anosov, Generic properties of closed geodesics, (Russian) *Izv. Akad. Nauk SSSR* Ser. Mat. 46 (1982), no. 4, 675–709, 896.
[Ang] S. Angenent, Parabolic equations for curves on surfaces. II. Intersections, blow up, and generalized solutions, *Ann. of Math.* (2) 133 no. 1 (1991) 171-215.
[Ch] K.-C. Chang, Infinite dimensional Morse theory and multiple solution problems, Birkhäuser, 1993.
[CH1] T.H. Colding and N. Hingston, Geodesic laminations with closed ends on surfaces and Morse index; Kupka-Smale metrics, preprint 2002.
[CH2] T.H. Colding and N. Hingston, in preparation.
[CM1] T.H. Colding and W.P. Minicozzi II, Examples of embedded minimal tori without area bounds, *International Mathematics Research Notices*, vol. 99, no. 20 (1999) 1097-1100.
[CM2] T.H. Colding and W.P. Minicozzi II, Embedded minimal surfaces without area bounds in 3-manifolds, *Proc. of conference on Geometry and Topology, Aarhus 1998. Contemporary Mathematics*, vol. 258 (2000) 107-120.
[Gr] M. Grayson, Shortening embedded curves, *Ann. of Math.* (2) 129, no. 1 (1989) 71-111.
[GgHa] M.J. Greenberg and J.R. Harper, Algebraic Topology, Benjamin/Cummings, Menlo Park, 1981.
[K] W.P.A. Klingenberg, Riemannian geometry. Second edition. de Gruyter Studies in Mathematics, 1. Walter de Gruyter and Co., Berlin, 1995.
[PiRu] J. Pitts and J.H. Rubinstein, Applications of minimax to minimal surfaces and the topology of three-manifolds, *Proc. of the CMA 12*(1987) 137-170.

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