Energy relaxation in galaxies induced by an external environment and/or incoherent internal pulsations

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ABSTRACT
This paper explores the phenomenon of energy relaxation for stars in a galaxy embedded in a high density environment that is subjected continually to perturbations reflecting the presence of other nearby galaxies and/or incoherent internal pulsations. The analysis is similar to earlier analyses of energy relaxation induced by binary encounters between nearby stars and between stars and giant molecular clouds in that the perturbations are idealised as a sum of near-random events which can be modeled as diffusion and dynamical friction. However, the analysis differs in one important respect: because the time scale associated with these perturbations need not be short compared with the characteristic dynamical time $t_D$ for stars in the original galaxy, the diffusion process cannot be modeled as resulting from a sequence of instantaneous kicks, i.e., white noise. Instead, the diffusion is modeled as resulting from random kicks of finite duration, i.e., coloured noise characterised by a nonzero autocorrelation time $t_c$. A detailed analysis of coloured noise generated by sampling an Ornstein-Uhlenbeck process leads to a simple scaling in terms of $t_c$ and an effective diffusion constant $D$. Interpreting $D$ and $t_c$ following early work by Chandrasekhar (1941) (the ‘nearest neighbour approximation’) implies that, for realistic choices of parameter values, energy relaxation associated with an external environment and/or internal pulsations could be important on times short compared with the age of the Universe.

Key words: galaxies: structure – galaxies: kinematics and dynamics

1 MOTIVATION
Dating back to the pioneering work of Chandrasekhar in the 1940’s (cf. Chandrasekhar 1943a), it has been recognised that discreteness effects will induce changes in the energy (and any other collisionless invariants) in a system of stars idealised in a first approximation as a collisionless equilibrium. In particular, by modeling interactions between individual stars as an incoherent sum of binary encounters (gravitational Rutherford scattering), Chandrasekhar computed a characteristic relaxation time $t_R$ on which the energy of a typical star will change by a factor of order unity. For small systems like open clusters and globular clusters, $t_R$ is typically short compared with $t_H$, the age of the Universe, and this energy relaxation has significant observable consequences, e.g., for the development of core-halo structures in globular clusters (cf. Binney & Tremaine 1987).

By contrast, for systems as large as entire galaxies, $t_R$ is typically long compared with $t_H$, so that it has become customary (cf. Binney & Tremaine 1987) to ignore interactions between individual stars. However, such interactions are not the only irregularities which can induce energy relaxation. For example, in the 1950’s Spitzer and Schwarzschild (1951, 1953) argued that interactions of stars with gas clouds in the Galactic disc will tend to increase their random velocities and their distance from the galactic plane on time scales short compared with $t_H$, thus providing an explanation of the observed correlations between kinematics and spectral type for Population I stars (cf. Mihalas & Binney 1981).

The basic physical idea underlying this earlier work is that interactions between stars and/or gas clouds can be idealised as a sum of instantaneous near-random events, resulting in a phase space diffusion where, at least in a first approximation, the root mean squared change in quantities like the energy grows as the square root of time, i.e., $\delta E_{rms} \propto t^{1/2}$. In particular, in his classic review article on ‘Stochastic Problems in Physics and Astronomy,’ Chandrasekhar (1943b) recast the problem in the language of nonequilibrium statistical mechanics, allowing for the combined effects of diffusion and dynamical friction, with the effects of diffusion being modeled as Gaussian white noise, i.e., a sequence of instantaneous random impulses.

The objective of this paper is to argue that there are at least two other sources of near-random irregularities which,
under certain circumstances, can induce significant energy relaxation on time scales short compared with \( t_d \).

(1) Real galaxies are all perturbed to a lesser or greater extent by other nearby galaxies and, under appropriate circumstances, one might expect that these perturbations can be approximated as a superposition of near-random interactions of finite duration, i.e., coloured noise. If the galaxy be situated in a comparatively low density environment, the dominant sources of irregularities may be a small number of satellite galaxies or, perhaps, a single large neighbour which execute near-periodic motions relative to the original galaxy. In this case, it is clearly inappropriate to approximate the perturbations as random events. However, a galaxy embedded in a high density rich cluster will be exposed to a more irregular set of perturbations, associated over the course of time with a larger number of different galaxies, and it would not seem unreasonable to suppose that these perturbations could be modeled as an incoherent sum of near-random interactions.

(2) There is also growing evidence, both numerical and observational, that real galaxies can exhibit finite amplitude oscillations which persist for times long compared with a characteristic dynamical time, \( t_d \). If these oscillations are dominated by a small number of normal or pseudo-normal modes, they should be well approximated as nearly periodic perturbations. If, however, the oscillations involve a comparatively large number of different modes, it would seem that their effects would be better modeled as a sum of near-random perturbations, involving a superposition of a large number of different frequencies. A collection of different forces characterised by a random combination of frequencies combined with random phases is equivalent mathematically to (in general) coloured noise (cf. van Kampen 1981), with a finite autocorrelation time.

Section 2 begins by formulating the problem of energy relaxation abstractly in terms of \( D \), the diffusion constant, \( \eta \), the coefficient of dynamical friction, and \( t_c \), the autocorrelation time, i.e., the characteristic time scale on which the random irregularities change. These quantities are then related to (1) the physical properties of a rich cluster, by assuming that the random perturbations acting on a given galaxy at given instant are associated with the effects of one or two particularly proximate galaxies (Chandrasekhar’s 1941 ‘nearest neighbour approximation’); and (2) the effects of incoherent internal pulsations, by exploiting simple dimensional arguments. Section 3 summarises a numerical investigation of energy relaxation in which diffusion is modeled as coloured noise sampling an Ornstein-Uhlenbeck process, demonstrating the existence of a simple scaling in terms of \( D \) and \( t_c \). Section 4 concludes by showing that, for reasonable choices of parameter values, energy relaxation induced by external perturbations and/or incoherent internal oscillations can be significant over the age of the Universe, and then speculating on possible implications for real galaxies.

2 THEORETICAL CONSIDERATIONS

2.1 The abstract problem

Begin by formulating the problem of energy relaxation abstractly for an ensemble of point masses moving in a time-independent potential \( V(\mathbf{r}) \), each of which is also subjected to dynamical friction and coloured noise, i.e., random kicks of finite duration. This leads naturally to the consideration of a Langevin equation of the form (cf. van Kampen 1981)

\[
\frac{d^2 x^a}{dt^2} = -\frac{\partial V(x^a)}{\partial x^a} - \eta v^a + F^a, \quad (a = x, y, z)
\]  

(1)

where \( \eta \) is the coefficient of dynamical friction and \( F \) is the random force, which is treated as a stochastic variable. Assuming in the usual fashion that \( F \) corresponds to homogeneous Gaussian noise, its statistical properties are characterised completely by its first two moments, namely

\[
\langle F_a(t) \rangle = 0 \quad \text{and} \quad \langle F_a(t_1) F_b(t_2) \rangle = \delta_{ab} K(t_1 - t_2), \quad (a, b = x, y, z).
\]  

(2)

For the special case of white noise, seemingly appropriate, e.g., for modeling gravitational Rutherford scattering of nearby stars, the autocorrelation function \( K \) is proportional to a Dirac delta, so that

\[
K(\tau) = 2\eta \Theta D(\tau).
\]  

(3)

Here the normalisation ensures that the friction and noise are related by a Fluctuation-Dissipation Theorem in terms of a ‘temperature’ \( \Theta \). It is well known that such a Langevin description is equivalent (cf. Riskin 1989) to a Fokker-Planck equation of the form

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} - \eta v \frac{\partial^2 f}{\partial v^2} = -\frac{\partial}{\partial v} \langle \eta v f \rangle + D \frac{\partial^2 f}{\partial v^2},
\]  

(4)

where

\[
D \equiv \int_{-\infty}^{\infty} d\tau K(\tau) = 2\Theta \eta
\]  

(5)

is the diffusion constant.

For coloured noise, the forces at nearby instants of time remain correlated so that \( K \) is no longer proportional to a Dirac delta. As a simple example, one can consider the so-called Ornstein-Uhlenbeck process (cf. Uhlenbeck & Ornstein 1930, van Kampen 1981), for which \( K \) decreases exponentially in time. Here

\[
K(\tau) = \frac{\eta}{t_c} \Theta \exp(-|\tau|/t_c),
\]  

(6)

where the autocorrelation time \( t_c \) sets the time scale on which the fluctuating random forces \( F \) change appreciably. The normalisation entering into this equation ensures that the effective diffusion constant \( D \), again defined by eq. (5), is fixed completely by \( \eta \) and \( \Theta \).

2.2 Intuitive expectations

The effects of energy diffusion induced by friction and white noise can be derived directly from the Fokker-Planck description associated with the Langevin equation (1). As shown, e.g., in Habib, Kandrup, & Mahon (1997), the Fokker-Planck equation (4) implies that the mean squared change in energy associated with multiple noisy integrations of the same initial condition satisfies

\[
\frac{d \langle \delta E^2 \rangle}{dt} = D \langle v^2 \rangle + \eta \langle \langle v^2 \rangle^2 \rangle
\]  

(7)

However, at early times one can approximate \( \langle v^2 \rangle \approx \langle v^2 \rangle^2 \) and \( \langle v^2 x^2 \rangle \approx \langle v^2 \rangle \langle x^2 \rangle \), so that

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\[ \frac{d(\delta E)^2}{dt} \approx D(v^2). \]  

(8)

Noting further that \( \langle v^2 \rangle \sim |E_0| \), where \( E_0 \) is the initial energy, one then infers that

\[ \frac{d(\delta E)^2}{dt} \sim D|E_0|, \]

(9)

which implies a fractional root mean squared change in energy

\[ \frac{\delta E_{\text{rms}}}{|E_0|} \sim \left( \frac{Dt}{|E_0|} \right)^{1/2}. \]  

(10)

Given that an ensemble of different initial conditions each evolved in the presence of friction and noise can be viewed as a sum of random samplings of Langevin simulations with different initial conditions, each satisfying eq. (10), it is clear as a sum of random samplings of Langevin simulations with evolved in the presence of friction and noise can be viewed.

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The obvious question is: how will this result be changed if one considers coloured noise with a finite autocorrelation time? Recent work indicates that coloured noise impacts orbits via a resonant coupling between the natural frequencies associated with the perturbation and the natural frequencies of the orbits (Pogorelov & Kandrup 1999, Kandrup, Pogorelov, & Sideris 2000). White noise is characterised by a flat spectral density with power at all frequencies (the square of the spectral density is the Fourier transform of the autocorrelation function \( K \)), so that it can couple to more or less anything. Replacing white noise by coloured Ornstein-Uhlenbeck noise with a nonzero autocorrelation time yields instead a spectral density

\[ S(\omega) \sim \frac{\alpha^2}{\omega^2 + \alpha^2}, \]

(11)

with \( \alpha = t_c^{-1} \), which cuts off for \( \omega \gg \alpha \). Given that the characteristic frequencies associated with the unperturbed orbits typically scale as \( t_D^{-1} \), with \( t_D \) a characteristic dynamical time, one might therefore expect (i) that coloured noise with \( t_c \ll t_D \) will have virtually the same effect as white noise; but (ii) that coloured noise with the same \( D \) but \( t_c > t_D \) will be significantly less efficient than white noise in inducing energy relaxation. This in fact turns out to be the case. The numerical experiments described in the following section lead to the simple scaling

\[ \frac{\delta E_{\text{rms}}}{|E|} \sim \left( \frac{Dt}{|E|} \right)^{1/2} \left\{ \begin{array}{ll} 1 & \text{for } t_c < t_D \\ \left( \frac{t_c}{t_D} \right) & \text{for } t_c > t_D. \end{array} \right. \]

(12)

2.3 Connection with physical parameters

It remains to relate \( D, \eta, \) and \( t_c \) to observable parameters characterising a real galaxy, considering first the effects of a high density environment.

Begin by observing that, on dimensional grounds (cf. eq. [6]), the diffusion constant

\[ D \sim F^2 t_c, \]

where \( F \) denotes the characteristic magnitude of the force per unit mass associated with the perturbing galaxies. Following Chandrasekhar (1941) or Chandrasekhar & von Neumann (1942), suppose now that these effects are associated primarily with one or two nearby galaxies. It then follows on dimensional grounds that

\[ F \sim \frac{G M_\delta}{R^2}, \]

(14)

and

\[ t_c \sim \frac{R_0}{V_\nu}, \]

(15)

where \( M_\delta \) represents the mass of a typical perturbing galaxy, \( R_0 \) the typical separation between galaxies, and \( V_\nu \) the typical relative velocity between galaxies. This implies that

\[ D \sim \frac{G^2 M_\delta^2}{R^3 V_\nu}. \]

(16)

However, by interpreting the ‘temperature’ \( \Theta \) as a characteristic squared velocity with which the perturbing galaxies move through space, one has \( \Theta \sim V^2 \), so that

\[ \eta \sim \frac{G^2 M_\delta^2}{R^3 V^2}. \]

(17)

It is easily seen that an application of a similar argument to stars moving within a galaxy leads essentially to the standard expression for the coefficient of dynamical friction associated with binary encounters. Specifically, if \( M_\delta \) is reinterpreted as the mass \( m \) of a typical star and \( V_\nu \) as a characteristic relative velocity \( V_s \) between nearby stars, and \( R_0 \) is replaced by the typical separation \( \sim n^{-1/3} \) between neighbouring stars, one concludes that

\[ \eta = t_r^{-1} \sim \frac{G^2 m^2 n}{V_s^2}. \]

(18)

where \( n \) represents a characteristic stellar density. Equation (18) agrees (cf. Binney & Tremaine 1987) with the standard expression for \( t_r \), modulo the absence of the Coulomb logarithm ln \( A \) which would ordinarily appear in the numerator. This additional factor, which arises in a more careful analysis of discreteness effects, reflects the fact that weaker, more distant interactions also contribute to energy relaxation, thus reducing the value of \( t_r \) somewhat. In this sense, one infers that an analysis of energy relaxation based on eqs. (15) - (17) will, if anything, underestimate the importance of environmental effects.

Estimating the effects of internal pulsations seems more uncertain, but one can proceed using dimensional analysis. Clearly one can write

\[ F \sim \alpha \frac{G M_\delta}{R^2}, \quad t_c \sim \beta t_D, \quad \text{and } \Theta \sim \gamma V^2, \]

(19)

where \( \alpha, \beta, \) and \( \gamma \) represent dimensionless constants. Here the constant \( \alpha \) essentially sets the characteristic amplitude of the irregularities associated with the internal pulsations as compared with the bulk force associated with the galaxy as a whole and, as such, should be small compared with unity. One would thus anticipate that \( \alpha \ll 1 \). Appropriate values for the constants \( \beta \) and \( \gamma \) are less clear, but it would at least seem reasonable to assume that \( \beta \) is not significantly larger than unity, i.e., the autocorrelation time \( t_c \) is not much larger than the dynamical time \( t_D \). The characteristic size of \( \gamma \), which enters into \( D \) but not \( \eta \), is more
difficult to estimate. Fortunately, however, it follows from the simulations described in Section 3 (cf. eq. [12]) that the rate of energy relaxation is fixed completely by $D$ and $t_c$, independent of $\eta$, so that its precise value is immaterial. In any event, given this scaling one infers that

$$D \sim \alpha^2 \beta \frac{G^2 M^2}{R^3 V} \sim \alpha^2 \beta \frac{G^2 M^2}{R^3 V},$$

where one has used the fact that $t_D \sim R_c/V_r$.

3 NUMERICAL EXPERIMENTS

3.1 What was computed

Solutions to the Langevin equation (1) for a constant coefficient of dynamical friction and coloured noise sampling the Ornstein-Uhlenbeck process (7) were obtained for three different classes of potentials. The most interesting astrophysically, but by far the most expensive computationally, were the triaxial generalisations of the Dehnen (1993) potentials, which have been considered extensively by Merritt and collaborators (e.g. Merritt & Fridman 1996). These correspond to potentials generated self-consistently from the triaxial mass density

$$\rho(m) = \frac{(3 - \gamma)}{4\pi abc} m^{-\gamma} (1 + m)^{-(4 - \gamma)}$$

with

$$m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

for various choices of axis ratios $a:b:c$ and cusp index $\gamma$. Because of the cost associated with integrations of this potential, a careful exploration of parameter space was prohibitive, so that more systematic investigations were performed instead for the toy potentials

$$V(x, y, z) = \frac{1}{2} \left( a^2 x^2 + b^2 y^2 + c^2 z^2 \right) - \frac{M_{BH}}{\sqrt{r^2 + c^2}},$$

with $r^2 = x^2 + y^2 + z^2$, which have been shown (Kandrup & Sideris 2000) to reproduce much of the observed behaviour for orbit ensembles evolved in the triaxial Dehnen potentials. To test the generality of the basic conclusions, integrations were also performed for a completely different class of computationally inexpensive potentials with

$$V(x, y, z) = -(x^2 + y^2 + z^2) + \frac{1}{4} (x^2 + y^2 + z^2)^2$$

$$- \frac{1}{4} (x^2 y^2 + ay^2 z^2 + b z^2 x^2).$$

These potentials, which were explored systematically in Kandrup, Pogorelov, & Sideris (2000), constitute three-dimensional generalisations of the two-dimensional dihedral potential first considered by Armbruster, Guckenheimer, & Kim (1989).

The experiments that were performed each involved selecting ensembles of 8000 initial conditions, all with the same energy, and, for given choices of $\eta$, $\Theta$, and $t_c$, integrating these initial conditions into the future for a time $t \sim 100 - 200 t_D$. In order to determine whether energy relaxation impacts regular and chaotic orbits in the same fashion, regular and chaotic ensembles were considered separately.

For the potentials considered, the typical particle speed $v$ and the dynamical time $t_D$ are both of order unity. For this reason, most of the experiments assumed that $\Theta \sim 1$. The coefficient of dynamical friction $\eta$, which defines the diffusion constant $D = 2\Theta \eta$, was allowed to vary in the range $10^{-8} \leq \eta \leq 10^{-2}$. The autocorrelation time $t_c$ was chosen to satisfy $10^{-2} \leq t_c \leq 10^3$. It was verified explicitly that integrations with autocorrelation times as short as $t_c = 0.01$ yield results essentially indistinguishable from integrations with white noise.

White noise integrations were performed using a fixed time step algorithm developed by Griner, Strittmatter, & Honerkamp (1988). Coloured noise integrations were performed using an extension of this algorithm developed by I. V. Pogorelov as part of his Ph. D. thesis (see Pogorelov & Kandrup 1999). The basic idea underlying this extension is to (i) use a pseudo-random number generator to generate white noise $X(t)$, and then (ii) transform $X(t)$ into coloured noise $Y(t)$ sampling the Ornstein-Uhlenbeck process by solving the stochastic differential equation

$$\frac{dY}{dt} + \alpha Y = X(t),$$

with $\alpha = t_c^{-1}$. Viewed as an initial value problem (cf. Chandrasekhar 1943b), eq. (25) has the property that any non-trivial dependence on initial conditions dies away exponentially on a time scale $\alpha^{-1}$, so that, at late times, the random variable $Y(t)$ satisfies

$$\langle Y(t) \rangle = 0 \quad \text{and} \quad \langle Y(t_1)Y(t_2) \rangle = 2 \exp(-\alpha |t_1 - t_2|).$$

In other words, the late time solution to eq. (25) for $X(t)$ given as white noise is coloured noise sampling the Ornstein-Uhlenbeck process $Y(t)$.

3.2 What was found

Perhaps the single most important conclusion derived from these simulations is that the observed behaviour of different orbit ensembles is independent of energy and choice of potential, i.e., the results appear to be universal. Modulo the numerical constant $A$ entering into the relation $(r^2) = A E$, which is exploited in the derivation of eqs. (9) and (10), different ensembles yielded very similar results. This is not surprising. When considering the problem of diffusion on a constant energy hypersurface, different types of orbits and/or different potentials can yield very different behaviour (cf. Siopis & Kandrup 2000): Regular orbits are confined to three-dimensional phase space tori whereas chaotic orbits move on a higher-dimensional phase space hypersurface which is impacted by a complex Arnold web, the details of which can depend sensitively on the choice of potential and initial condition. However, energy relaxation involves motion orthogonal to the hypersurfaces of constant energy, so that most of these details are irrelevant. As discussed more carefully below, for very long autocorrelation times $t_c$ it does appear that the qualitative results depend at all on the form of the potential or whether the orbits be regular as opposed to chaotic.

It was also found that, as would be expected, coloured noise acts diffusively, so that the root mean squared energy for an orbit ensemble grows as the square root of the integration time, i.e., $\delta E_{rms} \propto t^{1/2}$. This is, e.g., evident from
different choices of potential for a total time $t$ for orbits with $\Theta = 1$ for several different choices of $\eta$ and $t_c$. (a) The top curve has $\eta = 10^{-4}$ and $t_c = 10.0$. The lower has $\eta = 10^{-5}$ and $t_c = 1.0$. (b) The top curve has $\eta = 10^{-5}$ and $t_c = 0.1$. The lower has $\eta = 10^{-4}$ and $t_c = 1.0$.

FIG. 1, which exhibits data for an ensemble of 8000 chaotic orbits with $E = 4$ and $a = b = 1$ in the presence of coloured noise with $\Theta = 1$ for several different choices of $\eta$ and $t_c$. Similarly, as for the case of white noise, one finds that the individual values of $\Theta$ and $\eta$ are irrelevant. For fixed autocorrelation time $t_c$, all that matters is the diffusion constant $D = 2\Theta\eta$. Moreover, as for the case of white noise, the amount of energy relaxation, as probed by $\delta E_{\text{rms}}$, scales as $D^{1/2}$ for fixed $t_c$. This fact is illustrated in FIG. 2 which, for the ensemble of initial conditions used to generate FIG. 1, exhibits the fractional root mean squared change in energy, $|\delta E_{\text{rms}}|/|E_0|$, at time $t = 256$ as a function of $D$ for various choices of $t_c$.

For fixed diffusion constant $D$, the dependence on $t_c$ is less trivial. As expected, one discovers that, for $t_c \ll t_D$, the effects of coloured noise are virtually identical to the effects of white noise, which can be viewed as a singular limit of eq. (5) or (11). Alternatively, for $t_c \gg t_D$, the amount of energy relaxation for fixed $D$ is significantly reduced. This is, e.g., illustrated in FIG. 3, which, once again for the ensemble used in FIG. 1, exhibits $|\delta E_{\text{rms}}|/|E_0|$ at time $t = 256$ as a function of $t_c$ for various choices of $D$. It is evident that, even for $t_c \gg t_D$, the slopes of the curves in this Figure are not completely constant, but it does appear that, for $0 \leq \log_{10} t_c \leq 2$, i.e., for $t_c$ between roughly $t_D$ and $100t_D$, the amount of energy relaxation scales as $t_c^{-1}$. In this approximation, the results illustrated in FIGS. 1 - 3 yield the scaling summarised in eq. (12).

It is well known that, as far as energy relaxation is concerned, white noise impacts regular and chaotic orbits identically. This also appears to be the case for coloured noise, at least for values of the autocorrelation time $t_c < 30t_D$ or so. Interestingly, however, there are distinct differences for very large values of $t_c$. As illustrated in FIG. 4, it appears that, for $t_c > 30t_D$ or so, regular orbits are impacted more than chaotic orbits. In particular, for very large $t_c$ the scaling relation (12) drastically underestimates the importance.
and the dynamical time $t_D$ may be taken to satisfy
\[ t_D^2 \sim \frac{1}{G\rho} \sim \frac{R_g^3}{(GM_\phi^2)} \]  
(29)

Substitution of eqs (28) and (29) into eq. (27) then leads to the conclusion that
\[ \frac{\delta E_{rms}}{|E|} \sim \left( \frac{R_g}{R_e} \right)^2 \left( \frac{t_D}{t_c} \right)^{1/2} \left( \frac{t}{t_D} \right)^{1/2} \]  
(30)

In a similar fashion, one can allow for the effects of internal pulsations, assuming, as in Section 2.3, that $t_c$ is less than or comparable to $t_D$. In this case, $\delta E_{rms}$ satisfies
\[ \frac{\delta E_{rms}}{|E|} \sim \left( \frac{Dt}{|E|} \right) \sim \alpha^2 \beta \left( \frac{GM_\phi}{R_g^3} \right) t_D t, \]  
(31)

and by again exploiting eqs. (28) and (29), one concludes that
\[ \frac{\delta E_{rms}}{|E|} \sim \alpha \beta^{1/2} \left( \frac{t}{t_D} \right)^{1/2}, \]  
(32)

Equations (30) and (32) allow one to infer the typical degree of energy relaxation that can be expected over a time scale $\sim t_H$, the age of the Universe.

Consider, e.g., the effects of a dense cluster environment where, not unrealistically, the typical velocity dispersion associated with stars in an individual galaxy is comparable in magnitude to the velocity dispersion of galaxies moving in the cluster, and where the typical distance between galaxies is of order six times the size of a typical galaxy. In this case, $R_g \sim 6 R_\phi$ and $t_c \sim 6 t_D$, so that, assuming that $t_H \sim 100 t_D$, one would anticipate a fractional root mean squared energy $\delta E_{rms}/|E| \sim 0.1$ over the lifetime of the galaxy. Note that, as discussed in Section 2.3, this estimate likely underestimates by a small amount the efficacy of energy relaxation induced by random interactions with nearby galaxies.

Similarly, for the case of internal pulsations, it might seem plausible to assume that the characteristic fractional amplitude of the perturbations $\alpha \sim 10^{-2}$ and that $t_c$ is comparable to $t_D$, so that $\beta \sim 1$. One would then anticipate once again that, within a time $t_H \sim 100 t_D$, the fractional root mean squared change in energy will have grown to a value $\delta E_{rms}/|E| \sim 0.1$.

These estimates changes in energy by of order 10% over the age of the Universe – are not huge, but they could prove important in real galaxies. Recent numerical work suggests that, because of the role of chaos in certain key phase space regions, e.g., near corotation and other resonances in spiral galaxies (cf. Wozniak 1993) or in the central regions of cuspy-core ellipticals (cf. Merritt & Fridman 1996), it may be hard for a galaxy to settle down towards a true collisionless equilibrium. However, it would seem substantially easier (cf. Siopis & Kandrup 2000) for the system to evolve towards a configuration which, albeit not corresponding to a true equilibrium, could persist in isolation as a near-equilibrium for times long compared with the age of the Universe.

The important point, then, is that such near-equilibria, even if nearly stable as isolated entities over intervals $\sim t_H$, could be destabilised by low amplitude perturbations of the form to which real galaxies are necessarily subjected. There is compelling numerical evidence that chaotic orbits are very susceptible to perturbations which induce phase space
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transport on (or very nearly on) the constant energy hypersurfaces \((cf.\ Kandrup,\ Pogorelov,\ &\ Sideris\ 2000;\ Siopis\ &\ Kandrup\ 2000)\), thus facilitating extrinsic diffusion along the Arnold web. Such diffusion will occasion changes in the phase space density, which in turn alters the form of the gravitational potential, which could in principle triggering a nontrivial secular evolution.

One might, however, argue that this need not have significant implications for bulk evolution since most of the phase space is (presumably) dominated by regular orbits which \((cf.\ Binney\ 1978)\) support the skeleton of the system and which, significantly, are not susceptible to such phase space diffusion. The key recognition, therefore, is that energy relaxation impacts both regular and chaotic orbits; and that even comparatively small effects acting on regular orbits, combined with larger effects acting on the (presumably) smaller measure of chaotic orbits, could effectively destabilise a near-equilibrium on a time scale \(<\ t_H\).

One plausible scenario might involve the effects of internal oscillations triggered by a near-collision with another galaxy. Numerical simulations \((cf.\ Vesperini\ &\ Weinberg\ 2000)\) suggest that such near-collisions can serve to induce comparatively long lived, large amplitude oscillations which could, \(e.g.,\) account for the offset nuclei and/or lopsided and warped discs observed in certain galaxies \((cf.\ Kornreich,\ Haynes,\ &\ Lovelace\ 1998\ and\ references\ cited\ therein)\); and even if these oscillations are dominated by a small number of lower order normal modes, they should be accompanied by a larger number of higher order modes which could reasonably be modeled as near-random excitations.

In this context, it important to note that energy relaxation induced by near-random internal irregularities is likely to be more important for the bulk structure of a galaxy than more systematic, nearly periodic effects. Periodic, or near-periodic, driving, associated with a small number of companion objects or with a small number of normal or pseudo-normal modes, involves perturbations with a countable set of discrete frequencies. The obvious point, then, is that since these perturbations have their effect because of a resonant coupling to orbits in the galaxy, they can only have a large effect on orbits which have considerable power at or near these special frequencies and/or harmonics thereof. Chaotic orbits typically have broad band Fourier spectra \((cf.\ Tabor\ 1989)\) and, as such, can couple efficiently to such periodic disturbances \((cf.\ Kandrup,\ Abernathy,\ &\ Bradley\ 1995)\). However, for regular orbits the power is concentrated at a few special frequencies and, if these frequencies are not in a resonance or near-resonance with the frequencies of the perturbations, the perturbation will have a comparatively minimal effect. Periodic driving could perhaps prove important for chaotic orbits in general or for regular orbits in specific phase space regions. However, one might anticipate generically that it will be less important on the overall phase space structure which, presumably, is dominated by regular orbits with frequencies that are not in resonance with the periodic driving.

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REFERENCES

Armbruster, D., Guckenheimer, J., Kim, S. 1989, Phys. Lett. A 140, 416.
Binney, J., 1978, Comments Astrophys. 8, 27.
Binney, J., Tremaine, S. 1987, Galactic Dynamics. Princeton University press, Princeton.
Chandrasekhar, S. 1941, ApJ 94, 511
Chandrasekhar, S. 1943a, Principles of Stellar Dynamics. University of Chicago Press, Chicago.
Chandrasekhar, S. 1943b, Rev. Mod. Phys. 15, 1
Chandrasekhar, S., von Neumann, J. 1942, ApJ 95, 489
Dehnen, W. 1993, MNRAS 265, 250
Griner, A., Strittmatter, W., Honerkamp, J. 1988, J. Stat. Phys. 51, 95
Habib, S., Kandrup, H. E., Mahon, M. E. 1997, ApJ 480, 155
Kandrup, H. E., Abernathy, R. A., Bradley, B. O. 1995, Phys. Rev. E 51, 5287
Kandrup, H. E., Pogorelov, I. V., Sideris, I. V. 2000, MNRAS 311, 719
Kandrup, H. E., Sideris, I. V. 2000, preprint
Kornreich, D. A., Haynes, M. P., Lovelace, R. V. 1998, AJ 116, 2154
Meritt, D., Fridman, T. 1996, ApJ 460, 136
Mihalas, D., Binney, J. 1981, Galactic Astronomy. Freeman, San Francisco.
Pogorelov, I. V., Kandrup, H. E. 1999, Phys. Rev. E 60, 1657
Risken, H. 1989, The Fokker-Planck Equation. Springer, Berlin
Spitzer, L., Schwarzchild, M. 1951, ApJ 114, 385
Spitzer, L., Schwarzchild, M. 1953, ApJ 118, 106
Tabor, M. 1989, Chaos and Integrability in Nonlinear Dynamics. Wiley, New York.
Uhlenbeck, G. E., Ornstein, L. S. 1930, Phys. Rev. 36, 823
van Kampen, N. G. 1981, Stochastic Processes in Physics and Chemistry. North Holland, Amsterdam.
Vesperini, E., Weinberg, M. D. 2000, ApJ 534, 598
Wozniak, H. 1993, in Ergodic Concepts in Stellar Dynamics, ed. V. G. Gurzadyan & D. Pfenniger. Springer, New York, p. 264. 

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