On equivariant versions of higher order orbifold Euler characteristics

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Abstract

There are (at least) two different approaches to define equivariant analogue of the Euler characteristic for a space with a finite group action. The first one defines it as an element of the Burnside ring of the group. The second approach emerged from physics and includes the orbifold Euler characteristic and its higher order versions. Here we suggest a way to merge the two approaches together defining (in a certain setting) higher order Euler characteristics with values in the Burnside ring of a group. We give Macdonald type equations for these invariants. We also offer generalized (“motivic”) versions of these invariants and formulate Macdonald type equations for them as well.

1 Introduction

Let $X$ be a topological space (good enough, say, a real subanalytic variety) with an action of a finite group $G$. There are (at least) two different approaches to define equivariant analogue of the Euler characteristic for the
pair $(X, G)$. The first one ([13]) defines the equivariant Euler characteristic $\chi^G(X)$ as an element of the Burnside ring $A(G)$ of the group $G$. The second approach emerged from physics (the string theory of orbifolds: [7]). The orbifold Euler characteristic $\chi^{\text{orb}}(X, G)$ is defined through the fixed point sets of some subgroups of $G$ and is an integer. Higher order (orbifold) Euler characteristics were introduced in [1] and [6] (also as integers). They can be defined through the fixed point sets of collections of commuting elements in $G$. Here we suggest a way to merge the two approaches together.

Through this paper we consider the Euler characteristic $\chi(\cdot)$ defined as the alternating sum of the dimensions of the cohomology groups with compact support. This Euler characteristic is not a homotopy invariant in the usual sense, but an invariant of the homotopy type defined in terms of proper maps. It is an additive function on the the algebra of ("good") spaces. There is the universal additive invariant on the algebra of complex constructible sets. It takes values in the Grothendieck ring $K_0(\text{Var}_C)$ of complex quasi-projective varieties and can be regarded as a generalized ("motivic") Euler characteristic $\chi_g(X)$. There were defined generalized ("motivic") analogues of the orbifold Euler characteristic and of its higher order generalizations. First it was essentially made in [2] (for the Hodge-Deligne polynomial) and then formulated precisely in [10] and [12]. These higher order generalized Euler characteristics take values in a certain modification of the Grothendieck ring $K_0(\text{Var}_C)$ of complex quasi-projective varieties.

The usual Euler characteristic satisfies the Macdonald type equation

$$1 + \sum_{n=1}^{\infty} \chi(S^n X) \cdot t^n = (1 - t)^{-\chi(X)},$$

where $S^n X = X^n/S_n$ is the $n$th symmetric power of $X$ (see, e.g., [15]). (A Macdonald type equation for an invariant expresses the generating series of the values of the invariant for the symmetric powers of a space (or for their analogues) as a series does not depending on the space in the power equal to the value of the invariant for the space itself.) There are Macdonald type equations for the orbifold Euler characteristic and for its higher order analogues ([20], [19]) and also for the equivariant Euler characteristic with values in the Burnside ring $A(G)$ of $G$ (see Lemma [1] below). An analogue of these equations for the generalized ("motivic") higher order Euler characteristics was obtained in [12]. It was formulated in terms of the (natural) power structure over the Grothendieck ring $K_0(\text{Var}_C)$ of complex quasi-projective
Here we define (in a certain setting) higher order Euler characteristics with values in the Burnside ring of a group. We give Macdonald type equations for these invariants. We also offer generalized ("motivic") versions of these invariants and formulate Macdonald type equations for them as well.

2 Preliminaries

**Burnside ring** \( A(G) \) of a finite group \( G \) is the Grothendieck ring of finite \( G \)-sets: see, e.g., [5]). As an abelian group the Burnside ring \( A(G) \) is freely generated by the classes \([G/H]\) of the quotients \( G/H \) for representatives of the conjugacy classes \( h \) of subgroups of \( G \). The multiplication is defined by the cartesian product with the diagonal action of \( G \). There is a natural power structure over the Burnside ring \( A(G) \): see, e.g., [11]. A power structure over a ring \( R \) defines a meaning of an expression of the form \((1+a_1t+a_2t^2+\ldots)m\), where \( a_i \) and \( m \) are elements of \( R \), as an element of \( 1 + tR[[t]] \). In particular, for a finite \( G \)-set \( X \) one has

\[
(1 - t)^{-[X]} = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots ,
\]

where \( S^kX = X^k/S_k \) is the \( k \)th symmetric power of the \( G \)-set \( X \) with the natural ("diagonal") action of \( G \).

Let \( X \) be a \( G \)-space. For a point \( x \in X \), let \( G_x = \{g \in G : g \cdot x = x\} \) be the isotropy subgroup of the point \( x \). For a subgroup \( H \subset G \) let \( X^H = \{x \in X : Hx = x\} \) be the fixed point set of \( H \) (\( X^H = \{x \in X : H \subset G_x\} \)) and let \( X^{(H)} = \{x \in X : G_x = H\} \) be the set of points with the isotropy group \( H \). Let \( \text{Conjsub} G \) be the set of the conjugacy classes of subgroups of \( G \). For a conjugacy class \( h \in \text{Conjsub} G \), let \( X^h = \{x \in X : x \in X^H \text{ for a subgroup } H \in h\} \), \( X^{(h)} = \{x \in X : G_x \in h\} \).

The **equivariant Euler characteristic** of a (good enough) \( G \)-space \( X \) is defined by

\[
\chi^G(X) := \sum_{h \in \text{Conjsub} G} \chi(X^{(h)}/G)[G/H] \in A(G) ,
\]

where \( H \) is a representative of the conjugacy class \( h \).

The **orbifold Euler characteristic** \( \chi^{\text{orb}}(X, G) \) of the \( G \)-space \( X \) is defined,
e.g., in [1], [13]:
\[ \chi^{\text{orb}}(X, G) = 1 / |G| \sum_{(g_0, g_1) \in G \times G : g_0 g_1 = g_1 g_0} \chi(X^{(g_0, g_1)}) = \sum_{[g] \in G_*} \chi(X^{(g)}/C_G(g)) \in \mathbb{Z}, \]

where \( G_* \) is the set of the conjugacy classes of elements of \( G \), \( g \) is a representative of the class \([g]\), \( C_G(g) = \{ h \in G : h^{-1}gh = g \} \) is the centralizer of \( g \), and \( \langle g \rangle \) and \( \langle g_0, g_1 \rangle \) are the subgroups of \( G \) generated by the corresponding elements.

The higher order Euler characteristics of \((X, G)\) are defined recursively by:
\[ \chi^{(k)}(X, G) = 1 / |G| \sum_{g \in G^{k+1}, g_i g_j = g_j g_i} \chi(X^{(g)}) = \sum_{[g] \in G_*} \chi^{(k-1)}(X^{(g)}/C_G(g)), \]

where \( g = (g_0, g_1, \ldots, g_k) \), \( \langle g \rangle \) is the subgroup generated by \( g_0, g_1, \ldots, g_k \), and \( \chi^{(0)}(X, G) \) is defined as the usual Euler characteristic \( \chi(X/G) \) of the quotient. The usual orbifold Euler characteristic \( \chi^{\text{orb}}(X, G) \) is the Euler characteristic \( \chi^{(1)}(X, G) \) of order 1.

For a \( G \)-space \( X \), the cartesian power \( X^n \) carries the natural action of the wreath product \( G_n = G \wr S_n = G^n \rtimes S_n \) generated by the natural action of the symmetric group \( S_n \) (permutting the factors) and by the natural (componentwise) action of the cartesian power \( G^n \). The pair \((X_n, G_n)\) should be (or can be) considered as an analogue of the symmetric power of the pair \((X, G)\). One has the following Macdonald type equation (see [19, Theorem A])
\[ \sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n = \left( \prod_{r_1, \ldots, r_k \geq 1} (1 - t^{r_1 r_2 \cdots r_k} r_2 r_3 \cdots r_{k+1}^{k-1}) \right)^{-\chi^{(k)}(X, G)}. \]

When \( k = 0 \), one gets the equation (11) for the quotient \( X/G \).

Let \( K_0(\text{Var}_\mathbb{C}) \) be the Grothendieck ring of complex quasi-projective varieties (see, e.g., [3]) and let \( \mathbb{L} \) be the class in \( K_0(\text{Var}_\mathbb{C}) \) of the complex affine line. Let \( K_0(\text{Var}_\mathbb{C})[\mathbb{L}^s]_{s \in Q} \) be the modification of the ring \( K_0(\text{Var}_\mathbb{C}) \) obtained by adding all rational powers of \( \mathbb{L} \). (This includes \( \mathbb{L}^{-1} \) and thus \( K_0(\text{Var}_\mathbb{C})[\mathbb{L}^s]_{s \in Q} \) contains the localization \( K_0(\text{Var}_\mathbb{C})_{(\mathbb{L})} \) of the ring \( K_0(\text{Var}_\mathbb{C}) \) by \( \mathbb{L} \). Pay attention that \( \mathbb{L} \) is a zero divisor in \( K_0(\text{Var}_\mathbb{C}) \) (3) and therefore the natural map \( K_0(\text{Var}_\mathbb{C}) \to K_0(\text{Var}_\mathbb{C})_{(\mathbb{L})} \) is not an embedding.)
Now let $X$ be a smooth quasi-projective variety of dimension $d$ with an (algebraic) action of the group $G$. To define the higher order generalized (orbifold) Euler characteristics of the pair $(X, G)$, one has to use the so-called age (or fermion) shift $F^g_x$ of an element $g \in G$ at a fixed point $x$ of $g$ defined in [22]. The element $g$ acts on the tangent space $T_x X$ as an automorphism of finite order. This action on $T_x X$ can be represented by a diagonal matrix 
\[ \text{diag}(\exp(2\pi i \theta_1), \ldots, \exp(2\pi i \theta_d)) \] 
with $0 \leq \theta_j < 1$ for $j = 1, 2, \ldots, d$ ($\theta_j$ are rational numbers). The age shift is defined by $F^g_x = \sum_{j=1}^d \theta_j \in \mathbb{Q}$.

For $g \in G$, let the set of connected components of the fixed point set $X^{(g)}$ consist of $N_g$ orbits, and let $X^{(g)}_1, X^{(g)}_2, \ldots, X^{(g)}_{N_g}$ be the unions of the components of each of the orbits. For $1 \leq \alpha_g \leq N_g$ let $F^g_{\alpha_g}$ be the age shift $F^g_x$ at a point of $X^{(g)}_{\alpha_g}$ (this shift does not depend on the point $x \in X^{(g)}_{\alpha_g}$).

For a rational number $\varphi_1 \in \mathbb{Q}$, the generalized orbifold Euler characteristic of weight $\varphi_1$ of the pair $(X, G)$ is defined by

\[ [X, G]_{\varphi_1} := \sum_{(g) \in G} \sum_{\alpha_g = 1}^{N_g} [X^{(g)}_{\alpha_g} / C_G(g)] \cdot \mathbb{L}^{\varphi_1 F^g_{\alpha_g}} \in K_0(\text{Var}_C)[\mathbb{L}]_{s \in \mathbb{Q}}. \tag{6} \]

For $\varphi_1 = 1$ one gets the definition of the generalized orbifold Euler characteristic from [10] inspired by the definition of the orbifold Hodge-Deligne polynomial from [2]. (This generalized orbifold Euler characteristic maps to the orbifold Hodge-Deligne polynomial by the natural ring homomorphism $e : K_0(\text{Var}_C) \to \mathbb{Z}[u, v]$.) For $\varphi_1 = 1$ one gets the so called inertia stack class: see, e.g., [8].

Let $\underline{\varphi} = (\varphi_1, \varphi_2, \ldots)$ be a fixed sequence of rational numbers. The generalized (orbifold) Euler characteristics of order $k$ of weight $\underline{\varphi}$ of the pair $(X, G)$ can be defined recursively by

\[ [X, G]^k_{\underline{\varphi}} := \sum_{(g) \in G} \sum_{\alpha_g = 1}^{N_g} [X^{(g)}_{\alpha_g} / C_G(g)]^{k-1} \cdot \mathbb{L}^{\varphi_1 F^g_{\alpha_g}} \in K_0(\text{Var}_C)[\mathbb{L}]_{s \in \mathbb{Q}}, \tag{7} \]

where $[X, G]^1_{\underline{\varphi}} := [X, G]_{\varphi_1}$ is the generalized orbifold Euler characteristic given by (6). (Alternatively one can start from $k = 0$ using the definition $[X, G]^0_{\underline{\varphi}} = [X, G]^0 = [X/G]$.)

One has the following Macdonald type equations (12)

\[ \sum_{n \geq 0} [X^n, G_n]_{\underline{\varphi}}^k \cdot t^n = \left( \prod_{r_1, \ldots, r_k \geq 1} \left( 1 - \mathbb{L}^{\Phi_k(\underline{\varphi})d/2} \cdot t^{r_1 r_2 \cdots r_k} r_1 r_2 \cdots r_k^{k-1} \right) \right)^{-[X, G]^k_{\underline{\varphi}}}, \tag{8} \]
where
\[ \Phi_k(r_1, \ldots, r_k) = \varphi_1(r_1 - 1) + \varphi_2 r_1(r_2 - 1) + \ldots + \varphi_k r_1 r_2 \cdots r_{k-1}(r_k - 1). \]

3 Equivariant higher order Euler characteristics

Assume that \( X \) is a (good enough) topological space with \textit{commuting} actions of two finite groups \( G_O \) and \( G_B \) (or equivalently with an action of the product \( G_O \times G_B \)). The quotient \( X/G_O \) carries the natural \( G_B \)-action and thus one can define \( \chi^{(0,G_B)}(X;G_O,G_B) = \chi^B(X/G_O) \in A(G_B) \).

For an element \( g \in G_O \), the fixed point set \( X^{(g)} \) is \( G_B \)-invariant and the quotient \( X^{(g)}/C_{G_O}(g) \) by the centralizer \( C_{G_O}(g) \) carries the natural \( G_B \)-action.

**Definition:** The \textit{equivariant orbifold Euler characteristics} of \((X;G_O,G_B)\) is

\[
\chi^{(1,G_B)}(X;G_O,G_B) := \sum_{[g] \in G_O^\ast} \chi^B(X^{(g)}/C_{G_O}(g)) \]

\[
= \sum_{[g] \in G_O^\ast} \chi^{(0,G_B)}(X^{(g)}/C_{G_O}(g),G_B) \in A(G_B). \tag{9}
\]

The equivariant higher order Euler characteristics are defined recursively in the same way as the non-equivariant one.

**Definition:** The \textit{equivariant Euler characteristics of order} \( k \) of \((X;G_O,G_B)\) is

\[
\chi^{(k,G_B)}(X;G_O,G_B) := \sum_{[g] \in G_O^\ast} \chi^{(k-1,G_B)}(X^{(g)}/C_{G_O}(g),G_B) \in A(G_B). \tag{10}
\]

**Theorem 1** One has

\[
\sum_{n \geq 0} \chi^{(k,G_B)}(X^n; (G_O)_n,G_B) \cdot t^n = \left( \prod_{r_1, \ldots, r_k \geq 1} (1 - t^{r_1 r_2 \cdots r_k})^{r_1 r_2 \cdots r_k - 1} \right)^{-\chi^{(k,G_B)}(X;G_O,G_B)},
\]

where the exponent in the right hand side is defined by the power structure over the Burnside ring \( A(G_B) \).
The proof essentially repeats, e.g., the one in [19] (see also [12]). One has to pay attention to two facts. First, all subsets participating in the course of the proof in [19] are $G$-invariant. Second, one has to use the Macdonald type equation for the equivariant Euler characteristic $\chi^G(\cdot)$: see (11) below. Though it seems to be known, we have not found it in the literature. Therefore we put its proof here.

**Lemma 1** For a $G$-space $X$ one has

\[
1 + \sum_{n=1}^{\infty} \chi^G(S^n X) \cdot t^n = (1 - t)^{-\chi^G(X)} \in 1 + tA(G)[[t]],
\]

where the right hand side is defined by the power structure over the Burnside ring $A(G)$ (or by the pre-$\lambda$-structure on it).

**Remark.** Note that, for a finite $G$-set $X$, the equation (11) is just the definition of its right hand side. (In this case $\chi^G(X) = [X] \in A(G).$)

**Proof.** Let us denote the left hand side of (11) by $\chi^G\zeta_{X,G}(t)$. If $X = X_1 \coprod X_2$ is a decomposition of $X$ into two $G$-subspaces, one has

\[
\chi^G\zeta_{X,G}(t) = \chi^G\zeta_{X_1,G}(t) \cdot \chi^G\zeta_{X_2,G}(t).
\]

(This follows from the identities $S^n X = \coprod_{m=0}^{n} (S^m X_1) \times (S^{n-m} X_2)$, $\chi^G(X \times Y) = \chi^G(X) \chi^G(Y).$) Therefore it is sufficient to prove (11) for the elements of a decomposition of $X$ into $G$-invariant subspaces. A “good enough” $G$-space (say, a real subanalytic one) can be represented as the disjoint union of subspaces of the form $\sigma^d \times (G/H)$, where $H$ is a subgroup of $G$, $G/H$ is the corresponding $G$-set (the quotient), and $\sigma^d$ is the open cell of dimension $d$ with the trivial $G$-action. Let $\overline{\sigma}^d \supset \sigma^d$ be the closed $d$-dimensional ball. Since $\overline{\sigma}^d$ can be $G$-equivariantly contracted to a point, $S^k(\overline{\sigma}^d \times (G/H))$ can be contracted to $S^k(G/H)$. Therefore

\[
\chi^G(S^k(\overline{\sigma}^d \times (G/H))) = \chi^G(S^k(G/H)) = [S^k(G/H)]
\]

(see, e.g., [18] where this is formulated for finite $G$-CW-complexes) and thus

\[
\chi^G\zeta_{S^k(\overline{\sigma}^d \times (G/H))}(t) = 1 + \sum_{i=1}^{\infty} [S^i(G/H)] t^i = (1 - t)^{-[G/H]} = (1 - t)^{-\chi^G(\overline{\sigma}^d \times (G/H))}.
\]
The equation (11) obviously holds for $X = \sigma^d \times (G/H)$ with $d = 0$ (when $\sigma^d$ is a point). Assume that it holds for $X = \sigma^d \times (G/H)$ with $d < d_0$. One has a decomposition $\sigma^{d_0} = \sigma^{d_0} \coprod \sigma^{d_0 - 1}$. Therefore

$$\chi^G \zeta_{\sigma^{d_0} \times (G/H)}(t) = (\chi^G \zeta_{\sigma^{d_0} \times (G/H)}(t))^2 \cdot \chi^G \zeta_{\sigma^{d_0 - 1} \times (G/H)}(t),$$

$$\chi^G \zeta_{\sigma^{d_0} \times (G/H)}(t) = \left(\chi^G \zeta_{\sigma^{d_0 - 1} \times (G/H)}(t)\right)^{-1} = \left((1 - t)^{-(-1)^{(d_0 - 1)(G/H)}}\right)^{-1} = (1 - t)^{-(-1)^{d_0}[G/H]} = (1 - t)^{-\chi^G(\sigma^{d_0} \times (G/H)).$$

□

4 Equivariant generalized higher order Euler characteristics

For a finite group $G$, let $K^G_0(\text{Var}_C)$ be the Grothendieck ring of complex quasi-projective $G$-varieties. By that we mean the free abelian group generated by the $G$-isomorphism classes $[X, G]$ (or $[X]$ for short) of complex quasi-projective varieties $X$ with $G$-actions modulo the relation: $[X, G] = [Y, G] + [X \setminus Y, G]$ for a Zariski closed $G$-invariant subvariety $Y$ of $X$. The multiplication in $K^G_0(\text{Var}_C)$ is defined by the cartesian product with the diagonal $G$-action. Let $\mathbb{L} \in K^G_0(\text{Var}_C)$ be the class of the affine line $A_1^C$ with the trivial $G$-action.

Remark. Usually, in the definition of the Grothendieck ring of complex quasi-projective $G$-varieties, one adds one more relation: if $E \rightarrow X$ is a $G$-equivariant vector bundle of rank $n$, then $[E] = [\mathbb{A}^n_C \times X]$. We do not need this relation for the construction. One can say that we use the Grothendieck ring denoted by $K^G_0(\text{Var}_C)$ in [3]. The same definition was used in [16]. An equation which holds in the equivariant Grothendieck ring $K^G_0(\text{Var}_C)$ defined here, holds in the “traditional” one as well.

There is a natural power structure over the (equivariant) Grothendieck ring $K^G_0(\text{Var}_C)$. Its geometric definition is given in the same way as the usual power structure over the (non-equivariant) Grothendieck ring $K_0(\text{Var}_C)$ in [9].
for complex quasi-projective $G$-varieties $A_i, i = 1, 2, \ldots,$ and $M$ one has

$$(1 + [A_1]t + [A_2]t^2 + \ldots)[M] = 1 + \sum_{k=1}^{\infty} \sum_{\sum ik_i = k} \left( \prod_i M^{k_i} \setminus \Delta \right) \times \prod_i A_i^{k_i} / \prod_i S_{ik_i} t^k,$$ \hspace{1cm} (12)

where $\Delta$ is the “big diagonal” in $M^{\sum k_i}$, the symmetric groups $S_{ik_i}$ act by the permutations of both the corresponding factors in $M^{\sum k_i} = \prod_i M^{k_i}$ and of the factors in $A_i^{k_i}$. One has to take into account that all summands in the right hand side of the equation (12) are $G$-invariant spaces. The proof of the necessary properties of the power structure is the same as in [9]. This power structure is induced by the pre-$\lambda$-structure on $K_G^0(\text{Var}_{\mathbb{C}})$ defined by the Kapranov zeta-function. For a quasi-projective $G$-variety $X$, the series

$$(1 - t)^{-[X]} := 1 + [X] \cdot t + [S^2X] \cdot t^2 + [S^3X] \cdot t^3 + \ldots = (1 - t)^{-[X]}$$

where $S^kX = X^k / S_k$ is the $k$-th symmetric power of the $G$-variety $X$ with the natural $G$-action (see, e.g., [14], [16] for the non-equivariant case). The map $\chi^G : K_G^0(\text{Var}_{\mathbb{C}}) \to A(G)$ is a pre-$\lambda$-ring homomorphism.

In what follows we need the following statement.

**Lemma 2** Let $p : E \to X$ be a $G$-equivariant vector bundle of rank $n$ such that for each $x \in X$ the action of the isotropy subgroup $G_x$ on the fibre $E_x = p^{-1}(x)$ is trivial. Then $[E] = \mathbb{L}^n[X]$.

**Proof.** Factorizing by the action of $G$ one gets the map $\tilde{p} : E/G \to X/G$ which is a vector bundle (due to the triviality of the action of $G_x$ on $E_x$). According to [17] the quotient $X/G$ can be covered by Zariski open subsets $U_i$ such that over each $U_i$ the fibre bundle $\tilde{p}$ is trivial. Therefore

$$\tilde{p}^{-1}(U_i) \cong U_i \times \mathbb{A}_C^n.$$

If $V_i = \pi^{-1}(U_i)$, where $\pi : X \to X/G$ is the canonical factorization map, then the trivialization (13) gives a trivialization of the bundle $p$ over $V_i$, i.e. an isomorphism between $p^{-1}(V_i)$ and $V_i \times \mathbb{A}_C^n$ with the trivial $G$-action on $\mathbb{A}_C^n$. This gives the statement. \hfill $\Box$

In what follows we need the following properties of the power structure over the equivariant Grothendieck ring $K_G^0(\text{Var}_{\mathbb{C}})$. 
Proposition 1 Let $A_i$ and $M$ be $G$-varieties, and let $A(t) := 1 + [A_1]t + [A_2]t^2 + \ldots \in K_0^G(\text{Var}_C)$. Then, for $s \geq 0$,

$$
(A(\mathbb{L}^s t))^{[M]} = (A(t))^{[M]} |_{t \to \mathbb{L}^s t}.
$$

(14)

Proof. The coefficient at the monomial $t^k$ in the power series $(A(t))^{[M]}$ is a sum of the varieties of the form

$$
V = \left(\prod_i M^{k_i} \setminus \Delta\right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i}
$$

with $\sum ik_i = k$. The corresponding summand $\tilde{V}$ in the coefficient at the monomial $t^k$ in the power series $(A(\mathbb{L}^s t))^M$ has the form

$$
\tilde{V} = \left(\prod_i M^{k_i} \setminus \Delta\right) \times \prod_i (\mathbb{L}^{s_i} A_i)^{k_i} / \prod_i S_{k_i}.
$$

There is the natural map $\tilde{V} \to V$ which is a $G$-equivariant vector bundle of rank $sk$ satisfying the conditions of Lemma 2. By (2) one has $[\tilde{V}] = \mathbb{L}^{sk}[V]$, what implies (14). °

Proposition 1 together with the multiplicative property of the power structure $((A(t))^{[M][N]} = (A(t))^{[M]}([N])$ implies the following statement.

Proposition 2 For a complex quasi-projective $G$-variety $X$ one has

$$
\zeta_{\mathbb{L}[X]}(t) = \zeta_X(\mathbb{L} t).
$$

(15)

Here we suggest an equivariant version of the generalized higher order Euler characteristic with values in the modification $K_0^G(\text{Var}_C)[\mathbb{L}^s]_{s \in \mathbb{Q}}$ of the equivariant Grothendieck ring of complex quasi-projective varieties.

Let $X$ be a smooth quasi-projective variety of dimension $d$ with commuting actions of two finite groups $G_O$ and $G_B$ (or equivalently with an action of the product $G_O \times G_B$).

Let us define the zero order equivariant generalized Euler characteristic of $(X; G_O, G_B)$ by $[X; G_O, G_B]^{0, G_B} : = [X/G_O] \in K_0^{G_B}(\text{Var}_C)$.

For $g \in G_O$, let the set of connected components of the fixed point set $X^{(g)}$ consist of $N_g (G_0 \times G_B)$-orbits and let $X^{(g)}_1, X^{(g)}_2, \ldots, X^{(g)}_{N_g}$ be the unions
of the components of each of the orbits. For $1 \leq \alpha_g \leq N_g$ let $F_{\alpha_g}^g$ be the age shift $F_x^g$ at a point of $X_{\alpha_g}^{(g)}$ (this shift does not depend on the point $x \in X_{\alpha_g}^{(g)}$).

For a rational number $\varphi_1 \in \mathbb{Q}$, the generalized orbifold Euler characteristic of weight $\varphi_1$ of $(X; G_O, G_B)$ is defined by

$$[X; G_O, G_B]^{1,G_B}_{\varphi_1} := \sum_{[g] \in (G_O)_s} \sum_{\alpha_g = 1}^{N_g} [X_{\alpha_g}^{(g)} / C_{G_O}(g)] \cdot \mathbb{L}^{\varphi_1 F_{\alpha_g}^g}$$

$$= \sum_{[g] \in (G_O)_s} \sum_{\alpha_g = 1}^{N_g} [X; G_O, G_B]^{0,G_B}_{\varphi_1} \cdot \mathbb{L}^{\varphi_1 F_{\alpha_g}^g} \in K_0^{G_B}(\text{Var}_\mathbb{C})[\mathbb{L}^s]_{s \in \mathbb{Q}}.$$  (16)

Let $\varphi = (\varphi_1, \varphi_2, \ldots)$ be a fixed sequence of rational numbers. The equivariant generalized Euler characteristics of order $k$ of weight $\varphi$ of $(X; G_O, G_B)$ can be defined recursively by

$$[X; G_O, G_B]^{k,G_B}_{\varphi} := \sum_{[g] \in (G_O)_s} \sum_{\alpha_g = 1}^{N_g} [X_{\alpha_g}^{(g)} ; C_{G_G}(g), G_B]^{(k-1),G_B}_{\varphi} \cdot \mathbb{L}^{\varphi_k F_{\alpha_g}^g} \in K_0^{G_B}(\text{Var}_\mathbb{C})[\mathbb{L}^s]_{s \in \mathbb{Q}},$$

where $[X; G_O, G_B]^{1,G_B}_{\varphi_1} := [X; G_O, G_B]^{1,G_B}_{\varphi_1}$ is the equivariant generalized orbifold Euler characteristic given by (16).

**Theorem 2** Let $X$ be a (smooth) quasi-projective variety of dimension $d$ with commuting actions of two finite groups $G_O$ and $G_B$. Then

$$\sum_{n \geq 0} [X^n; (G_O)_n, G_B]^{k,G_B}_{\varphi} t^n = \left( \prod_{r_1 \leq \cdots \leq r_k \geq 1} \left( 1 - \mathbb{L}^{\Phi_k t^{d/2}} \cdot t^{r_1 \cdots r_k} r_2r_3 \cdots r_{k-1} \right) \right)^{- [X; G_O, G_B]^{k,G_B}_{\varphi}},$$

where

$$\Phi_k(r_1, \ldots, r_k) = \varphi_1(r_1 - 1) + \varphi_2 r_1(r_2 - 1) + \cdots + \varphi_k r_1 r_2 \cdots r_{k-1} (r_k - 1).$$

**Proof.** The proof in [12] was by induction started from $k = 1$, i.e. from the generalized orbifold case. The latter one was treated in [10]. One can easily see that the both proofs admit an action of an additional group $G_B$, i.e. all the subspaces are $G_B$-invariants. In particular, symmetric products participating in the proof of [10] carry the natural action of $G_B$. Using Propositions [11] and [2] the corresponding generating series can be written as exponents in terms of the power structure over the modification $K_0^{G_B}(\text{Var}_\mathbb{C})[\mathbb{L}^s]_{s \in \mathbb{Q}}$ of the equivariant Grothendieck ring of quasi-projective varieties. □
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