Resurgence, Operator Product Expansion, and Remarks on Renormalons in Supersymmetric Yang-Mills

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Abstract

I discuss similarities and differences between the resurgence program in quantum mechanics and the operator product expansion in strongly coupled Yang-Mills theories. In $\mathcal{N} = 1$ super-Yang-Mills renormalons possess peculiar features that make them different from renormalons in QCD. Their conventional QCD interpretation does not seem to be applicable in supersymmetric Yang-Mills in a straightforward manner.

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1 Introduction

The notion of resurgence and trans-series associated with it – a breakthrough discovery in constructive mathematics in the 1980s mostly associated with the name of Jean Ecalle – gradually spread in mathematical and theoretical physics. I was impressed by diverse and numerous applications of these ideas which were discussed at this conference in the excellent talks of J. Zinn-Justin, M. Berry, U. Jentschura, G. Dunne, M. Beneke, and others. The topic closest to my talk is resurgence in quantum mechanics. We could see that in quantum mechanics it works well, and trans-series of the type

\[ E(g^2) = E_{\text{PT, regularized}}(g^2) + \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{p=0}^{\infty} \left( \frac{1}{g^{2N+1}} \exp \left[ -\frac{c}{g^2} \right] \right)^k \left( \log \frac{c}{g^2} \right)^l c_{k,l,p} g^{2p} \]

regularized PT

can be derived for all energy eigenvalues \((g^2)\) is assumed to be small; the subscript PT stands for perturbation theory).

In weakly coupled field theories trans-series could be perhaps constructed, although conclusive arguments have not yet been presented. One of the tasks which I formulated for myself is to explain why resurgence – being conceptually close to the operator product expansion (OPE) does not work in strongly coupled field theories, for instance, quantum chromodynamics, (QCD). It is worth noting that OPE existed in QCD from mid-1970s, and in its general form the late 1960s. It grew from a formalism which had been suggested by K. Wilson before the the advent of QCD. The first part of my talk will be devoted to this issue.

In the second part I will focus on a more technical aspect: peculiarities of the factorial divergence of perturbation theory in \(N = 1\) super-Yang-Mills (SYM). Do far renormalons in SYM were scarcely discussed. We (I mean M. Ünsal, G. Dunne, A. Cherman, and myself) exchanged a few remarks on this subject almost a year ago, when we all participated in one and the same conference and shortly after. Later I returned to this issue but no final conclusion has been reached. To a large extent this question remains open.

\footnote{For a pedestrian review understandable to physicists (at least, in part) and an exhaustive list of references see \cite{1,2}.}
2 The simplest quantum-mechanical examples

2.1 Anharmonic oscillator

Let us consider one-dimensional anharmonic oscillator,

\[ H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 + g^2 x^4, \quad (2) \]

see Fig. 1. For definiteness we will focus on the ground state energy \( E_0 \).

![Figure 1: V(x) in the anharmonic oscillator problem (2).](image)

There exists a well-defined procedure of constructing \( E_0 \) order by order in perturbation theory, to any finite order,

\[ E_0 = \frac{\omega}{2} (1 + c_1 g^2 + c_2 g^4 + ....) . \quad (3) \]

Nevertheless, Eq. (3) does not define the ground state energy. Indeed, the coefficients \( c_k \) are factorially divergent at large \( k \)[3],

\[ c_k \sim (-1)^k B^{-k} k! , \quad k \gg 1 , \quad (4) \]

where \( B = \frac{1}{3}\omega^3 \) is the so-called bounce action[4]. Thus, the sum in (3) needs a regularization.

In the simplest case under consideration an appropriate (and exhaustive) regularization is provided by the Borel transformation \( \mathcal{B} \),

\[ \mathcal{B}E_0 \equiv \frac{\omega}{2} \left( 1 + \sum_{k=1}^{\infty} \frac{1}{k!} c_k g^{2k} \right) \equiv \frac{\omega}{2} f(g^2) . \quad (5) \]

[3]Equation (4) is slightly simplified. For a more precise formula see [3].
The Borel transformation introduces $1/k!$ in each term of the series rendering it convergent. Moreover, if the convergent series

$$1 + \sum_{k=1}^{\infty} \frac{1}{k!} c_k g^{2k} \equiv f(g^2)$$

(6)

which defines the Borel function $f(g^2)$ has no singularities on the real positive semi-axis $g^2 \geq 0$ then one can obtain the ground-state energy $E_0$ starting from the well-defined expression for $\mathcal{B}E_0$ and using the Laplace transformation,

$$E_0 = L(\mathcal{B}E_0) \equiv \frac{\omega}{2} \int_{0}^{\infty} da \ g^{-2} \exp \left( -\frac{a}{g^2} \right) f(a) .$$

(7)

This procedure is usually referred to as the Borel summation. Thus, the perturbative expansion in the anharmonic oscillator is Borel-summable due to the fact that the singularities of $f(a)$ are on the negative real semi-axis. Indeed, assume that $f(a)$ has a pole at $a = -B$ (see Fig. 2), namely,

$$f(a) = \frac{B}{a + B}, \quad B = \frac{\omega^3}{3} .$$

(8)

Then the integral (7) is well-defined. At the same time, expanding (8),

Figure 2: The perturbative series in the anharmonic oscillator problem is Borel-summable. The $g^2$ series for $E_0$ is sign alternating; $f(a)$ has a singularity on the real negative semi-axis in the Borel parameter complex plane. $a$ is the Borel parameter.

$$f(a) = \sum_{k=0}^{\infty} (-1)^k \left( \frac{a}{B} \right)^k ,$$

(9)

and substituting this series in (7) we immediately arrive at (4).
Figure 3: The same potential with the replacement $g^2 \rightarrow -g^2$, to be denoted as $\tilde{V}(x)$.

The fact that the position of the singularity in the a plane is to the left of the origin and that the series is sign-alternating are in one-to-one correspondence with each other.

Exactly fifty years ago Vainshtein identified the physical meaning of the factorial growth of the coefficients and explained why the underlying singularity in the Borel parameter plane is on the negative semi-axis. Changing the sign of $g^2$ from positive to negative, $g^2 \rightarrow -g^2$, one converts a stable potential $V(x)$ in (2) into an unstable potential $\tilde{V}(x)$ presented in Fig 3 allowing for the leakage of the wave function to large distances.

In the leaking potential $\tilde{V}$ the energy to the 0-th eigenvalue acquires an imaginary part (as well as other energy eigenvalues). This imaginary part can be easily determined. Indeed, after the Euclidean time rotation effectively the potential $\tilde{V}(x) \rightarrow -\tilde{V}(x)$, as shown in Fig. 4. Then the so-called bounce trajectory becomes classically accessible. The bounce trajectory starts at $x = 0$, slides to the right, bounces off at $x_\ast = \frac{\omega}{\sqrt{2g}}$ and then returns to

Figure 4: An effective potential in Euclidean time. This potential presents a sign reflection of that in Fig. 3 i.e. is $-\tilde{V}(x)$. It vanishes at $x = 0$ and at $x = \pm x_\ast$ where $x_\ast = \frac{\omega}{\sqrt{2g}}$.

\footnote{See e.g. Chapter 7 in [3].}
the point \( x = 0 \). The Euclidean action on the bounce trajectory is readily calculable,

\[
A_{\text{bounce}} = \frac{B}{g^2},
\]

where \( B \) is defined in Eq. (8). In this way we obtain that

\[
\text{Im} E_0 = \frac{\pi \omega}{2} \frac{B}{g^2} \exp \left( -\frac{B}{g^2} \right).
\]

Now one can calculate the ground state energy for the original potential in Fig. 1 by using (11) and a dispersion relation in the coupling constant \( \bar{g} \),

\[
E_0 = \frac{1}{\pi} \int_0^\infty \frac{g^2}{g^2 + \bar{g}^2} \text{Im} E_0 \left( \bar{g}^2 \right)
= \frac{\omega}{2} \int dz \frac{1}{1 + \frac{g^2}{B^2} z} \exp(-z).
\]

The last expression reproduces the series in (3) and (4) with its sign alternation and factorial divergence of the coefficients. Both features are explained by the imaginary part in (11) proportional to \( \exp(-B/g^2) \).

Summarizing, the perturbative expansion for the anharmonic oscillator is factorially divergent; however, the Borel summability allows one to find the closed, well-defined and exact expressions for the energy eigenvalues. The physical meaning of the factorial divergence, as well as the sign alternation, are fully understood. Now we will pass to a more complicated but more interesting non-Borel summable case.

### 2.2 Double-well potential

The double-well problem is described by the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} p^2 - \frac{1}{4} \omega^2 x^2 + g^2 x^4,
\]

i.e. the sign of the \( O(x^2) \) term is changed, and the point \( x = 0 \) becomes unstable. Instead, two stable minima develop at \( x_* = \pm x_* \) where

\[
x_* = \omega / 2\sqrt{2} g.
\]
The shape of the double well potential is depicted in Fig. 4. Classically each of the two minima \( x = \pm \omega/2\sqrt{2g} \) present stable solutions of the system at hand. Quantum-mechanically zero point oscillations about the minima occur. Taking into account anharmonicity near the minima, we generate a perturbative series for the ground state energy. This is in perturbation theory. In fact, the two minima are connected by the tunneling trajectory (instanton) in Euclidean time. The instanton action is

\[
S_{\text{inst}} = \frac{\omega^3}{12g^2},
\]

see e.g. \[6\]. In what follows it will be convenient to introduce

\[
B_{\text{inst}} = g^2 S_{\text{inst}} = \frac{\omega^3}{12}.
\]

At small \( g^2 \) the ground state energy is close to \( \omega/2 \) plus corrections in \( g^2 \) and nonperturbative corrections of the type \( \exp(-c/g^2) \). A crucial distinction from the anharmonic oscillator discussed in Sec. 2.1 is the fact the the \( g^2 \) series in this case will not be sign-alternating (although still factorially divergent), corresponding to a singularity in the Borel function at a real positive value of \( a = 2B_{\text{inst}} \), i.e. on the integration contour, see Fig. 5. Thus, one has to rethink the Borel summation procedure.

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\[
E_0 = L(BE_0) \equiv \frac{\omega}{2} \int_0^\infty da \, g^{-2} \exp \left( -\frac{a}{g^2} \right) f(a),
\]

\[7\]
where roughly speaking

\[ f(a) = \frac{-2B_{\text{inst}}}{a - 2B_{\text{inst}}}. \]  

(17)

Then, instead of Eq. (11) we obtain

\[ c_k = k! \left( 2B_{\text{inst}} \right)^{-k}. \]  

(18)

The perturbative series is not sign alternating, unlike the case of the anharmonic oscillator.

Let us pause here to have a closer look at the above results. In fact, the integral (16) is undefined: the integration along the real positive semi-axis cannot be performed since we hit a singularity. It must be circumvented either along the upper or lower small semicircles as shown in Fig. 5. Depending on whether we choose the upper or lower semicircle we get an imaginary contribution

\[ (\Delta E_0)_{\text{Borel}} = \pm \pi i \left( -\frac{2B_{\text{inst}}}{g^2} \right) \exp \left( -\frac{2B_{\text{inst}}}{g^2} \right). \]  

(19)

However, in the case of the double well potential the system is stable and does not decay, implying that the ground state energy must be strictly real. \((\Delta E_0)_{\text{Borel}}\) must be canceled by something, and, indeed, it is canceled by a contribution coming from the instanton-anti-instanton (IA) pair. The position of singularity at \(2B_{\text{inst}}\) in Fig. 5 prompts us that it is a pair of instantons which is important.

The IA pair is only an approximate saddle point. There is an attraction potential which is very shallow when they are far apart. As usual, approximate saddle points require a regularization. One of regularizations which is very helpful at least for qualitative purposes is considering the IA pair at a finite (rather than zero) energy \(E\), along the lines described e.g. in [9], Secs. 23.2 and 23.3. Then the imaginary part of the IA contribution reduces to \(\exp(-2S_{\text{inst}})\) with a known pre-exponent, and cancels the imaginary part in (19). The real part of the IA contribution is proportional to \(\omega T_\star \sim \log(S_{\text{inst}})(\omega/E)\) where \(T_\star\) is the critical IA separation and the value of \(E\) relevant to the problem is \(E \sim \omega\). Thus, the real part of the IA contribution reduces to \((\log S_{\text{inst}}) \times \exp(-2S_{\text{inst}})\) times a known power of \(S_{\text{inst}}\) in the pre-exponent. For a more careful calculation see [7, 8, 9].
If we write the ground-state energy in the form

\[ E_0 = \frac{\omega}{2} P \int_0^\infty da \, g^{-2} \exp \left( -\frac{a}{g^2} \right) f(a) \]

\[ + \quad (S_{\text{inst}})^p \log S_{\text{inst}} \exp(-2 S_{\text{inst}}), \]  

(20)

where P stands for the principle value of the integral this expression is well-defined and, being expanded, generates the perturbative series in its entirety.\(^4\) Strictly speaking the second line is oversimplified, since the pre-exponent in the second line will also be represented by an infinite \( g^2 \) series with factorially divergent coefficients. To amend this series we will have to include 2I-2A contribution, and so on. Let us refer to this formula as the minimal Borel procedure (MBP). The MBP formula contains all information one can squeeze from perturbation theory. It still lacks something. In order to understand what this something is let us make a digression.

As well-known, perturbation theory (PT) describes fluctuations of a quantal system around classical minima of the potential. In the case at hand we have two degenerate minima reflecting a \( Z_2 \) symmetry of the potential. Let us choose one of them for the “unperturbed” Hamiltonian, for instance,

\[ H_0 = \frac{p^2}{2} + \omega^2 (x - x_*)^2. \]  

(21)

All cubic and quartic terms from the expansion of the potential (13) are referred to \( H_{\text{int}} \). Perturbation theory in \( H_{\text{int}} \) is well defined in any order.

\( H_0 \) does not know about the second vacuum vacuum, but high-order corrections will reflect the existence of the second vacuum indirectly, through factorial divergence of the PT series. Perturbation theory in \( H_{\text{int}} \) requires only the knowledge of the unperturbed eigenfunctions and eigenvalues (i.e, those of the harmonic oscillator (21)). The eigenfunctions of \( H_0 \) should be square-normalizable, and no other requirements are imposed.

Next, we define the sum of the factorially divergent series as MBP plus IA. Using this procedure we would conclude that the system at hand has two degenerate ground states: the \( Z_2 \) restoration in the vacuum is still absent.

This fact – restoration of \( Z_2 \) – does not ensue with necessity from the amended PT series. It presents an additional information on the global vacuum structure: a \( Z_2 \) order parameter drastically changes compared to

\(^4\)Equation (20) is to be compared with the general trans-series formula (1).
its PT value, accordingly the degeneracy of the ground state is lifted. This effect is proportional to \( \exp(-S_{\text{inst}}) \), as opposed to \( \exp(-2S_{\text{inst}}) \) reflecting the corresponding singularity in the Borel plane at \( 2B_{\text{inst}} \).

Conceptually, this is similar to the chiral symmetry breaking in the chiral limit in QCD. No matter what one does with PT, one will not see any splitting between the axial and vector quark two-point functions. One has to infer the global vacuum structure of QCD other sources.

3 Asymptotically free field theories

We arrived at a point where it would be natural to pass from quantum mechanics to asymptotically free field theories. Up to a certain point we will be able to proceed along the lines outlined in Sec. 2. There are two cases allowing one to go all the way up to complete resurgence: (a) if a given field theory is exactly solvable (in which case this is a triviality), or (b) if it is weakly coupled (perhaps, after a certain deformation) and hence can be treated quasiclassically. In the latter case complications that arise are of a technical nature. Today we are aware of quite a few examples of this type which have been identified and studied in the past.

However, the most interesting theories are QCD and its relatives. They are special because QCD is the theory of Nature, describing quark-gluon dynamics. They are strongly coupled in the infrared domain (IR) where it is impossible to treat them quasiclassically – perturbation theory fails even qualitatively. It does not capture drastic rearrangement of the vacuum structure related to confinement.

I would like to discuss the following question: how far can we go in the resurgence program in these theories? We will see that a certain procedure suggested in the late 1960s [10] and implemented in QCD in the 1970s [11] allows one to advance rather far, although, unfortunately, not to the very end. This is as good as it gets...

The Lagrangian of QCD has the form (in the chiral limit)

\[
\mathcal{L} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{a\mu\nu} + \sum \bar{\psi} i \not{D} \psi. \tag{22}
\]

5 The instanton can leak to another minimum and then anti-instanton will return the system back to the original one. That is the origin of the \( \exp(-2S_{\text{inst}}) \) factor. The splitting between the ground state and the first excitation is due to a single instanton which connects two “prevacua.” This effect is proportional to \( \exp(-S_{\text{inst}}) \).
where the sum runs over the massless quark flavors, and $\psi$ is the quark field in the fundamental representation of SU($N$). In actual world $N = 3$, but in theoretical laboratory one is free to consider any value of $N$. If we drop the quark term we are left with the $G_{\mu\nu}^2$ gluon term. This is pure Yang-Mills theory. Moreover, $g^2$ in front of the gluon term is the asymptotically free gauge coupling.

As we know, this is a strongly coupled theory. The Lagrangian is defined at short distances in terms of gluons and quarks, while at large distances of the order of $\gtrsim \Lambda_{\text{QCD}}^{-1}$ we deal with hadrons, e.g. pions and protons. Certainly, the latter are connected with quarks and gluons in a divine way, but this connection is highly nonlinear, non-local and is not amenable to analytical description at the moment. Moreover, the very existence of massless pions and massive protons is due to a dramatic restructuring of the QCD vacuum reflecting spontaneous breaking of the chiral symmetry. The latter phenomenon is only possible at very strong coupling. Perhaps, in the future string theory will be able to provide an adequate description, but as they say “the future is not ours to see.”

Another – a much simpler – example is two-dimensional CP($N-1$) model with a varying degree of supersymmetry (or no supersymmetry at all). The action of the model is

$$S = \int d^2 x \left( \sum_{i,j=1}^{N-1} G_{ij} \partial_\mu \phi^i \partial^\mu \phi^j + \text{fermions} \right)$$

(23)

where

$$G_{ij} = \frac{2}{g^2} \left( \delta_{ij} - \frac{\phi^i \phi^j}{\chi^2} \right), \quad \chi = 1 + \sum_m \phi^m \phi^m,$$

(24)

and $g^2$ is the asymptotically free coupling constant. In the large-$N$ limit this model is exactly solvable \[12, 13, 14\]. To the leading order in $1/N$ the solution is known but cannot be expressed in terms of (11) because instantons are irrelevant at strong coupling. Since the solution is known, one can still present it in the form of a generic trans-series. In the first subleading $1/N$ correction we return to a generic contrived situation, to be discussed below, similar to that in QCD.

A common feature of both theories above as well as many others from this class, is the fact that the coupling constant is not a \textit{bona fide constant};
it runs. In more detail

\[ g^2(Q) = \frac{B_{\text{inst}}}{\beta_0 \log(Q/\Lambda)} \]  \hspace{1cm} (25)\]

where

\[ B_{\text{inst}} = \begin{cases} 
8\pi^2, & \text{QCD} \\
4\pi, & \text{CP}(N - 1) 
\end{cases}, \quad \beta_0 = \begin{cases} 
\frac{11}{3}N, & \text{YM} \\
N, & \text{CP}(N - 1) 
\end{cases}. \]  \hspace{1cm} (26)\]

and \( Q \) is an appropriate momentum scale (assuming \( Q \gg \Lambda \)). Here \( \beta_0 \) is the first coefficient of the \( \beta \) function. In the upper line on the right it is given for pure Yang-Mills. When characteristic values of \( Q \) become close to \( \Lambda \), the running constant is undefined and all calculations in terms of gluons and quarks become meaningless.

As we see, the genuine parameter of QCD is not dimensionless \( g^2 \) but, rather the dynamical QCD scale \( \Lambda \) invisible in the classical Lagrangian. That’s the phenomenon of dimensional transmutation inherent to all strongly coupled asymptotically free field theories. The series in \( g^2 \) becomes the series in \( 1/\log(Q/\Lambda) \), exponential terms \( \exp(-c/g^2(Q)) \) reduce to powers

\[ \left( \frac{\Lambda}{Q} \right)^{c\beta_0/8\pi^2}, \]

while exponential in \( Q \) terms \( \sim \exp(-cQ/\Lambda) \), which also appear in QCD and similar theories, in \( g^2 \) perturbation theory have to emerge from

\[ \exp(-c \exp(\tilde{c}/g^2(Q))). \]

Complete failure of quark-gluon calculations at \( Q \sim \Lambda \) blocks the program of “analytic” resurgence in terms of trans-series in QCD. However, some kind of resurgence is possible, known as the operator product expansion. Now I proceed to a more systematic (albeit brief) discussion of OPE.

4 OPE vs trans-series

Instead of a general introduction to Wilson’s operator product expansion which would require a lot of time\(^6\), I will briefly discuss OPE from a somewhat

\(^6\)For a review of OPE in QCD see [15].
nonstandard standpoint: following the logic of Sec. 2 devoted to resurgence in quantum mechanics.

Let us start our discussion from the two-point function

\[
\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \left\langle T \left( j_\mu(x) j_\nu(0) \right) \right\rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(Q^2),
\]

where \( j_\mu = \bar{\psi} \gamma_\mu \psi \) is the quark current, and we denote

\[ Q^2 = -q^2, \]

so that in the Euclidean domain \( Q^2 \) is positive. We will limit ourselves to large values of the Euclidean momentum, \( Q \gg \Lambda \), so that perturbation theory can be used. In fact, instead of \( \Pi(Q^2) \), for technical reasons it will be convenient analyze the so-called Adler function defined as

\[
D(Q^2) = -4\pi^2 Q^2 \frac{d\Pi(Q^2)}{dQ^2}.
\]

The first two terms in the expansion of the Adler function are defined by the diagrams in Fig. 6 where the coupling constant

\[ \alpha \equiv \frac{g^2}{4\pi} . \]

Given the external momentum \( Q \) flowing through the wavy line, it is easy to see that it is the running coupling \( \alpha(Q) \) that enters in Fig. 6b. Indeed, the momentum flowing through the gluon line in Fig. 6b) is \( k \sim Q \).

Moving to higher orders in \( \alpha \) we will find more and more complicated multiloop graphs. Among them a special role belongs to the bubble-chain diagrams, depicted in Fig. 7 These graphs (referred to as renormalons) were extensively studied in the late 1970s [16] (for reviews see [17, 18]).

When we say bubble chains, we should be careful. Generally speaking, the very definition of a bubble chain in the form of Fig. 7 is not quite accurate. The appropriate renormalon graphs cannot be isolated in the form of a bubble chain since in this form they are not even gauge invariant. An honest-to-god renormalon calculation is quite contrived.
Figure 6: The leading and the next-to leading terms in the expansion of the Adler function. The external current $j_\mu$ injecting the quark-antiquark pair in the vacuum (and then annihilating it) is denoted by wavy lines.

Figure 7: The bubble-chain diagrams representing renormalons. Solid lines denote quark propagators, while dashed lines are for gluons.

There is a useful trick, however. One adds to the theory $N_f$ flavors, where $N_f$ is treated as a free parameter. Then, instead of the full calculation of the genuine "bubble chain," with gluon degrees of freedom in the bubbles, one calculates only the matter bubbles (which are gauge invariant in the chain of Fig. 7), and then replaces

$$\beta_f^0 \equiv -\frac{2}{3} N_f \rightarrow \beta_0$$

where $\beta_0$ is the first coefficient in the $\beta$ function which includes everything: gluons (plus ghosts in the covariant gauges) and matter fields.\(^7\)

It is easy to see that the renormalon contribution into the $D$ function is sign-nonalternating and factorially divergent in high orders, $\Delta D_{\text{renorm}} \sim n! \alpha^n$ ($n \gg 1$). If $n$ is large, the estimate $k \sim Q$ is no longer valid. Both observations – the absence of sign alteration and factorial divergence – become obvious after a closer look at Fig. 6b before integrating over $k$. The exact result for fixed $k^2$ was found by Neubert.\(^{19}\) For illustrative purposes it is sufficient to use a simplified interpolating expression\(^{20}\) collecting all bubble insertions in the gluon propagator: no bubbles, one bubble, two bubbles, and

\(^7\) Note that $\beta_f^0$ and $\beta_0$ have opposite signs.
so on,

\[ D = C \times Q^2 \int dk^2 \frac{k^2 \alpha_s(k^2)}{(k^2 + Q^2)^3}, \quad (32) \]

which coincides with the exact expression [19] in the limits \( k^2 \ll Q^2 \) and \( k^2 \gg Q^2 \), up to minor irrelevant details. The coefficient \( C \) in Eq. (32) is a numerical constant and \( \alpha(k^2) \) is the running gauge coupling which can be represented as

\[ \alpha(k^2) = \frac{\alpha(Q^2)}{1 - \frac{\beta_0 \alpha(Q^2)}{4\pi} \ln(Q^2/k^2)}, \quad (33) \]

Let us focus on the infrared (IR) domain. Omitting the overall constant \( C \) we obtain

\[ D(Q^2) = \frac{1}{Q^4} \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha}{4\pi} \right)^n \int dk^2 k^2 \left( \ln \frac{Q^2}{k^2} \right)^n, \quad \alpha \equiv \alpha(Q^2) \quad (34) \]

which can be rewritten as

\[ D(Q^2) = \frac{\alpha}{2} \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha}{8\pi} \right)^n \int dy y^n e^{-y}, \quad y = 2 \ln \frac{Q^2}{k^2}. \quad (35) \]

The \( y \) integration in Eq. (35) represents all bubble-chain diagrams after integration over the loop momentum \( k \). The \( y \) integral from zero to infinity is \( n! \). A characteristic value of \( k^2 \) saturating the integral is

\[ y \sim n \text{ or } k^2 \sim Q^2 \exp\left(-\frac{n}{2}\right). \quad (36) \]

Thus, we observe the factorial divergence of the coefficients. The corresponding singularity in the Borel plane is depicted in Fig. 8.

If \( Q^2 \) is fixed and \( n \) is sufficiently large, \( n > n_* \) where

\[ n_* = 2 \ln \frac{Q^2}{\Lambda^2}, \quad (37) \]

the factorial divergence of the coefficients in (35) is purely formal and cannot be trusted. At small \( k^2 \ll \Lambda^2 \) the diagrams in Figs. 6b and 7 (in fact, any Feynman diagrams) cease to properly represent non-Abelian dynamics due to strong coupling in the IR. Equation (36) shows that if \( n > n_* \) the characteristic values of \( k^2 \) saturating the integral do fall off below \( \Lambda^2 \). The point
$n = n_*$ represents the optimal truncation point: at this point the terms of the asymptotic series are minimal. Formally, if we discard the domain $k^2 < \Lambda^2$ at $n > n_*$ the factorial growth is suppressed, see Fig. 9 and the series must be truncated,

$$D(Q^2) \to \frac{\alpha_s}{2} \sum_{n=0}^{n_*)} \left(\frac{\beta_0\alpha_s}{8\pi}\right)^n n!.$$

(38)

At this point the road we have to take in QCD and similar strongly coupled theories diverges from that in quantum mechanics. In the latter the validity of the quasiclassical approximation combined with the clear-cut picture of the vacuum structure allows one to achieve full resurgence. In field theories the vacuum structure is determined by infrared dynamics the theory of which is still lacking, and quasiclassical approximations are bound to fail. What can one do under the circumstances?

5 Operator product expansion

I remember that after the first seminar on the SVZ sum rules [11] in 1978 Eugene Bogomol’nyi used to ask me each time we met: “Look, how can you speak of power corrections in the two-point functions at large $Q^2$ when even the perturbative expansion (i.e. the expansion in $1/\ln(Q^2/\Lambda^2)$) is not well defined? Isn’t it inconsistent?”

Now, with the discussion of Sec. 4 in mind, I will be able to answer the above Bogomoln’yi question in a positive way, namely:
Figure 9: The plot of the integrand in Eq. (34) for two values of $n$, “small” and “large.” A sharp peak at $y \sim n$ saturates the integral. In the left plot $n < n_* = 2 \ln(Q^2/\Lambda^2)$ and the forbidden domain $k^2 \sim \Lambda^2$ does not contribute to the factorial factor. In the right plot $n > n_*$. The $y$ integration has to be cut off at $y = n_*$, which kills the factorial growth.

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Consistent use of Wilson’s OPE makes everything well-defined at the conceptual level. Technical implementation may not always be straightforward, however. Moreover, the resulting OPE formula contains unknown vacuum condensates in the form of power corrections. In turn, their summation presents an unsolved problem.

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The operator product expansion in asymptotically free theories is a bookkeeping device separating short-distance (weak-coupling) contributions from those coming from large distances (strong coupling domain). To this end one introduces an auxiliary parameter $\mu$, a separation scale between large and short distances. The latter contribution resides in the
OPE coefficient functions $C_i(Q, \mu)$, while the former contribution is encoded in the matrix elements of the corresponding operators $O_i(\mu, \Lambda)$,

$$D(Q, \Lambda) = \sum_{i=0}^{\infty} C_i(Q, \mu) \langle O_i(\mu, \Lambda) \rangle.$$  

(39)

Generally speaking, OPE is applicable whenever one deals with problems that can be formulated in the Euclidean space-time and in which one can regulate typical Euclidean distances by a varying large external momentum $Q$. Factorization (39) is technically meaningful (i.e. allows one to carry out constructive calculations of $C_i(Q, \mu)$) if one can choose

$$\mu \ll Q, \quad \text{but} \quad \mu \gg \Lambda \quad \text{(40)}$$

Then the coefficients $C_i(Q, \mu)$ can be found quasiclassically, even though they by no means reduce to PT. The matrix elements $\langle O_i(\mu, \Lambda) \rangle$ cannot be determined quasiclassically. As a bookkeeping device OPE cannot fail [11], provided no arithmetic mistake is made en route.

A remarkable observation was made in 1990s. Perturbative analysis (e.g. that of renormalons) prompts us that certain nonperturbative condensates must be present in QCD. Moreover, one can even determine their dimension from the the position of singularities in the Borel plane. Unfortunately, by far not all condensates are visible in the analysis of PT high orders. For instance, all condensates related to the spontaneous breaking of the chiral symmetry leave no trace in any order of perturbation theory, nor in its divergence.

The values of condensates which are visible in the PT divergence are not determined by the PT analysis either.

6 OPE and renormalons in QCD

After this brief digression let us return to the Adler function (27) at large Euclidean $q^2$ in which OPE can be consistently built through separation of large- and short-distance contributions.

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[8] Prevalent in the 1970s and early 80s was a misconception that the OPE coefficients are determined exclusively by perturbation theory while the matrix elements of the operators involved are purely nonperturbative. Attempts to separate perturbation theory from “purely nonperturbative” condensates gave rise to inconsistencies (see e.g. [21]).
For simplicity, taking into account that my purpose today is illustrative, I will ignore the second inequality in (40) and set the separation scale at $\mu = \Lambda$ rather than at $\mu \gg \Lambda$. This would be inappropriate in quantitative analyses; however, my task is to explain a qualitative situation. Being auxiliary the parameter $\mu$ will cancel from the master formula (42) anyway, see below.

Let us have a closer look at Eqs. (32) and (33). The unlimited factorial divergence in (35) is a direct consequence of integration over $k^2$ in (34) all the way down to $k^2 = 0$. Not only this is nonsensical because of the pole in (33) at $k^2 = \Lambda^2$, this is not what we should do in calculating coefficient functions in OPE. The coefficients must include $k^2 > \Lambda^2$ by construction. The domain of small $k^2$ (below $\Lambda^2$) must be excluded from $c_0$ and referred to the vacuum matrix element of the gluon operator $G^{\mu\nu}_{2\mu}$. Indeed, in the sum in Eq. (34) all terms with $n > n^*$ can be written as (see Figs. 9 and 10)

$$\Delta D(Q^2) = \frac{\alpha}{2} \sum_{n > n^*} \left( \frac{\beta_0 \alpha}{8\pi} \right)^n n^*_n e^{-n^*}$$

$$= \frac{\alpha}{2} \sum_{n > n^*} \frac{\Lambda^4}{Q^4} \quad (41)$$

where I used the fact that

$$\frac{\beta_0 \alpha(Q^2)}{8\pi} = \frac{1}{2 \ln(Q^2/\Lambda^2)} = 1/n^*.$$ 

Of course, we cannot calculate the gluon condensate from the above expression for the tail of the series (34) representing the large distance contribution, for a number of reasons. In particular, the value of the coefficient in front of $\Lambda^4/Q^4$ remains uncertain in (41) because Eq. (33) is no longer valid at such momenta. We do not expect the gluon Green functions used in calculation in Fig. 7 and in Eq. (33) to retain any meaning in the strong coupling non-perturbative domain. A qualitative feature – the power dependence $(\Lambda/Q)^4$ in (11) – is correct, however.

We note with satisfaction that the fourth power of the parameter $\Lambda/Q$ which we find from this tail exactly matches the OPE contribution of the operator $\langle G^{\mu\nu}_{2\mu} \rangle$, see (11).

Summarizing, we see that the consistent use of OPE cures the problem of the renormalon-related factorial divergence of the coefficients in the $\alpha$ series, absorbing the IR tail of the series in the vacuum expectation value of the
Figure 10: The PT expansion for the Adler function is asymptotic. One can trust it only up to a point of optimal truncation. A “tail” beyond this point tells us of the existence of an operator of dimension $2k$ representing this tail not accessible by PT calculation. (In the case at hand $k = 2$).

The gluon operator $G_{\mu\nu}^2$ and similar higher-order operators. Although the value of $\langle G_{\mu\nu}^2 \rangle$ cannot be calculated from renormalons, the very fact of its existence can be established.

7 Sources of factorials and master formula

From quantum mechanics we learned that the factorial divergence can arise from instantons. In QCD the instantons are ill-defined in the IR and, strictly speaking, nobody knows what to do with them. If one considers QCD in the ’t Hooft limit of large number of colors instantons decouple. The corresponding singularity in the Borel plane (see Fig. 8) moves to the right infinity. At the same time, none of the essential features of QCD disappears in the ’t Hooft limit. Therefore, in our simplified consideration we can forget about instantons. Perhaps, they will be needed later.

If so, we can write down a single (simplified) “master” formula for QCD and similar theories. At large Euclidean momenta the correlation functions

\[ \langle c_n Q^k \rangle \]

\[ \approx \Lambda^k \]

\[ n_\ast \]

\[ n \]

\[ ??? \]

\[ ??? \]

\[ (\Lambda/Q)^{2k} \]

---

This statement is a slight exaggeration. I refer to [22] for an alternative point of view on instantons in the QCD vacuum.
of the type (27) can be represented as

\[
D(Q^2) = \sum_{n=0}^{n_0^2} c_{0,n} \left( \frac{1}{\ln Q^2 / \Lambda^2} \right)^n \\
+ \sum_{n=0}^{n_1^2} c_{1,n} \left( \frac{1}{\ln Q^2 / \Lambda^2} \right)^n \left( \frac{\Lambda}{Q} \right)^{d_1} \\
+ \sum_{n=0}^{n_2^2} c_{2,n} \left( \frac{1}{\ln Q^2 / \Lambda^2} \right)^n \left( \frac{\Lambda}{Q} \right)^{d_2} + \ldots \\
+ "\text{exponential terms}".
\]

Equation (42) is simplified in a number of ways. First, it is assumed that the currents in the left-hand side have no anomalous dimensions, and so do the operators appearing on the right-hand side. They are assumed to have only normal dimensions given by \(d_i\) for the \(i\)-th operator. Second, I ignore the second and all higher coefficients in the \(\beta\) function so that the running coupling is represented by a pure logarithm. All these assumptions are not realistic in QCD. I stick to them to make the master formula more concise. Inclusion of higher orders in the \(\beta\) function and anomalous dimensions both on the left- and right-hand sides will give rise to rather contrived additional terms and factors containing \(\log \log\)'s, \(\log \log \log\)'s (\(\log \log / \log\)'s, etc. This is a purely technical, rather than conceptual, complication, however.

So far I discussed the divergence/convergence of the perturbative series explaining that the regulating parameter \(\mu\) in OPE allows one to make PT meaningful. The expansion (42) runs not only in powers of \(1/\ln Q^2\), but also in powers of \(\Lambda/Q\). This is a double expansion, and the power series in \(\Lambda/Q\) is also infinite in its turn. Does it have a finite radius of convergence?

Needless to say, this is an important question. The answer to it is negative. Twenty years ago I argued in [23] (see also [15]) that the power series in (42) are factorially divergent in high orders. This is a rather straightforward observation following from the analytic structure of \(D(Q^2)\). In a nutshell,

\[\text{They could be made somewhat more realistic in } \mathcal{N} = 2 \text{ super-Yang-Mills.}\]

\[\text{Multiple logarithms are elements of the trans-series analysis too, see [1].}\]

\[\text{Factorial divergence of PT series due to a factorially large number of Feynman graphs with many loops is suppressed in the 't Hooft limit.}\]
since the $Q^2$ singularities in $D(Q^2)$ run all the way to infinity along the positive real semi-axis of $q^2$, the $1/Q^2$ expansion cannot be convergent. The last line in Eq. (12) symbolically represents a divergent tail of the power series.

8 Supersymmetric Yang-Mills

Factorial divergence of the perturbative series in supersymmetric theories was only scarcely discussed in the past [24, 25, 26]. Meanwhile, this is an interesting question because renormalons in supersymmetric theories have peculiarities related to peculiarities of the operator product expansion in supersymmetric Yang-Mills.

As we already know, the renormalons are in one-to-one correspondence with particular gluon operators in OPE. There is a one-to-one correspondence between the given bubble chain graph and an appropriate operator in OPE (e.g. [18]).

The SYM Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a + i \bar{\lambda}^a \sigma^\mu D_\mu \lambda^a.$$ (43)

The only difference with the QCD Lagrangian (22) is in the fermion sector: the fundamental quarks are replaced by a Majorana spinor in the adjoint representation of the gauge group.

Supersymmetry of the model implies that an infinite class of gluonic operators cannot have nonvanishing vacuum expectation values (VEVs). This fact tells us that the conventional renormalon analysis must be modified. Below I will discuss a modification needed, but at first let us see why gluonic operator VEVs must vanish in SYM theory, in contradistinction to QCD.

8.1 Why gluon operators have vanishing VEVs in SYM?

The operator $G_{\mu\nu}^a G_{\mu\nu}^a + i \bar{\lambda}^a \sigma^\mu D_\mu \lambda^a$ is the highest component of $\text{Tr} W^2$ where

$$W_\alpha = i \left( \lambda_\alpha + i \theta_\alpha D - \theta^\beta G_{\alpha\beta} - i \theta^2 D_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \right).$$ (44)

Supersymmetry allows only the lowest components of super fields to develop a nonvanishing VEV. In pure SYM theory, without matter, $D = 0$ and,
therefore,

\[ G_{\alpha\beta} \sim D_{\{\beta W_{\alpha}\}} + \ldots \] (45)

where the braces denote symmetrization, \( D_{\beta} \) is the spinorial derivative, and the ellipses stand for higher components irrelevant for our purposes.

Gluonic operators in pure SYM theory must contain at least two \( G \) factors; in other words their generic form is

\[ O_g \propto G \ldots G \propto D_{\{\beta W_{\alpha}\}} \ldots D_{\{\tilde{\beta} W_{\tilde{\alpha}}\}} . \] (46)

The ellipses above represent any number of covariant derivatives and extra \( W \) factors provided that the overall number of the \( W \) factors must be even. Taking \( \text{Tr} \) (which singles out color-singlet parts) is implied but not explicitly indicated.

The right-hand side in (46) can be identically rewritten as

\[ D_{\{\beta W_{\alpha}\}} \ldots D_{\{\tilde{\beta} W_{\tilde{\alpha}}\}} = D_{\{\beta \left( W_{\alpha} \ldots D_{\{\tilde{\beta} W_{\tilde{\alpha}}\}} \right)} + W_{\alpha} \left( D_{\beta} \ldots D_{\{\tilde{\beta} W_{\tilde{\alpha}}\}} \right) . \] (47)

The first term is a full superderivative and, as such, can have no nontrivial VEV. The lowest component of second term contains at the very least \( \lambda \) and \( \mathcal{D}_{\alpha\tilde{\alpha}} \bar{\lambda}^\alpha \) (the last factor vanishes due to the equation of motion). Thus, the lowest-dimensional operator which could in principal appear in OPE is a two-\( \lambda \) operator. However, this cannot appear too because if we calculate the OPE coefficients perturbatively (and renormalons are perturbative objects) then two-\( \lambda \) operators have wrong \( R \) parity, while \( \lambda \tilde{\lambda} \) operators can have Lorentz spin zero only in the combination

\[ \text{Tr} \, \lambda_{\alpha} \mathcal{D}_{\alpha\gamma} G^{i\tilde{j} \tilde{\gamma}} \bar{\lambda}^{\tilde{i}} \] (48)

which reduces to a four-fermion operator by virtue of the equation of motion. Thus, in pure SYM we start OPE, in fact, from four-gluino operators of dimension 6, and the four-lambda operators with possible additional insertions of covariant derivatives and \( G \) or \( \lambda \lambda \) or \( \lambda \bar{\lambda} \) factors which have dimensions higher than 6. No purely gluonic operator can have a nonvanishing VEV in SYM.

The above argument based on \( R \) parity is applicable to the two-point functions of the type

\[ i \int d^4 x \, e^{iqx} \langle O(x) , O^\dagger(0) \rangle , \quad O = \text{Tr} \, \bar{\lambda}_\alpha \lambda_\alpha \quad \text{or} \quad \text{Tr} \, \lambda^\alpha \lambda_\alpha , \] (49)
Since the IR renormalons are in one-to-one correspondence with OPE, we conclude that the bubble chain in Fig. 7 which normally is responsible for non-Borel summable divergence of high orders (closest to the origin in the Borel plane) must be canceled by something else.

Let us note, however, that an easy way of identifying the “bubble chains” in QCD was through matter loops with the subsequent extraction of the $N_f$ factor plus substitution \((31)\). This trick does not work in SYM, see \((43)\). We are forced to introduce matter fields.

8.2 Matter loops in SYM

To identify the renormalon bubble chain through matter loops we should expand supersymmetric gluodynamics \((43)\) to include $N_f$ matter fields in the fundamental representation of $SU(N)$,

\[
\mathcal{L} = -\frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a + \frac{i}{g^2} \bar{\lambda}^a \sigma^\mu D_\mu \lambda^a \\
+ \sum_f \left( D^\mu \bar{q}_f D_\mu q_f + i \bar{\psi}_f \sigma^\mu D_\mu \psi_f \right) \\
+ \left[ -\frac{m}{2} \psi_f^\dagger \psi_f^\alpha + i \sqrt{2} (\psi_f^\alpha T^a) \bar{\tau}_f T^a + \text{H.c.} \right] - V(q_f),
\]

where

\[
V(q_f) = \frac{g^2}{2} \left( \sum_f \bar{q}_f T^a q_f \right)^2 + \sum_f |m|^2 |q_f|^2.
\]

Here $q$ and $\psi$ are the squark and quark fields, respectively. The mass terms in \((50)\) and \((51)\) are irrelevant and can be safely omitted\(^{13}\).

In addition to bubbles depicted in Fig. 7 the bubble chains develop elsewhere, e.g. Fig 11. Unlike the familiar QCD example (Fig. 7) in SYM theory the matter bubbles appear even in the diagrams without gluon insertions, such as the graph depicted in Fig. 12. Each bubble produces $N_f g^2 \log p^2$ where $p$ is the momentum flowing through the gluino line. However, this particular diagram would correspond to the operator

\[
\bar{\lambda}_\dot{\alpha}^\dagger D^{\dot{\alpha} \alpha} \lambda_\alpha
\]

\(^{13}\)One should remember that each flavor is represented by two squarks and two Weyl quarks, one in the fundamental and another in the antifundamental representation of $SU(N)$. 
Figure 11: Elementary bubble insertion in the gluino line.

which reduces through the equation of motion to another dimension-4 operator, namely,

$$\sum_f \phi^j (\lambda^j_i \bar{\psi}_i)$$

which has no analog in QCD. Note that in SYM with matter chiral symmetry is broken \textit{a priori}, and is replaced by $R$ symmetry of the U(1) type. In addition, there is an anomalous part in the equation of motion to be used in Eq. (52), which produces the operator $G_{\mu\nu}^2$.

Figure 12: Additional bubble diagrams in SYM, with matter insertions in the gluino line. $N_f$ matter bubbles produce the $N_f$ factor.

Calculation of the elementary bubble insertion in the gluino line is straightforward. It is determined by the graph in Fig. 11. In fact, there is no need for an explicit calculation of this diagram. It represents the $Z$ factor of the
gluino field. However, supersymmetry guarantees that the renormalization of the gluon and gluino fields are identical.

If we start from the Lagrangian \((\ref{50})\) normalized at a momentum scale \(Q\) (then, the corresponding coupling is \(g^2(Q)\)) and evolve it down to \(p\) then the operator \(\frac{i}{g^2(Q)}\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a\) in the Lagrangian evolves in the following way

\[
\frac{i}{g^2(Q)}\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a \rightarrow \frac{i}{g^2(p)}\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a. \tag{53}
\]

The corresponding \(Z\) factor can be easily read off, for instance, by proceeding to the canonically normalized gluino kinetic term. In this way we find

\[
Z^{-1} = \frac{g^2(Q)}{g^2(p)} = 1 - \frac{g^2}{4\pi} (3N - N_f) \log \frac{Q^2}{p^2}, \tag{54}
\]

where the matter bubble produces only the \(N_f\) part, of course. In other words, the (truncated) diagram in Fig. 11 produces

\[
(p_\mu\gamma^\mu) \frac{N_f g^2}{4\pi} \log \frac{Q^2}{p^2}. \tag{55}
\]

In summary, we see that the standard method of the renormalon analysis which works well in QCD is not so straightforward in supersymmetric gluodynamics (i.e. gluons plus gluinos), since introduction of matter dramatically changes the OPE operator basis. In pure YM and in QCD with massless quarks it is one and the same dimension-4 operator which acquires a VEV and is responsible for the leading renormalon singularity. This is in sharp contradistinction with what happens in SYM.

### 8.3 Renormalons, OPE, and diagrams in SYM

Let us elucidate the last statement in more detail. The role of the IR renormalon bubble chain in a given diagram is to make the line to which bubbles are attached soft \([17, 18]\). At a critical value of \(n\) the integration momentum flowing through the bubble line becomes of the order of \(\Lambda\). Hence, in the framework of OPE, this line must be “cut” and becomes a part of the operator with a VEV, which will represent the tail of the renormalon. For instance, if we consider the graph in Fig. 17 the solid lines carry large momentum, while the dashed one is soft. Correspondingly, this bubble chain is in one-to-one
correspondence with the operators $G^2$, $GD^2G \rightarrow G^3$ and so on. In Fig. 12 the upper part of the graph is soft, while the lower part is hard. One of the operators corresponding to this bubble chain is $\text{Tr} \lambda_\alpha D_{\alpha \beta} \bar{\lambda}^\beta$. Four-gluino operators are obtained from the chains depicted in Fig. 13.

![Diagram](image)

Figure 13: Lines with bubbles are soft. Those without bubbles are hard.

In this graph one should “cut” the gluino lines with the bubble insertions. A large external momentum is passed through the graph through the lines without bubbles.

The problem of interpretation arises only with the bubble chains attached to the gluon lines, because the corresponding operators which should conspire with the tail of such renormalons can have no VEVs. Basing on arguments which I have no time to discuss I am inclined to conjecture that the renormalon depicted in Fig. 7 is canceled by the renormalon depicted in Fig. 12. If the numerical coefficient $c$ is right, the operator which these two graphs (combined together) will give in OPE

$$- \frac{1}{4} G^a_{\mu \nu} G^a_{\mu \nu} + \frac{i c}{g^2} \bar{\lambda}^a \sigma^\mu \mathcal{D}_\mu \lambda^a \rightarrow 0.$$  (56)

This question should be explored further, however, see [27].

## 9 Conclusions

1) Resurgence in the sense it is carried out in quantum mechanics, encounters with conceptual difficulties in strongly coupled Yang-Mills theories.
2) The best we can do is to use Wilson’s operator product expansion adapted to QCD, which has conceptual similarities with the resurgence program.

3) Renormalons is a fancy way to say that at large distances perturbation theory requires matching with nonperturbative strong coupling regime (“conspiracy”). Large distance dynamics is intricate and is yet unsolved. No details of the matching procedure are known, except that it must be consistent with OPE. It does not have to be universal and may well differ in passing from QCD to supersymmetric Yang-Mills.

3) In SYM theories there are obvious problems with renormalons, not addressed in the past, which are not yet solved. They seemingly defy a conventional (QCD-like) conspiracy between OPE and the tails of the renormalon contributions which was discussed in detail in my talk. It is obvious that this question should become a focus of future studies.

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