BOUNDED BEREZIN-TOEPLITZ OPERATORS
ON THE SEGAL-BARGMANN SPACE

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Abstract. We discuss the boundedness of Berezin-Toeplitz operators on a generalized Segal-Bargmann space (Fock space) over the complex n-space. This space is characterized by the image of a global Bargmann-type transform introduced by Sjöstrand. We also obtain the deformation estimates of the composition of Berezin-Toeplitz operators whose symbols and their derivatives up to order three are in the Wiener algebra of Sjöstrand. Our method of proofs is based on the pseudodifferential calculus and the heat flow determined by the phase function of the Bargmann transform.

1. Introduction

We study the boundedness and the deformation estimates of Berezin-Toeplitz operators on a generalized Segal-Bargmann space (Fock space) introduced by Sjöstrand in [15]. This space is a reproducing kernel Hilbert space of square-integrable holomorphic functions on the complex n-space, and is characterized by the image of a global Bargmann-type transform. We begin with a review of Sjöstrand’s “linear” theory in [15] to introduce the setting of the present paper. Let $\phi(X,Y)$ be a quadratic form of $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^n$ of the form

$$\phi(X,Y) = \frac{1}{2} \langle X, AX \rangle + \langle X, BY \rangle + \frac{1}{2} \langle Y, CY \rangle,$$

where $A$, $B$ and $C$ are complex $n \times n$ matrices, $^t A = A$, $^t C = C$ and $\langle X, Y \rangle = X_1 Y_1 + \cdots + X_n Y_n$ for $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$. Set $i = \sqrt{-1}$, $C_R = (C + C)/2$ and $C_I = (C - C)/2i$. We denote by $I_n$ the $n \times n$ identity matrix. Assume that

(1) $\det \phi''_{XY} = \det B \neq 0,$

(2) $\Im \phi''_{YY} = C_I > 0.$

We remark that $\det C = \det C_I \det(C_I^{-1/2} C_R C_I^{-1/2} + i I_n) \neq 0$ since $C_I^{-1/2} C_R C_I^{-1/2}$ is a real symmetric matrix. Let $h \in (0, 1]$ be a semiclassical parameter, and let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class on $\mathbb{R}^n$. A global Bargmann-type transformation of $u \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$Tu(X) = C \phi h^{-3n/4} \int_{\mathbb{R}^n} e^{i\phi(X,y)/h} u(y) dy,$$

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where $C_\phi$ is a normalizing constant as

$$C_\phi = 2^{-n/2} n^{-3n/4} |\det B| (\det C_1)^{-1/4}.$$

The assumption (2) guarantees the existence of a function

$$\Phi(X) = \max_{y \in \mathbb{R}^n} \{- \Im \phi(X, y)\}$$

$$= \frac{1}{2} \langle \Im (tBX), C_1^{-1} \Im (tBX) \rangle - \frac{1}{2} \Im \langle X, AX \rangle$$

$$= \langle X, \Phi''_{XX} \rangle + \Re \langle X, \Phi''_{XX} X \rangle,$$

$$\Phi''_{X\bar{X}} = \frac{BC_1^{-1}tB}{4} > 0, \quad \Phi''_{XX} = - \frac{BC_1^{-1}tB}{4} - \frac{A}{2i}.$$ 

We denote the Lebesgue measure on $\mathbb{C}^n$ by $L$. Set $|X| = \sqrt{\langle X, X \rangle}$ for $X \in \mathbb{C}^n$. Let $L^2_\Phi$ be the set of all square-integrable functions on $\mathbb{C}^n$ with respect to $e^{-2\Phi(X)/h} L(dX)$, and let $H_\Phi$ be the set of all holomorphic functions in $L^2_\Phi$. We remark that

$$\Re \{ i\phi(X, y) \} = \Phi(X) - \frac{1}{2} |C_1|^{-1/2} (y + C_1^{-1} \Im (tBX))|^2.$$ 

The Bargmann transform $T$ is well-defined for any tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$. Moreover $Tu$ satisfies $e^{-\Phi(X)/h} Tu(X) \in \mathcal{S}'(\mathbb{C}^n)$, and is holomorphic on $\mathbb{C}^n$. In particular, $T$ gives a Hilbert space isomorphism of $L^2(\mathbb{R}^n)$ onto $H_\Phi$, where $L^2(\mathbb{R}^n)$ is the set of all Lebesgue square-integrable functions on $\mathbb{R}^n$. We here remark that $e^{-\Phi(X)/h} T(\mathcal{S}'(\mathbb{R}^n)) \subset \mathcal{S}'(\mathbb{C}^n)$, and $T(\mathcal{S}'(\mathbb{R}^n))$ is densely embedded in $H_\Phi$ and $T(\mathcal{S}'(\mathbb{R}^n))$ respectively since $\mathcal{S}'(\mathbb{R}^n)$ is densely embedded in $L^2(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ respectively. The Bargmann transform $T$ is interpreted as a Fourier integral operator associated with a linear canonical transform

$$\kappa_T : \mathbb{C}^n \times \mathbb{C}^n \ni (Y, -\phi'_Y(X, Y)) \mapsto (X, \phi'_X(X, Y)) \in \mathbb{C}^n \times \mathbb{C}^n,$$

$$\kappa_T(x, \xi) = (-iB^{-1}(Cx + \xi), Bx - A^tB^{-1}(Cx + \xi)).$$

If we set

$$\Lambda_\Phi = \left\{ \left( X, \frac{2}{i} \frac{\partial \Phi}{\partial X}(X) \right) \bigg| X \in \mathbb{C}^n \right\},$$

then $\Lambda_\Phi = \kappa_T(\mathbb{R}^{2n})$. This means that the singularities of $u \in \mathcal{S}'(\mathbb{R}^n)$ described in the phase space $\mathbb{R}^{2n}$ are translated into those of $Tu$ described in the Lagrangian submanifold $\Lambda_\Phi$.

Let $\Psi(X, Y)$ be a holomorphic quadratic function on $\mathbb{C}^n \times \mathbb{C}^n$ defined by the critical value of $\{ \phi(X, Z) - \phi(Y, Z) \}/2i$ for $Z \in \mathbb{C}^n$, that is,

$$\Psi(X, Y) = \langle X, \Phi''_{XX} Y \rangle + \frac{1}{2} \langle X, \Phi''_{XX} X \rangle + \frac{1}{2} \langle Y, \Phi''_{XX} Y \rangle.$$

Note that $\Psi(X, \bar{X}) = \Phi(X)$. $TT^*$ is an orthogonal projector of $L^2_\Phi$ onto $H_\Phi$, and given by

$$TT^* u(X) = \frac{C_\phi}{h^n} \int_{\mathbb{C}^n} e^{[2\Psi(X, Y) - 2\Phi(Y)]/h} u(Y) L(dY),$$

$$C_\phi = \left( \frac{2}{\pi} \right)^n \det(\Phi''_{XX}) = (2\pi)^{-n}|\det B|^2 (\det C_1)^{-1}.$$
Here we state the definition of Berezin-Toeplitz operators on $H_\Phi$. If we set $R = C_I^{-1/2} B/2$, then $R^* R = \Phi''_{XX}$. Let $\mathcal{T}$ be a class of symbols defined by

$$
\mathcal{T} = \left\{ b(X) \left| \int_{\mathbb{C}^n} e^{-2|R(X-Y)|^2/h} |b(Y)|^2 L(dY) < \infty \text{ for any } X \in \mathbb{C}^n \right. \right\}.
$$

A Berezin-Toeplitz operator $\tilde{T}_b$ associated with a symbol $b \in \mathcal{T}$ is defined by $\tilde{T}_b u = TT^* (bu)$ for $u \in H_\Phi$. Since

$$
\text{Re}\{2\Phi(X, \bar{Y}) - 2\Phi(Y)\} = \Phi(X) - \Phi(Y) - |R(X-Y)|^2,
$$

e $^{-\Phi(X)/h} \tilde{T}_b u(X)$ takes a finite value for each $X \in \mathbb{C}^n$ provided that $u \in L^2_\Phi$ and $b \in \mathcal{T}$. Historically, Berezin introduced this type of operators acting on a class of holomorphic functions over some complex spaces or manifolds, and established the foundation of geometric quantization in his celebrated paper [1]. Properties of such operators and related problems on the usual Segal-Bargmann space have been investigated in several papers. See [2], [3], [5], [9], [7], [17] and references therein.

Here we give two examples of $H_\Phi$.

**Example 1:** If $\phi(X, Y) = i\beta (X^2/2 - 2XY + Y^2)$, $\beta > 0$ and $XY = \langle X, Y \rangle$, then $H_\Phi$ is the usual Segal-Bargmann space (the Fock space), and

$$
\Psi(X, \bar{Y}) = \frac{\beta}{2} XY, \quad \kappa_T(x, \xi) = \left( x - \frac{i}{2\beta} \xi, -i\beta \left( x + \frac{i}{2\beta} \xi \right) \right).
$$

It is remarkable that $\Phi(X) = \beta |X|^2/2$ is strictly convex and $\Phi''_{XX} = 0$ in this case. The strict convexity justifies the change of quantization parameter. See [15] Proposition 1.3. These facts are effectively used in the analysis on the usual Segal-Bargmann space. See e.g., [8] for the detail.

**Example 2:** If we set $\phi(X, Y) = i(X - Y)^2/2$, then $T$ is the heat kernel transform, and

$$
\Psi(X, \bar{Y}) = -\frac{(X - \bar{Y})^2}{8}, \quad \Phi(X) = \frac{(\text{Im} X)^2}{2}, \quad \kappa_T(x, \xi) = (x - i\xi, \xi).
$$

In this case, the global FBI transform $e^{-\Phi(X)/h} T$ and the space $H_\Phi$ are used as strong tools for microlocal and semiclassical analysis of linear differential operators on $\mathbb{R}^n$. See [12] for the detail.

The purpose of the present paper is to study the boundedness and the deformation estimates of Berezin-Toeplitz operators on the generalized Segal-Bargmann space $H_\Phi$. To state our results, we introduce notation and review pseudodifferential calculus on $H_\Phi$ developed in [15].

We denote by $\mathcal{L}(H_\Phi)$ the set of all bounded linear operators of $H_\Phi$ to $H_\Phi$, and set

$$
Q(a, b) = \left\langle \frac{\partial a}{\partial X}, (\Phi''_{XX})^{-1} \frac{\partial b}{\partial X} \right\rangle, \quad \{a, b\} = iQ(a, b) - iQ(b, a)
$$

for $a, b \in C^1(\mathbb{C}^n)$. Pick up $\chi \in \mathcal{S}(\mathbb{C}^n)$ such that $\int_{\mathbb{C}^n} \chi(X) L(dX) \neq 0$. Sjöstrand’s Wiener algebra $S_W(\mathbb{C}^n)$ is the set of all tempered distributions
on \( \mathbb{C}^n \) satisfying
\[
U(\zeta; b) = \sup_{Z \in \mathbb{C}^n} |\mathcal{F}[u_{\tau Z}\chi](\zeta)| \in L^1(\mathbb{C}^n),
\]
where \( \mathcal{F} \) is the usual (not semiclassical) Fourier transform on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \), \( \tau Z\chi(X) = \chi(X - Z) \), and \( L^1(\mathbb{C}^n) \) is the set of all Lebesgue integrable functions on \( \mathbb{C}^n \). Set \( |b|_{S_W} = \|U(\cdot; b)\|_{L^1(\mathbb{C}^n)} \). We also denote by \( L^\infty(\mathbb{C}^n) \) the set of all essentially bounded functions on \( \mathbb{C}^n \). The definition of \( S_W(\mathbb{C}^n) \) is independent of the choice of \( \chi \), and \( S_W(\mathbb{C}^n) \) is invariant under linear transforms on \( \mathbb{C}^n \).

It is remarkable that
\[
\mathcal{B}^{2n+1}(\mathbb{C}^n) \subset S_W(\mathbb{C}^n) \subset \mathcal{B}^0(\mathbb{C}^n),
\]
and the Weyl quantization of any element of \( S_W \) is a bounded linear operator.

Set \( \mathcal{N}_0 = \{0, 1, 2, \ldots \} \) for short. \( \mathcal{B}^k(\mathbb{C}^n), k \in \mathcal{N}_0 \) is the set of all bounded \( C^k \)-functions on \( \mathbb{C}^n \) whose derivatives of any order up to \( k \) are also bounded on \( \mathbb{C}^n \).

Next we introduce the Weyl quantization on \( H_\Phi \). For fixed \( X \in \mathbb{C}^n \), set
\[
\Gamma(X) = \left\{ (Y, \theta) \mid Y \in \mathbb{C}^n, \theta = \frac{2}{\pi} \frac{\partial \Phi}{\partial X} \left( \frac{X + Y}{2} \right) \right\},
\]
and a volume of \( \Gamma(X) \) is defined by
\[
d\Omega = dY_1 \wedge \cdots \wedge dY_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n.
\]
For \( u \in H_\Phi \), the reproducing formula \( u = TT^*u \) has another expression
\[
u(X) = \frac{1}{(2\pi h)^n} \int_{\Gamma(X)} e^{i(X-Y, \theta)/h} u(Y) d\Omega.
\]

The right hand sides of (4) and (5) coincide to each other via the change of variables called the Kuranishi trick. The Weyl quantization of a symbol \( a(X, \theta) \in \mathcal{S}_W(\lambda_\Phi) = (\kappa_\tau^{-1})^* \mathcal{S}_W(\mathbb{R}^{2n}) \) is defined by
\[
\text{Op}_h^W(a)u(X) = \frac{1}{(2\pi h)^n} \int_{\Gamma(X)} e^{i(X-Y, \theta)/h} a \left( \frac{X + Y}{2}, \theta \right) u(Y) d\Omega.
\]

for \( u \in \mathcal{T}(\mathcal{S}(\mathbb{R}^{2n})) \). \( \text{Op}_h^W(a)u \) is holomorphic in \( \mathbb{C}^n \) since
\[
\frac{\partial}{\partial X} e^{i(X-Y, \theta)/h} a \left( \frac{X + Y}{2}, \theta \right) = \frac{\partial}{\partial Y} e^{i(X-Y, \theta)/h} a \left( \frac{X + Y}{2}, \theta \right)
\]
in the sense of distribution. The Weyl quantization of \( a \circ \kappa_T \) is defined by
\[
\text{Op}_h^W(a \circ \kappa_T)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y, \xi)/h} a \circ \kappa \left( \frac{x + y}{2}, \xi \right) u(y) dy d\xi
\]
for \( u \in \mathcal{S}(\mathbb{R}^{2n}) \). It is remarkable that \( \text{Op}_h^W(\mathcal{S}_W(\lambda_\Phi)) \) is extended on \( H_\Phi \) and a subalgebra of \( \mathcal{L}(H_\Phi) \), and the exact Egorov theorem
\[
\text{Op}_h^W(a) \circ T = T \circ \text{Op}_h^W(a \circ \kappa_T)
\]
holds for \( a \in \mathcal{S}_W(\lambda_\Phi) \). Moreover, Guillemin discovered in [9] that \( \tilde{T}_b = \text{Op}_h^W(b_{1/2}) \) for \( b(X) = b(X, X) \), where
\[
b_{1/2}(X, \theta) = b_{1/2} \left( X, (\Phi''_{XX}^{-1}) \left( \frac{i}{2} \theta - \Phi'_X X \right) \right),
\]
and \( \{b_t\}_{t \geq 0} \) is the heat flow of \( b \) defined by
\[
b_t(X) = e^{th} b(X)
\]
\[ \Delta = \frac{1}{2} \left( \frac{\partial}{\partial X}, (\Phi_{XX}^\mu)^{-1} \frac{\partial}{\partial X} \right). \]

\( b_t \) makes sense for \( b \in \mathcal{T} \) and \( t \in (0, 2) \). We use only \( t \in [0, 1] \) as a quantization parameter. \( b_1 \) is said to be the Berezin symbol of a Berezin-Toeplitz operator \( \widetilde{T}_b \). These facts show that pseudodifferential calculus (see e.g., [10], [12] and [16]) and the heat flow determined by the phase function play essential roles in the analysis of Berezin-Toeplitz operators.

Here we state our results.

**Theorem 1.** Suppose that \( b \in \mathcal{T} \). We have
(i) If \( \widetilde{T}_b \in \mathcal{L}(H_\Phi) \), then for any \( t \in (1/2, 1] \),

\[(7) \quad \|b_t\|_{L^\infty(\mathbb{C}^n)} \leq \frac{\|\widetilde{T}_b\|_{\mathcal{L}(H_\Phi)}}{(2t - 1)^n}.\]

(ii) If \( b \in L^\infty(\mathbb{C}^n) \) for some \( t \in [0, 1/2] \), then \( \widetilde{T}_b \in \mathcal{L}(H_\Phi) \).

(iii) Suppose that \( b \in \mathcal{L}''(\mathbb{C}^n) \) in addition. Set \( b^\lambda(X) = e^{\lambda \text{Re}(X, \lambda)} b(X) \) for \( \lambda \in \mathbb{C}^n \). Then, \( b_{1/2} \in S_W(\mathbb{C}^n) \) if and only if

\[(8) \quad \| (b^\lambda)_1 \|_{L^\infty(\mathbb{C}^n)} e^{-\lambda^2/8} \in L^1(\mathbb{C}^n).\]

In this case, \( \widetilde{T}_b \in \mathcal{L}(H_\Phi) \).

**Theorem 2.** Suppose that \( \partial_X^\alpha \partial_X^\beta a, \partial_X^\alpha \partial_X^\beta b \in S_W(\mathbb{C}^n) \) for any multi-indices satisfying \(|\alpha + \beta| \leq 3\). Then, there exists a positive constant \( C_0 \) which is independent of \( a, b \) and \( \hbar \), such that

\[ \left\| \partial^{\alpha} X \partial^{\beta} X a \right\|_{S_W} \leq \sum_{|\alpha + \beta| \leq 3} C_0 \hbar^2 \sum_{|\mu| \leq 3} \left\| \partial_X^\mu \partial_X^\nu b \right\|_{S_W}. \]

Here we explain the known results and the detail of our results. Theorem [1](i) is a refinement and a generalization of the results of Berger and Coburn in [3]. They proved that \( \|b_t\|_{L^\infty(\mathbb{C}^n)} \leq C(t) \|\widetilde{T}_b\|_{\mathcal{L}(H_\Phi)} \) for \( t \in (1/2, 1] \) with some function \( C(t) \) in case that \( H_\Phi \) is the usual Segal-Bargmann space. For a general \( H_\Phi \), we need some ideas to avoid difficulties coming from \( \Phi_{XX}^\mu \neq 0 \). Theorem [1](ii) is obvious by the \( L^2 \)-boundedness theorem of pseudodifferential operators of order zero with smooth symbols. The condition [8] is a special form of the condition for which \( b_{1/2} \in S_W(\mathbb{C}^n) \). This is given by a special choice of a Schwartz function \( \chi \) appearing in the definition of \( S_W(\mathbb{C}^n) \). Theorem [1](iii) seems to extend the known results by Berger and Coburn in [3, Theorem 13], that is, if \( b \geq 0 \) and \( b \in L^\infty(\mathbb{C}^n) \), then \( \widetilde{T}_b \in \mathcal{L}(H_\Phi) \).

Theorem [2] reminds us of the recent interesting results of Lerchner and Morimoto in [11] on the Fefferman-Phong inequality. Coburn proved in [5] the deformation estimates on the usual Segal-Bargmann space under the assumption

\[ a, b \in \text{the set of all trigonometric polynomials} + C_0^{2n+6}(\mathbb{C}^n), \]
where $C^{2n+6}(\mathbb{C}^n)$ is the set of all compactly supported $C^{2n+6}$-functions on $\mathbb{C}^n$. Roughly speaking, Theorem 2 asserts that the deformation estimates hold for $a, b \in \mathcal{B}^{2n+4}(\mathbb{C}^n)$. The relationship between Berezin-Toeplitz operators and Weyl pseudodifferential operators on $H_\Phi$ gives a formal identity

$$\tilde{T}_a \circ \tilde{T}_b = \tilde{T}_c, \quad c = e^{-h\Delta/2}(a_{1/2}' \# b_{1/2}')/2,$$

where $\#$ is the product of $S_W(\Lambda_{\Phi})$ in the sense of the Weyl calculus introduced later. Unfortunately, however, the backward heat kernel $e^{-h\Delta/2}$ can act only on a class of real-analytic symbols, and it is very hard to obtain the symbol $c$. We apply the forward heat kernel $e^{th\Delta}$ to the construction of the asymptotic expansion of the backward heat kernel

$$e^{-h\Delta/2} = 1 - \frac{h}{2} \Delta + O(h^2),$$

and give an elementary proof of Theorem 2.

The organization of the present paper is as follows. In Section 2 we prove (i) and (iii) of Theorem 1. In Section 3 we prove Theorem 2.

2. Boundedness of Berezin-Toeplitz operators

In this section we prove (i) and (iii) of Theorem 1. On one hand, to prove (i), we express the boundedness of $\tilde{T}_b$ in terms of a complete orthonormal system of $H_\Phi$. We introduce a trace class operator defined by $\tilde{T}_b$ and the complete orthonormal system, and take its trace which becomes $b_t(X)$ for any fixed $X \in \mathbb{C}^n$. This idea is basically due to Berger and Coburn in [3]. In our case, however, $\Phi(X)$ is not supposed to be strictly convex, nor $\Phi''_{XX}$ is not supposed to vanish. We need to be careful about these obstructions. On the other hand, the proof of (iii) is a simple computation. We choose a Schwartz function $\chi$ as a heat kernel at the time $t = 1/2$.

Here we give two lemmas used in the proof of (i). For $u, v \in H_\Phi$, the inner product $\langle \cdot, \cdot \rangle_{H_\Phi}$ is defined by

$$\langle u, v \rangle_{H_\Phi} = \int_{\mathbb{C}^n} u(X)\overline{v(X)}e^{-2\Phi(X)/h}L(dX),$$

which is the restriction of $\langle \cdot, \cdot \rangle_{L_2^\Phi}$ on $H_\Phi$. Set

$$u_\alpha(X) = \left\{ \frac{C_\Phi \cdot \alpha!}{h^n} \right\}^{1/2} (RX)^\alpha e^{\langle X, \Phi''_{XX}X \rangle/h},$$

for a multi-index $\alpha \in \mathbb{N}_0^n$. The first lemma is concerned with a complete orthonormal system of $H_\Phi$ which is naturally generated by the Taylor expansion of the reproducing kernel $e^{2\Phi(X,Y)/h}$.

**Lemma 3.** $\{u_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ is a complete orthonormal system of $H_\Phi$.

In case that $H_\Phi$ is the usual Segal-Bargmann space, the proof of Lemma 3 is given in [8, page 40, (1.63) Theorem]. In this case, $\{T^*u_\alpha\}_{\alpha \in \mathbb{N}_0^n}$ is said to be the family of Hermite functions. The general case can be proved in the same way, and we here omit the proof of Lemma 3.

Next lemma is concerned with the family of Weyl operators, which is a family of unitary operators on $H_\Phi$ and acts on symbols of Berezin-Toeplitz
operators as a group of shifts on \( \mathbb{C}^n \). The family of Weyl operators \( \{ W_\lambda \}_{\lambda \in \mathbb{C}^n} \) on \( H_\phi \) is defined by

\[
W_\lambda u(X) = e^{[2\varphi(X,\lambda) - \varphi(\lambda,\lambda)]/\hbar}(X - \lambda),
\]

where

\[
\varphi(X, \lambda) = \langle X, \Phi_X \tilde{\lambda} \rangle + \langle X, \Phi_X X, \lambda \rangle.
\]

We remark that \( \varphi(X, \lambda) \) is holomorphic in \( X \), and if \( u \) is holomorphic, then \( W_\lambda u \) is also. Properties of Weyl operators are the following.

**Lemma 4.** We have

(i) \( W_\lambda^* = W_{-\lambda} \) on \( H_\phi \).

(ii) \( W_\lambda^* W_\lambda = I \) on \( H_\phi \).

(iii) \( W_\lambda^* \tilde{T}_b W_\lambda = \tilde{T}_{b(-\lambda)} \) on \( H_\phi \) for \( b \in \mathcal{T} \).

**Proof.** A direct computation shows that

\[
2\varphi(X + \lambda, \lambda) - \varphi(\lambda, \lambda) - 2\Phi(X + \lambda) = -2\varphi(X + \lambda, \lambda) + \varphi(\lambda, \lambda) - 2\Phi(X)
\]

(9)

\[
= 2\varphi(X, -\lambda) - \varphi(-\lambda, -\lambda) - 2\Phi(X).
\]

(10)

Let \( u, v \in H_\phi \). Using a translation \( X \mapsto X + \lambda \) and (10), we deduce

\[
\langle W_\lambda u, v \rangle_{H_\phi} = \int_{\mathbb{C}^n} e^{[2\varphi(X, \lambda) - \varphi(\lambda, \lambda) - 2\Phi(X)]/\hbar} u(X - \lambda) \overline{v(X)} L(dX)
\]

\[
= \int_{\mathbb{C}^n} e^{[2\varphi(X + \lambda, \lambda) - \varphi(\lambda, \lambda) - 2\Phi(X + \lambda)]/\hbar} u(X) \overline{v(X + \lambda)} L(dX)
\]

\[
= \int_{\mathbb{C}^n} e^{[2\varphi(X, -\lambda) - \varphi(-\lambda, -\lambda) - 2\Phi(X)]/\hbar} u(X) \overline{v(X + \lambda)} L(dX)
\]

\[
= \langle u, W_{-\lambda} v \rangle_{H_\phi},
\]

which shows that \( W_\lambda^* = W_{-\lambda} \).

\( W_\lambda^* W_\lambda = I \) is also proved by a direct computation

\[
W_\lambda^* W_\lambda u(X) = W_{-\lambda} W_\lambda u
\]

\[
= e^{[2\varphi(X, -\lambda) - \varphi(-\lambda, -\lambda)]/\hbar}(W_\lambda u)(X + \lambda)
\]

\[
= e^{[-2\varphi(X, \lambda) - \varphi(\lambda, \lambda)]/\hbar}(W_\lambda u)(X + \lambda)
\]

\[
= e^{[-2\varphi(X, \lambda) - \varphi(\lambda, \lambda) + 2\varphi(X + \lambda, \lambda) - \varphi(\lambda, \lambda)]u(X)} = u(X),
\]

since \( \varphi(X + \lambda, \lambda) = \varphi(X, \lambda) + \varphi(\lambda, \lambda) \).

\( TT^* \) is self-adjoint on \( L^2_\phi \) and \( TT^* W_\lambda v = W_\lambda v \) for \( v \in H_\phi \). Using this and (9), we deduce

\[
\langle W_\lambda^* \tilde{T}_b W_\lambda u, v \rangle_{H_\phi} = \langle \tilde{T}_b W_\lambda u, W_\lambda v \rangle_{H_\phi}
\]

\[
= \langle TT^* (bW_\lambda u), W_\lambda v \rangle_{H_\phi}
\]

\[
= \langle bW_\lambda u, W_\lambda v \rangle_{L^2_\phi}
\]

\[
= \int_{\mathbb{C}^n} b(x) e^{[2\varphi(X, \lambda) + 2\varphi(X, \lambda) - \varphi(\lambda, \lambda) - \varphi(\lambda, \lambda) - 2\Phi(X)]/\hbar}
\]

\[
\times u(X - \lambda) \overline{v(X - \lambda)} L(dX)
\]
In particular, if we take $t$ \[ u(\lambda) e^{2\varphi(X+\lambda,\lambda)+2\varphi(X+\lambda,\lambda)-\varphi(\lambda,\lambda)-\varphi(\lambda,\lambda)-2\Phi(X+\lambda)/h} \]
\times u(X) v(X) L(dX)
\begin{align*}
= \int_{\mathbb{C}^n} b(X+\lambda) e^{-2\Phi(X)/h} u(X) v(X) L(dX) \\
= (\tilde{T}_b(\cdot, X) u, v)_{H_\Phi},
\end{align*}
which proves $W^*_\lambda \circ \tilde{T}_b \circ W_\lambda = \tilde{T}_b(\cdot, \lambda)$.

Here we prove Theorem H(i).

**Proof of Theorem H(i).** Suppose $\tilde{T}_b \in \mathcal{L}(H_\Phi)$, and set $M = \|\tilde{T}_b\|_{\mathcal{L}(H_\Phi)}$ for short. Lemma 3 shows that $\tilde{T}_b(\cdot, +X) \in \mathcal{L}(H_\Phi)$ and $M = \|\tilde{T}_b(\cdot, +X)\|_{\mathcal{L}(H_\Phi)}$ for any $X \in \mathbb{C}^n$. In terms of the complete orthonormal system given in Lemma 3, $\tilde{T}_b \in \mathcal{L}(H_\Phi)$ implies that $\|\tilde{T}_b u_\alpha, u_\beta\|_{H_\Phi} \leq M$ for any $\alpha, \beta \in \mathbb{N}_0^n$. Since $\Phi(Y) = |RY|^2 + \text{Re} \langle Y, \Phi'(X) Y \rangle$, we deduce that for any $X \in \mathbb{C}^n$
\[ (\tilde{T}_b(\cdot, +X) u_\alpha, u_\beta)_{H_\Phi} \]
\begin{align*}
= (TT^* (\cdot + X) u_\alpha, u_\beta)_{H_\Phi} \\
= (b(\cdot + X) u_\alpha, u_\beta)_{L^2_\Phi} \\
= C_\Phi \frac{1}{\alpha!} \beta! \int_{\mathbb{C}^n} b(X + Y) \\
\times \left\{ \left( \frac{2}{h} \right)^{1/2} RY \right\}^\alpha \left\{ \left( \frac{2}{h} \right)^{1/2} RY \right\}^\beta e^{-2|RY|^2/h} L(dY).
\end{align*}
In particular, if we take $\alpha = \beta$ and sum it up for $|\alpha| = k$, then we have
\[ \sum_{|\alpha| = k} (\tilde{T}_b(\cdot, +X) u_\alpha, u_\alpha)_{H_\Phi} \]
\begin{align*}
= C_\Phi \frac{1}{k!} \left( \frac{2|h|}{h} \right)^k \int_{\mathbb{C}^n} 1 b(X + Y) L(dY).
\end{align*}

Fix $(t, X) \in (1/2, 1] \times \mathbb{C}^n$. When $k = 0$, 11 shows that $\langle \tilde{T}_b(\cdot, +X) u_0, u_0 \rangle_{H_\Phi} = b_1(X)$, and $\|\tilde{T}_b(\cdot, +X) u_0, u_0\|_{H_\Phi} \leq M$ implies that $b_1 L_{\infty}(\mathbb{C}^n) \leq M$, which is (7) at $t = 1$. We consider $(t, X) \in (1/2, 1) \times \mathbb{C}^n$ below, and set $s = 1/t - 1 \in (0, 1)$. Here we introduce a trace class operator
\[ H_{s, X} u = \sum_{k=0}^{\infty} (-s)^k \sum_{|\alpha|=k} (u, u_\alpha)_{H_\Phi} \tilde{T}_b(\cdot, +X) u_\alpha \]
for $u \in H_\Phi$. Let $K_{s, X}(Y, Z)$ be the integral kernel of $H_{s, X}$, that is,
\[ K_{s, X}(Y, Z) = \sum_{k=0}^{\infty} (-s)^k \sum_{|\alpha|=k} \tilde{T}_b(\cdot, +X) u_\alpha (Y) u_\alpha (Z). \]
It is easy to see that $K_{s, X}(Y, Z) \in L^1(\mathbb{C}^n; e^{-2\Phi(Y)/h} L(dY))$ since
\[ \sum_{k=0}^{\infty} s^k \sum_{|\alpha|=k} \int_{\mathbb{C}^n} \tilde{T}_b(\cdot, +X) u_\alpha (Y) |u_\alpha (Y)| e^{-2\Phi(Y)/h} L(dY) \]
Hence, we obtain

$$M \sum_{|\alpha|} s^{|\alpha|} = M \left( \sum_{k=0}^{\infty} s^k \right)^n = M (1-s)^{-n} = \frac{Mt^n}{(2t-1)^n}.$$ 

Then, the Lebesgue convergence theorem and (11) imply that

$$t^{-n} \int_{C^n} \sum_{k=0}^{N} (-s)^k \sum_{|\alpha|=k} \tilde{T}_{b(\cdot+X)} u_\alpha(Y) u_\alpha(Y) e^{-2\Phi(Y)/h} L(dY)$$

converges as $N \to \infty$. Thus we have (7) for $t \in (1/2, 1)$ since the right hand side of (12) converges to $b_t(X)$.

Next we prove Theorem 1-(iii). Boulkhemair proved in [4] that (11) is equivalent to

$$\sup_{X \in C^n} |\mathcal{F}^{-1}[\mathcal{F}[b] \tau \tilde{\chi}](X)| \in L^1(C^n)$$

with some $\tilde{\chi} \in \mathcal{F}(C^n)$ satisfying $\int_{C^n} \tilde{\chi}(X) L(dX) \neq 0$, where $\mathcal{F}^{-1}$ is the usual inverse Fourier transform on $C^n$.

**Proof of Theorem 1-(iii).** We compute the condition (13). We choose $\mathcal{F}[\chi](X) = C_1 e^{-4|RX|^2/h}$ which is the heat kernel at the time $t = 1/2$, and expect a comprehensive expression coming from the parallelogram law. Let $X^* \in \mathbb{C}^n$ be the dual variable under the Fourier transform. We choose a constant $C_1 > 0$ so that $\chi(X^*) = e^{-h|\tilde{R}^{-1}X^*|^2/16}$. Set $\chi_\lambda = \tau \lambda \chi$ for short. The parallelogram law implies that

$$\mathcal{F}[b_{1/2}](X^*) \chi_2 \chi_3(X^*) = e^{-h|\tilde{R}^{-1}X^*|^2/16 - h|\tilde{R}^{-1}(X^* - 2\lambda)|^2/16} \mathcal{F}[b](X^*)$$

$$= e^{-h|\tilde{R}^{-1} \lambda|^2/8 - h|\tilde{R}^{-1}(X^* - \lambda)|^2/8} \mathcal{F}[b](X^*).$$

Taking the inverse Fourier transformation of the above, we deduce

$$\mathcal{F}^{-1}[\mathcal{F}[b_{1/2} \chi_2 \chi_3]](X)$$

$$= e^{-h|\tilde{R}^{-1} \lambda|^2/8} \frac{C_1}{h^{n/2}} \int_{C^n} e^{i \Re(X-Y, \lambda) - 2|\Re(X-Y)|^2/h} b(Y) L(dY)$$

$$= e^{i \Re(X, \lambda) - h|\tilde{R}^{-1} \lambda|^2/8} (b-\lambda)_1(X).$$

Hence, we obtain

$$\sup_{X \in C^n} |\mathcal{F}^{-1}[\mathcal{F}[b_{1/2} \chi_2 \chi_3]](X)| = e^{-h|\tilde{R}^{-1} \lambda|^2/8} \|b^\lambda_1\|_{L^\infty(C^n)}.$$ 

This completes the proof. □

### 3. Deformation Estimates for Compositions

Finally, we prove Theorem 2. We first review the composition of pseudo-differential operators on $H^\Phi$. Let $\sigma$ be a canonical symplectic form on $\mathbb{C}^{2n}$, that is,

$$\sigma = d\Xi \wedge dX = \sum_{j=1}^{n} d\Xi_j \wedge dX_j.$$
at \((X, \Xi) \in \mathbb{C}^n \times \mathbb{C}^n\). Split \(\sigma\) into real and imaginary parts, and denote \(\sigma = \sigma_\mathbb{R} + i\sigma_I\). \(\mathbb{R}^{2n}\) and \(\Lambda_\Phi\) are \(I\)-Lagrangian and \(\mathbb{R}\)-symplectic. Indeed, this is obvious for \(\mathbb{R}^{2n}\), and a direct computation shows that \(\sigma_{I|\Lambda_\Phi} = 0\) and
\[
\sigma_{\mathbb{R}|\Lambda_\Phi} = 2i \sum_{j,k=1}^{n} \frac{\partial^2 \Phi}{\partial X_j \partial X_k} dX_j \wedge dX_k \quad \text{for} \quad \theta = \frac{2}{i} \left(\frac{\partial \Phi}{\partial X}(X)\right),
\]
which is nondegenerate. We use this fact as \(\kappa_+^+ \sigma = \sigma_{\mathbb{R}}\) on \(\mathbb{R}^{2n}\).

Let \(a', b' \in S_W(\Lambda_\Phi)\). It is well-known that
\[
\text{Op}_h^W(a' \circ \kappa_T) \circ \text{Op}_h^W(b' \circ \kappa_T) = \text{Op}_h^W(a' \circ \kappa_T \# b' \circ \kappa_T),
\]
\[
a' \circ \kappa_T \# b' \circ \kappa_T(x, \xi) = \frac{1}{(2\pi h)^{2n}} \int_{\mathbb{R}^{4n}} e^{-2i\sigma_{\mathbb{R}}(y, \eta; z, \xi)/h} \times a'(x + y, \xi + \eta)b'(x, \xi + \eta)dy \eta dz d\xi.
\]

Set \(\theta(X) = -2i\Phi'(X)\) for \(X \in \mathbb{C}^n\). Using the exact Egorov theorem \([6]\) together with the symplectic transform \(\kappa_T\) or a direct computation, we have
\[
\text{Op}_h^W(a' \circ \kappa_T) \circ \text{Op}_h^W(b') = \text{Op}_h^W(a' \# b'),
\]
\[
a' \# b'(X, \theta(X)) = \left(\frac{2^n C_\Phi}{h^n}\right)^2 \int_{\mathbb{C}^{2n}} e^{-2i\sigma(Y; \theta(Y); Z, \theta(Z))/h} \times a'(X + Y, \theta(X + Y))b'(X + Z, \theta(X + Z))L(dY)L(dZ).
\]

Here we begin the proof of Theorem 2 Suppose that \(\partial^2_{XX} a, \partial^2_{XX} b \in S_W(\mathbb{C}^n)\) for any multi-indices satisfying \(|\alpha + \beta| \leq 3\). Set \(a_t = e^{t^2 \Delta} a, b_t = e^{t^2 \Delta} b, a_{1/2} = a_{1/2}(X, \theta), b_{1/2} = b_{1/2}(X, \theta)\),
\[
a'_{1/2}(X, \theta) = a_{1/2} \left( X, \frac{i}{2} \frac{\Phi''}{\Phi''_{XX}}(X) - \left( \theta - \frac{2}{i} \Phi''_{XX}(X) \right) \right),
\]
\[
b'_{1/2}(X, \theta) = b_{1/2} \left( X, \frac{i}{2} \frac{\Phi''}{\Phi''_{XX}}(X) - \left( \theta - \frac{2}{i} \Phi''_{XX}(X) \right) \right).
\]

Then, we have \(\hat{T}_a \circ \hat{T}_b = \text{Op}_h^W(a'_{1/2} \# b'_{1/2})\). Since \(a'_{1/2}(X, \theta(X)) = a_{1/2}(X, \hat{X})\), if we write \(a_t(X) = a_t(X, \hat{X})\) and \(b_t(X) = b_t(X, \hat{X})\) simply, then \(\hat{T}_a \circ \hat{T}_b = \text{Op}_h^W(a_{1/2} \# b_{1/2})\), and
\[
a_t \# b_t(X) = \left(\frac{2^n C_\Phi}{h^n}\right)^2 \int_{\mathbb{C}^{2n}} e^{-2i\sigma(Y; \theta(Y); Z, \theta(Z))/h} \times a_t(X + Y)b_t(X + Z)L(dY)L(dZ).
\]

To complete the proof of Theorem 2 we have only to show that
\[
(14) \quad a_{1/2} \# b_{1/2} \equiv e^{h^2/2} (ab) - \frac{h}{2} e^{h^2/2} Q(a, b) \mod h^2 S_W(\mathbb{C}^n).
\]

Here we remark that
\[
-2i\sigma(Y; \theta(Y); Z, \theta(Z)) = 4\langle Y, \Phi''_{XX} \hat{Z} \rangle - 4\langle Z, \Phi''_{XX} \hat{Y} \rangle = 8i \text{Im} \langle Y, \Phi''_{XX} \hat{Z} \rangle,
\]
\[
\frac{\partial Y e^{-2i\sigma(Y; \theta(Y); Z, \theta(Z))/h}}{\partial Z} = \frac{-h}{4} \Phi''_{XX}^{-1} \left( \frac{\partial}{\partial Z} e^{-2i\sigma(Y; \theta(Y); Z, \theta(Z))/h} \right),
\]
\[
\frac{\partial \bar{Y} e^{-2i\sigma(Y; \theta(Y); Z, \theta(Z))/h}}{\partial Z} = \frac{h}{4} \Phi''_{XX}^{-1} \left( \frac{\partial}{\partial Z} e^{-2i\sigma(Y; \theta(Y); Z, \theta(Z))/h} \right).
\]
From Taylor’s formula and the integration by parts we derive

$$a_t \# b_t(X) = a_t b_t(X) - \frac{h}{4} Q(a_t, b_t)(X) + \frac{h}{4} Q(b_t, a_t)(X) + h^2 r_t(X; h),$$

where \( \{r_t(X; h)\}_{h \in (0, 1]} \) is bounded in \( B^\infty(C^n) \) for fixed \( t > 0 \).

We approximate the main term of \( a_t \# b_t \) which is

$$c_t = a_t b_t - \frac{h}{4} Q(a_t, b_t) + \frac{h}{4} Q(b_t, a_t),$$

by constructing an approximate solution to the initial value problem for the heat equation satisfied by \( c_t \). In other words, we construct an asymptotic solution to the transport equation whose main term is given by the heat operator \( \partial_t - h \Delta \). It is easy to see that

$$\partial_X^\alpha \partial_{\bar{X}}^\beta a_t, \partial_X^\alpha \partial_{\bar{X}}^\beta b_t \in C([0, \infty); S_W(C^n))$$

for \( |\alpha + \beta| \leq 3 \). Set \( p_t = e^{th\Delta}(ab) + h p_t^{(1)} \) and

$$p_t^{(1)} = -\frac{1}{4} e^{th\Delta}Q(a, b) + \frac{1}{4} e^{th\Delta}Q(b, a)$$

$$- \frac{1}{2} \int_0^t e^{(t-s)\Delta} \{Q(a_s, b_s) + Q(b_s, a_s)\} ds.$$

Then, \( c_t \) and \( p_t \) solve

$$\left( \frac{\partial}{\partial t} - h \Delta \right) c_t = -\frac{h}{2} \{Q(a_t, b_t) + Q(b_t, a_t)\} + \frac{h^2}{4} Q_1(a_t, b_t),$$

$$c_0 = ab - \frac{h}{4} Q(a, b) + \frac{h}{4} Q(b, a),$$

$$Q_1(a, b) = \left\langle (\Phi''_XX)^{-1} \partial^2 a, (\Phi''_XX)^{-1} \partial^2 b \right\rangle,$$

$$\left( \frac{\partial}{\partial t} - h \Delta \right) p_t = -\frac{h}{2} \{Q(a_t, b_t) + Q(b_t, a_t)\},$$

$$p_0 = ab - \frac{h}{4} Q(a, b) + \frac{h}{4} Q(b, a),$$

respectively. Hence,

$$c_t - p_t = \frac{h^2}{4} \int_0^t e^{(t-s)\Delta} Q_1(a_s, b_s) ds \in h^2 C([0, \infty); S_W(C^n)).$$

We show that the main part of the second term in \( p_t^{(1)} \) is \(-te^{th\Delta}\{Q(a, b) + Q(b, a)\}/2\), that is,

$$\int_0^t e^{(t-s)\Delta} \{Q(a_s, b_s) + Q(b_s, a_s)\} ds = te^{th\Delta} \{Q(a, b) + Q(b, a)\} + O(h).$$

For this purpose, we estimate

$$\int_0^t e^{(t-s)\Delta} Q(a_s, b_s) ds - te^{th\Delta} Q(a, b) = \int_0^t \{e^{(t-s)\Delta} Q(a_s, b_s) - Q(a_s, b_s)\} ds$$

$$+ \int_0^t \{Q(a_s, b_s) - Q(a_t, b_t)\} ds$$

$$+ t\{Q(a_t, b_t) - e^{th\Delta} Q(a, b)\}$$
\[= F_t + G_t + tH_t.\]

We here remark that the heat kernel \( e^{th\Delta} \) is an even function in the space variable. Combining this fact and Taylor’s formula, we can obtain the desired estimates of \( F_t \) and \( G_t \). This technique has been frequently used for approximating symbols. Changing the variables in the explicit formula of the heat kernel, we have

\[
F_t(X) = \int_0^t \frac{C_\Phi}{(t-s)h^n} ds \int_{\mathbb{C}^n} e^{-2|R Y|^2/(t-s)h} \times \{ Q(a_s, b_s)(X + Y) - Q(a_s, b_s)(X) \} L(dy) \\
= C_\Phi \int_0^t ds \int_{\mathbb{C}^n} e^{-2|R Y|^2} \times \{ Q(a_s, b_s)(X + \sqrt{(t-s)h}Y) - Q(a_s, b_s)(X) \} L(dy).
\]

Substituting Taylor’s formula

\[
Q(a_s, b_s)(X + Y) = Q(a_s, b_s)(X) + \langle Y, \partial_X Q(a_s, b_s)(X) \rangle \\
+ \langle Y, \partial_X Q(a_s, b_s)(X) \rangle + Q_2(a_s, b_s)(X, Y),
\]

\[
Q_2(a_s, b_s)(X, Y) = \sum_{|\alpha + \beta| = 2} \frac{Y^\alpha \bar{Y}^\beta}{\alpha! \beta!} \int_0^1 (1 - \tau) \left( \frac{\partial^2 Q(a_s, b_s)}{\partial^2 X^\alpha \partial^2 X^\beta} \right) (X + \tau Y) d\tau,
\]

into (15), we have

\[
F_t(X) = C_\Phi \int_0^t ds \int_{\mathbb{C}^n} e^{-2|R Y|^2} Q_2(a_s, b_s)(X, \sqrt{(t-s)h}Y) L(dy),
\]

which belongs to \( hC([0, \infty); S_W(\mathbb{C}^n)) \).

We split \( G_t \) into two parts

\[
G_t = \int_0^t \{ Q(a_s, b_s) - Q(a_t, b_t) \} ds = \int_0^t \{ Q(a_s - a_t, b_s + Q(a_t, b_s - b_t) \} ds.
\]

Since

\[
a_s(X) - a_t(X) = C_\Phi \int_{\mathbb{C}^n} e^{-2|R Y|^2} \{ a(X + \sqrt{sh}Y) - a(X + \sqrt{th}Y) \} L(dy) \\
= C_\Phi \int_{\mathbb{C}^n} e^{-2|R Y|^2} \bar{a}(X, (\sqrt{s} - \sqrt{t})\sqrt{h}Y) L(dy),
\]

\[
\bar{a}(X, Y) = \sum_{|\alpha + \beta| = 2} \frac{Y^\alpha \bar{Y}^\beta}{\alpha! \beta!} \int_0^1 (1 - \tau) \left( \frac{\partial^2 a}{\partial^2 X^\alpha \partial^2 X^\beta} \right) (X + \tau Y) d\tau,
\]

we can show that \( G_t \in hC([0, \infty); S_W(\mathbb{C}^n)) \).

It follows that \( H_t \in hC([0, \infty); S_W(\mathbb{C}^n)) \) since

\[
\left( \frac{\partial}{\partial t} - h\Delta \right) H_t \in hC([0, \infty); S_W(\mathbb{C}^n)), \quad H_0 = 0.
\]

Combining the estimates of \( F_t, G_t \) and \( H_t \), we have

\[
\int_0^t e^{(t-s)h\Delta} Q(a_s, b_s) ds - te^{th\Delta} Q(a, b) \in hC([0, \infty); S_W(\mathbb{C}^n)) .
\]
Applying (16) to $p_t^{(1)}$, we obtain
\[ c_t = e^{th\Delta}(ab) - \frac{h}{2} \left( \frac{1}{2} + t \right) e^{th\Delta}Q(a,b) + \frac{h}{2} \left( \frac{1}{2} - t \right) e^{th\Delta}Q(b,a) + O(h^2). \]
If we take $t = 1/2$, we obtain (14). This completes the proof of Theorem 2.

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