Nonequilibrium stationary states and phase transitions in directed Ising models

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Abstract. We study the nonequilibrium properties of directed Ising models with non-conserved dynamics, in which each spin is influenced by only a subset of its nearest neighbours. We treat the following models: (i) the one-dimensional chain; (ii) the two-dimensional square lattice; (iii) the two-dimensional triangular lattice and (iv) the three-dimensional cubic lattice. We raise and answer the question: (a) under what conditions is the stationary state described by the equilibrium Boltzmann–Gibbs distribution? We show that, for models (i), (ii) and (iii), in which each spin ‘sees’ only half of its neighbours, there is a unique set of transition rates, namely with exponential dependence in the local field, for which this is the case. For model (iv), we find that any rates satisfying the constraints required for the stationary measure to be Gibbsian should satisfy detailed balance, ruling out the possibility of directed dynamics. We finally show that directed models on lattices of coordination number \(z \geq 8\) with exponential rates cannot accommodate a Gibbsian stationary state. We conjecture that this property extends to any form of the rates. We are thus led to the conclusion that directed models with Gibbsian stationary states only exist in dimensions one and two. We then raise the question: (b) do directed Ising models, augmented by Glauber dynamics, exhibit a phase transition to a ferromagnetic state? For the models considered above, the answers are open problems, with the exception of the simple cases (i) and (ii). For Cayley trees, where each spin sees only the spins further from the root, we show that there is a phase transition provided the branching ratio, \(q\), satisfies \(q \geq 3\).
1. Introduction

A long-standing theme of the statistical physics of nonequilibrium systems is the question of the nature of their stationary state, and in particular of the existence of phase transitions at stationarity. By nonequilibrium systems, we mean systems whose dynamics is non-reversible, i.e. such that the dynamical rules defining the model do not obey detailed balance. For example, systems which are driven, e.g. submitted to a field, do not obey detailed balance. The nature of the stationary state of driven systems has been well investigated for a number of models, among which are the zero range process [1], the asymmetric simple exclusion process [2] or the KLS model [3, 4].

In contrast, the same questions have not yet been addressed in a comprehensive way for directed Ising models with non-conserved dynamics. In one dimension, for example, a spin is influenced only by its right neighbour, while for a two-dimensional square lattice, for example, each spin is influenced only by the neighbouring spins above (‘north’) and
to the right (‘east’), and not by the spins below and to the left. The asymmetry of the interactions means that the detailed balance condition is not obeyed, and therefore there is no conventional equilibrium. Again one speaks instead in terms of a stationary state. However, the latter differs from the stationary state that prevails for driven systems in that it does not carry a macroscopic current. In the present work we endeavour a study of these questions, namely the nature of the stationary state measure and the existence of phase transitions, for directed Ising models with single spin-flip dynamics.

Usually, when the dynamics of a model does not fulfil detailed balance, one expects its stationary measure to be non-Gibbsian. Consider, for instance, the Ising chain with conserved dynamics and totally asymmetric rules: only the $+\to -$ bond is updated, not the $-\to +$ bond. As pointed by Katz et al [3], while generically the stationary measure of this model is non-Gibbsian, for special choices of the rates defining the dynamics, the stationary measure is the same as that of the model with symmetric dynamics, i.e. the equilibrium Boltzmann–Gibbs distribution. The one-dimensional KLS model, which can also be viewed as the hopping of one species of particles with exclusion on a lattice, has been further analysed and generalized to the case of two species of particles hopping with exclusion [4], where, again, the stationary measure is the same whether the system is driven or not. To date, for spin models with non-conserved dynamics, no systematic analysis of this kind has been performed. The main purpose of this paper is to provide such an analysis, for the Ising model, in one, two and three dimensions.

We treat the following models: (i) the one-dimensional chain; (ii) the two-dimensional square lattice; (iii) the two-dimensional triangular lattice and (iv) the three-dimensional cubic lattice. The question to be answered is how to choose the transition rates, for the single spin-flip dynamics, in such a way that they lead to a stationary state with the equilibrium measure (whether or not the dynamics is directed)? The method consists in writing the master equation at stationarity, then imposing the equilibrium measure [4]. One thus finds a set of constraint equations on the rates. Additional constraints come from the choice of dynamics (with or without spin symmetry, directed or not). For the undirected models (i.e. with symmetric dynamics) we find, as expected, that any rates leading to the Gibbs measure satisfy detailed balance. For the directed case, we show that for models (i), (ii) and (iii), in which each spin ‘sees’ only half of its neighbours, there is a unique set of transition rates, namely with exponential dependence in the local field, for which this is the case. By local field we mean the restricted field felt by the flipping spin. For a variant of model (ii), where each spin sees only its west, north and east neighbours, the property still holds but the set of rates have no longer a simple dependence on the local field. For model (iv), we find that any rates satisfying the constraints required for the stationary measure to be Gibbsian should satisfy detailed balance, ruling out the possibility of directed dynamics. We finally show that directed models on lattices of coordination number $z \geq 8$ with exponential rates cannot accommodate a Gibbsian stationary state. We conjecture that this property extends to any form of the rates. We are thus led to the conclusion that directed models with Gibbsian stationary states only exist in dimensions one and two.

The second purpose of this paper is to investigate whether directed Ising models, augmented by Glauber dynamics, can exhibit a phase transition to a ferromagnetic state. This question was motivated by a recent paper [5], where Lima and Stauffer considered Ising models with directed interactions in two to five space dimensions, evolving via
Glauber dynamics \cite{6}. They find in particular that for the two-dimensional square lattice where each spin is influenced only by the north and east neighbouring spins there is no ferromagnetic phase transition, in contrast to the usual equilibrium Ising model which has a second-order phase transition in two dimensions. In the present work we show how the absence of a phase transition arises both for the directed Ising chain and for the two-dimensional directed square lattice considered in \cite{5}. We then investigate Cayley trees, with the influence directed from the tips towards the root, and show that there is a phase transition for coordination number \( z \geq 4 \) but not for \( z = 3 \), i.e. for branching ratio \( q \geq 3 \), since \( q = z - 1 \).

The outline of this paper is as follows. We begin, in the following section, by giving an extensive analysis of the one-dimensional case. We show in particular that, for the directed Ising chain, there is a unique set of rates which produce stationary states described by the Gibbs measure. We then address in section 3 the case of two-dimensional and three-dimensional Ising models on regular lattices. Section 4 is devoted to another approach, which allows us to determine which lattices, with given coordination number, can sustain directed models with rates of exponential form. We then proceed, in section 5, by studying directed models with Glauber dynamics. We show that for the two-dimensional square lattice there is no phase transition to a ferromagnetic state, the latter result being in agreement with the numerical data of \cite{5}. The same result holds for the directed Ising chain. We show that, for the Cayley tree model, a phase transition occurs if the branching ratio is greater than or equal to three. Some generalizations are given in the appendix. Section 6 concludes with a discussion and summary of the results.

2. Equilibrium measures for nonequilibrium stationary states

In this section we pose the question: ‘under what conditions are the stationary states of Ising models described by the Boltzmann–Gibbs distribution?’. We start by the one-dimensional Ising model. We consider higher-dimensional models in sections 3 and 4.

2.1. The method illustrated in the case of the Ising chain

We derive a set of constraints between the transition rates that need to be satisfied in order for the stationary state measure to be Gibbsian. For symmetric dynamics, the constraints are satisfied if and only if the rates satisfy detailed balance. For directed dynamics, however, the constraints uniquely determine the rates (up to an overall timescale).

The energy of configuration \( C = \{\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N\} \) is given by

\[
E(C) = -J \sum_i \sigma_i \sigma_{i+1},
\]

and we choose periodic boundary conditions. The dynamics consist in flipping a spin, chosen at random, say spin \( i \), with a rate \( w(C_i | C) \) corresponding to the transition between configurations \( C \) and \( C_i = \{\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_N\} \). The change in energy due to the flip is

\[
\Delta E = E(C_i) - E(C) = 2\sigma_i h_i = 2\sigma_i J(\sigma_{i-1} + \sigma_{i+1}),
\]
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Table 1. List of moves and corresponding energy difference $\Delta E$ (equation (2.2)) for the Ising chain.

| Move       | Rate | $\Delta E$ | Move       | Rate | $\Delta E$ |
|------------|------|------------|------------|------|------------|
| $+++ \rightarrow +++$ | $w_{++}$ | 4$J$ | $+++ \rightarrow +++$ | $w_{++}$ | $-4J$ |
| $+++ \rightarrow +--$ | $w_{+-}$ | 0 | $+-- \rightarrow +++$ | $w_{+-}$ | 0 |
| $+-- \rightarrow +--$ | $w_{-+}$ | 0 | $+-- \rightarrow +++$ | $w_{-+}$ | 0 |
| $-+- \rightarrow +-+$ | $w_{-+}$ | $-4J$ | $-+- \rightarrow +--$ | $w_{-+}$ | 4$J$ |

where we denote the local field acting on spin $i$ by $h_i$. At stationarity, the master equation expresses that losses are equal to gains, and gives

$$P(\mathcal{C}) \sum_i w(\mathcal{C}_i | \mathcal{C}) = \sum_i w(\mathcal{C} | \mathcal{C}_i) P(\mathcal{C}_i),$$

(2.3)

where, by hypothesis,

$$P(\mathcal{C}) \propto e^{-\beta E(\mathcal{C})}.$$

(2.4)

After division of both sides by the weight $P(\mathcal{C})$, equation (2.3) can be rewritten as

$$\sum_i w(\mathcal{C}_i | \mathcal{C}) - w(\mathcal{C} | \mathcal{C}_i) e^{-\beta \Delta E} = 0,$$

(2.5)

which is also, in the present context, the general form of the stationary master equation in any dimension.

There are eight ($2^3$) possible rates $w(\mathcal{C}_i | \mathcal{C})$ denoted by $w(\sigma_{i-1} \sigma_i \sigma_{i+1})$, or, for short, by $w_{\sigma_{i-1} \sigma_{i+1}}$ if $\sigma_i = +1$, or by $\bar{w}_{\sigma_{i-1} \sigma_{i+1}}$ if $\sigma_i = -1$ (see table 1), corresponding to the flipping of spin $\sigma_i$ for the eight possible motifs (in the present case, the triplets appearing in table 1):

$$\sigma_{i-1} \sigma_i \sigma_{i+1} \rightarrow \sigma_{i-1} (-\sigma_i) \sigma_{i+1}.$$  

(2.6)

Let $N_{\sigma_{i-1} \sigma_i \sigma_{i+1}}$ be the number of such motifs in configuration $\mathcal{C}$, now considered as fixed. The balance equation (2.5) is finally recast as

\begin{align*}
N_{+++}(w_{++} - e^{-4K} \bar{w}_{++}) + N_{++-}(w_{+-} - \bar{w}_{++}) + N_{--+}(w_{-+} - \bar{w}_{++}) + N_{+++}(\bar{w}_{++} - w_{++}) \\
+ N_{+-+}(w_{--} - e^{4K} \bar{w}_{--}) + N_{++-}(\bar{w}_{++} - e^{4K} w_{++}) + N_{-+-}(\bar{w}_{-+} - w_{--}) + N_{++-}(\bar{w}_{-+} - e^{-4K} w_{--}) = 0,
\end{align*}

(2.7)

where we denoted the reduced coupling constant $\beta J$ by $K$. A shorter notation for the same equation is

$$\sum_{\alpha=1}^{8} N(\mu_\alpha) \left[ w(\mu_\alpha) - w(\bar{\mu}_\alpha)e^{-\beta(E(\bar{\mu}_\alpha)-E(\mu_\alpha))} \right],$$

(2.8)

where $N(\mu_\alpha)$ is the number of occurrences of the motif $\mu_\alpha$ in configuration $\mathcal{C}$ and $\bar{\mu}_\alpha$ is the motif obtained after flipping the central spin in the motif.

Equation (2.8) should be satisfied for any given configuration $\mathcal{C}$. Since the quantities inside the brackets are couples of rates related by detailed balance conditions, the left side of (2.8) vanishes identically as soon as these conditions are imposed on the rates. However,
imposing detailed balance is not \textit{a priori necessarily} the only way of solving (2.8), because the $N(\mu_\alpha)$ are not independent quantities. One should therefore express the latter on a basis of independent quantities, in the present case correlators defined as follows. Let us denote by $I_i = (1 + \sigma_i)/2$ the indicator variable for the presence of a $+$ spin on site $i$. Then, for example,

$$N_{++} = \sum_{i=1}^N I_i I_{i+1} (1 - I_{i+2}) = \frac{1}{8} \sum_{i=1}^N (1 + \sigma_i) (1 + \sigma_{i+1}) (1 - \sigma_{i+2})$$

$$= \frac{N}{8} (1 + c_1 + c_2 - c_3 + c_{12} - c_{13} - c_{23} - c_{123}),$$

where, for example,

$$c_1 = \frac{1}{N} \sum_{i=1}^N \sigma_i, \quad c_2 = \frac{1}{N} \sum_{i=1}^N \sigma_{i+1}, \quad c_{12} = \frac{1}{N} \sum_{i=1}^N \sigma_i \sigma_{i+1},$$

and so on. Translational invariance imposes the following identities between correlators:

$$c_1 = c_2 = c_3, \quad c_{23} = c_{12}.$$  \hspace{1cm} (2.10)

Rewriting the master equation (2.8) in terms of the remaining independent correlators yields

$$e_1 + c_1 e_2 + c_{12} e_3 + c_{13} e_4 + c_{123} e_5 = 0,$$

where the coefficients $e_1, \ldots, e_5$, are linear combinations of the expressions appearing in the brackets of equation (2.8). This equation should be satisfied for any configuration $C$, hence the coefficients of the correlators must vanish identically. This provides five constraint equations, $e_1 = \ldots = e_5 = 0$, not all independent, on the eight rates $w_\sigma \sigma$ and $\bar{w}_\sigma \sigma$. Reducing this set of equations yields the final constraint equations:

$$\bar{w}_{++} = e^{4K} w_{++}$$  \hspace{1cm} (2.13)

$$\bar{w}_{--} = e^{-4K} w_{--}$$  \hspace{1cm} (2.14)

$$\bar{w}_{+-} + \bar{w}_{-+} = w_{+-} + w_{-+}.$$  \hspace{1cm} (2.15)

The first two equations are detailed balance conditions. They involve rates for moves corresponding to a non-zero value of the local field $J(\sigma_{i-1} + \sigma_{i+1}) = \pm 2J$. The third equation involves the rates for moves that do not imply a change in the energy, i.e. with zero local field (motion of a domain wall).

The eight rates fulfilling the constraints (2.13)–(2.15) therefore depend on five independent parameters. A general expression of these rates is

$$w(\sigma_1 \sigma_2 \sigma_3) = \frac{\alpha}{2} \left\{ 1 + \lambda \sigma_1 - \lambda' \sigma_2 + \left( 2 \frac{\lambda'}{\gamma} - \lambda \right) \sigma_3 + \delta \sigma_1 \sigma_3 + \epsilon \sigma_1 \sigma_2 \\
- \left( \gamma (1 + \delta + \epsilon) \right) \sigma_2 \sigma_3 - \lambda' \sigma_1 \sigma_2 \sigma_3 \right\},$$

where $\lambda, \lambda', \alpha, \delta$ and $\epsilon$ are the five independent parameters, while $\gamma$ is given by

$$\gamma = \frac{e^{4K} - 1}{e^{4K} + 1} = \tanh 2K,$$

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where $\lambda, \lambda', \alpha, \delta$ and $\epsilon$ are the five independent parameters, while $\gamma$ is given by

$$\gamma = \frac{e^{4K} - 1}{e^{4K} + 1} = \tanh 2K,$$
and $\sigma_1, \sigma_2, \sigma_3$ are respectively the left, central and right spins. These rates correspond to the generic situation where the central spin is influenced unequally by the left and right spins.

Hereafter we restrict the discussion to the case of rates invariant by spin symmetry, that is
\[
    w_{\sigma\sigma} = \bar{w}_{-\sigma-\sigma}.
\]
(2.18)

In this case, the constraints (2.13)–(2.15) reduce to a single equation:
\[
    w_{--} = e^{4K} w_{++},
\]
(2.19)

hence three rates remain unknown, namely $w_{++}, w_{+-}, w_{-+}$. Setting $\lambda = \lambda' = 0$, in order to satisfy (2.18) we obtain
\[
    w(\sigma_1 \sigma_2 \sigma_3) = \frac{\alpha}{2} (1 + \delta \sigma_1 \sigma_3 + \epsilon \sigma_1 \sigma_2 - (\gamma(1 + \delta) + \epsilon) \sigma_2 \sigma_3),
\]
(2.20)
depending now on the three independent parameters, $\alpha, \delta$ and $\epsilon$.

We now address two extreme cases of interest, where the dynamics is symmetric, then when it is totally asymmetric, or directed.

2.2. Symmetric dynamics

The dynamics is spatially symmetric, or undirected, if the left and right spins have an equal influence on the central spin. This right–left symmetry imposes therefore that
\[
    w_{+-} = w_{-+}.
\]
(2.21)

This fixes one of the remaining unknown rates. By (2.21) we obtain $2\epsilon + \gamma(1 + \delta) = 0$, and therefore
\[
    w(\sigma_1 \sigma_2 \sigma_3) = \frac{\alpha}{2} \left(1 + \delta \sigma_1 \sigma_3 - \frac{1}{2} \gamma(1 + \delta) \sigma_2 (\sigma_1 + \sigma_3)\right),
\]
(2.22)

which is the most general form of the rates for symmetric dynamics, originally proposed by Glauber (see the appendix of [6]), which now depend on two parameters. This form satisfies the detailed balance conditions (2.19) and (2.21).

Simplified forms of the rates are obtained by fixing the parameters $\alpha$ and $\delta$. Examples of possible choices are the Glauber (or heat bath) rule:
\[
    w(\sigma_1 \sigma_2 \sigma_3) = \frac{1}{2} (1 - \sigma_2 \tanh(K(\sigma_1 + \sigma_3))) = \frac{1}{e^{\beta \Delta E}+1},
\]
(2.23)

or the Metropolis rule:
\[
    w(\sigma_1 \sigma_2 \sigma_3) = \min(1, e^{-\beta \Delta E}),
\]
(2.24)

or even the following rule:
\[
    w(\sigma_1 \sigma_2 \sigma_3) = e^{-\beta \Delta E/2} = e^{-K\sigma_2(\sigma_1 + \sigma_3)},
\]
(2.25)

where $\Delta E$, equation (2.2), is the change in energy of the system due to the flip.
2.3. Directed dynamics

Let us now consider the case where the dynamics is totally directed, i.e. totally asymmetric. Assume for instance that spin $\sigma_i$ only looks to the right, hence that the rates only depend on the right neighbour of the flipping spin:

$$w_{++} = w_{-+}, \quad w_{+-} = w_{--}. \quad (2.26)$$

Carrying these conditions into (2.20), we obtain $\delta = \epsilon = 0$ and therefore

$$w(\sigma_1 \sigma_2 \sigma_3) = \frac{\alpha}{2} (1 - \gamma \sigma_2 \sigma_3). \quad (2.27)$$

This expression can be equivalently obtained by suppressing all terms where the spin $\sigma_1$ appears in the general expression of the rates (2.16). We thus find a unique solution to the problem posed, up to the global timescale $\alpha$. Fixing this scale by the choice $\alpha = 2 \cosh 2K$, we obtain the compact form

$$w(\sigma_1 \sigma_2 \sigma_3) = e^{-2K \sigma_2 \sigma_3}, \quad (2.28)$$

while, with the choice $\alpha = 1$, we obtain the alternate form

$$w(\sigma_1 \sigma_2 \sigma_3) = \frac{1}{2} (1 - \sigma_2 \tanh(2K \sigma_3)). \quad (2.29)$$

The directed dynamics is more constrained than the undirected one, as previously exemplified for conserved dynamics of two or three species of particles in [4]. No kinetically constraint models (i.e. with vanishing rates) can be devised in this case, in contrast with the case of the undirected dynamics which allows more freedom.

Let us finally mention that, as a special case of the generic expression (2.20), where the central spin is unequally influenced by its neighbours, the following form of the rates has the virtue of interpolating between the undirected case (2.25) and directed case (2.28):

$$w(\sigma_1 \sigma_2 \sigma_3) = e^{-2K \sigma_2 (x \sigma_1 + (1-x) \sigma_3)}, \quad (2.30)$$

with $0 \leq x \leq 1$.

3. Ising models on regular lattices

In this section we consider two- and three-dimensional Ising models on regular lattices. The energy of configuration $C = \{\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N\}$ is now given by

$$E(C) = -J \sum_{(i,j)} \sigma_i \sigma_j, \quad (3.1)$$

where $(i, j)$ are nearest neighbours. We follow step by step the method used in section 2 for the determination of the transition rates leading to a stationary measure given by the Boltzmann–Gibbs distribution. We start by the two-dimensional Ising model on the square lattice. We proceed with the case of the two-dimensional triangular lattice, then of the three-dimensional cubic lattice.
3.1. Constraint equations on the rates for the square lattice

On the square lattice each spin has four neighbours. The change in energy when flipping the central spin, denoted by $\sigma$, is 

$$\Delta E = 2\sigma J(\sigma_E + \sigma_N + \sigma_W + \sigma_S),$$  

(3.2)

where $\sigma_E$ is the east spin, $\sigma_N$ the north spin, etc. The stationary master equation can be compactly written as in (2.8) with rates corresponding to the 32 motifs $\sigma\sigma_E\sigma_N\sigma_W\sigma_S$. We use the following abridged notations:

\begin{align*}
    w_{++++} &= w_1, & w_{++++} &= w_2, & w_{++++} &= w_3, & w_{++++} &= w_4, \\
    w_{++++} &= w_5, & w_{++++} &= w_6, & w_{++++} &= w_7, & w_{++++} &= w_8, \\
    w_{++++} &= w_9, & w_{++++} &= w_{10}, & w_{++++} &= w_{11}, & w_{++++} &= w_{12}, \\
    w_{++++} &= w_{13}, & w_{++++} &= w_{14}, & w_{++++} &= w_{15}, & w_{++++} &= w_{16},
\end{align*}

and similarly,

\begin{align*}
    w_{-+++} &= \tilde{w}_1, & w_{-+++} &= \tilde{w}_2, & w_{-+++} &= \tilde{w}_3, & w_{-+++} &= \tilde{w}_4, \\
    w_{-+++} &= \tilde{w}_5, & w_{-+++} &= \tilde{w}_6, & w_{-+++} &= \tilde{w}_7, & w_{-+++} &= \tilde{w}_8, \\
    w_{-+++} &= \tilde{w}_9, & w_{-+++} &= \tilde{w}_{10}, & w_{-+++} &= \tilde{w}_{11}, & w_{-+++} &= \tilde{w}_{12}, \\
    w_{-+++} &= \tilde{w}_{13}, & w_{-+++} &= \tilde{w}_{14}, & w_{-+++} &= \tilde{w}_{15}, & w_{-+++} &= \tilde{w}_{16}.
\end{align*}

The method then proceeds as in the one-dimensional case. The identities imposed by translational invariance on the correlators are

\begin{align*}
    c_1 &= c_2 = c_3 = c_4 = c_5, \\
    c_{12} &= c_{14}, & c_{13} &= c_{15}, & c_{25} &= c_{34}, & c_{23} &= c_{45},
\end{align*}

(3.3) 

(3.4)

where the indices 1, 2, ..., 5, correspond respectively to the central spin, the east, north, west and south spins. After reduction, the equations on the rates yield 14 constraint equations. If one restricts the study to situations where the rates are invariant by spin symmetry, then the following additional constraints must be taken into account:

\begin{align*}
    \tilde{w}_1 &= w_{16}, & \tilde{w}_2 &= w_{15}, & \tilde{w}_3 &= w_{14}, & \tilde{w}_4 &= w_{13}, \\
    \tilde{w}_5 &= w_{12}, & \tilde{w}_6 &= w_{11}, & \tilde{w}_7 &= w_{10}, & \tilde{w}_8 &= w_{9}, \\
    \tilde{w}_9 &= w_8, & \tilde{w}_{10} &= w_7, & \tilde{w}_{11} &= w_6, & \tilde{w}_{12} &= w_5, \\
    \tilde{w}_{13} &= w_4, & \tilde{w}_{14} &= w_3, & \tilde{w}_{15} &= w_2, & \tilde{w}_{16} &= w_1,
\end{align*}

(3.5)

which in turn reduce the former system of 14 constraint equations to a system of six equations, as follows:

\begin{align*}
    e^{4K} w_1 - \tilde{w}_1 &= 0, \\
    w_6 - \tilde{w}_6 &= 0, \\
    e^{4K} w_2 - \tilde{w}_2 + e^{4K} w_5 - \tilde{w}_5 &= 0, \\
    e^{4K} w_3 - \tilde{w}_3 - (w_8 - e^{4K} \tilde{w}_8) &= 0, \\
    e^{4K} w_2 - \tilde{w}_2 - (e^{4K} w_3 - \tilde{w}_3) + \frac{2e^{4K}}{1 + e^{4K}} (w_7 - \tilde{w}_7) &= 0, \\
    e^{4K} w_2 - \tilde{w}_2 + e^{4K} w_3 - \tilde{w}_3 - \frac{2e^{4K}}{1 + e^{4K}} (w_4 - \tilde{w}_4) &= 0.
\end{align*}

(3.6)

\[^{3}\] The notation $\sigma_N$, where $N$ stands for north, should not be confused with the notation for the spin with index $N$, the size of the system.

\[ \text{doi:10.1088/1742-5468/2009/12/P12016} \]
The first two equations involve couples of rates related by detailed balance conditions, the following ones linear combinations of such couples. The rates now depend on ten independent parameters.

3.2. Symmetric dynamics on the square lattice

The dynamics is spatially symmetric, or undirected, if the left and right spins, or up and down spins, have an equal influence on the central spin. For simplicity, we restrict the study to the most symmetric dynamics where the rates only depend on the number of up and down neighbouring spins of the central spin, i.e. rates which only depend on the value of the sum \( \sigma_E + \sigma_N + \sigma_W + \sigma_S \):

\[
\begin{align*}
w_2 &= w_3 = w_5 = w_9 = y, \\
w_4 &= w_6 = w_7 = w_{10} = w_{11} = w_{13} = z, \\
w_8 &= w_{12} = w_{14} = w_{15} = t.
\end{align*}
\] (3.7)

Carried in the constraint equations above, we find

\[
t = e^{4K} y, \quad w_{16} = e^{8K} w_1,
\] (3.8)

and \( z \) remains arbitrary. After fixing the overall scale of time the transition rates still depend on two independent parameters. Fixing these parameters leads to the following simplified forms of the rates, which are simple generalizations of the 1D expressions:

\[
w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = \frac{1}{2} (1 - \sigma \tanh(K(\sigma_E + \sigma_N + \sigma_W + \sigma_S))),
\] (3.9)

\[
w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = \min(1, e^{-\beta \Delta E}),
\] (3.10)

\[
w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = e^{-K\sigma(\sigma_E + \sigma_N + \sigma_W + \sigma_S)}.
\] (3.11)

3.3. Directed dynamics on the square lattice

Consider first the case where the central spin is only influenced by the north and east spins. This means that

\[
\begin{align*}
w_1 &= w_2 = w_3 = w_4 = x, \\
w_5 &= w_6 = w_7 = w_8 = z, \\
w_9 &= w_{10} = w_{11} = w_{12} = z', \\
w_{13} &= w_{14} = w_{15} = w_{16} = y.
\end{align*}
\] (3.12)

Carrying these relationships in the constraint equations (3.6) lead to

\[
y = e^{8K} x, \quad z' = z, \quad z = e^{4K} x.
\] (3.13)

Out of the four unknowns \( x, y, z \) and \( z' \), only one remains undetermined, i.e. there only remains one free parameter in the expression of the rates. The general expression of the rates satisfying these constraints is

\[
w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = \frac{\alpha}{2} (1 + \gamma^2 \sigma_E \sigma_N - \gamma \sigma(\sigma_E + \sigma_N)),
\] (3.14)

which is the unique solution of the question posed, up to the global timescale \( \alpha \). Fixing this timescale by the choice \( \alpha = 2 \cosh^2 2K \) allows us to recast (3.14) into the compact

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form:

$$w(\sigma E \sigma N \sigma W \sigma S) = e^{-2K\sigma(\sigma_E + \sigma_N)}. \tag{3.15}$$

Such rates violate detailed balance, but yet, as for the one-dimensional case, lead to a stationary state with Gibbs measure. This form of the rates for the two-dimensional directed Ising model on the square lattice where each spin sees only its north and east neighbours was first proposed in [7] (up to a seemingly missing factor of 2 in the exponent).

We also considered the more general case where the central spin is influenced by three of its neighbours, say, west, north and east. Introducing the simplifying notations $u_1, \ldots, u_8$, we have

$$w_1 = w_2 = u_1, \quad w_3 = w_4 = u_2, \quad w_5 = w_6 = u_3, \quad w_7 = w_8 = u_4, \quad w_9 = w_{10} = u_5, \quad w_{11} = w_{12} = u_6, \quad w_{13} = w_{14} = u_7, \quad w_{15} = w_{16} = u_8,$$

which carried into the six constraint equations (3.6) yield six new constraints:

$$u_3 = e^{4K} u_1, \quad u_4 = e^{4K} u_2, \quad u_5 = (1 + e^{4K}) u_1 - u_2, \quad u_6 = u_3, \quad u_7 = e^{4K} u_5, \quad u_8 = e^{8K} u_1. \tag{3.17}$$

The general expression of the rates satisfying these constraints depend on two independent parameters:

$$w(\sigma E \sigma N \sigma W \sigma S) = \frac{\alpha}{2} \left(1 + c \sigma_W \sigma_N + (\gamma^2 - c) \sigma_E \sigma_N - \sigma \left(\frac{c}{\gamma} \sigma_W + \gamma \sigma_N + \left(\gamma - \frac{c}{\gamma}\right) \sigma_E\right)\right). \tag{3.18}$$

Let us now consider two particular cases of this general expression.

- The previous case where the central spin is influenced only by the north and east spins is a particular case of the model just considered. In order to recover the results (3.13) and (3.14) it suffices to impose the additional conditions
  $$u_1 = u_2 = x, \quad u_3 = u_4 = z, \quad u_5 = u_6 = z', \quad u_7 = u_8 = y \tag{3.19}$$
  in (3.17), yielding $c = 0$ in (3.18), that is (3.14).

- If we impose the left–right spatial symmetry on the rates (3.18), i.e. if $u_2 = u_5$, which itself leads to $u_4 = u_7$, then we obtain the unique solution
  $$w(\sigma E \sigma N \sigma W \sigma S) = \frac{\alpha}{2} \left(1 + \frac{\gamma^2}{2} \sigma_N (\sigma_E + \sigma_W) - \frac{\gamma}{2} \sigma(\sigma_W + 2\sigma_N + \sigma_E)\right), \tag{3.20}$$
  up to the global scale $\alpha$. This expression cannot be recast in an exponential form analogous to (3.15).
3.4. Directed dynamics on the triangular lattice

We now consider the two-dimensional Ising model on the triangular lattice. There are 128 ($2^7$) unknown rates, corresponding to the 128 motifs where the central spin is surrounded by its six neighbours. Using the same method as above, we obtain the set of constraints between the transition rates that need to be satisfied in order for the stationary state measure to be Gibbsian. The result is rather lengthy and will not be written here.

Let us specialize to the directed model in which the subset of influential spins are any three consecutive spins at $\pi/3$ one from the other. The sites corresponding to the subset of influential spins and the sites corresponding to the complementary subset of neighbouring spins are thus related by spatial parity with respect to the central site. For this directed case, we find a unique solution, up to a global timescale, to the problem of the determination of rates leading to a Gibbsian stationary state. The solution found has again the exponential form

$$w(\sigma_1 \sigma_2 \cdots \sigma_6) = e^{-2K(\sigma_1 + \sigma_2 + \sigma_3)},$$

(3.21)

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the spins felt by the central spin, denoted by $\sigma$.

3.5. Directed dynamics on the cubic lattice

We finally consider the three-dimensional Ising model on the cubic lattice. Again there are 128 ($2^7$) unknown rates, corresponding to the 128 motifs where the central spin is surrounded by its six neighbours. We find that the set of constraints between the transition rates that need to be satisfied in order for the stationary state measure to be Gibbsian are the detailed balance relationships. In other words, for the three-dimensional Ising model on the cubic lattice, the only possible dynamics leading to the Boltzmann–Gibbs distribution at stationarity is the symmetric one.

The method used in this section is unfortunately cumbersome when applied to lattices of increasing coordination number. The next section provides an alternate approach.

4. Lattices with given coordination number

In this section we introduce a complementary viewpoint on the general problem of the existence of rates leading to a Gibbsian stationary state, when the dynamics is directed. We proceed in two steps. First, we check the results of section 3 for exponential rates: assuming the form of the rates, we show that they satisfy, or do not satisfy, the master equation at stationarity, depending on the lattice considered. Second, generalizing these observations, we give a method for the determination of which lattice, of given coordination number, is able or unable to sustain exponential rates for directed dynamics.

4.1. Check of the results of section 3

The idea stems from the observation that, once the form of the rates satisfying the master equation (2.5) have been determined by the method of section 3, it should be an easy matter to check the reverse, i.e. that the stationary master equation (2.5) is satisfied. For example, for the directed Ising chain with rates (2.28), equation (2.5) leads to the
following condition, for a given fixed configuration $C$:

$$\sum_i e^{-2K\sigma_i\sigma_{i+1}} = \sum_i e^{-2K\sigma_i\sigma_{i-1}}$$  \hspace{1cm} (4.1)$$

which indeed holds, due to the translational symmetry of the system.

We now consider the general case of a lattice of given coordination number $z$. We assume that the rates have the exponential form of equations (2.28), (3.15) and (3.21):

$$w(C_i|C) = e^{-2\beta\sigma_i h_i^+},$$  \hspace{1cm} (4.2)$$

where we denoted by $h_i^+$ the field felt by spin $i$ due to the subset $v^+(i)$ of its influential neighbours. Likewise we denote by $h_i^-$ the field due to the complementary subset $v^-(i)$ of neighbouring spins. We only consider the case where the subsets $v^+(i)$ and $v^-(i)$ are related by spatial parity with respect to the central spin. With the hypothesis (4.2), the master equation (2.5) gives

$$\sum_i e^{-2\beta\sigma_i h_i^+} = \sum_i e^{-2\beta\sigma_i h_i^-},$$  \hspace{1cm} (4.3)$$

or

$$\sum_i \prod_{j\in v^+(i)} e^{-2K\sigma_i\sigma_j} = \sum_i \prod_{j\in v^-(i)} e^{-2K\sigma_i\sigma_j},$$  \hspace{1cm} (4.4)$$

that is, with $\tau = -\tanh 2K$,

$$\sum_i \prod_{j\in v^+(i)} (1 + \tau\sigma_i\sigma_j) = \sum_i \prod_{j\in v^-(i)} (1 + \tau\sigma_i\sigma_j).$$  \hspace{1cm} (4.5)$$

This equation must be satisfied order by order in $\tau$. This leads to a set of geometrical constraints that a given lattice, of given coordination number, should satisfy, in order for the master equation to be satisfied.

Before using the method in all its generality we first check that, for the 2D square lattice, the 2D triangular lattice and the 3D cubic lattice, we recover the results found in section 3: the constraints are satisfied for the first two lattices, not for the last one, as we now show.

We start with the 2D square lattice. We have $h_i^+ = J(\sigma_E + \sigma_N)$ and $h_i^- = J(\sigma_W + \sigma_S)$. The constraint equation (4.5) is

$$\sum_i 1 + \tau\sigma_i(\sigma_E + \sigma_N) + \tau^2\sigma_E\sigma_N = \sum_i 1 + \tau\sigma_i(\sigma_W + \sigma_S) + \tau^2\sigma_W\sigma_S,$$  \hspace{1cm} (4.6)$$

which should be fulfilled for any given fixed configuration $C$. At order $\tau$ this imposes that the equation

$$\sum_i \sigma_i(\sigma_E + \sigma_N) = \sum_i \sigma_i(\sigma_W + \sigma_S)$$  \hspace{1cm} (4.7)$$

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be satisfied. This is obviously the case because of translation invariance. At order $\tau^2$ the constraint is
\[ \sum_i \sigma_i \sigma_N = \sum_i \sigma_W \sigma_S, \hspace{1cm} (4.8) \]
which again is easily seen to hold. This completes the proof for the 2D square lattice.

We proceed with the 2D triangular lattice, with $z = 6$. Let us denote by $e_1, e_2, e_3$ the unit vectors spanning the lattice. Then, at order $\tau$, the constraint equation reads
\[ \sum_i \sigma_i \sigma_{i+e_1} + \sigma_i \sigma_{i+e_2} + \sigma_i \sigma_{i+e_3} = \sum_i \sigma_i \sigma_{i+\bar{e}_1} + \sigma_i \sigma_{i+\bar{e}_2} + \sigma_i \sigma_{i+\bar{e}_3}, \hspace{1cm} (4.9) \]
where $\bar{e}_1 = -e_1$, and so on. Again, trivially it is possible to pair the terms of both sides by translation. At order $\tau^2$ the constraint is
\[ \sum_i \sigma_i \sigma_{i+e_1} + \sigma_i \sigma_{i+e_2} + \sigma_i \sigma_{i+e_3} = \sum_i \sigma_i \sigma_{i+\bar{e}_1} + \sigma_i \sigma_{i+\bar{e}_2} + \sigma_i \sigma_{i+\bar{e}_3}. \hspace{1cm} (4.10) \]
It is also easy to check that this equation is satisfied by translation invariance by pairing adequately the terms on both sides. At order $\tau^3$, the constraint equation is
\[ \sum_i \sigma_i \sigma_{i+e_1} \sigma_{i+e_2} \sigma_{i+e_3} = \sum_i \sigma_i \sigma_{i+\bar{e}_1} \sigma_{i+\bar{e}_2} \sigma_{i+\bar{e}_3}. \hspace{1cm} (4.11) \]
This equation is indeed satisfied for the triangular lattice because the following geometrical relationship between unit vectors holds:
\[ e_2 = e_1 + e_3. \hspace{1cm} (4.12) \]
Thus the quadrilateral formed by the spins on the left-hand side is related by translation to that appearing in the right-hand side. This completes the proof for the 2D triangular lattice.

Let us now consider the 3D cubic lattice, with $z = 6$. The constraint equations are the same as for the 2D triangular lattice, since these equations only depend on the coordination number and not on the structure of the lattice. In the present case, the constraints are satisfied at order $\tau$ and $\tau^2$, but not at order $\tau^3$ since (4.12) no longer holds. We thus confirm the result of section 3, namely that the 3D cubic lattice cannot sustain exponential rates for directed dynamics.

4.2. Which lattice of given coordination number can sustain exponential rates for directed dynamics?

As exemplified by the last two cases considered above, where $z = 6$, the constraints coming from equation (4.5) only depend on the coordination number $z$ of the lattice. Then, in all generality, for $z$ given, the question is whether there exists a lattice fulfilling these constraints or not.

Consider, for example, the case of a lattice of coordination $z = 8$. The constraints at order $\tau$ and $\tau^2$ are trivially satisfied, as above. At order $\tau^3$ the constraint imposes, between the four unit vectors spanning the lattice, equalities of the type (4.12). At order
the constraint imposes the relationship: \( e_1 + e_4 = e_2 + e_3 \). This relationship can be implemented for the 3D centred cubic lattice, for example. However the constraints imposed at order \( \tau^3 \) cannot be fulfilled either by the 3D centred cubic lattice or by any other one.

Some more work can convince the reader that any lattice with coordination number \( z \geq 8 \) does not satisfy the constraints, hence cannot sustain a directed model with rates (4.2).

This method assumes rates of the form (4.2). It is, however, plausible that if these rates do not produce stationary states described by the Gibbs measure, then no other form is able to do so, because the exponential form (4.2) is the best fitted to be a solution of the master equation (2.5). We are therefore led to conclude that directed models with Gibbsian stationary states only exist in dimensions one and two.

5. Directed Ising–Glauber models

In this section we investigate whether directed Ising models, augmented by Glauber dynamics, can exhibit a phase transition to a ferromagnetic state.

5.1. The one- and two-dimensional directed Glauber–Ising models

We begin by presenting the transition rates for directed Ising models with Glauber dynamics. Let \( h_i^+ \) denote, as above, the field felt by spin \( i \) due to its influential neighbours \( v^+(i) \). For example, for the directed Ising chain where each spin only sees the spin to its right, the field felt by spin \( i \) due to spin \( i + 1 \) is \( h_i^+ = J\sigma_{i+1} \). In the two-dimensional case where the spin \( \sigma_i \) sees the spins \( \sigma_N \) and \( \sigma_E \), the field felt by spin \( i \) is given by \( h_i^+ = J(\sigma_N + \sigma_E) \). We define the Glauber rate for the process in which spin \( \sigma_i \) is flipped in configuration \( C = \{\sigma_1, \ldots, \sigma_i, \ldots, \sigma_N\} \) resulting in configuration \( C_i = \{\sigma_1, \ldots, -\sigma_i, \ldots, \sigma_N\}, \) as

\[
w(C_i|C) = \frac{1}{2}(1 - \sigma_i \tanh \beta h_i^+).
\] (5.1)

We now consider the equation of motion for the magnetization, \( m_i = \langle \sigma_i \rangle \), of the spin at site \( i \), first for the directed Ising chain with periodic boundary conditions. This equation reads (where a dot indicates a time derivative)

\[
m_i = -m_i + \langle \tanh(K\sigma_{i+1}) \rangle = -m_i + \tanh K \ m_{i+1},
\] (5.2)

\((K = \beta J)\. The periodic boundary conditions, combined with translational invariance, imply that \( m_i = m \), independent of \( i \), giving

\[
m(t) = m(0) \exp[-(1 - \tanh K)t].
\] (5.3)

This shows that the one-dimensional system cannot support an ordered phase, since an initial magnetization will decay exponentially to zero on a timescale \( t_{eq} = (1 - \tanh K)^{-1} \). Note that the stationary measure for this case is the Boltzmann distribution with temperature doubled, as can be seen from the alternative form of the rate (2.29). Therefore the vanishing of the magnetization in the stationary state of the 1D directed Ising model is
not very surprising. However, a very similar analysis can be employed for the 2D directed Ising model defined above.

Let us label the sites with a pair of integers \((i, j)\), these being the coordinates in the two spatial directions, and suppose the site \(i, j\) is influenced only by the spins at sites \((i + 1, j)\) and \((i, j + 1)\). Then the equation of motion for \(m_{i,j}\) is

\[
\dot{m}_{i,j} = -m_{i,j} + \frac{1}{2} \tanh(2K)(m_{i,j+1} + m_{i+1,j}),
\]

(5.4)

where the last line follows from the fact that \(\sigma_{i,j+1} + \sigma_{i+1,j}\) can take only three values, \(-1, 0\) and \(1\). With periodic boundary conditions, translational invariance implies that \(m_{ij} = m\), independent of the site indices \(i, j\), giving

\[
m(t) = m(0) \exp[-(1 - \tanh 2K) t].
\]

(5.5)

It follows that the magnetization again vanishes in the stationary state.

One can also treat the case of open boundaries, where translational invariance no longer applies. In one dimension, for example, the rightmost spin, \(\sigma_N\), retains its initial value for all \(t\). Without loss of generality, we can fix \(\sigma_N = 1\). In the stationary state, equation (5.2) then gives \(m_i = (\tanh K)^{N-i}\), so the local magnetization decays exponentially as a function of distance from the fixed spin \(\sigma_N\), and the total magnetization per spin vanishes in the thermodynamic limit.

5.2. Cayley trees

We now discuss the case of directed Ising models on Cayley trees, for the case of Glauber dynamics. We take the influence to be directed from the tips towards the root, i.e. each spin only sees the spins that are further from the root.

We consider first a tree with branching ratio 2, i.e. coordination number 3. We assume that the initial condition is one in which all spins at a given level of the tree have an equivalent initial condition, e.g. all spins up, or all spins either up or down with probabilities that are the same for all the spins at that level. Then the Glauber equation for the magnetization \(m_n\) of a site at the \(n\)th level of the tree, counting the tips as level 1, is

\[
\dot{m}_n = -m_n + \langle \tanh(K(s_{1}^{(n-1)} + s_{2}^{(n-1)})\rangle,
\]

(5.6)

where, for a given spin at level \(n\), \(S_{1}^{(n-1)}\) and \(S_{2}^{(n-1)}\) are the two spins at the \((n - 1)\) th level that influence the given spin. Then, as for the \(d = 2\) case, one has immediately

\[
\dot{m}_n = -m_n + \tanh(2K) m_{n-1}.
\]

(5.7)

In the stationary state, this gives \(m_n = \tanh(2K) m_{n-1}\). Under iteration, the site magnetization is driven to zero, i.e. the interior of the tree remains unmagnetized, at any non-zero temperature, even when the spins at the tips are completely aligned (and recall that, if initially aligned, they will stay aligned forever, as nothing can change the tip spins in the directed model). We conclude that there is no phase transition to a ferromagnetic state in the directed Cayley tree with branching ratio 2.

It is instructive to contrast this with the properties of the undirected Cayley tree. In this case we will partially align the tip spins by applying a magnetic field \(h\) to these spins.
and ask whether, in the equilibrium state, the induced magnetization propagates into the interior. It is trivial to trace out the tip (i.e. level 1) spins to obtain a new model in which the new tip spins (now level 2) experience a field \( h' \) given by

\[
H' = q \tanh^{-1}[\tanh H \tanh K],
\]

where \( q \) is the branching ratio, \( H = h/T \), and \( K = J/T \), with \( T \) the temperature, now a physical variable. This recurrence relation has a trivial fixed point at \( H = 0 \), which is stable for \( q \tanh K < 1 \). In this regime, the interior of the tree is unmagnetized. For \( q \tanh K > 1 \), however, the recurrence relation has a stable non-trivial fixed point and spins in the interior of the tree have a non-zero mean value. These two regimes are separated by a critical coupling \( K_c \) given by \( \tanh K_c = 1/q \). We conclude that the undirected model has a phase transition for all \( q > 1 \), but for \( q = 2 \) the directed model does not.

To explore further the directed models we consider the case \( q = 3 \). Since each spin at the \( n \)th level is influenced by three spins at the \((n - 1)\)th level, the analogue of equation (5.6) is

\[
\dot{m}_n = -m_n + \langle \tanh K[S_1^{(n-1)} + S_2^{(n-1)} + S_3^{(n-1)}] \rangle.
\]

Exploiting the fact that \( S_1 + S_2 + S_3 \) can only take four values \((3, 1, -1, -3)\), and its odd parity under \( S_i \to -S_i \) for all \( i = 1, 2, 3 \), one can write

\[
\langle \tanh K[S_1^{(n-1)} + S_2^{(n-1)} + S_3^{(n-1)}] \rangle = 3A m_{n-1} - B \langle S_1^{(n-1)} S_2^{(n-1)} S_3^{(n-1)} \rangle,
\]

where

\[
A = [\tanh K + \tanh(3K)]/4, \quad B = [3 \tanh K - \tanh(3K)]/3.
\]

It is clear that spins at the same level are uncorrelated, since they are influenced by disjoint sets of spins at lower levels. It follows that \( \langle S_1^{(n-1)} S_2^{(n-1)} S_3^{(n-1)} \rangle = m_{n-1}^3 \). In the stationary state, therefore, equation (5.9) reduces to

\[
m_n = 3A m_{n-1} - B m_{n-1}^3.
\]

This recurrence relation has two fixed points: a trivial one \( m^* = 0 \), which is stable for \( A < 1/3 \), and a non-trivial one \( m^* = [(3A-1)/B]^{1/2} \), which is stable for \( A > 1/3 \). There is a critical coupling \( K_c \) obtained from the condition \( A = 1/3 \), i.e. \( \tanh K + \tanh(3K) = 4/3 \), with a solution \( K_c = 0.475 \, 3269 \). We deduce that the magnetization deep in the interior of the tree vanishes for \( K < K_c \), and is non-zero for \( K > K_c \), i.e. there is a phase transition at \( K = K_c \).

Higher values of the branching ratio \( q \) can be treated in a similar way. In each case one finds a phase transition at a critical coupling value \( K_c \), which decreases with increasing \( q \). For \( q = 4 \), for example, one finds \( K_c = 0.310 \, 2182 \).

It is interesting that the smallest integer \( q \) for which there is a phase transition on the undirected tree is \( q = 2 \), whereas on the directed tree it is \( q = 3 \). This is consistent with our observation that the 2D Ising model on a directed square lattice, in which each spin is influenced by only two other spins, has no phase transition and raises the question as

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to whether the 2D Glauber–Ising model on a triangular lattice, with each spin influenced by (say) three other spins, would have a phase transition.

6. Discussion and summary

In this paper we have addressed two open questions. The first, is under what circumstances the stationary states of directed Ising models have the conventional Boltzmann–Gibbs measure. In such cases, there will be a conventional ferromagnetic phase transition identical to that of the undirected Ising model, for space dimensions $d \geq 2$. We have shown that there is a unique set of rates for which the Boltzmann–Gibbs measure describes the stationary state in dimensions $d = 1$ and 2, both for the square and triangular lattices (where in the latter models each spin sees half of its neighbours). We have also shown that, for the cubic lattice in dimension $d = 3$, the only set of rates satisfying the constraints of stationary Gibbs measure are those satisfying detailed balance. In one and two dimensions it is actually quite simple to demonstrate that the rates (4.2) do lead to stationary states with the Boltzmann–Gibbs measure by simply substituting this measure into the stationary master equation. The advantage of the present approach is that it shows that these rates are unique. We have generalized this procedure and used it as a method for the determination of which lattice is able to sustain transition rates for directed dynamics with stationary Gibbs measure. We confirmed that the 2D triangular lattice is able to do so, while the 3D cubic lattice is not. More generally we claim that lattices of coordination number $z \geq 8$ cannot be directed, if their stationary measure is Gibbsian, and conclude that directed models with such measure only exist in dimensions one and two.

The second, related question, concerning the existence of phase transitions in directed Glauber–Ising models, was inspired by the numerical studies presented in [5], where it was demonstrated that a directed Glauber–Ising model on a square lattice, with each spin seeing only its north and east neighbours, does not exhibit a phase transition to a ferromagnetic state. Here we have provided a simple analytic proof that there is no transition. We have also shown that directed Cayley trees exhibit a phase transition when the branching ratio $q$ satisfies $q \geq 3$.

A number of open questions remain. A completion of the proof of the absence of rates producing a Gibbsian stationary state for directed lattices of coordination $z \geq 8$ would be desirable. It would be worth studying the 2D hexagonal lattice and see whether we can apply the method of sections 2 and 3 to this case. It would also be interesting to compare the dynamical properties of the directed and undirected models studied in the present work in one and two dimensions, when both share the same Gibbsian stationary measure.

Secondly, the results on the Cayley tree suggest that the coordination number plays an important role, and raises the question as to whether a 2D directed Glauber–Ising model in which each spin sees more than two neighbours (e.g. a square lattice in which each spin sees its north, east and west neighbours, or a triangular lattice) could have a phase transition. Another obvious question concerns the directed 3D model, with each spin seeing its ‘north’, ‘east’ and ‘Up’ neighbours. Does this model, with Glauber dynamics, have a ferromagnetic phase transition? The data of [5] suggest not. Unfortunately, an analytic study for this case does not seem straightforward.
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Appendix

In this appendix we give the form of the rates for the 2D directed Ising models on the square lattice with Hamiltonian defined with two coupling constants, $J_1$ and $J_2$.

For the model where each spin sees its north and east neighbours, equation (3.14) is replaced by

$$w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = \frac{\alpha}{2} (1 + \gamma_1 \gamma_2 \sigma_E \sigma_N - \sigma(\gamma_1 \sigma_E + \gamma_2 \sigma_N)), \quad (A.1)$$

where $\gamma_{1,2} = \tanh 2K_{1,2}$, which, after fixing the global timescale by the choice $\alpha = 2 \cosh 2K_1 \cosh 2K_2$, yields the unique solution:

$$w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = e^{-2\sigma(J_1 \sigma_E + J_2 \sigma_N)}. \quad (A.2)$$

For the model where each spin sees its west, north and east neighbours, equation (3.20) is replaced by

$$w(\sigma \sigma_E \sigma_N \sigma_W \sigma_S) = \frac{\alpha}{2} \left( 1 + \frac{\gamma_1 \gamma_2}{2} \sigma_N (\sigma_E + \sigma_W) - \frac{\sigma}{2} (\gamma_1 \sigma_W + 2 \gamma_2 \sigma_N + \gamma_1 \sigma_E) \right).$$

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