Nuclear Physics from lattice QCD
at strong coupling

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and in progress
This talk is about: hadron ↔ nuclear matter transition and $T = 0$ nuclear interactions
Nuclear physics from lattice QCD?

- Fundamental QCD theory is known:
  $$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i (i\gamma^\mu \partial_\mu + m_i) \psi_i$$

  (gluons + quarks) confined into hadrons

- One should be able to derive interactions between hadrons from QCD

- Lattice QCD:
  only known non-perturbative gauge-invariant regulator of QCD

- Derive nuclear physics from lattice QCD?
  Difficult but feasible when quark density is zero (ie. $\mu = 0$)

  Non-zero quark density $\Rightarrow$ sign problem

  State of the art: 2 nucleons  Hatsuda et al, Savage et al

\[ \beta = 0 \text{ LQCD} \]
**Motivation (1)**

**Strong coupling LQCD: why bother?**

Asymptotic freedom: \[ a(\beta_{\text{gauge}}) \propto \exp\left( -\frac{\beta_{\text{gauge}}}{4N_c b_0} \right) \]

ie. \( a \to 0 \) when \( \beta_{\text{gauge}} \equiv \frac{2N_c}{g^2} \to +\infty \). Here \( \beta_{\text{gauge}} = 0 \) !!

- Lattice “infinitely coarse”
- Physics not universal

Nevertheless:
- Properties similar to QCD: confinement and \( \chi_{\text{SB}} \)
- Include (perhaps) next term in strong coupling expansion, ie. \( \beta_{\text{gauge}} > 0 \)

When \( \beta_{\text{gauge}} = 0 \), sign problem is **manageable** \( \to \) **complete solution**

Valuable insight?
Motivation (2)

- 25+ years of analytic predictions:
  80’s: Kluberg-Stern et al., Kawamoto-Smit, Damgaard-Kawamoto
  \( \mu_c(T = 0) = 0.66, \quad T_c(\mu = 0) = 5/3 \)
  90’s: Petersson et al., \(1/\beta^2\) corrections
  00’s: detailed \((\mu, T)\) phase diagram: Nishida, Kawamoto,...

  now: Ohnishi et al. \( O(\beta) \) & \( O(\beta^2) \), Münster & Philipsen,...

  How accurate is mean-field \((1/d)\) approximation?

- Almost no Monte Carlo crosschecks:
  89: Karsch-Mütter \(\rightarrow\) MDP formalism \(\rightarrow\) \( \mu_c(T = 0) \sim 0.63 \)
  92: Karsch et al. \( T_c(\mu = 0) \approx 1.40 \)
  99: Azcoiti et al., MDP ergodicity ??

  06: PdF-Kim, HMC \(\rightarrow\) hadron spectrum \(\sim 2\%\) of mean-field

Can one trust the details of analytic phase-diagram predictions?
Phase diagram from Nishida (2004, mean field, cf. Fukushima)

- Very similar to conjectured phase diagram of $N_f = 2$ QCD
- But no deconfinement here: high density phase is nuclear matter
- Baryon mass $= M_{\text{proton}} \Rightarrow$ lattice spacing $a \sim 0.6$ fm not universal
Strong coupling $SU(3)$ with staggered quarks: why simpler?

$$Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(-\bar{\psi}(\mathcal{D}(U)+m)\psi),$$
no plaquette term ($\beta_{gauge} = 0$)

- One colored fermion field per site (6 d.o.f. – no Dirac indices, spinless)
- $\mathcal{D}(U) = \frac{1}{2} \sum_{x,v} \eta_v(x)(U_v(x) - U_v^\dagger(x-\mathbf{\hat{v}}))$, $\eta_v(x) = (-)^{x_1 + \ldots + x_{v-1}}$

$U(1)_V \times U(1)_A$ symmetry:

$$\begin{align*}
\psi(x) &\rightarrow e^{i\theta} \psi(x) \\
\bar{\psi}(x) &\rightarrow e^{-i\theta} \bar{\psi}(x)
\end{align*} \quad \text{unbroken} \quad \Rightarrow \quad \text{quark number} \quad \Rightarrow \quad \text{chem. pot.}$$

$$\begin{align*}
\psi(x) &\rightarrow e^{i\varepsilon(x)\theta} \psi(x) \\
\bar{\psi}(x) &\rightarrow e^{i\varepsilon(x)\theta} \bar{\psi}(x) \\
\varepsilon(x) &= (-)^{x_0 + x_1 + x_2 + x_3}
\end{align*} \quad \text{spont. broken ($m = 0$)} \quad \Rightarrow \quad \text{quark condensate}$$
Strong coupling $SU(3)$ with staggered quarks: why simpler?

\[ Z = \int D U \bar{\psi} D \psi \exp(-\bar{\psi}(\mathcal{D}(U) + m)\psi), \] no plaquette term ($\beta_{\text{gauge}} = 0$)

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- Chemical potential $\mu \rightarrow \exp(\pm a\mu) U_{\pm4}$

- Alternative 1: integrate over fermions
  \[ Z = \int D U \det(\mathcal{D}(U) + m) \rightarrow \text{HMC, severe sign pb. for } \mu \neq 0 \]
Strong coupling $SU(3)$ with staggered quarks: why simpler?

$$Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp(-\bar{\psi}(\not{D}(U) + m)\psi), \quad \text{no plaquette term (}\beta_{\text{gauge}} = 0\text{)}$$

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- **Alternative 1**: integrate over fermions
  $$Z = \int \mathcal{D}U \det(\not{D}(U) + m) \rightarrow \text{HMC, severe sign pb. for } \mu \neq 0$$

- **Alternative 2**: $\mathcal{D}U = \prod dU$ factorizes $\rightarrow$ integrate over links Rossi & Wolff
  $\quad \rightarrow$ **Color singlet** degrees of freedom:
  - **Monomer** (meson $\bar{\psi}\psi$) $M(x) \in \{0, 1, 2, 3\}$
  - **Dimer** (meson hopping), non-oriented $n_{\nu}(x) \in \{0, 1, 2, 3\}$
  - **Baryon** hopping, oriented $\bar{B}B_{\nu}(x) \in \{0, 1\}$ $\rightarrow$ **self-avoiding loops** $C$

**Point-like, hard-core baryons in pion bath**
Strong coupling $SU(3)$ with staggered quarks: why simpler?

\[
Z = \int \mathcal{D} U \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp(-\bar{\psi}(\mathcal{D}(U) + m)\psi), \text{ no plaquette term (} \beta_{\text{gauge}} = 0) \]

- One colored fermion field per site (6 d.o.f. – no Dirac indices, spinless)
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- Chemical potential $\mu \rightarrow \exp(\pm a\mu) U \pm 4$
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- Alternative 2: $\mathcal{D} U = \prod dU$ factorizes → integrate over links Rossi & Wolff → \textbf{Color singlet} degrees of freedom:
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\[
Z(m, \mu) = \sum_{\{M, n_\nu, C\}} \prod_x \frac{m^{M(x)}}{M(x)!} \prod_{x, \nu} \frac{(3 - n_\nu(x))!}{n_\nu(x)!} \prod_{\text{loops } C} \rho(C)
\]
with constraint $(M + \sum_{\pm} n_\nu)(x) = 3 \ \forall x \notin \{C\}$
**MDP Monte Carlo**

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\]

with **constraint** \( (M + \sum_{\pm n_v} n_v)(x) = 3 \forall x \notin \{C\} \)

- **sign** of \( \prod_C \rho(C) \): geometric factor \( \varepsilon(C) = \pm 1 \) for each loop \( C \); 4 types:

  - \( \varepsilon(C) \exp(+3 \frac{\mu}{T}) \)
  - \( \varepsilon(C) \exp(-3 \frac{\mu}{T}) \)
  - +1
  - +1

Karsch & Mütter: Resum into “MDP ensemble” \( \rightarrow \) sign pb. eliminated at \( \mu = 0 \)

\[
\begin{align*}
1 + \varepsilon(C) \cosh(3 \frac{\mu}{T}) & \quad 1 + \varepsilon(C) \cosh(3 \frac{\mu}{T}) \\
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  \[\rightarrow \text{“MDP ensemble”}\]

Further, algorithmic difficulties:

- changing monomer number difficult: weight \(\sim m^{\sum_x M(x)}\)
  monomer-changing update (Karsch & Mütter) restricted to \(m \sim O(1)\)
MDP Monte Carlo

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- \textbf{tight-packing constraint} \(\rightarrow\) local update inefficient, esp. as \(m \rightarrow 0\)
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Solved with **worm algorithm** (Prokof’ev & Svistunov 1998)
Worm algorithm for MDP

- Sample $G(x, y)$ rather than $Z$, ie. add source and sink
- Monte Carlo: guided random walk of sink, ie. *local* steps ("worm")
- When $y = x$, contribution to $Z$ $\rightarrow$ *global* change
- cf. “directed path” (Adams & Chandrasekharan) for $U(N)$
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- When $y = x$, contribution to $Z \rightarrow \text{global}$ change
- cf. “directed path” (Adams & Chandrasekharan) for $U(N)$
- Efficient even when $m_q = 0
[Non-trivial] consistency check with HMC

Worm–MDP vs. HMC (Forcrand and Kim ’06) $\beta = 0$, same volume ($\mu = T = 0$)
Sign problem? Monitor $-\frac{1}{V} \log \langle \text{sign} \rangle$

$\langle \text{sign} \rangle = \frac{Z_{\uparrow}}{Z} \sim \exp(-\frac{V}{T} \Delta f(\mu^2))$ as expected; $\Delta f \sim \mu^2 + O(\mu^4)$

Can reach $\sim 16^3 \times 4 \ \forall \mu$, ie. adequate
Phase diagram in the chiral limit $m_q = 0$

- Phase boundary for breaking/restoration of $U(1)$ chiral symmetry
- Mean field analysis: 2nd order at $\mu = 0$, $T_c = 5/3$
- If 2nd order, then expect 3d $O(2)$ universality class
- Monte Carlo: 2nd order at $\mu = 0$ (Karsch et al, 1992)
- 1st order at $T = 0$: $\rho_B$ jumps from 0 to 1 baryon per site $\rightarrow$ tricrit. pt. TCP
Phase diagram in the chiral limit

\[ \chi_\sigma = \frac{1}{V} \frac{\partial^2}{\partial m_q^2} \log Z = \langle \sum_x \bar{\psi}_x \psi(x) \bar{\psi}_0 \psi(0) \rangle \sim L^{\gamma/\nu} \tilde{\chi}(tL^{1/\nu}) \]

\[ \rightarrow \frac{\chi_\sigma}{L^{\gamma/\nu}} \text{ is universal function of } tL^{1/\nu} \]

- Data collapse using 3d O(2) exponents for \( a\mu = 0 \) and 0.30
• Data collapse using mean-field exponents (d=3 is TCP upper crit. dim.) for $a\mu = 0.33$

• 1st-order Borgs-Kotecky: $Z(T) = \exp(-\frac{V}{T}f_1(T)) + c\exp(-\frac{V}{T}f_2(T))$ for $a\mu = 0.36$
Compare with Nishida (2004) for $m_q = 0$

- TCP: $(\mu, T) = (0.33(3), 0.94(7))$ (Monte Carlo) vs $(0.577, 0.866)$ (mean-field)

- No reentrant phase diagram (caused by decreasing entropy in dense phase)  
  cf. Clausius-Clapeyron: $\frac{dT}{d\mu} = -\frac{\Delta n}{\Delta S} \rightarrow$ vertical at $T = 0$

- Beware of quantitative mean-field predictions for phase diagram
Phase diagram away from the chiral limit (in progress)

- Qualitatively similar to mean-field
- CEP moves fast with $m_q$, following tricritical scaling
Transition to nuclear matter: $T = 0, \mu = \mu_c$

**Puzzle:**
- Mean-field baryon mass is $\approx 3 \Rightarrow$ expect $\mu_c = \frac{1}{3} F_B(T = 0) \approx 1$
- Mean-field estimate $\mu_c \sim 0.55 - 0.66$ much smaller, ie. $\mu_c^B \sim 600$ MeV!

  Mean field gives wrong $M_B$? wrong $\mu_c$?

- Check $M_B$ and $\mu_c$ by Monte Carlo $\Rightarrow$ ok (next slide)

**Remaining explanation:** nuclear attraction $\sim 1/3$ baryon mass !!

Why so large? Nuclear potential? Nuclear spectroscopy??
\( \mu_c(T = 0) \)

- \( T = 0 \) dense phase is \textit{baryon crystal} (1 baryon per site)
- \( \mu_c \) is free energy necessary to add 1 baryon to dense phase

- Monte Carlo: \( a\mu_c^B = 1.78(1) \) compared with \( am_B = 2.88(1) \)
- Each baryon binds to 3 nearest-neighbours \( \rightarrow \) attraction
  \[ V_{NN}(r = a) \sim \frac{2.88-1.78}{3} a^{-1} \sim 120 \text{ MeV} !! \]
- \textbf{Surface tension}: first layer of dense phase \( \rightarrow \) 2 nearest-neighbours only
  \[ \sigma \approx \frac{1}{2}|V_{NN}(r = a)|a^{-2} \sim (200 \text{ MeV})^3 \]
- Complete nuclear potential ?
• Nucleons are point-like → no ambiguity with definition of static potential
  
• Nearest-neighbour attraction \( \sim 120 \text{ MeV} \) at distance \( \sim 0.5 \text{ fm} \): cf. real world  
  Baryon worldlines self-avoiding → no meson exchange here (just hard core)  
  Attraction due to bath of neutral pions: cf. Casimir effect (see later)
Nuclear spectroscopy

- Can compare masses of differently shaped “isotopes”
- \( E(B = 2) - 2E(B = 1) \approx -0.4 \), i.e. “deuteron” binding energy ca. 120 MeV
- \( a m(A) \approx a \mu_B^{\text{crit}} A + (36\pi)^{1/3} \sigma a^2 A^{2/3} \), i.e. (bulk + surface tension)
  - Bethe-Weizsäcker parameter-free (\( \mu_B^{\text{crit}} \) and \( \sigma \) fixed)
- “Magic numbers” with increased stability: \( A = 4, 8, 12 \)
Where does the mass of the nucleon come from?

- Point-like nucleon **distorts pion bath** cf. Casimir

  ![Diagram](image)

- Energy = nb. time-like pion lines

  Constraint: 3 pion lines per site ($m_q = 0$) $\rightarrow$ energy density $= 3/4$ in vacuum

  No spatial pion lines connecting to site occupied by nucleon $\rightarrow$ energy increase

  **Steric effect**

  - $am_B \approx 2.88 = (3 - 0.75) + \Delta E_\pi$, ie. "valence"(78%) + "pion cloud"(22%)
Where does the mass of the nucleon come from?

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![Graph showing energy dependence on R/a](image)

- Energy = nb. time-like pion lines
  
  Constraint: 3 pion lines per site \((m_q = 0)\) → energy density = 3/4 in vacuum

  No spatial pion lines connecting to site occupied by nucleon → energy increase **Steric effect**

- \(am_B \approx 2.88 = (3 - 0.75) + \Delta E_\pi\), ie. "valence"(78%) + "pion cloud"(22%)
Nuclear interaction as *steric* effect

- Here, baryons make self-avoiding loops → no direct meson exchange

- Interaction comes from **overlapping pion clouds**

- Example: nearest-neighbour nucleon attraction
  - Energy of pion cloud mostly in 6 nearest-neighbours of nucleon
  - Nearest-neighbour nucleon pair has only 10 nearest-neighbour sites

\[
\Delta E_\pi(2 \text{ nucleons}) < 2\Delta E_\pi(1 \text{ nucleon}) \rightarrow \text{attraction}
\]
Conclusions

Summary

- Take mean-field results with a grain of salt
- “Clean-up” of phase diagram justified
- [Crude] nuclear matter from QCD
- “Understand” nuclear interaction as steric effect

Outlook

- Non-zero quark mass:
  - Critical end-point as a function of $m_q$
  - Nuclear potential & spectroscopy as a function of $m_q$
- Include second quark species → isospin (degenerate masses or not)
- Include $O(\beta)$ effects?
Backup slide: Influence of pion mass on nuclear potential

- Nearest-neighbour attraction weakens (as expected). Not much else.
Backup slide: $\sigma(\beta)$ effects

**Phase Diagram**

- **Strong Coupling Limit**
- **Baryonic Effects**
- **1st order**
- **2nd order**
- **TCP**
- **CEP**
- **cross over**

**Reality** $(1/g^2, m_0, N_f, ...)$

**Forcrand-Philipsen**

**Fodor-Katz**

**MC / Real World**

$\beta = 0$ LQCD