Dynamical r-matrices
and the chiral WZNW phase space

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Abstract

The dynamical generalization of the classical Yang-Baxter equation that governs the possible Poisson structures on the space of chiral WZNW fields with generic monodromy is reviewed. It is explained that for particular choices of the chiral WZNW Poisson brackets this equation reduces to the CDYB equation recently studied by Etingof–Varchenko and others. Interesting dynamical r-matrices are obtained for generic monodromy as well as by imposing Dirac constraints on the monodromy.

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1 Introduction

The classical and quantum Yang-Baxter equations occupy a central position in the modern theory of integrable systems. Recently dynamical generalizations of these structures attracted considerable attention. Of particular interest for us is the classical dynamical Yang-Baxter (‘CDYB’) equation given by

\[ [\hat{r}_{12}(\omega), \hat{r}_{23}(\omega)] + H_i^i \frac{\partial}{\partial \omega^i} \hat{r}_{23}(\omega) + \text{cycl. perm.} = C \hat{f}, \]

where the variable \( \omega \) lies in a Cartan subalgebra \( \mathcal{H} \) of a simple Lie algebra \( G \), \( \{H^i\} \) is a basis of \( \mathcal{H} \), \( C \) is some constant, and \( \hat{f} \) is the canonical \( G \)-invariant element in \( G^{3\Lambda} \). It is usually assumed that \( \hat{r}(\omega) \) is \( \mathcal{H} \)-invariant and its symmetric part is proportional to the ‘tensor Casimir’. The CDYB equation is the classical limit of the Gervais-Neveu-Felder equation

\[ R_{12}(\omega + \hbar H_3)R_{13}(\omega)R_{23}(\omega + \hbar H_1) = R_{23}(\omega)R_{13}(\omega + \hbar H_2)R_{23}(\omega). \]

These equations govern the classical and quantum exchange algebras of the chiral Bloch waves in the conformal Toda and WZNW field theories on the cylinder [1, 2, 3]. They also appear in the description of the conformal blocks of the WZNW model on the torus [4] and in the study of Calogero-Moser models [5]. The solutions of these equations and the underlying abstract algebraic structures, the so called dynamical Poisson-Lie (PL) groupoids and dynamical quantum groups, have been studied recently in details by Etingof and Varchenko. See the review [6] and references therein, where further applications are described, too. In the paper [7] generalizations of the CDYB equation were introduced, which are obtained from (1) by replacing the Cartan subalgebra with an arbitrary subalgebra of \( G \). We here call this generalization the \( \mathcal{H} \)-CDYB equation allowing \( \mathcal{H} \subseteq G \) to be any subalgebra.

We have recently investigated the possible chiral extensions of the WZNW phase space and found that a new generalization of the CDYB equation naturally arises in this context [8]. This equation will be called the \( G \)-CDYB equation, since its dynamical variable lies in the group \( G \) associated with \( G \). The \( G \)-CDYB equation encodes the most general PBs of the chiral WZNW fields with generic monodromy. Any solution of this equations defines also a PL groupoid. Under some special circumstances, when the natural gauge transformations act on the chiral WZNW phase space as a classical \( G \)-symmetry, our \( G \)-CDYB reduces to \( G \)-CDYB (i.e. \( \mathcal{H} \)-CDYB for \( \mathcal{H} = G \)). Then Dirac reductions of the chiral WZNW phase space result in dynamical r-matrices that solve \( \mathcal{H} \)-CDYB for self-dual subalgebras \( \mathcal{H} \subset G \). For instance, we recover in this way the fundamental solution of the original CDYB equation that was first obtained in [3] by a different method.
It will be illustrated by this report that the chiral WZNW phase space serves as an effective source of dynamical r-matrices. The quantization of these r-matrices and their associated Poisson-Lie groupoids should contribute to a better understanding of the quantum group properties of the WZNW model, but this issue is not yet properly understood.

2 \textit{$G$-CDYB and PL groupoids from chiral WZNW}

The WZNW model as a classical field theory on the cylinder can be defined for any (real or complex) Lie group $G$ whose Lie algebra $\mathfrak{g}$ carries an invariant, non-degenerate bilinear form $\langle \ , \ \rangle$. The scalar product is proportional to $\text{tr}(XY)$ if $G$ is a simple Lie algebra, and to ease the notations we shall denote $\langle X,Y \rangle (\forall X,Y \in \mathfrak{g})$ as $\text{Tr}(XY)$ in general. The solution of the classical field equation for the $G$-valued WZNW field, which is $2\pi$-periodic in the space variable, turns out to be the product of left- and right-moving factors. The chiral WZNW fields are quasi-periodic, i.e., are elements in $M_G := \{ g \in C^\infty(\mathbb{R},G) | g(x+2\pi) = g(x)M \quad M \in G \}$. (3)

Since the chiral factors of the full WZNW field are determined only up to a gauge freedom, the symplectic structure of the WZNW model does not yield a unique Poisson bracket (PB) on $\mathcal{M}_G$. In fact, as explained in [10], $\mathcal{M}_G$ is equipped canonically only with a quasi-Poisson structure in the sense of [11]. To describe the system in terms of genuine PBs for the chiral fields and an associated chiral symplectic form [12], in general one needs to restrict oneself to a submanifold of $\mathcal{M}_G$, where the monodromy matrix $M$ belongs to some submanifold $\tilde{G} \subset G$. A condition on $\tilde{G}$ is that the canonical closed 3-form of $G = \{ M \}$, given by $\chi = \frac{1}{6} \text{Tr} (M^{-1}dM \wedge M^{-1}dM \wedge M^{-1}dM)$, must become exact upon restriction to $\tilde{G} \subset G$. One may then choose a 2-form $\rho$ on $\tilde{G}$ for which $d\rho = \chi_{\tilde{G}}$, where $\chi_{\tilde{G}}$ is the restriction of $\chi$ to $\tilde{G}$. Any such $\rho$, one can define a closed 2-form $\Omega^\rho$ on

$\mathcal{M}_{\tilde{G}} := \{ g \in C^\infty(\mathbb{R},G) | g(x+2\pi) = g(x)M \quad M \in \tilde{G} \}$

by the following formula:

\[ \frac{1}{\kappa} \Omega^\rho = -\frac{1}{2} \int_0^{2\pi} dx \text{Tr} \left( g^{-1}dg \right) \wedge \left( g^{-1}dg \right)^\prime - \frac{1}{2} \text{Tr} \left( (g^{-1}dg)(0) \wedge dM M^{-1} \right) + \rho(M), \] (5)

where $\kappa$ is a constant. If a further condition is satisfied, which we shall state below, then $\Omega^\rho$ is (weakly) non-degenerate, and thus it can be inverted to define PBs on a set of ‘admissible’ functions of the chiral WZNW field. The derivation of $\Omega^\rho$ from the symplectic structure of the full WZNW model is due to Gawedzki [12].

One may ensure the exactness of $\chi_{\tilde{G}}$ by choosing $\tilde{G} \subset G$ to be a topologically trivial open submanifold. In this case, the following description of the chiral PBs was obtained
in [8] by extending the results of [13]. In fact, the PBs of all admissible functions [8] are encoded by the ‘distribution valued’ PBs of the matrix elements of \( g(x) \), which have the form

\[
\left\{ g(x), g(y) \right\} = \frac{1}{\hbar} \left( g(x) \otimes g(y) \right) \left( \frac{i}{2} \hat{I} \text{sign}(y - x) + \hat{r}(M) \right), \quad 0 < x, y < 2\pi.
\] (6)

Here the interesting object is the ‘exchange r-matrix’ \( \hat{r}(M) = r^{ab}(M)T_a \otimes T_b \in \mathcal{G} \otimes \mathcal{G} \); \( \hat{I} = T_a \otimes T_a \) where \( \{ T_a \} \) and \( \{ T^a \} \) denote dual bases of \( \mathcal{G} \), \( \text{Tr}(T_a T^b) = \delta^b_a \), and summation over coinciding indices is understood. The Jacobi identity of the PB is equivalent to a dynamical generalization of the CYB equation, which we call the G-CDYB equation. To write it down, on functions \( \psi \) on \( G \) we introduce the derivations \( D^\pm_a = \mathcal{R}_a \pm \mathcal{L}_a \) by

\[
(\mathcal{R}_a \psi)(M) := \frac{d}{dt} \psi(e^{iTa}M)|_{t=0}, \quad (\mathcal{L}_a \psi)(M) := \frac{d}{dt} \psi(e^{iTa}M)|_{t=0}.
\] (7)

The G-CDYB equation [8] reads as

\[
\left[ \hat{r}_{12}(M), \hat{r}_{23}(M) \right] + T_1^a \left( \frac{1}{2} D^+_a + r^b(M)D^-_b \right) \hat{r}_{23}(M) + \text{cycl. perm.} = -\frac{1}{4} \hat{f},
\] (8)

where \( \hat{f} := f_{ab} cT^a \otimes T^b \otimes T_c \) with \( [T_a, T_b] = f_{ab} cT_c \) and the cyclic permutation is over the three tensorial factors with \( \hat{r}_{23} = r^{ab}(1 \otimes T_a \otimes T_b) \), \( T_1^a = T^a \otimes 1 \otimes 1 \) and so on. This equation becomes the modified classical YB equation if \( \hat{r} \) is an \( M \)-independent constant, and at the same time it is a generalization of the CDYB equation [8].

The exchange r-matrix that results from the inversion of a symplectic form in [8] automatically satisfies [8]. To describe its dependence on the 2-form \( \rho \), expand \( \rho \) as

\[
\rho(M) = \frac{1}{2} q^{ab}(M) \text{Tr}(T_a M^{-1}dM) \land \text{Tr}(T_b M^{-1}dM), \quad q^{ab} = -q^{ba}.
\]

Denote by \( q(M) \) and \( r(M) \) the linear operators on \( \mathcal{G} \) whose matrices are \( q^{ab}(M) \) and \( r^{ab}(M) \), respectively. Introduce also the operators \( q_{\pm}(M) := q(M) \pm \frac{i}{2} I \) and \( r_{\pm}(M) := r(M) \pm \frac{1}{2} I \), where \( I \) is the identity operator on \( \mathcal{G} \). It is proved in [8] that the inversion of \( \Omega^\rho \) leads to [8] with

\[
r_-(M) = -q_-(M) \circ (q_- - \text{Ad} M \circ q_+(M))^{-1}.
\] (9)

The condition on the pair \( (\hat{G}, \rho) \) that guarantees the non-degeneracy of the 2-form \( \Omega^\rho \) is that \( (q_-(M) - \text{Ad} M \circ q_+(M)) \in \text{End}(\mathcal{G}) \) must be an invertible operator for any \( M \in \hat{G} \). This can be ensured by restricting \( M \) to be near enough to \( e \in G \).

Any solution of (8) on some domain \( \hat{G} \) gives rise to a PB [8] on \( M_{\hat{G}} \) and any such PB implies that \( J := \kappa g g^{-1} \) satisfies the standard current algebra PBs and \( g(x) \) is a primary field with respect to the current algebra. The exchange r-matrix drops out from the PBs with any function of the current \( J \), and thus it encodes the ‘non-current-algebraic’ aspects of the infinite dimensional chiral WZNW phase space. Remarkably, the exchange r-matrix also defines the PBs of an associated Poisson-Lie groupoid, as described below.

Let \( \hat{r} \) be a solution of (8) on \( \hat{G} \) and define \( \mathcal{G} \otimes \mathcal{G} \)-valued functions on \( \hat{G} \) by

\[
\hat{\Theta}(M) = \hat{r}_+(M) - M_2^{-1} \hat{r}_-(M) M_2, \quad \hat{\Delta}(M) = \hat{\Theta}(M) - M_1^{-1} \hat{\Theta}(M) M_1
\] (10)
with $M_1 = M \otimes 1$, $M_2 = 1 \otimes M$. Then introduce on the manifold $P$ given by

$$P := \hat{G} \times G \times \hat{G} := \{ (M^F, g, M^I) \}$$

(11)
a PB $\{\ ,\ \}$ by the following formulas:

$$\kappa\{g_1, g_2\}_P = g_1 g_2 \hat{r}(M^I) - \hat{r}(M^F) g_1 g_2$$
$$\kappa\{g_1, M^I_2\}_P = g_1 M^I_2 \hat{\Theta}(M^I)$$
$$\kappa\{g_1, M^F_2\}_P = M^F_2 \hat{\Theta}(M^F) g_1$$
$$\kappa\{M^I_1, M^I_2\}_P = M^I_1 M^I_2 \hat{\Delta}(M^I)$$
$$\kappa\{M^F_1, M^F_2\}_P = -M^F_1 M^F_2 \hat{\Delta}(M^F)$$
$$\kappa\{M^I_1, M^F_2\}_P = 0.$$  

(12)

$P$ is an example of the simplest sort of groupoids [14]: the base is $\hat{G}$, the source and target projections operate as $s : (M^F, g, M^I) \mapsto M^I$ and $t : (M^F, g, M^I) \mapsto M^F$, and the partial multiplication is defined by $(M^F, g, M^I)(\bar{M}^F, \bar{g}, \bar{M}^I) := (M^F, g\bar{g}, \bar{M}^I)$ for $M^I = \bar{M}^F$. $P$ is a Poisson-Lie groupoid in the sense of [15]. This means that the graph of the partial multiplication, i.e. the subset of

$$P \times P \times P = \{ (M^F, g, M^I) \} \times \{ (\bar{M}^F, \bar{g}, \bar{M}^I) \} \times \{ (\hat{M}^F, \hat{g}, \hat{M}^I) \}$$

(13)
defined by the constraints $M^I = \bar{M}^F$, $\bar{M}^F = M^F$, $\hat{M}^I = \bar{M}^I$, $\hat{g} = g\bar{g}$, is a coisotropic submanifold of $P \times P \times P^-$, where $P^-$ denotes the manifold $P$ endowed with the opposite of the PB on $P$. In other words, the graph is defined by imposing first class constraints on the Poisson space $P \times P \times P^-$ equipped with the natural direct product PB. This would actually hold for any choice of the structure functions $\hat{r}(M)$, $\hat{\Theta}(M)$ and $\hat{\Delta}(M)$ in (12), and the choice (11) in terms of a solution of (8) guarantees the Jacobi identity for $\{\ ,\ \}$.

We have extracted a PL groupoid from any symplectic structure $\Omega^\rho$ on the chiral WZNW phase space. If the exchange r-matrix is constant, then the PL groupoid $P$ carries the same information as the group $G$ endowed with the corresponding Sklyanin bracket. It is an open problem to study these PL groupoids further in the general case, to understand their quantization and relate them to the quantized (chiral) WZNW conformal field theory.

3 $G$-CDYB from $G$-symmetry and $H$-CDYB from Dirac reductions

We next describe an interesting special case of the chiral WZNW symplectic structure $\Omega^\rho$, for which the corresponding exchange r-matrix becomes a solution of the $G$-CDYB
equation mentioned in the introduction, and then consider some Dirac reductions.

Let us suppose that \( \hat{G} \) is diffeomorphic to a domain \( \hat{G} \subset \hat{G} \) by the exponential parametrization, whereby we write \( \hat{G} \cong M = e^{\omega} \) with \( \omega \in \hat{G} \). The chiral WZNW fields whose monodromy lies in \( \hat{G} \) can be parametrized as

\[
g(x) = \eta(x)e^{\bar{\omega}x}, \quad \bar{\omega} := \frac{\omega}{2\pi}, \quad \eta \in \hat{G}. \tag{14}
\]

Here \( \hat{G} = \{ \eta \in C^{\infty}(\mathbb{R}, G) \mid \eta(x + 2\pi) = \eta(x) \} \) and equation (14) defines the identification \( \mathcal{M}_G = \hat{G} \times \hat{G} = \{(\eta, \omega)\} \). If we now choose the 2-form \( \rho \) on \( G \simeq \hat{G} \) to be

\[
\rho_0(\omega) = -\frac{1}{2} \int_0^{2\pi} dx \text{Tr} \left( d\bar{\omega} \wedge de^{x\bar{\omega}} e^{-x\bar{\omega}} \right), \tag{15}
\]

then in terms of the variables \( \eta \) and \( \omega \) we find

\[
\frac{1}{\kappa} \Omega^{\rho_0} = -\frac{1}{2} \int_0^{2\pi} dx \text{Tr} \left( \eta^{-1}d\eta \right) \wedge \left( \eta^{-1}d\eta \right) + d \int_0^{2\pi} dx \text{Tr} \left( \bar{\omega}\eta^{-1}d\eta \right). \tag{16}
\]

\( \Omega^{\rho_0} \) is invariant under the natural action of the group \( G \) on \( \mathcal{M}_G \) given by

\[
G \ni h : g(x) \mapsto g(x)h^{-1} \quad \text{i.e.} \quad \eta(x) \mapsto \eta(x)h^{-1}, \quad \omega \mapsto h\omega h^{-1}. \tag{17}
\]

Note that \( \hat{G} \) is assumed to be invariant under the action of \( G \), otherwise one has to consider the corresponding \( G \)-action. Since \( \Omega^{\rho_0} \) is symplectic, which may be ensured by taking \( \hat{G} \) to be a neighbourhood of 0, the \( G \)-symmetry obtained from (17) is generated by a classical momentum map. The value of this \( \hat{G} \simeq \hat{G}^* \) valued momentum map is proportional to \( \omega \). In fact, we can calculate that the PB \( \{ , \} \) corresponding to \( \Omega^{\rho_0} \) gives

\[
\{g(x), \omega_a\}_0 = \frac{1}{\kappa} g(x)T_a, \quad \{\omega_a, \omega_b\}_0 = -\frac{1}{\kappa} f_{ab}^c \omega_c. \tag{18}
\]

Moreover, we have

\[
\left\{ g(x) \otimes g(y) \right\}_0 = \frac{1}{\kappa} \left( g(x) \otimes g(y) \right) \left( \frac{1}{2} \hat{I} \text{sign} (y - x) + \hat{r}^0(\omega) \right), \quad 0 < x, y < 2\pi, \tag{19}
\]

where \( \hat{r}^0(\omega) \) denotes the exchange r-matrix associated with \( \rho_0(\omega) \) by (9). Now the Jacobi identity of the PB for the functions \( \omega_a, g(x), g(y) \ (x \neq y) \) and the relations in (18) imply that \( \hat{r}^0 \) is a \( G \)-equivariant function on \( \hat{G} \):

\[
\frac{d}{dt} \hat{r}^0(e^{T\omega} e^{-tT})|_{t=0} = [T \otimes 1 + 1 \otimes T, \hat{r}^0(\omega)] \quad \forall T \in \mathcal{G}. \tag{20}
\]

In the present case, the Jacobi identity for 3 evaluation functions \( g(x_i) \ (x_i \neq x_j) \) gives a simplified version of the \( G \)-CDYB equation (8). Namely, the Jacobi identity and (18) imply that

\[
[r_{12}^0(\omega), r_{23}^0(\omega)] + T^a_i \frac{\partial}{\partial \omega^a} r_{23}^0(\omega) + \text{cycl. perm.} = -\frac{1}{4} \hat{f} (\hat{G} \ni \omega = \omega^a T_a). \tag{21}
\]
This is nothing but the $\mathcal{G}$-CDYB equation mentioned in the introduction. We stress that this equation follows as a consequence of the Jacobi identity of the PBs $\mathcal{I}$ and $\mathcal{I}$.

We can now determine $r^0(\omega)$ explicitly from $\mathcal{I}$ and thereby find a solution of $\mathcal{I}$. The result $\mathcal{I}$ is given by $r^0(\omega) = f_0(\text{ad} \omega)$ with $f_0$ being the power series expansion of the complex analytic function

$$f_0(z) = \frac{1}{2} \coth \frac{z}{2} - \frac{1}{z} \quad (22)$$

around $z = 0$. This solution of $\mathcal{I}$ was found in a different context $\mathcal{I}$, too.

In $\mathcal{I}$ the CDYB equation $\mathcal{I}$ was generalized by allowing the dynamical variable to belong to the dual of an arbitrary subalgebra $\mathcal{H} \subset \mathcal{G}$. Next we explain that if $\mathcal{H} \subset \mathcal{G}$ is a ‘self-dual’ subalgebra, then some solutions of the $\mathcal{H}$-CDYB equation arise from the solutions of $\mathcal{I}$ and $\mathcal{I}$ upon applying Dirac reduction to the associated PB on $\mathcal{M}_G$.

We now start by considering a PB of the form $\mathcal{I}$ on $\mathcal{M}_G$ and also suppose that $\mathcal{I}$ holds where $\omega = \log M$ varies in a domain $\tilde{\mathcal{G}} \subset \mathcal{G}$. As we have seen, then the exchange r-matrix $\tilde{r}^0(\omega) \in \mathcal{G} \wedge \mathcal{G}$ satisfies $\mathcal{I}$ and $\mathcal{I}$. We choose a Lie subalgebra $\mathcal{H} \subset \mathcal{G}$ and assume that the restriction of the scalar product of $\mathcal{G}$ remains non-degenerate on $\mathcal{H}$, which means that $\mathcal{H}$ is ‘self-dual’. We have the linear direct sum decomposition $\mathcal{G} = \mathcal{H} + \mathcal{H}^\perp$ and can introduce an adapted basis of $\mathcal{G}$ in the form $\{T_a\} = \{H_i\} \cup \{E_\alpha\}$, $H_i \in \mathcal{H}$, $E_\alpha \in \mathcal{H}^\perp$ with dual basis $\{T^a\} = \{H^i\} \cup \{E_\alpha\}$. (The notation is motivated by the ‘principal example’ for which $\mathcal{H}$ is a Cartan subalgebra of a simple Lie algebra.) Correspondingly, we can write

$$\omega = \omega_{\mathcal{H}} + \omega_{\mathcal{H}^\perp} = \omega^i H_i + \omega^\alpha E_\alpha. \quad (23)$$

We wish to impose the Dirac constraint $\omega_{\mathcal{H}^\perp} = 0$ on the PB on $\mathcal{M}_G$. To calculate the resulting Dirac bracket, we need to invert the matrix $\mathcal{C}^{\alpha\beta}(\omega_{\mathcal{H}}) := \{\omega^\alpha, \omega^\beta\}_0|_{\omega_{\mathcal{H}^\perp} = 0}$. This is identified from $\mathcal{I}$ as the matrix of the linear operator $\mathcal{C}(\omega_{\mathcal{H}}) : \mathcal{H}^\perp \to \mathcal{H}^\perp$ that equals the restriction of a multiple of $\text{ad} \omega_{\mathcal{H}}$ to $\mathcal{H}^\perp$,

$$\mathcal{C}(\omega_{\mathcal{H}}) := \frac{1}{\kappa} \text{ad} \omega_{\mathcal{H}}|_{\mathcal{H}^\perp}. \quad (24)$$

Thus we also have to restrict ourselves to a submanifold of $\mathcal{M}_G$ where $\mathcal{C}(\omega_{\mathcal{H}})$ is invertible. We define the domain $\tilde{\mathcal{H}} \subset \tilde{\mathcal{G}}$ to contain the $\mathcal{H}$-projection of those elements $\omega \in \tilde{\mathcal{G}}$ for which the operator $\mathcal{C}^{-1}(\omega_{\mathcal{H}})$ exists. Then we can compute the Dirac bracket on the constrained manifold $\mathcal{M}_{\tilde{\mathcal{H}}} := \{g \in \mathcal{M}_G \mid \log M \in \tilde{\mathcal{H}}\}$ by using the standard formula $\{F_1, F_2\}_0 = \{F_1, F_2\}_0 - \{F_1, \omega^\alpha\}_0 \mathcal{C}^{-1}(\omega)\{\omega^\beta, F_2\}_0$. From $\mathcal{I}$ we obtain

$$\{g(x), \omega_i\}_0 = \frac{1}{\kappa} g(x) H_i, \quad \{\omega_i, \omega_j\}_0 = -\frac{1}{\kappa} f_{ij}^k \omega_k \quad ([H_i, H_j] = f_{ij}^k H_k), \quad (25)$$

where $g(x + 2\pi) = g(x) M$ with $\log M = \omega^i H_i \in \mathcal{H}$. Furthermore, $\mathcal{I}$ yields

$$\{g(x) \circ g(y)\}_0 = \frac{1}{\kappa} (g(x) \otimes g(y)) \left(\frac{1}{2} \hat{I} \text{sign} (y - x) + \hat{r}^*(\omega)\right), \quad 0 < x, y < 2\pi, \quad (26)$$
where
\[ \hat{r}^*(\omega) = r^0(\omega) + \frac{1}{\kappa} C^{-1}_{\alpha\beta}(\omega) E^\alpha \otimes E^\beta. \] (27)

Now the point is that—analogously to (20) and (21)—the Jacobi identities of the Dirac bracket imply that the function \( \hat{r}^* : \bar{\mathcal{H}} \to \mathcal{G} \wedge \mathcal{G} \) given by (27) is \( \mathcal{H} \)-equivariant in the natural sense and satisfies the \( \mathcal{H} \)-CDYB equation
\[ \left[ \hat{r}^*_{12}(\omega), \hat{r}^*_{23}(\omega) \right] + H_i \frac{\partial}{\partial \omega^i} \hat{r}^*_{23}(\omega) + \text{cycl. perm.} = -\frac{1}{4} \hat{f} \quad (\omega \in \bar{\mathcal{H}}). \] (28)

Examples may be obtained by taking \( \mathcal{H} \) to be the grade zero subalgebra in some integral gradation of \( \mathcal{G} \) and taking \( r^0(\omega) \) to be \( f_0(\text{ad} \omega) \). Then the operator \( r^*(\omega) (\omega \in \bar{\mathcal{H}}) \) associated with (27) is found to be
\[ r^*(\omega)(X) = f_0(\text{ad} \omega)(X) \quad \forall X \in \mathcal{H}, \quad r^*(\omega)(Y) = \frac{1}{2} \coth \left( \frac{1}{2} \text{ad} \omega \right)(Y) \quad \forall Y \in \mathcal{H}^\perp. \] (29)

We here use the Laurent series expansion of \( \frac{1}{2} \coth \left( \frac{z}{2} \right) \) in a punctured disc around \( z = 0 \), and the \( \frac{1}{z} \) term in the expansion corresponds to the operator \( (\text{ad} \omega)^{-1} \) on \( \mathcal{H}^\perp \). In the special case of the principal gradation of a simple Lie algebra \( \mathcal{G} \), for which \( \mathcal{H} \) is a Cartan subalgebra, this gives
\[ \hat{r}^*(\omega) = \frac{1}{2} \sum_{\alpha \in \Phi} \coth \left( \frac{1}{2} \alpha(\omega) \right) E^\alpha \otimes E^\alpha, \] (30)
where \( \Phi \) is the set of the roots and the \( E^\alpha \) are corresponding root vectors. This solution of the CDYB equation (1) was obtained in [3] by determining the PBs of the chiral WZNW Bloch waves with the aid of a different method.

4 Conclusion

We reported on our recent results [8] concerning the chiral WZNW phase space by focusing on the dynamical generalizations of the CYB equation that appear naturally in this context. Not only several variants of the CDYB equation but also some of their most interesting solutions were described. The Dirac reduction of certain solutions of the \( \mathcal{G} \)-CDYB equation to solutions of the \( \mathcal{H} \)-CDYB equation is treated in this report in a general manner for the first time. Other aspects of the chiral WZNW phase space that are not mentioned here for lack of space have been also analysed in [8, 10, 17], for instance we have found explicit solutions of the \( \mathcal{G} \)-CDYB equation that realize arbitrary PL symmetries. The open problems that arose from our investigation will hopefully be discussed in future publications.
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References

[1] J.-L. Gervais and A. Neveu, Nucl. Phys. B 238, 125 (1984).
[2] E. Cremmer and J.-L. Gervais, Commun. Math. Phys. 134, 619 (1990).
[3] J. Balog, L. Dąbrowski and L. Fehér, Phys. Lett. B 244, 227 (1990).
[4] G. Felder, Conformal field theory and integrable systems associated to elliptic curves, pp. 1247-1255 in: Proc. Int. Congr. Math. Zürich, 1994 (Birkhäuser, Zürich, 1994).
[5] J. Avan, O. Babelon and E. Billey, Commun. Math. Phys. 178, 281 (1996).
[6] P. Etingof and O. Schiffmann, Lectures on the dynamical Yang-Baxter equations, preprint math.QA/9908064.
[7] P. Etingof and A. Varchenko, Commun. Math. Phys. 192, 77 (1998).
[8] J. Balog, L. Fehér and L. Palla, Nucl. Phys. B 568, 503 (2000).
[9] E. Witten, Commun. Math. Phys. 92, 455 (1984).
[10] J. Balog, L. Fehér and L. Palla, The chiral WZNW phase space as a quasi-Poisson space, preprint hep-th/0007043.
[11] A. Alekseev and Y. Kosmann-Schwarzbach, Manin pairs and moment maps, preprint math.DG/9909176.
[12] K. Gawędzki, Commun. Math. Phys. 139, 201 (1991).
[13] F. Falceto and K. Gawędzki, J. Geom. Phys. 11, 251 (1993).
[14] K. Mackanzie, Lie Groupoids and Lie Algebroids in Differential Geometry (Cambridge University Press, Cambridge, 1987).
[15] A. Weinstein, J. Math. Soc. Japan 4, 705 (1988).
[16] A. Alekseev and E. Meinrenken, Invent. Math. 139, 135 (2000).
[17] J. Balog, L. Fehér and L. Palla, J. Phys. A. 33, 945 (2000).