A hybrid variational Allen-Cahn/ALE scheme for the coupled analysis of two-phase fluid-structure interaction

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Summary
We present a partitioned iterative formulation for the modeling of fluid-structure interaction (FSI) in two-phase flows. The variational formulation consists of a stable and robust integration of three blocks of differential equations, viz, an incompressible viscous fluid, a rigid or flexible structure, and a two-phase indicator field. The fluid-fluid interface between the two phases, which may have high density and viscosity ratios, is evolved by solving the conservative phase-field Allen-Cahn equation in the arbitrary Lagrangian-Eulerian coordinates. While the Navier-Stokes equations are solved by a stabilized Petrov-Galerkin method, the conservative Allen-Cahn phase-field equation is discretized by the positivity preserving variational scheme. Fully decoupled implicit solvers for the two-phase fluid and the structure are integrated by the nonlinear iterative force correction in a staggered partitioned manner and the generalized-$\alpha$ method is employed for the time marching. We assess the accuracy and stability of the phase-field/ALE variational formulation for two- and three-dimensional problems involving the dynamical interaction of rigid bodies with free surface. We consider the decay test problems of increasing complexity, namely, free translational heave decay of a circular cylinder and free rotation of a rectangular barge. Through numerical experiments, we show that the proposed formulation is stable and robust for high density ratios across fluid-fluid interface and for low structure-to-fluid mass ratio with strong added-mass effects. Overall, the proposed variational formulation produces results with high accuracy and compares well with available measurements and reference numerical data. Using unstructured meshes, we demonstrate the second-order temporal accuracy of the coupled phase-field/ALE method via decay test of a circular cylinder interacting with the free surface. Finally, we demonstrate the three-dimensional phase-field FSI formulation for a practical problem of internal two-phase flow in a flexible circular pipe subjected to vortex-induced vibrations due to external fluid flow.

KEYWORDS
ALE-FSI, Allen-Cahn, nonlinear iterative force correction, phase-field, partitioned staggered, vortex-induced vibration
1 | INTRODUCTION

Fluid-structure interaction (FSI) involving two-phase flows is omnipresent, for example, its applications include offshore pipelines conveying oil or gas,\textsuperscript{1,2} marine vessels exposed to free-surface waves, blood flow through veins and arteries, and multiphase flow inside heat exchangers.\textsuperscript{3} Of particular interest to the present study is the internal flow in offshore risers, which are subjected to turbulent external flow associated with the ocean currents as well as the motion of the offshore vessel induced by wave-structure interaction. These offshore structures may undergo self-excited vibrations and fluid-elastic instabilities,\textsuperscript{4,5} which may lead to structural failure and operational delay due to nonlinear dynamical effects of FSI. High-fidelity numerical simulations can play an important role to understand the nonlinear coupled physics as well as to provide guidelines for engineering design and optimization. The development of robust, efficient, and general integration procedure of two-phase flow interacting with freely moving rigid and flexible bodies poses serious challenges from a computational viewpoint. In particular, the accurate modeling of free-surface motion with topological changes and strong FSI effects at high Reynolds number remain a daunting task in computational science and engineering.

Two-phase FSI involves complex nonlinear interface dynamics and the difficulties associated with the boundary conditions, viz, the evolution of the fluid-fluid interface with the structural motion, the no-slip condition at the structure in the neighborhood of highly-deformable fluid-fluid interface, and the precise movement of the fluid-structure interface. A typical schematic of the problem is shown in Figure 1. It can be observed that fluid-fluid interface has to be evolved with the deformation of the structure while satisfying the no-slip condition at the fluid-structure interface. This can be achieved by either considering the structural domain as Lagrangian and then solving the two-phase flow equations in the arbitrary Lagrangian-Eulerian (ALE) coordinates or with the help of an immersed boundary technique, where the equations are solved in a Eulerian grid with boundary conditions represented by a fictitious force field.\textsuperscript{6,7} The modeling of boundary-layer vorticity flux and near-wall turbulence at the fluid-structure interface is accurate if one considers a boundary-conforming grid (eg, ALE framework) to track the fluid-structure interface. Due to the accuracy consideration along the interface for the two-phase FSI analysis, we employ the ALE framework for the moving fluid domain, ie, the physical properties vary as a sharp discontinuity at the fluid-structure interface, as shown in Figure 1C, and the Eulerian fluid mesh follows the moving interface at all time with the precise satisfaction of the boundary conditions.

The robust and efficient modeling of the three-dimensional fluid-fluid interface using a sharp interface description via ALE-type interface tracking technique is a nontrivial task, especially in problems involving any topological changes of the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Schematic of two-phase fluid-structure interaction at (A) the initial configuration \( t = 0 \) and (B) some deformed configuration of the structure at time \( t > 0 \). \( \Omega_f(0), \Omega_s, \Omega_f(t), \Omega_s(t) \) are the fluid and the structural domains at \( t = 0 \) and some time \( t > 0 \), respectively, with (C) sharp fluid-structure interface and (D) diffused fluid-fluid interface, smeared using the internal length scale parameter \( \varepsilon \). \( \Gamma_{fs} \) and \( \Gamma_{ff} \) denote the fluid-structure and fluid-fluid interfaces, respectively. [Colour figure can be viewed at wileyonlinelibrary.com]}
\end{figure}
interface. Instead, the fluid-fluid interface is described by a diffuse interface description-based phase-field method in the present study where the interface is distributed over a finite width ($O(\varepsilon)$) across which the physical properties vary gradually as a function of the order parameter $\phi$, as shown in Figure 1D. Unlike the widely used level-set, volume-of-fluid (VOF), and front-tracking methods, the phase-field method is based on the minimization of a free energy functional, which drives the evolution of the interface. The phase-field method is a type of interface-capturing method (i.e., interface is solved over a fixed Eulerian domain), which has an advantage in dealing with any topological change in the fluid-fluid interface without involving any complex reinitialization (level-set) or geometric reconstruction (VOF). A recent review of the phase-field methods for multiphase flows can be found in the work of Kim.

Most of the works carried out in the literature involving vortex-induced vibrations (VIVs) have focused on the FSI involving single-phase flows. Extensive experiments and numerical simulations have been performed to study the VIV dynamics of different kinds of structures subjected to external single-phase flows. However, there are very few works on the FSI involving internal flow, which are mainly concerned with the experiments. Some of the recent works based on the dynamics of different kinds of structures subjected to external single-phase flows. However, there are very few works on the FSI involving internal flow, which are mainly concerned with the experiments. Some of the recent works based on the fluid-fluid interface without involving any complex reinitialization (level-set) or geometric reconstruction (VOF). A recent review of the phase-field methods for multiphase flows can be found in the work of Kim.

The present work is an extension of the work carried out by Joshi and Jaiman, where the coupling of the conservative Allen-Cahn equation with the incompressible flow equations was discussed and analyzed with the help of various numerical tests. In this paper, we extend the two-phase flow developed framework in the aforementioned work to include the fluid-structure coupling and propose a robust and efficient variational formulation for the coupled two-phase FSI. The rigid/flexible body equations are solved in the Lagrangian framework with the two-phase flow equations written in the ALE reference coordinate system. The coupled two-phase fluid-structure equations are solved in the nonlinear partitioned iterative format by consistent variational finite element formulation in arbitrarily complex three-dimensional domains. The fluid-fluid interface is evolved by the conservative Allen-Cahn equation written in the ALE framework, which takes care of the moving fluid-structure interface due to the structural deformation. While the two-phase fluid domain with the Navier-Stokes equations is discretized using Petrov-Galerkin finite-element and semi-discrete time stepping, the conservative phase-field Allen-Cahn equation is discretized by the positivity preserving variational scheme, which provides boundedness, mass conservation, and energy stability in the underlying discretization. Identical order of interpolation has been used for all the fluid and phase-field variables, which implies their collocated arrangement at discrete nodes of unstructured finite element mesh. A nonlinear iterative force correction (NIFC) scheme is employed for updating the hydrodynamic forces from the fluid flow to the structure in strongly-coupled FSI. To achieve stability at low structure-to-fluid mass ratios, the approximate interface force generated through iterations is corrected via nonlinear force transformation in the predictor-corrector format. The temporal discretizations of the fluid, the structure, and the phase-field equations are performed by the generalized-a method together with the partitioned iterative solution strategy. The moving mesh/ALE characteristic of the fluid-structure interface with the interface capturing technique for the fluid-fluid interface forms our hybrid ALE/phase-field formulation. The salient features of the phase-field FSI formulation are (i) accurate and stable variational FSI formulation for low structure-to-fluid mass ratios, (ii) consistent two-phase flow formulation for high density and viscosity ratios, (iii) robustness to handle topological changes in the fluid-fluid interface with FSI, and (iv) ease of implementation and flexibility in existing variational solvers due to the partitioned block-iterative coupling. While the previous work by the authors Joshi and Jaiman covered the various benchmarks dealing with the high density and viscosity ratios and capturing of the evolving fluid-fluid interface, this work builds up the FSI capabilities of the framework including the ability to handle low structure-to-fluid mass ratios and its flexibility and ease of implementation. These desirable features of the proposed variational framework together with the increasing complexity of two-phase FSI problems are covered in this paper.
The organization of this paper is as follows. Section 2 reviews the governing equations for the two-phase fluid flow (Navier-Stokes and Allen-Cahn) and the structure with their corresponding variational formulations. The proposed partitioned iterative coupling of the FSI and the two-phase flows are presented in Section 3. The proposed formulation is then assessed numerically in Section 4 via decay tests involving translation of a circular cylinder and rotation of a rectangular barge. The results are compared with the experiments and simulations from the literature. For a practical application, we demonstrate the VIV of a flexible riser exposed to a uniform current with an internal two-phase flow in Section 5. Finally, we conclude the paper with some of the key findings in Section 6.

2 COUPLED FLUID-STRUCTURE FORMULATION FOR TWO-PHASE FLOWS

In this section, we describe the governing equations of the two-phase flow and the structure and their corresponding variational formulations. We first review the Navier-Stokes equations for two-phase flows along with the conservative Allen-Cahn equation, which evolves the fluid-fluid interface. Thereafter, we discuss the structural equation and the treatments of the fluid-structure and the fluid-fluid interfaces.

2.1 The Navier-Stokes equations for two-phase flow

We present the differential equation for the two-phase flow at the continuous level and then review its semi-discrete and variational formulation using the stabilized finite element framework.

2.1.1 Strong differential form

Consider a spatial domain \( \Omega^f(t) \) consisting of the spatial points \( \mathbf{x} \) at temporal coordinate \( t \). The boundary to the domain \( \Gamma^f(t) \) consists of three components, the Dirichlet boundary \( \Gamma^D(t) \), the Neumann boundary \( \Gamma^H(t) \), and the fluid-structure boundary \( \Gamma^{fs}(t) \) at time \( t \). We write the one-fluid formulation for two-phase incompressible and immiscible fluid flow in the arbitrary Lagrangian-Eulerian framework with the boundary conditions as

\[
\rho^f \frac{\partial \mathbf{u}^f}{\partial t} + \rho^f (\mathbf{u}^f - \mathbf{u}^m) \cdot \nabla \mathbf{u}^f = \nabla \cdot \mathbf{\sigma}^f + \mathbf{s}^f + \mathbf{b}^f, \quad \text{on } \Omega^f(t), \tag{1}
\]

\[
\nabla \cdot \mathbf{u}^f = 0, \quad \text{on } \Omega^f(t), \tag{2}
\]

\[
\mathbf{u}^f = \mathbf{u}^D, \quad \forall \mathbf{x}^f \in \Gamma^D(t), \tag{3}
\]

\[
\mathbf{\sigma}^f \cdot \mathbf{n}^f = h^f, \quad \forall \mathbf{x}^f \in \Gamma^H(t), \tag{4}
\]

\[
\mathbf{u}^f = \mathbf{u}^0, \quad \text{on } \Omega^f(0), \tag{5}
\]

where \( \mathbf{u}^f \) and \( \mathbf{u}^m \) represent the fluid velocity and the mesh velocity defined for each spatial point \( \mathbf{x} \) in \( \Omega^f(t) \); \( \rho^f \) is the density of the fluid; \( \mathbf{s}^f \) is the surface tension singular force replaced by the continuum surface force (CSF) in the diffuse interface description; \( \mathbf{b}^f \) is the body force on the fluid such as gravity \( (\mathbf{b}^f = \rho^f \mathbf{g}) \); \( \mathbf{g} \) being the acceleration due to gravity; \( \mathbf{u}^D \) and \( h^f \) denote the boundary conditions at the Dirichlet and Neumann boundaries, respectively; \( \mathbf{n}^f \) is the unit outward normal to the Neumann boundary, and \( \mathbf{u}^0 \) represents the initial velocity field at \( t = 0 \). The partial derivative of the velocity field with respect to time is evaluated with the ALE referential coordinate \( \mathbf{x} \) fixed. The Cauchy stress tensor for a Newtonian fluid is given as

\[
\mathbf{\sigma}^f = -p \mathbf{I} + \mathbf{T}^f, \quad \mathbf{T}^f = 2 \mu^f \mathbf{e}^f(\mathbf{u}^f), \quad \mathbf{e}^f(\mathbf{u}^f) = \frac{1}{2} \left[ \nabla \mathbf{u}^f + (\nabla \mathbf{u}^f)^T \right], \tag{6}
\]

where \( p \) is the pressure field; \( \mathbf{T}^f \) and \( \mathbf{e}^f \) represent the shear stress tensor and the fluid strain rate tensor, respectively; and \( \mu^f \) denotes the dynamic viscosity of the fluid. The physical parameters of the fluid such as \( \rho^f \) and \( \mu^f \) are dependent on the order parameter \( \phi \) (which evolves with the fluid-fluid interface) as

\[
\rho^f(\phi) = \frac{1 + \phi}{2} \rho^f_1 + \frac{1 - \phi}{2} \rho^f_2, \tag{7}
\]

\[
\mu^f(\phi) = \frac{1 + \phi}{2} \mu^f_1 + \frac{1 - \phi}{2} \mu^f_2, \tag{8}
\]

where \( \rho^f_1 \) and \( \mu^f_1 \) are the density and dynamic viscosity of the \( i \)th phase of the fluid, respectively.
2.1.2 | Semi-discrete variational form

The temporal discretization of the two-phase incompressible Navier-Stokes equations is carried out by the generalized-α method.\textsuperscript{28} It enables a user-controlled high frequency damping, which is desirable for coarser discretization in space and time. This is achieved by a single parameter called the spectral radius $\rho_\infty$. The following expressions are employed for the temporal discretization:

\begin{align}
\mathbf{u}^{f,n+1} &= \mathbf{u}^{f,n} + \Delta t \partial_t \mathbf{u}^{f,n} + \gamma_\ell \Delta t \left( \partial_t \mathbf{u}^{f,n+1} - \partial_t \mathbf{u}^{f,n} \right), \\
\partial_t \mathbf{u}^{f,n+\alpha_m} &= \partial_t \mathbf{u}^{f,n} + a_m^f \left( \partial_t \mathbf{u}^{f,n+1} - \partial_t \mathbf{u}^{f,n} \right), \\
\mathbf{u}^{f,n+\alpha_f} &= \mathbf{u}^{f,n} + a_f^f \left( \mathbf{u}^{f,n+1} - \mathbf{u}^{f,n} \right),
\end{align}

where $a_f^f$, $a_m^f$, and $\gamma_\ell$ are the generalized-α parameters dependent on the user-defined spectral radius $\rho_\infty$,\textsuperscript{28,29} which is selected such that $a_f^f = a_m^f = \gamma_\ell = 0.5$. The time step size is denoted by $\Delta t$ and $\partial_t$ denotes the partial differentiation with respect to time.

Suppose $S_{\mathbf{u}}^h$ and $S_p^h$ denote the space of trial solution such that

\begin{align}
S_{\mathbf{u}}^h &= \left\{ \mathbf{u}^f_h \mid \mathbf{u}^f_h \in (H^1(\Omega^f(t)))^d, \mathbf{u}^f_h \mid \Gamma^f_D(t) \right\}, \\
S_p^h &= \left\{ p_h \mid p_h \in L^2(\Omega^f(t)) \right\},
\end{align}

where $(H^1(\Omega^f(t)))^d$ denotes the space of square-integrable $\mathbb{R}^d$-valued functions with square-integrable derivatives on $\Omega^f(t)$ and $L^2(\Omega^f(t))$ is the space of the scalar-valued functions that are square-integrable on $\Omega^f(t)$. Similarly, we define $V_{\mathbf{u}}^h$ and $V_p^h$ as the space of test function such that

\begin{align}
V_{\mathbf{u}}^h &= \left\{ \psi^f_h \mid \psi^f_h \in (H^1(\Omega^f(t)))^d, \psi^f_h \mid \Gamma^f_D(t) = 0 \right\}, \\
V_p^h &= \left\{ q_h \mid q_h \in L^2(\Omega^f(t)) \right\}.
\end{align}

The variational statement of the Navier-Stokes equations can thus be written as follows. Find $[\mathbf{u}^f_h(t^{n+\alpha_f}), p_h(t^{n+1})] \in S_{\mathbf{u}}^h \times S_p^h$ such that, $\forall [\psi^f_h, q_h] \in V_{\mathbf{u}}^h \times V_p^h$, 

\begin{align}
\int_{\Omega^f(t)} \rho^f(\phi) \left( \partial_t \mathbf{u}^f_h + \left( \mathbf{u}^f_h - \mathbf{u}^m_h \right) \cdot \nabla \mathbf{u}^f_h \right) \cdot \psi^f_h \, d\Omega + \int_{\Omega^f(t)} \sigma^f_h : \nabla \psi^f_h \, d\Omega \\
- \int_{\Omega^f(t)} \mathbf{s}^f_h(\phi) \cdot \psi^f_h \, d\Omega + \sum_{e=1}^{n_t} \int_{\Gamma_e^f} t_m \rho^f(\phi) \left( \mathbf{u}^f_h - \mathbf{u}^m_h \right) \cdot \nabla \psi^f_h + \nabla q_h \cdot \mathbf{R}_m \, d\Gamma^e \\
+ \int_{\Omega^f(t)} q_h \left( \nabla \cdot \mathbf{u}^f_h \right) \, d\Omega + \sum_{e=1}^{n_t} \int_{\Gamma_e^f} \nabla \cdot \psi^f_h t_c \rho^f(\phi) \mathbf{R}_c \, d\Gamma^e \\
- \sum_{e=1}^{n_t} \int_{\Gamma_e^f} t_m \psi^f_h \cdot \left( \mathbf{R}_m : \nabla \mathbf{u}^f_h \right) \, d\Gamma^e - \sum_{e=1}^{n_t} \int_{\Gamma_e^f} \nabla \psi^f_h \cdot \left( t_m \mathbf{R}_m \otimes t_m \mathbf{R}_m \right) \, d\Gamma^e \\
= \int_{\Omega^f(t)} \mathbf{b}^f \left( t^{n+\alpha_f} \right) \cdot \psi^f_h \, d\Omega + \int_{\Gamma^f_h} \mathbf{h}^f \cdot \psi^f_h \, d\Gamma,
\end{align}

where the first and the second lines represent the Galerkin terms and the Petrov-Galerkin stabilization terms for the momentum equation, the third line depicts the Galerkin and the stabilization term for the continuity equation, the fourth line consists of the additional terms that are introduced as the approximation of the fine scale velocity on the element interiors based on the multiscale argument,\textsuperscript{30-32} and the fifth line represents the body forces and the Neumann boundary conditions. The element-wise residuals of the momentum and the continuity equations are represented by $\mathbf{R}_m$ and $\mathbf{R}_c$, respectively. The stabilization parameters $t_m$ and $t_c$ are the least-squares metrics added to the element-level integrals in the stabilized formulation\textsuperscript{33-36} and are defined as

\begin{equation}
\tau_m = \left[ \left( \frac{2}{\Delta t} \right)^2 + \left( \mathbf{u}^f_h - \mathbf{u}^m_h \right) \cdot \mathbf{G} \left( \mathbf{u}^f_h - \mathbf{u}^m_h \right) + C_f \left( \frac{\mu^f(\phi)}{\rho^f(\phi)} \right)^2 \mathbf{G} \cdot \mathbf{G} \right]^{-1/2}, \quad \tau_c = \frac{1}{\text{tr}(\mathbf{G})} \tau_m,
\end{equation}
where $C_l$ is a constant derived from the element-wise inverse estimates, $G$ is the element contravariant metric tensor, and $\text{tr}(G)$ is the trace of the contravariant metric tensor. This stabilization in the variational form circumvents the Babuška-Brezzi condition that is required to be satisfied by any standard mixed Galerkin method.

### 2.2 The Allen-Cahn equation

The modeling of the order parameter that distinguishes the two phases is achieved by solving the conservative Allen-Cahn equation and evolving the fluid-fluid interface. The governing equation in its strong form and the recently proposed positivity preserving variational formulation in the present context have been described in this section.

#### 2.2.1 Strong differential form

Consider the spatial domain $\Omega^i(t)$ with Dirichlet and Neumann boundaries for the order parameter denoted by $\Gamma_D^\phi(t)$ and $\Gamma_N^\phi(t)$, respectively. The phase-field order parameter $\phi$, which represents the different phases of the fluid, is evolved by solving the conservative Allen-Cahn equation in the ALE framework with the boundary conditions given as

$$\frac{\partial \phi}{\partial t} + (u^i - u^m) \cdot \nabla \phi - \gamma \left( \epsilon^2 \nabla^2 \phi - F'(\phi) + \beta(t) \sqrt{F'(\phi)} \right) = 0, \quad \text{on } \Omega^i(t), \quad (18)$$

$$\phi = \phi_D, \quad \forall x^i \in \Gamma_D^\phi(t), \quad (19)$$

$$\nabla \phi \cdot n^\phi = 0, \quad \forall x^i \in \Gamma_N^\phi(t), \quad (20)$$

$$\phi = \phi_0, \quad \text{on } \Omega^0(t), \quad (21)$$

where $\gamma$ is a mobility parameter (taken as 1 for the present study), $\epsilon$ is the parameter that represents the thickness of the interface between the phases, $F(\phi)$ is the double-well energy potential that represents the free energy of mixing or bulk energy, and $F'(\phi)$ is the derivative of the energy potential with respect to the order parameter. It has two minima corresponding to the two stable phases of the fluid. The value of the order parameter at the Dirichlet boundary is denoted by $\phi_D$, the initial condition is represented by $\phi_0$, and $n^\phi$ denotes the unit normal to the Neumann boundary, where zero flux condition is satisfied. The mass conservation is enforced in the Allen-Cahn equation by a Lagrange multiplier $\beta(t)\sqrt{F'(\phi)}$ (see the works of Brassel and Bretin and Kim et al), where $\beta(t) = \int_{\Omega^i(t)} F'(\phi) d\Omega / \int_{\Omega(t)} \sqrt{F'(\phi)} d\Omega$, which gives an advantage of accurate capturing of the geometrical features of the interface.

#### 2.2.2 Semi-discrete variational form

Following the generalized-$\alpha$ temporal discretization of the aforementioned equation for consistency, we employ the expressions in Equations (9-11) similarly for the order parameter. The temporal discretized Allen-Cahn equation can be transformed into a convection-diffusion-reaction equation as follows:

$$\partial_t \phi^{n+\alpha} + \hat{u} \cdot \nabla \phi^{n+\alpha} - \hat{k} \nabla^2 \phi^{n+\alpha} + \hat{s} \phi^{n+\alpha} - \hat{f} \left( \phi^{n+\alpha} \right) = 0, \quad \text{on } \Omega^i(t), \quad (22)$$

where $\hat{u}$, $\hat{k}$, $\hat{s}$, and $\hat{f}$ are the modified convection velocity, diffusion coefficient, reaction coefficient, and the source, respectively, given as

$$\hat{u} = u^i - u^m, \quad (23)$$

$$\hat{k} = \epsilon^2, \quad (24)$$

$$\hat{s} = \frac{1}{4} \left[ \frac{\alpha^2}{(a^i)^2} \left( - \frac{3}{(a^i)^2} - \frac{4}{(a^i)^2} \phi^n - \frac{3}{(a^i)^2} \phi^n + \frac{8}{(a^i)^2} \phi^n - \frac{2}{a^i} \phi^n \right) \right]$$

$$- \frac{3}{2} \left[ \frac{\phi^n (a^i)^2}{3(a^i)^2} + \frac{1}{3} \left( - \frac{2}{(a^i)^2} + \frac{3}{a^i} \right) \phi^n \right], \quad (25)$$

$$\hat{f} = - \frac{1}{4} \left[ \left( - \frac{1}{(a^i)^2} + \frac{4}{(a^i)^2} - \frac{6}{a^i} + 4 \right) \phi^n + \left( \frac{2}{a^i} - 4 \right) \phi^n \right]$$

$$+ \frac{\beta(t^{n+\alpha})}{2} \left[ \frac{1}{3} \left( - \frac{1}{(a^i)^2} - \frac{3}{a^i} + 3 \right) \phi^n - 1 \right]. \quad (26)$$
Defining the space of trial solution as \( S^h_\phi \) and that of the test function as \( Y^h_\phi \) such that

\[
S^h_\phi = \left\{ \phi_h \mid \phi_h \in H^1(\Omega^f(t)), \phi_n = \phi_D \text{ on } \Gamma^f_D(t) \right\},
\]

\[
Y^h_\phi = \left\{ \hat{w}_h \mid \hat{w}_h \in H^1(\Omega^s(t)), \hat{w}_n = 0 \text{ on } \Gamma^s_D(t) \right\},
\]

the variational statement for the Allen-Cahn equation is given as follows. Find \( \phi_h(\mathbf{x}^f, t^{n+\delta}) \in S^h_\phi \) such that, \( \forall \hat{w}_h \in Y^h_\phi \),

\[
\begin{align*}
\int_{\Omega^f(t)} (\hat{w}_h \partial_t \phi_h + \hat{w}_h (\hat{\mathbf{u}} \cdot \nabla \phi_h) + \nabla \hat{w}_h \cdot (k \nabla \phi_h) + \hat{w}_h \hat{\Delta} \phi_h - \hat{w}_h f) \, d\Omega + \\
+ \sum_{e=1}^{n_e} \int_{\Omega^e} \left( (\hat{\mathbf{u}} \cdot \nabla \hat{w}_h) \tau \left( \partial_t \phi_h + \hat{\mathbf{u}} \cdot \nabla \phi_h - \nabla \cdot (k \nabla \phi_h) + \hat{\Delta} \phi_h - f \right) \right) \, d\Omega^e + \\
+ \sum_{e=1}^{n_e} \int_{\Omega^e} \chi \frac{|R(\phi_h)|}{|\nabla \phi_h|} k^\text{add}_s \nabla \hat{w}_h \cdot \frac{\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}}{|\hat{\mathbf{u}}|^2} \cdot \nabla \phi_h \, d\Omega^e + \sum_{e=1}^{n_e} \int_{\Omega^e} \chi \frac{|R(\phi_h)|}{|\nabla \phi_h|} k^\text{add}_c \nabla \hat{w}_h \cdot \left( I - \frac{\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}}{|\hat{\mathbf{u}}|^2} \right) \cdot \nabla \phi_h \, d\Omega^e = 0,
\end{align*}
\]

where the first line represents the Galerkin terms for the Allen-Cahn equation, the second line is the streamline upwind Petrov-Galerkin terms, and the third line depicts the positivity preserving terms. Here, the stabilization parameter \( \tau \) is given by

\[
\tau = \left[ \left( \frac{2}{\Delta t} \right)^2 + \frac{\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} G + 9k^2G + \hat{\Delta}}{2} \right]^{-1/2}.
\]

The positivity preserving stabilization terms are derived for the multidimensional convection-diffusion-reaction equation by satisfying the positivity condition at the element level matrix of the variationally discretized equation in the work of Joshi and Jaiman.\(^{25}\) For one-dimensional explicit scheme, the positivity preserving property reduces to the conditional inequality of the Harten’s coefficients.\(^{39}\) This has been shown for some particular cases in the work of Joshi and Jaiman.\(^{40}\) For an implicit matrix form of the scheme, the positivity condition can be imparted by transforming the left-hand-side matrix \( \mathbf{A} = \{a_{ij}\} \) to an M-matrix, which ensures positivity and convergence\(^{41}\) satisfying the following properties:

\[
a_{ii} > 0, \forall i,
\]

\[
a_{ij} \leq 0, \forall j \neq i,
\]

\[
\sum_j a_{ij} = 0, \forall i.
\]

This transformation is done by the addition of the discrete upwind matrix, which renders the variational scheme positivity preserving and monotone.\(^{42}\) The factor \( \chi |R(\phi_h)|/|\nabla \phi_h| \) acts as a limiter to the upwinding near the regions of high solution gradients, which is adjusted by the nonlinear corrections. Several test cases have been performed to assess the effectiveness of the PPV technique in the work of Joshi and Jaiman.\(^{25}\) The details of the derivation of the added diffusions \( k^\text{add}_s \), \( k^\text{add}_c \) and \( \chi \) can be found in the aforementioned work,\(^{25}\) which are given for the present context by\(^{24}\)

\[
\chi = \frac{2}{|\hat{s}|h + 2|\hat{\mathbf{u}}|},
\]

\[
k^\text{add}_s = \max \left\{ \frac{|\hat{\mathbf{u}}|}{2} - \frac{2}{\tau} \frac{|\hat{s}|h}{2} - \left( \hat{k} + \frac{|\hat{\mathbf{u}}|^2}{6} \right), 0 \right\},
\]

\[
k^\text{add}_c = \max \left\{ \frac{|\hat{\mathbf{u}}|}{2} - \frac{2}{\tau} \frac{3h^2}{6}, 0 \right\},
\]

where \( |\hat{\mathbf{u}}| \) is the magnitude of the modified convection velocity and \( h \) is the characteristic element length defined in the work of Joshi and Jaiman.\(^{25}\)

### 2.3 The structural equation

We describe the structural equation and its weak formulation in this section. The structural equation is solved in the Lagrangian framework where the fluid-structure interface is a sharp boundary and the mesh surrounding the structure deforms in the ALE framework.
2.3.1 | Strong differential form

Consider a d-dimensional structural domain $\Omega^s \subset \mathbb{R}^d$ with a piecewise smooth boundary $\Gamma^s$ consisting of the material coordinates $x^s$ at time $t = 0$. The boundary $\Gamma^s$ can be decomposed into three disjoint sections consisting of the Dirichlet boundary $\Gamma^s_D$, the Neumann boundary $\Gamma^s_H$, and the fluid-structure interface $\Gamma^s$. We define a one-to-one mapping function $q^s(x^s, t) : \Omega^s \rightarrow \Omega^s(t)$, which denotes the position vector and maps the reference coordinates of the structure $x^s$ at $t = 0$ to its position in the deformed configuration $\Omega^s(t)$. Let $q(x^s, t)$ be the structural displacement due to the deformation. The position vector mapping is thus given by

$$q^s(x^s, t) = q(x^s, t) + x^s. \quad (37)$$

The velocity and acceleration of the body at the deformed configuration are given as follows:

$$u^s = \frac{\partial q^s}{\partial t}, \quad \frac{\partial u^s}{\partial t} = \frac{\partial^2 q^s}{\partial t^2}. \quad (38)$$

The structural equations can be written in the most general form as

$$\rho^s \frac{\partial^2 \sigma^s}{\partial t^2} + \nabla \cdot \sigma^s = b^s, \quad \text{on } \Omega^s, \quad (39)$$

$$u^s = u^s_D, \quad \forall x^s \in \Gamma^s_D, \quad (40)$$

$$\sigma^s \cdot n^s = h^s, \quad \forall x^s \in \Gamma^s_H, \quad (41)$$

$$q^s = q^s_0, \quad \text{on } \Omega^s, \quad (42)$$

$$u^s = u^s_0, \quad \text{on } \Omega^s, \quad (43)$$

where $\rho^s$, $\sigma^s$, and $b^s$ denote the density, stress tensor, and the body forces acting on the structure, respectively. The quantities $u^s_0$ and $h^s$ denote the Dirichlet and Neumann conditions on the structural velocity, respectively, and $q^s_0$ and $u^s_0$ represent the initial position vector and the initial velocity of the structure, respectively. The unit normal to the Neumann boundary is denoted by $n^s$.

2.3.2 | Semi-discrete variational form

Following the consistency in the time discretization using the generalized-$\alpha$ framework, we can write the following expressions:

$$q^{s,n+1} = q^{s,n} + \Delta t u^{s,n} + \Delta t^2 \left( \frac{1}{2} - \beta^s \right) \partial_t u^{s,n} + \beta^s \partial_t u^{s,n+1}, \quad (44)$$

$$u^{s,n+1} = u^{s,n} + \Delta t \left( (1 - \gamma^s) \partial_t u^{s,n} + \gamma^s \partial_t u^{s,n+1} \right), \quad (45)$$

$$\partial_t u^{s,n+1} = \partial_t u^{s,n} + \alpha^s_m (\partial_t u^{s,n+1} - \partial_t u^{s,n}), \quad (46)$$

$$u^{s,n+\alpha^s} = u^{s,n} + \alpha^s (u^{s,n+1} - u^{s,n}), \quad (47)$$

$$q^{s,n+\alpha^s} = q^{s,n} + \alpha^s (q^{s,n+1} - q^{s,n}), \quad (48)$$

where $\alpha^s$, $\alpha^s_m$, $\beta^s$, and $\gamma^s$ are the generalized-$\alpha$ parameters, which are selected as $\alpha^s = \alpha^s_m = \gamma^s = 0.5$ and $\beta^s = 0.25$.

Considering the space of trial solution $S_{\text{tr}}^h$ and that of the test function $V_{\text{tr}}^h$, which are defined as

$$S_{\text{tr}}^h = \{ u^s_h | u^s_h \in (H^1(\Omega^s))^d, u^s_h = u^s_D | \Gamma^s_D \} \quad (49)$$

$$V_{\text{tr}}^h = \{ \psi^s_h | \psi^s_h \in (H^1(\Omega^s))^d, \psi^s_h = 0 | \Gamma^s_D \} \quad (50)$$

the variational statement for the structural equation is given as follows. Find $u^s_h \in S_{\text{tr}}^h$ such that, $\forall \psi^s_h \in V_{\text{tr}}^h$,

$$\int_{\Omega^s} \left( \rho^s \partial_t u^{s,n+\alpha^s}_h \right) \cdot \psi^s_h \, d\Omega + \int_{\Omega^s} \sigma^s \cdot \nabla \psi^s_h \, d\Omega = \int_{\Gamma^s_H} h^s \cdot \psi^s_h \, d\Gamma + \int_{\Omega^s} b^s \cdot \psi^s_h \, d\Omega. \quad (51)$$

In the present study, we consider rigid and linear flexible motions of the structure. The respective equations and their matrix form are reviewed in Appendix A. We next describe the treatment of the fluid-structure and the fluid-fluid interfaces, i.e., how the boundary conditions are satisfied with the help of the equilibrium conditions at those interfaces.
2.4 | The fluid-structure interface

The coupling between the fluid and the structural equations is achieved by the velocity continuity and the equilibrium of the tractions along the fluid-structure interface. Suppose \( \Gamma^f = \Gamma^f(0) \cap \Gamma^s \) denotes the fluid-structure interface at \( t = 0 \). The interface at time \( t \) will then be denoted by \( \Gamma^f(t) = \varphi^f(\Gamma^f, t) \). The required conditions to be satisfied at the interface can be mathematically formulated as

\[
\mathbf{u}^f(\varphi^f(\mathbf{x}^f, t), t) = \mathbf{u}^s(\mathbf{x}^f, t), \quad \forall \mathbf{x}^f \in \Gamma^f,
\]

\[
\int_{\varphi^f(\Gamma, t)} \sigma^f(\mathbf{x}^f, t) \cdot \mathbf{n}^f d\Gamma + \int_{\gamma} \sigma^s(\mathbf{x}^s, t) \cdot \mathbf{n}^s d\Gamma = 0, \quad \forall \gamma \subset \Gamma^f,
\]

where \( \mathbf{n}^f \) and \( \mathbf{n}^s \) are the unit normals to the deformed fluid element \( \varphi^f(\gamma, t) \) and its corresponding undeformed structural element \( \gamma \), respectively. Here, \( \gamma \) is any part of the interface \( \Gamma^f \) in the reference configuration.

2.5 | The fluid-fluid interface

In the sharp fluid-fluid interface description, the velocity continuity and the pressure-jump condition are required to be satisfied at the interface

\[
\mathbf{u}^f_{\Omega^f} = \mathbf{u}^f_{\Omega^s}, \quad \forall \mathbf{x} \in \Gamma^f(t),
\]

\[
(\sigma^f_{\Omega^f} - \sigma^f_{\Omega^s}) \cdot \mathbf{n}_{\Omega^f} = \sigma_{\kappa} \mathbf{n}_{\Omega^s}, \quad \forall \mathbf{x} \in \Gamma^f(t),
\]

where \((\cdot)_{\Omega^f}\) denotes the argument in the fluid phase \( i \), \( \mathbf{n}_{\Omega^s} \) is the normal to the fluid-fluid interface, \( \sigma \) is the surface tension coefficient between the two fluid phases, and \( \kappa \) is the curvature of the interface denoted by \( \kappa = -\nabla \cdot \mathbf{n}_{\Omega^s} \). The surface tension singular force in the Navier-Stokes equations (Equation (1)), which models the surface tension, is thus written as \( \mathbf{sf} = \sigma \kappa \delta_{\Omega^s} \mathbf{n}_{\Omega^s} \), where \( \delta_{\Omega^s} \) is the one-dimensional Dirac delta function given as

\[
\delta_{\Omega^s} = \begin{cases} 
1, & \text{for } \mathbf{x} \in \Gamma^f(t), \\
0, & \text{otherwise}.
\end{cases}
\]

It was pointed out in the introduction that the sharp interface description based on the moving mesh framework is not trivial for complex three-dimensional fluid-fluid interfaces. Therefore, in the present formulation, we employ the diffuse fluid-fluid interface description in which the interface is assumed to have a finite thickness, i.e., \( \partial(\epsilon) \), on which the physical properties of the two phases vary gradually based on an indicator field \( \phi \). The diffuse interface description of the fluid-fluid interface recovers to the classical jump discontinuity conditions (Equations (54-55)) for the sharp interface description asymptotically as \( \epsilon \to 0 \). The singular force in the diffuse interface description is replaced by a CSF, which depends on the order parameter \( \phi \). Several forms of \( \mathbf{sf}(\phi) \) have been used in the literature, which are reviewed in the works of Kim. In this study, we employ the following definition:

\[
\mathbf{sf}(\phi) = \epsilon \alpha_{sf} \nabla \cdot (|\nabla \phi|^2 \mathbf{I} - \nabla \phi \otimes \nabla \phi),
\]

where \( \epsilon \) is the interface thickness parameter defined in the Allen-Cahn phase-field equation and \( \alpha_{sf} = 3\sqrt{2}/4 \) is a constant. Since \( \mathbf{sf}(\phi) \) consists of high-order derivatives of the order parameter, the third term in the variational form of Navier-Stokes equations in Equation (16) corresponding to the surface tension effects is reduced via integration by parts in the numerical implementation. This enables to model the surface tension effects numerically in the problem. We have considered the surface tension coefficient \( \sigma \) as a constant in the formulation. This completes the fully-coupled variational formulation for FSI with two-phase flow.

3 | THE NONLINEAR PARTITIONED ITERATIVE COUPLING

3.1 | Coupled linearized matrix form

We present the coupled linearized matrix form of the variationally discretized two-phase fluid-structure equations formulated in the previous section for nonoverlapping decomposition of the fluid and structure domains. The linear system of equations for the formulation can be written as \( \mathbf{A} \mathbf{u} = \mathbf{R} \), where \( \mathbf{u} \) and \( \mathbf{R} \) are the vector of unknowns and the
right-hand side vector, respectively. Corresponding to the domain decomposition, the set of degrees of freedom (DOFs) is decomposed into the interior DOFs for the two-phase fluid-structure system and the fluid-structure interface DOFs for the Dirichlet-to-Neumann (DtN) mapping. Using the Newton-Raphson type of linearization, the coupled two-phase fluid-structure system with the DtN mapping along the fluid-structure interface can be expressed as

\[
\begin{bmatrix}
A^{ss} & 0 & 0 \\
A^{sl} & I & 0 \\
0 & A^{fl} & 0 \\
0 & 0 & A^{fl} & I \\
\end{bmatrix}
\begin{bmatrix}
\Delta \eta^s \\
\Delta \eta^l \\
\Delta q^f \\
\Delta f^f \\
\end{bmatrix} =
\begin{bmatrix}
R^s \\
R^l \\
R^f \\
R_N^l \\
\end{bmatrix},
\]

(58)

where \(\Delta \eta^s\) denotes the increment in the structural displacement and \(\Delta \eta^l\) and \(\Delta f^f\) represent the increments in the displacement and the forces along the fluid-structure interface. The increment in the unknowns associated with the two-phase fluid domain is denoted by \(\Delta q^f = (\Delta u^f, \Delta p, \Delta \phi)\). On the right-hand side, \(R^s\) and \(R^l\) represent the weighted residual of the structural and stabilized two-phase flow equations, respectively, whereas \(R^f_D\) and \(R^f_N\) denote the residuals corresponding to the imbalances during the enforcement of the kinematic (Dirichlet) condition (Equation (52)) and the dynamic (Neumann) condition (Equation (53)) at the fluid-structure interface, respectively.

The block matrices on the left-hand side can be described as follows. \(A^{ss}\) represents the matrix consisting of the mass, damping, and stiffness matrices of the structural equation for the noninterface structural DOFs and \(A^{sl}\) is the transformation to obtain the structural force vector from the fluid-structure interface. \(A^{fl}\) maps the structural displacements to the fluid-structure interface, which satisfies the Dirichlet kinematic condition with \(I\) being an identity matrix. \(A^{fl}\) transfers the fluid forces to the fluid-structure interface to satisfy the Neumann dynamic equilibrium condition. \(A^{fl}\) associates the ALE mapping of the fluid spatial points and \(A^{fl}\) consists of the stabilized terms for the Navier-Stokes and the Allen-Cahn equations. It can be expanded as

\[
A^{fl} = \begin{bmatrix}
K_{fl} & G_{fl} & D_{fl} \\
-G_{fl}^T & C_{fl} & 0 \\
G_{AC} & 0 & K_{AC} \\
\end{bmatrix},
\]

(59)

where \(K_{fl}\) is the stiffness matrix of the momentum equation consisting of inertia, convection, viscous, and stabilization terms; \(G_{fl}\) is the gradient operator; \(G_{fl}^T\) is the divergence operator for the continuity equation; and \(C_{fl}\) is the pressure-pression stabilization term. On the other hand, \(D_{fl}\) contains the terms in the momentum equation having dependency on the order parameter, \(G_{AC}\) consists of the velocity coupled term in the Allen-Cahn equation; and \(K_{AC}\) is the left-hand side stiffness matrix for the Allen-Cahn equation consisting of inertia, convection, diffusion, reaction, and stabilization terms.

As derived in the work of Jaiman et al., the idea of partitioning is to eliminate the off-diagonal term \(A^{ls}\) to facilitate the staggered sequential updates for strongly coupled fluid-structure system. Through static condensation, Equation (58) can be written as

\[
\begin{bmatrix}
A^{ss} & 0 & 0 \\
A^{sl} & I & 0 \\
0 & A^{fl} & 0 \\
0 & 0 & A^{fl} & I \\
\end{bmatrix}
\begin{bmatrix}
\Delta \eta^s \\
\Delta \eta^l \\
\Delta q^f \\
\Delta f^f \\
\end{bmatrix} =
\begin{bmatrix}
R^s \\
R^l \\
R^f \\
R_N^l \\
\end{bmatrix},
\]

(60)

In the nonlinear interface force correction, we form the iterative scheme of the following matrix-vector product form:

\[
\Delta f^f = (A^{fl})^{-1} R^l_N,
\]

(61)

where \((A^{fl})^{-1}\) is not constructed explicitly. Instead, the force correction vector \(\Delta f^f\) at the nonlinear iteration (subiteration) \(k\) can be constructed by successive matrix-vector products. This process essentially provides the control for the interface fluid force \(f^f = \int \sigma^f \cdot \mathbf{n}^f \, df^f\) to stabilize strong FSI at low structure-to-fluid mass ratio. The scheme proceeds in a similar fashion as the predictor-corrector schemes by constructing the iterative interface force correction at each iteration.

Let the error in the interface fluid force between the initial and first nonlinear iteration be \(\Delta E^f_0 = f^f_1 - f^f_0\). Similarly, the force at the iteration \(k + 1\) is given by

\[
f^f_{(k+1)} = f^f_{(k)} + \Delta f^f_{(k)} = f^f_{(k)} + (A^{fl})^{-1} R^l_N_{(k)},
\]

(62)
For the iterative correction of the fluid forces, a power method is considered for the aforementioned matrix problem. We assume an iteration matrix $M$, which is diagonalizable in such a way that $M \psi(k) = \lambda(k) \psi(k)$ for each $k$ and eigenvalues $\lambda(k)$ are distinct and nonzero with the corresponding eigenvectors $\psi(k)$. The correction to the forces is then constructed with the aid of the error vector $\Delta E^{(0)}$ as

$$f^{(k+1)} = f^{(k)} + M^k \Delta E^{(0)},$$

which can be written in terms of successive estimates as

$$f^{(k+1)} = f^{(0)} + \sum_{l=0}^{k} M^l \Delta E^{(0)}, \quad \text{for } k = 1, 2, \ldots$$

The error vectors $\Delta E^{(k)}$, can then be expressed in terms of the eigenvalues and eigenvectors $\lambda$ and $\psi$, respectively, to obtain a sequence of transformation for the force vector $f^{(k+1)}$ similar to Aitken’s iterated $\Delta^2$ process.26

This interface force correction can also be interpreted as a quasi-Newton update

$$\Delta f^{(k+1)} = \Delta f^{(k)} + \Lambda(k) \Delta E^{(0)},$$

where $\Delta f^{(k+1)} = f^{(k+1)} - f^{(k)}$, $\Delta f^{(k)} = f^{(k)} - f^{(k-1)}$, and $\Lambda(k) = (M^k - M^{k-1})$ is an $n \times n$ matrix. There are three possible alternatives for the matrix $\Lambda(k)$, namely, scalar, diagonal, and full matrix. We consider $\Lambda(k) = a(k)I$ for the iterative quasi-Newton update, which can be considered as a minimal residual iteration method when $(y, \Delta f^{(k+1)}) = 0$ for some $y$, where $(\cdot, \cdot)$ denotes the standard inner product. Thus, we have

$$\left(y, \Delta f^{(k+1)}\right) = \left(y, \Delta f^{(k)}\right) + a(k) \left(y, \Delta E^{(0)}\right) = 0,$$

$$\Rightarrow a(k) = -\frac{(y, \Delta f^{(k)})}{(y, \Delta E^{(0)})}.$$ (67)

It can be observed that the choice of $y = \Delta E^{(0)}$, minimizes $\|\Delta f^{(k+1)}\|$ and this type of iterative procedure is similar to the minimal residual method.45,46

### 3.2 Implementation details

The two-phase flow system in Equation (59) is decoupled into two subsystems, i.e., Navier-Stokes and Allen-Cahn solves, for which the linear system of equations can be summarized as

$$\begin{bmatrix} K_{uf} & G_{uf} \\ -G_{uf} & C_{uf} \end{bmatrix} \{\Delta u\} + \{\Delta p\} = \{\tilde{R}_m\} + \{\tilde{R}_c\},$$

$$[K_{AC}] \{\Delta \phi\} = \{\tilde{R}(\phi)\},$$

where $\tilde{R}_m$, $\tilde{R}_c$, and $\tilde{R}(\phi)$ represent the weighted residuals of the stabilized momentum, continuity, and the Allen-Cahn equations, respectively. Notice that the terms forming the matrices $D_{uf}$ and $G_{AC}$ do not exist after the decoupling because we decouple the equations in a partitioned iterative manner, which is described later. Using a Newton-Raphson technique, the resulting two-phase flow variables and the ALE mesh displacement arising from the finite element discretization are evaluated by solving the linear system of equations via the Generalized Minimal RESidual (GMRES) algorithm proposed in the work of Saad and Schultz.27 To form the linear matrix system, we only construct the required matrix-vector products of each block matrix for the GMRES algorithm, instead of constructing the left-hand side matrix explicitly.

The algorithm for the partitioned iterative coupling of the implicit two-phase fluid structure solver is presented in Algorithm 1. It consists of seven steps in a nonlinear iteration for the exchange of data between the different blocks of the solver. In a typical nonlinear iteration $k$, the first step involves the solution of the structure equation to get the updated structural displacements $\eta^{n+1}_{(k)}$. These displacements are transferred to the Navier-Stokes solve by satisfying the ALE compatibility condition at the fluid-structure interface $\Gamma^{fs}$ in the second step. This is accomplished as follows. Let the updated mesh displacement be denoted by $\eta_{(k+1)}^{m,n+1}$. This mesh displacement is equated to the structural displacement at the interface $\Gamma^{fs}$ to prevent any overlaps between the fluid and the structural domains

$$\eta_{(k+1)}^{m,n+1} = \eta_{(k+1)}^{s,n+1}, \text{ on } \Gamma^{fs}.$$ (70)
Moreover, the conservation property between the moving elements in the fluid domain is satisfied by equating the fluid velocity to the mesh velocity at the interface, i.e.,

\[ u_{f,n+1}^{(k+1)} = u_{m,n+\alpha}^{(k+1)}, \text{ on } \Gamma_{fs}, \] (71)

where the mesh velocity is written as follows:

\[ u_{m,n+\alpha}^{(k+1)} = \frac{n_{m,n+1}^{(k+1)} - n_{m,n}^{(k+1)}}{\Delta t} = u_{s,n+\alpha}^{(k+1)} \text{ on } \Gamma_{fs}. \] (72)

This ensures that the no-slip condition is satisfied at the fluid-structure interface (Equation (52)). The mesh displacement for each spatial point \( x^f \in \Omega^f(t) \) is obtained by modeling the fluid mesh as a hyperelastic material in equilibrium

\[ \nabla \cdot \sigma_m = 0, \] (73)

where \( \sigma_m \) is the stress experienced by the fluid mesh due to the strain by the deformation of the fluid-structure interface and is modeled by the Ogden model.\(^{48,49} \) The mesh velocity for the spatial points \( x^f \in \Omega^f(t) \) is then evaluated using the first equality in Equation (72). The convection velocity is adjusted by subtracting the mesh velocity \( u_{m,n+\alpha}^{(k+1)} \) from the fluid velocity \( u_{f,n+\alpha}^{(k+1)} \) and transferred to the Navier-Stokes solve. In the third step, the Navier-Stokes equations are solved in the ALE reference coordinate system (Equation (68)), thus solving for updated velocity \( u_{f,n+1}^{(k+1)} \) and pressure \( p_{n+1}^{(k+1)}. \) This updated fluid velocity with the mesh velocity (to obtain the adjusted convection velocity) is then transferred to the Allen-Cahn solve in the fourth step. The Allen-Cahn equation (Equation (69)) is solved to evolve the fluid-fluid interface \( \Gamma_{ff} \) in the updated mesh configuration in the fifth step. The physical properties of the fluid such as its density, viscosity, and surface tension are then updated with the help of the updated order parameter values \( \phi_{n+1}^{(k+1)} \) in the sixth step. With the help of all the updated fluid variables, the hydrodynamic force on the fluid-structure interface \( \Gamma_{fs} \) is evaluated by integrating the stress tensor over the structural surface. The force corrected by the NIFC\(^{26,27} \) procedure, \( f_{s}^{(k+1)} \) at the fluid-structure interface is equated with the structural force in the final step, thus satisfying the dynamic equilibrium (Equation (53)) at the fluid-structure interface

\[ f_{s,n+\alpha}^{(k+1)} = f_{s}^{(k+1)} \text{ on } \Gamma_{fs}. \] (74)

---

**Algorithm 1 Partitioned coupling of implicit two-phase fluid-structure interaction solver**

Given \( u^0, p^0, \phi^0, \eta^0 \)

Loop over time steps, \( n = 0, 1, \ldots \)

Start from known variables \( u^{n,n}, p^n, \phi^n, \eta^n \)

Predict the solution:

\[ u_{0}^{n+1} = u_{0}^{n}, \quad p_{0}^{n+1} = p^n, \quad \phi_{0}^{n+1} = \phi^n, \quad \eta_{0}^{n+1} = \eta^n \]

Loop over the nonlinear iterations, \( k = 0, 1, \ldots \) until convergence

---

### 3.3 General remarks

The exact tracking of the fluid-structure interface via the ALE technique along with the interface capturing phase-field technique for the fluid-fluid interface renders the formulation hybrid. While the phase-field model approximates the interface by a smeared surface using the internal length scale parameter, the present Allen-Cahn–based phase-field
formulation is derived from the thermodynamic arguments and has a theoretical basis in the minimization of the Ginzburg-Landau energy functional. Unlike the level-set and VOF techniques, the interface evolution by the phase-field description simplifies the formulation by avoiding any reinitialization or geometric reconstruction of the interface. Furthermore, the PPV formulation to solve the nonlinear Allen-Cahn equation helps to establish the positivity condition nonlinearly at the local element matrix level, resulting in the positivity preserving and monotone scheme, which has been shown in the work of Joshi and Jaiman.40

The ability of the solver to handle low structure-to-fluid mass ratio can be attributed to the NIFC procedure based on quasi-Newton updates. The idea behind the procedure is to construct the cross-coupling effect of strong FSI along the interface without forming the off-diagonal Jacobian term ($A_{ij}$ in Equation (58)) via nonlinear iterations. The correction relies on an input-output relationship between the structural displacement and the force transfer at each nonlinear iteration. The input-output feedback process can be also considered as a minimal residual iteration method to transform a divergent fixed-point iteration to a stable and convergent update of the approximate forces associated with the interface DOF.26 Unlike the brute-force iterations in the strongly coupled FSI, which lead to severe numerical instabilities for low structure-to-fluid mass ratios, the NIFC procedure provides a desired stability to the partitioned fluid-structure coupling, without the explicit evaluation of the off-diagonal Jacobian term. Further details about the NIFC formulation can be found in the works of Jaiman et al.26,27 The aforementioned characteristics of the proposed partitioned coupling between the two-phase fluid and the structure lead to a robust and stable formulation. Moreover, the partitioned-block type feature of the solver leads to flexibility and ease in its implementation to the existing variational solvers. These desirable features of the proposed formulation are analyzed and assessed through various numerical tests in the next section.

4 | NUMERICAL TESTS

In this section, we present some numerical tests to assess the coupling between the two-phase Allen-Cahn–based solver and the structural solver. To accomplish this, we perform the decay tests by examining the interaction of the free surface with a rigid circular cylinder under translation and a rectangular barge under pure rotation.

4.1 | Heave decay test under translation

We herein consider the free heave motion of a circular cylinder at the free surface of water. The schematic of the computational domain $\Omega \in [0, 90D] \times [0, 14.6D] \times [0, 17D]$ considered in this study is shown in Figure 2, where a circular cylinder of diameter $D = 0.1524$ is placed initially at an offset of $0.167D$ m from the free surface of water. The density of the cylinder is half that of the denser fluid, ie, $\rho^s = 500$, $\rho^f_1 = 1000$, and $\rho^f_2 = 1.2$. The dynamic viscosities of the two phases are $\mu^f_1 = 10^{-3}$ and $\mu^f_2 = 1.8 \times 10^{-5}$. The acceleration due to gravity is $g = (0, -9.81, 0)$. Apart from the high density ratio between the two phases, ie, $\rho^* = \rho^f_1 / \rho^f_2 = 833.3$, a low structure-to-fluid density ratio ($\rho^s / \rho^f_1 = 0.5$) based on the denser fluid has been chosen. The initial condition for the order parameter is given as

$$\phi(x, y, 0) = -\tanh\left(\frac{y}{\sqrt{2\epsilon}}\right). \tag{75}$$

![FIGURE 2](image-url) Schematic of the decay response of a cylinder of diameter $D$ under gravity in the $X-Y$ cross-section. The computational domain extends a distance of $17D$ in the $Z$-direction.
We have employed the hybrid RANS/LES model discussed in the work of Joshi and Jaiman\textsuperscript{50} for modeling the turbulent effects. The Reynolds number is defined based on the maximum velocity achieved by the cylinder and its diameter with respect to the denser fluid, i.e., \( Re = \frac{\rho f U_{\text{cyl}} D}{\mu f} \approx 30,000 \).

A typical computational mesh prepared for the decay test is shown in Figure 3. A boundary layer covers the structural cylinder with the first layer at a distance from the cylindrical surface such as to maintain \( y^+ \sim 1 \) (Figure 3B). Moreover, a refined mesh consisting of a cylinder with radius 3.3\( D \) is constructed around the boundary layer region to capture the vortices produced due to the heave motion at the free surface (Figure 3A). To capture the air-water interface accurately, the interfacial region is refined in accordance with the suggestions in the work Joshi and Jaiman\textsuperscript{24} such that at least four elements lie in the equilibrium interface region. Therefore, the employment of a sharper fluid-fluid interface (smaller \( \epsilon \)) leads to a more refined mesh, assuming the interfacial region is sufficiently captured by the refinement. The mesh consists of 72,604 number of nodes with 144,566 triangular elements in the two-dimensional plane perpendicular to the axis of the cylinder. The mesh is then extruded in the Z-direction consisting of seven layers. The no-slip boundary condition is satisfied at the cylindrical surface, whereas the slip boundary condition is set on all other boundaries.

We first present the convergence studies for which we considered the two-dimensional domain for the current case. For the temporal convergence, a time step of \( \Delta t = 2 \times 10^{-2} \) is decreased by a factor of 2 till \( \Delta t = 6.25 \times 10^{-4} \). The interfacial thickness parameter is selected as \( \epsilon = 0.01 \) for the study. The decaying heave motion of the cylinder under different time steps is plotted in Figure 4A. A nondimensional error for quantifying the convergence is established, which is defined as

\[
e_1 = \frac{||\eta - \eta_{\text{ref}}||_2}{||\eta_{\text{ref}}||_2},
\]

where \( \eta \) represents the temporal evolution of the heave motion of the cylinder for corresponding time step, \( \eta_{\text{ref}} \) is the heave motion evolution with time for the finest time step (\( \Delta t = 6.25 \times 10^{-4} \)), and \( || \cdot ||_2 \) is the standard Euclidean \( L^2 \) norm. Figure 4B shows the plot for the error \( e_1 \) with the time step size \( \Delta t \), which gives a temporal convergence of 1.6.

The spatial convergence is studied based on the interface thickness parameter \( \epsilon \). Three values of the parameter are selected, viz., \( \epsilon \in [0.02, 0.01, 0.005] \). In accordance with the suggestion of four number of elements in the interfacial region, smaller \( \epsilon \) leads to a more refined mesh. The mesh for \( \epsilon = 0.02, 0.01 \) and 0.005 consists of 35,954 (71,300 elements), 72,604 (144,566 elements), and 205,094 (409,494 elements) nodes, respectively. The heave motion is shown in Figure 5 for different values of \( \epsilon \). We observe a minor difference in the response of the cylinder interacting with the free surface. Therefore, based on the convergence studies, we select the interface thickness parameter of \( \epsilon = 0.01 \) and time step of \( \Delta t = 0.0025 \) for further assessment.

In what follows, we perform the three-dimensional computation of the FSI problem with the selected spatial and temporal convergence parameters for validation with the experiment\textsuperscript{51} and the simulation.\textsuperscript{18} The three-dimensional mesh consists of 580,000 nodes with 1.01 million six-node wedge elements. The simulation is carried out with 48 processors,
FIGURE 4 Temporal convergence study for the decay test of a circular cylinder. A, Heave motion employing temporal refinement; B, The dependence of nondimensionalized $L^2$ error ($e_1$) as a function of uniform temporal refinement $\Delta t$.

FIGURE 5 Dependence of spatial grid convergence on the interfacial thickness parameter $\epsilon$ for the decay test of a circular cylinder at free surface.

FIGURE 6 Decay test of a circular cylinder under translation along the free surface. A, Validation of the heave motion of the cylinder at the free surface with the experimental51 and simulation18 studies; B, Z-vorticity contours around the cylinder at $t = 1.5$ s with the free surface indicated at $\phi = 0$.

which take a total computational time of 11.63 hours. The solver performs four nonlinear iterations to achieve a relative nonlinear convergence tolerance of $5 \times 10^{-4}$. The results of the evolution of the heave of the cylinder are shown in Figure 6A, where we find a very good agreement with the literature. The Z-vorticity contours with the interface shown at
4.2 Decay test under rotation along free surface

For further validation and robustness assessment, we consider the pure rotation of a rectangular barge of length $L = 0.3$, height $H = 0.2$, and width $W = 3L$ at the free surface of water. The computational domain $\Omega \in [0, 58.3L] \times [0, 6.3L] \times [0, 3L]$ with the barge inclined at an angle of $\theta = 15^\circ$ from the free-surface level is shown in Figure 7A. The center of gravity of the barge is at the free-surface level with its mass moment of inertia and the rotational damping matrices, respectively, as

$$I^s = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.236 \end{bmatrix} \text{kg \cdot m}^2, \quad C_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.275 \end{bmatrix} \text{kg \cdot m}^2 \cdot \text{s}^{-1}. \quad (77)$$

The physical properties of the fluid domain are $\rho_1 = 1000$, $\rho_2 = 1.2$, $\mu_1 = 10^{-3}$, $\mu_2 = 1.8 \times 10^{-5}$, and $g = (0, -9.81, 0)$. The initial condition for the order parameter is given as

$$\phi(x, y, 0) = -\tanh \left( \frac{y}{\sqrt{2\epsilon}} \right). \quad (78)$$

The Reynolds number is defined based on the maximum velocity achieved by the upper corner of the barge and its length with respect to the denser fluid, ie, $Re = \frac{\rho_1 U_{\text{corner}} L / \mu_1}{99, 500}$. The computational mesh employed for the simulation shown in Figure 7B is constructed with similar characteristics as the mesh in Section 4.1. A boundary layer envelops the rectangular barge with a refined mesh around the boundary layer to capture the vortices produced at the free surface. Moreover, the evolution in the free surface is captured by a refined region around the interfacial region. The mesh extrudes in the Z-direction and consists of 10 layers.

With the converged spatial and temporal parameters, we validate the rotational response of the rectangular barge with that of the experiment and the computational data. The mesh consists of 1 025 684 nodes with 1 856 930 six-node wedge elements. The simulation is carried out by 72 processors with a computational time of 22.2 hours. On an average, the solver performs four nonlinear iterations to achieve a relative nonlinear convergence tolerance of $5 \times 10^{-4}$. The rotational motion is plotted in Figure 8A, where we observe good agreement with the available results in the literature. The Z-vorticity contours are shown in Figure 8B at $t = 1.4$ s. This concludes the validation and convergence studies for the coupled FSI two-phase solver.
5 | APPLICATION TO FLEXIBLE RISER FSI WITH INTERNAL TWO-PHASE FLOW

We next demonstrate the capability of the developed phase-field FSI solver in handling a practical problem of a riser with an internal two-phase flow and expose to external uniform current flow. A typical schematic for the problem is shown in Figure 9. The riser has an outer diameter of $D$ and span of $L = 20D$. The inflow and outflow boundaries are at a distance of $10D$ and $30D$ from the center of the riser, respectively. The side walls are equidistant from the riser center at $15D$ on either side. The outer surface of the riser is exposed to a uniform inflow current of $\mathbf{u}_f = (U_\infty, 0, 0)$. The no-slip boundary condition is satisfied at the outer surface of the riser. All other boundaries are slip boundaries except the outflow where the stress-free condition is satisfied. The fluid domain exposed to the external part of the riser is denoted by $\Omega_f^1$. The interior...
of the riser has internal diameter $2r_2$ with an initial concentric profile for the two phases with the interface at a radius of $r_1$ from the riser axis separating the two phases $\Omega_2^f$ and $\Omega_1^f$. A prescribed profile for the Z-velocity is imposed at the inlet of the riser for the internal flow, whereas the stress-free condition is satisfied at the outlet of the internal flow. The velocity is such that no-slip condition is satisfied at the internal surface of the riser. We consider the profile of the velocity for a coannular, laminar, and fully developed flow regime consisting of immiscible Newtonian fluids given by Nogueira and Cotta.\(^5\)

$$\mathbf{u}^t = (0, 0, w)$$

(79)

$$w(R) = \begin{cases} \left[ \frac{c_{1-(r^2+y^2/R^2)^2}}{(r^2+y^2/R^2)} \right]^{\frac{1}{2}}, & 0 \leq R \leq r^*, \\ \frac{c_{1-(r^2+y^2/R^2)^2}}{(r^2+y^2/R^2)} \right]^{\frac{1}{2}}, & r^* \leq R \leq 1, \end{cases}$$

(80)

where $R = \sqrt{x^2 + y^2/r_2}$, $r^* = r_1/r_2$, $\mu^* = \mu_1^f/\mu_2^f$, and $C = (n + 3)/2$, where $n = 1$ for a circular tube. The physical parameters employed for the demonstration are $r_1 = 0.2$, $r_2 = 0.4$, and $g = (0, 0, 0)$. The initial condition for the order parameter is given as

$$\phi(x, y, 0) = \begin{cases} \tan \left( \frac{\sqrt{x^2+y^2-r_1}}{\sqrt{2x}} \right), & \sqrt{x^2+y^2} \leq r_2, \\ 1, & \text{elsewhere}. \end{cases}$$

(81)

In the present demonstration, we employ a linear flexible body solver for solving the structural equation. The Euler-Bernoulli beam equation is solved in the eigenspace with the structural displacement represented as a linear combination of the eigenmodes. This analysis has been explained in Appendix A.2. Following the notations from the analysis in Appendix A.2, the nondimensional parameters for the VIV of riser with the internal flow are defined as follows:

$$Re = \frac{\rho_1^f U_{\infty} D}{\mu_1^f}, \quad m^* = \frac{m^b}{\pi D^2 L \rho_1^f / 4}, \quad \rho^* = \frac{\rho_1^f}{\rho_2^f}, \quad \mu^* = \frac{\mu_1^f}{\mu_2^f},$$

$$U_r = \frac{U_{\infty}}{f_1 D}, \quad P^* = \frac{P}{\rho_1^f U_{\infty}^2 D^2}, \quad EI^* = \frac{EI}{\rho_1^f U_{\infty}^2 D^2},$$

(82)

where $Re$, $m^*$, $\rho^*$, $\mu^*$, $U_r$, $P^*$, and $EI^*$ denote the nondimensional quantities, viz, Reynolds number, mass ratio, density ratio, viscosity ratio, reduced velocity, axial tension, and flexural rigidity of the riser modeled as a beam. Here, $f_1$ denotes the frequency of the first eigenmode calculated by Equation (A10). We consider two cases with different Reynolds number in the present demonstration. The nondimensional numbers related to the cases are shown in Table 1.

In the following sections, we provide a brief physical insight on the response of the riser for the two cases and the effect of the external flow VIV on the internal flow regime. As the reduced velocity for the two cases lies within the “lock-in” range, we expect the maximum amplitude to be of the order of the riser diameter. We also shed some light on the response wave patterns and their frequencies. Further parametric studies need to be conducted to thoroughly understand the physics of this complex coupled problem, which is beyond the scope of this work.

| Cases | $Re$ | $m^*$ | $\rho^*$ | $\mu^*$ | $U_r$ | $P^*$ | $EI^*$ | $\rho_1^f/\rho_2^f$ |
|-------|------|-------|--------|--------|-------|-------|--------|-------------------|
| Case 1 | 100  | 2.89  | 100    | 100    | 5     | 0.34  | 5872.8 | 6.68              |
| Case 2 | 1000 | 2.89  | 100    | 100    | 5     | 0.34  | 5872.8 | 6.68              |
FIGURE 10  Computational mesh for the vortex-induced vibration of riser with internal two-phase flow. A, Three-dimensional view of the mesh; B, Two-dimensional cross-section of the mesh with refined wake region behind the riser and refined internal region of the riser to capture the interface between the two phases in the internal flow accurately [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 11  The response amplitude at the midpoint of the riser ($z/L = 0.5$) exposed to external uniform flow with internal two-phase flow. A, Case 1 ($Re = 100$); B, Case 2 ($Re = 1000$)

5.1 | Amplitude response and flow patterns

The amplitude of the riser is found maximum at the midpoint along its span and has been plotted in Figure 11 for the two Reynolds number cases. Cases 1 and 2 are simulated for a nondimensional time $tU_\infty/D$ of 90 and 50, respectively, so that at least four cycles of VIV are captured. The temporal variation of the response amplitude along the riser in the cross-flow and in-line directions are shown in Figures 12 and 13 for Cases 1 and 2, respectively. We observe a standing wave pattern along the riser from the plots. Furthermore, a spectral analysis of the amplitude response at $z/L = 0.5$ reveals a single nondimensional frequency ($fU_\infty/D$) of 0.1709 in the cross-flow and 0.3662 in the in-line directions for Case 1, indicating the lock-in phenomenon. On the other hand, the in-line oscillation in Case 2 shows a multimodal response comprising of $fU_\infty/D$ as 0.1953 and 0.3662 and the cross-flow vibration indicates a single frequency response with $fU_\infty/D = 0.1709$.

The flow contours of the Z-vorticity along the riser span with the visualization of the internal flow via the order parameter $\phi$ are shown in Figure 14 for Case 1 and Figure 15 for Case 2. The irregularity in the vortex patterns suggests the onset of turbulent wake for Case 2. It is found that the topological changes in the fluid-fluid interface are captured qualitatively in the current two-phase FSI simulation. The change in the internal flow pattern from coannular regime to bubble/slug flow pattern can be observed due to the vibrating riser.
5.2 Relationship between VIV and internal flow patterns

To investigate the effect of the VIV on the internal fluid flow and vice-versa, we simulate the VIV problem of the riser without the internal flow, i.e., a solid riser with the same nondimensional parameters considered for Case 1. The results show an insignificant difference in the vibrational amplitude along the riser. It can be inferred that the internal flow has negligible or no effect on the VIV response of the riser for the parameters considered in this study. This may be due to the dominant effect of the external flow on VIV compared to the inertia of the internal fluid. On the other hand, the effect of the VIV on the internal flow is evident from the flow contours in Figures 14C and 15C. It is observed that the coannular initial two-phase flow pattern is transitioning into an elongated bubble/slug flow pattern. This type of flow pattern prediction through a fully-coupled two-phase FSI can be advantageous to improve multiphase flow assurance. A rigorous parametric study is required to quantify the variation of the structural parameters, the reduced velocity, and the mass ratio on the internal flow regimes, which forms a topic for future study.
FIGURE 14 Contour plots for the vortex-induced vibration of a riser at $Re = 100$ with internal two-phase flow at $tU_\infty/D$. A, 80; B, 90; C, The internal flow along the riser. The $Z$-vorticity contours are shown at three cross-sections along the riser, viz, $z/L \in [0.25, 0.5, 0.75]$, and are colored with red for positive vorticity and blue for negative vorticity. The inset figure provides the velocity magnitude at the midsection of the riser. The interior two-phase flow of the riser is visualized by the contours of order parameter $\phi > 0$ at an arbitrary plane passing through the axis of the deformed riser.

FIGURE 15 Contour plots for the vortex-induced vibration of a riser at $Re = 1000$ with internal two-phase flow at $tU_\infty/D$. A, 40; B, 50; C, The internal flow along the riser. The $Z$-vorticity contours are shown at three cross-sections along the riser, viz, $z/L \in [0.25, 0.5, 0.75]$, and are colored with red for positive vorticity and blue for negative vorticity. The inset figure provides the velocity magnitude at the midsection of the riser. The interior two-phase flow of the riser is visualized by the contours of order parameter $\phi > 0$ at an arbitrary plane passing through the axis of the deformed riser.

6 | CONCLUSIONS

In this study, a variational phase-field FSI formulation has been developed for the coupled analysis of FSI in two-phase flow. The one-fluid formulation for the two-phase flows with a diffuse interface description of the fluid-fluid interface offers an advantage to capture the interface with topological changes without involving any remeshing or complex geometric manipulations of unstructured mesh. The conservative Allen-Cahn equation has been utilized to evolve the fluid-fluid interface. The fluid-structure interface is considered in the Lagrangian manner, whereas the two-phase flow equations
are formulated in the ALE framework. This produces a hybrid Allen-Cahn/ALE scheme that can accurately capture the fluid-structure interface with phase-field–based interface capturing of the fluid-fluid interface. The governing equations are solved via the nonlinear partitioned iterative technique, which can be easily implemented in variational solvers with little effort. This type of coupling, considering the Allen-Cahn equation to model the two-phases, has been carried out for the first time. Such FSI simulations via Allen-Cahn phase-field model can be very powerful for highly complex three-dimensional evolving fluid-fluid and fluid-structure interfaces.

The desirable features of the proposed phase-field/ALE formulation are examined for two-phase flow interacting with rigid and flexible structures. The robustness and generality of the proposed formulation have been demonstrated for increasing complexity of problems involving high density $\sim O(10^3)$ and viscosity ratios $\sim O(10^2)$ of the two fluid phases and low structure-to-fluid mass ratios. For the decay test, the structure-to-fluid density ratio is $\rho_s/\rho_1 = 0.5$, whereas the mass ratio for the three-dimensional flexible riser is $m^* = 2.89$. These tests establish the high density/viscosity ratio and low mass ratio handling capability of the coupling. The convergence tests reveal almost second order of the temporal accuracy for the phase-field FSI formulation. It is also noticed that the coupled two-phase FSI solver performs about four nonlinear iterations to achieve a relative nonlinear convergence tolerance of $5 \times 10^{-4}$. With regards to spatial accuracy, the interfacial thickness parameter $\epsilon = 0.01$ is observed to be sufficient to capture the topological changes in the fluid-fluid interface accurately. The numerical results are validated with the available experimental as well as computational results from the literature for the two decay test problems, namely, free translational heave decay of a circular cylinder and free rotation of a rectangular barge. Finally, a practical problem of an internal flow through a flexible riser exposed to external flow VIV has been demonstrated. A detailed physical investigation of the effect of the internal flow on the VIV of the riser or vice-versa is a subject for future study.

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APPENDIX A

STRUCTURAL EQUATIONS

A.1 | Rigid body

We first consider the six DOFs motion of a rigid body. Let the center of mass of the body in the reference configuration $x^s$ and the current configuration $\phi^s$ be $x_0^s$ and $\phi_0^s$, respectively, and $\eta_0^s$ denote the displacement of the center of mass due to the translation of the body. Therefore, the rigid body kinematics is given by

$$\phi^s = Q(x^s - x_0^s) + \phi_0^s = Q(x^s - x_0^s) + x_0^s + \eta_0^s,$$

(A1)

where $Q$ is a rotation matrix. Using Equations (37) and (A1),

$$\eta^s = (Q - I)(x^s - x_0^s) + \eta_0^s,$$

(A2)

$$\frac{\partial \eta^s}{\partial t} = \frac{\partial Q}{\partial t}(x^s - x_0^s) + \frac{\partial \eta_0^s}{\partial t},$$

(A3)

where $I$ is the identity matrix and Equation (A3) is obtained by differentiating Equation (A2) with respect to time. Suppose the rotational DOFs for the body are given by $\Theta^s$. Equation (A3) can be restructured in terms of the angular velocity of the body denoted by $\omega^s = \partial \Theta^s / \partial t$ as

$$\frac{\partial \eta^s}{\partial t} = \omega^s \times (\phi^s - \phi_0^s) + \frac{\partial \eta_0^s}{\partial t}.$$

(A4)

The rigid body equations are thus given by

$$M^s \frac{\partial^2 \eta_0^s}{\partial t^2} + C^s \frac{\partial \eta_0^s}{\partial t} + K^s \eta_0^s = f^s,$$

on $\Omega^s$,

(A5)

$$I^s \frac{\partial^2 \Theta^s}{\partial t^2} + C^s \frac{\partial \Theta^s}{\partial t} + K^s \Theta^s = \tau^s,$$

on $\Omega^s$,

(A6)

where $M^s$, $C^s$, and $K^s$ denote the mass, damping, and stiffness matrices for the translational DOF, respectively; $I^s$, $C^s$, and $K^s$ represent the moment of inertia, damping, and stiffness matrices for the rotational DOFs, respectively; and $f^s$ and $\tau^s$ denote the forces and the moments applied on the body, respectively.

A.2 | Linear flexible body

For modeling the flexible body dynamics, we consider a linear modal analysis by solving the Euler-Bernoulli beam equation. Suppose the axis of the beam is parallel to the $Z$-direction along which the coordinates are given by $z$. We solve for the lateral displacements (denoted by $\eta^s(z,t)$) along the beam as

$$m^s \frac{\partial^2 \eta^s}{\partial t^2} + \frac{\partial^2}{\partial z^2} \left( E I L \frac{\partial^2 \eta^s}{\partial z^2} \right) - P L \frac{\partial^2 \eta^s}{\partial z^2} = f^s,$$

(A7)

where $m^s$, $E$, $I$, $P$, and $f^s$ denote the mass, Young’s modulus, the second moment of the cross-sectional area of the beam, the applied axial tension, and the external applied force on the beam of span $L$, respectively.
A beam under pinned-pinned condition has to satisfy the following boundary conditions at its ends:

\[ \eta^s|_{z=0} = 0, \quad \left. \frac{\partial^2 \eta^s}{\partial z^2} \right|_{z=0} = 0. \]  
\[ (A8) \]

\[ \eta^s|_{z=L} = 0, \quad \left. \frac{\partial^2 \eta^s}{\partial z^2} \right|_{z=L} = 0. \]  
\[ (A9) \]

To solve Equation (A7), we employ a mode superposition procedure, where the frequency of \( n \)th mode is given by

\[ f_n = \frac{1}{2\pi} \sqrt{\frac{1}{L^4} \left( \frac{n^4EI}{m} + \frac{n^2PL^2}{\pi^2} \right)} . \]  
\[ (A10) \]

The structural displacements are represented by the superposition of linear eigenmodes, which are obtained using the eigenvalue analysis. For the current configuration, the eigenmodes are assumed to be sinusoidal so that the eigenmode shape for mode \( n \) is given by

\[ S^n(z) = \sin \left( \frac{n\pi z}{L} \right). \]  
\[ (A11) \]

Equation (A7) can be recast into a matrix form as

\[ M^s \frac{\partial^2 \eta^s}{\partial t^2} + K^s \eta^s = f^s, \]  
\[ (A12) \]

where \( M^s \) and \( K^s \) are the mass and stiffness matrices, respectively; \( \eta^s \) is the vector of unknown displacements along the beam; and \( f^s \) is the vector of the force applied on the beam. Now, we project the aforementioned equation in the eigenspace with eigenmodes defined by Equation (A11), which transforms Equation (A12) to a system of linear equations with \( n \) DOFs (modes) as

\[ \tilde{M}^s \frac{\partial^2 \xi^s}{\partial t^2} + \tilde{K}^s \xi^s = \tilde{f}^s, \]  
\[ (A13) \]

where \( \tilde{M}^s = \mathbf{S}^T M^s \mathbf{S} \) and \( \tilde{K}^s = \mathbf{S}^T K^s \mathbf{S} \) are the projected matrices on the eigenspace, where \( \mathbf{S} \) is the matrix containing the eigenvectors and \( \xi^s \) is the vector of the modal responses along the beam. The projected force vector \( \tilde{f}^s = \mathbf{S}^T f^s \) and \( \eta^s = \mathbf{S} \xi^s \).