NLO corrections to $e^+e^- \rightarrow W^+W^-Z$ and $e^+e^- \rightarrow ZZ\ Z$

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Abstract

We calculate the one-loop electroweak corrections to $e^+e^- \rightarrow W^+W^-Z$ and $e^+e^- \rightarrow ZZ\ Z$ and analyse their impacts on both the total cross section and some key distributions. These processes are important for the measurements of the quartic couplings of the massive gauge bosons which can be a window on the mechanism of spontaneous symmetry breaking. We find that even after subtracting the leading QED corrections, the electroweak corrections can still be large especially as the energy increases. We compare and implement different methods of dealing with potential instabilities in the routines pertaining to the loop integrals. For the real corrections we apply a dipole subtraction formalism and compare it to a phase-space slicing method.
1 Introduction

The LHC has just started running again and seems now to be on course for what it has been built for: discovery of the last remaining particle of the much successful Standard Model, the Higgs boson. It may well be that before this particle is uncovered we will have seen clear signs of New Physics that better encompasses an elementary Higgs boson. The conventional Minimal Supersymmetric Standard Model is the most popular example of such a scenario. It may however happen that the mechanism of electroweak symmetry breaking will remain elusive and that one has to look for subtle deviations in standard processes. Because of its clean environment a linear collider might be more suited for this purpose.

From this perspective the study of $e^+e^- \rightarrow W^+W^-Z$ and $e^+e^- \rightarrow ZZZ$ may be very instructive and would play a role similar to $e^+e^- \rightarrow W^+W^-$ at lower energies. Indeed it has been stressed that $e^+e^- \rightarrow W^+W^-Z$ and $e^+e^- \rightarrow ZZZ$ are prime processes for probing the quartic vector boson couplings [1]. In particular deviations from the gauge value in the quartic $W^+W^-ZZ$ and $ZZZZ$ couplings that are accessible in these reactions might be the residual effect of physics intimately related to electroweak symmetry breaking. Since these effects can be small and subtle, knowing these cross sections with high precision is mandatory. This calls for theoretical predictions taking into account loop corrections.

Apart from the physics motivations for performing such calculations, the other reason is that one-loop corrections, in particular the electroweak corrections, for such $2 \rightarrow 3$ processes are a good testing ground for the various ingredients and techniques that enter such one-loop multi-leg corrections. Although recently NLO corrections to $2 \rightarrow 4$ processes have set the technical frontier with a handful of processes in this category having been addressed [2], NLO corrections to $2 \rightarrow 3$ processes are far from straightforward. Not only the number of diagrams differs greatly from one process to another but perhaps more importantly the loop structure can also differ significantly. For the processes at hand one has to deal with high rank tensors, rank-4, for the pentagon diagrams, compared to at most a rank-2 for $e^+e^- \rightarrow \nu\bar{\nu}H$ [3, 4, 5, 6]. This might lead to much more severe numerical instabilities due to the appearance of higher powers of the inverse Gram determinants in the tensor reduction. Moreover different scales and masses may lead to sensitive issues related to Landau singularities in scalar integrals [9]. It is therefore important to conduct one-loop corrections to a variety of $2 \rightarrow 3$ processes.

Radiative corrections to $e^+e^- \rightarrow ZZZ$ have appeared recently in [10] and those to $e^+e^- \rightarrow W^+W^-Z$ in [11] while we were preparing this paper. We have made an independent calculation of the electroweak corrections to $e^+e^- \rightarrow W^+W^-Z$ and $e^+e^- \rightarrow ZZZ$. Preliminary results on $e^+e^- \rightarrow W^+W^-Z$ have been presented in [12] before those of [11] were made public. We perform two independent calculations and check further through non-linear gauge parameter independence tests. These help also identify potential instabilities in the routines pertaining to the loop integrals. We detail how some critical

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1 $ZZZZ$ is absent at tree-level in the Standard Model. Other photonic quartic couplings may be probed in the processes we study but they are best studied in other reactions. Moreover the latter, from the point of view symmetry breaking, are less relevant and would be of higher order [2].

2 As far as electroweak corrections are concerned only two such calculations have been performed [3, 4].
issues related to inverse Gram determinants have been tackled, how real corrections have been implemented and how checks were conducted. Our calculation is implemented in two independent Monte Carlo codes which can calculate total cross sections and arbitrary distributions. We compare our results with those in [10, 11] and comment also on the renormalisation scheme and input parameters.

2 Calculational details

At leading order $W^+W^-Z$ and $ZZZ$ final states are produced through the diagrams shown in Fig. 1. A common contribution to the two processes is the Abelian-like $t$-channel fermionic exchange akin to the QED process $e^+e^- \rightarrow 3\gamma$. Both processes include the Higgsstrahlung contribution where the splitting $H^* \rightarrow VV$ occurs. Apart from this contribution $e^+e^- \rightarrow W^+W^-Z$ can be built up from $e^+e^- \rightarrow W^+W^-$ through the addition of $Z$ radiation from either the initial or final state and the $s$-channel quartic $WWZZ$. Since the precision electroweak data suggest a Higgs mass below the $WW$ threshold, we restrict our study to the region $M_H < 160\,\text{GeV}$. This means that the Higgsstrahlung contribution can not be resonant and therefore in our calculation no width is introduced.

We also set the electron mass to zero whenever possible. The electron mass then appears only in mass singular logarithms. These arise in the virtual corrections from loop diagrams containing electrons and from photons radiated off electrons in the real corrections.

We have performed our calculation in at least two independent ways both for the virtual and the real corrections leading to two independent numerical codes. A comparison of both codes has shown full agreement at the level of the integrated cross sections as well.
as all the distributions that we have studied.

The phase-space integration is done by using the Monte Carlo integrator BASES \cite{13,14} in one code while the other code employs VEGAS \cite{15}.

2.1 Renormalisation

We adopt the on-shell renormalisation scheme as detailed in \cite{16,17}. By default, in this scheme the electromagnetic coupling is defined in the Thomson limit at $q^2 \to 0$. The counterterm $e \to Ye = (1 + \delta Z_e)e$ is related through a Ward identity to the wave function renormalisation constant of the photon and the wave function describing the $A \to Z$ transition defined at $q^2 = 0$ so that the photon propagator is defined with residue equal to one and no $A \to Z$ transition remains when the photon is on shell \cite{16,17}. In the conventions of \cite{16} this leads to

$$
\delta Z_e = -\delta Z_{AA}^{1/2} + \frac{s_W}{c_W} \delta Z_{VA}^{1/2}, \quad \delta Z_{AA}^{1/2} = \frac{1}{2} \frac{d}{dq^2} \Pi^{AA}_T(q^2)_{q^2=0}, \quad \delta Z_{VA}^{1/2} = -\frac{1}{M_Z^2} \Pi^{AZ}_T(0). \quad (1)
$$

$\Pi^{VV}_T$ is the transverse parts of the $VV$ self-energy. This particular definition of the charge at the scale $q^2 = 0$ is not the most appropriate since the weak processes take place at scales of order $M_Z$ or higher. The running of $\alpha$ from $q^2 = 0$ to $q^2 = M_Z^2$ alone amounts to a 6% correction. For a process of order $\alpha^n$, this running will amount to a correction of order $n \times 6\%$, thus hiding more interesting corrections. Moreover these corrections due to the running are sensitive to the light fermion masses through logarithms of the type $\ln(q^2/m_f^2)$. In fact the effective couplings of the $Z$ to fermions are also sensitive to isospin breaking effects and therefore virtual heavy top effects through $\Delta r$ \cite{18}. The combined effect of these two corrections is better parameterised if one uses the Fermi coupling in lieu of $\alpha(0)$. We will therefore use a variant of the $G_\mu$ scheme. This scheme absorbs a large universal part of $\mathcal{O}(\alpha)$ corrections into the Born contribution. The advantage of the scheme is that the final results are not sensitive to the light fermion masses, in particular the light quark masses, and some universal $m_t^2$ corrections are also absorbed. In this scheme we use $\{G_\mu, M_Z, M_W, \text{other masses}\}$ instead of $\{\alpha(0), M_Z, M_W, \text{other masses}\}$ as input parameters, from which the electromagnetic coupling constant is calculated as

$$
\alpha_{G_\mu} = \frac{\sqrt{2} G_\mu M_W^2}{\pi} s_W^2, \quad s_W^2 = \left(1 - \frac{M_W^2}{M_Z^2}\right). \quad (2)
$$

To avoid double counting we have to subtract the one-loop part of the universal correction from the explicit $\mathcal{O}(\alpha)$ corrections by using the counterterm

$$
\delta Z_{e_\mu} = \delta Z_e - \frac{1}{2} (\Delta r)_{1\text{-loop}}. \quad (3)
$$

In the ’t Hooft-Feynman gauge with the usual linear gauge fixing $(\Delta r)_{1\text{-loop}}$ is given
by \[17, 18\]

\[
(\Delta r)_{1\text{-loop}} = 2\delta Z_e - \frac{\delta s_W^2}{s_W^2} - \left( \Pi_{T}^{WW}(0) + \delta M_W^2 \right) - \frac{2}{s_W c_W} \delta Z^{1/2}_{ZA} \\
+ \frac{\alpha}{4\pi s_W^2} \left[ 6 + \frac{7 - 4s_W^2}{2s_W^2} \ln c_W^2 \right].
\] (4)

Using \(\alpha G_\mu\) in tree-level calculations will therefore take care of some universal higher-order contributions. In the \(G_\mu\) scheme the corrections are initially of order \(\alpha_3 G_\mu\). Considering however that the typical scale for both virtual photon exchange and real photon radiation is \(q^2 = 0\) we rescale our results so that the NLO results are of order \(\alpha_3^3 G_\mu^3\).

### 2.2 Virtual corrections

The virtual corrections have been evaluated using a conventional Feynman-diagram based approach using standard techniques in the two independent codes. The total number of diagrams in the ‘t Hooft-Feynman gauge is about 2700 including 109 pentagon diagrams for \(e^+e^- \rightarrow W^+W^-Z\) and about 1800 including 64 pentagons for \(e^+e^- \rightarrow ZZ\). This already shows that \(e^+e^- \rightarrow W^+W^-Z\) with as many as 109 pentagons is more challenging than \(e^+e^- \rightarrow ZZ\).

**Code 1**

The first code uses **FeynArts-3.4** \[19\] to generate all Feynman diagrams and amplitude expressions. **FormCalc-6.0** \[20, 21\] is used to simplify and generate a Fortran 77 code suited for the numerical evaluation of the differential cross sections. We also use **SloopS** \[22, 23, 24\] an automated code that uses a few modules from **FeynArts-3.4** but which implements the generalised non-linear gauge (NLG) \[25, 16\]

\[
\mathcal{L}_{GF} = \frac{1}{\xi_W} \left( (\partial_\mu - ie \tilde{\alpha} A_\mu - igc_W \tilde{\beta} Z_\mu) W^{\mu+} + \xi_W \frac{g}{2} (v + \tilde{\delta} H + i\tilde{\kappa} \chi_3) \chi^+ \right)^2 \\
- \frac{1}{2\xi_Z} (\partial.Z + \xi_Z \frac{g}{2c_W} (v + \tilde{\varepsilon} H) \chi_3)^2 - \frac{1}{2\xi_A} (\partial.A)^2 .
\] (5)

The \(\chi\) represents the Goldstone. We take the ‘t Hooft-Feynman gauge with \(\xi_W = \xi_Z = \xi_A = 1\) so that no “longitudinal” term in the gauge propagators contributes. Not only does this make the expressions much simpler and avoids unnecessary large cancellations, but it also avoids the need for higher tensor structures in the loop integrals. The use of the five parameters, \(\tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\kappa}, \tilde{\varepsilon}\) is not redundant as often these parameters check complementary sets of diagrams. At a few random points in phase space we exploit these parameters to perform powerful tests on the generated matrix elements. This test can reveal for example numerical instabilities that are due to the reduction algorithm in some points in phase space. While at a regular point in phase space the non-linear gauge check attains a 14 digit agreement in double-precision when changing a NLG parameter from 0 to 1, at non-regular points the same tests can fail.

Five-point one-loop integrals (up to rank 4) are reduced to four-point integrals by using the reduction method of Denner and Dittmaier \[26\]. By default, four-point and
three-point tensor integrals are reduced to scalar integrals by using the Passarino-Veltman reduction algorithm [27]. With the latter we have observed serious problems of numerical instability related to four-point tensor integrals. This occurs when the Gram determinants associated to these tensor integrals, defined by \( \det(G_3) = \det(2p_ip_j) \), become sufficiently small. We have solved this problem in two ways. In the first method, we use a simple extrapolation trick using the segmentation technique described in [28] when the Gram determinant is small enough. The condition implemented in our code is
\[
\frac{\det(G_3)}{(2p_{\text{max}}^2)^3} < 10^{-7},
\]
where \( p_{\text{max}}^2 \) is the maximum external mass of a box diagram. In this limit the \( N \)-point function of rank \( M \) is written as a combination of \((N - 1)\)-point functions of rank \( M \). This is done directly at the level of the loop integral in momentum space before introducing any Feynman parameters. This implementation requires that one supplements the standard libraries with the reduction of the tensors of rank \( M = N + 1 \), for certain \( N \)-point functions. Another way of tackling this problem in the first code is to calculate all the loop integrals in quadruple precision. For this study we have performed this everywhere in phase space and not just for the points that satisfy Eq. (6). The numerical integration becomes very stable even in the case of very small Gram determinant. The price to pay is that the computation speed is about 6 times slower than using the segmentation method. We have obtained agreement of cross sections within integration errors between the segmentation method and using quadruple precision.

We have also observed that the scalar one-loop four-point integral can show numerical problems and the library LoopTools [29, 30, 31] alone is not good enough for our calculations. While reverting to quadruple precision (everywhere in LoopTools) remedies the problem, in double precision we call other loop libraries to calculate scalar one-loop four-point integrals for some special cases where LoopTools fails. OneLoop [32] is used for some special cases with zero internal masses. Other specific cases that we have identified are treated with D0C [33], a code to calculate scalar one-loop four-point integrals with complex/real masses. Generically, numerical instabilities in scalar one-loop integrals can originate from the following two sources. One is related to an endpoint singularity manifested as a pole very close to the boundary of the integration interval. The other is called a pinch singularity where there are two poles sitting very close to each other with a very small imaginary part. These are both called Landau singularities in Feynman loop integrals (e.g. see [34, 9] for a detailed discussion). At NLO, these singularities (up to four-point) are integrable but they may cause numerical instability. For the case of the pinch singularity the sign of the imaginary part of a pole (in the complex plane) can be wrongly calculated and hence a regular case where both poles sitting on the same side of the real axis can be numerically misidentified as a pinch singularity or vice versa. We have observed that in our calculations, in particular the process \( e^+e^- \rightarrow W^+W^-Z \), numerical problems in the scalar four-point integrals are related to both sources. An efficient way to cross check the results of scalar integrals is therefore to introduce a tiny positive width for internal masses (\( \Gamma_i = 10^{-5}m_i \)). In this case, the masses become complex and the code D0C must be used. This is indeed what we did in our calculations to obtain the preliminary results [12] which agree within integration errors with our final results. In
$e^+e^- \rightarrow ZZZ$ an example where D0C is used is for $\sqrt{s} = 300$ GeV. Here the problem is related to the $tt$ threshold ($\sqrt{s} < 2m_t$ in this case) in the box diagram with four top quarks in the loop. Since all the important discriminants in D0C depend only on external momenta, the problem does not occur in D0C.

**Code 2**

The second code also uses **FeynArts-3.4** for Feynman-diagram and amplitude generation and **FormCalc-6.0** to evaluate the amplitudes. The analytical output of FormCalc-6.0 in terms of Weyl-spinor chains and coefficients containing the tensor one-loop integrals is then translated to C++ code after performing further optimizations of the expressions.

The evaluation of the one-loop tensor integrals is done by reducing them to a set of scalar integrals. The 5-point integrals are written in terms of 4-point functions following [35], which avoids leading inverse Gram determinants and the associated numerical instabilities. The remaining 3- and 4-point tensor integrals are recursively reduced to scalar integrals with the Passarino–Veltman algorithm [27]. For exceptional phase-space points this reduction scheme becomes numerically unstable. In this case we reevaluate both the scalar integrals and the reduction itself in higher precision using the QD library [36]. To determine when to switch to quadruple precision we use the condition number of the Gram matrix. This is a good estimator of the number of digits lost in the solution of the linear equation system appearing in Passarino–Veltman reduction. While this simple estimator is sufficient for triangle integrals in the case of 4- and 5-point integrals the numerical instabilities can also originate from small Gram determinants in the lower N-point integrals. We therefore use not only the condition number of the N-point Gram matrix but also the condition numbers of the (N-1)- and (N-2)-point integrals appearing in the tensor reduction in these cases.

Finally, the scalar integrals are calculated using the results of [38, 39] for the finite and [40, 41] for the IR singular integrals. The scalar integrals and the tensor reduction have been implemented as a C++ library allowing the calculation both in double and higher precision. Internally the library uses dimensional regularization for both UV and IR divergences. For this calculation the IR singularities are then translated to photon mass regularization [42].

### 2.3 Real corrections

In addition to the virtual corrections we also have to consider real photon emission, i.e. the processes $e^+e^- \rightarrow W^+W^-Z\gamma$ and $e^+e^- \rightarrow ZZZ\gamma$. The corresponding amplitudes are divergent in the soft and collinear limits. The soft singularities cancel against the ones in the virtual corrections while the collinear singularities are regularized by the physical electron mass. To extract the singularities from the real corrections and combine them with the virtual contribution we apply both the dipole subtraction scheme and a phase-space slicing method.

The dipole subtraction formalism is a process independent approach and was originally introduced for massless QCD [43]. We use a generalization of this method to include photon radiation from massive fermions [44]. Since photon radiation from a massive

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3The condition number of a symmetric matrix is defined as the ratio of the largest and smallest eigenvalues, see e.g. [37].
fermion and from a massive charged gauge boson has the same singular structure in the soft limit this formalism can directly be applied to our calculation.

In the dipole subtraction method a specially constructed function is subtracted from the real amplitude and then added back

$$\sigma_{\text{real}} = \int_4 (d\sigma_{\text{real}} - d\sigma_{\text{sub}}) + \int_4 d\sigma_{\text{sub}}. \tag{7}$$

The subscript 4 refers to the 4-body final state including photon radiation. The subtraction function has the same singular structure as the amplitude pointwise in phase space. The difference of the real amplitude and the subtraction function is therefore regular and the integration of the first term in Eq. (7) can be performed numerically. Introducing a photon mass as a regulator the subtraction function can be integrated analytically over the photon emission phase space up to a convolution

$$\int_4 d\sigma_{\text{sub}} = -\frac{\alpha}{2\pi} \int d^4x \sum_{i \neq j} Q_i Q_j G_{ij}(x) \int_3 d\sigma_{\text{Born}} + \sigma_{\text{endpoint}}, \tag{8}$$

where the charges $Q_i$ are counted as outgoing and the subscript 3 refers to the 3-body phase space without photon radiation. While the first term in Eq. (8) has only mass singularities, the endpoint contribution contains both soft and collinear singularities and is given by

$$\sigma_{\text{endpoint}} = -\frac{\alpha}{2\pi} \int_3 d\sigma_{\text{Born}} \sum_{i \neq j} Q_i Q_j G_{ij}. \tag{9}$$

The summations in Eq. (8) and Eq. (9) run over all initial and final state charged particles. The explicit expressions for the $G_{ij}$ and the $G_{ij}(x)$ can be found in [44]. The soft and collinear singularities cancel in the sum of the endpoint and virtual contributions. The mass singularities in the final result then originate only from the first term in Eq. (8).

The idea of phase-space slicing is to split the phase space of the photon emission contribution into a soft, a collinear and a remaining finite part. In the soft and collinear regions the real amplitude approximately factorizes into universal soft and collinear functions and the Born amplitude. In addition the phase space splits into the leading order phase space and a soft or collinear part. The phase-space integration over the photon degrees of freedom can then be performed analytically resulting in infrared and mass singular contributions. In the remaining part of phase space the amplitude is regular and the integration can be performed using numerical integration.

We have implemented a two-cutoff phase-space slicing method closely following [45, 46]. The differential cross section for the real contribution is decomposed as follows

$$d\sigma_{\text{real}} = d\sigma_{\text{soft}}(\delta_s) + d\sigma_{\text{hard}}(\delta_s),$$

$$d\sigma_{\text{hard}}(\delta_s) = d\sigma_{\text{coll}}(\delta_s, \delta_c) + d\sigma_{\text{fin}}(\delta_s, \delta_c) \tag{10}$$

using the soft and collinear cutoffs $\delta_s$ and $\delta_c$. The soft region is defined by $E_\gamma < \delta_s \sqrt{s}/2 = \Delta E$. The collinear but non-soft region is given by $\{E_\gamma \geq \Delta E, 1 - \cos \theta_{\gamma f} < \delta_c\}$ where $\theta_{\gamma f}$ is the polar angle of the photon with respect to the $e^\pm$ direction in the c.m. frame.
Figure 2: Dependence of $\sigma_{e^+e^-\rightarrow W^+W^-Z\gamma}$ on the soft cutoff $\delta_s$ in phase-space slicing with fixed $\delta_c = 7 \cdot 10^{-4}$. Only the non-singular part is shown, i.e. the IR singular $\ln(m^2_{\gamma})$ terms are set to zero. The result using dipole subtraction is shown for comparison with the error given by the width of the band.

Since the approximations used in the soft and collinear regions introduce errors of $O(\delta_s, \delta_c)$ the cutoffs have to be chosen sufficiently small. As can be seen from Fig. 2 the approximation errors are below the integration errors for $\delta_s \leq 10^{-3}$. For smaller values of the soft cutoff the errors grow larger due to cancellations between the soft and the hard contributions which both diverge as $\log \delta_s$. We have similarly verified the stability with respect to the variation of the collinear cutoff $\delta_c$. Fig. 2 also shows agreement between the slicing and dipole subtraction results, although the errors are typically a factor 10 smaller when using the subtraction formalism.

2.4 Defining the weak corrections

It is well known that the collinear QED correction related to initial state radiation (ISR) in $e^+e^-$ processes is large. The effect due to ISR can be treated along a structure function approach which resums the effects of higher orders, see for example [47]. Likewise for the linear collider the effect of beamstrahlung [48] needs to be convoluted over the genuine weak corrections. Therefore in a NLO computation such as ours it is best to subtract the ISR corrections in order to sum their effect to all orders or, put another way, once the weak correction has been defined to convolute its result within a structure function approach. Deconvolution is also possible from the experimental data to arrive at the non_ISR result as is done at LEP, for example [47]. The weak corrections encapsulate more interesting features. In order to see the effect of the weak corrections, one should separate the large QED corrections from the full NLO result. It means that we can define the weak correction as an infrared and collinear finite quantity. In our work we will subtract all of the QED corrections, not only the initial but, for $WWZ$, also the final and the interference QED corrections. The definition we adopt in this paper is based on the dipole subtraction...
formalism. In this approach, the sum of the virtual and endpoint contributions satisfies the above conditions and can be chosen as a definition for the weak correction
\[ \sigma_{\text{weak}} = \sigma_{\text{virt}} + \sigma_{\text{endpoint}}. \] (11)

For the numerical results shown in Section 3 and Section 4, we will make use of this definition.

If one uses the phase-space slicing approach, a definition for the weak correction can be found as well. In the neutral process \( e^+e^- \rightarrow ZZZ \) the QED corrections are confined to the initial state. The universal leading QED part of \( \sigma_{\text{virt}} + \sigma_{\text{soft}} \) can be extracted and is given by
\[ \sigma_{\text{QED}}^{V+S} = \frac{2\alpha}{\pi} \left( (L_e - 1) \ln \delta_s + \frac{3}{4} L_e - 1 \right) \sigma_{\text{Born}}, \quad L_e = \ln(s/m_e^2). \] (12)

Subtracting this from the sum of virtual and soft contributions we can define the weak corrections in phase-space slicing as
\[ \sigma_{\text{ZZZ weak, slicing}} = \sigma_{\text{virt}} + \sigma_{\text{soft}} - \sigma_{\text{QED}}^{V+S}. \] (13)

This procedure will lead to the same result for the weak correction as obtained by simply taking the sum of the virtual and endpoint parts.

For \( e^+e^- \rightarrow W^+W^-Z \) there is also final state radiation and its interference with the initial state radiation. Diagrammatically there is no unambiguous way to subtract this contribution in a gauge invariant way. Nonetheless after subtracting the initial state radiation in Eq. (12), the logarithms of infrared origin can be easily isolated and the weak correction defined by
\[ \sigma_{\text{virt}} + \sigma_{\text{soft}} - \sigma_{\text{QED}}^{V+S} = \sigma_{\text{WWZ weak, slicing}} + b \ln \delta_s. \] (14)

The coefficient \( b \) of \( \ln \delta_s \) can be extracted by choosing two different values of \( \delta_s \) (which should be sufficiently small). We have compared the weak correction obtained in this way with the one calculated by taking the sum of the virtual and endpoint parts. The results for the case of \( \sqrt{s} = 500 \text{ GeV} \) and \( M_H = 120 \text{ GeV} \) are: \( \delta_{\text{dipole}}^{\text{weak}} = -7.014(5)\% \), \( \delta_{\text{weak}}^{\text{slicing}} = -6.73(1)\% \). This is for the process \( e^+e^- \rightarrow W^+W^-Z \) and other input parameters as specified in Section 2.5.

### 2.5 Input parameters

We use the following set of input parameters [49, 50],
\[ G_\mu = 1.16637 \times 10^{-5} \text{ GeV}^{-2}, \quad \alpha(0) = 1/137.035999679, \]
\[ M_W = 80.398 \text{ GeV}, \quad M_Z = 91.1876 \text{ GeV}, \]
\[ m_e = 0.510998910 \text{ MeV}, \quad m_\mu = 105.658367 \text{ MeV}, \quad m_\tau = 1776.84 \text{ MeV}, \]
\[ m_u = 66 \text{ MeV}, \quad m_c = 1.2 \text{ GeV}, \quad m_t = 173.1 \text{ GeV}, \]
\[ m_d = 66 \text{ MeV}, \quad m_s = 150 \text{ MeV}, \quad m_b = 4.3 \text{ GeV}. \] (15)

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4 A similar discussion was given in [46].

5 This definition differs by the sub-sub-leading term \( 2\alpha/\pi \times \pi^2/6 \) to what is usually taken for the universal initial state radiation. Adding this term would give the same result as the one based on a diagrammatic approach as done in [10] for \( ZZZ \).
The masses of the light quarks\textsuperscript{6}, \textit{i.e.} all but the top mass, are effective parameters adjusted to reproduce the hadronic contribution to the photonic vacuum polarization of\textsuperscript{52} with $\alpha^{-1}(M_Z^2) = 128.907$. As discussed in Subsection 2.1 we use a variant of the $G_{\mu}$ scheme with $\alpha_{G_{\mu}}$ at leading order leading to NLO corrections that are of $O(\alpha_{3G_{\mu}}\alpha(0))$. Using $\alpha_{G_{\mu}}$ as coupling we calculate $\Delta r = 3.0792 \times 10^{-2}$ for $M_H = 120$ GeV and $\Delta r = 3.1577 \times 10^{-2}$ for $M_H = 150$ GeV. The Cabibbo-Kobayashi-Maskawa matrix is set to be diagonal. For the calculation we neglect the electron Yukawa coupling proportional to the electron mass, as mentioned at the beginning of Section 2. For both processes we apply no cuts at the level of the $W^{\pm}$ and $Z$, since these will decay.

3 $e^+e^- \rightarrow ZZZ$

As shown in Fig. 3 the tree-level cross section rises sharply once the threshold for production opens, reaches a peak of about 1.1 fb around a centre-of-mass energy of 600 GeV before very slowly decreasing with a value of about 0.9 fb at 1 TeV. Exact results are displayed in Table 1. The Higgsstrahlung contribution to the total cross section is about 10% at $\sqrt{s} = 600$ GeV and $\sqrt{s} = 1$ TeV.

The full NLO corrections are quite large and negative around threshold, $-35\%$, decreasing sharply to stabilise at a plateau around $\sqrt{s} = 600$ GeV with $-16\%$ correction. The sharp rise and negative corrections at low energies are easily understood. They are essentially due to initial state radiation (ISR) and the behaviour of the tree-level cross section. The photon radiation reduces the effective centre-of-mass energy and therefore explains what is observed in the figure. On the other hand the genuine weak corrections, in the $G_{\mu}$ scheme, are relatively small at threshold, $-7\%$. The magnitude of the corrections however increases steadily reaching a value as large as $-18\%$ at $\sqrt{s} = 1$ TeV. These large\textsuperscript{6}which are the same as those used in [51].

\textsuperscript{6}which are the same as those used in [51].
Table 1: Cross section for $e^+e^- \rightarrow ZZZ$ at tree-level, including only the weak corrections and at full next-to-leading order for $M_H = 120$ GeV. Also shown are the relative weak and full NLO corrections.

| $\sqrt{s}$ [TeV] | $\sigma_{\text{Born}}$ [fb] | $\sigma_{\text{weak}}$ [fb] | $\sigma_{\text{full}}$ [fb] | $\delta_{\text{weak}}$ [%] | $\delta_{\text{full}}$ [%] |
|-------------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.3               | 0.184524(4)    | 0.17132(1)      | 0.11916(2)      | -7.156(7)       | -35.420(8)      |
| 0.4               | 0.92437(3)     | 0.83637(9)      | 0.73502(9)      | -9.520(9)       | -20.484(9)      |
| 0.5               | 1.12353(1)     | 0.99338(9)      | 0.92820(9)      | -11.584(7)      | -17.386(8)      |
| 0.6               | 1.14203(6)     | 0.9900(2)       | 0.9546(2)       | -13.31(1)       | -16.41(1)       |
| 0.8               | 1.04796(8)     | 0.8801(2)       | 0.8786(2)       | -16.02(2)       | -16.16(2)       |
| 1.0               | 0.92962(10)    | 0.7609(2)       | 0.7759(2)       | -18.15(2)       | -16.54(2)       |

Negative corrections are typical of the electroweak Sudakov logarithms $-\log^2(s/M_W^2)$. In the usual $\alpha(0)$ on-shell scheme this important effect would be blurred and weakened unless one reaches much higher energies.

We have also studied distributions in some key kinematic variables and how they are affected by radiative corrections. Fig. 4 shows the invariant mass $M_{ZZ}$, the transverse momentum $p_T^Z$ and the rapidity $y^Z$ for $\sqrt{s} = 500$ GeV and $M_H = 120$ GeV. The important message is that the genuine weak corrections are almost just an overall rescaling of the leading-order distributions, in particular in the bulk of the region of phase space where the yield is largest. The full NLO corrections on the other hand show more structure with very large corrections at the edges of phase space where the cross section is smallest, for example the full NLO correction for $M_{ZZ} > 350$ GeV drops below $-50\%$. This again is essentially due to hard photon radiation. This shows that the effects of New Physics could be discovered in a less ambiguous way after subtracting the QED corrections.

4 $e^+e^- \rightarrow W^+W^-Z$

Compared to $e^+e^- \rightarrow ZZZ$, the cross section for $e^+e^- \rightarrow W^+W^-Z$ is almost 2 orders of magnitudes larger for the same centre-of-mass energy. For example at 500 GeV it is about 40 fb at tree level, compared to 1 fb for the $e^+e^- \rightarrow ZZZ$ cross section. For an anticipated luminosity of 1 ab$^{-1}$, this means that the cross section should be known at the permille level. In absolute terms the Higgsstrahlung contribution is a factor 2 (due to SU(2)) larger than in $e^+e^- \rightarrow ZZZ$, however its contribution to the total $e^+e^- \rightarrow W^+W^-Z$ cross section is much less than 1% for $M_H = 120$ GeV.

The behaviour of the $e^+e^- \rightarrow W^+W^-Z$ cross section as a function of energy resembles that of $e^+e^- \rightarrow ZZZ$. It rises sharply once the threshold for production opens, reaches a peak before very slowly decreasing as shown in Fig. 5 (exact results are also displayed in Table 2). However as already discussed the value of the peak is much larger, $\sim 50$ fb at NLO, moreover the peak is reached around $\sqrt{s} = 1$ TeV, much higher than in $ZZZ$. This explains the bulk of the NLO corrections at lower energies which are dominated by the QED correction, large and negative around threshold and smaller at higher energies. As the energy increases the weak corrections get larger reaching about $-18\%$ at $\sqrt{s} = 1$ TeV.
Figure 4: From top to bottom: distributions for the ZZ invariant mass, the rapidity of the Z and the transverse momentum of the Z for $e^+e^- \rightarrow ZZZ$ at $\sqrt{s} = 500$ GeV and $M_H = 120$ GeV. The panels on the left show the tree-level, the full NLO and the weak correction. The panels on the right show the corresponding relative (to the tree-level) percentage corrections. The distributions are obtained by entering for each event the corresponding observable, say $p_T^Z$, of each Z and then normalising by a factor $1/3$. 
Figure 5: Left: the total cross section for \(e^+e^- \rightarrow W^+W^-Z\) as a function of \(\sqrt{s}\) for the Born, full \(\mathcal{O}(\alpha)\) and genuine weak correction for \(M_H = 120\) GeV. Right: the corresponding relative percentage corrections \(\sigma_{NLO}/\sigma_{LO} - 1\).

Table 2: Cross section for \(e^+e^- \rightarrow W^+W^-Z\) at tree-level, including only the weak corrections and at full next-to-leading order for \(M_H = 120\) GeV. Also shown are the relative weak and full NLO corrections.

| \(\sqrt{s} [\text{TeV}]\) | \(\sigma_{\text{Born}} [\text{fb}]\) | \(\sigma_{\text{weak}} [\text{fb}]\) | \(\sigma_{\text{full}} [\text{fb}]\) | \(\delta_{\text{weak}} [%]\) | \(\delta_{\text{full}} [%]\) |
|---|---|---|---|---|---|
| 0.3 | 3.27055(4) | 3.1888(3) | 2.3880(3) | -2.500(8) | -26.986(9) |
| 0.5 | 39.7557(9) | 36.967(2) | 33.476(2) | -7.014(5) | -15.795(5) |
| 0.7 | 55.358(3) | 49.878(6) | 47.409(6) | -9.899(10) | -14.359(10) |
| 0.9 | 59.121(4) | 51.881(8) | 50.678(8) | -12.25(1) | -14.28(1) |
| 1.0 | 59.061(4) | 51.206(9) | 50.541(9) | -13.30(1) | -14.43(1) |
| 1.2 | 57.202(5) | 48.49(1) | 48.69(1) | -15.24(2) | -14.88(2) |
| 1.5 | 52.740(5) | 43.34(1) | 44.43(1) | -17.82(2) | -15.76(2) |

1.5 TeV. Again a large part of this correction seems to be of the Sudakov type.

More interesting than in the case of \(e^+e^- \rightarrow ZZZ\) are the distributions in some key variables like the invariant WW mass, the \(p_T^Z\) and the rapidity of the WW system. First, due to photon radiation, in the full NLO corrections some large corrections do show up at the edges of phase space, see Fig. 6. However when the QED corrections are subtracted, the weak corrections cannot be parameterised by an overall scale factor, for all the distributions that we have studied. Other distributions not shown here can be found in [53].
Figure 6: From top to bottom: distributions for the WW invariant mass, the rapidity of the WW system and the transverse momentum of the Z for $e^+e^- \rightarrow W^+W^-Z$ at $\sqrt{s} = 500$ GeV and $M_H = 120$ GeV. The panels on the left show the tree-level, the full NLO and the weak correction. The panels on the right show the corresponding relative (to the tree-level) percentage corrections.
5 Comparison with other calculations

The electroweak corrections to $e^+e^- \rightarrow ZZZ$ have also been calculated in [10]. They use the $\alpha(0)$ scheme and slightly different input parameters. We have performed a tuned comparison by adapting to the input parameters of [10] and switching to the $\alpha(0)$ scheme.

We find full agreement within the quoted statistical errors for $\sigma_{LO}$ and $\Delta \sigma_{tot}$ shown in Table 2 of [10]. A comparison of the results in Fig. 4 of [10] is shown in Table 3. The results for the Born cross section agree to within 0.01% while the NLO results agree to at least 0.1%.

Table 3: Born cross section and relative corrections for $e^+e^- \rightarrow ZZZ$ using the input parameter scheme of [10].

| $\sqrt{s}$ [GeV] | $M_H = 120$ GeV | $M_H = 150$ GeV |
|------------------|-----------------|-----------------|
|                  | $\sigma_{Born}$ [fb] | $\delta_{full}$ [%] | $\sigma_{Born}$ [fb] | $\delta_{full}$ [%] |
| 350              | Ref. [10] 0.58696 | -15.79          | 0.68422 | -13.91 |
|                  | This work 0.586955(2) | -15.850(1) | 0.684209(2) | -13.970(1) |
| 370              | Ref. [10] 0.70531 | -13.79          | 0.80821 | -12.00 |
|                  | This work 0.705303(2) | -13.822(1) | 0.808196(3) | -11.986(1) |
| 400              | Ref. [10] 0.83409 | -11.75          | 0.9375 | -9.98 |
|                  | This work 0.834083(4) | -11.765(2) | 0.937484(4) | -9.973(1) |
| 450              | Ref. [10] 0.95792 | -9.79           | 1.05294 | -8.06 |
|                  | This work 0.957904(5) | -9.763(3) | 1.052917(5) | -8.044(2) |
| 500              | Ref. [10] 1.01384 | -8.70           | 1.09754 | -7.09 |
|                  | This work 1.013806(6) | -8.682(4) | 1.097440(7) | -7.064(4) |
| 600              | Ref. [10] 1.03052 | -7.77           | 1.09370 | -6.36 |
|                  | This work 1.030489(9) | -7.714(6) | 1.093668(9) | -6.289(6) |
| 700              | Ref. [10] 0.99611 | -7.47           | 1.04437 | -6.20 |
|                  | This work 0.99607(1) | -7.438(9) | 1.04437(1) | -6.164(9) |
| 800              | Ref. [10] 0.94567 | -7.50           | 0.98647 | -6.61 |
|                  | This work 0.94563(1) | -7.46(1) | 0.98343(1) | -6.30(1) |
| 900              | Ref. [10] 0.89168 | -7.71           | 0.92196 | -6.65 |
|                  | This work 0.89164(1) | -7.62(1) | 0.92191(1) | -6.55(1) |
| 1000             | Ref. [10] 0.83892 | -7.94           | 0.86366 | -6.89 |
|                  | This work 0.83887(2) | -7.86(2) | 0.86362(2) | -6.86(2) |

We have also made a comparison for the process $e^+e^- \rightarrow W^+W^-Z$ with the results of [11]. In addition to different input parameters, [11] uses an unusual scheme/input parameter for the electromagnetic coupling constant. One can read from [11] that their renormalisation condition for the electric charge is the on-shell definition at $q^2 = 0$, see Eq. (1), yet the value of $\alpha_{MS}(M_Z^2)$ is used as a value for the coupling already at tree level and no shift of the electric charge counterterm at NLO is made in order to avoid double counting. This is just like taking a numerical value of $\alpha_{MS}(M_Z^2)$ for $\alpha(0)$ in the usual on-shell scheme. For the sake of comparison we have used the same scheme in our

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The data for Fig. 4 of [10] have been kindly provided by Prof. Ma.
calculation and show the comparison with Table 2 of [11] in Table 4. The agreement for the Born result is within integration errors as are the NLO corrections for \( M_H = 120 \text{ GeV} \). However for \( M_H = 150 \text{ GeV} \) we find agreement only for \( \sqrt{s} = 0.5 \text{ TeV} \). Close to threshold and especially at high energy the results differ by up to 5 times the integration errors.

Table 4: Born cross section and NLO correction for \( e^+ e^- \rightarrow W^+W^-Z \) using the input parameter scheme of [11].

| \( \sqrt{s} \) [TeV] | \( M_H = 120 \text{ GeV} \) | \( \Delta \sigma_{NLO} \) [fb] | \( M_H = 150 \text{ GeV} \) | \( \Delta \sigma_{NLO} \) [fb] |
|-----------------|----------------|----------------|----------------|----------------|
| 0.3             | Ref. [11]     | 3.6216(2)     | -0.683(2)     | 3.8856(2)     | -0.694(2)     |
|                 | This work     | 3.62165(5)    | -0.69013(3)   | 3.88558(5)    | -0.7010(3)    |
| 0.5             | Ref. [11]     | 44.026(5)     | -3.03(6)      | 44.303(5)     | -2.89(6)      |
|                 | This work     | 44.0235(10)   | -3.107(3)     | 44.301(1)     | -2.949(3)     |
| 0.8             | Ref. [11]     | 64.35(1)      | -3.48(7)      | 64.50(1)      | -3.57(9)      |
|                 | This work     | 64.345(4)     | -3.466(8)     | 64.488(4)     | -3.250(8)     |
| 1.0             | Ref. [11]     | 65.42(1)      | -3.74(9)      | 65.51(1)      | -3.90(9)      |
|                 | This work     | 65.401(5)     | -3.650(9)     | 65.499(5)     | -3.440(10)    |

Note Added
After our paper was made public, [11] was updated. The on-shell \( \alpha(0) \) scheme is now implemented properly. Moreover they have improved the numerical stability of the phase-space integration at high energies and the results for \( WWZ \) at \( \sqrt{s} = 800 \text{ GeV} \) and 1 \text{ TeV} \) have substantially changed. We now find agreement for these energies while the small discrepancy near threshold (\( \sqrt{s} = 300 \text{ GeV} \)) remains.

6 Conclusions
We have performed a calculation of the full next-to-leading order correction to the processes \( e^+ e^- \rightarrow W^+W^-Z \) and \( e^+ e^- \rightarrow ZZZ \) in the energy range of the international linear collider and for Higgs masses below the \( WW \) threshold. These processes would be the successor of \( e^+ e^- \rightarrow W^+W^- \) in that they would measure the quartic couplings \( WWZZ \) and \( ZZZZ \) which could retain residual effects of the physics of electroweak symmetry breaking. With this in mind we have subtracted the QED corrections and studied the genuine weak corrections in the \( G_\mu \) scheme. We find that the weak corrections can be large. For example, for a centre-of-mass energy of 1 \text{ TeV} these corrections reach \(-13\%\) for \( WWZ \) and \(-18\%\) for \( ZZZ \) and grow larger for higher energies. At lower energies around the production threshold the cross sections are small and the weak corrections are modest. However, in this energy range the QED corrections are largest due to the rapid rise of the cross section. We have also studied the effects of the genuine weak radiative corrections on various distributions. While for the \( ZZZ \) channel the effect might be described by an overall rescaling over most of the range of the kinematic variable under consideration, the corrections in the \( WWZ \) channel show more structure, pointing once again to the need for radiative corrections when looking for New Physics effects.
The calculations involved in our studies are also a contribution to the field of one-loop corrections for multi-leg processes and the techniques that one requires to control and further develop to make these calculations as automatic as possible. Since these processes involve three-vector boson production they are much more complex than say single Higgs production not only from just a counting of the one-loop diagrams involved but also the fact that the loop integrals are more challenging, in particular for \( e^+e^- \rightarrow W^+W^-Z \). We have shown that using higher precision arithmetic or a combination of different loop integral libraries can be very efficient to overcome numerical instabilities. These techniques are instrumental for a very precise computation, especially for the WWZ channel where the foreseen luminosity calls for a better than per-mil accuracy for the theoretical prediction. We have shown that our results agree well with those of a previous calculation of \( e^+e^- \rightarrow ZZZ \) in \([10]\) but not as well, especially for some Higgs masses and centre-of-mass energies, for the WWZ channel in \([11]\).

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