Laplace transformation of vector-valued distributions and applications to Cauchy-Dirichlet problems

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Abstract

We present two new proofs of the exchange theorem for the Laplace transformation of vector-valued distributions. We then derive an explicit solution to the Dirichlet problem of the polyharmonic operator in a half-space. Finally, we obtain explicit solutions to Cauchy-Dirichlet problems of iterated wave- and Klein-Gordon-operators in half-spaces.

Keywords: Laplace transformation, vector-valued distributions, poly- and metaharmonic operator, Dirichlet problem in half spaces, Poisson kernels, iterated wave- and Klein-Gordon operators, mixed (initial value and Dirichlet) problems in half-spaces, explicit solution formulae

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1 Introduction

The exchange theorem for the Laplace transformation $\mathcal{L}$ states that

$$\mathcal{L}(S * T) = \mathcal{L}S \cdot \mathcal{L}T \quad (S \in \mathcal{S}'(\Gamma), T \in \mathcal{O}_c(\Gamma))$$

with notation as explained in Section 2 below (cf. [29, Prop. 7, p. 308]). A first task of this study is to present two new proofs for the exchange theorem for vector-valued distributions $S$ and $T$. The original proof was given by L. Schwartz in his theory of vector-valued distributions. Our first proof follows an idea indicated at the beginning of L. Schwartz’ proof, namely to apply a proposition on the convolution of two vector-valued distributions $S \in \mathcal{H}(E)$, $T \in \mathcal{H}(F)$, in which both the space $\mathcal{H}$ and its strong dual $\mathcal{H}'$ are assumed to be nuclear. Our second proof relies on a theorem of R. Shiraishi on the convolution of vector-valued distributions that supposes only $\mathcal{H}$ (and not necessarily $\mathcal{H}'$) to be nuclear.

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Our second goal is to use the Laplace transform of vector-valued distributions to deduce explicit solution formulae for the Cauchy-Dirichlet problem of the operators
\[(\Delta_n + \partial_y^2 - \partial_t^2)^m\] (iterated wave operator)
\[(\Delta_{2n+1} + \partial_y^2 - \partial_t^2 - \xi^2)^m\] (iterated Klein-Gordon operator)
in the half-space \(y > 0\) (Propositions 14 and 15) by starting with the solutions of the Dirichlet problem of the elliptic operator
\[(\Delta_n + \partial_y^2 - p^2)^m\] (iterated metaharmonic operator)
(Propositions 9 and 11). We assume the Cauchy data \(u|_{t=0}, \partial_t u|_{t=0}, \ldots, \partial_t^{2m-1} u|_{t=0}\) to vanish. In the terminology of R. Courant and D. Hilbert such problems are called “transient response problems” [7, p. 224]. Compare also [14, p. 85]. The expression “metaharmonic” is borrowed from [11] and from [34].

In Proposition 7 we recall the distributionally formulated solution to the Dirichlet problem of the iterated Laplace (i.e., polyharmonic) operator \((\Delta_n + \partial_y^2)^m\) in the half-space \(y > 0\), which was presented for the first time in [8].

We note that our method could also be used, e.g., to derive explicit formulae for the solution to the Cauchy-Dirichlet problem of the operator
\[(\Delta_n + \partial_y^2 - \partial_t)^m\] (iterated heat operator)
in the half-space \(y > 0\).

For \(m = 1\), the solution of the Cauchy-Dirichlet problem can be found by an odd extension of the sought solution in the half-space, application of the distributional differentiation formula and convolution with the fundamental solution (in classical terms, by application of a representation theorem by means of Green’s function). If \(m > 2\) the solution by extension is not known to us.

Our notation is standard, mostly following [27, 28, 29].

2 New proofs of L. Schwartz’ exchange theorem for the Laplace transform of the convolution of vector-valued distributions.

Let us first recall L. Schwartz’ version [28, Prop. 43, p. 186]:

**Theorem 1.** Let \(\Gamma\) be a non-void open convex subset of \(\mathbb{R}^n\). Let \(E\) and \(F\) be separated locally convex topological vector spaces. Then there is a hypocontinuous (with respect to bounded sets) convolution mapping
\[\otimes: \mathscr{S}(\Gamma)(E) \times \mathscr{S}(\Gamma)(F) \to \mathscr{S}(\Gamma)(E \otimes_\alpha F).\]

For two Laplace-transformable distributions \(S \in \mathscr{S}(\Gamma)(E), T \in \mathscr{S}(\Gamma)(F)\) with values in \(E\) and \(F\), respectively, and their Laplace images \(\mathcal{L}S, \mathcal{L}T\) we have
\[\mathcal{L}(S \otimes T) = (\mathcal{L}S) \otimes (\mathcal{L}T).\]
We will now explain the notions appearing in this theorem. First, the mappings \( \ast \) and \( \circ \) in Theorem 1 are defined as in [28, Proposition 3, p. 37] and would be denoted by \( \ast_\pi \) and \( \circ_\pi \) there, respectively. Also, \( E \circ_\pi F \) denotes the quasi-completion of \( E \otimes_\pi F \).

**Definition 2** ([27, p. 58], [29, p. 303]). Let \( \Gamma \subseteq \mathbb{R}^n \) be non-void and convex. The space of Laplace transformable distributions \( \mathcal{S}'(\Gamma) \) is defined as

\[
\mathcal{S}'(\Gamma) = \bigcap_{\xi \in \Gamma} e^{\xi x} \mathcal{S}'_{x} = \{ S \in \mathcal{D}' | \forall \xi \in \Gamma : e^{-\xi x} S(x) \in \mathcal{S}'_{x} \},
\]

where \( \mathcal{S}' \) is the space of tempered distributions on \( \mathbb{R}^n \). \( \mathcal{S}'(\Gamma) \) is endowed with the projective topology with respect to the linear maps \( \mathcal{S}'(\Gamma) \rightarrow \mathcal{S}'_{x}, S(x) \mapsto e^{-\xi x} S(x) \) for \( \xi \in \Gamma \).

As usual, we denote by \( \mathcal{O}'_{C} \) the space of rapidly decreasing distributions on \( \mathbb{R}^n \). By defining \( \mathcal{O}'_{C}(\Gamma) = \bigcap_{\xi \in \Gamma} e^{\xi x} \mathcal{O}'_{x} \) analogously to \( \mathcal{S}'(\Gamma) \) we have \( \mathcal{O}'_{C}(\Gamma) = \mathcal{S}'(\Gamma) \) if \( \Gamma \neq \emptyset \) is convex and open [29, p. 303]. Also, for such \( \Gamma \) the space \( \mathcal{O}'_{C}(\Gamma) \) is a commutative algebra with respect to convolution, which in turn is continuous [29, Corollaire, p. 304].

Let us recall L. Schwartz’ definition of the Laplace transformation of scalar valued distributions and the Paley-Wiener-Schwartz theorem:

**Definition and Theorem 3** ([27, Prop. 22, p. 76], [29, Prop. 6, p. 306]). Let \( \emptyset \neq \Gamma \subseteq \mathbb{R} \) be open and convex and \( T^\Gamma := \Gamma + i\mathbb{R}^n \) the tube domain over \( \Gamma \). The Laplace transformation \( \mathcal{L} \) is the mapping

\[
\mathcal{L}: \mathcal{O}'_{C}(\Gamma) \rightarrow \mathcal{H}(T^\Gamma), \quad S \mapsto \mathcal{L}S(p) = \langle 1(x), e^{-px} S(x) \rangle, \quad p \in T^\Gamma.
\]

The vector-valued scalar product \( \langle \ , \ \rangle \) is defined on \( \mathcal{O}_{C} \times \mathcal{O}'_{C}(\mathcal{H}(T^\Gamma)) \) due to \( e^{-px} S(x) \in \mathcal{O}_{C,x}(\mathcal{H}(T^\Gamma_p)) \) [29, Cor., p. 302].

\( \mathcal{L} \) is an isomorphism if \( \mathcal{H}(T^\Gamma) \) is endowed with the projective topology (with respect to the compact subsets \( K \) of \( \Gamma \)) of the inductive limits

\[
\mathcal{H}(T^K) = \{ f: T^K \rightarrow \mathbb{C} \text{ holomorphic} | \exists m \in \mathbb{N}_0 : (1 + |p|^2)^{-m} f(p) \in L^\infty(T^K) \}.
\]

Concerning the proof of Theorem 1 L. Schwartz remarks that it could be realized by applying Proposition 3 in [28, p. 37] to the spaces \( \mathcal{S}'(\Gamma)(E) \) and \( \mathcal{S}'(\Gamma)(F) \). But this procedure would require the proof of the nuclearity of the space \( \mathcal{S}'(\Gamma) \) “which is easy” and of the nuclearity of its strong dual “which is not so easy”. Thus, he proceeds differently [28, p. 187]. However, we aim at performing the proof in such a manner as L. Schwartz remarked. As a byproduct we sharpen Theorem 1 slightly.

**Theorem 4.** Let \( \emptyset \neq \Gamma \subseteq \mathbb{R}^n \) be an open and convex set and \( E \) and \( F \) separated locally convex topological vector spaces. There exists a unique bilinear, continuous mapping

\[
\ast : \mathcal{O}'_{C}(\Gamma)(E) \times \mathcal{O}'_{C}(\Gamma)(F) \rightarrow \mathcal{O}'_{C}(\Gamma)(E \otimes_\pi F)
\]
which extends the mapping
\[ \otimes : (\mathcal{O}_c'(\Gamma) \otimes E) \times (\mathcal{O}_c'(\Gamma) \otimes F) \to \mathcal{O}_c'(\Gamma)(E \otimes \pi F) \]
\[ (S \otimes e, T \otimes f) \mapsto (S \ast T)e \otimes f, \]
wherein \( \ast : \mathcal{O}_c'(\Gamma) \times \mathcal{O}_c'(\Gamma) \to \mathcal{O}_c'(\Gamma) \) is the continuous convolution and \( \otimes : E \times F \to E \otimes \pi F \) the canonical bilinear and continuous mapping.

If \( \mathcal{L} : \mathcal{O}_c'(\Gamma)(E) \to \mathcal{H}(T^\Gamma)(E) \) is the Laplace transformation of \( E \)-valued distributions then we have
\[ \mathcal{L}(S \ast T) = \mathcal{L}S \otimes \mathcal{L}T \]
for \( S \in \mathcal{O}_c'(\Gamma)(E) \), \( T \in \mathcal{O}_c'(\Gamma)(F) \). Summarizing, we have the commutative diagram
\[ \begin{array}{ccc}
\mathcal{O}_c'(\Gamma)(E) \times \mathcal{O}_c'(\Gamma)(F) & \xrightarrow{\otimes} & \mathcal{O}_c'(\Gamma)(E \otimes \pi F) \\
\downarrow_{\mathcal{L} \times \mathcal{L}} & & \downarrow_{\mathcal{L}} \\
\mathcal{H}(T^\Gamma)(E) \times \mathcal{H}(T^\Gamma)(F) & \xrightarrow{\otimes} & \mathcal{H}(T^\Gamma)(E \otimes \pi F).
\end{array} \]

**Proof.** The claim about the map \( \otimes \) follows from Proposition 3 in [28, p. 37] because the space \( \mathcal{O}_c'(\Gamma) \) has the strict approximation property [27, Proposition 16, p. 59], is nuclear and its strong dual is nuclear (see Lemma 5 below), and because the convolution \( \mathcal{O}_c'(\Gamma) \times \mathcal{O}_c'(\Gamma) \to \mathcal{O}_c'(\Gamma) \) is continuous. Concerning \( \otimes \), we note that the multiplication \( \cdot : \mathcal{H}(T^\Gamma) \times \mathcal{H}(T^\Gamma) \to \mathcal{H}(T^\Gamma) \) is continuous, and since \( \mathcal{H}(T^\Gamma) \cong \mathcal{O}_c'(\Gamma) \), we may again apply Proposition 3 in [28, p. 37]. Finally, commutativity of the diagram follows from that of its scalar variant, which holds due to [29, Proposition 7, p. 308].

**Lemma 5.** Let \( \emptyset \neq \Gamma \subseteq \mathbb{R}^n \) be open and convex.

(i) The space \( \mathcal{O}_c'(\Gamma) \) is nuclear and complete.

(ii) The strong dual \( (\mathcal{O}_c'(\Gamma))^\prime \) of \( \mathcal{O}_c'(\Gamma) \) is nuclear.

**Proof.** (i) The nuclearity of \( \mathcal{O}_c'(\Gamma) \) is an immediate consequence of [13, Corollaire 2, p. 48] and the nuclearity of \( \mathcal{O}_c'(\Gamma) \) (13 Théorème 16, p. 131]).

By [25] Proposition 5.3, p. 52] the space \( \mathcal{O}_c'(\Gamma) \) is complete.

(ii) The projective limit \( \mathcal{O}_c'(\Gamma) = \mathcal{S}'(\Gamma) \) is countable due to
\[ \bigcap_{\xi \in \Gamma} e^{\xi x} \mathcal{S}'_x = \bigcap_{\xi \in \Gamma \cap \mathbb{Q}^n} e^{\xi x} \mathcal{S}'_x. \]
We only have to show the inclusion “\( \supseteq \)”, the rest being elementary. For this, given \( T \in \bigcap_{\xi \in \Gamma \cap \mathbb{Q}^n} e^{\xi x} \mathcal{S}'_x \) and \( \xi \in \Gamma \), choose \( \xi_1, \ldots, \xi_k \in \Gamma \cap \mathbb{Q}^n \) such that \( \xi \) is in the convex hull of \( \{\xi_1, \ldots, \xi_k\} \). By [29] p. 301],
\[ e^{-\xi x} = a(x, \xi) \sum_{j=1}^k e^{-\xi_j x}, \]
with \(\alpha(., \xi) \in \mathcal{B}\), so we have
\[
e^{-\xi x^T} = \alpha(x, \xi) \sum_{j=1}^{k} e^{-\xi_j x^T} \in \mathcal{S}'_x.
\]

It follows that \(\mathcal{S}'\) is given by the projective limit
\[
\mathcal{S}' = \lim_{\Gamma_f \subseteq \Gamma, \text{finite}} \mathcal{S}'(\Gamma_f).
\]

This limit is reduced because the inclusions \(\mathcal{D} \subseteq \mathcal{S}'(\Gamma) \subseteq \mathcal{S}'(\Gamma_f)\) imply that \(\mathcal{S}'\) also is dense. By [25, 4.4, p. 139] the dual \((\mathcal{S}'(\Gamma))')\) endowed with the Mackey topology \(\tau((\mathcal{S}'(\Gamma))', \mathcal{S}'(\Gamma))\) can be identified with the inductive limit of the spaces
\[
((\mathcal{S}'(\Gamma))', \tau((\mathcal{S}'(\Gamma))', \mathcal{S}'(\Gamma))).
\]

Because \(\mathcal{S}'\) is nuclear and complete it is semireflexive, hence the Mackey topology on its dual equals the strong topology [14 Prop. 4, p. 228 and Prop. 8, p. 218]. Hence, we have
\[
(\mathcal{S}'(\Gamma))' = \lim_{\Gamma_f \subseteq \Gamma, \text{finite}} (\mathcal{S}'(\Gamma_f))'_b
\]

and the nuclearity of \((\mathcal{S}'(\Gamma))'_b = (\mathcal{S}'(\Gamma))'_b\) follows by [13 Cor. 1, p. 48] from the nuclearity of \((\mathcal{S}'(\Gamma_f))'_b\). To see that the latter space is nuclear we note that \(\mathcal{S}'(\Gamma_f)\), as a finite projective limit of (DFS)-spaces, is itself a (DFS)-space because this class of spaces is stable under the formation of finite products and closed subspaces [20, Theorem A.5.13, p. 253]. Furthermore, \(\mathcal{S}'(\Gamma_f)\) is nuclear by [13 Cor. 1, p. 48], hence its strong dual is nuclear by [13 Théorème 7, p. 40].

Further properties of the space \(\mathcal{O}'(\Gamma)\) will be published in an upcoming paper.

**Remark 6.** A third proof of Theorem 4 can be given by means of [32 Theorem 2, p. 196], see also [41 Theorem 5, p. 18]. Compared to [29 Prop. 3, p. 37] it has the advantage that the nuclearity of \(\mathcal{O}'(\Gamma) = \mathcal{S}'(\Gamma)\) is sufficient (Lemma 5 (i)), while nuclearity of its strong dual \(\mathcal{O}'(\Gamma)_b\) need not be established. Instead, one has to show that \(\mathcal{O}'(\Gamma)\) is \(B\)-normal (which is straightforward) and that \(\mathcal{O}'(\Gamma) \otimes E\) is strictly dense in \(\mathcal{O}'(\Gamma)(E)\), which in turn is implied by the strict approximation property of \(\mathcal{S}'(\Gamma)\) ([29 Proposition 16, p. 59]). In fact, by using [4 Prop. 1, p. 19] we can even dispense with showing \(B\)-normality of \(\mathcal{O}'(\Gamma)\).

### 3 Poisson kernels for Dirichlet problems

Our next aim is to reformulate a known result on the Poisson kernels of the Dirichlet problems of polyharmonic operators in half-spaces and to apply the partial Fourier transformation in order to deduce the Poisson kernels of the Dirichlet problems of the iterated metaharmonic operators in half-spaces. Then the theory of vector-valued distributions is applied in order to continue the results analytically. This method goes back to H. G. Garnir [11].
We follow the terminology of [2] p. 635 and [31] p. 140: The Poisson kernel of the \( j \)-th Dirichlet problem for the operator

\[
(\Delta_n + \partial_y^2 - \xi^2)^m, \quad \Delta_n = \partial_1^2 + \cdots + \partial_n^2, \quad m, n \in \mathbb{N}, \quad \xi \in \mathbb{R}
\]

in the half-space \( \mathbb{H} = \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, \ y > 0\} \), \( j = 0, \ldots, m-1 \), is the distribution \( E_j \in \mathcal{D}'(\mathbb{H}) \) for which

\[
(\Delta_n + \partial_y^2 - \xi^2)^m E_j = 0 \quad E_j \in \mathcal{O}_M(\mathbb{H}) = \{ \varphi \in \mathcal{E}(\mathbb{H}) \mid \forall \alpha \in \mathbb{N}_0^n \exists k \in \mathbb{N}_0 : (1 + |x|^2)^{-k/2} \partial^\alpha \varphi \in C_0(\mathbb{H}) \} \quad (\text{for } j = 0, \ldots, m-1 \text{ in } \mathcal{D}'(\mathbb{R}^n)).
\]

Here, \( C_0(\mathbb{H}) = \{ \psi \in \mathcal{C}(\mathbb{H}) : \lim_{|x,y| \to \infty} \psi(x,y) = 0 \} \). The existence of the restrictions \( \partial_y^k E_j|_{y=0} = \delta(x)\delta_{jk}, \ k = 0, \ldots, m-1 \) will follow from the explicit form of \( E_j \) in Proposition 7 below (see also [16] Theorem 4.4.8, p. 115).

For a more general notion of Poisson kernel we refer to [33] Section 4.5, p. 137]

To begin with, we use [8, Satz 3] to derive the following result:

**Proposition 7.** The Poisson kernel of the \( j \)-th Dirichlet problem for the polyharmonic (i.e., iterated Laplace) operator \( (\Delta_n + \partial_y^2)^m \) is given by

\[
E_j = \frac{2}{\omega_{n+1}} \frac{y^m}{j!(m-1-j)!} (-\partial_y)^{m-j-1} \left( \frac{1}{(|x|^2 + y^2)^{\frac{n-1}{2}}} \right),
\]

where \( \omega_{n+1} = \frac{2\pi^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)} \) denotes the surface measure of the unit sphere in \( \mathbb{R}^{n+1} \).

**Proof.** Let \( \varphi \in \mathcal{D}(\mathbb{R}^n) \), \( \check{\varphi}(x) = \varphi(-x) \), and denote by \( \ast \) the convolution with respect to the \( x \)-variables. Then it follows from [8, Satz 3] that \( E_j \ast \check{\varphi} \in \mathcal{E}(\mathbb{H}) \) is the unique solution to

\[
(\Delta_n + \partial_y^2)^m (E_j \ast \check{\varphi}) = 0
\]

\[
\lim_{y \searrow 0} \partial_y^k (E_j \ast \check{\varphi})(x) = \check{\varphi}(x)\delta_{jk}, \quad k = 0, 1, \ldots, m-1.
\]

Consequently, \( (\Delta_n + \partial_y^2)^m E_j = 0 \) and \( \lim_{y \searrow 0} \partial_y^k (E_j, \varphi) = \varphi(0)\delta_{jk}, \ k = 0, 1, \ldots, m-1 \), i.e., \( \lim_{y \searrow 0} \partial_y^k E_j = \delta(x)\delta_{jk}, \ k = 0, 1, \ldots, m-1 \).

**Remark 8.** We point out the following particular cases of Proposition 7

(a) For \( m = 1, j = 0 \) we obtain the well-known Poisson kernel of the Dirichlet-problem for \( \Delta_n + \partial_y^2 \) in the half-space \( y > 0 \) to be

\[
\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2 + y^2)^{\frac{n-1}{2}}},
\]

cf., e.g., [31] (1.2), p. 163] or [9] p. 37, Th. 14].
(b) The choice $m = 2, j = 0, j = 1$ gives the Poisson kernels of the Dirichlet problem for the biharmonic operator $(\Delta_n + \partial_y^2)^2$ in the half-space $y > 0$:

$$E_0 = \frac{2\Gamma\left(\frac{n+4}{2}\right)}{\pi^{n-2}} \frac{y^3}{(|x|^2 + y^2)^{\frac{n+3}{2}}}, \quad E_1 = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{n-1}} \frac{y^2}{(|x|^2 + y^2)^{\frac{n+1}{2}}}.$$

In [8], J. Edenhofer cites [3] for this result.

The solution $u$ to

$$(\partial_x^2 + \partial_y^2)^2 u = 0 \quad \text{in} \quad y > 0$$

$$u|_{y=0} = g_0, \quad \partial_y u|_{y=0} = g_1 \quad (n = 1, m = 2, j = 0, 1),$$

in the form

$$u(x, y) = \frac{2y^3}{\pi} \int_\mathbb{R} g_0(x - \xi) \frac{d\xi}{(\xi^2 + y^2)^2} + \frac{y^2}{\pi} \int_\mathbb{R} g_1(x - \xi) \frac{d\xi}{\xi^2 + y^2}$$

can be found in [22] (2.14), p. 262] (where a sign should be corrected and where the formula is attributed to L. F. Richardson in [23]). For a more recent, direct treatment of the Poisson kernel $E_0$ for the biharmonic operator in the half-plane we refer to [1] p. 781).

(c) If $n = 2$, the Poisson kernel $E_{m-1}$ of the Dirichlet problem for $(\partial_x^2 + \partial_y^2)^m$ in $y > 0$ (i.e., with the boundary conditions $\partial_y^k E_{m-1}|_{y=0} = 0, k = 0, 1, \ldots, m - 2, \partial_y^{m-1} E_{m-1}|_{y=0} = \delta(x)$) is given by the formula

$$E_{m-1} = \frac{1}{\pi(m - 1)!} \frac{y^m}{x^2 + y^2},$$

see Example 5 in [30] p. 275).

Next, let us derive the Poisson kernel of the $j$-th Dirichlet problem, $j = 0, 1, \ldots, m - 1$, of the operator $(\Delta_n + \partial_y^2 - \xi^2)^m$ in $\mathbb{H}$ by Fourier transformation.

**Proposition 9.** Let $m, n \in \mathbb{N}$. The Poisson kernel of the $j$-th Dirichlet problem, $j = 0, 1, \ldots, m - 1$, for the iterated meta-harmonic operator $(\Delta_n + \partial_y^2 - \xi^2)^m$ in $\mathbb{H}$, $\xi \neq 0$, is given by

$$E_j = \frac{y^m |\xi|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \pi^\frac{n+1}{2} j! (m - 1 - j)!} (-\partial_y)^{m-j-1} \left( \frac{K_{\frac{n+1}{2}}(|\xi|\sqrt{|x|^2 + y^2})}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \right)$$  \hspace{1cm} (1)

or

$$E_j = \frac{-y^m |\xi|^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \pi^\frac{n+1}{2} j! (m - 1 - j)!} (-\partial_y)^{m-j-1} \left( \frac{K_{\frac{n-1}{2}}(|\xi|\sqrt{|x|^2 + y^2})}{(|x|^2 + y^2)^{\frac{n-1}{2}}} \right)$$  \hspace{1cm} (2)

Here, $K_\lambda$ is the modified Bessel function of the second kind of order $\lambda$.  \hspace{1cm} 7
Proof. Setting
\[ F_j = \frac{2}{\omega_{m+2} j! (m-1-j)!} (-\partial_y)^{m-j-1} \frac{1}{(|x|^2 + y^2 + s^2)^{\frac{m+1}{2}}} \]
we obtain by means of Proposition 7
\[ (\Delta_n + \partial_y^2 + \partial_s^2) F_j = 0 \]
\[ \partial_y^j F_j|_{y=0} = \delta(x,s)\delta_{jk}, \quad k = 0, 1, \ldots, m-1. \]

A partial Fourier transformation with respect to \( s \) yields for \( E_j = \int_{\mathbb{R}} e^{-i\xi s} F_j \, ds \):
\[ (\Delta_n + \partial_y^2 - \xi^2) F_j = 0 \]
\[ \partial_y^k E_j|_{y=0} = \delta(x)\delta_{jk}, \quad k = 0, 1, \ldots, m-1. \]

By [12] 8.432,5 we obtain
\[ E_j = \frac{\Gamma(\frac{n}{2} + 1)}{\pi^{\frac{n}{2}+1} j! (m-1-j)!} (-\partial_y)^{m-j-1} \left( \frac{2\sqrt{\pi} |\xi|^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(|\xi| \sqrt{|x|^2 + y^2})}{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}} j! (m-1-j)!} \right) \]
\[ = \frac{-y^m |\xi|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}} j! (m-1-j)!} (-\partial_y)^{m-j-1} \left( \frac{1}{y} \partial_y \right) \frac{K_{\frac{n+1}{2}}(|\xi| \sqrt{|x|^2 + y^2})}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \]

establishing (1). The second claim follows from the functional relation \( (\frac{\nu}{2} \partial_x)(z^{-\lambda} K_\lambda(z)) = -z^{-\lambda-1} K_{\lambda+1}(z) \) (cf. [12] 8.486,15):
\[ E_j = \frac{\Gamma(\frac{n}{2} + 1) y^m}{n \pi^{\frac{n}{2}+1} j! (m-1-j)!} (-\partial_y)^{m-j-1} \left( \frac{1}{y} \partial_y \right) \frac{2\sqrt{\pi} |\xi|^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(|\xi| \sqrt{|x|^2 + y^2})}{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}} j! (m-1-j)!} \]
\[ = \frac{-y^m |\xi|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}} j! (m-1-j)!} (-\partial_y)^{m-j-1} \left( \frac{1}{y} \partial_y \right) \frac{K_{\frac{n+1}{2}}(|\xi| \sqrt{|x|^2 + y^2})}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \]
\[ \square \]

Remark 10. Let us mention a particular case of Proposition 9. Setting \( m = 1, j = 0 \), the Poisson kernel \( E_0 \) of the metaharmonic operator \( \Delta_n + \partial_y^2 - \xi^2 \) in \( \mathbb{H} \) is given by
\[ E_0 = \frac{y |\xi|^{\frac{n-1}{2}}}{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}}} K_{\frac{n+1}{2}}(|\xi| \sqrt{|x|^2 + y^2}) \]
see [5] Rem. 2, p. 321].

Proposition 9 remains valid if we substitute \( p \in T^\Gamma = \mathbb{R}_+ + i\Gamma \) (\( \Gamma = \mathbb{R}_+ = (0, \infty) \), \( T^\Gamma \) the right half-plane) for \( \xi \in \mathbb{R} \setminus \{0\} \). We obtain by analytic continuation:
Proposition 11. The Poisson kernel \( E_j \) of the \( j \)-th Dirichlet problem, \( j = 0, 1, \ldots, m - 1 \), of the iterated meta-harmonic operator \( (\Delta_n + \partial_y^2 - p^2)^m, p \in T^p \), in the half-space \( \mathbb{H} \) is given by

\[
E_j = \frac{-y^m p^{\frac{n+1}{2}}}{2^{\frac{n+1}{2}} \pi^{\frac{n+1}{2}} j! (m - 1 - j)!} (-\partial_y)^{m-j-1} \left( \frac{1}{y} \right) \left( \frac{K_{n-1} \left( p \sqrt{|x|^2 + y^2} \right)}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \right)
\]

(For even \( n \) the square root in \( p^{\frac{n+1}{2}} \) is defined as usual.) Furthermore, \( E_j \in \mathcal{H}(T^p)(\mathcal{D}'(\mathbb{H}_{xy})) \).

Proof. The integral representation

\[
K_{n-1} \left( p \sqrt{|x|^2 + y^2} \right) \frac{1}{(|x|^2 + y^2)^{\frac{n+1}{2}}} = \frac{1}{2} \int_0^\infty t^{-\frac{n+1}{2}} e^{-\frac{p}{2} \left( t+\frac{|x|^2+y^2}{t} \right)} dt
\]

in [12, 8.432, 7] can be interpreted as a vector-valued scalar product

\[
\frac{1}{2} \left\langle 1(t), t^{-\frac{n+1}{2}} e^{-\frac{p}{2} t} \cdot e^{-\frac{p}{2} \left( |x|^2+y^2 \right)} \right\rangle
\]

on \( L^\infty(\mathbb{R}_+,t) \times L^1(\mathbb{R}_+,t) \langle \mathcal{H}(T^p)(\mathcal{D}'(\mathbb{H}_{xy})) \rangle \). Here, \( e^{-\frac{p}{2} t} \in \mathcal{H}(T^p)(L^1(\mathbb{R}_+,t)) \), and

\[
S(p,t,x,y) := e^{-\frac{p}{2} \left( |x|^2+y^2 \right)} t^{-\frac{n+1}{2}} \in \mathcal{H}(T^p)(C_0(\mathbb{R}_+,t))\mathcal{D}'(\mathbb{H}_{xy})) \quad = \mathcal{H}(T^p) \hat{\otimes} C_0(\mathbb{R}_+,t) \hat{\otimes} \mathcal{D}'(\mathbb{H}_{xy})
\]

To prove this, we first show the following two auxiliary results:

Lemma 12. Any complete, nuclear normal space of distributions has the \( \varepsilon \)-property.

Proof. By the Kömura Theorem [18, 21.7.1, p. 500], any such space \( F \) is isomorphic to a closed subspace of \( s' \) for some index set \( J \). Since the \( \varepsilon \)-property is preserved under taking products and subspaces, this implies that \( F \) has the \( \varepsilon \)-property.

Lemma 13. \( \mathcal{H}(T^p) \hat{\otimes} \mathcal{D}'(\mathbb{H}_{xy}) \) has the \( \varepsilon \)-property.

Proof. We have \( \mathcal{H}(T^p) \subseteq \mathcal{E}(\Gamma \times \mathbb{R}^n) \), with the topology induced by \( \mathcal{E} \). Both \( \mathcal{E} \) and \( \mathcal{D} \) are nuclear normal spaces of distributions, so \( \mathcal{E} \hat{\otimes} \mathcal{D}' \) is nuclear, normal, and complete and has the \( \varepsilon \)-property by Lemma [12]. As the \( \varepsilon \)-property is inherited by topological subspaces, the claim follows.

To establish \( S(p,t,x,y) \in (\mathcal{H}(T^p) \hat{\otimes} \mathcal{D}'(\mathbb{H}_{xy}))(C_0(\mathbb{R}_+,t)) \) it therefore suffices to show that for \( \mu \in \mathcal{M}(\mathbb{R}_+,t) \) we have

\[
\langle S(p,t,x,y), \mu(t) \rangle \in \mathcal{H}(T^p) \hat{\otimes} \mathcal{D}'(\mathbb{H}_{xy})
\]

Noting that \( \mathcal{H}(T^p) \) has the \( \varepsilon \)-property, to see this it suffices in turn to show that

\[
\langle \varphi(x,y), \langle S(p,t,x,y), \mu(t) \rangle \rangle \in \mathcal{H}(T^p)
\]

for each \( \varphi \in \mathcal{D}(\mathbb{H}_{xy}) \). By Fubini’s theorem ([27, p. 131, Corollaire]) this is equivalent to

\[
\langle \langle S(p,t,x,y), \varphi(x,y) \rangle, \mu(t) \rangle \in \mathcal{H}(T^p)
\]
In fact,
\[
\langle t^{-\frac{n+1}{2}} e^{-\frac{|x|^2+y^2}{t}} \varphi(x,y) \rangle = \int_{\mathbb{R}^{2n+1}} e^{-\frac{|x|^2+y^2}{t}} t^{-\frac{n+1}{2}} \varphi(x,y) \, dxdy
\]
\[
= \int_{\mathbb{R}^{2n+1}} e^{-\frac{y^2}{t}} \varphi(\sqrt{t} \xi, \sqrt{t} \eta) \, d\xi d\eta,
\]
so that
\[
\langle \mu(t), \langle S(p,t,x,y), \varphi(x,y) \rangle \rangle = \int_{\mathbb{R}^{2n+1}} e^{-\frac{y^2}{t}} \langle \mu(t), \varphi(\sqrt{t} \xi, \sqrt{t} \eta) \rangle \, d\xi d\eta.
\]
Since the map $\mathbb{H} \to \mathbb{C}$, $(\xi, \eta) \mapsto \langle \mu(t), \varphi(\sqrt{t} \xi, \sqrt{t} \eta) \rangle$ is bounded by $||\mu||_1 ||\varphi||_\infty$, (3) follows by dominated convergence. Note that this argument in fact also shows that $\mathcal{S}(p,t,x,y) \in \mathcal{H}(T^T_p)(\mathcal{BC}(\mathbb{R}^{n+1}, L^1(\mathbb{R}^{2n+1})))$. Here, the subscript $b$ refers to the Buck topology, so $\mathcal{BC}(\mathbb{R}^+) = \mathcal{M}(\mathbb{R}^+)$ (cf. [21] p. 6).

Due to the continuity of the bilinear multiplication $\mathcal{H}(T^T_p) \times \mathcal{H}(T^T_p) \to \mathcal{H}(T^T_p)$ and the continuity of the vector-valued multiplication $\mathcal{C}_0(\mathbb{R}^+)'(\mathcal{D}'(\mathbb{R}^{n+1})) \times L^1(\mathbb{R}^{n+1}, L^1(\mathbb{R}^{2n+1})) \to L^1(\mathbb{R}^{n+1}, L^1(\mathbb{R}^{2n+1}))$ (Definition and Theorem 3 and Lemma 5).

In virtue of
\[
\mathcal{H}(T^T_p)(L^1(\mathbb{R}^{n+1}, L^1(\mathbb{R}^{2n+1}))) = L^1(\mathbb{R}^{n+1}, \mathcal{H}(T^T_p)(\mathcal{D}'(\mathbb{R}^{2n+1}))),
\]
the final step consists in applying [26] Theorem 7.1, p. 31] to the vector-valued scalar product $\langle \cdot, \cdot \rangle : L^\infty \times L^1(E) \to E$, with $E = \mathcal{H}(T^T_p)(\mathcal{D}'(\mathbb{R}^{2n+1}))$. $
$

4 Transient response Dirichlet problems

In this section we study the transient response Dirichlet problem for the iterated wave operator $\left(\delta_n + \partial_y^2 - \partial^2 \right)^m$ and the iterated Klein-Gordon operator $\left(\Delta_n + \partial_y^2 - \partial^2 \right)^m$ in the half-space $y > 0$, more precisely in $\mathbb{H}_{yt} = \{(x, y, t) \in \mathbb{R}^{n+2} : y > 0, \ t > 0\}$.

We look for an explicit expression for the solution $E_j$ to the $j$-th, $j = 0, 1, \ldots, m - 1$, (mixed) Cauchy-Dirichlet problem
\[
\left(\Delta_n + \partial_y^2 - \partial^2 \right)^m E_j = 0 \quad \text{in} \ \mathcal{D}'(\mathbb{H}_{yt})
\]
\[
E_j|_{t=0} = \partial_t E_j|_{t=0} = \cdots = \partial^2 E_j|_{t=0} = 0 \quad \text{in} \ \mathcal{D}'(\mathbb{H}_1)
\]
\[
\partial_y^2 E_j|_{y=0} = \delta(x,t) \delta_{jk}, \quad k = 0, \ldots, m - 1, \quad \text{in} \ \mathcal{D}'(\mathbb{H}_2),
\]
where $\mathbb{H}_1 = \{(x, y) \in \mathbb{R}^{n+1} : y > 0\}, \ \mathbb{H}_2 = \{(x, t) \in \mathbb{R}^{n+1} : t > 0\}$.

For the general theory of the mixed problem for constant coefficient, linear partial differential operators see [15], 12.9, p. 162–179] and [24], p. 57–118]. We call $E_j$ Poisson kernel of the Cauchy-Dirichlet problem for the iterated wave operator and the iterated Klein-Gordon operator if $\xi = 0$ or $\xi \not= 0$, respectively ([21] p. 94)].
Proposition 14. The Poisson kernel $E_j$, $j = 0, 1, \ldots, m-1$, of the Cauchy-Dirichlet problem for the iterated wave operator $(\Delta_n + \partial_y^2 - \partial_t^2)^m$ in the half space $\mathbb{H}_{yt}$ is given by

$$E_j = \frac{-y^m}{2^{n-1}\pi^\frac{n}{2}\Gamma\left(\frac{n}{2}\right)j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right)\left(\frac{\partial_j^{n-1}(t^2 - |x|^2 - y^2)^\frac{n-1}{2}}{(|x|^2 + y^2)^\frac{n}{2} + t}Y(t)\right),$$

where $x_\perp := \langle x, y \rangle$. Furthermore, $E_j \in S'(\mathbb{H}_{yt})((\text{D}'(\mathbb{H}_{xy}))).$

Proof. Recall from Definition 2 that

$$S'(\mathbb{R}_{+,t}) := \bigcap_{\tau \in (0, \infty)} e^{\tau t}S',$$

so by Definition and Theorem 3 the Laplace transform

$$\mathcal{L} : S'(\mathbb{R}_{+,t})((\text{D}'(\mathbb{H}_{xy}))) \rightarrow \mathcal{H}(T^1_p)((\text{D}'(\mathbb{H}_{xy})))$$

is an isomorphism.

Thus the inverse Laplace transform $E_j := \mathcal{L}^{-1}E_j$ of the Poisson kernel $E_j$ of the $j$-th Dirichlet problem in the half-space of the iterated metaharmonic operator in Proposition 11 yields

$$(\Delta_n + \partial_y^2 - \partial_t^2)^m E_j = 0$$

$E_j|_{t=0} = \partial_t E_j|_{t=0} = \cdots = \partial_t^{2m-1}E_j|_{t=0} = 0$

$\partial_y^{k} E_j|_{y=0} = \delta(x, t)\delta_jk, \; \; k = 0, \ldots, m-1,$

and

$$E_j = \frac{-y^m}{2^{n-1}\pi^\frac{n}{2}\Gamma\left(\frac{n}{2}\right)j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right)\left(\frac{\mathcal{L}^{-1}\left(p^{\frac{n-1}{2}}K_{n-1}\left(p\sqrt{|x|^2 + y^2}\right)\right)}{(|x|^2 + y^2)^\frac{n}{2} + t}Y(t)\right).$$

By (6.15. (4), p. 172) we have for any $S \in \mathcal{H}(T^1_p)$: $\mathcal{L}^{-1}(p^{n-1} S) = \partial_t^{n-1}\mathcal{L}^{-1}S$. Hence, we obtain by means of the transform pair

$$\mathcal{L}^{-1}\left(p^{\frac{n-1}{2}}K_{n-1}\left(p\sqrt{|x|^2 + y^2}\right)\right) = \partial_t^{n-1}\mathcal{L}^{-1}\left(\frac{K_{n-1}\left(p\sqrt{|x|^2 + y^2}\right)}{p^{\frac{n-1}{2}}}\right) = \partial_t^{n-1}\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\frac{\sqrt{t^2 - |x|^2 - y^2}^\frac{n-1}{2}}{(2\sqrt{|x|^2 + y^2})^{\frac{n-1}{2}}}Y(t)\right).$$

In fact, (6.15. (4)) is the inverse relation of

$$\left\langle 1(t), e^{-pt}(t^2 - |x|^2 - y^2)^\frac{n-1}{2}Y(t - \sqrt{|x|^2 - y^2}^\frac{n-1}{2})\right\rangle = \frac{\sqrt{\pi}}{2^{\frac{n-1}{2}}\Gamma(n/2)} = \frac{K_{n-1}\left(p\sqrt{|x|^2 + y^2}\right)}{p^{\frac{n-1}{2}}}$$

(which can be seen using [12 8.432,3] or [33 §6.15. (4), p. 172]). That this identity is in fact valid in $\mathcal{H}(T^1_p) \otimes \text{D}'(\mathbb{H}_{xy})$ can be concluded similarly to the proof of Proposition 11. This
yields the formula stated in the Proposition. The fact that $E_j$ belongs to $S'(\mathbb{R}_{+,t})(\mathcal{D}'(\mathbb{H}_{1,xy}))$ now follows by inspection.

It remains to show that $\partial_t^k E_j|_{t=0} = 0$ for $0 \leq k \leq 2m - 1$. For this we first note that by [15 Th. 12.9.12, p. 176] we have $E_j \in C^\infty([0, \infty), \mathcal{D}'(\mathbb{H}_{xy}))$, so

$$\partial_t^k E_j(t) = \text{const} \cdot y^m (-\partial_y)^{m-j-1} \left( \frac{1}{y} \partial_y \right) \partial_t^{k+n-1} Y(t) \frac{(t^2 - |x|^2 - y^2)^{\frac{n-1}{2}}}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \in \mathcal{D}'(\mathbb{H}_{xy})$$

and for $\varphi \in \mathcal{D}(\mathbb{H}_{xy})$ we obtain

$$\langle \varphi, \partial_t^k E_j(t) \rangle = \text{const} \cdot \partial_t^{k+n-1} Y(t) \left\{ \partial_y \frac{1}{y} \partial_y^{m-j-1} (y^m \varphi), \frac{(t^2 - |x|^2 - y^2)^{\frac{n-1}{2}}}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \right\}$$

$$= \text{const} \cdot \partial_t^{k+n-1} \left( Y(t) \int_{|x|^2 + y^2 \leq t^2} \phi(x, y) \frac{(t^2 - |x|^2 - y^2)^{\frac{n-1}{2}}}{(|x|^2 + y^2)^{\frac{n+1}{2}}} \ dx \ dy \right),$$

where $\phi(x, y) := \partial_y \frac{1}{y} \partial_y^{m-j-1} (y^m \varphi)$. Applying the homothety $x = t\xi$, $y = t\eta$, $t > 0$ shows that the latter equals

$$\text{const} \cdot \partial_t^{k+n-1} \left( t^n \int_{|\xi|^2 + \eta^2 \leq 1} \phi(t\xi, t\eta) \frac{(1 - |\xi|^2 - \eta^2)^{\frac{n-1}{2}}}{(|\xi|^2 + \eta^2)^{\frac{n+1}{2}}} \ d\xi \ d\eta \right),$$

so

$$\lim_{t \downarrow 0} \langle \varphi, \partial_t^k E_j(t) \rangle = \text{const} \cdot \int_{|\xi|^2 + \eta^2 \leq 1} \left[ \lim_{t \downarrow 0} \partial_t^{k+n-1} t^n \phi(t\xi, t\eta) \right] \cdot \frac{(1 - |\xi|^2 - \eta^2)^{\frac{n-1}{2}}}{(|\xi|^2 + \eta^2)^{\frac{n+1}{2}}} \ d\xi \ d\eta.$$

As $\phi$ vanishes at $t = 0$, together with all its derivatives, we indeed arrive at $\partial_t^k E_j|_{t=0} = 0$ for all $k$.

**Remark 15.** We single out two important special cases:

(a) $n = 1$, $m = 1$, $j = 0$:

The Poisson kernel of the mixed problem

$$(\partial_x^2 + \partial_y^2 - \partial_t^2) E_0 = 0, \quad x \in \mathbb{R}, \quad y > 0, \quad t > 0,$$

$E_0|_{t=0} = \partial_t E_0|_{t=0} = 0$

$E_0|_{y=0} = \delta(x, t)$

in the half-space $y > 0$ is given by

$$E_0 = -\frac{1}{\pi} \partial_y \frac{Y(t)}{(t^2 - x^2 - y^2)^{\frac{3}{2}}} = -\frac{Y(t)}{\pi} \partial_y \left( \frac{(t^2 - x^2 - y^2)^{\frac{1}{2}}}{(t^2 - x^2 - y^2)^{\frac{3}{2}}} \right).$$
In [19] Ex. 405, p. 189 the solution $U$ to the mixed problem with the temporally constant boundary value $U|_{y=0} = \delta(x)$ is presented. It emerges from $E_0$ by convolution with $\delta(x) \otimes Y(t)$, i.e.,

$$U = -\frac{Y(t)}{\pi} \partial_y \left( \frac{t^2 - x^2 - y^2}{\sqrt{t^2 - x^2 - y^2}} \right) * (\delta(x) \otimes Y(t)) = -\frac{y t_+}{\pi(x^2 + y^2) (t^2 - x^2 - y^2)^{1/2}}.$$

Note that our derivation differs essentially from that proposed in [19], where Fourier- and Laplace transformation are suggested to be applied with respect to different variables.

(b) $n = 2$, $m = 1$, $j = 0$:

The Poisson kernel of the mixed Cauchy-Dirichlet problem

$$(\Delta_2 + \partial_y^2 - \partial_l^2) E_0 = 0, \quad x \in \mathbb{R}^2, \; y > 0, \; t > 0, \quad E_0|_{t=0} = \partial_t E_0|_{t=0} = 0 \quad E_0|_{y=0} = \delta(x, t)$$

in the half-space $y > 0$ is given by

$$E_0 = -\frac{Y(t)}{2\pi} \partial_y \left( \frac{1}{\sqrt{|x|^2 + y^2}} \partial_t (Y(t^2 - |x|^2 - y^2)) \right) = -\frac{1}{2\pi t} \partial_y (t - \sqrt{|x|^2 + y^2}).$$

Note that $E_0$ is the negative derivative in the direction normal to the boundary of the Green-function of the mixed problem of $\Delta_2 + \partial_y^2 - \partial_l^2$ in the half-space $y > 0$ (compare [10] p. 92)). We obtain the solution $U$ of the mixed problem with a temporally constant boundary value, $U|_{y=0} = \delta(x)$, by convolution of $E_0$ with $\delta(x) \otimes Y(t)$:

$$U = -\frac{Y(t)}{2\pi} \partial_y \left( \frac{Y(t^2 - |x|^2 - y^2)}{\sqrt{|x|^2 + y^2}} \right) = -\frac{1}{2\pi} \partial_y \left( \frac{Y(t - \sqrt{|x|^2 + y^2})}{\sqrt{|x|^2 + y^2}} \right),$$

which coincides with [4] p. 7.

The main idea in deriving the Poisson kernel of the Cauchy-Dirichlet problem $(\Delta_n + \partial_y^2 - \partial_l^2)^m$ in the half-space $y > 0$ (cf. the proof of Proposition 14) is the application of the inverse Laplace transformation to the Poisson kernel of the Dirichlet problem of $(\Delta_n + \partial_y^2 - p^2)^m$ in $y > 0$. The Poisson kernel of the Cauchy-Dirichlet problem of the iterated Klein-Gordon operator $(\Delta_{2n+1} + \partial_y^2 - \partial_l^2 - \xi^2)^m$ ($\xi > 0$) in the half-space $y > 0$ can be derived by the same method, using the Poisson kernel of $(\Delta_{2n+1} + \partial_y^2 - p^2 - \xi^2)^m$ in $y > 0$.

**Proposition 16.** The $j$-th Poisson kernel $E_j$, $j = 0, 1, \ldots, m - 1$, of the Cauchy-Dirichlet problem of the iterated Klein-Gordon operator $(\Delta_{2n+1} + \partial_y^2 - \partial_l^2 - \xi^2)^m$ ($m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\xi > 0$) in the half-space $\mathbb{H}$ is given by

$$E_j = \frac{-y^m}{(2\pi)^{n+\frac{3}{2}}} j!(m - 1 - j)! \left( -\partial_y \right)^{m-j-1} \left( \frac{1}{y} \right)^{\frac{1}{2}} \left( \frac{1}{|x|^2 + y^2} \right)^{\frac{1}{2}} \sum_{l=0}^{n} \left( \frac{n}{l} \right) \xi^{n-2l+1} \partial_l^2 \left( Y(t^2 - |x|^2 - y^2) \right)^{\frac{m}{2} - \frac{1}{4}} J_{\frac{m}{2} - \frac{1}{4}} \left( \xi \sqrt{t^2 - |x|^2 - y^2} \right),$$

where $J_\lambda$ denotes the Bessel function of the first kind of order $\lambda$. 

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Proof. Similar to the proof of Proposition \[14\] we have by means of Proposition \[11\]

\[
E_j = \frac{-y^m}{2n\pi^{n+1}j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right) \left(\mathcal{L}^{-1}\left(\frac{(p^2 + \xi^2)^{\frac{m}{2}} K_n(\sqrt{(p^2 + \xi^2)(|x|^2 + y^2)})}{(|x|^2 + y^2)^{m/2}}\right)\right)
\]

\[
= \frac{-y^m}{2n\pi^{n+1}j!(m-1-j)!}(-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right) \left(\sum_{l=0}^{n} \binom{n}{l} \xi^{2n-2l} \partial_l^{2l} \mathcal{L}^{-1}\left(\frac{K_n(\sqrt{(p^2 + \xi^2)(|x|^2 + y^2)})}{(p^2 + \xi^2)(|x|^2 + y^2)^{m/2}}\right)\right).
\]

By the formula

\[
\mathcal{L}^{-1}\left(\frac{K_n(\beta \sqrt{(p^2 + \xi^2)})}{(p^2 + \xi^2)^{m/2}}\right) = \sqrt{\frac{\pi}{2}} \int_0^t Y(t)\xi^{-n+\frac{1}{2}} \beta^{-n}(t^2 - \beta^2)^{-\frac{n}{2}} \frac{1}{2} J_{n-\frac{1}{2}}\left(\xi \sqrt{t^2 - \beta^2}\right)
\]

in \[6\] p. 125] we obtain

\[
E_j = \frac{-y^m}{(2\pi)^{n+1}j!(m-1-j)!} \sum_{l=0}^{n} \binom{n}{l} \xi^{n-2l+\frac{1}{2}} (-\partial_y)^{m-j-1}\left(\frac{1}{y}\partial_y\right) \partial_l^{2l} \left(\frac{Y(t)}{(|x|^2 + y^2)^{m/2}}(t^2 - |x|^2 - y^2)^{\frac{-n}{2}} \frac{1}{2} J_{n-\frac{1}{2}}\left(\xi \sqrt{t^2 - |x|^2 - y^2}\right)\right),
\]

establishing our claim.

Remark 17. (a) In the special case \(n = 0, m = 1, j = 0\) we obtain

\[
E_0 = \frac{Y(t)}{\sqrt{2\pi}} \xi^{1/2} \partial_y \left(\frac{J_{-1/2}\left(\xi \sqrt{t^2 - x^2 - y^2}\right)}{(t^2 - x^2 - y^2)^{1/4}}\right) = \frac{Y(t)}{\pi} \partial_y \left(\frac{\cos\left(\xi \sqrt{t^2 - x^2 - y^2}\right)}{(t^2 - x^2 - y^2)^{1/4}}\right)
\]

as the Poisson kernel of the Cauchy-Dirichlet problem of \(\partial_t^2 + \partial_y^2 - \partial_x^2 - \xi^2\) (\(\xi > 0\)) in \(y > 0\). The solution \(U\) to this problem in \(y > 0\) with the temporally constant boundary value \(U|_{y=0} = \delta(x)\) emerges from \(E_0\) by convolution with \(\delta(x) \otimes Y(t)\), i.e., \(U = E_0 \ast (Y(t) \otimes \delta(x, y))\).

Note that for the Cauchy-Dirichlet problem of the related operator \(\partial_t^2 + \partial_y^2 - \partial_x^2 - \beta^2\) in \(y > 0\) with a temporally constant boundary value, the solution is given explicitly in \[19\] Ex. 406, p. 189] in terms of elementary functions.

(b) The Poisson kernel of the Cauchy-Dirichlet problem of the iterated Klein-Gordon operator \((\Delta_{2n} + \partial_y^2 - \partial_x^2 - \xi^2)^m\) in odd space dimensions can be deduced from that in Proposition \[19\] by J. Hadamard’s method of descent, i.e., by integration with respect to the variable \(x_{2n+1}\).

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