Relative Spin Phase Modulated Radial Inverse Square Law Magnetic Force

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The nature of the magnetic interaction is investigated for two classical point charge particles separately undergoing circulatory motion at the speed of light to account for the particles’ intrinsic spin. It is found that the magnetic force between the particles has a radial component that depends inversely on the separation between the centers of motion of the two charges, is up to the same magnitude as the Coulomb force, and is modulated by the relative orientation of the spins, as well as the phase difference of the circular motion of one particle relative to the other at the retarded time.

I. INTRODUCTION

Schrödinger [1] called the electron oscillatory motion predicted in the Dirac theory of the electron “zitterbewegung,” and showed that its amplitude is of the order of the electron Compton wavelength. It can be further shown [2] that the zitterbewegung may be regarded as a circular motion at the speed of light around the direction of the electron spin, giving rise to the electron intrinsic magnetic moment. The phase of the zitterbewegung motion has been proposed [3] to provide a physical interpretation for the complex phase factor of the Dirac wave function.

An analysis of the electromagnetic structure of a luminally circulating charge by Rivas [4] shows that if the velocity of the circulatory motion is the speed of light, then the average electric field is purely radial, isotropic (except in the plane of the motion, where it is undefined) and, sufficiently far from the center of the motion, is consistent in magnitude with the field of a stationary charge of equal magnitude, falling off inverse square with distance. Rivas also shows, for radius of motion of half the reduced Compton wavelength, at sufficient distance the time averaged magnetic acceleration field of the circulating charge is that of an ideal magnetic dipole with moment one-half the Bohr magneton.

In this paper the analysis of the electromagnetic field structure of the luminally circulating charge is extended to consider the magnetic interaction between two particles each consisting of a luminally circulating charge, where the radii of the circular motions of the two particles are equal, that are widely separated compared to the radius. Because the second, test, particle is moving at the speed of light in a field that is of the same mean magnitude as a Coulomb field of a charge of equal value to the field source charge, the magnetic interaction has nominally the same strength as the Coulomb interaction. However, it is modulated by both the relative orientation and the relative phase of the circular motions. Furthermore, the relevant phase difference is distance dependent, because it is between the test particle present-time phase and the field source particle retarded-time phase.

II. ELECTROMAGNETIC FIELD OF A RELATIVISTICALLY CIRCULATING CHARGE

The electromagnetic field due to a point charge in arbitrary motion can be determined from the Liénard-Wiechert potentials. The electric and magnetic fields at a position \( \mathbf{r} \) and time \( t \) for an arbitrarily moving charge \( q_s \) may be expressed as

\[
E(\mathbf{r}, t) = q_s \left[ \frac{\mathbf{n} - \beta}{\gamma^2 (1 - \beta \cdot \mathbf{n})^3 R^2} \right]_{\text{ret}},
\]

and

\[
B(\mathbf{r}, t) = \left[ \mathbf{n} \times E \right]_{\text{ret}},
\]

where the subscript “ret” indicates that quantities in the brackets are evaluated at the retarded time \( t_r = t - R/c \), with \( R = c(t-t_r) \) the magnitude of the displacement from the charge position at the retarded time to the field point \( \mathbf{r} \) at time \( t \) in an inertial reference frame. \( \mathbf{n} \) is a unit vector in the direction of the field point from the position of the field-source charge at the retarded time, \( \beta = \beta_0 \) is the velocity of the charge divided by the speed of light, and \( \gamma = (1 - \beta^2)^{-1/2} \). The overdot represents differentiation with respect to time, \( t \), so \( \dot{\beta} = a/c \) where
\( \mathbf{a} = \alpha \dot{\mathbf{a}} \) is the acceleration.

The first term on the right hand side of Eq. (1), often referred to as the electric velocity field, vanishes in the limit of \( \beta \) approaching unity, except if \( \beta = \mathbf{n} \). The second term on the right hand side of Eq. (1), also undefined when \( \beta = \mathbf{n} \), can be referred to as the electric acceleration field, or as the radiative field owing to its inverse \( R \) rather than inverse \( R^2 \) dependence. However, it can also give rise to more rapidly decaying field terms that are dynamically relevant in the near field.

The point of the current paper can be made under conditions where Eq. (1) is well defined, so the behavior when \( \beta = \mathbf{n} \) will not be considered further.

A. Electric Field of a Classical Zitterbewegung Particle

To evaluate the magnetic force between two relativistically circulating particles, the Liénard-Wiechert retarded electric acceleration field is first evaluated for the source charge undergoing circular motion. (To avoid the singular condition where \( \beta = \mathbf{n} \), it is sufficient assume that the test particle center of circular motion is away from the plane of circular motion of the source charge. This also ensures that the electric velocity field vanishes, for the luminal field-source charge.) Next, the magnetic field can be evaluated using Eq. (2), and the magnetic force on the luminally-circulating test charge follows straightforwardly.

For brevity, and after [6], it will be convenient to refer to the luminal circular motion of the point charge as the “zitter” motion, and to a charge undergoing such motion as a zitter particle. In spite of the luminal circular motion of its charge, the zitter particle is considered stationary when its center of motion is fixed.

To evaluate the field for the stationary zitter particle, it is useful to suppose that the center of the zitter motion is the origin of a Cartesian coordinate system, so that \( \mathbf{r} \) is the vector displacement from the center of the zitter motion of the source charge to the field point. Then define \( \hat{k} \) as the vector displacement from the zitter center to the charge instantaneous position. Thus \( \hat{k} = \hat{R}_0 \hat{k} \) where \( \hat{k} \) is a unit vector and \( \hat{R}_0 \) is the radius of the circular motion.

Also, for the stationary zitter particle, the acceleration magnitude \( a = c|\beta| = c^2/R_0 \) obtains the electron intrinsic angular momentum \( h/2 \). The diameter of the circulatory motion that has angular momentum \( h/2 \) is thus one reduced Compton wavelength. Also, for the stationary zitter particle, the acceleration given as \( L = R_0 \gamma m v = c \), taking \( \gamma m = m_e \) where \( m_e \) is the observed electron mass, and \( R_0 = \hbar/2m_ec \), obtains the electron intrinsic angular momentum \( h/2 \). The diameter of the circulatory motion that has angular momentum \( h/2 \) is thus one reduced Compton wavelength. Also, for the stationary zitter particle, the acceleration magnitude \( a = c|\beta| = c^2/R_0 \) obtaining the electron intrinsic angular momentum \( h/2 \).

If \( \mathbf{R} = \mathbf{r} - \hat{k} = \mathbf{r} - \hat{R}_0 \hat{k} \) is the displacement from the instantaneous charge position to the field point, then \( \mathbf{R} = (r^2 - 2\hat{R}_0 \mathbf{r} \cdot \hat{k} + \hat{R}_0^2)^{1/2} = r(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)^{1/2} \) with \( \epsilon = \hat{R}_0/r \), and \( n \equiv \mathbf{R}/R \) becomes

\[
\mathbf{n} = \frac{\mathbf{r} - \hat{R}_0 \hat{k}}{r(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)^{1/2}}.
\]

With \( \dot{\mathbf{a}} = -\hat{k}, \) and \( \beta \times \dot{\mathbf{a}} = \hat{z}, \)

\[
(n - \beta) \times \dot{\mathbf{a}} = -\frac{\mathbf{r} \times \dot{\mathbf{k}}}{r(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)^{1/2}} - \hat{z}
\]

and, with \( \dot{\mathbf{k}} \times \hat{z} = -\beta, \) and using the vector identity \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}, \)

\[
n \times ((n - \beta) \times \dot{\mathbf{a}}) = -\frac{(\dot{\mathbf{r}} \cdot \hat{k}) (\dot{\mathbf{r}} - \epsilon \hat{k}) - \hat{k} - \epsilon \hat{r}}{(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)} - \frac{\dot{\mathbf{r}} \times \hat{z} + \epsilon \beta}{(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)^{1/2}}.
\]

Recalling that \( a = \epsilon^2/R_0 \) for the stationary zitter particle and \( R = r(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)^{1/2} \), the electric acceleration field term from Eq. (1) can now be written in terms of the distance \( r \) from the zitter particle center of motion (rather than in terms of the instantaneous distance \( R \) from the luminally-circulating charge position) as

\[
E_{\text{acc}}(r, t) = \frac{-q_a}{R_0 (1 - \beta \cdot n)^3 r} \left[ \frac{(\dot{\mathbf{r}} \cdot \mathbf{k}) (\dot{\mathbf{r}} - \epsilon \hat{k}) - \hat{k} - \epsilon \hat{r}}{(1 - 2\hat{c} \cdot \hat{k} + \epsilon^2)} + \dot{\mathbf{r}} \times \hat{z} + \epsilon \beta \right],
\]

where all quantities on the right hand side are evaluated at the retarded time.
Now it can be observed that although the electric acceleration field due to the zitter particle has an explicit inverse dependence on the distance \( r \), the factors of \( \epsilon = R_0/r \) give dependencies of inverse higher powers of \( r \), including, but not limited to, inverse \( r^2 \).

To see clearly the inverse square of distance character of the time-averaged electric acceleration field intensity, it is useful to consider the behavior of the field on the \( z \)-axis, i.e., perpendicular to the plane of the zitter motion of the source charge. When \( \hat{r} = \hat{z} \), then \( \hat{r} \cdot \hat{k} = \hat{r} \times \hat{z} = 0 \), \( \beta \cdot \hat{n} = 0 \), and so

\[
E_{\text{acc}}(r = z\hat{z}, t) = \frac{q_s}{R_0 z} \left[ \frac{\hat{k} + \epsilon \hat{z}}{(1 + \epsilon^2)^{1/2}} - \epsilon \beta \right].
\]  

(7)

The unit vectors \( \hat{k} \) and \( \beta \) average to zero over a cycle of the zitter motion, and on the \( z \)-axis \( \epsilon = R_0/z \). The average electric acceleration field on the \( z \)-axis is thus

\[
< E_{\text{acc}}(r = z\hat{z}) > = \frac{q_s}{z^2} \left[ \frac{\dot{\hat{z}}}{(1 + \epsilon^2)^{1/2}} \right].
\]  

(8)

As \( z \) becomes large, \( 1 + \epsilon^2 \) approaches unity, and so the long distance behavior of the time-averaged electric acceleration field of the fixed zitter particle, perpendicular to the plane of the zitter motion, is indistinguishable from the electric field of a static charge of the same magnitude.

Rivas [7] has numerically integrated the electric acceleration field to find the average at points off the \( z \)-axis, and at near as well as distant separation from center of the zitter motion. The average field is purely radial and isotropic at distances large compared to the zitter radius for all point off the \( x-y \) plane, where \( (1 - \beta \cdot \hat{n}) \) can vanish and the electric field average cannot be calculated. Rivas also shows that off the \( x-y \) plane the time-averaged magnetic acceleration field of the zitter particle with \( R_0 = \hbar/2me_c \), and charge \( e \) where \( e \) is the electron charge, is at large distance indistinguishable from the magnetic field of an ideal classical current-loop magnetic dipole with constant charge-to-mass rations of \( e/m_e \), and angular momentum \( \hbar/2 \).

B. Magnetic Field of a Classical Zitterbewegung Particle

The magnetic field due to the zitter particle is evaluated using Eqs. [2] and [4], but it is convenient to use the intermediate result of Eq. [6]. Taking the cross product of Eq. [6] with \( \hat{n} \) as given by Eq. [3] on the left obtains that

\[
n \times [n \times ((n - \beta) \times \hat{a})] = -\frac{\hat{r} \times \hat{k}}{(1 - 2\epsilon \hat{r} \cdot \hat{k} + \epsilon^2)^{1/2}} - \frac{\left( \hat{r} - \epsilon \hat{k} \right) \times (\hat{r} \times \hat{z}) + \epsilon \left( \hat{r} - \epsilon \hat{k} \right) \times \beta}{(1 - 2\epsilon \hat{r} \cdot \hat{k} + \epsilon^2)},
\]  

(9)

and with the vector identity \( a \times (b \times c) = (a \cdot c) b - (a \cdot b) c \),

\[
n \times [n \times ((n - \beta) \times \hat{a})] = \frac{\hat{r} \times \hat{k}}{(1 - 2\epsilon \hat{r} \cdot \hat{k} + \epsilon^2)^{1/2}} - \frac{(\hat{r} \cdot \hat{z}) \hat{r} - \hat{z} - \epsilon \left( (\hat{k} \cdot \hat{z}) \hat{r} - (\hat{k} \cdot \hat{r}) \hat{z} \right) + \epsilon \left( \hat{r} - \epsilon \hat{k} \right) \times \beta}{(1 - 2\epsilon \hat{r} \cdot \hat{k} + \epsilon^2)}.
\]  

(10)

With \( a = \epsilon^2/R_0 \) and \( \beta = a \hat{a} / c \), the magnetic acceleration field of the zitterbewegung particle can now be found from Eqs. [2] and [6] to be

\[
B_{\text{accel}} = \frac{q_s}{R_0 (1 - \beta \cdot \hat{n})^3 \hat{r}} \left[ \frac{\hat{r} \times \hat{k} - (\hat{r} \cdot \hat{z}) \hat{r} - \hat{z} - \epsilon \left( (\hat{k} \cdot \hat{z}) \hat{r} - (\hat{k} \cdot \hat{r}) \hat{z} \right) + \epsilon \left( \hat{r} - \epsilon \hat{k} \right) \times \beta}{(1 - 2\epsilon \hat{r} \cdot \hat{k} + \epsilon^2)^{1/2}} \right].
\]  

(11)

It can be noted that since for luminal charge velocity the electric velocity field vanishes everywhere that it is defined, and the acceleration field is undefined under the same conditions as the velocity field, the expression of Eq. [11], when it is well defined, describes the complete magnetic field of the zitter particle.
C. Radially-Directed Inverse-Square Law Magnetic Interaction Between Zitterbewegung Particles, where One Particle is Stationary

The force on a test particle with charge $q_t$ and velocity $\beta_t$ moving in a magnetic field $B$ is $F = \beta_t \times B$. The magnetic force on a test particle in arbitrary motion, due to the acceleration field of a source particle undergoing the circular zitter motion of radius $R_0$ can be evaluated using Eq. (10). Taking a cross product of Eq. (10) with $\beta_t$ on left obtains that

$$[\beta_t \times [n \times [n \times ((n - \beta) \times \dot{a})]] = \frac{(\beta_t \cdot \dot{k} - (\beta_t \cdot \hat{r}) \dot{k}}{(1 - 2\epsilon\hat{r} \cdot \dot{k} + \epsilon^2)^{1/2}} - \frac{r^2}{(1 - \beta \cdot n)^3 R(1 + \epsilon^2)} \left(\begin{array}{c} 1 + \epsilon \left(\dot{k} \cdot \dot{r} - (\beta_t \cdot \dot{r}) \beta_t \right) - \epsilon^2 \left(\beta_t \cdot \beta \right) \right) \right].$$

$$\hat{r} \times \dot{\hat{r}} - \beta_t \times \hat{z} + \epsilon \left(\dot{k} \cdot \dot{r} - (\beta_t \cdot \dot{r}) \beta_t \right) - \epsilon^2 \left(\beta_t \cdot \beta \right)$$

$$\frac{1 + \epsilon (1 + \epsilon)}{(1 - 2\epsilon\hat{r} \cdot \dot{k} + \epsilon^2)^{1/2}} \cdot \frac{\beta}{(1 - \beta \cdot n)^3}.$$ (12)

A detailed evaluation of the complete magnetic force on the test particle due to the field source zitterbewegung particle is outside the scope of the present contribution. However, it is apparent that the magnetic force will have a radially-directed term,

$$F_r = -\frac{q_t q_s}{R_0^2} \beta_t \cdot \left[ -\frac{n \cdot \dot{r}}{(1 - \beta \cdot n)^3 R(1 + \epsilon^2)} \right] \cdot \frac{1}{(1 - \beta \cdot n)^3} \frac{\beta}{(1 - 2\epsilon\hat{r} \cdot \dot{k} + \epsilon^2)^{1/2}}.$$ (13)

where on the right hand side, $\hat{r}$ and $r$ have been taken outside the brackets indicating retardation, because the center of motion of the zitterbewegung field-source particle is assumed to be stationary, and it has been used again that $\epsilon = R_0/r$. For large enough interparticle separation, $\epsilon << 1$ and so

$$F_r \approx -\frac{q_t q_s \beta_t \cdot \beta_{s,ret}}{(1 - \beta \cdot n)^3} \frac{\dot{r}}{r^2},$$ (13)

where all non-constant quantities on the right should be understood to be at the retarded time, except for $\beta_t$, and where $\beta$ is rewritten as $\beta_{s,ret}$ to emphasize its association with the source particle and at the retarded time. It can be noted at this point that the term given by Eq. (13) that contributes to the magnetic force caused by a stationary zitter particle on another zitter particle, differs from the Coulomb force caused by a stationary charge on another classical point charge only by the factor $-\beta_t \cdot \beta_{s,ret}$.

To make a preliminary assessment of how the magnetic force radial term given by Eq. (13) will affect the motion of a zitterbewegung test particle, the test particle may be supposed to be executing a circular motion with the same radius $R_0$ as that of the field source particle, and also at the speed of light so that $\beta_t = 1$. For simplicity, suppose both particles have their zitter motion in the x-y plane and in the same direction. Also, while strictly $\dot{r}$ is directed from the source particle center of motion toward a field point moving with the test particle, for interparticle separation large compared to $R_0$, i.e., half of a reduced Compton wavelength, it is reasonable to disregard the change in position of the test particle as it moves in its zitter orbit, and consider only its velocity direction time dependence. Then, with $\tau = \omega t$,

$$\beta_t \cdot \beta_{s,ret} = \left[-\sin \tau, \cos \tau, 0 \right] \cdot \left[-\sin(\tau + \psi), \cos(\tau + \psi), 0 \right]$$ (14)

where $\psi$ is the phase difference of the test particle present time zitter motion to the retarded-time source particle zitter motion. Carrying out the multiplication obtains that

$$\beta_t \cdot \beta_{s,ret} = \cos \psi.$$ (15)

Thus, depending on the zitter phase difference between the particles, for aligned spins, the magnetic force may either entirely cancel or effectively double the Coulomb force between the particles.

In the case of the test particle circularity motion opposite to that of the field source particle,

$$\beta_t \cdot \beta_{s,ret} = -\sin^2 \tau - \cos^2 \tau \cos \psi - \sin 2\tau \sin \psi,$$ (16)

which averages to zero over a zitter cycle. However, the magnetic force inverse square law radial component may be found to nonetheless have an influence on the motion, when the average is more properly carried out including the change in position of the test particle over its zitter orbit.

Finally, it’s worth a brief examination of how, for aligned zitter motions, the change in the source particle zitter phase induced by retardation might influence
the test particle motion under the combination of electric and magnetic forces. Suppose $\psi_0$ is the instantaneous phase difference between the zitter motions. Then,

$$\psi = \psi(r_{ret}) = \psi_0 + \frac{\omega R_{ret}}{c} = \psi_0 + \frac{R_{ret}}{R_0}$$

(17)

Therefore, over the course of an interparticle distance change by an amount equal to $\pi$ times the reduced Compton wavelength $2R_0$ of the particles, the magnetic force radial term of Eq. (12) will undergo a full cycle of modulation. Also, because the radial magnetic force can have a magnitude up to as large as the Coulomb force between two static charges of the same magnitude, it can be expected to significantly influence the motion.

Detailed analysis of the dynamical influence on classical zitterbewegung particles of the magnetic force is outside the scope of the present paper. Of course, there are many other magnetic force terms other than the radially acting term examined here. However many of them can be expected to average to zero over a zitter cycle, or become insignificant after averaging at some distance. Numerical averaging and other modeling should be straightforward. Rivas’ numerical technique for time averaging over a zitterbewegung cycle while accounting for retardation can be extended to include the motion of the test particle and applied here. Further, it should be possible to numerically propagate the motions of the two zitterbewegung particles given arbitrary initial conditions.

III. DISCUSSION

In Bohmian mechanics [8], the quantum Hamilton-Jacobi equation differs from the classical Hamilton-Jacobi equation by an additional term called the quantum potential. In the simplest case, this results in an added non-classical force equal but opposite the Coulomb force between two charges, similarly to the magnetic force term examined here under the condition of vanishing relative spin phase. Thus, it seems plausible that if spinning charged particles consist of luminally circulating point charges, then the Bohmian quantum potential and quantum force might be the consequence of a classical electromagnetic interaction.

Other authors [9–11] have also related the possible internal phase of the spin to quantum behavior.

The magnetic interactions found here are also found in the time symmetric electrodynamics picture where the total field acting is the mean of the retarded and advanced fields [12–14]. It may therefore also be relevant to more mathematically rigorous electrodynamic analyses taking precise and time-symmetric account of delay [15–17]. It has been suggested that time-symmetric interactions might account for the apparent non-locality of quantum mechanics [15].

IV. CONCLUSION

A preliminary evaluation of the magnetic force between two relativistically-circulating point charge particles has been performed. It was found that the magnetic interaction between such particles has a radially-directed component with a strength equal to that of the Coulomb interaction, except that it is modulated by the relative orientation of particle circulatory motions, and their propagation-delayed phase difference. This type of interaction might find application in efforts to explain quantum behavior using only classical physics. In any case, it is clear that if the zitterbewegung of the Dirac electron reflects a luminal circular motion of a classical point charge, then what classical physics says about atomic-scale electromagnetic interactions between particles possessing intrinsic magnetic moments has yet to be determined.

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