New Wallis- and Catalan-Type Infinite Products for $\pi$, $e$, and $\sqrt{2} + \sqrt{2}$

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Abstract

We generalize Wallis’s 1655 infinite product for $\pi/2$ to one for $(\pi/K) \csc(\pi/K)$, as well as give new Wallis-type products for $\pi/4$, 2, $\sqrt{2} + \sqrt{2}$, $2\pi/3\sqrt{3}$, and other constants. The proofs use a classical infinite product formula involving the gamma function. We also extend Catalan’s 1873 infinite product of radicals for $e$ to Catalan-type products for $e/4$, $\sqrt{e}$, and $e^{3/2}/2$. Here the proofs use Stirling’s formula. Finally, we find an analog for $e^{3/2}/\sqrt{3}$ of Pippenger’s 1980 product for $e/2$, and conjecture that they can be generalized to a product for a power of $e^{3/2}$.

1. INTRODUCTION. In 1655 Wallis [16] published his famous infinite product for pi:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \cdot 12 \cdot 12 \cdot 14 \cdot 14 \cdot 16 \cdot 16 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \cdot 11 \cdot 13 \cdot 13 \cdot 15 \cdot 15 \cdots}.$$ (1)

In 1873 Catalan [4] proved the Wallis-type formulas

$$\frac{\pi}{2\sqrt{2}} = \frac{4 \cdot 4 \cdot 8 \cdot 8 \cdot 12 \cdot 12 \cdot 16 \cdots}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdots}$$ (2)

and

$$\sqrt{2} = \frac{2 \cdot 2 \cdot 6 \cdot 6 \cdot 10 \cdot 10 \cdot 14 \cdot 14 \cdots}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdots}.$$ (3)

Together they give a beautiful factorization of Wallis’s formula, which we write symbolically as $(1) = (2) \times (3)$.

Catalan also found a product for $e$ similar to Wallis’s product for pi:

$$e = \frac{2}{1} \left( \frac{4}{3} \right)^{1/2} \left( \frac{6 \cdot 8}{5 \cdot 7} \right)^{1/4} \left( \frac{10 \cdot 12 \cdot 14 \cdot 16}{9 \cdot 11 \cdot 13 \cdot 15} \right)^{1/8} \cdots.$$ (4)

Even closer to Wallis’s formula is a product for $e$ discovered by Pippenger [12] in 1980:
While Catalan obtained (2), (3), and (4) as by-products of series and integrals for the gamma function $\Gamma(x)$, Pippenger's proof of (5) uses only Stirling's asymptotic formula

$$N! \sim \sqrt{2\pi N (N/e)^N} \quad (N \to \infty).$$

In this note, we offer several new products like Wallis's.

**Theorem 1.** The following Wallis-type formulas are valid:

$$\frac{\pi}{4} = \frac{2 \cdot 6 \cdot 8 \cdot 8}{3 \cdot 5 \cdot 7 \cdot 9} \cdot \frac{10 \cdot 14 \cdot 16 \cdot 16}{11 \cdot 13 \cdot 15 \cdot 17} \cdot \frac{18 \cdot 22 \cdot 24 \cdot 24}{19 \cdot 21 \cdot 23 \cdot 25} \cdots,$$

$$2 = \frac{2 \cdot 4 \cdot 4 \cdot 6}{1 \cdot 3 \cdot 5 \cdot 7} \cdot \frac{10 \cdot 12 \cdot 14}{9 \cdot 11 \cdot 13} \cdot \frac{18 \cdot 20}{17 \cdot 19} \cdot \frac{22}{21} \cdot \frac{23}{23} \cdots,$$

$$\frac{\pi}{4\sqrt{2 - \sqrt{2}}} = \frac{8 \cdot 8 \cdot 16}{7 \cdot 9} \cdot \frac{16 \cdot 24}{17 \cdot 23} \cdot \frac{24}{25} \cdots,$$

$$\sqrt{2 - \sqrt{2}} = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22}{3 \cdot 5 \cdot 11 \cdot 13 \cdot 19} \cdots,$$

$$\frac{\pi}{2\sqrt{2 + \sqrt{2}}} = \frac{2 \cdot 4 \cdot 6 \cdot 8 \cdot 8}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 9} \cdot \frac{10 \cdot 12 \cdot 14 \cdot 16 \cdot 16}{11 \cdot 11 \cdot 13 \cdot 15 \cdot 17} \cdot \frac{18 \cdot 20}{19 \cdot 21} \cdot \frac{22 \cdot 24 \cdot 24}{21 \cdot 23 \cdot 25} \cdots,$$

$$\sqrt{2 + \sqrt{2}} = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22}{1 \cdot 7 \cdot 9 \cdot 15 \cdot 17} \cdots,$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{3 \cdot 3 \cdot 6 \cdot 6 \cdot 9 \cdot 9}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 10} \cdots.$$
as well as the factorization \((3) = (9) \times (11)\) of Catalan's product for
\[
\sqrt{2} = \sqrt{2} - \sqrt{2} \cdot \sqrt{2} + \sqrt{2}.
\]
In particular, formulas (1), (3), (7), and (9) imply (6), (8), (11), and (10), by division.

On the other hand, the product (7) for 2 cannot be proved by simply squaring Catalan's product (3) for \(\sqrt{2}\). Much less can the square of the product (11) for \(\sqrt{2} + \sqrt{2}\) be obtained from (7) and (3) by addition!

We also give some new products of the same shape as Catalan's for \(e\).

**Theorem 2.** The following Catalan-type formulas hold:

\[
e = \left(\frac{2}{3}\right)^{1/2} \left(\frac{4}{5}\right)^{1/4} \left(\frac{6}{7}\right)^{1/8} \left(\frac{10}{11}\right)^{1/16} \left(\frac{12}{13}\right)^{1/32} \left(\frac{14}{15}\right)^{1/64} \cdots,
\]

\[
\sqrt{e} = 2 \left(\frac{2}{3}\right)^{1/2} \left(\frac{6}{7}\right)^{1/4} \left(\frac{10}{11}\right)^{1/8} \left(\frac{12}{13}\right)^{1/16} \left(\frac{14}{15}\right)^{1/32} \cdots,
\]

\[
\frac{e^{3/2}}{2} = \left(\frac{4}{3}\right)^{1/2} \left(\frac{8}{7}\right)^{1/4} \left(\frac{10}{9}\right)^{1/8} \left(\frac{12}{11}\right)^{1/16} \left(\frac{14}{13}\right)^{1/32} \cdots,
\]

\[
\frac{e^{2/3}}{\sqrt{3}} = \left(\frac{3}{2}\right)^{1/3} \left(\frac{6}{5}\right)^{1/6} \left(\frac{9}{8}\right)^{1/12} \left(\frac{12}{11}\right)^{1/24} \left(\frac{15}{14}\right)^{1/48} \cdots.
\]

Note that Pippenger's, Catalan's, and our first three products for \(e\) are related by the factorizations

\[(5)^2 = (4) \times (15) = \frac{1}{2} (16) \times (17).
\]

Also, notice that the ``geometric-series product''

\[2 = 2^{1/2} \cdot 2^{1/4} \cdot 2^{1/8} \cdots
\]

allows the factorization \((4) = (15) \times (20)^2\). Thus Catalan’s and Pippenger’s products are related by the surprisingly simple formula \((4) = (5) \times (20)^2\).

We built the product (18) for \(e^{2/3} / \sqrt{3}\) from the product (12) for \(2\pi / 3\sqrt{3}\) by analogy with Pippenger’s construction of his formula (5) for \(e/2\) from Wallis’s formula (1) for \(\pi/2\). Just as (13) generalizes (1) and (12), so too we conjecture that one can generalize (5) and (18) to a product for a power of \(e^{1/k}\).
Formulas (7), (3), and (1) for 2, $\sqrt{2}$, and $\pi$ show that a Wallis-type infinite product (in particular, a factor of Wallis's product) can be rational, algebraic irrational, or transcendental, respectively. As for Catalan-type infinite products, we know of no algebraic example; in fact, the known ones are all algebraically dependent on $e$.

Other types of infinite products can be found for $\pi$ in [2, p. 94], [3, p. 55], [5, Section 1.4.2], [7], [11], [19], for $e$ in [6], [8], [18], and for both $\pi$ and $e$ in [9], [10], [13].

We conclude the Introduction with some remarks on the use of modern symbolic algorithms, as implemented in computer algebra systems such as Mathematica. They can compute symbolically not only particular Wallis-type products such as (1) and (12), but also general ones like (13). This is useful for generating new Wallis-type formulas. On the other hand, Mathematica (version 7.0.0) cannot evaluate the Catalan-type infinite products symbolically. (Surprisingly, neither can it calculate them numerically. The reason given is "overflow.") In any case, such a computer-assisted approach is only intended for mechanical discovery and does not provide a rigorous mathematical proof. Thus the method is advisory, and provides us with a modern scope within which we can do more mathematics.

2. PROOF OF THEOREM 1. In view of the factorizations (14) and the fact that (12) is a special case of (13), it suffices to prove (7), (9), and (13). We use the following classical formula [17, Section 12.13], which is a corollary of the Weierstrass infinite product for the gamma function.

If $k$ is a positive integer and $a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_k$, where the $a_j$ and $b_j$ are complex numbers and no $b_j$ is zero or a negative integer, then

$$
\prod_{n=0}^{\infty} \frac{(n + a_1)(n + a_2)\cdots(n + a_k)}{(n + b_1)(n + b_2)\cdots(n + b_k)} = \frac{\Gamma(b_1)\cdots\Gamma(b_k)}{\Gamma(a_1)\cdots\Gamma(a_k)}.
$$

(21)

To prove (7), we write the product as

$$
\prod_{n=0}^{\infty} \frac{(8n + 2)(8n + 4)(8n + 4)(8n + 6)}{(8n + 1)(8n + 3)(8n + 5)(8n + 7)} = \prod_{n=0}^{\infty} \frac{(n + (1/8))(n + (1/2))(n + (1/2))(n + (3/4))}{(n + (1/8))(n + (3/8))(n + (5/8))(n + (7/8))}.
$$

Then from (21) and Euler’s reflection formula [17, Section 12.14]

$$
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},
$$

(22)

the product is equal to

$$
\frac{\Gamma(1/8)\Gamma(3/8)\Gamma(5/8)\Gamma(7/8)}{\Gamma(1/4)\Gamma(1/2)\Gamma(1/2)\Gamma(3/4)} = \frac{\sin(\pi/4)\sin(\pi/2)}{\sin(\pi/8)\sin(3\pi/8)} = \frac{1/\sqrt{2}}{\sqrt{4 + \sqrt{8} - \sqrt{4 - \sqrt{8}}}} = 2,
$$

$$
as claimed. (For the values of sine, see for example [15, Chap. 32].)

Similarly, the product in (9) can be written

\[
\begin{align*}
2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 & \cdots = \prod_{n=0}^{\infty} \frac{(8n+2)(8n+6)}{3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 21} = \prod_{n=0}^{\infty} \frac{(n + (1/4))(n + (3/4))}{(n + (3/8))(n + (5/8))},
\end{align*}
\]

and by (21) and (22) its value is

\[
\frac{\Gamma(3/8)\Gamma(5/8)}{\Gamma(1/4)\Gamma(3/4)} = \frac{\sin(\pi/4)}{\sin(3\pi/8)} = \frac{1/\sqrt{2}}{1/\sqrt{4} - \sqrt{2}} = \sqrt{2} - \sqrt{2},
\]
as desired.

Finally, to prove (13) we write the product as

\[
\prod_{n=0}^{\infty} \frac{(Kn)^2}{(Kn-1)(Kn+1)} = \prod_{n=0}^{\infty} \frac{(K(n+1))^2}{(K(n+1)-1)(K(n+1)+1)} = \prod_{n=0}^{\infty} \frac{(n+1)^2}{(n+1 - (1/K))(n+1 + (1/K))}.
\]

Then by (21), (22), and the factorial property \(\Gamma(x+1) = x\Gamma(x)\) (see [17, Section 12.12]), together with the value \(\Gamma(1) = 0! = 1\), the product is equal to

\[
\frac{\Gamma(1-(1/K))\Gamma(1+(1/K))}{\Gamma(1)^2} = \Gamma\left(1 - \frac{1}{K}\right) \frac{1}{K} \Gamma\left(1 + \frac{1}{K}\right) = \frac{\pi/K}{\sin(\pi/K)},
\]
as was to be shown. This completes the proof of Theorem 1.

In a similar way, one can also derive Catalan's products (2) and (3) for \(\pi/2\sqrt{2}\) and \(\sqrt{2}\) (compare [1, Section 4.4]). For a different way to prove (2), see [14, Example 7].

3. PROOF OF THEOREM 2. By the factorizations (19), to prove the first three formulas it suffices to prove (16). Equivalently, we show that

\[
\left(\frac{2}{3}\right)^{1/2} \left(\frac{6 \cdot 6}{5 \cdot 7}\right)^{1/4} \left(\frac{10 \cdot 10 \cdot 14 \cdot 14}{9 \cdot 11 \cdot 13 \cdot 15}\right)^{1/8} \cdots = \frac{\sqrt{6}}{2}.
\]

Notice the cancellations that occur in computing the partial products, which are simply

\[
\frac{2^{1/2}}{3^{1/2}}, \frac{2^{2/2}}{(5 \cdot 7)^{1/4}}, \frac{2^{3/2}}{(9 \cdot 11 \cdot 13 \cdot 15)^{1/8}} \cdots \frac{2^{n/2}}{(2^n + 1)(2^n + 3) \cdots (2^n + 1)^{1/2^n}} \cdots
\]

Using factorials, we can write the \(n\)th partial product as
\[2^n! \left( \frac{2^n!(2^n + 2)(2^n + 4) \cdots 2^{n+1}}{2^{n+1}} \right)^{1/2^n} = 2^n! \left( \frac{2^n!2^{n+1} - 1}{2^{n+1}} \right)^{1/2^n}\]

\[= 2 \frac{n+1}{2} \left( \frac{(2^n!)^2}{2^{n+1}} \right)^{1/2^n}.
\]

After applying Stirling’s formula to each factorial, the resulting expression simplifies to \(\sqrt{e/2}\), establishing (23).

To evaluate the infinite product in (18), note first that, for \(k \geq 2\), its \(k\)th factor is the \(1/3^k\) power of the product of \(2(3^{k-1} - 3^{k-2}) = 4 \cdot 3^{k-2}\) fractions. Factoring powers of 3 out of their numerators, we see that we can write the infinite product as

\[
\left( \frac{3^1}{2} \right)^{1/3} \left( \frac{3^8}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 8} \right)^{1/9} \left( \frac{3^{41}}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 10 \cdot 11 \cdot 13 \cdot 14 \cdot 16 \cdot 17 \cdot 19 \cdot 20 \cdot 22 \cdot 23 \cdot 25 \cdot 26} \right)^{1/27},
\]

and that by induction the exponents 1, 5, 17, … are equal to

\[2 \cdot 3^{k-1} - 1 = 4 \cdot 3^{k-2} + 2 \cdot 3^{k-2} - 1,
\]

for \(k = 1, 2, 3, \ldots\). Then the partial products are

\[
\left( \frac{3^1}{2} \right)^{1/3}, \left( \frac{3^8}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 8} \right)^{1/9}, \left( \frac{3^{41}}{2 \cdot 4 \cdot 5 \cdot 7 \cdot 8 \cdot 10 \cdot 11 \cdot 13 \cdot 14 \cdot 16 \cdot 17 \cdot 19 \cdot 20 \cdot 22 \cdot 23 \cdot 25 \cdot 26} \right)^{1/27},
\]

where the exponents 1, 8, 41, … are given by the formula

\[E_n := \sum_{k=1}^{n} 3^{n-k}(2 \cdot 3^{k-1} - 1) = 2n \cdot 3^{n-1} - \frac{1}{2}(3^n - 1),
\]

for \(n = 1, 2, 3, \ldots\). Thus the \(n\)th partial product is

\[
\left( \frac{3^{E_n}}{2 \cdot 4 \cdot 5 \cdot 7 \cdots (3^n - 4)(3^n - 2)(3^n - 1)} \right)^{1/3^n} = \left( \frac{3^{E_n}}{(3^n)!} \right)^{1/3^n} = 3^{n/3} \left( \frac{3^{E_n}3^{n-1}!}{3^n!} \right)^{1/3^n},
\]

which by Stirling’s formula is asymptotic to \(e^{2/3}/3^{1/2}\). This proves (18) and completes the proof of Theorem 2.

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