ON THE RATIONALITY OF THE SINGULARITIES OF THE
\(A_2\)-LOCI

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1. Introduction

In the last twenty years, a significant progress has been made in calculating Thom polynomials of contact singularities, see [9], [3], [15], and [2]. We note that the residue formulas for \(A_n\)-singularities obtained in [3] are reminiscent of the Jeffrey-Kirwan residue for reductive quotients [16]. According to Boutot [12], if a variety has rational singularities, then so does its quotient by the action of a reductive group. Thus, the natural question is whether the \(A_n\)-loci can be presented as a reductive quotient and, in particular, if they have rational singularities. The same question appears in the recent work of Rimányi and Szenes [10] on the \(K\)-theoretic invariants of the same loci. In this paper we show that, in general, the \(A_2\)-loci have singularities worse than rational, and therefore they can not be presented as a GIT quotient of a smooth variety with respect to a reductive group.

We begin with recalling some facts about smooth resolutions and a brief introduction to singularity theory.

Let \(X\) be an affine variety. If \(Y\) is smooth and there exists a proper birational map \(f: Y \to X\), then we say that \(Y\) is a smooth resolution of \(X\).

Proposition 1.1. The cohomology groups \(H^i(Y, O_Y)\) do not depend on the smooth resolution \(Y\), i.e. are invariants of \(X\).

This fact follows from the Elkik-Fujita Vanishing Theorem [1].

Proposition 1.2. \(H^0(X, O_X) = H^0(Y, O_Y)\) if and only if \(X\) is normal.

If \(X\) is not normal, there exists a unique normalisation of \(X\) – normal affine variety \(\tilde{X}\). In this case \(H^0(\tilde{X}, O_{\tilde{X}}) = H^0(Y, O_Y)\), but \(H^0(\tilde{X}, O_{\tilde{X}}) \neq H^0(X, O_X)\).

The proof of the proposition above is based on the universal property of the normalization and Zariski’s Main Theorem [6].

Definition 1.3. Let \(X\) be a normal affine variety, then \(X\) has rational singularities if \(H^i(Y, O_Y) = 0\) for all \(i > 0\).

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Suppose a compact Lie group $G$ acts on the affine space $\mathbb{A}^N$. Let $X \subset \mathbb{A}^N$ be a $G$-invariant subvariety. $Y$ is called an equivariant smooth resolution of $X$ if $Y$ is smooth, $G$ acts on $Y$, and the map $f: Y \to X$ is proper birational and $G$-equivariant.

Let $T$ be the maximal torus of $G$. One of the natural questions that arises in [10] is whether $\chi[H^0(X, \mathcal{O}_X)](t) = \chi[\sum (-1)^i H^i(Y, \mathcal{O}_Y)](t)$, $t \in T$. Note that while $X$ is an affine variety and therefore $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, this is not necessarily true for $H^i(Y, \mathcal{O}_Y)$.

**Proposition 1.4.** Let $G$ be a compact Lie group acting on $\mathbb{A}^N$. Let $X \subset \mathbb{A}^N$ be a $G$-invariant subvariety, and let $Y$ be its smooth $G$-equivariant resolution. Let $T$ be the maximal torus of $G$. The equality $\chi[H^0(X, \mathcal{O}_X)](t) = \chi[\sum (-1)^i H^i(Y, \mathcal{O}_Y)](t)$, $t \in T$ holds if and only if $X$ has rational singularities.

In this paper we study whether certain singularity loci have rational singularities. To give the definition of the main object of this paper, the $A_2$-locus, we recall the necessary notions of singularity theory. For a more detailed introduction see [3] or [11].

Denote by $x_1, \ldots, x_n$ coordinates on $\mathbb{C}^n$. We introduce the notation $J(n) = \{ h \in \mathbb{C}[[x_1, \ldots, x_n]] \mid h(0) = 0 \}$ for the algebra of power series without constant term, $(\mathbb{C}^{d+1})$ for the ideal generated by monomials in $x_1, \ldots, x_n$ of degree $d + 1$ and $J_d(n) = J(n)/(\mathbb{C}^{d+1})$ for the space of $d$-jets of holomorphic functions near the origin.

Let $J_d(n, k)$ be the space of $d$-jets of holomorphic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$:

$$J_d(n, k) = \text{Hom}(\mathbb{C}^k, J_d(n)).$$

An element of this space can be thought of as a $k$-tuple of elements of $J_d(n)$:

$$J_d(n, k) \cong \{(P_1, \ldots, P_k) \mid P_i \in J_d(n)\}.$$ 

$J_d(n, k)$ is a finite-dimensional complex vector space equipped with $\text{Gl}(n) \times \text{Gl}(k)$-action. In this paper we will consider $n \leq k$.

We will call an algebra $N$ nilpotent if it is finite dimensional and if there exists a natural number $m$ such that the product of each $m$ elements of the algebra vanishes, that is, $N^m = 0$. $J_d(n)$ is nilpotent: $(J_d(n))^{d+1} = 0$, the algebra $J_d(1)$ is often denoted by $A_d = t\mathbb{C}[t]/t^{d+1}$. 

Definition 1.5. An algebra $C$ is $(1,1,\ldots,1)$-filtered if $C$ has an increasing finite sequence of subspaces $\{0\} \subset F_m \subset F_{m-1} \subset \ldots \subset F_1 \subset F_0 = C$ such that $F_i \cdot F_j \subset F_{i+j}$ and $\dim F_i/F_{i+1} = 1$.

Nilpotent algebras have a natural filtration: $\{0\} \subset N^{m-1} \subset N^{m-2} \subset \ldots \subset N^2 \subset N$. In case of $A_d$, this filtration is a $(1,1,\ldots,1)$-filtration.

Definition 1.6. $A_d$-singularity locus is given by

$$\Theta_{A_d}^{n,k} = \{(P_1, \ldots, P_k) \in J_d(n,k) \mid J_d(n)/\langle P_1, \ldots, P_k \rangle \cong A_d\}.$$ 

$\Theta_{A_d}^{n,k}$ is a $\text{Gl}(n) \times \text{Gl}(k)$-invariant affine subvariety in $J_d(n,k)$.

This paper is devoted to the study of the rationality of the singularities of $\Theta_{A_d}^{n,k}$.

Let us briefly look at a simpler case, the $A_1$-locus:

$$\Theta_{A_1}^{n,k} = \{M \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^k) \mid \text{rk} M < n\},$$

i.e. for every $M \in \Theta_{A_1}^{n,k}$ there exists a non-zero eigenvector $v \in \mathbb{C}^n$ such that $Mv = 0$.

Proposition 1.7. The space

$$\{(M, v) \mid Mv = 0, \ M \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^k), \ v \in \mathbb{C}^n\} \subset \text{Hom}(\mathbb{C}^n, \mathbb{C}^k) \times \mathbb{P}^{n-1}$$

is an equivariant smooth resolution of $\Theta_{A_1}^{n,k}$.

This space can be understood as follows: let us fix an element $v \in \mathbb{P}^{n-1}$ and describe the set $\{M \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^k) \mid Mv = 0\}$.

There is a tautological sequence of vector bundles on $\mathbb{P}^{n-1}$:

$$\mathcal{O}(-1) = L \longrightarrow \mathbb{C}^n \longrightarrow Q \longrightarrow \mathbb{P}^{n-1}$$

We can apply $\text{Hom}(\ast, \mathbb{C}^k)$ to it and obtain the following sequence:

$$\text{Hom}(Q, \mathbb{C}^k) \longrightarrow \text{Hom}(\mathbb{C}^n, \mathbb{C}^k) \longrightarrow \text{Hom}(L, \mathbb{C}^k) \longrightarrow \mathbb{P}^{n-1}$$

The map $\text{Hom}(\mathbb{C}^n, \mathbb{C}^k) \to \text{Hom}(L, \mathbb{C}^k)$ can be interpreted as the evaluation map $M \mapsto Mv$ for a fixed $v \in \mathbb{P}^{n-1}$. Its kernel is exactly $\text{Hom}(Q, \mathbb{C}^k)$.

The equivariant smooth resolution of $\Theta_{A_1}^{n,k}$ defined above may be presented as the following vector bundle:
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$\text{Hom}(Q, C^k) \rightarrow \Theta_{A_2}^{n,k} \rightarrow \mathbb{P}^{n-1}$

It is well-known that $\Theta_{A_1}^{n,k}$ has rational singularities. In this paper we study the rationality of the singularities of $\Theta_{A_2}^{n,k}$ and prove the following theorems.

**Theorem 1.8.** $\Theta_{A_2}^{n,k}$ in general can have singularities worse than rational.

**Theorem 1.9.** $\Theta_{A_2}^{n,n}$ has rational singularities.

Before proving the main theorems, we recall the explicit construction for the equivariant smooth resolution of $\Theta_{A_2}^{n,k}$, the Borel-Weil-Bott theorem, and demonstrate the spectral sequences technique that will allow us to study the rationality of the singularities of the $A_2$-loci.

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2. Preliminaries

2.1. Equivariant smooth resolution of the $A_2$-locus. In this section we recall an explicit construction for the equivariant smooth resolution of the $A_2$-locus following [2]. The general case is discussed in [2] and [3].

Before we present the equivariant smooth resolution of $\Theta_{A_2}^{n,k}$, we need to introduce some preliminary notions.

**Definition 2.1.** The curvilinear Hilbert scheme of order 2 is defined as follows:

$$\text{Hilb}_{A_2}(\mathbb{C}^n) \cong \{I \subset J_2(n) \mid J_2(n)/I \cong A_2\}.$$  

Each ideal $I \in \text{Hilb}_{A_2}(\mathbb{C}^n)$ comes with the tautological sequence:

$$I \rightarrow J_2(n) \rightarrow N \cong J_2(n)/I$$

To construct a smooth equivariant resolution of $\Theta_{A_2}^{n,k}$ we start with the following vector bundle:
The fiber over $I \in \text{Hilb}_{A_2}(\mathbb{C}^n)$ is the space of all $k$-tuples of elements of $I$. The set of $k$-tuples of elements of $I$ that generate $I$ is Zariski open in $\text{Hom}(\mathbb{C}^k, I)$ and the projection $\text{Hom}(\mathbb{C}^k, I) \twoheadrightarrow J_d(n, k) \supset \Theta_{A_2}^{n,k}$ is proper.

This vector bundle is not a smooth equivariant resolution of $\Theta_{A_2}^{n,k}$ because $\text{Hilb}_{A_2}(\mathbb{C}^n)$ is not smooth. The next step is to find a smooth equivariant resolution of $\text{Hilb}_{A_2}(\mathbb{C}^n)$.

Since every $I \in \text{Hilb}_{A_2}(\mathbb{C}^n)$ is equipped with the tautological sequence mentioned above, we can rewrite $\text{Hilb}_{A_2}(\mathbb{C}^n)$ as

$$\text{Hilb}_{A_2}(\mathbb{C}^n) = \left\{ f : J_2(n) \to N \mid \dim N = 2, \quad f - \text{surj. alg. homomorphism}\right\} / \sim$$

The equivalence relation is defined as follows: $f \sim f'$ if the diagram commutes:

We will be interested in $(1,1)$-filtered 2-dimensional nilpotent algebras. There are two different types of them:

- $A_2$ with the natural $(1,1)$-filtration: $A_2^2 \subset A_2$,
- algebra $N$ generated by two elements, such that the product of any two elements of $N$ is 0. This algebra does not have a natural $(1,1)$-filtration, so we introduce an artificial $(1,1)$-filtration $F_1 \subset N$, where $F_1$ is any line in $N$.

Let us introduce the notation for filtered algebra homomorphisms. Suppose filtered algebras $N$ and $C$. We will denote a homomorphism compatible with the filtrations on $N$ and $C$ by

$$f : N \xrightarrow{\Delta} C$$

**Proposition 2.2.** The smooth equivariant resolution of $\text{Hilb}_{A_2}(\mathbb{C}^n)$ is given by

$$\tilde{\text{Hilb}}_{A_2}(\mathbb{C}^n) = \left\{ f : J_2(n) \xrightarrow{\Delta} N \mid N - 2\text{-dim. } (1,1)\text{-filt., } f - \text{surj.}\right\} / \sim,$$
The equivalence is taken up to a filtered algebra isomorphism:

![Diagram]

The following vector bundle is a smooth equivariant resolution of the $A_2$-locus:

$$\text{Hom}(C^k, I) \longrightarrow \Theta_{A_2}^{n,k}$$

$$\text{Hilb}_{A_2}(C^n)$$

Now we need to find a simpler interpretation of this resolution. Let $g$ be the inverse of the canonical map $J_2(n) \to J_2(n)/(J_2(n))^2 \cong \mathbb{C}^n$:

$$g: \mathbb{C}^n \to J_2(n)$$

Let us denote its image by $\text{Im}(g) = E^*$. $E^*$ is the linear part of $J_2(n)$.

Let $A^\Delta$ be a 2-dimensional algebra equipped with the $(1,1)$-filtration and $f \in \text{Hilb}_{A_2}(C^n)$. We can define two natural maps

$$\psi_1: E \to A^\Delta, \quad \psi_1 = f|_{E^*}$$

$$\psi_2: \text{Sym}^2 A^\Delta \to A^\Delta$$

**Proposition 2.3.** The linear map $\psi_1 \oplus \psi_2: E^* \oplus \text{Sym}^2 A^\Delta \to A^\Delta$ is surjective.

**Proposition 2.4.** Let $N$ be a 2-dimensional filtered vector space. $\text{Hilb}_{A_2}(C^n)$ is in one-to-one correspondence with the set of isomorphism classes of pairs $(\psi_1, \psi_2)$, where $\psi_2: \text{Sym}^2 N \to N$ is a map giving $N$ an associative commutative algebra structure and $\psi_1: (C^n)^* \to N$ is a linear map such that $\psi_1 \oplus \psi_2$ is surjective. Pairs $(\psi_1, \psi_2)$ are taken up to filtered algebra isomorphism.

Let us describe $\text{Hilb}_{A_2}(C^n)$ using this correspondence.

Suppose $N$ be a 2-dimensional vector space with a filtration $N_2 \subset N$, where $N_2$ is a line in $N$. 
The kernel of this map is defined by $\text{Ker}(\psi_1) = \{ V \subset (\mathbb{C}^n)^* \mid \dim V = n - 1 \} = \mathbb{P}^{n-1}(\mathbb{C}^n)^* \cong \mathbb{P}^{n-1}$. Let us denote $\mathcal{O}(-1)$ over $\mathbb{P}^{n-1}$ by $L_1$ and the quotient bundle by $Q_1$.

The kernel of $\psi_1 \oplus \psi_2$ is then a codimension 2 subspace in $\text{Sym}^2 L_1 \oplus (\mathbb{C}^n)^* \cong L_1^2 \oplus (\mathbb{C}^n)^*$, such that its projection is of codimension 1 in $(\mathbb{C}^n)^*$, that is:

$$\mathbb{P}^{n-1}(Q_1^* \oplus (L_1^*)^2) \cong \mathbb{P}(Q_1 \oplus L_1^2)$$

Let us fix a point $a$ in $\mathbb{P}^{n-1}$. The fiber over this point is $\mathbb{P}((Q_1 \oplus L_1^2)|_a) = \mathbb{P}V_a$. Let $V$ be an $n$-dimensional complex vector space. We have the following tautological sequence on $\mathbb{P}V_a$:

$$\mathcal{O}(-1) = L_2 \rightarrow V_a \rightarrow Q_2$$

This description allows us to present the smooth equivariant resolution of the $A_2$-locus in the following form:

$$\text{Hom} \left( (\mathbb{C}^k, \text{Sym}^2 \mathbb{C}^n \oplus Q_1) \right) \rightarrow \mathcal{O}_{A_2}^{n,k}$$

2.2. **The Borel-Weil-Bott theorem.** Let $V$ be an $n$-dimensional complex vector space. In this paper we use the Borel-Weil-Bott theorem to compute the cohomology of $\text{Gl}(V)$-equivariant vector bundles on $\mathbb{P}V$. 
The irreducible representations of $\text{Gl}(V)$ are parametrized by their highest weights—non-increasing integer partitions $\lambda$ of length $n$ (we allow the entries to be equal to 0): $\lambda_1 \geq \lambda_2 \geq \lambda_n \geq 0$. We will denote the irreducible representation of $\text{Gl}(V)$ of highest weight $\lambda$ by $\Sigma^\lambda V$.

Consider the canonical sequence of vector bundles on $\mathbb{P}V$:

$$
\begin{array}{c}
\mathcal{O}(-1) = L \\
V \\
Q \\
\mathbb{P}V
\end{array}
$$

We will be interested in computing the cohomology of $\text{Gl}(V)$-equivariant vector bundles of the form $\Sigma^\lambda Q \otimes L^m$ on $\mathbb{P}V$. Following the argument in [4], a vector bundle of this form may be presented as a pushforward of the corresponding line bundle on the flag variety of $\text{Gl}(V)$. Thus, we may compute its cohomology using the following interpretation of the Borel-Weil-Bott theorem.

**Theorem 2.5** (The Borel-Weil-Bott theorem, [4]). Consider an irreducible $\text{Gl}(V)$-equivariant vector bundle $\Sigma^\lambda Q \otimes L^m$ on $\mathbb{P}V$. Denote by $(\lambda, m)$ the concatenation of $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ and $m$, and by $\rho = (n, n-1, \ldots, 1)$ the half-sum of the positive roots of $\text{Gl}(V)$.

Consider $(\lambda, m) + \rho = (\lambda_1 + n, \lambda_2 + n - 1, \ldots, \lambda_{n-1} + 2, m + 1)$.

If two entries of $(\lambda, m) + \rho$ are equal, then

$$
H^i(\mathbb{P}V, \Sigma^\lambda Q \otimes L^m) = 0 \text{ for all } i.
$$

If all entries of $(\lambda, m) + \rho$ are distinct, then there exists a unique permutation $\sigma$ such that $\sigma((\lambda, m) + \rho)$ is strictly decreasing, i.e. dominant. The length of this permutation, $l(\sigma)$, is the number of strictly increasing pairs of elements of $(\lambda, m) + \rho$.

Then $H^i(\mathbb{P}V, \Sigma^\lambda Q \otimes L^m) = \begin{cases} 
\Sigma^{\sigma((\lambda, m) + \rho) - \rho} V & \text{if } i = l(\sigma) \\
0 & \text{otherwise}.
\end{cases}$

**Example 2.1.** Let us compute $H^i(\mathbb{P}^3, Q \otimes \text{Sym}^2 Q \otimes L^5)$.

First, we need to decompose $Q \otimes \text{Sym}^2 Q$ into the direct sum of irreducible representations. The algorithm is the same as in decomposing the product of two corresponding Schur polynomials into a sum of Schur polynomials, for the details see [7] or [8].

In the case of $Q \otimes \text{Sym}^2 Q$, we obtain the following:

$$
Q \otimes \text{Sym}^2 Q = \Sigma^{(1,0,0)} Q \otimes \Sigma^{(2,0,0)} Q = \Sigma^{(3,0,0)} Q + \Sigma^{(2,1,0)} Q.
$$
To compute the cohomology groups of the initial sheaf, we compute the cohomology groups of both irreducible summands:

\[ H^i(\mathbb{P}^3, Q \otimes \text{Sym}^2 Q \otimes L^5) = H^i(\mathbb{P}^3, \Sigma^{(3,0,0)} Q \otimes L^5) \oplus H^i(\mathbb{P}^3, \Sigma^{(2,1,0)} Q \otimes L^5). \]

Applying the Borel-Weil-Bott theorem to \( \Sigma^{(3,0,0)} Q \otimes L^5 \), we first construct the sequence \((\lambda,m)\): here \( \lambda = (3,0,0) \) and \( m = 5 \). We see that \((\lambda,m) + \rho = (3,0,0,5) + (4,3,2,1) = (7,3,2,6) \) has no repetitions. The unique permutation making \((7,3,2,6)\) decreasing is \( \sigma = (2,3,4) \). Since there are two increasing pairs in \((7,3,2,6)\), namely, \((3,6)\) and \((2,6)\), \( l(\sigma) = 2 \). Finally, \( \sigma((\lambda,m) + \rho) - \rho = (7,6,3,2) - (4,3,2,1) = (3,3,1,1) \), so the only non-zero cohomology group is

\[ H^2(\mathbb{P}^3, \Sigma^{(3,0,0)} Q \otimes L^5) = \Sigma^{(3,3,1,1)} \mathbb{C}^4. \]

The second irreducible summand is \( \Sigma^{(2,1,0)} Q \otimes L^5 \). Here we obtain \((\lambda,m) + \rho = (2,1,0,5) + (4,3,2,1) = (6,4,2,6) \) – there are repetitions, so

\[ H^i(\mathbb{P}^3, \Sigma^{(2,1,0)} Q \otimes L^5) = 0 \text{ for all } i. \]

The final answer is

\[ H^i(\mathbb{P}^3, Q \otimes \text{Sym}^2 Q \otimes L^5) = \begin{cases} \Sigma^{(3,3,1,1)} \mathbb{C}^4 & \text{if } i = 2 \\ 0 & \text{if } i \neq 2 \end{cases}. \]

3. Main results

In this section we show that \( \tilde{\Theta}^{n,m}_{A_2} \), the normalization of \( \Theta^{n,m}_{A_2} \), has rational singularities, and give an example, where \( \Theta^{n,k}_{A_2} \) has singularities worse than rational.

Consider the quasi-projective variety \( Y \) – Kazarian’s smooth resolution of \( \Theta^{n,k}_{A_2} \):

\[
\begin{array}{c}
\text{Hom} \left( \mathbb{C}^k, \frac{\text{Sym}^2 \mathbb{C}^n \oplus Q_1}{L_2} \right) \\
\downarrow \\
\mathbb{P}(Q_1 \oplus L_1^2) \\
\downarrow \\
\mathbb{P}^{n-1}
\end{array}
\]

\( Y = \text{Hom} \left( \mathbb{C}^k, \frac{\text{Sym}^2 \mathbb{C}^n \oplus Q_1}{L_2} \right) \rightarrow \Theta^{n,k}_{A_2} \)

\( p_1 \) and \( p_2 \)

By definition, \( \tilde{\Theta}^{n,k}_{A_2} \) has rational singularities if \( H^i(Y, \mathcal{O}_Y) = 0 \) for all \( i > 0 \). We will compute these cohomology groups step by step, by pushing forward along the tower.

Fix a point \( a \) in \( \mathbb{P}^{n-1} \), the fiber over this point is \( p_2^{-1}(a) = \mathbb{P}(Q_1 \oplus L_1^2)|_a \cong \mathbb{P}V_a \), where \( V_a \) is an \( n \)-dimensional complex vector space. Let us also denote the constant sheaf \( (\text{Sym}^2 \mathbb{C}^n \oplus Q_1)|_a \) on \( \mathbb{P}V_a \) by \( W \).
Since the fiber over a point \( b \) in \( PV_a \), \( \left( \text{Hom} \left( C^k, \frac{\text{Sym}^{n-m} \mathcal{O}_a}{L_2} \right) \right) \bigg|_b \), is affine, we have \( H^i(Y, \mathcal{O}_Y) = H^i(PV_a, (p_1)_*, \mathcal{O}_Y) \). Moreover, the \( \mathbb{C}^* \)-action on the fiber allows us to decompose \((p_1)_*, \mathcal{O}_Y\) into homogeneous components:

\[
(p_1)_* \mathcal{O}_Y = \mathcal{O}_Y \big|_{p_1^{-1}(b)} \cong \bigoplus_i \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right)
\]

This decomposition leads to the following identity on the level of cohomology:

\[
H^i(Y, \mathcal{O}_Y) = H^i(PV_a, (p_1)_*, \mathcal{O}_Y) = \bigoplus_i H^i \left( PV_a, \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right) \right).
\]

Let us compute \( H^i \left( PV_a, \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right) \right) \). We start with the Koszul resolution \( \mathcal{O}_Y \) of \( \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right) \):

\[
\Lambda^i(L_2 \otimes C^k) \rightarrow \Lambda^i-1(L_2 \otimes C^k) \otimes \text{Sym}^1(W \otimes C^k) \rightarrow \ldots
\]

\[
\ldots \rightarrow \Lambda^i-1(L_2 \otimes C^k) \otimes \text{Sym}^i(W \otimes C^k) \rightarrow \ldots
\]

\[
\ldots \rightarrow \Lambda^2(L_2) \otimes \text{Sym}^{i-1}(W \otimes C^k) \rightarrow \text{Sym}^i(W \otimes C^k) \rightarrow \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right)
\]

We are interested in the case when \( l \) is sufficiently large. Note that since \( L_2 \) is a line bundle, \( \Lambda^i(L_2 \otimes C^k) \) vanishes for \( i > k \). Using these facts we can rewrite the resolution as follows.

**Resolution 1:**

\[
L_2^k \otimes \Lambda^k(C^k) \rightarrow \ldots \rightarrow L_2^{k-1} \otimes \Lambda^{k-1}(C^k) \otimes \text{Sym}^i(W \otimes C^k) \rightarrow \ldots
\]

\[
\ldots \rightarrow \text{Sym}^i(W \otimes C^k) \rightarrow \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right)
\]

According to the Borel-Weil-Bott theorem,

- \( H^{n-1}(PV_a, \mathcal{O}(-m)) \cong \text{Sym}^{m-n} V_a \otimes \det V_a \) if \( m - n \geq 0 \),
- \( H^{n-1}(PV_a, \mathcal{O}(-m)) \cong 0 \) if \( m - n < 0 \),
- \( H^i(PV_a, \mathcal{O}(-m)) \equiv 0 \) if \( i \neq n - 1 \).

This knowledge allows us to write down the Leray spectral sequence, which is a collection of indexed pages, i.e. tables with arrows pointing in the direction \((n,n-1)\) on the \( n \)-th page. The Leray spectral sequence allows us to obtain the cohomology groups of \( \text{Sym}^i \left( \frac{W \otimes C^k}{L_2 \otimes C^k} \right) \) by computing successive approximations. On the first page of the Leray spectral sequence, to each sheaf in the resolution above corresponds a column of its cohomology groups:
According to Leray’s theorem, the spectral sequence for the exact sequence converges to zero. The only term in the first column that can be cancelled by the other terms in the spectral sequence is the term in the 0-th line. This means that $H^i \left( \mathbb{P} V_a, \text{Sym}^l \left( \frac{W \otimes C_k}{L_2 \otimes C_k} \right) \right)$ vanishes for $i > 0$.

Applying the pushforward $(p_2)_*$, we obtain

$$H^i(Y, O_Y) = H^i \left( \mathbb{P}^{n-1}, H^0 \left( \mathbb{P} V_a, \text{Sym}^l \left( \frac{W \otimes C_k}{L_2 \otimes C_k} \right) \right) \right).$$

Let us construct the resolution of $H^0 \left( \text{Sym}^l \left( \frac{W \otimes C_k}{L_2 \otimes C_k} \right) \right)$. In the spectral sequence above, whatever remains in the line number $n - 1$ after the first page goes exactly to $\text{Sym}^l(W \otimes C_k)$ in the line number 0 on the $n$-th page. This allows us to write down the following resolution:

$$\text{det} V_a \otimes \text{Sym}^{k-n} V_a \otimes \Lambda^k C_k \otimes \text{Sym}^{l-k}(W \otimes C_k) \rightarrow \ldots$$

$$\ldots \rightarrow \text{det} V_a \otimes \text{Sym}^{k-n-i} V \otimes \Lambda^{k-i} C_k \otimes \text{Sym}^{l-(k-i)}(W \otimes C_k) \rightarrow \ldots$$

$$\ldots \rightarrow \text{det} V_a \otimes \Lambda^n C_k \otimes \text{Sym}^{l-n}(W \otimes C_k) \rightarrow \text{Sym}^l(W \otimes C_k) \rightarrow H^0 \left( \text{Sym}^l \left( \frac{W \otimes C_k}{L_2 \otimes C_k} \right) \right)$$

Which can be presented in the following form.

**Resolution 2:**

$$\text{det} Q \otimes L_1^i \otimes \text{Sym}^{k-n} (Q_1 \oplus L_1^i) \otimes \Lambda^{k} C_k \otimes \text{Sym}^{l-k}((\text{Sym}^2 C^n \otimes Q_1) \otimes C_k) \rightarrow \ldots$$

$$\ldots \rightarrow \text{det} Q \otimes L_1^i \otimes \text{Sym}^{k-n-i} (Q_1 \oplus L_1^i) \otimes \Lambda^{k-i} C_k \otimes \text{Sym}^{l-(k-i)}((\text{Sym}^2 C^n \otimes Q_1) \otimes C_k) \rightarrow \ldots$$

$$\ldots \rightarrow \text{det} Q \otimes L_2^i \otimes \Lambda^n C_k \otimes \text{Sym}^{l-n}((\text{Sym}^2 C^n \otimes Q_1) \otimes C_k) \rightarrow \text{Sym}^l((\text{Sym}^2 C^n \otimes Q_1) \otimes C_k) \rightarrow H^0 \left( \text{Sym}^l \left( \frac{((\text{Sym}^2 C^n \otimes Q_1) \otimes C_k)}{L_2 \otimes C_k} \right) \right)$$

This allows us to formulate our first result.

**Theorem 3.1.** $\Theta_{A_2}^{n,n}$ has rational singularities.
Proof. If $k = n$ then Resolution 2 may be rewritten as follows:

\[
det Q \otimes L^2_1 \otimes \Lambda^k \mathbb{C}^k \otimes \text{Sym}^{l-k}((\text{Sym}^2 \mathbb{C}^k \oplus Q_1) \otimes \mathbb{C}^k) \rightarrow \\
\rightarrow \text{Sym}^l((\text{Sym}^2 \mathbb{C}^k \oplus Q_1) \otimes \mathbb{C}^k) \rightarrow \\
\rightarrow H^0 \left( \text{Sym}^l \left( \frac{(\text{Sym}^2 \mathbb{C}^k \oplus Q_1) \otimes \mathbb{C}^k}{L_2 \otimes \mathbb{C}^k} \right) \right)
\]

We will prove that, in the corresponding spectral sequence, there are no non-trivial terms above the 0-th line.

Lemma 3.2.

\[
\text{Sym}^N((\text{Sym}^2 \mathbb{C}^k \oplus Q_1) \otimes \mathbb{C}^k) = \\
= \bigoplus_{i=0}^{N} \left( \text{Sym}^{N-i}((\text{Sym}^2 \mathbb{C}^k \otimes \mathbb{C}^k)) \otimes \bigoplus_{(i_1,\ldots,i_k)} \text{Sym}^{i_1}Q_1 \otimes \cdots \otimes \text{Sym}^{i_k}Q_1 \right).
\]

Setting $N = l$, the lemma provides the decomposition of $\text{Sym}^l((\text{Sym}^2 \mathbb{C}^k \oplus Q_1) \otimes \mathbb{C}^k)$. The only non-constant sheaves here are the sheaves of the form $\text{Sym}^{i_1}Q_1 \otimes \cdots \otimes \text{Sym}^{i_k}Q_1$.

We decompose this tensor product into a sum of irreducible representations:

\[
\text{Sym}^{i_1}Q_1 \otimes \cdots \otimes \text{Sym}^{i_m}Q_1 = \bigoplus_{\lambda} a_{\lambda} \Sigma^{\lambda}Q_1,
\]

where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\sum \lambda_k = \sum i_j$, and $a_{\lambda}$ are non-negative integers.

Since there is no multiplication by a power of $L_1$ and $\lambda$ is already dominant, i.e. strictly decreasing, by the Borel-Weil-Bott theorem $H^i(\mathbb{P}^{n-1}, \text{Sym}^{i_1}Q_1 \otimes \cdots \otimes \text{Sym}^{i_k}Q_1) = 0$ for $i > 0$.

This proves that the term in the second line of the resolution \((*)\) does not have any higher cohomology.

However, the term in the first line of the resolution \((*)\) has $L^2_1$ as a multiplier. As before, we use the lemma above for $N = l - k$ to find the decomposition of this term. The non-trivial part in this case is the following:

\[
\det Q_1 \otimes L^2_1 \otimes \bigoplus_{\lambda} a_{\lambda} \Sigma^{\lambda}Q_1 = \det \mathbb{C}^n \otimes L_1 \otimes \bigoplus_{\lambda} a_{\lambda} \Sigma^{\lambda}Q_1.
\]

Let us apply the Borel-Weil-Bott theorem to $\Sigma^{\lambda}Q_1 \otimes L_1$:

\[
(\lambda_1, \ldots, \lambda_{n-1}, 1) + (n, \ldots, 1) = (\nu_1 + n, \ldots, \nu_{n-1} + 2, 2).
\]

Since $\nu_{n-1} \geq 0$, we either have a dominant sequence if $\nu_{n-1} > 0$, or a repetition if $\nu_{n-1} = 0$. In both cases there is no higher cohomology.
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So, there are no non-trivial entries in the corresponding Leray spectral sequence above the 0-th line, so $H^i(Y, \mathcal{O}_Y) = 0$ for $i > 0$, and $\widehat{\Theta}_{A_2}^{n,n}$ has rational singularities.

\begin{proof}
Consider the case $n = 5$, $k = 7$, $l = 7$.
We prove that $H^1(P^4, \text{Sym}^7 \left( \frac{(\text{Sym}^2 \mathcal{C}^5 \oplus \mathcal{Q}_1) \otimes \mathcal{C}^7}{L_2 \otimes \mathcal{C}^7} \right)) \not\cong 0$. In this particular case, Resolution 2 is the following:

\[
\det Q_1 \otimes L_1^2 \otimes \text{Sym}^2 (Q_1 \oplus L_1^2) \rightarrow \\
\rightarrow \det Q_1 \otimes L_1^2 \otimes (Q_1 \oplus L_1^2) \otimes \Lambda^6 \mathcal{C}^7 \otimes ((\text{Sym}^2 \mathcal{C}^5 \oplus Q_1) \otimes \mathcal{C}^7) \rightarrow \\
\rightarrow \det Q_1 \otimes L_1^2 \otimes \Lambda^5 \mathcal{C}^7 \otimes \text{Sym}^2 ((\text{Sym}^2 \mathcal{C}^5 \oplus Q_1) \otimes \mathcal{C}^7) \rightarrow \\
\rightarrow \text{Sym}^7 ((\text{Sym}^2 \mathcal{C}^5 \oplus Q_1) \otimes \mathcal{C}^7) \rightarrow \\
\rightarrow H^0 \left( \text{Sym}^7 \left( \frac{(\text{Sym}^2 \mathcal{C}^5 \oplus Q_1) \otimes \mathcal{C}^7}{L_2 \otimes \mathcal{C}^7} \right) \right) \\
\end{proof}

Consider the term in the first line of the resolution above.

\[
\det Q_1 \otimes L_1^2 \otimes \text{Sym}^2 (Q_1 \oplus L_1^2) = \det Q_1 \otimes L_1^2 \otimes (\text{Sym}^2 Q_1 \oplus Q_1 \otimes L_1^2 \oplus L_1^4) = \\
= \det Q_1 \otimes L_1^4 \oplus \det Q_1 \otimes L_1^2 \left( \text{Sym}^2 Q_1 \oplus Q_1 \otimes L_1^2 \right).
\]

Using the Borel-Weil-Bott theorem, one can easily check that

\[
H^4 \left( P^4, \det Q_1 \otimes L_1^6 \right) \not\cong 0,
\]

\[
H^0 \left( P^4, \text{Sym}^7 ((\text{Sym}^2 \mathcal{C}^5 \oplus Q_1) \otimes \mathcal{C}^7) \right) \not\cong 0,
\]

but all other terms of the resolution do not have any cohomology.

The corresponding Leray spectral sequence is the following:

\[
\begin{array}{ccc}
H^4 & \rightarrow & H^1 \\
\downarrow & & \downarrow \\
H^0 & \rightarrow & H^1
\end{array}
\]
Thus, we proved that
\[ H^1 \left( \mathbb{P}^4, \operatorname{Sym}^7 \left( \frac{(\operatorname{Sym}^2 \mathbb{C}^5 \oplus Q_1) \otimes \mathbb{C}^7}{L_2 \otimes \mathbb{C}^7} \right) \right) \neq 0, \]
and therefore \( \Theta_{A_2}^{5,7} \) has singularities worse than rational. \( \square \)

According to Boutot [12], the GIT quotient of a smooth variety with respect to a reductive group has rational singularities. Thus, we have the following corollary of the Theorem 3.3.

**Corollary 3.4.** \( \Theta_{A_2}^{n,k} \) can not be presented as a reductive quotient of a smooth variety.

For the recent results on the GIT quotient with respect to non-reductive groups, see the works of Kirwan and Bérczi [13], and Bérczi, Doran, Hawes and Kirwan [14].

**Remark 3.5.** In both Theorem 3.1 and Theorem 3.3 we consider the normalizations of the \( A_2 \)-loci. Let us show that the normalization is not redundant, i.e. that \( \Theta_{A_2}^{n,k} \) is not always normal.

Let \( V \) be a complex vector space equipped with the action of a compact Lie group \( G \), and let \( X \) be a closed \( G \)-invariant subvariety of \( V \). Suppose \( Y \) is a smooth \( G \)-equivariant resolution of \( X \).

Consider the following diagram:

\[ \begin{array}{ccc}
H^0(Y, \mathcal{O}_Y) & \xrightarrow{f} & H^0(V, \mathcal{O}_V) \\
\downarrow h & & \downarrow g \\
H^0(X, \mathcal{O}_X) & \xleftarrow{g} & \bigoplus_i \operatorname{Sym}^i V^* \\
\end{array} \]

We know that \( g \) is always surjective, and, according to Proposition 1.2, \( h \) is an isomorphism if and only if \( X \) is normal. Now, if \( f \) is not surjective, then \( h \) can not be an isomorphism, and therefore in this case \( X \) is not a normal variety.

Let \( V = J_2(n, k), G = \operatorname{Gl}(n) \times \operatorname{Gl}(k), X = \Theta_{A_2}^{n,k}, \) and let \( Y \) be the Kazarian’s smooth equivariant resolution of \( \Theta_{A_2}^{n,k} \).
Consider Resolution 2 in the general case:
\[
\det Q \otimes L_1^2 \otimes \text{Sym}^{k-n}(Q_1 \oplus L_1^2) \otimes \Lambda^k \mathbb{C}^4 \rightarrow \ldots
\]
\[
\ldots \rightarrow \det Q \otimes L_1^2 \otimes \Lambda^{k-i} \mathbb{C}^4 \otimes \text{Sym}^{l-(k-i)}((\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k) \rightarrow \ldots
\]
\[
\rightarrow \text{Sym}^l((\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k) \rightarrow H^0\left(\mathbb{P}^{n-1}, \text{Sym}^l\left(\frac{(\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k}{L_2 \otimes \mathbb{C}^k}\right)\right).
\]

Recall that
\[
H^0(Y, \mathcal{O}_Y) = \bigoplus_l H^0\left(\mathbb{P}^{n-1}, \text{Sym}^l\left(\frac{(\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k}{L_2 \otimes \mathbb{C}^k}\right)\right)
\]
and
\[
H^0(V, \mathcal{O}_V) = \bigoplus_l \text{Sym}^l((\text{Sym}^2 \mathbb{C}^n \oplus \mathbb{C}^n) \otimes \mathbb{C}^k) = \bigoplus_l H^0(\mathbb{P}^{n-1}, \text{Sym}^l((\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k)).
\]

Since the map \(f\) from the diagram above preserves the graded components, it is enough to prove that
\[
f_l: \text{Sym}^l((\text{Sym}^2 \mathbb{C}^n \oplus \mathbb{C}^n) \otimes \mathbb{C}^k) \rightarrow H^0\left(\mathbb{P}^{n-1}, \text{Sym}^l\left(\frac{(\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k}{L_2 \otimes \mathbb{C}^k}\right)\right)
\]
is not surjective for some fixed \(l\).

Note that \(f_l\) is the right arrow in the line \(H^0\) of the first page of the Leray spectral sequence corresponding to Resolution 2. That is, if we can find an example of a spectral sequence with a non-horizontal arrow pointing to the term
\[
H^0\left(\mathbb{P}^{n-1}, \text{Sym}^l\left(\frac{(\text{Sym}^2 \mathbb{C}^n \oplus Q_1) \otimes \mathbb{C}^k}{L_2 \otimes \mathbb{C}^k}\right)\right),
\]
we prove that \(f\) is not surjective.

Let \(n = 3, k = 4, l = 4\). In this case Resolution 2 is the following:
\[
\det Q \otimes L_1^2 \otimes (Q_1 \oplus L_1^2) \otimes \Lambda^4 \mathbb{C}^4 \rightarrow \\
\rightarrow \det Q \otimes L_1^2 \otimes \Lambda^3 \mathbb{C}^4 \otimes ((\text{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4) \rightarrow \\
\rightarrow \text{Sym}^4((\text{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4) \rightarrow H^0\left(\mathbb{P}^2, \text{Sym}^4\left(\frac{(\text{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4}{L_2 \otimes \mathbb{C}^4}\right)\right).
\]

A straightforward computation using the Borel-Weil-Bott theorem shows that the corresponding Leray spectral sequence is the following.

We see that there is a non-horizontal arrow pointing to \(H^0\left(\mathbb{P}^2, \text{Sym}^4\left(\frac{(\text{Sym}^2 \mathbb{C}^3 \oplus Q_1) \otimes \mathbb{C}^4}{L_2 \otimes \mathbb{C}^4}\right)\right)\),
thus \(\Theta^{3,4}_{A_2}\) is not a normal variety.
Remark 3.6. Since the equivariant resolutions for the $A_3$-loci given in \cite{3} and \cite{2} are smooth, the computational methods presented in this paper may be used to check the rationality of the singularities of $\Theta^{n,k}_{A_3}$.

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