Forces in electromagnetic field and gravitational field

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The force can be defined from the linear momentum in the gravitational field and electromagnetic field. But this definition can not cover the gradient of energy. In the paper, the force will be defined from the energy and torque in a new way, which involves the gravitational force, electromagnetic force, inertial force, gradient of energy, and some other new force terms etc. One of these new force terms can be used to explain why the solar wind varies velocity along the magnetic force line in the interplanetary space between the sun and the earth.

I. INTRODUCTION

The force is one important physical conception for the electromagnetic and gravitational fields. There is only the gravity [1], Lorentz force [2], and gradient of energy in classical field theory described by the vector. But these forces can not explain why the solar wind varies velocity along the magnetic force line in the interplanetary space and the coronal hole on the sun [3].

Since the antiquity, scientists have used the concept of force in the study of different physical objects. Modern description of force was introduced by I. Newton in the 17th century. Nowadays, the description of force covers all kinds of forces we found up to now can be described with one single formula, including the gravity, Lorentz force, and gradient of energy etc. Meanwhile all kinds of energies we knew at present can also be described with one single formula, including the work, kinetic energy, potential energy, field energy, proper energy, interacting energy among magnetic fields with magnetic moments, and some other new energy terms etc.

In the electromagnetic field and gravitational field, the force can be defined from the linear momentum [4]. The forces include the inertial force, gravitational force, Lorentz force, except for the gradient of energy. Further, the force can be defined from the energy and torque in a new way. The force is able to involve the gravity, inertial force, Lorentz force, gradient of energy, and some other new force terms etc. And one of these new force terms can explain the dynamical puzzle of solar wind.

II. FORCES FROM LINEAR MOMENTA

The forces can be defined from the linear momentum in the electromagnetic field and gravitational field. And the forces include the inertial force, gravitational force, electric force, and magnetic force, etc.

A. Gravitational field

The force in the gravitational field can be described by quaternions. In the quaternion space, the coordinates are \( r_0, r_1, r_2, \) and \( r_3 \), with the basis vector \( \mathbb{E}_g = (1, i_1, i_2, i_3) \). Herein, \( r_0 = v_0t \); \( t \) is the time; \( v_0 \) is the speed of gravitational intermediate boson, which is the first part of the photon. The radius vector is \( \mathbb{R}_g = r_0 + \Sigma (r_j i_j) \), and the velocity is \( \mathbb{V}_g = v_0 + \Sigma (v_j i_j) \), \( j = 1, 2, 3 \); \( i = 0, 1, 2, 3 \).

1. Linear momentum in gravitational field

In the quaternion space for the gravitational field, the gravitational potential \( \mathbb{A}_g \) is

\[
\mathbb{A}_g = a_0 + \Sigma (a_j i_j),
\]

and the strength \( \mathbb{B}_g \) of gravitational field

\[
\mathbb{B}_g = \dot{\Sigma} \circ \mathbb{A}_g = b_0 + \Sigma (b_j i_j),
\]

where \( a_0 \) is the scalar potential, while \( a \) is the vectorial potential. The \( \circ \) denotes the quaternion multiplication.

\[
\dot{\Sigma} = \partial_0 + \Sigma (i_j \partial_j); \quad \partial_1 = \partial / \partial r_1; \quad a = \Sigma (a_j i_j); \quad \nabla = \Sigma (i_j \partial_j).
\]

The gauge is selected as \( b_0 = \partial_0 a_0 + \nabla \cdot a = 0 \).

\[
\begin{array}{cccc}
1 & i_1 & i_2 & i_3 \\
1 & i_1 & i_2 & i_3 \\
i_1 & i_1 & -1 & i_3 \\
i_2 & i_2 & -i_1 & -1 \\
i_3 & i_3 & i_2 & -i_1 \\
\end{array}
\]

TABLE I: The quaternion multiplication table.
TABLE II: The operator and multiplication of the physical quantity in the quaternion space.

| definitions | meanings |
|-------------|-----------|
| $\nabla \cdot \mathbf{a}$ | $-(\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3)$ |
| $\nabla \times \mathbf{a}$ | $i_1(\partial_2 a_3 - \partial_3 a_2) + i_2(\partial_3 a_1 - \partial_1 a_3) + i_3(\partial_1 a_2 - \partial_2 a_1)$ |
| $\nabla a_0$ | $i_1 \partial_1 a_0 + i_2 \partial_2 a_0 + i_3 \partial_3 a_0$ |
| $\partial_0 \mathbf{a}$ | $i_1 \partial_1 a_1 + i_2 \partial_2 a_2 + i_3 \partial_3 a_3$ |

The gravitational strength $B_g$ includes two kinds of components $\mathbf{g}/\nu_0 = \partial_0 \mathbf{a} + \nabla \nu_0$ and $\mathbf{b} = \nabla \times \mathbf{a}$,

$$\mathbf{g}/\nu_0 = i_1(\partial_0 a_1 + \partial_1 a_0) + i_2(\partial_0 a_2 + \partial_2 a_0) + i_3(\partial_0 a_3 + \partial_3 a_0),$$

$$\mathbf{b} = i_1(\partial_2 a_3 - \partial_3 a_2) + i_2(\partial_3 a_1 - \partial_1 a_3) + i_3(\partial_1 a_2 - \partial_2 a_1),$$

where the $\mathbf{b}$ may be too weak to be detected presently; and there are $\mathbf{a} = 0$ and $\mathbf{b} = 0$ in the classical Newtonian gravitational theory.

The source $S$ of gravitational field includes the linear momentum density $S_g = m \mathbf{v}_g$ and an extra part $\nu_0 \Delta m$,

$$\mu S = -(B_g/\nu_0 + \mathcal{O})^* \circ B_g$$

$$= \mu g(S_g + \nu_0 \Delta m),$$

where $m$ is the mass density; $\mu$ denotes the quaternion conjugate; $\mu$ is one coefficient, and $\mu g$ is the gravitational constant; $B_g^* \circ B_g/(2 \mu g)$ is the energy density of gravitational field; $\Delta m = -B_g^* \circ B_g/(\mu g v_0^2)$.

The force density $F_g$ is defined from the linear momentum density $P_g = \mu S/\mu g$. And the latter is the extension of the $S_g$.

$$F_g = \nu_0 (B_g/\nu_0 + \mathcal{O})^* \circ B_g.$$

We introduce above definition so as to recover various forces associated with a gravitational field including the inertial force density and gravity density etc.

TABLE III: Some definitions and the gravitational force density in the quaternion space.

| definitions | meanings |
|-------------|-----------|
| $\nu_0 \partial_0 \mathbf{p}$ | inertial force density |
| $mg^*$ | gravity density |
| $\Delta mg^*$ | new force part |
| $p_0 b^*$ | new force part |
| $\nu_0 \nabla^* p_0$ | new force part |
| $\mathbf{g}^* \times \mathbf{p}/\nu_0$ | new force part |
| $b^* \times \mathbf{p}$ | new force part |
| $\nu_0 \nabla^* \times \mathbf{p}$ | new force part |

2. Gravitational force

In the quaternion space, the inertial mass density is $m$, and the gravitational mass density is $\tilde{m} = m + \Delta m$. The linear momentum density is $F_g = p_0 + \Sigma(p_i j_i)$, with $p_0 = \tilde{m} v_0$ and $p_i = m v_i$.

By Eq.(6), the gravitational force density $F_g$ is

$$F_g = f_0 + \Sigma(f_i j_i),$$

where $f_0 = \partial p_0/\partial t + v_0 \Sigma(\partial p_i/\partial v_i) + \Sigma(b_j p_j)$.

In the quaternion space, the vectorial part $f = \Sigma(f_i j_i)$ of force density $F_g$ can be decomposed from Eq.(7),

$$f = v_0 \partial_0 \mathbf{p} + p_0 \mathbf{g}^*/\nu_0 + p_0 \mathbf{b}^* + v_0 \nabla^* p_0 + (\mathbf{g}/\nu_0 + \mathbf{b} + v_0 \nabla)^* \times \mathbf{p},$$

where $p = \Sigma(p_i j_i); \nabla = \Sigma(i_i \partial_i); v_0 \partial_0 \mathbf{p}$ is the inertial force density; $p_0 \mathbf{g}^*/\nu_0 = mg^* + \Delta mg^*$; $mg^*$ is the gravity density; $\Delta mg^* + v_0 \nabla^* p_0 + p_0 \mathbf{b}^* + (\mathbf{g}/\nu_0 + \mathbf{b} + v_0 \nabla)^* \times \mathbf{p}$ is one new part of gravitational force density.

In the gravitational field, some new parts of above force density are too weak to be detected. As a particular case, these new parts of forces are equal to zero in Newtonian gravitational theory.

B. Electromagnetic field

The force in the electromagnetic field can be described by quaternions as well. In the quaternion space, the coordinates are $r_0$, $r_1$, $r_2$, and $r_3$, with the basis vector $E_q = (1, i_1, i_2, i_3)$. Where, $r_0 = v_0 t$; $t$ is the time; $v_0$ is the speed of electromagnetic intermediate boson, which is the second part of the photon. The radius vector is $r_q = r_0 + \Sigma(r_j j_i)$, and the velocity is $V_q = v_0 + \Sigma(v_j j_i)$.

1. Linear momentum in electromagnetic field

In the quaternion space for the electromagnetic field, the electromagnetic potential $A_q$ is

$$A_q = A_0 + \Sigma(A_j j_i),$$

and the strength $B_q$ of electromagnetic field

$$B_q = k_{eg} \mathcal{O} \circ A_q$$

$$= k_{eg} [B_0 + \Sigma(B_j j_i)],$$

where $A = \Sigma(A_j j_i)$, with $k_{eg}$ being a coefficient.

The electromagnetic strength $B_q$ includes two parts, where $E/\nu_0 = \partial_0 \mathbf{A} + \nabla \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, as follows

$$E/\nu_0 = i_1(\partial_0 A_1 + \partial_1 A_0) + i_2(\partial_0 A_2 + \partial_2 A_0) + i_3(\partial_0 A_3 + \partial_3 A_0),$$

$$B = i_1(\partial_2 A_3 - \partial_3 A_2) + i_2(\partial_3 A_1 - \partial_1 A_3) + i_3(\partial_1 A_2 - \partial_2 A_1),$$

where the gauge is selected as $B_0 = \partial_0 A_0 + \nabla \cdot \mathbf{A} = 0$. 

TABLE IV: Some definitions and the electromagnetic force density in the quaternion space.

| definitions               | meanings                                      |
|---------------------------|-----------------------------------------------|
| \(v_0\partial_0\mathbf{P}\) | (inertial force density)                      |
| \(q\mathbf{E}^*\)         | electric force density                        |
| \(q\mathbf{B}^* \times \mathbf{V}\) | magnetic force density                        |
| \(k_{eg}\Delta m\mathbf{E}^*/v_0\) | (new force part)                              |
| \(qv_0\mathbf{B}^*\)      | new force part                                |
| \(k_{eg}\Delta m\mathbf{B}^*\) | (new force part)                              |
| \(v_0\nabla^* \mathbf{P}_0\) | (new force part)                              |
| \(q\mathbf{E}^* \times \mathbf{V}/v_0\) | new force part                                |
| \(v_0\nabla^* \times \mathbf{P}\) | (new force part)                              |

The electromagnetic source \(S\) encompasses the electric current density \(S_\mathbf{q} = q\nabla \mathbf{q}\) and an extra part \(v_0\Delta m\),

\[
\mu S = - (\mathbf{E}_\mathbf{q}/v_0 + \nabla \mathbf{q}) \cdot \mathbf{B}_\mathbf{q} - \mathbf{B}_\mathbf{q}/v_0,
\]

where \(q\) is the charge density; \(\mu\) is the electromagnetic constant; \(\mathbf{E}_\mathbf{q}/v_0\) is the energy density of electromagnetic field; \(\Delta m = - \mathbf{B}_\mathbf{q}/(\mu_0 v_0^2)\).

The force density \(F_q\) is defined from the linear momentum density \(\mathbf{p}_q = \mu \mathbf{S}/\mu_g\). And the latter is one function of the \(S_q\).

\[
F_q = v_0(\mathbf{B}_\mathbf{q}/v_0 + \nabla \mathbf{q}) \cdot \mathbf{p}_q,
\]

where the force density includes the inertial force density and electromagnetic force density, etc.

2. Electromagnetic force

In the quaternion space, the linear momentum density is \(\mathbf{p}_q = \mathbf{P}_0 + \Sigma (\mathbf{P}_j i_j)\), with \(\mathbf{P}_j = MV_j\), \(M = k_{eg} q \mu_g/\mu_g\).

By Eq. (6), the electromagnetic force density \(\mathbf{f}_q\) is

\[
\mathbf{f}_q = \mathbf{F}_q = F_0 + \Sigma (\mathbf{F}_j i_j),
\]

where \(F_0 = \partial \mathbf{P}_0/\partial t + v_0 \Sigma (\partial \mathbf{P}_j/\partial r_j) + \Sigma (k_{eg} \mathbf{B}_j \mathbf{P}_j)\).

The vectorial part \(\mathbf{f}_q = \Sigma (\mathbf{F}_j i_j)\) of force density \(\mathbf{F}_q\) can be decomposed from Eq. (15).

\[
\mathbf{f}_q = v_0\partial_0\mathbf{P} + k_{eg} P_0 \mathbf{E}^*/v_0 + k_{eg} P_0 \mathbf{B}^* + v_0 \nabla^* P_0
\]

\[
+ (k_{eg} \mathbf{E}/v_0 + k_{eg} \mathbf{B} + v_0 \nabla^\times) \times \mathbf{P},
\]

where \(\mathbf{P} = \Sigma (\mathbf{P}_j i_j); \mathbf{V} = \Sigma (V_j i_j); v_0\partial_0\mathbf{P}\) is the inertial force density; \(k_{eg} P_0 \mathbf{E}^*/v_0 = k_{eg} \mathbf{B}^* \times \mathbf{V}\) is the electric force density; \(k_{eg} \mathbf{B}^* + \mathbf{E}^*/v_0 = \mathbf{Q}^*\) is the magnetic force density; \(k_{eg} \Delta m \mathbf{E}^*/v_0 + v_0 \nabla^* P_0 + k_{eg} P_0 \mathbf{B}^* + (k_{eg} \mathbf{E}/v_0 + v_0 \nabla^\times) \times \mathbf{P}\) is one new part of the electromagnetic force density.

In the quaternion space, the \(q v_0 \mathbf{B}^*\) is one new force term, which can be used to explain why the solar wind varies velocity along the magnetic force line. Except for the term \(q v_0 \mathbf{B}^*\), other new force terms may be too weak to be detected in weak electromagnetic field.

C. Gravitational and electromagnetic fields

The gravitational field and electromagnetic field both can be illustrated by the quaternion, and their quaternion spaces will be combined together to become the octonion space. In other words, the characteristics of gravitational field and electromagnetic field can be described with the octonion space simultaneously.

1. Linear momentum

In the quaternion space for the gravitational field, the basis vector \(\mathbf{E}_q = (1, i_1, i_2, i_3)\), and the radius vector \(\mathbf{R}_q = (r_0, r_1, r_2, r_3)\), with the velocity \(\mathbf{V}_q = (v_0, v_1, v_2, v_3)\).

For the electromagnetic field, the basis vector \(\mathbf{E}_e = (I_0, I_1, I_2, I_3)\), the radius vector \(\mathbf{R}_e = (R_0, R_1, R_2, R_3)\), and the velocity \(\mathbf{V}_e = (V_0, V_1, V_2, V_3)\), with \(\mathbf{E}_e = I_0 \circ I_0\).

The \(E_q\) is independent of the \(F_q\). Both of them can be combined together to become the basis vector \(\mathbf{E}\) of the octonion space. That is,

\[
\mathbf{E} = (1, i_1, i_2, i_3, 1, I_1, I_2, I_3).\]

The radius vector \(\mathbf{R}(r_0, r_1, r_2, r_3, R_0, R_1, R_2, R_3)\) in the octonion space is

\[
\mathbf{R} = r_0 + i_1 r_1 + i_2 r_2 + i_3 r_3 + I_0 R_0 + I_1 R_1 + I_2 R_2 + I_3 R_3,
\]

and the velocity \(\mathbf{V}(v_0, v_1, v_2, v_3, V_0, V_1, V_2, V_3)\) is

\[
\mathbf{V} = v_0 + i_1 v_1 + i_2 v_2 + i_3 v_3 + I_0 V_0 + I_1 V_1 + I_2 V_2 + I_3 V_3.
\]

where \(r_0 = v_0 t; t\) is the time; \(v_0\) is the speed of gravitational intermediate boson; the symbol \(\circ\) denotes the octonion multiplication.

When the electric charge is combined with the mass to become the electron or the proton etc., we obtain the \(R_i I_i = r_i i_i \circ I_0\) and \(V_i I_i = v_i i_i \circ I_0\), with \(i_0 = 1\).

TABLE V: The operator and multiplication of the physical quantity in the octonion space.

| definitions               | meanings                                      |
|---------------------------|-----------------------------------------------|
| \(\nabla \cdot \mathbf{a}\) | \((-a_1 \partial_1 + a_2 \partial_2 + a_3 \partial_3)\) |
| \(\nabla \times \mathbf{a}\) | \(i_1(a_2 \partial_3 - a_3 \partial_2) + i_2(a_3 \partial_1 - a_1 \partial_3) + i_3(a_1 \partial_2 - a_2 \partial_1)\) |
| \(\nabla a_0\)            | \(i_1 \partial_1 a_0 + i_2 \partial_2 a_0 + i_3 \partial_3 a_0\) |
| \(\partial_0 \mathbf{a}\) | \(i_1 \partial_1 a_1 + i_2 \partial_2 a_2 + i_3 \partial_3 a_3\) |
| \(\nabla \cdot \mathbf{P}\) | \((-a_1 \partial_1 + a_2 \partial_2 + a_3 \partial_3) I_0\) |
| \(\nabla \times \mathbf{P}\) | \(-I_0 (\partial_1 P_0 - \partial_2 P_2 - \partial_3 P_3) - I_2 (\partial_2 P_1 - \partial_1 P_3) - I_3 (\partial_3 P_1 - \partial_1 P_2)\) |
| \(\nabla \circ \mathbf{P}_0\) | \(I_0 \partial_1 P_0 + I_2 \partial_2 P_2 + I_3 \partial_3 P_3\) |
| \(\partial_0 \mathbf{P}\)   | \(I_0 \partial_1 P_1 + I_2 \partial_2 P_2 + I_3 \partial_3 P_3\) |
In the same way, the gravitational intermediate boson and electromagnetic intermediate boson can be combined together to become the photon.

The potential of the gravitational and electromagnetic fields are $A_g = (a_0, a_1, a_2, a_3)$ and $A_e = (A_0, A_1, A_2, A_3)$ respectively, with $A_e = A_g \circ I_0$. They can be combined together to become the potential $A = A_g + k_{eg} A_e$.

The strength $B(b_0, b_1, b_2, b_3, B_0, B_1, B_2, B_3)$ consists of gravitational strength $B_g$ and electromagnetic strength $B_e$. The gauge equations $b_0 = 0$ and $B_0 = 0$. And

$$
\mathbb{B} = \hat{\phi} \circ A
\quad = \mathbb{B}_g + k_{eg} \mathbb{B}_e .
$$

The gravitational strength $B_g$ in Eq.(2) includes two components, $g = (g_{01}, g_{02}, g_{03})$ and $b = (g_{23}, g_{31}, g_{12})$, while the electromagnetic strength $B_e$ involves two parts, $E = (B_{01}, B_{02}, B_{03})$ and $B = (B_{23}, B_{31}, B_{12})$.

$$
\frac{E}{v_0} = I_1(\partial_3 A_1 - \partial_2 A_0) + I_2(\partial_0 A_2 + \partial_2 A_0) + I_3(\partial_0 A_3 + \partial_3 A_0) ,
$$

$$
\mathbb{B}_e = I_1(\partial_3 A_2 - \partial_3 A_3) + I_2(\partial_1 A_3 - \partial_3 A_1) + I_3(\partial_2 A_1 - \partial_1 A_2) .
$$

In the octonion space, the electric current density $S_e = q V_g \circ I_0$ is the source for the electromagnetic field, and the linear momentum density $S_g = mg^\ast$ for the gravitational field. The source $S$ satisfies

$$
\mu S = - (\mathbb{B}/v_0 + \hat{\phi})^\ast \circ \mathbb{B}
\quad = \mu g S_g + k_{eg} \mu_e S_e - \mathbb{B}^\ast \circ \mathbb{B}/v_0 ,
$$

where $k_{eg} = \mu_g / \mu_e$; $q$ is the electric charge density; $\mu_e$ is the constant; $\ast$ denotes the conjugate of octonion. And

$$
\mathbb{B}^\ast \circ \mathbb{B}/\mu_g = \mathbb{B}_g^\ast \circ \mathbb{B}_g + \mathbb{B}_e^\ast \circ \mathbb{B}_e/\mu_e .
$$

The force density $\mathbb{F}$ is defined from linear momentum density $p = \mu S/\mu_g$, which is the extension of the $S_g$.

$$
\mathbb{F} = v_0 (\mathbb{B}/v_0 + \hat{\phi})^\ast \circ p ,
$$

where the force density includes Lorentz force density, gravity density, and inertial force density.

### TABLE VI: The octonion multiplication table.

|   | 1 | 1 | i | i | i | I | I | I |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | i | i | i | I | I | I | I |
| i | i | i | i | i | I | I | I | I |
| i | i | 1 | i | i | I | I | I | I |
| i | i | i | i | i | I | I | I | I |
| i | i | i | i | i | I | I | I | I |
| i | i | i | i | i | I | I | I | I |
| I | I | I | I | I | I | I | I | I |
| I | I | I | I | I | I | I | I | I |

### TABLE VII: The electromagnetic force and gravity in the quaternion space $\mathbb{E}_g$ of octonion space.

| definitions | meanings |
|-------------|----------|
| $v_0 \partial_0 p$ | inertial force density |
| $mg^\ast$ | gravity density |
| $qE^\ast \circ I_0$ | electric force density |
| $q \mathbb{F} \times V$ | magnetic force density |
| $q \nabla \mathbb{F} \circ I_0$ | new force part |
| $\Delta mg^\ast$ | new force part |
| $p_0 \mathbb{B}^\ast$ | new force part |
| $v_0 \nabla \mathbb{B}/p_0$ | new force part |
| $g^\ast \circ p/v_0$ | new force part |
| $b^\ast \circ p$ | new force part |
| $v_0 \nabla \mathbb{B}/p$ | new force part |

### 2. Electromagnetic force and gravity

In the octonion space, the gravitational mass density $\hat{m} = m + \Delta m$, with $\Delta m = - \mathbb{B}^\ast \circ \mathbb{B}/(\mu g v_0^2)$. The linear momentum density $\mathbb{P} = p_0 + \Sigma (p_j I_j) + \Sigma P_i I_i$. And, $m$ is the inertial mass density; $P_i = MV_i$, $M = k_{eg} \mu_e q / \mu_g$; $p_0 = \hat{m} v_0$, $p_j = mv_j$; $\mathbb{V} = \Sigma (v_j I_j)$, $\mathbb{V} = \Sigma (v_j I_j)$.

By Eq.(25), the force density $\mathbb{F}$ is

$$
\mathbb{F} = f_0 + \Sigma (f_j I_j) + \Sigma (F_i I_i) ,
$$

where $f_0 = \partial p_0 / \partial t + v_0 \Sigma (p_j / \partial r_j) + \Sigma (b_j + B_j P_j)$.

The vectorial part $f = \Sigma (f_j I_j)$ and $\mathbb{F} = \Sigma (F_i I_i)$ of force density $\mathbb{F}$ can be decomposed from Eq.(26).

$$
f = v_0 \partial_0 p + (g/v_0 + b + v_0 \nabla)^\ast \circ p + q (E/v_0 + B)^\ast \circ V + p_0 (g/v_0 + b)^\ast + v_0 \nabla^\ast \circ p_0 + \nabla^\ast \circ p_0 + (g/v_0 + b)^\ast \circ p_0 ,
$$

where $\mathbb{P} = \Sigma (P_i I_j)$, $P_0 = P_0 I_0$.

In the paper, the physical quantity in the space $\mathbb{E}_g$ can be detected, but in $\mathbb{E}_e$ can not presently.

In the physics world, the term $\mathbb{F}$ can be detected, which includes the inertial force, electromagnetic force, gravity and some other new kinds of forces. But the term $\mathbb{F}$ may not be detected at present. Meanwhile we can obtain the mass continuity equation from $f_0 = 0$, and the charge continuity equation from $F_0 = 0$ respectively.

Comparing Table IV with Table VII, we find that some terms of forces in the former table can not be detected in the physics space $\mathbb{E}_g$. The above means that the quaternion space $\mathbb{E}_g$ or $\mathbb{E}_q$ is not suitable for describing the electromagnetic force in the physics world.
III. FORCES FROM ENERGIES

More force terms can be defined from the energy in the electromagnetic and gravitational fields. Defining from the linear momentum, the force definition can not cover the gradient of energy sufficiently, so that we need one new definition for the force. The new definition of forces is defined from the energy. Those new forces include the inertial force, gravitational force, electric force, magnetic force, gradient of energy, and interacting force between the magnetic strength with magnetic moment, etc.

A. Gravitational field

To incorporate various kinds of energies within a single definition, the angular momentum and energy will both be extended to apply within gravitational fields.

The angular momentum density $L_g = l_0 + \Sigma (l_j \hat{t}_j)$ is defined from the linear momentum density $P_g$ and the radius vector $R_g$, and can be rewritten as follows

$$L_g = (R_g + k_{r \times X_g}) \circ P_g,$$

with

$$l_0 = (r_0 + k_{r \times x_0})p_0 + (r + k_{r \times x}) \cdot p,$$

$$l = (r_0 + k_{r \times x_0})p + p_0(r + k_{r \times x})$$

$$+ (r + k_{r \times x}) \times p,$$

where $l = \Sigma (l_j \hat{t}_j)$; $k_{r \times x}$ is a coefficient. The quaternion quantity $X_g = \Sigma (x_i \hat{t}_i)$ is similar to Hertz vector in the electrodynamics theory. The derivation of $X_g$ will yield the gravitational potential, with $k_{r \times x} X_g \ll R_g$.

The angular momentum density $L_g$ includes the orbital angular momentum density $r \times p$, spin angular momentum density $l_0$, and some other omissible terms etc.

1. Energy and torque

We choose the following definition of energy to include various energies in the gravitational field. In quaternion space, the quaternion energy density $W_g = w_0 + \Sigma (w_j \hat{t}_j)$ is defined from the angular momentum density $L_g$. 

$$W_g = v_0(\mathbb{B}_g / v_0 + \circ) \circ L_g,$$

where the $-w_0 / 2$ is the energy density, including the kinetic energy, potential energy, field energy, and work etc; the $w / 2 = \Sigma (w_j \hat{t}_j) / 2$ is the torque.

Expressing the energy density as

$$w_0 = v_0 \partial_0 l_0 + v_0 \nabla \cdot l + (g / v_0 + b) \cdot l$$

$$= (g / v_0 + b) \cdot (r_0 p + p_0 r + r \times p)$$

$$+ k \cdot (g / v_0 + b) \cdot (x_0 p + p_0 x + x \times p)$$

$$+ v_0 \nabla \cdot (r_0 p + p_0 r + r \times p)$$

$$+ v_0 \delta_0 (r_0 p_0 + r \cdot p) + v_0 k \cdot \delta_0 (x_0 p_0 + x \cdot p)$$

$$+ v_0 k \cdot \nabla \cdot (x_0 p + p_0 x + x \times p),$$

where the term $(g / v_0 + b) \cdot (x_0 p + p_0 x + x \times p)$ is one new kind of energy; $x = \Sigma (x_j \hat{t}_j)$; $k \cdot x = 1$.

In case of $r_0 = 0, x_0 = 0, b = 0, r \parallel p$, and $x \parallel p$, the above can be reduced to

$$w_0 / 2 \approx \frac{mv_0^2 + \Delta mv_0^2 - a \cdot p / 2 - a_0 p_0 / 2}{v_0 \delta_0 (r \cdot p) / 2 - (g / v_0) \cdot (p_0 r) / 2},$$

where $a_0 = v_0 \delta_0 (r_0 x_0 + \nabla \cdot x)$ is the scalar potential of gravitational field; the last three terms in the above are equal to the sum of kinetic energy and gravitational potential energy in classical field theory.

In a similar way, expressing the torque density as

$$w = v_0 \delta_0 l + v_0 \nabla l + v_0 \nabla \times l$$

$$+ l_0 (g / v_0 + b) + (g / v_0 + b) \times l$$

$$= (g / v_0 + b) \times (r_0 p + p_0 r + r \times p)$$

$$+ (r_0 p_0 + r \cdot p)(g / v_0 + b)$$

$$+ k \cdot (g / v_0 + b) \times (x_0 p + p_0 x + x \times p)$$

$$+ v_0 \nabla \times (r_0 p + p_0 r + r \times p)$$

$$+ v_0 \nabla (r_0 p_0 + r \cdot p) + v_0 \delta_0 (r_0 p + p_0 r + r \times p)$$

$$+ v_0 k \cdot \nabla (x_0 p + x \times x)$$

$$+ v_0 k \cdot \nabla \delta_0 (x_0 p + p_0 x + x \times p).$$

In case of $r_0 = 0, x_0 = 0, (v^* \cdot v) \ll v_0^2$, and $b = 0$, the above can be reduced to

$$w \approx \frac{w_0}{2} \approx \frac{(g / v_0) \times (p_0 r) + v_0 \delta_0 (r \cdot p + p_0 r)}{v_0 \partial_0 l}$$

$$- 2 v_0 p + v_0 r \cdot (\nabla \cdot p) + v_0 r \times (\nabla \times p)$$

$$+ a_0 p + p_0 a + a \times p,$$

where $a = v_0(\partial_0 x + \nabla x + \nabla \times x)$ is the vectorial potential of gravitational field; the first and second terms in the above are the torque caused by gravity and some other force terms respectively.

**TABLE VIII**: Some related physical quantities in the quaternion space.

| definitions | meanings |
|-------------|----------|
| $p_0$       | scalar part |
| $\mathbf{p}$ | linear momentum density |
| $l_0$       | scalar part |
| $l$         | angular momentum density |
| $w_0$       | energy density |
| $\mathbf{w}$ | torque density |
| $n_0$       | power density |
| $n$         | vectorial part |
2. Power and force

In the quaternion space, the quaternion power density \( N_g = n_0 + \Sigma (n_j i_j) \) is defined from the \( W_g \),
\[
N_g = v_0 (\mathbb{B}/v_0 + \mathcal{Q})^* \circ W_g ,
\]  
(35)
where the \( f_0 = -n_0/(2v_0) \) is the power density, while the vectorial part \( n = \Sigma (n_j i_j) \) is the function of forces.

Expressing the scalar \( n_0 \) as
\[
n_0 = v_0 \partial_0 w_0 + v_0 \nabla^* \cdot w + (g/v_0 + b)^* \cdot w .
\]  
(36)

In a similar way, expressing the vectorial part \( n \) of \( N_g \) as follows
\[
n = v_0 \nabla w_0 + v_0 \partial_0 w + v_0 \nabla \times w
+ (g/v_0 + b)^* \times w + w_0 (g/v_0 + b)^* .
\]
The force \( f = -n/(2v_0) \) in the gravitational field can be defined from the vectorial part \( n \) as
\[
-2f = \nabla^* w_0 + \partial_0 w + (g/v_0 + b)^* \times w/v_0
+ \nabla \times w + w_0 (g/v_0 + b)^*/v_0 ,
\]  
(37)
where the force \( f \) includes the gravity, inertial force, and gradient of energy etc.

In the gravitational field, the above means that Eq.(37) encompasses the force terms in Eq.(8) and some other terms of the gradient of energy etc.

In case of \( N_g = 0 \) in Eq.(35), we can obtain the mass continuity equation from \( n_0 = 0 \) in Eq.(36), as well as the force equilibrium equation from \( n = 0 \) in Eq.(37).

B. Gravitational and electromagnetic fields

In the case for coexistence of electromagnetic field and gravitational field, the octonion angular momentum density \( L = l_0 + \Sigma (L_j i_j) + L_0 j_0 + \Sigma (L_j j_j) \) is defined from the octonion linear momentum density \( P \) and octonion radius vector \( \mathbb{R} \) in Eqs.(18) and (19).
\[
L = (\mathbb{R} + k_{xx} X) \circ P \, ,
\]  
(38)
with
\[
l_0 = (r_0 + k_{xx} x_0) \partial_0 v_0 + \mathbb{R}_0 + k_{xx} X_0 \circ P_0
+ (r + k_{xx} x) \circ P + (\mathbb{R} + k_{xx} X) \circ P ,
\]  
(39)
\[
l = (r_0 + k_{xx} x_0) \partial_0 v_0 + \mathbb{R}_0 + k_{xx} X_0 \circ P_0
+ (r + k_{xx} x) \circ P + (\mathbb{R} + k_{xx} X) \circ P
+ p_0 (r + k_{xx} x) + (\mathbb{R} + k_{xx} X) \times P ,
\]  
(40)
\[
L_0 = (r_0 + k_{xx} x_0) \circ P_0 + (r + k_{xx} x) \circ P
+ p_0 (\mathbb{R} + k_{xx} X_0) + (\mathbb{R} + k_{xx} X) \circ P ,
\]  
(41)
\[
L = (r_0 + k_{xx} x_0) \circ P_0 + (r + k_{xx} x) \circ P
+ p_0 (\mathbb{R} + k_{xx} X_0) + (\mathbb{R} + k_{xx} X) \times P
+ p_0 (\mathbb{R} + k_{xx} X) + (\mathbb{R} + k_{xx} X) \times P ,
\]  
(42)
where \( X = \Sigma (x_j i_j) + \Sigma (X_j I_j) \); \( X_0 = X_0 \circ I_0 \); \( X = \Sigma (X_j I_j) \). \( L_0 = L_0 j_0 \); \( L = \Sigma (L_j I_j) \). \( P_0 = P_0 j_0 \); \( P = \Sigma (P_j I_j) \). \( R_0 = R_0 j_0 \); \( R = \Sigma (R_j I_j) \). Similarly, the derivation of octonion physical quantity \( X \) will yield the potentials of gravitational field and electromagnetic field.

| TABLE IX: Some physical quantity in the quaternion space and octonion space. |
|-------------------------------------|
| definitions                        |
| meaninngs                          |
|-------------------------------------|
| \( A \) = \( \mathcal{Q} \circ X \) |
| \( B \) = \( \mathcal{Q} \circ A \)  |
| \( \mu S = -(\mathbb{B}/v_0 + \mathcal{Q})^* \circ \mathbb{B} \) |
| \( P = \mu S / \mu_g \)            |
| \( P' = P / \mathbb{R} \)          |
| \( L = P' \circ \mathbb{P} \)      |
| \( W = v_0 (\mathbb{B}/v_0 + \mathcal{Q}) \circ \mathbb{L} \) |
| \( N = v_0 (\mathbb{B}/v_0 + \mathcal{Q})^* \circ W \) |
| \( F = -N/(2v_0) \)               |

1. Energy and torque

In the case for coexistence of electromagnetic field and gravitational field, the octonion energy density \( W = v_0 + \Sigma (w_j j_j) + \Sigma (W_j I_j) \) is defined from the octonion angular momentum density \( L \) and octonion field strength \( \mathbb{B} \) in Eqs.(20) and (38).
\[
W = v_0 (\mathbb{B}/v_0 + \mathcal{Q}) \circ L ,
\]  
(43)
where the \(-w_0/2\) is the energy density, including the kinetic energy, potential energy, field energy, and work etc; the \( w/2 = \Sigma (w_j j_j)/2 \) is the torque.

Expressing the energy density as
\[
w_0 = v_0 \partial_0 p_0 + v_0 \nabla \cdot l + (g/v_0 + b) \cdot l
+ k_{eg} (E/v_0 + B) \cdot L ,
\]

where the \(-w_0/2\) includes some new kinds of energies. In case of \( r_0 = 0 \), \( x_0 = 0 \), \( b = 0 \), \( R \parallel p \), and \( X \parallel p \), the
above can be reduced to

\[-w_0/2 \approx \text{terms that include torque, electromagnetic force, and other force terms etc.}\]

In a similar way, expressing the torque density \( w \) as

\[ w = v_0 \partial_0 l + v_0 \nabla l_0 + v_0 \nabla \times l \]

\[ + (g/v_0 + b) \times L + k_{eg} (E/v_0 + B) \cdot L \]

where \( A_0 + A = k_{eg} k_e \cdot A_0 = v_0 (\partial_0 X_0 + \nabla \cdot X) \) and \( A = v_0 (\partial_0 X + \nabla \cdot X_0 + \nabla \times X) \) are related to the scalar potential and vectorial potential of electromagnetic field respectively; the last four terms are the electric potential energy, magnetic potential energy, and interacting energy between dipole moment with electromagnetic strength in classical field theory.

In the electromagnetic and gravitational field, the torque \( w/2 \) in the space \( \mathbb{E}_g \) can be detected, but the other vectorial part \( \Sigma(W_I l) \) can not currently.

Other parts of torque can be rewritten as follows

\[ W_0 = v_0 \partial_0 l_0 + k_{eg} (E/v_0 + B) \cdot L \]

\[ + v_0 \nabla \cdot (g/v_0 + b) \cdot L \]

\[ W = v_0 \partial_0 L_0 + l_0 k_{eg} (E/v_0 + B) + v_0 \nabla \times L \]

\[ + v_0 \nabla \cdot L_0 + k_{eg} (E/v_0 + B) \times L \]

\[ + (g/v_0 + b) \times L + (g/v_0 + b) \circ L_0 \]

where \( W_0 = W_0 l_0, W = \Sigma(W_I l) \).

The torque \( \Sigma(W_I l) \) can not be detected in \( \mathbb{E}_g \), but it has still an effect on the power and forces.

### 2. Power and force

In the octonion space, the octonion power density \( N = v_0 + \Sigma (n_j i_j) + \Sigma (N_i l_i) \) is defined from the octonion energy density \( W \) and field strength \( B \).

\[ N = v_0 (B/v_0 + \phi) \cdot W, \]  \[ (46) \]

where \( f_0 = -n_0/(2v_0) \) is the power density, while one vectorial part \( n = \Sigma (n_j i_j) \) is the function of forces. But the other vectorial parts \( N_0 = N_0 l_0 \) and \( N = \Sigma (N_j l_j) \) may not be detected in the space \( \mathbb{E}_g \) at present.

Further expressing the scalar \( n_0 \) as

\[ n_0 = v_0 \partial_0 w_0 + v_0 \nabla^* \cdot w + (g/v_0 + b)^* \cdot w \]

\[ + k_{eg} (E/v_0 + B)^* \cdot W, \]  \[ (47) \]

In a similar way, expressing the vectorial part \( n \) of \( N \) as follows

\[ n = v_0 \nabla^* w_0 + v_0 \partial_0 w + v_0 \nabla^* \cdot w \]

\[ + (g/v_0 + b)^* \times w + w_0 (g/v_0 + b)^* \]

\[ + k_{eg} (E/v_0 + B)^* \times W \]

\[ + k_{eg} (E/v_0 + B)^* \circ W_0 \]  \[ (48) \]

The force \( f = -n/(2v_0) \) in the gravitational field and electromagnetic field can be defined from the \( n \).

\[ -2f = \nabla^* w_0 + \partial_0 w + (g/v_0 + b)^* \times w/v_0 \]

\[ + \nabla^* \cdot w + w_0 (g/v_0 + b)^*/v_0 \]

\[ + k_{eg} (E/v_0 + B)^* \times W/v_0 \]

\[ + k_{eg} (E/v_0 + B)^* \circ W_0/v_0, \]  \[ (49) \]

where the force \( f \) includes the inertial force, gravity, Lorentz force, gradient of energy, and interacting force between dipole moment with magnetic strength etc.
In the gravitational field and electromagnetic field, the above means that Eq.(49) is much more complicated than Eq.(27), and encompasses more new force terms about the gradient of energy etc.

In case of \( N = 0 \) in Eq.(46), we can achieve the mass continuity equation from \( u_0 = 0 \) in Eq.(47), and the force equilibrium equation from \( \mathbf{n} = 0 \) in Eq.(48) in the space \( \mathbb{E}_g \). In a similar way, we may deduce the charge continuity equation from \( N_0 = 0 \), and one new force equilibrium equation from \( \mathbf{N} = 0 \) in the space \( \mathbb{E}_e \).

IV. CONCLUSIONS

The force can be defined from the linear momentum in the gravitational field and electromagnetic field. The forces cover the gravity, Lorentz force, and inertial force etc, except for the gradient of energy. Of more important is that forces can be defined from the energy. The latter definition of force is able to include the gravity, Lorentz force, inertial force, gradient of energy, interacting energy between dipole moment with electromagnetic strength, and some other new force terms, etc.

In the gravitational field and electromagnetic field, there exist some new force terms in Eq.(27) and Eq.(49). The term \( qv_0 \mathbf{B}^\ast \cdot \mathbf{I}_0 \) will cause the charged particle to move along the magnetic force line. This is the reason why the solar wind varies velocity along the magnetic force line in the sun’s coronal hole as well as in the interplanetary space between the sun and the earth.

It should be noted that the study for the new force terms examined only some simple cases under the force definition from the linear momentum or energy. Despite its preliminary character, this study can clearly indicate that the gravity and Lorentz force are only two simple force terms in the electromagnetic field and gravitational field. Meanwhile, there exist more force terms from the definition of energy than that from the linear momentum. For the future studies, the research will concentrate on only the predictions about the velocity variation of solar wind and charged particles in the strong electric field and magnetic field.

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