W–Gravity and Generalized Lax Equations for (super) Toda Theory

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Abstract

We generalize the Lax pair and Bäcklund transformations for Toda and N=1 super Toda equations to the case of arbitrary worldsheet background geometry. We use the fact that the Toda equations express constant curvature conditions, which arise naturally from flatness conditions equivalent to the W–gravity equations of motion.

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1. Introduction

The Toda chain problem in mechanics, and Toda field theory have been known to be integrable systems for some time now [1][2][3], despite the fact that the precise origin of this integrability has not heretofore been elucidated.

In Liouville theory (the simplest Toda field theory), integrability can be traced to the fact that the field equation expresses a constant curvature condition. With constant curvature, the covering space of any Riemann surface is $A_1/U(1)$ and the constant curvature geometry arises from the reduction of the flat Maurer–Cartan form on the group $A_1$.

Thus the constant curvature condition is naturally equivalent to a flatness condition on the $A_1$ connection, and this is just the Lax pair [4]. Thus, Liouville theory is intrinsically related to the geometry of 2-dimensional Riemann surfaces, i.e. gravity. This geometrical framework for the understanding of integrability allowed for the generalization of the Lax and Bäcklund equations to arbitrary background geometries [4][5].

Within the context of 2-dimensional gravity alone, there does not seem to be enough room for an extension to include Toda field theory as well. We shall show in the present paper that the correct geometrical framework for the understanding of the integrability of Toda field theory is that of 2–dimensional W–gravity with gauge group a Lie group $G$.

In a topological field theory formulation of W–gravity [6][7][8], the action is given by

$$ S = \int \text{tr}(N \mathcal{F}) \quad \mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \quad (1.1) $$

where $\mathcal{F}$ is the curvature of the $G$-connection $\mathcal{A}$ and $N$ is an auxiliary field. Components of $\mathcal{A}$ corresponding to generators of $G$ with height $h$ have spin $|h|+1$. Gauge transformations in $G$ intertwine field components of different spins, very much like in supersymmetry theories. We shall show that Toda field equations arise naturally as the restriction of the W–gravity connection to the case of components of height 1 only. A flat connection always exists as the Maurer–Cartan form on the group $G$, and the Lax pair is the equation of parallel transport on the surface. In particular, we obtain Lax pair and Bäcklund equations in the presence of an arbitrary background geometry.

The above construction is generalized to the case of W–supergravity as well. For supergroups $G$ where all simple roots can be chosen to be of odd grading, we derive the Toda Lax pair on arbitrary supergravity background geometries. For supergroups where some simple roots must be even, supersymmetry is broken, and the Toda field theory for these supergroups is coupled only to background gravity, but not supergravity.

Before we move on to describe Toda field theory and its N=1 supersymmetric extension, we briefly recall some basics on Lie (super) algebras. We define a Lie algebra or a Lie superalgebra (not necessarily finite dimensional) by the following relations in the Chevalley basis [9]. We have

$$ [h_i, h_j] = 0, \quad [h_i, x_{\alpha_j}] = k_{ij}x_{\alpha_j}, \quad [h_i, x_{-\alpha_j}] = -k_{ij}x_{-\alpha_j}, \quad [x_{\alpha_i}, x_{-\alpha_j}] = \delta_{ij}h_i \quad (1.2) $$

1 By abuse of notation, groups and algebras are not distinguished; e.g. $A_1$ stands for SU(2) or any non–compact version of it.

2 Throughout, $[..,..]$ denotes the graded commutator.
Here the generators of the Cartan subalgebra $H$ are denoted $h_i$ for $i = 1, \ldots, r = \text{rank } G$, the system of all roots is denoted $\Delta$ and $x_\alpha$ is the generator of $G$ associated with the root $\alpha \in \Delta$. The system of (positive) simple roots is denoted $\Delta_s$, and $k_{ij}$ is the Cartan matrix of $G$. The full algebra is closed using the Jacobi identity and the Serre relations, or equivalently the restriction

$$[x_\beta, x_\gamma] = c_{\beta, \gamma} x_{\beta + \gamma} \quad \text{where} \quad c_{\beta, \gamma} = 0 \text{ if } \beta + \gamma \notin \Delta \quad (1.3)$$

We may interpolate between the various compact and non-compact versions of the algebra by rescaling the structure constants $c_{\beta, \gamma}$ using arbitrary real constants $\{\mu_i\}$ as

$$c_{\beta, \gamma} \mapsto c_{\beta, \gamma} \prod_{i} \mu_i^{(|\beta^i + \gamma^i| - |\beta^i| - |\gamma^i|)} \quad (1.4)$$

where the components of $\gamma \in \Delta$ are defined by $\gamma \equiv \gamma^i \alpha_i$. The height of a generator is defined as follows. Simple positive roots have height 1, their negatives have height $-1$ and the commutator preserves the grading. For example, elements in the Cartan subalgebra $H$ have height 0.

2. Toda Theories for Ordinary Lie Groups Coupled to Gravity

Two dimensional Riemannian geometry may be defined by the frame, $e^a = d\xi^m e_m^a$, and the $U(1)$–connection, $\omega = d\xi^m \omega_m$. Covariant derivatives acting on tensors of weight $n$ are defined by

$$D^{(n)}_a = e^m_a (\partial_m + in\omega_m) \quad (2.1)$$

The metric is given by $g_{mn} = e^m_a e^n_b \delta_{ab}$. As usual, the torsion and the curvature are defined by the relation

$$[D_a, D_b] = T_{ab}^c D_c - i n \epsilon_{ab} R_g \quad (2.2)$$

Weyl transformations are defined by

$$e^m_a = \exp\{\phi\} \hat{e}^m_a \quad \omega_m = \hat{\omega}_m + \epsilon_m^n \partial_p \phi \quad (2.3)$$

under which torsion and curvature transform as

$$T_{ab}^c = \exp\{-\phi\} \hat{T}_{ab}^c \quad R_g = \hat{R}_g \exp\{-2\phi\} - 2D(z)D(\bar{z}) \phi \quad (2.4)$$

Upon setting $R_g=$constant in the last equation, we recover the Liouville equation, for which a generalized Lax pair was obtained in [4] in the following way.

Topological gravity is defined by an $A_1$ gauge field [7]

$$A = -i \omega J_3 + e^z J_z + e^{\bar{z}} J_{\bar{z}} \quad (2.5)$$

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3 Here, $\xi$ is a set of local coordinates. We denote coordinate indices by $m, n, \ldots$ and the $U(1)$ frame indices by $a, b, \ldots$, where $a = z, \bar{z}$. Also, $\delta_{z\bar{z}} = \delta_{\bar{z}z} = 1$, $c_{z\bar{z}} = -c_{\bar{z}z} = i$. 

2
where the $J$’s are the generators of $A_1$. Flatness of $A$ is equivalent to zero torsion and constant curvature, i.e. the Liouville equation, and the Maurer–Cartan form on $A_1$ always provides with such a connection. The Lax pair is the equation for parallel transport.

Topological W–gravity is a construction similar to that of topological gravity, but in which the group $A_1$ is replaced with an arbitrary Lie group $G$. Thus, we introduce a $G$–valued connection $A$, which may be decomposed as follows

$$A \equiv \sum_i \omega^i h_i + \sum_{\gamma \in \Delta} e^\gamma x_\gamma \quad (2.6)$$

Here, $\omega^i$ are the components of the Abelian connection with gauge group $H$, and $e^\gamma$ are a generalization of the frame on the Riemann surface $e^a$. Actually, $A$ may be viewed as a connection in the bundle $G$ with structure group $H$ over the manifold $G/H$. The latter is always a Kähler manifold, so the field contents of W–gravity may be viewed as resulting from embedding a Riemann surface into the Kähler manifold $G/H$ or equivalently from dimensional reduction of the manifold $G/H$ to a Riemann surface. Analogous embedding problems and their relation to Toda systems were considered in [10].

The corresponding gauge field strength is

$$F = \sum_i d\omega^i h_i + \frac{1}{2} \sum_{\gamma \in \Delta} e^\gamma \wedge e^{-\gamma} [x_\gamma, x_{-\gamma}] + \sum_{\gamma \in \Delta} \left[ de^\gamma + \sum_{i,j} \omega^i k_{ij} \gamma^j \wedge e^\gamma + \frac{1}{2} \sum_{\gamma' + \gamma'' = \gamma} e^{\gamma'} \wedge e^{\gamma''} e_{\gamma', \gamma''} x_\gamma \right] x_\gamma \quad (2.7)$$

The dynamics of topological W–gravity is given by action of (1.1) and reduces to $F = 0$. This produces “torsion constraints” from the second line above and constant curvature equations for the connections $\omega^i$ from the first line. These are just the constant torsion and curvature formulas for the $H$–connection on the manifold $G/H$.

Toda field theory is now simply obtained by restriction to the spin 1 and spin 2 fields in the connection $A$:

$$e^{\alpha_i} = \exp\{\phi_i\} e^z, \quad e^{-\alpha_i} = \exp\{\phi_i\} e^{-z}, \quad e^\gamma = 0 \text{ if } |\text{height}(\gamma)| \geq 2 \quad (2.8)$$

as well as

$$i \sum_j \omega^j k_{ji} = \omega \rho_i + e^a \epsilon^{a,b} D_b \phi_i \quad (2.9)$$

Using this expression, we see that the height of the field is transformed into its spin. Here $(e^a, \omega)$ describes an arbitrary 2–dimensional Riemannian geometry as given above, and $\rho_i = 1$, $i = 1, \ldots, r$ for this embedding. Clearly, more general embeddings may be chosen, as can be seen by applying an arbitrary W–gravity i.e. $G$–gauge transformations to Ansatz (2.8). The expression for the field strength reduces to

$$F = \sum_i \bar{e}^z \wedge e^z \left[ -(D_z \omega^i_{\bar{z}}) - D_{\bar{z}} \omega^i_z + \exp\{2\phi_i\} h_i + \exp\{\phi_i\} T z^{-2} x_{\alpha_i} + \exp\{\phi_i\} T z^{-2} x_{-\alpha_i} \right] \quad (2.10)$$
The W–gravity field equations \( F = 0 \) are equivalent to the zero torsion constraints of ordinary gravity plus the curvature equations

\[
2D_z D_{\bar{z}} \phi_i - \sum_j \exp\{2\phi_j\} k_{ji} + R_g \rho_i = 0
\]

which are precisely the Toda field equations. When \( G \) is a Kac–Moody algebra, Cartan matrix \( k \) has an eigenvector \( n_i \) with eigenvalue zero. As a result, the linear combination \( \sum n_i \phi_i \) obeys a linear equation. Upon elimination of this field, conformal invariance is broken. The Sine–Gordon system, for example, is obtained in this way from \( G= \hat{A}_1 \).

The Lax pair is identified as the equation for parallel transport under the \( G \)-connection of \( G \)-valued functions \( \psi \) in some representation of the group.

\[
(\partial_m + A_m) \psi = 0 \quad \Leftrightarrow \quad (D_\alpha + A_\alpha) \psi = 0 \tag{2.12}
\]

or in terms of frame index notation, using the explicit form of \( A \)

\[
(D_z + \sum_i \omega^i_z h_i + \sum_i \exp\{\phi_i\} x_{\alpha_i}) \psi = 0
\]

These equations now provide a Lax pair for Toda field theory on an arbitrary Riemann surface background geometry, with the connections \( \omega^i_z \) given by (2.3). Spectral parameters arise in the same way as they did for Liouville theory.

When the Cartan matrix has an inverse \( l_{ij} \) and is symmetric in \( i, j \), we may write an action from which the Toda equations are obtained

\[
S_\phi = \int d^2 \xi \sqrt{g} \sum_i \left[ \sum_j \{l_{ij} D_z \phi_i D_{\bar{z}} \phi_j - R_g l_{ij} \rho_i \phi_j\} + \frac{1}{2} \exp\{2\phi_i\} \right] \tag{2.14}
\]

From the Lax pair, we construct the Bäcklund transformation by passing from homogeneous variables \( \psi \) in the Lax equation to projective (inhomogeneous) coordinates. Saveliev [3] has introduced a particularly natural way of passing to inhomogeneous coordinates, including when \( \psi \) is in an arbitrary (finite-dimensional) representation \( \mu \) of \( G \).

For a representation \( \mu \) with highest weight vector \( \mu \) all other weights can be built up by applying the generators corresponding to negative simple roots to the highest weight \( |0; \mu\rangle \)

\[
|j_1 \ldots j_p; \mu\rangle \equiv x_{-\alpha_{j_p}} \ldots x_{-\alpha_{j_1}} |0; \mu\rangle \tag{2.15}
\]

Here, it is always understood that \( |j_1 \ldots j_p; \mu\rangle = 0 \) if the corresponding weight does not belong to the weight diagram of \( \mu \). Application of Cartan generators and simple roots is straightforward and may be found using the structure relations as

\[
\begin{align*}
  h_j |j_1 \ldots j_p; \mu\rangle & = \lambda_{j; \mu}^{(p+1)} |j_1 \ldots j_p; \mu\rangle \\
  x_{-\alpha_j} |j_1 \ldots j_p; \mu\rangle & = |j_1 \ldots j_p \hat{j}; \mu\rangle \tag{2.16} \\
  x_{\alpha_j} |j_1 \ldots j_p; \mu\rangle & = \sum_{q=1}^p \delta_{j, j_q} \lambda_{j_q; \mu}^{(q)} |j_1 \ldots \hat{j} q \ldots j_p; \mu\rangle
\end{align*}
\]
Here, the hat denotes omission and we shall use the abbreviation

$$\lambda_{j;\mu}^{(q)} \equiv \mu_j - \sum_{m=1}^{q-1} k_{jjm} \quad (2.17)$$

We define the Bäcklund variables $\psi_{j_1 \ldots j_p;\mu}$ through the matrix elements

$$\langle 0; \mu | \psi | 0; \mu \rangle \exp\{\psi_{j_1 \ldots j_p;\mu}\} = \langle j_1 \ldots j_p; \mu | \psi | 0; \mu \rangle \quad (2.18)$$

From the Lax pair (2.13) and equations (2.16) it is easy to see that $\psi_{j_1 \ldots j_p;\mu}$ satisfy

$$D_z \psi_{j_1 \ldots j_p;\mu} + \sum_{i=1}^r \left[ \lambda_{i;\mu}^{(p+1)} \omega_i^z + \exp\{\phi_i + \psi_{j_1 \ldots j_p;\mu} \right] = 0$$

$$D_{\bar{z}} \psi_{j_1 \ldots j_p;\mu} + \sum_{i=1}^r \left[ \lambda_{\bar{i};\mu}^{(p+1)} \omega_{\bar{i}}^z + \sum_{q=1}^p \delta_{\bar{i},j_q} \lambda_{\bar{i};\mu}^{(q)} \exp\{\phi_i + \psi_{j_1 \ldots \hat{j}_q \ldots j_p;\mu} \right] = 0 \quad (2.19)$$

These are the Bäcklund transformations for Toda field theory on an arbitrary background geometry and for arbitrary (finite-dimensional) representations of groups $G$. For $G=A_n$, and $\mu$ the fundamental representation, we get a Bäcklund transformation in the usual sense i.e. there are as many $\psi$’s as $\phi$’s, and the integrability on both are the Toda equations. It is not known to us whether for other groups or other representations, the above Bäcklund system can be further reduced; we suspect it cannot in general.

3. Toda theories coupled to supergravity

Two dimensional supergeometry may be defined by the zweibein, $E^A = d\xi^M E_M^A$, and the $U(1)$–connection, $\Omega = d\xi^M \Omega_M$. The covariant derivative acting on a superfield of $U(1)$ weight $n$ is given by

$$D_A^{(n)} \equiv E_A^M (\partial_M + in\Omega_M) \quad (3.1)$$

The torsion and the curvature tensors are defined by

$$[D_A, D_B] = T_{AB}^C D_C + inR_{AB} \quad (3.2)$$

acting on a superfield of weight $n$. The standard torsion constraints are imposed [11].

Topological W–supergravity is defined with respect to a supergroup $G$, and an associated $G$–valued connection $A$. This construction generalizes topological supergravity which is constructed on the supergroup $B(0,1) = OSp(1,1)$ [12]. We may parametrize this connection as

$$A \equiv \sum_i \Omega^i h_i + \sum_{\gamma \in \Delta} E^\gamma x_\gamma \quad (3.3)$$
The corresponding gauge field strength is

\[ F = \sum_i d\Omega^i h_i - \frac{1}{2} \sum_{\gamma \in \Delta} E^{-\gamma} \wedge E^{\gamma} [x_\gamma, x_{-\gamma}] \]

\[ + \sum_{\gamma \in \Delta} \left( dE^\gamma + \sum_{i,j} \Omega^i k_{ij} \gamma^j \wedge E^{\gamma} - \frac{1}{2} \sum_{\gamma', \gamma'' = \gamma} E^{\gamma'} \wedge E^{\gamma''} c_{\gamma', \gamma''} \right) x_\gamma \]  \hspace{1cm} (3.4)

The equations of motion of topological supergravity are the vanishing of this curvature \( F = 0 \), and may again be viewed as constant torsion and curvature equations of the H–connection \( \Omega^i \) on the supermanifold \( G/H \).

Two classes of supergroups must be distinguished. First, we have supergroups for which all simple positive roots may be chosen to have odd grading. In the Kac classification, the finite dimensional groups that have this property are \( A(n, n-1) \), \( B(n-1, n) \), \( B(n, n) \), \( D(n+1, n) \), \( D(n, n) \), \( D(2, 1; \alpha) \) \[9\]. For these groups, the corresponding Toda superfield theory can be covariantly coupled to \( N=1 \) supergravity. Second, there are all the other supergroups for which at least one simple root must have even grading. For these supergroups, Toda field theory is not supersymmetric and can be coupled covariantly only to ordinary gravity. This distinction originates from Toda field theory on flat superspace \[13\] where global supersymmetry is preserved only in the first case (see \[13\] and references therein).

We treat the first case first, and use the convention

\[ c_{\alpha_i, \alpha_j} = c_{-\alpha_i, -\alpha_j} = 2, \quad \text{for } \forall \alpha_i, \alpha_j \in \Delta_s \]  \hspace{1cm} (3.5)

We recover \( N=1 \) Toda field theory by making the following reduction

\[ E^\alpha_i = \exp\left\{ \frac{1}{2} \Phi_i \right\} \left[ E^+ + \mathcal{D}_+ \Phi_i E^z \right], \quad E^{-\alpha_i} = \exp\left\{ \frac{1}{2} \Phi_i \right\} \left[ E^- + \mathcal{D}_- \Phi_i E^\overline{z} \right] \]

\[ E^\alpha_i + \alpha_j = \exp\left\{ \frac{1}{2} (\Phi_i + \Phi_j) \right\} E^z \times \left\{ \begin{array}{cl} 2 & i \neq j \\ 1 & i = j \end{array} \right\}, \quad E^{-\alpha_i - \alpha_j} = \exp\left\{ \frac{1}{2} (\Phi_i + \Phi_j) \right\} E^\overline{z} \times \left\{ \begin{array}{cl} 2 & i \neq j \\ 1 & i = j \end{array} \right\} \]

\[ E^\gamma = 0 \quad \text{if } |\text{height}(\gamma)| \geq 3 \]  \hspace{1cm} (3.6)

where \( (E^A, \Omega) \) describes an arbitrary two dimensional supergeometry. We set the H-connections to

\[ i \sum_j \Omega^j k_{ji} = \frac{1}{2} \Omega \rho_i + E^A J_A^B \mathcal{D}_B \Phi_i \]  \hspace{1cm} (3.7)

where the supercomplex structure \( J_A^B \) is defined as

\[ J_A^B = \delta_A^B \times \begin{cases} +i & A = z, + \\ -i & A = \overline{z}, - \end{cases} \]  \hspace{1cm} (3.8)

The flatness condition \( F_{\pm \pm} = 0 \) evaluated on (3.6) and (3.7) reduces to the torsion constraints of ordinary supergravity and \( F_{++} = 0 \) yields Toda field equations in the presence of \( N=1 \) supergravity.

\[ \mathcal{D}_- \mathcal{D}_+ \Phi_i + \sum_j \exp\{\Phi_j\} k_{ji} - \frac{i}{2} R_{+-} \rho_i = 0 \]  \hspace{1cm} (3.9)
The remaining flatness conditions on $F$ vanish by derivatives of the torsion constraints and the Toda field equations.

The Lax pair is again easily identified as the equation for parallel transport under the G–connection

$$\left( D^{(0)}_A + A_A \right) \Psi = 0 \quad (3.10)$$

or in terms of U(1)–frame index notation, we have the equations for the components $A = \pm$

$$(D_+^{(0)} + \sum_i \Omega^i h_i + \exp\left(\frac{1}{2} \Phi_i \right) x_{\alpha_i}) \Psi = 0 \quad \quad (3.11)$$

The compatibility of the Lax pair (3.10) for $A = +, -$ reduces to the Toda equations (3.9), using the definition of $\Omega^i$ in (3.7). Using the equation (3.10) for $A = +$ twice, we find $A_z$ and $A_{\bar{z}}$

$$A_z = D_{\bar{z}}^{(\frac{1}{2})} A_+ + A_+ \wedge A_+, \quad A_{\bar{z}} = D_{z}^{(-\frac{1}{2})} A_- + A_- \wedge A_- \quad (3.12)$$

This, of course, agrees with the previous expression for $A$ in (3.6). Then using the Toda equation (3.9), the torsion constraints of supergeometry and the Jacobi identity, we find that the equations (3.10) for all indices $A$ are compatible.

In the case of supergroups for which all simple roots cannot be chosen to be odd, we may only couple the Toda theory to gravity. Gravity is embedded in supergravity by setting the gravitino and auxiliary fields to zero in the Wess–Zumino gauge \[11\]

$$D_{\pm}^{(n)} = \partial_{\theta} + \theta D_{\pm}^{(n)} - \frac{i}{2} \theta \bar{\omega}_{\pm} \bar{\partial}_{\theta}, \quad D_+^{(n)} = \partial_{\bar{\theta}} + \bar{\theta} D_{\bar{z}}^{(n)} - \frac{i}{2} \bar{\theta} \omega_+ \partial_{\bar{\theta}} \quad (3.13)$$

where $D^{(n)}_a$ was defined in (2.1). The gauge field is of the following form

$$A_+ = \sum_i \Omega^i h_i + \sum_{\alpha_i \in \Delta_s^{odd}} \exp\left(\frac{1}{2} \Phi_i \right) x_{\alpha_i} + \sum_{\alpha_i \in \Delta_s^{even}} \theta \exp\left(\frac{1}{2} \Phi_i \right) x_{\alpha_i} \quad (3.14)$$

$$A_- = \sum_i \Omega^i h_i + \sum_{\alpha_i \in \Delta_s^{odd}} \exp\left(\frac{1}{2} \Phi_i \right) x_{-\alpha_i} + \sum_{\alpha_i \in \Delta_s^{even}} \bar{\theta} \exp\left(\frac{1}{2} \Phi_i \right) x_{-\alpha_i}$$

where $\Delta_s^{odd}, \Delta_s^{even}$ denote the set of positive odd and even simple roots, respectively. The H–connections $\Omega^j$ may be reexpressed in terms of $\Phi_i$ by exactly the same relation as before, namely equation (3.7). The rest of the gauge field, $A_a$ are defined using the relation (3.11).

The condition $\mathcal{F}_{+-} = 0$ is equivalent to the Toda equation coupled to gravity,

$$D_- D_+ \Phi_i + \sum_{\alpha_j \in \Delta_s^{odd}} \exp\left(\Phi_j \right) k_{ji} + \sum_{\alpha_j \in \Delta_s^{even}} \theta \bar{\theta} \exp\left(\Phi_j \right) k_{ji} - \frac{i}{2} R_+ - \rho_i = 0 \quad (3.15)$$

The torsion constraints, the Toda field equations and the Jacobi identity guarantee that the other components of $\mathcal{F}_{AB} = 0$ are satisfied.
The Bäcklund transformations may be obtained from the Lax equations in analogy with the methods used in the purely bosonic case. Since one has to divide by the \( \Psi \)-field corresponding to the highest weight, we must consider representations in which the highest weight has even grading. This is always possible, since the Lax equations are linear. The relation between \( A_a \) and \( A_\alpha \), (3.12), is automatically satisfied by the Lax equation, (3.10), since \( A = \mathcal{D} \Psi \Psi^{-1} \).

The action for the Toda field theory associated with a super Lie algebra may be obtained when the Cartan matrix is symmetric and invertible

\[
S_\Phi = \int d^2 \xi d^2 \theta E \left[ \sum_{i,j} \left\{ \frac{1}{2} l_{ij} D_i \Phi_j D_j - \frac{i}{2} R_{+l} l_{i} \rho_i \Phi_j \right\} + \sum_{\alpha_i \in \Delta \text{odd}} \exp\{\Phi_i\} + \sum_{\alpha_i \in \Delta \text{even}} \theta \bar{\theta} \exp\{\Phi_i\} \right]
\]

(3.16)

This action is written in superfield language, but is (locally) supersymmetric only when \( \Delta_{\text{even}} \) can be chosen to be empty.

4. Summary

In this paper, we have shown that Toda field theories coupled to gravity or supergravity provide a class of solutions to the W–gravity equations of motion. Also, we obtained the Lax pair for the Toda system on an arbitrary (super) Riemann surface. This was shown for arbitrary Lie algebras and superalgebras, including the infinite dimensional ones. It is natural to expect that Toda field theory can in fact be coupled to an arbitrary W–gravity, providing an integrable system as well.

Let us point out that the presence of (global) W–symmetries \([14]\) in Toda field systems was discussed in \([13]\). Also, some work on the relation between W–gravity and Toda field theory has been presented in \([16]\).
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