Multiplexing induced explosive synchronization in Kuramoto oscillators with inertia

Ajay Deep Kachhvah and Sarika Jalan

Complex Systems Lab, Physics Discipline, Indian Institute of Technology Indore - Simrol, Indore-453552, India

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Abstract – Explosive synchronization (ES) of coupled oscillators on networks is shown to be originated from the existence of correlation between natural frequencies of oscillators and degrees of corresponding nodes. Here, we demonstrate that ES is a generic feature of multiplex network of second-order Kuramoto oscillators and can exist in the absence of a frequency-degree correlation. A monoplex network of second-order Kuramoto oscillators bearing homogeneous (heterogeneous) degree distribution is known to display the first-order (second-order) transition to synchronization. We report that multiplexing of two such networks having homogeneous degree distribution support the first-order transition in both the layers thereby facilitating ES. More interesting is the multiplexing of a layer bearing heterogeneous degree distribution with another layer bearing homogeneous degree distribution, which induces a first-order (ES) transition in the heterogeneous layer which was incapable of showing the same in isolation. Further, we report that such induced ES transition in the heterogeneous layer of multiplex networks can be controlled by varying inter- and intra-layer coupling strengths. Our findings emphasize the importance of multiplexing or the impact of one layer on the dynamical evolution of other layers of systems having inherent multiplex or multilevel architecture.

Introduction. – Networks of coupled oscillators provide a useful paradigm for understanding diverse processes such as epidemic spreading [1], random walks [2], traffic congestion [1,3] and synchronization [4,5]. One of the important factors influencing the synchronization of coupled oscillators is the topology of the underlying network connecting them [5–9]. It is well known that Kuramoto oscillators display second-order transition to the synchronous state [5]. In recent years, a new phenomenon called explosive (discontinuous) synchronization has attracted the attention of many researchers. Explosive synchronization (ES) had been considered to be an outcome of microscopic correlation between the network topology and the natural frequencies of the Kuramoto phase oscillators [10–14]. However, Tanaka et al. [15,16], in their seminal work pointed out that a network of second-order Kuramoto oscillators, i.e., Kuramoto oscillators with an inertia term, is capable of displaying a discontinuous (first-order) transition to a synchronous state inherently in the absence of any correlation between natural frequencies and network degrees of oscillators. Furthermore, recently Peng et al. [17,18] demonstrated that scale-free networks of second-order Kuramoto oscillators with frequency-degree correlation exhibit cluster explosive synchronization. Additionally, similar to the first-order Kuramoto oscillators [10], the discontinuous transition can emerge in a strongly assortative scale-free network of the second-order Kuramoto oscillators in the presence of a positive correlation between their degrees and the natural frequencies [19].

Further, to fully investigate the properties of a divergent range of real-world networks such as transportation [20], social [21], brain [22] or infrastructure [23], a multilayer or multiplex framework [24–28] approach is becoming necessary. Recently it has been demonstrated that adaptive and multilayer networks of first-order Kuramoto oscillators display ES in the absence of a degree-frequency correlation [29]. In this paper, we study the phase transition in a two-layered multiplex network of second-order Kuramoto oscillators. It is shown that the second-order Kuramoto oscillators on a monoplex network bearing homogeneous degree distribution, such as globally connected, random
and small-world networks, exhibit the first-order synchronization transition. We show that when a network bearing homogeneous degree distribution is multiplexed with another network having the same type of degree distribution, both networks display first-order (explosive) synchronization transition. More interesting is the behavior depicted by the networks having heterogeneous degree distribution. A monoplex network of second-order Kuramoto oscillators having heterogeneous degree distribution such as a scale-free network displays a second-order phase transition, however, when multiplexed with a network having homogeneous degree distribution, it starts displaying a first-order (explosive) synchronization transition. Hence, our investigation reveals that the first-order transition in a network can be induced via multiplexing it with another layer which already exhibits first-order transition. Further, for the scale-free networks, such induced first-order transition upon multiplexing is weak and can be enhanced by controlling multiplexing properties such as inter- and intra-layer coupling strengths. Impact of the inter-layer coupling strength on synchronizability in a multiplex network is also reported by Dwivedi et al. [30]. We demonstrate that an increase in the inter-layer coupling leads to a gradual strengthening of the first-order (ES) nature of the transition. On the contrary, increasing the intra-layer coupling strength in the scale-free layer makes the already weak first-order transition further weaker and eventually reverting it to the second order.

Theoretical framework. – For a monoplex network of $N$ second-order Kuramoto phase oscillators, the evolution of each oscillator is ruled by

$$m\ddot{\theta}_i(t) + \dot{\theta}_i(t) = \omega_i + \lambda \sum_{j=1}^{N} a_{ij} \sin[\theta_j(t) - \theta_i(t)],$$  \hspace{1cm} (1)

where $i = 1, \ldots, N$, $\omega_i (\theta_i(t))$ is the natural frequency (the instantaneous phase) of the $i$-th oscillator, parameter $m$ is mass, $\lambda$ is homogeneous coupling strength and $\langle k \rangle$ is the average degree of the network. Elements of the network’s adjacency matrix are indicated by $a_{ij}$, $a_{ij} = 1$ when the nodes $i$ and $j$ are connected and $a_{ij} = 0$, otherwise. The extent of synchronization in a network can be measured by a global order parameter $R$ defined as $R e^{i\psi} = (1/N) \sum_{j=1}^{N} e^{i\theta_j}$, where $\psi$ is the network’s average phase, and $0 \leq R \leq 1$. The order parameter $R$ takes the value 0 for an incoherent state and 1 for a phase-synchronized state. Upon progressively increasing the coupling strength $\lambda$, the coupled oscillators undergo a change in phase-state from the incoherent to the synchronous one at a critical coupling strength. This process is referred to as phase transition to synchronization.

Next, we consider a multiplex network (fig. 1) consisting of two layers of the same size $N$. Each node is represented by a second-order Kuramoto oscillators with its evolution governed by

$$m\ddot{\theta}_{i,1} + \dot{\theta}_{i,1} = \omega_{i,1} + \frac{\lambda}{\langle k \rangle} \sum_{j=1}^{N} M_{ij} \sin[\theta_{j,1} - \theta_{i,1}],$$

$$m\ddot{\theta}_{i,2} + \dot{\theta}_{i,2} = \omega_{i,2} + \frac{\lambda}{\langle k \rangle} \sum_{j=1}^{N} M_{ij} \sin[\theta_{j,2} - \theta_{i,2}],$$  \hspace{1cm} (2)

where $i = 1, \ldots, N$, and subscripts 1, 2 stand for the first and second layers, respectively. $D_x$ is the inter-layer coupling strength. $M_{ij}$ is the element of the adjacency matrix of multiplex network $M$ denoted as

$$M = \begin{bmatrix} A_1 & D_x I \\ D_x I & A_2 E_y \end{bmatrix},$$  \hspace{1cm} (3)

where $E_y$ stands for the intra-layer coupling strength of the second layer, $I$ is the identity matrix and $A_1$ and $A_2$ are the adjacency matrices of the first and second layers, respectively.

We track the phase transition to synchronization in both the layers of a multiplex network by means of the order parameters $R_1$ and $R_2$, respectively, given by

$$R_1 e^{i\psi_1} = (1/N) \sum_{j=1}^{N} e^{i\theta_{j,1}},$$

$$R_2 e^{i\psi_2} = (1/N) \sum_{j=1}^{N} e^{i\theta_{j,2}},$$  \hspace{1cm} (4)

First, we adiabatically increase the coupling strength $\lambda$ from $\lambda_0$ (incoherent) to $\lambda_0 + n\delta \lambda_0$ (synchronous) state in steps of $\delta \lambda_0$. Second, we adiabatically decrease the coupling strength from $\lambda_0 + n\delta \lambda_0$ (synchronous) to $\lambda_0$ (incoherent) in steps of $\delta \lambda_0$. We compute the order parameters $R_1$ and $R_2$ for $\lambda_0, \lambda_0 + \delta \lambda_0, \ldots, \lambda_0 + n\delta \lambda_0$ for both the increasing (forward, i.e., $F$) and decreasing $\lambda$ (backward, i.e., $B$) directions. Before each $\delta \lambda_0$ step, we eliminate initial transients and integrate the system long enough ($10^4$ time steps) using the fourth-order Runge-Kutta method with time steps $dt = 0.01$, to arrive at the stationary states. For all simulations, the initial phases for oscillators in individual layer are selected from a random uniform distribution in the range $[0,2\pi]$, and the natural frequencies of nodes of each network are drawn from a random uniform distribution from $[-1,1]$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(Colour online) Schematic diagram of a 3-layered multiplex network comprised of globally connected and random networks. Blue solid lines denote intra-layer links while red dashed lines denote inter-layer links.}
\end{figure}
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Fig. 2: (Colour online) Synchronization transition plots ($R$ vs. $\lambda$) of a single-layer network for (a) globally connected, (b) ER, (c) WS and (d) SF network topologies. The size and average degree of each network is $N = 100$ and $\langle k \rangle = 12$.

**Synchronization transition in monoplex networks.** – Before presenting results for the synchronization transition (ST) in a multiplex network, we discuss ST in a monoplex network having different network architectures, such as globally connected, Erdős-Rényi (ER) random, small-world (SW) and scale-free (SF) networks generated using Barabási-Albert model\(^1\). Figure 2 depicts the behavior of the order parameter $R$ as a function of the coupling strength for these different network models. It is evident that the globally connected, ER and SW networks exhibit a first-order transition with an associated hysteresis loop for the order parameters $R^F$ and $R^B$ indicating an occurrence of ES. Values of the critical couplings $\lambda^F_c$ and $\lambda^B_c$ in the forward and the backward directions, respectively, for the onset of ES are slightly different for the globally connected, ER and SW networks, whereas the SF networks exhibit a usual second-order transition with critical coupling $\lambda_c = 2.1$.

**Synchronization transition in multiplex networks.** – Here we discuss investigation of ST in a multiplex network for various combinations of network architectures for two layers. To systematically investigate the effects of multiplexing, we fix the first layer of the multiplex network to a globally connected network and choose the architecture of the second layer from ER, SW and SF network models.

For a multiplex network with the first and second layers represented by the globally connected networks, both the layers exhibit the first-order transitions and the related hysteresis loops with values of $\lambda^F_c$ and $\lambda^B_c$ being 1.5 and 1.3, and 1.5 and 1.2, respectively (fig. 3). It indicates that upon multiplexing, the critical coupling $\lambda^F_c$ for both the layers increases and the hysteresis width either manifests an increase or remains the same. We further observe that when the second layer is comprised of either ER or SW networks, it exhibits the first-order (ES) transition and associated hysteresis loop for the forward and the backward directions in $\lambda$ (fig. 4(a) and fig. 5(a)), respectively. Moreover, the hysteresis widths corresponding to the ER and SW networks depict an enhancement upon multiplexing as compared to those for the corresponding monoplex networks (fig. 2(b) and fig. 2(c)). However, when the second layer is comprised of the SF network, multiplexing with a globally connected layer yields an interesting behavior. For this arrangement, the second layer also starts exhibiting a first-order (weak) transition with a hysteresis loop in the forward and in the backward direction (fig. 6(a)) as compared to the second-order transition manifested by the monoplex SF network (fig. 2(d)).

\(^1\)In Erdős-Rényi (ER) random network, nodes are connected by edges chosen at random with probability $0 < P < 1$ [31]. A Watts-Strogatz (SW) small-world network is generated by rewiring edges at random with probability $0 < p < 1$, in a ring lattice with nearest neighbors connections [32]. In the scale-free (BA model) networks, new nodes are added in accordance with preferential attachment and are characterized by a power-law degree distribution [33].

Fig. 3: (Colour online) Synchronization plot of a multiplex network comprising two layers of a globally connected network. The size of each layer is $N = 100$.

Fig. 4: (Colour online) $R_1, R_2$ vs. $\lambda$ plots of a multiplex network with the first layer fixed to a globally connected network and the second layer to an ER network: (a) $D_x = 1$, (b) $D_x = 4$ and (c) $D_x = 8$. The size of each layer is $N = 100$ and the average degree of the second layer is $\langle k \rangle = 12$.

Fig. 5: (Colour online) $R_1, R_2$ vs. $\lambda$ plots of a multiplex network with the first layer fixed to a globally connected network and the second layer to a SW network: (a) $D_x = 1$, (b) $D_x = 4$ and (c) $D_x = 8$. The size of each layer is $N = 100$ and the average degree of the second layer is $\langle k \rangle = 12$. 

Effect of the inter-layer coupling strength on ST. We have already witnessed that multiplexing two layers affects the synchronization properties of both the layers. It is therefore interesting to trace how variations in the
inter-layer coupling strength further affect the transition in the second layer. First, we investigate the impact of the inter-layer coupling strength on ST in a multiplex network with the first layer fixed to a globally connected network and the second layer represented by an ER network. For the usual case of inter-layer coupling strength being the same as that of intra-layer (i.e., $D_x = 1$), ST in an ER network exhibits the first-order transition with the existence of the hysteresis loop, and the dynamical evolution of the ER layer does not synchronize simultaneously with that of the globally connected layer (see fig. 4(a)). For a larger value of inter-layer coupling strength (say $D_x = 4$), a stronger first-order transition is observed for the second layer. Additionally, the hysteresis loop for the second layer now exhibits a coherent behavior with that of the first layer (see fig. 4(b)). For a further increase in the inter-layer coupling strength the first-order ST for the second layer, corresponding to a large $D_x = 8$, becomes as strong as that of the first layer and the hysteresis loop associated to it now is in complete coherence with that of the first layer (see fig. 4(c)). Second, we investigate ST in the SW networks upon its multiplexing with a layer represented by a globally connected network. It is found that for $D_x = 1$, the SW layer displays the first-order transition (though weak as compared to the ER layer case) with a hysteresis loop which is not in coherence with that of the first layer as depicted in fig. 5(a). However, when the inter-layer coupling $D_x$ is increased to 4, the first-order ST observed for the SW layer becomes stronger. Moreover, both the layers now synchronize with each other accompanied with a slight reduction in their hysteresis widths (fig. 5(b)). For a larger inter-layer coupling ($D_x = 8$), the first-order ST for the second layer becomes almost as strong as that of the globally connected layer. Both the layers synchronize simultaneously and their hysteresis widths are reduced further to a new low as displayed in fig. 5(c). On the basis of the above simulation results, we infer that a strong inter-layer coupling strengthens the ES transition in a non-globally connected multiplex layer.

Next, we investigate the effects of $D_x$ on ST in a multiplex network with the first layer fixed to a globally connected and the second layer to a SF network. As displayed by fig. 6(a), for the usual case of $D_x = 1$, there is a weak first-order transition with a hysteresis loop in the forward and in the backward directions. Furthermore, for the SF layer, $R_2^f$ and $R_2^b$ do not get synchronized simultaneously with those of the globally connected first layer. Upon increasing $D_x$ to 2, the first-order ST in the second layer gets enhanced or becomes stronger, and the associated hysteresis width widens and gets synchronized almost simultaneously with the first layer (fig. 6(b)). For $D_x = 4$, we observe an enhancement in ST of the second layer, and the second layer now exhibits a stronger first-order transition accompanied with a broader hysteresis loop (fig. 6(c)). Moreover, it gets synchronized simultaneously with the first layer. For a further increase in $D_x$, a much stronger first-order ST is observed for the second layer displaying simultaneous synchronization with the first layer (fig. 6(d)). However, such a large inter-layer coupling ($D_x = 8$) brings upon a reduction in the width of the synchronized hysteresis loops corresponding to both the first and the second layers. In this way, an increased $D_x$ with one layer having globally connected architecture (yielding first-order ST) helps another layer having a heterogeneous degree distribution (second-order ST) in achieving a first-order (explosive) ST. Therefore, choosing a high inter-layer coupling $D_x$ allows an enhancement in ST for multiplex networks.

To better understand the underlying dynamics behind the ES transition witnessed for the above case of a SF network multiplexed with a globally connected network, we perform detailed investigation of the nodes behavior. Since networks with homogeneous and heterogeneous degree distributions manifest different behavior, we calculate average frequency of all the nodes with degree $k$. The average frequency for a degree $k$ is defined as

$$\langle \omega \rangle_k = \sum_{i|k_i = k} \omega_i / N_k; \quad \omega_i = \int_{t_i}^{t_i + T} \dot{\theta}_i(t)dt / T, \quad (5)$$

where $N_k$ is the number of nodes with degree $k$, and $T$ is the total time of averaging after eliminating initial transients. Figure 7 plots the average frequency $\langle \omega_k \rangle_k$ of all the nodes of degree $k$ in the SF network for different values of $D_x$. It is quite apparent that not all the nodes gets synchronized to the large synchronous component at the critical coupling $\lambda_c^f = 1.4$. Figure 7(a) infact indicates that the nodes with large degree converge first to a common average frequency at $\lambda_c^f$, while the nodes with small degree achieve synchronization post $\lambda_c^f$. A weak explosive synchronization is observed for $D_x = 1$ as only a few nodes are synchronized subsequent to the critical coupling $\lambda_c^f$. The number of nodes synchronizing post the critical coupling $\lambda_c^f = 1.5$ gets further reduced for $D_x = 2$ (fig. 7(b)). For the stronger couplings $D_x = 4$ and $D_x = 8$, nodes of all the
degrees synchronize simultaneously at the critical coupling \( \lambda_F^c = 1.5 \) (as depicted in fig. 7(c) and fig. 7(d)), yielding an obvious first-order (explosive) transition. It is apparent that the larger \( D_x \) values do not influence the onset of synchronization as the value of \( \lambda_F^c \) remains fixed to 1.5 for higher values of \( D_x \) in a network of 100 nodes. From these observations we can conclude that strengthening the inter-layer coupling (or multiplexing impact) greatly enhances the underlying dynamics towards ES.

**Effect of intra-layer coupling strength on ST.** Next, we discuss impact of relative mis-match in intra-layer coupling for layer, on ST for multiplex networks. Let us first consider a multiplex network comprising a globally connected and SF networks. \( E_y \) is introduced in the SF layer of the multiplex network. As we have already seen that for a usual choice of \( D_x = 1 \) and \( E_y = 1 \) (fig. 8(b)), the first layer having globally connected and the second layer having SF topology display a strong discontinuous and weakly discontinuous transitions (seen at \( \lambda_F^c = 1.4 \)), respectively. However, when \( E_y \) is set to 0, which corresponds to each node of the second layer being connected merely to a single node in the first layer via the inter-layer couplings, this arrangement leads to an enhancement in ST in the second layer causing a strong first-order transition (see fig. 8(a)) which synchronizes simultaneously with the first layer. On the contrary, for an increased \( E_y \) (say \( E_y = 4 \), a weak discontinuous ST in the SF network becomes further weaker with no change in the onset of synchronization or \( \lambda_F^c = 1.4 \) (see fig. 8(c)). For a more strong \( E_y \), the weak first-order ST with its hysteresis loop gets disappeared and instead a second-order transition is observed. Therefore, a gradual increase in the intra-layer coupling strength weakens the first-order (explosive) ST further and eventually turns it into a second-order one with no hysteresis. Hence, a lower value of \( E_y \) in the heterogeneous layer is good for the enhancement of explosive synchronization in the same layer.

**Effect of number of layers and size of network on ST.** It is important to investigate how the nature of synchronization is affected by the change in both the number of layers and the number of nodes in multiplex networks. Figure 9 presents ST for 3-layered multiplex networks corresponding to two different network size of 100 and 500 nodes. We present the results for a multiplex network consisting of one globally connected and two scale-free layers, each of (a) 100 and (b) 500 nodes. The average degrees of the second and third layers are \( \langle k \rangle = 12 \).
inducing ES transition in both the SF layers (having heterogeneous degree distribution). Besides, the size of the network has no significant effect on the nature of ST and only affects the onset of synchronization, i.e., the critical coupling strength. For instance, fig. 9 indicates that the pairs of values of the critical coupling strengths \{\lambda^F, \lambda^B\} for network sizes 100 and 500 are \{1.5, 1.3\} and \{1.7, 1.3\}, respectively.

**Effect of non-globally connected layers on ST.** Next, we consider both layers of a multiplex network represented by non-globally connected networks. We consider a multiplex network consisting of the first layer fixed to an ER network and the second layer is chosen from an ER, SW or a SF network. However for these case, the behavior of ST is found to be similar to that of the respective cases of multiplex networks with first layer fixed to a globally connected network. Both the globally connected and ER monoplex networks bear homogeneous degree distributions and display a first-order (explosive) ST. From fig. 10(a) and fig. 10(b), it is quite apparent that the second layer comprising ER or SW network gets synchronized simultaneously with the first layer, with the forward and the backward transitions in \lambda exhibiting hysteresis loops. However, when the second layer is comprised of a SF network, it exhibits a first-order (weak) transition with associated hysteresis loop in the forward and the backward directions (fig. 10(c)). This is an interesting revelation as this combination of layers multiplexed with each-other displays ST behavior similar to that seen in the case of a multiplex network comprising the globally connected and the SF layers. Hence, we deduce that multiplexing a layer having homogeneous degree distribution to an another layer having homogeneous degree distribution induces a first-order (ES) transition in the former layer which was displaying a second-order transition in isolation. It remains to be investigated further why a first-order transition always gets induced in the heterogeneous layer and the reverse, i.e., a second-order transition induced in the homogeneous layer, does not occur.

**Discussions.** In this article, we explored the synchronization transition in second-order Kuramoto oscillators on multiplex network and focused on understanding how the network structure in one layer affects the synchronization transition in another layer. We observe that the second-order Kuramoto oscillators on a monoplex network having homogeneous degree distribution, such as a globally connected, ER or SW network, displays first-order (ES) transition, while the coupled dynamics on networks having heterogeneous degree distribution, such as SF networks, displays a second-order phase transition. We have reported that when a layer with homogeneous degree-distribution is multiplexed with a second layer bearing a similar type of degree distribution, both layers display a first-order ST. However, a network which follows a second-order transition in isolation such as the networks which are highly heterogeneous, upon multiplexing with another network having homogeneous degree distribution, starts displaying a weak first-order ST. Hence, the first-order transition in the second-order Kuramoto oscillators can be induced via multiplexing in those networks which exhibit a second-order ST in isolation. Therefore, in a two-layered multiplex network if one layer comprises a network with a homogeneous degree distribution another layer will display a first-order transition irrespective of its distribution type. It is further numerically demonstrated that such induced rather weak ES transition can be enhanced making it stronger by controlling the inter-layer coupling. Similarly, it can be weakened further eventually turning into a second-order transition by controlling the relative intra-layer couplings of the layers of the multiplex network.

It has been sought to explain the occurrence of ES. For instance, it has been shown that the coexistence of disassortativity in the degrees and the natural frequencies of the nodes leads to ES [34]. Furthermore, it was demonstrated that the dynamical origin of the hysteresis corresponding to ES in a heterogeneous network results from the change in the basin of attraction of the synchronized state [35]. Moreover, using the Ott-Antonsen approach [36], it was reported that in a low-dimensional dynamical space corresponding to the globally synchronized state in heterogeneous networks different bifurcations reveal various transitions among diverse collective states such as fixed points and limit cycles, and ES is determined by the bistable state [37]. The reason for witnessing the first- and second-order transitions for the SF layer in the multiplex network considered here lies in the heterogeneity of degrees, which we have explained by computing the average frequencies of nodes having the same degree in the SF layer. When a SF layer is multiplexed with a globally connected layer the hubs or higher-degree nodes in the SF layer get synchronized first. However, the inter-layer coupling strength \(D_x = 1\) is not strong enough for the lower-degree nodes, driven by globally connected nodes, to get synchronized. Therefore, we observe a weak ES transition. When \(D_x\) is sufficiently high (say \(D_x = 4\)), the lower-degree nodes also, along with the hubs, driven by the globally connected nodes, get synchronized leading to a strong ES transition.

To conclude, earlier studies have shown that coupled dynamical behaviors such as cluster synchronization of a layer can be governed by the network properties of other
layers in a multiplex network [38]. Here, we demonstrate that ES of one or more than one layer can be achieved by multiplexing them with an appropriate network structure.

The Kuramoto oscillators with inertia provide a more realistic model of many systems such as power grid [39–41], disordered arrays of Josephson junctions [42], etc. In a power grid, the first- and second-order Kuramoto oscillators correspond to loads and generators, respectively. These realization have lead a spurth in the activities of investigation of second-order Kuramoto oscillators on networks to get insights into the underlying mechanism behind emerging phenomena due to interactions of nonlinear dynamical units. All these investigations on second-order Kuramoto oscillators are restricted to monoplex networks. This article investigates the impact of multiplexing on the behavior of second-order Kuramoto oscillators, and shows that there is a relation between the strength of multiplexing (inter-layer coupling strength) and ST. The results are important to understand those transitions observed in real-world complex systems inherently having multi-layers architecture.

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REFERENCES

[1] Barrat A., Barthélemy M. and Vespignani A. (Editors), Dynamical Processes on Complex Networks (Cambridge University Press, Cambridge, England) 2008.
[2] Noh J. D. and Rieger H., Phys. Rev. Lett., 92 (2004) 118701.
[3] Boccaletti S.; Latora V., Moreno Y., Chavez M. and Hwang D., Phys. Rep., 424 (2006) 175.
[4] Pecora L. M. and Carroll T. L., Phys. Rev. Lett., 80 (1998) 2109.
[5] Arenas A., Díaz-Guilera A., Kurths J., Moreno Y. and Zhou C., Phys. Rep., 469 (2008) 93.
[6] Peron T. K. D., Rodrigues F. A. and Kurths J., Phys. Rev. E, 87 (2013) 032807.
[7] Moreno Y. and Pacheco A. F., Europhys. Lett., 68 (2004) 603.
[8] Lee D.-S., Phys. Rev. E, 72 (2005) 026208.
[9] Arenas A., Díaz-Guilera A. and Pérez-Vicente C. J., Phys. Rev. E., 96 (2006) 114102.
[10] Gómez-Gardeñes J., Moreno Y. and Arenas A., Phys. Rev. Lett., 98 (2007) 034101.
[11] Gómez-Gardeñes J., Gómez S., Arenas A. and Moreno Y., Phys. Rev. Lett., 106 (2011) 128701.
[12] Leyva I. et al., Phys. Rev. Lett., 108 (2012) 168702.
[13] Peron T. K. D. and Rodrigues F. A., Phys. Rev. E, 86 (2012) 016102.
[14] Peron T. K. D. and Rodrigues F. A., Phys. Rev. E, 86 (2012) 056108.
[15] Tanaka H., Lichtenberg A. J. and Oishi S., Phys. Rev. Lett., 78 (1997) 2104.
[16] Tanaka H., Lichtenberg A. J. and Oishi S., Physica D, 100 (1997) 279.
[17] Ji P., Peron T. K. DM., Menc P. J., Rodrigues F. A. and Kurths J., Phys. Rev. Lett., 110 (2013) 218701.
[18] Ji P., Peron T. K. DM., Rodrigues F. A. and Kurths J., Phys. Rev. E, 90 (2014) 062810.
[19] Peron T. K. DM., Peng Ji, Rodrigues F. A. and Kurths J., Phys. Rev. E, 91 (2015) 052805.
[20] Gallotti R. and Barthelemy M., Sci. Data, 2 (2015) 140056.
[21] Szell M., Lambiotte R. and Thurner S., Proc. Natl. Acad. Sci. U.S.A., 107 (2010) 13636.
[22] Bullmore E. and Sporns O., Nat. Rev. Neurosci., 10 (2009) 186.
[23] Kurant M. and Thiran P., Phys. Rev. Lett., 96 (2006) 138701.
[24] Boccaletti S. et al., Phys. Rep., 544 (2014) 1.
[25] Leyva I. et al., Sci. Rep., 7 (2017) 45475.
[26] Sarkar C., Yadav A. and Jalan S., EPL, 113 (2016) 18007.
[27] Shinde P. and Jalan S., EPL, 112 (2015) 58001.
[28] Singh A., Jalan S. and Boccaletti S., Chaos, 27 (2017) 043103.
[29] Zhang X., Boccaletti S., Guan S. and Liu Z., Phys. Rev. Lett., 114 (2015) 038701.
[30] Dwivedi S. K., Sarkar C. and Jalan S., EPL, 111 (2015) 10005.
[31] Erdős P. and Rényi A., Publ. Math. Instrum. Hung. Acad. Sci., 5 (1960) 17; Bollobas B. (Editor), Random Graphs (Academic Press, London) 1985.
[32] Watts D. J. and Strogatz S. H., Nature, 393 (1998) 440.
[33] Barabási A.-L. and Albert R., Science, 286 (1999) 509.
[34] Zhu L., Tian L. and Shi D., Phys. Rev. E, 88 (2013) 042921.
[35] Zou Y., Pereira T., Small M., Liu Z. and Kurths J., Phys. Rev. Lett., 112 (2014) 114102.
[36] Ott E. and Antonsen T. M., Chaos, 18 (2008) 037113.
[37] Xu C., Gao J., Sun Y., Huang X. and Zheng Z., Sci. Rep., 5 (2015) 12039.
[38] Jalan S. and Singh A., EPL, 113 (2016) 30002.
[39] Filatrella G., Nielsen A. H. and Pedersen N. F., Eur. Phys. J. B, 61 (2008) 485.
[40] Dörfler F., Chertkov M. and Bullo F., Proc. Natl. Acad. Sci. U.S.A., 110 (2005) 2013.
[41] Pinto R. S. and Saa A., Physica A, 463 (2016) 77.
[42] Trees B. R., Saranathan V. and Stroud D., Phys. Rev. E, 71 (2005) 016215.