A NOTE ON POSITIVE PARTIAL TRANSPOSE BLOCKS

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Abstract. In this article, we study the class of PPT blocks. We introduce several inequalities, related to this class, with emphasis on comparing the main diagonal and the off-diagonal components of a $2 \times 2$ PPT block.

1. Introduction

Let $M_n$ be the algebra of all $n \times n$ complex matrices. For $X \in M_n$, the notation $X \geq 0$ (resp. $X > 0$) will be used to mean that $X$ is positive semidefinite (resp. positive definite). If $X, Y \in M_n$ are two Hermitian matrices in $M_n$, we write $X \leq Y$ to mean $Y - X \geq 0$. The unitarily invariant norm of $X \in M_n$ is denoted by $\|X\|$. Recall that a norm $\| \cdot \|$ on $M_n$ is said to be unitarily invariant if it satisfies the property $\|UXV\|$ for all $X \in M_n$ and all unitaries $U, V \in M_n$.

Let $A, B, X \in M_n$. Throughout this note, we consider the $2 \times 2$ block matrix $H$ in the form

$$H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}.$$ 

It is well known that $H$ positive if and only if the Schur complement of $A$ in $H$ is positive semidefinite provided that $A$ is strictly positive. That is, $H \geq 0$ if and only if

$$H/A = B - X^*A^{-1}X \geq 0. \quad (1)$$

The $2 \times 2$ blocks play an important role in studying matrices and positive matrices in particular. Bhatia book [6] provides a comprehensive survey about block matrices. Furthermore, a positive $2 \times 2$ block can be a very useful tool in studying sectorial matrices, see for example [1], [2] and [3].

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The partial transpose of the block $H$ is defined by

$$H^\tau = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}.$$ 

It is quite clear that the positivity of $H$ does not, in general, imply the positivity of $H^\tau$. The block $H$ is said to be positive partial transpose, or PPT for short, if both $H$ and $H^\tau$ are positive semidefinite. The Schure criterion for positivity implies that $H$ is PPT if and only if

$$B - X^* A^{-1} X \geq 0 \quad \text{and} \quad B - X A^{-1} X^* \geq 0,$$

provided that $A$ is strictly positive.

The PPT criterion (also called Peres–Horodecki criterion) plays an important roll in the quantum information theory. For example, PPT condition is a necessary condition for a mixed quantum state to be separable. Moreover, in low dimensional composite spaces (two and three) this condition (PPT) is also sufficient. See [12] and [16].

The class of PPT matrices possess many interesting properties. Therefore, it has attracted a huge interest. See [4, 9, 10, 11, 13, 14, 15].

Given a PPT block $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$. It is well known that for any unitarily invariant norm

$$||H|| \leq ||A + B||. \quad (2)$$

This result due to Hiroshima [11]. See also [15] for related results. A stronger result can be obtained when $X$ is Hermitian. In fact it was shown in [8] that if $X$ is Hermitian, then

$$H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = \frac{1}{2} [U (A + B) U^* + V (A + B) V^*], \quad (3)$$

for some isometries $U, V \in \mathbb{M}_{2n \times n}$. Here $\mathbb{M}_{2n \times n}$ is the space of all $2n \times n$ complex matrices. The inequality (2) and the identity (3) present connections between the whole block $H$ and its main diagonal components $A$ and $B$. Recently, some interesting inequalities, connecting the main and the off-diagonal of the PPT Block $H$, have been established. For example, in [14], Lin proved that if $H$ is PPT, then

$$tr (X^* X) \leq tr (AB) \quad (4)$$

In the sense of Loewner, it has been proved, in [13], that for some unitary $U \in \mathbb{M}_n$

$$|X| \leq \frac{A \# B + U^* (A \# B) U}{2}. \quad (5)$$
An improvement of the inequality [3] was given in [17], the authors proved that if $H$ is PPT, then
\[ |X| \leq (A\#B)\#U^*(A\#B)U, \] (6)
for some unitary $U \in M_n$.

In this paper, we show that if $H$ is PPT and $t \in [0, 1]$, then
\[ |X| \leq (A\#tB)\#U^*((A\#_{1-t}B)U \leq \frac{A\#tB + U^*((A\#_{1-t}B)U}{2}. \] (7)
for some unitary $U \in M_n$. Then we present several consequences of (7) including inequalities such as (4), (5) and (6). Finally, we present some inequalities that connect the diagonal components to the real part of the off-diagonal components of $H$.

2. Preliminaries

In this section we present some basic properties of positive and PPT blocks. These properties are summarized in Proposition 2.1, 2.2 and 2.3. To make this note self-contained, we outline the proofs of these propositions. We also recall some important facts about weighted geometric mean of two positive matrices.

**Proposition 2.1.** If $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geq 0$, then

1. \( \begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0 \) and \( \begin{pmatrix} B & X^* \\ X & A \end{pmatrix} \geq 0. \)
2. \( \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \leq \frac{1}{2}H. \)

**Proof.** To see the first part, observe that
\[ \begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \geq 0 \]
and
\[ \begin{pmatrix} B & X^* \\ X & A \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & X^* \\ X & B \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \geq 0. \]

For the second part, notice that
\[ \frac{1}{2}H - \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} A & -X \\ -X^* & B \end{pmatrix} \geq 0. \]

\[ \square \]
Proposition 2.2. If \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, then the following blocks are positive semidefinite.
\[
\begin{pmatrix} A & \mp X \\ \mp X^* & B \end{pmatrix}, \quad \begin{pmatrix} A & \mp X^* \\ \mp X & B \end{pmatrix}, \quad \begin{pmatrix} B & \mp X \\ \mp X^* & A \end{pmatrix}, \quad \begin{pmatrix} B & \mp X^* \\ \mp X & A \end{pmatrix}.
\]

Proof. The semi positivity of the first two blocks follows from the definition of PPT and the first part of Proposition 2.1. The semi positivity of the second two blocks results from conjugating the first two blocks by the unitary \( \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \).

Proposition 2.3. Let \( H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) be PPT. Then
\[
\begin{pmatrix} A & e^{i\theta} X \\ e^{-i\theta} X^* & B \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{A+B}{2} & X \\ X^* & \frac{A+B}{2} \end{pmatrix}
\]
are PPT.

Proof. Let \( W = \begin{pmatrix} e^{i\theta} I & 0 \\ 0 & I \end{pmatrix} \). Notice that
\[
\begin{pmatrix} A & e^{i\theta} X \\ e^{-i\theta} X^* & B \end{pmatrix} = W \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} W^* \geq 0,
\]
and
\[
\begin{pmatrix} A & e^{i\theta} X^* \\ e^{-i\theta} X & B \end{pmatrix} = W^* \begin{pmatrix} A & X^* \\ X & B \end{pmatrix} W \geq 0.
\]
This implies that the first block is PPT.

Since \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, Proposition 2.2 implies that \( \begin{pmatrix} B & -X \\ -X^* & A \end{pmatrix} \geq 0 \). Therefore,
\[
H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \leq \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} + \begin{pmatrix} B & -X \\ -X^* & A \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix},
\]
and so
\[
\frac{1}{2} H \leq \begin{pmatrix} \frac{A+B}{2} & 0 \\ 0 & \frac{A+B}{2} \end{pmatrix}.
\]
(8)
The second part of Proposition 2.1 implies that
\[
\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \leq \frac{1}{2} H.
\]
(9)
Hence, combining (8) and (9) gives
\[
\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \leq \begin{pmatrix} \frac{A+B}{2} & 0 \\ 0 & \frac{A+B}{2} \end{pmatrix}.
\]
Consequently, \( \left( \frac{A+B}{2} \begin{pmatrix} X & \frac{X^*}{A+B} \end{pmatrix} \right) \geq 0 \), and then, by Proposition 2.1, we have \( \left( \frac{A+B}{2} \begin{pmatrix} X^* & \frac{X^*}{A+B} \end{pmatrix} \right) \geq 0 \). A similar argument implies that \( \left( \frac{A+B}{2} \begin{pmatrix} X & \frac{X}{A+B} \end{pmatrix} \right) \geq 0 \). This proves that \( \left( \frac{A+B}{2} \begin{pmatrix} X^* & \frac{X}{A+B} \end{pmatrix} \right) \) is PPT. □

In the following paragraph, we present the definition of the weighted geometric mean of two positive matrices and then we state some of its properties.

For positive definite \( X, Y \in \mathbb{M}_n \) and \( t \in [0, 1] \), the weighted geometric mean of \( X \) and \( Y \) is defined as follows

\[
X \#_t Y = X^{1/2}(X^{-1/2}YX^{-1/2})^t X^{1/2}.
\]

When \( t = \frac{1}{2} \), we drop \( t \) from the above definition, and we simply write \( X \# Y \) and call it the geometric mean of \( X \) and \( Y \). It is well-known that

\[
X \#_t Y \leq (1 - t)X + tY. \tag{10}
\]

See [6, Chapter 4].

When \( t = \frac{1}{2} \), an extremal property of the geometric mean of positive definite \( X, Y \in \mathbb{M}_n \) is given as follows

\[
X \# Y = \max \left\{ Z : Z = Z^*, \left( \begin{array}{cc} X & Z \\ Z & Y \end{array} \right) \geq 0 \right\}. \tag{11}
\]

See [6, Theorem 4.1.3].

For every unitarily invariant norm we have

\[
||X \#_t Y|| \leq ||X^{1-t}Y^t|| \\
\leq ||(1 - t)X + tY||. \tag{12}
\]

See [5].

3. Main results

We start this section by the following two lemmas.

**Lemma 3.1.** If \( \left( \begin{array}{cc} A_j & X \\ X^* & B_j \end{array} \right) \geq 0 \) \( (j = 1, 2) \), then

\[
\left( \begin{array}{cc} A_1 \#_t A_2 & X \\ X^* & B_1 \#_t B_2 \end{array} \right) \geq 0, \forall t \in [0, 1].
\]
Proof. Without loss of generality we may assume that for \( j = 1, 2 \) the block \( \begin{pmatrix} A_j & X \\ X^* & B_j \end{pmatrix} \) is positive definite, otherwise we use the well known continuous argument. Therefore, by Schure criterion (1), we have
\[
X^*A_1^{-1}X \leq B_1 \quad \text{and} \quad X^*A_2^{-1}X \leq B_2.
\]
Observe,
\[
X^*(A_1\#_t A_2)^{-1})X = X^*(A_1^{-1}\#_t A_2^{-1})X
= (X^*A_1^{-1}X)^t(X^*A_2^{-1}X)
\leq B_1\#_t B_2 \quad \text{(by the increasing property of means)}.
\]
And so \( B_1\#_t B_2 \geq X^*(A_1\#_t A_2)^{-1})X \). This implies the result. \( \square \)

Lemma 3.2. If \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, then for every \( t \in [0, 1] \) the block \( \begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t} B \end{pmatrix} \) is PPT.

Proof. The result follows directly from Lemma 3.1 Proposition 2.2 and the fact that \( B\#_t A = A\#_{1-t}B \). \( \square \)

Recall that the absolute value of \( X \in \mathbb{M}_n \) is defined as \( |X| = (X^*X)^{1/2} \).

The main result can be stated as follows.

Theorem 3.1. Let \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) be PPT and let \( X = U|X| \) be the polar decomposition of \( X \). Then
\[
|X| \leq (A\#_t B)\#U^*(A\#_{1-t}B)U, \quad \forall t \in [0, 1].
\]

Proof. Let \( X = U|X| \) be the polar decomposition of \( X \). Let \( W \) be the unitary defined as \( W = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix} \). Since \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, Lemma 3.2 implies that \( \begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t} B \end{pmatrix} \geq 0 \), for every \( t \in [0, 1] \).

Therefore,
\[
W^* \begin{pmatrix} A\#_t B & X \\ X^* & A\#_{1-t} B \end{pmatrix} W = \begin{pmatrix} U^*(A\#_t B)U & |X| \\ |X| & A\#_{1-t} B \end{pmatrix} \geq 0.
\]

By the extremal property of the geometric mean (11) we get
\[
|X| \leq (A\#_t B)\#U^*(A\#_{1-t}B)U.
\]

This proves the result. \( \square \)
Corollary 3.1. Let \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) be PPT and let \( X = U|X| \) be the polar decomposition of \( X \). Then for some unitary \( U \in \mathbb{M}_n \)

\[
|X| \leq \frac{(A\#_t B) + U^*(A\#_{1-t} B)U}{2}, \quad \forall t \in [0, 1].
\]

In particular,

\[
|X| \leq \frac{(A\# B) + U^*(A\# B)U}{2}.
\]

We remark that the particular case \( t = 1/2 \) of Theorem 3.1 and Corollary 3.1 can be found in [17] and [13], respectively.

Corollary 3.2. If \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, then for every unitarily invariant norm \( \| \cdot \| \) and for \( t \in [0, 1] \)

\[
\|X\| \leq \|(A\#_t B)\#U^*(A\#_{1-t} B)U\|
\leq \|(A\#_t B)^{1/2}U^*(A\#_{1-t} B)^{1/2}U\|
\leq \left\| \frac{(A\#_t B) + U^*(A\#_{1-t} B)U}{2} \right\|
\leq \frac{\|A\#_t B\| + \|A\#_{1-t} B\|}{2}
\leq \frac{\|A^{1-t} B^t\| + \|A^t B^{1-t}\|}{2}
\leq \frac{\|(1 - t)A + tB\| + \|tA + (1 - t)B\|}{2},
\]

for some unitary \( U \in \mathbb{M}_n \).

Proof. The first inequality follows directly from Theorem 3.1. The fourth is just the triangle inequality. The other inequalities follow from [12].

\[\square\]

In particular, when \( t = 1/2 \) we have the following result.
Corollary 3.3. If \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, then for every unitarily invariant norm \( \| \cdot \| \) and for \( t \in [0, 1] \)

\[
\| X \| \leq \|(A\#B)\#U^*(A\#B)\| \leq \|(A\#B)^{1/2}U^*(A\#B)^{1/2}\| \\
\leq \left\| \frac{(A\#B) + U^*(A\#B)U}{2} \right\|
\leq \| A\#B \| \\
\leq \| A^{1/2}B^{1/2} \| \\
\leq \left\| \frac{A + B}{2} \right\|,
\]

for some unitary \( U \in \mathbb{M}_n \).

If we square the inequalities in Corollary 3.3 and choose the Hilbert-Schmidt norm, \( \| \cdot \|_2 \), we get the following result, which is an improvement of the trace inequality (4). Recall that the Hilbert-Schmidt norm is defined as \( \| X \|_2^2 = \text{tr}(X^*X) \).

Corollary 3.4. If \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) is PPT, then

\[
\text{tr}(X^*X) \leq \text{tr}(A\#B)^2 \\
\leq \text{tr}(AB) \\
\leq \text{tr} \left( \frac{A + B}{2} \right)^2.
\]

Finally, we study the connection between the diagonal components and the real part of the off-diagonal components of the PPT block \( H \). Before doing so, we recall that every \( X \in \mathbb{M}_n \) admits what is called the cartesian decomposition

\[
X = \text{Re}(X) + i\text{Im}(X),
\]

where \( \text{Re}(X) \) and \( \text{Im}(X) \) are the Hermitian matrices defined as \( \text{Re}(X) = \frac{X + X^*}{2}, \text{Im}(X) = \frac{X - X^*}{2i} \) and are known, respectively, as the real and the imaginary parts of \( X \).

Theorem 3.2. Let \( \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \) be PPT. Then \( \forall t \in [0, 1] \)

\[
\text{Re}(X) \leq (A\#_tB)\#(A\#_{1-t}B) \leq \frac{(A\#_tB) + (A\#_{1-t}B)}{2}
\]
and
\[ Im(X) \leq (A\#_t B)\#(A\#_{1-t} B) \leq \frac{(A\#_t B) + (A\#_{1-t} B)}{2}. \]

**Proof.** In first part, the second inequality follows from (12). For the second inequality, notice that, by Lemma 3.2, we have
\[
\begin{pmatrix}
A\#_t B & X \\
X^* & A\#_{1-t} B
\end{pmatrix} \geq 0, \quad \text{and} \quad \begin{pmatrix}
A\#_t B & X^* \\
X & A\#_{1-t} B
\end{pmatrix} \geq 0,
\]
for \( t \in [0,1] \). Therefore,
\[
\begin{pmatrix}
A\#_t B & Re(X) \\
Re(X) & A\#_{1-t} B
\end{pmatrix} = \frac{1}{2} \left( \begin{pmatrix}
A\#_t B & X \\
X^* & A\#_{1-t} B
\end{pmatrix} + \begin{pmatrix}
A\#_t B & X^* \\
X & A\#_{1-t} B
\end{pmatrix} \right) \geq 0.
\]
Therefore, by the extremal property of the geometric mean we have
\[
Re(X) \leq (A\#_t B)\#(A\#_{1-t} B)
\]
This implies the first inequality. To prove the second inequality just applying the first inequality to the block \( G = \begin{pmatrix} A & -iX \\ iX^* & B \end{pmatrix} \). Note that \( G \) is PPT by Proposition 2.3. □

**Corollary 3.5.** Let \( \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0 \). If \( X \) is Hermitian, then
\[
X \leq (A\#_t B)\#(A\#_{1-t} B) \leq \frac{(A\#_t B) + (A\#_{1-t} B)}{2}, \quad \forall t \in [0,1].
\]

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