Rayleigh’s collapsing time of a spherical cavity: the $\Delta$-factor

J. A. S. Lima

Universidade de São Paulo, Instituto de Astronomia, Geofísica e Ciências Atmosféricas
Rua do Matão, 1226, CEP 05508-900, São Paulo, SP, Brazil

F. E. M. da Silveira

Universidade de São Paulo, Instituto de Física
Caixa Postal 66318, CEP 05315-970, São Paulo, SP, Brazil

(Dated: March 31, 2022)

Abstract

New corrections to the equation of motion and total collapsing time of an empty spherical cavity immersed in an infinite incompressible medium are proposed on the assumption of a non-uniform density. The dimensionless number quantifying the corrections with respect to the standard Rayleigh results (coined the $\Delta$-factor) is fully independent of other possible contributions like surface tension and viscous terms. The $\Delta$-factor effect advocated here can be seen as a direct consequence of a mass-shell non-trivial solution to the continuity equation. The consistency of the corrections with respect to the Bernoulli theorem and some physical consequences in the framework of the Rayleigh-Plesset equation are also discussed.

PACS numbers: 47.55.D-; 47.55.dd; 47.55.dp

Keywords: bubble dynamics; cavitation; non-uniform density

*Electronic address: limajas@astro.iag.usp.br
†Electronic address: feugenio@if.usp.br
I. INTRODUCTION

Ninety years ago, Lord Rayleigh published his seminal article on the problem of determining the collapsing time of an empty spherical cavity, starting from an arbitrary radius \( a \), suddenly formed in the bulk of an infinite medium \([1]\). Later on, that paper became the predecessor of a very active area of research, collectively referred to as bubble dynamics, and, as such, it can be regarded as a kind of paradigm to a vast class of spherically symmetric one-dimensional non-steady flows. To quote some examples, Rayleigh’s work is closely related to a large number of applications in the development of engineering devices \([2]\), sonoluminescence \([3]\), and, more generally, with the important problem of cavitation \([4]\). Nowadays, that classical approach is the starting point for a large number of investigations, based on the medium properties therein neglected, like surface tension, fluid viscosity, heat transfer, acoustic cavitation, bubble relaxation, diffusive terms, shock-wave induced collapse, chaos, and many others \([5-9]\)(see also \([10, 11]\) for reviews on such subjects).

Rayleigh achieved his celebrated result on the basis of a few number of hypothesis, thereby capturing the essence of the physics contained in the collapsing cavity. First, he considered the fluid to be incompressible, that is, to have a divergenceless flow. By adopting a spherical coordinate system, concentric with the cavity, this means that the negative radial component of the velocity field \( v(r, t) \) may be expressed as

\[
v = \frac{R^2 \dot{R}}{r^2},
\]

where \( R(t) \) denotes the instantaneous radius of the cavity, and the dot represents total time derivative. Further, by assuming a uniform fluid density \( \rho = \rho_\infty = \text{constant} \), and applying the work-kinetic energy theorem, he was able to relate the collapsing rapidity of the cavity’s boundary to its instantaneous radius through

\[
\dot{R}^2 = \frac{2p_\infty}{3\rho_\infty} \left( \frac{a^3}{R^3} - 1 \right),
\]

where \( a \) is the initial radius of the cavity and \( p_\infty \) stands for the constant fluid pressure at infinity. In the above expression, the instantaneous pressure on the cavity’s boundary is presumed to vanish, once effects provoked by surface tension and viscous terms are neglected. Finally, in order to determine the total collapsing time \( \tau_R \) of the cavity, Rayleigh integrated
Eq. (2) with respect to $R$ from the initial radius to zero, thereby obtaining

$$\tau_R = a\sqrt{\frac{3\pi \rho_{\infty} \Gamma(5/6)}{2p_{\infty} \Gamma(1/3)}},$$

(3)

where $\Gamma(m/n)$ is the gamma function. Here, we notice that Eq. (2) is easily shown to be a particular first integral of

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_{\infty}}{\rho_{\infty}} = 0,$$

(4)

which, from now on, will be referred to as Rayleigh’s equation of motion.

In what follows, we do not try to include any additional phenomenological effect into Rayleigh’s bubble dynamics, as the ones mentioned in the above quoted references. Instead, we seek fundamental modifications of his results by relaxing the hypothesis of uniform density assumed by Rayleigh, as well as in some of their important developments like in the so-called Rayleigh-Plesset equation [12].

The article is organized as follows. In section II, we derive an analytical, non-trivial solution to the continuity equation, which is consistent with the hypothesis of a divergenceless flow. In section III, the $\Delta$-factor is defined, leading to an extension of Rayleigh’s equation of motion and a correction to the total collapsing time of the cavity. In sections IV and V, we discuss the consistency of the corrections with the Bernoulli theorem and some physical consequences in the framework of the so-called Rayleigh-Plesset equation, respectively. Finally, in the conclusion section, the main results are summarized.

II. MASS-SHELL SOLUTION

For a spherically symmetric one-dimensional non-steady flow, the fluid density $\rho(r, t)$ is demanded to satisfy the continuity equation in the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0.$$

(5)

Now, once Eq. (1) is assumed to hold true, and changing coordinates from time $t$ to the instantaneous radius $R(t)$, Eq. (5) readily leads to

$$\frac{1}{R^2} \frac{\partial \rho}{\partial R} + \frac{1}{r^2} \frac{\partial \rho}{\partial r} = 0.$$

(6)
As it appears, one is tempted to take $\rho = \text{constant}$ as the solution to the above equation which is consistent with the divergenceless flow. However, by considering the region $a \leq r \leq b$ of the fluid, a more general solution to that equation may be written as

$$
\rho (r, R) = \frac{\rho_b - \rho_a e^{-\lambda}}{1 - e^{-\lambda}} + \frac{\rho_a - \rho_b}{1 - e^{-\lambda}} \exp \left[ - \left( \frac{r^3}{a^3} - \frac{R^3}{a^3} \right) \right],
$$

where we have chosen $\rho_a = \rho (a, a)$ and $\rho_b = \rho (b, a)$ as boundary conditions at the initial instant, whilst the dimensionless boundary factor $\lambda$ has been defined as

$$
\lambda = \frac{b^3}{a^3} - 1.
$$

Note that Eq. (7) actually describes a vast collection of radially stratified mass-shells at some fixed instant. Notwithstanding, since we are describing a finite class of solutions, Rayleigh’s one may be contained into this as a particular case. Indeed, by identifying $a$ to the initial radius of the cavity, and taking the limit $b \to \infty$, from Eq. (7), it directly follows that

$$
\rho (r, R) = \rho_\infty + (\rho_0 - \rho_\infty) \exp \left[ - \left( \frac{r^3}{a^3} - \frac{R^3}{a^3} \right) \right],
$$

where we have renamed $\rho_0 = \rho_a$ and $\rho_\infty = \rho_{b \to \infty}$. In that case, clearly, $\rho_0$ denotes the instantaneous density on the cavity boundary and $\rho_\infty$ represents the constant density at infinity. Of course, by putting $\rho_0 = \rho_\infty = \text{constant}$, Eq. (9) trivially recovers the solution adopted by Rayleigh. In principle, when a spherical bubble is formed in an uniform medium, one may expect a variation of the density with the radial coordinate. Since the mass is conserved and a hole has been somewhat created, the mass must be redistributed (compressed) close to the frontier of the cavity. In this way, the density near the cavity boundary becomes greater than far from the wall, and, solution (9), says that such a decay is actually very fast ($\propto e^{-\frac{r^3}{a^3}}$) for a divergenceless flow.

Here, for further purposes, we rewrite Eq. (9) in the form

$$
\rho (x) = \rho_\infty \left( 1 + A e^{-x} \right),
$$

where we have changed coordinates from radius $r$ to the dimensionless quantity $x$ through the transformation

$$
x = \frac{r^3}{a^3} - \frac{R^3}{a^3},
$$
for some fixed \( R(t) \), as well as defined the dimensionless \( A \)-factor by the relation

\[
A = \frac{\rho_0}{\rho_\infty} - 1. \tag{12}
\]

Given the above analytical possibility of a non-uniform fluid density, as the only new modification with respect to Rayleigh’s original formulation, and since the assumption of a vanishing instantaneous pressure on the cavity’s boundary still holds, one can anticipate that any correction to the total collapsing time of the cavity must involve only some dimensionless combination of both \( \rho_0 \) and \( \rho_\infty \). Let us now carefully examine the proposed generalization of the original Rayleigh equation of motion, which shall show consistency with our non-uniform density solution, as well as lead to a correction to the standard total collapsing time of the cavity.

III. GENERALIZED RAYLEIGH’S EQUATION: THE \( \Delta \)-FACTOR

The fluid pressure \( p(r, t) \) is required to satisfy Euler’s equation in the form

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \tag{13}
\]

for a spherically symmetric one-dimensional non-steady flow. Once Eqs. (1) and (10) are assumed to hold true and taking into account Eq. (11), it proves convenient to introduce the dimensionless functional

\[
y(x) = \frac{p(x)}{p_\infty}, \tag{14}
\]

for some fixed instant \( t \), or, equivalently, for some fixed instantaneous radius \( R(t) \). Now, by integrating Eq. (13) from the instantaneous boundary of the cavity to infinity, provided the instantaneous pressure on the cavity’s boundary vanishes, we obtain the extended Rayleigh equation of motion

\[
R \ddot{R} + \frac{3}{2} \dot{R}^2 + \frac{p_\infty}{\rho_\infty} \Delta = 0, \tag{15}
\]

where, from Eqs. (11) and (14), the time independent dimensionless \( \Delta \)-factor is defined by

\[
\Delta = \int_0^\infty \frac{1}{1 + Ae^{-x}} \frac{\partial y}{\partial x} \, dx. \tag{16}
\]
As one may check, by applying the initial condition on the surface of the empty cavity \( \dot{R}(0) = 0 \) for \( R = a \), a first integral of Eq. (15) reads

\[
\dot{R}^2 = \frac{2\Delta p_\infty}{3\rho_\infty} \left( \frac{a^3}{R^3} - 1 \right).
\]

(17)

As a result, the corrected total collapsing time \( \tau_\Delta \) of the cavity can be written as

\[
\tau_\Delta = \frac{\tau_R}{\sqrt{\Delta}}.
\]

(18)

which could be anticipated from the equation of motion (e.g., by replacing \( p_\infty \rightarrow \Delta p_\infty \)).

All the above results should be compared with Eqs. (2), (3) and (4) of the introduction. In the limit \( A \rightarrow 0 \), that is, by putting \( \rho_0 = \rho_\infty = \text{constant} \), we see that Eq. (16) implies \( \Delta = 1 \), thereby recovering the respective Rayleigh results. Naturally, in order to quantify an arbitrary \( \Delta \)-contribution, the integral given by Eq. (16) must somewhat be evaluated. Notwithstanding, we discuss now a simple estimate of \( \Delta (\rho_0/\rho_\infty) \), by assuming that \( A \) is sufficiently large, that is, \( \rho_0 \gg \rho_\infty \). An independent derivation based on the Bernoulli equation can be seen at the end of the next section.

To begin with, we observe that the functionals \( \rho(x) \) and \( p(x) \) over the domain of integration \( 0 < x < \infty \) suggest the definition of a large dimensionless quantity \( x_* \), defined as the value \( x \) takes such that

\[
y(x_*) = \frac{1}{2}.
\]

(19)

Next, we separate the integral in Eq. (16) in two parts, namely, the first over the interval \( 0 < x < x_* \), and the second one for \( x_* < x < \infty \). Since \( A \) is assumed to be large enough and the cavity very small as compared to the infinite medium, we may neglect the functional dependence of the integrating factor on \( x \) throughout the first interval, whilst keep it through the second one. However, since \( x \) is also large throughout the second interval, the whole integrating factor may be fairly well approximated to unity over there. As a consequence, we may approximate our Eq. (16) by

\[
\Delta = \frac{1}{1 + A \int_0^{x_*} \frac{\partial y}{\partial x} \, dx + \int_{x_*}^\infty \frac{\partial y}{\partial x} \, dx},
\]

(20)

from which it immediately follows that

\[
\Delta = \frac{\rho_0 + \rho_\infty}{2\rho_0}.
\]

(21)
Note that, as long as the instantaneous density on the cavity’s boundary severely exceeds the constant density at infinity, $\Delta$ approaches its lowest possible value, namely

$$\Delta_{\text{min}} = \frac{1}{2}. \quad (22)$$

As a consequence, the corrected total collapsing time of the cavity approaches its largest possible value, namely

$$\tau_{\text{max}} = \tau_R \sqrt{2}. \quad (23)$$

Let us now investigate the consistency of the $\Delta$-factor as given by Eqs. (16) and (21) with Bernoulli’s theorem. As we shall see, the foregoing independent argument leads to the same $\Delta$-factor and Rayleigh’s type equation of motion.

IV. $\Delta$-FACTOR AND THE BERNOULLI THEOREM

The fluid motion represented by Eq. (1) implies $\nabla \times \vec{v} = 0$, i.e., the velocity field in the Rayleigh dynamics yields a potential flow. As a consequence, we can immediately write

$$\vec{v} = \nabla \phi,$$

where the hydrodynamical potential $\phi(r, t)$ may be chosen to be

$$\phi = -\frac{R^2 \dot{R}}{r}. \quad (24)$$

Now, for an isentropic flow, viz., on the assumption of a uniform specific (per unit mass) entropy $s$ throughout the fluid, a generalized form of Bernoulli’s theorem shall be written as [13, 14]

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}v^2 + \omega = f(t), \quad (25)$$

where $\omega(r, t)$ denotes the fluid enthalpy, and $f(t)$ represents some arbitrary function of time. At this point, we can, simultaneously, evaluate the left-hand side of Eq. (25) on the cavity’s boundary, that is, at $r = R$, and, consistently with Eqs. (1) and (24), which ensure that both $v(r, t)$ and $\phi(r, t)$ vanish in the limit $r \to \infty$, choose $f(t)$ to be the constant value $\omega(r, t)$ takes at infinity. In that case, once $\omega(r, t)$ is related to the specific internal energy $\epsilon(r, t)$ of the fluid through

$$\omega = \epsilon + \frac{p}{\rho}, \quad (26)$$
by assuming a vanishing instantaneous pressure on the cavity’s boundary, Eqs. (25) and (26) lead to

\[ R \dddot{R} + \frac{3}{2} \ddot{R}^2 + \frac{p_\infty}{\rho_\infty} + (\epsilon_\infty - \epsilon_0) = 0, \]

(27)

where \( \epsilon_0 \) and \( \epsilon_\infty \) denote the constant values \( \epsilon(r, t) \) takes at the cavity’s boundary and at infinity, respectively. It thus follows that any correction to the original Rayleigh’s equation of motion may only come from the parenthesis in Eq. (27). In order to evaluate it, we shall make use of the well-known thermodynamical relation

\[ d\epsilon = T ds + p \frac{d\rho}{\rho^2}, \]

(28)

where \( T(r, t) \) denotes the thermodynamical temperature throughout the fluid. However, since we have an isentropic flow \( (ds = 0) \), we can rewrite Eq. (27) as

\[ R \dddot{R} + \frac{3}{2} \ddot{R}^2 + \frac{p_\infty}{\rho_\infty} + \int_{\rho_0}^{\rho_\infty} p \frac{d\rho}{\rho^2} = 0. \]

(29)

It is a trivial matter to show that the above equation is equivalent to the previously derived form of the generalized Rayleigh’s equation of motion, Eq. (15), provided the integral in Eq. (29) is carried out by parts,

\[ \int_{\rho_0}^{\rho_\infty} p \frac{d\rho}{\rho^2} = -\frac{p_\infty}{\rho_\infty} + \int_{\rho}^{\infty} \frac{1}{\rho} \frac{d\rho}{\rho} dr. \]

(30)

As a consequence, from Eq. (27), we arrive at an independent definition of the \( \Delta \)-factor with respect to the one given in Eq. (16),

\[ \Delta = 1 + \frac{p_\infty}{p_\infty} (\epsilon_\infty - \epsilon_0) = 1 + \frac{p_\infty}{p_\infty} \int_{\rho_0}^{\rho_\infty} p \frac{d\rho}{\rho^2}. \]

(31)

At this point, through a quite simple argument, consistently with a non-uniform density solution, we show that the same estimate of the \( \Delta \)-factor as given by Eq. (21) can be achieved in the framework of Bernoulli’s theorem. The unique necessary assumption is the monotonical growth of the fluid pressure throughout the increasing radial coordinate concentric with the cavity. We start by introducing an instantaneously averaged fluid pressure, \( \bar{p} \) say, defined throughout the bulk of the infinite medium as

\[ \bar{p} = \lim_{N \to \infty} \frac{1}{N} [\delta p + 2\delta p + 3\delta p + \ldots + (N - 1) \delta p + p_\infty], \]

(32)
where $N$ denotes a very large number of mass-shells, on the assumption that $p(\rho)$ on each radially stratified layer. In principle, one should expect $p$ to be dependent on the fluid entropy, as well. However, we have assumed the flow to be isentropic. In this proposed scheme, two neighboring surfaces, no matter if they stand close together or far apart throughout the radial direction, are labeled by two different values of $p(\rho)$, whose difference is exactly equal to a very small number, $\delta p$, say. Note that $p_\infty = N\delta p$, whilst the pressure on the cavity’s boundary presumably vanishes. In addition, consistently with the assumption that $p_\infty = \text{constant}$, $\delta p \to 0$ in the limit $N \to \infty$. Now, the problem of evaluating the sum in Eq. (32) is a trivial one. The answer is

$$
\delta p + 2\delta p + 3\delta p + \ldots + (N - 1)\delta p + p_\infty = \frac{N(\delta p + p_\infty)}{2}.
$$

(33)

In other words, the sought instantaneously averaged fluid pressure, actually, does not depend on time, it is a constant, and we can legitimately approximate the pressure throughout the volume of the fluid by

$$
\bar{p} = \frac{p_\infty}{2}.
$$

(34)

As one may check, by inserting Eq. (34) into Eq. (31), our previous expression for $\Delta$ as given by Eq. (21) is recovered.

V. $\Delta$-FACTOR AND RAYLEIGH-PLESSET EQUATION

It is also interesting to discuss how the $\Delta$-factor (or the variation of the density) may modify the description of other fluid properties, from the point of view of bubble dynamics theory. In this regard, an important analytical development was achieved by Plesset thus leading to what is now often referred to as Rayleigh-Plesset equation. That extended theory is remarkably simple and, in particular, explains several features related to single-bubble sonoluminescence experiments. In such an approach, the fluid density is also constant but one assumes a gas-filled cavity so that several boundary effects are taken into account as well. The modified equation of motion reads \cite{10, 12}

$$
R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_\infty - p(R)}{\rho_\infty} = 0,
$$

(35)
where \( p(R) \) is the pressure of the liquid at the bubble boundary. It includes terms like surface-tension, viscous stress and the pressure in the bubble, \( p_B \), and can be written as

\[
p(R) = p_B - \frac{2\sigma}{R} - 4\mu \frac{\dot{R}}{R},
\]

where \( \sigma \) is the surface-tension constant and \( \mu \) is the viscosity coefficient. From the above expression, the Rayleigh-Plesset equation assumes the following form

\[
R \ddot{R} + \frac{3}{2} \dot{R}^2 + \frac{1}{\rho_\infty} \left( p_\infty + \frac{2\sigma}{R} + 4\mu \frac{\dot{R}}{R} - p_B \right) = 0.
\]

(37)

An important observable difference between Eqs. (37) and (4) is the existence of an equilibrium (unstable) radius, \( R_E \), which is obtained by imposing the conditions \( \dot{R} = 0 \) and \( \ddot{R} = 0 \) to be simultaneously fulfilled. For further reference, it is given by

\[
R_E = \frac{2\sigma}{p_B - p_\infty},
\]

(38)

which does not depend explicitly on the fluid density. Additional properties of the Rayleigh-Plesset equation have been widely discussed in the literature [10, 11].

In the above context, the interesting question is: how the non-uniform density solution of section II will affect the Rayleigh-Plesset equation? Naturally, one may also expect a modification of the standard equilibrium radius.

Following the same approach presented in Section III, it is readily checked that the Rayleigh-Plesset relation, Eq. (37), must be replaced by

\[
R \ddot{R} + \frac{3}{2} \dot{R}^2 + \Delta \frac{p_\infty}{\rho_\infty} + \frac{2\sigma}{\rho_0 R} + 4\mu \frac{\dot{R}}{R} - \frac{p_B}{\rho_0} = 0.
\]

(39)

As one should expect, beyond the modification of the \( \Delta \)-factor, all boundary terms are now divided by the density of the liquid in the evolving bubble frontier (\( \rho_0 \)). In addition, the equilibrium conditions lead to a corrected equilibrium radius of the bubble

\[
R_E(\Delta) = \frac{2\sigma}{p_B - \Delta (\rho_0/\rho_\infty) p_\infty},
\]

(40)

which depends explicitly on the boundary value of the density. It is also worth notice that in the limit \( \Delta = 1 \), that is, for \( \rho_0 = \rho_\infty = \text{constant} \), Eq. (40) trivially recovers the standard equilibrium radius of the bubble as predicted by the standard Rayleigh-Plesset equation.
VI. CONCLUSION

In this work, we have discussed a new kind of contribution to Rayleigh’s equation of motion and the total collapsing time of an empty spherical cavity. As we have seen, a correction term, herein called the $\Delta$-factor, naturally appears if the fluid density is not uniform, as assumed by many authors including Rayleigh himself. Indeed, the $\Delta$-factor is a direct consequence of a divergenceless spherically symmetric non-steady flow, whose fluid density is an evolving function in space and time, as given by the general solution derived in section II [see Eq. (9)].

The inclusion of the $\Delta$-factor as a new parameter into Rayleigh’s dynamics necessarily leads to non-negligible corrections of a number of meaningful physical quantities, as the collapsing time. We have also shown how the extension usually named Rayleigh-Plesset equation is affected by the non-uniform time varying density. In particular, a new equilibrium radius has also been derived. Unlike the standard approach, it depends explicitly on the fluid density at the boundary and far from the bubble. All the results summarized above are analytical and have been justified from first principles. It should also be stressed that even more general formulations, as the one describing approximately the influence of the liquid compressibility [15], may be affected by the correction proposed here. A more detailed investigation of the $\Delta$-factor on bubble dynamics will be discussed in a forthcoming communication.

Acknowledgments

This work has been partially supported by CNPq, Brazilian research agency. JASL is also grateful to FAPESP No. 04/13668-0.

[1] L. Rayleigh, Philos. Mag. 34, 98 (1917).
[2] A. Vogel, W. Lauterborn and R. Timm, J. Fluid Mech. 206, 299 (1989).
[3] D. F. Gaitan and L. A. Crum, J. Acoust. Soc. Am. Suppl. 87, 1 (1990); B. P. Barber and S. J. Putterman, Nature 352, 318 (1991); Phys. Rev. Lett. 69, 3839 (1992); D. F. Gaitan, L. A. Crum, C. C. Church and R. A. Roy, J. Acoust. Soc. Am. 91, 3166 (1992).
[4] J. Brennen, *Cavitation and Bubble Dynamics*, Oxford University Press, Oxford (1995); J. Holzfuss and M. Rüeggeberg, Phys. Rev. E *69*, 056304 (2004); A. Prosperetti, Phys. Fluids *16*, 1852 (2004).

[5] L. A. Crum and R. A. Roy, Science *266*, 233 (1994).

[6] R. Clift, J. R. Grace, *Bubbles, Drops, and Particles* (Academic Press, New York); A. Tufaile and J. C. Sartorelli, Phys. Rev. E *66*, 056204 (2002).

[7] N. Xu and L. Wang and X. Hu, Phys. Rev. Lett. *83*, 2441 (1999).

[8] G. E. Vasquez and S. J. Putterman, Phys. Rev. Lett. *85*, 3037 (2000).

[9] A. J. Szeri, B. D. Storey, A. Pearson, and J. R. Blake, Phys. Fluids *15*, 2576 (2003).

[10] M. Plesset and A. Prosperetti, Ann. Rev. Fluid. Mech., *9*, 145 (1977).

[11] M. P. Brenner, Rev. Mod. Phys., *74*, 425 (2002).

[12] M. S. Plesset, J. Appl. Mech. *16*, 277 (1949); J. Appl. Phys. *25*, 96 (1954).

[13] L. Landau, I. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, NY (1982).

[14] G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge U. P., Cambridge (1994).

[15] J. B. Keller and M. Miksis, J. Acoust. Soc. Am. *68*, 628 (1980).