Bilinear Functional Equations in 2D Quantum Gravity

Ivan K. Kostov ∗†

C.E.A. - Saclay, Service de Physique Théorique
CE-Saclay, F-91191 Gif-Sur-Yvette, France

The microscopic theories of quantum gravity related to integrable lattice models can be constructed as special deformations of pure gravity. Each such deformation is defined by a second order differential operator acting on the coupling constants. As a consequence, the theories with matter fields satisfy a set of constraints inherited from the integrable structure of pure gravity. In particular, a set of bilinear functional equations for each theory with matter fields follows from the Hirota equations defining the KP (KdV) structure of pure gravity.

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∗ on leave of absence from the Institute for Nuclear Research and Nuclear Energy, Boulevard Tsarigradsko shosse 72, BG-1784 Sofia, Bulgaria
† kostov@amoco.saclay.cea.fr
1. Introduction

The simplest and perhaps the only possible microscopic realization of the two-dimensional quantum gravity is provided by the ensemble of planar graphs [1]. In its turn, the ensemble of planar graphs is generated by the (divergent) quasiclassical expansion of $N \times N$ matrix integrals, with the parameter $1/N$ playing the role of topological coupling constant. The formulation of 2D quantum gravity in terms of random matrix variables opened the possibility to apply powerful nonperturbative techniques as the method of orthogonal polynomials [2] and led to the discovery of unexpected integrable structures associated with its different scaling regimes. Originally, a structure related to the KdV hierarchy of soliton equations was found in the continuum limit of pure gravity [3]. The KdV hierarchy can be obtained as a 2-reduction of the KP hierarchy, which defines the integrable structure of the microscopic theory. Consequently, the partition functions of the $A$ series of models of matter coupled to 2D gravity were identified as $\tau$-functions of higher reductions of the KP integrable hierarchy [4]. Similar statement concerning the $D$ series was made in [5].

The description of interacting gravity and matter based on the higher reductions of the KP hierarchy is well suited for studying the spectroscopy and the correlation functions of local scaling operators. On the other hand, it becomes less efficient for evaluating macroscopic loop correlators, topology changing interactions and other infrared phenomena. Indeed, at distances much larger than the correlation length of the matter fields, the relevant integrable structure is this of pure gravity. Therefore it make sense to look for a complementary description in which the matter is considered as a perturbation of pure gravity. The simplest realization of this idea is given by the Kazakov’s multicritical points of the one-matrix model [6], which describe a series of nonunitary matter fields coupled to gravity.

A systematic construction of the matter fields as deformations of pure gravity will be the subject of this talk, which is partially based on the work [7]. Our main task will be to exhibit in any theory with matter fields the integrable structure inherited from pure gravity. Namely, we are going to derive bilinear functional equations that can be viewed as a deformation of the Hirota equations for the partition function of pure gravity. We are going to consider in details only the microscopic realization of the theory, but our construction survives without substantial changes in the continuum limit.

We start with the derivation of the Virasoro constraints and the Hirota bilinear equations for the matrix integral generating the ensemble of planar graphs. The Virasoro constraints follow from the loop equations, which relate boson-like quantities (traces), while
the Hirota equations follow from orthogonality relations involving fermion-like quantities (determinants). Then we show how both types of equations generalize for the matrix integrals describing theories of gravity with matter fields. For this purpose we will exploit the microscopic construction of 2D quantum gravity given by the random-lattice versions of the $sl(2)$-related statistical models: the Ising and the $O(n)$ models, the SOS and RSOS models and their $ADE$ and $\hat{A}\hat{D}\hat{E}$ generalizations. Each of these models can be reformulated as theory of one or several random matrices with interaction of the form $\text{tr} \ln(M_a \otimes 1 + 1 \otimes M_b)$. Such an interaction can be introduced by means of a differential operator of second order $H$ acting on the coupling constants; the partition function of the model is obtained by acting with the operator $e^H$ on the partition function of one or several decoupled one-matrix models. The operator $e^H$ defines a canonical transformation of the symplectic structure associated with the space of coupling constants.

The Virasoro constraints $L^a_n = 0$ for each of the one-matrix integrals transform to linear differential constraints $e^H L^a_n e^{-H} = 0$ for the interacting theory. These constraints are equivalent to the loop equations, which have been already derived by other means. The new point is that the Hirota bilinear equations that hold for each of the one-matrix integrals induce, by simply replacing the vertex operators as $V_\pm(z)^a \rightarrow e^H V_\pm(z)e^{-H}$, bilinear functional equations for the interacting theory. In this way the integrable structure associated with pure gravity is deformed but not destroyed by the matter fields. In particular, the deformed Virasoro and vertex operators satisfy the same algebra as the bare ones.

In the continuum limit, the loop equations and the bilinear equations are obtained as deformations of the Virasoro constraints and the Hirota equations in the KdV hierarchy. Their perturbative solution is given by the loop-space Feynman rules obtained in [13].

2. The one-matrix model

The partition function of the ensemble of all two-dimensional random lattices is given by the hermitian $N \times N$ matrix integral

$$Z_N[t] \sim \int dM \exp \left( \text{tr} \sum_{n=0}^{\infty} t_n M^n \right).$$

(2.1)

1 The $q$-state Potts needs a more elaborate construction and will not be considered here.
The integrand depends on the matrix variable $M$ only through its eigenvalues $\lambda_i$, $i = 1, \ldots, N$. Retaining, therefore, only the radial part of the integration measure $dM \sim \prod_{i=1}^{N} d\lambda_i \prod_{i \neq j} (\lambda_i - \lambda_j)$ we can write the integral (2.1) as the partition function for a system of $N$ Coulomb particles in a common potential

$$Z_N[t] = \int \prod_{i=1}^{N} d\lambda_i \exp \left( \sum_{n=0}^{\infty} t_n \lambda_i^n \right) \prod_{i<j} (\lambda_i - \lambda_j)^2. \tag{2.2}$$

Let us recall that the last factor is the square of the Vandermonde determinant

$$\Delta_N(\lambda_1, \ldots, \lambda_N) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j) = \det(\lambda_{ij}^{-1}). \tag{2.3}$$

In the following we will denote by $\langle \ldots \rangle_{N,t}$ the mean value defined by the partition function (2.2). For the time being we restrict the integration in $\lambda$'s to a finite interval $[\lambda_L, \lambda_R]$ on the real axis, so that the measure

$$d\mu_t(\lambda) = d\lambda \exp \left( \sum_{n=0}^{\infty} t_n \lambda^n \right) \tag{2.4}$$

is integrable for any choice of the coupling constants $t_n$. The limit $\lambda_L \to -\infty$, $\lambda_R \to \infty$ exists if the exponential $e^{\sum_n t_n \lambda^n}$ vanishes at $\pm \infty$.

2.1. Loop equations and Virasoro constraints

The loop equations determine the $1/N$ expansion of the integral (2.2) and represent an infinite set of identities satisfied by the correlation functions of the collective loop variable

$$W(z) = \sum_{i=1}^{N} \frac{1}{z - \lambda_i} = \text{tr} \left( \frac{1}{z - M} \right). \tag{2.5}$$

The derivation goes as follows. From the translational invariance of the integration measure $d\lambda_i$ we find, neglecting the boundary terms at $\lambda = \lambda_{L,R}$,

$$\left\langle \sum_{i=1}^{N} \left( \frac{\partial}{\partial \lambda_i} + 2 \sum_{j(\neq i)} \frac{1}{\lambda_i - \lambda_j} + \sum_{n \geq 0} nt_n \lambda_i^{n-1} \right) \frac{1}{z - \lambda_i} \right\rangle_{N,t} = 0. \tag{2.6}$$

Using the identity

$$\sum_{i} \frac{1}{(z - \lambda_i)^2} + 2 \sum_{i \neq j} \frac{1}{z - \lambda_i} \frac{1}{\lambda_i - \lambda_j} = \sum_{i,j} \frac{1}{z - \lambda_i} \frac{1}{z - \lambda_j} \tag{2.7}$$
we find
\[
\left\langle W^2(z) + \sum_{i=1}^{N} \frac{1}{z - \lambda_i} \sum_{n \geq 0} n t_n \lambda_i^{n-1} \right\rangle_{N,t} = 0.
\] (2.8)

The sum in the second term can be viewed as the result of a contour integration along a contour \( \mathcal{C} \) enclosing the interval \([\lambda_L, \lambda_R]\) but leaving outside the point \(z\):
\[
\left\langle W(z)^2 + \oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{1}{z - z'} \sum_n n (z')^{n-1} t_n \right\rangle_{N,t} = 0.
\] (2.9)

Introducing the collective field
\[
\Phi(z) = \frac{1}{2} \sum_{n \geq 0} t_n z^n - \text{tr} \log \left( \frac{1}{z - M} \right)
\] (2.10)
we write the last equation as
\[
\oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{1}{z - z'} \left( \partial \Phi(z') \right)^2 \bigg|_{N,t} = 0.
\] (2.11)

The insertion of the operator \( \text{tr} M^n = \sum \lambda_i^n \) can be realized by taking a partial derivative with respect to the coupling \( t_n \). The collective field (2.10) is therefore represented by the free field operator acting on the coupling constants
\[
\Phi(z) = \frac{1}{2} \sum_{n=0}^{\infty} t_n z^n - \ln \left( \frac{1}{z} \right) \frac{\partial}{\partial t_0} - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n}
\] (2.12)
and the loop equation (2.11) is equivalent to the linear condition
\[
\oint_{\mathcal{C}} \frac{dz'}{2\pi i} \frac{1}{z - z'} T(z') \cdot Z_N[t] \equiv \left( T(z) \cdot Z_N[t] \right)_< = 0
\] (2.13)
imposed on the partition function, where
\[
T(z) = \frac{1}{4} : (\partial \Phi(z))^2 :
\] (2.14)
denotes the energy-momentum tensor for the gaussian field (2.12). Expanding
\[
: (\partial \Phi(z'))^2 : = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}
\] (2.15)
we find a set of differential equations (Virasoro constraints)
\[
L_n \cdot Z_N[t] = 0 \quad (n \geq -1)
\] (2.16)
where
\[
L_n = \sum_{k=0}^{n} \frac{\partial}{\partial k} \frac{\partial}{\partial t_{n-k}} + \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}}
\]  \hspace{1cm} (2.17)

satisfy the algebra
\[
[L_m, L_n] = (m - n) L_{m+n}.
\]  \hspace{1cm} (2.18)

With respect to the trivial variable \( t_0 \), we have the additional equation
\[
\frac{\partial}{\partial t_0} Z_N(t) = N Z_N(t).
\]  \hspace{1cm} (2.19)

An elaborate analysis of the loop equations and explicit expressions of the lowest orders of the \( 1/N \) expansion of the free energy are given in ref. \[14\].

2.2. Orthogonal polynomials and Hirota equations

It is known \[15\] that the partition function (2.1) is a \( \tau \)-function of the KP hierarchy of soliton equations. The global form of this hierarchy is given by the Hirota’s bilinear equations \[16\] (for a review on the theory of the \( \tau \)-functions see, for example, \[17\].) Below we will derive the Hirota equations using the formalism of orthogonal polynomials.

Let us define for each \( N \) (\( N = 0, 1, 2, ... \)) the polynomial
\[
P_{N,t}(\lambda) = \langle \det(\lambda - M) \rangle_{N,t} = \left\langle \prod_{i=1}^{N} (\lambda - \lambda_i) \right\rangle_{N,t}
= \lambda^N - \langle \text{tr} M \rangle_{N,t} \lambda^{N-1} + ... + (-)^N \langle \det M \rangle_{N,t}.
\]  \hspace{1cm} (2.20)

It easy to prove that the polynomials (2.20) are orthogonal with respect to the measure \( d\mu_t(\lambda) \). Indeed,
\[
Z_N \int d\mu_t(\lambda_{N+1}) P_{N,t}(\lambda_{N+1}) \lambda_{N+1}^k
= \frac{1}{N+1} \int \prod_{i=1}^{N+1} d\mu_t(\lambda_i) \Delta_{N+1}(\lambda_1, ..., \lambda_{N+1}) \Delta_N(\lambda_1, ..., \lambda_N) \lambda_{N+1}^k
\]  \hspace{1cm} (2.21)
\[
\sum_{s=1}^{N+1} (-)^{N+1-s} \lambda_{s}^k \Delta_{N+1}(\lambda_1, ..., \hat{\lambda}_s, ..., \lambda_{N+1}).
\]
The sum in the integrand is the expansion of the determinant
\[
\begin{vmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{N-1} & \lambda_1^k \\
1 & \lambda_2 & \cdots & \lambda_2^{N-1} & \lambda_2^k \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \lambda_{N+1} & \cdots & \lambda_{N+1}^{N-1} & \lambda_{N+1}^k
\end{vmatrix}
\]
with respect to its last column. It vanishes for \(k = 1, \ldots, N-1\), which proves the statement.

For \(n = N\), one finds
\[
Z_N \int d\mu_t(\lambda) P_{N,t}(\lambda)\lambda^N = Z_{N+1}/(N+1).
\] (2.22)

Hence
\[
\int_{\lambda_L}^{\lambda_R} d\mu_t(\lambda) P_{N,t}(\lambda) P_{k,t}(\lambda) = \delta_{N,k} \frac{Z_{N+1}[t]}{(N+1)Z_N[t]}.
\] (2.23)

The orthogonality relations (2.23) can be written in the form of a contour integral, namely,
\[
\frac{1}{2\pi i} \oint_C dz \langle \det(z-M) \rangle_{N,t} \left\langle \frac{1}{\det(z-M)} \right\rangle_{[k+1,t]} = \delta_{N,k}
\] (2.24)
where the integration contour \(C\) encloses the point \(z = 0\) and the interval \([\lambda_L, \lambda_R]\). Indeed, the residue of each of the \(n\) poles is equal to the left hand side of (2.23) multiplied by \(Z_k[t]/Z_{k+1}[t]\).

A set of more powerful identities follow from the fact that the polynomial \(P_{N,t}(z)\) is orthogonal to any polynomial of degree less than \(N\) and in particular to the polynomials \(P_{k,t'}[\lambda], k = 1, 2, \ldots, N-1\) where \(t' = \{t'_n, n = 1, 2, \ldots\}\) is another set of coupling constants. Written in the form of contour integrals, these orthogonality relations state, for \(N' \leq N\),
\[
\oint_C dz e^{\sum_{n=1}^{\infty} (t_n-t'_n)z^n} \left\langle \det(z-M) \right\rangle_{N,t} \left\langle \frac{1}{\det(z-M)} \right\rangle_{N',t'} = 0.
\] (2.25)

The Hirota equations for the KP hierarchy are obtained from eq. (2.25) after expressing the mean values of \(\langle \det^{\pm 1} \rangle\) in terms of the vertex operators
\[
V_\pm(z) = \exp\left( \pm \sum_{n=0}^{\infty} t_n z^n \right) \exp\left( \mp \ln \frac{1}{z} \frac{\partial}{\partial t_0} \mp \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t_n} \right).
\] (2.26)

It follows from the definition (2.2) that
\[
V_\pm(z) \cdot Z_N[t] = e^{\pm \sum_{n=0}^{\infty} t_n z^n} \left\langle \det(z-M)^{\pm 1} \right\rangle_{N,t} Z_N[t]
\] (2.27)
and eq. (2.25) is therefore equivalent to
\[ \int_C dz \left( V_+(z) \cdot Z_N[t] \right) \left( V_-(z) \cdot Z_{N'}[t'] \right) = 0 \quad (N' \leq N), \]
which is one of the forms of the Hirota equation for KP [17].

After a change of variables
\[ x_n = \frac{t_n + t'_n}{2}, \quad y_n = \frac{t_n - t'_n}{2}, \]
the Hirota equations (2.28) take its canonical form
\[ \text{Res}_{z=0} z^{N-N'} e^{\sum_{n=1}^{\infty} 2y_n z^n} e^{-\sum_{n=1}^{\infty} \frac{z^n}{n} \frac{\partial}{\partial y_n} Z_N[x+y]} Z_{N'}[x-y] = 0 \]
where \( N' \leq N \). The differential equations of the KP hierarchy are obtained by expanding (2.30) in \( y_n \). For example, for \( N' = N \), the coefficient in front of \( y_1^2 \) is
\[ \left( \frac{\partial^4}{\partial y_1^4} + 3 \frac{\partial^2}{\partial y_2^2} - 4 \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_3} \right) Z_N[t + y] Z_{N'}[t - y] \bigg|_{y=0} = 0 \]
and one finds for the “free energy” \( u[t] = 2 \frac{\partial^2}{\partial t_1^2} \log Z_N \) the KP equation
\[ 3 \frac{\partial^2 u}{\partial t_1^2} + \frac{\partial}{\partial t_1} \left[ -4 \frac{\partial u}{\partial t_3} + 6u \frac{\partial u}{\partial t_1} + \frac{\partial^3 u}{\partial t_1^3} \right] = 0. \]
The lowest equation in the case \( N'-N = 1 \) (modified KP) is the so called Miura transformation relating the functions \( u = 2\partial_1^2 \log Z_N[t] \) and \( v = \log(Z_{N+1}[t]/Z_N[t]) \):
\[ u = \partial_2 v - \partial_1^2 v - (\partial_1 v)^2. \]
One can check directly that these differential equations are satisfied by the asymptotic expansion around the gaussian point
\[ Z_N[t] = (-t_2)^{-N^2/2} \exp \left( Nt_0 - \frac{N}{4} t_1^2 + \frac{N^2}{4} t_3 t_1 - \frac{N}{8} \frac{t_1^3 t_3}{t_2^2} + \ldots \right). \]

The Hirota equations themselves are not sufficient to determine uniquely the partition functions \( Z_N[t] \). The additional information comes from the “string equations”
\[ \sum_n n t_n \int d\mu(t) \lambda^{n-1} P_{N,t}(\lambda) P_{N-1+\sigma,t}(\lambda) = \delta_{\sigma,0} \frac{Z_N}{Z_{N-1}} \quad (\sigma = 0, 1) \]
obtained, with integrating by parts, from
\[ \int d\mu(t) P_{N-1+\sigma,t}(\lambda) \partial_\lambda P_{N,t}(\lambda) = \delta_{\sigma,0} \frac{Z_N}{Z_{N-1}} \quad (\sigma = 0, 1) \]
and therefore valid only if \( d\mu(\lambda)/d\lambda \) vanishes at the endpoints \( \lambda_L, \lambda_R \) of the eigenvalue interval.

In terms of vertex operators, the string equations (2.33) read
\[ \sum_n n t_n \int_C dz \ z^{n-1} (V_-(z) \cdot Z_{N+\sigma}[t]) (V_+(z) \cdot Z_N[t]) = \delta_{\sigma,0} N \ Z_N[t]^2 \]
with \( \sigma = 0, 1 \).
3. The $O(n)$ model

The partition function of the $O(n)$ matrix model is defined by the $N \times N$ matrix integral

$$Z_N^{O(n)}[t] \sim \int dM \exp \left[ \sum_{n=0}^{\infty} t_n \, \text{tr} M^n + \sum_{n+m \geq 1} \frac{T^{-n-m}}{n+m} \frac{(n+m)!}{n! \, m!} \, \text{tr} M^n \text{tr} M^m \right].$$

(3.1)

(the sum over $n$ and $m$ runs the set of nonnegative integers) and describes the ensemble of nonintersecting loops on a random graph. A loop of length $n$ is weighted by a factor $n T^{-n}$ where $T$ is the temperature of the loop gas. For $n \in [-2, 2]$, the model exhibits critical behavior with spectrum of the central charge $C = 1 - 6(g-1)^2/g$, $g = \frac{1}{\pi} \arccos(-n/2)$; the critical behavior for $|n| > 2$ is that of a branched polymer. The matrix integral (3.1) can be again interpreted as the partition function of a Coulomb gas with more complicated but still pairwise interaction between particles:

$$Z_N^{O(n)}[t] = T^{-\frac{1}{2} n N^2} \int \prod_{i=1}^{N} d\lambda_i \exp \left( \sum_{n=0}^{\infty} t_n \lambda_i^n \right) \prod_{1 \leq i < j \leq N} \left( \lambda_i - \lambda_j \right)^2 \prod_{i,j} \left( T - \lambda_i - \lambda_j \right)^{n/2}.$$ 

(3.2)

The interval of integration $[\lambda_L, \lambda_R]$ should be such that $\lambda_R \leq T/2$. With the last restriction the denominator in the integrand never vanishes. The choice of the integration interval does not influence the quasiclassical expansion and hence the geometrical interpretation in terms of a gas of loops on the random planar graph. The saddle point spectral density is automatically supported by an interval on the half-line $[-\infty, T/2]$. The most natural choice for the eigenvalue interval is therefore $\lambda_L \to -\infty, \lambda_R \to T/2$, with the conditions $d\mu(\lambda)/d\lambda|_{-\infty} = d\mu(\lambda)/d\lambda|_{T/2} = 0$.

The $O(n)$ model reduces to the hermitian matrix model in the limit $n \to 0$ and/or $T \to \infty$, and can be considered as a deformation of the latter in the following sense. Let us define the differential operator

$$H = \frac{n}{2} \left[ -\ln T \frac{\partial^2}{\partial t_0^2} + \sum_{n+m \geq 1} \frac{T^{-n-m}}{n+m} \frac{(n+m)!}{n! \, m!} \frac{\partial}{\partial t_n} \frac{\partial}{\partial t_m} \right]$$

(3.3)

acting on the coupling constants. It is easy to see that the partition function (3.2) is obtained from the partition function of the one-matrix model by acting with the operator $e^H:

$$Z_N^{O(n)}[t] = e^H \cdot Z_N[t].$$

(3.4)

We use a Roman letter for the parameter $n \in [-2, 2]$ to avoid confusion with the summation index $n$ running the set of natural numbers.
This simple observation will be of crucial importance for our further consideration. It means that the integrable structure of the one-matrix model survives in some form in the $O(n)$ model. The operator (3.3) defines a canonical transformation

$$
\frac{\partial}{\partial t_n} \rightarrow \frac{\partial}{\partial t_n},
$$

$$
t_n \rightarrow \hat{t}_n = t_n + n \sum_m T^{-n-m} \frac{(n+m-1)!}{n! m!} \frac{\partial}{\partial t_m}
$$

preserving the symplectic structure.

3.1. Loop equations and Virasoro constraints

From the translational invariance of the integration measure $d\lambda_i$ we find

$$
\left( \sum_i z - \lambda_i \right)^2 + n \sum_{i,j} \frac{1}{z - \lambda_i T - \lambda_i - \lambda_j} + \sum_{n \geq 0} \frac{1}{z - \lambda_i} n t_n \lambda_i^{-1} \right)_{N,t} = 0. \tag{3.6}
$$

or, expressing the sum as a contour integral,

$$
\left\langle \left( \frac{\partial \tilde{\Phi}(z')}{\partial z'} \right)^2 \right\rangle_{N,t} = 0. \tag{3.7}
$$

The integration contour $C_-$ encloses the interval $[\lambda_L, \lambda_R]$ and leaves outside the interval $[T - \lambda_R, T - \lambda_L]$ and the point $z$. Introducing the collective field

$$
\Phi(z) = \text{tr} \log(z - M) - \frac{n}{2} \text{tr} \log(T - z - M) + \frac{1}{2} \sum_n t_n z^n \tag{3.8}
$$

we write the last equation as

$$
\oint_{C_-} \frac{dz'}{2\pi i} \frac{1}{z - z'} \left\langle \left( \frac{\partial \Phi(z')}{\partial z'} \right)^2 \right\rangle_{N,t} = 0. \tag{3.9}
$$

The field (3.8) is represented by the linear operator

$$
\tilde{\Phi}(z) = \frac{1}{2} \sum_{n=0}^\infty t_n z^n - \ln \left( \frac{(T - z)^n}{z} \right) \frac{\partial}{\partial t_0} - \sum_{n=1}^\infty \frac{z^{-n} - \frac{n}{2} (T - z)^{-n}}{n} \frac{\partial}{\partial t_n} \tag{3.10}
$$

and is related to the “bare” field (2.12) by $\tilde{\Phi}(z) = e^H \Phi(z) e^{-H}$. The loop equation (3.9) is therefore equivalent to the linear differential constraints

$$
\tilde{L}_n \cdot Z^{O(n)}[t] = 0 \quad (n \geq -1) \tag{3.11}
$$

where the operators

$$
\tilde{L}_n = \sum_{k=0}^n \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k}} + \sum_k k t_k \frac{\partial}{\partial t_{n+k}}
$$

$$
+ n \sum_{k,m} T^{-k-m-1} \frac{(k+m)!}{k! m!} \frac{\partial}{\partial t_{n+k+1}} \frac{\partial}{\partial t_m} \tag{3.12}
$$

are related to the standard Virasoro generators (2.17) by $\tilde{L}_n = e^H L_n e^{-H}$ and therefore satisfy the same algebra.
3.2. Bilinear functional equations

In quite similar way the Hirota equations (2.28) provide, due to the relation (3.4), a set of bilinear equations for the partition function of the $O(n)$ model

$$
\oint_{C_-} dz \left( \tilde{V}_+(z) \cdot Z^{O(n)}_N[t] \right) \left( \tilde{V}_-(z) \cdot Z^{O(n)}_{N'}[t'] \right) = 0 \quad (N' \leq N)
$$

(3.13)

where $\tilde{V}_\pm(z)$ are the transformed vertex operators

$$
\tilde{V}_\pm = e^H V_\pm(z)e^{-H}
$$

$$
= (T - 2z)^{-n/2} \exp \left( \pm \sum_{n=1}^{\infty} t_n z^n \right)
$$

$$
\exp \left( \mp \ln \left[ \frac{(T - z)^n}{z} \right] \partial_{t_0} \mp \sum_{n=1}^{\infty} \frac{z^{-n} - n(T - z)^{-n}}{n} \partial_{t_n} \right).
$$

(3.14)

After being expanded in $t_n - t'_n$, eq. (3.13) generates a hierarchy of differential equations, each of them involving derivatives with respect to an infinite number of “times” $t_n$.

The functional relations (3.13) are equivalent to the bilinear relations

$$
\oint_{C_-} \frac{dz}{(T - 2z)^n} \exp \left( \sum_{n=1}^{\infty} (t_n - t'_n) z^n \right)
$$

$$
\left\langle \frac{\det(z - M)}{\det(T - z - M)^n} \right\rangle_{N,t} \left\langle \frac{\det(T - z - M)^n}{\det(z - M)} \right\rangle_{[N',t']} = 0 \quad (N' \leq N)
$$

(3.15)

where $\left\langle \right\rangle_{N,t}$ denotes the average corresponding to the partition function (3.2). These relations can be also proved directly by exchanging the order of the integration in the $\lambda$’s and the contour integration in $z$. The ”string equations”, needed to determine completely the partition function, are obtained from (2.37) by replacing $V_\pm \to \tilde{V}_\pm(z)$.

4. ADE and $\hat{A}\hat{D}\hat{E}$ models

The $ADE$ and $\hat{A}\hat{D}\hat{E}$ matrix models give a nonperturbative microscopic realization of the rational string theories with $C \leq 1$. Each one of these models is associated with a rank $r$ classical simply laced Lie algebra (that is, of type $A_r, D_r, E_{6,7,8}$) or its affine extension, and represents a system of $r$ coupled random matrices. The matrices $M_a$ of size
\(N_a \times N_a\) \((a = 1, 2, \ldots, r)\) are associated with the nodes of the Dynkin diagram of the simply laced Lie algebra, the latter being defined by its adjacency matrix \(G\) with elements

\[
G^{ab} = \begin{cases} 
1 & \text{if the two nodes are the extremities of a link } < ab > \\
0 & \text{otherwise.} 
\end{cases}
\]

The partition function \(Z_G^\vec{N}[\vec{t}]\) depends on \(r\) sets of coupling constants \(\vec{t} = \{t^a_n \mid a = 1, \ldots, r; \ n = 1, 2, \ldots\}\). The interaction is of nearest-neighbor type and the measure is a product of factors associated with the nodes \(a\) and the links \(< ab >\) of the Dynkin diagram

\[
Z_G^\vec{N}[\vec{t}] \sim \int dM \exp \left( \sum_{a=1}^{r} \sum_{n=1}^{\infty} t^a_n \text{tr} M^n_a + \frac{1}{2} \sum_{a,b} G^{ab} \sum_{m,n=1}^{\infty} \frac{T^{-m-n} (m+n)!}{m+n} \text{tr} M^m_a \text{tr} M^n_b \right). \tag{4.2}
\]

The target space of the \(A_r\) model is an open chain of \(r\) points and the one of the \(\hat{A}_{r-1}\) model is a circle with \(r + 1\) points. In this sense the \(O(2)\) model can be referred to as the \(\hat{A}_0\) model of the \(\hat{A}\) series. The continuum limit of the \(\hat{A}_r\) model is that of a free field coupled to gravity and compactified at the radius \(r\) in a scale where the self-dual radius is \(r = 2\).

Again, the only nontrivial integration is with respect to the eigenvalues \(\lambda_{ai}\) \((i = 1, \ldots, N_a)\) of the matrices \(M_a\):

\[
Z_G^\vec{N}[\vec{t}] = \prod_{a=1}^{r} \prod_{i=1}^{N_a} d\lambda_{ai} e^{\sum_n t^a_n \lambda_{ai}^n} \frac{\prod_a \prod_{i<j} (\lambda_{ai} - \lambda_{aj})^2}{\prod_{<ab>} \prod_{i,j} (T - \lambda_{ai} - \lambda_{bj})}. \tag{4.3}
\]

The domain of integration is assumed to be a compact interval \([\lambda_L, \lambda_R]\) with \(\lambda_R \leq T/2\).

The partition function (4.3) can be obtained by acting on the product of \(r\) one-matrix partition functions \(Z_{N_a}[t^a], a = 1, \ldots, r\), with the exponent of the second-order differential operator

\[
H = \frac{1}{2} \sum_{a,b} G^{ab} \left[ \ln T^{-1} \frac{\partial}{\partial t^a_0} \frac{\partial}{\partial t^b_0} + \sum_{n+m \geq 1} \frac{T^{-n-m}}{n+m} \frac{(n+m)!}{n! \ m!} \frac{\partial}{\partial t^a_n} \frac{\partial}{\partial t^b_m} \right], \tag{4.4}
\]

namely,

\[
Z_G^\vec{N}[\vec{t}] = e^{H} \cdot \prod_{a=1}^{r} Z_{N_a}[t^a]. \tag{4.5}
\]
4.1. Loop equations and Virasoro constraints

The Dyson-Schwinger equations are written in terms of the $r$-component collective variable
\[ W_a(z) = \sum_{i=1}^{N_a} \frac{1}{z - \lambda_{ai}}. \] (4.6)

They are obtained as in the previous cases, by shifting the variable $\lambda_{ai}$, and read
\[ \langle W_a(z)^2 + \oint_{C_{-}} \frac{dz'}{2\pi i} \frac{1}{z - z'} W(z') \left[ \sum_b G^{ab} W_b(T - z') + \sum_n n^{t_a}_n (z')^{n-1} \right] \rangle_{N,t} = 0. \] (4.7)

The loop equations are equivalent to the differential constraints
\[ \tilde{L}_n^a \cdot Z^G[t] = 0 \quad (n \geq -1, \ a = 1, ..., r), \] (4.8)
where the operators
\[ \tilde{L}_n^a = \sum_{k=0}^{n} \frac{\partial}{\partial t_k^a} \frac{\partial}{\partial t_{n-k}} + \sum_k k t_k^a \frac{\partial}{\partial t_{n+k}} + \sum_b G^{ab} \sum_{k,m} T^{-k-m-1} \frac{(k+m)!}{k! m!} \frac{\partial}{\partial t_{n+k+1}^a} \frac{\partial}{\partial t_m^b} \] (4.9)
are related to the “bare” Virasoro generators by $\tilde{L}_n^a = e^H L_n^a(z)e^{-H}$ and form $r$ commuting Virasoro algebras
\[ [\tilde{L}_m^a, \tilde{L}_n^b] = \delta_{a,b} (m - n) \tilde{L}_{m+n}^a. \] (4.10)

With respect to the trivial variables $\tilde{t}_0$, we have the additional equations
\[ \frac{\partial}{\partial \tilde{t}_0^a} Z^G[t] = N_a Z^G[t], \quad (a = 1, ..., r). \] (4.11)

4.2. Bilinear functional equations

Each of the one-matrix partition functions on the right hand side of (4.5) satisfies the Hirota equations (2.28). As a consequence, the left hand side satisfies a set of $r$ functional equations associated with the nodes $a = 1, ..., r$ of the Dynkin diagram. Introducing the transformed vertex operators
\[ \tilde{V}_\pm^a (z) = e^H V_\pm^a (z)e^{-H} \] (4.12)
we find, for each node $a = 1, \ldots, r$,

$$\oint_{\mathcal{C}_-} dz \left( \tilde{V}^a_+(z) \cdot Z^G_N[\bar{t}] \right) \left( \tilde{V}^a_-(z) \cdot Z^G_N[\bar{t}] \right) = 0 \quad (N'_a \leq N_a) \quad (4.13)$$

where the integration contour $\mathcal{C}_-$ in (4.13) encloses the interval $[\lambda_L, \lambda_R]$ but leaves outside the interval $[T - \lambda_R, T - \lambda_L]$.

Inserting (4.4) in the definition (4.12) we find the explicit form of the dressed vertex operators

$$\tilde{V}^a_\pm(z) = \prod_b (T - 2z)^{-\frac{1}{2}} G^{ab} e^{\pm \sum_{n=0}^{\infty} t^n^a z^n} \exp \left( \pm \left[ \ln z^{-1} \frac{\partial}{\partial t^0_b} + \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial}{\partial t^n_b} \right] \right) \quad (4.14)$$

and write the generalized Hirota equations (4.13) in terms of correlation functions of determinants:

$$\oint_{\mathcal{C}_-} dz \frac{e^{\sum_{n=1}^{\infty} [t^n_a - t'^n_a] z^n}}{\prod_b (T - 2z)^{G^{ab}}} \left\langle \frac{\det(z - M_a)}{\prod_b \det(T - z - M_b)^{G^{ab}}} \right\rangle_{\bar{N}, \bar{t}} \quad (4.15)$$

$$\left\langle \frac{\prod_b \det(T - z - M_b)^{G^{ab}}}{\det(z - M_a)} \right\rangle_{[\bar{N}', \bar{t}']_{\bar{N}, \bar{t}}} = 0 \quad (N'_a \leq N_a).$$

Here $\langle \cdots \rangle_{\bar{N}, \bar{t}}$ denotes the average in the ensemble described by the partition function (4.3). It is possible to prove eq. (4.15) directly by performing the contour integration. It is essential for the proof that the interval $[T - \lambda_R, T - \lambda_L]$ is outside the contour $\mathcal{C}_-.$

### 5. Continuum limit

We conclude with a brief description of the continuum limit of the above construction. In the scaling limit the field (2.12) is expanded in the half-integer powers of the (shifted and rescaled) variable $z$

$$\partial \Phi(z) = \frac{1}{2} \sum_{s > 0} t_s z^s - \sum_{s < 0} \frac{z^{-s}}{s} \frac{\partial}{\partial t_s} \quad (s \in \mathbb{Z} + \frac{1}{2}) \quad (5.1)$$
where $t_{1/2}, t_{3/2}, \ldots$ are the corresponding coupling constants. We adopted half-integer indices in order to follow more easily the analogy with the microscopic formulas. The loop equations (2.13) lead to the Virasoro constraints

$$L_n \cdot Z_N[t] = 0 \quad (n \geq -1)$$  \quad (5.2)

with

$$L_n = \sum_{s+s' = n} \frac{\partial}{\partial t_s} \frac{\partial}{\partial t_{s'}} + \sum_{s-s' = n} st_s \frac{\partial}{\partial t_{s'}} + \frac{1}{4} \sum_{-s-s' = n} ss't_{s}t_{s'}.$$  \quad (5.3)

Note that in the continuum limit the string equation is equivalent to the constraint defined by the operator $L_{-1}$ and the constraints (5.2) determine completely the partition function. The Hirota equations for the KdV hierarchy are formulated in terms of the vertex operators

$$V_{\pm}(z) = \exp \left( \pm \sum_{s \in \mathbb{Z} + \frac{1}{2}} t_s z^s \right) \exp \left( \mp \sum_{s \in \mathbb{Z} - \frac{1}{2}} \frac{z^{-s}}{s} \frac{\partial}{\partial t_s} \right).$$  \quad (5.4)

The loop equations and the bilinear functional equations in the continuum limit of the $O(n)$ model are obtained by transforming the Virasoro generators and the vertex operators with the operator $e^H$, where $H$ is given by

$$H = \frac{n}{2} \sum_{s+s' \geq 1} \frac{M^{-s-s'}}{s+s'} \frac{\Gamma(s+s'+1)}{\Gamma(s+1)^{2} \Gamma(s'+1)} \frac{\partial}{\partial t_{s}} \frac{\partial}{\partial t_{s'}}.$$  \quad (5.5)

The parameter $M$ measures the deviation from the critical point (we use the same notation as in [10]) so that $1/M$ is the inverse correlation length of the matter field. In the limit $M \to \infty$ the deformation operator $e^H$ becomes the identity operator.

Similarly, the statistical model associated with the adjacency matrix $G_{ab}$ is described by the deformation operator

$$H = \frac{1}{2} \sum_{a,b} G^{ab}_{a,b} \left[ \sum_{s+s' \geq 1} \frac{M^{-s-s'}}{s+s'} \frac{\Gamma(s+s'+1)}{\Gamma(s+1)^{2} \Gamma(s'+1)} \frac{\partial}{\partial t_{a}^{s}} \frac{\partial}{\partial t_{b}^{s'}} \right].$$  \quad (5.6)

acting on the set of coupling constants

$$t = t_{a}^{s}, \quad (a = 1, \ldots, r; \; s \in \mathbb{Z} + \frac{1}{2}).$$  \quad (5.7)

The partition function is determined by the Hirota equations and $r$ the subsidiary conditions (string equations)

$$\hat{L}_{-1}^{a} \cdot Z^{G}[\vec{t}] = 0 \quad (a = 1, \ldots, r).$$  \quad (5.8)
6. Concluding remarks

Our construction represents an alternative nonperturbative formulation of 2D gravity suited for investigating the infrared properties of the theory. At very large distances the fluctuations of the matter fields can be neglected and the theory decouples to \( r \) copies of pure gravity, where \( r \) is the number of points of the target space. At smaller distances the fluctuations become important, but still can be considered as a perturbation. A highly nontrivial fact, related to the specific realization of 2D gravity we are considering, is that the perturbation can be imposed by means of a differential operator acting on the coupling constants. As a consequence of this, the Virasoro constraints and the Hirota bilinear equations characterizing pure gravity survive in some form in the theory with matter fields.

The coupling constants (5.7) are related to the coupling constants associated with scaling operators as follows. The coupling constants \( t_a^a \) are the coefficients in the expansion of the loop fields (always in half-integer powers of \( z \)) at \( z = 0 \), which is convergent in the circle \( |z| < 2M \). The coupling constants associated with scaling operators represent the coefficients in the expansion (in fractional powers of \( z \)) of the same loop fields at \( z = \infty \), which is convergent in the domain \( |z + M| > M \).

Most likely the two realizations of the \( A_r \) series, the one which we are considering and and the one as a matrix chain with interaction \( \text{tr}M_aM_{a+1} \), have the same scaling limit. We therefore expect that both cases are described, in the continuum limit, by \( (r + 1) \)-reduced KP hierarchy and the differential constraints generating a \( W_{r+1} \) algebra.

The exact integrable structures associated with the \( \hat{A}_r \) models are discussed in [13]. It is shown there that the canonical partition function of the \( \hat{A}_0 \) model is a \( \tau \)-function of the KdV hierarchy. Thus the KdV hierarchy appears in two cases: in pure gravity (the \( A_1 \) model) and in the case of a gaussian field compactified at the Kosterlitz-Thouless radius (the \( O(2) \equiv \hat{A}_0 \) model). One can speculate that the partition functions of the \( A_r \) and \( \hat{A}_{r-1} \) models are \( \tau \)-functions of the \((r + 1)\)-reduced KP hierarchy and that, more generally, the integrable structures associated with the \( ADE \) and \( \hat{A}\hat{D}\hat{E} \) models are given by the exceptional hierarchies studied in [20].
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