LITTLEWOOD–PALEY–RUBIO DE FRANCIA INEQUALITY FOR
THE WALSH SYSTEM

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Abstract. Rubio de Francia proved the one-sided Littlewood–Paley inequality for arbitrary intervals in \( L^p \), \( 2 \leq p < \infty \). In this article, such an inequality is proved for the Walsh system.

1. Formulation of the result

First, we make some agreement about our notation. From now on, by the space \( L^p \) we mean the space \( L^p([0, 1]) \). Also, by \( L^p(l^2) \) we mean \( L^p([0, 1], l^2) \) (i.e., the space of \( l^2 \)-valued functions on the interval \([0, 1]) \).

Let \( I_m \) be mutually disjoint intervals in \( \mathbb{Z} \) (here and below, we assume that \( m \) runs over some finite or countable set). In 1983, Rubio de Francia proved (see [1]) that

\[
\left\| \left( \sum_m |(\hat{f} 1_{I_m})|^2 \right)^{1/2} \right\|_{L^p} \leq C_p \|f\|_{L^p}, \quad 2 \leq p < \infty,
\]

where the constant \( C_p \) does not depend on the intervals \( I_m \) or the function \( f \). It is worth noting that he considered the whole line \( \mathbb{R} \) rather than the interval \([0, 1]) \) (so \( I_m \) were intervals in \( \mathbb{R} \), not in \( \mathbb{Z} \)), but this fact did not play a significant role in his considerations.

By duality, estimate (1) is equivalent to the following:

\[
\left\| \sum_m f_m \right\|_{L^p} \leq C_p \|\{f_m\}\|_{L^p(l^2)}, \quad 1 < p \leq 2,
\]

where \( \{f_m\} \) is a sequence of functions such that \( \text{supp} \hat{f}_m \subset I_m \). In fact, it is already known that estimate (2) remains true for \( p \in (0, 1) \) (see [2] for \( p = 1 \) and [3] for all \( p \in (0, 1) \)).

Our goal is to prove an analogue of (2) for the situation where we use the Walsh system instead of the exponential functions. We give the corresponding definition.

Definition 1. The Walsh system \( \{w_n\}_{n \in \mathbb{Z}_+} \) consists of step functions on the interval \([0, 1]) \) that are defined as follows. First, we set \( w_0 = 1 \). Next, for any index \( n > 0 \) we consider its dyadic decomposition \( n = 2^{k_1} + \cdots + 2^{k_s}, \quad k_1 > k_2 > \cdots > k_s \geq 0 \), and set

\[
w_n(x) \overset{\text{def}}{=} \prod_{i=1}^{s} r_{k_i+1}(x),
\]
where \(r_k\) are the Rademacher functions, that is \(r_k(x) = \text{sign} \sin 2^k \pi x\).

The Walsh functions form an orthonormal basis in \(L^2\) (see, e.g., [4 IV.5]). In the next section, we will discuss their properties in more detail. Now we present the corresponding analogue of Rubio de Francia’s result.

**Theorem 1.** Let \(I_m\) be mutually disjoint intervals in \(\mathbb{Z}_+\). Let \(f_m\) be functions such that

\[
f_m = \sum_{n \in I_m} (f_m, w_n) w_n.
\]

If \(1 < p \leq 2\), then

\[
\left\| \sum_{m} f_m \right\|_{L^p} \leq C_p \left\| \{f_m\} \right\|_{L^p(\ell^2)},
\]

where \(C_p\) does not depend on the collections \(\{I_m\}\) and \(\{f_m\}\).

The proof of this theorem will be close in spirit to arguments in [1] or [3]. However there will be some interesting combinatorial considerations that do not occur in the case of the trigonometric basis. On the other hand, some parts of our proof will be much easier due to the discrete nature of the Walsh system.

2. Preliminaries

**Concerning the Walsh system.** Here we define a certain group operation on \(\mathbb{Z}_+\) and describe its connection with the Walsh functions.

**Definition 2.** Let \(a\) and \(b\) be numbers in \(\mathbb{Z}_+\). Consider their dyadic decompositions

\[
a = \sum_{k=0}^{\infty} \theta_k(a) 2^k \quad \text{and} \quad b = \sum_{k=0}^{\infty} \theta_k(b) 2^k,
\]

where the functions \(\theta_k\) can take the values 0 or 1. Then, we set

\[
a + b \overset{\text{def}}{=} \sum_{k=0}^{\infty} ((\theta_k(a) + \theta_k(b)) \mod 2) 2^k.
\]

**Fact 1.** The set \(\mathbb{Z}_+\), together with the operation \(\overset{\text{def}}{+}\), is an abelian group whose elements are inverse to themselves: \(a + a = 0, a \in \mathbb{Z}_+\).

**Fact 2.** The Walsh system is an abelian group with respect to multiplication that is isomorphic to the group \(\mathbb{Z}_+\) with operation \(\overset{\text{def}}{+}\). Namely, we have

\[
w_a(x) w_b(x) = w_{a+b}(x)
\]

for a.e. \(x \in [0,1]\) and for any \(a, b \in \mathbb{Z}_+\).

This two facts follow directly from Definitions [1] and [2]. A more detailed discussion of the Walsh functions and the operation \(\overset{\text{def}}{+}\) can be found, for example, in [4 IV.5].
Dyadic martingales. Let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots$ be the increasing sequence of $\sigma$-algebras on $[0,1]$ where each $\mathcal{F}_k$ is generated by the dyadic subintervals of length $2^{-k}$. We introduce the following notation:

$$E_k f \overset{\text{def}}{=} E[f|\mathcal{F}_k] = \sum_{i=0}^{2^k-1} \frac{1_{A_i}}{|A_i|} \int_{A_i} f(x) \, dx,$$

where $f$ is a function in $L^1$ and $A_i = [i2^{-k}, (i+1)2^{-k}]$.

**Definition 3.** Consider a collection $\mathcal{M} = \{M_k\}_{k \in \mathbb{Z}^+}$ of scalar-valued or $l^2$-valued integrable functions on $[0,1]$. We say that $\mathcal{M}$ is a dyadic martingale (scalar-valued or $l^2$-valued, respectively) if each $M_k$ is $\mathcal{F}_k$-measurable and $E_k M_{k+1} = M_k$.

From now on, by martingales we mean dyadic martingales. The general concept of vector-valued martingales (not only dyadic ones) is described in detail, for example, in [5]. We will use the following notation:

$$\Delta_0 M \overset{\text{def}}{=} M_0 \quad \text{and} \quad \Delta_k M \overset{\text{def}}{=} M_k - M_{k-1}, \ k > 0.$$

The $L^p$-norms for martingales are defined as follows:

$$\|\mathcal{M}\|_{L^p} \overset{\text{def}}{=} \sup_k \|M_k\|_{L^p}, \ 1 \leq p < \infty.$$

If $1 < p < \infty$, then each martingale $\mathcal{M} = \{M_k\}$ with $\|\mathcal{M}\|_{L^p} \leq \infty$ can be identified with some function $f \in L^p$ and vice versa: the functions $M_k$ have a limit $f$ in $L^p$ such that $\|\mathcal{M}\|_{L^p} = \|f\|_{L^p}$ (see, e.g., [5] V.2), and, on the other hand, if $f$ is a function in $L^p$, then the sequence $\{E_k f\}_{k \in \mathbb{Z}^+}$ is a martingale with the same norm.

As for the case $p = 1$, the condition $\|\mathcal{M}\|_{L^1} \leq \infty$ is not sufficient for the existence of the $L^1$-limit, but each function $f \in L^1$ can still be treated as the martingale $\mathcal{M} = \{E_k f\}$ and we have $\|\mathcal{M}\|_{L^1} = \|f\|_{L^1}$.

The above considerations justifies the following notation: for $f \in L^1$ we set

$$\Delta_0 f \overset{\text{def}}{=} E_0 f \quad \text{and} \quad \Delta_k f \overset{\text{def}}{=} E_k f - E_{k-1} f, \ k > 0.$$

We also introduce a collection of dyadic intervals in $\mathbb{Z}^+$:

$$\delta_0 \overset{\text{def}}{=} \{0\} \quad \text{and} \quad \delta_k \overset{\text{def}}{=} [2^{k-1}, 2^k - 1], \ k > 0.$$

The following fact shows the connection between dyadic martingales and the Walsh system.

**Fact 3.** For any $f \in L^1$, we have

$$E_k f = \sum_{n=0}^{2^k-1} (f, w_n) \, w_n \quad \text{and} \quad \Delta_k f = \sum_{n \in \delta_k} (f, w_n) \, w_n.$$

This simple and well-known fact follows, for example, from arguments in [4] IV.5.
Operators on martingales. Consider the space of simple martingales (we say that a martingale $\mathcal{M} = \{M_k\}$ is simple if $M_k = M_{k+1}$ for all sufficiently large $k$). We suppose it consists of martingales that are either all scalar-valued or all $l^2$-valued. Let $T$ be an operator (not necessarily linear) that is defined on this space and transforms martingales into scalar-valued measurable functions. Suppose it satisfies the following conditions:

(a) $|T(M_1 + M_2)| \leq C_1(|T M_1| + |T M_2|)$;
(b) $\|T M\|_{L^2} \leq C_2\|M\|_{L^2}$;
(c) if a martingale $\mathcal{M} = \{M_k\}_{k \in \mathbb{Z}^+}$ satisfies the relations $M_0 \equiv 0$ and

$$
\Delta_k \mathcal{M} = \mathbb{1}_{e_k} \Delta_k \mathcal{M} \quad \text{for} \quad k > 0,
$$

where $e_k \in \mathcal{F}_{k-1}$, then

$$
\{ |T \mathcal{M}| > \lambda \} \subset \bigcup_{k=1}^{\infty} e_k.
$$

For such an operator we can state the following theorem (it was proved for scalar-valued martingales in [6] and was modified for vector-valued martingales in [7]).

**Theorem 2.** If an operator $T$ satisfies conditions (a), (b), and (c), then for simple martingales $\mathcal{M}$ we have the weak type $(1, 1)$ estimate:

$$
\left| \left\{ |T \mathcal{M}| > \lambda \right\} \right| \leq \text{const} \lambda^{-1} \|\mathcal{M}\|_{L^1} \quad \text{for} \quad \lambda > 0,
$$

where the constant depends only on $C_1$ and $C_2$.

Note that it is presented in greater generality in [7]: martingales are $X$-valued (where $X$ is an arbitrary Banach space), they are not supposed to be dyadic, and a weaker condition is imposed instead of condition (b).

3. Auxiliary lemmas

Here we prove some auxiliary propositions. We start with a lemma that describes how the operation $\oplus$ transforms intervals in $\mathbb{Z}^+$.

**Lemma 1.** Let $N$ be some number in $\mathbb{Z}^+$. Consider its dyadic decomposition:

$N = 2^{k_1} + \cdots + 2^{k_s}$, where $k_1 > k_2 > \cdots > k_s \geq 0$. Also we introduce the collection

$$
\{ \kappa_j \}_{j=1}^{\infty} \overset{\text{def}}{=} \mathbb{Z}^+ \setminus \{ k_i \}_{i=1}^{s}
$$

ordered by ascending: $\kappa_1 < \kappa_2 < \cdots < \kappa_j < \cdots$. Then

$$
[0, N - 1] \oplus N = \bigcup_{i=1}^{s} \delta_{k_i+1} \quad \text{and} \quad [N, +\infty) \oplus N = \delta_0 \cup \left( \bigcup_{j=1}^{\infty} \delta_{\kappa_j+1} \right).
$$
More precisely, we have

$$
\begin{align*}
&[0, 2^{k_1} - 1] + N = \delta_{k_1+1}; \\
&[2^{k_1}, 2^{k_1} + 2^{k_2} - 1] + N = \delta_{k_2+1}; \\
&\vdots \\
&[\sum_{l=1}^{s-1} 2^{k_l}, N - 1] + N = \delta_{k_s+1}; \\
&\{N\} + N = \delta_0; \\
&[N + 1, N + 2^{\kappa_1}] + N = \delta_{\kappa_1+1}; \\
&\vdots \\
&[N + \sum_{l=1}^{j-1} 2^{\kappa_l} + 1, N + \sum_{l=1}^{j} 2^{\kappa_l}] + N = \delta_{\kappa_j+1}; \\
&\vdots
\end{align*}
$$

(4)

Proof. It is worth noting that the first of identities (3) and its proof can be found in [4, IV.5]. The corresponding identities for the intervals (the first \(s\) identities in (4)) can also be derived from that proof.

Here we provide a complete proof of the lemma. Consider the set

$$
Q_i \overset{\text{def}}{=} \left[ \sum_{l=1}^{i-1} 2^{k_l}, \sum_{l=1}^{i} 2^{k_l} - 1 \right] + N, \quad 1 \leq i \leq s.
$$

We denote

$$
\sigma \overset{\text{def}}{=} \sum_{l=1}^{i-1} 2^{k_l} \quad \text{and} \quad \gamma \overset{\text{def}}{=} \sum_{l=i+1}^{s} 2^{k_l}.
$$

By Definition 2 and Fact 1, we have

$$
Q_i = \left\{ (\sigma + v) + (\sigma + 2^{k_i} + \gamma) \right\}_{v=0}^{2^{k_i}-1} = [0, 2^{k_i} - 1] + 2^{k_i} + \gamma.
$$

Fact 1 implies that the set \([0, 2^{k_i} - 1] + \gamma\) consists of \(2^{k_i}\) numbers. On the other hand, all these numbers are lesser than \(2^{k_i}\) because \(\gamma < 2^{k_i}\) (see Definition 2 again). Thus, we obtain \([0, 2^{k_i} - 1] + \gamma = [0, 2^{k_i} - 1]\). Finally, we have

$$
Q_i = [2^{k_i}, 2^{k_i+1} - 1] = \delta_{k_i+1}.
$$

Next, we consider the set

$$
U_j \overset{\text{def}}{=} \left[ N + \sum_{l=1}^{j-1} 2^{\kappa_l} + 1, N + \sum_{l=1}^{j} 2^{\kappa_l} \right] + N, \quad j \geq 1.
$$

We denote

$$
\mu \overset{\text{def}}{=} \sum_{l: k_l > \kappa_j} 2^{k_l} \quad \text{and} \quad \eta \overset{\text{def}}{=} \sum_{l: k_l < \kappa_j} 2^{k_l}.
$$

By the definition of the sequence \(\{\kappa_j\}\), we have

$$
U_j = [2^{\kappa_j} + \mu, \sum_{k=0}^{\kappa_j} 2^k + \mu] + \mu + \eta
$$

We note that

$$
\sum_{k=0}^{\kappa_j} 2^k = 2^{\kappa_j+1} - 1
$$

and that for any integer \(v\) such that \(2^{\kappa_j} \leq v \leq 2^{\kappa_j+1} - 1\), we have \(v + \mu = v + \mu\). Thus, we can see that

$$
U_j = [2^{\kappa_j}, 2^{\kappa_j+1} - 1] + \eta.
Lemma 3. Let $A$ be a function in $L^p$. Then, passing to the limit we obtain the weak type $(1,1)$ estimate for $G$. Suppose for a while that the set $A$ is finite. Then, passing to the limit we obtain the weak type $(1,1)$ estimate for $G$, where $h$ is any function in $L^p(l^2)$. By the Marcinkiewicz interpolation theorem (see, for example, [8, I.4]), we obtain estimate (5). This implies that $U_j$ consists of $2^{2^{\kappa_j}}$ integers that are not less than $2^{\kappa_j}$, but are less than $2^{2^{\kappa_j}+1}$. Therefore, we have

$$U_j = [2^{\kappa_j}, 2^{2^{\kappa_j}+1} - 1] = \delta_{\kappa_j+1}.$$

Now we consider two auxiliary operators and obtain their $L^p$-boundedness as a consequence of Theorem 2.

**Lemma 2.** Suppose multi-index $(j, k)$ runs over some subset $A \subset \mathbb{Z}_+^2$. Consider a collection of numbers $\{a_{j,k}\}_{(j,k)\in A}$ in $\mathbb{Z}_+$ such that $\{a_{j,k} + \delta_k\}_{(j,k)\in A}$ is a collection of mutually disjoint subsets in $\mathbb{Z}_+$. Let $h = \{h_{j,k}\}_{(j,k)\in \mathbb{Z}_+^2}$ be a function in $L^p(l^2)$, $1 < p \leq 2$. Suppose the operator $S$ is defined by the formula

$$Gh \overset{\text{def}}{=} \sum_{(j,k)\in A} w_{a_{j,k}} \Delta_k h_{j,k}.$$

Then we have

$$\|Gh\|_{L^p} \leq C_p \|h\|_{L^p(l^2)},$$

where the constant $C_p$ depends only on $p$.

**Proof.** We recall that the Walsh system is an orthonormal basis in $L^2$. Using Parseval’s identity together with Facts 2 and 3, we can prove the $L^2$-boundedness of $G$. Indeed, since the sets $a_{j,k} + \delta_k$ are pairwise disjoint for $(j, k) \in A$, we have

$$\|Gh\|_{L^2}^2 = \sum_{(j,k)\in A} \|\Delta_k h_{j,k}\|_{L^2}^2 \leq \sum_{(j,k)\in \mathbb{Z}_+^2} \|h_{j,k}\|_{L^2}^2 = \|h\|_{L^2(l^2)}^2.$$ 

Since the operator $G$ is linear and satisfies conditions (b) and (c), Theorem 2 implies the weak type $(1,1)$ estimate for $Gh$, where $h$ is any function in $L^1(l^2)$. By the Marcinkiewicz interpolation theorem (see, for example, [8, I.4]), we obtain estimate (5). Passing to the limit one more time, we lift the assumption about the finiteness of $A$. \hfill \Box

**Lemma 3.** Let $h$ be a function in $L^p$ or in $L^p(l^2)$, $1 < p \leq 2$. Consider the operator $S$ defined by the formula

$$Sh \overset{\text{def}}{=} \left(\sum_{k=0}^{\infty} |\Delta_k h|^2\right)^{1/2}.$$ 

Then we have

$$\|Sh\|_{L^p} \leq C_p \|h\|_{L^p(l^2)}.$$ 

This estimate is well known for scalar-valued functions (moreover, in [9] it is proved that $\|Sh\|_{L^p} \asymp \|h\|_{L^p}$, $1 < p < \infty$). As for our situation, Lemma 3 is a simple consequence of Theorem 2 (the arguments are the same as in the proof of Lemma 2).

Also we will need the following simple fact.

**Fact 4.** Let $\{h_i\}$ and $\{v_j\}$ be sequences in $L^p(l^2)$, $1 \leq p \leq 2$. Then

$$\|\{h_i\}\|_{L^p(l^2)} + \|\{v_j\}\|_{L^p(l^2)} \leq \sqrt{2} \|\{h_i\} \cup \{v_j\}\|_{L^p(l^2)}.$$ 

Proof. This fact follows from the concavity of the functions \(x^{1/p}\) and \(x^{p/2}\) for \(x \geq 0\), i.e., we need to apply the inequality \(\frac{x^q}{x} \leq \left(\frac{x+y}{x}\right)^q\) twice: with \(q = 1/p\) and \(q = p/2\).

4. PROOF OF THEOREM \[\Box\]

Let \(I = [a, b) = [a, b - 1]\) be some interval in \(\mathbb{Z}_+\). We consider the dyadic decomposition of its left end: \(a = 2^{k_1} + \cdots + 2^{k_r}\), where \(k_1 > k_2 > \cdots > k_r \geq 0\). Also we consider the collection \(\{x_j\}_{j=1}^\infty = \mathbb{Z}_+ \setminus \{k_1\}_{i=1}^\infty\) ordered by ascending: \(x_1 < x_2 < \cdots < x_j < \cdots\). We split the right-unbounded interval \([a, \infty)\) into pairwise disjoint subintervals as follows:

\[
[a, +\infty) = \{a\} \cup \bigcup_{j=1}^{\infty} J_j,
\]

where

\[
J_j \overset{\text{def}}{=} [a + \sum_{i=1}^{j-1} 2^{x_i} + 1, a + \sum_{i=1}^{j} 2^{x_i}].
\]

By \(q\) we denote the index such that \(J_q \cap I \neq \emptyset\) and \(J_{q+1} \cap I = \emptyset\).

Next, we consider the dyadic decomposition of \(b\): \(b = 2^{\tilde{k}_1} + \cdots + 2^{\tilde{k}_r}\), where \(\tilde{k}_1 > \tilde{k}_2 > \cdots > \tilde{k}_r \geq 0\), and split the interval \([0, b - 1]\) into pairwise disjoint subintervals:

\[
[0, b - 1] = \bigcup_{i=1}^{r} \tilde{J}_i,
\]

where

\[
\tilde{J}_i \overset{\text{def}}{=} [\sum_{i=1}^{i-1} 2^{\tilde{k}_i}, \sum_{i=1}^{i} 2^{\tilde{k}_i} - 1].
\]

From the collection \(\{\tilde{k}_i\}_{i=1}^r\), we choose the exponent \(\tilde{k}_\rho\) such that \(\theta_{\tilde{k}_\rho}(a) = 0\) and \(\theta_k(a) = \theta_k(b)\) for \(k > \tilde{k}_\rho\). Note that \(\theta_{\tilde{k}_\rho}(b) = 1\).

Now we prove the identity

(6)

\[
I = \{a\} \cup \left(\bigcup_{j=1}^{q-1} J_j\right) \cup \left(\bigcup_{i=\rho+1}^{r} \tilde{J}_i\right)
\]

as well as the fact that all the intervals in this partition are pairwise disjoint. For this, it suffices to show that \(J_q \cap I = [\sum_{i=1}^{q} 2^{k_i}, b - 1]\), or, what is the same, that

(7)

\[
a + \sum_{i=1}^{q-1} 2^{x_i} + 1 = \sum_{i=1}^{\rho} 2^{\tilde{k}_i}.
\]

The number \(a \overset{\text{def}}{=} a + \sum_{i=1}^{q-1} 2^{x_i}\) is constructed from \(a\) as follows: we fill “empty” lower binary digits of \(a\) until we get the number that is smaller than \(b\), but that will become greater if we fill one more digit. So, since \(\theta_{\tilde{k}_\rho}(b) = 1\), we have \(\theta_k(\bar{a}) = 1\) for \(k < \tilde{k}_\rho\), \(\theta_{\tilde{k}_\rho}(\bar{a}) = 0\), and \(\theta_k(\bar{a}) = \theta_k(a) = \theta_k(b)\) for \(k > \tilde{k}_\rho\). This implies identity (7).

Therefore, we have proved relation (6) together with the fact that all the intervals in it are pairwise disjoint.

Now we apply the procedure just described to each interval \(J_m = [a_m, b_m]\). We assign the additional index \(m\) to all the objects arising from the application of this
procedure to $I_m$. Also we introduce the following notation:

$$f_{m,0} \overset{\text{def}}{=} (f_m, w_{a_m}) w_{a_m},$$

$$f_{m,j} \overset{\text{def}}{=} \sum_{n \in J_{m,j}} (f_m, w_n) w_n, \quad \text{and} \quad f_{m,i} \overset{\text{def}}{=} \sum_{n \in J_{m,i}} (f_m, w_n) w_n.$$  

Since the intervals in $\mathcal{I}$ are pairwise disjoint, we have

$$f_m = f_{m,0} + \sum_{j=1}^{q_m-1} f_{m,j} + \sum_{i=\rho_m+1}^{r_m} f_{m,i}. \tag{8}$$

Next, we set $g_{m,j} \overset{\text{def}}{=} w_{a_m} f_{m,j}$ and $g_{m,i} \overset{\text{def}}{=} w_{b_m} f_{m,i}$. Using Facts 1 and 2 we can rewrite identity (8) as follows:

$$f_m = w_{a_m} \left( g_{m,0} + \sum_{j=1}^{q_m-1} g_{m,j} \right) + w_{b_m} \sum_{i=\rho_m+1}^{r_m} g_{m,i}. \tag{9}$$

Therefore, by Lemma 1 and Fact 3 we have

$$f_m = w_{a_m} \left( \Delta_0 g_{m,0} + \sum_{j=1}^{q_m-1} \Delta_{1+\kappa_{m,j}} g_{m,j} \right) + w_{b_m} \sum_{i=\rho_m+1}^{r_m} \Delta_{1+\kappa_{m,i}} g_{m,i}.$$  

This identity, together with Lemma 2 implies that

$$\left\| \sum_m f_m \right\|_{L^p} \leq C_p \left\| \left( \sum_m \sum_{j=0}^{q_m-1} |g_{m,j}|^2 + \sum_m \sum_{i=\rho_m+1}^{r_m} |g_{m,i}|^2 \right)^{1/2} \right\|_{L^p}.$$  

Using the triangle inequality and applying Lemma 3 to one of the terms, we conclude that the last expression is not greater than

$$C_p \left\| \left( \sum_m \sum_{j=0}^{q_m-1} |g_{m,j}|^2 \right)^{1/2} \right\|_{L^p} + C_p' \left\| \left( \sum_m \sum_{i=\rho_m+1}^{r_m} |g_{m,i}|^2 \right)^{1/2} \right\|_{L^p}. \tag{9}$$

Next, we note that

$$g_{m,q_m} = w_{a_m} w_{b_m} \sum_{i=\rho_m+1}^{r_m} g_{m,i}.$$  

This identity and Fact 4 imply that expression (9) can be estimated by

$$C_p' \sqrt{2} \left\| \left( \sum_m \sum_{j=0}^{q_m} |g_{m,j}|^2 \right)^{1/2} \right\|_{L^p} = C_p' \sqrt{2} \left\| \left( \sum_m \sum_{k=0}^{\infty} |\Delta_k g_{m}|^2 \right)^{1/2} \right\|_{L^p},$$

where $g_m \overset{\text{def}}{=} w_{a_m} f_m$. Applying Lemma 3 once again, we see that the last expression is not greater than

$$C_p' \sqrt{2} \left\| \left\{ g_m \right\}_{L^p(\mathcal{I})} \right\|_{L^p(\mathcal{I})} = C_p' \sqrt{2} \left\| \left\{ f_m \right\}_{L^p(\mathcal{I})} \right\|_{L^p(\mathcal{I})}. \quad \Box$$

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