Genuine non-congruence subgroups of Drinfeld modular groups

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Abstract

Let $A$ be the ring of elements in an algebraic function field $K$ over a finite field $\mathbb{F}_q$ which are integral outside a fixed place $\infty$. In an earlier paper we have shown that the Drinfeld modular group $G = \text{GL}_2(A)$ has automorphisms which map congruence subgroups to non-congruence subgroups. Here we prove the existence of (uncountably many) normal genuine non-congruence subgroups, defined to be those which remain non-congruence under the action of every automorphism of $G$. In addition, for all but finitely many cases we evaluate $\text{ngncs}(G)$, the smallest index of a normal genuine non-congruence subgroup of $G$, and compare it to the minimal index of an arbitrary normal non-congruence subgroup.

Key words: Drinfeld modular group; genuine non-congruence subgroup; non-standard automorphism

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Introduction

Let $K$ be an algebraic function field of one variable with constant field $k = \mathbb{F}_q$. As usual we assume that $k$ is algebraically closed in $K$. Let $\infty$ be a fixed place of $K$ of degree $\delta$ and let $A$ be the ring of all those elements of $K$ which are integral outside $\infty$. The simplest example: If $K$ is a rational function field over $k$ and $\delta = 1$, then $A \cong k[t]$, a polynomial ring over $k$.

Our focus of attention here are the Drinfeld modular groups $G = \text{GL}_2(A)$. These
groups play a central role [G2] in the theory of Drinfeld modular curves, which are the analogues in characteristic $p$ of the classical modular curves arising from the action of the modular group $SL_2(\mathbb{Z})$ on the extended complex upper half-plane. The quotients of this plane by finite index subgroups of $SL_2(\mathbb{Z})$ are compact Riemann surfaces, some of which have automorphism groups that are big compared to their genus. However, in many interesting cases these finite index subgroups have to be non-congruence subgroups. Analogously, from the quotients of the Drinfeld upper half-plane by finite index subgroups $N$ of $G$ one obtains algebraic curves in positive characteristic. If $N$ is normal in $G$ and contains the center $Z$ of $G$, then $G/N$ is a subgroup of the automorphism group of that curve. Again, depending on what one would want $G/N$ to be, in most cases $N$ has to be a non-congruence subgroup of $G$. (See Lemma 2.4).

In an earlier paper [MS3] we have shown that $SL_2(A)$ has automorphisms which map (some) congruence subgroups to non-congruence subgroups. Such automorphisms can be readily extended to $G$. They were originally introduced for the special case $A = k[t]$ by Reiner [R]. We extend a terminology used in [MS3].

**Definition.** We call an automorphism $\Psi$ of an arithmetic group $X$ **non-standard** if there exists a congruence subgroup $C$ of $X$ such that $\Psi(C)$ is non-congruence. Otherwise $\Psi$ is called **standard**.

Although non-standard automorphisms exist for every Drinfeld modular group this is not true for other important arithmetic groups like $SL_2(\mathbb{Z})$ and the Bianchi groups $SL_2(O)$, where $O$ is the ring of integers in an imaginary quadratic number field. See [HR], [SV]. In this paper we are concerned with the existence of non-congruence subgroups of the Drinfeld modular groups which remain non-congruence under the action of every non-standard automorphism.

**Definition.** Let $S$ be a non-congruence subgroup of an arithmetic group $X$. We call $S$ **genuine** if and only if $\Psi(S)$ is a non-congruence subgroup, for all $\Psi \in \text{Aut}(X)$.

Non-congruence subgroups which are *not* genuine are subject to the same group-theoretic restrictions as congruence subgroups. For example, when $X = G$, the simple factors of the composition series of $G/N$ where $N$ is the normal core of such a subgroup are either cyclic (of prime order) or are isomorphic to $PSL_2(\mathbb{F}_{q^s})$, for some $s \geq 1$. (See Lemma 2.4.)

For groups like $SL_2(\mathbb{Z})$ and $SL_2(O)$ this definition is, of course, redundant since all their non-congruence subgroups are genuine, i.e. every automorphism maps a congruence subgroup to a congruence subgroup. Despite the existence of non-standard automorphisms however we can prove the following.

**Theorem A.** Every $G$ contains uncountably many normal genuine non-congruence
subgroups.

See Section 3 for more precise results.

In previous papers \cite{M3}, \cite{MS1}, \cite{MS2} we determined the smallest index of a non-congruence subgroup of $SL_2(A)$. Here we turn our attention to $\text{ngncs}(G)$, the smallest index of a normal genuine non-congruence subgroup of $G$. Our second main result (in Section 5) is the following.

**Theorem B.** In all but finitely many cases

$$\text{ngncs}(G) = m(G) = 2,$$

where $m(G)$ is the smallest index of a proper subgroup of $G$.

Note that this implies that the minimal index of a (not necessarily normal) genuine non-congruence subgroup is then also 2. We determine a number of cases for which Theorem B does not hold. In particular for the case $A = \mathbb{F}_q[t]$ we are able to show that $\text{ngncs}(G)$ is strictly bigger than the minimal index of a normal non-congruence subgroup of $G$. See Section 6 for more results.

Other known results in characteristic zero which are relevant to this paper include the following. It is known \cite[Proposition 4.3]{D} that $SL_2(\mathbb{Z})$ is a characteristic subgroup of $GL_2(\mathbb{Z})$ which, in view of \cite{HR}, implies that every non-congruence subgroup of $GL_2(\mathbb{Z})$ is genuine. It is worth noting that results for congruence subgroups of groups of type $SL_2$ and $GL_2$ do not necessarily apply to the corresponding projective groups $PSL_2$ and $PGL_2$. For example it is known that every non-congruence subgroup of $PSL_2(\mathbb{Z})$ is genuine by \cite[Corollary 4.4 (2)]{D}. On the other hand it is also known \cite{JT} that $PGL_2(\mathbb{Z})$ has a non-standard automorphism which therefore cannot fix $PSL_2(\mathbb{Z})$. Hence $PSL_2(\mathbb{Z})$ is not characteristic in $PGL_2(\mathbb{Z})$.

**Notation**

- $k = \mathbb{F}_q$ the finite field of order $q = p^n$;
- $K$ an algebraic function field of one variable with constant field $\mathbb{F}_q$;
- $g = g(K)$ the genus of $K$;
- $\infty$ a chosen place of $K$;
- $\delta$ the degree of the place $\infty$;
- $\nu$ the discrete valuation of $K$ defined by $\infty$;
- $T$ the Bruhat-Tits tree of $GL_2(K_\infty)$;
- $A$ the ring of all elements of $K$ that are integral outside $\infty$;
- $G$ the group $GL_2(A)$;
- $\Gamma$ the group $SL_2(A)$;
- $Z$ the centre of $G$;
- $X$ the groups $G, \Gamma$. 

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1. Subgroups defined by ideals

Let $q$ be an $A$-ideal. It is known that $A/q$ is finite, when $q$ is non-zero. We recall that by the well-known product formula $\nu(a) \leq 0$, for all $a \in A$, and that $\nu(a) = 0$ if and only if $a \in k^*$. For each $\alpha, \beta \in k^*, a \in A$, we put

$$L(\alpha, \beta, a) := \begin{bmatrix} \alpha & a \\ 0 & \beta \end{bmatrix}$$

and

$$T(a) := L(1, 1, a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$ 

For every subset $S$ of $A$ we put

$$T(S) := \{T(s) : s \in S\}.$$ 

For each subgroup $H$ of $G$ and $g \in G$ we denote the conjugate $gHg^{-1}$ by $H^g$.

**Definition.** We define quasi-level of $H$ to be

$$\text{ql}(H) := \{h \in A : T(h) \in H^g, \text{ for all } g \in G\}.$$ 

The level of $H$ is the biggest $A$-ideal contained in $\text{ql}(H)$.

Clearly $\text{ql}(H)$ is an additive subgroup of $A$ and, by considering conjugates by the elements $\text{diag}(\alpha, 1)$, where $\alpha \in k^*$, it is clear that it is also a vector space over $k$.

In [MS3] the “quasi-level” of $H \leq \Gamma$ is defined to be

$$\text{ql}(H)^* := \{h \in A : T(h) \in H^g, \text{ for all } g \in \Gamma\}.$$ 

It is easily shown that the definitions are equivalent.

**Lemma 1.1.** $\text{ql}(H)^* = \text{ql}(H)$.

**Proof.** Clearly $\text{ql}(H) \leq \text{ql}(H)^*$. By conjugating with the matrices $\text{diag}(\alpha, \alpha^{-1})$ and using the fact that every element in a finite field is a sum of two squares, it follows that $\text{ql}(H)^*$ is a vector space over $k$.

Now let $h \in \text{ql}(H)^*$ and $g \in G$. Then $g = dg'$, where $d = \text{diag}(\beta, 1)$ (with $\beta \in k^*$) and $g' \in \Gamma$. Then, from the above, $T(h\beta^{-1}) \in H^g$ and so $T(h) = dT(h\beta^{-1})d^{-1} \in H^g$. 

We define the Borel subgroup of $G$

$$B_2(A) = G_\infty = \{L(\alpha, \beta, a) : \alpha, \beta \in k^*, a \in A\}.$$
**Definition.** For every ideal $q$ of $A$ we define

$$\Delta(q) := \text{the normal subgroup of } \Gamma \text{ generated by } T(q).$$

A subgroup $H$ of $G$ is said to have non-zero level if $\Delta(q) \subseteq H$, for some $q \neq \{0\}$. Otherwise $H$ is said to have level zero.

As in the proof of Lemma 1.1 one shows that $\Delta(q)$ is also normal in $G$. So $\Delta(q)$ can also be defined as the normal subgroup of $G$ generated by $T(q)$.

**Definition.** Let

$$\Gamma(q) = \{ M \in \Gamma : M \equiv I_2 \ mod \ q \}.$$

A subgroup $C$ of $G$ is said to be a congruence subgroup if $\Gamma(q') \leq C$, for some $q' \neq \{0\}$. Such a subgroup is necessarily of finite index in $G$. A finite index subgroup of $G$ which is not congruence is called a non-congruence subgroup.

We will make use of the following properties of these subgroups. The first [B, (9.3) Corollary, p.267] plays an important role in determining whether or not a finite index subgroup of $G$ is congruence (the so-called congruence subgroup problem).

**Lemma 1.2.** Let $q_1, q_2$ be $A$-ideals, where $q_2 \neq \{0\}$. Then

$$\Delta(q_1) \Gamma(q_2) = \Gamma(q_1 + q_2).$$

**Lemma 1.3.** There exists an epimorphism

$$\Gamma(q) / \Delta(q) \to F_r,$$

where $F_r$ is the free group of (finite) rank $r = r(q) = r_{k_2}(\Gamma(q))$, the torsion-free rank of the abelianization of $\Gamma(q)$. Moreover

$$r(q) \to \infty \text{ as } A/q \to \infty.$$

**Proof.** Let $A(q)$ be the subgroup of $G$ generated by $\Gamma(q) \cap G_v$, for all $v \in \text{vert}(T)$. Then $A(q) \leq G$ and, from the theory of groups acting on trees [Se, Corollary 1, p.55], it is known that

$$\Gamma(q) / A(q) \cong F_{r(q)},$$

the fundamental group of the quotient graph $\Gamma(q) \backslash T$, which is known [Se, Corollary 4, p.108] to have finite rank $r(q)$. Again from the presentation [Se, p.42] of $\Gamma(q)$ derived from its action on $T$, together with [Se, Proposition 2, p.76], it follows that $A(q)$ is the subgroup of $\Gamma(q)$ generated by its torsion elements. Hence $\Delta(q) \leq A(q)$. Given $q_1$ and any $q_2$ for which $q_2 \leq q_1$, it is clear that $F_{r(q_2)} \leq F_{r(q_1)} \cap \Gamma(q_2)$. By means of the Schreier formula we can always choose $q_2$ such that $r(q_2) > r(q_1)$. □
2. Non-standard automorphisms

For each non-negative integer \( n \), we put

\[
A(n) := \{ x \in A : \nu(x) \geq -n \}.
\]

Then \( A(n) \) is a finite-dimensional vector space over \( k \), whose dimension is determined by the Riemann-Roch Theorem. (See [St, Theorem 1.5.15, p.30].) Obviously \( k \subseteq A(n) \). In particular \( A(0) = k \), by [St, Corollary 1.1.20, p.8].

We put

\[
G_n := \{ L(\alpha, \beta, a) : \alpha, \beta \in k^*, a \in A(n) \}.
\]

Then \( G_n \) is a finite subgroup of \( G \). We note that

\[
G_\infty = \bigcup_{n \geq 0} G_n.
\]

Serre’s decomposition theorem [Se, Theorem 10, p.119] shows that \( G_\infty \) is a (non-trivial) factor in a decomposition of \( G \) as an amalgamated product of a pair of its subgroups. See [MS3, Theorems 2.1, 2.2].

**Theorem 2.1.** Let \( n \) be the smallest non-negative integer for which

\[
\delta n \geq 2g - 1.
\]

Then there exists a subgroup \( H \) of \( G \), such that

\[
G = G_\infty \ast_L H,
\]

where \( L = G_{n_0} \). Moreover

\[
\dim_k(A(n_0)) = n_0 \delta + 1 - g.
\]

Note that \( n_0 = 0 \) when \( g = 0 \). Otherwise \( n_0 > 0 \).

**Definition.** Let \( \phi : A \to A \) by any \( k \)-automorphism of the \( k \)-vector space \( A \) which fixes the elements of \( A(n_0) \) (including \( k \)). Then \( \phi \) induces an automorphism \( \Phi \) of \( G_\infty \) defined by

\[
\Phi : L(\alpha, \beta, a) \mapsto L(\alpha, \beta, \phi(a)),
\]

where \( \alpha, \beta \in k^*, a \in A \).

The following is an immediate consequence of Theorem 2.1.
Theorem 2.2. Let $\phi$ be any $k$-automorphism of $A$ which fixes the elements of $A(n_0)$. The map
\[
\Phi(g) = \begin{cases} 
L(\alpha, \beta, \phi(a)) , & g = L(\alpha, \beta, a) \in G_\infty \\
g , & g \in H
\end{cases}
\]
extends to an automorphism of $G$ (and $\Gamma$).

As shown in [R] and [MS3] such an automorphism is non-standard. Standard automorphisms include inner automorphisms, the contragredient map $M \mapsto (M^T)^{-1}$, twists with certain determinant characters, i.e. $M \mapsto \chi(\det(M))M$ where $\chi : k^* \to k^*$ is a group homomorphism (with the property that $\chi(\alpha^2) = \alpha^{-1}$ if and only if $\alpha = 1$), or automorphisms derived from ring-automorphisms of $A$. Such a ring-automorphism $\psi$ of $A$ induces the automorphism $\Psi$ of $G$, defined by
\[
\Psi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \psi(a) & \psi(b) \\ \psi(c) & \psi(d) \end{bmatrix}.
\]
Clearly every such ring-automorphism maps $A$-ideals to $A$-ideals.

For the case $A = \mathbb{F}_q[t]$ it is known [R] that the standard automorphisms listed above together with the non-standard ones from Theorem 2.2 generate $\text{Aut}(G)$. This is the only case for which $\text{Aut}(G)$ is known.

We record two obvious properties.

Lemma 2.3. For every subgroup $S$ of $G$ and any non-standard automorphism $\Phi$ of $G$ defined by $\phi$ (as in Theorem 2.2) we have
\[
\text{ql}(\Phi(S)) = \phi(\text{ql}(S)).
\]
In particular
\[
\text{ql}(\Phi(S)) = A \iff \text{ql}(S) = A.
\]

Note however that for an arbitrary automorphism of $G$ or $\Gamma$ we do not know whether it maps subgroups of quasi-level $A$ to subgroups of quasi-level $A$.

Before proceeding we require a further definition.

Definition. For each ideal $q$ let
\[
Z(q) = \{ X \in G : X \equiv \alpha I_2 \pmod{q} \text{ for some } \alpha \in A \}.
\]
It is clear that $Z(q)/\Gamma(q^2)$ is abelian.

Lemma 2.4. Let $N$ be a normal congruence subgroup of index $n$ in $G$ and let
\( \Psi \) be any automorphism of \( G \). Then the (simple) factors in a composition series of \( G/\Psi(N) \) are either cyclic of prime order or are isomorphic to some \( PSL_2(\mathbb{F}_{q^s}) \), where \( s \geq 1 \).

Moreover if \( q > 3 \) and \( n \nmid (q - 1) \) then at least one factor in its composition series is of the latter type.

**Proof.** It is known [M2, Theorem 3.14] that, for some non-zero ideal \( q_0 \)

\[
\Gamma(q_0^2a^2) \leq N \leq Z(q_0),
\]

where \( a \) is the product of all prime ideals of \( A \) of index 2 or 3. If none exists we put \( a = A \). We may confine our attention therefore to the composition factors of the groups \( Z(q)/\Gamma(q^2a^2) \) and \( Z(q)/Z(qp) \), where \( p \) as usual is prime.

We write \( a = a_1a_2 \), where \( a_1 + q = A \) and \( a_2 \) divides \( q \). From standard results (like Lemma 1.2) it follows that

\[
\Gamma(q)/\Gamma(q^2a^2) \cong (\Gamma(q)/\Gamma(q^2a_2^2)) \times \prod_{p|a_1} \Gamma/\Gamma(p^2).
\]

Now the first group in the decomposition is metabelian from above and each \( \Gamma/\Gamma(p^2) \) is a soluble group of order \( m^4(m^2 - 1) \), where \( m = 2, 3 \).

We now consider the group \( \Gamma(q)/\Gamma(qp) \). If \( p \) divides \( q \) it is abelian. On the other hand if \( p + q = A \) then

\[
\Gamma(q)/\Gamma(qp) \cong \Gamma/\Gamma(p) \cong SL_2(\mathbb{F}_{q^s}),
\]

where \( \mathbb{F}_{q^s} = A/p \). The first part follows.

If \( q > 3 \) then \( a = A \) and so

\[
\Gamma(q_0^2) \leq N \leq Z(q_0).
\]

If \( q_0 = A \) then \( \Gamma \leq N \), which contradicts one of the hypotheses. Hence \( q_0 \neq A \) and so \( N \leq Z(p) \), for some \( p \). We note that

\[
G/Z(p) \hookrightarrow PGL_2(\mathbb{F}_{q^s}),
\]

where \( \mathbb{F}_{q^s} = A/p \). It is well-known that \( PSL_2(\mathbb{F}_{q^s}) \) is contained in this embedding. When \( q > 3 \) the latter group is simple. The second part follows. \( \square \)

We record a restricted version of Lemma 2.4.

**Lemma 2.5.** Let \( N \) be a proper normal congruence subgroup of \( \Gamma \) and let \( \Psi \) be any automorphism of \( \Gamma \). Then the factors in a composition series of \( \Gamma/\Psi(N) \) are as in Lemma 2.4. Moreover if \( q > 3 \) then at least one factor in its composition series is of the latter type.
3. Genuine non-congruence subgroups

**Notation.** For the remainder of this paper $X$ will always denote $G$ or $\Gamma$.

**Definition.** A finite index subgroup $S$ of $X$ is said to be a genuine non-congruence subgroup of $X$ if $\Psi(S)$ is a non-congruence subgroup, for all $\Psi \in \text{Aut}(X)$.

The following straightforward result enables us in many instances to assume that a given genuine non-congruence subgroup is normal.

**Lemma 3.1.** A finite index subgroup of $X$ is a genuine non-congruence subgroup if and only if its core in $X$ is a genuine non-congruence subgroup of $X$.

**Remark 3.2.** Note however that we cannot be sure whether a genuine non-congruence subgroup $H$ of $\Gamma$ automatically is a genuine non-congruence subgroup of $G$, as theoretically $G$ might have non-standard automorphisms (other than those discussed in Theorem 2.2) that do not respect $\Gamma$.

Similarly, if $N$ is a (normal) genuine non-congruence subgroup of $G$, we cannot be sure whether $N \cap \Gamma$ is a genuine non-congruence subgroup of $\Gamma$, as theoretically there might be automorphisms of $\Gamma$ that do not extend to automorphisms of $G$.

We now make use of Lemma 1.3 to prove the existence of genuine non-congruence subgroups.

**Theorem 3.3.** For all but finitely many $q$ there exist infinitely many normal genuine non-congruence subgroups $N$ of $X$ for which

\[ q \ell(N) = q. \]

**Proof.** By Lemma 1.3 there exists an epimorphism

\[ \Gamma(q)/\Delta(q) \twoheadrightarrow F_2, \]

for all but finitely many $q$. We note that if $M$ is a finite index subgroup of $X$ and

\[ \Delta(q) \leq M \nsubseteq \Gamma(q), \]

then $M$ is non-congruence by Lemma 1.2.

Let $H$ be any finite group that can be generated by 2 elements and that has a non-cyclic simple composition factor that is not isomorphic to any $\text{PSL}_2(\mathbb{F}_{q^*})$. For example, we can always take $H = S_n$ with $n \geq 7$. Then there exists $M \leq \Gamma(q)$, where $M \geq \Delta(q)$, for which

\[ \Gamma(q)/M \cong H. \]
Let $N$ be the core of $M$ in $X$. Then $\Delta(q) \leq N \leq \Gamma(q)$ so that $ql(N) = q$. In addition $N$ is genuine by Lemmas 2.4, 2.5. □

The subgroups in Theorem 3.3 have all non-zero level. By an earlier method we can prove the following.

**Theorem 3.4.** There exist uncountably many normal genuine non-congruence subgroups of $X$ of level zero.

**Proof.** As in the proof of Theorem 3.3 we choose $q$ and $N \triangleleft X$ so that $\Delta(q) \leq N \leq \Gamma(q)$ and $A_7$, say, is a factor in the composition series of $X/N$. Now choose an ideal $q_0$ so that $A = V_0 \oplus q_0$, where $V_0$ is a finite-dimensional space containing $A(n_0)$ (from Theorem 2.1). Now, if $q_1 \leq q_2$ and $q_1 \neq \{0\}$, then the natural map

$$\frac{\Gamma(q_1)}{\Delta(q_1)} \to \frac{\Gamma(q_2)}{\Delta(q_2)}$$

is surjective by Lemma 1.2. So replacing $q$ with $q \cap q_0$ we may assume that $q = q_0$. Let $W$ be one of the uncountably many subspaces of $A$ not containing any non-zero $A$-ideal for which $A = V_0 \oplus W$. Then we can find a non-standard isomorphism $\Phi$ of $X$ for which $ql(\Phi(N)) = W$. □

However as we now show many genuine non-congruence subgroups of $X$ of quasi-level $A$ do exist. Let

$$X_V = \langle X_v : v \in \text{vert}(T) \rangle.$$

By [Se, Proposition 2, p.76] $X_V$ is the subgroup of $X$ generated by all its torsion elements and so is invariant under every automorphism of $X$. It follows that $\Delta(A) \leq X_V$. In addition

$$X/X_V \cong F_r(X),$$

where $F_r(X)$ is the fundamental group of the quotient graph $X \backslash T$. See [Se, Corollary 1, p.55]. The rank $r(X)$ is known to be finite [Se, Corollary 4, p.108]. In particular $r(X) = 0$ if and only if $X \backslash T$ is a tree. Moreover there are formulae for $r(X)$ involving $\delta, q$ and values of the $L$-polynomial of $K$ ([G1], [G2, p.73], or see [MS1, p.33].) From these it is clear that, for any fixed $g$, $r(X) \to \infty$, as $\delta, q \to \infty$.

It is clear that $r(\Gamma) \geq r(G)$. The rank zero cases are known precisely [MS1, Theorem 2.10]. For convenience we record them.

**Theorem 3.5.**

(i) $r(G) = 0$ if and only if $(g, \delta) = (1, 1), (0, 1), (0, 2)$ or $(0, 3)$.

(ii) $r(\Gamma) = 0$ if and only if $(g, \delta) = (0, 1), (0, 2)$ or (when $q$ is even) $(0, 3), (1, 1)$. 

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Lemma 3.6. Let $S$ be any proper finite index subgroup of $X$ containing $X_V$. Then $S$ is a genuine non-congruence subgroup of $X$ with $ql(S) = A$.

Proof. Recall that $\Psi(X_V) = X_V$ for all $\Psi \in \text{Aut}(X)$. Any congruence subgroup containing $X_V$ must contain $\Gamma . X_V = X$ by Lemma 1.2. □

Corollary 3.7. Suppose that $r = r(X) > 0$ and that $H$ is any finite group with at most $r$ generators. Then there exists a normal genuine non-congruence subgroup $N$ of $X$ of level $A$ with

$$X/N \cong H.$$ 

When $r(X) = 0$ there need not be any non-congruence subgroups of level $A$. Consider, for example, the case where $(g, \delta) = (0, 1)$. Then $A = k[t]$, a euclidean ring. In this case then $\Delta(A) = \Gamma$ and $X_V = X$.

On the one hand Lemmas 2.4, 2.5 provide necessary conditions for a non-congruence subgroup to be not genuine. On the other hand Corollary 3.7 enables us to show that these conditions are not sufficient.

Example 3.8. We provide a simple illustration of Corollary 3.7. Consider one of the many $G$ with $r(G) \geq 4$. Let $p$ be any prime $A$-ideal. Then, by Lemma 1.2,

$$G/\Gamma(p) \cong SL_2(A/p) \rtimes k^*.$$ 

Now $A/p$ is a field $F_{q'}$, for some power $q'$ of $q$, and so $SL_2(A/p)$ is generated by all $T(a)$ and $T(a)^T$. Let $\lambda$ be a generator of $F_{q'}^*$. Then $G/\Gamma(p)$ is generated by $T(1)$, diag($\lambda, \lambda^{-1}$), diag($\mu, 1$), where $\mu$ generates $k^*$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Recall that every element in a finite field is a sum of 2 squares.) Then there exists $N \leq G$, with $G_V \leq N$, such that

$$G/N \cong G/\Gamma(p).$$ 

In some cases, of course, the rank restriction here can be weakened. For example, if $q' = q$, the group $G/\Gamma(p)$ is 3-generated.

4. Some immediate criteria for being genuine

In this section we show that sometimes the index of a subgroup in itself can show that it is genuine non-congruence.

Proposition 4.1. Let $N$ be a proper normal subgroup of index $n$ in $G$, where $\gcd(n, q) = 1$. Suppose further that there exists $S \leq N$ such that

$$S \cong k^* \times k^*.$$ 

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Then $N$ is a genuine non-congruence subgroup of $G$.

**Proof.** Clearly $\Delta(A) \leq \Psi(N)$ for any automorphism $\Psi$ of $G$. By Maschke’s theorem applied to (the abelian group) $\Psi(S)$, there exists $g \in GL_2(K)$ such that

$$g\Psi(S)g^{-1} = D,$$

where $D$ is the set of all diagonal matrices in $GL_2(k)$. Hence $\Psi(N) = G$ and so $\Psi(N)$ is a non-congruence subgroup of $G$ by Lemma 1.2. \hfill $\square$

**Remark 4.2.** If $q$ is even or if $n > 2$, the condition $S \leq N$ in Proposition 4.1 can be replaced by $Z \leq N$ as then $Z \leq \Psi(N)$ for any $\Psi \in \text{Aut}(G)$. However, it is not clear whether conditions like $\text{det}(N) = k^*$ or $N.\Gamma = G$ would suffice, as their behaviour under $\Psi$ is not obvious.

The version of Proposition 4.1 for $\Gamma$ is simpler.

**Proposition 4.3.** Let $N$ be a proper normal subgroup of index $n$ in $\Gamma$, where $\text{gcd}(n,q) = 1$. Then $N$ is a genuine non-congruence subgroup of $\Gamma$.

The following two results are easy conclusions of Lemmas 2.4 and 2.5.

**Proposition 4.4.** Suppose that $q > 3$ and that $N$ is a normal subgroup of index $n$ in $G$, where $n \nmid (q - 1)$. If $|\text{PSL}_2(\mathbb{F}_q)| \nmid n$, then $N$ is a genuine non-congruence subgroup of $G$.

**Proposition 4.5.** Suppose that $q > 3$ and that $N$ is a proper normal subgroup of index $n$ in $\Gamma$. If $|\text{PSL}_2(\mathbb{F}_q)| \nmid n$, then $N$ is a genuine non-congruence subgroup of $\Gamma$.

Note that Propositions 4.4, 4.5 hold in particular if $\text{gcd}(n,q) = 1$ or $\text{gcd}(n,q \pm 1) = 1$. The restrictions on $q$ are necessary. When $q = 2, 3$ and $A = k[t]$ it is well-known that $X$ has normal congruence subgroups of index $q$. Moreover it is known [MS1, Lemma 3.1] that for these cases $X$ has non-congruence subgroups of index $q$ which are *not* genuine.

**Notation.** For the case where a group $H$ has proper finite index subgroups, we denote by $m(H)$ ($> 1$) the smallest index of such a subgroup.

It is a classical result (originally due to Galois) that $m(SL_2(\mathbb{F}_q)) = q + 1$ for $q > 11$ and $q = 4, 8$. Otherwise this index is $q$ unless $q = 9$ in which case it is 6.

For non-normal subgroups of $\Gamma$ we can now prove the following.
Proposition 4.6. Suppose that $q > 3$ and that $H$ is a proper subgroup of $\Gamma$ for which

$$|\Gamma : H| < m(SL_2(\mathbb{F}_q)).$$

Then $H$ is a genuine non-congruence subgroup of $\Gamma$.

Proof. Let $S = SL_2(\mathbb{F}_q)$. Then, for each $g \in \Gamma$,

$$|S : S \cap H^g| \leq |\Gamma : H^g|.$$

It follows that $S \leq H^g$, and hence that $S$ is contained in the core of $H$. By [MS3, Lemma 3.2] this implies $\Delta(A) \leq H$. By Lemma 1.2, $H$ then is non-congruence. We can repeat the argument with $\Psi(H)$, for all $\Psi \in \text{Aut}(\Gamma)$. □

5. The minimum index of a genuine non-congruence subgroup

The first two of the following appear in [MS1] and [MS2].

Definitions.

(i) $\text{ncs}(X) = \min\{|X : S| : S \leq X, \ S \text{ noncongruence}\}$.

(ii) $\text{nnccs}(X) = \min\{|X : S| : S \leq X, \ S \text{ normal, noncongruence}\}$.

(iii) $\text{gncs}(X) = \min\{|X : S| : S \leq X, \ S \text{ genuine noncongruence}\}$.

(iv) $\text{ngncs}(X) = \min\{|X : S| : S \leq X, \ S \text{ normal, genuine noncongruence}\}$.

In [M3], [MS1], [MS2] we determined $\text{ncs}(\Gamma)$ in all cases, and also $\text{nnccs}(\Gamma)$ [MS2, Theorem 6.2]. In this section we evaluate $\text{ngncs}(X)$ and $\text{gncs}(X)$ in all but finitely many cases. It is clear that $\text{ngncs}(X) \geq \text{nnccs}(X)$ and that $\text{gncs}(X) \geq \text{ncs}(X)$.

An immediate consequence of Corollary 3.7 is the following.

Theorem 5.1. Suppose that $r(X) > 0$. Then

$$\text{ngncs}(X) = \text{gncs}(X) = \text{nnccs}(X) = \text{ncs}(X) = m(X) = 2.$$ 

Theorem 5.1 also holds for some but not all rank zero cases, which are listed in Theorem 3.5. For the remainder of this section we consider the cases $(g, \delta) = (1, 1), \ (0, 3)$ in detail. We recall from [MS4] the possible structures of the stabilizers in any $G$ of the vertices of $\mathcal{T}$. For any $v \in \text{vert}(\mathcal{T})$ it is known that one of the following holds...
(a) $G_v \cong GL_2(k)$ or $\mathbb{F}_{q^2}^*$.
(b) $G_v \cong k^* \times N$,
(c) $G_v/N \cong k^* \times k^*$,
where $N \cong V^+$, the additive group of a finite dimensional $k$-space $V$. See [MS4, Corollaries 2.2, 2.4, 2.7]. We require the following.

**Lemma 5.2.** Suppose that $(g, \delta) = (1, 1)$ or $(0, 3)$. Then there exist subgroups $P, Q$ of $G$ such that

$$G = P \ast_Z Q,$$

where

(i) $GL_2(k) \leq P$,
(ii) $\det(Q) = k^*$.

**Proof.** We recall from Theorem 3.5 that in both cases $G \backslash T$ is a *tree*. Let $T_0$ be a lift of $G \backslash T$ with respect to the natural projection of $T$ onto $G \backslash T$. Hence, by definition, $T_0$ is a subtree of $T$ isomorphic to $G \backslash T$. It is known that there exists $e \in \text{edge}(T_0)$ for which

$$G_e = Z.$$

The edge $e$ naturally partitions the vertices of $T_0$ into $V_1, V_2$, say. Let $P = \langle G_v : v \in \text{vert}(V_1) \rangle$ and $Q = \langle G_v : v \in \text{vert}(V_2) \rangle$. Then from standard Bass-Serre theory [Se, p.42]

$$G = P \ast_Z Q.$$

We can choose $T_0$ so that there exists $v \in \text{vert}(V_1)$ for which

$$G_v = GL_2(k).$$

In addition there exists $v \in \text{vert}(V_2)$ for which $G_v$ is of type (a) or (c). From the descriptions of the matrices in the stabilizers of these types given in [MS4, Theorems 2.1, 2.6] it is clear that in either case

$$\det(G_v) = k^*.$$

(For stabilizers of type (a) we require the fact that the *norm* map $N_{L/k}$, where $L = \mathbb{F}_{q^2}$, is surjective.)

For the elliptic case $(g, \delta) = (1, 1)$ we can choose $T_0$ so that the assertions hold. See Takahashi’s paper [T], in particular [T, Theorems 3, 5].

For the case $(g, \delta) = (0, 3)$ a detailed description of $T_0$ is given in [M4, Theorem...
2.26] (for the case $d = 3$). The edge with trivial stabilizer is, in the notation of [M4], the one joining $\bar{\Lambda}_0$ and $\bar{\Lambda}_1[1]$. It turns out we can choose $\mathcal{T}_0$ so that the stabilizer of $\bar{\Lambda}_0$ is $GL_2(k)$ and that all other stabilizers of the vertices of $\mathcal{T}_0$ are of type (c). □

(a) The elliptic case $(g, \delta) = (1, 1)$, $q \neq 2$.

(i) Suppose that $4 | q$. Then $r(\Gamma) = r(G) = 0$. By [MS2, Theorem 5.5] and Proposition 4.3

$$ngncs(\Gamma) = m(\Gamma) = p',$$

where, with two exceptions, $p' = 3$. For the exceptional cases (when $q = 4$) $p' = 5$.

Take $M \leq \Gamma$ with $|\Gamma : M| = p'$ and let $N = Z.M$. Then, since $Z.\Gamma = G$ it follows that $N \leq G$ and that $|G : N| = p'$. By Proposition 4.1 and Remark 4.2 $N$ is a normal genuine non-congruence subgroup of $G$. So $ngncs(G) \leq p'$.

Now let $H$ be any subgroup of $G$ with $|G : H| < p'$. Then $|\Gamma : H \cap \Gamma| < p'$, and hence $\Gamma \leq H$. We conclude then that, when $4 | q$,

$$ngncs(G) = p'.$$

With the two above exceptions $ngncs(G) = m(G) = 3$. For the two exceptional cases (when $q = 4$) we have $ngncs(G) = gnccs(G) = 5$ and $m(G) = 3$. The subgroup attaining the latter bound is (the congruence subgroup) $\Gamma$.

(ii) Suppose now that $q$ is odd. From the rank formulae ([G1] or see [MS1, p.33]) it follows that here $r(\Gamma) = q$ and so Theorem 5.1 applies here for the case $X = \Gamma$. However $r(G) = 0$. With the notation of Lemma 5.2 we define an epimorphism $\phi$ from $G$ to $\{\pm 1\}$ by

$$\phi(g) = \begin{cases} 
1, & g \in P \\
(det(g))^{\frac{q-1}{2}}, & g \in Q
\end{cases}$$

Hence there exists $N \leq G$, containing $GL_2(k)$ for which $|G : N| = 2$. By Proposition 4.1 therefore in this case

$$ngncs(G) = m(G) = 2.$$ 

We summarize the results for this case.

**Theorem 5.3.** Suppose that $(g, \delta) = (1, 1)$ and that $q \neq 2$.

(i) If $q$ is odd

$$ngncs(X) = gnccs(X) = m(X) = 2.$$ 

(ii) If $4 | q$ then with two exceptions

$$ngncs(X) = gnccs(X) = m(X) = 3.$$
For each of the exceptional cases \( q = 4 \) and
\[
\text{ngncs}(X) = \text{gncs}(X) = m(\Gamma) = 5, \ m(G) = 3.
\]

Less precise results appear to hold for the elliptic case when \( q = 2 \). When \( q = 2 \) it is known [MS1, Lemma 3.1(i)] that, for any \( A \),
\[
\text{ncs}(A) = 2.
\]

In addition, for some \( A \) of elliptic type, it is known [MS3, Lemma 5.2(c)] that \( G(= \Gamma) \) has a normal subgroup of index 3, in which case
\[
2 \leq \text{gn}(G) = \text{ngnc}(G) \leq 3,
\]
by Proposition 4.3. For such an example see [Se, 2.4.4, p.115].

(b) The case \((g, \delta) = (0, 3), \ q \neq 2\).

(i) Suppose that \( 4|q \). It follows from [MS2, Theorem 4.7, Theorem 6.2] and Proposition 4.3 that
\[
\text{ngnc}(\Gamma) = m(\Gamma) = p'
\]
where \( p' \) is the smallest prime dividing \( q - 1 \). Taking \( M \trianglelefteq \Gamma \) with \( |\Gamma : M| = p' \) and considering the subgroup \( Z \cdot M \) it can be shown as in (a)(i) above that
\[
\text{ngnc}(G) = m(G) = p'.
\]

(ii) Suppose that \( q \) is odd. Then as in the elliptic case from Lemma 5.2 it follows that
\[
\text{ngnc}(G) = m(G) = 2.
\]

We summarize the results for this case.

**Theorem 5.4.** Suppose that \((g, \delta) = (0, 3)\) and that \( q \neq 2 \). Then
\[
\text{ngncs}(X) = \text{gncs}(X) = m(X) = p',
\]
where \( p' \) is the smallest prime dividing \( q - 1 \).

6. The case \( A = \mathbb{F}_q[t] \)

Finally we look at the most important case \((g, \delta) = (0, 1)\), that is, \( A = \mathbb{F}_q[t] \). For most aspects of Drinfeld modular curves this is the case that is by far the best understood. Ironically, for this case we don’t know \( \text{ngnc}(G) \) exactly and can only
Theorem 6.1. Let $A = \mathbb{F}_q[t]$ with $q > 3$ and let $N$ be a normal genuine non-congruence subgroup of $\Gamma$. Then $|\Gamma : N|$ is divisible by $q|\text{PSL}_2(\mathbb{F}_q)|$. In particular

$$\text{ngncs}(\Gamma) > \text{ncs}(\Gamma) = |\text{PSL}_2(\mathbb{F}_q)|.$$

Proof. Actually, the index of any proper normal subgroup of $\Gamma$ is divisible by $|\text{PSL}_2(\mathbb{F}_q)|$. Namely, by [MS3, Lemma 3.2] we either have $\text{ql}(N) = A$ (and hence $N = \Gamma$) or $N \cap SL_2(\mathbb{F}_q) \leq \{ \pm I_2 \}$. In the latter case $SL_2(\mathbb{F}_q)/(N \cap SL_2(\mathbb{F}_q))$ is a subgroup of $\Gamma/N$.

So we only have to show that $q^2$ divides $|\Gamma : N|$. Assume not. Since $A/\text{ql}(N)$ is a subgroup of $\Gamma/N$, this implies that $\text{ql}(N)$ has codimension 1 (or 0) in $A$. From the previous paragraph we already know that $T(1) \notin N$, that is, $1 \notin \text{ql}(N)$. Hence there exists a non-standard automorphism $\Phi$ of $\Gamma$ with $\text{ql}(\Phi(N)) = tA$. But $\Delta(t) = \Gamma(t)$ ([M1, Corollary 1.4]); so $\Phi(N)$ is a congruence subgroup. □

The corresponding result for $q \leq 3$ requires a little bit of preparation.

Lemma 6.2. Let $q \leq 3$ and $A = \mathbb{F}_q[t]$. Denote the commutator group of $\Gamma$ by $\Gamma'$.

(i) $\Gamma'$ is the normal subgroup of $\Gamma$ generated by the unique subgroup of order $q^2 - 1$ of $SL_2(\mathbb{F}_q)$.

(ii) If $N$ is a normal subgroup of $\Gamma$ containing $\Gamma'$, then $\Gamma/N$ is naturally isomorphic to $A/\text{ql}(N)$.

(iii) If $N$ is a normal subgroup of $\Gamma$ with $1 \in \text{ql}(N)$, then $N$ contains $\Gamma'$.

Proof. For $q \leq 3$ the commutator of $SL_2(\mathbb{F}_q)$ is its unique subgroup $P$ of order $q^2 - 1$. So $\Gamma'$ contains the normal hull of $P$. For the converse we use Nagao’s Theorem

$$\Gamma = SL_2(\mathbb{F}_q) \ast_{SB(\mathbb{F}_q)} SB(A),$$

where $SB(R) = B_2(R) \cap \Gamma$. See for example [Se, p.88]. From this we see that the quotient of $\Gamma$ by the normal hull of $P$ is $T(A)$. This proves (a) and (b).

Part (c) follows from the simple fact that the normal subgroup of $SL_2(\mathbb{F}_q)$ generated by $T(1)$ is $SL_2(\mathbb{F}_q)$ itself. □

Theorem 6.3. Let $A = \mathbb{F}_q[t]$ with $q \leq 3$ and let $N$ be a normal genuine non-congruence subgroup of $\Gamma$. Then $q^2$ divides $|\Gamma : N|$. In particular

$$\text{ngncs}(\Gamma) > \text{ncs}(\Gamma) = q.$$
Proof. If $1 \not\in \text{ql}(N)$, the proof is exactly the same as for Theorem 6.1.

If $1 \in \text{ql}(N)$ and $q^2 \nmid |\Gamma : N|$, then by Lemma 6.2 necessarily $\Gamma' \leq N$ and $|\Gamma : N| = |A : \text{ql}(N)| = q$. Then there exists an $F_q$-vector space automorphism $\phi$ of $A$ (with corresponding non-standard automorphism $\Phi$) such that $\phi(1) = 1$ and $\phi(\text{ql}(N)) = t(t - 1)A \oplus F_q$. So $\Phi(N)$ has level $t(t - 1)A$ (and still contains $\Gamma'$).

To finish the proof we verify that $\Phi(N)$ is a congruence subgroup by showing

$$\Delta(t(t - 1)).\Gamma' = \Gamma(t(t - 1)).\Gamma'.$$

It is well-known that

$$\Gamma/\Gamma(t(t - 1)) \cong SL_2(A/(t)) \times SL_2(A/(t - 1)).$$

Under this isomorphism

$$(\Gamma(t(t - 1)).\Gamma')/\Gamma(t(t - 1)) \cong SL_2(F_q)' \times SL_2(F_q)'.$$

It follows that

$$|\Gamma : \Gamma(t(t - 1)).\Gamma'| = q^2 = |\Gamma : \Delta(t(t - 1)).\Gamma'|,$$

which proves the claim. $$\square$$

Theorem 6.4. Let $N$ be a normal genuine non-congruence subgroup of $G = GL_2(F_q[t])$. Then $|G : N|$ is divisible by

$$\begin{cases} q|\text{PSL}_2(F_q)|, & \text{if } q > 3, \\ q^2, & \text{if } q \leq 3. \end{cases}$$

In particular,

$$\text{ngncs}(G) > \text{mncs}(G).$$

Proof. First of all, $|G : N|$ is divisible by $|\Gamma : N \cap \Gamma|$, which is bigger than 1 unless $\Gamma \subseteq N$. If $q > 3$, then, as explained earlier, $|\Gamma : N \cap \Gamma|$ is divisible by $|\text{PSL}_2(F_q)|$.

Now assume that $q^2$ does not divide $|\Gamma : N \cap \Gamma|$. By the proofs of Theorems 6.1 and 6.3 then there exists a non-standard automorphism $\Phi$ of $\Gamma$ that maps $N$ to a congruence subgroup. By defining $\Phi$ to act as identity on diagonal matrices, $\Phi$ extends to a non-standard automorphism $\Phi$ of $G$. As $\Phi(N)$ contains a congruence subgroup, $N$ is not genuine.

To prove the last claim we exhibit a normal congruence subgroup of small index in $G$ and quasi-level $tA$. By a suitable non-standard automorphism this group can then be mapped to a normal (non-genuine) non-congruence subgroup of the same index.
If $q > 3$ we can take $Z.\Gamma(t)$, which has index $|SL_2(\mathbb{F}_q)|$. If $q = 2$, then $G = \Gamma$ anyway, and Theorem 6.3 applies. Finally, if $q = 3$ we observe that the 2-Sylow subgroup of $SL_2(\mathbb{F}_3)$ is normal in $GL_2(\mathbb{F}_3)$. Taking its inverse image under the (in this case surjective) natural map $G \to GL_2(A/(t))$, we obtain a normal subgroup of index 6. □

**Remark 6.5.** More precisely, Theorems 6.1, 6.3 and 6.4 show that in order for a normal subgroup $N$ to have a chance of being genuine ql$(N)$ must have at least codimension 2 in $A$, and hence $X/N$ must contain a subgroup isomorphic to $\mathbb{F}_q \oplus \mathbb{F}_q$.

We finish with a partial result on not necessarily normal genuine non-congruence subgroups.

**Corollary 6.6.** Let $q = p$ a prime, i.e. $A = \mathbb{F}_p[t]$. Then $\text{gncs}(X) \geq 2p$, and hence in particular $\text{gncs}(\Gamma) > \text{ncs}(\Gamma)$.

**Proof.** Let $N$ be the core of $H$ in $X$. If $|X : H| < 2p$, then $|X : N|$ divides $(2p - 1)!$ and is therefore not divisible by $p^2$. So $N$ cannot be genuine, and consequently neither can be $H$. □

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**References**

[B] H. Bass: *Algebraic K-Theory*, (Benjamin, New York, 1968).

[D] J. L. Dyer: Automorphism sequences of integer unimodular groups, *Illinois J. Math.* 22 (1978), 1-30.

[G1] E.-U. Gekeler: Le genre des courbes modulaires de Drinfeld, *C.R. Acad. Sci. Paris* 300 (1985), 647-650.

[G2] E.-U. Gekeler: *Drinfeld Modular Curves*, (Springer LNM 1231, Berlin Heidelberg New York, 1986).

[HR] L. K. Hua and I. Reiner: Automorphisms of the unimodular group, *Trans. Amer. Math. Soc.* 71 (1951), 331-348.

[JT] G. A. Jones and J. S. Thornton: Automorphisms and congruence subgroups of the extended modular group, *J. London Math. Soc.* (2) 34 (1986), 26-40.
[M1] A. W. Mason: Anomalous normal subgroups of $SL_2(K[x])$, *Quart. J. Math. Oxford (2)* **36** (1985), 345-358.

[M2] A. W. Mason: The order and level of a subgroup of $GL_2$ over a Dedekind ring of arithmetic type, *Proc. Roy. Soc. Edinburgh Sect. A* **119** (1991), 191-212.

[M3] A. W. Mason: On non-congruence subgroups of the analogue of the modular group in characteristic $p$: Rankin memorial issues. *Ramanujan J.* **7** (2003), 141-144.

[M4] A. W. Mason: The generalization of Nagao’s theorem to other subrings of the rational function field, *Comm. Algebra* **31** (2003), 5199-5242.

[MS1] A. W. Mason and A. Schweizer: The minimum index of a non-congruence subgroup of $SL_2$ over an arithmetic domain, *Israel J. Math.* **133** (2003), 29-44.

[MS2] A. W. Mason and A. Schweizer: The minimum index of a non-congruence subgroup of $SL_2$ over an arithmetic domain. II: The rank zero cases, *J. London Math. Soc.* **71** (2005), 53-68.

[MS3] A. W. Mason and A. Schweizer: Non-standard automorphisms and non-congruence subgroups of $SL_2$ over Dedekind domains contained in function fields, *J. Pure Appl. Algebra* **205** (2006), 189-209.

[MS4] A. W. Mason and A. Schweizer: The stabilizers in a Drinfeld modular group of the vertices of its Bruhat-Tits tree: an elementary approach, *Int. J. Algebr. Comput.* **23** (2013), 1653-1683.

[R] I. Reiner: A new type of automorphism of the general linear group over a ring, *Ann. of Math.* **66** (1957), 461-466.

[Se] J.-P. Serre, *Trees*, (Springer, Berlin, Heidelberg, New York, 1980).

[SV] J. Smillie and K. Vogtmann: Automorphisms of $SL_2$ over imaginary quadratic integers, *Proc. Amer. Math. Soc. (2)* **112** (1991), 691-699.

[St] H. Stichtenoth: *Algebraic Function Fields and Codes (Second Edition)*, (Springer GTM 254, Berlin Heidelberg, 2009).

[T] S. Takahashi: The fundamental domain of the tree of $GL(2)$ over the function field of an elliptic curve, *Duke Math. J.* **72** (1993), 85-97.