A Note on the Multivariate CLT and Convergence of Lévy Processes at Long and Short Times

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Abstract

We show that a necessary and sufficient condition for the sum of iid random vectors to converge (under appropriate shifting and scaling) to a multivariate Gaussian distribution is that the truncated second moment matrix is slowly varying at infinity. This is more natural than the standard conditions, and allows for the possibility that the limiting Gaussian distribution is degenerate (so long as it is not concentrated at a point). We also give necessary and sufficient conditions for a $d$-dimensional Lévy process to converge (under appropriate shifting and scaling) to a multivariate Gaussian distribution as time approaches zero or infinity.

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1 Introduction

Let $X$ be a random variable with distribution $\mu$ on $\mathbb{R}^d$. If $d = 1$, a necessary and sufficient condition for $X$ to belong to the domain of attraction of the Gaussian is that the truncated second moment function

$$f(t) = \int_{|x| \leq t} x^2 \mu(dx)$$

is slowly varying (see e.g. [4]). However, we have found no similar condition in the literature for the case $d \geq 2$. Instead, conditions are given in terms of regular variation of certain quadratic forms, see [9]. For this reason, conditions are only given for convergence to nondegenerate Gaussian distributions. We give necessary and sufficient conditions for $X$ to belong to the domain

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of attraction of any (possibly degenerate) Gaussian distribution in terms of
the slow variation of its truncated second moment matrix. We also give the
resulting result for the long and short time behavior of Lévy processes.

A Lévy process is a continuous time process that generalizes summation
of iid random variables, see [10]. Assume that \( \mu \) is an infinitely divisible
distribution and let \( \{X_t : t \geq 0\} \) be a Lévy processes with \( X_1 \sim \mu \). The long
(short) time behavior of this process is the weak limit of \( X_t \), under appro-
priate shifting and scaling, as \( t \) approaches infinity (zero). An alternate, but
Equivalent, definition, in terms of weak convergence of certain time rescaled
processes, is also sometimes used (see e.g. [8]). Necessary and sufficient con-
ditions for the long and short time behavior to be an infinite variance stable
distribution are given in [5]. In this paper, we give necessary and sufficient
conditions for the long and short time behavior of the proces s to be Gaussian.

In the next section we introduce our notation and give some back-
ground. In particular we define regular variation for matrix-valued functions. While
regular variation of invertible matrix-valued functions has been studied (see
E.g. [1] or [6]), our definition, which is valid even for certain non-invertible
matrix-valued functions, appears to be new. In Section 3 we state our main
results and in Section 4 we give the proofs.

2 Preliminaries

Let \( \mathbb{R}^d \) be the space of \( d \)-dimensional column vectors of real numbers, let \( | \cdot | \)
be the usual norm on \( \mathbb{R}^d \), and let \( \mathcal{B}(\mathbb{R}^d) \) denote the Borel sets on \( \mathbb{R}^d \). For
\( x \in \mathbb{R}^d \) we write \( x = (x_1, x_2, \ldots, x_d) \) and let \( x^T \) be the transpose of \( x \). We
write \( X \sim \mu \) to denote that \( X \) is a random variable on \( \mathbb{R}^d \) with distribution \( \mu \).
For a sequence \( X_1, X_2, \ldots \) of random variables we write \( \lim_{n \to \infty} X_n \) to denote
the limit in distribution. Let \( \mathbb{R}^{d\times d} \) be the collection of all \( d \times d \) dimensional
matrices with real entries. If \( A \in \mathbb{R}^{d\times d} \) let \( \text{tr} A \) be the trace of the matrix
\( A \). If \( f \) and \( g \) are real-valued functions and \( c \in \{0, \infty\} \), a Borel function \( f : (0, \infty) \to (0, \infty) \)
is called regularly varying at \( c \) with index \( \rho \) if
\[
\lim_{x \to c} \frac{f(tx)}{f(x)} = t^\rho.
\]
In this case we write \( f \in RV^c_\rho \). If \( h(x) = 1/f(1/x) \) then
\[
f \in RV^c_\rho \text{ if and only if } h \in RV^{1/c}_\rho. \tag{1}
\]
If \( f \in RV^c_\rho \) then \( f(x + t) \sim f(t) \) as \( t \to c \) (this follows from the Potter
Bounds, see e.g. Theorem 1.5.6 in [2]). If \( f \in RV^c_\rho \) with \( \rho > 0 \) and \( f^c(x) = \int_{0}^{x} f(t) dt \),
\[
\inf \{ y > 0 : f(y) > x \} \quad \text{then} \\
\quad f^{-} \in RV_{1/\rho}^c
\]

and \( f^{-} \) is an asymptotic inverse of \( f \) in the sense that
\[
f(f^{-}(x)) \sim f^{-}(f(x)) \sim x \quad \text{as} \quad x \to c.
\]

When \( c = \infty \) this result is given on page 28 of [2]. The case when \( c = 0 \) can be shown using an extension of those results and (1).

We now introduce a definition of regular variation for matrix valued functions. While, regular variation of invertible matrix-valued functions is defined in [1] and [6], we need a different definition to allow for regular variation of certain non-invertible matrix-valued functions.

**Definition 1.** Fix \( c \in \{0, \infty\} \) and \( \rho \in \mathbb{R} \). Let \( A_\bullet : (0, \infty) \to \mathbb{R}^{d \times d} \) and \( B \in \mathbb{R}^{d \times d} \). If \( \text{tr} A_\bullet \in RV^c_\rho \) and
\[
\lim_{t \to c} \frac{A_t}{\text{tr} A_t} = B
\]
we say that \( A_\bullet \) is matrix regularly varying at \( c \) with index \( \rho \) and limiting matrix \( B \). In this case we write \( A_\bullet \in MRV^c_\rho(B) \).

Before proceeding, we review some basic properties of infinitely divisible distributions. Recall that the characteristic function of an infinitely divisible distribution \( \mu \) can be written as \( \hat{\mu}(z) = \exp \{ C_\mu(z) \} \) where
\[
C_\mu(z) = \frac{1}{2} \langle z, Az \rangle + i \langle b, z \rangle + \int_{\mathbb{R}^d} \left( e^{i \langle z, x \rangle} - 1 - i \frac{\langle z, x \rangle}{1 + |x|^2} \right) M(dx)
\]
is the cumulant generating function of \( \mu \). Here \( A \) is a symmetric nonnegative-definite \( d \times d \) matrix, \( b \in \mathbb{R}^d \), and \( M \) is a Lévy measure, i.e. \( M(\{0\}) = 0 \) and
\[
\int_{\mathbb{R}^d} (|x|^2 \wedge 1) M(dx) < \infty. \tag{2}
\]
The measure \( \mu \) is uniquely identified by the so called Lévy triplet \((A, M, b)\) and we write \( \mu = ID(A, M, b) \). For more information about infinitely divisible distributions and their associated Lévy processes see [10].

### 3 Main Results

First, we give necessary and sufficient conditions for the multivariate central limit theorem.
Theorem 1. Let $B \neq 0$ be a symmetric nonnegative-definite $d \times d$ matrix, let $\mu$ be a probability measure on $\mathbb{R}^d$, let $X_1, X_2, \ldots \sim \mu$, and let

$$A_t = \int_{|x| \leq t} xx^T \mu(dx) - \int_{|x| \leq t} x \mu(dx) \int_{|x| \leq t} x^T \mu(dx).$$

(3)

There exist non-stochastic $a_n \in (0, \infty)$ and $\xi_n \in \mathbb{R}^d$ such that

$$a_n \sum_{i=1}^{n} X_i - \xi_n \xrightarrow{d} N(0, B) \text{ as } n \to \infty$$

(4)

if and only if $A_* \in MRV_{0}^\infty(kB)$, where $k = 1/\text{tr}B$. Moreover, when this holds $a_* \in RV^{c-1/2}$ and if $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$ then

$$a_n \sim k^{-1/2} \left[ \int_{\mathbb{R}^d} |x|^2 \mu(dx) - \int_{\mathbb{R}^d} x^T \mu(dx) \int_{\mathbb{R}^d} x \mu(dx) \right]^{-1/2} n^{-1/2}$$

(5)

and if $\int_{\mathbb{R}^d} |x|^2 \mu(dx) = \infty$ then

$$a_n \sim k^{-1/2}/h^{-c}(n) \text{ as } n \to \infty,$$

(6)

where

$$h(t) = \frac{t^2}{\int_{|x| \leq t} |x|^2 \mu(dx)}.$$  

(7)

Corollary 1. Let $\mu$ be a probability measure on $\mathbb{R}^d$ and let

$$A'_t = \int_{|x| \leq t} xx^T \mu(dx).$$

Then $\mu$ belongs to the domain of attraction of some multivariate normal distribution if and only if there exists a nonnegative definite matrix $B' \neq 0$ such that $A_* \in MRV_{0}^\infty(B')$.

We now give necessary and sufficient conditions for the long and short time behavior of a Lévy process to be Gaussian. In one dimension this was characterized in [3]. Since long and short time behavior of Brownian motion is straight-forward, without loss of generality, we focus on the case where the Gaussian part is zero.

Theorem 2. Fix $c \in \{0, \infty\}$. Let $B \neq 0$ be a symmetric nonnegative-definite matrix, let $\{X_t : t \geq 0\}$ be a Lévy process with $X_1 \sim ID(0, M, b)$, and let

$$A_t = \int_{|x| \leq t} xx^T M(dx).$$

(8)
There exist non-stochastic functions $a_t$ and $\xi_t$ such that
\[ a_t X_t - \xi_t \xrightarrow{d} N(0, B) \text{ as } t \to c \]  
(9)

if and only if $A_\bullet \in MRV_0^c(kB)$ where $k = 1/\text{tr} B$. Moreover, when this holds, $a_\bullet \in RV_{-1/2}$ and
\[ a_t \sim k^{-1/2}/h^+(t) \text{ as } t \to c, \]  
(10)

where
\[ h(t) = \frac{t^2}{\int_{|x| \leq t} |x|^2 M(dx)}. \]  
(11)

Combining Corollary 1 with Theorem 2 gives the following.

**Corollary 2.** Let $\mu = 1D(0, M, b)$. There is a nonnegative definite matrix $B \neq 0$ with
\[ \int_{|x| \leq \bullet} xx^T \mu(dx) \in MRV_0^\infty(B) \]
if and only if there is a nonnegative definite matrix $B' \neq 0$ with
\[ \int_{|x| \leq \bullet} xx^T M(dx) \in MRV_0^\infty(B'). \]

**4 Proofs**

**Lemma 1.** Fix $a, b \in (0, \infty)$ and let $\{M_n\}$ be a sequence of measures on $\mathbb{R}^d$ satisfying (2). If, for any $s > 0$, $\lim_{n \to \infty} M_n(x : |x| > s) = 0$ then
\[ \lim_{n \to \infty} \left( \int_{|x| \leq a} xx^T M_n(dx) - \int_{|x| \leq b} xx^T M_n(dx) \right) = 0. \]  
(12)

Proof. Without loss of generality assume that $a < b$. For all $1 \leq i, j \leq d$
\[ \left| \int_{|x| \leq b} x_i x_j M_n(dx) - \int_{|x| \leq a} x_i x_j M_n(dx) \right| = \left| \int_{a < |x| \leq b} x_i x_j M_n(dx) \right| \]
\[ \leq \int_{a < |x| \leq b} |x|^2 M_n(dx) \leq b^2 M_n(x : a < |x| \leq b) \to 0, \]
as required.  

The following is a specialization of Theorem 3.1.14 in [6] (or Theorem 8.7 in [10]). To put the result in this form we use Lemma 1.

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Lemma 2. If $\mu_n = ID(0, M_n, b_n)$ then $\mu_n \overset{w}{\rightarrow} N(b, A)$ if and only if $b_n \rightarrow b$,
\begin{equation}
M_n(x : |x| > s) \rightarrow 0 \text{ for every } s > 0,
\end{equation}
and
\begin{equation}
\lim_{n \rightarrow \infty} \int_{|x| \leq 1} xx^T M_n(dx) = A.
\end{equation}

Lemma 3. Let $\{X_t : t \geq 0\}$ be a Lévy process, let $Y \sim N(0, A)$ with $A \neq 0$, and let $a_t$ be a positive function. Assume that (9) holds with some function $\xi_t$.
1. If $c = 0$ then $\lim_{t \downarrow 0} a_t = \infty$ and $a_{1/t} \sim a_{1/(t+1)}$ as $t \rightarrow \infty$.
2. If $c = \infty$ then $\lim_{t \rightarrow \infty} a_t = 0$ and $a_t \sim a_{t+1}$ as $t \rightarrow \infty$.

Proof. First assume $c = 0$. Let $\ell := \liminf_{t \downarrow 0} a_t$ and assume for the sake of contradiction that $\ell < \infty$. This means that there is a sequence of positive real numbers $\{t_n\}$ converging to 0 such that $\lim_{n \rightarrow \infty} a_{t_n} = \ell$. Consider a further subsequence $\{t_{n_i}\}$ such that $\lim_{i \rightarrow \infty} \xi_{t_{n_i}}$ exists (although we allow it to be infinite). Stochastic continuity of Lévy processes implies that $X_{t_i} \overset{D}{\rightarrow} 0$ as $t \downarrow 0$, thus Slutsky’s Theorem implies that
\begin{align*}
Y &= \lim_{i \rightarrow \infty} (a_{t_{n_i}} X_{t_{n_i}} - \xi_{t_{n_i}}) \overset{d}{=} \ell 0 - \lim_{i \rightarrow \infty} \xi_{t_{n_i}},
\end{align*}
which contradicts the assumption that $Y \sim N(0, A)$. Thus $\lim_{t \downarrow 0} a_t = \infty$.

Let $C_{X_1}(z)$, $z \in \mathbb{R}^d$, be the cumulant generating function of $X_1$. The characteristic function of $a_{1/t} X_{1/t} - \xi_{1/t}$ is $\exp \left( \frac{1}{t} C_{X_1}(a_{1/t} z) - i \langle z, \xi_{1/t} \rangle \right)$. If $\tilde{\mu}_Y(z)$ is the characteristic function of $Y$ then
\begin{align*}
\tilde{\mu}_Y(z) &= \lim_{t \rightarrow \infty} \exp \left( \frac{1}{t} C_{X_1}(a_{1/t} z) - i \langle z, \xi_{1/t} \rangle \right) \\
&= \lim_{t \rightarrow \infty} \exp \left( \frac{1}{t+1} C_{X_1}(a_{1/t} z) - i \langle z, \frac{t}{t+1} \xi_{1/t} \rangle \right).
\end{align*}
This implies that
\begin{align*}
Y &\overset{d}{=} \lim_{t \rightarrow \infty} \left( a_{1/t} X_{1/t} - \xi_{1/t} \right) \\
&\overset{d}{=} \lim_{t \rightarrow \infty} \left( a_{1/t} X_{1/(t+1)} - \frac{t}{t+1} \xi_{1/t} \right) \\
&\overset{d}{=} \lim_{t \rightarrow \infty} \left( \frac{a_{1/t}}{a_{1/(t+1)}} (a_{1/(t+1)} X_{1/(t+1)} - \xi_{1/(t+1)}) + \frac{a_{1/t}}{a_{1/(t+1)}} \xi_{1/(t+1)} - \frac{t}{t+1} \xi_{1/t} \right).
\end{align*}
Since $(a_{1/(t+1)} X_{1/(t+1)} - \xi_{1/(t+1)}) \overset{d}{\rightarrow} Y$ as $t \rightarrow \infty$, the result follows.
Now assume \( c = \infty \). The Lévy measure of \( a_t X_t - \xi_t \) is \( M_t(\cdot) = tM(\cdot/a_t) \).

By Lemma 2 for any \( s > 0 \)

\[
\lim_{t \to \infty} tM(|x| > s/a_t) = \lim_{t \to \infty} M_t(|x| > s) = 0,
\]

which implies that \( \lim_{t \to \infty} a_t = 0 \). Now let \( X' \overset{d}{=} X_1 \) be independent of \( \{X_t : t \geq 0\} \) and note that \( a_t X' \overset{d}{\to} 0 \) as \( t \to \infty \). The facts that

\[
Y \overset{d}{=} \lim_{t \to \infty} (a_t X_t - \xi_t)
\]

and \( (a_t X_t - \xi_t) \overset{d}{=} Y \) as \( t \to \infty \) give the result. \( \square \)

**Lemma 4.** Fix \( c \in \{0, \infty\} \). Let \( M \) be a measure on \( \mathbb{R}^d \) satisfying (2), let \( A_t \) be defined by (8), and let \( a_t \) be defined by (10). Assume that \( A_\bullet \in MRV_0^c(B) \) for some \( B \neq 0 \) and

\[
M_t(D) = t \int_{\mathbb{R}^d} 1_A(a_t x) M(dx), \quad D \in \mathcal{B}(\mathbb{R}^d). \tag{15}
\]

If, for \( \eta \in [0, 2) \),

\[
\int_{|x| > 1} |x|^\eta M(dx) < \infty \tag{16}
\]

then \( \lim_{t \to c} \int_{|x| > s} |x|^\eta M_t(dx) = 0 \) for all \( s > 0 \). Moreover, when \( c = \infty \) \( \tag{16} \)

holds for every \( \eta \in [0, 2) \).

**Proof.** Let

\[
U(u) := \int_{|x| \leq u} |x|^2 M(dx) = \text{tr} A_u
\]

and

\[
U^t(u) := \int_{|x| \leq u} |x|^2 M_t(dx) = ta_t^2 U(u/a_t).
\]

Note that \( U \in RV^c_0, a_\bullet \in RV^c_{1/2} \), and \( \lim_{t \to c} a_t = 1/c \). By Fubini’s Theorem and the fact that \( t \sim h(1/(a_t \sqrt{k})) = [ka_t^2 U(1/(a_t \sqrt{k}))]^{-1} \sim [ka_t^2 U(1/a_t)]^{-1} \)
as \( t \to c \) it follows that for any \( s > 0 \)

\[
\lim_{t \to c} \int_{|x| > s} |x|^{\eta} M_t(dx) = \lim_{t \to c} \left[ (2 - \eta) \int_{s}^{\infty} u^{\eta - 3} U^t(u)du - s^{\eta - 2} U^t(s) \right]
\]

\[
= \lim_{t \to c} t a_t^2 \left[ (2 - \eta) \int_{s}^{\infty} u^{\eta - 3} U(u/a_t)du - s^{\eta - 2} U(s/a_t) \right]
\]

\[
= \lim_{t \to c} k^{-1} \left[ (2 - \eta) \int_{s/a_t}^{\infty} u^{\eta - 3} U(u)du \frac{U(s/a_t)}{U(1/a_t)} - s^{\eta - 2} U(s/a_t)(s/a_t)^{\eta - 2} s^{\eta - 2} \right]
\]

\[
= k^{-1} \left( s^{\eta - 2} - s^{\eta - 2} \right) = 0,
\]

where the fourth equality follows by change of variables and the fifth by Karamata’s Theorem (see e.g. Theorem 2.1 in [7]). Note that Karamata’s Theorem still holds when \( U \in RV_{0}^{c} \) since, in that case, \( U(1/\bullet) \in RV_{0}^{\infty} \). We now prove the last statement. Fubini’s theorem implies that for any \( s > 0 \)

\[
\int_{|x| > s} |x|^{\eta} M_t(dx) = (2 - \eta) \int_{s}^{\infty} u^{\eta - 3} U^t(u)du - s^{\eta - 2} U^t(s).
\]

The right side is finite by Lemma 2 on Page 277 in [4], and hence the left side must be finite as well.

**Proof of Theorem 2.** Let \( M_t \) be given by (10), this is the Lévy measure of \( a_t X_t - \xi_t \).

First assume that \( A_\bullet \in MRV_{0}^{c}(kB) \) and that \( a_t \) is given by (10). This implies that \( a_\bullet \in RV_{c-1/2} \) and

\[
\lim_{t \to c} \int_{|x| \leq 1} x x^T M_t(dx) = \lim_{t \to c} t a_t^2 \int_{|x| \leq 1/a_t} x x^T M(dx)
\]

\[
= \lim_{t \to c} k^{-1} \frac{\int_{|x| \leq 1/a_t} x x^T M(dx)}{\int_{|x| \leq 1/(a_t \sqrt{\eta})} |x|^2 M(dx)}
\]

\[
= \lim_{t \to c} k^{-1} \frac{\int_{|x| \leq 1/a_t} x x^T M(dx)}{\int_{|x| \leq a_t} |x|^2 M(dx)} = B.
\]

From here the result follows by Lemmas 1 and 2.
Now assume that (9) holds. Lemma 2 implies that for every $s > 0$
\[
\lim_{t \to c} M_t(x : |x| > s) = 0.
\]
This means that we can use Lemma 1, which combined with Lemma 2 says
that for any $s > 0$
\[
\lim_{t \to c} ta^2 \int_{|x| \leq s/a} xx^T M(dx) = \lim_{t \to c} \int_{|x| \leq s} xx^T M_t(dx) = B.
\]
Thus, for any $s > 0$, $\lim_{t \to c} ta^2 U(s/a) = \text{tr} B$, where $U(t) = \int_{|x| \leq t} |x|^2 M(dx)$. Lemma 3 implies that the sequential criterion for regular variation of monotone functions (see e.g. Propositions 2.3 in [7]) holds, and hence $U \in RV_c^\infty$. Note that when $c = 0$ we can use the sequential criterion because $U \in RV_0^\infty$, if and only if $U(1/\bullet) \in RV_0^\infty$. The fact that
\[
\lim_{t \to c} \int_{|x| \leq t} xx^T M(dx) = \lim_{t \to c} ta^2 \int_{|x| \leq 1/a} xx^T M(dx) = B \quad \text{tr} B,
\]
gives $A_* \in MRV_0^\infty(kB)$.

The following technical result is easily verified.

**Lemma 5.** If $\int_{\mathbb{R}^d} |x|^2 \mu(dx) = \infty$ and $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$ then
\[
\lim_{t \to \infty} \frac{\int_{|x| \leq t} xx^T \mu(dx)}{\int_{|x| \leq t} |x|^2 \mu(dx)} = \lim_{t \to \infty} \frac{\int_{|x| \leq t} xx^T \mu(dx) - \int_{|x| \leq t} x \mu(dx) \int_{|x| \leq t} x^T \mu(dx)}{\int_{|x| \leq t} |x|^2 \mu(dx) - \int_{|x| \leq t} x^T \mu(dx) \int_{|x| \leq t} x \mu(dx)}
\]
so long as at least one of the limits exists.

**Proof of Theorem 7.** By Corollary 3.2.15 in [8] (4) holds if and only if for any $\epsilon > 0$
\[
n\mu(\{x : |x| > \epsilon/a_n\}) \to 0 \quad (17)
\]
and
\[
\lim_{n \to \infty} n a_n^2 \left[ \int_{|x| \leq 1/a_n} xx^T \mu(dx) - \int_{|x| \leq 1/a_n} x \mu(dx) \int_{|x| \leq 1/a_n} x^T \mu(dx) \right] = B. \quad (18)
\]
We will show that this is equivalent to $A_* \in MRV_0^\infty(kB)$.  

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We begin with the case when $\int_{\mathbb{R}^d} |x|^2 \mu(dx) = \infty$. First assume that $A_\bullet \in MRV_0^\infty(kB)$ and let $a_n$ be defined by (6). In this case, Lemma 4 implies that (17) holds and $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$. By Lemma 3

$$\lim_{n \to \infty} \frac{na_n^2}{\text{tr} A_{1/a_n}} = B.$$ 

Conversely, assume that (11) holds for some sequence $a_n$. From univariate results (see e.g. [4]), it follows that $a_n \to 0$, $a_n/a_{n+1} \to 1$, and $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$. From (13) we get

$$\text{tr} B = \lim_{n \to \infty} \frac{na_n^2}{\text{tr} A_{1/a_n}} \left[ \int_{|x| \leq 1/a_n} |x|^2 \mu(dx) - \int_{|x| \leq 1/a_n} x^T \mu(dx) \int_{|x| \leq 1/a_n} x \mu(dx) \right]$$

$$= \lim_{n \to \infty} \frac{na_n^2}{\text{tr} A_{1/a_n}} \int_{|x| \leq 1/a_n} |x|^2 \mu(dx).$$

From here, by arguments similar to those in the proof of Theorem 2, we get $\int_{|x| \leq 1/a_n} x^T \mu(dx) \in MRV_0^\infty(kB)$. By Lemma 5 this implies that $A_\bullet \in MRV_0^\infty(kB)$ as well.

Now consider the case when $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$. Let

$$B' = \frac{\int_{\mathbb{R}^d} x x^T \mu(dx) - \int_{\mathbb{R}^d} x \mu(dx) \int_{\mathbb{R}^d} x^T \mu(dx)}{\int_{\mathbb{R}^d} |x|^2 \mu(dx) - \int_{\mathbb{R}^d} x^T \mu(dx) \int_{\mathbb{R}^d} x \mu(dx)}.$$ 

By dominated convergence $A_\bullet \in MRV_0^\infty(B')$. Now let $a_n \sim c^{1/2}n^{-1/2}$ where $c^{1/2} = \eta^{-1/2} \left[ \int_{\mathbb{R}^d} |x|^2 \mu(dx) - \int_{\mathbb{R}^d} x^T \mu(dx) \int_{\mathbb{R}^d} x \mu(dx) \right]^{-1/2}$ for some $\eta > 0$. By Markov’s inequality, for any $\epsilon > 0$

$$\lim_{n \to \infty} n \mu(x : |x| > \epsilon/a_n) \leq \lim_{n \to \infty} \frac{n a_n^2}{\epsilon^2} \int_{|x| > \epsilon/a_n} |x|^2 \mu(dx)$$

$$= \lim_{n \to \infty} \frac{c}{\epsilon^2} \int_{|x| > \epsilon/a_n} |x|^2 \mu(dx) = 0.$$

By dominated convergence

$$\lim_{n \to \infty} \frac{na_n^2}{\text{tr} A_{1/a_n}} \left[ \int_{|x| \leq 1/a_n} x x^T \mu(dx) - \int_{|x| \leq 1/a_n} x \mu(dx) \int_{|x| \leq 1/a_n} x^T \mu(dx) \right]$$

$$= \frac{\eta^{-1} \int_{\mathbb{R}^d} x x^T \mu(dx) - \int_{\mathbb{R}^d} x \mu(dx) \int_{\mathbb{R}^d} x^T \mu(dx)}{\int_{\mathbb{R}^d} |x|^2 \mu(dx) - \int_{\mathbb{R}^d} x^T \mu(dx) \int_{\mathbb{R}^d} x \mu(dx)} = \eta^{-1} B'.$$
This implies that (4) holds for $B = \eta^{-1}B'$ and some sequence $\xi_n$. Since $\text{tr}B' = 1$, $\eta^{-1} = \text{tr}B$ and the result follows. \hfill $\Box$

Proof of Corollary 1. It suffices to show that $A_\bullet \in \text{MRV}_0^\infty(B)$ if and only if $A_\bullet' \in \text{MRV}_0^\infty(B')$. If $\int_{\mathbb{R}^d} |x|^p \mu(dx) < \infty$ the result is immediate. When $\int_{\mathbb{R}^d} |x|^p \mu(dx) = \infty$ the result follows from Lemma 5. The fact that, in both directions, $\int_{\mathbb{R}^d} |x| \mu(dx) < \infty$ can be shown as in the proof of Theorem 1. \hfill $\Box$

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