CLASSIFICATION OF FLAG-TRANSITIVE
STEINER QUADRUPLE SYSTEMS

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Abstract. A Steiner quadruple system of order $v$ is a $3-(v, 4, 1)$ design, and will be denoted $SQS(v)$.

Using the classification of finite 2-transitive permutation groups all $SQS(v)$ with a flag-transitive automorphism group are completely classified, thus solving the "still open and longstanding problem of classifying all flag-transitive $3-(v, k, 1)$ designs" (cf. [5, p. 273], [6]) for the smallest value of $k$. Moreover, a generalization of a result of H. Lüneburg [14] is achieved.

1. Introduction

In the last decades, there has been a great interest in classifying $t-(v, k, \lambda)$ designs with certain transitivity properties. For example, all point 2-transitive $2-(v, k, 1)$ designs were classified by Kantor [12] and a few years later Buekenhout et al. [2] reached a classification of all flag-transitive $2-(v, k, 1)$ designs. Both results depend on the classification of finite simple groups. However, the classification of flag-transitive $3-(v, k, 1)$ designs is "a still open and longstanding problem" (cf. [5, p. 273], [6]).

In this article we use the classification of finite 2-transitive permutation groups to classify all flag-transitive $SQS(v)$, thus solving the above problem for the smallest value of $k$. Moreover, our result generalizes a theorem of Lüneburg [14] that characterizes all flag-transitive $SQS(v)$ under the additional strong assumption that every non-identity element of the automorphism group fixes at most two points. Our procedure as well as our proofs are independent of Lüneburg.

For positive integers $t \leq k \leq v$ and $\lambda$ we define a $t-(v, k, \lambda)$ design to be an incidence structure $D = (X, B, I)$, where $X$ is a set of points, $|X| = v$, and $B$ a set of blocks, $|B| = b$, with the properties that each block $B \in B$ is incident with $k$ points, and every $t$-subset of $X$ is incident with $\lambda$ blocks. A Steiner quadruple system of order $v$, which will be denoted by $SQS(v)$, is a $3-(v, 4, 1)$ design. Hanani [8] showed that a $SQS(v)$ exists if and only if $v \equiv 2$ or $4 \pmod{6}$ ($v \geq 4$).

In the following let $D = (X, B, I)$ be a non-trivial $SQS(v)$ and $G \leq Aut(D)$ a group of automorphisms of $D$. A flag is an incident point-block pair, that is $x \in X$ and $B \in B$ such that $xIB$, and we call $G \leq Aut(D)$ to be flag-transitive (resp. block-transitive) if $G$ acts transitively on the flags (resp. on the blocks) of $D$. For short, $D$ is called flag-transitive (resp. block-transitive, point $t$-transitive) if $D$
admits a flag-transitive (resp. block-transitive, point $t$-transitive) group of automorphisms.

Our result is the following

**Theorem 1.** Let $\mathcal{D} = (X, B, I)$ be a non-trivial $SQS(v)$. Then $G \leq \text{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ if and only if one of the following occurs:

1. $\mathcal{D}$ is isomorphic to the $SQS(2^d)$ whose points and blocks are the points and planes of the affine space $AG(d, 2)$, and one of the following holds:
   - $d \geq 3$, and $G \cong AGL(d, 2)$,
   - $d = 3$, and $G \cong AGL(1, 8)$ or $A\Gamma L(1, 8)$,
   - $d = 4$, and $G_0 \cong A_7$,
   - $d = 5$, and $G \cong A\Gamma L(1, 32)$,
2. $\mathcal{D}$ is isomorphic to a $SQS(3^d + 1)$ whose points are the elements of $GF(3^d) \cup \{\infty\}$ and whose blocks are the images of $GF(3) \cup \{\infty\}$ under $PGL(2, 3^d)$ with $d \geq 2$ (resp. $PSL(2, 3^d)$ with $d > 1$ odd) and the derived design is isomorphic to the $2 - (3^d, 3, 1)$ design whose points and blocks are the points and lines of $AG(d, 3)$, and $PSL(2, 3^d) \leq G \leq P\Sigma L(2, 3^d)$,
3. $\mathcal{D}$ is isomorphic to a $SQS(q + 1)$ whose points are the elements of $GF(q) \cup \{\infty\}$ with a prime power $q \equiv 7 \pmod{12}$ and whose blocks are the images of $\{0, 1, \infty, \varepsilon\}$ under $PSL(2, q)$, where $\varepsilon$ is a primitive sixth root of unity in $GF(q)$ and the derived design is isomorphic to the Netto triple system, and $PSL(2, q) \leq G \leq P\Sigma L(2, q)$.

A detailed description of the Netto triple system can be found in [7, Section 3].

2. Preliminaries

If $\mathcal{D} = (X, B, I)$ is a $t - (v, k, \lambda)$ design, and $x \in X$ arbitrarily, the derived design with respect to $x$ is $\mathcal{D}_x = (X_x, B_x, I_x)$, where $X_x = X \setminus \{x\}$, $B_x = \{b \in B : x \notin B\}$, and $I_x = I \setminus \{x\}$. We shall also speak of $\mathcal{D}$ as being an extension of $\mathcal{D}_x$. Obviously, a derived design is a $(t - 1) - (v - 1, k - 1, \lambda)$ design.

For $g \in G \leq \text{Sym}(X)$ let $\text{fix}(g)$ denote the set of fixed points and $\text{supp}(g)$ the support of $g$. If $\{x_1, \ldots, x_n\} \subseteq X$ let $G_{\{x_1, \ldots, x_n\}}$ be its setwise stabilizer and $G_{x_1, \ldots, x_n}$ its pointwise stabilizer. If $B \in B$ let $G_B$ be its block stabilizer and $G_{(B)}$ its pointwise block stabilizer. By $r \perp q^n - 1$ we mean that $r$ divides $q^n - 1$ but not $q^k - 1$ for all $1 \leq k < n$.

All other notation is standard.

If $\mathcal{D} = (X, B, I)$ is a $t - (v, k, 1)$ design then it is elementary that the point 2-transitivity of $G \leq \text{Aut}(\mathcal{D})$ implies its flag-transitivity when $t = 2$. However, for $t \geq 3$ the converse holds:

**Lemma 2.** Let $\mathcal{D} = (X, B, I)$ be a $t - (v, k, 1)$ design with $t \geq 3$. If $G \leq \text{Aut}(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ then $G$ also acts 2-transitively on the points of $\mathcal{D}$.

**Proof.** Let $x \in X$. As $G$ acts flag-transitively on $\mathcal{D}$, obviously $G_x$ acts block-transitively on $\mathcal{D}_x$. Since block-transitivity implies transitivity on points for $t \geq 2$ by Block’s Theorem [1], $G_x$ also acts transitively on the points of $\mathcal{D}_x$ and the claim follows. □
To classify all flag-transitive SQS($v$), we can therefore use the classification of finite 2-transitive permutation groups which itself relies on the classification of finite simple groups (cf. [3], [9], [11], [12]).

The list of groups is as follows:

Let $G$ be a finite 2-transitive permutation group of a non-empty set $X$. Then we have either

(A) **Affine type:** $G$ contains a regular normal subgroup $T$ which is elementary abelian of order $v = p^d$, where $p$ is a prime. Let $a$ be a divisor of $d$. Identify $G$ with a group of affine transformations $x \mapsto x^g + c$ of $V(d, p)$, where $g \in G_0$. Then one of the following occurs:

1. $G \leq \AGL(1, p^d)$
2. $G_0 \cong SL(\frac{d}{a}, p^a)$
3. $G_0 \cong Sp(\frac{2d}{a}, p^a)$, $d \geq 2a$
4. $G_0 \cong G_2(2^a)$, $d = 6a$
5. $G_0 \cong A_6$ or $A_7$, $v = 2^4$
6. $G_0 \cong SL(2, 3)$ or $SL(2, 5)$, $v = p^2$, $p = 5, 7, 11, 19, 23, 29$ or $59$, or $v = 3^4$
7. $G_0$ contains a normal extraspecial subgroup $E$ of order $2^{5\epsilon}$, and $G_0/E$ is isomorphic to a subgroup of $S_5$, where $v = 3^4$
8. $G_0 \cong SL(2, 13)$, $v = 3^6$

or

(B) **Semisimple type:** $G$ contains a simple normal subgroup $N$, and $N \leq G \leq \Aut(N)$. In particular, one of the following holds, where $N$ and $v = |X|$ are given:

1. $A_v$, $v \geq 5$
2. $PSL(d, q)$, $d \geq 2$, $v = \frac{q^d - 1}{q - 1}$, where $(d, q) \neq (2, 2), (2, 3)$
3. $PSU(3, q^2)$, $v = q^3 + 1$, $q > 2$
4. $Sz(q)$, $v = q^2 + 1$, $q = 2^{2c+1} > 2$ (Suzuki group)
5. $2G_2(q)$, $v = q^3 + 1$, $q = 3^{2c+1} > 3$ (Ree group)
6. $Sp(2d, 2)$, $d \geq 3$, $v = 2^{2d-1} \pm 2^{d-1}$
7. $PSL(2, 11)$, $v = 11$
8. $PSL(2, 8)$, $v = 28$ (N not 2-transitive)
9. $M_\nu$, $v = 11, 12, 22, 23, 24$ (Mathieu group)
10. $M_{11}$, $v = 12$
11. $A_7$, $v = 15$
12. $HS$, $v = 176$ (Higman-Sims group)
13. $Co_3$, $v = 276$. (smallest Conway group)

Let $r$ denote the number of blocks incident with a point. The following obvious observation is important for this paper:

**Lemma 3.** Let $\mathcal{D} = (X, B, I)$ be a $t - (v, k, 1)$ design, and $x \in X$ arbitrarily. If $G \leq \Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ then the division property

$$r \mid |G_x|$$

holds.

Counting in two ways easily yields that $r = (v-1)(v-2)/6$ when $\mathcal{D}$ is a SQS($v$).
3. Proof of the theorem

Using the notation as before, let \( D = (X, B, I) \) be a \( SQS(v) \). In this section we run through the list of finite 2-transitive permutation groups given in Section 2 and examine successively whether \( G \leq \text{Aut}(D) \) acts flag-transitively on \( D \).

3.1. Affine case. From Section 2 we know that a 2-transitive permutation group \( G \) of affine type has degree \( v = p^d \). As a \( SQS(v) \) exists if and only if \( v = 2 \) or \( 4 \mod 6 \) \( (v \geq 4) \) by Hanani’s theorem, we conclude that \( v = 2^d \) in this case. To avoid trivial \( SQS(v) \), let \( d \geq 3 \).

The following lemma is fundamental for this case.

**Lemma 4.** Let \( D = (X, B, I) \) be a \( SQS(2^d) \) with \( d \geq 3 \), and \( G \leq \text{Aut}(D) \) contains a regular normal subgroup \( T \) which is elementary abelian of order \( v = 2^d \). If \( G \) acts flag-transitively on \( D \) and \( |G_0| \equiv 1 \mod 2 \), then \( D \) is uniquely determined (up to isomorphism), and the points and blocks of \( D \) are the points and planes of \( AG(d, 2) \).

**Proof.** \( T \) contains subgroups of order 4 as it is elementary abelian of order \( 2^d \). Moreover, \( T \) is the only Sylow 2-group since \( |G_0| \equiv 1 \mod 2 \), and contains therefore all subgroups of \( G \) of order 4. By assumption, \( G_B \) acts transitively on the points of \( B \) for \( B \in \mathcal{B} \) arbitrarily. Thus 4 is a divisor of the order of \( G_B \), and \( G_B \) contains at least one subgroup \( S \) of \( T \) of order 4. Then \( B \in \mathcal{B} \) is an orbit of \( S \) and hence an affine plane. As \( G \leq \text{Aut}(D) \) is block-transitive, we can conclude that all blocks must be affine planes. Now identify the points of \( D \) with the elements of \( T \) and the assertion follows. \( \square \)

**Case (1):** \( G \leq AGL(1, 2^d) \).

Let \( D = (X, B, I) \) be a \( SQS(2^d) \), \( d \geq 3 \), and assume \( G \leq \text{Aut}(D) \) acts flag-transitively on \( D \). Lemma 3 and Lagrange’s theorem yield

\[
|G| = |AGL(1, 2^d)_{0}| = |GL(1, 2^d)| = d(2^d - 1).
\]

Thus \( d = 3, 5 \). First, assume \( d = 3 \). Then

\[
|AGL(1, 8)| = |T| |GL(1, 8)| = 8 \cdot 7 \cdot 3.\]

Since \( G \) is 2-transitive, we have \( 8 \cdot 7 \mod |G| \), hence \( |G| = 8 \cdot 7 \text{ or } 8 \cdot 7 \cdot 3 \). The latter implies \( G \cong AGL(1, 8) \), so assume \( |G| = 8 \cdot 7 \). Since \( AGL(1, 8) \) is solvable, we deduce from Hall’s theorem that \( G \cong AGL(1, 8) \) as \( G \) is a Hall \( \{2, 7\} \)-group. For \( d = 5 \) again \( |G| = 32 \cdot 31 \text{ or } 32 \cdot 31 \cdot 5 \). We conclude \( G \cong AGL(1, 32) \) as for \( |G| = 32 \cdot 31 \) lemma 3 yields a contradiction.

On the contrary, we have to show that \( G \cong AGL(1, 8), AGL(1, 8) \) resp. \( AGL(1, 32) \) acts flag-transitively on the \( SQS(8) \) resp. the \( SQS(32) \) given in the theorem. For \( v = 8 \) there exists (up to isomorphism) only the unique \( SQS(v) \) consisting of the points and planes of \( AG(3, 2) \). Since \( G \cong AGL(1, 8) \) acts transitively on the points, it is sufficient to show that \( G_0 \cong GL(1, 8) \) acts transitively on the blocks incident with 0. As these are exactly the 2-dimensional subspaces of the underlying vector space, we have

\[
B_1 := \{0, 1, t, t + 1\} \neq B_1^t = \{0, t, t^2, t^2 + 1\} \quad \text{for} \quad 1 \neq t \in GL(1, 8) \cong GF(8)^*.
\]

Thus \( |B_1^{GL(1, 8)}| \neq 1 \), and hence as \( r = 7 \), the claim follows by the orbit-stabilizer property. Obviously, \( G \cong AGL(1, 8) \) acts flag-transitively on \( D \) as well. For \( v = 32 \) we have by lemma 3 also only the unique \( SQS(v) \) consisting of the points and planes of \( AG(5, 2) \) because \( |G_0| = |GL(1, 32)| \equiv 1 \mod 2 \). To see that
$G_0 \cong \Gamma L(1,32)$ acts flag-transitively on the blocks incident with 0, examine as before that $|P_{G}^{GL(1,32)}| \neq 1$, thus $|GL(1,32)| = 1$ for any $0 \in B \in B$ by the orbit-stabilizer property. Hence $|B^{\Gamma L(1,32)}| = 31$ or $31 \cdot 5$. Assuming the first yields $|\Gamma L(1,32)| = 5$ by the orbit-stabilizer property again. Let $H$ be a cyclic group of order 5. Then $|H_B| \neq 1$ for any $0 \in B \in B$. On the other hand, 5 is a 2-primitive divisor of $2^4 - 1$. Thus $H$ has irreducible modules of degree 4 in view of [9] Theorem 3.5]. As the 5-dimensional $GF(32)H$-module is completely reducible by Maschke’s theorem, $H$ has as irreducible modules only the trivial module and one of degree 4. But if $H$ fixes any 2-dimensional vector subspace then, again by Maschke’s theorem, $H$ would have as irreducible modules two 1-dimensional modules, a contradiction. Therefore, $|B^{\Gamma L(1,32)}| = 31 \cdot 5$ must hold and the claim follows as $r = 31 \cdot 5$.

Case (2): $G_0 \geq SL(\frac{d}{a}, 2^a)$.

For $a = 1$ we have $G \cong AGL(d, 2)$. Here $G$ is 3-transitive and the only $SQS(v)$ on which $G$ acts are the ones whose points and blocks are the points and planes of $AG(d, 2)$, $d \geq 3$, by Kantor [12]. Obviously, $G$ is also flag-transitive. As $a = d$ has already been done in case (1) we can assume that $a$ is a proper divisor of $d$. We prove that here no flag-transitive $SQS(v)$ exists.

Because of lemma 3 it is enough to show that $r$ is no divisor of $|G_0|$. Clearly,

$$|SL(\frac{d}{a}, 2^a)| = 2^{d(\frac{d}{a} - 1)}/2 \prod_{i=2}^{\frac{d}{a}} (2^{a} - 1),$$

and

$$|\Gamma L(\frac{d}{a}, 2^a) : SL(\frac{d}{a}, 2^a)| = |Aut(GF(2^a))||GF(2^a)^*| = a \cdot (2^a - 1).$$

Thus it is sufficient to show that $r$ does not divide $a \cdot (2^a - 1) \cdot |SL(\frac{d}{a}, 2^a)|$.

By Zsigmondy’s theorem (cf. [15] p. 283)

$$2^{d-1} - 1$$

has a 2-primitive prime divisor $\tilde{r} \neq 1$ with $\tilde{r} \not\parallel 2^{d-1} - 1$. Obviously, $\tilde{r} \neq 2$. Furthermore, $\tilde{r} \not\parallel 3a$ since $\tilde{r} \equiv 1 \pmod{(d - 1)}$ (cf. [24] Theorem 3.5]) and $d$ is properly divisible by $a$.

Therefore,

$$2^{d-1} - 1 \not\parallel 3a \cdot 2^{d(\frac{d}{a} - 1)/2} \prod_{i=1}^{\frac{d}{a} - 1} (2^{a} - 1)$$

and the claim follows.

Cases (3)-(4): These cases can be eliminated analogous case (2) using lemma 3 and Zsigmondy’s theorem. (For $|Out(G_0)|$ see e.g. [23] Table 5.1 A).

Case (5): $G_0 \cong A_6$ or $A_7$, $v = 2^4$.

If $G \cong A_6$ then lemma 3 implies that $G$ cannot act flag-transitively on any $SQS(v)$.

As $G \cong A_7$ is 3-transitive and the only $SQS(v)$ on which $G$ acts is the one whose points and blocks are the points and planes of $AG(4, 2)$ by Kantor [12], we have also flag-transitivity in this case.

Cases (6)-(8): These cases cannot occur since $v$ is no power of 2.
3.2. Semisimple case. The cases (3), (5), (8), (12) from the list where $G$ is of semisimple type can easily be ruled out as above by using Lemma 3. Obviously, the cases (4), (7), (10), (11), (13) cannot occur by Hanani’s theorem. Before we proceed we indicate Lemma 5. Let $V(d, q)$ be a vector space of dimension $d > 3$ over $GF(q)$ and $PG(d−1, q)$ the $(d−1)$-dimensional projective space. Assume $G$ containing $PSL(d, q)$ acts on $PG(d−1, q)$ and for all $g \in G$ with $|M^g \cap M| \geq 3$ we have $M^g = M$, where $M$ is an arbitrary set of points of $PG(d−1, q)$ of cardinality $k$ with $3 \leq k \leq |H|$, and $H$ a hyperplane of $PG(d−1, q)$. If $|M \cap H| \geq 3$, then $M \cap H = M$ holds.

**Proof.** For $k = 3$ the assertion is trivial. So assume $3 < k \leq |H| = \frac{q^d−1−1}{q−1}$. In $PG(d−1, q)$ Desargues’ theorem holds and the translations $T(H)$ form an abelian group which is sharply transitive on the points of $PG(d−1, q) \setminus H$ by Baer’s theorem. But on $H$ the group $T(H)$ acts trivially since the central collineations fix each point of $H$. Thus the claim holds if all elements of $M$ lie in $H$. Therefore, assume that there is an element of $M$ which is not in $H$. Then $M$ contains all points of $PG(d−1, q) \setminus H$ as $T(H)$ is transitive. Thus

$$|M| \geq \frac{q^d−1}{q−1} − \frac{q^d−1−1}{q−1} = \frac{q^d−q^d−1}{q−1} = q^d−1 > \frac{q^d−1−1}{q−1} = |H|.$$ 

But this contradicts the assumption $|M| \leq |H|$, and the claim follows. □

**Case (1):** $N = A_v$, $v \geq 5$. Here, $G$ is $3$-transitive and does not act on any non-trivial $3 − (v, k, 1)$ design by Kantor [12].

**Case (2):** $N = PSL(d, q)$, $d \geq 2$, $v = \frac{q^d−1}{q−1}$, where $(d, q) \neq (2, 2), (2, 3)$.

We distinguish two subcases:

(i) $N = PSL(2, q)$, $v = q + 1$.

Here $q \geq 5$ as $PSL(2, 4) \cong PSL(2, 5)$, and $Aut(N) = PGL(2, q)$. First suppose that $G$ is $3$-transitive. According to Kantor [12], we have then only the $SQS(3d+1)$ described in (2) of theorem [4] and $PSL(2, 3^+d) \leq G \leq PGL(2, 3^d)$. Obviously, also flag-transitivity holds. As $PGL(2, q)$ is a transitive extension of $AGL(1, q)$, it is easily seen that the derived design at any point of $GF(3^d) \cup \{\infty\}$ is isomorphic to the $2 − (3d, 3, 1)$ design consisting of the points and lines of $AG(d, 3)$.

Now assume that $G$ is $3$-homogeneous but not $3$-transitive. As here $PSL(2, q)$ is a transitive extension of $AG^2L(1, q)$ we deduce from [4] that the derived design is either the affine space $AG(d, 3)$ or the Netto triple system. Thus (2) with the part in brackets or (3) of theorem [4] holds with $PSL(2, 3^d) \leq G \leq PSL(2, 3^d)$ (where $P\Sigma L(2, p^d) := PSL(2, p^d) \gg 3 < \tau_\alpha >$ with $\tau_\alpha \in Sym(GF(p^d) \cup \{\infty\}) \cong S_3$ of order $d$ induced by the Frobenius automorphism $\alpha : GF(p^d) \rightarrow GF(p^d), x \mapsto x^p$.) Conversely, as $G$ is $3$-homogeneous it is also block-transitive. In both cases we have $PSL(2, q)_B \cong A_4$ for any $B \in B$ since $PSL(2, q)_B$ has order 12 by the orbit-stabilizer property and $PSL(2, q)_B \rightarrow Sym(B) \cong S_4$ is a faithful representation. Thus, in each case flag-transitivity holds.

Finally, suppose $G$ is not $3$-homogeneous. As $PGL(2, q)$ is $3$-homogeneous the $PGL(2, q)$-orbit on $3$-subsets therefore splits under $PSL(2, q)$ into two orbits of same length. Let $M$ be an arbitrary $3$-subset. Then $|PSL(2, q)_M| = |PGL(2, q)_M| = 6$ by the orbit-stabilizer property. Thus, as $PGL(2, q)$ is $3$-transitive
we have $PSL(2, q)_M \cong S_4$ for each orbit. If $PSL(2, q)$ acts block-transitively on any $SQS(v)$ then $PSL(2, q)_B \cong A_4$ again for any $B \in \mathcal{B}$. But, by the definition of $SQS(v)$ this would imply that $PSL(2, q)_\tilde{B}$, where $\tilde{B}$ denotes the block uniquely determined by $M$, contains $PSL(2, q)_M$, a contradiction. Thus $PSL(2, q)$ does not act flag-transitively on any $SQS(v)$. We show now that $G$ cannot act flag-transitively on any $SQS(v)$. Without restriction choose $O_1$ to be the $PSL(2, q)$-orbit containing $\{0, 1, \infty\}$. Easy calculation shows that $PSL(2, q)_{O_1}$ acts flag-transitively on $PGL(2, q)$ and $PGL(2, q)$ is $3$-transitive. Therefore, we only have to consider $PSL(2, q) \leq G \leq PSL(2, q)$. Dedekind’s law yields $G = PSL(2, q) \triangleright (G \cap <\tau_\alpha>)$ and $G_{(B)} = PSL(2, q)_{(B)} \triangleright G \cap <\tau_\alpha> = G \cap <\tau_\alpha> \cong C_m$, the cyclic group of order $m \mid d$, for any $B \in \mathcal{B}$ since every non-identity element of $PSL(2, q)$ fixes at most two points. Assume $G$ acts block-transitively on any $SQS(v)$. Then we can choose $B \in \mathcal{B}$ such that $B$ contains $\{0, 1, \infty\}$. Since $G_{(B)}$ is the kernel of the representation $G_B \rightarrow Sym(B) \cong S_4$ and $PSL(2, q)_B \cong A_4$ we have therefore again by Dedekind’s law

$$G_B = PSL(2, q)_B \times (G \cap <\tau_\alpha>) \cong A_4 \times C_m.$$ 

However, as $PSL(2, q)_{\{0, 1, \infty\}} \cong S_3$ we get analogously

$$G_{\{0, 1, \infty\}} = PSL(2, q)_{\{0, 1, \infty\}} \times (G \cap <\tau_\alpha>) \cong S_3 \times C_m,$$

which leads again to a contradiction by the definition of $SQS(v)$.

(ii) $N = PSL(d, q), d \geq 3, v = \frac{q^d - 1}{q - 1}$.

Here $Aut(N) = PTL(d, q) \triangleright <\iota>$, where $\iota$ denotes a graph automorphism. We show that $G$ does not act on any $SQS(v)$. For $d = 3$ this is obvious since $v = q^3 + q + 1$ is always odd, a contradiction to Hanani’s theorem.

Consider $d > 3$ and let $H$ be a hyperplane of the projective space $PG(d - 1, q)$. Assume that the claim does not hold. Then there is a counterexample with $d$ minimal. Without restriction we can choose three arbitrary points $\alpha, \beta, \gamma$ from $H$. As for $d > 3$

$$|H| = \frac{q^d - 1}{q - 1} > 4$$

holds, the block uniquely determined by $\alpha, \beta, \gamma$ is contained in $H$ by lemma \[3\]. Thus $H$ induces a $SQS(\frac{q^d - 1}{q - 1})$ on which $G$ containing $PSL(d - 1, q)$ operates. By induction, we get the minimal counterexample for $d = 3$. So $G$ containing $PSL(3, q)$ acts on a $SQS(\frac{q^3 - 1}{q - 1})$. But, as above $\frac{q^3 - 1}{q - 1} = q^2 + q + 1$ is always odd yielding the desired contradiction.

Case (6): $N = Sp(2d, 2), d \geq 3, v = 2^{2d - 1} + 2^{d - 1}$.

Here $N = G$ since $|Out(N)| = 1$ (cf. [12] Table 5.1 A). We show that $G$ contains elements which fix exactly 3 points and hence cannot act on any $SQS(v)$ by definition.

Let $X^+$ respectively $X^-$ denote the set of points on which $G$ operates with $|X^+| = 2^{2d - 1} + 2^{d - 1}$ resp. $|X^-| = 2^{2d - 1} - 2^{d - 1}$, and define

$$m_p(G) := \min\{|\text{supp}(g)| : 1 \neq g \in G, g \text{ a } p\text{-element of } G\}$$

to be the minimal $p$-degree of a transitive permutation group $G$, $p$ a prime divisor of $|G|$ (cf. [10]).
First, suppose \(d\) is even. By Zsigmondy’s theorem
\[
2^{d-1} - 1
\]
has a 2-primitive prime divisor \(p\) with \(p \perp 2^{d-1} - 1\). Moreover, \(p\) divides \(|G|\) since \(|G| = 2^d \prod_{i=1}^{d} (2^{2i} - 1)\) (see e.g. Table 2.1 C]). Therefore, according to [10, Theorem 3.7] we get in \(X^+\)
\[
m_p(G) = 2^{2d-2(d-1)-1} (2^{2d-1} - 1) + 2^{d-(d-1)-1} (2^{d-1} - 1) = |X^+| - 3.
\]
Thus, there exists \(g \in G\) of prime order \(p\) that fixes 3 points in \(X^+\).

For \(d \neq 4\) Zsigmondy’s theorem yields the existence of a 2-primitive prime divisor \(p\) with \(p \perp 2^{2(d-1)} - 1\) and as \(p\) divides \(|G|\) we have in \(X^-\) again by [10, Theorem 3.7]
\[
m_p(G) = 2^{2d-2(d-1)-1} (2^{2d-1} - 1) - 2^{d-(d-1)-1} (2^{d-1} + 1) = |X^-| - 3.
\]
When \(d = 4\) then [3, p.123] yields \(|\text{fix}(g)| = 3\) in \(X^-\) for \(g \in 3D\), where \(3D\) denotes a conjugacy class in [3].

Now, suppose \(d\) is odd. Again by Zsigmondy’s theorem and [10, Theorem 3.7] there exists a 2-primitive prime divisor \(p\) with \(p \perp 2^{2(d-1)} - 1\), and \(m_p(G) = |X^-| - 3\) in \(X^-\).

If \(d \neq 7\) Zsigmondy’s theorem yields the existence of a 2-primitive prime divisor \(p\) with \(p \perp 2^{d-1} - 1\). Choose \(A = (\begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix}) \in S \in Syl_p(Sp(d-1, 2))\) and define
\[
h := (\begin{pmatrix} A_0 & A_1 & 0 \\ A_2 & A_3 & 1 \\ 0 & 1 \end{pmatrix})
\]

The proof of [10, Theorem 3.7] yields \(|\text{fix}(h)| = 3\) in \(X^+\) and \(|\text{fix}(h)| = 1\) in \(X^-\).

For \(d = 7\) choose \(A := (\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix})\) and define \(k := \text{diag}(A, A, 1, 4^{-1}, 4^{-1}, 4^{-1}, 1)\).

Again, \(|\text{fix}(k)| = 3\) in \(X^+\) and \(|\text{fix}(k)| = 1\) in \(X^-\). Thus the assertion is proved.

Case (9): \(M_v, v = 11, 12, 22, 23, 24\).

Here, only \(v = 22\) is possible by Hanani’s theorem. But as \(M_{22}\) is 3-transitive, Kantor [12] shows that the only \(3 - (v, k, 1)\) design on which \(M_{22}\) resp. \(\text{Aut}(M_{22})\) acts is the \(3 - (22, 6, 1)\) design. Therefore, this case cannot occur finishing the proof of theorem [10].

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