MEAN DIMENSION AND AN EMBEDDING THEOREM FOR REAL FLOWS

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Abstract. We develop mean dimension theory for $\mathbb{R}$-flows. We obtain fundamental properties and examples and prove an embedding theorem: Any real flow $(X, \mathbb{R})$ of mean dimension strictly less than $r$ admits an extension $(Y, \mathbb{R})$ whose mean dimension is equal to that of $(X, \mathbb{R})$ and such that $(Y, \mathbb{R})$ can be embedded in the $\mathbb{R}$-shift on the compact function space $\{f \in C(\mathbb{R}, [-1, 1]) \mid \text{supp}(\hat{f}) \subset [-r, r]\}$, where $\hat{f}$ is the Fourier transform of $f$ considered as a tempered distribution. These canonical embedding spaces appeared previously as a tool in embedding results for $\mathbb{Z}$-actions.

1. Introduction

Mean dimension was introduced by Gromov [Gro99] in 1999, and was systematically studied by Lindenstrauss and Weiss [LW00] as an invariant of topological dynamical systems (t.d.s). In recent years it has extensively been investigated with relation to the so-called embedding problem, mainly for $\mathbb{Z}^k$-actions ($k \in \mathbb{N}$). For $\mathbb{Z}$-actions, the problem is which $\mathbb{Z}$-actions $(X, T)$ can be embedded in the shifts on the Hilbert cubes $(([0, 1]^N)\mathbb{Z}, \sigma)$, where $N$ is a natural number and the shift $\sigma$ acts on $([0, 1]^N)\mathbb{Z}$ by $\sigma((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}$ for $x_n \in [0, 1]^N$. Under the conditions that $X$ has finite Lebesgue covering dimension and the system $(X, T)$ is aperiodic, Jaworski [Jaw74] proved in 1974 that $(X, T)$ can be embedded in the shift on $[0, 1]^\mathbb{Z}$. Using Fourier and complex analysis, Gutman and Tsukamoto showed that if $(X, T)$ is minimal and has mean dimension strictly less than $N/2$ then it can be embedded in $(([0, 1]^N)\mathbb{Z}, \sigma)$ (see a more general result in [GQT19]). We note that the value $N/2$ is optimal since a minimal system of mean dimension $N/2$ which cannot be embedded in $(([0, 1]^N)\mathbb{Z}, \sigma)$ was constructed in [LT14, Theorem 1.3]. More references for the embedding problem are given in [Aus88, Kak68, Lin99, Gut11, GT14, GLT16, Gut16, Gut17, GQS18].

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In this paper, we develop the mean dimension theory for \( \mathbb{R} \)-actions and investigate the embedding problem in this context. Throughout this paper, by a flow we mean a pair \((X, \mathbb{R})\), where \( X \) is a compact metric space and \( \Gamma : \mathbb{R} \times X \to X, (r, x) \mapsto rx \) is a continuous map such that \( \Gamma(0, x) = x \) and \( \Gamma(r_1, \Gamma(r_2, x)) = \Gamma(r_1 + r_2, x) \) for all \( r_1, r_2 \in \mathbb{R} \) and \( x \in X \). Let \((X, \mathbb{R}) = (X, (\varphi_r)_{r \in \mathbb{R}}) \) and \((Y, \mathbb{R}) = (Y, (\phi_r)_{r \in \mathbb{R}})\) be flows. We say that \((Y, \mathbb{R})\) can be \textbf{embedded} in \((X, \mathbb{R})\) if there is an \( \mathbb{R} \)-equivariant homeomorphism of \( Y \) onto a subspace of \( X \); namely, there is a homeomorphism \( f : Y \to f(Y) \subset X \) such that \( f \circ \phi_r = \varphi_r \circ f \) for all \( r \in \mathbb{R} \).

This paper is organized as follows: In Section 2 we present basic notions and properties of mean dimension theory for flows. In Section 3 we construct minimal real flows with arbitrary mean dimension. In Section 4 we propose an embedding conjecture for flows and discuss its relation to the Lindenstrauss-Tsukamoto embedding conjecture for \( \mathbb{Z} \)-systems. In Section 5 we state the main embedding theorem and prove it using a key proposition. In Section 6 we prove the key proposition.

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## 2. Mean dimension for real flows

We first introduce the definition of mean dimension for \( \mathbb{R} \)-actions. Let \((X, d)\) be a compact metric space. Let \( \epsilon > 0 \) and \( Y \) a topological space. A continuous map \( f : X \to Y \) is called a \( (d, \epsilon) \)-\textbf{embedding} if for any \( x_1, x_2 \in X \) with \( f(x_1) = f(x_2) \) we have \( d(x_1, x_2) < \epsilon \). Define

\[
\text{Widim}_\epsilon(X, d) = \min_{K \in \mathcal{K}} \dim(K),
\]

where \( \dim(K) \) is the Lebesgue covering dimension of the space \( K \) and \( \mathcal{K} \) denotes the collection of compact metrizable spaces \( K \) satisfying that there is a \( (d, \epsilon) \)-embedding \( f : X \to K \). Note that \( \mathcal{K} \) is always nonempty since we can take \( K = X \) which is a compact metric space and \( f = id \) which is the identity map from \( X \) to itself.
Let \((X, \mathbb{R})\) be a flow. For \(x, y \in X\) and a subset \(A\) of \(\mathbb{R}\) let
\[
d_A(x, y) = \sup_{r \in A} d(rx, ry).
\]
For \(R > 0\) denote by \(d_R\) the metric \(d_{[0,R]}\) on \(X\). Clearly, the metric \(d_R\) is compatible with the topology on \(X\).

**Proposition 2.1.** For any \(\epsilon > 0\), we have

1. \(\text{Widim}_\epsilon(X, d) \leq \dim(X)\);
2. if \(0 < \epsilon_1 < \epsilon_2\) then \(\text{Widim}_{\epsilon_1}(X, d) \geq \text{Widim}_{\epsilon_2}(X, d)\);
3. if \(0 \leq R_1 < R_2\) then \(\text{Widim}_\epsilon(X, d_{R_1}) \leq \text{Widim}_\epsilon(X, d_{R_2})\);
4. \(\text{Widim}_\epsilon(X, d_{[r_1,r_2]}) = \text{Widim}_\epsilon(X, d_{[r_0+r_1, r_0+r_2]})\) for any \(r_0, r_1, r_2 \in \mathbb{R}\);
5. \(\text{Widim}_\epsilon(X, d_{N+M}) \leq \text{Widim}_\epsilon(X, d_N) + \text{Widim}_\epsilon(X, d_M)\) for any \(N, M \geq 0\).

**Proof.** Since \((X, d)\) is a compact metric space that belongs to \(\mathcal{K}\), we have (1). Points (2) and (3) follow from the definition. Let \(\epsilon > 0\). If \(K\) is a compact metrizable space and \(f : X \to K\) is a continuous map such that for any \(x_1, x_2 \in X\) with \(f(x_1) = f(x_2)\) we have \(d_{[r_1,r_2]}(x_1, x_2) < \epsilon\), then \(f \circ r_0 : X \to K\) is a continuous map such that for any \(x_1, x_2 \in X\) with \(f \circ r_0(x_1) = f \circ r_0(x_2)\) we have \(d_{[r_0+r_1, r_0+r_2]}(r_0x_1, r_0x_2) < \epsilon\) which implies that \(d_{[r_0+r_1, r_0+r_2]}(x_1, x_2) < \epsilon\). This shows (4).

To see (5), let \(\epsilon > 0\), \(K\) (resp. \(L\)) be a compact metrizable space and \(f : X \to K\) (resp. \(g : X \to L\)) be a continuous map such that for any \(x_1, x_2 \in X\) with \(f(x_1) = f(x_2)\) (resp. \(g(x_1) = g(x_2)\)) we have \(d_N(x_1, x_2) < \epsilon\) (resp. \(d_M(x_1, x_2) < \epsilon\)). Define \(F : X \to K \times L\) by \(F(x) = (f(x), g(Nx))\) for every \(x \in X\). Clearly, \(K \times L\) is a compact metrizable space and the map \(F\) is continuous. For \(x, y \in X\), if \(F(x) = F(y)\) then \(f(x) = f(y)\) and \(g(Nx) = g(Ny)\), thus we have \(d_N(x, y) < \epsilon\) and \(d_M(Nx, Ny) < \epsilon\), and hence \(d_{N+M}(x, y) < \epsilon\). It follows that \(\text{Widim}_\epsilon(X, d_{N+M}) \leq \dim(K \times L) \leq \dim(K) + \dim(L)\). Thus, \(\text{Widim}_\epsilon(X, d_{N+M}) \leq \text{Widim}_\epsilon(X, d_N) + \text{Widim}_\epsilon(X, d_M)\). \(\square\)

We define the **mean dimension** of a flow \((X, \mathbb{R})\) by:
\[
\text{mdim}(X, \mathbb{R}) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{\text{Widim}_\epsilon(X, d_N)}{N}.
\]

The limit exists by the Ornstein-Weiss lemma \cite{LW00} Theorem 6.1 as subadditivity holds.

Next we recall the definition of mean dimension for \(\mathbb{Z}\)-actions in \cite{LW00} Definition 2.6]. Let \((X, T)\) be a \(\mathbb{Z}\)-action. For \(x, y \in X\) and \(N \in \mathbb{N}\), denote
\[
d_N^T(x, y) = \max_{n \in \mathbb{Z} \cap [0, N-1]} d(T^n(x), T^n(y)).
\]
Define the mean dimension of \((X, T)\) by:

\[
\text{mdim}(X, Z) = \text{mdim}(X, T) = \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{\text{Widim}_\epsilon(X, d^Z_N)}{N}.
\]

**Proposition 2.2.** Let \((X, \mathbb{R})\) be a flow. If \(X\) is finite dimensional then \(\text{mdim}(X, \mathbb{R}) = 0\).

**Proof.** We have \(\text{Widim}_\epsilon(X, d^Z_N) \leq \dim(X) < +\infty\). The result follows. \(\square\)

Although the definition of mean dimension for \(\mathbb{R}\)-actions depends on the metric \(d\), the next proposition shows that the mean dimension of a flow has the same value for all metrics compatible with the topology. Therefore mean dimension is an invariant of \(\mathbb{R}\)-actions.

**Proposition 2.3.** Let \((X, \mathbb{R})\) be a flow. Suppose that \(d\) and \(d'\) are compatible metrics on \(X\). Then \(\text{mdim}(X, \mathbb{R}; d) = \text{mdim}(X, \mathbb{R}; d')\).

**Proof.** Since \(d\) and \(d'\) are equivalent, the identity map \(id : (X, d') \to (X, d)\) is uniformly continuous. Thus, for every \(\epsilon > 0\) there is \(\delta > 0\) with \(\delta < \epsilon\) such that for any \(x, y \in X\) with \(d'(x, y) < \delta\) we have \(d(x, y) < \epsilon\) which implies that \(\text{Widim}_\epsilon(X, d^Z_N) \leq \text{Widim}_{\delta}(X, d'_N)\) for every \(N \in \mathbb{N}\). Noting that \(\epsilon \to 0\) yields \(\delta \to 0\) we obtain that \(\text{mdim}(X, \mathbb{R}; d) \leq \text{mdim}(X, \mathbb{R}; d')\). In the same way we also obtain \(\text{mdim}(X, \mathbb{R}; d') \leq \text{mdim}(X, \mathbb{R}; d)\). \(\square\)

**Proposition 2.4** ([LW00, Def. 2.6]). Let \((X, \mathbb{Z})\) be a t.d.s. If \(d\) and \(d'\) are compatible metrics on \(X\) then we have \(\text{mdim}(X, \mathbb{Z}; d) = \text{mdim}(X, \mathbb{Z}; d')\).

Note that a flow \((X, (\varphi_r)_{r \in \mathbb{R}})\) naturally induces a “sub-\(\mathbb{Z}\)-action” \((X, \varphi_1)\).

**Proposition 2.5.** Let \((X, (\varphi_r)_{r \in \mathbb{R}})\) be a flow. Then \(\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}(X, \varphi_1)\).

**Proof.** Recall that for any compatible metric \(D\) on \(X\) and \(R > 0\), we denote \(D_R = D_{[0, R]}\). For a flow \((X, d; \mathbb{R})\) and \(N \in \mathbb{N}\), we have

\[
(d^Z_1)_N = (d^Z_N)_1 = d_N.
\]

Thus,

\[
\text{mdim}(X, \mathbb{R}; d) = \text{mdim}(X, \mathbb{Z}; d_1).
\]

Since \(d_1\) and \(d\) are compatible metrics on \(X\), by Proposition 2.4 we have

\[
\text{mdim}(X, \mathbb{Z}; d_1) = \text{mdim}(X, \mathbb{Z}; d).
\]

Combining the two equalities we have as desired

\[
\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}(X, \varphi_1).
\]

\(\square\)
Thus if the space is not metrizable then we may take $\text{mdim}(X, \varphi_1)$ as the definition of mean dimension.

**Proposition 2.6.** Let $(X, (\varphi_r)_{r \in \mathbb{R}})$ be a flow. If the topological entropy of $(X, (\varphi_r)_{r \in \mathbb{R}})$ is finite then the mean dimension of $(X, (\varphi_r)_{r \in \mathbb{R}})$ is zero.

**Proof.** By [HK03, Proposition 8.3.6] we have $h_{\text{top}}(X, \varphi_1) = h_{\text{top}}(X, (\varphi_r)_{r \in \mathbb{R}})$ which is finite. By [LW00, Theorem 4.2] we have $\text{mdim}(X, \varphi_1) = 0$. By Proposition 2.5, $\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = 0$. □

The following proposition directly follows from the definition.

**Proposition 2.7.** For any flow $(X, (\varphi_r)_{r \in \mathbb{R}})$ and $c \in \mathbb{R}$,

$$\text{mdim}(X, (\varphi_{cr})_{r \in \mathbb{R}}) = |c| \cdot \text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}).$$

3. **Construction of minimal real flows with arbitrary mean dimension**

By definition $\text{mdim}(X, \mathbb{R})$ belongs to $[0, +\infty]$. In this section we will show that for every $r \in [0, +\infty]$, there is a minimal flow $(X, \mathbb{R})$ with $\text{mdim}(X, \mathbb{R}) = r$.

Recall that there are natural constructions for passing from a $\mathbb{Z}$-action to a flow, and vice versa [BS02, Section 1.11]. Let $(X, T)$ be a $\mathbb{Z}$-action and $f : X \to (0, \infty)$ be a continuous function (in particular bounded away from 0). Consider the quotient space (equipped with the quotient topology)

$$S_f X = \{(x, t) \in X \times \mathbb{R}^+ : 0 \leq t \leq f(x)\} / \sim,$$

where $\sim$ is the equivalence relation $(x, f(x)) \sim (Tx, 0)$. The suspension over $(X, T)$ generated by the roof function $f$ is the flow $(S_f X, (\psi_t)_{t \in \mathbb{R}})$ given by

$$\psi_t(x, s) = (T^n x, s')$$

for $t \in \mathbb{R}$ and $(x, s) \in S_f X$, where $n$ and $s'$ satisfy

$$\sum_{i=0}^{n-1} f(T^i x) + s' = t + s, \ 0 \leq s' \leq f(T^n x).$$

In other words, flow along $\{x\} \times \mathbb{R}^+$ to $(x, f(x))$ then continue from $(Tx, 0)$ (which is the same as $(x, f(x))$) along $\{Tx\} \times \mathbb{R}^+$ and so on. When $f \equiv 1$, then $S_f X$ is called the mapping torus over $X$.

Let $d$ be a compatible metric on $X$. Bowen and Walters introduced a compatible metric $\tilde{d}$ on $S_f X$ [BW72, Section 4] known today as the **Bowen-Walters metric**. Let us recall the construction. First assume $f \equiv 1$. We

1Note that in [BW72] it is assumed that $\text{diam}(X) < 1$ but this is unnecessary.
will introduce $\tilde{d}_{S_1}$ on the space $S_1X$. First, for $x, y \in X$ and $0 \leq t \leq 1$
define the length of the horizontal segment $((x, t), (y, t))$ by:

$$d_h((x, t), (y, t)) = (1 - t)d(x, y) + td(Tx, Ty).$$

Clearly, we have $d_h((x, 0), (y, 0)) = d(x, y)$ and $d_h((x, 1), (y, 1)) = d(Tx, Ty)$. Secondly, for $(x, t), (y, s) \in S_1X$ which are on the same orbit define the
length of the vertical segment $((x, t), (y, s))$ by:

$$d_v((x, t), (y, s)) = \inf \{|r| : \psi_r(x, t) = (y, s)|. $$

Finally, for any $(x, t), (y, s) \in S_1X$ define the distance $\tilde{d}_{S_1X}((x, t), (y, s))$ to be the infimum of the lengths of paths between $(x, t)$ and $(y, s)$ consisting of a finite number of horizontal and vertical segments. Bowen and Walters showed this construction gives rise to a compatible metric on $S_1X$. Now assume a continuous function $f : X \to (0, \infty)$ is given. There is a natural homeomorphism $i_f : S_1X \to S_fX$ given by $(x, t) \mapsto (x, tf(x))$. Define $\tilde{d}_{S_fX} = (i_f)_*(\tilde{d}_{S_1X})$.

Recall from [LW00, Definition 4.1] that for a $\mathbb{Z}$-action $(X, T)$, the **metric mean dimension** $\text{mdim}_M(X, d)$ of $X$ with respect to a metric $d$ compatible with the topology on $X$ is defined as follows. Let $\epsilon > 0$ and $n \in \mathbb{N}$. A subset $S$ of $X$ is called $(\epsilon, d, n)$-spanning if for every $x \in X$ there is $y \in S$ such that $d_n^\mathbb{Z}(x, y) \leq \epsilon$. Set

$$A(X, \epsilon, d, n) = \min\{\#S : S \subset X \text{ is } (\epsilon, d, n)\text{-spanning}\}$$

and define

$$\text{mdim}_M(X, T, d) = \liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \limsup_{n \to \infty} \frac{1}{n} \log A(X, \epsilon, d, n).$$

Similarly one may define metric mean dimension for flows but we will not pursue this direction.

**Theorem 3.1** (Lindenstrauss-Weiss [LW00, Theorem 4.2]). For any $\mathbb{Z}$-action $(X, T)$ and any metric $d$ compatible with the topology on $X$,

$$\text{mdim}(X, T) \leq \text{mdim}_M(X, T, d).$$

**Theorem 3.2** (Lindenstrauss [Lin99, Theorem 4.3]). If a $\mathbb{Z}$-action $(X, T)$ is an extension of an aperiodic minimal system then there is a compatible metric $d$ on $X$ such that

$$\text{mdim}(X, T) = \text{mdim}_M(X, T, d).$$

For related results we refer to [Gut17, Appendix A].
Proposition 3.3. Let \((Y, (\varphi_r)_{r \in \mathbb{R}})\) be the mapping torus over \((X, T)\) (the suspension generated by the roof function 1). Assume that there is a compatible metric \(d\) on \(X\) with \(\text{mdim}_M(X, T, d) = \text{mdim}(X, T)\). Then

\[
\text{mdim}(X, T) = \text{mdim}(Y, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}_M(Y, T, \bar{d}).
\]

Proof. By Proposition 2.5, we have \(\text{mdim}(Y, (\varphi_r)_{r \in \mathbb{R}}) = \text{mdim}(Y, \varphi_1)\). Since \((X, T)\) is a subsystem of \((Y, T) = (Y, \varphi_1)\), we have \(\text{mdim}(X, T) \leq \text{mdim}(Y, \varphi_1)\). Note that for every \(r \in [0, 1)\), \(\varphi_r(X)\) is a \(\varphi_1\)-invariant closed subset of \(Y\), and \((\varphi_r(X), \varphi_1)\) can be regarded as a copy of \((X, T)\). Let \(\epsilon > 0\) and \(n \in \mathbb{N}\). If \(d_{n+1}^s(x, y) \leq \frac{\epsilon}{2}\) and \(|t - t'| \leq \frac{\epsilon}{2}\) for \(0 \leq t, t' < 1\) then \(d_{n}^s((x, t), (y, t')) \leq \epsilon\). Thus it is easy to see \(A(Y, \epsilon, \bar{d}, n) \leq ([1/\epsilon] + 1) \cdot A(X, \epsilon/2, d, n + 1)\). In particular

\[
\limsup_{n \to \infty} \frac{1}{n} \log A(Y, \epsilon, \bar{d}, n) \leq \limsup_{n \to \infty} \frac{1}{n} \log A(X, \epsilon/2, d, n)
\]

and we obtain that \(\text{mdim}_M(Y, \bar{d}) \leq \text{mdim}_M(X, d)\). By Theorem 3.1, we know that \(\text{mdim}(Y, \varphi_1) \leq \text{mdim}_M(Y, \bar{d})\). Summarizing, we have

\[
\text{mdim}(X, T) \leq \text{mdim}(Y, \varphi_1) \leq \text{mdim}_M(Y, \varphi_1, \bar{d})
\]

\[
\leq \text{mdim}_M(X, T, d) = \text{mdim}(X, T).
\]

This ends the proof. \(\blacksquare\)

We note that for general roof functions Proposition 3.3 does not hold. Indeed Masaki Tsukamoto has informed us that he has constructed an example of a minimal topological dynamical system \((X, T)\) with compatible metric \(d\) and \(f \neq 1 : X \to (0, \infty)\) such that \(\text{mdim}(X, T) = \text{mdim}_M(X, d) = 0\) but \(\text{mdim}_M(S_f X, \varphi_1, \bar{d}) > 0\) ([Tsu]).

Problem 3.4. Is Proposition 3.3 always true without assuming that there is a compatible metric \(d\) on \(X\) with \(\text{mdim}_M(X, d) = \text{mdim}(X, T)\)?

Problem 3.5. Is it possible to find a topological dynamical system \((X, T)\) with compatible metric \(d\) and \(f : X \to (0, \infty)\) such that \(\text{mdim}(X, T) = 0\) and \(\text{mdim}(S_f X, (\varphi_r)_{r \in \mathbb{R}}) \neq 0\).  

In Proposition 3.3, if \((X, T)\) is minimal then \((Y, (\varphi_r)_{r \in \mathbb{R}})\) is minimal. In particular, by Theorem 3.2 we have the following:

Proposition 3.6. Suppose that \((X, T)\) is minimal and \((Y, \mathbb{R})\) is the mapping torus over \((X, T)\) (the suspension generated by the roof function 1). Then \((Y, \mathbb{R})\) is also minimal and \(\text{mdim}(X, T) = \text{mdim}(Y, \mathbb{R})\).

Proposition 3.7. For every \(c \in [0, +\infty]\) there is a minimal flow \((X, (\varphi_r)_{r \in \mathbb{R}})\) such that \(\text{mdim}(X, (\varphi_r)_{r \in \mathbb{R}}) = c\).
Proof. By the $\mathbb{Z}$-version result due to Lindenstrauss and Weiss [LW00, Proposition 3.5] there is a minimal $\mathbb{Z}$-action $(Y, \mathbb{Z})$ such that $\text{mdim}(Y, \mathbb{Z}) = c$. By Proposition 3.6 we obtain a minimal flow $(X, \mathbb{R})$ with $\text{mdim}(X, \mathbb{R}) = c$. □

4. AN EMBEDDING CONJECTURE

We now state the main embedding theorem of this paper. We recall some necessary notions and results in Fourier analysis. A $C^\infty$ function $f : \mathbb{R} \to \mathbb{C}$, is said to be rapidly decreasing if there are constants $M_{n,m} > 0$ such that $|f^{(m)}(x)| < M_{n,m}|x|^{-n}$ as $x \to \infty$, for all $n, m \in \mathbb{N}$. The space of such function is called the Schwartz space and is denoted by $\mathcal{S}$. For $f \in \mathcal{S}$ the definitions of the Fourier transform and its inverse are given by:

$$
\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} e^{-2\pi it\xi} f(t) dt, \quad \mathcal{F}^{-1}(f)(\xi) = \int_{-\infty}^{\infty} e^{2\pi it\xi} f(t) dt.
$$

One has $\mathcal{F}(\mathcal{S}) = \mathcal{S}$, $\mathcal{F}^{-1}(\mathcal{S}) = \mathcal{S}$ and for all $f \in \mathcal{S}$, $\mathcal{F}(\mathcal{F}(f)) = \mathcal{F}(\mathcal{F}^{-1}(f)) = f$.

The operators $\mathcal{F}$ and $\mathcal{F}^{-1}$ can be extended to tempered distributions in a standard way (for details see [Sch66, Chapter 7] and [Str03, Chapters 3 & 4]). The tempered distributions include in particular bounded continuous functions.

Let $a < b$ be real numbers. We define $V[a, b]$ as the space of bounded continuous functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $\text{supp} \mathcal{F}(f) \subset [a, b]$. We denote $B_1(V[a, b]) = \{ f \in V[a, b] : \|f\|_\infty \leq 1 \}$ and $B_1(V^\mathbb{R}[-a, a]) = \{ f \in B_1(V[-a, a]) : f(\mathbb{R}) \subset \mathbb{R} \}$. One may show that $B_1(V[a, b])$ is a compact metric space with respect to the distance:

$$
d(f_1, f_2) = \sum_{n=1}^{\infty} \frac{\|f_1 - f_2\|_{L^\infty([-n,n])}}{2^n}.
$$

This metric coincides with the standard topology of tempered distributions (for details see [Sch66, Chapter 7, Section 4]). Let $\mathbb{R} = (\tau_r)_{r \in \mathbb{R}}$ act on $B_1(V[a, b])$ by the shift: for every $r \in \mathbb{R}$ and $f \in B_1(V[a, b])$, $(\tau_r f)(t) = f(t + r)$ for all $t \in \mathbb{R}$. Thus we obtain a flow $(B_1(V[a, b]), \mathbb{R})$.

In [LT14, Conjecture 1.2], Lindenstrauss and Tsukamoto posed the following conjecture:

**Conjecture 4.1.** Let $(X, T)$ be a $\mathbb{Z}$ dynamical system and $D$ an integer. For $r \in \mathbb{N}$, define $P_r(X, T) = \{ x \in X : rx = x \}$. Suppose that for every $r \in \mathbb{N}$ it holds that $\dim P_r(X, T) < \frac{rD}{2}$ and $\text{mdim}(X, T) < \frac{D}{2}$. Then $(X, T)$ can be embedded in the system $(([0, 1]^D)^\mathbb{Z}, \sigma)$. 

By [LW00] Proposition 3.3, \( \text{mdim}(([0,1]^D, \sigma)) = D \). It is not hard to see that for \( r \in \mathbb{N} \),
\[
\dim P_r(([0,1]^D, \sigma)) = rD.
\]
Thus the above conjecture may be rephrased as if
\[
\dim P_r(X, T) < \frac{\dim P_r(([0,1]^D, \sigma))}{2}
\]
for all \( r \in \mathbb{N} \) and
\[
\text{mdim}(X, T) < \frac{\text{mdim}(([0,1]^D, \sigma))}{2}
\]
then \((X, T) \hookrightarrow ([0,1]^D, \sigma)\). We expect that a similar phenomenon holds for flows where the role of \(([0,1]^D, \sigma)\) is played by \((B_1(V^R[-a, a]), \mathbb{R})\). By [GQT19], Footnote 4, \( \text{mdim}(B_1(V^R[-a, a]), \mathbb{R}) = 2a \). For \( r \in \mathbb{R}_{>0} \) denote
\[
P_r(X, \mathbb{R}) = \{ x \in X : rx = x \}.
\]
We now calculate \( \dim P_r(B_1(V^R[-a, a]), \mathbb{R}) \).

**Proposition 4.2.** Let \( r > 0 \) then \( \dim P_r(B_1(V^R[-a, a])) = 2\lfloor ar \rfloor + 1 \).

**Proof.** Let \( f \in B_1(V^R[-a, a]) \) with \( f(x) = f(x + r) \) for all \( x \in \mathbb{R} \). In particular we have a periodic \( f \in C^\infty(\mathbb{R}, \mathbb{R}) \), being a restriction of a holomorphic function, and hence the Fourier series representation of \( f \), \( f(x) = \sum_{k=-\infty}^{\infty} c_k e^{\frac{2\pi ikx}{r}} \), converges uniformly to \( f \) and \( c_{-k} = \overline{c_k} \) for all \( k \). Since \( \mathcal{F}(f) = c_0 \mathcal{F}(1) + \sum_{k=1}^{\infty} c_k \mathcal{F}(e^{\frac{2\pi ikx}{r}}) + \overline{c_k} \mathcal{F}(e^{-\frac{2\pi ikx}{r}}) \) is supported in \([-a, a]\), we have \( c_k = 0 \) for \( |k| > ar \). Let \( N = \lfloor ar \rfloor \). Choose \( x_0 < x_1 < x_2 < \cdots < x_N \) so that \( e^{\frac{2\pi ikx_i}{r}} \neq e^{\frac{2\pi ikx_j}{r}} \) for \( i \neq j \). The Vandermonde matrix formula indicates that \( \det \left(e^{\frac{2\pi ikx_j}{r}}\right)_{j=0}^{N} \neq 0 \). This implies that the functions \( e^{\frac{2\pi ikx}{r}} \), \( 0 \leq k \leq N \) are linearly independent. Thus, we conclude that \( \dim P_r(B_1(V^R[-a, a])) = 2\lfloor ar \rfloor + 1 \). \( \square \)

We now conjecture:

**Conjecture 4.3.** Let \((X, \mathbb{R})\) be a flow and \( a > 0 \) a real number. Suppose that \( \text{mdim}(X, \mathbb{R}) < a \) and for every \( r \in \mathbb{R} \), \( \dim P_r(X, \mathbb{R}) < \lfloor ar \rfloor + \frac{1}{2} \). Then \((X, \mathbb{R})\) can be embedded in the flow \((B_1(V^R[-a, a]), \mathbb{R})\).

**Problem 4.4.** Does Conjecture 4.3 imply Conjecture 4.1? Does Conjecture 4.1 imply Conjecture 4.3?

We give a very partial answer:

**Proposition 4.5.** Assume Conjecture 4.3 holds. Let \((X, T)\) be a t.d.s such that:

1. \( \exists D \in \mathbb{N} \), \( \text{mdim}(X, T) < \frac{D}{2} \),
ii. \( \exists b \in \mathbb{R}, \ b < \frac{D}{2} \) and \( \forall r > \frac{3}{D-2b}, \ \dim P_r(X, T) < br \),

iii. \( \forall r \leq \frac{1}{D-2b} \), \( P_r(X, T) = \emptyset \).

iv. \( \mdim(S_1X, \mathbb{R}) = \mdim(X, T) \)

Then \((X, T)\) can be embedded in the system \(((0, 1)^D, \mathbb{Z}, \sigma)\).

**Proof.** Note that the periodic orbits of the suspension \((S_1X, \mathbb{R})\) have positive integer lengthes and orbits of length \(r \in \mathbb{N}\) in \(S_1X\) corresponds to the \(r\)-periodic points of \((X, T)\) so that \(P_r(X, T) = \emptyset\) implies \(P_r(S_1X, \mathbb{R}) = \emptyset\) and \(P_r(X, T) \neq \emptyset\) implies:

\[ \dim P_r(S_1X, \mathbb{R}) = \dim P_r(X, T) + 1. \]

Consider the following sequence of embeddings:

\[ (X, T) \xrightarrow{(1)} (S_1X, \psi_1) \xrightarrow{(2)} (B_1(V^R[-c, c]), \sigma) \xrightarrow{(3)} (([-1, 1]^D, \mathbb{Z}, \sigma). \]

Embedding (1) is the trivial embedding from \((X, T)\) into \((S_1X, \psi_1)\) where \(\psi_1\) is the time-1 map. Embedding (3) is a consequence of \([GQT19, \text{Lemma 2.4}]\) as long as \(c < \frac{D}{2}\). We now justify Embedding (2). This \(\mathbb{Z}\)-embedding is induced from an \(\mathbb{R}\)-embedding \((S_1X, \mathbb{R}) \hookrightarrow (B_1(V^R[-c, c]), \mathbb{R})\) whose existence follows from Conjecture 4.3 which we assume to hold. We need to verify the conditions appearing in Conjecture 4.3. Let \(c\) be a real number such that \(\mdim(X, T) < c < \frac{D}{2}\). Thus \(\mdim(S_1X, \mathbb{R}) = \mdim(X, T) < c\).

Let \(r \) be an integer such that \(r > \frac{3}{D-2b}\), then \(\dim P_r(X, \mathbb{R}) < br + 1\), whereas \(\frac{1}{2} \dim P_r(B_1(V^R[-c, c]), \text{shift}) = |rc| + \frac{1}{2} = cr - t_r + \frac{1}{2}\), where \(0 \leq t_r < 1\).

Note \(cr - t_r + \frac{1}{2} \geq br + 1\) if \((c - b)r \geq \frac{3}{2} > t_r + \frac{1}{2}\), i.e if \(r \geq \frac{3}{2(c-b)}\). Thus it is enough to check it for the minimal integer \(r_0\) such that \(r_0 > \frac{3}{D-2b} = \frac{3}{2(c-b)}\).

We thus choose \(b < c < \frac{D}{2}\) such that \(r_0 \geq \frac{3}{2(c-2b)} > \frac{3}{2(c-2b)}\) and this ends the proof. \(\square\)

5. **An Embedding Theorem**

For every \(n \in \mathbb{N}\) denote by \(S_n\) the circle of circumference \(n!\) (identified with \([0, n!])\). Let \(\mathbb{R}\) act on \(\prod_{n \in \mathbb{N}} S_n\) as follows: \((x_i)_i \mapsto (x_i + r \mod i)_i\), \(r \in \mathbb{R}\). Define the **solenoid** \([\text{NS60, V.8.15}]\)

\[ S = \{ (x_n)_n \in \prod_{n \in \mathbb{N}} S_n : x_n = x_{n+1} \mod n! \}. \]

It is easy to see that \((S, \mathbb{R})\) is a (minimal) flow.

The following definitions are standard: A continuous surjective map \(\psi : (X, \mathbb{Z}) \to (Y, \mathbb{Z})\) is called an **extension** (of t.d.s) if for all \(n \in \mathbb{Z}\) and \(x \in X\) it holds \(\psi(n.x) = n.\psi(x)\). A continuous surjective map \(\psi : (X, \mathbb{R}) \to (Y, \mathbb{R})\) is called an **extension** (of flows) if for all \(r \in \mathbb{R}\) and \(x \in X\) it holds \(\psi(r.x) = r.\psi(x)\).
The following embedding result, which is the main result of this paper, provides a partial positive answer to Conjecture 4.3. This result may be understood as an analog for flows of [GT14, Corollary 1.8] which states that Conjecture 4.1 is true for any $\mathbb{Z}$-system which is an extension of an aperiodic subshift, i.e. an aperiodic subsystem of a symbolic shift $(\{1, 2, \ldots, l\}^\mathbb{Z}, \sigma)$ for some $l \in \mathbb{N}$.

**Theorem 5.1.** Let $a < b$ be two real numbers. If $(X, \mathbb{R})$ is an extension of $(S, \mathbb{R})$ and $\text{mdim}(X, \mathbb{R}) < b - a$, then $(X, \mathbb{R})$ can be embedded in $(B_1(V[a,b]), \mathbb{R})$.

**Corollary 5.2.** Conjecture 4.3 holds for $(X, \mathbb{R})$ which is an extension of $(S, \mathbb{R})$.

**Proof.** Suppose $\text{mdim}(X, \mathbb{R}) < a$ for some $a > 0$. As $(X, \mathbb{R})$ is an extension of an aperiodic system, it is aperiodic and in particular for every $r \in \mathbb{R}$, $\dim P_r(X, \mathbb{R}) = 0$. We have to show that $(X, \mathbb{R})$ may be embedded in the flow $(B_1(V^{\mathbb{R}}[-a,a]), \mathbb{R})$. Indeed by Theorem 5.1 $(X, \mathbb{R})$ may be embedded in $(B_1(V[0,a]), \mathbb{R})$. It is now enough to notice that one has the following embedding:

$$B_1(V[0,a]) \to B_1(V^{\mathbb{R}}[-a,a]), \ \varphi \mapsto \frac{1}{2}(\varphi + \overline{\varphi}).$$

$\square$

Since for any flow $(X, \mathbb{R})$, the product flow $(X \times S, \mathbb{R} \times \mathbb{R})$ is an extension of the flow $(S, \mathbb{R})$, the following result is a direct corollary of Theorem 5.1.

**Theorem 5.3.** For every flow $(X, \mathbb{R})$ with $\text{mdim}(X, \mathbb{R}) < b - a$ (where $a < b$ are real numbers) there is an extension $(Y, \mathbb{R})$ with $\text{mdim}(Y, \mathbb{R}) = \text{mdim}(Y, \mathbb{R})$ that can be embedded in $(B_1(V[a,b]), \mathbb{R})$.

In our proof of Theorem 5.1, the key step is to embed $(X, \mathbb{R})$ in a product flow (Theorem 5.4):

**Theorem 5.4.** Suppose that $a < b$, $\text{mdim}(X, \mathbb{R}) < b - a$ and $\Phi : (X, \mathbb{R}) \to (S, \mathbb{R})$ is an extension. Then for a dense $G_\delta$ subset of $f \in C_{\mathbb{R}}(X, B_1(V[a,b]))$ the map

$$(f, \Phi) : X \to B_1(V[a,b]) \times S, \ x \mapsto (f(x), \Phi(x))$$

is an embedding.

**Remark 5.5.** It is possible to prove a similar theorem where $(S, \mathbb{R})$ is replaced by a solenoid defined by circles of circumference $r_n \to_{n \to \infty} \infty$ but we will not pursue this direction.

The proof is given in the next section. We start by an auxiliary result:
Proposition 5.6. There is an embedding of \((S, \mathbb{R})\) in \((B_1(\mathbb{V}[0, c]), \mathbb{R})\) for any \(c > 0\).

Proof. Define a continuous and \(\mathbb{R}\)-equivariant map
\[
\phi : (S, \mathbb{R}) \to (B_1(\mathbb{V}[0, c]), \mathbb{R})
\]
by:
\[
S \ni x = (x_n) \mapsto f_x(t) = \sum_{n \geq m(c)} \frac{1}{2^n} \cdot e^{2\pi i (t + x_n)/n!} \cdot e^{\frac{2\pi i}{m(c)} x_n} = \sum_{n \geq m(c)} \left( \frac{1}{2^n} \cdot e^{\frac{2\pi i}{m(c)} x_n} \right) \cdot e^{\frac{2\pi i}{m(c)} t}
\]
where \(m(c) \in \mathbb{N}\) it taken to be sufficiently large so that the (RHS) belongs to \(B_1(\mathbb{V}[0, c])\).

Assume \(f_x(t) = f_y(t)\) for some \(x = (x_n)_n, y = (y_n)_n \in S\). We claim \(x = y\). This implies that the map is an embedding. Indeed it is enough to show that for all \(n\), \(\frac{1}{2^n} \cdot e^{\frac{2\pi i}{m(c)} x_n} = \frac{1}{2^n} \cdot e^{\frac{2\pi i}{m(c)} y_n}\). This is a consequence of the following more general lemma:

Lemma 5.7. Let \(a_n\) be an absolutely summable series \((\sum |a_n| < \infty)\). Let \(\lambda_n\) be a pairwise distinct sequence of real numbers bounded in absolute value by \(M > 0\) \((|\lambda_n| \leq M)\). Then \(f(z) = \sum a_n e^{i \lambda_n z}, z \in \mathbb{C}\), defines an entire function such that \(f \equiv 0\) iff \(a_n = 0\) for all \(n\).

Proof. (Compare with the proof of [Man72, Theorem I.3.1]) We claim
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) e^{-i \lambda_m t} dt = a_m
\]
for all \(m\). Thus \(f \equiv 0\) implies \(a_m = 0\) for all \(m\). Indeed
\[
\frac{1}{T} \int_0^T f(t) e^{-i \lambda_m t} dt = \frac{1}{T} \int_0^T \sum_{n \neq m} a_n e^{i(\lambda_n - \lambda_m) t} dt + \frac{1}{T} \int_0^T a_n dt.
\]
For \(n \neq m\) as \(\lambda_n - \lambda_m \neq 0\), we have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{i(\lambda_n - \lambda_m) t} dt = 0.
\]
As absolute summability implies one may reorder the limiting operations one has
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \sum_{n \neq m} a_n e^{i(\lambda_n - \lambda_m) t} dt = \sum_{n \neq m} \lim_{T \to \infty} \frac{1}{T} \int_0^T a_n e^{i(\lambda_n - \lambda_m) t} dt = 0.
\]
This completes the proof.

Now we show Theorem 5.1 assuming Theorem 5.4.
Proof of Theorem 5.1 assuming Theorem 5.4. We take \( a < c_1 < c_2 < b \) with \( \text{mdim}(X, \mathbb{R}) < c_1 - a \). By Theorem 5.4, \((X, \mathbb{R})\) can be embedded in \((B_1(V[a, c_1]), S, \mathbb{R} \times \mathbb{R})\), which, by Proposition 5.6, can be embedded in \((B_1(V[a, c_2]), B_1(V[c_2, b]), \mathbb{R} \times \mathbb{R})\), and finally embedded in \((B_1(V[a, b]), \mathbb{R})\) by the following embedding:

\[
B_1(V[a, c_1]) \times B_1(V[c_2, b]) \to B_1(V[a, b]), \quad (\varphi_1, \varphi_2) \mapsto \frac{1}{2}(\varphi_1 + \varphi_2).
\]

This ends the proof. \(\square\)

6. Embedding in a product

Let \( C_{\mathbb{R}}(X, B_1(V[a, b])) \) be the space of \( \mathbb{R} \)-equivariant continuous maps \( f: X \to B_1(V[a, b]) \). This space is nonempty because it contains the constant 0. The metric on \( C_{\mathbb{R}}(X, B_1(V[a, b])) \) is chosen to be the uniform distance \( \sup_{x \in X} d(f(x), g(x)) \). This space is completely metrizable and hence is a Baire space (see [Mun00, Theorem 48.2]).

We denote by \( d \) the metric on \( X \). To prove Theorem 5.4, it suffices to show that the set

\[
\bigcap_{n=1}^{\infty} \{ f \in C_{\mathbb{R}}(X, B_1(V[a, b])) : (f, \Phi) \text{ is a } \frac{1}{n}\text{-embedding with respect to } d \}
\]

is a dense \( G_\delta \) subset of \( C_{\mathbb{R}}(X, B_1(V[a, b])) \). It is obviously a \( G_\delta \) subset of \( C_{\mathbb{R}}(X, B_1(V[a, b])) \). Therefore it remains to prove the following:

Proposition 6.1. For any \( \delta > 0 \) and \( f \in C_{\mathbb{R}}(X, B_1(V[a, b])) \), there is \( g \in C_{\mathbb{R}}(X, B_1(V[a, b])) \) such that:

1. for all \( x \in X \) and \( t \in \mathbb{R} \), \( |f(x)(t) - g(x)(t)| < \delta \);
2. \((g, \Phi): X \to B_1(V[a, b]) \times S \) is a \( \delta \)-embedding with respect to \( d \).

To show Proposition 6.1, we prove several auxiliary results. We start by quoting [GT14, Lemma 2.1]:

Lemma 6.2. Let \((X, d')\) be a compact metric space, and let \( F : X \to [-1, 1]^M \) be a continuous map. Suppose that positive numbers \( \delta' \) and \( \epsilon \) satisfy the following condition:

\[
d'(x, y) < \epsilon \implies ||F(x) - F(y)||_{\infty} < \delta',
\]

then if \( \text{Widim}(X, d') < M/2 \) then there is an \( \epsilon \)-embedding \( G : X \to [-1, 1]^M \) satisfying:

\[
\sup_{x \in X} ||F(x) - G(x)||_{\infty} < \delta'.
\]
We say that a holomorphic function $g$ in $S \subset \mathbb{C}$ is of **exponential type** if for all $z \in S$, $|g(z)| \leq Ce^{T|z|}$ for some $C, T > 0$. The following classical theorem is proven in [DM72, Section 3.1.7].

**Theorem 6.3** (Phragmén–Lindelöf principle). Let $g$ be a function of exponential type that is holomorphic in the sector $S = \{ z \in \mathbb{C} \mid \alpha < \arg z < \beta \}$ of angle $\beta - \alpha < \pi$, and continuous on its boundary. If $|g(z)| \leq 1$ for $z \in \partial S$ then $|g(z)| \leq 1$ for $z \in S$.

According to the classical Paley-Wiener theorem ([Rud87, Theorem 19.3]), if $f \in L^2(\mathbb{R})$ extends to an entire function $F$ such that there exist $A, C > 0$ such that for all $z = x + iy \in \mathbb{C}$, $|f(x+yi)| \leq Ce^{2\pi A|y|}$, then $F(f) \in L^2(\mathbb{R})$ is supported in $[-A,A]$. We will need a generalized version:

**Theorem 6.4.** Let $f \in L^\infty(\mathbb{R})$ be a function which extends to an entire function $F : \mathbb{C} \to \mathbb{C}$ ($F|_\mathbb{R} = f$) such that there exist $A, C > 0$ and $M \in \mathbb{N}$ such that for all $z = x + iy \in \mathbb{C}$

$$|F(z)| \leq C(1 + |z|)^M \cdot e^{2\pi A|y|}.$$  

Then $f \in V[-A,A]$.

**Proof.** See [Str03, Theorem 7.2.3].

Let $\rho > 0$ and $N \in \mathbb{N}$ so that $\rho N! \in \mathbb{N}$. Define:

$$L(\rho) = \left\{ \frac{k}{\rho} \right\}_{k \in \mathbb{Z}}, \quad L^*(\rho) = L(\rho) \setminus \{0\}.$$  

In the next lemma we write $x \lesssim y$ for two real numbers $x$ and $y$ if there exists a constant $C > 0$ which depends only on $\rho$ and $N$ such that $x \leq Cy$.

**Lemma 6.5.** Let

$$f(z) = \lim_{A \to \infty} \prod_{\lambda \in L(\rho), 0 < |\lambda| < A} \left( 1 - \frac{z}{\lambda} \right).$$  

Then $f$ defines a holomorphic function in $\mathbb{C}$ satisfying

$$f(0) = 1, \quad f(\lambda) = 0, \quad \forall \lambda \in L^*(\rho).$$  

Moreover, for all $z \in \mathbb{C}$ we have

$$|f(z)| \lesssim (1 + |z|)^{5\rho N!} \cdot e^{\pi \rho |y|},$$  

where $y$ is the imaginary part of $z$.

---

2While reading the proof in the reference one should note that in [Str03] the Fourier transform is defined as $\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} e^{it\xi} f(t)dt$. 

Proof. We first show the convergence of \( f(z) \). Notice
\[
f(z) = \lim_{A \to \infty} \prod_{\lambda \in L(\rho), 0 < \lambda < A} \left( 1 - \frac{z^2}{\lambda^2} \right)
\]
As \( \sum_{\lambda \in L(\rho), 0 < \lambda} \frac{1}{\lambda^2} \) converges, the limit above converges locally uniformly (see [Kno51, §29, Theorems 6 & 7]). Thus, \( f(z) \) is a holomorphic function which satisfies
\[
f(0) = 1, \quad f(\lambda) = 0, \quad \forall \lambda \in L^* (\rho).
\]
Next we shall estimate the growth of \( f \) on the real line. Suppose \( x > 0 \) and let \( k \) be the integer with \( kN! \leq x < (k + 1)N! \). We may assume \( k > 0 \), as the case \( k = 0 \) is easier and can be dealt with in a similar way. For \( n \in \mathbb{Z} \), set
\[
L_n = L(\rho) \cap [nN!, (n + 1)N!).
\]
For \( \lambda \in L_n \) with \( n \leq -2 \) or \( n \geq k + 1 \) we have
\[
|1 - x/\lambda| \leq 1 - x/(n + 1)N!
\]
and hence
\[
\prod_{\lambda \in L_n} \left| 1 - \frac{x}{\lambda} \right| \leq \left| 1 - \frac{x}{(n + 1)N!} \right|^{\rho N!}.
\]
For \( \lambda \in L_n \) with \( 1 \leq n < k \) we have
\[
|1 - x/\lambda| \leq x/(nN!) - 1
\]
and hence
\[
\prod_{\lambda \in L_n} \left| 1 - \frac{x}{\lambda} \right| \leq \left| 1 - \frac{x}{nN!} \right|^{\rho N!}.
\]
The factors for \( n = -1, 0, k \) need to be treated separately. Recall Euler's sine product formula ([Cia13]):
\[
\frac{\sin z}{z} = \lim_{A \to \infty} \prod_{0 < |n| < A} \left( 1 - \frac{z}{n\pi} \right)
\]
Using this it is easy to see that \( |f(x)| \) is bounded by
\[
\prod_{0 \neq \lambda \in L_{-1} \cup L_0 \cup L_k} \left| 1 - \frac{x}{\lambda} \right| \cdot \lim_{A \to \infty} \prod_{|n| < A, n \neq 0, k, k+1} \left| 1 - \frac{x}{nN!} \right|^{\rho N!}
\]
\[
= \prod_{0 \neq \lambda \in L_{-1} \cup L_0 \cup L_k} \left| 1 - \frac{x}{\lambda} \right| \cdot \frac{\sin \frac{\pi x}{N!} - \frac{x}{kN!} \left( 1 - \frac{x}{(k+1)N!} \right)}{\frac{\pi x}{N!} \left( 1 - \frac{x}{kN!} \right) \left( 1 - \frac{x}{(k+1)N!} \right)}^{\rho N!}.
\]
The first factor is easy to estimate:
\[
\prod_{0 \neq \lambda \in L_{-1} \cup L_0 \cup L_k} \left| 1 - \frac{x}{\lambda} \right| \lesssim (1 + x)^{3\rho N!}.
\]
Set \( t = x/N! \),

\[
\frac{\sin \frac{\pi x}{N!}}{\pi \frac{1 - x}{kN!} \left( 1 - \frac{x}{(k+1)N!} \right)} = \frac{k(k+1) \sin \pi t}{\pi t(k-t)(k+1-t)}.
\]

By the mean value theorem,

\[
\left| \frac{\sin \pi t}{t} \right| \leq \pi, \quad \left| \frac{\sin \pi t}{k-t} \right| \leq \pi, \quad \left| \frac{\sin \pi t}{k+1-t} \right| \leq \pi.
\]

Thus,

\[
\left| \frac{k(k+1) \sin \pi t}{\pi t(k-t)(k+1-t)} \right| \lesssim k(k+1) \lesssim (1+x)^2.
\]

Therefore

\[
|f(x)| \lesssim (1+x)^{5\rho N!}.
\]

The case \( x < 0 \) is similar so we get

\[
|f(x)| \lesssim (1+|x|)^{5\rho N!}.
\]

We now turn to estimating \(|f(yi)| \) for \( y \in \mathbb{R} \setminus \{0\} \). For \( r > 0 \) we set

\[
n(r) = \#(L^*(\rho) \cap (-r, r)).
\]

We have

\[
n(r) < 2\rho r.
\]

Note that for \( 0 < r \leq \frac{1}{\rho} \), one has \( n(r) = 0 \). Since

\[
|f(yi)|^2 = \prod_{\lambda \in L^*(\rho)} \left( 1 + y^2/\lambda^2 \right),
\]

As \( n(r) \) is monotonic increasing, we may use the Riemann-Stieltjes integral to write:

\[
\log |f(yi)| = \frac{1}{2} \sum_{\lambda \in L^*(\rho)} \log \left( 1 + \frac{y^2}{\lambda^2} \right) = \frac{1}{2} \int_{\frac{1}{\rho}}^{\infty} \log \left( 1 + \frac{y^2}{r^2} \right) dn(r).
\]

Using integration by parts for the Riemann-Stieltjes integral ([Gor94, Theorem 12.14]), we see that for all \( R \geq \frac{1}{\rho} \) it holds:

\[
\frac{1}{2} \int_{\frac{1}{\rho}}^{R} \log \left( 1 + \frac{y^2}{r^2} \right) dn(r) = \frac{1}{2} \cdot \left( \log \left( 1 + \frac{y^2}{\frac{R^2}{r^2}} \right) n(R) \bigg|_{\frac{1}{\rho}}^{R} - \int_{\frac{1}{\rho}}^{R} n(r) d \log \left( 1 + \frac{y^2}{r^2} \right) \right).
\]

Taking \( R \to \infty \), we conclude:

\[
\log |f(yi)| = y^2 \int_{\frac{1}{\rho}}^{\infty} \frac{n(r)}{r(r^2 + y^2)} dr
\]

Since \( n(r) \leq 2\rho r \), we deduce

\[
\log |f(yi)| \leq 2\rho y^2 \int_{\frac{1}{\rho}}^{\infty} \frac{dr}{r^2 + y^2}.
\]
It is a standard exercise to show:
\[
\int_0^\infty \frac{dr}{r^2 + y^2} = \frac{1}{|y|} \int_0^\infty \frac{dr}{1 + r^2} = \frac{\pi}{2|y|}.
\]
It follows that
\[
|f(yi)| \leq e^{\pi \rho|y|}.
\]
Finally we show that \(|f(z)|\) grows at most exponentially. Let \(z = x + yi\).
We may assume \(x, y > 0\), as all the other cases are similar. Let \(k\) be the integer with \(kN! \leq x < (k + 1)N!\). Set
\[
L' = L(\rho) \setminus (L_{k-1} \cup L_k \cup L_{k+1}).
\]
We estimate
\[
\prod_{0 \neq \lambda \in L_{k-1} \cup L_k \cup L_{k+1}} \left|1 - \frac{z}{\lambda}\right| \lesssim (1 + |z|)^{3\rho N!}.
\]
\[
\lim_{A \to \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left|1 - \frac{z}{\lambda}\right|^2 = \lim_{A \to \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left\{\left(1 - \frac{x}{\lambda}\right)^2 + \frac{y^2}{\lambda^2}\right\} = \left\{\lim_{A \to \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left(1 - \frac{x}{\lambda}\right)^2\right\} \cdot \prod_{0 \neq \lambda \in L'} \left\{1 + \frac{y^2}{(\lambda - x)^2}\right\}.
\]
As in the proof of \(|f(x)| \lesssim (1 + |x|)^{5\rho N!}\) we estimate
\[
\lim_{A \to \infty} \prod_{0 \neq \lambda \in L', |\lambda| < A} \left(1 - \frac{x}{\lambda}\right)^2 \lesssim (1 + x)^{12\rho N!}.
\]
As in \(|f(yi)| \leq e^{\pi \rho|y|}\),
\[
\prod_{0 \neq \lambda \in L'} \left\{1 + \frac{y^2}{(\lambda - x)^2}\right\} \leq e^{2\pi \rho|y|}.
\]
Thus, we deduce that \(|f(z)|\) grows at most exponentially.
We have thus shown that \(f(z)\) has exponential type and satisfies \(|f(x)| \lesssim (1 + |x|)^{5\rho N!}\) and \(|f(yi)| \leq e^{\pi \rho|y|}\). By the Phragmén–Lindelöf principle of Theorem 6.3 (e.g. in the first quadrant \(x, y \geq 0\)) applied to \((1 + z)^{-5\rho N!} e^{\pi \rho iz} f(z)\), the claim follows.

Next we construct an interpolation function based on [Beu89 pp. 351–365]:

**Proposition 6.6.** Let \(a < b\). Let \(\rho > 0\) with \(\rho \in \mathbb{Q}\) and \(\rho < b - a\). There exists \(\varphi \in V[a, b]\) rapidly decreasing so that \(\varphi(0) = 1\) and for all \(\lambda \in L^*(\rho)\), \(\varphi(\lambda) = 0\).
Proof. Fix \( \tau > 0 \) so that \( \rho + \tau < b - a \). Let \( \psi(\xi) \in S \) be a nonnegative smooth function in \( \mathbb{R} \) satisfying
\[
\text{supp}(\psi) \subset \left[ -\frac{\tau}{2}, \frac{\tau}{2} \right], \quad \int_{-\infty}^{\infty} \psi(\xi) d\xi = 1.
\]
Define the function \( h : \mathbb{C} \to \mathbb{C} \) by
\[
h(z) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \psi(\xi) e^{2\pi i z \xi} d\xi.
\]
It is easy to see that \( h \) is an entire function which satisfies:
\[
(6.2) \quad h|_{\mathbb{R}} = \mathcal{F}(\psi) \in S, \quad h(0) = 1, \quad |h(x + yi)| \leq e^{\pi|y|}, \quad \forall x, y \in \mathbb{R}.
\]
Let
\[
g(z) = \lim_{A \to \infty} \prod_{\lambda \in L(\rho), 0 < |\lambda| < A} \left( 1 - \frac{t}{\lambda} \right).
\]
By Lemma 6.5, \( g(z) \) is an entire function. Thus we may define the following entire functions:
\[
\tilde{\varphi}(z) = h(z)g(z), \quad \varphi(z) = e^{\pi iz(a+b)}\tilde{\varphi}(z).
\]
It is easy to see that \( \varphi(0) = 1 \) and for all \( \lambda \in L^*(\rho) \), \( \varphi(\lambda) = 0 \). By Lemma 6.5, \( g|_{\mathbb{R}} \) has polynomial growth. Therefore as \( \mathcal{F}(\psi) \) is rapidly decreasing, so are \( \varphi|_{\mathbb{R}} \) and \( \tilde{\varphi}|_{\mathbb{R}} \). By Lemma 6.5 and (6.2) (recall the convention \( z = x + iy \)):
\[
|\tilde{\varphi}(z)| \lesssim (1 + |z|)^{5pNt} \cdot e^{\pi(\rho+\tau)|y|}
\]
As in addition \( \tilde{\varphi}|_{\mathbb{R}} \) is bounded (as it is rapidly decreasing), it follows from Theorem 6.4 that \( \tilde{\varphi} \in V[\frac{b-a}{2}, \frac{b+a}{2}] \subset V[a-b, b-a] \). This immediately implies \( \varphi \in V[a,b] \) which finishes the proof. \( \square \)

Now we are ready to prove Proposition 6.1.

Proof of Proposition 6.1. We take \( \delta > 0 \) and \( f \in C_{\mathbb{R}}(X, B_1(V[a,b])) \). Without loss of generality, we assume that \( |f(x)(t)| \leq 1 - \delta \) for all \( x \in X \) and \( t \in \mathbb{R} \) (by replacing \( f \) with \( (1 - \delta)f \) if necessary). Fix \( \rho \in \mathbb{Q} \) with
\[
\text{mdim}(X, \mathbb{R}) < \rho < b - a.
\]
Let \( \varphi \) be the function constructed in Proposition 6.6. As \( \varphi \) is a rapidly decreasing function, we may find \( K > 0 \) such that:
\[
(6.3) \quad |\varphi(t)| \leq \frac{K}{1 + |t|^2}.
\]
Let \( \delta' > 0 \) be such that:
\[
(6.4) \quad \delta' \cdot \sum_{\lambda \in L(\rho)} \frac{K}{1 + |t - \lambda|^2} < \delta \text{ for all } t \in \mathbb{R}.
\]
Fix $\epsilon \in (0, \delta)$. Let $N \in \mathbb{N}$ be such that $\rho N! \in \mathbb{N}$, $\text{Widim}_e(X, d_{N!}) < \rho N!$, and such that

(6.5) $d_{N!}(x, y) < \epsilon$ implies $|f(x)(t) - f(y)(t)| < \frac{\delta'}{2}$ for all $t \in [0, N!]$.

Define:

$$F : X \to [0, 1]^{2\rho N!} = ([0, 1]^2)^{\rho N!}, \quad F(x) = (\text{Re}f(x)|_{L^{\rho, N}}, \text{Im}f(x)|_{L^{\rho, N}}).$$

$$F^C : X \to \mathbb{C}^{\rho N!}, \quad F^C(x) = f(x)|_{L^{\rho, N}}.$$ 

Let $M = 2\rho N!$, $d' = d_{N!}$. Equation (6.5) implies that Equation (6.1) holds, so Lemma 6.2 implies, there is an $(d_{N!}, \epsilon)$-embedding $G : X \to [-1, 1]^{2\rho N!}$ such that $\sup_{x \in X} ||F(x) - G(x)||_\infty < \frac{\delta'}{2}$. Similarly to $F^C(x)(k)$, we introduce the notation $G^C(x)(k), k = 0, \ldots, \rho N! - 1$ in the natural way. Notice it holds:

(6.6) $\sup_{x \in X} ||F^C(x) - G^C(x)||_\infty < \delta'.$

Take $x \in X$. Denote $\Phi(x) = (\Phi(x)_n)_{n \in \mathbb{N}}$, where $\Phi(x)_n \in S_{n!}$. For every $n \in \mathbb{Z}$ let

$$\Lambda(x, n) = nN! - \Phi(x)_n + L(\rho, N),$$

$$\Lambda(x) = \bigcup_{n \in \mathbb{N}} \Lambda(x, n) \subset \mathbb{R}.$$ 

![Figure 6.1. The set $\Lambda(x, n)$.](image)

Next we construct a perturbation $g$ of $f$:

$$g(x)(t) = f(x)(t) + h(x)(t),$$

where $h(x)(t)$ is defined by

$$\sum_{n \in \mathbb{Z}} \sum_{k=0}^{\rho N! - 1} \left( G^C(T^{nN!-\Phi(x)_N} x)(k) - F^C(T^{nN!-\Phi(x)_N} x)(k) \right) \varphi(t - \left( \frac{k}{\rho} + nN! - \Phi(x)_N \right)).$$

As $\varphi$ is rapidly decreasing the sum defining $g(x)$ for fixed $x$ converges in the compact open topology to a function in $V[a, b]$. Moreover the mapping $x \mapsto g(x)$ is continuous. In order to see that $g(x)$ is $\mathbb{R}$-equivariant, it suffices to deal with $h(x)$ (because $f$ is already $\mathbb{R}$-equivariant). To see that $h(x)$ is $\mathbb{R}$-equivariant we first note that for $0 \leq r < N! - \Phi(x)_N$ we
have $\Phi(T^r x)_N = \Phi(x)_N + r$ and hence from the definition of $h$ it follows that $h(T^r x)(t) = h(x)(t+r)$. Similarly, if $N! - \Phi(x)_N \leq r < N!$ then $\Phi(T^r x)_N = r - (N! - \Phi(x)_N)$ and hence $(T^{n+1})^{N! - \Phi(x)_N} T^r x)(k) = (T^{n})^{N! - \Phi(x)_N} - r T^r x)(k)$. Using such information in each summand in the sum over $k$’s appearing in the definition of $h(T^r x)(t)$, and then substituting $n+1$ by $n$ when summing over $n \in \mathbb{Z}$, we get as desired $h(T^r x)(t) = h(x)(t+r)$ for $r$’s in this range. If $r = sN!$ where $s \in \mathbb{Z}$ then $\Phi(T^r x)_N = r - (sN! - \Phi(x)_N)$ and hence $(T^{nN! - \Phi(T^r x)_N}) T^r x)(k) = (T^{n+sN! - \Phi(x)_N}) x)(k)$. Using this information in each summand in the sum over $k$’s appearing in the definition of $h(T^r x)(t)$, and substituting $n+s$ by $n$ when summing over $n \in \mathbb{Z}$, we obtain as desired $h(T^r x)(t) = h(x)(t+r)$ for $r$’s in this range. Finally if $r = sN! + r'$ where $s \in \mathbb{Z}$ and $0 < r' < N!$ we use the additivity properties of the terms involved in order to combine the two cases and get the desired result. Note that by Equations (6.3) and (6.4) for all $x \in X$ and $t \in \mathbb{R}$:

$$
\sum_{n \in \mathbb{Z}} \sum_{k=0}^{\rho N!-1} \varphi(t - \frac{k}{\rho} + nN! - \Phi(x)_N) < \frac{\delta}{\rho}.
$$

By Equation (6.6) for all $x \in X$, $k = 0, \ldots, \rho N! - 1$:

$$
|G^C(T^{nN!-\Phi(x)_N} x)(k) - F^C(T^{nN!-\Phi(x)_N} x)(k)| < \delta'.
$$

Combining the two last inequalities we have $|g(x)(t) - f(x)(t)| < \delta$ for all $x \in X$ and $t \in \mathbb{R}$. Since $|f(x)(t)| \leq 1 - \delta$, we have $g(x) \in B_1(V[a, b])$. Thus, $g \in C_\mathbb{B}(X, B_1(V[a, b]))$. It remains to check that the map

$$(g, \Phi) : X \to B_1(V[a, b]) \times S, \quad x \mapsto (g(x), \Phi(x))$$

is a $\delta$-embedding with respect to $d$. We take $x, x' \in X$ with $(g(x), \Phi(x)) = (g(x'), \Phi(x'))$. We calculate for $k = 0, \ldots, \rho N! - 1$:

$$
g(x)(-\Phi(x)_N + \frac{k}{\rho}) = f(x)(-\Phi(x)_N + \frac{k}{\rho}) + (G^C(T^{-\Phi(x)_N} x)(k) - F^C(T^{-\Phi(x)_N} x)(k)).
$$

As $F^C(T^{-\Phi(x)_N} x)(k) = f(T^{-\Phi(x)_N} x)(\frac{k}{\rho}) = f(x)(-\Phi(x)_N + \frac{k}{\rho})$, we conclude for $k = 0, \ldots, \rho N! - 1$ that $g(x)(-\Phi(x)_N + \frac{k}{\rho}) = G^C(T^{-\Phi(x)_N} x)(k)$.

Similarly $g(x')(\Phi(x')_N + \frac{k}{\rho}) = G^C(T^{-\Phi(x')_N} x')(k)$. Thus:

$$
g(x)(-\Phi(x)_N + \frac{k}{\rho}) = g(x')(\Phi(x')_N + \frac{k}{\rho}) = g(x')(\Phi(x')_N + \frac{k}{\rho})
$$

implies

$$
G^C(T^{-\Phi(x)_N} x)(k) = G^C(T^{-\Phi(x')_N} x')(k) = G^C(T^{-\Phi(x)_N} x')(k).
$$

Since $G^C : X \to [0, 1]^{\rho N!}$ is an $(d_{N!}, \epsilon)$-embedding, we have

$$
d_{N!}(T^{-\Phi(x)_N} x, T^{-\Phi(x)_N} x') < \epsilon < \delta
$$
which implies $d(x, x') < \epsilon < \delta$. This ends the proof. □

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