Wilson Loops for M2- and M5-brane spaces

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ABSTRACT: We calculate the quark and anti-quark interaction energy in different positions in spaces generated by $N$ coincident $M2$- and $M5$-branes. We use the Maldacena-Rey-Yee method for calculating this energy as a function of quark-antiquark separation. We obtain the solution for these problems as integrals of the metric elements. For limiting regimes we find simpler solutions for which some potentials exhibit a confinement behavior.
1 Introduction

The AdS/CFT correspondence is a duality between string/M theory in $AdS_{n+1} \times S^p$ and supersymmetric conformal $SU(N)$ Yang-Mills theories, for large $N$, in $n$-dimensional flat space. This correspondence was proposed by Maldacena [1] and soon after detailed by Gubser, Klebanov and Polyakov [2] and Witten [3]. For a review, see for instance [4, 5].

After the conjecture about the duality between M/string theory in AdS spaces and conformal gauge field theories, Maldacena [6] and independently Rey and Yee [7] (MRY) proposed a method to calculate expectation values of an operator similar to the Wilson loop for the large $N$ limit of field theories. The Wilson loop operator is $W(C) = \frac{1}{N} Tr Pe^{i \oint_C A}$, where $C$ denotes a closed loop in space-time and the trace is over the fundamental representation of the gauge field $A$. In the particular case of a rectangular loop (of sides $T$ and $L$), it is possible to calculate (in the limit $T \rightarrow \infty$) the expectation value for the Wilson loop: $< W(C) > = A(L)e^{-TE(L)}$, where $E(L)$ is the energy of the quark-antiquark pair. The MRY proposal claims that the expectation value of the Wilson loop correspond to the worldsheet area $S$ of a string whose boundary is the loop in question i.e. $< W(C) > \sim e^S$. 
In this way, Maldacena calculate the quark-antiquark potential for the case of $AdS_5 \times S^5$. The result is a nonconfining potential for the quark-antiquark interaction, which is consistent with the conformal symmetry of the dual super Yang-Mills theory. A discussion of the Wilson loops for strings on a certain class of curved 10 dimensional spacetimes were presented in [8], where a confinement criterion was obtained.

A finite temperature calculation of the Wilson loops following the MRY approach was given in [9, 10] by considering an AdS Schwarzschild background, where the temperature of the conformal dual theory is identified with the Hawking temperature of the black hole, as in the original Witten’s work [11]. This set up also behaves as a nonconfining potential for the quark-antiquark interaction.

An interesting modification of the zero temperature set up is to consider the string in a $D3$-brane background. This case was treated in [12], where different behaviors for the potentials were obtained, some of them confining and others not. This is understood based on the fact that the $D3$-brane does not have conformal symmetry, but in the near horizon approximation it reproduces asymptotically the $AdS$ space.

A simpler situation was considered calculating the Wilson loop for the string in some phenomenological holographic AdS/QCD models. In the case of the hard-wall model, a confining behavior was obtained [13, 14] with a match with the Cornell potential. A finite temperature version of this calculation was given in [15], where a second order phase transition takes place describing qualitatively a confinement/deconfinement phase transition. Actually, it was shown that a Hawking-Page phase transition [16] occurs for the hard- and soft-wall models at finite temperature [17–21] and in the case of the soft-wall model, a good estimate of the deconfinement temperature, compatible with QCD predictions, was found [18].

Here, in this work, we consider $M2$-branes and $M5$-branes supergravity solutions as the background spaces for classical string theory in order to investigate the behavior of the Wilson loops in this set up, following the MRY approach [6, 7]. The solution for the quark-antiquark potential in these spaces are expressed as integrals involving metric elements. We consider different limits for these potentials with highly curved or almost straight geodesics as well as far from or close to the branes approximations. In some limiting cases we obtain a confining behavior. This work can be understood as an extesion of [12] to the case of $M2$- and $M5$-brane backgrounds. A connection between M2-branes and Wilson loops have also appeared recently in [22] with a different perspective.

2 General set up and quark-antiquark potential

The space-time metric for M-branes supergravity solutions takes the general form\(^1\) [4, 5]

\[
    ds^2 = -G_{00}(s)dt^2 + G_{x||x||}(s)dx^2_{||} + G_{ss}(s)ds^2 + G_{x_TX_T}(s)dx^2_T
\]  

(2.1)

where $x_{||}$ are Euclidean ordinary space coordinates (for the $M2$-brane it has 2 dimensions while for the $M5$-brane it has 5 dimensions), $s$ is a radial coordinate and $x_T$ are the rest of coordinates that are transverse to the others.

\(^1\)This 11-dimensional metric is written here in analogy with the 10-dimensional case discussed in [8].
A convenient form for the string action to calculate quark-antiquark potential is given by the Nambu-Goto action:

\[ S = \int d\sigma d\tau \sqrt{\det [\partial_\alpha X^M \partial_\beta X^N G_{MN}]} \] (2.2)

where

\[ \partial_\alpha X^M \partial_\beta X^N G_{NM} = -\partial_\alpha X^0 \partial_\beta X^0 G_{00} + \partial_\alpha X^i \partial_\beta X^i G_{xx} + \partial_\alpha s \partial_\beta s G_{ss} + \partial_\alpha X^T \partial_\beta X^T G_{xx} \]. (2.3)

On the other hand

\[ \det [\partial_\alpha X \cdot \partial_\beta X] = \det \begin{vmatrix} \partial_0 X_0 & \partial_0 X_1 & \cdots & \partial_0 X_{x_1} & \cdots & \partial_0 X_s \\ \partial_1 X_0 & \partial_1 X_1 & \cdots & \partial_1 X_{x_1} & \cdots & \partial_1 X_s \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial_{x_1} X_0 & \partial_{x_1} X_1 & \cdots & \partial_{x_1} X_{x_1} & \cdots & \partial_{x_1} X_s \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial_s X_0 & \partial_s X_1 & \cdots & \partial_s X_{x_1} & \cdots & \partial_s X_s \\ \end{vmatrix} = (G_{00})(G_{x|x}) + G_{ss}(\partial_x s)^2, \] (2.4)

as long as the world sheet coordinates \( \tau = X^0 \) and \( \sigma = X^1 \) are chosen.

Since we are interested in static quark-antiquark potentials, we assume translational invariance in time, so that the Nambu-Goto action takes the form:

\[ S = T \int dx \sqrt{(G_{00})(G_{x|x}) + G_{ss}(\partial_x s)^2}, \] (2.5)

or even

\[ S = T \int dx \sqrt{f^2 + g^2(\partial_x s)^2}, \] (2.6)

where one defines

\[ f^2 = G_{00}G_{x|x}, \] (2.7)

\[ g^2 = G_{00}G_{ss}. \] (2.8)

From the Nambu-Goto action (2.6) we write a Lagrangian

\[ L(s, s') = \sqrt{f^2(s) + g^2(s)(s')^2}, \] (2.9)

and the conjugate momentum is

\[ p = \frac{\partial L}{\partial s'} = \frac{g^2(s)s'}{L}. \] (2.10)

Therefore the corresponding Hamiltonian results in

\[ H(s, p) = \frac{g^2(s)(s')^2}{L} - L = -\frac{f^2(s)}{L}. \] (2.11)

This Hamiltonian is a constant of motion \([6, 8]\), so that \( H(s, 0) = H(s, p) \), and therefore:

\[ -f(s_0) = \frac{g^2(s)(s')^2}{L} - L. \] (2.12)
Solving for \( s'(x) \) one has:
\[
s'(x) = \pm \frac{f(s)}{g(s)} \sqrt{\frac{f^2(s)}{f^2(s_0)}} - 1. \tag{2.13}
\]

The distance between the ends of the string for this configuration can then obtained from
\[
l = \int dx = \int \left( \frac{ds}{dx} \right)^{-1} ds = 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \frac{f(s_0)}{\sqrt{f^2(s) - f^2(s_0)}} ds, \tag{2.14}
\]
and the corresponding energy associated with this string is
\[
E' = \int Ldx = \int \left( \frac{ds}{dx} \right)^{-1} Lds = 2 \int_{s_0}^{s_1} \frac{g(s)f(s)}{\sqrt{f^2(s) - f^2(s_0)}} ds
\]
\[
= f(s_0)l + 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \sqrt{f^2(s) - f^2(s_0)} ds. \tag{2.15}
\]

As this expression for the energy is in general infinite, we need to renormalize it with a convenient subtraction. This is done with a term associated with the quark mass \( m_q = f_0 \frac{\pi}{s} g(s) ds \). Therefore the renormalized potential energy can be written as
\[
E' = f(s_0)l + 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \left( \sqrt{f^2(s) - f^2(s_0)} - f(s) \right) - 2 \int_0^{s_0} g(s) ds \tag{2.16}
\]
or simply
\[
E' = f(s_0)l - 2K(s_0), \tag{2.17}
\]
where
\[
K(s_0) = 2 \int_{s_0}^{s_1} \frac{g(s)}{f(s)} \left( \sqrt{f^2(s) - f^2(s_0)} - f(s) \right) - 2 \int_0^{s_0} g(s) ds. \tag{2.18}
\]

This expression for the renormalized energy \( E' \) can be interpreted as the quark-antiquark potential that we want to calculate in the M2- and M5-brane backgrounds. From now on, we will drop the prime and just call it \( E \).

### 3 Wilson loops in M2-brane space

The metric of the M2-brane space, which is generated by \( N \) coincident branes [4, 5], is defined as:
\[
ds^2 = H^{-2/3}dx_3^2 + H^{1/3}(dr^2 + r^2 d\Omega_7^2), \tag{3.1}
\]
where
\[
H = 1 + \frac{R^6}{r^6}, \tag{3.2}
\]
\[
R^6 = Ns_1^{12} 25 \pi^2, \tag{3.3}
\]
\[
dx_3^2 = -dt^2 + dx_1^2 + dx_2^2. \tag{3.4}
\]

From eqs. (2.7) and (2.8) we find that
\[
f^2 = G_{00}G_{x|x|x} = H^{-4/3}, \tag{3.5}
\]
\[
g^2 = G_{00}G_{ss} = H^{-1/3}, \tag{3.6}
\]

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where the coordinate "s" here is the radial coordinate \( r \). Therefore, from eqs. (2.14) and (2.15) we have that the formulas for distance and energy interaction between the pair \( q \bar{q} \) that live in 2-dimensional flat space are:

\[
L_2 = 2 \int_{r_0}^{r_1} dr \frac{H^{7/6}}{(H_0^{4/3} - H^{4/3})^{1/2}},
\]

(3.7)

\[
E_2 = 2 \int_{r_0}^{r_1} dr \frac{H^{-1/6}H_0^{2/3}}{(H_0^{4/3} - H^{4/3})^{1/2}} - 2m_q,
\]

(3.8)

where \( H_0 = 1 + R^6/r_0^6 \). Defining a new variable of integration \( y = r/r_0 \), and new parameters \( y_1 = r_1/r_0 \) and \( a = R^6/r_0^6 \), we can rewrite formulas (3.7) and (3.8) as:

\[
L_2(y_1) = 2r_0 \int_{y_1}^{y_1} dy \frac{H^{7/6}}{(H_0^{4/3} - H^{4/3})^{1/2}},
\]

(3.9)

\[
E_2(y_1) = 2r_0 \int_{y_1}^{y_1} dy \frac{H_0^{2/3}}{H_1^{1/6}(H_0^{4/3} - H^{4/3})^{1/2}} - 2m_q,
\]

(3.10)

where with this new variable and parameters \( H_0 = 1 + a \) and \( H = 1 + a/y^6 \).

The equations (3.7) and (3.8), or equivalently, (3.9) and (3.10), give the separation and the energy of the quark-antiquark pair in M2-brane space. However, in order to analyze the behavior of the quark-antiquark potential \( E_2 \) against the separation \( L_2 \), one needs to express \( E_2 \) as a function of \( L_2 \). In the present form of \( E_2 \) and \( L_2 \) it is not easy to obtain this behavior. So, we are going to proceed with an analysis where different approximations are considered, corresponding to different physical situations in M2-brane space. One possibility is to distinguish the cases of highly curved geodesics from the almost flat ones. Other possibility is to study the strings close to or far from the branes.

### 3.1 First case: highly curved geodesics

The situation of highly curved geodesics is defined by the condition \( r_1 \gg r_0 \), or simply \( y_1 \gg 1 \). As a result we can write asymptotic expansion for the quark-antiquark separation and interaction energy, Eqs. (3.9) and (3.10), so that we have:

\[
L_2 = c_1(a) + \frac{2r_0a}{(H_0^{4/3} - 1)^{1/2}} y_1 - \frac{2r_0a}{5(H_0^{4/3} - 1)^{1/2}} \left(\frac{7}{6} + \frac{2}{3(H_0^{4/3} - 1)}\right) \frac{1}{y_1^{11}} + O(y_1^{-11}),
\]

(3.11)

\[
E_2 = c_2(a) + \frac{2r_0H_0^{2/3}}{(H_0^{4/3} - 1)^{1/2}} y_1 - \frac{2r_0aH_0^{2/3}}{5(H_0^{4/3} - 1)^{1/2}} \left(\frac{1}{6} + \frac{2}{3(H_0^{4/3} - 1)}\right) \frac{1}{y_1^{11}} + O(y_1^{-11}) - 2m_q,
\]

(3.12)

where

\[
c_1(a) = \lim_{y_1 \to \infty} \left(2r_0 \int_{y_1}^{y_1} dy \frac{H^{7/6}}{(H_0^{4/3} - H^{4/3})^{1/2}} - \frac{2r_0a}{(H_0^{4/3} - 1)^{1/2}} y_1\right),
\]

(3.13)

and

\[
c_2(a) = \lim_{y_1 \to \infty} \left(2r_0 \int_{y_1}^{y_1} dy \frac{H_0^{2/3}}{H_1^{1/6}(H_0^{4/3} - H^{4/3})^{1/2}} - \frac{2r_0H_0^{2/3}}{(H_0^{4/3} - 1)^{1/2}} y_1\right).
\]

(3.14)
In order to simplify calculations, we disregard order $O(y_1^{-5})$ in formulas (3.11) and (3.12), so they become:

\[
L_2 = c_1(a) + \frac{2r_0a}{(H_0^{4/3} - 1)^{1/2}} y_1, \quad (3.15)
\]

\[
E_2 = c_2(a) + \frac{2r_0H_0^{2/3}}{(H_0^{4/3} - 1)^{1/2}} y_1 - 2m_q. \quad (3.16)
\]

These equations give the separation and the energy of the quark-antiquark pair in M2-brane space for highly curved geodesics. These expressions can be further simplified taking into account the position of the geodesics with respect to the branes, as we discuss in the following.

### 3.1.1 Geodesic minimum far from the branes

The geodesics that are solutions to this problem are symmetric with respect to $r_0$, which we call the minimum of the geodesic. The condition of the minimum far from the branes can be expressed as $R_6 << r_0^6$ which corresponds to $a << 1$. Then, in this case eqs. (3.13) and (3.14) reduce to:

\[
c_1(a) \approx \frac{\sqrt{3}r_0}{\sqrt{a}} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{y^3}{\sqrt{y^6 - 1}} - ay_1 \right] = \frac{\sqrt{3}r_0}{\sqrt{a}} c, \quad (3.17)
\]

\[
c_2(a) \approx \frac{\sqrt{3}r_0}{\sqrt{a}} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{y^3}{\sqrt{y^6 - 1}} - y_1 \right] = \frac{\sqrt{3}r_0}{\sqrt{a}} c, \quad (3.18)
\]

where

\[
c = -\frac{2\pi \sqrt{\pi}}{\Gamma(1/3)\Gamma(1/6)}. \quad (3.19)
\]

Calculations concerning integrals expressions in eqs. (3.17) and (3.18) are show in the Appendix. Therefore, the quark-antiquark separation and interaction energy, Eqs. (3.15) and (3.16), become

\[
L_2 \approx \sqrt{3} \frac{r_0^4}{R^3} c + \sqrt{3} \frac{R^3}{r_0^3} r_1 \quad (3.20)
\]

\[
E_2 \approx \sqrt{3} \frac{r_0^4}{R^3} c + \sqrt{3} \frac{r_0^3}{R^3} \left(1 + \frac{2R_6^6}{3 r_0^6}\right) r_1 - 2m_q, \quad (3.21)
\]

From the separation, Eq. (3.20), we solve for $r_1 = \frac{L_2 r_0^6}{R^3 \sqrt{3}} = \frac{r_0^6 c}{R^3}$ and then substitute in the interaction energy, Eq. (3.21), obtaining:

\[
E_2 = -\frac{\sqrt{3}r_0^{10} c}{R^6} \left(1 - \frac{R^6}{3 r_0^6}\right) + L_2 \frac{r_0^6}{R^6} \left(1 + \frac{2R_6^6}{3 r_0^6}\right) - 2m_q. \quad (3.22)
\]

We can simplify this expression further, since we are working in the regime $R^6 << r_0^6$. So,

\[
E_2 \approx -\frac{\sqrt{3}r_0^{10} c}{R^6} + L_2 \frac{r_0^6}{R^6} - 2m_q. \quad (3.23)
\]
We may identify the term $-\sqrt{3^{10} \over R^6}$ with the quark masses $2m_q$, so that the potential energy becomes

$$E_2 \approx L_2 \frac{r_0^6}{R^6}.$$  \hfill (3.24)

So, in this case of highly curved geodesics in M2-brane space with minimum far from the branes, we obtain a linear confining behavior for the quark-antiquark pair.

### 3.1.2 Geodesic minimum close to the branes

The condition of the geodesic minimum close to the branes is equivalent to $R^6 >> r_0^6$ ($a >> 1$) so that eqs. (3.13) and (3.14) reduce to:

$$c_1(a) \approx 2r_0\sqrt{a} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{1}{y^3 \sqrt{y^8 - 1}} - \frac{y_1}{a^{1/6}} \right] = 2r_0\sqrt{a}c' - 2r_0a^{1/3} \lim_{y_1 \to \infty} y_1$$  \hfill (3.25)

$$c_2(a) \approx \frac{2r_0}{a^{1/6}} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{y^5}{\sqrt{y^8 - 1}} - a^{1/6} y_1 \right] = -\frac{2r_0}{a^{1/6}} c'$$  \hfill (3.26)

where

$$c' = \frac{(2\pi)^{3/2}}{4\Gamma^2(1/4)}.$$  \hfill (3.27)

The computation of these integrals in eqs. (3.25) and (3.26) is shown in the Appendix, and the expressions for the quark-antiquark separation and interaction energy, Eqs. (3.15) and (3.16), become respectively

$$L_2 \approx \frac{2R^3}{r_0^2} c' - \frac{2R^2}{r_0^2} \lim_{r_1 \to \infty} r_1 + \frac{2R^2}{r_0^2} r_1$$  \hfill (3.28)

$$E_2 \approx -\frac{2r_0^2}{R} c' + 2r_1 - 2m_q$$  \hfill (3.29)

The divergences can be canceled out in the last expressions if we assume that $r_1 \to \infty$ and identify the quark mass as $m_q = r_1$. Then,

$$L_2 \approx \frac{2R^3}{r_0^2} c',$$  \hfill (3.30)

$$E_2 \approx -\frac{2r_0^2}{R} c'.$$  \hfill (3.31)

From (3.30) we solve for $r_0^2 = \frac{2R^3 c'}{L_2}$ and after substitution in (3.31) we find the quark-antiquark potential:

$$E_2 = -\frac{4R^2 c'^2}{L_2}.$$  \hfill (3.32)

This case presents a non-confining behavior and corresponds to the calculation of Wilson loops in $AdS_4 \times S^7$, since the position of the minimum of the geodesic is close to the branes. This result is in agreement with conformal field theory and is the analogous result of Maldacena for the $AdS_5 \times S^5$ space [6].
3.2 Second case: almost straight geodesics

The case of almost straight geodesics is characterized by the condition \( r_1 \sim r_0 \), where \( r_0 \) is the minimum of the geodesic, as before. This condition is equivalent to \( y_1 = 1 + \epsilon \), where \( \epsilon \ll 1 \), so that the quark-antiquark separation and interaction energy, Eqs. (3.9) and (3.10), become:

\[
L_2(1 + \epsilon) = \frac{r_0 \sqrt{\epsilon}}{\sqrt{a}} (1 + a)^{5/6} [1 + O(\epsilon)],
\]

\[
E_2(1 + \epsilon) = \frac{r_0 \sqrt{\epsilon}}{\sqrt{a}} (1 + a)^{5/6} [1 + O(\epsilon)] - 2m_q.
\]

Then, in this case we find that:

\[
E_2 = L_2 - 2m_q.
\]

This shows that the potential for the quark-antiquark pair in M2-brane space for almost straight geodesics has a linear confining behavior. This result is the one expected for a stretched string.

4 Wilson loops in M5-brane space

For the case of \( N \) coincident M5-branes we have that the metric of this space is given by [4, 5]:

\[
ds^2 = K^{-1/3} dx_6^2 + K^{2/3} (dr^2 + r^2 d\Omega_4^2),
\]

where

\[
K = 1 + \frac{\tilde{R}^3}{r^3},
\]

\[
\tilde{R}^3 = \pi N l_1^3,
\]

\[
dx_6^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2.
\]

Following eqs. (2.7) and (2.8) we find that

\[
f^2 = G_{00} G_{x|x} = K^{-2/3},
\]

\[
g^2 = G_{00} G_{ss} = K^{1/3}.
\]

As a result, the formulas for distance and energy interaction between the pair \( q\bar{q} \) that live in 5-dimensional flat space are:

\[
L_5 = 2 \int_{r_0}^{r_1} dr \frac{K^{5/6}}{\sqrt{K_0^{2/3} - K^{2/3}}},
\]

\[
E_5 = 2 \int_{r_0}^{r_1} dr \frac{K^{1/6} K_0^{1/3}}{\sqrt{K_0^{2/3} - K^{2/3}}} - 2m_q,
\]
where \( K_0 = 1 + \tilde{R}^3/r_0^3 \). In the new variable of integration \( y = r/r_0 \), and new parameters \( y_1 = r_1/r_0 \) and \( \tilde{a} = \tilde{R}^3/r_0^3 \), the \( q\bar{q} \) separation and interaction energy, Eqs. (4.7) and (4.8), become:

\[
L_5(y_1) = 2r_0 \int_1^{y_1} dy \frac{K_0^{5/6}}{\sqrt{K_0^{2/3} - K_0^{2/3}}} ,
\]

\[
E_5(y_1) = 2r_0 \int_1^{y_1} dy \frac{K_0^{1/6} K_0^{1/3}}{\sqrt{K_0^{2/3} - K_0^{2/3}}} - 2m_q ,
\]

where \( K_0 = 1 + \tilde{a} \) and \( K = 1 + \tilde{a}/y_3 \).

As in the previous section we are going to consider particular cases of these equations regarding the shape and location os the strings in the M5-brane space, in order to obtain simpler expressions for the \( q\bar{q} \) potential.

### 4.1 First case: highly curved geodesic

This situation is defined by the condition \( r_1 >> r_0 \) that means \( y_1 >> 1 \). The asymptotic expansions for the \( q\bar{q} \) separation and interaction energy, Eqs. (4.9) and (4.10), give:

\[
L_5 = c_1(\tilde{a}) + \frac{2r_0}{(K_0^{2/3} - 1)^{1/2}} y_1 - \frac{r_0 \tilde{a}}{(K_0^{2/3} - 1)^{1/2}} \left( \frac{5}{6} + \frac{1}{3(K_0^{2/3} - 1)} \right) \frac{1}{y_1} + O(y_1^{-5}) , \]

\[
E_5 = c_2(\tilde{a}) + \frac{2r_0 K_0^{1/3}}{(K_0^{2/3} - 1)^{1/2}} y_1 - \frac{r_0 K_0^{1/3}}{(K_0^{2/3} - 1)^{1/2}} \left( \frac{1}{6} + \frac{1}{3(K_0^{2/3} - 1)} \right) \frac{\tilde{a}}{y_1^2} + O(y_1^{-5}) - 2m_q ,
\]

where

\[
c_1(\tilde{a}) = \lim_{y_1 \to \infty} \left( 2r_0 \int_1^{y_1} dy \frac{K_0^{5/6}}{\sqrt{K_0^{2/3} - K_0^{2/3}}} - \frac{2r_0}{(K_0^{2/3} - 1)^{1/2}} y_1 \right) ,
\]

\[
c_2(\tilde{a}) = \lim_{y_1 \to \infty} \left( 2r_0 \int_1^{y_1} dy \frac{K_0^{1/6} K_0^{1/3}}{\sqrt{K_0^{2/3} - K_0^{2/3}}} - \frac{2r_0 K_0^{1/3}}{(K_0^{2/3} - 1)^{1/2}} y_1 \right).
\]

In order to simplify calculations, we disregard order \( O(y_1^{-2}) \) in the expressions for the \( q\bar{q} \) separation and interaction energy, Eqs. (4.11) and (4.12), so they become:

\[
L_5 = c_1(\tilde{a}) + \frac{2r_0}{(K_0^{2/3} - 1)^{1/2}} y_1 ,
\]

\[
E_5 = c_2(\tilde{a}) + \frac{2r_0 K_0^{1/3}}{(K_0^{2/3} - 1)^{1/2}} y_1 - 2m_q .
\]

Let us discuss in the following relevant particular cases of these equations regarding the position of the minimum of the geodesic with respect to the M5-branes and obtain the corresponding \( q\bar{q} \) potentials.
4.1.1 Geodesic minimum far from the brane

The geodesic minimum \( r_0 \) far from the M5-branes corresponds to the condition \( \tilde{R}^3 << r_0^3 \). We calculate formulas (4.13) and (4.14) in this approximation \( (\tilde{a} << 1) \), obtaining

\[
c_1(\tilde{a}) \approx \frac{\sqrt{6r_0}}{\sqrt{\tilde{a}}} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{y^{3/2}}{\sqrt{y^3 - 1}} - y_1 \right] = -\frac{6r_0}{\sqrt{\tilde{a}}} \tilde{c} \tag{4.17}
\]

\[
c_2(\tilde{a}) \approx \frac{\sqrt{6r_0}}{\sqrt{\tilde{a}}} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{y^{3/2}}{\sqrt{y^3 - 1}} - y_1 \right] = -\frac{6r_0}{\sqrt{\tilde{a}}} \tilde{c} \tag{4.18}
\]

where

\[
\tilde{c} = \frac{(2\pi)^{3/2}}{\sqrt{6}\Gamma(1/3)\Gamma(1/6)} \tag{4.19}
\]

As a result, the quark-antiquark separation and its interaction energy, Eqs. (4.15) and (4.16), become respectively

\[
L_5 \approx -\sqrt{6} \frac{r_0^{5/2}}{\tilde{R}^{5/2}} \tilde{c} + \sqrt{6} r_1 \frac{r_0^{3/2}}{\tilde{R}^{3/2}} \tag{4.20}
\]

\[
E_5 \approx -\sqrt{6} \frac{r_0^{5/2}}{\tilde{R}^{5/2}} \tilde{c} + \sqrt{6} r_1 \frac{r_0^{3/2}}{\tilde{R}^{3/2}} - 2m_q \tag{4.21}
\]

Then, we find

\[
E_5 = L_5 - 2m_q \tag{4.22}
\]

This potential energy between the quark-antiquark pair shows a linear confining behavior, compatible with the flat space limit far away from the M5-branes.

4.1.2 Geodesic minimum close to the branes

In the limit where the geodesic minimum \( r_0 \) is close to the M5-branes we have the condition \( \tilde{R}^3 >> r_0^3 \) so that \( \tilde{a} >> 1 \). Calculating the formulas (4.13) and (4.14) in this approximation give:

\[
c_1(\tilde{a}) \approx 2r_0 \sqrt{\tilde{a}} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{1}{y^{3/2} \sqrt{y^2 - 1}} - \frac{y_1}{\tilde{a}^{5/6}} \right] = 2r_0 \sqrt{\tilde{a}} \tilde{c} - \lim_{r_1 \to \infty} 2 \frac{r_0}{\tilde{R}} r_1 \tag{4.23}
\]

\[
c_2(\tilde{a}) \approx 2r_0 \tilde{a}^{1/6} \lim_{y_1 \to \infty} \left[ \int_1^{y_1} dy \frac{y^{1/2}}{\sqrt{y^2 - 1}} - \frac{y_1}{\tilde{a}^{1/6}} \right] = -2r_0 \tilde{a}^{1/6} \tilde{c} \tag{4.24}
\]

where

\[
\tilde{c} = \frac{(2\pi)^{3/2}}{\Gamma^2(1/4)} \tag{4.25}
\]

As a result, the quark-antiquark separation and its interaction energy, Eqs. (4.15) and (4.16), become:

\[
L_5 \approx \frac{2 \tilde{R}^{3/2}}{r_0^{1/2}} \tilde{c} + \frac{2r_0}{\tilde{R}} r_1 - \lim_{r_1 \to \infty} 2 \frac{r_0}{\tilde{R}} r_1 \tag{4.26}
\]
\[ E_5 \approx -2r_0^{1/2} \tilde{R}^{1/2} \hat{c} + 2r_1 - 2m_q \]  

The divergences are canceled out in the last expressions if we identify the quark mass as \( m_q = r_1 \) and take the limit \( r_1 \to \infty \). Then,

\[
L_5 \approx \frac{2\tilde{R}^{3/2}}{r_0^{1/2}} \hat{c},
\]

\[
E_5 \approx -2r_0^{1/2} \tilde{R}^{1/2} \hat{c},
\]

From eq. (4.28) we get \( r_0^{1/2} = \frac{2\tilde{R}^{3/2} \hat{c}}{L_5} \), and after substitution in eq. (4.29) we find:

\[
E_5 = -\frac{4\tilde{R}^2 \hat{c}^2}{L_5}.
\]

This potential energy between the \( q\bar{q} \) pair shows a nonconfining behavior compatible with the conformal symmetry since in this case the M5-branes tends asymptotically to the \( AdS_7 \times S^4 \) space. This is consistent with Maldacena’s results for the \( AdS_5 \times S^5 \) case.

4.2 Second case: almost straight geodesics

The situation that the geodesics are almost straight is characterized by the condition \( r_1 \sim r_0 \), which means \( y_1 \equiv r_1/r_0 = 1 + \epsilon \), with \( \epsilon << 1 \). So the formulas for the \( q\bar{q} \) separation and interaction energy, Eqs. (4.9) and (4.10), become:

\[
L_5(1 + \epsilon) = \frac{2r_0 \sqrt{\tilde{R}}}{\sqrt{\hat{a}}} (1 + \hat{a}) [1 + O(\epsilon)],
\]

\[
E_5(1 + \epsilon) = \frac{2r_0 \sqrt{\tilde{R}}}{\sqrt{\hat{a}}} (1 + \hat{a})^{2/3} [1 + O(\epsilon)] - 2m_q.
\]

Let us investigate further these expressions in the limiting cases of the geodesic minimum far from or close to the M5-branes in the following.

4.2.1 Geodesic minimum far from the branes

This case of geodesic minimum far away from the M5-branes corresponds to \( \tilde{R}^3 << r_0^3 \). So in this limit the quark-antiquark pair separation and interactions energy, Eqs. (4.31) and (4.32), become respectively:

\[
L_5 \approx \frac{2r_0^{5/2}}{\tilde{R}^{3/2}} \sqrt{\frac{r_1}{r_0}} - 1,
\]

\[
E_5 \approx \frac{2r_0^{5/2}}{\tilde{R}^{3/2}} \sqrt{\frac{r_1}{r_0}} - 1 - 2m_q,
\]

which implies that

\[
E_5 = L_5 - 2m_q.
\]

This is a linear confining potential for the \( q\bar{q} \) pair, compatible with the flat space limit far away from the M5-branes.
4.2.2 Geodesic minimum close to the branes

The case of geodesic minimum near the branes corresponds to $\tilde{R}^3 >> r_0^3$. Then, in this limit the quark-antiquark separation and interaction energy, Eqs. (4.31) and (4.32), become:

\[
L_5 \approx \frac{2\tilde{R}^{3/2}}{r_0^{1/2}} \sqrt{\frac{r_1}{r_0}} - 1 ,
\]

\[
E_5 \approx \frac{4\tilde{R}^{3/2}}{3r_0^{1/2}} \sqrt{\frac{r_1}{r_0}} - 1 - 2m_q .
\]

So, we find that:

\[
E_5 = \frac{2L_5}{3} - 2m_q .
\]

This $q\bar{q}$ potential presents a linear confinement, thanks to the almost straight shape of the string.

5 Conclusions

We have applied the MRY method to calculate the expectation value of a Wilson loop operator in $M2$- and $M5$-brane backgrounds. As a result of this, we have been able to compute the classical interaction energy between a pair of static (i.e. infinitely heavy) quarks, which depends on the distance between this pair. This calculation was made firstly for a space generated by $N$ coincident $M2$-branes and secondly, for a space generated by a stack of $N$ $M5$-branes. All calculations were analysed in some limiting regimes regarding the shape of the string and its location in the brane space. Confinement was found in some regimes and a non-confinement behavior was found in limiting regimes corresponding to AdS geometries (as expected).

In Refs. [23, 24] Wilson loops in $AdS_5$ were related to $D3$-branes. Based on these results we expect that analogous relations could be established for Wilson loops in $AdS_4$ and $AdS_7$ in relation to $M2$- and $M5$-branes.

Recently, a proposal to calculate $1/N$ corrections to the $D3$-brane Wilson loop was given in [25]. We expect that such an approach could also be applied to the case of $M2$- and $M5$-brane Wilson loops.

A Appendix: Computation of some integrals

In this appendix we show in detail the computation of irregular integrals that appear in sections 3 and 4. Some of them are infinite results that need to be regularized, others are not.

First, let us consider the following integral:

\[
I_1 = \int_1^\infty dy \frac{y^3}{\sqrt{y^6 - 1}} .
\]

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After the change $t = y^{-6}$ the integral $I_1$ become:

$$I_1 = \frac{1}{6} \int_0^1 dt t^{-\frac{5}{2}} (1 - t)^{-\frac{1}{2}} = \frac{1}{6} \int_0^{\epsilon} (...) + \frac{1}{6} \int_{\epsilon}^{1-\epsilon} (...) + \frac{1}{6} \int_{1-\epsilon}^1 (...) \quad (A.2)$$

When $\epsilon$ is very small ($\epsilon \to 0$), this integral has a divergent part:

$$I_1 = \frac{1}{6\epsilon^{1/6}} + \frac{1}{6} B(-1/6, 1/2) + \frac{\epsilon^{1/2}}{6} + \frac{1}{6} B(-1/6, 1/2) = \frac{1}{6\epsilon^{1/6}} - \frac{2\pi \sqrt{\pi}}{\Gamma(1/3) \Gamma(1/6)}, \quad (A.3)$$

so, when an infinite part $ay_1 = \frac{1}{6\epsilon^{1/6}}$ is subtracted from $I_1$ we obtain a finite result:

$$\lim_{y_1 \to \infty} (I_1 - ay_1) = -\frac{2\pi \sqrt{\pi}}{\Gamma(1/3) \Gamma(1/6)}, \quad (A.4)$$

this expression appears in eqs. (3.17) and (3.18).

Now we consider

$$I_2 = \int_1^\infty dy \frac{1}{y^{3/2} \sqrt{y^3 - 1}} \quad (A.5)$$

$$I_2 = \frac{1}{8} \int_0^1 dt t^{-\frac{1}{2}} (1 - t)^{-\frac{1}{2}} = \frac{1}{8} \int_0^{\epsilon} (...) + \frac{1}{8} \int_{\epsilon}^{1-\epsilon} (...) + \frac{1}{8} \int_{1-\epsilon}^1 (...) \quad (A.6)$$

As $\epsilon$ is very small ($\epsilon \to 0$), this integral is finite:

$$I_2 = \frac{\epsilon^{3/4}}{8} + \frac{1}{8} B(3/4, 1/2) + \frac{\epsilon^{1/2}}{8} = \frac{(2\pi)^{3/2}}{4\Gamma^2(1/4)} \quad (A.7)$$

This result was used in eq. (3.25)

Next we consider

$$I_3 = \int_1^\infty \frac{y^5}{\sqrt{y^3 - 1}} \quad (A.8)$$

This integral appears in eq. (3.26)

$$I_3 = \frac{1}{8} \int_0^1 dt t^{-\frac{1}{2}} (1 - t)^{-\frac{1}{2}} = \frac{1}{8} \int_0^{\epsilon} (...) + \frac{1}{8} \int_{\epsilon}^{1-\epsilon} (...) + \frac{1}{8} \int_{1-\epsilon}^1 (...) \quad (A.9)$$

As $\epsilon$ goes to zero, this integral has an infinite part:

$$I_3 = \frac{1}{8\epsilon^{1/4}} + \frac{1}{8} B(-1/4, 1/2) + \frac{1}{8} \epsilon^{1/2} = \frac{1}{8\epsilon^{1/4}} - \frac{(2\pi)^{3/2}}{4\Gamma^2(1/4)}, \quad (A.10)$$

so, when an infinite part $a^{1/6}y_1 = \frac{1}{8\epsilon^{1/4}}$ is subtracted from $I_3$ we obtain a finite quantity:

$$\lim_{y_1 \to \infty} (I_3 - y_1) = -\frac{(2\pi)^{3/2}}{4\Gamma^2(1/4)} \quad (A.11)$$

Now consider

$$I_4 = \int_1^\infty dy \frac{y^{3/2}}{\sqrt{y^3 - 1}} \quad (A.12)$$
This appears in eqs. (4.17) and (4.18)

\[ I_4 = \frac{1}{3} \int_0^1 dt t^{-\frac{3}{4}} (1-t)^{-\frac{1}{2}} = \frac{1}{3} \int_0^\epsilon (...) + \frac{1}{3} \int_\epsilon^{1-\epsilon} (...) + \frac{1}{3} \int_{1-\epsilon}^1 (...) \]  

(A.13)

As \( \epsilon \) goes to zero, this integral has an infinite part:

\[ I_4 = \frac{1}{3\epsilon^{1/3}} + \frac{1}{3} B(-1/3, 1/2) + \frac{1}{3} \epsilon^{1/2} = \frac{1}{3\epsilon^{1/3}} - \frac{(2\pi)^{3/2}}{\sqrt{6} \Gamma(1/3) \Gamma(1/6)}, \]  

(A.14)

so, when an infinite part \( y_1 = \frac{1}{3\epsilon^{1/3}} \) is subtracted from \( I_4 \) we obtain a finite term:

\[ \lim_{y_1 \to \infty} (I_4 - y_1) = -\frac{(2\pi)^{3/2}}{\sqrt{6} \Gamma(1/3) \Gamma(1/6)} \]  

(A.15)

Then, we consider

\[ I_5 = \int_1^\infty dy \frac{1}{y^{3/2} \sqrt{y^2 - 1}} \]  

(A.16)

This integral appears in eq. (4.23)

\[ I_5 = \frac{1}{2} \int_0^1 dt t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}}, \]  

(A.17)

then we realize that

\[ I_5 = 4I_2 = \frac{(2\pi)^{3/2}}{\Gamma^2(1/4)}, \]  

(A.18)

which is a finite number. Finally we consider

\[ I_6 = \int_1^\infty dy \frac{y^{1/2}}{\sqrt{y^2 - 1}}, \]  

(A.19)

which appears in eq. (4.24)

\[ I_6 = \frac{1}{2} \int_0^1 dt t^{-\frac{5}{4}} (1-t)^{-\frac{1}{2}} = 4I_3 = \frac{1}{2\epsilon^{1/4}} - \frac{(2\pi)^{3/2}}{\Gamma^2(1/4)}, \]  

(A.20)

where \( \epsilon \) goes to zero. So if we subtract an infinite term \( y_1 = \frac{1}{2\epsilon^{1/4}} \) from \( I_6 \) we obtain:

\[ \lim_{y_1 \to \infty} (I_6 - y_1) = -\frac{(2\pi)^{3/2}}{\Gamma^2(1/4)} \]  

(A.21)

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