Exact spectral densities of complex noise-plus-structure random matrices

Jacek Grela\textsuperscript{1} and Thomas Guhr\textsuperscript{2}

\textsuperscript{1}M. Smoluchowski Institute of Physics and Mark Kac Complex Systems Research Centre, Jagiellonian University, PL–30348 Kraków, Poland
\textsuperscript{2}Fakultät für Physik, Universität Duisburg–Essen, Duisburg, Germany

We use supersymmetry to calculate exact spectral densities for a class of complex random matrix models having the form \( M = S + LXR \), where \( X \) is a random noise part \( X \) and \( S, L, R \) are fixed structure parts. This is a certain version of the “external field” random matrix models. We found two–fold integral formulas for arbitrary structural matrices. We investigate some special cases in detail and carry out numerical simulations. The presence or absence of a normality condition on \( S \) leads to a qualitatively different behavior of the eigenvalue densities.

I. NOISE-PLUS-STRUCTURE RANDOM MATRICES

In the last 50 years, Random Matrix Theory (RMT) has been established as an impressively versatile approach \cite{4} of studying complex systems. In particular, applications include large data structures \cite{34}, machine learning algorithms \cite{1} and telecommunications \cite{14} arose recently. It is a common problem in these and many other areas to infer a signal or information from noisy data. In this work we study a type of RMT noise-plus-structure model suitable for this type of inference tasks. More specifically, let \( M \) be a matrix of the form:

\[
M = S + LXR, \tag{1}
\]

where \( S \) is a fixed matrix and \( L, R > 0 \) are diagonal positive definite covariance matrices. The matrix \( X \) is the source of noise drawn typically from a multi-dimensional Gaussian ensemble. Equation (1) thus comprises a simplest model combining both randomness (\( X \)) and structure (\( S, L, R \)). The matrix \( S \) is called a source and is interpreted as the signal/information matrix of the system in study. We add a structured noise \( LXR \) as every real–world data is contaminated, and only the resulting matrix \( M \) is attainable by experiment. The matrices \( L, R \) encode an anisotropic (or correlated) source of randomness — a single element of the source matrix \( S_{ij} \) is perturbed by a noisy term \( L_{ii}R_{jj}x_{ij} \), \textit{i.e.} with variance \( \sigma_{ij}^2 = (L_{ii}R_{jj})^2 \). Absence of any structure means setting \( S = 0 \) and \( L = R = 1 \) which reduces Eq. (1) to standard RMT models of pure randomness.

There are at least two strategies of studying the model (1) — we look at either the eigenvalues or the singular values of \( M \) (equivalently at the eigenvalues of \( M^\dagger M \)). The first approach is limited to square matrices whereas the second route is the main idea behind the Principal Component Analysis in which, in general, rectangular data matrices \( M \) are investigated. In this work we focus on the first approach and study the statistics of the eigenvalues. It is well-known that the symmetries of \( M \) constrain the position of its eigenvalues. Here, however, we drop any symmetry constraints and focus on the case where eigenvalues spread over the whole complex plane. In what follows we discuss a couple of instances which can be realized with the model (1) and which are interesting from a practical as well as from a theoretical perspective.

In finance, one studies the markets to make educated guesses of their future behaviour, including the search for possibly profitable correlations. To this end one typically considers \( N \) assets in \( T \) time slices which may be ordered in a rectangular \( N \times T \) matrix \( M \). We set \( S = 0 \) and interpret \( L, R \) as noise correlation matrices in both time and space. Because \( M \) is rectangular, the spectral density of \( M^\dagger M \) is studied and thus we arrive at the doubly correlated Wishart model \cite{38}. As a second example, in wireless telecommunication Eq. (1) arises in Multiple Input Multiple Output (MIMO) systems as a complex \( N_r \times N_t \) transmission matrix \( M \) between \( N_t \) transmitters and \( N_r \) receivers \cite{31}.

As a physics application, we consider a Hermitian Hamiltonian \( M \) which models an ensemble of charged spinless particles interacting with a strong external magnetic field \cite{29}. In this instance we set \( S = e^{-\tau}H_0 \), \( LR = \sqrt{1 - e^{-2\tau}} \) and both \( H_0 \) and \( X \) are random matrices drawn from the Gaussian Unitary Ensemble (GUE). The parameter \( \tau \) is proportional to the applied magnetic field. For moderate fields a different Random Matrix Model of (1) applies — a transition between a Gaussian Orthogonal Ensemble (GOE) and a GUE takes place due to the breaking of time reversal invariance. In this regime we set \( LR = i\alpha \) while the random matrices \( S \) and \( X \) are symmetric \( S = S^T \) and \( X \) antisymmetric \( X = -X^T \), respectively. Even though we drop the positivity condition of \( L, R \) and consider a random matrix \( S \), the model described is still of the form (1). As the parameter \( \alpha \) which is proportional to the field varies between 0 → 1, a transition between GOE and GUE takes place.

Independently, the rich mathematical structure of models of the type (1) has attracted a lot of attention in its own right. These ensembles are known in the RMT community as “external source models”. So far they were mostly considered for \( L = R = 1 \) and Hermitian \( X \) \cite{10, 11, 16, 21}. These models also have a natural
interpretation in terms of Dyson’s Brownian motion for the stochastic evolution in time $\tau$, when we set $LR = \sqrt{\tau}$ and view $S$ as the initial matrix $S$. 

All of the above examples contain either complex or real matrices $M$ with a purely real spectrum, there are situations where symmetry constraints are not present and the spectrum spreads over the whole complex plane. One of the main tenets of quantum mechanics for closed systems is the Hermiticity of the Hamiltonian, while dropping it is an often used effective way to describe open systems, i.e. to account for the environment. As a consequence, complex energies of the type $E = \varepsilon - i\Gamma$ arise which correspond to resonant states. Such an energy eigenstate $|\phi_E(i)\rangle = e^{-iEt} |\phi_E(0)\rangle$ does not only oscillate with a frequency $\varepsilon$ but also decays with a characteristic time $1/\Gamma$. Random Matrix Models of this type were used for studying quantum chaotic scattering in open cavities [18]. In this case, the matrix $S$ is drawn from the GUE, $LR = -i\tau$, and $X = W^\dagger W$ models a random interaction between the cavity and its surroundings, where $W$ is drawn from a complex Girko–Ginibre Ensemble.

As a second application of non–Hermitian matrices, we mention efforts in constructing mathematical models of neuronal behaviour. Moreover, a recent paper [3] argued that also the $S, L$ and $R$ matrices in the model [1] might be of significance.

In the sequel, we consider matrices $X$ drawn from the Girko–Ginibre Ensemble (i.e., a matrix with complex Gaussian random entries) as well as various types of structural matrices $S, L$ and $R$. In Sec. IV we compute an exact formula for the spectral density of $M$ and arbitrary matrices $S, L$ and $R$. In Sec. II we investigate particular cases: a normal matrix $S$ and arbitrary matrices $L, R$, a vanishing source $S = 0$ and trivial $L = R = 1$, and a rank–one non–normal source $S$ with $L = R = 1$. Eventually, we comment on the spectral formula for a related problem of eigenvalues of $M^{-1}$. We summarize and conclude in Sec. IV.

II. SPECTRAL DENSITY OF $M$

We now describe the model [1] in greater detail. Let $X$ be an $N \times N$ matrix drawn from a complex Girko–Ginibre Ensemble, 

$$P(X) dX = C^{-1} \exp \left( -n \text{Tr} X^\dagger X \right) dX,$$

where $n$ is an (inverse) variance parameter and $C = (\pi/n)^{N^2}$ is the normalization constant. The flat measure over the matrices $X$ is denoted $dX$. All matrices $S, L$ and $R$ are $N \times N$, with $L, R$ being positive definite and diagonal. The source matrix $S$ is in the most general form given by $S = D + T$ where $D$ is diagonal and $T$ is strictly upper triangular. These reduced forms are not restrictive because the spectrum of $M$ is unitarily invariant. In particular, the Schur decomposition of the source matrix reads $S = U^\dagger (D + T) U$ for a particular unitary matrix $U$. When $T = 0$ the source matrix is called normal, otherwise it is non-normal.

A basic statistical quantity characterizing the model [1] is the spectral density

$$\rho(z, \bar{z}) = \frac{1}{N} \left( \sum_{i=1}^{N} \delta^{(2)}(z - m_i) \right)_p,$$

depending on the complex variable $z$. The $m_i$ are the eigenvalues of $M$. We use the two-dimensional Dirac delta function due to complexity of the spectrum, the average is taken over the random measure [2].

Many authors have studied the spectral density [3] in the large $N$ limit [5, 7, 20]. In particular, convenient quaternionic/hermitization methods [15, 25] were developed to complete this task. For $L = R = 1$ and a general normal source $S$, spectral density in the large-$N$ limit was found in Ref. [27] whereas the $L, R \neq 1$ generalization was recently studied in Ref. [8]. For finite matrix size, a formula for the spectral density was calculated in Ref. [23] for $L = R = 1$ and a normal source term $S$ only. In this work we address the cases $L, R \neq 1$ as well as non–normal $S$.

A. Generating function

To find the spectral density, we define the averaged ratio of determinants

$$R_{L, R}(Z, V) = \left( \frac{\det(Z - M)}{\det(V - M)} \right)_p,$$

with the $2N \times 2N$ block matrices

$$M = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} L^2 u & \bar{z} 1_N \\ z 1_N & -R^2 w \end{pmatrix}, \quad V = \begin{pmatrix} L^2 u & v 1_N \\ \bar{v} 1_N & -R^2 w \end{pmatrix},$$

where $1_N$ denotes the $N \times N$ unit matrix. We notice that the matrices $Z$ and $V$ depend on the complex variables $z, u, v$ and $w$. For $u = w = 0$ we recover the special case

$$R_{L, R}(z, v) = \left( \frac{\det(\bar{z} - M)(\bar{v} - M)}{\det(v - M)(\bar{u} - M)} \right)_p.$$

Although the variables $u, w$ have an interesting interpretation in terms of the eigenvectors [12], we only use their regulatory properties – as long as $u, w \neq 0$, the ratio is finite for all complex $v$. Importantly, the spectral density
is generated by taking proper derivatives of the averaged ratio, equation
\[ \rho(z, \bar{z}) = \frac{1}{N^2 \pi} \lim_{w \to 0} \lim_{\tilde{z} \to z} \frac{\partial}{\partial \tilde{z}} \mathcal{R}_{L,R}(Z, V) \]
introduced in Ref. [1] for \( L = R = 1 \).

As a first step we make the change of variables \( Y = LX \), implying \( M = S + Y \) as well as \( M = S + Y \). The measure \( P(X) dX \) now reads
\[ P_{L,R}(Y) dY = C^{-1}_{L,R} \exp \left( -n \text{Tr} R^{-1} Y^T L^{-1} Y^T \right) dY, \]
where the normalization constant is given as \( C_{L,R} = \left( \pi/n \right)^{N^2} \det(LR)^2 \). We open the ratio of determinants with the help of complex Grassmann variables \( \chi_i \) and complex ordinary variables \( \phi_i \),
\[ \frac{\det(Z - M)}{\det(V - M)} = c \int \! d[\phi, \chi] e^{i q \cdot \text{diag}(V - M, Z - M) q}, \]
with a proper normalization constant \( c \). We introduced the supervector \( q = (\phi_1 \phi_2 \chi_1 \chi_2)^T \), and the joint measure
\[ d[\phi, \chi] = \prod_{i=1}^2 d[\phi_1] d[\phi_2] d[\chi_1] d[\chi_2]. \]
Averaging with the distribution \( P_{L,R} \) only affects the exponential terms proportional to \( Y \) which are given by
\[ e^{-i q \cdot \text{diag}(Y, V) q} = e^{-\left( \phi_1^T Y \phi_2 + \chi_1^T Y \chi_2 + \phi_2^T Y^T \phi_1 + \chi_2^T Y^T \chi_1 \right)} \]
where we set \( (E_1)_{ij} = (\phi_2)_i (\phi_1)_j - (\chi_2)_i (\chi_1)_j \) and \( (E_2)_{ij} = (\phi_1)_i (\phi_2)_j - (\chi_1)_i (\chi_2)_j \). The average is easily found to be
\[ \int \! dY P_{L,R}(Y) e^{-\text{Tr}(E_1 Y + E_2 Y^T)} = e^{-\frac{1}{8} \text{Tr}(E_1 L^2 + E_2 R^2)}. \]
To proceed further, we carry out a Hubbard–Stratonovich transformation
\[ e^{-\frac{1}{8} \text{Tr}(E_1 L^2 + E_2 R^2)} = c_0 \int \! [d\Sigma] e^{-F - q^T Q q}, \]
which reduces the fourth order supervector terms to second order. The supermatrix \( Q \) appearing in the exponent is given by
\[ Q = \left( \begin{array}{cc} L \text{diag}(\sigma_1 \mathbf{1}_N, -\sigma_1 \mathbf{1}_N) & L \text{diag}(\alpha \mathbf{1}_N, \beta \mathbf{1}_N) \\ L \text{diag}(\bar{\sigma} \mathbf{1}_N, -\bar{\sigma} \mathbf{1}_N) & L \text{diag}(\bar{\alpha} \mathbf{1}_N, \bar{\beta} \mathbf{1}_N) \end{array} \right), \]
with \( L = \text{diag}(L^2, R^2) \). It depends on four new complex integration variables, two ordinary variables \( \sigma \) and \( \rho \) as well as two anticommuting ones \( \alpha \) and \( \beta \). The corresponding measure
\[ [d\Sigma] = d^2 \sigma d^2 \rho d^2 \alpha d^2 \beta \]
is flat. We use the notation \( d^2 \sigma = d\sigma d\bar{\sigma} \). The normalization constant in Eq. (12) is given by \( c_0 = \pi^{-2} \). The function \( F = |\sigma|^2 + |\rho|^2 + \alpha \beta + \beta \alpha \) in the exponent yield the Gaussians needed for the supervector \( q \) to second order.

Thus, we can cast the generating function \( \mathcal{R}_{L,R} \) into the form
\[ \mathcal{R}_{L,R} = c c_0 \int \! [d[\phi, \chi]] [d\Sigma] e^{-n F + i q^T A q}, \]
where we introduced the supermatrix
\[ A = \text{diag}(V - S, Z - S) + iQ. \]
In the next step we interchange the order of integration \( d[\phi, \chi] \leftrightarrow [d\Sigma] \). This, however, has a subtle flaw: the resulting integral in the bosonic \( \sigma, \rho \) directions is no longer convergent, an issue addressed previously [22, 24]. To circumvent this problem, we make the change of variables
\[ \rho = \rho_1 + i \rho_2, \quad \sigma = \sigma_1 + i \sigma_2, \]
\[ \rho_1 = i \frac{w - \bar{w}}{2} + f \cos \phi, \quad \rho_2 = -\frac{w + \bar{w}}{2} + f \sin \phi, \]
\[ \sigma_1 = i \frac{u + \bar{u}}{2} - ig_+ \sinh \gamma, \quad \sigma_2 = -\frac{u - \bar{u}}{2} + g_- \cosh \gamma, \]
before swapping the order of integration. Here, we introduced real commuting variables \( f, g, \gamma \) and \( \phi \) as well as a small imaginary increment, \( g_- = g - i \epsilon \) with \( \epsilon > 0 \). The range of integration is \( f \geq 0, \phi \in (0, 2\pi], g \in \mathbb{R}, \gamma \in \mathbb{R} \). The anticommuting variables \( \alpha, \beta \) remain unchanged. The integral then becomes
\[ \int [d\Sigma] e^{-n F + i q^T A q} = \int [d\Sigma'] (-i g_-) e^{-n F' + i q'^T A' q}, \]
with \( [d\Sigma'] = df \! d\phi \! dg \! d\gamma \! d^2 \alpha \! d^2 \beta \) and
\[ F' = g_+^2 + f^2 + |w|^2 + |u|^2 + g_- (w \gamma - \bar{u} \gamma) + i f (w e^{i \phi} - \bar{u} e^{-i \phi}) + \alpha \beta + \beta \alpha. \]
We also introduced the transformed supermatrix
\[ A' = \left( \begin{array}{cc} A_{BB} & A_{BF}^T \\ A_{FB} & A_{FF} \end{array} \right), \]
with the \( 2N \times 2N \) blocks
\[ A_{BB} = \left( \begin{array}{cc} -L^2 \sigma e^{-s} & v \mathbf{1}_N - S \\ v \mathbf{1}_N - S^T & -R^2 \sigma e^{-s} \end{array} \right), \quad A_{BF} = \left( \begin{array}{cc} i \alpha L^2 & 0 \\ 0 & i \beta R^2 \end{array} \right), \]
\[ A_{FF} = \left( \begin{array}{cc} i L^2 \rho e^{-i \phi} & z \mathbf{1}_N - S \\ z \mathbf{1}_N - S^T & i R^2 \rho e^{i \phi} \end{array} \right), \quad A_{FB} = \left( \begin{array}{cc} i \bar{\alpha} L^2 & 0 \\ 0 & i \bar{\beta} R^2 \end{array} \right). \]
After this change of variables, we now may safely interchange the order of integration and arrive at
\[ \mathcal{R}_{L,R} = -ic_0 \int [d\Sigma'] g_- f e^{-n F'} s \det^{-1} A', \]
where the integral over the supervector yielded the superdeterminant as an extension of Eq. (10)
\[ c \int \! d[\phi, \chi] e^{i q^T A' q} = s \det^{-1} A'. \]
The superdeterminant is known to satisfy the formula
\[
\text{sdet}^{-x} A' = \frac{\text{det}^x A'_{EF}}{\text{det}^x A'_{BB}} \left( 1 + x^2 \text{Tr} A_0 + \frac{x^2}{2} \text{Tr} A_0^2 + \frac{x^2}{2} (\text{Tr} A_0)^2 \right)
\]
where \( A_0 = A_{BB}^{-1} A_{BF} A_{EF}^{-1} A_{EF} \) for any integer \( x \). This result enables us to integrate over the Grassmann variables \( \alpha, \beta \) in Eq. (20). The integral
\[
I(f, g, \phi, \gamma) = \int d\alpha \beta e^{-n(\bar{\alpha} \beta + \bar{\beta} \alpha)} \text{sdet}^{-1} A'
\]
can be written in the form
\[
I = -G (g_1 + (n - g_2)(n - g_3) + g_4),
\]
after some algebra and by utilizing the standard normalization of the Berezin integrals to one. The individual terms are
\[
G = \frac{\text{det}(-f^2 \Gamma_1 - \Gamma_2 \psi)}{\text{det}(g^2 \Gamma_1 - \Gamma_2 \psi)} \),
\]
\[
g_2 = \text{Tr} [\Omega \Gamma_1 P_v Q_u],
\]
\[
g_3 = \text{Tr} [\Omega \Gamma_2 Q_u P_v],
\]
\[
g_4 = f^2 \text{Tr} [\Omega \Gamma_2 P_v Q_u] + g^2 \text{Tr} [\Omega \Gamma_1 Q_u P_v Q_u],
\]
where we defined
\[
\begin{align*}
\Omega_x &= R^{-2}(x \Gamma_1 - S^\dagger), \\
\Gamma_x &= L^{-2}(x \Gamma_1 - S), \\
P_v &= (g^2 \Gamma_1 - \Gamma_v \psi)^{-1}, \\
P_v' &= (g^2 \Gamma_1 - \Gamma_v \psi)^{-1}, \\
Q_v &= (-f^2 \Gamma_1 - \Gamma_v \psi)^{-1}, \\
\end{align*}
\]
At this point we make the remarkable observation that the function \( I \) is independent of the variables \( \gamma \) and \( \phi \) such that \( I(f, g, \phi, \gamma) = I(f, g) \). Hence integrating over the fermionic variables effectively restores a certain invariance.

Assembling everything, the generating function (20) is given by
\[
\mathcal{R}_{L,R} = -\frac{4i}{\pi} e^{-n|u|^2 + n|v|^2} \int_{-\infty}^{\infty} dg - \int_0^\infty df J(f, g),
\]
with the integrand
\[
J(f, g) = g_- f e^{-n(g^2 + f^2)} I(f, g_-) I_0(2nf|w|) K_0(2in|u|g_-),
\]
depending on the modified Bessel functions \( I_0 \) and \( K_0 \) of the first and second type, respectively. They result from the following integrals over the \( \gamma, \phi \) variables,
\[
\begin{align*}
N_\gamma &= \int_{-\infty}^{\infty} d\gamma e^{-ng_- (ue^\gamma - \bar{u}e^{-\gamma})}, \\
N_\phi &= \int_0^{\infty} d\phi e^{-nf(we^{i\phi} - \bar{w}e^{-i\phi})}.
\end{align*}
\]
We set \( u = |u|e^{i\theta}, w = |w|e^{i\psi} \) and choose the argument of \( u \) to be \( \theta = \pi/2 \) to make the \( \gamma \) integral convergent. The angle of \( w \) is arbitrary since the \( \phi \) integral is periodic. We therefore set \( \psi = 0 \) and arrive at
\[
N_\gamma = \int_{-\infty}^{\infty} d\gamma e^{-2inng_- |u| \cosh \gamma} = 2K_0(2in|u|g_-),
\]
\[
N_\phi = \int_0^{2\pi} d\phi e^{2nf|w| \sin \phi} = 2\pi I_0(2nf|w|),
\]
which after taking care of the constants yields Eq. (24).

### III. PARTICULAR CASES

So far, the result (24) for the generating function is exact for any matrix dimension \( N \) and is valid for any structural matrices \( L, R \) and \( S \). Although the integrand (25) is, in general, rather complicated, the integral can be worked out explicitly for certain subclasses of \( L, R \) and \( S \). We are particularly interested in the three cases
1. normal source \( S \) and variance matrices \( L, R \) arbitrary,
2. vanishing source \( S = 0 \) and trivial \( L = R = 1 \),
3. non–normal source \( S \) of rank one and trivial variance matrices \( L = R = 1 \),
which we compute and discuss in the sequel.

#### A. Normal \( S \) and arbitrary \( L, R \)

In this case all structure matrices \( L, R \) and \( S \) are diagonal,
\[
\begin{align*}
S &= \text{diag}(s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots), \\
L &= \text{diag}(l_1, \ldots, l_1, l_2, \ldots, l_2, \ldots), \\
R &= \text{diag}(r_1, \ldots, r_1, r_2, \ldots, r_2, \ldots),
\end{align*}
\]
with three sets of multiplicities \( u_1, v_1, w_1 \) which should not be confused with the above employed complex variables \( u, v, w \). Here, \( x, y, z \) are the numbers of different entries in the structure matrices \( L, R \) and \( S \), respectively, thereby defining the sizes of the sets. The multiplicities in each set add up to \( N \). Because the integrand (23) only depends on the products \( (\Omega_x)_{ii}(\Gamma_y)_{ii} \), we introduce a structured source matrix of the form
\[
\alpha_{xy} = \Omega_x \Gamma_y = (LR)^{-2}(\bar{x} \Gamma_1 - S^\dagger)(y \Gamma_1 - S),
\]
which depends on all three matrices \( L, R \) and \( S \). It is accompanied by a merged multiplicity vector \( \vec{n} \). We define it by the following construction: we first form the
multiplicity vectors $\vec{u} = (u_1, ..., u_x)$, $\vec{v} = (v_1, ..., v_y)$ and $\vec{w} = (w_1, ..., w_z)$ corresponding to the matrices $S, L$ and $R$, respectively. The vectors $\vec{u}$ is graphically represented by a column of $N$ points which are ordered in $x$ groups according to the multiplicities $u_i$. The points within each of these $x$ groups are given the same (arbitrary) color which is only used to distinguish the different groups. We refer to the first and last points in each group as boundary. The vectors $\vec{u}, \vec{v}, \vec{w}$ are represented accordingly. The multiplicity vector $\vec{n} = (n_1, ..., n_k)$ is then constructed as a vector which has a boundary whenever at least one of the vectors $\vec{u}, \vec{v}, \vec{w}$ has one. We illustrate this by the example in Fig. 1 in which the vector $\vec{u}$ is represented by $N = 11$ points ordered in $x = 3$ groups with multiplicities $u_1 = 5, u_2 = 2$ and $u_3 = 4$ with $5 + 2 + 4 = 11$. As seen, the multiplicities for the other two vectors differ. We juxtapose the point sets of all three multiplicity vectors along with the constructed $\vec{n}$. From now on we only

![Diagram](image)

FIG. 1. Construction of the multiplicity vector $\vec{n} = (1, 1, 2, 1, 2, 2)$ from $\vec{u} = (5, 2, 4), \vec{v} = (2, 5, 4), \vec{w} = (1, 3, 5, 2)$. The points depict groups of sizes determined by the corresponding multiplicities. Horizontal lines (both solid and dashed) are drawn along the boundaries of the groups of any of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, visualizing the construction of the merged vector $\vec{n}$.

use the merged vector $\vec{n}$. We introduce the dimension $d(\vec{n})$ of the vector $\vec{n}$ as the number of differing groups, e.g. $d(\vec{n}) = 7$ in the above example. We also introduce the length $|\vec{n}| = \sum_{i=1}^{d(\vec{n})} n_i$. The generating function can then be cast into the form

$$\frac{1}{C^2} R_{L,R} = i\vec{n} J\vec{n} - \sum_{i=1}^{d(\vec{n})} n_i \left( \alpha_{0}^i + \alpha_{zz}^i + \frac{N}{n} \right) i\vec{n} - \vec{e}_i J\vec{n} + \vec{e}_i \vec{e}_i + \sum_{i,j=1}^{d(\vec{n})} n_i n_j \left( \alpha_{zz}^i \alpha_{zz}^j \vec{n} - \vec{e}_i - \vec{e}_j \right) J\vec{n} + \vec{e}_i + \vec{e}_j (\vec{n})$$

$$+ \sum_{i,j=1}^{d(\vec{n})} n_i n_j \left[ \alpha_{zz}^i \alpha_{zz}^j \vec{n} - \vec{e}_i - \vec{e}_j \right] J\vec{n} + \vec{e}_i + \vec{e}_j (\vec{n})$$

where the index $N$ is a short–hand notation for the one–dimensional multiplicity vector $\vec{n} = (N)$. We find from the formulas (28) and (30) for the fermionic and bosonic building blocks

$$i_{m}(w) = L_{m}(-N|w|^{2}), \quad j_{m}(u) = (x-1)!U_{m,1}(N|u|^{2}),$$

which reproduces the results of Ref. [19]. However, in the present study we are interested in the complementary limit, i.e., we set $u, w \to 0$ and look at $z, v \neq 0$.

We now wish to calculate the spectral density. We recall the formula (8) where the parameters $u$ and $w$ serve as regulators. It is desirable to set them to zero before computing the derivatives. Even though this does not pose a problem for the fermionic block (28), it turns out
to produce infinities in the bosonic block \([30]\). To control these emerging singularities, we use the identity

\[
k!U_{k+1,1}(n|u|^2) = e^{|n|u|^2} \Gamma(0, n|u|^2) L_k(-n|u|^2) + \tilde{L}_k(-n|u|^2),
\]

for the confluent hypergeometric function. Here, \(L_k\) are the Laguerre polynomials whereas \(\tilde{L}_k\) are defined by the same recurrence relations but with different initial conditions \(L_0(x) = 0, \tilde{L}_1(x) = -1\). The singular behavior for \(U\) as \(u \to 0\) is due to the incomplete Gamma function \(\Gamma(0, n|u|^2)\) in the first term. We therefore split the bosonic block into a singular and a regular part,

\[
j_m(v, u) = j_m^{(\text{sing})}(v, u) + j_m^{(\text{reg})}(v, u).
\]

To control the singularity, we set the singular part \(j_m^{(\text{sing})}\) to zero and take the limit \(u \to 0\) in the regular part \(j_m^{(\text{reg})}\). We formalize this procedure by introducing the regularized generating function

\[
\tilde{\mathcal{R}}_{L,R} = \mathcal{R}_{L,R} \left[ i_{\tilde{m}}(z, w) \to \tilde{i}_{\tilde{m}}(z), j_m(v, u) \to \tilde{j}_m(v) \right],
\]

with new building blocks \(\tilde{i}_{\tilde{m}}(z) = i_{\tilde{m}}(z, w = 0)\) and \(\tilde{j}_m(v) = j_m^{(\text{reg})}(v, u = 0)\) already in the \(w, u \to 0\) limit. We stress that this procedure is not an approximation — we have \(\tilde{\mathcal{R}}_{L,R} \neq \mathcal{R}_{L,R}\), the spectral densities obtained by Eq. \((33)\) agree exactly \(\tilde{\rho} = \rho\). We checked this numerically. This property is intuitively justified since we subtract the otherwise infinite part proportional to \(j_m^{(\text{sing})}\). The regularized building blocks are given by

\[
\tilde{i}_{\tilde{m}} = \frac{1}{\prod_{i=1}^{d(\tilde{m})} m_i!} \int_0^\infty d\rho e^{-\rho} \prod_{i=1}^{d(\tilde{m})} \left( \rho + n\alpha_{zz}^i \right)^{m_i},
\]

\[
\tilde{j}_m = -\frac{1}{\prod_{i=1}^{d(\tilde{m})} (m_i - 1)!} \int_\mathbb{R} dp \frac{e^{\nu}(\gamma + \Gamma(0, p) + \ln p)}{\prod_{i=1}^{d(\tilde{m})} (p + n\alpha_{zz}^i)^{m_i}},
\]

where we used the identity

\[
\sum_{m=0}^\infty \frac{1}{m!} \tilde{L}_m(0) p^m = -e^\nu(\gamma + \Gamma(0, p) + \ln p)
\]

for the modified Laguerre polynomials with \(\gamma\) denoting the Euler constant. This identity follows from the fact that \(\tilde{L}_m(0) = -\sum_{k=1}^m \tilde{1}\) are the (negative) harmonic numbers.

The final formula for the spectral density in the case of a normal source \(S\) and nontrivial \(L, R\) then reads

\[
\tilde{\rho} = -\frac{1}{N\pi} \frac{\partial}{\partial z} \lim_{v \to z} \frac{\partial}{\partial v} \tilde{\mathcal{R}}_{L,R}(z, v),
\]

together with the definitions \((27), (33)\) and \((34)\). We demonstrate the utility of our analytical result in Fig. 2 by comparing it with numerical simulations. Adding (structured) noise \(LXR\) produces an overall eigenvalues spreading with anisotropic features reflecting the \(L, R\) covariance matrices. The density is concentrated around the initial eigenvalues of \(S\) and varies smoothly as we change the noise level \(n\), i.e. the inverse variance of the ensemble \((2)\).

**B. Vanishing source \(S = 0\) and \(L = R = 1\)**

We now consider the case \(S = 0\) and \(L = R = 1\) in which a simple spectral density formula is known from the work of Ginibre \([20]\). The multiplicity vector is one-dimensional \(\tilde{n} = (N)\) and the source matrix has the simple form \(\alpha_{xy} = \bar{x}y1_N\). The regularized generating function \((33)\) reads

\[
\mathcal{R}_G = N (i_N \tilde{\gamma}_N - \tilde{i}_{N-1} \tilde{j}_{N+1} - n \tilde{j}_{N-1} \tilde{\gamma}_{N+1} \tilde{v} z + \tilde{v} \tilde{z} + n (i_N - 1) \tilde{j}_{N+1} |v|^2 + \tilde{i}_{N-2} \tilde{j}_{N+1} |z|^2),
\]

where we write \(\tilde{i}_N = \tilde{i}_{\tilde{n}}, \tilde{j}_N = \tilde{j}_{\tilde{n}}\). The building blocks are

\[
\tilde{i}_\alpha = \frac{1}{\alpha!} \int_0^\infty d\rho e^{-\rho} (\rho + n|z|^2)^\alpha,
\]

\[
\tilde{j}_\beta = -\frac{1}{2\pi i} \int_\mathbb{R} dp \frac{e^{\nu} \ln p}{(p + n|v|^2)^\beta},
\]

The bosonic block, when compared to Eq. \((34)\), lacks the term \(\gamma + \Gamma(0, p)\) since this contribution vanishes in the generating function \((33)\), as can be seen by a symbolic.
calculation. This observation holds more generally, not only in this simplest case. Directly from the definitions, we derive the iterative formulas
\[
\hat{i}_\alpha = \hat{i}_{\alpha-1} + (n|z|^2)^\alpha (\alpha!)^{-1}, \\
\hat{j}_\beta = \hat{j}_{\beta+1} - (\beta - 1)! (n|z|^2)^{-\beta} \frac{e^{-n|v|^2}}{|v|^{2N}} \hat{j}_{\beta-1}(v)
\]
and use them to re-express the generating function
\[
\hat{R}_G = n^2 N_{-1} N_{-1} |v - z|^2 + \frac{e^{-n|v|^2}}{|v|^{2N}} \left( \hat{i}_{N-1}(z)|v|^{2N} - \hat{i}_{N-1}(v)|z|^{2N} \right), \quad (40)
\]
where we have written out explicitly the argument of \(i\) to avoid confusion. At this point we observe that the generating function vanishes for \(z = v\), \(\hat{R}_G = 0\). It is thus evident that the derivative formula \((36)\) only produces contributions due to the second term. Lastly, by using \(\partial_z \hat{i}_{\alpha} = n z \hat{i}_{\alpha-1}\) and \(\partial_v \hat{j}_{\beta} = -n \hat{j}_{\beta+1}\), we recover the well-known formula
\[
\rho_G = \frac{n}{N \pi} e^{-n|z|^2} \sum_{k=0}^{N-1} \frac{(n|z|^2)^k}{k!}, \quad (41)
\]
for the spectral density, which often appears for \(n = N\).

C. Non-normal rank-1 \(S\) and \(L = R = 1\)

A major reason to study models of the type \((1)\) is the issue of spectral stability. — How far do the eigenvalues of \(S + Y\) spread around the eigenvalues of \(S\) for a small perturbation \(Y\). This is especially interesting for finite rank sources \(S\) where extremal (or outlier) eigenvalues emerge from the eigenvalue sea of the matrix \(Y\). This phenomenon was studied in a Hermitian \([6, 13, 33]\) as well as a non-Hermitian \([32, 36, 37]\) setting. Here, we examine how the normal or non-normal character of the source influences the eigenvalue distribution. We consider a rank-one source of the form
\[
S = \alpha |m\rangle \langle m|, \quad (42)
\]
for complex parameter \(\alpha\) and bras (kets) \(|m\rangle \langle m|\) denoting the canonical matrix basis – the source matrix \(S\) has one non-zero element \(\alpha\) placed on the off-diagonal. For the sake of simplicity we choose the trivial variance structure \(L = R = 1\). After a fair amount of algebra we find the result
\[
\mathcal{R}_{NN} = R_0 + |\alpha|^2 R_1 + |\alpha|^4 R_2 + |\alpha|^6 R_3 + |\alpha|^8 R_4 \quad (43)
\]
for the generating function. The formulas for the \(R_i\)'s are lengthy and thus were explicitly given only in the App. B. Although the terms in Eq. \((43)\) turn out to lack structure, they are still assembled from the bosonic and fermionic building blocks similar to Eq. \((28)\),
\[
\hat{i}_{k,l}(z, w) = \frac{(-1)^k}{n^{k+l+1} z^{2k} w^{2l}} \int_0^\infty dp e^{-p} \rho_o (2 \sqrt{n \rho} |w|) \times (\rho + n|z|^2)^k (\rho + n k_+^z)^l, \quad (44)
\]
and Eq. \((29)\),
\[
j_{q,r}(v, u) = \frac{2}{\pi} e^{n|u|^2} \int_{-\infty}^\infty dg e^{-n g^2} K_0(2i n|u| |g|) \times (g^2 - |v|^2)^{q} (g^2 - k_+^v)^{-r} (g^2 - k_-^v)^{-r}, \quad (45)
\]
where \(k_+^z = \frac{1}{\sqrt{2}} \left(|\alpha|^2 + 2 |x|^2 \pm |\alpha| \sqrt{4 |x|^2 + |\alpha|^2} \right)\). By investigating the terms in each of the \(R_i\)'s, we find the conditions \(l = -1, 0, 1, k \geq 0\) and \(q + r \geq 1, r = 1, 2, 3\) for the indices of \(i_{k,l}\) and \(j_{q,r}\), respectively. We employ the same regularization steps as in Sec. III A obtain the generating function \(\mathcal{R}_{NN}\) and construct the regularized fermionic block
\[
\hat{g}_{k,0} = \frac{(-1)^k k^!}{i^{k+1}} \hat{i}_{k}, \quad (46)
\]
\[
\hat{g}_{k,1} = \hat{i}_{k+2,0} - |\alpha|^2 (\hat{i}_{k+1,0} + |z|^2 \hat{g}_{k,0}), \quad (47)
\]
\[
\hat{g}_{k,-1} = \frac{(-1)^k k^!}{(k_+^z - k_-^z) n^k} \times \sum_{l=0}^{k} \frac{(n|z|^2)^l}{l!} \left[ U_{1,1+l-k}(n k_-^z) - U_{1,1+l-k}(n k_+^z) \right], \quad (48)
\]
where \(\hat{i}_{G}\) is the Ginibre block of Eq. \((38)\) and \(k \geq 0\). We relegate the derivation of Eq. \((48)\) to the App. B. The bosonic block reads
\[
j_{q,r}(v, u) = \frac{2}{\pi} e^{n|u|^2} \int_{-\infty}^\infty dg e^{-n g^2} K_0(2i n|u| |g|) \times (g^2 - |v|^2)^{q} (g^2 - k_+^v)^{-r} (g^2 - k_-^v)^{-r}, \quad (49)
\]
where \(q \geq 0, r \geq 1\) and the contour \(\Gamma\) encircles both \(-n|v|^2\) and \(-n k_+^z\). Lastly we obtain the formulas for \(q = -1, -2\),
\[
\hat{j}_{-1,2} = \frac{1}{2} \left( \hat{j}_{0,0} - \hat{j}_{0,2} + |\alpha|^2 \hat{j}_{0,2} \right), \quad (50)
\]
\[
\hat{j}_{-1,3} = \frac{1}{2} \left( \hat{j}_{0,3} - \hat{j}_{0,3} + |\alpha|^2 \hat{j}_{0,3} \right), \quad (51)
\]
\[
\hat{j}_{-2,3} = \frac{1}{4} \left( \hat{j}_{0,3} + \hat{j}_{0,3} + \hat{j}_{0,3} + |\alpha|^2 \hat{j}_{0,3} + 2 |\alpha|^2 \hat{j}_{0,3} \right), \quad (52)
\]
where the subscripts ± indicate that the underlying multiplicity vector \(\tilde{x} = (q, r-1, r)\) is applied with decrement to the source at \(nk_+^z\).

Finally, we obtain the spectral density \([11]\) analytically and plot it in Fig. 3. To facilitate a comparison, we juxtapose it with the analogous results for the case of a rank-one normal source \(S\) and for the Ginibre case \([11]\). A non-normal source \(S\) (third row in Fig. 3) does not produce, on average, outlier eigenvalues in the spectrum, in contrast normal source \(S\) (second row in Fig. 3) where we find an island around \(\alpha = 10\). Instead, in the non-normal case we observe something like a blow-up of the spectral bulk. The first row in Fig. 3 is devoted to the case of a vanishing source, \(S = 0\). Near \(z = 0\) both,
normal and vanishing source, produce similarly shaped spectral densities — the only difference between these cases is the presence or absence of the finite–rank island.

FIG. 3. Left hand side: complex plane of eigenvalues, from top to bottom for: unperturbed $S = 0$ (Ginibre), normal perturbation $S = 10|1\rangle\langle 1|$ and non–normal perturbation $S = 10|2\rangle\langle 1|$. Right hand side: numerical simulations and analytical results for the spectral densities $\rho_G, \rho_N$ and $\rho_{NN}$ along the real axis line (dashed lines on the left hand side). Numerical simulations are for matrices of size $N = 4$, $\alpha = 10$, we set $n = N$.

D. Spectrum of $M^{-1}$

As a last application we discuss how to infer somewhat gratuitously the spectrum of $(S + X)^{-1}$ from the results for the spectrum of $S + X$. For simplicity we deal with a normal source $S$ only and set $L = R = 1$. To this end we define a generating function $\mathcal{R}_{-1}$ for the inverse as

$$\mathcal{R}_{-1}(Z, V) = \frac{\det(Z - M_{-1})}{\det(V - M_{-1})} = \frac{\det Z}{\det V} \mathcal{R}_{1,1}(Z', V'),$$

and relate it to the generating function (53) previously considered. The matrices $M_{-1} = \left(\begin{array}{cc} 0 & M^{-1} \\ M^{-1} & 0 \end{array}\right)$ and $Z', V'$ are rearranged versions of the inverse matrices $Z^{-1}, V^{-1}$ of Eq. (6),

$$X' = \left(\begin{array}{cc} (X^{-1})_{22} & (X^{-1})_{21} \\ (X^{-1})_{12} & (X^{-1})_{11} \end{array}\right), \quad X = Z, V. \quad (54)$$

We thus conclude that the whole calculation discussed in Sec. IIIA can be repeated with only making the replacements $w \to -wG_{zw}, z \to zG_{zw}, u \to -uG_{wu}$ and $v \to vG_{vw}$ with $G_{xy} = (|x|^2 + |y|^2)^{-1}$. We again conduct the regularization procedure and eventually find that only the source matrix of Eq. (26) is modified according to

$$\alpha_{xy} \to (\alpha^{-1})_{xy} = \alpha_{x^{-1}y^{-1}} = (x^{-1}1_N - S^\dagger(y^{-1}1_N - S),$$

The regularized ratio for the problem of finding the spectrum of $(S + X)^{-1}$ reads

$$\tilde{R}_{-1} = \left(\frac{|z|^2}{|v|^2}\right) \tilde{R}_{1,1} [\alpha_{xy} \to (\alpha^{-1})_{xy}], \quad (55)$$

where the generating function $\tilde{R}_{1,1}$ is that of Eq. (33) and the constituent fermionic and bosonic blocks (34) are affected accordingly. In particular, we calculate the spectral density for an inverse matrix $X^{-1}$ as

$$\rho_{G, -1} = \frac{n!}{N\pi|z|^4} \sum_{k=0}^{N-1} \frac{1}{(k)!} \left(\frac{n}{|z|^2}\right)^k, \quad (56)$$

obtained from Eq. (40). This formula was also found in a recent work on the product of matrices [2]. In Fig. 4, the spectral density of $(S + X)^{-1}$ is depicted as calculated from the generating function (55) for non–zero external source $S$.

IV. CONCLUSIONS

We have calculated exact spectral densities for a class of complex random matrix models of the form $M = S + LX\mathcal{R}$ consisting of a noise part $X$ and structure parts
We found two-fold integral formulas for arbitrary structural matrices. In greater detail, we investigated the case of a normal source matrix $S$ and arbitrary diagonal matrices $L,R$ which are of particular interest. The resulting formulas are of a remarkably succinct form. We confirmed our analytical results by numerical simulations.

We showed how the presence or absence of the normality condition for $S$ leads to a qualitatively different behaviour of the eigenvalue densities. Our study was focused mainly on the finite rank source matrices where analytical solutions proved tractable. For a non-normal source, the most interesting feature is the lack of outliers, i.e., extreme values in the averaged spectral density. However, when imposing the normality condition on the source matrix $S$, the outliers are clearly present in the spectral density.

Lastly, we looked at the problem of finding spectra of an inverse matrix $M^{-1}$ which, by using the approach in this paper, proved to be trivially connected to the spectrum of $M$.

Among the open problems in the context of our study, the question remains on whether the normal vs. non-normal dichotomy has any counterpart relevant for applications. Secondly, the information on eigenvectors is encoded in the objects of study but was, due to the approach taken, completely omitted in our present work. Thirdly, issues related to universality seem feasible within our approach and are certainly worth future investigation.

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Appendix A: Derivation of (30)

We start from equation (29):

$$j_{\vec{m}}(v,u) = \frac{2in}{\pi} \prod_{i=1}^{k} (m_i - 1)! \cdot e^{n|u|^2} J_{\vec{m}}(v,u),$$

$$J_{\vec{m}}(v,u) = \int_{-\infty}^{\infty} dg_\varepsilon e^{-ng_\varepsilon^2} K_0(2in|u|g_-) \prod_{i=1}^{k} (g_i^2 - \alpha_{vi}^2)^{-m_i},$$

where we set $d(\vec{m}) = k$ for brevity. By Lagrange interpolation formula we find:

$$\prod_{i=1}^{k} (g_i^2 - \alpha_{vi}^2)^{-m_i} = \lim_{\gamma_1,...,\gamma_k \to 1} \sum_{i=1}^{k} D_i (g_i^2 - \gamma_i \alpha_{vi}^2)^{-1},$$

with the operator $D_i$ defined as

$$D_i = \prod_{i=1}^{k} (\alpha_{vi}^2)^{-m_i} \frac{d^{m_i-1}}{d\gamma_i^{m_i-1}} \prod_{j=1(j\neq i)}^{k} (\gamma_i \alpha_{vj}^2 - \gamma_j \alpha_{vi}^2)^{-1},$$

So that the whole integral $J_{\vec{m}}$ is expressed as

$$J_{\vec{m}} = \lim_{\gamma_1,...,\gamma_k \to 1} \sum_{l=1}^{k} D_l C_l.$$ (A3)

From now on we focus on the integral $C_l$:

$$C_l = \int_{0}^{\infty} dg_\varepsilon g \cdot e^{-ng_\varepsilon^2} K_0(2ni|u|g_-) = \int_{-\infty}^{\infty} ds \exp(-2ni|u|g_- \cosh s)$$

We re-introduce the representation $K_0(2ni|u|g_-) = \int_{0}^{\infty} ds \exp(-2ni|u|g_- \cosh s)$ and compute:

$$C_l = \frac{1}{2\sqrt{\gamma_i \alpha_{vi}^2}} \int_{0}^{\infty} ds \left(I_+(s) - I_-(s)\right),$$

$$I_{\pm}(s) = \int_{-\infty}^{\infty} dg \frac{f(g_{\pm} - s)}{g - (\pm \sqrt{\gamma_i \alpha_{vi}^2} + \mathrm{i}e)},$$

with $f(x,s) = xe^{-nx^2 - 2ni|u|x \cosh s}$. The integrals $I_{\pm}$ are calculable by Sokhotski–Plemelj formulas:

$$I_{\pm}(s) = i\pi f(\pm \sqrt{\gamma_i \alpha_{vi}^2},s) + \mathrm{PV} \int_{-\infty}^{\infty} \frac{dx f(x,s)}{x - (\pm \sqrt{\gamma_i \alpha_{vi}^2})}.$$ (A7)

The second part is the Hilbert transform [28]:

$$\frac{1}{\pi} \mathrm{PV} \int_{-\infty}^{\infty} \frac{dy e^{-ay^2 - by}}{y - x} = \frac{1}{\sqrt{a \pi}} e^{b^2/4a} + ixe^{-x^2a - bx} \text{erf} \left( \frac{i}{2\sqrt{a}} (b + 2ax) \right).$$ (A8)

Lastly, we need the identity:

$$\int_{x}^{\infty} dt e^{-at^2 - b^2/t^2} =$$

$$\frac{\sqrt{\pi}}{4a} \left( e^{2ab} \text{erfc}(ax + b/x) + e^{-2ab} \text{erfc}(ax - b/x) \right),$$

valid for $x > 0$. Combining the formulas of (A7)-(A9) result in

$$C_l = 2i\sqrt{\pi n|u|e^{-n\alpha_{vi}^2} \times}$$

$$\times \int_{0}^{\infty} ds \int_{1}^{\infty} dt \cosh s e^{\frac{n\alpha_{vi}^2 - \gamma^2 - n|u|^2 t^2 \cosh^2 s}}.$$ (A10)
In the next step we integrate over $s$ and change variables $t^2 = \tau + 1$:

$$C_i = \frac{i\pi}{2} \int_0^\infty \frac{d\tau}{\tau + 1} e^{-n|u|^2\tau} \frac{1}{\Gamma_+} \frac{1}{p + \sigma_+} \prod_{i=1}^n (q - \alpha_{i+}^+) m_i,$$

where the contour $\Gamma_+$ encircles all $\alpha_{i+}^+$'s counter-clockwise. This formula is a part of (??) which, after changing $p = -nq$ is equal to:

$$J_{\tilde{\omega}} = \frac{i\pi}{2} (-n) n|u|^2 e^{-n|u|^2} \times$$

$$\frac{1}{2\pi i} \int_0^\infty \frac{d\tau}{\tau + 1} e^{-n|u|^2 \tau} + \frac{\pi^\mp}{\sqrt{2\pi n}}$$

with appropriately modified contour $\Gamma_+$. Lastly, we use an integral representation of the Tricomi confluent hypergeometric function:

$$\int_0^\infty \frac{d\tau}{\tau + 1} e^{-n|u|^2 \tau} + \frac{\pi^\mp}{\sqrt{2\pi n}} = \sum_{k=0}^\infty U_{k+1,1}(n|u|^2)p^k,$$

and combine it with (A1) and (A12):

$$j_{\tilde{\omega}} = \frac{\prod_{i=1}^n |m_i - 1|}{2\pi i} \int_{\Gamma_+} \frac{dp}{p} \sum_{k=0}^\infty U_{k+1,1}(n|u|^2)p^k$$

which is exactly the formula (30).

Appendix B: Details of non-normal S case

The ratio for non-normal case is given by (43) with $R_4$ terms:

$$R_0 = 2(V_{iN-3,1}+Z_{iN-1,jN-3,2}) + 6(V_{iN-1,0,jN-4,3}+Z_{iN-4,1,jN-1,1}) - 4V_{jN-2,2}\sigma_+^+ - 4Z_{iN-2,0}\sigma_+^+ +$$

$$N^2 [jN-1,1\Delta_{N-3}^z + V_{iN-3,1}J_{1N,1}] + nd_1 [(N-2)jN-1,1iN-3,1 + 2iN-1,0jN-3,2] - n^2iN-2,1jN-2,1 +$$

$$N [2V_{jN-2,2}\sigma_+^+ - 2Z_{jN-1,1}\sigma_+^+ + 2jN-3,2\Delta_{N-1,0}\sigma_+^+ - 2jN\Delta_{N-1,1}\sigma_+^+ - Z_{jN-1,1}\sigma_+^+ - Z_{iN-4,1}\sigma_+^+ - 3ViN-3,1J_{1N,1}],$$

$$R_i = -\frac{N}{2} [\Delta_{N-2,0} \sigma_+^+ + 2\Delta_{N-2,0} \sigma_+^+] + n[2\Delta_{N-1,0} \sigma_+^+ + 2d_2iN-2,0jN-2,2] + iN-1,0(2V_{jN-1,2} + 3jN-4,3) +$$

$$+ d_1 [2NjN-2,2\Delta_{N-2,0} + iN-2,0(4V_{jN-3,3} - NjN-2,2)] + iN-2,0jN-4,3 - iN-1,jN-2,2 + V(N-2)iN-2,0jN-1,2 +$$

$$- Z(N+2)iN-3,0jN-2,2] + 2V_{jN-4,3}\delta_3 - Z_{iN-1}\sigma_2 - 2jN\Delta_{N-2,0} - 2jN\Delta_{N-2,1} +$$

$$+ 2ViN-1,0\Delta_{N-3,3} + jN-3,2(2Z_{iN-3,0} + iN-1) - (V_{iN-2,0} - Z_{iN-1,1})jN-4,3,$$

$$R_2 = d_1 [\Delta_{N-2,0} J_{N-2,1} + (iN-2,0 - 2iN-1,1)\Delta_{N-2,2}^j] + 2(N-2)\Delta_{N-2,0}\Delta_{N-2,2} + V_{iN-2,0}\Delta_{N-3,3} +$$

$$- 2(Z + V)jN-4,3\Delta_{N-2,1} - iN-1,0\Delta_{N-3,3} + jN-2,3\sigma_2 + 2d_1\Delta_{N-3,3},$$

$$R_3 = -\delta_3\Delta_{N-3,3} + \Delta_{N-2,2} [\sigma_2 + 2d_1\Delta_{N-3,3}],$$

$$R_4 = \Delta_{N-2,0,1}\Delta_{N-3,3},$$

where $V = |v|^2$, $Z = |z|^2$, $d_1 = \bar{v}v + \bar{z}z$, $d_2 = (\bar{v}v)^2 + (\bar{z}z)^2$ and the notation reads

$$\delta_1 = iN-1,0 \pm iN-3,1, \quad \delta_2 = iN-4,1 - iN-2,0, \quad \delta_3 = iN-1,1 - iN-2,0, \quad \delta_4 = jN-3,2 \pm jN-1,1, \quad \sigma_2 = jN-2,2 - jN-4,3,$$

$$\Delta_{x,y}^z = ix,y + zix-1,y, \quad \Sigma_{x,y}^z = jx,y + zjx+1,y.$$

Now we turn to the calculation of regularized bosonic block $i_{k,-1}$ of (48). We start from the definition (44):

$$i_{k,-1} = \frac{(-1)^k}{n^{k-1}} \frac{e^{-n|u|^2}}{\sqrt{2\pi n}} \times \int_0^\infty dp e^{-p} I_0(2\sqrt{n|u|^2}) \frac{(\rho + n|z|^2)^k}{\rho + nk_2^+ (\rho + nk_2^+)}.$$
Firstly, we express the denominator as an integral:

\[
\frac{1}{(\rho + n k_+)(\rho + n k_-)} = \frac{1}{2n\delta k} \int_0^\infty d\rho e^{-p\rho - pnk_0} \sinh(pn\delta k),
\]

with \(k_\pm = k_0 \pm \delta k\). We consider the integral:

\[
\mathcal{I}(p) = \int_0^\infty d\rho e^{-(1+p)\rho}(\rho + n|z|^2)^k I_0(2\sqrt{n|p|}|w|) = e^{\frac{n|w|^2}{p^2}} \left( \frac{n|z|^2}{(p + 1)^{k-l}} \right) L_l \left( \frac{n|w|^2}{p + 1} \right),
\]

and obtain the formula for \(i_{k,-1}\):

\[
i_{k,-1} = \frac{(-1)^k}{2n^k\delta k} e^{-n|w|^2} \int_0^\infty d\rho e^{-pnk_0} \sinh(pn\delta k) \mathcal{I}(p).
\]

It gets simplified in the regularization \(w \to 0\) limit:

\[
i_{k,-1} = \frac{(-1)^k k!}{2n^k\delta k} \times \sum_{l=0}^{k} \left( \frac{n|z|^2}{l!} \right) \left[ U_{1,1+l-k}(nk_-^+) - U_{1,1+l-k}(nk_+^-) \right],
\]

thus reproducing the equation \([18]\).

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