ON THE WEIGHTED FORWARD REDUCED ENTROPY OF RICCI FLOW

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ABSTRACT. In this paper, we introduce the weighted forward reduced volume of Ricci flow. The weighted forward reduced volume, which related to expanders of Ricci flow, is well-defined on noncompact manifolds and monotone non-increasing under Ricci flow. Moreover, we show that, just the same as the Perelman’s reduced volume, the weighted reduced volume entropy has the value $(4\pi)^n$ if and only if the Ricci flow is the trivial flow on flat Euclidean space.

1. Introduction

In [9], G.Perelman introduced the reduced entropy (i.e. reduced distance and reduced volume), which becomes one of powerful tools for studying Ricci flow. The reduced entropy enjoys very nice analytic and geometric properties, including in particular the monotonicity of the reduced volume. These properties can be used, as demonstrated by Perelman, to show the limit of the suitable rescaled Ricci flows is a gradient shrinking soliton.

Then M.Feldman, T.Ilmanen, L.Ni [3] observed that there is a dual version of G.Perelman’s reduced entropy, which related to the expanders of Ricci flow. Let $g(t)$ solves the Ricci flow

$$\frac{\partial g}{\partial t} = -2Rc.$$ 

(1.1)
on $M \times [0, T]$. Fix $x \in M^n$ and let $\gamma$ be a path $(x(\eta), \eta)$ joining $(x, 0)$ and $(y, t)$. They define the forward $L^+_+\text{-length}$ as

$$L_+(\gamma) = \int_0^t \sqrt{\eta}(R(\gamma(\eta)) + |\gamma'(\eta)|^2) d\eta.$$ 

(1.2)

Denote $L_+(y, t)$ be the length of a shortest forward $L^+_+\text{-length}$ joining $(x, 0)$ and $(y, t)$. Let

$$l_+(y, t) = \frac{L_+(y, t)}{2 \sqrt{t}}$$ 

(1.3)

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be the forward $l_+$-length. Note that the forward reduced distance (1.3) is defined under the forward Ricci flow (1.1), which is the only difference from Perelman’s reduced distance defined under the backward Ricci flow. The forward reduced volume is defined in [3] as

$$\theta_+(t) = \int_M (t)^{-\frac{3}{2}} e^{l_+(y,t)} dvol(y). \quad (1.4)$$

They also proved forward reduced volume defined in (1.4) is monotone non-increasing along the Ricci flow (1.1).

Unfortunately, the forward reduced volume defined in (1.4) may not well-defined on noncompact manifolds. In the first part of this paper, we introduce the weighted forward reduced volume based on the work in [3] and [9]. The weighted forward reduced volume is well-defined on noncompact manifolds and monotone non-increasing under the Ricci flow (1.1). Moreover, we show that, just the same as the Perelman’s reduced volume, the weighted reduced volume entropy has the value $(4\pi)^2 n^2$ if and only if the Ricci flow is the trivial flow on flat Euclidean space.

We define the weighted forward reduced volume as follows. First, we define the forward $L_+$-exponential map $L_+ \exp(V,t) : T_x M \to M$ at time $t \in [0, T)$. For $V \in T_x M$, let $\gamma_V$ denote the $L_+$-geodesic such that $\gamma_V(0) = p$, $\lim_{t \to 0} \sqrt{t} \gamma_V'(t) = V$. If $\gamma_V$ exists on $[0, t]$, we set

$$L_+ \exp(V,t) = \gamma_V(t). \quad (1.5)$$

Denote $\tau_V$ be the first time the $L_+$-geodesic $\gamma_V$ stop minimizing. Define

$$\Omega(t) = \{ V \in T_x M^n : \tau_V > t \}.$$

Obviously, $\Omega(t_2) \subset \Omega(t_1)$ if $t_1 < t_2$. Let $J^V_i(t), i = 1, \cdots, n$, be $L_+$-Jacobi fields along $\gamma_V(t)$ with

$$J^V_i(0) = 0, (\nabla_V J^V_i)(0) = E^0_i, \quad (1.6)$$

where $\{E^0_i\}_{i=1}^n$ is an orthonormal basis for $T_x M$ with respect to $g(0)$. Then $D(L_+ \exp(V,t))(E^0_i) = J^V_i(t)$. We define

$$L_+J^V(t) = \sqrt{\det(<J^V_i(t), J^V_j(t)>)} \quad (1.7)$$

and the weighted forward reduced volume as

$$\tilde{V}_+(t) = \int_{\Omega(t)} t^{-\frac{3}{2}} e^{l_+(\gamma_V(0),t)} e^{-2\|\gamma_V\|^2_{g(0)}} L_+J^V(t) d_{x_{g(0)}}(V), \quad (1.8)$$

where $d_{x_{g(0)}}$ is the standard Euclidean volume form on $(T_x M, g(x, 0))$, i.e. we define the weighted forward reduced volume as

$$\tilde{V}_+(t) = \int_{\Omega(t)} t^{-\frac{3}{2}} e^{l_+(\gamma_V(0),t)} e^{-2L_+ \exp^{-1}(y,t)_{g(0)}(V)} dvol(y), \quad (1.9)$$
We use the convention
\[ \mathcal{L}_+ J_V(t) = 0 \text{ for } t \geq \tau_V. \]
Then we can write the weighted forward reduced volume as
\[ \tilde{V}_+(t) = \int_{T_xM} t^{-\frac{d}{2}} e^{\mathcal{L}_+ J_V(t)} \mathcal{L}_+ J_V(t) e^{-2|V|_{g_0}^2} dx_{g(0)}(V). \]  
(1.10)

We remark that the density of forward reduced volume (1.4) (i.e. \( (t^{-\frac{d}{2}} e^{\mathcal{L}_+ J_V(t)} dv_{\text{vol}}(y)) \)) is not pointwise monotone non-increasing under the Ricci flow (1.1). So it is not easy for us to add the weighted term to the forward reduced volume such that it could be defined on noncompact manifolds. In order to overcome this problem, we employ the G.Perelman’s technique [9] that we pull the density of forward reduced volume back to the tangent space with the \( \mathcal{L}_+ \)-exponential map (here we define the \( \mathcal{L}_+ \)-exponential map as the similar way to [9]). Then we prove that the forward reduced volume density
\[ d\mathcal{V}_+ = t^{-\frac{d}{2}} e^{\mathcal{L}_+ J_V(t)} \mathcal{L}_+ J_V(t) dx_{g(0)}(V), \]
is pointwise monotone non-increasing under the Ricci flow (1.1) with respect to \( V \in T_xM \) (see Theorem 2.3). Just notice that (see the proof of Theorem 1.1)
\[ \lim_{t \to 0^+} t^{-\frac{d}{2}} e^{\mathcal{L}_+ J_V(t)} \mathcal{L}_+ J_V(t) = 2^n e^{2|V|_{g(0)}^2}. \]
So we only need add the weighted term \( e^{-2|V|_{g(0)}^2} \) to the density of forward reduced volume, which guarantees that the weighted forward reduced volume we defined in (1.8) is well defined on noncompact manifolds at \( t = 0 \). Moreover, the the weighted forward reduced volume is monotone non-increasing under the Ricci flow (1.1) since we have
\[ \frac{d}{dt} (t^{-\frac{d}{2}} e^{\mathcal{L}_+ J_V(t)} e^{-2|V|_{g(0)}^2}) = e^{-2|V|_{g(0)}^2} \frac{d}{dt} (t^{-\frac{d}{2}} e^{\mathcal{L}_+ J_V(t)} \mathcal{L}_+ J_V(t)) \leq 0 \]
for \( V \in \Omega(t) \).

We exactly have the following properties for the weighted forward reduced volume.

**Theorem 1.1.** The weighted forward reduced volume defined in (1.8) is monotone non-increasing under the Ricci flow (1.1) and well-defined on complete noncompact manifolds. Moreover, \( \tilde{V}_+(t) \leq \lim_{t \to 0^+} \tilde{V}_+(t) \leq (4\pi)^\frac{n}{2} \)
for \( t > 0 \). If \( \tilde{V}_+(t_1) = \tilde{V}_+(t_2) \) for some \( 0 < t_1 < t_2 \), then this flow is a gradient expanding soliton on \( 0 \leq t < \infty \) and hence is the trivial flow on flat Euclidean space. In particular, if \( \tilde{V}_+(\bar{t}) = (4\pi)^\frac{n}{2} \) for some time \( \bar{t} > 0 \), then this flow is the trivial flow on flat Euclidean space.
We also have the following rescaling property for the weighted forward
reduced volume.

**Theorem 1.2.** We have
\[ \tilde{\mathcal{V}}_j'(t) = \tilde{\mathcal{V}}_\lambda^{-1}(t) \]
under the rescaling
\[ g_j(t) = \lambda g(\lambda^{-1} t) \]
where \( \tilde{\mathcal{V}}_j \) and \( \tilde{\mathcal{V}}_\lambda \) denote the weighted forward reduced volume with respect to metric
\( g_j \) and \( g \) respectively.

The organization of the paper is as follows. In section 2, we first recall
some basic formulas and properties about forward reduced entropy in [3].
Then we study the properties of forward reduced volume density which
defined by forward \( L^+ \)-exponential map. Finally, we give the proofs of
Theorem 1.1 and Theorem 1.2.

2. Weighted Forward Reduced Volume and Expanders

Before we present the proofs of Theorem 1.1 and Theorem 1.2, we recall
some basic formulas and properties about forward reduced entropy in [3].
Clearly, one can show that the \( l^+ \)-length \( l^+(y, t) \) is locally lipschitz function
and the cut-Locus of \( L^+ \exp(V, t) \) is a closed set of measure zero by using
the similar methods in [12].

We need the following two theorems due to M. Feldman, T. Ilmanen, L. Ni
[3], which state the following adapted form.

**Theorem 2.1.** [3] Let \( \gamma \) be a path \((x(\eta), \eta)\) joining \((x, 0)\) and \((y, t)\). Denote
\( L_+ \doteq L_+(y, t) \) be the forward \( L_+ \)-length joining \((x, 0)\) and \((y, t)\). Set \( X = \gamma'(t) \)
and \( Y \) be a variational vector along \( \gamma \) such that \( Y(0) = 0 \). The first variation
of \( L_+ \) is that
\[
\delta L_+ = 2 \sqrt{t} \langle X, Y \rangle (t) + \int_0^t \sqrt{\eta} < Y, \nabla R - 2 \nabla_X X + 4 \Rc(X, \cdot) - \frac{1}{\eta} X > d\eta. 
\]  
(2.1)

If \( \gamma(t) \) is the minimal \( L_+ \)-geodesic, then
\[
\nabla L_+ = 2 \sqrt{t} X, 
\]
(2.2)

\[
t^\frac{3}{2} (R + |X|^2) = K + \frac{1}{2} L_+, 
\]
(2.3)

where \( K = \int_0^t \eta H(X) d\eta \), \( H(X) = \frac{\partial R}{\partial t} + 2 < \nabla R, X > + 2 \Rc(X, X) + \frac{8}{t} \). The
second variation of \( L_+ \) is that
\[
\delta^2 L_+ = 2 \sqrt{t} \langle X, Y \rangle (t) + \int_0^t \sqrt{\eta} (Hess R(Y, Y) - 2 R(X, Y, X, Y) \\
+ 2 |\nabla_X Y|^2 + 4 \nabla_Y Rc(Y, X) - 2 \nabla_X Rc(Y, Y)) d\eta. 
\]
(2.4)
Let $\tilde{Y}$ be a vector field along $\gamma$ satisfies the ODE
\[
\begin{cases}
\nabla_X \tilde{Y}(\eta) = \text{Rc}(\tilde{Y}(\eta)), \cdot + \frac{1}{2\eta} \tilde{Y}(\eta), \eta \in [0, t] \\
\tilde{Y}(0) = Y(0) = 0.
\end{cases}
\] (2.5)

Then
\[
\text{Hess}L_+(\tilde{Y}, \tilde{Y}) \leq \frac{|\tilde{Y}|^2}{\sqrt{t}} + 2 \sqrt{t} \text{Rc}(\tilde{Y}, \tilde{Y}) - \int_0^t \sqrt{\eta} H(X, \tilde{Y}) d\eta, \quad (2.6)
\]
where $H(X, \tilde{Y}) = -\text{Hess}R(X, \tilde{Y}) + 2\text{R}(X, \tilde{Y}, X, \tilde{Y}) + 2|\text{Rc}(X, \cdot)|^2 + \frac{\text{Rc}(\tilde{Y}, \tilde{Y})}{t} + 2\frac{\partial \text{Rc}}{\partial t}(\tilde{Y}, \tilde{Y}) - 4\nabla_{\tilde{Y}} \text{Rc}(\tilde{Y}, X) + 4\nabla_X \text{Rc}(\tilde{Y}, \tilde{Y})$. The equality holds in (2.6) if and only if the vector field $\tilde{Y}$ satisfying (2.16) is an $L_+-$Jacobi field.

**Theorem 2.2.** [3] Let $l_+ \doteq l_+(y, t) = l_+(y, t)$ be the $l_+-$length from $(x, 0)$ to $(y, t)$. If $(y, t)$ is not in the cut-Locus of $L_+\exp$, then at $(y, t)$
\[
\frac{\partial l_+}{\partial t} = R - \frac{l_+}{t} - \frac{K}{2t^2},
\]
\[
|\nabla l_+|^2 = \frac{l_+}{t} - R + \frac{K}{t^2},
\]
\[
\Delta l_+ \leq R + \frac{n}{2t} - \frac{K}{2t^2},
\]
\[
\frac{\partial l_+}{\partial t} + \Delta l_+ + |\nabla l_+|^2 - R - \frac{n}{2t} \leq 0,
\]
\[
2\Delta l_+ + |\nabla l_+|^2 - R - \frac{l_+ + n}{t} \leq 0.
\]

We first study properties of the forward reduced volume density defined as
\[
d^\ast V_+ = l_+^{-\frac{n}{2}} e^{\int_{1(t)}^{t}(yV(t), t)} L_+ J_V(t) dx_{g(0)}(V),
\] (2.12)
where $L_+ J_V(t)$ defined in (1.7).

Note that the weighted forward reduced volume
\[
\tilde{V}_+(t) = \int_{T^m, M^n} e^{-2l_+^{-\frac{n}{2}}(yV(t), t)} d^\ast V_+.
\]

Analogous to [9], we have the following theorem.

**Theorem 2.3.** The forward reduced volume density $d^\ast V_+$ defined in (2.12) is monotone non-increasing along the Ricci flow (1.7). Moreover, if $d^\ast V_+(t_1) = d^\ast V_+(t_2)$ for some $0 < t_1 < t_2$, then this flow is a gradient expanding soliton.
Proof. Let $\gamma_V(t)$ be the minimal $L_+\text{-geodesic}$ defined in (1.5) and $y = \gamma_V(t)$. We consider $(y, t)$ in the cut-Locus of $L_+\exp(V, t)$. Recall that $\nabla l_+(y, t) = \gamma'_V(t) = X(t)$. Then by (2.3) and (2.7), we get
\[
\frac{\partial l_+(y_V(t), t)}{\partial t} = \frac{\partial l_+(y, t)}{\partial t} + \nabla l_+ \cdot X = \frac{R - l_+(y, t)}{t} - \frac{K}{2t^2} + |X|^2
\]
(2.13)

For any fixed $t$, we choose an orthonormal basis $\{E_i(t)\}$ of $T_{\gamma_V(t)}M$. We extend $E_i(\eta), \eta \in [0, t]$ to an $L_+\text{-Jacobi field}$ along $\gamma_V$ with $E_i(0) = 0$. We write $J^V_i(t) = \sum^n_i A^j_i E_j(t)$ for same matrix $(A^j_i) \in GL(n, \mathbb{R})$. Then
\[
J^V_i(\eta) = \sum^n_i A^j_i E_j(\eta) \quad \text{for all} \quad \eta \in [0, t].
\]

Hence, by (2.6), we calculate at time $t$
\[
\frac{d}{d\eta}|_{\eta=t} \ln \mathcal{L}_+ J^V = \frac{d}{d\eta}|_{\eta=t} \ln \sqrt{\det(< \sum^n_{k=1} A^k_i E_k, \sum^n_{l=1} A^l_j E_l>)}
\]
\[
= \frac{1}{2} \frac{d}{d\eta}|_{\eta=t} \sum_i |E_i|^2
\]
\[
= \sum_i (-Rc(E_i, E_i) + < \nabla_{E_i} X, E_i >)
\]
\[
= \sum_i (-Rc(E_i, E_i) + \frac{1}{2t} Hess_{L_+}(E_i, E_i)) (2.14)
\]
\[
\leq \sum_i \left( \frac{1}{2t} + \frac{1}{2t} \int_0^t \sqrt{\eta} H(X, \bar{E}_i) d\eta \right) (2.15)
\]

where $\bar{E}_i(\eta)$ are the vector fields along $\gamma_V$ satisfying
\[
\begin{cases}
\nabla_X \bar{E}_i(\eta) = Rc(\bar{E}_i(\eta), \cdot) + \frac{\eta}{2t} \bar{E}_i(\eta), \eta \in [0, t] \\
\bar{E}_i(t) = E_i(t),
\end{cases}
\]
(2.16)

which in particular implies that
\[
< \bar{E}_i, \bar{E}_j > (\eta) = \frac{\eta}{t} < E_i, E_j > (t) = \frac{\eta}{t} \delta_{ij},
\]
(2.17)

It follows that
\[
H(X, \bar{E}_i)(\eta) = \frac{\eta}{t} K.
\]
Hence
\[
\frac{d}{d\eta} \ln L_+ J_V \leq \frac{n}{2t} - \frac{1}{2} t^{-\frac{3}{2}} K,
\]
and
\[
\frac{d}{dt} \ln d'V_+ = -\frac{n}{2t} + \frac{\partial l}{\partial t} + \frac{d \ln L_+ J_V}{dt} \leq 0.
\tag{2.18}
\]
If equality in (2.18) holds, then we have equality in (2.15) holds. By Theorem 2.1, we conclude that each $\tilde{E}_i(\eta)$ is an $L_+$-Jacobi field. Hence
\[
\frac{d}{d\eta} \big|_{\eta=0} |E_i|^2 = \frac{d}{d\eta} \big|_{\eta=0} |\tilde{E}_i|^2 = \frac{|E_i(t)|^2}{t}.
\tag{2.19}
\]
Combining with (2.14) and (2.19), we get
\[
Rc(E_i, E_i) - \frac{1}{2} \sqrt{t} \text{Hess}_{L_+} (E_i, E_i) = -\frac{|E_i|^2}{2t}.
\]

□

Now we can give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Theorem 2.3, we know that
\[
\frac{d}{dt} (t - \frac{n}{2} e + (\gamma V(t), t) L_+ J_V) \leq 0
\]
for $V \in \Omega(t)$. It follows that
\[
\frac{d}{dt} (t - \frac{n}{2} e + (\gamma V(t), t) L_+ J_V) e^{-2|V|_{0}^2} \leq 0
\]
for $V \in \Omega(t)$. Moreover, since we have $\Omega(t_2) \subset \Omega(t_1)$,
\[
\tilde{V}_+(t_2) \leq \tilde{V}_+(t_1)
\]
for $t_1 < t_2$. We calculate that
\[
\lim_{t \to 0^+} l_+(\gamma_V(t), t) = \lim_{t \to 0^+} \frac{1}{2} \sqrt{t} \int_0^t \sqrt{t} (R(\gamma_V(\eta), \eta) + |\frac{d\gamma_V}{d\eta}|^2) d\eta
\]
\[
= \lim_{t \to 0^+} t (R(\gamma_V(t), t) + |\frac{d\gamma_V}{dt}|^2)
\]
\[
= |V|_{0}^2.
\]
Let $J_Y^i(t), i = 1, \cdots, n$, be $L_+$-Jacobi fields along $\gamma_V(t)$ with
\[
J_Y^i(0) = 0, (\nabla_V J_Y^i)(0) = E_i^0,
\tag{2.20}
\]
where \( \{E_i^0\}_{i=1}^n \) is an orthonormal basis for \( T_xM \) with respect to \( g(0) \). Since \( (\nabla V)J^V)(0) = E_i^0 \) and \( V = \lim_{t \to 0} \sqrt{t}V(t) \), we get
\[
\lim_{t \to 0^+} \frac{\mathcal{L}_t J_V(t)}{t^2} = \lim_{t \to 0^+} \frac{\sqrt{\det (<2\sqrt{t}E_i(t), 2\sqrt{t}E_j(t)>_g(t))}}{t^2} = 2^n,
\]
we conclude that
\[
\lim_{t \to 0^+} t^{-\frac{n}{2}} e^{\int_{\gamma(t)}^y \mathcal{L}_t J_V(t)} = 2^n e^{\|V\|^2_g}.
\]
Hence
\[
\lim_{t \to 0^+} \tilde{V}_+(t) \leq \int_{T_pM} 2^n e^{-\|V\|^2_g} dx(V) = (4\pi)^{\frac{n}{2}}.
\]
If \( \tilde{V}_+(t_1) = \tilde{V}_+(t_2) \) for any \( 0 < t_1 < t_2 \), then \( d\tilde{V}_+(t_1) = d\tilde{V}_+(t_2) \) for any \( 0 < t_1 < t_2 \). So \( (M^n, g(t)) \) must be a gradient expanding soliton by Theorem 2.3, i.e. we have
\[
Rc + Hess(-l_+) = -\frac{g}{2t}
\]
for some smooth function \( l_+ \) on \( M^n \). Let \( \phi_t : M \to M, 0 < t \leq \bar{t} \) be the one-parameter family of diffeomorphisms obtained by
\[
\frac{d\phi_t}{dt} = \nabla l_+ \quad \text{and} \quad \phi_{\bar{t}} = Id.
\]
We consider \( h(t) = \frac{\bar{t}}{t} \phi_t^* g(t) \) and calculate
\[
\frac{dh}{dt} = -\frac{\bar{t}}{t^2} \phi_t^* g(t) + \frac{\bar{t}}{t} \phi_t^* \mathcal{L}_t \phi_t^* g(t) - 2\frac{\bar{t}}{t} \phi_t^* Rc(g(t))
\]
\[
= -\frac{\bar{t}}{t^2} \phi_t^* g(t) + \frac{\bar{t}}{t} 2Hess(l_+) + \frac{\bar{t}}{t} \phi_t^* (\frac{g}{t} - 2Hess(l_+)) = 0.
\]
It follows that
\[
g(t) = \frac{t}{\bar{t}} (\phi_{\bar{t}}^{-1})^* g(\bar{t}).
\]
Suppose that there is some \( (y, \bar{t}) \) with \( |Rm|(y, \bar{t}) = K > 0 \), we have \( |Rm|(\phi_{\bar{t}}^{-1}(y), t) = \frac{Kt}{\bar{t}} \), and these curvatures are not bounded as \( t \to 0 \), which is a contradiction. Then we have
\[
Hess(l_+) = \frac{1}{2t^2} g.
\]
Thus \( l_+ \) is strictly convex function. The similar arguments to Lemma 2.3 in [12] can show that
\[
l_+(y, t) \geq e^{-2\alpha} \frac{d_g(0)(x, y)}{4t} - \frac{nc}{3t},
\]
if \( Rc \geq -cg \) on \([0, t]\), so that \( l_+(y, t) \) have the only minimum point in \( M^n \). Hence \( M^n \) is diffeomorphic to \( \mathbb{R}^n \).

Since \( \tilde{V}^+(t) \) is monotone non-increasing, \( \tilde{V}^+(t) \) is independent of \( t \) if \( \tilde{V}^+ (\hat{t}) = (4\pi)^{\frac{2}{n}} \) for some time \( \hat{t} > 0 \). Then we derive that \( M^n \) is isometric to \( \mathbb{R}^n \). \( \Box \)

Finally, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2** We denote \( \gamma^i_j(t) \) be the minimal \( L^+ \)-geodesic with respect to \( g_j(t) \) which starting from \((x^i_j, 0)\) and satisfying

\[
\lim_{t \to 0} \sqrt{t} d \gamma^i_j(V(t)) dt = V.
\]

We have that

\[
\int_{T : M^n} (t)^{-\frac{n}{2}} e^{\int_{g_j(0)} (\gamma^i_j(t), L^+_j \sqrt{V_j(t)})} dx_{g_j(0)}(V) = (\lambda_j^{-1} t)^{-\frac{n}{2}} L^+_j (\lambda_j^{-1} t) dx_{g_j(0)}(\sqrt{\lambda_j} V).
\]

Hence

\[
\tilde{V}^+_j(t) = \int_{T : M^n} (t)^{-\frac{n}{2}} e^{\int_{g_j(0)} (\gamma^i_j(t), L^+_j \sqrt{V_j(t)})} dx_{g_j(0)}(V) = \int_{T : M^n} (\lambda_j^{-1} t)^{-\frac{n}{2}} e^{\int_{g_j(0)} (\gamma^i_j(t), L^+_j \sqrt{V_j(t)})} dx_{g_j(0)}(\sqrt{\lambda_j} V)
\]

\[
\times L^+_j \sqrt{V_j} (\lambda_j^{-1} t) e^{-2|\sqrt{\lambda_j} V_j(t)|} dx_{g_j(0)}(\sqrt{\lambda_j} V) = \tilde{V}^+_j(\lambda_j^{-1} t).
\]

\( \Box \)

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