Stacks of quantization-deformation modules on complex symplectic manifolds

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Abstract

On a complex symplectic manifold $\mathcal{X}$, we construct the stack of quantization-deformation modules, that is, (twisted) modules of microdifferential operators with an extra central parameter $\tau$, a substitute to the lack of homogeneity. We also quantize involutive submanifolds of contact manifolds.

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1 Introduction

Masaki Kashiwara [10] has constructed the stack of modules of microdifferential operators on a complex contact manifold \( \mathcal{Q} \). Let us explain briefly what this means. A local model for \( \mathcal{Q} \) is an open subset \( V \) of \( P^*X \), the projective cotangent bundle of a complex manifold \( X \). The manifold \( P^*X \) is endowed with the sheaf of rings \( \mathcal{E}_X \) of microdifferential operators of Sato-Kawai-Kashiwara [17]. Twisting this ring by half-densities on \( X \), we find another ring \( \mathcal{E}_\sqrt{v}X \) with an extra property: namely, it is endowed with an anti-involution \( * \). If \( \psi : P^*X \supset V_X \xrightarrow{\sim} V_Y \subset P^*Y \) is a contact transformation, one can locally “quantize” it as a ring isomorphism \( \Psi : \psi^*(\mathcal{E}_\sqrt{v}X|_{V_X}) \xrightarrow{\sim} (\mathcal{E}_\sqrt{v}Y|_{V_Y}) \) commuting with the anti-involution \( * \). This quantization is not unique and this fact makes it impossible to glue together the sheaves \( \mathcal{E}_\sqrt{v}X|_{V} \)'s and to get a globally defined sheaf of rings on \( \mathcal{Q} \). However, and this is the content of Kashiwara’s paper, it is possible to glue together the categories of abelian sheaves \( \text{Mod}(\mathcal{E}_\sqrt{v}X|_{V}) \) and to get a canonically defined stack (roughly speaking, a sheaf of categories) on \( \mathcal{Q} \). The proof has two aspects, one analytical, which consists essentially in noticing that the automorphisms of the ring \( \mathcal{E}_\sqrt{v}X \) preserving \( * \) are in bijection with a subgroup of its invertible elements, the other one purely algebraic, dealing with the machinery of stacks.

In this paper, we first recall Kashiwara’s proof and extend it to the case of regular involutive submanifolds of \( \mathcal{Q} \). Then, and this is our main result, we adapt it to the case of symplectic manifolds. As we shall see, new difficulties appear.

The local model is now an open subset \( U \) of the cotangent bundle \( T^*X \) of a complex manifold \( X \). Let \( \mathbb{C} \) denote the complex line with holomorphic coordinate \( t \), let \( \hat{T}^*\mathbb{C} \) denote the cotangent bundle to \( \mathbb{C} \) with the zero-section removed and with coordinates \((t, \tau)\). Denote by \( \hat{P}^*(X \times \mathbb{C}) \) the quotient of \( T^*X \times \hat{T}^*\mathbb{C} \) by the diagonal \( \mathbb{C}^* \)-action. Then \( \hat{P}^*(X \times \mathbb{C}) \) is an open subset of the projective cotangent bundle \( P^*(X \times \mathbb{C}) \) and there is a natural map \( \rho : \hat{P}^*(X \times \mathbb{C}) \to T^*X, (p, (t; \tau)) \mapsto pr^{-1} \). The manifold \( \hat{P}^*(X \times \mathbb{C}) \) is endowed with the sheaf of rings \( \mathcal{E}_{X \times \mathbb{C}, i} \) of microdifferential operators on \( P^*(X \times \mathbb{C}) \) which commute with \( D_i \) (i.e., which do not depend on the \( t \)-variable). We endow \( T^*X \) with the sheaf of rings \( \mathcal{W}_X := \rho_*\mathcal{E}_{X \times \mathbb{C}, i} \). Roughly speaking, \( \mathcal{W}_X \) is the sheaf of microdifferential operators in the \((x, D_x)\)-variables and a central extra parameter \( \tau \) of order 1, which kills the homogeneity of \( T^*X \). Such algebras in the formal case over real symplectic manifolds are called semi-classical star-algebras, or also, quantization-deformation algebras by
many authors and we refer to [2] for their study. Note that the link between the sheaves of rings \(E_X\) and \(W_X\) is well-known from the specialists and appears explicitly when \(\text{dim} X = 1\) in [4]. By reference to the WKB-method of the physicists, these authors call WKB-differential operators the sections of \(W_X\) and we shall follow this terminology.

Denote by \(\hat{k} := \mathbb{C}[\tau, \tau^{-1}]\) the field of formal Laurent series \(\sum_{j \in \mathbb{Z}} a_j \tau^j\) with \(a_j = 0\) for \(j \gg 0\), and by \(k\) the field \(W_{pt}\), a subfield of \(\hat{k}\) (see Definition 8.4).

The sheaf \(W_X\) is thus a sheaf of \(kT^*_X\)-central algebras and the center of \(W_X\) is now too large in order to apply Kashiwara’s method. We overcome this difficulty by showing that above a symplectic transformation \(\varphi : T^*X \supset U_X \simrightarrow U_Y \subset T^*Y\), there exists locally a contact transformation \(\psi : \hat{P}^*(X \times \mathbb{C}) \supset \rho^{-1}(U_X) \simrightarrow \rho^{-1}(U_Y) \subset \hat{P}^*(Y \times \mathbb{C})\) commuting with \(\tau\) and that this transformation may be quantized as an isomorphism of rings \(\Phi : \rho_*\psi_*(\mathcal{E}_{X \times \mathbb{C}}|_{\rho^{-1}(U_X)}) \simrightarrow \rho_*(\mathcal{E}_{Y \times \mathbb{C}}|_{\rho^{-1}(U_Y)})\), this isomorphism \(\Phi\) commuting with \(D_t\), hence interchanging \(W^\sqrt{\nu}_X\) and \(W^\sqrt{\nu}_Y\). In general, these isomorphisms do not allow us to glue the categories \(\text{Mod}(\rho_*\mathcal{E}_{X \times \mathbb{C}}|_{\rho^{-1}(U_X)})\) and to obtain a stack, due to a kind of translation operator which appears in the fibers of \(\rho\). But, fortunately, these translations act trivially on \(W^\sqrt{\nu}_X\), and we can glue the categories \(\text{Mod}(W^\sqrt{\nu}_X)\).

Note that Maxim Kontsevich [14] has recently announced a similar result in the much more general setting of (algebraic) Poisson manifolds, based on a different method. Also note that semi-classical star-algebras on complex symplectic manifolds are constructed under suitable hypotheses in [16], using Fedosov connections. We refer to [5] for a discussion on possible physical applications of such constructions.

The contents of this paper is as follows.

In Section 3 we recall some facts (well known from a few specialists) concerning the construction of stacks. References are made to [8], [3], [10], [13].

In Section 4 we explain how to construct \(\tau\)-preserving contact isomorphisms associated with symplectic isomorphisms.

In Section 5 we overview the theory of Sato-Kawai-Kashiwara [17] of microdifferential operators and modules over such rings (see also [18], [11] for a detailed exposition).

In Section 6 we give a detailed proof of Kashiwara’s quantization theorem of [10].

In Section 7 we treat the case of involutive submanifolds of complex
contact manifolds.

In Section 8 we construct the ring \( \mathcal{W}_X \) of WKB-differential operators on the cotangent bundle to a complex manifold \( X \).

In Section 9 we adapt Kashiwara’s proof of [10] to construct the stack of WKB-modules on a complex symplectic manifold.

All our results extend to the formal case, that is, to the ring \( \widehat{\mathcal{W}} \) associated with the ring of formal microdifferential operators \( \widehat{\mathcal{E}} \).

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2 Notations

We will mainly follow the notations of [12]. In this paper, unless otherwise specified, all manifolds are complex analytic.

Let \( X \) be a manifold. We denote by \( T^*X \) the cotangent bundle, \( \pi : T^*X \to X \) the projection, \( \tilde{T}^*X \) the bundle \( T^*X \setminus X \), where \( X \) is identified with the zero-section, \( a : T^*X \to T^*X \) the antipodal map. We shall also consider the projective cotangent bundle, \( P^*X = \tilde{T}^*X / \mathbb{C}^\times \), (\( \mathbb{C}^\times \) denotes the multiplicative group of non-zero complex numbers). We keep the notation \( \pi \) for the projection \( P^*X \to X \).

On a complex manifold \( X \), we consider the structure sheaf \( \mathcal{O}_X \), the sheaves \( \Omega^p_X \) of holomorphic \( p \)-forms and the sheaf \( \mathcal{D}_X \) of linear holomorphic differential operators of finite order. One sets \( \Omega_X := \Omega^n_X \), where \( n \) is the dimension of \( X \).

This paper deals with stacks, which roughly speaking, means sheaves of categories. The classical reference is [8] and a more popular one is [6]. This notion is also well explained in [10].

Recall that if \( \mathcal{R} \) is a sheaf of unital rings on \( X \), one denotes by \( \mathcal{R}^\times \) the sheaf of invertible sections, and if \( a \in \mathcal{R}^\times \) is a local section, one defines the ring automorphism \( \text{Ad}(a) \) of \( \mathcal{R} \) by setting for any local section \( b \in \mathcal{R} \)

\[
\text{Ad}(a)(b) = aba^{-1}.
\]

By an \( \mathcal{R} \)-module (resp., \( \mathcal{R}^{\text{op}} \)-module), we mean a sheaf of left (resp., right) \( \mathcal{R} \)-modules. We denote by \( \text{Mod}(\mathcal{R}) \) the abelian category of \( \mathcal{R} \)-modules and by \( \mathcal{M}\text{Mod}(\mathcal{R}) \) the corresponding abelian stack on \( X \) given by the assignement \( X \supset U \mapsto \text{Mod}(\mathcal{R} |_U) \).
3 Construction of stacks and functors

Let $X$ be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$. Define:

$$X_0 = \bigsqcup_{i \in I} U_i, \quad X_1 = \bigsqcup_{i,j \in I} U_{ij}, \quad X_2 = \bigsqcup_{i,j,k \in I} U_{ijk}, \text{ etc.}$$

where $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, etc. In other words,

$$X_1 := X_0 \times X_0, \quad X_n = X_0 \times X \cdots \times X (n + 1\text{-times}).$$

We introduce the projections

$$p^0 : X_n \to X, \quad 0 \leq n,$$

$$p_i^n : X_n \to X_0, \quad 0 \leq i \leq n,$$

$$p_{ij}^n : X_n \to X_1, \quad 0 \leq i < j \leq n,$$

$$p_{ijk}^n : X_n \to X_2, \quad 0 \leq i < j < k \leq n, \text{ etc.}$$

For example, $p_{ij}^n : X_n \to X_1$ is the projection to the $(i,j)$-factor. In the sequel, we set $p = p^0$ and, if there is no risk of confusion, we write $p_i$ instead of $p_i^n$, $p_{ij}$ instead of $p_{ij}^n$, etc. Hence we have the following diagram

\begin{equation}
(3.1) \quad \begin{array}{ccc}
X & \\
\downarrow & \downarrow & \\
X_1 & \rightarrow & X_0 \\
\downarrow & \downarrow & \\
X_2 & \rightarrow & X_0 \\
\downarrow & \downarrow & \\
X & \rightarrow & X \\
\end{array}
\end{equation}

**Notation 3.1.** (i) For a sheaf $\mathcal{F}$ or a stack $\mathcal{G}$ on $X_0$, we set $\mathcal{F}_i = p_i^{-1}\mathcal{F}$ and $\mathcal{G}_i = p_i^{-1}\mathcal{G}$ for $0 \leq i \leq n$, where $p_i : X_n \to X_0$, $n \geq 1$.

(ii) For a morphism of sheaves $f$ or a functor of stacks $F$ on $X_1$, we set $f_{ij} = p_{ij}^{-1}f$ and $F_{ij} = p_{ij}^{-1}F$ for $0 \leq i < j \leq n$, where $p_{ij} : X_n \to X_1$, $n \geq 2$.

(iii) For a section of a sheaf $a$ or an isomorphism of functors $\alpha$ on $X_2$, we set $a_{ijk} = p_{ijk}^{-1}a$ and $\alpha_{ijk} = p_{ijk}^{-1}\alpha$ for $0 \leq i < j < k \leq n$, where $p_{ijk} : X_n \to X_2$, $n \geq 3$.

Let $\kappa$ be a sheaf of commutative unital rings on $X$.

**Definition 3.2.** A $\kappa$-lien on $X_\bullet \to X$ is a triplet $\mathcal{R} = (\mathcal{R}, f, a)$, where $\mathcal{R}$ is a sheaf of central $p^{-1}\kappa$-algebras on $X_0$ (i.e., $p^{-1}\kappa$ is the center of $\mathcal{R}$), $f : \mathcal{R}_1 \xrightarrow{\sim} \mathcal{R}_0$ is an isomorphism of $\kappa_0$-algebras on $X_1$ and $a$ is section in $\Gamma(X_2, \mathcal{R}_0^\times)$, such that on $X_2$

\begin{equation}
(3.2) \quad f_{01} \circ f_{12} = \text{Ad}(a) \circ f_{02}.
\end{equation}
On $X_3$ one has
\[(f_{01} \circ f_{12}) \circ f_{23} = \text{Ad}(a_{012}) \circ f_{02} \circ f_{23} = \text{Ad}(a_{012}a_{023}) \circ f_{03},\]
and
\[f_{01} \circ (f_{12} \circ f_{23}) = f_{01} \circ \text{Ad}(a_{123}) \circ f_{13} = \text{Ad}(f_{01}(a_{123})) \circ f_{01} \circ f_{13} = \text{Ad}(f_{01}(a_{123})a_{013}) \circ f_{03}.\]

It follows that there exists a section $c \in \Gamma(X_3; \kappa^X)$, such that
\[
(3.3) \quad a_{012}a_{023} = f_{01}(a_{123})a_{013} \cdot c \quad \text{in} \quad \Gamma(X_3; \mathcal{R}_0^X).
\]

**Definition 3.3.** One says that a $\kappa$-lien $R = (\mathcal{R}, f, a)$ is effective if $c = 1$.

**Notation 3.4.** Let $R = (\mathcal{R}, f, a)$ be a $\kappa$-lien on $X_0$. We denote by $f^{-1}(\cdot)$ the equivalence of stacks $\mathcal{M}\text{od}(\mathcal{R}_1) \xrightarrow{\sim} \mathcal{M}\text{od}(\mathcal{R}_0)$ which associates to an $\mathcal{R}_1$-module $\mathcal{F}$ on $X_1$, the $\mathcal{R}_0$-module $f^{-1}\mathcal{F}$, i.e. the $p^{-1}\kappa$-module $\mathcal{F}$ endowed with the action induced by $f^{-1}: \mathcal{R}_0 \xrightarrow{\sim} \mathcal{R}_1$ : if $l \in \mathcal{F}$ and $r \in \mathcal{R}_0$ are local sections, the action of $r$ on $l$ is given by $(r, l) \mapsto f^{-1}(r)l$.

**Theorem 3.5.** (M. Kashiwara [10].) To an effective $\kappa$-lien $R = (\mathcal{R}, f, a)$ on $X_\bullet \to X$ one associates an abelian $\kappa$-stack $\mathcal{M}\text{od}(R)$ on $X$, an equivalence of $p^{-1}\kappa$-stacks $F_R: p^{-1}\mathcal{M}\text{od}(R) \xrightarrow{\sim} \mathcal{M}\text{od}(\mathcal{R})$ and an isomorphism of functors $\alpha_R: f^{-1}(\cdot) \circ (F_R)_1 \xrightarrow{\alpha_R} (F_R)_0$ such that $(\alpha_R)_01 \circ (\alpha_R)_12 = (\alpha_R)_02 \circ (a \cdot)$ (here $a \cdot$ denotes the isomorphism of functors defined by left multiplication by $a$).

Moreover, the datum of $(\mathcal{M}\text{od}(R), F_R, \alpha_R)$ is unique up to equivalence where the equivalence is unique up to unique isomorphism.

**Sketch of the proof.** For $V$ open in $X_0$, let $\mathcal{M}\text{od}_0(R)(V)$ be the category defined as follows: an object of $\mathcal{M}\text{od}_0(R)(V)$ is a pair $(\mathcal{F}, m)$ where $\mathcal{F}$ is an $\mathcal{R}$-module on $V$ and $m: f^{-1}\mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_0$ is an isomorphism of $\mathcal{R}_0$-modules on $V_1 = V \times_X X_1$ such that the following diagram of isomorphisms of $\mathcal{R}_0$-modules on $V_2 = V \times_X X_2$ commutes

\[
\begin{array}{ccc}
\mathcal{F}_2 & \xrightarrow{f_{01}^{-1}(f_{12}^{-1}\mathcal{F}_2)} & \mathcal{F}_2 \\
\downarrow m_{12} & & \downarrow m_{02} \\
\mathcal{F}_1 & \xrightarrow{f_{01}^{-1}\mathcal{F}_1} & \mathcal{F}_0.
\end{array}
\]
Here \( a \cdot \) denotes left multiplication by \( a \) in \( f_{02}^1 \). Morphisms \( \alpha \): \((\mathcal{F}, m) \rightarrow (\mathcal{F}', m')\) are morphisms of \( \mathcal{R}\)-modules on \( V \), \( \alpha \): \( \mathcal{F} \rightarrow \mathcal{F}' \) such that the following diagram of morphisms of \( \mathcal{R}_0\)-modules on \( V_1 \) commutes

\[
\begin{array}{ccc}
f_{02}^1 \mathcal{F}_1 & \xrightarrow{m} & \mathcal{F}_0 \\ \alpha_1 \downarrow & & \downarrow \alpha_0 \\ f_{02}^1 \mathcal{F}'_1 & \xrightarrow{m'} & \mathcal{F}'_0.
\end{array}
\]

The assignment \( V \mapsto \mathcal{M}od_0(\mathcal{R})(V) \) defines a \( p^{-1}\kappa\)-prestack \( \mathcal{M}od_0(\mathcal{R}) \) on \( X_0 \). Moreover, the position \((\mathcal{F}, m) \mapsto \mathcal{F}\) defines a natural functor of prestacks \( \mathcal{M}od_0(\mathcal{R}) \rightarrow \mathcal{M}od(\mathcal{R}) \).

One checks that \( \mathcal{M}od(\mathcal{R}) := p_* \mathcal{M}od_0(\mathcal{R}) \) is a \( \kappa\)-stack on \( X \), that the natural adjunction functor \( F_R: p^{-1}\mathcal{M}od(\mathcal{R}) \rightarrow \mathcal{M}od(\mathcal{R}) \) is an equivalence of \( p^{-1}\kappa\)-stacks on \( X_0 \) and that there exists an isomorphism of functors \( \alpha_R: f^{-1}(\cdot) \circ (F_R)_1 \sim \rightarrow (F_R)_0 \), these data satisfying the desired property. Moreover, one can show that the triplet \((\mathcal{M}od(\mathcal{R}), F_R, \alpha_R)\) is unique up to equivalence.

q.e.d.

**Remarks 3.6.**

(i) Roughly speaking, a \( \kappa\)-lien on \( X_0 \) is the data of sheaves of \( \kappa|_{U_i}\)-algebras \( \mathcal{R}_i \) on \( U_i \) and of isomorphisms of \( \kappa|_{U_{ij}}\)-algebras \( f_{ij}: \mathcal{R}_j|_{U_{ij}} \sim \rightarrow \mathcal{R}_i|_{U_{ij}} \) such that \( f_{ij} \circ f_{jk} = Ad(a_{ijk}) \circ f_{ik} \) on \( U_{ijk} \) for invertible sections \( a_{ijk} \) of \( \mathcal{R}_i|_{U_{ijk}} \). Hence the stack \( \mathcal{M}od(\mathcal{R}) \) constructed in Theorem 3.5 is a stack of twisted modules (see for example \[3\]), i.e., \( \mathcal{M}od(\mathcal{R})|_{U_i} \simeq \mathcal{M}od(\mathcal{R}_i) \).

(ii) The section \( c \) in \[3,3\] defines a Cech cohomology class in \( H^3(X, \kappa^\times) \). Indeed \( c \) is a 3-cocycle, since one has the following chain of equalities on \( \Gamma(X_4; \kappa^\times) \):

\[
c_{0124}c_{0234} = a_{014}^{-1}f_{01}(a_{124}^{-1})a_{012}a_{024}a_{023}^{-1}a_{024}^{-1}a_{023}a_{034} = a_{014}^{-1}f_{01}(a_{124}^{-1})f_{01} \circ f_{12}(a_{234}^{-1})a_{012}a_{023}a_{034} = a_{014}^{-1}f_{01}(a_{124}^{-1}f_{12}(a_{234}^{-1})a_{123})f_{01}(a_{123}^{-1})a_{012}a_{023}a_{034} = a_{014}^{-1}f_{01}(a_{124}^{-1}c_{123}^{-1}a_{013}c_{012}a_{034} = c_{123}c_{013}c_{012}.
\]

(iii) Definitions \[2\] and \[3\] are adapted from \[3\] (see also \[3\]). In particular, the notion of an effective lien is a restrictive version of that of a realizable lien, i.e., a lien with \( c \) cohomologous to 1.

(iv) For simplicity of the exposition, the definition of a lien given here deals with a fixed open covering \( \mathcal{U} = \{U_i\}_{i \in I} \) of the topological space \( X \). Let \( \mathcal{U}' = \{U'_j\}_{j \in J} \) be a finer open covering and define the corresponding diagram \( X' \rightarrow X \). Let \( R \) be a \( \kappa\)-lien on \( X' \rightarrow X \). Hence the natural map \( r_0: X_0' \rightarrow \)
$X_0$ induces a lien $R' = (r_0^{-1}R, r_1^{-1}f, r_2^{-1}a)$ on $X'_0$, where the $r_i$’s are the induced maps on the $X_i$’s. One checks that if $R$ is effective, then $R'$ is effective and they define equivalent stacks on $X$. More precisely, the equivalence $\mathcal{M}\text{od}(R) \sim \mathcal{M}\text{od}(R')$ is given by $(F, m) \mapsto (r_0^{-1}F, r_1^{-1}m)$.

**Definition 3.7.** Let $R = (R, f, a)$ and $S = (S, g, b)$ be $\kappa$-liens on $X \rightarrow X$. An isomorphism of $\kappa$-liens $u: R \rightarrow S$ is pair $u = (u, l)$ where $u : R \sim S$ is an isomorphism of $p^{-1}\kappa$-algebras on $X_0$ and $l$ is a section in $\Gamma(X_1, S_0^\times)$ such that on $X_1$

$$g \circ u_1 = \text{Ad}(l) \circ u_0 \circ f. \quad (3.4)$$

If $T = (T, h, e)$ is another $\kappa$-lien and $t = (t, n): S \rightarrow T$ is an isomorphism of $\kappa$-liens, the composition is defined by $t \circ u := (t \circ u, nt_0(l))$.

On $X_2$ one has

$$g_{01} \circ g_{12} \circ u_2 = \text{Ad}(b) \circ g_{02} \circ u_2 = \text{Ad}(bl_{02}) \circ u_0 \circ f_{02}$$

and

$$g_{01} \circ g_{12} \circ u_2 =
\begin{align*}
g_{01} \circ \text{Ad}(l_{12}) \circ u_1 \circ f_{12} &= \text{Ad}(g_{01}(l_{12})) \circ \text{Ad}(l_{01}) \circ u_0 \circ f_{01} \circ f_{12} =
\end{align*}$$

$$= \text{Ad}(g_{01}(l_{12})l_{01}) \circ \text{Ad}(u_0(a)) \circ u_0 \circ f_{02}$$

It follows that there exists a section $d \in \Gamma(X_2; \kappa^\times)$, such that

$$bl_{02} = g_{01}(l_{12})l_{01}u_0(a) \cdot d \quad \text{in} \ \Gamma(X_2; S_0^\times)$$

**Definition 3.8.** One says that an isomorphism of $\kappa$-liens $u$ is effective if $d = 1$.

**Proposition 3.9.** Let $R = (R, f, a)$ and $S = (S, g, b)$ be effective $\kappa$-liens on $X \rightarrow X$. To an effective isomorphism of $\kappa$-liens $u = (u, l): R \sim S$ one associates an equivalence of $\kappa$-stacks $F_u : \mathcal{M}\text{od}(R) \sim \mathcal{M}\text{od}(S)$ and an isomorphism of functors $\beta_u : F_S \circ p^{-1}F_u \sim F_R$ such that $\alpha_R \circ l \cdot \circ(\alpha_u)_1 = (\beta_u)_0 \circ \alpha_S$ (here $(\mathcal{M}\text{od}(R), F_R, \alpha_R)$ and $(\mathcal{M}\text{od}(S), F_S, \alpha_S)$ are as in Theorem 3.5, and $l \cdot$ denotes the isomorphism of functors defined by left multiplication by $l$). Moreover, the datum of $(F_u, \beta_u)$ is unique up to unique isomorphism.
Proof. To \((\mathcal{F}, m) \in \mathcal{M}od_0(R)(V)\) one associates the pair \((u^{-1}\mathcal{F}, m \circ l)\), where the isomorphism \(m \circ l\) stands for the following chain of isomorphisms of \(S_0\)-modules on \(V_1\)

\[
g^{-1}(u^{-1}\mathcal{F})_1 \simeq g^{-1}(u_1^{-1}\mathcal{F}_1) \rightarrow l^{-1}u_0^{-1}(f^{-1}\mathcal{F}_1) \xrightarrow{m} u_0^{-1}\mathcal{F}_0 \simeq (u^{-1}\mathcal{F})_0,
\]

where \(l\) denotes left multiplication by \(l\) in \(f^{-1}\mathcal{F}_1\). Let us check that this position gives a well defined functor of prestacks \((F_u)_0 : \mathcal{M}od_0(R) \to \mathcal{M}od_0(S)\).

It is enough to note that the following diagram is commutative

\[
s_{01}^{-1}(g_{12}^{-1}(u^{-1}\mathcal{F}_2)) \xrightarrow{l_{01}} s_{02}^{-1}(u_2^{-1}\mathcal{F}_2) \xrightarrow{b} s_{01}^{-1}(u_1^{-1}\mathcal{F}_2) \xrightarrow{l_{02}}
\]

\[
s_{01}^{-1}(u^{-1}\mathcal{F}_1) \xrightarrow{m_{12}} u_0^{-1}(f^{-1}\mathcal{F}_2) \xrightarrow{m_{01}} (u^{-1}\mathcal{F})_0.
\]

Set \(F_u := p_*(F_u)_0\). This is an equivalence of \(\kappa\)-stacks, since a quasi-inverse is given by \(F_{u^{-1}}\), and the natural morphism of functors \(\beta_u : F_{S} \circ p^{-1}F_u \to u^{-1}(\cdot) \circ F_R\) is an isomorphism satisfying the desired property. q.e.d.

**Remark 3.10.** In the commutative case, an effective \(\kappa\)-lien on \(X_\bullet \to X\) is nothing but the datum of a 2-cocycle \(a \in \Gamma(X_2, \kappa^\times)\). Then there exists an effective isomorphism between two 2-cocycles if and only if they are cohomologous.

**Remark 3.11.** Consider a diagram in the category of topological spaces

\[
X_\bullet := X_2 \xrightarrow{p_{01}} X_1 \xrightarrow{p_0} X_0
\]

such that

\[
\left\{
\begin{array}{l}
p_0 \circ p_{01} = p_0 \circ p_{02}, \\
p_1 \circ p_{01} = p_0 \circ p_{12}, \\
p_1 \circ p_{12} = p_1 \circ p_{02}.
\end{array}
\right.
\]

Then one can extend the notions of a lien and of an isomorphism of liens on \(X_\bullet\). Moreover, there is a natural notion of continuous map between diagrams \(r : Y_\bullet \to X_\bullet\) as above. Then, if \(R = (R, f, a)\) is a lien on \(X_\bullet\), one defines the lien \(r^{-1}R := (r_0^{-1}R, r_1^{-1}f, r_2^{-1}a)\) on \(Y_\bullet\).

These considerations suggest that Theorem 3.5 extends to the case of Lie groupoids or better, differentiable stacks (see [15], [4]).
4 Symplectic and contact geometry

Let $X$ be a complex manifold. Recall that a local model of a homogeneous symplectic manifold is an open subset $V$ of the cotangent bundle $T^*X$ equipped with the canonical 1-form $\alpha_{T^*X}$. If $x = (x_1, \ldots, x_n)$ is a local coordinate system on $X$, we denote by $(x; \xi) = (x_1, \ldots, x_n; \xi_1, \ldots, \xi_n)$ the associated coordinates on $T^*X$. Then $\alpha_{T^*X} = \sum_{i=1}^n \xi_i dx_i$. A local model of a contact manifold is an open subset of the projective cotangent bundle $\mathcal{P}^*X = \dot{T}^*X/C\times$, equipped with its canonical 1-form. A local model of a symplectic manifold is an open subset $U$ of $T^*X$ equipped with the symplectic 2-form $\omega_{T^*X} = d\alpha_{T^*X}$.

We shall use the terminology “a symplectic isomorphism” to denote an isomorphism of complex symplectic manifolds, and similarly for “a contact isomorphism” or “a homogeneous symplectic isomorphism”.

We denote by $\mathbb{C}$ the complex line endowed with a holomorphic coordinate $t$ and by $(t; \tau)$ the associated coordinates on $T^*\mathbb{C}$. Hence, $\tau$ defines a map $\tau : \dot{T}^*\mathbb{C} \to \mathbb{C}^\times$.

The embedding $T^*X \times \dot{T}^*\mathbb{C} \hookrightarrow \dot{T}^*(X \times \mathbb{C})$ defines the open embedding $(T^*X \times \dot{T}^*\mathbb{C})/\mathbb{C}^\times \hookrightarrow P^*(X \times \mathbb{C})$. Set for short:

$$\dot{P}^*(X \times \mathbb{C}) = (T^*X \times \dot{T}^*\mathbb{C})/\mathbb{C}^\times.$$ 

**Remark 4.1.** There is a natural isomorphism $\dot{P}^*(X \times \mathbb{C}) \simeq T^*X \times \mathbb{C}$ given by $(p, (t; \tau)) \mapsto (p\tau^{-1}, t)$. Hence, one can identify $\dot{P}^*(X \times \mathbb{C})$ with the space $\mathcal{J}^1X$ of 1-jets of holomorphic functions on $X$.

We shall consider the maps:

\[
\begin{array}{ccc}
\mathbb{C}^\times & \hookrightarrow & T^*X \times \dot{T}^*\mathbb{C} \\
\gamma & \hookrightarrow & \dot{P}^*(X \times \mathbb{C}) \\
\rho & \rightarrow & T^*X
\end{array}
\]

where $\rho$ is the map $(p, (t; \tau)) \mapsto p\tau^{-1}$.

**Definition 4.2.** Consider a contact isomorphism $\psi : V_X \xrightarrow{\sim} V_Y$, where $V_X$ (resp. $V_Y$) is an open subset of $\dot{P}^*(X \times \mathbb{C})$ (resp. of $\dot{P}^*(Y \times \mathbb{C})$). We say that $\psi$ is a $\tau$-preserving contact isomorphism, if it lifts as a homogeneous
symplectic isomorphism $\tilde{\psi}$ making the diagram below commutative:

$$
\begin{array}{ccc}
\dot{P}^*(X \times \mathbb{C}) & \ni & V_X \\
\uparrow & & \uparrow \\
T^*X \times \dot{T}^*\mathbb{C} & \ni & \gamma^{-1}(V_X) \\
\uparrow & & \uparrow \\
t^*X \times t^*\mathbb{C} & \ni & \gamma^{-1}(V_X) \\
\end{array}
$$

\[
\psi \rightarrow \psi \\
\gamma \rightarrow \gamma
\]

Lemma 4.3. Let $U_X$ (resp. $U_Y$) be an open subset of $T^*X$ (resp. $T^*Y$) and let $\varphi : U_X \rightarrow U_Y$ be a symplectic isomorphism. Then, locally on $U_X$, there exists a $\tau$-preserving contact isomorphism $\psi : \rho^{-1}(U_X) \rightarrow \rho^{-1}(U_Y)$ making the diagram below commutative

$$
\begin{array}{ccc}
T^*X & \ni & U_X \\
\uparrow & & \uparrow \\
\dot{P}^*(X \times \mathbb{C}) & \ni & \rho^{-1}(U_X) \\
\uparrow & & \uparrow \\
T^*Y & \ni & \rho^{-1}(U_Y) \\
\end{array}
$$

(4.2)

One shall be aware that $\psi$ commutes with $\tau$, but not with $t$ in general.

Proof. Let us denote by $(x; u)$ a local symplectic coordinate system on $U_X$ and let $(y; v) = \varphi(x; u)$. Then $du \wedge dx = dv \wedge dy$, hence $d(udx) = d(vdy)$ (we write for short $udx$ instead of $(u, dx)$). It follows that $udx = vdy + da(y,v)$ for some locally defined function $a(y,v)$.

Set $\xi = \tau u, \eta = \tau v$ and consider the map $\psi$ such that

$$(y, s; \eta, \tau) = \psi(x, t; \xi, \tau)$$

for an $s$ to be calculated below. We have

$$\tau dt + \xi dx = \tau(dt + udx) = \tau(dt + vdy + da) = \tau d(t + a) + \eta dy.$$ 

Hence, choosing $s = t + a(y, \eta^{-1})$, the map $\psi$ is a $\tau$-preserving contact isomorphism.

Note that, if $\varphi$ is an homogeneous symplectic isomorphism, then the function $a$ constructed above is globally defined and locally constant on $U_X$. In particular, if $\varphi$ is the identity on $U_X$, then a $\tau$-preserving contact isomorphism $\psi$ is nothing but a translation on the $t$-variable by a locally constant function.

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5 $\mathcal{E}$-modules

On $T^*X$ we consider the sheaves $\mathcal{E}_X$ and $\widehat{\mathcal{E}}_X$ of microdifferential operators and formal microdifferential operators, respectively. We refer the reader to [17] for more details (see also [18] or [9] for an exposition).

These sheaves are constant on the fibers of the projection $\dot{T}^*X \to P^*X$, and we shall keep the same notation to denote their direct images on $P^*X$. Hence we regard them as sheaves on $\dot{T}^*X$ as well as sheaves on $P^*X$.

The sheaf $\mathcal{E}_X$ is filtered over $\mathbb{Z}$, and one denotes by $\mathcal{E}_X(m)$ the sheaf of operators of order less than or equal to $m$. We denote by $\sigma_m(\cdot): \mathcal{E}_X(m) \to \mathcal{E}_X(m)/\mathcal{E}_X(m-1)$ the symbol map. Recall that $\mathcal{E}_X(m)/\mathcal{E}_X(m-1) \simeq \mathcal{O}_{T^*X}(m)$, the sheaf of holomorphic functions homogeneous of order $m$ in the fiber variable. A section $f$ of $\mathcal{O}_{T^*X}(m)$ is a holomorphic function solution of the differential equation $\left( \sum_j x_j \partial x_j - m \right)f = 0$. Hence

$$\text{gr}(\mathcal{E}_X) \simeq \bigoplus_{j \in \mathbb{Z}} \mathcal{O}_{T^*X}(j).$$

The same result holds for $\widehat{\mathcal{E}}_X$.

In a local coordinate system $x$ on $X$, with associated coordinates $(x; \xi)$ on $T^*X$, a formal microdifferential operator $P$ of order $m$ (i.e., a section of $\widehat{\mathcal{E}}_X(m)$) defined on an open subset $V$ of $T^*X$ has a total symbol $\sigma(P)$:

$$\sigma(P) = \sum_{j=-\infty}^m p_j(x, \xi), \quad p_j \in \mathcal{O}_{T^*X}(j)(V).$$

**Notation 5.1.** In a local coordinate system $x = (x_1, \ldots, x_n)$ on $X$, one denotes by $\partial_{x_j}$ (or else, $D_{x_j}$) the microdifferential operator with total symbol $\xi_j$. Hence, $P$ is written as

$$P = \sum_{j=-\infty}^m p_j(x, \partial_x).$$

The product structure on $\widehat{\mathcal{E}}_X$ is then given by the Leibniz formula. If $Q$ is a formal microdifferential operator of total symbol $\sigma(Q)$, then

$$\sigma(P \circ Q) = \sum_{\alpha \in \mathbb{N}_0} \frac{1}{\alpha!} \partial_x^\alpha \sigma(P) \partial_x^\alpha \sigma(Q).$$

In particular, a section $P$ in $\widehat{\mathcal{E}}_X$ is invertible on an open subset $V$ of $T^*X$ if and only if its principal symbol is nowhere vanishing on $V$. 

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The ring $\mathcal{E}_X$ is the subring of $\hat{\mathcal{E}}_X$ of operators whose total symbol satisfies the estimates
\begin{equation}
\text{(5.3)} \quad \left\{ \begin{array}{l}
\text{for any compact subset } K \text{ of } V \text{ there exists a constant } C_K > 0 \text{ such that for all } j < 0, \sup_K |p_j| \leq C_K^{-j}(-j)!.
\end{array} \right.
\end{equation}

Let $\lambda$ be a complex number. Replacing $O_{T^*X}(j)$ with $O_{T^*X}(\lambda + j)$, one defines by the same procedure the sheaves $\mathcal{E}_X(\lambda)$ and $\hat{\mathcal{E}}_X(\lambda)$ on $T^*X$ of operators of order $m + \lambda$. Note that in the $P^*X$ case, one obtains twisted sheaves.

From now on, we shall concentrate our study on $\mathcal{E}_X$, but all results extend unchanged to $\hat{\mathcal{E}}_X$.

A volume form on $X$ defines an anti-automorphism $\ast : \mathcal{E}_X \to a_*\mathcal{E}_X$ (recall that $a$ is the antipodal map on $T^*X$). This leads to consider the sheaf of rings
\begin{equation}
\mathcal{E}_X^{\sqrt{\pi}} := \Omega_X^{\otimes 1/2} \otimes_{\mathcal{O}_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1/2}.
\end{equation}
(Here, we write $\mathcal{O}_X$ instead of $\pi^{-1}\mathcal{O}_X$ and similarly for $\Omega_X$.) Note that $\Omega_X^{\otimes 1/2}$ and $\Omega_X^{\otimes -1/2}$ are not globally defined as sheaves but are globally defined as twisted sheaves. On the other-hand $\mathcal{E}_X^{\sqrt{\pi}}$ is a well-defined sheaf of rings on $T^*X$, locally isomorphic to $\mathcal{E}_X$. This sheaf on $P^*X$ (not on $T^*X$) is thus endowed with an anti-automorphism $\ast$ such that $\ast\ast = \text{id}$.

Quantized contact transformations

Let us briefly recall the constructions of “quantized contact transformations” of [17].

Assume to be given open subsets $V_X$ of $T^*X$, $V_Y$ of $T^*Y$ and a homogeneous symplectic isomorphism $\psi : V_X \to V_Y$. Let $\Lambda \subset V_X \times V_Y^\circ$ denote the Lagrangian submanifold associated with $\psi$, that is, the image of the graph of $\psi$ by the antipodal map $a : V_Y \to V_Y^\circ$. Locally on $\Lambda$, there exists a left ideal $\mathcal{I}$ of $\mathcal{E}_{X \times Y}$ such that its symbol ideal is reduced and coincides with the defining ideal of $\Lambda$ in $\bigoplus_{j \in \mathbb{Z}} O_{T^*X}(j)$.

Then for each $P \in \mathcal{E}_X$ (defined in a neighborhood of $p \in V_X$) there exists a unique $Q \in \mathcal{E}_Y$ (defined in a neighborhood of $\psi(p) \in V_Y$) such that $(P - Q) \in \mathcal{I}$. The correspondance $P \mapsto Q$ is an anti-isomorphism of $\mathbb{C}$-algebras. Composing it with the anti-isomorphism $Q \mapsto Q^*$ associated with a volume form on $Y$, we find an isomorphism of $\mathbb{C}$-algebras
\begin{equation}
\Psi : \psi_* (\mathcal{E}_X|_{V_X}) \to \mathcal{E}_Y|_{V_Y}.
\end{equation}
The same construction holds on projective cotangent bundles. It also holds replacing \( E \) with \( E^{\sqrt{v}} \).

**Definition 5.2.** (i) Let \( \psi \) be a homogeneous symplectic isomorphism. The isomorphism \( \Psi \) in (5.5) is called a homogeneous quantized contact transformation above \( \psi \) (a homogeneous QCT, for short).

(ii) If \( \psi \) is a contact transformation (i.e. \( V_X \) and \( V_Y \) are open subsets of projective cotangent bundles), one calls \( \Psi \) a quantized contact transformation (a QCT, for short).

(iii) We keep the same terminology when \( E \) is replaced with \( E^{\sqrt{v}} \).

**Lemma 5.3.** \([17], [9], [10]\).

(i) Let \( \Psi \) be a QCT above the identity. Locally there exists \( \lambda \in \mathbb{C} \) and an invertible operator \( P \in E_X(\lambda) \) such that \( \Psi = \text{Ad}(P) \).

(ii) Let \( P \in E_X \) be of order \( \lambda \) and invertible. There exists a unique (up to sign) invertible operator \( Q \) of order \( \frac{1}{2} \lambda \) such that \( P = Q \circ Q \). If \( P \in E_X^{\sqrt{v}} \), \( P \) is of order 0 and \( P = P^* \), then \( Q = Q^* \).

(iii) Let \( \psi \) be a contact isomorphism. Locally there exists a QCT

\[
\Psi : \psi^*(E_X^{\sqrt{v}}|_{V_X}) \xrightarrow{\sim} E_Y^{\sqrt{v}}|_{V_Y}
\]

commuting with \( * \).

**Proof.** (i) We shall not give the proof here and refer to [9].

(ii) (a) Unicity. First assume \( P \) has order 0. Let \( P = Q^2 = Q_0^2 \). Then \( \sigma_0(Q_0) = \pm \sigma_0(Q) \) and we may assume there is equality. Set \( Q_0 = Q + R \) and let \( m \) denote the order of \( R \). One has \( Q_0^2 = (Q + R)^2 \) and therefore \( QR + RQ + R^2 = 0 \). Since \( 2m < m \), it follows that \( \sigma_m(R) = 0 \), hence \( R = 0 \). The general case follows by adding a dummy variable \( \partial_t \), and applying the preceding result to \( \partial_t^{-\lambda}P \).

(ii) (b) Existence. First assume \( P \) has order 0. Let \( p_0 \in T^*X \) and assume \( \sigma_0(P)(p) \neq 0 \). Consider a dummy variable \( t \) and set \( t_0 = (\sigma_0(P)(p_0))^\frac{1}{2} \). Since \( t^2 - \sigma_0(P) \) has a simple root at \( (t_0, p_0) \), the Weierstrass Preparation Theorem for microdifferential operators of \([17]\) (see also \([18]\) Chapter I \S 2 for an exposition) allows us to write uniquely

\[
t^2 - P = G(t - R)
\]

with \( G \) invertible at \( (t_0, p_0) \) and \( R \) not depending on \( t \). Therefore \( P - R^2 = (t - R)(t + R - G) \), and this operator not depending on \( t \), we find \( G = t + R \). Hence, \( P = R^2 \).
The general case follows by adding a dummy variable $\partial_t$. Then $\partial_t^{-\lambda} P = R^2$.

Set $Q = \partial_t^2 R$. Then $P = Q^2$, and the unicity of this decomposition shows that $Q$ does not depend on $\partial_t$.

(ii) (c) Since $P = P^*$, we get $Q^* Q = Q Q$, and by the unicity of the decomposition, $Q = \pm Q^*$. Since $\sigma_0(Q) = \sigma_0(Q^*)$, we get $Q = Q^*$.

(iii) First choose a QCT $\Psi^\dagger$. Set $Q = \Psi_{\dagger-1}^* \circ \Psi_{\dagger} \circ \Psi_{\dagger}$. By (i), $\Psi_{\dagger} = \text{Ad}(R)$ for an invertible operator $R$ of order $\lambda$. Hence

$$
\Psi_{\dagger} = \Psi_{\dagger} \circ \text{Ad}(R)
$$

Therefore, $R = c R^*$ for some non-zero constant $c$, that we may assume equal to 1. Hence using (ii), we may write $R = Q \circ Q^*$ and we get $\Psi_{\dagger} = \text{Ad}(Q) \text{Ad}(Q^*)$. Let $\Psi = \Psi_{\dagger} \circ \text{Ad}(Q)$. Then, one has

$$
\Psi_{\dagger} \circ \Psi_{\dagger} = \Psi \circ \text{Ad}(Q^*)
$$

Since $\text{Ad}(Q) \circ \Psi_{\dagger} = \text{Ad}(Q^*) \circ \Psi_{\dagger}$, one gets

$$
\Psi_{\dagger} \circ \Psi_{\dagger} = \Psi.
$$

q.e.d.

Denote by $\mathcal{A}ut^*_{\lambda}(\mathcal{E}^\sqrt{v}_X)$ the sheaf of QCT above the identity commuting with $\ast$ and set

$$(\mathcal{E}^\sqrt{v}_X)^* = \{ P \in \mathcal{E}^\sqrt{v}_X; P \text{ has order } 0, \sigma_0(P) = 1, PP^* = 1 \} \subset (\mathcal{E}^\sqrt{v}_X)^X.
$$

Lemma 5.4. [10] The morphism $\text{Ad}$ induces an isomorphism of groups on $T^*X$

$$
(\mathcal{E}^\sqrt{v}_X)^* \cong_{\text{Ad}} \mathcal{A}ut_{\lambda}(\mathcal{E}^\sqrt{v}_X).
$$

Proof. Let $\Psi$ be a QCT above the identity commuting with $\ast$. By Lemma 5.3 locally there exists an invertible $P \in \mathcal{E}^\sqrt{v}_X$ of order $\lambda$ such that $\Psi = \text{Ad}(P)$. One has

$$
\ast \circ \text{Ad}(P) \circ \ast = \text{Ad}(P^*).\]

Since $\ast \circ \Psi = \Psi \circ \ast$, we get $\text{Ad}(P^*) = \text{Ad}(P)$, and $C := P^* P$ is an invertible element in the center of $\mathcal{E}^\sqrt{v}_X$. Therefore, $P$ has order 0 and $\sigma_0(P)$ is a non-zero constant. Then we may suppose $P$ of principal symbol 1. q.e.d.
Remark 5.5. The results of this section still hold when replacing $\mathcal{E}$ with $\hat{\mathcal{E}}$.

6 Quantization of complex contact manifolds

In this section, we recall Kashiwara’s theorem.

Theorem 6.1. (M. Kashiwara [10].) Let $\mathcal{Y}$ be a complex contact manifold. There exists canonically a $\mathbb{C}$-abelian stack $\mathcal{M}(\mathcal{E}^{\sqrt{\sigma}}, \mathcal{Y})$ on $\mathcal{Y}$ such that if $V \subset \mathcal{Y}$ is an open subset isomorphic by a contact transformation $\psi$ to an open subset $V_X \subset P^*X$, then $\mathcal{M}(\mathcal{E}^{\sqrt{\sigma}}, \mathcal{Y})|_V$ is equivalent by $\psi$ to the stack $\mathcal{M}(\mathcal{E}^{\sqrt{\sigma}}|_{V_X})$.

Definition 6.2. We call $\mathcal{M}(\mathcal{E}^{\sqrt{\sigma}}, \mathcal{Y})$ the stack of microdifferential modules on $\mathcal{Y}$.

Proof. There exists an open covering $\mathcal{V} = \{V_i\}_{i \in I}$ of $\mathcal{Y}$ and for each $i \in I$, a contact open embedding $\psi_i : V_i \hookrightarrow P^*X_i$ for some projective cotangent bundle $P^*X_i$. Set $\mathcal{E}^{\sqrt{\sigma}}_i = \psi_i^{-1}\mathcal{E}^{\sqrt{\sigma}}_X$. Then $\mathcal{E}^{\sqrt{\sigma}}_i$ is a sheaf of $\mathbb{C}$-algebras on $V_i$ endowed with a filtration and an anti-involution $\ast$.

Consider the contact isomorphism $\psi_{ij} = \psi_i \circ \psi^{-1}_j : \psi_j(V_{ij}) \sim \psi_i(V_{ij})$. After shrinking the covering $\mathcal{V}$, we may assume by Lemma 5.3 that there exist QCT’s above the $\psi_{ij}$’s and hence isomorphisms of $\mathbb{C}$-algebras $\Psi_{ij} : \mathcal{E}^{\sqrt{\sigma}}_j|_{V_{ij}} \sim \mathcal{E}^{\sqrt{\sigma}}_i|_{V_{ij}}$ commuting with $\ast$.

Now we follow the notations of Section 3 for $X = \mathcal{Y}$. Set $\mathcal{Y}_0 = \bigsqcup_i V_i$, $\mathcal{Y}_1 = \mathcal{Y}_0 \times_\mathcal{Y} \mathcal{Y}_0$, etc. Let $j_{Vi} : V_i \hookrightarrow \mathcal{Y}_0$ be the natural map. Set

$$\mathcal{E}^{\sqrt{\sigma}} = \bigoplus_{i \in I} j_{Vi}^! \mathcal{E}^{\sqrt{\sigma}}_i.$$  

Then $\mathcal{E}^{\sqrt{\sigma}}$ is a sheaf of central $\mathbb{C}$-algebras on $\mathcal{Y}_0$ endowed with an anti-involution $\ast$. The $\Psi_{ij}$’s induce a $\mathbb{C}$-algebra isomorphism $\Psi : \mathcal{E}^{\sqrt{\sigma}}_1 \sim \mathcal{E}^{\sqrt{\sigma}}_0$ commuting with $\ast$. Hence, by Lemma 5.4 after shrinking again the covering $\mathcal{V}$ there exists a unique $P \in \Gamma(\mathcal{Y}_2; \mathcal{E}^{\sqrt{\sigma}}_0)$ of order 0 such that $\sigma_0(P) = 1$, $PP^* = 1$ and

$$\Psi_{01} \circ \Psi_{12} = \text{Ad}(P) \circ \Psi_{02}.$$  

Since $P$ is unique, $\mathcal{E} := (\mathcal{E}^{\sqrt{\sigma}}, \Psi, P)$ is an effective $\mathbb{C}$-lien on $\mathcal{Y}_\bullet \to \mathcal{Y}$, and it remains to apply Theorem 5.5 to get an abelian $\mathbb{C}$-stack $\mathcal{M}(\mathcal{E})$ on $\mathcal{Y}$.  

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Let $\mathcal{V}' = \{V'_j\}_{j \in J}$ be another open covering of $\mathcal{Y}$ and $\psi'_j : V'_j \hookrightarrow P^*X'_j$ be a contact open embedding for each $j \in J$. By Remark 3.6 (iii), it is not restrictive to assume $\mathcal{V} = \mathcal{V}'$. Setting $E'E' = \psi'^{-1}_j E\sqrt{\nu}_i$ and proceeding as above, we get another effective $\mathbb{C}$-lien $E' := (E\sqrt{\nu}, \Psi', P')$ on $\mathcal{Y}_1 \to \mathcal{Y}$. Consider the contact isomorphism $\psi'_i \circ \psi'^{-1}_i : \psi_i(V_i) \overset{\sim}{\to} \psi'_i(V_i)$. After shrinking the covering $\mathcal{V}$, we may assume by Lemma 5.3 that there exist QCT’s above the $\psi'_i \circ \psi'^{-1}_i$’s and hence an isomorphism of $\mathbb{C}$-algebras

(6.3) \[ \Upsilon : E\sqrt{\nu} \overset{\sim}{\to} E\sqrt{\nu}' \]

commuting with *. By Lemma 5.4, after shrinking again the covering $\mathcal{V}$, there exists a unique $Q \in \Gamma(\mathcal{Y}_1; E\sqrt{\nu})$ of order 0 such that $\sigma_0(Q) = 1$, $QQ^* = 1$ and

(6.4) \[ \Psi' \circ \Upsilon_1 = \text{Ad}(Q) \circ \Upsilon_0 \circ \Psi. \]

Since $Q$ is unique, the pair $(\Upsilon, Q)$ defines an effective isomorphism of $\mathbb{C}$-liens $\mathcal{E} \overset{\sim}{\to} \mathcal{E}'$. By Proposition 3.9, the stacks $\text{Mod}(E)$ and $\text{Mod}(E')$ are equivalent. Hence the stack above constructed depends only on $\mathcal{Y}$ and on the algebra $E\sqrt{\nu}$, and it makes sense to denote it by $\text{Mod}(E\sqrt{\nu}, \mathcal{Y})$. q.e.d.

**Remark 6.3.** The results of this section still hold when replacing $\mathcal{E}$ with $\hat{\mathcal{E}}$.

### 7 Quantization of involutive submanifolds

We keep the notations of §6 and consider a complex contact manifold $\mathcal{Y}$. In this section, we consider a smooth regular involutive submanifold $\Lambda$ and we denote by

(7.1) \[ \iota : \Lambda \hookrightarrow \mathcal{Y} \]

the inclusion morphism. Recall that one says that $\Lambda$ is involutive if for any pair of holomorphic functions $(f, g)$ vanishing on $\Lambda$, their Poisson bracket $\{f, g\}$ vanishes on $\Lambda$ and recall that $\Lambda$ is regular involutive if moreover the canonical 1-form on $\mathcal{Y}$ does not vanish on $\Lambda$.

It is well-known that locally, two smooth regular involutive submanifolds of the same codimension may be interchanged by a complex contact isomorphism. In particular, locally on $\Lambda$, we may assume that

(7.2) \[ \mathcal{Y} = P^*X, \quad X = Y \times Z, \quad \Lambda = P^*Y \times Z. \]
Let $b$ denote the bicharacteristic relation on $\Lambda$, which identifies a bicharacteristic leaf to a point. We denote by

$$
\beta : \Lambda \to \Lambda/b
$$

(7.3)

the projection, and if $U$ is open in $\Lambda$ we denote by $\beta_U : U \to U/b$ the restriction of $\beta$ to $U$. Note that for $U$ small enough, $U/b$ has the structure of complex contact manifold.

We can now formulate a variant of Theorem 6.1.

Proposition 7.1. Let $\mathcal{Y}$ be a complex contact manifold and let $\Lambda$ be a smooth regular involutive submanifold of $\mathcal{Y}$. There exists canonically a $\mathbb{C}$-abelian stack $\text{Mod}(\beta^{-1}\mathcal{E}^\sqrt{v}, \Lambda)$ on $\Lambda$ such that if $U \subset \Lambda$ is an open subset and $U/b$ is a contact manifold isomorphic by a contact transformation $\psi$ to an open subset $V \subset P^*Y$, then $\text{Mod}(\beta^{-1}\mathcal{E}^\sqrt{v}, \Lambda)|_U$ is equivalent to the stack $\text{Mod}(\beta^{-1}\psi^{-1}\mathcal{E}^\sqrt{v}|_V)$.

Proof. Consider an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $\Lambda$ such that for each $i \in I$, $U_i/b$ is a smooth complex contact manifold and there exists a contact embedding $\psi_i : U_i/b \to P^*Y_i$ for some projective cotangent bundle $P^*Y_i$. Set $V_i = U_i/b$, $V_{ij} = (U_i \cap U_j)/b$, $V_{ijk} = (U_i \cap U_j \cap U_k)/b$, $\mathcal{Y}_0 = \bigsqcup V_i$, $\mathcal{Y}_1 = \bigsqcup V_{ij}$, $\mathcal{Y}_2 = \bigsqcup V_{ijk}$. We find a diagram $\mathcal{Y}_*$ as in (3.5) and after shrinking the covering $\mathcal{U}$, we find an effective $\mathbb{C}$-lien $E := (\mathcal{E}^\sqrt{v}, \Phi, P)$ on $\mathcal{Y}_*$. Set $\Lambda_0 = \bigsqcup U_i$, $\Lambda_1 = \bigsqcup U_{ij}$, $\Lambda_2 = \bigsqcup U_{ijk}$, and define the diagram $\Lambda_* \to \mathcal{Y}_*$ similarly. The projection $\beta$ defines a continuous map of diagrams $\beta : \Lambda_* \to \mathcal{Y}_*$, and it remains to apply Theorem 3.5.

The proof of the canonicity of the stack $\text{Mod}(\beta^{-1}\mathcal{E}^\sqrt{v}, \Lambda)$ goes along the same lines as in the proof of Theorem 6.1 q.e.d.

Consider the situation of (7.2): $\mathcal{Y} = P^*X$, $X = Y \times Z$, $\Lambda = P^*Y \times Z$ and denote by $\beta : P^*Y \times Z \to P^*Y$ the projection. Let $\mathcal{L}_\Lambda := \mathcal{E}_Y \boxtimes \mathcal{O}_Z$ where the external product in taken in the category of $\mathcal{E}$-modules. Such an $\mathcal{E}_X$-module is called simple along $\Lambda$. Then $\text{End}_{\mathcal{E}_X}(\mathcal{L}_\Lambda) \simeq \beta^{-1}\mathcal{E}_Y$ and $\mathcal{L}_\Lambda$ is faithfully flat over $\beta^{-1}\mathcal{E}_Y$. This suggests another method to quantize the involutive submanifold $\Lambda$, namely by glueing the categories of modules over $\text{End}_{\mathcal{E}_X}(\mathcal{L}_\Lambda)$, for local choices of simple modules $\mathcal{L}_\Lambda$.

Remark 7.2. The results of this section still hold when replacing $\mathcal{E}$ with $\hat{\mathcal{E}}$. 

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8 \(\mathcal{W}\)-modules

Let \(X\) be a complex manifold, and let \(\mathbb{C}\) denote as above the complex line endowed with the holomorphic coordinate \(t\). Recall the map \(\rho\) constructed in (4.1)

\[
\rho : \dot{P}^*(X \times \mathbb{C}) \to T^*X.
\]

**Definition 8.1.** (i) On \(\dot{P}^*(X \times \mathbb{C})\), we denote by \(E_{X \times \mathbb{C},i}\) the subsheaf of rings of \(E_{X \times \mathbb{C}}\) consisting of microdifferential operators \(P\) such that

\[
[P, D_t] = 0.
\]

For \(m \in \mathbb{Z}\), we set \(E_{X \times \mathbb{C},i}(m) = E_{X \times \mathbb{C},i} \cap E_{X \times \mathbb{C}}(m)\).

(ii) On \(T^*X\) we define the sheaf of rings

\[
\mathcal{W}_X = \rho_* E_{X \times \mathbb{C},i}.
\]

We set \(\mathcal{W}_X(m) = \rho_* E_{X \times \mathbb{C},i}(m)\).

(iii) Replacing \(E\) with \(E^{\sqrt{\sigma}}, \hat{E}\) and \(\hat{E}^{\sqrt{\sigma}}\), one constructs similarly the sheaves of rings \(\hat{\mathcal{W}}^{\sqrt{\sigma}}_X, \hat{\mathcal{W}}_X\) and \(\hat{\mathcal{W}}^{\sqrt{\sigma}}_X\) on \(T^*X\).

After choosing a local coordinate system \(x\) on \(X\), the microdifferential operator \(P(x, t; D_x, D_t) \in E_{X \times \mathbb{C},i}\) does not depend on \(t\) and its total symbol may be written as a series

\[
\sigma(P) = \sum_{j=-\infty}^{m} p_j(x; \xi, \tau)
\]

where the \(p_j\) are holomorphic functions defined on some conic open subset \(V\) of \(T^*(X \times \mathbb{C})\), homogeneous of order \(j\) with respect to \((\xi, \tau)\). Setting \(u = \xi \tau^{-1}\), \(\tilde{p}_j(x; u) = p_j(x; \xi \tau^{-1}, 1)\), we get that a section \(P\) of \(\mathcal{W}_X\) on an open subset \(U\) of \(T^*X\) has a total symbol

\[
\sigma(P) = \sum_{j=-\infty}^{m} \tilde{p}_j(x; u) \tau^j
\]

where the \(\tilde{p}_j\)'s are holomorphic (but no more homogeneous) on \(U\). These functions should satisfy the follow estimates

\[
\text{for any compact subset } K \text{ of } U \text{ there exists a constant } C_K > 0 \text{ such that for all } j < 0, \sup_K |\tilde{p}_j| \leq C_K^{-j}(-j)!.
\]
Notation 8.2. In a local coordinate system $x = (x_1, \ldots, x_n)$ on $X$, one denotes by $\tau^{-1}\partial_{x_j}$ (or else, $\tau^{-1}D_{x_j}$) the operator with total symbol $u_j$. Hence, an operator $P$ is written as

$$P = \sum_{j=-\infty}^{m} p_j(x, \tau^{-1}\partial_x)\tau^j. \quad (8.5)$$

The ring $\mathcal{W}_X$ is filtered and

$$gr(\mathcal{W}_X) \simeq \mathcal{O}_{T^* X}[\tau, \tau^{-1}] \quad (8.6)$$

As in the case of microdifferential operators, the symbol of order $m$ of $P$ is denoted $\sigma_m(P)$. This function does not depend on the local coordinate system on $X$. If $\sigma_m(P)$ is not identically zero, then one says that $P$ has order $m$ and $\sigma_m(P)$ is called the principal symbol of $P$.

The product in $\mathcal{W}_X$ (and in $\hat{\mathcal{W}}_X$) is given by the Leibniz formula not involving the $\tau$-derivatives. If $Q$ is an operator of total symbol $\sigma(Q)$, then

$$\sigma(P \circ Q) = \sum_{\alpha \in \mathbb{N}^n} \frac{\tau^{-|\alpha|}}{\alpha!} \partial_{x}^{\alpha} \sigma(P) \partial_{x}^{\alpha} \sigma(Q).$$

In particular, a section $P$ in $\mathcal{W}_X$ is invertible on an open subset $U$ of $T^* X$ if and only if its principal symbol is nowhere vanishing on $U$.

Remarks 8.3. (i) The rings $\mathcal{W}_X$ and $\hat{\mathcal{W}}_X$ have already been introduced (when $X = \mathbb{C}$) in [1]. They are denoted $\mathcal{E}_{WKB}$ and $\hat{\mathcal{E}}_{WKB}$ by these authors who call their sections, WKB-differential operators. We shall keep this last terminology.

(ii) The ring $\hat{\mathcal{W}}_X$ is a semi-classical star-algebra in the sense of [2]: it is locally isomorphic to $\mathcal{O}_{T^* X}[\tau, \tau^{-1}]$ as a $\mathcal{C}_{T^* X}$-module (via the total symbol) and it is equipped with an unital associative product (the Leibniz rule) which induces a star-product on $\mathcal{O}_{T^* X}[\tau, \tau^{-1}]$ not involving the $\tau$-derivatives.

Definition 8.4. (i) One denotes by $\hat{k}$ the field $\mathbb{C}[\tau, \tau^{-1}]$, that is, the field of formal series $\sum_{j \in \mathbb{Z}} a_j \tau^j$ with $a_j = 0$ for $j \gg 0$.

(ii) One denotes by $k$ the subfield of $\hat{k}$ consisting of series $\sum_j a_j \tau^j$ which satisfies the estimate:

$$\left\{ \begin{array}{l} \text{there exists a constant } C > 0 \text{ such that} \\ \text{for all } j < 0, \ |a_j| \leq C^{-j}(-j)!. \end{array} \right. \quad (8.7)$$

In other words, $k = \mathcal{W}_{pt}$ and $\hat{k} = \hat{\mathcal{W}}_{pt}$.

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The following result follows immediately from the fact that the center of $\mathcal{E}_X$ and $\hat{\mathcal{E}}_X$ is $C_{T^\ast X}$.

**Lemma 8.5.**

(i) The center of $\mathcal{W}_X$ is the constant sheaf $k_{T^\ast X}$.

(ii) The center of $\hat{\mathcal{W}}_X$ is the constant sheaf $\hat{k}_{T^\ast X}$.

**Remark 8.6.** Replacing the sheaf of rings $\mathcal{E}$ with the sheaf of rings $\mathcal{E}_\infty$ of infinite order microdifferential operators, one constructs similarly the sheaf of rings $\mathcal{W}_\infty$ on $T^\ast X$ of infinite order WKB-differential operators.

**Quantized symplectic transformations**

**Lemma 8.7.** Assume to be given an open subset $U_X$ (resp. $U_Y$) of $T^\ast X$ (resp. $T^\ast Y$), a symplectic isomorphism $\varphi : U_X \sim \to U_Y$ and a $\tau$-preserving contact isomorphism $\psi : \rho^{-1}(U_X) \to \rho^{-1}(U_Y)$ making the diagram commutative. Denote by $x = (x_1, \ldots, x_n)$ a local coordinate system on $X$, by $(x, t; \xi, \tau)$ the associated local homogeneous coordinate system on $\rho^{-1}(U_X)$ and by $(y, s; \eta, \tau)$ its image in $\rho^{-1}(U_Y)$ by the $\tau$-preserving contact isomorphism $\psi$ (hence, $y_j = f_j(x, \xi, \tau)$, $\eta_j = g_j(x, \xi, \tau)$ and $s = t + a(x, \xi, \tau)$).

Then, there locally exists a QCT above $\psi$

\[
\Psi : \psi_{\ast}(\mathcal{E}_{X \times \mathbb{C}|\rho^{-1}(U_X)}) \sim \to \mathcal{E}_{Y \times \mathbb{C}|\rho^{-1}(U_Y)}
\]

satisfying:

\[
\Psi \text{ commutes with the anti-involution } *,
\]

\[
\Psi(\partial_t) = \partial_t, \quad \Psi(t) = t + S(x, \partial_x, \partial_t), \quad \sigma_0(S) = a,
\]

\[
\Psi(x_j) = P_j(x, \partial_x, \partial_t), \quad \sigma_0(P_j) = f_j,
\]

\[
\Psi(\partial_{x_j}) = Q_j(x, \partial_x, \partial_t), \quad \sigma_1(Q_j) = g_j.
\]

**Proof.** Quantizing the contact transformation $\psi$ means finding microdifferential operators $P_j(x, t, \partial_x, \partial_t)$ and $S(x, t, \partial_x, \partial_t)$ of order 0, $Q_j(x, t, \partial_x, \partial_t)$ and $T(x, t, \partial_x, \partial_t)$ of order 1, satisfying:

\[
[P, Q_j] = -\delta_{ij}, \quad [P, P_j] = 0, \quad [Q_i, Q_j] = 0
\]

\[
[P, S] = 0, \quad [P, T] = 0, \quad [Q_i, T] = 0, \quad [S, T] = -1,
\]

\[
\sigma_0(P_j) = f_j, \quad \sigma_1(Q_j) = g_j, \quad \sigma_1(T) = \tau.
\]

One may choose $T = \partial_t$ and it follows that $P_j, Q_j, S - t$ will be independent of $t$ and $\sigma_0(S) = a$. q.e.d.
Recall that \( \rho : \hat{P}^*(X \times \mathbb{C}) \to T^*X \) is given by \( \rho(p, (t, \tau)) = (p\tau^{-1}) \).
Taking the direct image by \( \rho \), the isomorphism \( \Psi \) induces an isomorphism
\[ \Phi : \varphi_\ast \rho_\ast (\mathcal{E}_{X \times \mathbb{C}}^\sqrt{v}|_{\rho^{-1}(U_Y)}) \cong \rho_\ast (\mathcal{E}_{Y \times \mathbb{C}}^\sqrt{v}|_{\rho^{-1}(U_Y)}). \]

We identify \( \mathcal{W}_X \) with the subsheaf of algebras of \( \rho_\ast \mathcal{E}_{X \times \mathbb{C}} \) consisting of sections commuting with \( \partial_t \). Since \( \partial_t \) is central in \( \mathcal{W}_X \), we denote it by \( \tau \), that is, we identify the operator and its symbol. Since \( \Psi(\partial_t) = \partial_t \), the isomorphism \( \Phi \) induces an isomorphism of filtered \( k \)-algebras (which we denote by the same symbol):
\[ \Phi : \varphi_\ast (\mathcal{W}_X^\sqrt{v}|_{U_X}) \cong \mathcal{W}_Y^\sqrt{v}|_{U_Y}. \]

**Definition 8.8.** We call the isomorphism \( \Phi \) constructed in \( \text{(8.10)} \) a quantized symplectic transformation (a QST, for short) above \( \varphi \).

Note that \( \Phi \) depends on the \( \tau \)-preserving contact isomorphism \( \psi \).

**Definition 8.9.** Let \( c \) be a section of \( \mathbb{C}_{T^*X} \). We denote by \( \delta_c \) the automorphism of \( \mathbb{C} \)-algebras on \( T^*X \)
\[ \delta_c : \rho_\ast \mathcal{E}_{X \times \mathbb{C}}^\sqrt{v} \to \rho_\ast \mathcal{E}_{X \times \mathbb{C}}^\sqrt{v} \]
\[ P \mapsto \exp(c\partial_t) \circ P \circ \exp(-c\partial_t). \]

In a local coordinates system \( (x, t; \xi, \tau) \),
\[ \delta_c(P(x, t, \partial_x, \partial_t)) = P(x, t + c, \partial_x, \partial_t). \]

Note that \( \delta_c \) induces the identity on \( \mathcal{W}_X^\sqrt{v} \).

**Lemma 8.10.** Let \( \Phi : \rho_\ast \mathcal{E}_{X \times \mathbb{C}}^\sqrt{v} \to \rho_\ast \mathcal{E}_{X \times \mathbb{C}}^\sqrt{v} \) be a QCT above the identity on \( U \subset T^*X \) satisfying the commutation properties of Lemma 8.7. Then there exist \( c \in \mathbb{C} \) and a section \( P \in \mathcal{W}_X^\sqrt{v} \) of order 0 with \( \sigma_0(P) = 1 \) and satisfying \( PP^* = 1 \) such that
\[ \Phi = \delta_c \circ \text{Ad}(P). \]

Moreover, such a pair \( (c, P) \) is unique.

**Proof.** (i) In a local coordinate system, the \( \tau \)-preserving contact transformation \( \psi \) above the identity on \( T^*X \) is given by
\[ \psi(x, t; \xi, \tau) = (x, t + c; \xi, \tau). \]
for a locally constant function \( c \). The transformation \( \Phi^\dagger = \Phi \circ \delta_c \) is thus a QCT above the identity on \( \rho^{-1}(U) \subset \hat{P}^\ast(X \times \mathbb{C}) \) commuting with \( * \) and preserving \( \partial_t \). Therefore, by Lemma 5.41 there exists a unique section \( P \) of \( \rho_* \mathcal{E}^{\sqrt{\nabla}}_{X \times \mathbb{C}} \) of order 0 satisfying \( PP^* = 1 \) such that \( \Phi^\dagger = \text{Ad}(P) \). Since \( \text{Ad}(P)(\partial_t) = \partial_t \), \( P \) does not depend on \( t \), i.e. \( P \) is a section of \( W^{\sqrt{\nabla}}_X \).

(ii) Assume there exist \( P \) as above, not depending on \( t \) and \( c \in \mathbb{C} \) such that \( \text{Ad}(P) = \delta_c \). Then \( P = \exp(c\tau) \). Hence \( P \) cannot belong to \( \mathcal{E}^{\sqrt{\nabla}}_{X \times \mathbb{C}} \), except if \( c = 0 \). q.e.d.

Denote by \( \text{Aut}_*(\rho_* \mathcal{E}^{\sqrt{\nabla}}_{X \times \mathbb{C}}) \) the sheaf of QCT’s above the identity on \( T^*X \) commuting with \( * \) and preserving \( \partial_t \) and \( \text{Aut}_*(W^{\sqrt{\nabla}}_X) \) the sheaf of QST’s above the identity on \( T^*X \) commuting with \( * \). Denote by \( \delta \) the morphism of groups \( \mathbb{C}T^*X \to \text{Aut}_*(\rho_* \mathcal{E}^{\sqrt{\nabla}}_{X \times \mathbb{C}}) \) given by \( c \mapsto \delta_c \) and set

\[
(\mathcal{W}^{\sqrt{\nabla}}_X)_* = \{ P \in W^{\sqrt{\nabla}}_X; \text{P has order 0}, \sigma_0(P) = 1, PP^* = 1 \},
\]

\[
(k_*)_* = \{ a \in k; \text{a has order 0}, \sigma_0(a) = 1, a(\tau)a(-\tau) = 1 \} = (W^{\sqrt{\nabla}}_{pt})_*.
\]

Then the diagram below commutes and has exact rows

\[
\begin{array}{cccc}
1 & \to & \mathbb{C}T^*X \times (\mathcal{W}^{\sqrt{\nabla}}_X)_* & \to & \text{Aut}_*(\rho_* \mathcal{E}^{\sqrt{\nabla}}_{X \times \mathbb{C}}) & \to & 1 \\
\downarrow & & \delta \circ \text{Ad} & & & \downarrow & \\
1 & \to & (k_*)_{T^*X} & \to & (W^{\sqrt{\nabla}}_X)_* & \to & \text{Aut}_*(W^{\sqrt{\nabla}}_X) & \to & 1.
\end{array}
\]

**Remark 8.11.** The results of this section still hold when replacing \( W \) with \( \hat{W} \).

### 9 Quantization of complex symplectic manifolds

**Theorem 9.1.** Let \( \mathfrak{X} \) be a complex symplectic manifold. There exists canonically a \( k \)-abelian stack \( \text{Mod}(W^{\sqrt{\nabla}}_X, \mathfrak{X}) \) on \( \mathfrak{X} \) such that if \( U \subset \mathfrak{X} \) is an open subset isomorphic by a symplectic transformation \( \varphi \) to an open subset \( U_X \subset T^*X \), then \( \text{Mod}(W^{\sqrt{\nabla}}_X|_U, \mathfrak{X}) \) is equivalent by \( \varphi \) to the stack \( \text{Mod}(W^{\sqrt{\nabla}}_{U_X}, \mathfrak{X}) \).

**Definition 9.2.** We call \( \text{Mod}(W^{\sqrt{\nabla}}_X, \mathfrak{X}) \) the stack of WKB-differential modules on \( \mathfrak{X} \).

**Proof.** There exists an open covering \( \mathcal{U} = \{ U_i \}_{i \in I} \) of \( \mathfrak{X} \) and, for each \( U_i \in \mathcal{U} \), a symplectic open embedding \( \varphi_i : U_i \hookrightarrow T^*X_i \) for some cotangent bundle
Consider the symplectic isomorphism $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_{ij}) \xrightarrow{\sim} \varphi_i(U_{ij})$. After shrinking the covering $\mathcal{U}$, by Lemma 4.3 there exists a $\tau$-preserving contact isomorphism $\psi_{ij}$ making the diagram below commutative:

$$
\begin{array}{ccc}
T^*X_j & \xrightarrow{\varphi_{ij}} & \varphi_i(U_{ij}) \\
\downarrow \rho_j & & \downarrow \rho_i \\
\hat{P}^*(X_j \times \mathbb{C}) & \ni \rho_j^{-1}(\varphi_j(U_{ij})) & \ni \rho_i^{-1}(\varphi_i(U_{ij})) \subset \hat{P}^*(X_i \times \mathbb{C}).
\end{array}
$$

We use the notations of Lemma 8.7. Set $\rho_*\mathcal{E}_U^\sqrt{\varpi} = \varphi_i^{-1}\rho_*\mathcal{E}_{X_i \times \mathbb{C}}$ and $\mathcal{W}_U^\sqrt{\varpi} = \varphi_i^{-1}\mathcal{W}_{X_i}^\sqrt{\varpi}$. Hence, $\mathcal{W}_U^\sqrt{\varpi}$ is the subsheaf of $\mathcal{C}$-algebras of $\rho_*\mathcal{E}_U^\sqrt{\varpi}$ consisting of sections commuting with $\partial_t$. By shrinking the covering $\mathcal{U}$ again, we may assume by Lemma 8.7 that there exist QCT’s above the $\psi_{ij}$’s and hence isomorphisms of $\mathcal{C}$-algebras

$$
\Phi_{ij} : \rho_*\mathcal{E}_U^\sqrt{\varpi} \ni \rho_*\mathcal{E}_{U_{ij}}^\sqrt{\varpi},
$$

these isomorphisms commuting with $\partial_t$ and $\ast$.

With the notations of Section 4 for $X = \mathfrak{x}$, set $\mathfrak{x}_0 = \bigsqcup U_i$, $\mathfrak{x}_1 = \mathfrak{x}_0 \times \mathfrak{x}_0$, etc. and let $j_{U_i} : U_i \hookrightarrow \mathfrak{x}_0$ be the natural map. Then $\rho_*\mathcal{E}_U^\sqrt{\varpi} = \oplus_{i \in U_i} \rho_*\mathcal{E}^\sqrt{\varpi}_{U_i}$ is a sheaf of central $\mathcal{C}$-algebras on $\mathfrak{x}_0$ endowed with an anti-involution and $\mathcal{W}_U^\sqrt{\varpi} = \oplus_{i \in U_i} \mathcal{W}_{U_i}^\sqrt{\varpi}$ is a sub-sheaf of algebras. The $\Phi_{ij}$’s induce a $\mathcal{C}$-algebra isomorphism $\Phi : \rho_*\mathcal{E}_1^{\sqrt{\varpi}} \xrightarrow{\sim} \rho_*\mathcal{E}_0^{\sqrt{\varpi}}$ commuting with $\partial_t$ and with the anti-involution.

By Lemma 8.10 there exist $P \in \Gamma(\mathfrak{x}_2; \mathcal{W}_0^{\sqrt{\varpi}})$ of order $0$ with $\sigma_0(P) = 1$, $PP^* = 1$ and $c \in \Gamma(\mathfrak{x}_2; \mathbb{C}\mathfrak{x}_2)$ such that on $\mathfrak{x}_2$

$$
\Phi_{01} \circ \Phi_{12} = \text{Ad}(P) \circ \delta_c \circ \Phi_{02}.
$$

Now we make a computation similar to that leading to Definition 3.2.

On $\mathfrak{x}_3$ one has

$$
\begin{align*}
(\Phi_{01} \circ \Phi_{12}) \circ \Phi_{23} &= \text{Ad}(P_{012}) \circ \delta_c \circ \Phi_{02} \circ \Phi_{23} \\
&= \text{Ad}(P_{012}P_{023}) \circ \delta_c \circ \Phi_{03},
\end{align*}
$$

and

$$
\begin{align*}
\Phi_{01} \circ (\Phi_{12} \circ \Phi_{23}) &= \Phi_{01} \circ \text{Ad}(P_{123}) \circ \delta_c \circ \Phi_{13} \\
&= \text{Ad}(\Phi_{01}(P_{123})) \circ \delta_c \circ \Phi_{01} \circ \Phi_{13} \\
&= \text{Ad}(\Phi_{01}(P_{123})P_{013}) \circ \delta_c \circ \Phi_{03}.
\end{align*}
$$
Here, we have used the fact that the isomorphisms \( \Phi \) commutes with \( \delta_c \). It follows that

\[
\begin{align*}
R_{012}P_{023} &= \Phi_{01}(P_{123})P_{013} \quad \text{in } \Gamma(X_3; W_0^{√v^x}), \\
c_{012} + c_{023} &= c_{123} + c_{013} \quad \text{in } \Gamma(X_3; C_{X_3}).
\end{align*}
\]

The \( \mathbb{C} \)-algebra isomorphism \( \Phi : \rho_∗E_1^{√v} \sim \rho_∗E_0^{√v} \) induces an isomorphism of \( k \)-algebras (which we denote by the same symbol):

\[
\Phi : W_1^{√v} \sim W_0^{√v}.
\]

Since the isomorphisms \( \delta_c \) induce the identity on \( W_0^{√v} \), we get

\[
\begin{align*}
\Phi_{01} \circ \Phi_{12} &= \text{Ad}(P) \circ \Phi_{02} \quad \text{on } X_2, \\
P_{012}P_{023} &= \Phi_{01}(P_{123})P_{013} \quad \text{in } \Gamma(X_3; W_0^{√v^x}).
\end{align*}
\]

Hence \( W := (W_0^{√v}, \Phi, P) \) is an effective \( k \)-lien on \( X_• \), and it remains to apply Theorem \( 3.5 \).

To prove the canonicity of the stack \( \mathcal{M}_{\text{Mod}}(W_0^{√v}, X) \), one argues as in the proof of Theorem \( 6.1 \). More precisely, for each \( U_i ∈ U \) consider another symplectic open embedding \( ϕ'_i : U_i ↪ T^*X_i' \). Setting \( ρ_∗E_{U_i} = ϕ'^{-1}_i ∗ ρ_∗E_{X_i'} \) and \( W_{U_i} = ϕ'^{-1}_i W_0^{√v} \), and proceeding as above, we get a \( \mathbb{C} \)-algebra isomorphism \( \Phi' : ρ_∗E_1^{√v'} \sim ρ_∗E_0^{√v'} \) commuting with \( ∂_t \) and with the anti-involution and thus an effective \( k \)-lien \( W' := (W_0^{√v'}, \Phi', P') \) on \( X_• \) and a 2-cocycle \( c' \in Γ(X_1; C_{X_1}) \).

By Lemma \( 4.3 \) there exists a \( τ \)-preserving contact isomorphism \( v_i \) making the corresponding diagram \( 4.2 \) commutative. By Lemma \( 5.3 \) there exist QCT’s above the \( v_i \)'s and hence an isomorphism of \( \mathbb{C} \)-algebras

\[
\begin{align*}
\Upsilon : ρ_∗E^{√v} &\sim ρ_∗E^{√v'}
\end{align*}
\]

commuting with \( ∂_t \) and \( ∗ \). By Lemma \( 8.10 \) there exist a unique pair \( (Q, b) \), where \( Q ∈ Γ(X_1; W_0^{√v}) \) of order 0 with \( σ_0(Q) = 1, QQ^∗ = 1 \) and \( b ∈ Γ(X_1; C_{X_1}) \) are such that

\[
\Phi' \circ \Upsilon_1 = \text{Ad}(Q) \circ ∂_t \circ \Upsilon_0 \circ \Phi.
\]

After some computations and using the unicity of the pair \( (Q, b) \), we get

\[
\begin{align*}
P_{021}Q_{012} &= \Phi'_{01}(Q_{12})Q_{01} \Upsilon_0(P) \quad \text{in } \Gamma(X_2; W_0^{√v^x}), \\
c' + b_{02} &= b_{12} + b_{01} + c \quad \text{in } \Gamma(X_2; C_{X_2}).
\end{align*}
\]
Hence the 2-cocycle $c$ and $c'$ are cohomologous and the pair $(\Phi, Q)$ defines an effective isomorphism of $k$-liens $W \sim \rightarrow W'$. 

**Remarks 9.3.** (i) The Cech cohomology class in $H^2(\mathcal{X}; \mathbb{C})$ defined by the 2-cocycle $c$ in (9.1) coincides with the class $-\lbrack \omega \rbrack$ of the symplectic form $\omega$ on $\mathcal{X}$ under the de Rham isomorphism. Indeed, let $\mathcal{U} = \{U_i\}_{i \in I}$ be the open covering of $\mathcal{X}$ as in the proof of Theorem 9.1 and let $\alpha_{T^*X_i}$ be the canonical 1-form on $T^*X_i$. Then $\omega|_{U_i} = d(\varphi_i^*\alpha_{T^*X_i})$ and $(\varphi_i^*\alpha_{T^*X_j})|_{U_{ij}} = (\varphi_i^*\alpha_{T^*X_i})|_{U_{ij}} + da_{ij}$, for a function $a_{ij}$ on $U_{ij}$. By Lemma 4.3, the contact isomorphisms $\psi_{ij}$ in (4.2) are given by 

$$(p, (t; \tau)) \mapsto (\varphi_{ij}(p\tau^{-1}), (t + a_{ij}(p\tau^{-1}); \tau)).$$

Hence the $c_{ijk}$ coincide with $a_{ij} + a_{jk} - a_{ik}$, i.e. the Cech class given by the $c_{ijk}$’s corresponds to the class $-\lbrack \omega \rbrack$ under the de Rham isomorphism.

(ii) Recall that the group $k_*$ has been defined in (8.14). For each 2-cocyle $a \in \Gamma(\mathcal{X}; (k_*)\mathcal{X})$ we may associate a $k$-abelian stack $\textnormal{Mod}(\sqrt{W}, a, \mathcal{X})$ on $\mathcal{X}$ which is locally equivalent to the stack $\textnormal{Mod}(\sqrt{W}, \mathcal{X})$. Indeed, $\textnormal{Mod}(\sqrt{W}, a, \mathcal{X})$ is the stack associated with the effective lien $W_a := (\sqrt{W}, \Phi, aP)$. It is straightforward to check that, if $b$ is another 2-cocyle cohomologous to $a$, there is an effective isomorphism of $k$-liens $W_a \sim \rightarrow W_b$ and then an equivalence of $k$-stacks $\textnormal{Mod}(\sqrt{W}, a, \mathcal{X}) \simeq \textnormal{Mod}(\sqrt{W}, b, \mathcal{X})$. We refer to [7] for a similar construction in the framework of deformation algebras on real symplectic manifolds.

**Remark 9.4.** One can extend the results of Section 7 to the symplectic case and treat smooth involutive submanifolds of complex symplectic manifolds. But one should recall that Kontsevich [14] has obtained a much more general result, quantizing arbitrary complex Poisson manifolds.

**Remark 9.5.** The results of this section still hold when replacing $W$ with $\hat{W}$.

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