Topology and pion correlators – a study in the $N_f=2$ Schwinger model

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I readdress the issue whether the topological charge of the gauge background has an influence on a hadronic observable. To this end pion correlators in the Schwinger model with 2 dynamical flavours are determined on subensembles with a fixed topological charge. It turns out that the answer depends on a specific function of the sea-quark mass and the box volume which is in close analogy to the Leutwyler-Smilga parameter in full QCD.

1. INTRODUCTION

Does topology matter? Or, more explicitly: Does the global topological charge of a QCD configuration have an influence on a typical hadron correlator determined on that background and does thus the charge distribution of an ensemble affect physical measurements?

In a first round, one has to distinguish between observables which relate to the $U(1)_A$ issue and those which do not. The mass of the $\eta'$ is known to be sensitive, since it depends directly on the distribution of topological charges via the explicit breaking of the $U(1)_A$ symmetry through instantons. On the other hand, standard observables like $M_\pi$, $M_K$, $M_\eta$, $M_\rho$ or the heavy-quark potential $V_{q\bar{q}}$ are generally known to be independent of the index of the background. Below, I address the latter category and the precise conditions (regarding the four-volume $V$ and the sea-quark mass $m$) that have to be met for their supposed insensitivity on topology to become true.

Leutwyler and Smilga have shown that for the specific case of pionic observables much can be said on analytical grounds, if the parameter

$$x \equiv \frac{V}{\Sigma m},$$  \hspace{1cm} (1)

is known. Its role is to discriminate ‘small’ from ‘large’ boxes in the sense that the magnitude of $x$, decides whether the system prefers to show symmetry restoration or spontaneous symmetry breaking phenomena ($N_f \geq 2$): For $x \ll 1$ the chiral symmetry is effectively restored and a good description is in terms of quarks and gluons. For $x \gg 1$ the $SU(N_f)_A$ symmetry is effectively broken (though the box-volume is formally finite), meaning that pions represent appropriate degrees of freedom. Below it is important to keep in mind that the classification w.r.t. $x$ is independent of the standard classification w.r.t. the ratio ‘pion correlation length to box size’ $(1/M_{\pi}L)$.

For $x \ll 1$ the partition function is dominated by the topologically trivial sector $Z_\nu \propto \exp\left(\frac{\nu^2}{2}\right)$, and one expects a clear sectoral dependence of all observables, including the standard $M_\pi$.

In the opposite regime $x \gg 1$ and with the auxiliary condition that the quark masses are so light that the sea-pion substantially overlaps the box $(1/\Lambda_{\text{eff}} \ll L \ll 1/M_{\pi})$, the path-integral in the effective description is dominated by the constant mode $Z_\nu \propto \exp\left(-\frac{\nu^2}{2}\right)$ with $\langle \bar{\psi}\psi \rangle = V \Sigma m$, and as a result standard observables are expected to be approximately independent of $\nu$.

From a lattice perspective, the main problem with the Leutwyler-Smilga (LS) analysis is that it involves the wrong ‘auxiliary’ condition in the large $x$ regime (one would like the pion to fit into the box rather than to spread itself uniformly in three-space). Furthermore, it would be desirable to investigate the issue directly at the level of observables rather than to deduce their sensitivity on $x$ from that of the partition function.
2. SIMULATION DETAILS

As the issue is peculiar to the dynamical theory, a pilot study in a suitable model seems justified. The massive multiflavour Schwinger model (QED$_2$ with $N_f \geq 2$) shares the qualitative features needed \cite{3}. The ensemble is generated with the Wilson gauge action $S_{\text{gauge}} = \beta \sum (1 - \cos \theta \Box)$ and a pair of staggered fermions. The plan is to compare the regimes $x \ll 1$, $x \simeq 1$, and $x \gg 1$ to each other using three dedicated simulations: Working at fixed $\beta = 1/g^2 = 3.4$ and $m = 0.09$, the regimes are represented by the volumes $V = 8 \times 4, 18 \times 6, 40 \times 10$. Given the approximate solution of the bosonized model, the LS parameter is expected to take the values $x \simeq 0.33, 1.12, 4.16$, respectively, and the ‘pion’ (pseudo-scalar isotriplet) to have a mass $M_x \simeq 0.329$ and hence a correlation length $\xi_x \simeq 3.04$ as to fit into the box (see \cite{3} for details and \cite{4} for references to and an assessment of the bosonized solution).

A configuration is assigned an index only if the geometric ($\nu_{\text{geo}} = \frac{1}{g^2} \sum \log U$) and the field-theoretic definition ($\nu_{\text{th}} = \kappa \nu_{\text{nai}}, \nu_{\text{nai}} = \sum \sin \theta \Box$ with $\kappa \simeq 1/(1 - (S_{\text{gauge}})/3V)$, after rounding to the nearest integer, agree. Since this turned out to be the case for 99.9%, 98.8%, 88.1% of the configurations in the small/intermediate/large lattice, it means that for the majority of configurations an assignment can be done without cooling. Fig. 1 shows that the associate partition function obeys the LS prediction: For $x \ll 1$ it essentially consists of the charge zero contribution, for $x \gg 1$ it gets broad and gaussian – even though the ‘auxiliary’ condition $M_x L \ll 1$ in the LS analysis has been reversed into what is usual on the lattice.

3. SECTORAL PION PROPAGATORS

The final step is to evaluate the pseudoscalar two-point correlator

$$\langle (\bar{u} \gamma_5 d)(0)(\bar{d} \gamma_5 u)(x) \rangle$$  \hspace{1cm} (4)

at spacelike separations (note that in the chiral limit the multiflavour Schwinger model shows a second order phase-transition with $T_c = 0$ \cite{4}, hence the rectangular shape of the manifolds). For practical reasons, I have built the propagator from Wilson fermions, even though there is an imminent risk that this ‘hybrid’ formulation (sea-quarks staggered, current-quarks Wilson) suffers from serious field-theoretic problems (e.g. it’s not clear whether there exists a bounded, symmetric, positive transfer matrix \cite{4}). To avoid a ‘partially quenched’ situation, $\kappa$ is chosen such that $\frac{1}{2}(\kappa^{-1} - \kappa_{\text{crit}}^{-1}) \simeq m$. The result for the correlators is shown in Fig. 2, together with fits to the sum of two cosh-functions.

As expected, a pronounced sectoral dependence of the pseudoscalar propagator shows up in the small LS regime. For $x \simeq 1$ there is a remnant sensitivity: the sectoral pion masses as extracted from the fits still have a tendency to decrease with $|\nu|$, but none of them is far from the physical value. In the large LS regime ($x \gg 1$) the sectoral correlators (and hence the sectoral pion masses)\footnote{Assuming universality in $x$, it seems that for $m \rightarrow 0$ at fixed $V$ the dominant long-range contribution to \cite{4} stems from $\nu = \pm 1$, and this is at variance with what happens if one works directly at $m = 0$ in a finite volume, since there only $\nu = 0, \pm 2$ contribute \cite{4} i.e. our findings support previous skepticism about the mass perturbation approach in the Schwinger model with $N_f \geq 2$ \cite{4}.}.
agree surprisingly well with each other. In summary, our findings confirm the LS prediction [1], even though their ‘auxiliary’ condition at large $x$ has been reversed as to guarantee that the pion would fit into the box (what is the usual situation on the lattice). It would be interesting to see this type of investigation repeated in full QCD.

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