Strong Matching of Points with Geometric Shapes

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Abstract

Let $P$ be a set of $n$ points in general position in the plane. Given a convex geometric shape $S$, a geometric graph $G_S(P)$ on $P$ is defined to have an edge between two points if and only if there exists an empty homothet of $S$ having the two points on its boundary. A matching in $G_S(P)$ is said to be strong, if the homothets of $S$ representing the edges of the matching, are pairwise disjoint, i.e., do not share any point in the plane. We consider the problem of computing a strong matching in $G_S(P)$, where $S$ is a diametral-disk, an equilateral-triangle, or a square. We present an algorithm which computes a strong matching in $G_S(P)$; if $S$ is a diametral-disk, then it computes a strong matching of size at least $\lceil n - \frac{11}{17} \rceil$, and if $S$ is an equilateral-triangle, then it computes a strong matching of size at least $\lceil n - \frac{1}{9} \rceil$. If $S$ can be a downward or an upward equilateral-triangle, we compute a strong matching of size at least $\lceil n - \frac{1}{4} \rceil$ in $G_S(P)$. When $S$ is an axis-aligned square we compute a strong matching of size $\lceil n - \frac{1}{4} \rceil$ in $G_S(P)$, which improves the previous lower bound of $\lceil \frac{n}{5} \rceil$.

1 Introduction

Let $S$ be a compact and convex set in the plane that contains the origin in its interior. A homothet of $S$ is obtained by scaling $S$ with respect to the origin by some factor $\mu \geq 0$, followed by a translation to a point $b$ in the plane: $b + \mu S = \{b + \mu a : a \in S\}$. For a point set $P$ in the plane, we define $G_S(P)$ as the geometric graph on $P$ which has a straight-line edge between two points $p$ and $q$ if and only if there exists a homothet of $S$ having $p$ and $q$ on its boundary and whose interior does not contain any point of $P$. If $S$ is a disk whose center is the origin, then $G_S(P)$ is the Delaunay triangulation of $P$. If $S$ is an equilateral triangle $\triangle$ whose barycenter is the origin, then $G_\triangle(P)$ is the triangular-distance Delaunay graph of $P$ which is introduced by Chew [10].

A matching in a graph $G$ is a set of edges which do not share any vertices. A maximum matching is a matching with maximum cardinality. A perfect matching is a matching which matches all the vertices of $G$. Let $M$ be a matching in $G_S(P)$. $M$ is referred to as a matching of points with shape $S$, e.g., a matching in $G_\triangle(P)$ is a matching of points with with disks. Let $S_M$ be a set of homothets of $S$ representing the edges of $M$. $M$ is called a strong matching if there exists a set $S_M$ whose elements are pairwise disjoint, i.e., the objects in $S_M$ do not share any point in the plane. Otherwise, $M$ is a weak matching. See Figure 1. To be consistent with the definition of the matching in the graph theory, we use the term “matching” to refer to a weak matching. Given a point set $P$ in the plane and a shape $S$, the (strong) matching problem

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is to compute a (strong) matching of maximum cardinality in $G_S(P)$. In this paper we consider the strong matching problem of points in general position in the plane with respect to a given shape $S \in \{\odot, \triangle, \square\}$ (see Section 2 for the definition), where by $\odot$ we mean the line segment between the two points on the boundary of the disk is a diameter of that disk.

Figure 1: Point set $P$ and (a) a perfect weak matching in $G_\odot(P)$, (b) a perfect strong matching in $G_\triangle(P)$, and (c) a perfect strong matching in $G_\square(P)$.

1.1 Previous Work

The problem of computing a maximum matching in $G_S(P)$ is one of the fundamental problems in computational geometry and graph theory [1, 2, 3, 5, 6, 7, 11]. Dillencourt [11] and Æárego et al. [1] considered the problem of matching points with disks. Let $S$ be a closed disk $\odot$ whose center is the origin, and let $P$ be a set of $n$ points in the plane which is in general position with respect to $\odot$. Then, $G_\odot(P)$ is the graph which has an edge between two points $p, q \in P$ if there exists a homothet of $\odot$ having $p$ and $q$ on its boundary and does not contain any point of $P \setminus \{p, q\}$. $G_\odot(P)$ is equal to the Gabriel graph on $P$, $DG(P)$. Dillencourt [11] proved that $G_\odot(P)$ contains a perfect (weak) matching. Æárego et al. [1] proved that $G_\odot(P)$ has a strong matching of size at least $\lceil (n - 1)/8 \rceil$. They also showed that there exists a set $P$ of $n$ points in the plane with arbitrarily large $n$, such that $G_\odot(P)$ does not contain a strong matching of size more than $\frac{39}{38}n$.

For two points $p$ and $q$, the disk which has the line segment $pq$ as its diameter is called the diametral-disk between $p$ and $q$. We denote a diametral-disk by $\odot$. Let $G_\odot(P)$ be the graph which has an edge between two points $p, q \in P$ if the diametral-disk between $p$ and $q$ does not contain any point of $P \setminus \{p, q\}$. $G_\odot(P)$ is equal to the Gabriel graph on $P$, $DG(P)$. Biniaz et al. [6] proved that $G_\odot(P)$ has a matching of size at least $\lceil (n - 1)/4 \rceil$, and this bound is tight.

The problem of matching of points with equilateral triangles has been considered by Babu et al. [3]. Let $S$ be a downward equilateral triangle $\blacktriangle$ whose barycenter is the origin and one of its vertices is on the negative $y$-axis. Let $P$ be a set of $n$ points in the plane which is in general position with respect to $\blacktriangle$. Let $G_\blacktriangle(P)$ be the graph which has an edge between two points $p, q \in P$ if there exists a homothet of $\blacktriangle$ having $p$ and $q$ on its boundary and does not contain any point of $P \setminus \{p, q\}$. $G_\blacktriangle(P)$ is equal to the triangular-distance Delaunay graph on $P$, which was introduced by Chew [10]. Bonichon et al. [8] showed that $G_\blacktriangle(P)$ is equal to the half-theta six graph on $P$, $\frac{1}{2}\Theta_6(P)$. Babu et al. [3] proved that $G_\blacktriangle(P)$ has a matching of size at least $\lceil (n - 1)/3 \rceil$, and this bound is tight. If we consider an upward triangle $\blacktriangle$, then $G_\blacktriangle(P)$ is defined similarly. Let $G_\blacktriangle(P)$ be the graph on $P$ which is the union of $G_\blacktriangle(P)$ and $G_\blacktriangle(P)$. Bonichon et al. [8] showed that $G_\blacktriangle(P)$ is equal to the theta six graph on $P$, $\Theta_6(P)$. Since $G_\blacktriangle(P)$ is a subgraph of $G_\blacktriangle(P)$, the lower bound of $\lceil (n - 1)/3 \rceil$ on the size of maximum matching in $G_\blacktriangle(P)$ holds for $G_\blacktriangle(P)$.
Theorem 2

Theorem 5

Theorem 6

Theorem 7

Theorem 8

Theorem 9

Theorem 10

Table 1: Lower bounds on the size of weak and strong matchings in $G_S(P)$.

| $S$     | weak matching | reference | strong matching | reference |
|---------|---------------|-----------|----------------|-----------|
| □       | $\left\lceil \frac{n}{2} \right\rceil$ | [11] 2 | $\left\lceil \frac{n}{4} \right\rceil$ | [11] 2 |
| ◦       | $\left\lceil \frac{n-1}{2} \right\rceil$ | [6] 3 | $\left\lceil \frac{n+1}{2} \right\rceil$ | [11] 1 |
| ▽       | $\left\lceil \frac{n-1}{2} \right\rceil$ | [3] 4 | $\left\lceil \frac{n-1}{2} \right\rceil$ | [11] 1 |
| ▽ or △ | $\left\lceil \frac{n-1}{2} \right\rceil$ | [3] 4 | $\left\lceil \frac{n-1}{2} \right\rceil$ | [11] 1 |

The problem of strong matching of points with axis-aligned rectangles is trivial. An obvious algorithm is to repeatedly match the two leftmost points. The problem of matching points with axis-aligned squares was considered by Ábrego et al. [2]. Let $S$ be an axis-aligned square whose center is the origin. Let $P$ be a set of $n$ points in the plane which is in general position with respect to □. Let $G_\Box(P)$ be the graph which has an edge between two points $p, q \in P$ if there exists a homothet of □ having $p$ and $q$ on its boundary and does not contain any point of $P \setminus \{p, q\}$. $G_\Box(P)$ is equal to the $L_\infty$-Delaunay graph on $P$. Ábrego et al. [11 2] proved that $G_\Box(P)$ has a perfect (weak) matching and a strong matching of size at least $\lceil n/5 \rceil$. Further, they showed that there exists a set $P$ of $n$ points in the plane with arbitrarily large $n$, such that $G_\Box(P)$ does not contain a strong matching of size more than $\frac{5}{2n}$. Table 1 summarizes the results.

Bereg et al. [3] concentrated on matching points of $P$ with axis-aligned rectangles and squares, where $P$ is not necessarily in general position. They proved that any set of $n$ points in the plane has a strong rectangle matching of size at least $\lceil \frac{n}{9} \rceil$, and such a matching can be computed in $O(n \log n)$ time. As for squares, they presented a $\Theta(n \log n)$ time algorithm that decides whether a given matching has a weak square realization, and an $O(n^2 \log n)$ time algorithm for the strong square matching realization. They also proved that it is NP-hard to decide whether a given point set has a perfect strong square-matching.

1.2 Our results

In this paper we consider the problem of computing a strong matching in $G_S(P)$, where $S \in \{\Box, \triangledown, \square\}$. In Section 2, we provide some observations and prove necessary Lemmas. Given a point set $P$ in which is in general position with respect to a given shape $S$, in Section 3 we present an algorithm which computes a strong matching in $G_S(P)$. In Section 4, we prove that if $S$ is a diametral-disk, then the algorithm of Section 3 computes a strong matching of size at least $\lceil (n-1)/17 \rceil$ in $G_\Box(P)$. In Section 5, we prove that if $S$ is an equilateral triangle, then the algorithm of Section 3 computes a strong matching of size at least $\lceil (n-1)/9 \rceil$ in $G_\triangledown(P)$. In Section 6 we compute a strong matching of size at least $\lceil (n-1)/4 \rceil$ in $G_\Box(P)$. In Section 7 we compute a strong matching of size at least $\lceil (n-1)/4 \rceil$ in $G_\Box(P)$; this improves the previous lower bound of $\lceil n/5 \rceil$. A summary of the results is given in Table 1. In Section 8 we discuss a possible way to further improve upon the result obtained for diametral-disks in Section 4. Concluding remarks and open problems are given in Section 9.

2 Preliminaries

Let $S \in \{\Box, \triangledown, \square\}$, and let $S_1$ and $S_2$ be two homothets of $S$. We say that $S_1$ is smaller than $S_2$ if the area of $S_1$ is smaller than the area of $S_2$. For two points $p, q \in P$, let $S(p, q)$ be a smallest
homothet of \( S \) having \( p \) and \( q \) on its boundary. If \( S \) is a diametral-disk, a downward equilateral-triangle, or a square, then we denote \( S(p, q) \) by \( D(p, q) \), \( t(p, q) \), or \( Q(p, q) \), respectively. If \( S \) is a diametral-disk, then \( D(p, q) \) is uniquely defined by \( p \) and \( q \). If \( S \) is an equilateral-triangle or a square, then \( S \) has the shrinkability property: if there exists a homothet \( S' \) of \( S \) that contains two points \( p \) and \( q \), then there exists a homothet \( S'' \) of \( S \) such that \( S'' \subseteq S' \), and \( p \) and \( q \) are on the boundary of \( S'' \). If \( S \) is an equilateral-triangle, then we can shrink \( S'' \) further, such that each side of \( S'' \) contains either \( p \) or \( q \). If \( S \) is a square, then we can shrink \( S'' \) further, such that \( p \) and \( q \) are on opposite sides of \( S'' \). Thus, we have the following observation:

**Observation 1.** For two points \( p, q \in P \),

- \( D(p, q) \) is uniquely defined by \( p \) and \( q \), and it has the line segment \( pq \) as a diameter.
- \( t(p, q) \) is uniquely defined by \( p \) and \( q \), and it has one of \( p \) and \( q \) on a corner and the other point is on the side opposite to that corner.
- \( Q(p, q) \) has \( p \) and \( q \) on opposite sides.

![Figure 2: Illustration of Observation 2](image)

Given a shape \( S \in \{\ominus, \triangledown, \square\} \), we define an order on the homothets of \( S \). Let \( S_1 \) and \( S_2 \) be two homothets of \( S \). We say that \( S_1 \prec S_2 \) if the area of \( S_1 \) is less than the area of \( S_2 \). Similarly, \( S_1 \leq S_2 \) if the area of \( S_1 \) is less than or equal to the area of \( S_2 \). We denote the homothet with the larger area by \( \max\{S_1, S_2\} \). As illustrated in Figure 2, if \( S(p, q) \) contains a point \( r \), then both \( S(p, r) \) and \( S(q, r) \) have smaller area than \( S(p, q) \). Thus, we have the following observation:

**Observation 2.** If \( S(p, q) \) contains a point \( r \), then \( \max\{S(p, r), S(q, r)\} \prec S(p, q) \).

**Definition 1.** Given a point set \( P \) and a shape \( S \in \{\ominus, \triangledown, \square\} \), we say that \( P \) is in “general position” with respect to \( S \) if

- \( S = \ominus \): no four points of \( P \) lie on the boundary of any diametral disk defined by any two points of \( P \).
- \( S = \triangledown \): the line passing through any two points of \( P \) does not make angles 0°, 60°, or 120° with the horizontal. This implies that no four points of \( P \) are on the boundary of any homothet of \( \triangledown \).
- \( S = \square \): (i) no two points in \( P \) have the same \( x \)-coordinate or the same \( y \)-coordinate, and (ii) no four points of \( P \) lie on the boundary of any homothet of \( \square \).

Given a point set \( P \) which is in general position with respect to a given shape \( S \in \{\ominus, \triangledown, \square\} \), let \( K_S(P) \) be the complete edge-weighted geometric graph on \( P \). For each edge \( e = (p, q) \) in \( K_S(P) \), we define \( S(e) \) to be the shape \( S(p, q) \), i.e., a smallest homothet of \( S \) having \( p \) and \( q \) on
its boundary. We say that \( S(e) \) represents \( e \), and vice versa. Furthermore, we assume that the weight \( w(e) \) (resp. \( w(p,q) \)) of \( e \) is equal to the area of \( S(e) \). Thus,

\[
w(p,q) < w(r,s) \quad \text{if and only if} \quad S(p,q) < S(r,s).
\]

Note that \( G_S(P) \) is a subgraph of \( K_S(P) \), and has an edge \((p,q)\) if \( S(p,q) \) does not contain any point of \( P \setminus \{p,q\} \).

**Lemma 1.** Let \( P \) be a set of \( n \) points in the plane which is in general position with respect to a given shape \( S \in \{\emptyset, \triangledown, \square\} \). Then, any minimum spanning tree of \( K_S(P) \) is a subgraph of \( G_S(P) \).

**Proof.** The proof is by contradiction. Assume there exists an edge \( e = (p,q) \) in a minimum spanning tree \( T \) of \( K_S(P) \) such that \( e \notin G_S(P) \). Since \((p,q)\) is not an edge in \( G_S(P) \), \( S(p,q) \) contains a point \( r \) such that \( r \in P \setminus \{p,q\} \). By Observation 2, \( \max\{S(p,r),S(q,r)\} < S(p,q) \). Thus, \( w(p,r) < w(p,q) \) and \( w(q,r) < w(p,q) \). By replacing the edge \((p,q)\) in \( T \) with either \((p,r)\) or \((q,r)\), we obtain a spanning tree in \( K_S(P) \) which is smaller than \( T \). This contradicts the minimality of \( T \).

**Lemma 2.** Let \( G \) be an edge-weighted graph with edge set \( E \) and edge-weight function \( w : E \to \mathbb{R}^+ \). For any cycle \( C \) in \( G \), if the maximum-weight edge in \( C \) is unique, then that edge is not in any minimum spanning tree of \( G \).

**Proof.** The proof is by contradiction. Let \( e = (u,v) \) be the unique maximum-weight edge in a cycle \( C \) in \( G \), such that \( e \) is in a minimum spanning tree \( T \) of \( G \). Let \( T_u \) and \( T_v \) be the two trees obtained by removing \( e \) from \( T \). Let \( e' = (x,y) \) be an edge in \( C \) which connects a vertex \( x \in T_u \) to a vertex \( y \in T_v \). By assumption, \( w(e') < w(e) \). Thus, in \( T \), by replacing \( e \) with \( e' \), we obtain a tree \( T' = T_u \cup T_v \cup \{(x,y)\} \) in \( G \) such that \( w(T') < w(T) \). This contradicts the minimality of \( T \).

Recall that \( t(p,q) \) is the smallest homothet of \( \triangledown \) which has \( p \) and \( q \) on its boundary. Similarly, let \( t'(p,q) \) denote the smallest upward equilateral-triangle \( \triangle \) having \( p \) and \( q \) on its boundary. Note that \( t'(p,q) \) is uniquely defined by \( p \) and \( q \), and it has one of \( p \) and \( q \) on a corner and the other point is on the side opposite to that corner. In addition the area of \( t'(p,q) \) is equal to the area of \( t(p,q) \).

\[ G_{\triangledown}(P) \] is equal to the triangular-distance Delaunay graph \( TD-DG(P) \), which is in turn equal to a half theta-six graph \( \frac{1}{3} \Theta_6(P) \) \cite{3}. A half theta-six graph on \( P \), and equivalently \( G_{\triangledown}(P) \), can be constructed in the following way. For each point \( p \) in \( P \), let \( l_p \) be the horizontal line through \( p \). Define \( l_p^\gamma \) as the line obtained by rotating \( l_p \) by \( \gamma \)-degrees in counter-clockwise direction around \( p \). Thus, \( l_p^0 = l_p \). Consider three lines \( l_p^0 \), \( l_p^60 \), and \( l_p^{120} \) which partition the plane into six disjoint cones with apex \( p \). Let \( C_p^1, \ldots, C_p^6 \) be the cones in counter-clockwise order around \( p \) as shown in Figure 3 \cite{3}. \( C_p^1, C_p^3, C_p^5 \) will be referred to as odd cones, and \( C_p^2, C_p^4, C_p^6 \) will be referred to as even cones. For each even cone \( C_p^i \), connect \( p \) to the “nearest” point \( q \) in \( C_p^i \). The distance between \( p \) and \( q \), is defined as the Euclidean distance between \( p \) and the orthogonal projection of \( q \) onto the bisector of \( C_p^i \). See Figure 3 in \cite{3}. In other words, the nearest point to \( P \) in \( C_p^i \) is a point \( q \) in \( C_p^i \) which minimizes the area of \( t(p,q) \). The resulting graph is the half theta-six graph which is defined by even cones \cite{3}. Moreover,
the resulting graph is $G_{\triangledown}(P)$ which is defined with respect to the homothets of $\triangledown$. By considering the odd cones, $G_{\triangle}(P)$ is obtained. By considering the odd cones and the even cones, $G_{\Box}(P)$—which is equal to $\Theta_0(P)$—is obtained. Note that $G_{\Box}(P)$ is the union of $G_{\triangledown}(P)$ and $G_{\triangle}(P)$.

Let $X(p, q)$ be the regular hexagon centered at $p$ which has $q$ on its boundary, and its sides are parallel to $t_{p}^{0}$, $t_{p}^{60}$, and $t_{p}^{120}$. Then, we have the following observation:

**Observation 3.** If $X(p, q)$ contains a point $r$, then $t(p, r) \prec t(p, q)$.

## 3 Strong Matching in $G_{S}(P)$

Given a point set $P$ in the plane which is in general position with respect to a given shape $S \in \{\ominus, \triangledown, \square\}$, in this section we present an algorithm which computes a strong matching in $G_{S}(P)$. Recall that $K_{S}(P)$ is the complete edge-weighted graph on $P$ with the weight of each edge $e$ is equal to the area of $S(e)$, where $S(e)$ is a smallest homothet of $S$ representing $e$. Let $T$ be a minimum spanning tree of $K_{S}(P)$. By Lemma 1, $T$ is a subgraph of $G_{S}(P)$. For each edge $e \in T$ we denote by $T(e^+)$ the set of all edges in $T$ whose weight is at least $w(e)$. Moreover, we define the **influence set** of an edge, as the set of all edges in $T(e^+)$ whose representing shapes overlap with $S(e)$, i.e.,

$$\text{Inf}(e) = \{ e^\prime : e^\prime \in T(e^+) \wedge S(e^\prime) \cap S(e) \neq \emptyset \}.$$ 

Note that $\text{Inf}(e)$ is not empty, as $e \in \text{Inf}(e)$. Consequently, we define the **influence number** of $T$ to be the maximum size of a set among the influence sets of edges in $T$, i.e.,

$$\text{Inf}(T) = \max\{|\text{Inf}(e)| : e \in T\}.$$ 

Algorithm 1 receives $G_{S}(P)$ as input and computes a strong matching in $G_{S}(P)$ as follows. The algorithm starts by computing a minimum spanning tree $T$ of $G_{S}(P)$, where the weight of each edge is equal to the area of its representing shape. Then it initializes a forest $F$ by $T$, and a matching $M$ by an empty set. Afterwards, as long as $F$ is not empty, the algorithm adds to $M$, the smallest edge $e$ in $F$, and removes the influence set of $e$ from $F$. Finally, it returns $M$.

**Algorithm 1 Strong-matching($G_{S}(P)$)**

1. $T \leftarrow \text{MST}(G_{S}(P))$
2. $F \leftarrow T$
3. $M \leftarrow \emptyset$
4. **while** $F \neq \emptyset$ **do**
   5. $e \leftarrow \text{smallest edge in } F$
   6. $M \leftarrow M \cup \{e\}$
   7. $F \leftarrow F - \text{Inf}(e)$
5. **return** $M$

**Theorem 1.** Given a set $P$ of $n$ points in the plane and a shape $S \in \{\ominus, \triangledown, \square\}$, Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{\text{Inf}(T)} \rceil$ in $G_{S}(P)$, where $T$ is a minimum spanning tree of $G_{S}(P)$.

**Proof.** Let $M$ be the matching returned by Algorithm 1. First we show that $M$ is a strong matching. If $M$ contains one edge, then trivially, $M$ is a strong matching. Consider any two edges $e_1$ and $e_2$ in $M$. Without loss of generality assume that $e_1$ is considered before $e_2$ in the while loop. At the time $e_1$ is added to $M$, the algorithm removes from $F$, the edges in $\text{Inf}(e_1)$,
4 Strong Matching in $G_{\ominus}(P)$

In this section we consider the case where $S$ is a diametral-disk $\ominus$. Recall that $G_{\ominus}(P)$ is an edge-weighted geometric graph, where the weight of an edge $(p,q)$ is equal to the area of $D(p,q)$. $G_{\ominus}(P)$ is equal to the Gabriel graph, $GG(P)$. We prove that $G_{\ominus}(P)$, and consequently $GG(P)$, has a strong diametral-disk matching of size at least $\left\lceil \frac{n}{17} \right\rceil$.

We run Algorithm 1 on $G_{\ominus}(P)$ to compute a matching $\mathcal{M}$. By Theorem 1, $\mathcal{M}$ is a strong matching of size at least $\left\lceil \frac{n-1}{\text{Inf}(T)} \right\rceil$, where $T$ is a minimum spanning tree in $G_{\ominus}(P)$. By Lemma 1, $T$ is a minimum spanning tree of the complete graph $K_{\ominus}(P)$. Observe that $T$ is a Euclidean minimum spanning tree for $P$ as well. In order to prove the desired lower bound, we show that $\text{Inf}(T) \leq 17$. Since $\text{Inf}(T)$ is the maximum size of a set among the influence sets of edges in $T$, it suffices to show that for every edge $e$ in $T$, the influence set of $e$ contains at most 17 edges.

**Lemma 3.** Let $T$ be a minimum spanning tree of $G_{\ominus}(P)$, and let $e$ be any edge in $T$. Then, $|\text{Inf}(e)| \leq 17$.

We will prove this lemma in the rest of this section. Recall that, for each two points $p,q \in P$, $D(p,q)$ is the closed diametral-disk with diameter $pq$. Let $\mathcal{D}$ denote the set of diametral-disks representing the edges in $T$. Since $T$ is a subgraph of $G_{\ominus}(P)$, we have the following observation:

**Observation 4.** Each disk in $\mathcal{D}$ does not contain any point of $P$ in its interior.

Recall that, for each two points $p,q \in P$, $D(p,q)$ is the closed diametral-disk with diameter $pq$. Let $\mathcal{D}$ denote the set of diametral-disks representing the edges in $T$. Since $T$ is a subgraph of $G_{\ominus}(P)$, we have the following observation:

**Observation 5.** Each disk in $\mathcal{D}$ does not contain any point of $P$ in its interior.

**Lemma 4.** For each pair $D_i$ and $D_j$ of disks in $\mathcal{D}$, $D_i$ (resp. $D_j$) does not contain the center of $D_j$ (resp $D_i$).

**Proof.** Let $(a_i,b_i)$ and $(a_j,b_j)$ respectively be the edges of $T$ which correspond to $D_i$ and $D_j$. Let $C_i$ and $C_j$ be the circles representing the boundary of $D_i$ and $D_j$. W.l.o.g. assume that $C_j$ is the bigger circle, i.e., $|a_i b_i| < |a_j b_j|$. By contradiction, suppose that $C_j$ contains the center $c_i$ of $C_i$. Let $x$ and $y$ denote the intersections of $C_i$ and $C_j$. Let $x_i$ (resp. $x_j$) be the intersection of $C_i$ (resp. $C_j$) with the line through $y$ and $c_i$ (resp. $c_j$). Similarly, let $y_i$ (resp. $y_j$) be the intersection of $C_i$ (resp. $C_j$) with the line through $x$ and $c_i$ (resp. $c_j$).
As illustrated in Figure 4, the arcs \( \overline{x_i x}, \overline{y_i y}, \overline{x_j x} \), and \( \overline{y_j y} \) are the potential positions for the points \( a_i, b_i, a_j, \) and \( b_j \), respectively. First we will show that the line segment \( x_i x_j \) passes through \( x \) and \( |a_i a_j| \leq |x_i x_j| \). The angles \( \angle x_i x y \) and \( \angle x_j x y \) are right angles, thus the line segment \( x_i x_j \) goes through \( x \). Since \( \overline{x_i x} < \pi \) (resp. \( \overline{x_j x} < \pi \)), for any point \( a_i \in \overline{x_i x}, |a_i x| \leq |x_i x| \) (resp. \( a_j \in \overline{x_j x}, |a_j x| \leq |x_j x| \)). Therefore,

\[
|a_i a_j| \leq |a_i x| + |x a_j| \leq |x_i x| + |x_j x| = |x_i x_j|.
\]

Consider triangle \( \triangle x_i x_j y \) which is partitioned by segment \( c_i x_j \) into \( t_1 = \triangle x_i x_j c_i \) and \( t_2 = \triangle c_i x_j y \). It is easy to see that \( |x_i c_i| \) in \( t_1 \) is equal to \( |c_i y| \) in \( t_2 \), and the segment \( c_i x_j \) is shared by \( t_1 \) and \( t_2 \). Since \( c_i \) is inside \( C_j \) and \( \overline{y c_j} = \pi \), the angle \( \angle y c_i x_j > \frac{\pi}{2} \). Thus, \( \angle x_i c_i x_j \) in \( t_1 \) is smaller than \( \frac{\pi}{2} \) (and hence smaller than \( \angle y c_i x_j \) in \( t_2 \)). That is, \( |x_i x_j| \) in \( t_1 \) is smaller than \( |x_j y| \) in \( t_2 \). Therefore,

\[
|a_i a_j| \leq |x_i x_j| < |x_j y| = |a_j b_j|.
\]

By symmetry \( |b_i b_j| < |a_j b_j| \). Therefore \( \max\{|a_i a_j|, |b_i b_j|\} < \max\{|a_i b_i|, |a_j b_j|\} \). Therefore, the cycle \( a_i, a_j, b_j, b_i, a_i \) contradicts Lemma \( \Box \)

Let \( \epsilon = (u, v) \) be an edge in \( T \). Without loss of generality, we suppose that \( D(u, v) \) has radius 1 and centered at the origin \( o = (0, 0) \) such that \( u = (-1, 0) \) and \( v = (1, 0) \). For any point \( p \) in the plane, let \( ||p|| \) denote the distance of \( p \) from \( o \). Let \( D(\epsilon^+) \) be the disks in \( \mathcal{D} \) representing the edges of \( T(\epsilon^+) \). Recall that \( T(\epsilon^+) \) contains the edges of \( T \) whose weight is at least \( w(\epsilon) \), where \( w(\epsilon) \) is equal to the area of \( D(u, v) \). Since the area of any circle is directly related to its radius, we have the following observation:

**Observation 6.** The disks in \( \mathcal{D}(\epsilon^+) \) have radius at least 1.

Let \( C(x, r) \) (resp. \( D(x, r) \)) be the circle (resp. closed disk) of radius \( r \) which is centered at a point \( x \) in the plane. Let \( \mathcal{I}(\epsilon^+) = \{D_1, \ldots, D_k\} \) be the set of disks in \( \mathcal{D}(\epsilon^+) \setminus \{D(u, v)\} \) intersecting \( D(u, v) \). We show that \( \mathcal{I}(\epsilon^+) \) contains at most sixteen disks, i.e., \( k \leq 16 \).
For $i \in \{1, \ldots, k\}$, let $c_i$ denote the center of the disk $D_i$. In addition, let $c'_i$ be the intersection point between $C(o, 2)$ and the ray with origin at $o$ which passing through $c_i$. Let the point $p_i$ be $c_i$, if $\|c_i\| < 2$, and $c'_i$, otherwise. See Figure 5. Finally, let $P' = \{o, u, v, p_1, \ldots, p_k\}$.

**Observation 7.** Let $c_j$ be the center of a disk $D_j$ in $\mathcal{I}(e^+)$, where $\|c_j\| \geq 2$. Then, the disk $D(c_j, \|c_j\|-1)$ is contained in the disk $D_j$. Moreover, the disk $D(p_j, 1)$ is contained in the disk $D(c_j, \|c_j\|-1)$. See Figure 5.

**Lemma 5.** The distance between any pair of points in $P'$ is at least 1.

**Proof.** Let $x$ and $y$ be two points in $P'$. We are going to prove that $|xy| \geq 1$. We distinguish between the following three cases.

- $x, y \in \{o, u, v\}$. In this case the claim is trivial.

- $x \in \{o, u, v\}, y \in \{p_1, \ldots, p_k\}$. If $\|y\| = 2$, then $y$ is on $C(o, 2)$, and hence $|xy| \geq 1$. If $\|y\| < 2$, then $y$ is the center of a disk $D_i$ in $\mathcal{I}(e^+)$. By Observation 5, $D_i$ does not contain $u$ and $v$, and by Lemma 4, $D_i$ does not contain $o$. Since $D_i$ has radius at least 1, we conclude that $|xy| \geq 1$.

- $x, y \in \{p_1, \ldots, p_k\}$. Without loss of generality assume $x = p_i$ and $y = p_j$, where $1 \leq i < j \leq k$. We differentiate between three cases:
  - $\|p_i\| < 2$ and $\|p_j\| < 2$. In this case $p_i$ and $p_j$ are the centers of $D_i$ and $D_j$, respectively. By Lemma 4 and Observation 6, we conclude that $|p_ip_j| \geq 1$.
  - $\|p_i\| < 2$ and $\|p_j\| = 2$. By Observation 7 the disk $D(p_j, 1)$ is contained in the disk $D_j$. By Lemma 4, $p_i$ is not in the interior of $D_j$, and consequently, it is not in the interior of $D(p_j, 1)$. Therefore, $|p_ip_j| \geq 1$.
  - $\|p_i\| = 2$ and $\|p_j\| = 2$. Recall that $c_i$ and $c_j$ are the centers of $D_i$ and $D_j$, such that $\|c_i\| \geq 2$ and $\|c_j\| \geq 2$. Without loss of generality assume $\|c_i\| \leq \|c_j\|$. For
Lemma 6. Let \( T \) be a minimum spanning tree of \( G_\triangle(P) \), and let \( e \) be any edge in \( T \). Then, \( |\text{Inf}(e)| \leq 9 \).

By Lemma 5, the points in \( P' \) has mutual distance 1. Moreover, the points in \( P' \) lie in (including the boundary) \( C(o, 2) \). Bateman and Erdős [4] proved that it is impossible to have 20 points in (including the boundary) a circle of radius 2 such that one of the points is at the center and all of the mutual distances are at least 1. Therefore, \( P' \) contains at most 19 points, including \( o, u, \) and \( v \). This implies that \( k \leq 16 \), and hence \( I(e^+) \) contains at most sixteen edges. This completes the proof of Lemma 3.

Theorem 2. Algorithm 1 computes a strong matching of size at least \( \lceil \frac{n-1}{16} \rceil \) in \( G_\triangle(P) \).

5 Strong Matching in \( G_\triangle(P) \)

In this section we consider the case where \( S \) is a downward equilateral triangle \( \triangle \), whose barycenter is the origin and one of its vertices is on the negative y-axis. In this section we assume that \( P \) is in general position, i.e., for each point \( p \in P \), there is no point of \( P \setminus \{p\} \) on \( t^0, t^60, \) and \( t^{120} \). In combination with Observation 1 this implies that for two points \( p, q \in P \), no point of \( P \setminus \{p, q\} \) are on the boundary of \( t(p, q) \) (resp. \( t'(p, q) \)). Recall that \( t(p, q) \) is the smallest homothet of \( \triangle \) having of \( p \) and \( q \) on a corner and the other point on the side opposite to that corner. We prove that \( G_\triangle(P) \), and consequently \( \frac{1}{2}G_\triangle(P) \), has a strong triangle matching of size at least \( \lceil \frac{n-1}{9} \rceil \).

We run Algorithm 1 on \( G_\triangle(P) \) to compute a matching \( \mathcal{M} \). Recall that \( G_\triangle(P) \) is an edge-weighted graph with the weight of each edge \( (p, q) \) is equal to the area of \( t(p, q) \). By Theorem 1 \( \mathcal{M} \) is a strong matching of size at least \( \lceil \frac{n-1}{3m(T)} \rceil \), where \( T \) is a minimum spanning tree in \( G_\triangle(P) \). In order to prove the desired lower bound, we show that \( \text{Inf}(T) \leq 9 \). Since \( \text{Inf}(T) \) is the maximum size of a set among the influence sets of edges in \( T \), it suffices to show that for every edge \( e \) in \( T \), the influence set of \( e \) has at most nine edges.

Lemma 6. Let \( T \) be a minimum spanning tree of \( G_\triangle(P) \), and let \( e \) be any edge in \( T \). Then, \( |\text{Inf}(e)| \leq 9 \).
Lemma 8 (Biniaz et al. [7]). Let configuration, such that Lemma 9.
Let Without loss of generality assume that left of right of t1(s3). Refer to Figure 7(a). Let
Proof. Refer to Figure 7(a). Let t1(s3) be the part of the line segment t1(s3) which is to the left of t2(s2), and let t2(s2) be the part of the line segment t2(s2) which is to the right of t1(s3). Without loss of generality assume that t1(s3) is larger than t2(s2). Let t' be an upward triangle
four triangles. We are going to show that the side in each of with Observation 8, this implies that $q$ can be either in $e$ edge $e$ is on a corner of $I$ in $I$, such that every triangle $t$ is on a corner, or in $I$. Therefore, $t(p, q) \leq t'$; which completes the proof.

Because of the symmetry, the statement of Lemma holds even if $p$ is above $t(s_1)$ and $q$ is on $t_2(s_1)$. Consider the six cones with apex at $p$, as shown in Figure 3.

**Lemma 10.** Let $T$ be a minimum spanning tree in $G_\triangledown(P)$. Then, in $T$, every point $p$ is adjacent to at most one point in each cone $C_i^p$, where $1 \leq i \leq 6$.

**Proof.** If $i$ is even, then by the construction of $G_\triangledown(P)$, which is given in Section 2, $p$ is adjacent to at most one point in $C_i^p$. Assume $i$ is odd. For the sake of contradiction, assume in $T$, the point $p$ is adjacent to two points $q$ and $r$ in a cone $C_i^p$. Then, $t(p, q)$ has $q$ on a corner, and $t(p, r)$ has $r$ on a corner. Without loss of generality assume $t(p, r) \prec t(q, r)$. Then, the hexagon $X(q, r)$ has $r$ in its interior. Thus, $t(q, r) \prec t(p, q)$. Then the cycle $r, p, q, r$ contradicts Lemma 2. Therefore, $p$ is adjacent to at most one point in each of the six cones.

In Algorithm 1, in each iteration of the while loop, let $T(e^+)$ be the triangles representing the edges of $F$. Recall that $e$ is the smallest edge in $F$, and hence, $t(e)$ is a smallest triangle in $T(e^+)$. Let $e = (p, q)$ and let $\mathcal{I}(e^+)$ be the set of triangles in $T(e^+)$ (excluding $t(e)$) which intersect $t(e)$. We show that $\mathcal{I}(e^+)$ contains at most eight triangles. We partition the triangles in $\mathcal{I}(e^+)$ into $\mathcal{I}_1, \mathcal{I}_2$, such that every triangle $\tau \in \mathcal{I}_1$ shares only $p$ or $q$ with $t = t(e)$, i.e., $\mathcal{I}_1 = \{ \tau : \tau \in \mathcal{I}(e^+), \tau \cap t \in \{p, q\} \}$, and every triangle $\tau \in \mathcal{I}_2$ intersects $t$ either through a side or through corner which is not $p$ nor $q$.

By Observation 1, for each triangle $t(p, q)$, one of $p$ and $q$ is on a corner of $t(p, q)$ and the other one is on the side opposite to that corner. Without loss of generality assume that $p$ is on the corner $t(v_1)$, and hence, $q$ is on the side $t(s_2)$. See Figure 8. Note that the other cases, where $p$ is on $t(v_2)$ or on $t(v_3)$ are similar. Since the intersection of $t$ with any triangle $\tau \in \mathcal{I}_1$ is either $p$ or $q$, $\tau$ has either $p$ or $q$ on its boundary. In combination with Observation 8, this implies that $\tau$ represent an edge $e'$ in $T$, and hence, either $p$ or $q$ is an endpoint of $e'$. As illustrated in Figure 8, the other endpoint of $e'$ can be either in $C_1^p, C_2^p, C_6^p$, or in $C_4^q$, because otherwise $\tau \cap t \notin \{p, q\}$. By Lemma 10, $p$ has at most one neighbor in each of $C_1^p, C_2^p, C_6^p$, and $q$ has at most one neighbor in $C_4^q$. Therefore, $\mathcal{I}_1$ contains at most four triangles. We are going to show that $\mathcal{I}_2$ also contains at most four triangles.
The point \( q \) divides \( t(s_2) \) into two parts. Let \( t(s'_2) \) and \( t(s''_2) \) be the parts of \( t(s_2) \) which are below and above \( q \), respectively; see Figure 8. The triangles in \( \mathcal{I}_2 \) intersect \( t \) either through \( t(s_1) \cup t(s''_2) \) or through \( t(s_3) \cup t(s'_2) \); which are shown by red and blue polylines in Figure 8. We show that most two triangles in \( \mathcal{I}_2 \) intersect \( t \) through each of \( t(s_1) \cup t(s''_2) \) or \( t(s_3) \cup t(s'_2) \). Because of symmetry, we only prove for \( t(s_3) \cup t(s'_2) \). When a triangle \( t' \) intersects \( t \) through both \( t(s_3) \) and \( t(s'_2) \) we say \( t' \) intersects \( t \) through \( t(v_3) \). In the next lemma, we prove that at most one triangle in \( \mathcal{I}_2 \) intersects \( t \) through each of \( t(s_3) \), \( t(s'_2) \). Again, because of symmetry, we only prove for \( t(s_3) \).

**Lemma 11.** At most one triangle in \( \mathcal{I}_2 \) intersects \( t \) through \( t(s_3) \).

**Proof.** The proof is by contradiction. Assume two triangles \( t_1(p_1,q_1) \) and \( t_2(p_2,q_2) \) in \( \mathcal{I}_2 \) intersect \( t \) through \( t(s_3) \). Without loss of generality assume that \( p_i \) is on \( t_i(s_1) \) and \( q_i \) is on \( t_i(s_2) \) for \( i = 1,2 \). Recall that the area of \( t_1 \) and the area of \( t_2 \) are at least the area of \( t \). If \( t_1(v_2) \) is in the interior of \( t_2 \) (as shown in Figure 9(a)) or \( t_2(v_2) \) is in the interior of \( t_1 \), then we get a contradiction to Corollary 1. Thus, assume that \( t_1(v_2) \notin t_2 \) and \( t_2(v_2) \notin t_1 \).

Without loss of generality assume that \( t_1(s_1) \) is above \( t_2(s_1) \); see Figure 9(b). By Lemma 9 we have \( t(p,p_1) < \max\{t,t_1\} \leq t_1 \). If \( q_1 \) is in \( X(p,q) \), then by Observation 3, \( t(p,q_1) < t \). Then, the cycle \( p,p_1,q_1,p \) contradicts Lemma 2. Thus, assume that \( q_1 \notin X(p,q_1) \). In this case \( t_2(s_3) \) is to the left of \( t_1(s_3) \), because otherwise \( q_1 \) lies in \( t_2 \) which contradicts Observation 8. Since both \( t_1 \) and \( t_2 \) are larger than \( t \), \( t_2 \) intersects \( t_1 \) through \( t(s_2) \), and hence \( t_2(v_1) \) is in the interior of \( t_1 \). This implies that \( q_2 \) is on \( t_2(v_3) \). In addition, \( p_2 \) is on the part of \( t_2(s_1) \) which lies in the interior of \( X(p,q) \). By Observation 3 and Lemma 9 we have \( t(p,p_2) < t \) and \( t(q_1,q_2) < \max\{t_1,t_2\} \), respectively. Thus, the cycle \( p,p_1,q_1,q_2,p_2,p \) contradicts Lemma 2.

**Lemma 12.** At most two triangles in \( \mathcal{I}_2 \) intersect \( t \) through \( t(v_3) \).

**Proof.** For the sake of contradiction assume three triangles \( t_1,t_2,t_3 \in \mathcal{I}_2 \) intersect \( t \) through \( t(v_3) \). This implies that \( t(v_3) \) belongs to four triangles \( t,t_1,t_2,t_3 \), which contradicts Lemma 8.

**Lemma 13.** If two triangles in \( \mathcal{I}_2 \) intersect \( t \) through \( t(v_3) \), then no other triangle in \( \mathcal{I}_2 \) intersects \( t \) through \( t(s_3) \) or through \( t(s'_2) \).

**Proof.** The proof is by contradiction. Assume two triangles \( t_1(p_1,q_1) \) and \( t_2(p_2,q_2) \) in \( \mathcal{I}_2 \) intersect \( t \) through \( t(v_3) \), and a triangle \( t_3(p_3,q_3) \) in \( \mathcal{I}_2 \) intersects \( t \) through \( t(s_3) \) or \( t(s'_2) \). Let \( p_i \) be the point which lies on \( t_i(s_1) \) for \( i = 1,2,3 \). By Lemma 12, \( t_3 \) cannot intersect both \( t(s_3) \) and
Figure 10: Illustration of Lemma 13: (a) \( p_2 \) is to the right of \( t_1(s_3) \), (b) \( q_1 \in C_{t(v_3)}^5 \), (c) \( q_1 \in C_{t(v_3)}^6 \), and (d) \( q_1 \in C_{t(v_3)}^1 \).

\( t(s'_2) \). Thus, \( t_3 \) intersects \( t \) either through \( t(s_3) \) or through \( t(s'_2) \). We prove the former case; the proof for the latter case is similar. Assume that \( t_3 \) intersects \( t \) through \( t(s_3) \). By Lemma 9, \( t(p, p_3) \prec t_3 \). See Figure 10. In addition, both \( t_1(s_3) \) and \( t_2(s_3) \) are to the right of \( t_3(s_3) \), because otherwise \( q_3 \) lies in \( t_1 \cup t_2 \cup X(p, q) \). If \( q_3 \in t_1 \cup t_2 \) we get a contradiction to Observation 8. If \( q_3 \in X(p, q) \) then by Observation 3 we have \( t(p, q_3) \prec t \), and hence, the cycle \( p, p_3, q_3, p \) contradicts Lemma 2.

Without loss of generality assume that \( t_1(s_1) \) is above \( t_2(s_1) \); see Figure 10. If \( t_1(v_3) \) is in \( t_2 \) or \( t_2(v_3) \) is in \( t_1 \), then we get a contradiction to Corollary 1. Thus, assume that \( t_1(v_3) \not\in t_2 \) and \( t_2(v_3) \not\in t_1 \). This implies that either (i) \( t_2(s_3) \) is to the right of \( t_1(s_3) \) or (ii) \( t_2(s_2) \) is to the left of \( t_1(s_2) \). We show that both cases lead to a contradiction.

In case (i), \( p_2 \) lies in the interior of \( X(p, q) \), and then by Observation 3 we have \( t(p, p_2) \prec t \); see Figure 10(a). In addition, Lemma 9 implies that \( t(p_2, q_3) \prec \max\{t, t_3\} \leq t_3 \). Thus, the cycle \( p, p_3, q_3, p_2, p \) contradicts Lemma 2.

Now consider case (ii) where \( t_1(s_1) \) is above \( t_2(s_1) \) and \( t_2(s_2) \) is to the left of \( t_1(s_2) \). If \( p_1 \) is to the right of \( t \), then as in case (i), the cycle \( p, p_3, q_3, p_1, p \) contradicts Lemma 2. Thus, assume that \( p_1 \) is to the left of \( t \), as shown in Figure 10(b). By Lemma 9, we have \( t(q, p_1) \prec \max\{t, t_1\} \leq t_1 \). Each side of \( t_1 \) contains either \( p_1 \) or \( q_1 \), while \( p_1 \) is on the part of \( t_1(s_1) \) which is to the left of \( t \), thus, \( q_1 \) is on \( t_1(s_3) \). Consider the six cones around \( t(v_3) \); see Figure 10(b). We have three cases: (a) \( q_1 \in C_{t(v_3)}^5 \) \( q_1 \in C_{t(v_3)}^6 \) or (c) \( q_1 \in C_{t(v_3)}^1 \).

In case (a), which is shown in Figure 10(b), by Lemma 7, we have \( \max\{t(p_1, p_2), t(q_1, q_2)\} \prec \max\{t_1, t_2\} \). Thus, the cycle \( p_1, p_2, q_2, q_1, p_1 \) contradicts Lemma 2. In Case (b), which is shown
Lemma 14. If three triangles intersect $t$ through $t(s'_2), t(v_3)$ and $t(s_3)$. Then, at least one of the three triangles is not in $T_2$.

Proof. The proof is by contradiction. Assume that three triangles $t_1(p_1, q_1), t_2(p_2, q_2), t_3(p_3, q_3)$ in $T_2$ intersect $t$ through $t(s'_2), t(v_3), t(s_3)$, respectively. Let $p_i$ be the point which lies on $t_i(s_i)$ for $i = 1, 2, 3$. See Figure 11(a). By Lemma 9, we have $t(p_1, p_3) < t_3$ and $t(q_1, p_3) < t_1$. If $q_3$ is in the interior of $X(p, q)$, then by Observation 3 $t(p, q_3) < t$, and hence, the cycle $p, p_3, q_3, p$ contradicts Lemma 2. If $q_1$ is in $X(q, p)$, then by Observation 3 $t(q, q_1) < t$, and hence, the cycle $q, q_1, p_1, q$ contradicts Lemma 2 (see Figure 11(b)). Thus, assume that $q_3 \notin X(p, q)$ and $q_1 \notin X(q, p)$. Let $t_2(s'_1)$ and $t_2(s''_1)$ be the parts of $t_2(s_1)$ which are to the right of $t(s_3)$ and to the left of $t(s_2)$, respectively. Consider the point $p_2$ which lies on $t_2(s_1)$. If $p_2 \in t_2(s'_1)$, then $p_2 \in X(p, q)$ and by Observation 3 $t(p, p_2) < t$. In addition, Lemma 9 implies that $t(p_2, q_3) < t_3$. Thus, the cycle $p, p_3, q_3, p, p$ contradicts Lemma 2 (see Figure 11(a)). If $p_2 \in t_2(s''_1)$, then $p_2 \in X(q, p)$ and by Observation 3 $t(q, p_2) < t$. In addition, Lemma 9 implies that $t(p_2, q_1) < t_2$. Thus, the cycle $q, p_2, q_1, p_1, q$ contradicts Lemma 2 (see Figure 11(b)).

Putting Lemmas 11, 12, 13 and 14 together, implies that at most two triangles in $T_2$ intersect $t$ through $t(s_3) \cup t(s'_2)$, and consequently, at most two triangles in $T_2$ intersect $t$ through $t(s_1) \cup t(s''_2)$. Thus, $T_2$ contains at most four triangles. Recall that $T_1$ contains at most four triangles. Then, $T(e^+) \cup T(e^-)$ has at most eight triangles. Therefore, the influence set of $e$, contains at most 9 edges (including $e$ itself). This completes the proof of Lemma 6.

Theorem 3. Algorithm 4 computes a strong matching of size at least $\lceil \frac{n-1}{3} \rceil$ in $G_{\gamma}(P)$.

The bound obtained by Lemma 6 is tight. Figure 12 shows a configuration of 10 points in general position such that the influence set of a minimal edge is 9. In Figure 12 $t = t(p, q)$ represents a smallest edge of weight 1; the minimum spanning tree is shown in bold-green line segments. The weight of all edges—the area of the triangles representing these edges—is at least 1. The red triangles are in $T_1$ and share either $p$ or $q$ with $t$. The blue triangles are in $T_2$ and intersect $t$ through $t(s_1) \cup t(s''_2)$ or through $t(s_3) \cup t(s'_2)$; as shown in Figure 12 two of them share only the points $t(v_2)$ and $t(v_3)$. 

15
Figure 12: Four triangles in $\mathcal{I}_1$ (in red) and four triangles in $\mathcal{I}_2$ (in blue) intersect with $t(p, q)$.

6 Strong Matching in $G_{\vartriangle}(P)$

In this section we consider the problem of computing a strong matching in $G_{\vartriangle}(P)$. Recall that $G_{\vartriangle}(P)$ is the union of $G_{\bigtriangledown}(P)$ and $G_{\triangle}(P)$, and is equal to the graph $\Theta_6(P)$. We assume that $P$ is in general position, i.e., for each point $p \in P$, there is no point of $P \setminus \{p\}$ on $l_p^0$, $l_p^{120}$, and $l_p^{240}$. A matching $M$ in $G_{\vartriangle}(P)$ is a strong matching if for each edge $e$ in $M$ there is a homothet of $\bigtriangledown$ or a homothet of $\triangle$ representing $e$, such that these homothets are pairwise disjoint. See Figure 1(b). Using a similar approach as in [2], we prove the following theorem:

**Theorem 4.** Let $P$ be a set of $n$ points in general position in the plane. Let $S$ be an upward or a downward equilateral-triangle that contains $P$. Then, it is possible to find a strong matching of size at least $\lceil \frac{n-1}{4} \rceil$ for $G_{\vartriangle}(P)$ in $S$.

**Proof.** The proof is by induction. Assume that any point set of size $n' \leq n-1$ in a triangle $S'$, has a strong matching of size $\lceil \frac{n'-1}{4} \rceil$ in $S'$. Without loss of generality, assume $S$ is an upward equilateral-triangle. If $n$ is 0 or 1, then there is no matching in $S$, and if $n \in \{2, 3, 4, 5\}$, then by shrinking $S$, it is possible to find a strongly matched pair; the statement of the theorem holds. Suppose that $n \geq 6$, and $n = 4m + r$, where $r \in \{0, 1, 2, 3\}$. If $r \in \{0, 1, 3\}$, then $\lceil \frac{n-1}{4} \rceil = \lceil \frac{(n-1)-1}{4} \rceil$, and by induction we are done. Suppose that $n = 4m + 2$, for some $m \geq 1$. We prove that there are $\lceil \frac{n-1}{4} \rceil = m+1$ disjoint equilateral-triangles (upward or downward) in $S$, each of them matches a pair of points in $P$. Partition $S$ into four equal area equilateral triangles $S_1, S_2, S_3, S_4$ containing $n_1, n_2, n_3, n_4$ points, respectively; see Figure 13(a). Let $n_i = 4m_i + r_i$, where $r_i \in \{0, 1, 2, 3\}$. By induction, in $S_1 \cup S_2 \cup S_3 \cup S_4$, we have a strong matching of size at least

$$A = \left\lfloor \frac{n_1-1}{4} \right\rfloor + \left\lfloor \frac{n_2-1}{4} \right\rfloor + \left\lfloor \frac{n_3-1}{4} \right\rfloor + \left\lfloor \frac{n_4-1}{4} \right\rfloor. \tag{3}$$

**Claim 1:** $A \geq m$.

**Proof.** By Equation (3), we have

$$A = \sum_{i=1}^{4} \left\lfloor \frac{n_i-1}{4} \right\rfloor \geq \sum_{i=1}^{4} \frac{n_i-1}{4} = \frac{n-1}{4} - 1 = \frac{4m+2}{4} - 1 = m - \frac{1}{2}.$$

Since $A$ is an integer, we argue that $A \geq m$. \qed
If \( A > m \), then we are done. Assume that \( A = m \); in fact, by the induction hypothesis we have an strong matching of size \( m \) for \( P \). In order to complete the proof, we have to get one more strongly matched pair. Let \( R \) be the multiset \( \{r_1, r_2, r_3, r_4\} \).

**Claim 2:** If \( A = m \), then either (i) one element in \( R \) is equal to 3 and the other elements are equal to 1, or (ii) two elements in \( R \) are equal to 0 and the other elements are equal to 1.

**Proof.** Let \( \alpha = r_1 + r_2 + r_3 + r_4 \), where \( 0 \leq r_i \leq 3 \). Then \( n = 4(m_1 + m_2 + m_3 + m_4) + \alpha \). Since \( n = 4m + 2, \alpha = 4k + 2 \), for some \( 0 \leq k \leq 2 \). Thus, \( n = 4(m_1 + m_2 + m_3 + m_4 + k) + 2 \), where \( m = m_1 + m_2 + m_3 + m_4 + k \).

By induction, in \( S_1 \), we get a matching of size at least \( [\frac{(4m + r_1) - 1}{4}] = m_i + [\frac{r_1 - 1}{4}] \). Hence, in \( S_1 \cup S_2 \cup S_3 \cup S_4 \), we get a matching of size at least

\[
A = m_1 + m_2 + m_3 + m_4 + \left[\frac{r_1 - 1}{4}\right] + \left[\frac{r_2 - 1}{4}\right] + \left[\frac{r_3 - 1}{4}\right] + \left[\frac{r_4 - 1}{4}\right].
\]

Since \( A = m \) and \( m = m_1 + m_2 + m_3 + m_4 + k \), we have

\[
k = \left[\frac{r_1 - 1}{4}\right] + \left[\frac{r_2 - 1}{4}\right] + \left[\frac{r_3 - 1}{4}\right] + \left[\frac{r_4 - 1}{4}\right]. \tag{4}
\]

Note that \( 0 \leq k \leq 2 \). We go through some case analysis: (i) \( k = 0 \), (ii) \( k = 1 \), (iii) \( k = 2 \). In case (i), we have \( \alpha = 4k + 2 = r_1 + r_2 + r_3 + r_4 = 2 \). In order to have \( k \) equal to 0 in Equation (4), no element in \( R \) should be more than 1; this happens only if two elements in \( R \) are equal to 0 and the other two elements are equal to 1. In case (ii), we have \( \alpha = r_1 + r_2 + r_3 + r_4 = 6 \). In order to have \( k \) equal to 1 in Equation (4), at most one element in \( R \) should be greater than 1; this happens only if three elements in \( R \) are equal to 1 and the other element is equal to 3 (note that all elements in \( R \) are smaller than 4). In case (iii), we have \( \alpha = r_1 + r_2 + r_3 + r_4 = 10 \). In order to have \( k \) equal to 2 in Equation (4), at most two elements in \( R \) should be greater than 1; which is not possible. \( \square \)

We show how to find one more matched pair in each case of Claim 2.

We define \( S_i^x \) as the smallest upward equilateral-triangle contained in \( S_1 \) and anchored at the top corner of \( S_1 \), which contains all the points in \( S_1 \) except \( x \) points. If \( S_1 \) contains less than \( x \) points, then the area of \( S_i^x \) is zero. We also define \( S_i^{-x} \) as the smallest upward equilateral-triangle that contains \( S_1 \) and anchored at the top corner of \( S_1 \), which has all the points in \( S_1 \) plus \( x \) other points of \( P \). Similarly we define upward triangles \( S_i^x \) and \( S_i^{-x} \) which are anchored at the left corner of \( S_2 \). Moreover, we define upward triangles \( S_i^x \) and \( S_i^{-x} \) which are anchored at the right corner of \( S_3 \). We define downward triangles \( S_i^{x+1} \), \( S_i^{-x-1} \), \( S_i^{-x} \), which are anchored at the top-left corner, top-right corner, and bottom corner of \( S_3 \), respectively. See Figure 13(a).

**Case 1:** One element in \( R \) is equal to 3 and the other elements are equal to 1.

In this case, we have \( m = m_1 + m_2 + m_3 + m_4 + 1 \). Because of the symmetry, we have two cases: (i) \( r_3 = 3 \), (ii) \( r_j = 3 \) for some \( j \in \{1, 2, 4\} \).

- \( r_3 = 3 \).

In this case \( m_3 = 4m_3 + 3 \). We differentiate between two cases, where all the elements of the multiset \( \{m_1, m_2, m_4\} \) are equal to zero, or some of them are greater than zero.

  - All elements of \( \{m_1, m_2, m_4\} \) are equal zero. In this case, we have \( m = m_3 + 1 \). Consider the triangles \( S_2^{x+1} \) and \( S_2^{-x-1} \). See Figure 13(a). Note that \( S_2^{x+1} \) and \( S_2^{-x-1} \) are disjoint, \( S_2^{x+1} \) contains two points, and \( S_2^{-x-1} \) contains \( 4m_3 + 2 \) points. By induction, we get a matched pair in \( S_2^{x+1} \) and a matching of size at least \( m_3 + 1 \) in \( S_2^{-x-1} \). Thus, in total, we get a matching of size at least \( 1 + (m_3 + 1) = m + 1 \) in \( S \).
Some elements of \{m_1, m_2, m_4\} are greater than zero. Consider the triangles $S_{1}^3$, $S_{2}^3$, and $S_{4}^3$. Note that the area of some of these triangles—but not all—may be equal to zero. See Figure 13(b). By induction, we get matchings of size $m_1$, $m_2$, and $m_4$ in $S_{1}^3$, $S_{2}^3$, and $S_{4}^3$, respectively. Without loss of generality, assume $S_{2}^3$ is larger than $S_{1}^3$ and $S_{4}^3$. Consider the half-lines $l_1$ and $l_2$ which are parallel to $l^0$ and $l^{60}$, and have their endpoints on the top corner and right corner of $S_{2}^3$, respectively. We define $S_{2}^3_1$ as the downward equilateral-triangle which is bounded by $l_1$, $l_2$, and the right side of $S_{2}^3$; the dashed triangle in Figure 13(b). Note that $l_1$ and $l_2$ do not intersect $S_{1}^3$ and $S_{4}^3$. In addition, $S_{1}^3$, $S_{2}^3$, $S_{4}^3$, and $S_{2}^3_1$ are pairwise disjoint. If any point of $S_{1}^3 \cup S_{2}^3 \cup S_{3}$ is to the right of $l_2$, then consider $S_{2}^3_1$ and $S_{4}^3$. By induction, we get a matching of size $m_1 + m_2 + (m_3 + 1) + (m_4 + 1)$ in $S_{1}^3 \cup S_{2}^3 \cup S_{3} \cup S_{4}^3$, and hence a matching of size $m + 1$ in $S$. If any point of $S_{2}^3 \cup S_{3} \cup S_{4}$ is above $l_1$, then consider $S_{4}^3$ and $S_{4}^3$. By induction, we get a matching of size $(m_1 + 1) + (m_2 + (m_3 + 1) + m_4$ in $S_{1}^3 \cup S_{2}^3 \cup S_{4}^3$, and hence a matching of size $m + 1$ in $S$. Otherwise, $S_{2}^3$ contains $n_3 + 3 = 4(m_3 + 1) + 2$ points. Thus, by induction, we get a matching of size $m_1 + m_2 + (m_3 + 2) + m_4$ in $S_{1}^3 \cup S_{2}^3 \cup S_{4}^3$, and hence a matching of size $m + 1$ in $S$.

- $r_j = 3$, for some $j \in \{1, 2, 4\}$.

Without loss of generality, assume that $r_j = r_2$. Then, $n_2 = 4m_2 + 3$. Consider the triangles $S_{1}^3$, $S_{2}^3$, and $S_{4}^3$. See Figure 14(a). By induction, we get matchings of size $m_1$, $m_2 + 1$, and $m_4$ in $S_{1}^3$, $S_{2}^3$, and $S_{4}^3$, respectively. Now we consider the largest triangle among $S_{1}^3$, $S_{2}^3$, and $S_{4}^3$. Because of the symmetry, we have two cases: (i) $S_{2}^3$ is the largest, or (ii) $S_{4}^3$ is the largest.

- $S_{2}^3$ is larger than $S_{1}^3$ and $S_{4}^3$. Define the half-lines $l_1$, $l_2$, and the triangle $S_{2}^3$ as in the previous case. See Figure 14(a). If any point of $S_{1}^3 \cup S_{2}^3 \cup S_{3}$ is to the right of $l_2$, then consider $S_{4}^3$ and $S_{4}^3$. By induction, we get a matching of size $m_1 + (m_2 + 1) + m_3 + (m_4 + 1)$ in $S_{1}^3 \cup S_{2}^3 \cup S_{4}^3 \cup S_{4}^3$. If any point of $S_{2}^3 \cup S_{3} \cup S_{4}$ is above $l_1$, then consider $S_{4}^3$ and $S_{4}^3$. By induction, we get a matching of size $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S_{1}^3 \cup S_{2}^3 \cup S_{4}^3 \cup S_{4}^3$. Otherwise, $S_{2}^3$ contains $n_3 + 1 = 4m_3 + 2$ points. Thus, by induction, we get a matching of size $m_1 + (m_2 + 1) + (m_3 + 1) + m_4$ in $S_{1} \cup S_{2}^3 \cup S_{2}^3 \cup S_{4}$. As a result, in all cases we get a matching of size $m + 1$ in $S$.

- $S_{4}^3$ is larger than $S_{1}^3$ and $S_{2}^3$. Define the half-lines $l_1$, $l_2$, and the triangle $S_{4}^3$ as in
two cases: (i) S^3_m elements in \{r_1, r_2, r_3, r_4\}. Without loss of generality assume that S^3_m is disjoint. By induction, we get a matched pair in m_1 = 0, then we have n_3 = 4(m_2 + 1) + 2 points. By induction, we get a matching of size m_1 + (m_2 + 2) + m_3 + m_4 in S^3_1 \cup S^3_2 \cup S^3_3 \cup S^3_4. Otherwise, S^3_4 contains at least n_3 + 1 = 4m_3 + 2 points. Thus, by induction, we get a matching of size m_1 + (m_2 + 1) + (m_3 + 1) + m_4 in S_1 \cup S_2 \cup S'_3 \cup S^3_4. As a result, in all cases we get a matching of size m + 1 in S.

Case 2: Two elements in R are equal to 0 and the other elements are equal to 1.

In this case, we have m = m_1 + m_2 + m_3 + m_4. Again, because of the symmetry, we have two cases: (i) r_3 = 0, (ii) r_3 ≠ 0.

- r_3 = 0.

Without loss of generality assume that r_2 = 0 and r_1 = r_4 = 1. Thus, n_1 = 4m_1 + 1, n_2 = 4m_2, n_3 = 4m_3, and n_4 = 4m_4 + 1. If all elements of \{m_1, m_2, m_4\} are equal to zero, then we have m = m_3, where m_3 ≥ 1. Consider the triangles S^3_{r_1} and S^3_{r_2}, which are disjoint. By induction, we get a matched pair in S^3_{r_3} and a matching of size at least m_3 in S^3_{r_1}. Thus, in total, we get a matching of size at least 1 + m_3 = m + 1 in S. Assume some elements in \{m_1, m_2, m_4\} are greater than zero. Consider the triangles S^3_{r_1}, S^3_{r_2}, and S^3_{r_4}. See Figure 15(a). By induction, we get a matching of size m_1, m_2, and m_4 in S^3_{r_1}, S^3_{r_2}, and S^3_{r_3}, respectively. Now we consider the largest triangle among S^3_{r_1}, S^3_{r_2}, and S^3_{r_3}. Because of the symmetry, we have two cases: (i) S^3_{r_2} is the largest, or (ii) S^3_{r_3} is the largest.

- S^3_{r_2} is larger than S^3_{r_1} and S^3_{r_3}. Define l_1, l_2, S'_2 as in Figure 15(a). If any point of S_1 \cup S_2 \cup S_3 is to the right of l_2, then by induction, we get a matching of size m_1 + m_2 + m_3 + (m_4 + 1) in S^3_1 \cup S^3_2 \cup S^3_3 \cup S^3_4. If any point of S_2 \cup S_3 \cup S_4 is above l_1, then by induction, we get a matching of size (m_1 + 1) + m_2 + m_3 + m_4 in S^3_{r_1} \cup S^3_{r_2} \cup S^3_{r_3} \cup S^3_{r_4}. Otherwise, S'_2 contains n_3 + 2 = 4m_3 + 2 points. Thus, by induction, we get a matching of size m_1 + m_2 + (m_3 + 1) + m_4 in S_1 \cup S_2 \cup S'_3 \cup S'_4. In all cases we get a matching of size m + 1 in S.

- S^3_{r_3} is larger than S^3_{r_1} and S^3_{r_2}. Define l_1, l_2, S'_3 as in Figure 15(b). If any point of S_1 \cup S_3 \cup S_4 is above l_1, then by induction, we get a matching of size (m_1 + 1) + m_2 + m_3 + m_4 in S^3_{r_1} \cup S^3_{r_2} \cup S^3_{r_3} \cup S^3_{r_4}. If at least two points of S_1 \cup S_3 \cup S_4 are to the
Figure 15: (a) $S_2^3$ is larger than $S_1^3$ and $S_4^3$. (b) $S_4^3$ is larger than $S_1^3$ and $S_2^3$.

left of $l_2$, then by induction, we get a matching of size $m_1 + (m_2 + 1) + m_3 + m_4$ in $S_1^3 \cup S_2^3 \cup S_3^2 \cup S_4^3$. Otherwise, $S'_1$ contains at least $n_3 + 2 = 4m_3 + 2$ points. Thus, by induction, we get a matching of size $m_1 + m_2 + (m_3 + 1) + m_4$ in $S_1 \cup S_2 \cup S'_1 \cup S_4^3$. In all cases we get a matching of size $m + 1$ in $S$.

• $r_3 \neq 0$. In this case $r_3 = 1$, and without loss of generality, assume that $r_2 = 1$; which means $r_1 = r_4 = 0$. Thus, $n_1 = 4m_1$, $n_2 = 4m_2 + 1$, $n_3 = 4m_3 + 1$, and $n_4 = 4m_4$. If all elements of $\{m_1, m_2, m_4\}$ are equal to zero, then we have $m = m_3$, where $m_3 \geq 1$. Consider the triangles $S_2^{r_1}$ and $S_3^{r_1}$, which are disjoint. By induction, we get a matched pair in $S_2^{r_1}$ and a matching of size at least $m_3$ in $S_{u_3}^3$. Thus, in total, we get a matching of size at least $1 + m_3 = m + 1$ in $S$. Assume some elements in $\{m_1, m_2, m_4\}$ are greater than zero. Consider the triangles $S_1^3$, $S_2^3$, and $S_4^2$. See Figure 16(a). By induction, we get matchings of size $m_1$, $m_2$, and $m_4$ in $S_1^2$, $S_2^3$, and $S_4^2$, respectively. Now we consider the largest triangle among $S_1^2$, $S_2^3$, and $S_4^2$. Because of symmetry, we have two cases: (i) $S_2^3$ is the largest, or (ii) $S_1^2$ is the largest.

Figure 16: (a) $S_2^3$ is larger than $S_1^2$ and $S_4^2$. (b) $S_4^2$ is larger than $S_1^2$ and $S_2^3$.

- $S_2^3$ is larger than $S_1^2$ and $S_4^2$. Define $l_1$, $l_2$, $S'_2$ as in Figure 16(a). If at least two points of $S_1 \cup S_2 \cup S_3$ are to the right of $l_2$, then by induction, we get a matching of
size \( m_1 + m_2 + m_3 + (m_4 + 1) \) in \( S_i^2 \cup S_3^3 \cup S_4^2 \cup S_4^4 \). If at least two points of \( S_2 \cup S_3 \cup S_4 \) are above \( l_1 \), then by induction, we get a matching of size \( (m_1 + 1) + m_2 + m_3 + m_4 \) in \( S_i^2 \cup S_2^3 \cup S_3^2 \cup S_4^4 \). Otherwise, \( S'_i \) contains \( n_3 + 1 = 4m_3 + 2 \) points, and we get a matching of size \( m_1 + m_2 + (m_3 + 1) + m_4 \) in \( S_1 \cup S_2^3 \cup S'_2 \cup S_4 \). In all cases we get a matching of size \( m + 1 \) in \( S \).

- \( S_i^2 \) is larger than \( S_i^a \) and \( S_i^b \). Define \( l_1, l_2, S'_4 \) as in Figure 16(b). If at least two points of \( S_2 \cup S_3 \cup S_4 \) are above \( l_1 \), then by induction, we get a matching of size \( (m_1 + 1) + m_2 + m_3 + m_4 \) in \( S_i^2 \cup S_2^3 \cup S_3^2 \cup S_4^4 \). If any point of \( S_1 \cup S_3 \cup S_4 \) is to the left of \( l_2 \), then by induction, we get a matching of size \( m_1 + (m_2 + 1) + m_3 + m_4 \) in \( S_i^2 \cup S_2^3 \cup S_3^2 \cup S_4^4 \). Otherwise, \( S'_i \) contains at least \( n_3 + 1 = 4m_3 + 2 \) points, and we get a matching of size \( m_1 + m_2 + (m_3 + 1) + m_4 \) in \( S_1 \cup S_2 \cup S'_3 \cup S'_4 \). In all cases we get a matching of size \( m + 1 \) in \( S \).

\( \square \)

7 Strong Matching in \( G_\Box(P) \)

In this section we consider the problem of computing a strong matching in \( G_\Box(P) \), where \( \Box \) is an axis-aligned square whose center is the origin. We assume that \( P \) is in general position, i.e., (i) no two points have the same \( x \)-coordinate or the same \( y \)-coordinate, and (ii) no four points are on the boundary of any homothet of \( \Box \). Recall that \( G_\Box(P) \) is equal to the \( L_\infty \)-Delaunay graph on \( P \). Ábrego et al. [12] proved that \( G_\Box(P) \) has a strong matching of size at least \( \lceil n/5 \rceil \). Using a similar approach as in Section 6, we prove that \( G_\Box(P) \) has a strong matching of size at least \( \lceil n/4 \rceil \).

**Theorem 5.** Let \( P \) be a set of \( n \) points in general position in the plane. Let \( S \) be an axis-parallel square that contains \( P \). Then, it is possible to find a strong matching of size at least \( \lceil n/4 \rceil \) for \( G_\Box(P) \) in \( S \).

**Proof.** The proof is by induction. Assume that any point set of size \( n' \leq n - 1 \) in an axis-parallel square \( S' \), has a strong matching of size \( \lceil n'/4 \rceil \) in \( S' \). If \( n \) is 0 or 1, then there is no matching in \( S \), and if \( n \in \{2, 3, 4, 5\} \), then by shrinking \( S \), it is possible to find a strongly matched pair. Suppose that \( n \geq 6 \), and \( n = 4m + r \), where \( r \in \{0, 1, 2, 3\} \). If \( r \in \{0, 1, 3\} \), then \( \lceil n-1 \rceil = \lceil (n-1)/4 \rceil \), and by induction we are done. Suppose that \( n = 4m + 2 \), for some \( m \geq 1 \). We prove that there are \( \lceil n/4 \rceil = m + 1 \) disjoint squares in \( S \), each of them matches a pair of points in \( P \). Partition \( S \) into four equal area squares \( S_1, S_2, S_3, S_4 \) which contain \( n_1, n_2, n_3, n_4 \) points, respectively; see Figure 17(a). Let \( n_i = 4m_i + r_i \) for \( 1 \leq i \leq 4 \), where \( r_i \in \{0, 1, 2, 3\} \). Let \( R \) be the multiset \( \{r_1, r_2, r_3, r_4\} \). By induction, in \( S_1 \cup S_2 \cup S_3 \cup S_4 \), we have a strong matching of size at least

\[
A = \left\lceil \frac{n_1 - 1}{4} \right\rceil + \left\lceil \frac{n_2 - 1}{4} \right\rceil + \left\lceil \frac{n_3 - 1}{4} \right\rceil + \left\lceil \frac{n_4 - 1}{4} \right\rceil.
\]

In the proof of Theorem 4, we have shown the following two claims:

**Claim 1:** \( A \geq m \).

**Claim 2:** If \( A = m \), then either (i) one element in \( R \) is equal to 3 and the other elements are equal to 1, or (ii) two elements in \( R \) are equal to 0 and the other elements are equal to 1.

If \( A > m \), then we are done. Assume that \( A = m \); in fact, by the induction hypothesis we have a strong matching of size \( m \) in \( S \). We show how to find one more strongly matched pair in each case of Claim 2.

21
We define $S^*_1$ as the smallest axis-parallel square contained in $S_1$ and anchored at the top-left corner of $S_1$, which contains all the points in $S_1$ except $x$ points. If $S_1$ contains less than $x$ points, then the area of $S^*_1$ is zero. We also define $S^*_2$ as the smallest axis-parallel square that contains $S_1$ and anchored at the top-left corner of $S_1$, which has all the points in $S_1$ plus $x$ other points of $P$. See Figure 17(a). Similarly we define the squares $S^*_3$, $S^*_4$, and $S^*_5$ which are anchored at the top-right corner of $S_2$, and the bottom-left corner of $S_3$, and the bottom-right corner of $S_4$, respectively.

**Case 1:** One element in $R$ is equal to 3 and the other elements are equal to 1.

In this case, we have $m = m_1 + m_2 + m_3 + m_4 + 1$. Without loss of generality, assume that $r_1 = 3$ and $r_2 = r_3 = r_4 = 1$. Consider the squares $S^*_1$, $S^*_2$, $S^*_3$, and $S^*_4$. Note that the area of some of these squares—but not all—may be equal to zero. See Figure 17(b). By induction, we get a matching of size $m_1 + 1$, $m_2$, $m_3$, and $m_4$, in $S^*_1$, $S^*_2$, $S^*_3$, and $S^*_4$, respectively. Now consider the largest square among $S^*_1$, $S^*_2$, $S^*_3$, and $S^*_4$. Because of the symmetry, we have only three cases: (i) $S^*_1$ is the largest, (ii) $S^*_2$ is the largest, and (iii) $S^*_3$ is the largest.

![Figure 17: (a) Split $S$ into four equal area squares. (b) $S^*_1$ is larger than $S^*_2$, $S^*_3$, and $S^*_4$. (c) $S^*_2$ is larger than $S^*_1$, $S^*_3$, and $S^*_4$.](image)

- **$S^*_1$ is the largest square.** Consider the lines $l_1$ and $l_2$ which contain the bottom side and right side of $S^*_1$, respectively; the dashed lines in Figure 17(b). Note that $l_1$ and $l_2$ do not intersect any of $S^*_2$, $S^*_3$, and $S^*_4$. If any point of $S_1$ is to the right of $l_2$, then by induction, we get a matching of size $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S^*_1 \cup S^*_2 \cup S_3 \cup S^*_4$. Otherwise, by induction, we get a matching of size $(m_1 + 1) + m_2 + (m_3 + 1) + m_4$ in $S^*_1 \cup S^*_2 \cup S^*_3 \cup S^*_4$. In all cases we get a matching of size $m + 1$ in $S$.

- **$S^*_2$ is the largest square.** Consider the lines $l_1$ and $l_2$ which contain the bottom side and left side of $S^*_2$, respectively; the dashed lines in Figure 17(c). Note that $l_1$ and $l_2$ do not intersect any of $S^*_1$, $S^*_3$, and $S^*_4$. If any point of $S_2$ is below $l_1$, then by induction, we get a matching of size $(m_1 + 1) + m_2 + m_3 + (m_4 + 1)$ in $S^*_1 \cup S^*_2 \cup S^*_3 \cup S^*_4$. Otherwise, by induction, we get a matching of size $(m_1 + 2) + m_2 + m_3 + m_4$ in $S^*_1 \cup S^*_2 \cup S^*_3 \cup S^*_4$; see Figure 17(c). In all cases we get a matching of size $m + 1$ in $S$.

- **$S^*_3$ is the largest square.** Consider the lines $l_1$ and $l_2$ which contain the top side and left side of $S^*_3$, respectively. If any point of $S_4$ is above $l_1$, then by induction, we get a matching of size $(m_1 + 1) + (m_2 + 1) + m_3 + m_4$ in $S^*_1 \cup S^*_2 \cup S^*_3 \cup S^*_4$. Otherwise, by induction, we get a matching of size $(m_1 + 1) + m_2 + (m_3 + 1) + m_4$ in $S^*_1 \cup S^*_2 \cup S^*_3 \cup S^*_4$. In all cases we get a matching of size $m + 1$ in $S$.

**Case 2:** Two elements in $R$ are equal to 0 and two elements are equal to 1.
In this case, we have $m = m_1 + m_2 + m_3 + m_4$. Because of the symmetry, only two cases may arise: (i) $r_1 = r_2 = 1$ and $r_3 = r_4 = 0$, (ii) $r_1 = r_4 = 1$ and $r_2 = r_3 = 0$.

- $r_1 = r_2 = 1$ and $r_3 = r_4 = 0$. Consider the squares $S^{13}_1$, $S^3_1$, $S^3_2$, and $S^4_2$. By induction, we get matchings of size $m_1$, $m_2$, $m_3$, and $m_4$, in $S^{13}_1$, $S^3_1$, $S^3_2$, and $S^4_2$, respectively. Now consider the largest square among $S^{13}_1$, $S^3_1$, $S^3_2$, and $S^4_2$. Because of the symmetry, we have only two cases: (a) $S^{13}_1$ is the largest, (b) $S^3_2$ is the largest. If case (a) we get one more matched pair either in $S^{13}_2$ or in $S^3_2$. In case (b) we get one more matched pair either in $S^{13}_1$ or in $S^3_2$.

- $r_1 = r_4 = 1$ and $r_2 = r_3 = 0$. Consider the squares $S^{13}_1$, $S^2_1$, $S^2_2$, and $S^3_4$. By induction, we get matchings of size $m_1$, $m_2$, $m_3$, and $m_4$, in $S^{13}_1$, $S^2_1$, $S^2_2$, and $S^3_4$, respectively. Now consider the largest square among $S^{13}_1$, $S^2_1$, $S^2_2$, and $S^3_4$. Because of the symmetry, we have only two cases: (a) $S^{13}_1$ is the largest, (b) $S^2_2$ is the largest. In case (a) we get one more matched pair either in $S^{13}_2$ or in $S^3_2$. In case (b) we get one more matched pair either in $S^{13}_1$ or in $S^3_2$.

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8 A Conjecture on Strong Matching in $G_\circ(P)$

In this section, we discuss a possible way to further improve upon Theorem 2 as well as a construction leading to the conjecture that Algorithm 1 computes a strong matching of size at least $\lceil \frac{n-1}{8} \rceil$; unfortunately we are not able to prove this.

In Section 4 we proved that $I(e^+)$ contains at most 16 edges. In order to achieve this upper bound we used the fact that the centers of the disks in $I(e^+)$ should be far apart. We did not consider the endpoints of the edges representing these disks. By Observation 5 the disks representing the edges in $I(e^+)$ cannot contain any of the endpoints. We applied this observation only on $u$ and $v$. Unfortunately, our attempts to apply this observation on the endpoints of edges in $I(e^+)$ have been so far unsuccessful.

Recall that $T$ is a Euclidean minimum spanning tree of $P$, and for every edge $e = (u, v)$ in $T$, $\deg(e)$ is the degree of $e$ in $T(e^+)$, where $T(e^+)$ is the set of all edges of $T$ with weight at least $w(e)$. Note that $w(e)$ is directly related to the Euclidean distance between $u$ and $v$. Observe that the discs representing the edges adjacent to $e$ intersect $D(u, v)$. Thus, these edges are in $\text{Inf}(e)$. We call an edge $e$ in $T$ a minimal edge if $e$ is not longer than any of its adjacent edges. We observed that the maximum degree of a minimal edge is an upper bound for $\text{Inf}(e)$. We conjecture that,

**Conjecture 1.** $\text{Inf}(T)$ is at most the maximum degree of a minimal edge.

Monma and Suri [12] showed that for every point set $P$ there exists a Euclidean minimum spanning tree, $\text{MST}(P)$, of maximum vertex degree five. Thus, the maximum edge degree in $\text{MST}(P)$ is 9. We show that for every point set $P$, there exists a Euclidean minimum spanning tree, $\text{MST}(P)$, such that the degree of each node is at most five and the degree of each minimal edge is at most eight. This implies the conjecture that $\text{Inf}(\text{MST}(P)) \leq 8$. That is, Algorithm 1 returns a strong matching of size at least $\lceil \frac{n-1}{8} \rceil$.

**Lemma 15.** If $uv$ and $uw$ are two adjacent edges in $\text{MST}(P)$, then the triangle $\triangle uvw$ has no point of $P \setminus \{u, v, w\}$ in its interior or on its boundary.
Figure 18: In MST($P$), the triangle $\triangle uvw$ formed by two adjacent edges $uv$ and $uw$, is empty.

Proof. If the angle between $uv$ and $uw$ is equal to $\pi$, then there is no other point of $P$ on $uv$ and $uw$. Assume that $\angle vw < \pi$. Refer to Figure 18. Since MST($P$) is a subgraph of the Gabriel graph, the circles $C_1$ and $C_2$ with diameters $uv$ and $uw$ are empty. Since $\angle vw < \pi$, $C_1$ and $C_2$ intersect each other at two points, say $u$ and $p$. Connect $u$, $v$ and $w$ to $p$. Since $uv$ and $uw$ are the diameters of $C_1$ and $C_2$, $\angle upv = \angle wpu = \pi/2$. This means that $vw$ is a straight line segment. Since $C_1$ and $C_2$ are empty and $\triangle uvw \subset C_1 \cup C_2$, it follows that $\triangle uvw \cap P = \{u, v, w\}$. \qed

Figure 19: Illustration of Lemma 16: $|ab| \leq |bc| \leq |ad|$, $\angle abc \geq \pi/3$, $\angle bad \geq \pi/3$, and $\angle abc + \angle bad \leq \pi$.

Lemma 16. Follow Figure 19. For a convex-quadrilateral $Q = a, b, c, d$ with $|ab| \leq |bc| \leq |ad|$, if $\min\{\angle abc, \angle bad\} \geq \pi/3$ and $\angle abc + \angle bad \leq \pi$, then $|cd| \leq |ad|$.

Proof. Let $\alpha_1 = \angle cad$, $\alpha_2 = \angle bac$, $\beta_1 = \angle cbd$, $\beta_2 = \angle abd$, $\gamma_1 = \angle acd$, $\gamma_2 = \angle acb$, $\delta_1 = \angle bdc$, and $\delta_2 = \angle adb$; see Figure 19. Since $|ab| \leq |bc| \leq |ad|$, $\gamma_2 \leq \alpha_2$ and $\delta_2 \leq \beta_2$.

Let $\ell$ be a line passing through $c$ which is parallel to $ad$. Since $\angle abc + \angle bad \leq \pi$, $\ell$ intersects the line segment $ab$. This implies that $\alpha_1 \leq \gamma_2$. If $\beta_1 < \delta_1$, then $|cd| < |bc|$, and hence $|cd| < |ad|$ and we are done. Assume that $\delta_1 \leq \beta_1$. In this case, $\delta \leq \beta$. Now consider the two triangles $\triangle abc$ and $\triangle acd$. Since $\delta \leq \beta$ and $\alpha_1 \leq \gamma_2$, $\alpha_2 \leq \gamma_1$. Then we have

$$\alpha_1 \leq \gamma_2 \leq \alpha_2 \leq \gamma_1.$$
Since $\alpha_1 \leq \gamma_1$, $|cd| \leq |ad|$, where the equality holds only if $\alpha_1 = \gamma_2 = \alpha_2 = \gamma_1$, i.e., $Q$ is a diamond. This completes the proof.

Lemma 17. Every finite set of points $P$ in the plane admits a minimum spanning tree whose node degree is at most five and whose minimal-edge degree is at most nine.

Proof. Consider a minimum spanning tree, $\text{MST}(P)$, of maximum vertex degree 5. The maximum edge degree in $\text{MST}(P)$ is 9. Consider any minimal edge, $uv$. If the degree of $uv$ is 8, then $\text{MST}(P)$ satisfies the statement of the lemma. Assume that the degree of $uv$ is 9. Let $u_1, u_2, u_3, u_4$ and $v_1, v_2, v_3, v_4$ be the the neighbors of $u$ and $v$ in clockwise and counterclockwise orders, respectively. See Figure 20. In $\text{MST}(P)$, the angles between two adjacent edges are at least $\pi/3$. Since $\angle uu_{i+1} \geq \pi/3$ and $\angle vv_{i+1} \geq \pi/3$ for $i = 1, 2, 3$, either $\angle uu_{1} + \angle vv_{1} \leq \pi$ or $\angle uu_{4} + \angle vv_{4} \leq \pi$. Without loss of generality assume that $\angle uu_{1} + \angle vv_{1} \leq \pi$ or $\angle uu_{4} + \angle vv_{4} \leq \pi$. We prove that the spanning tree obtained by swapping the edge $uv$ with $u_1v_1$ is also a minimum spanning tree, and it has one fewer minimal-edge of degree 9. By repeating this procedure at each minimal-edge of degree 9, we obtain a minimum spanning tree which satisfies the statement of the lemma. Let $Q = u, v, v_1, u_1$. By Lemma 15, $v_1$ is outside the triangle $\triangle uu_{1}uv$, and $u_1$ is outside the triangle $\triangle uu_{1}v_1$. In addition, $u_1$ and $v_1$ are on the same side of the line subtended from $uv$. Thus, $Q$ is a convex quadrilateral. Without loss of generality assume that $|vv_{1}| \leq |uu_{1}|$. By Lemma 16, $|u_{1}v_{1}| \leq |uu_{1}|$. If $|u_{1}v_{1}| < |uu_{1}|$, we get a contradiction to Lemma 2. Thus, assume that $|u_{1}v_{1}| = |uu_{1}|$. As shown in the proof of Lemma 16, this case happens only when $Q$ is a diamond. This implies that $\angle uu_{1} + \angle vv_{1} = \pi$, and consequently $\angle uu_{4} + \angle vv_{4} = \pi$. In addition, $\angle uu_{i+1} = \pi/3$ and $\angle vv_{i+1} = \pi/3$ for $i = 1, 2, 3$. To establish the validity of our edge-swap, observe that the nine edges incident to $u$ and $v$ are all equal in length. Therefore, swapping $uv$ with $u_1v_1$ does not change the cost of the spanning tree and, furthermore, the resulting tree is a valid spanning tree since $u_1v_1$ is not an edge of the original spanning tree $\text{MST}(P)$; otherwise $u, v, v_1, u_1$ would form a cycle. We have removed a minimal edge $uv$ of degree 9, but it remains to show that the degree of $u_1$ and
v_1 does not increase to six and new minimal edge of degree 9 is not generated. Note that u_1u_2 and v_1v_2 are not the edges of MST(P), and hence, deg(u_1) and deg(v_1) are still less than six. In order to show that no new minimal edge is generated, we differentiate between two cases:

- \( \min \{ \angle v_1u_1, \angle v_1u_2 \} > \pi/3 \). Since \( \angle v_1u_1 > \pi/3 \) and \( \angle uu_1u_2 = \pi/3 \), \( u_1 \) can be adjacent to at most two vertices other than \( u \) and \( v_1 \), and hence \( \text{deg}(u_1) \leq 4 \); similarly \( \text{deg}(v_1) \leq 4 \). Thus, \( u, v, u_1 \), and \( v_1 \) are of degree at most four, and hence no new minimal edge of degree 9 is generated.

- \( \min \{ \angle v_1u_1, \angle v_1u_2 \} = \pi/3 \). W.l.o.g. assume that \( \angle v_1u_1 = \pi/3 \). This implies that \( \angle v_1u_1 = 2\pi/3 \). Since \( \angle v_1u_1 = \pi/3 \) and \( \angle uu_1u_2 = \pi/3 \), \( u_1 \) is adjacent to at most three vertices other than \( u \) and \( v_1 \). Let \( u, v_1, w_1, w_2, w_3 \) be the neighbors of \( u_1 \) in clockwise order. Note that \( v_1 \) is not adjacent to \( u, v_2 \) nor \( w_1 \). But \( v_1 \) can be connected to another vertex, say \( x \), which implies that \( \text{deg}(v_1) \leq 3 \). We prove that the spanning tree obtained by swapping the edge \( u_1v_1 \) with \( v_1w_1 \) is also a minimum spanning tree of node degree at most five, which has one fewer minimal edge of degree 9. The new tree is a legal minimum spanning tree for \( P \), because \( |v_1w_1| = |v_1u_1| \). In addition, \( \text{deg}(u_1) \leq 4 \) and \( \text{deg}(v_1) \leq 4 \). Since \( w_1w_2 \) and \( w_1x \) are illegal edges, \( \text{deg}(w_1) \leq 4 \). Thus, \( u, v, u_1, v_1, \) and \( w_1 \) are of degree at most four and no new minimal edge of degree 9 is generated. This completes the proof that our edge-swap reduces the number of minimal-edges of degree nine by one.

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9 Conclusion

Given a set of \( n \) points in general position in the plane, we considered the problem of strong matching of points with convex geometric shapes. A matching is strong if the objects representing whose edges are pairwise disjoint. In this paper we presented algorithms which compute strong matchings of points with diametral-disks, equilateral-triangles, and squares. Specifically we showed that:

- There exists a strong matching of points with diametral-disks of size at least \( \lfloor \frac{n-1}{4} \rfloor \).
- There exists a strong matching of points with downward equilateral-triangles of size at least \( \lfloor \frac{n-1}{17} \rfloor \).
- There exists a strong matching of points with downward/upward equilateral-triangles of size at least \( \lfloor \frac{n-1}{4} \rfloor \).
- There exists a strong matching of points with axis-parallel squares of size at least \( \lfloor \frac{n-1}{8} \rfloor \).

The existence of a downward/upward equilateral-triangle matching of size at least \( \lfloor \frac{n-1}{4} \rfloor \), implies the existence of either a downward equilateral-triangle matching of size at least \( \lfloor \frac{n-1}{8} \rfloor \) or an upward equilateral-triangle matching of size at least \( \lfloor \frac{n-1}{8} \rfloor \). This does not imply a lower bound better than \( \lfloor \frac{n-1}{4} \rfloor \) for downward equilateral-triangle matching (or any fixed oriented equilateral-triangle).

A natural open problem is to improve any of the provided lower bounds, or extend these results for other convex shapes. The specific open problem is to prove that Algorithm \( \Pi \) computes a strong matching of points with diametral-disks of size at least \( \lfloor \frac{n-1}{8} \rfloor \) as discussed in Section 8.
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