THE BEST SIMULTANEOUS APPROXIMATION IN LINEAR 2-NORMED SPACES

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Abstract. In this paper, we shall investigate and analyse a new study on the best simultaneous approximation in the context of linear 2-normed spaces inspired by Elumalai and his coworkers in [10]. The basis of this investigation is to extend and refinement the definition of the classical approximation, best approximation and some related concepts to linear 2-normed spaces.

1. Introduction

The problem of best and simultaneous best approximation has been studied by several mathematicians (for more informations, see [5], [6], [7], [8], [9], [18], [19], [20], [26]). Most of these works have dealt with the existence, uniqueness and characterization of best approximations in spaces of continuous functions with values in Banach spaces. Recently, many works on approximation has been done on 2-structures such as 2-normed spaces, generalized 2-normed spaces (for details, see [4], [23]) and 2-Banach spaces (see [2], [3], [10], [11], [12], [13], [14], [15]). Diaz and McLaughlin [7] and Dunham [9] have considered simultaneously approximating of two real-valued continuously functions \( f_1, f_2 \) defined on \([a, b]\), by elements of set \( C[a, b] \). Several results of best simultaneous approximation in the context of linear space were obtained by Goel et al. (for details, see [19], [20]). The subject of approximation theory has attracted the attention of several mathematicians during the last 130 years or so. This theory is now an extremely extensive branch of mathematical analysis. It has many applications in many areas, especially in engineering.

The concept of linear 2-normed spaces has been investigated by Gahler in 1965 [17] and given many important properties and examples for these spaces. After, these spaces have been developed extensively in different subjects by other researchers from many points of view and then the field has considerably grown. Z. Lewandowska published some of papers on 2-normed sets and generalized 2-normed spaces in 1999-2003 (see [23]). In [10], Elumalai and his coworkers published a series of papers related this subject. They have developed best approximation theory in the context of linear 2-normed spaces. These are some works on characterization of 2-normed spaces, extension of 2-functionals and approximation in 2-normed spaces (see [1], [2]). Also, the author has some works in \( \varepsilon \)-approximation theory [3] and Rezapour has also such studies in [25]. The essential aim of this paper is to derive new different definitions of approximation and obtain some results related to these...
definitions. The essential results of the set of best simultaneous approximation are given in the fourth section of this paper.

Throughout this paper, we first fix some notations. Let $X$ be a linear space and $L \{y\}$ be the subspace of $Y$ generated by $y$. It is also let $(X, \|\|)$ and $(X, \|\|)$ denote a normed space and 2-normed space with the corresponding norms, respectively. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{C}$ denotes the set of complex numbers. Throughout this work, $K$ is variously considered as an indeterminate, as a real number $K \in \mathbb{R}$, or as a complex number $K \in \mathbb{C}$.

We now summarize our work in four section as follows:

In first section, we gave history of normed and 2-normed spaces and motivation of our work. In section 2 and 3, we specify definitions and properties of normed and 2-normed spaces, respectively. In section 4, we gave suitable a definition for studying in linear 2-normed spaces and so we derived three lemma and two proposition by using our definition.

Thus, we are now ready in order to begin with the second section as follows.

2. Some Definitions of Normed Spaces

Definition 1. Let $(X, \|\|)$ be a normed space and $K \subset X$. For $u \in X$,

$$\inf_{v \in K} \{\|u - v\|\}$$

is a general best approximation.

Mohebi and Rubinov ([24]) and Rezapour in [25] gave the main preliminaries on the approximation theory in the usual sense as follows:

Definition 2. Let $(X, \|\|)$ be a linear normed space. For a nonempty subset $A$ of $X$ and $x \in X$,

$$d(x, A) = \inf_{a \in A} \{\|u - a\|\}$$

denotes the distance from $x$ to the set $A$. If

$$\|x - a_0\| = d(x, A).$$

Then, we say that a point $a_0 \in A$ is called a best approximation for $x \in X$. If each $x \in X$ has at least one best approximation $a_0 \in A$, then $A$ is called a proximinal subset of $X$. If each $x \in X$ has a unique best approximation $a_0 \in A$, then $A$ is called a Chebyshev subset of $X$.

Definition 3. Let $A \subset X$. For $x \in X$,

$$P_A(x) = \{a \in A : \|x - a\| = d(x, A)\}$$

where $P_A(x)$, the set of all best approximations of $x$ in $A$. We know that $P_A(x)$ is a closed and bounded subset of $X$. For $x \notin A$, $P_A(x)$ is located in the boundary of $A$.

Definition 4. Let $(X, \|\|)$ be a linear normed space. For a nonempty subset $A$ of $X$ and a nonempty set $W \subset X$,

$$d(A, W) = \inf_{w \in W} \sup_{a \in A} \{\|a - w\|\}$$
denotes the distance from the set \( A \) to the set \( W \). If
\[
\inf_{w \in W} \sup_{a \in A} \{\|a - w\|\} = \sup_{a \in A} \{\|a - w_0\|\}.
\]
Then, we say that a point \( w_0 \in W \) is called a best approximation from \( A \) to \( W \).

### 3. Properties of 2-Normed Spaces

In [16], Cho et al. defined linear 2-normed spaces and gave interesting properties of them. After, Lewandowska defined generalized 2-normed spaces and derived properties of these spaces in [23]. Now, let us give the definition of 2-normed space.

**Definition 5.** Let \( X \) be a linear space over \( F \), where \( F \) is the real or complex numbers field, \( \dim X > 1 \), and let \( \|., .\| : X \times X \to \mathbb{R}^+ \cup \{0\} \) be a non-negative real-valued function on \( X \times X \) with the following properties:

1. \( \|x, y\| = 0 \) if and only if \( x \) and \( y \) are linearly dependent vectors,
2. \( \|x, y\| = \|y, x\| \) for all \( x, y \in X \),
3. \( \|\alpha x, y\| = |\alpha| \|x, y\| \) for all \( \alpha \in K \) and all \( x, y \in X \),
4. \( \|x + y, z\| \leq \|x, z\| + \|y, z\| \) for all \( x, y, z \in X \).

Then, \( \|., .\| \) is called a 2-norm on \( X \) and \( (X, \|., .\|) \) is called a linear 2-normed space.

Every 2-normed space is a locally convex topological linear space. In fact, for a fixed \( b \in X \), for all \( x \in X \),
\[
p_b(x) = \|x, b\|
\]
which is a seminorm and the family of \( P \), that is
\[
P = \{p_b : b \in X\}
\]
generates a locally convex topology on \( X \). This space will be denoted by \( (X, p_b) \).

In each 2-normed space \( (X, \|., .\|) \). For all \( x, y \in X \) and for every real \( \alpha \), we have non-negative norm,
\[
\|x, y\| \geq 0 \text{ and } \|x + \alpha y\| = \|x, y\|.
\]

Also, if \( x, y \) and \( z \) are linearly dependent, this occurs for \( \dim X = 2 \). Then,
\[
\|x + y + z\| = \|x, y\| + \|x, z\| \text{ or } \|x, y - z\| = \|x, y\| + \|x, z\|.
\]

**Example 1.** ([27]) Let \( P_n \) denotes the set of real polynomials of degree less than or equal to \( n \), on the interval \([0, 1]\). By considering usual addition and scalar multiplication, \( P_n \) is a linear vector space over the reals. Let \( \{x_1, x_2, \cdots, x_{2n}\} \) be distinct fixed points in \([0, 1]\) and define the 2-norm on \( P_n \) as
\[
\|f, g\| = \sum_{k=1}^{2n} |f(x_k)g'(x_k) - f'(x_k)g(x_k)|.
\]

Then, \( (P_n, \|., .\|) \) is a 2-normed space.

Let \( (X, \|., .\|) \) be a 2-normed space. Under this assumption, we can give the following definitions:
Definition 6. \((27)\) A sequence \(\{x_n\}_{n \geq 1}\) in a linear 2-normed space \(X\) is called Cauchy sequence if there exist independent elements \(y, z \in X\) such that
\[
\lim_{n, m \to \infty} \|x_n - x_m, y\| = 0 \quad \text{and} \quad \lim_{n, m \to \infty} \|x_n - x_m, z\| = 0.
\]

Definition 7. \((27)\) A sequence \(\{x_n\}_{n \geq 1}\) in a linear 2-normed space \(X\) is called convergent if there exists an element \(x \in X\) such that
\[
\lim_{n \to \infty} \|x_n - x, z\| = 0
\]
for all \(z \in X\).

Proposition 1. \((6)\) Let \((X, \|\cdot\|)\) be 2-normed space and \(W\) be a subspace of \(X\), \(b \in X\) and \(L\{b\}\) be the subspace of \(X\) generated by \(b\). If \(x_0 \in X\) is such that
\[
\delta = \inf_{w \in W} \{\|x_0 - w, b\|\} > 0.
\]
Then, there exists a bounded bilinear functional as follows
\[
f : X \times L\{b\} \to K
\]
such that
\[
F|_{w \times L\{b\}} = 0, \quad F(x_0, b) = 1 \quad \text{and} \quad \|F\| = \frac{1}{\delta}.
\]

Definition 8. A 2-normed space \((X, \|\cdot\|)\) is which every Cauchy sequence \(\{x_n\}\) converges to some \(x \in X\) then \(X\) is said to be complete with respect to the 2-norm.

Definition 9. A complete 2-normed space \((X, \|\cdot\|)\) is called a 2-Banach space.

The examples 1 and 2 are 2-Banach spaces while the example 3 does not (For details, see [27]).

Lemma 1. \((27)\) (i) Every 2-normed space of dimension 2 is a 2-Banach space, when the underlying field is complete.

(ii) If \(\{x_n\}\) is a sequence in 2-normed space \((X, \|\cdot\|)\) and if
\[
\lim_{n \to \infty} \|x_n - x, y\| = 0.
\]
then, we have
\[
\lim_{n \to \infty} \|x_n, y\| = \|x, y\|.
\]

4. Fundamental Results
In this section, let us also consider a definition, and however, we give Lemma and Proposition for the best simultaneous approximation in linear 2-normed spaces.

Definition 10. Let \((X, \|\cdot\|)\) be a linear 2-normed space and \(W\) be any bounded subset of \(X\). An element \(g^* \in G\) is said to be a best approximation to the set \(W\), if
\[
\sup_{f \in W} \|f - g^*, b\| = \inf_{g \in G} \left\{ \sup_{f \in W} \|f - g, b\| \right\}
\]
where \(b \in X\setminus L\{f, g^*\}\) is the subspace of \(X\) generated by \(f\) and \(g^*\).
Lemma 2. Let \((X, \|\cdot\|)\) be a linear 2-normed space, \(G \subseteq X\) and \(W\) be bounded subset of \(X\). Then,

\[
\Phi (g, b) = \sup_{f \in W} \{\|f_1 - g, b\|, \|f_2 - g, b\|\}
\]

is a continuous functional on \(X\), where \(b \in X \setminus L \{f, g^*\}\).

Proof. Since the norms \(\|f_1 - g, b\|, \|f_2 - g, b\|\)are continuous functionals of \(g\) on \(X\), \(\phi (g, b)\) is clearly a continuous functional. To show this, for any \(f_1, f_2 \in W\) and \(g, g \in X\), we have

\[
\{\|f_1 - g, b\|, \|f_2 - g, b\|\} \leq \{\|f_1 - g', b\| + \|g - g', b\|, \|f_2 - g', b\|, \|g - g', b\|\}.
\]

Then

\[
\sup_{f_1, f_2 \in W} \{\|f_1 - g, b\|, \|f_2 - g, b\|\}
\]

\[
\leq \sup_{f_1, f_2 \in W} \{\|f_1 - g', b\| + \|g - g', b\|, \|f_2 - g', b\|, \|g - g', b\|\}.
\]

Now, if

\[
\|g - g', b\| < \frac{\varepsilon}{2}, \text{ then } \phi (g, b) \leq \phi (g', b) + \varepsilon.
\]

By interchanging \(g\) and \(g'\), proof of Theorem will be completed. \(\square\)

Lemma 3. Let \((X, \|\cdot\|)\) be a linear 2-normed space, \(G \subseteq X\) and \(W\) be bounded subset of \(X\). Then there exists a best simultaneous approximation \(g^* \in G\) to any given compact subset \(W \subset X\).

Proof. By using the proof of Elumalai and his coworkers in same manner, we can make the proof, using the definition of the continuous functional

\[
\phi (g, b) = \sup_{f \in W} \{\|f_1 - g, b\|, \|f_2 - g, b\|\}.
\]

\(\square\)

Lemma 4. Let \((X, \|\cdot\|)\) be a linear 2-normed space, \(G \subseteq X\) and \(W\) be bounded subset of \(X\). If \(g_1, g_2 \in G\) are best simultaneous approximations to \(W\) by elements of \(G\). Then \(g = \lambda_1 g_1 + \lambda_2 g_2\) is also a best simultaneous approximation to \(f_1\) and \(f_2\), where \(0 \leq \lambda \leq 1\) and \(\lambda_1 + \lambda_2 = 1\).

Proof. By using expression of \(\sup_{f \in W} \{\|f_1 - g, b\|, \|f_2 - g, b\|\}\), we discover the followings

\[
= \sup_{f \in W} \{\|f_1 - \lambda_1 g_1 - \lambda_2 g_2, b\|, \|f_2 - \lambda_1 g_1 - \lambda_2 g_2, b\|\}
\]

\[
= \sup_{f \in W} \{\|\lambda (f_1 - g_1) + (1 - \lambda) (f_1 - g_2), b\|, \|\lambda (f_2 - g_1) + (1 - \lambda) (f_2 - g_2), b\|\}.
\]
From last equality, we easily derive as
\[
\sup_{f \in W} \left\{ \| \lambda (f_1 - g_1) + (1 - \lambda) (f_1 - g_2), b \|, \| \lambda (f_2 - g_1) + (1 - \lambda) (f_2 - g_2), b \| \right\}
\]
\[
\sup_{f \in W} \left\{ \| \lambda (f_1 - g_1) + (1 - \lambda) (f_1 - g_2), b \|, \| \lambda (f_2 - g_1) + (1 - \lambda) (f_2 - g_2), b \| \right\}
\]

By using definition of 2-norm and definition 10, we deduce as follows
\[
\inf_{g \in G} \left\{ \sup_{f \in W} \| f_1 - g, b \|, \| f_2 - g, b \| \right\}.
\]

Subsequently, we complete the proof of Lemma. □

**Proposition 2.** Let \((X, \|\cdot\|)\) be a linear 2-normed space, \(G\) is a non-empty strictly convex subset of \(X\) and \(Y\) be a compact subset of \(X\). Then there is only one \(y_0 \in Y\) such that
\[
\| x_0 - y_0, z \| = \inf_{y \in Y} \left\{ \| x_0 - y, z \| \right\}
\]
for \(x_0 \in X \setminus Y\) and for every \(z \in X \setminus L \{ x \in G \}\) and \(y_0 \in Y\).

**Proof.** If \(x_0 \in Y\). Then, we have \(\| x_0 - y_0, z \| = 0\). Hence, assume that
\(x_0 \in X \setminus Y\).

If we say
\[
d_0 = \inf_{y \in Y} \left\{ \| x_0 - y, z \| \right\}
\]
and
\[
d_0 = \inf_{y \in Y} \left\{ \left\| x_0 - y, y' \right\| \right\}.
\]

Then, there are linearly independent elements \(y'\) and \(z\) in \(X\). So, there is a Cauchy sequence \(\{y_n\}\) such that
\[
\lim_{n \to \infty} \| x_0 - y_n, z \| = d_0, \quad \lim_{m \to \infty} \| x_0 - y_m, z \| = d_0
\]
and
\[
\lim_{n \to \infty} \left\| x_0 - y_n, y' \right\| = d_0, \quad \lim_{m \to \infty} \left\| x_0 - y_m, y' \right\| = d_0.
\]

Thus, we procure the following
\[
\| x_0 - y_0, y \| = d_0 \quad \text{and} \quad \| x_0 - y_0, z \| = d_0.
\]

By using the following inequalities
\[
d_0 \leq \| x_0 - y_0, z \| \leq \| x_0 - y_n, z \| + \| y_n - y_0, z \|
\]
and
\[
d_0 \leq \| x_0 - y_0, y \| \leq \| x_0 - y_n, y \| + \| y_n - y_0, y \|.
\]

From this, we see that
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\[ \|y_0 - y_0', z\|_2^2 = \|(y_0 - x_0) + (x_0 - y_0'), z\|_2^2 \]
\[ = \|(y_0 - x_0) + (x_0 - y_0'), z\|_2^2 + \|(y_0 - x_0) + (x_0 - y_0'), z\|_2^2 \]
\[ = \|y_0 + y_0' - 2x_0, z\|_2^2 \]
\[ \leq 2 \left( \|x_0 - y_0, z\|_2^2 + \|y_0' - x_0, z\|_2^2 \right) - 4 \left\| \frac{y_0 + y_0'}{2} - x_0, z \right\|_2^2 \]
\[ \leq 2 (2d_0^2) - 4d_0^2 = 0. \]

We find \( y_0 = y_0' \). In similar way, we again obtain \( y_0 = y_0' \) with respect to \( y \)

\[ \|y_0 - y_0', y\|_2^2 = \|(y_0 - x_0) + (x_0 - y_0'), y\|_2^2 \]
\[ = \|(y_0 - x_0) + (x_0 - y_0'), y\|_2^2 + \|(y_0 - x_0) + (x_0 - y_0'), y\|_2^2 \]
\[ = \|y_0 + y_0' - 2x_0, y\|_2^2 \]
\[ \leq 2 \left( \|x_0 - y_0, y\|_2^2 + \|y_0' - x_0, y\|_2^2 \right) - 4 \left\| \frac{y_0 + y_0'}{2} - x_0, y \right\|_2^2 \]
\[ \leq 2 (2d_0^2) - 4d_0^2 = 0 \]

Then, we complete the proof of theorem. \( \square \)

Let \( X \) be a linear 2-normed space and \( W_1 \) and \( W_2 \) are linear subspaces in \( X \), and \( f \) be a 2-functional with domain \( W_1 \times W_2 \). If \( \|\cdot,\| \) denotes 2-norm, then the problem is to find an element \( g^* \in G \), if it exists for which

\[ \sup_{f_1, f_2 \in W} \{\|f_1 - g^*, b\|, \|f_2 - g^*, b\|\} = \inf_{g \in G} \left\{ \sup_{f_1, f_2 \in W} (\|f_1 - g^*, b\|, \|f_2 - g^*, b\|) \right\}. \]

Thus, we reach the following proposition which is interesting and worthwhile for studying in linear 2-normed spaces.

**Proposition 3.** Let \( (X, \|\cdot,\|) \) be a linear 2-normed space over \( R \) and \( G \) be a linear subspace of \( X \). Let \( f_1, f_2 \in X \setminus G \) such that \( f_1, f_2 \) and \( b \in X \) are linearly independent. Then there exists a best simultaneous approximation by elements of \( G \) to \( f_1, f_2 \in W \) such that

\[ \inf_{g \in G} \left\{ \sup_{f_1, f_2 \in W} (\|f_1 - g, b\|, \|f_2 - g, b\|) \right\} = \sup_{f_1, f_2 \in W} \{\|f_1 - g^*, b\|, \|f_2 - g^*, b\|\} \]

where \( W = \{f_1, f_2\} \).

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