Canonical formulation of Poincaré BFCG theory and its quantization

Aleksandar Miković\textsuperscript{a,b} and Miguel A. Oliveira\textsuperscript{b,1}

\textsuperscript{a}Departamento de Matemática
Universidade Lusófona de Humanidades e Tecnologias
Av. do Campo Grande, 376, 1749-024 Lisboa, Portugal

\textsuperscript{b}Grupo de Fisica Matemática da Universidade de Lisboa
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

Abstract

We find the canonical formulation of the Poincaré BFCG theory in terms of the spatial 2-connection and its canonically conjugate momenta. We show that the Poincaré BFCG action is dynamically equivalent to the BF action for the Poincaré group and we find the canonical transformation relating the two. We study the canonical quantization of the Poincaré BFCG theory by passing to the Poincaré-connection basis. The quantization in the 2-connection basis can be then achieved by performing a Fourier transform. We also briefly discuss how to approach the problem of constructing a basis of spin-foam states, which are the categorical generalization of the spin-network states from Loop Quantum Gravity.

1 Introduction

Canonical formulation of General Relativity (GR) is suitable for performing a non-perturbative and background-metric independent quantization of GR, see [1, 2]. When using the spatial metric and its canonically conjugate momentum as the degrees of freedom for the gravitational field, one obtains a non-polynomial Hamiltonian Constraint (HC). Consequently the corresponding operator in the canonical quantization yields the Wheeler-DeWitt (WdW) equation, which is difficult to solve.

\textsuperscript{1}E-mails: amikovic@ulusofona.pt; masm.oliveira@gmail.com
The situation improves if the Ashtekar variables are used [3]. These are given by an $SU(2)$ complex connection on the spatial manifold and its canonically conjugate momentum. One then obtains a polynomial HC, but since the connection is complex, this introduces an additional non-polynomial constraint, the reality condition, which makes the quantization complicated. One can also use the real Ashtekar connection [4], but then the HC becomes again non-polynomial. Still, the fact that the basic canonical variables are the same as the $SU(2)$ Yang-Mills gauge theory canonical variables, makes it possible to use the holonomy and the electric-field flux variables, which leads to spin-network variables and Loop Quantum Gravity, (LQG) see [2].

The difficulties of solving the HC in the canonical LQG have led to the development of a path-integral quantization approach known as spin-foam models (SF), see [5, 6]. Although the HC problem can be solved in the SF approach by using the path-integral construction of the evolution operator, there is the problem of the semi-classical limit of a SF model and the problem of the coupling of fermionic matter [7, 8]. These problems are related to the fact that the edge-lengths, or the tetrads, are not explicitly present in a spin-foam formulation.

In order to introduce the tetrads in the SF formalism, a formulation of GR based on the Poincaré 2-group was given in [7]. The idea is to reformulate GR as a constrained topological theory of the BFCG type [9]. This approach is a categorical generalization of the constrained BF-theory formulation of GR which is used in SF models, see [10].

The BFCG reformulation of GR is useful for a path-integral quantization [8], where one obtains the spin-cube models, which represent a categorical generalization of the SF models. As far as the canonical quantization (CQ) is concerned, the progress has been hindered because the constrained BFCG theory has a complicated canonical structure. A reasonable strategy is to study first a simpler theory, which is the unconstrained BFCG theory. This is a topological gravity theory, and we will show that its canonical formulation is simple to understand. Another feature of this theory is that it is equivalent to the Poincaré group BF theory, so that one can perform a canonical quantization in terms of the BF theory variables. This is mathematically simpler than performing a canonical quantization in terms of the BFCG theory variables and it can also help to understand the quantization based on a spin-foam basis, which is a categorical generalization of the spin-network basis from LQG.

In section 2 we review the Poincaré BFCG theory and its relation to GR. In section 3 we perform a canonical analysis of the BFCG theory by using a shortened Dirac procedure. In section 4 we reformulate the BFCG theory...
as a BF theory for the Poincaré group and find the canonical transformation which relates the two canonical formulations. In section 5 we study the canonical quantization of the BFCG theory and by using the canonical transformation from the previous section we find a relation between the 2-connection basis and the Poincare-connection basis. We also indicate how to construct the spin-network and the spin-foam wavefunctions. In section 6 we present our conclusions.

2 Poincaré BFCG theory

Poincaré BFCG theory is a theory of flat 2-connections for a Poincaré 2-group, see [9, 7]. A 2-group is a 2-category with one object where all the 1-morphisms and all the 2-morphisms are invertible. This is equivalent to having a pair of groups \((G, H)\) with a group action \(\triangleright : G \times H \to H\) and a homomorphism \(\partial : H \to G\). The morphisms are the elements of \(G\), while the 2-morphisms are the elements of the semi-direct product group \(G \times_s H\).

In the Poincaré 2-group case \(G = SO(1,3)\) and \(H = \mathbb{R}^4\), while the group action is given by a Lorentz transformation of a four vector from \(\mathbb{R}^4\) and \(\partial\) is trivial. The 2-morphisms form the Poincaré group \(ISO(1,3)\).

One can define a notion of a 2-connection for a Lie 2-group, in analogy to the connection on a principal bundle for a manifold \(M\) and a Lie group \(G\). The 2-connection is a pair \((A, \beta)\), where \(A\) is a one-form taking values in the Lie algebra \(\mathfrak{g}\) of \(G\), while \(\beta\) is a 2-form taking values in the Lie algebra \(\mathfrak{h}\) of \(H\). The gauge transformations of \((A, \beta)\) are given by the usual gauge transformations

\[
A \to g^{-1}(A + d)g, \quad \beta \to g^{-1} \triangleright \beta, \tag{1}
\]

where \(g : M \to G\). These transformations correspond to local 1-morphisms, while the 2-morphisms generate a new gauge transformation

\[
A \to A, \quad \beta \to \beta + d\epsilon + A \wedge^b \epsilon, \tag{2}
\]

where \(\epsilon\) is a one-form from \(\mathfrak{h}\). By writing

\[
A(x) = \omega^{ab}(x) J_{ab}, \quad \beta(x) = \beta^a(x) P_a, \tag{3}
\]

where \(J\) are the Lorentz group generators and \(P\) are the translation generators, we obtain for the infinitesimal gauge transformations

\[
\delta \lambda \omega^{ab} = d\lambda^{ab} + \omega^{[a}_c \lambda^{b]c}, \quad \delta \lambda \beta^a = \lambda^a_c \beta^c, \tag{4}
\]
while for the infinitesimal 2-morphism gauge transformations we obtain
\[
\delta_s \omega = 0, \quad \delta_s \beta^a = de^a + \omega^a_c \wedge e^c.
\]

(5)

The curvature for a 2-connection \((A, \beta)\) is a pair of a 2-form \(F \in \mathfrak{g}\) and a 3-form \(G \in \mathfrak{h}\), given by
\[
F = dA + A \wedge A - \partial \beta, \quad G = d\beta + A \wedge^b \beta.
\]

(6)

In components we have
\[
R_{ab} = d\omega_{ab} + \omega^c_a \wedge \omega^b_c, \quad G^a = \nabla \beta^a = d\beta^a + \omega^a_c \wedge \beta^c,
\]

(7)\hspace{1cm}(8)

so that \(R_{ab}\) is the usual spin-connection curvature.

The dynamics of flat 2-connections for the Poincaré 2-group is given by the BFCG action
\[
S = \int_M (B_{ab} \wedge R_{ab} + e_a \wedge G^a)
\]

(9)

where \(B_{ab}\) is a 2-form and \(e_a\) are the tetrads [7]. The Lagrange multipliers \(B\) and \(e\) transform under the usual gauge transformations as
\[
B \to g^{-1} B g, \quad e \to g \triangleright e,
\]

(10)

while they are invariant under the 2-morphism transformations. The action (9) is also invariant under the diffeomorphism transformations.

If a constraint
\[
B_{ab} = \epsilon_{abcd} e^c \wedge e^d,
\]

(11)

is imposed in the action (9), one obtains a theory which is equivalent to the Einstein-Cartan formulation of GR
\[
S_{EC} = \int_M \epsilon^{abcd} e_a \wedge e_b \wedge R_{cd}.
\]

(12)

More precisely
\[
S_{EC} \cong \int_M \left[ B_{ab} \wedge R_{ab} + e_a \wedge G^a - \phi^{ab} \wedge (B_{ab} - \epsilon_{abcd} e^c \wedge e^d) \right],
\]

(13)

see [7].
3 Canonical analysis of BFCG theory

The canonical analysis of the BFCG action can be performed by using the Dirac procedure (DP). This is generally a laborious procedure, since it requires the introduction of the canonically conjugate momenta for every variable in the action \((9)\) and then executing the DP steps, see [11] in the case of a BF theory. However, in certain cases one can obtain a desired result in an easier fashion. Namely, given an action for variables \(Q\)

\[
S = \int_I L(Q, \dot{Q}) \, dt, \tag{14}
\]

where \(\dot{Q} = dQ/dt\), then the end-result of the Dirac procedure will be described by the action

\[
S_D = \int_I dt \left[ P \dot{Q} - H_0(P, Q) - \lambda^a G_a(P, Q) - \mu^\alpha \theta_\alpha(P, Q) \right], \tag{15}
\]

where \(P\) are the canonically conjugate momenta for the coordinates \(Q\), \(G_a\) are the First Class (FC) constraints, \(\theta^\alpha\) are the Second Class (SC) constraints and \(\lambda\) and \(\mu\) are the corresponding Lagrange multipliers\(^2\).

The FC constraints will satisfy

\[
\{G_a, G_b\}_D = f_{abc}(P, Q) G_c, \tag{16}
\]

and

\[
\{G_a, H_0\}_D = h_a^b(P, Q) G_b, \tag{17}
\]

where

\[
\{A, B\}_D = \{A, B\} - \{A, \theta_\alpha\} \Delta^{\alpha\beta}\{\theta_\beta, B\}, \tag{18}
\]

is the Dirac bracket. \(\Delta^{\alpha\beta}\) is the inverse matrix of \(\{\theta_\alpha, \theta_\beta\}\) and the Poisson Bracket (PB) is defined as

\[
\{A, B\} = \frac{\partial A}{\partial Q} \frac{\partial B}{\partial P} - \frac{\partial A}{\partial P} \frac{\partial B}{\partial Q}. \tag{19}
\]

In particular, if one can write the action (14) in the form

\[
S = \int_I dt \left[ p \dot{q} - \lambda^k G_k(p, q) \right], \tag{20}
\]

\(^2\)Here \(Q\) denotes both the set of the coordinates and the corresponding vector. Hence \(P\dot{Q}\) denotes the scalar product of vectors \(P\) and \(\dot{Q}\).
where \( p \cup q \cup \lambda = Q \) and

\[
\{ G_k, G_l \}^* = f_{ml}^k (p, q) \ G_m ,
\]

where \( \{ , \}^* \) is the \((p, q)\) Poisson bracket, then from (15) it follows that (20) is a gauge-fixed form of \( S_D \) where the second-class constraints have been eliminated and some of the phase-space coordinates have been set to zero. Hence the remaining FC constraints are given by \( G_k \) and \( H_0 \equiv 0 \).

This approach works in the BFCG case, which can be seen by splitting all the fields into the temporal and the spatial components via the coordinate splitting

\[
x^\mu = (x^0, x^i) = (t, \vec{x}) ,
\]

which corresponds to spacetime manifold \( M \) having the topology \( \Sigma \times I \), where \( \Sigma \) is a spatial 3-manifold.

We can then decompose the tensor fields from the action (9) as

\[
X^\mu \ldots Y^\nu \ldots = X^0 \ldots Y^0 \ldots + X^i \ldots Y^i \ldots .
\]

For example

\[
\epsilon^{\mu\nu\rho\sigma} B_{\mu
u}^a R_{\rho\sigma}^{cd} = 2 \epsilon^{ijk} (B_{0i}^a R_{jk}^{cd} + B_{ij}^a R_{0k}^{cd}) ,
\]

where

\[
R_{\mu\nu}^{ab} = \partial_{[\mu} \omega_{\nu]}^{ab} + \omega_{[\mu}^{ac} \omega_{\nu]}^{cb} .
\]

Similarly

\[
\epsilon^{\mu\nu\rho\sigma} e_{\mu}^{a} G_{a \nu\rho\sigma} = \epsilon^{ijk} (3 \epsilon_{i}^{0} G_{a 0jk} - \epsilon_{0}^{a} G_{a ijk}) ,
\]

where

\[
G_{a \mu\nu\rho} = \partial_{[\mu} \beta_{\nu]}^{a} + \omega_{[\mu}^{ab} \beta_{\nu]}^{b} .
\]

The Lagrangian density \( \mathcal{L} \) of the BFCG action (9) can be written as

\[
\mathcal{L} = \epsilon^{\mu\nu\rho\sigma} \left( \frac{1}{4} B_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} + \frac{1}{6} e_{\mu}^{a} G_{a \nu\rho\sigma} \right) ,
\]

so that

\[
\mathcal{L} = \pi_{a}^{b} \omega_{k}^{a} + \Pi_{a}^{ij} \beta_{ij}^{a} - \mathcal{H} ,
\]

where

\[
\pi_{ab}^{i} = \frac{1}{2} \epsilon^{ijk} B_{jk}^{ab} , \quad \Pi_{a}^{ij} = - \frac{1}{2} \epsilon^{ijk} e_{k}^{a} ,
\]

and

\[
\mathcal{H} = - \left[ \frac{1}{2} B_{0i}^{ab} \epsilon^{ijk} R_{ab}^{jk} + \frac{1}{2} e_{0}^{a} \epsilon^{ijk} \nabla_{i} \beta_{a}^{jk} + \cdots \right] .
\]
\[ + \omega_{ab}^0 \left( \nabla_i \pi_{ab}^i + \Pi_{[a}^{ij} \beta_{b]ij} \right) + \beta_{a0k} \epsilon^{ijk} \nabla_i e_{aj} \right). \] (31)

We have discarded a total divergence term in \( \mathcal{H} \) because a total divergence vanishes when \( \Sigma \) is compact. For non-compact \( \Sigma \) we assumed that all fields vanish at a spatial infinity.

The expression (31) implies that the constraints are given by

\[ C_{1ab}^i \equiv \frac{1}{2} \epsilon^{ijk} R_{abjk} = 0, \] (32)

\[ C_2^a \equiv \frac{1}{2} \epsilon^{ijk} \nabla_i \beta_{jk}^a = 0 \] (33)

\[ G_{1ab} \equiv \nabla_i \pi_{ab}^i - \beta_{[a|ij} \Pi_{b]ij}^j = 0, \] (34)

\[ G_{2a}^k \equiv \nabla_i \Pi_{ia}^k = \frac{1}{2} \epsilon^{ijk} T_{a}^k = 0, \] (35)

whose PB algebra is given by

\[ \{ C_2^a (\vec{x}), G_{2b}^i (\vec{y}) \} = -4 \epsilon_{[a}^i b \delta^{[3]}(\vec{x} - \vec{y}) \]

\[ \{ C_2^a (\vec{x}), G_{1cd} (\vec{y}) \} = \delta_{[a}^b \epsilon_{c d]} \delta^{[3]}(\vec{x} - \vec{y}) \]

\[ \{ C_{1ab}^i (\vec{x}), G_{1cd} (\vec{y}) \} = -4 \delta_{[a}^i \epsilon_{b] c d} \delta^{[3]}(\vec{x} - \vec{y}) \]

\[ \{ G_{1ab} (\vec{x}), G_{1cd} (\vec{y}) \} = 4 \delta_{[a}^c \epsilon_{b] d} \delta^{[3]}(\vec{x} - \vec{y}) \]

\[ \left\{ G_{1ab} (\vec{x}), G_{2c}^i (\vec{y}) \right\} = -\delta_{[a}^i \epsilon_{b] c} \delta^{[3]}(\vec{x} - \vec{y}) \]. (36)

Hence the constraints \( C_k \) and \( G_k \) are first class and \( H_0 \equiv 0 \).

## 4 BF formulation of Poincaré BFCG theory

Note that the \( e \land \nabla \beta \) term in the BFCG action (9) can be integrated by parts, so that

\[ S = \int_M \left( B^{ab} \land R_{ab} + e^a \land \nabla \beta_a \right) = \int_M \left[ B^{ab} \land R_{ab} + e^a \land \left( d \beta_a + \omega_a^b \land \beta_b \right) \right] \]

\[ = \int_M \left[ B^{ab} \land R_{ab} + \left( de^a + \omega_c^a \land e_b \right) \land \beta_a \right] - \int_M d(e^a \land \beta_a) \]. (37)

Hence

\[ S \cong \int_M \left( B^{ab} \land R_{ab} + T^a \land \beta_a \right), \] (38)

where

\[ T^a = de^a + \omega_c^a \land e_c. \] (39)
is the torsion.

The action (38) represents a BF-theory action for the Poincaré group, which can be seen by introducing a Poincaré-group connection
\[
A(x) = A^I(x)X_I = \omega^{ab}(x) J_{ab} + e^a(x) P_a,
\]
where \(J\) and \(P\) satisfy the Poincaré Lie algebra
\[
[J_{ab}, J_{cd}] = \eta_{[a][c]J_{d][b]}, \quad [P_a, J_{bc}] = \eta_{a[b]P_{c]}, \quad [P_a, P_b] = 0.
\]

The corresponding curvature is given by
\[
F = F^I X_I = (dA^I + f_{JK}^I A^J \wedge A^K) X_I,
\]
so that
\[
F = R^{ab} J_{ab} + T^a P_a.
\]
The action (38) can be then written in a BF form as
\[
S = \int_M B^I \wedge R_I,
\]
where
\[
B^I = \begin{pmatrix} B^{ab}, \beta^a \end{pmatrix}, \quad F_I = \begin{pmatrix} R_{ab}, T_a \end{pmatrix}.
\]
The canonical analysis can be performed by using the same method as in the BFCG case. The Lagrangian density can be written as
\[
\mathcal{L} = \pi_{ab}^i \dot{\omega}_{ab}^i + p^i_a \dot{e}_a^i - \tilde{\mathcal{H}},
\]
where
\[
\pi_{ab}^i = \frac{1}{2} \epsilon^{ijk} B_{ab}^j, \quad p^i_a = \frac{1}{2} \epsilon^{ijk} \beta_{a j k},
\]
and
\[
\tilde{\mathcal{H}} = \left[ \frac{1}{2} \epsilon^{ijk} B_{0i}^{ab} R_{ab}^j k + e^a_0 \nabla_i p^i_a \right] + \omega_{0}^{ab} \left( \nabla_i \pi_{ab}^i - e_{[a][i} P_{b]}^i \right) + \frac{1}{2} \epsilon^{ijk} \beta_{a 0i} T_{jk}^a.
\]
Therefore the constraints are given by
\[
\tilde{C}_1^{abi} \equiv \frac{1}{2} \epsilon^{ijk} R_{ab}^j k = 0,
\]
\[
\tilde{C}_2^{ai} \equiv \frac{1}{2} \epsilon^{ijk} T_{jk}^a = 0
\]
\[
\tilde{G}_1^{ab} \equiv \nabla_i \pi_{ab}^i - e_{[a][i} P_{b]}^i = 0,
\]
\[
\tilde{G}_2^a \equiv \nabla_i p^i_a = 0.
\]

The PB algebra of these constraints is given by

\[
\begin{align*}
\{ \tilde{C}^a_2(x), \tilde{G}^i_2(y) \} &= -4 \tilde{C}^a_1 b i \delta(3)(x - y) \\
\{ \tilde{C}^a_2(x), g_{1 c d}(y) \} &= \delta_{[c}^a \tilde{C}_{2 d]} \delta(3)(x - y) \\
\{ \tilde{C}^{ab}_1(x), \tilde{G}^c_1(y) \} &= -4 \delta_{[c}^a \tilde{C}^b_{1 d]} i \delta(3)(x - y) \\
\{ \tilde{G}^{ab}_1(x), \tilde{G}^{cd}_1(y) \} &= 4 \delta_{[a}^c \tilde{G}^{b d}_{2]} \delta(3)(x - y), \\
\{ \tilde{G}^{ab}_1(x), \tilde{G}^{c d}_2(y) \} &= -\delta^c_{[a} \tilde{G}^{b d}_{2]} \delta(3)(x - y).
\end{align*}
\]

(53)

Hence the constraints \( \tilde{C}_k \) and \( \tilde{G}_k \) are first class and \( H_0 \equiv 0 \).

Note that the BF constraint algebra (53) is the same as the BFCG constraint algebra (36). This is because there is a canonical transformation which relates the canonical pairs \((\beta, \Pi)\) and \((e, p)\). It is given by

\[
\begin{align*}
\beta^{a}_{ij} &= \varepsilon^{ijk} p^{ak}, \\
\Pi^{ij}_a &= -\varepsilon^{ijk} e^{ak},
\end{align*}
\]

(54)

so that \( \tilde{C}_k = C_k \) and \( \tilde{G}_k = G_k \). Hence (54) transforms the Poincaré BFCG theory into the BF theory for the Poincaré group.

5 Canonical quantization

Given a set of canonical variables \( \{(p_k, q_k) | k \in K\} \), one can define a quantization based on a representation of the corresponding Heisenberg algebra in the Hilbert space \( H_0 = L_2(\mathbb{R}^{|K|}) \) such that

\[
\begin{align*}
\hat{p}_k \Psi(q) &= i \frac{\partial \Psi(q)}{\partial q^k} , \\
\hat{q}_k \Psi(q) &= q_k \Psi(q).
\end{align*}
\]

(55)

We will refer to the representation (55) as the quantization in the \( q \) basis.

The results of the previous section imply that the canonical quantization of the Poincaré BFCG theory in the 2-connection basis \((\omega, \beta)\), can be related to the canonical quantization of the Poncare BF theory in the \((\omega, e)\) basis.

Since \( \beta \) is canonically conjugate to \( e \), by performing a functional Fourier transform, we obtain

\[
\Psi(\omega, \beta) = \int D e \Phi(\omega, e) \exp \left( i \int \omega a \wedge e_a \right).
\]

(56)

On the other hand, \( \Phi(\omega, e) \equiv \Phi(A) \) is a solution of a quantum version of the Poincare BF constraints. For any BF theory, the canonical pair \((A^I_I, E^J_I)\)
can be represented by the operators
\[
\hat{E}^i_I(x) \Phi(A) = i \frac{\delta \Phi}{\delta A^i_I(x)}, \quad \hat{A}^I_i(x) \Phi(A) = A^I_i(x) \Phi(A),
\]
so that the Gauss constraint
\[
\hat{G}^I_i \Phi(A) = \partial_i \left( \frac{\delta \Phi}{\delta A^i_I(x)} \right) + f^K_{IJ} A^J_i(x) \frac{\delta \Phi}{\delta A^K_I(x)} = 0,
\]
is equivalent to
\[
\Phi(A) = \Phi(\tilde{A})
\]
where \(\tilde{A} = A + d\lambda + [A, \lambda]\) is the infinitesimal gauge-transform of \(A\). This implies that \(\Phi(A)\) must be a gauge-invariant functional, while the vanishing curvature constraint
\[
F(A(x)) \Phi(A) = 0
\]
implies
\[
\Phi(A) = \prod_x \delta(F_x) \phi(A),
\]
i.e. \(\Phi(A)\) has a non-zero support on flat connections.

Consequently any gauge-invariant functional of flat Poincaré connections on \(\Sigma\), \(\Phi(\omega_0, e_0)\), is a solution. The space of \(\Phi(\omega_0, e_0)\), which we denote as \(\mathcal{H}_0\), is the space of functions on the moduli space of flat connections on \(\Sigma\) for the Poincare group \(ISO(1,3)\), which we denote as \(MS(ISO(3,1))\). It is easy to see that
\[
MS(ISO(3,1)) = VB[MS(SO(3,1))],
\]
where \(VB\) is the vector bundle such that the fiber at a point \(\omega_0\) of \(MS(SO(3,1))\) is the solution space of the vanishing torsion \(de_0 + \omega_0 \wedge e_0 = 0\).

In \(\mathcal{H}_0\) we can introduce a basis of spin-network wavefunctions. Let \(A\) be a connection for a Lie group \(G\) on \(\Sigma\), and let \(\gamma\) be a graph in \(\Sigma\). Given the irreps \(\Lambda_l\) of \(G\) associated to the edges of \(\gamma\) and the corresponding intertwiners \(\iota_v\) associated to the vertices of \(\gamma\), one can construct the spin-network wavefunctions
\[
W_{\tilde{\gamma}}(A) = Tr \left( \prod_l D^{(\Lambda_l)}(A) \prod_v C^{(\iota_v)} \right) \equiv \langle A|\tilde{\gamma}\rangle,
\]
where \(D^{(\Lambda_l)}(A)\) is the holonomy for the line-segment \(l\) and \(\tilde{\gamma} = (\gamma, \Lambda, \iota)\) denotes a spin network associated to a graph \(\gamma\).
Note that when $A$ is a flat connection, than (63) is invariant under a homotopy of the graph $\gamma$, so that we can label the spin-network wavefunctions by combinatorial (abstract) graphs $\gamma$.

In the case of a non-compact group there is a technical difficulty when constructing the spin-network wavefunctions. Namely, if one uses the unitary irreps (UIR), these are infinite-dimensional, and one has to insure that the trace in (63) is convergent. In the Poincare group case, we will consider the massive UIRs, which are labelled by a pair $(M, j)$, where $M > 0$ is the mass and $j \in \mathbb{Z}_{+}/2$ is an $SU(2)$ spin. In this case

$$D^{(M,j)}_{q,m';p,m}(\omega, a) = e^{i(\Lambda\omega p \cdot a)} D^{(j)}_{m'm}(W(\omega, p)) \delta^3(q - \Lambda\omega p), \quad (64)$$

where $p = (p_0, \vec{p}) = (\sqrt{\vec{p}^2 + M^2}, \vec{p})$, $D^{(j)}$ is a spin-$j$ rotation matrix and $W(\omega, p)$ is the Wigner rotation, see [12].

By requiring that $W_\gamma(A)$ form a basis in $\mathcal{H}_0$, we obtain

$$|\Psi\rangle = \int DA |A\rangle \langle A|\Psi\rangle = \sum_\gamma |\hat{\gamma}\rangle \langle \hat{\gamma}|\Psi\rangle \quad (65)$$

and

$$\langle \hat{\gamma}|\Psi\rangle = \int DA \langle \hat{\gamma}|A\rangle \langle A|\Psi\rangle = \int DA W^*_\gamma(A) \Psi(A). \quad (66)$$

The last formula is known as the loop transform.

Since we are dealing with a Lie 2-group, one would like to generalize the spin-network wavefunctions for the case of a 2-connection $(\omega, \beta)$. The categorical nature of a 2-group implies that one can associate 2-group representations to a 2-complex. Namely, if $(\omega, \beta)$ is a 2-connection for a Lie 2-group $(G, H)$ on $\Sigma$, then given a 2-complex $\Gamma$ in $\Sigma$, one can associate the 2-group representations $L_f$ to the faces $f$ of $\Gamma$. The corresponding 1-intertwiners $\Lambda_l$ can be associated to the edges of $\Gamma$, while the corresponding 2-intertwiners $\iota_v$ can be associated to the vertices of $\Gamma$. Hence we obtain a spin foam $\hat{\Gamma} = (\Gamma, L, \Lambda, \iota)$.

For example, in the 2-Poincare group case, there is a class of representations labelled by a positive number $L$, see [13]. The intertwiners for 3 such representations, $L_1, L_2, L_3$, are labeled by integers $m$ if $L_k$ satisfy the triangle inequalities strongly. The $m$'s label the irreps of an $SO(2)$ group, which leaves the triangle $(L_1, L_2, L_3)$, embedded in $\mathbb{R}^4$, invariant. The 2-intertwiners for the $m$'s are trivial and $L_k$ in this case can be identified with the edge-lengths of a triangle, see [7]. If $L_k$ are collinear, i.e. $L_1 = L_2 + L_3$, the invariance group is $SO(3)$ and the corresponding intertwiners are the
SU(2) spins $j$ while the 2-intertwiners are the SU(2) intertwiners. In this case the $L_k$ look like particle masses, but then it is not clear what would be the geometrical interpretation of these masses.

A spin-foam wavefunction should be an appropriate generalization of the trace of a surface-holonomy, so that

$$W_\Gamma(\omega, \beta) = \text{Tr} \left( \prod_{f \in \Gamma} D^{(L_f, \Lambda_l(f))}(\omega, \beta) \right),$$

where $l(f)$ is an edge from $\Gamma$. However, the problem with (67) is that it still is not clear what is the function $D^{(L, \Lambda)}(\omega, \beta)$. If $g_l = \exp(\omega_l J)$ and $h_f = \exp(\beta_f P)$ then

$$D^{(L_f, \Lambda_l)}(\omega, \beta) \sim D^{(L_f)}(g_{l(f)} \triangleright h_f).$$

This is based on the formula for the surface holonomy $h_p$ for the surface of a polyhedron $p$, given by

$$h_p = \prod_{f \in \partial p} g_{l(f)} \triangleright h_f,$$

where $g_{l(f)}$ can be calculated by representing the $\partial p$ surface as a composition of 2-morphisms $(g_l, h_f)$ from some 1-morphism $g_{l'} (l' \in p)$ to itself.

Another difficulty is that is not known what is the 2-group analog of the Peter-Weyl theorem

$$\phi(g) = \sum_{\Lambda} \sum_{\alpha_\Lambda, \beta_\Lambda} \tilde{\phi}_\Lambda^{\alpha_\Lambda \beta_\Lambda} D^{(\Lambda)}_{\alpha_\Lambda \beta_\Lambda}(g) = \sum_{\Lambda} \langle \tilde{\phi}_\Lambda, D^{(\Lambda)}(g) \rangle,$$

where $\phi$ is a function on a Lie group $G$ and

$$\tilde{\phi}_\Lambda^{\alpha_\Lambda \beta_\Lambda} = \int_G dg D^{(\Lambda)}_{\alpha_\Lambda \beta_\Lambda}(g) \phi(g).$$

Note that in the case of the Poincaré 2-group, the relation (56) can give some clues. Given a Poincaré group holonomy $g_l = \exp(\omega_l J + e_l P)$ for an edge $l$, then a function $\phi(g_l) = \Phi(\omega_l, e_l)$ can be expanded by using the PW theorem for the Poincaré group

$$\Phi(\omega_l, e_l) = \int_0^\infty dM \sum_j \langle \tilde{\Phi}_{M,j}, D^{(M,j)}(\omega_l, e_l) \rangle.$$
Consequently

\[ \Psi(\omega_l, \beta_f) = \int_{\mathbb{R}^4} d^4 e_l \mu(e_l) e^{i(\beta_f, e_l)} \Phi(\omega_l, e_l) \]

\[ = \sum_{\Lambda} \left( \tilde{\Phi}_\Lambda, \int_{\mathbb{R}^4} d^4 e_l \mu(e_l) e^{i(\beta_f, e_l)} D^{(\Lambda)}(\omega_l, e_l) \right) \]

\[ = \sum_{\Lambda} \left( \tilde{\Phi}_\Lambda, \tilde{D}^{(\Lambda)}(\omega_l, \beta_f) \right), \tag{73} \]

where \( f \) is the face dual to the edge \( l \) and \( \mu \) is some appropriately chosen measure.

We then obtain

\[ \Psi(\omega_l, \beta_f) = \int_0^\infty dM \sum_j \langle \tilde{\Phi}_{M,j}, \tilde{D}^{(M,j)}(\omega_l, \beta_f) \rangle. \tag{74} \]

This suggests \( L = M, \Lambda = j \) so that

\[ D^{(L,\Lambda)}(\omega_l, \beta_f) \sim \tilde{D}^{(M,j)}(\omega_l, \beta_f). \tag{75} \]

6 Conclusions

We have found a canonical formulation of the BFCG action for the Poincaré 2-group where the phase-space variables are the 2-connection \((\omega^{ab}_i, \beta^a_i)\) on a 3-manifold \(\Sigma\) and its canonically conjugate pair \((\pi^{ab}_i, \Pi^{a}_i)\). This canonical formulation is suitable for the canonical quantization where the physical Hilbert space is spanned by the spin-foam states, which are the categorical generalization of the spin-network states from LQG. By using the fact that the BFCG action for the 2-Poincaré group is equivalent to the BF action for the Poincaré group, we obtain a canonical transformation which relates the two canonical formulations. In the BF canonical formulation, the basic variable is the Poincare connection \((\omega^{ab}_i, e^a_i)\) on \(\Sigma\) and its canonically conjugate pair \((\pi^{ab}_i, p^a_i)\), and the corresponding canonical transformation is given by (54).

There is a mathematical difficulty when trying to construct the spin-foam basis, that comes from the lack of knowledge of what is the exact form of the Peter-Weyl theorem for 2-groups. However, in the Poincaré 2-group case, we can use the relation to the Poincaré BF theory, which gives important clues how to construct the spin-foam wavefunctions. We believe that those clues will be sufficient to complete the spin-foam basis construction.
On the other hand, one can quantize the theory in the BF formulation, and in this case the physical Hilbert space is given by the space of square-integrable functions on the moduli space of flat connections. One can proceed further, and introduce the spin-network basis, by constructing the spin-network wave functions for the Poincare group. An interesting problem will be to investigate the relation between the spin-network basis and the spin-foam basis.

As far as the canonical quantization of GR in the spin-foam basis is concerned, this requires a canonical formulation of the constrained BFCG theory based on the 2-connection variables (ω, β) and their momenta (π, Π). However, the structure of the GR constraints is such that the short-cut procedure based on the space-time decomposition of the fields in the action does not work, and one has to perform the full Dirac procedure. Given the corresponding action $S_D$, one has to eliminate the second-class constraints and use the gauge-fixing procedure in order to obtain the action (20) for the 2-connection canonical variables (ω, π) and (β, Π). Another possibility is to gauge fix $S_D$ such that the remaining canonical variables are the Poincaré-connection variables (ω, π) and (ε, p) and the corresponding set of the first-class constraints. We expect that the gauge fixing will reduce the manifest local Lorentz group invariance to the local $SO(3)$ invariance.

Acknowledgements

We would like to thank M. Vojinović for discussions. A. Miković was partially supported by the FCT grants PEst-OE/MAT/UI0208/2011 and EXCL/MAT-GEO/0222/2012, while M. Oliveira was supported by the FCT PhD grant SFRH/BD/79285/2011.

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