DIMENSION COUNTS FOR SINGULAR RATIONAL CURVES VIA SEMIGROUPS

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Abstract. We study singular rational curves in projective space, deducing conditions on their parametrizations from the value semigroups S of their singularities. In particular, we prove that a natural heuristic for the codimension of the space of nondegenerate rational curves of arithmetic genus \( g > 0 \) and degree \( d \) in \( \mathbb{P}^n \), viewed as a subspace of all degree-\( d \) rational curves in \( \mathbb{P}^n \), holds whenever \( g \) is small.

Introduction

Rational curves are essential tools for classifying complex algebraic varieties. Establishing dimension bounds for families of embedded rational curves that admit (analytic isomorphism classes of) singularities of a particular type arises naturally in this context; see, for example [19] and [9], where such bounds are used to infer (dimension-theoretic) information about parameter spaces of rational curves embedded in general hypersurfaces. It is also a basic fact [22] that any curve singularity occurs along a rational curve.

For our purposes, it will be most useful to view a rational curve as a morphism: given a choice of ambient dimension \( n \) and degree \( d \geq n \), any such is the image of an \((n+1)\)-tuple of holomorphic functions \( f = (f_0,\ldots,f_n) \) with \( f_i \in H^0(O_{\mathbb{P}^1}(d)) \). Explicitly, we may write

\[
 f_i = a_{i,0}t^d + a_{i,1}t^{d-1}u + \cdots + a_{i,d}u^d
\]

where \( t, u \) are homogeneous coordinates on \( \mathbb{P}^1 \), and \( a_{i,j} \) are complex coefficients for every \( i = 0,\ldots,n \) and \( j = 0,\ldots,d \). From this point of view, the natural parameter space for rational curves is thus the Grassmannian \( G(n,d) \).

Viewing rational curves as points of the Grassmannian is the point of departure for Griffiths and Harris’ proof [17] of the Brill–Noether theorem for general curves of genus \( g \geq 2 \). In that work, they show that rational curves with \( g \) cusps serve as dimension-theoretic surrogates for general curves of genus \( g \), in the sense that the dimensions of the spaces of linear series on such curves are the (generically) expected ones. Eisenbud and Harris subsequently reworked (and simplified the proof of) the Brill–Noether result using rational curves with \( g \) cusps [12]. The fact that cuspidal, as opposed to nodal, rational curves are the most-amenable to dimension estimates in the Grassmannian is a recurring theme in this work as well: the value semigroups of unibranch, i.e. cuspidal, singularities have a simple structure, while value semigroups of multibranch singularities do not.

1991 Mathematics Subject Classification. Primary 14H20, 14H45, 14H51, 20Mxx.

Key words and phrases. linear series, rational curves, singular curves, semigroups.
A key invariant of cusps is their \textit{ramification}, or inflection, in a point along their underlying curve. In each point $P \in \mathbb{P}^1$, the morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ is determined by local sections $(\sigma_0, \ldots, \sigma_n)$ vanishing at orders $a_0 < \cdots < a_n$ in $P$. Generically, we have $(a_0, a_1, \ldots, a_n) = (0, 1, \ldots, n)$, and the deviation

\begin{equation}
\alpha = (a_0, a_1, \ldots, a_n) - (0, 1, \ldots, n)
\end{equation}

indexes a Schubert variety $\Sigma_\alpha \subset G(n, d)$ of codimension $|\alpha|$ associated to certain incidence conditions with respect to an osculating flag of a rational normal curve in $\mathbb{P}^d$ (and canonically specified by (2)). A fundamental fact is that $m$ Schubert varieties $\Sigma_{\alpha_i}$, for $1 \leq i \leq m$, obtained by ramification in $m$ distinct points $P_i \in \mathbb{P}^1$ intersect dimensionally transversely; i.e., their codimensions are additive.

The ramification of a singularity imposes linear conditions on the coefficients of the underlying morphism $f = (f_0, \ldots, f_n) : \mathbb{P}^1 \to \mathbb{P}^n$, namely the vanishing of partial derivatives of the $f_i$, $i = 0, \ldots, n$. In general, a singularity is not specified by its ramification. However, the semigroup of a singularity provides a natural set of additional, non-linear conditions on coefficients of the $f_i$, which arise from the multiplicative structure of the local ring of the singularity. We conjecture that, taken together, these two sets always impose the required $(n-2)g$ independent conditions on morphisms. Our method for producing dimension estimates for singular rational curves, which uses the natural stratification of singularities by semigroups, should be viewed as a first attempt to systematically classify these objects.

0.1. \textbf{Roadmap.} We prove two main theorems in this paper. The first result, Thm 1.1, establishes that the expected codimension $(n-2)g$ is achieved for morphisms whose images have at-worst unibranch singularities, provided that $g \leq 8$. The proof uses the stratification of singularities according to their value semigroups; in the unibranch case, these are \textit{numerical semigroups}, i.e. subsemigroups of $\mathbb{N}$, and thus form a tree.

Our second result, Thm 2.1, establishes that our codimension heuristic holds in full generality when $g \leq 4$. The proof of Thm 2.1 also uses the stratification of morphisms according to the semigroups of their singularities. The analysis in the multibranch setting is significantly more involved, however, since there is no obvious combinatorial structure to index the semigroups. Our strategy is to filter subsemigroups of $\mathbb{N}^r$ according to their $r$-tuple sums, thereby exploiting the tree structure of numerical semigroups. For every given semigroup, we then reverse-engineer a set of possible geometric configurations, from which we may count conditions. To our knowledge, our reverse-engineering technique has not appeared elsewhere.

Our theorems 1.1 and 2.1 generalize results obtained in the papers [19], [8], and [9], where they are applied towards dimension-counting for rational curves on very general hypersurfaces.

0.2. \textbf{Conventions.} We work over $\mathbb{C}$. By \textit{rational curve} we always mean a projective curve of geometric genus zero. We denote by $M^g_d$ the space of nondegenerate morphisms $f : \mathbb{P}^1 \to \mathbb{P}^n$ of degree $d > 0$. Here each morphism is identified with the set of coefficients of its homogeneous parametrizing polynomials, so $M^g_d$ is a space of frames over $G(n, d)$. We denote by $M^g_{d,g}$ the subvariety of morphisms whose images have arithmetic genus $g > 0$. These curves are necessarily singular. Clearly, $M^g_{d,g}$ contains all curves with $g$ simple nodes or $g$ simple cusps.
It will also be useful to consider parameter spaces associated to a fixed choice of value semigroup $S$. Accordingly, let $V_S \subset \mathcal{M}^n_{d,g}$ denote the space of nondegenerate morphisms $f : \mathbb{P}^1 \to \mathbb{P}^n$ with (fixed) degree $d > 0$, image of arithmetic genus (at least) $g$, and a singularity with value semigroup $S$ of genus (exactly) $g$. Here the genus of a value semigroup encodes the contribution of the underlying singularity to the arithmetic genus of the underlying projective curve. We use $a_{i,j}$ to denote the coefficient $[t^i u^{d-i}] f_j$ of $t^i u^{d-i}$ in $f_j$, $j = 0, \ldots, n$. We reserve the letter $c$ for the conductor of a semigroup $S$. We also follow the convention of [15] and denote by $S^*$ the truncation of $S$ in $c$, i.e., the ordered sequence of values in $S$ in the critical range between $0 = (0,\ldots,0)$ and the conductor.

0.3. A heuristic for dimension counts. Requiring a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ to map distinct points $p_1, p_2 \in \mathbb{P}^1$ to the same image $q \in \mathbb{P}^n$ imposes $2n$ linear conditions on the coefficients of $f$. Allowing the preimages and image to vary yields $(n-2)$ linear conditions. Since a simple node has arithmetic genus 1, it might seem reasonable to expect more generally that singularities of arithmetic genus $g$ impose at least $(n-2)g$ conditions on morphisms $f : \mathbb{P}^1 \to \mathbb{P}^n$. In other words, we’d expect that $\text{cod}(\mathcal{M}^n_{d,g}, \mathcal{M}^n_d) \geq (n-2)g$, when $d$ is sufficiently large relative to $g$.

It is worth noting that $(n-2)g$ is the best possible codimension estimate in $n$ and $g$ that is uniform across all singularity types. This follows from an elementary analysis of $g$-nodal curves that we will carry out in detail in the proof of Thm 2.1.

Acknowledgements. We would like to thank Maksym Fedorchuk, Joe Harris, Steve Kleiman, Karl-Otto Stöhr, Fernando Torres, and Filippo Viviani for illuminating conversations, as well as the mathematics department at UFMG for making this collaboration possible. The first and third authors are partially supported by CNPq grant numbers 309211/2015-8 and 306914/2015-8, respectively. The second author is supported by FAPEMIG.

1. Counting conditions for unibranch singularities

Unibranch singularities form a naturally distinguished (simple) class of singularities. Accordingly, it makes sense to ask for dimension estimates for rational curves with at-worst-unibranch singularities. We will prove that our na"{i}ve codimension estimate holds for rational curves of arithmetic genus at most 8:

**Theorem 1.1.** Let $\mathcal{V} := \bigcup_{S \subset \mathbb{N}} V_S \subset \mathcal{M}^n_{d,g}$ be the subvariety consisting of rational curves with at-worst-unibranch singularities. Suppose, moreover, that $g \leq 8$ and $d \geq \max(n,2g-2)$. Then

$$\text{cod}(\mathcal{V}, \mathcal{M}^n_d) \geq (n-2)g.$$ 

**Remarks 1.2.** The choice of genus threshold in Thm 1.1 is essentially arbitrary (indeed, our codimension estimate seems to hold in every computable case) but our method of proof requires us to consider a certain number of exceptional cases, whose number grows as the genus increases.

On the other hand, the condition $d \geq n$ is imposed by the requirement that our rational curves be nondegenerate. It is less clear what a reasonable lower threshold for the degree as a function of the genus should be, but the assumption that $d \geq 2g-2$ is well-adapted to the analysis of conditions beyond ramification in the proof to follow; it also includes the (canonical) case in which $d = 2g-2$ and $n = g-1$. 
The proof of Thm 1.1 invokes a number of standard tools from linear series and singularities, which we review now. Accordingly, let \( P \in C := f(\mathbb{P}^1) \subset \mathbb{P}^n \) be a unibranch singularity. Then \( P \) admits a local parametrization \( \psi : t \mapsto (\psi_1(t), \ldots, \psi_n(t)) \) corresponding to a map of rings \( \phi : R := \mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[t] \). Let \( v : \mathbb{C}[t] \to \mathbb{N} \) denote the standard valuation induced by the assignment \( t \mapsto 1 \). Let \( S := v(\phi(R)) \) denote the numerical value semigroup of \( P \). The (local) genus of the singularity at \( P \) is \( \delta_P := \#(\mathbb{N} \setminus S) \), and the (global arithmetic) genus of \( C \) is the sum of all of these local contributions:

\[
g = \sum_{P \in C} \delta_P.
\]

It will be convenient in what follows to think of the genus of a unibranch singularity as an invariant of the associated numerical semigroup \( S \); the definition remains unchanged. The weight of a unibranch singularity (or equivalently, of its semigroup \( S \)) of genus \( g \) is \( W_S = \sum_{\ell \in \mathbb{N} \setminus S} \ell - \frac{g(g+1)}{2} \). Here \( G_S := \mathbb{N} \setminus S \) is the gap set of \( S \).

### 1.1. Beginning of the proof of Thm 1.1.

Let \( m_i \) denote the \( i \)th positive integer in \( S \) (ordered from smallest to largest), and let \( a_0 < a_1 < \cdots < a_n \) be the vanishing orders of the sections of \( f \) at \( f^{-1}(P) \). Then \( a_0 = 0 \), and clearly

\[
a_i \geq m_i
\]

for every \( i = 1, \ldots, n \), so we see that \( C \) ramifies at \( P \) to order at least

\[
r_P = \sum_{i=1}^{n} (m_i - i).
\]

As the codimension of Schubert varieties associated to the ramification at distinct points of \( \mathbb{P}^1 \) is additive, it suffices to prove that

\[
r_P - 1 \geq (n - 2)g
\]

where the -1 on the left hand side arises from varying the preimage of \( P \) along \( \mathbb{P}^1 \).

We will show that (3) holds whenever \( W_S \leq 2g - 1 \).

To this end, we use the fact [4] that any numerical semigroup \( S \) of genus \( g \) may be represented as a Dyck path \( \tau = \tau(S) \) on a \( g \times g \) square grid \( \Lambda_S \) with axes labeled by \( 0, 1, \ldots, g \). Each path starts at \((0,0)\), ends at \((g,g)\), and has unit steps upward or to the right. Namely, the \( i \)th step of \( \tau \) is up if \( i \notin S \), and is to the right otherwise.

The weight \( W = W_S \) of \( S \) is then equal to the total number of boxes in the Young tableau traced between the upper and left borders of the grid and the Dyck path \( \tau \).

Denote by \( W^\alpha_S \) the contribution of the first \( n \) elements \( m_1, \ldots, m_n \) of \( S \). Diagrammatically, this is precisely the area above the Dyck path in the subgrid determined by the first \( n \) columns of the \( \Lambda_S \); equivalently,

\[
W^\alpha_S = ng - r_P.
\]

On the other hand, we clearly have

\[
W^\alpha_S \leq W_S \leq 2g - 1.
\]

Combining (4) with (5) now yields (3).
1.2. The semigroup tree. It is well-known that all numerical semigroups are indexed by the vertices of an infinite tree. More precisely, this semigroup tree has vertices indexed by sets of minimal generators; the unique common ancestor of all vertices is the semigroup \( (1) = \mathbb{N} \) of genus 0. Any semigroup \( S \) of genus \( g \geq 0 \) may be presented by minimal generators as \( S = \langle a_1, \ldots, a_N; b_1, \ldots, b_M \rangle \) in which \( a_1 < \cdots < a_N < b_1 < \cdots < b_M \) and the conductor \( c = c(S) \) satisfies \( a_N < c \leq b_1 \). (Here the set \( \{b_1, \ldots, b_M\} \) might be empty; recall that the conductor of a semigroup \( S \subset \mathbb{N}^r \) is the unique minimal element \( c \) for which \( c + \mathbb{N}^r \subset S \).) A child of \( S \), if it exists, is the unique completion of some subset \( S - \{b_i\}, i = 1, \ldots, M \) to a semigroup of genus \( g + 1 \); for an illustration of the tree showing all vertices of genus \( g \leq 4 \), see [5, Fig. 1].

The infinite branches of the semigroup tree distinguish infinite subfamilies of singularities, beginning with hyperelliptic singularities, i.e., those whose semigroups contain 2. Using the semigroup tree, it is easy (though tedious) to check that all nonhyperelliptic numerical semigroups \( S \) of genus \( g < 7 \) verify \( W_S \leq 2g - 1 \).

1.3. Conditions beyond ramification. In this subsection, we consider nonhyperelliptic singularities whose semigroups \( S \) fail to verify the weight restriction \( W_S \leq 2g - 1 \). Specifically, we will show how to produce additional conditions in such cases, using multiplicative constraints imposed by \( S \). In this subsection we will do so for exactly those semigroups of genus \( g \leq 8 \) for which \( W_S \geq 2g \).

- **Case:** \( S = (3,8) \). The semigroup has genus 7, weight 14, and gap set \( G_S = \{1, 2, 4, 5, 7, 10, 13\} \). To estimate the codimension of the corresponding parameter space \( V_S \) of morphisms \( f \) with singularities of semigroup \( S \), we work analytically locally near such a singularity \( P \). We may also assume that the lowest vanishing orders of global sections of \( f \) in \( f^{-1}(P) \) match the initial entries in \( S^* = \{0, 3, 6, 8, 9, 11, 12, 14\} \). Indeed, it is straightforward to check that the space of morphisms that fail this minimality hypothesis is of codimension at least equal to \( (n - 2)g + 1 \). (In fact, the only cases in which ramification fails to produce the required \( (n - 2)g + 1 \) conditions are those in which \( n \geq 6 \).) It suffices to exhibit a single condition beyond (and independent of) ramification. Accordingly, let 

\[
 f_1(t) = t^3 + \alpha_1 t^4 + O(t^5) \quad \text{and} \quad f_2(t) = t^6 + \alpha_2 t^7 + O(t^8)
\]

denote power series representatives for two local sections of minimal vanishing orders. Here we use local affine coordinates with respect to which \( P \) is the origin in \( \mathbb{C}^n \), and \( t \) is a local coordinate centered in \( f^{-1}(P) \). We then have

\[
 f_1^2 - f_2 = (2\alpha_1 - \alpha_2)t^7 + O(t^8).
\]

In particular, since \( 7 \in G_S \), it follows necessarily that \( 2\alpha_1 - \alpha_2 = 0 \). So we obtain an additional linear condition on the coefficients of \( f \) that is distinct from (and independent of) the ramification conditions.

- **Case:** \( S = (3,10,17) \). Here \( g(S) = 8 \), \( W_S = 16 \), and \( G_S = \{1, 2, 4, 5, 7, 8, 11, 14\} \). As in the preceding case, we work analytically in an adapted local coordinate \( t \) and we may assume that local vanishing orders of \( f \) in the preimage of the singularity are minimal. It suffices to exhibit a single condition beyond ramification. Once more, the linear condition \( 2\alpha_1 - \alpha_2 = 0 \) arises from the
Figure 1. Dyck path and Young tableau corresponding to $S = \langle 3, 8 \rangle$.

Figure 2. Dyck path and Young tableau corresponding to $S = \langle 3, 10, 17 \rangle$.

Figure 3. Dyck path and Young tableau corresponding to $S = \langle 4, 6, 13 \rangle$.

quadratic polynomial $f_2^2 - f_2$ in the two local sections $f_1 = t^4 + \alpha_1 t^4 + O(t^5)$ and $f_2 = t^6 + \alpha_2 t^7 + O(t^8)$ of minimal vanishing orders.

• Case: $S = \langle 4, 6, 13 \rangle$. Here $g(S) = 8$, $W_S = 17$, and $G_S = \{1, 2, 3, 5, 7, 9, 11, 15\}$.

For dimension reasons, we may assume local vanishing orders of sections in the preimage of the singularity are minimal in $S^* = \{0, 4, 6, 8, 10, 12, 13, 14, 16\}$.

We may further assume that $n \geq 6$. It will suffice to exhibit two linear conditions beyond ramification. For this purpose, consider three local (analytic) sections of minimal vanishing orders:

$f_1(t) = t^4 + \alpha_1 t^5 + O(t^6)$, $f_2(t) = t^6 + \alpha_2 t^7 + O(t^8)$, and $f_3(t) = t^8 + \alpha_3 t^9 + O(t^{10})$.

We then have

$f_1^2 - f_3 = (2\alpha_1 - \alpha_3)t^9 + O(t^{10})$ and $f_1^2 f_2 - f_2 f_3 = (2\alpha_1 - \alpha_2 - \alpha_3)t^{15} + O(t^{16})$.

Since 9 and 15 belong to $G_S$, it follows that

$2\alpha_1 - \alpha_3 = 0$ and $2\alpha_1 - \alpha_2 - \alpha_3 = 0$,

which gives the two required conditions.

Remark 1.3. In light of the preceding discussion, it is natural to ask for $g$-asymptotic estimates for the preponderance of semigroups of weight at least $2g$, i.e. refinements of Zhai’s result [29] that the number $n_g$ of all semigroups of given genus has Fibonacci-like asymptotics. Results of this type have recently been obtained by Kaplan and Ye [20], who show among other things that the proportion
of genus-\(g\) semigroups with weight between (approximately) \(0.035g^2\) and \(0.089g^2\) is asymptotically equal to 1. In particular, this shows that semigroups of \(g\)-linearly bounded weight form a set of measure zero for \(g \gg 0\).

1.4. Dimension counts for rational curves with hyperelliptic singularities.

There is a unique hyperelliptic semigroup \(S^h = S^h(g)\) of genus \(g\), namely \((2, 2g + 1)\), and it has weight \(W_{S^h} = \binom{g}{2}\). In particular, we have \(W_{S^h} \ge 2g\) for \(g \ge 5\). As in the preceding section, we must use the arithmetic structure of \(S^h\) to produce conditions beyond ramification. It will also be convenient to codify the notion of the ramification arising from a numerical semigroup itself (as opposed to one of its truncations). In general, given a numerical semigroup presented as

\[
S = \{s_i\}_{i \ge 0}
\]

where \(s_i < s_{i+1}\) for all \(i \ge 0\) and \(s_0 = 0\), we denote by

\[
R_S := \{s_i - i\}_{i \ge 0} \quad \text{and} \quad TR_S := \left\{ \sum_{j=0}^{i} (s_j - j) \right\}
\]

the ramification and total ramification sequences of \(S\), respectively. We now specialize to the case in which \(S\) is hyperelliptic. It is then easy to see that \(R_{S^h} = \{r_i\}_{i \ge 0}\) and \(TR_{S^h} = \{\tilde{r}_i\}_{i \ge 0}\) are characterized by

\[
r_i = i, 0 \le i \le g; \quad r_i = g, i \ge g; \quad \text{and} \quad \tilde{r}_i = \left(\frac{i + 1}{2}\right), 0 \le i \le g; \quad \tilde{r}_i = \left(\frac{g + 1}{2}\right) + (i - g)g = ig - \left(\frac{g}{2}\right), i \ge g.
\]

In particular, we see that

\[
\tilde{r}_i \le (i - 2)g \quad \text{if and only if} \quad g \ge \left\lceil \frac{(i + 1)i}{2(i - 2)} \right\rceil.
\]

On the other hand, the number of conditions beyond ramification required (to match \((n - 2)g + 1\) total conditions) is bounded above by \(N_R = N_R(g)\), where

\[
N_R(g) := g(g - 2) + 1 - \left(\frac{g + 1}{2}\right) = \frac{g(g - 5)}{2} + 1.
\]

**Case: \(g = 5\).** Here (7) yields \(N_R = 1\), so we must produce a single condition beyond ramification. Moreover, applying (6), we may assume \(n \ge 4\). Accordingly, let

\[
f_1(t) = t^2 + \alpha_1 t^3 + O(t^4) \quad \text{and} \quad f_2(t) = t^4 + \alpha_2 t^5 + O(t^6)
\]

denote two analytic sections of minimal nontrivial vanishing orders in a local coordinate \(t\) adapted to the singularity. Then

\[
f_1^2 - f_2 = (2\alpha_1 - \alpha_2)t^5 + O(t^6).
\]

As \(5 \in G_{S^h} = \{1, 3, 5, 7, 9\}\), it follows that

\[
2\alpha_1 - \alpha_2 = 0
\]

which is the required condition.
- **Case:** $g = 6$. Here $N_R = 4$. More precisely, we must produce (at least) one, three, or four conditions beyond ramification when $n = 3$, $n = 4$, or $n \geq 5$, respectively. If $n = 3$, we conclude using the same condition (8) as in the analysis of the $g = 5$ case. If $n = 4$, we further let

$$f_3(t) = t^6 + \alpha_3 t^7 + O(t^8) \quad \text{and} \quad f_4(t) = t^8 + \alpha_4 t^9 + O(t^{10}).$$

Now consider the following quadratic polynomials in $f_1$, $f_2$, $f_3$, and $f_4$:

$$f_1 f_3 - f_4 = (\alpha_1 + \alpha_3 - \alpha_4) t^9 + O(t^{10}) \quad \text{and} \quad f_2^2 - f_4 = (2\alpha_2 - \alpha_4) t^9 + O(t^{10}).$$

As $9 \in G_{sa} = \{1, 3, 5, 7, 9, 11\}$, it follows that

$$\alpha_1 + \alpha_3 - \alpha_4 = 0 \quad \text{and} \quad 2\alpha_2 - \alpha_4 = 0.$$

(9)

Taken together, conditions (8) and (9) allow us to conclude when $n = 4$, provided that the sections of $f$ vanish minimally to orders 2, 4, 6, and 8. The remaining possibility to be treated when $n = 4$ is that the global sections of $f$ vanish to orders 2, 4, 6, and 10. However, in that instance only a single condition beyond ramification is required, and (8) is still operative.

Now assume $n \geq 5$. We begin by considering the situation in which global sections of $f$ vanish to orders 0, 2, 4, 6, 8, and 10 in $f^{-1}(P)$. Accordingly, we let

$$f_5(t) = t^{10} + \alpha_5 t^{11} + O(t^{12}).$$

Then

$$f_1 f_4 - f_5 = (\alpha_1 + \alpha_4 - \alpha_5) t^{11} + O(t^{12})$$

and as $11 \in G_{sa}$, we deduce that

$$\alpha_1 + \alpha_4 - \alpha_5 = 0.$$

(10)

The independent conditions (8), (9), and (10) allow us to conclude when $n \geq 5$, provided that the sections of $f$ vanish minimally to orders 0, 2, 4, 6, 8, and 10 in $f^{-1}(P)$. Similarly, if global sections of $f$ vanish minimally to orders 0, 2, 4, 6, 8, and $\gamma$ with $\gamma \geq 12$, we need only produce two conditions beyond ramification, and (8) and (9) remain operative. In all other cases, ramification gives the required $(n-2)g + 1$ conditions. Note that conditions (8), (9), and (10) are all of the form

$$\alpha_j = j\alpha_1$$

(11)

where $j \geq 1$.

- **Case:** $g = 7$. This time $N_R = 8$. Specifically, we must produce two, five, seven, or eight conditions beyond ramification when $n = 3$, $n = 4$, $n = 5$, or $n \geq 6$, respectively. If $n = 3$, we may assume that sections of $f$ vanish minimally to orders 0, 2, 4, and 6 in $f^{-1}(P)$, since otherwise ramification yields $(n-2)g + 1$ conditions. Imposing that the lowest-order terms of $f_1^2 - f_2$ and $f_2^2 - f_1 f_3$ vanish gives the required two additional independent conditions, namely (8) and

$$2\alpha_2 - \alpha_1 - \alpha_3 = 0.$$

(12)

Of course, (8) and (12) simply translate to the instances $j = 2, 3$ of the linear constraint (11).
Now say \( n = 4 \), and assume that sections of \( f \) vanish minimally to orders 0, 2, 4, 6, and 8 in \( f^{-1}(P) \). Much as before, we obtain three linear conditions \((11)\), \( j = 2, 3, 4 \) beyond ramification. These may obtained by imposing that the leading terms of \( F_1 := f_1^2 - f_2, \) \( F_2 := f_1f_2 - f_3, \) and \( F_3 := f_1f_3 - f_4, \) vanish. Thus \( F_1, F_2, \) and \( F_3, \) are power series with lowest terms of orders 6, 8, 10, and 12, respectively, in the absence of further nontrivial conditions. More precisely, rewriting

\[
f_j(t) = t^{2j} + \alpha_j t^{2j+1} + \beta_j t^{2j+2} + \gamma_j t^{2j+3} + O(t^{2j+4}), \quad j \geq 1
\]
to account for higher-order terms, we have

\[
F_1 = (2\beta_1 + \alpha_2^2 - \beta_2)t^6 + (2\gamma_1 + 2\alpha_1\beta_1 - \gamma_2)t^7 + O(t^8),
\]
\[
F_2 = (\beta_1 + \beta_2 + \alpha_1\alpha_2 - \beta_3)t^8 + (\gamma_1 + \gamma_2 + \alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_3)t^9 + O(t^{10}), \quad \text{and}
\]
\[
F_3 = (\beta_1 + \beta_3 + \alpha_1\alpha_3 - \beta_4)t^{10} + (\gamma_1 + \gamma_3 + \alpha_1\beta_3 + \alpha_3\beta_1 - \gamma_4)t^{11} + O(t^{12}).
\]

Imposing that

\[
[t^7](F_1 - ([t^6]F_1)f_3) = [t^9](F_2 - ([t^8]F_2)f_4) = 0
\]
yields two further independent conditions as required, namely

\[
2\gamma_1 + 2\alpha_1\beta_1 - \gamma_2 - \alpha_3(2\beta_1 + \alpha_1^2 - \beta_2) = 0 \quad \text{and}
\gamma_1 + \gamma_2 + \alpha_1\beta_2 + \alpha_2\beta_1 - \gamma_3 - \alpha_4(\beta_1 + \beta_2 + \alpha_1\alpha_2 - \beta_3) = 0.
\]

Note that the independence of these conditions is witnessed by the appearance of the variables \( \gamma_2 = [t^7]f_2 \) and \( \gamma_3 = [t^7]f_3 \) in the first and second of these equations, respectively, which in turn comes about because \( f_2 \) and \( f_3 \) are the “linear” terms in their respective expansions

\[
F_1 - ([t^6]F_1)f_3 = f_1^2 - f_2 - ([t^6]F_1)f_3 \quad \text{and} \quad F_2 - ([t^8]F_2)f_4 = f_1f_2 - f_3 - ([t^8]F_2)f_4.
\]

Finally, if instead sections of \( f \) vanish to orders \( (0, 2, 4, 6, 10) \), we need only exhibit three conditions beyond ramification, and it is easy to see that \((11)\) holds for \( j = 2, 3, 5 \). Similarly, if sections of \( f \) vanish to orders \((0, 2, 4, 8, 10)\) or \((0, 2, 4, 6, 12)\), only one condition beyond ramification is required, and this may be chosen to be of type \((11)\). In all other cases, ramification furnishes the required number of conditions.

The analysis is similar when \( n = 5 \). Assume that sections of \( f \) vanish to orders \((0, 2, 4, 6, 8, 10)\). We must then produce seven conditions beyond ramification. It is easy to exhibit four of linear type, namely \((11)\) with \( 2 \leq j \leq 5 \). We obtain three further (independent, but nonlinear) conditions by imposing

\[
[t^7](F_1 - ([t^6]F_1)f_3) = [t^9](F_2 - ([t^8]F_2)f_4) = [t^{11}](F_3 - ([t^{10}]F_3)f_5) = 0.
\]

The argument is analogous, but simpler, if sections of \( f \) vanish to orders larger than \((0, 2, 4, 6, 8, 10)\).

Finally, say \( n \geq 6 \). Assume that sections of \( f \) vanish to orders \((0, 2, 4, 6, 8, 12)\). We must produce eight conditions beyond ramification. Of these, five will be of linear type, namely \((11)\) with \( 2 \leq j \leq 6 \). The same three further nonlinear conditions \((14)\) as in the \( n = 5 \) case remain operative.
• **Case:** $g = 8$. Here $N_R = 13$. More precisely, we must produce three, seven, ten, twelve, or thirteen conditions beyond ramification depending upon whether $n = 3$, $n = 4$, $n = 5$, $n = 6$, or $n \geq 7$. We assume that sections of $f$ vanish to minimal possible orders, the remaining possibilities being analogous (and easier). Once the $(n-1)$ linear conditions (11) with $2 \leq j \leq n$ have been taken into account, we are left to produce one, four, six, seven, or seven nonlinear conditions. If $n = 3$, we impose $[t^7](F_1 - ([t^6]F_1)f_3) = 0$, which we interpret as a condition on $[t^7]f_2$. Similarly, if $n = 4$, we let

$$Q_1 := F_1 - ([t^6]F_1)f_3 	ext{ and } Q_2 := F_2 - ([t^8]F_2)f_4.$$  

We impose

$$[t^7]Q_1 = [t^9]Q_2 = 0;$$  

the first is the condition on $[t^7]f_2$ obtained in the $n = 3$ case, while the second is a condition on $[t^9]f_3$. Now let

$$\tilde{Q}_1 := Q_1 - ([t^8]Q_1)f_4 \text{ and } \tilde{Q}_2 := Q_2 - ([t^{10}]Q_2)f_1f_4.$$  

We then impose

$$[t^9]\tilde{Q}_1 = [t^{11}]\tilde{Q}_2 = 0$$  

which give conditions on $[t^9]f_2$ and $[t^{11}]f_3$, respectively. Taken together, (15) and (16) give the required four nonlinear conditions.

Now say $n = 5$. Let $Q_3 := F_3 - ([t^{12}]F_3)f_1f_5$; we impose that

$$[t^{13}]Q_3 = 0;$$  

this is a condition on $[t^{13}]f_4$. Further, let $\tilde{Q}_3 := Q_3 - ([t^{14}]Q_3)f_5f_4$; we then impose that

$$[t^{15}]\tilde{Q}_3 = 0$$  

which is a condition on $[t^{15}]f_4$. The conditions (15), (16), (17), and (18) give the required six nonlinear conditions.

Finally, say $n \geq 6$. The six nonlinear conditions obtained in the $n = 5$ case remain operative, and we obtain an additional condition, by setting $Q_4 := F_4 - ([t^{14}]F_4)f_1f_5$ where $F_4 := f_1f_5 - f_6$ (having implicitly imposed that $[t^{13}]F_4 = 0$, this being one of our linear conditions), and imposing that

$$[t^{15}]Q_4 = 0$$  

which is a condition on $[t^{15}]f_6$.

**Remarks 1.4.** We have not attempted here to make a systematic count of constraints on the coefficients of a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ arising from a hyperelliptic singularity in its image, but we anticipate that a careful implementation of the basic strategy for obtaining conditions on coefficients $[t^p]f_\rho$ for $\rho \in G_{2g}$ as carried out when $g \leq 8$ above should at least yield a proof that morphisms with semigroup $S^h = \langle 2, 2g + 1 \rangle$ of arbitrary genus verify our uniform codimension heuristic. We plan to return to this in future work.

Note that when $n = g - 1$, the number of conditions beyond ramification required to validate our original heuristic is exactly $N_R$. If $n = g - 1$, our morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ has a unique singularity which is hyperelliptic, and if the image of $f$ is
not (globally) hyperelliptic, then $f$ describes a canonical model for $f$. This is a consequence of the Gorensteinness of a hyperelliptic singularity, which as explained in [2] is manifested by the symmetry

$$m_i = 2g - 1 - \ell_{g-i}$$

between values $m_i$ and gaps $\ell_{g-i}$ of the semigroup $S^h$ for every $i = 0, \ldots, g - 1$, where by convention we set $m_0 = 0 \in S^h$.

1.5. End of the proof of Thm 1.1. The arguments of the preceding subsections show that Thm 1.1 holds for all rational curves with a single cusp, so in this subsection we focus on rational curves with multiple cusps. The dimensional transversality result of Eisenbud–Harris is extremely useful in this regard, as it establishes that that ramification conditions in distinct points of (the domain of) a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ are independent. However, because of the presence of conditions beyond ramification, it isn’t quite sufficient. Here we analyze the finite number of corresponding special cases, which we classify according to the partitions of $6 \leq g \leq 8$ induced by the the local genera of their cusps.

- **Case:** $g = 6$. Without loss of generality, we may assume the image of $f$ is of type $(5, 1)$, where the entries of the partition refer to the genera of the singularities in the image of $f$. Further, we may assume the singularities are supported in $P_1 = (0 : \cdots : 0 : 1)$ and $P_2 = (1 : 0 \cdots : 0)$ in $\mathbb{P}^n$ and are the images of $(0 : 1)$ and $(1 : 0)$ in $\mathbb{P}^1$, respectively; and that the corresponding semigroups are $(2, 5)$ and $(2, 3)$, respectively. The independence of the (linear) conditions beyond ramification of the genus-5 hyperelliptic singularity in $P_1$ with respect to the simple cusp in $P_2$ is clear. Indeed, considering the matrix of coefficients of our degree-$d$ morphism as a $(n + 1) \times (d + 1)$ box, the vanishing conditions imposed by the singularities in $P_1$ and $P_2$ define diagonally opposite Young tableaux inside the box.

- **Case:** $g = 7$. Without loss of generality, we may assume $f$ to be of partition type $(5, 2)$, $(6, 1)$ or $(5, 1, 1)$. Of these possibilities, the first two may be handled just as in the $g = 6$ case. Similarly, if $f$ is of type $(5, 1, 1)$, we may assume its singularities are the images of $(0 : 1)$, $(1 : 0)$, and $(1 : 1)$, respectively. Likewise, we may assume that $f$ maps $(0 : 1)$ to $(0 : \cdots : 0 : 1)$ and $(1 : 0)$ to $(1 : 0 : \cdots : 0)$. The remaining singularity, a simple cusp, is specified by requiring rank($Df(1 : 1)$) $\leq 1$, where $Df = (\frac{\partial f}{\partial t}, \frac{\partial f}{\partial u})$. To count conditions explicitly, we assume the orders of vanishing of $f$ in $(0 : 1)$, the preimage of the singularity of genus 5, are minimal (as usual, this is the crucial case). Applying a diagonal automorphism if necessary, we may then assume that $f = (f_0, \ldots, f_n)$ is of the form

$$f_0 = u^{d} + a_{0,1}tu^{d-1} + \cdots + a_{0,d-n-1}t^{d-n-1}u^{n+1}$$

$$f_1 = t^2u^{d-2} + a_{1,3}t^3u^{d-3} + \cdots + a_{1,d-n}t^{d-n}u^n$$

$$\cdots$$

$$f_{n-1} = t^{2n-2}u^{d-2n+2} + a_{n-1,2n-1}t^{2n-1}u^{d-2n+1} + \cdots + a_{n-1,d-2}t^{d-2}u^2$$

$$f_n = t^{2n}u^{d-2n} + a_{n,2n+1}t^{2n+1}u^{d-2n-1} + \cdots + a_{n,d}t^d$$

This is a canonical model for $f$. This is a consequence of the Gorensteinness of a hyperelliptic singularity, which as explained in [2] is manifested by the symmetry
with $a_{i,j} \in \mathbb{C}$ for all $0 \leq i \leq n$ and $0 \leq j \leq d$. Further, (8) yields

$$2(a_{1,3} - a_{0,1}) = a_{2,5} - a_{0,1},$$

i.e.

$$a_{0,1} = 2a_{1,3} - a_{2,5}.$$  \hspace{1cm} (20)

Note that

$$\begin{align*}
\frac{\partial f_j}{\partial t}(1,1) &= \sum_{i=j}^{d-n-1} (i+j)a_{j,i+j} \quad \text{for} \quad 0 \leq j \leq n-1, \\
\frac{\partial f_n}{\partial t}(1,1) &= \sum_{i=n}^{d-n-1} (n+i)a_{n,n+i}
\end{align*}$$

and

$$\begin{align*}
\frac{\partial f_j}{\partial u}(1,1) &= \sum_{i=j}^{d-n-1} (d-i-j)a_{j,i+j} \quad \text{for} \quad 0 \leq j \leq n-1, \\
\frac{\partial f_n}{\partial u}(x,1) &= \sum_{i=n}^{d-n-1} (d-n-i)a_{n,n+i}
\end{align*}$$

where $a_{0,0} = a_{1,2} = \cdots = a_{n-1,2n-2} = a_{n,2n} := 1$. The requirement that

$$\text{rank}(Df(1:1)) \leq 1$$

translates to

$$\frac{\partial f_j}{\partial t}(1,1) \frac{\partial f_k}{\partial u}(1,1) - \frac{\partial f_k}{\partial t}(1,1) \frac{\partial f_j}{\partial u}(1,1) = 0$$

for all $j \neq k \in \{0, \ldots, n\}$. Using (21) and (22), is clear that we obtain (in fact more than) the required $(n-2)$ independent conditions on the coefficients $a_{i,j}$ from the determinantal equations (23).

**Case:** $g = 8$. Invoking the “box principle” of the preceding two items, we may assume without loss of generality that $f$ is associated to a partition with at least four nontrivial parts. The unique such possibility is $(5,1,1,1)$. But because each simple cusp imposes $(n-1)$ instead of the required $(n-2)$ ramification conditions (and because the genus-5 singularity is at worst hyperelliptic, in which case its ramification fails by at most a single condition to meet the codimension heuristic in its genus) it follows by Eisenbud–Harris dimensional transversality that ramification arising from singularities of type $(5,1,1,1)$ produce (more than) the required $8(n-2)$ conditions. (Note that this argument also serves in the $(5,1,1)$ case.)

Remark 1.5. Given that our study focuses on the case $n \geq 3$, it complements the largely topological investigations of (the existence of) planar rational cuspidal curves carried out, e.g., in [14], [16], [25], and [27] (our bibliography is nonexhaustive; we recommend the overview given in [24]). In particular, Płonkiewski conjectured in [26] that a rational plane curve of degree at least six may have at most three cusps (and checked the validity of his conjecture in degrees $\leq 20$) and Koras–Palka [21] have announced that a proof of this result is forthcoming. It is natural to wonder whether a similar boundedness result holds for rational cuspidal curves in $\mathbb{P}^n$, $n \geq 3$.

2. Multibranch singularities of arithmetic genus at most 4

Over the course of the next two subsections, we will prove the following result, which extends Thm 1.1 when $g \leq 4$.

**Theorem 2.1.** Suppose that $g \leq 4$ and $d \geq \max(n, 2g - 2)$. Then

$$\text{cod}(M_d^n, M_n^n) \geq (n-2)g.$$
To obtain dimension estimates for rational curves with multi-branch singularities, our plan is as follows. We primarily focus on rational curves with a single multi-branch singularity, which we view as elements of $M^n_m$. In general, singularities with $r \geq 1$ branches are classified up to analytic isomorphism by ring maps of the form

$\phi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[[t_1]] \times \cdots \times \mathbb{C}[[t_r]].$

Their value semigroups [2] are obtained by composing the ring maps (24) with the product of the valuations $v_i : \mathbb{C}[[t_i]] \to \mathbb{N}$, $1 \leq i \leq r$, and thus are subsets of $\mathbb{N}^r$. Accordingly, we begin by producing a list of possible semigroups.

2.1. **Step one: Taxidermy of value semigroups.** We stratify singularities according to their number $r$ of branches. In practice, it will be useful to distinguish the following subclass of these, which are easily dispensed with.

**Definition 2.2.** Let $C$ be a genus-$g$ singularity with $r$ branches $C_i$ of genera $g_i$, $i = 1, \ldots, r$. We say that $C$ is maximally transverse (MT) if and only if $\sum_{i=1}^{r} g_i + r - 1 = g$.

Maximally-transverse singularities are characterized by the fact that the genera of their respective branches are maximal. Maximally-transverse singularities are also easy to identify on the basis of their value semigroups. Indeed, the $i$th branch of a given singularity $C$ corresponds to a copy of $\mathbb{N}$ that lies along the $x_i$-axis inside $\mathbb{Z}^r$, and the genus of the branch is precisely the number of gaps associated to the numerical semigroup obtained as the projection of the value semigroup of $C$ to the $x_i$-axis. Any singularity whose value semigroup is minimally generated by $(0, \ldots, 0)$ and its conductor is maximally-transverse.

We now describe, in concrete terms, our stratification scheme for semigroups of singularities. We begin by recalling some standard facts about value semigroups associated to local rings, following [2, Sec. 2]. The first crucial fact, a consequence of [2, Prop. 2.11.(iii)], is that the genus of a singularity is computed by the number of unoccupied places in an arbitrary saturated path linking $0 = (0, \ldots, 0)$ with the conductor. Here a saturated path of length $n$ between endpoints $\alpha, \beta \in S \subset \mathbb{N}^r$ is a chain of strict inequalities

$\alpha = a_0 < \cdots < a_n = \beta$

with respect to the natural partial order on $\mathbb{N}^r$ that may not be extended to any strictly larger chain via insertions of elements in $S$; the result [2, Prop. 2.8] establishes that any two saturated paths with the same endpoints have the same length. An unoccupied place denotes any lattice point belonging to the complement of $S$ in $\mathbb{N}^r$.

Moreover, every value semigroup $S$ verifies the following rules.

(i) *(Closure under addition.)* If $a, b \in S$, then $a + b \in S$.

(ii) *(SM1: Strong closure under min, I.)* If $a, b \in S$, then $\min(a, b) := (\min(a_i, b_i)) \in S$.

(iii) *(SM2: Strong closure under min, II.)* If $a, b \in S$ and $a_i = b_i$ for some component index $i$, then there is some $\gamma \in S$ such that $\gamma_i > a_i$ and $\gamma_j \geq \min(a_j, b_j)$ for all component indices $j \neq i$, with $\gamma_j = \min(a_j, b_j)$ whenever $a_j \neq b_j$.

(iv) *(Locality.)* 0 is the unique element of $S$ with at least one vanishing coordinate.
Remark 2.3. Properties (i)-(iii) above show, in particular, that $S$ is a tropical subsemiring of $(\mathbb{N} \cup \{\infty\})^r$, in the sense of [23]. Namely, $S$ is closed under the operations of

$$a \oplus_T b := \min(a, b) \quad \text{and} \quad a \otimes_T b := a + b,$$

which should be thought of as tropical addition and tropical multiplication, respectively. Here $\infty$ is an idempotent with respect to $\oplus_T$, satisfies $a \oplus_T \infty = \infty$, and arises as the value associated to the trivial element $0 \in \mathbb{C}[[t]]$. In [6], Carvalho and Hernandes show that the tropical semiring structure associated with any value semigroup is always finitely generated.

Recall that $S^*$ denotes the ordered sequence of elements in $S$ less than or equal to the conductor $c \in S$, with respect to the partial order inherited from $(\mathbb{N} \cup \{\infty\})^r$; just as in the unibranch case, a value semigroup of a multibranch singularity is uniquely prescribed by its truncation $S$. Further, any $S$ is naturally related to a numerical semigroup $|S|$ defined as follows. Namely, given $\gamma \in S \subset \mathbb{N}^r$, let $|\gamma| := \sum_{i=1}^r \gamma_i$ denote the sum of its coordinates. We then set

$$|S| := \{|\gamma| : \gamma \in S \text{ and } \gamma \leq c\} \cup \{n \in \mathbb{N} : n \geq |c|\}$$

It follows easily from the definitions that

$$g(|S|) \leq g(S). \tag{26}$$

For every choice of genus $1 \leq g \leq 4$ and number of branches $r \geq 1$, we now classify value semigroups according to $|S|$, the possibilities for which are described, in turn, by the numerical semigroup tree. Here a few comments about our definition (25) of the modulus are in order. Ideally, we would have equality in (26); and indeed this often happens. Slightly more precisely, since the genus of $S$ is computed by the sequence $\gamma$, we would like the genus of $|S|$, suitably defined, to be computed by the sequence $\{|s_i|\}$. Pursuing this line of thought naturally leads to three questions. Namely,

1. Is the modulus of a saturated sequence $\{|s_i|\}$ in $S$ unique?
2. Is the set $\{|s_i|\}$ a numerical semigroup?
3. Does every $S$ admit a saturated sequence whose modulus is a semigroup?

The answer is negative for the first two questions, while the third seems to be an open problem. The definition (25) we have opted for, then, represents a compromise, with the virtue that it unambiguously assigns a numerical semigroup to each semigroup that satisfies the axioms (i)-(iv) listed previously.

(a) **Case: $g=1$.** The unique possibility for $|S|$ is $\langle 2, 3 \rangle$. We claim that $r = 1$ or $r = 2$. Indeed, any singularity $C$ with $r$ branches has genus greater than or equal to $r-1$, with equality if and only if $C$ is maximally transverse and each of its components is smooth. If $r = 1$, then $S = \langle 2, 3 \rangle$ and the corresponding singularity is a simple cusp. If $r = 2$, we have $S = \langle (1, 1), (1, 2) \rangle$ or $S = \langle (1, 1), (2, 1) \rangle$, which is maximally transverse: the corresponding singularity is a simple double point, i.e., a node.

(b) **Case: $g=2$.** Possibilities for $|S|$ are $\langle 3, 4, 5 \rangle$ and $\langle 2, 5 \rangle$. This time $r \leq 3$. If $r = 1$, then $S = \langle 3, 4, 5 \rangle$ or $S = \langle 2, 5 \rangle$. Now say $r = 2$. If $|S| = \langle 3, 4, 5 \rangle$, then up to an $S_2$-action, we have either $S^* = \{0, (1, 2)\}$ or

$$S^* = \{0, (1, 2), (1, 4), (1, 5), (2, 2), (2, 4), (2, 6)\}. \tag{27}$$
Note that the latter case is one of only two possible value semigroups $S$ of genus at most 4 for which the modulus $|S|$ is of strictly smaller genus; here $g(S) = 4$. Indeed, the genus of $S$ is computed by two distinct saturated sequences. Namely,

\begin{equation}
0 < (1, 2) < (1, 4) < (2, 4) < (2, 6)
\end{equation}

may be completed to a saturated chain by inserting $(1, 0) < (1, 1) < (1, 3) < (2, 5)$; and

\begin{equation}
0 < (1, 2) < (2, 2) < (2, 4) < (2, 6)
\end{equation}

may be completed by inserting $(1, 0) < (1, 1) < (2, 3) < (2, 5)$. The moduli of the elements in sequence (28) yields form a saturated chain \{0, 3, 5, 6, 8, …\} for the numerical semigroup \langle 3, 5 \rangle. On the other hand, the moduli of the second sequence (29) comprise the set

\[ \{0, 3, 4, 6, 8\} \]

which is not additively closed, so in particular not a semigroup.

When $r = 2$ and $|S| = \langle 2, 5 \rangle$, we stratify possibilities according to the smallest nonzero element $\alpha \in S$; by SM1, the latter is well-defined. If $\alpha = (1, 1)$, then applying closure under addition and SM1, we have $S^* = \{0, (1, 1), (2, 2)\}$. The remaining possibility, namely $\alpha = (0, 2)$, is precluded by locality.

Finally, say that $r = 3$. If $|S| = \langle 3, 4, 5 \rangle$, then either $S^* = \{0, (1, 1, 1)\}$ or $S^* = \{0, (1, 1, 1), (1, 2, 2), (1, 2, 3), (1, 3, 2), (2, 1, 1), (2, 2, 2), (2, 3, 3)\}$ which is of genus 4.

On the other hand, if $|S| = \langle 2, 5 \rangle$, then $\alpha$ necessarily has at least one vanishing coordinate, which is precluded by locality.

(c) **Case:** $g = 3$. Possibilities for $|S|$ are $\langle 4, 5, 6, 7 \rangle$, $\langle 3, 5, 7 \rangle$, $\langle 3, 4 \rangle$, and $\langle 2, 7 \rangle$. We have $r \leq 4$. We ignore the unibranch case $r = 1$. Now say $r = 2$. Possibilities with $|S| \neq \langle 3, 4 \rangle$ are easy to classify. Namely, up to an $S_2$-action, we have $S^* = \{0, (1, 3)\}$ or $S^* = \{0, (2, 2)\}$ (corresponding to $|S| = \langle 4, 5, 6, 7 \rangle$); $S^* = \{0, (1, 2), (1, 4)\}$ or $S^* = \{0, (1, 2), (2, 3)\}$ (corresponding to $|S| = \langle 3, 5, 7 \rangle$); or $S^* = \{0, (1, 1, 1), (2, 2), (3, 3)\}$ (corresponding to $|S| = \langle 2, 7 \rangle$). If $r = 2$ and $|S| = \langle 3, 4 \rangle$, assume $\alpha = (1, 2)$ without loss of generality, and let $\alpha_2$ denote any minimal element in $S$ strictly larger than $\alpha$. If $\alpha_2 = (1, 3)$, then SM2 implies that $S$ contains an element other than $\alpha$ along the line $x_2 = 2$, but then as $2\alpha = (2, 4)$ also belongs to $S$, SM1 forces $(2, 2) \in S$. Continuing in this vein, we deduce that $S^* = \{0, (1, 2), (1, 3), (2, 2), (2, 4)\}$. An analogous argument yields the same conclusion upon choosing $\alpha_2 = (2, 2)$.

We now consider possibilities when $r = 3$. If $|S| = \langle 4, 5, 6, 7 \rangle$, then up to an $S_2$-action, we have $S^* = \{0, (2, 1, 1)\}$. If $|S| = \langle 3, 5, 7 \rangle$, then $\alpha = (1, 1, 1)$. Let $\alpha_2$ denote a minimal element in $S$ strictly larger than $\alpha$; up to $S_3$, we have $\alpha_2 = (2, 2, 1)$ or $\alpha_2 = (3, 1, 1)$. In the former case, genus considerations force $S^* = \{0, (1, 1, 1), (2, 2, 1)\}$. The remaining case, $\alpha_2 = (3, 1, 1)$, is precluded: otherwise, SM1 would force $\min(2\alpha, \alpha_2) = (2, 1, 1) \in S$, which is absurd. The case of $|S| = \langle 3, 4 \rangle$ is similar: without loss of generality,
assume $\alpha = (1, 1, 1)$ and $\alpha_2 = (2, 1, 1)$ lie in $S$; then SM1 and genus considerations force $S^* = \{0, (1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2), (2, 2, 2)\}$. Locality precludes $|S^*| = 2, 7$.

Finally, when $r = 4$, locality forces $S^* = \{0, (1, 1, 1, 1)\}$.

(d) Case: $g=4$. Possibilities for $|S|$ are $\langle 5, 6, 7, 8, 9 \rangle$, $\langle 4, 6, 7, 9 \rangle$, $\langle 4, 5, 7 \rangle$, $\langle 4, 5, 6 \rangle$, $\langle 3, 7, 8 \rangle$, $\langle 3, 5 \rangle$, and $\langle 2, 9 \rangle$. We have $r < 5$. As before, we ignore unibranch possibilities; those that remain (up to obvious actions of symmetric groups) are as follows.

i. $|S| = \langle 5, 6, 7, 8, 9 \rangle$. All possibilities are MT. If $r = 2$, then $S^* = \{0, (1, 4)\}$ or $S^* = \{0, (2, 3)\}$. If $r = 3$, then $S^* = \{0, (1, 1, 3)\}$ or $S^* = \{0, (1, 2, 2)\}$. If $r = 4$, then $S^* = \{0, (1, 1, 1, 2)\}$. If $r = 5$, then $S^* = \{0, (1, 1, 1, 1)\}$.

ii. $|S| = \langle 4, 6, 7, 9 \rangle$. If $r = 2$, then $S^* = \{0, (1, 3), (1, 5)\}$ (MT) or $S^* = \{0, (1, 3), (2, 4)\}$ (non-MT). If $r = 3$, then $S^* = \{0, (1, 1, 2), (1, 1, 4)\}$ (MT) or $S^* = \{0, (1, 1, 2), (1, 2, 3)\}$ (non-MT). If $r = 4$, then $S^* = \{0, (1, 1, 1, 1), (1, 1, 2, 2)\}$ (non-MT).

iii. $|S| = \langle 4, 5, 7 \rangle$. If $r = 2$, then

$S^* = \{0, (1, 3), (1, 4), (1, 6)\}$ (MT),
$S^* = \{0, (1, 3), (1, 4), (2, 3), (2, 5)\}$ (non-MT), or
$S^* = \{0, (2, 2), (2, 3), (3, 2), (3, 4)\}$ (non-MT).

If $r = 3$, then $S^* = \{0, (1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 4)\}$ (non-MT). If $r = 4$, then $S^* = \{0, (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 1, 4)\}$ (MT).

iv. $|S| = \langle 4, 5, 6 \rangle$. All possibilities are non-MT. If $r = 2$, then

$S^* = \{0, (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 6)\}$ or
$S^* = \{0, (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (4, 2), (4, 4)\}$.

If $r = 3$, then

$S^* = \{0, (1, 1, 2), (1, 1, 3), (1, 1, 4), (1, 2, 2), (1, 2, 3), (2, 1, 2), (2, 1, 3), (2, 2, 2), (2, 2, 4)\}$.

If $r = 4$, then

$S^* = \text{Sym} \{0, (1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2)\}$

where Sym, for symmetrization, denotes the smallest orbit under the natural $S_4$-action.

v. $|S| = \langle 3, 7, 8 \rangle$. All possibilities are non-MT. If $r = 2$, then $S^* = \{0, (1, 2), (2, 4)\}$. If $r = 3$, then $S^* = \{0, (1, 1, 1), (2, 2, 2)\}$.

vi. $|S| = \langle 3, 5 \rangle$. All possibilities are non-MT. If $r = 2$, then

$S^* = \{0, (1, 2), (2, 3), (2, 4), (3, 3), (3, 5)\}$

vii. $|S| = \langle 2, 9 \rangle$. The unique possibility is $S^* = \{0, (1, 1), (2, 2), (3, 3), (4, 4)\}$.

2.2. Step two: Reverse engineering and dimension estimates. We now describe local parametrizations associated to each of the value semigroups obtained in Step one; doing so will allow us to associate geometric conditions to each value semigroup, which we count in turn. We focus on non-maximally transverse cases first.
Any singularity with \( r \) branches is specified by an \( r \)-tuple of maps
\[
\varphi_i : \mathbb{C} \to \mathbb{C}^n, \quad 1 \leq i \leq r;
\]
indeed, (30) is a dual reformulation of (24). In particular, there is a bijection between monomial maps and value semigroups \( S \). To produce a map \( \varphi \) associated to a prescribed semigroup \( S \) (or equivalently, to \( S^* \)), we will produce a distinguished set of generators for \( S \). Here we view \( S \) as a subset of \( (\mathbb{N} \cup \{\infty\})^r \). We consider an \( r \)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_r) \) with \( \alpha_{i_1}, \ldots, \alpha_{i_l} \) infinite for some number of indices \( 1 \leq i_1, \ldots, i_l \leq r \) as belonging to \( S \) provided:

- There exist finitely-valued \( r \)-tuples \( \tilde{\alpha} \in S \) with values of \( \alpha_{i_1}, \ldots, \alpha_{i_l} \) arbitrarily large natural numbers; and
- \( \tilde{\alpha} \) and \( \alpha \) agree in every finitely-valued coordinate.

Each semigroup \( S \) admits a minimal set of \( r \)-tuple generators \( \alpha \), where “generation” is with respect to the rules (i)-(iv) of Step one above and where we allow coordinates of \( \alpha \) to be infinitely-valued. By “minimality” we mean minimal number of minimal generating sets associated to each \( S^* \) below which are unique up to \( S^* \)-symmetries. Each such minimal generating set is dual to a monomial map that we call a model for the corresponding \( S \). To motivate this terminology, note that any given model may be interpreted as a collection of \( n \)-tuples \( \varphi \) of initial terms \( \varphi_i = \varphi_i(t) \), \( i = 1, \ldots, n \) of more general power series \( \tilde{\varphi}_i = \tilde{\varphi}_i(t) \) for which \( v(\tilde{\varphi}_i) = v(\varphi_i) \), where \( v : \mathbb{C}[\![t]\!] \to \mathbb{N} \) denotes the standard valuation. This implies that \( \tilde{\varphi} \) and \( \varphi \) ramify to equal orders in \( t = 0 \); in particular, the model is a minimal representative of a class of \( n \)-tuples of power series with fixed ramification in \( t = 0 \). Each model, in turn, determines a certain set of geometric conditions that are satisfied by rational curves belonging to the S-stratum, which place explicit constraints on the coefficients \( a_{ij} \) of the parametrizations (1) of the corresponding rational curves.

- **Case 1:** \( S^* = \{(0, (1, 1), (2, 2))\} \). See Figure 4 for a graphical representation.

  The corresponding set of minimal generators \( \Gamma = \Gamma_3 \) is \( \Gamma = \{(1, 1), (2, \infty)\} \).

  The corresponding model is the pair of maps \( \varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}^2 \) defined by
  \[
  \varphi_1 : t_1 \mapsto (t_1, t_1^2) \quad \text{and} \quad \varphi_2 : t_2 \mapsto (t_2, 0).
  \]

  It follows that the corresponding genus-2 singularity \( C \) has two smooth branches, which necessarily share a common tangent line: \( C \) is a tacnode.

  Finally, to estimate the codimension of the corresponding stratum of tacnodal rational curves, consider the generic parametrization of a degree-\( d \) rational curve \( f = (f_0, \ldots, f_n) : \mathbb{P}^1 \to \mathbb{P}^n \), where \( f \) is as in (1). Assume that \( f \) has a tacnode in its image. Assuming, as we may, that the tacnode is supported in the point \( (0 : \cdots : 0 : 1) \) in \( \mathbb{P}^n \) and that its preimages are the points \( 0 = (0 : 1) \) and \( \infty = (1 : 0) \) in \( \mathbb{P}^1 \); the coefficients \( a_{ij} \) of \( f \) satisfy \( 3n \) linearly independent conditions. Namely, we have
  \[
  a_{i,d} = a_{i,0} = 0 \quad \text{(incidence); and} \quad a_{i,d-1} = a_{i,1} \quad \text{(common tangency)}, \quad 0 \leq i \leq n - 1.
  \]
Varying the choice of target and preimages, we obtain \(2^n - 2\) independent conditions.

- **Case 2:** \(S^* = \{0, (1, 2), (2, 3)\}\). See Figure 5 for a graphical representation. The corresponding genus-3 singularity \(C\) consists of two branches, of genera 0 and 1, respectively. The set of minimal generators for \(S\) is \(\Gamma = \{(1, 2), (2, \infty), (\infty, 3)\}\), and the corresponding model is the pair \(\varphi_1, \varphi_2 : \mathbb{C} \rightarrow \mathbb{C}^3\) defined by

\[
\varphi_1 : t_1 \mapsto (t_1, t_1^2, 0) \quad \text{and} \quad \varphi_2 : t_2 \mapsto (t_2^3, 0, t_2^2).
\]

Here \(m(S^*) = 3\), but in fact it is not hard to see that \(m(S^*)\) also calculates the embedding dimension of \(C\). Indeed, the only other possibility, that \(\text{emb. dim}(C) = 2\), would force \(C\) to admit a local parametrization of the form

\[
\rho_1 : t_1 \mapsto (t_1 + O(t_1^3), t_1^2 + O(t_1^3)), \rho_2 : (t_2^3 + O(t_2^3), t_2^2 + O(t_2^3)).
\]

However it is easy to see that the genus of (32) is strictly larger than 3. Further, it is easy to check that the analogue of the model (31) with generic coefficients, namely

\[
\varphi_1 : t_1 \mapsto (t_1 + O(t_1^3), t_1^2 + O(t_1^3), 0) \quad \text{and} \quad \varphi_2 : t_2 \mapsto (t_2^3 + O(t_2^3), 0, t_2^2 + O(t_2^3)),
\]

has the same value semigroup as (31). It follows that the contacts of the branches are encoded by the geometric conditions

\[
\varphi_1(0) = \varphi_2(0) \quad \text{and} \quad \varphi_1'(0) = \varphi_2''(0).
\]

To estimate the codimension of the corresponding stratum of singular curves, consider a generic parametrization \(f = (f_0, \ldots, f_n) : \mathbb{P}^1 \rightarrow \mathbb{P}^n\) of
a degree-$d$ rational curve, and as in Case 1, assume that the singularity is supported in $(0 : \cdots : 0 : 1) \in \mathbb{P}^n$ and that its preimages are the points $0 = (0 : 1), \infty = (1 : 0) \in \mathbb{P}^1$. The conditions (33) then translate to the following conditions on the coefficients $a_{ij}$ of $f$:

\[ a_{i,d} = a_{i,0} = 0 \text{ (incidence)}; \quad \text{and} \quad a_{i,d-1} = a_{i,2} \text{ (common tangency)}, 0 \leq i \leq n - 1. \]

Moreover, the fact that $\infty$ is the preimage of the cusp imposes

\[ a_{i,1} = 0, 0 \leq i \leq n - 1. \]

Consequently, we obtain $3n - 2$ independent conditions.

- **Case 3**: $S^* = \{0, (1, 1), (2, 2), (3, 3)\}$. See Figure 6 for a graphical representation. The singularity is of genus 3, with two smooth branches. We have $\Gamma = \{(1, 1), (\infty, 3)\}$. The corresponding model is the pair $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}^2$ defined by

\[ \varphi_1 : t_1 \mapsto (t_1, 0) \quad \text{and} \quad \varphi_2 : t_2 \mapsto (t_2, t_2^3). \]

The contact conditions are

\[ \varphi_1(0) = \varphi_2(0), \varphi_1'(0) = \varphi_2'(0) \text{ and } \varphi_1''(0) = \varphi_2''(0). \] (34)

The fact that there are 3 sets of equalities between derivatives of the branches $\varphi_1$ and $\varphi_2$ implies, much as in Cases 1 and 2 above shows that the codimension of the corresponding singularity stratum is at least $3n - 2$.

- **Case 4**: $S^* = \{0, (1, 2), (1, 3), (2, 2), (2, 4)\}$. See Figure 7 for a graphical representation. The singularity is of genus 3. One branch is smooth, while the other is of genus one, i.e., is a simple cusp. We have $\Gamma = \{ (\infty, 2), (1, 3) \}$; the corresponding model is the pair $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}^2$ defined by

\[ \varphi_1 : t_1 \mapsto (0, t_1) \quad \text{and} \quad \varphi_2 : t_2 \mapsto (t_2^2, t_2^3). \]

The contact conditions are

\[ \varphi_1(0) = \varphi_2(0) \text{ (incidence), and } \text{rank}(\varphi_1'(0), \varphi_1''(0), \varphi_2''(0)) = 2. \] (35)

The second condition listed in (35) expresses the fact that the embedding dimension of the singularity is 2, rather than (the generic value of) 3. To analyze conditions on the coefficients $a_{ij}$ of the parametrization $f$, we assume that the singularity is supported in $(0 : \cdots : 0 : 1) \in \mathbb{P}^n$ and that
its preimages are 0 = (0 : 1), ∞ = (1 : 0) ∈ \mathbb{P}^1. The conditions (35) then translate to

\[ a_{i,d} = a_{i,0} = 0, \quad 0 \leq i \leq n - 1 \] 

(incidence); and det

\[
\begin{bmatrix}
    a_{i,d-1} & \tilde{a}_{i,d-2} & \tilde{a}_{i,2} \\
    a_{j,d-1} & \tilde{a}_{j,d-2} & \tilde{a}_{j,2} \\
    a_{k,d-1} & \tilde{a}_{k,d-2} & \tilde{a}_{k,2}
\end{bmatrix} = 0
\]

for all triples of distinct \( i, j, k \in \{0, \ldots, n - 1\} \). Here we use the notation \( \tilde{a}_{ij} \) to denote a deformation of \( a_{ij} \) that arises when formally inverting (a dehomogenization of) the nonzero global section \( f_n \). The crucial point is that the determinantal conditions above constrain the coefficients \( a_{ij} \). When discussing determinantal conditions in the sequel, we will abusively write \( a_{ij} \) in place of \( \tilde{a}_{ij} \), but the reader should be aware that an analogous adjustment (arising from passing between local power series in the neighborhood of a singularity, and global sections \( f_i \)) will be needed; the outcomes of the analyses remain unchanged.

Finally, the cuspidality of the branch corresponding to \( \varphi_2 \) imposes

\[ a_{i,1} = 0, \quad 0 \leq i \leq n - 1. \]

Despite the nonlinearity of the \( \binom{n}{3} \) determinantal conditions above, we clearly obtain at least \( 3n - 2 \) independent conditions when \( n \geq 4 \). When \( n = 3 \), there is only a single determinantal condition, which is independent of the \( 2n - 2 = 4 \) remaining conditions, which beats the required \( (n - 2)g = 3 \).

**Case 5:** \( S^* = \{0, (1, 1, 1), (2, 2, 1)\} \). Projections show that the corresponding genus-3 singularity is the union of three smooth branches, two of which (corresponding to the first two coordinates) share a common tangent line.

To count conditions, assume the singularity is supported in \( (0 : \cdots : 0 : 1) \in \mathbb{P}^n \), that its preimages are the points \( 0 = (0 : 1), 1 = (1 : 1) \) and \( \infty = (1 : 0) \) in \( \mathbb{P}^1 \), and that the branches corresponding to 0 and \( \infty \) are tangent to one another. The conditions arising from our singularity are then

\[
\sum_{j=0}^{d} a_{i,j} = a_{i,d} = a_{i,0} = 0, \quad 0 \leq i \leq n - 1 \] 

(incidence), and \( a_{i,d-1} = a_{i,1} \) (tangency).
These are $4n$ independent conditions on the coefficients $a_{i,j}$ of rational curves $f = (f_0, \ldots, f_n)$. Accounting for variation of preimages and target leaves us with $3n - 3$ conditions, which beats the required $3(n - 2)$.

- **Case 6:** $S^* = \{0, (1, 1), (2, 1, 1), (1, 1, 2), (2, 2, 1)\}$. Projections show the corresponding genus-3 singularity $C$ has 3 smooth branches, no two of which are tangent to one another. Moreover, $\Gamma = \{(1, 1, \infty), (\infty, 1, 1)\}$, so $m(S^*) = 2$. It follows that $C$ has embedded dimension two: $C$ is a planar triple point.

To count conditions, assume as in case 5 that the singularity is supported in $(0 : \cdots : 0 : 1) \in \mathbb{P}^n$, and that its preimages are $0 = (0 : 1), 1 = (1 : 1)$ and $\infty = (1 : 0) \in \mathbb{P}^1$. The conditions that specify the planar triple point with respect to these choices are

\[ \sum_{j=0}^{d} a_{i,j} = a_{i,d} = a_{i,0} = 0, \quad 0 \leq i \leq n - 1 \text{ (incidence)} \]

and

\[ \det \begin{bmatrix} a_{i,d-1} & a_{i,1} & \sum_{t=1}^{d} la_{i,d-t} \\ a_{j,d-1} & a_{j,1} & \sum_{t=1}^{d} la_{j,d-t} \\ a_{k,d-1} & a_{k,1} & \sum_{t=1}^{d} la_{k,d-t} \end{bmatrix} = 0 \text{ (linear degeneracy of tangent lines).} \]

After variations of preimage and target have been accounted for, we count at least $3n - 2$ conditions, exactly as in case 4.

- **Case 7:** $S^* = \{0, (1, 3), (2, 4)\}$. See Figure 8 for a graphical representation. The corresponding genus-4 singularity $C$ has branches of genera 0 and 2; the genus-2 branch has multiplicity three. We have $\Gamma = \{(1, 3), (2, \infty), (\infty, 4), (\infty, 5)\}$; the corresponding model is defined by maps $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}^4$ that verify the following contact conditions:

\[ (36) \quad \varphi_1(0) = \varphi_2(0) \text{ and } \varphi_1'(0) = \varphi_2^{(3)}(0). \]

In fact, irrespective of the actual embedding dimension of $C$ (which, in principle, might be less than $m(S^*) = 4$), it is clear that the conditions (36) remain operative.

Now assume the singularity is supported in $(0 : \cdots : 0 : 1) \in \mathbb{P}^n$, and that its preimages, corresponding respectively to the branches $\varphi_1$ and $\varphi_2$,
are the points $0 = (0 : 1)$ and $\infty = (1 : 0)$ in $\mathbb{P}^1$. The conditions arising from the singularity, with respect to these choices, are

\begin{align*}
a_{i,d} &= a_{i,0} = 0 \text{ (incidence)}, \\
a_{i,1} &= a_{i,2} = 0 \text{ (higher-cuspidality of the } \infty \text{ branch)}, \text{ and} \\
a_{i,d-1} &= a_{i,3} \text{ (higher-order contact)}, 0 \leq i \leq n - 1.
\end{align*}

After adjusting for variations of preimage and target, we count $4n - 2$ conditions.

There are 15 remaining non-MT cases to analyze, all of which are associated to singularities of genus 4. The analysis of each case proceeds along lines analogous to the preceding 7 cases. Detailed dimension counts may be found in \[10\].

### 2.3. End of the proof of Thm 2.1.

The analysis of the cases above shows that Thm 2.1 holds for the space of rational curves with a unique singular point of non-dimensionally transverse type of genus $1 \leq g \leq 4$. It follows easily that Thm 2.1 holds for the space of rational curves with a unique singular point of genus $1 \leq g \leq 4$. Indeed, by definition any dimensionally transverse multibranch singularity $C$ decomposes as a transverse gluing of branches $C_i$, $1 \leq i \leq r$ of strictly lower genera $g_i$ with $g = \sum g_i + r - 1$. In particular, to count the conditions each imposed by such a singularity $C$, we compute the sum of the conditions imposed by $C_i$, $1 \leq i \leq r$ individually, plus $r(m - 1)$ incidence conditions; by induction, this sum will be at least $\sum (n - 2)g_i + n(r - 1) = (n - 2)g + 2(r - 1)$, which beats the required estimate.

It remains to prove Thm 2.1 holds for rational curves with at least two distinct singularities whose local genera sum to $g \leq 4$. To do so, we filter according to the partition types determined by the local genera of their singularities; since our argument shows that any entry of a given partition contribute conditions independent of those arising from the partition’s other entries, we may assume without loss of generality that $g = 4$. Further, in the interest of space we will treat here only the cases $(1, 1, 1, 1)$ and $(2, 2)$, deferring the (analogous) arguments in the remaining cases to \[10\].

**Case:** $(1, 1, 1, 1)$. We will in fact show that the subspace of rational curves with partition type $(1^4)$ inside $M_n^d$ is exactly $(n - 2)g$ whenever $d \geq 2g - 2$.

A rational curve of this type has $g$ singularities, each of which is either a simple node or a simple cusp. To begin, say all are simple nodes. Consider one of these; writing down the conditions it imposes on coefficients of the corresponding morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ requires that certain explicit choices of preimages and image be made, with respect to affine charts in the domain and target. Assume the preimages are $(x : 1)$ and $(y : 1)$ with $x \neq y$, and that their common image $\alpha = (\alpha_i)$ in $\mathbb{P}^n$ has nonvanishing $l$th coordinate, where $0 \leq l \leq n$. We then require that

\begin{equation}
\sum_{j=0}^{d} a_{i,j} x^{d-j} = \alpha_i \text{ and } \sum_{j=0}^{d} a_{i,j} y^{d-j} = \alpha_i
\end{equation}

for all $i \neq l \in \{0, \ldots, n\}$. The theory of Vandermonde matrices implies that for every $i$, the two corresponding conditions (37) are linearly independent.
More precisely, any \(2m\) distinct preimages \((x_k : 1)\) and \((y_k : 1)\) that map to points \(\alpha^{(1)}, \ldots, \alpha^{(m)} \in \mathbb{P}^n\) with nonvanishing \(l\)th coordinate for \(1 \leq k \leq m\) induce linear independent sets of conditions, as long as \(d \geq 2m\). In particular, if \(d \geq 2g - 1\), all conditions (37) arising from distinct preimages \((x_k : 1)\) and \((y_k : 1)\) mapping to points with nonvanishing \(l\)th coordinate are independent. Clearly the analysis applies irrespective of the assumption on \(l\); and \(l\) need not be fixed, either. Indeed, when \(d = 2g - 2\), \(l\) cannot be fixed. But our Vandermonde-based analysis remains unchanged, because any pair of conditions (37) associated with a particular value of \(i\) are automatically linearly independent relative to conditions associated with any other value of \(i\), and for each coordinate index \(i\) any number \(2m \leq d - 1\) of (pairs of) conditions are independent.

Now say that a node with nonvanishing \(l\)th coordinate has \(\infty = (1 : 0)\) as a preimage; then one of the sets of conditions in (37) is substituted by the requirement that \(a_{i,0} = 0\) for all \(i \neq l \in \{0, \ldots, n\}\). Such substitutions leave our analysis unchanged. Indeed, their associated matrices of coefficients (whose rows are of the form \([x^d, x^{d-1}, \ldots, 1]\) as in (37), together with a single row \([1, 0, \ldots, 0]\)) may be row-reduced to Vandermondes with nonvanishing maximal minors. We count \(g(2n) = 2gn\) conditions from (37), with respect to fixed choices of preimages and images. Allowing the latter to vary leaves us with \(2gn - gn - g(1+1) = g(n-2)\) independent conditions, as required.

We now explain how to incorporate simple cusps into the above analysis. A simple cusp supported in a point \(\alpha \in \mathbb{P}^n\) with nonvanishing \(l\)th coordinate and preimage \((x, 1)\) (resp, \(\infty = (1 : 0)\)) imposes that

\[
\sum_{j=0}^{d-1} (d-j)a_{i,j}x^{d-j-1} = 0; \quad \text{resp., that } a_{i,1} = 0
\]

for all \(i \neq l \in \{0, \ldots, n\}\). One upshot of (38) is that each simple cusp imposes exactly \((n-1)\) conditions on morphisms \(f \in M^n_d\), once variations of preimage and image are taken into account (of course, we already knew this!). In particular, Eisenbud–Harris dimensional transversality implies that \(g\) simple cusps impose \((n-1)g\) independent conditions. On the other hand, a second upshot of (38) is that the matrices of coefficients of conditions imposed by \textit{mixed} collections of simple nodes and simple cusps are (evaluations of) partial derivatives \(\frac{\partial}{\partial x}\) of Vandermondes, where each row of a given Vandermonde matrix is viewed as a function of an indeterminate variable \(x\) independent of those of the other rows. In particular, their (sub)determinants are partial derivatives of Vandermonde (sub)determinants. It is unclear (to us, anyway) which positivity properties such polynomials satisfy in general. So at this stage we specialize to the case \(g = 4\).

Let \(f \in M^n_d\) be a rational curve of arithmetic genus 4 with \(q\) simple nodes and \(4 - q\) simple cusps, where \(1 \leq q \leq 3\). Say \(q = 1\); without loss of generality, we may assume the cusps have preimages \(\infty = (1 : 0)\), \(0 = (0 : 1)\), and \(1 = (1 : 1)\). For the sake of exposition, suppose that all
three cusps have nonvanishing \( l \)th coordinate; we then require that
\[
a_{i,1} = 0, a_{i,d-1} = 0, \text{ and } \sum_{j=0}^{d-1} (d-j)a_{i,j} = 0
\]
for all \( i \neq l \in \{0, \ldots, n\} \). Assuming that our simple node has preimages \((x : 1)\) and \((y : 1)\) and is supported in a point \( \alpha \in \mathbb{P}^n \) with nonvanishing \( l \)th coordinate, the conditions (37) are also operative. For a particular choice of \( i \neq l \), the \( 5 \times d \) matrix of coefficients \( A \) associated with these conditions is
\[
A = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 \\
x^d & d-d-1 & d-2 & \ldots & 1 & 0 \\
y^d & y_{d-1} & y_{d-2} & \ldots & x & 1 \\
\end{bmatrix}
\]
where \( x \neq y \) and \( x, y \notin \{0, 1\} \). It suffices to exhibit a nonvanishing \( 5 \times 5 \) minor of \( A \); for this, it suffices in turn to exhibit a nonvanishing \( 3 \times 3 \) minor of the submatrix \( A_0 \) obtained by omitting rows \( \{1, 2\} \) and columns \( \{2, d\} \). So consider
\[
A_1 = \begin{bmatrix}
3 & 2 & 1 \\
x^3 & x^2 & x \\
y^3 & y^2 & y \\
\end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix}
4 & 3 & 2 \\
x^4 & x^3 & x^2 \\
y^4 & y^3 & y^2 \\
\end{bmatrix}
\]
each of which is a \( 3 \times 3 \) submatrix of \( A_0 \). Note that
\[
\det(A_1) = xy \det \begin{bmatrix} x^2 - 3 & x - 2 \\ y^2 - 3 & y - 2 \end{bmatrix} = xy(x-y)((x-1)(y-1)-(x-1)-(y-1)),
\]
while
\[
\det(A_2) = 2x^2y^2 \det \begin{bmatrix} x^2 - 2 & x - \frac{1}{2} \\ y^2 - 2 & y - \frac{1}{2} \end{bmatrix} = x^2y^2(x-y)(2(x-1)(y-1)-(x-1)-(y-1)).
\]
It is easy to see that the determinants (41) and (42) cannot vanish simultaneously.

The analysis when \( q = 2 \) or \( q = 3 \) is similar, but simpler, so we merely sketch it. In these cases, there are at most two cusps, which we may assume are supported in (a subset of) \( \{0 = (0 : 1), \infty = (1 : 0)\} \). We must then show that a matrix of conditions \( A \) has maximal rank, but because the cusps at 0 and \( \infty \) each correspond to rows with a single nonzero entry, the fact that \( A \) is of maximal rank reduces to the nondegeneracy of the usual Vandermonde matrix.

**Case:** \((2, 2)\). A rational curve of this partition type has two singularities, and by Eisenbud–Harris dimensional transversality we may assume \( q \) of these is multibranch, where \( q = 1 \) or \( q = 2 \).

Now say \( q = 1 \). We may filter possibilities according to the semigroup \( S \) of the unique multibranch singularity \( C_1 \), as in the preceding item. To begin, say that \( S^* = \{0, (1, 1), (2, 2)\} \). We assume that both singularities of our rational curve are supported in points of \( \mathbb{P}^n \) with nonvanishing zeroth coordinate, the preimages of the branches of \( C_1 \) are \( x = (x : 1) \) and \( y = \).
(y : 1) with x, y ∈ C and that the unique genus-2 cusp C₂ has preimage ∞ = (1 : 0). Recall that C₂ may have semigroup (3, 4, 5) or (2, 5). When S(C₂) = (3, 4, 5), it suffices to show the following coefficient matrix is of maximal rank:

\[
A = \begin{bmatrix}
x^d & x^{d-1} & x^{d-2} & \ldots & x^2 & x & 1 \\
y^d & y^{d-1} & y^{d-2} & \ldots & y^2 & y & 1 \\
dx^d & (d - 1)x^{d-2} & (d - 2)x^{d-3} & \ldots & 2x & 1 & 0 \\
dy^d & (d - 1)y^{d-2} & (d - 2)y^{d-3} & \ldots & 2y & 1 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 & 0
\end{bmatrix}.
\]

This follows from an easy modification of the argument used in the analysis of the corresponding (2, 1, 1) subcase above. Similarly, when S(C₂) = (2, 5), the ramification conditions satisfied by the coefficients of each \(f_i\) with \(2 \leq i \leq n\) are independent, because \(A\) has maximal rank. One additional condition beyond ramification applies when \(i = 1\) (in place of the higher-cuspidal condition corresponding to the last row in \(A\)), and it is clearly independent of ramification; see (20). The analysis when \(S^* = \{0, (1, 1, 1)\}\), or \(S^* = \{0, (1, 2)\}\) is analogous.

Finally, say \(q = 2\). The corresponding rational curves have two multi-branch singularities \(C₁\) and \(C₂\) of genus 2; it suffices to show the corresponding coefficient matrices are of maximal rank. We filter according to possible pairs \(S₁ = S(C₁), S₂ = S(C₂)\).

- \(S₁^* = \{0, (1, 1), (2, 2)\}, S₂^* = \{0, (1, 1), (2, 2)\}\). Here we may assume without loss of generality that the branches \(x = (x : 1), y = (y : 1)\) and \(z = (z : 1), w = (w : 1)\) of \(C₁\) and \(C₂\), respectively, all belong to the same toric chart (the adaptation of the proof to the general case is immediate). Our coefficient matrix is then

\[
A = \begin{bmatrix}
x^d & x^{d-1} & x^{d-2} & \ldots & x^2 & x & 1 \\
y^d & y^{d-1} & y^{d-2} & \ldots & y^2 & y & 1 \\
dx^d & (d - 1)x^{d-2} & (d - 2)x^{d-3} & \ldots & 2x & 1 & 0 \\
dy^d & (d - 1)y^{d-2} & (d - 2)y^{d-3} & \ldots & 2y & 1 & 0 \\
z^d & z^{d-1} & z^{d-2} & \ldots & z^2 & z & 1 \\
w^d & w^{d-1} & w^{d-2} & \ldots & w^2 & w & 1 \\
dz^d & (d - 1)z^{d-2} & (d - 2)z^{d-3} & \ldots & 2z & 1 & 0 \\
dw^d & (d - 1)w^{d-2} & (d - 2)w^{d-3} & \ldots & 2w & 1 & 0
\end{bmatrix}.
\]

We claim the rightmost 8 × 8 submatrix \(A₀\) of \(A\) is of maximal rank. Up to a sign, we have

\[
\det(A₀) = \frac{\partial^4 P}{\partial x\partial y\partial z\partial w} = \frac{P}{(x - x₁)(y - y₁)(z - z₁)(w - w₁)} \bigg|_{x=x₁, y=y₁, z=z₁, w=w₁}
\]

where \(P = V(x, y, z, w, x₁, y₁, z₁, w₁)\) is the usual Vandermonde determinant. It follows immediately that \(\det(A₀) \neq 0\).

- \(S₁^* = \{0, (1, 1), (2, 2)\}, S₂^* = \{0, (1, 1, 1)\}\). The analysis is similar to that of the preceding subcase. Recall that \(C₂\) is necessarily a triple point. Assuming that the branches \(x = (x : 1), y = (y : 1)\) and \(v = (v : 1), w = (w : 1), z = (z : 1)\) of \(C₁\) and \(C₂\), respectively, all lie
in the same toric chart, the corresponding coefficient matrix $A$ has a (maximal, rightmost) $7 \times 7$ submatrix $A_0$ for which (up to a sign)

$$\det(A_0) = \left. \frac{\partial^2 P}{\partial x \partial y} \right|_{x=x_1,y=y_1} = \left. \frac{P}{(x-x_1)(y-y_1)} \right|_{x=x_1,y=y_1},$$

where $P = V(x,y,v,z,w,x_1,y_1)$ is the usual Vandermonde determinant. Again $\det(A_0) \neq 0$.

- $S_1^* = \{(0,1,1), (2,2)\}$, $S_2^* = \{(0,1,2)\}$. Recall that $C_2$ is the transverse union of smooth and simple-cuspidal branches. Assuming that all branches of $C_1$ and $C_2$ lie in the same toric chart, the corresponding coefficient matrix $A$ has a maximal minor $A_0$ for which

$$\det(A_0) = \left. \frac{\partial^3 P}{\partial x \partial y \partial z} \right|_{x=x_1,y=y_1,z=z_1} = \left. \frac{P}{(x-x_1)(y-y_1)(z-z_1)} \right|_{x=x_1,y=y_1,z=z_1},$$

where $P = V(x,y,v,z,w,x_1,y_1)$ is the usual Vandermonde determinant. Hence $\det(A_0) \neq 0$.

- The analyses of the remaining cases are analogous. In each instance, there is a maximal minor of the coefficient matrix $A$ equal to the restriction of the $n$th partial derivative of a Vandermonde determinant to the corresponding diagonal locus, which in particular means it is nonzero.

\[ \square \]

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