ON EXTREMELY AMENABLE GROUPS OF HOMEOMORPHISMS

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Abstract. A topological group $G$ is extremely amenable if every compact $G$-space has a $G$-fixed point. Let $X$ be compact and $G \subseteq \text{Homeo}(X)$. We prove that the following are equivalent: (1) $G$ is extremely amenable; (2) every minimal closed $G$-invariant subset of $\text{Exp} R$ is a singleton, where $R$ is the closure of the set of all graphs of $g \in G$ in the space $\text{Exp}(X^2)$ (Exp stands for the space of closed subsets); (3) for each $n = 1, 2, \ldots$ there is a closed $G$-invariant subset $Y_n$ of $(\text{Exp} X)^n$ such that $\bigcup_{n=1}^{\infty} Y_n$ contains arbitrarily fine covers of $X$ and for every $n \geq 1$ every minimal closed $G$-invariant subset of $\text{Exp} Y_n$ is a singleton. This yields an alternative proof of Pestov’s theorem that the group of all order-preserving self-homeomorphisms of the Cantor middle-third set (or of the interval $[0, 1]$) is extremely amenable.

1. Introduction

With every topological group $G$ one can associate the greatest ambit $\mathcal{S}(G)$ and the universal minimal compact $G$-space $\mathcal{M}(G)$. To define these objects, recall some definitions. A $G$-space is a topological space $X$ with a continuous action of $G$, that is, a map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $1x = x$ ($g, h \in G$, $x \in X$). A map $f : X \to Y$ between two $G$-spaces is $G$-equivariant, or a $G$-map for short, if $f(gx) = gf(x)$ for every $g \in G$ and $x \in X$.

A semigroup is a set with an associative multiplication. A semigroup $X$ is right topological if it is a topological space and for every $y \in X$ the self-map $x \mapsto xy$ of $X$ is continuous. (Sometimes the term left topological is used for the same thing.) A subset $I \subset X$ is a left ideal if $XI \subset I$. If $G$ is a topological group, a right topological semigroup compactification of $G$ is a right topological compact semigroup $X$ together with a continuous semigroup morphism $f : G \to X$ with a dense range such that the map $(g, x) \mapsto f(g)x$ from $G \times X$ to $X$ is jointly continuous (and hence $X$ is a $G$-space).

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1All spaces are assumed to be Tikhonov, and all maps are assumed to be continuous
The greatest ambit $S(G)$ for $G$ is a right topological semigroup compactification which is universal in the usual sense: for any right topological semigroup compactification $X$ of $G$ there is a unique morphism $S(G) \to X$ of right topological semigroups such that the obvious diagram commutes. Considered as a $G$-space, $S(G)$ is characterized by the following property: there is a distinguished point $e \in S(G)$ such that for every compact $G$-space $Y$ and every $a \in Y$ there exists a unique $G$-map $f : S(G) \to Y$ such that $f(e) = a$.

We can take for $S(G)$ the compactification of $G$ corresponding to the $C^*$-algebra $RUCB(G)$ of all bounded right uniformly continuous functions on $G$, that is, the maximal ideal space of that algebra. (A complex function $f$ on $G$ is right uniformly continuous if

$$\forall \epsilon > 0 \exists V \in \mathcal{N}(G) \forall x, y \in G (xy^{-1} \in V \Rightarrow |f(y) - f(x)| < \epsilon),$$

where $\mathcal{N}(G)$ is the filter of neighbourhoods of unity.) The $G$-space structure on $S(G)$ comes from the natural continuous action of $G$ by automorphisms on $RUCB(G)$ defined by $gf(h) = f(g^{-1}h)$ ($g, h \in G, f \in RUCB(G)$). We shall identify $G$ with a subspace of $S(G)$. Closed $G$-subspaces of $S(G)$ are the same as closed left ideals of $S(G)$.

A $G$-space $X$ is minimal if it has no proper $G$-invariant closed subsets or, equivalently, if the orbit $Gx$ is dense in $X$ for every $x \in X$. The universal minimal compact $G$-space $M(G)$ is characterized by the following property: $M(G)$ is a minimal compact $G$-space, and for every compact minimal $G$-space $X$ there exists a $G$-map of $M(G)$ onto $X$. Since Zorn’s lemma implies that every compact $G$-space has a minimal compact $G$-subspace, it follows that for every compact $G$-space $X$, minimal or not, there exist a $G$-map of $M(G)$ to $X$. The space $M(G)$ is unique up to a $G$-space isomorphism and is isomorphic to any minimal closed left ideal of $S(G)$, see e.g. [1], [9] Section 4.1], [11] Appendix], [10] Theorem 3.5.

A topological group $G$ is extremely amenable if $M(G)$ is a singleton or, equivalently, if $G$ has the fixed point on compacta property: every compact $G$-space $X$ has a $G$-fixed point, that is, a point $p \in X$ such that $gp = p$ for every $g \in G$. Examples of extremely amenable groups include Homeo$_+[0, 1] = \text{the group of all orientation-preserving self-homeomorphisms of } [0, 1]; U_s(H) = \text{the unitary group of a Hilbert space } H, \text{ with the topology inherited from the product } H^H; \text{ Iso}(U) = \text{the group of isometries of the Urysohn universal metric space } U$. See Pestov’s book [9] for the proof. Note that a locally compact group $\neq \{1\}$ cannot be extremely amenable, since every locally compact group admits a free action on a compact space [12], [9] Theorem 3.3.2.

We refer the reader to Pestov’s book [9] for various intrinsic characterizations of extremely amenable groups. These characterizations reveal a close connection between Ramsey theory and the notion of extreme amenability. The aim of the present paper is to give another characterization of extremely amenable groups, based on a different approach. For a compact space $X$ let $H(X)$ be the group of all self-homeomorphisms of $X$, equipped with the compact-open topology. Let $G$ be a topological subgroup.
of \( H(X) \). There is an obvious necessary condition for \( G \) to be extremely amenable: every minimal closed \( G \)-subset of \( X \) must be a singleton. However, this condition is not sufficient. For example, let \( X \) be the Hilbert cube, and let \( G \subset H(X) \) be the stabilizer of a given point \( p \in X \). Then the only minimal closed \( G \)-subset of \( X \) is the singleton \( \{ p \} \), but \( G \) is not extremely amenable \( \text{[1]} \), since \( G \) acts without fixed points on the compact space \( \Phi_p \) of all maximal chains of closed subsets of \( X \) starting at \( p \). The space \( \Phi_p \) is a subspace of the compact \( G \)-space \( \text{Exp Exp} X \), where for a compact space \( K \) we denote by \( \text{Exp} K \) the compact space of all closed non-empty subsets of \( K \), equipped with the Vietoris topology.\(^2\) It was indeed necessary to use the second exponent in this example, the first exponent would not work. One can ask whether in general for every group \( G \subset H(X) \) which is not extremely amenable there exists a compact \( G \)-space \( X' \) derived from \( X \) by applying a small number of simple functors, like powers, probability measures, exponents, etc., such that \( X' \) contains a closed \( G \)-subspace (which can be taken minimal) on which \( G \) acts without fixed points. We answer this question in the affirmative.

Consider the action of \( G \) on \( \text{Exp} (X^2) \) defined by the composition of relations: if \( g \in G \), \( F \subset X^2 \), and \( \Gamma_g \subset X^2 \) is the graph of \( g \), then \( gF = \Gamma_g \circ F = \{(x, gy) : (x, y) \in F\} \). This amounts to considering \( X^2 \) as the product of two different \( G \)-spaces: the first copy of \( X \) has the trivial \( G \)-structure, and the second copy is the given \( G \)-space \( X \). If \( G \) is not extremely amenable, then there is a closed minimal \( G \)-subspace \( Y \) of \( \text{Exp Exp} (X^2) \) that is not a singleton (and hence fixed point free). This follows from:

**Theorem 1.1.** Let \( X \) be compact, \( G \) a subgroup of \( H(X) \). Denote by \( R \) the closure of the set \( \{ \Gamma_g : g \in G \} \) of the graphs of all \( g \in G \) in the space \( \text{Exp} (X^2) \). Then \( G \) is extremely amenable if and only if every minimal closed \( G \)-subset of \( \text{Exp} R \) is a singleton.

Here \( X^2 \) is the product of the trivial \( G \)-space and the given \( G \)-space \( X \), as in the paragraph preceding Theorem [1] and \( R \) is considered as a \( G \)-subspace of \( \text{Exp} (X^2) \).

For example, let \( X = I = [0,1] \) be the closed unit interval. Consider the group \( G = H_+([0,1]) \) of all orientation-preserving self-homeomorphisms of \( I \). The space \( R \) in this case consists of all curves \( \Gamma \) in the square \( I^2 \) that connect the lower left and upper right corners and “never go down”: if \( (x, y) \in \Gamma \), \( (x', y') \in \Gamma \) and \( x < x' \), then \( y \leq y' \) (see the picture in [3, Example 2.5.4]). It can be verified that the only minimal compact \( G \)-subsets of \( \text{Exp} R \) are singletons (they are of the form \{a closed union of \( G \)-orbits in \( R \})\). The proof depends on the following lemma:

**Lemma 1.2.** Let \( \Delta^n \) be the \( n \)-simplex of all \( n \)-tuples \( (x_1, \ldots, x_n) \in I^n \) such that \( 0 \leq x_1 \leq \cdots \leq x_n \leq 1 \). Equip \( \Delta^n \) with the natural action of the group \( G = H_+([0,1]) \).

\(^2\)If \( F \) is closed in \( K \), the sets \( \{A \in \text{Exp} K : A \subset F\} \) and \( \{A \in \text{Exp} K : A \text{ meets } F\} \) are closed in \( \text{Exp} K \), and the Vietoris topology is generated by the closed sets of this form. If \( K \) is a \( G \)-space, then so is \( \text{Exp} K \), in an obvious way.
Then every minimal closed $G$-subset of $\text{Exp} \Delta^n$ is a singleton ($= \{\text{a union of some faces of } \Delta^n\}$).

The idea to consider the action of $G = H_+([0, 1])$ on $\Delta^n$ is borrowed from [2], where it is shown that the geometric realization of any simplicial set can be equipped with a natural action of $G$. We shall not prove Lemma 1.2, since this lemma follows from Pestov’s theorem that $G$ is extremely amenable, and I am not aware of a short independent proof of the lemma. The essence of the lemma is that every subset of $\Delta^n$ can be either pushed (by an element of $G$) into the $\epsilon$-neighbourhood of the boundary of the simplex or else can be pushed to approximate the entire simplex within $\epsilon$. Some Ramsey-type argument seems to be necessary for this. Actually Lemma 1.2 may be viewed as a topological equivalent of the finite Ramsey theorem [9, Theorem 1.5.2], since Pestov showed that this theorem has an equivalent reformulation in terms of the notion of a “finely oscillation stable” dynamical system [9, Section 1.5], and extremely amenable groups are characterized in the same terms [9, Theorem 2.1.1].

An important example of an extremely amenable group is the Polish group $\text{Aut} (\mathbb{Q})$ of all automorphisms of the ordered set $\mathbb{Q}$ of rationals [6], [9, Theorem 2.3.1]. This group is considered with the topology inherited from $(\mathbb{Q}_d)^2$, where $\mathbb{Q}_d$ is the set of rationals with the discrete topology. Let $K \subset [0, 1]$ be the usual middle-third Cantor set. The topological group $\text{Aut} (\mathbb{Q})$ is isomorphic to the topological group $G = H_+(K) \subset H(K)$ of all order-preserving self-homeomorphisms of $K$. To see this, note that pairs of the endpoints of “deleted intervals” (= components of $[0, 1] \setminus K$) form a set which is order-isomorphic to $\mathbb{Q}$, whence a homomorphism $G \to \text{Aut} (\mathbb{Q})$ which is easily verified to be a topological isomorphism. One can prove that the group $G \simeq \text{Aut} (\mathbb{Q})$ is extremely amenable with the aid of Theorem 1.1. The proof is essentially the same as in the case of the group $G = H_+([0, 1])$. The space $R$ considered in Theorem 1.1 again is the space of “curves”, this time in $K^2$, that go from $(0, 0)$ to $(1, 1)$ and “look like graphs”, with the exception that they may contain vertical and horizontal parts. The evident analogue of Lemma 1.2 holds for “Cantor simplices” of the form $(x_1, \ldots, x_n) \in K^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 1$.

Theorem 1.1 may help to answer the following:

**Question 1.3.** Let $P$ be pseudoarc, $G = H(P)$, and let $G_0$ be the stabilizer of a given point $x \in P$. Is $G_0$ extremely amenable?

As explained in [11], this question is motivated by the observation that the argument involving maximal chains, which shows that the stabilizer $G_0 \subset H(X)$ of a point $p \in X$ is not extremely amenable if $X$ is either a Hilbert cube or a compact manifold of dimension $> 1$, does not work for the pseudoarc. A positive answer to Question 1.3 would imply that the pseudoarc $P$ can be identified with $\mathcal{M}(G)$ for $G = H(P)$. The problem whether this is the case was raised in [11] and appears as Problem 6.7.20 in [9].

The suspension $\Sigma X$ of a space $X$ is the quotient of $X \times I$ obtained by collapsing the “bottom” $X \times \{0\}$ and the “top” $X \times \{1\}$ to points. Let $q : \Sigma X \to I$ be the
natural projection. The inverse image under $q$ of the maximal chain $\{[0, x] : x ∈ I}\}$ of closed subsets of $I$ is a maximal chain of closed subsets of $ΣX$.

**Question 1.4.** Let $Q = I^ω$ be the Hilbert cube, and $C$ be the maximal chain of subcontinua of $ΣQ$ considered above. If $G = H(ΣQ)$ and $G_0 ⊂ G$ is the stabilizer of $C$, is $G_0$ extremely amenable?

This question is motivated by the search for a good candidate for the space $M(G)$, where $G = H(Q)$. The space $Φ_c$ of all maximal chains of subcontinua of $Q$, proved to be minimal by Y. Gutman [5], may be such a candidate [9, Problem 6.4.13]. Recall that for the group $G = H(K)$, where $K = 2^ω$ is the Cantor set, $M(G)$ can be identified with the space $Φ ⊂ ExpExp K$ of all maximal chains of closed subsets of $K$ [4], [9, Example 6.7.18].

There is another characterization (Theorem 1.5) of extremely amenable groups in the spirit of Theorem 1.1 which, in combination with Lemma 1.2, readily implies Pestov’s results that $H_+([0, 1])$ and $Aut(Q)$ are extremely amenable. Let $X$ be compact, $Y_n ⊂ (Exp X)^n$ for $n = 1, 2, \ldots$. We say that $\bigcup_{n=1}^∞ Y_n$ contains arbitrarily fine covers if for every open cover $α$ of $X$ there are $n ≥ 1$ and $(F_1, \ldots, F_n) ∈ Y_n$ such that $\bigcup_{i=1}^n F_i = X$ and the cover $\{F_i\}_{i=1}^n$ of $X$ refines $α$.

**Theorem 1.5.** Let $X$ be compact, $G$ a subgroup of $H(X)$. Let $Y_n$ be a closed $G$-invariant subset of $(Exp X)^n$ ($n = 1, 2, \ldots$) such that $\bigcup_{n=1}^∞ Y_n$ contains arbitrarily fine covers of $X$. Then $G$ is extremely amenable if and only if for every $n ≥ 1$ every minimal closed $G$-invariant subset of $Exp Y_n$ is a singleton.

Observe that Pestov’s theorem asserting that $G = H_+([0, 1])$ is extremely amenable follows from Theorem 1.5 and Lemma 1.2. It suffices to take for $Y_{n+1}$ the collection of all sequences

$$([0, x_1], [x_1, x_2], \ldots, [x_n, 1]),$$

where $0 ≤ x_1 ≤ \cdots ≤ x_n ≤ 1$. The $G$-space $Y_{n+1}$ is isomorphic to the $n$-simplex $Δ^n$ considered in Lemma 1.2. The argument for $Aut(Q) ≃ H_+(K)$ is similar.

The proof of Theorems 1.1 and 1.5 depends on the notion of a representative family of compact $G$-spaces. We introduce this notion in Section 2 and observe that a topological group $G$ is extremely amenable if (and only if) there exists a representative family $\{X_α\}$ such that any minimal closed $G$-subset of any $X_α$ is a singleton (Theorem 2.2). In Section 3 we prove that the single space $Exp R$ considered in Theorem 1.1 constitutes a representative family (Theorem 3.1). The conjunction of Theorems 2.2 and 3.1 proves Theorem 1.1. In Section 4 we prove that under the conditions of Theorem 1.5 the sequence $\{Exp Y_n\}$ is representative (Theorem 4.2). The conjunction of Theorems 2.2 and 4.2 proves Theorem 1.5.

2. Representative families of $G$-spaces

Let $G$ be a topological group, $X$ a compact $G$-space. For $g ∈ G$ the $g$-translation of $X$ is the map $x ↦ gx$, $x ∈ X$. The **enveloping semigroup** (or the **Ellis semigroup**
$E(X)$ of the dynamical system $(G, X)$ is the closure of the set of all $g$-translations, $g \in G$, in the compact space $X^X$. This is a right topological semigroup compactification of $G$, as defined in Section 1. The natural map $G \to E(X)$ extends to a $G$-map $S(G) \to E(X)$ which is a morphism of right topological semigroups.

**Definition 2.1.** A family $\{X_\alpha : \alpha \in A\}$ of compact $G$-spaces is representative if the family of natural maps $S(G) \to E(X_\alpha)$, $\alpha \in A$, separates points of $S(G)$ (and hence yields an embedding of $S(G)$ into $\prod_{\alpha \in A} E(X_\alpha)$).

**Theorem 2.2.** Let $G$ be a topological group, $\{X_\alpha\}$ a representative family of compact $G$-spaces. Then $G$ is extremely amenable if (and only if) every minimal closed $G$-subset of every $X_\alpha$ is a singleton.

This is a special case of a more general theorem:

**Theorem 2.3.** If $\{X_\alpha\}$ is a representative family of compact $G$-spaces, the universal minimal compact $G$-space $M(G)$ is isomorphic (as a $G$-space) to a $G$-subspace of a product $\prod Y_\beta$, where each $Y_\beta$ is a minimal compact $G$-space isomorphic to a $G$-subspace of some $X_\alpha$.

**Proof.** By definition of a representative family, the greatest ambit $S(G)$ can be embedded (as a $G$-space) into the product $\prod E(X_\alpha)$ and hence also into the product $\prod X_\alpha^{X_\alpha}$. Consider $M(G)$ as a subspace of $S(G)$ and take for the $Y_\beta$ ’s the projections of $M(G)$ to the factors $X_\alpha$. □

We now give a sufficient condition for a family of compact $G$-spaces to be representative. Let us say that two subsets $A, B$ of $G$ are far from each other with respect to the right uniformity if one of the following equivalent conditions holds: (1) the neutral element $1_G$ of $G$ is not in the closure of the set $BA^{-1}$; (2) for some neighbourhood $U$ of $1_G$ the sets $A$ and $UB$ are disjoint; (3) there exists a right uniformly continuous function $f : G \to [0, 1]$ such that $f = 0$ on $A$ and $f = 1$ on $B$; (4) $A$ and $B$ have disjoint closures in $S(G)$.

**Proposition 2.4.** Let $\mathcal{F}$ be a family of compact $G$-spaces. Suppose that the following holds:

(*) if $A, B \subset G$ are far from each other with respect to the right uniformity, then there exists $X \in \mathcal{F}$ and $p \in X$ such that the sets $Ap$ and $Bp$ have disjoint closures in $X$.

Then $\mathcal{F}$ is representative.

**Proof.** Consider the natural map $G \to \prod \{E(X) : X \in \mathcal{F}\}$. It defines a compactification $bG$ of $G$. We must prove that this compactification is equivalent to $S(G)$.

Let $A, B$ be any two subsets of $G$ with disjoint closures in $S(G)$. Then $A$ and $B$ are far from each other with respect to the right uniformity. According to the condition (*), there exists $X \in \mathcal{F}$ and $p \in X$ such that the sets $Ap$ and $Bp$ have disjoint closures in $X$. It follows that the images of $A$ and $B$ in $E(X)$ have disjoint closures, and a
fortiori the images of $A$ and $B$ in $bG$ have disjoint closures. It follows that $\mathcal{S}(G)$ and $bG$ are equivalent compactifications of $G$ [3, Theorem 3.5.5].

3. Proof of Theorem 1.1

Recall the setting of Theorem 1.1. $X$ is compact, $G$ is a topological subgroup of $H(X)$. For $g \in G$ let $\Gamma_g = \{(x, gx) : x \in X\} \subset X^2$ be the graph of $g$, and let $R$ be the closure of the set $\{\Gamma_g : g \in G\}$ in the compact space $\text{Exp}(X^2)$. We consider the action of $G$ on $\text{Exp}(X^2)$ defined by $gF = \{(x, gy) : (x, y) \in F\}$ ($g \in G, F \in \text{Exp}(X^2)$), and consider $R$ as a $G$-subspace of $\text{Exp}(X^2)$.

**Theorem 3.1.** Let $X$ be a compact space, $G \subset H(X)$. Let $R \subset \text{Exp}(X^2)$ be the compact $G$-space defined above. The family consisting of the single compact $G$-space $\text{Exp}R$ is representative.

In other words, $\mathcal{S}(G)$ is isomorphic to the enveloping semigroup of $\text{Exp}R$.

**Proof.** Let $A, B \subset G$ be far from each other (that is, $1_G$ is not in the closure of $BA^{-1}$).

In virtue of proposition 2.4, it suffices to find $p \in Y = \text{Exp}R$ such that $Ap$ and $Bp$ have disjoint closures in $Y$.

Let $p$ be the closure of the set $\{\Gamma_g : g \in A^{-1}\}$ in the space $\text{Exp}(X^2)$. Then $p$ is a closed subset of $R$ and hence $p \in Y$. We claim that $p$ has the required property: $Ap$ and $Bp$ have disjoint closures in $Y$ or, which is the same, in $\text{ExpExp}(X^2)$.

There exist a continuous pseudometric $d$ on $X$ and $\delta > 0$ such that

\[ \forall f \in A \forall g \in B \exists x \in X \left( d(gf^{-1}(x), x) \geq \delta \right). \]

Let $\Delta \subset X^2$ be the diagonal. Let $C \subset X^2$ be the closed set defined by

\[ C = \{(x, y) \in X^2 : d(x, y) \geq \delta\}. \]

Let $K \subset \text{Exp}X^2$ be the closed set defined by

\[ K = \{F \subset X^2 : F \text{ meets } C\}. \]

Consider the closed sets $L_1, L_2 \subset \text{ExpExp}(X^2)$ defined by

\[ L_1 = \{q \subset \text{Exp}(X^2) : q \text{ is closed and } \Delta \in q\} \]

and

\[ L_2 = \{q \subset \text{Exp}(X^2) : q \text{ is closed and } q \subset K\}. \]

Since $\Delta \notin K$, the sets $L_1$ and $L_2$ are disjoint. It suffices to verify that $Ap \subset L_1$ and $Bp \subset L_2$.

The first inclusion is immediate: if $g \in A$, then for $h = g^{-1}$ we have $\Delta = g\Gamma_h \in gp$, hence $gp \in L_1$. Thus $Ap \subset L_1$. We now prove that $Bp \subset L_2$. Let $g \in B$. If $f \in A$ and $h = f^{-1}$, there exists $x \in X$ such that $d(gh(x), x) \geq \delta$, which means that $\Gamma_{gh}$ meets $C$. Hence $g\Gamma_h = \Gamma_{gh} \in K$. It follows that the closed set $g^{-1}K$ contains the set $\{\Gamma_h : h \in A^{-1}\}$ and hence also its closure $p$. In other words, $gp \subset K$ and hence $gp \in L_2$. □
As noted in Section 1, Theorem 1.1 follows from Theorems 2.3 and 3.1.

Combining Theorems 2.3 and 3.1 we obtain the following generalization of Theorem 1.1:

**Theorem 3.2.** Let \( X \) be a compact space, \( G \) a subgroup of \( H(X) \). Let \( R \) be the same as in Theorems 1.1 and 3.1. Let \( F \) be the family of all minimal closed \( G \)-subspaces of \( \text{Exp} R \). Then \( \mathcal{M}(G) \) is isomorphic to a subspace of a product of members of \( F \) (some factors may be repeated).

4. **Proof of Theorem 1.5**

Theorem 3.1 implies that for any subgroup \( G \subset H(X) \) the one-point family \( \{\text{Exp Exp} (X^2)\} \) is representative (recall that we consider the trivial action on the first factor \( X \)). I do not know whether \( X^2 \) can be replaced here by \( X \). On the other hand, the following holds:

**Theorem 4.1.** Let \( X \) be a compact space, \( G \) a subgroup of \( H(X) \). The sequence \( \{\text{Exp} ((\text{Exp} X)^n)\}^\infty_{n=1} \) of compact \( G \)-spaces is representative.

This is a special case of a more general theorem:

**Theorem 4.2.** Let \( X \) be a compact space, \( G \) a subgroup of \( H(X) \). Let \( Y_n \) be a closed \( G \)-invariant subset of \( (\text{Exp} X)^n \) \( (n = 1, 2, \ldots) \) such that \( \bigcup_{n=1}^\infty Y_n \) contains arbitrarily fine covers of \( X \). Then the sequence \( \{\text{Exp} Y_n\}^\infty_{n=1} \) of compact \( G \)-spaces is representative.

**Proof.** Let \( A, B \subset G \) be two sets that are far from each other with respect to the right uniformity. In virtue of proposition 2.3 it suffices to find \( n \) and a point \( p \in \text{Exp} Y_n \) such that \( Ap \) and \( Bp \) have disjoint closures in \( \text{Exp} Y_n \) or, which is the same, in \( Z_n = \text{Exp} ((\text{Exp} X)^n) \).

There exist a continuous pseudometric \( d \) on \( X \) and \( \delta > 0 \) such that \( A \) and \( B \) are \((d, 2\delta)\)-far from each other, in the sense that

\[
\forall f \in A \ \forall g \in B \ \exists x \in X \ (d(f(x), g(x)) > 2\delta).
\]

The assumption that \( \bigcup_{n=1}^\infty Y_n \) contains arbitrarily fine covers implies that we can find \( n \geq 1 \) and closed sets \( C_1, \ldots, C_n \subset X \) of \( d \)-diameter \( \leq \delta \) such that \( (C_1, \ldots, C_n) \in Y_n \) and \( \bigcup_{i=1}^n C_i = X \). For each \( g \in G \) let \( F_g = (g^{-1}(C_1), \ldots, g^{-1}(C_n)) \in (\text{Exp} X)^n \). Since \( Y_n \) is \( G \)-invariant, we have \( F_g \in Y_n \). Let \( p \) be the closure of the set \( \{F_g : g \in A\} \) in the space \( (\text{Exp} X)^n \). Then \( p \in \text{Exp} Y_n \). We claim that \( p \) has the required property: \( Ap \) and \( Bp \) have disjoint closures in \( Z_n \).

Let \( D_i = \{x \in X : d(x, C_i) \geq \delta\}, i = 1, \ldots, n \). Consider the closed sets \( K_1, K_2 \subset (\text{Exp} X)^n \) defined by

\[
K_1 = \{(F_1, \ldots, F_n) \in (\text{Exp} X)^n : F_i \subset C_i, \ i = 1, \ldots, n\}
\]

and

\[
K_2 = \{(F_1, \ldots, F_n) \in (\text{Exp} X)^n : F_i \text{ meets } D_i \text{ for some } i = 1, \ldots, n\}.
\]
Consider the closed sets \( L_1, L_2 \subset \mathbb{Z}^n \) defined by

\[
L_1 = \{ q \subset (\text{Exp} X)^n : q \text{ is closed and } q \text{ meets } K_1 \}
\]

and

\[
L_2 = \{ q \subset (\text{Exp} X)^n : q \text{ is closed and } q \subset K_2 \}.
\]

Clearly \( K_1 \) and \( K_2 \) are disjoint, hence \( L_1 \) and \( L_2 \) are disjoint as well. It suffices to verify that \( Ap \subset L_1 \) and \( Bp \subset L_2 \).

The first inclusion is immediate: if \( g \in A \), then \( F^g \in p \) and \( gF^g = (C_1, \ldots, C_n) \in K_1 \cap gp \), hence \( gp \) meets \( K_1 \) and \( gp \in L_1 \). We now prove that \( Bp \subset L_2 \). Let \( h \in B \). If \( g \in A \), we can find \( x \in X \) such that \( d(g(x), h(x)) > 2\delta \) and an index \( i, 1 \leq i \leq n \), such that \( g(x) \in C_i \). Since \( \text{diam} C_i \leq \delta \), we have \( h(x) \in D_i \) and therefore \( h(x) \in hg^{-1}(C_i) \cap Di \neq \emptyset \). It follows that \( hF^g = (hg^{-1}(C_1), \ldots, hg^{-1}(C_n)) \in K_2 \). This holds for every \( g \in A \), and thus we have shown that the closed set \( h^{-1}K_2 \subset (\text{Exp} X)^n \) contains the set \( \{ F^g : g \in A \} \) and hence also its closure \( p \). In other words, \( hp \subset K_2 \) and hence \( hp \in L_2 \).\( \square \)

Theorem 1.5 follows from Theorems 4.2 and 2.2.

Combining Theorems 2.3 and 1.2 we obtain the following generalization of Theorem 1.5:

**Theorem 4.3.** Let \( X \) be a compact space, \( G \) a subgroup of \( H(X) \). Let \( Y_n \) be a closed \( G \)-invariant subset of \( (\text{Exp} X)^n \) \((n = 1, 2, \ldots)\) such that \( \cup_{n=1}^{\infty} Y_n \) contains arbitrarily fine covers of \( X \). Let \( \mathcal{F} \) be the family of all (up to an isomorphism) minimal closed \( G \)-subspaces of \( \text{Exp} Y_n \), \( n = 1, 2, \ldots \). Then \( \mathcal{M}(G) \) is isomorphic to a subspace of a product of members of \( \mathcal{F} \) (some factors may be repeated).
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