An equation of state for dark matter in the Milky Way

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ABSTRACT Dark matter, believed to be present in many galaxies, is interpreted as a hydrodynamical system in interaction with the gravitational field and with nothing else. The gravitational field of our Galaxy can be inferred from observation of orbital velocities of the visible stars, in a first approximation in which the field is taken to be due to the distribution of dark matter only. An equation of state is determined by the gravitational field via the equations of motion.

To arrive at an estimate of the distribution of dark matter in our galaxy, and simultaneously learn something about the gravitational field in the inner regions, the following strategy was adopted: 1. The observed rotation curves suggest an expression for the newtonian potential, valid in the outer region. 2. The assumption of a quasi stationary, spherically symmetric distribution of dark matter then leads to a unique equation of state. 3. This equation of state is assumed to be valid all the way to the center (though of course the newtonian approximation is not). 4. Using this equation of state, together with Einstein’s equations and the relativistic hydrostatic condition, we calculate the metric and the matter density throughout the galaxy. The solutions are regular all the way to the center; there is no indication of a structure of the type of a Black Hole.

The equation of state that is thus determined experimentally is of the type used by Chandrasekhar and others for the degenerate Fermi gas. In the approximation of weak fields the associated ”sinh-Emden” equation, $\Delta \mu = a \sinh^4 \mu$ has a global, nonsingular solution.

I. Introduction

One of the enduring problems of astrophysics is to place an upper limit on the mass of a star. Let us agree from the outset that the observed, gravitational “mass” $M$ of a spherically symmetric object is a limiting value or asymptotic value of the function $M(r)$ that appears in the asymptotic Schwarzschild metric,

$$g_{00}(r) = c^2 \left(1 - \frac{2M(r)G}{r}\right). \quad (1.1)$$

A locally observed mass is defined in a region where this function is slowly varying. The Great Attractor near the center of the Milky Way is observed, at a distance from the center of around $10^{16}$ cm, to have a value for $M$ that is several million solar masses (Ghez 2008). This is several orders of magnitude greater than “reasonable” physical models (Hartle 1978). The observation has been interpreted as being the effect of a black hole, with the horizon at about $10^{12}$ cm from the center.
Analysis of the distribution of velocities of orbiting stars show that the newtonian potential cannot be attributed to visible sources; the locally observed mass increases far too rapidly with the distance from the center. (Vioillier et al, 1993, Bilic et al 1998, Munyaneza et al, 1998, Gobar et al 2006, Ahmed et al 2009, Genzel et al 2010). Both problems can be qualitatively explained in terms of ‘dark matter’, the high value of $M(r)$ because the equation of state of dark matter is unknown and not subject to the physical constraints of known forms of matter, the unexpected variation of $M(r)$ with distance because dark matter may be present in regions that appear to be empty.

All that is known about dark matter is that it does not interact directly with ordinary matter. It does not interact with electrodynamics, it is not in thermal equilibrium with ordinary matter or with radiation; the temperature is therefore not observable and in consequence it is not defined. This puts the theoretician in the same position as the one that he confronts in hydrodynamics when the temperature is eliminated from the theory by means of the ideal gas equation and the equation of state reduces to a relation between density and pressure. The free energy density $f(T, \rho)$ is a function of density alone, the entropy density $s = \partial f/\partial T$ is zero and the pressure is

$$p = \rho \frac{d}{d\rho}f - f. \quad (1.2)$$

The system is thus determined by the expression chosen for the function $f(\rho)$; this expression, or the inferred relation between pressure and density, is the equation of state.

A simple, analytic expression for the newtonian potential accounts for the main features of the rotation curves of the outer region of our Galaxy. In this domain it is not necessary to invoke General Relativity, accordingly the equations of newtonian gravity are used to determine the equation of state. This equation of state turns out to have an interesting relation to equations of state proposed long ago in connection with the degenerate Fermi gas. The “sinh-Emden equation” for the potential

$$\Delta \mu = a \sinh^4 \mu, \quad a \text{ constant},$$

is solved exactly.

Once an equation of state has been determined it is natural to assume that it is valid everywhere, but this assumption is contradicted by observation; it predicts a velocity for the inner orbiters of the Milky Way that is too slow by almost one order of magnitude. We propose to interpret this as evidence for 2 types of Dark Matter or, alternatively as a change of phase.

A most fascinating problem is the interpretation of the heavy object in the center of the Galaxy. Attempts to interpret the data in terms of a Black Hole are interesting but if taken literally lack experimental support since there is no evidence as yet of the existence of an event horizon. The Schwarzschild Black Hole is a solution of Einstein’s equations FOR EMPTY SPACE. * We shall show that there are well behaved solutions

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* That is, a space that is empty in the region outside the horizon.
of Einstein’s equations that account for the existence of very large concentrations of galactic masses and that do not have a horizon.

This paper presents a simple equation of state that can account for large masses and for strong gravitational fields in the inner region of galaxies. We have used the same type of equation of state that was used for the outer region of the Milky Way, exploring a wide range of the values of the parameters.

Since a Schwarzschild Black Hole metric is a solution for empty space one expects that the actual metric, given the presence of matter, can be quite different. We have at hand a family of models (equations of state) for the description of an object with adjustable mass and size. As the size is reduced the outer, nearly empty region in which the Schwarzschild expression for the metric is approximately valid, extends further inwards, eventually so far as to reach a distance from the center of the order of the Schwarzschild radius, at which point a horizon might be expected to appear.

But no horizon was found, even when the size of the galaxy was reduced by 4 orders of magnitude, the mass fixed. The solutions are regular all the way to the center and no inner structure appears.

Notation and some data. The radius of the Milky Way is about $R = 10^{23} \text{cm}$ and the mass is about $2MG = 2 \times 10^{17} \text{cm}$. This measure of the mass of a star, in units of centimeters, is the value of the nominal Schwarzschild radius. The innermost, observed satellite has a nearest approach of about $2 \times 10^{15} \text{cm}$ and it moves in the newtonian field of a mass of about $2MG = 10^{12} \text{cm}$. The cgs system is used throughout; $1kpc = 3 \times 10^{21} \text{cm}$ and $8\pi G = 1.863 \times 10^{-27} \text{cm}/g$.

Summary

Section 2. Knowledge of the equation of state gives information about the nature of Dark Matter that is not available by other means. To properly interpret the evidence we begin with a discussion of equations of state in the context of General Relativity. The only thing that is known about the nature of Black Matter is that it is difficult to obtain information about it; we interpret this fact as evidence that it has very little to tell us, that the entropy is negligible. With this insight the observed orbital velocities can be directly related to an equation of state for Dark Matter. This equation of state is reminiscent of one proposed long ago for a degenerate Fermi gas.

Sections 3 and 4. Our model for the gravitational potential in the outer region, in the approximation in which the contribution of visible matter is ignored, is

$$\frac{2MG}{r} = -\phi(r) = k \ln \frac{r + b}{r}, \quad k, b \text{ constants.}$$

The equations of motion include the hydrostatic condition in integrated form,

$$\frac{\epsilon^2}{2} \phi = \frac{df}{d\rho},$$

where $f$ is the free energy density. Einstein’s equations give a unique equation of state represented parameterically as follows

$$f(\rho) = B\psi \sinh^4 \psi - p, \quad \rho = A \sinh^4 \psi, \quad p = B \int \sinh^4 \psi d\psi,$$
Section 5. Using the same equation of state, and the full structure of General Relativity, we explore the inner region of a star or a galaxy that consists of Dark Matter. Using the same equation of state, with the parameters $b$ (the size) and $kb$ (the mass) we solve the equations of General Relativity numerically. Fine tuning the initial values (at a large distance) we run the computer calculations all the way between a distance of 30 kiloparsecs and the center, literally to a distance of 1 cm from the center. The main conclusion is that the presence of the matter prevents the appearance of a horizon.

Section 6. Conclusions and suggestions

2. On equations of state in stellar models

Theories of stellar structure and evolution must postulate some properties of the matter that makes up the star. To be precise, what is needed is a relation between density and pressure. In the early models of Lane (1870), Ritter (1870-1880), Emden (1907) and Eddington (1926) the polytropic relation was used,

$$p = a\rho^\gamma, \quad \gamma = 1 + \frac{1}{n},$$  \hspace{1cm} (2.1)

$a$ and $n$ constants. Since these are relations taken from thermodynamics one has a right to inquire about the temperature, it was obtained from the ideal gas law

$$p = \mathcal{R}\rho T.$$  \hspace{1cm} (2.2)

Together, the two relations imply that $\rho/T^n$ is constant. The fact that stellar matter seems to behave like an ideal gas was noted with amazement by Eddington. But this approach, its apparent naivitè notwithstanding, is characterized by strong internal coherence; it may serve as a paradigm for parallel developments.

In the first place, since the temperature is relevant to understanding the phenomena, it is not simply a question of the hydrodynamic relation between $p$ and $\rho$, one needs both of the relations that characterize a simple fluid or, alternatively, an expression for a thermodynamic potential. In the case of an ideal gas of particles with adiabatic index $n$, the free energy density is expressed in terms of its natural variables as

$$f(T, \rho) = \mathcal{R}\rho T \ln \frac{\rho}{T^n}.$$  \hspace{1cm} (2.3)

The entropy density is defined as

$$s = -\frac{\partial f}{\partial T} = -\mathcal{R}\rho (\ln \frac{\rho}{T^n} - n).$$  \hspace{1cm} (2.4)
That is: using the polytropic relation (2.1) with a constant implies that the specific entropy density \( s/\rho \) is a constant throughout.

The thermodynamic pressure and the chemical potential are defined by

\[
p(\mu, T) = \rho \frac{\partial f}{\partial \rho} \bigg|_T - f(\rho, T), \quad \mu := \frac{\partial f(\rho, T)}{\partial \rho}.
\]

The pressure is thus a thermodynamic potential defined by a Legendre transformation; the natural variables are \( T \) and the chemical potential \( \mu \), with

\[
\frac{\partial p}{\partial T} \bigg|_\mu = s, \quad \frac{\partial p}{\partial \mu} \bigg|_T = \rho.
\]

In the case of an ideal gas this gives the ideal gas law

\[
p = \mathcal{R}_n \rho T, \quad \mu = \mathcal{R} T \exp(\ln \frac{\rho}{T_n} + 1).
\]

and

\[
p(T, \mu) = \mathcal{R} T^{n+1} \exp\left(\frac{\mu}{\mathcal{R} T_n} - 1\right).
\]

The Gibbsean action principle (minimum energy) leads to the following equation of motion,

\[
\frac{\partial}{\partial \rho} [\phi \rho + f(T, \rho)] = \text{constant.} \tag{2.5}
\]

where \( \phi \) is the gravitational potential. By making use of the relations (2.1) and (2.2), or more generally by postulating that the specific entropy density is uniform, one can derive the famous hydrostatic condition

\[
\rho \, \text{grad} \phi + \text{grad} p = 0.
\]

The necessary condition on the entropy reflects the fact that no energy source is being taken into account. The problem of incorporating the effect of radiation is solved in the Eddington approximation by adding the Stefan-Boltzmann term \( \dot{a} T^4 \) to the (free) energy, which generates the addition of \( (\dot{a}/3)T^4 \) to the pressure. But we shall leave that refinement aside in our discussion.

The usual system of equations that is used to describe a stellar atmosphere consists of a relation between \( p \) and \( \rho \), with the continuity equation and the hydrostatic condition and, eventually, Einstein’s equations. The relation between \( p \) and \( \rho \) must be interpreted as the result of the elimination of \( T \) in favor of the entropy, which is usually taken to be uniform. If the entropy is not specified the eventual success of an equation of state, as usually understood to mean a relation between \( p \) and \( \rho \), can have many different thermodynamical interpretations.

For example, given any expression \( p(\rho) \), one is free to postulate that the entropy is zero, implying that the free energy is independent of the temperature. In this paper we
shall assume that the entropy of dark matter is zero; that goes a long way towards explaining why it is difficult to establish any communication with dark matter, other than via the intervention of gravity; it has no information to give us. But we stress that the equation of state that we shall use may have other interpretations; interpretations that can be distinguished when we gain some independent knowledge about the temperature or the entropy of dark matter.

The search for an understanding of very dense stars led Chandrasekhar (1935), following a suggestion by Stoner, to an equation of state for partly or fully degenerate Fermi gas. The relation between $p$ and $\rho$ is given in parametric form, the following expressions taken from Landau and Lifschitz (1959),

$$p = \frac{2}{3m^2} \frac{(kT)^{5/2}}{m^2} A \int_0^\infty \frac{\zeta^{3/2}d\zeta}{e^{\zeta-t}+1},$$

$$\rho = (kT)^{3/2} A \int_0^\infty \frac{\zeta^{1/2}d\zeta}{e^{\zeta-t}+1}, \quad A = \text{constant},$$

with the real parameter $t$. Since the natural variables of $p$ are $T$ and the chemical potential $\mu$, we expect that $t$ must be related to $\mu$ and indeed the identification

$$t = mc^2/2\hbar T$$

leads, after a rescaling of the integration variable, to the formulas

$$p(\mu, T) = \frac{2}{3} B \int \frac{\zeta^{3/2}d\zeta}{e^{\frac{\mu}{kT}(\zeta-\mu)}+1},$$

$$\rho(\mu, T) = B \int \frac{\zeta^{1/2}d\zeta}{e^{\frac{\mu}{kT}(\zeta-\mu)}+1} d\zeta = \frac{\partial p}{\partial \mu} \bigg|_T.$$  

where $B$ is a constant. These expressions are in Chandrasekhar (1957) page 400. The first relation is enough to define the thermodynamical system.

To use this equation of state in the context of stellar atmospheres more information is required. One could, for example, fix the temperature.* More in line with the earlier work based on the ideal gas would be to fix the entropy density

$$s = \frac{\partial p}{\partial T} \bigg|_\mu.$$  

Numerical calculations show that the entropy is in fact very low for temperatures that are not extremely high; therefore, fixing the temperature is roughly equivalent to setting the entropy to zero.

Let us examine this theory in the degenerate limit that Chandrasekhar (1957, page 358) describes thus: “a completely degenerate electron gas is one in which all the lowest

* See for example the discussion in a footnote in Oppenheimer and Volkov (1939).
quantum states are occupied”. Taking the limit in this sense Chandrasekhar ends up with a hydrodynamical equation of state, also given parameterically, as follows*

\[ p(T, \mu) = \frac{B}{3} \int_0^{\theta_0} \sinh^4 \theta \, d\theta, \]

\[ \rho = B \sinh^3 \theta, \quad \cosh \theta_0 := \mu. \]

This is the result of an approximation that involves replacing the exponential term in the denominator by a cutoff at \( \zeta = \mu \). What is important here is that

\[ \frac{dp(\mu)}{d\mu} = \rho(\mu), \]

since it identifies the parameter \( \mu = \cosh \theta \) as the chemical potential. In this approximation the dependence of \( p \) on \( T \) is neglected; so that the entropy is effectively zero. In the next section we shall propose an equation of state that bears a strong resemblance to these relations:

\[ \rho(\mu) = A \sinh^4 \mu, \quad p(\mu) = A \int_0^\mu d\nu \sinh^4 \nu \, d\nu. \]

Here too, \( \mu \) is the chemical potential.

3. The equations of motion

We shall calculate static, spherically symmetric solutions of Einstein’s equations,

\[ G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}, \quad G = .7414 \times 10^{-28} \, \text{cm} \, \text{g}, \]

with a metric of the form

\[ ds^2 = e^{\nu}(c dt)^2 - e^\lambda dr^2 - r^2 d\Omega, \quad g_{00} = c^2 e^{\nu(r)}, \quad g_{rr} = -e^{\lambda(r)}, \]

and a matter energy momentum tensor of the form

\[ T_{00} = \rho U_0 U_0, \quad T_{rr} = p g_{rr}, \quad (3.1) \]

all other components zero. Besides Einstein’s equation we invoke the hydrostatic condition in integrated form\(^1\)

\[ \frac{c^2}{2} (e^{-\nu} - 1) = \frac{df}{d\rho}. \quad (3.2) \]

* Oppenheimer and Volkov used a variant of this equation of state in their pioneering work on neutron stars. We were not able to relate their parameter (“\( t \)”) to a chemical potential.

\(^1\) See Section 6.
The reduced form of Einstein’s equations given in the textbooks, beginning with that of Tolman (1934), is

\[ G_t^t = -e^{-\lambda} \left( \frac{-\lambda'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = 8\pi G \left( e^{-\nu} \rho - \frac{p}{c^2} \right), \]
\[ G_r^r = -e^{-\lambda} \left( \frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = -8\pi G \frac{p}{c^2}. \]

With the notation

\[ H(r) = e^{-\lambda} = 1 - \frac{m(r)}{r}, \quad K(r) = e^{\nu+\lambda} = 1 - \frac{u(r)}{r}, \quad HK = e^{\nu} = 1 - \frac{\phi}{r}; \]

they are

\[ H' = \frac{1 - H}{r} - 8\pi G r \rho \left( e^{-\nu} - \frac{p}{c^2} \rho \right), \]
\[ K' = 8\pi G H^{-2} r \rho. \]

To continue we need an expression for the free energy that will allow us to express the density and the pressure in terms of the fields, with the help of Eq.(3.2). To determine the free energy we shall work, provisionally, with the weak field approximation.

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1 The prime stands for differentiation with respect to \( r \).
2 The choice of the letter \( m \) in the first expression is traditional, but unfortunate, in as much the locally observed mass defined in (1.1) is \( 2M(r)G = m(r) + u(r) \). (See below.)
4. The equation of state

A weak field approximation will be used to determine an approximate equation of state. In this approximation we replace Eq.s (3.4) by

\[ m'(r) = r^2 w \rho, \]

(4.1)

\[ K'(r) = r w \rho. \]

(4.2)

The primes, as before, denote the derivative with respect to \( r \). The newtonian potential is \(-\phi/2\), where

\[ 1 - e^{-r} \approx (1 - H) + (1 - K) = \frac{m}{r} + \frac{u}{r} =: \phi. \]

Combining (3.1-2) we get

\[ m = -r^2 \phi', \quad w \rho = m'/r^2 \]

The full set of equations is thus

\[ \frac{c^2}{2} \phi = \frac{\partial f}{\partial \rho}, \quad -r^{-2}(r^2 \phi')' = w \rho. \]

(4.3)

Something is known about \( \phi \), from observation of radial acceleration of orbiting stars. A family of satellites moving in circular orbits with radius \( r \) in a radial, newtonian potential \( V \) have orbital speed \( v \) given by \( v^2 = r V' \). Observation has revealed that there is a wide interval in which the speed is nearly constant, independent of the distance, which implies that, in this interval, the potential is approximated by \( V = -\phi/2 = (k/2) \ln(r) \). We shall model the function \( \phi(r) \), then calculate the equation of state. In other words, when the distribution \( \phi(r) \) is known from observation, then the last pair of equations provides a parametric representation of the relation between the free energy density \( f \) and the density \( \rho \). Finally we shall use this equation of state in the exact, relativistic field equations.

Example

Taking

\[ \phi = k \ln \frac{r + b}{r}, \quad r = e^x, \]

we obtain the velocity distribution shown in Fig.1 with \( k = 10^{-6}, b = e^{54} \). It is very nearly constant for \( x < 52 \) and very nearly newtonian outside. In the flat region the value is

\[ \frac{v}{c} = \sqrt{-\frac{1}{2} r \phi'(r)} \approx \sqrt{-k/2}, \quad v = 212 \text{ km/sec}. \]

The value of the constant \( k \) is thus determined by observation of the orbital velocities.

In the region of the innermost orbiters \( r \approx r_0 = 10^{16} \) and the local mass is

\[ 2mG = r \phi = r_0 k \ln \frac{r_0 + b}{r_0}. \]
With $b = e^{54} = 2.83 \times 10^{23}$ we obtain $2mG = 1.7 \times 10^{11} cm$ which is close enough to the observed value $10^{12} cm$.

![Graph](image)

Fig.1. The orbital velocity distribution that was used as a model of the observations in the Milky way, for $k = 1.044 \times 10^{-6}$. The velocity is shown in km/sec.

From (3.4) we get

$$m(r) = -r^2 \phi' = kb \frac{r}{r+b} \quad (= kb - \frac{kb}{r} + \ldots, \quad r > b),$$

and the density

$$w\rho(r) = -\Delta \phi = -r^{-2}(r^2 \phi')' = \frac{k}{r^2(1 + r/b)^2}.$$  \hspace{1cm} (4.4)

The parametric representation of the equation of state is thus, in this case,

$$\frac{2}{c^2} \frac{\partial f}{\partial \rho} = \phi(t) = k \ln \frac{t+b}{t}, \quad w\rho(t) = \frac{k}{t^2(1 + t/b)^2}.$$  \hspace{1cm} (4.5)

Equivalently,

$$\rho = \frac{16}{b^2} w k \sinh^4 \psi, \quad \psi := \phi/2k.$$  \hspace{1cm} (4.5)

The free energy is obtained by integrating the hydrostatic equation,

$$\frac{2}{c^2} \frac{df}{d\rho} = \phi, \quad \frac{2}{c^2} \frac{df}{d\psi} = \phi \frac{d\rho}{d\psi} = 2k \left( \frac{d}{d\psi} (\psi \rho) - \rho \right).$$

Thus

$$\hat{f}(\rho) = \frac{b^2 w}{16c^2 k^2} f(\rho) = \int \psi \frac{d}{d\psi} \sinh^4 \psi d\psi = \psi \sinh^4 \psi - \hat{\rho}.$$  \hspace{1cm} (4.6)

The last term,

$$\hat{\rho} := \int \sinh^4 \psi d\psi = \frac{1}{32} \left( \sinh(4\psi) - 8 \sinh(2\psi) + 12 \psi \right),$$  \hspace{1cm} (4.6)
is the pressure,
\[ p = \frac{16 k^2 c^2}{b^2 \omega} \tilde{p} = \rho \frac{\partial}{\partial \rho} f - f. \]  
(4.7)

**Remark.** The function
\[ \psi = \frac{1}{2} \ln(1 + b/r) \]
is an exact solution of the modified Emden equation (4.4)
\[ \Delta \psi + \frac{8}{b^2} \sinh^4 \psi = 0, \quad \Delta = r^{-2} \frac{d}{dr} r^2 \frac{d}{dr}. \]
The original Emden equation has \( \psi^n \) instead of \( \sinh^4 \psi \); it has an exact solution in the case that \( n = 5 \) only.

Returning to the equations (3.1-2) - General Relativity - we now apply the equation of state in the form (4.5-6), fix the appropriate initial values,
\[ x = a = 52, \quad m(a) = \frac{bk}{2} = .2 \times 10^{17}, \quad k = 1.044 \times 10^{-6}, \quad K(a) = 1 - k \ln 2 + \frac{k}{2}, \quad \phi = k \ln 2 \]
and run the equations in Mathematica. The program runs from \( x = 52(r = 4 \times 10^{22}) \) in both directions, covering 36 orders of magnitude of the radius, correctly reproducing the exact solution (3.4) from \( x = 0 \) to \( x = 83 \) (\( r = 0 \) to \( r = 10^{36} \) cm).

5. Solutions of the relativistic equations of motion

It is customary, whenever it is difficult to construct a physically reasonable model of a stellar object, to conclude that one must be dealing with a Black Hole, which in the astrophysical context, if taken literally, means a Schwarzschild Black Hole. It is a conjecture that is difficult to verify, since it is impossible to receive information from the horizon, let alone from beyond. And then there is the very thorny issue of the creation of a Black Hole, since it is difficult to describe a reasonable scenario of matter “falling into” it. But the main difficulty is that a Black Hole is a property of empty space.

All “theorems” that claim to limit the size of any kind of stellar object rely on some assumption about the equation of state. Early work was entirely based on polytropes that are notorious for unphysical properties. Already the work of Emden, 100 years ago, show that no choice of the polytropic index is satisfactory, either because the mass is infinite or because the density (or the pressure) turns negative at a finite distance from the center. The impracticality of carrying out a thorough program of numerical calculations was a limiting factor 50 years ago as is exemplified by the well known paper by Oppenheimer and Volkov (1939), the first attempt to use a much more sophisticated equation of state. And even today it is difficult to determine the initial conditions that
characterize the very elusive global solutions. This paper adopts an equation of state that resembles that of Oppenheimer and Volkov (1939), especially appropriate for Dark Matter. A powerful computer, and a novel strategy for discovering the elusive initial values that are needed for numerical calculations allow us to do calculations that have not been feasible in the past. What follows is the result of a study of thousands of solutions.

It is not difficult to find equations of state that give rise to regular, generally relativistic gas spheres of very high mass. In this paper we are making use of the fact that the equation of state of Dark Matter is not known from other evidence. The result is of general interest, for if we can visualize a reasonable object consisting mostly of Dark Matter then there should be no difficulty in imagining that ordinary matter can stick to it, or fall into it.

Imagine traveling towards the center of an object of a certain mass $M$. This mass is determined by measuring the metric at great distances from the center of the object, where to a good approximation the density of matter is zero. The mass $M$ determines the Schwarzschild radius. In the case of a galaxy like our own this radius is $10^{16} - 10^{17}$ cm. But here we are closer to the center than the greater part of the mass. Measuring the metric at this neighborhood we find that the Schwarzschild radius has shrunk to $10^{11}$ cm. And if we should have any prospect of finding a horizon at that place then there must be little or no matter in the interval.

Given a value of the total mass, to encounter a phenomenon that resembles a horizon we must arrange for most of the mass to be squeezed into a configuration that has a very low density everywhere except for a small region very close to the center. A sequence of configurations, of increasing concentration near the center, is provided by our equation of state: by keeping the total mass $bk/2$ fixed and gradually decreasing the effective radius ($b$). Once we have a family of models of stars with a fixed total mass, indexed by a parameter that controls the effective size, then we have a tool for trying to understand if and how a star can acquire a horizon. This sequence is described in detail in Subsection 5a.

Every effort to understand stellar evolution describes evolution, a relatively slow process, as a progression of adiabatic equilibria, configurations that are equilibria in the short term. Without describing the dynamics of evolution we propose to form an evolutionary sequence from our family of static models. The principal constraint is that each adiabatic equilibrium configuration must be a stationary solution of Einstein’s equations. Now the successive equilibria have to be plausibly linked by evolution, and that brings us to the important question of mass and conservation laws.

Although we shall not attempt to describe the evolutionary process, it is possible to speculate about the behavior of the principal parameters of the evolving stellar object. Thus it seems at first quite natural that the total mass remain unchanged. An argument can be made to show that the mass is likely to diminish, and with some cost of plausibility one can imagine conditions under which it may be increasing. The neutral point among these contrasting scenarios is the case that the mass remains fixed during the evolution; this will serve as a natural bench mark and it is natural to investigate this possibility first.
But why focus on the mass? And what exactly is the mass? If the change of some quantity is characteristic of evolution, then it must be conserved by the adiabatic dynamics.

Tolman’s phenomenological approach (Tolman 1934) incorporates a density that is tentatively interpreted either as a density of mass or a density of particles, but this density is not conserved. Indeed, the theory abandons one of the two principal features of hydrodynamics. The total mass, as defined asymptotically by the limiting Schwarzschild metric can be expressed, in Tolman’s theory as

\[ M = 4\pi \int_0^\infty m(r)r^2 dr, \]

where the function \( m(r) \) is defined as

\[ \frac{m(r)}{r} = \frac{1}{2} \left( \frac{1}{g_{rr}} - 1 \right), \]

in terms of the radial component of the metric. The identification of this function as density is common, although, as remarked by Kippenhahn and Weigert (1990) and others, it is hazardous, since its transformation properties are not correct.

In this paper we are using an approach to Relativistic Thermodynamics that, at first sight, and in a restricted context, is indistinguishable from Tolman’s theory. It is based on Gibb’s thermodynamical action principle and incorporates hydrodynamics with a conserved density. In the newtonian approximation it is possible to interpret this as a mass density, but in the relativistic context we have adopted another use of the word “mass”, and there is no compelling reason to relate the two concepts. We shall abandon the concept of “mass density” and we shall refer to the conserved density \( \rho \) as the particle number density. In the case of the model galaxy, with parameters \( b = 52 \) (radius \( R = 3.8 \times 10^{22} \) cm) and total mass \( bk/2 = 2 \times 10^{16} \) cm, we find

\[ N := 2Ge^2 \int \rho g^{00} \sqrt{-g} r^2 dr \rho = 1.999 \times 10^{16} \text{cm}. \]

The close agreement of this number with the gravitational mass (when \( \ln b = 52 \)) is due to the fact that the newtonian approximation is valid in the greater part of the galaxy.

As we reduce the dimension (the value of the parameter \( b \)), we now have to reduce \( k \) as well, to keep constant the number of particles. The sequence that results from this procedure is described in Subsection 5b.

5a. A sequence with fixed Schwarzschild mass

We need the full dynamics of General Relativity; the equations were summarized in Section 2. To begin, we solve the non relativistic problem and note the initial values at \( x = 52 \). This gives us initial values to be used in Einstein’s equations, otherwise very difficult to determine. No assumptions are made initially concerning the behavior of the

\[^3\text{Introduction, Eq. (1.1)}.\]
solution near the center, for all experimental information pertains to the outer region. Once it has been established that the computation can be extended to \( r = 0 \) we can start at either end.

To reach the center it is necessary to fine tune the initial values to more than 10 significant digits, although the adjustment is tiny, never more than 1 percent and in most cases less than one part in 10,000. With a fair amount of labor we succeeded in running from \( r = e^{52} \) cm or more to within 1 cm of the center and outward to about \( e^{70} \).

We started with the parameters that fit the Milky Way and fixed the mass \( kb/2 \). With initial value \( \ln b = 52(R = 3.8 \times 10^{22}) \) we reduced this parameter by steps to \( \ln b = 40(R = 2.4 \times 10^{17}) \), reducing the radius of the model galaxy by 5 orders of magnitude. Here is what we found.

In an outer region the newtonian approximation is surprisingly accurate and the parameters \( b \) and \( k \) determine, via the sinh-Emden equation, an essentially unique solution. As the mass is compressed, by reduction of the extent (radius) \( b \) with the mass \( bk/2 \) kept fixed, this region grows. That is, the density of mass (particles) in the outer region, the region that is accessible to observation, becomes very small, and this region grows.

![Fig.2](chart.png)

Fig.2. The metric function \( K - 1 \), obtained by integrating from the center outwards, with different initial values. From \( x = 30 \) the solutions are indistinguishable. In this example \( \ln b = 52 \) and \( m(0) = 0 \).

In an inner, complimentary region there is a great disparity between solutions. Fig. 2 shows a typical example of the confluence of two solutions that start off very different near the center. Conversely, when a solution is extended inwards its course cannot be predicted, for it depends on extremely minute fine tuning of the initial values of \( m \) and \( K \).

A behavior hinting at the development of a Black Hole would be a rise in the value of the potential \( \phi \) near the Schwarzschild radius. No tendency towards such development was observed. Fig. 3 illustrates the dramatic rise in the values of the potential, by a factor of 500, as \( \ln b \) is reduced from 52 to 46.
Fig. 3. The metric function $\phi$ (the gravitational potential) in the case that $\ln b = 46$ (upper curve) compared with $100\phi$ in the case that $\ln b = 52$ (lower curve).

All integration was first done from the outside in, then the final values of the metric functions at $x = 0$ were used as initial values for the reverse run along an identical track.

The orbital velocity changes in an interesting manner, as illustrated in Fig. 4 in the case $\ln b = 46 (R = 10^{20})$. The high velocities that were introduced by the choice of equation of state last only as far as $x = 20 (R = 5 \times 10^8)$, but return to high values very near the center. This behavior of the orbital velocity suggests that the very high velocities of the inner orbiters of the Milky Way may be attributed to General Relativity, in a model with a more refined equation of state.

Fig. 4. The orbital velocity in the case $b = 46$. The horizontal line is the result of the newtonian model. The drop at around $x = 20$ is characteristic, the increase very close to the center is not - it depends on the initial values and it may be spurious.

Finally there is the important question of the density. Fig. 5 shows the profile in a log-log presentation (natural logarithms). In the innermost region the density is high, about $10^7 g/cm^3$. But at $x = 30$, which is far closer than the inner orbiters, it is $5 \times 10^{-6}$.

As we squeeze the mass into a smaller region the density near the center grows, we obtain density profiles that vary greatly with the boundary conditions at the center. The densities can get very high.
The first column of the following table give some values of \(N\) in the case of the galactic model with \(b = 52\) and Schwarzshild mass \(bk/2 = 2 \times 10^{16}\text{ cm}\). The integration runs from \(x = 0\) to \(x = X\). Each of the other columns show the result of reducing the size, to \(R = e^{50}, e^{48}, e^{46}, e^{44}\text{ cm}\), respectively.

\[
\begin{array}{cccccccc}
X & \ln b = 52 & \ln b = 50 & \ln b = 48 & \ln b = 46 & \ln b = 44 & \ln b = 42 & \ln b = 40 \\
52 & 2 \times 10^{16} & 3.5 \times 10^{16} & 3.9 \times 10^{16} & 4.0 \times 10^{16} & 4 \times 10^{16} & 8.3 \times 10^{17} & 9.3 \times 10^{17} \\
48 & 7.2 \times 10^{14} & 4.8 \times 10^{15} & 2.0 \times 10^{16} & 3.5 \times 10^{16} & 3.9 \times 10^{16} & 4.5 \times 10^{16} & 4.4 \times 10^{16} \\
44 & 1.3 \times 10^{13} & 9.9 \times 10^{13} & 7.2 \times 10^{14} & 4.8 \times 10^{15} & 2.0 \times 10^{16} & 3.2 \times 10^{16} & 3.4 \times 10^{16} \\
40 & 2.5 \times 10^{11} & 1.8 \times 10^{12} & 1.3 \times 10^{13} & 9.8 \times 10^{13} & 7.1 \times 10^{14} & 4.4 \times 10^{15} & 4.1 \times 10^{15} \\
\end{array}
\]

All our solutions were checked by integrating from the center outwards as well as inward from a large distance. The last column gives an indication of why further reduction of size (beyond \(\ln b = 44\)) is not possible. All attempts to reduce \(b\) below \(e^{40}, (R = 2 \times 10^{17})\) were unfruitful.

The main conclusion is that (this model of) the Milky Way can contract by at least five orders of magnitude of linear dimension without the need to develop a horizon.

5b. An evolutionary sequence

With the same starting point, at \(b = 52\) and \(bk = 4 \times 10^{16}\), we now reduce both parameters while holding the number \(N\) fixed at \(x = \ln r = 52, N(52) = 2.5 \times 10^{17}\text{ cm}\). The second table shows a significant departure from the first one. The “mass”, in cm, is the \(2G\) times the Schwarzshild or asymptotic mass.

\[
\begin{array}{cccccccc}
X & \ln b = 52 & \ln b = 50 & \ln b = 48 & \ln b = 46 & \ln b = 44 & \ln b = 42 \\
52 & 2.0 \times 10^{16} & 2 \times 10^{16} & 2 \times 10^{16} & 2 \times 10^{16} & 2 \times 10^{16} & 2 \times 10^{16} \\
48 & 7.2 \times 10^{14} & 2.7 \times 10^{15} & 1.0 \times 10^{16} & 1.8 \times 10^{16} & 1.9 \times 10^{16} & 8.5 \times 10^{15} \\
44 & 1.3 \times 10^{13} & 5.7 \times 10^{13} & 3.7 \times 10^{14} & 2.4 \times 10^{15} & 1.0 \times 10^{16} & 7.4 \times 10^{15} \\
40 & 2.5 \times 10^{11} & 1.0 \times 10^{12} & 6.8 \times 10^{13} & 4.9 \times 10^{13} & 3.5 \times 10^{14} & 9.8 \times 10^{14} \\
\end{array}
\]

\[
\begin{array}{cccccccc}
Mass & 2 \times 10^{16} & 1.15 \times 10^{16} & 1.01 \times 10^{16} & 1.0 \times 10^{16} & 1.0 \times 10^{16} & 4.2 \times 10^{15} \\
\end{array}
\]

16
The most remarkable feature is that, if the current conservation law is maintained by the evolutionary dynamics, then the mass, the observed Schwarzschild or asymptotic mass of the galaxy, will be reduced by a factor of 2 as the size of the galaxy is reduced by about 1 order of magnitude. It is not sure that the number of particles is preserved during evolution, but we do not expect it to increase; the estimate of a factor of 2 is thus a lower limit.

6. Conclusions

In this paper we have sought to determine a principal characteristic of the distribution of dark matter that is believed to be largely responsible for the observed distribution of stellar velocities within our galaxy.

An equation of state in hydrodynamics is a relation between density and pressure. The contribution of the pressure to Einstein's equation is not as important as the role that is played by the pressure in the hydrostatic equation. In our approach the equation of state is expressed by the free energy density. It is completely phenomenological, extracted from experimental data. It is very reminiscent of the degenerate fermion equation of state proposed by Chandrasekhar in another context. Interpretation of the data thus suggests a similar physical model for Dark Matter. It is in the nature of the problem that the basic physics is not known, it must be discovered by interpretation of the experiments.

Ultimately, the model of our galaxy must be improved by including the contribution of visible matter. *

The shape of the dark matter distribution is not known, but to make a start the overall structure was studied in terms of an idealized, continuous, spherically symmetric distribution that makes an important contribution to the source of the gravitational field. We used the observed data in the outer region to determine an equation of state. We assumed that the same type of dark matter is dominant in the central region, but this is inconsistent with observations in a small region near the galactic center, generally associated with the compact radio source Sgr A*. The stellar velocities observed in this region do not conform to the predictions of our model in its present form. We interpret this as evidence for a second type of dark matter in the central region, or as the result of a phase change taking place at high pressures.

In a subsequent paper we hope to report on an attempt to account for all the data by a two-component model of Dark Matter. In addition we try to find an efficient way to include the visible component with its characteristic, disk-like shape.

* The fact that there is no interaction between dark and visible matter will make this very straightforward; for in this case the free energy density is additive. Note that it is essential to recognize the roles of two quite different density fields. In this paper the contribution of the visible component is not taken into account.
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