SOME DONALDSON INVARIANTS OF CP²

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In memory of the victims of the Kobe earthquake

INTRODUCTION

For an integer \( n \geq 2 \), let \( q_{4n-3} \) be the coefficient of the Donaldson polynomial of degree \( 4n - 3 \) of \( P = \mathbb{CP}² \). An interpretation of \( q_{4n-3} \) in an algebro-geometric context is the following. Let \( M_n \) denote the Gieseker-Maruyama moduli space of semistable coherent sheaves on \( P \) with rank 2 and Chern classes \( c_1 = 0 \) and \( c_2 = n \). For such a sheaf \( F \), the Grauert-Mülich theorem implies that the restriction of \( F \) to a general line \( L \subseteq P \) splits as \( F_L \cong \mathcal{O}_L \oplus \mathcal{O}_L \), and that the exceptional lines form a curve \( J(F) \) of degree \( n \) in the dual projective plane \( P^\vee \). The association \( F \mapsto J(F) \) is induced from a morphism of algebraic varieties, called the Barth map, \( f_n : M_n \to P_n \). Here \( P_n = \mathbb{P}^{n(n+3)/2} \) is the linear system parameterizing all curves of degree \( n \) in \( P^\vee \). Let \( H \in \text{Pic}(P_n) \) be the hyperplane class and let \( \alpha = f_n^*H \). The interpretation of the Donaldson invariant is:

\[
q_{4n-3} = \int_{M_n} \alpha^{4n-3}.
\]

Thus \( q_{4n-3} \) is the degree of \( f_n \) times the degree of its image. From [4] it follows that \( f_n \) is generically finite for all \( n \geq 2 \), that \( f_2 \) is an isomorphism and \( q_5 = 1 \), and that \( f_3 \) is of degree 3 and \( q_9 = 3 \). Le Potier [8] proved that \( f_4 \) is birational onto its image and that \( q_{13} = 54 \). The value of \( q_{13} \) has also been computed independently by Tikhomirov and Tyurin [5, prop. 4.1] and by Li and Qin [6, thm. 6.29].

The main result in the present note is the following

**Theorem 0.1.** \( q_{17} = 2540 \) and \( q_{21} = 233208 \).

The proof consists of two parts. The first part, treated in this note, is to express \( q_{4n-3} \) in terms of certain classes on the Hilbert scheme of length-(\( n + 1 \)) subschemes of \( P \). This is theorems 0.2 and 0.3 below.

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The second part is to evaluate these classes numerically. This has been carried out in [4, prop. 4.2].

Let $H_{n+1} = \text{Hilb}_{n+1}^{-1}$ denote the Hilbert scheme parameterizing closed subschemes of $P$ of length $n+1$. There is a universal closed subscheme $Z \subseteq H_{n+1} \times P$. Consider the vector bundles

$$E = R^1 p_1^*(\mathcal{I}_Z \otimes p_2^* \mathcal{O}_P(-1))$$

and

$$G = R^1 p_1^* \mathcal{I}_Z$$

on $H_{n+1}$ of ranks $n+1$ and $n$, respectively, and the linebundle

$$L = \text{det}(G) \otimes \text{det}(E)^{-1}.$$

**Theorem 0.2.** Let the notation be as above. Then

$$q_{17} = \int_{H_6} s_{12}(E \otimes L) \quad \text{and} \quad q_{21} = \frac{2}{5} \int_{H_7} s_{14}(E \otimes L).$$

This result was obtained both by Tikhomirov and Tyurin [12], using the method of “geometric approximation procedure” and by Le Potier [7], using “coherent systems”. We present in this note what we believe is a considerably simplified proof, which is strongly hinted at on the last few pages of [12].

The formula for $q_{17}$ is a special case of the following formula:

**Theorem 0.3.** For $2 \leq n \leq 5$, we have

$$q_{4n-3} = \frac{1}{2^{5-n}} \int_{H_{n+1}} c_1(L)^{5-n} s_{3n-3}(E \otimes L).$$

With this it is also easy to recompute $q_5$, $q_9$, and $q_{13}$ using similar techniques as in [4].

**Notation**. We let $h$, $h^\vee$, and $H$ be the hyperplane classes in $P$, $P^\vee$, and $P_n$, respectively. In general, if $\omega$ is a divisor class, we denote by $\mathcal{O}''(\omega)$ the corresponding linebundle and its natural pullbacks.

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1. **Hulsbergen sheaves**

Barth [4] used the term Hulsbergen bundle to denote a stable rank-2 vector bundle $F$ on $P$ with $c_1(F) = 0$ and $H^0(P, F(1)) \neq 0$. We modify this definition a little as follows:
Definition 1.1. A Hulsbergen sheaf is a coherent sheaf $F$ on $P$ which admits a non-split short exact sequence (Hulsbergen sequence)

$$0 \rightarrow \mathcal{O}_P \rightarrow F(1) \rightarrow \mathcal{I}_Z(2) \rightarrow 0,$$

(1.1)

where $Z \subseteq P$ is a closed subscheme of finite length (equal to $c_2(F) + 1$).

Note that a Hulsbergen sheaf is not necessarily semistable or locally free. However:

Lemma 1.2. Let $F$ be a Hulsbergen sheaf with $c_2(F) = n > 0$. Then the set $J(F) \subseteq P^\vee$ of exceptional lines for $F$ is a curve of degree $n$, defined by the determinant of the bundle map

$$m: H^1(P, F(-2)) \otimes \mathcal{O}_{P^\vee}(-1) \rightarrow H^1(P, F(-1)) \otimes \mathcal{O}_{P^\vee}$$

induced by multiplication with a variable linear form.

Proof. First note from the Hulsbergen sequence that the two cohomology groups have dimension $n$. It is easy to see that any Hulsbergen sheaf is slope semistable, in the sense that it does not contain any rank-1 subsheaf with positive first Chern class. Thus by [2, thm. 1], $F_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L$ for a general line $L$. On the other hand, it is clear that a line $L$ is exceptional if and only if $m$ is not an isomorphism at the point $[L] \in P^\vee$. 

It is straightforward to construct a moduli space for Hulsbergen sequences. For any length-$(n+1)$ subscheme $Z \subseteq P$, the isomorphism classes of extensions $(1.1)$ are parameterized by $\mathbf{P}(\text{Ext}^1_P(\mathcal{I}_Z(2), \mathcal{O}_P)^\vee)$. By Serre duality,

$$\text{Ext}^1_P(\mathcal{I}_Z(2), \mathcal{O}_P)^\vee \simeq H^1(P, \mathcal{I}_Z(-1)).$$

For varying $Z$, these vector spaces glue together to form the vector bundle $\mathcal{E}$ over $H_{n+1}$, hence $D_n = \mathbf{P}(\mathcal{E})$ is the natural parameter space for Hulsbergen sequences. Let $\mathcal{O}(\tau)$ be the associated tautological quotient linebundle. For later use, note that for any divisor class $\omega$ on $H_{n+1}$, we have $\pi_*(\tau + \pi^*\omega)^{k+n} = s_k(\mathcal{E}(\omega))$, where $\pi: D_n \rightarrow H_{n+1}$ is the natural map [3].

The tautological quotient $\pi^*\mathcal{E} \rightarrow \mathcal{O}(\tau)$ gives rise to a short exact sequence on $D_n \times P$:

$$0 \rightarrow \mathcal{O}(\tau) \rightarrow \mathcal{F}(h) \rightarrow (\pi \times 1)^*\mathcal{I}_Z(2h) \rightarrow 0$$

which defines a complete family $\mathcal{F}$ of Hulsbergen sheaves.

As we noted earlier, a Hulsbergen sheaf is not necessarily semistable. On the other hand, the generic Hulsbergen sheaf is stable if $n \geq 2$. It follows that the family $\mathcal{F}$ induces a rational map $g_n: D_n \rightarrow M_n$. By
lemma 1.2 above, there is also a Barth map $b_n: D_n \to P_n$, defined everywhere, and by construction, the following diagram commutes:

\[
\begin{array}{ccc}
D_n & \xrightarrow{b_n} & P_n \\
\downarrow{g_n} & & \downarrow{||} \\
M_n & \xrightarrow{f_n} & P_n
\end{array}
\]

(1.2)

**Proposition 1.3.** Put $\lambda = c_1(\pi^*L)$. Then $b_n^*H = \tau + \lambda$.

**Proof.** Let $L \subseteq P$ be a line. Twist the universal Hulsbergen sequence by $-2h$ and $-3h$ respectively. Multiplication by an equation for $L$ gives rise to the vertical arrows in a commutative diagram with exact rows on $D_n \times P$:

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & \mathcal{O}(\tau - 3h) & \longrightarrow & \mathcal{F}(-2h) & \longrightarrow & (\pi \times 1)^*\mathcal{I}_Z(-h) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(\tau - 2h) & \longrightarrow & \mathcal{F}(-h) & \longrightarrow & (\pi \times 1)^*\mathcal{I}_Z & \longrightarrow & 0
\end{array}
\]

Pushing this down via the first projection, we get the following exact diagram on $D_n$:

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & R^1p_1*\mathcal{F}(-2h) & \longrightarrow & \pi^*\mathcal{E} & \longrightarrow & \mathcal{O}(\tau) & \longrightarrow & 0 \\
& & \downarrow{m_L} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^1p_1*\mathcal{F}(-h) & \xrightarrow{\simeq} & \pi^*\mathcal{G} & \longrightarrow & 0
\end{array}
\]

Here the last map of the top row is nothing but the tautological quotient map on $P(\mathcal{E})$. Let $A(L) \subseteq D_n$ be the set of Hulsbergen sequences where $L$ is an exceptional line for the middle term. Clearly, $A(L)$ is the degeneration locus of the left vertical map $m_L$ above. Hence the divisor class of $A(L)$ is

\[
\begin{align*}
&= c_1(R^1p_{1*}\mathcal{F}(-h)) - c_1(R^1p_{1*}\mathcal{F}(-2h)) \\
&= \pi^*c_1(\mathcal{G}) - \pi^*c_1(\mathcal{E}) + \tau \\
&= \tau + \lambda.
\end{align*}
\]

On the other hand, $A(L)$ is the inverse image of a hyperplane in $P_n$ under $b_n$, so its divisor class is $b_n^*H$. \hfill \Box

2. The case $n \leq 5$

**Proposition 2.1.** For $2 \leq n \leq 5$, the rational map $g_n$ is dominating, and the general fiber is isomorphic to $P^{n-5}$. For $n \geq 5$, the map $g_n$ is generically injective with image of codimension $n - 5$. In particular, $g_5$ is birational.
Proof. Everything follows from the observation that the fiber over a point \([F] \in M_n\) in the image of \(g_n\) is the projectivization of \(H^0(P, F(1))\), and that for general such \(F\), this vector space has dimension \(h^0(F(1)) = \max(1, 6 - n)\), which is easily seen from (2.1). The assertion about the codimension follows from a dimension count: \(\dim(M_n) = 4n - 3\) and \(\dim(D_n) = 3n + 2\).

The first half of theorem 0.2 now follows: First of all, since \(g_n\) is birational, the two morphisms \(f_5\) and \(b_5\) have the same image and the same degree. Therefore \(q_{17}\) can be computed as

\[
q_{17} = \int_{D_n} H_1^{17} = \int_{D_n} (\tau + \lambda)^{17} = \int_{H_n} s_{12}(E \otimes L).
\]

For theorem 0.3, let \(L_1, \ldots, L_{5-n}\) be general lines in \(P\), and let \(B_n \subseteq D_n\) be the locus of Hulsbergen sequences where the closed subscheme \(Z\) meets all these \(5-n\) lines. The cohomology class of \(B_n\) in \(H^*(D_n)\) is \(\lambda^{5-n}\).

**Lemma 2.2.** Let \(2 \leq n \leq 5\). The general nonempty fiber of \(g_n\) meets \(B_n\) in \(2^{5-n}\) points, hence the rational map \(g_n|_{B_n}: B_n \to M_n\) is dominating and generically finite, of degree \(2^{5-n}\).

**Proof.** The general nonempty fiber is of the form \(P(H^0(P, F(1)))^\vee\). It suffices to show that the restriction of \(\mathcal{L}\) to this fiber has degree 2 (if \(n < 5\)). For this, it suffices to consider a linear pencil in the fiber. So let \(\sigma_0\) and \(\sigma_1\) be two independent global sections of \(F(1)\), and consider the pencil they span. Now \(\sigma_0 \wedge \sigma_1 \in H^0(P, \wedge^2 F) = H^0(P, \mathcal{O}_P(2))\) is the equation of a conic \(C \subseteq P\) which contains the zero scheme \(V(t_0\sigma_0 + t_1\sigma_1)\) of each section in the pencil, \((t_0, t_1) \in \mathbb{P}^1\). Since \(C\) meets a general line in two points, it follows that there are exactly two members of the pencil whose zero set meets a general line.

To complete the proof of theorem 0.3, by lemma 2.2 we now have for \(2 \leq n \leq 5\):

\[
2^{5-n} q_{4n-3} = 2^{5-n} \int_{M_n} H^{4n-3} = \int_{B_n} (\tau + \lambda)^{4n-3} = \int_{D_n} \lambda^{5-n} (\tau + \lambda)^{4n-3} = \int_{H_{n+1}} c_1(\mathcal{L})^{5-n} s_{3n-3}(\mathcal{E} \otimes \mathcal{L}).
\]

This completes the proof of the theorems for \(n \leq 5\).
3. The case \( n = 6 \)

For \( n \geq 6 \) the techniques above will say something about the restriction of the Barth map to the Brill-Noether locus \( B \subseteq M_n \) of semistable sheaves whose first twist admit a global section. For general \( n \) this locus is too small to carry enough information about \( M_n \), but in the special case \( n = 6 \), it is actually a divisor, whose divisor class \( \beta = [B] \) we can determine. Now \( \Pic(M_n) \otimes \mathbb{Q} \) has rank 2, generated by \( \alpha \) and \( \delta = [\Delta] \), the class of the locus \( \Delta \subseteq M_n \) corresponding to non-locally free sheaves [8].

**Proposition 3.1.** In \( \Pic(M_6) \otimes \mathbb{Q} \), the following relation holds:

\[
\beta = \frac{5}{2} \alpha - \frac{1}{2} \delta.
\]

**Proof.** Let \( \xi: X \rightarrow M_6 \) be a morphism induced by a flat family \( \mathcal{F} \) of semistable sheaves on \( P \), parameterized by some variety \( X \). For certain divisor classes \( a \) and \( d \) on \( X \), the second and third Chern classes of \( \mathcal{F} \) can be written in the form

\[
\begin{align*}
c_2(\mathcal{F}) &= ah + 6h^2, \\
c_3(\mathcal{F}) &= dh^2
\end{align*}
\]

modulo higher codimension classes on \( X \). The Grothendieck Riemann-Roch theorem for the projection \( p: X \times P \rightarrow X \) easily gives (for example using [3]) that

\[
-c_1(p_*\mathcal{F}(h)) = \frac{5}{2}a - \frac{1}{2}d.
\]

The locus \( \xi^{-1}B \subseteq X \) is set-theoretically the support of \( R^1p_*\mathcal{F}(h) \). It is not hard to see that one can take the family \( X \) in such a way that the 0-th Fitting ideal of \( R^1p_*\mathcal{F}(h) \) is actually reduced. Therefore the left hand side of the equation above is \( \xi^*\beta \). On the other hand, \( a = \xi^*\alpha \) by the usual definition of the \( \mu \) map of Donaldson [3], and \( d = \xi^*\delta \). Since the family \( \mathcal{F}/X \) was arbitrary, the required relation is actually universal, and so holds also in \( \Pic(M_6) \otimes \mathbb{Q} \). (It suffices to take a family with the properties that (i) \( \xi^*: \Pic(M_6) \rightarrow \Pic(X) \) is injective, (ii) the Fitting ideal above is reduced, and (iii) the general non-locally free sheaf in the family has colength 1 in its double dual.)

With this, we complete the proof of the second part of theorem [3] in the following way. The general fiber of \( f_6 \) restricted to \( \Delta \) has dimension...
1, so $f_6(\Delta)$ has dimension 19, see e.g. [11]. Therefore we get

$$\int_{H_7} s_{14}(\mathcal{E} \otimes \mathcal{L}) = \int_{D_6} (\lambda + \tau)^{20}$$

$$= \int_{M_6} \beta \alpha^{20}$$

$$= \int_{M_6} \left(\frac{5}{2} \alpha - \frac{1}{2} \delta\right) \alpha^{20}$$

$$= \frac{5}{2} \int_{M_6} \alpha^{21} - \frac{1}{2} \int_{\Delta} \alpha^{20} = \frac{5}{2} q_{21}.$$  

4. A geometric interpretation

**Definition 4.1.** A **Darboux configuration** in $P^\vee$ consists of a pair $(\Pi, C)$ where $\Pi \subseteq P^\vee$ is the union of $n+1$ distinct lines, no three concurrent, and $C \subseteq P^\vee$ is a curve of degree $n$ passing through all the nodes of $\Pi$.

If we let $Z \subseteq P$ consist of the $n+1$ points dual to the components of $\Pi$, we have by Hulsbergen’s theorem [11, thm. 4] a natural 1-1 correspondence between Hulsbergen sequences [11] and Darboux configurations $(\Pi, C)$, by letting $C = J(F)$. Therefore $D_n$ can be used as a compactification of the set of Darboux configurations, and the intersection number

$$\int_{D_n} \lambda^i (\tau + \lambda)^{3n+2-i} = \int_{H_{n+1}} c_1(\mathcal{L})^i s_{2n+2-i}(\mathcal{E} \otimes \mathcal{L})$$

can be interpreted as the number of Darboux configurations $(\Pi, C)$ where $\Pi$ passes through $i$ given points and $C$ passes through $3n+2-i$ given points.

It is not known whether the Barth map has degree 1 for $n \geq 5$. A related question is the following: Let $(\Pi, C)$ be a general Darboux configuration ($n \geq 5$). Is the inscribed polygon $\Pi$ uniquely determined by $C$?

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