D-Branes and Spin$^c$ Structures

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It was recently pointed out by E. Witten that for a D-brane to consistently wrap a submanifold of some manifold, the normal bundle must admit a Spin$^c$ structure. We examine this constraint in the case of type II string compactifications with vanishing cosmological constant, and argue that in all such cases, the normal bundle to a supersymmetric cycle is automatically Spin$^c$.

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1 Introduction

It was recently pointed out by E. Witten [1, 2] that for a D-brane to consistently wrap a submanifold of some manifold, the normal bundle must admit a Spin$^c$ structure.

For type II string compactifications of vanishing cosmological constant, other conditions for a consistently wrapped D-brane were worked out several years ago [3, 4].

In this technical note we shall argue that that for compactifications of vanishing cosmological constant, the Spin$^c$ constraint is redundant, that the conditions previously known for supersymmetric cycles always imply that the normal bundle admits a Spin$^c$ structure.

What are the options? In a Calabi-Yau, D-branes can be (supersymmetrically) wrapped on complex submanifolds and on special Lagrangian submanifolds [3]. In a $G_2$ holonomy 7-manifold, D-branes can be wrapped on associative (3-)submanifolds and coassociative (4-)submanifolds [4]. In a Spin(7) 8-manifold, D-branes can be wrapped on Cayley (4-)submanifolds [4].

In each case, a supersymmetric cycle corresponds to a calibrated submanifold. Recall [5, 6] a calibration is a closed $p$-form $\phi$ on a Riemannian manifold $M$ such that $\phi$ restricts to each tangent $p$-plane to be less than or equal to the volume form of that $p$-plane. Oriented submanifolds for which $\phi$ restricts to be equal to the Riemannian volume form (with respect to the induced metric) are said to be calibrated by the form $\phi$.

Calibrated submanifolds have an orientation, so, as the ambient space is oriented, we know that the structure group of the normal bundle to a calibrated submanifold is $SO(n)$, as opposed to $O(n)$. Thus, it makes sense to ask whether the normal bundle admits a Spin$^c$ structure, or even a Spin structure.

We shall begin by making some general observations concerning Spin$^c$ structures on normal bundles, then we shall examine each type of calibrated submanifold on a case-by-case basis.

Throughout this note we shall assume that we have compactified on a smooth manifold, and that the D-brane is wrapped on a smooth submanifold. In general if these conditions are not satisfied then there can be nonobvious subtleties. Even in complex geometry, if one tries to wrap a brane on a singular subvariety or on a “subvariety” that is actually a subscheme, one will encounter a number of apparent difficulties [7].

We shall also assume the reader is acquainted with the notion of a Spin$^c$ structure. If not, consult for example [8, appendix D] for expository material.
2 Generalities

Before we begin examining individual cases, we will make a few comments that will greatly simplify the analysis.

Let $M$ be either a Calabi-Yau, a $G_2$ holonomy 7-manifold, or a Spin(7) holonomy 8-manifold, and let $P$ be a calibrated submanifold. Let $TP$ denote the tangent bundle of $P$, and $N$ its normal bundle inside $M$.

Bundles with structure group $SU(n)$, $G_2$, and Spin(7) have no characteristic classes living in $H^1(M, \mathbb{Z}_2)$ or $H^2(M, \mathbb{Z}_2)$. Thus, the Stiefel-Whitney classes satisfy $w_1(TP \oplus N) = w_2(TP \oplus N) = 0$. The Whitney sum formula then yields

$$
w_1(N) = w_1(TP) \\
w_2(N) = w_2(TP) + w_1(TP)^2
$$

Since $P$ is oriented, $w_1(TP) = 0$, and so $w_1(N) = 0$. Thus, the structure group of the normal bundle $N$ can be reduced to $SO(n)$, not merely $O(n)$, as remarked in the introduction. The second relation now implies that $w_2(N) = w_2(TP)$. It is a standard fact that an $SO(n)$ bundle $N$ will admit a Spin$^c$ structure if and only if $w_2(N)$ is the mod 2 reduction of an element of $H^2(P, \mathbb{Z})$. Thus, since $w_2(N) = w_2(TP)$, we see immediately that the normal bundle $N$ will admit a Spin$^c$ structure if and only if the tangent bundle to the calibrated submanifold admits a Spin$^c$ structure. (For that matter, $N$ will admit a Spin structure if and only if $TP$ admits a Spin structure.)

We shall see, on a case-by-case basis, that for associative and coassociative submanifolds of $G_2$ holonomy 7-manifolds, Cayley submanifolds of Spin(7) holonomy 8-manifolds, complex submanifolds of Calabi-Yau’s, and special Lagrangian submanifolds of Calabi-Yau’s of complex dimension less than five, the tangent bundle to the calibrated submanifold admits a Spin$^c$ structure, and so the normal bundle necessarily admits a Spin$^c$ structure.

In addition, in each case we will also try to give additional information regarding normal bundles. For example, in several cases normal bundles are necessarily trivializable, not just Spin$^c$. We shall also give relevant technical pointers regarding Spin$^c$ structures in general.

3 Calabi-Yau manifolds

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1Because for $G = SU(n)$, $G_2$, and Spin(7), $\pi_1(BG) = \pi_2(BG) = 0$. 

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3
3.1 Complex submanifolds

Complex submanifolds of a Calabi-Yau can be viewed as calibrated submanifolds \([5\), section 0\]; simply choose \(\phi = \frac{1}{p!} \omega^p\) for a complex \(p\)-fold submanifold, where \(\omega\) is the Kähler form on the Calabi-Yau.

As mentioned in section (2), the normal bundle to a calibrated submanifold will admit a Spin\(^c\) structure if and only if the tangent bundle to the calibrated submanifold admits a Spin\(^c\) structure. Now, it is a standard fact that the tangent bundle of any complex manifold (in fact, any almost complex manifold) admits a Spin\(^c\) structure \([8\), appendix D\], thus the normal bundle to a complex submanifold must admit a Spin\(^c\) structure.

There is a more direct way to get this result. The normal bundle to a complex submanifold of a Calabi-Yau is a \(U(n)\) bundle, and so always admits a Spin\(^c\) structure \([8\), appendix D\].

3.2 Special Lagrangian submanifolds

Special Lagrangian submanifolds are calibrated submanifolds defined by a calibration \(\phi\) equal to the real part of the holomorphic \(n\)-form trivializing the canonical bundle of the Calabi-Yau. (It can be shown \([6\), section III.1\] that a submanifold is special Lagrangian if and only if it is Lagrangian and the restriction of the imaginary part of the holomorphic \(n\)-form vanishes.)

In general one does not expect the special Lagrangian submanifolds of any Calabi-Yau to have Spin\(^c\) normal bundles, but in the special case of low-dimensional Calabi-Yau’s (in particular, Calabi-Yau’s of complex dimension less than five) we shall see that the normal bundle to any special Lagrangian submanifold is Spin\(^c\).

First, note that this is trivial to check for Calabi-Yau’s of complex dimension one or two, so we shall only consider Calabi-Yau’s of dimensions three and four.

For Calabi-Yau’s of complex dimension three, the special Lagrangian submanifold will be a compact oriented 3-manifold. It is known that the tangent bundle of any compact oriented 3-manifold is trivializable \([9\), problem 12-B\], thus it admits a Spin\(^c\) structure and so by the arguments of section (2), the normal bundle must also admit a Spin\(^c\) structure.

In fact, in this case we can make a much stronger statement. It can be shown \([5\), corollary 3-3\] that the normal bundle of a special Lagrangian submanifold is isomorphic to the tangent bundle of the special Lagrangian submanifold. Thus, since the tangent bundle of a compact oriented 3-manifold is trivial, we know that the normal bundle to a special Lagrangian submanifold of a Calabi-Yau 3-fold is trivializable, not just Spin\(^c\).

For Calabi-Yau’s of complex dimension four, we use the standard fact that the tangent
bundle of any oriented compact four-manifold admits a Spin\(^c\) structure \([10, \text{lemma 3.1.2}]\). Thus, by either the arguments of section \((2)\) or since the normal bundle of a special Lagrangian submanifold is isomorphic to the tangent bundle, we see that the normal bundle to a special Lagrangian submanifold necessarily admits a Spin\(^c\) structure.

For Calabi-Yau’s of higher dimension, it is not clear that in general normal bundles to their special Lagrangian submanifolds will admit Spin\(^c\) structures. However, as higher dimensional Calabi-Yau’s cannot be used in type II string compactifications, this is not a relevant issue.

4 \(G_2\) 7-manifolds

On a 7-manifold \(M\) of \(G_2\) holonomy, there exists a 3-form \(\phi\) that is compatible with the \(G_2\) structure \([11]\). For example, on \(\mathbb{R}^7\) let \(y_1, y_2, \cdots, y_7\) denote an oriented, orthonormal basis, then define

\[
\phi = y_1 \wedge y_2 \wedge y_7 + y_1 \wedge y_3 \wedge y_6 + y_1 \wedge y_4 \wedge y_5 + y_2 \wedge y_3 \wedge y_5 - y_2 \wedge y_4 \wedge y_6 + y_3 \wedge y_4 \wedge y_7 + y_5 \wedge y_6 \wedge y_7
\]

Then the subgroup of \(GL_+(7, \mathbb{R})\) (the orientation-preserving subgroup of \(GL(7, \mathbb{R})\)) that preserves \(\phi\) is precisely \(G_2\). The Hodge dual to \(\phi\), namely \(\ast \phi\), has analogous properties.

On a \(G_2\) holonomy 7-manifold, there are two natural sets of calibrated submanifolds \([12]\). One set is defined by taking the calibration to be \(\phi\), and consists of three-dimensional submanifolds known as associative submanifolds. The other set is defined by taking the calibration to be \(\ast \phi\), and consists of four-dimensional submanifolds known as coassociative submanifolds. We shall study normal bundles to each type of calibrated submanifold separately.

4.1 Associative (3-)submanifolds

We shall argue that the normal bundle to an associative submanifold is not just Spin\(^c\), but actually trivializable.

First, since the tangent bundle of an oriented compact 3-manifold is trivial \([1, \text{problem 12-B}]\) and so trivially Spin\(^c\), we know from the arguments given in section \((2)\) that the normal bundle to an associative submanifold admits a Spin\(^c\) structure.

In fact, we can make a much stronger statement concerning the normal bundle to an associative submanifold, namely that it is trivializable. We shall need a few general facts concerning associative submanifolds \([3, \text{section 5}]\). First, the restriction of the tangent bundle of a \(G_2\) holonomy 7-manifold to an associative submanifold is a principal \(SO(4) = Sp(1) \times \mathbb{Z}_2\).
$Sp(1)$ bundle. Write $\mathbb{R}^7 = \mathbb{H} \oplus \text{im } \mathbb{H}$, where $\mathbb{H}$ denotes the quaternions. (Intuitively, if we locally identify tangent directions to the $G_2$ holonomy 7-manifold with vectors in $\mathbb{R}^7$, then directions in $\text{im } \mathbb{H}$ will be tangent to the associative submanifold, and directions in $\mathbb{H}$ will be normal to the associative submanifold.) Then the structure group of the restriction of the tangent bundle of the $G_2$ holonomy 7-manifold to an associative submanifold can be written as a subgroup of $GL(\mathbb{H}) \times GL(\text{im } \mathbb{H}) \subset GL(7, \mathbb{R})$ of the form

$$
\left[ \begin{array}{cc}
R_q L_p & 0 \\
0 & \rho(R_q)
\end{array} \right]
$$

where $p, q \in \mathbb{H}$, where $R_q$ and $L_q$ denote right and left multiplication, respectively, by a quaternion $q \in \mathbb{H}$, and where $\rho$ is the irreducible representation of the action of $Sp(1)$ in $\text{im } \mathbb{H}$ (defined by $\rho(R_q) = L_\mathbb{T} R_q$). The lower right block, a subset of $GL(\text{im } \mathbb{H})$, corresponds to the structure group of the tangent bundle to the associative submanifold, and the upper left block, a subset of $GL(\mathbb{H})$, corresponds to the structure group of the normal bundle to the associative submanifold.

Now, the tangent bundle of a compact oriented 3-manifold is topologically trivial \cite[problem 12-B]{[9]}, so the action of $Sp(1)$ given in the form above by $R_q$ can be gauged away. Thus, the structure group of the normal bundle to an associative submanifold is actually $Sp(1) = SU(2)$.

Finally, $SU(2)$ bundles on a 3-manifold are topologically trivial\cite[4.2]{[10]}, so we see that the normal bundle to an associative (3-)submanifold is topologically trivial. Thus, it trivially admits a $Spin^c$ structure, and even a $Spin$ structure.

### 4.2 Coassociative (4-)submanifolds

A standard fact concerning 4-manifolds is that the tangent bundle to any compact oriented 4-manifold admits a $Spin^c$ structure \cite[lemma 3.1.2]{[12]}, so by the arguments given in section (2), we know that the normal bundle to a coassociative submanifold is $Spin^c$.

As a check, we shall derive this result an alternative way. It can be shown \cite[proposition 4-2]{[5]} that the normal bundle to a coassociative submanifold $M$ is isomorphic to the bundle of anti-self-dual two-forms, $\Lambda^2_- T^* M$. Also, as mentioned above the tangent bundle to any compact oriented 4-manifold admits a $Spin^c$ structure.

We claim these two facts imply that $\Lambda^2_- T^* M$ admits a $Spin^c$ structure. It is somewhat easier to explain why if $M$ were Spin then $\Lambda^2_- T^* M$ would be Spin, so we shall do this first. If $M$ were Spin, the the structure group of $TM$ (and more relevantly, $T^* M$) could be lifted from $SO(4)$ to $Spin(4) = SU(2) \times SU(2)$. Now, the structure groups of $\Lambda^2_- T^* M$ and $\Lambda^2_+ T^* M$

\footnote{This is essentially because $\pi_1(BSU(2) = HP^\infty) = \pi_2(HP^\infty) = \pi_3(HP^\infty) = 0.$}
are in general $SO(3)$ groups associated to the two $SU(2)$ factors of the structure group of $T^*M$, via the decomposition $SO(4) = [SU(2) \times SU(2)]/\mathbb{Z}_2$. If the structure group of $T^*M$ can be lifted to $Spin(4) = SU(2) \times SU(2)$, then clearly the structure groups of $\Lambda^2 T^*M$ and $\Lambda^2 T^*M$ can both be lifted from $SO(3)$ to $Spin(3) = SU(2)$.

Now, in the case at hand, $M$ is $Spin^c$, but not necessarily $Spin$. An argument similar to the one above shows that $\Lambda^2 T^*M$ and $\Lambda^2 T^*M$ are necessarily $Spin^c$ also. The point is simply that if the structure group of $TM$, and more to the point $T^*M$, can be lifted from $SO(4)$ to $Spin^c(4) = SU(2) \times SU(2)$, then the projection into the self-dual and anti-self-dual pieces of $SO(4)$ yields a lift from $SO(3)$ to $Spin^c(3) = SU(2) \times SU(2)$, consequently both $\Lambda^2 T^*M$ and $\Lambda^2 T^*M$ are $Spin^c$.

Thus, since any compact oriented 4-manifold is $Spin^c$, we know that $\Lambda^2 T^*M$ is $Spin^c$, hence the normal bundle to a coassociative (4-)submanifold is necessarily $Spin^c$.

5 Cayley (4-)submanifolds of $Spin(7)$ 8-manifolds

On a $Spin(7)$ holonomy 8-manifold, there is a natural 4-form $[3]$. This 4-form can be used as a calibration, defining real four-dimensional submanifolds known as Cayley submanifolds.

As the tangent bundle to any compact oriented 4-manifold necessarily admits a $Spin^c$ structure, we know from the arguments of section (2) that the normal bundle to a Cayley submanifold admits a $Spin^c$ structure.

We can also study normal bundles to Cayley submanifolds directly. The restriction of the tangent bundle of a $Spin(7)$ holonomy 8-manifold to a Cayley submanifold is associated to a principal $[Sp(1) \times Sp(1) \times Sp(1)]/\mathbb{Z}_2$ bundle $[3]$, section 6]. Write $\mathbb{R}^8 = H \oplus H$, where $H$ denotes the quaternions. (Intuitively, if we locally identify tangent directions to the $Spin(7)$ holonomy 8-manifold with vectors in $\mathbb{R}^8$, then directions in one copy of $H$ will be tangent to the Cayley submanifold and directions in the other $H$ will be normal to the Cayley submanifold.) Then the structure group of the restriction of the tangent bundle of the $Spin(7)$ holonomy 8-manifold to a Cayley submanifold can be written as a subgroup of $GL(H) \times GL(H) \subset GL(8, \mathbb{R})$ of the form

$$\begin{bmatrix}
L_q R_{p_1} & 0 \\
0 & L_q R_{p_2}
\end{bmatrix}$$

where $p_1, p_2, q \in H$, and other notation is as in section (4.1). One block corresponds to the structure group of the tangent bundle of the Cayley submanifold; the other block corresponds to the structure group of the normal bundle. It should be clear that the normal bundle is $Spin$ or $Spin^c$ if and only if the tangent bundle to the Cayley submanifold is $Spin$ or $Spin^c$, respectively.
In passing we shall also mention that if \( X \) is an oriented simply-connected 4-manifold, then all elements of \( H^2(X, \mathbb{Z}_2) \) are mod 2 reductions of elements of \( H^2(X, \mathbb{Z}) \) \[14\], section 1.1, p. 6. The condition for an \( SO(n) \) bundle to admit a \( \text{Spin}^c \) structure is that its second Stiefel-Whitney class \( w_2 \in H^2(X, \mathbb{Z}_2) \) be the mod 2 reduction of an element of \( H^2(X, \mathbb{Z}) \), consequently all \( SO(n) \) bundles on \( X \) admit \( \text{Spin}^c \) structures. Thus, if a compact Cayley submanifold is simply-connected, then one can see immediately that its normal bundle necessarily admits a \( \text{Spin}^c \) structure.

6 Conclusions

In this technical note we have argued that the normal bundles to all calibrated submanifolds encountered in (supersymmetric) wrapped branes in type II string compactifications with vanishing cosmological constant admit \( \text{Spin}^c \) structures. More precisely, all associative and coassociative submanifolds of \( G_2 \) holonomy 7-manifolds, all complex submanifolds of Calabi-Yau’s, all special Lagrangian submanifolds of Calabi-Yau’s of complex dimension less than five, and all Cayley submanifolds of Spin(7) 8-manifolds have normal bundles admitting a \( \text{Spin}^c \) structure.

Thus, for type II string compactifications with vanishing cosmological constant, the recent observation by E. Witten \[1, 2\] that normal bundles must be \( \text{Spin}^c \) for consistent wrapped branes, is redundant. For wrapped branes in \( AdS \) compactifications, by contrast, normal bundles are not automatically \( \text{Spin}^c \), and so in that case this constraint is much more interesting.

In retrospect this is not very surprising. First, as a rule of thumb it is relatively easy to satisfy the \( \text{Spin}^c \) constraint. Second, for several years now various authors have studied wrapped branes in theories with vanishing cosmological constant without running into any unexpected anomalies. Thus, one should not be too surprised that for compactifications with vanishing cosmological constant, the \( \text{Spin}^c \) condition is always satisfied automatically. For wrapped branes in \( AdS \) compactifications, the \( \text{Spin}^c \) constraint is doubtless more interesting.

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