On the eigenvalues of some non-Hermitian Hamiltonians with space-time symmetry

Paolo Amore

Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, Colima, Colima, Mexico.

Francisco M. Fernández and Javier García

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica,
Bvd. 113 y 64 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

We calculate the eigenvalues of some two-dimensional non-Hermitian Hamiltonians by means of a pseudospectral method and straightforward diagonalization of the Hamiltonian matrix in a suitable basis set. Both sets of results agree remarkably well but differ considerably from the eigenvalues obtained some time ago by other authors. In particular, we do not observe the multiple phase transitions claimed to occur in one of the anharmonic oscillators.

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I. INTRODUCTION

In a recent paper, Klaiman and Cederbaum[1] studied the spectrum of non-Hermitian Hamiltonians $H = H_0 + i\lambda W$ by means of the point-group symmetries of the Hermitian $H_0$ and non-Hermitian $W$ parts. They showed that, in principle, the symmetry properties of the Hamiltonian are responsible for the appearance of real eigenvalues in the spectrum of the non-Hermitian Hamiltonian $H$. To this end they constructed an effective energy-dependent Hermitian Hamiltonian that exhibits the same real spectrum as the non-Hermitian one. They illustrated the main theoretical results by means of suitable chosen examples. One of them is of great interest because it exhibits multiple phase transitions. As the parameter $\lambda$ increases two real eigenvalues approach each other, coalesce and become a pair of complex conjugate numbers. That is to say, the space-time symmetry is broken beyond the coalescence point. However, on increasing $\lambda$ those same complex eigenvalues become real again, separate, just to approach each other again and coalesce at a larger value of $\lambda$.

The purpose of this paper is a critical discussion of those results. In Sec. II we outline the point-group symmetry of the models considered by Klaiman and Cederbaum[1]. In Sec. III we compare present results with the ones of those authors and briefly discuss an example not considered by them. In Sec. IV we draw conclusions.

*Electronic address: paolo.amore@gmail.com
†Corresponding author: fernande@quimica.unlp.edu.ar
‡Electronic address: jgarcia@inifta.unlp.edu.ar
II. THE MODELS

Three of the examples considered by Klaiman and Cederbaum are based on the Hamiltonian

\[
H = H_0 + i \lambda W,
\]

\[
H_0 = -\frac{1}{2} \left( \partial_x^2 + \partial_y^2 \right) + \alpha_x x^4 + \alpha_y y^4,
\]

(1)

where \( \alpha_x = 1 \) and \( \alpha_y = \sqrt{2} \). They wrote the eigenvectors of \( H_0 \) formally as

\[
|n_x, n_y\rangle = |n_x\rangle \otimes |n_y\rangle,
\]

(2)

where \( n_x, n_y = 0, 1, \ldots \) and \( |n_x\rangle, |n_y\rangle \) denote the eigenvectors of the \( x \)- and \( y \)-quartic oscillators, respectively.

They described the symmetry of \( H_0 \) by means of the point group \( D_{2h} \) (isomorphic to \( C_{2v} \)) with symmetry operations \( \{E, P, P_x, P_y\} \) that are given by the coordinate transformations

\[
E : \{x, y\} \to \{x, y\},
\]

\[
P : \{x, y\} \to \{-x, -y\},
\]

\[
P_x : \{x, y\} \to \{-x, y\},
\]

\[
P_y : \{x, y\} \to \{x, -y\}.
\]

(3)

It follows from

\[
E |n_x, n_y\rangle = |n_x, n_y\rangle,
\]

\[
P |n_x, n_y\rangle = (-1)^{n_x+n_y} |n_x, n_y\rangle,
\]

\[
P_x |n_x, n_y\rangle = (-1)^{n_x} |n_x, n_y\rangle,
\]

\[
P_y |n_x, n_y\rangle = (-1)^{n_y} |n_x, n_y\rangle,
\]

(4)

that the eigenvectors are bases for the irreducible representations \( A_g, B_g, A_u \) or \( B_u \) when \( (n_x, n_y) \) is \( (\text{even, even}) \), \( (\text{odd, odd}) \), \( (\text{even, odd}) \) or \( (\text{odd, even}) \), respectively.

III. RESULTS

Before discussing the non-Hermitian Hamiltonians considered by Klaiman and Cederbaum we first focus on the Hermitian Hamiltonian \( H_0 \). Since \( \alpha_y > \alpha_x \) it is clear that \( E_{10}^{(0)} (B_u) < E_{01}^{(0)} (A_u) \). Surprisingly their figures 3 and 4 show exactly the reverse order. Besides, the same level order appears in Fig. 5 where the authors labelled the eigenvalues by means of the point group \( C_i \) instead of \( D_{2h} \).

We calculated the lowest eigenvalues \( E_{n_x,n_y}^{(0)} \) of \( H_0 \) by three completely different approaches: the Riccati-Padé method (RPM) \[2, 3\], a pseudospectral method \[4\] and the straightforward diagonalization method (DM) using a basis set of products of eigenfunctions of the harmonic oscillator \( H_{HO} = p^2 + q^2 \). The three methods agree remarkably well for \( \lambda = 0 \) and the latter two ones for all \( \lambda \) (the RPM does not apply to nonseparable problems).

Table I shows the lowest eigenvalues of \( H_0 \) as well as the symmetry of the corresponding eigenfunctions according to the point groups \( C_i \) and \( D_{2h} \). By simple inspection it is clear that the results of this table do not agree with those for \( \lambda = 0 \) in figures 3, 4, and 5 of Ref. \[1\] in agreement with the discussion above.
We first consider the non-Hermitian perturbation $W = xy$ that is invariant with respect to $P$: $P WP = W$. On the other hand, the whole Hamiltonian is invariant under two antiunitary transformations $A_x = TP_x$ and $A_y = TP_y$, where $T$ is the time-reversal operator. According to the authors it exhibits two space-time symmetries that are a generalization of the well known PT symmetry. In this case the states that transform as $A_g (A_u)$ couple to states that transform as $B_g (B_u)$. The authors illustrate such couplings in their Fig. 3 but, as discussed above, some of the labels of the lines $E_{mn} (\lambda)$ appear to exhibit a reverse order and the numerical values of the eigenvalues $E_{mn} (0)$ do not appear to agree with present calculation displayed in Table I.

The second example is given by $W = x^2 y$. In this case the states $A_g (B_g)$ couple with the $A_u (B_u)$ ones as shown in Fig. 4 in the paper by Klaiman and Cederbaum. The eigenvalues of $H_0$ exhibit the discrepancy already discussed above.

The non-Hermitian perturbation $W = x^2 y + xy^2$ is of special interest because the authors identified pairs of states that are real for $0 < \lambda < \lambda_b$, coalesce at $\lambda_b$ and become complex conjugate for $\lambda_b < \lambda < \lambda_c$, then real again for $\lambda_c < \lambda < \lambda_f$ and coalesce again at $\lambda = \lambda_f$ to become complex conjugate once more. Bender et al. have recently discussed such consecutive phase transitions in the case of classical and quantum-mechanical linearly-coupled harmonic oscillators (see also [1]). We calculated the same eigenvalues $E_{mn} (\lambda)$ in the same range of values of $\lambda$ and did not find any of the multiple phase transitions mentioned by Klaiman and Cederbaum. Fig. 1 shows present results that exhibit the customary phase transitions for multidimensional oscillators.

Finally, we want to discuss a problem that was not considered by Klaiman and Cederbaum but may be of interest. When $\alpha_x = \alpha_y = 1$ the Hamiltonian $H_0$ is invariant under the unitary transformations of the point group $C_{4v}$. This group exhibits a degenerate irreducible representation $E$ and, therefore, is beyond the discussion of the paper of Klaiman and Cederbaum. However, we deem it worth mentioning it here as another example of those discussed by Fernández and Garcia. In this case the non-Hermitian perturbation $W = xy$ (with point group $C_{2v}$) couples the degenerate eigenvectors $|2m, 2n + 1\rangle$ and $|2m + 1, 2n\rangle$ and the $ST$-symmetric non-Hermitian operator exhibits complex eigenvalues for all $\lambda > 0$. More precisely, some of the eigenvectors of $H_0$ belonging to the irreducible representation $E$ with real eigenvalues are coupled by the non-Hermitian perturbation and become eigenvectors of $H$ belonging to the irreducible representations $B_1$ and $B_2$ with complex eigenvalues. As argued by Fernández and García the $ST$ symmetry is not as robust as the $PT$ one (were $P$ is the inversion operation in the point group).

IV. CONCLUSIONS

In this paper we carried out three completely different calculations of the eigenvalues and eigenfunctions of the Hermitian operator $H_0$ and two of them for the eigenvalues and eigenfunctions of the non-Hermitian operator with three non-Hermitian perturbations $W$. The agreement of the results provided by those methods makes us confident of their accuracy. Present results do not agree with those of Klaiman and Cederbaum. Straightforward comparison of the results in Table I with those in figures 3, 4 and 5 of Ref. I shows that the magnitude of the eigenvalues and the level ordering are quite different. Present eigenvalues $E_{mn} (\lambda)$ for $W = x^2 y + xy^2$ displayed in Fig. 1 do not exhibit the multiple phase transitions discussed by those authors but the well-known symmetry breaking at exceptional points common to other two-dimensional PT-symmetric oscillators.

In addition to all that, we have also shown that the $ST$ symmetry proposed by Klaiman and Cederbaum is
not as robust as the $PT$ one. In two recent papers Fernández and Garcia have already discussed two other $ST$-symmetric cases that exhibit phase transitions at the trivial Hermitian limit. Those authors also argued that the coupling of the degenerate states of $H_0$ to produce complex eigenvalues will not take place when $PW_P = -W$.

[1] S. Klainman and L. S. Cederbaum, Phys. Rev. A 78, 062113 (2008).
[2] F. M. Fernández, Q. Ma, and R. H. Tipping, Phys. Rev. A 39, 1605 (1989).
[3] F. M. Fernández, Q. Ma, and R. H. Tipping, Phys. Rev. A 40, 6149 (1989).
[4] P. Amore and F. M. Fernandez, Phys. Scr. 81, 045011 (2010).
[5] C. M. Bender and D. J. Weir, J. Phys. A 45, 425303 (2012).
[6] F. M. Fernández and J. Garcia, Ann. Phys. 342, 195 (2014).
[7] F. M. Fernández and J. Garcia, PT-symmetry broken by point-group symmetry, arXiv:1308.6179v2 [quant-ph]
[8] C. M. Bender, M. Gianfreda, S. K. Özdemir, B. Peng, and L. Yang, Phys. Rev. A 88, 062111 (2013).
[9] F. M. Fernández, Algebraic treatment of PT-symmetric coupled oscillators, arXiv:1402.4473 [quant-ph]
FIG. 1: First eight eigenvalues of the non-Hermitian Hamiltonian \( H \) with \( W = xy^2 + x^2y \). The continuous blue lines and dashed red ones indicate states with symmetry \( A_g \) and \( A_u \), respectively.