ON REGULARITY OF $\bar{\partial}$-SOLUTIONS ON $a_q$ DOMAINS WITH $C^2$ BOUNDARY IN COMPLEX MANIFOLDS

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Abstract. We study regularity of solutions $u$ to $\bar{\partial}u = f$ on a relatively compact $C^2$ domain $D$ in a complex manifold of dimension $n$, where $f$ is a $(0, q)$ form. Assume that there are either $(q + 1)$ negative or $(n - q)$ positive Levi eigenvalues at each point of boundary $\partial D$. Under the necessary condition that a locally $L^2$ solution exists on the domain, we show the existence of the solutions on the closure of the domain that gain $1/2$ derivative when $q = 1$ and $f$ is in the Hölder-Zygmund space $\Lambda^r(D)$ with $r > 1$. For $q > 1$, the same regularity for the solutions is achieved when $\partial D$ is either sufficiently smooth or of $(n - q)$ positive Levi eigenvalues everywhere on $\partial D$.

1. Introduction

Let $D$ be a relatively compact domain in a complex manifold $X$ of dimension $n$. We say that $D$ satisfies the condition $a_q$ if boundary $\partial D \in C^2$ and its Levi-form on complex tangent space $T^{(1,0)}(\partial D)$ has either $(q + 1)$ negative or $(n - q)$ positive eigenvalues at each point $\zeta \in \partial D$. We are interested in the regularity of solutions $u$ to the $\bar{\partial}$-equation, $\bar{\partial}u = f$, on $D$. We will study the case where $f$ is a $V$-valued $(0, q)$ form for a holomorphic vector bundle $V$ on $X$. Denote by $\Lambda^r_{(0,q)}(D,V)$ the space of $V$-valued $(0,q)$ forms whose coefficients are in Hölder–Zygmund space $\Lambda^r(D)$ defined in Sect. 5; note that $\Lambda^r(D)$ is the Hölder space $C^r(D)$ when $r \in (0, \infty) \setminus \mathbb{N}$. To ensure the existence of solutions, we impose a minimum requirement that there is an $L^2_{\text{loc}}$ solution $u_0$ on $D$ and seek a possibly different solution of better regularity.

Our main results are the following.

Theorem 1.1. Let $r \in (1, \infty)$ and $q \geq 1$. Let $D$ be a relatively compact domain with $C^2$ boundary in a complex manifold $X$ satisfying the condition $a_q$. Let $V$ be a holomorphic vector bundle on $X$. Then there exists a bounded linear $\bar{\partial}$-solution operator $H_q: \Lambda^r_{(0,q)}(D,V) \cap \bar{\partial}L^2_{\text{loc}}(D) \to \Lambda^{r+1/2}_{(0,q)}(D,V)$, provided (a) $q = 1$ or $\partial D$ has $(n - q)$ positive Levi eigenvalues at each point on $\partial D$; or (b) $\partial D \in \Lambda^{r+5/2}$.

Note that $H_q$ is independent of $r$ and it provides a smooth ($C^\infty$) linear $\bar{\partial}$-solution operator for smooth forms in both cases. When $\partial D \in C^2$, case (a) provides a satisfactory regularity result for $\bar{\partial}$-solutions in the Hölder–Zygmund spaces for $q = 1$. For $q > 1$, we have the following.
Theorem 1.2. Let $q \geq 2$ and keep notations in Theorem 1.1 with $\partial D \in C^2$. Then there exists a finite $r_0$ such that if $r > r_0$, there exists a bounded linear $\bar{\partial}$-solution operator

$$H^r_q : \Lambda^{(0,q)}_{\partial}(D,V) \cap \mathcal{I}L^2_{\text{loc}}(D) \to \Lambda^{r-5/2}_{(0,q-1)}(D).$$

Further, $H^r_q$ maps $C^\infty(\partial D,V) \cap \mathcal{I}L^2_{\text{loc}}(D)$ into $C^\infty(\partial D,V)$.

The value $r_0$ is stated in a detailed version of Theorem 1.2 in Theorem 9.1.

We first state some closely related results on $\bar{\partial}$-solutions $u$ to $\bar{\partial}u = f$ on strictly pseudoconvex domains $D$ in $\mathbb{C}^n$: After work of Lieb–Grauert [24] and Kerzman [35], Henkin–Romanov [30] achieved the sharp $C^{1/2}$ solutions for $f \in L^\infty$ by integral formulas. The $C^{k+1/2}$ solutions for $f \in C^k (k \in \mathbb{N})$ were obtained by Shi [59] for $(0,1)$ forms and by Lieb–Range [46] for forms with $q \geq 1$ when $\partial D$ is sufficiently smooth. For $\partial D \in C^2$, Theorem 1.1 and analogous results for a homotopy formula were proved in [19] through the construction of a homotopy formula. These results were extended by Shi [59] to a weighted Sobolev spaces with a gain less than $1/2$ derivative and by Shi–Yao [60, 61] to $H^{r+1/2,p}$ space gaining $1/2$ derivative for $s > 1/p$ when $\partial D \in C^2$ and for $s \in \mathbb{R}$ when $\partial D$ is sufficiently smooth. It is worthy to point out that Shi–Yao achieved the first regularity result for negative order $s$, and Yao [68] further showed that a similar result holds for convex finite multitype smooth domains in $\mathbb{C}^n$. Also, Gong–Lanzani [21] obtained $\Lambda^{r+1/2}$ (with $r > 1$) regularity gaining $1/2$ derivative on strongly $C^1$-linear convex $C^{1,1}$ domains $D$.

Next, we mention results on $\bar{\partial}$ solutions on $a_q$ domains in complex manifolds, which are also called $Z(q)$ domains in [16, p. 57]). The basic estimate for $\bar{\partial}$ was proved by Morrey [51] for $(0,1)$-forms and by Kohn [37, 38] for forms of any type on strongly pseudoconvex manifolds with smooth boundary. These results lead to regularity of the $\bar{\partial}$-Neumann operator for strictly pseudoconvex manifolds and more general for compact manifolds whose boundary satisfies property $Z(q)$ (see [37, 38] and [40, Thm. 3.9]). By work of Hörmander [31], the basis estimate is equivalent to the condition $Z(q)$ on $D$ (see [40, Prop. 3.12] and [16, Thm. 3.2.2] for details). When $\partial D \in C^\infty$, sharp regularity results for $\bar{\partial}$ solutions were obtained by Greiner–Stein [20] for $(0,1)$ forms and Beals–Greiner–Stanton [7] for $(0,q)$ forms under condition $Z(q)$ through the study of the regularity of $\bar{\partial}$-Neumann operator in $L^{k,p}$ and $\Lambda^s$ spaces. The condition $a_q$ also ensures the stability of the solvability of the $\bar{\partial}$-equation on $(0,q)$ forms; namely if $\tilde{f} = \bar{\partial}u_0 + \tilde{f}$ on $D$ while $\tilde{f}$ is a $\bar{\partial}$-closed form on a larger domain and $u_0$ has the regularity in the sought-after class, then $\tilde{f} = \bar{\partial}\tilde{u}$ for some $L^2$ form $\tilde{u}$. This stability is useful to obtain regularity for $\bar{\partial}$ solutions as shown by Kerzman [35]; first one seeks regularity for $u_0$ without solving the $\bar{\partial}$-equation. Then $u_0 + \tilde{u}$ provides a desired solution based on regularity of $u_0$ and the interior regularity of $\tilde{u}$ from the elliptic theory on systems of partial differential equations.

To prove our results, we will use integral formulas to obtain local solutions near each boundary point of $D$. We then use the Grauert bumping method as in [35] to construct $\tilde{f}$. To provide background for our results, let us mention regularity results for $\bar{\partial}$-solutions on the transversal intersection of domains in $\mathbb{C}^n$. Range–Siu [55] obtained $C^{1/2-\epsilon}$ estimate with any positive $\epsilon$ for a real transversal intersection of sufficiently smooth strictly pseudoconvex domains. For higher order derivatives, J. Michel [47] obtained $C^{k+1/2-\epsilon}$ estimate for $\bar{\partial}$ solutions on a certain intersection
of smooth strictly pseudoconvex domains. J. Michel and Perotti [48] extended the result to real transversal intersection of strictly pseudoconvex domains with sufficiently smooth boundary. We should also mention that the local version of Theorem 1.1 was proved by Laurent-Thiébaut and Leiterer [42] when $\partial D \in C^\infty$ and $k \in \mathbb{N}$. Ricard [56] obtained regularity for concave wedge with $C^{k+2}$ boundary and convex wedges with $C^2$ boundary. The reader is referred to Barkatou [5] and Barkatou-Khidr [6] for further results in this direction. However, all existing integral formulas for $\overline{\partial}$ solutions, including ours for $q > 1$, require boundary to be sufficiently smooth when concavity is present.

It is well-known that concavity of the domains is useful; the classical Hartogs theorem says that a holomorphic function on a 1-concave domain extends to a holomorphic function across the boundary. Therefore, on a 1-concave domain in a complex manifold, all $\overline{\partial}$-solutions for $(0,1)$ forms must have the same regularity regardless the smoothness of the boundary of the domain. In Section 7 we will successfully implement this idea to prove Theorem 1.1 (i) with $q = 1$.

When $q > 1$, not all $\overline{\partial}$ solutions have the same regularity. To prove Theorem 1.2, we first derive an estimate for a solution operator when $\partial D$ is sufficiently smooth. Then we apply a Nash-Moser iteration method by solving the $\overline{\partial}$-equation on the subdomains $D_k \subset D_{k+1}$ that have smooth boundary and in the limit, we obtain a desired solution on the closure of $D = \cup D_k$.

We organize the paper as follows.

In Section 2, we formulate an approximate local homotopy formula on a suitable neighborhood of a boundary point $\zeta_0 \in \partial \Omega$. In Sections 3 and 4, we derive (genuine) local homotopy formulas for $\overline{\partial}$-closed $(0,q)$ forms near $(n-q)$ convex and $(q+1)$ concave boundary points of an $a_q$ domain. There we follow approaches developed in Lieb–Range [46] and Henkin–Leiterer [29]. While a homotopy formula for forms that are not necessary $\overline{\partial}$-closed can be derived for $(n-q)$ convex points without extra conditions, such a formula on the concave side of the boundary turns out to be subtle. In fact, Laurent-Thiébaut and Leiterer [43, Prop. 0.7] proved that there is no local homotopy formula near a $(q+1)$ concave boundary for $(0,q)$ forms that has good estimates, and see also Nagel–Rosay [53] on $\overline{\partial}_b$ for domains in sphere in $\mathbb{C}^3$. Such phenomena leads to difficulties for strictly pseudoconvex hypersurfaces for local CR embedding problem in Webster [67] and Polykov [54] on global CR embeddings of concave compact CR manifolds in critical dimensions.

Section 5 contains some elementary facts on Hölder–Zygmund spaces, where we derive an equivalence characterization on the Hölder–Zygmund norms solely relying on a version of Hardy-Littlewood lemma.

Section 6 contains main local estimates for homotopy operators that appear in the local homotopy formula. One of main purposes of the section is to derive precise estimates that reflex the convexity of the Hölder–Zygmund norms; see (6.1) for strictly $(n-q)$ convex $C^2$ domains and (6.15) for strictly $(q+1)$-concave domains. We emphasize that the estimates do not require the forms to be $\overline{\partial}$-closed; in fact, the estimates hold for $(q+1)$ concave boundary points. These estimates immediately give us the desired regularity stated in Theorem 1.1 for local solutions.

In Section 7 we show as a novelty how the Hartogs extension theorem can be used to study the regularity of $\overline{\partial}$ solutions for $(0,1)$ forms. A local version of Theorem 1.1 (a) for $q = 1$ is proved in this section.
In Section 8 we show the existence of global solutions with the desired regularity by using local solutions in Sections 6 and 7 and the interior regularity of elliptic systems. We also derive a global estimate for $\bar{\partial}$-solutions. Using this global estimate, we employ the Nash–Moser smoothing operator to prove a detailed version of Theorem 1.2 in Section 9.

The paper has two appendices. In Appendix A, we recall the existing regularity on the signed distance function near a $C^2$ hypersurface in a Riemannian manifold. In Appendix B, we describe a stability result of solvability of $\bar{\partial}$-equation on $a_q$ domains with $C^2$ boundary using results in Hörmander [31].

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2. A local approximate homotopy formula

Let $X$ be a complex manifold of dimension $n$. Let $D$ be a relatively compact domain in $X$ defined by $\rho < 0$, where $\rho$ is a $C^2$ defining function with $\nabla \rho(\zeta) \neq 0$ when $\rho(\zeta) = 0$. Throughout the paper, $\nabla^k \rho$ is the set of $k$-th partial derivatives in local coordinates. We recall the following definitions in Henkin–Leiterer [29] and Hörmander [31].

**Definition 2.1.** Let $D \subset X$ be a $C^2$ domain defined by $\rho < 0$ as above.

(a) For $\zeta \in \partial D$, the Levi-form $L_\zeta \rho$ in local holomorphic coordinates $z$ on $X$ is the complex Hessian

$$H_\zeta \rho(w) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\zeta)w_j \bar{w}_k, \quad w \in \mathbb{C}^n$$

restricted on the complex tangent space $T^{1,0}_\zeta(\partial D)$, where the latter is identified with $\{w \in \mathbb{C}^n : \sum \frac{\partial \rho}{\partial z_j}(\zeta)w_j = 0\}$.

(b) $\partial D$ is strictly $q$-convex (resp. $q$-concave) at $\zeta \in \partial D$, if $L_\zeta \rho$ has at least $q$ positive (resp. negative) eigenvalues.

(c) $D$ satisfies the condition $a_q$ if $L_\zeta \rho$ has at least either $(q + 1)$ negative eigenvalues or $(n - q)$ positive eigenvalues for every $\zeta \in \partial D$.

Therefore, a domain is strictly pseudoconvex if and only if it is strictly $(n - 1)$-convex. To see a domain satisfying the condition $a_q$, let $\mathbb{P}^n$ be the complex projective space. Let $B'_q \subset \mathbb{P}^n$ be defined by

$$B'_q : |z'|^2 + \cdots + |z^{q-1}|^2 < r|z|^2 + \cdots + r|z^{q-1}|^2$$

with $1 \leq q \leq n$ and $r > 0$. Then $B'_q$ is both strictly $(n - q)$-convex and $(q - 1)$-concave. When $\partial D$ is Levi non-degenerate, the condition $a_q$ for $\partial D$ is equivalent to the number of negative Levi eigenvalue not being $q$ at every point $\zeta \in \partial D$. Thus $B'_q \cap \overline{B_{q-1}'}$ satisfies the condition $a_q$ if $r_2 > r_1$, $q_2 \geq q_1$, and

$$q \in \{0, \ldots, n - 1\} \setminus \{q_1 - 1, n - q_2 - 1\}.$$
\textbf{Definition 2.2.} Let \( k \geq 1 \) and \( r > 1 \). Let \( k \geq 1 \) and \( r > 1 \). Let \( D^1, \ldots, D^\ell \) be open sets in a complex manifold \( X \). We say that \( \omega := D^1 \cdots D^\ell \) is a \( C^k \) (resp. \( \Lambda^r \) with \( r > 1 \)) transversal intersection (of \( D^1, \ldots, D^\ell \)), if \( \overline{\omega} \) is compact, there are \( C^k \) (resp. \( \Lambda^r \)) real functions \( \rho_1, \ldots, \rho_\ell \) on a neighborhood \( U \) of \( \overline{\omega} \) such that \( D^j \subset U : \rho_j < 0 \), and for any \( 1 \leq j_1 < \cdots < j_\ell \leq \ell \),
\[
d \rho_{j_1}(\zeta) \wedge \cdots \wedge d \rho_{j_\ell}(\zeta) \neq 0, \quad \text{when} \quad \rho_{j_1}(\zeta) = \cdots = \rho_{j_\ell}(\zeta) = 0.
\]

We will need Leray maps, following notation in [29].

\textbf{Definition 2.3.} Let \( D \) be a domain in \( \mathbb{C}^n \). Let \( S \subset \mathbb{C}^n \setminus D \) be a \( C^1 \) submanifold in \( \mathbb{C}^n \). We say that \( g : D \times S \to \mathbb{C}^n \) is a Leray map if \( g \in C^1(D \times S) \) and
\[
g(z, \zeta) \cdot (\zeta - z) \neq 0, \quad \zeta \in S, \quad z \in D.
\]

Throughout the paper, we use
\[
(2.1) \quad g_0(z, \zeta) = \overline{z} - \zeta.
\]

Let \( g^j : D \times S^j \to \mathbb{C}^n \) be \( C^1 \) Leray mappings for \( j = 1, \ldots, \ell \). Let \( w = z - \zeta \). Define
\[
\omega^j = \frac{1}{2\pi i} \frac{dg^j}{g^j} \cdot dw, \quad \Omega^j = \omega^j \wedge (\overline{\partial} \omega^j)^{n-1},
\]
\[
\Omega^{01} = \omega^0 \wedge \omega^1 \wedge \sum_{\alpha + \beta = n-2} (\overline{\partial} \omega^0)^\alpha \wedge (\overline{\partial} \omega^1)^\beta.
\]

Here both differentials \( d \) and \( \overline{\partial} \) are in \( z, \zeta \) variables. In general, define
\[
\Omega^{1\cdots \ell} = \omega^{g_1} \wedge \cdots \wedge \omega^{g_\ell} \wedge \sum_{\alpha_1 + \cdots + \alpha_\ell = n-\ell} (\overline{\partial} \omega^{g_1})^{\alpha_1} \wedge \cdots \wedge (\overline{\partial} \omega^{g_\ell})^{\alpha_\ell}.
\]

Decompose \( \Omega^* = \sum \Omega^*_{(0,q)} \) where \( \Omega^*_{(0,q)} \) has type \((0,q)\) in \( z \). Hence \( \Omega^*_{(0,q)} \) has type \((n,n-\ell-q)\) in \( \zeta \). Set \( \Omega^*_{0,-1} = 0 \) and \( \Omega^*_{0,0+1} = 0 \). The Koppelman lemma says that
\[
\overline{\partial}_\zeta \Omega^*_{(0,q)} + \overline{\partial}_z \Omega^*_{(0,q-1)} = \sum (-1)^j \Omega^*_{(j,q)} + \cdots.
\]

See Chen-Shaw [10, p. 263] for a proof. We will use special cases:
\[
(2.2) \quad \overline{\partial}_\zeta \Omega^1_{(0,q)} + \overline{\partial}_z \Omega^1_{(0,q-1)} = 0,
\]
\[
(2.3) \quad \overline{\partial}_\zeta \Omega^01_{(0,q)} + \overline{\partial}_z \Omega^01_{(0,q-1)} = -\Omega^01_{(0,q)} + \Omega^00_{(0,q)},
\]
\[
(2.4) \quad \overline{\partial}_\zeta \Omega^102_{(0,q)} + \overline{\partial}_z \Omega^102_{(0,q-1)} = -\Omega^12_{(0,q)} + \Omega^02_{(0,q)} - \Omega^01_{(0,q)}.
\]

Here each identity holds in the sense of distributions on the set where the forms are non-singular.

To integrate on submanifolds of \( \mathbb{C}^n \), let us see how a sign changes when an exterior differentiation interchanges with integration. Following [10, p. 263], define
\[
\int_{y \in M} u(x,y) dy^j \wedge dx^j = \left\{ \int_{y \in M} u(x,y) dy^j \right\} dx^j
\]
for a function \( u \) on a manifold \( M \) with boundary. For the exterior differential \( d_x \), we have
\[
d_x \int_{y \in M} \phi(x,y) = (-1)^{\dim M} \int_{y \in M} d_x \phi(x,y).
\]
Then, Stokes’ formula has the form
\begin{equation}
\int_{y \in \partial M} \phi(x, y) \wedge \psi(y) = \int_{y \in M} \{d_y \phi(x, y) \wedge \psi(y) + (-1)^{\text{deg } \phi} \phi(x, y) \wedge d\psi(y)\},
\end{equation}
where \( \text{deg } \phi \) is the total degree of \( \phi \) in \((x, y)\). When \( D^{1 \ldots \ell} \) is a \( C^1 \) transversal intersection, we choose orientations so that
\[ \int_{D^{1 \ldots \ell}} f = \sum_{i=1}^{\ell} \int_{D^{1 \ldots \ell} \cap \partial D^i} f. \]
Suppose \( D^{12} \) is a \( C^1 \) transversal intersection. Then we define
\[ S^i := D^{12} \cap \partial D^i, \quad \partial D^{12} = S^1 \cup S^2; \quad S^{12} := \partial S^1, \quad S^{21} := \partial S^2. \]
Thus Stokes’ formula has the following special cases
\[ \int_{D^1 \cap D^2} f = \int_{S^1} f + \int_{S^2} f; \quad \int_{S^1} df = \int_{S^{12}} f; \quad \int_{S^2} df = \int_{S^{21}} f. \]
Next, we introduce integrals on domains and lower-dimensional sets:
\begin{equation}
\int_{D_{\rho_1 \ldots \rho_\ell}} f(z) := \int_D \Omega^{(1 \ldots \ell)}(z, \zeta) \wedge f(\zeta), \quad \int_{S_{\rho_1 \ldots \rho_\ell}} f := \int_{S_{\rho_1 \ldots \rho_\ell}} \Omega^{(1 \ldots \ell)}(z, \zeta) \wedge f.
\end{equation}
Let \( E_D : C^0(D) \to C^0(R^n) \) be the Stein extension operator such that
\[ \|E_D u\|_{\Lambda'(R^n)} \leq C(D)\|u\|_{\Lambda'(D)} \]
where the Hölder–Zygmund norm and its equivalent norms are defined in Section 5.
Note that the extension exists for any bounded Lipschitz domain \( D \). See [19, Prop. 3.11] for a proof of the extension property and references therein.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.1.png}
\caption{\( \partial D^{12} = S^1 \cup S^2 \) and \( \partial U^1 = S^1 \cup S^1_+ \)}
\end{figure}

The main purpose of this section is to derive the following approximate homotopy formula on a bounded domain that is a \( C^1 \) transversal intersection.

**Notation 2.4.** In all figures, when a domain is denoted by a letter such as \( D^1, U^1 \) in Figure 2.1, the letter is always placed inside the domain and next to the boundary of the domain. In all figures, \( D^2 \) is the ball \( B_{r_2} := \{ z \in C^n : |z|^2 < r_2^2 \} \).
Proposition 2.5 (see Figure 2.1). Let \( D^{12} \subset \mathbb{C}^n \) be a bounded domain of \( C^1 \) transversal intersection. Let \( S^1, S^2 \) be given by (2.6). Let \( U^1 \subset \mathbb{C}^n \setminus D^{12} \) be a bounded domain with piecewise \( C^1 \) boundary such that \( \partial U^1 = S^1 \cup S^1_n \) with \( S^1_n = \partial U^1 \setminus S^1 \). Suppose that \( g^1(\cdot, \cdot) \) is a \( C^1 \) Leray map on \( D^{12} \times \overline{U^1} \) and \( g^2 \) is a \( C^1 \) Leray map on \( D^{12} \times S^2 \). Let \( f \) be a \((0,q)\)-form such that \( f \) and \( \partial f \) are in \( C^1(\overline{D^{12}}) \). Then on \( D^{12} \) we have

\[
(2.8) \quad f = L^1_{1; q} f + L^2_{2; q} f + L^1_{12; q} f + \overline{\partial H q} f + H_{q+1} \overline{\partial f}, \quad \text{if } q \geq 1,
\]

\[
(2.9) \quad f = L^1_{1; 0} f + L^2_{2; 0} f + L^1_{12; 0} f + \overline{H} \partial f, \quad \text{if } q = 0
\]

where for \( s = q, q+1 \) with \( q \geq 1 \) and \( E = E_{D^{12}} \)

\[
(2.10) \quad H_s f := H_s^{(1)} f + H_s^{(2)} f,
\]

\[
(2.11) \quad H_s^{(1)} f := R_{0; 1:s-1}^1 \partial f + R_{0; 1:s-1}^1 \overline{\partial, E} f,
\]

\[
(2.12) \quad H_s^{(2)} f := -R_{0; 1:s-1}^1 \partial f + L_{1; 1:s-1}^1 \partial f + L_{2; 1:s-1}^2 \partial f + L_{1; 12; 1:s-1}^1 \partial f,
\]

\[
(2.13) \quad L_{1; 1:s-1}^1 \partial f := \int_{S^1_{s-1}} \Omega^0_{0, s-1} \wedge \partial f,
\]

\[
(2.14) \quad L_{0; 1:s-1}^1 \partial f := \int_{D^{12}} \Omega^0_{0, s-1} \wedge \partial f - \int_{U^1} \Omega^0_{1, s-1} \wedge \overline{\partial, E} f = \int_{U^1} \Omega^0_{1, s-1} \wedge \overline{\partial, E} f.
\]

Remark 2.6. Formula (2.8) is called an approximate homotopy formula due to the presence of the boundary integrals \( L^1_{1; q}, L^2_{2; q}, L^1_{12; q} \). These boundary integrals will be transformed under further conditions on the Levi-form of \( \partial D^1 \).

Proof. Consider first \( q \geq 1 \). We recall the Bochner–Martellini–Koppelman formula [10, p. 273] and a version for domains with piecewise \( C^1 \) boundary [29, Def. 3.1, p. 46; Thm. 3.12, p. 53]:

\[
f(z) = \overline{\partial} \int_{D^{12}} \Omega^0_{0; q-1} (z, \zeta) \wedge f(\zeta) + \int_{D^{12}} \Omega^0_{0; q} (z, \zeta) \wedge \overline{\partial f}(\zeta)
\]

\[
+ \int_{D^{12}} \Omega^0_{0; q} (z, \zeta) \wedge f(\zeta).
\]

Here and in what follows, we assume \( z \in D^{12} \). To ease notation, we drop the degree indicator in \( L^1_{1; q} \) and write it as \( L^1 \) and do the same for other operators. To apply Stokes’ formula on \( \partial D^{12} \), we use notation in (2.2)-(2.4) and rewrite the last term as

\[
\int_{\partial D^{12}} \Omega^0_{0; q} (z, \zeta) \wedge f(\zeta) = \int_{S^1} \Omega^0_{0; q-1} (z, \zeta) \wedge f(\zeta) + \int_{S^2} \Omega^0_{0; q} (z, \zeta) \wedge f(\zeta).
\]

Let us transform two boundary integrals via Koppelman’s lemma. By (2.3), we get

\[
\int_{S^1} \Omega^0_{0; q} (z, \zeta) \wedge f(\zeta) = L^1_{1} f(\zeta) + \int_{S^1} (\overline{\partial} \Omega^0_{0; q-1} (z, \zeta) + \zeta \Omega^0_{0; q-1} (z, \zeta)) \wedge f(\zeta)
\]

\[
= L^1_{1} f(\zeta) + L^1_{12} f(\zeta) - \int_{S^1} \Omega^0_{0; q-1} (z, \zeta) \wedge \zeta f(\zeta) - \zeta \int_{S^1} \Omega^0_{0; q-1} (z, \zeta) \wedge f(\zeta),
\]

where the last identity is obtained by Stokes’ formula for \( S^1 \) with \( \partial S^1 = S^{12} \). Analogously, we get

\[
L^0_{0; q} = L^2_{2} f + L^0_{2} f - L^0_{2} \zeta f - \zeta L^0_{2} f.
\]
Using $L_{12}^0 = -L_{12}^0 f$, we get

$$L_{12}^0 f(z) + L_{21}^0 f(z) = -L_{12}^0 f(z) + \int_{S_{12}} \overline{\partial}_z \Omega_{(0,q-1)}^0(z,\zeta) \wedge f(\zeta) + \int_{S_{12}} \overline{\partial}_\zeta \Omega_{(0,q)}^0(z,\zeta) \wedge f(\zeta).$$

Using Stokes’ theorem to the last term and $\partial(S^1 \cap S^2) = \emptyset$, we obtain from (2.5)

$$L_{12}^0 f + L_{21}^0 f = -L_{12}^0 f + \overline{\partial} L_{12}^0 f + L_{12}^0 f.$$

This shows that

$$f(z) = -\overline{\partial}_z \int_{S_1} \Omega_{(0,q-1)}^0(z,\zeta) \wedge f(\zeta) + \overline{\partial}_z \int_{D_{12}} \Omega_{(0,q-1)}^0(z,\zeta) \wedge f(\zeta)$$

$$- \int_{S_0} \Omega_{(0,q-1)}^0(z,\zeta) \wedge \overline{\partial} f(\zeta) + \int_{D_{12}} \Omega_{(0,q)}^0(z,\zeta) \wedge \overline{\partial} f(\zeta)$$

$$- L_{12}^0 f(z) + \overline{\partial} L_{12}^0 f(z) + L_{12}^0 \overline{\partial} f(z) - L_{12}^0 \overline{\partial} f(z) - \overline{\partial} L_{12}^0 f(z).$$

Next, we transform both integrals on $S^1$ into volume integrals using Stokes’ formula. Here we need to modify the methods in Lieb–Range [46] and [19], since $S^1$ has boundary.

For the rest of the proof let $E$ be the Stein extension operator $E_{D_{12}}$.

With orientations, we have $\partial U^1 = S^1 \cap S^1$. By Stokes’ formula and [19, (2.12)], we have

$$- \int_{\zeta \in S^1} \Omega_{(0,q-1)}^0(z,\zeta) \wedge f(\zeta) + L_{12}^0 f(z) =$$

$$\int_{U_1} \Omega_{(0,q-1)}^0(z,\zeta) \wedge \overline{\partial} f(\zeta) + \int_{U_1} \Omega_{(0,q-1)}^0(z,\zeta) \wedge f(\zeta)$$

$$- \int_{U_1} \Omega_{(0,q-1)}^0(z,\zeta) \wedge E f(\zeta) + \int_{U_1} \overline{\partial} \Omega_{(0,q-2)}^0(z,\zeta) \wedge f(\zeta).$$

This shows that on $D_{12}$

$$R_{D_{12}}^0 f - L_{12}^0 f = -L_{12}^0 f + R_{U_1}^0 \overline{\partial} f$$

$$+ R_{U_1 \cup D_{12}}^0 f - R_{U_1}^0 f + \overline{\partial} R_{U_1}^0 f.$$

After applying $\overline{\partial}$, the last term will be dropped. This shows that

$$\overline{\partial} R_{D_{12}}^0 f - \overline{\partial} L_{12}^0 f = -L_{12}^0 \overline{\partial} f - \overline{\partial} R_{U_1}^0 f$$

$$+ \overline{\partial} R_{U_1 \cup D_{12}}^0 f.$$ 

We apply (2.15) in which $f$ is replaced by $\overline{\partial} f = E \overline{\partial} f$ to obtain

$$R_{D_{12}}^0 \overline{\partial} f - L_{12}^0 \overline{\partial} f = -L_{12}^0 \overline{\partial} f + R_{U_1 \cup D_{12}}^0 \overline{\partial} f$$

$$+ \overline{\partial} R_{U_1 \cup D_{12}}^0 \overline{\partial} f.$$

We can pair the last term with the second last term in (2.16) to form the desired commutator $[E, \overline{\partial}] f$. Finally, we have $\overline{\partial} E \overline{\partial} f = [\overline{\partial}, E] \overline{\partial} f$. This completes the proof of (2.8). The above proof is still valid for (2.9) (case $q = 0$), as the Koppelman lemma holds with $\Omega_{0,-1} = 0$.

Strictly speaking, the above computation is only valid when $\partial D \in C^3$, since the Koppelman lemma can be verified easily when all Leray maps $W_j \in C^2$. When $\partial D \in C^2$, one can still verify the integral formula on the domain $D^1 \cap D^2$ by smoothing $g^j$. For instance, see [21, p. 6808] for details. □
3. A local homotopy formula for \((n-q)\) convex configuration

The main purpose of this section is to construct a local homotopy formula near a boundary point of strictly \((n-q)\) convex.

In Proposition 2.5, we have derived a local approximate homotopy formula (2.8):

\[ f = L_{1;\varrho} f + L_{2;\varrho} f + L_{12;\varrho} f + \overline{\partial}H_{\varrho} f + H_{\varrho + 1} \overline{\partial} f. \]

To obtain a genuine local homotopy formula, we will show that the boundary integrals \(L_{1;\varrho} f, L_{2;\varrho} f, L_{12;\varrho} f\) vanish when the boundary is \((n-q)\) convex and the Leray mappings \(g^1, g^2\) are chosen appropriately.

The constructions in this section and the next are inspired by Henkin–Leiterer [29].

We first transform a \((n-q)\)-convex domain \(D\) into a new form \(D^1\).

**Lemma 3.1.** Let \(D \subset U_0\) be defined by \(\rho^0 < 0\) with \(\rho^0 \in C^2(U_0)\). Suppose that \(\partial D\) is \((n-q)\)-convex at \(\zeta \in \partial D\) and \(\nabla \rho^0(\zeta) \neq 0\).

(a) There are an open set \(U_1 \subset U_0\) containing \(\zeta\) and a biholomorphic mapping \(\psi\) defined on a neighborhood of \(\overline{U_1}\) such that \(\psi(\zeta) = 0\), \(U := \psi(U_1)\) is a polydisc, and \(D^1 := \psi(U_1 \cap D)\) is defined by

\[ \rho^1(z) = -y_n + \lambda_1 |z_1|^2 + \cdots + \lambda_{q-1} |z_{q-1}|^2 + |z_q|^2 + \cdots + |z_n|^2 + R(z) < 0, \]

where \(|\lambda_j| < 1/4\) and \(R(z) = o(|z|^2)\). There exists \(r_1 = r_1(\nabla \rho^0, \nabla^2 \rho^0) > 0\) such that the boundary \(\partial D^1\) intersects the sphere \(\partial B_{r_1}\) transversally when \(0 < r < r_1\). Furthermore, the function \(R\) in (3.1) is in \(C^a(B_{r_1})\) (resp. \(A^a(B_{r_1})\)), when \(\rho^0 \in C^a(U_0)\) with \(a \geq 2\) (resp. \(A^a(U_0)\) with \(a > 2\)).

(b) Let \(\psi\) be as above. There exists \(\delta(\partial) > 0\) such that if \(\tilde{D}\) is defined by \(\tilde{\rho}^0 < 0\) and \(\|\tilde{\rho}^0 - \rho^0\|_{C^2(U_0)} < \delta(\partial)\), then \(\psi(U_1 \cap \tilde{D})\) is given by

\[ \tilde{\rho}^1(z) = -y_n + \lambda_1 |z_1|^2 + \cdots + \lambda_{q-1} |z_{q-1}|^2 + |z_q|^2 + \cdots + |z_n|^2 + \tilde{R}(z) < 0 \]

with \(\|\tilde{R} - R\|_{A^a(B_{r_1})} \leq C_a \|\tilde{\rho}^0 - \rho^0\|_{A^a(U_0)}\) for \(a > 2\) and \(\|\tilde{R} - R\|_{C^a(B_{r_1})} \leq C_a \|\tilde{\rho}^0 - \rho^0\|_{C^a(U_0)}\) for \(a \geq 2\). There exists \(r_1 > 0\) such that the boundary \(\overline{\partial}(\psi(U_1 \cap \tilde{D}))\) intersects the sphere \(\partial B_{r_2}\) transversally when \(r_1/2 < r_2 < r_1\).

Here \(\delta(\partial)\) depends on the modulus of continuity of \(\nabla^2 \rho^0\).

**Remark 3.2.** (a) Throughout the paper, we denote by \(C(\nabla \rho^0, \nabla^2 \rho^0)\), such as the above \(r_1(\nabla \rho^0, \nabla^2 \rho^0)\), a constant depending on \(\nabla \rho^0, \nabla^2 \rho^0\) in local coordinates. In particular, when \(\rho_0\) is a defining function of an \(a_q\) domain \(D\), the constant \(C(\nabla \rho^0, \nabla^2 \rho^0)\) depends on the Levi-form \(L_\zeta \rho^0\) and the low and upper bounds of \(|\nabla \zeta \rho^0|\) for \(\zeta \in \partial D\).

(b) We will refer to \((D^1, \rho^1)\) as \((D^1, \rho^1)\) indicating the various constants in estimates are uniform in \(\rho^0\) or stable under small \(C^2\) perturbations of \(\partial D\) via \(\rho^0\). Set

\[ D^2: \rho^2(z) = |z|^2 - r_2^2 < 0 \]

with restriction \(r_1 > r_2 > r_1/2\). We will assume that \(D^2\) is contained in the polydisc \(U\) in Lemma 3.1 (a).

**Proof.** (a) We may assume that \(\zeta = 0\). Permuting coordinates yields \(\rho_{\tilde{z}_n}^0 \neq 0\). Let \(\tilde{z}_n = 2 \rho_0 \cdot (\zeta - z) - \sum \rho_{\zeta_k}^0 (\zeta_k - z_k)(z_k - \zeta_k)\) and \(\tilde{z}' = z'\). Then \(\rho_1(\tilde{z}) := \rho^0(\tilde{z})\), where the new domain has a defining function

\[ \rho_1(z) = -y_n + \sum a_j \tilde{z}_j \tilde{z}_k + o(|z|^2). \]
Choose a nonsingular matrix $A$ for a linear change of coordinates

\[(3.3) \quad A\tilde{z} = z \quad \text{with } \tilde{z}_n = z_n.\]

Set $\rho_2(\tilde{z}) := \rho_1(z)$. The new domain has the definition function

$$
\rho_2(z) = -y_n + \sum_{j < q} \lambda_j |z_j|^2 + \sum_{j = q}^{n-1} |z_j|^2 + \sum_{j=1}^n \text{Re}\{a_j z_j \overline{\tilde{z}_n}\} + o(|z|^2),
$$

where $a_1, \ldots, a_{n-1} \in \mathbb{C}$ and $a_n \in \mathbb{R}$. Set $\rho_3 := \rho_2 + \mu \rho_2^2$ with $\mu = a_n + 1$. We get

$$
\rho_3(z) = -y_n + \sum_{j < q} \lambda_j |z_j|^2 + \sum_{j = q}^{n-1} |z_j|^2 + \text{Re}\{\mu z_n^2 + \sum_{j=1}^{n-1} a_j z_j \overline{\tilde{z}_n}\} + o(|z|^2).
$$

Using new coordinates $\tilde{z}_n = z_n - i\mu z_n^2 + iz_n \sum_{j=1}^{n-1} a_j z_j$ and $\tilde{z}' = z'$, we get the defining function of a new domain:

$$
\rho_4(z) = -y_n + \lambda_1 |z|^2 + \cdots + \lambda_{q-1} |z_{q-1}|^2 + |z_q|^2 + \cdots + |z_n|^2 + y_n \text{Re}\{\sum_{j=1}^n b_j z_j\} + o(|z|^2).
$$

We get $|\lambda_j| < 1/4$ after a dilation. Then $\rho_4 + \rho_4 \text{Re}\{\sum b_j z_j\}$, renamed as $\rho_1$, has the form (3.1). As usual, the implicit function theorem can be proved by using the inverse mapping theorem [57, pp. 224-225]. Then the inverse mapping theorem for Zygmund spaces [17] yields the desired smoothness of $R$. The details are left to the reader. The transversality of $\partial D^1$ and $\partial D^2$ also follows from the computation below.

(b) The above construction of $\psi$ is explicit in $\rho^0$ with the exception of the linear change of coordinates $\tilde{z} = A^{-1}z$ in (3.3) that is fixed for all small perturbations $\tilde{\rho}$ of $\rho^0$. Thus, it is easily to check that $\|\tilde{R} - R\|_{C^a(B_{r_1})} \leq C_a \|\rho^0 - \tilde{\rho}\|_{C^a(U_0)}$ for $a \geq 2$ and $\|\tilde{R} - R\|_{C^a(B_{r_1})} \leq C_a \|\rho^0 - \tilde{\rho}\|_{C^a(U_0)}$ for $a > 2$.

We want to show that $\nabla \tilde{\rho}^1$ is not proportional to $\nabla \rho^2$ on the common zero set of $\tilde{\rho}^1, \rho^2$. Suppose that $\nabla \rho^2 = \mu \nabla \tilde{\rho}^1$ when $\tilde{\rho}^1(z) = \rho^2(z) = 0$. We get $2y_n = \mu(-1 + 2y_n + \tilde{R}_n)$. When $r < 1/4$ and $\delta(D)$ are sufficient small, by $|z| = r$ we obtain $| - 1 + 2y_n + \tilde{R}_n | < 1/2$. Hence $-\mu^{-1} y_n \in (1/4, 3/4)$ as

$$
\|\tilde{R}\|_{C^2(B_{r_1})} < 1/C.
$$

For $j < n$, we have $2y_j = 2\mu \tilde{\rho}_j$. This shows that $|y_j| \leq C|y_n|$. Also, $|x_k| \leq C|y_n|$. Thus $\tilde{\rho}^1(z) = 0$ implies $|y_n| \leq C' |y_n|^2 + |\tilde{R}(z)|$. In view of $C'|y_n| < 1/2$, we get

$$
|z| \leq C\|\tilde{R}\|_{C^2(B_{r_1})}|z|^2 + C\delta(D).
$$

By choosing $\delta(D)$ depending on $r_1$, we get $|z| < r_1^2/C$. The latter contradicts the vanishing of $\rho^2(z) = |z|^2 - r_2^2$ since $r > r_1/2$. \qed

Recall that our original domain $D$ is normalized as $D^1$. We now fix notation. Let $(D^1, U, \phi, \rho^1)$ be as in Lemma 3.1. Thus $\rho^1$ is given by (3.1) (or (3.2)). Recall that

\[(3.4) \quad \rho^2(z) = |z|^2 - r_2^2\]
where $0 < r_2 < r_1$ and $r_1/2 < r_2 < r_1$ for Lemma 3.1 (a), (b). Let us define

\begin{align}
(3.5) \quad D^1: \rho^1 < 0, \quad D^2: \rho^2 < 0, \quad D^{12} = D^1 \cap D^2, \\
(3.6) \quad \partial D^{12} = S^1 \cup S^2, \quad S^i \subset \partial D^i, \\
(3.7) \quad g^2(z, \zeta) = \left(\frac{\partial \rho^2}{\partial \zeta_1}, \ldots, \frac{\partial \rho^2}{\partial \zeta_n}\right) = \overline{\zeta}.
\end{align}

It is well-known that

\begin{align}
(3.8) \quad |g^2(z, \zeta) \cdot (\zeta - z)| > 0, \quad (z, \zeta) \in D^2 \times \partial D^2.
\end{align}

**Lemma 3.3.** Let $(D^1, U, \phi, \rho^1)$ be as in Lemma 3.1. Define

\begin{align}
(3.9) \quad g_j^1(z, \zeta) = \left(\frac{\partial \rho^1}{\partial \zeta_1}, \ldots, \frac{\partial \rho^1}{\partial \zeta_n}\right) + (\zeta_j - \bar{z}_j), \quad 1 \leq j < q.
\end{align}

Then for $\zeta, z \in U$ and by shrinking $U$ if necessary, we have

\begin{align}
(3.10) \quad 2 \text{Re}\{g^1(z, \zeta) \cdot (\zeta - z)\} & \geq \rho^1(\zeta) - \rho^1(z) + \frac{1}{2}|\zeta - z|^2, \\
(3.11) \quad |g^1(z, \zeta) \cdot (\zeta - z)| & \geq 1/C_*, \quad \forall \zeta \in S^{12}, z \in B_{\tilde{r}_2}
\end{align}

where $\tilde{r}_2 < r_2/C$. Here $C_*$ depend $\nabla \rho^1, \nabla^2 \rho^1$ (and hence only on $\nabla \rho^0, \nabla^2 \rho^0$), and $r_2; C_*$ is independent of $\tilde{r}_2$.

**Proof.** We have $\text{Re}\{g^1(z, \zeta) \cdot (\zeta - z)\} = \text{Re}\{\rho^1(z, \zeta) \cdot (\zeta - z)\} + \sum_{j < q} |\zeta_j - \bar{z}_j|^2$. By Taylor’s theorem, we have

\begin{align}
(3.12) \quad \rho^1(z) - \rho^1(\zeta) = 2 \text{Re}\{\rho^1_\zeta(z - \zeta)\} + \text{Re}\{\rho^1_{\zeta_1, \zeta_k}(z_j - \zeta_j)(z_k - \zeta_k)\} \\
&+ H_{\zeta^2}\rho^1(z - \zeta) + R(z, \zeta).
\end{align}

Note that $H_{\zeta^2}\rho^1$, restricted to the $(z_2, \ldots, z_n)$ subspace, is a positive definite quadratic form. Also, $\rho^1_{\zeta_1, \zeta_k}$ and second-order derivatives of $R$ are small. We can show that

\begin{align}
\rho^1(z) - \rho^1(\zeta) \geq 2 \text{Re}\{\rho^1_\zeta(z - \zeta)\} + \sum_{j < q} |\zeta_j - \bar{z}_j|^2 - c|\zeta - z|^2
\end{align}

where $c < 1/2$. Now (3.10) follows from (3.1). Note that (3.11) follows from (3.10) as $\rho^1(\zeta) = 0, |\zeta - z|^2 \geq r_2^2/4$ for $\zeta \in S^{12} \subset (\partial D^1) \cap \partial D^2_{\tilde{r}_2}$, and $|\rho^1(\zeta)| < C\tilde{r}_2$ for $z \in B_{\tilde{r}_2}$. By (3.12) and (3.10) for case (3.1), we also get (3.10) for the $\tilde{\rho}^1$ in (3.2) when $\|\tilde{\rho}^0 - \rho^0\|_2 \leq C\delta(D)$ is small. \qed

**Definition 3.4.** (a) As in [29, p. 80], the $(U, D^1, D^2, \psi, \rho^1, \rho^2)$ in Lemma 3.1 (a) and (3.4)-(3.5) is called an $(n - q)$-convex configuration (of type I). Lemma 3.1 (b) is referred to as the stability of the configuration. In brevity, we call $(D^1, D^2)$ an $(n - q)$-convex configuration. See Figure 3.1, i.e. the figure for type I in [29, p. 80].

(b) The $g^1, g^2$ given by (3.7) and (3.9) are called the canonical Leray maps for the $(n - q)$-convex configuration $(D^1, D^2)$. 
Figure 3.1. Convex configuration: $D^{12} = D^1 \cap D^2$

Note that $g^2(z, \zeta)$ is holomorphic in $z$ and $g^1(z, \zeta)$ is anti-holomorphic in merely $q - 1$ variables of $z$. Checking the types, we have

\begin{align}
\Omega_{(0,k)}^2(z, \zeta) &= 0, \quad k \geq 1; \quad \overline{\partial} \Omega_{(0,0)}^2(z, \zeta) = 0; \\
\Omega_{(0,k)}^1(z, \zeta) &= 0, \quad k \geq q; \quad \overline{\partial} \Omega_{(0,q-1)}^1(z, \zeta) = 0; \\
\Omega_{(0,k)}^{12}(z, \zeta) &= 0, \quad k \geq q; \quad \overline{\partial} \Omega_{(0,q-1)}^{12}(z, \zeta) = 0.
\end{align}

By (3.13)-(3.15), we have for $z \in D^{12}$

\begin{align}
L_i^q f(z) &= \int_{S_i} \Omega_{(0,q)}^i(z, \zeta) \wedge f(\zeta) = 0, \quad i = 1, 2; \quad L_{12}^q f(z) = 0; \\
\overline{\partial} \int_{U_i} \Omega_{(0,q-1)}^1(z, \zeta) \wedge Ef(\zeta) = 0, \quad \int_{U_i} \Omega_{(0,q)}^1(z, \zeta) \wedge E\overline{f}(\zeta) = 0.
\end{align}

This shows that for $s = q, q + 1$, the $H_s^{(2)}$ in (2.12) can be replaced by

\begin{equation}
H_s^{(2)} f = L_{1+q,s-1}^1 Ef + L_{2,s-1}^2 Ef + L_{12,s-1}^{12} f.
\end{equation}

Therefore, we have obtained the following local homotopy formula.

**Theorem 3.5.** Let $0 < q \leq n$. Let $(D^1, D^2)$ be a $(n-q)$-convex configuration with Leray maps (3.7)-(3.9). Let $U^1 = D^2 \setminus D^1$ and $S_+^1 = \partial D^2 \setminus D^1$. Suppose that $f$ is a $(0,q)$ form such that $f$ and $\overline{\partial} f$ are in $C^1(D^2)$. Then on $D^{12}$

$$f = \overline{\partial} H_q f + H_{q+1} \overline{\partial} f$$

with $H_s = H_s^{(1)} + H_s^{(2)}$, for $H_s^{(1)}$ defined by (2.11) and $H_s^{(2)}$ defined by (3.18).

We remark that the kernel in $H_s^{(2)}$ is smooth, since $S_+^1, S^2$ and $S^{12}$ do not intersect small neighborhoods of the origin in $\partial D^1$. Therefore, terms in $H_s^{(2)}$ can be estimated easily, while the main term $H_s^{(1)}$ will be estimated in Section 6.

4. A local $\overline{\partial}$ solution operator for $(q + 1)$-concave configuration

We recall from Proposition 2.5 the approximate local homotopy formula

\begin{equation}
f = L_{1,q}^1 f + L_{2,q}^2 f + L_{12,q}^{12} f + \overline{\partial} H_q f + H_{q+1} \overline{\partial} f.
\end{equation}
As the strictly \((n-q)\)-convex case, we will show that \(L_{1,q}^2 f, L_{2,q}^2 f\) vanish when the boundary is \((q+1)\)-concave and the Leray mappings \(g_1, g_2\) are chosen appropriately. However, the boundary integral \(L_{12,q}^3 f\) may not vanish. We shall show that this term is \(\overline{\partial}\)-closed for a \(\overline{\partial}\)-closed \(f\), and this allows us to use a third Leray mapping to transform it into a genuine \(\overline{\partial}\) solution operator \(f = \overline{\partial}H_q f\) for possibly different \(H_q\). Thus, the \((q + 1)\) concavity is sufficiently to construct a \(\overline{\partial}\) solution operator.

The presence of \(L_{12,q}^3 f\) will lead to a subtlety. For a local homotopy formula for forms which are not necessarily \(\overline{\partial}\)-closed, we however need an extra negative Levi eigenvalue, which will be assumed at the end of the section.

The following is a restatement of Lemma 3.1, by considering the complement \((D^1)^c\) and \(-\rho_0\) where \(\rho_0\) defines \(D^1\).

**Lemma 4.1.** Let \(D \subset U_0\) be defined by \(\rho_0 < 0\) with \(\rho_0 \in C^2(U_0)\). Suppose that \(\partial D\) is \((q + 1)\)-concave at \(\zeta \in \partial D\) and \(\nabla \rho^0(\zeta) \neq 0\). Then all assertions, including dependence of various constants on \(\nabla \rho^0, \nabla^2 \rho^0\), in Lemma 3.1 on \(\rho^1, \psi\) are valid, provided (3.1)-(3.2) are replaced by

\[
\begin{align*}
\rho^1(z) &= -y_{q+2} - |z|^2 -\ldots - |z_{q+2}|^2 + \lambda_{q+3}|z_{q+3}|^2 + \ldots + \lambda_n|z_n|^2 + R(z), \\
\tilde{\rho}^2(z) &= -y_{q+2} - |z|^2 -\ldots - |z_{q+2}|^2 + \lambda_{q+3}|z_{q+3}|^2 + \ldots + \lambda_n|z_n|^2 + \tilde{R}(z)
\end{align*}
\]

with \(|\lambda_j| < 1/4\) for \(j > q + 2\).

As in Lemma 3.1, we assume \(\psi(U_1)\) is a polydisc \(U\).

When \(\rho^1\) has the form (4.2), as in [29, pp. 118-120] define

\[
g^j_1(z, \zeta) = \begin{cases} \\
\frac{\partial \rho^1}{\partial z_j} + \zeta_j - \zeta, & 1 \leq j \leq q + 2, \\
\frac{\partial \rho^1}{\partial z_j} + \zeta_j - \zeta, & q + 3 \leq j \leq n.
\end{cases}
\]

Note that this kind of Leray mappings was used by Hortmann [33, 34] for strictly concave domains. Then we have

\[
-2 \text{Re}\{g^1(z, \zeta) \cdot (\zeta - z)\} \geq \rho(\zeta) - \rho(z) + |\zeta - z|^2/C.
\]

An essential difference between the \(q\)-convex and \((q+1)\) concave cases is that \(g^1(z, \zeta)\) is no longer \(C^\infty\) in \(z\) when \(\partial D\) is only finitely smooth. Nevertheless, a useful feature is that \(g^1(z, \zeta)\) is holomorphic in \(\zeta_1, \ldots, \zeta_{q+2}\).

As \((n-q)\)-convex case, we use \(D^2_{\tilde{z}}: \tilde{\rho}^2(z) := |z|^2 - r_2^2 < 0\). We still take Leray maps \(\rho^0(z, \zeta) = \overline{\zeta} - \zeta\) and \(g^0(z, \zeta) = \frac{\partial \rho^0}{\partial \zeta_1}, \ldots, \frac{\partial \rho^0}{\partial \zeta_n} = \overline{\zeta}\). Denote by \(\deg_\zeta\) the degree of a form in \(\zeta\). We get

\[
\deg_\zeta \Omega^1_{(0,1)}(z, \zeta) \leq 2n - q - 2.
\]

Therefore, we still have (3.16). Thus (4.1) becomes

\[
f = L_{12,q}^2 f + \overline{\partial}H_q f + H_{q+1} f .
\]

However, unlike the \((n-q)\) convex case, \(L_{12,q}^3 f = \int_{S^12} \Lambda_{(0,q)} \wedge f\) may not be identically zero. Let us transform this integral on \(S^12\) via Stokes’ formula. We intersect \(D^1 \cap D^2\) with a third domain

\[
D^3: \rho^3 < 0, \quad 0 \in D^3
\]

where

\[
\rho^3(z) := -y_{q+2} + \sum_{j=q+3}^n 3|z_j|^2 - r_3^2
\]
with $0 < r_3 < r_2/C_n$; see Lemma 4.2 below for further restrictions. As in [29, p. 120] define

$$g^3_j(z, \zeta) = \begin{cases} 0, & 1 \leq j < q + 2, \\ i, & j = q + 2, \\ 3(\zeta_j + \pi_j), & q + 3 \leq j \leq n. \end{cases}$$

We can verify

$$\text{Re}\{g^3(z, \zeta) \cdot (\zeta - z)\} = \rho^3(\zeta) - \rho^3(z).$$

Note that $0 \in D^{23}$. See Figure 4.1 for relations of $D^1, D^2, D^3$.

Then we have the following.

**Lemma 4.2.** Let $\rho^i, g^i$ be defined by (3.4), (3.7), (4.2), (4.4), (4.9)-(4.10) for $i = 1, 2, 3$.

(a) There exists $r_1$, depending on $\nabla \rho^0, \nabla^2 \rho^0$, such that $\partial D^1, \partial D^2, \partial D^3$ pairwise intersect transversally when $0 < C_n r_3 < r_2 < r_1$ and $r_1 < 2r_2$.

(b) Let $\bar{\rho}^0, \bar{\rho}^1$ be as in Lemma 4.1 and $\bar{D}^3$ be defined by $\bar{\rho}^1 < 0$. If $\delta(D)$ is sufficiently small, $\|\bar{\rho}^0 - \rho^0\|_2 < \delta(D)$, and $1/C'_{n} < C_n r_3 < r_2 < r_1$ and $r_2 < r_1/2$, then $\partial \bar{D}^1, \partial \bar{D}^2, \partial \bar{D}^3$ pairwise intersect transversally.

(c) Let $r_1, r_2, r_3$ be as in (b). Here $C'_n$ depends only on $n$. Then

$$\partial \bar{D}^1 \cap \partial D^2 \cap D^3 = \emptyset, \quad \partial \bar{D}^1 \cap \partial D^3 \cap \partial D^2 = \emptyset,$$

$$|g^i(z, \zeta) \cdot (\zeta - z)| \geq 1/C_n, \quad \forall (z, \zeta) \in B_{r_2} \times \overline{D^2 \setminus (D^1 \cup D^3)}, \quad i = 0, 2, 3,$$

$$S^{12} \subset \overline{D^2 \setminus (D^1 \cup D^3)}.$$

Here $C_n, C'_n$ depend only on $\nabla \rho^1, \nabla^2 \rho^1$ (and hence only on $\nabla \rho^0, \nabla^2 \rho^0$).

**Proof.** (a) Suppose $\nabla \rho^1(z) = \mu \nabla \rho^3(z)$, and $\rho^3(z) = \rho^3(0) = 0$. We have $-1 + 2y_{q+j} + R_{y_{q+j}} = \mu$. This shows that $-3/2 < \mu < -1/2$. We also have $z_j + o(|z|) = 0$ for $j = 1, \ldots, q + 1$, $x_{q+j} + o(|z|) = 0$, and $\lambda_j z_j + o(|z|) = 3\mu z_j$ for $j > q + 2$. The latter implies that $|z_j| = o(|z|)$ since $|3\mu| - |\lambda_j| > 1/4$. Hence, $\rho^1(z) = 0$ yields $y_{q+j} = o(|z|)$. This shows that $z = 0$, which contradicts $\rho^3(0) < 0$. 
To show $\partial D^2$ and $\partial D^3$ intersect transversally, suppose that at an intersection point $z$ we have $\nabla\rho^2 = \mu \nabla\rho^3$. We first get $2y_{q+2} = -\mu$. Then $\nabla\rho^2 = \mu \nabla\rho^3$ implies that $|z| \leq C_n |y_{q+2}|$. We get $|y_{q+2}| \geq r_3/C_n$ and $|y_{q+2} + z| \leq r_3 + C_n^2 |y_{q+2}|^2$. Then $2r_3 \geq |y_{q+2}| \geq r_3/C_n$, a contradiction to the assumption $r_3 < r_2/C_n$. Note that $\partial D^1, \partial D^2$ intersect transversally was proved in Lemma 3.1.

(b) We leave the details to the reader.

(c) Suppose $\tilde{\rho}^2(z) = 0$ and $\rho^3(z) < 0$. Then we have

$$|z|^2 + \cdots + |z_{q+2}|^2 + \sum_{j>q+2} (\lambda_j + 3)|z_j|^2 - \tilde{R}(z) < r_3^2.$$ 

This shows that $|z| < 2r_3$. We obtain the first identity in (4.12). The proof of the second identity in (4.12) is similar.

We now verify (4.13). Cases for $i = 0, 2$ are trivial. Case $i = 3$ follows from (4.8). Finally, (4.14) follows from (4.12).

**Definition 4.3.** (a) As in [29, p. 119], the $(U, D^1, D^2, D^3, \psi, \rho^1, \rho^2, \rho^3)$ in Lemma 4.2 is called a $(q + 1)$-concave configuration, while the stability in Lemma 3.1 for the corresponding $(q + 1)$-concave case will be called as the stability of the configuration. In short, $(D^1, D^2, D^3)$, shown in Figure 4.1 or figure for type 1 in [29, p. 119], is called a $(q + 1)$-concave configuration.

(b) The $g^1, g^2, g^3$ in (4.4), (3.7) and (4.10) are called the canonical Leray maps of the configuration.

Note that the anti-holomorphic differentials $d\zeta_j, d\zeta$ appear in $\Omega^3$ as a wedge product in some of

$$d(\zeta_j - \zeta_j), \quad q + 3 \leq j \leq n.$$ 

Consequently, $d\zeta$ and $d\zeta_j$, having the same index, cannot appear in $\Omega^3_{(0,q)}$ simultaneously. Therefore

$$\text{deg}_\zeta\Omega^3_{(0,q)}(z, \zeta) \leq n + ([n - (q + 3) + 1] - \ell) = 2n - q - \ell - 2, \quad \forall \ell.$$

We now derive a result analogous to [29, Lem. 13.6 (iii), p. 122] but for different boundary integrals $L_{ij}$.

**Lemma 4.4.** Let $0 < q \leq n - 2$. Let $(D^1, D^2, D^3)$ be a $(q+1)$ concave configuration with Leray maps $(g^1, g^2, g^3)$. Let $U^1 = D^2 \setminus D^3$ and $S^1_+ = \partial D^2 \setminus D^1$. Then

$$\Omega^3_{(0,q)}(z, \zeta) = 0, \quad \ell < q: \quad \overline{\partial}_\zeta \Omega^3_{(0,q)}(z, \zeta) = 0.$$ 

(a) Suppose that $f \in C^1_{(0,q)}(D^1)$ is $\overline{\partial}$-closed on $D^1$. Then on $D^1_{123}$

$$L^1_{12,q} f = 0,$$

(b) If $(D^1, D^2, D^3)$ is a $(q+2)$-concave configuration and $0 < q \leq n - 3$, then (4.17) and (4.18) are valid on $D^1_{123}$ for any $f \in C^1_{(0,q)}(D^1)$.

Proof. To verify (4.16), we note that $\Omega^3_{(0,q)}(z, \zeta)$ has type $(0, q)$ in $z$. It has type $(n, n - 2 - q)$ in $\zeta$ and it is holomorphic in $\zeta_1, \ldots, \zeta_{q+2}$. After taking $\overline{\partial}_\zeta$, it has type $(n, n - q - 1)$ in $\zeta$ for anti-holomorphic variables $\zeta_{q+3}, \ldots, \zeta_n$. However, the number of these anti-holomorphic variables is $< n - (q + 3) + 2 = n - q - 1$. We have verified (4.16).
(a) To verify (4.17), we need an approximation theorem of Henkin–Leiterer [29, Lemma 13.5 (iii), p. 122] in which $r = n - q - 2$ for our $(q + 1)$ concave domain $D^1$.

Fix $z \in D^{123}$. Let $K = \overline{D^2 \setminus (D^1 \cup D^3)}$. See Figure 4.1 for $K$ and $D^{123}$. Note that $D^{123}$ is open, $K$ is compact, and they are disjoint.

By (4.5), $g^1(z, \zeta) \cdot (\zeta - z) \neq 0$ for $\zeta \in \overline{D^2 \setminus D^1}$. By (4.13), we know that $\Omega^{13}$ is a continuous $(n, n - q - 2)$-forms in $z \in U \supset \overline{D^2}$ such that $\omega^\nu_\zeta$ are $\overline{\partial}$-closed in $U$ and converge to $\Omega^{13}(z, \cdot)$ uniformly on $K$ as $\nu \to \infty$. Using a standard smoothing, we may assume that $\omega^\nu_\zeta$ are $C^1$ in $z \in \overline{D^2}$, $\overline{\partial}$-closed and approximate $\Omega^{13}(z, \cdot)$ uniformly on $K$. By (4.14), $S^{12}$ is contained in $K$. We obtain for each fixed $z \in D^{123}$

$$L_{12;\partial}^{13} f(z) = \lim_{\nu \to \infty} \int_{S^{12}} \omega^\nu_\zeta(\zeta) \wedge f(\zeta).$$

Since $S^1 \subset \partial D^{12} \subset U$ and $\overline{\partial} \omega^\nu_\zeta = 0$ on $U$, Stokes’ formula implies

$$\int_{S^{12}} \omega^\nu_\zeta(\zeta) \wedge f(\zeta) = \int_{S^1} \omega^\nu_\zeta(\zeta) \wedge \partial \zeta f = 0.$$

Hence $L_{12;\partial}^{13} f(z) = 0$. By $\Omega^{12}_{(0,q)} = \Omega^{13}_{(0,q)} - \Omega^{23}_{(0,q)} + \overline{\partial} \Omega^{12}_{(0,q)} + \overline{\partial} \Omega^{13}_{(0,q)}$, we obtain (4.18) through integration on $S^{12}$.

(b) Note that when $(D^1, D^2, D^3)$ is a $(q + 2)$ concave configuration. We have $L_{12;\partial}^{13} f = 0$ for any $(0, q)$ forms $f$ that are not necessarily $\overline{\partial}$-closed by (4.16) that now holds for $\ell < q + 1$. We get the desired conclusion immediately. □

**Remark 4.5.** In the proof for case (i), we do not have any control on $\omega^\nu$ outside $D^2 \setminus (D^1 \cup D^3)$ other than the uniform convergence. Therefore, in (a) it is crucial that $f$ is $\overline{\partial}$-closed.

So far, we have been following Henkin–Leiterer [29]. We could derive a homotopy formula on $D^1 \cap D^2 \cap D^3$ as in [29]. However, since we only need a local homotopy formula near a boundary point, we now departure from the approach in [29]. Let us still use the approximate homotopy formula on $D^{12}$. Modify it only for $z \in D^1 \cap D^2 \cap B_{r_4}$, using mainly (4.12)-(4.14) to construct a $\overline{\partial}$ solution operator on this smaller domain. To this end, we fix $0 < r_4 < r_3/c_n$ so that

\begin{equation}
B_{r_4} \subset D^2 \cap D^3.
\end{equation}

Thus our starting point is still the approximate homotopy formula (4.1). We will however use Koppelman’s lemma for $g^1, g^2, g^3$ on the set $S^{12}$.

We recall the Koppelman homotopy formula for the ball

$$g = \overline{\partial T_{B_{r_4}} g} + T_{B_{r_4}} g \partial g$$

for $g \in C^1_{(0,q)}(\overline{B_{r_4}})$; see [28, Cor. 1.12.2, p. 60; Cor. 2.1.4, p. 68]. We now transform $L_{12;\partial}^{23} f$ in (4.18).

The following is analogous to [29, Lem. 13.7, p. 125] for $L_{23;\partial}^{23}$.

**Lemma 4.6.** Let $1 \leq q \leq n - 2$. Let $(D^1, D^2_r, D^3_r)$ be a $(q + 1)$-concave configuration. Let $L_{12;\partial}^{23}$ be defined by (2.7). Assume $r_4$ satisfies (4.19). For $f \in C^1_{(0,q)}(\overline{D^1})$,
we have for $z \in \overline{B_{r_4}}$

$$\overline{D}L^{23}_{12}f(z) = \int_{S^{12}} \Omega^{23}_{(0,q+1)}(z,\zeta) \wedge \overline{D}f(\zeta).$$

$$L^{23}_{12}f = \overline{D}T_{B_{r_4},q}L^{23}_{12}f + T_{B_{r_4},q+1}L^{23}_{12}f,$$

$$L^{23}_{12}f : C^1(0,q)(\overline{D}^3) \cap \ker \overline{D} \to C^\infty(0,q)(\overline{B_{r_4}}) \cap \ker \overline{D}.$$

Proof. By (4.13)-(4.14), we have $g^i(z,\zeta) \cdot (\zeta - z) \neq 0$ for $i = 2,3$, $z \in \overline{B_{r_4}}$ and $\zeta \in S^{12}$. Thus, the form $\Omega^{23}(z,\zeta)$ is smooth in $z \in B_{r_4}$ and $\zeta \in S^{12} \subset \partial D^2$.

We have

$$\overline{D}_\zeta \Omega_{(0,q+1)} + \overline{D}_\zeta \Omega^{23}_{(0,q)} = \Omega^2_{(0,q+1)} - \Omega^3_{(0,q+1)}.$$

By (3.13), $\Omega^2_{(0,q+1)} = 0$. Thus $\int_{S^{12}} \Omega^{23}_{(0,q+1)}(z,\zeta) \wedge f(\zeta) = 0$. By (4.15), the $\zeta$-degree of $f(\zeta) \wedge \Omega^{23}_{(0,q+1)}(z,\zeta)$ is less than $2n - 3$, which is less than $\dim(S^1 \cap S^2)$. This shows that

$$\int_{S^{12}} \Omega^{23}_{(0,q+1)}(z,\zeta) \wedge f(\zeta) = 0, \quad q > 0.$$

By Stokes’ formula and $\partial (S^{12}) = \emptyset$, we obtain

$$\overline{D}L^{23}_{12}f(z) = - \int_{S^{12}} \overline{D}_\zeta \Omega^{23}_{(0,q+1)}(z,\zeta) \wedge f(\zeta) = \int_{S^{12}} \Omega^{23}_{(0,q+1)}(z,\zeta) \wedge \overline{D}f(\zeta). \quad \square$$

In summary, we have the following local $\overline{D}$-solution operator for the concave case.

**Theorem 4.7.** Let $1 \leq q \leq n - 2$. Let $(D^1, D^2, D^3)$ be a $(q+1)$-concave configuration. Let $U^1 = D^1 \setminus \overline{D^3}$ and $S^1 = \partial D^2 \setminus D^1$. Assume $0 < r_4 < r_3/C_n$. Let $f \in C^1(0,q)(\overline{D}^3)$ be $\overline{D}$-closed. On $D^1 \cap B_{r_4}$, we have

$$f = \overline{D}H_qf$$

with $H_q = H^{(1)}_q + H^{(2)}_q + H^{(3)}_q$. Here $H^{(1)}_q$ and $H^{(2)}_q$ are given by (2.11), (3.18), and

$$H^{(3)}_q f = L^{23}_{12}f - T_{B_{r_4},q}L^{23}_{12}f.$$

Although it is not used in this paper, for potential applications, it is worthy to state the following local homotopy formula if we have an extra negative Levi eigenvalue: if $\partial D^1$ is strictly $(q + 1)$ concave, then $L^{23}_{12}f = 0$ by (4.16) for a $(0,q)$-form $f$. Therefore, we have the following.

**Theorem 4.8.** Let $1 \leq q \leq n - 3$. Let $(D^1, D^2, D^3)$ be a $(q+2)$-concave configuration. Assume $r_4$ satisfies (4.19). Let $U^1 = D^2 \setminus \overline{D^3}$ and $S^1 = \partial D^2 \setminus D^1$. Let $f \in C^1(0,q)(\overline{D}^3)$ with $\overline{D}f \in C^1(\overline{D}^3)$. On $D^1 \cap B_{r_4}$, we have

$$f = \overline{D}H_qf + H_{q+1}f$$

with $H_q = H^{(1)}_q + H^{(2)}_q + H^{(3)}_q$. Here $H^{(1)}_q$ and $H^{(2)}_q$ are given by (2.11), (3.18) and

$$H^{(3)}_q f = L^{23}_{12}f - T_{B_{r_4},q}L^{23}_{12}f.$$

As in the convex case, the integral kernels in $H^{(2)}_q$, $H^{(3)}_q$ have smooth kernels, since $S^1_1, S^2, S^{12}$ do not intersect small neighborhood of the origin in $\partial D^3$. There is another main difference in kernels between the convex and concave cases. For the concave case, when the boundary $\partial D$ is in $\Lambda^m$, the Leray functions (4.4) is $\Lambda^{m-1}$ in $z$ when $\rho \in \Lambda^m$. Anyway $H^{(1)}_q$ is the main term to be estimated.
5. Hölder-Zygmund spaces and a Hardy-Littlewood lemma

In this section, we recall the Hölder–Zygmund norms and indicate how a Hardy-Littlewood lemma can be used to derive the estimates. We will also discuss various equivalent norms.

By a bounded Lipschitz domain $D$ in $\mathbb{R}^n$, we mean that there are a finite open covering $\{U_i\}_{i=1}^N$ of $\overline{D}$, rigid affine transformations $A_i$, positive numbers $\delta_0, \delta_1, L$ such that $A_i(\overline{D} \cap \overline{U_i})$ is defined $\delta_1 \geq x_n \geq R_i(x')$ with $x' \in [0, \delta_0]^{n-1}$ and $|R_i(x') - R_i(x')| \leq L|x' - x'|$. We say that a constant $C(D)$ depending on $D$ is stable under small perturbations of $D$, if $C(\tilde{D})$ can be chosen independent of $\tilde{D}$ if $A_i(\overline{D} \cap \overline{U_i})$ is defined by $\delta_1 \geq x_n \geq \tilde{R}_i(x')$, where $|\tilde{R}_i(x') - R_i(x')| < \epsilon$ and $|\tilde{R}_i(x') - \tilde{R}_i(x')| \leq (L + \epsilon)|x' - x'|$ for some $\epsilon > 0$.

Let $N$ (resp. $N_+$) be the set of nonnegative (resp. positive) integers. For $k \in N$, let $C^k(\overline{D})$ be the space of $C^k$ functions on $\overline{D}$ with the standard norm $\| \cdot \|_{C^k(\overline{D})}$. When $a = k + \alpha$ with $0 < \alpha < 1$, let $C^a(\overline{D})$ be the space of $C^k$ functions $f$ on $\overline{D}$ such that
\[
\|f\|_{C^a(D)} := \|f\|_{C^k(D)} + \sup_{x,y \in D, x \neq y} \frac{\|\nabla^k f(y) - \nabla^k f(x)\|}{|y - x|^\alpha} < \infty.
\]
Define $\Delta^h f(x) = f(x + h) - f(x)$ and $\Delta^h_2 f(x) = f(x + 2h) + f(x) - 2f(x + h)$. When $k \in N_+$, define $\Lambda^k(\mathbb{R}^n)$ to be the space of all functions $f \in C^{k-1}(\mathbb{R}^n)$ such that
\[
\|f\|_{\Lambda^k(\mathbb{R}^n)} := \|f\|_{C^{k-1}(\mathbb{R}^n)} + \sup_{h \neq 0, x \in \mathbb{R}^n} \frac{|\Delta^h_2 \nabla f(x)|}{|h|} < \infty.
\]
We then define for $k \in N_+$
\[
\|f\|_{\Lambda^k(D)} := \inf_{\tilde{f} \circ \partial D = f} \|\tilde{f}\|_{\Lambda^k(\mathbb{R}^n)},
\]
\[
\Lambda^k(D) := \{f \in C^k(\overline{D}); \|f\|_{\Lambda^k(D)} < \infty\}.
\]
For convenience, set $\Lambda^r(D) := C^r(\overline{D})$ and $\|\cdot\|_{\Lambda^r(D)} := \|\cdot\|_{C^r(\overline{D})}$ when $r \in (0, \infty) \setminus N$.

Following [13, Def. 3.15], define an intrinsic norm in difference operators
\[
\|f\|_{\Lambda^r(D)} = \|f\|_{C^0(D)} + \sup_{x + jh \in D, j = 0, \ldots, \langle r \rangle + 1; h \neq 0} \left\{ \frac{|\Delta^r_j f(x)|}{|h|^r} \right\}, \quad \forall f \in \Lambda^r(D).
\]
It is known that when $D = \mathbb{R}^n$, $|\cdot|_{\mathbb{R}^{n,r}}$ and $|\cdot|_{\mathbb{R}^{n,r}}$ are equivalent [65, Thm. 2.5.12, p. 110; Thm. 2.5.13 (i), p. 115]. A classical theorem [8, Thm. 18.5, p. 63] says that given any function $f$ on a bounded Lipschitz domain satisfying $\|f\|_{\Lambda^r(D)} < \infty$ (without assuming $f \in \Lambda^r(D)$), there is an extension $\tilde{f} \in \Lambda^r(\mathbb{R}^n)$ such that $\tilde{f}\big|_D = f$; consequently, $f \in \Lambda^r(D)$.

Notation 5.1. To simplify notation, sometimes we denote the norm $\| \cdot \|_{C^a(D)}$ by $\| \cdot \|_{D,a}$ or simply $\| \cdot \|_a$, and also we denote $\| \cdot \|_{\Lambda^a(D)}$ by $\| \cdot \|_{D,a}$ or simply $\| \cdot \|_a$.

Lemma 5.2. Let $0 < \beta \leq 1$. Let $D \subset \mathbb{R}^n$ be a bounded and connected Lipschitz domain. Suppose that $f$ is in $C^0(D)$ and
\[
|\nabla^{|\beta|+1} f(x)| \leq A \text{dist}(x, \partial D)^{\beta - 1 + |\beta|}.
\]
Fix $x_0 \in D$. Then $\|f\|_{\Lambda^{|\beta|+1}(D)} \leq \|f\|_{C^0(D)} + C_{\beta} A$, where constants $C_0, C_\beta$ are stable under small perturbations of $D$. 
Proof. Suppose $\beta = 1$. We need to estimate $\Delta^2_h f(x)$ when $x$ is close to the boundary and $h \in \mathbb{R}^n$ is small. Take a boundary point $x_1$ of $\partial D$. We may assume that $x_1 = 0$. By definition, we may assume that $D$ is defined by $x_n > g(x')$ where $g$ is a Lipschitz function satisfying $|g(\tilde{x'}) - g(x')| \leq L|x' - \tilde{x}'|$ with $L > 1$. Then

$$\text{dist}(x, \partial D) \geq (x_n - g(x'))/\sqrt{L^2 + 1}.$$  

Suppose that $x, x - h, x + h$ are in $D$ and close to the origin. We first consider the special case when $x, h$ satisfy

$$x + th \in D, \quad \text{dist}(x + th, \partial D) \geq |h|, \quad \forall t \in [0, 2]. \quad (5.1)$$

Set $u(s) = f(x + h + sh) + f(x + h - sh) - 2f(x + h)$. Thus $u(0) = u'(0) = 0$ and $|u''(s)| \leq C_n LA|h|$. This shows that $|u(1)| \leq C_n LA|h|$.

For the general case, set $\hat{h} = (0', 4\sqrt{L^2 + 1}|h|)$. When $y \in D$ is close to the origin and $|\hat{h}| < 1$, we have

$$y + \hat{t}h \in D, \quad \text{dist}(y + \hat{t}h, \partial D) \geq |\hat{h}|/\sqrt{L^2 + 1}, \quad \forall t \in [0, 2]. \quad (5.2)$$

Rewrite $\Delta^2_h f(x) = f(x + 2h) + f(x) - 2f(x + h)$ as

$$\Delta^2_h f(x) = 2f(x + 2h + \hat{h}) + 2f(x + \hat{h}) - 4f(x + h + \hat{h})$$

$$- f(x + 2\hat{h}) - f(x + 2h + 2\hat{h}) + 2f(x + 2h + 2\hat{h})$$

$$+ f(x) + f(x + 2\hat{h}) - 2f(x + \hat{h})$$

$$+ f(x + 2h) + f(x + 2h + 2\hat{h}) - 2f(x + 2h + 2\hat{h})$$

$$- 2f(x + h) - 2f(x + h + 2\hat{h}) + 4f(x + h + \hat{h}). \quad (5.3)$$

We estimate each row on the right-hand side of (5.3). Let us denote by $[a, b]$ the line segment connecting two points $a, b$ in $\mathbb{R}^n$. By (5.2), we have

$$[x + \hat{h}, x + \hat{h} + 2\hat{h}] \subset D, \quad \text{dist}(x + \hat{h} + th, \partial D) > |h|,$$

for $t \in [0, 2]$. Therefore, we can estimate the first row using the estimation for the special case (5.1) in which $x$ is replaced by $x + \hat{h}$. The second row is estimated similarly. For the third row, by (5.2) and $x \in D$, we get

$$\text{dist}(x + \hat{h} + sh, \partial D) > (1 + s)|h| \geq (1 - |s|)|h|, \quad s \in [-1, 1].$$
Take $u(s) = f(x + \tilde h + sh) + f(x + \tilde h - sh) - 2f(x + \tilde h)$. This yields $|u''(s)| \leq C_n AL|h|/(1-s)$ for $s \in [0,1]$ and $|u(1)| \leq C_n AL|h|$ by $u(1) = \int_0^1 (1-s)u''(s)\,ds$. This gives us the desired estimate for the third row. The last two rows in (5.3) can be estimated similarly.

The proof is standard for $0 < \beta < 1$, by using the three-term decomposition $\Delta_h f(x) = \Delta_h f(x) + \Delta_h f(x + h) - \Delta_h f(x + h)$ and (5.2).

Very recently, Shi–Yao [62, Def. 3.3, Thm. 1.1] show among other results that when $r = k + \beta$ with $k \in \mathbb{N}$ and $0 < \beta \leq 1$, $\|f\|_{\Lambda_r'(D)}$ and $\sum_{|\alpha| \leq k} \|\nabla^\alpha f\|_{\Lambda_r(D)}^{in}$ are equivalent using Rychkov’s universal extension for Besov spaces $B^r_{\eta,q}(D)$. They also show the stability of constants in their estimates under small deformation of Lipschitz domains [62, Rnk. 6.9]. Let us also observe that Dispa [13, Thm. 3.18] shows that there exists a constant $C_r(D)$ such that

$$\|f\|_{\Lambda_r'(D)} \leq C_r(D)\|f\|_{\Lambda_r(D)}, \quad r > 0.$$  

By definition and Taylor formulae, it is easy to see that $|f|_{\Lambda_r'(D)} \leq C_r|f|_{\Lambda_r(D)}$ and hence $\|f\|_{\Lambda_r'(D)} \leq C_r\|f\|_{\Lambda_r(D)}$. Therefore, we have

**Corollary 5.3.** The norms $\|\cdot\|_{\Lambda_r'(D)}$ and $\|\cdot\|_{\Lambda_r(D)}$ are equivalent.

The proof in [13] uses only the non-universal part of Rychkov extension in [58, Thms. 2.2-2.3]. Thus one can see that the constant $C_r(D)$ in (5.4) depends only on the Lipschitz norm of the graph function of $\partial D$ and hence it is stable under small perturbations of $D$.

We also need the following result.

**Lemma 5.4.** Let $r = k + \beta$ with $k \in \mathbb{N}$ and $0 < \beta \leq 1$. Let $D$ be a bounded Lipschitz domain. Then

$$\|f\|_{\Lambda_r'(D)}^{in} \leq C_r(D)(\|f\|_{C^0(D)} + \|\nabla f\|_{\Lambda_r(D)}^{in}).$$

Further, $C_r(D)$ is stable under a small perturbation of Lipschitz domains.

**Proof.** It suffices to verify that if $x + jh$ are in $D$ for $j = 0, \ldots, (k+1)$,

$$|\Delta_h f(x)| \leq C_r|h|\|\nabla f\|_{\Lambda_r(D)}^{in}.$$  

The case $k = 0$ follows from definition. Suppose $k > 0$. By a simpler analogue of (5.3), we have a three-term decompose $\Delta_h f(x) = \sum_{i=1}^3 \Delta_h f(y^i)$ where $y^i = x + h^i$, $h^i$ depend only on $h$ such that $[y^i, y^i + h^i] \subset D$ and $|h^i| \leq C|h|$. Thus with $\partial_i f(x) = \frac{\partial}{\partial x_i} f(x)$ we have

$$\Delta_h f(y^i) = \sum_{j=1}^h h_j \int_0^1 \partial_j f(y^i + sh^i)\,ds.$$  

Repeat this. We can write $\Delta_h f(x)$ as a linear combination of

$$\int_{[0,1]^k} \partial^\alpha f(\tilde y^\alpha + s_1 h^{(1)} + \cdots + s_k h^{(k)})\,ds \times \prod_{i=1}^k h^{(i)}_\alpha$$

where $|\alpha| = k$, and $\tilde y^\alpha = x + h^\alpha$, $h^\alpha, h^{(i)}$ depend only on $h$. Further, $|h^{(i)}| \leq C_k(D)|h|$. Thus we obtain (5.5) applying a three-term decomposition to the integrand in (5.6) when $0 < \beta < 1$ or applying (5.3) when $\beta = 1$. \hfill $\Box$
We will need interpolation.

**Proposition 5.5.** Let \( E : C^0(D) \to C^0(D) \) be the Stein extension. Then
\[
\| f \|_{C^r(D)} \leq \| Ef \|_{C^r(D)} \leq C_r(D) \| f \|_{C^r(D)}, \quad r \in [0, \infty); \\
\| f \|_{\Lambda^r(D)} \leq \| Ef \|_{\Lambda^r(D)} \leq C_r(D) \| f \|_{\Lambda^r(D)}, \quad r \in (0, \infty).
\]

**Proof.** The first and the third inequalities follow from definitions. The other two are properties of the Stein extension operator; see [19, Prop. 3.11] for details. \( \square \)

**Definition 5.6** ([9, p. 167]). Let \((X_0, \| \cdot \|_0), (X_1, \| \cdot \|_1)\) be two Banach spaces contained in a Banach space \( X \). Set
\[
K(t, f; X_0, X_1) := \inf_{f = f_0 + f_1} \{ t \| f_0 \|_0 + t \| f_1 \|_1 \}, \quad t > 0,
\]
\[
\| f \|_{\theta; X_0, X_1} := \inf_{t > 0} t^{-\theta} K(t, f; X_0, X_1), \quad 0 < \theta < 1.
\]
We now specialize the intermediate spaces via Hölder spaces and set
\[
\| f \|_{\Lambda^r(D); r_0, r_1} := \| f \|_{\theta; C^{r_0}(D), C^{r_1}(D)}, \quad r_\theta = (1 - \theta) r_0 + \theta r_1,
\]
where \( r_0, r_1 \) are given and \( 0 < \theta < 1 \).

**Lemma 5.7.** \( \| f \|_{\Lambda^r(D); r_0, r_1} \) and \( \| Ef \|_{\Lambda^r(D)} \) are equivalent.

**Proof.** It is well-known that \( \| Ef \|_{\Lambda^r(D)} \) is equivalent to \( \| Ef \|_{\Lambda^r(D); r_0, r_1} \). Let us write \( \| \cdot \|_{C^r(D)} \), \( \| \cdot \|_{\Lambda^r(D)} \) respectively. Thus it suffices to show
\[
(5.7) \quad \| f \|_{\Lambda^r(D); r_0, r_1} \leq \| Ef \|_{\Lambda^r(D); r_0, r_1} \leq C_r(D) \| f \|_{\Lambda^r(D); r_0, r_1}.
\]
Set \( K(t, f; D) := K(t, f; C^{r_0}(D), C^{r_1}(D)) \) and
\[
K(t, E f) := K(t, E f; C^{r_0}(D), C^{r_1}(D)).
\]
Suppose \( f = f_0 + f_1 \) on \( D \). Since \( E \) is linear, then \( E f = f_0 + E f_1 \). By definition, \( K(t, E f) \leq \| Ef_0 \|_{r_0} + t \| E f_1 \|_{r_1} \leq C_{r_0, r_1}(D) (\| f_0 \|_{r_0} + t \| f_1 \|_{r_1}) \). Thus
\[
K(t, E f) \leq C_{r_0, r_1}(D) K(t, f; D), \quad \| Ef \|_{\Lambda^r(D); r_0, r_1} \leq C_r(D) \| f \|_{\Lambda^r(D); r_0, r_1}.
\]
If \( E f = \tilde{f}_0 + \tilde{f}_1 \), we have \( f = \tilde{f}_0 + \tilde{f}_1 \). By definition, \( K(t, f; D) \leq \| \tilde{f}_0 \|_{D, r_0} + t \| \tilde{f}_1 \|_{D, r_1} \). Thus \( K(t, f; D) \leq K(t, E f) \) and hence \( \| f \|_{\Lambda^r(D); r_0, r_1} \leq \| Ef \|_{\Lambda^r(D); r_0, r_1} \). We have verified (5.7) and hence the lemma. \( \square \)

In summary, we have the following.

**Corollary 5.8.** The norms \( \| Ef \|_{\Lambda^r(D)} \), \( \| f \|_{\Lambda^r(D)} \), \( \| f \|_{\Lambda^r(D); r_0, r_1} \) are equivalent with constant factors that are stable under small perturbations of \( D \) by Lipschitz domains.

**Proposition 5.9** ([9, Thm. 3.2.23, p. 180]). Let \((X_0, \| \cdot \|_0), (X_1, \| \cdot \|_1)\) (resp. \((Y_0, \| \cdot \|_0), (Y_1, \| \cdot \|_1)\)) be two Banach spaces continuously embedded in a Banach space \( X \) (resp. \( Y \)). If \( L : X \to Y \) is linear and
\[
\| L f_i \|_{Y_i} \leq M_i \| f_i \|_{X_i}, \quad i = 0, 1
\]
then \( \| L f \|_{\theta; Y_0, Y_1} \leq M_0^{1-\theta} M_1^\theta \| f \|_{\theta; X_0, X_1} \) for \( 0 < \theta < 1 \).
Corollary 5.10 ([17, Prop. 3.4]). Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $E : C^0(\overline{D}) \to C^0_0(\mathbb{R}^n)$ be the Stein extension. Then
\[ \|\nabla E f(x)\| \leq C_r(D) \|f\|_{\Lambda^r(D)} \text{dist}(x, D)^{-1}, \quad r > 1. \]
Further, $C_r(D)$ is stable under small perturbations of $\partial D$.

6. $\frac{1}{2}$-GAIN ESTIMATES FOR LOCAL HOMOTOPY OPERATORS

In this section, we derive the estimates for homotopy operators. We will give precise estimates which are potentially useful for applications. We remark that the local estimates do not require the forms to be $\overline{\partial}$ closed. Throughout the paper, we denote by $A(z, \zeta, \nabla \rho^1, \ldots, \nabla^k \rho^1)$ a polynomial in $z, \zeta, \nabla \rho^1, \ldots, \nabla^k \rho^1$.

We first consider the $(n-q)$ convex case. In this case the result is essentially in [19]. We simplified proof by using Lemmas 5.2, 5.4 and equivalent norms, which are applicable for $C^2$ domains.

Theorem 6.1. Let $r \in (1, \infty)$ and $1 \leq q \leq n-1$. Let $(D^1, D^2)$ be an $(n-q)$-convex configuration. The homotopy operator $H_q$ in Theorem 3.5 satisfies
\[
(6.1) \quad \|H_q \varphi\|_{\Lambda^{r+1/2}((D^1 \cup D^2)_r)} \leq C_r(\nabla \rho^1, \nabla^2 \rho^1)\|\varphi\|_{\Lambda^r(D^2)}, \quad r_{1/2} < r < 3r/4,
\]
\[
(6.2) \quad \|H_q \varphi\|_{\Lambda^{r+1/2}((D^1 \cup D^2)_r)} \leq C(\nabla \rho^1, \nabla^2 \rho^1)\|\varphi\|_{\Lambda^r(D^2)}, \quad r_{1/2} < r < 3r/4.
\]
Moreover, $C_r(\nabla \rho^1, \nabla^2 \rho^1)$ is stable under small $C^2$ perturbations of $\rho^1$.

Proof. To simplify notation, write $\|\varphi\|_{\Lambda^r(D^2)}$, $\|H_q \varphi\|_{\Lambda^{r+1/2}((D^1 \cup D^2)_r)}$ as $|\varphi|_r$, $|H_q \varphi|_{r+1/2}$. Recall the homotopy operator $H_q \varphi = (H_q^{(1)} + H_q^{(2)}) \varphi$ with
\[ H_q^{(1)} \varphi = R_{U^1 \cup \partial D^2}^0 E \varphi + R_{U^1, \overline{\partial}}^0 \overline{E} \varphi. \]
Near the origin, each term in $H_q^{(2)} \varphi$ given by (3.18) is a linear combination of integrals of the form
\[ Kf(z) := \int_{S^1} \frac{A(\nabla \zeta \rho^1, \nabla^2 \rho^1, \zeta, z)f(\zeta)}{(g^1 \cdot (\zeta - z))^a (g^2 \cdot (\zeta - z))^b |\zeta - z|^{2r}} dV \]
where $A$ is a polynomial, $f$ is the coefficients of $\varphi$, and $a, b, c$ are integers. Note that $S^1$ is one of $S^1_+, S^1_-, S^2$. For $z$ close to the origin and $\zeta \in S^1$ we have $|g^1(\zeta, z)| \cdot (\zeta - z) > c$. Thus $|Kf|_{r+1/2} \leq C_r \|f\|_0$. Here and in what follows $C_r$ denotes a constant depending on $\nabla \rho^1, \nabla^2 \rho^1$.

We now estimate the main term $H^{(1)} \varphi$. Decompose it as
\[
(6.3) \quad H^{(1)} \varphi(z) = \int_{D^1 \cup U^1} \Omega_{[0,q]}^1(z, \zeta) \wedge E \varphi(z) + \int_{U^1} \Omega_{[0,q]}^1(z, \zeta) \wedge [\overline{E}] \varphi(z).
\]
Denote the first integral by $K_1 \varphi$. Since $D^1_{12}$ is contained in a relatively compact ball in $D^1 \cup U^1$, by estimates on the Newtonian potential [66, p. 316], we have
\[ |K_1 \varphi|_{r+1/2} \leq C |\varphi|_{r-1/2}. \]
Note that the above is proved in [66] when $r$ is not an integer. When it is an integer, the estimate follows the interpolation for the Zygmund spaces. The last integral in
(6.3) can be written as a linear combination of

$$K_2 f(z) := \int_{U^1} f(\zeta) \frac{A(\nabla \zeta \rho^1, \nabla_\zeta \rho^1, \zeta, z) N_1(\zeta - z)}{\Phi^{n-j}(z, \zeta) |\zeta - z|^{2j}} \, dV(\zeta), \quad 1 \leq j < n,$$

$$\Phi(z, \zeta) = g^1(z, \zeta) \cdot (\zeta - z),$$

where $A$ is a polynomial, $N_m(\zeta)$ denotes a monomial in $\zeta, \zeta$ of degree $m$, and $f$ is a coefficient of the form $\overline{\partial} E |\varphi|$ and hence by Corollary 5.10

$$|f(\zeta)| \leq C_r |\varphi|, \text{dist}(\zeta, D^1)^{r-1}.$$ 

Here and in what follows, $N_m(\zeta)$ denotes a monomial of $\zeta$ with degree $m \geq 0$ whose coefficient is a smooth function in $z, \zeta$.

Fix $\zeta_0 \in \partial D^1$. We first choose local coordinates such that $s_1(\zeta), s_2(\zeta)$ and $\tau(\zeta) = (t_1, \ldots, t_{2n})(\zeta)$ vanish at $\zeta_0$, $D^1$ is defined by $s_1 < 0$, and

1. $|\Phi(z, \zeta)| \geq c_0(\text{dist}(z, \partial D^1) + s_1(\zeta) + |s_2(\zeta)| + |\tau(\zeta)|^2)$,
2. $C|z - \zeta| \geq |\Phi(z, \zeta)| \geq c_0|z - \zeta|^2$,
3. $|\zeta - z| \geq c_0(\text{dist}(z, \partial D^1) + s_1(\zeta) + |s_2(\zeta)| + |\tau(\zeta)|)$,

for $z \in D^1, \zeta \notin D^1$.

Let $r = k + \alpha$ with integer $k \geq 1$ and $0 < \alpha \leq 1$. Assume $z \in D_{r_3}^1$. Consider first the case $0 < \alpha < 1/2$. We have $|\partial^{k+1} K f(z)| \leq C_r |\varphi| I(z)$, where

$$I(z) := \int_{[0,1] \times [0,1]^{2n-1}} \frac{s_1^{r-1} (s_1 + |s_2| + |t|)^{-(2j+1-b-1)}}{\text{dist}(z, \partial D^1) + s_1 + |s_2| + |t|^2} \, ds_1 ds_2 dt$$

with $a + b = k + 1$ and $1 \leq j < n$. The worst term occurs when $j = n - 1$ and $a = k + 1$. Therefore, using polar coordinates for $t(\zeta)$ we obtain $I(z) \leq CI_1(z)$ for

$$I_1(z) := \int_{[0,1]^3} \frac{s_1^{r-1} t^{2n-3}}{\text{dist}(z, \partial D^1) + s_1 + s_2 + t^2} \, ds_1 ds_2 dt$$

Using polar coordinates for $(s_1, s_2)$, we obtain $I_1(z) \leq CI_2(z)$ with

$$I_2(z) = \int_{[0,1]^2} \frac{s^{a+1} \, dsla}{\text{dist}(z, \partial D^1) + s + t^2} \leq C \text{dist}(z, \partial D^1)^{(a+1/2)-1}$$

where the last inequality is proved in [19, Lem. 4.3 (i)].

Note that the estimate is valid for $r = 1$ and $\alpha = 0$, when we replace $|\varphi|$ by $||\varphi||_2$. This gives us (6.2).

The case $1/2 \leq \alpha \leq 1$ is obtained by interpolation via Proposition 5.9 in which $X_i = \Lambda^{r_i}(D)$, $Y_i = \Lambda^{r_i+1/2}(D)$ for $i = 0, 1$ with $r_0 = k + \alpha/3$, $r_1 = k + 1 + \alpha/3$, $r_0 = k + \alpha$, and $\theta = 2\alpha/3$.

To be used later, we can also give a direct estimate as follows. When $1/2 \leq \alpha \leq 1$, we use $|\nabla^{k+2} K f(z)| \leq C_r |\varphi| I(z)$ defined by (6.7) in which $a + b = k + 2$. Then $I(z) \leq CI_3(z)$ for

$$I_3(z) := \int_{[0,1]^3} \frac{s_1^{r-1} t^{2n-3} ds_1 ds_2 dt}{\text{dist}(z, \partial D^1) + s_1 + s_2 + t^2} \leq C \text{dist}(z, \partial D^1)^{(a+1/2)-2}$$

which is less than $C \text{dist}(z, \partial D^1)^{(a+1/2)-2}$ because $a + 1/2 < 2$. □
Lemma 6.2. Let $\beta, \mu_1 \in [0, \infty)$, $0 \leq \lambda \leq \mu_1$, and $0 < \delta < 1$. Set

$$\beta' := \beta - \frac{\mu_1 + \lambda - 3}{2}.$$ 

Then for $m \geq 0$,

$$\int_{[0,1]^3} s_1^\beta (\delta + s_1 + s_2 + t^2)^{-1 - \mu_1 t^m} ds_1 ds_2 dt < \begin{cases} C \delta^{\beta'}, & \beta' < 0; \\ C, & \beta' > 0. \end{cases}$$

Proof. In our application, we have $m = 2(n-1) - 1 \geq 1$. It suffices to consider $m = 0$. We consider the integral in the following regions.

(i) $s_2 > \max\{s_1, \delta, t\}$. On this region the integral is less than

$$\int_{s_2=\delta}^1 \int_{s_1=0}^{s_2} s_1^\beta ds_1 ds_2 \leq \int_{\delta}^1 s_2^{\beta + 1 - \lambda} ds_2 < \int_{\delta}^1 s_2^{\beta - 1} ds_2 < C \delta^{\beta'}$$

if $\beta' < 0$. Also the integral bounded by a constant if $\beta' > 0$. The same bounds can be obtained for the integral on regions (ii) $s_1 > \max\{\delta, s_2, t\}$ and (iii) $\delta > \max\{s_1, s_2, t\}$.

(iv) $t^2 > \max\{\delta, s_1, s_2\}$. On this region, the integral is less than

$$\int_{t=\sqrt{\delta}}^1 \int_{s_2=t}^1 \int_{s_1=0}^{s_2} s_1^\beta ds_1 ds_2 ds_3 < \int_{t=\sqrt{\delta}}^1 s_2^{2\beta + 2 - \mu_1 - \lambda} ds_2 < C \delta^{\beta -(\mu_1 + \lambda - 3)/2}$$

when $\beta' < 0$. When $\beta' > 0$, the integral is bounded by a constant.

(v) $t^2 < \delta + s_1 + s_2 < t$. On this region, the integral is less than

$$\int_{(s_1, s_2) \in [0,1]^2} \int_{t=s_1+s_2}^{s_1+s_2} s_1^\beta ds_1 ds_2 ds_3 < C \delta^{\beta'}.$$ 

Suppose $\beta' < 0$. Since $\lambda \leq \mu_1$, the latter is less than

$$\int_{(s_1, s_2) \in [0,1]^2} \frac{s_1^\beta ds_1 ds_2}{(\delta + s_1 + s_2)^{1 + \mu_1 + (\lambda - \mu_1 - 1)/2}} < C \delta^{\beta'},$$

where the last inequality is obtained by computing integrals for $\delta \geq \max\{s_1, s_2\}$, $s_1 \geq \max\{\delta, s_2\}$, and $s_2 \geq \max\{\delta, s_1\}$. When $\beta' > 0$, it is straightforward that the integral is bounded above by a constant $C$. \qed

We now treat the concave case. We start with the following.

Lemma 6.3. Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $|\cdot|_a = |||\cdot|||_{L^\infty(D)}$ and $||\cdot||_a = ||\cdot||_{C^\infty(D)}$. Then

(6.12) $||u||_{a+b}||v||_{c+d} \leq C_{a,b,c,d}(||u||_a+||v||_d, a+b+||u||_a||v||_{c+d}), \quad a,b,c,d \geq 0$;

(6.13) $|uv|_a \leq C_a(||u||_a||v||_a, a \in (0, \infty), a > 0$;

(6.14) $|uv|_{a+b}||v||_{c+d} \leq C_{a,b,c,d}(||u||_a+||v||_a||v||_{c+d}), \quad a,d > 0, b,c \geq 0$.

Further, $C_{a,b,c,d}$ and $C_a$ depend on the Stein extension $E_D$ and are stable under small perturbations of the Lipschitz domain $D$.

Proof. A proof of (6.12) is in [22, Prop. A.4 (iii)]. When $D = \mathbb{R}^n$, (6.13) is proved in [2, Cor. 2.86, p. 104]. For the $D$, we use the Stein extension $E_D$ and get $|uv|_a \leq |(E_D u)(E_D v)|_a \leq C_a(||E_D u||_a||E_D v||_a, a \in (0, \infty)$. For the $D$, we use the Stein extension $E_D$ and get $|uv|_a \leq |(E_D u)(E_D v)|_a \leq C_a(||E_D u||_a||E_D v||_a, a \in (0, \infty)$. For the $D$, we use the Stein extension $E_D$ and get $|uv|_a \leq |(E_D u)(E_D v)|_a \leq C_a(||E_D u||_a||E_D v||_a, a \in (0, \infty)$.
We now prove (6.14). Let us first consider the case that \( D = \mathbb{R}^n \) and \( u, v \) have compact support in \( \mathbb{R}^n \). Let us use a Littlewood-Paley decomposition

\[
 u = \sum_{i \geq 0} u_i, \quad u_0 = \Phi * u, \quad u_j = \Psi_{2^{-j}} * u, \quad j > 0
\]

where \( \Psi_{t^{-1}}(x) = t^n \Psi(tx) \); for the definition of \( \Phi \) and \( \Psi \), see [23, sect. 1.4.3]. By [23, Thm. 1.4.3, p. 46], there is a constant \( C(n, r) \) such that

\[
 |u|/C(n, r) \leq \sup_{j \geq 0} 2^{jr} \|f_j\|_0 \leq C(n, r)|u|_r.
\]

Decompose \( v = \sum_{k \geq 0} v_k \) analogously. Then

\[
 |u|_{a+b+c+d} \leq C_{a,b,c,d} \sup_{j,k \geq 0} 2^{j(a+b)+k(c+d)} \|u_j\|_a \|v_k\|_0
\]

\[
 \leq C_{a,b,c,d} \sup_{j,k \geq 0} (2^{j(a+b+c)+kd} + 2^{j(a+b+c+d)}) \|u_j\|_a \|v_k\|_0
\]

\[
 \leq C_{a,b,c,d}' \sup_{j,k \geq 0} (|u|_{a+b+c} \|v\|_d + |u|_a \|v|_{b+c+d}).
\]

Note that we have used \( b \geq 0 \) and \( c \geq 0 \) for the second inequality. By a similar proof of (6.13), the general case of (6.14) is obtained through the extension \( E_D \). □

We now derive our main estimates, using Lemmas 5.2, 5.4 and equivalent norms.

**Theorem 6.4.** Let \( r \in (1, \infty) \) and \( 1 \leq q \leq n - 2 \). Let \( (D^1, D^2, D^3) \) be a \((q+1)\)-concave configuration. The homotopy operators \( H_q \) in Theorems 4.7 and 4.8 satisfy

\[
 H_q \varphi \|_{\Lambda^{s+\theta/2}(D^1)} \leq C_q \|\nabla \rho^1 \|_{\Lambda^{s+\theta/2}} \|\varphi\|_{C^0(D^1)},
\]

\[
 H_q \varphi \|_{\Lambda^{s+\theta}(D^2)} \leq C_q \|\nabla \rho^2 \|_{\Lambda^{s+\theta}} \|\varphi\|_{C^0(D^2)},
\]

\[
 H_q \varphi \|_{\Lambda^{s+\theta}(D^3)} \leq C_q \|\nabla \rho^3 \|_{\Lambda^{s+\theta}} \|\varphi\|_{C^0(D^3)}.
\]

Here \( 0 \leq \theta \leq 1/2 \). Moreover, \( C_q \|\nabla \rho^1, \nabla \rho^2, \nabla \rho^3\| \) are stable under small \( C^2 \) perturbations of \( \rho^1 \).

**Proof.** To ease notation, write

\[
 \|\varphi\|_{\Lambda^{s}(D^1)}, \quad \|\varphi\|_{C^0(D^1)}, \quad \|H_q \varphi\|_{\Lambda^{s}(D^2)} , \quad \|H_q \varphi\|_{\Lambda^{s}(D^3)}
\]

as \( \|\varphi\|_a, \|\varphi\|_{a}, \|H_q \varphi\|_{a}, \|H_q \varphi\|_{a} \) respectively. We also write \( \|\rho^1\|_{\Lambda^{s}(U_1)}, \|\rho^1\|_{\Lambda^{s}(U_1)} \) as \( \|\rho^1\|_a, \|\rho^1\|_a \). Recall the homotopy operator

\[
 H_q \varphi = (H_q^{(1)} + H_q^{(2)} + H_q^{(3)}) \varphi
\]

where \( \varphi \) has type \((0,q)\) and

\[
 H_q^{(1)} \varphi = R_{U_1 \cup D^1 E}^0 \varphi + R_{U_1 \cup D^1 E} \varphi.
\]

Near the origin, each term in \( H_q^{(2)} \varphi, H_q^{(3)} \varphi \) is a linear combination of integrals of \( K_0 f_0 \) with

\[
 K_0 f_0(z) = A(z, \nabla \rho^1, \nabla \rho^3) \tilde{K}_0 f_0(z)
\]

with \( f_0 \) being a coefficient of \( \varphi \). Here \( \tilde{K}_0 \), which involves only \( \nabla \rho^1 \), is defined by

\[
 \tilde{K}_0 f_0(z) := \int_{S^1} \sum_{a,b,c,d} f_0(\zeta, (\zeta - z)^{2d}) \cdot \left( g^1(z, \zeta) \cdot (\zeta - z)^{2d} (g^2(z, \zeta) \cdot (\zeta - z)^{2d} (g^3(z, \zeta) \cdot (\zeta - z)^{2d}) \right) dz
\]

where \( a,b,c,d \) are nonnegative integers, \( e \in \mathbb{N}^{2n} \), and \( S^1 \) is one of \( S_1^1, S_1^2, S_1^2 \). Therefore, for the \( g^1, g^2, g^3 \) that appears in the kernel, we have by (4.13)

\[
 |g^1(z, \zeta) \cdot (\zeta - z)| \geq c_0
\]
when \( z \in D^{12}_{r_5/2} \) and \( \zeta \in S^1 \). By (6.13), we have \( |\tilde{K}_0 f_0|_{r+1/2} \leq C_r |\rho^1|_{r+3/2} \| f_0 \|_0 \) and \( |K_0 f_0|_{r+1/2} \leq C_r |\rho^1|_{r+5/2} \| \tilde{K}_0 f_0 \|_0 + C_r |\tilde{K}_0 f_0|_{r+1/2} \).

Here and in what follows \( C_r \) denotes a constant depending on \( \rho, \partial \rho, \partial^2 \rho \). Therefore, we have
\[
|K_0 f_0|_{r+1/2} \leq C_r |\rho^1|_{r+5/2} \| f \|_0.
\]

We now estimate the main term \( H^{(1)} \varphi \), which is decomposed as
\[
H^{(1)} \varphi(z) = \int_{D^{12}_{r_5/2}} \Omega^0_{(0,q)}(z, \zeta) \wedge E \varphi(\zeta) + \int_{U^1} \Omega^0_{(0,q)}(z, \zeta) \wedge [\bar{\partial}, E] \varphi(\zeta).
\]

Denote the first integral by \( K_1 \varphi \). By an estimate on Newtonian potential \cite[p. 316]{66}, we have
\[
|K_1 \varphi|_{r+1/2} \leq C_r |\varphi|_{r-1/2}.
\]

The last integral in (6.17) can be written as a linear combination of
\[
K_2 f(z) := A(z, \nabla_2 \rho^1, \nabla_2^2 \rho^1) \tilde{K}_2 f(z)
\]
where \( f \) is a coefficient of the form \([\bar{\partial}, E] \varphi\). Cors. 5.8 and 5.10 imply
\[
|f|_{r-1} \leq C_r (D^1) \| \varphi \|_{D^1, r}, \quad |f(\zeta)| \leq C_r (D^1) \| \varphi \|_r \text{ dist}(\zeta, D^1)^{r-1}.
\]

Note that \( f \) vanishes on \( \bar{D} \). Further
\[
\tilde{K}_2 f(z) := \int_{U^1} f(\zeta) \frac{N_1(\zeta - z)}{\Phi^{n-j}(\zeta, \zeta) \gamma |\zeta - z|^{2j}} dV(\zeta), \quad 1 \leq j < n,
\]
\[
\Phi(\zeta, \zeta) = g^1(z, \zeta) \cdot (\zeta - z).
\]

We have
\[
|K_2 f|_{r+1/2} \leq C_r (\|\rho^1\|_2 (\|\rho^1\|_{r+3/2} \| \tilde{K}_2 f \|_0 + |\tilde{K}_2 f|_{r+1/2}), \quad r > 1;
\]
(6.19) \[
\|K_2 f\|_{1+\gamma} \leq C_r (\|\rho^1\|_2 (1 + \|\rho^1\|_{3+\gamma} \| \tilde{K}_2 f \|_0 + \| \tilde{K}_2 f\|_{1+\gamma}), \quad 0 \leq \gamma \leq \frac{1}{2}.
\]

The rest of the proof is devoted to the proof of
\[
|\tilde{K}_2 f|_{r+1/2} \leq C_r (\|\rho^1\|_{r+3/2} \| \varphi \|_1 + |\varphi|_r), \quad r > 1,
\]
or (6.18) directly for easy cases. When \( r = 1 \), we simply replace \( |\varphi|_1 \) by \( \| \varphi \|_1 \) in the proof below. Then combining above estimates yields the proof for (6.15)-(6.16).

A technical difficulty to prove (6.20) is that our approach relies on (6.13) and (6.12) for Zygmund norms, instead of (6.14). See for instance (6.26) below. The computation is tedious and our main observation is that the kernel of \( \tilde{K}_2 \) involves only \( \nabla^1 \rho \) instead of \( \nabla^2 \rho \).

Let \( r = k + \alpha \) with integer \( k \geq 1 \) and \( 0 < \alpha \leq 1 \). In the following cases, we will apply Lemma 6.2 several times. For clarity, we will specify values \((\beta, \beta')\) in Lemma 6.2 via \((\beta_i, \bar{\beta}_i)\) when we use the lemma.

(i) \( 0 < \alpha < 1/2 \). Recall that the kernel of \( \tilde{K}_2 \) involves only first-order derivatives of \( \rho^1 \) in \( z \)-variables and it does not involve \( \zeta \)-derivatives of \( \rho^1 \). Further, \( \Phi \) is a linear combination of \( \zeta_j - z_j \). Since \( \rho^1 \in \Lambda^{k+\alpha+5/2} \subset C^{k+2}, \) we can express \( \nabla^{k+1} \tilde{K}_2 f \) as a sum of
\[
K^{(k+1)}_{\mu,\nu} f := \nabla^{1+\nu_1} \rho^1 \ldots \nabla^{1+\nu_{k+1}} \rho^1 K_{\mu} f.
\]
Here and in what follows $\nabla^\ell \rho^1$ stands for a partial derivatives of $\rho^1(z)$ in $z, \varpi$ of order $\ell$ and

\[
K_\mu f(z) := \int_{U^1} \frac{f(\zeta)N_1 - \mu_0 + \mu_1 - \nu''_i + \cdots - \nu''_{\nu'_1} + \nu_2 (\zeta - z)}{(\Phi(z, \zeta) + \mu_1 |\zeta - z|^{2n+2\mu_2})} dV.
\]

(6.22)

Here $1 \leq j < n, \nu''_i = 0, 1,$ and

$\mu_0 + \mu_2 + \sum (\nu'_i + \nu''_i) \leq k + 1, \quad \nu'_i + \nu''_i \geq 1.$

To estimate (6.21), we use (6.4)-(6.6): For $z \in D^1, \varpi \in U \setminus D^1$ and $D^1$ defined by $s_1 < 0,$ we have

$|\Phi(z, \zeta)| \geq c_\varpi (\text{dist}(z, \partial D^1) + s_1(\zeta) + |s_2(\zeta)| + |t(\zeta)|^2),$

$C|\zeta - z| \geq |\Phi(z, \zeta)| \geq c_\varpi |\zeta - z|^2,$

$|\zeta - z| \geq c_\varpi (\text{dist}(z, \partial D^1) + s_1(\zeta) + |s_2(\zeta)| + |t(\zeta)|).$

Consequently, the worst term for $K_\mu f$ occurs when $j = n - 1.$ Note that we use $|\nabla^{\mu} \rho^1(z)| \geq c$ in (6.21). Thus the worst term for $K_{\nu''_0, \nu''_1} f$ also occurs when $\mu_0 + \mu_1$ is absorbed into $\sum \nu''_i.$ Therefore, it suffices to estimate terms with $\mu_0 = \mu_2 = 0$ and $j = n - 1,$ which are assumed now.

Throughout the proof, we assume that $\nu''_i = 0$ and hence $\nu'_i \geq 1$ for $i \leq \mu_1$ and $\nu'_i = 1$ for $i > \mu_1.$ Thus

$\mu'_1 + \mu''_1 = \mu_1, \quad \sum_{i \leq \mu'_1} \nu'_i = \sum_{i \leq \mu_1} \nu'_i.$

Thus (6.22) is simplified in the form

\[
K_\mu f(z) := \int_{U^1} \frac{f(\zeta)N_1 + \mu'_1 (\zeta - z)}{(\Phi(z, \zeta) + \mu_1 |\zeta - z|^{2(n+1)})} dV,
\]

(6.23)

$\mu_1 \leq \mu''_1 + \sum_{i=1}^{\mu_1} \mu'_i \leq \sum (\nu'_i + \nu''_i) \leq k + 1.$

(6.24)

We will apply Lemma 6.2 a few times, using

$\lambda := \mu_1 - \mu'_1, \quad m = 2n - 3.$

We need $\beta_1 \geq 0$ and $\beta'_1 \leq \beta_1 - \frac{\mu_1 + \lambda - 3}{2}.$ Thus we take

$\beta'_1 = \alpha - 1/2 < 0, \quad \beta_1 = \max\{0, \alpha - 1/2 + \frac{\mu_1 + \lambda - 3}{2}\}.$

We can verify that $\beta_1 \leq r - 1.$ By Lemma 6.2, we obtain for $z \in D_{r/2}^{12},$

\[
|K_\mu f(z)| \leq |\varphi|_{\beta_1 + 1} \text{ dist}(z, \partial D^1)^{\beta_1'}.
\]

(6.25)

Thus,

\[
|K_{\nu''_0, \nu''_1} f(z)| \leq C\|\rho^1\|_{1 + \nu''_1} \cdots \|\rho^1\|_{1 + \nu''_{\nu'_1}} |\varphi|_{\beta_1 + 1} \text{ dist}(z, \partial D^1)^{\alpha - 1/2}.
\]

(6.26)

When $\beta_1 = 0,$ we actually need to replace the above $|\varphi|_{\beta_1 + 1}$ by $\|\varphi\|_1.$ Then we the desired estimate easily since $\|\rho^1\|_{1 + \nu''_1} \cdots \|\rho^1\|_{1 + \nu''_{\nu'_1}} \leq C\|\rho^1\|_{1 + \nu''_1} \cdots \|\rho^1\|_{1 + \nu''_{\nu'_1}}$ and $\sum \nu''_i < r + 1.$ The case all $\nu''_i \leq 1$ can be estimated directly via (6.9) to get $|K_\mu f(z)| \leq C|\varphi|_r \text{ dist}(z, D^1)^{\alpha + 1/2 - 1}$ and hence (6.20).
Assume now $\beta_1 > 0$ and $\nu'_1 \geq 2$. Thus $\mu'_1 \geq 1$. Let $x_+ = \max\{0, x\}$ and

$$\gamma := \sum (\nu'_i - 1)_+ = -\mu'_1 + \sum \nu'_i.$$  

Then $\|\rho^1\|_{1+\nu'_1} \cdots \|\rho^1\|_{1+\nu'_{l+1}} \leq \|\rho^1\|_{2+r}$. Then

$$\beta_1 + \gamma \leq \left[ -2 + \frac{\mu'_1 + \lambda}{2} \right] - \mu'_1 + \sum \nu'_i = -2 - \frac{\mu'_1}{2} + \mu'_1 + \sum \nu'_i \leq -2 - \frac{\nu'_i}{2} + k - 1 \leq r - \frac{3}{2},$$

where the second inequality is obtained by (6.24) and $\mu'_1 \geq 1$. Therefore,

$$\|\rho^1\|_{1+\nu'_1} \cdots \|\rho^1\|_{1+\nu'_{l+1}} \|\varphi|_{\beta_1+1} \leq C_T \|\rho^1\|_{2+r} \|\varphi\|_{\beta_1+1} \leq C^\prime_T \|\rho^1\|_{2+r} + C_T \|\rho^1\|_{2+s+\gamma} \|\varphi\|_{1} \leq C^\prime_T \|\rho^1\|_{r+1} \|\varphi\|_{1} + C_T \|\rho^1\|_{r+1} \|\varphi\|_{1}.$$

By an estimation similar to (6.2), we have $\|\tilde{K} f\|_{1} \leq C(\|\rho^1\|_{2}) \|\varphi\|_{1}$. The above estimate also works for $r = k = 1$ (and $\alpha = 0$), when $|\varphi|_{1}$ is replaced by $|\varphi|_{1}$. Namely, we have $\|\tilde{K} f\|_{1+\gamma} \leq C(\|\rho^1\|_{2}) \|\rho^1\|_{3+\gamma} \|\varphi\|_{1}$. Thus, (6.19) yields (6.16).

(ii) $1/2 < \alpha \leq 1$. In this case we can take an extra derivative since $\rho^1 \in \Lambda^{k+\alpha+5/2} \subset \Lambda^{k+3}$. Write $\nabla^{k+2} \tilde{K} f$ as a sum of

$$K_{\mu, \nu}^{(k+2)} f := \nabla^{1+\nu'_1} \cdots \nabla^{1+\nu'_{l+1}} \rho^1 \rho^1 K_{\mu, \nu} f$$

where $K_{\mu, \nu}$ is defined by (6.22). As before, the worst term occurs for $j = n - 1$ and $\mu_0 = \mu_2 = 0$ in $K_{\mu, \nu}$, which are assumed now. Then (6.24) in which $k$ is replaced by $k + 1$ becomes

$$\mu_1 \leq \mu'_1 + \sum_{i=1}^{\mu_1} \nu'_i \leq k + 2.$$

To apply Lemma 6.2, we need to $\beta_2 \geq 0$ and $\beta'_2 \leq \beta_2 - \frac{\mu'_1 + \lambda - 3}{2}$. Thus we take

$$\beta'_2 = \alpha - 3/2 < 0, \quad \beta_2 = \max\{0, \alpha - 3/2 + (\mu'_1 + \lambda - 3)/2\}.$$

Recall $\lambda = \mu_1 - \mu'_1$. We can verify that $\beta_2 \leq r - 1$. We obtain for $z \in D_{r_4}^{12}$

$$|K_{\mu, \nu}^{(k+2)} f(z)| \leq C_T \|\rho^1\|_{1+\nu'_1} \cdots \|\rho^1\|_{1+\nu'_{l+1}} \|\varphi\|_{\beta_2+1} \text{dist}(z, \partial D) \alpha^{-3/2}.$$

When $\beta_2 = 0$, we can replace $|\varphi|_{\beta_2+1}$ by $\|\varphi\|_{1}$. Then, using

$$\|\rho^1\|_{1+\nu'_1} \cdots \|\rho^1\|_{1+\nu'_{l+1}} \leq C \|\rho^1\|_{k+3} \leq C_T \|\rho^1\|_{r+5/2},$$

we get the estimate immediately. When all $\nu'_i \leq 1$, using (6.10) we get $|K_{\mu} f(z)| \leq C_T \|\varphi\|_{r} \text{dist}(z, \partial D) \alpha^{-3/2}$ and hence (6.20).

Assume now $\beta_2 > 0$ and $\nu'_i \geq 2$. Thus $\mu'_i \geq 1$. Let $\gamma$ be given by (6.27). Then we have

$$\beta_2 + \gamma \leq \left[ -3 + \frac{\mu'_1 + \lambda}{2} \right] - \mu'_1 + \sum \nu'_i = -3 - \frac{\mu'_1}{2} + \mu'_1 + \sum \nu'_i \leq -3 - \frac{\nu'_i}{2} + k \leq r - \frac{3}{2}.$$

As in previous case, we get

$$\|\rho^1\|_{1+\nu'_1} \cdots \|\rho^1\|_{1+\nu'_{l+1}} \varphi|_{\beta_2+1} \leq C_T |\varphi|_{r} + \|\rho^1\|_{r+1/2} \|\varphi\|_{1}.$$

(iii) $\alpha = 1/2$. We need to estimate $|\tilde{K} f|_{r+1/2}$ with $r + 1/2 = k + 1$.

Recall that $\rho^1 \in \Lambda^{k+3}$. On $D_{r_4/2}^{12}$, we write $\nabla^{k} \tilde{K} f$ as a sum of

$$K_{\mu, \nu}^{(k)} f := \nabla^{1+\nu'_1} \cdots \nabla^{1+\nu'_{l+1}} \rho^1 K_{\mu, \nu} f.$$

To estimate $|K_{\mu, \nu}^{(k)} f|_{1}$, we will use Lemma 5.2 to estimate $|K_{\mu} f|_{1}$. 

As before, the worst term occurs for \( j = n - 1 \) and \( \mu_0 = \mu_2 = 0 \) in \( K_{\nu} \), which are assumed now. Then \( K_{\nu}f \) has the form (6.23), while (6.24) has the form

\begin{equation}
\mu_1 \leq \mu''_1 + \sum_{i=1}^{\mu_1} \nu_i' \leq k.
\end{equation}

By (6.13), we have up to a constant factor

\begin{equation}
|K_{\nu}^{(k)} f|_1 \leq |\nabla^{1 + \nu'_1} \rho \cdot \nabla^{1 + \nu'_1} \rho|_1 \|K_{\nu}f\|_0 + \|\nabla^{1 + \nu'_1} \rho \cdot \nabla^{1 + \nu'_1} \rho\|_0 |K_{\nu}f|_1 =: I + II.
\end{equation}

We first estimate \( I \). Assume first \( \frac{\mu_1 + \lambda - 3}{2} \geq 0 \). We set
\[
\beta_3' = \epsilon \in (0, 1), \quad \beta_3 = \beta_3' + \frac{\mu_1 + \lambda - 3}{2} \geq 0.
\]
Recall \( \lambda = \mu_1 - \mu'_1 \). Then \( \beta_3 \leq r - 1 \). Then by Lemma 6.2 in which \( \beta_3' > 0 \) we obtain
\[
\|K_{\nu}f\|_0 \leq C\|f\|_{\beta_3}.
\]
We also have
\[
\|\nabla^{1 + \nu'_1} \rho \cdot \nabla^{1 + \nu'_1} \rho\|_1 \leq C_r \sum_{\beta \neq i} \|\nabla^{2 + \nu'_1} \rho\|_0 \prod_{\beta \neq i} \|\nabla^{2 + \nu'_1} \rho\|_0 \leq C'_r \|\rho\|_{2 + \gamma}
\]
with \( \gamma = \sum \nu_i' \). Thus
\[
\beta_3 + \gamma = \frac{\mu_1 + \lambda - 3}{2} + \epsilon + \sum \nu_i' = \epsilon - \frac{\mu_1' - \gamma}{2} - \frac{3}{2} + k \leq r - 2 + \epsilon.
\]
Thus we get
\[
I \leq C_r \|\rho\|_{2 + \gamma} \|\varphi\|_{1 + \beta_3} \leq C_r' \|\rho\|_{2} \|\varphi\|_{r - 1 + \epsilon} + C'_r \|\rho\|_{r + 2} \|\varphi\|_1.
\]
If \( \frac{\mu_1 + \lambda - 3}{2} < 0 \), we set \( \beta_3 = 0 \) and \( \beta_3' = -\frac{\mu_1 + \lambda - 3}{2} \). By Lemma 6.2 with \( \beta_3' > 0 \), we get \( \|K_{\nu}f\|_0 \leq C\|f\|_{\beta_3} = C\|\varphi\|_1 \). Hence \( I \leq C_r \|\rho\|_{2 + \gamma} \|\varphi\|_1 \leq C_r \|\rho\|_{r + 2} \|\varphi\|_1 \).

Finally, we estimate \( II \) in (6.29). Lemma 5.2 says that we can estimate \( |K_{\nu}f|_1 \) via the pointwise estimate of \( \nabla^2 K_{\nu}f \). Write \( \nabla^2 K_{\nu}f \) as a sum of
\[
K_{\nu}^{(k)} f := \nabla^{1 + \nu'_1} \rho \cdot \nabla^{1 + \nu'_1} \rho K_{\nu}f
\]
with
\[
\tilde{\mu}_1 \leq \mu''_1 + \sum \tilde{\nu}_i' \leq 2.
\]
The worst terms occur in the forms
\[
\Pi' := \nabla^3 \rho K_{\nu + \tilde{\mu}} f, \quad \Pi'' := \nabla^2 \rho K_{\nu + \tilde{\mu}} f
\]
for \( \tilde{\mu} = 1 \) and \( \tilde{\mu} = 2 \), respectively.

We first estimate \( \Pi' \) where \( \tilde{\mu} = 1 \). We have
\begin{equation}
K_{\nu}f(z) := \int_{U_1} \frac{f(\zeta) N_{1 + \mu' + \tilde{\mu}}(\zeta - z)}{(\Phi(z, \zeta))^{1 + \mu' + \tilde{\mu}} |\zeta - z|^{2(n - 1)}} dV(\zeta).
\end{equation}
Note that the worst term occurs when \( \tilde{\mu}_1 = \tilde{\mu} = 1 \). Let us combine the indices as follows. Set \( \tilde{\mu} = \mu + \tilde{\mu} = \mu + 1, \tilde{\mu}_1 = \mu_1 + \tilde{\mu}_1 = \mu_1 + 1 \). Set \( \tilde{\nu}'_i = \nu_i' \) for \( i \leq \mu_1, \tilde{\nu}'_{\mu_1} = 2 \). The latter implies
\[
\mu''_1 = \mu'_1 + 1 \geq 1.
\]
Set $\tilde{\lambda} = \tilde{\mu}''_1 = \tilde{\mu}_1 - \tilde{\mu}'_1$. Then $\tilde{\mu}''_1 = \mu''_1$. We get
\[
\tilde{\mu}_1 \leq \tilde{\mu}''_1 + \sum \tilde{\nu}'_i = \mu''_1 + 2 + \sum \nu'_i \leq k + 2.
\]
Thus, we set
\[
\beta_4 = -1, \quad \beta_4 = \max\left\{0, -1 + \frac{\tilde{\mu}_1 + \tilde{\lambda} - 3}{2}\right\}.
\]
Then
\[
|II'(z)| \leq C_r \|\rho^4 \|_{1+\tilde{\nu}'_1} |\varphi|_{\beta_4+1} \text{ dist}(z, \partial D^1)^{\beta_4}, \quad \tilde{\nu}'_1 = 2,
\]
where $|\varphi|_{\beta_4+1}$ is replaced by $|\varphi|_1$ when $\beta_4 = 0$. For the new indices, let $\hat{\gamma} = \sum (\hat{\nu}'_i - 1)$. We have
\[
\|\nabla^{1+\hat{\gamma}} \rho^1 \cdots \nabla^{1+\hat{\nu}'_1} \rho^1\|_0 \leq C_r \|\rho^1\|_{2+\hat{\gamma}}.
\]
Assume first that $\beta_4 > 0$. Then by $\tilde{\mu}_1 \geq 1$, we get from $(\tilde{\mu}_1 + \lambda)/2 - \tilde{\nu}'_1 = \mu''_2 - \tilde{\nu}'_1/2$ and $r = k + 1/2$
\[
\beta_4 + \hat{\gamma} \leq \left[\frac{-5}{2} + \frac{\tilde{\mu}_1 + \lambda}{2}\right] - \tilde{\nu}'_1 + \sum \tilde{\nu}'_i \leq -\frac{\mu''_1}{2} - \frac{5}{2} + (k + 2) \leq r - \frac{3}{2}
\]
Thus we obtained the desired estimate for $|II'(z)|$ from
\[
\|\rho^4 \|_{2+\hat{\gamma}} |\varphi|_{1+\beta_4} \leq C_r \|\rho^4 \|_{r+1/2} |\varphi|_1 + C_r \|\rho^4 \|_{2} |\varphi|_{r-1/2}.
\]
When $\beta_4 = 0$, with $k \geq 1$ and (6.28) we simply use $\hat{\gamma} = 1 + \sum (\nu'_i - 1)_+ \leq k \leq r - 1/2$. Thus $\|\rho^4 \|_{2+\hat{\gamma}} |\varphi|_{1+\beta_4} \leq C_r \|\rho^4 \|_{r+3/2} |\varphi|_1$, which gives us the desired estimate.

We now estimate $II''$. Then we have $\bar{\mu} = 2$, and hence $K_{\mu+\bar{\mu}} f(z)$ has the form (6.30) in which $\tilde{\mu}, \tilde{\mu}_1, \tilde{\mu}_2$ are replaced by $\tilde{\mu}, \tilde{\mu}_1, \tilde{\mu}$ respectively with
\[
\tilde{\mu} \leq \tilde{\mu}''_1 + \sum \tilde{\nu}'_i \leq 2.
\]
The worst term occurs when $\tilde{\mu}_1 = 2$. Let us combine the indices as before. Set $\tilde{\mu} = \mu + 2$, $\tilde{\mu}_1 = \mu_1 + \tilde{\mu}_1$, $\tilde{\mu}_2 = \mu'_1 + \tilde{\mu}'$, and $\tilde{\lambda} = \tilde{\mu}''_1 = \mu''_1 + \tilde{\lambda}'$. Let $\tilde{\nu}'_i = \nu'_i$ for $i \leq \mu_1$ and $\tilde{\nu}'_{\mu_1} = 2$. The latter implies $\tilde{\mu}'_i = \mu'_i + 1 \geq 1$. Then
\[
(6.31)
\tilde{\mu}''_1 + \sum \tilde{\nu}'_i \leq k + 2.
\]
Thus, we set
\[
\beta_5 = -1, \quad \beta_5 = \max\left\{0, -1 + \frac{\tilde{\mu}_1 + \tilde{\lambda} - 3}{2}\right\}.
\]
Then
\[
|K_{\mu+\bar{\mu}} f(z)| \leq C_r |\varphi|_{\beta_5+1} \text{ dist}(z, \partial D^1)^{\beta_5},
\]
where $|\varphi|_{\beta_5+1}$ is replaced by $|\varphi|_1$ when $\beta_5 = 0$. The values of $\beta_5, \beta'_5, \tilde{\mu}'_1$, and condition (6.31) are the same as in the previous case. The same computation yields the desired estimate for $II''$.

In summary, we have obtained desired estimates for $|\nabla^k \partial \partial_1 H_{\varphi}\partial^{\text{in}}_{\partial_1^r/2}|r_{1/2}$ where $r + 1/2 = k_1 + \beta$ with $k_1 \in \mathbb{N}$ and $0 < \beta \leq 1$, by using Lemma 5.2. Then Lemma 5.4 yields the estimate for $|\nabla^k \partial \partial_1 H_{\varphi}\partial^{\text{in}}_{\partial_1^r/2}|r_{1/2}$. The proof is complete by the equivalence of norms in Corollary 5.3. $\square$
7. AN ESTIMATE OF \( \overline{\partial} \) SOLUTION FOR (0,1) FORMS VIA HARTOGS’S EXTENSION

In this section we obtain the regularity of functional \( \overline{\partial} \)-solutions for (0,1)-forms can be achieved via Hartogs’s extension for concavity domains that require merely \( C^2 \) boundary.

We formulate the following result.

**Proposition 7.1.** Let \( D \subset U \subset \mathbb{C}^n \) be defined by \( \rho < 0 \) with \( \rho \in C^2 \). Suppose that \( U \cap \partial D \) is strictly \( 2 \)-concave for each \( \zeta \in \partial D \cap U \). Assume that \( f \in \Lambda_r^c(D) \) with \( r > 1 \) and \( \partial u_0 = f \) on \( D \) with \( u_0 \in C^0(\overline{D}) \). Then for open sets \( U', U'' \) satisfying \( U'' \Subset U' \Subset U \), we have

\[
\|u_0\|_{\Lambda^r+1/2(U'' \cap D)} \leq C_r(U', U'', \nabla \rho, \nabla^2 \rho)(\|f\|_{\Lambda^r(D)} + \|u_0\|_{C^0(\overline{D})}).
\]

Furthermore, \( C_r(U', U'', \nabla \rho, \nabla^2 \rho) \) is stable under a small \( C^2 \) perturbation of \( \rho \).

*Proof.* For each \( \zeta \in U'' \cap \partial D \), by Proposition 4.1 we find a local biholomorphic map \( \psi_\zeta \) such that \( \psi_\zeta(0) = \zeta \), and \( \rho \circ \psi_\zeta = \alpha \rho \zeta \) has the form

\[
\rho_\zeta(z) = -y_n - 3|z_1|^2 - 3|z_2|^2 + \sum_{j>2} \lambda_j |z_j|^2 + o(|z|^2), \quad |\lambda_j| < 1/2.
\]

Then \( D_{\delta_0, \epsilon_0} := (\Delta_{\delta_0}^{n-1} \times \Delta_{\epsilon_0}) \cap \psi_\zeta^{-1}(D \cap U) \) contains \( \tilde{D}_{\delta_0, \epsilon_0} = \{ z \in \Delta_{\delta_0}^{n-1} \times \Delta_{\epsilon_0} : \tilde{\rho}(z) < 0 \} \), where

\[
\tilde{\rho}(z) = -y_n - 2|z_1|^2 - 2|z_2|^2 + \frac{3}{4} \sum_{j=3}^n |z_j|^2.
\]

When \( \delta, \epsilon \) are sufficiently small and \( \delta < \epsilon \), both \( D_{\delta_0, \epsilon_0}, \tilde{D}_{\delta_0, \epsilon_0} \) contain the Hartogs’s domain

\[
H_{\delta, \epsilon} := (0, \ldots, 0, \sqrt{-1} \delta) + (\Delta_{\epsilon} \times \Delta_{\epsilon}) \times \Delta_{\delta}^{n-1} \cup \Delta_{\delta}^{n}.
\]

Here \( H_{\delta, \epsilon} \) is the shaded region in Figure 7.1. Note that \( \tilde{D}_{\delta_0, \epsilon_0} \) has smooth boundary and \( \partial D_{\delta_0, \epsilon_0} \cap \partial \tilde{D}_{\delta_0, \epsilon_0} = \{ 0 \} \).

By Lemma 4.1, we know that the \( C^2 \) norm of \( \psi_\zeta \) is bounded uniformly and \( \epsilon \) can be chosen uniform in \( \zeta \in \partial D \). Further, \( \epsilon \) can be chosen uniformly for small \( C^2 \) perturbation of \( \rho \).

Since \( \partial \tilde{D}_{\delta, \epsilon} \) is smooth, by Theorem 6.4 we have a solution \( u \) on \( \tilde{D}_{\delta, \epsilon} \) for a possibly smaller \( \delta, \epsilon \) such that \( \tilde{\partial} u = \psi_\zeta^* f \) and \( |u|_{\Delta_{\epsilon}^{n-1} \times \Delta^2_{\delta}} \leq C_r |f|_{D_{\rho}} \). Then \( u_0 \circ \psi_\zeta - u \) admits a holomorphic extension \( h \) to

\[
\tilde{D}_{\delta, \epsilon} := \tilde{D}_{\delta, \epsilon} \cup \Delta_{\delta}^{n}
\]

via the Cauchy formula:

\[
h(z) = \frac{1}{2\pi} \int_{|\zeta_1| = \epsilon} h(\zeta_1, z') \frac{d\zeta_1}{\zeta_1 - z_1}.
\]

Here \( \Delta_{\epsilon} \times \{ z' \} \) is the disk indicated by a dotted line in Figure 7.1.

Let \( m \) be the largest integer less than \( r + 1/2 \). By the Cauchy inequalities, we obtain

\[
\|h\|_{\Delta_{\delta} \times \{ z' \} \cup D_{\rho}} \leq C(\|h\|_{L^\infty(H_{\delta, \epsilon})} \leq C_m(\|u_0\|_{D_{\rho}} + |f|_{D_{\rho}}). \]

In particular,

\[
\|u_0\|_{\Lambda^r+1/2(D_{\rho})} \leq C_r(\|u_0\|_{C^0(D)} + \|f\|_{\Lambda^r(D)}), \quad D_{\rho} := \psi_\zeta(\tilde{D}_{\delta', \epsilon'}).
\]
Figure 7.1. $D_{\delta, \epsilon} \subset \tilde{D}_{\delta, \epsilon}$ with $\partial D_{\delta, \epsilon} \in C^2$ and $\partial \tilde{D}_{\delta, \epsilon} \in C^\infty$

Figure 7.2. Lift $z \pm w \in D_{\delta, \epsilon}^* \subset D_{\delta, \epsilon}^*$ into $D_{\delta, \epsilon}^*$

Here $\delta', \epsilon'$ are chosen so that $\tilde{D}_{\delta', \epsilon'}$ are contained in $\Delta^n_{\delta/2}$. Note that $\delta', \epsilon'$ can be chosen uniformly in $\zeta \in U'' \cap \partial D$. The constant $C_r$ is independent of $\zeta \in U'' \cap \partial D$ and stable under a small $C^2$ perturbation of $\rho$.

Next we need to estimate the $\Lambda^\alpha$ norms of $m$-th derivatives of $u_0$. Set $v$ be a derivative of $u_0$ of order $m$. It remains to estimate $v(z + w) + v(z - w) - 2v(z)$ when $z, z \pm w \in D$.

We may assume that $z$ is sufficiently close to the origin and $|w|$ is small. Let $z^* \in \partial D$ be the closed point to $z$. Suppose $\delta = |w|$ is small. Let $\tilde{w} = \delta (z - z^*)/(z - z^*)$. Since $\partial D_z^*$ is tangent to $\partial D$ at $z^*$, then $z + t \tilde{w}$ and $z \pm w + t' \tilde{w} \in D_z^*$ for $t \in (0, 2)$ and $t' \in [1, 2]$. Here we cannot claim any concavity of the intersection of $D$ with the complex line through $z, z - w$, as shown in Figure 7.2 for $\partial D$. Nevertheless, for $\partial D_z^*$ is tangent to $\partial D$ at $z^*$. Consequently, the smooth domain $D_z^*$ contains a cone $V_z^*$ with vertex $z^*$. We now use decomposition (5.3) for $u_0$. We can estimate each row in (5.3) via (7.2) because the triple points in each row are in same smooth domain $D_z^*$ for $\zeta = (z - w)^*, z^*$ or $(z + w)^*$. With the estimate for the Zygmund ratio, we can conclude that $v \in \Lambda^\alpha$ when $0 < \alpha < 1$.

We now consider $\alpha = 1$. Since our solution operator $H_1$ in Theorem 6.4 is linear, by interpolation we conclude that when $f \in \Lambda^r$, $u_0 = H_1 f$ is in $\Lambda^{r+1/2}(D_1^{12})$. Therefore, the estimate for Zygmund ratio is valid too.

To conclude this section, we note that Theorem 6.1, (6.16) in Theorem 6.4 (for $C^2$ domains $D$) and Proposition 7.1 yield the local version of Theorem 1.1 (a). We also remark that the $C^{1/2}$ estimate in [29, Thm. 14.1] seems to require $\partial D \in C^{5/2}$ to repeat the proof of [29, Thm. 9.1].
8. Proof of Theorem 1.1 via canonical solutions

The proof of regularity of the solutions from local to global uses some standard approaches. See Kerzman [35] for the case when $D$ is a domain in $C^n$. We will also derive a global estimate reflecting the norm convexity and this estimate will be used in the next section to prove Theorem 1.2.

Let $D \subset X$ be a relatively compact $C^1$ domain with a $C^1$ defining function on $X$. Fix a relatively compact neighborhood $U$ of $\partial D$. Using a partition of unity on $\overline{D}$ and (6.13), we can define a Stein extension $E: C^0(D) \to C^0_0(U) \cap C^0(X)$ such that

$$\|Ef\|_{C^0(X)} \leq C_0\|f\|_{C^0(\overline{D})}, \quad \|Ef\|_{\Lambda^a(X)} \leq C_0\|f\|_{\Lambda^a(D)}$$

where $C_0$ depends on $\rho$ and but is independent of small $C^1$ perturbations of $\rho$. Using local coordinates on $U$, we can define a Moser smoothing operator $S_t: C^0_0(U) \cap C^0(X) \to C^0_0(X)$ such that

$$\|S_tf - f\|_{C^0(\overline{U})} \leq C_{a,b}t^{a-b}\|f\|_{C^0(\overline{U})}, \quad 0 \leq a < b \leq a + L;$$

$$\|S_tf\|_{C^0(\overline{U})} \leq C_{a,b}t^{a-b}\|f\|_{C^0(\overline{U})}, \quad \forall b \geq a \geq 0;$$

$$\|S_tf - f\|_{\Lambda^a(\overline{U})} \leq C_{a,b}t^{a-b}\|f\|_{\Lambda^a(\overline{U})}, \quad 0 < a \leq b \leq L + a;$$

$$\|S_tf\|_{\Lambda^a(\overline{U})} \leq C_{a,b}t^{a-b}\|f\|_{\Lambda^a(\overline{U})}, \quad \forall b \geq a > 0.$$ See [17, (3.19)-(3.22)] for details when $X = C^n$, and the general case follows from a partition of unity and (6.13).

We start with the following.

**Lemma 8.1.** Let $D \subset X$ be a domain defined by a $C^2$ function $\rho < 0$ and let $D_a$ be defined by $\rho < a$. Let $\rho_t = S_t\rho$, where $S_t$ is the Moser smoothing operator. Suppose that $\partial D$ is an $a_q$ domain. Let $D^a_q$ be defined by $\rho_t < a$. There exists $t_0 = t_0(\nabla \rho, \nabla^2 \rho) > 0$ and $C > 1 > c > 0$ such that if $0 \leq t < t_0$, then

(8.1) \[ \|\rho_t - \rho\|_{C^0(X)} \leq t/4, \]

(8.2) \[ \partial D^a_q \subset D_{-ct} \setminus D_{-Ct} \quad \partial D^a_q \subset D_{Ct} \setminus D_{ct} \]

while $D_a$ and $D^a_q$ still satisfy the condition $a_q$ for $b \in (-t_0, t_0)$. Here $t_0, C, C$ are independent of small $C^2$ perturbations of $\rho$.

**Proof.** Let $\rho_t = S_t\rho$. We have $\|\rho_t - \rho\|_{C^0(X)} \leq C_2t^2\|\rho\|_{C^2(X)}$. This shows that

(8.3) \[ \text{dist}(\partial D^a_q, \partial D_a) \leq C_2\|\rho\|_{C^2(X)}t^2, \]

Using a local $C^1$ diffeomorphism whose first component is $\rho$, we can verify that $\max\{|s|, |s'|\} < \|\rho\|_{C^1(X)}/C_1$ and $s' > s$, we also have

$c_1(s' - s) \leq \text{dist}(D_a, \partial D_{s'}) \leq C_1(s' - s), \quad c_1(s' - s) \leq \text{dist}(D^a_q, \partial D^a_q) \leq C_1(s' - s).$ Thus (8.3) implies that

$$\text{dist}(\partial D^a_q, \partial D_a) \geq \|\rho\|_{C^2(X)}t^2 > c_1t/2.$$

Suppose $L\zeta\rho$ has $(q+1)$ negative Levi eigenvalues bounded above by $-\lambda$ or $(n-q)$ positive Levi eigenvalues bounded below by $\lambda$ for $\zeta \in U$, where $U$ is neighborhood of $\partial D$ and $\lambda$ is positive number. We find a subspace $W$ of $T^{(1,0)}_{\zeta}\partial D$ such $L\rho(\zeta, v) \leq -\lambda$ for $v$ in the unit sphere of $W$. Projecting $W$ onto $\tilde{W} \subset T^{(1,0)}_{\zeta}\partial D_a$...
when \( \tilde{\zeta} \in \partial D^n \) is sufficiently close to \( \zeta \) and \( t \) is close to zero. Then \( \text{dim} W \geq q + 1 \) and \( L\rho(\zeta, \nu) \leq -\lambda/2 \) for \( \nu \) in the unit sphere of \( \tilde{W} \). One can also verify that if \( L_{\zeta}\rho \) has at least \( (n - q) \) positive eigenvalues, so is \( L_{\tilde{\zeta}}\rho \) when \( \tilde{\zeta} \) is sufficiently close to \( \zeta \).

Therefore, \( D^t_\rho \) still satisfies the condition \( a_q \). \( \square \)

We need to deal with domains of which different boundary components may have different smoothness. Let \( D \) be a relatively compact \( a_q \) domain in a complex manifold \( X \). Assume that the \( (n - q) \) strictly convex components \( b^+_n D \) are of class \( C^2 \) and \( (q + 1) \) strictly concave components \( b^-_{q+1} D \) are of class \( \Lambda^s \) with \( s > 2 \). Using a partition of unity, we can find a defining function \( \rho \) on \( X \) such that \( D \) is defined by \( \rho < 0 \) and \( \nabla \rho(\zeta) \neq 0 \) for \( \zeta \in \partial D \), \( \rho \in C^2(\overline{U}) \) and \( \rho \in \Lambda^s(U' \cap U'' \cap D \cap V) \), where \( U' \) (resp. \( U'' \)) is a neighborhood of \( b^+_n D \) (resp. \( b^-_{q+1} D \)) in \( X \).

**Definition 8.2.** Set \( \| \rho \|_{C^0} := \max\{\|\rho\|_{C^2(D)}, \|\rho\|_{C^2(V)}\} \) and \( \| \rho \|_{\Lambda^s} := \max\{\|\rho\|_{C^2(D)}, \|\rho\|_{\Lambda^s(U' \cap U'')}}\} \).

We know formulate the main result of this paper in details.

**Theorem 8.3.** Let \( r \in (1, \infty) \) and \( q \geq 1 \). Let \( D := \{ \rho < 0 \} \) be a relatively compact domain with \( C^2 \) boundary in a complex manifold \( X \) satisfying the condition \( a_q \). Let \( V \) be a holomorphic vector bundle of finite rank over \( X \). Then there exists a linear \( \overline{\partial} \) solution operator \( H_q : \Lambda_{(0,q)}^{(r+1/2)}(D, V) \cap \overline{\partial}L^2(\rho)(D, V) \rightarrow \Lambda_{(0,q-1)}^{(r+1/2)}(D, V) \) satisfying the following

(a) When \( q = 1 \) or \( \partial D \) is strictly \((n - q)\) convex, we have \( \| H_q f \|_{\Lambda^{r+1/2}(D)} \leq C_{r}\left(\nabla \rho, \nabla^2 \rho\right)\| f \|_{\Lambda^r(D)} \).

(b) When \( q > 1 \) and the components of \( \partial D \) that are \((q + 1)\)-concave are in \( \Lambda^{r+1/2} \), we have \( \| H_q f \|_{\Lambda^{r+1/2}(D)} \leq C_{r}\left(\nabla \rho, \nabla^2 \rho\right)\left((\| \rho \|_{C^1})^m\| f \|_{\Lambda^{r+1/2}(D)}\right) \).

(c) In both cases, \( H_q \) is independent of \( r \) and hence \( H_q f \in C^\infty(\overline{D}) \) when \( f \in C^\infty(\overline{D}) \).

Further, \( m \) is independent of \( r, s \) and remain the same under small \( C^2 \) perturbation of \( \rho \). Furthermore, the constant \( C_{r}\left(\nabla \rho, \nabla^2 \rho\right) \) is stable under small \( C^2 \) perturbations of \( \rho \); more precisely there exists \( \delta(\nabla \rho, \nabla^2 \rho) > 0 \) such that if \( \| \tilde{\rho} - \rho \|_{C^2(X)} < \delta(\nabla \rho, \nabla^2 \rho) \), then

\[
C_{r}\left(\nabla \rho, \nabla^2 \rho\right) < C'_{r}C_{r}\left(\nabla \rho, \nabla^2 \rho\right).
\]

We emphasize that \( C_{r}\left(\nabla \rho, \nabla^2 \rho\right) \) involves an unknown constant that is \( C_* \) from Theorem B.10.

**Remark 8.4.** The stability of estimates on \( \overline{\partial} \) solutions has been discussed extensively in literature; see Greene–Krantz [25] for strictly pseudocovex domains in \( C^n \), Lieb-Michel [45] strictly pseudocovex domains with smooth boundary in a complex manifold. The stability in terms (8.4) is called upper-stability in Gan–Gong [17] where the reader can find a version of lower stability and its use.

**Proof.** The proof is a combination of the following: the local regularity results obtained, Grauert’s bumping method, the stability of solvability of the \( \overline{\partial} \)-equation after the bumping is applied [31, Thm. 3.4.1] (see Theorem B.10 for the vector.
bundle version), and the interior estimates of $\partial$-solutions on Kohn’s canonical solutions.

We will complete the proof in three steps.

**Step 1. Reduction to interior regularity.** Let $D$ be a relatively compact subset of $\mathcal{U}$, defined by $\rho < 0$ in $\mathcal{U}$ in $X$ with $\|\rho\|_{C^{0,2}(\mathcal{U})} < \infty$ and $\nabla \rho \neq 0$ when $\rho = 0$. For each $p \in \partial D$, we have a configuration $(U_p, D^1_p, D^2_p, \psi_p, \rho^1_p, \rho^2_p)$ with $\rho^1_p = \rho \circ \psi_p^{-1}$ as in Definition 3.4. Recall that $\Lambda^r = \Lambda^r(\mathcal{U})$, $\Lambda^r(D^1_p \cap D^2_p)$ is $\overline{\partial}$-closed and in $\Lambda^r$. We have $\Lambda^r(\mathcal{U}) \subset X$, for $\chi \psi_p^{-1}(\partial(\chi u_p)) = (1 - \chi)(\psi_p^{-1})^* f - \overline{\partial} = \chi u_p$. Let $\hat{f}_1 := (\psi_p^{-1})^* f - \overline{\partial} = \chi u_p + (1 - \chi)(\psi_p^{-1})^* f - \overline{\partial} \chi \wedge u_p$.

Recall that

$$
\|f\|_{\Lambda^{r+1/2}(\mathcal{U})} \leq C_r(\|f\|_{D^r} + \|\rho\|_{\Lambda^{r+3/2}}(\mathcal{U})}, \quad r > 1,
$$

$$
\|\hat{f}\|_{\Lambda^{1}(\mathcal{U})} \leq C_r(1 + \|\rho\|_{\Lambda^1}(\mathcal{U})},
$$

In fact, setting $f_1 = 0$ on $X \setminus D$, we have $f_1 \in \Lambda^r(D \cup \hat{B}_{r_{4/3}}(p))$ for $\hat{B}_a(p) = \psi^{-1}(B_a)$. Let $D_p$ be defined by $\rho_p := \rho - \epsilon \chi \circ \psi_p < 0$. We have $D \cup \omega_p \subset D \cup \hat{B}_{r_{4/3}}(p)$, where $\omega_p$ is an open set containing $p$; for instance we can take $\omega_p = \{\rho < \epsilon/C\} \cap B_{r/3}$, where $C$ depends only on $\|\nabla \rho\|_0$. Then $\omega_p \cap \partial D$ contains $\partial D \cap \hat{B}_{r_{4/3}}(p)$ no matter how small $\epsilon$ is. As in [20, 45], we find finitely many $p_1, \ldots, p_m \in \partial D$ independent of $\epsilon$ so that $\{\omega_{p_1}, \ldots, \omega_{p_m}\}$ covers $\partial D$. Also $\sum \chi \circ \psi_{p_j} > 0$ on $\partial D$.

**Figure 8.1.** Stability of bumping: $\partial D_0 \subset \bigcup \omega_j \subset D_m$
With \( \rho_0 = \rho, D_0 = D \) and \( \epsilon > 0 \), set
\[
(8.5) \quad \rho_j = \rho_{j-1} - \epsilon \chi \circ \psi_{p_j} = \rho_0 - \epsilon \sum_{\ell \leq j} \chi \circ \psi_{p_\ell}
\]
and \( D_j := (D_{j-1})_{p_j} \); \( \rho_j < 0 \) for \( j \geq 1 \). We have \( D_j \setminus D_{j-1} \subset \tilde{B}_{\epsilon/2}(p_j) \). Also, \( D_j \)
contains \( D \cup \omega_{p_j} \), and \( D_j \subset D_{j+1} \). Thus \( D_m \) contains \( D_0 \) and \( \text{dist}(D_0, \partial D_m) > 1/C_\epsilon \).

Let \( \tilde{D}_0 \) be a perturbed domain of \( D_0 \), which is defined by \( \tilde{\rho} < 0 \), and let \( \tilde{D}_m \) be
defined by \( \tilde{\rho}_m < 0 \), where \( \tilde{\rho}_m \) is defined by (8.5) with \( \rho_0 \) being replaced by \( \tilde{\rho}_0 = \tilde{\rho} \).

Fix \( \epsilon > 0 \) and then fix \( \delta_1 \) such that
\[
0 < \max\{\|\rho_m - \rho_0\|_{C^2(X)}, \delta_1\} < \min \left\{ c_*, \frac{1}{2} \delta(\nabla \rho, \nabla^2 \rho), \frac{1}{2} \delta_*, \frac{1}{2} t_0(\nabla \rho, \nabla^2 \rho) \right\},
\]
\[
\{\rho < 2\delta_1\} \subset \{\rho_m < -\delta_1\},
\]
where \( t_0(\nabla \rho, \nabla^2 \rho) \) is given by Lemma 8.1, \( \delta(\nabla \rho, \nabla^2 \rho) \) is given by Theorems 8.3 and B.10, and \( c_*, \delta_* \) are given by Corollary B.11.

Suppose \( \|\tilde{\rho} - \rho\|_2 \leq \delta_1 \). Thus
\[
\{\tilde{\rho} < a - \delta_1\} \subset \{\rho < a\} \subset \{\tilde{\rho} < a + \delta_1\}.
\]

Since \( \tilde{\rho}_m - \rho_m = \tilde{\rho} - \rho \), then \( \|\tilde{\rho}_m - \rho_m\|_2 \leq \delta_1 \) and hence
\[
\{\tilde{\rho}_m < a - \delta_1\} \subset \{\rho_m < a\} \subset \{\tilde{\rho}_m < a + \delta_1\}.
\]

Therefore,
\[
\{\rho < -c_*\} \subset \{\tilde{\rho} < 0\} \subset \{\rho < \delta_1\} \subset \{\rho < 2\delta_1\} \subset \{\rho_m < -\delta_1\} \subset \{\tilde{\rho}_m < 0\}.
\]

Set \( \rho' = S_{\delta_1, \rho} \). Then by (8.1)
\[
(8.6) \quad \|\rho' - \rho\|_{C^0(X)} \leq \frac{\delta_1}{4}.
\]

Define
\[
\Omega' = \{\rho' < \frac{5\delta_1}{4}\}, \quad \Omega = \{\rho' < \frac{7\delta_1}{4}\}.
\]

Then (8.6) implies that \( \{\rho < \delta_1\} \subset \Omega' \) and \( \Omega \subset \{\rho < 2\delta_1\} \). Therefore,
\[
(8.7) \quad \{\rho < -c_*\} \subset \{\rho < 0\} \subset \{\rho < \delta_1\} \subset \Omega' \subset \Omega \subset \{\tilde{\rho}_m < 0\}.
\]

For the configuration \((D_1, D_2, U_p, \psi_p)\) or \((D_{1, p}, D_{2, p}, D_{3, p}, U_{p, j}, \psi_{p, j})\) as in Definitions 3.4 or 4.3, we have the operator \( T_j \) satisfying \( \mathcal{D}T_j(\psi^{-1}_{p+1})^*f_j = f_j \),
where \( f_0 := f \). Define
\[
f_{j+1} = f_j = \mathcal{D}(\psi_{p+1}^*(\chi \cdot (T_j(\psi_{p+1}^{-1})^*)f_j)).
\]

Then \( f_m \in \Lambda'(D_m) \) is \( \mathcal{D} \) closed on \( D_m \). We remark that \( f \mapsto f_m \) is a linear operator \( \mathcal{G}_D : \Lambda'(D) \cap ker \mathcal{D} \to \Lambda'(D_m) \cap ker \mathcal{D} \), and \( \mathcal{G}_D \) is independent of \( r \).

We write
\[
f = \mathcal{D}u_m + f_m, \quad u_m = \sum_{j=1}^m \psi_{p,j}^*(\chi \cdot (T_{j-1}(\psi_{p,j}^{-1})^*)f_{j-1}).
\]

Therefore, we can focus on the \( \mathcal{D} \) equation for a fixed \( a_q \) domain \( \Omega \) with smooth boundary. It is important that \( \Omega \subset \tilde{D}_m \) as long as \( \|\tilde{\rho} - \rho\|_{C^2(X)} < \delta_1 \). We will apply Theorem B.10 to the domain \( D_m, \tilde{D}_m \) respectively.
To ease notation, we write
\[ \| \cdot \|_{D,a} = \| \cdot \|_{C^0(\Omega)}, \quad \| \cdot \|_{D,a} = \| \cdot \|_{\Lambda^2(\Omega)}, \quad \| \rho_j \|_{\Lambda^a} = |\rho_j|_{\bar{a}}. \]
Recall that \( |\rho_j|_{\bar{a}} \) is defined in Definition 8.2. We also need to estimate the norms for \( f_m \). We have for \( r > 1 \)
\begin{align*}
(8.8) \quad |f_j|_{D_{j+1};r} & \leq C_r(D_j)(|f_j|_{D_{j};r} + |\rho_j|^{-1}_{r+2}f_j(D_{j};1)), \\
(8.9) \quad |f_j|_{D_{j+1};1} & \leq C_r(D_j)(1 + |\rho_j|^{-1})\| f_j \|_{D_{j};1}, \\
(8.10) \quad |u_j|_{D_{j+1};r+1/2} & \leq C_r(D_j)(|f_j|_{D_{j};r} + |\rho_j|^{-1}_{r+5/2}\| f_j \|_{D_{j};1}).
\end{align*}
By (8.5), we have \(|\rho_j|^{-1}_{r+5/2} \leq C_{r,r}(1 + |\rho|^{-1}_{r+5/2}) \leq 2C_{r,r}|\rho|^{-1}_{r+5/2} / r \). Thus,
\begin{align*}
(8.11) \quad |f_m|_{D_{m};r} & \leq C_r(D)(|f_m|_{D_{m};r} + (|\rho_j|^{-1}_{r+2})^{m}|f_m|_{D_{m};r}), \\
(8.12) \quad |u_m|_{D_{m+1};r+1/2} & \leq C_r(D)(|f_m|_{D_{m};r} + (|\rho_j|^{-1}_{r+5/2})^{m}|f_m|_{D_{m};r}).
\end{align*}

Step 2. Smoothing for interior regularity. To obtain the interior regularity, we will use regularity in Sobolev spaces. We need to avoid the loss in H"older exponent from the Sobolev embedding. To this end, we will again use a partition of unity to overcome the loss. We can make \( f_m \) to be \( C^\infty \) on any relatively compact subdomain \( U' \) of \( \Omega \) via local solutions as follows. Fix \( x_0 \in \overline{U} \) where \( \Omega' \) is given in Step 1 with \( D \subset \Omega' \subset D_m \). We solve \( \bar{\partial} u = f_m \) on an open set \( \Omega \) in \( D_m \) that contains \( x_0 \). Let \( \chi \) be a smooth function with compact support in \( \omega \) such that \( \chi = 1 \) on a neighborhood \( \omega' \) of \( x_0 \). Then \( \hat{f} = f_m - \bar{\partial} u = (1 - \chi) f_m + \bar{\partial} \chi \wedge u \) is still in \( \Lambda^r \), where \( \hat{f} = 0 \) on \( \omega' \). In particular, \( \hat{f} \in C^\infty(\omega') \). Repeating this finitely many times, we can find \( \hat{u} \in \Lambda^{r+1}(\Omega) \) with compact support in \( D_m \) such that
\[ \hat{f} = f_m - \bar{\partial} \hat{u} \in C^\infty(\overline{U}), \]
for any \( r' > r \). Recall that \( \Omega' \) still satisfies the condition \( a_q \) and \( \partial \Omega' \in C^\infty \).

By (8.7), we have \( D_{r-c} \subset \hat{D} \subset \Omega' \subset D_{r-c} \subset \hat{D} \). Since \( 2\delta < \delta \), Cor. B.11 implies that \( \bar{\partial} u = \hat{f} \) admits a \( L^2 \) solution \( u \) on \( \Omega' \) and hence it admits an \( L^2 \) solution \( u \) on \( \Omega' \). Furthermore, the defining function \( \rho' := \rho - 5\delta \) of \( \Omega' \) satisfies \( \|\rho'|_a \leq C_\rho \|\rho\|^2 \).

Step 3. An application of canonical solutions. We fix a smooth hermitian metric on \( T^{1,0}X \) and a smooth hermitian metric on \( V \). Let \( \varphi = e^{\rho'} \), where \( \rho(z) = -\text{dist}(z, \partial\Omega') \). Let \( L_{p,q}(\Omega', V, \varphi) \) be the space of \( V \)-valued \( (p,q) \) forms \( f \) on \( \Omega' \) such that
\[ \| f \|_{\varphi}^2 := \int_{\Omega'} |f(x)|^2 e^{-\varphi(x)} \, dv(x) < \infty, \]
where \( dv \) is the volume form on \( X \) with respect to the hermitian metric on \( X \). Write \( \| f \|_{\varphi} \) as \( \| f \| \) when \( \varphi = 0 \).

We want to apply [31, Thm. 3.4.6], i.e. Theorem B.10 for the vector bundle version to \( \Omega' \). Then \( \varphi \) satisfies the condition \( a_q \) on \( \Omega' \). Let \( k = k_\chi(\varphi) \). Let \( k_a \) to be the integer in Theorem B.10 and let \( \tilde{\varphi} = \varphi_{k_a} \).
We now consider \( T^\varphi_{\tilde{\varphi}} = \hat{\varphi} \) as densely defined from \( L^2(\Omega', \varphi) \) into \( L^2(\Omega', \varphi) \) and \( T^\varphi_{\tilde{\varphi}} \) as its adjoint. Let \( f_m \) be the \((0,q)\) form derived in Step 1. By Corollary B.11, we find a solution \( u_0 \) satisfying \( \bar{\partial} u_0 = f_m \) on \( \Omega' \) and
\[ \| u_0 \|_{\varphi} \leq C_s \| f_m \|_{\varphi}. \]
For the estimate, we need Kohn’s canonical solutions. By [31, Thm. 1.1.1], $R_{(T_q^\mathbf{\hat{\varphi}})^*}$ is also closed and $R_{(T_q^\mathbf{\hat{\varphi}})^*} = N_{(T_q^\mathbf{\hat{\varphi}})^*}$. We now apply the decomposition

\begin{equation}
(8.13)
\quad u_0 = u + h, \quad u \in N_{T_q^\mathbf{\hat{\varphi}}}, \quad h \in N_{T_q^\mathbf{\hat{\varphi}}}.
\end{equation}

Thus, $u = (T_q^\mathbf{\hat{\varphi}})^* v$ and $u$ satisfies

\begin{equation}
(8.14)
\quad \bar{\partial}u = \bar{f} \quad \text{on } \Omega',
\end{equation}

\begin{equation}
(8.15)
\quad \|u\|_{\mathbf{\hat{\varphi}}} \leq \|u_0\|_{\mathbf{\hat{\varphi}}} \leq C_{\mathbf{f}_m}\|f_{\mathbf{m}}\|_{\mathbf{\hat{\varphi}}}.
\end{equation}

In particular, in the sense of distributions, we have $\bar{\partial} q^\mathbf{\hat{\varphi}} v = u$ on $\Omega'$. Here $q^\mathbf{\hat{\varphi}}$ is the formal adjoint (acting on test forms) of $\bar{\partial} q$ in the $L^2$ spaces with weight $\mathbf{\hat{\varphi}}$. Since $\bar{\partial}_{q-1} q^\mathbf{\hat{\varphi}} = 0$, we get in the sense of distributions

\begin{equation}
(8.16)
\quad \bar{\partial} q^\mathbf{\hat{\varphi}}_{q-1} u = 0 \quad \text{on } \Omega'.
\end{equation}

Our last step is to use the system of elliptic equations for $u$ to derive the interior estimates, using (8.14)-(8.16). For its independent interest, the last step follows from Proposition 8.5 below.

**Proposition 8.5.** Let $\Omega'$ be an open set in $X$. Let $f \in L^2_{(0,q-1)}(\Omega', V, \mathbf{\hat{\varphi}})$ and $u \in L^2_{(0,q-1)}(\Omega', V, \mathbf{\hat{\varphi}})$ with $\mathbf{\hat{\varphi}} \in C^\infty(\overline{\Omega'})$. Suppose

\begin{equation}
(8.17)
\quad \bar{\partial} u = f, \quad \bar{\partial} q^\mathbf{\hat{\varphi}} u = h \quad \text{on } \Omega'.
\end{equation}

If $\Omega'' \subset \subset \Omega'$, then for $2 \leq p < \infty, 0 < \alpha < 1$ and $k \in \mathbb{N}$, we have

\begin{equation}
(8.18)
\quad \|u\|_{W^{k+2,p}(\Omega'')} \leq C_{k,p}(\Omega', \Omega'')(\|u\|_{W^{0,p}(\Omega')} + \|f\|_{W^{k+1,p}(\Omega')});
\end{equation}

\begin{equation}
(8.19)
\quad \|u\|_{C^{k+\alpha+1}(\Omega'')} \leq C_{k,\alpha}(\Omega', \Omega'')(\|u\|_{L^2(\Omega')} + \|f\|_{C^{k+\alpha}(\Omega')}), \quad k > 1.
\end{equation}

Here $C_{k,\alpha}(\Omega', \Omega''), C_{k,p}(\Omega', \Omega'')$ also depend on $\|\varphi\|_{C^{k+3}(\Omega')}$.

**Proof.** Let $\|\cdot\|_{W^{k,p}} := \|\cdot\|_{W^{k,p}(\Omega')}$ denote the norm for space $W^{k,p}(\Omega')$. Recall that $T^{(1,0)}X$ and $V$ are endowed with smooth hermitian metrics. Let $\omega_1, \ldots, \omega_n$ be a local unit frame for $(1,0)$ forms and let $e_1, \ldots, e_m$ be the local unit frame of $V$. Following notation in Appendix B, we have $u = \sum u_j^\nu \overline{\omega}^\nu \otimes e_\nu$ and

\begin{equation}
\bar{\partial} u = \sum_\nu \sum_j \sum_k \nu \frac{\partial u_j^\nu}{\partial x_k} \overline{\omega}^\nu \otimes \bar{\omega}^j \otimes e_\nu + R u,
\end{equation}

\begin{equation}
\bar{\partial} q^\mathbf{\hat{\varphi}}_{q-1} u = - \sum_\nu \sum_j \sum_k \nu \frac{\partial u_j^\nu}{\partial \nu^k} \overline{\omega}^\nu \otimes \bar{\omega}^j \otimes e_\nu + B u,
\end{equation}

where $R u$ and $B u$ are of order zero in $u$. Write $\bar{\partial}^\varphi$ as $\bar{\partial}$ when $\varphi = 0$. Let $\chi$ be a smooth function with support in a ball $B_R \subset \subset \Omega'$ of radius $R$ centered at a point in $x_0 \in \Omega'$. Let $\bar{\partial} u = \chi u.\text{ We abbreviate } (f,h) \text{ by } \bar{f}. \text{ We have}$

\begin{equation}
\quad ||\bar{\partial} u||_{W^{0,2}} + ||\bar{\partial} \bar{u}||_{W^{0,2}} \leq C(\|\bar{f}\|_{W^{0,2}} + \|u\|_{W^{0,2}}).
\end{equation}

By [32, Lem. 4.2.3, p. 86], for any relatively compact subset $\Omega''$ of $\Omega'$, we have

\begin{equation}
(8.20)
\quad ||u||_{W^{1,2}(\Omega'')} \leq C(\Omega'', \Omega'^2)C_{s}(\|\bar{f}\|_{W^{0,2}} + \|u\|_{W^{0,2}}).
\end{equation}

Recall the Sobolev compact embedding of $W^{j,q}$ in $W^{j+1,p}$ for $q = \frac{p}{1-\frac{p}{n}}$ when $1 \leq p < 2n$. For the following, we take $p_0 = 2$ and fix any $2 < q < \infty$. We then fix $2 < p_1 < \cdots < p_n$ such that $p_n-1 < 2n$, $p_{j+1} \leq \frac{p_j}{1-\frac{p}{2n}}$ and $p_n > q$. 


Let $\Box_{\varphi} = \overline{\partial}_{q-1} \varphi^{2}_{q-1} + \varphi^{2}_{q} \overline{\partial}_{q}$. Then the principal part $\Delta_{q} = -\sum g^{ij} \frac{\partial^{2}}{\partial x^{i} \partial x^{j}}$ of $\Box_{\varphi}$ has smooth coefficients. Further, $\Delta_{g}$ is independent of $\varphi$, diagonal and elliptic, where $g$ is the smooth hermitian metric $X$ (see [36, p. 154, p. 160] or Appendix B).

In the sense of distribution, we have

$$\Delta_{g} u = b(\nabla \varphi) \nabla f + c_{1}(\nabla \varphi) \nabla u + c_{0}(\nabla^{2} \varphi) u,$$

where $b, c_{1}, c_{0}$ and $c_{1}, c_{2}$ below are matrices of polynomials whose coefficients depending on $g$. Then as a weak solution, $u$ satisfies $\Delta_{g} \tilde{u} = v$ for

$$(8.21) \quad v := \chi[b(\nabla \varphi) \nabla \tilde{f} + c_{1}(\nabla \varphi) \nabla u + c_{0}(\nabla^{2} \varphi) u] + c_{1}(\nabla \chi)) \nabla u + c_{2}(\nabla^{2} \chi) u.$$

Next, we recall two interior estimates on systems of elliptic equations from Morrey [52, Thm 6.4.4., p. 246]:

$$(8.22) \quad \| \tilde{u} \|_{W^{k+p,2} \Omega} \leq C_{k,p} \| v \|_{W^{k,2} \Omega} + C_{R} \| \tilde{u} \|_{W^{0,1} \Omega}, \quad 1 < p < \infty;$$

$$(8.23) \quad \| \tilde{u} \|_{W^{k+2,0} \Omega} \leq C_{k,a} \| v \|_{W^{k,a} \Omega} + C_{R} \| \tilde{u} \|_{L^{p} \Omega}, \quad \text{supp} \tilde{u} \subset B_{R}$$

provided the right-hand sides are finite. We may assume that $c_{0} < R < C_{0}$.

By the Sobolev inequality, $\| \tilde{u} \|_{W^{k,2} \Omega} \leq C_{k} \| \tilde{u} \|_{W^{k+1,2} \Omega}$. Estimating the latter via (8.22) with $k = 0$ and $p = 2$ and (8.20), we get $\| \tilde{u} \|_{W^{0,2} \Omega} \leq C_{2}(\| v \|_{W^{0,2} \Omega} + \| u \|_{W^{0,2} \Omega})$. Note that all constants depend on $\text{dist}(\Omega', \partial \Omega')$. Repeating this bootstrapping argument, we can show that for any $\ell \leq \infty$, we have $\| \tilde{u} \|_{W^{1,\ell} \Omega} \leq C_{\ell}(\| f \|_{W^{1,\ell} \Omega} + \| u \|_{W^{0,2} \Omega})$.

By (8.22) and the latter, we get $\| \tilde{u} \|_{W^{2,0} \Omega} \leq C_{2}(\| v \|_{W^{0,2} \Omega} + \| u \|_{W^{0,2} \Omega}) \leq C_{2}'(\| \tilde{u} \|_{W^{0,2} \Omega} + \| f \|_{W^{1,\ell} \Omega}) \leq C_{2}''(\| \tilde{u} \|_{W^{0,2} \Omega} + \| f \|_{W^{1,\ell} \Omega})$ for $\Omega' \in \Omega''' \in \Omega$. Repeating this for higher order derivatives, we obtain (8.18).

Recall Sobolev inequality $C_{k,a} \subset W^{k+1,\ell}$ for $a := 1 - \frac{2m}{p} > 0$. We obtain

$$(8.24) \quad \| u \|_{W^{\ell,1} \Omega} \leq C \| u \|_{W^{\ell+1,0} \Omega} \leq C_{a}(\| \tilde{f} \|_{W^{1,\ell} \Omega} + \| u \|_{W^{0,2} \Omega}) \leq C_{a}'(\| \tilde{f} \|_{L^{p} \Omega} + \| u \|_{W^{0,2} \Omega}).$$

Next, we prove by induction that

$$(8.25) \quad \| u \|_{W^{\ell,k+1,0} \Omega} \leq C_{k,a} \| \varphi \|_{W^{\ell+1,0} \Omega} \| \tilde{f} \|_{W^{1,\ell} \Omega} + \| u \|_{W^{0,2} \Omega})$$

By (8.24), the above holds for $k = 1$. Suppose the above hold and we want to verify it when $k$ is replaced by $k + 1$. We have for $v$ in (8.21)

$$\| v \|_{W^{\ell,k+1,0} \Omega} \leq C \{ \| \tilde{f} \|_{W^{\ell,k+1,0} \Omega} + C_{a} \| \varphi \|_{W^{\ell,k+1,0} \Omega} (\| \tilde{f} \|_{W^{1,\ell} \Omega} + \| u \|_{W^{0,2} \Omega}) \leq C \| \tilde{f} \|_{W^{1,\ell} \Omega} \}.$$

By (8.25) and (8.24), we get $\| v \|_{W^{\ell,k+1,0} \Omega} \leq C_{a} \| \tilde{f} \|_{W^{1,\ell} \Omega} + C_{a} \| \varphi \|_{W^{\ell,k+1,0} \Omega} + C_{a} \| u \|_{W^{0,2} \Omega} \leq C_{a}'(\| \tilde{f} \|_{L^{p} \Omega} + \| u \|_{W^{0,2} \Omega}).$

This proves (8.25) and hence (8.19). \hfill \Box

We conclude this section with an isomorphism theorem on cohomology groups with bounds. Define

$$H_{r,loc}^{r,1/2}(0,q) = \frac{A_{r}(0,q)(\Omega, V) \cap \ker \overline{\partial}}{A_{r}(0,q)(\Omega, V) \cap \overline{\partial} A_{r+1/2}^{r+1/2}(0,q-1)}(\Omega, V),$$

$$H_{r,loc}^{r}(0,q) = \frac{A_{r}(0,q)(\Omega, V) \cap \ker \overline{\partial}}{A_{r}(0,q)(\Omega, V) \cap \overline{\partial} L^{2,local}(0,q-1)}(\Omega, V).$$
Theorem 8.6. Let \( r \in (1, \infty] \). Let \( \Omega \) be relatively compact \( a_q \) domain in \( X \) and let \( V \) be a holomorphic vector bundle on \( X \). There exists \( c > 0 \) such that the restriction \( H^{r,r+1/2}_{(0,q)}(\Omega, V) \to H^{r,loc}_{(0,q)}(\Omega, V) \) is an isomorphism for the following cases.

(a) \( q = 1 \) or \( \partial \Omega \) is strictly \((n - q)\) convex.

(b) \( \partial \Omega \in \Lambda^{r+5/2} \) and \( r \in (1, \infty] \).

Proof. The injectivity follows from the stability result proved in Appendix B. The surjectivity is obtained by the Grauert bumping method.

We remark that using the \( C^{1/2} \) local solution operators in [29, Thms. 9.1 and 14.1], we can also show that the restriction

\[
\begin{align*}
\frac{C^0_{(0,q)}(\Omega, V) \cap \ker \partial}{C^0_{(0,q)}(\Omega, V) \cap \overline{\partial C^{1/2}_{(0,q-1)}(\Omega, V)} } & \to \frac{C^0_{(0,q)}(\Omega, V) \cap \ker \overline{\partial}}{C^0_{(0,q)}(\Omega, V) \cap \overline{\partial L^{2,loc}_{(0,q-1)}(\Omega, V)} }
\end{align*}
\]

is an isomorphism, provided (a) \( \Omega \) is \((n - q)\) strictly convex or \( q = 1 \), or (b) \( \Omega \) is \((q + 1)\) concave with \( \partial \Omega \in C^{5/2} \).

9. Proof of Theorem 1.2 via a Nash-Moser Method

In this section, we will prove Theorem 1.2 for \( q \geq 2 \) when the domains have negative Levi eigenvalues by using the Nash-Moser smoothing operators. Our approach was inspired by a method of Dufresnoy [14] for the \( \overline{\partial} \)-equation on a compact set that can be approximated from outside by strictly pseudoconvex domains of which the Levi eigenvalues are well controlled. It is interested that V. Michel [50] showed that if the number of non-negative Levi eigenvalues of \( \partial \Omega \) is exact \( n - q' \) near \( z_0 \in \partial \Omega \), then \( \overline{\partial} u = f_{(0,q)} \) has a solution in \( C^\infty(U \cap \Omega) \) when \( \partial \Omega \in C^2 \) for all \( q \geq q' \). When \( \partial \Omega \in C^4 \), there is (a possibly different) solution \( u \in C^\infty(U \cap U) \). For pseudoconvex domains with \( C^2 \) boundary in \( C^n \), the \( C^\infty \) regularity of \( \overline{\partial} \) solutions under suitable assumptions on the Levi-form has been proved by Zampieri [70, 71] and Baracco–Zampieri [3, 4]. The reader is referred to the thesis of Yie [69] for the global regularity of \( \overline{\partial} \) solutions with \( \partial D \in C^4 \). When \( \partial D \in C^\infty \) additionally, the existence of \( u \) was proved by Kolm [39]. Michel–Shaw [49] obtained smooth regularity of the \( \overline{\partial} \) solutions on annulus domain \( D_1 \setminus \overline{D_2} \) where \( D_1 \) is a pseudoconvex domain with piecewise smooth boundary and \( D_2 \) is the intersection of bounded pseudoconvex domains. The \( H^s \) solutions was proved by Harrington [27] for pseudoconvex domains \( D \) with \( \partial D \in C^{k-1,1} \), \( k > s + 1/2, k \geq 2 \), and \( s \geq 0 \).

As observed in [19], to the author’s best knowledge it remains an open problem if \( C^\infty(\overline{\partial}) \) solutions \( u \) to \( \overline{\partial} u = f \) exist on a bounded weakly pseudoconvex domain \( D \) in \( C^n \) with \( C^2 \) boundary.

We now state a detailed version of Theorem 1.2.

Theorem 9.1. Let \( q > 1 \). Let \( D \) be a relatively compact domain with \( C^s \) boundary in a complex manifold \( X \) satisfying the condition \( a_q \). Let \( V \) be a holomorphic vector bundle on \( X \). Fix \( \hat{r} \) as follows. Let \( m \) be as in Theorem 8.3, which is the number of times the Grauert bumping is used in Step 1 of the proof of Theorem 8.3.

(i) When \( s = 2 \) and \( r > r_0 := \max\{2m - 1, s + 5/2\} \), fix \( \hat{r} < r - 5/2 + \epsilon \) for some sufficiently small \( \epsilon > 0 \).

(ii) When \( s \geq 7/2 \) and \( r > s + 5/2 \), fix

\[
\hat{r} < r + \frac{1}{2} - \frac{1}{2} \frac{s - 1}{s - 1} \frac{r - 1}{r}.
\]
There exists a linear $\overline{\partial}$-solution operator

$$H^r_q : \Lambda^r_{(p,q)}(D,V) \cap \overline{\partial}L^2_{\text{loc}}(D,V) \rightarrow \Lambda^r_{(p,q-1)}(D,V)$$

satisfying $|H^r_q f| \leq C_{r,r'}(D)|f|$ for all $r' \leq r$. Furthermore,

(a) $C_{r,r'}(D)$ is stable under small $C^2$ perturbations of $D$, and $H^r_q f \in C^\infty(D)$ if $f \in C^\infty((\overline{D},V) \cap \overline{\partial}L^2_{\text{loc}}(D,V))$ additionally.

(b) The $r_0$ as the $m$ is stable under small $C^2$ perturbations of $\partial D$.

Proof. It suffices to prove the theorem when $r, r+1/2, r', r'+1/2$ are not integers, by replacing $r$ by a smaller number and $r'$ by a larger number. This allows us to identity the Zygmund spaces with the Hölder spaces for these orders. This also allows us to use the Taylor theorem and the global estimates for smooth domains.

Let $D$ be a $(q+1)$ concave domain with $C^2$ boundary defined by $\rho \leq 0$. Denote by $D^n$ the domain defined by $\rho < -a$. Suppose that $f \in C^\infty$. Using the Moser smoothing operator $S_t$, we define

$$\tilde{S}_t u = S_tE_{D^t}u.$$

We remark that we need to use values of $u$ on $D^t \subset D$ and the value of $t$ in $S_t$ is much larger than $\epsilon$. Thus we need to use $S_tE_{D^t}$. We have

$$\|\tilde{S}_tu - u\|_{D^t,a} = \|S_tE_{D^t}u - E_{D^t}u\|_{D^t,a} \leq C_{a,b}t^{-a-b}a\|u\|_{D^t,b}, \quad a < b < L + a;$$

$$\|\tilde{S}_tu\|_{D^t,b} \leq C_{a,b}t^{a-b}\|u\|_{D^t,a}.$$

The $L$ is fixed for the rest of the proof.

Assume that $\partial D \in C^s$ with

$$s \geq 2, \quad r \in (s+5/2, \infty), \quad t_{i+1} = t_i^d, \quad i \geq 1$$

with $d > 1$ and $t_1 \in (0,1)$ to be determined. As in Yie [69], we apply the above to the defining function $\rho$ of $D$ by setting

$$\rho_i = \rho \ast \chi_{t_i}, \quad t_i := (c_i \epsilon_i)^{1/s}, \quad D_i^t = \{\rho_i < -\epsilon_i\}, \quad D_i^t = \{\rho < -\epsilon_i\}.$$

Here $c_i \in (0,1)$ is to be determined. Also we need $\epsilon_1 = t_1^d/c_1 < t_0$ for the $t_0$ in Lemma 8.1. We now have

$$\|\rho_i - \rho\|_0 \leq C_s t_i^d\|\rho\|_{s};$$

$$\|\rho_i - \rho\|_2 \leq \epsilon_1, \quad t_i < t^*(\nabla^2 \rho, \epsilon_1);$$

$$\|\rho_i\|_b \leq C_{b,s}t_i^{s-b}\|\rho\|_{s}, \quad b \geq s.$$

We emphasize that $t^*(\nabla^2 \rho, \epsilon_1)$ in (9.3) depends on the modulus of continuity of $\nabla^2 \rho$. Note that $t_i^d = c_i \epsilon_i$. When $c_i$ is sufficiently small, we have by (9.2) and (8.2)

$$\partial D_i^t \subset D_i^{t^d - \epsilon_1} \setminus D_i^{t^{d+\epsilon}} \subset \partial D_i^t \subset D_i^{t^d - \epsilon_1} \setminus D_i^{t^{d+\epsilon}}$$

where $c \in (0,1/2)$ is independent of $c_i$. This however does not cause any difficulty for domains satisfying the condition $a_q$, because the Levi eigenvalues do not decay towards the boundary. This is decisive for the Nash–Moser iteration to succeed in our proof for $C^2$ domains.

We now determine $\epsilon_1$ and $t_1$. Choose $c_1$ so that $\tilde{\rho} := \rho_i$ meet the requirements to apply the stability of constants in Theorem B.10 and Theorem 8.3. We then fix $t_1 < t_0$ so that $\epsilon_1 < t_0$ for $t_0$ in Lemma 8.1 and (9.3) hold.

We will apply Taylor’s theorem for Hölder spaces. Our estimates of gaining 1/2 derivatives for the homotopy formula are for Zygmund spaces. Therefore, in the
following argument, we will work on Hölder spaces $C^a$ with $a \notin \frac{1}{2}\mathbb{N}$. By (9.4) and Theorem 8.3 applied to the domain $D^i_1$, we have a $\partial$ solution operator $H_{q,D^i_1}$ on $D^i_1$ satisfying

$$|H_{q,D^i_1}\varphi|_{D^i_1,r+1/2} \leq C_{r,e'}(\nabla \rho_{t_1}, \nabla^2 \rho_{t_1})(|\varphi|_{D^i_1,r} + |\rho_{t_1}|_{r+5/2}|\rho_{t_1}|_{3+e'}|\varphi|_{D^i_1,1+e'})$$

where $e' > 0$ is a small positive number. Set $f_1 = f$ and $v_1 = H_{q,D^i_1}f_1$ and $\tilde{v}_1 = E_{D^i_1}v_1$. Therefore, we obtain

$$|v_1|_{D^i_1,r+1/2} \leq C_{r,e'}(|f_1|_{D^i_1,r} + t^{4-s}\rho_{t_1}|f_1|_{D^i_1,1+e'})$$

with

$$s_* = \max\{0, 5/2 - s\} + m \max\{0, 3 + e - s\}.$$

Smoothing with a different parameter $\epsilon_1$, we define $w_1 = S_{\epsilon_1}\tilde{v}_1$ on $X$. On $D$ define

$$f_2 = f_1 - \partial w_1.$$

We iterate this. We also find a solution $v_i = H_{q,D^i_1}f_i$ so that

$$f_i = \partial v_i,$$

on $D^i_1$.

Define $\tilde{v}_i = E_{D^i_1}v_i$, $w_i = S_{\epsilon_1}\tilde{v}_i$ and $f_{i+1} = f_i - \partial w_i$. Then we want to show that $u := \sum w_i$ is the desired solution to $\partial u = f_1$ on $D$, where $f_j$ tends to zero on $D$ as $j \to \infty$.

We have

$$\partial v_2 = f_1 - \partial w_1,$$ on $D^2_1$.

On $D^i_1$, $\tilde{v}_1 = E_{D^i_1}v_1 = v_1$ and hence $\partial v_1 = \partial \tilde{v}_1$. Therefore

$$f_2 = f_1 - \partial w_1 = \partial (\tilde{v}_1 - w_1) = \partial (\tilde{v}_1 - S_{\epsilon_1}\tilde{v}_1),$$ on $D^i_1$.

We have

$$\|w_1 - \tilde{v}_1\|_{D^i_1,a} = \|S_{\epsilon_1}\tilde{v}_1 - \tilde{v}_1\|_{D^i_1,a} \leq C_{b,a}b^{-a}\|\tilde{v}_1\|_b, \quad a < b < L + a.$$

Hence by Taylor’s theorem, we get for $b \geq 1 + e^*$

$$\|f_2\|_{D,1+e^*} \leq C_b \sum_{k=1}^{[b]-1} \epsilon_1^{k-1}\|f_2\|_{D^i_1,k} + C_b\epsilon_1^{b-1-e^*}\|f_2\|_{D,b}$$

$$= C_b \sum_{k=1}^{[b]-1} \epsilon_1^{k-1}\|\partial \tilde{v}_1 - \partial w_1\|_{D^i_1,k} + C_b\epsilon_1^{b-1-e^*}\|f_2\|_{D,b}$$

$$\leq C_b \sum_{k=1}^{[b]-1} C_b^{k-1}\epsilon_1^{b-k-e^*}\|v_1\|_{D^i_1,b+k} + C_b\epsilon_1^{b-1-e^*}\|f_2\|_{D,b}$$

$$= C_b\epsilon_1^{b-e^*}\|v_1\|_{D^i_1,b+\frac{1}{2}} + C_b\epsilon_1^{b-1-e^*}\|f_2\|_{D,b}.$$

Recall that $\partial D \in C^a$. By analogy of (9.5) and (9.7), we have

$$|v_1|_{D^i_1,r+1/2} \leq C_{r,e'}(|f_1|_{D^i_1,r} + t^{4-s}\rho_{t_1}|f_1|_{D^i_1,1+e'})$$

$$\|f_{i+1}\|_{D,1+e^*} \leq C_{r,e'}\epsilon_i^{b-e^*}\|v_i\|_{D^i_1,r+\frac{1}{2}} + C_r\epsilon_i^{b-1-e^*}\|f_{i+1}\|_{D,b}.$$
To simplify notation, let \( \| f_i \|_r = \| f_i \|_{D, r} \) and \( \| v_i \|_r = \| v_i \|_{D_i, r} \). Thus the \( r \)-norm is estimated by
\[
\| f_{i+1} \|_r \leq \| f_i \|_r + C_r \| w_i \|_{r+1} \leq \| f_i \|_r + C_r \epsilon_i^{1/2} \| v_i \|_{r+1/2}.
\]
Therefore, by (9.8), we obtain
\[
(9.10) \quad \| f_{i+1} \|_r \leq 2C_{r, \epsilon} \epsilon_i^{1/2} (\| f_i \|_r + t_i^{s-r} \| f_i \|_{1+\epsilon^*}), \quad r \in (1, \infty)^*.
\]
Here \( (1, \infty)^* = (1, \infty) \setminus \{1\} \). We exclude this discrete set of values since we need Taylor theorem for Hölder spaces while our estimates for the homotopy formula need Zygmund spaces.

We now define
\[
(9.11) \quad \hat{B} = \hat{C} \hat{t}^{-1/2} B_i,
\]
with \( B_0 \geq 1 \) being fixed and \( \hat{C} \geq 1 \) to be determined. Fix \( r \in (s + \frac{5}{2}, \infty)^* \). Since our solution \( u \) to \( \partial u = f_i \) is linear in \( f \), by rescaling we may assume that
\[
(9.12) \quad \max \{ \| f_i \|_r, t_i^{s-r} \| f_i \|_{1+\epsilon^*}, \| v_i \|_{r+1/2} \} \leq 1.
\]

By induction, let us show that
\[
(9.13) \quad \| f_i \|_r \leq B_i,
\]
\[
(9.14) \quad t_i^{s-r} \| f_i \|_{1+\epsilon^*} \leq \hat{C} B_i,
\]
\[
(9.15) \quad \| v_i \|_{r+s} \leq \hat{C}_r^2 B_i.
\]
Suppose that the three inequalities hold. We want to verify them when \( i \) is replaced by \( i+1 \). Clearly, (9.13)_{i+1} follows from (9.10), (9.11), (9.13), and \( \hat{C} \geq 4C_{r, \epsilon^*} \). By (9.1), (9.9), (9.15) and (9.13)_{i+1}, we obtain
\[
(9.16) \quad -d(r+s-s) + s(r-1-\epsilon^*) > 0.
\]
The latter is assumed now. Then (9.15)_{i+1} follows from (9.8)_{i+1}, (9.13)_{i+1} and (9.14)_{i+1} and \( \hat{C} \geq 2C_{r, \epsilon^*} \). Therefore, it suffices to take
\[
(9.17) \quad \hat{C} := \max \{ 4C_{r, \epsilon^*}, (C'_{r, \epsilon^*} + C_{r, \epsilon^*}) \hat{c}_i^{s-r} \}.
\]

By interpolation, we get
\[
|f_i|_{(1-\theta)(1+\epsilon^*) \theta} \leq C_{r, \theta} |f_i|^{1-\theta}_{1+\epsilon^*} |f_i|^\theta_r \leq C_{r, \theta} \hat{C}^{\theta} t_i^{1-\theta (r+s-s)} B_i.
\]

By definitions, we have
\[
B_i = \hat{C}^{1-\theta} (t_1 \cdots t_i-1)^{-\frac{2}{d}} B_i = \hat{C}^{1-\theta} t_i^{-\frac{2}{d} \frac{d-1}{4}} B_i \leq \hat{C}^{1-\theta} t_i^{-\frac{2}{d} \frac{d-1}{4}} B_i.
\]
Therefore, \( |f_i|_{1-\theta+\theta \epsilon} \leq C_{r, \theta} \hat{C}_i^2 B_i t_i^{\lambda} \) converges rapidly if (9.16) holds and
\[
\lambda := (1-\theta)(r+s-s) - \frac{s}{2(d-1)} > 0.
\]
By (9.8), we have for $\theta r - \theta \in (0, \infty)^*$

$$|v_i|_{D^{3/2-\theta+\theta r}} \leq C_{r,\epsilon^*}(|f_i|_{D^{3/2-\theta+\theta r}} + \|t_i^s-s-1+\theta-\theta r||f_i|_{D^{3/2-\theta+\theta r}})$$

$$\leq B_1 C_{r,\epsilon^*} C_{r,\theta} \hat{C}_i^* t_i + C_{r,\epsilon^*} \hat{C}_i^* t_i^{r-1+\theta-\theta r} \sup_{s} B_1.$$  

This shows that

$$|v_i|_{3/2-\theta+\theta r} \leq 2 B_1 C_{r,\epsilon^*} C_{r,\theta} \hat{C}_i^* t_i^{\lambda_0}$$

converges rapidly if

$$\lambda_* := (1-\theta)r_* - \frac{s}{2(d-1)} > 0, \quad r_* := r - 1 + \min\{0, s_s - s + 1\}.$$  

We want to maximize $\theta r$ or $\theta$. Now, $\lambda_* > 0$ implies that

$$\theta < 1 - \frac{s}{2(d-1)r_*}.$$  

The latter is an increasing function of $d$. Assume

$$1 - \frac{s}{2(d-1)r_*} > 0.$$  

We now specify the parameters. Note that (9.16) and (9.20) are equivalent to

$$1 + \frac{s}{2r_*} < d < d_*, \quad d_* := \frac{s(r-1-\epsilon^*)}{r+s_s - s}.$$  

Under the above restriction on $d$, $\frac{3}{2} - \theta + \theta r$ has maximum value

$$\hat{r} = r + \frac{1}{2} - \frac{s(r-1)}{2(d_* - 1)r_*}.$$  

Then $|v_i|_{d} < C_{r,\epsilon^*} t_i^{\lambda_*}$ with $\lambda_* > 0$ where $\lambda'$ depends on $r'$ and $r' < \hat{r}$. Consequently the solution $u = \sum S_t E_t v_i$ to $\hat{u} = f$ is also in $\Lambda^r(D)$.

We now compute the value of $\hat{r}$. When $s = 2$, we have $s_* = m(1+\epsilon^*)$ and $r_* = r - 1$. Thus for

$$r > \max\{2m - 2, s + 5/2\},$$  

we have $d_* > 4/3$. Then we choose $\epsilon^*$ sufficiently small so that (9.21) is satisfied and

$$\hat{r} > r - \frac{5}{2}.$$  

Suppose that $s > 7/2$, $r > s + 5/2$ and $\epsilon^*$ is sufficiently small. We have $s_* = 0$ and $r_* = r - s$. One can check that there exists $d$ satisfying (9.21). Then

$$\hat{r} = r + \frac{1}{2} - \frac{1}{2s-1} r - \frac{r-1}{s-1} \epsilon^*.$$  

Note that from $s \geq 7/2$ and $r > s + 5/2$, we see that

$$\hat{r} > r - 1/12.$$  

Further, we still have $\hat{r} > r - 1/12$ for $s = 7/2$.

As stated in the theorem, the dependence of $H^{r,r'}$ on $r,r'$ for $r' < \hat{r}$ stated in arises from the restriction for $r \in (1, \infty)^*$. When $r$ is already in $(1, \infty)^*$, we obtain a solution operator $H^{r,r'}$ depends only on $r$ for any $r' < \hat{r}$.

Finally, we show that if $f \in C^\infty\overline{(D)}$, then the constructed $u$ is smooth on $\overline{D}$. Fix $s \geq 2$. For notations, we fix $r,r'$ and rename them as $r_0,r'_0$. Fix any $r > r_0$
satisfying \( r, r + \frac{1}{2} \not\in N \). We may assume (9.12), by the linearity of the solution \( u \) in \( f_1 \). Thus we have found solutions \( \sum w_i \) with

\[
\|w_i\|_0 \leq C \hat{C}^{-i} \lambda_i, \quad i = 1, 2, \ldots
\]

by (9.18). Here for \( \lambda_i \) we have fixed \( d \in (1, 2), \theta, t_0 \) so that (9.16) and (9.19) in which \( r \) is replaced by \( r_0 \) are satisfied and hence by the larger \( r \). With (9.12), we have (9.13)-(9.15) for \( i = 1 \). For the \( \hat{C}^{-i} \) defined by (9.17), the same proof shows that (9.13)-(9.15) hold for all \( i \) since (9.16) and (9.19)-(9.20) hold for the fixed \( \theta, d \) (depending on \( r_0, r' \)) and the \( r \). This shows that \( |v_i|_{1-\theta+\theta r+1/2} \) converges rapidly. Therefore, \( u := \sum w_i \in \Lambda^{1-\theta+\theta r+1/2} \). We conclude that \( u \in C^\infty(\overline{D}) \). \( \Box \)

**Appendix A. Distance function to \( C^2 \) boundary**

The following is proved in Li-Nirenberg [44] for \( \partial \Omega \in C^{k,\alpha} \) for \( k \geq 2 \) and \( 0 < \alpha \leq 1 \) and stated in Spruck [64] for \( C^2 \) case and proved in Crasta-Malusa [11] for \( C^2 \) boundary. We provide a proof here, including a stability property for our purpose. See Gilbarg–Trudinger [18] and Krantz–Park [41] when domains are in \( \mathbb{R}^N \).

**Proposition A.1** ([44, Thm. 1], [64, Prop. 4.1]). Let \( s \in [2, \infty] \). Let \( M \) be a smooth Riemannian manifold. Let \( \Omega \subset M \) be a bounded domain with \( C^s \) (resp. \( \Lambda^s \)) with \( s > 2 \) boundary. Let \( \rho \) be the signed distance function of \( \partial \Omega \). There is \( \delta = \delta(\Omega) > 0 \) so that \( \rho \in C^s \) (resp. \( \Lambda^s \)) with \( s > 2 \) in \( B_\delta(\partial \Omega) = \{ x \in M : \text{dist}(x, \partial \Omega) < \delta \} \). Furthermore, \( \delta(\Omega) \) is upper-stable under small \( C^2 \) perturbations of \( \partial \Omega \).

**Proof.** We are given a smooth Riemannian metric on \( M \). Let \( N \) be a subset of \( M \). Let \( \gamma_p : [0, d] \to M \) be a geodesic connecting \( \gamma(0) = p \in M \setminus \overline{N} \) and \( p^* = \gamma(d) \in \overline{N} \). Suppose that \( \gamma \) is normal, i.e. \( |\gamma_p'| = 1 \) and length \( |\gamma_p| \) equals \( \text{dist}(p, N) \). Then

\[
|\gamma_p(t) - \gamma_p(t')| = t' - t, \quad \text{dist}(\gamma_p(t), N) = d - t.
\]

Thus if \( N \) is a \( C^1 \) hypersurface near \( p^* \in M \), then \( \gamma_p \) is orthogonal to \( N \) at \( p^* \).

We recall some facts about geodesic balls; see [12, Chap. 3, Sect. 4]:

(i) For \( p \in M \), there is \( 0 < r(p) \leq \infty \) so that the geodesic ball \( B := B_r(p) \), centered at \( p \) with radius \( r \), is strictly geodesic convex for \( 0 < r < r(p) \). Specifically, any two points \( p_0, p_1 \in B \) are connected by a unique shortest normal geodesic in \( M \) and the geodesic is contained in \( B \). Here the uniqueness is up to a reparameterization \( t \to \pm t + c \).

(ii) \( \partial B \) is a smooth compact hypersurface.

Let \( K \) be a compact set in \( M \). By a), we have \( r(K) := \inf_{p \in K} r(p) > 0 \). We cover \( K \) by finitely many open sets \( U_i \) and choose coordinate chart \( x_i \) on \( U_i \) such that if a normal geodesic \( \gamma \) is contained in \( K \) then \( |(x_i \circ \gamma)(t)| \leq C_j(K) \) whenever \( x_i \circ \gamma(t) \) is defined. This implies that if \( p \in N \), and \( N \cap B \) is closed in \( B \), then any point \( q \in B_{r/2}(p) \) is connected to a point \( q^* \in N \cap B \) via a geodesic \( \gamma_q \) in \( B \) with length \( |q, N| \). However, \( q^* \) may not be unique.

Let us use the above facts for \( N = \partial \Omega \). Set \( r_0 = r(\partial \Omega) \). Let \( \nu \) be the unit inward normal of \( \partial \Omega \) with respect to the Riemannian metric and choose \( \rho > 0 \) so that \( \rho < 0 \) on \( \Omega \). Then \( \nu \) is \( C^{s-1} \) on \( \partial \Omega \). Let \( \gamma(t, p) \) be the geodesic through \( p \in \partial \Omega \) with \( \partial_t \gamma(0, p) = \nu(p) \) and \( |t| < r_0 \). Then \( \gamma \) is \( C^{s-1} \) on \( (-r_0, r_0) \times \partial \Omega \). Since \( s \geq 2 \), \( \partial_t \gamma(0, p) \) is non zero, normal to \( \partial \Omega \), and \( \gamma \) fixes \( \{0\} \times \partial \Omega \) pointwise, then
the Jacobian of $\gamma$ is non-singular. By the inverse mapping theorem, there exists a unique solution $(\tilde{\rho}, P) \in \mathbb{R} \times \partial \Omega$ satisfying

$$
\gamma(\tilde{\rho}(x), P(x)) = x, \quad |\tilde{\rho}(x)| < r_1
$$

for $x \in B_{r_2}(\partial \Omega)$. Here $0 < r_2 < r_1 < r_0$ and $r_1, r_2$ are sufficiently small. Furthermore, $\tilde{\rho}, P$ are in $C^{s-1}(B_{r_2}(\partial \Omega))$.

We want to show that $\tilde{\rho} = \rho$ on $B_{r_2}(\partial \Omega)$. Fix $x \in B_{r_2}(\partial \Omega)$ and take $x^* \in \partial \Omega$ with $\text{dist}(x, x^*) = \rho(x)(= \text{dist}(x, \partial \Omega))$. Since $r_2 < r(\partial \Omega)$, then $x$ is connected to the center $x^*$ by a geodesic $\tilde{\gamma}$ in the geodesic ball $B_{r_2}(x^*)$. Since $\text{dist}(x, x^*) = \rho(x)$, then $\tilde{\gamma}$ is orthogonal to $\partial \Omega$ at $x^*$. Then $\tilde{\gamma}$ must be contained in the normal geodesic $\gamma(\cdot, x^*)$ with tangent vector $\nu(x^*)$. Next we choose the parametrization of $\tilde{\gamma}$ so that $\tilde{\gamma}'(0) = \nu(x^*)$. We get

$$
\gamma(\rho(x), x^*) = \gamma(\rho(x)) = x, \quad \gamma(0) = x^* = \gamma(0, x^*), \quad |\rho(x)| = \text{dist}(x, x^*) < r_2.
$$

By the uniqueness of solution to (A.1), we conclude that $\tilde{\rho}(x) = \rho(x)$ (and $x^* = P(x)$, $|\tilde{\rho}(x)| < r_2$).

Next we verify $\rho \in C^s$. Recall that the vector field $X_1(x) := \partial_1 \gamma(\rho(x), P(x))$ is $C^{s-1}$ in $x = \gamma(\rho(x), P(x))$. Fix $x_0 \in B_{r_2}(\partial \Omega)$. In a small neighborhood $U$ of $x_0$ in $B_{r_2}(\partial \Omega)$, we use the Gram–Schmidt orthogonalization to find pointwise linearly independent vector fields $X_2, \ldots, X_n$ of class $C^{s-1}$ that are orthogonal to $X_1$. We already know that $\rho = \tilde{\rho} \in C^{s-1} \subset C^1$. This allows us to compute a directional derivative of $\rho$ via any $C^1$ curve that is tangent to the direction. Since $\gamma$ is normal, then $X_1 \rho = 1$. Let $j > 1$ and we want to show that $X_j \rho = 0$. At $x \in U \setminus \partial \Omega$, Gauss lemma says that $X_2, \ldots, X_n$ are tangent to the smooth geodesic sphere $\partial B_{|\rho(x)|}(P(x))$. We have $|\rho(x)| = \text{dist}(x, \partial \Omega) > 0$ and

$$
|\rho(q)| = \text{dist}(q, \partial \Omega) \leq \text{dist}(q, P(x)) = |\rho(x)|, \quad \forall q \in \partial B_{|\rho(x)|}(P(x)).
$$

Hence on this geodesic sphere, the $C^{s-1}$ function $\rho$ attains a local extreme at $x$. This shows that $X_j \rho = 0$ on $U \setminus \partial \Omega$. By continuity, $X_j \rho(x) = 0$ on $U$. Therefore all $X_i \rho$ are $C^\infty$ functions on $U$. As observed by Spruck [44], since all $X_i$ are $C^{s-1}$, then $\rho \in C^s(U)$. Therefore, $\rho \in C^s(B_{r_2}(\partial \Omega))$.

Finally, the stability of $\text{dist}(\cdot, \partial \Omega)$ near $\partial \Omega$ is a consequence of the geodesic equations of the second-order ODE system. We leave the details to the reader. □

**Appendix B. Stability of $L^2$ solutions on pseudoconvex manifolds with $C^2$ boundary satisfying condition $a_q$**

Let $\Omega$ be a relatively compact domain in a complex manifold $X$. Let $V$ be a holomorphic vector bundle on $X$. Let $f$ be a $V$-valued $(0, q)$ form on $\Omega$. Suppose that $\overline{\partial} u = f$ can be solved on $\Omega$ for some $u \in L^2_{\text{loc}}(\Omega)$ and $f = \tilde{f} + \overline{\partial} v$ for $v \in L^2_{\text{loc}}(\Omega)$ and $\tilde{f}$ is a closed form on a larger domain $\Omega'$ containing $\Omega$. We want to know if there exists a neighborhood $\Omega$ of $\overline{\Omega'}$, that is independent of $f$ such that $\tilde{f} = \overline{\partial} \tilde{u}$ for some $\tilde{u} \in L^2_{\text{loc}}(\Omega')$. If such a domain $\Omega$ exists, we say the solvability of the $\overline{\partial}$-equation on $\Omega$ is stable. This stability was proved by Hörmander [31] when $\Omega$ is an $a_q$ domain and $V$ is the trivial bundle and by Andreotti–Vesentini [1, Lem. 29, p. 122] for vector bundles on domains that are strictly $(n - q)$ convex with smooth boundary. For completeness, we sketch a proof for the case of vector bundle. We also take this opportunity to relax the boundary condition $\partial \Omega \in C^3$ to the minimum $C^2$ smoothness and we also formulate a stability for the $L^2$ bounds of the $\overline{\partial}$ equation.
on small $C^2$ perturbations of $a_q$ domains. For the reader’s convenience, we will give our statements for the vector bundle case the references in [31].

We fix smooth hermitian metrics on $X$ and $V$. Cover $\Omega$ by finitely open sets $U_1, \ldots, U_{m_0}$ of $X$. We assume that each $U_j$ is biholomorphic to the unit ball in $\mathbb{C}^n$ by a coordinate map $z_j$ which is biholomorphic on $\overline{U}_j$. In what follows, we will denote by $U$ one of $U_1, \ldots, U_{m_0}$ or their subdomains.

Let $\{e_1, \ldots, e_m\}$ be a smooth unitary basis of $V$ on $U$. Let $u = \sum u^\nu e_\nu \in C^1((p,q-1)(\Omega, V, loc))$. We have

$$\mathcal{D}u = Au + Ru, \quad A(u^\nu e_\nu) = (\overline{\partial}u^\nu)e_\nu,$$

where $Au^\nu = \overline{\partial}u^\nu$ is as in [31] for the scalar case, and $Ru$ involves no derivatives of $u$, i.e. $Ru$ is of order zero in $u$ with smooth coefficients. Therefore, the principal part $A$ of $\mathcal{D}$ is locally diagonal. This is an important property allowing the proofs for the scalar case can be adapted to the vector bundles case without difficulty.

Let $\omega_1, \ldots, \omega_n$ be unitary smooth $(1, 0)$ forms on $X$. For a $V$-valued $(p, q)$-forms $f = \sum f^\nu e_\nu$, define

$$\left| \sum f^\nu e_\nu(x) \right| = \sum_{\nu} \sum_{I,J} f^\nu_{I,J}(x)^2, \quad f^\nu_{I,J} = \sum_{I,J} f^\nu_{I,J} \omega^I \wedge \overline{\omega}^J.$$

The volume form on $X$ will be

$$dv = (\sqrt{-1})^n \omega^1 \wedge \cdots \wedge \omega^n \wedge \overline{\omega}^1.$$

For $q \geq 0$, let $L^2_{(p,q)}(\Omega, V, loc)$ and $\Lambda^r_{(p,q)}(\Omega, V)$ be the spaces of $V$-valued $(p, q)$ forms of which the coefficients on $U$ are in $L^2_{loc}(\Omega \cap U)$, $\Lambda^r(\Omega \cap U)$, respectively. Let $\mathcal{D}^{(p,q)}(\Omega, V)$ be the space of smooth $V$-valued $(p, q)$ forms of which the coefficients are in $\mathcal{D}(\Omega)$, i.e. smooth functions with compact support. Let $\mathcal{D}^\prime_{(p,q)}(\Omega, V)$ be the space of $V$-valued $(p, q)$ forms of which the coefficients are distributions in $\Omega$.

If $\varphi$ is a real $L^\infty$ function in $\Omega$, let $L^2_{(p,q)}(\Omega, V, \varphi)$ be the space of sections of $V$-valued $(p, q)$ forms satisfying

$$\| f \|_\varphi^2 = \int_\Omega |f(x)|^2 e^{-\varphi(x)} \, dv(x) < \infty.$$

We will write $\langle \cdot, \cdot \rangle_\varphi$ for the induced hermitian product and $\| \cdot \|_\varphi$ for the norm on $L^2_{(p,q)}(\Omega, V, \varphi)$. The norms are equivalent for all weights $\varphi \in L^\infty$.

Throughout the appendix, we assume $q \geq 1$. The operator $\overline{\partial}$ defines a linear, closed, densely defined operator

$$T : L^2_{(p,q-1)}(\Omega, \varphi) \to L^2_{(p,q)}(\Omega, \varphi)$$

while $Tu = f$ holds if $\overline{\partial}u = f$ in the sense of distributions. We abbreviate $T = T_q, S = T_{q+1}$. We will write $T_q$ for $T$ if needed. The domain $D_T$ and range $R_T$ are independent of $\varphi \in L^\infty$. For $f \in L^2_{(p,q)}(\Omega, V, \varphi)$, write $v = T^* f$ if $\langle u, v \rangle_\varphi = \langle \overline{\partial}u, f \rangle_\varphi$ for all $u \in D_T$.

Throughout the section, we assume that $\partial \Omega \in C^2$. By Proposition A.1, $\Omega$ has a $C^2$ defining function $\rho$ in $X$ satisfying

$$2|\partial \rho| = 1 \quad \text{on } \partial \Omega.$$

We also assume that $\varphi \in Lip(\Omega)$. Then $D_{T^*}$ is independent of $\varphi$, while $R_{T^*}$ depends on $\varphi$. 

**Remark B.1.** As in [31], \(|T^* f|_\psi\) is always referred to as the dual with respect to \(\psi\), where \(\psi\) will be chosen appropriately. For clarity, we write \(T^*_\psi\) for \(T^*\) when \(\psi\) needs to be specified.

Using integration by parts, we can verify that if \(f \in C^1_{(p,q)}(\Omega, V)\) has compact support in \(U \cap \Omega\), then \(f \in D_{T^*}\) if and only if

\[
\sum_{j=1}^n f'_i \partial_j k \partial_i \omega_j = 0, \quad \text{on} \ U \cap \partial \Omega, \quad \nu = 1, \ldots, m.
\]

Define \(D^1_{T^*}(\Omega) := C^1(\Omega, V) \cap D_{T^*}\). We have from [31, p. 148]

\[
T^* f = B f + R^* f, \quad B f := (B f') e_\nu, \quad \text{on} \ U
\]

with \(B f' = -\sum_{j} \sum_{I,K} f_{i,j,K} \omega_i \wedge I^K\). Thus \(R^*\), independent of \(\varphi\), is an operator of the zero-th order with smooth coefficients, and the \(B\) is diagonal and its principle part is also independent of \(\varphi\). Thus the boundary condition is principle and of order zero.

In summary, we have

**Proposition B.2.** Let \(\Omega\) be a relatively compact \(C^2\) domain in \(X\) and let \(\varphi \in \text{Lip}(\Omega)\). Then \(D_{T^*}\) is independent of \(\varphi\). Let \(\psi \in L^\infty(\Omega)\) be a real function.

(a) For all \(f = \sum f' \varphi \in C^1_{(p,q)}(U \cap \Omega, V)\),

(B.1) \[\|S f\|_{\psi}^2 - \|A f\|_{\psi}^2 \leq C(\Omega)\|f\|_{\psi}^2.\]

(b) For all \(f = \sum f' \varphi \in C^1_{(p,q)}(\Omega, V) \cap D_{T^*}\) with compact support in \(U \cap \Omega\),

(B.2) \[\|T^* f\|_{\psi}^2 - \|B f\|_{\psi}^2 \leq C(\Omega)\|f\|_{\psi}^2.\]

(c) \(D^1_{T^*}(\Omega)\) is dense in \(D_{T^*} \cap D_S\) w.r.t. the graph norm \(|f|_\psi + |S f|_\psi + |T^* f|_\psi\).

Further, \(C(\Omega)\), independent of \(\varphi, \psi\), depends only on the diameter of \(\Omega\).

Here the last assertion follows from [31, p. 121]. We also have

**Proposition B.3.** ([31, eq. (3.1.9)]). Let \(\Omega\) be a relatively compact \(C^2\) domain in \(X\). Let \(\rho\) be the signed distance function of \(\partial \Omega\). Let \(\varphi \in C^{1,1}(\Omega)\). For all \(f \in C^1_{(p,q)}(\Omega, V)\) with compact support in \(U \cap \Omega\), we have

\[
\|A f\|_{\varphi}^2 + \|B f\|_{\varphi}^2 = \sum_{\nu=1}^m \|A f'\|_{\varphi}^2 + \|B f'\|_{\varphi}^2
\]

\[
= \sum_{\nu=1}^m (Q_1 + Q_2 + t_1 + t_2 + t_3 + t_4)(f''(f''), f''),
\]
with

\[ Q_1(f'', f'') := \sum_{k,j} \sum_{j=1}^{n} \int_{\Omega} \left| \frac{\partial f''_{i,j,k}}{\partial x} \right|^2 e^{-\varphi} \, dv, \]

\[ Q_2(f'', f'') := \sum_{k,n} \sum_{k,j} \int_{\Omega} \frac{\partial f''_{i,j,k}}{\partial x} f''_{i,j,k} e^{-\varphi} \, dv, \]

\[ t_1(f'', f'') := \sum_{k,n} \sum_{j=1}^{n} \int_{\Omega} \left( f''_{i,j,k} \frac{\partial f''_{i,j,k}}{\partial x} - f''_{i,j,k} \delta_{i,j,k} \right) e^{-\varphi} \, dv, \]

\[ t_2(f'', f'') := \sum_{k,n} \sum_{j=1}^{n} \int_{\Omega} \left( f''_{i,j,k} \sigma_{j,k} f''_{i,j,k} - f''_{i,j,k} \sigma_{j,k} \delta_{i,j,k} \right) e^{-\varphi} \, dv, \]

\[ t_3(f'', f'') := \sum_{k,n} \sum_{j=1}^{n} \int_{\Omega} f''_{i,j,k} \delta_{i,j,k} e^{-\varphi} \, dv, \]

\[ t_4(f'', f'') := -\sum_{k,n} \sum_{j=1}^{n} \int_{\Omega} f''_{i,j,k} \delta_{i,j,k} e^{-\varphi} \, dv. \]

Proof. The proof for \( f \in C^2 \) and \( \varphi \in C^2 \) is in [31]. The case for \( f \in C^1 \) can be obtained by \( C^1 \) approximation of \( C^2 \) forms as in [31, p. 101]. \( \square \)

Let \( \varphi \) be a \( C^2 \) real function defined in a neighborhood of \( z_0 \in X \). Let \( 1 \leq q < n \). Suppose \( \varphi \) satisfies the condition \( a_q \) at \( z_0 \). Let \( \mu_1(z) \leq \mu_2(z) \leq \cdots \leq \mu_{n-1}(z) \) be the eigenvalues of the Levi form \( L_{z_0} \varphi \) and let \( \lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_n(z) \) be the eigenvalue of the hermitian form \( H_{z_0} \varphi(t) := \sum_{i,j,k} \frac{\partial^2 \varphi}{\partial x^i \partial x^j} t_{j,k} \). The minimum-maximum principle for the eigenvalues says that

\[ \lambda_j(z) = \min_{W = \{ \mu \}} \left\{ \max_{v \in W, |v| = 1} \{ H_{z_0} \varphi(v) \} \right\}. \]

Thus \( \lambda_1(z) \leq \mu_1(z) \leq \cdots \leq \mu_{n-1}(z) \leq \lambda_n(z) \). Let \( r^- = \max(-r, 0) \) for a real \( r \). Then at \( z_0 \), the condition \( a_q \) is valid if and only if

\[ \mu_1 + \cdots + \mu_q + \sum_{j=1}^{n-1} \mu_j > 0. \]

If \( \psi < \psi(z_0) \) is strictly pseudoconvex at \( z_0 \) then \( \psi \) satisfies the \( a_q \) condition for \( q = 0, \ldots, n-1 \). As in [31, Def. 3.3.2], we say \( \psi \) satisfies the condition \( A_q \) at \( z_0 \), if \( \nabla \varphi(z_0) \neq 0 \) and

\[ \lambda_1(z_0) + \cdots + \lambda_q(z_0) + \sum_{j=1}^{n-1} \mu_j > 0. \]

When needed, we denote the above eigenvalues \( \lambda_j, \mu_k \) by \( \lambda_j(z_0, \varphi), \mu_k(z_0, \varphi) \). Let us prove the following estimate for weighted eigenvalues.

**Lemma B.4** ([31, Lem. 3.3.3]). Suppose that \( \varphi \) satisfies the condition \( a_q \) at \( z_0 \). Then \( e^{\varphi} \) satisfies the condition \( A_q \). More specifically, there exist \( c(\varphi) > 0 \) and \( \tau_0(\varphi) \) such that for \( \tau > \tau_0 \)

\[ e^{\tau \varphi} \left\{ \lambda_1(\zeta, e^{\tau \varphi}) + \cdots + \lambda_q(\zeta, e^{\tau \varphi}) + \sum_{j=1}^{n-1} \mu_j(\zeta, e^{\tau \varphi}) \right\} > c(\varphi) \tau. \]
Furthermore, $c(\varphi), \tau_0(\varphi)$ are stable under small $C^2$ perturbation of $\varphi$.

Proof. The proof in [31] uses a proof-by-contradiction argument. For stability, we need a direct proof. We have

$$λ_1(z_0) + \cdots + λ_q(z_0) + \sum_{j=1}^{n-1} μ_j^−(z_0) ≥ λ_1(z_0) + \cdots + λ_q(z_0) + \sum_{j=1}^{n-1} μ_j^+(z_0).$$

For $t$, decompose $t \cdot \frac{2}{\kappa} = t' + t''$ where $t'$ is in the complex tangent space $T_0^\varphi$ and $t''$ is in its orthogonal complement. We have

$$\hat{H}_{\xi}(t) := τ^{−1}e^{−τ\varphi}He_{\xi}^τ(t) = H_{\xi}\varphi(t) + τ|\partial_\xi\varphi(\xi)|^2|t''|^2.$$

Restricted on $T_0^\varphi$, the above is still the Levi form $L_0\varphi$ of which the eigenvalues are $λ_1 ≤ \cdots ≤ μ_{n-1}$. Let $λ_1(τ), \ldots, λ_n(τ)$ be the eigenvalues of the above quadratic form. We still have $λ(τ) ≤ μ_1 ≤ \cdots ≤ μ_n ≤ λ_n(τ)$. For any $δ > 0$, we choose $τ_0$ so that

$$|τ|\partial_\xi\varphi(\xi)|^2|δ^2 > λ_n(0) + 1, \ \forall τ > τ_0.$$

Then $λ_1(τ) ≥ H_{\xi}\varphi(t) ≥ μ_1 - ε$ when $δ$ is sufficiently small. Analogously, we get $λ_j(τ) ≥ μ_j - ε$ for $j = 1, \ldots, n - 1$ when $δ$ is sufficiently small. We can choose $ε$ depending on $μ_1 + \cdots + μ_n + \sum_{j=1}^{n-1} μ_j^+$ and modulus of continuity of $|τ|\partial_\xi\varphi(\xi)$ to obtain (B.3).

Theorem B.5 ([31, Thm. 3.3.1]). Let $Ω$ be a relatively compact $C^2$ domain in $X$. Let $z_0 ∈ Ω$. Suppose that $φ ∈ C^2(Ω)$. Then

$$(B.4) \quad τ||f||^2_{τ,φ} ≤ C^*_{φ} \left\{ ||T^*_{τ,φ}f||^2_{τ,φ} + ||Sf||^2_{τ,φ} + |f|^2_{τ,φ} \right\}$$

holds for some neighborhood $U ⊂ Ω$ of $z_0$, some $C^1_{φ,τ,φ}$, and all $τ > τ_φ$ and all $f ∈ C^1_{p,q}(\overline{U}, V)$ with compact support in $U$, if and only if the hermitian form

$$\sum_{j=1}^q ϕ_j(z_0) t^j \overline{t_j}$$

on $C^1$ has either at least $q + 1$ negative or at least $n - q + 1$ positive eigenvalues. Furthermore, we can take

$$C^*_{φ} = \frac{C(Ω)}{\min_{z_0 ∈ Ω \cap U} \left( \sum_{j=1}^{n-1} μ_j(z_0, φ) + \sum_{j=1}^q μ_j(z_0, φ) \right)}$$

where $μ_1(z_0, φ) ≤ \cdots ≤ μ_{n-1}(z_0, φ)$ are eigenvalues of $L_{z_0,φ}$ with respect the hermitian metric on $X$, while $U$ depends on the modulus of continuity of $|τ|\partial_\xi\varphi$. The constants $C^*_{φ,τ,φ}$ are stable under $C^2$ perturbation of $\partial Ω$.

Proof. Take any $g ∈ C^1_{p,q}(\overline{U} \cup D_U)$ with compact support in $U \cap \overline{U}$. Apply (B.4) to $\tilde{f} = ge_1$, which is actually proved in [31] for the $g$; see (3.3.4)-(3.3.6) in [31]. By (B.1) we get

$$τ||g||^2_{τ,φ} ≤ C_{φ}(||T^*_{φ,τ}(ge_1)||^2_{τ,φ} + ||\overline{D}(ge_1)||^2_{τ,φ}) ≤ C_{φ}(||T^*_{φ,τ}g||^2_{τ,φ} + ||\overline{D}g||^2_{τ,φ} + C_1||g||^2_{τ,φ}),$$

where $C_{φ}$ depends on the eigenvalues of $φ$ and $C_1$ is independent of $τ$ and $φ$. Assume further that $τ > 2C_{φ}C_1$. Then we get (B.4) in which $f, C_{φ}$ are replaced by $g, 2C_0$. Note that the constant $C$ in (B.1) is independent of $τ$. By [31, Thm 3.3.1], we get the eigenvalue condition. Assume that the eigenvalue condition holds. Then (B.4) holds when $f$ is replaced by $f^ν$ for each $ν$. By (B.1) again, we get (B.4) by adjusting $τ_0$ and $C_0$. □
Theorem B.6 ([31, Thm. 3.3.5]). Let $\Omega$ be a relatively compact $C^2$ domain in $X$. Let $\varphi$ satisfy the condition $A_q$ at $z_0 \in \overline{\Omega}$. If $z_0 \in \partial \Omega$ assume further that $\varphi < \varphi|_{\partial \Omega} = \varphi(z_0)$ in $\Omega$. Then there are a neighborhood $U$ of $z_0$ and a constant $C^*_\varphi$ such that for all convex increasing function $C^2$ function $\chi$ in $\mathbb{R}$ we have

$$\int \chi(\varphi)|f|^2 e^{-\chi(\varphi)} \, dv \leq C^*_\varphi\left(||T^*f||_{\chi(\varphi)}^2 + ||\mathcal{D}f||_{\chi(\varphi)}^2 + ||f||_{\chi(\varphi)}^2\right)$$

for all $f \in C^1_{(p,q+1)}(\overline{\Omega}, V) \cap D_{T^*}$ with compact support in $U \cap \overline{\Omega}$.

Proof. We apply the scalar version of the result as in the proof of Theorem B.5. The proof in [31] is valid via $C^1$ approximations for $f$. \hfill \Box

By partition of unity, the above yields the following.

Proposition B.7 ([31, Prop. 3.4.4]). Let $\Omega$ be a relatively compact $C^2$ domain in $X$. Let $\varphi < 0$ in $\Omega$ and vanish in $\partial \Omega$ with $\varphi \in C^2(\overline{\Omega})$. Let $\Omega_a = \{z \in \Omega : \varphi(z) < a\}$. Suppose that $\varphi$ satisfies the condition $A_q$ in $\overline{\Omega} \setminus \Omega_{-c}$ for some $c > 0$. Then there are a compact subset $K$ of $\Omega_{-c}$ and a constant $C^*_\varphi$ such that for all convex increasing function $\chi \in C^2(\mathbb{R})$

$$(B.5) \quad \int_{\Omega \cap K} \chi(\varphi)|f|^2 e^{-\chi(\varphi)} \, dv \leq C^*_\varphi\left(||T^*f||_{\chi(\varphi)}^2 + ||\mathcal{D}f||_{\chi(\varphi)}^2 + ||f||_{\chi(\varphi)}^2\right)$$

holds for all $f \in C^1_{(p,q)}(\overline{\Omega}, V) \cap D_{T^*}$.

Theorem B.8 ([31, Thm. 3.4.1]). Let $\Omega$ be a relatively compact $C^2$ domain in $X$. Suppose that $\partial \Omega$ satisfies the condition $a_q$. Fix $C^2$ defining function $\rho$ of $\Omega$ such that $\rho$ is the signed distance function to $\partial \Omega$ and fix $\varphi = e^{\lambda \rho} - 1$ with $\lambda$ sufficiently large. Then there exist compact subset $K$ of $\Omega$ and constant $\tau_\varphi$ such that if $\tau > \tau_\varphi$ and $f \in D_S \cap D_{T^*} \cap L_{p,q}^2(\Omega, V)$ we have

$$(B.6) \quad \int_{\Omega \cap K} |f|^2 e^{-\tau \varphi} \, dv \leq ||T^*f||_{\tau_\varphi}^2 + ||\mathcal{D}f||_{\tau_\varphi}^2 + \int_K |f|^2 e^{-\tau \varphi} \, dv.$$

The latter implies that $R_T$ is closed and finite codimension in $N_S$.

Proof. Here we need to go through the proof of [31, Thm. 3.4.1]. There is a compact set $K$ in $\Omega$ such that

$$\tau \int_{\Omega \cap K} |f|^2 e^{-\tau \varphi} \, dv \leq C^*_\varphi\left(||T^*f||_{\tau_\varphi}^2 + ||\mathcal{D}f||_{\tau_\varphi}^2 + ||f||_{\tau_\varphi}^2\right),$$

where $C^*_\varphi$ is independent of $\tau$. The above is proved in [31, Thm. 3.4.1] when $V$ is trivial. Thus it also holds for any $V$ by (B.1) and (B.2). We get (B.6) for $f \in C^1(\overline{\Omega}, V) \cap D_{T^*}$ when $\tau > 2C^*_\varphi$. By the density theorem, it holds for $f \in D_{T^*} \cap D_S$. The proof for the other direction in [31, Thm. 3.4.1] is valid without any change. \hfill \Box

So far, all the constants in the estimates are stable under $C^2$ perturbations of the domain $\Omega$ and these constants are explicit to some extent. The next constant is however not explicit since it comes from a proof by contradiction. Nevertheless, it leads no essential difficulty in our applications.

Fix $\gamma > 2C^*_\varphi$, where $C^*_\varphi$ is in (B.6). Let $\chi_k \in C^2$ be an increasing sequence of convex increasing functions such that

$$(B.7) \quad \chi_k(\tau) = \gamma \tau, \quad \text{when } \tau < -c; \quad \chi_k'(\tau) \to \infty, \quad \text{as } k \to \infty, \tau > -c.$$
Set $\varphi_k = \chi_k(\varphi)$. Note that $\varphi_k \in C^2(\overline{\Omega})$. Define

$$N_{(p,q)}(\Omega_{-c},V,\gamma \varphi) := N_{S_c} \cap N_{T^*},$$

where $T^*_c$ is the adjoint of $T_c = \overline{\partial}; L^2_{(p,q-1)}(\Omega_{-c}, V, \gamma \varphi) \to L^2_{(p,q)}(\Omega_{-c}, V, \gamma \varphi)$, while $S_c$ is the operator $\overline{\partial}; L^2_{(p,q)}(\Omega_{-c}, V, \gamma \varphi) \to L^2_{(p,q+1)}(\Omega_{-c}, V, \gamma \varphi)$. We have the following.

**Proposition B.9** ([31, Prop. 3.4.5]). Fix $q > 0$. Let $\Omega, \Omega_{-c}, \varphi$ satisfy the hypotheses in Proposition B.7. In particular, $\varphi$ satisfies the condition $A_q$ in $\overline{\Omega} \setminus \Omega_{-c}$. There exist constants $C_*$ and $k_*$, depending on $\varphi, c, \gamma$, and the sequence $\chi_k$ such that for $k > k_*$

$$\frac{\|f\|^2_k}{\|\varphi^k\|} \leq C_*\left(\|T^* f\|^2_k + \|S f\|^2_k\right)$$

provided $f \in D_{T^*} \cap D_{S} \cap L^2_{(p,q)}(\Omega, V)$ with $q \geq 1$ and

$$\int_{\Omega_{-c}} \langle f, h \rangle e^{-\gamma \varphi} \, dv = 0, \quad \forall h \in N_{(p,q)}(\Omega_{-c}, V, \gamma \varphi).$$

Further, $k_*, c, C_*$ are stable under $C^2$ perturbation of $\partial \Omega$ as follows: There is $\delta_* = \delta_*(\nabla \rho, \nabla^2 \rho) > 0$, depending on $\rho, c$ such that if a real function $\varphi^0$ satisfies $\|\varphi^0 - \varphi\|_{C^2} < \delta$, then there exists a real function $\tilde{\varphi}$ satisfying the following

(a) $\|\tilde{\varphi} - \varphi\|_2 < C(\|\varphi\|_2)\delta_*$ and

$$\tilde{\varphi} = \varphi \quad \text{on} \quad \Omega_{-c}, \quad \tilde{\varphi} = \varphi^0 \quad \text{on} \quad \{z \in X: \varphi^0(z) < -c/2\}$$

(b) For $\tilde{\Omega} = \{z \in X: \tilde{\varphi} < 0\}, \tilde{\varphi}_k = \chi_k(\tilde{\varphi})$ and $k \geq k_*$, we have

$$\frac{\|f\|^2_k}{\|\tilde{\varphi}^k\|} \leq C_*\left(\|T^* f\|^2_k + \|S f\|^2_k\right)$$

provided $f \in D_{T^*} \cap D_{S} \cap L^2_{(p,q)}(\tilde{\Omega}, V)$ with $q \geq 1$ and $f$ satisfies (B.9).

**Proof.** Fix a smooth function $\chi$ on $X$ such that $0 \leq \chi \leq 1, \chi = 1$ on $\Omega_{-8c/9}$ and $\text{supp} \chi \subset \Omega_{-7c/9}$. Define

$$\tilde{\varphi} = \chi \varphi + (1 - \chi)\varphi^0.$$  

Then $\tilde{\varphi}$ satisfies (a).

Fix $\gamma > 2C^*_{\varphi}$ for the constant $C^*_{\varphi}$ in (B.5).

Assume that estimate (B.8) is false. Then we can find a sequence $\varphi_j$ of $C^2$ defining functions satisfying the following. For each $j$, we have the following.

(i) $\varphi_j \in C^2(X)$ and $\|\varphi_j - \varphi\|_2 < \frac{1}{j}$, $\Omega^j = \{\varphi_j < 0\}$, the operators $T_j, S_j$ for $\Omega^j$.

(ii) $\Omega^j_{-c} = \{\varphi_j < -c\} = \Omega_{-c}$.

(iii) $f_j \in D_{T_j} \cap D_{S_j} \cap L^2_{(p,q)}(\Omega^j, V)$ with $q \geq 1$ and for some $k_j > j$

$$\frac{\|f_j\|_{(\varphi_j^k)_{k_j}}}{\|\varphi^k\|} = 1, \quad \frac{\|T_j^* f_j\|_{(\varphi_j^k)_{k_j}}}{\|S_j f_j\|_{(\varphi_j^k)_{k_j}}} < \frac{1}{j}.$$  

$$\int_{\Omega_{-c}} \langle f_j, h \rangle e^{-\gamma \varphi} \, dv = 0, \quad \forall h \in N_{(p,q)}(\Omega_{-c}, V, \gamma \varphi).$$

By the density theorem, we may assume that $f_j \in C^1_{(p,q)}(\overline{\Omega^j}, V) \cap DT^*$, while (B.11) still holds and (B.12) is, however, weakened to

$$\int_{\Omega_{-c}} \langle f_j, h \rangle e^{-\gamma \varphi} \, dv < \frac{1}{j}, \quad \forall h \in N_{(p,q)}(\Omega_{-c}, V, \gamma \varphi), \quad \|h\|_{\Omega_{-c}, \gamma \varphi} \leq 1.$$
(Compare (B.11) and (B.13) with [31, eq. (3.4.5)].) Here we have used \((\varphi_j)_j = \gamma \varphi\) on \(\Omega_{-c}\) and Cauchy-Schwarz inequality
\[
\left| \int_{\Omega_{-c}} \langle \tilde{f}_j, h \rangle e^{-\gamma \varphi} \, dv \right| = \left| \int_{\Omega_{-c}} \langle \tilde{f}_j - f_j, h \rangle e^{-(\varphi_j)k_j} \, dv \right| \leq C \left\{ \int_{D'_{\gamma}} |\tilde{f}_j - f_j|^2 e^{-(\varphi_j)k_j} \, dv \right\}^{1/2}
\]
for \(\tilde{f}_j \in C_{(p,q)}^1(\overline{\Omega}, V) \cap D_{T_j} \). We still call the approximation \(\tilde{f}_j\) satisfying (B.11) and (B.13) by \(f_j\).

The rest of proof is based on some minor changes of the proof in [31]. We give details for completeness. The increasing function \(\chi'\) is bounded below by \(\gamma\). Let \(K\) be any compact subset of \(\Omega_{-c}\). By (B.5) we have
\[
\int_{\Omega \setminus K} |f_j|^2 e^{-(\varphi_j)k_j} \, dv \leq C^* \gamma^{-1}(1 + j^{-1}),
\]
\[
1 \geq \int_K |f_j|^2 e^{-\gamma \varphi} \, dv \geq 1 - C^* \gamma^{-1}(1 + j^{-1}).
\]
Thus
\[
\int_{\Omega_{-c} \setminus K} |f_j|^2 e^{-\gamma \varphi} \, dv \leq C^* \gamma^{-1}(1 + j^{-1}),
\]
\[
1 \geq \int_K |f_j|^2 e^{-\gamma \varphi} \, dv \geq 1 - C^* \gamma^{-1}(1 + j^{-1}).
\]
By (B.14)-(B.15) and the Banach-Alaoglu theorem, we can find a subsequence of \(f_j\), still called it \(f_j\), that converges weakly to \(f\) in \(L^2(\Omega_{-c}, V)\). Hence, by (B.13) and the weak convergence, we get (B.9). By (B.11), we have \(\overline{\partial}f_j\) tends to 0 in \(L^2\) norm on each compact subset of \(\Omega_{-c}\). Therefore, \(\overline{\partial}f = 0\) on \(\Omega_{-c}\).

Take any function \(\psi \in C_{0}^\infty(\Omega_{-c})\) with \(\psi = 1\) on \(K\). Then by (B.11),
\[
\|T^*_{\gamma}(\partial f_j)\|_{\gamma \varphi} + \|S_{\gamma}(\psi f_j)\|_{\gamma \varphi} + \|\psi f_j\|_{\gamma \varphi} \leq C_{\varphi}.
\]
Further, by (B.11) and [32, Lem. 4.2.3, p. 86], we have \(\|Df_j\|_K \leq C'\) for all \(f_j\).

By Rellich–Kondrachov compactness theorem [15, p. 272], there is a subsequence \(f_j\) converges strongly to \(f\) on \(K\). Still denote the subsequence by \(f_j\). By (B.15), we get from \(\gamma > 2C_{\varphi}\)
\[
\int_K |f|^2 e^{-\gamma \varphi} \, dv = 1 - C^* \gamma^{-1} > \frac{1}{2}.
\]

Next we want to show that \(f \in N_{T'_{\gamma}}\). Set \(g_j = f_j e^{-(\varphi_j)k_j}\). By (B.7) and the convexity of \(\chi_{k_j}\), we have \((\varphi_j)_j \geq \gamma \varphi_j \geq \gamma (\varphi - \frac{1}{j})\). Thus, for any relatively compact subdomain \(\Omega'\) of \(\Omega\), \(\Omega'\) is contained in \(\Omega'_{\gamma}\) for large \(j\) and hence
\[
\int_{\Omega'} |g_j|^2 e^{\gamma \varphi} \, dv \leq e^\gamma \int_{\Omega} |g_j|^2 e^{(\varphi_j)k_j} \, dv \leq e^\gamma \|f_j\|^2_{(\varphi_j)k_j} = e^\gamma.
\]
Fix \(\Omega_i \in \Omega\) with \(\Omega_i \subset \Omega_{j+1}\) and \(\Omega = \cup \Omega_i\). Take a subsequence \(g_{k_1,j}\) of \(g_j\) converging weakly to \(g_{k_1,\infty}\) on \(\Omega_1\). Inductively, take a subsequence \(\{g_{k_{j+1},j}\}_{j=1}^\infty\) of \(\{g_{k_j}\}_{j=1}^\infty\) converging weakly to \(g_{k_{j+1},\infty}\) on \(\Omega_{j+1}\). Then \(g_{k_{j+1},\infty} = g_{k_{j+1},\infty}\) on \(\Omega_i\), which is denoted by \(g\) now. Then \(g_{k_{j+1},j}\), still denoted by \(g_j\), converges to \(g\) weakly on \(L^2(\Omega_i, V, \gamma \varphi)\) for each \(i\). By the weak convergence of \(g_j\), we have
\[ \int_{\Omega} |g|^{2} e^{\gamma \varphi} \, dv = \lim_{j \to \infty} \int_{\Omega} (g, g^{j}) e^{\gamma \varphi} \, dv. \] By (B.17) and the Cauchy-Schwarz inequality, we obtain \( \int_{\Omega} |g|^{2} e^{\gamma \varphi} \, dv \leq e^{\gamma/2} \). Hence \( g \in L^{2}(\Omega, V, \gamma \varphi) \).

On \( \Omega_{-c} \), we have \( g = f e^{-\gamma \varphi} \). As \( j \to \infty \),
\[ |\int_{\Omega} \langle g, \chi_{\Omega \setminus \Omega_{-c}} \rangle e^{\gamma \varphi} \, dv| \leq \int_{\Omega} |g|^{2} e^{(\varphi_{j})_{kj}} \, dv \left[ \int_{\Omega_{\setminus \Omega_{-c}}} |g|^{2} e^{2\gamma \varphi - (\varphi_{j})_{kj}} \, dv \right]^{1/2} \leq \left[ \int_{\Omega_{\setminus \Omega_{-c}}} |g|^{2} e^{2\gamma \varphi - (\varphi_{j})_{kj}} \, dv \right]^{1/2} \to 0. \]

Hence \( \int_{\Omega_{\setminus \Omega_{-c}}} |g|^{2} e^{\gamma \varphi} \, dv = \lim_{j \to \infty} \int_{\Omega} \langle g, \chi_{\Omega \setminus \Omega_{-c}} g \rangle e^{\gamma \varphi} \, dv = 0. \) Therefore, \( g = 0 \) on \( \Omega \setminus \Omega_{-c} \).

In the sense of distribution, we have \( \vartheta g_{j} = e^{-(\varphi_{j})_{kj}} T_{j}^{*} f_{j} \) where \( \vartheta \) is defined by
\[ \int_{\Omega} \langle \vartheta g_{j}, u \rangle \, dv = \int_{\Omega} \langle g_{j}, \vartheta u \rangle \, dv \]
for all \( u \in C_{(p,q-1)}(\Omega^{j}, V) \) with compact support in \( \Omega^{j} \). Now
\[ (B.18) \quad \int_{\Omega} |\vartheta g_{j}|^{2} e^{\gamma \varphi} \, dv \leq \int_{\Omega} |\vartheta g_{j}|^{2} e^{(\varphi_{j})_{kj}} \, dv = \| T_{j}^{*} f_{j} \|_{(\varphi_{j})_{kj}}^{2} \to 0, \quad j \to \infty. \]

For any \( u \in C_{(p,q-1)}(\Omega_{-c}, V) \), we extend it to \( \tilde{u} \in C_{(p,q-1)}(\Omega \cup \Omega^{j}, V) \) with compact support in \( \Omega \setminus \Omega^{j} \) when \( j \) is sufficiently large. Recall that on \( \Omega_{-c}, g = f e^{-\gamma \varphi} \), \( g = 0 \) on \( \Omega \setminus \Omega_{-c} \). Thus
\[ \int_{\Omega_{-c}} \langle f, \vartheta u \rangle e^{-\gamma \varphi} \, dv = \int_{\Omega} \langle g, \vartheta u \rangle \, dv = \lim_{j \to \infty} \int_{\Omega} \langle g_{j}, \vartheta \tilde{u} \rangle \, dv. \]

We have for large \( j \)
\[ \int_{\Omega} \langle g_{j}, \vartheta \tilde{u} \rangle \, dv = \int_{\Omega} \langle g_{j}, \vartheta \tilde{u} \rangle \, dv = \int_{\Omega} \langle \vartheta g_{j}, \tilde{u} \rangle \, dv. \]

By (B.18) the last integral tends to 0 as \( j \to \infty \). This shows \( \int_{\Omega_{-c}} \langle f, \vartheta u \rangle e^{-\gamma \varphi} \, dv = 0. \) Since \( C_{(p,q-1)}(\Omega_{-c}, V) \) is dense in \( D_{T_{c}^{*}} \), we conclude that \( f \in N_{T_{c}^{*}} \). By (B.11), \( \vartheta f_{j} \to 0 \) in \( L^{2} \) norm on any compact subset of \( \Omega_{-c} \). Thus \( \vartheta f = 0 \) on \( \Omega_{-c} \). Therefore, \( f \in N_{(p,q)}(\Omega_{-c}, V, \gamma \varphi) \). Then \( f = 0 \) on \( \Omega_{-c} \), which contradicts (B.16).

**Theorem B.10** ([31, Thm. 3.4.6]). Let \( \Omega \) be a relatively compact \( C^{2} \) domain in \( X \). Let \( \Omega_{-c}, \varphi, k, \varphi_{k} \) be as in Proposition B.7. Let \( V \) be a holomorphic vector bundle in \( X \). Assume that \( \varphi \) satisfies the condition \( a_{q} \) in \( \overline{\Omega} \setminus \Omega_{-c} \). There exist \( k_{\ast} \) and \( C_{\ast} \) satisfying the following.

(a) If \( f \in L_{(p,q)}^{2}(\Omega, V) \) and the equation \( \overline{\vartheta} u_{0} = f \) has a solution \( u_{0} \) in \( L_{2}(\Omega_{-c}, V) \), then it has a solution \( u \) in \( L_{(p,q-1)}^{2}(\Omega, V) \). In other words, the restriction \( \overline{H}_{(p,q)}(\Omega, V) \to \overline{H}_{(p,q)}(\Omega_{-c}, V) \) is injective. Moreover,
\[ \| u \|_{\varphi_{k}} \leq C_{\ast} \| f \|_{\varphi_{k}}, \quad k \geq k_{\ast}. \]
where \( C_{\ast}, c, k_{\ast} \) are the constants in (B.8).
(b) Furthermore, $C_*, c, k_*$ are stable under $C^2$ perturbations of $\partial \Omega$ in the following sense: Let $\Omega, \tilde{\Omega}$ be defined by $\rho < 0, \tilde{\rho} < 0$ respectively. There exists $\delta_* = \delta_* (\nabla \rho, \nabla^2 \rho) > 0$, depending on $c, \rho$ such that with $\varphi = e^{\lambda \rho} - 1, \tilde{\varphi} = e^{\lambda \tilde{\rho}} - 1$ for some $\lambda > 0$ depending only on $\rho$ and $\| \tilde{\rho} - \rho \|_2 < \delta_*$, if $f \in L^2(\Omega)$ is $C^2$ closed and $f = \tilde{\partial} u_0$ for some $u_0 \in L^2_{loc}(\Omega_{c})$, then there is a solution $u \in L^2(\Omega)$ such that $\partial u = f$ on $\tilde{\Omega}$ and

$$|u|_{\tilde{\Omega}, \tilde{\varphi}_k} \leq C_* |f|_{\Omega, \varphi_k}, \quad k \geq k_*.$$

Here $C_*, k_*, \delta_*$ are independent of $\tilde{\rho}$ and unknown.

Proof. Assertion (a) is proved in [31, Thm. 3.4.6], using (B.8) and [31, Lem. 3.3.3] and Thm. 4.1.4. The same proof is valid for (b) via (B.10).

Denote the $c$ in Theorem B.10 by $c_*$. Taking $\tilde{\rho} = \rho - a$, we obtain

**Corollary B.11.** Let $\rho, c_*, \delta_*, \Omega$ be as above. Then the restriction $\Pi_{(p,q)}(\Omega_{a}, V) \to \Pi_{(p,q)}(\Omega_{c_*}, V)$ is injective for any $|a| < \delta_*$.  

There is a detailed study in Lieb–Michel [45, Chapt. VIII, Sect. 8] on the stability of estimates for the $\overline{\partial}$-Neumann operator on $\Omega_{c}$, when $\Omega$ is a strictly pseudoconvex manifold with smooth boundary. In our case, we must treat a slightly more general situation where $\Omega$ can be any small $C^2$ perturbations of $\Omega$. We do not know if $C_*, k_*$ are stable under $C^2$ perturbations in the sense of (8.4); nevertheless, using Grauert’s bumping method for $a_q$ domains, Corollary B.11 suffices our purposes.

**References**

[1] A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 81–130. MR0175148

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011. MR2768550

[3] L. Baracco and G. Zampieri, *Regularity at the boundary for $\overline{\partial}$ on Q-pseudoconvex domains*, J. Anal. Math. **95** (2005), 45–61. MR2145559

[4] **Boundary regularity for $\overline{\partial}$ on Q-pseudoconvex wedges of $\mathbb{C}^N$**, J. Math. Anal. Appl. **313** (2000), no. 1, 262–272. MR2178735

[5] M.-Y. Barkatou, *C^k estimates for $\overline{\partial}$ on q-conved wedges*, Math. Z. **239** (2002), no. 2, 335–352. MR1888228

[6] M.-Y. Barkatou and S. Khidr, *Global solution with C^k-estimates for $\overline{\partial}$-equation on q-convex intersections*, Math. Nachr. **284** (2011), no. 16, 2024–2031. MR2844676

[7] R. Beals, P. C. Greiner, and N. K. Stanton, *$L^p$ and Lipschitz estimates for the $\overline{\partial}$-equation and the $\overline{\partial}$-Neumann problem*, Math. Ann. **277** (1987), no. 2, 185–196. MR884618

[8] O.V. Besov, V.P. Il’in, and S.M. Nikolski˘ı, *Integral representations of functions and imbedding theorems. Vol. II* (Mitchell H. Taibleson, ed.), Scripta Series in Mathematics, V. H. Winston & Sons, Washington, DC; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1979.

[9] P.L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Die Grundlehren der mathematischen Wissenschaften, vol. Band 145, Springer-Verlag New York, Inc., New York, 1967. MR0230022

[10] S.-C. Chen and M.-C. Shaw, *Partial differential equations in several complex variables*, AMS/IP Studies in Advanced Mathematics, vol. 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001. MR1800297

[11] G. Crasta and A. Malusa, *The distance function from the boundary in a Minkowski space*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 5725–5759. MR2336304
[38] Harmonic integrals on strongly pseudo-convex manifolds. II, Ann. of Math. (2) 79 (1964), 450–472. MR0208200
[39] Global regularity for $\partial$ on weakly pseudo-convex manifolds, Trans. Amer. Math. Soc. 181 (1973), 273–292. MR044703
[40] J.J. Kohn and H. Rossi, On the extension of holomorphic functions from the boundary of a complex manifold, Ann. of Math. (2) 81 (1965), 451–472, DOI 10.2307/1970624. MR0177135
[41] S.G. Krantz and H.R. Parks, Distance to $C^k$ hypersurfaces, J. Differential Equations 40 (1981), no. 1, 116–120. MR614221
[42] C. Laurent-Thiébaut and J. Leiterer, The Andreotti-Vesentini separation theorem with $C^k$ estimates and extension of CR-forms, Several complex variables (Stockholm, 1987/1988), 1993, pp. 416–439. MR1207871
[43] The Andreotti-Vesentini separation theorem and global homotopy representation, Math. Z. 227 (1998), no. 4, 711–727. MR1621967
[44] Y. Li and L. Nirenberg, Regularity of the distance function to the boundary, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5) 29 (2005), 257–264. MR2305073
[45] I. Lieb and J. Michel, The Cauchy-Riemann complex, Aspects of Mathematics, E34, Friedr. Vieweg & Sohn, Braunschweig, 2002. Integral formulae and Neumann problem. MR1900133
[46] I. Lieb and R.M. Range, Lösungenoperatoren für den Cauchy-Riemann-Komplex mit $C^k$-Abschätzungen, Math. Ann. 253 (1980), no. 2, 145–164. MR597825
[47] J. Michel, Randregularität des $\bar{\partial}$-Problems für stückweise streng pseudokonvexe Gebiete in $C^k$, Math. Ann. 280 (1988), no. 1, 45–68. MR928297
[48] J. Michel and A. Perotti, $C^k$-regularity for the $\bar{\partial}$-equation on strictly pseudoconvex domains with piecewise smooth boundaries, Math. Z. 203 (1990), no. 3, 415–427. MR1038709
[49] J. Michel and M.-C. Shaw, The $\bar{\partial}$ problem on domains with piecewise smooth boundaries with applications, Trans. Amer. Math. Soc. 351 (1999), no. 11, 4365–4380. MR1675218
[50] V. Michel, Sur la régularité $C^\infty$ du $\bar{\partial}$ au bord d’un domaine de $C^k$ dont la forme de Levi a exactement $s$ valeurs propres strictement négatives, Math. Ann. 295 (1993), no. 1, 135–161. MR1198845
[51] C.B. Morrey Jr., The analytic embedding of abstract real-analytic manifolds, Ann. of Math. (2) 68 (1958), 159–201. MR0090960
[52] Multiple integrals in the calculus of variations, Die Grundlehren der mathematischen Wissenschaften, Band 130, Springer-Verlag New York, Inc., New York, 1966. MR0205711
[53] A. Nagel and J.-P. Rosay, Nonexistence of homotopy formula for $(0,1)$ forms on hypersurfaces in $C^k$, Duke Math. J. 58 (1989), no. 3, 823–827. MR1016447
[54] P.L. Polyakov, Versal embeddings of compact $3$-pseudoconvex CR submanifolds, Math. Z. 248 (2004), no. 2, 267–312. MR2088929
[55] R.M. Range and Y.-T. Siu, Uniform estimates for the $\bar{\partial}$-equation on domains with piecewise smooth strictly pseudoconvex boundaries, Math. Ann. 206 (1973), 325–354. MR338450
[56] H. Ricard, Estimations $C^k$ pour l’opérateur de Cauchy-Riemann sur des domaines à coins $q$-convexes et $q$-concaves, Math. Z. 244 (2003), no. 2, 349–398. MR1992543
[57] W. Rudin, Principles of mathematical analysis, Third, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York-Auckland-Düsseldorf, 1976. MR0385023
[58] V.S. Rychkov, On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. (2) 60 (1999), no. 1, 237–257. MR1721827
[59] Z. Shi, Weighted Sobolev $L^p$ estimates for homotopy operators on strictly pseudoconvex domains with $C^2$ boundary, J. Geom. Anal. 31 (2021), no. 5, 4398–4446. MR4244873
[60] Z. Shi and L. Yao, A solution operator for the $\bar{\partial}$ equation in Sobolev spaces of negative index, Trans. Amer. Math. Soc. 377 (2024), 1111–1139. MR4688544
[61] Sobolev $1/2$ estimates for $\bar{\partial}$ equations on strictly pseudoconvex domains with $C^2$ boundary, to appear in Amer. J. Math. (2021).
[62] New estimates of Rychkov’s universal extension operator for Lipschitz domains and some applications, Math. Nachr. 207 (2024), 1407–1443. MR4734977
[63] Y.-T. Siu, The $\bar{\partial}$ problem with uniform bounds on derivatives, Math. Ann. 207 (1974), 163–176. MR330515
[64] J. Spruck, Interior gradient estimates and existence theorems for constant mean curvature graphs in $M^n \times \mathbb{R}$, Pure Appl. Math. Q. 3 (2007), no. 3, Special Issue: In honor of Leon Simon. Part 2, 785–800. MR2351645
[65] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983. MR0781540

[66] S.M. Webster, *A new proof of the Newlander-Nirenberg theorem*, Math. Z. **201** (1989), no. 3, 303–316. MR999729

[67] ———, *On the proof of Kuranishi’s embedding theorem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), no. 3, 183–207. MR995504

[68] L. Yao, *Sobolev and H"older estimates for homotopy operators of the $\overline{\partial}$-equation on convex domains of finite multitype*, J. Math. Anal. Appl. **538** (2024), no. 2, Paper No. 128238, 41, DOI 10.1016/j.jmaa.2024.128238. MR4739361

[69] S.L. Yie, *Solutions of Cauchy-Riemann equations on pseudoconvex domain with non-smooth boundary*, ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Purdue University. MR2693230

[70] G. Zampieri, *q-pseudoconvexity and regularity at the boundary for solutions of the $\overline{\partial}$-problem*, Compositio Math. **121** (2000), no. 2, 155–162. MR1757879

[71] ———, *Solvability of the $\overline{\partial}$ problem with $C^\infty$ regularity up to the boundary on wedges of $\mathbb{C}^N$*, Israel J. Math. **115** (2000), 321–331. MR1749685

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