Dynamical selforganization of atoms in cavities

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Laser-cooled atoms can spontaneously form spatially-ordered structures in optical resonators when the intracavity field energy exceeds a threshold value. This behaviour results from the mechanical forces associated with superradiant scattering of laser photons into the cavity. We treat the atomic motion semiclassically and show that, while the onset of spatial ordering depends on the intracavity photon number, the stationary momentum distribution follows a Maxwell-Boltzmann distribution whose width is determined by the rate of photon losses. Above threshold, the dynamics leading to the stationary state is characterized by two time scales: after a violent relaxation, the system slowly reaches the stationary state over time scales which exceed the cavity lifetime by several orders of magnitude. In this transient regime the atoms form non-Gaussian metastable distributions, reminiscent of the quasi-stationary states of long-range interacting systems. We argue that the dynamics of selforganization of atoms in cavities offers as a testbed for studying the predictions of the statistical mechanics of long-range interacting systems.

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Long-range interactions characterize the dynamics of systems from microscopic to macroscopic scales, ranging from nuclear to astrophysical distances [1]. In these systems the individual components can interact with a long-range potential that decays with the interparticle distance $r$ slower than $r^{-d}$ in $d$ dimensions. This property leads, to mention some, to ensemble inequivalence and to the existence of quasi-stationary states, i.e., metastable states with non-thermal distributions [1].

Cold atoms driven by laser light constitute a promising laboratory realization of long-range interacting systems [2–5]. Here, multiple scattering of photons by atoms gives rise to mechanical forces which are infinitely long ranged when the atoms couple to a single-mode high finesse cavity [6]. In the overdamped regime this long-ranged potential lies at the origin of synchronization [7] and collective atomic recoil lasing [8]. When the cavity mode is a standing wave and the atoms are transversally pumped, as in the setup sketched in Fig. 1, spontaneous ordering in spatially periodic structures is observed [3, 9–11]. The phenomenon can be described in terms of formation of atomic gratings which maximize coherent scattering of laser photons into the cavity mode. These "Bragg gratings" are stably trapped by the mechanical effects of the light they scatter, provided that the laser pump compensates the cavity losses so that the number of intracavity photons is sufficiently large. This occurs when the strength of the laser coupling exceeds a threshold value $\Omega_c$ depending, amongst others, on the rate of photon losses and the number of atoms $N$ that couple with the cavity mode [12, 13]. This spatial selforganization was first predicted in Refs. [4, 9] and then reported in a series of experiments at laser-cooling temperatures [10, 14] and in the ultracold [11, 15].

In this Letter we theoretically analyse the dynamics leading to the formation of spatial structures and their stationary properties in one dimension. For this purpose we resort to a Fokker-Planck equation (FPE) derived when the atoms are classical polarizable particles, their center-of-mass motion is treated semiclassically, while the cavity field is a full quantum mechanical variable [16]. Our approach complements the one applied in [9, 12, 13, 17] and based on the assumption that the cavity field is a semiclassical variable. By treating the cavity field quantum mechanically, we determine its state for any value of the laser amplitude and in particular at threshold, where quantum fluctuations are
important. This information is extracted provided that retardation effects in the scattering processes are perturbations, such that at leading order the field is determined by the instantaneous atomic distribution within the cavity field \[13\]. Thus, for \(N\) identical atoms confined in one dimension along the cavity axis, the total scattering amplitude depends on their positions \(x_1, \ldots, x_N\) within the cavity standing wave \(\cos(kx)\) and the cavity electric field at time \(t = E_c(t) \propto \sqrt{N\tilde{n}(\Theta)_t}\). Here, \(\tilde{n}\) is the maximum intracavity-photon number per atom, and is thus the quantity controlled by the strength of the external laser pump \[10\], while

\[
\Theta = \sum_{j=1}^{N} \cos(kx_j)/N
\]

characterizes spatial ordering in the cavity field \[10\]. The field reaches its maximum when \(|\Theta| = 1\), namely, when the atoms form a Bragg grating. The corrections to the field due to the atomic motion are systematically included in the following but treated as perturbation, assuming that the Doppler shifts of the atoms are smaller than the cavity linewidth \(\kappa\) \[16\].

The averages \(\langle \rangle_t\) are taken over the normalized distribution \(f(x_1, p_1; \ldots; x_N, p_N; t)\) at time \(t\), where \(p_1, \ldots, p_N\) are the atomic momenta and \(f\) obeys the FPE \[16\]

\[
\partial_t f + \{f, H\} \simeq -\tilde{n}\Gamma \sum_i \sin(kx_i) \partial_{p_i} \frac{1}{N} \sum_j \sin(kx_j) \left( p_j + \frac{m}{\beta} \partial_{p_j} \right) f.
\]

Here, the left-hand side (LHS) contains the Poisson brackets with the Hamiltonian \(H\) governing the coherent dynamics, that originate from the conservative mechanical forces of light, while the right-hand side (RHS) contains the friction coefficient due to retardation and the diffusion, which arises from fluctuations of the cavity field due to photon losses \[22\]. These terms are scaled by \(\tilde{n}\) and by the rate \(\Gamma = 8\omega_r\kappa\Delta_c/\left(\Delta_c^2 + \kappa^2\right)\), with \(\Delta_c = \omega_L - \omega_c\) the detuning between laser and cavity-mode frequencies, such that \(\tilde{n}\Gamma\) is the maximum cooling rate of a single atom \((N = 1)\). In addition, \(h\beta = -4\Delta_c/(\Delta_c^2 + \kappa^2)\). The Hamiltonian

\[
H = \sum_j \frac{p_j^2}{2m} + h\Delta_c\tilde{n}N\Theta^2 + O(U)
\]

contains the cavity-mediated potential, which scales with \(\tilde{n}\) and is attractive when \(\Delta_c\) is negative. Hence, this detuning determines whether the formation of Bragg gratings is energetically favoured. Equation \(2\) summarizes in a compact way a property which was observed in several previous works \[9, 10, 12\]. It is reported at leading order in \(|NU/\Delta_c|\), where \(U\) accounts for scattering processes involving the virtual absorption and emission of cavity photons \[16, 17\], and whose effect is systematically included in the numerical simulations.

Remarkably, at leading order in \(|NU/\Delta_c|\) Eq. \(2\) allows one to draw a direct connection with the Hamiltonian Mean Field (HMF) model, the workhorse of the statistical mechanics of systems with long-range interaction, which in a canonical ensemble exhibits a second-order phase transition from a paramagnetic to a ferromagnetic phase controlled by the temperature \[1\]. This analogy becomes explicit writing \(\Theta = (\sum_{i,j}(\cos(k(x_i + x_j)) + \cos(k(x_i - x_j)))/(2N^2)\), which shows that \(H\) is extensive as it satisfies Kac prescription \[1\], and suggests to identify \(\Theta\) with the \(x\)-component of a two-dimensional magnetization.

Differing from the HMF model, the term \(\cos(k(x_i + x_j))\) originates from the underlying cavity standing-wave potential that breaks translational invariance. Moreover, our system is characterized by an out-of-equilibrium dynamics, where the cavity forces at higher order in \(|NU/\Delta_c|\) give rise to deviations from the Hamiltonian dynamics due to additional terms in the LHS of Eq. \(1\) (see, e.g., \[20\]) which are responsible of bistable behaviour \[21\]. Retardation effects and cavity losses, moreover, give rise not only to friction and diffusion, but also to long-range correlations between the atoms, as visible by inspecting the RHS. In fact, diffusion is here due to global quenches of the cavity potential. Similarly, retardation effects modify the global potential \[23\]. When the density is uniform, the terms in the RHS reduce to the Langevin terms of a FPE which fulfills detailed balance and the model is analogous to the Brownian Mean Field model \[24\]. However, this is valid only well below the self-organization threshold. Indeed, selforganization is here controlled by \(\tilde{n}\), and thus by the laser intensity, which thus scales both the strength of the long-range coherent and incoherent forces, and thus the atomic spatial distribution. This becomes evident when studying the dynamics at the asymptotics. A solution of \(\partial_t f_{\infty} = 0\) is the thermal distribution \(f_{\infty} = f_0 \exp(-\beta H)\) for \(\Delta_c < 0\), with \(f_0\) normalizing factor. The temperature is independent of the laser pump strength and its minimum \(k_B T_{\text{min}} = \hbar\kappa/2\) is achieved for \(\Delta_c = -\kappa\), as also found in Ref. \[12, 13\] using different approaches. In \[13\] the selforganization threshold \(\tilde{n}_c = (1 + \kappa^2/\Delta_c^2)/4\) was estimated by means of a kinetic theory based on treating the cavity field semiclassically. This value is consistent with our results.

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We first discuss the predictions of Eq. \(1\) at the asymptotics. Figure \(2a\) displays the stationary distri-
bution of the magnetization, \( P(\Theta_0) = \langle \delta(\Theta_0 - \Theta) \rangle_\infty \), for
different values of \( n \). For \( n < n_c \), \( P(\Theta_0) \) is approximately
a Gaussian centered at zero. At threshold it broadens
and becomes increasingly localized at the values \( \pm 1 \) as
\( n \) grows. Typical trajectories \( \Theta(t) \) fluctuates about zero (corresponding to a uniform spatial
distribution), as \( n \) is increased above threshold it takes
either positive or negative values, in which it remains
trapped for time intervals which grow with \( n \). Jumps
between the two values correspond to quenches of the
intracavity photon number following losses, as shown in
(c) for \( n = 1.1n_c \), and take place over intervals of the
order of the cavity decay rate. Note that these jumps
correspond to a simultaneous jump of all atomic trajec-
tories out of the Bragg gratings. For \( n = 4n_c \), the
residence time is infinite: photon losses give rise to small
fluctuations of the potential depth and the atoms remain
locked in a Bragg grating. These features determine the
light amplitude at the cavity output and can be mea-
sured by heterodyne detection. Additional information
is contained in the power spectrum of the light
intensity, which is Fourier transform \( S(\omega) \) of the
correlation function \( g^{(2)}(\tau) = \lim_{t \to \infty} \langle \Theta(t + \tau)\Theta(t) \rangle / \langle |\Theta(t)|^2 \rangle \)
and is displayed in Fig. (d) for different values of \( n \).\( S(\omega) \) exhibits a narrow peak at the laser frequency as
the threshold is approached, and is associated with the
creation of Bragg gratings coherently scattering light into
the resonator. The broad background spectrum is pro-
gressively suppressed, while at threshold two broad side-
bands appear whose maximum moves away from \( \omega = \omega_L \)
as \( n \) increases from \( n_c \). Similar features have been observed
at the selforganization transition in the ultracold
and have been interpreted in terms of density
waves which drive the instability. Figure (e) di-
plays the second-order correlation function of the emit-
ted light at zero-time delay \( g^{(2)}(0) \) as a function of \( n \),
where \( g^{(2)}(\tau) = \lim_{t \to \infty} \langle \Theta(t + \tau)\Theta(t) \rangle / \langle |\Theta(t)|^2 \rangle \).
Below threshold \( g^{(2)}(0) \) \( \to 3 \). This value is also found
analytically after discarding correlations between the atoms.
It monotonously decreases with \( n \) and reaches unity
above threshold, \( g^{(2)}(0) \) \( \to 1 \), corresponding to a coher-
ent state inside the resonator. The crossover be-
 tween these two regimes narrows as the number of atoms
is increased, suggesting a jump at \( n_c \) in the thermody-
namic limit, consisting in keeping \( n_c \) constant as
\( N \to \infty \). These features are consistent with the con-
jecture that selforganization is a second-order phase tran-
sition controlled by \( n_c \). This is also supported by the
behaviour of the susceptibility, \( \chi = \langle |\Theta(t)|^2 \rangle - \langle |\Theta(t)| \rangle^2 \),
as a function of \( n \), which suggests a divergence at \( n_c \) for
\( N \to \infty \). We remark that the typical understanding of
spatial domain formation at a second-order phase transition
is here meaningless due to the non-additivity of the energy:
mesoscopic Bragg gratings with \( \Theta = \pm 1 \) cannot
stably coexist in space, since the resulting cavity field
vanishes and with it the interatomic potential.

Fig. 2. (color online) (a) Distribution \( P(\Theta) \) of the magnetization \( \Theta \) at steady state for \( n/n_c = 0.1, 0.9, 1, 1.1, 4 \) (see box for
color code). (b) Typical trajectories at the asymptotics for \( N = 200 \) atoms are shown in (b) as a function of time (in
units of \( 1/\kappa \)) and for \( n/n_c = 0.1, 1, 1.1, 4 \) (from top to bottom). (c) \( g^{(2)}(0) \) as a function of \( n \) for different atoms numbers. The dots correspond
to numerical results obtained by integrating the SDE. The cavity parameters are rescaled with \( N \) so that \( n_c \) is independent on
\( N \) and finite (see [20]). The atomic transition is the D_2-line of ^87Rb at half linewidth \( \gamma = 2\pi \times 3 \) MHz. The laser detuning
from the atomic frequency is \( \Delta_n = -500\gamma \). Here, \( \Delta_c = -\kappa \) with \( \kappa = 0.5\gamma \).
We now turn to the dynamics leading to selforganization, assuming that the initial distribution is spatially uniform, while the momentum distribution is a Maxwell-Boltzmann at width $\hbar \kappa / 2$. For $\Delta_c = -\kappa$, well below threshold this distribution is stationary. For any value of $\bar{n}$ the uniformly spatial distribution gives vanishing cross-correlations, even though for a transient time. Peculiar features are observed well above threshold. Figure 3 displays a sample of 500 trajectories of $\Theta(t)$ and as a function of time when $\bar{n} = 4 \bar{n}_c$ and $N = 200$. The trajectories are bunched and their behaviour can be ordered into three regimes, characterized by different time scales. First, a fast relaxation occurs over the time scale of dozens cavity lifetimes, in which the magnetization reaches an intermediate value of about 0.6 (Fig. 3(b)) where it remains for a time scale exceeding the cavity lifetime by four orders of magnitude. At this stage, part of the atoms form a Bragg grating (Fig. 3(c)) while the momentum distribution is non-Gaussian (Fig. 3(d)). We denote this regime by prethermalization. Then, the magnetization slowly grows to the stationary value over time scales which are 6 orders of magnitude the cavity lifetime. Remarkably, for times of the order of $t \sim 10^5 \kappa^{-1}$ the momentum distribution exhibits clear deviations from a Gaussian, and hence from a thermal state, even though the spatial distribution is very close to the asymptotic one. This behaviour can be understood considering that the diffusion is a function of the spatial distribution: As visible in the RHS of Eq. 1, the strength of noise (and thus the relaxation rate) decreases the more the atoms are localized in the Bragg gratings, and thus at the nodes of the $\sin(kxz)$ function. Over the time scale in which the system prethermalizes, we verified that spatial diffusion follows a power law according to $\langle x(t)^2 \rangle \propto t^{2 \alpha}$, where $\alpha$ is monotonously decreasing as $\bar{n}$ increases. In particular, it is superdiffusive ($\alpha > 1/2$) below $\bar{n}_c$, while above $\bar{n}_c$ it becomes increasingly subdiffusive. In this latter case, in the long tails of relaxation it becomes normal again, $\alpha \to 1/2$. These characteristic features can be experimentally observed in the properties of the light at the cavity output. Figure 3(e) displays $g^2(\tau)$ for different values of $\bar{n}$. While below threshold it rapidly decays from 3 to unity on a time scale of the order of cavity decay, at threshold its relaxation is orders of magnitude slower and exhibits damped oscillations, which can be associated with the density waves that become unstable and determine the Bragg grating (cf. Fig. 2(d)). Well above threshold, instead, it remains locked to unity, corresponding to coherent light.

The prethermalization behaviour, followed by the slow rate at which the steady state is approached, is typical above the selforganization threshold. We argue that it is a manifestation of quasi-stationary states, namely, metastable solutions of the dynamics, which are non-Gaussian and whose existence is intrinsically related to the long-range nature of the interaction. A feature that is usually associated with their existence are relaxation times which increase with $N$. In our case we could not find clear signatures of this behaviour. Even though our analysis in this respect is not conclusive, it is consistent with studies of relaxation of quasi-stationary state of the HMF in presence of an external environment, which shows that its action can make these states dynamically unstable. In this work we discarded the effect of spontaneous decay, assuming it is negligible as the laser field is far off resonance. Its role is expected to be-
come more important as $\bar{n}$ is increased above threshold, and thus to enforce the dynamical instability of quasi-stationary states. Our model is also valid for any optically polarizable particles which can be confined within the resonator $[30]$. Its validity extends to deep in the selforganized phase, where the atoms are tightly trapped in the potentials, as long as the effective trap frequency $\nu$ of the resulting lattice is smaller than the cavity linewidth $[31]$. The description breaks down when $\nu \approx \kappa$: In this limit robust quantum mechanical coherence between the motional levels can be observed $[32, 33]$.

In view of these results, it is worthwhile to consider the selforganization transition observed in the ultracold regime $[11]$ by taking into account the thermodynamic peculiarity of long-range interacting systems. Preliminary studies in this direction have appeared in $[34, 35]$. Further work shall aim at developing a kinetic equation following the lines of the treatments in $[13, 36, 37]$. To conclude, our study shows that photonic systems offer a promising platform to test the physics of long-range interacting systems.

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