Work extraction exploiting thermalization with a single bath

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We propose a protocol which exploits the collective thermalisation of a bipartite system to extract work from another system. The protocol is based on a recently proposed work definition not requiring measurements and involving the presence of a single bath. A general description of the protocol is provided without specifying the characteristics of the bipartite system. We quantify both the extracted work and the ideal efficiency of the process also giving a maximum bound to the extracted work. Then, we apply the protocol to the case when the bipartite system is governed by the Rabi Hamiltonian while using a zero temperature bath. For very strong couplings, an extraction of work comparable with the typical energies of the subsystems and an efficiency greater than one half can be obtained.

I. INTRODUCTION

In recent years, the interest on quantum thermodynamics has been growing (for a review, see [1]). Several topics are included in such research area, ranging from typicality [2,3] to maximum entropy production principle [4,5]. One of the most intriguing problems concerns the realisation of thermodynamic processes at a quantum level [6–12]. In this context, a question immediately raises, whether quantities generally exploited in the macroscopic thermodynamics are still meaningful and properly defined at a microscopic scale. For example, in recent years there has been an increasing interest in the way the thermodynamic quantity work can be defined and manipulated in the quantum domain [13–22]. Classically, one can measure the amount of work done on an isolated system during a protocol by measuring the energy of this system at the start and at the end of the protocol. This procedure, which requires two measurements, is not directly transferable from classical to quantum physics because a measurement act has generally a big impact on the state of the measured system in quantum mechanics. However, the two-projective-energy-measurement scheme is one of the most used approaches to deal with work in quantum mechanics, despite leading to counter-intuitive results when applied to initial quantum states that have coherences in the energy basis [22]. An extensive review on this approach can be found in Ref. [21].

A recently developed theoretical framework, the Thermodynamic Resource Theory (TRT), provides new ways of quantifying work [13,15]. Among all the proposals we choose to use that of Ref. [13] defining work without requiring measurements. In this approach, both system $R$, which does work, and system $S$, on which the work is done, are included in the description and the work is treated like a quantity that can be transferred between the two systems. The work done on $S$ is defined on the basis of its initial density matrix and of its final one, together with the temperature at which the protocol is performed. Thus, to obtain the work done on a system during a protocol it is sufficient to know the initial density matrix of the system and how the protocol acts on this density matrix. If system $S$ has gained a quantity of work $W$ from $R$, $R$ has lost at least a quantity of work $W$, i.e. work is a not conserved quantity that can only decrease. This definition allows the utilisation of different quantifiers, among which we choose to use the Von Neumann free energy because of the property of being valid even if systems $S$ and $R$ are correlated at the end of the protocol [12].

Most of the results of TRT are given in terms of bounds which have to be respected, but these results are hard to test experimentally because this would require a very fine control of the system [23,24]. In the protocol we propose, a specific part of the procedure is studied in detail and it is a simple thermalisation process, while the other parts consist of turning on and off the interaction between the two parts of system $S$. This should be easily implementable even if not in the ideal way (i.e. without dissipation). Moreover, in this paper the protocol is applied to a bipartite system described by the Rabi Hamiltonian, which is an Hamiltonian used to describe various physical setups, such as in cavity QED [25] and in circuit QED [26]. The Rabi Hamiltonian describes one of the most common interactions between a two-level system and a quantum harmonic oscillator. If the interaction between the two subsystems is very weak, the Jaynes-Cummings approximation, which disregards the counter-rotating terms, can be used [25]. For a larger value of the relevant coupling constant, the counter-rotating terms cannot be neglected and it is possible to take their effect into account, at some level, through the Bloch-Siegert approximation [27]. However, when the interaction is of the same order of magnitude of the typical energies of the two systems other approaches are needed [28]. Despite its simplicity, the Rabi model has been solved analytically only recently [29,30]. Exploitation of such solutions will
allow us for an exploration of our thermodynamics protocol in a wide range of the relevant parameters.

The paper is organised as follows. In section II using the definition of work based on the free energy, we propose a protocol whose aim is to exploit the collective thermalisation of two subsystems \(A\) and \(B\) (which together form \(S\)), induced by some thermal bath, to extract work from another system \(R\). This is done in a quite general way, without considering the explicit forms both of the bipartite system \(S\) and of system \(R\). In section III we apply the protocol to a system \(S\) described by the Rabi Hamiltonian \(\text{III}\), when the bath is at zero temperature. We show that when the interaction between the two parts of \(S\) is sufficiently strong, the protocol returns an extraction of work of the same magnitude of the typical energies of the subsystems and an efficiency higher than 1/2. Finally, in section IV we provide some final remarks on our results.

II. WORK EXTRACTION PROTOCOL

As said in the introduction, we choose to use a definition of work developed in the framework of TRT. In general, a resource theory is a theory in which the possible operations that can be done on a system are restricted by some constraints and it is supposed that to perform a given protocol some special states belonging to other systems are unlimitedly available for free. These external systems in these particular states are called “free resources”\(^3\). An operation \(T(\rho)\) permitted by the TRT and addressed as a Thermal Operation (TO) has the following structure:

\[
T(\rho) = \text{Tr}_{\text{bath}}\left(U(\rho \otimes \rho_{\text{bath}})U^\dagger\right),
\]

\[
\left[\mathcal{H} + \mathcal{H}_{\text{bath}}, U\right] = 0,
\]

where \(\mathcal{H}\) is the Hamiltonian of the system described by the density operator \(\rho\), \(\mathcal{H}_{\text{bath}}\) is the Hamiltonian of the free resource (usually called bath) and \(U\) is a generic unitary operator that connects initial and final states of the total system (free resource included). The free resource (bath) is a system with an arbitrary Hamiltonian, assumed to be in a thermal state \(\rho_{\text{bath}}\) at a given temperature. In a thermal operation the bath is used only for the duration of the protocol so that the interaction between system and bath at the end and at the start of the protocol is identically zero. Then, the commutator \(\left[\mathcal{H} + \mathcal{H}_{\text{bath}}, U\right] = 0\) assures us that the global system has kept its energy unchanged (not only the mean value but also the entire distribution). This is needed because the aim of the TRT is to keep track of all the energy terms involved in a possible thermal process and to find what bounds this constraint generates.

In Ref. \(\text{III}\) the work is defined inside this theory by dividing the total system to which the thermal operation is applied into two parts or subsystems: a resource system \(R\), from which we want to extract work, and a storage system \(S\), in which we want to store the extracted work from \(R\). Following Ref. \(\text{III}\), there are various quantities which can serve as work quantifier but cannot be used if the resource and the storage are correlated at the end of the process. However, the quantifier based on the Von Neumann free energy does not have this problem \(\text{III}\), therefore we choose to use it. The work stored in system \(S\) during the process is then equal to

\[
W = \Delta F(\rho'_S, H'_S) - \Delta F(\rho_S, H_S),
\]

where

\[
\Delta F(\rho, H) = F(\rho, H) - F\left(\rho^\text{th}, H\right),
\]

\[
F(\rho, H) = \text{Tr}(H\rho) - k_B TS(\rho),
\]

\[
S(\rho) = - \text{Tr}(\rho \log(\rho)).
\]

Here \(F(\rho, H)\) is the free energy of the state \(\rho\) when the system is governed by the Hamiltonian \(H\), \(\rho^\text{th}\) is the thermal state of the system at temperature \(T\) equal to the temperature of the thermal bath which is used in the process, \(k_B\) is the Boltzmann’s constant and \(S(\rho)\) is the Von Neumann entropy of the state \(\rho\). The quantities marked with the symbol ′ are related to the end of the process, while those not marked are related to the start of the process. If \(H'_S = H_S\), Eq. (2) simplifies to:

\[
W = F(\rho'_S, H_S) - F(\rho_S, H_S).
\]

Contrary to other definitions of work, this one does not require measurements.

In this framework, we are going to describe a protocol aimed at transferring exploitable energy (work extraction) from a resource \(R\) to a storage system \(S\) composed by two subsystems \(A\) and \(B\) whose collective thermalisation is a key ingredient. In general, if a system described by the Hamiltonian \(\mathcal{H}\) is weakly coupled to a thermal bath, in some finite time it will typically be in a state practically indistinguishable from the thermal state \(\rho^\text{th}\) characterised by the same temperature of the bath \(\text{III}\). The thermal state is equal to

\[
\rho^\text{th} = \frac{e^{-\beta \mathcal{H}}}{\text{Tr}[e^{-\beta \mathcal{H}}]}, \quad \beta = 1/k_B T.
\]

However, if a system is made of different interacting subsystems, each of such parts will not be in a local thermal state, \(\rho^\text{th}_{A(B)}\) defined by Eq. (4) where \(\mathcal{H} = H_{A(B)}\). Thus, at thermal equilibrium, for each of the two subsystems composing \(S\) we have:

\[
\rho^\text{th}_{A(B)} \equiv \text{Tr}_{B(A)}[\rho^\text{th}_S] \neq \rho^\text{th}_{A(B)}
\]

where the symbol \(\rho^\text{th}_{A(B)}\) denotes what we call reduced thermal state of system \(A(B)\).

The possibility of extracting work by exploiting the difference, in terms of free energy, between reduced and local thermal states is immediately clear at zero temperature where the thermal state is just the ground state (if
the interaction between $A$ and $R$ during a time interval $S$. This process could bring system $S$ to a state with a higher free energy at time $t_2$ than at time $t_1$. If this is the case, the work has been done by system $R$ as will be explained later. We suppose that the state of system $S$ at time $t_2$ is still $\rho_{S}^{th} \otimes \rho_{R}^{th} = \rho_{S}(t_1)$. This can be achieved if the turning on of the interaction is doable in a time interval much smaller than the typical evolution time of system $S$ coupled to the bath ($\tau_r$). Thus, we have:

$$H(t_2) = H_A + H_B + H_I = H_0 + H_I,$$

$$\rho_{S}(t_2) = \rho_{A}^{th} \otimes \rho_{B}^{th} = \rho_{S}(t_1).$$

Here, $H_I$ is the interaction Hamiltonian between subsystems $A$ and $B$.

From time $t_2$ to time $t_3$, $(t_3 - t_2 \gg \tau_r)$, $A$ and $B$ thermalise as a whole so that the description of system $S$ at time $t_3$ is given by

$$H(t_3) = H(t_2) = H_0 + H_I,$$

$$\rho_{S}(t_3) = \rho_{S}^{th} = \frac{e^{-\beta(H_0+H_I)}}{\text{Tr}[e^{-\beta(H_0+H_I)}]}.$$

Finally, from time $t_3$ to time $t_4 = t_3 + \delta_2 t$ we use again system $R$ to turn off the interaction term between subsystems $A$ and $B$ by spatially separating them [34]. We suppose again that their state does not change during the time interval in which the interaction term goes to zero ($\delta_2 t \ll \tau_r$). However, the reduced states of systems $A$ and $B$ are now different from the initial ones, which were the thermal states. These are the reduced states obtained from the composite thermal state, i.e. the ones that we named reduced thermal states (see Eq. (6)). The situation at time $t_4$ is thus given by

$$H(t_4) = H(t_1) = H_0,$$

$$\rho_{S}(t_4) = \rho_{S}(t_3) = \rho_{S}^{th},$$

$$\rho_{A(B)}(t_4) = \text{Tr}_{B(A)}[\rho_{A(B)}^{th}] = \rho_{A(B)}^{th}.$$  

From the single-shot thermodynamic point of view, the extraction of work is completed at time $t_4$. Nevertheless, from a practical point of view we could be interested either in extracting the available energy saved in $S$ or in iterating the process in order to increase the amount of stored work. If this is the case, it becomes important to pay attention on the evolution of system $S$ after the time instant $t_4$. At this point some considerations are due. Let us suppose that the coupling with the bath is interrupted before the turning off of the $A-B$ interaction (i.e. before $t_4$). Under this condition the quantities such as average energy and entropy of the single subsystems $A$ and $B$ will not change after the time instant $t_4$. Thus, also the free energy of the two subsystems will be a conserved quantity and we could expend this work later. On the other hand, if we want to iterate the process by using the same subsystems $A$ and $B$, we firstly have to transfer the stored work to another external system. If the transfer of work from system $S$ to the external storage system is done without decoupling $S$ from the bath, then it has to be done immediately after the time instant $t_4$ and in a time interval $\delta_t t \ll \tau_r$. Theorems of TRT assure us that it is possible to transfer the free energy without losses under certain conditions [13, 32, 33] and that, in every

non-degenerate), while a different state, such as the reduced thermal state, surely has an average energy higher than that of the ground state.

System $R$ is not specified in this protocol and it will be mostly kept out of the equations. Namely, system $R$ could be a very large and complicate macroscopic system.

### A. The phases of the protocol

The protocol we are going to examine can be divided into different phases (see Fig. 1).

First of all, at the start of the protocol ($t = t_1$) $A$ and $B$ are non-interacting, non correlated and spatially separated [34]. They are both in contact with the same thermal bath at temperature $T$ and it holds:

$$H(t_1) = H_A + H_B = H_0,$$

$$\rho_{S}(t_1) = \rho_{A}^{th} \otimes \rho_{B}^{th},$$

$$\rho_{A(B)}(t_1) = \rho_{A(B)}^{th} = \frac{e^{-\beta H_{A(B)}}}{\text{Tr}[e^{-\beta H_{A(B)}}]},$$

where $H_{A(B)}$ is the Hamiltonian of system $A(B)$. Then, the interaction between $A$ and $B$ is turned on exploiting $R$ during a time interval $\delta t$ from $t_1$ to $t_2 = t_1 + \delta t$. This process could bring system $S$ in a state with higher free energy at time $t_2$ than at time $t_1$. If this is the case, the work has been done by system $R$ as will be explained later. We suppose that the state of system $S$ at time $t_2$ is still $\rho_{S}^{th} \otimes \rho_{R}^{th} = \rho_{S}(t_1)$. This can be achieved if the turning on of the interaction is doable in a time interval much smaller than the typical evolution time of system $S$ coupled to the bath ($\tau_r$). Thus, we have:

$$H(\delta t) = H_A + H_B + H_I = H_0 + H_I,$$

$$\rho_{S}(\delta t) = \rho_{A}^{th} \otimes \rho_{B}^{th} = \rho_{S}(t_1).$$

Here, $H_I$ is the interaction Hamiltonian between subsystems $A$ and $B$.

From time $t_2$ to time $t_3$, $(t_3 - t_2 \gg \tau_r)$, $A$ and $B$ thermalise as a whole so that the description of system $S$ at time $t_3$ is given by

$$H(t_3) = H(t_2) = H_0 + H_I,$$

$$\rho_{S}(t_3) = \rho_{S}^{th} = \frac{e^{-\beta(H_0+H_I)}}{\text{Tr}[e^{-\beta(H_0+H_I)}]}.$$

Finally, from time $t_3$ to time $t_4 = t_3 + \delta_2 t$ we use again system $R$ to turn off the interaction term between subsystems $A$ and $B$ by spatially separating them [34]. We suppose again that their state does not change during the time interval in which the interaction term goes to zero ($\delta_2 t \ll \tau_r$). However, the reduced states of systems $A$ and $B$ are now different from the initial ones, which were the thermal states. These are the reduced states obtained from the composite thermal state, i.e. the ones that we named reduced thermal states (see Eq. (6)). The situation at time $t_4$ is thus given by

$$H(t_4) = H(t_1) = H_0,$$

$$\rho_{S}(t_4) = \rho_{S}(t_3) = \rho_{S}^{th},$$

$$\rho_{A(B)}(t_4) = \text{Tr}_{B(A)}[\rho_{A(B)}^{th}] = \rho_{A(B)}^{th}.$$
case, the maximum transfer happens if the final state of system $S$ is the thermal state. Therefore, we assume that at time $t_5 = t_4 + \delta T$, system $S$ is described by:

$$H(t_5) = H(t_4) = H_A + H_B = H_0,$$
$$\rho_S(t_5) = \rho_S(t_4) = \rho_A^{th} \otimes \rho_B^{th}. \tag{11}$$

After having charged the external storage system there could be a time $t_r$ which serves to reinitialise the process. It is worth emphasising that the time $t_5 + t_r$ can be viewed as the time $t_1$ of the next iteration of the process and that for $t_3 \leq t \leq t_5 + t_r$ the system is described equivalently by equations (9) or (11). The power of the process increases as times $t_3 - t_2$ and $t_r$ decrease (we consider $\delta t, \delta t_r$ negligible).

### B. Analysis of free energies

In order to understand how the work flows in this protocol we have to follow the evolution of the free energies through the whole process. We refer to the previous section and see how the free energies of system $S$ and of its parts change over time. At time $t = t_1$ the system is described by Eq. (7) so that the free energy of system $S$ is

$$F_1 = F(\rho_A^{th} \otimes \rho_B^{th}, H_0) = F(\rho_A^{th}, H_A) + F(\rho_B^{th}, H_B). \tag{12}$$

Then, at time $t = t_2$ (see Eq. (9)) we have, defining $\langle H \rangle_{t_n} = \text{Tr}[\rho_S(t_n) H]$, $F_2 = F(\rho_A^{th} \otimes \rho_B^{th}, H_0 + H_1)$

$$F_2 = F(\rho_A^{th} \otimes \rho_B^{th}, H_0 + H_1) = F_1 + \text{Tr}[\rho_A^{th} \otimes \rho_B^{th} H_1] \tag{13}.$$ 

There are three possible cases: $\langle H_1 \rangle_{t_2}$ can be positive, negative or equal to zero. If $F_2 > F_1$, then the resource $R$ is required to do work on system $S$, from time $t_1$ to time $t_2$, greater or equal to $\langle H \rangle_{t_2}$, otherwise the process could be implemented without expense of work. We stress that, during this phase, system $R$ could lose a certain quantity of free energy due to dissipative effects, independently of the required variations of free energy. The same consideration holds for the phase of switching off (from time $t_3$ to time $t_4$).

Referring to Eq. (9), we can write at time $t = t_3$

$$F_3 = F(\rho_S^{th}, H_0 + H_1) = F(\rho_S^{th}, H_0) + \text{Tr}[\rho_S^{th} H_1] \tag{14}.$$ 

What can we say about the free energy at time $t_3$? First of all, this quantity is the minimum attainable free energy at the bath temperature $T$. That means that $F_3 \leq F_2$.

By exploiting the subadditivity property of the entropy $S(\rho)^{32}$,

$$S(\rho_S) \leq S(\rho_A) + S(\rho_B), \tag{15}$$

we can write

$$S(\rho_S) = S(\rho_A) + S(\rho_B) - D, \tag{16}$$

where $D$ is a real non-negative quantity. So, by using the definition of reduced thermal state [6], $F_3$ can be written as

$$F_3 = F(\rho_A^{th}, H_A) + F(\rho_B^{th}, H_B) + k_B T D + \langle H_1 \rangle_{t_3}, \tag{17}$$

so that,

$$F_3 - F_2 = \Delta F(\rho_A^{th}, H_A) + \Delta F(\rho_B^{th}, H_B) + k_B T D + \Delta \langle H_1 \rangle_{t_3}. \tag{18}$$

Eventually, the free energy has the following value at the end of the process (see Eq. (10)):

$$F_4 = F(\rho_S^{th}, H_0) = F_3 - \langle H_1 \rangle_{t_3}. \tag{19}$$

If $\langle H_1 \rangle_{t_3}$ is negative, then system $R$ has to do work on $S$ from time $t_3$ to time $t_4$, otherwise it could not. By using the definition of work (being $H_S^0 = H_S$, see Eq. (4)), the work extracted from $R$ and stored into $S$ is equal to

$$W = F_4 - F_1 = \Delta F(\rho_A^{th}, H_A) + \Delta F(\rho_B^{th}, H_B) + k_B T D. \tag{20}$$

The total extracted work is then given by the sum of two local quantities and a non local one. The two subsystems, though possibly correlated, are now non interacting and can be considered as separated. Therefore, it is physically reasonable to think that we can only use the local free energies of subsystems $A$ and $B$, for introducing the local extracted work:

$$W_I = \Delta F(\rho_A^{th}, H_A) + \Delta F(\rho_B^{th}, H_B). \tag{21}$$

Then, we have

$$W_I \leq W \leq -\Delta \langle H_1 \rangle_{t_4}. \tag{22}$$

The quantities $\Delta F(\rho_A^{th}, H_A)$ can be written as

$$\Delta F(\rho_A^{th}, H_A) = k_B T S(\rho_A^{th} || p_A^{th}), \tag{23}$$

where $S(\rho || \sigma) = \text{Tr}[\rho \log \rho - \rho \log \sigma]$.

The quantity $S(\rho || \sigma)$ is called relative entropy and, even if from a mathematical point of view it does not have all
the properties of a distance measure, it may quantify how much two density operators are different. Therefore, the more the reduced thermal states are different from the local ones the more should be the work extracted. One then expects that the work extracted increases as the strength of the interaction between the subsystems increases. When $T = 0$, if the population of an excited state of $\rho^R$ is not zero, the relative entropy goes to infinity leaving us with an indeterminate form. Fortunately, at $T = 0$ the quantities of Eq. (21) turn out to be simply differences of average energies thanks to Eq. (6).

Following the theorems of the TRT it is in principle possible to always transfer, without losses, a certain quantity of free energy from one system to another one through a thermal operation, if the starting and ending states of the systems involved are in block-diagonal form ($[\rho_S \otimes \rho_R, H_R] = 0$, $H_R$ is the Hamiltonian of system $R$). Achieving the maximum efficiency for this transfer may require, for example, the use of one or more catalysts [13, 14]. The initial state of $S$ is block diagonal given that it is the product of two thermal states and we can suppose that $R$ is also block-diagonal. Thus, we choose to define the ideal efficiency of the process as the work stored in system $R$ divided by the minimum free energy lost by system $S$, i.e.

$$\eta = \frac{W}{\max(0, \langle H_I \rangle_{t_2}) + \max(0, -\langle H_I \rangle_{t_1})},$$  
(24)

In other words, we compare the work that system $S$ gains with the work that system $R$ would lose in the best-case scenario. This comparison makes sense because TRT assures us that it exists a thermal operation such that all the work lost by $R$ is gained by $S$. So, one could equally say that we compare the work extracted with our process with the work that the best thermal operation would have extracted from system $R$ when this spends the same amount of free energy as in our process. Nonetheless, in the non-ideal case system $R$ would spend more work during the process because of dissipative effects. Moreover, we remark that if we use the local work (see Eq. (21)) instead of the total one, the efficiency is given by

$$\eta = \frac{W_I}{\max(0, \langle H_I \rangle_{t_2}) + \max(0, -\langle H_I \rangle_{t_1})},$$  
(25)

where $\eta \leq \eta_I$.

Summing up, this procedure permits the extraction of work from a system $R$ through the exploitation of a thermal bath. Even if it is always possible in principle to transfer all the work from system $R$ to system $S$ through a thermal operation, it can be difficult to find the correct thermal operation and a way to implement it in practice. In this process, it is required only the ability to switch on and off the $A - B$ interaction quickly enough to prevent changes of the local states of the subsystems during the switching processes. However, because of non negligible dissipation effects during the action of system $R$, the real efficiency of the process could be much lower than the ideal one.

With respect to processes involving usual thermal machines, here one has to deal with only one thermal bath. This is an interesting feature of this and other protocols described by TRT [13, 14].

In the next section we analyse in detail how this protocol works in the case of a system described by the Rabi Hamiltonian.

### III. THE PROTOCOL APPLIED TO THE RABI MODEL AT ZERO TEMPERATURE

The aim of this section is to specialise the protocol previously discussed to a physical model which effectively describes realistic situations in different physical contexts. In particular, we will show that both the amount of extracted work and the efficiency of the process are satisfactory.

The system we want to study is composed by a two level system (subsystem $A$) interacting with an harmonic oscillator (subsystem $B$). Subsystem $A$ is described by the Hamiltonian ($\hbar = 1$):

$$H_A = \Delta \sigma_z, \quad \text{where } \sigma_z = |e\rangle \langle e| - |g\rangle \langle g|,$$
(26)

and $\Delta$ is half of the energy distance between the ground state $|g\rangle$ and the excited state $|e\rangle$. Subsystem $B$ is described by

$$H_B = \omega \hat{n}, \quad \text{where } \hat{n} = a^\dagger a.$$  
(27)

Here, $\omega$ is the frequency of the harmonic oscillator, $\hat{n}$ is the number operator (with the number basis given by $|n\rangle = |n\rangle$ and $a^\dagger$ and $a$ are the usual creation and annihilation operators. Finally, the interaction between the two subsystems is given by

$$H_I = g \sigma_x (a^\dagger + a), \quad \text{where } \sigma_x = \sigma_+ + \sigma_- = |e\rangle \langle g| + |g\rangle \langle e|$$  
(28)

and $g$ is the coupling parameter. Thus, the total Hamiltonian of system $S$ is the Rabi Hamiltonian [25]:

$$H_{Rb} = H_A + H_B + H_I = \Delta \sigma_z + \omega \hat{n} + g \sigma_x (a^\dagger + a),$$  
(29)

where the frequency of the harmonic oscillator is typically comparable with that of the two-level system ($\omega \sim 2\Delta$).

The Rabi Hamiltonian is an ubiquitous model that describes the interaction between a two-state system and an harmonic oscillator. It is very effective for example in cavity QED [28] and in circuit QED [26]. In the past decades this model has been mainly treated under suitable approximations allowing for an analytical resolution. For example, the rotating wave approximation leading to the Jaynes-Cummings model [25, 33] and the Bloch-Siegert approximation [25] are two very well-known ones. Anyway, their validity is limited to the weak coupling
regime \((g \ll \omega)\). Recently, a lot of attention has been devoted, both theoretically and experimentally \([24, 28-31, 32, 33]\), to the study of the Rabi Hamiltonian beyond the weak coupling regime, namely, the ultra strong coupling \((0.1 \lesssim g/\omega \lesssim 1)\) \([28]\) and the deep strong coupling \((g/\omega \gtrsim 1)\) \([33]\). The interest in the strong coupling regimes is augmented by the recent remarkable experimental realisations of physical situations characterised by high values of \(g/\omega\) \([24, 27, 33, 38]\), for example \(g/\omega \sim 1.34\) \([20]\). Among the methods proposed to solve the model beyond the rotating wave approximation \([36, 39]\) a special attention deserves the recent complete analytical resolution found by Braak \([27]\), whose result has also been obtained with a different approach in Ref. \([31]\) (see also \([30]\) for a review).

A. The evolution of free energy

Now we follow the same steps of section \([13]\) but in a specific physical situation described by the Rabi model. We will focus on the zero temperature environment case, in which, anyway, we will find interesting results. At zero temperature, the entropy terms do not contribute to the free energies and, then, we can deal with average energies only. Moreover, the local thermal states coincide with the ground states so that \(\rho^h_A = |g\rangle\langle g|\) and \(\rho^h_B = |0\rangle\langle 0|\). By substituting the specific quantities of this system to the Eqs. \([12, 13, 14, 19]\) we can write:

\[
\begin{align*}
F_1 &= F(\rho^h_A \otimes \rho^h_B, H_0) = -\Delta, \\
F_2 &= F(\rho^h_A \otimes \rho^h_B, H_0 + H_1) = F_1, \\
F_3 &= F(\rho^h_B, H_0 + H_1) = \langle H_0 \rangle_{t^2} + \langle H_1 \rangle_{t^3} = E_0, \\
F_4 &= F(\rho^h_B, H_0) = E_0 - \langle H_1 \rangle_{t^3},
\end{align*}
\]

where \(E_0\) is the lowest eigenvalue of the Rabi Hamiltonian and we recall that \(\langle H_1 \rangle_{t^2} = 0\).

At zero temperature, \(W = W_t\) (see Eqs. \([20]\) and \([21]\)), which takes the form

\[
W = \langle H_A \rangle_{t^3} + \langle H_B \rangle_{t^3} + \Delta = E_0 - \langle H_1 \rangle_{t^3} + \Delta.
\]

The efficiency is then

\[
\eta = \frac{\langle H_A \rangle_{t^3} + \langle H_B \rangle_{t^3} + \Delta}{-\langle H_1 \rangle_{t^3}}.
\]

In order to calculate the quantities in Eqs. \([31]\) and \([32]\) we need to study the ground state and how it is decomposed in the bare basis. To this end we will mainly follow the approach and the formalism of Ref. \([30]\).

B. Study of the ground state

In this section all the quantities with the tilde are in units of \(\omega\) to lighten the notation \((\tilde{X} \equiv X/\omega)\). Following Ref. \([30]\), the ground energy \(E_0\) in \([30]\) can be calculated by searching for the first zero of the Braak’s function \(G_{-}(x)\) \([29, 31]\), defined by:

\[
G_{-}(x) = \sum_{n=0}^{\infty} \left(1 + \frac{\Delta}{x - n}\right) f_n \tilde{g}^n = 0,
\]

where \(x = \tilde{E} + \tilde{g}^2\). The factors \(f_n\) are calculated by recurrence through the following formulas:

\[
f_n = \frac{1}{n} [\Omega(n - 1)f_{n-1} - f_{n-2}], \quad f_0 = 1, \quad f_1 = \Omega(0),
\]

\[
\Omega(n) = \frac{1}{2\tilde{g}} \left( n + 3\tilde{g}^2 - \tilde{E} - \frac{\Delta^2}{n - \tilde{g}^2 - \tilde{E}} \right).
\]

The values of \(E\) for which the Braak’s functions are zero are the eigenvalues of the Rabi Hamiltonian, \(H_{Rb}\). The lowest of these eigenvalues is the ground energy of the system.

According to \([30]\), after some easy but lengthy calculations the ground state can be written as follows:

\[
|\psi_g\rangle = \frac{1}{2\sqrt{N}} (|e\rangle \langle \phi_1 | + | \phi_2 \rangle) + |g\rangle (|\phi_1| - |\phi_2|))
\]

where \(N\) is a normalisation constant,

\[
\langle n|\phi_1\rangle = e^{-\tilde{g}^2/2}\sqrt{n!} \sum_{m=0}^{\infty} m! \ v_m \left[ \sum_{k=\max(0,n-m)}^{n} \frac{(-1)^k}{(m - (n - k))! (n - k)! k!} \tilde{g}^{m-(n-2k)} \right], \quad v_m = -\frac{\Delta}{m - \tilde{g}^2 - E_0} f_m,
\]

and \(\langle n|\phi_2\rangle\) is equal to \(\langle n|\phi_1\rangle\) if one replaces in its expression \(v_m\) with \(f_m\).

The parity operator \(\Pi = -\sigma_z (-1)^n\) commutes with \(H_{Rb}\). Thus, it is easy to show that the ground state of the Rabi Hamiltonian has to be of the form

\[
|\psi_g\rangle = \sum_{n=0}^{\infty} c_{2n} |g, 2n\rangle + \sum_{n=0}^{\infty} c_{2n+1} |e, 2n + 1\rangle.
\]
Indeed, for low values of \( \tilde{g} \) the ground state has to contain the component \(|g, 0\rangle\) so that all the other components have to be of the same parity. Moreover, for every value of \( \tilde{g} \) the ground eigenvalue does not cross with the others eigenvalues, therefore the ground state has the same parity for each value of \( \tilde{g} \). By taking the scalar product of both sides of \( (35) \) with \(|n\rangle\), odd or even, and checking \( (36) \) one can easily infer the following equalities:

\[
\sqrt{N} c_n = \langle n| \phi_1 \rangle = (-1)^{n+1} \langle n| \phi_2 \rangle, \quad \forall n. \tag{38}
\]

Now we can easily calculate the reduced states and write down the quantities of interest. First of all, let us observe that

\[
\rho_A^{\text{th}} = \text{Tr}_B[|\psi_g\rangle\langle\psi_g|] = \frac{1}{N} \left[ \left( \sum_{n=0}^{\infty} |(2n+1)|\phi_1\rangle|^2 \right) |\epsilon\rangle\langle\epsilon| + \left( \sum_{n=0}^{\infty} |(2n)|\phi_1\rangle|^2 \right) |g\rangle\langle g| \right]. \tag{41}
\]

and the average energy is

\[
\langle H_A \rangle_{t_3} = \Delta \left( 2 \sum_{n=0}^{\infty} |c_{2n+1}|^2 - 1 \right). \tag{42}
\]

Concerning the average of the interaction energy \( \langle H_I \rangle_{t_3} \), it can be directly calculated with the formula:

\[
\langle H_I \rangle_{t_3} = \frac{2g}{N} \sum_{n=0}^{\infty} \sqrt{n+1} \langle n+1| \phi_1 \rangle \langle n| \phi_1 \rangle \tag{43}
\]

or, alternatively, it can be inferred from Eq. \( (31) \) as we already know \( \langle H_A \rangle_{t_3} \) and \( \langle H_B \rangle_{t_3} \) which we calculated through the knowledge of \( E_0 \).

For very low values of \( g/\omega \) the Jaynes-Cummings approximation can be used \( (33) \). In this case Eq. \( (35) \) becomes

\[
|\psi_g\rangle \simeq |g, 0\rangle. \tag{44}
\]

The Bloch-Siegert approximation holds good for higher values of \( g/\omega \) (still \( g/\omega \ll 1 \)). In this case, choosing the resonant case \( 2\Delta = \omega \), Eq. \( (35) \) becomes

\[
|\psi_g\rangle \simeq \left( 1 - \frac{\Lambda^2}{2} \right) |g, 0\rangle - \Lambda |\epsilon, 1\rangle + \Lambda^2 \sqrt{2} |g, 2\rangle, \tag{45}
\]

where \( \Lambda = g/(2\omega) \).

### C. Analysis of the results

The results of section \( III \) allow us to compute numerically the amount of extracted work and the efficiency of the protocol.

Figure 2 shows how the extracted work depends on the value of the coupling parameter \( g \) for different values of the dimensionless quantity \( \delta = (2\Delta - \omega)/\omega \). If the two systems are resonant (\( \delta = 0 \)), when the coupling parameter is of the order of \( \omega \), even the extracted work \( W \) is comparable to \( \omega \) and it grows quite fast with the increase of the coupling parameter. This is in accordance with what said in section \( III \) that is the fact that one normally expects that the stronger the interaction is the greater the extracted work is. For \( \delta \neq 0 \), one can see that the more is the frequency of the two-level system (\( 2\Delta \)) the more is the extracted work when the interaction is very intense. On the contrary, when \( g \) is smaller than \( \omega \) more work is extracted for low values of \( 2\Delta \). Moreover,
for very low values of $g$ the work extracted is nearly zero independently of $\delta$, as one expects from Eq. (44).

This process has also a satisfying efficiency $\eta$, as shown in Fig. 3 where we see that $\eta$ is always higher than 0.5. This means that the extraction of work obtained is of the same order of magnitude of the extraction performed with the optimal thermal operation (which has $\eta = 1$). In particular, we observe that the value of $g$ for which there is a peak of efficiency is an increasing function of $\Delta$. Furthermore, we notice that for low values of $g/\omega$ ($g/\omega \rightarrow 0.1$) the efficiency goes to $1/2$ as one would obtain with a simple calculation through Eq. (45).

We can divide the work extracted into two pieces (see eq (31)):

$$W = W_A + W_B, \quad \text{where} \quad W_A = \langle H_A \rangle_{t_3} + \Delta \quad \text{and} \quad W_B = \langle H_B \rangle_{t_3}. \quad (46)$$

Then, we can also define the two efficiencies (see eq (32))

$$\eta_A = \frac{W_A}{\langle H \rangle_{t_3}} \quad \text{and} \quad \eta_B = \frac{W_B}{\langle H \rangle_{t_3}}, \quad (47)$$

where $\eta_A + \eta_B = \eta$. Looking at $\eta_B$ (inset of Fig. 3), we note that there is an inversion with respect to the total extraction situation described by Fig. 3 while for the total extraction the more is $2\Delta$ the higher is the peak of the efficiency, if one only considers the work stored in subsystem $B$ the best situation is found for $2\Delta \ll \omega$, when $g \lesssim \omega$. So, the detuning $\delta$ plays a non negligible role in affecting the amount of extracted work and the efficiency of the protocol. In addition, when $2\Delta \ll \omega$ ($\delta \rightarrow -1$), the most relevant amount of the extracted work must be stored in the harmonic oscillator. This could be useful from an experimental point of view because it could be easier to use only the work stored in the harmonic oscillator.

The plots shown until now deal with values of $g/\omega$ which are near the experimental values actually available. Fig. 4 shows the total extracted work for higher values of $g/\omega$, though such values have not been yet reached in experiments. We can see an asymptotic behaviour compatible with the following law:

$$W \simeq \frac{g^2}{\omega} + \Delta. \quad (48)$$

Systems with a lower $\Delta$ reach their asymptotic behaviour for lower values of $g/\omega$ with respect to systems with an higher $\Delta$.

It is a known result [29, 31] that in the limit $g \gg \omega, \Delta$ the ground eigenvalue of the Rabi Hamiltonian is given by $E_0 = -g^2/\omega$. This means that, by using Eq. (48) in Eq. (31) in this limit, the interaction energy at time $t_3$ has the following behaviour:

$$\langle H_1 \rangle_{t_3} \rightarrow -\frac{2g^2}{\omega}, \quad (49)$$

whereas the efficiency (see Eq. (32)) is given by:

$$\eta \rightarrow \frac{1}{2}. \quad (50)$$

**D. Entanglement and work extraction**

In this subsection we study possible connections between the amount of extracted work and of entanglement between subsystems $A$ and $B$ in the total thermal state ($\rho_S(t_4)$) at zero temperature. The presence of entanglement is a sufficient condition for work extraction when the ground state of at least one between $A$ and $B$ is non-degenerate because in this case at least one non-ground state is populated at the end of the protocol (see for example Eq. (47)). However, as we will see in subsec. III E the presence of entanglement is not a necessary condition for the extraction of work.

Choosing to use the Von Neumann entropy (with...
We now make an example of an interaction Hamiltonian that could give maximal efficiency. Let us consider two discrete spectra. We take

\[ N \]

\[ M \]

which makes the efficiency arbitrarily close to 1. The work the faster is the system at thermalising, therefore the efficiency takes the form

\[ \eta = \frac{A_n + B_m}{\text{max}(0, G) + \text{max}(0, -L)} \quad (56) \]

There are two cases:

\[ G \leq 0 \quad \Rightarrow \quad \eta = \frac{W}{W + \epsilon} \quad (57) \]

\[ 0 \leq G \leq L \quad \Rightarrow \quad \eta = \frac{W}{W + \epsilon - G} \]

So, there is an entire family of interaction Hamiltonians which makes the efficiency arbitrarily close to 1. The first of the above equations (57) can be written in the alternative form

\[ G \geq 0 \quad \Rightarrow \quad \eta = \frac{1}{1 + \epsilon/W} \quad (58) \]

Given that \( \epsilon \) is the energy difference between the ground state and the first excited state it is clear that the more \( \epsilon \) is near to zero the more time the system takes to fully thermalise. However, in the form of Eq. (58) it is also clear that for a given efficiency the more is the extracted work the faster is the system at thermalising, therefore the power \( P = W/(t_4 - t_3) \) of this process increases as the extracted work \( W \) increases.

Let us now consider the case \( \epsilon = 0 \). In such a situation, the ground state of the total Hamiltonian is degenerate and the bath will divide the population into the two ground states. At \( t = t_4 \) (end of the protocol), we can consider that the state will be of the form

\[ \rho_S(t_4) = \frac{1}{2} (|A_0\rangle\langle A_0| \otimes |B_0\rangle\langle B_0| + |A_n\rangle\langle A_n| \otimes |B_m\rangle\langle B_m|) \quad (59) \]

That means that the extracted work will be the half of before, as the work required to \( R \) in order to turn off the interaction. Therefore

\[ G \geq 0 \quad \Rightarrow \quad \eta = \frac{1}{2}(A_n + B_m) \quad (60) \]

Notice that in this toy model subsystems \( A \) and \( B \) are not entangled in the final state \( \rho_S(t_4) \).

E. A Toy Model for optimal efficiency

We now make an example of an interaction Hamiltonian of the form \( (n, m > 0) \)

\[ H_I = G |A_0\rangle\langle A_0| \otimes |B_0\rangle\langle B_0| + L |A_n\rangle\langle A_n| \otimes |B_m\rangle\langle B_m| \quad (53) \]

When the two systems start interacting \((t = t_2)\) the total free energy is \( G \quad (G < \min\{A_1, B_1\}) \). We impose that

\[ L = G - A_n - B_m - \epsilon, \quad \epsilon > 0, \quad n, m \neq 0 \quad (54) \]

which implies that there is only one ground state \((|A_n\rangle |B_m\rangle)\) to which the system is driven by the bath. By definition, \( L < 0 \). When we turn off the interaction system \( R \) has performed some work on system \( S \) equal to

\[ W = A_n + B_m \quad (55) \]

FIG. 5. This graph shows the work extracted, \( W/\omega \) (continuous black line), and the entanglement between \( A \) and \( B \) (dashed blue line) as functions of the coupling parameter \( g/\omega \) for \( \delta = (2\Delta - \omega)/\omega = 0 \). Similar results are obtained for different values of the detuning.

Log \( \log_2 \) of the reduced states of \( A \) and \( B \) as entanglement quantifier, we see in Fig. 5 that in our system the extracted work increases as the entanglement increases (the same behaviour is obtained for different values of the detuning \( \delta \)). Nonetheless, the link between entanglement and work extraction in this protocol is non trivial and, in general, depends heavily on the form of the Hamiltonian.

We think that this is a topic that merits to be studied deeper.

A way to improve this protocol is to find systems for which the efficiency of the protocol is higher than in the quantum Rabi model case. In the next section we apply the protocol to a toy model which gives the highest possible efficiency \((\eta = 1)\) without the presence of entanglement in the final state \( \rho_S(t_4) \).
IV. CONCLUSIONS

In this paper, we have proposed a protocol of work extraction from a system \( R \) based on the thermalization of a composite system \( S \). We first described the protocol in the general case without need of referring to a specific implementation. The protocol described should be easily implementable because the only requirement is to turn on and off the interaction between subsystems \( A \) and \( B \) in a short amount of time without changing their states. Results from thermodynamic resource theory have been then used to define the optimal extracted work \( W \) and the optimal efficiency \( \eta \).

Then, we applied this protocol to a system \( S \) described by the Rabi Hamiltonian at zero temperature. In this limit the extraction of work is identified with the increase of the average energy of subsystems \( A \) and \( B \) playing the role of work storage \( S \) at the expenses of system \( R \). We showed that with this realistic Hamiltonian, when the interaction is of the same order of magnitude of the typical energies of the subsystems, the protocol allows for an extraction of work which is comparable with those energies with an efficiency higher than 1/2.

We believe it would be interesting to generalise our study to the case of \( N \) two-level systems interacting with an harmonic oscillator, always using the Rabi model. This would allow one to study if the coherences that would be built among the two-level systems allow for a greater work extraction. It is reasonable to expect that our protocol can also be applied to a system described by the Rabi Hamiltonian at arbitrary temperature and that the link between entanglement and work extraction could be investigated in more detail. Finally, further studies could lead to a complete characterisation of all the parts of the protocol and of its possible experimental implementation with specific systems.

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[1] S. Vinjanampathy and J. Anders, Contemp. Phys. 57, 545 (2016)
[2] S. Popescu, A. J. Short, and A. Winter, Nat. Phys. 2, 754-758 (2006)
[3] I. T. D. Rickles and N. Y. Halpern, Phys. Rev. Lett. 106, 031105 (2011)
[4] S. Popescu, A. J. Short, and A. Winter, Nat. Phys. 2, 754-758 (2006)
[5] G. P. Beretta, Reports Math. Phys. 64, 139 (2009)
[6] G. P. Beretta, Phys. Rev. E 90, 042113 (2014)
[7] B. Militello, Phys. Rev. E 97, 052113 (2018)
[8] H. T. Quan, Y.-X. Liu, C. P. Sun, and F. Nori, Phys. Rev. E 96, 031105 (2007)
[9] H. T. Quan, Y. D. Wang, Y. X. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. 97, 180402 (2006)
[10] S. W. Kim, T. Sagawa, S. De Liberato, and M. Ueda, Phys. Rev. Lett. 106, 070401 (2011)
[11] T. E. Humphrey, R. Newbury, R. P. Taylor, and H. Linke, Phys. Rev. Lett. 89, 116801 (2002)
[12] B. Leggo, B. Bellomo, and M. Antezza, Phys. Rev. A 91, 012117 (2015)
[13] R. Gallego, J. Eisert, and H. Wilming, New J. Phys. 18, 103017 (2016)
[14] M. Horodecki and J. Oppenheim, Nat. Commun. 4, 1 (2013)
[15] P. Skrzypczyk, A. J. Short, and S. Popescu, Nat. Commun. 5, 1 (2014)
[16] A. E. Allahverdyan, R. Balian, and T. M. Nieuwenhuizen, Europhys. Lett. 67, 565 (2004)
[17] A. Mukherjee, A. Roy, S. S. Bhattacharya, and M. Banik, Phys. Rev. E 93, 052140 (2016)
[18] H. Terças, S. Ribeiro, M. Pezzutto, and Y. Omar, Phys. Rev. E 95, 022135 (2017)
[19] L. Fusco, M. Paternostro, and G. De Chiara, Phys. Rev. E 94, 052122 (2016)
[35] H. Wilming, R. Gallego, and J. Eisert, *Entropy* **19** (2017).

[36] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano, *Phys. Rev. Lett.* **105**, 263603 (2010).

[37] N. K. Langford, R. Sagastizabal, M. Kounalakis, C. Dickel, A. Bruno, F. Luthi, D. J. Thoen, A. Endo, and L. Dicarlo, *Nat. Commun.* **8** (2017).

[38] C. Maissen, G. Scalari, F. Valmorra, M. Beck, J. Faist, S. Cibella, R. Leoni, C. Reichl, C. Charpentier, and W. Wegscheider, *Phys. Rev. B* **90**, 205309 (2014).

[39] E. K. Irish, *Phys. Rev. Lett.* **99**, 173601 (2007).