Clustering, intermittency and scaling for passive particles on fluctuating surfaces

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We show that a scaling approach successfully characterizes clustering and intermittency in space and time, in systems of noninteracting particles driven by fluctuating surfaces. We study both the steady state and the approach to it, for passive particles sliding on one-dimensional Edwards-Wilkinson or Kardar-Parisi-Zhang surfaces, with particles moving either along or against the growth direction in the latter case. Extensive numerical simulations are supplemented by analytical results for a sticky slider model in which particles coalesce when they meet. Results for single particle displacement versus time show to what extent particle dynamics is slaved to the surface, while scaling properties of the probability distribution of the separation of two particles have important implications for replica symmetry breaking for a pair of trajectories. For the many-particle system, clustering in steady state is studied via moments of particle number fluctuations in a single stretch, revealing different degrees of spatial multiscaling with different driving. Temporal intermittency in steady state is established by showing that the scaled flatness diverges. Finally we consider the approach to the steady state, and study both the flatness and the evolution of equal-time correlation functions, as in coarsening of phase ordering systems. Our studies give clear evidence for a simple scaling description of the approach to steady state, with a diverging length scale. An investigation of aging properties reveals that flatness is nonmonotonic in time with two distinct branches, and that a scaling description holds for each one.

I. INTRODUCTION

The motion of passive particles advected by a fluid field has long been of interest in nonequilibrium statistical mechanics [11]. More generally, when subjected to a random force field with long-ranged correlations, passive particles exhibit strong clustering in space and time [3–5], characterized by pronounced inhomogeneties in density. The formation of a large cluster increases the local density in that region and decreases the density around it. Visits of the cluster leads to intermittency, characterized by a sudden burst of activity followed by a long period of stasis.

In this paper we study clustering and intermittency in a system of passive particles driven by a fluctuating surface. The passive particles follow active surface slopes without affecting the surface dynamics. Evidently, the degree of clustering and intermittency depends strongly on the nature of driving; the purpose of this paper is to quantify this dependence and study associated scaling properties. We study driving by a Kardar-Parisi-Zhang (KPZ) surface [6, 7] with particles moving either along or against the direction of surface growth [8], and also by an Edwards-Wilkinson (EW) surface [9], and show that there are pronounced differences in the three cases.

The KPZ equation itself today occupies a central place in nonequilibrium statistical physics, in part because of its relationship to other systems, such as directed polymers in a random medium [10] and the Burgers equation [11]. Interestingly, the latter correspondence implies that the problem under study here maps onto that of passive scalars in a highly compressible fluid [3–5, 12, 13].

Due to the high compressibility, passive sliders show strong clustering in steady state. This was first studied by Drossel and Kardar [8, 14] and thereafter by Nagar et al. [15, 16] who focused on the two-point density-density correlation function and single-site probability distribution. Here, our central focus is on clustering and intermittency in the steady state and especially their characterization during the approach to the steady state starting from a homogeneous distribution.

The density profile in typical steady state configurations is shown in Fig. 1 for the three different drivings to illustrate the different degrees of clustering. This type of signal which is distinct from random, self-similar or self-affine signals, arises in contexts ranging from fluid turbulence [17] to studies of cell biology [18–20]. Self-similar or self-affine signals obey regular scaling implying that the exponent of the $q$th-order structure function varies linearly with $q$. By contrast, in the case of interest, there is a deviation away from linearity indicating intermittency. A simple diagnostic is the flatness, the ratio of the fourth moment to the square of the second moment; there is a deviation away from linearity indicating intermittency. A simple diagnostic is the flatness, the ratio of the fourth moment to the square of the second moment; a divergence of flatness for small scaling argument is the hallmark of intermittency.

![FIG. 1. Spatial profiles of the passive particle density in steady state. Typical configurations with KPZ, EW and AA driving are shown.](image-url)
Although the passive particles are noninteracting, they develop strong correlations as they move, as they are subject to the same history of driving by the fluctuating surface. This leads ultimately to anomalously large fluctuations of particle density, which are the central concern of this work. In order to develop this theme, it is useful to ask a number of questions.

- How far does a single particle move in time $t$? If the typical distance moved is $r \sim t^{1/z}$, how is $z$ related to the dynamical exponent $z_s$ of surface fluctuations?

- How does the separation of two particles which start together, vary with time? In the long time limit, what fraction of the time would they be found within a specified finite range?

- For the many-particle system, is there spatial intermittency in steady state and does the density profile show multiscaling? Do fluctuations in steady state exhibit temporal intermittency?

- Starting from a random distribution, how do clustering and intermittency build up during the coarsening regime describing the approach to steady state? Is there a growing length scale, and if so how do scaling functions differ from those in phase ordering systems, for both coarsening and aging?

The answers to these questions depend on whether the driving surface obeys EW or KPZ dynamics, and if the latter, then whether particles move along (advection) or against (antiadvection, AA) the direction of surface growth. For advection, earlier work has shown that the driving surface obeys EW or KPZ dynamics, and if the latter, then whether particles move along (advection) or against (antiadvection, AA) the direction of surface growth. For advection, earlier work has shown that the driving surface obeys EW or KPZ dynamics, and if the latter, then whether particles move along (advection) or against (antiadvection, AA) the direction of surface growth.

The local density evolves through

$$\frac{\partial \rho}{\partial t} = \kappa \nabla^2 \rho + a \nabla (\rho \nabla h) + \nabla \xi(x,t)$$

(1)

In this work, we are interested in the time dependent statistical properties of a scalar field, namely, the density of particles $\rho(x,t)$, driven by a fluctuating height field $h(x,t)$, but which has no back effect on the driving field. The local density evolves through

$$\frac{\partial \rho}{\partial t} = \kappa \nabla^2 \rho + a \nabla (\rho \nabla h) + \nabla \xi(x,t)$$

(1)
FIG. 3. World lines of passive particles with KPZ, EW and AA drivings are shown from left to right, respectively.

where \( \kappa \) is the diffusivity and \( \xi(x, t) \) is a Gaussian noise, uncorrelated in space and time. The second term in Eq. (1) indicates that particles follow the surface slopes with a coupling strength \( a \), the value of which is +1 or −1, for advection or antiadvection dynamics, respectively. The surface evolution is taken to follow the KPZ equation

\[
\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, t)
\]  

(2)

which describes a growing, fluctuating interface, where \( \nu \) and \( \eta(x, t) \) are the surface tension and spatio-temporal uncorrelated Gaussian noise, respectively. \( \lambda \) is the strength of the nonlinearity which arises for a growing, fluctuating KPZ interface. On setting \( \lambda = 0 \), Eq. (2) reduces to the Edwards-Wilkinson (EW) equation, in which the average surface height does not change in time. On substituting \( \nabla h(x, t) = -u(x, t) \) and setting \( \lambda = 1 \) in Eq. (2), one obtains the vorticity-free noisy Burgers equation [7], where \( u(x, t) \) is the velocity of Burgers fluid.

1. Passive Slider Model on a Lattice

In this work, we consider a discrete lattice model in one dimension, with bonds inclined upward (\( / \)) or downward (\( \backslash \) ). The two ends of a finite lattice of length \( L \) are connected via periodic boundary conditions so that \( h(0) = h(L) \) and we have an equal number of upward and downward slopes. The dynamics involves local hills (\( / \)) stochastically transforming into local valleys (\( \backslash \)) at rate \( u_1 \) and local valleys (\( \backslash \)) into local hills (\( / \)) at rate \( u_2 \). This is the single step model for a discrete surface[26], which can be mapped to the asymmetric simple exclusion process (ASEP) by associating an upward-tilted bond (\( / \)) with a particle and a downward-tilted bond (\( \backslash \)) with a hole. On large enough length and time scales, the surface is described by the KPZ equation if \( u_1 \neq u_2 \) and by the EW equation if \( u_1 = u_2 \).

FIG. 4. A schematic diagram to show the elementary moves of the surface and particles for KPZ, EW and AA drivings, shown from left to right, respectively.

Initially, the passive particles are distributed randomly over the surface sites between successive bonds. Particles move independently and there is no restriction on the number of particles on each site. Each particle moves stochastically down the fluctuating surface, one step at a time, following the local surface slope. The total number of particles \( N \) is taken to be equal to the total number of sites \( L \) which makes the average particle density \( \langle \rho \rangle = 1 \).

In order to mimic Eq. (1), the rules for particle update are the following. A randomly chosen particle slides down a bond; if atop a local hill (\( / \)), it slides down one of the two bonds with equal likelihood, and if in a valley (\( \backslash \)), it does not move. Antiadvection corresponds to the case \( u_1 < u_2 \), in which case the surface grows upwards, opposite to the direction of particle sliding.

In short, particles always slide down the surface slopes,
whereas the surface itself grows downward for advection, fluctuates around the mean in the EW case, and grows upwards for antiadvection. In Fig. 3 we show the world-lines of passive particles with KPZ, EW, and AA dynamics to illustrate the typical evolution. Significant differences in the amount of clustering are apparent in Fig. 2; the quantification of this feature and how it develops in time is the major concern of this paper.

2. Sticky Slider Model

We also introduce a simpler model, namely a sticky slider model (SSM): once the particles come to the same site, they stick together and then move on the surface as a single entity. Updation rules for the numerical simulation of the SSM are the following. Instead of randomly choosing an individual particle (as in the PSM), for the SSM a randomly chosen individual cluster slides on the surface and eventually, in the steady state, they merge to form a single cluster which then moves on the fluctuating surfaces.

III. SINGLE PARTICLE ON A FLUCTUATING SURFACE

In this section, we consider a single particle driven by the three types of fluctuating surfaces. In particular, we seek to elucidate the connection between the dynamical exponents for the driving surface and the particle displacement.

The equation of motion of a passive particle is

$$\frac{dr(t)}{dt} = -a \frac{\partial h}{\partial x} \big|_{x=r(t)} + \eta(t)$$

where $r(t)$ is the position of the particle and the slope $\partial h/\partial x$ of the surface is evaluated at $r(t)$. The particle moves either along (when $a$ is positive) or opposite to (when $a$ is negative) the growth direction of the surface. In Eq. 2, $\eta(t)$ is a white noise which acts on the particle at $r(t)$.

The most common and useful measure of the motion of particles is the root mean-squared displacement (RMSD)

$$\mathcal{R}(t) = \langle [r(t + t') - r(t')]^2 \rangle^{1/2}$$

where $t'$ is the starting time. Evidently, $\mathcal{R}(t)$ depends on the surface dynamics; however, it is not so clear whether the surface dynamical exponent $z_s$ alone dictates the growth of $\mathcal{R}(t)$. In the asymptotic limit of system size $L \to \infty$, the RMSD is expected to follow

$$\mathcal{R}(t) \sim t^{1/z}$$

in the limit $t \ll L^{z_s}$ and $L \to \infty$. As we shall see below, the exponent $z$ may differ from $z_s$.

As discussed in the introduction, a central role is played by a growing length scale $\mathcal{L}(t)$ in a scaling description of coarsening and aging, during the approach to steady state. Later, we will see in fact $\mathcal{L}(t) \sim \mathcal{R}(t)$, which underscores the importance of estimating the growth law for $\mathcal{R}(t)$.

A. KPZ driving

Let us summarize earlier predictions for the growth of $\mathcal{R}(t)$.

Bohr and Pikovsky [22] studied the RMSD of passive particles advected by the noisy Burgers fluid, within a mean-field approach to the scaling form of the two-point velocity. They obtained a self-consistent asymptotic solution $\mathcal{R}(t) \sim t^{1/z}$ with $z = 3/2$.

Drossel and Kardar performed a numerical simulation of a restricted solid-on-solid model belonging to the KPZ universality class, coupled with the passive particle dynamics; they found $z = 3/2$ [14].

Later, the same value of the dynamical exponent was obtained by modeling the surface dynamics through the Kim-Kosterlitz model [21] and the single step model [13]. As emphasized in [21], with KPZ driving the particle motion becomes slaved to the fluctuations of the surface so that $z = z_s$. Our results from numerical simulations are consistent with $z = z_s = 3/2$.

B. EW driving

Let us turn to EW surface dynamics, in which case the motion of the particle is less strongly coupled to the surface fluctuations.

A numerical simulation of particles driven by an unbiased single step model was carried out by Manoj [27]. He found $\mathcal{R}(t) \sim t^{1/z}$ where $z$ shows an apparent dependence on the ratio $\omega$ of update rates for the surface and particle evolution, varying from $1/z \simeq 0.67$ for $\omega \gg 1$ (rapid surface motion) to $1/z \simeq 0.56$ for $\omega = 1$, and to $1/z \simeq 0.50$ for $\omega \ll 1$ (rapid particle motion) [27].

However, the apparent dependence on $\omega$ may result from crossover effects, with the true form involving multiplicative logarithms:

$$\mathcal{R}^2(t) \sim t f \left( \frac{t}{t_0} \right)$$

where $f(t/t_0)$ indicates a multiplicative logarithmic correction. In fact, such logarithmic corrections have been proposed earlier. Bohr and Pikovsky [22] studied a linear version of the noisy Burgers equation (tantamount to EW dynamics) for which the velocity-velocity correlation function is known. Within a mean-field approach, they concluded $f(t/t_0) \sim \ln[1+(t/t_0)^{1/2}]$ where $t_0$ depends on the model parameters of the system. This would imply $\mathcal{R}^2(t) \sim t^{3/2}$ for $t \ll t_0$, and $\mathcal{R}^2(t) \sim t \ln(t)$ as $t \to \infty$. 
In recent work, Huveneers suggests that $f(t)$ may follow $\sim |\ln(t)|^\alpha$ with $0 \leq \alpha < 1$ but raises the possibility that $\alpha > 0$ may be a transient effect.\[29\].

The results of our Monte Carlo simulations are shown in Fig. 5. To distinguish between competing predictions\[22, 23, 27\], we have plotted $R^2(t)/(tf(t/t_0))$ with (a) $f(t/t_0) \sim t^\theta$ with $\theta \approx 0.12$ as in\[24\] for $\omega = 1$ (b) $f(t/t_0) = \ln(t/t_0)$ as in\[22\] (c) $f(t/t_0) = [\ln(t/t_0)]^\alpha$ with $\alpha = 1/2$. The results in Fig. 5 seem to indicate $\alpha = 1/2$, but there is a degree of uncertainty. In the remainder of this work, for convenience we use an effective dynamical exponent $1/z \approx 0.56$ for EW driving as proposed by Manoj\[27\], recognizing that the different estimates of $f(t)$ do not lead to substantial differences in the ranges to be considered.

### C. Antiadvection

For the case of antiadvection (AA), the exponent $z$ of the particles was estimated numerically by Drossel and Kardar\[8\] and found to be nonuniversal, changing continuously with $a$\[8, 14\]. For $a = 1$ they obtained $z \approx 1.74$ consistent with the numerical findings in\[15, 24\] as also in the current work. As the coupling constant $a$ decreases from 1 to 0, the exponent $z$ increases from approximately 1.74 to 2, with $a = 0$ corresponding to a simple random walk.

Note that single particle dynamics with different sorts of driving also determines the dynamics of SSM in the steady state, as in that case there is a single cluster. Evidently, the motion of this cluster is exactly that of a single particle.

### IV. TWO PARTICLES

Here we study how the separation of two passive particles changes with time when they start from the same location. This is quantified by studying the time evolution of the probability distribution of their separation. A slight variant of this problem, namely with two-second class particles, was studied by Derrida et al.\[28\] as discussed below. More recently, the overlap of the trajectories of two passive scalars with KPZ driving was studied in\[29\] from the viewpoint replica symmetry breaking (RSB). In this section, we show that scaling properties of the probability distribution of the separation has strong implications for the evolution of the average of the overlap function of trajectory pairs driven by KPZ, EW, and AA driving.

### A. PDF of separation between two particles

The interplay of advection and independent noise cause a pair of trajectories to overlap during one part of the evolution and deviate from each other during other parts. A measure of the closeness of the trajectories is the PDF of the interparticle separation

$$r_s(t) \equiv |x^{(1)}(t) - x^{(2)}(t)|,$$

which has been studied in\[29\], where an interesting coexistence of a high-overlap and low-overlap regimes in space-time were found. We studied the PDF of $r_s$ for KPZ, EW and AA drivings and found power-law decays with different exponents for different drivings. Interestingly, the PDFs turn out to be scaling functions of $r_s$ and $t$. Data for different times can be collapsed for each of the drivings, as shown in Fig. 6 when $r_s$ is scaled by $\mathcal{L}(t)$ and $P(r_s, t)$ is scaled with $[\mathcal{L}(t)]^{1+\theta}$ with $\theta \approx 1/2$ for KPZ and $\theta = 0$ for EW and AA drivings. The scaled PDF can then be expressed as

![FIG. 6. Probability distribution of separation $r_s$ between two particles with KPZ, EW, and AA driving. The PDFs for different times collapse when the separation $r_s$ is scaled by the corresponding $\mathcal{L}(t)$ and $P(r_s, t)$ with $[\mathcal{L}(t)]^{1+\theta}$.](image-url)
\[ P(r_s, t) \approx \frac{1}{[\mathcal{L}(t)]^{1+\theta}} Y \left( \frac{r_s}{\mathcal{L}(t)} \right) \quad (7) \]

where the scaling function follows \( Y(y) \sim y^{-\nu} \) as \( y \to 0 \), and falls exponentially as \( y \to \infty \). The exponent values corresponding to the power-law decay are estimated as \( \nu \approx \frac{2}{3}, \frac{4}{3}, \frac{1}{3} \) for KPZ, EW and AA drivings, respectively. These estimated values are consistent with the steady state values \([15]\), if \( \mathcal{L}(t) \) in Eq. \( 7 \) is replaced by system size \( L \).

Ueda and Sasa \([29]\) had numerically found that for KPZ driving, the mean-squared separation follows

\[ \langle r_s^2 \rangle \sim t, \quad (8) \]

while at the same time \( P(r_s, t) \) approaches a constant value for \( r_s < r_s^* \) where \( r_s^* \) is fixed. It should be noted that both these properties are an immediate consequence of the scaling form of Eq. \( 7 \) \([30, 31]\). On the other hand, for EW and AA drivings, we see that the time dependence of \( \langle r_s^2 \rangle \) follows \( \mathcal{R}^2(t) \) which is numerically verified as shown in Fig. \( 7 \).

From the scaling form of Eq. \( 7 \) along with the corresponding values of \( \nu \) and \( \theta \), we conclude that there exists a limiting form for the PDF for KPZ driving for large \( t \), whereas PDFs decay with time and eventually vanish for EW and AA drivings (Fig. \( 8 \)). Thus, for KPZ driving, given any value of separation \( r_s \), the distribution \( P(r_s, t) \) approaches a time-independent value \( P_{ss}(r_s) \sim r_s^{-3/2} \) for times \( t \gg r_s^* \). For instance, we consider PDFs \( P(r_s = 0, t) \), shown in Fig. \( 8 \) which quickly approach a constant value for KPZ driving and decay as a power with different exponents for EW and AA driving.

It should be noted that the problem of two passive particles on a fluctuating KPZ surface is closely related to the problem of two second-class particles in the ASEP which has been exactly solved in \([28]\). The probability of finding two second-class particles at distance \( r_s \) apart has been found to follow \( P(r_s) \sim \frac{1}{r_s^2} \) for large \( r_s \) and the limit \( L \to \infty \). Let us first discuss the differences and similarities between the models. The rules of hopping of a single second class particle in the usual ASEP are the same as the rules of advection of a passive particle on a KPZ surface (on mapping particle hole occupancies in the ASEP to uphills and downhills in the KPZ dynamics). Two second-class particles cannot overlap, and behave like two passive hard core particles sliding on a KPZ interface of length \( L - 2 \).

### B. Overlap function

A good way to quantify the closeness of the trajectories of two particles, is to follow the ‘overlap’ of the trajectories up to time \( t \). To this end, we follow \([29]\) and consider
an overlap function

\[ q_o(t) = \frac{1}{t} \int_0^t dt' \Theta(t - |r_o(t')|) \]  

(9)

where \( \Theta(x) \) is a theta function, \( r_o(t') \) is the separation of two particles at time \( t' \) [29] and \( l \) is a length which is used to qualify whether or not the trajectories do overlap.

Now consider the average overlap up to time \( t \)

\[ \langle q_o(t) \rangle = \frac{1}{t} \int_0^t dt' (\Theta(t - |r_o(t')|)) \]  

(10)

where \( \langle \rangle \) indicates average over independent realizations. Rewriting in terms of \( P(r_o, t) \), we obtain

\[ \langle q_o(t) \rangle = \frac{1}{t} \int_0^t dt' \int_0^t dr_o P(r_o, t'). \]  

(11)

Substituting the scaling form of Eq. 7 in Eq. 11, we obtain

\[ \langle q_o(t) \rangle \sim \left( \frac{t}{l} \right)^{\frac{\nu - 1}{\nu}} t^{-\frac{\nu}{2}}. \]  

(12)

For KPZ driving, substituting \( \nu \approx 3/2 \) and \( \theta \approx 1/2 \) in Eq. 12 we obtain

\[ \langle q_o(t) \rangle \approx \text{const.} \]  

(13)

which is independent of time in the large-distance and long-time limit.

For EW and AA drivings, we have \( \theta \approx 0 \). Therefore Eq. 12 reduces to

\[ \langle q_o(t) \rangle \sim \left( \frac{t}{l} \right)^{\frac{\nu - 1}{\nu}} \]  

(14)

in the asymptotic limit of time. For EW driving, on substituting \( \nu = 0.67 \) and \( z = 2 \) (omitting the logarithmic correction to \( z \) given by Eq. 6), we obtain \( \langle q_o(t) \rangle \sim (t/l)^{-\phi_{EW}} \) with \( \phi_{EW} \approx 0.17 \). Similarly, for AA driving, on substituting \( \nu = 0.33 \) and \( z = 1.75 \) in Eq. 14 we obtain \( \langle q_o(t) \rangle \sim (t/l)^{-\phi_{AA}} \) with \( \phi_{AA} \approx 0.38 \).

In order to verify the dependence of time and localization length \( l \) on \( \langle q_o(t) \rangle \), we carried out a numerical simulation, and find fair agreement with Eq. 12 for all three drivings, as shown in Fig. 10.

V. STEADY STATE : STATICS

In this section, we give an overview of the two-point spatial and temporal correlation functions. Further, we study the structure functions and thereby multiscaleing of particle number in space for the three types of driving.

A. Static Correlation Function

The steady states obtained for the three types of driving show interesting similarities and differences as illustrated in Fig. 11. These are reflected in the two-point density-density correlation function

\[ G_{ss}(r, L) = \langle n_i n_{i+r} \rangle \]  

(15)

where \( n_i \) denotes the total number of particles at \( i \)-th site. Numerical simulations [15, 16] reveal

\[ G_{ss}(r, L) = \frac{1}{L^D} Y_{ss}(r/L), \]  

(16)

i.e. \( G_{ss} \) is a scaling function of separation \( r \) scaled by system size \( L \). This unusual behavior is reminiscent of phase ordering, but the new point is that the scaling function \( Y_{ss}(y) \) is divergent in this case. In Eq. 16, \( Y_{ss}(y) \sim y^{-\nu} \) as \( y \to 0 \) where the exponent \( \nu \) is estimated to be \( \approx \frac{2}{3}, \approx \frac{2}{3} \), and \( \approx \frac{1}{3} \) for KPZ, EW, and AA drivings, respectively. The exponent \( \theta \) is \( \approx \frac{1}{2} \) for KPZ driving ensuring normalizability, while \( \theta \approx 0 \) for EW and AA drivings.

B. Static structure functions and flatness

As is evident from the density profiles shown in Fig. 11, there is a good deal of clustering for all three types of surface driving, though the degree of surface clustering seems to vary substantially from one case to the other. In order to quantify this, we study the moments of particle numbers \( N_i \) in a stretch of \( l \) successive sites in steady state. A good idea of clustering is obtained by studying the dependence of the \( q \)th order moment on the stretch length \( l \):

\[ R_q^{ss}(l) = \langle N_i^q \rangle \sim l^{\zeta(q)}, \]  

(17)

where \( \langle \rangle \) indicates the average over steady state configurations. We choose the stretch length to be a finite fraction of system size with \( l/L = 1/2^7, 1/2^6, 1/2^5, 1/2^4 \) for \( L = 4096 \) and 8192.

In Fig. 11 we plot \( R_q^{ss} \) versus \( l/L \) for \( q = 1/2, 1/4, 1/2, 3/4, 2 \) and 4 for PSMs with KPZ, EW, and AA drivings. Measuring the slope of \( R_q^{ss}(l) \) with \( l/L \), we numerically determine \( \zeta(q) \) for PSMs.
with \( q \) which implies that the smaller signals are self-similar while \( \zeta(q) \) for \( q > 1 \) is not linear with \( q \) indicating multiscaling.

It is interesting that though the \( \zeta(q) \) versus \( q \) plot shows a significant difference between KPZ and EW drivings, the flatness varies in a similar way for both. If we define the exponent \( \alpha \) through

\[
\kappa_4 = \frac{R_{ss}^{4s}}{(R_{ss}^{2s})^2} \sim \left( \frac{l}{L} \right)^{-\sigma}
\]  

the value of \( \sigma \) is \( \simeq 1 \) for KPZ and EW driving whereas \( \sigma \simeq 0.75 \) in the for AA case.

![Figure 11: Variation of \( q \)-th order structure functions \( R_{ss}^{q}(l) \) with stretch length \( l \). For the purpose of presentation, we rescaled \( R_{ss}^{q}(l) \) by a factor of \((8192)^{1/2}, (8192)^{2} \) and \((8192)^{4}\) for \( q = 1/2, 2 \) and \( 4 \), respectively.](image)

![Figure 12: \( q \)-dependence of exponents \( \zeta(q) \) which define the growth of structure functions. Particles show a larger degree of clustering with KPZ driving than in the EW case. The two dashed lines depict the limits of the no clustering (random) and extreme clustering (sticky slider model). Error-bars are smaller than the size of the symbols.](image)

![Figure 13: Flatness \( \kappa_4 \) in steady state as a function of \( l/L \). \( \kappa_4 \) diverges as a power law with exponent \( \sigma \simeq 1 \) for KPZ and EW driving, and \( \sigma \simeq 0.75 \) for AA driving.](image)

VI. STEADY STATE: DYNAMICS

Clustering of particles is intimately related with intermittency in space and time as clustering at some spatial regions causes depletion of particles in other regions, leading to strong temporal fluctuations in the number of particles within a fixed spatial stretch. Typical time series of particle number \( N_i \) in a stretch \( l \) is shown in Fig. 2

A. Dynamic correlation functions

The time dependent density-density auto-correlation function

\[
G_s(t,L) = \langle n_i(0)n_i(t) \rangle
\]  

has been studied [16], and found to follow the scaling form

\[
G_s(t,L) \sim \tilde{Y}(t/L^z)
\]

where \( \tilde{Y}(y) \sim \tilde{y}^{-\nu} \) as \( \tilde{y} \to 0 \). The estimated values of the exponent \( \nu \) are \( \simeq \frac{2}{3}, \simeq \frac{1}{4} \) and \( \simeq 0.19 \) for KPZ, EW, and AA drivings, respectively [16].
B. Dynamic structure functions and intermittency

We now present numerical results for particle number fluctuations in a stretch of the lattice for PSMs and SSMs. We show that a scaling description holds for structure functions, and support this through analytical arguments for the SSM. We also compare our numerical results for PSMs with KPZ, EW and AA dynamics with analytical and numerical results for the corresponding SSMs.

The time-dependent $q$-th order structure function of particle number fluctuations in steady state is given by

$$S_q^{ss}(t_0, t, l) = ⟨[N_l(t_0 + t) - N_l(t_0)]^q⟩$$

where the condition $t_0 \gg L^z$ is imposed to guarantee for steady state. Here $N_l(t)$ is the total number of particles at time $t$ in a stretch length $l$, which we take to be a finite fraction of the system size. We consider a large value of $l$ as clusters may be spread out, both in the steady state and coarsening regime (which is discussed in the next section). Associated with the stretch length $l$, there is a time scale $\tau_l$ beyond which particle number fluctuations are uncorrelated. Consequently the structure function $S_q^{ss}(t_0, t, l)$ saturates for $t > \tau_l$.

Representative time series of particle number $N_l$ for the three types of dynamics are shown in Fig. 2. We monitor the ratio of fourth ($S_4$) and square of second moment ($S_2$), namely flatness $\kappa^{ss} \equiv S_4^{ss}/(S_2^{ss})^2$. Intermittency is indicated by the divergence of $\kappa^{ss}$ in the limit $t/l^z \to 0$ [17].

![FIG. 14. The divergence of flatness indicates temporal intermittency in steady state for both PSM and SSM. For ease of display, $\kappa_4^{ss}$ is multiplied by constant factors, namely 1.5, 2.5, 2 and 3 for PSM-EW, PSM-KPZ, SSM-EW and SSM-KPZ, respectively.](image)

Passive slider model: We study the flatness for different values of system size $L$ and stretch length $l$, i.e., $l/L = 1/8, 1/16$ and $1/32$ for $L = 1024$ and $2048$ for the PSM with KPZ, EW and AA dynamics. In the scaling limit, the flatness diverges with a power law $t/\tau_l \to 0$ and saturates as $t/\tau_l \to \infty$. The saturation values depend on $L/l$, but collapse when time $t$ is scaled by $\tau_l$ and $\kappa^{ss}$ is scaled by $(L/l)^q$. Since particles are non-interacting, the time scale $\tau_l$ is determined by the time taken by a single particle to cover a distance $l$, i.e., $\tau_l \sim l^z$, where the dynamical exponent $z$ depends on the surface driving as we have seen in Section 2.

This results in the compact scaling form

$$\kappa^{ss} \sim (L/l)^q F_{PSM}\left(\frac{t}{\tau_l}\right)$$

where $F_{PSM}(y) \sim y^{-\gamma}$ as $y \to 0$ and $F_{PSM}(y) \to \text{const}$ as $y \to \infty$. As seen in Fig. 14, our numerical simulations estimate $\phi \approx 1$ for both KPZ and EW driving, and $\phi \approx 0.75$ for AA. Similarly, exponents corresponding to $F_{PSM}(y)$ have the values $\gamma \approx 0.67$, $0.50$ and $0.40$ for KPZ, EW and AA driving, respectively (see Table I).

Sticky slider model: To get some insight into the occurrence of scaling, we study the SSM [25], defined by the rule that particles which find themselves on the same site undergo irreversible aggregation and do not separate once they are together. Starting from a configuration with random placement of particles, the number of clusters decreases in time in the coarsening regime, finally reaching a single cluster which moves all over the system in steady state. By design, the SSM is a model of extreme clustering which is simple enough that one can understand the origin of scaling through analytic arguments. On the quantitative front, the SSM for KPZ driving resembles the corresponding PSM fairly closely, whereas the SSM and PSM for both EW and AA cases differ substantially from each other. This is borne out by the results shown in Table I.

In the steady state of the SSM, a single aggregate $A_N$ with $N$ particles slides stochastically on a 1D stochastically evolving surface of size $L$. Its motion is identical to that of a single walker, so in time $t$, its typical displacement $R(t) \sim t^{1/z}$ implying that $A_N$ takes time $\tau_N \sim l^z$ to traverse the stretch length $l$. In order to estimate $S_4^{ss}$ given in Eq. 21 for SSMs, let us consider the location $R_0$ and $R$ of $A_N$ at times $t_0$ and $(t_0 + t)$ respectively. The probability that $R_0$ is inside the stretch $l$ is $l/L$ in which case the probability of $R$ falling outside $l$ is of the order of $p_1 = \frac{R(t)}{L}$. Likewise, when $R_0$ is outside $l$, the probability of $R$ falling inside $l$ is $p_2 = (1 - \frac{R(t)}{L})$. Hence, the $q$-th order structure function is given by

$$S_q^{ss} = p_1 N^q + p_2 N^q.$$  

Hence, by considering $N = L$ (which guarantees unit global density), we find that the flatness $\kappa^{ss}(t) = S_4^{ss}/(S_2^{ss})^2$ is given by

$$\kappa^{ss} \approx \frac{L}{R(t)} \sim \frac{L}{t^{1/z}}.$$  

Thus for the SSM, the distinction between different surface driving enters only through the values of $z$ for the different models. Similarly, the higher order normalized cumulants can be calculated straightforwardly.
Figure 14(b) shows that the flatness for the SSM for different surface drivings is given by

\[ \kappa_{ss}^* \sim (L/l)^\phi F_{SSM} \left( \frac{t}{\tau} \right) \]  

(25)
as for the PSM, but with different exponents (Table I). The scaling function \( F_{SSM}(y) \sim y^{-1/2} \) as \( y \to 0 \) and \( F_{SSM}(y) \to \text{const} \) as \( y \to \infty \). The exponent \( 1 + z \simeq 0.67 \), \( \simeq 0.56 \) and \( \simeq 0.57 \) in the KPZ, EW and AA cases respectively. Further, the scaling functions for \( \kappa^* \) are different for different drivings.

For KPZ driving, the decay exponent \( \gamma \) of \( F_{PSM}(y) \) in Eq. (22) is the same as \( 1/z \) of \( F_{SSM}(y) \), but it is substantially different for PSMs with EW and AA drivings. Exponent values for PSMs and SSMs for the three types of driving are given for PSMs and SSMs in Table I.

### VII. COARSENING REGIME

#### A. Correlation function

We now turn to an important question: How is the intermittent steady state approached in time, starting from a state in which passive particles are distributed randomly in the system? Initially, particles tend to move to the closest local minima (for KPZ and EW driving) or maxima (for AA) of the co-evolving surface. As time \( t \) passes, each passive particle typically move to a deeper valley (or hill) a distance \( \mathcal{R}(t) \sim t^{1/2} \) from its starting point, as discussed in Section 2. Therefore, in time \( t \), particles from a catchment region of length \( \mathcal{L}(t) \) collect near the valley bottom. Evidently, \( \mathcal{L}(t) \) is of order of \( \mathcal{R}(t) \), and since we start with random placement of the particles, the typical number of particles in the catchment region is \( \rho \mathcal{L}(t) \). With unit density, this reduces to \( \mathcal{L}(t) \).

The typical number of particles in a cluster increases with time, reminiscent of phase ordering dynamics, where
ordered domains grow in time. We study the two-point density density correlation $G_c(r,t) = \langle n_i(t)n_{i+r}(t) \rangle$ where $n_i(t)$ denotes the number of particles at $i$-th site at time $t$. Numerical simulations of $G_c(r,t)$ for KPZ, EW, and AA drivings (fig. 16) show that data for different times exhibit a scaling collapse when the separation $r$ is scaled by the growing length scale $\mathcal{L}(t)$ and $G_c(r,t)$ is scaled with $\mathcal{L}^{\theta}(t)$. Our numerical simulations indicate $\theta \simeq 0.50$ for KPZ and $\simeq 0$ for both EW and AA drivings.

![FIG. 15. Time dependent flatness in the adiabatic limit. The divergence of $\kappa^a_d$ indicates intermittency. However, the corresponding exponents differ from those of the corresponding PSMs.](image)

![FIG. 16. Correlation function in the coarsening regime. $G(r,t)$ diverges in the limit $r/\mathcal{L}(t) \to 0$ in all cases.](image)

![FIG. 17. Time evolution of the density-density autocorrelation function in the coarsening regime.](image)

![FIG. 18. In the coarsening regime, flatness diverges as $t$ increases. For ease of display, $\kappa_4$ is multiplied by 1.5 for PSM-EW and SSM-EW, and 2.5 for PSM-KPZ and SSM-KPZ.](image)

**TABLE II.** Exponents values corresponding to the correlation function

| System    | $\theta$ | $\nu$ | $\delta/z$ |
|-----------|----------|-------|------------|
| PSM-KPZ   | 0.50     | 1.50  | 0.67       |
| PSM-EW    | 0        | 0.67  | 0.32       |
| PSM-AA    | 0.33     | 0.33  | 0.17       |

The scaling form of the correlation can be written as

$$G_c(r,t) \sim \frac{1}{\mathcal{L}^\theta(t)} Y_{\nu}(\frac{r}{\mathcal{L}(t)})$$  \hspace{1cm} (27)
where \( Y_c(y) \sim y^{-\nu} \) as \( y \to 0 \) which indicates the divergence of \( G_c(r, t) \), with \( \nu \simeq 1.50 \), \( \simeq 0.67 \), and \( \simeq 0.33 \) for the PSM with KPZ, EW, and AA driving, respectively; the different exponent values quantify the spreading of the clusters for different drivings. AA driving leads to a relatively large spread while KPZ shows the least and EW lies in between.

The scaling form for two-point density correlation function in Eq. [27] is consistent with the steady state two-point density density correlation given in Eq. 15 of Ref. [14] when the system size \( L \) in [14] is replaced by \( L(t) \). As for the steady state, we find \( G_c(0, t) \sim L^2(t) \) with \( \delta/z \simeq 0.67, 0.33, \) and \( 0.17 \) for KPZ, EW and AA, respectively as shown in Fig. [17] In Table II we present the exponent values associated with \( G_c(r, t) \). On comparing the values of steady state \( \delta \) with our estimated \( \delta/z \) (in the coarsening regime), we find good agreement for KPZ and for AA cases, while \( \delta \) for EW is smaller than its corresponding value in the steady state.

**B. Structure functions and Intermittency**

To track the intermittent signal as the system evolves from an initially random state towards the clustered steady state, we monitor the flatness of the distribution of particle number fluctuation \( [N_i(t) - N_i(0)] \) in a stretch length \( l \). We study the \( q \)th order structure function \( S_q^r(t, l) = \langle [N_i(t) - N_i(0)]^q \rangle \) for \( q = 2 \) and \( q = 4 \), and the flatness \( \kappa_4 \) for both the PSM and SSM models with KPZ, EW and AA driving.

For the SSM, we argue that \( S_q^r(t, l) \) obeys scaling, and compare with numerical simulations. Recalling that particles are drawn into basins of typical size \( L(t) \), let us assume that there is a single SSM cluster with \( \overline{L(t)} \) particles in each such basin. At early time, when \( L(t) \ll l \), particle movements within \( l \) do not affect \( N_i \) and number fluctuations arise from random motion of particles in and out of the edges. Thus the statistics of the number fluctuations are Gaussian (and the value of the flatness \( \kappa^c = 3 \)). This ceases to hold once \( t \) is large enough that \( L(t) \sim l \). Once \( L(t) \gg l \), the probability that the SSM cluster falls within the \( l \)-stretch is \( \frac{1}{L(t)} \) which implies

\[
S_q^r \sim \frac{l}{L(t)} L(t)^q \quad \text{for} \quad L(t) \gg l
\]  (28)

and leads to the estimate

\[
\kappa^c(t, l) \sim \frac{L(t)}{l}. \quad (29)
\]

Thus, in the ‘coarsening’ regime, in contrast to the steady state, the flatness diverges in the limit \( t/\tau_i \to \infty \). These two limits are captured by the scaling form

\[
\kappa^c \sim h \left( l/\overline{L(t)} \right)
\]  (30)

with \( h(y) \sim y^{-\psi} \) as \( y \to 0 \) and \( h(y) \to \text{const.} \) as \( y \to \infty \). In view of Eq. [29] we expect \( \psi = 1 \). This analytical prediction is well confirmed by numerical simulations of \( \kappa^c(t, l) \) for SSMs with all three drivings (see fig. [18] and Table II).

Motivated by the success of scaling for the SSM, we performed simulations for the PSMs and SSMs for the three types of surface driving and have plotted the scaled data in Fig. [18] We see that scaling holds in all three cases. However, the exponent \( \psi \) coincides with the corresponding SSM value only for KPZ and EW driving. It differs substantially from the SSM prediction for AA driving, reflecting the smaller degree of clustering. Results are summarized in Table II.

**VIII. AGING**

During the process of coarsening, the system exhibits aging, namely changes in the pattern of dynamical evolution as time passes. Traditionally, these changes are studied by monitoring a two-time correlation function between an initial time \( t_0 \) and a final time \( t_0 + t \). This approach has been used in diverse contexts, e.g., phase ordering kinetics [24], and interface evolution [35–38].

Our motivation in studying aging is to investigate the growth of the intermittent signals. To this end, we study \( q \)th order structure function \( S_q(t_0, t, l) = \langle [N_i(t_0 + t) - N_i(t)]^q \rangle \) where \( t_0 \) is the waiting time and \( t \) is a further time difference. We restrict ourselves to \( (t_0 + t) \ll L^2 \) so that the stationary state is not reached. In our numerical study, we work with \( L = 2^{15} \), so that the size effects do not interfere with the study of aging correlations.

With a stretch of length \( l \) is associated a time scale \( \tau_i \sim l^2 \) which becomes very important when it competes with waiting time scale \( t_0 \). This connection leads to two different behaviors of flatness which are discussed below.

**A. \( \tau_i < t_0 \)**

If \( \tau_i \ll t_0 \), the flatness shows two distinct wings (left and right) separated by an intermediate plateau regime shown in Fig. [19] as will be discussed below. The left wings of the curves in Fig. [19] corresponds to a quasi-steady state (QSS) regime, while the right wings correspond to long-time aging (LTA). To understand the nonmonotonicity of flatness, we first study \( \kappa_4(t_0, t, l) \) for SSMs via a probabilistic arguments. Within the SSM, when \( t \ll \tau_i \ll t_0 \), a typical catchment of size \( L(t_0) \) typically contains a single aggregate with \( L(t_0) \) particles. The position of the single aggregate can be anywhere within \( L(t_0) \) (reminiscent of the steady state where a single aggregate moves over the system size \( L \)) which implies that within \( L(t_0) \), a local steady state is reached. Therefore, the structure functions and corresponding flatness in QSS can be estimated by replacing \( L \) by \( L(t_0) \) in Eqs. [23] and
The flatness for SSM in QSS is thus
\[ \kappa_4 \sim (t_0/t)^{1/z}. \]  
(31)

On the other hand, when \( t \gg t_0 \), the right wing of the nonmonotonic flatness where a typical catchment \( L(t_0 + t) \) increases with \( t \) and accordingly, number of particles in an aggregate increases because of the increasing basin size. This process continues until the difference time \( t \ll L^2 \). The \( q \)-th order structure function, defined by Eq. 21 is estimated as
\[ S_q(t_0, t, \ell) \approx \frac{1}{L(t)} [L(t_0 + t)]^q \]  
(32)
where \( t \gg t_0 \). Therefore, Eq. 32 can be approximated and the corresponding flatness is obtained as
\[ \kappa(t_0, t, \ell) \approx \left( \frac{t}{t_0} \right)^{1/z} \]  
(33)
which diverges as \( t \) increases. The extent of the plateau regime is \( \sim (t_0 - \tau_1) \). Substituting \( t = \tau_1 \) in Eq. 31 we get \( \kappa_4 \sim (t_0/\tau_1)^{1/z} \) which smoothly matches with the LTA regime by setting \( t = t_0 \) in Eq. 33.

Using the SSM results as a guide, we now discuss the numerical simulations of the PSMs. Figure 20 shows results of the numerical simulation of PSMs for different values of \( \tau_1 \) and \( t_0 \). In the QSS, the data for several values of \( \tau_1 \) and \( t_0 \) collapse in the limit \( t/t_0 \to 0 \) when \( t \) is scaled by \( t_0 \) shown in Fig. 20(a). The flatness then can be estimated by replacing \( L \) by \( L(t_0) \) in Eq. 22 leading to
\[ \kappa_4(t, t_0, \tau_1) \sim \left( \frac{L(t_0)}{L} \right)^{\phi} \left( \frac{t}{\tau_1} \right). \]  
(34)

For the LTA regime (corresponding to the right hand branch) numerical results for different values of \( \tau_1 \) and \( t_0 \) collapse when separation time \( t \) is scaled by \( \tau_1 \) shown in Fig. 20(b). The flatness diverges in the limit \( t/\tau_1 \to \infty \) for an infinite system. The numerical results for PSM-KPZ follow the scaling form predicted by the SSM, whereas results for the PSM in the EW and AA cases deviate from the corresponding SSMs.

B. \( \tau_1 > t_0 \)

In the less interesting case \( \tau_1 > t_0 \), the number fluctuations in \( t \) increase with \( t \) and consequently, \( \kappa_4 \) increases monotonically with \( t \) as shown in Fig. 21. This monotonic behavior of \( \kappa_4 \) can be identified with the LTA regime in Fig. 19(b).

IX. CONCLUSION

In this work, we have characterized the intermittent steady state of passive particles driven by fluctuating surfaces, and the manner in which it is approached. We have given strong evidence that a scaling description holds for
the three types of surface driving considered (KPZ advection, EW, and KPZ antiadvection), although the scaling functions differ considerably in the three cases, reflecting the different degrees of clustering in both space and time.

It was known earlier that the single particle dynamical exponent $z$, defined through $R(t) \sim t^{1/z}$, coincides with the surface dynamical exponent for KPZ advection $14$, $15$, $21$, $22$, and differs from it for antiadvection $8$, $11$, $15$, indicating that particles are slaved to the surface dynamics in the former case. For EW driving, logarithmic corrections to $z$ were considered earlier $22$, $23$, and our study corroborates this finding. The exponent $z$ is significant for our study of the many-particle system, as it enters in scaling descriptions of correlations, both in steady state and in the coarsening and aging regimes.

In order to understand the correlation between two passive particles, we studied the evolution of the probability distribution $P(r_s,t)$ for their separation for the three different drivings. In $29$ it was found numerically that $P(0,0)$ approaches a nonzero constant as $t \to \infty$ even though $\langle r_s^2 \rangle \sim t$ for large $t$. We showed that these features follow from the fact that $P(r_s,t)$ scales, and is a function of $r_s/\mathcal{L}(t)$ with $\mathcal{L}(t) \sim t^{1/2}$. Interestingly, the time evolution of the average overlap of trajectory pairs, which enters into the discussion of replica symmetry breaking in trajectory space $29$, is also obtained from the scaling form of $P(r_s,t)$.

In the many-particle system, clustering of particles leads to intermittency in space and time. Our numerical study of the phenomenon was supplemented by analytic arguments for a simplified sticky slider model, which suggested scaling forms for the passive particle model with all three drivings. Spatial multiscale was demonstrated numerically in the steady state for all three drivings, with KPZ advection showing the strongest effect, EW driving being intermediate, and antiadvection displaying the weakest effect of the three, but still quite different from the usual scaling. Further, intermittency was also quantified by monitoring the divergence of flatness as a function of scaled distance or time, confirming the sequence of relative strengths.

We studied the approach to the steady state through the time evolution of the two-point density-density correlation function. It is a function of the separation $r$ scaled by $\mathcal{L}(t) \sim t^{1/2}$ where the growing length scale $\mathcal{L}(t)$ describes the spatial extent of the basin from which particles are drawn to form clusters. It also enters in the scaling properties of the time-dependent flatness. We also investigated aging by monitoring the flatness with different waiting times $t_0$ within the coarsening regime, and found that in a broad region, it is a nonmonotonic function, with two separate scaling regimes.

We conclude by pointing out some questions not studied so far. For antiadvection, we studied only the case $a = -1$ for the coupling of the driving surface to the passive particles. However, variation of $a$ seems to induce nonuniversality, in that the exponent $z$ has been found to depend on $a$ $14$. How this would affect the measures of intermittency studied here remains an open question. Likewise, it may be interesting to study variation of $\omega$, the ratio of particle to surface updates. Finally, it would be interesting to know whether a scaling description of intermittency remains valid in other models, as in systems with long-ranged correlated noise $3$, $11$ or the zero range process $19$.

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[1] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. 73, 913 (2001).
[2] B. I. Shraiman and E. D. Siggia, Nature London 405, 639 (2000).
[3] J. M. Deutsch, J. Phys. A 18, 1449 (1985).
[4] M. Wilkinson and B. Mehlig, Phys. Rev. E 68, 040101(R) (2003).
[5] R. H. Kraichnan, Phys. Rev. Lett. 72, 1016 (1994).
[6] M. Kardar, G. Parisi and Y.-C. Zhang, Phys. Rev. Lett. 56, 889 (1986).
[7] E. Medina, T. Hwa, and M. Kardar, Yi-cheng Zhang, Phys. Rev. A 39, 3053 (1989).
[8] B. Drossel and M. Kardar, Phys. Rev. Lett. 85, 614 (2000).
[9] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A 381, 17 (1982).
[10] M. Kardar, Phys. Rev. Lett. 55, 2923 (1985).
[11] J. M. Burgers, The nonlinear Diffusion Equation (Reidal, Boston 1974)
[12] M. R. Maxey, J. Fluid Mech. 174, 441 (1987).
[13] E. Balkovsky, G. Falkovich, and A. Fouxon, Phys. Rev. Lett. 86, 2790 (2001).
[14] B. Drossel and M. Kardar, Phys. Rev. B 66, 195414 (2002).
[15] A. Nagar, S. N. Majumdar, and M. Barma, Phys. Rev. E 74, 021124 (2006).
[16] A. Nagar, M. Barma, and S. N. Majumdar, Phys. Rev. Lett. 94, 240601 (2005).
[17] U. Frisch, Turbulence: The Legacy of A.N. Kolmogorov, (Cambridge University Press, Cambridge, England, 1995).
[18] C. Huepe and M. Aldana, Phys. Rev. Lett. 92, 168701 (2004).
[19] A. Das, A. Polley, and Madan Rao, Phys. Rev. Lett. 116, 068306 (2016).
[20] H. Sachdeva, M. Barma, and Madan Rao, Phys. Rev. Lett. 110, 150601 (2013).
[21] Chen-Shan Chin, Phys. Rev. E 66, 201104 (2002)
[22] T. Bohr and A. Pikovsky, Phys. Rev. Lett. 70, 2892 (1993)
[23] F. Huveneers, Phys. Rev. E 97, 042116 (2018)
[24] A. J. Bray, Adv. Phys. 43, 357 (1994).
[25] T. Singha and M. Barma, Phys. Rev. E 97 010105(R) (2018)
[26] T. M. Liggett, Interacting Particle Systems Springer-Verlag, New York, (1985).
[27] M. Gopalakrishnan, Phy. Rev. E 69, 011105 (2004)
[28] B. Derrida, S. A. Janowsky, J. L. Lebowitz, and E. R. Speer, J. Stat. Phys. 73, 813 (1993)
[29] M. Ueda and S. Sasa, Phys. Rev. Lett. 115, 080605 (2015).
[30] T. Singha and M. Barma, arXiv:1806.06824
[31] M. Ueda and S. Sasa, arXiv:1805.10474
[32] J. P. Bouchaud, M. Mézard, and G. Parisi, Phys. Rev. E 52 3656 (1995)
[33] Y. G. Sinai, Theor. Probab. Appl. 27, 256 (1982).
[34] A. Comtet and C. Texier, Supersymmetry and Integrable Models Proceedings, Chicago, IL, edited by H. Aratyn, T. Imbo, W. Y. Keung, and U. Sukhatme Springer, Berlin, (1998).
[35] J. J. Ramasco, J. M. Lopez, and M. A. Rodriguez, Euro. Phys. Lett 76 554 (2006)
[36] S. Chatterjee and M. Barma, Phys. Rev. E 73, 011107 (2006)
[37] S. Bustingorry, J. Stat. Mech. (2007) P10002
[38] M. Henkel, J. D. Noh, and M. Pleimling, Phys. Rev. E 85, 030102(R) (2012)
[39] M. R. Evans and T. Hanney, J. Phys. A 38, R195 (2005)