CLASSICAL DYNAMICS FROM A UNITARY REPRESENTATION OF THE GALILEI GROUP

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Abstract

We give a formulation of classical mechanics in the language of operators acting on a Hilbert space. The formulation given comes from a unitary irreducible representation of the Galilei group that is compatible with the basic postulates of classical mechanics, particularly the absence of an uncertainty principle between the position and momentum of a particle. It is shown that the theory exposed in the article contains the Koopman-von Neumann formalism of classical mechanics as a particular case.

1 Introduction

The study of the realizations of the Galilei group in classical mechanics is different compared to the one given to quantum mechanics. The disparity in the approaches is due to the differences in the usual mathematical structures used for both theories. The Hilbert space formalism of quantum mechanics leads naturally to a study of the unitary representations of the Galilei group. There is a large literature on this subject, we will cite the classic works [1, 2, 3, 4, 5, 6] and the more recent ones [7, 8]. On the other hand, classical mechanics have been derived from the structure of the Galilei group in the context of Lagrangian mechanics [9, 10] and as a canonical representation in terms of Poisson brackets in Hamiltonian mechanics [11].

However, to study the interplay between quantum and classical mechanics is desirable to express both theories in mathematical formalism that are similar to
each other. In this regard, there are two ways to proceed. One way is the rewrite quantum mechanics in the same language of classical mechanics as it is done in the versions of the theory known as quantum mechanics in phase space \[11, 12\] and geometrical quantum mechanics \[13, 14\]. Conversely, there is an old, but perhaps not widely known, mathematical formalism due to of Koopman and von Neumann that puts classical Hamiltonian mechanics in the same mathematical language of quantum mechanics, i.e., as a theory of operators acting on a Hilbert space and a statistical interpretation given by a Born rule \[15, 16\] (Ref. \[17\] gives a good review on the topic).

The Koopman-von Neumann formalism (hereafter abbreviated as KvN) has been used to investigate the differences and similarities between classical and quantum mechanics and the overall relation between both theories. The classical limit of quantum mechanics in phase space is discussed in \[18\], where it is shown that the Wigner functions go to KvN wavefunctions in a suitable limit. Other works include the quantum-classical correspondence for integrable and chaotic systems in the $\hbar \to 0$ limit \[19\] and geometric dequantization \[20\]. On the other hand, studies in the opposite direction can also be found, references \[21, 22\] give procedures to obtain quantum mechanics from the operational classical mechanics. Moreover, the KvN theory has also been used to derive purely classical results \[23, 24, 25, 26, 27\].

The KvN theory starts with the Liouville equation, and from it a Hilbert space and a set of relevant operators are built. We take a different approach to get an operational formulation of classical mechanics. Namely, in this article we derive an operational formulation from a unitary, irreducible representation of the Galilei group. Unlike the KvN theory, our approach is independent of any previous results from analytical mechanics. From the beginning we will postulate a complex Hilbert space $\mathcal{H}_C$, and then we will look for a realization of the Galilei group where the space-time transformations are represented by unitary transformations acting on $\mathcal{H}_C$. We will show, by direct construction, that this program is compatible with the basic postulates of classical mechanics and that it contains the KvN theory as a particular case. Our presentation will be close to the one given in \[28, 29\] for the derivation of quantum mechanics from the Galilei algebra.

The organization of this article is as follows. In the following section we give a brief summary of the Galilei group, and we present the conventions that will be followed in the rest of this work. For the purpose of this article we shall only need the most basic properties of the Galilei group and algebra. For reasons of necessity, our notation for the generators of the Galilei algebra is not the standard one used in quantum mechanics.

In section 3 we introduce the Hilbert space $\mathcal{H}_C$ where the operators from the Galilei group act upon, and we state the action of these operators over the vectors of $\mathcal{H}_C$. There we also define operators outside the generators of the Galilei group that are necessary to give a physical interpretation of the theory, namely the position and velocity operators. We will postulate the basic relations between the operators of the theory that are consistent with the requirements of a classical dynamic. We will then proceed to find an irreducible unitary repre-
sentation of the Galilei group for the free particle and for a particle interacting with an external force.

In section 4 we give an operational approach to concepts from analytical mechanics like the Lagrangian and the canonical momentum. We will show that the definition of a canonical momentum allows for an alternative, but unitary equivalent, representation of the Galilei group. This alternative representation is the Koopman-von Neumann theory. In section 5 we point out the relation between the operational approach to dynamics with the Hamiltonian mechanics.

In section 6 we compare the irreducible unitary representations of the Galilei group for classical and quantum mechanics. We expect that the ab initio construction of a Hilbert space associated with classical mechanics will help to reveal more about the fundamental differences and similarities between the two theories.

The Einstein summation convention is used through all this work.

2 The Galilei Group and Algebra

The proper Galilei group is a ten parameter group that consists of space and time translations, rotations and pure Galilei transformation (boosts). The general transformation \((x, t) \rightarrow (x', t')\) can be written as

\[
  x' = R x + b t + a,
  
  t' = t + \tau,
\]

where \(R\) is a rotation, \(a\) is a space displacement, \(b\) is the velocity of a moving frame and \(\tau\) is a time displacement. The generators of the basic group transformations will be associated with Hermitian operators as follows: \(\hat{J}_\alpha\) stands for the rotations around \(\alpha\)-axis \((\alpha = 1, 2, 3)\); \(\hat{\lambda}_{x_\alpha}\) is the space displacement generator in the \(\alpha\)-direction; \(\hat{G}_\alpha\) correspond to the Galilean boost along the \(\alpha\)-axis; \(\hat{L}\) will be the time displacement generator. All these operators will act on a suitable Hilbert space to be described in the next section.

The space-time transformation of the Galilei group will be realized by unitary operators with the following convention:
| Space – Time Transformations | Unitary Operator |
|------------------------------|------------------|
| Rotations                    | $e^{-i\theta_{\alpha}\hat{J}_{\alpha}}$. |
| x → $R_{\alpha}(\theta_{\alpha})x$ |                  |
| Spatial Displacement         | $e^{-i\alpha_{\alpha}\hat{\lambda}_{\alpha}}$. |
| x → x + a                    |                  |
| Galilean Boost               | $e^{ib_{\alpha}\hat{G}_{\alpha}}$. |
| x → x + bt                   |                  |
| Time Displacement            | $e^{i\tau\hat{L}}$. |
| t → t + $\tau$               |                  |

The derivation of the Lie algebra associated to the Galilei group can be found in many places, for example [28]. It will be useful to divide the Galilei algebra relations in two sets. The first set does not involve the time displacement generator in the commutation relations

\[
\left[\hat{\lambda}_{\alpha},\hat{\lambda}_{\beta}\right] = 0, \quad (1a)
\]

\[
\left[\hat{G}_{\alpha},\hat{G}_{\beta}\right] = 0, \quad (1b)
\]

\[
\left[\hat{J}_{\alpha},\hat{J}_{\beta}\right] = i\varepsilon_{\alpha\beta\gamma}\hat{J}_{\gamma}, \quad (1c)
\]

\[
\left[\hat{J}_{\alpha},\hat{\lambda}_{\beta}\right] = i\varepsilon_{\alpha\beta\gamma}\hat{\lambda}_{\gamma}, \quad (1d)
\]

\[
\left[\hat{J}_{\alpha},\hat{G}_{\beta}\right] = i\varepsilon_{\alpha\beta\gamma}\hat{G}_{\gamma}, \quad (1e)
\]

\[
\left[\hat{G}_{\alpha},\hat{\lambda}_{\beta}\right] = i\delta_{\alpha\beta}\hat{M}. \quad (1f)
\]

The second is composed by the brackets that do involve $\hat{L}$

\[
\left[\hat{J}_{\alpha},\hat{L}\right] = 0, \quad (2a)
\]

\[
\left[\hat{G}_{\alpha},\hat{L}\right] = i\hat{\lambda}_{\alpha}, \quad (2b)
\]

\[
\left[\hat{\lambda}_{\alpha},\hat{L}\right] = 0. \quad (2c)
\]

In (1f) $\hat{M}$ is the central charge of the algebra and can be thought just as a real number whose value is not determined a priori by any equation.

We are using the unusual symbols ($\hat{J}$, $\hat{\lambda}$, $\hat{G}$, $\hat{L}$, $\hat{M}$) for the Galilei algebra, instead of the more common ones ($\hat{J}$, $\hat{P}$, $\hat{G}$, $\hat{H}$, $\hat{M}$) used in quantum mechanics, because in our representation of the Galilei group the elements of the Lie algebra, although related, will not be identified with usual physical quantities.
For example, the generator of rotation $\hat{J}$ is not an angular momentum operator, and $\hat{L}$ is not an energy operator. Perhaps the most dramatic difference is in the number $M$ which will not be the mass of the system, and, as we will show later, the situation for the allowed value for $M$ is completely different than in quantum mechanics where the mass $M$ has to be a positive number in order to have a physical representation of the Galilei group.

3 Classical Representation of the Galilei Algebra

We posit that the position $\mathbf{r}$ and the velocity $\mathbf{v}$ of a point particle can be simultaneously measured to any degree of accuracy, and our formalism will reflect this. The state of the classical particle will be described by a vector in a suitable complex Hilbert space. To account for the lack of an uncertainty principle between the classical particle’s position and velocity (or later on, the momentum) the state vectors should contain information of both the particle’s position and velocity\(^1\); hence, we postulate that the Hilbert space $\mathcal{H}_C$ of the classical point particle is composed of vectors $|\psi\rangle$ of the form

$$|\psi\rangle = \int \langle \mathbf{r}, \mathbf{v} | \psi \rangle |\mathbf{r}, \mathbf{v}\rangle \, d\mathbf{r}d\mathbf{v},$$

such that $\psi(\mathbf{r}, \mathbf{v}) = \langle \mathbf{r}, \mathbf{v} | \psi \rangle$ is a square integrable function and the kets $|\mathbf{r}, \mathbf{v}\rangle$ obey the orthornormality condition

$$\langle \mathbf{r}', \mathbf{v}' | \mathbf{r}, \mathbf{v} \rangle = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{v} - \mathbf{v}').$$

The probability $P(\mathbf{r}, \mathbf{v})$ of finding the particle with position $\mathbf{r}$ and velocity $\mathbf{v}$ is given by the Born rule

$$P(\mathbf{r}, \mathbf{v}) = |\langle \mathbf{r}, \mathbf{v} | \Psi \rangle|^2.$$  

Geometrically, the effect of $\hat{\lambda}_{\mathbf{r}}$, $\hat{G}$ and $\hat{J}$ represent a translation in the spatial coordinates, a Galilean boost and a rotation respectively. Accordingly, we will demand the action of these operators on the base kets to be

$$e^{-i\mathbf{a} \cdot \hat{\lambda}_{\mathbf{r}}} |\mathbf{r}, \mathbf{v}\rangle = |\mathbf{r} + \mathbf{a}, \mathbf{v}\rangle,$$  

$$e^{ib \cdot \hat{G}} |\mathbf{r}, \mathbf{v}\rangle \propto |\mathbf{r} + b(t, \mathbf{v} + \mathbf{b})\rangle,$$  

$$e^{-i\theta \mathbf{n} \cdot \hat{J}} |\mathbf{r}, \mathbf{v}\rangle = |\mathbf{r} + \theta \mathbf{n} \times \mathbf{r}, \mathbf{v} + \theta \mathbf{n} \times \mathbf{v}\rangle,$$  

where in (6b) we have used the proportionality sign to indicate that a phase factor can be present due to commutation relation (1f). The effect of $\hat{L}$ is temporal displacement in the state vectors, namely

$$e^{-it\hat{L}} |\Psi(0)\rangle = |\Psi(t)\rangle.$$  

\(^1\)In this paper we ignore the possibility of internal degree of freedom that come from the spin.
As in quantum mechanics, the time displacement equation (7) implies a Schrödinger equation
\[ \frac{d}{dt} |\Psi(t)\rangle = -i\hat{L} |\Psi(t)\rangle. \] (8)

It is natural to introduce position and velocity operators \( \hat{R} = (\hat{X}_1, \hat{X}_2, \hat{X}_3) \) and \( \hat{V} = (\hat{V}_1, \hat{V}_2, \hat{V}_3) \) such that, by definition, we have
\[ \hat{X}_\alpha |r, v\rangle = x_\alpha |r, v\rangle, \] (9a)
\[ \hat{V}_\alpha |r, v\rangle = v_\alpha |r, v\rangle. \] (9b)

We will also assume the existence of operators \( \hat{\lambda}_v = (\hat{\lambda}_{v_1}, \hat{\lambda}_{v_2}, \hat{\lambda}_{v_3}) \) that act as translation operators in the velocity coordinates
\[ e^{-ib\cdot\hat{\lambda}_v} |r, v\rangle = |r, v + b\rangle. \] (10)

As \( \hat{X}_x \) and \( \hat{\lambda}_v \) are translation operators, they are conjugated in the quantum sense to \( \hat{X}_\alpha \) and \( \hat{V}_\alpha \) respectively. The following commutation relations are postulated to be satisfied
\[ \left[ \hat{X}_\alpha, \hat{X}_\beta \right] = 0, \] (11a)
\[ \left[ \hat{V}_\alpha, \hat{V}_\beta \right] = 0, \] (11b)
\[ \left[ \hat{\lambda}_v, \hat{X}_\beta \right] = i\delta_{\alpha\beta}, \] (11c)
\[ \left[ \hat{\lambda}_v, \hat{V}_\beta \right] = i\delta_{\alpha\beta}. \] (11d)

Equations (11a) summarize the physical statement of classical mechanics that the position and the velocity of a particle can be known with any desired degree of accuracy.

The operators \( \hat{R}, \hat{V}, \) and \( \hat{\lambda}_v \) have to be vector operators; this is, their components have to transform under rotations according to
\[ \left[ \hat{J}_\alpha, \hat{X}_\beta \right] = i\varepsilon_{\alpha\beta\gamma} \hat{X}_\gamma, \] (12a)
\[ \left[ \hat{J}_\alpha, \hat{V}_\beta \right] = i\varepsilon_{\alpha\beta\gamma} \hat{V}_\gamma, \] (12b)
\[ \left[ \hat{J}_\alpha, \hat{\lambda}_{v_\beta} \right] = i\varepsilon_{\alpha\beta\gamma} \hat{\lambda}_{v_\gamma}. \] (12c)
Furthermore, the effect of a Galilean boost on $\hat{\mathbf{R}}$ and $\hat{\mathbf{V}}$ is the same as in quantum mechanics [29], i.e., the boost generates a displacement in the operators as follows

\[ e^{i b \cdot \hat{\mathbf{G}}} \hat{\mathbf{V}} e^{-i b \cdot \hat{\mathbf{G}}} = \hat{\mathbf{V}} - b, \quad (13a) \]
\[ e^{i b \cdot \hat{\mathbf{G}}} \hat{\mathbf{R}} e^{-i b \cdot \hat{\mathbf{G}}} = \hat{\mathbf{R}} - b t. \quad (13b) \]

Let us note that (13a) and (13b) imply that

\[ [\hat{\mathbf{X}}_{\alpha}, \hat{\mathbf{G}}_{\beta}] = i \delta_{\alpha \beta} t, \quad (14a) \]
\[ [\hat{\mathbf{V}}_{\alpha}, \hat{\mathbf{G}}_{\beta}] = i \delta_{\alpha \beta}. \quad (14b) \]

We postulate the same dynamical relation between the position operator $\hat{\mathbf{X}}_{\alpha}$ and $\hat{\mathbf{V}}_{\alpha}$ as in quantum mechanics, namely [29]

\[ \frac{d}{dt} \langle \hat{\mathbf{R}} \rangle = \langle \hat{\mathbf{V}} \rangle. \quad (15) \]

Due to equation (8), equation (15) reduces to

\[ \hat{\mathbf{V}} = i \left[ \hat{\mathbf{L}}, \hat{\mathbf{R}} \right]. \quad (16) \]

Finally, consider the acceleration operator $\hat{\mathbf{a}}$ defined by

\[ \frac{d}{dt} \langle \hat{\mathbf{V}} \rangle = \langle \hat{\mathbf{a}} \rangle. \quad (17) \]

We demand the acceleration operator to be function of $\hat{\mathbf{R}}$ and $\hat{\mathbf{V}}$, and to be independent of $\hat{\mathbf{L}}$ and $\hat{\mathbf{L}}$, i.e, $\hat{\mathbf{a}} = \hat{\mathbf{a}}(\hat{\mathbf{R}}, \hat{\mathbf{V}})$. In view of this last condition, equation (17) imposes the following on $\hat{\mathbf{L}}$

\[ i \left[ \hat{\mathbf{L}}, \hat{\mathbf{V}} \right] = \frac{1}{m} \hat{\mathbf{F}}(\hat{\mathbf{R}}, \hat{\mathbf{V}}), \quad (18) \]

for some function $\hat{\mathbf{F}}$ where $m$ is to be identified with the mass of the particle[2].

We shall see in section 5 the relation of $\hat{\mathbf{L}}$ with the Liouville equation of classical statistical mechanics; for this reason, we will call $\hat{\mathbf{L}}$ the Liouvillian of the system.

As we are ignoring internal degrees of freedom, the set of six operators \{\hat{\mathbf{R}}, \hat{\mathbf{V}}\} form a complete set of commuting operators in the Hilbert space of

\[ \begin{array}{l}
\text{The introduction here of a mass } m \text{ is rather artificial compared to the quantum case where the mass appears in the Galilei algebra and acts as a superselection operator. In classical mechanics the concept of mass becomes important because, under the same force, it is possible to observe different responses from different particles. As we are dealing here with only one particle, it is of no surprise that we could make } m \text{ disappear by a redefinition of the force as the same can be done in Newtonian mechanics.}
\end{array} \]
a single classical particle. Therefore, due to commutation relations (11a) to (11d) and Schur’s lemma [28] the collection \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \} forms an irreducible set of operators on the Hilbert space of the square integrable wave functions \psi(r, v) = \langle r, v | \psi \rangle. In the following section we are going to use the commutation relations of the Galilei algebra together with the definitions presented in this section to find a realization for \hat{G}, \hat{J} and \hat{L} in terms of \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \}.

3.1 Free Particle

The dynamic of the free particle has to be invariant under the full Galilei group of space-time transformations. As it is done in quantum mechanics, we will use this invariance to identify the generators of the group.

We will find first the representation of the rotation operator \hat{J}. It can be checked by direct computation that the following operators \[ \hat{J}_\alpha = \varepsilon_{\alpha \beta \gamma} \left( \hat{X}_\beta \hat{\lambda}_{x\gamma} + \hat{V}_\beta \hat{\lambda}_{v\gamma} \right) \] (19) satisfy each of the conditions (1c), (1d), (12a), (12b) and (12c); signifying that we have a viable representation for \hat{J}. Now, let us write \[ \hat{G}_\alpha = -\hat{\lambda}_{x\alpha} t - \hat{\lambda}_{v\alpha} + W, \] (20) where \( W_\alpha = W_\alpha(\hat{R}, \hat{V}) \) is an arbitrary function. Equation (20) satisfies each of the conditions (6b), (13a) and (13b). Equation (1f) constrains \( W_\alpha \) to be linear in \( \hat{V}_\alpha \). The only vector function of \( \hat{V} \) is itself, and the only rotational invariant function of \( \hat{V} \) is \( \hat{V}^2 \). Thus, the possible \( W_\alpha \) has to be of the form

\[ W_\alpha = \mathcal{M} \hat{X}_\alpha + w(\hat{V}^2) \hat{V}_\alpha. \] (21)

We postpone further investigations of \( \hat{G}_\alpha \) until we have examined the time displacement generator \( \hat{L} \).

Equation (10) will be fulfilled if the Liouvillian is of the form

\[ \hat{L} = \hat{V}_\alpha \hat{\lambda}_{x\alpha} + f(\hat{R}, \hat{V}, \hat{\lambda}_v), \] (22)



where \( f \) is, so far, an arbitrary function. We can use others commutators to investigate the allowed functions \( f \). For example, from (26) we can deduce that \( f \) can not be a function of \( \hat{R} \), and from (18) \( f \) can be at most linear in \( \hat{\lambda}_v \), so we can write \( \hat{L} \) as

\[ \hat{L} = \hat{V}_\alpha \hat{\lambda}_{x\alpha} + f_\alpha(\hat{V}) \hat{\lambda}_{v\alpha}. \] (23)

Now, \( \hat{G}_\alpha \) and \( \hat{L} \) are related by (25). Equations (24) and (26) are incompatible with \( \hat{G} \) having a linear term in \( \hat{X}_\alpha \). The only possibility to avoid a contradiction is to have \( \mathcal{M} = 0 \) in equation (21). Moreover, at least one of \( f_\alpha \) and \( w_\alpha \) have
to vanish in order to fulfill (25). We will prove that \( f_\alpha \) vanishes. We can write (2b) as

\[
[-\hat{\lambda}_{v_\alpha}, \hat{\mathcal{L}}] = i\hat{\lambda}_{x_\alpha},
\]  

or

\[
[\hat{\lambda}_{v_\alpha}, \hat{\mathcal{L}} - \hat{V}_\alpha \hat{\lambda}_{x_\alpha}] = 0.
\]  

We can see in (25) that \( \hat{\mathcal{L}} - \hat{V}_\alpha \hat{\lambda}_{x_\alpha} \) is not a function of \( \hat{\mathcal{V}} \); therefore, \( \hat{\mathcal{L}} \) have to be of the form

\[
\hat{\mathcal{L}} = \hat{V}_\alpha \hat{\lambda}_{x_\alpha} + B_\alpha \hat{\lambda}_{v_\alpha},
\]  

where the \( B_\alpha \) are constants. However, the term \( B_\alpha \hat{\lambda}_{v_\alpha} \) breaks the rotational invariance of \( \hat{\mathcal{L}} \) given by equation (2a); hence, it can not be allowed. The final form of \( \hat{\mathcal{L}} \) turns out to be

\[
\hat{\mathcal{L}} = \hat{V}_\alpha \hat{\lambda}_{x_\alpha}.
\]  

It can be immediately checked that (27) gives the expected value of the acceleration for a free particle

\[
\hat{a} = i \left[ \hat{\mathcal{L}}, \hat{\mathcal{V}} \right] = 0.
\]  

Under the action of \( \hat{\mathcal{L}} \), the kets \( |\mathbf{r}, \mathbf{v}\rangle \) transform as expected for the free particle

\[
e^{-it\hat{\mathcal{L}}} |\mathbf{r}, \mathbf{v}\rangle = e^{-it\hat{V}} \hat{\lambda}_x |\mathbf{r}, \mathbf{v}\rangle, \\
e^{-it\hat{\lambda}_x} |\mathbf{r}, \mathbf{v}\rangle = |\mathbf{r} + \mathbf{v}t, \mathbf{v}\rangle.
\]  

There is no commutation relation in the Lie Algebra of the Galilei group, nor in the postulated relations for \( \hat{\mathcal{R}}, \hat{\mathcal{V}} \) and \( \hat{\lambda}_x \) that allow an unequivocally determination of the function \( w \) in (21). The existence of \( w \) would not affect any of the commutation relations already defined, nor the dynamical equations that we will find later. Since it seems that we have the freedom to choose any \( w \) we want, we will take the simplest choice \( w = 0 \). The above imply the following commutation relation

\[
\left[ \hat{\mathcal{G}}_\alpha, \hat{\lambda}_{v_\beta} \right] = 0.
\]  

To recapitulate the results so far, as long as \( \mathcal{M} = 0 \), the elements of the Galilei algebra for the free particle can be written in terms of the irreducible set
of operators $\{\hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v\}$ as

\begin{align}
\hat{J}_\alpha &= \varepsilon_{\alpha\beta\gamma} \left( \hat{X}_\beta \hat{\lambda}_x + \hat{V}_\beta \hat{\lambda}_v \right), \\
\hat{G}_\alpha &= -\hat{\lambda}_{x,\alpha} t - \hat{\lambda}_{v,\alpha}, \\
\hat{L} &= \hat{V}_\alpha \hat{\lambda}_{x,\alpha}.
\end{align}

### 3.2 A Particle Interacting With External Forces

An interaction with an external force modifies the time evolution of the state vector compared to the free particle case. As in quantum mechanics, we retain the equation of motion

$$\frac{d}{dt} |\Psi(t)\rangle = -i\hat{L} |\Psi(t)\rangle,$$

but we modify $\hat{L}$ so that it will accounts for the interactions. The operator $\hat{L}$ is now to be understood as the operator of dynamical evolution in time. As $\hat{L}$ is changed, the commutation relationships (2a), (2b) and (2c) can no longer be used.

The transformations generated by $\hat{G}$ and $\hat{J}$ are understood to be purely geometrical in nature, so the expressions for them found in the previous section remain the same.

To identify the form of $\hat{L}$, we will use the dynamical equations

\begin{align}
i \left[ \hat{L}, \hat{X}_\alpha \right] &= \hat{V}_\alpha, \\
i \left[ \hat{L}, \hat{V}_\alpha \right] &= \frac{1}{m} \hat{F}_\alpha (\hat{R}, \hat{V}).
\end{align}

It can be readily checked that

$$\hat{L} = \hat{V}_\alpha \hat{\lambda}_{x,\alpha} + \frac{1}{2m} \left( \hat{F}_\alpha \hat{\lambda}_{v,\alpha} + \hat{\lambda}_{v,\alpha} \hat{F}_\alpha \right)$$

fulfills both equations (33) and (34). Using equation (35), the commutators of $\hat{L}$ with $\hat{\lambda}_r$ and $\hat{\lambda}_v$ can be computed as

\begin{align}
i \left[ \hat{L}, \hat{\lambda}_{x,\alpha} \right] &= -\frac{1}{2m} \left( \frac{\partial \hat{F}_\beta}{\partial \hat{X}_\alpha} \hat{\lambda}_{v,\beta} + \hat{\lambda}_{v,\beta} \frac{\partial \hat{F}_\beta}{\partial \hat{X}_\alpha} \right), \\
i \left[ \hat{L}, \hat{\lambda}_{v,\alpha} \right] &= -\hat{\lambda}_{x,\alpha} - \frac{1}{2m} \left( \frac{\partial \hat{F}_\beta}{\partial \hat{V}_\alpha} \hat{\lambda}_{v,\beta} + \hat{\lambda}_{v,\beta} \frac{\partial \hat{F}_\beta}{\partial \hat{V}_\alpha} \right).
\end{align}
As we already have expressed \( \hat{G} \) and \( \hat{J} \) in terms of the irreducible set \( \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \} \), their commutators with \( \hat{L} \) can be computed, and the result will depend on the given \( \hat{F} \).

The form of the force operator is not given a priori, \( \hat{F} \) has to be investigated from the experiments. Once \( \hat{F} \) is found, the evolution in the state of the particle can be known at any time solving the Schrödinger-like equation.

**Example: Projectile Motion**

As an example of the formalism developed above, let us consider the case of projectile motion. For this case the force operator is

\[
\hat{F} = -mg\hat{j},
\]

where \( g \) is a constant.

Using the formula, the Liouvillian of this simple system takes the form

\[
\hat{L} = \hat{V}_x\hat{\lambda}_x + \hat{V}_y\hat{\lambda}_y + \hat{V}_z\hat{\lambda}_z - g\hat{\lambda}_v.
\]

As the Liouvillian is time independent, we can write the evolution operator as follows

\[
U(t) = e^{-it\hat{L}} = e^{-it(\hat{V}_x\hat{\lambda}_x + \hat{V}_y\hat{\lambda}_y + \hat{V}_z\hat{\lambda}_z - g\hat{\lambda}_v)} = e^{-it(\hat{\lambda}_x + \hat{\lambda}_z)} e^{-itg\hat{\lambda}_v} e^{it\hat{\lambda}_y},
\]

where in the last step we used the well known operator formula

\[
e^{t(A+B)} = e^{tA} e^{tB} e^{\frac{t}{2}[A,B]} \text{ valid when } A \text{ and } B \text{ commute with } [A,B].
\]

The effect of the evolution operator on a given base ket is

\[
|r(t); v(t)\rangle = U(t) |r_0; v_0\rangle = |r_0 + vt - \frac{t^2}{2}g\hat{j}; v_0 - gt\hat{j}\rangle,
\]

which is the expected behavior from a projectile motion. An arbitrary wave function \( \psi(r, v) = \langle r, v | \psi \rangle \) will evolve in time according to the rule

\[
\psi(r, v, t) = \langle r - vt + \frac{t^2}{2}g\hat{j}; v + gt\hat{j} | \psi \rangle = \psi(r - vt + \frac{t^2}{2}g\hat{j}; v + gt\hat{j}).
\]

**Example 2: Harmonic Oscillator**

For the sake of simplicity, we only consider the movement along the \( x \)-axis of a particle under the effect of a linear restoring force. Hence, the base kets to be considered are of the form \( |x, v_x\rangle \) and the force operator is given by

\[
\hat{F} = -k\hat{X}.
\]
For the force (43) the Liouvillian takes the form

$$\hat{L} = \hat{V}_x \hat{\lambda}_x - \omega^2 \hat{X} \hat{\lambda}_{v_x},$$

where $\omega^2 = k/m$.

By means of the following transformation

$$\hat{V}'_x = \frac{1}{\omega} \hat{V}_x, \quad \hat{\lambda}'_{v_x} = \omega \hat{\lambda}_{v_x},$$

we can write the evolution operator as

$$U(t) = \exp \left[ -i\omega t \left( \hat{V}'_x \hat{\lambda}_x - \hat{X} \hat{\lambda}'_{v_x} \right) \right].$$

Algebraically, the combination $\hat{V}'_x \hat{\lambda}_x - \hat{X} \hat{\lambda}'_{v_x}$ have the same commutation relations with $\hat{X}$ and $\hat{V}_x$ as the $z$ component of the orbital angular momentum $J_z = (\hat{r} \times \hat{p})_z$ has with the position operators $\hat{x}$ and $\hat{y}$ in quantum mechanics. Therefore, $\hat{V}'_x \hat{\lambda}_x - \hat{X} \hat{\lambda}'_{v_x}$ acts as a generator of rotations in a plane where the velocity plays the role of a perpendicular axis to the $x$-axis, i.e., the tangent bundle of configuration space, given in this case by the Cartesian product $X \times V$.

Hence, we can write the evolution of a base ket as follows

$$U(t) |x_0; v_0\rangle = |x_0 \cos (\omega t) + \frac{v_0}{\omega} \sin (\omega t) ; -\omega x_0 \sin (\omega t) + v_0 \cos (\omega t)\rangle.$$

### 3.3 System of Mutually Interacting Particles.

We obtained the operators that represent the classical dynamical variables of a single particle, and now we are going to generalize the result for the case of a system of interacting particles. We work out in detail the case of two interacting particles. The case of $N$ particles will be just a trivial generalization.

Let the Hilbert spaces of particle 1 and particle 2 be $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The set $\left\{ \hat{R}^{(1)}, \hat{V}^{(1)}, \hat{\lambda}_{r}^{(1)}, \hat{\lambda}_{v}^{(1)} \right\}$ acts on $\mathcal{H}_1$ while $\left\{ \hat{R}^{(2)}, \hat{V}^{(2)}, \hat{\lambda}_{r}^{(2)}, \hat{\lambda}_{v}^{(2)} \right\}$ acts on $\mathcal{H}_2$. Just as in quantum mechanics, and for the same reasons, we postulate that the Hilbert space of the composite system is given by the tensor product

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$
where $W$ is an interaction term. Now, we demand that the interaction term has to give rise to the force equation for the individual particles, more precisely

$$i \left[ \hat{L}, \hat{\mathbf{V}}^{(1)} \right] = \frac{1}{m_1} \hat{\mathbf{F}}^{(1)}, \quad (52)$$

$$i \left[ \hat{L}, \hat{\mathbf{V}}^{(2)} \right] = \frac{1}{m_2} \hat{\mathbf{F}}^{(2)}, \quad (53)$$

with $\hat{\mathbf{F}}^{(1)}$ and $\hat{\mathbf{F}}^{(2)}$ function of $\{ \hat{\mathbf{R}}^{(1)}, \hat{\mathbf{V}}^{(1)}, \hat{\mathbf{R}}^{(2)}, \hat{\mathbf{V}}^{(2)} \}$ only. It can be checked that (52) and (53) are satisfied by

$$\hat{L} = \hat{\mathbf{V}}^{(1)} \hat{\lambda}_{\alpha}^{(1)} + \hat{\mathbf{V}}^{(2)} \hat{\lambda}_{\alpha}^{(2)} + \frac{1}{2m_1} \left( \hat{\mathbf{F}}^{(1)} \hat{\lambda}_{\alpha}^{(1)} + \hat{\lambda}_{\alpha}^{(1)} \hat{\mathbf{F}}^{(1)} \right)$$

$$+ \frac{1}{2m_2} \left( \hat{\mathbf{F}}^{(2)} \hat{\lambda}_{\alpha}^{(2)} + \hat{\lambda}_{\alpha}^{(2)} \hat{\mathbf{F}}^{(2)} \right). \quad (54)$$

We want to know what are the constraints imposed on $\hat{\mathbf{F}}^{(1)}$ and $\hat{\mathbf{F}}^{(2)}$ by Galilean invariance. As in quantum mechanics, the other elements of the Galilei algebra are the sum of the individual generators for each particle, namely

$$\hat{\lambda}_r = \hat{\lambda}_r^{(1)} + \hat{\lambda}_r^{(2)}, \quad (55)$$

$$\hat{\mathcal{J}}_\alpha = \hat{\mathcal{J}}_\alpha^{(1)} + \hat{\mathcal{J}}_\alpha^{(2)}, \quad (56)$$

$$\hat{\mathcal{G}}_\alpha = \hat{\mathcal{G}}_\alpha^{(1)} + \hat{\mathcal{G}}_\alpha^{(2)}. \quad (57)$$

Although it does not belong to the Galilei algebra, we also define an operator for the total translation in the velocities given by

$$\hat{\lambda}_v = \hat{\lambda}_v^{(1)} + \hat{\lambda}_v^{(2)}. \quad (58)$$

Instead of the operators for the individual particles, let us consider the center of mass and the velocity of the center of mass operators given by (where $M = m_1 + m_2$)

$$\hat{\mathbf{R}}_{CM} = \frac{1}{M} \left( m_1 \hat{\mathbf{R}}^{(1)} + m_2 \hat{\mathbf{R}}^{(2)} \right), \quad (59)$$

$$\hat{\mathbf{V}}_{CM} = \frac{1}{M} \left( m_1 \hat{\mathbf{V}}^{(1)} + m_2 \hat{\mathbf{V}}^{(2)} \right), \quad (60)$$

the relative position and relative velocity operators according to

$$\hat{\mathbf{R}}_{Rel} = \hat{\mathbf{R}}^{(1)} - \hat{\mathbf{R}}^{(2)}, \quad (61)$$

$$\hat{\mathbf{V}}_{Rel} = \hat{\mathbf{V}}^{(1)} - \hat{\mathbf{V}}^{(2)}., \quad (62)$$
and the relative translation operators

$$\hat{\lambda}_r^* = \frac{1}{M} \left( m_2 \hat{\lambda}_r^{(1)} - m_1 \hat{\lambda}_r^{(2)} \right),$$  \hspace{1cm} (63)

$$\hat{\lambda}_v^* = \frac{1}{M} \left( m_2 \hat{\lambda}_v^{(1)} - m_1 \hat{\lambda}_v^{(2)} \right).$$ \hspace{1cm} (64)

It can be easily checked that the only non-vanishing commutators between the elements of the set \( \{ \hat{R}_{CM}, \hat{V}_{CM}, \hat{R}_{Rel}, \hat{V}_{Rel}, \hat{\lambda}_r, \hat{\lambda}_v, \hat{\lambda}_r^*, \hat{\lambda}_v^* \} \) are

$$\left[ \hat{X}_{CM,\alpha}, \hat{\lambda}_{x,\alpha} \right] = \left[ \hat{V}_{CM,\alpha}, \hat{\lambda}_{v,\alpha} \right] = \left[ \hat{X}_{Rel,\alpha}, \hat{\lambda}_{x,\alpha}^* \right] = \left[ \hat{V}_{Rel,\alpha}, \hat{\lambda}_{v,\alpha}^* \right] = i.$$  \hspace{1cm} (65)

We will use the set \( \{ \hat{R}_{CM}, \hat{V}_{CM}, \hat{R}_{Rel}, \hat{V}_{Rel}, \hat{\lambda}_r, \hat{\lambda}_v, \hat{\lambda}_r^*, \hat{\lambda}_v^* \} \) to study the allowed forms of \( \hat{F}^{(1)} \) and \( \hat{F}^{(2)} \).

In view of the relations (2a), (2b) and (2c), the interaction term \( W \) has to obey the following commutation relations

$$\left[ \hat{\lambda}_r, W \right] = 0, \hspace{1cm} (66a)$$

$$\left[ \hat{G}_\alpha, W \right] = 0, \hspace{1cm} (66b)$$

$$\left[ \hat{J}_\alpha, W \right] = 0. \hspace{1cm} (66c)$$

Furthermore, the forces \( \hat{F}^{(1)} \) and \( \hat{F}^{(2)} \) should satisfy the commutation relations

$$\left[ \hat{\lambda}_r, \hat{F}^{(1)} \right] = \left[ \hat{\lambda}_r, \hat{F}^{(2)} \right] = 0, \hspace{1cm} (67)$$

$$\left[ \hat{G}_\alpha, \hat{F}^{(1)} \right] = \left[ \hat{G}_\alpha, \hat{F}^{(2)} \right] = 0. \hspace{1cm} (68)$$

Equation (66a) implies that the forces cannot be function of \( \hat{R}_{CM} \), as \( \hat{R}_{CM} \) does not commute with \( \hat{\lambda}_r \). Using (67), equation (66b) implies that the forces cannot be function of \( \hat{V}_{CM} \). Therefore, \( \hat{F}^{(1)} \) and \( \hat{F}^{(2)} \) can only be function of \( \hat{R}_{Rel} \) and \( \hat{V}_{Rel} \). Finally, the condition (66c) is met only if the forces are vector operators.

The generalization for the case of \( N \) interacting particles is straightforward. The Hilbert space of the composite system is the tensor product of the individual Hilbert spaces. The interaction forces are limited to be functions of the scalar combination of the relative positions \( \hat{R}^{(i)} - \hat{R}^{(j)} \) and the relative velocities \( \hat{V}^{(i)} - \hat{V}^{(j)} \).
4 Lagrangian Operator

In the last section we completed our task of giving a complete formulation of the classical dynamics of a point particle. The developments of this section are aimed to justify the definition of a canonical momentum operator, as this new operator will help us to give an alternative operational formulation of the classical dynamics.

Let us start by noting that commutator (11d) allows us to rewrite equation (34) as

\[- \left[ \hat{L}, \left[ \hat{\lambda}_v, \frac{m}{2} \hat{\mathbf{V}}^2 \right] \right] = \hat{F}_\alpha.\]  

(69)

Moreover, let us decompose the forces as follows

\[\hat{F} = \hat{F}^{(C)} + \hat{F}^{(NC)},\]  

(70)

where, by definition, the components of the conservative forces can be computed from a potential \(U = U(\hat{\mathbf{R}}, \hat{\mathbf{V}})\) according to

\[\hat{F}^{(C)}_\alpha = -i \left[ \hat{\lambda}_{x,\alpha}, \hat{U} \right] - \left[ \hat{L}, \left[ \hat{\lambda}_v, \hat{U} \right] \right].\]  

(71)

With the help of (71), equation (69) can be further rewritten as

\[\Phi[\hat{L}] = \hat{F}^{(NC)}_\alpha,\]  

(72)

where \(\hat{L}\) is the Lagrangian operator

\[\hat{L} = \frac{m}{2} \hat{\mathbf{V}}^2 - \hat{U} ,\]  

(73)

and \(\Phi\) is the superoperator given by

\[\Phi = - \left[ \hat{L}, \left[ \hat{\lambda}_v, \mathbf{\cdot} \right] \right] - i \left[ \hat{\lambda}_{x,\alpha}, \mathbf{\cdot} \right],\]  

(74)

where the superoperator acts on \(\hat{L}\) as \(\Phi[\hat{L}] = - \left[ \hat{L}, \left[ \hat{\lambda}_v, \hat{L} \right] \right] - i \left[ \hat{\lambda}_{x,\alpha}, \hat{L} \right].\)

Equation (72) is the equivalent of the Lagrange equations in analytical mechanics. From (71), the force \(\hat{F}^{(C)}\) will be independent of the acceleration \(\hat{a}_\alpha = \left[ \hat{L}, \hat{V}_\alpha \right]\) only if \(\hat{U}\) is at most linear in the velocities. The generalized potential is then of the form

\[\hat{U} = \phi - \hat{V}_\alpha \hat{A}_\alpha,\]  

(75)

where both the scalar and vector potentials \(\phi\) and \(\hat{A}_\alpha\) are functions of \(\hat{\mathbf{R}}\) only

\[\phi = \phi\left( \hat{\mathbf{R}} \right),\]  

(76)

\[\hat{A} = \hat{A}\left( \hat{\mathbf{R}} \right).\]  

(77)
Inserting the generalized potential (75) into equation (71) yields

\[
\hat{F}_{\alpha}^{(C)} = -i \left[ \hat{\lambda}_{x\alpha}, U \right] - \left[ \hat{L}, \left[ \hat{\lambda}_{v\alpha}, U \right] \right] + i \frac{\partial}{\partial t} \left[ \hat{\lambda}_{v\alpha}, U \right] \\
= - \frac{\partial \hat{\phi}}{\partial X_\alpha} + i \left[ \hat{L}, \hat{A}_\alpha \right] \\
= - \frac{\partial \hat{\phi}}{\partial X_\alpha} - \frac{\partial \hat{A}_\alpha}{\partial t} \frac{\partial \hat{A}_\beta}{\partial X_\alpha} + \hat{V}_\beta \frac{\partial \hat{A}_\alpha}{\partial X_\beta} \\
= \hat{E}_\alpha + \left( \hat{V} \times \hat{B} \right)_\alpha ,
\]

where

\[
\begin{align*}
\hat{E}_\alpha & = - \frac{\partial \hat{\phi}}{\partial X_\alpha} - \frac{\partial \hat{A}_\alpha}{\partial t}, \\
\hat{B}_\alpha & = \left( \nabla \times \hat{A} \right)_\alpha .
\end{align*}
\]

There is a result in classical mechanics [32], rediscovered by Feynman [33] in a different context, that states that all the velocity dependent but acceleration independent forces that can be derived from a Lagrangian have the form of the Lorentz force. We have obtained here the same result starting from different premises.

4.1 Momentum Representation

Having the Lagrangian operator (73), it is natural to look for a definition of a canonical momentum operator \( \hat{P} \). We define the components of the canonical momentum operator by

\[
\hat{P}_\alpha = i \left[ \hat{\lambda}_{v\alpha}, \hat{L} \right] = m \hat{V}_\alpha + \hat{A}_\alpha .
\]

It is straightforward to check the following commutation relations for \( \hat{P} \)

\[
\begin{align*}
\left[ \hat{X}_\alpha, \hat{P}_\beta \right] & = 0, \\
\left[ \hat{P}_\alpha, \hat{\lambda}_{x\beta} \right] & = im \delta_{\alpha\beta}, \\
\left[ \hat{P}_\alpha, \hat{\lambda}_{v\beta} \right] & = i \frac{\partial \hat{A}_\alpha}{\partial X_\beta}.
\end{align*}
\]

The introduction of a canonical momentum allows for a change of irreducible
representation of the Galilei group via the definitions

\[ \hat{\lambda}_{pa} = \frac{1}{m} \hat{\lambda}_{va}, \]  

(83a)

\[ \hat{\lambda}'_{x,\beta} = \hat{\lambda}_{x,\beta} - \frac{\partial \hat{A}_\alpha}{\partial \hat{X}_\beta} \hat{\lambda}_{pa}. \]  

(83b)

The commutation relations between \( \hat{R}, \hat{P}, \hat{X}'_r, \) and \( \hat{\lambda}'_p \) are

\[ [\hat{X}_\alpha, \hat{X}_\beta] = [\hat{P}_\alpha, \hat{P}_\beta] = 0, \]  

(84a)

\[ [\hat{X}_\alpha, \hat{\lambda}'_{x,\beta}] = 0, \]  

(84b)

\[ [\hat{P}_\alpha, \hat{\lambda}'_{x,\beta}] = i \delta_{\alpha\beta}, \]  

(84c)

\[ [\hat{P}_\alpha, \hat{\lambda}_p] = i \delta_{\alpha\beta}. \]  

(84d)

The set of operators \( \{ \hat{R}, \hat{P}, \hat{X}'_r, \hat{\lambda}'_p \} \) is irreducible in the Hilbert space we are considering in view of the preceding set of equations (84a) to (84d). Therefore, the transformations given by (81), (83a) and (83b) allow us to pass from an irreducible representation of the Galilei group in terms of \( \{ \hat{R}, \hat{V}, \hat{\lambda}'_r, \hat{\lambda}'_v \} \) to one given in terms of the operators \( \{ \hat{R}, \hat{P}, \hat{\lambda}'_r, \hat{\lambda}'_p \} \).

The change from \( \{ \hat{R}, \hat{V}, \hat{\lambda}'_r, \hat{\lambda}'_v \} \) to \( \{ \hat{R}, \hat{P}, \hat{\lambda}'_r, \hat{\lambda}'_p \} \) is a quantum canonical transformation as discussed in [34]. This transformation can be done in two steps. First, we make the change

\[ \hat{V} \rightarrow m \hat{V}, \]  

(85a)

\[ \hat{\lambda}'_v \rightarrow \frac{1}{m} \hat{\lambda}'_v. \]  

(85b)

Second, we will show that the set \( \{ \hat{R}, m \hat{V}, \hat{\lambda}'_r, \frac{1}{m} \hat{\lambda}'_v \} \) can be transformed into \( \{ \hat{R}, \hat{P}, \hat{\lambda}'_r, \hat{\lambda}'_p \} \) by an unitary transformation using the unitary operator \( \hat{C} \) given by

\[ \hat{C} = e^{\frac{i}{m} \hat{A} \hat{\lambda}}. \]  

(86)

The above assertion can be checked by direct computation. The unitary transformation given by the operator (86) leaves \( \hat{R} \) and \( \frac{1}{m} \hat{\lambda}'_v \) unchanged,

\[ e^{\frac{i}{m} \hat{A} \hat{\lambda}} \hat{X}_\alpha e^{-\frac{i}{m} \hat{A} \hat{\lambda}} = \hat{X}_\alpha, \]  

(87a)

\[ e^{-\frac{i}{m} \hat{A} \hat{\lambda}} \left( \frac{1}{m} \hat{\lambda}'_{va} \right) e^{\frac{i}{m} \hat{A} \hat{\lambda}} = \frac{1}{m} \hat{\lambda}'_{va} = \hat{\lambda}_{pa}. \]  

(87b)
The effect of $\hat{C}$ on $m\hat{V}$ and $\hat{\lambda}_r$ can be computed using the well known Baker-Campbell-Hausdorff formula $e^X e^{-Y} = Y + [X,Y] + \frac{1}{2!} [X,[X,Y]] + \ldots$ as follows

\[
e^{\frac{1}{i} \hat{A} \cdot \hat{\lambda}_r} \left( m\hat{V}_\alpha \right) e^{-\frac{1}{i} \hat{A} \cdot \hat{\lambda}_r} = m\hat{V}_\alpha + i[\hat{A} \cdot \hat{\lambda}_r, \hat{V}_\alpha] - \frac{1}{2m}[\hat{A} \cdot \hat{\lambda}_r, \hat{A} \cdot \hat{\lambda}_r, \hat{V}_\alpha] + \ldots
= m\hat{V}_\alpha + \hat{A}_\alpha = \hat{P}_\alpha, \quad (88a)
\]

\[
e^{\frac{i}{2} \hat{\lambda}_x \hat{A}_x} e^{-\frac{i}{2} \hat{\lambda}_x \hat{A}_x} = \hat{\lambda}_{x_\alpha} + \frac{i}{m}[\hat{A} \cdot \hat{\lambda}_x, \hat{\lambda}_{x_\alpha}] - \frac{1}{2m^2}[\hat{A} \cdot \hat{\lambda}_x, \hat{A} \cdot \hat{\lambda}_x, \hat{\lambda}_{x_\alpha}] + \ldots
= \hat{\lambda}_{x_\alpha} - \frac{1}{2m} \frac{\partial \hat{A}_\beta}{\partial \hat{X}_\alpha} \hat{\lambda}_{x_\beta} = \hat{\lambda}_{x_\alpha} - \frac{\partial \hat{A}_\beta}{\partial \hat{X}_\alpha} \hat{\lambda}_{p_\beta}. \quad (88b)
\]

The transformation on the operators is accompanied by a transformation on the base ket $|\mathbf{r},\mathbf{v}\rangle$ given by

\[
|\mathbf{r},\mathbf{p}\rangle = e^{\frac{i}{m} \hat{A} \cdot \hat{\lambda}_r} |\mathbf{r},\mathbf{v}\rangle = e^{\frac{i}{2} \hat{\lambda}_x \hat{A}_x} |\mathbf{r},\mathbf{v}\rangle = |\mathbf{r},\mathbf{v} - \frac{1}{m} \mathbf{A}(\mathbf{r})\rangle. \quad (89)
\]

By doing the unitary transformation $\mathbf{89}$ we are changing the description of the state of the particle from the tangent bundle of configuration space to the phase space.

The operators $\hat{R}$ and $\hat{P}$ act as multiplication operators on the kets $|\mathbf{r},\mathbf{p}\rangle$ defined above

\[
\hat{X}_\alpha |\mathbf{r},\mathbf{p}\rangle = \hat{X}_\alpha \left( e^{\frac{i}{m} \hat{A} \cdot \hat{\lambda}_r} |\mathbf{r},\mathbf{v}\rangle \right) = e^{\frac{i}{m} \hat{A} \cdot \hat{\lambda}_r} \left( \hat{X}_\alpha |\mathbf{r},\mathbf{v}\rangle \right)
= x_\alpha e^{\frac{i}{m} \hat{A} \cdot \hat{\lambda}_r} |\mathbf{r},\mathbf{v}\rangle = x_\alpha |\mathbf{r},\mathbf{p}\rangle, \quad (90a)
\]

\[
\hat{P}_\alpha |\mathbf{r},\mathbf{p}\rangle = \left( m\hat{V}_\alpha + \hat{A}_\alpha \right) \left( e^{\frac{i}{2} \hat{\lambda}_x \hat{A}_x} |\mathbf{r},\mathbf{v}\rangle \right)
= e^{\frac{i}{2} \hat{\lambda}_x \hat{A}_x} \left( m\hat{V}_\alpha |\mathbf{r},\mathbf{v}\rangle \right) = mv_\alpha |\mathbf{r},\mathbf{p}\rangle, \quad (90b)
\]

where in the last equality we used the following identity

\[
m\hat{V}_\alpha = e^{-\frac{i}{m} \hat{A} \cdot \hat{\lambda}_r} \left( m\hat{V}_\alpha + \hat{A}_\alpha \right) e^{\frac{i}{m} \hat{A} \cdot \hat{\lambda}_r}. \quad (91)
\]

On the other hand, the operators $\hat{\lambda}'_v$ and $\hat{\lambda}'_p$ act as translation operators on
In terms of the momentum operator, the Liouvillian reads

\[
\hat{L}' = \frac{1}{m} \left( \hat{P}_\alpha - \hat{A}_\alpha \right) \hat{\lambda}'_\alpha + \frac{1}{2} \left( \hat{P}_\alpha \hat{\lambda}'_\alpha + \hat{\lambda}'_\alpha \hat{P}_\alpha \right) \\
+ \frac{1}{2m} \frac{\partial \hat{A}_\beta}{\partial \hat{X}_\alpha} \left\{ \left( \hat{P}_\beta - \hat{A}_\beta \right) \hat{\lambda}'_\beta + \hat{\lambda}'_\beta \left( \hat{P}_\beta - \hat{A}_\beta \right) \right\}.
\]

(93)

The procedure to obtain equation (93) explains from first principles the
origin of the minimal coupling in the KvN theory given in [25].

We can check that the Liouvillian (93) is consistent with the basic dynamic
definitions (16) and (18)

\[
i \left[ \hat{L}', \hat{\lambda}'_\alpha \right] = i \left[ \hat{L}', \frac{1}{m} \left( \hat{P}_\alpha - \hat{A}_\alpha \right) \right] = \frac{1}{m} \hat{P}_\alpha (\hat{R}, \hat{V}),
\]

(94a)

\[
i \left[ \hat{L}', \hat{X}_\alpha \right] = \frac{1}{m} \left( \hat{P}_\alpha - \hat{A}_\alpha \right) = \hat{V}_\alpha.
\]

(94b)

The equation of motion for any state \(|\Psi(t)\rangle\) is given by

\[
\frac{d}{dt} |\Psi(t)\rangle = -i\hat{L}' |\Psi(t)\rangle.
\]

The probability \(P(\mathbf{r}, \mathbf{p})\) of finding the particle at the point \((\mathbf{r}, \mathbf{p})\) in the phase space is

\[
P(\mathbf{r}, \mathbf{p}) = |\langle \mathbf{r}, \mathbf{p} | \Psi \rangle|^2.
\]

(95)

The remaining elements of the Galilei algebra in terms of \(\{ \hat{R}, \hat{P}, \hat{X}_\alpha, \hat{\lambda}_\alpha \} \) are

\[
\hat{J}'_\alpha = \varepsilon_{\alpha\beta\gamma} \left( \hat{X}_\beta \left( \hat{\lambda}'_\gamma + \frac{\partial \hat{A}_\gamma}{\partial \hat{X}_\gamma} \hat{\lambda}'_\alpha \right) + \left( \hat{P}_\beta - \hat{A}_\beta \right) \hat{\lambda}'_\alpha \right),
\]

(96)

\[
\hat{G}'_\alpha = - \left( \hat{\lambda}'_\alpha + \frac{\partial \hat{A}_\alpha}{\partial \hat{X}_\alpha} \hat{\lambda}'_\alpha \right) t - m \hat{\lambda}'_\alpha.
\]

(97)
The Hilbert space spanned by the kets $|r, p\rangle$, the set of operators $\{\hat{R}, \hat{P}, \hat{\lambda}_r, \hat{\lambda}_p\}$, the Liouvilian \[\text{(93)},\] and the elements \[\text{(94)}\text{ and (97)}\] form a unitary and irreducible, though gauge dependent, representation of the Galilei group. This representation of the Galilei group together with the Born rule \[\text{(95)}\] is the KvN formulation of classical mechanics\[\text{[17]}\].

Let us end this section with a comparison between the wavefunctions in the velocity vs the momentum representation. Consider the two wavefunctions given by.

$$\psi(r, v) = \langle r, v | \Psi \rangle,$$
$$\varphi(r, p) = \langle r, p | \Psi \rangle. \quad (98)$$

Using Eq. \[\text{(89)}\] we can write

$$\varphi(r, p) = \int \langle r, p | r', v' \rangle \langle r', v' | \Psi \rangle \, dr' \, dv'$$
$$= \int \langle r, v | e^{-\frac{\hat{A}}{m} \cdot \hat{\lambda} v} | r', v' \rangle \psi(r', v') \, dr' \, dv'$$
$$= \int \langle r, v | r', v' - \frac{1}{m} A(r') \rangle \psi(r', v') \, dr' \, dv'$$
$$= \int \delta(r' - r) \delta(v' - \frac{1}{m} A(r') - v) \psi(r', v') \, dr' \, dv'$$
$$= \psi(r, v + \frac{1}{m} A).$$

The relation between a wavefunction expressed in terms of the velocity with the one expressed in terms of the momentum is then

$$\varphi(r, p) = \psi(r, v + \frac{1}{m} A). \quad (99)$$

5 Relation with Hamiltonian Mechanics

For sake of completeness, we will show the relation between the operational and the Hamiltonian version of classical mechanics. Since, as shown in section 4, the operational version of classical mechanics derived in section 3 is unitary equivalent to the KvN theory, it is enough to show the derivation of Hamiltonian mechanics from the KvN formalism. Let us define a wave function $\psi$ by

$$\psi(r, p) = \langle r, p | \psi \rangle. \quad (100)$$

On $\psi(r, p)$ the position and momentum operators act as multiplication operators

$$\hat{X}_\alpha \psi(r, p) = x_\alpha \psi(r, p), \quad (101)$$
$$\hat{P}_\alpha \psi(r, p) = p_\alpha \psi(r, p). \quad (102)$$
On the other hand, the operators $\hat{\lambda}_p$ and $\hat{\lambda}_r'$ act as derivatives

\[
\begin{align*}
\hat{\lambda}_r' \psi(r, p) &= -i \nabla_r \psi(r, p), \\
\hat{\lambda}_p \psi(r, p) &= -i \nabla_p \psi(r, p).
\end{align*}
\]  

(103)  

(104)

With the help of the Poisson bracket defined by

\[
\{a, b\} = \frac{\partial a}{\partial x_\alpha} \frac{\partial b}{\partial p_\alpha} - \frac{\partial b}{\partial x_\alpha} \frac{\partial a}{\partial p_\alpha},
\]

we can write the components of $\hat{\lambda}_r'$ and $\hat{\lambda}_p$ as

\[
\begin{align*}
\hat{\lambda}_{x_\alpha} &= -i \{\cdot, p_\alpha\}, \\
\hat{\lambda}_{p_\alpha} &= i \{\cdot, x_\alpha\},
\end{align*}
\]  

(106)  

(107)

where the dot indicates where to put the function acted upon. For example, for any function $f$ the operator $\hat{\lambda}_{x_\alpha}$ acts according to $\hat{\lambda}_{x_\alpha} f = -i \{f, p_\alpha\}$.

The Liouvillian operator \((93)\) can be written as

\[
\hat{L} = -i \{\cdot, H\},
\]

(108)

where $H$ is the classical Hamiltonian given by

\[
H = \frac{(p - A(r))^2}{2m} + V(r).
\]

The Schrödinger-like equation \((8)\) becomes

\[
\frac{\partial \psi}{\partial t} + \{\psi, H\} = 0,
\]

(109)

or written for the complex conjugate $\psi^*$

\[
\frac{\partial \psi^*}{\partial t} + \{\psi^*, H\} = 0.
\]

(110)

Equations \((109)\) and \((110)\) are linear in the derivatives and they can be combined into the single equation

\[
\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0,
\]

(111)

where $\rho = |\psi|^2$. Equation \((111)\) is the classical Liouville equation for the evolution of the probability distribution in phase space. Thus, the operational formulation of classical mechanics is equivalent to classical statistical mechanics.

Let us note that the following probability density, known as the Klimontovich distribution for the particle \([35]\), given by

\[
\rho(t) = \delta(r - r(t)) \delta(p - p(t)),
\]

(112)
is a solution to Liouville’s equation as long as \( x_\alpha(t) \) and \( p_\alpha(t) \) are solution to Hamilton equations

\[
\begin{align*}
\frac{dx_\alpha}{dt} &= \frac{\partial H}{\partial p_\alpha} \\
\frac{dp_\alpha}{dt} &= -\frac{\partial H}{\partial x_\alpha}.
\end{align*}
\]

(113) (114)

The probability distribution represents a point in phase space and that point will move along a trajectory given by the solution of the Hamilton equations.

6 Comparison with the Quantum Case

For the sake of comparing the classical and quantum unitary representations of the Galilei group, let us restate the commutation relations of the algebra in terms of the usual symbols used in quantum mechanics (\( \hat{J}, \hat{P}, \hat{G}, \hat{H}, M \)). The Galilei algebra for the vanishing commutators read

\[
\begin{align*}
\left[ \hat{P}_\alpha, \hat{P}_\beta \right] &= \left[ \hat{G}_\alpha, \hat{G}_\beta \right] = \left[ \hat{J}_\alpha, \hat{H} \right] = \left[ \hat{P}_\alpha, \hat{H} \right] = 0, \\
\left[ \hat{J}_\alpha, \hat{J}_\beta \right] &= i\varepsilon_{\alpha\beta\gamma} \hat{J}_\gamma; \\
\left[ \hat{J}_\alpha, \hat{P}_\beta \right] &= i\varepsilon_{\alpha\beta\gamma} \hat{P}_\gamma; \\
\left[ \hat{J}_\alpha, \hat{G}_\beta \right] &= i\varepsilon_{\alpha\beta\gamma} \hat{G}_\gamma; \\
\left[ \hat{G}_\alpha, \hat{H} \right] &= i\hat{P}_\alpha.
\end{align*}
\]

(115)

(116)

As was done for the classical case, the idea is to look for a Hilbert space \( H_Q \) where the space-time transformation of the Galilei group are represented by

| Space – Time Transformations | Unitary Operator |
|-----------------------------|------------------|
| Rotations \( x \to R_\alpha(\theta_\alpha)x \) | \( e^{-i\theta_\alpha \hat{J}_\alpha} \) |
| Spatial Displacement \( x \to x + a \) | \( e^{-ia_\alpha \hat{P}_\alpha} \) |
| Galilean Boost \( x \to x + bt \) | \( e^{ib_\alpha \hat{G}_\alpha} \) |
| Time Displacement \( t \to t + \tau \) | \( e^{i\tau \hat{H}} \) |

Ignoring the spin degrees of freedom, the components of the position operator \( \hat{R} \) are assumed to form a complete set of commuting observables and the Hilbert
space $\mathcal{H}_Q$ is build using kets of the form $|r\rangle$. With this choice of Hilbert space, and assuming relation (15) for the position and velocity, the usual relations for quantum mechanics for the particle can be found [28, 29].

$$\begin{align*}
\left[ \hat{X}_\alpha, \hat{P}_\beta \right] &= i\delta_{\alpha\beta}, \\
\hat{J} &= \hat{R} \times \hat{P}, \\
\hat{G} &= M\hat{R} - i\hat{P}, \\
\hat{H} &= \frac{\hat{P}^2}{2M}.
\end{align*}$$

A difference of the quantum representation over the classical one is that all the operators involved in the algebra have direct physical significance: $M$ is the mass, $\hat{P}$ is the momentum, $\hat{J}$ is the angular momentum, $\hat{H}$ is the energy and $\hat{G}$ is a physical quantity sometimes called the dynamic mass moment. The above is in contrast to the classical case where it is hard to say what is the physical significance of $\hat{L}$ or $\hat{\lambda}_x$ [17]. On the other hand, due to the non-vanishing of $M$ (quantum representation with vanishing $M$ are unphysical [5]) the quantum case end up being a projective (or “up to a phase”) representation. In the classical case, as we shown in section 3, the central charge $\mathcal{M}$ is not only allowed to vanish but it is actually required to do so.

The origin of the difference between the two theories lies in the amount of information that a state can contain in relation to the algebra of operators that act on the Hilbert space $\mathcal{H}$. In quantum mechanics the number of mutually commuting operators is restricted to three compared with the classical case where complete set consist of six operators, three from the position and three from the momentum (or velocity).

From the point of view of “wave mechanics”, the choice of the different algebra of operators acting on the Hilbert space now means that the classical $\psi$ and $\rho$ are functions of phase space (or the tangent bundle of configuration space), unlike the quantum case where $\psi$ and $\rho$ are functions of configuration space alone. The structure of the two theories is markedly different. As an example, we have seen that the classical Schrödinger-like equation in phase space is the Liouville equation (111), a equation of first derivatives in contrast with the second derivatives appearing in the Schrödinger equation. As a result in quantum mechanics the wave functions squared is the probability density in volume whereas in the classical case it is the probability density in phase space. Moreover, going to the polar representation

$$\psi = \sqrt{\rho} e^{iS},$$

the two theories differ in the sense that by replacing (121) in the Schrödinger equation a set of two coupled differential equations is obtained giving rise to

---

3 The difference can not be in the mathematical structure of the Hilbert space used because both classical and quantum mechanical Hilbert spaces are isomorphic to the space of square-integrable functions.
the Madelung fluid \[36\] which sparked a possible new interpretation of quantum mechanics (see, e.g., \[37\]). For the classical case, after replacing \[124\] into the Liouville equation the resulting equations for \(\rho\) and \(S\) are decoupled \[4\].

Finally, let us mention another aspect of the operational version of classical mechanics that is different from the quantum case, the superposition of states. Coherent superposition of states has been studied in the context of the KvN theory \[24, 38\] (the same analysis holds if we use the velocity operator instead the momentum) and there seems to be two possibilities. (1) If the observables of the theory are only the operators corresponding to phase space variables (position, momentum and any combination of them) then a superselection rule gets triggered, coherent superposition cannot be distinguished from a mixed density matrix and relative phases cannot be detected. This is somewhat strange compared to the quantum case, but it makes some sense if we consider that the pure state of classical mechanics are delta functions in phase space and all other densities are statistical states. (2) To avoid the superselection mechanism the auxiliary operators have to be included as observables. As a consequence, this will make the phases in the KvN wavefunctions to be observable also. It is known, for example, that the lack of degeneracy of the zero eigenvalue of the Liouvillian implies ergodicity \[39\], so it is tempting to prefer this possibility. However, since we are increasing the number of observables, this path seems to imply a more general theory that just classical mechanics. In the end, the question of which of the above two choices is the better seems to be unanswered \[5\].

7 Discussion and Concluding Remarks

The operational formulation of classical mechanics given in the article is completely independent from any quantum result. The equations given in the present work were not derived as a classical limit of some quantum equations. Just as the non-relativistic quantum mechanics can be derived, under certain assumptions, from a unitary representation of the Galilei group, we have shown that a Hilbert space formulation of classical mechanics (leading later to the Koopman-von Neumann theory) can be obtained as a unitary representation of the same group. The derivations, albeit similar in spirit, differ in a crucial assumption. In the classical case we demand the base kets \(|r, v\rangle\) to be eigenvectors of the position and the velocity operators simultaneously, in contrast with the quantum mechanical case.

Our formalism stresses the fact that in classical mechanics, and in contrast with quantum mechanics, the potentials are auxiliary optional quantities. We can see from equations \[8\] and \[34\] that the force operator is sufficient to give a complete description of the evolution of the state of the classical particle. Moreover, the formalism allows forces that cannot be derived from any generalized potential.

\[4\] It has been shown that in the classical case \(S\) can be related to the classical action \[22\].

\[5\] A clue to this question might be obtained from a measurement theory of the KvN formalism \[19\] that includes analysis of operators like the Liouvillian.
There are several aspects that can be analyzed in further investigations. For example, through the article we only used Cartesian coordinates but we know classical mechanics gives plenty of freedom to choose other charts. Hence, the following question arises: what is the role of constraints and generalized coordinates in the operational version of classical mechanics?

Another relevant result that requires further clarification is the relation between the quantum canonical transformation we used to go from the irreducible set of operators \( \{ \hat{R}, \hat{V}, \hat{\lambda}_r, \hat{\lambda}_v \} \) to the set \( \{ \hat{R}, \hat{P}, \hat{\lambda}'_r, \hat{\lambda}_p \} \) and the Legendre transformation used to pass from Lagrangian mechanics to Hamiltonian mechanics. From a more abstract point of view, the rays \( \{ |r, v \rangle \} \) and \( \{ |r, p \rangle \} \) are in a one to one correspondence with points in the tangent and the cotangent bundles of configuration space, respectively. We demonstrated that the set of kets \( |r, v \rangle \) and \( |r, p \rangle \) are related by a unitary transformation. It remains to be seen what is the exact relation between our treatment and the geometrical one of analytical mechanics.

Besides the purely classical results, we hope that our work here helps in the advancement of interesting research topics that probe into the interface between quantum mechanics and the classical physics, such like decoherence, quantum chaos, the hydrodynamical interpretation of quantum mechanics, or quantum-classical hybrid systems [41, 42]. With a Hilbert space at hand closely related to classical mechanics such research are enriched with new perspectives. In this paper we laid a solid ground for this kind of investigations by showing that both quantum and classical mechanics can be based on unitary representation of the Galilei group.

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