A central extension of $U_q\mathfrak{sl}(2|2)^{(1)}$ and $R$-matrices with a new parameter

Hiroyuki Yamane

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka 560-0043, Japan

e-mail:yamane@ist.osaka-u.ac.jp

Abstract

In this paper, using a quantum superalgebra associated with the universal central extension of $\mathfrak{sl}(2|2)^{(1)}$, we introduce new $R$-matrices having an extra parameter $x$. As $x \to 0$, they become those associated with the symmetric and anti-symmetric tensor products of the copies of the vector representation of $U_q\mathfrak{sl}(2|2)^{(1)}$.

(Received:
I. INTRODUCTION

The Yang-Baxter equation (YBE for short) has played important roles in study of statistical mechanics, knot theory, conformal field theory etc., and many of its solutions are associated with finite dimensional irreducible representations of quantum affine algebras and superalgebras. We call the solutions of the YBE the $R$-matrices.

If a finite dimensional simple Lie superalgebra is $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $F(4)$, $G(3)$ or $D(2,1;\alpha)$ ($\alpha \neq 0$, $-1$), it is called a basic classical Lie superalgebra (BCLS for short). We first recall that $A(m,n)$ coincides with $\text{sl}(m+1|n+1)$ if and only if $m \neq n$, and that $\text{sl}(m+1|m+1)$ is a one dimensional central extension of $A(m,m)$. Let $\mathfrak{g}$ be a BCLS and $\overline{\mathfrak{g}}$ the universal central extension (UCE for short) of $\mathfrak{g}$. We also recall that $\overline{\mathfrak{g}} = \mathfrak{g}$ if $\mathfrak{g} \neq A(m,m)$ for any $m$, and that $\overline{A(m,m)} = \text{sl}(m+1|m+1)$ ($m \geq 2$) and $\overline{A(1,1)} = \mathfrak{d}$. Here $\mathfrak{d}$ is the Lie superalgebra called $D(2,1;-1)$.

The $\mathfrak{d}$ is a two (resp. three) dimensional central extension of $\text{sl}(2|2)$ (resp. $A(1,1)$). The UCE of $\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}]$ is given by the affine version $\overline{\mathfrak{g}}^{(1)} = \overline{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus Cc$ of $\overline{\mathfrak{g}}$. Motivated by this fact, we direct our attention to
the quantum superalgebra $U_q\mathfrak{sh}(1)$ (strictly speaking, $\tilde{U} = \tilde{U}_q\mathfrak{sh}(1)$) in order to give new $R$-matrices $\hat{R}(u, v; x)$ satisfying the (twisted) YBE:

\[
(\hat{R}(v, w; x) \otimes I)(I \otimes \hat{R}(u, w; q^n x)) (\hat{R}(u, v; x) \otimes I) = (I \otimes \hat{R}(u, v; q^n x))(\hat{R}(u, w; x) \otimes I))(I \otimes \hat{R}(v, w; q^n x))
\]

for some integer $n$, where $u, v, x \in \mathbb{C}$ are continuous parameters. This can be viewed as a quantum dynamical YBE (see Appendix). The $R$-matrices we will give are such that as $x \to 0$, they become the $U_q\mathfrak{sl}(2|2)^{(1)} R$-matrices$^{4,8-10}$ associated with the symmetric and anti-symmetric tensor products of the copies of the vector representation $\varphi$ of $U_q\mathfrak{sl}(2|2)^{(1)}$. One of our tools is a four dimensional irreducible representation $\rho_x$ of $\tilde{U}$ with the parameter $x$ such that $\rho_0 = \varphi \circ p$, where $p : \tilde{U} \to U_q\mathfrak{sl}(2|2)^{(1)}$ is the natural epimorphism.

The paper is organized as follows. In Section 1, we introduce $\tilde{U}$ and $\rho_x$. In Section 2, we give $\hat{R}(u, v; x)$ associated with $\rho_x$. In Section 3, we give all the $\hat{R}(u, v; x)$’s mentioned above using the fusion process.
II. A CENTRAL EXTENSION OF $\mathfrak{u}_q\mathfrak{sl}(2|2)^{(1)}$

Let $E = \bigoplus_{i=0}^{4} \mathbb{C}\varepsilon_i$ be the five dimensional vector space. Define the symmetric bilinear form $(\ , \ )$ on $E$ by $(\varepsilon_0, \varepsilon_0) = 0$, $(\varepsilon_1, \varepsilon_1) = (\varepsilon_2, \varepsilon_2) = 1$, $(\varepsilon_3, \varepsilon_3) = (\varepsilon_4, \varepsilon_4) = -1$ and $(\varepsilon_i, \varepsilon_j) = 0$ if $i \neq j$. Let $\alpha_0 := \varepsilon_0 - \varepsilon_1 + \varepsilon_4$ and $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq 3)$. Define the parity $p(\alpha_i)$ to be $(4 - (\alpha_i, \alpha_i)^2)/4$.

Then the Cartan matrix of $A(1,1)^{(1)}$ is given by the $4 \times 4$ matrix $(a_{ij})$, where $a_{ij} = 2(\alpha_i, \alpha_j)/((\alpha_i, \alpha_i) + 2p(\alpha_i))$.

Throughout this paper, we assume $q \in \mathbb{C}$ to be such that $q \neq 0$ and $q' \neq 1$ for every positive integer $r$. Let $\tilde{U} = \tilde{U}_q\mathfrak{o}^{(1)}$ be the associative $\mathbb{C}$-algebra presented by the generators $s, K_i^\pm, E_i, F_i$ $(0 \leq i \leq 3)$ and the defining relations:

\begin{align*}
s^2 &= 1, \quad sK_is = K_is, \quad sE_is = (-1)^{p(\alpha_i)}E_is, \quad sF_is = (-1)^{p(\alpha_i)}F_is, \\
K_iK_{i}^{-1} &= 1, \quad K_iK_j = K_jK_i, \\
K_iE_jK_{i}^{-1} &= q^{(\alpha_i, \alpha_j)}E_j, \quad K_iF_jK_{i}^{-1} = q^{-\alpha_i, \alpha_j}F_j, \\
[E_i,F_j] &= \delta_{ij}\frac{K_i - K_{i}^{-1}}{q - q^{-1}} \quad \text{if } (i,j) \text{ is neither } (2,0) \text{ nor } (0,2), \\
K_2[E_2,F_0] &\in Z(\tilde{U}), \quad K_2^{-1}[E_0,F_2] \in Z(\tilde{U}),
\end{align*}
where \([E_i, F_j] := E_i F_j - (-1)^{p(\alpha_i)p(\alpha_j)} F_j E_i\) and \(Z(\tilde{U})\) is the center of \(\tilde{U}\).

We view \(\tilde{U}\) as the (non-\(\mathbb{Z}_2\)-graded) Hopf algebra with the comultiplication 
\[
\Delta : \tilde{U} \to \tilde{U} \otimes \tilde{U}
\]
satisfying:

\[
\Delta(s) = s \otimes s, \quad \Delta(K_i) = K_i \otimes K_i,
\]
\[
\Delta(E_i) = E_i \otimes 1 + K_i s^{p(\alpha_i)} \otimes E_i + \delta_{i0}(q - q^{-1})s[E_0, F_2] \otimes E_2,
\]
\[
\Delta(F_i) = F_i \otimes K_i^{-1} + s^{p(\alpha_i)} \otimes F_i - \delta_{i0}(q - q^{-1})F_2 \otimes [E_2, F_0].
\]

We do not give the antipode and the counit; we do not need them. We define \(\Delta^{(n-1)} : \tilde{U} \to \tilde{U} \otimes^n\) by letting \(\Delta^{(1)} = \Delta\) and \(\Delta^{(m)} = (\text{id}_{\tilde{U}} \otimes \Delta^{(m-1)}) \circ \Delta\) \((m \geq 2)\).

Remark: (1) The above comultiplication is not standard. Taking the twisting\(^{11}\) for the \(\tilde{U}\), we get the standard comultiplication of a quantum superalgebra defined for a Dynkin diagram other than the one associated with the Cartan matrix \((a_{ij})\) (see above); the \(A(1,1)\) has the two Dynkin diagrams.

(2) Let \(\tilde{U}'\) be the subalgebra of \(\tilde{U}\) generated by \(K_i^\pm, E_i, F_i\). Then \(\tilde{U} = \tilde{U}' \oplus \tilde{U}'s\). There exists a nonzero ideal \(J\) of \(\tilde{U}'\) such that \(\tilde{U}' / J\)
can be regarded as $U_q \mathfrak{d}^{(1)}$. We can get generators of $J$ in the same way as in Ref. 12. By the same argument as in the proof of Theorem 8.4.3 of Ref. 12, we can get the natural epimorphism from $U_q \mathfrak{d}^{(1)}$ to $U_q \mathfrak{sl}(2|2)^{(1)}$.

Let $V_x = \mathbb{C}^4$ be the four dimensional vector space, where $x \in \mathbb{C}$ is a parameter. Put $\theta(i) := (1 - (\varepsilon_i, \varepsilon_i))/2$. Define the irreducible representation $\rho_x : \widetilde{U} \to \text{End}(V_x)$ by:

$$\rho_x(s) = \sum_{j=1}^{4} (-1)^{\theta(j)} E_{jj}, \quad \rho_x(K_i) = \sum_{j=1}^{4} q^{(\alpha_i, \varepsilon_j)} E_{jj},$$

$$\rho_x(E_0) = E_{41}, \quad \rho_x(E_1) = E_{12},$$

$$\rho_x(E_2) = E_{23} + x E_{41}, \quad \rho_x(E_3) = E_{34},$$

$$\rho_x(F_0) = -E_{14} - xq^{-1} E_{32}, \quad \rho_x(F_1) = E_{21},$$

$$\rho_x(F_2) = E_{32}, \quad \rho_x(F_3) = -E_{43}.$$

Let $\widetilde{U}_0 = \widetilde{U}_q \mathfrak{d}$ be the subalgebra of $\widetilde{U}$ generated by $s, K_i^\pm, E_i, F_i$ $(1 \leq i \leq 3)$. Define the vector subspaces $V_x^{(i)} = V_{x,y}^{(i)}$ $(i = 1, 2)$ of $V_x \otimes V_y$.
by

\[ V'_x(1) := \mathbb{C}(e_3 \otimes e_3) \oplus \mathbb{C}(e_4 \otimes e_4) \]
\[ \oplus \bigoplus_{i<j} \mathbb{C}(e_i \otimes e_j - (-1)^{\theta(i)\theta(j)}qe_j \otimes e_i \]
\[ + \delta_{i1}\delta_{j2}(q^2 ye_3 \otimes e_4 + xe_4 \otimes e_3)) \]

and

\[ V'_x(2) := \mathbb{C}(e_1 \otimes e_1) \oplus \mathbb{C}(e_2 \otimes e_2) \oplus \bigoplus_{i<j} \mathbb{C}(e_i \otimes e_j + (-1)^{\theta(i)\theta(j)}q^{-1}e_j \otimes e_i). \]

**Lemma 1:** \( V'_x(1) \) is an irreducible \( \tilde{U}^0 \)-module. Moreover, \( V_x \otimes V_y \) is a completely reducible \( \tilde{U}^0 \)-module if and only if \( y = qx \). If this is the case, \( V'_x(2) \) is an irreducible \( \tilde{U}^0 \)-module which is not isomorphic to \( V'_x(1) \); in particular, \( V_x \otimes V_{qx} \) has an irreducible \( \tilde{U}^0 \)-submodule decomposition \( V'_x(1) \oplus V'_x(2) \).

**Proof:** For each 1 \( \leq i \leq 4 \), the weight space including \( e_i \otimes e_i \) is one dimensional. Hence, if \( V_x \otimes V_y \) is a completely reducible \( \tilde{U}^0 \)-module, there exists an irreducible \( \tilde{U}^0 \)-module including \( e_i \otimes e_i \). Using this fact, we can check the lemma directly. \( \square \)
III. R-MATRIX FOR THE VECTOR REPRESENTATION

Define \( P^{(i)}_x \in \text{End}(V_x \otimes V_{q^x}) \) \((i = 1, 2)\) by \( P^{(i)}_x(v) = \delta_{ij} v \) \((v \in V^{(j)}_x)\).

Put
\[
\tilde{R}(u, v; x) := (q^2 u - v)P^{(1)}_x + (q^2 v - u)P^{(2)}_x,
\]
where \( u, v \in \mathbb{C} \). Then:
\[
\tilde{R}(u, v; x) = (q^2 v - u) \left( \sum_{i=1}^2 E_{ii} \otimes E_{ii} \right) + (q^2 v - u) \left( \sum_{i=3}^4 E_{ii} \otimes E_{ii} \right) \\
+ (q^2 - 1) \sum_{i<j} (vE_{ii} \otimes E_{jj} + uE_{jj} \otimes E_{ii}) \\
- q(u - v) \sum_{i \neq j} (-1)^{\theta(i)\theta(j)} E_{ij} \otimes E_{ji} \\
+ x(q^2 - 1)(u - v)(qE_{31} \otimes E_{42} - q^2 E_{32} \otimes E_{41}) \\
- E_{41} \otimes E_{32} + qE_{42} \otimes E_{31}).
\]

For \( u \in \mathbb{C}^\times \), define \( \chi_u \in \text{Aut}(U) \) by \( \chi_u(s) = s, \chi_u(K_i) = K_i, \chi_u(E_i) = u^{-\delta_{ij}} E_i \) and \( \chi_u(F_i) = u^{\delta_{ij}} F_i \). Put \( \rho_{u,v,x} := (\rho_x \otimes \rho_{q^x}) \circ (\chi_u \otimes \chi_v) \circ \Delta \). Using
and Lemma 1, we can directly check that:

\[
\hat{R}(u, v; x) \rho_{u,v,x}(X) = \rho_{v,u,x}(X) \hat{R}(u, v; x)
\]

(4)

for \( X \in \tilde{U} \).

**Theorem 1:** The \( \hat{R}(u, v; x) \) satisfies the YBE in the form of (4) with \( n = 1 \).

**Proof:** Let \( E'_i, F'_i, H'_i \) \((0 \leq i \leq 3)\) be the Chevalley generators of \( \text{sl}(2|2)^{(1)} \). Then there exists a representation \( \hat{\psi}_u : \text{sl}(2|2)^{(1)} \to \text{End}(C^4) \) sending \( E'_i, F'_i, H'_i \) to the limits of \( \rho_x \circ \chi_u(E_i), \rho_x \circ \chi_u(F_i), (q - q^{-1})^{-1} \rho_x \circ \chi_u(K_i - K_i^{-1}) \) as \((q, x) \to (1, 0)\), respectively. Notice that \( \psi := \hat{\psi}_{1|\text{sl}(2|2)} \) is an irreducible representation of \( \text{sl}(2|2) \) and that there exist a highest root vector \( E'_{\alpha_1 + \alpha_2 + \alpha_3} \) and a lowest root vector \( E'_{-(\alpha_1 + \alpha_2 + \alpha_3)} \) of \( \text{sl}(2|2) \) such that

\[
u \psi(E'_{\alpha_1 + \alpha_2 + \alpha_3}) = \hat{\psi}_u(F_0), \quad u^{-1} \psi(E'_{-(\alpha_1 + \alpha_2 + \alpha_3)}) = \hat{\psi}_u(E_0).
\]

(5)

Then, using (4), together with the same argument used in the proof of Proposition 3 in Ref. 3, we get the theorem. \( \square \)
IV. R-MATRIX FOR THE (ANTI-)SYMMETRIC TENSORS

Here we use a similar process to the fusion process.\textsuperscript{13−15} To begin with, we recall some facts\textsuperscript{16,17} about the Hecke algebra $H_n(q^2)$ associated with the symmetric group $S_n$; the $H_n(q^2)$ is the associative $\mathbb{C}$-algebra presented by the generators $h_i$ ($1 \leq i \leq n - 1$) and the defining relations:

$$(h_i - q^2)(h_i + 1) = 0, \ h_i h_{i+1} = h_{i+1} h_i, \text{ and } h_i h_j = h_j h_i \ (|i - j| \geq 2).$$

We abbreviate $H_n(q^2)$ to $H$. We know that there exists a $\mathbb{C}$-basis $\{h(\sigma) | \sigma \in S_n\}$ of $H$ such that $h(1) = 1$, $h(\sigma_i) = h_i$ and $h(\sigma'\sigma) = h(\sigma') h(\sigma)$ if $\ell(\sigma'\sigma) = \ell(\sigma') + \ell(\sigma)$. Here $\sigma_i$ is the simple transposition $(i, i+1)$ and $\ell(\sigma)$ is the length of $\sigma$ with respect to $\sigma_i$’s.

Put

$$e_+ := \sum_{\sigma \in S_n} h(\sigma), \quad e_- := \sum_{\sigma \in S_n} (-q^{-2})^{\ell(\sigma)} h(\sigma).$$

Then $h_i e_+ = q^2 e_+$, $h_i e_- = -e_-$ and

$$e_+^2 = \sum_{\sigma \in S_n} q^{2\ell(\sigma)} e_+, \quad e_-^2 = \sum_{\sigma \in S_n} q^{-2\ell(\sigma)} e_-.$$  \hspace{1cm} (6)
Now we treat $R$-matrices. Let $W_x^{(n)} := V_x \otimes V_{q x} \otimes \cdots \otimes V_{q^{n-1} x}$. Put:

$$
\check{R}_i(u, v; x) := I \otimes I^{n-i-1} \otimes \check{R}_i(u, v; q_i^{-1} x) \otimes I^{n-i+1} \in \text{End}(W_x^{(n)}).
$$

By Theorem 1, we can define $\check{R}(a; x|\sigma) \in \text{End}(W_x^{(n)})$, $a \in (\mathbb{C}^\times)^n$ and $\sigma \in S_n$, inductively by

$$
\check{R}(a; x|1) = I^n, \quad \check{R}(a; x|\sigma_i) = \check{R}(a_i, a_{i+1}; x)
$$

and

$$
\check{R}(a; x|\sigma') = \check{R}(\sigma[a]; x|\sigma') \check{R}(a; x|\sigma) \quad \text{if} \quad \ell(\sigma') = \ell(\sigma') + \ell(\sigma),
$$

where $\sigma[a] := (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$. By Theorem 1 and [2], there exists a unique representation $\pi_x^{(n)} : H \to \text{End}(W_x^{(n)})$ such that $\check{R}_i(u, v; x) = \pi_x^{(n)}(uh_i - vq^2h_i^{-1})$.

Let $p_\pm := (1, q^\mp 2, \ldots, q^\mp 2(n-1)) \in \mathbb{C}^n$. Let $\gamma_n \in S_n$ be such that $\gamma_n(i) = n - i + 1$.

**Lemma 2:** Let $u \in \mathbb{C}$. Then:

$$
\check{R}(up_\pm; x|\gamma_n) = u^{\ell(\gamma_n)} a_\pm(q) \pi_x^{(n)}(e_\pm)
$$

for some $a_\pm(q) \in \mathbb{C}^\times$.  

11
This can be checked directly; a similar formula has been given in Section 5 in Ref. 14.

Let $V_{\pm,x} := \pi_{\pm,x}^{(n)}(e_{\pm})W_{x}^{(n)}$. By (6), $d_{\pm}(n) := \dim V_{x,x}$ does not depend on $q$ or $x$. For $a \in (\mathbb{C}^\times)^n$, define the representation $\rho_{a,x} : \tilde{U} \to \text{End}(W_{x}^{(n)})$ by

$$\rho_{a,x} := (\rho_{x} \otimes \cdots \otimes \rho_{q^{n-1}x}) \circ (\chi_{a_{1}} \otimes \cdots \otimes \chi_{a_{n}}) \circ \Delta^{(n-1)}.$$  

By (14), we have:

$$\tilde{R}(a; x|\sigma)\rho_{a,x}(X) = \rho_{\sigma[a],x}(X)\tilde{R}(a; x|\sigma)$$  \hspace{1cm} (7)

for $X \in \tilde{U}$. By Lemma 2 and (7), we may define the representation $\rho_{u,x}^{\pm,(n)} : \tilde{U} \to \text{End}(V_{\pm,x})$ by $\rho_{u,x}^{\pm,(n)}(X) = \rho_{\gamma_{u}[u\pm],x}(X)|_{V_{\pm,x}}$. Notice that

$$\rho_{u,x}^{\pm,(n)} = \rho_{1,x}^{\pm,(n)} \circ \chi_{u}. \hspace{1cm} (8)$$

We have a representation $\hat{\psi}_{u}^{\pm,(n)} : \mathfrak{sl}(2|2)^{(1)} \to \text{End}(\mathbb{C}d_{\pm}(n))$ sending $E_{i}'$, $F_{i}'$, $H_{i}'$ to the limits of $\rho_{u,x}^{\pm,(n)}(E_{i})$, $\rho_{u,x}^{\pm,(n)}(F_{i})$, $(q - q^{-1})^{-1}\rho_{u,x}^{\pm,(n)}(K_{i} - K_{i}^{-1})$ as $(q, x) \to (1, 0)$. Define the representation $\psi^{\pm,(n)}$ of $\mathfrak{sl}(2|2)$ to be $$(\hat{\psi}_{u}^{\pm,(n)})|_{\mathfrak{sl}(2|2)}.$$  Then $\psi^{+, (n)}$ (resp. $\psi^{-,(n)}$) is the $n$-fold symmetric (resp.
anti-symmetric) tensor product of the vector representation \( \psi \) of \( \text{sl}(2|2) \).

By Ref. 18, we have:

**Lemma 3:** The \( \psi^{\pm,(n)} \) is irreducible. Moreover \( d_{\pm}(n) \neq 0. \)

Define \( \tau \in S_{2n} \) by \( \tau(i) = i + n, \tau(n + i) = i \) (\( 1 \leq i \leq n \)). For \( g, h \in \mathbb{C}^n \), let \( g \cup h := (g_1, \ldots, g_n, h_1, \ldots, h_n) \in \mathbb{C}^{2n} \). Let \( S_n \) be embedded into \( S_{2n} \) in the natural way. By Lemma 2, we have:

\[
\tilde{R}(\gamma_n[up_\pm] \cup \gamma_n[vp_\pm]; x|\tau)(\pi^{(n)}_x(e_\pm) \otimes \pi^{(n)}_{q^n_x}(e_\pm))
= \frac{(uv)^{-\ell(\gamma_n)}}{a_{\pm}(q)^2} \tilde{R}(\gamma_n[up_\pm] \cup \gamma_n[vp_\pm]; x|\gamma_n) \tilde{R}(vp_\pm; x|\gamma_n \tau) \tilde{R}(up_\pm \cup vp_\pm; x|\gamma_n) \\
= \frac{(uv)^{-\ell(\gamma_n)}}{a_{\pm}(q)^2} \tilde{R}(vp_\pm; x|\gamma_n \tau) \tilde{R}(up_\pm \cup vp_\pm; x|\gamma_n \tau) \\
= \frac{(uv)^{-\ell(\gamma_n)}}{a_{\pm}(q)^2} \tilde{R}(vp_\pm; x|\gamma_n \tau) \tilde{R}(up_\pm \cup vp_\pm; x|\gamma_n \tau) \\
= \frac{(uv)^{-\ell(\gamma_n)}}{a_{\pm}(q)^2} \tilde{R}(up_\pm \cup vp_\pm; x|\gamma_n \tau) \\
= \frac{(uv)^{-\ell(\gamma_n)}}{a_{\pm}(q)^2} \tilde{R}(up_\pm \cup vp_\pm; x|\gamma_n \tau) \\
= (\pi^{(n)}_x(e_\pm) \otimes \pi^{(n)}_{q^n_x}(e_\pm)) \tilde{R}(up_\pm \cup vp_\pm; x|\tau).
\]

Hence we may put:

\[
\tilde{R}^{\pm,(n)}(u, v; x) := \tilde{R}(\gamma_n[up_\pm] \cup \gamma_n[vp_\pm]; x|\tau)|_{V_{\pm,x} \otimes V_{\pm,q^n,x}} \\
\in \text{End}(V_{\pm,x} \otimes V_{\pm,q^n,x}).
\]

Let \( \rho_{u,v}^{\pm,(n)} := (\rho_{u,x}^{\pm,(n)} \otimes \rho_{v,q^n_x}^{\pm,(n)}) \circ \Delta \). Notice that

\[
\rho_{u,v}^{\pm,(n)}(X) = (\rho_{\gamma_n[up_\pm] \cup \gamma_n[vp_\pm], x}(X))|_{V_{\pm,x} \otimes V_{\pm,q^n,x}}. \quad (9)
\]

13
By (7) and (9), we have:

\[
\hat{R}^{\pm}_{\nu}(u, v; x) \rho_{\nu}^{\pm}(X) = \rho_{\nu}^{\pm}(X) \hat{R}^{\pm}_{\nu}(u, v; x).
\]  

(10)

for \( X \in \tilde{U} \).

**Theorem 2:** The \( \hat{R}^{\pm}_{\nu}(u, v; x) \) satisfies the YBE in the form of (7).

**Proof:** By (5), we have \( u \psi^{\pm}_{\nu}(E'_{\alpha_1 + \alpha_2 + \alpha_3}) = \hat{\psi}^{\pm}_{\nu}(F_0) \) and

\[
u^{-1} u \psi^{\pm}_{\nu}(E'_{\alpha_1 + \alpha_2 + \alpha_3}) = \hat{\psi}^{\pm}_{\nu}(E_0).\]

Noting this fact and using (8), (10) and Lemma 3, together with the same argument as in the proof of Proposition 3 in Ref. 3, we have the theorem. □

**ACKNOWLEDGMENTS**

The author thanks E. Date, M. Okado and Y. Koga for valuable comments. He also thanks Y.Z. Zhang for telling him about face-type dynamical \( R \)-matrices.
APPENDIX: A QUANTUM DYNAMICAL $R$-MATRIX

Here we show that the $\tilde{R}^{\pm,(n)}(u, v; x)$ can be viewed as a dynamical $R$-matrix. Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra. Let $V$ be a finite dimensional diagonalizable $\mathfrak{h}$-module, i.e., $V = \oplus_{\mu \in \mathfrak{h}} V_{\mu}$, where $V_{\mu} := \{ v | h.v = \mu(h)v \}$. We say that a (meromorphic) function $\tilde{R}' : \mathbb{C}^2 \times \mathfrak{h}^* \to \text{End}(V \otimes V)$ is a quantum dynamical $R$-matrix if it satisfies the quantum dynamical YBE (see Ref. 19 for example):

$$(\tilde{R}'(v, w, \lambda) \otimes I)\tilde{R}'_{23}(u, w, \lambda - h^{(1)})(\tilde{R}'(u, v, \lambda) \otimes I)$$

$$= \tilde{R}'_{23}(u, v, \lambda - h^{(1)})(\tilde{R}'(u, w, \lambda) \otimes I)\tilde{R}'_{23}(v, w, \lambda - h^{(1)}),$$

where $\tilde{R}'_{23}(u, v, \lambda - h^{(1)}) \in \text{End}(V \otimes^3)$ is defined by

$$\tilde{R}'_{23}(u, v, \lambda - h^{(1)})|_{V_{\mu} \otimes V \otimes V} = (I \otimes \tilde{R}'(u, v, \lambda - \mu))|_{V_{\mu} \otimes V \otimes V}.$$

Let $\mathfrak{h}'' = \mathbb{C}$ and let $\mathfrak{h}''$ act on $\mathbb{C}^{d_{\pm}(n)}$ by $z.v = -nzv$. Let $a \in \mathbb{C}$ be such that $e^a = q$. Define $\tilde{R}'' : \mathbb{C}^2 \times (\mathfrak{h}'')^* \to \text{End}(\mathbb{C}^{d_{\pm}(n)} \otimes \mathbb{C}^{d_{\pm}(n)})$ by $\tilde{R}''(u, v, \lambda) = \tilde{R}^{\pm,(n)}(u, v; e^{a\lambda(1)})$. By Theorem 2, $\tilde{R}''$ is a quantum dynamical $R$-matrix.
1C. N. Yang and M. L. Ge(eds), *Braid Groups, Knot theory and Statistical Mechanics*, (World Scientific, Singapore, 1989).

2V. G. Drinfeld, *ICM Proceedings*, (Berkeley 1986), p. 798.

3M. Jimbo, *Commun. Math. Phys.* **102**, 537 (1986).

4A. J. Bracken, M. D. Gould and R. B. Zhang,, *Mod. Phys. Lett. A* **5**, 831 (1989).

5V. G. Kac, *Adv. in. Math.* **26**, 8 (1977).

6V. G. Kac, *Lect. Note in Math.* **676**, 597 (1978).

7K. Iohara and Y. Koga, *Comment. Math. Helv.* **76**, 110 (2001)

8V. V. Bazhanov and A. G. Shadrikov, *Theoret. Math. Phys.* **73**, 1302 (1987).

9R. M. Gade, *J. Phys. A.* **31**, 4909 (1998).

10J. H. H. Perk, and C. L. Schultz, *Non-linear Integrable systems–Classical theory and quantum theory* ed, by M. Jimbo and T. Miwa, (World Scientific, Singapore, 1983), p. 135.

11S. M. Khoroshkin and V. N. Tolstoy, *Twisting of quantum (super-
1)algebras, in Generalized symmetries in physics (Claustal, 1993), (World Sci. Publishing, River Edge, NJ, 1994) p. 42.

12H. Yamane, Publ. RIMS Kyoto Univ. 35, 321 (1999), (Errata) Publ. RIMS Kyoto Univ. 37, 615 (2002).

13I. V. Cherednik, Sov. Math. Dokl. 33, 507 (1986).

14M. Jimbo, Lett. Math. Phys. 11, 247 (1986).

15P. P. Kulish, N. Yu Reshetikhin and E. K. Sklyanin, Lett. Math. Phys. 5, 393 (1981).

16C. W. Curtis and I. Reiner, Methods of Representation Theory with Applications to Finite Groups and Orders, Volume II, (A Wiley International Publication, New York. 1987).

17A. Gyoja, Osaka J. Math. 23, 841 (1986).

18A. N. Sergeev, Math. USSR-Sb. 51, 419 (1985).

19P. Etingof and A. Varchenko, Commun. Math. Phys. 196, 591 (1998).