SOLUTIONS AND STABILITY OF A GENERALIZATION
OF WILSON’S EQUATION

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Abstract. In this paper we study the solutions and stability of the generalized Wilson’s functional equation
\[ \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), \quad x, y \in G, \]
where \( G \) is a locally compact group, \( \sigma \) is a continuous involution of \( G \) and \( \mu \) is an idempotent complex measure with compact support and which is \( \sigma \)-invariant. We show that
\[ \int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)g(y), \quad x, y \in G \]
as long as \( f \neq 0 \) and \( \int_G f(t)d\mu(t) \neq 0 \).

We also study some stability theorems of that equation and we establish the stability on noncommutative groups of the classical Wilson’s functional equation
\[ f(xy) + \chi(y)f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G, \]
where \( \chi \) is a unitary character of \( G \).

1. Introduction and preliminaries

Although d’Alembert’s functional equation
\[ f(x+y) + f(x-y) = 2f(x)f(y) \quad \text{for all } x, y \in \mathbb{R} \]
for functions \( f: \mathbb{R} \to \mathbb{C} \) on the real line has it roots back in d’Alembert’s investigation of vibrating strings \([10]\) from 1975. Furthermore, one solution of \( (1.1) \) is \( f(x) = \cos(x) \), another \( f(x) = \cosh(x) \). The obvious extension of \( (1.1) \) from \( \mathbb{R} \) to an abelian group \((G, +)\) is the functional equation
\[ f(x+y) + f(x-y) = 2f(x)f(y) \quad \text{for all } x, y \in G, \]
where \( f: G \to \mathbb{C} \) is the unknown. The non-zero solution of equation \( (1.2) \) are of the form \( f(x) = \frac{\chi(x) + \chi(-x)}{2} \), \( x \in G \), where \( \chi \) is a character in \( G \). The result is obtained by Kannappan in \([18]\). If the group \( G \) is not assumed abelian the solutions of equation
\[ f(xy) + f(xy^{-1}) = 2f(x)f(y) \quad \text{for all } x, y \in G \]
are obtained by Davison \([11, 12]\). There are of the form \( f = \frac{1}{2}tr(g) \), where \( g \) is continuous algebraically irreducible representation of \( G \) on \( \mathbb{C}^2 \).

In \([31]\) Wilson dealt with functional equations related to and generalizing

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(1.1) on the real line. He generalized the d’Alembert’s functional equation (1.1) to

\[(1.4) \quad f(x + y) + f(x - y) = 2f(x)g(y) \text{ for all } x, y \in \mathbb{R}.\]

Let us note that if \(f \neq 0\) is a solution of equation (1.4), then \(g\) satisfies equation (1.1).

Some general properties of the solutions of equation

\[(1.5) \quad f(xy) + f(x\sigma(y)) = 2f(x)g(y) \text{ for all } x, y \in G\]

on a topological monoid equipped with a continuous involution \(\sigma\) can be found in [26].

Recently, Ebanks, Stetkær [13] and Stetkær [28] proved a natural interesting relation between Wilson’s functional equation (1.5) and d’Alembert’s functional equation

\[(1.6) \quad f(xy) + f(x\sigma(y)) = 2f(x)f(y) \text{ for all } x, y \in G\]

and for \(\sigma(x) = x^{-1}\). That is if \(f \neq 0\) is a solution of equation (1.5), the \(g\) is a solution of equation (1.6).

The Hyers-Ulam stability of d’Alembert’s functional equation (1.2) was investigated by J.A. Baker in [4]. In [5], J. Baker, J. Lawrence and F. Zorzitto introduced the superstability of the exponential equation \(f(xy) = f(x)f(y), x, y \in G\). Badory [11] gave a new, shorter proof of Baker’s result.

A different generalization of the result of Baker, Lawrence and Zorzitto was given by L. Székelyhidi [30].

On abelian groups, the stability of d’Alembert’s functional equation (1.2) and Wilson’s functional equation (1.4) and other functional equation has been investigated by several authors. The interested reader should refer to [2], [3], [6], [9], [15], [16], [17], [20], [21], [22], [23], [24], [25], [29] and [32], for a thorough account on the subject of stability of functional equations.

The aim of this paper is to study some properties of the solutions and Hyers-Ulam stability of some generalization of d’Alembert’s and Wilson’s functional equations which has been introduced in [14]. As an application we obtain the Hyers-Ulam stability of Wilson’s functional equation (1.5) on groups that need not be abelian.

Throughout this paper, we let \(G\) be a locally compact group, \(C(G)\) the complex algebra of all continuous complex valued functions on \(G\). \(M(G)\) the Banach algebra of the complex bounded measures on \(G\). It’s the topological dual of \(C_0(G)\) : The Banach space of continuous functions vanishing at infinity. Let \(\sigma: G \rightarrow G\) be a continuous involution of \(G\), that is \(\sigma(xy) = \sigma(y)\sigma(x)\) and \(\sigma(\sigma(x)) = x\) for all \(x, y \in G\). If \(\mu \in M(G)\) is a measure with copmact support, we let \(\mu_\sigma\) denote the complex measure with compact support and defined by the relation: \(<\mu_\sigma, f> = <\mu, f \circ \sigma>_\sigma, f \in C(G)\), where \(<\mu, f> = \int_G f(t)d\mu(t)\). We will say that \(\mu\) is \(\sigma\)-invariant if \(\mu = \mu_\sigma\). We recall that the convolution measure \(\mu * \mu\) is the measure defined on \(C(G)\) by \(<\mu * \mu, f> = \int_G \int_G f(ts)d\mu(t)d\mu(s)\). Finally, for a
continuous function \( f \) we let \( f_\mu(x) = \int_G f(tx)d\mu(t), \ x \in G \) and we say that \( f \) is \( \mu \)-bi-invariant, if \( \int_G f(tx)d\mu(t) = \int_G f(tx)d\mu(t) = f(x) \), for all \( x \in G \).

2. Relations between Wilson’s and d’Alembert’s functional equations

In the special case where \( \mu \) is a Gelfand measure or \( f \) satisfies some Kan-nappan type condition Elqorachi and Akkouchi [14], Proposition 3.2 obtained a natural relation between the generalized Wilson’s functional equation

\[
(2.1) \quad \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)g(y), \ x, y \in G
\]

and the generalized d’Alembert’s short functional equation

\[
(2.2) \quad \int_G g(xty)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) = 2g(x)g(y), \ x, y \in G
\]

That is if the pair \( f, g: G \to \mathbb{C} \), where \( f \neq 0 \), is a solution of generalized Wilson’s functional equation (2.1) then \( g \) is a solution of the generalized d’Alembert’s functional equation (2.2). In more general setting the authors [8, Corollary 2.7 (iii)] got a weaker result, that is if the pair \( f, g: G \to \mathbb{C} \), where \( f \neq 0 \), is a solution of generalized Wilson’s functional equation (2.1) then \( g \) is a solution of the generalized d’Alembert’s long functional equation

\[
(2.3) \quad \int_G g(xty)d\mu(t) + \int_G g(ytx)d\mu(t) + \int_G g(xt\sigma(y))d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) = 4g(x)g(y), \ x, y \in G.
\]

The following theorem is a generalization of the result obtained by Ebanks, Stetkær [13].

**Theorem 2.1.** Let \( \sigma \) be a continuous involution of \( G \). Let \( \mu \) be a complex measure with compact support and which is \( \sigma \)-invariant. If the pair \( f,g: G \to \mathbb{C} \), where \( f \neq 0 \) is a continuous solution of the generalized Wilson’s functional equation (2.1) and \( f \) is odd: \( f(\sigma(x)) = -f(x) \) for all \( x \in G \). Then \( g \) is a solution of the generalized d’Alembert’s short functional equation (2.2).

**Proof.** The proof is closely related to the one obtained by Ebanks, Stetkær [13]. Let us assume that the pair \( f, g \) is a solution of equation (2.1) with \( f \neq 0 \) and \( f(\sigma(x)) = -f(x) \) for all \( x \in G \). By replacing \( y \) by \( \sigma(y) \) in (2.1) we get \( g(\sigma(x)) = g(x) \) for all \( x \in G \). Now, we consider the new function

\[
\Psi(x,y) = \int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) - 2g(x)g(y), \ x, y \in G.
\]

By using (2.1) we obtain

\[
2f(z)\Psi(x,y) + 2f(y)\Psi(x,z) = 2f(z)[\int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) - 2g(x)g(y)]
\]
\[+2f(y)[\int_G g(xtz)\,d\mu(t) + \int_G g(\sigma(z)tx)\,d\mu(t) - 2g(x)g(z)]\]

\[= \int_G \int_G f(zsxtxy)\,d\mu(t)\,d\mu(s) + \int_G \int_G f(zs\sigma(y)\sigma(t)\sigma(x))\,d\mu(t)\,d\mu(s)\]

\[+ \int_G \int_G f(zs\sigma(y)tx)\,d\mu(t)\,d\mu(s) + \int_G \int_G f(zs\sigma(x)\sigma(t)y)\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(ztxsxy)\,d\mu(t)\,d\mu(s) - \int_G \int_G f(zsx\sigma(y))\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(zt\sigma(x)sy)\,d\mu(t)\,d\mu(s) - \int_G \int_G f(zt\sigma(x)s\sigma(y))\,d\mu(t)\,d\mu(s)\]

\[+ \int_G \int_G f(ytsxz)\,d\mu(t)\,d\mu(s) + \int_G \int_G f(y\sigma(z)\sigma(t)\sigma(x))\,d\mu(t)\,d\mu(s)\]

\[+ \int_G \int_G f(y\sigma(z)tx)\,d\mu(t)\,d\mu(s) + \int_G \int_G f(y\sigma(x)\sigma(t)z)\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(ytxsz)\,d\mu(t)\,d\mu(s) - \int_G \int_G f(ytx\sigma(z))\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(yt\sigma(x)sz)\,d\mu(t)\,d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))\,d\mu(t)\,d\mu(s)\]

\[= \int_G \int_G f(zt\sigma(y)\sigma(x))\,d\mu(t)\,d\mu(s) + \int_G \int_G f(zt\sigma(y)sx)\,d\mu(t)\,d\mu(s)\]

\[+ \int_G \int_G f(yt\sigma(z)\sigma(x))\,d\mu(t)\,d\mu(s) + \int_G \int_G f(yt\sigma(z)sx)\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(zsx\sigma(y))\,d\mu(t)\,d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(ytsx\sigma(z))\,d\mu(t)\,d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))\,d\mu(t)\,d\mu(s)\]

where the last identity is due to our assumption that \(\mu\) is \(\sigma\)-invariant. Using equation (2.1) and the above computation to obtain

\[2f(z)\Phi(x, y) + 2f(y)\Phi(x, z)\]

\[= 2g(x) \int_G f(zt\sigma(y))\,d\mu(t) + 2g(x) \int_G f(yt\sigma(z))\,d\mu(t)\]

\[+ \int_G \int_G f(ytxsz)\,d\mu(t)\,d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))\,d\mu(t)\,d\mu(s)\]

\[- \int_G \int_G f(zsx\sigma(y))\,d\mu(t)\,d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))\,d\mu(t)\,d\mu(s)\]

\[= 2g(x) \int_G f(zt\sigma(y))\,d\mu(t) - 2g(x) \int_G f(zt\sigma(y))\,d\mu(t)\]

\[+ \int_G \int_G f(zs\sigma(x)t\sigma(y))\,d\mu(t)\,d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))\,d\mu(t)\,d\mu(s)\]

\[+ \int_G \int_G f(yt\sigma(x)s\sigma(z))\,d\mu(t)\,d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))\,d\mu(t)\,d\mu(s) = 0,\]
which is due to assumptions that \( f \) is odd and \( \mu \) is \( \sigma \)-invariant. So, this implies that \( f(z)\Psi(x,y) + f(y)\Psi(x,z) = 0 \) for all \( x, y, z \in G \) and then we conclude that there exists \( c_x \in \mathbb{C} \) such that \( \Psi(x,y) = c_x f(y) \), \( c_x f(z)f(y) + c_x f(y)f(z) = 0 \). We get for any \( x, y \in G \) \( \Psi(x,y) = 0 \). Now, since \( g \) is even: \( g(\sigma(x)) = g(x) \) for all \( x \in G \) and \( \mu \) is \( \sigma \)-invariant we obtain

\[
2g(x)g(y) = \int_G g(xty)d\mu(t) + \int_G g(ytx)d\mu(t)
\]

This means that

\[
\int_G g(xty)d\mu(t) + \int_G g(ytx)d\mu(t) = 2g(x)g(y)
\]

for all \( x, y \in G \) and this completes the proof.

**Theorem 2.2.** Let \( \sigma \) be a continuous involution of \( G \). Let \( \mu \) be a complex measure with compact support such \( \mu * \mu = \mu \) and \( \mu \) is \( \sigma \)-invariant. If the pair \( f, g : G \rightarrow \mathbb{C} \), where \( f \neq 0 \) is a continuous solution of the generalized Wilson’s functional equation (2.1) such that \( f_\mu \neq 0 \). Then \( g \) is a solution of the generalized d’Alembert’s short functional equation (2.2).

In the proof we use the ideas of Stetkær [28]. Let \( g \) be a non zero fixed solution of equation (2.1).

We put \( W_g = \{ f : G \rightarrow \mathbb{C} | f \) is continuous, satisfies (2.1), \( f_\mu = f \) and \( f(e) = 0 \} \). A continuous solution \( f \) of equation (2.1) such that \( f_\mu \neq 0 \). The function \( f_\mu \) is also a non zero solution of equation (2.1). Since \( W_g = \{ 0 \} \), \( (f_\mu)_\mu = f_\mu \), then \( f_\mu(e) \neq 0 \). Replacing \( f_\mu \) by \( f_\mu/f_\mu(e) \) we may assume that \( f_\mu(e) = 1 \). If \( h \) is a continuous solution of equation (2.1), then \( h_\mu = h_\mu(e)f_\mu \) \( W_g = \{ 0 \} \), \( h_\mu = h_\mu(e)f_\mu \). Let \( x \in G \), since \( \delta(x)f_\mu(y) = \int_G f_\mu(xty)d\mu(t) \) is a solution of equation (2.1), \( \mu * \mu = \mu \) and \( (\delta(x)f_\mu)_\mu = (\delta(x)f_\mu) \mu \), there exists \( \psi(x) \) such that \( \delta(x)f_\mu = \psi(x)f_\mu \). In particular for \( y = e \) we have \( \psi(x)f_\mu(e) = \psi(x) = (\delta(x)f_\mu)(e) = \int_G f_\mu(xtd\mu(t) \), so we get

\[
\int_G f_\mu(xty)d\mu(t) = f_\mu(y) \int_G f_\mu(xtd\mu(t)
\]

for all \( x, y \in G \).

Now, we will show that \( \int_G f_\mu(xtd\mu(t) = f_\mu(x) \) for all \( x \in G \). Since \( f_\mu \)

\[
\int_G f_\mu(xty)d\mu(t) + \int_G f_\mu(x(t\sigma(y))d\mu(t) = 2f_\mu(x)g(y)
\]

\[
\int_G \int_G f_\mu(xtysd\mu(s)d\mu(t) + \int_G \int_G f_\mu(xst\sigma(y))d\mu(s)d\mu(t) = 2 \int_G f_\mu(xs)d\mu(s)g(y),
\]
which implies that \( \int_G f_\mu(xs) d\mu(s) = f_\mu(x) \), for all \( x \in G \) and then equation \([2.4]\) can be written as follows. \( \int_G f_\mu(xty) d\mu(t) = f_\mu(x)f_\mu(y) \) for all \( x,y \in G \). Substituting this result into

\[
\int_G f_\mu(xty) d\mu(t) + \int_G f_\mu(xt\sigma(y)) d\mu(t) = 2f_\mu(x)g(y)
\]

we get \( g(y) = \frac{f_\mu(y) + f_\mu(\sigma(y))}{2} \) and we can easily verify that \( g \) satisfies the generalized d’Alembert’s short functional equation \([2.2]\). This ends the proof of theorem.

\( \square \)

3. HYERS-ULAM STABILITY OF WILSON’S FUNCTIONAL EQUATION

In [8], Corollary 2.7 (iii) the authors proved that if the function

\( (x,y) \rightarrow \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y) \)

is bounded and \( f \) is unbounded then \( g \) is a solution of the generalized d’Alembert’s long functional equation \([2.3]\). In the following theorem under another kind of assumption we get that \( g \) is a solution of the generalized d’Alembert’s short functional equation \([2.2]\).

**Theorem 3.1.** Let \( \sigma \) be a continuous involution of \( G \). Let \( \mu \) be a discrete complex measure with compact support such that \( \mu \) is \( \sigma \)-invariant and \( \mu \ast \mu = \mu \). Let \( \delta \geq 0 \). If the pair \( f,g: G \rightarrow \mathbb{C} \), where \( f \) is an unbounded \( \mu \)-biinvariant continuous solution of the following inequality

\[
|\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)| \leq \delta
\]

for all \( x,y \in G \). Then \( g \) is a solution of the generalized d’Alembert’s short functional equation \([2.2]\).

**Proof.** Assume that \( f,g \) satisfy inequality \([3.1]\) where \( f \) is unbounded on \( G \). So, for all \( x,y \in G \) we have

\[
|\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)| \leq \delta,
\]

\[
|\int_G f(xt\sigma(y))d\mu(t) + \int_G f(xty)d\mu(t) - 2f(x)g(\sigma(y))| \leq \delta,
\]

and by triangle inequality we find

\[|2f(x)||g(y) - g(\sigma(y))| \leq 2\delta\]

for all \( x,y \in G \). Since \( f \) is assumed to be unbounded, then we get \( g(\sigma(y)) = g(y) \) for all \( y \in G \). In the rest of the proof we use some ideas of the proof of Theorem 2.1 and Theorem 2.2.

First Case: We assume that the function \( x \mapsto f(x) + f(\sigma(x)) \) is a bounded function on \( G \), that is \( |f(x) + f(\sigma(x))| \leq \beta \) for some \( \beta \geq 0 \) and for all \( x \in G \). Let \( \Psi(x,y) = \int_G g(xty)d\mu(t) + \int_G g(\sigma(y)tx)d\mu(t) - 2g(x)g(y), \) \( x,y \in G \). We
bounded function when $g$ is bounded. The above computations show that

$$2f(z)\Psi(x,y) + 2f(y)\Psi(x,z)$$

$$= 2g(x)\left[\int_G f(zt\sigma(y))d\mu(t) + \int_G f(yt\sigma(z))d\mu(t)\right]$$

$$- \int_G \int_G f(ytxs\sigma(z))d\mu(t)d\mu(s) - \int_G \int_G f(zs\sigma(x)t\sigma(y))d\mu(t)d\mu(s)$$

$$- \int_G \int_G f(zsxt\sigma(y))d\mu(t)d\mu(s) - \int_G \int_G f(yt\sigma(x)s\sigma(z))d\mu(t)d\mu(s).$$

So, we get

$$|2f(z)\Psi(x,y) + 2f(y)\Psi(x,z)| \leq 2\beta\|\mu\|\|g(x)\| + 2\beta\|\mu\|^2$$

for all $x, y, z \in G$, where $\|\mu\| = \text{Sup}\{|< f, \mu >|, f \in C(G) \|f\|_\infty = 1\}$. There are two possibilities. One is: $g$ is unbounded, then from [8, Corollary 2.7 (iii)] and Theorem 2.2, we conclude that $g$ is a solution of the generalized d’Alembert’s short functional equation (2.1).

The other possibility is: $g$ is a bounded function, then from (3.2) the function $(x, y, z) \mapsto 2f(z)\Psi(x,y) + 2f(y)\Psi(x,z)$ is a bounded function on $G$. Having assumed $f$ unbounded, this implies that then there exists a sequence $(z_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to +\infty} |f(z_n)| = +\infty$. By using (3.2), there exists $c_x \in \mathbb{C}$ such that $\Psi(x,y) = c_x f(y)$, and the function $(x, y, z) \mapsto 2f(z)c_x f(y) + 2f(y)c_x f(z)$ is bounded. By using again the unboundedness of $f$ we get $c_x = 0$ for all $x \in G$. This means that $2f(z)\Psi(x,y) + 2f(y)\Psi(x,z) = 0$ for all $x, y, z \in G$. The computations above shows that $g$ is a solution of the generalized d’Alembert’s short functional equation (2.2).

Now, let $f, g$ be functions such that $f$ is unbounded on $G$ and the function $(x,y) \mapsto \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)$ is bounded on $G \times G$. One can verify that the function: $(x,y) \mapsto \int_G f(\mu)(xt\sigma(y))d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)\mu(y)$ is bounded on $G \times G$. If $f_\mu$ is unbounded and $f_\mu(e) = 0$, then since $\mu * \mu = \mu$ we have $f_\mu(x) + f_\mu(\sigma(x))$ is a bounded function, so by using the precedent proof, we get $g$ satisfies equation (2.2).

For the rest of the proof we fixe $g$ and we assume that $f_\mu(e) \neq 0$. Replacing $f_\mu$ by $f_\mu/f_\mu(e)$ we may assume that $f_\mu(e) = 1$. Consider the function $\delta_\alpha f(x) = \int_G f(atx)d\mu(t), x \in G$. We can easily verify that the function $(x,y) \mapsto \int_G \delta_\alpha f(xty)d\mu(t) + \int_G \delta_\alpha f(xt\sigma(y))d\mu(t) - 2\delta_\alpha f(x)g(y)$ is bounded on $G \times G$.

If there exists $a \in G$ such that $h = (\delta_\alpha f)_\mu - (\delta_\alpha f)_\mu(e)f_\mu$ is unbounded on $G$, since $h(e) = 0, h_\mu = h$, the function $(x,y) \mapsto \int_G h(xty)d\mu(t) + \int_G h(xt\sigma(y))d\mu(t) - 2h(x)g(y)$ is bounded we get $x \mapsto h(x) + h(\sigma(x))$ is a bounded function on $G$. Then from above computations we get that $g$ is a solution of equation (2.2). Now, assume that for all $x \in G$ the function $y \mapsto (\delta_x f)_\mu(y) - (\delta_x f)_\mu(e)f_\mu(y)$ is bounded, that is there exists $M(x)$ such
that
\[(3.3) \quad |\int_G f(xty)d\mu(t) - \int_G f(xt)d\mu(t)\int_G f(ty)d\mu(t)| \leq M(x)\]
for all \(x, y \in G\). Since \(f\) is assumed to be \(\mu\)-biinvariant, that is \(\int_G f(xt)d\mu(t) = \int_G f(x)y\mu(t) = f(x)\) for all \(x \in G\) then inequality (3.3) can be replaced by
\[(3.4) \quad |\int_G f(xty)d\mu(t) - f(x)f(y)| \leq M(x)\]
for all \(x, y \in G\). Indeed in view of triangle inequality we get for all \(x, y, z \in G\) that
\[
|f(z)||\int_G f(xty)d\mu(t) - f(x)f(y)| \leq |\int_G f(xtysz)d\mu(t)d\mu(s) + \int_G f(xty)d\mu(t)f(z)|
\]
\[
+ |\int_G f(xtysz)d\mu(t)d\mu(s) - f(x)| \int_G f(ysz)d\mu(s) + |f(x)| \int_G f(ysz)d\mu(s) - f(y)f(z)|
\]
\[
\leq \int_G M(xty)d|m|(t) + ||\mu||M(x) + |f(x)|M(y).
\]
Since \(f\) is assumed to be unbounded, then we get \(\int_G f(xty)d\mu(t) = f(x)f(y)\) for all \(x, y \in G\). Substituting this result into inequality (3.4) we get that
\[
|f(x)||f(y) + f(\sigma(y)) - 2g(y)| \leq \delta
\]
for all \(x, y \in G\). Since \(f\) is unbounded then we obtain \(g(y) = \frac{f(y) + f(\sigma(y))}{2}\) for all \(y \in G\), from which a simple computation shows that \(g\) is a solution of the generalized d’Alembert’s short functional equation (2.2). This completes the proof. \(\square\)

The first part of the above proof proves the following corollary.

**Corollary 3.2.** Let \(\sigma\) be a continuous involution of \(G\). Let \(\mu\) be a complex measure with compact support such that \(\mu\) is \(\sigma\)-invariant and \(\mu \ast \mu = \mu\). Let \(\delta \geq 0\). If the pair \(f, g : G \rightarrow \mathbb{C}\), where \(f\) is an unbounded continuous solution of the following inequality
\[(3.5) \quad |\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)| \leq \delta\]
for all \(x, y \in G\) and such that \(x \rightarrow f(x) + f(\sigma(x))\) is a bounded function. Then \(g\) is a solution of the generalized d’Alembert’s short functional equation (2.2).

**Corollary 3.3.** Let \(\sigma\) be a continuous involution of \(G\). Let \(\mu\) be a complex measure with compact support such that \(\mu\) is \(\sigma\)-invariant and \(\mu \ast \mu = \mu\). Let \(\delta \geq 0\). If the pair \(f, g : G \rightarrow \mathbb{C}\), where \(g\) is an unbounded continuous solution of the following inequality
\[(3.6) \quad |\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)g(y)| \leq \delta\]
for all \(x, y \in G\). Then the pair \(f, g\) is a continuous solution of the functional equation (2.7). Furthermore, if \(f \neq 0\) and \(f_\mu \neq 0\), then \(g\) satisfies the generalized d’Alembert’s short functional equation (2.2).

**Proof.** We use [8, Corollary 2.7, (iii)] and Theorem 2.2. \(\Box\)

In [7] on general groups and under the hypotheses that the function \((x, y) \rightarrow f(xy) + f(x\sigma(y)) - 2f(x)g(y)\) is a bounded function on \(G \times G\), the function \(g\) satisfies d’Alembert’s long functional equation

\[
g(xy) + g(x\sigma(y)) + g(yx) + g(\sigma(y)x) = 4g(x)g(y), \quad x, y \in G
\]

The following corollaries finishes the work on the Hyers Ulam stability of Wilson’s functional equation

\[
f(xy) + f(x\sigma(y)) = 2f(x)g(y), \quad x, y \in G
\]

on groups when the solutions are unbounded functions.

If we let \(\mu = \delta_e\): The dirac measure concentrated at the identity element of \(G\), we apply Theorem 2.3 to \(\mu = \delta_e\) to obtain the following result which has been proved by several authors in the case where \(G\) is an abelian group.

**Corollary 3.4.** Let \(G\) be a group. Let \(\sigma\) be an involution of \(G\). Let \(\delta \geq 0\). If the pair \(f, g\): \(G \rightarrow \mathbb{C}\), where \(f\) is an unbounded solution of the following inequality

\[
|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta
\]

for all \(x, y \in G\). Then \(g\) is a solution of d’Alembert’s short functional equation

\[
g(xy) + g(x\sigma(y)) = 2g(x)g(y), \quad x, y \in G
\]

Now, we can ready to formulate the Hyers-Ulam stability of the classical Wilson’s functional equation (3.8) on groups. The following result was obtained by Kannappan and Kim [19] under the condition that \(f\) is even and \(f\) satisfies the Kannappan condition \(f(xyz) = f(yxz)\) for all \(x, y, z \in G\).

**Corollary 3.5.** Let \(\delta \geq 0\), \(G\) a group, \(\sigma\) an involution of \(G\). Suppose that the pair \(f, g\): \(G \rightarrow \mathbb{C}\) satisfies

\[
|f(xy) + f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \quad \text{for all } x, y \in G.
\]

Under these assumptions the following statements hold:

(1) If \(f\) is unbounded, then \(g\) satisfies d’Alembert’s short functional equation (3.10).

(2) If \(g\) is unbounded and \(f \neq 0\) then the pair \((f, g)\) satisfies Wilson’s functional equation (3.8) and \(g\) satisfies d’Alembert’s short functional equation (3.10).

**Proof.** We use [7, Theorem 2.2] and Theorem 2.2. \(\Box\)

Recently, the authors proved the following result.
Theorem 3.6. [7] Let $\delta \geq 0$, $G$ a group, $\chi$ a unitary character of $G$ and $\sigma$ an involution of $G$ such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. Suppose that the pair $f, g : G \rightarrow \mathbb{C}$ satisfies
\begin{equation}
|f(xy) + \chi(y)f(x\sigma(y)) - 2f(x)g(y)| \leq \delta, \forall x, y \in G.
\end{equation}

Under these assumptions the following statements hold:
(a) If $f$ is unbounded then $g$ satisfies d’Alembert’s long functional equation
\begin{equation}
g(xy) + \chi(y)g(x\sigma(y)) + g(yx) + \chi(y)g(\sigma(y)x) = 4g(x)g(y), \quad x,y \in G
\end{equation}
(b) If $g$ is unbounded and $f \neq 0$, then the pair $(f,g)$ satisfies Wilson’s functional equation
\begin{equation}
f(xy) + \chi(y)f(x\sigma(y)) = 2f(x)g(y), \quad x,y \in G.
\end{equation}

The purpose in the following is to prove that in Theorem 3.6, case (a), the function $g$ satisfies d’Alembert’s short functional equation (3.15). We notice here that the solutions of equation (3.15) are obtained by Stetkær in [27].

Proposition 3.7. Let $G$ be a group. Let $\delta \geq 0$. Let $\chi$ a unitary character of $G$ and $\sigma$ an involution of $G$ such that $\chi(x\sigma(x)) = 1$ for all $x \in G$. If the pair $f, g : G \rightarrow \mathbb{C}$, where $f$ is an unbounded solution of the following inequality
\begin{equation}
|f(xy) + \chi(y)f(x\sigma(y)) - 2f(x)g(y)| \leq \delta
\end{equation}
for all $x, y \in G$. Then $g$ is a solution of d’Alembert’s short functional equation (3.15).

Proof. In the proof we use similar reasoning to that in the proof of Theorem 2.3 and some author’s computations [7]. Assume that the pair $f, g$ satisfies inequality (3.16), where $f$ is an unbounded function on $G$. First Case: We assume that the function $x \mapsto f(x) + \chi(x)f(\sigma(x))$ is a bounded function on $G$, that is $|f(x) + \chi(x)f(\sigma(x))| \leq \beta$ for some $\beta \geq 0$ and for all $x \in G$. Let
\[
\Psi(x, y) = g(xy) + \chi(y)g(\sigma(y)x) - 2g(x)g(y), \quad x, y \in G.
\]
We will show that the function $(x, y, z) \mapsto 2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)$ is a bounded function in the special case when $g$ is bounded. The computations in [7] show that
\[
2f(z)\Psi(x, y) + 2f(y)\Psi(x, z) = \mu(y)2f(z\sigma(z))g(x) + \mu(z)2f(y\sigma(z))g(x) - \mu(z)f(yx\sigma(z)) - \mu(x)f(zx\sigma(y)) + \mu(z)f(y\sigma(x)\sigma(z)) - \mu(x)f(\sigma(z)\sigma(y))
\]
So, we get
\begin{equation}
|2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)| \leq 2\beta\|f\|\|g(x)\| + 2\beta\|\mu\|^2
\end{equation}
for all $x, y, z \in G$. Now, we will discuss two subcases:
If $g$ is unbounded then from [7, Theorem 2.2, (b)], we conclude that $g$ is a
solution of the generalized d’Alembert’s short functional equation (3.15).
In the rest of the proof we examine the case of $g$ a bounded function on $G$.
Then from (3.17) the function $(x, y, z) \mapsto -2f(z)\Psi(x, y) + 2f(y)\Psi(x, z)$ is
a bounded function on $G$. By using similar computations in the proof of
Theorem 2.3 we get our result and this completes the proof.

□

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