ON THE EXISTENCE OF CURVES WITH $A_k$-SINGULARITIES ON $K3$ SURFACES

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Abstract. Let $(S, H)$ be a general primitively polarized $K3$ surface. We prove the existence of curves in $|\mathcal{O}_S(nH)|$ with $A_k$-singularities and corresponding to regular points of the equisingular deformation locus. Our result is optimal for $n = 1$. As a corollary, we get the existence of elliptic curves in $|\mathcal{O}_S(nH)|$ with a cusp and nodes or a simple tacnode and nodes. We obtain our result by studying the versal deformation family of the $m$-tacnode. Finally, we give a regularity condition for families of curves with only $A_k$-singularities in $|\mathcal{O}_S(nH)|$.

1. Introduction

Let $S$ be a complex smooth projective $K3$ surface and let $H$ be a globally generated line bundle of sectional genus $p = p_a(H) \geq 2$ and such that $H$ is not divisible in $\text{Pic}^S$. The pair $(S, H)$ is called a primitively polarized $K3$ surface of genus $p$. It is well-known that, for every $p$, the moduli space $\mathcal{K}_p$ of primitively polarized $K3$ surfaces is non-empty, smooth and irreducible of dimension 19. Moreover, if $(S, H) \in \mathcal{K}_p$ is a general element, then $\text{Pic}^S \cong \mathbb{Z}[H]$. Now, if $(S, H)$ is a general primitively polarized $K3$ surface, using non-standard notation, we denote by $\mathcal{V}^S_{nH,1^{\delta}} \subset |\mathcal{O}_S(nH)| = [nH]$ the so-called Severi variety of $\delta$-nodal curves, defined as the closure in the Zariski topology, of the locus of irreducible and reduced curves with exactly $\delta$ nodes as singularities. More in general, we will denote by $\mathcal{V}^S_{nH,1^2,2^j,\ldots,n_{-1}^{\delta_n}}$, the Zariski closure of the locus in $[nH]$ of irreducible curves with exactly $d_j$ singularities of type $A_{j-1}$, for every $2 \leq j \leq n$ and no further singularities. We recall that an $A_k$-singularity is a plane singularity of analytic equation $y^2 = -x^{k+1}$. Every plane singularity of multiplicity 2 is an $A_k$-singularity, for a certain $k$. To shorten notation, if $|C| \in V \subset \mathcal{V}^S_{nH,1^2,2^j,\ldots,n_{-1}^{\delta_n}}$, is a general point in an irreducible component $V$ of $\mathcal{V}^S_{nH,1^2,2^j,\ldots,n_{-1}^{\delta_n}}$, we will set $V = ES(C)$, being $V$ the equisingular deformation locus of $C$, (see Section 2).

Now, the Severi variety $\mathcal{V}^S_{nH,1^\delta} \subset |\mathcal{O}_S(nH)|$ is a well-behaved variety. By [20], we know that it is smooth of expected dimension at every point $|C|$, corresponding to a $\delta$-nodal curve. In particular, every irreducible component of $\mathcal{V}^S_{nH,1^\delta}$ has codimension $\delta$, for every $\delta \leq \dim([nH]) = p_a(nH)$, where $p_a(nH)$ is the arithmetic genus of $nH$, as expected. The existence of nodal curves of every allowed genus in $[H]$ has been proved first by Mumford, cf. [10]. Non-emptiness of $\mathcal{V}^S_{nH,1^\delta}$, for every $n \geq 1$ and $\delta \leq \dim([nH])$, has been proved later by Chen in [6]. Chen’s existence theorem is obtained by degeneration techniques. A general primitively polarized $K3$ surface $S_t \subset \mathbb{P}^5$ of genus $p$ is degenerated in $\mathbb{P}^p$ to the union of two rational scrolls.

Date: 20.01.12.
1991 Mathematics Subject Classification. 14B07, 14H10, 14J28.
Key words and phrases. versal deformations, tacnodes, Severi varieties, $K3$ surfaces, $A_k$-singularities.

Both authors want to thank the Department of Mathematics of the University of Calabria and the Department of Mathematics of the University of Bergen for hospitality and for financial support. The first author was also supported by GNSAGA of INdAM and by the PRIN 2008 "Geometria delle varietà algebriche e dei loro spazi di moduli", co-financed by MIUR.
$S_0 = R_1 \cup R_2$, intersecting transversally along a smooth elliptic normal curve $E$. Now, rational nodal curves on $S_0$ are obtained by deformation from suitable reduced curves $C_0 = C^1 \cup C^2 \subset S_0$, having tacnodes at points of $E$ and nodes elsewhere. A key ingredient in the proof of Chen’s theorem is the Caporaso-Harris description of the locus of $(m - 1)$-nodal curves in the versal deformation space $\Delta_m$ of the $m$-tacnode (or $A_{2m-1}$-singularity). The question we ask in this paper is the following.

**Main Problem** With the notation above, assume that $C = C_1 \cup C_2 \subset R_1 \cup R_2$ is any curve having an $m$-tacnode at a point $p$ of $E$. Then, which kind of curve singularities on $S_t$ may be obtained by deforming the $m$-tacnode of $C$ at $p$?

Theorem 1.1, which is to be considered the main result of this article, completely answers this question. It proves that, under suitable hypotheses, the $m$-tacnode of $C$ at $p$ deforms to $d_j$ singularities of type $A_{j - 1}$, for every $j \leq m$ and $d_j$ such that

$$\sum_j d_j (j - 1) = n = 1 - 1.$$

By trivial dimensional reasons, no further singularities on $S_t$ may be obtained by deforming of the $m$-tacnode of $C \subset R_1 \cup R_2$. The result is a local result, obtained by studying the versal deformation family of the $m$-tacnode, with the same approach as [2], Section 2.4. In particular, the result holds for any smooth family of regular surfaces, with smooth general fibre and reducible special fibre. Section 3 is completely devoted to the proof of Theorem 1.1. In Section 4, we apply Theorem 1.1 to the family of $K^3$ surfaces introduced above, using as limit curves the same curves $C_0 = C^1 \cup C^2 \subset R_1 \cup R_2$ as Chen in [5]. We obtain the following result.

**Theorem 1.1.** Let $(S, H)$ be a general primitively polarized $K^3$ surface of genus $p = p_0(H)$ as above. Then, for every $n \geq 1$ and for every $(m - 1)$-tuple of non-negative integers $d_2, \ldots, d_m$ such that

1. $\sum_{j=2}^{m} (j - 1) d_j = n(p - 2) + 1 = n(n^2/2 - 1) + 1 = 2nl - 2n + 1$

if $p = 2l$ is even, or

2. $\sum_{j=2}^{m} (j - 1) d_j = n(p - 1) = nH^2/2 = 2nl,$

if $p = 2l + 1$ is odd, there exist reduced irreducible curves $C$ in the linear system $|nH|$ on $S$ such that:

- $C$ has $p_0(nH) - \sum_{j=2}^{m} (j - 1) d_j = \dim(|nH|) - \sum_{j=2}^{m} (j - 1) d_j$ nodes and $d_j$ singularities of type $A_{j - 1}$, for every $j = 2, \ldots, m$, and no further singularities;
- $C$ corresponds to a regular point of the equisingular deformation locus $ES(C)$. Equivalently, $\dim(T_{C} ES(C)) = 0$.

Finally, the singularities of $C$ may be smoothed independently. In particular, under the hypotheses (1) and (2), for every $\delta_j \leq d_j$ and for every $\delta \leq \dim(|nH|) - \sum_{j=2}^{m} (j - 1) d_j$, there exist curves $C$ in the linear system $|nH|$ on $S$ with $\delta_j$ singularities of type $A_{j - 1}$, for every $j = 2, \ldots, m$, and $\delta$ nodes as further singularities and corresponding to regular points of their equisingular deformation locus.

The notion of regularity and other terminology in the theorem above will be introduced in Section 2. In Corollaries 1.2 and 1.3, we observe that Theorem 1.1 is optimal if $n = 1$ and, for $n \geq 1$, it implies the existence of curves of every geometric genus $g \geq 1$ with a cusp and nodes or a 2-tacnode and nodes as further singularities, in accordance with the Chen’s result that all rational curves in $|H|$ are nodal, see [5]. Finally, in the next section, we recall standard results and terminology of deformation theory, focusing our attention on properties of equisingular deformations of curves with only $A_k$-singularities on $K^3$ surfaces. Moreover, in Section 2, we provide the following regularity condition.
Proposition 1.2. Let $S$ be a K3 surface with $\text{Pic} \, S \cong \mathbb{Z}[H]$ and $n \geq 1$ an integer. Assume that $C \in |nH|$ is a reduced and irreducible curve on $S$ having precisely $a_k \geq 0$ singularities of type $A_k$, for each $k \geq 1$, and no further singularities, such that

\begin{align*}
(3) \quad \sum_k ka_k &= \deg T_C^1 < \frac{p_a(H) + 2}{2} = \frac{H^2}{4} + 2, \quad \text{if } n = 1 \quad \text{or} \\
(4) \quad \sum_k ka_k &= \deg T_C^1 < 2(n - 1)(p_a(H) - 1) = (n - 1)H^2, \quad \text{if } n \geq 2,
\end{align*}

where $T_C^1$ is the first cotangent bundle of $C$. Then $[C]$ is a regular point of $ES(C)$ and the singularities of $C$ may be smoothed independently.

Acknowledgments. The first author is indebted with J. Harris for invaluable conversations on deformation theory of curve singularities. She also benefited from conversation with F. van der Wyck. Both authors want to express deep gratitude to C. Ciliberto for many useful conversations, stimulating questions and suggestions. Finally, the authors want to thank T. Dedieu for very useful observations on the preliminary version of this paper.

2. Tangent spaces and a new regularity condition

In this section we recall some properties of the equisingular and equigeneric deformation loci of a reduced curve on an arbitrary smooth projective K3 surface $S$ and, in particular, of a curve with only $A_k$-singularities. Finally, at the end of the section, we provide a regularity condition for curves with only $A_k$-singularities on a primitively polarized K3 surface.

Let $S$ be a smooth projective K3 surface and let $D$ be a Cartier divisor on $S$ of arithmetic genus $p_a(D)$. Assume that $|D| = |O_S(D)|$ is a Bertini linear system, i.e. a linear system without base points and whose general element corresponds to a smooth curve. (In fact, by [13], every irreducible curve $D$ on $S$ such that $D^2 \geq 0$ defines a Bertini linear system on $S$.) If $C \in |D|$ is a reduced curve, we consider the following standard exact sequence of sheaves on $C$

\begin{equation}
0 \longrightarrow \Theta_C \longrightarrow \Theta_{S|C} \longrightarrow \alpha_N_{C|S} \longrightarrow T_C^1 \longrightarrow 0,
\end{equation}

where $\Theta_C \simeq \text{Hom}((\Omega^1_C), \mathcal{O}_C)$ is the tangent sheaf of $C$, defined as the dual of the sheaf of differentials of $C$, $\Theta_{S|C}$ is the tangent sheaf of $S$ restricted to $C$, $N_{C|S} \simeq \mathcal{O}_C(C)$ is the normal bundle of $C$ in $S$ and $T_C^1$ is the first cotangent sheaf of $C$, which is supported at the singular locus $\text{Sing}(C)$ of $C$ and whose stalk $T_{C,p}^1$ at every singular point $p$ of $C$ is the versal deformation space of the singularity (see [19], [11] or [2]). Identifying $H^0(C, N_{C|S})$ with the tangent space $T_{[C]}|D|$, the induced map

\begin{equation}
da : H^0(C, N_{C|S}) \longrightarrow H^0(C, T_C^1) = \bigoplus_{p \in \text{Sing}(C)} T_{C,p}^1
\end{equation}

is classically identified with the differential at $[C]$ of the versal map from an analytic neighborhood of $[C]$ in $|D|$ to an analytic neighborhood of the origin in $H^0(C, T_C^1)$. By this identification and by the fact that the versal deformation space of a singularity parametrizes equivalence classes of singularities in the analytic topology, we have that, if $N'_{C|S}$ is the kernel of the sheaf map $\alpha$ in (5), then the global sections of $N'_{C|S}$ are infinitesimal deformations of $C$ which are equisingular in the analytic topology, i.e. infinitesimal deformation of $C$ preserving the analytic class of every singularity of $C$. For this reason, $N'_{C|S}$ is usually called the equisingular deformation sheaf of $C$ in $S$. By a straightforward computation, if $J$ is the Jacobian ideal of $C$, then $J \otimes N'_{C|S} := N''_{C|S}$ and, consequently, $\dim(H^0(C, T_C^1)) = \deg(J) = \sum_{p \in C} \deg(J_p)$, where $J_p$ is the localization at $p$ of $J$. Keeping in mind the versal property of $T_C^1$, the following definition makes sense.
Definition 2.1. We say that the singularities of $C$ may be smoothed independently if the map $dx$ introduced in \((\ref{eq:jacobian})\) is surjective or, equivalently, if $h^0(C, N_C|_S) = h^0(C, N_C|_S) - \deg(J).$ If this happens, we also say that the Jacobian ideal imposes linearly independent conditions to the linear system $|D|$.

Remark 2.2. If $C$ is an irreducible reduced curve in a Bertini linear system $|D|$ on a smooth projective K3 surface $S$ then $h^1(C, N_C|_S) = h^1(C, \mathcal{O}_C(C)) = h^1(C, \omega_C) = 1$, where $\omega_C$ denotes the dualizing sheaf of $C$. Using that the equisingular ideal $\text{ES}(C)$ of $C$ coincides with the Jacobian ideal $I(C)$ introduced in \((\ref{eq:jacobian})\) of \cite{[7]}. The equisingular deformation locus $\text{ES}(C)$ of $C$ in $|D|$ has a natural structure of scheme, representing a suitable deformation functor. The tangent space $T_{[C]}\text{ES}(C)$ at $ES(C)$ at the point $[C]$, corresponding to $C$, is well understood. In particular, there exists an ideal sheaf $I$, named the equisingular ideal of $C$, such that $J \subset I$ and

$$T_{[C]}\text{ES}(C) \simeq H^0(C, I \otimes \mathcal{O}_C(C)).$$

Definition 2.3. We say that $[C]$ is a regular point of $\text{ES}(C)$ if $\text{ES}(C)$ is smooth of the expected dimension at $[C]$, equivalently if

$$\dim(T_{[C]}\text{ES}(C)) = \dim(H^0(C, I \otimes \mathcal{O}_C(C))) = \dim(H^0(C, \mathcal{O}_C(C))) - \deg I.$$

In this case, we also say that the equisingular ideal imposes linearly independent conditions to curves in $|D|$.

We also recall the inclusion $J \subset I \subset A$, where $A$ is the conductor ideal.

Through all this paper we will be interested in curves with $A_k$-singularities. An $A_k$-singularity has analytic equation $y^2 = x^{k+1}$. Every plane curve singularity of multiplicity 2 is an $A_1$-singularity for a certain $k \geq 1$. In particular, two singularities of multiplicity 2 are analytically equivalent if and only if they are equisingular.

Remark 2.4. Using that the equisingular ideal $I$ of an $A_k$-singularity of equation $y^2 = x^{k+1}$ coincides with the Jacobian ideal $J = I = (y, x^k)$ (cf. \cite{[2]} Proposition (5.6)), we find that, if $C \in |D|$ is a reduced curve on $S$ with only $A_k$-singularities, then the tangent space

$$T_{[C]}\text{ES}(C) = H^0(C, N_C|_S \otimes I) = H^0(C, N_C|_S \otimes J) = H^0(C, N_C'|_S)$$

to $\text{ES}(C)$ at the point $[C]$ consists of the linear system of curves in $|D|$ passing through every $A_k$-singularity $p \in C$ and tangent there to the reduced tangent cone to $C$ at $p$ with multiplicity $k$. In particular, every $A_k$-singularity imposes at most $k = \dim(\mathbb{C}[x, y]/(y, x^k))$ linear conditions to $|D|$ and the equisingular deformation locus $\text{ES}(C)$ of $C$ in $|D|$ is regular at $[C]$ if and only if the singularities of $C$ may be smoothed independently.

If $k$ is odd, an $A_k$-singularity is also called a $\frac{k+1}{2}$-tacnode whereas, if $k$ is even, an $A_k$-singularity is said to be a cusp. Moreover, by classical terminology, $A_1$-singularities are nodes, $A_2$-singularities are ordinary cusps and $A_3$-singularities are called simple tacnodes. If $C \subset S$ be an irreducible curve with $\delta$ nodes as singularities and such that $C \sim D$, then $\text{ES}(C)$ consists of an open set in one irreducible
component of the so called Severi Variety $\mathcal{V}_{D,1}^S$ of $\delta$-nodal curves on $S$. By [20], we know that, as for Severi Varieties of plane curves, $\mathcal{V}_{D,1}^S$ is smooth of the expected dimension at $[C]$. More generally, the Zariski dimensional characterization theorem for plane curves also holds for families of curves on $K3$ surfaces of positive dimension.

**Lemma 2.5** (Lemma 3.1 of [6]). Let $S$ be a $K3$ surface and let $|D|$ be a Bertini linear system on $S$. Suppose that $U_{D,g}(S)$ is a closed subset of $|\mathcal{O}_S(D)| = |D|$ whose general element $[C]$ corresponds to an irreducible and reduced curve $C \subset S$ of geometric genus $g$. Then $\dim(U_{D,g}(S)) \leq g$. Moreover, if $g > 0$, then $\dim(U_{D,g}(S)) = g$ if and only if $C$ is a nodal curve.

**Remark 2.6.** Let $S$ and $D$ as in the previous lemma. Let $C \in |D|$ be an irreducible and reduced curve on $S$ of geometric genus $g > 0$. We denote by $\text{EG}(C) \subset |D|$ the equigeneric deformations locus of $C$, defined as the Zariski closure of the the locus of deformations of $C$ in $|D|$ preserving the geometric genus. Then $\dim(\text{EG}(C)) = g$. Indeed, by the estimation (5.1) and observation (ii) of p. 323 of [17], we know that $\dim(\text{EG}(C)) \geq g$. The other inequality follows from the previous lemma. This implies, in particular, that $C$ may be deformed in $|D|$ to curves of the same geometric genus with only nodes as singularities. More precisely, the point $[C] \in |D|$ is in the Zariski closure of one irreducible component of the Severi variety $\mathcal{V}_{D,1}^S$, where $\delta = p_a(D) - g$. We finally observe that, under the hypotheses above, we have that

$$T_{[C]}\text{EG}(C)) = H^0(C, A \otimes \mathcal{O}_C(C)),$$

where $A$ is the conductor ideal of $C$. Indeed, by [17], we know that $T_{[C]}\text{EG}(C)) \subset H^0(C, A \otimes \mathcal{O}_C(C))$. On the other hand, since $S$ is a $K3$ surface we have that, if $\tilde{\phi} : \tilde{C} \to C$ is the normalization map of $C$, then $\tilde{\phi}^*(A \otimes \mathcal{O}_C(C)) \simeq \omega_{\tilde{C}}$, and hence $g = p_a(D) - \deg(A) \leq h^0(C, A \otimes \mathcal{O}_C(C)) \leq h^0(\tilde{C}, \omega_{\tilde{C}}) = g$.

We now prove our regularity condition for curves with only $A_k$-singularities on a general primitive polarized $K3$ surface.

**Proof of Proposition 1.2**. Assume that $[C]$ is not a regular point of $ES(C)$. Then, by Remarks 2.2 and 2.3, we must have $h^1(N_{C|S}^g) \geq 2$. Now consider $N_{C|S}^g$ as torsion free sheaf on $C$ and a torsion sheaf on $S$ and define $A := \text{Ext}^1(N_{C|S}^g, \mathcal{O}_S)$. Then $A$ is a rank one torsion free sheaf on $C$ and a torsion sheaf on $S$. Moreover, by [10] Lemma 2.3], being $S$ a $K3$ surface, we have that $h^0(A) = h^1(N_{C|S}^g) \geq 2$ and

$$\deg A = C^2 - \deg N_{C|S}^g = \deg T_C^1 = \sum_k k\alpha_k.$$

By [10] Proposition 2.5 and proof of Theorem I at p. 749], the pair $(C', A')$ may be deformed to a pair $(C', A')$ where $C' \sim C$ is smooth, and $A'$ is a line bundle on $C'$ with $h^0(A') \geq h^0(A)$ and $\deg A' = \deg A$. In other words, there is a smooth curve in $|nH|$ carrying a $g^1_{\deg T_C}$. Now, if $n = 1$ then, by Lazarsfeld’s famous result [14] Corollary 1.4, no curve in $|nH|$ carries any $g^1_k$ with $k < |\frac{p_a(H)+1}{2}|$. If $n \geq 2$, then, by [13] Theorem 1.3], the minimal gonality of a smooth curve in $|nH|$ is

$$\min\{D,E \mid nH \sim D + E \text{ with } h^0(D) \geq 2, h^0(E) \geq 2\},$$

which is easily seen to be $(n - 1)H^2 = 2(n - 1)(p_a(H) - 1)$. The result follows. □

**Remark 2.7.** As far as we know, the previously known regularity condition for curves $C$ like in the statement of the proposition above is given by

$$\sum_k (k + 1)^2 \alpha_k \leq n^2 H^2,$$

where $\alpha_k$ are the values of the Hilbert function of $A$ at $k$. In particular, this conditions is better than the one given in the proposition above if $n > 1.$
and it has been deduced from [12, Corollary 2.4]. This result is very different from Proposition [13] and we will not compare them here.

We conclude this section with a naive upper-bound on the dimension of the equisingular deformation locus of an irreducible curve with only $A_k$-singularities on a smooth $K3$ surface. This result is a simple application of Clifford’s theorem and for nodal curves it reduces to Tannenbaum’s proof that Severi varieties of irreducible nodal curves on $K3$ surfaces have expected dimension.

Lemma 2.8. Let $|D|$ be a Bertini linear system on a smooth projective $K3$ surface $S$. Let $C \in |D|$ be a reduced and irreducible genus $g$ curve with only $A_k$-singularities, $\tau$ of which are (not necessarily ordinary) cusps. Then, if $ES(C) \subset |D|$ is the equisingular deformation locus of $C$ in $|D|$, we have that
\[
\dim T_{|C|}ES(C) \leq g - \tau/2.
\]

Proof. Let $C$ and $S$ as in the statement. By Remark 2.4 since $C$ has only $A_k$-singularities, we have that $T_{|C|}ES(C) = H^0(C, N'_C|S)$. Moreover, by standard deformation theory (see e.g. [19], (3.51)), if $\phi : \tilde{C} \to C \subset S$ is the normalization map, we have the following exact sequence of line bundles on $C$
\[
0 \longrightarrow \Theta_{\tilde{C}}(Z) \xrightarrow{\phi_*} \phi^*\Theta_S \longrightarrow N'_\phi \longrightarrow 0.
\]
Here $\phi_* : \Theta_S \to \phi^*\Theta_S$ is the differential map of $\phi$, having zero divisor $Z$, and $N'_\phi \simeq N_\phi/K_\phi$ is the quotient of the normal sheaf $N_\phi$ of $\phi$ by its torsion subsheaf $K_\phi$ (with support on $Z$). By (8), using that $S$ is a $K3$ surface, we have that $h^1(C, N'_\phi) = h^1(C, \mathcal{O}_C(Z)) \geq 1$. Moreover, again by [19, p. 174], it follows that
\[N'_\phi \simeq \phi^*N'_C|S \quad \text{and hence} \quad h^0(C, N'_C|S) \leq h^0(C, N'_\phi).
\]
Finally, by applying Clifford’s theorem, we deduce that
\[T_{|C|}ES(C) = h^0(C, N'_C|S) \leq h^0(C, N'_\phi) \leq \frac{1}{2} \deg N'_\phi + 1 = \frac{1}{2} (2g - 2 - \tau) + 1 = g - \frac{\tau}{2}
\]
as desired. \qed

3. Smoothing tacnodes

In this section, by using classical deformation theory of plane curves singularities, we want to find sufficient conditions for the existence of curves with $A_k$-singularities on smooth projective complex surfaces we may obtain as deformations of a ”suitable” reducible surface.

Let $\mathcal{X} \to \mathbb{A}^1$ be a smooth family of surfaces, whose general fiber $\mathcal{X}_t$ is a smooth regular surface and whose special fiber $\mathcal{X}_0 = A \cup B$ is reducible, consisting of two irreducible components $A$ and $B$ with $h^1(\mathcal{O}_A) = h^1(\mathcal{O}_B) = h^1(\mathcal{O}_{X_t}) = 0$ and intersecting transversally along a smooth curve $E = A \cap B$. Let $D$ be a Cartier divisor on $\mathcal{X}$. We denote by $D_t = D \cap \mathcal{X}_t$ the restriction of $D$ to the fiber $\mathcal{X}_t$. Notice that, since $\mathcal{X}_0 = A \cup B$ is a reducible surface, the Picard group $\text{Pic}(\mathcal{X}_0)$ of $\mathcal{X}_0$ is the fiber product of the Picard groups $\text{Pic}(A)$ and $\text{Pic}(B)$ over $\text{Pic}(E)$. In particular, we have that
\[|D_0| = |\mathcal{O}_{\mathcal{X}_0}(D)| = |\mathcal{O}_A(D)| \times |\mathcal{O}_B(D)|.
\]
From now on, for every curve $C \subset \mathcal{X}$, we will denote by $C_A$ and $C_B$ the union of its irreducible components on $A$ and $B$, respectively. Let $p$ be a general point of $E$. Choose local analytic coordinates $(x, z)$ of $A$ at $p$ and $(y, z)$ of $B$ at $p$, in such a way that the equation of $\mathcal{X}$ at $p$, by using coordinates $(x, y, z, t)$, is given by $xy = t$.

Now assume there exists a divisor $C = C_A \cup C_B \subset \mathcal{X}_0$, with $[C] \in |D_0|$, such that $C_A$ and $C_B$ are both smooth curves, tangent to $E$ at a point $p \in E$ with multiplicity
Denote by \( V \) a regular point of \( C \) if the linear system \( m, n \) of non-negative integers such that 
\[
\begin{align*}
\sum_{j=2}^{n} (j-1)d_j &= m-1.
\end{align*}
\]
with \( m \geq 2 \). Since the singularity of \( C \) at \( p \) is analytically equivalent to the tacnode of local equation 
\[
 f(y, z) = (y - z^m)y = 0,
\]
we say that \( C \) has an \( m \)-tacnode at \( p \).

**Definition 3.1.** We say that the \( m \)-tacnode of \( C \) at \( p \) imposes linearly independent conditions to \(|D_0|\) if the linear system \( W_{p,m} \subset |D_0| \), parametrizing curves \( F_A \cup F_B \subset X_0 \), such that \( F_A \) and \( F_B \) are tangent to \( E \) at \( p \) with multiplicity \( m \), has codimension equal to \( m \) (which is the expected codimension).

**Remark 3.2.** We remark that, if the \( m \)-tacnode of \( C \) at \( p \) imposes independent linear conditions to \(|D_0|\), then, for every \( r \leq m \), the locus \( W_{p,r} \subset |D_0| \), parametrizing curves with an \( r \)-tacnode at \( p \) is non-empty of codimension exactly \( r \). In particular, the general element of an analytic neighborhood of \( |C| \) in \(|D_0|\), transversally intersects \( E \) at \( m \) points close to \( p \).

We now introduce the main result of this paper.

**Theorem 3.3.** Let \( m, n \) be integers such that \( m \geq n \geq 2 \) and let \( \{ j \} \) and \( \{ d_j \} \) be non-negative integers such that 
\[
\sum_{j=2}^{n} (j-1)d_j = m-1.
\]
Moreover, using the notation above, assume that:
1) \( \dim(|D_0|) = \dim(|D_t|) \);
2) the tacnode of \( C \) at \( p \) imposes \( m \) independent linear conditions to the linear system \(|D_0|\).

Then, if \( V_{D_t,1^{d_2},2^{d_3},...,n-1^{d_n}} \subset |D_t| \) is the Zariski closure of the locus in \(|D_t|\) of irreducible curves with exactly \( d_j \) singularities of type \( A_{j-1} \), for every \( 2 \leq j \leq n \) and no further singularities, then, for a general \( t \neq 0 \), there exists a non-empty irreducible component \( V_t \) of \( V_{D_t,1^{d_2},2^{d_3},...,n-1^{d_n}} \subset |D_t| \), whose general element \( |C_t| \in V_t \) is a regular point of \( V_t \). In particular, \( \dim(T_{|C_t|}V_t) = \dim(|D_t|) - \sum_{j=2}^{n} (j-1)d_j \).

To prove the theorem above we need the following auxiliary result, whose proof is postponed to Appendix A.

**Lemma 3.4.** Let \( m \geq n \geq 2 \) be integers and \((d^+_2, d^-_2, \ldots, d^+_n, d^-_n)\) be a \( 2(n-1) \)-tuple of non-negative integers satisfying 
\[
 m = \sum_{j=2}^{n} (j-1)(d^+_j + d^-_j) + 1 \geq 2 \quad \text{and} 
\]
\[
 m \geq \sum_{j=2}^{n} jd^\pm_j.
\]

Denote by \( \Sigma_m \) the symmetric group of all permutation of \( m \) indices. Then, up to the action of \( \Sigma_m \) by conjugation, there exists a unique triple of permutations \( (\tau^+, \tau^-, \sigma) \) of \( m \) indices, such that \( \tau^\pm \) has cyclic structure \( \Pi_{j=2}^{n} d^\pm_j \), \( \sigma \) is a cycle of order \( m \) and, finally, \( \sigma \tau^+ \tau^- = 1 \).
Definition 3.5. We call a 2(n − 1)-tuple of integers \((d_1^+ , d_2^+ , \ldots , d_n^+ , d_n^-)\) satisfying (11) and (12) an admissible 2(n − 1)-tuple.

Proof of Theorem 3.3. We want to obtain curves with \(A_k\) singularities as deformations of \(C\). The moduli space of deformations of \(C\) in \(\mathcal{X}\) is an irreducible component \(\mathcal{H}\) of the relative Hilbert scheme \(\mathcal{H}^{n|A_1}\) of the family \(\mathcal{X} \to \mathbb{A}^1\). Let \(\pi_\mathcal{H}: \mathcal{H} \to \mathbb{A}^1\) be the natural map from \(\mathcal{H}\) to \(\mathbb{A}^1\). By the hypothesis of regularity on the fibres of the family \(\mathcal{X}\), we have that the general fiber \(\mathcal{H}_t\) of \(\pi_\mathcal{H}\) coincides with the linear system \([\mathcal{O}_\mathcal{X}(D_0)]\), whereas, in general, the special fiber \(\mathcal{H}_0\) of \(\pi_\mathcal{H}\) consists of several irreducible components of the Hilbert scheme of \(\mathcal{X}_0\), only one of which coincides with \([\mathcal{O}_\mathcal{X}(D_0)]\).

Furthermore, by the standard identifications of the tangent space \(\mathcal{T}_x\mathcal{X}\) with its tangent space at the origin, the natural map

\[
d\phi_{[C]}: H^0(C, \mathcal{N}_{[C], \mathcal{X}}) \to H^0(C, \mathcal{T}_C^1) = \oplus_{q \in \text{Sing}(C)} T_{C,q}^1
\]
induced by $\mathfrak{L}$, is identified with the differential at $[C]$ of the versal map

$$\phi = \oplus_{q \in \text{Sing}(C)} \phi_q : \cap_{q \in \text{Sing}(C)} U_q \subset \mathcal{H} \rightarrow H^0(C, T^1_C).$$

What we want to prove is that the locus, in the image of $\phi$, of curves with $d_j$ singularities of type $A_{j-1}$, for every $j$ as in the statement, is non-empty, deducing the existence of the desired curves on $X_t$ in $|D_t|$ by versality. We begin by describing the kernel and the image of $d\phi_{[C]}$.\[1\]

By Section 2 we know that $\ker(d\phi_{[C]}) = H^0(C, \mathcal{N}'_{C|X})$, where $\mathcal{N}'_{C|X}$ is the equisingular deformation sheaf of $C$ in $X$. Moreover, since $C$ has only nodes and tacnodes as singularities, the global sections space of $\mathcal{N}'_{C|X}$ is the tangent space at the point $[C] \in \mathcal{H}$ to the equisingular deformation locus of $C$ in $X$. Now, using notation as in Definition 3.1, we will show that

$$\ker(d\phi_{[C]}) = H^0(C, \mathcal{N}'_{C|X}) = W_{p,m-1} \subset |O_{X_0}(D_0)|,$$

as we may expect. To prove $\ker(d\phi_{[C]})$, after localizing at $p$, consider the exact sequence

$$0 \rightarrow \mathcal{N}'_{C|X,p} \rightarrow \mathcal{N}_{C|X,p} \rightarrow T^1_{C,p} \rightarrow 0.$$\[16\]

Now, using local analytic coordinates $x, y, z, t$ at $p$ as in $[9]$, we may identify:

- the local ring $O_{C,p} = O_{X,p}/I_{C|X,p}C$ of $C$ at $p$ with $\mathbb{C}[x, y, z]/(f_1, f_2)$, where
  $$f_1(x, y, z) = x + y + z^m$$
  and $f_2(x, y, z) = xy$,

- the $O_{C,p}$-module $N_{C|X,p}$ with the free $O_{X,p}$-module $\text{Hom}_{O_{X,p}}(I_{C|X,p}, O_{C,p})$, generated by morphisms $f_1^*$ and $f_2^*$, defined by
  $$f_i^*(s_1(x, y, z)) = f_1(x, y, z)s_1(x, y, z) + s_2(x, y, z)f_2(x, y, z)$$
  for $i = 1, 2$ and finally,

- the $O_{C,p}$-module
  $$(\Theta_{X|C})_p \cong \Theta_{X,p}/(I_{C,p} \otimes \Theta_{X,p})$$
  with the free $O_{X,p}$-module generated by the derivatives $\partial/\partial x, \partial/\partial y, \partial/\partial z$.

With these identifications, the localization $\alpha_p : (\Theta_{X|C})_p \rightarrow \mathcal{N}_{C|X,p}$ at $p$ of the sheaf map $\alpha : \Theta_{X|C} \rightarrow \mathcal{N}_{C|X}$, is defined by

$$\alpha_p(\partial/\partial x) = \begin{cases} s = s_1f_1 + s_2f_2 \mapsto s/\partial x = o_{C,p} s_1\partial f_1/\partial x + s_2\partial f_2/\partial x \\ = f_1^*(s) + yf_2^*(s), \\
\alpha_p(\partial/\partial y) = f_1^*(s) + xf_2^*(s) \text{ and} \\
\alpha_p(\partial/\partial z) = zm^{m-1}f_1^*(s). \end{cases}$$

Now, by using exactness of sequence $[13]$, a section $s \in \mathcal{N}_{C,p}$ is equisingular at $p$, i.e. $s \in \mathcal{N}'_{C,p}$, if and only if there exists a section

$$u = u_x(x, y, z)\partial/\partial x + u_y(x, y, z)\partial/\partial y + u_z(x, y, z)\partial/\partial z \in \Theta_{X|C}_p$$

such that $s = \alpha_p(u)$. Hence, locally at $p$, equisingular deformations of $C$ in $X$ have equations

$$\begin{cases} x + y + zm^{m-1}u_x = 0 \\ xy + e(yu_x + xu_y) = 0. \end{cases}$$\[18\]

The first equation above gives an infinitesimal deformation of the Cartier divisor cutting $C$ on $X_0$, while the second equation gives an infinitesimal deformation of $X_0$ in $X$. More precisely, by $[3]$, $xy + e(yu_x + xu_y) = 0$ is the local equation at $p$ of an

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1 The kernel and the image of $d\phi_{[C]}$ are also computed in Theorem 2.3 of $[6]$. We give a different and more detailed proof of these two results. This will make the proof of Theorem $[14]$ shorter.
equisingular deformation of $\mathcal{X}_0$ in $\mathcal{X}$ preserving the singular locus $E$. But, $\mathcal{X}_0$ may be deformed in $\mathcal{X}$ only to a fiber and $\mathcal{X}_0$ is the only singular fiber of $\mathcal{X}$. It follows that the polynomial $yu_x(x, y, z) + xu_y(x, y, z)$ in the second equation of (18) must be identically zero. In particular, we get the inclusion

\begin{equation}
H^0(C, N^1_{C|\mathcal{X}}) \subset |\mathcal{O}_\mathcal{X}(D_0)| = H^0(C, N_{C|\mathcal{X}_0}),
\end{equation}

holding independently from the kind of singularities of $C$ on $E$. This also implies the very well-known fact that there do not exist deformations of $C$ in $\mathcal{X}$, preserving the nodes of $C$, except for deformations of $C$ in $\mathcal{X}_0$ (see [8, Section 2] for another proof). We now consider the first equation of (18). By the fact that the polynomial $yu_x(x, y, z) + xu_y(x, y, z)$ is identically zero, we deduce that

- $u_x(0, 0, 0, 0) = u_y(0, 0, 0, 0) = 0$
- for every $n \geq 1$, no $z^n$-terms appear in $u_x(x, y, z, t)$ and $u_y(x, y, z, t)$, no $y^n$-terms and $y^n z^m$-terms appear in $u_x(x, y, z, t)$ and, finally, no $x^n$-terms and $x^n z^m$-terms appear in $u_y(x, y, z, t)$.

In particular, local equations at $p$ on $A$ of equisingular infinitesimal deformations of $C$ are given by

\begin{equation}
\begin{cases}
yq(y, z) + zm + em z^{m-1} u_z(x, y, z) = 0 \\
x = 0,
\end{cases}
\end{equation}

where $q(y, z)$ is a polynomial with variables $y$ and $z$, and similarly on $B$. This proves that $H^0(C, N^1_{C|\mathcal{X}}) \subset W_{p, m-1}$. The other inclusion follows by a naive dimensional computation. Indeed, by Remark 5.2, $\dim(W_{p, m-1}) = \dim(|D_0|) - m + 1$. On the other hand, the family $W_{E, m}$ of curves in $|D_0|$ with an $m$-tacnode on $E$ close to $p$, having dimension $\dim(W_{E, m}) = \dim(|D_0|) - m + 1$ by hypothesis 2), is contained in the equisingular deformation locus of $E$ in $\mathcal{X}$. Thus $W_{E, m}$ coincides with the equisingular deformation locus of $C$ in $\mathcal{X}$ and (16) is proved.

We now describe the image of

$$d\phi_{[\mathcal{C}]} : H^0(C, N_{C|\mathcal{X}}) \to H^0(C, T^1_C) = \oplus_{q \in \text{Sing}(\mathcal{C})} T^1_{C, q}.$$ 

Actually, since no matter how we deform $C \subset \mathcal{X}_0$ to a curve on $\mathcal{X}$, the nodes of $C$ are necessarily smoothed, we may restrict our attention to the versal map $\phi_p$ of (15) and its differential

$$d\phi_p : H^0(C, N_{C|\mathcal{X}}) \to H^0(C, T^1_C) \to T^1_{C, p}$$

at $[\mathcal{C}]$. By (16) we deduce that, if $H_p \subset T^1_{C, p}$ is the image of $d\phi_p$, then $H_p$ is a linear space of dimension

$$\dim(H_p) = h^0(C, N_{C|\mathcal{X}}) - \dim(W_{p, m-1}) = m,$$

containing the image of $H^0(C, N_{C|\mathcal{X}_0})$ with respect to $d\phi_p$ as a codimension 1 linear subspace. By the hypothesis 2) and Remark 5.2 the $(m - 1)$-linear space $d\phi_p(H^0(C, N_{C|\mathcal{X}_0}))$ is contained in the tangent space $T_{\mathcal{O}} V^1_m \subset T^1_{C, p}$ at $[\mathcal{O}]$ to the Zariski closure $V^1_m \subset T^1_{C, p}$ of the locus of $m$-nodal curves. Now it is easy to verify that, using the coordinates (14) on $T^1_{C, p}$, the equations of $V^1_m$ are $\beta_0 = \cdots = \beta_{m-1} = 0$. Hence,

\begin{equation}
d\phi_p(H^0(C, N_{C|\mathcal{X}_0})) = T_{\mathcal{O}} V^1_m = V^1_m = \phi_p(|D_0| \cap U_p).
\end{equation}
By \((22)\), in order to find a base of \(H_p\), it is enough to find the image by \(d\phi_p\) of the infinitesimal deformation \(\sigma \in H^0(C, N_{C/X}) \setminus H^0(C, N_{C/X_0})\) having equations

\[
\begin{cases}
x + y + z^m = 0 \\
xy = \epsilon.
\end{cases}
\]

The image of \(\sigma\) is trivially the point corresponding to the curve \(y(y + z^m) = \epsilon\). Thus, the equations of \(H_p \subset T_{C,p}^1\) are given by

\[
H_p = d\phi_p(H^0(C, N_{C/X})) : \beta_1 = \cdots = \beta_{m-1} = 0.
\]

Now fix \(n > 2\) and \(\{d_j\}\) non-negative integers such that \(m = \sum_{j=2}^n (j-1)d_j + 1\), as in the statement. Denote by

\[
V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \subset T_{C,p}^1
\]

the Zariski closure of the locus in \(T_{C,p}^1\) of curves with exactly \(d_j\) singularities of type \(A_j\), for every \(j\), and no further singularities. What we want to prove is that \(V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap H_p \neq \emptyset\). More precisely, we want to prove that

\[
V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap H_p = B_1 \cup B_2 \cup \cdots \cup B_k
\]

where every \(B_i\) is an irreducible and reduced affine curve, whose general element corresponds to a curve with exactly \(d_j\) singularities of type \(A_j\), for every \(j\), and no further singularities. Observe that the equality \((23)\) implies the proposition.

Indeed, by \((16)\), we know that the image \(\phi_p(U_p) \subset T_{C,p}^1 \subset T_{C,p}^2\) of \(U_p \subset \mathcal{H}\) by \(\phi_p\) is a subvariety of \(T_{C,p}^1\) smooth at \(\emptyset\) and of dimension \(m - 1\). Now, by Remark \((24)\) it follows that \(V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \subset T_{C,p}^1\) is a variety of pure dimension \(\sum_j (j-1)d_j = m - 1\), which we are going to prove to be non-empty and containing \(\emptyset\) in every irreducible component. Hence \(V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap \phi_p(U_p)\) is not empty and each of its irreducible components has dimension \(\geq 2m - 1 - 2m + 2 = 1\). By recalling that \(H_p = d\phi_p(H^0(C, N_{C/X})) = T_{\phi_p}(U_p)\) and \(\phi_p(U_p)\) is smooth at \(\emptyset\), we have that, if \((25)\) holds then \(V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap \phi_p(U_p)\) has pure dimension 1. Finally, using that \(\phi^{-1}(\emptyset)\) has codimension \(m\) in \(\mathcal{H}\), we find that \(\phi^{-1}_{p=1}(V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap \phi_p(U_p)) \subset \mathcal{H}\) has codimension \(m - 1 = \sum_j (j-1)d_j\) in \(|\mathcal{O}_{\mathcal{S}_t}(D)|\), proving that \(V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}}\) has at least one non-empty irreducible component of the expected dimension.

To prove \((25)\), we use the same approach as Caporaso and Harris in \([2\) Section 2.4]. First, notice that the equation \((14)\), defining the versal family \(C_p^D\) of the \(m\)-tacnode, has degree 2 in \(y\). In particular, for every point \(x = (\alpha, \beta) \in T_{C,p}^1\), the corresponding curve \(C_{p,x} : F(y, z; \alpha, \beta) = 0\) is a double cover of the \(z\)-axis. Moreover, the discriminant map

\[
\Delta : T_{C,p}^1 \to \mathbb{P}^{2m-1} = \left\{ z^{2m} + \sum_{i=0}^{2m-2} a_i z^i, a_i \in \mathbb{C} \right\}
\]

from \(T_{C,p}^1\) to the affine space of monic polynomials of degree \(2m\) with no \(2m - 1\) term, defined by

\[
\Delta(\alpha, \beta)(z) = \sum_{i=0}^{2m-2} a_i z^i + z^m - 4 \left( \sum_{i=0}^{m-1} \beta_i z^i \right).
\]

is an isomorphism (see \([2\) p. 179]). Thus, we may study curves in the versal deformation family of the \(m\)-tacnode in terms of the associated discriminant polynomial. In particular, for every point \((\alpha, \beta_0, 0, \ldots, 0) \in H_p\), which we will shortly denote by \((\alpha, \beta_0) := (\alpha, \beta_0, 0, \ldots, 0)\), the corresponding discriminant polynomial is given by

\[
\Delta(\alpha, \beta_0)(z) = \sum_{i=0}^{2m-2} a_i z^i + z^m - 4 \beta_0 \left( \nu(z) - 2 \sqrt{\beta_0} \right) \left( \nu(z) + 2 \sqrt{\beta_0} \right),
\]

where \(\nu(z) = (z - 2 \sqrt{\beta_0}) \left( \nu(z) + 2 \sqrt{\beta_0} \right)\).
where we set \( \nu(z) := \sum_{i=0}^{m-2} \alpha_i z^i + z^m \). From now on, since we are interested in deformations of the \( m \)-tacnodal \( C \subset X \) to curves on \( X \), with \( t \neq 0 \), we will always assume \( \beta_0 \neq 0 \), being \( V_{\nu} : \beta_0 = \beta_1 = \ldots = \beta_m = 0 \) the image in \( T^1_{C,p} \) of infinitesimal (and also effective) deformations of \( C \) in \( X \). Writing down explicitly the derivatives of the polynomial \( F(y, z; \alpha, \beta) \) and using that the polynomials \( \nu(z) - 2\sqrt[3]{\beta_0} \) and \( \nu(z) + 2\sqrt[3]{\beta_0} \) may have no common factors if \( \beta_0 \neq 0 \), one may verify that a point \( x = (\alpha, \beta_0) \) parametrizes a curve \( C_{p,x} \) with an \( \ell \) singularity at the point \((x_0, y_0)\) if and only if \( x_0 \) is a root of multiplicity \( k+1 \) of the discriminant polynomial \( \Delta(\alpha, \beta_0)(z) \). Thus our existence problem reduces to proving that the locus of points \((\alpha, \beta_0) \in H_p \) such that the discriminant polynomial \( \Delta(\alpha, \beta_0)(z) \) has exactly \( d_j \) roots of multiplicity \( j \), for every \( 2 \leq j \leq n \), and no further multiple roots, is non-empty of pure dimension 1.

Assume that there exist a point \((\alpha, \beta_0) \in H_p \) with discriminant polynomial as desired. Then, since \( \nu(z) - 2\sqrt[3]{\beta_0} \) and \( \nu(z) + 2\sqrt[3]{\beta_0} \) may no have common factors for \( \beta_0 \neq 0 \), there exist non-negative integers \( d_j^+ \) and \( d_j^- \) such that \( \nu(z) - 2\sqrt[3]{\beta_0} = \sum_j d_j^+ \) and \( \nu(z) + 2\sqrt[3]{\beta_0} = \sum_j d_j^- \) roots of the discriminant \( \Delta(\alpha, \beta_0)(z) \) are distributed as \( d_j^+ \) roots of \( \nu(z) - 2\sqrt[3]{\beta_0} \) and \( d_j^- \) roots of \( \nu(z) + 2\sqrt[3]{\beta_0} \). The obtained \( 2(n-1) \)-tuple of non-negative integers \((d_1^+, d_2^+, \ldots, d_n^+, d_n^-)\) is admissible (see Definition \[8\]). Now, the polynomial \( \nu(z) = z^m + \sum_{i=0}^{m-2} \alpha_i z^i \) defines a degree \( m \) covering \( \nu : \mathbb{P}^1 \to \mathbb{P}^1 \), totally ramified at \( \infty \) and with further \( d_j \) ramification points of order \( j \) over \( \pm 2\sqrt[3]{\beta_0} \), for every \( 2 \leq j \leq n \). We get in this way

\[
(m-1) + \sum_{j=2}^{n} (j-1)d_j^+ + \sum_{j=2}^{n} (j-1)d_j^- = 2m - 2
\]

ramification points of \( \nu \). Hence \( \nu \) has no further ramification by the Riemann-Hurwitz formula. Now, the branch points of \( \nu \) are three, consisting of \( \infty, -2\sqrt[3]{\beta_0} \) and \( 2\sqrt[3]{\beta_0} \), if both sums \( \sum_j d_j^+ \) and \( \sum_j d_j^- \) are positive, while the branch points of \( \nu \) are only two if \( \sum_j d_j^+ = 0 \) or if \( \sum_j d_j^- = 0 \).

Consider first the case that \( \nu \) has only two ramifications points, say \( \infty \) and \( 2\sqrt[3]{\beta_0} \). Then, we have that \( \sum_j d_j^+ = 0 \) and \( \sum_j d_j^- = 0 \), for every \( j \). In particular, using the conditions \[11\] and \[12\], we find that \( d_j = 0 \) for \( j \neq m \) and \( d_m = 1 \). It follows that \( \nu(z) - 2\sqrt[3]{\beta_0} = (z - \lambda)^m \), for a certain \( \lambda \). But the only \( \lambda \) such that \( (z - \lambda)^m \) has not the degree \( m - 1 \) term, is \( \lambda = 0 \). Thus, we get \( \nu(z) = z^m + 2\sqrt[3]{\beta_0} \). On the other hand, for every fixed \( \beta_0 \neq 0 \), if \( \nu(z) = z^m + 2\sqrt[3]{\beta_0} \) then the associated discriminant \( \Delta(\alpha, \beta_0) = z^m(z^m + 4\sqrt[3]{\beta_0}) \) has a root of multiplicity \( m \) and no further multiple roots. This proves that the Zariski closure of the locus in \( H_p \subset T^1_{C,p} \) of curves with an \( A_{m-1} \) singularity is the smooth curve of equations

\[
V_{(m-1)} \cap H_p : \begin{cases} 
\alpha_i = 0, & \text{for every } 1 \leq i \leq m - 2, \\
\alpha_0^2 = 4\beta_0.
\end{cases}
\]

Now consider the general case, i.e. assume that \( \sum_j d_j^\pm > 0 \). Then the polynomial \( \nu(z) \) defines an \( m \)-covering of \( \nu : \mathbb{P}^1 \to \mathbb{P}^1 \) with branch points \( \infty, -2\sqrt[3]{\beta_0} \) and \( 2\sqrt[3]{\beta_0} \), with monodromy permutations \( \sigma, \tau^+ \) and \( \tau^- \) respectively, where \( \sigma \) is an \( m \)-cycle while \( \tau^\pm \) have cícic structure \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). Finally, the group \( \langle \sigma, \tau^+, \tau^- \rangle \) is trivially transitive and, by the theory of coverings of \( \mathbb{P}^1 \), we have that \( \sigma \tau^+ \tau^- = 1 \). In fact, the isomorphism class of \( \nu \) is uniquely determined by the conjugacy classes of the triple of permutation \( (\sigma, \tau^+, \tau^-) \) (cf. e.g. [15 Corollary 4.10]), whence, by Lemma \[8\], the isomorphism class of \( \nu \) is uniquely determined by the ordered \( 2(n-1) \)-tuple \((d_1^+, d_2^+, \ldots, d_n^+, d_n^-)\). Now an isomorphism class of \( \nu \) is an automorphism of the domain \( \mathbb{P}^1 \) preserving the fibers. Let \( \Phi : \mathbb{P}^1 \to \mathbb{P}^1 \) be one of these. Then \( \Phi(x) = ax + b \), as \( \nu(\infty) = \infty \). Moreover, the facts that \( \nu(x) \) is monic and has
no $z^{m-1}$-term yield $a^m = 1$ and $b = 0$, respectively, proving that $\nu(x)$ is unique up to replacing $\nu(x)$ with $\nu(\zeta x)$, where $\zeta$ is an $m$-th root of unity. Thus, if we assume that there exists a point $(\alpha, \beta_0)$ such that the discriminant polynomial $\Delta(\alpha, \beta_0)(z) = \nu(z)^2 - 4\beta_0$ has $d_j$ roots of multiplicity $j$, distributing as $d_j^+$ roots of $\nu(z) + 2\sqrt{\beta_0}$ and $d_j^-$ roots of $\nu(z) - 2\sqrt{\beta_0}$, for every $j$, then the polynomial $\nu(z)$ is uniquely determined up to replacing $\nu(x)$ with $\nu(\zeta x)$, where $\zeta$ is an $m$-th root of unity. On the other hand, by Lemma 3.4 we know that, up to the action of $\mathfrak{S}_m$ by conjugation, for every admissible $(2n-2)$-tuple $(d_2^+, d_2^-, \ldots, d_n^+, d_n^-)$ of non-negative integers there exists a unique triple of permutations $(\tau^+, \tau^-, \sigma)$ of $m$ indices such that $\tau^\pm$ has cyclic structure $\Pi_n^m$, $\sigma$ is a cycle of order $m$ and $\sigma\tau^+\tau^- = 1$. Since the group $\langle \sigma, \tau^+\tau^- \rangle$ is trivially transitive, again by the general theory of coverings of $\mathbb{P}^1$, for every $\gamma \in \mathbb{C}$ there exists an $m$-covering $\nu : \mathbb{P}^1 \to \mathbb{P}^1$ with branch points $\infty, \gamma, -\gamma$ and monodromy permutations $\sigma, \tau^+, \tau^-, \tau^\pm$ respectively. Up to a change of variable in the domain, we may always assume that the associated polynomial $\nu_{(d_2^+, d_2^-, \ldots, d_n^+, d_n^-), \gamma}(z)$ is monic with no $z^{m-1}$-term. Finally, if

\begin{align*}
\nu_{(d_2^+, d_2^-, \ldots, d_n^+, d_n^-), 1}(z) = z^m + \sum_{j=0}^{m-2} c_i(d_2^+, d_2^-, \ldots, d_n^+, d_n^-) z^j
\end{align*}

is a monic polynomial of degree $m$ with no $z^{m-1}$ term, such that $\nu(x) + 1$ (resp. $\nu(x) - 1$) has $d_j^+$ (resp. $d_j^-$) roots of multiplicity $j$, for each $j \in \{2, \ldots, n\}$, and no further multiple roots, then, for every $\gamma \in \mathbb{C} \setminus \{0\}$ and for every $u$ such that $u^m = \gamma$, the polynomial

\begin{align*}
z^m + \sum_{i=0}^{m-2} \alpha_i z^i = \nu_{(d_2^+, d_2^-, \ldots, d_n^+, d_n^-)}(z) = u^m \nu_{(d_2^+, d_2^-, \ldots, d_n^+, d_n^-), 1}(\frac{z}{u})
\end{align*}

is a monic polynomial without $z^{m-1}$ term and such that $(\nu_{(d_2^+, d_2^-, \ldots, d_n^+, d_n^-)}(z))^2 - \gamma^2$ has the same kind of multiple roots, with the desired distribution. We have thus proved that the reduced and irreducible curve of parametric equations

\begin{align*}
\alpha_i = u^{m-i} c_i(d_2^+, d_2^-, \ldots, d_n^+, d_n^-), \quad \text{for } i = 0, 1, \ldots, m - 2, \quad \text{and } \beta_0 = \frac{u^{2m}}{4}, \quad u \in \mathbb{C},
\end{align*}

is an irreducible component of $V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap H_p$. In particular,

\begin{align*}
V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap H_p = B_1 \cup \ldots \cup B_k
\end{align*}

is a reduced curve with $k$ irreducible components, where $k$ is the number of admissible $2(n - 1)$-tuples $(d_2^+, d_2^-, \ldots, d_n^+, d_n^-)$ such that $d_j^+ + d_j^- = d_j$, for every $j$.

The proof of the following result is an immediate consequence of the proof of the previous proposition.

**Corollary 3.6.** With the notation above, let $C' \subset |D_0|$ be any reduced curve with an $m$-tacnode at a general point $p \in E$ and possibly further singularities. Then, the image $H_{p'}$ of the morphism

\begin{align*}
H^0(C', N_{C'/x}) \longrightarrow T_{C', p}
\end{align*}

is contained in the $m$-plane $H_p$ of equations $H_p : \beta_1 = \ldots = \beta_{m-1} = 0$. If $H_{p'} = H_p$ and $d_1, \ldots, d_n$ are integers such that $\sum_j (j - 1) d_j = m - 1$, then $V_{1^{d_2}, 2^{d_3}, \ldots, n-1^{d_n}} \cap H_p$ is the curve give by (29).

**Remark 3.7.** Using the same notation as in the proof of Theorem 3.2, we observe that the parametric equations (29) may take a simpler form in many cases. By [2], Section 2.4), we know that, in the case $n = 2$, i.e. if we consider deformations of the $m$-tacnode to $(m - 1)$-nodal curves, then the polynomial $\nu_{(d_2^+, d_2^-), 1}(z)$ is odd if
m odd and it is even if m is even. This result extends to the general case for even m or if m is odd but \( d_j^+ = d_j^- \), for every j. Indeed, if m is even, referring to (28), the polynomial \( \nu(d_j^+, d_j^-,..., d_n^+, d_n^-)\),1\((z)\) is monic of degree m, with no \( z^{m-1}\)-term. Moreover, the discriminant polynomial \( (\nu(d_j^+, d_j^-,..., d_n^+, d_n^-),1\((z)\))^2 - 4 \) has \( d_j \) roots of multiplicity \( j \), distributed as \( d_j^+ \) roots of \( \nu(d_j^+, d_j^-,..., d_n^+, d_n^-)\),1\((z)\) + 1 and \( d_j^- \) roots of \( \nu(d_j^+, d_j^-,..., d_n^+, d_n^-),1\((z)\) - 1, for every j, and no further multiple roots. By unicity, there exists \( \zeta \in \mathbb{C} \), with \( \zeta^m = 1 \) such that

\[
\nu(d_j^+, d_j^-,..., d_n^+, d_n^-),1\((z)\) = \nu(d_j^+, d_j^-,..., d_n^+, d_n^-),1\((\zeta z)\),
\]

from which we deduce that \( c_i, (d_j^+, d_j^-, ... , d_n^+, d_n^-) = 0 \), for every odd \( i \) and \( \zeta = 1 \). In particular, the parametric equations (30), for m even, become

\[
\begin{align*}
\alpha_0 &= t^l c_0, (d_j^+, d_j^-, ... , d_n^+, d_n^-), \\
\alpha_2 &= t^{l-1} c_2, (d_j^+, d_j^-, ... , d_n^+, d_n^-), \\
\vdots &= \vdots \\
\alpha_{m-2} &= t c_{m-2}, (d_j^+, d_j^-, ... , d_n^+, d_n^-), \\
\end{align*}
\]

(30)

When \( m = 2l + 1 \) is odd and \( d_j^+ = d_j^- = d_j/2 \), for every j, by applying the same argument to \( \nu(d_j/2, d_j/2,..., d_n/2, d_n/2),1\((z)\) \) and \( -\nu(d_j/2, d_j/2,..., d_n/2, d_n/2),1\((z)\) \), the equations (29) become

\[
\begin{align*}
\alpha_0 &= 0, & \alpha_1 &= t^l c_1, (d_j/2, d_j/2,..., d_n/2, d_n/2), \\
\alpha_2 &= 0, & \alpha_3 &= t^{l-1} c_3, (d_j/2, d_j/2,..., d_n/2, d_n/2), \\
\vdots &= \vdots \\
\alpha_{m-2} &= 0 & \beta_0 = \frac{t^m}{4}, \ t = u^2 \in \mathbb{C}.
\end{align*}
\]

(31)

Remark 3.8 (Multiplicities and base changes). In the same way as in [8] Section 1 for families of curves with only nodes and ordinary cusps, it is possible to define a relative Severi-Enriques variety \( \mathcal{V}_{X, D, 1}^{X_1^{1^2}, 2^{m_2}, ..., 1^{d_n}} \subset \mathcal{H} \), whose general fibre is variety \( \mathcal{V}_{D_1, 1}^{X_1^{1^2}, 2^{m_2}, ..., 1^{d_n}} \subset |D_1| = \mathcal{H}_t \). Using the same notation as above, Theorem 2.3 proves that, whenever the hypotheses of the theorem are verified, then the equisingular deformation locus \( ES(C) = W_{E,m} \subset |D_0| \) of C in \( |D_0| \) is one of the irreducible components of the special fibre \( \mathcal{V}_0 \) of \( \mathcal{V}_{D, 1}^{X_1^{1^2}, 2^{m_2}, ..., 1^{d_n}} \to \mathbb{A}^1 \). The multiplicity \( m_C \) of \( ES(C) \), as irreducible component of \( \mathcal{V}_0 \), coincides with the intersection multiplicity of the curve

\[
\mathcal{V}_{1^2, 2^{m_2}, ..., 1^{d_n}} \cap H_p = B_1 \cup ... \cup B_k
\]

with

\[
H_p = d\phi_p(H^0(C, N_{C_X}(x_0))) = d\phi_p(T|_C|D_0) : \beta_0 = ... = \beta_{m-1} = 0.
\]

Furthermore, the minimal multiplicity intersection of each irreducible component \( B_i \) with \( d\phi_p(T|_C|D_0) \) is the geometric multiplicity \( m_g \) of C, (see [8] Problem 1 and Definition 1). Notice also that saying that, under the hypotheses of Theorem 2.3, the \( m \)-tacnodal curve \( C \subset X_0 \) deforms to a curve \( C_t \in |\mathcal{O}_{X_1}(D_0)| \) with \( d_j \) singularities of type \( A_j-1 \), for every j, is a terminology abuse. What is true is that, if

\[
\begin{tikzcd}
\mathcal{X}' \arrow[r, h] & \mathcal{X} \\
\mathbb{A}^1 \arrow[u] \arrow[r, h] & \mathbb{A}^1 \\
\end{tikzcd}
\]

is the smooth family of surfaces obtained from \( \mathcal{X} \to \mathbb{A}^1 \) by a base change of order \( m_g \) and blow-ups of the total space, then the pull-back curve \( h^*(C) \) of C to \( \mathcal{X}' \) may by deformed to a curve \( C_t \in |\mathcal{O}_{X_1}(h^*(D))| \) with the wished singularities. Equivalently,
the divisor $m_p C \in |O_{X_0}(m_p D_0)|$ deforms to a reduced curve in $|O_X(m_p D_0)|$ having $m_p$ irreducible components, each of which is a curve with the desired singularities.

When $d_2 = m - 1$ and $d_j = 0$ for $j \neq 2$, we know by [2] that $k = 1$ and $B_1$ is smooth at 0 and tangent to $d_2 \gamma (H^0 \langle C, N_{C/X_0} \rangle)$ with multiplicity $m_C = m$. In the general case, controlling the non-vanishing of the terms $c_i, \ldots, c_{16}$ in ([30], [31]) or in ([20]), is useful in order to find $m_C$ and $m_p$. We always expect $m_C = mk$.

**Example 3.9.** In the previous remark, take $m = 5$, $d_3 = 2$ and $d_j = 0$, if $j \neq 3$. Then $m_C = m_p = 5$. Roughly speaking, a 5-tacnode at a general point $p \in E \subset X_0$ appears as limit of curves with two ordinary cusps on $X_t$ with multiplicity 5.

**Proof.** If $m = 5$, $d_3 = 2$ and $d_j = 0$, if $j \neq 3$, then we have only one admissible 2-tuple $(d_3^+, d_3^-) = (1, 3)$. Thus the parametric equations of the curve $V_{22} \cap H_p$ are

$$\alpha_0 = \alpha_2 = 0, \quad \alpha_1 = \frac{t^2 c_1(d_3^+ = 1, d_3^- = 1)}{c_3(d_3^+ = 1, d_3^- = 1)} \text{ and } \beta_0 = \frac{t^5}{4} t \in \mathbb{C}.$$

The curve above intersects $\beta_0 = \beta_1 = \ldots = \beta_5 = 0$ with multiplicity 5, independently from the vanishing of $c_1(d_3^+ = 1, d_3^- = 1)$ or $c_3(d_3^+ = 1, d_3^- = 1)$. Actually, by the fact that the polynomial

$$\nu(d_3^+ = 1, d_3^- = 1)(z) = z^5 + c_3(d_3^+ = 1, d_3^- = 1)z^3 + c_1(d_3^+ = 1, d_3^- = 1)z \pm 1$$

has a triple root, we find $c_3(d_3^+ = 1, d_3^- = 1) \neq 0$ and $V_{22} \cap H_p$ is smooth at 0. \hfill $\square$

The corollary below follows directly from equations ([27]). Observe also that it was an expected result by [8] Lemmas 2 and 6.

**Corollary 3.10.** Independently from $m$, if $d_m = 1$ and $d_j = 0$, if $j \neq m$, then $m_C = m_p = 2$. In other words, the $m$-tacnode on $X_0$ appears as limit of an $A_m$ singularity on $X_t$ with multiplicity 2.

4. AN APPLICATION TO GENERAL K3 SURFACES

In this section $(S, H)$ will denote a general primitively polarized K3 surface of genus $p = p_n(H)$. Our aim is to show to existence of curves on $S$ with $A_k$ singularities as a straightforward application of Theorem [3,8]. We will degenerate $S$ to a union of two rational normal scrolls $R_1 \cup R_2$ and we will deform the same curves $C \subset R_1 \cup R_2$ introduced by Chen in [8] to prove the existence of rational nodal curve on $S$.

We briefly explain the degeneration argument, which is very classical and has been introduced in [4]. Fix an integer $p \geq 3$ and set $l := \lfloor \frac{p}{2} \rfloor$. Let $E \subset \mathbb{P}^p$ be a smooth elliptic normal curve of degree $p + 1$. Consider two general line bundles $L_1, L_2 \in \operatorname{Pic}^2(E)$. We denote by $R_1$ and $R_2$ the (unique) rational normal scrolls of degree $p + 2$ in $\mathbb{P}^p$ defined by $L_1$ and $L_2$, respectively. Note that $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, if $p$ is odd whereas $R_1 \cong R_2 \cong \mathbb{F}_1$ if $p$ is even. Moreover, $R_1$ and $R_2$ intersect transversally along the curve $E$ which is anticanonical in each $R_i$ (cf. [4] Lemma 1]). More precisely, for odd $p$, where $R_1 \cong R_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$, we let $\sigma_i = \mathbb{P}^1 \times \{pt\}$ and $F_i = \{pt\} \times \mathbb{P}^1$ on $R_i$ be the generators of $\operatorname{Pic} R_i$, for $i = 1, 2$. For even $p$, where $R_1 \cong R_2 \cong \mathbb{F}_1$, we let $\sigma_i$ be the section of negative self-intersection and $F_i$ be the class of a fiber. Then the embedding of $R_i$ into $\mathbb{P}^p$ is given by the line bundle $\sigma_i + I F_i$ for $i = 1, 2$. Let now $R := R_1 \cup R_2$ and let $U_p$ be the component of the Hilbert scheme of $\mathbb{P}^p$ containing $R$. Then we have that $\dim(U_p) = p^2 + 2p + 19$ and, by [4] Theorems 1 and 2], the general point $[S] \in U_p$ represents a smooth, projective K3 surface $S$ of degree $2p - 2$ in $\mathbb{P}^p$ such that $\operatorname{Pic} S \cong \mathbb{Z}[O_{S}(1)] = \mathbb{Z}[H]$. We denote by $S \to T$ a general deformation of $S_0 = S_0 = R$ over a one-dimensional disc $T$ contained in $U_p$. In particular, the general fiber is a smooth projective K3 surface $S_t = S_t \in \mathbb{P}^p$ with $\operatorname{Pic} S_t \cong \mathbb{Z}[O_{S_t}(1)]$. Now, $S$ is smooth except for 16 rational double points $\{\xi_1, \ldots, \xi_16\}$ lying on $E$. Blowing-up $S$ at these points and contracting the obtained exceptional curves (all isomorphic to $\mathbb{F}_0$) on $R_2$, we
get a smooth family of surfaces $\mathcal{X} \to T$, such that $\mathcal{X}_t \simeq S_t$ and $\mathcal{X}_0 = R_1 \cup \tilde{R}_2$, where $R_2$ is the blowing-up of $R_2$ at the points $\{\xi_1, \ldots, \xi_6\}$, with new exceptional curves $E_1, \ldots, E_6$. We will call $\{\xi_1, \ldots, \xi_6\}$ the special points of $E$.

Now fix $n \geq 1$. By [6, §3.2], we know that there exist finitely many curves $C \subset R = R_1 \cup R_2 \subset \mathbb{P}^p$ defined by

\[
C = C_1 \cup C_2 \cup \cdots \cup C_{n-1} \cup C_n \cup C_2^1 \cup C_2^2 \cup \cdots \cup C_{n-1}^2 \cup C_n^2 \in |\mathcal{O}_R(nH)|
\]

with:

- $C_j^1 \subset R_j$ and $C_j^2$ irreducible, for $i = 1, 2$ and $1 \leq j \leq n$;
- $C_1^i \in |\sigma_i|$ for $1 \leq j \leq n - 1$ and $C_n^i \in |\sigma_i + nF|$ if $p$ is odd; $C_j^i \in |\sigma_i + F|$ for $1 \leq j \leq n - 1$ and $C_n^i \in |\sigma_i + (n\ell - n + 1)F|$ if $p$ is even, for $i = 1, 2$;

where the curves $C_j^i$, with $1 \leq i \leq 2$ and $1 \leq j \leq n$, are determined by the relations

\[
\begin{align*}
C_1^1 \cap E &= q_{2j-2} + q_{2j-1}, & C_2^1 \cap E &= q_{2j-1} + q_{2j}, & C_j^1 \cap E &= q_{2n-2} + (2nl + 1)r, & C_j^2 \cap E &= q_0 + (2nl + 1)r, \\
C_1^2 \cap E &= q_{2j-2} + 2q_{2j-1}, & C_2^2 \cap E &= 2q_{2j-1} + q_{2j}, & C_j^2 \cap E &= q_{2n-2} + (2nl - 2n + 2)r, & C_j^2 \cap E &= q_0 + (2nl - 2n + 2)r,
\end{align*}
\]

if $p$ is odd and

\[
\begin{align*}
C_1^1 \cap E &= q_{2j-2} + 2q_{2j-1}, & C_2^1 \cap E &= 2q_{2j-1} + q_{2j}, & C_j^1 \cap E &= q_{2n-2} + (2nl - 2n + 2)r, & C_j^2 \cap E &= q_0 + (2nl - 2n + 2)r,
\end{align*}
\]

if $p$ is even, where $q_0$ is one of the sixteen special points of $E$ and $q_1, q_2, \ldots, q_{2n-2}$ and $r$ are distinct non-special points on $E$. As explained in [6, §3.2], the points $q_1, q_2, \ldots, q_{2n-2}$ and $r$ are uniquely determined by the conditions above and we may assume that all the curves $C_j$ intersect each other transversally and no three of them meet at a point.

We may now prove our existence theorem on $K3$ surfaces.

**Proof of Theorem 1.1.** Let $C = C_1 \cup C_2 \cup \cdots \cup C_{n-1} \cup C_n \cup C_2^1 \cup C_2^2 \cup \cdots \cup C_{n-1}^2 \cup C_n^2$ on $R = R_1 \cup R_2$, with $C \sim nH$, the curve introduced above.

We first consider the case $p = 2l + 1$ odd. In this case the singularities of $C$ are given by

- $2(n-1)nl$ nodes $y_1, \ldots, y_{(n-1)nl}$, on $R \setminus E$, arising from the intersection of the curves $C_j^i$, for $1 \leq j \leq n - 1$, with $C_n^i$, for every $i = 1, 2$;
- a special node at $q_0 \in E$;
- $2n - 1$ nodes $x_1, \ldots, x_{2n-1}$ at non-special points of $E$;
- an $(2nl + 1)$-tacnode at $r$.

Now, by abusing terminology (see Remark 3.8), we would like to deform the curve $C \in |\mathcal{O}_{S_0(nH)}|$ by smoothing the $(2nl + 1)$-tacnode at $r$ to $d_j$ singularities of type $A_j$, for every $(m - 1)$-tuple of integers $d_2, \ldots, d_m$ verifying (2), and by preserving, at the same time, the node of $C$ at $q_0$ and the $2nl(n-1)$ nodes on $R \setminus E$, obtaining an irreducible curve $C_t \in |\mathcal{O}_{S_0(nH)}|$ with the desired singularities. (As in [6], the irreducibility of $C_t$ follows from the fact that, by using that the hyperplane divisor class $H_t$ on $S_t$ is indivisible, it easy to see that every curve $C_t$ which is deformation of the curve $C \subset R_1 \cup R_2$ above must be irreducible.) In order to see that this is possible, let $E_{q_0}$ be the $(-1)$-curve of $\tilde{R}_2 \subset X_0$ passing through $q_0$ and let $\pi : \mathcal{X} \to S$ be the normalization morphism. Moreover, let $\bar{C}$ be the strict transform of $C$ and $\pi^*(C) = \bar{C} \cup E_{q_0}$ be the pull-back of $C$ with respect to $\pi$. Now, $\pi^*(C)$ has one more node at $x_0 = E_{q_0} \cap \bar{C}$ on $\tilde{R}_2 \setminus E$. By Section 2 and by the proof of Theorem 3.3, the tangent space $T[\pi^*(C)]ES(\pi^*(C))$ to the equisingular deformation locus of $\pi^*C$ in $|\mathcal{O}_{X_0}(nH)|$, parametrizes divisors $D = D^1 \cup D^2 \in |\mathcal{O}_{X_0}(nH)|$, passing through nodes of $\pi^*(C)$ on $X_0 \setminus E$, $x_0$ included, and having a $(2nl)$-tacnode at the $(2nl + 1)$-tacnode $r$ of $\pi^*(C)$. By applying Bezout theorem, we find that $T[\pi^*(C)]ES(\pi^*(C)) = \{[\pi^*(C)]\}$. Now, using that the nodes of $C$ at the points $x_i$
are trivially preserved by every section of $H^0(C, N_{\pi^*(C)|x_0})$, we have that the kernel of the standard morphism

$$H^0(\pi^*(C), N_{\pi^*(C)|x}) \longrightarrow \oplus_j T_{\pi^*(C), y_j}^1 \oplus T_{\pi^*(C), x_0}^1 \oplus T_{\pi^*(C), r}^1$$

coincides with $T_{\pi^*(C)ES(\pi^*(C))}$. Then, the morphism above is injective, with image of dimension $h^0(\pi^*(C), N_{\pi^*(C)|x}) = 2nl + 2 = \dim(|nH|) + 1$. Since, by Corollary 4.1, the image of the induced morphism

$$H^0(\pi^*(C), N_{\pi^*(C)|x}) \longrightarrow T_{\pi^*(C), r}^1$$

always has dimension $\leq 2nl + 1$, then, with the notation of Corollary 4.6 we find that the image of the morphism (33) is $\oplus_j T_{\pi^*(C), y_j}^1 \oplus T_{\pi^*(C), x_0}^1 \oplus H_r$, where $H_r : \beta_1 = \ldots = \beta_{2nl} = 0$. By versality and again by Corollary 4.6 we get the curves $C_t \in |O_{S, (nH)}|$ with the desired singularities. Finally, by the injectivity of (33), we have that $\dim(T_{|C_t|ES(C_t)}) = 0$ and the singularities of $C_t$ may be smoothed independently by Remark 2.2.

Now consider the case that $p = 2l$ is even. In this case $C$ has the following singularities:

- a $(2nl - 2n + 2)$-tacnode at the (non-special) point $r$ of $E$;
- a node at a special point $q_0$ of $E$;
- $n - 1$ simple tacnodes at the (non-special) points $q_{2j-1}$, with $1 \leq j \leq n - 1$;
- $n - 2$ nodes at the (non-special) points $q_{2j-2}$, with $2 \leq j \leq n - 1$;
- $(2(n - 1)(nl - n + 1) + (n - 2)(n - 1))$ nodes on the smooth locus of $R$ arising from the intersection of the curves $C^0_j$, for $1 \leq j \leq n - 1$, with $C_n^k$, for every $i = 1, 2$ and from the intersection of $C_j^0$ with $C_k^0$, for $k \neq j$.

The proposition follows as in the previous case, by smoothing every simple tacnode of $C$ to a node, the $(2nl - 2n + 2)$-tacnode at $r$ to $d_j$ singularities of type $A_j$, for every $(m - 1)$-tuple of integers $d_2, \ldots, d_m$ verifying 11 and, by preserving, at the same time, the node of $C$ at $q_0$ and the $(2(n - 1)(nl - n + 1) + (n - 2)(n - 1))$ nodes on $R \setminus E$.

Theorem 1.1 is optimal for $n = 1$. The precise statement is the following.

**Corollary 4.1.** Let $(S, H)$ be a general primitively polarized K3 surface of genus $p = p_a(H)$. Then, for every $(m - 1)$-tuple of non-negative integers $d_2, \ldots, d_m$ such that

$$\sum_{j=2}^m (j - 1)d_j \leq \dim(|H|),$$

there exist reduced irreducible curves $C$ in the linear system $|H|$ on $S$ having $d_j$ singularities of type $A_{j-1}$ for every $j = 2, \ldots, m$, and no further singularities and corresponding to regular points of their equisingular deformation locus $ES(C_t)$. Equivalently, $\dim(T_{|C_t|ES(C_t)}) = \dim(|H|) - \sum_{j=2}^m (j - 1)d_j$.

We finally point out the following consequence of Theorem 1.1.

**Corollary 4.2.** Let $(S_t, H)$ be a general primitively polarized K3 surface of genus $p = p_a(H)$. Then, there exist irreducible curves in $|nH|$ of every geometric genus $1 \leq g \leq p_a(nH)$ having 1 ordinary cusp and nodes or 1 simple tacnode and nodes as singularities.
Appendices

A. Proof of Lemma 3.4

The uniqueness statement follows from the well-known fact that two permutations with the same cyclic structure are conjugated.

We first prove the existence part in the special case of an admissible 2(\(n-1\))-tuple satisfying \(\sum_{j=2}^{n} d_j^- = 1\) or \(\sum_{j=2}^{n} d_j^+ = 1\). So assume, by symmetry, that \(\sum_{j=2}^{n} d_j^- = 1\). Then there is an index \(i\) such that \(d_i^- = 1\) and \(d_j^- = 0\) for all \(j \neq i\). Then the question is whether there is a permutation \(\tau^+\) of cyclic structure \(\Pi_{j=2}^{n} d_j^+\) and a cycle \(\tau^- = \sigma_i\) of order \(i\) such that \(\tau^+ \sigma_i = (1 \ 2 \ \cdots \ m)\). Let \(\sigma_i = (1 \ 2 \ \cdots \ i)\). By (11), we have \(\sum_{j=2}^{n} (j-1) d_j^+ = m - i\). This implies that we can construct a permutation \(\tau^+\) of the desired cyclic structure such that each cycle contains precisely one integer in the set \(\{1, 2, \ldots, i\}\). It is then easily seen that \(\tau^+ (1 \ 2 \ \cdots \ i) = (1 \ 2 \ \cdots \ m)\).

We now prove the existence part in general by induction on \(m\). The base cases of the induction are all cases where \(\sum_{j=2}^{n} d_j^- = 1\) or \(\sum_{j=2}^{n} d_j^+ = 1\), which have been treated above. Assume now that \((d_2^-, d_2^+, \ldots, d_n^-, d_n^+)\) is an admissible \((n-1)\)-tuple such that both \(\sum_{j=2}^{n} d_j^- \geq 2\). By symmetry we may assume that \(\sum_{j=2}^{n} j d_j^- \geq \sum_{j=2}^{n} j d_j^+ \geq 2\). Let \(i := \min\{j \mid j d_j^- > 0\}\). We now claim that the \((n-1)\)-tuple

\[(d_2^-, d_2^+, \ldots, d_i^-, d_i^+, d_{i+1}^- = d_{i+1}^+, \ldots, d_n^-, d_n^+) = (d_2^+, d_2^-, \ldots, d_i^+, d_i^-, d_{i+1}^+, \ldots, d_n^+, d_n^-) \]

is admissible. Indeed, set \(m' := \sum_{j=2}^{n} (j-1)(d_j^+ + d_j^-) + 1 = m - i + 1\). Then \(m' \geq 2\) since \(\sum_{j=2}^{n} d_j^+ = \sum_{j=2}^{n} d_j^- \geq 2\). Clearly,

\[\sum_{j=2}^{n} j d_j^- = \sum_{j=2}^{n} j d_j^+ - i \leq m - i < m - i + 1 = m' \]

by (12). Now, assume that \(\sum_{j=2}^{n} j d_j^+ = \sum_{j=2}^{n} j d_j^- > m' = m - i + 1\). Then we have that

\[m - i + 2 \leq \sum_{j=2}^{n} j d_j^+ \leq \sum_{j=2}^{n} j d_j^-, \text { whence} \]

\[2m - 2i + 4 \leq \sum_{j=2}^{n} j (d_j^+ + d_j^-) = 2 \sum_{j=2}^{n} (j-1)(d_j^+ + d_j^-) - \sum_{j=2}^{n} (j-2)(d_j^+ + d_j^-) \]

\[= 2(m - 1) - \sum_{j=2}^{n} (j-2)(d_j^+ + d_j^-), \]

by (13). It follows that \(0 \leq \sum_{j=2}^{n} (j-2)d_j^- \leq \sum_{j=2}^{n} (j-2)(d_j^+ + d_j^-) \leq 2i - 6\). In particular, we obtain that \(i \geq 3\). Moreover, by definition of \(i\), we must have \(2(i-2) \leq (i-2) \sum_{j=2}^{n} d_j^- \leq \sum_{j=2}^{n} (j-2)d_j^- \leq 2i - 6\), getting a contradiction. Therefore, we have proved our claim that the \((n-1)\)-tuple

\[(d_2^+, d_2^-, \ldots, d_i^+, d_i^-, d_{i+1}^+, \ldots, d_n^+, d_n^-) = (d_2^+, d_2^-, \ldots, d_i^+, d_i^-, d_{i+1}^+, \ldots, d_n^+, d_n^-) \]

is admissible. By induction, there exist permutations \(\tau^\pm\) in \(S_{m-i+1}\) of cyclic structures \(\Pi_{j=2}^{n} d_j^\pm\), respectively, such that

\[\tau^+ \tau^- = (1 \ 2 \ \cdots \ (m - i + 1))\]
Now, the number of distinct integers from \( \{1, 2, \ldots, m - i + 1\} \) appearing in the permutation \( \tau^- \) is \( \sum_{j=2}^{n} j \alpha_j^- \), which is less than \( m - i + 1 \) by \( \boxed{\text{34}} \). Hence there exists an \( x \in \{1, 2, \ldots, m - i + 1\} \) not appearing in \( \tau^- \). Then the permutation
\[
\alpha^- = \tau^- \left( (m - i + 2) (m - i + 3) \cdots m \right)
\]
has cyclic structure \( \Pi_{j=2}^{d} j \) and
\[
\tau^+ \alpha^- = \left( 1 2 \cdots (m - i + 1) \right) \left( (m - i + 2) (m - i + 3) \cdots m \right) = \left( 1 \cdots m \right),
\]
as desired.

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