Spine-tensor decomposition of nuclear transition matrix elements for neutrinoless double-$\beta$ decay of $^{76}\text{Ge}$ and $^{82}\text{Se}$ nuclei within PHFB approach

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Employing the PHFB model, nuclear transition matrix elements $M^{(K)}$ for the neutrinoless double-$\beta^-$ decay of $^{76}\text{Ge}$ and $^{82}\text{Se}$ isotopes are calculated within mechanisms involving light as well as heavy Majorana neutrinos, and classical Majorons by considering the spine-tensor decomposition of realistic KUO and empirical JUN45 effective two-body interaction. It is noticed that the effects due to the SRC on NTMEs $M^{(0\nu)}$ and $M^{(0N)}$ due to the exchange of light and heavy Majorana neutrinos, respectively, is maximally incorporated by the central part of the effective two-body interaction, which varies by a small amount with the inclusion of spin-orbit and tensor components. The maximum uncertainty in the average NTMEs $M^{(0\nu)}$ and $M^{(0N)}$ turns out to be about 10% and 37%, respectively.

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I. INTRODUCTION

In any gauge theoretical model, the possible occurrence of neutrinoless double beta ($0\nu\beta\beta$) decay is intimately associated with the violation of lepton number $L$ conservation. Out of several possible mechanisms involving left-right symmetry, $R_L$-violating supersymmetry, Majorons, sterile neutrinos, leptoquarks, compositeness and extra-dimensional scenarios [1, 2], the Majorana neutrino mass mechanism is considered as the standard one to ascertain the Dirac or Majorana nature of neutrinos. The half-lives $T^{0\nu}_{1/2}$ of the $0\nu\beta\beta$ decay is a product of phase space factors, nuclear transition matrix elements (NTMEs) and corresponding gauge theoretical parameters. As the phase space factors have been calculated to good accuracy in the recent past [3, 4], the accuracy of the extracted limits on the parameters of a particular gauge theoretical model depends on the reliability of model dependent NTMEs. Specifically, the effective mass of the light and heavy Majorana neutrinos are extracted in the standard mass mechanism. Over the past years, the theoretical studies devoted to the calculation of NTMEs have been excellently reviewed in Ref. [5] and references there in.

In the evaluation of NTMEs, different theoretical approaches, namely interacting shell-model (ISM) calculations based on direct diagonalization [7, 10], QRPA [11, 12] and its extensions [13, 14], deformed QRPA, [15, 16], QRPA with isospin restoration [17], projected-Hartree-Fock-Bogoliubov (PHFB) [18, 21], interacting boson model (IBM) [22, 23] with isospin restoration [24], the generator coordinate method (GCM) [25], and beyond mean field covariant density functional theory (BMFCDFT) [26] have been employed. In spite of the fact that, several alternatives are available for the choice of model space, effective two-body residual interactions, model dependent form factors to include the finite size of nucleons (FNS), short range correlations (SRC) with Miller- Spencer parametrization [27], unitary operator method (UCOM) [28] parametrization based on coupled cluster method (CCM) [29], and the value of axial vector current coupling constant $g_A$ [24, 30–32], the calculated NTMEs $M^{(0\nu)}$ interestingly differ by factor of 2–3.

In addition to these exciting developments in the theoretical front, the remarkable experimental studies of the $\beta^+\beta^-$ decay [33] have resulted in measuring half lives $T_{1/2}^{\beta^-\beta^-}$ of $0\nu\beta^-\beta^-$ decay of $^{76}\text{Ge}$, $^{100}\text{Mo}$, $^{130}\text{Te}$ and $^{136}\text{Xe}$ isotopes to be $\geq 10^{25}$ yr by the combined data of the Heidelberg-Moscow experiment [34], international germanium experiment (IGEX) [35] and GREDAS-I [36], $> 1.1 \times 10^{24}$ yr by NEMO-3 [37], $> 4.0 \times 10^{24}$ yr by CUORE [38] and $> 1.1 \times 10^{26}$ yr by KamLAND-Zen [39] ($> 1.6 \times 10^{25}$ by EXO [40]), respectively. Our present concern is to calculate NTMEs for the $0\nu\beta^-\beta^-$ decay of $^{76}\text{Ge}$, and $^{82}\text{Se}$ isotopes, which in turn requires the reliable wave functions of $^{76}\text{Ge}$, $^{76,82}\text{Se}$ and $^{82}\text{Kr}$ nuclei. As the wave functions are model dependent, the employed model should be versatile enough to reproduce as many observed properties of nuclei as possible.

An important observed characteristic feature of nuclei in the Ge region is the shape transitions at $N = 40$. The onset of deformation at $N = 40$ necessitates to adopt a calculational framework treating the interplay of pairing and deformation degrees of freedom simultaneously, and on equal footing [41]. Calculations have already been performed by using ISM in a valance space spanned by the $1p_{1/2}$, $1p_{3/2}$, $0f_{5/2}$ and $0g_{9/2}$ orbits treating the doubly even $^{56}\text{Ni}$ as an inert core. The present calculation is performed employing the PHFB approach in the above mentioned valance space with a realistic and an empiri-

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cal two body effective interaction, namely KUO \[42\] and JUN45 \[43\], respectively. The purpose of such a calculation is to demonstrate the difference between the two approaches and infer about the role of neglected configurations and quasiparticle interactions.

II. THEORETICAL FORMALISM

The detailed theoretical formalism required for the study of the $0
\nu\beta^-\beta^-$ decay within the Majorana neutrino mass mechanism has been given in Refs. \[44\], \[45\]. Within the PHFB model, the calculation of NTMEs due to the exchange of light \[19\] and heavy Majorana \[20\] neutrinos has already been reported. In the following, we present a brief outline of the required formalism for the clarity in notation used in the present paper.

Within the Majorana neutrino mass mechanism, the half-life $T_{1/2}^{(0\nu)}$ for the $0^+ \rightarrow 0^+$ transition of $0\nu\beta^-\beta^-$ decay is given by

$$[T_{1/2}^{(0\nu)} (0^+ \rightarrow 0^+)]^{-1} = G_{01} \left[ \frac{m_\nu}{m_e} M^{(0\nu)} + \frac{m_p}{M_N} M^{(0N)} \right]^2$$

where

$$\langle m_\nu \rangle = \sum_i U_{e_i}^2 m_i, \quad m_i < 10 \text{ eV},$$

$$\langle M_N \rangle^{-1} = \sum_i U_{e_i}^2 m_i^{-1}, \quad m_i > 10 \text{ GeV},$$

$$M^{(0K)} = \frac{-g^{(0K)}_F}{g_A^{(0)}} + M^{(0K)}_{GT} + M^{(0K)}_F$$

and the $K = 0\nu (0N)$ denotes mass mechanism due to the exchange of light (heavy) Majorana neutrinos. The phase space factors

$$G_{01} = \left[ \frac{2 (G_F g_A)^4 m_\nu}{64\pi^5 (m_e R)^2 \ln 2} \right] \int_1^{T+1} f^{(0)}_{11} p_1 p_2 \varepsilon_1 \varepsilon_2 \, d\varepsilon_1$$

have been recently calculated with good accuracy incorporating the screening correction \[42\] \[43\] and the calculation of the NTMEs $M^{(K)}$ of the $0\nu\beta^-\beta^-$ decay within the PHFB model, has already been discussed in Ref. \[18\], \[20\].

Employing the HFB wave functions, one obtains the following expression for the NTME $M^{(K)}_\alpha$ of the $0\nu\beta^-\beta^-$ decay corresponding to an operator $O^{(K)}_\alpha$:

$$M^{(K)}_\alpha = \left\langle 0_1^+ \parallel O^{(K)}_\alpha \parallel 0_1^+ \right\rangle = \left[ \int_{t=0}^\pi n_{i=0} n_{\gamma=0} \right]^{-1/2}$$

$$\times \int_0^{\pi} n_{(Z,N)},(Z+2,N-2) (\theta) \sum_{i \alpha \gamma \delta} \left( \alpha \beta \right) \left\langle 0_1^+ \parallel O^{(K)}_\alpha \parallel 0_1^+ \right\rangle$$

$$\times \sum_{\varepsilon n} \left[ 1 + F^{(\nu)}_{01} (\theta) F^{(\nu)^*}_{Z,N}(\theta) \right]_{\gamma \delta} \sin \theta d\theta, \quad (6)$$

where

$$n^J = \int_0^{\pi} [\det (1 + F^{(\nu)} f^{(\nu)^*})]^{1/2}$$

$$\times \left[ \det (1 + F^{(\nu)} f^{(\nu)^*}) \right]^{1/2} d\Phi_0 (\theta) \sin \theta d\theta, \quad (7)$$

The required amplitudes $(u_{im}, v_{im})$ and expansion coefficients $C_{ij,m}$ of axially symmetric HFB intrinsic state $\Phi_0$ with $K = 0$ to evaluate the expressions $n^J, n_{(Z,N),(Z+2,N-2)} (\theta), F_{Z,N}$ and $F_{Z,N}(\theta) \[18\]$ are obtained by minimizing the expectation value of the effective Hamiltonian in a basis constructed by using a set of deformed states.

III. RESULT AND DISCUSSIONS

Two different set of wave functions are generated using two distinct effective interactions, namely KUO \[42\] and JUN45 due to Honma et al. \[43\]. The former is a realistic interaction while the latter is an empirical one. The wave functions obtained by using KUO and JUN45 effective two-body interactions are referred to as HFB1 and HFB2, respectively. The single particle energies (SPE) used in HFB1 (HFB2) calculation are $\varepsilon_{p_{3/2}} = 0.0 (-9.828)$ MeV, $\varepsilon_{p_{1/2}} = 0.78 (-9.048)$ MeV, $\varepsilon_{f_{5/2}} = 1.08 (-8.7480)$ MeV and $\varepsilon_{g_{9/2}} = 3.0 (-6.828)$ MeV. However, the SPE of $\varepsilon_{g_{9/2}} = 4.0 (-6.828)$ MeV for $^{76}\text{Ge}$ isotope. Usually, a mass dependent term of the type $(58/A)^{1/3}$ is introduced \[46\] in the effective two body interaction to compensate for the noticed over attractiveness of the interaction for the nuclei with high neutron number occurring towards the end of the shell \[47\]. The above mentioned effective interactions, namely KUO and JUN45 are renormalized to reproduce the excitation energies $E_{2+}$ of the yrast $2^+$ states.

To ascertain the reliability of the generated wave functions HFB1 and HFB2, the calculated and experimentally observed excitation energies $E_{2+}$ of the yrast $2^+$
In Table 2, the calculated occupation numbers are given for neutrons \( g(2^+)^{[51]} \) and g-factors \( g(2^+) \) \([50]\) of \(^{76}\)Ge, \(^{76,82}\)Se and \(^{82}\)Kr isotopes with (a) HFB1 and (b) HFB2.

| Nuclei | \( E_{2^+} \) | \( \beta_2 \) | g-factor |
|--------|--------------|--------------|-----------|
| \(^{76}\)Ge | \(0.563\) | \(0.2610\) | \(0.353\) |
| \(^{76}\)Se | \(0.559\) | \(0.2991\) | \(0.394\) |
| \(^{82}\)Se | \(0.659\) | \(0.1988\) | \(0.580\) |
| \(^{82}\)Kr | \(0.767\) | \(0.2048\) | \(0.489\) |

**TABLE II:** Calculated (Theo.) and observed (Expt.) occupation numbers for neutrons \([52]\) and protons \([53]\) in \(^{76}\)Ge, \(^{76,82}\)Se and \(^{82}\)Kr isotopes with (a) HFB1 and (b) HFB2.

| Orbits | \(^{76}\)Ge | \(^{76}\)Se | \(^{82}\)Se | \(^{82}\)Kr |
|--------|-----------------|-----------------|-----------------|-----------------|
| \(1p_{1/2}+1p_{3/2}\) | \(1.60\) | \(2.54\) | \(5.76\) | \(2.98\) |
| \(0f_{5/2}\) | \(2.77\) | \(3.42\) | \(5.72\) | \(5.05\) |
| \(0g_{9/2}\) | \(0.26\) | \(0.70\) | \(5.97\) | \(5.49\) |

**TABLE I:** Comparison of calculated and observed excited energies \( E_{2^+} \) of yrast \( 2^+ \) states \([48]\), deformation parameters \( \beta_2 \) \([49]\) and g-factors \( g(2^+) \) \([50]\) are presented in Table 1. The deformation parameters \( \beta_2 \) are calculated with effective charges \( e_p=+1+e_{eff} \) and \( e_n=-e_{eff} \). The effective charge \( e_{eff}=0.5 \) for \(^{82}\)Kr while for other nuclei it is 0.78. The g-factors \( g(2^+) \) are calculated with two different prescriptions. In the first prescription, the \( g(2^+) \) values are calculated with \( g_{1\pi}=1.0, g_{2\pi}=0.0, g_{\pi/\mu}=0.6(g_{\pi/\mu})_barc \). In the second prescription, effective operators calculated with a set of first and second order diagrams are \((g_1, g_s, g_p)^s=(0.89, 3.18, 0.73)\) and \((g_2, g_s, g_p)^{\mu}=(0.07, -1.52, -0.89) \). In Table 2, the calculated occupation numbers are given along with the experimentally observed data \([52,53]\). It is noticed that the overall agreement between the calculated spectroscopic properties of \(^{76}\)Ge, \(^{76,82}\)Se and \(^{82}\)Kr isotopes and the experimentally observed data is reasonably good. Although the closure approximation is not valid for the \(2\nu\beta^-\beta^-\) decay, an estimate of \( M_{2\nu} \) with closure for \(^{76}\)Ge and \(^{82}\)Se provides 0.157 (0.132) and 0.155 (0.147) with HFB1 and HFB2, respectively. This implies \( g_{A,eff}=0.667 \) (0.729) and 0.576 (0.592) for \(^{76}\)Ge and \(^{82}\)Se isotopes with HFB1 and HFB2, respectively.

In addition, the two body effective interaction is further decomposed into central (C), spin-orbit (S) and tensor (T) components \([54]\) and the effect of these components on NTMEs \( M^{(K)} \) involved in \( 0\nu\beta^-\beta^- \) decay is studied. In spin-tensor decomposition, the most general two-body interaction is written as

\[
V(1,2) = \sum_{k=0,1,2} \left[ X^{(k)} \times S^{(k)} \right]^{(0)}
\]

\[
= \sum_{k=0,1,2} V^{(k)}
\]

where the most general two-particle spin operators are written as \( S_1^{(0)} = 1, S_2^{(0)} = [\sigma_1 \times \sigma_2]^{(0)}, S_3^{(1)} = [\sigma_1 + \sigma_2]^{(1)}, S_4^{(1)} = [\sigma_1 - \sigma_2]^{(1)} \), \( S_5^{(1)} = [\sigma_1 \times \sigma_2]^{(1)} \), and \( S_6^{(2)} = [\sigma_1 \times \sigma_2]^{(2)} \).

The central and tensor part of the effective two-body interaction are represented by \( V^{(0)} \) and \( V^{(2)} \), respectively. The \( V^{(1)} \) term contains the symmetric \( S_3^{(1)} \) as well as antisymmetric \( S_4^{(1)} \) and \( S_5^{(1)} \) spin-orbit operators. Three sets of wave functions are generated with central (C), central plus spin-orbit (CS) and central plus spin orbit plus tensor (CST) parts of the effective two-body interaction, which are subsequently employed to calculate the required NTMEs \( M^{(K)} \).
TABLE III: NTMEs for the 0ν β+β− decay of 76Ge and 82Se due to the light and heavy Majorana neutrino exchange with three sets of wave functions having central (C), central plus spin-orbit (CS) and central plus spin-orbit plus tensor (CST) for both (a) HFB1 and (b) HFB2.

| Nucleus | Case | HFB1 | HFB2 |
|---------|------|------|------|
|         | C    | CS   | CST  | C    | CS   | CST  |
| Light neutrino |
| 76Ge    | FNS  | 1.574| 3.982| 5.628| 1.321| 3.560| 5.346|
|         | SRC1 | 1.277| 3.490| 4.858| 1.060| 3.024| 4.507|
|         | SRC2 | 1.560| 3.945| 5.564| 1.311| 3.515| 5.270|
|         | SRC3 | 1.646| 4.087| 5.785| 1.386| 3.669| 5.511|
| 82Se    | FNS  | 3.575| 5.991| 6.415| 2.872| 2.176| 5.846|
|         | SRC1 | 3.003| 5.175| 5.494| 2.371| 1.882| 5.049|
|         | SRC2 | 3.542| 5.793| 6.344| 2.843| 2.158| 5.786|
|         | SRC3 | 3.706| 6.172| 6.609| 2.986| 2.242| 6.015|
| Heavy neutrino |
| 76Ge    | FNS  | 121.44| 194.58| 298.33| 109.50| 204.92| 320.41|
|         | SRC1 | 45.40 | 69.89 | 104.34| 42.35 | 70.36 | 110.10|
|         | SRC2 | 75.62 | 118.68| 179.84| 69.26 | 122.77| 191.67|
|         | SRC3 | 100.85| 160.12| 244.35| 91.52 | 167.51| 261.63|
| 82Se    | FNS  | 256.85| 322.09| 355.63| 233.28| 119.80| 307.09|
|         | SRC1 | 110.74| 114.61| 123.04| 105.32| 44.88 | 105.49|
|         | SRC2 | 170.22| 196.09| 213.70| 157.98| 74.50 | 183.98|
|         | SRC3 | 218.65| 265.01| 291.01| 200.31| 99.38 | 250.99|

Employing these reliable wave functions, various NTMEs, namely Fermi $M_{F}^{(K)}$, Gamow-Teller (GT) $M_{GT}^{(K)}$ consisting of $M_{GT-AA}^{(K)}$, $M_{GT-AP}^{(K)}$, $M_{GT-PP}^{(K)}$, $M_{GT-MM}^{(K)}$ and tensor $M_{T}^{(K)}$ consisting of $M_{T-AP}^{(0ν)}$, $M_{T-PP}^{(0ν)}$, $M_{T-MM}^{(0ν)}$ are calculated with $g_{v} = 1.0$, $g_{A} = 1.2701$, $\kappa = \mu_{p} - \mu_{n} = 3.70$, $\Lambda_{V} = 0.850$ GeV and $\Lambda_{A} = 1.086$ GeV. Three sets of NTMEs are calculated by considering a Jastrow type of SRC simulating the effects of Argonne V18 and CD-Bonn potentials in the self-consistent coupled cluster method (CCM) \cite{29}, given by

$$f(r) = 1 - \alpha e^{-ar^2} (1 - br^2)$$ (9)

where $a = 1.1 \text{ fm}^{-2}$, $1.59 \text{ fm}^{-2}$, $1.52 \text{ fm}^{-2}$, $b = 0.68 \text{ fm}^{-2}$, $1.45 \text{ fm}^{-2}$, $1.88 \text{ fm}^{-2}$ and $c = 1.0$, 0.92, 0.46 for Miller-Spencer parameterization, Argonne NN, CD-Bonn Potentials, and are denoted by SRC1, SRC2 and SRC3, respectively. In Table III, the NTMEs $M^{(0ν)}$ and $M^{(0NN)}$ due to the exchange of light and heavy Majorana neutrinos, respectively, are displayed.

Relative changes (in \%) of the NTMEs $M^{(0ν)}$ ($M^{(0NN)}$) with the inclusion of SRC1, SRC2, and SRC3, and average energy denominator $\Lambda/2$ are given in Table IV. In the case of light neutrino exchange, the contributions of C and CS parts of the two-body interaction to the total NTMEs $M^{(0ν)}$ of 76Ge calculated within HFB1 and HFB2 are about 24\%–29\% and 67\%–72\%, respectively. However, the contribution of the C part in the case of $^{82}$Se turn out to be 47\% and 56\% for HFB1 and HFB2, respectively. In the case of CS part, the contributions to the total NTMEs $M^{(0ν)}$ for HFB1 and HFB2 are about 37\% and 94\%, respectively. The maximum relative change in NTMEs $M^{(0ν)}$, when the energy denominator is taken as $\Lambda/2$ instead of $\Lambda$ is of the order of 10 \%, which confirms that the dependence of NTMEs on the average excitation energy $\Lambda$ is small and thus, the validity of the closure approximation for the $0ν β^− β^−$ decay is supported. In comparison to the case FNS, the NTMEs $M^{(0ν)}$ are approximately reduced by 14\%–16\%, 1\%–1.4\% and 2.8\%–3.0\% with the addition of SRC1, SRC2 and SRC3, respectively.

The contributions of C and CS parts of the two-body interaction to the total NTMEs $M^{(0NN)}$ of 76Ge due to the heavy neutrino exchange, are about 34\%–42\% and 64\%–67\%, within HFB1 and HFB2, respectively. However, the maximum contribution of the C part in the case of $^{82}$Se turn out to be 94\% and 56\% for HFB1 and HFB2, respectively. In the case of CS part, the contributions to the total NTMEs $M^{(0ν)}$ for HFB1 and HFB2 are about 49\% and 63\%, respectively. With the addition of SRC1, SRC2 and SRC3, the NTMEs $M^{(0NN)}$ are approximately reduced by 64\%–66\%, 40\% and 15\%–18\%, respectively in comparison to the case FNS. It is worth mentioning that the effects due to SRC on NTMEs $M^{(0ν)}$ and $M^{(0NN)}$ is maximally incorporated by the C part of the effective two-body interaction, which varies by a small amount with the inclusion of spin-orbit and tensor parts.

Limits on the effective neutrino mass ($m_{ν}$) and $M_{N}$ are extracted from the available limits on experimental half-lives $T^{0ν}_{1/2}$ using NTMEs $M^{(0ν)}$ and $M^{(0NN)}$ calculated within the PHFB model (Table V). In the case of 76Ge isotope using the HFB1 (HFB2) wave functions, one obtains the best limit on the effective neutrino mass ($m_{ν}$) < 0.24 eV, 0.21 eV, 0.20 eV (0.26 eV, 0.22 eV,
TABLE VII: Extracted parameters from the observed limits of wave functions HFB1 and HFB2 and (a) SRC1, (b) SRC2 and (c) SRC3.

| Nuclei | Cases | HFB1 | HFB2 |
|--------|-------|------|------|
|        |       | C    | CS   | CST  | C    | CS   | CST  |
| $^{16}$Ge | SRC1 | 18.9 (62.6) | 12.4 (64.1) | 13.7 (65.0) | 19.8 (61.3) | 15.0 (65.7) | 15.7 (65.6) |
|         | SRC2 | 0.9 (37.7)  | 0.9 (39.0)  | 1.1 (39.7)  | 0.8 (36.8)  | 1.3 (40.1)  | 1.4 (40.2)  |
|         | SRC3 | 4.5 (17.0)  | 2.6 (17.7)  | 2.8 (18.1)  | 4.9 (16.4)  | 3.0 (18.3)  | 3.1 (18.3)  |
|         | SRC1(A/2) | 6.2 | 9.7 | 9.4 | 6.0 | 9.1 | 8.8 |
|         | SRC2(A/2) | 6.0 | 9.1 | 9.4 | 6.0 | 9.1 | 9.4 |
|         | SRC3(A/2) | 5.9 | 8.9 | 8.6 | 5.9 | 8.9 | 8.6 |
| $^{82}$Se | SRC1 | 16.0 (56.9) | 13.6 (64.4) | 14.4 (65.4) | 17.4 (54.8) | 13.5 (62.5) | 13.6 (65.6) |
|         | SRC2 | 0.9 (33.7)  | 0.9 (39.1)  | 1.1 (39.9)  | 1.0 (32.3)  | 0.8 (37.8)  | 1.0 (40.1)  |
|         | SRC3 | 3.7 (14.9)  | 3.0 (17.7)  | 3.0 (18.2)  | 4.0 (14.1)  | 3.1 (17.0)  | 2.9 (18.3)  |
|         | SRC1(A/2) | 6.4 | 9.4 | 9.5 | 6.4 | 9.4 | 9.5 |
|         | SRC2(A/2) | 6.2 | 8.8 | 8.9 | 6.2 | 8.8 | 8.9 |
|         | SRC3(A/2) | 6.1 | 8.6 | 8.7 | 6.1 | 8.6 | 8.7 |

TABLE V: Extracted limits on the effective mass of neutrino $<m_\nu>$, $<M_N>$ and predicted half-lives $T_{1/2}^{(0\nu)}$ (yrs) with two sets of wave functions HFB1 and HFB2 and (a) SRC1, (b) SRC2 and (c) SRC3.

| Nuclei | $T_{1/2}^{(0\nu)}$ (yr) Ref. | SRC | $<m_\nu>$ (eV) | $T_{1/2}^{(0\nu)}$ (yr) | $<m_\nu>$ = 0.01 eV | $<M_N>$ (GeV) |
|--------|---------------------|-----|-----------------|---------------------|---------------------|----------------|
| $^{16}$Ge | $3.0 \times 10^{-2}$ [36] | (a) | 0.24 | 0.26 | 1.80 | 2.00 | 4.20 | 4.44 |
|         |                     | (b) | 0.21 | 0.22 | 1.37 | 1.53 | 7.25 | 7.72 |
|         |                     | (c) | 0.20 | 0.22 | 1.27 | 1.40 | 9.85 | 1.05 |
| $^{82}$Se | $3.6 \times 10^{-23}$ [37] | (a) | 0.95 | 1.04 | 3.27 | 3.87 | 1.13 | 9.66 |
|         |                     | (b) | 0.82 | 0.90 | 2.45 | 2.95 | 1.96 | 1.68 |
|         |                     | (c) | 0.79 | 0.87 | 2.26 | 2.73 | 2.66 | 2.30 |

TABLE VI: Extracted limits on the Majoron-neutrino coupling constant $g_M$ from the observed limits on $T_{1/2}^{(0\nu \phi)}$ (yr) with two sets of wave functions HFB1 and HFB2 and (a) SRC1, (b) SRC2 and (c) SRC3.

| Nuclei | $T_{1/2}^{(0\nu \phi)}$ (yr) Ref. | SRC | $<g_M>$ |
|--------|---------------------|-----|--------|
| $^{16}$Ge | $6.4 \times 10^{-24}$ [34] | (a) | $7.59\times10^{-5}$ | 8.18 | 9.90 |
|         |                     | (b) | $6.62\times10^{-5}$ | 6.99 | 1.05 |
|         |                     | (c) | $6.37\times10^{-5}$ | 6.69 | 1.05 |
| $^{82}$Se | $1.5 \times 10^{-22}$ [37] | (a) | $4.85\times10^{-5}$ | 5.28 | 1.05 |
|         |                     | (b) | $4.20\times10^{-5}$ | 4.60 | 1.05 |
|         |                     | (c) | $4.03\times10^{-5}$ | 4.43 | 1.05 |

TABLE VII: Extracted parameters from the observed limits on $T_{1/2}^{(0\nu)}$ and $T_{1/2}^{(0\nu \phi)}$ using average NTMEs $\overline{M}^{(0\nu)}$ and $\overline{M}^{(0\nu \phi)}$.

| Parameters | $^{16}$Ge | $^{82}$Se |
|------------|----------|----------|
| $M^{(0\nu)}$ | 5.249±0.481 | 5.883±0.568 |
| $<m_\nu>$ (eV) | 0.227 | 0.890 |
| $T^{(0\nu)}$ (yr) | $1.54 \times 10^{28}$ | $2.85 \times 10^{27}$ |
| $\overline{M}^{(0\nu \phi)}$ | 181.99±65.61 | 194.70±72.13 |
| $<M_N>$ (GeV) | $7.33 \times 10^{7}$ | $1.78 \times 10^{7}$ |
| $<g_M>$ | $7.02 \times 10^{-5}$ | $4.53 \times 10^{-5}$ |

0.22 eV) and $<M_N> \geq 4.20 \times 10^{-7} - 9.85 \times 10^{-7}$ GeV (4.44 $\times 10^{-7} - 10.5 \times 10^{-7}$ GeV) due to SRC1, SRC2 and SRC3, respectively. In the classical Majoron model, the inverse half-life $T_{1/2}^{(0\nu \phi)}$ for the $0^+ \rightarrow 0^+$ transition of Majoron emitting $0\nu\beta\beta$ decays is given by $\overline{M}^{(0\nu \phi)}$.

\[ T_{1/2}^{(0\nu \phi)} = 1/(g_M)^2 \cdot G_{\beta\beta}\phi |\overline{M}^{(0\nu \phi)}|^2 \] (10)

where $g_M$ is the effective Majoron-neutrino coupling constant, and the NTME $M^{(0\nu \phi)}$ is same as the $M^{(0\nu)}$ for the exchange of light Majorana neutrinos. The phase space factors $G_{\beta\beta}\phi$ for the $0^+ \rightarrow 0^+$ transition of $0\nu\beta\beta$ decays have been given by Kotila and Iachello [51]. The extracted limits on the effective Majoron-neutrino coupling parameter ($g_M$) form the largest limits on the half-lives $T_{1/2}^{(0\nu \phi)}$ are given in Table VI. The most stringent extracted limit on $g_M = (6.37 - 8.18) \times 10^{-5}$.

In spite of the fact that there are only a set of six NTMEs $M^{(0\nu \phi)}$ and $\overline{M}^{(0\nu \phi)}$ for a statistical analysis to estimate uncertainties therein, the calculated average NTMEs are given in Table VII. The maximum uncertainty in the average NTMEs $\overline{M}^{(0\nu \phi)}$ and $\overline{M}^{(0\nu \phi)}$ turns out to be about 10% and 37%, respectively. Using the estimated average NTMEs $\overline{M}^{(0\nu)}$ and $\overline{M}^{(0\nu \phi)}$ calculated
in the PHFB model, the most stringent extracted limits on the effective neutrino mass \( \langle m_\nu \rangle \) and \( \langle M_N \rangle \) from the available limit on experimental half-live \( T^{0\nu}_{1/2} \) of \(^{76}\text{Ge} \) are 0.23 eV and 7.33 \times 10^7 \text{ GeV} \), respectively. Further, the extracted limit on the effective Majoron-neutrino coupling parameter \( \langle g_M \rangle \) is 7.02 \times 10^{-5}.

IV. CONCLUSIONS

Within the PHFB approach, the required NTMEs \( M^{(0K)} \) for the study the \( 0\nu\beta^-\beta^- \) decay of \(^{76}\text{Ge} \) and \(^{82}\text{Se} \) isotopes in the Majorana neutrino mass mechanism are calculated using a two sets of HFB intrinsic wave functions, generated with KUO and JUN45 effective two-body interactions. The reliability of the wave functions has been tested by calculating the yrast spectra, deformation parameter \( \beta_2 \) and and \( g \)-factors \( g(2^+) \) as well as \( M_{2\nu} \) of nuclei participating in the \( 2\nu\beta^-\beta^- \) decay and comparing them with the available experimental data. An overall agreement between the calculated and observed spectroscopic properties as well as \( M_{2\nu} \) suggests that the PHFB wave functions generated by reproducing the \( E_{2^+} \) are quite reliable. Further, the contributions of the central, spin-orbit and tensor components of the effective two-body interaction to the total \( M^{(0K)} \) have been obtained by performing a spin-tensor decomposition of KUO and JUN45 two-body matrix elements.

The NTMEs \( M^{(0\nu)} \) have a weak dependence on the average excitation energy \( \bar{A} \) of intermediate nucleus and as expected, the closure approximation is quite valid. In comparison to the case FNS, the NTMEs \( M^{(0\nu)} (M^{(0N)}) \) are approximately reduced by 15\% (65\%), 1\% (40\%) and 3\% (16\%) with the addition of SRC1, SRC2 and SRC3, respectively. Specifically, the strong dependence of \( M^{(0N)} \) in the case of heavy neutrino exchange on the SRC is a major source of uncertainty in the calculation of NTMEs. It has been noticed that the C part of the effective two-body interaction picks up maximally the effects due to SRC on NTMEs \( M^{(0\nu)} \) and \( M^{(0N)} \), which varies by a small amount with the inclusion of spin-orbit and tensor parts.

Limits on the effective light neutrino mass \( \langle m_\nu \rangle \), effective heavy neutrino mass \( \langle M_N \rangle \) and neutrino-Majoron coupling constant \( \langle g_M \rangle \) of the classical Majoron model have been extracted from the available limits on experimental half-lives \( T^{(0\nu)}_{1/2} \) and \( T^{(0\nu\phi)}_{1/2} \), respectively. The most stringent extracted limits on \( \langle m_\nu \rangle \), \( \langle M_N \rangle \) and \( \langle g_M \rangle \) from the available experimental limit on \( T^{(0\nu)}_{1/2} \) of \(^{76}\text{Ge} \) are 0.23 eV and 7.33 \times 10^7 \text{ GeV}, and 7.02 \times 10^{-5}, respectively.

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