Modular Bosonic Subsystem Codes

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We introduce a formalism for encoding a qubit into a bosonic mode as a discrete subsystem. Using a modular decomposition of the position operator, we divide the bosonic mode into two subsystems: a logical qubit and a gauge mode. This formalism enables the analysis of continuous-variable quantum information using standard qubit-based quantum information tools. We apply the formalism to approximate Gottesman-Kitaev-Preskill (GKP) states and show that the logical qubit experiences decoherence due to entanglement with the gauge mode. We also identify and disentangle a qubit cluster state hidden inside of a Gaussian continuous-variable cluster state.

Introduction—Continuous-variable (CV) quantum computing is experiencing considerable theoretical [1, 2] and experimental [7–13] development due to the promise of substantial scalability. Practical quantum computing is experiencing considerable theoretical [1–6] development due to the promise of fault tolerance [19].

Bosonic codes [16, 17] restore the notion of a qubit within the framework of CV quantum computing by identifying a discrete-variable Hilbert space (C^2 for qubits) within one or more bosonic modes. The standard approach begins by selecting two wavefunctions that define the logical subspace, with the remaining “wilderness space” serving as a resource for error detection. This corresponds to a subspace decomposition of the CV Hilbert space, H_{CV} = C^2 ⊕ H_{wild}. A notable example is the Gottesman-Kitaev-Preskill (GKP) encoding [18], which can be combined with measurement-based quantum computing using CV cluster states to achieve fault tolerance [19].

Subspace encodings suffer from several problems. In contrast to qubit-based subspace codes, a bosonic-code subspace is vanishingly small compared to the full CV Hilbert space. Further, the vastness of the wilderness space outside means that many CV states can represent the same logical information, but a formal way to describe this has been lacking. A step towards a solution was given in Refs [20, 21], where a continuous direct-sum decomposition of the CV Hilbert space using a set of eigenstates for modular position and momentum was presented.

In this Letter we describe how to encode a qubit in an entirely different way: by identifying a discrete, logical subsystem within the CV Hilbert space. This encoding hinges on a modular decomposition of the position operator. A bosonic subsystem code gives a precise description both of the logical qubit hiding inside the CV Hilbert space and also of the complementary gauge mode. In contrast to subspace codes [17, 22, 25], a bosonic subsystem code endows every CV state with logical-qubit information. This is more practical, as one is not required to construct operations (measurements and gates) that select only a particular subspace. Once a trace over the gauge mode is performed, the original CV nature of the mode can be forgotten, and one can work entirely at the qubit level.

Modular position basis—A real number s ∈ R can be split into its integer and fractional parts with respect to a “spacing” α ∈ R as s = [s]_α + {s}_α, where [s]_α := α [s/α] + 1/2 indicates closest (centered) integer multiple of α and {s}_α := s − [s]_α is the (centered) remainder, see Fig. 1(a). Any quadrature operator can be decomposed in this way using the spectral theorem. Without loss of generality we decompose the position operator \( \hat{q} = \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger) \) and its eigenstates, \( \hat{q} |s\rangle = s |s\rangle \). Specifically, \( \hat{q} = \alpha \hat{m} + \hat{u} \), where \( \alpha \hat{m} := |\hat{q}\rangle_\alpha \) and \( \hat{u} := \{\hat{q}\}_\alpha \). Simultaneous eigenvectors of \( \hat{m} \) and \( \hat{u} \) arising from the decomposition are \( |s\rangle_q = |\alpha m + u\rangle_q := |m, u\rangle \) with \( m \in \mathbb{Z} \) and \( u \in [-\alpha/2, \alpha/2) \) [26]. These states form a modular position basis, which is convenient for representing states with discrete translation symmetry — notably the code-words of the GKP encoding of a qubit in an oscillator [18]. Several examples are given in Table I. If a Fourier series over \( m \) is performed, the result is the kq-representation introduced by Zak in other contexts [27, 28] and used more recently for analysis of GKP codes [21, 29, 30].

Modular bosonic subsystem codes—We explicitly identify a two-dimensional qubit subsystem using
| CV state | Modular position basis | Modular subsystem basis |
|---------|------------------------|------------------------|
| $|0\rangle_q$ | $|0,0\rangle$ | $|0\rangle \otimes |0\rangle_{GKP}$ |
| $|0\rangle_p$ | $\sum_{m} fdu \langle m,u \rangle$ | $|+\rangle \otimes |0\rangle_{p,G}$ |
| $|\psi_{GKP}\rangle$ | $\sum_{m} \langle 2m + \ell, 0 \rangle$ | $|0\rangle_{\ell} \otimes |+\rangle_{GKP_G}$ |

TABLE I. Modular subsystem representations of various (un-normalized) CV states. Sums are taken over $\mathbb{Z}$, and integrals are over the interval $[-\alpha/2, \alpha/2]$.

the discrete spectrum of the closest-integer operator $\hat{m}$ as follows. An integer $m \in \mathbb{Z}$ can be written as $m = P(m) + 2m_{G}$, where $P(m) \in \{0,1\}$ is its parity (even or odd), and $2m_{G}$ is an even integer with $m_{G} \in \mathbb{Z}$. Decomposing the position operator (and its eigenstates) accordingly, $\hat{q} = \alpha \hat{\ell} + 2\alpha \hat{m}_{G} + \tilde{u}_{G}$ with $\hat{\ell} := P(m)$, $\tilde{u}_{G} := \tilde{u}$, and $\hat{m}_{G} := \frac{1}{\alpha} (m - \ell)$, allows us to identify the basis states,

$$|s\rangle_q = |m, u\rangle = |\ell\rangle_G \otimes |m_G, u_G\rangle_G .$$ (1)

A logical qubit (denoted by subscript $L$) emerges from the $m$-parity states $|0\rangle_L$ and $|1\rangle_L$ [21]. The $m$-parity operator $\hat{\ell}$ acts as a projector: $\hat{\ell} = |1\rangle_L \langle 1 | = \frac{1}{\alpha} (\hat{I}_L - \hat{Z}_L)$, where $\hat{I}_L$ and $\hat{Z}_L$ are logical identity and Pauli-Z operators, respectively. The remaining eigenstates of $\hat{m}_G$ and $\hat{u}_G$, $|m_G, u_G\rangle_G$, span the infinite-dimensional Hilbert space of the “gauge” mode (denoted by subscript $G$). We recognize from earlier that the gauge mode is a bona fide CV mode, expressed here in the modular position basis.

In Eq. (1) we have suggestively inserted a tensor product to emphasize the decomposition of the original CV Hilbert space into two subsystems—a two-dimensional logical qubit and a CV mode: $H_{CV} = C^2 \otimes H_{CV}$. The states in Eq. (1) form a basis for a modular subsystem decomposition (referred to hereafter simply as a subsystem decomposition). CV unitaries and more general quantum operations can be decomposed in order to interpret their actions on the two subsystems. Operations that affect only the gauge mode do not disturb encoded information at all, although physical CV operations typically affect both subsystems and may entangle them.

In the subsystem decomposition, any pure CV state $|\Psi\rangle$ has a Schmidt decomposition

$$|\Psi\rangle = \sqrt{P_0} |\psi_0\rangle_L \otimes |\phi_0\rangle_G + \sqrt{P_1} |\psi_1\rangle_L \otimes |\phi_1\rangle_G ,$$ (2)

with Schmidt coefficients $\sqrt{P_j}$. When only one of these coefficients is nonzero, the right-hand side of Eq. (2) is a tensor-product state; see examples in Table I. Such states encode a pure logical qubit. More generally, entanglement between the subsystems acts as decoherence that damages the logical state.

The logical qubit $\hat{\rho}_L$ associated with a (potentially mixed) CV state $\hat{\rho}$ is isolated by tracing over the gauge mode:

$$\hat{\rho}_L = \text{Tr}_G[\hat{\rho}] = \sum_{m_G \in \mathbb{Z}} \int_{-\alpha/2}^{\alpha/2} du_G \langle m_G, u_G | \hat{\rho} | m_G, u_G \rangle_G .$$ (3)

Whereas every CV state contains a logical subsystem qubit, standard subspace codes can catastrophically fail to capture encoded logical information: most obvious is the case where a CV state (typically a damaged codeword) has no support in the code’s subspace. The subsystem decomposition shows that encoded quantum information can be accessed even in the presence of errors without constructing an infinite set of error subspaces [21].

Modular bosonic subsystem codes differ from standard subspace codes [32-34] in several ways. First, the gauge subsystem is a CV mode. More importantly, the division into subsystems is a flexible mathematical construction that can be performed in many ways on the same CV mode, most simply by choosing different values of $\alpha$ but also by decomposing any other quadrature.

Subsystem description of the GKP encoding—The subsystem decomposition generalizes the GKP encoding. In its original form the GKP code employs computational-basis codewords, $|0_{GKP}\rangle$ and $|1_{GKP}\rangle$, whose position-space wavefunctions are Dirac combs with spacing $2\alpha$. Due to their periodic structure, each of these states is compactly represented in a modular decomposition with spacing $\alpha$; see Table I. In the modular subsystem decomposition, GKP states are product states between the logical and gauge subsystems: $|\psi_{GKP}\rangle = |\psi\rangle_L \otimes |+\rangle_{GKP_G}$. In fact, any $2\alpha$-periodic state is a product state in this basis—the logical and gauge subsystems are unentangled. An example is the momentum eigenstate, $|0\rangle_p = \int ds |s\rangle_q = |+\rangle_L \otimes |0\rangle_{p,G}$, which encodes the same logical information as $|+\rangle_{GKP_G}$ but with a different gauge-mode state. That is, both states encode the correct qubit amplitudes and probabilities for the computational-basis measurements laid out in the original GKP formulation: binned homodyne detection [18]. The differences appear when the states undergo evolution or suffer errors—keeping the gauge mode close to $|+\rangle_{GKP_G}$ makes the state resilient to errors (see below).

GKP Pauli $X$- and $Z$-gates are implemented by position and momentum shifts, $\hat{X}(s) = e^{is\hat{p}}$ and $\hat{Z}(t) = e^{it\hat{q}}$, for $s = \alpha$ and $t = \pi/\alpha$, respectively. These shifts act as the expected Pauli gates directly on the logical qubit:

$$\hat{X}(\alpha) |\psi_{GKP}\rangle = \hat{X}_L |\psi\rangle_L \otimes |+\rangle_{GKP_G} ,$$ (4)

$$\hat{Z}(\pi/\alpha) |\psi_{GKP}\rangle = \hat{Z}_L |\psi\rangle_L \otimes |+\rangle_{GKP_G} .$$ (5)

There are several consequences of the fact that we defined the logical qubit by decomposing in the $q$-basis. First, any position shift on a GKP state that changes the value of $\ell$ implements a logical $X_L$, with differences
arising only in how the gauge mode is affected. However, such shifts on an arbitrary state (not a GKP state), involve logical-gauge interactions that in general entangle the subsystems and spoil the tensor-product form present for GKP states, as in Eq. (4). Second, all momentum shifts have a tensor-product structure: \( \hat{Z}(t) = e^{i\alpha t/2} \hat{R}_z^G(t\alpha) \otimes e^{i\pi(\alpha x_0 + \alpha z_0)}, \) where \( \hat{R}_z^G(\theta) := e^{-\theta\hat{Z}_L/2} \) is a rotation of the qubit around the logical \( z \)-axis by \( \theta \). Although small momentum shifts have logical consequences, shifted GKP states still maintain their tensor-product structure.

The shift freedom of the GKP encoding is the foundation for its resistance to errors. Shift errors in position by less than \( \alpha/2 \) (and in momentum by less than \( |\pi/\alpha| \)) are perfectly correctible if the codewords are high quality. The modular subsystem description tells us why: GKP error correction corresponds to measuring the gauge mode in the modular representation of Ref. [21] and returning it to \( |+\rangle_{\text{GKP}} \) through physical displacements. Small position shifts do not disturb the logical subsystem at all—they act solely on the gauge mode, and so do their correction. Small momentum shifts rotate the qubit by an amount that is revealed by measuring the gauge mode; displacing the full state back undoes this rotation.

**Approximate GKP states**—The subsystem decomposition allows us to identify the logical state in imperfect bosonic encodings, such as finite-energy approximations to GKP states [18, 30, 35]. These approximate GKP states, \( |\psi_{\text{GKP}}\rangle \), have a wavefunction given by a 2\( \alpha \)-periodic superposition of Gaussian spikes, each with variance \( \Delta^2 \). The complex amplitudes of every other spike are determined by the encoded qubit state, \( |\psi\rangle = a|0\rangle + b|1\rangle \). A comb of normalized Gaussians can be written as \( \sum_{m\in\mathbb{Z}} G_{2\alpha}(s - 2m\alpha) = \frac{1}{\sqrt{2\alpha}} \hat{\vartheta}(\frac{s}{2\alpha}, \frac{2\alpha\Delta^2}{4\pi}), \) where \( \hat{\vartheta}(z, \tau) \) is a Jacobi theta function of the third kind [35]. Including a broad Gaussian envelope with variance \( \kappa^{-2} \) damps spikes far from the origin and makes the state physical [37]. We compactly express an approximate-GKP wavefunction as

\[
\hat{\psi}_{\text{GKP}}(s) = e^{-\frac{\Delta^2}{2\alpha^2}} \sqrt{\mathcal{N}} \left[ a \vartheta\left(\frac{s}{2\alpha}, \tau \Delta \right) + b \vartheta\left(\frac{s - \alpha}{2\alpha}, \tau \Delta \right) \right],
\]

where \( \tau \Delta = \frac{i\pi\Delta^2}{2\alpha^2} \), and \( \mathcal{N} \) is the normalization. The limit \( \Delta, \kappa \to 0 \) gives a normalized, ideal GKP state [35].

The subsystem decomposition reveals that an approximate GKP state encodes a logical state whose fidelity with the intended qubit state depends on the quality of the encoding. Logical-qubit decoherence due to entanglement with the gauge mode is investigated numerically in Fig. 2 for \( |+\rangle_{\text{GKP}} \) states \((a = b = \frac{1}{\sqrt{2}})\). For very good position spikes, \( \Delta \ll \alpha \), we find a convenient analytic form for the logical state:

\[
\hat{\rho}_L = \frac{1}{2\pi(\alpha\tau')} \begin{pmatrix}
\vartheta(0, \tau') & e^{-\frac{\Delta^2}{\alpha^2}} \vartheta(\frac{1}{4}, \tau') \\
\vartheta(-\frac{\alpha}{2}, \tau') & \vartheta(\frac{1}{4}, \tau')
\end{pmatrix},
\]

where \( \tau' = \frac{1}{2} \tau \Delta (\Delta^2 + \kappa^{-2}) \). This state limits to \( |+\rangle_L \) for small \( \kappa \) (large envelope) and to \( |0\rangle_L \) for large \( \kappa \) (small envelope), in agreement with Fig. 2(a).

**Hidden qubit cluster states**—Continuous-variable cluster states (CVCSs) are resources for quantum computing, enabling universal fault-tolerant QC when used in conjunction with encoded GKP qubits [19]. Here we decompose a CVCS to reveal that it contains a logical qubit cluster state (CS) entangled with the gauge modes. Modular momentum measurements on the gauge modes disentangle the logical qubit CS.

A CVCS is constructed by entangling pairs of modes using the two-mode gate \( \hat{C}_Z |g\rangle = e^{i\pi \hat{X} \hat{Z}} |g\rangle \). Decomposing
with respect to the decomposition, and empty nodes are zero eigenstates (see Fig. 3 legend).

The logical qubits can be disentangled one by one by resetting the modular-position part of the gauge mode to a particular state—similar to (partial) gauge-fixing [34, 39]. Regardless of the gauge-mode state, measurement of modular gauge position, $u_G$, on mode 1 followed by a $u_G$-shift to $u_G = 0$ disconnects the middle node, Fig. 3(b). This has an interesting action on the corresponding CV mode as a whole: it is projected into $|+\text{GKP}\rangle$. One way to realize this measurement is using a GKP-encoded ancilla, such as in GKP error correction [35]. Performing this modular measurement on both modes fully disentangles the logical qubit CS, Fig. 3(c). Additionally, using this method the CV state produced is a GKP-encoded CS. Due to gauge freedom, however, this is not the only way to disentangle the qubit CS: For instance, perform the inverse of the logical-gauge entangling gate followed by any unitary gauge transformation. This isolates the qubit CS but does in general not correspond to a GKP CS in the CV modes.

Decomposing an ideal CVCS follows the same prescription. An ideal $N$-mode CVCS is produced by applying two-mode gates pairwise according to an $N \times N$ real, symmetric adjacency matrix $V$ that encodes the connections and interaction strengths [40]. An ideal CVCS is given by $\hat{C}_{A,B}^{\pi}[V] |0\rangle_N^\otimes$, where $\hat{C}_{A,B}^{\pi}[V] := \exp(i \hat{T}_{A,B}^{\pi} V\hat{T}_{A,B}^{\pi})$, and $A, B$ are $N \times 1$ column vectors of single-mode operators, e.g., $\hat{A}$. Decomposing the controlled-$Z$ operator in the same way, we find a hidden $N$-qubit CS associated with $V$, $|\text{CS}_V\rangle_N := \hat{C}_{A,B}^{\pi}[V] |+\rangle_N^\otimes$. It is entangled with an $N$-mode gauge state $|\Phi_{N,G}\rangle$ via the interaction operator $\hat{C}_{A,B}^{\pi}[V]$. Figure 3(d) shows a graphical representation for a linear CVCS with $V_{i,i+1} = 1$. Modular gauge measurements of $u_G$ (and subsequent correction), performed via GKP error correction on the $\hat{q}$ quadrature [38], can be used to disentangle the logical qubit CS just as for the two-mode squeezed state.

**Conclusion**—Continuous-variable (CV) quantum information has been plagued with the curse of infinity—with *ad hoc* tools built to accommodate subspace-encoded qubits. We have presented a procedure that encodes a logical qubit into a bosonic mode as a subsystem. With a modular bosonic subsystem code, the logical qubit is isolated by tracing over the gauge mode, and any CV quantum operation can be decomposed to uncover its action on the encoded logical information. The whole toolbox of qubit-based quantum information can then be applied.

The subsystem decomposition gives a new perspective on the GKP encoding and reveals its connection with CV cluster state (CVCS) quantum computing. While quantum computing with CVCSs has no built-in definition of a qubit [41], we have shown that the momentum eigenstates comprising the CVCS each indeed encode a
logical $|+\rangle$ state. This is the same logical state encoded by $|+\rangle_{\text{GKP}}$, and in this sense $|0\rangle_p$ are poor-quality GKP states.

This formalism also has applications beyond quantum computing. For example, decomposing a CV mode into interacting subsystems can serve as a platform to study non-Markovian reduced-state dynamics of the finite-dimensional subsystem [42].

Finally, while we focused here on encoding a logical qubit, the Hilbert space of a CV mode is immense and allows for many other discrete subsystem encodings. Encoding a logical qudit is straightforward: define the logical basis states using $m$ coding a logical qudit is straightforward: define the logical basis states using $m \mod d$, where $d$ is the dimension of the qudit. A different extension allows the encoding of multiple qubits in a “Russian nesting doll” fashion: decompose the gauge mode using the parity of $m_G$ and then repeat this process as desired.

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36. We use the definition \( \vartheta(\zeta, \tau) := \sum_{m \in \mathbb{Z}} \exp(i m \tau + 2 \pi i m \zeta). \)
37. For $|+\rangle_{\text{GKP}}$ this parameterization has the nice property that the momentum-space wavefunction has $2\pi/\tau$-periodic spikes, each with variance $\kappa^2$, and an envelope characterized by $\Delta^{-2}$.
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