On The Quantization Of Constraint Systems:  
A Lagrangian Approach

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Abstract

It is possible to introduce external time dependent background fields in the formulation of a system as fields whose dynamics can not be deduced from Euler Lagrange equations of motion. This method leads to singular Lagrangians for real systems. We discuss quantization of constraint systems in these cases and introduce generalized Gupta-Bleuler quantization. In two examples we show explicitly that this method of quantization leads to true Schrödinger equations.

1 Introduction

Obtaining Euler Lagrange equations of motion for a system given by a Lagrangian, $L = L(x_i, \dot{x}_i; t)$, it may happen that some of the accelerations $\ddot{x}_i$s remain unsolved. In these cases the corresponding coordinates become arbitrary functions of time. These systems are called constraint systems [1, 2]. In general, constraint systems possess gauge degrees of freedom. It is believed that gauge degrees of freedom are not physical. They should be eliminated for example by imposing gauge fixing conditions and defining reduced phase space. Gauge invariance should be completely eliminated because one deals only with gauge invariant objects [3]. In other words, observables are independent of gauges. Although eliminating gauge degrees of freedom is sufficient to fulfill the above considerations but it is not a necessary condition and causes some ambiguities in the general formulation

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of constraint systems. Part of these ambiguities are due to the fact that in principle it is possible to introduce external time dependent back ground fields in the formulation of a system as fields (coordinates) whose dynamics can not be determined by Euler Lagrange equations of motion (EL). Obviously such formulation leads to singular Lagrangians and physical degrees of freedom (may) become gauged. Consequently it is not reasonable to eliminate them. In this article we introduce a consistent approach for quantization by generalizing Gupta-Bleuler quantization [4]. The paper is organized as follows. Section 2 is a brief review of the reduced phase space quantization. In Section 3 we introduce generalized Gupta-Bleuler quantization and show that it is equivalent to Schrödinger approach when applied to unconstrained systems given by regular Lagrangians. In Section 4 we define reduced Hilbert space and examine the generalized Gupta-Bleuler quantization in two examples. Finally in section 5 we summarize our results.

2 Reduced Phase Space Quantization

Euler Lagrange equations of motion are not sufficient to determine dynamic of a constraint system. In other words, given a Lagrangian \( L(x_i, \dot{x}_i) \) of a constrained system, the Hessian matrix \( W_{ij} = \frac{\partial^2 L}{\partial x_i \partial \dot{x}_j} \) is singular, \( \det W = 0 \). The direct consequence of this singularity is that, momenta conjugate to coordinates are not independent and satisfy some relations called primary constraints (PC). The dynamic of the system should be consistent with PCs. Here, consistency means that these constraints should be preserved in time. This may generate some new constraints called secondary constraints (SC). If the system possesses gauge degrees of freedom, one fixes the gauges by imposing new constraints known as gauge fixing conditions. The classical trajectory of the system lies on a subspace of the phase space, the reduced phase space (RPS), which is the intersection of the subspaces defined by PCs, SCs and gauge fixing conditions. In principle one can reparameterize the phase space by new coordinates \((X_i, X_\alpha)\), where \(X_i\)'s are coordinates of the RPS and \(X_\alpha\)'s are the remaining. These new coordinates satisfy the following algebra,

\[
\{X_i, X_j\}^* = \sigma_{ij}, \quad \{X_i, X_\alpha\}^* = 0, \quad \{X_\alpha, X_\beta\}^* = 0,
\]

(1)

where \(\{ , \}^*\) is the Dirac bracket and \(\sigma_{ij}\) is a non-trivial two form. The classical equations of motion are given by the Hamilton equation, \(\dot{F}(X_i) = \{F(X_i), H\}^*\), in which \(H = H(X_i)\) is the well defined Hamiltonian on the RPS. For quantization one replaces Dirac brackets with commutators,

\[
\{X_i, X_j\}^* = \sigma_{ij} \rightarrow [\hat{X}_i, \hat{X}_j] = i\hbar\sigma_{i,j},
\]

(2)

where \(\hat{X}_i\) are operators corresponding to coordinates \(X_i\) and dynamic is given by the Hamiltonian \(\hat{H} = H(\hat{X}_i)\). This is the well known Dirac quantization. In this article we call
it RPS quantization to remind its structure. Most of the other methods for quantization of constrained systems follow the same ideas as those in the RPS method. For example in the Faddeev-Popov method the measure of path integral is written such that the path integral formulation of constrained systems turns out to be similar to the formulation of unconstrained systems on the RPS.

3 Generalized Gupta-Bleuler Quantization

Although PCs are generated by definition of momenta but SCs have their roots in dynamic. They can be (partly) obtained from Euler Lagrange equations of motion. It is natural to ask why these classical equations of motion should be satisfied in quantum level. Why do one allow a semi-classical object to fluctuate around its classical trajectory on the surface of SCs but not perpendicular to it? Consequently, it seems more reasonable to relax the restrictive conditions of the RPS quantization and only demand that in the classical limit the trajectory of the quantized system coincides with the classical trajectory.

One identifies a system classically with a number of coordinates \( q_i(t) \)s where \( t \) is the parameter of the evolution, the time. In principle if \( q_i(t) \)'s are known we know everything about the system. Dynamic of the system is given by the principle of least action or equivalently by Euler Lagrange equations of motion. In quantum mechanics one deals with coordinate operators \( \hat{q}(t) \)s which act on a definite Hilbert space. Dynamic is given by a time evolution operator. Given a system with classical trajectory \( q(t) \), we say it is quantum mechanically described by operators \( \hat{q}(t) \)s if and only if in the classical limit, \( \langle \hat{q}(t) \rangle = q(t) \), where \( \langle \hat{q}(t) \rangle \) is the expectation value of \( \hat{q}(t) \). We call this method, generalized Gupta-Bleuler (GGB) quantization. In the GGB method one deals with operators \( \hat{q}(t) \)s which are quantum versions of coordinates \( q(t) \)s. Conjugate to each \( \hat{q} \) one defines an operator \( \hat{p} \) which is the generator of translation in the \( q \) representation of the Hilbert space, i.e. \( \langle q|\hat{p}|\psi \rangle = -i\hbar \frac{\partial}{\partial q} \langle q|\psi \rangle \) or equivalently \([\hat{q}, \hat{p}] = i\hbar\). The operator \( \hat{p} \) is not considered to be a quantum version of the momentum \( p = \frac{\partial}{\partial \dot{q}} L \). One constructs a Hamiltonian \( \hat{H} = \hat{H}(\hat{q}, \hat{p}) \) and gives the dynamic by Schrödinger equation or by Heisenberg equation. In general there is no relation between operators \( \hat{p} \) and \( \hat{H} \) and classical quantities \( p = \frac{\partial}{\partial \dot{q}} L \) and \( H = p\dot{q} - L \). Quantum mechanics gives \( \langle \hat{q}(t) \rangle \) and Euler Lagrange equations give \( q(t) \). One only demands that the equality \( \langle \hat{q}(t) \rangle = q(t) \) be satisfied in the classical limit. The following example exhibits the equivalence between the GGB quantization and Schrödinger approach in the case of nonsingular Lagrangians. In fact this is the Ehrenfest theorem reversed.
Consider a system given by the Lagrangian,
\[ L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 - V(q). \]  
(3)

The Euler Lagrange equation of motion is
\[ \ddot{q} + \frac{\partial}{\partial q} V(q) = 0. \]  
(4)

In the GGB quantization one says this system is quantum mechanically described by the operator \( \hat{q}(t) \) if in the Heisenberg picture \( \hat{q} \) satisfies the same equation of motion
\[ \ddot{\hat{q}} + \frac{\partial}{\partial \hat{q}} V(\hat{q}) = 0. \]  
(5)

This can be achieved if one defines the Hamiltonian \( \hat{H} = \frac{1}{2} \hat{p}^2 + V(\hat{q}) \), where \( \hat{p} \) is the generator of translation in the \( q \)-representation of the Hilbert space i.e. \( [\hat{q}, \hat{p}] = i\hbar \). In the next section we show that this Hamiltonian could be uniquely determined by considering two conditions. The equations of motion are given by the Heisenberg equations,
\[ \dot{\hat{q}} = \frac{-i}{\hbar} [\hat{q}, \hat{H}], \quad \dot{\hat{p}} = \frac{-i}{\hbar} [\hat{p}, \hat{H}]. \]  
(6)

It is very important to note that \( \hat{p} \) is not the operator version of the momentum \( p = \frac{\partial}{\partial \dot{q}} L \). The operator \( \hat{p} \) is only the generator of translation in the \( q \)-representation of the Hilbert space, \( \langle q|p|\psi \rangle = -i\hbar \frac{\partial}{\partial q} \langle q|\psi \rangle \). The above observed similarities between the operators \( \hat{p} \) and \( \hat{H} \) and the classical quantities \( p = \frac{\partial}{\partial \dot{q}} L \) and \( H = p\dot{q} - L \) are only due to the particular form of the Lagrangian (3) which is quadratic with respect to the velocity. Most of the Lagrangians used to formulate ordinary classical models are in this form.

4 Reduced Hilbert Space in the GGB Quantization

Although the GGB quantization is equivalent to the RPS quantization (Schrödinger approach) when it is applied to unconstrained systems given by regular Lagrangian but results could be completely different for constraint systems. As an example consider the Lagrangian
\[ L = \dot{x}\dot{y} + yz, \]  
(7)

which leads to the following ELs
\[ \ddot{x} - z = 0, \quad y = 0. \]  
(8)

Since \( z \) remains undetermined, given \( z = E(t) \) one can interpret these equations to be equations of motion for a one dimensional charged particle subjected to an external electric field, \( E = E(t)e_x \). One can easily verify that the RPS quantization leads to no dynamic
for this system because it is given by a singular Lagrangian and the corresponding reduced phase space is null. In the GGB quantization one gives dynamic by the Hamiltonian
\[ \hat{H} = \frac{1}{2} \hat{p}^2_x - xz + \hat{E}(t)\hat{p}(z). \] (9)
where \( \hat{p}_x \) and \( \hat{p}_z \) are generators of translation in the corresponding directions. The Hamiltonian (9) is uniquely determined regarding two conditions:

i) Considering Eqs (8), the Hamiltonian should be quadratic in \( \hat{p}_x \) and linear in all other momenta. It should contain kinetic terms corresponding to coordinates which their accelerations appear in ELs. No kinetic term corresponding to other coordinates that their accelerations are not appeared in ELs should be included in the Hamiltonian.

ii) This Hamiltonian leads to Heisenberg equations equivalent to ELs in which coordinates \( x_i \) are replaced by operators \( \hat{x}_i \).

We do not consider the coordinate \( y \) in quantization similar to the RPS quantization. Consequently the GGB quantization leads to a dynamic that coincides with the classical dynamic in the classical limit. This is an important result but it is not sufficient yet. The Hamiltonian (9) is defined on a two dimensional Hilbert space expanded for example by \( |x\rangle |z\rangle \). Since we are studying a one dimensional system, the Hilbert space should be one dimensional. Considering Eq.(8), \( x \) is the spatial coordinate and \( z \) is only an auxiliary coordinate playing role of external electric field. Consequently, physical Hilbert space is the subspace expanded by \( |x\rangle \). How can one reduce the Hilbert space \( |x\rangle |z\rangle \) to realize the true subspace \( |x\rangle \)? Quantum invariants [9, 10] provide a suitable answer. A Hermitian operator \( \hat{I}(t) \) is called an invariant if it satisfies
\[ \frac{\partial \hat{I}}{\partial t} - \frac{i}{\hbar} [\hat{I}, \hat{H}] = 0. \] (10)
Considering the eigen value equation of \( \hat{I}(t) \)
\[ \hat{I}(t) |\lambda_n, t\rangle = \lambda_n |\lambda_n, t\rangle , \] (11)
one can show that the eigen values \( \lambda_n \) are independent of time and the particular solution \( |\lambda_n, t\rangle_S \) of the Schrödinger equation \( \hat{H} |\psi\rangle_S = i\hbar \frac{\partial}{\partial t} |\psi\rangle_S \) is different from the eigen state \( |\lambda_n, t\rangle \) of \( \hat{I}(t) \) only by a phase factor, \( |\lambda_n, t\rangle_S = \exp(i\phi_n(t)) |\lambda_n, t\rangle \). Given the Hamiltonian (9), an invariant is
\[ \hat{I}(t) = \hat{z} - \hat{E}(t). \] (12)
The null eigen function of this invariant is
\[ \psi(x, P_z) = \exp \left[ -\frac{i}{\hbar} E(t)p_z \right] \phi(x). \] (13)
Since
\[ \hat{H}\psi(x, p_z) = \exp \left[ -\frac{i}{\hbar} E(t)p_z \right] \left( \frac{1}{2} \hat{p}_x^2 - E(t)\hat{x} + \hat{E}(t)p_z \right) \phi(x) \] (14)
and
\[ i\hbar \frac{\partial}{\partial t} \psi(x, p_z) = \exp \left[ -\frac{i}{\hbar} E(t)p_z \right] \left( \dot{E}(t)p_z + i\hbar \frac{\partial}{\partial t} \right) \phi(x), \]
the Schrödinger equation becomes effectively
\[ \hat{H}_e \phi(x) = i\hbar \frac{\partial}{\partial t} \phi(x), \]
where \( \hat{H}_e \) is the effective Hamiltonian
\[ \hat{H} = \frac{1}{2}\hat{p}_x^2 - E(t) \hat{x}. \]

It is interesting to note that if one gives the classical dynamic of the charged particle by the regular Lagrangian
\[ L = \frac{1}{2}\dot{x}^2 + E(t)x \]
instead of the singular Lagrangian (7), Schrödinger approach leads to the same Schrödinger equation Eq.(16).

Quantum invariants provide a suitable framework for reducing Hilbert space. Physical states are (null) eigen states of the quantum invariant \( \hat{I}(t) \). This is a familiar statement. In the Dirac quantization, one defines physical states as null eigen states of the generator of gauge transformation \( \hat{G} \)
\[ \hat{G} |\text{Phys} \rangle = 0. \]

It is shown that \( G \) satisfies the following condition [11,12],
\[ \frac{\partial}{\partial t} G + \{G, H\} = PC, \]
where \( PC \) stands for any linear combination of primary constraints and \( H \) is the classical Hamiltonian. Comparing Eq.(19) with Eq.(10), one verifies that on the surface of primary constraints, \( G \) is an invariant.

As one further example, we study briefly the formulation of a charged time dependent one dimensional oscillator coupled to an external electric field. Consider the Lagrangian
\[ L = \dot{q}\dot{y} - qz\dot{y} - \dot{q}z y + qz^2 y + x y, \]
that leads to the following equations of motion
\[ \ddot{q} - q(\dot{z} + z^2) - x = 0, \]
\[ \dot{y} = 0. \]
The coordinates \( x \) and \( z \) remain arbitrary functions of time. Given \( x = E(t) \) and \( z = A(t) \) one verifies that Eq.(21) can be equivalently obtained from the regular Lagrangian
\[ L = \frac{1}{2}(\dot{q} - A(t)q)^2 + E(t)q. \]

For quantization we introduce operators \( \hat{q}, \hat{\dot{x}}, \hat{\dot{z}} \) and the corresponding generators of translation \( \hat{p}, \hat{p}_x \) and \( \hat{p}_z \). Again we do not consider \( \dot{y} \) in quantization. Considering Eq.(21)
the Hamiltonian should be a sum of the kinetic term $\frac{1}{2}p^2$ and a linear combination of other momenta. In addition, Heisenberg equations of motion should be equivalent to Eq.(21) written for operators. A Hamiltonian (maybe our unique choice) is

$$\hat{H} = \frac{1}{2}\hat{p}^2 - \hat{q}\hat{x} + \frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\hat{z} + \dot{A}(t)\hat{p}_x + \dot{E}(t)\hat{p}_z.$$  

(24)

Quantum invariants are

$$\hat{I}_1 = \hat{x} - E(t), \quad \hat{I}_2(t) = \hat{z} - A(t).$$  

(25)

Physical states are simultaneous (null) eigen states of these invariants,

$$\psi(q, p_x, p_z) = \exp\left[-\frac{i}{\hbar}E(t)p_x\right] \exp\left[-\frac{i}{\hbar}A(t)p_z\right] \phi(q).$$  

(26)

Consequently Schrödinger equation could be effectively given as follows,

$$\hat{H}_e\phi(q) = i\hbar\frac{\partial}{\partial t}\phi(q),$$  

(27)

where

$$\hat{H}_e = \frac{1}{2}(\dot{p} + \dot{q}A(t))^2 - \dot{q}E(t) - \frac{1}{2}\dot{q}^2A^2(t).$$  

(28)

This is what we expected from Eq.(23).

Although the above two models are very simple, but they reflect the main aspects of the method. One can show that for familiar gauge field theories like QED, the GGB quantization is completely equivalent to the RPS quantization. This result is a direct consequence of simple constraint-structures in these models. In general, classification of Lagrangians for which the GGB quantization and the RPS quantization are equivalent is not trivial and should be studied.

## 5 Conclusions

A system is identified by its dynamic given classically by Euler Lagrange equations of motion. Given a Lagrangian, there may exist other Lagrangians, particularly singular Lagrangians, that lead to equivalent dynamic. Consequently, quantization should be independent of the particular form of the Lagrangian. Quantization approach should be formulated such that in the classical limit quantum mechanic lead to classical dynamic given by Euler Lagrange equations of motion. The generalized Gupta-Bleuler quantization is based on these considerations. We showed that this method is equivalent to Schrödinger method for unconstrained Lagrangians. In the case of constraint systems we defined the reduced Hilbert space which is the true physical subspace of the Hilbert space by using the concept of quantum invariants. We studied two simple systems given by singular
Lagrangians and showed that the generalized Gupta-Bleuler quantization leads to true Schrödinger equations.

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References

[1] P.A. M. Dirac, Can. J. Math. 2, (1950) 129 ; Proc. R. Soc. London Ser. A 246, (1958) 326; "Lectures on Quantum Mechanics" New York: Yeshiva University Press, 1964,

[2] J. L. Anderson and P. G. Bergman, Phys Rev. 83, (1951) 1018.

[3] M. Henneaux and C. Teitelboim ”Quantization of Gauge System”, Princeton University Press, Princeton, New Jersey, 1992.

[4] C. Itzykson and J. B. Zuber, ”Quantum Field Theory”, McGraw-Hill Inc., 1980.

[5] D. M. Gitman and I. V. Tyutin, Quantization of Fields with Constraints, Springer-Verlag Berlin Heidelberg, 1990.

[6] J. Govaerts, Hamiltonian Quantization and Constraines Dynamics, Leuven Notes in Mathematical and Theoretical Physics, Vol. 4, 1991.

[7] J. R. Klauder, Lect.Notes Phys. 572 (2001) 143.

[8] C. Batlle, J. Gomis, J. M. Pons, N. Roman Roy, J. Math. Phys. 27 (12), (1986) 2953.

[9] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. 10, (1969) 1458.

[10] X. C. Gao, J. B. Xu and T. Z. Qian, Phys. Rev. A 44, (1991) 7016.

[11] C. Batlle, J. Gomis, X. Gracia and J. M. Pons, J. Math. Phys. 30 (6), (1986) 1345.

[12] J. M. Pons and J. A. Garcia, Int. J. Mod. Phys. A 15 (2000) 4681.