Light deflection in Kerr field for off-equatorial source

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Deflection angle for a light ray travelling in the equatorial plane of a rotating Kerr mass has been already calculated by various investigators. Considering the light ray to be travelling only slightly above the equatorial plane, calculations have been made for such a ray for its deflection angle. In this paper, we calculate deflection angles for the light ray at various heights, which are small compared to the impact parameter and derive corresponding analytical expressions for deflection angle.

I. INTRODUCTION

In general relativity, Schwarzschild line element gives the solution of Einstein’s field equation for the exterior of an uncharged, static body \cite{1}. Bending angle for Schwarzschild field was calculated by many authors up to second and higher order terms. Keeton et al. \cite{2} calculated the light deflection angle for Schwarzschild black-hole in weak field approximation. Iyer and Petters \cite{3} calculated light deflection angle for strong field approximation which, under weak field approximation is an exact match with that of Keeton et al. \cite{2}.

The solution of Einstein’s field equation which describes the exterior of an uncharged, rotating body was obtained by Kerr in the year of 1965 \cite{4}. According to this line element, frame dragging, an unusual prediction of the general relativity is exhibited by such rotating bodies. The prediction of this effect is that all objects coming close to a rotating mass would be entrained to participate in its rotation, not because of any applied force or torque that can be felt, but because of the curvature of space time associated with the body. At close enough distance all objects even light must rotate with the body. So the light deflection angle in Schwarzschild geometry and the Kerr geometry is not same. But there are some extra terms, which arise due to the effect of rotation.

Bending angle for Kerr field in equatorial plane (i.e $\theta = \frac{\pi}{2}$) was calculated by Iyer et al. \cite{5, 6}. According to their result, deflection produced in presence of a rotating black hole explicitly depends on direction of motion of the light. If the light ray is moving in the direction of rotation, then the deflection angle is higher than the zero rotation Schwarzschild field. If the light ray is moving in the opposite direction of rotation, the bending angle will be smaller than that for Schwarzschild field. All the above mentioned calculations (for Schwarzschild and Kerr) were done using the null geodesic for light ray.

On the other hand, another technique has been developing by some authors called effective refractive index of material medium. Here the effect of gravitation is seen as the change in the refractive index of the medium through which light is travelling. In the year 1958, Balaz used this method to calculate the effect of gravity due to a rotating body on the direction of polarization vector of an electromagentic wave \cite{7}. Atkinson used this method to study the trajectory of light ray near a very massive, static and spherically symmetric star \cite{8}. Fischbach and Freeman derived the second order term of gravitational deflection using the same method \cite{9}. Sen calculated the light deflection angle for Schwarzschild body without any weak field approximation using this method \cite{10}.

All these above mentioned work for Kerr field were done in equatorial plane. But there are authors who have worked on non-equatorial plane also. Bozza \cite{11} obtained the lensing formula, and calculated the relativistic image position for a light ray trajectory close to equatorial plane of a Kerr black hole. However, he did not calculate the light deflection angle explicitly in non-equatorial plane.

Aazami et al. \cite{12, 13} worked on quasi-equatorial regime of Kerr black hole and calculated the two components of light bending angle, along the direction of equatorial plane and perpendicular to the equatorial plane of a Kerr black hole. Their expression for light deflection angle along the direction of equatorial plane is in exact match with the result of Iyer \cite{6}.

For a static charged gravitational field, the solution of Einstein field equation was given by Reissner \cite{14} and then by Nordström \cite{15} independently, known as Reissner-Nordström solution. Eiroa et al \cite{16} calculated the light deflection angle for this line element and showed that charge itself has some effect on curvature of space-time.

Vibhadra et al. \cite{17} obtained the light bending angle for Janis-Newman-Winicour pace-time (another solution for static, charge body) and it was shown that for zero static charge, it reduces to Schwarzschild bending angle. The exact solution of Einstein’s field equation for a rotating charged body was found by Newman et. al, which is known as Kerr-Newman metric \cite{18, 19}.

In this paper we consider an uncharged, rotating Kerr body where the light ray is coming from an off-equatorial source i.e. the source at infinity is not exactly on equatorial plane, but it is at a very small height $l$ above the equatorial plane. For a rotating body, rotation drags the trajectory of light ray towards the rotation plane i.e. the equatorial plane, where the rotation has its maximum effect. In this paper we consider a slowly rotating, massive gravitational body, where gravity dominates over the rotation.
So the light ray remains on the gravitational plane i.e. all over its trajectory light ray maintains a height \( l \) above the equatorial plane. Direct consequence of this assumption is that the latitude angle \( \theta \) will be a constant. With this assumption we calculated the off-equatorial light deflection angle for a slowly rotating body.

II. NULL GEODESIC IN KERR FIELD

The Kerr line element in Boyer-Lindquist co-ordinate \((r, \theta, \phi, t)\) can be written as [20, page 313]

\[
ds^2 = (1 - \frac{2mr}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{\rho^2}{\Delta})d\phi^2
\]

From this equation, the energy and angular momentum integrals cab be written as [21, page 346],

\[
E = P_t = \frac{\partial \mathcal{L}}{\partial (ct)} = (1 - \frac{2Mr}{\rho^2})ct + \frac{2amr \sin^2 \theta}{\rho^2} c\phi - \frac{\rho^2}{\Delta} i^2 - \rho^2 \dot{\theta}^2 - (r^2 + a^2 + \frac{2amr \sin^2 \theta}{\rho^2}) \sin^2 \theta \dot{\phi}^2
\]

\[
L_z = -P_\phi = -\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{2amr \sin^2 \theta}{\rho^2} ct + \frac{2a^2 m r \sin^2 \theta}{\rho^2} \sin^2 \theta \dot{\phi}
\]

where \( L_z \) is the angular momentum along the direction of rotation, \( E \) is the energy of the particle. It is possible to show that impact parameter \( \xi_s = \frac{\dot{r}}{\rho^2} = s \xi \) [21, page 123], \( s \) defines the direction of motion of the light ray. For \( s = + \), the light ray is in the direction of black hole rotation and for \( s = - \), the light ray is in the opposite direction of rotation [6]. Here \( \xi_s \) is the first constant of motion. The second constant of motion is \( \eta = \frac{\dot{\phi}}{\rho^2} \), where \( \chi = K - (L_z - aE)^2 \) is the Carter constant [20] and \( K \) is the conserved quantity along the direction \( \theta \).

Following Carter [20, page 348 and 22, equation no. 62, 63, 64, 65], we can have the four geodesic equations governing the motion of light ray in Kerr geometry as:

\[
\dot{\phi} = \frac{2amr + (\rho^2 - 2mr) \xi \csc^2 \theta}{(a^2 + r^2 - 2mr)\rho^2}
\]

\[
c\dot{t} = \frac{(r^2 + a^2) - \Delta a^2 \sin^2 \theta - 2amr \xi}{\Delta \rho^2}
\]

\[
\dot{\theta} = \frac{(\eta + a^2 \cos \theta - \xi^2 \cot \theta) \frac{\dot{\phi}}{\rho^2}}{\rho^2}
\]

\[
\frac{2mra^2 \sin^2 \theta}{\rho^2} \sin^2 \theta \dot{\phi}^2 + \frac{4amr \sin^2 \theta}{\rho^2} c\dot{t} \dot{\phi}
\]

where

\[
\frac{\rho}{r^2 + a^2 \cos^2 \theta}
\]

\[
\Delta = a^2 + r^2 - 2mr
\]

and \( m = \frac{GM}{c^2} \) and \( a = \frac{J}{mc} \), further \( c \) is the velocity of light in free space, \( M \) is the mass of the gravitating body, G is the gravitational constant and \( J \) is the angular momentum of the gravitating body. Lagrangian (\( \mathcal{L} \)) can be written as, [20, page 96]

\[
\mathcal{L} = g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu
\]

where the dot indicates derivative with respect to some affine parameter. So the lagrangian (\( \mathcal{L} \)) can be written as [21],

\[
\dot{\psi} = \frac{[r^4 - (\eta + \xi^2 - a^2)r^2 + 2mr(a - \xi^2) + 2m\eta - a^2 \eta] \frac{\dot{\phi}}{\rho^2}}{r^2}
\]

Let us consider that the source of the light ray is not contained in the equatorial plane, but slightly above it by a distance \( l \). If there is no gravitational field, the light ray will follow a straight line. The projection of this straight line on equatorial plane will be at a distance \( u \) from the origin of the gravitating body, which we can call the projected impact parameter. At this minimum projected distance, the light ray has a height \( l \) from the equatorial plane and \( \psi \) is the angle that the light ray makes with equatorial plane [11]. So,

\[
\tan \psi = \frac{l}{u}
\]

The value of \( \psi \) is very small, as \( l \) is only slightly above the equatorial plane. In other words \( l \) is much smaller then \( u \) (\( l \ll u \)).

These two parameters \( (l, u) \) can be used to level the light ray coming from infinity. The relation between the two parameters \( (l, u) \) and two constants of motion \( (\xi_s, \eta) \) can be written as [11]:

\[
\xi_s = u \cos \psi
\]

and

\[
\eta = l^2 \cos^2 \psi + (u^2 - a^2) \sin^2 \psi
\]
Now the relation between colatitude angle $\theta$ and the angle $\psi$ can be written as,

$$\theta = \frac{\pi}{2} - \psi$$  \hspace{1cm} (15)$$

On the equatorial plane (i.e. $l = \psi = 0$), $\xi_s = u$ and $\eta = 0$.

Now geodesic equations can be written in terms of $\psi$ by using equation (15) in equation (8, 9, 10, 11)

$$\dot{\phi} = \frac{2mr + (r^2 + a^2 \sin^2 \psi - 2mr)\xi_s \sec^2 \psi}{(a^2 + r^2 - 2mr)(r^2 + a^2 \sin^2 \psi)}$$  \hspace{1cm} (16)

Or,

$$\dot{\phi} = \frac{2mra + r(r - 2m)\xi_s \sec^2 \psi + a^2 \xi_s \tan^2 \psi}{(a^2 + r^2 - 2mr)(r^2 + a^2 \sin^2 \psi)}$$  \hspace{1cm} (17)

$$\dot{\psi} = \frac{(\eta + a^2 \sin \psi - \xi_s \tan \psi)\frac{\eta}{r} + 2mr \xi_s}{r^2 + a^2 \sin^2 \psi}$$  \hspace{1cm} (18)

$$\dot{ct} = \frac{(r^2 + a^2)^2 - \Delta a^2 \cos^2 \psi - 2mra \xi_s}{\Delta (r^2 + a^2 \sin^2 \psi)}$$  \hspace{1cm} (19)

$$\dot{r} = \frac{[r^4 - (\eta + \xi_s^2 - a^2)r^2 + 2mra(a - \xi_s)^2 + 2mr \eta - a^2 \eta]^\frac{1}{2}}{r^2 + a^2 \sin^2 \psi}$$  \hspace{1cm} (20)

Here, we considered $\psi$ is very small i.e. $l \ll u$. So we can expand the trigonometric functions in the above geodesic equations up to second order terms as:

$$\tan^2 \psi \simeq \sin^2 \psi \simeq \psi^2$$  \hspace{1cm} (21a)

and

$$\sec^2 \psi \simeq 1 + \psi^2$$  \hspace{1cm} (21b)

we can have the new format of geodesic equations by using equations (20), (21) in equations (16) to (19) as:

$$\dot{\phi} = \frac{2mra + r(r - 2m)\xi_s (1 + \psi^2) + a^2 \xi_s \psi^2}{(a^2 + r^2 - 2mr)(r^2 + a^2 \psi^2)}$$  \hspace{1cm} (22)

$$\dot{\psi} = \frac{(\eta + a^2 \psi) - \xi_s^2 \psi}{r^2 + a^2 \psi^2}$$  \hspace{1cm} (23)

$$\dot{ct} = \frac{(r^2 + a^2)^2 - \Delta \psi^2 (1 - \psi^2) - 2mra \xi_s}{\Delta (r^2 + a^2 \psi^2)}$$  \hspace{1cm} (24)

$$\dot{r} = \frac{[r^4 - (\eta + \xi_s^2 - a^2)r^2 + 2mra(a - \xi_s)^2 + 2mr \eta - a^2 \eta]^\frac{1}{2}}{r^2 + a^2 \psi^2}$$  \hspace{1cm} (25)

$r$ obtains a local extremum for the closest approach $r_0$, so that we can write:

$$\dot{r}|_{r=r_0} = 0$$

Further from equation (25), we can write :

$$r_0^4 - (\eta + \xi_s^2 - a^2)r_0^2 + 2mra(a - \xi_s)^2 + 2mr \eta - a^2 \eta = 0$$

or,

$$\frac{r_0^2}{\xi_s} = (1 - \frac{a^2}{\xi_s^2}) - \frac{2m}{r_0 \xi_s} (1 - \frac{a}{\xi_s} + \frac{\eta}{\xi_s})$$

$$- \frac{2mr_0 \eta}{r_0 \xi_s^2} + \frac{\eta a^2}{r_0 \xi_s^2}$$  \hspace{1cm} (26)

### III. DEFLECTION ANGLE FOR OFF EQUATORIAL SOURCE

Under the above kind of geometry, one may write the expression for the light deflection angle as : [22, page 189]

$$\Delta \phi = 2 \int_{r_0}^{\infty} \frac{d\phi}{dr} dr - \pi$$  \hspace{1cm} (27)

Using equation (22) and (25) in equation (27), one may write

$$\Delta \phi = 2 \int_{r_0}^{\infty} \frac{[2mra + r(r - 2m)\xi_s (1 + \psi^2) + a^2 \xi_s \psi^2] dr}{[a^2 + r(r - 2m)][r^4 - (\eta + \xi_s^2 - a^2)r^2 + 2mra(a - \xi_s)^2 + 2mr \eta - a^2 \eta]^\frac{1}{2}} - \pi$$  \hspace{1cm} (28)

or,

$$\Delta \phi = 2 \int_{r_0}^{\infty} \frac{\xi_s [2mra \xi_s + r(r - 2m)(1 + \psi^2) + a^2 \psi^2] dr}{[a^2 + r(r - 2m)]\xi_s r^2 [\frac{1}{\xi_s} - \frac{1}{r^2} + \frac{2m}{r} (1 - \frac{a}{\xi_s})^2 + \frac{\eta}{r^2 \xi_s} + \frac{\eta}{r} (2mr - a^2)]^\frac{1}{2}} - \pi$$  \hspace{1cm} (29)
As was done by some previous authors [14, 15], here we substitute \( G = 1 - (\frac{\psi}{\xi})^2 = 1 - \hat{a}^2(\frac{\psi}{\xi})^2 \) and \( F = 1 - (\frac{\psi}{\xi})^m = 1 - s\hat{a}^2 \xi^2 \), where \( \hat{a} = \frac{a}{m} \). So the new form of the above deflection expression becomes:

\[
\Delta \phi = 2 \int_{r_0}^{\infty} \frac{[1 - \frac{2mF + \psi^2(1 - \frac{2m}{r} + \frac{\psi^2}{r^2})]}{[1 - \frac{g}{F} + \frac{G}{r^2} - \frac{\psi^2}{r^2} + \frac{\psi^2}{r^2}(2mr - a^2)]} - \pi
\]

We substitute, \( h = \frac{m}{r_0}, x = \frac{r_0}{r} \) and \( n = \frac{\eta}{\xi} \). Accordingly we have,

\[
dx = -\frac{r_0}{r^2} \, dr
\]

and the limits will change, when \( r \to \infty \), we have \( x \to 0 \) and when \( r \to r_0 \), then \( x \to 1 \). Using these substitutions in above equation we have,

\[
\Delta \phi = 2 \int_{0}^{1} \frac{[1 - 2hxF + \psi^2(1 - 2hx + \hat{a}^2hx^2)]}{[1 - 2hx + \hat{a}^2hx^2][G(1 - x^2) - 2hF^2(1 - x^2)]} - \pi
\]

Using the above substitutions we can rewrite the equation (26) as:

\[
r_0^2 \xi^2 = (1 - \frac{a^2}{\xi^2}) - \frac{2m}{r_0} (1 - \frac{a}{\xi^2})^2 + \frac{\eta}{\xi^2} - \frac{2mn}{r_0^2 \xi^2} + \frac{\eta n^2 a^2}{\xi^2 r_0^2}
\]

\[
= G - 2hF^2 + n - 2hn + n\hat{a}^2h^2
\]

Combining equation (31) and (32), we can write

\[
\Delta \phi = 2 \int_{0}^{1} \frac{[1 - 2hxF + \psi^2(1 - 2hx + \hat{a}^2hx^2)]}{[1 - 2hx + \hat{a}^2hx^2][G(1 - x^2) - 2hF^2(1 - x^2)]} - \pi
\]

or,

\[
\Delta \phi = 2 \int_{0}^{1} \frac{[1 - 2hxF + \psi^2(1 - 2hx + \hat{a}^2hx^2)]}{[1 - 2hx + \hat{a}^2hx^2][G(1 - x^2) - 2hF^2(1 - x^2)]} - \pi
\]

Let us assume \( G + n = g \) and \( F^2 + n = f \). If the light ray is contained in the equatorial plane, then \( n = 0 \) and we have, \( G = g \) and \( f = F^2 \). So the new form of the equation (34) will be,

\[
\Delta \phi = 2 \int_{0}^{1} \frac{[1 - 2hxF + \psi^2(1 - 2hx + \hat{a}^2hx^2)]}{[1 - 2hx + \hat{a}^2hx^2][g(1 - x^2) - 2hf(1 - x^2)]} - \pi
\]

or,

\[
\Delta \phi = 2 \int_{0}^{1} \frac{[1 - 2hxF + \psi^2(1 - 2hx + \hat{a}^2hx^2)]}{[1 - 2hx + \hat{a}^2hx^2][g(1 - x^2) - 2hf(1 - x^2) + \frac{n\hat{a}^2h^2(1 + x^2)}{g}]} - \pi
\]

or,

\[
\Delta \phi = 2 \int_{0}^{1} \frac{[1 - 2hxF + \psi^2(1 - 2hx + \hat{a}^2hx^2)]}{[1 - 2hx + \hat{a}^2hx^2][g(1 - x^2) - 2hf(1 - x^2) + \frac{n\hat{a}^2h^2(1 + x^2)}{g}]} - \pi
\]
or,

\[
\Delta \phi = 2 \int_0^1 \frac{dx}{\sqrt{g(1-x^2)}} \left[ 1 - 2hx F + \psi^2(1 - 2hx + \hat{a}^2h^2x^2) \right] \left[ 1 - 2hx + \hat{a}^2h^2x^2 \right]^{-1} \left[ 1 - 2h \frac{f(1 - x^3)}{g(1-x^2)} \right]^{-\frac{1}{2}}
\]

\[
[1 + \frac{n\hat{a}^2h^2}{g}(1 + x^2)\left[ 1 - 2h \frac{f(1 - x^3)}{g(1-x^2)} \right]^{-1}]^{\frac{1}{2}} - \pi
\]

For weak deflection limit, following [12, 13] one can assume, \(m \ll r_o\), in other words, \(h \ll 1\). So the above equation can be expanded in Taylor series in terms of \(h\). By considering terms up to second order only, we can write:

\[
\Delta \phi = 2 \int_0^1 \frac{dx}{\sqrt{g(1-x^2)}} \left[ 1 - 2hx F + \psi^2(1 - 2hx + \hat{a}^2h^2x^2) \right] \left[ 1 + 2hx + x^2h^2(4 - \hat{a}^2) \right]
\]

\[
[1 + \frac{fh}{g}\left( \frac{1 - x^3}{1 - x^2} \right) + \frac{3}{2} \frac{f^2h^2}{g^2}\left( \frac{1 - x^3}{1 - x^2} \right)^2 \left[ 1 - \frac{n\hat{a}^2h^2}{2g}(1 + x^2) \right] - \pi]
\]

Multiplying term by term and retaining only up to second order of \(h\) we get,

\[
\Delta \phi = 2 \int_0^1 \frac{dx}{\sqrt{g(1-x^2)}} \left[ 1 + h\left( 2x(1 - F) + \frac{f}{g} \frac{(1 - x^3)}{(1 + x^2)}(1 + \psi^2) \right) + \psi^2
\]

\[
+h^2\left( x^2(4 - \hat{a}^2 - 4F) + \frac{2fx(1 - F)}{g} \frac{(1 - x^3)}{(1 - x^2)} + \frac{3f^2(1 - x^3)^2}{2g^2}(1 + \psi^2) - \frac{n\hat{a}^2h^2}{2g}(1 + x^2)(1 + \psi^2) \right) \right] - \pi
\]

Now integrating term by term and retaining terms only up to second order in \(h\), we can write:

\[
\Delta \phi = \pi \left( \frac{1 + \psi^2}{\sqrt{g}} - 1 \right) + 4h\left[ \frac{f(1 + \psi^2) + g - Fg}{g^2} \right] + h^2\left[ - \frac{f}{g^2} \left( f(1 + \psi^2) + g - Fg \right) \right]
\]

\[
+ \frac{15\pi}{4} \frac{1}{15g^2} \left( 15f^2(1 + \psi^2) - 4g(F - 1)(3f + 2g) - 2g^2\hat{a}^2 \right) - \frac{3\pi n\hat{a}^2}{4g^2}(1 + \psi^2)
\]

The above equation (38) represents the off-equatorial deflection of light due to rotating sphere in weak field limit. This expression is a function of mass, rotation and \(\psi\). We verify our result under some limiting conditions below:

If the light ray is contained in equatorial plane only, then \(\psi, n\) will be equal to zero and \(g = G, f = F^2\). Under these conditions we find, the above equation (38) will reduce to the expression of equatorial deflection of light by Kerr mass obtained by Aazami et al. [13, equation no. (B17)].

IV. DISCUSSION AND CONCLUSION

The main focus of the paper is to study the light deflection angle (\(\Delta \phi\)) as a function of the height of the light source above the equatorial plane in Kerr geometry. As obtained in equation (38), the deflection angle (\(\Delta \phi\)) is a function of \(\psi\) and in our expressions we consider \(\psi \simeq \frac{1}{h}\). So by varying the source height and projected impact parameter, the main purpose of the paper can be achieved,
to see the effect on the deflection angle. 
In order to study the relation between \( \Delta \phi \) and \( l \) we obtained the values of \( \Delta \phi \) as a function of \( \psi \approx \frac{1}{u} \) in Table 1, by considering, the radius of Sun \( 6.955 \times 10^8 \) meter as the closest approach of light ray for both prograde and retrograde motions. The same was repeated, by considering PSR J 1748-2446 [23] for prograde and retrograde in Fig.1 and Fig.2 respectively, where the closest approach of the light ray is taken to be the radius of the pulsar i.e 20km. Subsequently, to understand the relation between \( \Delta \phi \) and \( u \), the plots were repeated in all the above mentioned figures for three different values of \( u \). Before proceeding further, it is to be mentioned that while plotting the graphs, the physical parameters (\( u \)) was normalized by replacing \( u \) by \( u/2m \). 

By studying the nature of the graphs, it is clear that the light deflection angle increases with the \( \psi(\approx \frac{1}{u}) \) in both prograde and retrograde motion. For Sun (Table 1) and pulsar (Fig1, Fig2) the variation of \( \Delta \phi \) as a function of \( \psi \) is giving a clear idea about relation between them. On the other hand, it is quite obvious that the deflection angle decreases with the increase of projected impact parameter \( \frac{u}{2m} \) in every graph and the Table 1. All the above mentioned calculations were done by considering \( \psi \) to be very small \( (l \ll u) \). Since from our expression we know that \( \Delta \phi \) is a function of both \( r \) and \( \psi \), therefore if we consider \( l \) to be in the same order as \( u \), then instead of \( \psi \) being a constant, it would be a function of \( r \) itself and then we have to write the expression of light deflection angle as:

\[
\Delta \phi = 2 \int_{r_o}^\infty \left[ \frac{d\varphi}{dr} \right] dr - \pi
\]
or,

\[
\Delta \phi = 2 \int_{r_o}^\infty \left[ \frac{\partial \varphi}{\partial r} + \frac{\partial \varphi}{\partial \psi} \frac{\partial \psi}{\partial r} \right] dr - \pi
\]

We may try to deduce the the values of \( \frac{\partial \varphi}{\partial \psi} \) and \( \frac{\partial \psi}{\partial r} \), utilizing equation (16), (17) and (19). However, this we postpone for our future work and in the present work we report our work under the condition of \( l \ll u \) only.

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TABLE I. Variation of bending angle of light ($\Delta \phi$) as a function of $\psi \approx \frac{l}{u}$ for three different values of $u/2m$ with $\hat{a} = 0.5$. To note, at the location of solar radius, value of $u/2m = 2.35 \times 10^5$ at $\psi = 0$.

| $u/2m$ | $\psi \approx \frac{l}{u}$ | bending angle for prograde motion (arcsec) | bending angle for retrograde motion (arcsec) |
|--------|-----------------|------------------------------------------|-------------------------------------------|
| $2.35 \times 10^5$ | $1 \times 10^{-9}$ | 1.753929 | 1.753933 |
|        | $2 \times 10^{-5}$ | 1.754125 | 1.754129 |
|        | $3 \times 10^{-5}$ | 1.754451 | 1.754455 |
|        | $4 \times 10^{-5}$ | 1.754906 | 1.754910 |
| $2.82 \times 10^5$ | $1 \times 10^{-9}$ | 1.461618 | 1.461621 |
|        | $2 \times 10^{-5}$ | 1.461814 | 1.461817 |
|        | $3 \times 10^{-5}$ | 1.462140 | 1.462142 |
|        | $4 \times 10^{-5}$ | 1.462595 | 1.462597 |
| $3.29 \times 10^5$ | $1 \times 10^{-9}$ | 1.252825 | 1.252826 |
|        | $2 \times 10^{-5}$ | 1.253021 | 1.253023 |
|        | $3 \times 10^{-5}$ | 1.253346 | 1.253803 |
|        | $4 \times 10^{-5}$ | 1.253801 | 1.253803 |
FIG. 1. Bending angle (arcsec) as a function of $\psi = \frac{1}{2}$ with constant rotation ($\dot{\psi} = 0.5$) for prograde motion of PSR J1748-2446 for different values of projected impact parameter ($u/2m$). Here $u$ is the radius of the pulsar. To note, at the location of physical radius of pulsar ($r = 20\text{km}$), value of $u/2m = 5$ at $\psi = 0$. 
FIG. 2. Bending angle (arcsec) as a function of $\psi = \frac{\dot{t}}{u}$ with constant rotation ($\dot{a} = 0.5$) for retrograde motion of PSR J 1748-2446 for different values of projected impact parameter ($u/2m$). Here $u$ is the radius of the pulsar. To note, at the location of physical radius of pulsar ($r = 20$ km), value of $u/2m = 5$ at $\psi = 0$. 