Fast algorithm for generating random bit strings and multispin coding for directed percolation

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We present efficient algorithms to generate a bit string in which each bit is set with arbitrary probability. By adopting a hybrid algorithm, i.e., a finite-bit density approximation with correction techniques, we achieve 3.8 times faster random bit generation than the simple algorithm for the 32-bit case and 6.8 times faster for the 64-bit case. Employing the developed algorithm, we apply the multispin coding technique to one-dimensional bond-directed percolation. The simulations are accelerated by up to a factor of 14 compared with an optimized scalar implementation. The random bit string generation algorithm proposed here is applicable to general Monte Carlo methods.

I. INTRODUCTION

A computer performs computation in the unit of words, which consist of a fixed number of bits. Multispin coding (MSC) is a technique to reduce the memory space and execution time in Monte Carlo simulations by packing information of spins into a computer word. In modern CPUs, the width of a computer word is 32 or 64, and bit operations between words are usually executed in one cycle. Therefore, a simulation of a system with discrete degrees of freedom can be markedly accelerated by effectively utilizing MSC and bit operations. The MSC technique was first mentioned by Friedberg and Cameron in 1970 [1], but Rebbi and co-workers proposed MSC in the form widely used today [2,3]. Since then, the MSC of spin systems, especially for the Ising model, has been extensively studied, mainly in the 1980s [4,11]. A special-purpose computer for Ising models was developed on the basis of the vector-parallelism. The vector-parallel MSC is the non-trivial parallelism where many replicas are encoded in one integer, where $N_{\text{bit}}$, the bit-length of the integer of the computer. By using the same random numbers on each replica, they updated $N_{\text{bit}}$ replicas simultaneously. The special-purpose hardware for generating random numbers was also developed and it was also applied to DP [17,18].

In this manuscript, we present efficient algorithms to generate a bit string in which each bit is set with arbitrary probability. These algorithms allow us to implement MSC for one-dimensional bond-directed percolation on the basis of the vector-parallelism. The vector-parallel MSC is the non-trivial parallelism where $N_{\text{bit}}$ sites in one system are encoded in one integer. The MSC technique accelerates the simulations by up to a factor of 14 compared with an optimized scalar implementation. The rest of the article is organized as follows. The random bit string generation algorithm is described in the next section. The application of the algorithm to 1d-BDP with the MSC technique is described in Sec. IV. Section V is devoted to a summary and discussion. The associated code is available at [19].

II. RANDOM BIT STRING GENERATION ALGORITHM

We consider algorithms to generate a bit string with length $N_{\text{bit}}$ such that each bit is 1 with a given probability $p$ and otherwise 0. A simple algorithm is shown in Algorithm 1. Here, $U_r(a,b)$ is a stochastic real variable distributed uniformly from $a$ to $b$. The logical left shift is denoted by $\ll$. The operation $\ll k$ results in a bit shift by $k$ bits, which is multiplication by $2^k$. Excess bits shifted off to the left are discarded. The operation $a \lor b$ denotes the bitwise OR between bit strings $a$ and $b$. This simple algorithm involves generating random numbers $N_{\text{bit}}$ times regardless of the value of $p$, and therefore, it is expensive to use it for the MSC technique. In order to reduce the computational cost to generate a random bit string, we propose two approaches. The first approach is effective when the number of set bits in the bit string...
is small. We propose two such algorithms, one is the binomial-shuffle algorithm described next section and the other is the Poisson-OR algorithm which is described in Sec. II B. The second approach is to utilize the fact that the bit string generated by the standard pseudo-random generator contains set bits with probability 0.5. This algorithm, which is described in Sec. II C, is effective when the number of digits in binary notation of a probability $p$ is small. In Sec. II D we propose a hybrid algorithm of the two approaches.

Algorithm 1 Simple Algorithm

1: $s \leftarrow 0$
2: for $i = 0$ to $N_{\text{bit}} - 1$ do
3: if $U_r(0, 1) < p$ then
4: $s \leftarrow s \lor (1 \ll i)$
5: end if
6: end for

A. Binomial-Shuffle Algorithm

In the simple algorithm, it is necessary to generate $N_{\text{bit}}$ random numbers regardless of the probability $p$. Since the number of set bits is about $pN_{\text{bit}}$, we can reduce the number of random numbers generated by first determining the number of set bits and then shuffling their positions. We refer to this algorithm as the binomial-shuffle algorithm. First, we determine the number of set bits $m$ out of $N_{\text{bit}}$ with probability $p$. This is a random number following the binomial distribution of $N_{\text{bit}}$ trials with probability of success $p$. After determining the number of set bits, we choose their positions. Adopting Floyd’s sampling algorithm, this selection process involves random number generation $m$ times. The pseudocode of the binomial-shuffle algorithm is shown in Algorithm 2 The operation $a \land b$ denotes the bitwise AND. Here, $B(n, p)$ is a stochastic integer variable following the binomial distribution with parameters $n$ and $p$, where $n$ is the number of trials and $p$ is the probability of success. We can implement the function $B(n, p)$ with $O(1)$ complexity by adopting Walker’s method of aliases [20] [21]. $U_d(a, b)$ is a stochastic integer variable distributed uniformly from $a$ to $b$. Since we need only one random number to determine the number of set bits and $pN_{\text{bit}}$ random numbers to shuffle the positions of the bits on average, this algorithm involves the generation of $pN_{\text{bit}} + 1$ random numbers.

Algorithm 2 Binomial-Shuffle Algorithm

1: $s \leftarrow 0$
2: $m \leftarrow B(N_{\text{bit}}, p)$
3: for $i = N_{\text{bit}} - m$ to $N_{\text{bit}} - 1$ do
4: $k \leftarrow U_d(0, i)$
5: $t \leftarrow 1 \ll k$
6: if $(s \land t) \neq 0$ then
7: $s \leftarrow s \lor (1 \ll i)$
8: else
9: $s \leftarrow s \lor t$
10: end if
11: end for

B. Poisson-OR Algorithm

The binomial-shuffle algorithm reduces the cost of generating a bit string by first determining the number of set bits in $O(1)$ complexity and then determining their position in $O(pN_{\text{bit}})$ complexity. Todo and Suwa proposed an efficient algorithm to generate a bit string in which each bit is set with some probability $p_k$, where $k$ is the index of the bits [22]. They employed the space-time interchange technique together with Walker’s method of aliases and achieved the generation of such a bit string in $O(k p_k)$ complexity. This algorithm was applied to spin systems with long-range interactions and achieved $O(N)$ complexity for the cluster Monte Carlo method without introducing any cutoff [23]. We need a bit string in which all bits are set independently with identical probability. Since this is a special case of Todo and Suwa’s case, we can apply their algorithm to our problem as follows. Consider an $N_{\text{bit}}$-length bit string with one of the bits is set randomly. We generate such bit strings $k$ times and take the bitwise OR between them. Then each bit of the resulting bit string is 1 with probability $1 - (N_{\text{bit}} - 1)^k / N_{\text{bit}}$. We choose the number $k$ following the Poisson distribution with parameter $\lambda$. Since the probability that the number of events is $k$ in the Poisson process with parameter $\lambda$ is $\lambda^k e^{-\lambda} / k!$, the probability that each bit in the resulting bit string is set is given by

$$p = \sum_{k=0}^{\infty} \left[ 1 - \left( \frac{N_{\text{bit}} - 1}{N_{\text{bit}}} \right)^k \right] \frac{\lambda^k e^{-\lambda}}{k!},$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} - \sum_{k=0}^{\infty} \left( \frac{N_{\text{bit}} - 1}{N_{\text{bit}}} \right)^k \frac{\lambda^k e^{-\lambda}}{k!},$$

$$= 1 - \exp \left( -\lambda / N_{\text{bit}} \right).$$

Therefore, when we choose $\lambda = N_{\text{bit}} \log(1 - p)$ then each bit in the resulting bit string is set with probability $p$. Moreover, each of the bits is mutually independent, which is proved by mathematical induction and the fact that the probability that specified $n$ bits in the resulting bit string are zero is given by

$$\sum_{k=0}^{\infty} \left( \frac{N_{\text{bit}} - n}{N_{\text{bit}}} \right)^k \frac{\lambda^k e^{-\lambda}}{k!} = (1 - p)^n.$$
is shown in Algorithm 3. Here, Poisson(λ) is a stochastic integer variable following the Poisson distribution with parameter λ. We can generate such an integer with \( O(1) \) complexity by adopting Walker’s algorithm. We need one random number to determine \( k \) and \(-N_{\text{bit}} \log(1 - p)\) random numbers to generate the bit strings for disjunction, and therefore, the Poisson-OR algorithm involves the generation of \(-N_{\text{bit}} \log(1 - p) + 1\) random numbers. While the number of random numbers generated in the Poisson-OR algorithm is always larger than that in the binomial-shuffle algorithm since \( pN_{\text{bit}} \leq N_{\text{bit}} \log(1 - p) \), they are close to each other when \( p \) is small. Additionally, the loop body of the Poisson-OR algorithm is simpler than that of the binomial-shuffle algorithm. Therefore the Poisson-OR algorithm can be faster than the binomial-shuffle algorithm.

Algorithm 3 Poisson-OR Algorithm

```c
1: \( s \leftarrow 0 \)
2: \( k \leftarrow \text{Poisson}(-N_{\text{bit}} \log(1 - p)) \)
3: for \( i = 1 \) to \( k \) do
4: \( j \leftarrow U_d(0, N_{\text{bit}} - 1) \)
5: \( s \leftarrow s \lor (1 \ll j) \)
6: end for
```

C. Finite-Digit Algorithm

Suppose there are two bit strings \( s_1 \) and \( s_2 \) in which each bit is set with probability \( p_1 \) and \( p_2 \). If we make a new random bit string \( y = s_1 \lor s_2 \), then \( y \) is a bit string in which each bit is set with a probability \( p = p_1 p_2 \). Similarly, we can have a bit string \( y = s_1 \lor s_2 \) in which each bit is set with probability \( p = 1 - (1 - p_1)(1 - p_2) \). By taking the bitwise AND or the bitwise OR of two random bit strings, we can generate a new bit string in which each bit is set with a new probability. Consider two random numbers which are generated by the standard random number generator, such as `std::mt19937`. The random numbers can be regarded as random bit strings in which each bit is set with a probability \( p = 0.5 \). Then we can generate a random bit string with a probability \( p = 0.25 \) by taking the bitwise AND of them. We can also generate a random bit string with a probability \( p = 0.75 \) by taking the bitwise OR. In this manner, we can generate a random bit string with a probability \( p_n \) by using \( n \) random numbers, where \( p_n \) is an \( n \)-digit number in the binary notation.

Suppose \( x_k \ (k = 1, 2, \cdots, n) \) is a random bit string such that each bit is set with probability 0.5 and the target probability \( \tilde{p}_n \) is given by,

\[
\tilde{p}_n = (0, b_n b_{n-1} \cdots b_2 1)_{\text{bin}},
\]

where \((\cdots)_{\text{bin}}\) denotes the binary notation. Note that we should truncate the digits so that the right-most bit (the least significant bit) is one, i.e., \( b_1 = 1 \). We can generate a bit string \( \tilde{y}_n \) such that each bit is set with a probability \( \tilde{p}_n \) by combining \( n \) bit strings as follows.

\[
\tilde{y}_1 = x_1,
\]

\[
\tilde{y}_k = \begin{cases} x_k \lor \tilde{y}_{k-1} & \text{if } b_k = 1, \\ x_k \land \tilde{y}_{k-1} & \text{otherwise.} \end{cases}
\]

The above procedure involves the generation of \( n \) random bit strings and \( n - 1 \) bit operations. See Fig. 1 for the schematic illustration of the algorithm. We refer to this algorithm the finite-digit algorithm. This algorithm is effective when the number of digits in binary notation of \( p \) is small. For example, consider to generate a random bit string which length is \( N_{\text{bit}} = 32 \) and in which each bit is set with a probability \( p = 0.5 \). To generate a such bit string, the binomial-shuffle algorithm involves random number generations \( 1 + pN_{\text{bit}} = 17 \) times and the Poisson-OR algorithm involves \( 1 - N_{\text{bit}} \log(1 - p) \sim 23 \) times in average. However, we can generate a such bit string by calling `std::mt19937` once.

D. Hybrid Algorithm

The binomial-shuffle and the Poisson-OR algorithms are effective when the number of set bits is small while the finite-digit algorithm is effective when the number of digits in binary notation of a probability \( p \) is small. Combining the two approaches, we can construct a new algorithm which is effective for arbitrary probability. As described in the previous section, we can generate a random bit string \( \tilde{y}_n \) such that each bit is set with a probability \( \tilde{p}_n \). The bit string \( \tilde{y}_n \) can be generated by using \( n \) random numbers. Suppose a target probability \( p \) is expressed as \( p = \tilde{p}_n + \varepsilon \), where \( \varepsilon \) is a small positive number. We generate the desired bit string \( y \) by

\[
y = \tilde{y}_n \lor z,
\]

where \( z \) is a bit string such that each bit is set with probability \( p_z \). A bit of \( y \) is zero if and only if the bits of \( \tilde{y}_n \)
The probability of the correction $z$ at the corresponding position are zero. Therefore, the following identity is fulfilled as

$$1 - p = (1 - \tilde{p}_n)(1 - p_e). \quad (7)$$

Solving the above equation with respect to $p_e$, we have

$$p_e = (p - \tilde{p}_n)/(1 - \tilde{p}_n). \quad (8)$$

Similarly, we can consider the case $p = \tilde{p}_n - \varepsilon$. We correct $\tilde{y}_n$ as

$$y = \tilde{y}_n \land \neg z,$$  

where $\neg z$ denotes the bitwise NOT of the bit string $z$. We generate $z$ so that each bit in $z$ is set with the probability $p_e$, therefore, each bit in $\neg z$ is set with the probability $1 - p_e$. A bit of $y$ is set when the bits of $\tilde{y}_n$ and $\neg z$ at the corresponding position are set. Therefore, we have

$$p = \tilde{p}_n(1 - p_e). \quad (9)$$

In both cases, we generate a bit string for the correction, $z$, by the binomial-shuffle or the Poisson-OR algorithms. The probability of the correction $p_e$ is $O(|p - \tilde{p}_n|) = O(\varepsilon)$. Therefore, the number of random numbers that must be generated for the correction becomes small when $\tilde{p}_n$ is close to $p$.

As an example, consider the probability $p = 0.6447$, which is the critical point of 1d-BDP [13], for the case of $N_{\text{bit}} = 32$. If we adopt the binomial-shuffle algorithm without the correction, $pN_{\text{bin}} + 1 \sim 21.6$ random numbers must be generated on average. The binary notation of $p$ is

$$p = 0.6447 = (0.10100101010 \cdots)_{\text{bin}}. \quad (10)$$

First, consider the one-digit correction $p = \tilde{p}_1 + \varepsilon$, where $\tilde{p}_1 = 0.5$ and $\varepsilon = 0.1447$, respectively. The probability for the correction is $p_e = (0.6447 - 0.5)/(1 - 0.5) = 0.2894$. Then the number of random numbers that must be generated to generate a bit string for the correction $z$ is $p_eN_{\text{bin}} + 1 \sim 10.26$. Therefore, the total number of random numbers generated is 11.26, which is much smaller than that for the algorithm without correction.

The two-digit correction $p = \tilde{p}_2 + \varepsilon$ is identical to the one-digit correction since $\tilde{p}_2 = (0.10)_{\text{bin}} = 1/2 = \tilde{p}_1$. Therefore, we have to consider the three-digit correction $p = \tilde{p}_3 + \varepsilon = 0.625 + 0.0197$ for the next step. The bit string $\tilde{y}_3$ is obtained by

$$\tilde{y}_3 = x_3 \lor (x_2 \land x_1). \quad (11)$$

Since the probability of the correction is $p_e = (0.6447 - 0.625)/(1 - 0.625) \sim 0.053$, the number of random numbers to generate the bit string for the correction $z$ is $p_eN_{\text{bin}} + 1 \sim 2.68$. Since we need three random numbers to generate $\tilde{y}_3$, the total number of random numbers generation is 5.68. A finite-digit probability $p_n$ is meaningful when the right-most (least significant) bit is 1. Therefore, the next meaningful step is the six-digit correction which will require at least seven random numbers. Since the three-digits approximation requires 5.68 random number generations, the three-digits approximation is the most efficient for $p = 0.6447$.

We can also consider the correction from the other side, i.e., $p = \tilde{p}_n - \varepsilon$. The two-digit correction is $p = \tilde{p}_2 - \varepsilon = 0.75 - 0.1053$. The probability for the correction is $p_e = (0.75 - 0.6447)/0.6447 \sim 0.1633$. The average number of random numbers generation is $p_eN_{\text{bin}} + 1 + 2 = 8.23$. Next meaningful step is the four-digit correction $p = 0.6875 - 0.0428$, and the average number of random numbers generation is 7.12. The next step is the five-digit correction $p = 0.65625 + 0.01155$ which involves 6.57 random number generations. Since the next step is the seven-digit correction, the five-digit correction is most effective. Comparing two approaches, $p = \tilde{p}_n + \varepsilon$ and $p = \tilde{p}_n - \varepsilon$, the most effective correction is $p = \tilde{p}_3 + \varepsilon$ for $p = 0.6447$.

If the number of digits in the approximation increases, the number of random numbers that must be generated for the correction decreases, whereas that required to generate the initial bit string increases. Therefore, there is an optimal number of digits for approximation. Additionally, there are two choices for correction, $p = \tilde{p}_n + \varepsilon$ and $p = \tilde{p}_n - \varepsilon$. The optimal number of digits for the approximation and the expected number of random numbers generated are shown in Fig. 2. This figure shows the
case where $0 \leq p \leq 0.5$. One can generate a bit string for $p > 0.5$ by first generating a bit string with probability $1 - p$ and inverting it. The expected number of random numbers generated to generate a 32-bit string in which each bit is set with arbitrary probability $p$ is at most 7. In the case of 64 bits, the expected number of random number generations is at most 8.

\[ \text{FIG. 3: (Color online) Generation speed of random bits set} \]

\[ \text{with probability } p = 0.6447 \text{ in the unit of MBPS (millions} \]

\[ \text{of bits per second) for (a) 32-bit and (b) 64-bit cases. The results for the simple algorithm (Simple), the binomial-shuffle algorithm (BS), the binomial-shuffle algorithm with the correction from } \tilde{p}_3 = 5/8 \text{ (BS from 5/8), the Poisson-OR algorithm (PO), and the Poisson-OR algorithm with correction (PO from 5/8) are shown.} \]

E. Benchmark Results

We performed benchmark tests on HPE SGI 8600 system with Intel Xeon Gold 6184 CPU at the Institute for the Solid State Physics of the University of Tokyo. The program was compiled using Intel C++ compiler 18.0.1 with the option -O3 -xHOST and executed as a single-threaded process on a single CPU core. The generation speed of random bits is shown in Fig. 3. Here, we adopt the unit MBPS (millions of bits per second), which is 1 when one million bits are generated in one second. When $k$ bit strings of $N_{\text{bit}}$ width are generated in $t$ [s], the generation speed is $kN_{\text{bit}} \cdot 10^{-6} / t$ [MBPS]. We estimate the generation speed by observing the time required to generate 4000000 bit strings in which each bit is set with probability $p = 0.6447$. The Poisson-OR algorithm with the correction from the three-digit approximation was the fastest. Compared with the simple algorithm, the generation speed was about 3.8 times faster for the 32-bit case and 6.8 times for the 64-bit case. Adopting the 64-bit implementation, the performance of the binomial-shuffle algorithm is improved by 24% where that of the Poisson-OR algorithm is improved by 38%.

The time required to generate a random bit string is expected to be roughly proportional to the number of random numbers generated. When the probability $p$ is close to the $n$-digit-approximated probability $\tilde{p}_n$, then the number of random numbers generated for correction becomes small as shown in Fig. 2. Consider the region $0.125 < p < 0.1875$. When $p$ is close to 0.125, then three-digit-approximation from 1/8 will exhibit the best performance, while four-digit-approximation from 3/16 will exhibit the best performance when $p$ is close to 0.1875. Therefore, it is expected that the performance inversion will occur between the three- and four-digit approximations in the region $0.125 < p < 0.1875$. To demonstrate this, the probability dependence of the time required to generate random bit strings is shown in Fig. 4. The performance of the three- and four-digit approximations is reversed at $p = 0.1805$, which is close to the expected value of $p = 0.181$. 

\[ \text{FIG. 4: (Color online) Time required to generate random bit} \]

\[ \text{strings together with the expected number of random numbers generated } N_r, \text{ which is shown in Fig. 2 (b). The time} \]

\[ \text{required to generate 32-bit random bit strings } 10^7 \text{ times is denoted by the dashed lines. We adopt the Poisson-OR algo-} \]

\[ \text{rithm with correction. The color corresponds to the number} \]

\[ \text{of digits } n \text{ used in the approximation. One can see that the} \]

\[ \text{performance inversion occurs near the theoretically expected} \]

\[ \text{probabilities, i.e., the intersection points of } N_r. \]

\[ \text{k} \text{bit strings of } N_{\text{bit}} \text{ width are generated in } t \text{ [s], the gen-} \]

\[ \text{eration speed is } kN_{\text{bit}} \cdot 10^{-6} / t \text{ [MBPS]. We estimate the} \]

\[ \text{generation speed by observing the time required to gen-} \]

\[ \text{erate 4000000 bit strings in which each bit is set with} \]

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\[ \text{64-bit implementation, the performance of the binomial-} \]

\[ \text{shuffle algorithm is improved by 24% where that of the} \]

\[ \text{Poisson-OR algorithm is improved by 38%}. \]
III. APPLICATION TO DIRECTED PERCOLATION

Employing the fast random bit-string generation algorithms developed in the previous section, we apply MSC to 1d-BDP. The time evolution of 1d-BDP is shown in Fig. 5 (a). Each bond is open with probability \( p \) and blocked otherwise. The state of the \( i \)-th site at time \( t \) is denoted by \( \sigma^t_i \). The site is active when \( \sigma^t_i = 1 \) and inactive when \( \sigma^t_i = 0 \). If a site at time \( t+1 \) is connected to an active site at time \( t \) with an open bond, then that site becomes active. The states of the sites at time \( t+1 \) are determined by the states at time \( t \) and the time evolution is performed by iterating this process. The scalar implementation to determine the states of the sites at time \( t+1 \) from the states at time \( t \) is shown in Algorithm 4.

Here, we omit the processing of the boundary conditions.

We pack the information of \( N_{\text{bit}} \) sites in a bit string, where \( N_{\text{bit}} = 32 \) or 64. If a bit is set, then the corresponding site is active and inactive otherwise. The \( k \)-th bit string \( s^t_k \) contains the information of \( \{ \sigma^t_i \} \), where \( i = (k-1)N_{\text{bit}} + 1, \ldots, kN_{\text{bit}} \). We generate a bit string \( x_1 \) in which each bit is set with probability \( p \) and take the bitwise AND between \( s^t_k \) and \( x_1 \) as \( t_1 = s^t_k \land x_1 \). Then each bit of \( s^t_k \) survives to \( t_1 \) with probability \( p \). Therefore, \( t_1 \) can be considered as the active sites at time \( t+1 \) connected to the active sites at time \( t \) with the lower left bond. Similarly, we generate \( t_2 = s^t_k \land x_2 \), which denotes the active sites at time \( t+1 \) connected to the active sites at time \( t \) with the lower right bond. Then the site configuration at time \( t+1 \) is obtained by taking the bitwise OR between \( t_1 \) and \( t_2 \) (\( t_2 \ll 1 \)). Note that the most signifi-

Algorithm 4 Scalar Implementation of 1d-BDP

| Line | Code |
|------|------|
| 1:   | for \( i = 1 \) to \( L \) do |
| 2:   | if \( \sigma^t_i = 1 \) then |
| 3:   | if \( U(r(0,1)) < p \) then |
| 4:   | \( \sigma^{t+1}_i = 1 \) |
| 5:   | end if |
| 6:   | if \( U(r(0,1)) < p \) then |
| 7:   | \( \sigma^{t+1}_i = 1 \) |
| 8:   | end if |
| 9:   | end if |
| 10:  | end for |

FIG. 6: (Color online) (a) Cluster growth of 1d-BDP from a single seed at the criticality (log-log axes). The system size is \( L = 32768 \). The time evolution is performed for 32768 steps and \( 10^3 \) independent samples are averaged. The solid line denotes \( \Theta^* \), where \( \Theta = 0.313 \). The results with the scalar code (Scalar) and bit-operation implementation with the binomial-shuffle algorithm (BS) and Poisson-OR algorithm (PO) are shown. We adopt the correction from the three-digit approximation \( p_3 = 5/8 \) for both the BS and PO algorithms. (b) Time required for the simulations.
Independent runs are averaged. Power-law behavior is observed for the density of the active sites $\rho(t) = n(t)/L$, where $\alpha = 0.159$ [24]. The time required to perform these simulations is shown in Fig. 7(b). While the fastest algorithm is the Poisson-OR algorithm with the correction, in this case, the increase in speed is only 4.5 times, which is much smaller than that in the cluster growth simulation.

### IV. SUMMARY AND DISCUSSION

We have developed efficient algorithms to generate a bit string in which each bit is set with arbitrary probability. The binomial-shuffle algorithm first determines the number of set bits and then determines their positions. While the expected number of generated random numbers is smaller in the binomial-shuffle algorithm than in the Poisson-OR algorithm, the Poisson-OR algorithm is faster owing to the simple loop structure. The finite-digit algorithm allows us to generate a random bit string with a finite-digit probability in the binary notation. Combining two algorithms, the number of the random numbers that must be generated for the correction markedly decreases.

We developed the MSC technique for 1d-BDP using the random bit string generation algorithms and achieved a marked increase in speed. The MSC was more effective in the simulation of cluster growth from a single seed than in that of the relaxation from the fully active state. This is due to the local density of the active sites. The number of random numbers generated for the scalar algorithm is proportional to the density of active sites, while that for MSC is independent of the density. Therefore, the efficiency of MSC decreases as the density of active sites decreases. Since the density of active sites in the relaxation process decreases monotonically, the efficiency of MSC decreases over time. In the case of cluster growth process.
simulation, we only update the region between the leftmost to rightmost active sites. Then the local density of active sites hardly changes and MSC works effectively for this case. Our random bit generating algorithm can be applied to replica-parallel MSC. While the previous implementation of MSC for DP used same random numbers among replicas, the replicas do not share the random numbers with our algorithm.

The random bit string generation algorithm is expected to be applicable to general Monte Carlo simulations on lattice systems. In the present work, we did not consider the use of SIMD instructions. SIMD stands for single instruction multiple data and it allows data-level parallelism with the SIMD register. For example, a 512-bit register is available in the Intel Advanced Vector Extensions (AVX-512). By using SIMD instructions, the efficiency of MSC can be further improved. In recent years, general-purpose computing on graphics processing units (GPGPU) has attracted the interests of many researchers. The MSC of the Ising model was implemented on GPGPU [25], and Komura and Okabe implemented the Swendsen–Wang algorithm on GPGPU [26]. The implementation of the MSC algorithm presented in the manuscript on GPGPU should also be attempted in the future.

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Appendix

In this appendix, we show the equivalence of the Poisson-OR algorithm (Algorithm 2) and the simple algorithm (Algorithm 1). In other words, we prove that the probability of observing a particular length bit string is given by

\[
P(s) = p^m (1-p)^{N-m},
\]

where \(m\) is the number of set bits in \(s\). This fact indicates that each bit is set with probability \(p\) mutually independently.

First, let us consider the number of ways to classify \(k\) labeled elements to \(m\) unlabeled groups so that no empty group exists. This number is denoted by \(S(k, m)\) which is called the Stirling numbers of the second kind. The explicit expression of \(S(k, m)\) is given by,

\[
S(k, m) = \frac{1}{m!} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^k,
\]

where \(\binom{m}{j}\) is a binomial coefficient. We define \(S(k, m) = 0\) when \(k < m\). It is useful to derive the exponential generating function of \(S(k, m)\). According to the binomial theorem, the following identity holds.

\[
(e^x - 1)^m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} e^{jx}.
\]

Differentiating both sides \(k\) times with respect to \(x\) and then substituting \(x = 0\), we have

\[
\frac{d^k}{dx^k} (e^x - 1)^m \bigg|_{x=0} = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^k = m! S(k, m).
\]

Therefore, we have the exponential generating function of the Stirling numbers as

\[
\sum_{k=0}^{\infty} S(k, m) \frac{x^k}{k!} = \frac{1}{m!} (e^x - 1)^m.
\]
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