LIMIT THEOREMS FOR HULL-WHITE MODEL WITH HAWKES JUMPS

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Abstract. In the present paper, we obtain limit theorems for a category of Hull-White models with Hawkes jumps including law of large numbers, central limit theorem, and large deviations. In the field of interest rate modeling, it is meaningful in characterizing a long-term rate of return.

1. Introduction

1.1. Interest rate modelling and the classical Hull-White model. Hull-White model is a famous model in interest rate modeling. It was originally based on the assumption of specific one-dimensional dynamics for the instantaneous spot rate process \( r_t \).

In the field of interest rate modelling, the most classical model is the Vasicek model\(^3\). It is defined in the following dynamics,

\[
dr_t = k(\theta - r_t)dt + \sigma dW_t,
\]

where \( k, \theta > 0 \) are constants and \((W_t)\) is a standard Brownian motion in the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). This equation is linear and can be solved explicitly, the distribution of short rate is Gaussian, and both the expressions and the distributions of several useful quantities related to the interest-rate world are easily obtainable. And the risk-neutral pricing of the bond can be computed as a simple expression depending on \( k, \theta, \sigma \) and \((r_t)\). However, the model also has some drawbacks. For example, rates can assume negative values with positive probability.

Another famous interest rate model is Cox-Ingersoll-Ross model\(^5\). A Cox-Ingersoll-Ross process is a stochastic process \((r_t)\) satisfying the stochastic differential equation

\[
dr_t = b(c - r_t)dt + \sigma \sqrt{r_t}dW_t,
\]

where \( b, c, \sigma > 0 \) are constants, and \((W_t)\) is a standard Brownian motion in the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). It was first used in short-term interest rate model inspired by Feller’s proof on a singular diffusion problem\(^6\). Feller proved that the process is nonnegative if \( 2bc \geq \sigma^2 \). Given \( r_0 \), it is well known that \( 4br_t/\sigma^2(1 - e^{-bt}) \) follows a noncentral \( \chi^2 \) distribution with degree of freedom \( 4bc/\sigma^2 \) and noncentrality parameter \( 4br_0e^{-bt}/\sigma^2(1 - e^{-bt}) \). As \( t \to \infty \), \( r_t \to r_\infty \), where \( r_\infty \) follows a Gamma distribution with shape parameter \( 2bc/\sigma^2 \) and scale parameter \( \sigma^2/2b \). This model effectively covers the shortage of Vasicek model to some extent.

In some cases, we have the initial zero-coupon curve \( T \mapsto P(0, T) \) from the market, and we wish our model to incorporate this curve. Denote \( F(0, T) \) to be the instantaneous forward rate for a maturity \( T \) as seen at time zero. Hull and White\(^14, 15\) develop the following dynamics

\[
dr_t = (\theta(t) - ar_t)dt + \sigma dW_t,
\]

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where
\[ \theta(t) = F_t(0, t) + a F(0, t) + \frac{\sigma^2}{2a} (1 - e^{-2at}) \] (1.1)
is a deterministic function to match the initial yield curve, \( a \) and \( \sigma \) are positive constants, \((W_t)\) is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). It can be shown that on average, \( r \) follows the slope of the initial instantaneous forward rate curve. When \( r \) deviates from that curve, it reverts back at rate \( a \).

1.2. Jump-diffusion models and Hawkes processes. In real world, it is often observed that interest rate has jumps. And usually they tend to occur in clusters. For example, in 2021, the US economy was hit hard by the spread of Covid-19, so the Federal Reserve System unwinded the quantitative easing as a response. In 2022, due to the improvement of epidemic situation and high inflation caused by QE, the Federal Reserve System raised the interest rates by 25 and 75 basis points in March and June. Mathematically, it can be recognized as clusters, because the majority of continuous time models for interest rates are driven by Brownian motion or other Lévy processes which cannot capture cluster effects because of the independence of their increments. A useful approach to characterize cluster effects is using the original diffusion process with Hawkes jumps.

A Hawkes process is a simple point process \( N \) admitting an intensity
\[ \lambda_t := \lambda(\int_{-\infty}^{t} h(t-s)N(ds)), \]
where \( \lambda(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) is locally integrable, left continuous, \( h(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \), with \( ||h||_{L^1} = \int_0^\infty h(t)dt < \infty \). \( \int_{-\infty}^{t} h(t-s)N(ds) = \sum_{\tau < t} h(t-\tau) \), here \( \tau \)'s are the occurrences of the points before time \( t \). We always assume that \( N(-\infty, 0] = 0 \), i.e. the Hawkes process has empty history. \( h(\cdot) \) and \( \lambda(\cdot) \) are usually referred to as an exciting function and an intensity function, respectively.

A Hawkes process is linear if \( \lambda(\cdot) \) is linear and is nonlinear otherwise. The linear Hawkes process is actually the classical Hawkes process, which named after Hawkes later, was firstly presented in [20]. The nonlinear Hawkes process was firstly introduced by Brémaud and Massoulié[22].

Recently, Zhu[7, 8, 9, 10, 11] investigated several results for both linear and nonlinear model. We are concerned that the central limit theorem was in [12] and the large deviation principles have been discussed in [10].

1.3. Large deviation theory. Large deviation theory is an important theory which studies the asymptotic computation of small probabilities on exponential scale. It characterizes the limiting behavior of a family of probability measures \( \{\mu_\epsilon\} \) in terms of a rate function (as \( \epsilon \to 0 \)). We say \( \{\mu_\epsilon, \epsilon > 0\} \) satisfies the large deviation principle with a rate function \( I \) if, for all \( A \in \mathcal{B} \),
\[- \inf_{x \in A^c} I(x) \leq \liminf_{\epsilon \to 0} \{ \epsilon \log \mu_\epsilon(A) \} \leq \limsup_{\epsilon \to 0} \{ \epsilon \log \mu_\epsilon(A) \} \leq - \inf_{x \in \hat{A}} I(x),\]
where \( \hat{A} \) denotes the interior of \( A \) and \( \hat{A} \) denotes its closure.

There are many ways to calculate the rate function of a family of random variables. In this paper, we mainly use the Gärtner-Ellis theorem. We refer the readers to [4] for general background on the theory and applications of large deviations.

1.4. Hull-White model with Hawkes jumps. In this paper, we consider Hull-White model with jumps as follows,
\[ r_t = \int_0^t (\theta(u) - ar_u)du + \sigma W_t + \sum_{i=1}^{N_t} Y_i, \] (1.2)
where \( \{Y_i, i \geq 1\} \) are i.i.d. \( L^4 \)-random variables on \( (\Omega, \mathcal{F}, \mathbb{P}) \), \( (N_t) \) is a Hawkes process with the following intensity function
\[
\lambda_t := \alpha + \beta r_t,
\]
where \( \alpha, \beta > 0 \) are constants.

**Assumption 1.1.** In (1.2), we assume that
1. \( \theta(t) \in C^1[0, \infty); \)
2. \( \theta(t) \) has bounded variation in \([0, \infty);\)
3. There exists a constant \( \gamma > 0 \), such that
\[
\lim_{t \to \infty} \theta(t) = \gamma.
\]

**Remark 1.2.** Let’s explain the rationality of (3) above. In the classical Hull-White model, the risk-neutral pricing is given as follows:
\[
P(0, T) = \mathbb{E}[e^{-\int_0^T r_s ds}],
\]
and the instantaneous forward rate
\[
F(0, T) = -\frac{\partial}{\partial T} \log P(0, T).
\]

If \( r_s \) has mean reverting property, \( \frac{1}{T} \log P(0, T) \approx c + o(1/T) \) as \( T \) is sufficiently large. Hence, \( F_T(0, T) = -\frac{\partial^2}{\partial T^2} \log P(0, T) \to 0 \) as \( T \to \infty \). It is natural to assume \( \theta(t) \) converging to a constant \( \gamma \) as \( t \to \infty \).

For the mean reverting property of our model, we establish the following assumption.

**Assumption 1.3.** The constants \( a \) and \( \beta \) satisfy
\[
a > \beta \mathbb{E}[Y].
\]

2. Main results

In this section, we would consider our main results.

**Proposition 2.1.** For any given initial value \( r_0 \in \mathbb{R}^+ \), we have the conditional expectation
\[
\mathbb{E}[r_t | r_0] = e^{(\beta \mathbb{E}[Y] - a)t} \left( \int_0^t (\alpha \mathbb{E}[Y] + \theta(u)) e^{(\alpha - \beta \mathbb{E}[Y])u} du + r_0 \right)
\]
\[
= \frac{\alpha \mathbb{E}[Y] + \theta(t)}{a - \beta \mathbb{E}[Y]} - \frac{e^{(\beta \mathbb{E}[Y] - a)t}}{a - \beta \mathbb{E}[Y]} \int_0^t e^{(a - \beta \mathbb{E}[Y])u} d\theta(u) + e^{(\beta \mathbb{E}[Y] - a)t} r_0.
\]

Proof. From (1.2), we obtain
\[
\frac{d\mathbb{E}[r_t]}{dt} = \alpha \mathbb{E}[Y] + \theta(t) + (\beta \mathbb{E}[Y] - a) \mathbb{E}[r_t].
\]
Solving this ODE, we obtain the conditional expectation
\[
\mathbb{E}[r_t | r_0] = e^{(\beta \mathbb{E}[Y] - a)t} \left( \int_0^t (\alpha \mathbb{E}[Y] + \theta(u)) e^{(a - \beta \mathbb{E}[Y])u} du + r_0 \right)
\]
\[
= \frac{\alpha \mathbb{E}[Y] + \theta(t)}{a - \beta \mathbb{E}[Y]} - \frac{e^{(\beta \mathbb{E}[Y] - a)t}}{a - \beta \mathbb{E}[Y]} \int_0^t e^{(a - \beta \mathbb{E}[Y])u} d\theta(u) + e^{(\beta \mathbb{E}[Y] - a)t} r_0.
\]

\( \square \)
By using Itô’s formula in (1.2),
\[ d(r_t^2) = 2r_t((θ(t) - a)r_t)dt + σdW_t + YdN_t + (σ^2dt + Y^2dN_t). \]

Taking expectation on both sides, we obtain
\[ \frac{dE[r_t^2]}{dt} = -2aE[r_t^2] + 2θ(t)E[r_t] + 2E[Y]E[r_t](α + βE[r_t]) + E[Y^2](α + βE[r_t]) + σ^2 \]
\[ = -2aE[r_t^2] + (2θ(t) + 2αE[Y] + 2βE[Y^2])E[r_t] + αE[Y^2] + σ^2 \]

Since \( θ(t) \) is differential and has bounded variation in \([0, ∞)\), we obtain the boundedness of \( \int_0^t e^{(a-βE[Y])u}dθ(u), E[r_t|r_0] \) and \( E[r_t^2|r_0] \). Hence, the following theorem is established.

**Theorem 2.2. (Law of large numbers)** For any given initial value \( r_0 \in \mathbb{R}^+ \),

(1) \[ \frac{1}{t} \int_0^t r_u du \rightarrow \frac{αE[Y] + γ}{α - βE[Y]} \text{ as } t \rightarrow ∞. \]

(2) \[ \frac{N_t}{t} \rightarrow \frac{αα + βγ}{α - βE[Y]} \text{ as } t \rightarrow ∞. \]

**Proof.** (1) \[ E[(\frac{1}{t} \int_0^t r_u du - \frac{αE[Y] + γ}{α - βE[Y]}]^2) \]
\[ = E[(\frac{1}{t} \int_0^t r_u du)^2] - \frac{2}{t} \int_0^t E[r_u|r_0]du \cdot \frac{αE[Y] + γ}{α - βE[Y]} + (\frac{αE[Y] + γ}{α - βE[Y]})^2. \]

From (2.1), we obtain
\[ \lim_{u \rightarrow ∞} E[r_u|r_0] = \frac{αE[Y] + γ}{α - βE[Y]}, \]
and so
\[ \lim_{t \rightarrow ∞} \frac{1}{t} \int_0^t E[r_u|r_0]du = \frac{αE[Y] + γ}{α - βE[Y]}. \]

We just need to show
\[ \lim_{t \rightarrow ∞} E[(\frac{1}{t} \int_0^t r_u du)^2] = (\frac{αE[Y] + γ}{α - βE[Y]})^2. \]

In fact,
\[ E[(\frac{1}{t} \int_0^t r_u du)^2] = \frac{2}{t^2} \int_{0<s_1<s_2<t} E[r_{s_1}|E[r_{s_2}|r_{s_1}]]ds_1ds_2 \]
\[ = \frac{2}{t^2} \int_{0<s_1<s_2<t} \{ E[Y] - \frac{E[Y]}{α - βE[Y]} \cdot \frac{e^{(βE[Y] - a)(s_2-s_1)}}{α - βE[Y]} \int_{s_1}^{s_2} e^{(α-βE[Y])(u-s_1)}dθ(u)E[r_{s_1}] \}
+ e^{(βE[Y]-α)(s_2-s_1)}E[r_{s_1}^2]]ds_1ds_2. \]

By the boundedness of \( \int_0^t e^{(a-βE[Y])u}dθ(u), E[r_t|r_0] \) and \( E[r_t^2|r_0] \), we obtain
\[ \lim_{t \rightarrow ∞} E[(\frac{1}{t} \int_0^t r_u du)^2] = \lim_{t \rightarrow ∞} \frac{2}{t^2} \int_{0<s_1<s_2<t} \frac{αE[Y] + θ(s_2)}{α - βE[Y]} ds_1ds_2 \]
\[ = (\frac{αE[Y] + γ}{α - βE[Y]})^2. \]
Hence,

\[ \mathbb{E}\left( \frac{N_t - \int_0^t \lambda_s ds}{t} \right)^2 = \frac{1}{t^2} \int_0^t \lambda_s ds = \frac{\alpha}{t} + \frac{\beta}{t} \int_0^t \mathbb{E}[r_s] ds \]

\[ = \frac{\alpha}{t} + \frac{\beta}{t^2} \int_0^t \left( \frac{\alpha \mathbb{E}[Y] + \theta(s)}{a - \beta \mathbb{E}[Y]} \right) \left( \frac{e^{(\beta \mathbb{E}[Y] - a)s} - e^{(\beta \mathbb{E}[Y] - a)s_0}}{a - \beta \mathbb{E}[Y]} \right) ds \rightarrow 0. \]

Hence,

\[ \frac{N_t}{t} - \frac{\alpha}{t} \frac{t}{\int_0^t r_s ds} \rightarrow 0. \]

By the law of large numbers of \( \frac{1}{t} \int_0^t r_s ds \), we complete the proof. \( \square \)

**Theorem 2.3.** (Central limit theorem) For any given initial value \( r_0 \in \mathbb{R}^+ \),

\[ \frac{1}{\sqrt{t}} \left( \sum_{i=1}^{N_t} Y_i - \frac{a(\alpha t + \beta \gamma t)}{a - \beta \mathbb{E}[Y]} \right) \rightarrow N\left( 0, \frac{\sigma^2 \beta^2 (\mathbb{E}[Y])^2}{(a - \beta \mathbb{E}[Y])^2} + \frac{(\alpha + \beta \gamma) \mathbb{E}[Y]^2}{(a - \beta \mathbb{E}[Y])^2} \right) \]

in distribution as \( t \rightarrow \infty \).

**Proof.** (1) Take \( f(t, r) = Kr \), with \( K \) a constant, \( f(t, r_t) - f(0, r_0) - \int_0^t (\mathcal{A}_t f)(r_s) ds \) is a martingale. So the corresponding generator is

\[ (\mathcal{A}_t f)(r) = K((\beta \mathbb{E}[Y] - a)r + \alpha \mathbb{E}[Y] + \theta(t)). \]

If \( K = 1/(a - \beta \mathbb{E}[Y]) \), then

\[ (\mathcal{A}_t f)(r) = -r + \frac{\alpha \mathbb{E}[Y] + \theta(t)}{a - \beta \mathbb{E}[Y]]. \]

Hence,

\[ \int_0^t (r_u - \frac{\alpha \mathbb{E}[Y] + \theta(u)}{a - \beta \mathbb{E}[Y]} \) du = \int_0^t (\mathcal{A}_t f)(r_u) du \]

\[ = f(t, r_t) - f(0, r_0) - \int_0^t (\mathcal{A}_t f)(r_u) du + f(0, r_0) - f(t, r_1). \]

By the boundedness of \( \theta(t) \),

\[ \mathbb{E}[f(t, r_t)] \]

\[ \frac{1}{\sqrt{t}} = \frac{K}{\sqrt{t}} \left( \frac{\alpha \mathbb{E}[Y] + \theta(t)}{a - \beta \mathbb{E}[Y]} \right) \frac{e^{(\beta \mathbb{E}[Y] - a)t} - 1}{a - \beta \mathbb{E}[Y]} \int_0^t e^{(\beta \mathbb{E}[Y] - a)t} d\theta(u) + e^{(\beta \mathbb{E}[Y] - a)t} r_0 \]

\[ \rightarrow 0, \]

as \( t \rightarrow \infty \). The quadratic variation of \( f(t, r_t) - f(0, r_0) - \int_0^t (\mathcal{A}_t f)(r_s) ds \) is the same as that of \( f(t, r_t) \), which is just the quadratic variation of

\[ \frac{1}{(a - \beta \mathbb{E}[Y])^2} \left( \sum_{i=1}^{N_t} Y_i^2 \right) + \sigma^2 t. \]

By the law of large numbers,

\[ \frac{N_t}{t} \sim \frac{a \alpha + \beta \gamma}{a - \beta \mathbb{E}[Y]} \text{ as } t \rightarrow \infty, \]

\[ \frac{1}{N_t} \mathbb{E}\left( \sum_{i=1}^{N_t} Y_i^2 \right) \sim \mathbb{E}[Y^2] \text{ as } t \rightarrow \infty, \]
so
\[
\frac{1}{t} \sum_{i=1}^{N_t} Y_i^2 = \frac{N_t}{t} \frac{1}{N_t} \sum_{i=1}^{N_t} Y_i^2 \xrightarrow{p} \frac{a\alpha + \beta\gamma}{a - \beta \mathbb{E}[Y]} E[Y^2]
\]
\[
\frac{1}{(a - \beta \mathbb{E}[Y])^2} (\sum_{i=1}^{N_t} Y_i^2 + \sigma^2 t) \xrightarrow{p} \frac{(a\alpha + \beta\gamma) E[Y^2] + \sigma^2 (a - \beta \mathbb{E}[Y])}{(a - \beta \mathbb{E}[Y])^3} t.
\]
Hence, by using the usual central limit theorem for martingales, we complete the first proof.

(2) From (1.2),
\[
\sum_{i=1}^{N_t} Y_i = r_t - \int_0^t (\theta(u) - a r_u) du - \sigma W_t.
\]
We have
\[
\sum_{i=1}^{N_t} Y_i - \frac{a\alpha t + \beta \int_0^t \theta(u) du}{\mathbb{E}[Y]} = r_t - \sigma W_t - a \int_0^t (-r_u + \frac{\alpha \mathbb{E}[Y] + \theta(u)}{a - \beta \mathbb{E}[Y]}) du
\]
\[
= r_t - \sigma W_t - a \int_0^t (\omega_u f)(r_u) du
\]
\[
= r_t - a f(t, r_t) + a f(0, r_0) + a (f(t, r_t) - f(0, r_0)) - \int_0^t (\omega_u f)(r_u) du - \sigma W_t.
\]
From the argument in (1), we notice
\[
\frac{1}{\sqrt{t}} (r_t - a f(t, r_t) + a f(0, r_0)) \xrightarrow{p} 0
\]
as \(t \to \infty\) and
\[
a (f(t, r_t) - f(0, r_0) - \int_0^t (\omega_u f)(r_u) du) - \sigma W_t
\]
is a martingale whose quadratic variation is the same as that of
\[
a (f(t, r_t) - \sigma W_t = \frac{a}{a - \beta \mathbb{E}[Y]} r_t - \sigma W_t,
\]
and the latter is the same as the quadratic variation of
\[
\frac{\beta \mathbb{E}[Y]}{a - \beta \mathbb{E}[Y]} \sigma W_t + \frac{a}{a - \beta \mathbb{E}[Y]} \sum_{i=1}^{N_t} Y_i.
\]
By the strong law of numbers,
\[
\frac{1}{n} \sum_{i=1}^{N_t} Y_i^2 \xrightarrow{a.s.} E[Y^2].
\]
Hence, by the law the large numbers of \(N_t/t\), we obtain
\[
\frac{1}{t} \sum_{i=1}^{N_t} Y_i^2 = \frac{N_t}{t} \frac{1}{N_t} \sum_{i=1}^{N_t} Y_i \xrightarrow{p} \frac{(a\alpha + \beta\gamma) E[Y^2]}{a - \beta \mathbb{E}[Y]}.
\]
The quadratic variation of
\[
\frac{1}{\sqrt{t}} \left( \frac{\beta \mathbb{E}[Y]}{a - \beta \mathbb{E}[Y]} \sigma W_t + \frac{a}{a - \beta \mathbb{E}[Y]} \sum_{i=1}^{N_t} Y_i \right)
\]
is
\[ \frac{\sigma^2 \beta^2 (E[Y])^2}{(a - \beta E[Y])^2} + \frac{(a \alpha + \beta r)E[Y^2]}{a - \beta E[Y]} . \]
Hence, by the usual central limit theorem for martingales, we complete the proof.

Set \( y_c \) satisfies
\[ E[Y e^{y_c Y}] = a/\beta, \]
and \( \Pi(dx) \) the probability measure of \( Y \).

**Theorem 2.4.** (Large deviation principle) \(((1/t) \int^t_0 u_s ds \in \cdot) \) satisfies large deviation principle with rate function
\[ I(x) = \sup_{\lambda \leq \lambda_c} \{ \lambda x - y(\lambda) - \frac{1}{2} \sigma^2 y^2(\lambda) - a \mathbb{E}[e^{y(\lambda)Y}] + \alpha \}, \]
where, for \( \lambda < \lambda_c \), \( y = y(\lambda) \) is the smaller solution of
\[ -ay + \beta \mathbb{E}[e^{yY}] - \beta + \lambda = 0, \]
and
\[ \lambda_c = ay_c - \beta \mathbb{E}[e^{y_c Y}] + \beta. \]

**Proof.** For any \( r_0 := r \in \mathbb{R}^+ \), we first calculate the generator of the process
\[ (\mathcal{A} f)(r) = \frac{\partial f}{\partial r}(\theta(t) - ar) + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial r^2} + (\alpha + \beta r)(E[f(t, r + Y)] - f(t, r)). \] (2.2)
Let \( u(\lambda, t, r) = \mathbb{E}[\exp\{\lambda \int^t_0 r_s ds\}] \), using Feynman-Kac formula, we obtain
\[ \frac{\partial u}{\partial t}|_{t=T-s} = \frac{\partial u}{\partial r}|_{t=T-s}(\theta(s) - ar) + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial r^2}|_{t=T-s} \]
\[ + (\alpha + \beta r)(\mathbb{E}[u(\lambda, T - s, r + Y)] - u(\lambda, T - s, r)) + \lambda ru(\lambda, T - s, r), s \in [0, T] \]
with initial condition
\[ u(\lambda, 0, r) = 1. \]
Let us choose
\[ u(\lambda, t, r) = e^{Q(t)r + R(t)}, \]
then
\[ \begin{cases} Q'(T - s) = -aQ(T - s) + \beta \mathbb{E}[e^{Q(T - s)Y}] - \beta + \lambda, \\ R'(T - s) = Q(T - s)\theta(s) + \frac{1}{2} \sigma^2 Q^2(T - s) + a \mathbb{E}[e^{Q(T - s)Y}] - \alpha, \\ Q(0) = R(0) = 0. \end{cases} \]
Set \( s = \frac{1}{2} T \), and let \( T \to \infty \). It is easy to see that \( \lim_{s \to \infty} Q(s) = y \), where \( y \) satisfies
\[ -ay + \beta \mathbb{E}[e^{yY}] - \beta + \lambda = 0 \] (2.3)
if the equation has a solution, and \( \lim_{s \to \infty} Q(t) = \infty \) otherwise.
Let
\[ F(x) := -ax + \beta \mathbb{E}[e^{xY}] - \beta + \lambda. \]
Obviously, \( F(x) \) is a convex function. We note that
\[ \lambda_c := \max_{y \in \mathbb{R}^+} \{ ay - \beta \mathbb{E}[e^{yY}] + \beta \} = ay_c - \beta \mathbb{E}[e^{y_c Y}] + \beta. \]
(1) When $\lambda < \lambda_c$, the equation has two solutions. Especially,
   (i) If $\lambda < 0$, due to $F(0) = \lambda, F(-\infty) = F(\infty) = \infty$, the equation has a positive solution and a negative solution. So $Q(t)$ will converge to the negative one. In this case, denote the negative solution as $y(\lambda)$.
   (ii) If $0 < \lambda < \lambda_c$, the equation has two positive solutions. $Q(t)$ will converge to the smaller one. In this case, denote the smaller positive solution as $y(\lambda)$.

(2) When $\lambda = \lambda_c$, the equation has a unique solution.

(3) When $\lambda > \lambda_c$, the equation has no solution. In this case, $Q(t)$ will go to infinity.

Hence, we obtain

$$ \lim_{t \to \infty} \frac{1}{t} \log u(\lambda, t, r) = \begin{cases} 
   y(\lambda)\gamma + \frac{1}{2}\sigma^2 y^2(\lambda) + \alpha E[e^{y(\lambda)Y}] - \alpha & \text{if } \lambda \leq \lambda_c, \\
   +\infty & \text{otherwise}. 
\end{cases} $$

Write the above RHS as $\Gamma(\lambda)$ for $\lambda \leq \lambda_c$, in this case it is obviously differential, and differentiating \( (2.3) \) with respect to $\lambda$ we obtain

$$ \frac{dy}{d\lambda} = \frac{1}{a - \beta \hat{\mu}} \to \infty $$

as $\lambda \uparrow \lambda_c$, since $y \uparrow y_c$ as $\lambda \uparrow \lambda_c$. Therefore, we have the essential smoothness and, by G"artner-Ellis theorem, ((1/t) $\int_0^t r_u ds \in \cdot$) satisfies large deviation principle. The rate function is

$$ I(x) = \sup_{\lambda \leq \lambda_c} \{\lambda x - y(\lambda)\gamma - \frac{1}{2}\sigma^2 y^2(\lambda) - \alpha E[e^{y(\lambda)Y}] + \alpha\}. $$

\[\square\]

### 3. Examples

In this section, we give two examples for random variable $Y$ to show quantities of several limit behaviors.

#### 3.1. $Y$ follows a normal distribution.

Let us assume that $Y$ follows a normal distribution $N(\hat{\mu}, \hat{\sigma}^2)$. That is

- $\mathbb{E}[Y] = \hat{\mu}$, $\text{Var}[Y] = \hat{\sigma}^2$, $\mathbb{E}[Y^2] = \hat{\mu}^2 + \hat{\sigma}^2$.

1. (Law of large numbers)

   $$ \frac{1}{t} \int_0^t r_u du \xrightarrow{L^2} \frac{\alpha \hat{\mu} + \gamma}{a - \beta \hat{\mu}} \text{ as } t \to \infty. $$

   $$ N_t \xrightarrow{L^2} \frac{\alpha a + \beta \gamma}{a - \beta \hat{\mu}} \text{ as } t \to \infty. $$

2. (Central limit theorem)

   $$ \frac{\int_0^t r_u du - \frac{\alpha \hat{\mu} + \gamma}{a - \beta \hat{\mu}}}{\sqrt{t}} \xrightarrow{\text{in distribution}} N(0, \frac{(\hat{\mu}^2 + \hat{\sigma}^2)(a \alpha + \beta \gamma) + \sigma^2(a - \beta \hat{\mu})}{(a - \beta \hat{\mu})^3}) $$

   in distribution as $t \to \infty$.

   $$ \frac{1}{\sqrt{t}} \sum_{i=1}^{N_t} Y_i - \frac{\alpha a + \beta \gamma t}{a - \beta \hat{\mu}} \xrightarrow{\text{in distribution}} N(0, \frac{\alpha^2 \beta^2 \hat{\mu}^2}{(a - \beta \hat{\mu})^2} + \frac{(a \alpha + \beta \gamma)(\hat{\mu}^2 + \hat{\sigma}^2)}{a - \beta \hat{\mu}}) $$

   in distribution as $t \to \infty$. 
(3) (Large deviation principle) \(((1/t) \int_0^t r_s ds = \cdot)\) satisfies large deviation principle, for \(\lambda \leq \lambda_c\) with
\[
\lambda_c = ay_c - \beta \exp\left\{\frac{2y_c\mu\sigma^2 + y_c^2\sigma^4}{2\sigma^2}\right\} + \beta,
\]
and \(y_c\) satisfies
\[
(\mu + \sigma^2 y_c) \exp\left\{\frac{2\mu\sigma^2 y_c + \sigma^4 y_c^2}{2\sigma^2}\right\} = \frac{\alpha}{\beta}.
\]
The rate function is
\[
I(x) = \sup_{\lambda \leq \lambda_c} \{\lambda x - y(\lambda)\gamma - \frac{1}{2}\sigma^2 y^2(\lambda) - \alpha \exp\left\{\frac{2y(\lambda)\mu\sigma^2 + y(\lambda)^2\sigma^4}{2\sigma^2}\right\} + \alpha\},
\]
where \(y = y(\lambda)\) is the smaller solution of the following equation
\[
-ay + \beta \exp\left\{\frac{2y\mu\sigma^2 + y^2\sigma^4}{2\sigma^2}\right\} - \beta - \lambda = 0.
\]

3.2. \(Y\) follows a double exponential distribution. Let us assume that \(Y\) follows a double exponential distribution with \(f(x|\mu, \sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}\). That is
\[
E[Y] = \mu, \ Var[Y] = 2\sigma^2, E[Y^2] = \mu^2 + 2\sigma^2.
\]

(1) (Law of large numbers)
\[
\frac{1}{t} \int_0^t r_s du \xrightarrow{L^2} \frac{\alpha\mu + \gamma}{a - \beta\mu} \text{ as } t \to \infty.
\]
\[
\frac{N_t}{t} \xrightarrow{L^2} \frac{\alpha\sigma + \beta\gamma}{a - \beta\mu} \text{ as } t \to \infty.
\]

(2) (Central limit theorem)
\[
\frac{\int_0^t r_s du - \frac{\alpha\hat{\mu} + \gamma t}{a - \beta\hat{\mu}}}{\sqrt{t}} \xrightarrow{D} N(0, \frac{(\mu^2 + 2\sigma^2)(a\alpha + \beta\gamma) + \sigma^2(a - \beta\mu)}{(a - \beta\mu)^3})
\]
in distribution as \(t \to \infty\).
\[
\frac{1}{\sqrt{t}} \sum_{i=1}^{N_t} Y_i - \frac{a\alpha \gamma + \gamma t}{a - \beta\mu} \xrightarrow{D} N(0, \frac{\sigma^2\beta^2\mu^2}{(a - \beta\mu)^2} + \frac{(a\alpha + \beta\gamma)(\mu^2 + 2\sigma^2)}{a - \beta\mu})
\]
in distribution as \(t \to \infty\).

(3) (Large deviation principle) \(((1/t) \int_0^t r_s ds = \cdot)\) satisfies large deviation principle, for \(\lambda \leq \lambda_c\) with
\[
\lambda_c = ay_c - \beta \exp\left\{\frac{\beta e^{y_c} \mu}{1 - \sigma^2 y_c^2}\right\} + \beta,
\]
and \(y_c\) satisfies
\[
e^{\hat{y}_c} \frac{\sigma^2}{2\sigma^2} \frac{1}{(1 - \sigma^2 y_c)^2} - \frac{\sigma^2}{(1 + \sigma^2 y_c)^2} + \frac{\hat{\sigma}\mu}{1 - \sigma y_c} + \frac{\hat{\sigma}\mu}{1 + \sigma y_c} = a/\beta.
\]
The rate function is
\[
I(x) = \sup_{\lambda \leq \lambda_c} \{\lambda x - y(\lambda)\gamma - \frac{1}{2}\sigma^2 y^2(\lambda) - \frac{\alpha e^{y(\lambda)\mu}}{1 - \sigma^2 y(\lambda)^2} + \alpha\},
\]
where \(y = y(\lambda)\) is the smaller solution of the following equation
\[
-ay + \frac{\beta e^{y\mu}}{1 - \sigma^2 y^2} - \beta + \lambda = 0.
\]
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