ON LUSTERNIK-SCHNIRELMANN CATEGORY OF CONNECTED SUMS

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ABSTRACT. In this paper we estimate the Lusternik-Schnirelmann category of the connected sum of two manifolds through their categories. We achieve a more general result regarding the category of a quotient space $X/A$ where $A$ is a suitable subspace of $X$.

1. INTRODUCTION

1.1. Definition. The Lusternik-Schnirelmann category (LS category) of a space $X$ is the smallest nonnegative integer $n$ such that there exists $\{A_0, A_1, ..., A_n\}$, an open cover of $X$ with each $A_i$ contractible in $X$. This is denoted by $n = \text{cat } X$.

Following this definition, spaces with LS category 0 are contractible.

The goal of the paper is to prove the inequality

$$\max\{\text{cat } M, \text{cat } N\} - 1 \leq \text{cat}(M \# N) \leq \max\{\text{cat } M, \text{cat } N\},$$

where $M$ and $N$ are closed manifolds.

To prove the inequality, we consider a more general problem about the relation of $\text{cat } X$ and $\text{cat}(X/A)$. This problem is indeed more general: in fact, put $X = M \# N$ and $A$ be an $(n - 1)$ sphere that separates $M$ and $N$ (with removed discs). Then $X/A = M \lor N$ and $\text{cat}(M \lor N) = \max\{\text{cat } M, \text{cat } N\}$.

All spaces are assumed to be CW spaces.

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2. PRELIMINARIES

2.1. Definition. For a path-connected space $X$ with basepoint $x_0$, we define $PX$ to be the set of all continuous functions $\gamma : I \to X$ satisfying $\gamma(0) = x_0$ topologized by the compact-open topology.

We then define $p : PX \to X$ given by $p(\gamma) = \gamma(1)$, a fibration with base space $X$ and fiber $\Omega(X)$, the loop space of $X$.

Given $f : Y \to X$ and $g : Z \to X$ we can define $Y *_X Z = \{(y, z, t) \in Y * Z | f(y) = g(z)\}$ and $(f *_X g) : Y *_X Z \to X$ by $(f *_X g)(y, z, t) = f(y)$.

From this, we define $P_n X$ to be the fiberwise join of $n$ copies of $PX$ over $p : PX \to X$ defined above and denote the fiberwise map as

$$p_n^X : P_n X \to X.$$
Note that the (homotopy) fiber of $p^n_X$ is $\Omega(X)^n$, the $n$-fold join of $\Omega(X)$.

We need the following theorem of Schwarz, see [2, 16].

2.2. **Theorem.** The inequality $\text{cat}(X) \leq n$ holds iff there exists a section $s : X \to P_{n+1}X$ to $p : P_{n+1}X \to X$.

2.3. **Remark.** These claims are well-known, we list them here for reference.

(1) $\text{cat}(X \vee Y) = \max\{\text{cat } X, \text{cat } Y\}$;

(2) $\text{cat}(X \cup Y) \leq \text{cat}(X) + \text{cat}(Y) + 1$;

(3) $\text{cat}(X/A) - 1 \leq \text{cat } X$. This follows from the fact that $X/A$ has homotopy type of $X \cup CA$, the union of $X$ with the cone over $A$, and item (2).

It should be noted that Berstein and Hilton explored the changes in category of a space after attaching a cone in [1] following Hilton’s exploration of what’s now known as the Hilton-Hopf invariant in [8].

3. **Main results**

3.1. **Proposition.** The inequality (1.1) holds whenever $\max\{\text{cat } M, \text{cat } N\} \leq 2$

*Proof.* If $\text{cat } M = 1 = \text{cat } N$ then $M$ and $N$ are homotopy spheres, and so $M\#N$ is. Conversely, if $\text{cat } M\#N = 1$ then $M$ and $N$ must be homotopy spheres. Thus, we proved that $\text{cat}(M\#N) = \max\{\text{cat } M, \text{cat } N\}$ if $\max\{\text{cat } M, \text{cat } N\} \leq 2$. □

3.2. **Theorem.** Suppose $X$ is an $n$-dimensional space with $m$-connected subspace $A$, with $3 \leq \text{cat}(X/A) \leq k$, and $k + m - 1 \geq n$. Then $\text{cat } X \leq k$.

*Proof.* For sake of simplicity, put $p = p^X_{k+1}$ and $p' = p^X_{k+1}$, cf. (2.2). As $\text{cat}(X/A) \leq k$, and by Theorem 2.2 there exists the following section $s$ with $ps = 1_{X/A}$.

\[
\begin{array}{c}
P_{k+1}(X/A) \\
p \\
X/A
\end{array}
\]

Now we consider the collapsing map $q : X \to X/A$, and get the fiber-pullback diagram.

\[
(3.1)
\begin{array}{ccc}
E & \xrightarrow{f} & P_{k+1}(X/A) & \xrightarrow{P_{k+1}(X/A)} \\
\downarrow & & \downarrow p & \uparrow s \\
X & \xrightarrow{p} & X/A & \xrightarrow{s} X/A
\end{array}
\]

Now consider $P_{k+1}X$. We already have $p' : P_{k+1}X \to X$, and the collapsing map $q : X \to X/A$ induces a map $q' : P_{k+1}X \to P_{k+1}(X/A)$. Since $pq' = gp'$ and the square is the pull-back diagram, we get a map $h : P_{k+1} \to X$ such that the following diagram commutes.
Recall that our goal is to prove $\text{cat} X \leq k$. Because of Schwarz’s Theorem \[2.2\] it suffices to construct a section of $p'$. To do this, it suffices in turn to construct a section of the map $h : P_{k+1}(X) \to E$. Moreover, since $\dim X = n$, it suffices to construct a section of $h$ over the $n$-skeleton $E^{(n)}$ of $E$, i.e., to construct a map $\phi : E^{(n)} \to P_{k+1}(X)$ with $h\phi = 1_E$.

By homotopy excision \[7\ Prop. 4.28\], and because $A$ is $m$-connected, the quotient map $q : X \to X/A$ induces isomorphisms $q_* : \pi_n(X) \to \pi_n(X/A)$ for $n \leq m$ and epimorphism for $n = m + 1$. So, $\pi_n(\Omega X) \to \pi_n(\Omega(X/A))$ is an isomorphism for $n \leq (m-1)$ and epimorphism for $n = m$. Therefore $(\Omega X)^{(k+1)} \to (\Omega(X/A))^{(k+1)}$ is an isomorphism for $n \leq m+k$ because of \[3\ Prop. 5.7\].

The long exact sequence of homotopy groups for a fibration yields the following commutative diagram

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_i(\Omega X)^{(k+1)} & \longrightarrow & \pi_i(P_{k+1}X) & \longrightarrow & \pi_i(X) & \longrightarrow & \cdots \\
| & & | & & | & & | & & | \\
\cong & & \downarrow h_* & & \cong & & \downarrow h_* & & \cong \\
\cdots & \longrightarrow & \pi_i(\Omega(\Omega X)^{(k+1)}) & \longrightarrow & \pi_i(E) & \longrightarrow & \pi_i(X) & \longrightarrow & \cdots
\end{array}
$$

By the 5-lemma, the map $h_*$ is an isomorphism for $i \leq (m+k-1)$ and epimorphism for $n = m+k$. So by Whitehead’s theorem, there exists a map $\phi : E^{(n)} \to P_{k+1}X$. Now, the composition $(\phi \circ s')$ is a section to $p' : P_{k+1} \to X$. Thus $\text{cat} X \leq k$. \qed

Combining this with the previous Remark \[2.3\] gives the following inequality:

$$\text{cat}(X/A) - 1 \leq \text{cat}(X) \leq \text{cat}(X/A)$$

under the dimension-connectivity conditions from Theorem \[3.2\].

Consider the case where $X = M \# N$, the connected sum of $n$-dimensional manifolds $M$ and $N$, and $A = S^{n-1}$ is the separating sphere between $M$ and $N$. Then $X/A = M \lor N$, and $\text{cat}(X/A) = \max\{\text{cat} M, \text{cat} N\}$. We have $A$ is $(n-2)$-connected, and we can assume $\text{cat} M, \text{cat} N \geq 3$ because of Proposition \[3.1\]. Then $\text{cat}(X/A) \geq 3$, and so as $(n-2)+3-1 \geq n$, we are in the case of Theorem \[3.2\] and get the following corollary.

**3.3. Corollary.** There is a double inequality

$$\max\{\text{cat} M, \text{cat} N\} - 1 \leq \text{cat}(M \# N) \leq \max\{\text{cat} M, \text{cat} N\}.$$  

**Proof.** Consider the case where $X = M \# N$, the connected sum of $n$-dimensional manifolds $M$ and $N$, and $A = S^{n-1}$ is the separating sphere between $M$ and $N$. Then $X/A = M \lor N$, and $\text{cat}(X/A) = \max\{\text{cat} M, \text{cat} N\}$. We have $A$ is $(n-2)$-connected, and we can assume
cat $M$, cat $N \geq 2$ because of Proposition 3.1. Then cat$(X/A) \geq 3$, and so as $(n-2)+3-1 \geq n$, we are in the case of Theorem 3.2 and get the corollary. \hfill \square

3.4. Remark. In [6], an upper bound is given for the LS category of a double mapping cylinder. If we consider the connected sum of $n$-manifolds $M$ and $N$ as such a double mapping cylinder, then the following inequality is obtained:

\[
(3.2) \quad \text{cat } M \# N \leq \min\{1 + \text{cat } M' + \text{cat } N', 1 + \max\{\text{cat } M', \text{cat } N'\}\}.
\]

Here $M'$ and $N'$ are $M \smallsetminus \text{pt}$ and $N \smallsetminus \text{pt}$, respectively. Rivadeneya proved in [12] that category of a manifold without boundary does not increase when a point is removed. If the categories of $M$ and $N$ do decrease by one when a point is removed, then (3.2) has already established the main result here. However, in [11] a closed manifold is constructed so that the category remains unchanged after the deletion of a point, and Theorem 3.2 gives an improvement of the category estimate for such a case.

3.5. Remark. It is unknown if there is an example of two manifolds $M$ and $N$ such that cat $M \# N = \max\{\text{cat } M, \text{cat } N\} - 1$.

4. Connected sum and Toomer invariant

4.1. Definition. The Toomer invariant of $X$ $\epsilon(X)$ is the least integer $k$ for which the map $p^*_n : H^*(X) \to H^*(P_n(X))$ is injective, see [2]. It follows that $\epsilon(X) \leq \text{cat } X$.

4.2. Proposition. For closed and oriented manifolds $M$ and $N$, $\epsilon(M \# N) \geq \max\{\epsilon(M), \epsilon(N)\}$.

Proof. Consider $f : M \# N \to M$ the collapsing map onto $M$. Then we have the following diagram.

\[
\begin{array}{ccc}
H^*(P_n M) & \longrightarrow & H^*(P_n (M \# N)) \\
\uparrow & & \uparrow \\
H^*(M) & \longrightarrow & H^*(M \# N)
\end{array}
\]

This map has degree 1, and so $f^* : H^*(M) \to H^*(M \# N)$ is injective [13, Theorem V, 2.13]. Also suppose $p^*_n : H^*(M \# N) \to H^*(P_n (M \# N))$ is injective. Consider $u \in H^*(M)$. As $f^*$ and $p^*_n$ are injective, $p^*_n(u) \in H^*(P_n M)$ is nonzero, and so $p^*_n : H^*(M) \to H^*(P_n M)$ is injective, and similarly for $N$. And so $\epsilon(M \# N) \geq \max\{\epsilon(M), \epsilon(N)\}$.

\hfill \square

4.3. Proposition. For closed, oriented manifolds $M$ and $N$, if cat $M = \epsilon(M)$ and cat $N = \epsilon(N)$, then cat$(M \# N) = \max\{\text{cat } M, \text{cat } N\}$.

Proof. Combining the assumptions $\epsilon(M) = \text{cat } M$ and $\epsilon(N) = \text{cat } N$ with the inequality $\max\{\epsilon(M), \epsilon(N)\} \leq \text{cat}(M \# N) \leq \max\{\text{cat } M, \text{cat } N\}$, we have the claim. \hfill \square

Rudyak asked if the existence of a map $f : M \to N$, of degree 1, implies the inequality cat $M \geq \text{cat } N$ [14, 2 Open problem 2.48]. While not achieving the full result, he was able to prove some partial results. In particular it follows from the same injective property of $f^*$ (4.2) that $\epsilon(M) \geq \epsilon(N)$, when such a map exists [14].

4.4. Remark. We know $\epsilon(M \times S^n) \geq \epsilon(M)+1$, and there exist examples where $\text{cat}(M \times S^n) = \text{cat } M$ for suitable $M$ and $n$, [9], [10].
5. Rationalizations

Here we assume $X$ to be simply connected and denote by $X_\mathbb{Q}$ the rationalization of $X$, see [5], [15]. We define $e_\mathbb{Q}(X)$ to be the least integer $n$ such that the $n$th fibration $P_nX \to X$ induces an injection in cohomology with coefficients in $\mathbb{Q}$. For $X$ simply connected and of finite type, we have that $e_\mathbb{Q}(X) = e(X_\mathbb{Q})$, [2].

5.1. Proposition. For simply connected, CW spaces $X$ and $Y$, $(X \vee Y)_\mathbb{Q} \cong X_\mathbb{Q} \vee Y_\mathbb{Q}$.

Proof. In the following diagram, the map $l$ is the localization map of $X \vee Y$, and $k$ is given by the wedge of localization maps on $X$ and $Y$. The map $j$ exists by the universal property of $(X \vee Y)_\mathbb{Q}$, and induces isomorphisms in homology. Hence $X_\mathbb{Q} \vee Y_\mathbb{Q} \cong (X \vee Y)_\mathbb{Q}$.

\[
\begin{array}{ccc}
X \vee Y & \xrightarrow{l} & (X \vee Y)_\mathbb{Q} \\
\downarrow k & & \downarrow j \\
X_\mathbb{Q} \vee Y_\mathbb{Q} & \xleftarrow{j} & \\
\end{array}
\]

In [4] it is shown that for a closed, simply connected manifold $M$, $e(M) = e_\mathbb{Q}(M) = \text{cat}(M_\mathbb{Q})$, and hence $\text{cat} M_\mathbb{Q} \leq \text{cat} M$.

5.2. Proposition. For $M$ and $N$, closed and simply connected manifolds, $\text{cat}(M \# N)_\mathbb{Q} = \max\{\text{cat} M_\mathbb{Q}, \text{cat} N_\mathbb{Q}\}$.

Proof. As $M$ and $N$ are closed and simply connected, $M \# N$ is closed and simply connected, and $e_\mathbb{Q}(M \# N) = \text{cat}(M \# N)_\mathbb{Q}$. Combining (3.3) and (4.2) establishes on the left hand side, $\max\{\text{cat} M_\mathbb{Q}, \text{cat} N_\mathbb{Q}\} = \max\{e_\mathbb{Q}(M), e_\mathbb{Q}(N)\} \leq e_\mathbb{Q}(M \# N) = \text{cat}(M \# N)_\mathbb{Q}$.

While on the right hand side we have, $\text{cat}(M \# N)_\mathbb{Q} = \text{cat}_\mathbb{Q}(M \# N) \leq \max(\text{cat}_\mathbb{Q} M, \text{cat}_\mathbb{Q} N) = \max\{\text{cat} M, \text{cat} N\}$, where the middle inequality comes from (3.3).

Returning to Rudyak’s question on a possible relation between degree and category, we can settle it in the rational context.

5.3. Proposition. For closed and simply connected $m$-manifolds $M$ and $N$ with $f : M \to N$ of nonzero degree, we have $\text{cat} M_\mathbb{Q} \geq \text{cat} N_\mathbb{Q}$.

Proof. It suffices to show $e_\mathbb{Q}(M) \geq e_\mathbb{Q}(N)$. That is, suppose $p^* : H^*(M; \mathbb{Q}) \to H^*(P_n(M); \mathbb{Q})$ in the following diagram is injective.

\[
\begin{array}{ccc}
H^*(P_n(M); \mathbb{Q}) & \xrightarrow{p^*} & H^*(P_n(N); \mathbb{Q}) \\
h^* & & f^* \\
H^*(M; \mathbb{Q}) & \xleftarrow{f^*} & H^*(N; \mathbb{Q}) \\
\end{array}
\]

By [13] V.2.13, the map $f^*$ is injective. Since $p^*$ and $f^*$ are injective, the composition $p^* \circ f^*$ is injective, and $p^* : H^*(N; \mathbb{Q}) \to H^*(P_n(N); \mathbb{Q})$ is injective. Thus $e_\mathbb{Q}(N) \leq n$. □
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