ON KNOTS AND LINKS IN LENS SPACES

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We shortly review some recent results about knots and links in lens spaces. A disk diagram is described together with a Reidemeister-type theorem concerning equivalence. The lift of knots/links in the 3-sphere is discussed, showing examples of different knots and links having equivalent lift. The essentiality respect to the lift of classical invariants on knots/links in lens spaces is discussed.

Keywords: knot, link, lens space, lift, fundamental quandle, group of the link, twisted Alexander polynomial.

1. Introduction

The interest on knots and links in lens space arises from several topological reasons. One of the most important is the Berge conjecture about knots in $S^3$ admitting lens space surgeries, that can be translated into a conjecture about knots in lens spaces admitting $S^3$ surgeries [1; 2]. Moreover the interest does not come only from geometric topology, but also from physics [3] and biology [4].

The first step on the study of knots and links in lens spaces is to find a suitable representation: there are several possible diagrams for links in lens spaces, as mixed link diagrams [5], band diagrams [6] and grid diagrams [1] among the others. Using band diagrams Gabrovsek obtained in [7] a tabulation of prime knots up to 4 crossings. For a detailed introduction about knots and links in lens spaces, together with a vast bibliography, see [8].

Let $L$ be a link in the lens space $L(p,q)$ and let $P: S^3 \to L(p,q)$ be the (universal) cyclic covering, the lift of $L$ is the link $\tilde{L} = P^{-1}(L) \subset S^3$. In [9] is described an algorithm producing a diagram of $\tilde{L}$, starting from a disk diagram of $L$. This paper aims to investigating the behavior of the lift with respect to other invariants that have already been defined in [10; 11], namely: the fundamental quandle, the group of the link, the first homology group and the twisted Alexander polynomials. To be more precise, exploiting the different knots and links with equivalent lift described in [9], we show whether the considered invariants for $L$ are or not essential, that is to say, whether they cannot or can be defined directly on the lift $\tilde{L}$. A draft of this work can be found in [8].

The work about essential invariants extends also to the HOMFLY-PT invariant of Cornwell [12], the Link Floer Homology [1] (both of these results can be found in [13]) and the Kauffman Bracket Skein Module of [6] (the result can be found in [8]).

The setting of this paper is the $\text{Diff}$ category (of smooth manifolds and smooth maps). Every result also holds in the $\text{PL}$ category, and in the $\text{Top}$ category if we consider only tame links, that is to say, we exclude wild knots.

2. Diagrams and equivalence of links in lens spaces

In this section, we describe two equivalent definitions of lens spaces that we are going to use through the paper. Then we introduce links in lens spaces and their equivalence. At last we describe a representation of them by disk diagrams and in this context we prove a Reidemeister-type theorem.
2.1. Lens spaces

Apart from $S^3$, lens spaces are the simplest class of closed connected 3-manifolds. Usually they are regarded as quotients of the 3-sphere as follows. Let $p, q$ be coprime integers such that $0 \leq q < p$. Regard $S^3$ as the unit sphere in $\mathbb{C}$. Consider the diffeomorphism that sends

\[(z_1, z_2) \rightarrow (e^{2\pi i} z_1, e^{2\pi i} z_2),\]

and the cyclic group $G_{p,q}$ generated by this diffeomorphism. Clearly $G_{p,q}$ is isomorphic to $\mathbb{Z}_p$ and it acts freely and in a properly discontinuous way on $S^3$. Therefore the quotient space is a 3-manifold, the lens space $L(p,q)$.

Another possible definition of lens spaces is the following one. Consider $B^3 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^3 \leq 1\}$ and let $E_+$ and $E_-$ be respectively the upper and the lower closed hemisphere of $\partial B^3$. Call $B^3_0$ the equatorial disk, defined by the intersection of the plane $x_3 = 0$ with $B^3$. Let $g_{p,q} : E_+ \rightarrow E_+$ be the rotation of $\frac{2\pi q}{p}$ around the $x_3$ axis as in Fig. 2.1, and let $f_{\nu} : E_- \rightarrow E_-$ be the reflection with respect to the plane $x_3 = 0$. The lens space $L(p,q)$ is the quotient of $B^3$ by the equivalence relation on $\partial B^3$ which identifies $x \in E_+$ with $f_{\nu} \circ g_{p,q}(x) \in E_-$. We denote by $F : B^3 \rightarrow B^3/\sim$ the quotient map. Usually on $\partial B^3$ there are two points in each equivalence class, with the exception of the equator $\partial B^3_0 = E_+ \cap E_-$ where each class contains $p$ points.

\[\text{Fig. 2.1. Lens model of } L(p,q)\]

It is easy to see that $L(1,0) \cong S^3$ since $g_{1,0} = \text{Id}_{E_+}$. Furthermore, $L(2,1)$ is $\mathbb{R}P^3$, since we obtain the usual model of the projective space where opposite points of $\partial B^3$ are identified.

**Proposition 1.** [14]. The lens spaces $L(p,q)$ and $L(p',q')$ are diffeomorphic (as well as homeomorphic) if and only if $p' = p$ and $q' \equiv \pm q \pmod{p}$.

2.2. Links in lens spaces and their equivalence

A link $L$ in a lens space $L(p,q)$ is a pair $(L(p,q), L)$, where $L$ is a submanifold of $L(p,q)$ diffeomorphic to the disjoint union of $v$ copies of $S^1$, with $v \geq 1$. We call component of $L$ each connected component of $L$. When $v = 1$ the link is called a knot. We usually refer to $L \subset L(p,q)$ meaning the pair $(L(p,q), L)$. A link $L \subset L(p,q)$ is trivial if its components bound embedded pairwise disjoint 2-disks in $L(p,q)$.

We consider on the set of links in $L(p,q)$ two different definitions of equivalence. The stronger one is the equivalence up to ambient isotopy: two links $L, L' \subset L(p,q)$ are called isotopy-equivalent if there exists a smooth map $H : L(p,q) \times [0,1] \rightarrow L(p,q)$ where, if we define $h_t(x) := H(x, t)$, then $h_0 = \text{Id}_{L(p,q)}$, $h_1(L) = L'$ and $h_t$ is a diffeomorphism of $L(p,q)$ for each $t \in [0,1]$.

The weaker equivalence is up to diffeomorphism of pairs: two links $L$ and $L'$ in $L(p,q)$ are diffeo-equivalent if there exists a diffeomorphism of pairs $h : (L(p,q), L) \rightarrow (L(p,q), L')$, that is to say a diffeomorphism $h : L(p,q) \rightarrow L(p,q)$ such that $h(L) = L'$. This diffeomorphism is
not necessarily orientation-preserving. Two isotopy-equivalent links \( L \) and \( L' \) in \( L(p,q) \) are necessarily diffeo-equivalent, since from the ambient isotopy \( H: L(p,q) \times [0,1] \to L(p,q) \), the map \( h: (L(p,q), L) \to (L(p,q), L') \) is a diffeomorphism of pairs.

The two definitions coincide for links in \( S^3 \) if only orientation preserving diffeomorphisms are considered. For the lens spaces, this fact is no longer true, as we can see from the structure of the groups of the isotopy classes of diffeomorphisms of lens spaces obtained in [15] and [16].

When necessary we will specify whether an orientation on the links is considered.

2.3. Disk diagrams

For the case \( L(1,0) = S^3 \) links may be represented by the usual diagram coming from the regular projection onto a plane. This idea has been generalized in [11] for every \( p > 1 \): if the lens model of \( L(p,q) \) is considered and the link is regularly projected onto the equatorial disk, then a disk diagram for links in lens spaces is this projection together with overpasses and underpasses specifications for each crossing. The case \( \mathbb{R}^3 = L(2,1) \) is described also in [17]. In order to have a more comprehensible diagram, we index the boundary points of the projection, so that \(+i\) and \(-i\) represent identified endpoints respectively in \( E_+ \) and in \( E_- \). An example is shown in Fig. 2.2.

2.4. Generalized Reidemeister moves

The equivalence of links in \( S^3 \) can be studied through the Reidemeister theorem. We generalize this theorem for unoriented links in lens spaces up to isotopy equivalence. The oriented case is analogous. The generalized Reidemeister moves on a diagram of a link \( L \subset L(p,q) \), are the moves \( R_1, R_2, R_3, R_4, R_5, R_6 \) and \( R_7 \) of Fig. 2.3. Observe that, when \( p = 2 \) the moves \( R_5 \) and \( R_6 \) are equal, and \( R_7 \) is trivial, thus we re-obtain the result of [17].

**Theorem 1.** [11]. Two links \( L_0 \) and \( L_1 \) in \( L(p,q) \) are isotopy-equivalent if and only if their diagrams can be joined by a finite sequence of generalized Reidemeister moves \( R_1, \ldots, R_7 \) and diagram isotopies, when \( p > 2 \). If \( p = 2 \), moves \( R_1, \ldots, R_5 \) are sufficient.

2.5. Standard form of the disk diagram

A disk diagram is defined standard if the labels on its boundary points, read according to the orientation on \( \partial B^2_0 \), are \((+1, \ldots, +t, -1, \ldots, -t)\).

**Proposition 2.** [9]. Every disk diagram can be reduced to a standard disk diagram.

Indeed, if \( p = 2 \), the signs of the boundary points of the disk diagram can be exchanged by performing an isotopy on the link (that preserves the projection); if \( p > 2 \), a finite sequence of \( R_6 \) moves can be applied to the disk diagram in order to bring all the plus-type boundary points close to each other. An example is shown in Fig. 2.4.
3. Lift and essential invariants

In this section we deal with the following powerful invariant: let $L$ be a link in the lens space $L(p,q)$, the lift $\tilde{L}$ is the counterimage $P^{-1}(L)$ in $S^3$ under the universal cyclic covering $P : S^3 \to L(p,q)$. Clearly the lift is an isotopy-equivalence invariant for the homotopy lifting property. The main result of this section is the construction of a diagram for the lift $\tilde{L}$ from a standard disk diagram of $L$. This result is the key to find different links with equivalent lift. Thus the lift is not a complete invariant for links, but it becomes complete with some further assumption. We conclude by defining precisely what is an essential invariant of links in lens spaces.

3.1. Lift component number

Let $L$ be a link in $L(p,q)$, denote with $v$ its number of components, and with $\delta_1, \ldots, \delta_v$ the homology class in $H_1(L(p,q)) \cong \mathbb{Z}_p$ of the $i$-th component $L_i$ of $L$. In Lemma 1 it will be described how to compute the homology classes directly from a disk diagram.

**Proposition 3.** [9] Given a link $L \subset L(p,q)$, the number of components of $\tilde{L}$ is $\sum_{i=1}^v \gcd(\delta_i, p)$.

A knot $K \subset L(p,q)$ is **primitive-homologous** if its homology class $\delta$ is coprime with $p$; clearly its lift is still a knot.
3.2. Diagram for the lift via disk diagrams

The construction of a diagram for \( \hat{L} \subset S^3 \) starting from a disk diagram of \( L \subset L(p, q) \) is explained by the following two theorems. The case of \( L(2,1) \cong \mathbb{R}P^3 \) is outlined in [17]. As usual, the generators of the braid group on \( t \) strands are \( \sigma_1, \ldots, \sigma_{t-1} \). The Garside braid \( \Delta_t \) on \( t \) strands is defined by \( (\sigma_{t-1}\sigma_{t-2}\cdots\sigma_1)(\sigma_{t-1}\sigma_{t-2}\cdots\sigma_2)(\cdots\sigma_{t-1}) \) and it is illustrated in Fig. 3.1.

Theorem 1. [9]. Let \( L \) be a link in the lens space \( L(p, q) \) and let \( D \) be a standard disk diagram for \( L \); then a diagram for the lift \( \hat{L} \subset S^3 \) can be found as follows (refer to Fig. 3.2):

- consider \( p \) copies \( D_1, \ldots, D_p \) of the standard disk diagram \( D \);
- for each \( i = 1, \ldots, p-1 \), using the braid \( \Delta_{t-1} \), connect the diagram \( D_{i+1} \) with the diagram \( D_i \) joining the boundary point \( -j \) of \( D_{i+1} \) with the boundary point \( +j \) of \( D_i \);
- connect \( D_1 \) with \( D_p \) via the braid \( \Delta_{t^2-1} \), where the boundary points are connected as in the previous case.

The proof can be found in [9] and the example of Fig. 3.4 gives the main idea of the construction.

The planar diagram of the lift of Theorem 2 has not the least possible number of crossings. Indeed if we reverse upside down \( D_2 \), reverse twice \( D_3 \), and so on, the braid \( \Delta_{t-1} \) between the disks becomes the trivial one, moving the crossings close to \( \Delta_{t^2-1} \) so that a simplification to \( \Delta_{t^2-p} \) is possible. In order to describe this construction, we define the reverse disk diagram \( \overline{D} \) of \( D \): it is the diagram that can be obtained by considering the image of \( D \) under a symmetry with respect to an external line and then exchanging all overpasses / underpasses.

Theorem 3. [9]. Let \( L \) be a link in the lens space \( L(p, q) \) and let \( D \) be a standard disk diagram for \( L \); then a diagram for the lift \( \hat{L} \subset S^3 \) can be found as follows (refer to Fig. 3.3):

- consider \( p \) copies \( D_1, \ldots, D_p \) of the standard disk diagram \( D \), then denote \( F_i = D_i \) if \( i \) is odd, and \( F_i = \overline{D_i} \) if \( i \) is even;
- for each \( i = 1, \ldots, p-1 \), using a trivial braid, connect the diagram \( F_{i+1} \) with the diagram \( F_i \) joining the boundary point \( -j \) of \( F_{i+1} \) with the boundary point \( +j \) of \( F_i \);  
- connect \( F_1 \) with \( F_p \) via the braid \( \Delta_{t^2-p} \), where the boundary points are connected as in the previous case.
3.3. Different links with equivalent lift

In [9] a short tabulation of the lifts of a particular class of links in lens spaces — that can be easily described by a braid — is performed in order to investigate the existence of different links with equivalent lift. Two pairs of links with such a property are shown in Example 1 and 2. In order to distinguish the links of each pair we will use invariants such as the group of the link and the Alexander polynomial that will be respectively defined in Sections 5 and 6.

Example 1. Different knots in \( L(p, \frac{p+1}{2}) \) with trivial knot lift

The knots \( K_1 \) and \( K_2 \) in \( L(p, \frac{p+1}{2}) \) with \( p \) odd, depicted in Table 1, both lift to the unknot.

The homology class \( [K] = \delta \in H_1(L(p,q)) \cong \mathbb{Z}_p \) of a knot in \( L(p,q) \) can be \( 0, 1, \ldots, p-1 \), but since we do not consider the orientation of the knots, we have to identify \( \pm\delta \), so that the knots are partitioned into \( |p/2| + 1 \) classes: \( \delta = 0, 1, \ldots, |p/2| \), where \( \lfloor x \rfloor \) denotes the integer part of \( x \). If two knots have different homology classes, they are necessarily not isotopy-equivalent.

Since \( [K_1] = 1 \) and \( [K_2] = 2 \), the knots are not isotopy-equivalent when \( p > 3 \). An interesting fact is that \( K_1 \) and \( K_2 \) turn out to be diffeo-equivalent for \( L(5,2) \). The diffeomorphism realizing this equivalence is the generator \( \sigma_2 \) of the group of isotopy classes of diffeomorphisms of the lens space \( L(p, \frac{p+1}{2}) \), described in [15]. It is possible to show that for \( p > 5 \), the knots \( K_1 \) and \( K_2 \) are also not diffeo-equivalent.
Table 1

Geometric invariants of $K_1$ and $K_2$ in $L_p^{(p,\frac{p+1}{2})}$

|       | $K_1$ | $K_2$ |
|-------|-------|-------|
| $\nu$ | 1     | 1     |
| $[K] \subset H_1(L(p,q))$ | 1 | 2 |
| lift | unknot | unknot |
| $\pi_1(L(p,q) \setminus K)$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $H_1(L(p,q) \setminus K)$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $\overline{\Delta}(t)$ | 1 | 1 |

Example 2. Different links in $L(4,1)$ lifting to the Hopf link

The knot $L_A$ and the link $L_B$ in $L(4,1)$ described in Table 2 have a different number of components, hence they are not diffeo-equivalent (and consequently also not isotopy-equivalent); beside this, they both lift to the Hopf link.

Table 2

Geometric invariants of $L_A$ and $L_B$ in $L(4,1)$

|       | $L_A$ | $L_B$ |
|-------|-------|-------|
| $\nu$ | 1     | 2     |
| $[K] \subset H_1(L(p,q))$ | 2 | 1,1 |
| lift | Hopf link | Hopf link |
| $\pi_1(L(p,q) \setminus L)$ | $(a, f^{-1}af^{-3} = 1)$ | $(a, f = fa)$ |
| $H_1(L(p,q) \setminus L)$ | $\mathbb{Z} \oplus \mathbb{Z}_a$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
| $\overline{\Delta}(t)$ | $t + 1$ | $t - 1$ |
| $\overline{A}(t)$ | 1 | 1 |

The previous examples consist of pairs of links that are easy to distinguish, because they have different numbers of components or different homology classes. In [9] a wide family of links that have got equivalent lift is shown. This family can be found by cabling Example 2 with particular braids. The simplest example that we can extract from this family, with the same number of components and the same homology class for each component, is the following one.
Example 3. Different links in $L(4,1)$ with cables of Hopf link as lift

The links $A_{2,2}$ and $B_{2,2}$ of Table 3 have the same number of components $n = 2$ and each of these components has the same homology class $\delta = 2$, anyway the computation of the Alexander polynomial of $A_{2,2}$ and $B_{2,2}$ (see Section 6 for details) shows that the links are not diffeo-equivalent. Their lift is equivalent because from the construction, we insert a braid into each arc of $L_\alpha$ and $L_\beta$ so that each component of the Hopf link resulting from the lift has the same cabling.

Their lift is equivalent because from the construction, we insert a braid into each arc of $L_A$ and $L_B$ so that each component of the Hopf link resulting from the lift has the same cabling.

Table 3

| Geometric invariants of $A_{2,2}$ and $B_{2,2}$ in $L(4,1)$ |
|-------------------------------------------------------------|
| $A_{2,2}$                                                   |
| $B_{2,2}$                                                   |
| $\nu$                                                      | 2 | 2 |
| $[K] \subset H_1(L(p,q))$                                  | 2,2 | 2,2 |
| $H_1(L(p,q)\setminus L)$                                   | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$ |
| $\bar{\Delta}(t)$                                          | $t^2 + t^6 - t - 1$ | $t^2 + t^6 - t^4 + t^2 + t - 1$ |
| $\bar{\Delta}'(t)$                                         | $t^6 + 1$ | $t^6 + t^4 + t^2 + 1$ |

3.4. When the lift is a complete invariant

Since the lift of links in lens spaces comes from a cyclic covering, it is a natural question to ask if it is a complete invariant, at least for some family of knots.

As a consequence of Examples 1, 2 and 3, the lift is not a complete invariant for unoriented knots and links in $L(p,q)$, both up to diffeomorphism and up to isotopy. The problem of understanding whether the lift is a complete invariant can be referred also to oriented links. The answer is slightly different.

First of all, an orientation on the previous counter-examples allows us to find new examples, consisting of different oriented links in lens spaces having equivalent oriented lift. Moreover another family of counter-examples arises in the case of isotopy-equivalence. If we take an oriented knot $K \subset L(p,q)$ such that $\bar{K}$ is invertible (i.e., it is equivalent to the knot with reversed orientation), then also the knot $-K \subset L(p,q)$ with reversed orientation has the same oriented lift. Usually $-K$ is not isotopy-equivalent to $K$ because the homology class changes. A really simple example is provided by the two knots in $L(3,1)$ illustrated in Fig. 3.5: they both lift to the trivial knot, nevertheless they have different homology classes ($[K] = 1$ and $[-K] = 2$). For links something similar happens, but you have to be careful to the orientation of each component.

We can say something more in the case of oriented primitive-homologous knots when they are considered up to diffeo-equivalence, by the following theorem of Sakuma — also proved by Boileau and Flapan — about freely periodic knots.
A knot $K$ in $S^3$ is said to be freely periodic if there is a free cyclic action on $S^3$ that fix $K$. Clearly this action produces a lens space and a knot inside it, that lifts to $K$. In this case, if $\text{Diff}(S^3, K)$ is the group of diffeomorphisms of the pair $(S^3, K)$, which preserves the orientation of both $S^3$ and $K$, up to isotopy, then a symmetry $G$ of a knot $K$ in $S^3$ is a finite subgroup of $\text{Diff}(S^3, K)$, up to conjugation.

**Theorem 4.** [18; 19]. Suppose that a knot $K \subset S^3$ has free period $p$. Then there is a unique symmetry $G$ of $K$ realizing it, provided that (i) $K$ is prime, or (ii) $K$ is composite and the slope is specified.

If we translate this theorem into the language of knots in lens spaces, the specification of the slope is equivalent to fixing the parameter $q$ of the lens space. As a consequence, two primitive-homologous knots $K$ and $K'$ in $L(p, q)$ with equivalent non-trivial lift are necessarily diffeo-equivalent in $L(p, q)$.

From [15] and [16], we know that the group of the isotopy classes of diffeomorphisms of $L(p, q)$ is not trivial, thus Theorem 4 does not provide a complete answer about the equivalence of $K$ and $K'$ up to ambient isotopy.

### 3.5. Essential invariants

An invariant of links in $S^3$ turns out to be an invariant of links in lens spaces when it is computed on their lifts. This operation produces a lot of invariants. On the contrary, an invariant for links in lens spaces which can not merely be computed in the lift is called an essential invariant.

Different links with equivalent lift are an useful tool to check whether an invariant is an essential diffeo-invariant. The fundamental quandle is an inessential diffeo-invariant.

### 4. Fundamental quandle

The fundamental quandle is a very strong invariant of links in the 3-sphere: in fact it is a complete diffeo-invariant. The fundamental quandle can be defined also for links in lens spaces [10; 20]: is it still a complete invariant? This question is strictly related also to the essentiality of the invariant.

Given an oriented link $L \subset L(p, q)$, let $N(L)$ denote an open tubular neighborhood of $L$, consider the manifold $Q = L(p, q) \setminus N(L)$ and fix a base point $*$ in it. Let $\Gamma_L$ be the set of based homotopy classes of paths from $*$ to $\partial N(L)$ (the homotopy endpoint can move freely on $\partial N(L)$). We can define an operation $\circ$ on this set: for every $a$ and $b$ in $\Gamma_L$, consider the toric component of $\partial N(L)$ containing the starting point of $b$ and let $m$ be a meridian of this torus, the operation $a \circ b$ gives the class of the path $mbm^{-1}a$. The set $\Gamma_L$ with the operation $\circ$ is a distributive groupoid or equivalently, a quandle (see [10]). The algebraic structure $(\Gamma_L, \circ)$ is the fundamental quandle of an oriented link $L$ in $L(p, q)$.

**Proposition 4.** The fundamental quandle of a link in a lens space is isomorphic to the fundamental quandle of its lift in $S^3$.

**Proof.** According to [20, Lemma 5.4], the fundamental quandle is invariant under cyclic coverings, and if we consider the cyclic covering $P: (S^3 \setminus \tilde{L}) \to (L(p, q) \setminus L)$, we get the assertion. □

A consequence of this result is the following corollary.

**Corollary 1.** The fundamental quandle of links in lens spaces is an inessential diffeo-invariant.

The fundamental quandle of a link in a 3-manifold is a geometric invariant that can be explicitly computed on a diagram only for links in $S^3$ [10] and in $\mathbb{R}P^3$ [21]. Proposition 4 allows
us to compute the fundamental quandle of a link $L$ in lens spaces by computing the fundamental quandle of the lift $\tilde{L}$.

Theorem 4 can be combined to Proposition 4 to get the following statement.

**Corollary 2.** The fundamental quandle of oriented primitive-homologous knots in lens spaces classifies them up to diffeo-equivalence, unless the fundamental quandle is trivial.

For the case $\mathbb{R}^3 = L(2,1)$, this result is directly stated in [22], where it is generalized also for non primitive-homologous knots.

**Theorem 5.** [22. Theorem 2]. Two knots in $\mathbb{R}^3$ are diffeo-equivalent if and only if the corresponding fundamental quandles are isomorphic.

As a direct consequence of Proposition 4 and Theorem 5 we get the following theorem.

**Theorem 6.** Two knots in $\mathbb{R}^3$ are diffeo-equivalent if and only if the corresponding lifts are equivalent.

We cannot generalize Corollary 2 to knots in all lens spaces up to diffeomorphism (and hence also up to isotopy) because of Example 1.

**Remark 1.** The fundamental quandle of knots in lens spaces is not a complete diffeo-invariant for $L\left(p, \frac{p+1}{2}\right)$ with odd $p > 5$. Moreover it is not a complete isotopy-invariant also in the case of $L(5,2)$.

A similar statement for $L(4,1)$ follows from Examples 2 and 3.

By [23], we can compute other invariants of links in lens spaces derived from the quandle theory, such as quandle co-cycles invariants. If they are an invariant of the quandle, then they are inessential. If we consider bi-quandles instead, there is an example in [21] for links in the projective space where the co-cycle invariant seems more significant.

If we want a quandle-like structure that results essential we should turn to the oriented augmented fundamental rack [20], which is a complete invariant of framed links in 3-manifolds, and can be computed using mixed link diagrams.

### 5. Group of the link and homology

In this section we focus on the properties of the group of the link $L$ in lens spaces, that is to say, the fundamental group of the complement $L(p,q) \setminus L$. After giving a presentation on disk diagram of the group, we compute it on several examples, in order to show that the group is an essential diffeo-invariant and that Norwood theorem about knots in $S^3$ holds no longer in $L(p,q)$.

#### 5.1. Group of the link

We follow the presentation given in [11], that is a generalization of the Wirtinger theorem as the presentation of [24] for the case of the projective space $L(2,1)$.

Let $L$ be a link in $L(p,q)$ described by a disk diagram. Assume $p > 1$. Fix an orientation for $L$, which induces an orientation on the projection of the link. We can prove that if we reverse the orientation, the corresponding group is isomorphic to the former one. In order to find a presentation, perform an $R_1$ move on each overpass of the diagram having both endpoints on the boundary of the disk; in this way every overpass has at most one boundary point. Then label the overpasses as follows: $A_1, \ldots, A_t$ are the ones ending in the upper hemisphere, namely in $+1, \ldots, +t$, while $A_{t+1}, \ldots, A_{2t}$ are the overpasses ending in $-1, \ldots, -t$. The overpasses with no boundary points are labelled by $A_{2t+1}, \ldots, A_r$. For each $i = 1, \ldots, t$, let $e_i = +1$ if, according to the link orientation, the overpass $A_i$ starts from the point $+i$; otherwise, if $A_i$ ends in the point $+i$, let $e_i = -1$. 
Associate to each overpass $A_i$, a generator $a_i$, which is a loop around the overpass as in the classical Wirtinger theorem, oriented following the left hand rule. Moreover let $f$ be the generator of the fundamental group of the lens space illustrated in Fig. 5.1. The relations are the following:

**W:** $w_1, \ldots, w_s$ are the classical Wirtinger relations for each crossing, that is to say $a_i a_j a_i^{-1} a_k^{-1} = 1$ or $a_i a_j^{-1} a_i^{-1} a_k = 1$, according to Fig. 5.2;

**L:** $l$ is the lens relation $a_1^o \cdots a_t^o = f^p$;

**M:** $m_1, \ldots, m_t$ are relations (of conjugation) between loops corresponding to overpasses with identified endpoints on the boundary. If $t=1$ the relation is $a_2^o f a_1^o a_2^{-1} = a_1^o f^{-1} a_2^o f^{-1} a_1^o$. Otherwise, consider the point $-i$ and, according to equator orientation, let $+j$ and $+j+1$ (mod $t$) be the plus-type points adjacent to it. We distinguish two cases:

- if $-i$ lies on the diagram between $-1$ and $+1$, then the relation $m_i$ is
  \[
  a_{i,i}^o \left( \prod_{k=1}^{j} a_k^o \right)^{-1} f^q \left( \prod_{k=1}^{j} a_k^o \right) a_{i,i}^o \left( \prod_{k=1}^{j} a_k^o \right)^{-1} f^{-q} \left( \prod_{k=1}^{j} a_k^o \right);
  \]

- otherwise, the relation $m_i$ is
  \[
  a_{i,i}^o \left( \prod_{k=1}^{j} a_k^o \right)^{-1} f^{q-p} \left( \prod_{k=1}^{j} a_k^o \right) a_{i,i}^o \left( \prod_{k=1}^{j} a_k^o \right)^{-1} f^{q-p} \left( \prod_{k=1}^{j} a_k^o \right).
  \]

Consider the lens space model depicted in Fig. 2.1, let $N$ be the point $(0,0,1)$ of $B^3$ and $F: B^3 \to L(p,q)$ be the quotient map.

![Fig. 5.1. Example of overpasses labelling for a link in $L(6,1)$](image)

![Fig. 5.2. Wirtinger relations](image)

**Theorem 7.** [11]. Let $* = F(N)$, then the group of the link $L \subset L(p,q)$ is:

\[
\pi_1(L(p,q) \setminus L,*) = \langle a_1, \ldots, a_r, f \mid w_1, \ldots, w_s, l, m_1, \ldots, m_t \rangle.
\]

In the special case of $L(2,1) = \mathbb{R}P^3$, the presentation is equivalent (via Tietze transformations) to the one given in [24].
5.2. First homology group

At first, a way to compute the homology class of a knot directly from a disk diagram is shown and it is really useful because it is the easiest isotopy invariant. Then the method to determine, directly from the diagram, the first homology group of links in \( L(p, q) \) is found. Differently from the \( S^3 \) case, a non-trivial torsion part may appears and it is useful for the computation of twisted Alexander polynomials.

Consider a diagram of an oriented knot \( K \subset L(p, q) \) and let \( e_i \) be as defined in the previous section. Define \( \delta_k = \sum_{i=1}^t e_i^k \).

**Lemma 1.** If \( K \subset L(p, q) \) is an oriented knot and \( [K] \) is the homology class of \( K \) in \( H_1(L(p, q)) \), then \( [K] = \delta_k \).

By abelianizing the presentation of the group of the link of Theorem 7, we get the first homology group of the complement of a link in lens spaces.

**Corollary 3.** [11]. Let \( L \) be a link in \( L(p, q) \) with components \( L_1, \ldots, L_v \). For each \( j = 1, \ldots, v \), let \( \delta_j = [L_j] \in \mathbb{Z}_p = H_1(L(p, q)) \). Then \( H_1(L(p, q) \setminus L) \cong \mathbb{Z}^v \oplus \mathbb{Z}_d \), where \( d = \gcd(\delta_1, \ldots, \delta_v, p) \).

5.3. Norwood theorem

A theorem of Norwood [25] states that every knot in the 3-sphere admitting a presentation for its group with only two generators is prime. For every lens space \( L(p, q) \) with \( p > 1 \), we now show a knot that has a minimal presentation of the group with two generators, but it is not prime; as a consequence, the Norwood theorem cannot be generalized to lens spaces.

**Example 4.** Let \( T \) be the trefoil knot in \( S^3 \). Let \( K_1 \) be the knot of the previous example and consider the connected sum \( K_1 \# T \) in \( L(p, q) \), as Fig. 5.3 shows.

\[
\pi_1(L(p, q) \setminus (K_1 \# T), \ast) = \langle a_1, a_2, a_3, a_4, f \mid a_1 a_2^{-1} a_3^{-1} a_4^{-1} = 1, a_1 a_2^{-1} a_3^{-1} a_4^{-1} = 1, a_2 a_4^{-1} a_2^{-1} a_3^{-1} = 1, a_4 = f^n, a_2 = a_2^{-1} f a_4 f^{-1} a_3 \rangle = \langle a_3, f \mid f^{-n} a_3 f^{-1} a_3 f^{-n} a_3 = 1 \rangle.
\]

5.4. Essentiality of the group and the homology

**Theorem 8.** The group of the link is an essential diffeo-invariant, as a consequence of Example 2.

**Theorem 9.** The homology group \( H_1(L(p, q) \setminus L) \) is an essential diffeo-invariant too, as we can see from Table 2. Moreover the homology class of a knot is an essential isotopy-invariant (see Table 1).

**Example 5.** The last example consists of the two links \( M_1 \) and \( M_2 \) in \( L(5, 2) \) on Table 4. They have not diffeo-equivalent lift, more precisely, one lift is the link \( L4_1 \) of the Knot Atlas, while the other one is its mirror image. In this case the groups are isomorphic, hence sometimes the lift may be stronger than the group of the link.

6. Twisted Alexander polynomials

In this section we describe a class of twisted Alexander polynomials of links in lens spaces. This class consists of those polynomials with 1-dimensional representation over particular Noetherian unique factorization domains that take into account the torsion part of the group of the link. The goal is to investigate whether they are an essential invariant.
The twisted Alexander polynomials are essential invariants (see Table 4). Moreover they are not complete invariants (see Table 1).

### Geometric invariants of \( M_1 \) and \( M_2 \) in \( L(3,2) \)

| \( M_1 \) | \( M_2 \) |
|---|---|
| \( -4 \) | \( -2 \) |
| \( -3 \) | \( 2 \) |
| \( -2 \) | \( 1 \) |
| \( -1 \) | \( 0 \) |

| \( \nu \) | 2 |
|---|---|
| \( [K] \subset H_1(L(p,q)) \) | 2,2 |
| \( \pi_1(L(p,q) \setminus L) \) | \( \langle a,f | a f^2 = f^2 a \rangle \) |
| \( H_1(L(p,q) \setminus L) \) | \( \mathbb{Z} \oplus \mathbb{Z} \) |
| \( \bar{\Delta}'(t) \) | \( t^2 - 1 \) |

#### 6.1. The computation of the twisted Alexander polynomials

The twisted Alexander polynomials are defined in the following way (for further references see [11; 26; 27]). Given a finitely generated group \( \pi \), denote with \( H = \pi / \pi' \) its abelianization and let \( G = H / \text{Tors}(H) \). Take a presentation \( \pi = \langle x_1, \ldots, x_m | r_1, \ldots, r_n \rangle \) and consider the Alexander–Fox matrix \( A \) associated to the presentation, that is

\[
A_i = P \left[ \frac{\partial r_i}{\partial x_j} \right],
\]

where \( P \) is the natural projection \( \mathbb{Z}[F(x_1, \ldots, x_m)] \to \mathbb{Z}[\pi] \to \mathbb{Z}[H] \) and \( \frac{\partial r_i}{\partial x_j} \) is the Fox derivative of \( r_i \). Moreover let \( E(\pi) \) be the first elementary ideal of \( \pi \), which is the ideal of \( \mathbb{Z}[H] \) generated by the \((m-1)\)-minors of \( A \). For each homomorphism \( \sigma: \text{Tors}(H) \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \) we can define a twisted Alexander polynomial \( \Delta'(\pi) \) of \( \pi \) as follows: fix a splitting \( H = \text{Tors}(H) \times G \) and consider the ring homomorphism that we still denote with \( \sigma: \mathbb{Z}[H] \to \mathbb{C}[G] \) sending \( (f,g) \), with \( f \in \text{Tors}(H) \) and \( g \in G \), to \( \sigma(f)g \), where \( \sigma(f) \in \mathbb{C}^* \). The ring \( \mathbb{C}[G] \) is a unique factorization domain and we set \( \Delta'(\pi) = \gcd(\sigma(E(\pi))) \). This is an element of \( \mathbb{C}[\pi] \) defined up to multiplication by elements of \( G \) and non-zero complex numbers. If \( \Delta(\pi) \) denotes the classical Alexander polynomial we have \( \Delta'(\pi) = \alpha \Delta(\pi) \), with \( \alpha \in \mathbb{C}^* \).

#### 6.2. Twisted Alexander polynomials are essential invariants

If \( L \subset L(p,q) \) is a link in a lens space then the \( \sigma \)-twisted Alexander polynomial of \( L \) is \( \Delta_L^\sigma = \Delta'(\pi_1(L(p,q) \setminus L)) \). Since in this case \( \text{Tors}(H) = \mathbb{Z}_d \) then \( \sigma(\text{Tors}(H)) \) is contained in the cyclic group generated by \( \zeta \), where \( \zeta \) is a \( d \)-th primitive root of the unity. Note that \( \Delta_L^\sigma \in \mathbb{C}[G] \) is defined up to multiplication by \( \zeta^d g \), with \( g \in G \).

If \( L \) has at least two components we can consider the projection \( \varphi: \mathbb{C}[G] = \mathbb{C}[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}] \to \mathbb{C}[t, t^{-1}] \), sending each variable \( t_i \) to \( t \). The one-variable twisted Alexander polynomial of \( L \) is \( \bar{\Delta}^\sigma_L = \varphi(\Delta_L^\sigma) \). The computation of \( \bar{\Delta}^\sigma_L \) for knots in arbitrary lens spaces has been implemented in a program using Mathematica code: the input is a knot diagram in \( L(p,q) \) given through a generalization of the Dowker–Thistlewaite code (see [28; 29]). Thanks to this program we obtained the results listed in Tables 1, 2, 3 and 4.

**Theorem 10.** The twisted Alexander polynomials are essential diffeo-invariants (see Tables 2 and 3 even the lift may be stronger than the Alexander polynomial (see Table 4). Moreover they are not complete invariants (see Table 1).
6.3. Relationship between Alexander invariants of $L$ and $\tilde{L}$

The lift of links in the lens space $L(p, q)$ can be seen as a freely $p$-periodic link (also said a $(p,q)$-lens links in $S^3$ [30]). Exploiting known results about freely periodic links, we can relate the invariants of the link to the corresponding invariants of its lift. The first question that deserves our interest is the following one: does the Alexander polynomial of the lift depend on the twisted Alexander polynomials of the link in lens spaces? Hartley provided the answer for the Alexander polynomial of freely periodic knots: in [31] there is a formula connecting the twisted Alexander polynomials in the case that both $K \subset L(p, q)$ and $\tilde{K} \subset S^3$ are knots. Furthermore, Chbili has shown in [32] an interesting characterization for the multi-variable Alexander polynomial of the lift of braid links in lens spaces.

Can we find pieces of information about the twisted Alexander polynomials of a link $L \subset L(p, q)$ from the Alexander polynomial of its lift? From Tables 2 and 3 we see that this is not possible, neither for knots nor for links. Another interesting counter-example for this question is the next one: considering the unknot and the local trefoil in $L(2, 1)$, their lifts are the unlink with two components and two split trefoils respectively. The twisted Alexander polynomials of these links in $L(2, 1)$ are different, their lifts in $S^3$ are different, but their lifts have the same Alexander polynomial (equal to zero).

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**ОБ УЗЛАХ И ЗАЦЕПЛЕНИЯХ В ЛИНЗОВЫХ ПРОСТРАНСТВАХ**

**Э. Манфреди, М. Мулаццани**

Дается короткий обзор некоторых недавних результатов об узлах и зацеплениях в линзовских
направления. Описываются дисковые диаграммы вместе с касающимся эквивалентности анало-
гом теоремы Райдемайстера. Рассмотрено поднятие узлов и зацеплений в трехмерную сферу,
приводится несколько примеров различных узлов и зацеплений, обладающих эквивалентными
поднятиями. Обсуждается существенность относительно поднятия классических инвариантов
узлов и зацеплений в линзовидных пространствах.

Ключевые слова: узел, зацепление, линзовое пространство, поднятие, фундаментальный
квандл, группа зацепления, скрученный полином Александра.

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