Two Types of Second Order Fractional Differential Equations

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Abstract. In this paper, we make use of a new multiplication of fractional functions, variable separation method and chain rule for fractional derivatives, regarding the Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives to solve two types of second order fractional differential equations. Moreover, some examples are proposed to illustrate our results.

Keywords: new multiplication, variable separation method, chain rule, modified R-L fractional derivatives, second order fractional differential equations.

1. Introduction
The fractional differential equations have acquired great importance. The study of fractional differential equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. Fractional differential equations appear naturally in many fields such as physics, polymer rheology, biophysics, aerodynamics, electrodynamics of complex medium, viscoelasticity, capacitor theory, electrical circuits, electron analytical chemistry, biology, control theory, etc. Excellent data in the study of fractional differential equations can be found in [1-4], and more details and examples, see [5-7].

Unlike traditional calculus, there are many different definitions of derivation and integration in fractional calculus. The commonly used definitions are the Riemann-Liouville (R-L) fractional derivative, the Caputo definition of fractional derivative, the Grunwald-Letnikov (G-L) fractional derivative, and the Jumarie’s modified R-L fractional derivatives. In this article, we obtain the general solutions of two types of second order fractional differential equations by using a new multiplication of fractional functions, variable separation method and chain rule for fractional derivatives, regarding the Jumarie type of modified R-L fractional derivative. In fact, the results we obtained are the generalizations of classical second order ordinary differential equations. Furthermore, we provide two examples to demonstrate the applications of our results.

2. Preliminaries and definitions
Firstly, we introduce the fractional calculus used in this paper.

Definition 2.1: If \( \alpha \) is a real number and \( m \) is a positive integer. Then the modified R-L fractional derivatives of Jumarie type is defined by ([8])
\[aD_x^\alpha[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_x^\infty (x-\tau)^{-\alpha-1} f(\tau) d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty (x-\tau)^{-\alpha} f(\tau) - f(\alpha) d\tau & \text{if } 0 \leq \alpha < 1 \\ \frac{d^m}{dx^m} \left( aD_x^{\alpha-m} \right) [f(x)], & \text{if } m \leq \alpha < m + 1. \end{cases}\]  

(1)

If \( \left( aD_x^\alpha \right)^n [f(x)] = \left( aD_x^\alpha \right) \left( aD_x^\alpha \right) \cdots \left( aD_x^\alpha \right) [f(x)] \) exists, then \( f(x) \) is called \( n \)-th order \( \alpha \)-fractional differentiable function, and \( \left( aD_x^\alpha \right)^n [f(x)] \) is the \( n \)-th order \( \alpha \)-fractional derivative of \( f(x) \).

On the other hand, we define the fractional integral of \( f(x) \), \( aI_x^\alpha[f(x)] = aD_x^{-\alpha} [f(x)] \), where \( \alpha > 0 \), and \( f(x) \) is called \( \alpha \)-fractional integrable function.

**Proposition 2.2:** Suppose that \( \alpha, \beta, c \) are real constants and \( 0 < \alpha \leq 1 \), then

\[0D_x^\alpha \left[ x^\beta \right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \text{ if } \beta \geq \alpha\]  

(2)

\[0D_x^\alpha [c] = 0.\]  

(3)

**Definition 2.3:** The Mittag-Leffler function is defined by

\[E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},\]  

(4)

Where \( \alpha \) is a real number, \( \alpha > 0 \), and \( z \) is a complex variable.

Next, we define a new multiplication of fractional functions.

**Definition 2.4** ([9]): Let \( \lambda, \mu, z \) be complex numbers, \( 0 < \alpha \leq 1 \), \( j, l, k \) be non-negative integers, and \( a_k, b_k \) be real numbers, \( p_k(z) = \frac{1}{\Gamma(\alpha k + 1)} z^k \) for all \( k \). The \( \otimes \) multiplication is defined by

\[p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha) = \frac{1}{\Gamma(j+\alpha+1)} (\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l+\alpha+1)} (\mu y^\alpha)^l = \frac{1}{\Gamma((j+l)+\alpha+1)} \left( j + l \right)! \frac{1}{j! l!} (\lambda x^\alpha)^j (\mu y^\alpha)^l,\]  

(5)

Where \( \left( j + l \right)! \frac{1}{j! l!} = \binom{j+l}{j} \). If \( f_\alpha(\lambda x^\alpha) \) and \( g_\alpha(\mu y^\alpha) \) are two fractional functions,

\[f_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} a_k p_k(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(\alpha k + 1)} (\lambda x^\alpha)^k,\]  

(6)

\[g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} b_k p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(\alpha k + 1)} (\mu y^\alpha)^k,\]  

(7)

Then we define

\[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} a_k p_k(\lambda x^\alpha) \otimes \sum_{k=0}^{\infty} b_k p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \binom{k}{m} a_k b_m (\lambda x^\alpha)^{k-m} (\mu y^\alpha)^m.\]  

(8)

**Proposition 2.5:**

\[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda x^\alpha)^{k-m} (\mu y^\alpha)^m.\]  

(9)
Definition 2.6: Let \((f_{\alpha}(\lambda x^{\alpha}))\) be the \(n\) times product of the fractional function \(f_{\alpha}(\lambda x^{\alpha})\). If \(f_{\alpha}(\lambda x^{\alpha}) \otimes g_{\alpha}(\lambda x^{\alpha}) = 1\), then \(g_{\alpha}(\lambda x^{\alpha})\) is called the \(\otimes\) reciprocal of \(f_{\alpha}(\lambda x^{\alpha})\), and is denoted by \((f_{\alpha}(\lambda x^{\alpha}))^{\otimes -1}\).

Definition 2.7: If \(f(z) = \sum_{k=0}^{\infty} a_k z^k\), \(g(x^{\alpha}) = \sum_{k=0}^{\infty} b_k p_k(x^{\alpha})\), then
\[
f_{\otimes\alpha}(g_{\alpha}(x^{\alpha})) = \sum_{k=0}^{\infty} a_k \left(g_{\alpha}(x^{\alpha})\right)^{\otimes k}.
\]

Theorem 2.8 (chain rule for fractional derivatives) ([9]): Suppose that \(f(z) = \sum_{k=0}^{\infty} a_k z^k\), \(g(x^{\alpha}) = \sum_{k=0}^{\infty} b_k p_k(x^{\alpha})\). If \(f_{\otimes\alpha}(g_{\alpha}(x^{\alpha})) = \sum_{k=0}^{\infty} a_k \left(g_{\alpha}(x^{\alpha})\right)^{\otimes k}\) and \(f_{\otimes\alpha}(g_{\alpha}(x^{\alpha})) = \sum_{k=1}^{\infty} a_k (g_{\alpha}(x^{\alpha}))^{\otimes(k-1)}\), then
\[
\left(\frac{D_{\alpha}^\alpha}{D_{\alpha}^\alpha}[f_{\otimes\alpha}(g_{\alpha}(x^{\alpha}))]\right) = f_{\otimes\alpha}(g_{\alpha}(x^{\alpha})) \otimes \left(\frac{D_{\alpha}^\alpha}{D_{\alpha}^\alpha}[g_{\alpha}(x^{\alpha})]\right).
\]

3. Main results and discussions

3.1. The fractional differential equation
\[
\left(\frac{D_{\alpha}^\alpha}{D_{\alpha}^\beta}[y]\right)^2 = f\left(x^{\alpha}, \frac{D_{\alpha}^\alpha}{D_{\alpha}^\beta}[y]\right). \ (0 < \alpha \leq 1)
\]

The following is the method for solving Eq. (12):

Let \(\frac{D_{\alpha}^\alpha}{D_{\alpha}^\beta}[y] = p\), then \(\left(\frac{D_{\alpha}^\alpha}{D_{\alpha}^\beta}[y]\right)^2 = \frac{D_{\alpha}^\alpha}{D_{\alpha}^\beta}[p]\), and Eq. (12) becomes the following first order \(\alpha\)-fractional differential equation
\[
\frac{D_{\alpha}^\beta}{D_{\alpha}^\beta}[p] = f\left(x^{\alpha}, p\right).
\]

Assume that Eq. (13) has the general solution \(p = \varphi(x^{\alpha}, C_1)\), then
\[
\frac{D_{\alpha}^\beta}{D_{\alpha}^\beta}[y] = \varphi(x^{\alpha}, C_1).
\]

And hence, we obtain the general solution of Eq. (12),
\[
y = \frac{D_{\alpha}^\alpha}{D_{\alpha}^\alpha}[\varphi(x^{\alpha}, C_1)] + C_2.
\]

3.2. The fractional differential equation
\[
\left(\frac{D_{\alpha}^\alpha}{D_{\beta}^\beta}[y]\right)^2 = f\left(y^{\alpha}, \frac{D_{\alpha}^\alpha}{D_{\beta}^\beta}[y]\right). \ (0 < \alpha \leq 1)
\]

We solve Eq. (16) as follows:

Let \(\frac{D_{\alpha}^\alpha}{D_{\beta}^\beta}[y] = p\), then by chain rule for fractional derivatives, we have
\[
\left(\frac{D_{\alpha}^\alpha}{D_{\beta}^\beta}[y]\right)^2 = \frac{D_{\alpha}^\beta}{D_{\beta}^\beta}[p] = \frac{D_{\alpha}^\beta}{D_{\beta}^\beta}[p] \otimes \frac{D_{\alpha}^\beta}{D_{\beta}^\beta}[y] = p \otimes \frac{D_{\alpha}^\beta}{D_{\beta}^\beta}[p] = p \otimes \frac{D_{\alpha}^\beta}{D_{\beta}^\beta}[p].
\]

Therefore,
\[
p \otimes \frac{D_{\alpha}^\beta}{D_{\beta}^\beta}[p] = f\left(y^{\alpha}, p\right).
\]
Suppose that the general solution of Eq. (18) is \( \alpha D_x^\alpha [y] = p = \psi(y^\alpha, C_1) \), then by variable separation method ([10]), we have the general solution of Eq. (16)

\[
\alpha I_y^\alpha \left( \psi(y^\alpha, C_1) \right)^{-1} = x + C_2.
\]  

(19)

4. Applications

4.1. Example

Consider the second order \( 1/2 \)-fractional differential equation

\[
\left( \alpha D_x^{1/2} [y] \right)^2 = \left( \Gamma \left( \frac{3}{2} \right) x^{3/2} \otimes \left( 1 + \frac{1}{\Gamma(2)} x \right)^{-1} \right) \otimes \alpha D_x^{1/2} [y].
\]  

(20)

By Eq. (13), we have

\[
\alpha D_x^{1/2} [p] = \left( \Gamma \left( \frac{3}{2} \right) x^{3/2} \otimes \left( 1 + \frac{1}{\Gamma(2)} x \right)^{-1} \right) \otimes p.
\]  

(21)

Using variable separation method yields

\[
\alpha D_x^{1/2} [y] = p = C_1 \left( 1 + \frac{1}{\Gamma(2)} x \right).
\]  

(22)

Hence, the general solution of Eq. (20) is

\[
y = \alpha I_x^{1/2} \left[ C_1 \left( 1 + \frac{1}{\Gamma(2)} x \right) \right] + C_2
\]

\[
= \frac{1}{\Gamma \left( \frac{1}{2} \right)} x^{3/2} + C_1 \frac{1}{\Gamma \left( \frac{1}{2} \right)} x^{3/2} + C_2.
\]  

(23)

4.2. Example

Consider the second order \( 1/3 \)-fractional differential equation

\[
\left( \alpha D_x^{1/3} [y] \right)^2 = y^{\otimes -1} \otimes \left( \alpha D_x^{1/3} [y] \right)^{\otimes 2}.
\]  

(24)

Using Eq. (18) yields

\[
p \otimes \alpha D_y^{1/3} [p] = y^{\otimes -1} \otimes p^{\otimes 2}.
\]  

(25)

It follows that

\[
p \otimes \left( p - y \otimes \alpha D_y^{1/3} [p] \right) = 0.
\]  

(26)

And hence
\[ p = 0 \text{ or } p - y \otimes D_{y}^{1/3}[p] = 0. \]  

(27)

If \( p = 0 \), then \( y \) is a constant.

If \( p - y \otimes D_{y}^{1/3}[p] = 0 \), then \( p = D_{x}^{1/3}[y] = C_{1}y \). Therefore ([9]),

\[ y = C_{2}E_{1/3}\left(C_{1}\frac{1}{\Gamma(\frac{1}{3})}x^{1/3}\right). \]  

(28)

In Eq. (28), if \( C_{1} = 0 \), then \( y = C_{2} \) is a constant. Thus, the general solution of \( p = 0 \) has been included in Eq. (28).

5. Conclusions

As mentioned above, the general solutions of two types of second order fractional differential equations studied in this paper can be obtained by variable separation method and chain rule for fractional derivatives. In fact, the applications of these two methods are extensive, and can be used to easily solve many fractional differential equations. On the other hand, our result is the generalization of classical second order differential equations. Moreover, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the Jumarie type of modified R-L fractional derivatives and the new multiplication to extend the research topics to the problems of applied mathematics and fractional calculus.

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