VARIATIONAL ANALYSIS
FOR NONLOCAL YAMABE-TYPE SYSTEMS

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Dedicated to Professor Patrizia Pucci with deep esteem and admiration

Abstract. The paper is concerned with existence, multiplicity and asymptotic behavior of (weak) solutions for nonlocal systems involving critical nonlinearities. More precisely, we consider

\[
\begin{align*}
M \left( |u|^2 - \frac{\mu}{\|x\|^{2s}} \right) & \left| (-\Delta)^s u - \mu V(x) u - \phi |u|^{2^*_s - 2} u \right| \\
& = \lambda h(x) |u|^{p - 2} u + |u|^{2^*_s - 2} u \quad \text{in } \mathbb{R}^3 \\
(-\Delta)^t \phi &= |u|^{2^*_t - 1} \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \((-\Delta)^s\) is the fractional Laplacian, \([u]_s\) is the Gagliardo seminorm of \(u\), \(M : \mathbb{R}^+_0 \to \mathbb{R}^+_0\) is a continuous function satisfying certain assumptions, \(V(x) = |x|^{-2s}\) is the Hardy potential function, \(2^*_s = (3 + 2t)/(3 - 2s)\), \(s, t \in (0, 1)\), \(\lambda, \mu\) are two positive parameters, \(1 < p < 2^*_s = 6/(3 - 2s)\) and \(h \in L^{2^*_t/(2^*_s - p)}(\mathbb{R}^3)\). By using topological methods and the Krasnoleskii’s genus theory, we obtain the existence, multiplicity and asymptotic behaviour of solutions for above problem under suitable positive parameters \(\lambda\) and \(\mu\). Moreover, we also consider the existence of nonnegative radial solutions and non-radial sign-changing solutions. The main novelties are that our results involve the possibly degenerate Kirchhoff function and the upper critical exponent in the sense of Hardy–Littlewood–Sobolev inequality. We emphasize that some of the results contained in the paper are also valid for nonlocal Schrödinger–Maxwell systems on Cartan–Hadamard manifolds.

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1. Introduction and main results. The existence of solutions, compactness and stability properties for Kirchhoff-type systems on manifolds have been investigated by many authors. We mention here, among others, the recent results obtained by Druet and Hebey [19], Hebey and Thizy [26], Heby, Robert and Wen [27]. On the other hand, critical equations involving nonlocal operators have been studied also in connection with the celebrated Yamae problem (see [41, Chapter III]).

Motivated by this wide interest in the current literature, in this paper, we study the following critical nonlocal system:

\[
\begin{cases}
M(|u|^2)\left((-\Delta)^s u - \mu \frac{u}{|x|^{2s}}\right) - \phi |u|^{2^*_s} - 2 u = \lambda h(x) |u|^{p-2} u + |u|^{2^*_s-2} u & \text{in } \mathbb{R}^3 \\
(-\Delta)^t \phi = |u|^{2^*_t} & \text{in } \mathbb{R}^3,
\end{cases}
\]

where

\[
\|u\| = \left(\int_{\mathbb{R}^3} \frac{|u|^2}{|x|^{2s}} \, dx\right)^{1/2}, \quad [u]_s = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x-y|^{3+2s}} \, dx \, dy\right)^{1/2},
\]

\(s, t \in (0, 1), M : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) is a Kirchhoff function, \(0 < \mu, \lambda > 0, h \in L^{\frac{2^*_s}{3+2s}}(\mathbb{R}^3)\), \(1 < p < 2^*_s, 2^*_s = \frac{3+2s}{3-2s}\) is the upper critical exponent in the sense of Hardy–Littlewood–Sobolev inequality, \(2^*_s = 6/(3-2s)\) is the critical fractional exponent and \((-\Delta)^s\) is the fractional Laplace operator which, up to a normalization constant, is defined as

\[(-\Delta)^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^3 \setminus B_{\varepsilon}(x)} \frac{\varphi(x) - \varphi(y)}{|x-y|^{3+2s}} \, dy, \quad x \in \mathbb{R}^3,
\]

along functions \(\varphi \in C_0^\infty(\mathbb{R}^3)\). Henceforward \(B_{\varepsilon}(x)\) denotes the ball of \(\mathbb{R}^3\) centered at \(x \in \mathbb{R}^3\) and radius \(\varepsilon > 0\). For details on the introduction to fractional Laplace operator we refer the readers to [14] and the references therein. In recent years, nonlocal problems involving the fractional Laplacian have received a great attention. For instance, fractional Laplace operators could be viewed as the infinitesimal generators of Lévy stable diffusion processes, see e.g. [1]. In [31], Laskin introduced the fractional Schrödinger equation in the study of particles on stochastic fields modeled by Lévy processes. All in all, nonlocal fractional Laplacian models allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media, for more details see [1, 10] and the references therein. Indeed, the literature on nonlocal fractional operators and their applications is quite large, for example, we refer to [22, 41, 50, 55] and the references therein.

Throughout the paper, without further mention, the following hypothesis is imposed on \(h\):

\((H_1)\) \(h : \mathbb{R}^3 \to \mathbb{R}_0^+, h \neq 0\) and \(h \in L^{\frac{2^*_s}{3+2s}}(\mathbb{R}^3)\).

For the function \(M\), we assume that \(M : \mathbb{R}_0^+ \to \mathbb{R}_0^+\) is a continuous function verifying

\((M_1)\) For any \(\tau > 0\) there exists \(\kappa = \kappa(\tau) > 0\) such that \(M(t) \geq \kappa\) for all \(t \geq \tau\).

\((M_2)\) There exists \(\theta \in \left[1, \frac{2^*_s}{2}\right]\) such that \(tM(t) \leq \theta \tilde{M}(t)\) for all \(t \geq 0\), where \(\tilde{M}(t) = \int_0^t M(\tau) \, d\tau\).

\((M_3)\) There exists \(m_0 > 0\) such that \(M(t) \geq m_0 t^{\theta - 1}\) for all \(t \in [0, 1]\).
A typical example of $M$ is given by $M(t) = a + b t^{\theta - 1}$ for $t \geq 0$, where $a, b \geq 0$ and $a + b > 0$, if $\theta > 1$, and $M(t) = a > 0$ if $\theta = 1$. For $\theta > 1$, when $M$ is of this type, problem (1) is said to be non–degenerate if $a > 0$, while it is called degenerate if $a = 0$.

Evidently, assumptions $(M_1)$–$(M_3)$ cover the degenerate case and $(M_2)$–$(M_3)$ are automatic in the non–degenerate case. In [46], condition $(M_3)$ was also applied to investigate the existence of entire solutions for the stationary Kirchhoff type equations driven by the fractional $p$–Laplacian operator in $\mathbb{R}^N$, see [11] for more results. It is worth mentioning that the degenerate case is rather interesting and is treated in well–known papers in Kirchhoff theory, see for example [16]. In the large literature on degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of $u$ via $M(||u||^2)$. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero.

Actually, the study of Kirchhoff–type problems, which arise in various models of physical and biological systems, have received more and more attention in recent years. More precisely, Kirchhoff [30] established a model given by

$$\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{p_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where $\rho$, $p_0$, $h$, $E$, $L$ are constants which represent some physical meanings respectively. Here (3) extends the classical D’Alembert wave equation by considering the effects of the changes in the length of the strings during the vibrations. Recently, Fiscella and Valdinoci in [20] first proposed a stationary Kirchhoff model involving the fractional Laplacian by taking into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see [20, Appendix A] for more details. Since then, many papers have been dedicated to investigating existence of solutions for the fractional Kirchhoff type problems, see [2, 11, 23, 54, 42, 46, 56] and the references therein for the degenerate case of Kirchhoff–type problems. We also collect some recent existence results for fractional non–degenerate Kirchhoff problems, see [20, 45, 47].

On the one hand, the study of system like (1) has been motivated by the following Schrödinger–Poisson type system

$$\left\{ \begin{array}{l}
- \Delta u + V(x)u + \phi u = f(x, u) \quad \text{in } \mathbb{R}^3 \\
- \Delta \phi = u^2 \quad \text{in } \mathbb{R}^3,
\end{array} \right.$$

which was studied in [17] as a physical model describing solitary waves for nonlinear Schrödinger type equations interacting with an unknown electrostatic field; see also [6, 18] for related topics. The first equation of (4) is coupled with a Poisson equation, which means that the potential is determined by the charge of the wave function. The term $\phi u$ is nonlocal and concerns the interaction with the electric field. For more details on the physical background of system (4), we refer the readers to [7, 17] and the references cited there.

In the last decades, many researchers have devoted to the existence and multiplicity of solutions for system like (4) via critical point theory under various assumptions on the potential $V$ and the nonlinearity, see e.g. [4, 5, 29, 36, 37]. In particular, Azzollini and D’Avenia in [4] first considered the following Schrödinger–Poisson
system with critical nonlocal term
\[
\begin{aligned}
-\Delta u - q|u|^3 u &= \lambda u \quad \text{in } B_R \\
-\Delta \phi &= q|u|^5 \quad \text{in } B_R \\
u = \phi = 0 \quad &\text{in } \partial B_R,
\end{aligned}
\]  
\tag{5}

where \( q > 0 \). Although the second equation in (5) can be solved by a Green’s function, the term \( p|u|^5 \) will lead to a nonlocal critically growing nonlinearity in (5), see [5] for more details. After that, by using a monotonic trick introduced by Jeanjean in [28], Li et al. in [32] studied the following case
\[
\begin{aligned}
-\Delta u + bu - q\phi|u|^3 u &= f(u) \quad \text{in } \mathbb{R}^3 \\
-\Delta \phi &= q|u|^5 \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]  
\tag{6}

where \( b \geq 0 \) and \( q \in \mathbb{R} \) are parameters. The authors proved that system (6) possesses at least one positive radially symmetric solution under certain conditions. Recently, Liu in [36] considered the existence of positive solutions for the following generalized Schrödinger–Poisson system
\[
\begin{aligned}
-\Delta u + V(x)u - K(x)\phi|u|^3 u &= f(x,u) \quad \text{in } \mathbb{R}^3 \\
-\Delta \phi &= K(x)|u|^5 \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]  
\tag{7}

where \( V, K, f \) are asymptotically periodic functions of \( x \). To the best of our knowledge, there is no paper to deal with the case \( f(x,u) = \lambda h(x)|u|^{p-2} u + |u|^{2s-2} u \) in (7).

Note that, by the reduction method proposed by Benci and Fortunato in [6], system (7) reduces to the well–known Choquard type equation
\[
-\Delta u + V(x)u - (I_\mu * (K|u|^5)) K|u|^3 u = f(x,u) \quad \text{in } \mathbb{R}^3,
\]  
\tag{8}

which was elaborated by Pekar [43] in the framework of quantum mechanics. Subsequently, it was adopted as an approximation of the Hartree–Fock theory, see [8]. Recently, Penrose in [44] settled it as a model of self–gravitational collapse of a quantum mechanical wave function. Here \( I_\mu : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R} \) denotes the Riesz potential. The first investigations for existence and symmetry of the solutions to (8) go back to the works of Lieb [34] and Lions [35]. For the critical case in the sense of Hardy–Littlewood–Sobolev inequality, we refer the interested reader to [24] for recent results in a smooth bounded domain of \( \mathbb{R}^N \). In the setting of the fractional Laplacian, Wu in [53] investigated the existence and stability of solutions for the following equation
\[
(\lambda)^s u + \omega u = (I_\mu * |u|^7)|u|^{7-2} u + \lambda f(x,u) \quad \text{in } \mathbb{R}^N,
\]  
\tag{9}

where \( q = 2, \lambda = 0 \) and \( \mu \in (N - 2s, N) \). Subsequently, D’Avenia and Squassina in [15] studied (9) with \( \lambda = 0 \), such as, the existence, nonexistence, and regularity, decays properties of solutions. In particular, they claimed the nonexistence of solutions as \( q \in (2 - \mu/N, 2s) \). The existence of ground states for fractional Choquard equations like (9) was investigated by [12] and [51].

On the other hand, Zhao et al. in [58] studied the following Kirchhoff–Schrödinger–Poisson system
\[
\begin{aligned}
&\left\{ 
\begin{align*}
a + b \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2 \, dx\\
-\Delta \phi = \lambda l(x) u^2, \quad x \in \mathbb{R}^3,
\end{align*}
\right.
\end{aligned}
\]

where constants \( a > 0, b \geq 0 \) and \( \lambda \geq 0 \). The authors obtained infinitely many solutions of (10) by using the symmetric mountain pass theorem. Liu in [39] studied the following Kirchhoff–type equation

\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V_\lambda(x) u = (\tilde{K}_\mu * u^q)|u|^{q-2} u \quad \text{in} \quad \mathbb{R}^3, \tag{11}
\]

where \( a > 0, b \geq 0 \) are given numbers, \( V_\lambda(x) = 1 + \lambda g(x), \lambda \in \mathbb{R}^+ \) is a parameter and \( g(x) \) is a continuous potential function on \( \mathbb{R}^3, q \in (2, 6 - \mu) \). By using the Nehari manifold and the concentration compactness principle, Liu obtained the existence of ground state solutions for (11) if the parameter \( \lambda \) is large enough. In [57], Zhang and Squassina considered the following fractional Schrödinger–Poisson system

\[
\begin{aligned}
&\left\{ 
\begin{align*}
(-\Delta)^s u + \lambda \phi u &= g(u) \quad \text{in} \quad \mathbb{R}^3 \\
(-\Delta)^t \phi &= \lambda u^2 \quad \text{in} \quad \mathbb{R}^3,
\end{align*}
\right.
\end{aligned}
\]

where \( \lambda > 0 \) and \( g \) satisfies subcritical or critical growth conditions. By using a perturbation approach, the authors obtained the existence of positive solutions for small \( \lambda \) and studied the asymptotic of solutions for \( \lambda \to 0^+ \). In [52], Teng studied the following fractional Schrödinger–Poisson system

\[
\begin{aligned}
&\left\{ 
\begin{align*}
(-\Delta)^s u + V(x) u + \phi u &= \mu |u|^{q-1} u + |u|^{2^*_s - 2} u \quad \text{in} \quad \mathbb{R}^3 \\
(-\Delta)^t \phi &= u^2 \quad \text{in} \quad \mathbb{R}^3,
\end{align*}
\right.
\end{aligned}
\]

where \( \mu > 0 \) is a parameter, \( 1 < q < 2^*_s - 1 \) and \( 2s + 4t > 3 \). When \( \mu \) is large enough, the existence of a nontrivial ground state solution was obtained in [52] by using the method of Pohozaev–Nehari manifold and the arguments of Brézis–Nirenberg, the monotonic trick and global compactness Lemma.

Motivated by the above works, we investigate the existence, multiplicity and asymptotic behavior of solutions for system (1) and overcome the lack of compactness due to the presence of critical terms as well as the possibly degenerate nature of the Kirchhoff coefficient. To the best of our knowledge, there are no results which studied the system (1) in the literature.

Before stating our main results, we introduce some notations. The natural solution space for (1) is \( D^s(\mathbb{R}^3) \), which is defined as the completion of \( C_0^\infty(\mathbb{R}^3) \) with respect to the norm \( \| \cdot \| \) defined in (2). In terms of Theorem 2 of [40], we know that

\[
\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^{2\alpha}} \, dx \leq \frac{1}{\mu^*} [u]_s^2 \quad \text{for all} \quad u \in D^s(\mathbb{R}^3),
\]

where \( \mu^* = \bar{c} \frac{(3 - 2s)^2}{s(1-s)} \) and \( \bar{c} > 0 \) is a positive constant. If \( 0 < \mu < \mu^* \), it follows from (13) that the norm \( \| \cdot \| \) is equivalent to \( \| \cdot \| \).

When \( \lambda \) is small enough, we obtain the following result.

**Theorem 1.1.** Assume that \( M \) fulfills (M₁)–(M₃), \( h \) satisfies (H₁) with \( 1 < p < 2 \), and \( \mu \in (0, \mu^*) \). Then there exists \( \lambda^{**} > 0 \) such that for all \( \lambda \in (0, \lambda^{**}) \) system (1) admits a nontrivial nonnegative solution \( (u_\lambda, \phi_{u_\lambda}) \) in \( D^s(\mathbb{R}^3) \times D^t(\mathbb{R}^3) \). Moreover, \( \lim_{\lambda \to 0^+} \| u_\lambda \| = 0 \).
The proof of Theorem 1.1 is mainly based on a minimization argument and takes inspiration from Theorem 1.4 of [21] and Theorem 1.2 of [48]. Obviously, Theorem 1.1 cannot cover the range \( p \in [2, 2^*_s) \) because of the geometry of the energy functional. For this, we consider the special case \( M(\tau) = \tau^{\theta-1} \). Hence the assumption of \( p \in (1, 2) \) can be relaxed to \( p \in (1, 2\theta) \). More precisely, we have the following result.

**Corollary 1.** Let \( M(\tau) = \tau^{\theta-1} \) for all \( \tau \geq 0 \), where \( 1 < \theta < 2^*_s/2 \). Assume that \( \mu \in (0, \mu^*) \) and that \( h \) satisfies \((H_1)\) with \( 1 < p < \theta \). Then there exists \( \lambda^{**} > 0 \) such that for all \( \lambda \in (0, \lambda^{**}) \) system (1) admits a nontrivial nonnegative solution \((u_\lambda, \phi_{u_\lambda})\) in \( D^*(\mathbb{R}^3) \times D'(\mathbb{R}^3) \). Moreover, \( \lim_{\lambda \to 0^+} \|u_\lambda\| = 0 \).

Moreover, if \( h \) is a radially symmetric function in \( \mathbb{R}^3 \), i.e., \( h(x) = h(|x|) \) for all \( x \in \mathbb{R}^3 \), then we have

**Theorem 1.2.** Assume that \( M \) fulfills \((M_1)-(M_3)\), \( h : \mathbb{R}^3 \to \mathbb{R}^+ \) is a radial function satisfying \((H_1)\) with \( 1 < p < 2 \), and \( \mu \in (0, \mu^*) \). Then there exists \( \lambda^{**} > 0 \) such that for all \( \lambda \in (0, \lambda^{**}) \) system (1) admits a nontrivial nonnegative radial solution \((u_\lambda, \phi_{u_\lambda})\) in \( D^*(\mathbb{R}^3) \times D'(\mathbb{R}^3) \). Moreover, \( \lim_{\lambda \to 0^+} \|u_\lambda\| = 0 \).

Under the same assumptions of Theorem 1.2, the Palais principle ensures the existence of a non–radial and sign–changing solution.

**Theorem 1.3.** Assume that \( M \) fulfills \((M_1)-(M_3)\), \( h : \mathbb{R}^3 \to \mathbb{R}^+ \) is a radial function satisfying \((H_1)\) with \( 1 < p < 2 \), and \( \mu \in (0, \mu^*) \). Then there exists \( \lambda^{**} > 0 \) such that for all \( \lambda \in (0, \lambda^{**}) \) system (1) admits a non–radial sign–changing solution \((u_\lambda, \phi_{u_\lambda})\) in \( D^*(\mathbb{R}^3) \times D'(\mathbb{R}^3) \). Moreover, \( \lim_{\lambda \to 0^+} \|u_\lambda\| = 0 \).

Now we turn to investigate the multiplicity of solutions for problem (1). Consequently, we have the following results.

**Theorem 1.4.** Assume that \( M \) fulfills \((M_1)-(M_3)\), \( h \) satisfies \((H_1)\) with \( 1 < p < 2 \), and \( \mu \in (0, \mu^*) \). Then there exists \( \lambda_* > 0 \) such that system (1) has infinitely many solutions for all \( \lambda \in (0, \lambda_*) \).

**Corollary 2.** Let \( M(\tau) = \tau^{\theta-1} \) for all \( \tau \geq 0 \), where \( 1 < \theta < 2^*_s/2 \). Assume that \( \mu \in (0, \mu^*) \) and that \( h \) satisfies \((H_1)\) with \( 1 < p < \theta \). Then there exists \( \lambda_* > 0 \) such that for all \( \lambda \in (0, \lambda_*) \) system (1) has infinitely many solutions for all \( \lambda \in (0, \lambda_*) \).

Let us simply describe the main approach to obtain Theorem 1.4. Since system (1) contains a critical nonlinearity, it is difficult to get the global Palais-Smale condition. To overcome the lack of compactness, we follow some techniques from [48], where a critical Kirchhoff problem involving the fractional Laplacian has been studied. See also [21] for a critical fractional \( p \)-Kirchhoff problem. We first show that the energy functional associated with system (1) satisfies the local Palais-Smale condition at suitable levels \( c < 0 \). To show the existence of infinitely many critical points for the energy functional associated with system (1), we shall use Krasnoleskii’s genus theory. Here we use an idea similar to that of [3]. On the Cartan–Hadamard manifold setting, a Rubik–cube technique and suitable variational arguments allow us to obtain further multiplicity results that will be presented in a forthcoming paper.

The paper is organized as follows. In Section 2, we recall some necessary definitions and properties for the functional setting and give some basic results. In Section 3, we obtain the existence and the asymptotic behavior of nontrivial nonnegative solutions for system (1) by using Ekeland’s variational principle. Moreover,
we also investigate the existence of nonnegative radial solutions and non-radial sign-changing solutions. In Section 4, we get the existence of infinitely many solutions for (1) by using Krasnoleskii’s genus theory and a truncated argument.

2. Preliminaries. We first provide some basic functional setting that will be used in the next sections. Let $s \in (0, 1)$. The critical exponent $2^*_s$ is defined as $6/(3 - 2s)$. We define the fractional Sobolev space $H^s(\mathbb{R}^3)$ as

$$H^s(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : |u|_s < \infty \},$$

endowed with the norm

$$||u||_s = \left( ||u||_2^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |u(x) - u(y)|^2 |x - y|^{3 + 2s} dxdy \right)^{1/2},$$

where $||u||_2 := ||u||_{L^2(\mathbb{R}^N)} = (\int_{\mathbb{R}^3} |u|^2 dx)^{1/2}$. The embeddings $H^s(\mathbb{R}^3) \subset D^s(\mathbb{R}^3) \hookrightarrow L^{2^*_s}(\mathbb{R}^3)$ are continuous by [14, Theorem 6.7]. Set

$$S = \inf_{u \in D^s(\mathbb{R}^3) \setminus \{0\}} \frac{|u|_{2^*_s}^2}{||u||_2^2}. \quad (14)$$

Clearly, $S > 0$. Hereafter we denote by $|| \cdot ||_q$ the norm of Lebesgue space $L^q(\mathbb{R}^3)$.

By the Hardy inequality (13), $D^s(\mathbb{R}^3)$ can be equipped with the inner product

$$\langle u, v \rangle_s = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3 + 2s}} dxdy - \mu \int_{\mathbb{R}^3} \frac{uv}{|x|^{2s}} dx,$$

for all $u, v \in D^s(\mathbb{R}^3)$, where $0 < \mu < \mu^* = \frac{\overline{c}(3 - 2s)^2}{s(1 - s)}$ and $\overline{c} > 0$ is a positive constant.

Let us now recall the Hardy–Littlewood–Sobolev inequality, see [33, Theorem 4.3].

**Theorem 2.1.** Assume that $1 < r_1, r_2 < \infty$, $0 < \mu < 3$ and

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{\mu}{3} = 2.$$

Then there exists $C(\mu, r_1, r_2) > 0$ such that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)| |v(y)|}{|x - y|^\mu} dxdy \leq C(\mu, r_1, r_2) ||u||_{r_1} ||v||_{r_2}$$

for all $u \in L^{r_1}(\mathbb{R}^3)$ and $v \in L^{r_2}(\mathbb{R}^3)$.

Note that, by the Hardy–Littlewood–Sobolev inequality, the integral

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^q |v(y)|^q}{|x - y|^\mu} dxdy$$

is finite, whenever $|u|^q \in L^r(\mathbb{R}^3)$ for some $r > 1$ satisfying

$$\frac{2}{r} + \frac{\mu}{3} = 2, \quad \text{that is} \quad r = \frac{6}{6 - \mu}. $$

Hence, by the fractional Sobolev embedding theorem, if $u \in H^s(\mathbb{R}^3)$ this occurs provided that $rq \in [2, 2^*_s]$. Thus, $q$ has to satisfies

$$\tilde{2}_{\mu,s} = \frac{6 - \mu}{3} \leq q \leq \frac{6 - \mu}{3 - 2s} = 2^*_{\mu,s}.$$

Hence, $\tilde{2}_{\mu,s}$ is called the lower critical exponent and $2^*_s$ is said to be the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality.
Let \( t \in (0, 1) \) and \( \mu = 3 - 2t \). By the Hardy–Littlewood–Sobolev inequality, there exists \( C(t) > 0 \) such that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\mu,t}}|u(y)|^{2^*_{\mu,t}}}{|x-y|^{3+2t}} \, dx \, dy \leq \tilde{C}(t)\|u\|^{2^*_{\mu,t}}_{2^*_{\mu,t}} \leq C(t)\|u\|^{2^*_{\mu,t}}_{2^*_{\mu,t}},
\]
for all \( u \in D^s(\mathbb{R}^3) \), where \( C(t) = \tilde{C}(t)S^{-2^*_{\mu,t}} \) and \( 2^*_{\mu,t} = \frac{3+2t}{3-2t} \). Let us define
\[
S_{H,L} := \inf_{u \in D^s(\mathbb{R}^3) \setminus \{0\}} \frac{\|u\|^2}{\left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^{2^*_{\mu,t}}|u(y)|^{2^*_{\mu,t}}}{|x-y|^{3+2t}} \, dx \, dy \right)^{1/2^*_{\mu,t}}}. \tag{15}
\]
Clearly, \( S_{H,L} > 0 \).

Fix \( u \in D^s(\mathbb{R}^3) \), we define a linear functional \( L_u : D^t(\mathbb{R}^3) \to \mathbb{R} \) as
\[
L_u(v) = \int_{\mathbb{R}^3} |u|^{2^*_{\mu,t}}v \, dx,
\]
for all \( v \in D^t(\mathbb{R}^3) \). By Hölder’s inequality and the fractional Sobolev inequality, one has
\[
|L_u(v)| \leq \|u\|^{2^*_{\mu,t}}_{2^*_{\mu,t}} \|v\|_{2^*_{\mu,t}} \leq C\|u\|^{2^*_{\mu,t}}_{2^*_{\mu,t}} \|v\|_{2^*_{\mu,t}},
\]
which implies that \( L_u \) is a continuous linear functional. By the Lax-Milgram theorem, the exists a unique \( \phi_u \in D^t(\mathbb{R}^3) \) such that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\phi_u(x) - \phi_u(y))(v(x) - v(y)) \frac{\, dx}{|x-y|^{3+2t}} = \int_{\mathbb{R}^3} |u|^{2^*_{\mu,t}}v \, dx, \quad \forall v \in D^t(\mathbb{R}^3),
\]
which means that \( \phi_u \) is a weak solution of
\[
(-\Delta)^t \phi_u = |u|^{2^*_{\mu,t}}, \quad x \in \mathbb{R}^3.
\]
Moreover, the representation formula holds
\[
\phi_u = C_t \int_{\mathbb{R}^3} \frac{|u(y)|^{2^*_{\mu,t}}}{|x-y|^{3-2t}} \, dy, \quad \forall x \in \mathbb{R}^3, \tag{16}
\]
where \( C_t > 0 \) is a constant. Let \( \tilde{K}_t(x) = |x|^{-3+2t} \) for \( x \in \mathbb{R}^3 \setminus \{0\} \). Then the convolution \( \tilde{K}_t * (\cdot) \) is called \( t \)-Riesz potential and \( \phi_u = C_t \tilde{K}_t * (|u|^{2^*_{\mu,t}}) \). In the sequel, we omit the constant \( C_t \) for convenience in (20). Substituting \( \phi_u \) in (1), it reduces to the following fractional Schrödinger–Kirchhoff type equation
\[
M(|u|^2)(-\Delta)^s u + V(x)u
= \lambda h(x)|u|^{p-2}u + \left( \tilde{K}_t * |u|^{2^*_{\mu,t}} \right) |u|^{2^*_{\mu,t}-2}u + |u|^{2^*_{\mu,t}-2}u, \tag{17}
\]
for all \( x \in \mathbb{R}^3 \).

**Definition 2.2.** We say that \((u, \phi) \in D^s(\mathbb{R}^3) \times D^t(\mathbb{R}^3)\) is a (weak) solution of problem (1), if \( u \) is a (weak) solution of equation (17), namely,
\[
M(|u|^2)(u, \phi)_s
= \lambda \int_{\mathbb{R}^3} h(x)|u|^{p-2}u \varphi \, dx + \int_{\mathbb{R}^3} \left( \tilde{K}_t * |u|^{2^*_{\mu,t}} \right) |u|^{2^*_{\mu,t}-2}u \varphi \, dx + \int_{\mathbb{R}^3} |u|^{2^*_{\mu,t}-2}u \varphi \, dx,
\]
for all \( \varphi \in D^s(\mathbb{R}^3) \).
To study solutions of equation (17), we define the functional \( I_\lambda : D^s(\mathbb{R}^3) \to \mathbb{R} \) as
\[
I_\lambda(u) = \frac{1}{2} \tilde{M} \| u \|^2 - \frac{\lambda}{p} \int_{\mathbb{R}^3} h(x) |u|^p dx - \frac{1}{2s} \int_{\mathbb{R}^3} |u|^2 \, dx.
\]

By Hölder’s inequality, one can deduce that
\[
\int_{\mathbb{R}^3} (K_t * |u|^{2^*}) |u|^{2^*} \, dx \leq \left( \int_{\mathbb{R}^3} (|u|^{2^*}) \frac{2^*}{2^*-r} \, dx \right)^{2^*-1} \left( \int_{\mathbb{R}^3} (K_t * |u|^{2^*}) \frac{2^*}{2^*-r} \, dx \right)^{\frac{1}{2^*-r}} \leq C \| u \|^{2^*} \| \tilde{K}_t * |u|^{2^*} \|_{2^*},
\]
which together with assumptions \((H_1)\) and \((V_1)\) implies that \( I_\lambda \) is well–defined and of class \( C^1(D^s(\mathbb{R}^3), \mathbb{R}) \). Moreover,
\[
\langle I'_\lambda(u), v \rangle = M \langle \| u \|^2, v \rangle_s - \lambda \int_{\mathbb{R}^3} h(x) |u|^{p-2} u v dx - \int_{\mathbb{R}^3} (K_t * (|u|^{2^*})) |u|^{2^*-2} uv dx - \int_{\mathbb{R}^3} |u|^{2^*-2} u v dx,
\]
for all \( u, v \in D^s(\mathbb{R}^3) \). From now on, \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( (D^s(\mathbb{R}^3))^* \) and \( D^s(\mathbb{R}^3) \). Here \( (D^s(\mathbb{R}^3))^* \) is the dual space of \( D^s(\mathbb{R}^3) \).

Let us recall that \( I_\lambda \) satisfies the \((PS)_c\) condition for \( c \in \mathbb{R} \) in \( D^s(\mathbb{R}^3) \), if any \((PS)_c\) sequence \( \{u_n\}_n \subset D^s(\mathbb{R}^3) \), namely a sequence such that \( I_\lambda(u_n) \to c \) and \( I'_\lambda(u_n) \to 0 \) as \( n \to \infty \), admits a strongly convergent subsequence in \( D^s(\mathbb{R}^3) \).

Since the system in presence of critical nonlinearities, it is not easy to get the global \((PS)_c\) condition. However, for \( c < 0 \) we obtain the following compactness result.

**Lemma 2.3** (The \((PS)_c\) condition). Assume that \( M \) fulfills \((M_1)\)–\((M_3)\), \( h \) satisfies \((H_1)\) with \( 1 < p < 2 \), and \( \mu \in (0, \mu^*) \). There exists \( \lambda^* > 0 \) such that \( I_\lambda \) satisfies the \((PS)_c\) condition on \( D^s(\mathbb{R}^3) \) for all \( \lambda \in (0, \lambda^*) \) and \( c < 0 \).

**Proof.** Fix any sequence \( \{u_n\}_n \subset D^s(\mathbb{R}^3) \) such that \( I_\lambda(u_n) \to c \) and \( I'_\lambda(u_n) \to 0 \) as \( n \to \infty \). In the following, we divide the proof into two parts.

- **Case** \( \inf_{n \in \mathbb{N}} \| u_n \| = \theta > 0 \). We first show that \( \{u_n\}_n \) is bounded. By \((M_1)\), \( c < 0 \) and the assumptions that \( \theta < \frac{2^*_s}{2} \) and \( 1 < p < 2 \), we get
\[
o(1) + o(1) \| u_n \| \geq I_\lambda(u_n) - \frac{1}{2s} \langle I'_\lambda(u_n), u_n \rangle
\]
\[
= \frac{1}{2} \tilde{M} \| u_n \|^2 - \frac{1}{2s} M \| u_n \|^2 \| u_n \|^2 + \lambda \left( \frac{1}{\frac{1}{2s} - 1} \right) \int_{\mathbb{R}^3} h |u_n|^p \, dx + \left( \frac{1}{\frac{1}{2s} - 1} \right) \int_{\mathbb{R}^3} (K_t * |u_n|^{2^*}) |u_n|^{2^*} \, dx
\]
\[
\geq \left( \frac{1}{\frac{1}{2} - 1} \right) M \| u_n \|^2 \| u_n \|^2 + \lambda \left( \frac{1}{\frac{1}{2s} - 1} \right) \int_{\mathbb{R}^3} h |u_n|^p \, dx \geq \left( \frac{1}{\frac{1}{2} - 1} \right) \kappa \| u_n \|^2 - \lambda \left( \frac{1}{\frac{1}{2s} - 1} \right) S^{-\frac{2^*_s}{2^*_s-p}} \| h \|_{\frac{2^*_s}{2^*_s-p}} \| u_n \|^p.
\]

This yields at once that \( \{u_n\}_n \) is bounded in \( D^s(\mathbb{R}^3) \). It is easy to verify that
\[
\int_{\mathbb{R}^3} \left| u_n \right|^{2^* - 2} u_n \left| \frac{2^*}{2^* - s} \right| dx = \int_{\mathbb{R}^3} \left| u_n \right|^{2^*} dx \leq S^{-\frac{2^*}{2^* - s}} \|u_n\|^{2^*} \leq C
\]
and
\[
\int_{\mathbb{R}^3} \left| u_n \right|^{2^* - 2} \left| \frac{2^*}{2^* - s} \right| dx = \int_{\mathbb{R}^3} \left| u_n \right|^{2^*} dx \leq C.
\]
Therefore, an application of Theorem 4.9 of [9] gives the existence of some \( u_\lambda \in D^s(\mathbb{R}^3) \) and \( \alpha, \delta \geq 0 \) such that, up to a subsequence, still denoted by \( \{u_n\}_n \),
\[
u_n \rightharpoonup u_\lambda \text{ weakly in } D^s(\mathbb{R}^3), \quad \|u_n\| \to \alpha, \quad \int_{\mathbb{R}^3} \left| u_n - u_\lambda \right|^{2^*} dx \to \delta, \quad u_n \to u_\lambda \text{ a.e. in } \mathbb{R}^3,
\]
\[
\left| u_n \right|^{2^* - 2} u_n \rightharpoonup |u_\lambda|^{2^* - 2} u_\lambda \text{ weakly in } L^{\frac{2^*}{2^* - s}}(\mathbb{R}^3)
\]
\[
\left| u_n \right|^{2^* - 2} \rightharpoonup |u_\lambda|^{2^* - 2} \text{ weakly in } L^{\frac{2^*}{2^* - s}}(\mathbb{R}^3).
\]
Since \( h \in L^{\frac{2^*}{2^* - p}}(\mathbb{R}^3) \), for any \( \varepsilon > 0 \) there exists \( R_\varepsilon > 0 \) such that
\[
\int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} h \left( \frac{\cdot}{|x|} \right) (x) dx \leq \varepsilon.
\]
Also, for any measurable subset \( U \subset B_{R_\varepsilon} \), we have
\[
\int_U h(x) |u_n|^p dx \leq \left( \int_U h \left( \frac{\cdot}{|x|} \right) (x) dx \right)^{\frac{2^* - p}{2^*}} \left( \int_U |u_n|^{2^*} dx \right)^{\frac{p}{2^*}} \leq C \left( \int_U h \left( \frac{\cdot}{|x|} \right) (x) dx \right)^{\frac{2^* - p}{2^*}}
\]
which implies that \( \{h(x)|u_n|^p\}_n \) is equi-integrable in \( B_{R_\varepsilon} \). By \( u_n \to u_\lambda \) a.e. in \( \mathbb{R}^3 \), we obtain that \( h(x)|u_n|^p \to h(x)|u_\lambda|^p \) a.e. in \( \mathbb{R}^3 \). Then Vitali convergence theorem yields
\[
\lim_{n \to \infty} \int_{B_{R_\varepsilon}} h(x) |u_n|^p - |u_\lambda|^p dx = 0.
\]
Note that
\[
\int_{\mathbb{R}^3} h(x) \|u_n\|^p - |u_\lambda|^p dx \leq \int_{B_{R_\varepsilon}} h(x) |u_n|^p - |u_\lambda|^p dx + \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}} h(x) |u_n|^p - |u_\lambda|^p dx \leq \int_{B_{R_\varepsilon}} h(x) |u_n|^p - |u_\lambda|^p dx + C\varepsilon^{\frac{2^* - p}{2^*}}.
\]
Letting \( n \to \infty \) and using the arbitrary of \( \varepsilon \), one has
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} h(x) |u_n|^p - |u_\lambda|^p dx = 0. \tag{20}
\]
This together with the Brézis–Lieb lemma yields that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} h(x)|u_n - u_\lambda|^p \, dx = 0. \tag{21}
\]

Next we claim that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} (K_t * |u_n|^{2^*})|u_n|^{2^* - 2} u_n u_\lambda \, dx = \int_{\mathbb{R}^3} (K_t * |u_\lambda|^{2^*})|u_\lambda|^{2^*} \, dx, \tag{22}
\]
and
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n|^{2^* - 2} u_n u_\lambda \, dx = \int_{\mathbb{R}^3} |u_\lambda|^{2^*} \, dx. \tag{23}
\]

Indeed, by the Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{6}{3+2t}}(\mathbb{R}^3)$ to $L^{\frac{6}{3-t}}(\mathbb{R}^3)$. Then by (19), it yields
\[
K_t * |u_n|^{2^*} \to K_t * |u_\lambda|^{2^*} \quad \text{in} \quad L^{\frac{6}{3+2t}}(\mathbb{R}^3), \tag{24}
\]
as $n \to \infty$. Note that for any measurable subset $U \subset \mathbb{R}^3$, we have
\[
\int_U \left| u_n \right|^{2^* - 2} v \, dx \leq \|u_n\|_{L^{2^*}(U)}^{2^* - 2} \|v\|_{L^{2^*}(U)} \leq C \|u_n\|_{L^{2^*}(U)},
\]
which implies that $\{|u_n|^{2^* - 2} u_n u_\lambda\}_{n \in \mathbb{N}}$ is equi-integrable in $\mathbb{R}^3$. Observe that $|u_n|^{2^* - 2} u_n u_\lambda \to |u_\lambda|^{2^*}$ a.e. in $\mathbb{R}^3$, then the Vitali convergence theorem yields that as $n \to \infty$
\[
|u_n|^{2^* - 2} u_n u_\lambda \to |u_\lambda|^{2^*} \quad \text{in} \quad L^{2^*}(\mathbb{R}^3). \tag{25}
\]

Combining (24) with (25) and $2^*/2^* = 6/(3 + 2t)$, we get the desired result (22).

Let us now introduce, for simplicity, the bi-linear functional $L(\cdot, \cdot)$ on $D^s(\mathbb{R}^3) \times D^s(\mathbb{R}^3)$ defined by
\[
L(v, w) = \langle v, w \rangle_s
\]
for all $v, w \in D^s(\mathbb{R}^3)$. The Hölder inequality gives
\[
|L(v, w)| \leq [v]_s[w]_s - \mu \frac{1}{\mu^*} [v]_s[w]_s \leq \left(1 - \frac{\mu}{\mu^*}\right) [v]_s[w]_s.
\]
Then by $\mu < \mu^*$, we know that the bi-linear functional $L(\cdot, \cdot)$ is continuous on $D^s(\mathbb{R}^3) \times D^s(\mathbb{R}^3)$. Hence, the weak convergence of $\{u_n\}$ in $D^s(\mathbb{R}^3)$ gives that
\[
\lim_{n \to \infty} L(u_\lambda, u_n - u_\lambda) = 0. \tag{26}
\]
Since $\{u_n\}$ in bounded in $D^s(\mathbb{R}^3)$, we have
\[
\lim_{n \to \infty} L(u_n, v) = L(u_\lambda, v) \tag{27}
\]
for any $v \in D^s(\mathbb{R}^3)$. Since $I^*_s(u_n) \to 0$ as $n \to \infty$, we have $\lim_{n \to \infty} \langle I^*_s(u_n), u_\lambda \rangle = 0$. Then by (22), (23) and (26), we deduce
\[
M(\alpha^2)\|u_\lambda\|^2 = \lambda \int_{\mathbb{R}^3} h|u_\lambda|^p \, dx + \int_{\mathbb{R}^3} (K_t * |u_\lambda|^{2^*})|u_\lambda|^{2^*} \, dx + \int_{\mathbb{R}^3} |u_\lambda|^{2^*} \, dx,
\]
which means that $\langle I^*_s(u_\lambda), u_\lambda \rangle = 0$. Here $I^*_s$ is defined as follows:
\[
I^*_s(v) = \frac{1}{2} M(\alpha^2)\|v\|^2 - \lambda \int_{\mathbb{R}^3} h|v|^p \, dx
\]
and By the definition of $S$.

By the Brézis–Lieb lemma, one can obtain

$$
\lim_{n \to \infty} \int_{\mathbb{R}^3} (|u_n|^2 - |u_n - u_\lambda|^2) \, dx = \int_{\mathbb{R}^3} |u_\lambda|^2 \, dx,
$$

(28)

Also, by using a Brézis–Lieb type result [24, Lemma 2.2], one has

$$
\int_{\mathbb{R}^3} (\tilde{K}_t * |u_n|^{2^*_t})|u_n|^{2^*_t} \, dx - \int_{\mathbb{R}^3} (\tilde{K}_t * |u_\lambda|^{2^*_t})|u_\lambda|^{2^*_t} \, dx
$$

$$
= \int_{\mathbb{R}^3} (\tilde{K}_t * |u_n - u_\lambda|^{2^*_t})|u_n - u_\lambda|^{2^*_t} \, dx + o(1).
$$

(29)

Since $\{u_n\}_n$ is a $(PS)$ sequence, we deduce from (22), (23) and (26)–(29) that

$$
o(1) = (I'_\lambda(u_n) - I'_{\alpha_\lambda}(u_\lambda), u_n - u_\lambda)
$$

$$
= M(\|u_n\|^2 - M(\|u_\lambda\|^2)\langle u_n, u_\lambda \rangle) - M(\langle u_n, u_\lambda \rangle)\langle u_n, u_\lambda \rangle
$$

$$
- \int_{\mathbb{R}^3} \left( (\tilde{K}_t * |u_n|^{2^*_t})|u_n|^{2^*_t} - (\tilde{K}_t * |u_\lambda|^{2^*_t})|u_\lambda|^{2^*_t} \right) (u_n - u_\lambda) \, dx
$$

$$
- \lambda \int_{\mathbb{R}^3} h(x)|u_n|^{p-2}u_n - |u_\lambda|^{p-2}u_\lambda (u_n - u_\lambda) \, dx
$$

$$
- \int_{\mathbb{R}^3} (|u_n|^{2^*_t-2}u_n - |u_\lambda|^{2^*_t-2}u_\lambda)(u_n - u_\lambda) \, dx
$$

$$
= M(\alpha_\lambda^2 - \|u_\lambda\|^2) - \int_{\mathbb{R}^3} \left( (\tilde{K}_t * (|u_n - u_\lambda|^{2^*_t}))|u_n - u_\lambda|^{2^*_t} \right) \, dx
$$

$$
- \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^*_t} \, dx + o(1)
$$

$$
= M(\alpha_\lambda^2)\|u_n - u_\lambda\|^2 - \int_{\mathbb{R}^3} \left( (\tilde{K}_t * (|u_n - u_\lambda|^{2^*_t}))|u_n - u_\lambda|^{2^*_t} \right) \, dx
$$

$$
- \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^*_t} \, dx + o(1).
$$

(30)

Here we use the fact that $\|u_n\|^2 = \|u_n - u_\lambda\|^2 + \|u_\lambda\|^2 + o(1)$.

It follows from (30) that

$$
\lim_{n \to \infty} M(\|u_n\|^2)\|u_n - u_\lambda\|^2
$$

$$
= \lim_{n \to \infty} \left( \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^*_t} \, dx + \int_{\mathbb{R}^3} \left( (\tilde{K}_t * |u_n - u_\lambda|^{2^*_t})|u_n - u_\lambda|^{2^*_t} \right) \, dx \right).
$$

(31)

By the definition of $S$ and $S_{H,L}$, we get

$$
\|u_n - u_\lambda\|_{2^*_t}^2 \leq S^{-1}\|u_n - u_\lambda\|^2
$$

(32)

and

$$
\int_{\mathbb{R}^3} (\tilde{K}_t * |u_n - u_\lambda|^{2^*_t})|u_n - u_\lambda|^{2^*_t} \, dx \leq C\|u_n - u_\lambda\|_{2^*_t}^{2^*_t}.
$$

(33)
Inserting (32) and (33) in (31), we have
\[ SM(\alpha^2) \lim_{n \to \infty} \left( \int |u_n - u_\lambda|^{2^*_s} \right)^{2/2^*_s} \]
\[ \leq \lim_{n \to \infty} \int |u_n - u_\lambda|^{2^*_s} dx + C \left( \int |u_n - u_\lambda|^{2^*_s} dx \right)^{2^*_s/2^*_s}. \]

Hence, it follows from (19) that
\[ SM(\alpha^2) \delta^{2^*_s/2^*_s}_\lambda \leq \delta \lambda + C \delta^{2^*_s/2^*_s}_\lambda. \]

On the other hand, we have by \((M_2), (21), (28)\) and (29) that
\[ c + o(1) = I_\lambda(u_n) - \frac{1}{2\theta} \langle T'_\alpha(u_\lambda), u_\lambda \rangle \]
\[ \geq \frac{1}{2\theta} M(\alpha^2_\lambda) \|u_n - u_\lambda\|^2 - \frac{\lambda}{p} \int h |u_n|^p dx + \frac{\lambda}{2\theta} \int h |u_\lambda|^p dx \]
\[ - \frac{1}{2^*_s} \int |u_n|^{2^*_s} dx + \frac{1}{2^*_s} \int |u_\lambda|^{2^*_s} dx - \frac{1}{22^*_s,t} \int (\tilde{K}_t * |u_n|^{2^*_r}) |u_n|^{2^*_r} dx \]
\[ + \frac{1}{2\theta} \int (\tilde{K}_t * |u_\lambda|^{2^*_r}) |u_\lambda|^{2^*_r} dx + o(1) \]
\[ \geq \frac{1}{2\theta} M(\alpha^2_\lambda) \|u_n - u_\lambda\|^2 + \lambda \left( \frac{1}{2\theta} - \frac{1}{p} \right) \int h |u_\lambda|^p dx \]
\[ - \frac{1}{22^*_s,t} \int (\tilde{K}_t * |u_n - u_\lambda|^{2^*_r}) |u_n - u_\lambda|^{2^*_r} dx \]
\[ - \frac{1}{2^*_s} \int |u_n - u_\lambda|^{2^*_s} dx + \left( \frac{1}{2\theta} - \frac{1}{2^*_s} \right) \int |u_\lambda|^{2^*_s} dx + o(1), \]

thanks to \(\theta < 2^*_s,t\) by assumption \(\theta < 2^*_s/2\). Using (30), we obtain
\[ c + o(1) \geq \lambda \left( \frac{1}{2\theta} - \frac{1}{p} \right) \int h |u_\lambda|^p dx \]
\[ + \left( \frac{1}{2\theta} - \frac{1}{22^*_s,t} \right) \int (\tilde{K}_t * |u_n - u_\lambda|^{2^*_r}) |u_n - u_\lambda|^{2^*_r} dx \]
\[ + \left( \frac{1}{2\theta} - \frac{1}{2^*_s} \right) \int |u_n - u_\lambda|^{2^*_s} dx + \left( \frac{1}{2\theta} - \frac{1}{2^*_s} \right) \int |u_\lambda|^{2^*_s} dx + o(1) \]

By the Hölder and Young inequalities, for any \(\varepsilon > 0\), we have
\[ \lambda \left( \frac{1}{p} - \frac{1}{2\theta} \right) \int h |u_\lambda|^p dx \leq \lambda \left( \frac{1}{p} - \frac{1}{2\theta} \right) \|u_\lambda\|^p_{\mu_\lambda} \|h\|_{\frac{2^*_s}{2^*_s-p}}^{\frac{2^*_s}{2^*_s-p}} \]
\[ \leq \varepsilon \|u_\lambda\|^{\frac{2^*_s}{2^*_s-p}} + \varepsilon^{-\frac{p}{2^*_s-p}} \left( \left( \frac{\lambda}{p} - \frac{\lambda}{2\theta} \right) \|h\|_{\frac{2^*_s}{2^*_s-p}} \right)^{\frac{2^*_s}{2^*_s-p}}, \]

thanks to \(p < 2\theta\). Taking \(\varepsilon = 1/(2\theta) - 1/2^*_s\) in above inequality and putting it in (35), we arrive at
\[ c + o(1) \geq - \left( \frac{1}{2\theta} - \frac{1}{2^*_s} \right)^{-\frac{p}{2^*_s-p}} \left[ \left( \frac{\lambda}{p} - \frac{\lambda}{2\theta} \right) \|h\|_{\frac{2^*_s}{2^*_s-p}} \right]^{\frac{2^*_s}{2^*_s-p}} \]
Thus, we have
\[
\left(\frac{1}{2\theta} - \frac{1}{2\theta_s}\right) \int_{\mathbb{R}^3} (\tilde{K}_t \ast |u_n - u_\lambda|^{2_s^*}) |u_n - u_\lambda|^{2_s^*} dx \\
+ \left(\frac{1}{2\theta} - \frac{1}{2\theta_s}\right) \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2_s^*} dx + o(1)
\]
which together with (39) gives that
\[
\frac{1}{2\theta} \int_{\mathbb{R}^3} (\tilde{K}_t \ast |u_n - u_\lambda|^{2_s^*}) |u_n - u_\lambda|^{2_s^*} dx \\
+ \frac{1}{2\theta} \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2_s^*} dx \leq c + o(1),
\]
Thus, we conclude that
\[
\lim_{\lambda \to 0} \delta_\lambda = \lim_{\lambda \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2_s^*} dx = 0
\]
and
\[
\lim_{\lambda \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^3} (\tilde{K}_t \ast |u_n - u_\lambda|^{2_s^*}) |u_n - u_\lambda|^{2_s^*} dx = 0
\]
Now we claim that
\[
\lim_{\lambda \to \infty} \alpha_\lambda = 0.
\]
Otherwise, there exists sequence \(\lambda_k\), with \(\lambda_k \to \infty\) as \(k \to \infty\), such that \(\alpha_{\lambda_k} \to \alpha_0 > 0\) as \(k \to \infty\). Note that
\[
c = \lim_{n \to \infty} \left( I_\lambda(u_n) - \frac{1}{2\theta_s} \langle I'_\lambda(u_n), u_n \rangle \right).
\]
A similar discussion as in (18) gives that
\[
c \geq \left(\frac{1}{2\theta} - \frac{1}{2\theta_s}\right) M(\alpha_\lambda^2) \alpha_{\lambda_k}^2 - \lambda_k C \left(\frac{1}{p} - \frac{1}{2\theta_s}\right) S^{-\frac{\theta}{2\theta_s}} \|h\|^{\frac{2_s^*}{2}}.
\]
Letting \(k \to \infty\) in above inequality and using \(c < 0\), we get
\[
0 \geq \left(\frac{1}{2\theta} - \frac{1}{2\theta_s}\right) M(\alpha_\lambda^2) \alpha_0^2 > 0,
\]
which is impossible. Thus, (39) holds true.
By \(u_n \to u_\lambda\) weakly in \(D^*(\mathbb{R}^3)\) and the weak lower semi-continuity of the norm, we get \(\|u_\lambda\| \leq \lim_{n \to \infty} \|u_n\|\), this together with (39) gives that
\[
\lim_{\lambda \to \infty} \|u_\lambda\| = 0.
\]
We claim that there exists \( \lambda^* > 0 \) such that \( \delta_\lambda = 0 \) for all \( \lambda \in (0, \lambda^*) \). Arguing by contradiction, we assume that \( \delta_\lambda > 0 \) for all \( \lambda > 0 \). Since \( \delta_\lambda \to 0 \) as \( \lambda \to 0 \), there exists a sequence \( \{\lambda_k\}_k \), with \( \delta_{\lambda_k} \subset (0, 1) \), such that \( \lambda_k \to 0 \) as \( k \to \infty \). By (34), without loss of generality, we assume that \( \alpha_{\lambda_k} \subset (0, 1) \) for all \( k \geq 1 \). Note that by (34) there holds

\[
SM(\alpha_{\lambda_k}^2)\delta_{\lambda_k}^{2/2^*} \leq \delta_{\lambda_k} + C\delta_{\lambda_k}^{22^*/(2^* + 1)} \leq (C + 1)\delta_{\lambda_k}.
\]

Thus,

\[
\delta_{\lambda_k} \geq \left[ \frac{SM(\alpha_{\lambda_k}^2)}{C + 1} \right]^{\frac{2}{2^*}}.
\]

Then, from (30) we have

\[
M(\alpha_{\lambda_k}^2)\delta_{\lambda_k}^2 \geq M(\alpha_{\lambda_k}^2)(\alpha_{\lambda_k}^2 - \|u_{\lambda_k}\|^2) \geq \delta_{\lambda_k} \geq \left[ \frac{SM(\alpha_{\lambda_k}^2)}{C + 1} \right]^{\frac{2}{2^*}}.
\]

Hence, \((M_3)\) gives

\[
\frac{2(2^* - 2)}{\alpha_{\lambda_k}^{2(2^* - 2)}} \geq \left( \frac{Sm_0}{C + 1} \right)^{\frac{2}{2^*}}.
\]

This is impossible, since \( \theta < 2^*/2 \) and \( \alpha_{\lambda_k} \to 0 \) as \( k \to \infty \). Therefore, \( \delta_\lambda = 0 \) for all \( \lambda \in (0, \lambda^*) \), that is,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^*} dx = 0
\]

for all \( \lambda \in (0, \lambda^*) \). This, together with (31) and (33), yields that \( u_n \to u_\lambda \) strongly in \( D^*(\mathbb{R}^3) \) as \( n \to \infty \).

- Case \( \inf_{n \in \mathbb{N}} \|u_n\| = 0 \). If 0 is an isolated point for the real sequence \( \{\|u_n\|\}_n \), then there is a subsequence \( \{u_{n_k}\}_k \) such that \( \inf_{k \in \mathbb{N}} \|u_{n_k}\| = d > 0 \), and we can proceed as before. Otherwise, 0 is an accumulation point of the sequence \( \{\|u_n\|\}_n \) and so there exists a subsequence \( \{u_{n_k}\}_k \) of \( \{u_n\}_n \) such that \( u_{n_k} \to 0 \) strongly in \( D^*(\mathbb{R}^3) \) as \( n \to \infty \).

In conclusion, \( I_\lambda \) satisfies the \((PS)_c\) condition in \( D^*(\mathbb{R}^3) \) for all \( \lambda \in (0, \lambda^*) \) and any \( c < 0 \).

**Remark 1.** Obviously, if \( M(\tau) = \tau^{\theta - 1} \) and \( 1 < p < 2\theta \), then Lemma 2.3 still holds true.

3. **Existence of solutions.** In this section, we always assume that \( M \) satisfies \((M_1)-(M_3)\) and \( h \) verifies \((H_1)\) with \( 1 < p < 2 \). Before going to prove Theorem 1.4, we first give some auxiliary lemmas.

**Lemma 3.1.** There exist \( \rho \in (0, 1) \) independent of \( \lambda, \lambda_0 > 0 \) and \( \alpha > 0 \), such that \( I_\lambda(u) \geq \alpha > 0 \) for any \( u \in D^*(\mathbb{R}^3) \), with \( \|u\| = \rho \), and for all \( \lambda \in (0, \lambda_0) \).

**Proof.** For all \( u \in D^*(\mathbb{R}^3) \), with \( \|u\| \leq 1 \), we obtain by \((M_2)\) that

\[
I_\lambda(u) \geq \frac{\widetilde{M}(1)}{2} \|u\|^{2^*} - \frac{1}{22^*_sH} S_{H,L}^{2^*_s} \|u\|^{22^*_s} - \frac{1}{p} \lambda S_{H,L}^{-\frac{2}{2^*}} \|h\| \|u\|^{\frac{2^*_s}{2^* - p}} - \frac{1}{2^*_s} S_{H,L}^{-\frac{2^*_s}{2^* - p}} \|u\|^{2^*_s}.
\]

(41)
By Young’s inequality, for any $\varepsilon > 0$ we have
\[ \lambda S^{-\frac{\varepsilon}{2}} \|h\|_{2\frac{s}{2s-p}} \|u\|^p \leq \varepsilon \|u\|^2 + \varepsilon^{-\frac{p}{2s-p}} \left( \lambda S^{-\frac{\varepsilon}{2}} \|h\|_{2\frac{s}{2s-p}} \right)^{\frac{2p}{2s-p}}, \]
since $p < 2\theta$. Thus, for $\varepsilon = p\bar{M}(1)/4$,
\[ I_\lambda(u) \geq \frac{M(1)}{4} \|u\|^2 - \left( \frac{4}{pM(1)} \right)^{\frac{p}{2s-p}} \left( \lambda \|h\|_{2\frac{s}{2s-p}} S^{-\frac{\varepsilon}{2}} \right)^{\frac{2p}{2s-p}} - \frac{1}{2s} S^{-\frac{\varepsilon}{2}} \|u\|^{2\varepsilon}. \]
Define $g(\tau) = \frac{p\bar{M}(1)}{4} \tau^2 - \frac{1}{2s} S^{-\frac{\varepsilon}{2}} \tau^{2\varepsilon} - \frac{1}{2s} S^{-\frac{\varepsilon}{2}} \tau^{2\varepsilon}$ for all $\tau \geq 0$. Since $2\theta < 2^* < 2s$,

there exists $\rho \in (0,1]$ small enough such that $g(\rho) > 0$. Set
\[ \lambda_0 = \frac{1}{2} g(\rho)^{\frac{2s-p}{2s-p}} \left( \frac{p\bar{M}(1)}{2} \right)^{\frac{p}{2s-p}} \left( \|h\|_{2\frac{s}{2s-p}} S^{-\frac{\varepsilon}{2}} \right)^{\frac{2p}{2s-p}} \lambda^{\frac{2s}{2s-p}}. \]
Then, for all $u \in D^s(\mathbb{R}^3)$, with $\|u\| = \rho \leq 1$, and for all $\lambda \leq \lambda_0$ we get
\[ I_\lambda(u) \geq g(\rho) - \left( \frac{4}{pM(1)} \right)^{\frac{p}{2s-p}} \left( \|h\|_{2\frac{s}{2s-p}} S^{-\frac{\varepsilon}{2}} \right)^{\frac{2p}{2s-p}} \lambda^{\frac{2s}{2s-p}} \geq \frac{1}{2} g(\rho) : = \alpha > 0. \]
The proof is therefore complete.

\begin{lemma}
Define
\[ c_\lambda = \inf \{ I_\lambda(u) : u \in \overline{B}_\rho \}, \]
where $B_\rho = \{ u \in D^s(\mathbb{R}^3) : \|u\| < \rho \}$ and $\rho \in (0,1]$ is given by Lemma 3.1. Then $c_\lambda < 0$ for all $\lambda \in (0,\lambda_0]$.
\end{lemma}

\begin{proof}
Choose a function $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that $0 \leq \varphi \leq 1$, $\|\varphi\| = 1$ and $\int_{\mathbb{R}^3} h(x)\varphi^p dx > 0$. Fix $\lambda \in (0,\lambda_0]$. Then, by $(H_1)$ and $(M_2)$, for all $\tau$, with $0 < \tau < 1$, we have
\[ I_\lambda(\tau \varphi) = \frac{1}{2} \bar{M}(\|\varphi\|^2 \tau^2) - \frac{\tau^{22s}}{2s} \int_{\mathbb{R}^3} (K_t \ast (\varphi^{2s})) \varphi^{2s} dx - \frac{\tau^p}{p} \int_{\mathbb{R}^3} h(x)\varphi^p dx - \frac{1}{2} \left( \sup_{0 \leq \tau \leq 1} M(\tau) \right) \|\varphi\|^2 \tau^2 - \lambda \tau^p \int_{\mathbb{R}^3} h(x)\varphi^p dx. \]
Since $1 < p < 2$, taking $\tau > 0$ so small such that $\tau \varphi \in B_\rho$ and $I_\lambda(\tau \varphi) < 0$. This gives that $c_\lambda < 0$ for all $\lambda \in (0,\lambda_0]$, as desired.
\end{proof}

\begin{remark}
If $M(\tau) = \tau^{\theta - 1}$ ($\theta > 1$) for all $\tau \geq 0$, then we deduce from the proof of Lemma 3.2 that
\[ I_\lambda(\tau \varphi) \leq \frac{1}{2\theta} \|\varphi\|^2 \tau^{2\theta} - \lambda \tau^p \int_{\mathbb{R}^3} h(x)\varphi^p dx. \]
Hence, Lemma 3.2 still holds true if $p \in (1,2\theta)$.
\end{remark}
Proof of Theorem 1.1. Let $\lambda \in (0, \lambda_0]$ be fixed. By Lemmas 3.1–3.2, together with the Ekeland variational principle which is applied in $\overline{B}_\rho$, there exists a sequence $\{u_n\}_n$ such that
\[
c_\lambda \leq I_\lambda(u_n) \leq c_\lambda + \frac{1}{n}
\]  
and
\[
I_\lambda(v) \geq I_\lambda(u_n) - \frac{\|u_n - v\|}{n} \tag{42}
\]
for all $v \in \overline{B}_\rho$. We show first that $\|u_n\| < \rho$ for $n$ sufficiently large.Arguing by contradiction, we assume that $\|u_n\| = \rho$ for infinitely many $n$. Without loss of generality, we may assume that $\|u_n\| = \rho$ for any $n \in \mathbb{N}$. In view of Lemma 3.1, we deduce
\[
I_\lambda(u_n) \geq \alpha > 0.
\]
This, together with (42), implies that $c_\lambda \geq \alpha > 0$, which contradicts Lemma 3.2.

Next we show that $I_\lambda'(u_n) \to 0$ in $(D^s(\mathbb{R}^3))^*$. Set
\[
w_n = u_n + \tau v, \quad \forall v \in B_1 := \{v \in D^s(\mathbb{R}^3) : \|v\| = 1\},
\]
where $\tau > 0$ small enough such that $2\tau \rho + \tau^2 \leq \rho^2 - \|u_n\|^2$ for fixed $n$ large. Then
\[
\|w_n\|^2 = \|u_n\|^2 + 2\tau \rho(u_n, v)_s + \tau^2 \\
\leq \|u_n\|^2 + 2\rho \tau + \tau^2 \\
\leq \rho^2,
\]
which means that $w_n \in \overline{B}_\rho$. Thus, from (43), we obtain
\[
I_\lambda(w_n) \geq I_\lambda(u_n) - \frac{\tau}{n} \|u_n - w_n\|
\]
that is,
\[
\frac{I_\lambda(u_n + \tau v) - I_\lambda(u_n)}{\tau} \geq -\frac{1}{n}.
\]
By letting $\tau \to 0^+$, we get $\langle I_\lambda'(u_n), v \rangle \geq -1/n$ for any fixed $n$ large. Similarly, choosing $\tau < 0$ and $|\tau|$ small enough and repeating the process above, one can obtain
\[
\langle I_\lambda'(u_n), v \rangle \leq \frac{1}{n}
\]
for any fixed $n$ large.

Thus, we conclude
\[
\lim_{n \to \infty} \sup_{v \in B_1} |\langle I_\lambda(u_n), v \rangle| = 0,
\]
which yields that $I_\lambda(u_n) \to 0$ in $(D^s(\mathbb{R}^3))^*$ as $n \to \infty$. Therefore, $\{u_n\}_n$ is a $(PS)_{c_\lambda}$ sequence for the functional $I_\lambda$. Furthermore, by Lemma 2.3 there exists $\lambda^{**} = \min\{\lambda_0, \lambda^*\} > 0$ such that for all $\lambda \in (0, \lambda^{**})$, up to a subsequence still denoted by $\{u_n\}_n$, $u_n$ strongly converges to $u_\lambda$ as $n \to \infty$. Moreover, $c_\lambda = I_\lambda(u_\lambda) < 0$ and $I_\lambda'(u_\lambda) = 0$. Thus, $u_\lambda$ is a nontrivial solution of equation (17). Furthermore, $(u_\lambda, \phi_{u_\lambda})$ is a nontrivial solution of system (1). Note that
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x - y|^{3+2s}} \, dx \, dy \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_\lambda(x) - u_\lambda(y)|^2}{|x - y|^{3+2s}} \, dx \, dy,
\]
Thus, one can easily verify that \(|u_\lambda| \in D^s(\mathbb{R}^3)\). Moreover, \(I_\lambda(|u_\lambda|) \leq I_\lambda(u_\lambda)\), which, together with the local minimality of \(u_\lambda\), yields that \(c_\lambda = I_\lambda(|u_\lambda|) = I_\lambda(u_\lambda)\). Therefore, we obtain that \((\{u_\lambda\}, \phi_{u_\lambda})\) is a nontrivial nonnegative solutions of system (1).

Next we give an estimate for \(\|u\|_\lambda\). Note that \(\|u_n\|^2 = \|u_n - u_\lambda\|^2 + \|u_\lambda\|^2 + o(1)\). By (M2) and (31), we have

\[
c_\lambda \geq \frac{1}{2\theta}M(\|u_n\|^2)\|u_n\|^2 - \frac{1}{22_{s,t}} \int_{\mathbb{R}^3} (\bar{K}_t * |u_n|^{2^{*}_t}) |u_n|^{2^{*}_t} dx \\
- \frac{\lambda}{p} \int_{\mathbb{R}^3} h|u_n|^p dx - \frac{1}{22_{s,t}} \int_{\mathbb{R}^3} |u_n|^{2^{*}_t} dx + o(1)
\]

\[
= \frac{1}{2\theta}M(\|u_n\|^2)\|u_n - u_\lambda\|^2 + \frac{1}{2\theta}M(\|u_n\|^2)\|u_\lambda\|^2 \\
- \frac{1}{22_{s,t}} \int_{\mathbb{R}^3} (\bar{K}_t * |u_n - u_\lambda|^{2^{*}_t}) |u_n - u_\lambda|^{2^{*}_t} dx \\
- \frac{1}{22_{s,t}} \int_{\mathbb{R}^3} (\bar{K}_t * |u_\lambda|^{2^{*}_t}) |u_\lambda|^{2^{*}_t} dx - \frac{1}{2s} \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^{*}_t} dx \\
- \frac{1}{2s} \int_{\mathbb{R}^3} |u_\lambda|^{2^{*}_t} dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} h|u_\lambda|^p dx + o(1)
\]

\[
= \frac{1}{2\theta}M(\alpha_\lambda^2)\|u_\lambda\|^2 + \left(\frac{1}{2\theta} - \frac{1}{22_{s,t}}\right) \int_{\mathbb{R}^3} |u_n - u_\lambda|^{2^{*}_t} dx \\
+ \left(\frac{1}{2\theta} - \frac{1}{22_{s,t}}\right) \int_{\mathbb{R}^3} (\bar{K}_t * |u_\lambda|^{2^{*}_t}) |u_\lambda|^{2^{*}_t} dx \\
- \frac{1}{22_{s,t}} \int_{\mathbb{R}^3} (\bar{K}_t * |u_\lambda|^{2^{*}_t}) |u_\lambda|^{2^{*}_t} dx \\
- \frac{1}{2s} \int_{\mathbb{R}^3} |u_\lambda|^{2^{*}_t} dx - \frac{\lambda}{p} \int_{\mathbb{R}^3} h|u_\lambda|^p dx + o(1).
\]

Note that \(\langle I'_{\alpha_\lambda}(u_\lambda), u_\lambda \rangle = 0\), we obtain

\[
M(\alpha_\lambda^2)\|u_\lambda\|^2 = \lambda \int_{\mathbb{R}^3} h|u_\lambda|^p dx + \int_{\mathbb{R}^3} (K_t * |u_\lambda|^{2^{*}_t}) |u_\lambda|^{2^{*}_t} dx + \int_{\mathbb{R}^3} |u_\lambda|^{2^{*}_t} dx.
\]

Combining this fact with (44), we get

\[
c_\lambda \geq \left(\frac{1}{2\theta} - \frac{1}{22_{s,t}}\right) M(\alpha_\lambda^2)\|u_\lambda\|^2 + \left(\frac{1}{2\theta} - \frac{1}{22_{s,t}}\right) \int_{\mathbb{R}^3} |u_\lambda|^{2^{*}_t} dx \\
- \frac{\lambda}{p} \int_{\mathbb{R}^3} h|u_\lambda|^p dx + o(1)
\]

It follows from \(\|u_\lambda\| < \rho\) and \(\rho\) does not depend on \(\lambda\) that

\[
c_\lambda \geq \left(\frac{1}{2\theta} - \frac{1}{22_{s,t}}\right) M(|u_\lambda|^2)\|u_\lambda\|^2 - \lambda \left(\frac{1}{p} - \frac{1}{22_{s,t}}\right) S^{-\frac{p}{2^{*}_t}} \|h\|_{\frac{2^{*}_t}{2^{*}_t - p}} \|u_\lambda\|^p + o(1).
\]

By Lemma 2.3, we know that \(\alpha_\lambda = \|u_\lambda\|\), this together with \(\tilde{c}_\lambda < 0\) yields

\[
\left(\frac{1}{2\theta} - \frac{1}{22_{s,t}}\right) M(|u_\lambda|^2)\|u_\lambda\|^2 \leq \lambda \left(\frac{1}{p} - \frac{1}{22_{s,t}}\right) S^{-\frac{p}{2^{*}_t}} \|h\|_{\frac{2^{*}_t}{2^{*}_t - p}} \|u_\lambda\|^p.
\]
Furthermore, from $0 < \rho \leq 1$ and $(M_2)$, we deduce
\[
\left( \frac{1}{2\theta} - \frac{1}{2s} \right) m_0 \|u_\lambda\|^{2p - p} \leq \left( \frac{1}{p} - \frac{1}{2s} \right) \eta \|h\| \|x\| \lambda,
\]
which together with $\theta < 2s/2$ and $p < 2\theta$ implies that
\[
\|u_\lambda\| \leq \left( \frac{2\theta(2s - p)}{p(2s - 2\theta)} \right) \eta \|h\| \|x\| \lambda.
\]
Hence, the proof is complete.

Proof of Corollary 1. The proof is similar to that of Theorem 1.1 by combining Lemma 3.1 with Remark 2.

Proof of Theorem 1.2. Define
\[
D^*_s(\mathbb{R}^3) = \{ v \in D^*(\mathbb{R}^3) : v(x) = v(|x|) \forall x \in \mathbb{R}^3 \},
\]
and consider the restriction of $I_\lambda$ on $D^*_s(\mathbb{R}^3)$. Then Lemma 2.3 and Lemma 3.1 still hold on $D^*_s(\mathbb{R}^3)$. By choosing $\varphi$ as a radial function satisfying the same conditions as in Lemma 3.2, we know that the result of Lemma 3.2 still holds true. Repeating the process of proving Theorem 1.1, we get the proof of Theorem 1.2.

Proof of Theorem 1.3. Set $H = O(1) \times O(1) \times O(1) \subset O(3)$, where
\[
O(3) = \{ A \in \mathbb{R}^{3 \times 3} : A^T A = I_3 \text{ and } \det A = 1 \}.
\]
Let $\tau$ be the involution defined on $\mathbb{R}^3$ by $\tau(x_1, x_2, x_3) = (x_2, x_1, x_3)$, that is,
\[
\tau x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x,
\]
where $x = (x_1, x_2, x_3)^T$. Clearly, $\tau \notin H$ and $\tau H = H\tau$. Now, consider the group $G = H \cup \{ \tau \} \supseteq H \cup \{ \eta \tau, \eta \in H \}$. Define the action of $G$ on $D^*(\mathbb{R}^3)$ by
\[
\eta v(x) = v(\eta^{-1} x), \quad \eta \tau v(x) = -v(\tau \eta^{-1} x), \quad \forall x \in \mathbb{R}^3, \forall v \in D^*(\mathbb{R}^3), \forall \eta \in H.
\]
Since $h$ is radially symmetric, $I_\lambda$ is $G$–invariant, i.e. $I_\lambda(gv) = I_\lambda(v)$, $\forall v \in D^*(\mathbb{R}^3)$, $\forall g \in G$. Indeed, fixed $v \in D^*(\mathbb{R}^3)$, for all $\eta \in H$, we have
\[
\|\eta v\|^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v(\eta x) - v(\eta y)|^2}{|x - y|^{3+2s}} dx dy - \mu \int_{\mathbb{R}^3} \frac{|v(\eta x)|^2}{|x|^{2s}} dx
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v(x') - v(y')|^2}{|x' - y'|^{3+2s}} dx' dy' - \mu \int_{\mathbb{R}^3} \frac{|v(x')|^2}{|x'|^{2s}} dx' = \|v\|^2,
\]
being $|x - y| = |\eta (x - y)| = |\eta x - \eta y| = |x' - y'| \quad \text{and } \det \eta = 1$. For $\tau \eta$, we can prove that $\|\tau \eta v\|^2 = \|v\|^2$. Moreover, since $h$ is radially symmetric, we know that $I_\lambda$ is $G$–invariant.

Set
\[
D^*_G(\mathbb{R}^3) = \{ v \in D^*(\mathbb{R}^3) : gv = v, \forall g \in G \}.
\]
Obviously, 0 is the only radial function on $D^*_G(\mathbb{R}^3)$. Thus, all the functions in $D^*_s(\mathbb{R}^3) \setminus \{ 0 \}$ are sign-changing. Denote by $I_{\lambda,G}$ the restriction of $I_\lambda$ on $D^*_G(\mathbb{R}^3)$. Considering the functional $I_{\lambda,G}$ on $D^*_G(\mathbb{R}^3)$ and using a similar discussion as Theorem 1.1, we can obtain the proof of Theorem 1.3.
4. Multiplicity of solutions. We begin by recalling some basic notions on the
Krasnoselskii genus which will be used in the proof of Theorem 1.4.

Let \( X \) be a Banach space and \( A \) a subset of \( X \). \( A \) is said to be symmetric if
\( u \in A \) implies \( -u \in A \). Let us denote by \( \Gamma \) the family of closed symmetric subsets
\( A \subset X \setminus \{0\} \).

**Definition 4.1.** Let \( A \in \Gamma \). The Krasnoselskii genus \( \gamma(A) \) of \( A \) is defined as being the
least positive integer \( k \) such that there is an odd mapping \( \varphi \in C(A, \mathbb{R}^k) \) such that \( \varphi(x) \neq 0 \) for all \( x \in A \). If \( k \) does not exist we set \( \gamma(A) = \infty \). Furthermore, by definition, \( \gamma(\emptyset) = 0 \).

In the sequel we list some properties of the genus that will be used later. For
more details on this subject, we refer to [49].

**Proposition 1.** Let \( A, B \) be sets in \( \Gamma \).

1. If there exists an odd map \( \varphi \in C(A, B) \), then \( \gamma(A) \leq \gamma(B) \).
2. If \( A \in \Gamma \) and \( \gamma(A) \geq 2 \), then \( A \) has infinitely many points.
3. If \( A \subset B \), then \( \gamma(A) \leq \gamma(B) \).
4. If \( A \cup B \in \Gamma \), then \( \gamma(A \cup B) \leq \gamma(A) + \gamma(B) \).
5. If \( S \) is a sphere centered at the origin in \( \mathbb{R}^k \), then \( \gamma(S) = k \).
6. If \( A \) is compact, then \( \gamma(A) < \infty \) and there exists \( \delta > 0 \) such that \( N_{\delta}(A) \in \Gamma \) and
\( \gamma(N_{\delta}(A)) = \gamma(A) \), where \( N_{\delta}(A) = \{x \in X : ||x - A|| \leq \delta\} \).

Following the idea of [3] (see also [25]), we construct a truncated functional \( J_\lambda \)
such that critical points \( u \) of \( J_\lambda \) with \( J_\lambda(u) < 0 \) are also critical points of \( I_\lambda \). Since
the system (1) contains a nonlocal coefficient \( M(||u||^2) \) and the operator \((-\Delta)^s) \)
is nonlocal, our job is complicated. To overcome these difficulties, we divide the
discussion into two cases \( ||u|| \leq 1 \) and \( ||u|| > 1 \) in the building of \( J_\lambda \).

**Case 1.** \( ||u|| \leq 1 \). By the definition of \( I_\lambda \) and (M3), we deduce
\[
I_\lambda(u) \geq \frac{m_0}{2} \|u\|^{2\theta} - \frac{\lambda}{p} \int_{\mathbb{R}^3} h|u|^{p} \, dx
- \frac{1}{2^{2^*_s,t}_*} \int_{\mathbb{R}^3} (\widetilde{K} * |u|^{2^*_s,t}) |u|^{2^*_s,t} \, dx
- \frac{1}{2^{2^*_s,t}_*} \int_{\mathbb{R}^3} |u|^{2^*_s} \, dx,
\]
for all \( u \in D^s(\mathbb{R}^3) \) with \( ||u|| \leq 1 \). From the definitions of \( 2^*_s \) and \( 2^*_s,t \), we have
\[
\int_{\mathbb{R}^3} |u|^{2^*_s} \, dx \leq S^{-2^*_s} ||u||^{2^*_s},
\]
and
\[
\int_{\mathbb{R}^3} (\widetilde{K} * |u|^{2^*_s,t}) |u|^{2^*_s,t} \, dx \leq S^{-2^*_s,t} ||u||^{2^*_s,t}.
\]
By these two inequalities and Hölder’s inequality, we obtain
\[
I_\lambda(u) \geq \frac{m_0}{2} \|u\|^{2\theta} - \frac{\lambda}{p} S^{-\frac{\theta}{2}} \|h\|^{\frac{2^*_s}{2^{2^*_s}_*}} \|u\|^{2^*_s}
- \frac{1}{2^{2^*_s,t}_*} S^{-2^*_s,t} ||u||^{2^*_s,t}
- \frac{1}{2^{2^*_s,t}_*} S^{-\frac{\theta}{2}} \|u\|^{2^*_s},
\]
:= \( F(||u||) \),
(45)
where
\[
F(\tau) = \frac{m_0}{2} \tau^{2\theta} - \frac{\lambda}{p} S^{-\frac{\theta}{2}} \|h\|^{\frac{2^*_s}{2^{2^*_s}_*}} \tau^p
- \frac{1}{2^{2^*_s,t}_*} S^{-2^*_s,t} \tau^{2^*_s,t}
- \frac{1}{2^{2^*_s,t}_*} S^{-\frac{\theta}{2}} \tau^{2^*_s}.
\]
Since \( p < 2\theta < 2^*_s < 22^*_s,t \), there exists \( \lambda_s \in (0, \lambda^* \) small enough such that \( F \)
attains its positive maximum for \( \lambda \in (0, \lambda_s) \). Here \( \lambda^* > 0 \) is given by Lemma 2.3.
Denote by $0 < T_0(\lambda) < T_1(\lambda)$ the unique two roots of $F(\tau) = 0$. Indeed, to get the solutions of $F(\tau) = 0$, we consider $\bar{F}$ defined as

$$\bar{F}(\tau) = \frac{m_0}{2} \tau^{2\theta - p} - \frac{1}{22s_t} S^{-\frac{2\tau}{p}} T_{0\tau}^2 - \frac{1}{22s_t} S^{-\frac{2\tau}{p}} T_{0\tau}^2$$

for all $\tau \geq 0$.

Actually, $T_0(\lambda)$ has the following property.

**Lemma 4.2.** \( \lim_{\lambda \to 0^+} T_0(\lambda) = 0. \)

**Proof.** By $F(T_0(\lambda))$ and $F'(T_0(\lambda)) > 0$, we have

\[
\frac{m_0}{2} T_0(\lambda)^{2\theta} = \frac{\lambda}{p} \| h \|_{2s_t}^{2s_t} T_0(\lambda)^p + \frac{1}{22s_t} S^{-\frac{2}{p}} T_0(\lambda)^2 + \frac{1}{22s_t} S^{-\frac{2}{p}} T_0(\lambda)^2
\]

and

\[
\theta m_0 T_0(\lambda)^{2\theta - 1} > \lambda \| h \|_{2s_t}^{2s_t} T_0(\lambda)^{p - 1} + S^{-\frac{2}{p}} T_0(\lambda)^{2\theta}.
\]

It follows from (46) that

$$T_0(\lambda) \leq \left( \frac{2^s m_0}{2} S^{\frac{m_0}{2}} \right)^{\frac{1}{2\theta - 2p}},$$

which means that $T_0(\lambda)$ is uniformly bounded with respect to $\lambda$. Fix any sequence $\{\lambda_k\} \subset (0, \infty)$, with $\lambda_k \to 0$ as $k \to \infty$. Assume that $T_0(\lambda_k) \to T_0$ as $k \to \infty$. Then by (46) and (47), we have

\[
\frac{m_0}{2} T_0^{2\theta} = \frac{1}{22s_t} S^{-\frac{2}{p}} T_0^{2\theta} + \frac{1}{22s_t} S^{-\frac{2}{p}} T_0^{2\theta},
\]

and

\[
\theta m_0 T_0^{2\theta - 1} \geq S^{-\frac{2}{p}} T_0^{2\theta - 1} + S^{-\frac{2}{p}} T_0^{2\theta - 1}.
\]

Combining (48) with (49), we deduce

\[
\left( \frac{1}{2\theta} - \frac{1}{22s_t} \right) S^{-\frac{2}{p}} T_0^{2\theta} + \left( \frac{1}{2\theta} - \frac{1}{22s_t} \right) S^{-\frac{2}{p}} T_0^{2\theta} \leq 0,
\]

which implies that $T_0 = 0$, thanks to $2\theta < 2s_t < 22s_t$. The arbitrary of $\{\lambda_k\}$ yields that \( \lim_{\lambda \to 0^+} T_0(\lambda) = 0. \) This completes the proof. \( \Box \)

By Lemma 4.2, we can assume that $T_0(\lambda) < 1$ for small $\lambda$. Thus, $T_0(\lambda) < \min\{T_1(\lambda), 1\}$. Take $\Phi \in C^0_\infty([0, \infty)), 0 \leq \Phi(\tau) \leq 1$ for all $\tau \geq 0$ and

$$\Phi(\tau) = \begin{cases} 1, & \tau \in [0, T_0(\lambda)], \\
0, & \tau \in [\min\{T_1(\lambda), 1\}, \infty). \end{cases}$$
Then we define the functional
\[
J_\lambda(u) = \frac{1}{2} \tilde{M} \left( \|u\|^2 \right) - \frac{\lambda}{p} \int_{\mathbb{R}^3} h(x) |u|^p dx \\
- \frac{1}{2\gamma s} \Phi(||u||) \int_{\mathbb{R}^3} \left( \tilde{K}_t * |u|^{2^*_s} \right) |u|^{2^*_s} dx - \frac{1}{2\gamma s} \Phi(||u||) \int_{\mathbb{R}^3} |u|^{2^*_s} dx.
\]

One can easily verify that \( J_\lambda \in C^1(D^*(\mathbb{R}^3), \mathbb{R}) \) and \( J_\lambda(u) \geq \tilde{F}(\|u\|) \) for all \( u \in D^*(\mathbb{R}^3) \) with \( \|u\| < 1 \), where
\[
G(\tau) := \frac{m_0}{2} \tau^{2\theta} - \frac{\lambda}{p} S^{-\theta} \|h\|_{\frac{2\theta}{2\theta - p}} \tau^p - \frac{1}{2\gamma s} \Phi(\tau) \tau^{2^*_s} - \frac{1}{2\gamma s} \Phi(\tau) \tau^{2^*_s}.
\]

Clearly, \( G(\tau) \geq F(\tau) \geq 0 \) for all \( \tau \in (T_0(\lambda), \min \{ T_1(\lambda), 1 \}) \). By the definitions of \( I_\lambda \) and \( J_\lambda \), we know that \( I_\lambda(u) = J_\lambda(u) \) for all \( \|u\| \leq T_0(\lambda) \). Let \( u \) be a critical point of \( J_\lambda \) with \( J_\lambda(u) < 0 \). If \( \|u\| < T_0(\lambda) \), then \( u \) is also a critical point of \( I_\lambda \). To show that \( \|u\| < T_0(\lambda) \) is important to ensure that \( J_\lambda(u) \geq 0 \) when \( \|u\| \geq 1 \).

**Case 2.** \( \|u\| > 1 \). Note that in this case we always have \( \Phi(||u||) = 0 \). Hence, for all \( \|u\| > 1 \), we obtain by (M1) and (M2) that
\[
J_\lambda(u) = \frac{1}{2} \tilde{M} \left( \|u\|^2 \right) - \frac{\lambda}{p} \int_{\mathbb{R}^3} h(x) |u|^p dx \\
\geq \frac{1}{2\gamma} \kappa(1) \|u\|^2 - \frac{\lambda}{p} S^{-\theta} \|h\|_{\frac{2\theta}{2\theta - p}} \|u\|^p \\
= \tilde{g}(\|u\|),
\]
where \( \tilde{g} : [0, \infty) \to \mathbb{R} \) defined as
\[
\tilde{g}(\tau) = \frac{1}{2\gamma} \kappa(1) \tau^2 - \frac{\lambda}{p} S^{-\theta} \|h\|_{\frac{2\theta}{2\theta - p}} \tau^p.
\]

It is easy to check that \( \tilde{g} \) has a global minimum point at \( \tau_\lambda = \left( \frac{\lambda S^{-\theta}}{\kappa(1)} \|h\|_{\frac{2\theta}{2\theta - p}} \right)^\frac{1}{2\gamma} \) and
\[
\tilde{g}(\tau_\lambda) = \left( \frac{\theta}{\kappa(1)} \right)^\frac{2\gamma}{2\gamma - p} \left( \lambda \|h\|_{\frac{2\theta}{2\theta - p}} \right)^\frac{2\gamma}{2\gamma - p} \left( \frac{1}{2} - \frac{1}{p} \right) < 0,
\]
being \( p < 2 \). Observe that \( \tilde{g}(\tau) \geq 0 \) if and only if \( \tau \geq \left( \frac{2\lambda S^{-\theta}}{\kappa(1)} \|h\|_{\frac{2\theta}{2\theta - p}} \right)^\frac{1}{2\gamma} \). Thus, to ensure that \( J_\lambda(u) \geq 0 \) for all \( \|u\| \geq 1 \), we let 1 \( > \left( \frac{2\lambda S^{-\theta}}{\kappa(1)} \|h\|_{\frac{2\theta}{2\theta - p}} \right)^\frac{1}{2\gamma} \), that is, \( \lambda \leq \frac{2\gamma}{\kappa(1)} S^{-\theta} \). Hence, we take \( \lambda_* = \frac{2\gamma}{\kappa(1)} S^{-\theta} \). Then for each \( \lambda \in (0, \lambda_*) \) we have \( J_\lambda(u) \geq 0 \) for any \( \|u\| \geq 1 \).

**Lemma 4.3.** Let \( \lambda \in (0, \lambda_*) \). If \( J_\lambda(u) < 0 \), then \( \|u\| < T_0(\lambda) \) and \( J_\lambda(v) = I_\lambda(v) \) for all \( v \) in a small enough neighbourhood of \( u \). Moreover, \( J_\lambda \) satisfies a local (FS)\(_c\) condition for all \( c < 0 \).
Lemma 4.4. For any fixed $\lambda \in (0, \lambda_*)$, $J_\lambda(u) \geq 0$ for all $\|u\| \geq 1$. Thus, if $J_\lambda(u) < 0$ we have $\|u\| < 1$ and consequently $G(\|u\|) \leq J_\lambda(u) < 0$. Therefore $\|u\| < T_0(\lambda)$ and $J_\lambda(u) = I_\lambda(u)$. Moreover, $J_\lambda(v) = I_\lambda(v)$ for all $v$ satisfying $\|v - u\| < T_0(\lambda) - \|u\|$. Let $\{u_n\}_n$ be a sequence such that $J_\lambda(u_n) \to c < 0$ and $I_\lambda'(u_n) \to 0$. Then for $n$ sufficiently large, we have $I_\lambda(u_n) = J_\lambda(u_n) \to c < 0$ and $I_\lambda'(u_n) = J_\lambda'(u_n) \to 0$. Note that $J_\lambda$ is coercive in $D^s(\mathbb{R}^3)$. Thus, $\{u_n\}_n$ is bounded in $D^s(\mathbb{R}^3)$. By Lemma 2.3, up to a subsequence, $\{u_n\}_n$ is strongly convergent in $D^s(\mathbb{R}^3)$. 

Remark 3. Set $K_\epsilon = \{u \in D^s(\mathbb{R}^3) : J_\lambda'(u) = 0, J_\lambda(u) = c\}$. If $\lambda \in (0, \lambda_*)$ and $c < 0$, it follows from Lemma 4.3 that $K_\epsilon$ is compact.

Now, we will construct an appropriate mini-max sequence of negative critical values for the functional $J_\lambda$. For $\epsilon > 0$, we define

$$J_\lambda^{\epsilon^\ast} = \{u \in D^s(\mathbb{R}^3) : J_\lambda(u) \leq -\epsilon\}.$$

Lemma 4.4. For any fixed $k \in \mathbb{N}$ there exists $\epsilon_k > 0$ such that

$$\gamma(J_\lambda^{\epsilon^\ast}) \geq k.$$

Proof. Denote by $E_k$ an $k$-dimensional subspace of $D^s(\mathbb{R}^3)$. For any $u \in E_k$, $u \neq 0$, set $u = r_k v$ with $v \in E_k$, $\|v\| = 1$ and $r_k = \|u\|$. By the assumption of $h$, we know that $(\int_{\mathbb{R}^3} h(x)|v|^pdx)^{1/p}$ is a norm of $E_k$. Since all norms are equivalent in a finite Banach space, for each $v \in E_k$ with $\|v\| = 1$, there exists $C_k > 0$ such that

$$\int_{\mathbb{R}^3} h(x)|v|^pdx \geq C_k.$$

Thus, for $r_k \in (0, T_0(\lambda))$, we have

$$J_\lambda(u) = I_\lambda(u) = \frac{1}{2} \widetilde{M}(\|u\|^2) - \frac{\lambda}{p} \int_{\mathbb{R}^3} h(x)|u|^pdx - \frac{1}{22^s_{s_t}} \int_{\mathbb{R}^3} (K_t * |u|^{2^s_{s_t}})|u|^{2^s_{s_t}}dx - \frac{1}{2} \int_{\mathbb{R}^3} |u|^{2^s_{s_t}}dx \leq \frac{1}{2} \max_{0 \leq \tau \leq 1} M(\tau) \|u\|^2 - \frac{\lambda}{p} C_k r_k^p \int_{\mathbb{R}^3} |u|^{2^s_{s_t}}dx \leq \frac{1}{2} \left( \max_{0 \leq \tau \leq 1} M(\tau) \right) r_k^p - \frac{\lambda}{p} C_k r_k^p.$$

Since $p \in (1, 2)$, we can choose $r_k \in (0, T_0(\lambda))$ so small such that $J_\lambda(u) \leq \epsilon_k < 0$. Set $S_{\epsilon_k} = \{u \in D^s(\mathbb{R}^3) : \|u\| = r_k\}$. Then $S_{\epsilon_k} \cap E_k \subset J_\lambda^{\epsilon^\ast}$. Hence, from Proposition 1 it follows that $\gamma(J_\lambda^{\epsilon^\ast}) \geq \gamma(S_{\epsilon_k} \cap E_k) = k$. 

Remark 4. If $\theta > 1$, $M(\tau) = \tau^{\theta - 1}$ and $p \in (1, 2\theta)$, then

$$J_\lambda(u) = I_\lambda(u) \leq \frac{1}{2\theta} \|u\|^{2\theta} - \frac{\lambda}{p} C_k r_k^p = \frac{1}{2\theta} r_k^{2\theta} - \frac{\lambda}{p} C_k r_k^p.$$

From which it is easy to see that the result of Lemma 4.4 still holds.

Set $\Gamma_k = \{A \in \Gamma : \gamma(A) \geq k\}$ and let

$$c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} J_\lambda(u). \quad (51)$$

Then,

$$-\infty < c_k \leq \epsilon_k < 0, \forall k \in \mathbb{N},$$
since $J_{\lambda}^{c_k} \in \Gamma_k$ and $J_\lambda$ is bounded from below. By (51), we have $c_k < 0$. Since $J_\lambda$ satisfies $(PS)_c$ condition by Lemma 4.3, it follows from a standard argument that all $c_k$ are critical values of $J_\lambda$.

**Lemma 4.5.** Let $\lambda \in (0, \lambda_*)$. If $c = c_k = c_{k+1} = \cdots = c_{k+m}$ for some $m \in \mathbb{N}$, then $\gamma(K_c) \geq m + 1$.

**Proof.** Arguing by contradiction, we assume that $\gamma(K_c) \leq m$. By Remark 3, we know that $K_c$ is compact and $K_c \in \Gamma$. It follows from Proposition 1 that there exists $\delta > 0$ such that

$$\gamma(K_c) = \gamma(N_\delta(K_c)) \leq m.$$  

From the deformation lemma ([49, Theorem A.4]), there exist $0 < \epsilon < -c$, and an odd homeomorphism $\eta : D^*(\mathbb{R}^3) \to D^*(\mathbb{R}^3)$ such that

$$\eta(J^{c+\epsilon}_\lambda \setminus N_\delta(K_c)) \subset J^{\epsilon-\epsilon}_\lambda.$$  

(52)

On the other hand, by the definition of $c = c_k + m$, there exists $A \in \Gamma_{k+m}$ such that $\sup_{u \in A} J_\lambda(u) < c + \epsilon$, which means that

$$A \subset J^{c+\epsilon}_\lambda.$$  

It follows from Proposition 1 that

$$\gamma(A \setminus N_\delta(K_c)) \geq \gamma(A) - \gamma(N_\delta(K_c)) \geq k$$

and

$$\gamma(\eta(A \setminus N_\delta(K_c))) \geq k.$$  

Thus,

$$\eta(A \setminus N_\delta(K_c)) \in \Gamma_k,$$

which contradicts (52). This completes the proof. \qed

**Proof of Theorem 1.4.** Let $\lambda \in (0, \lambda_*)$. If $-\infty < c_1 < c_2 < \cdots < c_k < \cdots < 0$, since $c_k$ are critical values of $J_\lambda$, we obtain infinitely many critical points of $J_\lambda$. From Lemma 4.3, $I_\lambda = J_\lambda$ if $J_\lambda < 0$. Hence system (1) has infinitely many solutions.

If there exist $c_k = c_{k+m}$, then $c = c_k = c_{k+1} = \cdots = c_{k+m}$. By Lemma 4.5, we have $\gamma(K_c) \geq m + 1 \geq 2$. From (2) of Proposition 1, $K_c$ has infinitely many points. Thus, system (1) has infinitely many solutions. Thus, the proof is complete. \qed

**Proof of Corollary 2.** The proof is similar to Theorem 1.4, so we omit it. \qed

REFERENCES

[1] D. Applebaum, Lévy processes-from probability to finance quantum groups, Notices Amer. Math. Soc., 51 (2004), 1336–1347.
[2] G. Autuori, A. Fiscella and P. Pucci, Stationary Kirchhoff problems involving a fractional operator and a critical nonlinearity, Nonlinear Anal., 125 (2015), 699–714.
[3] J. García Azorero and I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, Trans. Amer. Math. Soc., 323 (1991), 877–895.
[4] A. Azzollini and P. d’Avenia, On a system involving a critically growing nonlinearity, J. Math. Anal. Appl., 387 (2012), 433–438.
[5] A. Azzollini, P. d’Avenia and V. Luisi, Generalized Schrödinger-Poisson type systems, Commun. Pure Appl. Anal., 12 (2013), 867–879.
[6] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods. Nonlinear Anal., 11 (1998), 283–293.
VARIATIONAL ANALYSIS FOR NONLOCAL YAMABE-TYPE SYSTEMS

[7] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations, Rev. Math. Phys., 14 (2002), 409–420.

[8] A. Bongers, Existenzaussagen für die Choquard-Gleichung: Ein nichtlineares eigenwertproblem der plasma-physics, Z. Angew. Math. Mech., 60 (1980), T240–T242.

[9] H. Brézis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitytext, Springer, New York, 2011.

[10] L. Caffarelli, Non-local diffusions, drifts and games, Nonlinear Partial Differential Equations, Abel Symposia, Springer, Heidelberg, 7 (2012), 37–52.

[11] M. Caponi and P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations, Ann. Mat. Pura Appl., 195 (2016), 2099–2129.

[12] Y.-H. Chen and C. G. Liu, Ground state solutions for non-autonomous fractional Choquard equations, Nonlinearity, 29 (2016), 1827–1842.

[13] A. Bongers, Existenzaussagen für die Choquard-Gleichung: Ein nichtlineares eigenwertproblem der plasma-physics, Z. Angew. Math. Mech., 60 (1980), T240–T242.

[14] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math., 136 (2012), 521–573.

[15] P. d’Avenia, G. Siciliano and M. Squassina, On fractional Choquard equations, Math. Models Methods Appl. Sci., 25 (2015), 1447–1476.

[16] P. d’Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math., 108 (1992), 247–262.

[17] E. Hebey and P.-D. Thizy, Stationary Kirchhoff systems in closed high dimensional manifolds, Invent. Math., 205 (2016), 2099–2129.

[18] F. Colasuonno and P. Pucci, Multiplicity of solutions for p(x)-polyharmonic Kirchhoff equations, Nonlinear Anal., 74 (2011), 5962–5974.

[19] T. D’Aprile and D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell systems with critical exponent, Commun. Contemp. Math., 18 (2016), 1550028, 53 pp.

[20] A. Fiscella and P. Pucci, p-fractional Kirchhoff equations involving critical nonlinearities, Nonlinear Anal. Real World Appl., 35 (2017), 350–378.

[21] A. Fiscella, G. Molica Bisci and R. Servadei, Multiplicity results for fractional Laplace problems with critical growth, Manuscripta Math., 155 (2018), 369–388.

[22] G. M. Figueiredo, G. Molica Bisci and R. Servadei, On a fractional Kirchhoff-type equation via Krasnoselskii’s genus, Asymptot. Anal., 94 (2015), 347–361.

[23] F. Gao and M. Yang, On the Brézis-Nirenberg type critical problem for nonlinear Choquard equation, Sci. China Math., 61 (2018), 1219–1242.

[24] X. M. He and W. M. Zou, Existence and concentration of ground states for Schrödinger-Poisson equations with critical growth, J. Math. Phys., 53 (2012), 023702, 19 pp.

[25] E. Hebey and P.-D. Thizy, Stationary Kirchhoff systems in closed high dimensional manifolds, Commun. Contemp. Math., 18 (2016), 1550028, 53 pp.

[26] E. Hebey, F. Robert and Y. L. Wen, Compactness and global estimates for fourth order equation of critical Sobolev growth arising from conformal geometry, Commun. Contemp. Math., 8 (2006), 9–65.

[27] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on \( \mathbb{R}^N \), Proc. Roy. Soc. Edinburgh Sect. A, 129 (1999), 787–809.

[28] Y. S. Jiang and H.-S. Zhou, Schrödinger-Poisson system with steep potential well, J. Differential Equations, 251 (2011), 582–608.

[29] G. Kirchhoff, Vorlesungen Über Mathematische Physik, B.G. Teubner, 1876.

[30] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E (3), 66 (2002), 056108, 7 pp.

[31] F. Y. Li, Y. H. Li and J. P. Shi, Existence of positive solutions to Schrödinger-Poisson type systems with critical exponent, Commun. Contemp. Math., 16 (2014), 1450036, 28 pp.

[32] E. Lieb and M. Loss, Analysis, 2nd edn. Graduate Studies in Mathematics, vol. 14, AMS, Providence, Rhode island, 2001.

[33] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, Stud. App. Math., 57 (1976/77), 93–105.

[34] P.-L. Lions, The Choquard equation and related questions, Nonlinear Anal., 4 (1980), 1063–1072.
[36] H. D. Liu, Positive solutions of an asymptotically periodic Schrödinger-Poisson system with critical exponent, Nonlinear Anal. Real World Appl., 32 (2016), 198–212.

[37] Z. L. Liu, Z.-Q. Wang and J. J. Zhang, Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system, Ann. Mat. Pura Appl., 195 (2016), 775–794.

[38] J. Liu, J.-F. Liao and C.-L. Tang, Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$, J. Math. Anal. Appl., 429 (2015), 1153–1172.

[39] D. F. Lü, A note on Kirchhoff-type equations with Hartree-type nonlinearities, Nonlinear Anal., 99 (2014), 35–48.

[40] V. Maz’ya and T. Shaposhnikova, On the Bourgain, Brézis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces, J. Funct. Anal., 195 (2002), 230–238.

[41] G. Molica Bisci, V. D. Radulescu and R. Servadei, Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications, 162, Cambridge University Press, Cambridge, 2016.

[42] G. Molica Bisci and L. Vilasi, On a fractional degenerate Kirchhoff-type problem, Commun. Contemp. Math., 19 (2017), 1550088, 23 pp.

[43] S. Pekar, Untersuchung Uber Die Elektronentheorie der Kristalle, Akademie Verlag, 1954.

[44] R. Penrose, Quantum computation, entanglement and state reduction, Philos. Trans. Roy. Soc., 356 (1998), 1927–1939.

[45] P. Pucci, M. Q. Xiang and B. L. Zhang, Existence and multiplicity of entire solutions for fractional $p$-Kirchhoff equations, Adv. Nonlinear Anal., 5 (2016), 27–55.

[46] P. Pucci, M. Q. Xiang and B. L. Zhang, Multiplicity of solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional $p$-Laplacian in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, 54 (2015), 2785–2806.

[47] P. Pucci, M. Q. Xiang and B. L. Zhang, Existence results for Schrödinger-Choquard-Kirchhoff equations involving the fractional $p$-Laplacian, J. Differential Equations, 261 (2016), 3061–3106.

[48] D. Wu, Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity, J. Math. Anal. Appl., 411 (2014), 530–542.

[49] M. Q. Xiang, G. Molica Bisci, G. H. Tian and B. L. Zhang, Infinitely many solutions for the stationary Kirchhoff problems involving the fractional $p$-Laplacian, Nonlinear Anal., 29 (2016), 357–374.

[50] M. Q. Xiang, B. L. Zhang and V. D. Rădulescu, Existence of solutions for perturbed fractional $p$-Laplacian equations, J. Differential Equations, 260 (2016), 1392–1413.

[51] M. Q. Xiang, B. L. Zhang and V. D. Rădulescu, Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional $p$-Laplacian, Nonlinearity, 290 (2016), 3186–3205.

[52] J. J. Zhang, J. M. do Ó and M. Squassina, Fractional Schrödinger-Poisson system with a general subcritical or critical nonlinearity, Adv. Nonlinear Stud., 16 (2016), 15–30.

[53] G. L. Zhao, X. L. Zhu and Y. H. Li, Existence of infinitely many solutions to a class of Kirchhoff-Schrödinger-Poisson system, Appl. Math. Comput., 256 (2015), 572–581.

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