Family of scalar-nonmetricity theories of gravity

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We extend the class of recently formulated scalar-nonmetricity theories by considering a five-parameter quadratic nonmetricity scalar and including a boundary term. The symmetric teleparallel constraint is invoked by Lagrange multipliers or by inertial variation. The equivalents for the general relativity and ordinary (curvature based) scalar-tensor theories are obtained as particular cases. We derive the field equations, discuss some technical details, e.g., debraiding, and formulate the Hamilton-like approach.

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I. INTRODUCTION

Both the success and failure of general relativity motivate community to conduct the study of gravity theories in two directions. The first direction focuses on finding alternative formulations of general relativity, and a well-known example of this kind is teleparallel gravity [1]. The latter imposes a zero curvature constraint which yields to an alternative interpretation of gravity: it is torsion [1, 2] or nonmetricity [3, 4] rather than curvature that causes attraction. Though a mere rephrasing should not extend the scope of the theory, it might give new insights and deeper understanding than the original formulation. For example in classical mechanics the Noether theorem does not reveal anything that could not be deduced from the equations of motion. The theorem is nevertheless useful as it points out what to look for.

The second direction in the study of gravity theories involves extensions of general relativity. Perhaps the simplest extension is given by adding a scalar field into the gravity sector yielding to scalar-tensor gravity [5, 6]. The first generation of scalar-tensor theories without derivative couplings or higher derivative terms involves a non-minimal coupling between the scalar field and the curvature scalar and therefore they are dubbed also as scalar-curvature theories. Although one could consider multiple scalar fields [7] and higher generations of scalar-tensor theories such as Horndeski [8] and beyond [9], the simplest scalar-curvature theories exhibit inflationary solutions [10], and are powerful enough to explain phenomenologically the early inflationary epoch [11] or the current accelerated expansion of the universe.

In this paper our route encompasses both of the aforementioned directions: we reformulate general relativ-
ity using the symmetric teleparallel connection and extend the theory by allowing arbitrary coefficients in the quadratic nonmetricity scalar (referred to as the newer general relativity in [4]) which is nonminimally coupled to a scalar field. This generalises the theories formulated in [12] where the quadratic non-metricity scalar was simply the quadratic Einstein Lagrangian, which without nonminimal coupling would yield to the symmetric teleparallel equivalent of general relativity.

Considering affine connection as an independent variable in addition to the metric is referred to as the so-called Palatini variation or working in the metric-affine framework. The research directions involving nonmetricity are not new and there are several studies in this field mainly in the context metric-affine gravity and possible microstructure of spacetime [13–18]. General affine connection contains additional structures to the Levi-Civita connection such as torsion and nonmetricity. As the latter are tensorial, one can argue at a textbook level that including them yields to just a theory with some additional fields [19]. However, from the gauge theory perspective one may ascribe to torsion and nonmetricity more fundamental meaning and thus provide a further motivation for their inclusion [20]. A related issue is whether the connection is coupled to other matter fields and whether it is constrained. A well-known example with the gravitational Lagrangian given by the Ricci scalar is the case where connection is neither coupled to matter fields nor invoking any constraints, then the Palatini variation yields to a trivial result for the connection. One can motivate the introduction of constraints from similar considerations in mechanics where constraints play a very useful role (e.g. describing the motion of a simple pendulum). In the current work we thus impose the symmetric teleparallel constraint, for previous studies involving symmetric teleparallelism consider [3, 4, 21–26].

The symmetric teleparallel connection relies only on nonmetricity and does not possess neither curvature nor torsion which yields to some interesting corollaries. One can transform to a zero connection gauge and thereby covariantise the partial derivatives as well as the split of the Einstein-Hilbert action into the Einstein Lagrangian denoting variables from the connection equation [12]. Instead of introducing the Lagrange multipliers, one could alternatively assume the symmetric inertial connection from the beginning and perform the so-called inertial variation, both methods yield the same equations for the connection (for similar calculations in the torsion-based teleparallel framework see [27, 28]).

As this paper accompanies the work of [12] we look in more detail some of the issues discussed there but also use a different perspective. Thus in addition to the nonminimally coupled quadratic nonmetricity scalar we add to the action a possible boundary term and discuss its role in relation to scalar-curvature theories. In fact the presence of the boundary term indicates that we are actually dealing with a disguised scalar-tensor theory. It is worth to pay attention that in principle one could consider modified or exotic matter fields which are coupled to symmetric teleparallel connection and yield to nonvanishing hyperstress. In the latter case we would not obtain a simple scalar-tensor (or general relativity) equivalent since the matter sector is deformed.

A new perspective is the classical mechanics viewpoint of the quadratic nonmetricity theory. One can interpret the metric $g$ as “the generalised coordinates” and its covariant derivative $Q$, which by definition is the nonmetricity, as “the generalized velocity”. In the simplest case, by “lowering the index” with the geometric object $\mathcal{G}$, which is “the metric” in the kinetic term, one obtains the conjugate momentum (or superpotential). One can further transform to the Hamilton-like formulation and define the field space metric $\mathcal{G}$. It is noteworthy that the objects $\mathcal{G}$ and $\mathcal{G}$ possess several interesting properties from which one could obtain some physical insights (e.g. the initial value formulation).

We adopt the conventions

$$K_{[\mu\nu]} \equiv \frac{1}{2} (K_{\mu\nu} - K_{\nu\mu}) , \quad (1a)$$
$$K_{[\mu|\lambda\nu]} \equiv \frac{1}{2} (K_{\mu\lambda\nu} - K_{\nu\lambda\mu}) , \quad (1b)$$
$$K_{(\mu\nu)} \equiv \frac{1}{2} (K_{\mu\nu} + K_{\nu\mu}) , \quad (1c)$$
$$K_{(\mu|\lambda\nu)} \equiv \frac{1}{2} (K_{\mu\lambda\nu} + K_{\nu\lambda\mu}) \quad (1d)$$

for (anti)symmetrization. We use the mostly plus signature of the metric and set $c = 1$.

The paper is organized as follows. In the Section II we revise the concepts of nonmetricity and symmetric teleparallel connection (in that Section stressed by STP on top of quantities, e.g., $\nabla \nabla$), write down the quadratic kinetic term for the metric, and recall the contracted second Bianchi identity, reformulated in terms of nonmetricity and symmetric teleparallel connection. Section III is devoted to postulating the action and deriving the field equations for the metric tensor $g^{\mu\nu}$, the scalar field $\Phi$, and for the connection $\Gamma^\lambda_{\mu\nu}$. In the Section IV we make use of $\nabla^\lambda g_{\mu\nu} \neq 0$ in order to formulate a manifestly covariant Hamilton-like approach. Section V concludes the paper. The body of the paper is followed by Appendixes.

II. FOREKNOWLEDGE

A. Nonmetricity $Q_{\omega\mu\nu}$

The nonmetricity

$$Q_{\omega\mu\nu} = \nabla^\omega g_{\mu\nu} = Q_{\omega(\mu\nu)} , \quad Q_{\omega} = -\nabla^\omega g^\rho , \quad (2)$$
enters the coefficients of affine connection as
\[
\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + L^\lambda_{\mu\nu} + K^\lambda_{\mu\nu},
\]  
where
\[
\tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} \xi^{\lambda} \left( 2 \partial_{[\mu}g_{\nu]\rho} - \partial_{\rho}g_{\mu\nu} \right) = \tilde{\Gamma}^\lambda_{(\mu\nu)},
\]  
is the Levi-Civita part of the connection,
\[
L^\lambda_{\mu\nu} = -\frac{1}{2} g^{\lambda\omega} \left( 2Q_{(\mu|\nu|\omega)} - Q_{\omega\mu\nu} \right) = L^\lambda_{(\mu\nu)},
\]  
and
\[
K^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\omega} \left( 2T_{(\mu|\nu|\omega)} + T_{\omega\mu\nu} \right) = g^{\lambda\omega} K_{(\mu|\nu|\omega)}. \tag{3c}
\]
Here \( T^\lambda_{\mu\nu} = T^\lambda_{[\mu\nu]} \) is the torsion. (Note that the torsion has been included for completeness. Actually, in the following sections we assume it to vanish.)

The nonmetricity tensor (2) possesses two independent contractions
\[
Q_{\omega} \equiv Q_{\omega\mu\nu} g^{\mu\nu}, \quad \tilde{Q}_\mu \equiv Q_{\omega\mu\nu} g^{\omega\nu}. \tag{4}
\]
The first of them is related to the invariant volume form as
\[
\nabla_\omega \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \nabla_\mu g_{\nu\omega} = \frac{1}{2} \sqrt{-g} Q_{\omega}. \tag{5}
\]
A straightforward calculation leads us further to
\[
\nabla_\mu \sqrt{-g} = -2 \nabla_\nu g_{\nu\mu} \sqrt{-g} - T^\lambda_{\mu\nu} \nabla_\lambda \sqrt{-g} = \sqrt{-g} \nabla_\nu g_{\nu\mu} = \sqrt{-g} \partial_{[\mu} Q_{\nu]} \tag{6a'}
\]
which is the homothetic or segmental curvature (cf. Eq. (1.3.34) in Ref. [29]).

\section{Symmetric teleparallel connection}

In the current paper we shall utilize the symmetric teleparallel (STP) connection \( \Gamma^\lambda_{\mu\nu} \) by imposing, in addition to symmetricity
\[
\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)} \iff \Gamma^\lambda_{\mu\nu} \equiv 2\Gamma^\lambda_{(\mu|\nu|)} \equiv 0, \tag{7a}
\]
also flatness
\[
\Gamma^\lambda_{\mu\nu} \equiv 2\partial_{[\mu}\Gamma^\lambda_{\nu]|\rho} + 2\Gamma^\lambda_{[\mu|\nu|} \Gamma^\lambda_{\nu]|\rho} \equiv 0. \tag{7b}
\]
In that case, based on Proposition 10.4.1. in Ref. [30], there exists a coordinate system \( \{\xi^\sigma\} \) where the connection coefficients \( \Gamma^\lambda_{(\mu\nu)} \) vanish, i.e.,
\[
\exists \{\xi^\sigma\} : \Gamma^\lambda_{(\mu\nu)}(\xi^\sigma) = 0 \Rightarrow \Gamma^\lambda_{\mu\nu}|_{\{\xi^\sigma\}} = \partial_\mu(), \tag{8}
\]
provided that the considered covariant derivative is partial derivative plus additive terms multiplied by the coefficients \( \Gamma^\lambda_{\mu\nu} \). The result (8) leads us to interesting corollaries. In particular, firstly, the covariant derivatives commute \( \{\xi^\sigma\} \) (cf. Eqs. (1.28) and (1.29) in Ref. [31])
\[
\nabla_\mu \nabla_\nu T|_{\{\xi^\sigma\}} = \partial_\mu \partial_\nu T = \partial_\nu \partial_\mu T = \nabla_\nu \nabla_\mu T|_{\{\xi^\sigma\}}, \tag{9}
\]
where \( T \) is a tensor (density of arbitrary rank (and weight). Secondly, in an arbitrary coordinate system \( \{x^\mu\} \), the connection coefficients read \( \Gamma^\lambda_{\mu\nu} \)
\[
\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial x^\alpha} \frac{\partial}{\partial x^\mu} \left( \frac{\partial x^\alpha}{\partial x^\nu} \right), \tag{10}
\]
where \( \{x^\alpha\} \) are the coordinates for which (8) holds.

Thirdly, one can covariantize the split \( \{\xi^\sigma\} \)
\[
\sqrt{-g} R = \sqrt{-g} \mathcal{L}_E - \partial_\rho \left( \sqrt{-g} \mathcal{B}^\rho \right) \tag{11}
\]
where (see Eq. (8) in Ref. [3], and also, e.g., Eq. (24) in Ref. [4])
\[
\mathcal{L}_E = \Gamma^\lambda_{\rho\sigma} g^{\lambda\nu} T^\nu_{\rho\sigma} = \Gamma^\lambda_{\rho\sigma} g^{\lambda\nu} T^\nu_{\rho\sigma} \tag{11a}
\]
\[
= \partial_\rho g_{\mu\nu} \left( -\frac{1}{4} g^{\lambda\omega} g^{\nu\sigma} g^{\rho\nu} + \frac{1}{4} g^{\lambda\rho} g^{\nu\sigma} g^{\nu\omega} \right) \partial_\omega g_{\rho\nu} \tag{11a'}
\]
\[
+ \partial_\mu g_{\rho\nu} \left( \frac{1}{4} g^{\rho\mu} g^{\lambda\omega} g^{\nu\sigma} - \frac{1}{2} g^{\mu\nu} g^{\lambda\rho} g^{\nu\omega} \right) \partial_\omega g_{\rho\mu} \tag{11a''}
\]
is the quadratic Einstein Lagrangian, and
\[
\mathcal{B}^\rho = g^{\rho\mu} \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\nu\rho} g^{\nu\sigma} \tag{11b}
\]
\[
= g^{\rho\mu} \left( \partial_\mu g_{\rho\nu} \right) g^{\nu\sigma} - g^{\rho\mu} \left( \partial_\nu g_{\rho\mu} \right) g^{\nu\sigma} \tag{11b'}
\]
is the boundary term, hosting the second derivatives of the metric that reside in \( \tilde{R} \). From the viewpoint of the Levi-Civita connection, neither (11a) nor (11b) is a tensor. However, both terms can be covariantized, by considering the symmetric teleparallel connection and promoting the partial derivatives in (11a') and (11b') to covariant ones, thus reversing the line of thought that underlies (8). The Einstein quadratic Lagrangian yields (see, e.g., Eq. (17) in Ref. [4], as well as Eq. (18) in Ref. [12])
\[
\mathcal{Q} \equiv -\frac{1}{4} \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\mu\nu} + \frac{1}{2} \Gamma^\mu_{\lambda\rho} \mathcal{Q}^\rho\mu\lambda \tag{12}
\]
\[
+ \frac{1}{4} \mathcal{Q}_{\mu\nu} \mathcal{Q}^{\mu\nu} - \frac{1}{2} \mathcal{Q}_{\mu\rho} \mathcal{Q}^{\mu\rho}, \tag{12}
\]
while (cf. Eq. (17) in Ref. [12])
\[
\mathcal{B}^\rho = \mathcal{Q}^\rho - \mathcal{Q}^\rho \tag{13}
\]
is the covariantized version of the boundary term (11b').
C. Kinetic term for the metric \( g^{\mu\nu} \)

The nonvanishing covariant derivative of the metric \( g^{\mu\nu} \) allows us to consider the kinetic term for the metric indeed analogously to the kinetic energy in classical mechanics. Let us define

\[
Q = Q_{\lambda}^{\mu\nu} g^{\lambda}_{\mu\nu} \omega_{\sigma\rho} Q_\omega^{\sigma\rho},
\]

where

\[
g^{\lambda}_{\mu\nu} \omega_{\sigma\rho} = c_1 \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} g_\omega^{(\sigma)} + c_2 \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} g_\omega^{(\sigma)} + c_3 g_{\mu\nu} g^{\omega}_{\sigma\rho} + c_4 \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} g_\omega^{(\sigma)} + c_5 g_{\mu\nu} Q_\omega^{\sigma\rho},
\]

with constants \( c_1, \ldots, c_5 \), and definitions (2), (4) contracts Eq. (14) to [4]

\[
Q = c_1 Q_{\lambda\mu\nu} Q^{\lambda\mu\nu} + c_2 Q_{\lambda\mu\nu} Q^{\mu\nu\lambda} + c_3 Q_{\mu\nu\lambda} Q^{\mu\nu\lambda} + c_4 Q_{\mu\nu\lambda} Q^{\lambda\mu\nu} + c_5 Q_{\mu\nu\lambda} Q^{\lambda\mu\nu}.
\]

Let us point out that in addition to the symmetries

\[
\begin{align*}
G^{\lambda}_{\mu\nu} \omega_{\sigma\rho} &= G^{\lambda}_{\nu\mu} \omega_{\sigma\rho} = G^{\lambda\mu}_{\nu}(\omega_{\sigma\rho} ) \quad \text{(17a)} \\
&= G^{\lambda\mu}_{\nu}(\omega_{\sigma\rho} ) \quad \text{(17c)}
\end{align*}
\]

the tensor \( G^{\lambda}_{\mu\nu} \omega_{\sigma\rho} \) is symmetric

\[
G^{\lambda\mu\nu} \omega_{\sigma\rho} = G^{\mu\nu\lambda} \omega_{\sigma\rho} \quad \text{(17c)}
\]

in the sense of the Definition 3.9 in Ref. [32]. Precisely the quality (17c) furnishes the result (Defs. (12) in Ref. [25] and (18) in Ref. [4]).

\[
P^{\lambda}_{\mu\nu} = \frac{1}{2} \frac{\partial Q}{\partial Q_{\lambda\mu\nu}} = G^{\lambda}_{\mu\nu} \omega_{\sigma\rho} Q_\omega^{\sigma\rho}
\]

\[
= c_1 Q^{\lambda}_{\mu\nu} + c_2 Q^{\lambda}_{\mu\nu} + c_3 Q^{\lambda}_{\mu\nu} + c_4 \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} + c_5 \frac{1}{2} \left( Q^{\lambda\mu\nu} + \delta^{(\mu}_{(\rho} Q^{\nu)}_{\sigma)} \right).
\]

From (18a) one can clearly see a similarity with classical mechanics. In terms of analogy, for the simplest case, the free particle, the “generalized momentum” \( P^{\lambda}_{\mu\nu} \) is obtained by taking the derivative of the “kinetic energy” \( \frac{1}{2} Q \) w.r.t. the “generalized velocity” \( Q_{\lambda\mu\nu} \). “Lowering the index” of the “generalized velocity” with the “metric” \( G^{\lambda}_{\mu\nu} \omega_{\sigma\rho} \) yields the “generalized momentum”.

1. Varying \( G^{\lambda}_{\mu\nu} \omega_{\sigma\rho} \)

A straightforward calculation shows that the variation of (15) yields

\[
\delta G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} = \left( \Delta G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} \right)_{\beta\alpha} \delta g^{\alpha\beta},
\]

where

\[
\left( \Delta G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} \right)_{\beta\alpha} = \frac{1}{2} \left( \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} \delta^{(\lambda}_{(\delta} g^{\gamma)}_{\beta)} + \delta^{(\mu}_{(\rho} g^{\nu)}_{(\sigma)} \delta^{(\lambda}_{(\delta} g^{\gamma)}_{\beta)} - \frac{2g_{(\mu}(\delta_{\rho)}(\gamma_{\beta})(\sigma) \right) .
\]

The positioning of the indices emphasizes that the variation respects the symmetries (17) of \( G^{\lambda}_{\mu\nu} \omega_{\sigma\rho} \), i.e.,

\[
\left( \Delta G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} \right)_{\beta\alpha} = \left( \Delta G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} \right)_{(\beta\alpha)},
\]

and therefore there is no need to invoke the symmetrizing brackets. Analogously

\[
\nabla_{\xi} G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} = - \left( \Delta G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} \right)_{\beta\alpha} Q_{\xi}^{\alpha\beta}.
\]

2. General relativity equivalent

By comparing Eqs. (12) and (16), we conclude, that general relativity is covered by the coefficients

\[
\begin{align*}
c_1 &= -\frac{1}{4} , \quad c_2 = \frac{1}{2} , \quad c_3 = \frac{1}{4} , \\
c_4 &= 0 , \quad c_5 = \frac{1}{2}.
\end{align*}
\]

Expression (15) reduces to

\[
G^{\lambda\mu\nu}_\omega \omega_{\sigma\rho} = -\frac{1}{4} \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} \delta^{(\lambda}_{(\delta} g^{\gamma)}_{\beta)} + \frac{1}{2} \delta^{(\mu}_{(\rho} g^{\nu)}_{(\sigma)} \delta^{(\lambda}_{(\delta} g^{\gamma)}_{\beta)} - \frac{1}{4} g_{(\mu}(\delta_{\rho)}(\gamma_{\beta})(\sigma) \right),
\]

and (18b) yields

\[
P^{\lambda\mu\nu} = -\frac{1}{4} Q^{\lambda\mu\nu} + \frac{1}{2} Q^{\lambda}_{\mu\nu}
\]

\[
+ \frac{1}{4} \left( Q^{\lambda\mu\nu} - Q^{\lambda}_{\mu\nu} \right) g_{\alpha\beta} - \frac{1}{4} \delta^{(\mu}_{(\rho} Q^{\nu)}_{\sigma)}.
\]

Compare with definition (24) in Ref. [12].

---

1. Note that in this section we actually do not need to assume the symmetric teleparallel connection, but just need the nonmetricity. Thus, the quantities \( Q_{\lambda\mu\nu} \), etc., will not be equipped with ‘\( STP \)’ on top.
2. The form \( \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} \delta^{(\lambda}_{(\delta} g^{\gamma)}_{\beta)} \) (multiplied by \( c_1 \) in the first line of Eq. (15) emphasizes the symmetry (17c) but for practical calculations \( g_{\mu(\rho\sigma)} g^{\lambda\mu\nu} = \delta^{(\mu}_{(\rho} g^{\nu)}_{\sigma)} g^{\lambda\nu} \) is more suitable.
D. Bianchi identity

If we impose (7) then

\[
R^\nu_{\rho\mu\sigma} - \nabla^\nu \left( \nabla_\mu g^\rho_\sigma + \nabla_\rho g^\mu_\sigma \right) - \nabla^\rho \left( \nabla_\sigma g^\mu_\nu + \nabla_\nu g^\sigma_\mu \right) = 0, \quad (23a)
\]

\[
\nabla^\nu g^\mu_\sigma = \frac{2}{\sqrt{-g}} \nabla^\nu \left( \sqrt{-g} g^\mu_\sigma \right) + \frac{1}{2} \nabla^\sigma \left( \sqrt{-g} g^\nu_\mu \right) + \frac{1}{2} g^\sigma_\mu \nabla^\nu \omega_\lambda, \quad (23b)
\]

\[
\tilde{R} = g^{\nu\rho} \nabla_\mu \nabla_\nu \left( \sqrt{-g} g^\rho_\sigma \right) - \frac{1}{2} g^\nu_\mu \nabla^\sigma \omega_\lambda, \quad (23c)
\]

Therefore, by making use of the definitions (3b), (12) and (22)

\[
G^\nu_\mu = R^\nu_\rho \mu - \frac{1}{2} g^{\nu\rho} \tilde{R}
\]

\[
= \frac{2}{\sqrt{-g}} \nabla^\nu \left( \sqrt{-g} g^\mu_\lambda \sigma_\nu \right) - \frac{1}{2} \nabla^\nu \omega_\sigma + \nabla^\nu \omega_\lambda \sigma_\nu, \quad (24)
\]

is the Einstein tensor.

One can show that for a symmetric tensor \( G_{\mu\nu} = G_{(\mu\nu)} \)

\[
\nabla^\sigma \left( \sqrt{-g} G^\mu_{\nu} \right) = \nabla^\sigma \left( \sqrt{-g} G^\nu_{\mu} \right) + \sqrt{-g} \nabla_\rho \lambda G_{\sigma\lambda}
\]

\[
= \nabla^\sigma \left( \sqrt{-g} G^\mu_{\nu} \right) - \frac{1}{2} \sqrt{-g} \nabla_\nu \omega_\lambda G_{\sigma\lambda}. \quad (25)
\]

By a straightforward calculation

\[
\nabla^\sigma \left( \sqrt{-g} g^\mu_\nu \right) = 2 \nabla^\sigma \nabla^\lambda \left( \sqrt{-g} g^\mu_\nu \right) + \frac{1}{2} \nabla^\sigma g^\mu_\nu \nabla^\nu G_{\mu\nu},
\]

where in addition to (19) we made use of

\[
\nabla^\sigma \left( \sqrt{-g} g^\mu_\nu \right) = \nabla^\sigma \left( \sqrt{-g} g^\mu_\nu \right)
\]

\[
= \nabla^\sigma \left( \sqrt{-g} g^\mu_\nu \right) + \sqrt{-g} \nabla_\rho \lambda G_{\sigma\nu}.
\]

\[
\nabla^\mu Q_{\nu\sigma\rho} = \nabla^\nu Q_{\mu\sigma\rho}, \quad \nabla^\nu Q_{\nu\sigma\rho} = \nabla^\mu Q_{\mu\sigma\rho}. \quad (27a)
\]

Hence

\[
\nabla^\sigma \left( \sqrt{-g} G^\mu_{\nu} \right) = 2 \nabla^\sigma \nabla^\lambda \left( \sqrt{-g} g^\mu_\nu \right) = 0. \quad (28)
\]

Yielding another possibility for obtaining the GR motivated coefficients (20) is the following. Let us consider generic coefficients \( a_1, \ldots, a_5 \) and the definition (18b). By imposing

\[
\nabla^\sigma \nabla_\lambda \left( \sqrt{-g} g^\sigma_\nu \right) = 0 \quad (30)
\]

we obtain 57 different terms, which vanish identically, if

\[
2c_1 + c_2 = 0, \quad 2c_3 + c_5 = 0, \quad (31a)
\]

\[
c_2 + c_3 = 0, \quad c_4 = 0. \quad (31b)
\]

Hence, up to an overall multiplier, we obtain the general relativity motivated coefficients (20).

One can loosen the conditions by demanding only the second derivatives of \( Q_{\lambda\mu\nu} \) to vanish. The explicit terms in (30) are

\[
\frac{1}{2} (2c_1 + c_2 + c_4) \sqrt{-g} g^\mu_\nu g^\sigma_\rho \nabla^\mu \nabla^\nu Q_{\lambda\mu\nu} = 0, \quad (32)
\]

\[
\frac{1}{2} (c_2 + c_4 + c_5) \sqrt{-g} g^\mu_\nu g^\sigma_\rho \nabla^\mu \nabla^\nu Q_{\lambda\rho\nu} = 0, \quad (33)
\]

\[
\frac{1}{2} (2c_3 + c_5) \sqrt{-g} g^\mu_\nu g^\sigma_\rho \nabla^\mu \nabla^\nu Q_{\lambda\sigma\rho} = 0, \quad (34)
\]

which are the three independent possibilities for placing indices. Hence, we slightly deform the system (31) to yield

\[
2c_1 + c_2 = 0, \quad 2c_3 + c_5 = 0, \quad (35a)
\]

\[
c_2 + c_3 = 0, \quad (35b)
\]

where

\[
\tilde{c}_2 = c_2 + c_4. \quad (36)
\]

It is interesting to note that the sum (36) is mentioned in [4] after Eq. (23). Note however that whatever deviation from the coefficients (20) instantly introduces several dozens of terms into (30).

III. ACTION AND FIELD EQUATIONS

A. Action

Let us postulate an action for the metric \( g^{\mu\nu} \), scalar field \( \Phi \), connection \( \Gamma^\nu_{\sigma\rho} \), and matter fields, collectively denoted by \( \chi \), as

\[
S = \int_{M_4} d^4 x \sqrt{-g} \left( \mathcal{L}_g + \mathcal{L}_\Phi + \mathcal{L}_b + \mathcal{L}_b + \mathcal{L}_m \right), \quad \text{(37)}
\]

composed of the following components. The kinetic term for the metric \( g^{\mu\nu} \)

\[
\mathcal{L}_g = \mathcal{L}_g \left[ g^{\mu\nu}, \Gamma^\nu_{\sigma\rho}, \Phi \right] = \frac{1}{2\kappa^2} \mathcal{A}(\Phi) \mathcal{Q}, \quad \text{(38a)}
\]
contains in addition to the nonmetricity scalar \( Q \), defined by (14), also the dimensionless nonminimal coupling function \( \mathcal{A}(\Phi) \). Roughly speaking, as in scalar-curvature theories [5], the latter introduces a scalar field dependent gravitational “constant” \( \propto \kappa^2 / \mathcal{A}(\Phi) \). Here the constant \( \kappa^2 \) wields the dimension, and its numerical value must be determined from the Newtonian limit.

The scalar field \( \Phi \), as well as the functions \( \mathcal{B}(\Phi) \) and \( \mathcal{V}(\Phi) \), are considered to be dimensionless. Note that we have introduced yet another dimensionful constant \( \ell^2 = g^2 \).

In addition to pure kinetic terms, one can include mixed term for the metric \( g \)

\[
L_\Phi = L_\Phi [g_{\mu\nu}, \Phi] = -\frac{1}{2\kappa^2} \left( \mathcal{B}(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2\ell^{-2} \mathcal{V}(\Phi) \right). 
\]

The scalar field \( \Phi \), as well as the functions \( \mathcal{B}(\Phi) \) and \( \mathcal{V}(\Phi) \)

in principle, by making use of \( \mathcal{B} = \mathcal{B}^i \) and \( \mathcal{V} = \mathcal{V}^i \) we do not have to consider any boundary terms explicitly when postulating the action (37). The term (38c) has been introduced with a constant parameter \( \epsilon \).

If the matter Lagrangian \( L_m \) is directly imported from general relativity, i.e., without any alterations \(^3\), then there are two particularly interesting subcases

i) If \( \epsilon = 0 \), and the coefficients \( c_1, \ldots, c_5 \) are given by (20), then the action (37) is equivalent to the action (20) in Ref. [12].

ii) If \( \epsilon = 1 \), and the coefficients are again those originating from general relativity (20), then the action (37) is equivalent to the action in scalar-curvature theories, see, e.g., action (2.2) in Ref. [6], but without the problematic boundary term.

The symmetric teleparallel conditions (7) are enforced by making use of the Lagrange` multipliers

\[
\mathcal{L}_L \equiv \mathcal{L}_L [\Gamma^\lambda_{\sigma\rho}, \lambda^\rho_{\mu\nu}, \lambda^\mu_{\nu}] = \lambda^\rho_{\mu\nu} R^\sigma_{\rho\mu\nu} + \lambda^\mu_{\nu} T^\sigma_{\mu\nu}, \tag{38d}
\]

where by assumption

\[
\lambda^\rho_{\mu\nu} = \lambda^\rho_{\nu\mu}, \quad \lambda^\mu_{\nu} = \lambda^\mu_{\nu}. \tag{38d'}
\]

Finally,

\[
\mathcal{L}_m \equiv \mathcal{L}_m [g_{\mu\nu}, \Gamma^\lambda_{\sigma\rho}, \chi], \quad S_m = \int_{M^4} d^4x \sqrt{-g} \mathcal{L}_m, \tag{38e}
\]

describes the matter fields \( \chi \). Note that \( \mathcal{L}_m \) may depend on the connection coefficients \( \Gamma^\lambda_{\sigma\rho} \).

1. Concerning notation

First, we vary the action (37) w.r.t. the Lagrange multipliers and in what follows, we already assume the symmetric teleparallel connection (7), unless stated otherwise. Therefore, due to narrower scope, we will omit some of the notational specifications used in [12] and also in the previous parts of the current paper. In particular, we omit the STP on top of quantities, and keep the notation somewhat simpler. Nevertheless, occasionally it is nearer to use the Levi-Civita connection, which in that case would be denoted by LC on top of the quantities.

Second, we drop the arguments of the functions \( \mathcal{A} \), \( \mathcal{B} \) and \( \mathcal{V} \). In addition to taking spacetime derivatives of these functions, we also may introduce derivative w.r.t. the scalar field

\[
A' \equiv \frac{dA}{d\Phi}, \quad B' \equiv \frac{dB}{d\Phi}, \quad V' \equiv \frac{dV}{d\Phi}. \tag{39}
\]

B. Field equation for the metric \( g^{\mu\nu} \)

Varying the action (37) w.r.t. the metric \( g^{\mu\nu} \) leads us to the expression

\[
\delta_g S = \frac{1}{2\kappa^2} \int_{M^4} d^4x \left\{ \sqrt{-g} E_{\mu\nu}^{(g)} \delta g^{\mu\nu} + \partial_\sigma \left( \sqrt{-g} B_{\sigma}^{(g)} \right) \right\}. \tag{40}
\]

Therefore, the equation of motion for the metric \( g^{\mu\nu} \) is

\[
E^{(g)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \nabla_\lambda \left( \sqrt{-g} \mathcal{A} \Gamma^\lambda_{\mu\nu} \right) - \frac{1}{2} g_{\mu\nu} \mathcal{A} \mathcal{Q} + \mathcal{A} (\mathcal{P}_{\mu\nu\rho} Q^{\rho} - 2Q_{\mu\rho\sigma} P^\rho_{\nu\sigma}) \nonumber \\
+ \left( \mathcal{P}_{\mu\nu} Q^\sigma_{\rho} - \mathcal{P}_{\nu \mu} Q^\rho_{\sigma} - 2P^\rho_{\mu\nu} \partial_\lambda \mathcal{A} \right) \nonumber \\
+ \frac{1}{2} g_{\mu\nu} \mathcal{B} g^{\rho\sigma} \partial_\rho \Phi \partial_\sigma \Phi - \mathcal{B} \partial_\mu \Phi \partial_\nu \Phi \nonumber \\
+ \ell^{-2} g_{\mu\nu} \mathcal{V} - \kappa^2 T_{\mu\nu} = 0, \tag{41}
\]

where the energy-momentum tensor \( T_{\mu\nu} \) is defined as

\[
T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \delta S_m / \delta g^{\mu\nu}. \tag{41a}
\]

Let us point out that on the third line, \( P^\lambda_{\mu\nu} \) is indeed the quantity (22), corresponding to general relativity, and not the generic \( \mathcal{P}^\lambda_{\mu\nu} \), defined by (18b). This, and also

\(^3\) Note that invoking the usual minimal coupling principle in general relativity would yield to an additional nonminimal coupling in the teleparallel framework [53].
the appearance of the Levi-Civita covariant derivatives on the same line, is due to the fact that Eq. (38c), the Lagrangian $L_b$ is motivated by comparison to general relativity. One can write down different versions of the same equation and some of those can be found in Appendix E.

For completeness, we include the boundary term

$$\mathcal{R}^\sigma_{(g)} \equiv - \left( \epsilon \left( g_{\mu\nu} g^{\sigma\lambda} \partial_\lambda A - \delta^\sigma_\mu \partial_\nu A \right) + 2A P^\sigma_{\mu\nu} \right) \delta g^{\mu\nu} + \mathcal{R}^\sigma_{(m,g)},$$  

(42)

where $\mathcal{R}^\sigma_{(m,g)}$ is the part that in principle may arise from the unspecified matter action $S_m$. The boundary term (42) does not contribute to the field equations. The boundary term only contains the variation $\delta g^{\mu\nu}$ of the metric, and not its derivative (cf. Eq. (6) in Ref. [34]).

Note that due to (19c),

$$P_{[\mu[\sigma\rho]}(Q_{\nu]}^\sigma - 2Q_{\rho[\mu}^\rho P^\rho_{\nu]\sigma]} = 0,$$

(42b)

and hence the equation (41) is symmetric by construction. We are stressing this result because in torsion-based teleparallel theories it has been shown that essentially the equations obtained by varying w.r.t. the connection and w.r.t. the metric have almost identical structure, the only difference being that the equation for the metric is symmetric, and the equation for the connection is antisymmetric. Therefore one can combine these equations into one, and extract the necessary information by taking the symmetric or antisymmetric part, respectively. However, the result (19c) states that no such prescription can be invoked in the symmetric teleparallel case studied in the current paper.

1. Further comments on eom-s for $g^{\mu\nu}$

From (41), the equation of motion for the metric tensor $g^{\mu\nu}$, one obtains that the second order derivatives of the metric are contracted by $G^\lambda_{\mu\nu} \omega^\omega_{\sigma\rho} \nabla_\lambda \nabla_\omega g^\sigma_{\rho} + \ldots.$

(43)

It remains for further study, how this observation is related to the initial value problem. See Theorem on page 13 in Ref. [35].

Contracting (41) yields

$$g^{\mu\nu} E_{\mu\nu}^{(g)} = \partial_\lambda A \left( (2C_1 - \epsilon) Q^\lambda + (2C_2 + \epsilon) \tilde{Q}^\lambda \right) + 2A \nabla_{\lambda} \left( C_1 Q^\lambda + C_2 \tilde{Q}^\lambda - A Q \right) + 3\nabla_{\lambda} \nabla_{\omega} A + B g^{\sigma\rho} \partial_\sigma A \partial_\rho \Phi + 4 \epsilon^{-2} \nabla - 5^2 \nabla,$$

(44)

where $T = g^{\mu\nu} T_{\mu\nu}$, and the constants $C_1$ and $C_2$ are defined by (A2a).

C. Field equation for the scalar field $\Phi$

Varying action (37) w.r.t. the scalar field $\Phi$ reads

$$\delta_S = \frac{1}{2\kappa^2} \int_{M^4} d^4 x \left\{ \sqrt{-g} E^{(\Phi)} \delta \Phi + \partial_\sigma \left( \sqrt{-g} \mathcal{R}^{(\Phi)} \right) \right\}.$$

(45)

Hence, the dynamics for the scalar field is governed by

$$E^{(\Phi)} = 2B \nabla_\sigma \nabla^\sigma \Phi + B' g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2\epsilon^{-2} \nabla + A' Q - A' \nabla_\sigma \left( Q^\sigma - \tilde{Q}^\sigma \right) = 0,$$

(46)

while

$$\mathcal{R}^{(\Phi)} = \left[ \epsilon A' \left( Q^\sigma - \tilde{Q}^\sigma \right) - 2B g^{\sigma\rho} \partial_\rho \Phi \right] \delta \Phi.$$

(47)

Compare with Eq. (7) in Ref. [34]. Adding (44) to (46) yields

$$AE^{(\Phi)} + A' g^{\mu\nu} E_{\mu\nu}^{(g)} =$$

$$= 4A^2 \mathcal{F}(\epsilon) \nabla^\omega \nabla_\omega \Phi + (2A^2 \mathcal{F}(\epsilon))' g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2\epsilon^{-2} \left( \mathcal{V}' A - 2A' \mathcal{T} \right) - \kappa^2 A' \mathcal{T}$$

$$+ (A')^2 \partial_\lambda \Phi \left[ (2C_1 - \epsilon) Q^\lambda + (2C_2 + \epsilon) \tilde{Q}^\lambda \right] + A A' \nabla_\lambda \left[ (2C_1 - \epsilon) Q^\lambda + (2C_2 + \epsilon) \tilde{Q}^\lambda \right],$$

(48)

where

$$4A^2 \mathcal{F}(\epsilon) = 2AB + \epsilon (A')^2.$$

(48a)

D. Debraiding the equations (41) and (46)

For solving the field equations (41), and (46) or equivalently (48), it would be good to have them debraded [36]. Let us consider two distinct cases

i) If

$$\epsilon = 0,$$

(49a)

then the equations (41) and (46) are naturally debraded. Let us recall that this means dropping the boundary-term-motivated Lagrangian $L_b$, defined by (38c). This observation holds for each choice of the coefficients $c_1, \ldots, c_5$. In the scalar-tensor extension of general relativity (corresponding to the coefficients (20), and $\epsilon = 1$), one would have to transform to the Einstein frame, in order to obtain the situation, where the equations are debraded [34]. Thus, one could argue, that if $\epsilon = 0$, then the theory under consideration is postulated in the Einstein frame. On the other hand, the matter fields couple to the metric residing in geometry Lagrangian, and hence, it is the Jordan frame. Therefore, contrary to the scalar-curvature
case, one could say that for the theory with \( \epsilon = 0 \) (see, e.g., [12]), the Einstein and Jordan frames coincide, exactly as in general relativity. In other words, the matter fields couple to the propagating tensorial degree of freedom. However, to be more conservative, we follow Ref. [37] and refer to the frame as the debriding frame (see Section VI.C in Ref. [37]).

Let us point out that in this case, adding (41) and (46) to yield (48) actually introduces second derivatives of the metric to the equation for the scalar field.

ii) If \( \epsilon \neq 0 \), \( (50) \)

then the equation for the metric \( g^{\mu \nu} \) inevitably contains the second derivatives of the scalar field (note the \( \nabla_\mu \nabla_\nu \Phi \) term, which is not a scalar). One may, however, ease finding solutions by trying to debraid the equation for the scalar field \( \Phi \). From (48) it follows that the sufficient conditions are
\[
2C_1 - \epsilon = 0, \quad 2C_2 + \epsilon = 0. \quad (51)
\]

E. Field equation for connection \( \Gamma^\lambda_{\mu \nu} \)

Varying the action (37) w.r.t. the connection reveals
\[
\delta_\Gamma S = \frac{1}{2\kappa^2} \int_{M_4} d^4x \left\{ \sqrt{-g} \left( E^{(\Gamma)} \right)^{\mu \nu} \frac{\partial \Gamma^\lambda_{\mu \nu}}{\partial \Gamma^\lambda_{\mu \nu}} + \partial_\sigma \left( \sqrt{-g} R^\sigma_\Gamma \right) \right\}. \quad (52)
\]

Thus,
\[
\frac{\sqrt{-g}}{4} \left( E^{(\Gamma)} \right)^{\lambda \mu \nu} \equiv \nabla_\rho \left( \sqrt{-g} \Gamma^\lambda_{\rho \mu \nu} \right) + \sqrt{-g} \Gamma^\lambda_{\mu \nu} - \sqrt{-g} A P^\mu \nu \lambda - H_\lambda^{\mu \nu} \nonumber
\]
\[
- \sqrt{-g} \frac{\epsilon}{2} \left( \partial_\omega A g^{\omega \nu \delta \lambda} + \partial_\omega A g^{\mu \nu \delta \lambda} \right) = 0, \quad (53)
\]
and
\[
R^\sigma_\Gamma = -4\Gamma^\rho_{\mu \sigma} \frac{\partial \Gamma^\rho_{\mu \nu}}{\partial \Gamma^\rho_{\nu \lambda}} + R^\sigma_{(m, \Gamma)}, \quad (54)
\]

where, as in the variation w.r.t. the metric, \( R^\sigma_{(m, \Gamma)} \) is the part which in principle may arise from the unspecified matter Lagrangian (38e). The hypermomentum density is defined as
\[
H_\lambda^{\mu \nu} = -\frac{1}{2} \frac{\delta S_m}{\partial \Gamma^\lambda_{\mu \nu}}, \quad (55)
\]
Due to (9) and (38d')
\[
-\frac{1}{4} \nabla_\nu \nabla_\mu \left[ \sqrt{-g} \left( E^{(\Gamma)} \right)^{\lambda \mu \nu} \right] = \nabla_\nu \nabla_\mu \left( \sqrt{-g} A \left( P^{(\mu \nu)}_\lambda - \epsilon P^{(\mu \nu)}_\lambda + \kappa^2 H_\lambda^{\mu \nu} \right) \right) = 0,
\]
which can be easily proven, if one takes into account (cf. Eq. (30) in Ref. [12])
\[
\left( \nabla_\nu \nabla_\mu A \right) \sqrt{-g} P^{(\mu \nu)}_\lambda + 2 \left( \nabla_\nu A \right) \nabla_\mu \left( \sqrt{-g} P^{(\mu \nu)}_\lambda \right) = -\frac{1}{2} \nabla_\mu \left[ \left( \partial_\nu A \right) \nabla_\omega \left( \sqrt{-g} g^{\nu \delta \omega \lambda} \right) \right], \quad (57)
\]
and the Bianchi identity
\[
\nabla_\nu \nabla_\mu \left( \sqrt{-g} P^{(\mu \nu)}_\lambda \right) = 0 \quad (58)
\]
(see Subsec. II.D). The result (57) is easily derived from (29) and
\[
\nabla_\mu \nabla_\omega \left( \sqrt{-g} g^{\nu \delta \omega \lambda} \right) = 0. \quad (59)
\]

1. Varying w.r.t. \( \xi^\sigma \)

Instead of varying the action (37) w.r.t. the generic connection \( \Gamma^\lambda_{\mu \nu} \), and imposing flat and torsionless conditions via the Lagrange multipliers (38d), one may assume the form (10) and vary w.r.t. the coordinates \( \xi^\sigma \) (see also discussion following Eq. (13) in Ref. [25]). Note that if this approach has been chosen, then the Lagrangian (38d) vanishes and therefore no derivatives of the connection appear in the action (up to the possibility for introducing exotic matter). Let us note that
\[
\delta_\xi \left( \frac{\partial x^\lambda}{\partial \xi^\sigma} \right) = -\frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial x^\sigma}{\partial \xi^\rho} \frac{\partial \xi^\omega}{\partial x^\rho}, \quad (60a)
\]
\[
\delta_\xi \Gamma^\lambda_{\mu \nu} = -\frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial \xi^\rho}{\partial x^\mu} \frac{\partial \xi^\omega}{\partial x^\nu} + \frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial \xi^\rho}{\partial x^\mu} \frac{\partial \xi^\omega}{\partial x^\nu}, \quad (60b)
\]

Therefore
\[
\delta_\xi S = \int_{M_4} d^4x \left\{ \sqrt{-g} \left( E^{(\Gamma)} \right)^{\lambda \mu \nu} \frac{\partial \xi^\lambda}{\partial \xi^\sigma} \frac{\partial \xi^\mu}{\partial \xi^\rho} \frac{\partial \xi^\nu}{\partial \xi^\omega} \right\} = \int_{M_4} d^4x \left\{ \nabla_\nu \nabla_\mu \left[ \sqrt{-g} \left( E^{(\Gamma)} \right)^{\lambda \mu \nu} \right] \frac{\partial x^\lambda}{\partial \xi^\sigma} \frac{\partial \xi^\mu}{\partial \xi^\rho} \frac{\partial \xi^\nu}{\partial \xi^\omega} \right\} \nonumber
\]
\[
+ \delta_\xi \left( \sqrt{-g} R^\sigma_\Gamma \right), \quad (61)
\]

\footnote{As previously, we will not use the STP notation, but we only consider the symmetric teleparallel connection.}
where

\[
\sqrt{-g} \mathcal{B}_\mu^\sigma(\xi) \equiv \sqrt{-g} \left( E^{(\Gamma)} \right)_{\lambda}^{\sigma} \partial_\mu \partial_\mu \partial_\xi^\rho \frac{\partial \xi^\rho}{\partial x^\nu} \left( -\sqrt{-g} \left( E^{(\Gamma)} \right)_{\lambda}^{\mu} \right) - \left[ \sqrt{-g} \left( E^{(\Gamma)} \right)_{\lambda}^{\mu \nu \lambda} \partial_\mu \left( \sqrt{-g} \left( E^{(\Gamma)} \right)_{\lambda}^{\sigma} \right) \right] - \left[ \sqrt{-g} \left( E^{(\Gamma)} \right)_{\lambda}^{\sigma} \Gamma^\nu_{\mu \lambda} \frac{\partial \xi^\lambda}{\partial x^\rho} \delta^\rho_{\beta} + \mathcal{B}_\mu^\sigma(\xi) \right]. \tag{62}
\]

First, varying w.r.t. \( \xi^\sigma \) indeed gave us Eq. (56). Second, from (61) \( \partial \delta x^\lambda / \partial \xi(x) \partial \xi^\sigma = \delta x^\lambda \), which means that varying w.r.t. \( \xi \) is varying w.r.t. the coordinates. Third, the boundary term (62) contains \( \partial_\mu \delta^\rho_{\beta} \). Let us point out that the procedure was based on varying the connection coefficients \( \Gamma^\lambda_{\mu \nu} \) w.r.t. \( \xi \), and hence the idea holds for arbitrary action.

### 2. Equation with GR motivated coefficients

Let us consider the coefficients (20), originating from general relativity, and matter action which does not contain generic connection. Then \( P^\mu_{(\mu \nu \lambda)} = P^\mu_{(\mu \nu \lambda)} \), and the equation for connection simplifies to

\[
(1 - \epsilon) \nabla_\mu \left[ (\partial_\nu A) \nabla_\nu \left( \sqrt{-g} g^{\mu \nu} \delta^\lambda_{\beta} \right) \right] = 0. \tag{63}
\]

Hence, for the action where \( \epsilon = 0 \), i.e., without the boundary-term-motivated Lagrangian (38c), we obtain the equation (30) in Ref. [12]. However, if \( \epsilon = 1 \) and we are thus considering an action that is equivalent to the action in scalar-curvature tensor theories (see action (2.2) in Ref. [6]), then the symmetric teleparallel connection is not constrained by this equation.

The connection equation (63) can be expressed as

\[
(1 - \epsilon) \partial_\nu \left[ \left( \partial_\mu A \right) \partial_\lambda \left( \sqrt{-g} g^{\mu \nu} \delta^\lambda_{\beta} \right) \right] = (1 - \epsilon) \det \left[ \frac{\partial x^\lambda}{\partial \xi^\nu} \nabla_\mu \left[ \left( \partial_\nu A \right) \nabla_\lambda \right] \left( \sqrt{-g} g^{\mu \nu} \right) \right], \tag{64}
\]

where the left hand side is evaluated in \( \xi^\nu \) coordinates, stressed (only in this Subsection) by adding a bar on top of \( \bar{g} \), and a prime along the indices. The result (64) just transforms the right hand side under a change of coordinates, convincing us that \( \xi^\nu \) are coordinates in which the connection coefficients vanish.

In such theory, for particular ansätze of the metric \( g_{\mu \nu} \) and the scalar field \( \Phi \), Eq. (64) provides us a differential equation for determining the Jacobian matrix \( \partial x^\mu / \partial \xi^\nu \) as

\[
\frac{\partial x^\mu}{\partial \xi^\nu} \frac{\partial x^\lambda}{\partial \xi^\nu} \partial_\nu \left[ \left( \partial_\nu A \right) \frac{\partial x^\lambda}{\partial \xi^\nu} \right] \times \delta_{\lambda} \left[ \det \left[ \frac{\partial x^\lambda}{\partial \xi^\nu} \sqrt{-g} \frac{\partial \xi^\nu}{\partial x^\sigma} \frac{\partial \xi^\nu}{\partial x^\rho} \right] \right] = 0. \tag{65}
\]

### 3. Simple example of \( \Gamma^\lambda_{\mu \nu} \neq 0 \)

Let us consider the GR motivated coefficients (20). The equation for the connection is then (63) or analogously (64). In Ref. [12] we studied spatially (Levi-Civita) flat Friedmann cosmology as an example (see Section V in Ref. [12]). It turned out that vanishing connection coefficients \( \Gamma^\lambda_{\mu \nu} \) lead to consistent results, if firstly the (Levi-Civita) flat Friedmann-Lemaître-Robertson-Walker line element is expressed in Cartesian coordinates \( \xi^\nu \equiv t, \xi^1 \equiv x, \xi^2 \equiv y, \xi^3 \equiv z \), i.e.,

\[
\text{ds}^2 = - \left( \text{d}\xi^0 \right)^2 + \left( a(\xi^0) \right)^2 \delta_{ij} \text{d}\xi^i \text{d}\xi^j, \quad (66a)
\]

and secondly the scalar field is assumed to depend only on cosmological time, i.e.,

\[
\Phi \equiv \Phi \left( \xi^0 \right) \Rightarrow A \equiv A \left( \xi^0 \right). \tag{66b}
\]

Equation (64) verifies that result immediately. Namely, both the metric \( g_{\mu \nu} \) and the scalar field \( \Phi \) only depend on the cosmological time \( t \) and hence the antisymmetrization on the first line yields zero. Reducing covariant derivatives to partial ones is in this case a consistent procedure. The nonvanishing components of the nonmetricity are

\[
\nabla_\nu \bar{g}_{\nu \nu} = \partial_\nu \bar{g}_{\nu \nu} = 2H \bar{g}_{\nu \nu}, \tag{67}
\]

where \( H \equiv \dot{a}/a \), and \( \dot{a} \equiv da/dt \).

Perhaps the simplest example of nonvanishing symmetric teleparallel connection coefficients arise, if one evaluates (66a) in spherical coordinates

\[
\begin{align*}
x^0 & \equiv t, \quad (68a) \\
x^1 & \equiv r = \sqrt{x^2 + y^2 - z^2}, \quad (68b) \\
x^2 & \equiv \theta = \arctan \left( \frac{\sqrt{x^2 + y^2}}{z} \right), \quad (68c) \\
x^3 & \equiv \varphi = \arctan \left( \frac{y}{x} \right), \quad (68d)
\end{align*}
\]

resulting in

\[
\text{ds}^2 = - \left( \text{d}x^0 \right)^2 + \left( a(\xi^0) \right)^2 g_{ij} \text{d}x^i \text{d}x^j, \quad (69)
\]

\[
(g_{ij}) = \begin{pmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}. \quad (70)
\]

The corresponding Jacobi matrix

\[
\left( \frac{\partial x^i}{\partial \xi^j} \right) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \sin \theta \cos \varphi & \sin \theta \sin \varphi & 0 & 0 & 0 \\
0 & \cos \theta \cos \varphi & \cos \theta \sin \varphi & 0 & 0 & 0 \\
0 & 0 & 0 & \sin \theta \\
\end{pmatrix}
\]

\[
\quad \begin{pmatrix}
r \sin \varphi \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \\
\begin{pmatrix}
0 \\
r \cos \varphi \\
0 \\
0
\end{pmatrix} \\
\begin{pmatrix}
0 \\
0 \\
r \sin \theta \\
0
\end{pmatrix}
\tag{71}
\]
and its inverse
\[
\left( \frac{\partial \gamma^j}{\partial x^k} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \vartheta \cos \varphi & r \cos \vartheta \cos \varphi - r \sin \vartheta \sin \varphi \\ 0 & \sin \vartheta \sin \varphi & r \cos \vartheta \sin \varphi + r \sin \vartheta \cos \varphi \\ 0 & \cos \vartheta & -r \sin \vartheta \end{pmatrix}, \tag{72}
\]

obviously satisfy (65). Calculating the connection coefficients via (10) yields
\[
\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r \sin^2 \vartheta, \quad \Gamma^2_{12} = \frac{1}{r}, \tag{73a}
\]
\[
\Gamma^2_{33} = -\sin \vartheta \cos \vartheta, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{32} = \cot \vartheta. \tag{73b}
\]

Expressions (73) are nothing else than the nonvanishing Christoffel symbols for (70) and thus possess metric compatibility w.r.t. (70). Applying the prescription (10) on the Jacobian matrix (71) does not generate temporal components of the connection coefficients, such as \( \Gamma^0_{01} \) (cf. the Christoffel symbols for whole FLRW metric given for example by Eqs. (8.44) in Ref. [19]). The covariant derivative w.r.t. the time direction thus reveals nonmetricity as
\[
\nabla_0 g_{ij} = \partial_0 g_{ij} = 2H g_{ij}, \tag{74}
\]
which corresponds to (67). All other covariant derivatives yield zero also in the spherical coordinates.

F. Continuity equation

Let us consider the diffeomorphism invariance on the action (37)
\[
\delta \zeta S = \frac{1}{2\kappa^2} \int_{M_4} d^4x \left\{ \sqrt{-g} E^{(g)}_\mu \left( \tilde{\zeta}_\mu \right) + \sqrt{-g} E^{(\Phi)} \tilde{\zeta}_\Phi \right. \\
+ \sqrt{-g} \left( E^{(\Gamma)} \right)_\lambda^{\mu \nu} \tilde{\zeta}_\lambda \Gamma^\lambda_{\mu \nu} + \delta S_{\delta m} \tilde{\zeta}_\chi \right\} = 0. \tag{75}
\]

By calculating the Lie derivatives, i.e., \( \mathcal{L}_\zeta g^{\mu \nu}, \mathcal{L}_\zeta \Phi \) and \( \mathcal{L}_\zeta \Gamma^\lambda_{\mu \nu} \) (see Ref. [38], in particular Eq. (10) for the Lie derivative of the connection), integrating by parts, neglecting matter equations and boundary terms, we obtain
\[
\delta \zeta S = \frac{1}{2\kappa^2} \int_{M_4} d^4x \left\{ \sqrt{-g} \left[ 2\sqrt{-g} \left( g^{\mu \nu} E^{(g)}_\mu \right) + E^{(\Phi)} \right] \partial_\nu \Phi \right. \\
+ \nabla_\nu \nabla_\lambda \left[ \sqrt{-g} \left( E^{(\Gamma)} \right)_\nu^{\lambda \omega} \right] \zeta^\nu \right\} = 0. \tag{76}
\]

In order to calculate the first line
\[
2\sqrt{-g} \left( g^{\mu \nu} E^{(g)}_\mu \right) + \sqrt{-g} E^{(\Phi)} \partial_\nu \Phi = 4\sqrt{-g} \left[ \sqrt{-g} A \left( P^{\lambda \omega}_\nu - \epsilon P^{\lambda \omega}_\nu \right) - \sqrt{-g} \kappa^2 \nabla_\nu T^\omega_\nu \right], \tag{77}
\]
we made use of (E1), (25), (23b), and (58). If the coefficients \( c_1, \ldots, c_5 \) are GR-motivated (20), then for two particular cases the usual continuity equation \( \tilde{\nabla}_\nu T^\omega_\nu = 0 \) is manifestly fulfilled. First, if \( A = 1 \), i.e., we consider the symmetric teleparallel equivalent of general relativity (with minimally coupled scalar), second, if \( \epsilon = 1 \), i.e., the equivalent to scalar-curvature theories (see, e.g., Ref. [6]). If this is not the case, then let us also include the third component. Combining (77), (76) and (56) yields
\[
-2\kappa^2 \left( \sqrt{-g} \nabla_\nu T^\omega_\nu + 2\nabla_\omega \nabla_\lambda H^\omega_\lambda \right) = 0, \tag{78}
\]
which also follows from
\[
2\kappa^2 \delta \zeta S_m = 0, \tag{79}
\]
i.e., from the diffeomorphism invariance of the matter action (38e)

IV. HAMILTON-LIKE APPROACH

A. Field space metric \((g^{\lambda \omega})\)

Let us define
\[
(g^{\lambda \omega}) \equiv \begin{pmatrix} A_{G^{\lambda \omega}} & \epsilon A^\mu g^{\lambda \omega} \\ \epsilon A_{G^{\lambda \omega}} & -B g^{\lambda \omega} \end{pmatrix}, \tag{80}
\]
where in order to suppress some indices, we have used a convention where, e.g.,
\[
G^{\lambda \omega} \equiv g^{\lambda \mu \nu} \delta_{\sigma \rho}, \quad \Phi^{\lambda \omega} \equiv \Phi^{\lambda \omega \mu \nu}. \tag{81}
\]
The capital Greek letter indicates the first small Greek letter. Here
\[
\Phi^{\lambda \omega} = \Phi^{\lambda \omega}_{\mu \nu}, \quad \Phi^{\lambda \omega}_{\mu \nu} = \Phi^{\lambda \omega \mu \nu}, \tag{82}
\]
and thus the field space metric (80) only depends on the usual metric \( g_{\mu \nu} \) and on the scalar field \( \Phi \) but not on their derivatives. By introducing
\[
\Psi = \begin{pmatrix} g^{\mu \nu} \\ \Phi \end{pmatrix}, \tag{83}
\]
we may write the kinetic terms in the action (37) as
\[
A Q - B(\Phi) g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi + \epsilon \partial_\mu A(\Phi) \left( Q^\mu - \tilde{Q}^\mu \right) = (\nabla_\lambda g^{\mu \nu} - \nabla_\lambda \Phi)(A_{G^{\lambda \omega \mu \nu}} + \epsilon A_{G^{\lambda \omega \mu \nu}} \epsilon A^\sigma \Phi^{\lambda \omega \sigma \nu} - B_{g^{\lambda \omega \mu \nu}})(\nabla_\omega g^{\sigma \rho}) \nabla_\nu \Phi. \tag{84}
\]
Here, in order to simplify the notation, we consider
\[
\nabla_\omega \Psi = \begin{pmatrix} \nabla_\omega g^{\sigma \rho} \\ \nabla_\omega \Phi \end{pmatrix} \cdot \begin{pmatrix} \partial_\omega \Phi \end{pmatrix}. \tag{85}
\]
One can thus write the whole Lagrangian (38), a function of the metric $g^{\mu\nu}$, its “generalized velocity” $\nabla_\lambda g^{\mu\nu} \equiv -Q_\lambda^{\mu\nu}$, the scalar field $\Phi$ and $\partial_\lambda \Phi$ as

$$\sqrt{-g} L = \frac{1}{2\kappa^2} \sqrt{-g} \nabla_\lambda \lambda (g^{\nu\rho}) \nabla_\omega \lambda^\nu \Psi - \kappa^{-2} \epsilon^2 \sqrt{-g} V + \sqrt{-g} L_m .$$

(86)

Note that we have not included the Lagrangian (38d) for the Lagrange multipliers. We assume the connection to have the symmetric teleparallel form (10), and in that case $\xi$ resides entirely in the “generalized velocity” $\nabla_\lambda g^{\mu\nu}$. Hence, the whole Lagrangian is indeed only a function of the scalar field and the metric along with their “generalized velocities”, and matter Lagrangian $L_m$.

B. Generalized momenta

Based on analogy, let us define generalized momenta as

$$\Pi_\lambda^{(g)} \equiv \frac{\partial \sqrt{-g} L}{\partial \nabla_\lambda g} = \sqrt{-g} \kappa^{-2} \left( A G^{\lambda\Omega} \nabla_\Omega g + \epsilon A' (\Theta^{\lambda\omega} \partial_\omega \Phi) \right)$$

$$= \sqrt{-g} \kappa^{-2} \left( -A \rho^\lambda + \epsilon A' (\Theta^{\lambda\omega} \partial_\omega \Phi) \right) ,$$

(87a)

$$\Pi_\lambda^{(\Phi)} \equiv \frac{\partial \sqrt{-g} L}{\partial \partial_\lambda \Phi} = \sqrt{-g} \kappa^{-2} \left( \epsilon A' (\Theta^{\lambda\omega} \nabla_\Omega g + \Theta^\lambda g^{\omega\rho} \partial_\omega \Phi) \right) .$$

(87b)

Here, for simplicity, we have assumed that the matter Lagrangian $L_m$ depends on the metric only algebraically. In principle one could also consider more generic cases, where these momenta also include, e.g., the Levi-Civita connection contribution to the matter Lagrangian $L_m$. The details of such calculations are beyond the scope of the current paper, but there does not seem to be any obvious reason, why the following results should not hold for the generic cases as well.

In order to construct a Hamiltonian, one should invert $(g^{\lambda\omega})$. This fails in only two distinct cases. First, if the condition (B4) does not hold, and hence $G^{\lambda\Omega}$ is not reversible. Second, if the multiplier (C4) vanishes. Of course we also assume that $A \neq 0$. For all other cases $(g^{\lambda\omega})$ is reversible. See Appendix C.

1. Generalized momenta in distinct cases

First, let us consider the case $\epsilon = 0$, then

$$\Pi_\lambda = \sqrt{-g} \kappa^{-2} \left( A G^{\lambda\nu}(\Theta^{\lambda\omega}\partial_\nu g^{\omega\rho}) - B g^{\nu\rho} \partial_\nu \Phi \right) .$$

(88)

and we see that the fields are debrained as suggested in Subsection III.D.

Second, in the case of the coefficients (20) and $\epsilon = 1$, corresponding to the scalar-curvature [6] equivalent,

$$\Pi_\lambda = \sqrt{-g} \kappa^{-2} A \left( -2F(1)g^{\lambda\nu} \nabla_\nu \Psi \nabla^\nu \Psi + A' (\Theta^{\lambda\omega} \partial_\rho g^{\omega\rho}) \right) ,$$

(89)

where in addition to the quantities (21), (48a), (82), we also defined

$$\hat{g}_{\mu\nu} \equiv A g_{\mu\nu} , \quad \hat{g}^{\rho\sigma} = A^{-1} g^{\rho\sigma}$$

which is the Einstein frame (invariant) metric (see Eq. (18) in Ref. [39], and Eq. (8) in Ref. [40]). Moreover

$$I_3 \equiv \int \sqrt{F(1)} d\Phi$$

(91)

is the Einstein frame (invariant) scalar field (see Eq. (15) in Ref. [39] and Eq. (5b) in Ref. [40], also Eqs. (55), (60) in Ref. [39]). Note that in that case we can transform to the Einstein frame, where $A = 1$, and debraid the variables.

C. Hamilton-like equations

The Hamiltonian is

$$H = \frac{\kappa^2}{2\sqrt{-g}} \Pi^\lambda (g^{\lambda\nu}) \Pi^\nu + \kappa^{-2} \kappa^{-2} \kappa^{-2} g^{\nu\rho} \nabla_\nu \Psi \nabla^\rho \Psi \nabla^\mu \nabla_\mu g^{\nu\rho} \nabla^\rho \Psi$$

(92)

where

$$\Pi_\lambda \equiv \frac{\Pi_\lambda^{(g)}}{\Pi_\lambda^{(\Phi)}}$$

(93)

gathers the “generalized momenta”, and is transposed if necessary. A straightforward calculation verifies

$$\nabla_\lambda \Psi = \frac{\partial H}{\partial \Pi_\lambda} .$$

(94)

Calculating the equations for $\nabla_\lambda \Pi_\lambda$, and checking the consistency with Eqs. (41) and (46), namely showing that up to choice of variables

$$\nabla_\lambda \Pi_\lambda^{(g)} + \frac{\partial H}{\partial g^{\mu\nu}} \nabla^\mu \nabla_\lambda g^{\nu\rho} + \frac{\partial H}{\partial \Phi} \Pi_\lambda^{(\Phi)} \nabla_\lambda g^{\nu\rho} = - \frac{\sqrt{-g}}{2\kappa^2} E^{(g)}$$

(95a)

$$\nabla_\lambda \Pi_\lambda^{(\Phi)} + \frac{\partial H}{\partial \Phi} \nabla^\mu \nabla_\lambda g^{\mu\nu} - \frac{\sqrt{-g}}{2\kappa^2} E^{(\phi)}$$

(95b)

if one makes use of the result

$$\delta (g^{\lambda\nu}) = - (g^{\lambda\nu}) \delta g^{\rho\sigma} (g^{\rho\sigma}) .$$

(96)

Note that we do not need to calculate the expression explicitly, because the inverses $(g^{\lambda\nu})$ contract with “generalized momenta”, thus yielding up to a multiplier the

\[5\] Note that by convention we vary w.r.t. $g^{\mu\nu}$ and thus the “generalized velocity” and also “generalized momentum” gain a minus sign. One could also vary w.r.t. $g_{\mu\nu}$ and then the “generalised velocity” would be $\nabla_\lambda g_{\mu\nu} \equiv +Q_\lambda^{\mu\nu}$. 


In Eqs. (56) after one has chosen particular ansätze for the metric and connection is permitted, and consistency must be checked only partially. In the generic case such a transformation of coordinates can be transformed to ordinary partial derivative. In the symmetric teleparallel framework and study their consequences. Unfortunately one cannot use Poisson brackets like structure because the chain rule can not be invoked due to the fact that in the equations of motion there already is contraction, i.e., via the equations of motion one can not calculate neither the connection equation (97) the nonmetricity scalar to a scalar field. This coupling resembles scalar-tensor theories where the scalar field is coupled to the metric tensor degree of freedom. As our previous work indicates, when one considers as the quadratic nonmetricity scalar the GR equivalent, one obtains a different theory then a simple scalar-curvature extension of GR. The current work at one hand extends this extension by allowing the quadratic nonmetricity scalar to differ from the GR equivalent by including five arbitrary coefficients (the newer GR), on the other hand the inclusion of the boundary term allows us to obtain an ordinary scalar-curvature theory as a particular case.

There are different directions for future work. One could study some specific applications, e.g. in order to distinguish the simplest scalar-nonmetricity and scalar-torsion theories one could study perturbations on a cosmological background. Another direction would be to consider more general actions in the symmetric teleparallel framework and study their consequences.

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APPENDIXES

Appendix A: Contractions of $G^λ_{\mu\nu} ω_{\sigma\rho}$

Let us calculate the contractions of $G^λ_{\mu\nu} ω_{\sigma\rho}$, defined by (15). A straightforward calculation yields

\[
g^μνG^λ_{\mu\nu} ω_{\sigma\rho} = C_1 g_σg_λ^λω + C_2 δ^λ_σδ_ρ^λ, \quad (A1a)
\]
\[
δ^λ_σδ_ρ^λ g^λ_{\mu\nu} ω_{\sigma\rho} = C_3 g_μ(σδ_ρ^λ + δ_μ^σg_σ), \quad (A1b)
\]
\[
g_λωG^λ_{\mu\nu} ω_{\sigma\rho} = C_5 g_μg_νg_σ + C_6 δ_λ^μg_σ(μg_σg_λ), \quad (A1c)
\]
\[
g^μσG^λ_{\mu\nu} ω_{\sigma\rho} = C_7 g^λ_{\nuσ}g_νσ + C_8 δ_ρ^μg_ν + C_9 δ_ρ^λδ_σ^μ, \quad (A1d)
\]

where

\[
C_1 ≡ c_1 + 4c_3 + \frac{1}{2}c_5, \quad C_2 ≡ c_2 + c_4 + 2c_5, \quad (A2a)
\]
\[
C_3 ≡ c_1 + \frac{1}{2}c_2 + \frac{5}{2}c_4 + \frac{1}{2}c_5, \quad C_4 ≡ \frac{1}{2}c_2 + c_3 + \frac{5}{4}c_5, \quad (A2b)
\]
\[
C_5 ≡ 4c_3 + c_5, \quad C_6 ≡ 4c_1 + c_2 + c_4, \quad (A2c)
\]
\[
C_7 ≡ \frac{5}{2}c_1 + \frac{1}{2}c_2 + c_3 + \frac{1}{4}c_4, \quad C_8 ≡ \frac{3}{2}c_2 + \frac{1}{2}c_5, \quad (A2d)
\]
\[
C_9 ≡ \frac{3}{2}c_1 + \frac{1}{2}c_5. \quad (A2e)
\]

V. SUMMARY

In recent years teleparallel theories have gained more attention as alternative theories of gravity. While one mostly works in the torsion-based setting, there has been interest in the direction of symmetric teleparallelism, where instead of curvature or torsion gravity is effectively described by nonmetricity. In the current paper we extended the class of scalar-nonmetricity theories by coupling the quadratic five-parameter nonmetricity scalar to a scalar field. This coupling resembles scalar-tensor theories where the scalar field is coupled to the metric tensor degree of freedom. As our previous work indicates, when one considers as the quadratic nonmetricity scalar the GR equivalent, one obtains a different theory then a simple scalar-curvature extension of GR. The current work at one hand extends this extension by allowing the quadratic nonmetricity scalar to differ from the GR equivalent by including five arbitrary coefficients (the newer GR), on the other hand the inclusion of the boundary term allows us to obtain an ordinary scalar-curvature theory as a particular case.

There are different directions for future work. One could study some specific applications, e.g. in order to distinguish the simplest scalar-nonmetricity and scalar-torsion theories one could study perturbations on a cosmological background. Another direction would be to consider more general actions in the symmetric teleparallel framework and study their consequences.
The coefficients $C_2$, $C_3$, $C_4$, $C_5$, $C_7$ form a basis, as
\begin{align}
C_1 &= \frac{5}{2} C_2 + C_3 + 4 C_4, \\
C_6 &= -9 C_2 + 4 C_3 + 16 C_4 - 4 C_5,
\end{align}
while $C_8$ and $C_9$ are more complicated combinations, also including $C_7$.
The first four of these coefficients enter the theory through (see definition (18b))
\begin{align}
P^\lambda &= P^\lambda_{\mu\nu} g^{\mu\nu} = C_1 Q^\lambda + C_2 \bar{Q}^\lambda, \\
\bar{P}_\nu &= P^\lambda_{\mu\nu} g^{\mu\lambda} = C_4 Q_\nu + C_3 \bar{Q}_\nu.
\end{align}
Also, if one considers the local Weyl rescaling on the metric
\[\bar{g}_{\mu\nu} = e^{\Omega(\Phi)} g_{\mu\nu}, \quad g^{\mu\nu} = e^{-\Omega(\Phi)} \bar{g}^{\mu\nu}\]
the non-metricity tensor $Q_{\lambda\mu\nu}$ and its two contractions transform as
\begin{align}
\bar{Q}_{\lambda\mu\nu} &= \nabla_\lambda \bar{g}_{\mu\nu} = e^{\Omega} (Q_{\lambda\mu\nu} + g_{\mu\nu} \partial_\lambda \Omega), \\
\bar{Q}_\lambda &= \bar{Q}_{\lambda\mu\nu} \bar{g}^{\mu\nu} = Q_\lambda + 4 \partial_\lambda \Omega, \\
\bar{Q}_\lambda &= \bar{Q}_{\lambda\mu\nu} \bar{g}^{\mu\nu} = \bar{Q}_\lambda + 3 \partial_\lambda \Omega.
\end{align}
Thus, based on the definition (16), it follows that
\[Q = e^{-\Omega} \bar{Q} + 2 e^{-\Omega} C_1 Q^{\mu\nu} \partial_\mu \partial_\nu + 2 e^{-\Omega} C_2 \bar{Q}^{\mu\nu} \partial_\mu \partial_\nu + e^{-\Omega} (4 C_1 + C_2) g^{\mu\nu} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma.
\]
For GR motivated values (B9a) Eq. (A7) yields Eq. (33) in Ref. [12].

Appendix B: Inverting $G^\lambda_{\mu\nu}\omega_{\sigma\rho}$

In order to invert $G^\lambda_{\mu\nu}\omega_{\sigma\rho}$, defined via (15), w.r.t. the Einstein product (see definition (2.1) in Ref. [32]), i.e.,
calculate
\[\left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} : \left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} G^\lambda_{\mu\nu}\omega_{\sigma\rho} = \delta_\tau^{\xi} \delta_\sigma^{\mu} \delta_\rho^{\nu}.\]
explicitly, we make an ansatz as
\[\left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} \equiv k_1 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda} + k_2 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda} + k_3 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda} + k_4 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda} + k_5 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda} + k_6 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda} + k_7 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} g_{\tau\lambda}.
\]
A straightforward calculation leads us to the following system of linear algebraic equations
\[\begin{pmatrix}
c_1 & c_2 & c_3 & c_4 & c_5 \\
c_2 & c_1 + c_2 & c_3 & c_4 & c_5 \\
c_3 & c_4 & c_0 & c_2 & c_4 \\
c_4 & c_5 & c_2 & c_3 & c_0 \\
c_5 & c_6 & c_3 & c_4 & c_0
\end{pmatrix}
\begin{pmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4 \\
k_5
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]
The matrix of the coefficients is regular, if
\[\det = \left( c_1^2 + \frac{1}{2} c_1 c_2 - \frac{1}{2} c_2^2 \right) (C_1 C_3 - C_2 C_4)^2 \neq 0.\]
More thorough inspection reveals, that (if determinant is nonvanishing then) always $k_5 = k_6$. Such a result enforces the symmetry
\[\left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} \left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} = \left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} \left( G^{-1} \right)_{\tau\lambda}^{\xi\mu},\]
as in (17c).
For later use, let us define
\[K_1 \equiv k_1 + 4 k_3 + \frac{1}{2} k_5, \quad K_2 \equiv k_2 + 4 k_4 + 2 k_5, \quad K_3 \equiv k_1 + \frac{1}{2} k_2 + \frac{5}{2} k_4 + \frac{1}{2} k_5, \quad K_4 \equiv \frac{1}{2} k_2 + k_3 + \frac{5}{4} k_5\]
amalogously to (A2a)-(A2b).

1. Inverting GR motivated $G^\lambda_{\mu\nu}\omega_{\sigma\rho}$

For the general relativity case (21)
\[\left( G^{-1} \right)_{\tau\lambda}^{\xi\mu} = 4 \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} + \frac{2}{3} g^{\xi\lambda} g^{(\mu} g^{\nu)} \]
\[- \frac{4}{3} \delta^{(\xi} g^{\lambda(\mu} g^{\nu)} - \frac{4}{3} g^{\xi(\mu} \delta^{(\tau\lambda)\nu) - \frac{4}{3} g^{(\xi\mu} \delta^{(\tau\lambda)\nu),\]
i.e.,
\[k_1 = 0, \quad k_2 = 4, \quad k_3 = \frac{2}{3}, \quad k_4 = -\frac{4}{3}, \quad k_5 = -\frac{4}{3}.
\]

2. Coefficients $C_i$ and $K_i$ in GR motivated case

Based on definitions (A2a) and (A2b) and numerical values (20), let us calculate
\[C_1|_{GR} = \frac{1}{2}, \quad C_2|_{GR} = \frac{1}{2}, \quad C_3|_{GR} = -\frac{1}{4}, \quad C_4|_{GR} = -\frac{1}{8}, \quad K_1|_{GR} = \frac{4}{3}, \quad K_2|_{GR} = -\frac{8}{3}, \quad K_3|_{GR} = -\frac{8}{3}, \quad K_4|_{GR} = \frac{2}{3}.
\]

Appendix C: Inverting the field space metric ($g^{\lambda\omega}$)

In order to invert (80), i.e., the field space metric ($g^{\lambda\omega}$), let us recall, how block matrices are inverted.
From Wikipedia \cite{41}

$$(A \ B)^{-1} = \left( A^{-1} + A^{-1}BF^{-1}CA^{-1} - A^{-1}BF^{-1} \right)^{-1}$$

where

$$F = D - CA^{-1}B.$$  \hfill(C2)

In our case

$$F^\xi\zeta = -B^\xi\zeta - c^2(A')^2 \xi\zeta (g^{-1})^{\lambda\rho} \phi^\lambda \phi^\rho$$

$$= -2A^3 g^{\xi\zeta},$$  \hfill(C3)

which is invertible, if the multiplier

$$\delta = \frac{2AB \mp c^2(A')^2}{4A^2} \left[ 6(K_1 - K_4) - 3(K_2 - K_3) \right]$$  \hfill(C4)

in front of $g^{\xi\zeta}$ is nonvanishing. In terms of the coefficients

$$c_1, \ldots, c_5$$

$$\frac{1}{8} \left[ 6(K_1 - K_4) - 3(K_2 - K_3) \right]$$

$$= \frac{9(c_1 + c_2 + 2c_3 + 2c_4 + 3c_5)}{C_1C_3 - C_2C_4}.$$  \hfill(C5)

Hence, we see that dividing by zero can only occur, when \hfill(B4) vanishes, but in that case the coefficients $k_i$ cannot be determined via \hfill(B3).

In the GR motivated case \hfill(B8), or analogously \hfill(20) we obtain

$$F^\xi\zeta = \frac{-2AB \mp c^2(A')^2}{2A} g^{\xi\zeta}.$$  \hfill(C6)

\begin{itemize}
\item[i)] If $\epsilon = 0$ then this result accommodates the multiplier of the d’Alembert operator in the scalar field equation of motion \hfill(46).
\item[ii)] If $\epsilon = 1$, then the multiplier is consistent with \hfill(48a), i.e., the multiplier of the d’Alembert operator in \hfill(48). Under the assumptions this particular equation does not contain second derivatives of the metric tensor, because the conditions \hfill(51) are fulfilled. Note that this case corresponds to the scalar-curvature theory \hfill(6), and hence one can transform to the Einstein frame and decouple the “generalized momenta” \hfill(89), which then also contain \hfill(C6).
\item[iii)] If $\epsilon \neq 0$ and $\epsilon \neq 1$, then \hfill(C6) differs from \hfill(48a) by \hfill$\epsilon^2$ multiplier. This suggests that neither \hfill(46) nor \hfill(48) describe the pure propagating scalar.
\end{itemize}

The inverse for the field space metric \hfill(80) reads

$$\left( \mathcal{G}^{-1} \right)^{\omega_\xi} \equiv \left( \begin{array}{ll} \left( \mathcal{G}^{-1} \right)_1^{\omega_\xi} & \left( \mathcal{G}^{-1} \right)_2^{\omega_\xi} \\ \left( \mathcal{G}^{-1} \right)_1^{\omega_\xi} & \left( \mathcal{G}^{-1} \right)_2^{\omega_\xi} \end{array} \right).$$  \hfill(C7)

where

$$\left( \mathcal{G}^{-1} \right)_1^{\omega_\xi} \equiv \frac{A^{\lambda\rho}}{\left( g^{-1} \right)_{\Omega\Xi}} + c^2 \left( A'^\lambda \right)^2 \phi^\lambda \phi^{\mu T} (g^{-1})_{\mu\xi}$$

\hfill(C8a)

$$\left( \mathcal{G}^{-1} \right)_2^{\omega_\xi} \equiv -c^2 \left( A'^\lambda \right)^2 \phi^{\lambda T} (g^{-1})_{\mu\xi}$$

\hfill(C8b)

$$\left( \mathcal{G}^{-1} \right)_1^{\omega_\xi} \equiv -c^2 \left( A'^\lambda \right)^2 \phi^{\lambda T} (g^{-1})_{\mu\xi}$$

\hfill(C8c)

$$\left( \mathcal{G}^{-1} \right)_2^{\omega_\xi} \equiv (g^{-1})_{\mu\xi}.$$  \hfill(C8d)

A straightforward calculation verifies that indeed

$$\left( \mathcal{G}^{\lambda\omega} \right) \left( \mathcal{G}^{\xi}_\omega \right) = \left( \Delta^\lambda_2 \ 0 \ \delta^\lambda_\xi \right).$$  \hfill(C9)

where

$$\Delta^\lambda_2 \equiv \delta^\lambda_\xi \delta^\eta_\zeta \frac{(g^\zeta_\eta)}{\delta^\zeta_\kappa}.$$  \hfill(C10)

Note that the prescription \hfill(C1) could be used recursively, and hence, if the momenta \hfill(87) would also include contributions from the matter Lagrangian $\mathcal{L}_m$, then the inverse \hfill(C7) could be used in the later steps of the recursion.

**Appendix D: Block diagonal partitioning of \hfill(\mathcal{G}^{\lambda\omega})**

From the definition \hfill(80) of the field space metric \hfill(\mathcal{G}^{\lambda\omega})

one can observe that

$$\left( \mathcal{G}^{\lambda\omega} \right)^T \equiv \left( \begin{array}{ll} A \phi^{\lambda\rho} & cA' \phi^{\lambda\rho} \\ cA' \phi^{\lambda\rho} & -B^\rho_{\lambda\rho} \end{array} \right)^T$$

$$\equiv \left( \begin{array}{ll} A \phi^{\lambda\rho} & cA' \phi^{\lambda\rho} \\ cA' \phi^{\lambda\rho} & -B^\rho_{\lambda\rho} \end{array} \right),$$  \hfill(D1)

and due to that symmetry it is natural to seek for some diagonal partitioning procedure for such an object.

A visit to Mathematics Stack Exchange site \hfill[42] reveals the following. Let

$$M \equiv \left( \begin{array}{ll} A & B \\ C & D \end{array} \right)$$  \hfill(D2)

be a block matrix, then

$$\left( \begin{array}{ll} I_1 & 0 \\ -CA^{-1}I_2 \end{array} \right) \left( \begin{array}{ll} A & B \\ C & D \end{array} \right) \left( \begin{array}{ll} I_1 & -A^{-1}B \\ 0 & I_2 \end{array} \right)$$

$$= \left( \begin{array}{ll} A & 0 \\ 0 & D - CA^{-1}B \end{array} \right),$$  \hfill(D3)

where $I_1$ and $I_2$ are some suitable unit matrices. In our case $A = B^T$ and \hfill$A^{-1}$ is $A^{-1}$. Under these conditions \hfill(D3) turns out to be a congruence transformation $P^TMP$ where

$$P = \left( \begin{array}{ll} I_1 & -A^{-1}B \\ 0 & I_2 \end{array} \right).$$  \hfill(D4)
Due to

\[ P^{-1} = \begin{pmatrix} I_1 & A^{-1}B \\ 0 & I_2 \end{pmatrix} \]  

(D5)

Eq. (D3) is not a similarity transformation and thus the term diagonalization would not be suitable. However, for tensor components with two indices at the same vertical position, it is exactly the congruence transformation that corresponds to change of basis.

In our case

\[ P \equiv \begin{pmatrix} \Delta \Omega \xi \delta \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \xi \x
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