Distribution of proper delay times in quantum chaotic scattering:  
A crossover from ideal to weak coupling

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The probability distribution of the proper delay times during scattering on a chaotic system is derived in the framework of the random matrix approach and the supersymmetry method. The result obtained is valid for an arbitrary number of scattering channels as well as arbitrary coupling to the energy continuum. The case of statistically equivalent channels is studied in detail. In particular, the semiclassical limit of infinite number of weak channels is paid appreciable attention.

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The temporal aspect of collision and transport phenomena has, starting with the pioneering works \cite{1,2}, repeatedly attracted attention. Analysis of the duration of a process gives an interesting and important information complementary to that delivered by the energy representation. The temporal approach becomes especially elucidating in the cases of chaotic resonance scattering and transport in disordered media when many complicated long-lived intermediate states are involved \cite{3–16}; see \cite{17,18} for recent reviews.

In this Letter we show that representation \cite{18} allows us to extend at least part of the results mentioned above to the case of arbitrary coupling. Namely, we calculate the probability distribution of the proper delay times $q_c$ following (henceforth $\bar{\hbar} = 1$) \cite{10}:

$$Q(E) = V^\dagger \frac{1}{(E - \mathcal{H})^\dagger} \frac{1}{E - \mathcal{H}} V \equiv b'(E) b(E). \quad (1)$$

The Hermitian part $\mathcal{H}$ determines the appropriate basis for the internal motion whilst the amplitudes $V_n^\dagger$ describe the coupling between $N$ interior and $M$ channel states. We adopt below the standard statistical approach \cite{19} based on the random-matrix theory (RMT) to simulate the complicated intrinsic motion; see \cite{20,21}.

Recently, an appreciable success has been achieved by the authors of Refs. \cite{12}. By quite a distinguished way of reasoning, they managed to calculate in the framework of RMT the joint probability distribution of the eigenvalues $q_c$ of the Wigner-Smith matrix, which are called the proper delay times. Using then the method of orthogonal polynomials they calculated also the density of proper delay times. These findings received later experimentally relevant applications to quantum dots \cite{12} and optics of random media \cite{23}. However, all the results are restricted to the case of perfect coupling to the continuum (all transmission coefficients are equal to 1).

In this Letter we show that representation \cite{18} allows us to extend at least part of the results mentioned above to the case of arbitrary coupling. Namely, we calculate the probability distribution of the proper delay times $q_c$:

$$P_M(q) = \frac{1}{M} \sum_{c=1}^{M} \delta(q - q_c) = \frac{1}{\pi} \text{Im} G(q - i0), \quad (2)$$

where $\langle \ldots \rangle$ denotes the statistical average, and

$$G(z) = M^{-1} (\text{tr} (z - Q)^{-1}) \quad (3)$$

is the trace of the Green’s function of the Wigner-Smith matrix. The positive definite form \cite{18} of $Q$ provides the analyticity of $G(z)$ throughout the complex $z$-plane save the discontinuity line along the positive part of the real axis $q = \text{Re} z > 0$. Function \cite{18} is normalized to unity.

The function $G(z)$ can be readily obtained from the generating function $(z_{\pm} = z \pm \omega / 2)$

$$Z(z, \omega) = \left\langle \frac{\det(z_+ - Q)}{\det(z_- - Q)} \right\rangle = \left( \frac{z_+}{z_-} \right)^M \left\langle \frac{\det(A_+)}{\det(A_-)} \right\rangle \quad (4)$$

as $G(z) = M^{-1} \partial Z(z, \omega) / \partial \omega |_{\omega = 0}$. We have exploited here the explicit form \cite{18} to cast $Z(z, \omega)$ in terms of determinants of $N \times N$ matrices $A_{\pm}$ (rather than those of $M \times M$ ones $z_{\pm} - Q$). Now, we use the identity

$$A_{\pm} \equiv (E - \mathcal{H})(E - \mathcal{H})^\dagger - V V^\dagger / z_{\pm} = (E - \mathcal{H} - i / z_{\pm})(E - \mathcal{H}^\dagger + i / z_{\pm}) - 1 / z_{\pm}^2 \quad (5)$$

and double the dimension, introducing Pauli matrices $\sigma_i$ in the doubling space. As a result, the determinants get the following simple form:
Applying (2), we get finally the following expression for the whole complex \( \zeta \) imaginary axis in the half-plane \( \Re \zeta < -1 \):

\[
Z(z, \omega) = \left( \frac{z}{-z} \right)^M \int [d\bar{\sigma}] \exp \left[ \frac{\pi}{2\Delta} \text{str}(U(\Lambda - i\Lambda_1)\bar{\sigma}) \right] \\
\times \prod_{c=1}^{M} \text{sdet}[1 + \gamma_c \Lambda(iE/2 + \pi\nu(E))\bar{\sigma}]^{-1/2}
\]  

(7)

as the integral over a noncompact saddle-point manifold \( \bar{\sigma} \). The 8x8 supermatrices \( \Lambda, \Lambda_1 \) appearing above are the supermatrix analog of the Pauli matrices \( \sigma_z, \sigma_1 \), and \( U = \text{diag}(z^{-1/2}, z^{-1/2}, z^{1/2}, z^{1/2}, z^{-1/2}, z^{-1/2}, z^{1/2}, z^{1/2}) \). Definitions of the superalgebra as well as the explicit parameterization of \( \bar{\sigma} \) and the invariant measure can be found in [19, 24]. The average density of states \( \nu(E) = \pi^{-1} \sqrt{1 - (E/2)^2} \) determines the mean level spacing \( \Delta(\nu N)^{-1} \) of the closed system. Phenomenological constants \( \gamma_c > 0 \) originate from the coupling amplitudes: \( V1 V_{cc} = 2 \gamma_c \delta_{cc} \); they enter final expressions only by means of the transmission coefficients \( T_c = 1 - |S_{cc}|^2 = 2(1 + (\gamma_c + \gamma_c^{-1})/2\pi\nu(E))^{-1} \leq 1 \). The latter are the conventional characteristics of coupling of an unstable system to the energy continuum.

Suspending all technical details of quite standard calculations [for the sake of simplicity, we restricted them to the unitary symmetry class which corresponds to systems with broken time-reversal symmetry (TRS)] to a more extended publication, we proceed with the following result for the Green function:

\[
G(\zeta) = \frac{1}{\zeta} + \frac{1}{M_2} \frac{1}{2M_2} \int_1^{\infty} d\lambda_1 \int_1^{\infty} d\lambda_2 \int_{-1}^{1-\lambda_1-\lambda_2} \prod_{c=1}^{M} \frac{g_c+\lambda_1}{g_c+\lambda_2} \\
\times \left( f(\lambda_2) \frac{\partial b(\lambda_1)}{\partial \zeta} - b(\lambda_1) \frac{\partial f(\lambda_2)}{\partial \zeta} \right)
\]  

(8)

Henceforth, we scale the original variable \( z = \zeta t_H \) [and correspondingly the Green’s function (3)] in the natural units of the Heisenberg time \( t_H = 2\pi \hbar / \Delta \). The constants \( g_c = 2/T_c - 1 \) are related to the transmission coefficients \( T_c \). At last, we denote \( b(\lambda_1) = e^{\lambda_1 / \zeta} I_0(\lambda_1 - 1) \) and \( f(\lambda_2) = e^{-\lambda_2 / \zeta} J_0(\lambda_2 - 1) \), where \( I_0(x) \) (\( J_0(x) \)) being the modified (usual) Bessel function of zero’th order. As it stands, the Green’s function (3) is an analytic function of the complex variable \( \zeta \) in the half-plane \( \Re \zeta < 0 \). One can deform the original \( \lambda_1 \)-integration to that along the imaginary axis \( [1, +i\infty) \) or \( [1, -i\infty) \) for \( \text{Im}(\zeta^{-1}) > 0 \) [or \( \text{Im}(\zeta^{-1}) < 0 \)], so that \( G(\zeta) \) is analytically continued to the whole complex \( \zeta \)-plane with a cut along positive \( \Re \zeta \).

Applying (3), we get finally the following expression for the probability distribution of the proper delay times

\[
\mathcal{P}_M(t = q/t_H) = \frac{1}{M} \sum_{M=1}^{M} \left( F_c^B \frac{\partial F_B}{\partial t} - F_c^F \frac{\partial F_F}{\partial t} \right)
\]  

(9)

for \( t > 0 \), and \( \mathcal{P}_M(t = 0) \equiv 0 \) for negative times. Here

\[
F_c^B = e^{-g_c/t} I_0(t^{-1} \sqrt{g_c^2 - 1}) \prod_{a(\neq c)} \frac{1}{g_a - g_c},
\]  

(10)

\[
F_c^F = \frac{1}{2} \int_{-1}^{+1} d\lambda e^{-\lambda/t} J_0(t^{-1} \sqrt{1 - \lambda^2}) \prod_{a(\neq c)} (g_a + \lambda).\n\]  

(11)

The function (11) is, in fact, a polynomial in \( 1/t \). The formulae (9)–(11) are valid for an arbitrary number of scattering channels \( M \) and arbitrary constants \( g_c \geq 1 \).

In the situation when all chaotic states of the target system are alike, the channels get statistically equivalent. Setting all \( g_c = g \), we arrive in such a case at

\[
\mathcal{P}^\text{eq}_M(t) = \frac{1}{M} \sum_{M=0}^{M-1} \left( F_M \frac{\partial B_M}{\partial t} - B_M \frac{\partial F_M}{\partial t} \right)
\]  

(12)

with

\[
B_M = \frac{1}{M!} \left( e^{-\frac{1}{\bar{\sigma}}} \right)^M e^{-g/t} I_0(t^{-1} \sqrt{g^2 - 1}),
\]  

(13)

\[
F_M = \frac{1}{2} \int_{-1}^{+1} d\lambda e^{-\lambda/t} J_0(t^{-1} \sqrt{1 - \lambda^2}) (g + \lambda)^M \\
= \sum_{m=0}^{M} \frac{1}{(2m+1)!} \left( \frac{\partial^2}{\partial g^2} - \frac{2}{t} \frac{\partial}{\partial g} \right)^m g^M.
\]  

(14)

It is worth noting that, due to similar analytical structure of the functions \( F_c^B \) and \( F_c^F \), distribution (3) decays as \( t^{-M-2} \) in the limit \( t \to \infty \). This is in agreement with the known universal law \( t^{-\beta M/2-2} \) [10, 13] valid for all Dyson’s symmetry classes \( \beta = 1, 2, 4 \) (\( \beta = 2 \) in the present case of broken TRS).

In the single-channel case, \( M = 1 \), Eqs. (3) and (12) confirm the result \( \mathcal{P}_1(t) = t^{-1} \partial B_0 / \partial t \) found earlier in [10]. When the constant \( g \approx 2/T \gg 1 \) this distribution shows a narrow maximum at the small time \( t \sim (2g)^{-1} \ll 1 \) because of the cooperative influence of the tails of many remote resonances. The far asymptotics \( \mathcal{P}_1(t) \propto t^{-3}, t \gg g \), is due to fluctuations of the widths of many narrow resonances [10, 17]. At last, in the parametrically wide domain \( (2g)^{-1} \ll t \ll g \) the distribution decreases as \( t^{-3/2} \) [1, 11]. Just this domain saturates in the case \( g \gg 1 \) the well known [3] sum rule \( \langle t \rangle = \langle \text{tr} Q \rangle / t_H = 1 \), which is, actually, satisfied regardless of the coupling strength to the continuum [see Eq. (3)] below.

At the same time, the average time spent by a spatially small wave packet in the interaction region (collision time) is equal to \( \langle \tau \rangle = (Q / T) = t_H/t \) [10, 11, 17]. This relation suggests splitting the transmission coefficient off the delay time of an almost monochromatic wave: \( q = T \tau \). The new variable \( \tau \) is then the time the projectile is delayed inside the interaction region after having
penetrated there with the probability $T$. In agreement with the above consideration, the most probable duration $\tau$ of the stay inside is given by the Heisenberg time $t_H$.

In the case of an arbitrary number of channels $M$ an additional scaling $t_s = M t$ provides a similar condition

$$
\langle t_s = q M / t_H = \Gamma \rangle = \int_0^\infty dt_s t_s \mathcal{P}_s(t_s) = 1 \, .
$$

Here $\mathcal{P}_s(t_s) \equiv \mathcal{P}_s(t = t_s / M)$ and the mean transmission coefficient $\mathcal{T} = M^{-1} \sum_s T_s$ has been set off, $q = T \tau$. The scaled variable $t_s$ measures the time $\tau$ in units of a typical life time. Indeed, the characteristic width of a state which decays through $M$ open channels is estimated as $\Gamma = \sum_s T_s \Delta / 2\pi$.

![FIG. 1. The distribution $\mathcal{P}_s(t_s)$ of the scaled proper delay times at $M = 2$ and $\mathcal{T} = 0.25$ for the case of equivalent, $\delta T = 0$, and strongly nonequivalent channels, $\delta T = 0.49$, (dashed and solid lines, respectively).](image)

Figure 1 demonstrates, using the example of the two-channel distribution $\mathcal{P}_2(t)$, the dependence on the strengths of individual channels. To fetch out the influence of channel nonequivalence, we keep the openness characterized by $T_1 + T_2 = 2 \mathcal{T}$ fixed and change the difference $|T_1 - T_2| = \delta T$. Two distinct time scales with similar probabilities are well seen when $\delta T$ is close enough to its maximal value $\min(2\mathcal{T}, 2-2\mathcal{T})$. Fast processes appear when one of the transmission coefficients becomes small.

We consider now in detail the case of large number of statistically equivalent channels, $M \gg 1$. The regime of isolated resonances, $\Gamma \ll \Delta$, corresponds to the condition $g \gg M \gg 1$. In this limit the widths of resonances become statistically independent and expression (12) reduces to the probability distribution of the partial delay times investigated in much detail in [13], see also [14,16].

Here we focus our attention rather on the semiclassical limit of strongly overlapping resonances excited through and decaying into a number $M$ of channels, which is scaled with $N$, the strength constant $g$ being kept arbitrary. Similar to Refs. [23,4], an additional saddle-point approximation can be used in this case to perform the integration in (12). Under the physically justified condition $M/N \ll 1$ (though both $M, N \to \infty$) one finds that $P(t_s) = \lim_{M \to \infty} \mathcal{P}_s(t_s) = (\pi t_s^2)^{-1} \text{Im} K(t_s)$, where the function $K(t_s) \equiv t_s^2 (G(t_s) - t_s^{-1})$ satisfies a cubic equation of the form

$$
K(K^2 + 2g K + 1) - t_s (K^2 - 1) = 0 \, .
$$

Being derived for the case of the unitary ensemble this result remains actually valid for all tree symmetry classes. The calculation just described sets a new intrinsic time scale of the problem [24]: the empty gap $\Gamma_g$ between the real axis in the complex energy plane and the cloud of the resonances in its lower half plane. In the limit considered $\Gamma_g$ coincides exactly with the well-known Weisskopf width $\Gamma_W = M \mathcal{T} \Delta / 2\pi$.

The case of ideal coupling, $g=1$, is especially simple since the cubic equation (14) readily reduces to a quadratic one which immediately leads to the result

$$
P_{id}(t_s) = (2\pi t_s^2)^{-1} \sqrt{(t_s^2 - t_s)(t_s - t_s^2)} \, ,
$$

with $t_s^2 = 3 \pm \sqrt{8}$, obtained earlier in [12] from the Laguerre ensemble. For an arbitrary value of $g$ the searched distribution is expressed as

$$
P(t_s) = \frac{\sqrt{3}}{2\pi t_s^2} \left( (|r| + \sqrt{D})^{1/3} - (|r| - \sqrt{D})^{1/3} \right) \, .
$$

in terms of two polynomials: $D = -3\eta^4 + 2g\eta^3 + 11\eta^2/3 - 4g\eta + g^2 + 3^{-3}$ and $r = \eta^3 - 2\eta + g$, where $\eta \equiv (2g - t_s)/3$. The density does not vanish in the domain where the discriminant $D > 0$. It is obvious that the forth order polynomial $D$ always has two real roots $\eta_+ < 0$ and $\eta_- > 0$ and $D$ is positive only inside the region $\eta_- < \eta < \eta_+$. Therefore, similar to the case of perfect coupling, the time $t_s$ is always restricted to a finite domain $t_- < t_s < t_+$. Near the edges $t_s = 2g \pm 3\eta$ the discriminant $D$ is small and expression (18), as in (17), simplifies to

$$
P(t_s) \approx (\pi t_s^2)^{-1} |r|^{-2/3} \sqrt{D/3} \propto \sqrt{|t_s - t_s^2|} \, .
$$

A more detailed investigation is possible in the case of weak individual channels, $g \gg 1$. All the roots of $D$ in this case can be found as expansion due to the smallness of $1/g$. Near the lower edge $t_- \approx 1/8g$ Eq. (19) gives

$$
P(t_s) \approx \frac{g}{\pi (2g t_s)^{3/2}} 4 - (2g t_s)^{-1} \, .
$$

In this case the term $|r|$ dominates in Eq. (18) and therefore Eq. (20) is valid in the parametrically large domain $1/8g \lesssim t_s < g - 3/(4g)^{1/3}$ where the influence of other (three) roots remains negligibly weak [26]. When $t_s \gg 1/8g$ this distribution displays the universal behavior $t_s^{-3/2}$. Such a law is a typical and most robust feature of the time-delay distributions in a weakly open chaotic
system and may be expected from quite a general argumentation [10, 11]. Within a relatively narrow domain $-\frac{4}{3}(g/2)^{1/3} \leq t_s - 2g \leq 3(g/2)^{1/3}$ the distribution develops a local minimum and maximum, which are due to a pair of the complex roots $t^\pm \approx 2g - \frac{3}{2}(g/2)^{1/3}(1 \pm i\sqrt{3})$. At last, very close to the upper edge $t_s \approx 2g + 3(g/2)^{1/3}$ the distribution sharply changes to the form

$$P(t_s) \approx \frac{2}{\sqrt{3\pi t_s^2}} \frac{(2g/2)^{1/6}}{\sqrt{t_s - t}} \quad (21)$$

provided by Eq. (19). Figure 2 illustrates these results.

![Figure 2](image-url)  
**FIG. 2.** The distribution $P(t_s)$ in the limit $M \to \infty$. The exact [10] and approximate expressions [20] (dotted and [21]) (dashed line, see inset) are shown at $g=10$. An intermediate $t_s^{-3/2}$ behavior is clearly seen.

With the help of the distribution found the Green’s function can be represented in the spectral form $G(z) = \int_{t_+}^{t_+} dt_s P(t_s)/(z - t_s)$. This function is analytical both at the origin and infinity and therefore can be expanded near these points in power series. Such an expansion yields the general formulae for the negative and positive moments of the distribution $(n \geq 0)$

$$(t_s^{-(n+1)}) = -\frac{1}{n!} \left. \frac{d^n G(z)}{dz^n} \right|_{z=0}, \quad (t_s^n) = \frac{1}{n!} \left. \frac{d^n G(z^{-1})}{d\zeta^n} \zeta \right|_{\zeta=0}.$$  

All the moments are finite. The derivatives can directly be calculated by making use of the cubic equation [17].

In conclusion, we have derived the distribution of proper delay times in chaotic scattering at arbitrary number of scattering channels and arbitrary coupling to continua. The appearance of distinct time scales is traced to the properties of the delay time distribution with increasing degree of channel nonequivalence. The physically interesting case of many equivalent channels weakly coupled to continua is studied in much detail.

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[26] We note that (20) can easily be derived from (10) if one neglects there the term $K^3$ as long as $t_s \ll g$. 

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