Dynamical barriers of pure and random ferromagnetic Ising models on fractal lattices

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Abstract. We consider the stochastic dynamics of the pure and random ferromagnetic Ising model on a hierarchical diamond lattice of branching ratio $K$ with fractal dimension $d_f = (\ln(2K))/\ln 2$. We adapt the real-space renormalization procedure introduced in our previous work (Monthus and Garel, 2013 J. Stat. Mech. P02037) to study the equilibrium time $t_{eq}(L)$ as a function of the system size $L$ near zero temperature. For the pure Ising model, we obtain the behavior $t_{eq}(L) \sim L^{d_s} e^{\beta J L^{d_s}}$, where $d_s = d_l - 1$ is the interface dimension, and we compute the prefactor exponent $\alpha$. For the random ferromagnetic Ising model, we derive the renormalization rules for dynamical barriers $B_{eq}(L) \equiv (\ln t_{eq}/\beta)$ near zero temperature. For the fractal dimension $d_l = 2$, we obtain that the dynamical barrier scales as $B_{eq}(L) = cL + L^{1/2}u$, where $u$ is a Gaussian random variable of non-zero mean. While the non-random term scaling as $L$ corresponds to the energy cost of the creation of a system-size domain-wall, the fluctuation part scaling as $L^{1/2}$ characterizes the barriers for the motion of the system-size domain-wall after its creation. This scaling corresponds to the dynamical exponent $\psi = 1/2$, in agreement with the conjecture $\psi = d_s/2$ proposed by Monthus and Garel (2008 J. Phys. A: Math. Theor. 41 115002). In particular, it is clearly different from the droplet exponent $\theta \simeq 0.299$ involved in the statics of the random ferromagnet on the same lattice.

Keywords: disordered systems (theory), dynamical processes (theory), slow relaxation and glassy dynamics, kinetic Ising models

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1. Introduction

Among real-space renormalization procedures [1], Migdal–Kadanoff block renormalizations [2] play a special role because they can be considered in two ways, either as approximate renormalization procedures on hypercubic lattices or as exact renormalization procedures on certain hierarchical lattices [3]–[5]. Besides the study of pure models, these hierarchical lattices have also been widely used to study the equilibrium of disordered classical spin models, such as the diluted Ising model [6], the random bond Potts model [7]–[10], the random field Ising model [11, 12] and spin-glasses [13]. For spin-glasses, these hierarchical lattices have also been used to study dynamical properties [14]–[17]. The equilibrium properties of disordered polymer models have been considered too, in particular the wetting on a disordered substrate [18]–[20] and the directed polymer model [20]–[29].

In the present paper, we consider the pure and the random ferromagnetic Ising model on the hierarchical diamond lattice in order to study its dynamical properties near zero temperature. We adapt the real-space renormalization procedure introduced in [30]: using the standard mapping between the detailed balance dynamics of classical Ising models and some quantum Hamiltonian, we obtain the appropriate real-space renormalization rules for the quantum Hamiltonians associated with single-spin-flip dynamics. This approach has been used previously to characterize the dynamics of the random ferromagnetic chain [30], of the pure Ising model on the Cayley tree [30], of the random Ising model on the Cayley tree [32] and for the hierarchical Dyson Ising model [31].

The paper is organized as follows. In section 2, we introduce the notations for the stochastic dynamics of Ising models defined on the hierarchical diamond model. In section 3, we analyze via real-space renormalization the dynamics of the pure Ising model. In section 4, we study via real-space renormalization the dynamics of the random ferromagnetic Ising model. Our conclusions are summarized in section 5. In appendix A, we derive some renormalization formulae that are used in the text.

2. Model and notations

2.1. Hierarchical diamond lattice of branching ratio $K$

As shown in figure 1, a hierarchical diamond lattice of branching ratio $K$ is constructed recursively from a single link called generation $n = 0$. Generation $n = 1$ consists of $K$ branches, each branch containing two bonds in series and generation $n = 2$ is obtained by applying the same transformation to each bond of generation $n = 1$. At generation $n$, the length $L_n$ between the two extreme sites is

$$L_n = 2^n$$

whereas the total number of bonds $B_n$ is

$$B_n = (2K)^n = L_n^{df}$$

so that

$$df = \frac{\ln(2K)}{\ln 2}$$

represents the fractal dimension.
As recalled in section 1, the equilibrium of many statistical physics models have been studied on this lattice. Here we consider the pure and the random ferromagnetic Ising model with the classical energy

$$ U(C) = -\sum_{i<j} J_{ij} S_i S_j $$

(4)

to study the properties of stochastic dynamics satisfying detailed balance.

2.2. Dynamics satisfying detailed balance

The stochastic dynamics is defined by the master equation

$$ \frac{dP_t(C)}{dt} = \sum_{C'} P_t(C') W(C' \to C) - P_t(C) W_{\text{out}}(C) $$

(5)

that describes the time evolution of the probability $P_t(C)$ to be in configuration $C$ at time $t$. The notation $W(C' \to C)$ represents the transition rate per unit time from configuration $C'$ to $C$, and

$$ W_{\text{out}}(C) \equiv \sum_{C'} W(C \to C') $$

(6)

represents the total exit rate out of configuration $C$.

The convergence towards Boltzmann equilibrium at temperature $T = 1/\beta$ in any finite system

$$ P_{\text{eq}}(C) = \frac{e^{-\beta U(C)}}{Z} $$

(7)

where $Z$ is the partition function

$$ Z = \sum_{C} e^{-\beta U(C)} $$

(8)

can be ensured by imposing the detailed balance property

$$ e^{-\beta U(C')} W(C \to C') = e^{-\beta U(C)} W(C' \to C) . $$

(9)
It is thus convenient to parametrize the transition rates as

\[ W(C \rightarrow C') = G(C, C') e^{-\beta/2[U(C') - U(C)]} \]  

(10)

where

\[ G(C, C') = G(C', C) = \sqrt{W(C \rightarrow C') W(C' \rightarrow C)} \]  

(11)

is a symmetric positive function of the two configurations. Near zero temperature, it is convenient to introduce the notion of a dynamical barrier \( B \) defined by the asymptotic behavior

\[ G(C, C') \propto e^{-\beta B(C, C')} \]  

(12)

2.3. Associated quantum Hamiltonian

The standard similarity transformation (see for instance the textbooks [33]–[35] or the works concerning spin models [30], [36]–[42])

\[ P_t(C) \equiv e^{-\beta/2 U(C)} \psi_t(C) = e^{-\beta/2 U(C)} \langle C|\psi_t \rangle \]  

(13)

transforms the master equation of equation (5) into the imaginary-time Schrödinger equation for the ket \( |\psi_t \rangle \)

\[ \frac{d}{dt}|\psi_t \rangle = -H|\psi_t \rangle \]  

(14)

with the quantum Hamiltonian

\[ H = \sum_{C, C'} G(C, C') \left[ e^{-\beta/2[U(C') - U(C)]}|C\rangle\langle C'| - |C'|\langle C|\right] . \]  

(15)

The groundstate energy is \( E_0 = 0 \), and the corresponding eigenvector corresponding to the Boltzmann equilibrium reads

\[ |\psi_0 \rangle = \sum_C \frac{e^{-\beta/2 U(C)}}{\sqrt{Z}} |C\rangle . \]  

(16)

The other energies \( E_n > 0 \) determine the relaxation towards equilibrium. In particular, the lowest non-vanishing energy \( E_1 \) determines the largest relaxation time (1/\( E_1 \)) of the system, i.e. the ‘equilibrium time’ needed to converge towards equilibrium,

\[ t_{eq} \equiv \frac{1}{E_1} \]  

(17)

2.4. Single-spin-flip dynamics of Ising models

We will focus here on single-spin-flip dynamics satisfying the detailed balance of equation (9), where the transition rate corresponding to the flip of a single spin \( S_k \) reads

\[ W(S_k \rightarrow -S_k) = G^{\text{ini}}_{\text{h}_k = \sum_{i \neq k} J_{ik} S_i} e^{-\beta S_k |\sum_{i \neq k} J_{ik} S_i|} . \]  

(18)
Here \( G^{\text{ini}}[h_k] \) is an arbitrary positive even function of the local field \( h_k = \sum_{i \neq k} J_{ik} S_i \). For instance, the Glauber dynamics corresponds to the choice

\[
G^{\text{Glauber}}_{\text{Glauber}}[h] = \frac{1}{2 \cosh (\beta h)}.
\] (19)

The quantum Hamiltonian of equation (15) reads in terms of Pauli matrices \( (\sigma^x, \sigma^z) \) [30], [36]–[42]

\[
\mathcal{H} = \sum_k G^{\text{ini}} \left[ \sum_{i \neq k} J_{ik} \sigma^z_i \right] \left( e^{-\beta \sigma^z_k (\sum_{i \neq k} J_{ik} \sigma^z_i) - \sigma^z_k} \right).
\] (20)

For ferromagnetic models near zero temperature, more precisely when the temperature is much smaller than any ferromagnetic coupling \( J_{ij} \)

\[
0 < T \ll J_{ij}
\] (21)

the thermal equilibrium is dominated by the two ferromagnetic groundstates where all spins take the same value, and the largest relaxation time \( t_{eq} \simeq 1/E_1 \) corresponds to the time needed to go from one groundstate (where all spins take the value +1) to the opposite groundstate (where all spins take the value −1). The aim of the renormalization procedure introduced in [30] is to preserve the lowest non-vanishing energy \( E_1 \) of the quantum Hamiltonian. We have already explained the application of this renormalization to the random ferromagnetic chain [30], to the random ferromagnetic Cayley tree [32], and to the Dyson hierarchical Ising model [31]. In the following, we derive the appropriate renormalization rules for the diamond hierarchical lattice described above.

3. Dynamics of the pure Ising model

3.1. Principle of renormalization for the dynamics

To analyze the statics of the ferromagnetic Ising model on the diamond lattice, one has to renormalize bonds to obtain the renormalization group (RG) rules for renormalized couplings. However, here, to analyze the single-spin-flip dynamics, we wish to define renormalized spins that represent ferromagnetic clusters of spins flipping together.

We start from the diamond lattice with \( n \geq 2 \) generations, of length \( L_n = 2^n \), containing \( B_n = (2K)^n \) bonds. All bonds have the same initial ferromagnetic coupling \( J \). All elementary spins have a dynamics characterized by the function \( G^{\text{ini}}[h] \) of the local field \( h \) (equation (20)).

After one RG step, we wish to have a diamond graph of generation \( (n - 1) \), where there is an alternation of initial spins, with a dynamics still characterized by the function \( G^{\text{ini}}[h] \), and of renormalized spins \( S_{R1} \) (representing clusters of \( (2K + 1) \) initial spins), whose dynamics is represented by some renormalized amplitude \( G_{R1} \). The renormalized bonds have for renormalized couplings \( J_{R1} = KJ \).

More generally, after \( p \) RG steps with \( 1 \leq p \leq n - 1 \), we wish to have a diamond graph of generation \( (n - p) \), where there is an alternation of initial spins, with a dynamics still characterized by the function \( G^{\text{ini}}[h] \), and of renormalized spins \( S_{Rp} \) (representing clusters of \( (2K + 1)^p \) initial spins), whose dynamics is represented by some renormalized amplitude \( G_{Rp} \). The renormalized bonds have for renormalized couplings

\[
J_{Rp} = K^p J.
\] (22)

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In particular, after \( p = n - 1 \) RG steps, we wish to have a diamond graph of generation 1, where the two boundary spins are initial spins, with a dynamics still characterized by the function \( G^{\text{ini}}[h] \), and where the \( K \) internal renormalized spins \( S_{R_{n-1}} \) (representing clusters of \( (2^K+1)^{n-1} \) initial spins), whose dynamics is represented by some renormalized amplitude \( G_{R_{n-1}} \). The renormalized bonds have for renormalized couplings \( J_{R_{n-1}} = K^{n-1} J \).

The last RG \( p = n \) RG step is thus special: the \( (K + 2) \) remaining spins are grouped together into a single renormalized spin \( S_{R_{n}} \) (representing the whole sample of \( (K(2^K+1)^{n-1} + 2) \) initial spins) whose dynamics is represented by some renormalized amplitude \( G_{\text{last}}^{R_{n}} \).

Let us now explain how the RG rule for the renormalized amplitude \( G_R \) describing the dynamics can be obtained for the special last RG step \( p = n \) and for the bulk RG steps \( 1 \leq p \leq n - 1 \), respectively.

### 3.2. RG rule for the last RG step \( p = n \)

The last RG step \( p = n \) shown on figure 2 involves two boundary spins \( S_{e_1} \) and \( S_{e_2} \) and \( K \) internal spins \( (S_{a_1}, S_{a_2}, \ldots, S_{a_K}) \) with the following classical energy (equation (4))

\[
U(C) = -J_{R_{n-1}} \sum_{i=1}^{K} S_{a_i} (S_{e_1} + S_{e_2}).
\]  

(23)

The quantum Hamiltonian of equation (20) associated with the single-spin-flip dynamics reads

\[
H_{K+2} = G^{\text{ini}} \left[ J_{R_{n-1}} \sum_{i=1}^{K} \sigma_{a_i}^z \right] \left( e^{-\beta J_{R_{n-1}} \sigma_{e_1}^z \sum_{i=1}^{K} \sigma_{a_i}^x} - \sigma_{e_1}^x \right) \\
+ G^{\text{ini}} \left[ J_{R_{n-1}} \sum_{i=1}^{K} \sigma_{a_i}^z \right] \left( e^{-\beta J_{R_{n-1}} \sigma_{e_2}^z \sum_{i=1}^{K} \sigma_{a_i}^x} - \sigma_{e_2}^x \right) \\
+ \sum_{i=1}^{K} G_{R_{n-1}} \left[ J_{R_{n-1}} \sigma_{e_1}^z \sigma_{e_2}^z \right] \left( e^{-\beta J_{R_{n-1}} \sigma_{a_i}^z (\sigma_{e_1}^z + \sigma_{e_2}^z)} - \sigma_{a_i}^x \right). 
\]  

(24)

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We are interested in the lowest non-vanishing eigenvalue $E_1 > 0$. The eigenvalue equation

$$0 = (H_{K+2} - E_1)|\psi_1\rangle$$

for the corresponding eigenvector in the basis

$$|\psi_1\rangle = \sum_{s_{e_1}=\pm} \sum_{s_{e_2}=\pm} \sum_{s_{a_1}=\pm} \cdots \sum_{s_{a_K}=\pm} \psi_1(s_{e_1}, s_{e_2}, s_{a_1} \cdots s_{a_K}) |s_{e_1}, s_{e_2}, s_{a_1} \cdots s_{a_K}\rangle$$

reads

$$0 = \left[G^{\text{ini}}_R \sum_{i=1}^{K} S_a \right] e^{-\beta J_{Rn-1} S_{e_1} \sum_{i=1}^{K} S_a}
+ \left[G^{\text{ini}}_R \sum_{i=1}^{K} S_a \right] e^{-\beta J_{Rn-1} S_{e_2} \sum_{i=1}^{K} S_a}
+ \sum_{i=1}^{K} G_{Rn-1} \left[J_{Rn-1} (s_{e_1} + s_{e_2}) \right] e^{-\beta J_{Rn-1} s_{a_1} (s_{e_1} + s_{e_2}) - E_1}
\times \psi_1(s_{e_1}, s_{e_2}, s_{a_1} \cdots s_{a_K})
- \left[G^{\text{ini}}_R \sum_{i=1}^{K} S_a \right] \psi_1(-s_{e_1}, s_{e_2}, s_{a_1} \cdots s_{a_K})
- \left[G^{\text{ini}}_R \sum_{i=1}^{K} S_a \right] \psi_1(s_{e_1}, -s_{e_2}, s_{a_1} \cdots s_{a_K})
- \sum_{i=1}^{K} G_{Rn-1} \left[J_{Rn-1} (s_{e_1} + s_{e_2}) \right] \psi_1(s_{e_1}, s_{e_2}, s_{a_1}, \ldots, -s_{a_i}, \ldots, s_{a_K}). \tag{27}$$

Let us now use the symmetry between the $K$ spins $S_a$ to note $\phi(s_{e_1}, k, s_{e_2})$ the components $\psi_1(s_{e_1}, s_{e_2}, s_{a_1} \cdots s_{a_K})$ where $k \in 0, 1, \ldots, K$ spins among $(s_{a_1}, \ldots, s_{a_K})$ take the value $(-)$

$$\psi_1(s_{e_1}, s_{e_2}, s_{a_1} \cdots s_{a_K}) = \phi_1\left(s_{e_1}, k = \sum_{i=1}^{K} \frac{1 - s_{a_i}}{2}, s_{e_2}\right). \tag{28}$$

Equation (27) becomes

$$0 = \left[G^{\text{ini}}_R \left[J_{Rn-1} (K - 2k) \right] \right] e^{-\beta J_{Rn-1} s_{e_1} (K - 2k)} + \left[G^{\text{ini}}_R \left[J_{Rn-1} (K - 2k) \right] \right] e^{-\beta J_{Rn-1} s_{e_2} (K - 2k)}
+ kG_{Rn-1} \left[J_{Rn-1} (s_{e_1} + s_{e_2}) \right] e^{\beta J_{Rn-1} (s_{e_1} + s_{e_2})}
+ (K - k)G_{Rn-1} \left[J_{Rn-1} (s_{e_1} + s_{e_2}) \right] e^{-\beta J_{Rn-1} (s_{e_1} + s_{e_2}) - E_1}\phi_1(s_{e_1}, k, s_{e_2})
- \left[G^{\text{ini}}_R \left[J_{Rn-1} (K - 2k) \right] \right] \phi_1(-s_{e_1}, k, s_{e_2})
- G^{\text{ini}}_R \left[J_{Rn-1} (K - 2k) \right] \phi_1(s_{e_1}, k - 1, s_{e_2})
- (K - k)G_{Rn-1} \left[J_{Rn-1} (s_{e_1} + s_{e_2}) \right] \phi_1(s_{e_1}, k + 1, s_{e_2}). \tag{29}$$

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For $E_0 = 0$, the (non-normalized) groundstate is known to be given by equation (16) with the classical energy of equation (23)

$$\phi_0(S_{e_1}, k, S_{e_2}) = e^{(\beta/2)J_{R_n-1}(K-2k)(S_{e_1} + S_{e_2})}$$

so it is convenient to look for the solution of equation (27) via the amplitude

$$A(S_{e_1}, k, S_{e_2}) = \frac{\phi_1(S_{e_1}, k, S_{e_2})}{\phi_0(S_{e_1}, k, S_{e_2})}$$

that satisfies

$$0 = [G^{\text{ini}} [ J_{R_n-1}(K-2k) ] e^{-\beta J_{R_n-1}(K-2k)S_{e_1}} + G^{\text{ini}} [ J_{R_n-1}(K-2k) ] e^{-\beta J_{R_n-1}(K-2k)S_{e_2}}$$

$$+ kG_{R_n-1} [ J_{R_n-1}(S_{e_1} + S_{e_2}) ] e^{\beta J_{R_n-1}(S_{e_1} + S_{e_2})}$$

$$+ (K-k)G_{R_n-1} [ J_{R_n-1}(S_{e_1} + S_{e_2}) ] e^{-\beta J_{R_n-1}(S_{e_1} + S_{e_2}) - E_1} A(S_{e_1}, k, S_{e_2})$$

$$- G^{\text{ini}} [ J_{R_n-1}(K-2k) ] e^{-\beta J_{R_n-1}(K-2k)S_{e_1}} A(-S_{e_1}, k, S_{e_2})$$

$$- G^{\text{ini}} [ J_{R_n-1}(K-2k) ] e^{-\beta J_{R_n-1}(K-2k)S_{e_2}} A(S_{e_1}, k, -S_{e_2})$$

$$- kG_{R_n-1} [ J_{R_n-1}(S_{e_1} + S_{e_2}) ] e^{\beta J_{R_n-1}(S_{e_1} + S_{e_2})} A(S_{e_1}, k-1, S_{e_2})$$

$$- (K-k)G_{R_n-1} [ J_{R_n-1}(S_{e_1} + S_{e_2}) ] e^{-\beta J_{R_n-1}(S_{e_1} + S_{e_2})} A(S_{e_1}, k+1, S_{e_2})] \text{ (32)}$$

Let us now consider the dynamical path associated with the swap of a domain-wall between the two ferromagnetic groundstates with the following notations for configurations:

$$C_0 = \{ S_{e_1} = +1, 0, S_{e_2} = +1 \}$$

$$C_1 = \{ S_{e_1} = +1, 0, S_{e_2} = +1 \}$$

$$C_2 = \{ S_{e_1} = +1, -1, S_{e_2} = +1 \}$$

$$C_p = \{ S_{e_1} = -1, p - 1, S_{e_2} = +1 \}$$

$$C_K = \{ S_{e_1} = -1, K - 1, S_{e_2} = +1 \}$$

$$C_{K+1} = \{ S_{e_1} = -1, K, S_{e_2} = +1 \}$$

$$C_{K+2} = \{ S_{e_1} = -1, K, S_{e_2} = -1 \}.$$  

(33)

The physical meaning is that the transition from $C_0$ to $C_1$ corresponds to the entrance of a domain-wall at the boundary $e_1$, the transitions between the configurations $(C_1, C_2, \ldots, C_{K+1})$ correspond to the displacement of this domain-wall, and finally the transition from $C_{K+1}$ to $C_{K+2}$ corresponds to the exit of the domain-wall at the boundary $e_2$.

The corresponding amplitudes $A(C_q)$ for this dynamical path satisfy (equation (32)) for $2 \leq q \leq K$

$$0 = (((q - 1)G_{R_n-1}[0] + (K + 1 - q)G_{R_n-1}[0] - E_1) A(C_q)$$

$$- (q - 1)G_{R_n-1}[0] A(C_{q-1}) - (K + 1 - q)G_{R_n-1}[0] A(C_{q+1})$$

(34)

and for $q = 0, 1, K, K+1$ and $K+2$

$$0 = [G^{\text{ini}} [ K J_{R_n-1} ] e^{-\beta K J_{R_n-1} - E_1} A(C_0) - G^{\text{ini}} [ K J_{R_n-1} ] e^{-\beta K J_{R_n-1}} A(C_1)$$

$$0 = [G^{\text{ini}} [ K J_{R_n-1} ] e^{\beta K J_{R_n-1}} + K G_{R_n-1}[0] - E_1] A(C_1)$$

$$- G^{\text{ini}} [ K J_{R_n-1} ] e^{\beta K J_{R_n-1}} A(C_0) - K G_{R_n-1}[0] A(C_2)$$

$$\text{doi:10.1088/1742-5468/2013/06/P06007}$$
0 = [G^{ini} [KJ_{R_{n-1}}] e^{\beta K J_{R_{n-1}}} + KG_{R_{n-1}}[0] - E_1] A(C_{K+1})
- G^{ini} [KJ_{R_{n-1}}] e^{\beta K J_{R_{n-1}}} A(C_{K+2}) - KG_{R_{n-1}}[0] A(C_K)
0 = [G^{ini} [KJ_{R_{n-1}}] e^{-\beta K J_{R_{n-1}}} - E_1] A(C_{K+2}) - G^{ini} [KJ_{R_{n-1}}] e^{-\beta K J_{R_{n-1}}} A(C_{K+1}).

(35)

For this effective one-dimensional problem with the effective transition rates

\begin{align*}
W^{\text{eff}}(C_0 \rightarrow C_1) &= G^{ini} [KJ_{R_{n-1}}] e^{-\beta K J_{R_{n-1}}} \\
W^{\text{eff}}(C_q \rightarrow C_{q+1}) &= (K + q - 1)G_{R_{n-1}}[0] \quad \text{for } 1 \leq q \leq K \\
W^{\text{eff}}(C_{K+1} \rightarrow C_{K+2}) &= G^{ini} [KJ_{R_{n-1}}] e^{-\beta K J_{R_{n-1}}}
\end{align*}

(36)

and

\begin{align*}
W^{\text{eff}}(C_1 \rightarrow C_0) &= G^{ini} [KJ_{R_{n-1}}] e^{\beta K J_{R_{n-1}}} \\
W^{\text{eff}}(C_q \rightarrow C_{q-1}) &= (q - 1)G_{R_{n-1}}[0] \quad \text{for } 2 \leq q \leq K + 1 \\
W^{\text{eff}}(C_{K+2} \rightarrow C_{K+1}) &= G^{ini} [KJ_{R_{n-1}}] e^{-\beta K J_{R_{n-1}}}
\end{align*}

(37)

we may use equation (A.21) of the appendix to obtain the renormalized amplitude

\[ G^{(e_1,e_2)}_{R,\text{last}} = G_{R}(C_0,C_{K+2}) \]

when the domain-wall enters by the boundary \( e_1 \) and exits by the boundary \( e_2 \)

\[ \frac{1}{G^{(e_1,e_2)}_{R,\text{last}}} = \frac{e^{\beta(U(C_0)-U(C_{K+2}))}}{W^{\text{eff}}(C_0 \rightarrow C_1)^{m+1} \sum_{q=1}^{m+1} W^{\text{eff}}(C_q \rightarrow C_{q-1}) \prod_{q=1}^{m+1} W^{\text{eff}}(C_q \rightarrow C_{q+1})} 
\]

\[ = \frac{2e^{\beta K J_{R_{n-1}}}}{G^{ini} [KJ_{R_{n-1}}][0]} \sum_{m=1}^{K} \frac{(m-1)!(K-m)!}{K!}. \]

(38)

Taking into account the other case where the domain-wall enters by the boundary \( e_2 \) and exits by the boundary \( e_1 \), which actually gives the same contribution

\[ G^{(e_2,e_1)}_{R,\text{last}} = G^{(e_2,e_1)}_{R,\text{last}} \]

(39)

we obtain that the final total amplitude \( G^{\text{last}}_{R} = G^{(e_1,e_2)}_{R,\text{last}} + G^{(e_2,e_1)}_{R,\text{last}} = 2G^{(e_1,e_2)}_{R,\text{last}} \) reads

\[ \frac{1}{G^{\text{last}}_{R}} = \frac{e^{\beta K J_{R_{n-1}}}}{G^{ini} [KJ_{R_{n-1}}]} + \frac{2e^{2\beta K J_{R_{n-1}}}}{2G_{R_{n-1}}[0]} \sum_{k=0}^{K-1} k!(K-1-k)! \]

(40)

3.3. RG rule for the bulk RG steps \( 1 \leq p \leq n - 1 \)

The bulk RG step \( p \) shown in figure 3 involves two external spins \( S^{ext}_A \) and \( S^{ext}_B \), and \((2K+1)\) internal spins \((S_{a_1}, S_{a_2}, \ldots, S_{a_K}, S_c, S_{b_1}, S_{b_2}, \ldots, S_{b_K})\) with the following classical energy (equation (4))

\[ U(C) = -J_{R_{p-1}} \sum_{i=1}^{K} S_{a_i}(S_c + S^{ext}_A) - J_{R_{p-1}} \sum_{i=1}^{K} S_{b_i}(S_c + S^{ext}_B). \]

(41)
The quantum Hamiltonian of equation (20) associated with the single-spin-flip dynamics reads

\[
H_{2K+1} = G^{\text{ini}} \left[ J_{R_{p-1}} \sum_{i=1}^{K} (\sigma_{a_i}^z + \sigma_{b_i}^z) \left( e^{-\beta J_{R_{p-1}} (\sigma_{c}^z + S_{A}^{\text{ext}})} - \sigma_{c}^x \right) \right] \\
+ \sum_{i=1}^{K} G_{R_{p-1}} \left[ J_{R_{p-1}} (\sigma_{c}^z + S_{A}^{\text{ext}}) \right] \left( e^{-\beta J_{R_{p-1}} (\sigma_{a_i}^z + S_{A}^{\text{ext}})} - \sigma_{a_i}^x \right) \\
+ \sum_{i=1}^{K} G_{R_{p-1}} \left[ J_{R_{p-1}} (\sigma_{c}^z + S_{B}^{\text{ext}}) \right] \left( e^{-\beta J_{R_{p-1}} (\sigma_{b_i}^z + S_{B}^{\text{ext}})} - \sigma_{b_i}^x \right). \tag{42}
\]

Let us now focus on the external domain-wall conditions

\[
\begin{align*}
S_{A}^{\text{ext}} &= -1 \\
S_{B}^{\text{ext}} &= +1
\end{align*} \tag{43}
\]

and take into account the symmetry between the \( K \) spins \( S_{a_i} \), and the symmetry between the \( K \) spins \( S_{b_i} \) to note \( \phi(k_a, k_b) \) the components of \( \psi_1 \), where \( k_a \in 0, 1, \ldots, K \) spins among \( (S_{a_1}, \ldots, S_{a_K}) \) take the value \((-)\) and where \( k_b \in 0, 1, \ldots, K \) spins among \( (S_{b_1}, \ldots, S_{b_K}) \) take the value \((-)\):

\[
\psi_1(S_{A}^{\text{ext}} = -1, S_{a_1}, \ldots, S_{a_K}, S_{c}, S_{b_1}, \ldots, S_{b_K}, S_{B}^{\text{ext}} = +1) = \phi_1 \left( k_a = \sum_{i=1}^{K} \frac{1 - S_{a_i}}{2}, S_{c}, k_b = \sum_{i=1}^{K} \frac{1 - S_{b_i}}{2} \right). \tag{44}
\]
Then the eigenvalue equation $0 = (H_{2K+1} - E_1)|\psi_1\rangle$ reads

$$0 = (G^{\text{ini}}_{J_{R_{p-1}}} (2K - 2k_a - 2k_b)) e^{-\beta J_{R_{p-1}} (2K - 2k_a - 2k_b)} S_c$$
$$+ k_a G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] e^{\beta J_{R_{p-1}} (S_c - 1)}$$
$$+ (K - k_a) G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] e^{-\beta J_{R_{p-1}} (S_c - 1)}$$
$$+ k_b G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] e^{\beta J_{R_{p-1}} (S_c + 1)}$$
$$+ (K - k_b) G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] e^{-\beta J_{R_{p-1}} (S_c + 1)} - E_1) \phi_1(k_a, S_c, k_b)$$
$$- G^{\text{ini}}_{J_{R_{p-1}}} (2K - 2k_a - 2k_b) \phi_1(k_a - S_c, k_b)$$
$$- k_a G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] \phi_1(k_a - 1, S_c, k_b)$$
$$- (K - k_a) G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] \phi_1(k_a + 1, S_c, k_b)$$
$$- k_b G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] \phi_1(k_a, S_c, k_b - 1)$$
$$- (K - k_b) G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] \phi_1(k_a, S_c, k_b + 1).$$

(45)

For $E_0 = 0$, the (non-normalized) groundstate is known to be given by equation (16) with the classical energy of equation (41)

$$\phi_0(k_a, S_c, k_b) = e^{(\beta/2)J_{R_{p-1}}} [S_c (2K - 2k_a - 2k_b) + (2k_a - 2k_b)]$$

(46)

so it is convenient to look for the solution of equation (27) via the amplitude

$$A(k_a, S_c, k_b) = \frac{\phi_1(k_a, S_c, k_b)}{\phi_0(k_a, S_c, k_b)}$$

(47)

that satisfies

$$0 = (G^{\text{ini}}_{J_{R_{p-1}}} (2K - 2k_a - 2k_b)) e^{-\beta J_{R_{p-1}} (2K - 2k_a - 2k_b)} S_c$$
$$+ k_a G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] e^{\beta J_{R_{p-1}} (S_c - 1)}$$
$$+ (K - k_a) G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] e^{-\beta J_{R_{p-1}} (S_c - 1)}$$
$$+ k_b G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] e^{\beta J_{R_{p-1}} (S_c + 1)}$$
$$+ (K - k_b) G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] e^{-\beta J_{R_{p-1}} (S_c + 1)} - E_1) A(k_a, S_c, k_b)$$
$$- G^{\text{ini}}_{J_{R_{p-1}}} (2K - 2k_a - 2k_b) e^{-\beta J_{R_{p-1}} (2K - 2k_a - 2k_b)} S_c A(k_a - S_c, k_b)$$
$$- k_a G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] e^{\beta J_{R_{p-1}} (S_c - 1)} A(k_a - 1, S_c, k_b)$$
$$- (K - k_a) G_{R_{p-1}} [J_{R_{p-1}} (S_c - 1)] e^{-\beta J_{R_{p-1}} (S_c - 1)} A(k_a + 1, S_c, k_b)$$
$$- k_b G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] e^{\beta J_{R_{p-1}} (S_c + 1)} A(k_a, S_c, k_b - 1)$$
$$- (K - k_b) G_{R_{p-1}} [J_{R_{p-1}} (S_c + 1)] e^{-\beta J_{R_{p-1}} (S_c + 1)} A(k_a, S_c, k_b + 1).$$

(48)
Let us consider the dynamical path along the following \((2K + 2)\) configurations

\[
\begin{align*}
C_0 &= \{0, S_c = +1, 0\} \\
C_1 &= \{1, S_c = +1, 0\} \\
C_2 &= \{2, S_c = +1, 0\} \\
C_K &= \{K, S_c = +1, 0\} \\
C_{K+1} &= \{K, S_c = -1, 0\} \\
C_{K+2} &= \{K, S_c = -1, 1\} \\
C_{K+3} &= \{K, S_c = -1, 2\} \\
C_{2K} &= \{K, S_c = -1, K - 1\} \\
C_{2K+1} &= \{K, S_c = -1, K\}
\end{align*}
\]

in order to describe the motion of a domain-wall corresponding to the boundary conditions of equation (43). Then equation (45) becomes for \(0 \leq q \leq K - 1\)

\[
0 = (qG_{R_{p-1}}[0] + (K - q)G_{R_{p-1}}[0] - E_1)A(C_q) - qG_{R_{p-1}}[0]A(C_{q-1})
- (K - q)G_{R_{p-1}}[0]A(C_{q+1}).
\]

For \(q = K\), one has

\[
0 = (G_{\text{ini}}[0] + KG_{R_{p-1}}[0] - E_1)A(C_K) - G_{\text{ini}}[0]A(C_{K+1}) - KG_{R_{p-1}}[0]A(C_{K-1}).
\]

For \(q = K + 1\), one has

\[
0 = (G_{\text{ini}}[0] + KG_{R_{p-1}}[0] - E_1)A(C_{K+1}) - G_{\text{ini}}[0]A(C_K) - KG_{R_{p-1}}[0]A(C_{K+2}).
\]

And finally, for \(q = K + 1 + k\) for \(1 \leq k \leq K\), one has

\[
0 = (kG_{R_{p-1}}[0] + (K - k)G_{R_{p-1}}[0] - E_1)A(C_{K+1+k}) - kG_{R_{p-1}}[0]A(C_{K+k})
- (K - k)G_{R_{p-1}}[0]A(C_{K+2+k}).
\]

For this effective one-dimensional problem with the effective transition rates

\[
\begin{align*}
W^{\text{eff}}(C_q \rightarrow C_{q+1}) &= (K - q)G_{R_{p-1}}[0] \quad \text{for } 0 \leq q \leq K - 1 \\
W^{\text{eff}}(C_K \rightarrow C_{K+1}) &= G_{\text{ini}}[0] \\
W^{\text{eff}}(C_q \rightarrow C_{q+1}) &= (2K + 1 - q)G_{R_{p-1}}[0] \quad \text{for } K + 1 \leq q \leq 2K
\end{align*}
\]

and

\[
\begin{align*}
W^{\text{eff}}(C_q \rightarrow C_{q-1}) &= qG_{R_{p-1}}[0] \quad \text{for } 1 \leq q \leq K \\
W^{\text{eff}}(C_{K+1} \rightarrow C_K) &= G_{\text{ini}}[0] \\
W^{\text{eff}}(C_q \rightarrow C_{q-1}) &= (q - K - 1)G_{R_{p-1}}[0] \quad \text{for } K + 2 \leq q \leq 2K + 1
\end{align*}
\]

we may use equation (A.21) of the appendix to obtain the renormalized amplitude \(G_{R_p}[0] = G_R(C_0, C_{2K+1})\)

\[
\frac{1}{G_{R_p}[0]} = \frac{e^{(\beta/2)[U(C_0) - U(C_{2K+1})]}}{W^{\text{eff}}(C_0 \rightarrow C_1)} \left[ 1 + \sum_{m=1}^{2K} \prod_{q=1}^{m} W^{\text{eff}}(C_q \rightarrow C_{q-1}) \right]
= \frac{1}{G_{\text{ini}}[0]} + \frac{2}{KG_{R_{p-1}}[0]} \sum_{m=0}^{K-1} \frac{(m)!(K - 1 - m)!}{(K - 1)!}.
\]
3.4. Conclusion

Using the combinatorial formula [43] concerning the sum of the inverse of binomial coefficients

$$
\sum_{m=0}^{K-1} \frac{(m)!}{(K-1)!} = \sum_{m=0}^{K-1} \frac{1}{C_{K-1}^m} = \frac{K}{2^k} \sum_{k=1}^{K} \frac{2^k}{k}
$$

(57)

and the form of renormalized couplings of equation (22)

$$
J_{R_{n-1}} = K^{n-1} J
$$

(58)

we may rewrite the last RG step of equation (40) as

$$
\frac{1}{G^\text{last}_{R_n}} = e^{\beta K^n J} \frac{1}{G^{\text{ini}}[K^n J]} + e^{2\beta K^n J} \frac{1}{2K+1} \sum_{k=1}^{K} \frac{2^k}{k}
$$

(59)

and the bulk RG steps $1 \leq p \leq n-1$ of equation (56) as

$$
\frac{1}{G_{R_p}[0]} = \frac{1}{G^{\text{ini}}[0]} + \frac{1}{2K+1} \sum_{k=1}^{K} \frac{2^k}{k}
$$

(60)

with the initial condition

$$
G_{R_0}[0] = G^{\text{ini}}[0].
$$

(61)

In terms of the numerical constant

$$
c_K \equiv \frac{1}{2K-1} \sum_{k=1}^{K} \frac{2^k}{k}
$$

(62)

the solution of the recurrence of equation (60) reads for $0 \leq p \leq n-1$

$$
\frac{1}{G_{R_p}[0]} = \frac{1}{G^{\text{ini}}[0]} \sum_{q=0}^{p} (c_K)^q
$$

(63)

so that the final renormalized amplitude of equation (59) for the whole sample reads

$$
\frac{1}{G^\text{last}_{R_n}} = e^{\beta K^n J} \frac{1}{G^{\text{ini}}[K^n J]} + e^{2\beta K^n J} \frac{1}{4 \sum_{q=1}^{n} (c_K)^q}.
$$

(64)

The final conclusion is thus that the equilibrium time $t_{eq}^{(n)}$ needed to go from one groundstate (where all spins take the value $+1$) to the opposite groundstate (where all spins take the value $-1$) reads for the Ising model on the diamond lattice of branching ratio $K$ with $n$ generations as

$$
t_{eq}^{(n)} \approx \frac{1}{G^\text{last}_{R_n}} = e^{\beta K^n J} \frac{1}{G^{\text{ini}}[K^n J]} + e^{2\beta K^n J} \frac{1}{4 \sum_{q=1}^{n} (c_K)^q}.
$$

(65)

In particular, the dynamical barrier $B^{(n)}$ defined by the low-temperature exponential behavior reads in terms of the length $L_{n}^{2n}$ and of the fractal dimension $d_{f}$ (equations (1)}
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and (3) as

\[ B^{(n)} \equiv \lim_{\beta \to +\infty} \frac{t^{(n)}_{\text{eq}}}{\beta} = 2J^d n^{d-1} L_n \]

in agreement with the expected scaling \( L^d \) of the energy cost of an interface of dimension \( d \), where \( d = d - 1 \) in a space of dimension \( d \).

Here, we have in addition computed explicitly the prefactors in equation (65): the leading behavior is a power-law of the length \( L_n \)

\[ t^{(n)}_{\text{eq}} \approx e^{\beta JK_n} \left( c_K \right)^n = e^{\beta J L_n^{d-1}} L_n^{\alpha_K} \]

where the exponent reads in terms of the constant \( c_K \) of equation (62) as

\[ \alpha_K = \frac{\ln c_K}{\ln 2} = \frac{\ln \left( 1/2^{K-1} \sum_{k=1}^{K} 2^k/k \right)}{\ln 2} \]

In particular, for the first values of \( K \), we obtain

- \( \alpha_{K=1} = 1 \)
- \( \alpha_{K=2} = 1 \)
- \( \alpha_{K=3} = \frac{\ln 5/3}{\ln 2} \)
- \( \alpha_{K=4} = \frac{\ln 4/3}{\ln 2} \).

For \( K = 1 \) corresponding to the one-dimensional chain, the result \( \alpha_{K=1} = 1 \) is in agreement with previous studies (see [30] and references therein).

4. Dynamics of the random ferromagnetic model

4.1. RG rule for the last RG step \( p = n \)

The last RG step \( p = n \) shown in figure 4 involves two boundary spins \( S_{e_1} \) and \( S_{e_2} \) and \( K \) internal spins \( \left( S_{a_1}, S_{a_2}, \ldots, S_{a_K} \right) \) with the following classical energy (equation (4))

\[ U(C) = - \sum_{i=1}^{K} S_{a_i} (J_{a_i} S_{e_1} + J'_{a_i} S_{e_2}). \]

The quantum Hamiltonian of equation (20) associated with the single-spin-flip dynamics reads

\[ H_{K+2} = G^{\text{ini}} \left[ \sum_{i=1}^{K} J_{a_i} \sigma_{a_i}^z \right] \left( e^{-\beta \sigma_{e_1}^z \sum_{i=1}^{K} J_{a_i} \sigma_{a_i}^z - \sigma_{e_1}^x} \right) + G^{\text{ini}} \left[ \sum_{i=1}^{K} J'_a \sigma_{a_i}^z \right] \left( e^{-\beta \sigma_{e_2}^z \sum_{i=1}^{K} J'_a \sigma_{a_i}^z - \sigma_{e_2}^x} \right) \]

\[ + \sum_{i=1}^{K} G_{a_i} \left[ J_{a_i} \sigma_{a_i}^z + J'_{a_i} \sigma_{a_i}^z \right] \left( e^{-\beta \sigma_{a_i}^z (J_{a_i} \sigma_{a_i}^z + J'_{a_i} \sigma_{a_i}^z) - \sigma_{a_i}^x} \right). \]
Let us first consider the dynamical path where the spins are flipped in the order $S_{e_1}, S_{a_1}, S_{a_2}, \ldots, S_{a_K}, S_{e_2}$. It is convenient to introduce the following notations for the corresponding $(K+3)$ configurations:

\begin{align*}
C_0 &= \{S_{e_1} = +1, S_{a_1} = +1, S_{a_2} = +1, \ldots, S_{a_K} = +1, S_{e_2} = +1\} \\
C_1 &= \{S_{e_1} = -1, S_{a_1} = +1, S_{a_2} = +1, \ldots, S_{a_K} = +1, S_{e_2} = +1\} \\
C_2 &= \{S_{e_1} = -1, S_{a_1} = -1, S_{a_2} = +1, \ldots, S_{a_K} = +1, S_{e_2} = +1\} \\
C_p &= \{S_{e_1} = -1, S_{a_1} = -1, \ldots, S_{a_{p-1}} = -1, S_{a_p} = +1, \ldots, S_{a_K} = +1, S_{e_2} = +1\} \quad (72) \\
C_K &= \{S_{e_1} = -1, S_{a_1} = -1, \ldots, S_{a_{K-1}} = -1, S_{a_K} = +1, S_{e_2} = +1\} \\
C_{K+1} &= \{S_{e_1} = -1, S_{a_1} = -1, \ldots, S_{a_K} = -1, S_{e_2} = +1\} \\
C_{K+2} &= \{S_{e_1} = -1, S_{a_1} = -1, \ldots, S_{a_K} = -1, S_{e_2} = -1\}.
\end{align*}

The physical meaning is that the transition from $C_0$ to $C_1$ corresponds to the entrance of a domain-wall at the boundary $e_1$, the transitions between the configurations $(C_1, C_2, \ldots, C_{K+1})$ correspond to the displacement of this domain-wall, and finally the transition from $C_{K+1}$ to $C_{K+2}$ corresponds to the exit of the domain-wall at the boundary $e_2$.

The two boundary configurations are the two ferromagnetic groundstates of classical energy (equation (70))

\[ U(C_0) = U(C_{K+2}) = -\sum_{i=1}^{K} (J_{a_i} + J'_{a_i}). \] \quad (73)

The intermediate configurations $C_p$ for $1 \leq p \leq K+1$ have a higher classical energy (equation (70))

\[ U(C_p) = \sum_{i=1}^{p-1} (-J_{a_i} + J'_{a_i}) + \sum_{i=p}^{K} (J_{a_i} - J'_{a_i}). \] \quad (74)
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So we may use the formula equation (A.23) derived in the appendix for the renormalized amplitude along this dynamical path

\[
\frac{1}{G_R(e_1,a_1,e_2,a_2,\ldots,e_K,e_{K+1})} = e^{-(\beta/2)[U(C_0)+U(C_{K+1})]} \sum_{m=0}^{K+1} \frac{e^{(\beta/2)[U(C_m)+U(C_{m+1})]}}{G(C_m,C_{m+1})}.
\]  

(75)

Taking into account the local field on the spin that is flipped between two consecutive configurations, one obtains for the two boundary flips

\[
G(C_0,C_1) = G^{\text{ini}} \left[ \sum_{i=1}^{K} J_{a_i} \right]
\]

(76)

\[
G(C_{K+1},C_{K+2}) = G^{\text{ini}} \left[ \sum_{i=1}^{K} J'_{a_i} \right]
\]

and for the intermediate flips \(1 \leq m \leq K\)

\[
G(C_m,C_{m+1}) = G_{am} \left[ J_{am} - J'_{am} \right].
\]

(77)

The renormalization formula of equation (75) for the last RG step finally reads

\[
\frac{1}{G_{R\text{last}}(e_1,a_1,e_2,a_2,\ldots,e_K,e_{K+2})} = e^{\beta \sum_{i=1}^{K} J_{a_i}} \sum_{m=1}^{K+1} \frac{e^{\beta \sum_{i=1}^{m-1} J'_{a_i} + J_{am} + J'_{am} + \sum_{i=m+1}^{K} J_{a_i}}}{G_{am} \left[ J_{am} - J'_{am} \right]} + e^{\beta \sum_{i=1}^{K} J'_{a_i}}.
\]

(78)

4.2. RG rule for the bulk RG steps \(1 \leq p \leq n - 1\)

The bulk RG step \(p\) shown in figure 5 involves two external spins, \(S_A^{\text{ext}}\) and \(S_B^{\text{ext}}\), and \((2K+1)\) internal spins, \((S_{a_1}, S_{a_2}, \ldots, S_{a_K}, S_c, S_{b_1}, S_{b_2}, \ldots, S_{b_K})\), with the following classical energy (equation (4))

\[
U(C) = -\sum_{i=1}^{K} S_{a_i} (J_{a_i} S_c + J'_{a_i} S_A^{\text{ext}}) - \sum_{i=1}^{K} S_{b_i} (J_{b_i} S_c + J'_{b_i} S_B^{\text{ext}}).
\]

(79)

The quantum Hamiltonian of equation (20) associated with the single-spin-flip dynamics reads

\[
H_{2K+1} = G^{\text{ini}} \left[ \sum_{i=1}^{K} (J_{a_i} \sigma_{a_i}^z + J_{b_i} \sigma_{b_i}^z) \right] (e^{-\beta \sigma_{a_i}^z \sum_{i=1}^{K} (J_{a_i} \sigma_{a_i}^z + J_{b_i} \sigma_{b_i}^z)} - \sigma_{c}^x)
\]

\[
+ \sum_{i=1}^{K} G_{a_i} \left[ J_{a_i} \sigma_{c}^z + J'_{a_i} S_A^{\text{ext}} \right] (e^{-\beta \sigma_{a_i}^z (J_{a_i} \sigma_{c}^z + J'_{a_i} S_A^{\text{ext}})} - \sigma_{a_i}^x)
\]

\[
+ \sum_{i=1}^{K} G_{b_i} \left[ J_{b_i} \sigma_{c}^z + J'_{b_i} S_B^{\text{ext}} \right] (e^{-\beta \sigma_{a_i}^z (J_{b_i} \sigma_{c}^z + J'_{b_i} S_B^{\text{ext}})} - \sigma_{b_i}^x).
\]

(80)
Let us first consider the dynamical path where the spins are flipped in the order $S_{a1}, S_{a2}, \ldots, S_{a_K}, S_c, S_{b1}, S_{b2}, \ldots, S_{b_K}$. More precisely, for the external domain-wall conditions

$$\begin{align*}
S^\text{ext}_A &= -1 \\
S^\text{ext}_B &= +1
\end{align*}$$

(81)

it is convenient to introduce the following notations for the corresponding $(2K + 2)$ configurations that describe the motion of the domain-wall

$$\begin{align*}
C_0 &= \{ S_{a1} = +1, S_{a2} = +1, \ldots, S_{a_K} = +1, S_c = +1, S_{b1} = +1, S_{b2} = +1 \} \\
C_1 &= \{ S_{a1} = -1, S_{a2} = +1, \ldots, S_{a_K} = +1, S_c = +1, S_{b1} = +1 \} \\
C_2 &= \{ S_{a1} = -1, S_{a2} = -1, S_{a3} = +1, \ldots, S_{a_K} = +1, S_c = +1, S_{b1} = +1 \} \\
C_K &= \{ S_{a1} = -1, S_{a2} = -1, \ldots, S_{a_K} = -1, S_c = +1, S_{b1} = +1, S_{b2} = +1 \} \\
C_{K+1} &= \{ S_{a1} = -1, S_{a2} = -1, \ldots, S_{a_K} = -1, S_c = -1, S_{b1} = +1, S_{b2} = +1 \} \\
C_{K+2} &= \{ S_{a1} = -1, S_{a2} = -1, \ldots, S_{a_K} = -1, S_c = -1, S_{b1} = +1, S_{b2} = +1 \} \\
C_{K+3} &= \{ S_{a1} = -1, S_{a2} = -1, \ldots, S_{a_K} = -1, S_c = -1, S_{b1} = -1, S_{b2} = -1, S_{b3} = +1 \}
\end{align*}$$

*Figure 5.* Bulk RG step for the dynamics of the disordered Ising model: notations for the quantum Hamiltonian of equation (80).
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\[ C_{2K} = \{ S_{a_1} = -1, S_{a_2} = -1, \ldots, S_{a_K} = -1, S_c = -1, S_{b_1} = -1, S_{b_2} = -1, \ldots, S_{b_{K-1}} = -1, S_{b_K} = +1 \} \]

\[ C_{2K+1} = \{ S_{a_1} = -1, S_{a_2} = -1, \ldots, S_{a_K} = -1, S_c = -1, S_{b_1} = -1, S_{b_2} = -1, \ldots, S_{b_{K-1}} = -1, S_{b_K} = -1 \}. \]

(82)

The classical energy (equation (79)) of these configurations reads for \( 0 \leq m \leq K \)

\[
U(C_m) = \sum_{i=1}^{m} (J_{a_i} - J'_{a_i}) - \sum_{i=m+1}^{K} (J_{a_i} - J'_{a_i}) - \sum_{i=1}^{K} (J_{b_i} + J'_{b_i})
\]

\[
U(C_{K+1+m}) = \sum_{i=1}^{K} (-J_{a_i} - J'_{a_i}) + \sum_{i=1}^{m} (-J_{b_i} + J'_{b_i}) - \sum_{i=m+1}^{K} (-J_{b_i} + J'_{b_i}).
\]

(83)

So we may used the formula equation (A.23) derived in the appendix for the renormalized amplitude along this dynamical path

\[
\frac{1}{G_{(a_1,a_2,\ldots,a_K,c,b_1,b_2\ldots,b_K)}(C_0, C_{2K+1})} = e^{-(\beta/2)[U(C_0)+U(C_{2K+1})]} \sum_{m=0}^{2K} \frac{e^{(\beta/2)[U(C_m)+U(C_{m+1})]}}{G(C_m, C_{m+1})}. \]

(84)

Taking into account the local field on the spin that is flipped between two consecutive configurations, one obtains for the two boundary flips for \( 1 \leq m \leq K - 1 \)

\[
G(C_m, C_{m+1}) = G_{a_m} [J_{a_m} - J'_{a_m}]
\]

\[
G(C_{K}, C_{K+1}) = G_{a_1} \sum_{i=1}^{K} (J_{a_i} - J_{b_i})
\]

\[
G(C_{K+m}, C_{K+m+1}) = G_{b_m} [J_{b_m} - J'_{b_m}].
\]

(85)

For the renormalized spin \( S_R \), the two renormalized ferromagnetic couplings read

\[
J_R \equiv \sum_{i=1}^{K} J'_{a_i}
\]

\[
J'_R \equiv \sum_{i=1}^{K} J'_{b_i}
\]

(86)

and the absolute value of the renormalized local field \( h_R \) in the domain-wall configurations takes the single value

\[
|h_R| = |J_R - J'_{R}| = \left| \sum_{i=1}^{K} (J'_{a_i} - J'_{b_i}) \right|.
\]

(87)

So one does not have to renormalize a function \( G[h] \) of an arbitrary local field \( h \), but only a set of three correlated variables \((J_R, J'_R, G_{R(a_1,a_2,\ldots,a_K,c,b_1,b_2\ldots,b_K)}[h_R = J_R - J'_R])\) representing the two ferromagnetic couplings \((J_R, J'_R)\) of equation (86) and the corresponding numerical
amplitude $G_{R}^{(a_1,a_2,...,a_K,c,b_1,b_2,...,b_K)}[h_R = J_R - J'_R]$ given by the final formula (equation (84))

$$
\frac{1}{G_{R}^{(a_1,a_2,...,a_K,c,b_1,b_2,...,b_K)}[h_R = J_R - J'_R]} = \sum_{m=1}^{K} e^{\beta \sum_{i=1}^{m-1} (2J_{a_i} - J'_{a_i} + J_{b_i} - J'_{b_i}) + J_{a_m} - J'_{a_m} + \sum_{i=m+1}^{K} (J_{a_i} - J'_{a_i})}
+ e^{\beta \sum_{i=1}^{K} (J_{a_i} - J'_{a_i})} \sum_{i=1}^{K} G_{a_i}^{ini} \left[ J_{a_i} - J'_{a_i} \right] \left[ J_{a_i} - J'_{a_i} \right] + \sum_{m=1}^{K} e^{\beta \sum_{i=1}^{m-1} (2J_{a_i} - J'_{a_i} + J_{b_i} - J'_{b_i}) + J_{a_m} - J'_{a_m} + \sum_{i=m+1}^{K} (J_{a_i} - J'_{a_i})}
+ e^{\beta \sum_{i=1}^{K} (J_{a_i} - J'_{a_i})} \sum_{i=1}^{K} G_{b_m}^{ini} \left[ J_{b_m} - J'_{b_m} \right] \left[ J_{b_m} - J'_{b_m} \right].
$$

(88)

4.3. Analysis of the random ferromagnetic chain ($K = 1$)

For the random ferromagnetic chain corresponding to the branching ratio $K = 1$, the renormalization formula for the last step (equation (78)) becomes

$$
\frac{1}{G_{last}^{(e_1,e_2)}} = \frac{e^{\beta J_{e_1}}}{G_{ini}^{ini} \left[ J_{a_1} \right]} + \frac{e^{\beta \left[ J_{a_1} + J_{b_1} \right]}}{G_{a_1}^{ini} \left[ J_{a_1} - J'_{a_1} \right] + G_{b_1}^{ini} \left[ J_{b_1} - J'_{b_1} \right]}.
$$

(89)

For the bulk, we may rewrite equation (88) as

$$
\frac{e^{\beta \left[ J'_{a_1} + J'_{b_1} \right]}}{G_{R}^{[J'_{a_1} - J'_{b_1}]} = \frac{e^{\beta \left[ J_{a_1} + J_{b_1} \right]}}{G_{a_1}^{ini} \left[ J_{a_1} - J'_{a_1} \right] + G_{b_1}^{ini} \left[ J_{b_1} - J'_{b_1} \right]},
$$

(90)

to make clearer that the combination $e^{\beta \left[ J'_{a_1} + J'_{b_1} \right]} / G_{R}^{[J'_{a_1} - J'_{b_1}]}$ has a simple renormalization rule. By iteration, we finally obtain that the final amplitude $G_{last}^{(e_1,e_2)}(L_n = 2^n)$ for a system of size $L_n = 2^n$ of $n$ generations reads in terms of the initial function $G_{ini}^{ini}[h]$ (with the notations $J_{-1,0} = 0 = J_{L_n,L_n+1}$)

$$
\frac{1}{G_{last}^{(e_1,e_2)}(L_n = 2^n)} = \sum_{i=1}^{L_n=2^n} e^{\beta \left( J_{i-1,i} + J_{i,i+1} \right)} \sum_{i=1}^{L_n=2^n} G_{ini}^{ini} \left[ J_{i-1,i} - J_{i,i+1} \right]
$$

(91)

when the domain-wall enters by the boundary $e_1$ and exits by the boundary $e_2$.

We should now take into account the other case where the domain-wall enters by the boundary $e_2$ and exits by the boundary $e_1$, which actually gives the same contribution

$$
G_{last}^{(e_2,e_1)}(L_n = 2^n) = G_{last}^{(e_1,e_2)}(L_n = 2^n).
$$

(92)

The total amplitude $G_{last}(L_n = 2^n)$ is the sum of these two contributions

$$
G_{last}(L_n = 2^n) = G_{last}^{(e_1,e_2)}(L_n = 2^n) + G_{last}^{(e_2,e_1)}(L_n = 2^n) = 2G_{last}^{(e_1,e_2)}(L_n = 2^n)
$$

(93)

so that the final result is

$$
\frac{1}{G_{last}(L_n = 2^n)} = \frac{1}{2} \sum_{i=1}^{L_n=2^n} e^{\beta \left( J_{i-1,i} + J_{i,i+1} \right)} \sum_{i=1}^{L_n=2^n} G_{ini}^{ini} \left[ J_{i-1,i} - J_{i,i+1} \right]
$$

(94)

\[\text{doi:10.1088/1742-5468/2013/06/P06007}\]
in agreement with the results of equation (104) obtained in our previous work [30] via the boundary renormalization procedure. This agreement shows the validity of the bulk renormalization procedure within the domain-wall approximation.

4.4. Dynamical barriers for branching ratios $K > 1$

For $K > 1$, we have to compare the various dynamical paths that display different dynamical barriers as a consequence of the disorder. As explained before equation (88), the important quantity is the numerical amplitude $G_R[h_R = J_R - J'_R]$ which is correlated with the two renormalized ferromagnetic couplings $(J_R, J'_R)$. It is thus convenient to introduce the corresponding dynamical barrier $B_R^{(J_R, J'_R)}$ defined by the low-temperature behavior

$$G_R[J_R - J'_R] \simeq e^{-\beta B_R^{(J_R, J'_R)}} \quad \text{(95)}$$

where the notation $B_R^{(J_R, J'_R)}$ has been chosen to remind us that this barrier is correlated with the two couplings $(J_R, J'_R)$.

Let us now focus on the Glauber dynamics of equation (19) with the following low-temperature behavior:

$$G_{\text{Glauber}}[h] = \frac{1}{2 \cosh (\beta h)} \simeq e^{-\beta |h|}. \quad \text{(96)}$$

4.4.1. Optimization of the dynamical path for the last RG step $p = n$. In terms of dynamical barriers, equation (78) with (96) yields that the final dynamical barrier $B_{\text{last}}^{(a_1, a_2, ..., a_K)}$ associated with the given dynamical path $(a_1, a_2, \ldots, a_K)$ reads

$$B_{\text{last}}^{(a_1, a_2, ..., a_K)} = \max \left[ 2 \sum_{i=1}^{K} J_{a_i}; 2 \sum_{i=1}^{K} J'_{a_i}; \right. \left. \max_{1 \leq m \leq K} \left( B_{a_m}^{(J_{a_{m}}, J'_{a_{m}})} + 2 \sum_{i=1}^{m-1} J'_{a_i} + J_{a_m} + J'_{a_m} + 2 \sum_{i=m+1}^{K} J_{a_i} \right) \right]. \quad \text{(97)}$$

We now have to consider the $K!$ possible dynamical paths: for a given permutation $\pi$ of the $K$ renormalized spins, the dynamical barrier $B_{\text{last}}^{(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(K)})}$ associated with the path $(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(K)})$ is, by adapting equation (97),

$$B_{\text{last}}^{(a_{\pi(1)}, \ldots, a_{\pi(K)})} = \max \left[ 2 \sum_{i=1}^{K} J_{a_{\pi(i)}}; 2 \sum_{i=1}^{K} J'_{a_{\pi(i)}}; \right. \left. \max_{1 \leq m \leq K} \left( B_{a_{\pi(m)}}^{(J_{a_{\pi(m)}}, J'_{a_{\pi(m)}})} + 2 \sum_{i=1}^{m-1} J'_{a_{\pi(i)}} + J_{a_{\pi(m)}} + J'_{a_{\pi(m)}} + 2 \sum_{i=m+1}^{K} J_{a_{\pi(i)}} \right) \right]. \quad \text{(98)}$$

We now have to choose the dynamical path, i.e. the permutation $\pi$ leading to the smallest barrier. So the final renormalized barrier $B_{\text{last}}$ is given by the minimum of
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4.4.2. Optimization of the dynamical path for the bulk RG steps $1 \leq p \leq n - 1$. Similarly for the bulk RG step, equation (88) yields that the renormalized dynamical barrier $B_R^{(a_1, a_2, \ldots, a_K, c, b_1, b_2, \ldots, b_K)}$ correlated with the renormalized couplings (equation (86))

$$J_R \equiv \sum_{i=1}^{K} J'_a,$$

$$J'_R \equiv \sum_{i=1}^{K} J'_b,$$

and associated with the dynamical path $(a_1, a_2, \ldots, a_K, c, b_1, b_2, \ldots, b_K)$ reads

$$B_R^{(a_1, a_2, \ldots, a_K, c, b_1, b_2, \ldots, b_K)} = \max \left( \max_{1 \leq m \leq K} \left[ B_{a_m}^{(J'_a + J'_b, J'_b)} + \sum_{i=1}^{K} (2J_a - J'_a - J'_b) + J_m - J'_b \right] + \sum_{i=m+1}^{K} (J_a + J_b - J'_a - J'_b) \right).$$

We now have to consider the $(K!)^2$ possible dynamical paths: for a given permutation $(\pi_a)$ of the $a_m$ renormalized spins, and for a given permutation $(\pi_b)$ of the $b_m$ renormalized spins, the dynamical barrier $B_R^{(a_{\pi_a(1)}, a_{\pi_a(2)}, \ldots, a_{\pi_a(K)}, c, b_{\pi_b(1)}, b_{\pi_b(2)}, \ldots, b_{\pi_b(K)})}$ associated with the path $(a_{\pi_a(1)}, a_{\pi_a(2)}, \ldots, a_{\pi_a(K)}, c, b_{\pi_b(1)}, b_{\pi_b(2)}, \ldots, b_{\pi_b(K)})$ reads, by adapting equation (101),

$$B_R^{(a_{\pi_a(1)}, a_{\pi_a(2)}, \ldots, a_{\pi_a(K)}, c, b_{\pi_b(1)}, b_{\pi_b(2)}, \ldots, b_{\pi_b(K)})} = \max \left( \max_{1 \leq m \leq K} \left[ B_{a_{\pi_a(m)}}^{(J'_a, J'_b, J'_b)} + \sum_{i=1}^{K} (2J_{a_{\pi_a(i)}} - J'_a - J'_b) + J_{b_{\pi_b(i)}} - J'_b \right] + \sum_{i=m+1}^{K} (J_{a_{\pi_a(i)}} + J_{b_{\pi_b(i)}} - J'_a - J'_b) \right).$$
4.5.2. Bulk RG rules for dynamical barriers when

\[ \sum_{i=1}^{K} (J_{a_i} - J_{b_i}) + \sum_{i=1}^{K} (J_{a_i} + J_{b_i} - J'_{a_i} - J'_{b_i}); \]

\[ \max_{1 \leq m \leq K} \left[ B_{b_{\ell}(m)}^{(J_{a_{\ell}}(m)-J'_{a_{\ell}}(m))} + \sum_{i=1}^{m-1} (J'_{a_{\ell}(i)} - J'_{a_{\ell}(i)}) + J_{b_{\ell}(m)} - J'_{a_{\ell}(m)} \right] \]

\[ + \sum_{i=m+1}^{K} (2J_{b_{\ell}(i)} - J'_{a_{\ell}(i)} - J'_{b_{\ell}(i)}). \] (102)

We now have to choose the dynamical path, i.e. the permutations \((\pi_a, \pi_b)\) leading to the smallest barrier. So the final renormalized barrier \(B_R^{(J_R, J'_R)}\) is given by the minimum of equation (102) over the \(K!\) permutations \((\pi_a)\) and over the \(K!\) permutations \((\pi_b)\)

\[ B_R^{(J_R, J'_R)} \equiv \min_{\pi_a, \pi_b} \left( B_R^{(a_{\pi_a(1)}, a_{\pi_a(2)}, \ldots, a_{\pi_a(K)}; b_{\pi_b(1)}, b_{\pi_b(2)}, \ldots, b_{\pi_b(K)})} \right). \] (103)

To see the structure more clearly, let us now focus on the case \(K = 2\).

4.5. Case \(K = 2\) corresponding to the fractal dimension \(d_f = 2\)

4.5.1. Last step RG rule for dynamical barriers when \(K = 2\). For \(K = 2\), equation (97) involves a maximum over four terms

\[ B_{\text{last}}^{a_1, a_2} = \max \left[ 2(J_{a_1} + J_{a_2}); 2(J'_{a_1} + J'_{a_2}); B_{a_1}^{(J_{a_1}, J'_{a_1})} + J_{a_1} + J'_{a_1} + 2J_{a_2}; B_{a_2}^{(J_{a_2}, J'_{a_2})} \right. \]

\[ \left. + 2J'_{a_2} + J_{a_2} + J'_{a_2} \right] \] (104)

and equation (99) involves a minimum over two permutations

\[ B_{\text{last}} = \min \left( \max \left[ 2(J_{a_1} + J_{a_2}); 2(J'_{a_1} + J'_{a_2}); B_{a_1}^{(J_{a_1}, J'_{a_1})} + J_{a_1} + J'_{a_1} + 2J_{a_2}; B_{a_2}^{(J_{a_2}, J'_{a_2})} \right. \right. \]

\[ \left. \left. + 2J'_{a_2} + J_{a_2} + J'_{a_2} \right] \right); \]

\[ \max \left[ 2(J_{a_1} + J_{a_2}); 2(J'_{a_1} + J'_{a_2}); B_{a_2}^{(J_{a_2}, J'_{a_2})} + J_{a_2} + J'_{a_2} + 2J_{a_1}; B_{a_1}^{(J_{a_1}, J'_{a_1})} \right. \]

\[ \left. \left. + 2J'_{a_1} + J_{a_1} + J'_{a_1} \right] \right). \] (105)

4.5.2. Bulk RG rules for dynamical barriers when \(K = 2\). Equation (101) involves a maximum over five terms

\[ B_R^{a_1, a_2, b_1, b_2} = \max \left[ B_{a_1}^{(J_{a_1}, J'_{a_1})} + J_{a_1} - J'_{b_1} - J_{b_1} \right. \right. \]

\[ \left. \left. B_{a_2}^{(J_{a_2}, J'_{a_2})} + (2J_{a_1} - J'_{a_1} - J'_{b_1}) + J_{a_2} - J'_{b_2}; \right] \right. \right.

\[ \left. \sum_{i=1}^{2} (J_{a_i} - J_{b_i}) \right. \right. \]

\[ \left. \left. + \sum_{i=1}^{2} (J_{a_i} + J_{b_i} - J'_{a_i} - J'_{b_i}); \right. \right. \]

\[ B_{b_1}^{(J_{b_1}, J'_{b_1})} + J_{b_1} - J'_{a_1} + (2J_{b_1} - J'_{a_1} - J'_{b_1}); \]

\[ B_{b_2}^{(J_{b_2}, J'_{b_2})} + (J'_{b_1} - J'_{a_1}) + J_{b_2} - J'_{a_2}. \] (106)
and equation (103) involves a minimum over four terms
\[ B_R^{(J_R,J'_R)} = \min \left[ B_R^{(a_1,a_2,c,b_1,b_2)}; B_R^{(a_2,a_1,c,b_1,b_2)}; B_R^{(a_1,a_2,c,b_2,b_1)}; B_R^{(a_2,a_1,c,b_2,b_1)} \right]. \] (107)

4.5.3. Numerical results obtained via the pool method. The pool method is very useful for studying renormalization rules for disordered models defined on trees [32], [44]–[46] and on hierarchical lattices [8, 20, 22, 47]. The idea of the pool method is the following: at each generation, one keeps the same number \( M_{\text{pool}} \) of random variables to represent probability distributions. Within our present framework, the joint probability distribution \( P_p(B,J,J') \) of the dynamical barrier \( B \) and of the two renormalized couplings of a renormalized spin at generation \( n \) will be represented by a pool of \( M_{\text{pool}} = 10^6 \) triplets \( (B_i,J_i,J'_i) \). To construct a new triplet \( (B_R,J_R,J'_R) \) of generation \( (p+1) \), one draws \( 2^K \) triplets \( (B_i,J_i,J'_i) \) within the pool of generation \( p \) and applies the rule of equation (106) and (107). At the last RG step \( p = n \), one draws instead \( K \) triplets \( (B_i,J_i,J'_i) \) within the pool of generation \( (n-1) \) and applies the rule of equations (104) and (105) to obtain the final barrier \( B_{\text{last}} \) between the two ferromagnetic groundstates of the whole sample of length \( L_n = 2^n \) (equation (1)).

For the Glauber dynamics satisfying equation (96), the initial condition at generation \( n = 0 \) reads in terms of the initial disorder distribution \( \rho(J) \) of the ferromagnetic coupling as
\[ P_{n=0}^{\text{bulk}}(B,J,J') = \rho(J)\rho(J')\delta(B-|J-J'|). \] (108)
We have chosen the box distribution of width \( \Delta = 1 \)
\[ \rho(J) = \theta(1 \leq J \leq 2). \] (109)

In figure 6, we present our numerical results concerning the probability distribution of the bulk dynamical barrier \( B \) at generation \( n \leq 100 \)
\[ P_n^{\text{bulk}}(B) = \int dJ \int dJ' P_n^{\text{bulk}}(B,J,J'). \] (110)
As explained in previous sections, this bulk dynamical barrier \( B \) characterizes the dynamics of a domain-wall crossing the system after its creation. We find that both the averaged value
\[ B_{\text{av}}^{\text{bulk}}(n) \equiv \int dB B P_n^{\text{bulk}}(B) \] (111)
and the width
\[ B_{\text{width}}^{\text{bulk}}(n) \equiv \left( \int dB B^2 P_n^{\text{bulk}}(B) - (B_{\text{av}}^{\text{bulk}}(n))^2 \right)^{1/2} \] (112)
grow with the same power-law of the length \( L_n = 2^n \)
\[ B_{\text{av}}^{\text{bulk}}(n) \propto L_n^{\psi} \]
\[ B_{\text{width}}^{\text{bulk}}(n) \propto L_n^{\psi} \] (113)
with the dynamical exponent (see figure 6(a))
\[ \psi \simeq 0.5. \] (114)
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Figure 6. Glauber dynamics of the random ferromagnetic model defined on the diamond hierarchical lattice of branching ratio $K = 2$ corresponding to the fractal dimension $d_f = 2$: (a) log–log plot of the averaged value $B_{\text{av}}^\text{bulk}(n)$ and of the width $B_{\text{width}}^\text{bulk}(n)$ of the probability distribution $P_n(B)$ of bulk dynamical barriers at generation $n$ corresponding to the length $L_n = 2^n$: the slope yields the dynamical exponent $\psi \simeq 0.5$. (b) The distribution of the rescaled barrier $u \equiv (B - B_{\text{av}}^\text{bulk}(n))/B_{\text{width}}^\text{bulk}(n)$ is the Gaussian distribution $g(u)$ of equation (116).

As shown in figure 6(b), the corresponding rescaled barrier

$$u \equiv \frac{B - B_{\text{av}}^\text{bulk}(n)}{B_{\text{width}}^\text{bulk}(n)}$$

(115)

follows the Gaussian distribution

$$g(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}.$$ 

(116)

We have also computed the probability distribution of the last barrier $B_{\text{last}}$ as a function of the generation $n$. As explained in previous sections, this last dynamical barrier $B$ characterizes the dynamics where a domain-wall is created near a boundary and then crosses the system. We find that the averaged value grows linearly

$$B_{\text{av}}^\text{last}(n) \propto L_n$$

(117)

as expected from the energy cost $L_d^s$ of the creation of an interface of dimension $d_s = d_f - 1 = 1$ in a space of dimension $d_f = 2$, in agreement with the result of equation (66) corresponding to the pure case. The width around this averaged value is found to scale as the width of the bulk barrier of equation (113)

$$B_{\text{width}}^\text{last}(n) \propto L_n^\psi$$

(118)

with $\psi \simeq 0.5$.

The result $\psi = 1/2$ for the dynamical exponent $\psi$ is in agreement with the conjecture $\psi = d_s/2$ proposed in our previous work [48]. In particular, $\psi = 1/2$ is clearly different from the droplet exponent $\theta \simeq 0.299$ involved in the statics of the random ferromagnet on the diamond lattice [49], which coincides with the directed polymer droplet exponent.
θ_{DP} \simeq 0.299 \ [21], since the optimization of the position of the interface in the random ferromagnet corresponds to the optimization of the position of a directed polymer in a random medium. We refer to [48] for a detailed discussion of the physical meaning of the conjecture \( \psi = d_s/2 \) with respect to other alternative proposals.

5. Conclusion

To characterize the stochastic single-spin-flip dynamics near zero temperature of the pure and random ferromagnetic Ising model on the hierarchical diamond lattice of branching ratio \( K \) with fractal dimension \( d_t = (\ln(2K))/\ln 2 \), we have adapted the real-space renormalization procedure introduced in our previous work [30].

For the pure Ising model, we have obtained that the equilibrium time behaves as

\[
t_{eq}(L) \sim L^{\alpha} e^{2JL^{d_s}}
\]

where \( d_s = d_t - 1 \) is the expected interface dimension. We have computed the prefactor exponent \( \alpha \) as a function of \( K \).

For the random ferromagnetic Ising model, we have derived the renormalization rules for dynamical barriers \( B_{eq}(L) \equiv (\ln t_{eq}/\beta) \) near zero temperature. For the fractal dimension \( d_t = 2 \) (corresponding to the branching ratio \( K = 2 \)), we have studied numerically these renormalization rules via the pool method to obtain

\[
B_{eq}(L) \sim cL + L^{1/2} u
\]

where \( u \) is a \( O(1) \) Gaussian random variable of non-zero mean. The non-random term scaling as \( L \) corresponds to the energy cost of the creation of an interface of dimension \( d_s = d_t - 1 \) as in the pure case of equation (119). The dynamical exponent \( \psi \) governing the fluctuation part characterizes the barriers for the motion of a domain-wall after its creation. The result \( \psi = 1/2 \) is in agreement with the conjecture \( \psi = d_s/2 \) proposed in [48]. In particular, the dynamical exponent \( \psi = 1/2 \) is clearly different from the droplet exponent \( \theta = 0.299 \) involved in the statics of the random ferromagnet on the same lattice [49].

Appendix A. Renormalization rule for an effective one-dimensional dynamics

A.1. First excited quantum state

Since the exact groundstate \( |\psi_0\rangle \) of zero energy \( E_0 = 0 \) is exactly known to be given by equation (16), it is natural to look for the first excited state through an amplitude \( A(C) \)

\[
|\psi_1\rangle = \sum_C A(C) e^{-(\beta/2)U(C)} \sqrt{Z} |C\rangle.
\]

(A.1)

Then the eigenequation for the quantum Hamiltonian of equation (15)

\[
0 = (H - E_1)|\psi_1\rangle
\]

(A.2)

can be rewritten for the amplitude \( A(C) \) as

\[
[W_{out}(C) - E_1] A(C) = \sum_{C'} W(C \rightarrow C') A(C').
\]

(A.3)
A.2. Explicit first non-vanishing energy $E_1$ for an effective one-dimensional dynamics

Let us consider an effective one-dimensional dynamics between configurations $(C_0, C_1, C_2, \ldots, C_n, C_{n+1})$ described by the system (equation (A.3)) for $1 \leq i \leq n$

$$0 = [W(C_i \rightarrow C_{i-1}) + W(C_i \rightarrow C_{i+1}) - E_i] A(C_i) - W(C_i \rightarrow C_{i-1}) A(C_{i-1}) - W(C_i \rightarrow C_{i+1}) A(C_{i+1})$$

(A.4)

and by the two boundary equations for $i = 0$ and $n + 1$

$$0 = [W(C_0 \rightarrow C_1) - E_1] A(C_0) - W(C_0 \rightarrow C_1) A(C_1)$$

$$0 = [W(C_{n+1} \rightarrow C_n) - E_n] A(C_{n+1}) - W(C_{n+1} \rightarrow C_n) A(C_n).$$

(A.5)

Let us assume that the intermediate configurations $C_i$ for $i = 1, 2, \ldots, n$ have higher classical energies $U(C_i)$ with respect to the two boundary configurations $C_0$ and $C_{n+1}$. Then the groundstate $|\psi_0\rangle$ of zero energy $E_0 = 0$ of equation (16) can be approximated at low temperature by its two leading components

$$|\psi_0\rangle \overset{\beta \to +\infty}{\approx} \frac{1}{\sqrt{e^{-\beta U(C_0)} + e^{-\beta U(C_{n+1})}}} \left( e^{-(\beta/2)U(C_0)}|C_0\rangle + e^{-(\beta/2)U(C_{n+1})}|C_{n+1}\rangle \right).$$

(A.6)

Then the small energy $E_1$ can be neglected in equation (A.4) to become for $1 \leq i \leq n$

$$A(C_i) = p_-(C_i) A(C_{i-1}) + p_+(C_i) A(C_{i+1})$$

(A.7)

with the notations

$$p_-(C_i) \equiv \frac{W(C_i \rightarrow C_{i-1})}{W(C_i \rightarrow C_{i-1}) + W(C_i \rightarrow C_{i+1})}$$

$$p_+(C_i) \equiv \frac{W(C_i \rightarrow C_{i+1})}{W(C_i \rightarrow C_{i-1}) + W(C_i \rightarrow C_{i+1})} = 1 - p_-(C_i).$$

(A.8)

The leading components of the first excited state at low temperature

$$|\psi_1\rangle \overset{\beta \to +\infty}{\approx} \psi_1(C_0)|C_0\rangle + \psi_1(C_{n+1})|C_{n+1}\rangle$$

(A.9)

are then fixed by orthogonality with the groundstate of equation (A.6)

$$\psi_1(C_0) \overset{\beta \to +\infty}{\approx} \frac{e^{-(\beta/2)U(C_{n+1})}}{\sqrt{e^{-\beta U(C_0)} + e^{-\beta U(C_{n+1})}}}$$

$$\psi_1(C_{n+1}) \overset{\beta \to +\infty}{\approx} \frac{e^{-(\beta/2)U(C_0)}}{\sqrt{e^{-\beta U(C_0)} + e^{-\beta U(C_{n+1})}}}$$

(A.10)

so that the amplitude $A(C)$ satisfies the boundary conditions

$$A(C_0) \overset{\beta \to +\infty}{\approx} -e^{-(\beta/2)[U(C_{n+1})-U(C_0)]}$$

$$A(C_{n+1}) \overset{\beta \to +\infty}{\approx} e^{(\beta/2)[U(C_{n+1})-U(C_0)]}.$$

(A.11)

The solution of equation (A.7) with the boundary conditions of equation (A.11) can be obtained by recurrence [50] and reads

$$A(C_i) = A(C_0) \frac{R_0(i, n)}{R_0(0, n)} + A(C_{n+1}) \frac{R_{n+1}(1, i)}{R_{n+1}(1, n+1)}$$

(A.12)
Using equations (A.8) and (A.14), one finally obtains

\[ E \text{ obtains} \]

\[ R_0(n + 1, n) = 0 \]
\[ R_0(n, n) = 1 \]
\[ R_0(k \leq n - 1, n) = 1 + \sum_{m=k+1}^{n} \prod_{i=m}^{n} \frac{p_+(i)}{p_-(i)} \]  \hfill (A.13)
\[ R_0(0, n) = 1 + \sum_{m=1}^{n} \prod_{i=m}^{n} \frac{p_+(i)}{p_-(i)} = 1 + p_+(n) + \cdots + \frac{p_+(n)p_+(n-1)\cdots p_+(1)}{p_-(n)p_-(n-1)\cdots p_-(1)} \]

and

\[ R_{n+1}(1, 0) = 0 \]
\[ R_{n+1}(1, 1) = 1 \]
\[ R_{n+1}(1, k \geq 2) = 1 + \sum_{m=1}^{k-1} \prod_{i=1}^{m} \frac{p_-(i)}{p_+(i)} \]  \hfill (A.14)
\[ R_{n+1}(1, n + 1) = 1 + \sum_{m=1}^{n} \prod_{i=1}^{m} \frac{p_-(i)}{p_+(i)} = 1 + \frac{p_-(1)}{p_+(1)} + \frac{p_-(1)p_-(2)}{p_+(1)p_+(2)} + \cdots + \frac{p_-(1)p_-(2)\cdots p_-(n)}{p_+(1)p_+(2)\cdots p_+(n)}. \]

The energy \( E_1 \) can be now computed from equation (A.5) at the boundary \( C_0 \) (or equivalently at the other boundary \( C_{n+1} \)), and using equations (A.12) and (A.11), one obtains

\[ E_1 \simeq W(C_0 \to C_1) \left[ 1 - \frac{A(C_1)}{A(C_0)} \right] \]
\[ \simeq W(C_0 \to C_1) \left[ 1 - \frac{R_0(1, n)}{R_0(0, n)} - \frac{A(C_{n+1})}{A(C_0)} \frac{R_{n+1}(1, 1)}{R_{n+1}(1, n)} \right] \]
\[ \simeq W(C_0 \to C_1) \left[ 1 - \frac{A(C_{n+1})}{A(C_0)} \right] \]
\[ \simeq W(C_0 \to C_1) \frac{R_{n+1}(1, n + 1)}{R_{n+1}(1, n + 1)} \left[ 1 + e^{\beta[U(C_{n+1})-U(C_0)]]} \right]. \]  \hfill (A.15)

Using equations (A.8) and (A.14), one finally obtains

\[ \frac{1}{E_1} = \frac{R_{n+1}(1, n + 1)}{W(C_0 \to C_1) [1 + e^{\beta[U(C_{n+1})-U(C_0)]]}} \]
\[ = \frac{1}{W(C_0 \to C_1) [1 + e^{\beta[U(C_{n+1})-U(C_0)]]}} \left[ 1 + \sum_{m=1}^{n} \prod_{i=1}^{m} \frac{p_-(i)}{p_+(i)} \right] \]
\[ = \frac{1}{W(C_0 \to C_1) [1 + e^{\beta[U(C_{n+1})-U(C_0)]]}} \left[ 1 + \sum_{m=1}^{n} \prod_{i=1}^{m} \frac{W(C_i \to C_{i-1})}{W(C_i \to C_{i+1})} \right]. \]  \hfill (A.16)
A.3. Renormalized amplitude $G_R$ for an effective one-dimensional dynamics

The renormalized quantum Hamiltonian is given by the projection onto the two lowest eigenstates $E_0 = 0$ and $E_1$

$$H^{\text{eff}} \simeq E_1 |\psi_1 \rangle \langle \psi_1|$$

(A.17)

where the first excited state $|\psi_1 \rangle$ is given by equation (A.10) near zero temperature. So equation (A.17) becomes

$$H^{\text{eff}} \simeq \frac{E_1}{\beta \to +\infty} e^{-\beta U(C_0)} + e^{-\beta U(C_{n+1})} [e^{-\beta U(C_{n+1})} |C_0 \rangle \langle C_0| + e^{-(\beta/2)U(C_0)} |C_{n+1} \rangle \langle C_{n+1}|]
- e^{-(\beta/2)U(C_{n+1})} (|C_0 \rangle \langle C_{n+1}| + |C_{n+1} \rangle \langle C_0|)].$$

(A.18)

So it is of the form of equation (15) with only the two configurations $C_0$ and $C_{n+1}$

$$H^{\text{eff}} \simeq \frac{E_1}{\beta \to +\infty} G_R(C_0, C_{n+1})
\times \left[ e^{-(\beta/2)[U(C_{n+1})-U(C_0)]} |C_0 \rangle \langle C_0| + e^{-(\beta/2)[U(C_0)-U(C_{n+1})]} |C_{n+1} \rangle \langle C_{n+1}|]
- |C_0 \rangle \langle C_{n+1}| - |C_{n+1} \rangle \langle C_0| \right]$$

(A.19)

where the renormalized amplitude reads

$$G_R(C_0, C_{n+1}) \simeq \frac{E_1 e^{-(\beta/2)[U(C_0)+U(C_{n+1})]}}{e^{-\beta U(C_0)} + e^{-\beta U(C_{n+1})}} = \frac{E_1 e^{(\beta/2)[U(C_0)+U(C_{n+1})]}}{e^{\beta U(C_0)} + e^{\beta U(C_{n+1})}}.$$ (A.20)

Using equation (A.16), the final formula for the renormalized amplitude reads

$$\frac{1}{G_R(C_0, C_{n+1})} = \frac{2 \cosh(\beta/2) [U(C_0) - U(C_{n+1})]}{E_1}$$

$$= \frac{e^{(\beta/2)[U(C_0)-U(C_{n+1})]}}{W(C_0 \to C_1)} \left[ 1 + \sum_{m=1}^{n} \prod_{i=1}^{m} \frac{W(C_i \to C_i-1)}{W(C_i \to C_i+1)} \right].$$

(A.21)

This formula is used to obtain equations (38) and (56) of the main text.

A.4. Example of application

Let us now consider the case where the transition rates $W(C \to C')$ of the effective one-dimensional problem of equations (A.4) and (A.5) satisfy the detailed balance form of equation (10). Then we may rewrite the products as

$$\prod_{i=1}^{m} \frac{W(C_i \to C_{i-1})}{W(C_i \to C_{i+1})} = \prod_{i=1}^{m} \left( \frac{G(C_i, C_{i-1}) e^{-(\beta/2)[U(C_{i-1})-U(C_i)]}}{G(C_i, C_{i+1}) e^{-(\beta/2)[U(C_{i+1})-U(C_i)]}} \right)$$

$$= \frac{G(C_0, C_1)}{G(C_m, C_{m+1})} e^{(\beta/2)[U(C_m)+U(C_{m+1})-U(C_0)-U(C_1)]}.$$ (A.22)
so that the renormalized amplitude of equation (A.21) reads
\[
\frac{1}{G_R(C_0, C_{n+1})} = e^{\beta/2[U(C_0) - U(C_{n+1})]} G(C_0, C_1) e^{-\beta/2[U(C_1) - U(C_0)]} 
\times \left[ 1 + \sum_{m=1}^{n} \frac{G(C_0, C_1)}{G(C_m, C_{m+1})} e^{\beta/2[U(C_m) + U(C_{m+1})] - U(C_0) - U(C_1)} \right] 
\]
\[
= e^{\beta/2[U(C_0) + U(C_{n+1})]} \sum_{m=0}^{n} \frac{e^{\beta/2[U(C_m) + U(C_{m+1})]}}{G(C_m, C_{m+1})}. 
\]  
(A.23)

This formula is used to obtain equations (75) and (84) of the main text.

References

[1] Niemeijer Th and van Leeuwen J M J, *Renormalization theories for Ising spin systems*, 1976 *Phase Transitions and Critical Phenomena* ed C Domb and M S Green (London: Academic) 30 (Berlin: Springer) 1982 Phys. Rep. 91 233

[2] Migdal A A, 1976 Sov. Phys.—JETP 42 743

[3] Kadanoff L P, 1976 Ann. Phys. 100 359

[4] Kaufman M and Griffiths R B, 1981 Phys. Rev. B 24 496

[5] Kaufman M and Griffiths R B, 1982 Phys. Rev. B 26 5022

[6] Kaufman M and Griffiths R B, 1984 Phys. Rev. B 30 244

[7] Derrida B, De Seze L and Itzykson C, 1983 J. Stat. Phys. 33 559

[8] Derrida B, Itzykson C and Luck J M, 1984 Commun. Math. Phys. 94 115

[9] Derrida B and Giacomin G, 2013 arXiv:1303.5971

[10] Jayaprakash C, Riedel E K and Wortis M, 1978 Phys. Rev. B 18 2244

[11] Kaufman M and Griffiths R B, 1984 Phys. Rev. B 29 2630

[12] Monthus C and Garel T, 2008 Phys. Rev. B 77 134416

[13] Monthus C and Garel T, 2008 Phys. Rev. B 80 134201

[14] Angles d’Auriac J Ch and Igloi F, 2008 Phys. Rev. E 87 022103

[15] Monthus C and Garel T, 2008 Phys. Rev. B 77 134416

[16] Angles d’Auriac J Ch and Igloi F, 2013 Phys. Rev. E 87 022103

[17] Cao M S and Machta J, 1993 Phys. Rev. B 48 3177

[18] Dayan I, Schwartz M and Young A P, 1993 J. Phys. A: Math. Gen. 26 3093

[19] see for instance Young A P and Stinchcombe R B, 1976 J. Phys. C: Solid State Phys. 9 4419

[20] Southern B W and Young A P, 1977 J. Phys. C: Solid State Phys. 10 2179

[21] Bray A J and Moore M A, 1984 J. Phys. C: Solid State Phys. 17 L463

[22] Gardner E, 1984 J. Physique 45 1575

[23] Banavar J R and Bray A J, 1987 Phys. Rev. B 35 8888

[24] Nifle M and Hilhorst H J, 1992 Phys. Rev. Lett. 68 2992

[25] Ney-Nilffe M and Hilhorst H J, 1993 Physica A 194 462

[26] Moore M A, Bokil H and Drossel B, 1998 Phys. Rev. Lett. 134 4252

[27] Aspelmeier T, Bray A J and Moore M A, 2002 Phys. Rev. Lett. 89 197202

[28] Riera R and Hertz J A, 1991 J. Phys. A: Math. Gen. 24 2625

[29] Ricci-Tersenghi F and Ritort F, 2000 J. Phys. A: Math. Gen. 33 3727

[30] Scheffler F, Yoshino H and Maass P, 2003 Phys. Rev. B 68 060404(R)

[31] Drossel B and Moore M A, 2004 Phys. Rev. B 70 064412

[32] Derrida B, Hakim V and Vannimenus J, 1992 J. Stat. Phys. 66 1189

[33] Tang L H and Chaté H, 2001 Phys. Rev. Lett. 86 830

[34] Monthus C and Garel T, 2008 Phys. Rev. E 77 021132

[35] Derrida B and Griffiths R B, 1989 Eur. Phys. Lett. 8 111

[36] Cook J and Derrida B, 1989 J. Stat. Phys. 57 89

doi:10.1088/1742-5468/2013/06/P06007
Dynamical barriers of pure and random ferromagnetic Ising models on fractal lattices

[23] Halpin-Healy T, 1989 Phys. Rev. Lett. 63 917
Halpin-Healy T, 1990 Phys. Rev. A 42 711
[24] Roux S, Hansen A, da Silva L R, Lucena L S and Pandey R B, 1991 J. Stat. Phys. 65 183
[25] Balents L and Kardar M, 1992 J. Stat. Phys. 67 1
Medina E and Kardar M, 1993 J. Stat. Phys. 71 967
[26] Cao M S, 1993 J. Stat. Phys. 71 51
Tang L H, 1994 J. Stat. Phys. 77 581
[27] Halpin-Healy T, 1990 Phys. Rev. A 42 711
[24] Roux S, Hansen A, da Silva L R, Lucena L S and Pandey R B, 1991 J. Stat. Phys. 65 183
[25] Balents L and Kardar M, 1992 J. Stat. Phys. 67 1
Medina E and Kardar M, 1993 J. Stat. Phys. 71 967
[26] Cao M S, 1993 J. Stat. Phys. 71 51
Tang L H, 1994 J. Stat. Phys. 77 581
[29] da Silveira R A and Bouchaud J P, 2004 Phys. Rev. Lett. 93 015901
[30] Monthus C and Garel T, 2013 J. Stat. Mech. P02037
Monthus C and Garel T, 2013 J. Stat. Mech. P02023
Monthus C and Garel T, 2013 J. Stat. Mech. P05012
[33] Gardiner C W, 1985 Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences (Springer Series in Synergetics) (Berlin: Springer)
[34] Van Kampen N G, 1992 Stochastic Processes in Physics and Chemistry (Amsterdam: Elsevier)
[35] Risken H, 1989 The Fokker–Planck Equation: Methods of Solutions and Applications (Berlin: Springer)
[36] Gardiner C W, 1985 Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences (Springer Series in Synergetics) (Berlin: Springer)
[34] Van Kampen N G, 1992 Stochastic Processes in Physics and Chemistry (Amsterdam: Elsevier)
[35] Risken H, 1989 The Fokker–Planck Equation: Methods of Solutions and Applications (Berlin: Springer)
[36] Glauber R J, 1963 J. Math. Phys. 4 294
[37] Roux S, Hansen A, da Silva L R, Lucena L S and Pandey R B, 1991 J. Stat. Phys. 65 183
[24] Roux S, Hansen A, da Silva L R, Lucena L S and Pandey R B, 1991 J. Stat. Phys. 65 183
[25] Balents L and Kardar M, 1992 J. Stat. Phys. 67 1
Medina E and Kardar M, 1993 J. Stat. Phys. 71 967
[26] Cao M S, 1993 J. Stat. Phys. 71 51
Tang L H, 1994 J. Stat. Phys. 77 581
[29] da Silveira R A and Bouchaud J P, 2004 Phys. Rev. Lett. 93 015901
[30] Monthus C and Garel T, 2013 J. Stat. Mech. P02037
Monthus C and Garel T, 2013 J. Stat. Mech. P02023
Monthus C and Garel T, 2013 J. Stat. Mech. P05012
[33] Gardiner C W, 1985 Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences (Springer Series in Synergetics) (Berlin: Springer)
[34] Van Kampen N G, 1992 Stochastic Processes in Physics and Chemistry (Amsterdam: Elsevier)
[35] Risken H, 1989 The Fokker–Planck Equation: Methods of Solutions and Applications (Berlin: Springer)
[36] Glauber R J, 1963 J. Math. Phys. 4 294
[37] Roux S, Hansen A, da Silva L R, Lucena L S and Pandey R B, 1991 J. Stat. Phys. 65 183
[24] Roux S, Hansen A, da Silva L R, Lucena L S and Pandey R B, 1991 J. Stat. Phys. 65 183
[25] Balents L and Kardar M, 1992 J. Stat. Phys. 67 1
Medina E and Kardar M, 1993 J. Stat. Phys. 71 967
[26] Cao M S, 1993 J. Stat. Phys. 71 51
Tang L H, 1994 J. Stat. Phys. 77 581
[29] da Silveira R A and Bouchaud J P, 2004 Phys. Rev. Lett. 93 015901
[30] Monthus C and Garel T, 2013 J. Stat. Mech. P02037
Monthus C and Garel T, 2013 J. Stat. Mech. P02023
Monthus C and Garel T, 2013 J. Stat. Mech. P05012
[33] Gardiner C W, 1985 Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences (Springer Series in Synergetics) (Berlin: Springer)
[34] Van Kampen N G, 1992 Stochastic Processes in Physics and Chemistry (Amsterdam: Elsevier)
[35] Risken H, 1989 The Fokker–Planck Equation: Methods of Solutions and Applications (Berlin: Springer)